Exponentiation of conditional expectations under stochastic volatility

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Abstract

We use the Itô Decomposition Formula (see [Alò12]) to express certain conditional expectations as exponentials of iterated integrals. As one application, we compute an exact formal expression for the leverage swap for any stochastic volatility model expressed in forward variance form. As another, we show how to extend the Bergomi-Guyon expansion to all orders in volatility of volatility. Finally, we compute exact expressions under rough volatility, obtaining in particular the fractional Riccati equation for the rough Heston characteristic function. As a corollary, we compute a closed-form expression for the leverage swap in the rough Heston model which can be used for fast calibration.

1 Introduction

Computing quantities of practical interest under realistic stochastic volatility models is in general hard; many asymptotic approximations are available in the literature [FGG+15]. As an example, in an influential paper [BG12], Bergomi and Guyon derive a low-noise expansion of conditional expectations of solutions of the Black-Scholes equation for any stochastic volatility model written in forward variance curve form up to second order in the volatility of volatility. This expansion is in terms of covariance functionals involving the asset and its variance which are in principle computable directly from the formulation of the model as in [BFG16].

Separately, the *exact* Malliavin Decomposition Formula and Itô Decomposition Formula [Alò06, Alò12] also express such conditional expectations explicitly in terms of covariance functionals involving the asset and its variance. However, the terms in these exact decompositions are not in general easy to compute.

In this paper, we extend both of these strands of the literature by providing an exact expansion for conditional expectations of solutions of the Black-Scholes equation which is in principle computable in any stochastic volatility model written in forward variance form. We show how this new expansion may be applied to compute an exact expression for the leverage swap in any such model. Using our new expansion, many quantities of interest may now be computed exactly without knowing the characteristic function in closed- or semi-closed form.

In Section 2, we rederive the Itô Decomposition Formula [Alò12]. Then in Section 3, we present and prove the main result of the paper, an expression for the conditional expectation of any solution of the Black-Scholes equation as an exponential of iterated integrals. As a corollary, we derive an elegant general expression for the cumulant generating function, showing how it can be applied to compute various quantities of interest such as the leverage swap. In Section 4, we note that our result yields (at least in principle) an extension of the Bergomi-Guyon expansion to all orders in volatility of volatility. In Section 5, in the context of the rough Heston model and its extensions, we compute terms in our expansion explicitly and use these to derive the fractional Riccati equation for the characteristic function. We use this to derive a closed-form expression for the leverage swap in the rough Heston model for general forward variance curves, allowing for fast calibration of the model. Finally in Section 6, we summarize and conclude.

2 The Itô Decomposition Formula

Consider the price process

$$\frac{dS_t}{S_t} = \sigma_t \, dZ_t$$

where Z is a Brownian motion defined in a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and σ is a positive and square-integrable process adapted to the filtration generated by another Brownian motion B possibly correlated with Z. Let $X_t = \log S_t$. Then

$$dX_t = \sigma_t \, dZ_t - \frac{1}{2} \sigma_t^2 \, dt. \tag{2.1}$$

Now let $H(x, w)$ be some function that solves the Black-Scholes equation where w depends on σ. That is [1],

$$-\partial_w H(x, w) + \frac{1}{2} \left(\partial_{xx} - \partial_x\right) H(x, w) = 0. \tag{2.2}$$

[1]Once w is defined, this will correspond to the gamma-vega relationship.

Now, define $w_t(T)$ as the integral of the expected future variance:

$$w_t(T) := \mathbb{E}\left[\int_t^T \sigma_s^2 ds \middle| \mathcal{F}_t\right].$$

Notice that

$$w_t(T) = M_t - \int_0^t \sigma_s^2 ds,$$

where the martingale $M_t := \mathbb{E}\left[\int_0^T \sigma_s^2 ds \middle| \mathcal{F}_t\right]$. Then it follows that

$$dw_t(T) = -\sigma_t^2\, dt + dM_t. \tag{2.3}$$

Define $H_t := H(X_t, w_t(T))$. Then, an application of Itô's Formula gives

$$\begin{aligned} dH_t &= \partial_x H_t\, dX_t + \partial_w H_t\, dw_t(T) + \frac{1}{2}\,\partial_{xx} H_t\, \sigma_t^2\, dt \\ &\quad + \partial_{xw} H_t\, d\langle X, w(T)\rangle_t + \frac{1}{2}\,\partial_{ww} H_t\, d\langle w(T), w(T)\rangle_t. \end{aligned} \tag{2.4}$$

We may then simplify (2.4) using (2.1), (2.2) and (2.3) to obtain

$$\begin{aligned} dH_t &= \partial_x H_t \left(\sigma_t\, dZ_t - \frac{1}{2}\,\sigma_t^2\, dt\right) + \partial_w H_t \left(-\sigma_t^2\, dt + dM_t\right) + \frac{1}{2}\,\partial_{xx} H_t\, \sigma_t^2\, dt \\ &\quad + \partial_{xw} H_t\, d\langle X, M\rangle_t + \frac{1}{2}\,\partial_{ww} H_t\, d\langle M, M\rangle_t \\ &= \partial_x H_t\, \sigma_t\, dZ_t + \partial_w H_t\, dM_t + \partial_{xw} H_t\, d\langle X, M\rangle_t + \frac{1}{2}\,\partial_{ww} H_t\, d\langle M, M\rangle_t. \end{aligned} \tag{2.5}$$

Assume that H is a function such that all the integrands in the following expression are processes in $L_a^2([0,T] \times \Omega)$. Integrating and taking conditional expectations of (2.5), we obtain

$$\begin{aligned} \mathbb{E}\left[H_T | \mathcal{F}_t\right] &= H_t + \mathbb{E}\left[\int_t^T \partial_{xw} H_s\, d\langle X, M\rangle_s \middle| \mathcal{F}_t\right] \\ &\quad + \frac{1}{2}\,\mathbb{E}\left[\int_t^T \partial_{ww} H_s\, d\langle M, M\rangle_s \middle| \mathcal{F}_t\right], \end{aligned} \tag{2.6}$$

Equation (2.6) is the Itô Decomposition Formula [Alò12] (abbreviated as "Decomposition Formula" in the foregoing), in marginally different notation. Note in particular that (2.6) is *exact*.

Remark 2.1. *Freezing the derivatives in* (2.6) *gives us the approximation*

$$\begin{aligned} \mathbb{E}\left[H_T | \mathcal{F}_t\right] &\approx H_t + \mathbb{E}\left[\int_t^T d\langle X, M\rangle_s \middle| \mathcal{F}_t\right] \partial_{xw} H_t \\ &\quad + \frac{1}{2}\,\mathbb{E}\left[\int_t^T d\langle M, M\rangle_s \middle| \mathcal{F}_t\right] \partial_{ww} H_t. \end{aligned}$$

In Theorem 3.3 of [Alò12] for example the error in this approximation is bounded in the context of European option pricing.

The essence of Theorem 3.1 of our paper is that we may express $\mathbb{E}\left[H_T|\mathcal{F}_t\right]$ as an exact expansion consisting of infinitely many terms, with derivatives in each such term frozen.

2.1 Diamond and dot notation

To simplify our exposition, let us introduce the following notation for various covariance functionals that arise in the rest of the paper.

Let A_t and B_t be stochastic processes (some combinations of X_t and M_t). Then

$$(A \diamond B)_t(T) = \mathbb{E}\left[\int_t^T d\langle A, B\rangle_s \middle| \mathcal{F}_t\right], \tag{2.7}$$

provided this expectation is finite. In particular, we will use repeatedly the fact that, apart from terms that do not contribute to the conditional expectation (abbreviated as 'martingale terms'),

$$d(A \diamond B)_t(T) = -d\langle A, B\rangle_t + \text{ martingale terms.} \tag{2.8}$$

Also, when $(A \diamond B)_t(T)$ appears before a solution of the Black-Scholes equation (denoted by H_t), the dot $\cdot$ is to be understood as representing the action of differential operators ∂_x and ∂_w respectively applied to H_t, each appearance of X introducing a ∂_x and each appearance of M introducing a ∂_w. So for example

$$\begin{aligned}
(X \diamond M)_t(T) \cdot H_t &= \mathbb{E}\left[\int_t^T d\langle X, M\rangle_s \middle| \mathcal{F}_t\right] \partial_{xw} H_t \\
(M \diamond M)_t\,(T) \cdot H_t &= \mathbb{E}\left[\int_t^T d\langle M, M\rangle_s \middle| \mathcal{F}_t\right] \partial_{ww} H_t \\
(X \diamond (X \diamond M))_t\,(T) \cdot H_t &= \mathbb{E}\left[\int_t^T d\langle X, (X \diamond M)\rangle_s \middle| \mathcal{F}_t\right] \partial_{xxw} H_t,
\end{aligned}$$

and so on. For ease of notation, we will often drop the explicit dependence on T or even on t. So for example, it is to be understood that

$$(X \diamond M) = \mathbb{E}\left[\int_t^T d\langle X, M\rangle_s \middle| \mathcal{F}_t\right].$$

Remark 2.2. *In terms of our diamond notation, the autocovariance functional $C^{x\xi}$ of [BG11] becomes $(X \diamond M)$, $C^{\xi\xi}$ becomes $(M \diamond M)$, and C^{μ} becomes $(X \diamond (X \diamond M))$.*

2.1.1 Autocovariance functionals as covariances

There is an intimate connection between our diamond functionals and conventional covariances as becomes clear from the following simple lemma.

Lemma 2.1. *Let A and B be martingales in the same filtered probability space. Then*

$$(A \diamond B)_t(T) = \mathbb{E}\left[A_T B_T | \mathcal{F}_t\right] - A_t\, B_t = \operatorname{cov}\left[A_T, B_T | \mathcal{F}_t\right].$$

Proof. An application of Itô's Formula gives

$$d(A\,B)_s = A_s\, dB_s + B_s\, dA_s + d\langle A, B\rangle_s.$$

Integrating and taking expectations gives

$$\begin{aligned}(A \diamond B)_t(T) &= \mathbb{E}\left[\int_t^T d\langle A, B\rangle_s \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}\left[A_T\, B_T | \mathcal{F}_t\right] - A_t\, B_t.\end{aligned}$$

□

By finding the appropriate martingales, it is thus always possible to re-express autocovariance functionals in terms of covariances of terminal quantities. For example, it is easy to show that $(M \diamond M)_t(T) = \operatorname{var}\left[\langle X\rangle_T | \mathcal{F}_t\right]$. Covariances are typically easy to compute by simulation. On the other hand, autocovariance functionals may be easily obtained from the model specification, as shown in Section 5 for the rough Heston model, for which simulation techniques are not as yet known.

2.2 The Decomposition Formula in differential form

In what follows, it will prove convenient to rewrite the Decomposition Formula (2.6), apart from terms that do not contribute to the conditional expectation (abbreviated as 'martingale terms'), in the differential form (2.5) as follows.

$$\begin{aligned}dH_t &= d\langle X, M\rangle_t\, \partial_{xw} H_t + \frac{1}{2}\, d\langle M, M\rangle_t\, \partial_{ww} H_t + \text{ martingale terms} \\ &= -d(X \diamond M)_t \cdot H_t - \frac{1}{2}\, d(M \diamond M)_t \cdot H_t + \text{ martingale terms.}\end{aligned} \tag{2.9}$$

To simplify notation, we further define

$$\mathbb{A}_t = (X \diamond M)_t + \frac{1}{2}\,(M \diamond M)_t$$

in terms of which (2.9) becomes

$$dH_t = -d\mathbb{A}_t \cdot H_t + \text{ martingale terms.} \tag{2.10}$$

2.3 Forward variance models

In what follows, we will focus on stochastic volatility models written in forward variance form. Specifically, following [BG12], let

$$\begin{aligned}\frac{dS_t}{S_t} &= \sqrt{v_t}\, dZ_t \\ d\xi_t(u) &= \lambda(t,u,\xi_t)\, dW_t.\end{aligned} \tag{2.11}$$

where $v_t = \sigma_t^2$ denotes instantaneous variance and the $\xi_t(u) = \mathbb{E}\left[v_u|\mathcal{F}_t\right], u \in [t,T]$ are forward variances.

Remark 2.3. *As noted earlier in [BG11], all conventional finite-dimensional Markovian stochastic volatility models may be cast as forward variance models.*

2.4 Some applications of the Decomposition Formula

Recall that $w_t(T)$ is given by

$$w_t(T) = \int_t^T \mathbb{E}\left[v_u|\mathcal{F}_t\right] du = \int_t^T \xi_t(u)\, du.$$

Thus $w_t(T)$ is the integral of the forward variance curve ξ_t. In particular, $w_t(T)$ represents the value of the static part of the hedge portfolio (the log-strip) for a variance swap and is thus a *tradable asset* in the terminology of [Fuk14]. For each u, $\xi_t(u)$ is a martingale in t.

Example 2.1 (Variance swap)**.** *Consider the quantity*

$$H(X_t, w_t(T)) = \log S_t - \frac{1}{2}\, w_t(T) = X_t - \frac{1}{2}\, w_t(T).$$

where as before

$$w_t(T) = \int_t^T \mathbb{E}\left[\sigma_u^2 \middle| \mathcal{F}_t\right] du = \int_t^T \xi_t(u)\, du$$

is the fair value of a variance swap. Then

$$-\partial_w H(X_t, w_t(T)) + \frac{1}{2}\,(\partial_{xx} - \partial_x)\, H(X_t, w_t(T)) = 0.$$

Also, $\partial_{xw} H(X_t, w_t(T)) = \partial_{ww} H(X_t, w_t(T)) = 0$. *Then* (2.6) *gives*

$$\mathbb{E}\left[\log S_T \middle| \mathcal{F}_t\right] = \log S_t - \frac{1}{2}\, w_t(T),$$

the well-known expression for the value of the log-strip.

Example 2.2 (Gamma swap)**.** *Consider the quantity*

$$H(X_t, w_t(T)) = S_t \, \log S_t + \frac{1}{2} \, w_t(T) \, S_t = X_t \, e^{X_t} + \frac{1}{2} \, w_t(T) \, e^{X_t}.$$

Then

$$-\partial_w H(X_t, w_t(T)) + \frac{1}{2} \, (\partial_{xx} - \partial_x) \, H(X_t, w_t(T)) = 0,$$

$\partial_{xw} H(X_t, w_t(T)) = \frac{1}{2} \, e^{X_t} = \frac{1}{2} \, S_t$*, and* $\partial_{ww} H(X_t, w_t(T)) = 0$*.* (2.6) *gives*

$$\begin{aligned} \mathbb{E}\left[S_T \, \log S_T \middle| \mathcal{F}_t \right] &= S_t \, \log S_t + \frac{1}{2} \, w_t(T) \, S_t + \frac{1}{2} \, \mathbb{E}\left[\int_t^T S_u \, d\langle X, M \rangle_u \middle| \mathcal{F}_t \right] \\ &= S_t \, \log S_t + \frac{1}{2} \, w_t(T) \, S_t + \frac{1}{2} \, \mathbb{E}\left[\int_t^T d\langle S, M \rangle_u \middle| \mathcal{F}_t \right]. \end{aligned}$$

It follows that the fair value of a gamma swap:

$$\begin{aligned} 2 \, \mathbb{E}\left[\frac{S_T}{S_t} \, \log \frac{S_T}{S_t} \middle| \mathcal{F}_t \right] &= \frac{2}{S_t} \, \mathbb{E}\left[S_T \, (\log S_T - \log S_t) \middle| \mathcal{F}_t \right] \\ &= \frac{2}{S_t} \, \mathbb{E}\left[S_T \, \log S_T \middle| \mathcal{F}_t \right] - 2 \, \log S_t \\ &= w_t(T) + \frac{1}{S_t} \, \mathbb{E}\left[\int_t^T d\langle S, M \rangle_u \middle| \mathcal{F}_t \right] \end{aligned} \tag{2.12}$$

consistent with Corollary 2.3 of [Fuk14]. (2.12) *says that the fair value of a leverage swap, defined as the difference between the gamma swap and the variance swap, is given by the quadratic covariation of the underlying and the variance swap, independent of any (diffusion) model of the underlying.*

Example 2.3 (Conditional variance of X_T)**.** *Consider*

$$H(X_t, w_t(T)) = X_t^2 + w_t(T) \, (1 - X_t) + \frac{1}{4} \, w_t(T)^2.$$

This also satisfies the Black-Scholes equation and $H(X_T, w_T(T)) = X_T^2$*. In this case,* $\partial_{xw} H(X_t, w_t(T)) = -1$ *and* $\partial_{ww} H(X_t, w_t(T)) = \frac{1}{2}$*. Plugging into* (2.6) *(wlog with* $X_t = 0$*) gives*

$$\begin{aligned} \mathbb{E}\left[X_T^2 \middle| \mathcal{F}_t \right] &= w_t(T) + \frac{1}{4} \, w_t(T)^2 + \mathbb{E}\left[\int_t^T \partial_{xw} H_s \, d\langle X, M \rangle_s \middle| \mathcal{F}_t \right] \\ &\quad + \frac{1}{2} \, \mathbb{E}\left[\int_t^T \partial_{ww} H_s \, d\langle M, M \rangle_s \middle| \mathcal{F}_t \right] \\ &= w_t(T) + \frac{1}{4} \, w_t(T)^2 - (X \diamond M)_t(T) + \frac{1}{4} \, (M \diamond M)_t(T). \end{aligned}$$

Then

$$\operatorname{var}[X_T | \mathcal{F}_t] = \mathbb{E}\left[X_T^2 \middle| \mathcal{F}_t \right] - \mathbb{E}\left[X_T | \mathcal{F}_t \right]^2 = w_t(T) - (X \diamond M)_t(T) + \frac{1}{4} \, (M \diamond M)_t(T) \tag{2.13}$$

which reminds us explicitly that in a stochastic volatility model, the variance of the terminal distribution of log-spot is not in general equal to the expected quadratic variation.

Of course, in the Black-Scholes model, $(X \diamond M)_t = (M \diamond M)_t = 0$ *and in that case, the variance of the final log-spot is equal to the expectation of quadratic variation.*

3 Exponentiation

Terms such as $X \diamond (X \diamond M)$ are naturally indexed by trees reminiscent of the trees in for example [Röß10], each of whose leaves corresponds to either X or M and whose internal vertices are the diamond operators. Though it will not be necessary for us to draw such trees explicitly, we will adopt the language of forests $\mathbb{F}_k$ of trees.

Definition 3.1. *Let* $\mathbb{F}_0 = M$. *Then the higher order forests* $\mathbb{F}_k$ *are defined recursively as follows:*

$$\mathbb{F}_k = \frac{1}{2} \sum_{i,j=0}^{k-2} \mathbb{1}_{i+j=k-2}\, \mathbb{F}_i \diamond \mathbb{F}_j + X \diamond \mathbb{F}_{k-1}. \tag{3.1}$$

Applying this definition to compute the first few terms, we obtain

$$\begin{aligned}
\mathbb{F}_0 &= M \\
\mathbb{F}_1 &= X \diamond \mathbb{F}_0 = (X \diamond M) \\
\mathbb{F}_2 &= \frac{1}{2}(\mathbb{F}_0 \diamond \mathbb{F}_0) + X \diamond \mathbb{F}_1 = \frac{1}{2}(M \diamond M) + X \diamond (X \diamond M) \\
\mathbb{F}_3 &= (\mathbb{F}_0 \diamond \mathbb{F}_1) + X \diamond \mathbb{F}_2 = M \diamond (X \diamond M) + \frac{1}{2} X \diamond (M \diamond M) + X \diamond (X \diamond (X \diamond M)) \\
\mathbb{F}_4 &= \frac{1}{2}(\mathbb{F}_1 \diamond \mathbb{F}_1) + \mathbb{F}_0 \diamond \mathbb{F}_2 + X \diamond \mathbb{F}_3 \\
&= \frac{1}{2}(X \diamond M) \diamond (X \diamond M) + \frac{1}{2} M \diamond (M \diamond M) + M \diamond (X \diamond (X \diamond M)) \\
&\qquad + X \diamond (M \diamond (X \diamond M)) + \frac{1}{2} X \diamond (X \diamond (M \diamond M)) + X \diamond (X \diamond (X \diamond (X \diamond M))).
\end{aligned} \tag{3.2}$$

Again, the differential operators ∂_x and ∂_w are to be understood whenever a forest appears with a dot $\cdot$ before a solution of the Black-Scholes equation. Thus, for example, if $\mathbb{F}_2$ precedes with a dot a solution $H_t = H(X_t, w_t(T))$ of the Black-Scholes equation,

$$\begin{aligned}
\mathbb{F}_2 \cdot H_t &= \frac{1}{2}(M \diamond M)_t(T) \cdot H_t + (X \diamond (X \diamond M))_t\,(T) \cdot H_t \\
&= \frac{1}{2}\mathbb{E}\left[\int_t^T d\langle M, M\rangle_s \middle| \mathcal{F}_t\right] \partial_{ww} H(X_t, w_t(T)) \\
&\qquad + \mathbb{E}\left[\int_t^T dX_s\, d(X \diamond M)_s(T) \middle| \mathcal{F}_t\right] \partial_{xxw} H(X_t, w_t(T)).
\end{aligned} \tag{3.3}$$

The following example will help to clarify the shorthand notation we use in the proof of our main theorem 3.1.

Example 3.1 (Itô's Lemma applied to forests and trees)**.** *First from* (3.3)*,*

$$d\left(\mathbb{F}_2 \cdot H_t\right) = \frac{1}{2} d\left((M \diamond M)_t(T) \cdot H_t\right) + d\left(\left(X \diamond (X \diamond M)\right)_t (T) \cdot H_t\right).$$

Then applying Itô's Lemma in differential form to the first tree above, we obtain

$$\begin{aligned} d\left((M \diamond M)_t(T) \cdot H_t\right) &= d\left(\mathbb{E}\left[\int_t^T d\langle M, M\rangle_s \middle| \mathcal{F}_t\right] \partial_{ww} H(X_t, w_t(T))\right) \\ &= d\left(\mathbb{E}\left[\int_t^T d\langle M, M\rangle_s \middle| \mathcal{F}_t\right]\right) \partial_{ww} H(X_t, w_t(T)) \\ &\quad + \mathbb{E}\left[\int_t^T d\langle M, M\rangle_s \middle| \mathcal{F}_t\right] d\left(\partial_{ww} H(X_t, w_t(T))\right) \\ &\quad + d\left\langle \mathbb{E}\left[\int_t^T d\langle M, M\rangle_s \middle| \mathcal{F}_t\right], \partial_{ww} H(X_t, w_t(T)) \right\rangle. \end{aligned}$$

This last expression can be conveniently recast in terms of trees as

$$\begin{aligned} d\left((M \diamond M)_t(T) \cdot H_t\right) &= d\left((M \diamond M)_t(T)\right) \partial_{ww} H(X_t, w_t(T)) + (M \diamond M)_t(T)\, d\left(\partial_{ww} H(X_t, w_t(T))\right) \\ &\quad + d\left\langle (M \diamond M)_t(T), \partial_{ww} H(X_t, w_t(T)) \right\rangle \\ &= d\left((M \diamond M)_t(T)\right) \cdot H(X_t, w_t(T)) + (M \diamond M)_t(T)\, d\left(\cdot H(X_t, w_t(T))\right) \\ &\quad + d\left\langle (M \diamond M)_t(T), \cdot H(X_t, w_t(T)) \right\rangle. \end{aligned}$$

For economy of notation, we now drop all explicit references to t and T to obtain the formal expression

$$d\left((M \diamond M) \cdot H\right) = d(M \diamond M) \cdot H + (M \diamond M)\, d \cdot H + d\langle (M \diamond M), \cdot H\rangle.$$

Remark 3.1. *Clearly, the previous example may be repeated for the second tree in $\mathbb{F}_2$ and similarly for any tree in any forest.*

The following theorem constitutes the main result of this paper. Though this theorem is in some sense purely formal, when the expansion exists, it effectively corresponds to a small-time expansion. For example, as shown in Section 5, in the rough Heston model, provided the forward variance curve ξ_t is bounded on finite intervals, each tree in the forest $\mathbb{F}_k$ exists and is of order $(T-t)^{k\left(H+\frac{1}{2}\right)+1}$.

Theorem 3.1. *Let H_t be any solution of the Black-Scholes equation such that $\mathbb{E}\left[H_T \middle| \mathcal{F}_t\right]$ is finite and the integrals contributing to each forest $\mathbb{F}_k, k \geq 0$ exist. Then*

$$\mathbb{E}\left[H_T \middle| \mathcal{F}_t\right] = e^{\sum_{k=1}^{\infty} \mathbb{F}_k} \cdot H_t. \tag{3.4}$$

Here and in the following, the exponential is to be understood as a formal power series.

Proof. Suppressing time indices for ease of notation, an application of Itô's Formula gives

$$d(e^{\sum_{k=1}^{\infty} \mathbb{F}_k} \cdot H) = d(e^{\sum_{k=1}^{\infty} \mathbb{F}_k}) \cdot H + e^{\sum_{k=1}^{\infty} \mathbb{F}_k}\, d \cdot H + d\langle e^{\sum_{k=1}^{\infty} \mathbb{F}_k}, \cdot H\rangle \tag{3.5}$$

Applying the differential formulation (2.10) of the Decomposition Formula, and using the fact that $\cdot$ and d commute,

$$e^{\sum_{k=1}^{\infty} \mathbb{F}_k}\, d \cdot H = e^{\sum_{k=1}^{\infty} \mathbb{F}_k} \cdot dH = -e^{\sum_{k=1}^{\infty} \mathbb{F}_k} \cdot d\mathbb{A} \cdot H = -e^{\sum_{k=1}^{\infty} \mathbb{F}_k}\, d\mathbb{A} \cdot H + \text{ martingale terms,}$$

where the dots in the last step combine as they represent the actions of differential operators specified by all the trees that come before. Next, for any given forest $\mathbb{F}_k$, since H is a function of X and w only, we have

$$\begin{aligned} d\mathbb{F}_k dH &= d\mathbb{F}_k \{\partial_x H\, dX + \partial_w H\, dM\} + \text{ martingale terms} \\ &= d\langle \mathbb{F}_k, X\rangle\, \partial_x H + d\langle \mathbb{F}_k, M\rangle\, \partial_w H + \text{ martingale terms.} \end{aligned}$$

Thus

$$d\langle \mathbb{F}_k, \cdot H\rangle = d\mathbb{F}_k\, d \cdot H = d\langle \mathbb{F}_k, X\rangle \cdot H + d\langle \mathbb{F}_k, M\rangle \cdot H + \text{ martingale terms}$$

where, as before, the dots combine to include the differential operators ∂_x and ∂_w induced by X and M respectively. An obvious generalization gives

$$\begin{aligned} d\langle e^{\sum_{k=1}^{\infty} \mathbb{F}_k}, \cdot H\rangle &= \sum_{k=1}^{\infty} d\langle \mathbb{F}_k, X\rangle\, e^{\sum_{k=1}^{\infty} \mathbb{F}_k} \cdot H + \sum_{k=1}^{\infty} d\langle \mathbb{F}_k, M\rangle\, e^{\sum_{k=1}^{\infty} \mathbb{F}_k} \cdot H \\ &\qquad + \text{ martingale terms.} \end{aligned}$$

Substituting into (3.5) gives

$$\begin{aligned} d(e^{\sum_{k=1}^{\infty} \mathbb{F}_k} \cdot H) &= d(e^{\sum_{k=1}^{\infty} \mathbb{F}_k}) \cdot H - e^{\sum_{k=1}^{\infty} \mathbb{F}_k}\, d\mathbb{A} \cdot H + \sum_{k=1}^{\infty} d\langle \mathbb{F}_k, X\rangle\, e^{\sum_{k=1}^{\infty} \mathbb{F}_k} \cdot H \\ &\quad + \sum_{k=1}^{\infty} d\langle \mathbb{F}_k, M\rangle\, e^{\sum_{k=1}^{\infty} \mathbb{F}_k} \cdot H + \text{ martingale terms.} \end{aligned} \tag{3.6}$$

Another application of Itô's Formula gives

$$d(e^{\sum_{k=1}^{\infty} \mathbb{F}_k}) = e^{\sum_{k=1}^{\infty} \mathbb{F}_k} \sum_{k=1}^{\infty} d\mathbb{F}_k + \frac{1}{2} e^{\sum_{k=1}^{\infty} \mathbb{F}_k} \sum_{i,j=1}^{\infty} d\langle \mathbb{F}_i, \mathbb{F}_j\rangle. \tag{3.7}$$

Substituting (3.7) into (3.6) gives

$$\begin{aligned} d(e^{\sum_{k=1}^{\infty} \mathbb{F}_k} \cdot H) &= e^{\sum_{k=1}^{\infty} \mathbb{F}_k} \sum_{k=1}^{\infty} d\mathbb{F}_k \cdot H + \frac{1}{2} e^{\sum_{k=1}^{\infty} \mathbb{F}_k} \sum_{i,j=1}^{\infty} d\langle \mathbb{F}_i, \mathbb{F}_j\rangle \cdot H - e^{\sum_{k=1}^{\infty} \mathbb{F}_k}\, d\mathbb{A} \cdot H \\ &\quad + \sum_{k=1}^{\infty} d\langle \mathbb{F}_k, X\rangle\, e^{\sum_{k=1}^{\infty} \mathbb{F}_k} \cdot H + \sum_{k=1}^{\infty} d\langle \mathbb{F}_k, M\rangle\, e^{\sum_{k=1}^{\infty} \mathbb{F}_k} \cdot H + \text{ martingale terms.} \end{aligned} \tag{3.8}$$

Now from the recursion relation, for $k \geq 2$, and using (2.8),

$$\begin{aligned} d\mathbb{F}_k &= \frac{1}{2}\sum_{i,j=0}^{k-2} \mathbb{1}_{i+j=k-2}\, d(\mathbb{F}_i \diamond \mathbb{F}_j) + d(X \diamond \mathbb{F}_{k-1}) \\ &= -\frac{1}{2}\sum_{i,j=0}^{k-2} \mathbb{1}_{i+j=k-2}\, d\langle \mathbb{F}_i, \mathbb{F}_j\rangle - d\langle X, \mathbb{F}_{k-1}\rangle + \text{ martingale terms.} \end{aligned}$$

Also

$d\mathbb{F}_1 + d\mathbb{F}_2 = d\mathbb{A} + d\,(X \diamond (X \diamond M)) = d\mathbb{A} + d\,(X \diamond \mathbb{F}_1) = d\mathbb{A} - d\langle X, \mathbb{F}_1\rangle +$ martingale term.

Summing then gives

$$\begin{aligned} \sum_{k=1}^{\infty} d\mathbb{F}_k &= d\mathbb{A} - d\langle X, \mathbb{F}_1\rangle + \sum_{k=3}^{\infty} d\mathbb{F}_k + \text{ martingale terms} \\ &= d\mathbb{A} - d\langle X, \mathbb{F}_1\rangle + \sum_{k=1}^{\infty} d\mathbb{F}_{k+2} + \text{ martingale terms} \\ &= d\mathbb{A} - d\langle X, \mathbb{F}_1\rangle + \sum_{k=1}^{\infty} \left\{ -\frac{1}{2}\sum_{i,j=0}^{k} \mathbb{1}_{i+j=k}\, d\langle \mathbb{F}_i, \mathbb{F}_j\rangle - d\langle X, \mathbb{F}_{k+1}\rangle \right\} + \text{ martingale terms} \\ &= d\mathbb{A} - \sum_{k=1}^{\infty} d\langle X, \mathbb{F}_k\rangle - \frac{1}{2}\sum_{k=1}^{\infty}\sum_{i,j=0}^{k} \mathbb{1}_{i+j=k}\, d\langle \mathbb{F}_i, \mathbb{F}_j\rangle + \text{ martingale terms} \\ &= d\mathbb{A} - \sum_{k=1}^{\infty} d\langle X, \mathbb{F}_k\rangle - \frac{1}{2}\sum_{i,j=1}^{\infty} d\langle \mathbb{F}_i, \mathbb{F}_j\rangle - \sum_{i=1}^{\infty} d\langle \mathbb{F}_i, \mathbb{F}_0\rangle + \text{ martingale terms.} \end{aligned} \tag{3.9}$$

Substituting (3.9) into (3.8) and noting that $\mathbb{F}_0 = M$ gives

$$d(e^{\sum_{k=1}^{\infty}\mathbb{F}_k} \cdot H) = 0 + \text{ martingale terms.}$$

Integrating this last expression (noting that $\mathbb{F}_k(T) = 0$ for all $k \geq 0$) gives

$$\mathbb{E}\left[H_T | \mathcal{F}_t\right] = e^{\sum_{k=1}^{\infty}\mathbb{F}_k} \cdot H_t + \mathbb{E}\left[\int_t^T d(e^{\sum_{k=1}^{\infty}\mathbb{F}_k} \cdot H)_s \middle| \mathcal{F}_t \right] = e^{\sum_{k=1}^{\infty}\mathbb{F}_k} \cdot H_t$$

as required. □

Remark 3.2. *Theorem 3.1 can be regarded as a non-Markovian extension of classical series expansions of conditional expectations around the Black-Scholes model such as that of [BG11]. Such classical expansions are in terms of the infinitesimal generator of the Itô diffusion corresponding to the stochastic volatility PDE. In contrast, Theorem 3.1 does not assume Markovianity of the stochastic volatility model as can be seen explicitly in Section 5 when we apply the theorem to the rough Heston model. In particular, we don't assume that there is any PDE in the conventional sense. Also, whereas classical expansions are typically expansions in terms of some small parameter, valid only in some asymptotic regime, Theorem 3.1 gives an exact representation of a conditional expectation in terms of iterated integrals that may in principle be computed from the formulation* (2.11) *of the model in forward variance form.*

3.1 An example: Skewness of X_T

As an example of the application of Theorem 3.1, consider the third central moment of the terminal log-asset price X_T, the skewness. Recall the following two processes previously introduced in Examples 2.1 and 2.3:

$$\begin{aligned} H_t^{(1)} &= X_t - \frac{1}{2}\, w_t(T) \\ H_t^{(2)} &= \left(X_t - \frac{1}{2}\, w_t(T) \right)^2 + w_t(T), \end{aligned}$$

and consider

$$H_t = 3\, H_t^{(1)}\, H_t^{(2)} - 2\, \left(X_t - \frac{1}{2}\, w_t(T) \right)^3 .$$

This also satisfies the Black-Scholes equation. Moreover $H_T = X_T^3$. The only nonzero partial derivatives are

$$\begin{aligned} \partial_{xw} H_t &= 3 + \frac{3}{2}\, w_t(T) - 3\, X_t \\ \partial_{ww} H_t &= -3 - \frac{3}{4}\, w_t(T) + \frac{3}{2}\, X_t \\ \partial_{xxw} H_t &= -3 \\ \partial_{xww} H_t &= \frac{3}{2} \\ \partial_{www} H_t &= -\frac{3}{4}. \end{aligned}$$

Applying Theorem 3.1, noting that $\mathbb{F}_k \cdot H_t = 0$ for $k > 4$, gives

$$\mathbb{E}\left[X_T^3 \middle| \mathcal{F}_t \right] = \exp\left\{ \mathbb{F}_1 + \mathbb{F}_2 + \mathbb{F}_3 + \mathbb{F}_4 \right\} \cdot H_t. \tag{3.10}$$

Wlog set $X_t = 0$. Applying the computations in Equation (3.2), we find

$$\begin{aligned} \mathbb{F}_1 \cdot H_t &= \left(3 + \frac{3}{2}\, w_t(T) \right)\, (X \diamond M)_t(T) \\ \mathbb{F}_2 \cdot H_t &= \frac{1}{2}\, \left(-3 - \frac{3}{4}\, w_t(T) \right)\, (M \diamond M)_t(T) - 3\, (X \diamond (X \diamond M))_t(T) \\ \mathbb{F}_3 \cdot H_t &= \frac{3}{2}\, (M \diamond (X \diamond M))_t(T) + \frac{3}{4}\, (X \diamond (M \diamond M))_t(T) \\ \mathbb{F}_4 \cdot H_t &= -\frac{3}{8}\, (M \diamond (M \diamond M))_t(T). \end{aligned}$$

Then expanding $(3.10)^2$ and noting that $H_t = -\frac{3}{2}\, w_t(T)^2 - \frac{1}{8} w_t(T)^3$ gives

$$\begin{aligned}
\mathbb{E}\left[X_T^3\middle|\,\mathcal{F}_t\right] &= \{1 + \mathbb{F}_1 + \mathbb{F}_2 + \mathbb{F}_3 + \mathbb{F}_4\} \cdot H_t \\
&= -\frac{1}{8}\, w_t(T)^3 - \frac{3}{2}\, w_t(T)^2 - \frac{3}{2}\, w_t(T)\, \left[-(X \diamond M)_t(T) + \frac{1}{4}\,(M \diamond M)_t(T)\right] \\
&\quad +3\,(X \diamond M)_t(T) - \frac{3}{2}\,(M \diamond M)_t(T) - 3\,(X \diamond (X \diamond M))_t(T) \\
&\quad +\frac{3}{2}\,(M \diamond (X \diamond M))_t(T) + \frac{3}{4}\,(X \diamond (M \diamond M))_t(T) - \frac{3}{8}\,(M \diamond (M \diamond M))_t(T).
\end{aligned}$$

Noting from (2.13) that $-(X \diamond M)_t(T) + \frac{1}{4}\,(M \diamond M)_t(T) = \mathrm{var}[X_T|\mathcal{F}_t] - w_t(T)$ and defining $\bar{X}_T = \mathbb{E}\,[X_T|\,\mathcal{F}_t] = -\frac{1}{2}\, w_t(T)$ and rearranging gives

$$\begin{aligned}
& \mathbb{E}\left[(X_T - \bar{X}_T)^3\middle|\,\mathcal{F}_t\right] \qquad (3.11)\\
=\ & \mathbb{E}\left[X_T^3 - \bar{X}_T^3\middle|\,\mathcal{F}_t\right] - 3\,\bar{X}_T\mathrm{var}[X_T|\mathcal{F}_t] \\
=\ & 3\,(X \diamond M)_t(T) - \frac{3}{2}\,(M \diamond M)_t(T) - 3\,(X \diamond (X \diamond M))_t(T) \\
& +\frac{3}{2}\,(M \diamond (X \diamond M))_t(T) + \frac{3}{4}\,(X \diamond (M \diamond M))_t(T) - \frac{3}{8}\,(M \diamond (M \diamond M))_t(T).
\end{aligned} \qquad (3.12)$$

Remark 3.3. *The exact expression* (3.12) *for skewness applies to any stochastic volatility model written in forward variance form. There are numerous references in the literature [BG11, BDLVC13, VDB15] to the connection between the skewness of the distribution of X_T and the implied volatility skew. Our explicit expression may help clarify this connection.*

3.2 If H_t is a characteristic function

Consider the Black-Scholes characteristic function

$$\Phi_t^T(a) = e^{\mathrm{i}\, a\, X_t - \frac{1}{2} a\,(a+\mathrm{i})\, w_t(T)}.$$

It is straightforward to verify that

$$\partial_w \Phi_t^T(a) = \frac{1}{2}\,(\partial_{xx} - \partial_x)\,\Phi_t^T(a).$$

Then an application of $\mathbb{F}_k$ to Φ multiplies Φ by some deterministic factor. It follows that

$$e^{\sum_{k=1}^{\infty} \mathbb{F}_k} \cdot \Phi_t^T(a) = e^{\sum_{k=1}^{\infty} \tilde{\mathbb{F}}_k(a)}\, \Phi_t^T(a)$$

where $\tilde{\mathbb{F}}_k(a)$ is $\mathbb{F}_k$ with each occurrence of ∂_x replaced with $\mathrm{i}\, a$ and each occurrence of ∂_w replaced with $-\frac{1}{2}\, a\,(a + \mathrm{i})$. Then from Theorem 3.1, we have the following lemma.

[2] Higher order terms do not contribute. For example $\mathbb{F}_1{}^2 \cdot H = ((X \diamond M)_t(T))^2\ \partial_{xxww} H = 0$.

Lemma 3.1. *Let*

$$\varphi_t^T(a) = \mathbb{E}\left[e^{\mathrm{i}\,a\,X_T}\middle|\,\mathcal{F}_t\right]$$

be the characteristic function of the log stock price. Then

$$\varphi_t^T(a) = e^{\sum_{k=1}^{\infty}\tilde{\mathbb{F}}_k(a)}\,\Phi_t^T(a). \tag{3.13}$$

Corollary 3.1. *The cumulant generating function (CGF) for the log stock price X is given by*

$$\psi_t^T(a) = \log\varphi_t^T(a) = \mathrm{i}\,a\,X_t - \frac{1}{2}a\,(a+\mathrm{i})\,w_t(T) + \sum_{k=1}^{\infty}\tilde{\mathbb{F}}_k(a). \tag{3.14}$$

3.3 Moment computations

As is well-known, the first three central moments are easily computed from cumulants by differentiation. For example, substituting explicit expressions for the $\mathbb{F}_k$ from (3.2) into the expression (3.14) for the CGF and differentiating, we obtain

$$\begin{aligned}
\mathbb{E}\left[X_T|\mathcal{F}_t\right] = (-\mathrm{i})\,\psi_t^{T\prime}(0) &= X_t - \frac{1}{2}\,w_t(T)\\
\mathbb{E}\left[(X_T-\bar{X}_T)^2\middle|\,\mathcal{F}_t\right] = (-\mathrm{i})^2\,\psi_t^{T\prime\prime}(0) &= w_t(T) - (X\diamond M)_t(T) + \frac{1}{4}\,(M\diamond M)_t(T)\\
\mathbb{E}\left[(X_T-\bar{X}_T)^3\middle|\,\mathcal{F}_t\right] = (-\mathrm{i})^3\,\psi_t^{T\prime\prime\prime}(0) &= -\frac{3}{2}\,(M\diamond M)_t(T) - \frac{3}{8}\,(M\diamond(M\diamond M))_t(T)\\
&\quad+\frac{3}{2}\,(M\diamond(X\diamond M))_t(T) + 3\,(X\diamond M)_t(T)\\
&\quad+\frac{3}{4}\,(X\diamond(M\diamond M))_t(T) - 3\,(X\diamond(X\diamond M))_t(T).
\end{aligned} \tag{3.15}$$

The first two of these moments correspond to Examples 2.1 and 2.3. The third moment agrees with the result of the more complicated computation in Section 3.1.

Remark 3.4. *The point is that we can in principle compute such moments for any stochastic volatility model written in forward variance form, whether or not there exists a closed-form expression for the characteristic function.*

Finally, we revisit the gamma swap computation of Example 2.2 using (3.14) by computing

$$\mathbb{E}\left[X_T\,e^{X_T}\middle|\,\mathcal{F}_t\right] = (-\mathrm{i})\,\psi_t^{T\prime}(-\mathrm{i}).$$

It is easy to see that only trees containing a single M leaf will survive in the sum after differentiation when $a=-\mathrm{i}$ so that

$$\sum_{k=1}^{\infty}\tilde{\mathbb{F}}_k'(-\mathrm{i}) = \frac{\mathrm{i}}{2}\sum_{k=1}^{\infty}(X\diamond)^k M$$

where $(X\diamond)^k M$ is defined recursively for $k > 0$ as $(X\diamond)^k M = X \diamond (X\diamond)^{k-1} M$. Then the fair value of a gamma swap (again assuming $X_t = 0$, wlog) is given by

$$\mathcal{G}_t(T) = 2\,\mathbb{E}\left[X_T\, e^{X_T} \middle| \mathcal{F}_t\right] = w_t(T) + \sum_{k=1}^{\infty} (X\diamond)^k M. \tag{3.16}$$

Though Equation (2.12) makes explicit that the fair value of a gamma swap is related to the covariance of the asset price and the variance process, this latter covariance is typically hard (or impossible) to compute. In contrast, equation (3.16) allows for explicit computation of the gamma swap for any model written in forward variance form. Moreover, the fair value of a leverage swap is given by

$$\mathcal{L}_t(T) = \mathcal{G}_t(T) - w_t(T) = \sum_{k=1}^{\infty} (X\diamond)^k M. \tag{3.17}$$

As an application, we will use (3.17) later in Section 5.3 to compute an explicit expression for the value of a leverage swap in the rough Heston model.

Remark 3.5. *Comparing equation* (3.17) *with equation* (2.12)*, since $S_T = S_t\, e^{X_T}$, it is natural to conjecture that*

$$\left(e^X \diamond M\right)_t(T) = \sum_{k=1}^{\infty} \left((X\diamond)^k M\right)_t(T).$$

Consequently, we have the following lemma.

Lemma 3.2. *Let $X_u = \log(S_u/S_t)$. Then*

$$(S \diamond M)_t(T) = S_t\, \left(e^X \diamond M\right)_t(T) = S_t \sum_{k=1}^{\infty} \left((X\diamond)^k M\right)_t(T).$$

Proof. Note that

$$(S \diamond M)_t(T) = \mathbb{E}\left[\int_t^T d\langle S, M\rangle_u \middle| \mathcal{F}_t\right] = \mathbb{E}\left[\int_t^T S_u\, d\langle X, M\rangle_u \middle| \mathcal{F}_t\right].$$

Now define $U_t := \mathbb{E}\left[\int_t^T d\langle X, M\rangle_u \middle| \mathcal{F}_t\right] = (X \diamond M)_t(T)$. Applying Itô's Formula to the process $S\,U$ we get

$$\begin{aligned}
&\mathbb{E}\left[\int_t^T S_u\, d\langle X, M\rangle_u \middle| \mathcal{F}_t\right] \\
&= S_t\, \mathbb{E}\left[\int_t^T d\langle X, M\rangle_u \middle| \mathcal{F}_t\right] + \mathbb{E}\left[\int_t^T S_u\, d\langle X, U\rangle_u \middle| \mathcal{F}_t\right] \\
&= S_t\, (X \diamond M)_t(T) + \mathbb{E}\left[\int_t^T S_u\, d\langle X, (X \diamond M)\rangle_u \middle| \mathcal{F}_t\right].
\end{aligned}$$

Then a recursive argument gives us that

$$\mathbb{E}\left[\int_t^T d\langle S, M\rangle_u \middle| \mathcal{F}_t\right] = S_t \sum_{k=1}^{\infty} \left((X\diamond)^k M\right)_t (T).$$

□

4 The Bergomi-Guyon expansion

To expand a model written in the forward variance form (2.11) for small volatility of volatility, we scale the volatility of volatility function $\lambda(\cdot)$ by a dimensionless parameter ϵ so that $\lambda \mapsto \epsilon\,\lambda$. Setting $\epsilon = 1$ at the end then gives the required expansion. According to equation (13) of [BG12] the conditional expectation of a solution of the Black-Scholes equation then satisfies the following equation to second order in ϵ (in our notation)

$$\begin{aligned} & \mathbb{E}\left[H_T|\mathcal{F}_t\right] \\ = & \left\{1 + \epsilon\,(X \diamond M)_t + \frac{\epsilon^2}{2}\,(M \diamond M)_t + \frac{\epsilon^2}{2}\,\left[(X \diamond M)_t\right]^2 + \epsilon^2\,(X \diamond (X \diamond M))_t\right\} \cdot H_t. \end{aligned} \tag{4.1}$$

We may immediately identify Equation (4.1) as our forest expansion (3.4) to the second order.

$$\begin{aligned} \mathbb{E}\left[H_T|\mathcal{F}_t\right] &= e^{\sum_{k=1}^{\infty} \mathbb{F}_k} \cdot H_t \\ &= \left\{1 + \epsilon\,\mathbb{F}_1 + \frac{\epsilon^2}{2}\,{\mathbb{F}_1}^2 + \epsilon^2\,\mathbb{F}_2\right\} \cdot H_t + \text{ higher order terms.} \end{aligned}$$

Note however that the two expansions are not in general identical, even to this order, because Bergomi and Guyon compute the forests $\mathbb{F}_1$ and $\mathbb{F}_2$ to lowest order in ϵ. It turns out that the two expansions are identical in the case of the rough Heston model of [EER18b] and its d-factor extensions as we will show explicitly in Sections 5.1 and 5.5. We will also show in Section 5.6 that the two expansions differ in the case of the rough Bergomi model of [BFG16] for example, where the forests have higher order contributions.

The Bergomi-Guyon expansion (Equation (14) of [BG12]) reads

$$\sigma_{\mathrm{BS}}(k,T) = \hat{\sigma}_T + \mathcal{S}_T\,k + C_T\,k^2 + O(\epsilon^3) \tag{4.2}$$

where the coefficients $\hat{\sigma}_T$, $\mathcal{S}_T$ and C_T are complicated combinations of terms up to order 2 appearing in our forest expansion. In our notation, as mentioned earlier, the $C^{x\xi}$ of Bergomi-Guyon becomes $(X \diamond M)$, $C^{\xi\xi}$ becomes $(M \diamond M)$, and C^{μ} becomes $(X \diamond (X \diamond M))$.

Separately, given an explicit characteristic function, Jacquier and Lorig show in [JL15] how to generate the smile algorithmically to any desired expansion order. One can therefore in principle extend the Bergomi-Guyon expansion to any order using our formal expression (3.14) for the CGF.

5 Explicit computations under rough volatility

As a second application of our forest expansion of Theorem 3.1, we perform explicit computations of various trees in the rough Heston model.

5.1 Tree computations in the rough Heston model

The rough Heston model of [EER18b] may be written as

$$\frac{dS_t}{S_t} = \sqrt{v_t}\,\left\{\rho\,dW_t + \sqrt{1-\rho^2}\,dW_t^{\perp}\right\} = \sqrt{v_t}\,dZ_t$$

with

$$v_u = \xi_t(u) + \frac{\nu}{\Gamma(\alpha)}\int_t^u \frac{\sqrt{v_s}}{(u-s)^\gamma}\,dW_s, \quad u \geq t \tag{5.1}$$

where $\xi_t(u) = \mathbb{E}\left[v_u | \mathcal{F}_t\right]$ is the forward variance curve, $\gamma = \frac{1}{2} - H$ and $\alpha = 1 - \gamma = H + \frac{1}{2}$. In forward variance form,

$$d\xi_t(u) = \frac{\nu}{\Gamma(\alpha)}\,\frac{\sqrt{v_t}}{(u-t)^\gamma}\,dW_t. \tag{5.2}$$

Remark 5.1. (5.2) *is a natural fractional generalization of the classical Heston model which reads, in forward variance form [BG12],*

$$d\xi_t(u) = \nu\,\sqrt{v_t}\,e^{-\kappa(u-t)}\,dW_t.$$

Apart from $\mathcal{F}_t$ measurable terms (abbreviated as 'drift'), we have

$$\begin{aligned}
dX_t &= \sqrt{v_t}\,dZ_t + \text{drift}\\
dM_t &= \int_t^T d\xi_t(u)\,du\\
&= \frac{\nu}{\Gamma(\alpha)}\,\sqrt{v_t}\,\left(\int_t^T \frac{du}{(u-t)^\gamma}\right)\,dW_t\\
&= \frac{\nu\,(T-t)^\alpha}{\Gamma(1+\alpha)}\,\sqrt{v_t}\,dW_t.
\end{aligned}$$

There is only one tree in the forest $\mathbb{F}_1$.

$$\begin{aligned}
\mathbb{F}_1 = (X \diamond M)_t(T) &= \mathbb{E}\left[\int_t^T d\langle X, M\rangle_s \middle| \mathcal{F}_t\right]\\
&= \frac{\rho\,\nu}{\Gamma(1+\alpha)}\,\mathbb{E}\left[\int_t^T v_s\,(T-s)^\alpha\,ds \middle| \mathcal{F}_t\right]\\
&= \frac{\rho\,\nu}{\Gamma(1+\alpha)}\int_t^T \xi_t(s)\,(T-s)^\alpha\,ds.
\end{aligned} \tag{5.3}$$

To compute higher order terms, it will be convenient for us to define for $j \geq 0$

$$I_t^{(j)}(T) := \int_t^T ds\, \xi_t(s)\,(T-s)^{j\,\alpha}. \tag{5.4}$$

Then[3]

$$\begin{aligned}
dI_s^{(j)}(T) &= \int_s^T du\, d\xi_s(u)\,(T-u)^{j\,\alpha} + \text{ drift terms} \\
&= \frac{\nu\,\sqrt{v_s}}{\Gamma(\alpha)}\, dW_s \int_s^T \frac{(T-u)^{j\,\alpha}}{(u-s)^{\gamma}}\, du + \text{ drift terms} \\
&= \frac{\Gamma(1+j\,\alpha)}{\Gamma(1+(j+1)\,\alpha)}\, \nu\,\sqrt{v_s}\,(T-s)^{(j+1)\,\alpha}\, dW_s + \text{ drift terms}.
\end{aligned} \tag{5.5}$$

With this notation,

$$(X \diamond M)_t(T) = \frac{\rho\,\nu}{\Gamma(1+\alpha)}\, I_t^{(1)}(T).$$

There are two trees in $\mathbb{F}_2$:

$$\begin{aligned}
(M \diamond M)_t(T) &= \mathbb{E}\left[\int_t^T d\langle M, M\rangle_s \middle| \mathcal{F}_t\right] \\
&= \frac{\nu^2}{\Gamma(1+\alpha)^2} \int_t^T \xi_t(s)\,(T-s)^{2\,\alpha}\, ds \\
&= \frac{\nu^2}{\Gamma(1+\alpha)^2}\, I_t^{(2)}(T)
\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}
(X \diamond (X \diamond M))_t\,(T) &= \frac{\rho\,\nu}{\Gamma(1+\alpha)}\, \mathbb{E}\left[\int_t^T d\langle X, I^{(1)}\rangle_s \middle| \mathcal{F}_t\right] \\
&= \frac{\rho^2\,\nu^2}{\Gamma(1+2\,\alpha)}\, I_t^{(2)}(T).
\end{aligned} \tag{5.7}$$

One can be easily convinced that each tree in the level-k forest $\mathbb{F}_k$ is $I^{(k)}$ multiplied by a simple prefactor. For example, continuing to the forest $\mathbb{F}_3$, we have the following.

$$\begin{aligned}
(M \diamond (X \diamond M))_t\,(T) &= \frac{\rho\,\nu^3}{\Gamma(1+\alpha)\,\Gamma(1+2\,\alpha)}\, I_t^{(3)}(T) \\
(X \diamond (X \diamond (X \diamond M)))_t\,(T) &= \frac{\rho^3\,\nu^3}{\Gamma(1+3\,\alpha)}\, I_t^{(3)}(T) \\
(X \diamond (M \diamond M))_t\,(T) &= \frac{\rho\,\nu^3\,\Gamma(1+2\,\alpha)}{\Gamma(1+\alpha)^2\,\Gamma(1+3\,\alpha)}\, I_t^{(3)}(T).
\end{aligned}$$

[3] Note that ‘drift terms‘ do not contribute to covariation terms.

5.2 The CGF of the rough Heston model from the forest expansion

The rough Heston characteristic function relevant to our formulation (5.1) was originally derived in semi-closed form in [EER18b] as a limit of characteristic functions of nearly-unstable Hawkes processes. This characteristic function can be obtained as the solution of a fractional Riccati equation. We now show how to derive this fractional Riccati equation for the rough Heston characteristic function from our forest expansion.

Theorem 5.1. *The CGF of the rough Heston model can be written in the form*

$$\psi_t^T(a) = \int_t^T D^\alpha h(a, T-u)\, \xi_t(u)\, du \tag{5.8}$$

where h solves the fractional Riccati equation

$$D^\alpha h(a,t) = -\frac{1}{2}\, a\,(a+\mathrm{i}) + \mathrm{i}\,\rho\,\nu\, a\, h(a,t) + \frac{1}{2}\,\nu^2\, h^2(a,t). \tag{5.9}$$

Proof. From the computations of Section 5.1, recalling the definition of the $\tilde{\mathbb{F}}$ in Section 3.2, we may write

$$\tilde{\mathbb{F}}_k(a) = \beta_k(a)\, \nu^k\, I_t^{(k)}(T) \tag{5.10}$$

for some coefficients $\beta_k(a)$ and

$$I_t^{(k)}(T) := \int_t^T ds\, \xi_t(s)\, (T-s)^{k\,\alpha}.$$

Applying the recursion formula (3.1) to the characteristic function, we obtain

$$\tilde{\mathbb{F}}_k(a) = \frac{1}{2} \sum_{i,j=0}^{k-2} \mathbb{1}_{i+j=k-2}\, \tilde{\mathbb{F}}_i(a) \diamond \tilde{\mathbb{F}}_j(a) + \tilde{X}(a) \diamond \tilde{F}_{k-1}(a), \tag{5.11}$$

with $\tilde{\mathbb{F}}_0(a) = -\frac{1}{2}\, a(a+\mathrm{i})\, I_t^0(T)$ and $\tilde{X}(a) = \mathrm{i}\, a\, X_t$. Plugging expression (5.10) into the recursion formula (5.11) yields a relationship between the coefficient functions $\beta_j(a)$ and the $\beta_i(a)$ for $i < j$.

We first compute

$$\begin{aligned}
\tilde{\mathbb{F}}_i(a) \diamond \tilde{\mathbb{F}}_j(a) &= \beta_i(a)\, \nu^i\, \beta_j(a)\, \nu^j\, \mathbb{E}\left[\int_t^T dI_s^{(i)}(T)\, dI_s^{(j)}(T) \middle| \mathcal{F}_t \right] \\
&= \nu^{i+j+2}\, \beta_i(a) \beta_j(a)\, \frac{\Gamma(1+i\,\alpha)}{\Gamma(1+(i+1)\,\alpha)}\, \frac{\Gamma(1+j\alpha)}{\Gamma(1+(j+1)\alpha)} \int_t^T ds\, \xi_t(s)\, (T-s)^{(i+j+2)\,\alpha} \\
&= \nu^{i+j+2}\, \beta_i(a) \beta_j(a)\, \frac{\Gamma(1+i\,\alpha)}{\Gamma(1+(i+1)\,\alpha)}\, \frac{\Gamma(1+j\alpha)}{\Gamma(1+(j+1)\alpha)}\, I_t^{(i+j+2)}(T).
\end{aligned}$$

Next we compute

$$\begin{aligned}
\tilde{X}(a) \diamond \tilde{\mathbb{F}}_{k-1}(a) &= \mathrm{i}\, a\, \beta_{k-1}(a)\, \nu^{k-1}\, \mathbb{E}\left[\int_t^T dX_s\, dI_s^{(k-1)}(T) \middle| \mathcal{F}_t \right] \\
&= \mathrm{i}\, \rho\, \nu^k\, a\, \frac{\Gamma(1+(k-1)\,\alpha)}{\Gamma(1+k\,\alpha)} \beta_{k-1}(a).
\end{aligned}$$

Substituting into (5.11) then gives the following relationship between the coefficient $\beta_k(a)$:

$$\begin{aligned}
\beta_0(a) &= -\frac{1}{2}\, a(a+\mathrm{i}) \\
\beta_k(a) &= \frac{1}{2}\sum_{i,j=0}^{k-2} \mathbb{1}_{i+j=k-2}\,\beta_i(a)\,\beta_j(a)\frac{\Gamma(1+i\,\alpha)}{\Gamma(1+(i+1)\,\alpha)}\,\frac{\Gamma(1+j\,\alpha)}{\Gamma(1+(j+1)\,\alpha)} \\
&\qquad +\mathrm{i}\,\rho\,a\,\frac{\Gamma(1+(k-1)\,\alpha)}{\Gamma(1+k\,\alpha)}\beta_{k-1}(a). \qquad (5.12)
\end{aligned}$$

We now define[4]

$$h(a,t) = \sum_{j=0}^{\infty} \frac{\Gamma(1+j\,\alpha)}{\Gamma(1+(j+1)\,\alpha)}\beta_j(a)\,\nu^j\,t^{(j+1)\,\alpha}. \qquad (5.13)$$

Then

$$D^\alpha h(a,t) = \sum_{j=0}^{\infty} \beta_j(a)\,\nu^j\,t^{j\,\alpha}, \qquad (5.14)$$

where D^α denote the fractional derivative of order α (see for example [EER18a] for a definition). Then from (3.14) and (5.10),

$$\psi_t^T(a) = \int_t^T D^\alpha h(a, T-u)\,\xi_t(u)\,du.$$

Now it remains for us to show that h thus defined satisfies the fractional Riccati equation (5.9). Notice that the recursion formulae (5.12) imply

$$\left(\sum_{j=0}^{\infty} \frac{\Gamma(1+j\,\alpha)}{\Gamma(1+(j+1)\,\alpha)}\beta_j(a)\,\nu^{j+1}\,t^{(j+1)\,\alpha} + \mathrm{i}\,\rho\,a\right)^2 = -a^2\,\rho^2 + 2\sum_{j=1}^{\infty} \beta_j(a)\,\nu^j\,t^{j\,\alpha}. \qquad (5.15)$$

Then (5.15) may be rewritten as

$$(\nu\,h(a,t) + \mathrm{i}\,\rho\,a)^2 = -a^2\,\rho^2 + 2\,D^\alpha h(a,t) + a\,(a+\mathrm{i}).$$

and (5.9) follows.

□

Remark 5.2. *The rough Heston characteristic function and its associated fractional Riccati equation are thus obtainable directly from the forward variance formulation* (5.2) *of the rough Heston model using* (3.14)*, with no need for recourse to complicated limiting arguments. For another derivation see [GKR18].*

[4]This short time expansion is used to construct an accurate approximation to the rough Heston CGF in [GR18].

5.3 Computation of the leverage swap in the rough Heston model

We now apply Theorem 5.1 to the computation of the leverage swap in the rough Heston model.

Corollary 5.1. *The fair value of the leverage swap in the rough Heston model is given by*

$$\mathcal{L}_t(T) = -2\,\mathrm{i}\,\rho\,\nu \int_t^T h'(-\mathrm{i}, T-u)\,\xi_t(u)\,du$$

where $h'(a,t) = \partial_a h(a,t)$.

Proof. For ease of notation, we abbreviate $\psi_t^T(a)$ by $\psi(a)$. Wlog let $X_t = 0$. We have that

$$\mathcal{L}_t(T) = \mathcal{G}_t(T) - w_t(T) = -2\,\mathrm{i}\,\psi'(-\mathrm{i}) - 2\,\mathrm{i}\,\psi'(0). \tag{5.16}$$

Moreover,

$$\psi(a) = \int_t^T D^\alpha h(a, T-u)\,\xi_t(u)\,du$$

so

$$\psi'(a) = \int_t^T D^\alpha h'(a, T-u)\,\xi_t(u)\,du.$$

Note that for all $j \geq 0, \beta_j(0) = \beta_j(-\mathrm{i}) = 0$. Then, from the fractional Riccati equation (5.9), we have

$$\begin{aligned} D^\alpha h'(0, T-u) &= -\frac{\mathrm{i}}{2} \\ D^\alpha h'(-\mathrm{i}, T-u) &= \frac{\mathrm{i}}{2} + \rho\,\nu\,h'(-\mathrm{i}, T-u). \end{aligned}$$

Substituting into (5.16) gives

$$\mathcal{L}_t(T) = -2\,\mathrm{i}\,\rho\,\nu \int_t^T h'(-\mathrm{i}, T-u)\,\xi_t(u)\,du. \tag{5.17}$$

□

Lemma 5.1.

$$h'(-\mathrm{i}, t) = \frac{\mathrm{i}}{2}\,\frac{1}{\rho\,\nu} \sum_{j=1}^{\infty} \frac{(\rho\,\nu\,t^\alpha)^j}{\Gamma(1 + j\,\alpha)}.$$

Proof. Differentiating (5.13) gives

$$h'(a,t) = \sum_{j=0}^{\infty} \frac{\Gamma(1 + j\,\alpha)}{\Gamma(1 + (j+1)\,\alpha)} \beta_j'(a)\,\nu^j\,t^{(j+1)\,\alpha}. \tag{5.18}$$

Then from the recursion formula (5.12), we have

$$\begin{aligned} \beta_0'(-\mathrm{i}) &= \frac{\mathrm{i}}{2} \\ \beta_k'(-\mathrm{i}) &= \rho\,\frac{\Gamma(1+(k-1)\,\alpha)}{\Gamma(1+k\,\alpha)}\beta_{k-1}'(-\mathrm{i}), \quad k>0. \end{aligned}$$

Then

$$\beta_j'(-\mathrm{i}) = \frac{\mathrm{i}}{2}\,\frac{1}{\Gamma(1+j\,\alpha)}\,\rho^j.$$

Substituting back into (5.18) gives

$$\begin{aligned} h'(-\mathrm{i},t) &= \frac{\mathrm{i}}{2}\sum_{j=0}^{\infty}\frac{(\rho\,\nu)^j\,t^{(j+1)\,\alpha}}{\Gamma(1+(j+1)\,\alpha)} \\ &= \frac{\mathrm{i}}{2}\,\frac{1}{\rho\,\nu}\sum_{j=1}^{\infty}\frac{(\rho\,\nu\,t^{\alpha})^j}{\Gamma(1+j\,\alpha)}. \end{aligned}$$

□

Putting Corollary 5.1 and Lemma 5.1 together, we obtain

Corollary 5.2.

$$\mathcal{L}_t(T) = \int_t^T f(T-u)\,\xi_t(u)\,du \tag{5.19}$$

with

$$f(\tau) = \sum_{j=1}^{\infty}\frac{(\rho\,\nu\,\tau^{\alpha})^j}{\Gamma(1+j\,\alpha)} = E_\alpha(\rho\,\nu\,\tau^\alpha) - 1$$

where $E_\alpha(\cdot)$ is the Mittag-Leffler function.

Rather than using the CGF, we could have computed $\mathcal{L}_t(T)$ using our general result (3.17). From the computations of Section 5.1,

$$(X\diamond)^k M_t(T) = \frac{(\rho\,\nu)^k}{\Gamma(1+k\,\alpha)}\,I_t^{(k)}(T)$$

where $I_t^{(k)}(T) = \int_t^T du\,\xi_t(u)\,(T-u)^{k\,\alpha}$. Then from (3.17), we have

$$\begin{aligned} \mathcal{L}_t(T) &= \sum_{k=1}^{\infty}(X\diamond)^k M_t(T) \\ &= \sum_{k=1}^{\infty}\frac{(\rho\,\nu)^k}{\Gamma(1+k\,\alpha)}\int_t^T du\,\xi_t(u)\,(T-u)^{k\,\alpha} \\ &= \int_t^T du\,\xi_t(u)\,\{E_\alpha(\rho\,\nu\,(T-u)^\alpha) - 1\} \end{aligned} \tag{5.20}$$

consistent with (5.19).

Remark 5.3. *The classical Heston model with no mean reversion corresponds to the special case of the rough Heston model where* $\alpha = H + \frac{1}{2} = 1$ *and* $v_t = \theta$. *In this case,* $E_\alpha(x) = e^x$ *and integrating* (5.20) *explicitly gives*

$$\mathcal{L}_t(T) = \theta\,\tau \left\{ \frac{e^{\rho\,\nu\,\tau} - 1}{\rho\,\nu\,\tau} - 1 \right\} \tag{5.21}$$

where $\tau = T - t$, *which is exactly consistent with the classical result as computed for example in [Fuk14].*

5.4 A numerical example

We now perform a numerical computation of the value of the leverage swap using the forest expansion in the rough Heston model with the following parameters, calibrated to the SPX options market as of April 24, 2017:

$$H = 0.0236; \quad \nu = 0.3266; \quad \rho = -0.6510.$$

Given the forms (5.19) and (5.20), it is natural to normalize the leverage swap by the variance swap. We therefore define

$$L_t(T) = \frac{\mathcal{L}_t(T)}{w_t(T)}. \tag{5.22}$$

In the special case of the rough Heston model with a flat forward variance curve,

$$L_t(T) = E_{\alpha,2}(\rho\,\nu\,\tau^\alpha) - 1,$$

where $E_{\alpha,2}(\cdot)$ is a generalized Mittag-Leffler function, independent of the reversion level θ. We further define an nth order approximation to $L_t(T)$ as

$$L_t^{(n)}(T) = \sum_{k=1}^{n} \frac{(\rho\,\nu\,\tau^\alpha)^k}{\Gamma(2 + k\,\alpha)}.$$

In Figure 5.4, we plot the normalized leverage swap $L_t(T)$ and successive approximations $L_t^{(n)}(T)$ to it as a function of τ. We note that three terms are enough to get a very good approximation to the normalized leverage swap for all expirations traded in the listed market. Moreover, leverage swaps are straightforward to estimate from volatility smiles.

Remark 5.4. *In practice,* (5.22) *can be used for very fast and efficient calibration of the three parameters of the rough Heston model by minimizing the distance between model and empirical normalized leverage swap estimates* [5].

[5] See [Fuk12] for a particularly nice and robust way to estimate leverage swaps from data.

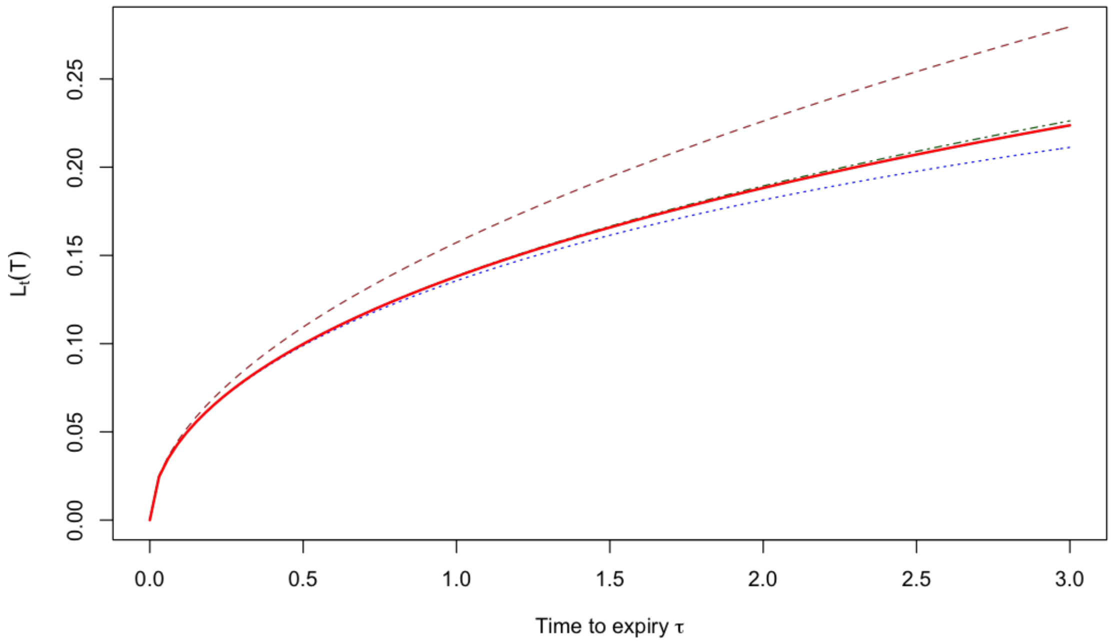

Figure 5.1: Successive approximations to the (absolute value of) the normalized rough Heston leverage swap. The solid red line is the exact expression $L_t(T)$; the successive approximations $L_t^{(1)}(T)$, $L_t^{(2)}(T)$, and $L_t^{(3)}(T)$ are brown dashed, blue dotted and dark green dash-dotted lines respectively.

5.5 A rough d-factor Heston model

Consider the Double Heston model of [CHJ09]. In forward variance form, this model can be written as

$$
\begin{aligned}
\frac{dS_t}{S_t} &= \sqrt{v_t^{(1)}}\, dZ_t^{(1)} + \sqrt{v_t^{(2)}}\, dZ_t^{(2)} \\
d\xi_t^{(1)}(u) &= e^{-\kappa_1 (u-t)}\, \nu_1 \sqrt{v_t^{(1)}}\, dW_t^{(1)} \\
d\xi_t^{(2)}(u) &= e^{-\kappa_2 (u-t)}\, \nu_2 \sqrt{v_t^{(2)}}\, dW_t^{(2)}
\end{aligned}
\tag{5.23}
$$

with $d\langle W^{(i)}, Z^{(i)}\rangle_t = \rho_i\, dt$, $i = 1, 2$ and all other correlations zero. One possible rough generalization of this model is

$$
\begin{aligned}
\frac{dS_t}{S_t} &= \sum_{i=1}^{d} \sqrt{v_t^{(i)}}\, dZ_t^{(i)} \\
d\xi_t^{(i)}(u) &= \frac{\nu_i}{\Gamma(\alpha_i)} \sqrt{v_t^{(i)}}\, \frac{dW_t^{(i)}}{(u-t)^{\gamma_i}}
\end{aligned}
\tag{5.24}
$$

where $\alpha_i = 1 - \gamma_i$, $i = 1, ..., d$ and the correlation structure is the obvious generalization of that of the classical Double Heston model.

We may compute the various forests $\mathbb{F}_k$ as before. For example,

$$(X \diamond M) = X \diamond \left(\sum_{i=1}^{d} M^{(i)} \right) = \sum_{i=1}^{d} \frac{\rho_i \, \nu_i}{\Gamma(1+\alpha_i)} \int_t^T \xi_t^{(i)}(u) \, (T-u)^{\alpha_i} \, du.$$

It is easy to see that this decomposition into d terms applies to all possible trees. Thus the CGF for this model is exactly equal to the sum of the CGFs for the corresponding one factor rough Heston model. Indeed we can immediately generalize Equation (3.17) to obtain for the leverage swap

$$\mathcal{L}_t(T) = \mathcal{G}_t(T) - w_t(T) = \sum_{k=1}^{\infty} (X\diamond)^k M = \sum_{k=1}^{\infty} (X\diamond)^k \left(\sum_{i=1}^{d} M^{(i)} \right). \tag{5.25}$$

We may then deduce from (5.20) the fair value of the leverage swap in the d-factor rough Heston model as

$$\mathcal{L}_t(T) = \sum_{i=1}^{d} \int_t^T \left[E_{\alpha_i}(\rho_i \, \nu_i \, \tau^{\alpha_i}) - 1 \right] \, \xi_t^{(i)}(u) \, du. \tag{5.26}$$

It is straightforward to check consistency with the classical result, by setting $d = 2$ and differentiating the classical Double Heston CGF from Equation (7) of [CHJ09] in the special case of no mean reversion.

Once again for emphasis, we were able to obtain the exact expression (5.26) for the leverage swap in the d-factor rough Heston model without knowing the characteristic function for this model.

Remark 5.5. *Theorem 5.1 has the following obvious extension to the d-factor rough Heston model:*

$$\psi_t^T(a) = \sum_{i=1}^{d} \int_t^T D^{\alpha} h^{(i)}(a, T-u) \, \xi_t^{(i)}(u) \, du$$

where the h^i satisfy the fractional Riccati equations

$$D^{\alpha} h^{(i)}(a,t) = -\frac{1}{2} \, a\,(a + \mathrm{i}) + \mathrm{i} \, \rho_i \, \nu_i \, a \, h^{(i)}(a,t) + \frac{1}{2} \, \nu_i^2 \, h^{(i)}(a,t)^2.$$

5.6 The Rough Bergomi model

As promised earlier, we will now show by explicit example that our forest expansion and the Bergomi-Guyon expansion are not in general identical. Recall from [BFG16] that in the Rough Bergomi (rBergomi) model,

$$\begin{aligned} \frac{dS_t}{S_t} &= \sqrt{\xi_t(t)} \, dZ_t \\ \frac{d\xi_t(u)}{\xi_t(u)} &= \tilde{\eta} \, \frac{dW_t}{(u-t)^{\gamma}} \end{aligned}$$

where $\gamma = \frac{1}{2} - H$ and $\tilde{\eta} = \eta\,\sqrt{2\,H}$. Thus

$$
\begin{aligned}
dX_t &= \sqrt{\xi_t(t)}\,dZ_t + \text{drift} \\
dM_t &= \int_t^T d\xi_t(u)\,du = \tilde{\eta}\,\left(\int_t^T \frac{\xi_t(u)}{(u-t)^\gamma}\,du\right)\,dW_t.
\end{aligned}
$$

Explicit computation of $\mathbb{F}_1$ gives

$$
\begin{aligned}
(X \diamond M)_t &= \mathbb{E}\left[\int_t^T d\langle X, M\rangle_s \middle| \mathcal{F}_t\right] \\
&= \rho\,\tilde{\eta}\int_t^T ds\int_s^T \mathbb{E}\left[\sqrt{\xi_s(s)}\,\xi_s(u)\middle|\mathcal{F}_t\right]\,\frac{du}{(u-s)^\gamma}.
\end{aligned}
$$

Bergomi and Guyon further approximate

$$
\mathbb{E}\left[\sqrt{\xi_s(s)}\,\xi_s(u)\middle|\mathcal{F}_t\right] = \sqrt{\xi_t(s)}\,\xi_t(u) + O(\epsilon^2).
$$

This stands in contrast to the rough Heston model where

$$
(X \diamond M)_t = \frac{\rho\,\nu}{\Gamma(1+\alpha)}\int_t^T \xi_t(s)\,(T-s)^\alpha\,ds
$$

exactly with no higher order contributions. Thus our forest expansion and the Bergomi-Guyon expansion are not in general identical, though they are in the special case of the rough Heston model.

6 Summary and conclusions

Starting from a generic formulation in forward variance form of a stochastic volatility model, we applied the Itô Decomposition Formula to derive an exact expression for the conditional expectation of any solution of the Black-Scholes equation as an exponential of iterated integrals. This result extends both the Bergomi-Guyon expansion and the exact decomposition formulae of [Alò06, Alò12]. We applied our new expansion to the computation of various quantities of interest, including the leverage swap. In the context of the rough Heston model and its extensions, we computed terms in our expansion explicitly and used these to derive the fractional Riccati equation for the characteristic function. We used this to derive a closed-form expression for the leverage swap in the rough Heston model for general forward variance curves, allowing for fast calibration of the model.

Exact computations of trees in our forest expansion in other models such as the rough Bergomi model of [BFG16] are left for further research.

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