Cheyette/Interest rate notes

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February 23, 2021

Some notes on the approach of Cheyette, this will connect with the wiener-chaos expansion approach by Funahashi to arrive at the approximation equation for the vanilla swaption price under Multi-factor Local Stochastic Volatility qG model

First we need to establish a couple things, such as the forward rate dynamic (in terms of the markovian variable, the centered short-rate x_t, then the bond reconstitution formula, and finally the SDE setup of the model. The material taken are from the Hoores, Funahashi-Kijima and Andersen-Piterbarg

1 Cheyette as the HJM representation with Seperable volatility

The starting term is the HJM setup:

$$df(t,T) = \sigma_f(t,T)\left(\int_t^T \sigma_f(t,s)\,ds\right)dt + \sigma_f(t,T)\,dW_t^Q \tag{1}$$

The volatility specification of the Cheyette is a general gaussian-short-rate model where it can be seperated into the instantaneous part $h(t)$ and the term-structure part $g(t,T) = \exp\left(-\int_t^T \kappa_u du\right)$

Some of the useful identities that will be used later is given here:

$\sigma_f(t,T) = h(t)\,g(t,T) = h(t)\,g(t,s)\,g(s,T) = \sigma_f(t,s)g(s,T)$

$\frac{\partial}{\partial T}\sigma_f(t,T) = -\kappa_T\sigma_f(t,T)$

The main motivation of this proof is to express the forward rate dynamic in terms of the markovian centered short-rate $x_t = r_t - f(0,t) = f(t,t) - f(0,t)$

1.1 Forward rate dynamic

This is the result from Hoores

$$\begin{aligned}
f(t,T) &= f(0,T) + \textstyle\int_0^t \sigma_f(s,T)\left(\int_s^T \sigma_f(s,u)\,du\right)ds + \int_0^t \sigma_f(s,T)\,dW_s^Q \\
&= f(0,T) + g(t,T)\textstyle\int_0^t \sigma_f(s,t)\left(\int_s^T \sigma_f(s,u)\,du\right)ds + g(t,T)\int_0^t \sigma_f(s,t)\,dW_s^Q \\
&= f(0,T) + g(t,T)\left(\textstyle\int_0^t \sigma_f(s,t)\left(\int_s^T \sigma_f(s,u)\,du\right)ds + \int_0^t \sigma_f(s,t)\,dW_s^Q\right) \\
&= f(0,T) + g(t,T)\left(\textstyle\int_0^t \sigma_f(s,t)\left(\int_s^t \sigma_f(s,u)\,du + \int_t^T \sigma_f(s,u)\,du\right)ds + \int_0^t \sigma_f(s,t)\,dW_s^Q\right) \\
&= f(0,T) + g(t,T)\left(x_t + \textstyle\int_0^t \sigma_f(s,t)\left(\int_t^T \sigma_f(s,u)\,du\right)ds\right) \\
&= f(0,T) + g(t,T)\left(x_t + \textstyle\int_0^t \sigma_f(s,t)\left(\int_t^T g(t,u)\,\sigma_f(s,t)\,du\right)ds\right) \\
&= f(0,T) + g(t,T)\left(x_t + \textstyle\int_0^t \sigma_f^2(s,t)\left(\int_t^T g(t,u)\,du\right)ds\right) \\
&= f(0,T) + g(t,T)\left(x_t + \left(\textstyle\int_t^T g(t,u)\,du\right)\int_0^t \sigma_f^2(s,t)\,ds\right) \\
&= f(0,T) + g(t,T)\left(x_t + B(t,T)\,y_t\right)
\end{aligned} \tag{2}$$

We have use the HJM representation for $x_t = f(t,t) - f(0,T) = \int_0^t \sigma_f(s,t)\left(\int_s^t \sigma_f(s,u)\,du\right)ds + \int_0^t \sigma_f(s,t)\,dW_s^Q$, and we have recovered the affine structure of the forward dynamic, where $B(t,T) = \int_t^T g(t,u)\,du$, and $y_t = \int_0^t \sigma_f^2(s,t)\,ds$, we can see that the state variables (x_t, y_t) are the information up to time t. Where the function $g(t,T)$ and $B(t,T)$ contains information of the forward terms from time t to future tenor T.

1.2 Bond Price Reconstitution formula

As with other interest rate model, we can get the zero-coupon bond price $P(t,T)$ from the forward rate dynamic

$$
\begin{aligned}
P(t,T) &= \exp\left(-\int_t^T f(t,s)\,ds\right) \\
&= \exp\left(-\int_t^T \left(f(0,s)+g(t,s)\left(x_t+B(t,s)\,y_t\right)\right)ds\right) \\
&= \frac{P(0,T)}{P(0,t)}\exp\left(-\int_t^T \left(g(t,s)\left(x_t+B(t,s)\,y_t\right)\right)ds\right) \\
&= \frac{P(0,T)}{P(0,t)}\exp\left(-x_t\int_t^T g(t,s)\,ds - y_t\int_t^T g(t,s)\,B(t,s)\,ds\right) \\
&= \frac{P(0,T)}{P(0,t)}\exp\left(-B(t,T)\,x_t-\int_t^T g(t,s)\,B(t,s)\,ds\cdot y_t\right) \\
&= \frac{P(0,T)}{P(0,t)}\exp\left(-B(t,T)\,x_t-\int_t^T g(t,s)\left(\int_t^T g(t,u)\,du\right)ds\cdot y_t\right) \\
&= \frac{P(0,T)}{P(0,t)}\exp\left(-B(t,T)\,x_t-\frac{1}{2}\left(\int_t^T g(t,s)\,ds\right)^2\cdot y_t\right) \\
&= \frac{P(0,T)}{P(0,t)}\exp\left(-B(t,T)\,x_t-\frac{1}{2}B(t,T)^2\,y_t\right)
\end{aligned}
\tag{3}
$$

Here we use the following identity: $\int\left(u^2\right)'=\int 2u'u$, so $\int u'u=\frac{1}{2}u^2$, and set $u=\int g(t,\cdot)$

1.3 Centered Short-rate dynamic

As a way to simulate the qG model, we also need to get the SDE for the qG pair (x_t, y_t):

$$
\begin{aligned}
f(t,T) &= f(0,T)+\int_0^t \sigma_f(s,T)\left(\int_s^T \sigma_f(s,u)\,du\right)ds+\int_0^t \sigma_f(s,T)\,dW_s^Q \\
r_t=\lim_{T\downarrow t} f(t,T) &= f(0,t)+\int_0^t \sigma_f(s,t)\left(\int_s^t \sigma_f(s,u)\,du\right)ds+\int_0^t \sigma_f(s,t)\,dW_s^Q \\
dr_t=dx_t &= \sigma_f(t,t)\left(\int_t^t \sigma_f(t,u)\,du\right)dt+\int_0^t \frac{\partial}{\partial t}\left(\sigma_f(s,t)\left(\int_s^t \sigma_f(s,u)\,du\right)\right)ds\cdot dt+\sigma_f(t,t)\,dW_t^Q-\kappa_t\int_0^t \sigma_f(s,t)\,dW_s^Q \\
&= \int_0^t \frac{\partial}{\partial t}\left(\sigma_f(s,t)\left(\int_s^t \sigma_f(s,u)\,du\right)\right)ds\cdot dt+\sigma_f(t,t)\,dW_t^Q-\kappa_t\int_0^t \sigma_f(s,t)\,dW_s^Q \\
&= -\kappa_t\int_0^t \sigma_f(s,t)\left(\int_s^t \sigma_f(s,u)\,du\right)ds\cdot dt+\int_0^t \sigma_f(s,t)\,\sigma_f(s,t)ds\cdot dt+\sigma_f(t,t)\,dW_t^Q-\kappa_t\int_0^t \sigma_f(s,t)\,dW_s^Q \\
&= -\kappa_t\int_0^t \sigma_f(s,t)\left(\int_s^t \sigma_f(s,u)\,du\right)ds\cdot dt-\kappa_t\int_0^t \sigma_f(s,t)\,dW_s^Q+\int_0^t \sigma_f(s,t)\,\sigma_f(s,t)ds\cdot dt+\sigma_f(t,t)\,dW_t^Q \\
&= -\kappa_t x_t dt+\int_0^t \sigma_f(s,t)\,\sigma_f(s,t)ds\cdot dt+\sigma_f(t,t)\,dW_t^Q \\
&= -\kappa_t x_t dt+y_t dt+h(t)\,dW_t^Q \\
&= \left(y_t-\kappa_t x_t\right)dt+h(t)\,dW_t^Q
\end{aligned}
\tag{4}
$$