Time-dependent skews and smiles in interest rate and hybrid modeling

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1 Non-time-stationary models

- In general we prefer time-stationary models, but...
- For practical reasons, it has been accepted that the volatility needs to be time-dependent
- What about skew parameters?
- Time-dependence arises in many contexts
  - Conscious choice to be able to calibrate to vanilla options across multiple expiries
  - Naturally arises in calibration of "big" models in IR/hybrids
- Need efficient methods to handle time-dependence
- PDE/MC – always available, not very suitable for calibration
- For calibration – need fast pricing of European options or, better yet, direct relationship between time-dependent and time-independent parameters
- Idea of averaging

2 Example of averaging formula

- For motivation, consider a log-normal model with time-dependent volatility,

$$dS(t) = \sigma(t) S(t)\, dW(t).$$

- It is known that, an option value with expiry T_n in this model is equal to the Black-Scholes option value with "effective" volatility

$$\sigma_n = \left(\frac{1}{T_n} \int_0^{T_n} \sigma^2(t)\, dt \right)^{1/2}.$$

- In particular, if we had "market" volatilities (σ_n^*), $n = 1, \ldots, N$, then $\sigma(t)$ can be calibrated by solving the following equations

$$\int_0^{T_n} \sigma^2(t)\, dt = \sigma_n^2 T_n, \quad n = 1, \ldots, N.$$

  Linear in $\sigma^2(t)$, trivial to solve.

- Direct link between "model" parameter $\sigma(t)$ and "market" parameters (σ_n^*)
- Much faster that using Black-Scholes for valuation of options and implied calculations

3 Simple SV model

- SV model with constant coefficients:

$$\begin{aligned} dz(t) &= \theta(1 - z(t))\, dt + \eta\sqrt{z(t)}\, dV(t), \\ dS(t) &= \lambda(bS(t) + (1-b)L)\sqrt{z(t)}\, dW(t), \\ z(0) &= 1, \quad \langle dV, dW \rangle = 0. \end{aligned}$$

- Collection of expiries $T_1 < T_2 < \ldots < T_N$
- For each expiry T_n, have market parameters

$$(\lambda_n^*, b_n^*, \eta_n^*), n = 1, \ldots, N.$$

- Meaning that market prices of options with expiry T_n, across all strikes, are well-approximated by the model

$$\begin{aligned} dz(t) &= \theta(1 - z(t))\, dt + \eta_n^*\sqrt{z(t)}\, dV(t), \\ dS(t) &= \lambda_n^*(b_n^* S(t) + (1-b_n^*)L)\sqrt{z(t)}\, dW(t). \end{aligned}$$

- Separate model for each expiry

4 Calibration problem, simple SV model

- The problem: Find time-dependent parameters

$$(\sigma(t), \beta(t), \gamma(t)), \quad t \geq 0, \tag{1}$$

in the model

$$\begin{aligned} dz(t) &= \theta(1 - z(t))\, dt + \gamma(t) \sqrt{z(t)}\, dV(t), \\ dS(t) &= \sigma(t)(\beta(t) S(t) + (1 - \beta(t)) L) \sqrt{z(t)}\, dW(t), \end{aligned} \tag{2}$$

such that for each T_n, option prices for all strikes K in the model (2) match "market" prices, ie produced by the model

$$\begin{aligned} dz(t) &= \theta(1 - z(t))\, dt + \eta_n^* \sqrt{z(t)}\, dV(t), \\ dS(t) &= \lambda_n^* (b_n^* S(t) + (1 - b_n^*) L) \sqrt{z(t)}\, dW(t) \end{aligned}$$

- Main idea: instead of matching the market **prices**, match market **SV parameters**, ie find formulas relating (1) to $(\lambda_n^*, b_n^*, \eta_n^*)$, $n = 1, \ldots, N$, **directly**

- "Averaging", "homogenization", or "effective media" approach, see [Pit05b], [Pit05c]

5 Averaging volatility of variance I

- Integrate the SDE ($b = 1$ for brevity),

$$S(T) \stackrel{D}{\sim} S(0) \exp\left(\sqrt{r(T)}\xi - \frac{1}{2} r(T)\right),$$

$$r(T) = \int_0^T \sigma^2(t)\, z(t)\, dt, \quad \xi \sim \mathcal{N}(0,1).$$

- $r(T)$ is ”“realized variance”. Black-Scholes with “random” variance
- Curvature of the smile depends on the variance of realized variance (kurtosis, 4-th moment)
- Averaged vol of variance η (to T) is obtained by solving

$$\mathbf{E}\left(\int_0^T \sigma^2(t)\, z(t)\; dt\right)^2 = \mathbf{E}\left(\int_0^T \sigma^2(t)\, \bar{z}(t)\; dt\right)^2,$$

where

$$\begin{aligned} dz(t) &= \boldsymbol{\theta}(1 - z(t))\; dt + \gamma(t)\sqrt{z(t)}\, dV(t), \\ d\bar{z}(t) &= \boldsymbol{\theta}(1 - \bar{z}(t))\; dt + \eta\sqrt{\bar{z}(t)}\, dV(t). \end{aligned}$$

6 Averaging volatility of variance II

- Formula:

$$\eta^2 = \frac{\int_0^T \gamma^2(t)\,\rho(t)\; dt}{\int_0^T \rho(t)\; dt}, \tag{3}$$

where the weight function $\rho(\cdot)$ is given by

$$\rho(r) = \int_r^T ds \int_s^T dt\, \sigma^2(t)\, \sigma^2(s)\, e^{-\theta(t-s)} e^{-2\theta(s-r)}.$$

- Somewhat ad-hoc but works well (especially after the proper choice of θ, see above)

7 Averaging skew

- Fixed T, vol of variance already averaged (use constant η)
  - Time-dependent skew

$$dS(t) = \sigma(t)\left(\beta(t)\,S(t) + (1-\beta(t))\,S(0)\right)\sqrt{z(t)}\,dW(t),$$

  - Constant skew

$$d\bar{S}(t) = \sigma(t)\left(b\bar{S}(t) + (1-b)\,\bar{S}(0)\right)\sqrt{z(t)}\,dW(t).$$

  - Given $\beta(\cdot)$, find b such that option prices for different strikes (same expiry T) are matched between two models

8 Skew averaging in the small-slope limit I

- As a tool to relate $\{\beta(t)\}_{t=0}^{T}$ to b we use a method of small slope expansion
- Let $g(t,x)$ be a time-dependent, and $\bar{g}(x)$ a time-independent local volatility functions, assuming without loss of generality that

$$g(t,x_0) \equiv 1, \quad \bar{g}(x_0) = 1, \quad t \in [0,T],$$

- Define

$$\begin{aligned} g_\varepsilon(t,x) &= g(t, x_0 + (x - x_0)\varepsilon), \\ \bar{g}_\varepsilon(x) &= \bar{g}(x_0 + (x - x_0)\varepsilon), \end{aligned}$$

- Define two families of diffusions indexed by ε,

$$\begin{aligned} dX_\varepsilon(t) &= g_\varepsilon(t, X_\varepsilon(t)) \sqrt{z(t)}\sigma(t)\, dW(t), \quad X_\varepsilon(0) = x_0, \\ dY_\varepsilon(t) &= \bar{g}_\varepsilon(Y_\varepsilon(t)) \sqrt{z(t)}\sigma(t)\, dW(t), \quad Y_\varepsilon(0) = x_0. \end{aligned}$$

- Define

$$q(\varepsilon) = \mathbf{E}\left(X_\varepsilon(T) - Y_\varepsilon(T)\right)^2.$$

- We look for conditions on $\bar{g}(\cdot)$ that minimize $q(\varepsilon)$ for small ε
- The condition ensures that options will all strikes are recovered as best as possible

9 Skew averaging in the small-slope limit II

- The main result: Any function $\bar{g}$ that minimizes $q(\varepsilon)$ for small ε satisfies the condition

$$\frac{\partial \bar{g}(x_0)}{\partial x} = \int_0^T \frac{\partial g(t, x_0)}{\partial x} w(t)\, dt,$$

where

$$\begin{aligned} w(t) &= \frac{v^2(t)\,\sigma^2(t)}{\int_0^T v^2(t)\,\sigma^2(t)\, dt}, \\ v^2(t) &= \mathbf{E}\left(z(t)\,(X_0(t) - x_0)^2 \right). \end{aligned}$$

- Comments:
  - “Total skew” $\frac{\partial \bar{g}(x_0)}{\partial x}$ is the average of “local skews” $\frac{\partial g(t,x_0)}{\partial x}$ with weights $w(t)$
  - Weights proportional to total variance, i.e. local slope further away matters more
  - Can get the same result under different criteria, i.e. robust

10 Average skew formula

- Apply the general skew homogenization result to the model
- The effective skew b for the equation

$$dS(t) = \sigma(t)\left(\beta(t) S(t) + (1 - \beta(t)) S(0)\right) \sqrt{z(t)}\, dW(t)$$

over a time horizon $[0, T]$ is given by

$$b = \int_0^T \beta(t)\, w(t)\; dt,$$

with

$$\begin{aligned} w(t) &= \frac{v^2(t)\,\sigma^2(t)}{\int_0^T v^2(t)\,\sigma^2(t)\; dt}, \\ v^2(t) &= \mathbf{E}\left[\left(X_0(t) - x_0\right)^2 u(t)\right] \\ &= z_0^2 \int_0^t \sigma^2(s)\; ds + z_0 \eta^2 e^{-\theta t} \int_0^t \sigma^2(s)\, \frac{e^{\theta s} - e^{-\theta s}}{2\theta}\, ds. \end{aligned}$$

- Example: No SV ($\eta = 0$), constant volatility $\sigma(t) \equiv \sigma$,

$$b = \left(T^2/2\right)^{-1} \int_0^T t\beta(t)\; dt.$$

11 Averaging volatility I

- Having averaged the vol of variance and the skew, the problem is reduced to a well-known one
- Approximate the dynamics of

$$dS(t) = \sigma(t)\left(bS(t) + (1-b)\,S(0)\right)\sqrt{z(t)}\,dW(t)$$

with

$$d\bar{S}(t) = \lambda\left(b\bar{S}(t) + (1-b)\,\bar{S}(0)\right)\sqrt{z(t)}\,dW(t)\,.$$

- Can extend the constant-parameter Fourier formula as in [Lew00], Andersen [ABR01]. Do Fourier integral with integrand a solution to Riccati ODEs. Workable but slow.
- We propose our own method – simple, fast, intuitive and accurate.
- Idea: approximate a European option payoff locally with a function whose expectation can be computed in both models above; choose λ to match the two.

12 Averaging volatility II

- Recall
$$\mathbf{E}\left(S\left(T\right)-S_0\right)^+=\mathbf{E}\left(\mathbf{E}\left(\left(S\left(T\right)-S_0\right)^+\middle|\,z\left(\cdot\right)\right)\right). \tag{4}$$

- The distribution of $S\left(T\right)$ conditioned on a particular path $\left\{z\left(t\right)\right\}_{t=0}^{T}$ is a shifted lognormal.

- The inside condition expectation in (4) can be evaluated easily to yield
$$\mathbf{E}\left(S\left(T\right)-S_0\right)^+=\mathbf{E}g\left(\int_0^T\sigma^2\left(t\right)z\left(t\right)\,dt\right),$$
where g is a known function (ATM Black price as a function of variance),
$$\begin{aligned}g\left(x\right)&=\frac{S_0}{b}\left(2\Phi\left(b\sqrt{x}/2\right)-1\right),\\ \Phi\left(y\right)&=\mathbf{P}\left(\xi<y\right),\quad \xi\sim\mathcal{N}\left(0,1\right).\end{aligned}$$

- The problem of finding the "effective" variance can then be represented as finding such λ that
$$\mathbf{E}g\left(\int_0^T\sigma^2\left(t\right)z\left(t\right)\,dt\right)=\mathbf{E}g\left(\lambda^2\int_0^T z\left(t\right)\,dt\right).$$

13 Averaging volatility III

- Moment-generating function in both models can be computed easily, so approximate g with an exponential

$$g(x) \approx a + be^{-cx}$$

by matching the value and first two derivatives at

$$\zeta = \mathbf{E} \int_0^T \sigma^2(t)\, z(t)\, dt$$

- The problem reduced to finding λ such that

$$\mathbf{E} \exp\left(\frac{g''(\zeta)}{g'(\zeta)} \int_0^T \sigma^2(t)\, z(t)\, dt \right) = \mathbf{E} \exp\left(\lambda^2 \frac{g''(\zeta)}{g'(\zeta)} \int_0^T z(t)\, dt \right).$$

- Very fast and easy numerical search for λ (starting with a good initial guess $\lambda^2 = T^{-1} \int_0^T \sigma^2(t)\, dt$).

14 Typical calibration results I

Market parameters (see next Figure for smiles)

Expiry (years), T_n	0.02	1	5	10	15	20	25	30
Market volatility λ_n^*	20.0%	19.0%	18.0%	17.0%	16.0%	15.5%	15.0%	14.5%
Market skew b_n^*	100%	90%	80%	70%	60%	55%	50%	45%
Market vol of var η_n^*	100%	110%	120%	130%	140%	145%	150%	155%

Calibrated model parameters

Expiry (years), t	0.02	1	5	10	15	20	25	30
Model volatility $\sigma(t)$	20.0%	18.6%	16.8%	14.8%	12.7%	13.4%	12.3%	11.2%
Model skew $\beta(t)$	100%	88%	75%	55%	31%	38%	26%	12%
Model vol of var $\gamma(t)$	100%	121%	139%	174%	166%	165%	178%	186%

15 Typical calibration results II

Market smiles

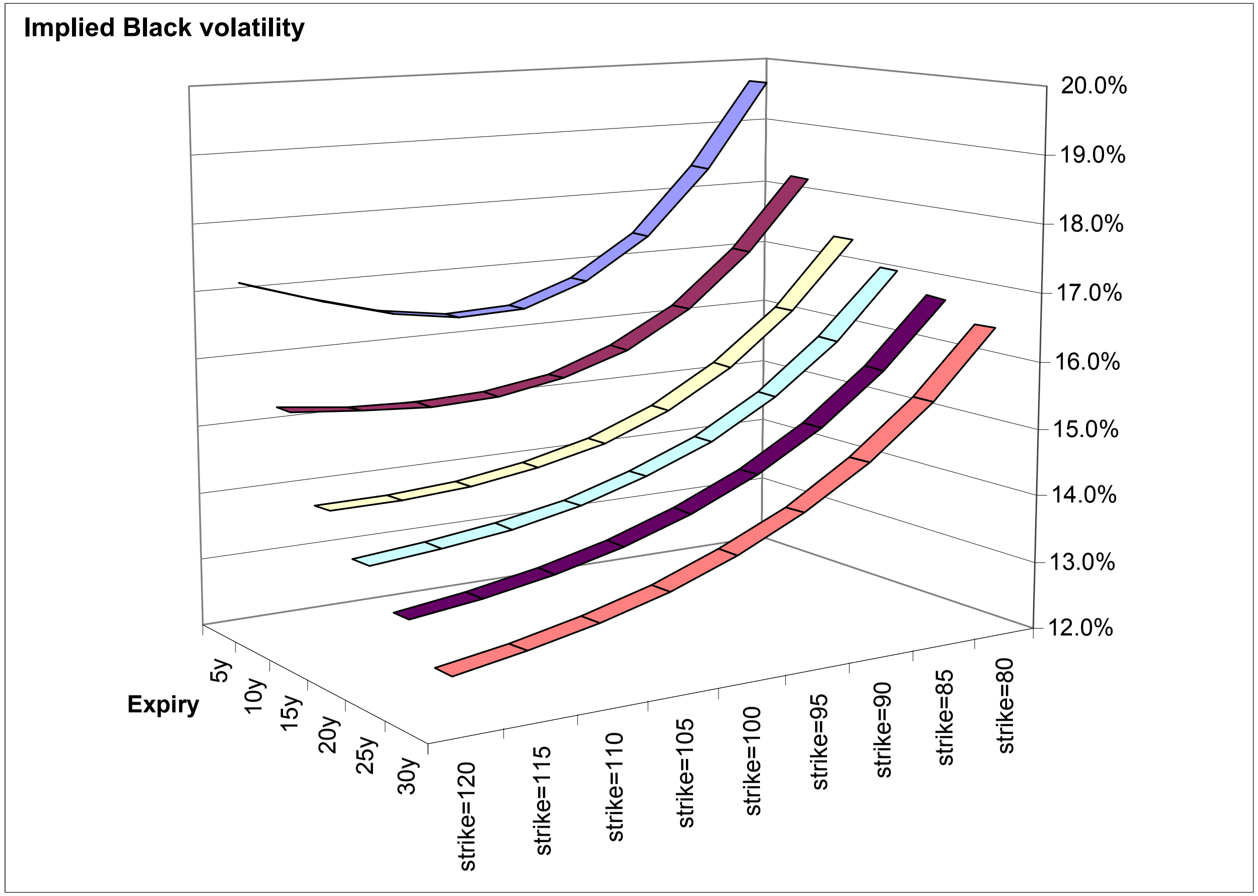

Figure 1:

16 Typical calibration results III

- Errors in 10,000th of Black volatilities between
  - "market" option values computed via Fourier methods from constant "market" parameters versus
  - PDE-computed option values in the model with calibrated (using the formulas above) time-dependent parameters.

Strike	60	70	80	90	100	110	120	130	140
Expiry 5y	-14.54	-11.10	-6.53	-2.60	-0.79	-0.26	-1.62	-3.67	-5.64
Expiry 10y	-13.79	-9.96	-5.90	-2.69	-0.88	0.11	-0.13	-1.07	-2.25
Expiry 15y	-8.86	-5.57	-2.24	0.31	1.71	2.43	2.18	1.34	0.27
Expiry 20y	-7.41	-4.60	-1.94	0.06	1.22	1.92	1.92	1.44	0.74
Expiry 25y	-3.94	-2.32	-0.69	0.53	1.23	1.67	1.63	1.27	0.77
Expiry 30y	-0.07	0.47	1.19	1.73	1.96	2.08	1.89	1.51	1.05

17 Improving BGM swaption approximation I

- Methods we developed is useful even in non-SV contexts
- Consider the standard log-normal BGM model. Typical approximation (see [AA00]) assumes log-normality of the swap rate.
- Well-known that there is an "implied smile" in swap rates.
- Becomes important for out-of-the-money, long-expiry, long-maturity swap rates
- Let $L_n(t)$ be spanning forward Libor rates

$$dL_n(t)/L_n(t) = \sum_{k=1}^{K} \sigma_k(t;n)\, dW_k^{n+1}(t), \quad n = 1, \ldots, N-1.$$

Improving BGM swaption approximation II

- Swap rate dynamics (under swap measure)

$$
\begin{aligned}
dS(t)/S(t) &= \sum_{k=1}^{K} \sum_{i=n}^{m} w_i(t)\,\sigma_k(t;i)\; dW_k^{n,m}(t)\,, \\
w_i(t) &= \frac{L_i(t)}{S(t)} \frac{\partial S(t)}{\partial L_i(t)}.
\end{aligned}
$$

- As follows from Dupire ([Dup94]), the value of all European options on $S(T)$ in the model above is the same as in the local volatility model

$$dS(t)/S(t) = \varphi(t, S(t))\; dW(t)\,,$$

where

$$\varphi^2(t,x) = \mathbf{E}\left(\sum_{k=1}^{K} \left(\sum_{i=n}^{m} w_i(t)\,\sigma_k(t;i) \right)^2 \middle| S(t) = x \right)$$

- To calibrate to caplets and swaptions, apply averaging formulas to Libor and swap rates

19 Improving BGM swaption approximation III

- Plan:
  - use approximations to compute $\varphi(t, x)$
  - Use skew-averaging to find an "effective" skew
  - Use displaced-diffusion model with effective skew to price swaptions

Improving BGM swaption approximation IV

- In computing $\varphi(t,x)$ we can be fairly crude, as we know the adjustment to lognormality is "small"
- We first write

$$\mathbf{E}\left(\sum_{k=1}^{K}\left(\sum_{i=n}^{m} w_i(t)\,\sigma_k(t;i)\right)^2 \middle| S(t)\right) \approx \sum_{k=1}^{K}\left(\sum_{i=n}^{m}\mathbf{E}\left(w_i(t)\middle| S(t)\right)\sigma_k(t;i)\right)^2$$

- Clearly we need $\mathbf{E}\left(w_i(t)\middle| S(t)\right)$ which we split as (recall definition of w_i)

$$\mathbf{E}\left(w_i(t)\middle| S(t)\right) \approx \frac{1}{S(t)}\mathbf{E}\left(L_i(t)\middle| S(t)\right)\mathbf{E}\left(\frac{\partial S(t)}{\partial L_i(t)}\middle| S(t)\right)$$

- The value $\mathbf{E}\left(L_i(t)\middle| S(t)\right)$ is computed by assuming joint log-normality of $L_i(t)$, $S(t)$ under the swap measure
- The value $\mathbf{E}\left(\frac{\partial S(t)}{\partial L_i(t)}\middle| S(t)\right)$ is computed by second-order expansion

$$\mathbf{E}^A\left(\frac{\partial S(t)}{\partial L(t)}\middle| S(t)\right) \approx \mathbf{E}\left(\frac{\partial S(t)}{\partial L_j(t)}\right) + \frac{\partial\left(\frac{\partial S}{\partial L_j}\right)}{\partial S}\left(S(t)-S(0)\right)$$

and approximating various derivatives by their values at time $t=0$.

21 Improving BGM swaption approximation V

- Once all the approximations are (carefully) computed, the model is reduced to the form

$$dS(t) = \varphi(t, S(t))\, S(t)\; dW(t),$$

which can be further simplified to

$$\begin{aligned} dS(t) &= S(t)\,(\alpha(t) + \beta(t)\, S(t))\; dW(t), \\ \alpha(t) &= \varphi(t, S(0)), \quad \beta(t) = \left.\frac{\partial \varphi(t, x)}{\partial x}\right|_{x=S(0)}. \end{aligned}$$

- Then "total" effective skew can be computed from $\beta(t)$ by the averaging results presented above. Details in the upcoming book [AP05]

22 Multi-currency model with skew I

- Consider a model for PRDCs, two interest rate processes in two currencies and a process for FX
- One-factor Gaussian for interest rates, but skew for FX. Important for PRDC!
- The model (under domestic risk-neutral measure)

$$\begin{aligned} dP_d(t,T)/P_d(t,T) &= r_d(t)\,dt + \sigma_d(t,T)\,dW_d(t), \qquad (5)\\ dP_f(t,T)/P_f(t,T) &= r_f(t)\,dt - \rho_{fS}\sigma_f(t,T)\,\gamma(t,S(t))\,dt + \sigma_f(t,T)\,dW_f(t),\\ dS(t)/S(t) &= (r_d(t) - r_f(t))\,dt + \gamma(t,S(t))\,dW_S(t), \end{aligned}$$

where a parametrized local volatility model for FX is used

$$\gamma(t,x) = \nu(t)\left(\frac{x}{L(t)}\right)^{\beta(t)-1}. \qquad (6)$$

- Need time-dependence of $\beta(t)$ to match different smiles of FX options for different expiries
- Valuation – same 3-factor PDE. Need fast calibration methods.

23 Multi-currency model with skew II

- Vanilla FX market – options on forward FX. Forward FX rate satisfies (under domestic T-forward measure)

$$\frac{dF\left(t,T\right)}{F\left(t,T\right)} = \Lambda\left(t, F\left(t,T\right) D\left(t,T\right)\right)\, dW_F\left(t\right), \tag{7}$$

where

$$\begin{aligned}
\Lambda\left(t,x\right) &= \left(a\left(t\right) + b\left(t\right)\gamma\left(t,x\right) + \gamma^2\left(t,x\right)\right)^{1/2}, \\
a\left(t\right) &= \left(\sigma_f\left(t,T\right)\right)^2 + \left(\sigma_d\left(t,T\right)\right)^2 - 2\rho_{df}\sigma_f\left(t,T\right)\sigma_d\left(t,T\right), \\
b\left(t\right) &= 2\rho_{fS}\sigma_f\left(t,T\right) - 2\rho_{dS}\sigma_d\left(t,T\right),
\end{aligned}$$

and $D\left(t,T\right) \triangleq P_d\left(t,T\right)/P_f\left(t,T\right)$

- SDE not closed in F. Has an extra stochastic process $D\left(\cdot,T\right)$. Approximate with $D_0\left(t,T\right) \triangleq P_d\left(0,t,T\right)/P_f\left(0,t,T\right)$? Not accurate
- Use Dupire again.

- For the purposes of European option valuation we can replace (7) with

$$\frac{dF(t,T)}{F(t,T)} = \tilde{\Lambda}(t, F(t,T))\, dW_F(t), \tag{8}$$

where

$$\tilde{\Lambda}^2(t,x) = \mathbf{E}_0^T\left(\Lambda^2(t, F(t,T)\, D(t,T)) \middle| F(t,T) = x\right).$$

- After some computations

$$\begin{aligned}\hat{\Lambda}(t,x) &\approx \left(a(t) + b(t)\,\hat{\gamma}(t,x) + \hat{\gamma}^2(t,x)\right)^{1/2},\\ \hat{\gamma}(t,x) &= \nu(t)\left(x\frac{D_0(t,T)}{L(t)}\right)^{\beta(t)-1}\left(1 + (\beta(t)-1)\, r(t)\left(\frac{x}{F(0,T)} - 1\right)\right).\end{aligned}$$

- Here $r(t)$ is a “regression coefficient” between $F(\cdot,T)$ and $D(\cdot,T)$.
- The “skew correction”, induced by the stochasticity of $D(\cdot,T)$, can be seen to be $(\beta(t)-1)\, r(t)$

25 Multi-currency model with skew IV

- Once the “local” skew is computed in

$$\frac{dF(t,T)}{F(t,T)} = \tilde{\Lambda}(t, F(t,T))\ dW_F(t)$$

it is just a matter of applying the skew averaging formula to come up with the “effective” skew and a fast calibration to European options

- Details in [Pit05a]

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