Pricing and Hedging Libor Exotics in Forward Libor Models

Risk Magazine's Quant Congress
November 2004

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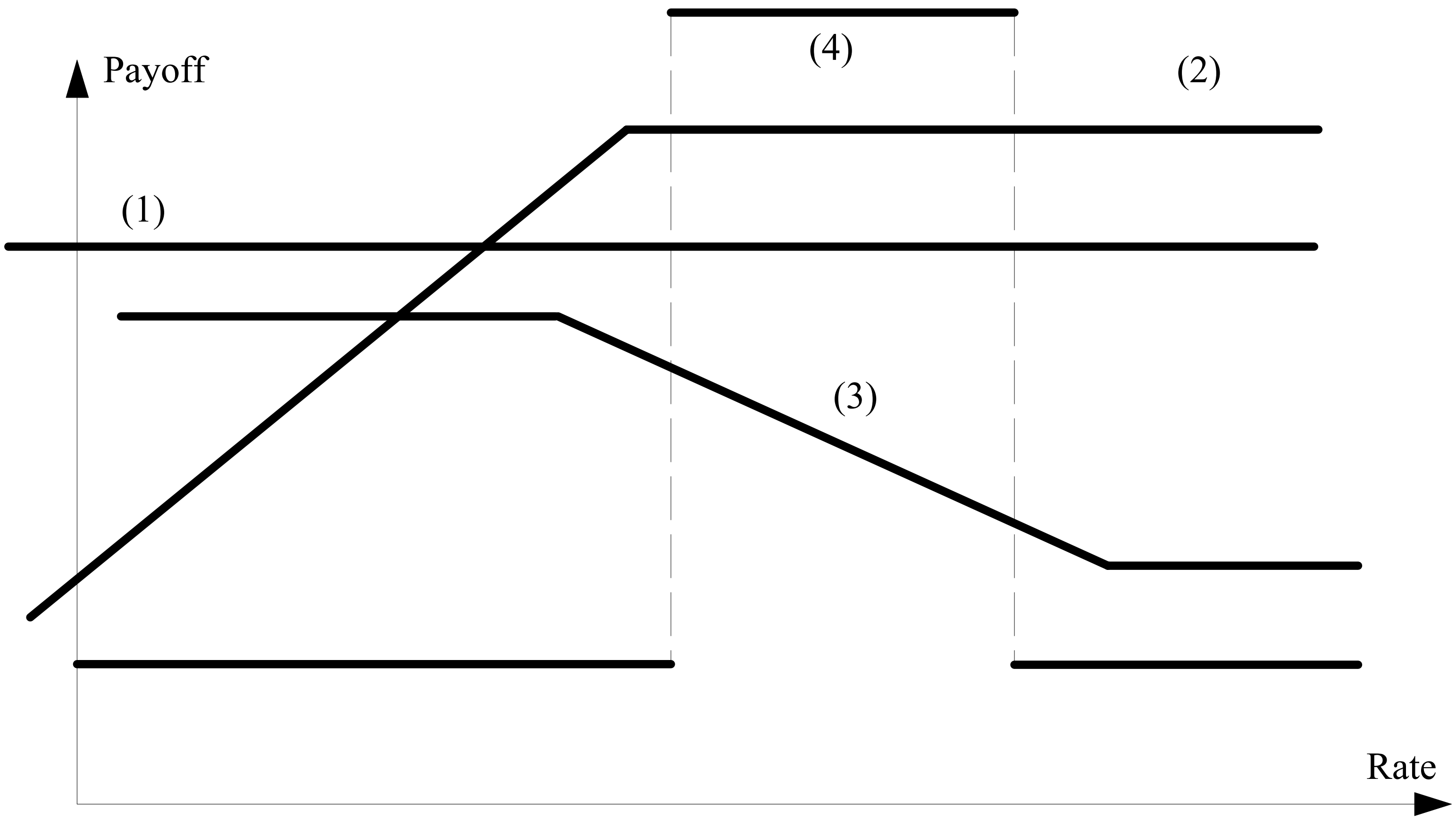
1 Market for callable Libor exotics

- Callable Libor exotic products (CLE):
  - Notes with coupons linked to interest rates in non-linear ways
  - The issuer has the right to call (cancel) the note after a lockout period
- Investors motivation: Can get a large, above-market current coupon in year one, and tailor future cupons to expectations of interest rates
- Why would the bank pay high coupons? The Bermuda-style option to call the note (cancel the deal after the lockout) is valuable!
- In effect, the investor sells an option to the bank for current yield (premium). The bank can monetize the option by delta hedging ("gamma scalping") it.
- New structures get invented all the time. Make the coupon appealing to customers + make the option to cancel as valuable as possible

2 Coupon payoffs I



(1) Bermuda swaption
(2) Callable capped floater
(3) Callable inverse floater
(4) Callable range accrual

3 Formal definition of CLE

Tenor structure

$$\begin{aligned} 0 &= T_0 < T_1 < \ldots < T_N, \\ \delta_n &= T_{n+1} - T_n. \end{aligned}$$

Value of coupon that fixes at time T_i and pays at T_{i+1},

$$C_i(t).$$

Libor (bank funding) rate that fixes at time T_i,

$$L_i(t).$$

A callable Libor exotic is a Bermuda style option with exercise dates $T_1, \ldots, T_{N-1}$. Exercised at time T_n – get all coupons fixing on or after T_n.

The exercise value for exercise opportunity n, at time t, is equal to

$$E_n(t) = \sum_{i=n}^{N-1} B_t \mathbf{E}_t \left(B_{T_{i+1}}^{-1} \times (C_i(T_i) - L_i(T_i)) \times \delta_i \right).$$

So the bank can exercise the option, and if it does so on date T_n, the PV of the swap it gets is equal to $E_n(T_n)$.

Define

$$X_i(T_i) = (C_i(T_i) - L_i(T_i)) \times \delta_i.$$

4 Pricing and hedging challenges

- Highly nontrivial dependence of CLEs on volatility structure require BGM-type models
- Only Monte-Carlo available.
- Computing values of Bermuda-style options is hard.
- Risk parameters – even harder!
- Solutions to these problems are discussed in the talk.
- See [Pit03], [Gla03]

5 Some useful notations

- Probability space $(\Omega, \mathbf{P}, \mathcal{F})$
- Zero-coupon bonds $P(t, T)$
- Numeraire B_t
- Filtration of sigma-algebras $\{\mathcal{F}_t\}$
- Pricing operator π. For arbitrary $X \sim \mathcal{F}_T$,

$$\pi_t(X) = B_t \mathbf{E}_t \left(B_T^{-1} X \right)$$

- Tenor structure

$$0 = t_0 < t_1 < \ldots < t_M$$

- Primary Libor rates

$$\bar{F}(t) = (F_0(t), \ldots, F_{M-1}(t)), \quad F_n(t) = \frac{P(t, t_n) - P(t, t_{n+1})}{(t_{n+1} - t_n) P(t, t_{n+1}).}.$$

6 BGM model and extensions

$W^{T_{n+1}}(\cdot)$ a Brownian motion under the T_{n+1}-forward measure

Stochastic Volatility BGM:

$$\begin{aligned} dz(t) &= \theta(z_0 - z(t))\, dt + \varepsilon\sqrt{z(t)}\, dB(t), \\ dF_n(t) &= \sqrt{z(t)}\phi(F_n(t))\,\lambda_n(t)\, dW^{T_{n+1}}(t), \end{aligned}$$

Use discrete money-market numeraire B_t by

$$\begin{aligned} B_{t_0} &= 1, \\ B_{t_{n+1}} &= B_{t_n} \times (1 + (t_{n+1} - t_n) F_n(t_n)), \quad 0 \leq n < M, \\ B_t &= P(t, t_{n+1}) B_{t_{n+1}}, \quad t \in [t_n, t_{n+1}]. \end{aligned}$$

Spot Libor measure: use B as the numeraire.

7 Pricing callable Libor exotics in BGM

- Define $H_n(t)$ to be the value of a "sub-CLE" with the exercise dates $T_n, \ldots, T_{N-1}$. The same as the "hold" value of the original CLE if it has not been exercised up to and including the date T_n.
- Main recursion for exercise and hold values

$$\begin{aligned}
E_n(T_n) &= P(T_n, T_{n+1}) X_n(T_n) + \pi_{T_n} E_{n+1}(T_{n+1}), \\
H_{n-1}(T_{n-1}) &= \pi_{T_{n-1}} \max\{H_n(T_n), E_n(T_n)\}, \\
H_{N-1} &\equiv 0, \\
E_N &\equiv 0, \\
n &= N-1, \ldots, 1.
\end{aligned}$$

For $n = 1$ we obtain the value $H_0(0)$, the value of the CLE that we are after.

Pricing callable Libor exotics as barriers with an optimized barrier I

- Formally,

$$H_n\left(T_n\right) = \operatorname*{ess\,sup}_{\xi \in \mathcal{T}_n} \mathbf{E}_{T_n} B_{T_\xi}^{-1} E_\xi\left(T_\xi\right).$$

$\mathcal{T}_n$ is a set of all stopping times that exceed n.

- Supremum over all barrier options.
- The solution to this series of problems is given by the optimal exercise time index $\eta = \eta\left(\omega\right)$,

$$\eta\left(\omega\right) = \min\left\{n \geq 1 : \omega \in R_n\right\} \wedge N,$$

where R_n are exercise region at time T_n,

$$R_n = \left\{\omega \in \Omega : H_n\left(T_n, \omega\right) \leq E_n\left(T_n, \omega\right)\right\}, \quad 1 \leq n \leq N - 1.$$

- The CLE value

$$H_0\left(0\right) = \mathbf{E}_0\left(B_{T_\eta}^{-1} E_\eta\left(T_\eta\right)\right) = \mathbf{E}_0\left(\sum_{n=\eta}^{N-1} B_{T_{n+1}}^{-1} X_n\right).$$

9 Pricing callable Libor exotics as barriers with an optimized barrier II

- Recall

$$H_0(T_0) = \operatorname*{ess\,sup}_{\xi \in \mathcal{T}_0} \mathbf{E}_0 B_{T_\xi}^{-1} E_\xi(T_\xi) = \mathbf{E}_0 \left(B_{T_\eta}^{-1} E_\eta(T_\eta) \right).$$

- Replacing optimal exercise regions with estimates $\tilde{R}_n$ and η with $\tilde{\eta}$ we get a lower bound on CLE value.
- The closer the estimated exercise region $\tilde{R}_n$ to the actual one, the tighter the lower bound on the value.
- Pricing in Monte-Carlo (lower bound):
  1. Pre-simulate some paths
  2. Estimate exercise regions for each exercise time
     (a) Replace expectations with regressions using Longstaff-Schwartz (LS)
  3. Simulate additional paths (main simulation), and compute the CLE value as the value of a barrier option with a given set of exercise regions
  4. Extension of LS from Bermuda swaptions to CLEs is fairly straightforward. In addition to regressing holds values, need to regress exercise

10 Exercise boundary and risk sensitivities

- Risk sensitivities – bump one of the inputs and revalue.
- Keep the exercise boundary when bumping and revaluing!
- Risk to parameter x with current value x_0. For each x there is an optimal exercise boundary $\zeta(x)$. The value of a CLE for x is then given by $h(\zeta(x), x)$ where $h(\zeta, x)$ is the value of a barrier option for the boundary ζ and parameter value x. Then the risk number is equal to

  $$\left.\frac{\partial}{\partial x} h(\zeta(x), x)\right|_{x=x_0} = \left.\frac{\partial}{\partial \zeta} h(\zeta, x_0)\right|_{\zeta=\zeta(x_0)} \times \left.\frac{\partial}{\partial x}\zeta(x)\right|_{x=x_0} + \left.\frac{\partial}{\partial x} h(\zeta(x_0), x)\right|_{x=x_0}.$$

  Important:

  $$\left.\frac{\partial}{\partial \zeta} h(\zeta, x_0)\right|_{\zeta=\zeta(x_0)} = 0,$$

  because $\zeta(x_0)$ is the *optimal* one for the CLE.
- Faster (do not need to recompute)
- More accurate (no noise from boundary calculations)
- The full derivative of h with respect to x is equal to the partial one while keeping the exercise boundary constant.

11 Deltas and why they are hard to obtain

- Interest rate deltas – changes in the value of the CLE with respect to $F_n(0)$, $n = 1, \ldots, M-1$. Natural bucketing.
- Recall that he value is computed as a sum over simulated paths ω_j, $j = 1, \ldots, J$,

$$\tilde{H}_0 = J^{-1} \sum_{j=1}^{J} \sum_{i=1}^{N-1} \left[B_{T_{i+1}}^{-1}(\omega_j) \, X_i(\omega_j) \, 1_{i \geq \tilde{\eta}(\omega_j)} \right]$$

- Two effects as we bump one of the rates slightly
  - Smooth: change in the values X_i or $B_{T_n}^{-1}$.
  - Jumpy: change in $\tilde{\eta}(\omega_j)$, can add/delete a whole coupon for a path affected. The quantities $1_{i \geq \tilde{\eta}(\omega_j)}$ do not depend smoothly on initial interest rate curve
- It it the second effect that makes simulation error large for Greeks (less smooth – higher error).

12 Pathwise deltas I

- Define Δ_m to be the delta with respect to $F_m(0)$, *for each realization* ω,

$$\Delta_m X(\omega) = \frac{\partial X(\omega)}{\partial F_m(0)}.$$

- Valuation recursion:

$$B_{T_{n-1}}^{-1} H_{n-1}(T_{n-1}) = \mathbf{E}_{T_{n-1}} \max\left\{B_{T_n}^{-1} H_n(T_n), B_{T_n}^{-1} E_n(T_n)\right\}$$

  Differentiate through (technical conditions to change the order of expectation and differentiation)

$$\begin{aligned}\Delta_m\left(B_{T_{n-1}}^{-1} H_{n-1}(T_{n-1})\right) &= \mathbf{E}_{T_{n-1}} 1_{\{H_n(T_n)>E_n(T_n)\}} \Delta_m\left(B_{T_n}^{-1} H_n(T_n)\right) \\ &\quad + \mathbf{E}_{T_{n-1}} 1_{\{E_n(T_n)>H_n(T_n)\}} \Delta_m\left(B_{T_n}^{-1} E_n(T_n)\right).\end{aligned}$$

- Unwrap the recursion to obtain (see [Pit04a])

$$\begin{aligned}\Delta_m H_0(0) &= \sum_{n=1}^{N-1} \mathbf{E}_0\left(\prod_{i=1}^{n-1} 1_{\{H_i(T_i)>E_i(T_i)\}} \times 1_{\{E_n(T_n)>H_n(T_n)\}} \times \Delta_m\left(B_{T_n}^{-1} E_n(T_n)\right)\right) \\ &= \mathbf{E}_0\left(\sum_{i=\eta}^{N-1}\left(\Delta_m\left(B_{T_{i+1}}^{-1} X_i\right)\right)\right).\end{aligned}$$

13 Pathwise deltas II

- Valuation:

$$H_0(0) = \mathbf{E}_0 \left(\sum_{i=\eta}^{N-1} B_{T_{i+1}}^{-1} X_i \right).$$

- Can differentiate through, and keep the exercise boundary constant:

$$\Delta_m H_0(0) = \mathbf{E}_0 \left(\sum_{i=\eta}^{N-1} \Delta_m \left(B_{T_{i+1}}^{-1} X_i \right) \right).$$

- Replace η with an estimate $\tilde{\eta}$, get an estimate of the delta,

$$\tilde{\Delta}_m H_0(0) = \mathbf{E}_0 \left(\sum_{i=\tilde{\eta}}^{N-1} \Delta_m \left(B_{T_{i+1}}^{-1} X_i \right) \right).$$

- Significant reduction in noise (the problem discussed in the slide above eliminated).
- Time savings because deltas are computed in the same simulation as the value.
- Valuation as barriers, yet pathwise deltas can be used (only for *optimal* boundary!)

14 Pathwise deltas III

- Apply chain rule to get $\Delta_m \left(B_{T_{i+1}}^{-1} X_i \right)$. For example

$$\begin{aligned}\Delta_m P(t, t_m, t_{m+1}) &= \Delta_m \frac{1}{1+\tau_m F_m(t)} = \frac{\partial}{\partial F_m(t)} \left(\frac{1}{1+\tau_m F_m(t)} \right) \times \frac{\partial F_m(t)}{\partial F_m(0)} \\ &= -\frac{\tau_m}{(1+\tau_m F_m(t))^2} \Delta_m F_m(t) .\end{aligned}$$

- The values $\Delta_m F_n(t)$ can be simulated in the same simulation as $F_n(t)$. Recall (under the spot measure)

$$dF_n(t) = \lambda_n(t)\, \phi(F_n(t)) \left(\mu\left(t, \bar{F}(t)\right) dt + dW(t) \right) . \qquad (1)$$

Differentiate through,

$$\begin{aligned}d\Delta_m F_n(t) &= \lambda_n(t) \left[\sum_k \frac{\partial \phi(F_n(t))}{\partial F_k(t)} (\Delta_m F_k(t)) \right] \left(\mu\left(t, \bar{F}(t)\right) dt + dW(t) \right) \\ &\quad + \lambda_n(t)\, \phi(F_n(t)) \left[\sum_k \frac{\partial \mu\left(t, \bar{F}(t)\right)}{\partial F_k(t)} (\Delta_m F_k(t)) \right] dt.\end{aligned}$$

15 "Sausage" Monte-Carlo I

- Pathwise differentiation method not always applicable (discontinuity of coupons, system constraints, inaccurate exercise boundary estimates).
- "Digital" features in CLE payoff: use payoff smoothing via conditional expectations. Tailored to the callable structure.
- Pre-integrate the payoff (approximately) along each simulated path of interest rates ("sausage"). Instead of

$$\tilde{H}_0 \approx J^{-1} \sum_{j=1}^{J} v_j, \quad v_j = \sum_{i=1}^{N-1} \left[B_{T_{i+1}}^{-1} (\omega_j) \, X_i (\omega_j) \, 1_{i \geq \tilde{\eta}(\omega_j)} \right]$$

use

$$\tilde{H}_0 \approx J^{-1} \sum_{j=1}^{J} v_j^{\varepsilon}, \quad v_j^{\varepsilon} = \mathbf{E} \left(\sum_{i=1}^{N-1} \left[B_{T_{i+1}}^{-1} (\omega) \, X_i (\omega) \, 1_{i \geq \tilde{\eta}(\omega)} \right] \middle| A_j^{\varepsilon} \right),$$

$$A_j^{\varepsilon} = \left\{ \omega : \left\| \bar{F} (T_i, \omega) - \bar{F} (T_i, \omega_j) \right\| < \varepsilon \quad \forall i = 1, \ldots, N-1 \right\}$$

- Instead of "hard" exercise/no exercise rule, have a concept of a fuzzy, or partial, exercise (conditioned on being in the sausage).
- The probability of exercise on each date, conditioned on being in the sausage, can be analytically estimated (use approximate conditional in-

- Final formula: For each path ω, instead of

$$v_j = \sum_{i=1}^{N-1} \left[B_{T_{i+1}}^{-1}\left(\omega_j\right) X_i\left(\omega_j\right) 1_{i \geq \tilde{\eta}(\omega_j)} \right]$$

we get

$$\begin{aligned} v_j &= \sum_{i=1}^{N-1} B_{T_{i+1}}^{-1}\left(\omega_j\right) \times X_i\left(\omega_j\right) \times \left(1 - q_i\left(\omega_j\right)\right), \\ q_i &= q_{i-1} \times \left(1 - p_i\right), \\ p_i &= \min\left(\max\left(\frac{\hat{E}_i - \hat{H}_i + \delta_i}{2\delta_i}, 0 \right), 1 \right). \end{aligned}$$

  – $\hat{E}_i$, $\hat{H}_i$ are proxy exercise and hold values (from pre-simulation),
  – p_i is a marginal exercise probability for date T_i,
  – q_i is a probability of not exercising up to time T_i,
  – δ_i is the smoothing parameter, a known function of ε.

- Instead of “all or nothing” for each path we get a weighted average of cashflows with weights being smooth functions of the initial parameters

17 Deltas – comparing different methods I

- Delta slide. An interest rate curve is shifted in parallel, parallel delta computed
- Bermuda swaption (similar for other CLEs)
- 2,048 pre-paths, 4,097 valuation paths

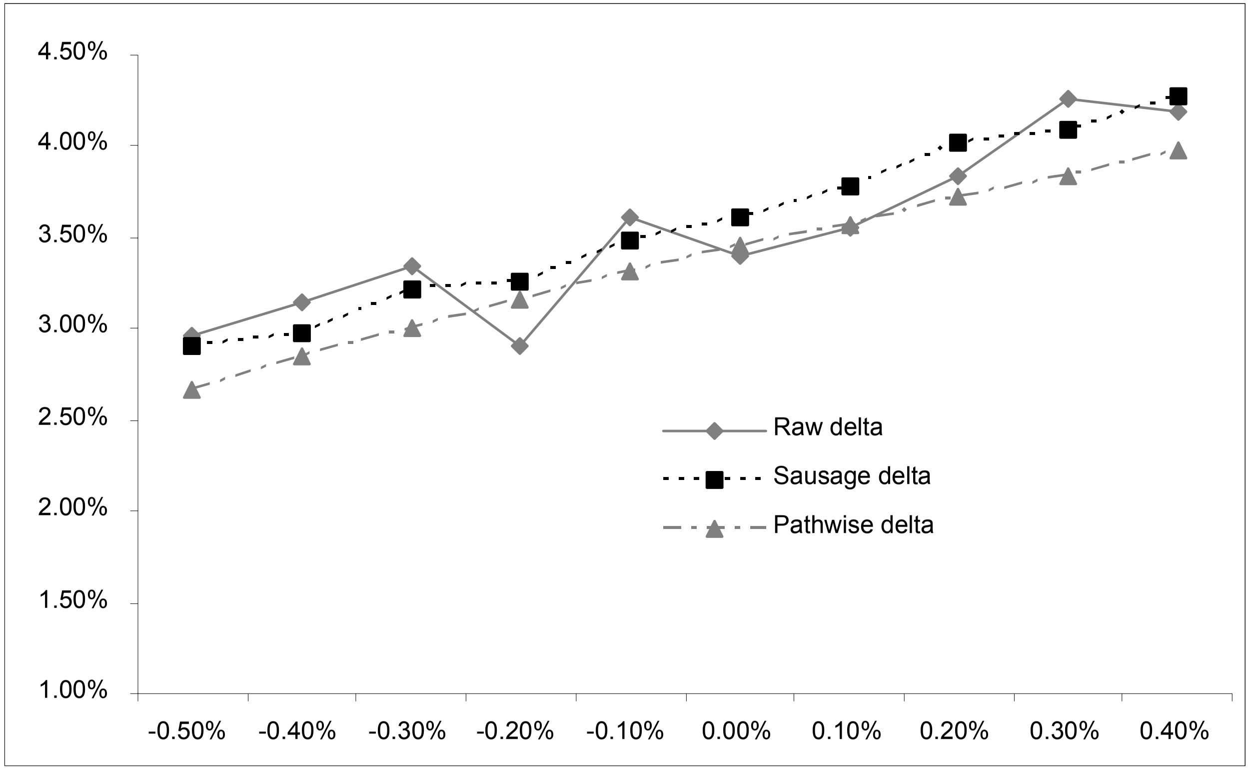


Figure 1:

18 Deltas – comparing different methods II

- Empirical results. Deltas computed 24 times with different seeds, and standard errors of estimates are computed

N paths for pre-simulation	512	2048	8192
N paths for valuation	1024	4096	16384
Raw delta	*3.62%*	*3.74%*	*3.63%*
Sausage delta	*3.62%*	*3.69%*	*3.65%*
Pathwise delta	*3.48%*	*3.51%*	*3.52%*
Raw delta, std err	0.921%	0.237%	0.132%
Sausage delta, std err	0.177%	0.122%	0.052%
Pathwise delta, std err	0.156%	0.069%	0.028%

- Sausage deltas are about as accurate as raw deltas computed with 4 times as many paths.
- Pathwise deltas are about as accurate as sausage deltas computed with 3-4 times as many paths

19 New exotics: TARNs

- Targeted Return Note. Not callable. A swap that knocks out as soon as the sum of paid coupons exceeds a threshold.
  - An investor receives a structured coupon (i.e. an inverse floater) and pays Libor
  - A sum of structured coupons paid to date is kept
  - As soon as the sum exceeds a pre-agreed upon amount (targeted return) the swap disappears
- Very popular. Why?
  - Easier for investors to understand than callable
  - Investors know exactly when it goes away
  - They know exactly how much money they are getting (but not when)
  - Any swap (inverse floaters, range accruals, CMS spreads, event multi-currency) can be TARNed
- For details see [Pit04b]

20 Formal definition

- Inverse floating coupon

$$C_n(t) = (s - 2F_n(t))^+ .$$

- Value

$$\begin{aligned} v &= \mathbf{E}_0\left(\sum_{n=1}^{N-1} B_{T_{n+1}}^{-1} \times X_n(T_n) \times \chi\{Q_n < R\}\right), \qquad (2) \\ X_n(t) &= \delta_n \times (C_n(t) - F_n(t)), \\ Q_n &= \sum_{i=1}^{n-1} \delta_i C_i(T_i), \quad Q_1 = 0, \\ \chi\{A\} &= \begin{cases} 1, & \text{if } A, \\ 0, & \text{if not } A. \end{cases} \end{aligned}$$

21 "Sausage" Monte-Carlo for TARNs

Same idea as before. Replace point estimates

$$\tilde{v} \approx J^{-1} \sum_{j=1}^{J} \tilde{v}_j,$$

$$\tilde{v}_j = \sum_{n=1}^{N-1} B_{T_{n+1}}^{-1}(\omega_j) \times X_n(T_n, \omega_j) \times \chi\{Q_n(\omega_j) < R\},$$

with averages over small sausages

$$\tilde{v}_j = \mathbf{E}\left(\sum_{n=1}^{N-1} B_{T_{n+1}}^{-1}(\omega_j) \times X_n(T_n, \omega_j) \times \chi\{Q_n(\omega_j) < R\} \middle| A_j^{\varepsilon} \right),$$

$$A_j^{\varepsilon} = \left\{\omega : \left\| \bar{F}(T_i, \omega) - \bar{F}(T_i, \omega_j) \right\| < \varepsilon \quad \forall i = 1, \ldots, N-1 \right\}.$$

After some tinkering,

$$\tilde{v}_j = \sum_{n=1}^{N-1} B_{T_{n+1}}^{-1}(\omega_j) \times X_n(T_n, \omega_j) \times p_n(\omega_j),$$

$$p_n(\omega) = \min\left(\max\left(\frac{R - Q_n(\omega_j) + \eta_n}{2\eta_n}, 0 \right), 1 \right).$$

22 Local projection method

- Local projection method
  - Calibrate global model to ”everything”
  - Compute relevant term volatilities and inter-temporal correlations *from the global model*;
  - Calibrate a simpler model to volatilities/correlations above + relevant smiles. This is the local model.
- Which “local” model to use?

23 Analyzing correlation exposures

- Value

$$
\begin{aligned}
v \;=\; & \mathbf{E}_0 \sum_{n=1}^{N-1} B_{T_{n+1}}^{-1} \delta_n \left(\left(s - 2F_n\left(T_n\right) \right)^+ - F_n\left(T_n\right) \right) \\
& \times \chi \left\{ \sum_{i=1}^{n-1} \delta_i \left(s - 2F_i\left(T_i\right) \right)^+ < R \right\}.
\end{aligned}
$$

- Function of

$$\tilde{F} = \left\{ F_1\left(T_1\right), F_2\left(T_2\right), \ldots, F_{N-1}\left(T_{N-1}\right) \right\}$$

- Only values of Libors on their fixing dates, NOT “along the path”
- Need to match
  - Term volatilities of Libor rates $\left\{ \operatorname{stdev}\left(\log F_n\left(T_n\right)\right) \right\}_n$;
  - Inter-temporal correlations of Libor rates $\left\{ \operatorname{corr}\left(\log F_n\left(T_n\right), \log F_m\left(T_m\right)\right) \right\}_{n,m}$
- The Hull-White model with time-dependent coefficients has enough flexibility.

24 Analyzing skew exposures

- Hull-White is great, but what about the skew?
- Skew exposure:
  - Big digital at first knockout, known strike. But also non-linearity in the coupon. Exposure to all strikes
  - Digitals at subsequent knockouts, *at unknown strikes*.
- Suggestion 1: Calibrate the Hull-White model to ceratin strikes. Bad (do not know which strikes to use)
- Suggestion 2: Enhance HW with smiles, to match vols *at all strikes* for all Libors

25 Local model for TARNs

- Hull-White with skew: the SV-Cheyette model, see [AA02]

$$\begin{aligned} dx(t) &= (-v(t)\,x(t) + y(t))\,dt + \sqrt{V(t)}\eta(t, x(t), y(t))\,dW(t), \\ dy(t) &= \left(V(t)\,\eta^2(t, x(t), y(t)) - 2v(t)\,y(t)\right)\,dt,, \\ dV(t) &= \varkappa(\theta - V(t))\,dt + \varepsilon\psi(V(t))\,dZ(t). \end{aligned}$$

- Mean reversion $v(t)$ calibrated to correlations

$$\operatorname{corr}(\log F_n(T_n), \log F_m(T_m)) \approx \left(\frac{\int_0^{\min(T_n,T_m)} e^{2\int_0^u v(s)ds} du}{\int_0^{\max(T_n,T_m)} e^{2\int_0^u v(s)ds} du} \right)^{1/2}.$$

- Volatility $\eta(t, 0, 0)$ calibrated to term vols of forward Libors;
- Skew function $\eta(\cdot, x, y)$ and SV vol of variance ε calibrated to Libor smiles;
- Mean reversion of variance $\varkappa$: fine-tune the relationship between smiles for different Libors

References

[AA02] Leif B.G. Andersen and Jesper Andreasen. Volatile volatilities. *Risk*, 15(12), December 2002.

[Gla03] Paul Glasserman. *Monte Carlo Methods in Financial Engineering*. Springer, 2003.

[GZ99] Paul Glasserman and Xiaoling Zhao. Fast greeks in forward Libor models. *Journal of Computational Finance*, 3:5–39, 1999.

[Pit03] Vladimir V. Piterbarg. A Practitioner's guide to pricing and hedging callable LIBOR exotics in forward LIBOR models. SSRN Working paper, 2003.

[Pit04a] Vladimir V. Piterbarg. Computing deltas of callable Libor exotics in forward Libor models. *Journal of Computational Finance*, 7(3):107–144, 2004.

[Pit04b] Vladimir V. Piterbarg. TARNs: models, valuation, risk sensitivities. To appear in Willmott Magazine, 2004.