Roughening Heston

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Abstract

Rough volatility models are known to fit the volatility surface remarkably well with very few parameters. On the other hand, the classical Heston model is highly tractable allowing for fast calibration. We present here the rough Heston model which offers the best of both worlds. Even better, we find that we can accurately approximate rough Heston model values by scaling the volatility of volatility parameter of the classical Heston model.

1 Introduction

Rough volatility models have succeeded in capturing the imagination of both practitioners and academics with their remarkable ability to consistently model both historical and implied volatilities with very few parameters. However, even with the introduction of the efficient hybrid BSS scheme of [BLP17], practical implementation has proved to be difficult. In the recent advances [EEFR18, EER18, EER19], the authors show how a natural rough generalization of the Heston model emerges as the macroscopic limit of a simple high frequency trading model which reflects the persistence of order flow, the high degree of endogeneity of the market, and liquidity asymmetry between bid and ask sides of the limit order book. In addition, they derive the characteristic function of the log-price as well as hedging strategies in this model.

In this note, we present the rough Heston model and explain how to use it in practice. As in the rough Bergomi model of [BFG16], the forward variance curve $\xi_t(u) = \mathbb{E}\left[V_u \middle| \mathcal{F}_t\right]$, where V_u is the spot variance at time u, is a state variable so that the model can be made to match at-the-money volatilities exactly. We are left with only three parameters to calibrate, defined in Section 2: the Hurst exponent H, the volatility of volatility ν and the correlation ρ between spot moves and volatility moves.[1]

Taking the parameter H as an example, (historical) H may be estimated from scaling properties of historical volatility estimated from the time series of high-frequency asset log-returns. On the other hand (implied) H may be estimated from implied volatilities at a fixed time by looking at the scaling of the short-dated implied volatility skew with time to expiration. The two values of H thus obtained are consistently small, typically of the order of 0.1. We can analyze the two other parameters, ρ and ν similarly finding that both of these are consistent between the time series of log-returns and the implied volatility surface on a given date. Moreover as is shown in [BFG16], the forward variance curve on a given date is consistent with the historical time series. Finally, consistency of rough volatility models with the dynamics of implied volatilities is formally proved in [Fuk17].

2 The rough Heston model

The rough Heston model of [EER19] for a one-dimensional asset price S takes the form

$$\frac{dS_t}{S_t} = \sqrt{V_t}\left\{\rho\, dB_t + \sqrt{1-\rho^2}\, dB_t^{\perp}\right\}$$

with

$$V_u = V_t + \frac{\lambda}{\Gamma(H+\frac{1}{2})}\int_t^u \frac{\theta^t(s) - V_s}{(u-s)^{\frac{1}{2}-H}}\, ds + \frac{\nu}{\Gamma(H+\frac{1}{2})}\int_t^u \frac{\sqrt{V_s}}{(u-s)^{\frac{1}{2}-H}}\, dB_s, \quad u \geq t, \tag{1}$$

where $H \in (0, 1/2)$ is the Hurst exponent, ν is the volatility of volatility, $\rho \in [-1, 1]$ is the correlation between spot and volatility moves, $\lambda \geq 0$ and Γ denotes the Gamma function. The mean reversion level parameter $\theta^t(\cdot)$ is allowed to be an $\mathcal{F}_t$-measurable function which makes the model time consistent as explained in [EER18]. It is straightforward to verify that Equation (1) gives back the classical Heston model with time-dependent mean reversion level in the limit $H \to 1/2$. From [EER19], volatility sample paths have Hölder regularity $H - \varepsilon$ for any $\varepsilon > 0$, hence the name *rough Heston* for this model.

It is also shown in [EER18] that $\lambda\, \theta^t(\cdot)$ can be directly inferred from the forward variance curve $(\xi_t(u))_{u\geq t}$ observed at time t. By doing so, the model may be rewritten in the asymptotic setting $\lambda \to 0$ in forward variance form as

$$\frac{dS_t}{S_t} = \sqrt{V_t}\left\{\rho\, dB_t + \sqrt{1-\rho^2}\, dB_t^{\perp}\right\}$$

with

$$V_u = \xi_t(u) + \frac{\nu}{\Gamma(H+\frac{1}{2})}\int_t^u \frac{\sqrt{V_s}}{(u-s)^{\frac{1}{2}-H}}\, dB_s, \quad u \geq t. \tag{2}$$

Remark 2.1. *Recall that the forward variance curve may in principle be obtained from the variance swap curve by differentiation. More practically, assuming continuous sample paths, it is well-known that the fair value of variance swaps can be obtained from an infinite log-strip of out-of-the-money options, see for example [Gat06].*

[1] From the rough volatility perspective, smiles flatten consistently with the scaling properties of historical volatility and as discussed in [GJR18], there is no need for an extra mean reversion parameter.

3 Pricing and hedging

Just as in the classical case, we can compute in quasi-closed form the characteristic function of the log-price of the stock in the rough Heston model. This makes the model highly tractable and easy to calibrate.

We define the fractional integral of order $r \in (0,1]$ of a function f as

$$I^r f(t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f(s) ds,$$

whenever the integral exists, and its fractional derivative of order $r \in [0,1)$ as

$$D^r f(t) = \frac{1}{\Gamma(1-r)} \frac{d}{dt} \int_0^t (t-s)^{-r} f(s) ds,$$

whenever it exists. From [EER18], the rough Heston model is Markovian in $X_t = \log(S_t)$ and the forward variance curve $(\xi_t(u))_{u \geq 0}$, in the same spirit as the Bergomi model [Ber05]. In forward-variance form, the characteristic function of the terminal log-spot X_T conditional on the time t initial state (X_t, ξ_t) is given by

$$\phi_t(T,a) = \mathbb{E}_{X_t,\xi_t}\left[\exp\left\{i\,aX_T\right\}\right] = \exp\left\{i\,a\,X_t + \int_t^T D^\alpha h(a, T-u)\,\xi_t(u)\,du\right\}, \tag{3}$$

where $\alpha = H + \frac{1}{2}$ and $h(a,.)$ is the unique continuous solution of the fractional Riccati equation

$$D^\alpha h(a,t) = -\frac{1}{2}\,a\,(a+i) + i\,\rho\,\nu\,ah(a,t) + \frac{1}{2}\,\nu^2\,h^2(a,t); \quad I^{1-\alpha}h(a,0) = 0. \tag{4}$$

Equation (4) is a rough version of the Riccati equation arising in the classical Heston model (with zero mean reversion). Indeed the only difference is that the time derivative is replaced by a fractional derivative. In contrast to the classical Heston case, there is no explicit solution to (4). This equation can be solved efficiently using numerical methods for fractional ordinary differential equations. We present one such method, the Adams scheme, in Appendix A, see [AJEE18, CGP18, GR19] for newly developed alternative numerical methods. Moreover, as we explain in Section 5, the true solution may also be accurately approximated in closed-form by a scaled version of the classical Heston solution. Then European option prices may be obtained from the characteristic function using standard Fourier techniques, see for example [Gat06].

With the characteristic function now in the Markovian form (3), hedging European options becomes obvious. Let $C_t(T) = \mathbb{E}\left[f(X_T)|\,\mathcal{F}_t\right]$. Then, a European option with payoff $f(X_T)$ can be perfectly replicated. As of time t, the hedge portfolio has $\partial_{S_t} C_t(T)$ of stock and $\partial_{\xi_t} C_t(T)$ of the forward variance curve $\xi_t(s)$ for each $s \in (t,T]$, where ∂_{ξ_t} represents the Fréchet derivative, roughly speaking the portfolio corresponding to bumping each of the forward variances $\xi_t(s)$, see [EER18]. From the above expressions, it is clear that perfect replication is in theory only. In practice, as with interest rates, one holds a finite number of variance contracts. Note however, that the rough Heston model has only one volatility factor. Thus one can hedge with only one European option, as in the classical Heston case, provided the value of the option component of the hedge portfolio coincides with the theoretical value of the forward variance component.

4 Calibration of the rough Heston model

In this section we present SPX volatility surface calibration results for two dates: August 14, 2013, to compare with the rough Bergomi calibration given for that day in [BFG16], and May 19, 2017, to show that the model continues to fit the market very well.

Note that the only parameters that we calibrate are the Hurst parameter H, the volatility of volatility ν and the correlation ρ. The forward variance curve, being a state variable in the model, is fixed to match at-the-money volatilities.

4.1 SPX calibration on August 14, 2013

From Bloomberg, on August 14, 2013, there were 19 expirations from 1 day to over 2.5 years, for a total of 1,809 options quoted. After eliminating options for which the bid price (of either the put or the call) was zero, we are left with 1,290 strike-expiration pairs.

For each such strike-expiration pair, we compute bid and ask implied volatilities $\sigma_i^{\pm} := \sigma_{\mathrm{BS}}^{\pm}(k_i, T_i)$, where k_i denotes the log-strike. Given model parameters $\{H, \nu, \rho\}$, we can obtain model implied volatilities $\sigma_i^M(H, \nu, \rho)$. We calibrate the parameters by minimizing[2]

$$\sum_i [\sigma_i^M(H, \nu, \rho) - \sigma_i^+]^2 + [\sigma_i^M(H, \nu, \rho) - \sigma_i^-]^2,$$

subject to the constraints

$$H \in (0, 1/2], \quad \nu \geq 0, \quad \rho \in [-1, 1].$$

We obtain the following optimal parameters:

$$H = 0.1216; \quad \nu = 0.2910; \quad \rho = -0.6714.$$

Figure 1 shows a remarkable fit to the SPX volatility surface, which is as good as with the rough Bergomi model in [BFG16].

4.2 SPX calibration on May 19, 2017

From Bloomberg, on May 19, 2017, there were 27 expirations from 1 day to over 2.5 years, for a total of 2,743 options quoted. By applying the same calibration procedure, we obtain the following optimal parameters:

$$H = 0.0474; \quad \nu = 0.4061; \quad \rho = -0.6710.$$

On this particular day, the calibrated value of H is closer to zero and so corresponds to very rough volatility.

[2] Alternatively, the parameters of the rough Heston model may be efficiently calibrated to the term structure of leverage swaps using a closed-form formula, see [AGR17].

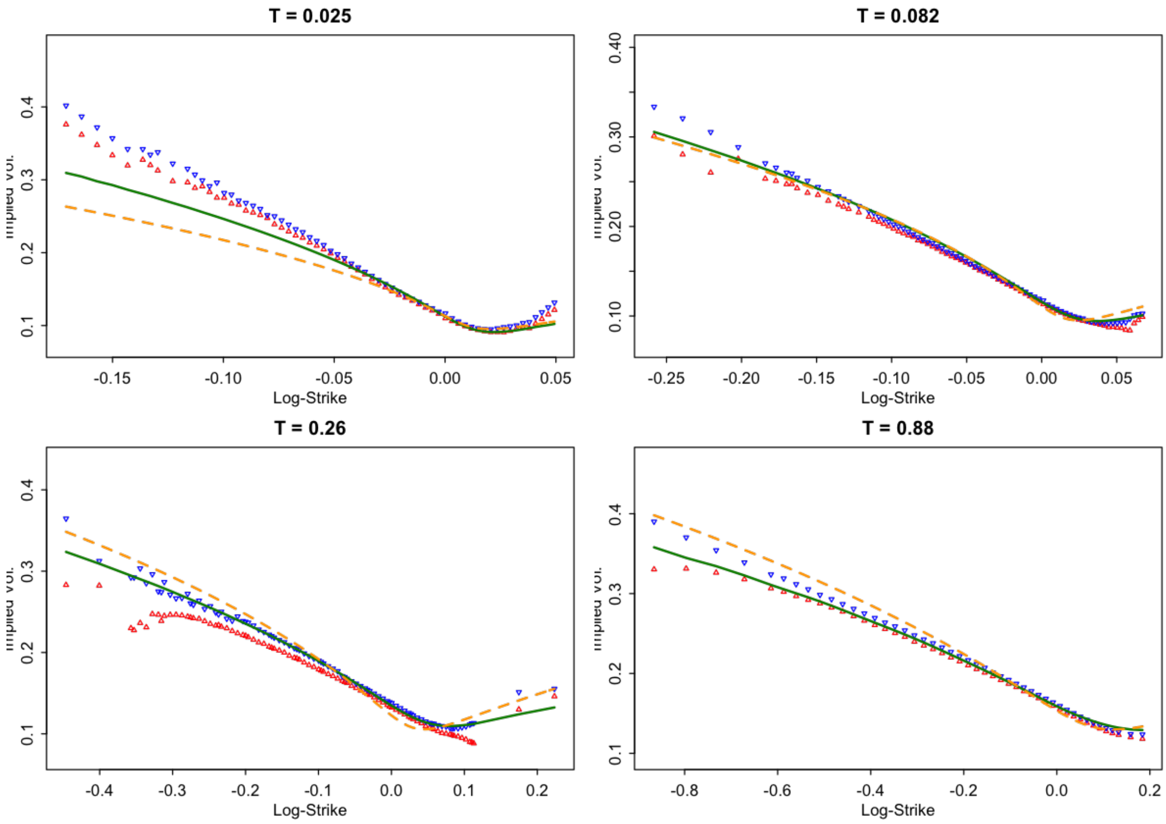

Figure 1: Representative SPX volatility smiles as of August 14, 2013, with time to expiration in years. Red and blue points represent bid and ask SPX implied volatilities; green smiles are from the rough Heston model calibrated using the Adams scheme; dashed orange lines are from the classical Heston model fitted to these four smiles.

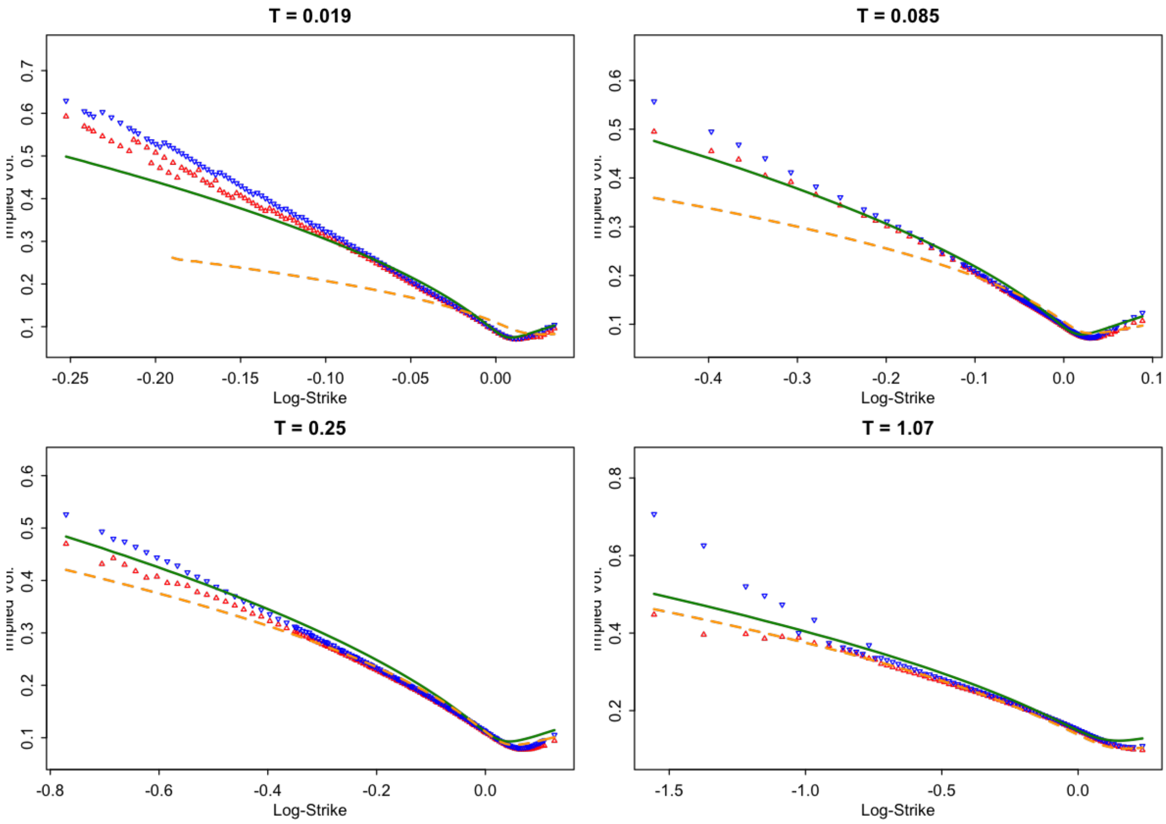

Figure 2: Representative SPX volatility smiles as of May 19, 2017, with time to expiration in years. Red and blue points represent bid and ask SPX implied volatilities; green smiles are from the rough Heston model calibrated using the Adams scheme; dashed orange lines are from the classical Heston model fitted to these four smiles.

4.3 Consistency with historical data

In addition, we have that the calibrated Hurst parameter on August 14, 2013 (see Section 4.1) is consistent with the one computed from historical data. Indeed, in [GJR18], the authors show that the behavior of historical log-volatility of the SPX index is close to that of a fractional Brownian motion with small Hurst parameter of order 0.1. As explained in detail in [GJR18], estimating the moment of order q of a log-volatility increment over a time interval of length Δ by

$$m(\Delta, q) = \frac{1}{N}\sum_{k=1}^{N} |\log(\sigma_{k\Delta}) - \log(\sigma_{(k-1)\Delta})|^q,$$

where the $(\sigma_{k\Delta})_{0\le k\le N}$ are historical measurements of volatility, we obtain a strong linear relationship between $\log(m(\Delta, q))$ and $\log(\Delta)$. In summary, we find

$$\mathbb{E}[|\sigma_\Delta - \sigma_0|^q] \approx K_q \Delta^{qH}; \quad H \approx 0.14,$$

which is in theory what we have if the log-volatility is a fractional Brownian motion with Hurst parameter $H = 0.14$.

Four representative SPX smiles as of August 14, 2013 and May 19, 2017 are plotted in Figures 1 and 2 respectively. We note that on both of these days, the calibrated values of H are small, consistent with the small H estimated from historical data. The rough Heston smiles are significantly closer to the market than the classical Heston smiles with optimized Heston parameters. Indeed, rough volatility models, though very parsimonious, are known to be consistent with historical data (see for example [GJR18]) and, as seen in these Figures, also with implied volatility data.

Of course, the rough Heston model, with only 3 parameters, cannot possibly generate implied volatilities within the bid-ask spread for each strike and expiration. One way to keep the desirable features of a rough volatility model and yet ensure a perfect fit to the volatility surface would be to construct a stochastic local volatility model with the rough Heston model as the stochastic volatility backbone.

5 A poor man's rough Heston model

In this section, we present respectively fast and almost instantaneous approximation methods to compute the implied volatility for a given expiry T and parameters (H, ρ, ν). The realized variance of the rough Heston model is given from (2) by

$$\int_0^T V_u du = \int_0^T \xi_0(u)du + \frac{\nu}{\Gamma(H+\frac{3}{2})}\int_0^T (T-u)^{\frac{1}{2}+H}\sqrt{V_u}\, dB_u. \tag{5}$$

The variance of (5) is equal to

$$\nu^2 \int_0^T \frac{(T-s)^{2H+1}}{\Gamma(H+\frac{3}{2})^2}\xi_0(s)ds. \tag{6}$$

This suggests that we approximate the rough Heston smile with the smile generated by a classical Heston-like model (2) with $H = 1/2$ and with a scaled volatility of volatility parameter $\widetilde{\nu}(T)$ matching the variance (6), that is

$$\nu^2 \int_0^T \frac{(T-s)^{2H+1}}{\Gamma(H+\frac{3}{2})^2} \xi_0(s)ds = \widetilde{\nu}(T)^2 \int_0^T (T-s)^2 \xi_0(s)ds.$$

Thus,

$$\widetilde{\nu}(T) = \frac{\nu}{\Gamma(H+\frac{3}{2})} \sqrt{\frac{\int_0^T (T-s)^{2H+1}\xi_0(s)ds}{\int_0^T (T-s)^2 \xi_0(s)ds}}. \tag{7}$$

For each expiry T, the characteristic function formula (3) is then approximated by the classical one

$$\exp\left\{ i\, a\, X_0 + \int_0^T \partial_u h^{(T)}(a, T-u)\, \xi_0(u)\, du \right\} \tag{8}$$

where $h^{(T)}(a,.)$ is now a solution of the classical Riccati equation:

$$\partial_u h^{(T)}(a,u) = -\frac{1}{2}\, a\,(a+i) + i\,\rho\, \widetilde{\nu}(T)\, a h^{(T)}(a,u) + \frac{1}{2}\, \widetilde{\nu}(T)^2\, (h^{(T)})^2(a,t); \quad h^{(T)}(a,0) = 0.$$

This equation can be solved explicitly as on page 18 of [Gat06]. The solution may be written as

$$h^{(T)}(a,t) = r_-(T) \frac{1 - e^{-A\,\widetilde{\nu}(T)\,t}}{1 - \frac{r_-(T)}{r_+(T)}\, e^{-A\,\widetilde{\nu}(T)\,t}}$$

with

$$A = \sqrt{a\,(a+i) - \rho^2\, a^2}; \quad r_\pm(T) = -\frac{1}{\widetilde{\nu}(T)}\, (i\,\rho\, a \pm A)\,.$$

On the other hand, for a given expiry T, a poor man's almost instantaneous approximation of the rough Heston characteristic function is obtained by approximating the forward variance curve as flat with $\xi_0(u) = v_0(T)$, $u \geq 0$. In practice, the obvious choice $v_0(T) = \frac{1}{T}\int_0^T \xi_0(s)ds$, the fair value of the variance swap, works fine. In that case, (7) becomes

$$\widetilde{\nu}(T) = \sqrt{\frac{3}{2H+2}}\, \frac{\nu}{\Gamma\left(H+\frac{3}{2}\right)}\, \frac{1}{T^{\frac{1}{2}-H}}. \tag{9}$$

With this choice of forward variance curve, the approximate characteristic function (8) is identical to the characteristic function of the classical Heston model with initial variance $v_0(T)$, mean reversion $\lambda = 0$, correlation ρ and volatility of volatility $\widetilde{\nu}(T)$ given by (9). Option prices under rough Heston may thus be almost instantaneously approximated using any existing implementation of the classical Heston pricing model.

To demonstrate the quality of approximations (7) and (9), in Figure 3, we have replotted the rough Heston smiles of Figure 2 generated using the Adams scheme together with those generated using the above approximations. We note that the smiles generated using the approximate characteristic function (8) with approximation (7) appear to be closer to true rough Heston smiles than those generated by the poor man's Heston approximation, especially close to at-the-money. Nevertheless, the poor man's smiles are surprisingly good. An accurate rough Heston approximation with no quants required!

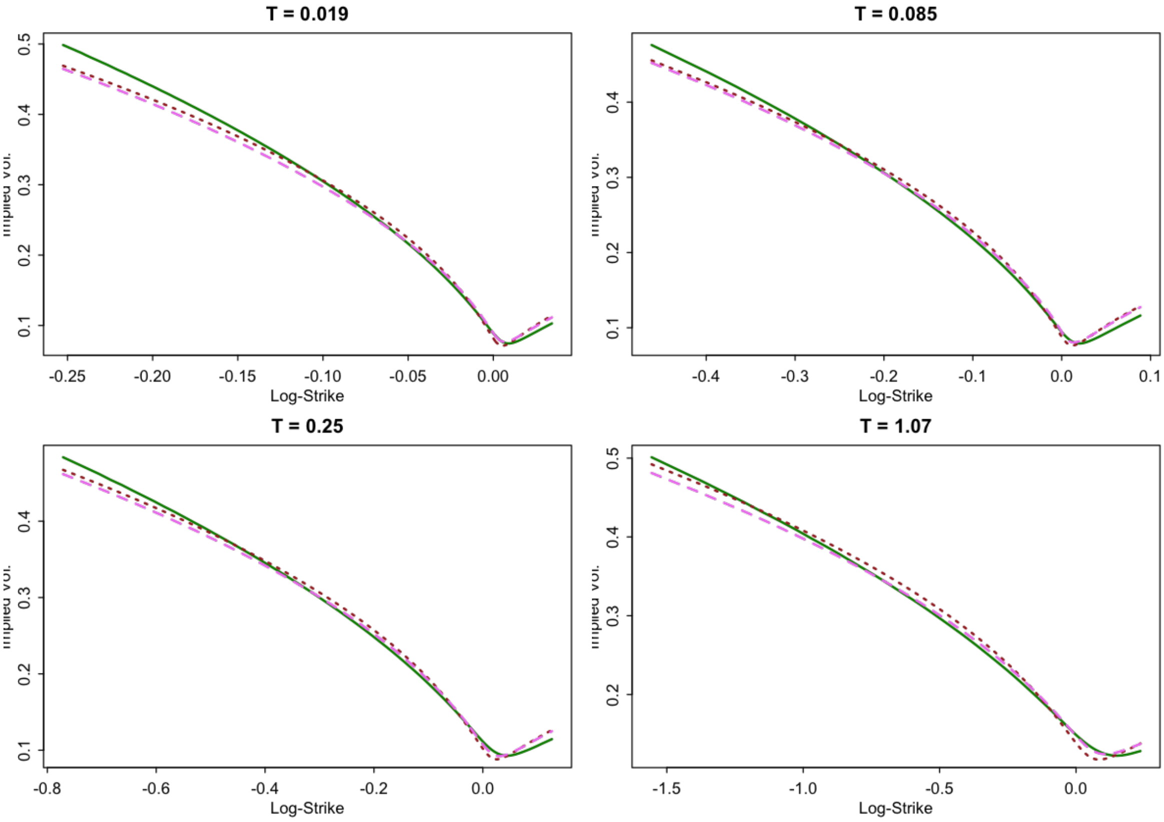

Figure 3: Representative SPX volatility smiles as of May 19, 2017, with time to expiration in years. Green smiles are from the rough Heston model calibrated using the Adams scheme; violet dashed smiles are generated using the approximate rough Heston characteristic function (8); the brown dotted smiles are generated from the poor man's existing Heston model with scaled volatility of volatility parameter.

6 Summary

In this article, we have presented the rough Heston model, a particularly tractable rough volatility model with a quasi-closed form characteristic function analogous to that of the classical Heston model. This characteristic function is in terms of the solution of a fractional Riccati ODE which we show how to solve numerically. Pricing and hedging using the rough Heston model are then straightforward. Calibration to the implied volatility surface is illustrated with two examples; as with other rough volatility models, the shape of the observed surface is faithfully reproduced. Finally, we show how to accurately approximate rough Heston model values by scaling the volatility of volatility parameter of the classical Heston model – we call this the poor man's rough Heston model. We thus generate accurate approximations of rough Heston model values with no quants required!

A Numerical solution of the fractional Riccati equation

We recall here how to solve fractional ordinary differential equations like (4). This is needed in order to compute the characteristic function. Specifically, again with $\alpha = H + \frac{1}{2}$, let

$$D^{\alpha} h(a,t) = F(a, h(a,t)), \quad h(a,0) = 0. \tag{10}$$

Several schemes for solving (10) numerically are available in the literature. Most of these are based on the idea that (10) implies the following Volterra equation:

$$h(a,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(a, h(a,s)) ds. \tag{11}$$

One way to solve (11) is to use the classical fractional Adams method presented in [DFF04]. The idea goes as follows. Let us write $g(a,t) = F(a, h(a,t))$. Over a regular discrete time-grid with mesh Δ, $0 \leq t_0, \ldots \leq t_n \leq t$, we approximate

$$h(a, t_{k+1}) = \frac{1}{\Gamma(\alpha)} \int_0^{t_{k+1}} (t_{k+1} - s)^{\alpha-1} g(a,s) ds$$

by

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \hat{g}(a,s) ds,$$

where

$$\hat{g}(a,t) = \frac{t_{j+1} - t}{t_{j+1} - t_j} \hat{g}(a, t_j) + \frac{t - t_j}{t_{j+1} - t_j} \hat{g}(a, t_{j+1}), \quad t \in [t_j, t_{j+1}], \quad 0 \leq j \leq k.$$

This corresponds to a trapezoidal discretization and leads to the following scheme:

$$\hat{h}(a, t_{k+1}) = \sum_{0 \leq j \leq k} a_{j,k+1} F(a, \hat{h}(a, t_j)) + a_{k+1,k+1} F\big(a, \hat{h}(a, t_{k+1})\big), \tag{12}$$

with

$$a_{0,k+1} = \frac{\Delta^{\alpha}}{\Gamma(\alpha+2)} [k^{\alpha+1} - (k-\alpha)(k+1)^{\alpha}],$$

$$a_{j,k+1} = \frac{\Delta^{\alpha}}{\Gamma(\alpha+2)} [(k-j+2)^{\alpha+1} + (k-j)^{\alpha+1} - 2(k-j+1)^{\alpha+1}]; \quad 1 \leq j \leq k \tag{13}$$

and

$$a_{k+1,k+1} = \frac{(\Delta t)^{\alpha}}{\Gamma(\alpha+2)}.$$

However, $\hat{h}(a, t_{k+1})$ being on both sides of (12), this scheme is implicit. Thus, in a first step, we compute a pre-estimation (or *predictor*) of $\hat{h}(a, t_{k+1})$ based on a Riemann sum that we then plug into the trapezoidal quadrature. We define this predictor $\hat{h}^P(a, t_{k+1})$ as follows.

$$\hat{h}^P(a, t_{k+1}) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \tilde{g}(a,s) ds,$$

with

$$\tilde{g}(a,t) = \hat{g}(a, t_j); \quad t \in [t_j, t_{j+1}), \quad 0 \leq j \leq k.$$

Therefore,

$$\hat{h}^P(a, t_{k+1}) = \sum_{0 \le j \le k} b_{j,k+1} F\big(a, \hat{h}(a, t_j)\big),$$

where

$$b_{j,k+1} = \frac{\Delta^\alpha}{\Gamma(\alpha+1)}\big((k-j+1)^\alpha - (k-j)^\alpha\big), \quad 0 \le j \le k.$$

Thus, the final explicit numerical scheme is given by

$$\hat{h}(a, t_{k+1}) = \sum_{0 \le j \le k} a_{j,k+1} F(a, \hat{h}(a, t_j)) + a_{k+1,k+1} F\big(a, \hat{h}^P(a, t_j)\big),$$

where the weights $a_{j,k+1}$ are defined in (13).

B Call option prices using Fourier techniques

We explain now the way to deal numerically with the Fast Fourier Transform technique of [CM99] for the computation of call option prices in the specific case of the rough Heston model. Recall that the price at time t, $C_t(T, S_0 e^x)$, of the call with expiration time $T > t$ and strike $S_0 e^x$ is related to the characteristic function of the log price X_T through the following expression:

$$C_t(T, S_0 e^x) = \frac{\exp(-\beta x)}{\pi} \int_0^\infty \Re[\psi_t(T, a) e^{-iax}] da, \tag{14}$$

where

$$\psi_t(T, a) = \frac{\phi_t(T, a - (\beta+1)i)}{\beta^2 + \beta - a^2 + i(2\beta+1)a}$$

and ϕ_t is defined in (3). Such a method includes the choice of $\beta > 0$ such that

$$\mathbb{E}_t[S_T^{\beta+1}] < \infty. \tag{15}$$

In [EER18], a sufficient condition for finite moments is given. In particular when $\rho < 0$, the existence of $\beta > 0$ satisfying (15) is guaranteed.

In practice, to apply the Fast Fourier Transform algorithm, we need to tackle the issue of the infinite upper limit of integration in (14) by looking for $a_{max} > 0$ such that

$$\frac{\exp(-\beta x)}{\pi} \int_{a_{max}}^\infty |\psi_t(T, a)| da < \varepsilon,$$

where $\varepsilon > 0$ is the expected truncation error. In [EER18], it is shown that

$$\Re[\int_t^T D^\alpha h(a - i(\beta+1), T-u) \xi_t(u) du]$$

is asymptotically dominated as $|a|$ goes to infinity by

$$-|a| \frac{\sqrt{1-\rho^2}}{\nu \Gamma(1-\alpha)} \int_t^T (T-u)^{-\alpha} \xi_t(u) du.$$

Hence from (3), it is enough to choose $a_{max} > 0$ such that

$$\frac{\exp(-\beta x)}{\pi} S_t^{\beta+1} \int_{a_{max}}^\infty \frac{\exp\big(-a \frac{\sqrt{1-\rho^2}}{\nu\Gamma(1-\alpha)} \int_t^T (T-u)^{-\alpha} \xi_t(u) du\big)}{|\beta^2 + \beta - a^2 + i(2\beta+1)a|} da < \varepsilon.$$

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