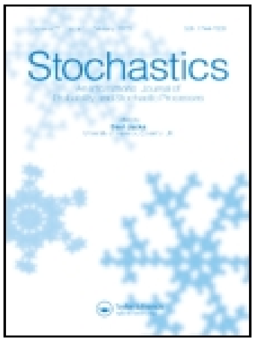

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ON THE MOMENTS OF A MULTIPLE WIENER–ITO INTEGRAL AND THE SPACE INDUCED BY THE POLYNOMIALS OF THE INTEGRAL

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In the first part of this note we derive a lower bound for $E|I_p(f)|^m$ where $I_p(f)$ is the multiple Wiener–Ito integral of the kernel $f(t_1,\ldots,t_p)$, $t\in T$ and $T=[0,\infty)$. In the second part we consider the linear space $H(I_p(f))$ of the L^2 functionals induced by the random variables $\{(I_p(f))^m, m\in\mathbb{N}\}$. For $p\geqq 3$ it is not known whether $H(I_p(f))$ is the same space of L^2 functionals which are measurable with respect to the subsigma field induced by the random variable $I_p(f)$. In the final part of this note some special results concerning the conditional expectation on the Wiener space are derived. In particular, it is shown that the conditional expectation of a random variable belonging to an odd chaos given the even chaos is zero.

KEY WORDS: Wiener–Ito integral, homogeneous chaos calculus on Wiener space.

0. INTRODUCTION

Let X and Y be two random variables possessing moments of all orders and consider the conditional expectation $\psi(y)=E(X\,|\,Y=y)$. Obviously $E((X-\psi(Y))Y^N)=0$ for every integer N. Now, if the linear span of the polynomials $Y^0, Y, Y^2, Y^3,\ldots$ (denoted $H(Y)$) is dense in the space of the L^2 random variables

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adapted to subsigma algebra generated by Y (denoted $L^2(Y)$) then $E((X-\psi(Y))Y^N)=0$ characterizes $E(X|Y)$. In general $H(Y)\subseteq L^2(Y)$ and it is possible to have $H(Y)\neq L^2(Y)$. It is well known (cf. Section 2) that if the characteristic function associated with the probability law of Y is analytic in a small domain then $L^2(X)=H(X)$. In the first part of this note we derive a lower bound on the moments of multiple Wiener–Ito integrals from which it follows that the characteristic function of any multiple Wiener–Ito integral $I_p(f)$ of order p greater than two is nonanalytic and we do not know whether $L^2(I_p(f))=H(I_p(f))$ for $p\geqq 3$ (equality holds for $p=1,2$). A definition of analytic random variables is introduced in Section 2 and compared with the notion of analytic characteristic functions. In the final section of the paper we evaluate the conditional expectation in some special cases. We show that the conditional expectation of a multiple Wiener–Ito integral of odd order given another one of even order (or given the even chaos) is zero. Also, we evaluate explicitly the conditional expectation of a multiple Wiener–Ito integral $I_p(f)$ for the special case where the conditioning is with respect to $I_q(h^{\otimes q})$ with $h^{\otimes q}=h(t_1)h(t_2)\cdots h(t_q)$.

Remark A case where the evaluation of the conditional expectation of one multiple integral given another becomes trivial is indeed when they are independent. Necessary and sufficient conditions for the independence of two random variables on Wiener space and, in particular, for the independence of two families of multiple Wiener–Ito integrals were given in Ustunel and Zakai [5, 6].

1. A LOWER BOUND ON THE ABSOLUTE MOMENTS OF MULTIPLE WIENER–ITO INTEGRALS

Let $T=[0,\infty)$ and let $f(t_1,\ldots,t_p)\in L^2(T^p)$, $I_p(f)$ will denote the multiple Wiener–Ito integral of the kernel f. As pointed out by McKean [3]

$$E|I_p(f)|^{2n}\leq\left(\frac{(2n)!}{n!2^n}\right)^p(E(I_p(f))^2)^n$$

and therefore

$$E|I_p(f)|^m\leqq\exp\left[\frac{pm}{2}\log m+0(m)\right]. \tag{1}$$

This means that as $m\to\infty$, the upper bound on $E|I_p(f)|^m$ increases as $2^{1/2}(m/e)^{mp/2}\cdot(E|I_p(f)|^2)^{m/2}$ which, as will be discussed in the next section, is quite fast. By the Nelson hypercontractivity theorem (cf. e.g. Watanabe [7]) it follows that

$$E|I_p(f)|^m\leqq(m-1)^{mp/2}(EI_p^2(f))^{m/2}$$

which, since as $m\to\infty$, $(m-1)^{mp/2}\approx(m^m/e)^{p/2}$, yields the same asymptotic growth.

This raises the question whether these upper bounds are tight and whether there exist $L^2(T^p)$ kernels f for which the rate of growth of $E|I_p(f)|^m$ is slower. These questions are answered by the following converse inequality.

PROPOSITION 1 *Given* $I_p(f)$, $\|f\|_{L^2(T^p)} \neq 0$ then there exists a constant $c > 0$ such that for all $mp \geqq 2$

$$E|I_p(f)|^m \geqq \left(\frac{mp}{2c}\right)^{mp/2} \cdot \exp - \frac{mp}{2}$$

and therefore

$$E|I_p(f)|^m \geqq \exp\left[\frac{mp}{2} \log m + 0(m)\right] \tag{2}$$

which is the same as (1) but with the inequality sign reversed.

Proof The proof is based on the following Eidlin–Linnik McKean bound (p. 202 of McKean [3]): There exists a constant $c > 0$ such that

$$\operatorname{prob}\{|I_p(f)| > \alpha\} \geqq \exp - c\alpha^{2/p}.$$

Set $X = |I_p(f)|$, $Q(\alpha) = \operatorname{prob}\{X > \alpha\}$, then, for $x_0 \geqq 0$ and integrating by parts

$$EX^m \geqq -\int_{x_0}^{\infty} \alpha^m Q(d\alpha)$$

$$= x_0^m Q(x_0) + \int_{x_0}^{\infty} m\alpha^{m-1} Q(\alpha)\, d\alpha$$

$$\geqq \int_{x_0}^{\infty} m\alpha^{m-1} Q(\alpha)\, d\alpha.$$

Setting $\alpha^{2/p} = y$ and substituting the lower bound for $Q(\alpha)$ yields

$$EX^m \geqq \frac{mp}{2} \int_{y = x^{2/p}}^{\infty} y^{(mp-2)/2} \exp - cy\, dy \tag{3}$$

$$\geqq \frac{mp}{2} x_0^{(mp-2)/p} \int \exp - cy\, dy$$

$$= \frac{mp}{2c} x_0^{(m-(2/p))} \exp - cx_0^{2/p}. \tag{3a}$$

Since x_0 is arbitrary, we may choose the particular x_0 which optimizes the right hand side of (3a). Differentiation yields that the bound is maximized for

$$x_0^{2/p}=\left(\frac{mp}{2}-1\right)\frac{1}{c}.$$

Substituting the approximation $x_0^{2/p}=mp/2c$ into (3a) yields (2). Remark: The integral of Eq. (3) with $x_0=0$ yields the gamma function.

2. ANALYTIC CHARACTERISTIC FUNCTIONS AND ANALYTIC RANDOM VARIABLES

A characteristic function $f(u)$ associated with a probability distribution $F(\theta)$, i.e. $f(u)=\int_{-\infty}^{\infty}\exp iu\theta F(d\theta)$, is said to be analytic if the following two conditions are satisfied (cf. Lukacs [2], p. 136):

i) $F(\alpha)$ has all the moments $\alpha_m=\int_{-\infty}^{\infty}\theta^m F(d\theta), m\in\mathbb{N}$

ii) $\limsup_{m\to\infty}[|\alpha_m|/m!]^{1/m}=1/\rho$ is finite

Part (ii) states that the power series expansion of $f(z)$ around the origin (i.e. $\sum\alpha_k(iz)^k/k!$) has a nonzero circle of convergence ρ. By the Stirling approximation, (ii) can be replaced by

ii') $\limsup_{m\to\infty}(|\alpha_m|^{1/m})/m<\infty$.

It follows from (1) that for $p=1,2$, the characteristic function of $I_p(f)$ is analytic. Furthermore, from (2) (or from McKean [3] and from Theorem 7.2.1 of Lukacs [2]) it follows that for $p\geqq 3$ the characteristic function of $I_p(f)$ is not analytic. Set:

$$J_p(f)=|I_p(f)|^{2/p}\cdot\operatorname{sgn} I_p(f) \tag{4}$$

then, by (1) $J_p(f)$ possesses an analytic characteristic function.

Let X be a random variable on a probability space (Ω,β,μ) such that $E|X|^n<\infty$ for all $n\in\mathbb{N}$, then in general

$$\overline{\operatorname{span}\{X^n, n\in\mathbb{N}\}}\subseteqq L^2(X)=L^2(\Omega,\sigma(X),\mu).$$

Let $H(X)$ denote $H(X)=\overline{\operatorname{span}}\{X^n, n\in\mathbb{N}\}$. A random variable will be said to be analytic if $H(X)=L^2(X)$ If the Hamburger moment problem for the moments induced by X is determinate (namely, unique) then $H(X)=L^2(X)$ (Corollary 2.3.3 of Akhiezer [1]). Consequently, if the characteristic function associated with the random variable X is analytic then X is an analytic random variable. We do not know whether $I_p(f)$, $p\geqq 3$, is analytic or if $I_p(f)$, $p\geqq 3$ is analytic provided some conditions are satisfied by f. Note, however, that $J_p(f)$ as defined by (4) is analytic.

If X is not an analytic random variable, we can define a "weak conditional expectation with respect to X" as the projection of $L^2(\Omega, \beta, \mu)$ on $H(X)$. We will denote by $\pi(Y|H(X))$ as the projection of $Y \in L^2(\Omega, \beta, \mu)$ on $H(X)$.

3. THE CONDITIONAL EXPECTATION OF SOME ELEMENTS OF THE WIENER CHAOS

Let G_{2m} and F_{2k+1} be L^2 random variables in the $2m$th and $(2k+1)$th Wiener chaos respectively, i.e. $G_{2m} = I_{2m}(g_{2m})$, $F_{2k+1} = I_{2k+1}(f_{2k+1})$. Let H_e denote the sigma-algebra generated by all the L^2 random variables which are in the Wiener chaos of even order.

PROPOSITION 2

$$E(F_{2k+1}|H_e) = 0 \quad a.s. \tag{5}$$

and in particular

$$E(F_{2k+1}|G_{2m}) = 0. \tag{6}$$

However $E(G_{2m}|F_{2k+1})$ *is, in general, a nondegenerate random variable.*

Proof Starting with the last assertion, set $k=0$, $m=1$, $F_1 = \int_T f(s) W(ds)$, where $W(s)$ is the standard Wiener process. Setting $g_2(t_1, t_2) = f(t_1) f(t_2)$ yields $G_2 = I_2(g_2) = F_1^2 - \int_T f^2(s)\, ds$. Hence

$$E(G_2|F_1) = F_1^2 - \|f\|^2_{L^2(T)}$$

which, in general, is not a degenerate random variable. In order to prove (5) it suffices to prove that

$$E(I_{2k+1}(f)|H_e) = 0 \quad \text{for all} \quad k \geqq 0.$$

We fix a complete orthonormal basis $\{e_i, i \geqq 1\}$ on $L^2(0,1)$.

i) Set $\delta_i = I_1(e_i) = \int_0^1 e_i(s) W(ds)$ and denote by τ_{ij} the sigma-field induced by the product $\delta_i \cdot \delta_j$, i.e.: $\tau_{ij} = \sigma(\delta_i \cdot \delta_j)$.

We know that the polynomials on the random variable $\delta_i \cdot \delta_j$ are dense in $L^2(\tau_{ij})$, because the characteristic function of this random variable is analytic in a neighborhood of the origin. Moreover, as we will show now, for any $p \geqq 2$ the polynomials on the random variable $\delta_i \cdot \delta_j$ are dense in $L^p(\tau_{ij})$: in fact, we have $E[\exp t|\delta_i \cdot \delta_j|] < \infty$ for $t < 1$ if $i \neq j$ and for $t < 1/2$ if $i = j$. Set $(1/p + 1/p') = 1$ and take $F \in L^{p'}(\tau_{ij})$. Asume that $E(F(\delta_i \cdot \delta_j)^n) = 0$ for any $n \geqq 0$. We want to show that $F = 0$. It suffices to prove that $E(F e^{it\delta_i \cdot \delta_j}) = 0$ for t small enough, because this means that if $F = f(\delta_i \cdot \delta_j)$ the Fourier transform of the measure $f(x)\nu(dx)$ (ν = law of $\delta_i \cdot \delta_j$) vanishes in a neighborhood of the origin. We have

$$\left|\exp(ixt) - \sum_{n=0}^{M} \frac{(ixt)^n}{n!}\right| = \left|\sum_{n=M+1}^{\infty} \frac{(ixt)^n}{n!}\right| \leqq \exp(|x|\,|t|)$$

and

$$E\left(|F|^r \left|\exp(it\delta_i \cdot \delta_j) - \sum_{n=0}^{M} \frac{(it\delta_i \cdot \delta_j)^n}{n!}\right|^r\right) \leqq E(|F|^r \exp(|t|\,|\delta_i \cdot \delta_j|^r))$$

$$\leqq (E(|F|^{p'}))^{r/p'} \left(E\left(\exp\left(|t|\,|\delta_i \cdot \delta_j| \frac{rp'}{p'-r}\right)\right)\right)^{(p'-r)/p'} < \infty$$

if $1 < r < p'$, and

$$\frac{|t| rp'}{p'-r} < \frac{1}{2}.$$

Consequently, the sequence

$$F\left(\exp(it\delta_i \cdot \delta_j) - \sum_{n=0}^{M} \frac{(it\delta_i \cdot \delta_j)^n}{n!}\right)$$

converges to zero in $L^1(\Omega)$, by the dominated convergence theorem, as $M \to \infty$. Hence $E(F \cdot (\delta_i \cdot \delta_j)^n) = 0$ for all $n \geqq 0$ implies that $EF e^{it\delta_i \cdot \delta_j} = 0$ for t sufficiently small and consequently $F = 0$ a.s.

ii) Define $\tau_N = \bigvee_{1 \leqq i, j \leqq N} \tau_{ij}$.

The multinomials in $\{\delta_i \cdot \delta_j,\ 1 \leqq i, j \leqq N\}$ are dense in $L^2(\tau_N)$. In fact, an element of $L^2(\tau_N)$ can be approximated in $L^2(\tau_N)$ by a finite linear combination of finite products of terms like $f_{ij}(\delta_i \cdot \delta_j)$ where f_{ij} are bounded and measurable functions. Then we approximate $f_{ij}(\delta_i \cdot \delta_j)$ in L^p by a polynomial in $\delta_i \cdot \delta_j$, and if p is large enough by Hölder's inequality, this gives an approximation in $L^2(\Omega)$ of $\prod_{i,j} f_{ij}(\delta_i \cdot \delta_j)$ by the corresponding product of polynomials.

iii) $E(I_{2k+1}(f) \mid H_e) = E(E(I_{2k+1}(f) \mid \bigvee_N \tau_N)$.

This formula follows from the fact that the sigma-field H_e is equal to $\bigcup_N \tau_N$. Indeed, it is clear that that the random variables $\delta_i \cdot \delta_j$ are H_e measurable. Therefore $\bigcup_N \tau_N \subseteqq H_e$. Conversely, any stochastic integral of even order $I_{2n}(g)$ can be approximated in $L_2(\Omega)$ by finite linear combinations of products like $W(A_1) \cdots W(A_{2n})$ where the A_i are pairwise disjoint Borel substitutes of T. Finally, any product of the form $W(A)W(B)$, $A \cap B = \varnothing$, can be decomposed into

$$\sum_{i,j} \delta_i \delta_j \int_A e_i(s)\, ds \int_B e_j(s)\, ds$$

and therefore $H_e \subseteqq \bigcup_N \tau_N$. Notice that, by the product formula for multiple

Wiener–Ito integrals (cf. e.g. Shigakawa, [4]), the multinomials in $\delta_i \cdot \delta_j$, $1 \leqq i, j \leqq N$ belong to the Wiener chaos of even order and are, therefore, orthogonal to $I_{2k+1}(f)$. Moreover, as pointed out above, they are dense in $L^2(\tau_N)$. Hence, by the martingale convergence theorem

$$E(I_{2k+1}(f) \mid H_e) = \lim_N E(E(I_{2k+1}(f) \mid \tau_N)) = 0,$$

and this completes the proof of (5).

Let $h(t) \in L^2(T)$ and consider $h^{\otimes q}$: the qth tensor product of $h(h^{\otimes q}(t_1, \ldots, t_q) = h(t_1) \cdot h(t_2) \cdots h(t_q))$ in this case we have:

PROPOSITION 3 *For the case where either $q=1$ or $q=p$ we have:*

$$E(I_p(f) \mid I_q(h^{\otimes q}) = K I_p(h^{\otimes p}) \tag{7}$$

where

$$K = \frac{\langle f, h^{\otimes p} \rangle_{L^2(T^p)}}{\langle h^{\otimes p}, h^{\otimes p} \rangle_{L^2(T^p)}} = \frac{\langle f, h^{\otimes p} \rangle_{L^2(T^p)}}{(\|h\|^2_{L^2(T)})^p}. \tag{8}$$

Proof Note that by the product formula for multiple Wiener–Ito integrals (cf. e.g. Shigakawa, [4]):

$$I_m(h^{\otimes m}) I_n(h^{\otimes n}) = \sum_{k=0}^{\min(m,n)} \alpha_k I_{m+n-2k}(h^{\otimes m+n-2k}).$$

Consequently,

$$(I_q(h^{\otimes q}))^N = \sum_0^{[qN/2]} \alpha_{N,k}(h^{\otimes qN-2r})).$$

Therefore

$$\begin{aligned} &E\{(I_p(f) - K I_p(h^{\otimes p}))(I_q(h^{\otimes q}))^N\} \\ &= \sum_{r=0}^{[qN/2]} \alpha_{N,r} E\{(I_p(f) - K I_p(h^{\otimes p})) I_{qN-2r}(h^{\otimes qN-2r})\}. \end{aligned} \tag{9}$$

Now,

$$E(I_p(f) \mid I_q(h^{\otimes q})) = E[E(I_p(f) \mid I_1(h)) \mid I_q(h^{\otimes q})]. \tag{10}$$

Consider (9) with $q=1$. by the orthogonality of the multiple integrals of different order, the right hand side of (9) is zero unless the integers N and r are such that

$p = N - 2r$. The term in the right hand side of (9) with r satisfying $p = N - 2r$ is also zero because of (8). Hence

$$E(I_p(f)|I_1(h)) = KI_p(h^{\otimes p}) \tag{11}$$

where K is as defined by (8). This proves (7) for $q = 1$ and substituting (11) into (10) yields (7) for $q = p$.

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