Stochastic volatility
Recent developments and future directions

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Introduction

- SV models are an established part of a quant toolbox.
- Many tools have been development, many more need to be developed
- Fundamental tools for calibration
  - Markovian projection (MP)
  - Parameter averaging (PA)
  - Direct calibration to market
- Applications
  - Short rate and forward Libor models of interest rates
  - Hybrids models
  - Basket and spread models
- Fundamental properties
  - Moment explosions
  - Smile asymptotics

Calibration

- For calibration, need fast methods for European option pricing
- A number of SV models for interest rates and hybrids have been put forward recently, with various approaches to calibration
- Many of these approaches can be aggregated into what we call the Markovian Projection method:

  a generic, powerful framework for deriving closed-form approximations to European option prices

Step 1 Apply Markovian projection to $S(\cdot)$, a technique to replace a complicated process with a simple one, preserving European option prices

Step 2 Approximate conditional expected values required

Step 3 Apply parameter averaging techniques to obtain time-independent coefficients from time-dependent

Step 4 Hopefully a simple model is obtained, use known results.

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The Markovian projection

Theorem (Dupire 97, Gyongy 86) Let $X(t)$ be given by

$$dX(t) = \alpha(t)\,dt + \beta(t)\,dW(t), \qquad (1)$$

where $\alpha(\cdot)$, $\beta(\cdot)$ are adapted bounded stochastic processes such that (1) admits a unique solution. Define $a(t,x)$, $b(t,x)$ by

$$a(t,x) = E(\alpha(t)|\, X(t) = x),$$
$$b^2(t,x) = E\left(\beta^2(t)\middle|\, X(t) = x\right),$$

Then the SDE

$$dY(t) = a(t, Y(t))\,dt + b(t, Y(t))\,dW(t), \qquad (2)$$
$$Y(0) = X(0),$$

admits a weak solution $Y(t)$ that has the same one-dimensional distributions as $X(t)$.

- See [Dup97], [Gyö86]

The Markovian projection, cont

Remark 1 Since $X(\cdot)$ and $Y(\cdot)$ have the same one-dimensional distributions, the prices of European options on $X(\cdot)$ and $Y(\cdot)$ for all strikes K and expiries T will be the same. Thus, for the purposes of European option valuation and/or calibration to European options, we can replace a potentially very complicated process $X(\cdot)$ with a much simpler Markov process $Y(\cdot)$, which we call the Markovian projection of $X(\cdot)$.

Remark 2 The process $Y(\cdot)$ follows what is known as a "local volatility" process. The function $b(t, x)$ is often called "Dupire's local volatility"

The Markovian projection, cont

- If $X(\cdot)$ itself came from a local volatility model (perhaps complicated), then replacing it with a (simpler) local vol model is probably the right thing to do. But:
- For SV models, better to replace "like for like". Replace a (complicated) SV model with a (simpler) SV model.
  - Approximations to conditional expected values (the hard part – see eg [Atl06]) may be simpler
  - Errors of approximations will tend to "cancel out"
- Dupire-Gyongy theorem still works

Corollary If two processes have the same Dupire's local volatility, the European option prices on both are the same for all strikes and expiries

- Future directions: better methods for conditional expected value calculations

Simple SV model

- After applying the MP method, often get the SDEs of the form

$$\begin{aligned} dz(t) &= \theta(1 - z(t))\,dt + \gamma(t)\sqrt{z(t)}\,dV(t), \\ dS(t) &= (\beta(t)S(t) + (1 - \beta(t))S(0))\,\sigma(t)\sqrt{z(t)}\,dW(t), \end{aligned} \tag{3}$$

- Or, rather, we apply the MP method with the goal of obtaining the SDEs in this form
  - Choose z to be the square root process
  - Linearize the volatility term of S
- Why? When parameters are constant, this is the (shifted) Heston model, a model with very efficient numerical methods for European option valuation, see [AA02].
- How to replace time-dependent parameters with constant? Parameter averaging. Proofs and details in [Pit05b], [Pit05a]

Example of a simple averaging formula

- For motivation, a log-normal model with time-dependent volatility,
$$dS(t) = \sigma(t)\, S(t)\, dW(t).$$
- It is known that, an option value with expiry T_n in this model is equal to the Black-Scholes option value with "effective" volatility
$$\sigma_n = \left(\frac{1}{T_n} \int_0^{T_n} \sigma^2(t)\, dt \right)^{1/2}.$$
- Direct link between "model" parameter $\sigma(t)$ and "market" parameters (σ_n)
- Much faster that using Black-Scholes for valuation of options and implied calculations

Non-trivial example: averaging skew

- Time-dependent skew

$$dS(t) = \sigma(t)\left(\beta(t)S(t) + (1-\beta(t))S(0)\right)dW(t),$$

- Constant skew

$$d\bar{S}(t) = \sigma(t)\left(b\bar{S}(t) + (1-b)\bar{S}(0)\right)dW(t).$$

- Given $\beta(\cdot)$, find b such that option prices for different strikes (same expiry T) are matched between two models
- The main result. In the "small skew" limit,

$$b = \int_0^T \beta(t)\,w(t)\,dt,$$

where

$$w(t) = \frac{v^2(t)\,\sigma^2(t)}{\int_0^T v^2(t)\,\sigma^2(t)\,dt}, \quad v^2(t) = \int_0^t \sigma^2(s)\,ds.$$

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Averaging skew, cont

- Comments:
  - "Total skew" b is the average of "local skews" $\beta(t)$ with weights $w(t)$
  - Weights proportional to total variance, i.e. local slope further away matters more
- Example: constant volatility $\sigma(t) \equiv \sigma$,

$$b = \left(T^2/2\right)^{-1} \int_0^T t\beta(t)\, dt.$$

- SV extenstions available, see [Pit05b], [Pit05a]
- Future directions:
  - Averaging methods for other models/parameters
  - More accurate/rigorous averaging methods

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Direct calibration to market

- In equity/FX: Let $\sigma_{\mathrm{mkt}}(\mathrm{T},\mathrm{K})$ be market volatilities for all expiries T and strikes K (assumed known). Given an exogenous SV process $z(t)$, find $b(t,x)$ such that the model

$$dS(t) = b(t, S(t))\sqrt{z(t)}\,dW(t), \quad S(0) = S_0,$$

matches the market

  - Define Dupire's market local volatility $b_{\mathrm{mkt}}(t,x)$ by the requirement that the local volatility model with $b_{\mathrm{mkt}}(t,x)$ matches the whole market. Easy to compute

$$b_{\mathrm{mkt}}(t,x) = \frac{2\partial C/\partial t}{\partial^2 C/\partial x^2}.$$

  - Then, from Theorem and Corollary,

$$b^2(t,x) = \frac{b^2_{\mathrm{mkt}}(t,x)}{E(z(t)|\,S(t)=x)}. \tag{4}$$

  - In practice $E(z(t)|\,S(t)=x)$ is often computed numerically in a forward PDE in (S,z). Slow and noisy.

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Direct calibration to market, cont

- Define a "proxy" process $X(t)$ by
$$dX(t) = \tilde{b}(t, X(t))\sqrt{z(t)}\,dW(t), \quad X(0) = S_0, \qquad (5)$$
where $\tilde{b}(t, x)$ is such that European options on X are easy to compute
- Define the "proxy" Dupire's local volatility $b_{proxy}(t, x)$ as before but for European options on X (not on market). Then
$$E(z(t)|X(t) = x) = \frac{b^2_{proxy}(t, x)}{\tilde{b}^2(t, x)}, \qquad (6)$$
thus having a stochastic volatility model which cheaply-computable European option prices allows us to compute the conditional expected values easily.
- Combining the two results we get
$$b(t, x) = \tilde{b}(t, x) \times \frac{b_{mkt}(t, x)}{b_{proxy}(t, x)} \times \left(\frac{E(z(t)|X(t) = x)}{E(z(t)|S(t) = x)}\right)^{/12}.$$

Direct calibration to market, cont

- Choice 1: Approximate
  $\mathrm{E}\left(\mathrm{z}\left(\mathrm{t}\right)\middle|\,\mathrm{X}\left(\mathrm{t}\right)=\mathrm{x}\right)=\mathrm{E}\left(\mathrm{z}\left(\mathrm{t}\right)\middle|\,\mathrm{S}\left(\mathrm{t}\right)=\mathrm{x}\right)$
- Choice 2: Link $\mathrm{S}\left(\mathrm{t}\right)$ and $\mathrm{X}\left(\mathrm{t}\right)$.
  - Define $\mathrm{H}\left(\mathrm{t},\mathrm{s}\right)$ by the requirement that $\mathrm{H}\left(\mathrm{t},\mathrm{S}\left(\mathrm{t}\right)\right)$ has the same dW term as dX (H a function of b, $\tilde{\mathrm{b}}$)
  - Then approximate

$$
\begin{aligned}
\mathrm{X}\left(\mathrm{t}\right) &\approx \mathrm{H}\left(\mathrm{t},\mathrm{S}\left(\mathrm{t}\right)\right),\\
\mathrm{E}\left(\mathrm{z}\left(\mathrm{t}\right)\middle|\,\mathrm{S}\left(\mathrm{t}\right)=\mathrm{x}\right) &\approx \mathrm{E}\left(\mathrm{z}\left(\mathrm{t}\right)\middle|\,\mathrm{X}\left(\mathrm{t}\right)=\mathrm{H}\left(\mathrm{t},\mathrm{x}\right)\right).
\end{aligned}
$$

  - Functional equation on b,

$$
\mathrm{b}\left(\mathrm{t},\mathrm{x}\right)=\tilde{\mathrm{b}}\left(\mathrm{t},\mathrm{H}\left(\mathrm{t},\mathrm{x}\right)\right)\frac{\mathrm{b}_{\mathrm{mkt}}\left(\mathrm{t},\mathrm{x}\right)}{\mathrm{b}_{\mathrm{proxy}}\left(\mathrm{t},\mathrm{H}\left(\mathrm{t},\mathrm{x}\right)\right)}. \tag{7}
$$

- Last derivation is an example of a clever way of computing conditional expectations. Need more of these in the future!
- Original result due to Forde ([For06]). More details in [Pit06a].

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Local volatility short rate model

- Simplest interest rate model: one-factor Gaussian ("Hull-White")

$$r(t) = f(0,t) + x(t), \quad dx(t) = (\theta(t) - ax(t))\, dt + \sigma(t)\, dW(t).$$

- Local-volatility extension: quasi-Gaussian ("Cheyette")

$$\begin{aligned} dx(t) &= (y(t) - ax(t))\, dt + \sigma(t, x(t), y(t))\, dW(t), \\ dy(t) &= \left(\sigma^2(t, x(t), y(t)) - 2ay(t)\right) dt. \end{aligned}$$

- Swap rate (under swap measure), $S(t) = S(t, x(t), y(t))$ for a known function $S(t, x, y)$,

$$dS(t) = \left.\frac{\partial S(t, x, y)}{\partial x}\right|_{x=x(t), y=y(t)} \sigma(t, x(t), y(t))\, dW^A(t).$$

Local volatility short rate model, cont

- Markovian projection (preserves European swaptions)

$$
\begin{aligned}
dS(t) &= \eta(t, S(t))\, dW^A(t), \\
\eta^2(t, S) &= E^A\left(\left(\frac{\partial S(t, x(t), y(t))}{\partial x}\right)^2 \sigma^2(t, x(t), y(t)) \middle| S(t) = S\right)
\end{aligned}
$$

- Let $y^*(t) = E^A(y(t))$, $\xi(t, s)$ is the inverse of $S(t, x, y^*(t))$ in x. Then

$$
\eta^2(t, S) \approx \left(\left.\frac{\partial S(t, x, y^*(t))}{\partial x}\right|_{x=\xi(t,S)}\right) \sigma(t, \xi(t, S), y^*(t)).
$$

- Local-volatility model for S with a known η. Apply parameter averaging (on skew and vol), then shifted-lognormal formula to get option prices.
- Similar approaches for SV extensions, forward Libor models with SV, and IR/Equity, IR/FX hybrids models. See details in [And05], [Pit05a], [Pit06b].

Multi-stochastic volatility

- When modelling multiple underlyings, natural to have different SV processes for them: baskets, spreads
- Also in current models: in interest rates, turns out variance decorrelation is important for exotics linked to CMS spreads, see [Pit06c]
- Consider a "simple" multi-SV model for a basket $\mathrm{S}(\mathrm{t}) = \sum \mathrm{w_i} \mathrm{S_i}(\mathrm{t})$,

$$\begin{aligned} \mathrm{dS_i}(\mathrm{t}) &= \varphi_{\mathrm{i}}(\mathrm{S_i}(\mathrm{t}))\sqrt{\mathrm{z_i}(\mathrm{t})}\,\mathrm{dW_i}(\mathrm{t}), \\ \mathrm{dz_i}(\mathrm{t}) &= \theta(1-\mathrm{z_i}(\mathrm{t}))\,\mathrm{dt} + \eta_{\mathrm{i}}\sqrt{\mathrm{z_i}(\mathrm{t})}\,\mathrm{dW_{I+i}}(\mathrm{t}), \quad \mathrm{z_i}(0) = 1, \\ \mathrm{i} &= 1, \ldots, \mathrm{I}. \end{aligned}$$

- Options on index $\mathrm{S}(\cdot)$. Apply MP to write SDE for S. Start

$$\begin{aligned} \mathrm{dS}(\mathrm{t}) &= \sigma(\mathrm{t})\,\mathrm{dW}(\mathrm{t}), \\ \sigma^2(\mathrm{t}) &= \sum_{\mathrm{n,m}=1}^{\mathrm{N}} \mathrm{w_n w_m} \varphi_{\mathrm{n}}(\mathrm{S_n}(\mathrm{t}))\,\varphi_{\mathrm{m}}(\mathrm{S_m}(\mathrm{t}))\sqrt{\mathrm{z_n}(\mathrm{t})\,\mathrm{z_m}(\mathrm{t})}\rho_{\mathrm{nm}}. \end{aligned}$$

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Multi-stochastic volatility, cont

- What to use for $z(t)$, the SV of the index? Inspired by constant case $\varphi_i(S_i(t)) \equiv \varphi_i(S_i(0)) = p_i$,

$$
\begin{aligned}
z(t) &= p^{-2} \sum w_n w_m p_n p_m \sqrt{z_n(t)\, z_m(t)}\rho_{nm}, \\
p^2 &= \sum w_n w_m p_n p_m \rho_{nm}.
\end{aligned}
$$

  Then

$$
\begin{aligned}
dS(t) &= \varphi(t, S(t))\sqrt{z(t)}\, dW(t), \\
\varphi^2(t,x) &= \mathrm{E}\left(\sigma^2(t) \middle| S(t) = x\right) / \mathrm{E}\left(z(t) \middle| S(t) = x\right).
\end{aligned}
$$

- Use Gaussian approximation to (S_i, z_i) to compute $\varphi^2(t,x)$
- Approximate the dynamics of $z(\cdot)$ by the mean reverting square root process – also use MP to come up with the coefficients!
- See more in [Pit06a], and an alternative approach in [DK06]

Moment explosions

- SV models widely used, but do we know their fundamental properties?

$$
\begin{aligned}
\mathrm{d}X(t) &= \lambda X(t)\sqrt{z(t)}\,\mathrm{d}W(t), \\
\mathrm{d}z(t) &= \varkappa\left(\theta - z(t)\right)\mathrm{d}t + \varepsilon z^{p}(t)\,\mathrm{d}B(t), \quad \langle \mathrm{d}W, \mathrm{d}B \rangle = \rho.
\end{aligned}
$$

- Well-known failure of the martingale property: When $p \leq \frac{1}{2}$ or $p > \frac{3}{2}$, X is a martingale. When $\frac{3}{2} > p > \frac{1}{2}$, X is a martingale for $\rho \leq 0$ and a strict supermartingale for $\rho > 0$.
- Many tools not valid anymore (Girsanov's theorem, etc)
- Even worse. Moments of $X(\cdot)$ may explode in finite time. See full details in [AP06]

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Moment explosions, cont

p	ρ	ω	Result
$0 < p < 1/2$	$-1 < \rho < 1$	≥ 0	$EX^{\omega}(T) < \infty$ for $\forall T$
$p = 1/2$	$-1 < \rho < 1$	> 1	$\exists T^{*}$: $EX^{\omega}(T) = \infty$ for $\forall T > T^{*}$
$1/2 < p \leq 1$	$-1 < \rho < 0$	small	$EX^{\omega}(T) < \infty$ for $\forall T > 0$.
$1/2 < p \leq 1$	$-1 < \rho < 0$	large	$\exists T^{*}$: $EX^{\omega}(T) = \infty$ for $\forall T > T^{*}$
$1/2 < p \leq 1$	$\rho = 0$	> 1	$EX^{\omega}(T) = \infty$ for $\forall T$
$1/2 < p < 3/2$	$0 < \rho < 1$	large	$EX^{\omega}(T) = \infty$ for $\forall T$

- Second moment of X often important, in particular in interest rate applications
  - If X is a Libor rate ,then the price of Libor-in-arrears involves $EX^{2}(T)$.
  - If X is a swap rate, the price of CMS involves $EX^{2}(T)$.
  - Numerical tricks to make them finite? A lot of issues.

Moment explosions, cont

- Consider SABR model

$$\begin{aligned} dX(t) &= \lambda X^c(t)\sqrt{z(t)}\,dW(t), \\ dz(t) &= \frac{1}{4}\varepsilon^2 z(t)dt + \varepsilon z(t)\,dB(t), \end{aligned}$$

- Case $c < 1$: all moments finite, X is a non-negative martingale;
- Case $c = 1$: (same classification as before, despite the positive drift)
  - for small ω, $EX^\omega(T) < \infty$,
  - for large ω, $\exists T^*$: $EX^\omega(T) = \infty$ for $\forall T > T^*$.

Smile asymptotics

- Roger Lee ([Lee04]) was the first to link the number of finite moments to the asymptotics of the smile.
- Define $\mathrm{I}(\mathrm{k})$ the implied BS volatility as a function of log strike, $\mathrm{k} = \log(\mathrm{K}/\mathrm{S}_0)$. Define $\beta = \limsup_{\mathrm{k}\to\infty} \frac{\mathrm{I}^2(\mathrm{k})}{\mathrm{k}/\mathrm{T}}$. Then

$$
\begin{aligned}
\beta &= 2 - 4\left[\sqrt{\omega_{\max}^2 + \omega_{\max}} - \omega_{\max}\right], \\
\omega_{\max} &= \arg\sup\left\{\omega : \mathrm{E}\mathrm{X}^{1+\omega}(\mathrm{T}) < \infty\right\}.
\end{aligned}
$$

- Together with moment explosion results, gives smile asymptotics in many cases
- However, if all moments exist (eg SABR), not very informative: $\limsup_{\mathrm{k}\to\infty} \frac{\mathrm{I}^2(\mathrm{k})}{\mathrm{k}/\mathrm{T}} = 0$, so which one is it
  - $\mathrm{I}(\mathrm{k})$ grows slower than $\sqrt{\mathrm{k}}$?
  - $\mathrm{I}(\mathrm{k})$ approaches non-zero limit?
  - $\mathrm{I}(\mathrm{k}) \to 0$?

Smile asymptotics, cont

- A refinement (stated by not proven in [Pit04], simpler case $v(k) = 1$ proven in [For06])
- Let $v(k)$ be such that $\delta^2 = \lim_{k \to -\infty} v^2(k)/k$ exists. Then

$$\limsup_{k \to \infty} \frac{I(k)}{v(k)} = a^*,$$

where a^* is a solution to

$$a\sqrt{2q_{max}T} = 1 - \frac{a^2\delta^2}{2}$$

and

$$q_{max} = \arg\sup \left\{ q : E \exp\left(q \left[\frac{\log(1 + X(T))}{v(\log(1 + X(T)))} \right]^2 \right) < \infty \right\}.$$

- Lee's result: $v(k) = \sqrt{k/T}$.
- $v(k) = 1$: the limit of $I(k)$

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Smile asymptotics, cont

- Recently, Benaim and Friz (see [BF06]) completely characterized implied vol smile asymptotics. Define
  - $\psi(x) = 2 - 4\left[\sqrt{x^2 + x} - x\right]$,
  - $f \sim g \Leftrightarrow f/g \to 1$ as $x \to \infty$,
  - R_α to be regularly varying functions of index α at $+\infty$ (eg "similar" to x^α)
  - $f(x)$ the density of $\log X(T)$.
- Main result:
  - If $-\log f(k) \in R_\alpha$, then $I^2(k)/k \sim \psi(-1 - \log f(k)/k)$;
  - If $-\log f(k)/k \to \infty$ (ie all moments exist) then $I^2(k) \sim -1/(2\log f(k))$
- Future challenge: find smile asymptotics for SABR. Conjecture: for cev<1, the limit is finite and non-zero.

Future challenges

Would like to see progress in the following directions

- New ways to compute/approximate $E(z(t)|X(t))$ found: will lead to advances in SV calibration
- Parameter averaging methods extended to new models and improved
- More types of models handled by MP+PA
- Calibration and approximation tools developed for multi-stochastic volatility models (baskets, spreads)
- Volatility smile asymptotics for the SABR model found

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