Stochastic Volatility Modelling: A Practitioner's Approach

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Outline

- Motivation
- Traditional models – the Heston model as an example
- Practitioner's approach – an example
- Conclusion

Papers *Smile Dynamics I, II, III, IV* are available on SSRN website

Motivation

- Why don't we just delta-hedge options ?
- Daily P&L of delta-hedged short option position is:

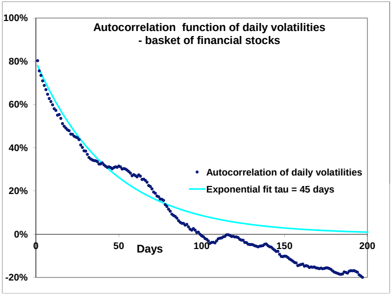
$$P\&L = -\frac{1}{2} S^2 \frac{d^2 P}{dS^2} \left[\frac{\delta S^2}{S^2} - \hat{\sigma}^2 \delta t \right]$$

- Write daily return as: $\frac{\delta S_i}{S_i} = \sigma_i Z_i \sqrt{\delta t}$. Total P&L reads:

$$P\&L = -\frac{1}{2} \sum S_i^2 \left. \frac{d^2 P}{dS^2} \right|_i \left(\sigma_i^2 Z_i^2 - \hat{\sigma}^2 \right) \delta t$$

- Variance of daily P&L has two sources:
  - the Z_i have thick tails
  - the σ_i are correlated and volatile
  - Delta-hedging not sufficient in practice

▷ **Options are hedged with options !**



- Implied volatilities of market-traded options (vanilla, ...) appear in pricing function $P(t, S, \hat{\sigma}, p, \dots)$.

▷ Other sources of P& L:

$$P\&L = -\frac{1}{2}S^2\frac{d^2P}{dS^2}\left[\frac{\delta S^2}{S^2} - \hat{\sigma}^2\delta t\right] - \frac{dP}{d\hat{\sigma}}\delta\hat{\sigma} - \left[\frac{1}{2}\frac{d^2P}{d\hat{\sigma}^2}\delta\hat{\sigma}^2 + \frac{d^2P}{dSd\hat{\sigma}}\delta S\delta\hat{\sigma}\right] + \cdots$$

  - Dynamics of implied parameters generates P&L as well
  - Vanilla options should be considered as hedging instruments in their own right

▷ Using options as hedging instruments:

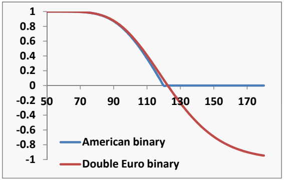
  - lowers exposure to dynamics of *realized* parameters, e.g. volatility
  - generates exposure to dynamics of *implied* parameters

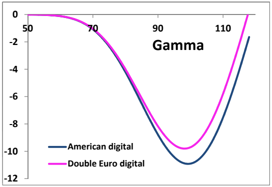
Example 1: barrier option

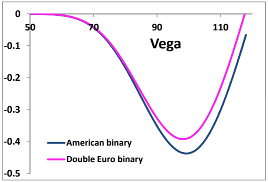
In the Black-Scholes model, a barrier option with payoff f can be statically replicated by a European option with payoff g given by:

$$\text{Barrier: } \begin{cases} f(S) & \text{if } S < L \\ 0 & \text{if } S > L \end{cases} \quad \text{European payoff: } \begin{cases} f(S) & \text{if } S < L \\ -\left(\frac{L}{S}\right)^{\frac{2r}{\sigma^2}-1} f\left(\frac{L^2}{S}\right) & \text{if } S > L \end{cases}$$

In our example $f(S) = 1$ and $L = 120$. European payoff is approximately double European Digital.







- Gamma / Vega well hedged by double Euro digital – are there any residual risks ?

- When S hits 120, unwind double Euro digital. Value of Euro digital depends on implied skew at barrier.
- Value of double Euro digital:

$$D = \frac{\mathrm{Put}_{L+\epsilon} - \mathrm{Put}_{L-\epsilon}}{2\epsilon} = \left.\frac{d\mathrm{Put}_K}{dK}\right|_L$$

$$\frac{d\mathrm{Put}_K}{dK} = \frac{d\mathrm{Put}_K^{BS}(K, \hat{\sigma}_K)}{dK} = \frac{d\mathrm{Put}_K^{BS}}{dK} + \frac{d\mathrm{Put}_K^{BS}}{d\hat{\sigma}}\frac{d\hat{\sigma}_K}{dK}$$

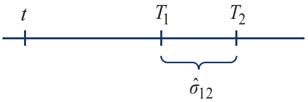
$$D = \underbrace{D^{BS}(\hat{\sigma}_L)}_{\simeq \text{ no sensitivity}} + \left.\frac{d\mathrm{Put}_L^{BS}}{d\hat{\sigma}}\frac{d\hat{\sigma}_K}{dK}\right|_L$$

▷ **Barrier option price depends on scenarios of implied skew at barrier !**

Example 2 : cliquet

- A cliquet involves ratios of future spot prices – ATM forward option pays:

$$\left(\frac{S_{T_2}}{S_{T_1}} - k\right)^+$$



- In Black-Scholes model, price is given by: $P_{BS}(\hat{\sigma}_{12}, r, \ldots)$
  - S does not appear in pricing function ??
  - Cliquet is in fact an option on forward volatility. For ATM cliquet ($k = 100\%$):

$$P_{BS} \simeq \frac{1}{\sqrt{2\pi}}\hat{\sigma}_{12}\sqrt{T_2 - T_1}$$

▷ **Price of cliquet depends on dynamics of forward implied volatilities**

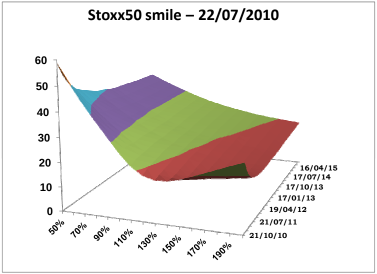
Modelling the full volatility surface

- Natural approach: write dynamics for prices of vanilla options as well:

$$\begin{cases} dS = (r-q)Sdt + \sigma S dW_t^S \\ dC^{KT} = rC^{KT}dt + \bullet\, dW_t^{KT} \end{cases}$$

Better: write dynamics on implied vols directly (P. Schönbucher)

$$\begin{cases} dS = (r-q)Sdt + \sigma S dW_t^S \\ d\hat{\sigma}^{KT} = \star dt + \bullet\, dW_t^{KT} \end{cases}$$



- drift of $\hat{\sigma}^{KT}$ imposed by condition that C^{KT} be a (discounted) martingale
  - How do we ensure no-arb among options of different K/T ??
- Other approach: model dynamics of local (implied) volatilities (R. Carmona & S. Nadtochiy, M. Schweizer & J. Wissel)
  - drift of local (implied) vols is non-local & hard to compute

▷ **So far inconclusive – try with simpler objects: Var Swap volatilities**

Forward variances

- Variance Swaps are liquid on indices – pay at maturity

$$\frac{1}{T-t}\sum_{t}^{T}\ln\left(\frac{S_{i+1}}{S_i}\right)^2 - \hat{\sigma}_t^{T\,2}$$

  - $\hat{\sigma}_t^T$: Var Swap implied vol for maturity T, observed at t
  - If S_t diffusive $\hat{\sigma}_t^T$ also implied vol of European payoff $-2\ln\left(\frac{S_T}{S_t}\right)$

- Long $T_2 - t$ VS of maturity T_2, short $T_1 - t$ VS of maturity T_1. Payoff at T_2:

$$\sum_{T_1}^{T_2}\ln\left(\frac{S_{i+1}}{S_i}\right)^2 - \left((T_2-t)\ \hat{\sigma}_t^{T_2\,2} - (T_1-t)\ \hat{\sigma}_t^{T_1\,2}\right) = \sum_{T_1}^{T_2}\ln\left(\frac{S_{i+1}}{S_i}\right)^2 - (T_2-T_1)V_t^{T_1T_2}$$

where *discrete* forward variance $V_t^{T_1T_2}$ is defined as:

$$V_t^{T_1T_2} = \frac{(T_2-t)\ \hat{\sigma}_t^{T_2\,2} - (T_1-t)\ \hat{\sigma}_t^{T_1\,2}}{T_2-T_1}$$

- Enter position at t, unwind at $t+\delta t$. P&L at T_2 is:

$$P\&L = (T_2-T_1)\left(V_{t+\delta t}^{T_1T_2} - V_t^{T_1T_2}\right)$$

No δt term in P&L: $\triangleright$ $V^{T_1T_2}$ **has no drift.**

- Replace finite difference by derivative: introduce *continuous* forward variances ζ_t^T :

$$\zeta_t^T = \frac{d}{dT}\left((T-t)\,\hat{\sigma}_t^{T\,2}\right)$$

ζ^T is driftless:

$$d\zeta_t^T \; = \; \bullet \; dW_t^T$$

- ζ^T easier to model than $\hat{\sigma}^{KT}$??
  - The ζ^T are driftless
  - Only no-arb condition: $\zeta^T > 0$

▷ **Model dynamics of foward variances**

Full model

- Instantaneous variance is $\zeta_t^{T=t}$. Simplest diffusive dynamics for S_t is:

$$dS_t = (r-q)S_t dt + \sqrt{\zeta_t^t} S_t dZ_t^S$$

- Pricing equation is:

$$\frac{dP}{dt} + (r-q)S\frac{dP}{dS} + \frac{\zeta^t}{2}S^2\frac{d^2P}{dS^2}$$
$$+\frac{1}{2}\int_t^T\int_t^T \frac{\langle d\zeta_t^u d\zeta_t^v\rangle}{dt}\frac{d^2P}{\delta\zeta^u\delta\zeta^v}dudv + \int_t^T \frac{\langle dS_t d\zeta_t^u\rangle}{dt}\frac{d^2P}{dSd\zeta^u}du \quad = \quad rP$$

- Dynamics of S / ζ^T generates joint dynamics of S and $\hat{\sigma}^{KT}$
  - ▷ Even though VSs may not be liquid, we can use forward variances to drive the dynamics of the full volatility surface.

- Can we come up with non-trivial low-dimensional examples of stochastic volatility models ?
- How do we specify a model – what do require from model ?

Historical motivations

Traditionally other motivations put forward – not always relevant from practitioner's point of view – for example:

- *Stoch. vol. needed because realized volatility is stochastic, exhibits clustering, etc.*

▷ We don't care about dynamics of realized vol – we're hedged. What we need to model is the dynamics of implied vols.

- *Stoch. vol. needed fo fit vanilla smile*

▷ Not always necessary to fit vanilla smile – usually mismatch can be charged as hedging cost

▷ Beware of calibration on vanilla smile:

  - OK if one is able to pinpoint vanillas to be used as hedges.
  - Letting vanilla smile – through model filter – dictate dynamics of implied vols may not be reasonable.

Connection to traditional approach to stochastic volatility modelling

Traditionally stochastic volatility models have been specified using the instantaneous variance:

- Start with historical dynamics of instantaneous variance:

$$dV \;=\; \mu(t, S, V, p)dt + \alpha()dW_t$$

- in "risk-neutral dynamics", drift of V_t is altered by "market price of risk":

$$dV \;=\; (\mu(t, S, V, p) + \lambda(t, S, V))dt + \alpha()dW_t$$

- a few lines down the road, jettison "market price of risk" and conveniently decide that risk-neutral drift has same functional form as historical drift – except parameters now have stars:

$$dV \;=\; \mu(t, S, V, p^{\star})dt + \alpha()dW_t$$

- eventually calibrate (starred) parameters on smile and live happily ever after.

▷ V is in fact wrong object to focus on – drift issue is pointless:

$$V_t = \zeta_t^t \quad \rightarrow \quad dV_t = \left.\frac{d\zeta_t^T}{dT}\right|_{T=t} dt \;+ \bullet\; dW_t^t$$

The Heston model

Among traditional models, the Heston model (Heston, 1993) is the most popular:

$$\begin{cases} dV_t = -k(V_t - V_0)dt + \sigma\sqrt{V_t}dZ_t \\ dS_t = (r-q)S_t dt + \sqrt{V_t}S_t dW_t \end{cases}$$

- It is an example of a 1-factor Markov-functional model of fwd variances: ζ^T and $\hat{\sigma}^T$ are *functions* of V_t:

$$\begin{aligned} \zeta_t^T &= E_t[V_T] &&= V_0 + (V_t - V_0)e^{-k(T-t)} \\ \hat{\sigma}_t^{T\,2} &= \frac{1}{T-t}\int_t^T \zeta_t^\tau d\tau &&= V_0 + (V_t - V_0)\frac{1-e^{-k(T-t)}}{k(T-t)} \end{aligned}$$

- Look at term-structure of volatilities of $\hat{\sigma}_t^T$. Dynamics of $\hat{\sigma}_t^T$ is given by:

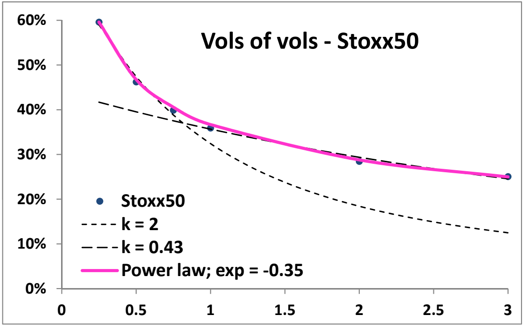
$$d[\,\hat{\sigma}_t^{T\,2}] = \star dt + \frac{1-e^{-k(T-t)}}{k(T-t)}\sigma\sqrt{V_t}dZ_t$$

Volatilities of volatilities

- Term-structure of volatilities of volatilities:

$$\begin{aligned} T-t \ll \frac{1}{k} \quad & \mathrm{Vol}(\sigma_t^T) \simeq 1 - \frac{k(T-t)}{2} \\ T-t \gg \frac{1}{k} \quad & \mathrm{Vol}(\sigma_t^T) \simeq \frac{1}{k(T-t)} \end{aligned}$$

- Term-structure of historical volatilities of volatilities for the Stoxx50 index:



Term-structure of skew

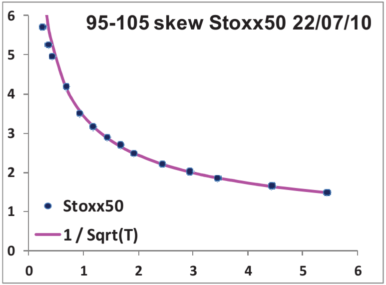
- ATM skew in Heston model: at order 1 in volatility-of-volatility σ:

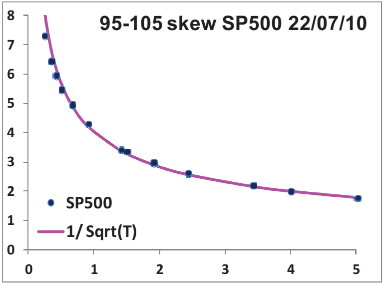
$$T-t \ll \frac{1}{k} \qquad \left.\frac{d\hat{\sigma}^{KT}}{d\ln K}\right|_{K=F} = \frac{\rho\sigma}{4\sqrt{V_t}}$$

$$T-t \gg \frac{1}{k} \qquad \left.\frac{d\hat{\sigma}^{KT}}{d\ln K}\right|_{K=F} = \frac{\rho\sigma}{2\sqrt{V_0}}\frac{1}{k(T-t)}$$

  - ▷ Short-term skew is flat, long-term skew decays like $1/(T-t)$

- Market skews of indices display $\simeq 1/\sqrt{T-t}$ decay:





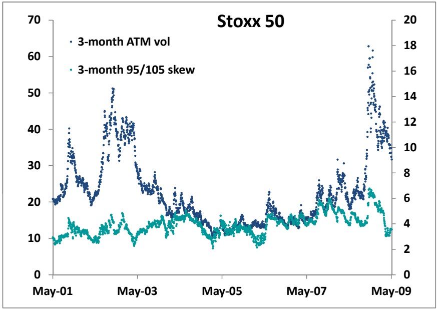
Relationship of skew to volatility

- ATM skew in Heston model at order 1 in volatility-of-volatility σ :

$$T - t \ll \frac{1}{k}: \qquad \left.\frac{d\hat{\sigma}^{KT}}{d\ln K}\right|_{K=F} = \frac{\rho\sigma}{4\sqrt{V_t}} \simeq \frac{\rho\sigma}{4\hat{\sigma}_{\mathrm{ATM}}}$$

  - ▷ In Heston model short-term skew is inversely proportional to short-term ATM vol

- Historical behavior for Stoxx50 index: (left-hand axis: $\hat{\sigma}_{\mathrm{ATM}}$, right-hand axis: $\hat{\sigma}_{K=95} - \hat{\sigma}_{K=105}$)



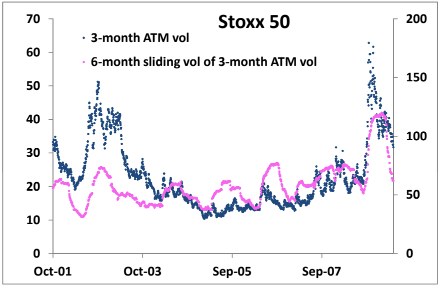
▷ Maybe not reasonable to hard-wire inverse dependence of skew on $\hat{\sigma}_{\mathrm{ATM}}$.

Smile of vol-of-vol

- In Heston model short ATM vol is normal:

$$\hat{\sigma}_{\mathrm{ATM}} \simeq \sqrt{V} \quad \longrightarrow \quad d\hat{\sigma}_{\mathrm{ATM}} = \star dt + \frac{\sigma}{2} dZ$$

- Historical behavior for Stoxx50 index: (left-hand axis: $\hat{\sigma}_{\mathrm{ATM}}$, right-hand axis: 6-month vol of $\hat{\sigma}_{\mathrm{ATM}}$)



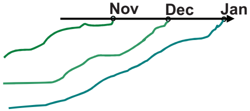
▷ $\hat{\sigma}_{\mathrm{ATM}}$ seems log-normal – or more than log-normal – rather than normal.

- Other issue: in Heston model VS variances are floored:

$$\hat{\sigma}_t^{T\,2} = V_0 + (V_t - V_0)\frac{1 - e^{-k(T-t)}}{k(T-t)} \quad \geq \quad V_0 \frac{k(T-t) - 1 + e^{-k(T-t)}}{k(T-t)}$$

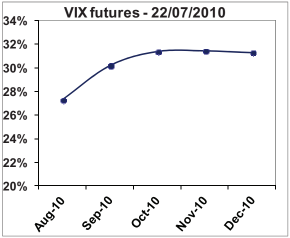
Smile of vol-of-vol – VIX market

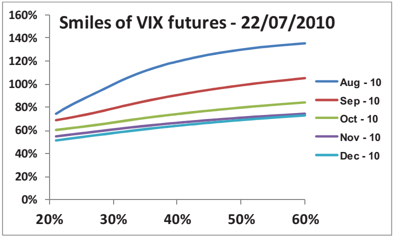
- VIX index is published daily: it is equal to the 30-day VS volatility of the S&P500 index: $\text{VIX}_t = \hat{\sigma}_t^{t+30 \text{ days}}$
- VIX *futures* have monthly expiries - their settlement value is the VIX index at expiry



- VIX options have same expiries as futures

$$F_t^i = E_t[\hat{\sigma}_i^{i+30d}] \qquad C_t^{iK} = E_t[(\hat{\sigma}_i^{i+30d} - K)^+]$$





So what do we do ?

- From a practitioner's point of view, question is: what do we require from a model ?
- Which risks would we like to have a handle on ?
  - forward skew
  - volatilities-of-volatilities, smiles of vols-of-vols
  - correlations between spot and implied volatilities
  - ...
- In next few slides an example of how to proceed to build model that satisfies (some of) our requirements

Practitioner's approach – an example

- Start with dynamics of fwd variances – we would like a time-homogeneous model
  - Start with 1-factor model:

$$d\zeta_t^T = \omega(T-t)\zeta_t^T dU_t \quad \rightarrow \ln\left(\frac{\zeta_t^T}{\zeta_0^T}\right) = \bullet + \int_0^t \omega(T-\tau)dU_\tau$$

  - For general volatility function ω, curve of ζ^T depends on *path* of U_t
  - Choose exponential form: $\omega(T-t) = \omega e^{-k(T-t)}$

$$\int_0^t \omega(T-\tau)dU_\tau = \omega e^{-k(T-t)} \int_0^t e^{-k(t-\tau)}dU_\tau$$

  - Model is now one-dimensional – curve of ζ^T is a *function* of one factor
  - For $T-t \gg \frac{1}{k}$, at order 1 in ω:

$$\mathrm{vol}(\hat{\sigma}_t^T) \propto \frac{1}{k(T-t)} \quad \text{and} \quad \left.\frac{d\hat{\sigma}^{KT}}{d\ln K}\right|_{K=F} \propto \frac{1}{k(T-t)}$$

▷ No flexibility on term-structure of vols-of-vols and term-structure of ATM skew

- Try with 2 factors:

$$d\zeta_t^T = \omega \zeta_t^T \left[(1-\theta) e^{-k_1(T-t)} dW_t^X + \theta e^{-k_2(T-t)} dW_t^Y \right]$$

  - Expression of fwd variances:

$$\zeta_t^T = \zeta_0^T e^{\omega x_t^T - \frac{\omega^2}{2} E[{x_t^T}^2]}$$

  with x_t^T given by:

$$\begin{aligned} x_t^T &= (1-\theta) e^{-k_1(T-t)} X_t + \theta e^{-k_2(T-t)} Y_t \\ dX_t &= -k_1 X_t dt + dW_t^X \\ dY_t &= -k_2 Y_t dt + dW_t^Y \end{aligned}$$

- Dynamics is low-dimensional Markov – fwd variances are functions of 2 easy-to-simulate factors:

$$V_t^{T_1 T_2} = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \zeta_t^T dT$$

- Log-normality of ζ^T can be relaxed while preserving Markov-functional feature

By suitably choosing parameters, it is possible to mimick power-law behavior for:

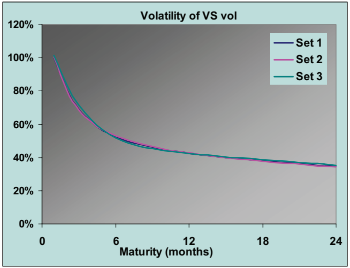
- Term-structure of vol-of-vol
  - for flat term-structure of VS vols, volatility of VS volatility is given by:

$$\mathrm{vol}(\hat{\sigma}^T)^2 = \frac{\omega^2}{4}\left[(1-\theta)^2\left(\frac{1-e^{-k_1 T}}{k_1 T}\right)^2 + \theta^2\left(\frac{1-e^{-k_2 T}}{k_2 T}\right)^2 + 2\rho_{XY}\theta(1-\theta)\frac{1-e^{-k_1 T}}{k_1 T}\frac{1-e^{-k_2 T}}{k_2 T}\right]$$

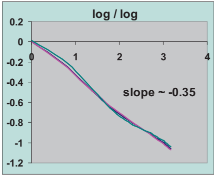
- Term-structure of ATM skew
  - for flat term-structure of VS vols, at order 1 in ω, skew is given by:

$$\left.\frac{d\hat{\sigma}^{KT}}{d\ln K}\right|_F = \frac{\omega}{2}\left[(1-\theta)\rho_{SX}\frac{k_1 T-(1-e^{-k_1 T})}{(k_1 T)^2} + \theta\rho_{SY}\frac{k_2 T-(1-e^{-k_2 T})}{(k_2 T)^2}\right]$$

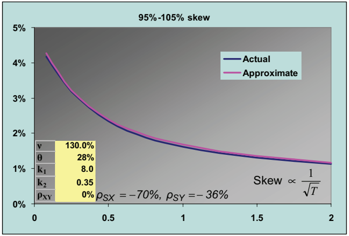
- Term-structure of volatilities of VS vols



	Set1	Set 2	Set 3
ν	130.0%	137.0%	125.0%
θ	28%	29%	32%
k_1	8.0	12.0	4.5
k_2	0.35	0.30	0.60
ρ_{XY}	0%	90%	-70%



- Term-structure of ATM skew



▷ Note that factors have no intrinsic meaning – only vol/vol and spot/vol correlation functions do have physical significance.

▷ It is possible to get slow decay of vol-of-vol and skew

Conclusion

- Models for exotics need to capture joint dynamics of spot and implied volatilities
- Calibration on vanilla smile not always a criterion for choosing model & model parameters
  - We need to have direct handle on dynamics of volatilities
  - Some parameters cannot be locked with vanillas: need to be able to choose them
- Availability of closed-form formulæ not a criterion either
  - Wrong / unreasonable dynamics too high a price to pay
  - What's the point in ultrafast mispricing ?
- So far, models for the (1-dimensional) set of forward variances. Next challenge: add one more dimension.
- One fundamental issue: in what measure does the initial configuration of asset prices – e.g. implied volatilities – restrict their dynamics ?