Quadratic Gaussian Models

For CMS Spread Options

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1 Quadratic Gaussian Models

- If a short rate is a quadratic form of a multi-dim linear Gaussian process, then bonds are exponentials of quadratic forms of the same
- Define, under risk-neutral measure,

$$dZ(t) = \{-M(t) Z(t)\ dt + \}\Sigma(t)\ dW(t),$$

where $Z(t)$ is $N \times 1$, $M(t)$ is $N \times N$, $\Sigma(t)$ is $N \times N$, and $W(t)$ is $N \times 1$. Further define

$$r(t) = Z(t)^\top \Gamma(t) Z(t) + b(t)^\top Z(t) + a(t)$$

Here $\Gamma(t)$ is $N \times N$, $b(t)$ is $N \times 1$ are model inputs, and $a(t)$ is a scalar to fit the initial yield curve. Then

$$\begin{aligned} -\log P(t,T) = Z(t)^\top \Gamma(t,T) Z(t) + b(t,T)^\top Z(t) \\ + \alpha(t,T) - \log P(0,t,T), \end{aligned}$$

where $\Gamma(t,T)$, $l(t,T)$, $\alpha(t,T)$ are obtained by solving ODEs.
- The model is Markovian in N state variables

2 Exotics Model

- Potentially an attractive choice for exotic interest rate derivatives
- Rich multi-factor dynamics
- Ability to generate and control volatility smile (quadratic term)
- Much faster to simulate than Libor market models
  - Gaussian state vector could be simulated with arbitrarily large steps with little effort, bonds have closed-form formulas

3 Tools: Riccati

- For the bond reconstruction formulas, we have

$$\begin{aligned} -\Gamma'(t,T) + 2\Gamma(t,T)\Sigma\Sigma^\top\Gamma(t,T) &= \Gamma(t) \\ -b'(t,T) + 2\Gamma(t,T)\Sigma\Sigma^\top b(t,T) &= b(t) \end{aligned} \tag{1}$$

- The scalar (although also satisfies an equation) is best obtained from the no-arb condition,

$$\mathsf{E}_0^t P(t,T) = P(0,t,T),$$

which implies

$$\alpha(t,T) = \log \mathsf{E}_0^t \exp\left(-\left(Z(t)^\top \Gamma(t,T) Z(t) + b(t,T)^\top Z(t)\right)\right) \tag{2}$$

4 Tools: Measure Changes

- Need to know E, Var of Z under forward measures (eg to compute the scalar for bonds)
- We have

  $$dP(t,T)/P(t,T) = -\left(2Z(t)^\top \Gamma(t,T) + b(t,T)^\top\right)\Sigma(t)\,dW(t) + \ldots,$$

  so

  $$dW^T(t) = dW(t) + \Sigma(t)^\top (2\Gamma(t,T)Z(t) + b)\,dt$$

  is a BM under P^T.
- Hence

  $$dZ(t) = -\Sigma(t)\Sigma(t)^\top (2\Gamma(t,T)Z(t) + b(t,T))\,dt + \Sigma(t)\,dW^T(t)$$

- Linear SDE, so
  - Z is Gaussian under any forward measure
  - E, Var are obtained by standard formulas

5 Swaption Pricing: Basics

- Fast swaption pricing formula is key to efficient calibration. Define swap rate, annuity

$$S(t) = \frac{P(t, T_1) - P(t, T_M)}{A(t)}, \quad A(t) = \sum_{m=1}^{M-1} \delta_m P(t, T_{m+1}).$$

- Need $\mathsf{E}^A (S(T) - K)^+$, ie 1d distribution of $S(T)$ only
- Distribution of $Z(T)$ under P^A? Gaussian mixture: for any ψ,

$$\mathsf{E}^A (\psi(Z(T))) = \sum p_m \mathsf{E}^{T_{m+1}} (\psi(Z(T))), \quad p_m = \delta_m P(0, T_{m+1}) / A(0).$$

- In principle, one-step Monte-Carlo in $N+1$ dimensions for Gaussian mixture is fast
- Can get faster by
  - simplifying distribution of $Z(T)$ under P^A
  - simplifying $S = S(Z)$

6 Swaption Pricing: Quadratic Approximation

- Quadratic model. Short rate quadratic in Z. Naturally: swap rate approx. quadratic in Z (all at time T)

$$S(Z) \approx Z^{\top}\Gamma_S Z + b_S^{\top} Z + a_S.$$

- Find Γ_S, b_S by numerical approximation to $S(Z)$ around $Z = 0$ (or $Z = \mathsf{E}^A Z(T)$)
- Find a_S from no-arbitrage (major advantage of using swap measure):

$$a_S = S(0) - \mathsf{E}^A\left(Z^{\top}\Gamma_S Z + b_S^{\top} Z\right).$$

- Curvature ($\Gamma_S \neq 0$) a function of two sources:
  - Non-linearity of S wrt factors Z
  - Quadratic terms in the model
- Even in the linear model, S would be approximated by a quadratic function, ie a "better" approximation than just linearize S (which would be a poor approximation)

7 Swaption Pricing: Fourier Methods

- Also possible:

$$\mathsf{E}\left(S_0 + Z^\top QZ + u^\top Z - \mathsf{E}\left(Z^\top QZ + u^\top Z\right) - K\right)^+ = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{(\lambda - i\omega)(S_0 - K)}}{(\lambda - i\omega)^2} F(\lambda - i\omega)\, d\omega,$$

where

$$\begin{aligned}\log F(\xi) &= \frac{1}{2}\xi^2 \left(2m^\top Q + u^\top\right)\left(V^{-1} - 2\xi Q\right)^{-1}(2Qm + u) \\ &\quad -\frac{1}{2}\log\det(I - 2\xi QV) - \xi \mathrm{tr}(QV)\end{aligned}$$

for $Z \sim \mathcal{N}(m, V)$.

- 1d example $(m = 0,\ u = 0)$

$$\mathsf{E}e^{\xi\left(qZ^2\right)} = \frac{1}{\sqrt{2\pi V}}\int e^{\xi q z^2} e^{-z^2/(2V)} dz = \frac{1}{\sqrt{2\pi V}}\int e^{-z^2(1/(2V) - \xi q)} dz = \sqrt{\frac{V^*}{V}}$$

where $V^* = 1/\left(2\left(1/(2V) - \xi q\right)\right) = V/(1 - 2\xi qV)$.

8 Spread Option Pricing

- Two swap rates, S_1 and S_2, of different tenors
- Spread option pays $(S_1(T) - S_2(T) - K)^+$ at $T_p > T$
- Valuation convenient under T_p-forward measure:

$$V = P(0, T_p)\mathsf{E}^{T_p}\left((S_1(T) - S_2(T) - K)^+\right)$$

- !!! Under quadratic approximation to swap rates spread option valuation is as easy as swaptions
- Assume (all at time T) $\quad S_i(Z) \approx Z^\top \Gamma_{S_i} Z + b_{S_i}^\top Z + a_{S_i}, \quad i = 1, 2.$
- Then

$$V = P(0, T_p)\mathsf{E}^{T_p}\left(Z^\top\left(\Gamma_{S_1} - \Gamma_{S_2}\right)Z + \left(b_{S_1} - b_{S_2}\right)^\top Z + \left(a_{S_1} - a_{S_2}\right) - K\right)^+$$

  and the distribution of Z under P^{T_p} is Gaussian with known moments
- So we can apply e.g. Fourier integration
- Availability of simple valuation formulas for spread option pricing facilitates their inclusion in the calibration set

9 Suitable for Spread Options?

- While it is easy to price CMS spread options in quadratic models, is our parametrization suitable?
- In [Pit09] we used "single-factor stochastic volatility" analogy to parametrize the model.
  - Allows us to control swaption smiles
  - No separate control over spread option smile
- Yet the (general) QG model seems to have enough degrees of freedom to allow for independent smile control of the swap rates **and** the spread between them
- What is the most suitable parametrization of QG model for exotics on spread options (e.g. CMS spread callable snowballs, in case anybody ever wants these things again)
- Start with a vanilla (i.e. non term structure) quadratic model

10 Step Back: How is Smile Generated?

- Consider $Z^\top \Gamma Z$ for some Gaussian Z with variance V. Then $Z = V^{1/2}X$ for $X \sim \mathcal{N}(0, I)$, and

$$Z^\top \Gamma Z = X^\top \tilde{\Gamma} X$$

for $\tilde{\Gamma} = \left(V^{1/2}\right)^\top \Gamma V^{1/2}$.

- Moreover there exists an orthogonal O such that $\tilde{\Gamma} = O\hat{\Gamma}O^\top$ where $\hat{\Gamma}$ is **diagonal**. Define $Y = O^\top X$, so that

$$Z^\top \Gamma Z = Y^\top \hat{\Gamma} Y = \sum \hat{\Gamma}_{i,i} Y_i^2,$$

where

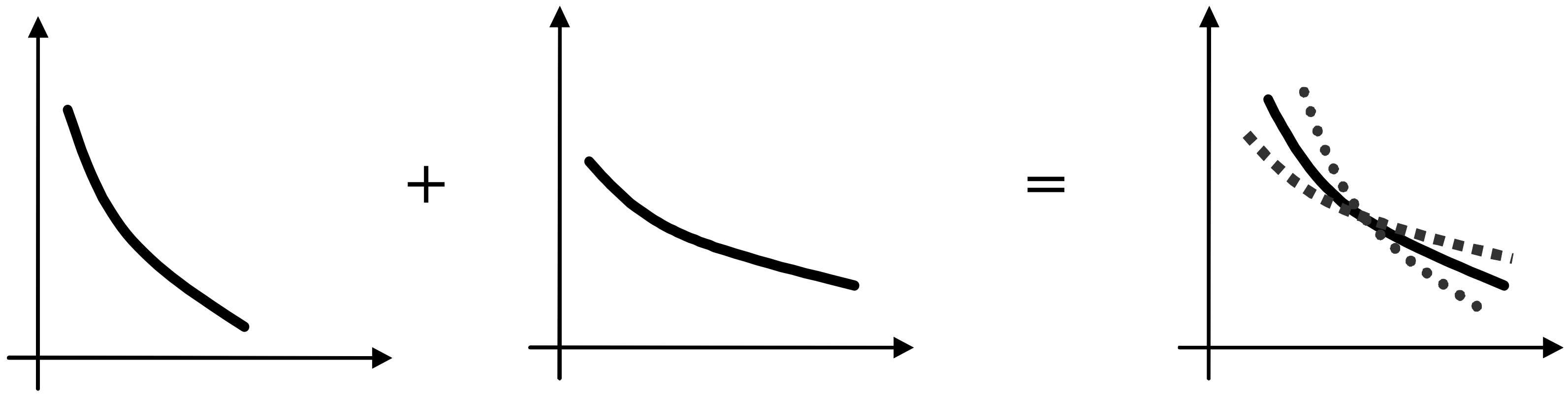
- $\hat{\Gamma}$ is diagonal
- Y is still $\mathcal{N}(0, I)$ (orthogonal transformation maps standard Gaussian variables into standard Gaussian variables)
- Hence, without loss of generality, we can consider quadratic terms that are sums of squares of independent Gaussian variables

11 Step Back: How is Smile Generated?

- Consider a 1d mapping function
$$\kappa(z;q) = z + q\left(z^2 - 1\right)/2$$
and a random variable $\kappa(Z;q)$ where Z is now a 1d standard Gaussian. What can we say about its distribution?
- We recall that the displaced log-normal SDE
$$dX_t = (1 + qX_t)\,dW_t, \quad X_0 = 0,$$
has a solution
$$X_t = \left(e^{qW_t - q^2 t/2} - 1\right)/q.$$
- Using Taylor expansion
$$\left(e^{qz} - 1\right)/q \approx z + qz^2/2, \quad X_t \approx \kappa(W_t;q).$$
- Therefore, the distribution of $\kappa(Z;q)$ is approximately displaced log-normal with the skew parameter q!
- Let us use $\kappa(Z;q)$ as building blocks for our distributions. Call the distribution QM ("quadratic mapping")

12 Sums of QMs

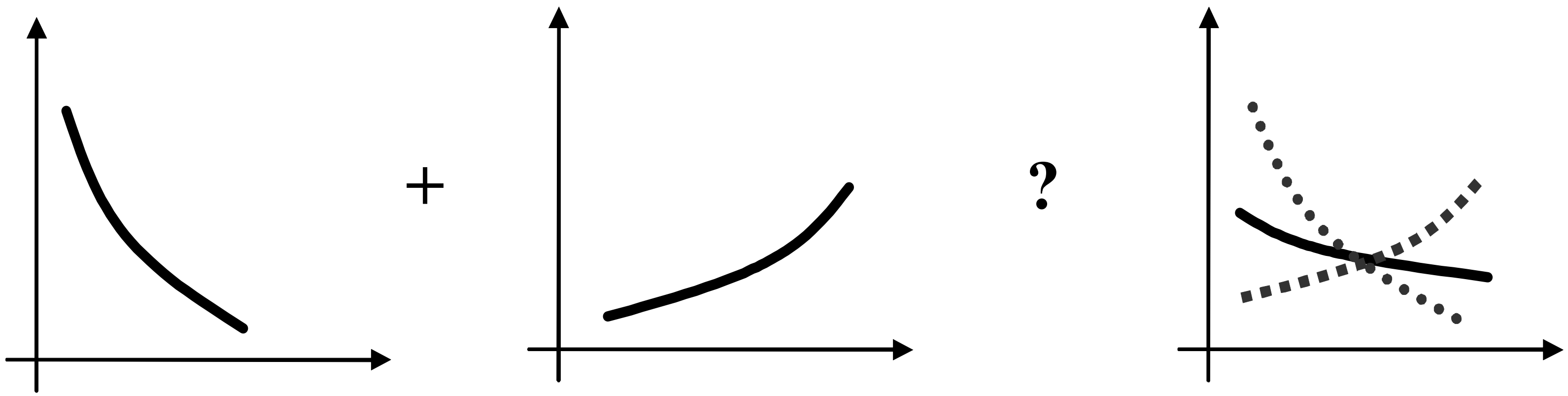
- Let Z_1, Z_2 be two independent standard Gaussians. What is the distribution of $\kappa(Z_1; q_1) + \kappa(Z_2; q_2)$?
- Skews have the same sign:



- Result is a skewed distribution with the skew of the same sign, which is some sort of average obtained by e.g. linear approximation [Pit07] or third-order moment (skewness) matching

13 Sums of QMs

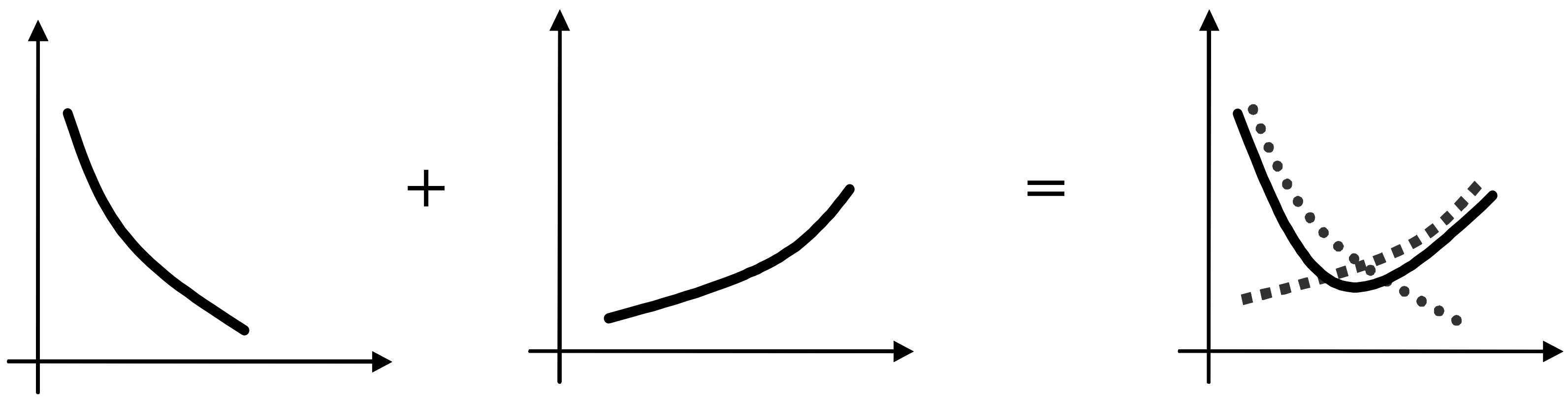
- Skews have different signs. Is it this?



- No! Sum of skewed distributions of different signs is not a skewed distribution

14 Sums of QMs

- In fact sum of skewed distributions of different signs gives rise to a genuine smile:



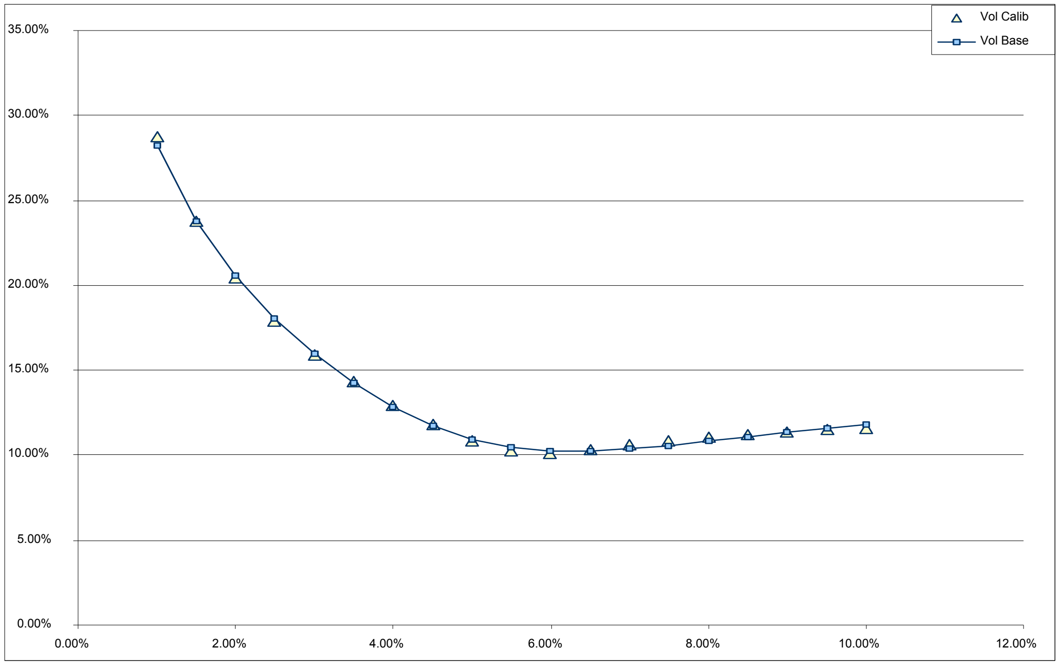
- Linear approximation no longer works
- However: a 4-parameter family of distributions

$$\kappa(\sigma_1 Z_1; q_1) + \kappa(\sigma_2 Z_2; q_2),$$

where normally $q_1 < 0 < q_2$, provides a flexible parametrization for volatility smiles

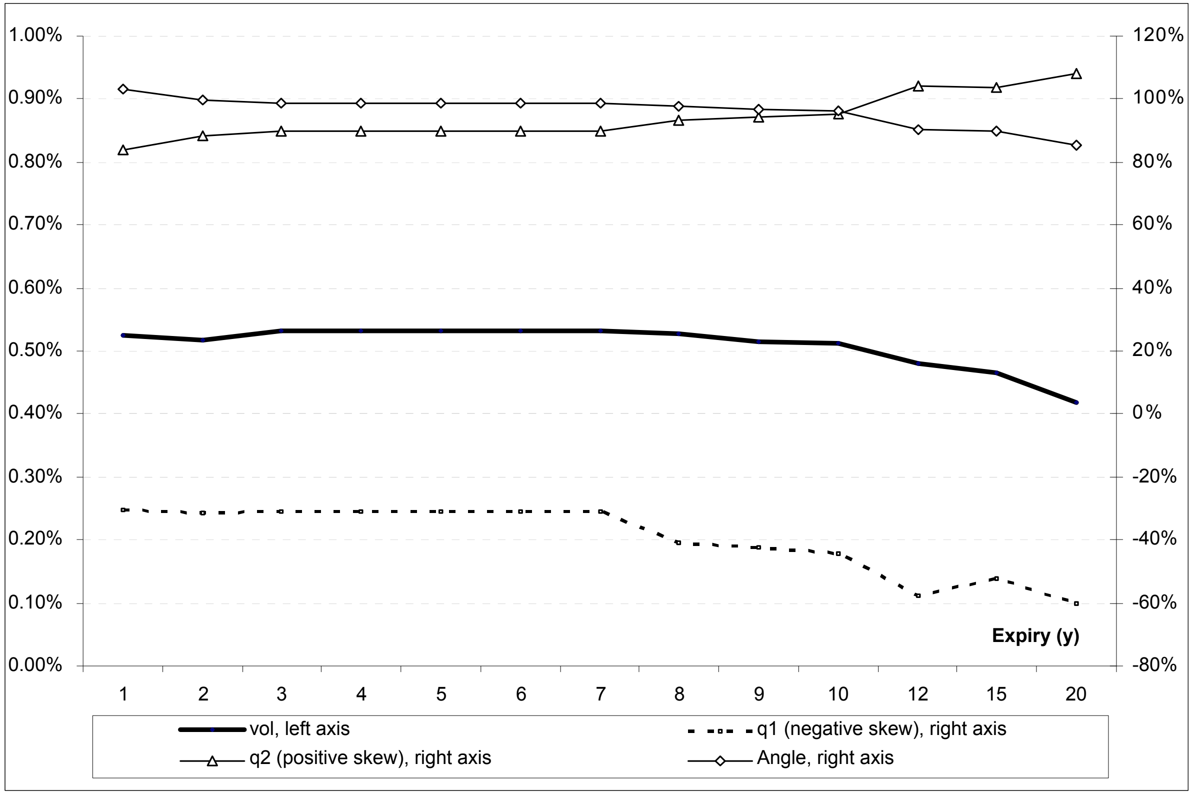
15 IQM2, Sample Fit

- Call this an IQM2 family ("independent quadratic mapping with 2 terms")
  - q_1: left tail up/down, q_2: right tail up/down, $(\sigma_1^2 + \sigma_2^2)^{1/2}$ overall level, and $\alpha = \arctan(\sigma_2/\sigma_1)$ tilt left/right
- Here is a fit to market smile for a 5y30y swaption with IQM2. $q_1 = -0.37$, $q_2 = 0.9$, $\sigma = 0.53\%$, $\alpha = 1.15$



16 IQM2, Fit Across Expiries

- And here are parameters as functions of expiry for a 30y tenor swaptions



17 Spread Options

- Suppose we use IQM2 for S_1 and S_2:

$$S_i = \kappa(\sigma_{i,1} Z_{i,1}; q_{i,1}) + \kappa(\sigma_{i,2} Z_{i,2}; q_{i,2}), \quad i = 1, 2.$$

- Then the spread $S_1 - S_2$ will have *four* QM terms. Spread distribution would be controlled by correlations between $Z_{1,j}$ and $Z_{2,j}$?

- Let us stay with independent QMs. Start with 4 independent Gaussians X_j, and 4 skews β_i, $i = 1, \ldots, 4$. Then

$$S_i = \sum_{j=1}^{4} \theta_{i,j} \kappa(X_j; \beta_j), \quad i = 1, 2,$$

  and the spread is of the same type:

$$S_1 - S_2 = \sum_{j=1}^{4} (\theta_{1,j} - \theta_{2,j})\, \kappa(X_j; \beta_j).$$

- Call this *IQM4 distribution*.

18 IQ4

- IQ4 distribution (2-dim) has a total of 12 parameters (4 β's, 4 θ_1's, 4 θ_2's). A lot?
- Not really: 4 for first marginal (IQM2), 4 for second marginal (IQM2), and 4 for the spread distribution. Have enough flexibility to match spread smile – but not more.
- Problem. Given
  - IQM2 parameters for both marginals S_1 and S_2
  - Market spread option volatilities (or correlations) across strikes
- Find
  - 12 IQM4 parameters to describe the 2-dim distribution of (S_1, S_2) consistently with market info
    * Options on S_1, S_2
    * Options on spread $S_1 - S_2$

19 IQ4

- Recall

$$S_i = \sum_{j=1}^{4} \theta_{i,j} \kappa(X_j; \beta_j),, \quad i = 1, 2.$$

Let $\beta_1, \beta_2 < 0$ and $\beta_3, \beta_4 > 0$. Observation: We can match the distribution of $\sum_{j=1}^{4} \theta_j \kappa(X_j; \beta_j)$ to IQM2$(q_1, q_2, \sigma, \alpha)$ closely if we match the two tails *separately*:

$$\begin{aligned} \theta_1 \kappa(X_1; \beta_1) + \theta_2 \kappa(X_2; \beta_2) &\stackrel{d}{=} \kappa(\sigma_1 Z_1; q_1), \qquad (3) \\ \theta_3 \kappa(X_3; \beta_3) + \theta_4 \kappa(X_4; \beta_4) &\stackrel{d}{=} \kappa(\sigma_2 Z_2; q_2). \end{aligned}$$

- As both equations involve skews of the same sign, linear-approximation type methods are accurate!
- In fact we moment-match second and third moments to link IQM4 and IQM2 parameters

20 IQ4, Calibration Algorithm

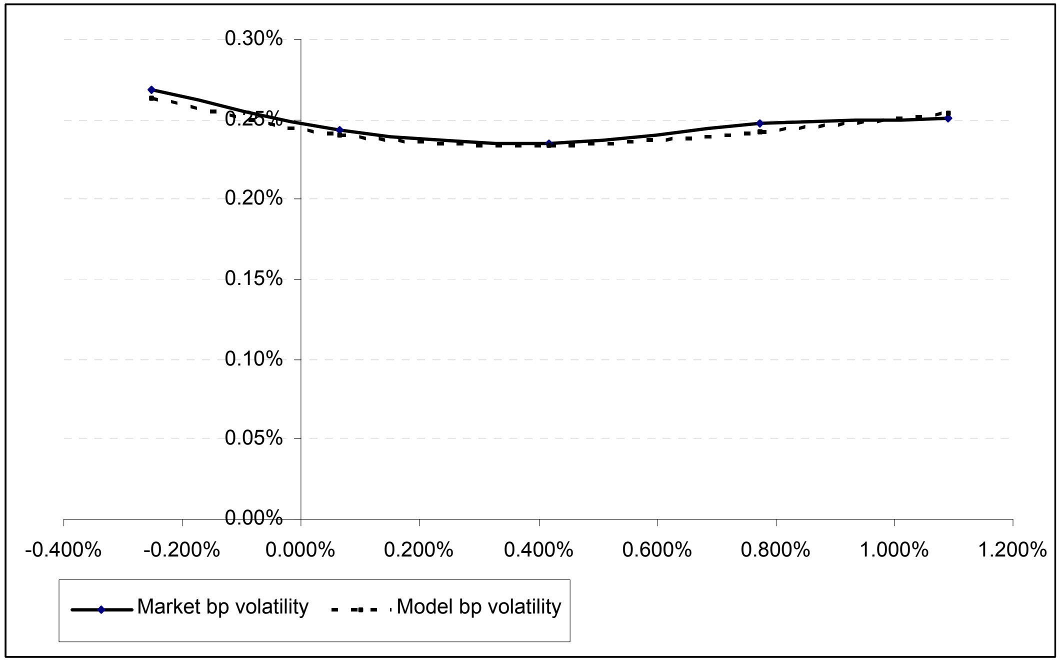
- Algorithm

  1. Iterate over $\arctan(\theta_{1,2}/\theta_{1,1})$, $\arctan(\theta_{1,4}/\theta_{1,3})$, $\arctan(\theta_{2,2}/\theta_{2,1})$, $\arctan(\theta_{2,4}/\theta_{2,3})$
  2. Find $\beta_1, \beta_2, \theta_{1,1}, \theta_{2,1}$ by moment-matching left (negative) tails of IQM2 distributions of both swap rates per (3) (two moments per swap rate so 4 equations)
  3. Find $\beta_3, \beta_4, \theta_{1,3}, \theta_{2,3}$ by moment-matching right (positive) tails of IQM2 distributions of both swap rates per (3) (four equations)
  4. All 12 parameters of IQM4 distribution are now given. Value spread options (using Fourier integration) across strikes
  5. Compare model prices of spread options to market. Adjust inputs and repeat

- The algorithm matches the market smile of the spread while keeping marginals calibrated

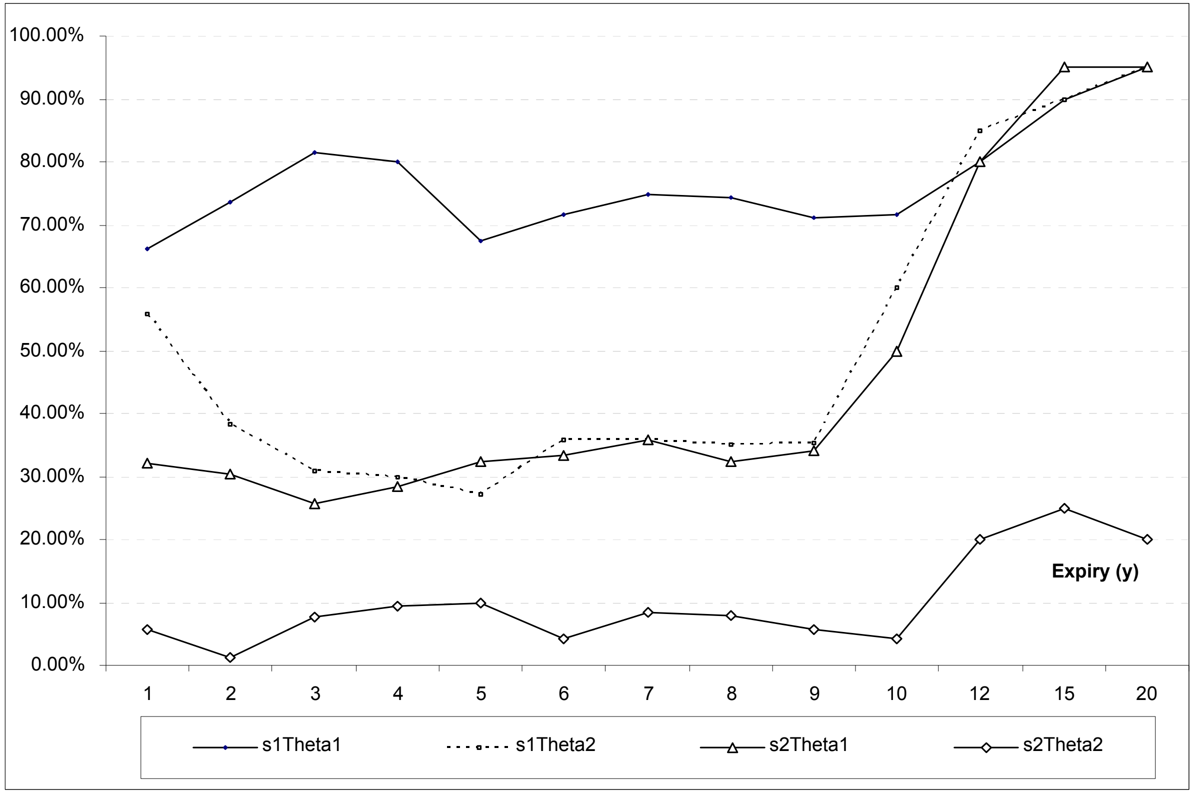
21 IQ4, Sample Fit

- Here is an example, a volatility smile for 30y2y CMS spread in 5 years time



22 IQ4, Fit Across Expiries

- And here are parameters for the 30y2y pair with different expiries



23 Back to Term Structure Models

- Recap: Have a 2-dim distribution
  - Built of squares of Gaussians
  - Flexible specification for marginals
  - Ability to control the smile of the spread independently of the marginals
- A good basis for a term structure model
- Let us come up with a QG parametrization suitable for exotics on spread options
- Minimalist linear-Gaussian model for spread options needs at least two factors:
- Start with $x(t) = (x_1(t), x_2(t))^\top$, where

$$dx_i(t) = -a_i x_i(t)dt + O(dW_i(t)), \quad i = 1, 2,$$

and

$$r(t) = 1^\top x(t) + O(t).$$

- Choose two benchmark tenors τ_1 and τ_2, define $f(t) = (f(t, t+\tau_1), f(t, t+\tau_2))^\top$. Then

$$f(t) = Bx(t) + O(t)$$

where

$$B = \begin{pmatrix} e^{-a_1\tau_1} & e^{-a_2\tau_1} \\ e^{-a_1\tau_2} & e^{-a_2\tau_2} \end{pmatrix}.$$

- In particular

$$r(t) = 1^\top B^{-1} f(t) + O(t),$$

and, parametrizing the diffusion term with $\lambda_i(t)$, $i = 1, 2$, inst fwd rate volatilities, and inst fwd rate correlation $\rho(t)$ (a-la LMM), we get

$$df_i(t) = -(M f(t))_i dt + \lambda_i(t) dW_i(t), \quad i = 1, 2, \tag{4}$$

where $M = B\,\mathrm{diag}(a_1, a_2)\,B^{-1}$ and $\langle dW_1(t), dW_2(t)\rangle = \rho(t)$.

- We can calibrate the model to ATM swaptions of tenors τ_1 and τ_2, and spread options on their difference, for each expiry (two swaption columns + spread options)

25 QG model for Spread Options

- Main idea: take LG model for spread options and replace

$$
\begin{aligned}
f_1(t) &\rightarrow \kappa(z_{1,1}(t);\beta_{1,1}(t)) + \kappa(z_{1,2}(t);\beta_{1,2}(t)),\\
f_2(t) &\rightarrow \kappa(z_{2,1}(t);\beta_{2,1}(t)) + \kappa(z_{2,2}(t);\beta_{2,2}(t)),
\end{aligned}
$$

  where $z_{1,1}$, $z_{1,2}$ follow the same dynamics as f_1 and $z_{2,1}$, $z_{2,2}$ the same as f_2. Defining $z(t) = (z_{1,1}, z_{2,1}, z_{1,2}, z_{2,2})^\top$, we have

$$
dz(t) = -\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} z(t)dt + \mathrm{diag}(\lambda_1(t), \lambda_2(t), \lambda_1(t), \lambda_2(t))d\tilde{W}(t)
$$

  and $d\tilde{W}$ is a 4-dim (correlated) BM. The short rate is

$$
r(t) = 1^\top B^{-1} \begin{pmatrix} \kappa(z_{1,1}(t);\beta_{1,1}(t)) + \kappa(z_{1,2}(t);\beta_{1,2}(t)) \\ \kappa(z_{2,1}(t);\beta_{2,1}(t)) + \kappa(z_{2,2}(t);\beta_{2,2}(t)) \end{pmatrix} f(t) + O(t).
$$

- We replace a Gaussian distribution for each f_i with an IQM2 distribution so can control distribution tails
- Each rate or spread is a type of an IQM4 distribution
- When β's are set to zero we recover (4) (up to rescaling of volatilities)

- We still use λ's and ρ to match ATM swaption and spread option vols. But now have other "knobs" to deal with smiles in swaptions and spread options
  - Skews (quadratic terms) $\beta_{i,j}(t)$'s
  - Correlations $\left\langle d\tilde{W}_i, d\tilde{W}_j \right\rangle$, $i, j = 1, \ldots 4$. These can be interpreted as "tail correlations" i.e. positive tail of f_1 vs negative tail of f_2.
- Calibration outline:
  - Choose "smile" parameters
  - Bootstrap-calibrate to ATM swaptions and ATM spread options for a given pair of swap rates S_1, S_2 for all expiries
  - Iterate over smile parameters
- Enough parameters to calibrate to three smiles
- Reasonably intuitive parametrization
- !! Can calibrate on parameters (IQM2/4) and not option values

27 Future Work

- Plenty of scope for future work:
  - Better understanding of impact of various parameters
  - More efficient calibration strategies, e.g. separation of spread calibration from marginals calibration?
  - Smile dynamics including spread smile dynamics
  - Impact on exotics

References

[Pit07] Vladimir V. Piterbarg. Markovian projection for volatility calibration. *Risk Magazine*, 20(4):84–89, April 2007.

[Pit09] Vladimir V Piterbarg. Rates squared. *Risk*, 22(1):100–105, January 2009.