Modern Approaches to Stochastic Volatility Calibration

Vladimir Piterbarg
Barclays Capital

Plan

- Generic method for volatility calibration
  - Markovian projection (MP)
  - Parameter averaging (PA)
- Examples
  - Options on baskets in local volatility models
  - Options on spreads in multi-stochastic volatility models
  - Short rate models
  - Forward Libor models
  - Long-dated FX

Calibration

- Need fast methods for European options for calibration
- A number of SV models for interest rates and hybrids have been put forward recently, with various approaches to calibration
- Many of these approaches can be aggregated into what we call the Markovian Projection method:

  a generic, powerful framework for deriving closed-form approximations to European option prices

  Step 1 Apply Markovian projection to $S(\cdot)$, a technique to replace a complicated process with a simple one, preserving European option prices

  Step 2 Approximate conditional expected values required

  Step 3 Apply parameter averaging techniques to obtain time-independent coefficients from time-dependent

  Step 4 Hopefully a simple model is obtained, use known results.

The Markovian projection

Theorem (Dupire 97, Gyongy 86) Let $X(t)$ be given by

$$dX(t) = \alpha(t)\,dt + \beta(t)\,dW(t), \tag{1}$$

where $\alpha(\cdot)$, $\beta(\cdot)$ are adapted bounded stochastic processes such that (1) admits a unique solution. Define $a(t,x)$, $b(t,x)$ by

$$a(t,x) = E\left(\alpha(t)\middle|\,X(t)=x\right),$$
$$b^2(t,x) = E\left(\beta^2(t)\middle|\,X(t)=x\right),$$

Then the SDE

$$dY(t) = a(t,Y(t))\,dt + b(t,Y(t))\,dW(t), \tag{2}$$
$$Y(0) = X(0),$$

admits a weak solution $Y(t)$ that has the same one-dimensional distributions as $X(t)$.

- See [Dup97], [Gyö86]

BARCLAYS CAPITAL

The Markovian projection, cont

Remark 1 Since $X(\cdot)$ and $Y(\cdot)$ have the same one-dimensional distributions, the prices of European options on $X(\cdot)$ and $Y(\cdot)$ for all strikes K and expiries T will be the same. Thus, for the purposes of European option valuation and/or calibration to European options, we can replace a potentially very complicated process $X(\cdot)$ with a much simpler Markov process $Y(\cdot)$, which we call the Markovian projection of $X(\cdot)$.

Remark 2 The process $Y(\cdot)$ follows what is known as a "local volatility" process. The function $b(t, x)$ is often called "Dupire's local volatility"

BARCLAYS CAPITAL

The Markovian projection, cont

- If $X(\cdot)$ itself came from a local volatility model (perhaps complicated), then replacing it with a (simpler) local vol model is probably the right thing to do. But:
- Any process (including a stochastic volatility one) can be replaced by a local volatility process for the purposes of European option valuation. Is it a good idea?
- Requires calculations of conditional expected values. This is the hard bit. Approximations often necessary
- In approximations, better to replace "like for like". Replace a (complicated) SV model with a (simpler) SV model.
  - Approximations to conditional expected values may be simpler
  - Errors of approximations will tend to "cancel out"
- Dupire-Gyongy theorem still works

Corollary If two processes have the same Dupire's local volatility, the European option prices on both are the same for all strikes and expiries

The Markovian projection for SV

- Let $X_1(t)$ follow
$$dX_1(t) = b_1(t, X_1(t))\sqrt{\zeta_1(t)}\,dW(t),$$
where $\zeta_1(t)$ is some variance process.
- We would like to match the European option prices on $X_1(\cdot)$ (for all expiries and strikes) in a model of the form
$$dX_2(t) = b_2(t, X_2(t))\sqrt{\zeta_2(t)}\,dW(t),$$
where $\zeta_2(t)$ is a different, and potentially simpler, variance process.
- Then the Corollary and the Theorem imply that we need to set
$$b_2^2(t,x) = b_1^2(t,x)\frac{E(\zeta_1(t)|X_1(t)=x)}{E(\zeta_2(t)|X_2(t)=x)}. \tag{3}$$
- Error cancellation – whatever approximations are used for conditional expected values in (3), hopefully they will tend to cancel when we take the ratio

Simple SV model

- After applying the MP method, often get the SDEs of the form

$$\begin{aligned} \mathrm{dz}(\mathrm{t}) &= \theta\,(1-\mathrm{z}(\mathrm{t}))\,\mathrm{dt} + \gamma(\mathrm{t})\sqrt{\mathrm{z}(\mathrm{t})}\,\mathrm{dV}(\mathrm{t}), \\ \mathrm{dS}(\mathrm{t}) &= (\beta(\mathrm{t})\,\mathrm{S}(\mathrm{t}) + (1-\beta(\mathrm{t}))\,\mathrm{S}(0))\,\sigma(\mathrm{t})\sqrt{\mathrm{z}(\mathrm{t})}\,\mathrm{dW}(\mathrm{t}), \end{aligned} \tag{4}$$

- Or, rather, we apply the MP method with the goal of obtaining the SDEs in this form
  - Choose z to be the square root process
  - Linearize the volatility term of S
- Why? When parameters are constant, this is the (shifted) Heston model, a model with very efficient numerical methods for European option valuation, see [AA02].
- How to replace time-dependent parameters with constant? Parameter averaging. Proofs and details in [Pit05b], [Pit05a]

Example of averaging formula

- For motivation, consider a log-normal model with time-dependent volatility,

$$dS(t) = \sigma(t) S(t)\, dW(t).$$

- It is known that, an option value with expiry T_n in this model is equal to the Black-Scholes option value with "effective" volatility

$$\sigma_n = \left(\frac{1}{T_n} \int_0^{T_n} \sigma^2(t)\, dt \right)^{1/2}.$$

- Calibration by solving the following equations

$$\int_0^{T_n} \sigma^2(t)\, dt = \sigma_n^2 T_n, \quad n = 1, \ldots, N.$$

  Linear in $\sigma^2(t)$, trivial to solve.
- Direct link between "model" parameter $\sigma(t)$ and "market" parameters (σ_n)

Averaging volatility of variance

- $\int_0^T \sigma^2(t)\, z(t)\, dt$ is ''realized variance''
- Curvature of the smile depends on the variance of realized variance (kurtosis, 4-th moment)
- Averaged vol of variance η (to T) is obtained by solving

$$E\left(\int_0^T \sigma^2(t)\, z(t)\, dt\right)^2 = E\left(\int_0^T \sigma^2(t)\, \bar{z}(t)\, dt\right)^2,$$

where

$$\begin{aligned} dz(t) &= \theta\,(1 - z(t))\, dt + \gamma(t)\,\sqrt{z(t)}\, dV(t)\,, \\ d\bar{z}(t) &= \theta\,(1 - \bar{z}(t))\, dt + \eta\sqrt{\bar{z}(t)}\, dV(t)\,. \end{aligned}$$

BARCLAYS CAPITAL

Averaging skew

- Fixed T, vol of variance already averaged (use constant η)
  - Time-dependent skew

$$\mathrm{dS}(\mathrm{t}) = \sigma(\mathrm{t})\left(\beta(\mathrm{t})\,\mathrm{S}(\mathrm{t}) + (1-\beta(\mathrm{t}))\,\mathrm{S}(0)\right)\sqrt{\mathrm{z}(\mathrm{t})}\,\mathrm{dW}(\mathrm{t}),$$

  - Constant skew

$$\mathrm{d\bar{S}}(\mathrm{t}) = \sigma(\mathrm{t})\left(\mathrm{b}\bar{\mathrm{S}}(\mathrm{t}) + (1-\mathrm{b})\,\bar{\mathrm{S}}(0)\right)\sqrt{\mathrm{z}(\mathrm{t})}\,\mathrm{dW}(\mathrm{t}).$$

  - Given $\beta(\cdot)$, find b such that option prices for different strikes (same expiry T) are matched between two models

Averaging skew, cont

- The main result. In the "small skew" limit,

$$b = \int_0^T \beta(t)\, w(t)\, dt,$$

where

$$\begin{aligned} w(t) &= \frac{v^2(t)\,\sigma^2(t)}{\int_0^T v^2(t)\,\sigma^2(t)\,dt}, \\ v^2(t) &= E\left(z(t)\left(S_0(t) - x_0\right)^2\right). \end{aligned}$$

- Comments:
  - "Total skew" b is the average of "local skews" $\beta(t)$ with weights $w(t)$
  - Weights proportional to total variance, i.e. local slope further away matters more
- Example: No SV ($\eta = 0$), constant volatility $\sigma(t) \equiv \sigma$,

$$b = \left(T^2/2\right)^{-1} \int_0^T t\beta(t)\, dt.$$

BARCLAYS CAPITAL

Averaging volatility

- Approximate the dynamics of

$$dS(t) = \sigma(t)(bS(t) + (1-b)S(0))\sqrt{z(t)}\,dW(t)$$

with

$$d\bar{S}(t) = \lambda\left(b\bar{S}(t) + (1-b)\bar{S}(0)\right)\sqrt{z(t)}\,dW(t).$$

- Can do numerically as in [Lew00], [AA02]: Do Fourier integral with integrand a solution to Riccati ODEs. Slow.
- Can use moment-matching

$$E(S(T) - S_0)^2 = E\left(\bar{S}(T) - S_0\right)^2, \quad \int_0^T \sigma^2(t)\,dt = \lambda^2 T.$$

Not always accurate
- Better: approximate a European option payoff locally with a function whose expectation can be computed in both models above; choose λ to match the two.

BARCLAYS CAPITAL

Averaging volatility, cont

- By conditioning on the realized variance
$$\mathrm{E}\left(\mathrm{S}(\mathrm{T})-\mathrm{S}_0\right)^+ = \mathrm{E}g\left(\int_0^{\mathrm{T}} \sigma^2(\mathrm{t})\, \mathrm{z}(\mathrm{t})\, \mathrm{dt}\right),$$
where g is a known function.
- Approximate
$$\mathrm{g}(\mathrm{x}) \approx \mathrm{a} + \mathrm{b}e^{-\mathrm{cx}}$$
by matching the value and first two derivatives at
$$\zeta = \mathrm{E}\int_0^{\mathrm{T}} \sigma^2(\mathrm{t})\, \mathrm{z}(\mathrm{t})\, \mathrm{dt}$$
- The problem reduced to finding λ such that
$$\mathrm{E}\exp\left(\frac{\mathrm{g}''(\zeta)}{\mathrm{g}'(\zeta)} \int_0^{\mathrm{T}} \sigma^2(\mathrm{t})\, \mathrm{z}(\mathrm{t})\, \mathrm{dt}\right) = \mathrm{E}\exp\left(\lambda^2 \frac{\mathrm{g}''(\zeta)}{\mathrm{g}'(\zeta)} \int_0^{\mathrm{T}} \mathrm{z}(\mathrm{t})\, \mathrm{dt}\right).$$
- Very fast and easy numerical search for λ (starting with a good initial guess $\lambda^2 = \mathrm{T}^{-1}\int_0^{\mathrm{T}} \sigma^2(\mathrm{t})\, \mathrm{dt}$).

Direct calibration to market

- In equity/FX: Let $\sigma_{\mathrm{mkt}}(T, K)$ be market volatilities for all expiries T and strikes K (assumed known). Given an exogenous SV process $z(t)$, find $b(t, x)$ such that the model

$$dS(t) = b(t, S(t)) \sqrt{z(t)}\, dW(t), \quad S(0) = S_0,$$

matches the market

  - Define Dupire's market local volatility $b_{\mathrm{mkt}}(t, x)$ by the requirement that the local volatility model with $b_{\mathrm{mkt}}(t, x)$ matches the whole market. Easy to compute

$$b_{\mathrm{mkt}}(t, x) = \frac{2\partial C/\partial t}{\partial^2 C/\partial x^2}.$$

  - Then, from Theorem and Corollary,

$$b^2(t, x) = \frac{b^2_{\mathrm{mkt}}(t, x)}{E(z(t)|\, S(t) = x)}. \tag{5}$$

  - In practice $E(z(t)|\, S(t) = x)$ is often computed numerically in a forward PDE in (S, z). Slow and noisy.

Direct calibration to market, cont

- Define a "proxy" process $X(t)$ by
$$dX(t) = \tilde{b}(t, X(t))\sqrt{z(t)}\,dW(t), \quad X(0) = S_0, \tag{6}$$
where $\tilde{b}(t, x)$ is such that European options on X are easy to compute
- Define the "proxy" Dupire's local volatility $b_{proxy}(t, x)$ as before but for European options on X (not on market). Then
$$E(z(t)|\,X(t) = x) = \frac{b^2_{proxy}(t, x)}{\tilde{b}^2(t, x)}, \tag{7}$$
thus having a stochastic volatility model with cheaply-computable European option prices allows us to compute the conditional expected values easily.
- Combining the two results we get
$$b(t, x) = \tilde{b}(t, x) \times \frac{b_{mkt}(t, x)}{b_{proxy}(t, x)} \times \left(\frac{E(z(t)|\,X(t) = x)}{E(z(t)|\,S(t) = x)}\right)^{/12}.$$

Direct calibration to market, cont

- Choice 1: Approximate
  $\mathrm{E}\left(\mathrm{z}\left(\mathrm{t}\right)\middle|\,\mathrm{X}\left(\mathrm{t}\right)=\mathrm{x}\right)=\mathrm{E}\left(\mathrm{z}\left(\mathrm{t}\right)\middle|\,\mathrm{S}\left(\mathrm{t}\right)=\mathrm{x}\right)$
- Choice 2: Link $\mathrm{S}\left(\mathrm{t}\right)$ and $\mathrm{X}\left(\mathrm{t}\right)$.
  - Define $\mathrm{H}\left(\mathrm{t},\mathrm{s}\right)$ by the requirement that $\mathrm{H}\left(\mathrm{t},\mathrm{S}\left(\mathrm{t}\right)\right)$ has the same dW term as dX (H a function of b, $\tilde{\mathrm{b}}$)
  - Then approximate

$$\begin{aligned} \mathrm{X}\left(\mathrm{t}\right) &\approx \mathrm{H}\left(\mathrm{t},\mathrm{S}\left(\mathrm{t}\right)\right), \\ \mathrm{E}\left(\mathrm{z}\left(\mathrm{t}\right)\middle|\,\mathrm{S}\left(\mathrm{t}\right)=\mathrm{x}\right) &\approx \mathrm{E}\left(\mathrm{z}\left(\mathrm{t}\right)\middle|\,\mathrm{X}\left(\mathrm{t}\right)=\mathrm{H}\left(\mathrm{t},\mathrm{x}\right)\right). \end{aligned}$$

  - Functional equation on b,

$$\mathrm{b}\left(\mathrm{t},\mathrm{x}\right)=\tilde{\mathrm{b}}\left(\mathrm{t},\mathrm{H}\left(\mathrm{t},\mathrm{x}\right)\right)\frac{\mathrm{b}_{\mathrm{mkt}}\left(\mathrm{t},\mathrm{x}\right)}{\mathrm{b}_{\mathrm{proxy}}\left(\mathrm{t},\mathrm{H}\left(\mathrm{t},\mathrm{x}\right)\right)}. \tag{8}$$

- Last derivation is an example of a clever way of computing conditional expectations
- Original result due to Forde ([For06]). More details in [Pit06a].

Basket modeling

- Consider a "simple" local volatility model for a basket $S(t) = \sum w_i S_i(t)$,

$$dS_i(t) = \varphi_i(S_i(t))\, dW_i(t), \quad i = 1, \ldots, I.$$

- Options on index $S(\cdot)$. Apply MP to write SDE for S. Start

$$\begin{aligned} dS(t) &= \sigma(t)\, dW(t), \\ \sigma^2(t) &= \sum_{n,m=1}^{N} w_n w_m \varphi_n(S_n(t))\, \varphi_m(S_m(t))\, \rho_{nm}. \end{aligned}$$

- Then

$$\begin{aligned} dS(t) &= \varphi(t, S(t))\, dW(t), \\ \varphi^2(t,x) &= E\left(\sigma^2(t) \middle| S(t) = x\right). \end{aligned}$$

BARCLAYS CAPITAL

Basket modeling, cont

- To compute $\mathrm{E}\left(\sigma^2(t)\,\middle|\, S(t)\right)$ use Gaussian approximation $S_i \approx \bar{S}_i$, $S \approx \bar{S}$,

$$\begin{aligned} d\bar{S}_i(t) &= p_i\, dW_i(t), \quad d\bar{S}(t) = \sigma(0)\, dW(t), \\ p_i &= \varphi_i(S_i(0)), \quad \sigma(0) = \sum_{n,m=1}^{N} w_n w_m p_n p_m \rho_{nm}, \end{aligned}$$

  and linearization

$$\varphi_i(x) \approx p_i + q_i(x - S(0)), \quad q_i = \varphi_i'(S_i(0)).$$

- Then

$$\mathrm{E}\left(\bar{S}_n(t) - S_n(0)\,\middle|\, \bar{S}(t) = x\right) = \rho_n \frac{p_n}{p}(x - S(0)),$$

$$\rho_n \triangleq \left\langle d\bar{W}(t), dW_n(t)\right\rangle / dt = \frac{1}{p}\sum_{m=1}^{N} w_m p_m \rho_{nm}.$$

- See more in [Pit06a]. More accurate method in [ABOBF02].

BARCLAYS CAPITAL

Spread options in SV model

- For spread options, important to use different SV process for each variable (see [Pit06c]),

$$\begin{aligned} dS_i(t) &= \varphi_i(S_i(t))\sqrt{z_i(t)}\, dW_i(t), \quad i = 1, 2, \\ dz_i(t) &= \theta(1 - z_i(t))\, dt + \eta_i \sqrt{z_i(t)}\, dW_{2+i}(t), \quad z_i(0) = 1, \end{aligned}$$

  with the correlations given by

$$\langle dW_i(t), dW_j(t) \rangle = \rho_{ij} \quad i, j = 1, \ldots, 4.$$

- Denote
$$p_i = \varphi_i(S_i(0)), \quad q_i = \varphi_i'(S_i(0)).$$
- Write down $dS(\cdot)$ for spread $S = S_1 - S_2$
- Identify a suitable "spread variance" process $z(\cdot)$
- Compute the skew function $\varphi(\cdot)$ of the spread using the Markovian projection ideas above
- "Massage" $z(\cdot)$ into the Heston form

BARCLAYS CAPITAL

Process for the spread

- We have
$$dS_i(t) = \varphi_i(S_i(t))\sqrt{z_i(t)}\,dW_i(t),$$
- $S = S_1 - S_2$, then $dS(t) = \sigma(t)\,dW(t)$, where

$$\begin{aligned}
\sigma^2(t) &= (\varphi_1(S_1(t))\,u_1(t))^2 \\
&\quad -2(\varphi_1(S_1(t))\,u_1(t))(\varphi_2(S_2(t))\,u_2(t))\,\rho_{12} \\
&\quad +(\varphi_2(S_2(t))\,u_2(t))^2, \\
dW(t) &= \frac{1}{\sigma(t)}(\varphi_1(S_1(t))\,u_1(t)\,dW_1(t) \\
&\quad -\varphi_2(S_2(t))\,u_2(t)\,dW_2(t)), \\
u_i(t) &= \sqrt{z_i(t)}, \quad i = 1, 2.
\end{aligned}$$

BARCLAYS CAPITAL

Process for the variance of the spread

- Try to find a stochastic volatility process $z(\cdot)$ such that the curvature of the smile of the spread $S(\cdot)$ is explained by it, and the local volatility function is only used to induce the volatility skew
- To identify a suitable candidate for $z(\cdot)$, consider what the expression for $\sigma^2(t)$ would be if $\varphi_i(x)$, $i = 1, 2$, were constant functions.
- In this case, the expression for $\sigma^2(t)$ above would not involve the processes $S_i(\cdot)$, $i = 1, 2$ and this is a good candidate for the stochastic variance process.
- We define (the division by $\sigma^2(0)$ is to preserve the scaling $z(0) = 1$)

$$z(t) = \frac{1}{p^2}\left((p_1 u_1(t))^2 - 2p_1 p_2 u_1(t) u_2(t) \rho_{12} + (p_2 u_2(t))^2\right), \tag{9}$$

where

$$p = \sigma(0) = \left(p_1^2 - 2p_1 p_2 \rho_{12} + p_2^2\right)^{1/2}. \tag{10}$$

BARCLAYS CAPITAL

Skew function of the spread

- By Corollary,

$$\varphi^2(\mathrm{t}, \mathrm{x}) = \frac{\mathrm{E}\left(\sigma^2(\mathrm{t}) \middle| \mathrm{S}(\mathrm{t}) = \mathrm{x}\right)}{\mathrm{E}\left(\mathrm{z}(\mathrm{t}) \middle| \mathrm{S}(\mathrm{t}) = \mathrm{x}\right)}. \tag{11}$$

- The expression for $\mathrm{E}\left(\sigma^2(\mathrm{t}) \middle| \mathrm{S}(\mathrm{t}) = \mathrm{x}\right)$ is a linear combinations of the conditional expected values of the terms

$$\varphi_{\mathrm{i}}(\mathrm{S}_{\mathrm{i}}(\mathrm{t}))\, \varphi_{\mathrm{j}}(\mathrm{S}_{\mathrm{j}}(\mathrm{t}))\, \mathrm{u}_{\mathrm{i}}(\mathrm{t})\, \mathrm{u}_{\mathrm{j}}(\mathrm{t}),$$

- Approximate to the first order by

$$\mathrm{p}_{\mathrm{i}}\mathrm{p}_{\mathrm{j}}\left(1 + \frac{\mathrm{q}_{\mathrm{i}}}{\mathrm{p}_{\mathrm{i}}}(\mathrm{S}_{\mathrm{i}}(\mathrm{t}) - \mathrm{S}_{\mathrm{i}}(0)) + \frac{\mathrm{q}_{\mathrm{j}}}{\mathrm{p}_{\mathrm{j}}}(\mathrm{S}_{\mathrm{j}}(\mathrm{t}) - \mathrm{S}_{\mathrm{j}}(0)) + \ldots\right).$$

- Use Gaussian approximation to compute conditional expected values

BARCLAYS CAPITAL

Gaussian approximation

- Use $\bar{X}$ to denote a Gaussian approximation to X for a generic X, then

$$\begin{aligned} E\left(S_i(t) - S_i(0) \middle| S(t) = x\right) &\approx E\left(\bar{S}_i(t) - \bar{S}_i(0) \middle| \bar{S}(t) = x\right) \\ E\left(u_i(t) - 1 \middle| S(t) = x\right) &\approx E\left(\bar{u}_i(t) - 1 \middle| \bar{S}(t) = x\right), \end{aligned}$$

- Here (we ignore dt terms for du, although they may be included for more accurate approximations)

$$\begin{aligned} d\bar{S}_i(t) &= p_i \, dW_i(t), \quad d\bar{S}(t) = p \, d\bar{W}(t), \qquad (12) \\ d\bar{u}_i(t) &= \frac{\eta_i}{2} dW_{2+i}(t), \quad d\bar{W}(t) = \frac{1}{p}\left(p_1 \, dW_1(t) - p_2 \, dW_2(t)\right). \end{aligned}$$

- Then

$$\begin{aligned} E\left(\bar{S}_i(t) - \bar{S}_i(0) \middle| \bar{S}(t) = x\right) &= \frac{p_i \rho_i}{p}\left(x - S(0)\right), \\ E\left(\bar{u}_i(t) - 1 \middle| \bar{S}(t) = x\right) &= \frac{\eta_i \rho_{2+i}}{2p}\left(x - S(0)\right), \end{aligned}$$

Skew function of the spread

- Combining the results, we get the following approximation to the spread dynamics,

$$dS(t) = \varphi(S(t))\sqrt{z(t)}\,dW(t),$$

- Here $\varphi(x)$ is a function of the same type as $\varphi_i(x)$ (linear or CEV) with

$$\varphi(S(0)) = p, \quad \varphi'(S(0)) = q,$$

where

$$\begin{aligned} p &= \left(p_1^2 - 2p_1p_2\rho_{12} + p_2^2\right)^{1/2} \\ q &\triangleq \frac{1}{p}\left(p_1\rho_1^2q_1 - p_2\rho_2^2q_2\right). \end{aligned}$$

BARCLAYS CAPITAL

Variance process for the spread

- The process for S is in a nice form. But z is not:

$$z(t) = \frac{1}{p^2}\left(p_1^2 z_1(t) - 2p_1p_2\sqrt{z_1(t)\, z_2(t)}\rho_{12} + p_2^2 z_2(t)\right).$$

- Compute dz,

$$\begin{aligned} dz(t) &= \delta_1(t)\, dt + \delta_2(t)\, dt + \delta_3(t)\, dt \\ &\quad + \xi_1(t)\, dW_3(t) + \xi_2(t)\, dW_4(t), \end{aligned}$$

- dW terms

$$\begin{aligned} \xi_1(t) &= \eta_1 \frac{p_1^2}{p^2}\left(\sqrt{z_1(t)} - \frac{p_2}{p_1}\rho_{12}\sqrt{z_2(t)}\right), \\ \xi_2(t) &= \eta_2 \frac{p_2^2}{p^2}\left(\sqrt{z_2(t)} - \frac{p_1}{p_2}\rho_{12}\sqrt{z_1(t)}\right). \end{aligned}$$

Variance process for the spread, cont

- dt terms

$$\begin{aligned}
\delta_1(\mathrm{t}) &= \theta\frac{\mathrm{p}_1^2}{\mathrm{p}^2}\left(1-\frac{\mathrm{p}_2}{\mathrm{p}_1}\rho_{12}\sqrt{\frac{\mathrm{z}_2(\mathrm{t})}{\mathrm{z}_1(\mathrm{t})}}\right)(1-\mathrm{z}_1(\mathrm{t})),\\
\delta_2(\mathrm{t}) &= \theta\frac{\mathrm{p}_2^2}{\mathrm{p}^2}\left(1-\frac{\mathrm{p}_1}{\mathrm{p}_2}\rho_{12}\sqrt{\frac{\mathrm{z}_1(\mathrm{t})}{\mathrm{z}_2(\mathrm{t})}}\right)(1-\mathrm{z}_2(\mathrm{t})),\\
\delta_3(\mathrm{t}) &= \frac{\mathrm{p}_1\mathrm{p}_2\rho_{12}}{4\mathrm{p}^2}\left(\sqrt{\frac{\mathrm{z}_2(\mathrm{t})}{\mathrm{z}_1(\mathrm{t})}}\eta_1^2-2\eta_1\eta_2\rho_{34}+\sqrt{\frac{\mathrm{z}_1(\mathrm{t})}{\mathrm{z}_2(\mathrm{t})}}\eta_2^2\right).
\end{aligned}$$

- Complicated expression, Not "closed" in $\mathrm{z}(\cdot)$

BARCLAYS CAPITAL

Variance process for the spread, cont

- The curvature of the volatility smile (of options on $S(\cdot)$) is driven by the variance of the stochastic variance
- It is preserved under the Markovian projection of $z(\cdot)$ so can apply the Theorem again, now to the process for $z(\cdot)$!
- Formulas getting unwieldy: need to compute conditional expected values of the type $E\left(\sqrt{z_i(t)\, z_j(t)}\,\middle|\, z(t) = x\right)$ and $E\left(\sqrt{z_i(t)/z_j(t)}\,\middle|\, z(t) = x\right)$, for which we would apply the Gaussian approximations
- Try something simpler:
  - replace $\sqrt{z_1(t)}$, $\sqrt{z_2(t)}$ in the dW terms with $\sqrt{z(t)}$;
  - replace $\sqrt{\frac{z_2(t)}{z_1(t)}}$, $\sqrt{\frac{z_1(t)}{z_2(t)}}$ in dt terms with 1.
- $\delta_1(t) + \delta_2(t)$ becomes $\theta(1 - z)$,

BARCLAYS CAPITAL

Variance process for the spread, simple approximation

- $\delta_3(t)$ becomes

$$\gamma \triangleq \frac{p_1 p_2 \rho_{12}}{4p^2}\left(\eta_1^2 - 2\eta_1\eta_2\rho_{34} + \eta_2^2\right). \tag{13}$$

- The dW terms can be re-written as $\eta\sqrt{z(t)}\,dB(t)$, where

$$\begin{aligned}
\eta^2 &= \frac{1}{p^2}\left((p_1\eta_1\rho_1)^2 - 2(p_1\eta_1\rho_1)(p_2\eta_2\rho_2)\rho_{34} + (p_2\eta_2\rho_2)^2\right),\\
dB(t) &= \frac{1}{\eta}\left(p_1\eta_1\rho_1\,dW_3(t) - p_2\eta_2\rho_2\,dW_4(t)\right).
\end{aligned}$$

- Altogether

$$\begin{aligned}
dS(t) &= \varphi(S(t))\sqrt{z(t)}\,dW(t),\\
dz(t) &= \theta\left(1 + \frac{\gamma}{\theta} - z(t)\right)dt + \eta\sqrt{z(t)}\,dB(t).
\end{aligned}$$

- Linearize φ and apply Heston valuation formula to options on the spread S!

BARCLAYS CAPITAL

Local volatility short rate model

- Simplest interest rate model: one-factor Gaussian ("Hull-White")

$$\mathrm{r}(\mathrm{t}) = \mathrm{f}(0,\mathrm{t}) + \mathrm{x}(\mathrm{t}), \quad \mathrm{dx}(\mathrm{t}) = (\theta(\mathrm{t}) - \mathrm{ax}(\mathrm{t}))\,\mathrm{dt} + \sigma(\mathrm{t})\,\mathrm{dW}(\mathrm{t}).$$

- Local-volatility extension: quasi-Gaussian ("Cheyette")

$$\begin{aligned}
\mathrm{dx}(\mathrm{t}) &= (\mathrm{y}(\mathrm{t}) - \mathrm{ax}(\mathrm{t}))\,\mathrm{dt} + \sigma(\mathrm{t},\mathrm{x}(\mathrm{t}),\mathrm{y}(\mathrm{t}))\,\mathrm{dW}(\mathrm{t}), \\
\mathrm{dy}(\mathrm{t}) &= \left(\sigma^2(\mathrm{t},\mathrm{x}(\mathrm{t}),\mathrm{y}(\mathrm{t})) - 2\mathrm{ay}(\mathrm{t})\right)\mathrm{dt}.
\end{aligned}$$

- Swap rate (under swap measure), $\mathrm{S}(\mathrm{t}) = \mathrm{S}(\mathrm{t},\mathrm{x}(\mathrm{t}),\mathrm{y}(\mathrm{t}))$ for a known function $\mathrm{S}(\mathrm{t},\mathrm{x},\mathrm{y})$,

$$\mathrm{dS}(\mathrm{t}) = \left.\frac{\partial \mathrm{S}(\mathrm{t},\mathrm{x},\mathrm{y})}{\partial \mathrm{x}}\right|_{\mathrm{x}=\mathrm{x}(\mathrm{t}),\mathrm{y}=\mathrm{y}(\mathrm{t})} \sigma(\mathrm{t},\mathrm{x}(\mathrm{t}),\mathrm{y}(\mathrm{t}))\,\mathrm{dW}^{\mathrm{A}}(\mathrm{t}).$$

Local volatility short rate model, cont

- Markovian projection (preserves European swaptions)

$$\begin{aligned} dS(t) &= \eta(t, S(t))\, dW^A(t), \\ \eta^2(t, S) &= E^A\left(\left(\frac{\partial S(t, x(t), y(t))}{\partial x}\right)^2 \sigma^2(t, x(t), y(t)) \middle| S(t) = S\right) \end{aligned}$$

- Let $y^*(t) = E^A(y(t))$, $\xi(t, s)$ is the inverse of $S(t, x, y^*(t))$ in x. Then

$$\eta^2(t, S) \approx \left(\left.\frac{\partial S(t, x, y^*(t))}{\partial x}\right|_{x=\xi(t,S)}\right) \sigma(t, \xi(t, S), y^*(t)).$$

- Local-volatility model for S with a known η. Apply parameter averaging (on skew and vol), then shifted-lognormal formula to get option prices.

Stochastic volatility short rate model

- Stochastic-volatility extension: quasi-Gaussian SV

$$\begin{aligned} dx(t) &= (y(t) - ax(t))\, dt + \sqrt{z(t)}\sigma(t, x(t), y(t))\, dW(t), \\ dy(t) &= \left(z(t)\sigma^2(t, x(t), y(t)) - 2ay(t)\right) dt, \\ dz(t) &= \theta(1 - z(t))\, dt + \gamma(t)\sqrt{z(t)}\, dV(t). \end{aligned}$$

- Same results (use the same $z(\cdot)$ in (3)), after MP:

$$\begin{aligned} dS(t) &= \eta(t, S(t))\sqrt{z(t)}\, dW^A(t), \\ dz(t) &= \theta(1 - z(t))\, dt + \gamma(t)\sqrt{z(t)}\, dV(t). \end{aligned}$$

- Linearize $\eta(t, S)$, apply PA on skew, vol, vol of vol.
- See [And05]

BARCLAYS CAPITAL

Forward Libor model with time-dependent skews

- $L_n(t)$ are spanning forward Libor rates

$$dL_n(t) = \psi_n(t, L_n(t))\, dW_n^{T_{n+1}}(t), \quad n = 1, \ldots, N-1.$$

- Swap rate ($S = S_{n,m}$) dynamics

$$\begin{aligned} dS(t) &= \sum_{k=n}^{n+m-1} \frac{\partial S(t)}{\partial L_k(t)} \psi_k(t, L_k(t))\, dW_k^A(t) \\ &= \Sigma\left(t, \bar{L}(t)\right) dW_n^A(t), \\ \Sigma^2\left(t, \bar{L}(t)\right) &= \sum_{k,k'} \frac{\partial S(t)}{\partial L_k(t)} \frac{\partial S(t)}{\partial L_{k'}(t)} \psi_k(t, L_k(t))\, \psi_{k'}(t, L_{k'}(t))\, \rho_{kk'}. \end{aligned}$$

- By MP

$$\begin{aligned} \eta(t, S) &= \left(E^A\left(\Sigma^2\left(t, \bar{L}(t)\right) \middle| S(t) = S\right)\right)^{1/2} \\ &\approx E^A\left(\Sigma\left(t, \bar{L}(t)\right) \middle| S(t) = S\right) \end{aligned}$$

Forward Libor model with time-dependent skews, cont

- Linearize

$$\Sigma\left(t, \bar{L}(t)\right) = \Sigma\left(t, E^A \bar{L}(t)\right) + \left[\nabla\Sigma\left(t, E^A \bar{L}(t)\right)\right]^\top \left(\bar{L}(t) - E^A \bar{L}(t)\right)$$

- Approximate $\bar{L}(t), S(t)$ with Gaussian processes (use "hats")

$$E^A\left(\bar{L}(t) - E^A \bar{L}(t) \middle| S(t)\right) \approx \left\langle \hat{L}(t), \hat{L}(t) \right\rangle^{-1} \left\langle \hat{L}(t), \hat{S}(t) \right\rangle \left(S(t) - S\right.$$

- Then

$$\begin{aligned} dS(t) &= \left(a(t) + b(t)\left(S - S(0)\right)\right) dW_n^A(t) \\ a(t) &= \Sigma\left(t, E^A \bar{L}(t)\right), \\ b &= \left[\nabla\Sigma\left(t, E^A \bar{L}(t)\right)\right]^\top \left\langle \hat{L}(t), \hat{L}(t) \right\rangle^{-1} \left\langle \hat{L}(t), \hat{S}(t) \right\rangle. \end{aligned}$$

- Shifted lognormal process for S with time-dependent coefs (skew is the weighted average of Libor skews), apply PA, and we are done.

Forward Libor model with SV

- Use the same SV process $z(\cdot)$ for all Libor rates

$$dL_n(t) = \psi_n(t, L_n(t)) \sqrt{z(t)}\, dW_n^{T_{n+1}}(t), \quad n = 1, \ldots, N-1.$$

- Same results, get

$$\begin{aligned} dS(t) &= (a(t) + b(t)(S - S(0))) \sqrt{z(t)}\, dW_n^A(t) \\ dz(t) &= \theta(1 - z(t))\, dt + \gamma(t) \sqrt{z(t)}\, dV(t). \end{aligned}$$

  same $a(\cdot)$, $b(\cdot)$.
- See [Pit05a]

Interest-rate/FX hybrids

- Interest rates in two currencies + a process for FX

$$
\begin{aligned}
dP_d(t,T)/P_d(t,T) &= r_d(t)\,dt + \sigma_d(t,T)\,dW_d(t), \qquad (14)\\
dP_f(t,T)/P_f(t,T) &= r_f(t)\,dt + \sigma_f(t,T)\,dW_f(t),\\
dS(t)/S(t) &= (r_d(t) - r_f(t))\,dt + \gamma(t,S(t))\,dW_S(t),
\end{aligned}
$$

- The “standard” Gaussian framework is recovered by choosing the function $\gamma(t,x)$ that is independent of x, $\gamma(t,x) = \gamma(t)$.
- FX skew via the local volatility function $\gamma(t,x)$.
- Skew very important for FX hybrids, eg PRDC
- Use a parametric form of the local volatility function

$$
\gamma(t,x) = \nu(t)\left(\frac{x}{L(t)}\right)^{\beta(t)-1}. \qquad (15)
$$

- $\nu(t)$ is the relative volatility function, $\beta(t)$ is a time-dependent constant elasticity of variance parameter and $L(t)$ is a time-dependent scaling constant (“level”).

BARCLAYS CAPITAL

Interest-rate/FX hybrids, cont

- Market – options on forward FX, $S(T) = F(T,T)$,
$$F(t,T) = S(t)/D(t,T), \quad D(t,T) = P_d(t,T)/P_f(t,T).$$
- Under domestic T-forward measure,
$$\begin{aligned} dF(t,T)/F(t,T) &= \sigma_f(t,T)\, dW_d^T(t) - \sigma_d(t,T)\, dW_d^T(t) \\ &\quad + \gamma(t, F(t,T)\, D(t,T))\, dW_S^T(t). \end{aligned} \tag{16}$$
- Single stochastic driver
$$dF(t,T)/F(t,T) = \Lambda(t, F(t,T)\, D(t,T))\, dW_F(t), \tag{17}$$
where
$$\begin{aligned} \Lambda(t,x) &= \left(a(t) + b(t)\,\gamma(t,x) + \gamma^2(t,x)\right)^{1/2}, \\ a(t) &= \ldots, b(t) = \ldots. \end{aligned}$$
- If $\gamma(t,x)$ is a function of time t only, then the $\Lambda(t, F(t,T)\, D(t,T)) = \Lambda(t)$ is also a deterministic function of time, and $F(T,T)$ is lognormal

BARCLAYS CAPITAL

Interest-rate/FX hybrids, cont

- In general case – use MP:

$$\tilde{\Lambda}^2(t,x) = E_0^T\left(\Lambda^2(t, F(t,T)\,D(t,T)) \middle| F(t,T) = x\right).$$

- Approximate :

$$\begin{aligned}
\hat{\Lambda}(t,x) &\approx \left(a(t) + b(t)\,\hat{\gamma}(t,x) + \hat{\gamma}^2(t,x)\right)^{1/2}, \\
\hat{\gamma}(t,x) &= \nu(t)\left(x\frac{D_0(t,T)}{L(t)}\right)^{\beta(t)-1} \\
&\quad \times \left(1 + (\beta(t)-1)\,r(t)\left(\frac{x}{F(0,T)} - 1\right)\right),
\end{aligned}$$

  here $r(t)$ is a "regression" coefficient of discount bond ratio to the forward FX.

- Local volatility model with time-dependent skew, use PA. FX forward approximately shifted-lognormal. See details in [Pit06b].

Conclusions

- We have presented a generic method for calibrating models with smile, consisting of
  - Markovian projection, and
  - Parameter averaging
- The method can be applied to a wide variety of models: baskets, spreads, interest rate models, interest rate/FX models, interest rate/equity models, etc
- While the application of the method can be more, or less, successful depending on the technical difficulties encountered on each step, at least we have a plan of attack applicable to any model

Bibliography I

Leif B.G. Andersen and Jesper Andreasen.
Volatile volatilities.
Risk, 15(12):163–168, December 2002.

Marco Avellaneda, Dash Boyer-Olson, Jrme Busca, and Peter Friz.
Reconstructing volatility.
Risk, 15(10), October 2002.

Jesper Andreasen.
Back to the future.
Risk Magazine, 18(9):104–109, September 2005.

Bruno Dupire.
A unified theory of volatility.
Banque Paribas working paper, 1997.

Bibliography II

Martin Forde.
Calibrating local-stochastic volatility models.
University of Bristol working paper,
http://www.maths.bris.ac.uk/ mamsf/, February 2006.

I. Gyöngy.
Mimicking the one-dimensional distributions of processes having an Ito differential.
Prob. Th. Rel. Fields, 71:501 – 516, 1986.

Alan L. Lewis.
Option Valuation under Stochastic Volatility : with Mathematica Code.
Finance Press, 2000.

Bibliography III

Vladimir V. Piterbarg.
Stochastic volatility model with time-dependent skew.
Applied Mathematical Finance, 12(2):147–185, June 2005.

Vladimir V. Piterbarg.
Time to smile.
Risk Magazine, 18(5):71–75, May 2005.

Vladimir V. Piterbarg.
Markovian projection method for volatility calibration.
SSRN working paper, June 2006.

Vladimir V. Piterbarg.
Smiling hybrids.
Risk Magazine, 19(5):66–71, May 2006.

Bibliography IV

Vladimir V. Piterbarg.
Towards a multi-stochastic volatility model for CMS spread exotics.
Global Derivatives Trading and Risk Management Conference, Paris, May 2006.

BARCLAYS CAPITAL