Cheyette/Interest rate notes

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Some notes on the approach of Cheyette, this will connect with the wiener-chaos expansion approach by Funahashi to arrive at the approximation equation for the vanilla swaption price under Multi-factor Local Stochastic Volatility qG model

First we need to establish a couple things, such as the forward rate dynamic (in terms of the markovian variable, the centered short-rate x_t, then the bond reconstitution formula, and finally the SDE setup of the model. The material taken are from the Hoores, Funahashi-Kijima and Andersen-Piterbarg

1 Cheyette as the HJM representation with Seperable volatility

The starting term is the HJM setup:

$$df\left(t,T\right)=\sigma_f(t,T)\left(\int_t^T\sigma_f\left(t,s\right)ds\right)dt+\sigma_f\left(t,T\right)dW_t^Q \tag{1}$$

The volatility specification of the Cheyette is a general gaussian-short-rate model where it can be seperated into the instantaneous part $h\left(t\right)$ and the term-structure part $g\left(t,T\right)=\exp\left(-\int_t^T\kappa_u du\right)$

Some of the useful identities that will be used later is given here:

$\sigma_f\left(t,T\right)=h\left(t\right)g\left(t,T\right)=h\left(t\right)g\left(t,s\right)g\left(s,T\right)=\sigma_f(t,s)g\left(s,T\right)$

$\frac{\partial}{\partial T}\sigma_f\left(t,T\right)=-\kappa_T\sigma_f\left(t,T\right)$

The main motivation of this proof is to express the forward rate dynamic in terms of the markovian centered short-rate $x_t=r_t-f\left(0,t\right)=f\left(t,t\right)-f\left(0,t\right)$

1.1 Forward rate dynamic

This is the result from Hoores

$$\begin{aligned}
f\left(t,T\right)&=f\left(0,T\right)+\int_0^t\sigma_f\left(s,T\right)\left(\int_s^T\sigma_f\left(s,u\right)du\right)ds+\int_0^t\sigma_f\left(s,T\right)dW_s^Q\\
&=f\left(0,T\right)+g\left(t,T\right)\int_0^t\sigma_f\left(s,t\right)\left(\int_s^T\sigma_f\left(s,u\right)du\right)ds+g\left(t,T\right)\int_0^t\sigma_f\left(s,t\right)dW_s^Q\\
&=f\left(0,T\right)+g\left(t,T\right)\left(\int_0^t\sigma_f\left(s,t\right)\left(\int_s^T\sigma_f\left(s,u\right)du\right)ds+\int_0^t\sigma_f\left(s,t\right)dW_s^Q\right)\\
&=f\left(0,T\right)+g\left(t,T\right)\left(\int_0^t\sigma_f\left(s,t\right)\left(\int_s^t\sigma_f\left(s,u\right)du+\int_t^T\sigma_f\left(s,u\right)du\right)ds+\int_0^t\sigma_f\left(s,t\right)dW_s^Q\right)\\
&=f\left(0,T\right)+g\left(t,T\right)\left(x_t+\int_0^t\sigma_f\left(s,t\right)\left(\int_t^T\sigma_f\left(s,u\right)du\right)ds\right)\\
&=f\left(0,T\right)+g\left(t,T\right)\left(x_t+\int_0^t\sigma_f\left(s,t\right)\left(\int_t^T g\left(t,u\right)\sigma_f\left(s,t\right)du\right)ds\right)\\
&=f\left(0,T\right)+g\left(t,T\right)\left(x_t+\int_0^t\sigma_f^2\left(s,t\right)\left(\int_t^T g\left(t,u\right)du\right)ds\right)\\
&=f\left(0,T\right)+g\left(t,T\right)\left(x_t+\left(\int_t^T g\left(t,u\right)du\right)\int_0^t\sigma_f^2\left(s,t\right)ds\right)\\
&=f\left(0,T\right)+g\left(t,T\right)\left(x_t+B\left(t,T\right)y_t\right)
\end{aligned} \tag{2}$$

We have use the HJM representation for $x_t=f\left(t,t\right)-f\left(0,T\right)=\int_0^t\sigma_f\left(s,t\right)\left(\int_s^t\sigma_f\left(s,u\right)du\right)ds+\int_0^t\sigma_f\left(s,t\right)dW_s^Q$, and we have recovered the affine structure of the forward dynamic, where $B\left(t,T\right)=\int_t^T g\left(t,u\right)du$, and $y_t=\int_0^t\sigma_f^2\left(s,t\right)ds$, we can see that the state variables $\left(x_t,y_t\right)$ are the information up to time t. Where the function $g\left(t,T\right)$ and $B\left(t,T\right)$ contains information of the forward terms from time t to future tenor T.

Also just from the HJM representation we can work out the increment of the forward rate, this will be useful for the fixed tenor $f\left(t,t+\delta_i\right)$

$$
\begin{aligned}
df\left(t,T_i\right) &= \sigma_f(t,T_i)\left(\int_t^{T_i}\sigma_f\left(t,s\right)ds\right)dt+\sigma_f\left(t,T_i\right)dW_t^Q\\
&= h\left(t\right)g\left(t,T_i\right)\left(\int_t^{T_i}h\left(t\right)g\left(t,s\right)ds\right)dt+h\left(t\right)g\left(t,T_i\right)dW_t^Q\\
&= h^2\left(t\right)g\left(t,T_i\right)\left(\int_t^{T_i}g\left(t,s\right)ds\right)dt+h\left(t\right)g\left(t,T_i\right)dW_t^Q\\
&= h^2\left(t\right)g\left(t,T_i\right)B\left(t,T_i\right)dt+h\left(t\right)g\left(t,T_i\right)dW_t^Q
\end{aligned}
$$

1.2 Bond Price Reconstitution formula

As with other interest rate model, we can get the zero-coupon bond price $P\left(t,T\right)$ from the forward rate dynamic

$$
\begin{aligned}
P\left(t,T\right) &= \exp\left(-\int_t^T f\left(t,s\right)ds\right)\\
&= \exp\left(-\int_t^T \left(f\left(0,s\right)+g\left(t,s\right)\left(x_t+B\left(t,s\right)y_t\right)\right)ds\right)\\
&= \frac{P(0,T)}{P(0,t)}\exp\left(-\int_t^T \left(g\left(t,s\right)\left(x_t+B\left(t,s\right)y_t\right)\right)ds\right)\\
&= \frac{P(0,T)}{P(0,t)}\exp\left(-x_t\int_t^T g\left(t,s\right)ds-y_t\int_t^T g\left(t,s\right)B\left(t,s\right)ds\right)\\
&= \frac{P(0,T)}{P(0,t)}\exp\left(-B\left(t,T\right)x_t-\int_t^T g\left(t,s\right)B\left(t,s\right)ds\cdot y_t\right)\\
&= \frac{P(0,T)}{P(0,t)}\exp\left(-B\left(t,T\right)x_t-\int_t^T g\left(t,s\right)\left(\int_t^T g\left(t,u\right)du\right)ds\cdot y_t\right)\\
&= \frac{P(0,T)}{P(0,t)}\exp\left(-B\left(t,T\right)x_t-\frac{1}{2}\left(\int_t^T g\left(t,s\right)ds\right)^2\cdot y_t\right)\\
&= \frac{P(0,T)}{P(0,t)}\exp\left(-B\left(t,T\right)x_t-\frac{1}{2}B\left(t,T\right)^2y_t\right)
\end{aligned}
\tag{3}
$$

Here we use the following identity: $\int\left(u^2\right)'=\int 2u'u$, so $\int u'u=\frac{1}{2}u^2$, and set $u=\int g\left(t,\cdot\right)$

1.3 Centered Short-rate dynamic

As a way to simulate the qG model, we also need to get the SDE for the qG pair (x_t,y_t):

$$
\begin{aligned}
f\left(t,T\right) &= f\left(0,T\right)+\int_0^t\sigma_f\left(s,T\right)\left(\int_s^T\sigma_f\left(s,u\right)du\right)ds+\int_0^t\sigma_f\left(s,T\right)dW_s^Q\\
r_t=\lim_{T\downarrow t}f\left(t,T\right) &= f\left(0,t\right)+\int_0^t\sigma_f\left(s,t\right)\left(\int_s^t\sigma_f\left(s,u\right)du\right)ds+\int_0^t\sigma_f\left(s,t\right)dW_s^Q\\
dr_t=dx_t &= \sigma_f\left(t,t\right)\left(\int_t^t\sigma_f\left(t,u\right)du\right)dt+\int_0^t\frac{\partial}{\partial t}\left(\sigma_f\left(s,t\right)\left(\int_s^t\sigma_f\left(s,u\right)du\right)\right)ds\cdot dt+\sigma_f\left(t,t\right)dW_t^Q-\kappa_t\int_0^t\sigma_f\left(s,t\right)dW_s^Q\\
&= \int_0^t\frac{\partial}{\partial t}\left(\sigma_f\left(s,t\right)\left(\int_s^t\sigma_f\left(s,u\right)du\right)\right)ds\cdot dt+\sigma_f\left(t,t\right)dW_t^Q-\kappa_t\int_0^t\sigma_f\left(s,t\right)dW_s^Q\\
&= -\kappa_t\int_0^t\sigma_f\left(s,t\right)\left(\int_s^t\sigma_f\left(s,u\right)du\right)ds\cdot dt+\int_0^t\sigma_f\left(s,t\right)\sigma_f\left(s,t\right)ds\cdot dt+\sigma_f\left(t,t\right)dW_t^Q-\kappa_t\int_0^t\sigma_f\left(s,t\right)dW_s^Q\\
&= -\kappa_t\int_0^t\sigma_f\left(s,t\right)\left(\int_s^t\sigma_f\left(s,u\right)du\right)ds\cdot dt-\kappa_t\int_0^t\sigma_f\left(s,t\right)dW_s^Q+\int_0^t\sigma_f\left(s,t\right)\sigma_f\left(s,t\right)ds\cdot dt+\sigma_f\left(t,t\right)dW_t^Q\\
&= -\kappa_t x_t dt+\int_0^t\sigma_f\left(s,t\right)\sigma_f\left(s,t\right)ds\cdot dt+\sigma_f\left(t,t\right)dW_t^Q\\
&= -\kappa_t x_t dt+y_t dt+h\left(t\right)dW_t^Q\\
dx_t &= \left(y_t-\kappa_t x_t\right)dt+h\left(t\right)dW_t^Q
\end{aligned}
\tag{4}
$$

Now we will also differentiate to get he incremental of the y_t:

$$
\begin{aligned}
y_t &= \int_0^t\sigma_f^2\left(s,t\right)ds\\
dy_t &= \sigma_f^2\left(t,t\right)dt-2\kappa_t\left(\int_0^t\sigma_f^2\left(s,t\right)ds\right)dt\\
dy_t &= \sigma_f^2\left(t,t\right)dt-2\kappa_t\left(\int_0^t\sigma_f^2\left(s,t\right)ds\right)dt\\
dy_t &= h^2\left(t\right)dt-2\kappa_t y_t dt
\end{aligned}
$$

So we now have the SDE pair:

$$
\begin{aligned}
dx_t &= \left(y_t-\kappa_t x_t\right)dt+h\left(t\right)dW_t^Q\\
dy_t &= h^2\left(t\right)dt-2\kappa_t y_t dt
\end{aligned}
$$

And the full version:

$$
\begin{aligned}
x_t &= \int_0^t g\left(s,t\right)y_s ds+\int_0^t h dW_s^Q\\
y_t &= \int_0^t h^2\left(s\right)g^2\left(s,T\right)ds
\end{aligned}
$$

2 Approximation Schemes

2.1 Basic set up

The approach to the approximation for calibration is pretty similar for all models, is to find a robust way to calculate the vanilla swaption as that is the instrument that can calibrate the market (3 dimensions, the maturity, tenor and strikes). The end goal is similar, is to get to either the Black-Scholes, or for the case with stochastic volatility, the Black-Scholes volatility with Heston correction term.

The main ingredient as suggested in Andreasen (add source here), the state-variables SDEs with the Bond Price formula, along with the specification of the volatility term $h(t)$ is all it is needed for the model. In the most extensive case, this will be a multi-factor qGLSV model, where the correlation structure is incorporated directly into the local volatility term in matrix form, this is the idea of Andreasen (2002), and adopted by Funahashi-Kijima.

For basic model to local volatility model (qGLV), is to use Markovian Projection and Time-Averaging (a la Piterbarg) to project it to the Displaced-Diffusion Black-Scholes model with the effective skewness, effective volatility, then uses the Displaced Diffusion BS equation to price the swaption. For qG model with Stochastic Volatility (along with the local volatility), several procedures are needed, (i) first step is to change from risk-neutral measure (bond price) to lognormal swap measure, this allows us to employ Displaced-Diffusion BS (or Fourier pricing with stochastic volatility a la Lewis-Lipton). (ii) Since the swap-dynamic is a complicated dynamic (multi-factor, thus not exactly (quasi)Gaussian, involving shift/skew/correlation), we employ Markovian Projection to project it onto a zero-drift, displaced-diffusion with Heston-type volatility model, with time-dependent parameter. In original paper suggested by Andersen and Andreasen (2002), the direct numerical fourier transform (with known form given in Lewis-Lipton) to arrive at price. (iii) further step can be done here, to use the time-averaging technique proposed by Piterbarg to transform the time-dependent SDE into a constant SDE, here one of the parameter (the λ) has to be solved numerically by the Riccati ODE, now with the parameters are all in constant. One can employ the closed-form Lewis-Lipton formula to get the option price.

The Funahashi-Kijima scheme is slightly difference only in the treatment of (ii), they proposed a parametric form of the local volatility term of the lognormal swap rate dynamic, and use the Ito-Chaos Expansion to arrive at the approximation with high-degree of the proposed parametric parameters (time-dependent) in terms of the original forward-rate dynamic volatility specification (the asymptoptic expansion of them, expanding around the non-random function, and with higher terms in iterated stochastic integral and their conditional expectation). The parameters are then render as constant using the same time-averaging technique proposed by Piterbarg.

2.2 Change of Measure to Lognormal Swap Model

The argument is made that the swap rate is the differential two zero coupon bond dividend by the numeraire (annuity), such that in the swap measure it would be martingale. This would allow for much easier derivation since only the volatility part is needed, there are ways to model this. For e.g. in the Piterbarg (2003) model, which models the forward rates $F_j(t)$ as shifted-lognormal diffusion with Heston type volatility. Where in the Hoores and Kijima-Funahashi approach the centered-short-rate is the state variables, where for Hoores, the local volatility term is displaced-diffusion based on the short-rate r_t (see Hoores (4.13)), and the Kijima-Funahashi approach has the local volatility terms based on forward-rates

$$\begin{aligned} dS_{\alpha,\beta}(t) &= \frac{\partial S_{\alpha,\beta}(t)}{\partial x_t} dx_t \\ &= \frac{\partial S_{\alpha,\beta}(t)}{\partial x_t} h(t)\, dW_t^Q + \cdots dt \\ &= \frac{\partial S_{\alpha,\beta}(t)}{\partial x_t} h(t)\, dW_t^{\alpha,\beta} \end{aligned}$$

Where the partial derivative:

$$\begin{aligned} \frac{\partial S_{\alpha,\beta}(t)}{\partial x_t} &= \frac{\partial\left(\frac{P(t,T_\alpha)-P\left(t,T_\beta\right)}{A_{\alpha,\beta}(t)}\right)}{\partial x_t} \\ &= -\frac{B(t,T_\alpha)P(t,T_\alpha)-B(t,T_\beta)P(t,T_\beta)}{A_{\alpha,\beta}(t)} - \frac{P(t,T_\alpha)-P(t,T_\beta)}{A^2_{\alpha,\beta}(t)}\left(-\sum_{i=\alpha}^{\beta}\tau_i B(t,T_i)\,P(t,T_i)\right) \\ &= -\left(\frac{B(t,T_\alpha)P(t,T_\alpha)-B(t,T_\beta)P(t,T_\beta)}{A_{\alpha,\beta}(t)} - \frac{S_{\alpha,\beta}(t)\sum_{i=\alpha}^{\beta}\tau_i B(t,T_i)P(t,T_i)}{A_{\alpha,\beta}(t)}\right) \end{aligned} \tag{5}$$

Another way to represents:

$$\begin{aligned} &-\frac{B(t,T_\alpha)P(t,T_\alpha)-B(t,T_\beta)P(t,T_\beta)}{A_{\alpha,\beta}(t)} + \frac{P(t,T_\alpha)-P(t,T_\beta)}{A^2_{\alpha,\beta}(t)}\sum_{i=\alpha}^{\beta}\tau_i B(t,T_i)\,P(t,T_i) \\ &= -1\cdot\left(\frac{B(t,T_\alpha)P(t,T_\alpha)}{A_{\alpha,\beta}(t)} - \frac{P(t,T_\alpha)\sum_{i=\alpha}^{\beta}\tau_i B(t,T_i)P(t,T_i)}{A^2_{\alpha,\beta}(t)}\right) \\ &\quad +1\cdot\left(\frac{B(t,T_\beta)P(t,T_\beta)}{A_{\alpha,\beta}(t)} - \frac{P(t,T_\beta)\sum_{i=\alpha}^{\beta}\tau_i B(t,T_i)P(t,T_i)}{A^2_{\alpha,\beta}(t)}\right) \\ &= \sum_{j=\alpha,\beta} s(j)\left\{\frac{B(t,T_j)P(t,T_j)}{A_{\alpha,\beta}(t)} - \frac{P(t,T_j)\sum_{l=\alpha+1}^{\beta}\tau_l B(t,T_l)P(t,T_l)}{A^2_{\alpha,\beta}(t)}\right\} \end{aligned} \tag{6}$$

Where we use the bond formula $P(t,T) = \exp\left(-B(t,T)x_t - \frac{1}{2}B^2(t,T)y_t\right)$, $\frac{dP(t,T_i)}{dx_t} = -B(t,T_i)P(t,T)$, $\frac{dA_{\alpha,\beta}(t)}{dx_t} = \frac{d\sum_{i=\alpha}^{\beta}\tau_i P(t,T_i)}{dx_t} = -\sum_{i=\alpha}^{\beta}\tau_i B(t,T_i)P(t,T_i)$

3 Andreasen and Andersen (2002), Piterbarg (2003) approach

For the Andreasen and Andersen (2002), Piterbarg (2003) approach and Funahashi-Kijima both have the same starting point and forward volatility specification (see Andreasen (2006) eq 11(a)(b)), here we adapt the multi-factor model

$$dF_i(t) = g(t,t+\delta_i)^T\left(h(t)h(t)^T\right)B(t,t+\delta_i)\,dt + \left\{\Gamma(t)h(t)dW^Q(t)\right\}_{i^{th}row}$$

Since the forward dynamic is used for projecting to the swap rate dynamic mainly, we can ignore the drift term (since Swap rate is MTG under swap measure), and we further specify the volatility measure in multi-dimensional sense

$$\begin{aligned}
dF(t) &= \Gamma(t)h(t)dW^Q(t) \\
&= \sqrt{z(t,\omega_t)}\left(I_{m(t)}I_{F(t)+\nu} + \left(I - I_{m(t)}\right)I_{F(0)+\nu}\right)I_{\lambda(t)}R(t)\,dW^Q(t) \\
dz(t,\omega_t) &= \alpha(t)(\theta(t) - z(t))\,dt + \epsilon(t)\sqrt{z(t)}dZ(t) \\
dZ\cdot dW &= 0
\end{aligned}$$

Where

$$\Gamma(t) = \begin{pmatrix} e^{\int_t^{t+\delta_1}\kappa_1(s)ds} & \dots & e^{\int_t^{t+\delta_1}\kappa_d(s)ds} \\ \vdots & \ddots & \vdots \\ e^{\int_t^{t+\delta_d}\kappa_1(s)ds} & \dots & e^{\int_t^{t+\delta_d}\kappa_d(s)ds} \end{pmatrix}$$

and

$$h(t) = \sqrt{z(t,\omega_t)}\Gamma^{-1}(t)\left(I_{m(t)}I_{F(t)+\nu} + \left(I - I_{m(t)}\right)I_{F(0)+\nu}\right)I_{\lambda(t)}R(t)\,dW^Q(t)$$

The inverse of the reversion matrix Γ appears in the $h(t)$ here as a compensation for the original forward dynamic (one can compare the multi-dimensional case with (2) , both the centered-short-rate and swap-rate dynamic (as a function of x_t do not have this Γ (or $g(t,T)$) term in the SDE.

We can see the d-dimension decays, for e.g. the first tenor $t+\delta_i$ where the rows $\left\{e^{\int_t^{t+\delta_1}\kappa_1(s)ds}\dots e^{\int_t^{t+\delta_1}\kappa_d(s)ds}\right\}$are multiplied with the d-dimension Brownian motion $\left\{dW^1\cdots dW^d\right\}^T$to get the accumulated effects.

So now we can write the swap-rate as following:

$$dS_{\alpha,\beta}(t) = h(t)\frac{\partial S_{\alpha,\beta}(t)}{\partial x_t}dW_t^{\alpha,\beta}$$

As we can see previously, the dynamic of the equation above is highly complciated given the multi-dimension displacement, skewness, vol-of-vol. So Piterbarg employ the Markovian Projection, onto the following LSM:

$$\begin{cases} dS^*_{\alpha,\beta}(t) &= \sqrt{z(t)}\sigma_t\left\{b_t\left(S^*_{\alpha,\beta}(t)+c\right) + (1-b_t)\left(S^*_{\alpha,\beta}(0)+c\right)\right\}dW_t^{\alpha,\beta} \\ dz^*(t) &= \alpha(t)(\theta(t) - z^*(t))\,dt + \epsilon(t)\sqrt{z^*(t)}dZ(t) \end{cases}$$

Where

$$W$$

4 Funahashi-Kijima Approach