Short-term at-the-money asymptotics under stochastic volatility models

Omar El Euch
École Polytechnique

Masaaki Fukasawa
Graduate School of Engineering Science, Osaka University
1-3 Machikaneyama, Toyonaka, Osaka, JAPAN
fukasawa@sigmath.es.osaka-u.ac.jp

Jim Gatheral
Baruch College, The City University of New York

Mathieu Rosenbaum
École Polytechnique

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Abstract

A small-time Edgeworth expansion of the density of an asset price is given under a general stochastic volatility model, from which asymptotic expansions of put option prices and at-the-money implied volatilities follow. A limit theorem for at-the-money implied volatility skew and curvature is also given as a corollary. The rough Bergomi model is treated as an example.

1 Introduction

A stochastic volatility model is an extension of the Black-Scholes model that incorporates an empirical evidence that the volatility of an asset price is not constant in its time-series data as well as in its option price data. The Heston and SABR models among others are popular in financial practices owing to (semi-)analytic (approximation) formulas for the vanilla option prices or the option-implied volatilities. See e.g., [11] for a practical guide on stochastic volatility modeling.

Recently, attracting much attention is a class of stochastic volatility models where the volatility is driven by a fractional Brownian motion. This is due to their consistency to a power law of the term structure in the implied volatility skew which has been empirically recognized; see [1, 3, 5, 7, 9, 10, 12, 13]. To be consistent, the fractional Brownian motion must be correlated with a Brownian motion driving the asset price and its Hurst parameter must be smaller than 1/2. The latter means in particular that the volatility path is rougher than a Brownian motion and so, this class of the models is often referred as the rough volatility models. Since the models do not admit of explicit expressions for option prices or implied volatilities, the above mentioned consistency has been discussed through asymptotic analyses.

The aim of this paper is to provide a general framework under which the short-term asymptotics of the at-the-money implied volatility is studied. The framework is for a general continuous stochastic volatility model. The rough Bergomi model introduced by [3] is treated as an example. The asymptotic expansion of the at-the-money implied volatility is given up to the second-order, while the first order expansion was already given in [9] by a different method. For the SABR model Osajima [17] gave the expansion based on the Watanabe-Yoshida theory; see e.g., [14, 19]. The same expansion formula was also obtained in Medvedev and Scaillet [15] by a formal computation. Friz et al. [6] derived the asymptotic skew and curvature of the implied volatility that correspond to the first and the second order terms by assuming the asymptotic behavior of the density function of the underlying asset price. Here, we introduce a novel approach based on a conditional Gaussianity of the stochastic volatility model to prove the validity of a second order density expansion for a general stochastic volatility model. From this density expansion follow expansions of the option prices and the implied volatility as well as the asymptotic skew and curvature formula. In contrast to [14, 17, 19], we do not rely on the Malliavin calculus, which enables us to treat effectively the rough volatility models. In contrast to the elementary method of [9], our approach can be extended to higher-order expansions without any additional theoretical difficulty. We choose the square root of the forward variance, that is, the fair strike of a variance swap, as the leading term of our asymptotic expansion, while a recent work [2] studies the difference between the implied volatility and the fair strike of a volatility swap in terms of the Malliavin derivatives.

The paper is organized as follows. In Section 2, we describe the model framework. In Section 3, we derive the asymptotic expansions of the characteristic function, the density function, the put option prices and the at-the-money implied volatility function. In Section 4, we derive the asymptotic behavior of the at-the-money implied volatility skew and curvature. In Section 5, we show that the rough Bergomi model fits the framework and compute the coefficients of the expansion for this particular model.

2 Framework

2.1 Assumptions

Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a filtration $\{\mathcal{F}_t; t \in \mathbb{R}\}$ satisfying the usual assumptions. A log price process Z is assumed to follow

$$\mathrm{d}Z_t = r\mathrm{d}t - \frac{1}{2}v_t\mathrm{d}t + \sqrt{v_t}\mathrm{d}B_t$$

under an equivalent measure Q, where $r \in \mathbb{R}$ stands for an interest rate and v is a positive continuous process adapted to a smaller filtration $\{\mathcal{G}_t; t \in \mathbb{R}\}$, of which the square root is called the volatility of Z. The Brownian motion B is decomposed as

$$\mathrm{d}B_t = \rho_t\mathrm{d}W_t + \sqrt{1-\rho_t^2}\mathrm{d}W'_t,$$

where W' is an $\{\mathcal{F}_t\}$-Brownian motion independent of $\mathcal{G}_t$ for all $t \in \mathbb{R}$, W is a $\{\mathcal{G}_t\}$-Brownian motion and ρ is a progressively measurable processes with respect to $\{\mathcal{G}_t\}$ and taking values in $[-1, 1]$. A typical situation for stochastic volatility models, including the Heston, SABR and rough Bergomi models, is that (W, W') is a two dimensional $\{\mathcal{F}_t\}$-Brownian motion and $\{\mathcal{G}_t\}$ is the filtration generated by W, that is,

$$\mathcal{G}_t = \mathcal{N} \vee \sigma(W_s - W_r; r \le s \le t),$$

where $\mathcal{N}$ is the null sets of $\mathcal{F}$.

An arbitrage-free price $p(K, \theta)$ of a put option at time 0 with strike $K > 0$ and maturity $\theta > 0$ is given by

$$p(K,\theta) = e^{-r\theta}E^Q[(K - \exp(Z_\theta))_+|\mathcal{F}_0] = e^{-r\theta}\int_0^K Q(\log x \ge Z_\theta|\mathcal{F}_0)\mathrm{d}x.$$

Denote by E_0 and $\|\cdot\|_p$ respectively the expectation and the L^p norm under the regular conditional probability measure of Q given $\mathcal{F}_0$, of which the existence is assumed. We impose the following technical condition: for any $p > 0$,

$$\sup_{\theta\in(0,1)}\left\|\frac{1}{\theta}\int_0^\theta v_t\mathrm{d}t\right\|_p < \infty, \quad \sup_{\theta\in(0,1)}\left\|\left\{\frac{1}{\theta}\int_0^\theta v_t(1-\rho_t^2)\mathrm{d}t\right\}^{-1}\right\|_p < \infty. \tag{1}$$

The forward variance curve $v^0(t)$ is defined by

$$v^0(t) = E_0[v_t] = E^Q[v_t|\mathcal{F}_0].$$

Let

$$\sigma_0(\theta) = \sqrt{\int_0^\theta v^0(t)\mathrm{d}t}.$$

Changing variable as

$$x = F\exp\left(\zeta\sigma_0(\theta)\right), \;\; F = \exp(r\theta + Z_0),$$

we have

$$\frac{p(Fe^{z\sigma_0(\theta)}, \theta)}{F\sigma_0(\theta)} = e^{-r\theta}\int_{-\infty}^{z} Q\left(\zeta \geq X_\theta|\mathcal{F}_0\right) e^{\sigma_0(\theta)\zeta}\mathrm{d}\zeta,$$

where

$$X_\theta = -\frac{1}{2\sigma_0(\theta)}\langle M\rangle_\theta + \frac{1}{\sigma_0(\theta)}M_\theta, \;\; M_\theta = \int_0^\theta \sqrt{v_t}\mathrm{d}B_t, \;\; \langle M\rangle_\theta = \int_0^\theta v_t\mathrm{d}t.$$

We assume the following asymptotic structure: there exists a family of random vectors

$$\left\{(M_\theta^{(0)}, M_\theta^{(1)}, M_\theta^{(2)}, M_\theta^{(3)}); \theta \in (0,1)\right\}$$

such that

1. the law of $M_\theta^{(0)}$ is standard normal for all $\theta > 0$,
2. $$\sup_{\theta\in(0,1)} \|M_\theta^{(i)}\|_p < \infty, \;\; i = 1,2,3 \tag{2}$$

   for all $p > 0$ and
3. for some $H \in (0, 1/2]$ and $\epsilon \in (0, H)$,

$$\begin{aligned} &\lim_{\theta\to 0} \theta^{-2H-2\epsilon}\left\|\frac{M_\theta}{\sigma_0(\theta)} - M_\theta^{(0)} - \theta^H M_\theta^{(1)} - \theta^{2H} M_\theta^{(2)}\right\|_{1+\epsilon} = 0, \\ &\lim_{\theta\to 0} \theta^{-H-2\epsilon}\left\|\frac{\langle M\rangle_\theta}{\sigma_0(\theta)^2} - 1 - \theta^H M_\theta^{(3)}\right\|_{1+\epsilon} = 0. \end{aligned} \tag{3}$$

Further, we assume the existence of the derivatives

$$\begin{aligned} a_\theta^{(i)}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{E_0[M_\theta^{(i)}|M_\theta^{(0)} = x]\phi(x)\right\}, \;\; i = 1,2,3, \\ b_\theta(x) &= \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left\{E_0[M_\theta^{(1)}|M_\theta^{(0)} = x]\phi(x)\right\} \\ c_\theta(x) &= \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left\{E_0[|M_\theta^{(1)}|^2|M_\theta^{(0)} = x]\phi(x)\right\} \end{aligned} \tag{4}$$

in the Schwartz space (i.e., the space of the rapidly decreasing smooth functions), where ϕ is the standard normal density.

2.2 Regular stochastic volatility models

Here we briefly discuss that regular stochastic volatility models satisfy all the above assumptions. Let us consider the volatility process $v_t = v(X_t)$, where X is a Markov process satisfying a stochastic differential equation

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + c(X_t)\mathrm{d}W_t$$

and v is a smooth positive function defined on the state space of X. Let $\rho \in (-1, 1)$ be a constant and $\{\mathcal{G}_t\}$ be the augmented filtration generated by W. We assume (1), which is satisfied in the usual cases including the log-normal SABR and Heston models. Denote by L the generator of X. Put $f = \sqrt{v}$, $g = f'c$ and $h = v'c$. Then, by Itô's formula, we have

$$\begin{aligned}
M_\theta &= f(X_0)B_\theta + \int_0^\theta \int_0^t g(X_s)\mathrm{d}W_s\mathrm{d}B_t + \int_0^\theta \int_0^t Lf(X_s)\mathrm{d}s\mathrm{d}B_t, \\
\langle M\rangle_\theta &= v(X_0)\theta + \int_0^\theta \int_0^t h(X_s)\mathrm{d}W_s\mathrm{d}t + \int_0^\theta \int_0^t Lv(X_s)\mathrm{d}s\mathrm{d}t.
\end{aligned}$$

Let $\bar{B}^\theta_t = \theta^{-1/2}B_{\theta t}$, $\bar{W}^\theta_t = \theta^{-1/2}W_{\theta t}$ and $X^\theta_t = X_{\theta t}$. Then

$$\begin{aligned}
\frac{M_\theta}{\sqrt{\theta}} &= f(X_0)\bar{B}^\theta_1 + \sqrt{\theta}\int_0^1 \int_0^u g(X^\theta_v)\mathrm{d}\bar{W}^\theta_v\mathrm{d}\bar{B}^\theta_u + \theta\int_0^1 \int_0^u Lf(X^\theta_v)\mathrm{d}v\mathrm{d}\bar{B}^\theta_u, \\
\frac{\langle M\rangle_\theta}{\theta} &= v(X_0) + \sqrt{\theta}\int_0^1 \int_0^u h(X^\theta_v)\mathrm{d}\bar{W}^\theta_v\mathrm{d}u + \theta\int_0^1 \int_0^u Lv(X^\theta_v)\mathrm{d}v\mathrm{d}u.
\end{aligned}$$

It would follow that

$$\frac{\sigma_0(\theta)^2}{\theta} = \frac{E_0[\langle M\rangle_\theta]}{\theta} = v(X_0) + \frac{1}{2}Lv(X_0)\theta + O(\theta^{3/2}),$$

and so

$$\frac{\sigma_0(\theta)}{\sqrt{\theta}} = f(X_0) + \frac{1}{4}\frac{Lv(X_0)}{f(X_0)}\theta + O(\theta^{3/2})$$

under a mild regularity condition. Then, we have (3) with $H = 1/2$, $M^{(0)}_\theta = \bar{B}^\theta_1$ and

$$\begin{aligned}
M^{(1)}_\theta &= \frac{g(X_0)}{f(X_0)}\int_0^1 \bar{W}^\theta_u\mathrm{d}\bar{B}^\theta_u, \\
M^{(2)}_\theta &= -\frac{Lv(X_0)}{4v(X_0)}\bar{B}^\theta_1 + \frac{g'(X_0)c(X_0)}{f(X_0)}\int_0^1\int_0^u \bar{W}^\theta_v\mathrm{d}\bar{W}^\theta_v\mathrm{d}\bar{B}^\theta_u + \frac{Lf(X_0)}{f(X_0)}\int_0^1 u\mathrm{d}\bar{B}^\theta_u, \\
M^{(3)}_\theta &= \frac{h(X_0)}{v(X_0)}\int_0^1 \bar{W}^\theta_u\mathrm{d}u
\end{aligned}$$

again under a mild regularity condition. Further, the derivatives (4) exist in the Schwartz space because $E_0[M^{(i)}_\theta | M^{(0)}_\theta = x]$ and $E_0[|M^{(1)}_\theta|^2 | M^{(0)}_\theta = x]$ are polynomials of x; see e.g., Nualart et al. [16] or Appendix A below.

3 Asymptotic expansions

3.1 Characteristic function expansion

Here we give an asymptotic expansion of the characteristic function of X_θ. Let

$$Y_\theta = M_\theta^{(0)} + \theta^H M_\theta^{(1)} + \theta^{2H} M_\theta^{(2)} - \frac{\sigma_0(\theta)}{2}\left(1 + \theta^H M_\theta^{(3)}\right).$$

Lemma 3.1 *For any $\alpha \in \mathbb{N} \cup \{0\}$,*

$$\sup_{|u|\le\theta^{-\epsilon}} |E_0[X_\theta^\alpha e^{iuX_\theta}] - E_0[Y_\theta^\alpha e^{iuY_\theta}]| = o(\theta^{2H+\epsilon}).$$

Proof: Since $|e^{ix} - 1| \le |x|$, we have

$$\begin{aligned}|E_0[X_\theta^\alpha e^{iuX_\theta}] - E_0[Y_\theta^\alpha e^{iuY_\theta}]| &\le E_0[|X_\theta^\alpha - Y_\theta^\alpha|] + uE_0[|Y_\theta|^\alpha |X_\theta - Y_\theta|] \\ &\le C(\alpha,\epsilon)(1+|u|)\|X_\theta - Y_\theta\|_{1+\epsilon}\end{aligned}$$

for some constant $C(\alpha,\epsilon) > 0$ by (2). Since $\sigma_0(\theta) = O(\theta^{1/2})$, we obtain the result from (3). ////

Lemma 3.2 *For any $\delta \in [0,(H-\epsilon)/3)$,*

$$\sup_{|u|\le\theta^{-\delta}} \left| E_0[Y_\theta^\alpha e^{iuY_\theta}] - E_0\left[e^{iuM_\theta^{(0)}}\left((M_\theta^{(0)})^\alpha + A(\alpha,u,M_\theta^{(0)}) + B(\alpha,u,M_\theta^{(0)})\right)\right]\right| = o(\theta^{2H+\epsilon}),$$

where

$$\begin{aligned}A_\theta(\alpha,u,x) &= \left(iux^\alpha + \alpha x^{\alpha-1}\right)(E_0[Y_\theta|M_\theta^{(0)} = x] - x),\\ B_\theta(\alpha,u,x) &= \left(-\frac{u^2}{2}x^\alpha + iu\alpha x^{\alpha-1} + \frac{\alpha(\alpha-1)}{2}x^{\alpha-2}\right)\\ &\quad\times\left(\theta^{2H}E_0[|M_\theta^{(1)}|^2|M_\theta^{(0)} = x] - \sigma_0(\theta)\theta^H E_0[M_\theta^{(1)}|M_\theta^{(0)} = x] + \frac{\sigma_0(\theta)^2}{4}\right).\end{aligned}$$

Proof: This follows from the fact that

$$\left|e^{ix} - 1 - ix + \frac{x^2}{2}\right| \le \frac{|x|^3}{6}$$

for all $x \in \mathbb{R}$. ////

Lemma 3.3 *Define $q_\theta(x)$ by*

$$\begin{aligned}q_\theta(x) =&\phi(x) - \theta^H a_\theta^{(1)}(x) - \theta^{2H}a_\theta^{(2)}(x) - \frac{\sigma_0(\theta)}{2}(x\phi(x) - \theta^H a_\theta^{(3)}(x))\\ &+ \frac{\theta^{2H}}{2}c_\theta(x) - \frac{\theta^H\sigma_0(\theta)}{2}b_\theta(x) + \frac{\sigma_0(\theta)^2}{8}(x^2-1)\phi(x)\end{aligned} \tag{5}$$

where $a_\theta^{(i)}$, b_θ and c_θ are defined by (4). Then,

$$\int_{\mathbb{R}} e^{iux} x^\alpha q_\theta(x)\mathrm{d}x = E_0\left[e^{iuM_\theta^{(0)}}\left((M_\theta^{(0)})^\alpha + A(\alpha, u, M_\theta^{(0)}) + B(\alpha, u, M_\theta^{(0)})\right)\right].$$

Proof: Since the density of $M_\theta^{(0)}$ is ϕ by the assumption, this simply follows from integration by parts. ////

3.2 Density expansion

Here we derive an asymptotic expansion of the density of X_θ.

Lemma 3.4 *There exists a density of X_θ under $Q(\cdot|\mathcal{F}_0)$ and for any $\alpha, j \in \mathbb{N} \cup \{0\}$,*

$$\sup_{\theta\in(0,1)} \int |u|^j |E_0[X_\theta^\alpha e^{iuX_\theta}]| \mathrm{d}u < \infty$$

Proof: Note that the distribution of X_θ is Gaussian conditionally on $\mathcal{G}_\theta$ under $Q(\cdot|\mathcal{F}_0)$, with conditional mean

$$-\frac{1}{2\sigma_0(\theta)}\langle M\rangle_\theta + \frac{1}{\sigma_0(\theta)}\int_0^\theta \sqrt{v_t}\rho_t \mathrm{d}W_t$$

and conditional variance

$$\frac{1}{\sigma_0(\theta)^2}\int_0^\theta v_t(1-\rho_t^2)\mathrm{d}t.$$

Therefore, X_θ admits a density $p_\theta(x)$ under $Q(\cdot|\mathcal{F}_0)$. Furthermore, the density function is in the Schwartz space $\mathcal{S}$ and each Schwartz semi-norm is uniformly bounded in θ by (1). Therefore,

$$\begin{aligned}\sup_{\theta\in(0,1)} \int |u|^j |E_0[X_\theta^\alpha e^{iuX_\theta}]| \mathrm{d}u &= \sup_{\theta\in(0,1)} \int \left| \int u^j x^\alpha e^{iux} p_\theta(x)\mathrm{d}x \right| \mathrm{d}u \\ &= \sup_{\theta\in(0,1)} \int \left| \int e^{iux} \partial_x^j (x^\alpha p_\theta(x))\mathrm{d}x \right| \mathrm{d}u < \infty\end{aligned}$$

since the Fourier transform is a continuous linear mapping from $\mathcal{S}$ to $\mathcal{S}$. ////

Theorem 3.1 *Denote by p_θ the density of X_θ under $Q(\cdot|\mathcal{F}_0)$. Then, for any $\alpha \in \mathbb{N}\cup\{0\}$,*

$$\sup_{x\in\mathbb{R}} (1+x^2)^\alpha |p_\theta(x) - q_\theta(x)| = o(\theta^{2H}) \tag{6}$$

as $\theta \to 0$, where q_θ is defined by (5).

Proof: As seen in the proof of Lemma 3.4, the density p_θ exists in the Schwartz space. By the Fourier identity,

$$(1+x^2)^\alpha(p_\theta(x)-q_\theta(x)) = \frac{1}{2\pi}\int\int e^{iuy}(1+y^2)^\alpha(p_\theta(y)-q_\theta(y))\mathrm{d}ye^{-iux}\mathrm{d}u$$

Combining the lemmas in the previous section, taking $\delta \in (0, \min\{\epsilon, (H-\epsilon)/3\})$, we have

$$\int_{|u|\leq\theta^{-\delta}}\left|\int e^{iuy}(1+y^2)^\alpha(p_\theta(y)-q_\theta(y))\mathrm{d}y\right|\mathrm{d}u = o(\theta^{2H}).$$

On the other hand,

$$\begin{aligned}\int_{|u|\geq\theta^{-\delta}}\left|\int e^{iuy}(1+y^2)^\alpha p_\theta(y)\mathrm{d}y\right|\mathrm{d}u &\leq \theta^{j\delta}\int_{|u|\geq\theta^{-\delta}}|u|^j|E_0[(1+X_\theta^2)^\alpha e^{iuX_\theta}]|\mathrm{d}u \\ &= O(\theta^{j\delta})\end{aligned}$$

for any $j \in \mathbb{N}$ by Lemma 3.4. The remainder

$$\int_{|u|\geq\theta^{-\delta}}\left|\int e^{iuy}(1+y^2)^\alpha q_\theta(y)\mathrm{d}y\right|\mathrm{d}u$$

is handled in the same manner. ////

3.3 Put option price expansion

Here we consider put option prices. Denote by p_θ the density of X_θ as before and consider a normalized put option price

$$\frac{p(Fe^{\sigma_0(\theta)z},\theta)}{F\sigma_0(\theta)} = e^{-r\theta}\int_{-\infty}^{z}\int_{-\infty}^{\zeta}p_\theta(x)\mathrm{d}xe^{\sigma_0(\theta)\zeta}\mathrm{d}\zeta.$$

Lemma 3.5 *Let $q_\theta(x)$, $\theta > 0$ be a family of functions on $\mathbb{R}$ (not necessarily the one given by (5)). If*

$$\sup_{x\in\mathbb{R}}(1+x^2)^\alpha|p_\theta(x)-q_\theta(x)| = o(\theta^\beta)$$

for some $\alpha > 5/4$ and $\beta > 0$, then for any $z_0 \in \mathbb{R}$,

$$\frac{p(Fe^{\sigma_0(\theta)z},\theta)}{F\sigma_0(\theta)} = e^{-r\theta}\int_{-\infty}^{z}\int_{-\infty}^{\zeta}q_\theta(x)\mathrm{d}xe^{\sigma_0(\theta)\zeta}\mathrm{d}\zeta + o(\theta^\beta)$$

uniformly in $z \leq z_0$.

Proof: By the Cauchy-Schwarz inequality,

$$\begin{aligned}&e^{-r\theta}\int_{-\infty}^{z}\int_{-\infty}^{\zeta}|p_\theta(x)-q_\theta(x)|\mathrm{d}ze^{\sigma_0(\theta)\zeta}\mathrm{d}\zeta \\ &\leq e^{-r\theta}\int_{-\infty}^{z}\sqrt{\int_{-\infty}^{\zeta}\frac{\mathrm{d}x}{(1+x^2)^{2\alpha-1}}}\sqrt{\int_{-\infty}^{\zeta}(1+x^2)^{2\alpha-1}|p_\theta(x)-q_\theta(x)|^2\mathrm{d}ze^{\sigma_0(\theta)\zeta}\mathrm{d}\zeta} \\ &\leq \sqrt{\pi}e^{-r\theta+\sigma_0(\theta)z}\sup_{x\in\mathbb{R}}(1+x^2)^\alpha|p_\theta(x)-q_\theta(x)|\int_{-\infty}^{z}\sqrt{\int_{-\infty}^{\zeta}\frac{\mathrm{d}x}{(1+x^2)^{2\alpha-1}}}\mathrm{d}\zeta,\end{aligned}$$

which is $o(\theta^\beta)$ if $\alpha > 5/4$. ////

Proposition 3.1 *Suppose we have (6) with q_θ of the form*

$$\begin{aligned} q_\theta(x) =& \phi\left(x + \frac{\sigma_0(\theta)}{2}\right)\left\{1 + \kappa_3(\theta)\left(H_3\left(x + \frac{\sigma_0(\theta)}{2}\right) - \sigma_0(\theta)H_2\left(x + \frac{\sigma_0(\theta)}{2}\right)\right)\theta^H\right\} \\ &+ \phi(x)\left(\kappa_4(\theta)H_4(x) + \frac{\kappa_3(\theta)^2}{2}H_6(x)\right)\theta^{2H} \end{aligned} \tag{7}$$

with bounded functions $\kappa_3(\theta)$ and $\kappa_4(\theta)$ of θ, where H_k is the kth Hermite polynomial:

$$H_1(x) = x, \;\; H_2(x) = x^2 - 1, \;\; H_3(x) = x^3 - 3x, \;\; H_4(x) = x^4 - 6x^2 + 3, \dots$$

Then, for any $z_0 \in \mathbb{R}$,

$$\begin{aligned} \frac{p(Fe^{\sigma_0(\theta)z}, \theta)}{Fe^{-r\theta}\sigma_0(\theta)} = & \frac{1}{\sigma_0(\theta)}\left(\Phi\left(z + \frac{\sigma_0(\theta)}{2}\right)e^{\sigma_0(\theta)z} - \Phi\left(z - \frac{\sigma_0(\theta)}{2}\right)\right) \\ &+ \kappa_3(\theta)\phi\left(z + \frac{\sigma_0(\theta)}{2}\right)H_1\left(z + \frac{\sigma_0(\theta)}{2}\right)e^{\sigma_0(\theta)z}\theta^H \\ &+ \phi(z)\left(\kappa_4(\theta)H_2(z) + \frac{\kappa_3(\theta)^2}{2}H_4(z)\right)\theta^{2H} + o(\theta^{2H}) \end{aligned}$$

uniformly in $z \le z_0$.

Proof: This is a direct consequence of the previous lemma. For example,

$$\frac{\mathrm{d}}{\mathrm{d}z}\left\{e^{-\sigma_0(\theta)z}\frac{\mathrm{d}}{\mathrm{d}z}\left\{\frac{1}{\sigma_0(\theta)}\left(\Phi\left(z + \frac{\sigma_0(\theta)}{2}\right)e^{\sigma_0(\theta)z} - \Phi\left(z - \frac{\sigma_0(\theta)}{2}\right)\right)\right\}\right\} = \phi\left(z + \frac{\sigma_0(\theta)}{2}\right).$$

The derivative of $H_k(z)\phi(z)$ is $-H_{k+1}(z)\phi(z)$. Recall also $\sigma_0(\theta) = O(\sqrt{\theta})$. ////

3.4 Implied volatility expansion

Here we give an expansion formula for the Black-Scholes implied volatility. Denote by $p_{\mathrm{BS}}(K, \theta, \sigma)$ the put option price with strike price K and maturity θ under the Black-Scholes model with volatility parameter $\sigma > 0$. Given a put option price $p(K, \theta)$, $K = Fe^k$, the implied volatility $\sigma_{\mathrm{BS}}(k, \theta)$ is defined through

$$p_{\mathrm{BS}}(K, \theta, \sigma_{\mathrm{BS}}(k, \theta)) = p(K, \theta).$$

Theorem 3.2 *Suppose we have (6) with q_θ of the form (7). Then, for any $z \in \mathbb{R}$,*

$$\begin{aligned} &\sigma_{\mathrm{BS}}(\sqrt{\theta}z, \theta) \\ &= \kappa_2\left\{1 + \kappa_3\left(\frac{z}{\kappa_2} + \frac{\kappa_2\sqrt{\theta}}{2}\right)\theta^H + \left(\frac{3\kappa_3^2}{2} - \kappa_4 + (\kappa_4 - 3\kappa_3^2)\frac{z^2}{\kappa_2^2}\right)\theta^{2H}\right\} + o(\theta^{2H}), \end{aligned}$$

where $\kappa_2 = \kappa_2(\theta) = \sigma_0(\theta)/\sqrt{\theta}$, $\kappa_3 = \kappa_3(\theta)$ and $\kappa_4 = \kappa_4(\theta)$.

Proof: Step 1). Fix $z \in \mathbb{R}$. Note that

$$P_\theta(\sigma) := \frac{p_{\mathrm{BS}}(Fe^{\sqrt{\theta}z}, \theta, \sigma)}{Fe^{-r\theta}\sqrt{\theta}} = \frac{1}{\sqrt{\theta}}\left(\Phi\left(\frac{z}{\sigma} + \frac{\sigma\sqrt{\theta}}{2}\right)e^{\sqrt{\theta}z} - \Phi\left(\frac{z}{\sigma} - \frac{\sigma\sqrt{\theta}}{2}\right)\right) \tag{8}$$

and that

$$P_\theta : [0, \infty] \to \left[\frac{(e^{\sqrt{\theta}z} - 1)_+}{\sqrt{\theta}}, \frac{e^{\sqrt{\theta}z}}{\sqrt{\theta}}\right]$$

is a strictly increasing function. From (8) and Proposition 3.1, we have

$$\begin{aligned}\frac{p(Fe^{\sqrt{\theta}z}, \theta)}{Fe^{-r\theta}\sqrt{\theta}} =& P_\theta(\kappa_2) + \kappa_2\kappa_3\phi\left(\frac{z}{\kappa_2} + \frac{\kappa_2\sqrt{\theta}}{2}\right)H_1\left(\frac{z}{\kappa_2} + \frac{\kappa_2\sqrt{\theta}}{2}\right)e^{\sqrt{\theta}z}\theta^H \\ &+ \kappa_2\phi\left(\frac{z}{\kappa_2}\right)\left(\kappa_4 H_2\left(\frac{z}{\kappa_2}\right) + \frac{\kappa_3^2}{2}H_4\left(\frac{z}{\kappa_2}\right)\right)\theta^{2H} + o(\theta^{2H}) \\ =& P_\theta(\kappa_2) + O(\theta^H).\end{aligned}$$

Therefore

$$\sigma_{\mathrm{BS}}(\sqrt{\theta}z, \theta) = P_\theta^{-1}(P_\theta(\kappa_2) + O(\theta^H)).$$

By (1), κ_2 is bounded in θ, say, by $L > 0$. The function P_θ converges as $\theta \to 0$ to

$$P_0(\sigma) := z\Phi\left(\frac{z}{\sigma}\right) + \sigma\phi\left(\frac{z}{\sigma}\right)$$

pointwize, and by Dini's theorem, this convergence is uniform on $[0, L]$. Since the limit function P_0 is strictly increasing, the inverse functions P_θ^{-1} converges to P_0^{-1}. Again by Dini's theorem, this convergence is uniform and in particular, P_θ^{-1} are equicontinuous. Thus we conclude $\sigma_{\mathrm{BS}}(\sqrt{\theta}z, \theta) - \kappa_2 \to 0$ as $\theta \to 0$. Then, write $\sigma_{\mathrm{BS}}(\sqrt{\theta}z, \theta) = \kappa_2 + \beta(\theta)$ and substitute this to the equation $P_\theta(\sigma_{\mathrm{BS}}(\sqrt{\theta}z, \theta)) = P_\theta(\kappa_2) + O(\theta^H)$. The Taylor expansion gives $\beta(\theta) = O(\theta^H)$.

Step 2). From (8) we have

$$P_\theta(\sigma) = \sigma F_1\left(\frac{z}{\sigma}\right) + \frac{\sigma^2\sqrt{\theta}}{2}F_2\left(\frac{z}{\sigma}\right) + \frac{\sigma^3\theta}{6}F_3\left(\frac{z}{\sigma}\right) + o(\theta),$$

where

$$F_1(x) = x\Phi(x) + \phi(x), \quad F_2(x) = x^2\Phi(x) + x\phi(x), \quad F_3(x) = x^3\Phi(x) + \left(x^2 - \frac{1}{4}\right)\phi(x).$$

Using that

$$\partial_\sigma\left\{\sigma F_1\left(\frac{z}{\sigma}\right)\right\} = \phi\left(\frac{z}{\sigma}\right),$$

we have

$$
\begin{aligned}
&\kappa_2 F_1\left(\frac{z}{\kappa_2}\right)+\frac{\kappa_2^2\sqrt{\theta}}{2}F_2\left(\frac{z}{\kappa_2}\right)+\kappa_2\phi\left(\frac{z}{\kappa_2}\right)\kappa_3 H_1\left(\frac{z}{\kappa_2}\right)e^{\sqrt{\theta}z}\theta^H\\
&=\sigma_{\mathrm{BS}}(\sqrt{\theta}z,\theta)F_1\left(\frac{z}{\sigma_{\mathrm{BS}}(\sqrt{\theta}z,\theta)}\right)+\frac{\sigma_{\mathrm{BS}}(\sqrt{\theta}z,\theta)^2\sqrt{\theta}}{2}F_2\left(\frac{z}{\sigma_{\mathrm{BS}}(\sqrt{\theta}z,\theta)}\right)+O(\theta^{2H})\\
&=\kappa_2 F_1\left(\frac{z}{\kappa_2}\right)+\frac{\kappa_2^2\sqrt{\theta}}{2}F_2\left(\frac{z}{\kappa_2}\right)+\phi\left(\frac{z}{\kappa_2}\right)(\sigma_{\mathrm{BS}}(\sqrt{\theta}z,\theta)-\kappa_2)+O(\theta^{2H}),
\end{aligned}
$$

from which we conclude $\sigma_{\mathrm{BS}}(\sqrt{\theta}z,\theta)=\kappa_2+\kappa_3 z e^{\sqrt{\theta}z}\theta^H+O(\theta^{2H})$.

Step 3). Using that

$$
\partial_\sigma^2\left\{\sigma F_1\left(\frac{z}{\sigma}\right)\right\}=\frac{z^2}{\sigma^3}\phi\left(\frac{z}{\sigma}\right),\ \ \partial_\sigma\left\{\sigma^2 F_2\left(\frac{z}{\sigma}\right)\right\}=z\phi\left(\frac{z}{\sigma}\right),
$$

we obtain

$$
\begin{aligned}
&\kappa_2\phi\left(\frac{z}{\kappa_2}+\frac{\kappa_2\sqrt{\theta}}{2}\right)\left(\kappa_3 H_1\left(\frac{z}{\kappa_2}+\frac{\kappa_2\sqrt{\theta}}{2}\right)e^{\sqrt{\theta}z}\theta^H+\left(\kappa_4 H_2\left(\frac{z}{\kappa_2}\right)+\frac{\kappa_3^2}{2}H_4\left(\frac{z}{\kappa_2}\right)\right)\theta^{2H}\right)\\
&=\frac{p(Fe^{\sqrt{\theta}z},\theta)}{Fe^{-r\theta}\sqrt{\theta}}-P_\theta(\kappa_2)+o(\theta^{2H})\\
&=P_\theta(\sigma_{\mathrm{BS}}(\sqrt{\theta}z,z))-P_\theta(\kappa_2)+o(\theta^{2H})\\
&=\partial_\sigma\left\{\sigma F_1\left(\frac{z}{\sigma}\right)\right\}\Big|_{\sigma=\kappa_2}(\sigma_{\mathrm{BS}}(\sqrt{\theta}z,\theta)-\kappa_2)+\frac{1}{2}\partial_\sigma^2\left\{\sigma F_1\left(\frac{z}{\sigma}\right)\right\}\Big|_{\sigma=\kappa_2}(\sigma_{\mathrm{BS}}(\sqrt{\theta}z,\theta)-\kappa_2)^2\\
&\quad+\frac{\sqrt{\theta}}{2}\partial_\sigma\left\{\sigma^2 F_2\left(\frac{z}{\sigma}\right)\right\}\Big|_{\sigma=\kappa_2}(\sigma_{\mathrm{BS}}(\sqrt{\theta}z,\theta)-\kappa_2)+o(\theta^{2H})\\
&=\phi\left(\frac{z}{\kappa_2}\right)(\sigma_{\mathrm{BS}}(\sqrt{\theta}z,\theta)-\kappa_2)+\frac{\sqrt{\theta}}{2}z\phi\left(\frac{z}{\kappa_2}\right)(\sigma_{\mathrm{BS}}(\sqrt{\theta}z,\theta)-\kappa_2)\\
&\quad+\frac{z^2}{2\kappa_2^3}\phi\left(\frac{z}{\kappa_2}\right)(\sigma_{\mathrm{BS}}(\sqrt{\theta}z,\theta)-\kappa_2)^2+o(\theta^{2H})
\end{aligned}
$$

from Proposition 3.1 and Step 2. The left hand side is further expanded as

$$
\begin{aligned}
\kappa_2\phi\left(\frac{z}{\kappa_2}\right)\Bigg\{&\kappa_3 H_1\left(\frac{z}{\kappa_2}\right)e^{\sqrt{\theta}z}\theta^H-\kappa_3 H_2\left(\frac{z}{\kappa_2}\right)\frac{\kappa_2}{2}\theta^{H+1/2}\\
&+\left(\kappa_4 H_2\left(\frac{z}{\kappa_2}\right)+\frac{\kappa_3^2}{2}H_4\left(\frac{z}{\kappa_2}\right)\right)\theta^{2H}\Bigg\}+o(\theta^{2H}).
\end{aligned}
$$

Denote $\gamma(\theta) = \sigma_{\mathrm{BS}}(\sqrt{\theta}z, \theta) - \kappa_2 - \kappa_3 z e^{\sqrt{\theta}z}\theta^H$ and substitute this to obtain

$$\begin{aligned}
\gamma(\theta) =& -\kappa_3 H_2\left(\frac{z}{\kappa_2}\right)\frac{\kappa_2^2}{2}\theta^{H+1/2} + \kappa_2\left(\kappa_4 H_2\left(\frac{z}{\kappa_2}\right) + \frac{\kappa_3^2}{2}H_4\left(\frac{z}{\kappa_2}\right)\right)\theta^{2H} \\
&- \frac{\kappa_3}{2}z^2\theta^{H+1/2} - \frac{\kappa_3^2}{2\kappa_2^3}z^4\theta^{2H} + o(\theta^{2H}) \\
=& \left(\frac{\kappa_2^2}{2} - z^2\right)\kappa_3\theta^{H+1/2} + \kappa_2\left((\kappa_4 - 3\kappa_3^2)\frac{z^2}{\kappa_2^2} + \frac{3}{2}\kappa_3^2 - \kappa_4\right)\theta^{2H} + o(\theta^{2H}),
\end{aligned}$$

from which we conclude the result. ////

4 Asymptotics for at-the-money skew and curvature

Here we derive the asymptotic behavior of at-the-money implied volatility skew and curvature. They are defined respectively as the first and the second derivatives of the implied volatility at $k = 0$. The skew behavior is especially important in order to argue the consistency of a model to the empirically observed power law.

Theorem 4.1 *Suppose we have (6) with q_θ of the form (7). Then,*

$$\begin{aligned}
\partial_k \sigma_{\mathrm{BS}}(0, \theta) &= \kappa_3(\theta)\theta^{H-1/2} + o(\theta^{2H-1/2}), \\
\partial_k^2 \sigma_{\mathrm{BS}}(0, \theta) &= 2\frac{\kappa_4(\theta) - 3\kappa_3(\theta)^2}{\kappa_2(\theta)}\theta^{2H-1} + o(\theta^{2H-1}).
\end{aligned}$$

Proof: It is known (see e.g., Fukasawa [8]) that

$$\begin{aligned}
\partial_k \sigma_{\mathrm{BS}}(k, \theta) &= \frac{Q(k \geq \sigma_0(\theta)X_\theta|\mathcal{F}_0) - \Phi(f_2(k, \theta))}{\sqrt{\theta}\phi(f_2(k, \theta))}, \\
\partial_k^2 \sigma_{\mathrm{BS}}(k, \theta) &= \frac{p_\theta(k/\sigma_0(\theta))}{\sigma_0(\theta)\sqrt{\theta}\phi(f_2(k, \theta))} - \sigma_{\mathrm{BS}}(k, \theta)\partial_k f_1(k, \theta)\partial_k f_2(k, \theta),
\end{aligned} \tag{9}$$

where

$$f_1(k, \theta) = \frac{k}{\sqrt{\theta}\sigma_{\mathrm{BS}}(k, \theta)} - \frac{\sqrt{\theta}\sigma_{\mathrm{BS}}(k, \theta)}{2}, \quad f_2(k, \theta) = \frac{k}{\sqrt{\theta}\sigma_{\mathrm{BS}}(k, \theta)} + \frac{\sqrt{\theta}\sigma_{\mathrm{BS}}(k, \theta)}{2}.$$

Since the condition of Proposition 3.1 is met, we have

$$Q(0 \geq X_\theta|\mathcal{F}_0) = \Phi\left(\frac{\sigma_0(\theta)}{2}\right) + \kappa_3(\theta)\phi\left(\frac{\sigma_0(\theta)}{2}\right)\theta^H + o(\theta^{2H}).$$

On the other hand, by Theorem 3.2,

$$f_2(0, \theta) = \frac{\sqrt{\theta}}{2}\kappa_2(\theta) + O(\theta^{2H+1/2})$$

and so,

$$\Phi(f_2(0,\theta)) = \Phi\left(\frac{\sigma_0(\theta)}{2}\right) + O(\theta^{2H+1/2}),$$
$$\phi(f_2(0,\theta)) = \phi(0) - \phi(0)\frac{\theta}{8}\kappa_2(\theta)^2 + O(\theta^{2H+1}).$$

Then, it follows from (9) that

$$\partial_k \sigma_{\mathrm{BS}}(0,\theta) = \kappa_3(\theta)\theta^{H-1/2} + o(\theta^{2H-1/2}). \tag{10}$$

Further, under the condition, we have

$$p_\theta(0) = \phi\left(\frac{\sigma_0(\theta)}{2}\right)\left\{1 - \frac{\kappa_3(\theta)}{2}\sigma_0(\theta)\theta^H + \left(3\kappa_4(\theta) - 15\frac{\kappa_3(\theta)^2}{2}\right)\theta^{2H}\right\} + o(\theta^{2H}).$$

On the other hand, by Theorem 3.2 and (10),

$$\begin{aligned}
&\sigma_{\mathrm{BS}}(0,\theta)\partial_k f_1(0,\theta)\partial_k f_2(0,\theta) \\
&\quad= \frac{1}{\sigma_{\mathrm{BS}}(0,\theta)\theta} + O(\theta^{2H}) \\
&\quad= \frac{1}{\kappa_2(\theta)\theta}\left(1 - \frac{1}{2}\kappa_2(\theta)\kappa_3(\theta)\theta^{H+1/2} - \left(\frac{3}{2}\kappa_3(\theta)^2 - \kappa_4(\theta)\right)\theta^{2H}\right) + o(\theta^{2H-1}).
\end{aligned}$$

Then, it follows from (9) that

$$\partial_k^2 \sigma_{\mathrm{BS}}(0,\theta) = \frac{2\kappa_4(\theta) - 6\kappa_3(\theta)^2}{\kappa_2(\theta)}\theta^{2H-1} + o(\theta^{2H-1}),$$

which completes the proof. ////

5 The rough Bergomi model

Here we show that the rough Bergomi model proposed by [3] fits the framework and compute the expansion terms. Let $\rho_t = \rho \in (-1, 1)$ be a constant and

$$\mathrm{d}\log v_t = \eta \mathrm{d}W_t^H,$$

where $\eta > 0$ is a constant and W^H is a fractional Brownian motion with the Hurst parameter $H \in (0, 1/2)$, given as

$$W_t^H = c_H \left\{ \int_0^t (t-s)^{H-1/2} \mathrm{d}W_s + \int_{-\infty}^0 (t-s)^{H-1/2} - (-s)^{H-1/2} \mathrm{d}W_s \right\}$$

with a normalizing constant $c_H > 0$. Since v_t is log-normally distributed, (1) holds by Jensen's inequality. We have

$$v_t = v^0(t) \exp\left\{ \eta_H \sqrt{2H} \int_0^t (t-s)^{H-1/2} \mathrm{d}W_s - \frac{\eta_H^2}{2} t^{2H} \right\},$$

where $\eta_H = \eta c_H / \sqrt{2H}$. Now we state the main result of this section.

Theorem 5.1 *We have (6) for q_θ given by (7) with*

$$\begin{aligned}
\kappa_3(\theta) &= \rho \eta_H \sqrt{\frac{H}{2}} \frac{1}{\theta^H \sigma_0(\theta)^3} \int_0^\theta \exp\left\{ -\frac{\eta_H^2}{8} t^{2H} \right\} \int_0^t (t-s)^{H-1/2} \sqrt{v^0(s)} \mathrm{d}s v^0(t) \mathrm{d}t, \\
\kappa_4(\theta) &= \frac{(1+2\rho^2)\eta_H^2 H}{(2H+1)^2(2H+2)} + \frac{\rho^2 \eta_H^2 H \beta(H+3/2, H+3/2)}{2(H+1/2)^2},
\end{aligned}$$

where β is the beta function.

Proof: The conditions (2) and (3) follow from Lemma 5.1 below. The functions $a_\theta^{(i)}$ and c_θ are computed in Lemmas 5.2, 5.3, 5.4 and 5.5 below. The function b_θ is obtained as the the derivative of $a_\theta^{(1)}$. They are apparently rapidly decreasing smooth functions. Then, by Theorem 3.1, it suffices to show that q_θ defined by (5) has the form (7) up to $o(\theta^{2H})$ with $\kappa_3(\theta)$ and $\kappa_4(\theta)$ specified above.

By the Taylor expansion, using that the derivative of $a_\theta^{(1)}$ is b_θ and that $\sigma_0(\theta) = O(\sqrt{\theta})$, it is easy to verify

$$\begin{aligned}
q_\theta(x) =& \phi\left(x + \frac{\sigma_0(\theta)}{2} \right) - \theta^H a_\theta^{(1)} \left(x + \frac{\sigma_0(\theta)}{2} \right) - \theta^{2H} a_\theta^{(2)}(x) + \frac{\theta^{2H}}{2} c_\theta(x) \\
&+ \frac{\sigma_0(\theta)\theta^H}{2} a_\theta^{(3)} \left(x + \frac{\sigma_0(\theta)}{2} \right) + O(\theta^{1+H})
\end{aligned}$$

in the Schwarz space. The rest is straightforward. ////

Proposition 3.1 and Theorems 3.2 and 4.1 are therefore valid here. The resulting formula of the implied volatility expansion turns out reduce the Bergomi-Guyon expansion formula formally derived in [3] when $H < 1/2$ and the forward variance curve is flat, that is, when v^0 is constant. Note however that there is a typo in the second order term in [3]. Numerical experiments are given in that paper. When v^0 is constant, the same formula can be formally obtained also by expanding the rate function appeared in the large deviation result of [5]; see [4] for a rigorous treatment in this approach.

In order to prove Lemmas below, we need some preparation. Let H_k, $k = 0, 1, \dots$ be the Hermite polynomials as before:

$$H_k(x) = (-1)^k e^{x^2/2} \frac{\mathrm{d}^k}{\mathrm{d}x^k} e^{-x^2/2}$$

and $H_k(x, a) = a^{k/2} H_k(x/\sqrt{a})$ for $a > 0$. As is well-known, we have

$$\exp\left\{ux - \frac{au^2}{2}\right\} = \sum_{k=0}^{\infty} H_k(x, a) \frac{u^k}{k!}$$

and for any continuous local martingale M and $n \in \mathbb{N}$,

$$\mathrm{d}L_t^{(n)} = n L_t^{(n-1)} \mathrm{d}M_t, \tag{11}$$

where $L^{(k)} = H_k(M, \langle M \rangle)$ for $k \in \mathbb{N}$. See, e.g., Revuz and Yor [18].

Define $\hat{W}$, $\hat{W}'$, $\hat{B}$ by

$$\hat{W}_t = \frac{1}{\sigma_0(\theta)} \int_0^{\tau^{-1}(t)} \sqrt{v^0(s)} \mathrm{d}W_s, \quad \hat{W}'_t = \frac{1}{\sigma_0(\theta)} \int_0^{\tau^{-1}(t)} \sqrt{v^0(s)} \mathrm{d}W'_s$$

and $\hat{B} = \rho \hat{W} + \sqrt{1 - \rho^2} \hat{W}'$, where

$$\tau(s) = \frac{1}{\sigma_0(\theta)^2} \int_0^s v^0(t) \mathrm{d}t.$$

Then, $(\hat{W}, \hat{W}')$ is a 2-dimensional Brownian motion under E_0 and for any square-integrable function f,

$$\int_0^a f(s) \mathrm{d}W_s = \sigma_0(\theta) \int_0^{\tau(a)} \frac{f(\tau^{-1}(t))}{\sqrt{v^0(\tau^{-1}(t))}} \mathrm{d}\hat{W}_t.$$

Therefore,

$$M_\theta = \sigma_0(\theta) \int_0^1 \exp\left\{\theta^H F_t^t - \frac{\eta_H^2}{4} |\tau^{-1}(t)|^{2H}\right\} \mathrm{d}\hat{B}_t$$

where

$$F_u^t = \eta_H \sqrt{\frac{H}{2}} \frac{\sigma_0(\theta)}{\theta^H} \int_0^u \frac{(\tau^{-1}(t) - \tau^{-1}(s))^{H-1/2}}{\sqrt{v^0(\tau^{-1}(s))}} \mathrm{d}\hat{W}_s, \quad u \in [0, t].$$

Let

$$G_t^{(k)} = H_k(F_t^t, \langle F^t \rangle_t).$$

Then, we have

$$\begin{aligned}
M_\theta &= \sigma_0(\theta) \int_0^1 \exp\left\{-\frac{\eta_H^2}{8}|\tau^{-1}(t)|^{2H}\right\} \exp\left\{\theta^H F_t^t - \frac{\theta^{2H}}{2}\langle F^t \rangle_t\right\} \mathrm{d}\hat{B}_t \\
&= \sigma_0(\theta) \int_0^1 \exp\left\{-\frac{\eta_H^2}{8}|\tau^{-1}(t)|^{2H}\right\} \sum_{k=0}^{\infty} G_t^{(k)} \frac{\theta^{Hk}}{k!} \mathrm{d}\hat{B}_t.
\end{aligned}$$

Lemma 5.1 *We have (3) with*

$$\begin{aligned}
M_\theta^{(0)} &= \hat{B}_1, \\
M_\theta^{(1)} &= \int_0^1 h_\theta(t) G_t^{(1)} \mathrm{d}\hat{B}_t, \\
M_\theta^{(2)} &= \int_0^1 \left\{\frac{h_\theta(t) - 1}{\theta^{2H}} + h_\theta(t)\frac{G_t^{(2)}}{2}\right\} \mathrm{d}\hat{B}_t, \\
M_\theta^{(3)} &= 2\int_0^1 F_t^t \mathrm{d}t,
\end{aligned}$$

where

$$h_\theta(t) = \exp\left\{-\frac{\eta_H^2}{8}|\tau^{-1}(t)|^{2H}\right\}.$$

Proof: For $M_\theta^{(i)}$, $i = 0, 1, 2$, it suffices to show

$$\left\| \int_0^1 h_\theta(t) \sum_{k=J}^{\infty} G_t^{(k)} \frac{\theta^{Hk}}{k!} \mathrm{d}\hat{B}_t \right\|_2 = O(\theta^{HJ})$$

for any $J \geq 3$. The proof for $M_\theta^{(3)}$ is similar and so omitted. It suffices to show

$$E_0\left[\int_0^1 \left|\sum_{k=J}^{\infty} G_t^{(k)} \frac{\theta^{Hk}}{k!}\right|^2 \mathrm{d}t\right] = O(\theta^{2HJ}).$$

By the Cauchy-Schwarz inequality, the left hand side is dominated by

$$\sum_{k=J}^{\infty} \theta^{Hk} \sum_{k=J}^{\infty} \frac{\theta^{Hk}}{(k!)^2} \int_0^1 E_0[|G_t^{(k)}|^2] \mathrm{d}t$$

Let

$$G_{t,s}^{(k)} = H_k(F_s^t, \langle F^t \rangle_s), \quad s \in [0, t].$$

Then, by (11),

$$\begin{aligned}
E_0[|G_t^{(k)}|^2] &= E_0[|G_{t,t}^{(k)}|^2] \\
&= k^2 \int_0^t E_0[|G_{t,s}^{(k-1)}|^2] \mathrm{d}\langle F^t \rangle_s \\
&= k^2(k-1)^2 \int_0^t \int_0^{s_1} E_0[|G_{t,s_2}^{(k-2)}|^2] \mathrm{d}\langle F^t \rangle_{s_2} \mathrm{d}\langle F^t \rangle_{s_1} \\
&\leq (k!)^2 \langle F^t \rangle_t^k = (k!)^2 \left(\frac{\eta_H^2}{4} \frac{|\tau^{-1}(t)|^{2H}}{\theta^{2H}} \right)^k.
\end{aligned}$$

Note that $\tau^{-1}(t) \leq \tau^{-1}(1) = \theta$. Therefore, for sufficiently small θ,

$$\sum_{k=J}^{\infty} \theta^{Hk} \sum_{k=J}^{\infty} \frac{\theta^{Hk}}{(k!)^2} \int_0^1 E_0[|G_t^{(k)}|^2] \mathrm{d}t \leq \left(\frac{\eta_H^2}{4} \right)^J \frac{\theta^{2HJ}}{(1-\theta^H)(1-\theta^H \eta_H^2/4)},$$

which completes the proof. ////

Now we compute $a_\theta^{(i)}$, b_θ and c_θ based on Lemma 5.1. The following lemmas follow from the results in Section A by straightforward computations.

Lemma 5.2

$$\begin{aligned}
a_\theta^{(1)}(x) &= -H_3(x)\phi(x)\rho\eta_H \sqrt{\frac{H}{2}} \frac{\sigma_0(\theta)}{\theta^H} \int_0^1 h_\theta(t) \int_0^t \frac{(\tau^{-1}(t) - \tau^{-1}(s))^{H-1/2}}{\sqrt{v^0(\tau^{-1}(s))}} \mathrm{d}s\mathrm{d}t \\
&= -H_3(x)\phi(x)\rho\eta_H \sqrt{\frac{H}{2}} \\
&\qquad \times \frac{1}{\theta^H \sigma_0(\theta)^3} \int_0^\theta \exp\left\{ -\frac{\eta_H^2}{8} t^{2H} \right\} \int_0^t (t-s)^{H-1/2} \sqrt{v^0(s)} \mathrm{d}s v^0(t) \mathrm{d}t \\
&\sim -H_3(x)\phi(x) \frac{\rho\eta_H \sqrt{2H}}{2(H+1/2)(H+3/2)}.
\end{aligned}$$

Lemma 5.3

$$\begin{aligned}
a_\theta^{(2)}(x) &= -H_2(x)\phi(x) \int_0^1 \frac{h_\theta(t) - 1}{\theta^{2H}} \mathrm{d}t \\
&- H_4(x)\phi(x)\rho^2 \frac{\eta_H^2 H}{4} \frac{\sigma_0(\theta)^2}{\theta^{2H}} \int_0^1 h_\theta(t) \left(\int_0^t \frac{(\tau^{-1}(t) - \tau^{-1}(s))^{H-1/2}}{\sqrt{v^0(\tau^{-1}(s))}} \mathrm{d}s \right)^2 \mathrm{d}t \\
&\sim H_2(x)\phi(x) \int_0^1 \frac{\eta_H^2}{8\theta^{2H}} |\tau^{-1}(t)|^{2H} \mathrm{d}t - H_4(x)\phi(x)\rho^2 \frac{\eta_H^2 H}{(2H+1)^2(2H+2)}.
\end{aligned}$$

Lemma 5.4

$$
\begin{aligned}
a_\theta^{(3)}(x) &= -2H_2(x)\phi(x)\rho\eta_H\sqrt{\frac{H}{2}}\frac{\sigma_0(\theta)}{\theta^H}\int_0^1\int_0^t \frac{(\tau^{-1}(t)-\tau^{-1}(s))^{H-1/2}}{\sqrt{v^0(\tau^{-1}(s))}}\mathrm{d}s\mathrm{d}t \\
&\sim -2H_2(x)\phi(x)\frac{\rho\eta_H\sqrt{2H}}{2(H+1/2)(H+3/2)}.
\end{aligned}
$$

Lemma 5.5

$$
\begin{aligned}
c_\theta(x) &= H_6(x)\phi(x)\rho^2\frac{\eta_H^2 H}{2}\frac{\sigma_0(\theta)^2}{\theta^{2H}}\left(\int_0^1 h_\theta(t)\int_0^t \frac{(\tau^{-1}(t)-\tau^{-1}(s))^{H-1/2}}{\sqrt{v^0(\tau^{-1}(s))}}\mathrm{d}s\mathrm{d}t\right)^2 \\
&\quad + H_4(x)\phi(x)\rho^2\frac{\eta_H^2 H}{2}\frac{\sigma_0(\theta)^2}{\theta^{2H}}\int_0^1 h_\theta(t)^2\left(\int_0^t \frac{(\tau^{-1}(t)-\tau^{-1}(s))^{H-1/2}}{\sqrt{v^0(\tau^{-1}(s))}}\mathrm{d}s\right)^2\mathrm{d}t \\
&\quad + H_4(x)\phi(x)\rho^2\eta_H^2 H\frac{\sigma_0(\theta)^2}{\theta^{2H}}\int_0^1 h_\theta(t)\int_0^t \frac{(\tau^{-1}(t)-\tau^{-1}(s))^{H-1/2}}{\sqrt{v^0(\tau^{-1}(s))}}\mathrm{d}s \\
&\qquad\qquad \times\int_t^1 h_\theta(u)\frac{(\tau^{-1}(u)-\tau^{-1}(t))^{H-1/2}}{\sqrt{v^0(\tau^{-1}(t))}}\mathrm{d}u\mathrm{d}t \\
&\quad + H_4(x)\phi(x)\frac{\eta_H^2 H}{2}\frac{\sigma_0(\theta)^2}{\theta^{2H}}\int_0^1 h_\theta(t)^2\left(\int_s^1 \frac{(\tau^{-1}(t)-\tau^{-1}(s))^{H-1/2}}{\sqrt{v^0(\tau^{-1}(s))}}\mathrm{d}t\right)^2\mathrm{d}s \\
&\quad + H_2(x)\phi(x)\frac{\eta_H^2 H}{2}\frac{\sigma_0(\theta)^2}{\theta^{2H}}\int_0^1 h_\theta(t)^2\int_0^t \frac{(\tau^{-1}(t)-\tau^{-1}(s))^{2H-1}}{v^0(\tau^{-1}(s))}\mathrm{d}s\mathrm{d}t \\
&\sim H_6(x)\phi(x)\rho^2\frac{\eta_H^2 H}{2(H+1/2)^2(H+3/2)^2} + H_4(x)\phi(x)\frac{2(1+\rho^2)\eta_H^2 H}{(2H+1)^2(2H+2)} \\
&\quad + H_4(x)\phi(x)\frac{\rho^2\eta_H^2 H\beta(H+3/2,H+3/2)}{(H+1/2)^2} \\
&\quad + H_2(x)\phi(x)\int_0^1 \frac{\eta_H^2}{4\theta^{2H}}|\tau^{-1}(t)|^{2H}\mathrm{d}t.
\end{aligned}
$$

A Conditional expectations of Wiener-Itô integrals

Here we collect results on the conditional expectations of Wiener-Itô integrals that follow from Proposition 3 of Nualart et al [16]. Let $x \in \mathbb{R}$ and B be a standard Brownian motion ($B_0 = 0$). Let f be a continuous function on

$$\left\{(s,t)\in(0,1)^2; s<t\right\}$$

with

$$\int_0^1\int_0^t |f(s,t)|^2\mathrm{d}s\mathrm{d}t < \infty.$$

Lemma A.1

$$
\begin{aligned}
E\left[\int_0^1 \int_0^t f(s,t)\mathrm{d}B_s\mathrm{d}t \,\Big|\, B_1 = x\right] &= H_1(x)\int_0^1 \int_0^t f(s,t)\mathrm{d}s\mathrm{d}t,\\
E\left[\int_0^1 \int_0^t f(s,t)\mathrm{d}B_s\mathrm{d}B_t \,\Big|\, B_1 = x\right] &= H_2(x)\int_0^1 \int_0^t f(s,t)\mathrm{d}s\mathrm{d}t,\\
E\left[\int_0^1 \left(\int_0^t f(s,t)\mathrm{d}B_s\right)^2 \mathrm{d}B_t \,\Big|\, B_1 = x\right] &= H_3(x)\int_0^1 \left(\int_0^t f(s,t)\mathrm{d}s\right)^2 \mathrm{d}t\\
&\quad + H_1(x)\int_0^1 \int_0^t f(s,t)^2\mathrm{d}s\mathrm{d}t,\\
E\left[\int_0^1 \left(\int_s^1 f(s,t)\mathrm{d}B_t\right)^2 \mathrm{d}s \,\Big|\, B_1 = x\right] &= H_2(x)\int_0^1 \left(\int_s^1 f(s,t)\mathrm{d}t\right)^2 \mathrm{d}s\\
&\quad + \int_0^1 \int_s^1 f(s,t)^2\mathrm{d}t\mathrm{d}s.
\end{aligned}
$$

Lemma A.2

$$
\begin{aligned}
&E\left[\left(\int_0^1 \int_0^t f(s,t)\mathrm{d}B_s\mathrm{d}B_t\right)^2 \,\Big|\, B_1 = x\right] - \int_0^1 \int_0^t f(s,t)^2\mathrm{d}s\mathrm{d}t\\
&= H_4(x)\left(\int_0^1 \int_0^t f(s,t)\mathrm{d}s\mathrm{d}t\right)^2 + H_2(x)\int_0^1 \left(\int_0^t f(s,t)\mathrm{d}s + \int_t^1 f(t,u)\mathrm{d}u\right)^2 \mathrm{d}t.
\end{aligned}
$$

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