Efficient Calibration to FX Options by Markovian Projection in Cross-Currency LIBOR Market Models

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Abstract

We revisit the cross-currency LIBOR Market Model armed with the technique of Markovian projection. We derive an efficient approximation for FX options and show how the FX skew can be modeled consistently with the interest rate skew in a common multifactor model.

Introduction

Extension of the LIBOR Market Model (LMM) framework to cross-currency instruments is conceptually transparent and was addressed by Mikkelsen (2001) and Schlögl (2002) at a time of widespread acceptance of the benefits of single-currency LMMs in the valuation of fixed-income exotics. A parallel improvement of the tools for simulation-based treatments of the early exercise (see *e.g.*, recent reviews by Piterbarg (2004) and Fries (2005)) alleviated the associated computational difficulties and led to a displacement of short-rate models, especially those with one factor, from the position of choice in single-currency applications. However, with regard to cross-currency exotics such as PRDCs, the mainstream approach is still to employ one-factor short-rate models for the interest rate components. A particularly tractable and therefore popular model arises when both interest rates are described by normal models (such as the Hull-White model) while the FX rate is described by the log-normal Black-Scholes model. The calibration to long-dated European FX options in this model is fully analytical, but the skew of implied volatilities for single-currency options and FX options cannot be captured.

In a recent work, Piterbarg (2006a) advanced a local volatility model for the FX component and developed efficient methods for the calibration to FX skew. The interest rate components remained short-rate Gaussian. In this work we start with a skewless log-normal process for the spot exchange rate but with skewed—also known as shifted log-normal or displaced-diffusion—LMM interest rate components. This approach allows for a consistent treatment of interest rate skew and consistent pricing of single-currency and cross-currency instruments within a common multifactor model. A notable technical difficulty which has prevented a wider roll-out

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of cross-currency LMMs is the simultaneous calibration to long-dated European FX options at several maturities. Despite possible optimizations (Amin (2003)), a simulation-based calibration remains insufficient. We tackle the problem of efficient analytical approximation using the methods of Markovian projection and parameter averaging in the footsteps of Piterbarg (2006b).

We begin by setting up the cross-currency LIBOR market model in the spot-LIBOR measure. Then we proceed to the description of the calibration procedure and the derivation of the analytical approximation for the FX option, followed by numerical results and conclusions. Key technical steps of the derivation are outlined in the Appendix.

Cross-currency LMM setup

All components of the cross-currency model are defined on the same probability space equipped with a filtration $\mathcal{F}_t$ which is a suitable augmentation of the filtration generated by a standard F-dimensional Brownian motion $Z(t)$. We assume a common set of maturities $0 = T_0 < T_1 < \ldots < T_N$ for both single-currency rate LMM components, and define domestic and foreign LIBORs, $L_n(t)$ and $\tilde{L}_n(t)$, as forward rates starting at T_n and ending at T_{n+1}, generally reserving the tilde for foreign-currency quantities. In terms of T-maturity zero-coupon bonds, $P(t,T)$ and $\tilde{P}(t,T)$, the LIBORs are given by

$$L_n(t) = \frac{1}{\delta_n}\left(\frac{P(t,T_n)}{P(t,T_{n+1})} - 1\right), \qquad \tilde{L}_n(t) = \frac{1}{\delta_n}\left(\frac{\tilde{P}(t,T_n)}{\tilde{P}(t,T_{n+1})} - 1\right), \tag{1}$$

where δ_n is the daycount fraction from T_n to T_{n+1}.

Stochastic evolution of the LIBORs is determined by the volatility functions $\gamma_n(t)$ and $\tilde{\gamma}_n(t)$, assumed deterministic. Each volatility function and each Brownian motion have F components by the number of factors, but the factor index is always omitted. Where no confusion is possible, we will write scalar products in the space of factors simply as fg (such as $\gamma_n(t)dZ_n(t)$ in the formula below). In more complicated cases, we will use the notation $(f \cdot g)$.

We assume a displaced diffusion for the evolution of each LIBOR in its martingale measure,

$$\begin{aligned} dL_n(t) &= (b_n(t)L_n(t) + (1-b_n(t))L_n(0))\gamma_n(t)dZ_n(t), \\ d\tilde{L}_n(t) &= (\tilde{b}_n(t)\tilde{L}_n(t) + (1-\tilde{b}_n(t))\tilde{L}_n(0))\tilde{\gamma}_n(t)d\tilde{Z}_n(t) \end{aligned} \tag{2}$$

($b_n \equiv 1$ corresponds to the purely log-normal case while $b_n \equiv 0$ is the normal limit). Thus we start with a skew model for each single-currency market. The calibration of the time-dependent shifts $b_n(t)$, $\tilde{b}_n(t)$ can be efficiently done using Piterbarg's averaging formula quoted in the next section as Eq. (19).

An important part in the analytics will be played by bond ratios,

$$\begin{aligned} R_n(t) &= \frac{P(t,T_n)}{P(t,T_{n+1})} = 1 + \delta_n\, L_n(t), \\ \tilde{R}_n(t) &= \frac{\tilde{P}(t,T_n)}{\tilde{P}(t,T_{n+1})} = 1 + \delta_n\, \tilde{L}_n(t). \end{aligned} \tag{3}$$

The SDEs for the bond ratios in the martingale measure have the same form as those for the LIBORs,

$$\begin{aligned} dR_n(t) &= (\beta_n(t)R_n(t) + (1-\beta_n(t))R_n(0))\sigma_n(t)dZ_n(t), \\ d\tilde{R}_n(t) &= (\tilde{\beta}_n(t)\tilde{R}_n(t) + (1-\tilde{\beta}_n(t))\tilde{R}_n(0))\tilde{\sigma}_n(t)d\tilde{Z}_n(t), \end{aligned} \tag{4}$$

with

$$\beta_n(t) = \frac{b_n(t) R_n(0)}{R_n(0) - 1}, \qquad \sigma_n(t) = \gamma_n(t) \frac{R_n(0) - 1}{R_n(0)}. \tag{5}$$

We take for the numeraire in the domestic currency the rolling spot account, corresponding to Jamshidian's (1997) spot-LIBOR measure,

$$N(t) = \frac{1}{P(T_0, T_1)} \cdots \frac{1}{P(T_{n-1}, T_n)} \frac{P(t, T_{n+1})}{P(T_n, T_{n+1})} \qquad \text{for } T_n < t \leq T_{n+1}, \tag{6}$$

and set the volatility of the short bond $P(t, T_{n+1})$ to 0. As was shown by Schlögl (2002), this numeraire is equivalent to the continuously compounded savings account in the sense that they lead to the same measure. Another property shared by $N(t)$ and the savings account is diffusion-free dynamics,

$$dN(t) = N(t)\, \mu_N(t) dt. \tag{7}$$

The foreign-currency numeraire $\tilde{N}(t)$ is chosen similarly.

The FX dynamics linking the single-currency models into a cross-currency model are defined by postulating log-normal dynamics for a numeraire-discounted asset $Y(t)$ related to the foreign exchange rate $X(t)$ by

$$Y(t) = \frac{X(t)\, \tilde{N}(t)}{N(t)}. \tag{8}$$

We set

$$dY(t) = Y(t) \sigma_Y(t) dZ(t) \tag{9}$$

in the spot-LIBOR measure with a deterministic F-component volatility, σ_Y. The dynamics of the FX rate $X(t)$ follow from Eqs. (8) and (9) and the deterministic character of the numeraire dynamics given by Eq. (7),

$$dX(t)/X(t) = (\mu_N(t) - \tilde{\mu}_N(t))\, dt + \sigma_Y(t) dZ(t). \tag{10}$$

Note that we have not introduced any skew in the process for $Y(t)$. We will see that the FX options skew will be naturally generated by the interest rate components even if we started with purely log-normal LIBOR dynamics.

We complete the operational definition of the model by restating the dynamics of the LIBORs in the domestic spot-LIBOR measure,

$$\begin{aligned} dL_n(t) &= (L_n(t)\, b_n(t) + L_n(0)\,(1 - b_n(t)))\gamma_n(t) \left(\sum_{k=\eta(t)}^{n} \alpha_k(t)\, dt + dZ(t) \right), \\ d\tilde{L}_n(t) &= (\tilde{L}_n(t)\, \tilde{b}_n(t) + \tilde{L}_n(0)\,(1 - \tilde{b}_n(t)))\tilde{\gamma}_n(t) \left(\sum_{k=\eta(t)}^{n} \tilde{\alpha}_k(t)\, dt - \sigma_Y\, dt\, + dZ(t) \right). \end{aligned} \tag{11}$$

Here $\eta(t) = n + 1$ for $T_n < t \leq T_{n+1}$, and

$$\begin{aligned} \alpha_k(t) &= \delta_n \frac{b_k(t) L_k(t) + (1 - b_k(t)) L_k(0)}{1 + \delta_k L_k(t)} \gamma_k(t), \\ \tilde{\alpha}_k(t) &= \delta_n \frac{\tilde{b}_k(t) \tilde{L}_k(t) + (1 - \tilde{b}_k(t)) \tilde{L}_k(0)}{1 + \delta_k \tilde{L}_k(t)} \tilde{\gamma}_k(t). \end{aligned} \tag{12}$$

Note that the drift terms in this equation are of the second order in volatilities.

The model can be simulated by inductively repeating the following steps after initialization at the origin of time.

- Extend the paths of the F random factors to the next date.
- Get domestic and foreign states for the next date from a discretization of Eqs. (11).
- Get domestic and foreign numeraires, $N(t)$ and $\tilde{N}(t)$, from Eq. (6) and its foreign-currency counterpart.
- Get the value of the process $Y(t)$ for the next date from a discretization of Eq. (9).
- Restore the FX rate at the next date as $X(t) = Y(t)N(t)/\tilde{N}(t)$.

The valuation of early-exercise options embedded in the exotic instruments can be done using the techniques reviewed by Piterbarg (2004) and Fries (2005). In the next section, we turn to the model calibration.

Calibration of FX volatility

The calibration of the volatilities $\gamma_n(t)$ and $\tilde{\gamma}_n(t)$—or, equivalently, $\sigma_n(t)$ and $\tilde{\sigma}_n(t)$—to single-currency caps and swaptions, and correlations between single-currency rates, can be done independently for the domestic and foreign components using the techniques developed for the single-currency LMM. The overlap in the factor structure of volatilities can be calibrated to cross-correlations of foreign and domestic rates in a straightforward way. Here, we address the key issue of the calibration to long-dated European FX options. Schlögl (2002) studied the linkage between domestic and foreign rate measures and pointed out that, with a choice of log-normal LIBOR models in both domestic and foreign markets, a log-normal distribution was possible for the forward FX rates at one maturity only, which precluded efficient calibration. Here we dispense with the log-normality desideratum for the forward FX rate at any maturity, and instead develop an efficient analytical approximation valid at all maturities.

To compute the option price with maturity T_M and strike K, we change the measure to the T_M-forward measure having as a numeraire the domestic bond $P(t, T_M)$. The forward FX rate

$$F(t, T_M) = \frac{X(t)\,\tilde{P}(t, T_M)}{P(t, T_M)} \tag{13}$$

is a martingale under the T_M-forward measure. The option price reduces to an expectation under this measure,

$$E\left[\frac{(X(T_M) - K)^+}{N(T_M)}\right] = P(0, T_M) E_{T_M}\big[(F(T_M, T_M) - K)^+\big]\,. \tag{14}$$

Using Eqs. (3), (6), (8), and (13), the forward FX-rate process can be expressed in terms of the bond ratios, $R_n(t)$ and $\tilde{R}_n(t)$, and the process $Y(t)$,

$$F(t, T_M) = \prod_{k=0}^{\eta(t)-1} \frac{\tilde{P}(T_k, T_{k+1})}{P(T_k, T_{k+1})}\, Y(t) \prod_{n=\eta(t)}^{M-1} R_n(t) \tilde{R}_n^{-1}(t). \tag{15}$$

Below we assume that the maturity T_M of the forward FX rate is fixed and omit the corresponding argument, writing $F(t)$ instead of $F(t, T_M)$. An SDE for $F(t)$ can be obtained by Ito's lemma from Eq. (15) using the diffusion terms from Eqs. (4) and (9). Note that we know without calculation that the drift terms will cancel because we now work in a martingale measure for $F(t)$. We get

$$dF(t) = F(t)\Lambda(t)dW(t), \tag{16}$$

$$\begin{aligned}\Lambda(t) &= \sigma_Y(t) \\ &+ \sum_{n=\eta(t)}^{M-1} \frac{\beta_n(t)\, R_n(t) + (1-\beta_n(t))\, R_n(0)}{R_n(t)} \sigma_n(t) \\ &- \sum_{n=\eta(t)}^{M-1} \frac{\tilde{\beta}_n(t)\, \tilde{R}_n(t) + (1-\tilde{\beta}_n(t))\, \tilde{R}_n(0)}{\tilde{R}_n(t)} \tilde{\sigma}_n(t).\end{aligned} \tag{17}$$

The forward-rate process is obviously non-Markovian. Our strategy is to replace it with an *effective* Markovian process $F^*(t)$ following a time-dependent displaced diffusion

$$dF^*(t) = (F^*(t)\beta(t) + (1-\beta(t))F^*(0))\, \sigma(t)dW(t), \qquad F^*(0) = F(0), \tag{18}$$

such that its marginal distributions well approximate those of the initial true process $F(t)$. Once this is done, we invoke the formula for shift averaging derived by Piterbarg (2005),

$$\bar{\beta}_{T_M} = \frac{\int_0^{T_M} \beta(t)\sigma^2(t) \int_0^t \sigma^2(\tau)d\tau dt}{\int_0^{T_M} \sigma^2(t) \int_0^t \sigma^2(\tau)d\tau dt}, \tag{19}$$

which reduces the pricing of an FX option with maturity T_M to a simple modification of the Black-Scholes formula.

We now solve the problem of finding the optimal shift $\beta(t)$ and volatility $\sigma(t)$ in Eq. (18) using an elaboration on Piterbarg's (2006) technique of Markovian projection. The technique is based on a theorem by Gyöngy (1986) and Dupire (1997) which asserts that the process

$$dF^{**}(t) = \Sigma(t, F^{**}(t))dW(t), \qquad F^{**}(0) = F(0) \tag{20}$$

has exactly the same marginal distributions as $F(t)$ provided

$$|\Sigma(t,x)|^2 = E\left[F^2(t)|\Lambda(t)|^2 \,\middle|\, F(t) = x\right]. \tag{21}$$

For every fixed t, the conditional expectation in the right hand side can be characterized as a function of state $|\Sigma(\cdot, x)|^2$ which minimizes the L_2-distance from the true variance,

$$\chi^2 = E\left[\left(F^2(t)|\Lambda(t)|^2 - |\Sigma(t, F(t))|^2\right)^2\right] \to \min. \tag{22}$$

We minimize the functional in Eq. (22) over a subspace of affine linear functions of state, expressed in the form

$$\Sigma(t) = (F(t)\beta(t) + (1-\beta(t))F(0))\, \sigma(t). \tag{23}$$

This corresponds to a minimization over $\beta(t)$ and $\sigma(t)$ of a function

$$\chi_L^2(\sigma(t),\beta(t)) = E\left[\left(F^2(t)|\Lambda(t)|^2 - (F(0)+\beta(t)\Delta F(t))^2\,|\sigma(t)|^2\right)^2\right] \tag{24}$$

for every fixed t. Here, $\Delta F(t) = F(t) - F(0)$. Equating to zero the variations of $\chi_L^2(\sigma(t),\beta(t))$ over $\sigma(t)$ and $\beta(t)$ gives a pair of equations,

$$E\left[F^2(t)|\Lambda(t)|^2(F(0)+\beta(t)\Delta F(t))^2\right] = |\sigma(t)|^2 E\left[(F(0)+\beta(t)\Delta F(t))^4\right], \tag{25}$$

$$E\left[F^2(t)|\Lambda(t)|^2(F(0)+\beta(t)\Delta F(t))\Delta F(t)\right] = |\sigma(t)|^2 E\left[(F(0)+\beta(t)\Delta F(t))^3\Delta F(t)\right]. \tag{26}$$

Solving these in the lowest leading order in volatilities $\sigma_Y(t)$, $\sigma_n(t)$, $\tilde{\sigma}_n(t)$ gives the result

$$\sigma(t) = \sigma_Y(t) + \sum_{n=\eta(t)}^{M-1}\left(\sigma_n(t) - \tilde{\sigma}_n(t)\right), \tag{27}$$

$$\beta(t) = 1 - \frac{\sum\limits_{n=\eta(t)}^{M-1}(1-\beta_n(t))(\sigma_n(t)\cdot\sigma(t))\int_0^t(\sigma_n(\tau)\cdot\sigma(\tau))d\tau}{|\sigma(t)|^2\int_0^t|\sigma(\tau)|^2 d\tau} + \frac{\sum\limits_{n=\eta(t)}^{M-1}(1-\tilde{\beta}_n(t))(\tilde{\sigma}_n(t)\cdot\sigma(t))\int_0^t(\tilde{\sigma}_n(\tau)\cdot\sigma(\tau))d\tau}{|\sigma(t)|^2\int_0^t|\sigma(\tau)|^2 d\tau}. \tag{28}$$

Additional technical details of the derivation can be found in the Appendix.

Recall that the skew parameter $\beta(t)$ identically equal to 1 corresponds to a purely log-normal distribution and the absence of implied volatility skew. Eq. (28) produces no skew in the FX option if there is no skew in all bond ratios, $\beta_n(t) = \tilde{\beta}_n(t) \equiv 1$. This, however, would not be the case if we started with log-normal LIBORs, $b_n(t) = \tilde{b}_n(t) \equiv 1$, as can be seen from Eq. (5).

The pricing of the European FX option (14) with maturity T_M and strike K finally reduces to a variant of the Black-Scholes formula,

$$E\left[\frac{(X(T_M)-K)^+}{N(T_M)}\right] = P(0,T_M)\left(\frac{F}{\bar{\beta}_{T_M}}\mathcal{N}(d_+) - \left(K + \frac{F(1-\bar{\beta}_{T_M})}{\bar{\beta}_{T_M}}\right)\mathcal{N}(d_-)\right), \tag{29}$$

$$d_\pm = \frac{\ln F/(K\bar{\beta}_{T_M} + F(1-\bar{\beta}_{T_M})) \pm V/2}{\sqrt{V}}, \quad V = \bar{\beta}_{T_M}^2\int_0^{T_M}|\sigma(t)|^2 dt, \tag{30}$$

where $F = F(0,T_M)$ is the forward rate, and $\bar{\beta}_{T_M}$ is the effective shift obtained by applying the averaging formula (19) to the time-dependent shift (28).

Numerical results

In this section we check numerically our approximation for the European FX option pricing. We used recent data for swap rates (Fig. 1) and ATM caplet volatilities in EUR (domestic) and USD (foreign) markets, and for long-dated European EUR/USD options to run a test with a 3-factor log-normal LIBOR market model. The absolute value of the instantaneous FX volatility $|\sigma_Y(t)|$ is plotted in Fig. 2 as a function of time. The absolute values of time-independent LIBOR volatilities $|\gamma_n|$ and $|\tilde{\gamma}_n|$ are plotted in Fig. 3 as functions of semi-annual LIBOR maturity T_n. Each volatility has three components with the weights chosen to obtain specific time-independent values of effective correlations,

$$\frac{(\sigma_Y \cdot \gamma_n)}{|\sigma_Y||\gamma_n|} = -15\%, \quad \frac{(\sigma_Y \cdot \tilde{\gamma}_n)}{|\sigma_Y||\tilde{\gamma}_n|} = -20\%, \quad \frac{(\gamma_n \cdot \tilde{\gamma}_n)}{|\gamma_n||\tilde{\gamma}_n|} = 25\%. \tag{31}$$

Table 1 shows the values of maturities and strikes of FX options used in the test. Table 2 shows implied BS volatilities computed analytically using the approximation based on the Markovian projection formulas (27, 28) and the parameter-averaging formula (19). The errors of the approximation with respect to a simulation with a large number of paths are in Table 3.

Implied volatility skews plotted in Figs. 4–6 show an excellent ability of the analytics to follow the exact pattern of skew computed using a simulation with a large number of paths. We observe a small systematic bias in the absolute level of the implied volatility curve, the correction of which would require taking higher-order corrections to the effective volatility (27) into account. Note that a noticeable skew of FX options is obtained starting from purely log-normal models for the LIBORs and without assuming any explicit skew in the process of the FX rate, which suggests that consistent log-normal approximations for FX options are neither possible nor desirable.

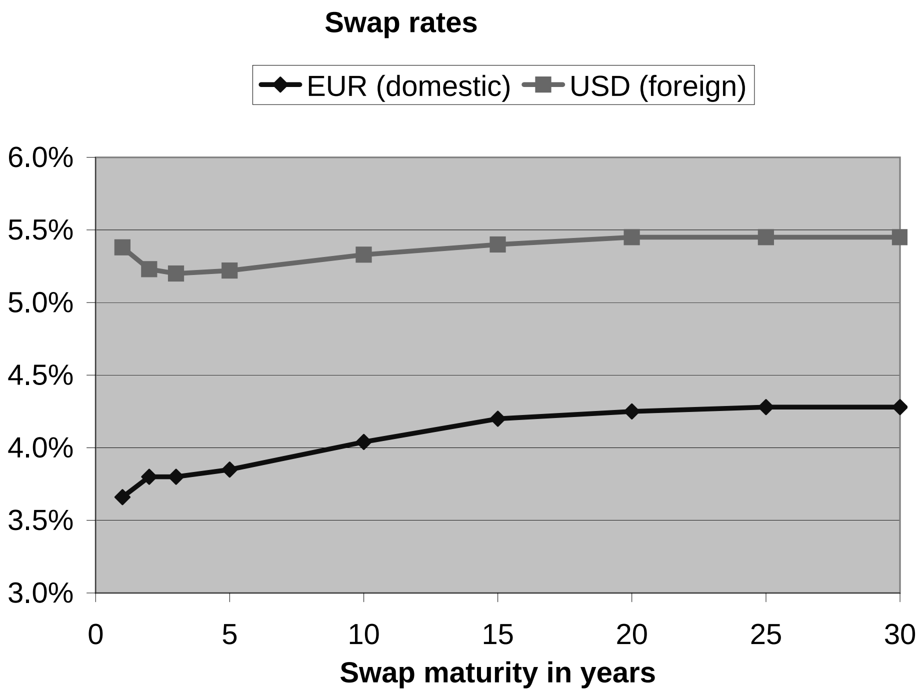

Figure 1: Swap rate curves in EUR (domestic) and USD (foreign) markets used in the test.

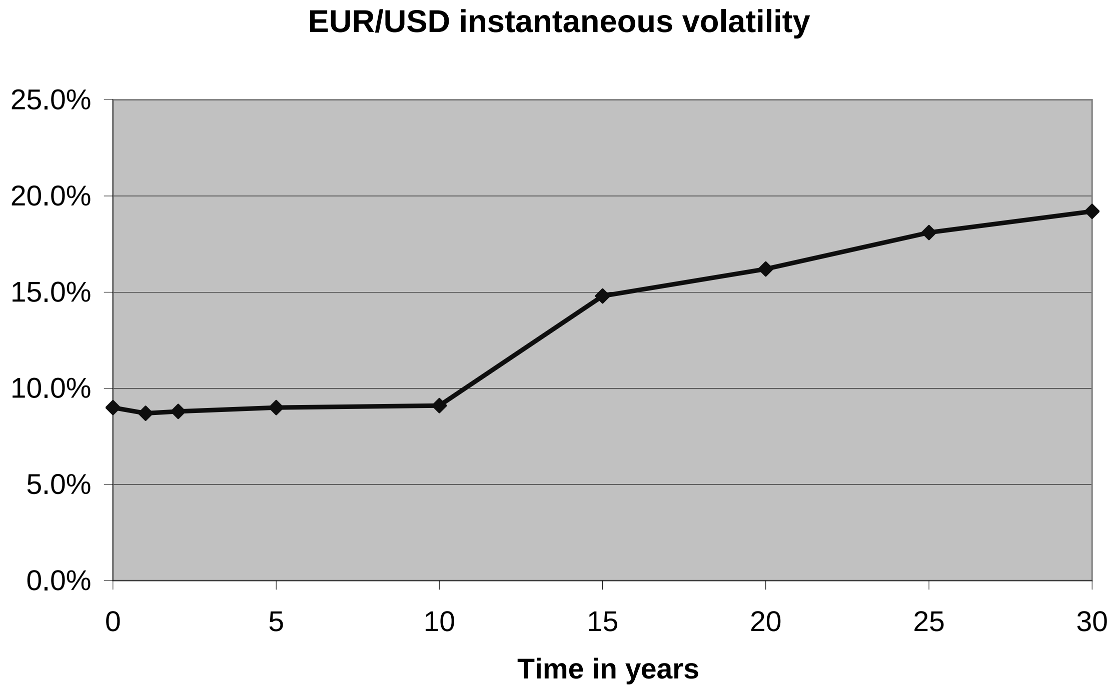

Figure 2: Absolute value of instantaneous volatility $|\sigma_Y(t)|$ of EUR/USD rate used in the test.

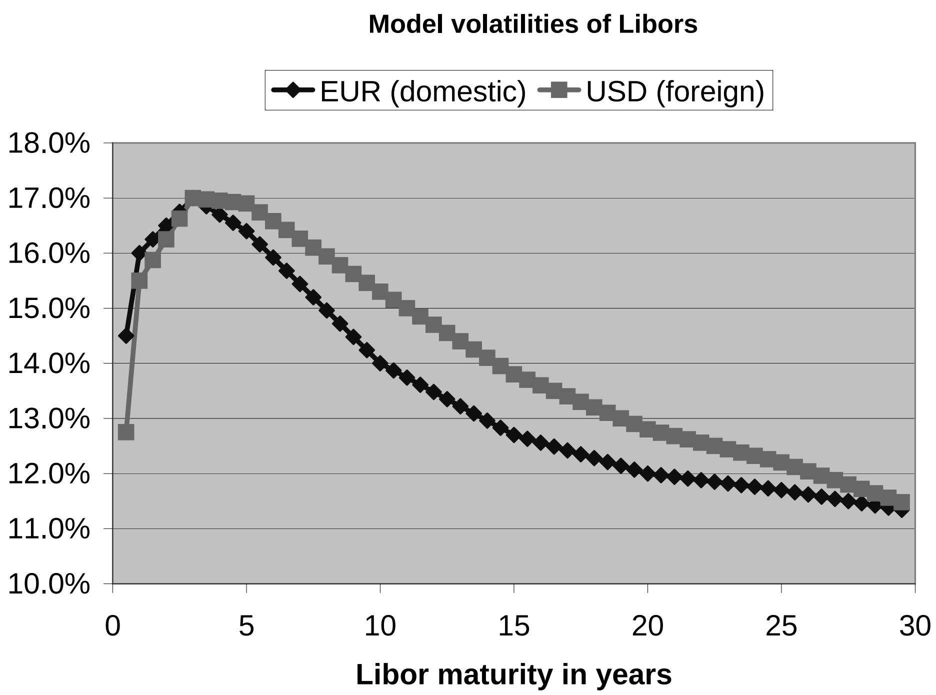

Figure 3: Absolute values of the LMM volatilities for domestic and foreign markets, $|\gamma_n(0)|$ and $|\tilde{\gamma}_n(0)|$, calibrated to interest rate caps in the corresponding currencies and plotted as a function of T_n. The volatilities are chosen to be time-independent, $\gamma_n(t) \equiv \gamma_n(0)$.

Maturity	set 1	set 2	set 3	set 4	set 5	set 6	set 7
5Y	66.77	74.67	83.50	**93.38**	104.43	116.78	130.59
10Y	54.70	64.07	75.04	**87.90**	102.95	120.59	141.25
15Y	46.72	56.71	68.82	**83.53**	101.37	123.04	149.32
20Y	40.22	50.30	62.90	**78.66**	98.37	123.03	153.85
25Y	35.26	45.27	58.13	**74.64**	95.84	123.06	158.01
30Y	30.96	40.71	53.53	**70.40**	92.58	121.74	160.09

Table 1: Maturities and strikes of European FX options used in the numerical test. Strikes are expressed in % of spot. The column with values in bold face corresponds to ATM strikes.

Maturity	set 1	set 2	set 3	set 4	set 5	set 6	set 7
5Y	9.43	9.40	9.38	**9.36**	9.34	9.32	9.31
10Y	11.03	10.90	10.78	**10.68**	10.58	10.50	10.42
15Y	13.65	13.43	13.24	**13.07**	12.92	12.78	12.66
20Y	16.47	16.19	15.95	**15.74**	15.55	15.39	15.26
25Y	18.95	18.60	18.31	**18.07**	17.85	17.67	17.52
30Y	21.25	20.84	20.50	**20.21**	19.97	19.77	19.60

Table 2: Implied volatilities of European FX options computed using the analytical approximation based on Markovian projection.

Maturity	set 1	set 2	set 3	set 4	set 5	set 6	set 7
5Y	-1	-1	-1	**-1**	-1	-1	-1
10Y	-8	-6	-5	**-4**	-5	-6	-7
15Y	-10	-8	-6	**-6**	-7	-9	-13
20Y	-7	-7	-6	**-7**	-9	-11	-15
25Y	-1	-2	-4	**-6**	-9	-12	-16
30Y	11	5	1	**-4**	-7	-10	-15

Table 3: Basis point errors of the analytical approximation expressed in terms of deviations in the the values of implied BS volatilities.

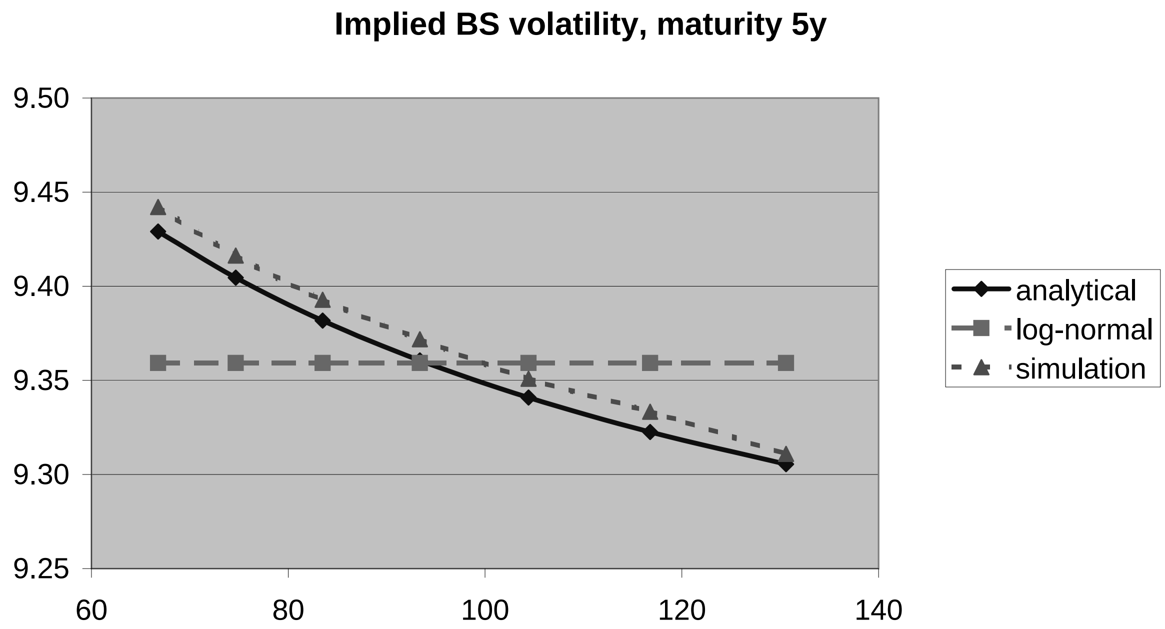

Figure 4: Comparison of analytics and simulation for implied BS volatility skew at 5y.

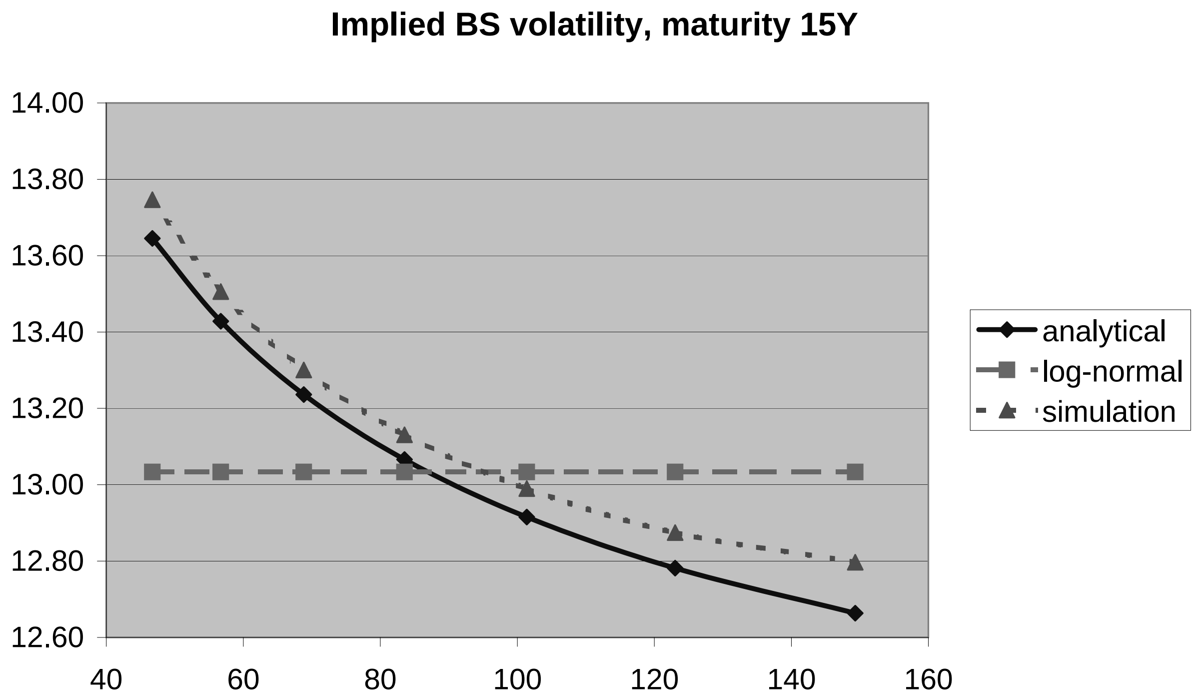

Figure 5: Comparison of analytics and simulation for implied BS volatility skew at 15y.

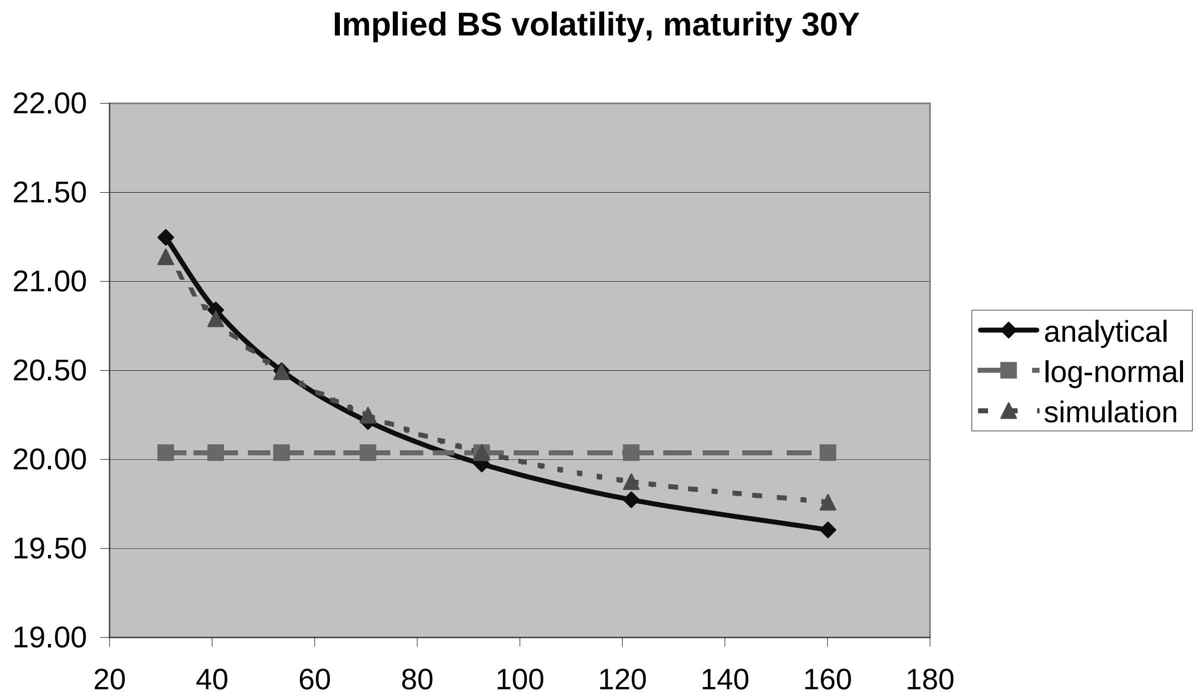

Figure 6: Comparison of analytics and simulation for implied BS volatility skew at 30y.

Conclusions and future directions

We revisited the setup of cross-currency LIBOR Market Models and showed how the technique of Markovian projection can be applied to the problem of calibration to FX options in the case when the presence of skew makes an assumption of log-normal forward FX rates inappropriate. In doing so, we developed a new approach to the Markovian projection, based on an explicit analytical minimization of the functional-space distance from the exact stochastic volatility to the low-dimensional subspace of displaced diffusion volatilities, given by Eq. (22).

We found that the skew in implied volatilities of FX options is present even if it was not built explicitly into the exchange rate model. The FX skew generated in this way from single-currency LMMs may not be sufficient to fit the market data. To obtain a more flexible model, we could start with a displaced-diffusion version of Eq. (9), which is addressed in a separate work.

We are indebted to Vladimir Piterbarg for an introduction into the method of Markovian projections. AA acknowledges a stimulating discussion with Peter Jäckel as a result of which we became aware of a forthcoming unpublished work by Jäckel and Kawai where the problem of FX skew in cross-currency LMM modeling is addressed using a different set of analytical tools. We are grateful to our colleagues at NumeriX and especially to Gregory Whitten for supporting our work.

Appendix: Perturbation theory for effective skew

We assume that an expansion in the powers of volatilities is possible in Eqs. (25) and (26) and measure the order of every term as a power of the dominant volatility $\sigma_+ = \max\{\sigma_n(t), \tilde{\sigma}_n(t), \sigma_Y(t)\}$. Keeping only the terms of the order $O(\sigma_+^2)$ in Eq. (25), we get

$$F(0)^2 E[F(t)^2 |\Lambda(t)|^2] = |\sigma(t)|^2 F(0)^4 + O(\sigma_+^4). \tag{32}$$

In the leading order, the expected value in the left hand side can be obtained by freezing the process $F(t) \to F(0)$ as well as the bond ratios $R_n(t) \to R_n(0)$, $\tilde{R}_n(t) \to \tilde{R}_n(0)$ in the definition (17) of $\Lambda(t)$. This leads to Eq. (27) as one of the solutions (the others are related by an orthogonal rotation in the factor space and lead to equivalent dynamics).

The lowest order balance in Eq. (26) is of order $O(\sigma_+^4)$,

$$\begin{aligned} F(0)E[F(t)^2|\Lambda^2(t)|^2\Delta F(t)] + \beta(t)E[F^2|\Lambda^2(t)|^2\Delta F(t)^2] = \\ 3\beta(t)F(0)^2|\sigma(t)|^2 E[\Delta F(t)^2] + O(\sigma_+^6), \end{aligned} \tag{33}$$

which entails

$$\beta(t) = \frac{F(0)E[F^2|\Lambda^2(t)|^2\Delta F(t)]}{3|\sigma(t)|^2F(0)^2E[\Delta F(t)^2] - E[F^2|\Lambda^2(t)|^2\Delta F(t)^2]} + O(\sigma_+^2). \tag{34}$$

The leading order of the averages in this expression cannot be directly found by a replacement of the processes by their initial values because the result would vanish. A general strategy is to use Ito's lemma to take consecutive differentials of the processes appearing under the expectation sign until the replacement by the initial values becomes possible. For example,

$$d(\Delta F(t)^2) = 2F(t)\Delta F(t)\Lambda(t)dW(t) + F(t)^2|\Lambda(t)^2|dt. \tag{35}$$

Taking the expectations, we obtain an ODE

$$dE[\Delta F(t)^2] = E[F(t)^2 |\Lambda(t)|^2] dt, \tag{36}$$

which is easily solved in the leading order,

$$E[\Delta F(t)^2] = F(0)^2 \int_0^t |\sigma(\tau)|^2 d\tau + O(\sigma_+^4). \tag{37}$$

This approach also works for a more difficult average

$$\begin{aligned} E[F(t)^2|\Lambda^2(t)|^2\Delta F(t)] &= 2|\sigma(t)|^2 \int_0^t |\sigma(\tau)|^2 d\tau \\ -2\sum_{n=\eta(t)}^{M-1} (1-\beta_n(t))(\sigma_n(t)\cdot\sigma(t)) &\int_0^t (\sigma_n(\tau)\cdot\sigma(\tau)) d\tau \\ +2\sum_{n=\eta(t)}^{M-1} (1-\tilde\beta_n(t))(\tilde\sigma_n(t)\cdot\sigma(t)) &\int_0^t (\tilde\sigma_n(\tau)\cdot\sigma(\tau)) d\tau. \end{aligned} \tag{38}$$

The derivation will be presented elsewhere and can be obtained from the authors on request. From this, Eq. (28) for effective skew follows.

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