Practical Multi-Factor Quadratic Gaussian Models

of Interest Rates

Vladimir Piterbarg

Barclays Capital

1 Forward Libor models

- Many exotics require a multi-factor model
- Forward Libor models a decent choice
  - Rich volatility structure, global calibration
  - Flexibility in decorrelation
  - Relatively straightforward calibration with fast formulas for swaptions
  - Various choices for volatility smile including stochastic volatility
- Problems? Slow
  - Markovian in a number of rates (40×4)
  - Relatively complicated time-dependent drifts
  - Zero coupon bond calculations involve large number of rates

2 Life after FLM?

- Want a model that is
  - Multi-factor, rich volatility structure
  - Easy to calibrate (global calibration to a swaption grid)
  - Control over smile
  - Fast to simulate – Markovian in small number of state variables
  - State variables with simple dynamics
- Choices
  - Multi-factor Cheyette (see [And05]) – but need extra state variables
  - Multi-factor linear Gaussian (subset of Cheyette) – but no smile control
  - Multi-factor quadratic Gaussian!

3 Quadratic Gaussian Models

- Not particularly well-known; more popular in academia than in banks
- Fairly strong links to affine models
  - Every quadratic model is affine in the extended set of variables
  - Yet looking at quadratic models is useful as the parametrization is more parsimonious
- Main idea: if a short rate is a quadratic form of a multi-dimensional linear Gaussian process, then bonds are exponentials of quadratic forms of the same
- See [CFP03], [AD99], [Ass07].

4 Quadratic Gaussian Models

- Notations: $P(t,T)$ are ZCBs, $f(t,T)$ are instantaneous cc forward rates, $r(t) = f(t,t)$ a short rate.
- Define, under risk-neutral measure,

$$dZ(t) = M(t)\,Z(t)\,dt + \Sigma(t)\,dW(t),$$

where $Z(t)$ is $N \times 1$, $M(t)$ is $N \times N$, $\Sigma(t)$ is $N \times N$, and $W(t)$ is $N \times 1$. Further define

$$r(t) = Z(t)^\top \Gamma(t)\,Z(t) + b(t)^\top Z(t) + a(t).$$

Here $\Gamma(t)$ is $N \times N$, $b(t)$ is $N \times 1$ are model inputs, and $a(t)$ is a scalar to fit the initial yield curve. Then

$$-\log P(t,T) = Z(t)^\top \tilde{\Gamma}(t,T)\,Z(t) + \tilde{l}(t,T)^\top Z(t) + \tilde{a}(t,T),$$

where $\tilde{\Gamma}(t,T)$, $\tilde{l}(t,T)$, $\tilde{a}(t,T)$ are obtained by solving ODEs.

- In particular, the model is Markovian in N state variables and can generate a volatility smile (because of the quadratic term).
- The lowest number of state variables (equal to the number of Brownian drivers) for any model with volatility smile.

5 Quadratic Gaussian Models

- Practical?
  - Monte-Carlo is easy. Gaussian state vector could be simulated with arbitrarily large steps with no effort, bonds have closed-form formulas.
  - Calibration? Two issues
    * Little is known about good approximations for swaptions
    * Parametrization of the model needs to be thought through
- With a linear change of variables, $M(t)$ disappears but still a lot of parameters $(\Sigma(t), \Gamma(t), b(t))$ with limited or no intuition

6 1-Factor Example

- We have for $N = 1$, constant coefficients

$$\begin{aligned} r(t) &= cz(t)^2 + bz(t) + a, \\ dz(t) &= \sigma(t)\, dW(t). \end{aligned}$$

- Then

$$dr(t) = (2cz(t) + b)\,\sigma(t)\, dW(t) + \ldots dt.$$

- Since

$$z(t) = -\frac{b}{2c} + \frac{1}{2c}\sqrt{b^2 - 4c(a - r(t))},$$

  we have

$$dr(t) = \sqrt{4cr(t) + (b^2 - 4ac)}\,\sigma(t)\, dW(t) + \ldots dt,$$

  and we obtain a model of square root type.

- Can control the slope of the volatility smile.

7 2-Factor Example

- Idea: use 1 factor to define the rate, the other its "volatility" (inspired by [Tez05]). Set $\Sigma_{12} = 0$.
- Recall

$$r(t) = \gamma_{11} z_1(t)^2 + 2\gamma_{12} z_1(t) z_2(t) + \gamma_{22} z_2(t)^2 \\ + b_1(t) z_1(t) + b_2(t) z_2(t) + \ldots.$$

- Set $b_2(t) = 0$, $b_1(t) = e^{-\varkappa t}$. Then, if all $\gamma_{ij} = 0$, we have a 1-factor linear Gaussian model,

$$r(t) = e^{-\varkappa t} z_1(t),$$

with mean reversion $\varkappa$:

$$dr(t) = -\varkappa r(t)\, dt + e^{-\varkappa t} \Sigma_{11}(t)\, dW_1(t).$$

- Use $b_1(t) z_1(t)$ is the "curve" factor.

8 2-Factor Example

- Choose γ_{ij} such that the square term is equal to the curve factor times some stochastic variable,

$$r(t) = (1 + \varepsilon v(t))(b_1(t) z_1(t)) + \ldots,$$

where

$$v(t) = \rho \times b_1(t) z_1(t) + \bar{\rho} \times z_2(t), \quad \bar{\rho} = \sqrt{1 - \rho^2},$$

so that

$$\gamma_{11} = \varepsilon \rho b_1(t)^2, \quad \gamma_{12} = \frac{\varepsilon}{2} \bar{\rho} b_1(t), \quad \gamma_{22} = 0.$$

- The process $v(t)$ plays the role of "stochastic volatility". The parameter ρ is the correlation between the volatility and the curve factor. The parameter ε is "volatility of volatility".

- Model produces U-shaped smiles with intuitive controls provided by ρ, ε.

- "Spanned" volatility. Zero coupon bonds depend on both z_1 and z_2. More on that later.

9 N+1 Factor Quadratic Model

- 2d example suggests a parametrization.
- Use N factors to define the "volatility structure" of the model, use 1 factor for "stochastic vol".
- The N-factor model – make it linear Gaussian.

10 Linear Gaussian Model a-la FLM

- (Main idea from [And05])
  - Global calibration
  - Parametrized in terms of "observable" rate volatilities and correlations (not vols/correls of factors or state variables)
- N curve factors

$$dZ_{1:N}(t) = \Sigma_{1:N}(t)\ dW_{1:N}(t).$$

- $b_{1:N}(t)$ is a (column) vector of dimension N, the loadings vector for rate states.
- Bond loading vector

$$b_{1:N}(t,T) = \int_t^T b_{1:N}(u)\ du.$$

- In a linear Gaussian model,

$$\begin{aligned} f(t,T) &= b_{1:N}(T)^\top Z_{1:N}(t) + \ldots, \\ -\log P(t,T) &= b_{1:N}(t,T)\, Z_{1:N}(t) + \ldots. \end{aligned}$$

11 Linear Gaussian Model a-la FLM

- Benchmark tenors $\{\tau_1, \ldots, \tau_N\}$ and benchmark rates
$$f(t) = (f(t, t+\tau_1), \ldots, f(t, t+\tau_N))^\top.$$
- Parametrize the model in terms of
  - $\lambda_n(t)$, the instantaneous volatility of $f(t, t+\tau_n)$, and
  - $C(t) = \{c_{ij}(t)\}$ the instantaneous correlation matrix of $f(t)$.
- Link to factor vols: the variance-covariance matrix of $f(t)$ is given by $\Lambda(t) = \{\lambda_i(t)\lambda_j(t)c_{ij}(t)\}$. On the other hand, in the linear Gaussian model,
$$df(t) = B_{1:N}(t)\Sigma_{1:N}(t)\, dW_{1:N}(t), \tag{1}$$
where B is a matrix whose n-th row is $b_{1:N}(t+\tau_n)^\top$, ie
$$B_{1:N}(t) = \begin{pmatrix} b_{1:N}(t+\tau_1)^\top \\ \ldots \\ b_{1:N}(t+\tau_N)^\top \end{pmatrix}.$$
- Hence, $\Sigma_{1:N}(t)$ is recovered from $\Lambda(t)$ by solving
$$B_{1:N}(t)\Sigma_{1:N}(t) = \sqrt{\Lambda(t)}.$$

12 Linear Gaussian Model a-la FLM

- Summary: the model is parametrized by $b_{1:N}(t)$, $\{\lambda_n(t)\}$ and $C(t)$
- Strong conceptual links to FLM
  - $\{\lambda_n(t)\}$ are the volatilities of benchmark rates ($\approx$ forward Libor rate volatilities)
  - $C(t)$ is the instantaneous correlation matrix of benchmark rates ($\approx$ instantaneous Libor rate correlations)
  - $b_{1:N}(t)$ essentially define interpolation rules, ie how to get the volatilities (and correlations) of non-benchmark rates from the benchmark ones
- Calibration a-la FLM
  - Choose τ_n's and $\{b_{1:N}(t)\}$ (eg $b_{1:N}(t) = \left(e^{-\varkappa_1 t}, \ldots, e^{-\varkappa_N t}\right)^\top$ for the familiar "mean reversion" specification)
  - Parametrize $C(t)$ with a functional form (any from FLM literature)
  - Iterate over $\{\lambda_n(t_m)\}$ for all n, m until match all target swaptions
  - Can match as many columns in the swaption matrix as there are benchmark rates, ie N.

13 Back to Quadratic

- Use what would have been a short rate in the linear model, $b_{1:N}(t)^\top Z_{1:N}(t)$, as the curve driver
- Add a "vol" factor:

$$Z(t) = \begin{pmatrix} Z_{1:N}(t) \\ Z_{N+1}(t) \end{pmatrix}, \quad dW(t) = \begin{pmatrix} dW_{1:N}(t) \\ dW_{N+1}(t) \end{pmatrix},$$

$$b(t) = \begin{pmatrix} b_{1:N}(t) \\ 0 \end{pmatrix}, \quad \Sigma(t) = \begin{pmatrix} \Sigma_{1:N}(t) & 0 \\ 0 & \Sigma_{N+1,N+1}(t) \end{pmatrix}.$$

- Want the quadratic term to be the curve driver times the vol factor:

$$r(t) = \left(1 + \varepsilon\left(\rho\left[b(t)^\top Z(t)\right] + \bar{\rho} Z_{N+1}(t)\right)\right) \times \left[b(t)^\top Z(t)\right] + \ldots.$$

so that

$$\Gamma(t) = \varepsilon \begin{pmatrix} \rho b_{1:N}(t)\, b_{1:N}(t)^\top & \frac{1}{2}\bar{\rho} b_{1:N}(t) \\ \frac{1}{2}\bar{\rho} b_{1:N}(t)^\top & 0 \end{pmatrix}.$$

- The model is fully specified. Two SV parameters ε, ρ ($\Sigma_{N+1,N+1}$ is superfluous?) that control the smile shape, plus all the "linear" parameters to control the volatility structure of rates.

- For the bond reconstruction formulas, we have

$$-\log P(t,T) = Z(t)^\top \Gamma(t,T) Z(t) + b(t,T)^\top Z(t) \\ + \alpha(t,T) - \log P(0,t,T),$$

where

$$\begin{aligned} -\Gamma'(t,T) + 2\Gamma(t,T)\Sigma\Sigma^\top\Gamma(t,T) &= \Gamma(t), \\ -b'(t,T) + 2\Gamma(t,T)\Sigma\Sigma^\top b(t,T) &= b(t). \end{aligned} \tag{2}$$

- (As pointed out by Elkouby [Elk07], if the model is re-formulated under some forward measure, the equations can actually be solved explicitly. We do not pursue this here.)
- The scalar (although also satisfies an equation) is best obtained from the no-arb condition,

$$\mathsf{E}_0^t P(t,T) = P(0,t,T),$$

which implies

$$\alpha(t,T) = \log \mathsf{E}_0^t \exp\left(-\left(Z(t)^\top \Gamma(t,T) Z(t) + b(t,T)^\top Z(t)\right)\right). \tag{3}$$

15 Measure Changes

- Need to know E, Var of Z under forward measures (eg to compute the scalar for bonds)
- We have

$$dP(t,T)/P(t,T) = -\left(2Z(t)^\top \Gamma(t,T) + b(t,T)^\top\right)\Sigma(t)\ dW(t) + \ldots,$$

so

$$dW^T(t) = dW(t) + \Sigma(t)^\top (2\Gamma(t,T)Z(t) + b)\ dt$$

is a BM under P^T.

- Hence

$$dZ(t) = -\Sigma(t)\Sigma(t)^\top (2\Gamma(t,T)Z(t) + b(t,T))\ dt + \Sigma(t)\ dW^T(t).$$

- Linear SDE, so
  - Z is Gaussian under any forward measure
  - E, Var are obtained by standard formulas

16 Swaption Pricing

- Fast swaption pricing formula is key to efficient calibration. Define swap rate, annuity

$$S(t) = \frac{P(t,T_1) - P(t,T_M)}{A(t)}, \quad A(t) = \sum_{m=1}^{M-1} \delta_m P(t, T_{m+1}).$$

- "FLM" approach:

$$\begin{aligned}
dS(t) &= \sum_{m=1}^{M} \frac{\partial S(t)}{\partial P(t,T_m)} \sum_{n=1}^{N} \frac{\partial P(t,T_m)}{\partial Z_n(t)} dZ_n(t) \\
&= -\sum_{m=1}^{M} \frac{\partial S(t)}{\partial P(t,T_m)} P(t,T_m) \left(2Z(t)^\top \Gamma(t,T_m) + b(t,T_m)^\top\right) \Sigma(t)\, dW^A(t) \\
&= \left(2Z(t)^\top \Gamma^A(t) + b^A(t)^\top\right) \Sigma(t)\, dW^A(t),
\end{aligned}$$

where

$$(\Gamma, b)^A(t) = \sum_{m=1}^{M} w_m(t) (\Gamma, b)(t, T_m), \quad w_m(t) = -P(t,T_m) \frac{\partial S(t)}{\partial P(t,T_m)}.$$

17 Swaption Pricing a-la FLM

- Probably not going to work
  - Are weights w_m constant enough to freeze at zero?
  - Dynamics of $Z(t)^{\top}$ in the diff coefficient are pretty complicated under P^A
  - Replacing the diff coefficient with its expected value is probably not accurate enough (smile!)
  - Approximating

$$\mathsf{E}^A\left(Z^{\top}\Gamma^A\Sigma\Sigma^{\top}\Gamma^A Z \middle| S(t)\right)$$

    to the first (second? smile!) order in S seems daunting.

18 Swaption Pricing by Integration

- Need $\mathsf{E}^A\left(S\left(T\right)-K\right)^+$, ie marginal (time $T = T_1$) distribution of $S\left(\cdot\right)$ only
- Distribution of $Z\left(T\right)$ under P^A? Gaussian mixture: for any ψ,

$$\mathsf{E}^A\left(\psi\left(Z\left(T\right)\right)\right)=\sum_{m=1}^{M-1} p_m \mathsf{E}^{T_{m+1}}\left(\psi\left(Z\left(T\right)\right)\right), \quad p_m=\delta_m P\left(0,T_{m+1}\right)/A\left(0\right).$$

- In principle, one-step Monte-Carlo in $N+1$ dimensions for Gaussian mixture is fast. But, let's try to get something faster
- Plan:
  - simplify distribution of $Z\left(T\right)$ under P^A.
  - simplify $S=S\left(Z\right)$

19 Approximate Distribution of Factors under Annuity Measure

- First idea – use Gaussian approximation,

$$Z(T) \sim \mathcal{N}\left(\mathsf{E}^A(Z), \mathsf{Var}^A(Z)\right).$$

- As the true distribution is a mixture, can compute moments easily

$$\begin{aligned}
\mathsf{E}^A(Z) &= \sum_{m=1}^{M-1} p_m \mathsf{E}^{T_{m+1}}(Z), \\
\mathsf{Var}^A(Z) &= \sum_{m=1}^{M-1} p_m \left(\mathsf{Var}^{T_{m+1}}(Z) + \left(\mathsf{E}^{T_{m+1}}(Z)\right)^2\right) - \left(\mathsf{E}^A(Z)\right)^2.
\end{aligned}$$

- Turns out good enough.
- Refinement – approximate with a mixture but fewer terms (2 or 3). Match some skew/kurtosis measure. Straightforward but cumbersome.

20 Quadratic Approximation for the Swap Rate

- Quadratic model. Short rate quadratic in Z. Naturally: swap rate quadratic in Z (all at time T)

$$S(Z) \approx Z^\top \Gamma_S Z + b_S^\top Z + a_S.$$

- Find Γ_S, b_S by numerical approximation to $S(Z)$ around $Z = 0$ (or $Z = \mathsf{E}^A Z(T)$)
- Find a_S from no-arbitrage (major advantage of using swap measure):

$$a_S = S(0) - \mathsf{E}^A \left(Z^\top \Gamma_S Z + b_S^\top Z \right).$$

- Curvature ($\Gamma_S \neq 0$) a function of two sources:
  - Non-linearity of S wrt factors Z
  - Quadratic terms in the model
- Even in the linear model, S would be approximated by a quadratic function, ie a “better” approximation than just linearize S (which would be a poor approximation)

21 2d Quadratic Approximation for the Swap Rate

- Still $N+1$ dimensional problem. Quadratic form for the short rate is rank-2
- In other words, the short rate is a function of two aggregated variables,

$$r(t) = \left(1 + \varepsilon\left(\rho\left[b(t)^\top Z(t)\right] + \bar{\rho} Z_{N+1}(t)\right)\right) \times \left[b(t)^\top Z(t)\right],$$

  rather than $N+1$.
- So, swap rate is approximately rank 2?
- Maybe, maybe not, but approximate with rank 2.
- Use two variables, $\hat{Z} = \left(\hat{Z}_1, \hat{Z}_2\right)^\top$

$$\hat{Z}_1 = b_S^\top Z(T), \quad \hat{Z}_2 = Z_{N+1}(T),$$

  or, formally

$$\begin{pmatrix} \hat{Z}_1 \\ \hat{Z}_2 \end{pmatrix} = RZ(T), \quad R = \begin{pmatrix} b_{S,1} & \dots & b_{S,N} & b_{S,N+1} \\ 0 & \dots & 0 & 1 \end{pmatrix}.$$

- Find 2×2 matrix $\hat{\Gamma}_S$ such that (eg using Frobenius norm)

$$\left\| \Gamma_S - R^\top \hat{\Gamma}_S R \right\|^2 \to \min .$$

- Then

$$S(Z) \approx \hat{Z}^\top \hat{\Gamma}_S \hat{Z} + \hat{Z}_1 + \hat{a}_S,$$

where

$$\begin{aligned} \mathsf{E}^A \hat{Z} &= R \mathsf{E}^A Z(T), \\ \mathsf{Var}^A \hat{Z} &= R \left(\mathsf{Var}^A Z(T) \right) R^\top, \\ \hat{a}_S &= S(0) - \mathsf{E}^A \left(\hat{Z}^\top \hat{\Gamma}_S \hat{Z} + \hat{Z}_1 \right). \end{aligned}$$

- It turns out that he approximation is good, so the model more or less preserves the rank-2 form for swap rates, and we can continue to interpret $Z_{N+1}(\cdot)$ is the SV factor.

23 Recap

- We have

$$S = \hat{Z}^\top \hat{\Gamma}_S \hat{Z} + \hat{Z}_1 + \hat{a}_S$$

- $\hat{Z}$ is a 2d, (approximately) Gaussian random variable with known moments.
- Option price – very easy. Condition on one variable, obtain analytic 1d formula, integrate using Gauss-Hermite.
- If the distribution is mixture of Gaussians, still easy – for K terms in the mixture apply the formula K times.
- Using different toolset – integration vs stochastic calculus

24 Fourier Methods?

- Possible:

$$\mathsf{E}\left(S_0 + Z^\top QZ + u^\top Z - \mathsf{E}\left(Z^\top QZ + u^\top Z\right) - K\right)^+ = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{(\lambda - i\omega)(S_0 - K)}}{(\lambda - i\omega)^2} F(\lambda - i\omega)\, d\omega,$$

where

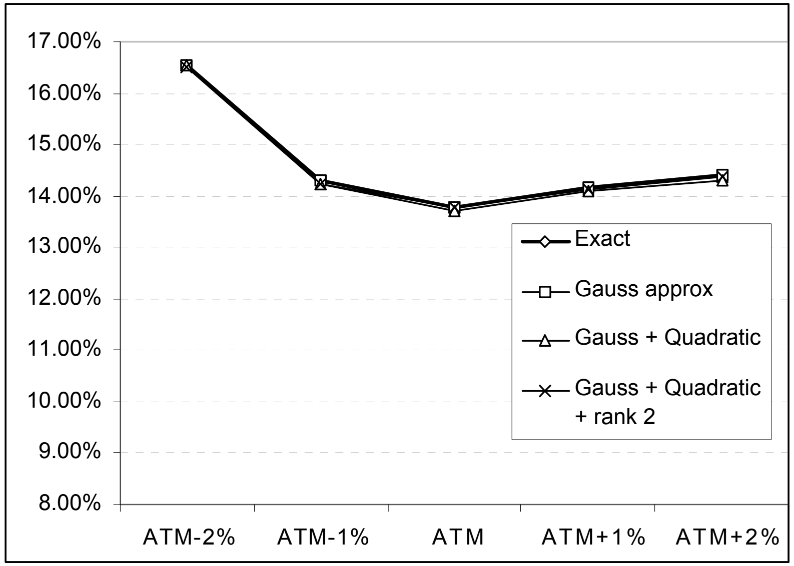
$$\log F(\xi) = \frac{1}{2}\xi^2 \left(2m^\top Q + u^\top\right)\left(V^{-1} - 2\xi Q\right)^{-1}(2Qm + u) - \frac{1}{2}\log\det(I - 2\xi QV) - \xi \mathrm{tr}(QV)$$

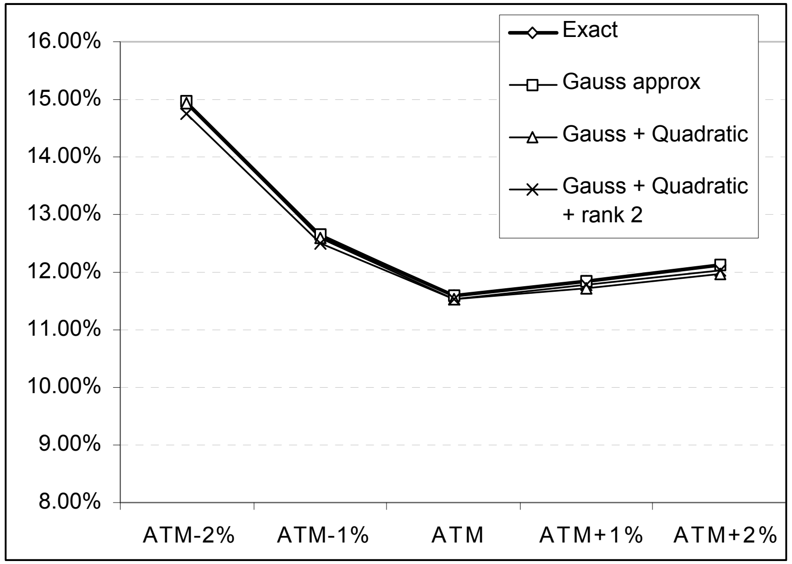
for $Z \sim \mathcal{N}(m, V)$.

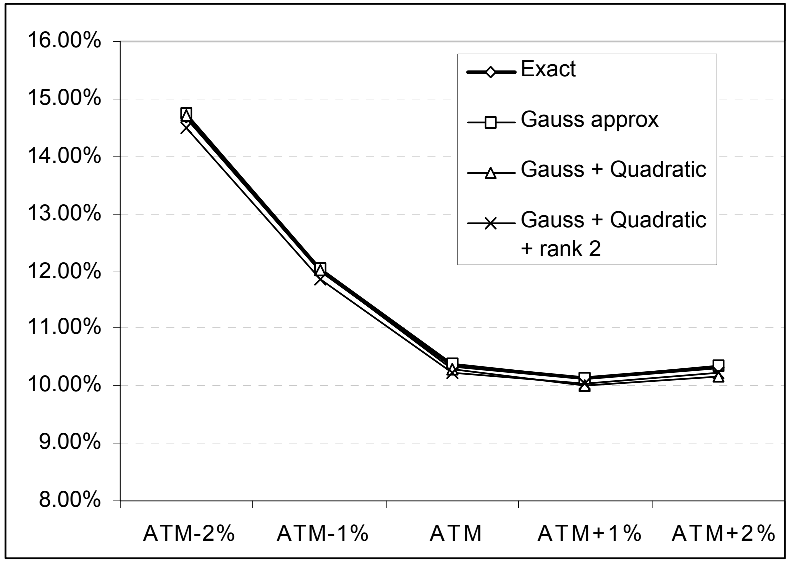
- Can use after approximating a swap rate with $(N+1)$ quadratic form.
- For mixtures of K Gaussians, F is a weighted sum of terms of this type.
- Expensive? Each value of F requires a linear system solution and a determinant (need $K \times N_\omega$ values)
- No integration issues unlike some claims, eg [BL07]

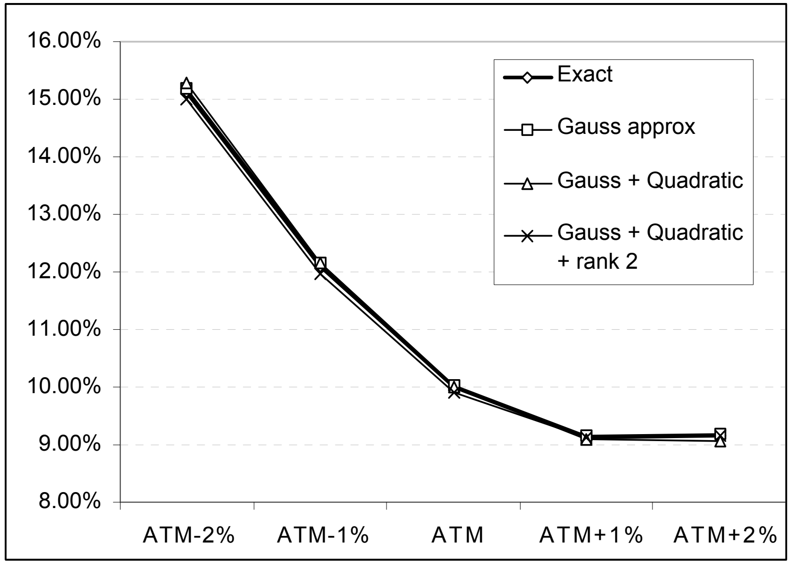
25 Quality of Approximations

- 5x5, 10x10, 15x15, 20x20 swaptions









- Calibration sequenced forward in expiry time: k-th calibration to match k-th row of swaptions by changing $\lambda_n(t_k)$, $n = 1, \ldots, N$.
- Instead of one large optimization – many small ones
- Linear case – one pass (swaption prices with expiry T depend on volatility parameters to T only)
- Quadratic case – some tail dependence (through bond reconstruction formulas)
- Dependence minor – can use a multi-pass bootstrap calibration
- For initial pass(es) use fast approximations, for the last one use more accurate one.

27 SV or not SV?

- Is it a stochastic volatility model?
- Not in the strict sense. If there are $N+1$ swap rates, then the equations $S_i = S(t, Z(t))$, $i = 1, \ldots, N+1$, could(in theory!) be solved to express all factors Z in terms of S, giving us

$$d\vec{S}(t) = f\left(t, \vec{S}(t)\right) dW(t),$$

  ie a multi-dimensional local volatility model.
- However, the dependence of S_i's on Z_{N+1} is relatively limited (mostly through the cross products in the quadratic term), so it is a "proxy" for the real SV
- Also, when looking at products that depend on N rates or less, there is extra randomness in the diffusion coefficient not captured by the "relevant" rates, also indicating SV behavior

28 Volatility smile dynamics

- More important question – what is the volatility smile dynamics in the model.
  - "SV" like when the vol smile moves with spot; or
  - Local vol like, where it stays fixed?
- This defines the impact on exotics (that depend on forward volatility or forward smile dynamics)
- Some evidence that it is somewhere in between, but more to do (watch this space)

29 Comparison to SV-FLM and Cheyette

All models with N non-SV Brownian motions

	FLM	Cheyette	QG
Non-SV state variables	160	$N + N(N+1)/2$	N
Large MC steps	no	no	yes
Simulation speed	**slow**	**fast**	**fastest**
Calibrate on params/values	params	params	values
Smile flexibility	good	good	average
Calibration speed	**fast**	**fastest**	**slower**
True SV	yes	yes	no
Dynamics understood	**well**	**average**	**poorly**

References

[AD99] D. Ahn and R. Dittmar. Quadratic term structure models: Theory and evidence, 1999. University of North Carolina working paper.

[And05] Jesper Andreasen. Back to the future. *Risk Magazine*, 18(9):104–109, September 2005.

[Ass07] Samson Assefa. Calibration and pricing in a multi-factor quadratic Gaussian model. Research Paper Series 197, Quantitative Finance Research Centre, University of Technology, Sydney, May 2007.

[BL07] Nina Boyarchenko and Sergei Z. Levendorski. On errors and bias of Fourier transform methods in quadratic term structure models. *International J. Theor. Appl. Finan.*, 10:273–306, 2007.

[CFP03] L. Chen, D. Filipović, and H. Vincent Poor. Quadratic term structure models for risk-free and defaultible rates. *Math. Finan.*, 14(4):515–536, 2003.

[Elk07] Simon Elkouby. Revisiting short rate models. Barclays Capital working paper, June 2007.

[Tez05] Christophe Tezier. Short rate models. Linear and quadratic Gaussian models. Barclays Capital working paper, September 2005.