Implied dynamics of the swaption skew surface

Fixed Income Conference 2004
Prague, Czech Republic

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1 Plan

- Build a model that incorporates a rich skew surface
- Calibrate it to market
- Compare the implied versus actual dynamics of the skew surface
- Draw conclusions

2 Interest rate vanilla options

- A swaption grid indexed by expiry/maturity
- Each “grid point” traded in a range of strikes: implied volatility smile
- Stochastic volatility model applied to each expiry/maturity
- Consistency with swaptions required for exotics pricing
- For exotics traders – a collection of SV parameters for the whole swaption grid is “the market”. Need to calibrate to that

3 Notations

- Tenor structure

$$\begin{aligned} 0 &= T_0 < T_1 < \ldots < T_N, \\ \tau_n &= T_n - T_{n-1}. \end{aligned}$$

- Zero coupon bonds $P(t, T)$
- Swap rate $S_{n,m}(t)$ for a swap with a first fixing date T_n and the last payment date T_m,

$$S_{n,m}(t) = \frac{P(t, T_n) - P(t, T_m)}{\sum_{i=n+1}^{m} \tau_i P(t, T_i)}.$$

4 Simple SV model for swaptions

- Use $S = S_{n,m}$. Dynamics given by the SV model (see e.g. [AA02])

$$\begin{aligned} dz(t) &= \theta (z_0 - z(t))\, dt + \eta \sqrt{z(t)}\, dV(t), \\ dS(t) &= \lambda (bS(t) + (1-b) S(0)) \sqrt{z(t)}\, dW(t), \\ z(0) &= z_0, \quad \langle dV, dW \rangle = 0. \end{aligned}$$

- The process $z(t)$ is the stochastic variance. Usually choose $z_0 = 1$
- The smile is parametrized by the SV volatility λ, the skew parameter b and the volatility of variance parameter η
- The mean reversion of variance θ appears superfluous at this point, but will be important later
- Effect of SV parameters on the smile (in Black volatility terms)
  - λ moves the smile up and down
  - b tilts it left or right
  - η makes it more convex
- The swaption grid, “the market” $\left\{ \left(\lambda_{n,m}, b_{n.m}, \eta_{n,m} \right) \right\}_{n,m=1}^{N}$

5 Forward Libor model with Stochastic Volatility

- The BGM model has been extended with Stochastic Volatility (see [ABR01])
- Define Libor rates

$$L_n(t) \triangleq L(t, T_n, T_{n+1}) = \frac{P(t, T_n) - P(t, T_{n+1})}{\tau_{n+1} P(t, T_{n+1})},$$

- SV process $z(t)$

$$dz(t) = \theta(z_0 - z(t))\, dt + \eta \sqrt{z(t)}\, dV(t), \tag{1}$$

- FL-SV model $(n = 1, \ldots, N-1)$

$$dL_n(t) = (\beta L_n(t) + (1-\beta) L_n(0)) \sqrt{z(t)} \sum_{k=1}^{K} \sigma_k(t; n)\, dW_k^{n+1}(t),$$

- Good enough?
  - Almost: $\{\sigma_k(t; n)\}$ can be chosen to match the volatility grid $\{(\lambda_{n,m})\}_{n,m=1}^{N}$
  - But β and η are **universal**, i.e. the smile grid $\left\{\left(b_{n.m}, \eta_{n,m}\right)\right\}_{n,m=1}^{N}$ can only be calibrated as best-fit; not sufficient
  - Match swaption smiles and miss caplets's or vice versa?

6 Smile curvatures are not a problem I

- Turns out $\left\{\left(\eta_{n,m}\right)\right\}_{n,m=1}^{N}$ has simple structure. Can be well-matched with a universal parameter η and (important!) θ in our model

$$dz\left(t\right)=\theta\left(z_0-z\left(t\right)\right)\,dt+\eta\sqrt{z\left(t\right)}\,dV\left(t\right),$$

- Curvature of the smile is proportional to the total variance

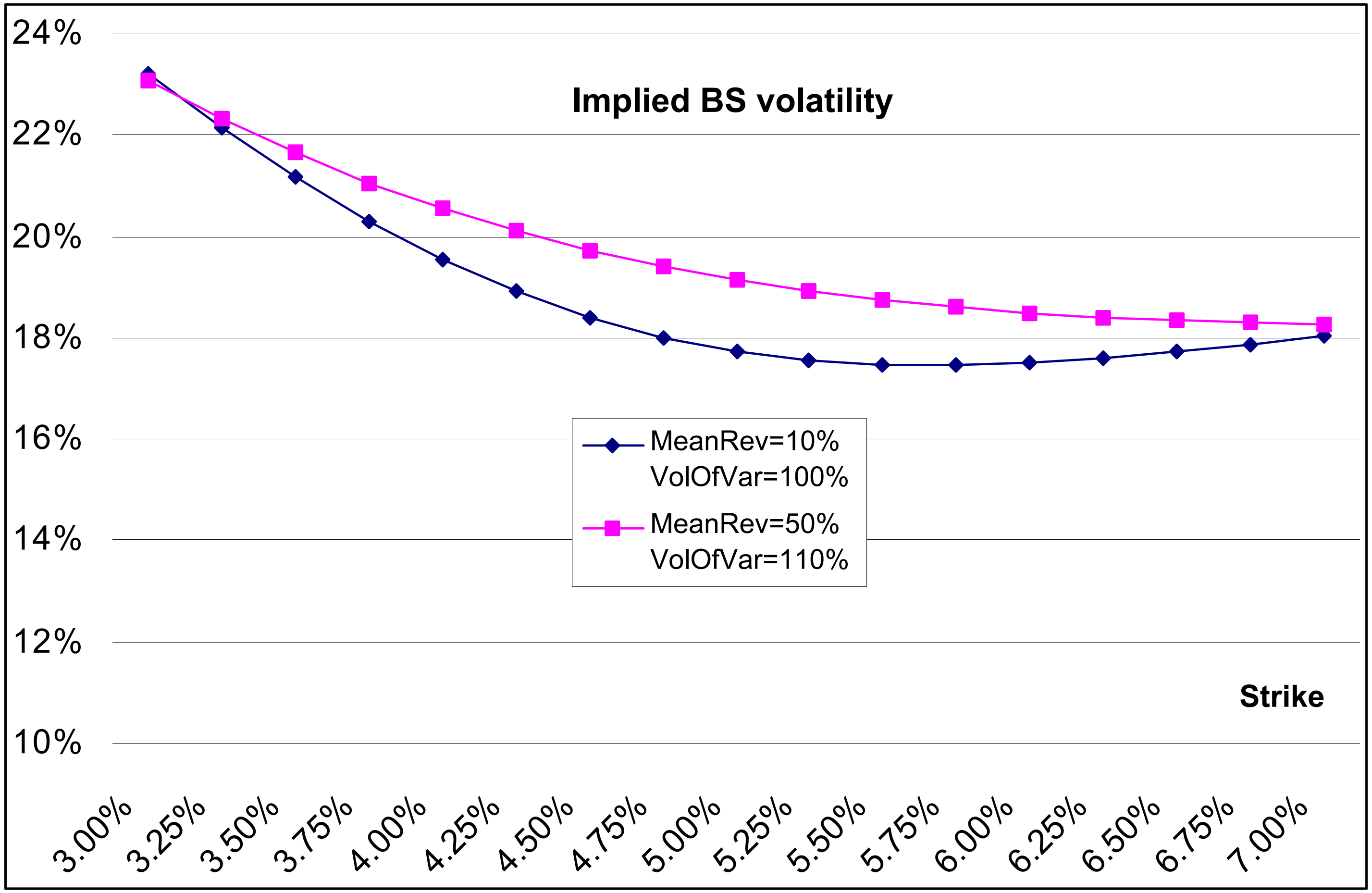
$$c\left(T;\eta,\theta\right)=\mathbf{Var}\int_0^T z\left(t\right)\,dt,$$

  where T is the option's expiry

- Using θ and η we can control relative curvature of short vs. long dated smiles
- (see Figure)

7 Smile curvatures are not a problem II

- For two models, volatility of variance η and mean reversion of variance θ were chosen to produce the same smile for 1 year expiry. The Figure shows the smiles these two models produce in for 5y expiry. Higher mean reversion makes for a smile that flattens out faster.



8 Extending the model with term structure of smiles

- The same blending β for all swaptions is a problem.
  - Skews of caplets are different than those of swaptions
  - Skews of long-dated swaptions are different from that of short-dated ones
- Make them time-dependent and Libor-specific, $\{(\beta(t;n))\}$
- FL-TSS model (see [Pit04] for details)

$$\begin{aligned} dL_n(t) &= (\beta(t;n)\, L_n(t) + (1-\beta(t;n))\, L_n(0)) \sqrt{z(t)} \\ &\quad \times \sum_{k=1}^{K} \sigma_k(t;n)\; dW_k^{n+1}(t), \\ n &= 1, \ldots, N-1. \end{aligned}$$

- Monte-Carlo pricing – trivial extension. Calibration is the problem

9 Calibration

- Market information (term parameters of swap rates): $\{(\lambda_{n,m}, b_{n.m})\}_{n,m=1}^{N}$.
- Model parameters: (instantaneous parameters of Libor rates) $\{(\sigma_k(t,n), \beta(t;n))\}$.
- How to obtain the latter from the former? The plan
  - Derive the dynamics of a swap rate in the model
  - Derive instantaneous parameters of swap rates in the model from those of Libor rates
  - Relate instantaneous time-dependent parameters of each swap rate to its term (market) parameters
  - Design an efficient algorithm for solving for the former from the latter
- Effective volatility/skew technique (parameter averaging):
  - Direct link between model parameters and market's
  - Do not compute European option prices or invert then – very important for calibration speed!

10 Approximating swap rate dynamics in the model I

- In virtually all BGM-type models, one starts by showing that swap rates follow SDEs of the same form as Libor rates
- Same here: swap rate $S_{n,m}(t)$ under the swap measure

$$\begin{aligned} dS_{n,m}(t) &= \left(\beta(t;n,m)\,S_{n,m}(t) + (1-\beta(t;n,m))\,S_{n,m}(0)\right)\sqrt{z(t)} \\ &\quad \times \sum_{k=1}^{K} \sigma_k(t;n,m)\, dW_k^{n,m}(t), \end{aligned}$$

- Here

$$\begin{aligned} \sigma_k(t;n,m) &= \sum_{i=n}^{m-1} q_i(n,m)\,\sigma_k(t;i), \\ \beta(t;n,m) &= \sum_{i=n}^{m-1} p_i(n,m)\,\beta(t;i), \end{aligned}$$

- Obtained by matching local volatility level and slope for swap rate SDE
- Very important – parameters of each swap rate $\sigma_k(t;n,m)$, $\beta(t;n,m)$ are linked directly to those of Libor rates.

11 Relating time-dependent volatility and skew to constant ones

- Both swap rates and Libor rates follow the same SDE. Let $S(t)$ be either, with T being the expiry
- Dropping unneeded subscripts we have
  - In the model

$$dS(t) = \sigma(t)\left(\beta(t) S(t) + (1-\beta(t)) S(0)\right) \sqrt{z(t)}\, dU(t),$$

  - In the market

$$d\bar{S}(t) = \lambda\left(b\bar{S}(t) + (1-b)\bar{S}(0)\right) \sqrt{z(t)}\, dU(t).$$

- We want to relate $\{\sigma(t), \beta(t)\}_{t=0}^{T}$ to λ, b
- Easy in no-SV ($\eta = 0$), constant skew case($\beta(t) \equiv b$):

$$\lambda^2 = \frac{1}{T}\int_0^T \sigma^2(t)\, dt.$$

- Looking for formulas of this type in the general case.

12 Skew homogenization in the small-slope limit I

- As a tool to relate $\{\beta(t)\}_{t=0}^{T}$ to b we use a method of small slope expansion
- Let $g(t,x)$ be a time-dependent, and $\bar{g}(x)$ a time-independent local volatility functions, assuming without loss of generality that

$$g(t,x_0) \equiv 1, \quad \bar{g}(x_0) = 1, \quad t \in [0,T],$$

- Define

$$\begin{aligned} g_\varepsilon(t,x) &= g(t, x_0 + (x - x_0)\varepsilon), \\ \bar{g}_\varepsilon(x) &= \bar{g}(x_0 + (x - x_0)\varepsilon), \end{aligned}$$

- Define two families of diffusions indexed by ε,

$$\begin{aligned} dX_\varepsilon(t) &= g_\varepsilon(t, X_\varepsilon(t))\sqrt{z(t)}\sigma(t)\, dW(t), \\ dY_\varepsilon(t) &= \bar{g}_\varepsilon(Y_\varepsilon(t))\sqrt{z(t)}\sigma(t)\, dW(t), \\ X_\varepsilon(0) &= x_0, \quad Y_\varepsilon(0) = x_0. \end{aligned}$$

- Define

$$q(\varepsilon) = \mathbf{E}\left(X_\varepsilon(T) - Y_\varepsilon(T)\right)^2.$$

- We look for conditions on $\bar{g}(\cdot)$ that minimize $q(\varepsilon)$ for small ε
- The condition ensures that options will all strikes are recovered as best as

13 Skew homogenization in the small-slope limit II

- The main result: Any function $\bar{g}$ that minimizes $q(\varepsilon)$ for small ε satisfies the condition

$$\frac{\partial \bar{g}(x_0)}{\partial x} = \int_0^T \frac{\partial g(t, x_0)}{\partial x} w(t)\, dt,$$

where

$$\begin{aligned} w(t) &= \frac{v^2(t)\,\sigma^2(t)}{\int_0^T v^2(t)\,\sigma^2(t)\, dt}, \\ v^2(t) &= \mathbf{E}\left(z(t)\,(X_0(t) - x_0)^2\right). \end{aligned}$$

- Comments:
  - "Total skew" $\frac{\partial \bar{g}(x_0)}{\partial x}$ is the average of "local skews" $\frac{\partial g(t,x_0)}{\partial x}$ with weights $w(t)$
  - Weights proportional to total variance, i.e. local slope further away matters more
  - Can get the same result under different criteria, i.e. robust

14 Proof of skew homogenization I

- Idea of the proof:
  - For small ε,write

$$q(\varepsilon) = q(0) + q'(0)\,\varepsilon + \frac{1}{2}q''(0)\,\varepsilon^2$$

  - Shown that

$$q(0) = q'(0) = 0$$

  and compute

$$q''(0) = \int_0^T v^2(t)\,\sigma^2(t)\left(\frac{\partial g}{\partial x}(t, x_0) - \frac{\partial \bar{g}}{\partial x}(x_0)\right)^2 dt$$

  - Set $q''(0)$ to minimum (a least-squares type problem).

15 Total skew for swaptions

- Apply the general skew homogenization result to the model
- The effective skew b for the equation

$$dS(t) = \sigma(t)\left(\beta(t) S(t) + (1-\beta(t)) S(0)\right) \sqrt{z(t)}\, dU(t)$$

over a time horizon $[0, T]$ is given by

$$b = \int_0^T \beta(t)\, w(t)\, dt,$$

with

$$\begin{aligned} w(t) &= \frac{v^2(t)\,\sigma^2(t)}{\int_0^T v^2(t)\,\sigma^2(t)\, dt}, \\ v^2(t) &= \mathbf{E}\left[\left(X_0(t) - x_0\right)^2 z(t)\right] \\ &= z_0^2 \int_0^t \sigma^2(s)\, ds + z_0 \eta^2 e^{-\theta t} \int_0^t \sigma^2(s) \frac{e^{\theta s} - e^{-\theta s}}{2\theta}\, ds. \end{aligned}$$

- Example: No SV $(\eta = 0)$, constant volatility $\sigma(t) \equiv \sigma$,

$$b = \left(T^2/2\right)^{-1} \int_0^T t\beta(t)\, dt.$$

Volatility homogenization I

- Having averaged the skew, the problem is reduced to a well-known one
- Approximate the dynamics of

$$dS(t) = \sigma(t)\left(bS(t) + (1-b)S(0)\right)\sqrt{z(t)}\,dU(t)$$

with

$$d\bar{S}(t) = \lambda\left(b\bar{S}(t) + (1-b)\bar{S}(0)\right)\sqrt{z(t)}\,dU(t)\,.$$

- Known methods: Lewis [Lew00], Andersen [ABR01], Hagan [SKSE02], Zhou [Zho03]
- We propose our own method – simple, intuitive and accurate.
- Idea: approximate a European option payoff locally with a function whose expectation can be computed in both models above; choose λ to match the two.

17 Volatility homogenization II

- Recall

$$\mathbf{E}\left(S\left(T\right)-S_0\right)^+=\mathbf{E}\left(\mathbf{E}\left(\left(S\left(T\right)-S_0\right)^+\middle|\, z\left(\cdot\right)\right)\right). \tag{2}$$

- The distribution of $S\left(T\right)$ conditioned on a particular path $\left\{z\left(t\right)\right\}_{t=0}^{T}$ is a shifted lognormal.
- The inside condition expectation in (2) can be evaluated easily to yield

$$\mathbf{E}\left(S\left(T\right)-S_0\right)^+=\mathbf{E}g\left(\int_0^T\sigma^2\left(t\right)z\left(t\right)\,dt\right),$$

where g is a known function (ATM Black price as a function of variance),

$$\begin{aligned} g\left(x\right) &= \frac{S_0}{b}\left(2\Phi\left(b\sqrt{x}/2\right)-1\right), \\ \Phi\left(y\right) &= \mathbf{P}\left(\xi<y\right), \quad \xi\sim\mathcal{N}\left(0,1\right). \end{aligned}$$

- The problem of finding the "effective" variance can then be represented as finding such λ that

$$\mathbf{E}g\left(\int_0^T\sigma^2\left(t\right)z\left(t\right)\,dt\right)=\mathbf{E}g\left(\lambda^2\int_0^T z\left(t\right)\,dt\right).$$

18 Volatility homogenization III

- Moment-generating function in both models can be computed (see next slide), so approximate g with an exponential

$$g(x) \approx a + be^{-cx}$$

by matching the value and first two derivatives at

$$\zeta = \mathbf{E} \int_0^T \sigma^2(t)\, z(t)\, dt$$

- The problem reduced to finding λ such that

$$\mathbf{E} \exp\left(\frac{g''(\zeta)}{g'(\zeta)} \int_0^T \sigma^2(t)\, z(t)\, dt \right) = \mathbf{E} \exp\left(\lambda^2 \frac{g''(\zeta)}{g'(\zeta)} \int_0^T z(t)\, dt \right).$$

- Very fast and easy numerical search for λ (starting with a good initial guess $\lambda^2 = T^{-1} \int_0^T \sigma^2(t)\, dt$).

19 Moment-generating functions

Denote $Z(T) = \int_0^T \sigma^2(t)\, z(t)\, dt$ and define

$$\varphi(\mu) \triangleq \mathbf{E} \exp(-\mu Z(T)), \quad \varphi_0(\mu) \triangleq \mathbf{E} \exp\left(-\mu \int_0^T z(t)\, dt\right),$$

The function $\varphi(\mu)$ can be represented as $\varphi(\mu) = \exp(A(0,T) - z_0 B(0,T))$,where the functions $A(t,T)$, $B(t,T)$ satisfy the Riccati system of ODEs

$$\begin{aligned} A'(t,T) - \theta z_0 B(t,T) &= 0, \\ B'(t,T) - \theta B(t,T) - \frac{1}{2}\eta^2 B^2(t,T) + \mu\sigma^2(t) &= 0, \end{aligned}$$

with terminal conditions $A(T,T) = 0$, $B(T,T) = 0$. The system of ODEs is trivial to solve numerically.

The function $\varphi_0(\mu)$ satisfies the same system of equations with $\sigma(t) \equiv 1$. In this case, the equations can be solved explicitly, to yield

$$\begin{aligned} B_0(0,T) &= \frac{2\mu\left(1 - e^{-\gamma T}\right)}{(\theta+\gamma)\left(1 - e^{-\gamma T}\right) + 2\gamma e^{-\gamma T}}, \\ A_0(0,T) &= \frac{2\theta z_0}{\eta^2} \log\left(\frac{2\gamma}{\theta + \gamma\left(1 - e^{-\gamma T}\right) + 2\gamma e^{-\gamma T}}\right) - 2\theta z_0 \frac{\mu}{\theta+\gamma} T, \\ \gamma &= \sqrt{\theta^2 + 2\eta^2\mu}. \end{aligned}$$

20 Volatility and skew calibration – summary

- The formulas developed above:

$$\begin{aligned}\lambda_{n,m} &= \lambda_{n,m}\left(\{\sigma(\cdot;\cdot)\},\{\beta(\cdot;\cdot)\}\right),\\ b_{n,m} &= b_{n,m}\left(\{\sigma(\cdot;\cdot)\},\{\beta(\cdot;\cdot)\}\right).\end{aligned}$$

- Let $\left\{\left(\lambda^*_{n,m}, b^*_{n,m}\right)\right\}_{n,m}$ be the market-impled parameters. Then calibration problem can be expressed as a problem of finding $\{\sigma(\cdot;\cdot)\},\{\beta(\cdot;\cdot)\}$ such that

$$\begin{aligned}\sum_{n,m}\left(b_{n,m}-b^*_{n,m}\right)^2 + \text{smoothness penalty} &\rightarrow \min,\\ \sum_{n,m}\left(\lambda_{n,m}-\lambda^*_{n,m}\right)^2 + \text{smoothness penalty} &\rightarrow \min.\end{aligned}$$

- It is better to separate volatility calibration from skew calibration.
- What to do about joint dependence?

21 Volatility and skew calibration – the algorithm

- Because of "weak" coupling between $\{\sigma(\cdot;\cdot)\}$ and $\{\beta(\cdot;\cdot)\}$, can use "relaxation"-type fit:

Step 1 Set $\{\beta(\cdot;\cdot)\}$ to the same value $\bar{\beta}$ (simple average of $b_{n,m}$ over the whole swaption grid). Calibrate $\{\sigma(\cdot;\cdot)\}$ to $\{\lambda_{n,m}\}$

Step 2 Use $\{\sigma(\cdot;\cdot)\}$ obtained on the previous step, calibrate $\{\beta(\cdot;\cdot)\}$ to $\{b_{n,m}\}$

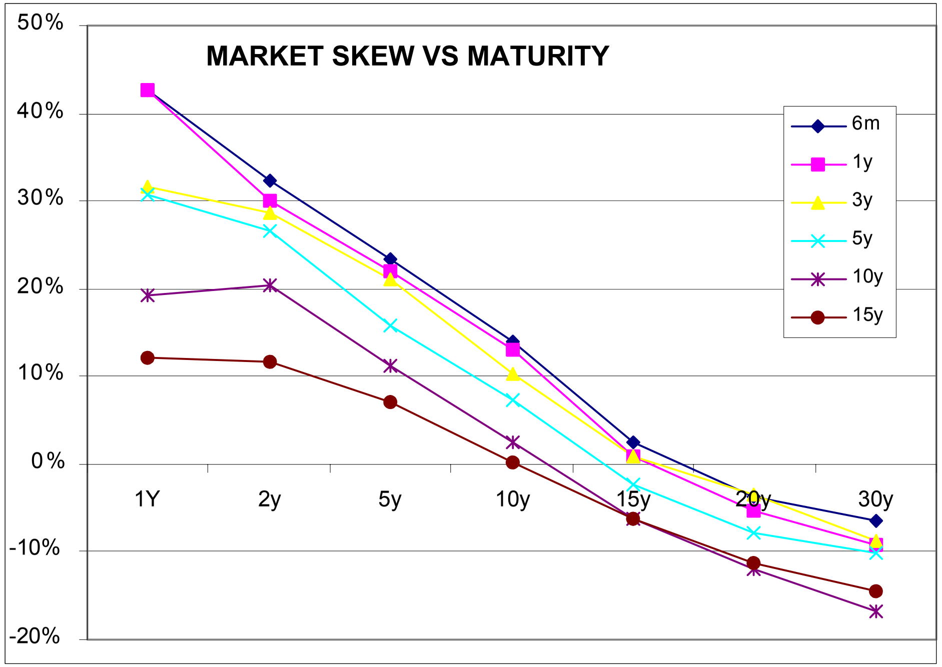
Step 3 (Optional) Re-calibrate $\{\sigma(\cdot;\cdot)\}$ to $\{\lambda_{n,m}\}$ with $\{\beta(\cdot;\cdot)\}$ from the previous step

Test results: Typical market skew (table)

- Implied market skew λ_{nm}, parametrized by the time to first expiry of the swaption (columns) and the length of the swap (rows)

Market skew	**1y**	**2y**	**5y**	**10y**	**15y**	**20y**	**30y**
6m	43%	32%	23%	14%	2%	-4%	-6%
1y	43%	30%	22%	13%	1%	-5%	-9%
3y	32%	29%	21%	10%	1%	-3%	-9%
5y	31%	27%	16%	7%	-2%	-8%	-10%
10y	19%	20%	11%	3%	-6%	-12%	-17%
15y	12%	12%	7%	0%	-6%	-11%	-15%

23 Test results: Typical market skew(figures)

- 

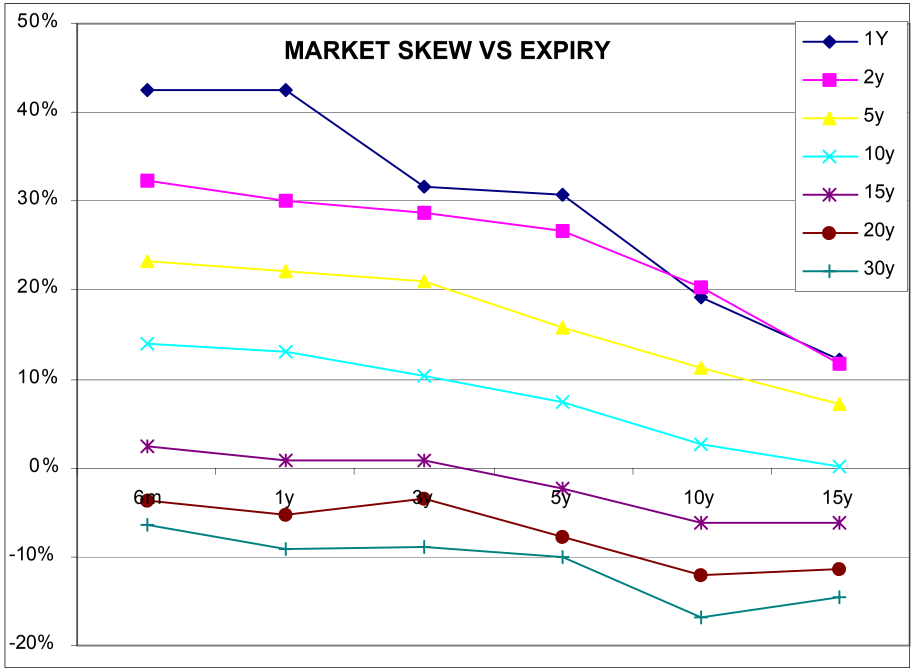


Figure 1:

Test results: Typical calibrated model skew (table)

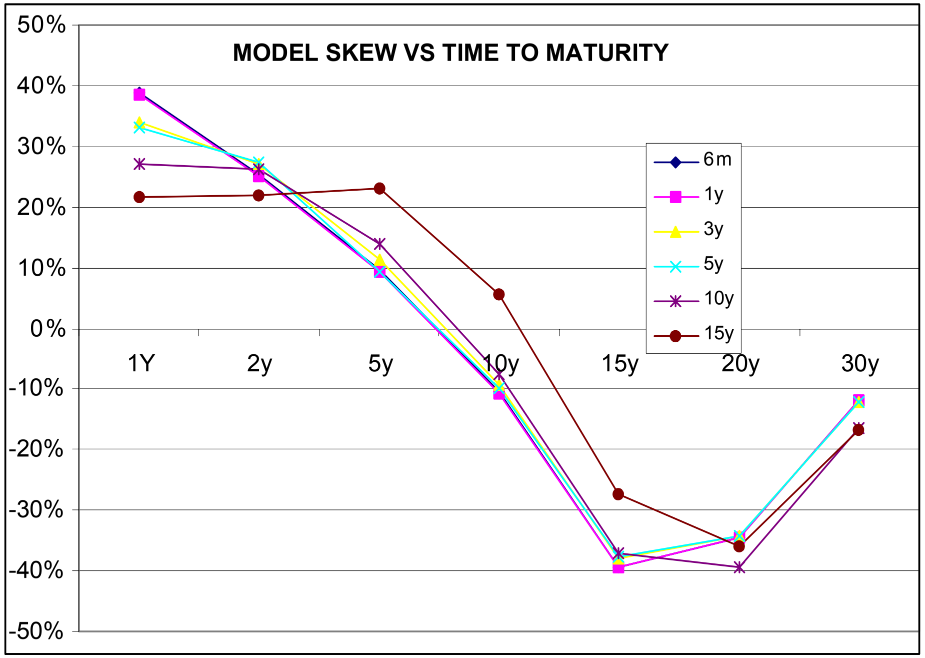
- Calibrated model skew, parametrized by

$$\beta(t;n) = f(t, T_n - t)$$

model skew	**1Y**	**2y**	**5y**	**10y**	**15y**	**20y**	**30y**
6m	39%	25%	10%	-10%	-39%	-34%	-12%
1y	39%	25%	9%	-11%	-39%	-35%	-12%
3y	34%	27%	11%	-9%	-38%	-34%	-12%
5y	33%	27%	9%	-10%	-38%	-34%	-12%
10y	27%	26%	14%	-8%	-37%	-39%	-16%
15y	22%	22%	23%	6%	-27%	-36%	-17%

25 Test results: Typical calibrated model skew (figures)

Note almost perfect time-homogeneity of skew – the second graph shows that $\beta(t;n)$ is a function of $T_n - t$ only!



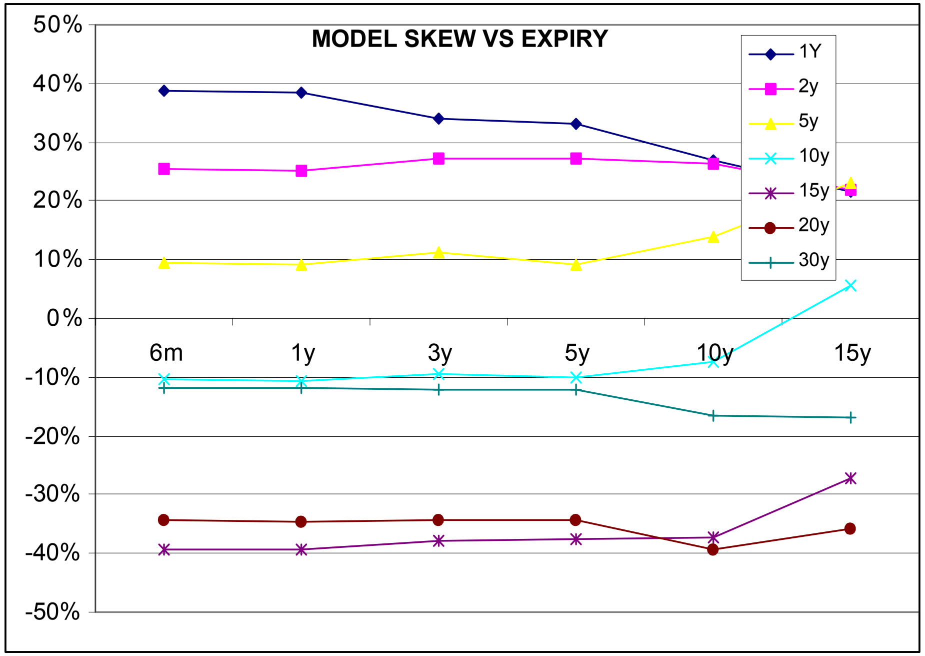
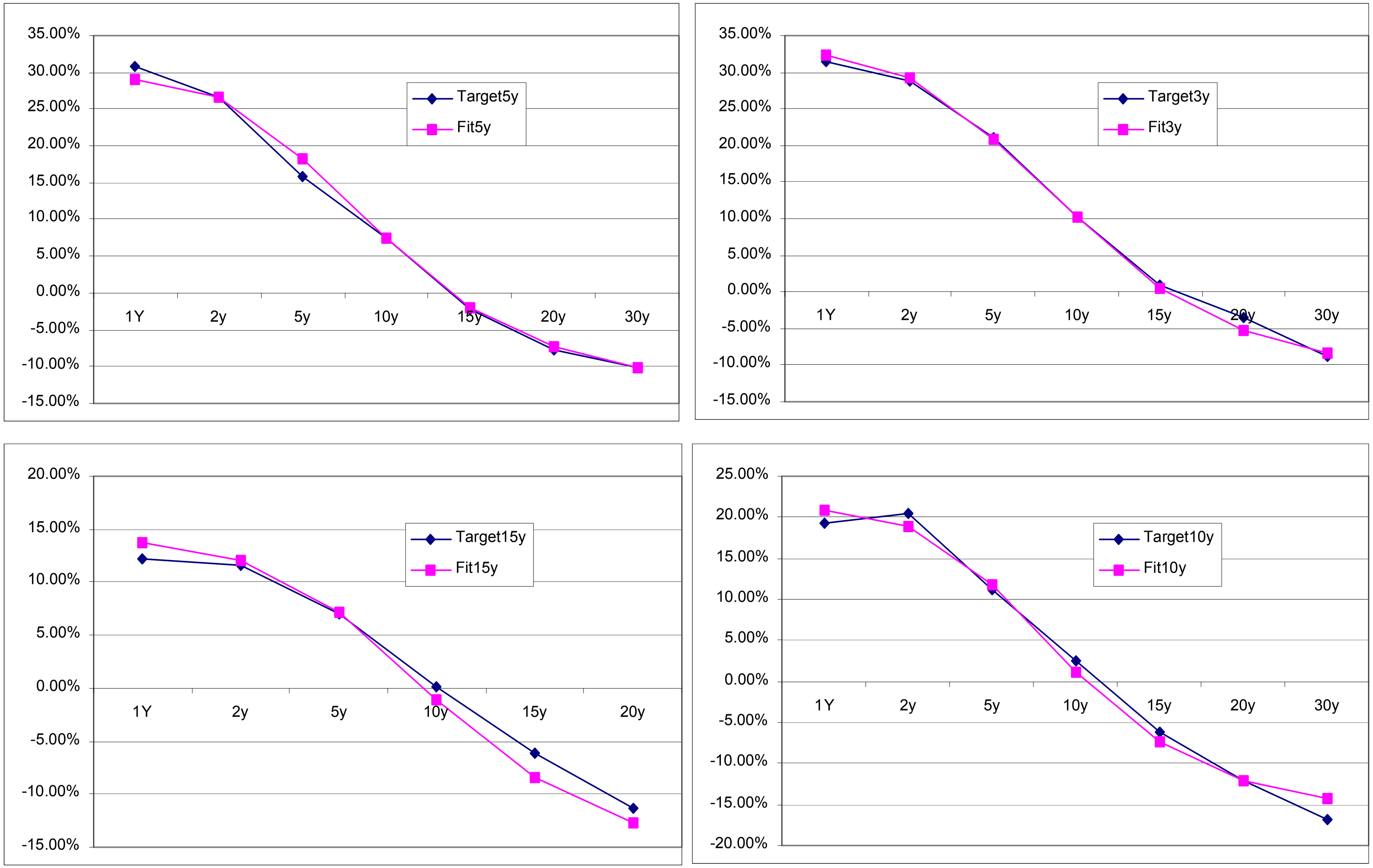


Figure 2:

26 Test results III: Typical skew calibration fit

These figures show the quality of skew fit by the model. “TargetXX” are implied from the market,“FitXX” are given by the (calibrated) model, where XX is the expiry.



27 Smile dynamics I

- The “implied” forward skew is time-homogeneous. A persistent feature
- Interesting to see whether the actual evolution of the forward skew is consistent with its “implied” dynamics
- For each date, compute forward skews, compare through time
- Since for each date, skews for different expiries match closely (implied skew is time-homogeneous), we plot 6m expiry skew (into X years) for different dates

28 Smile dynamics II

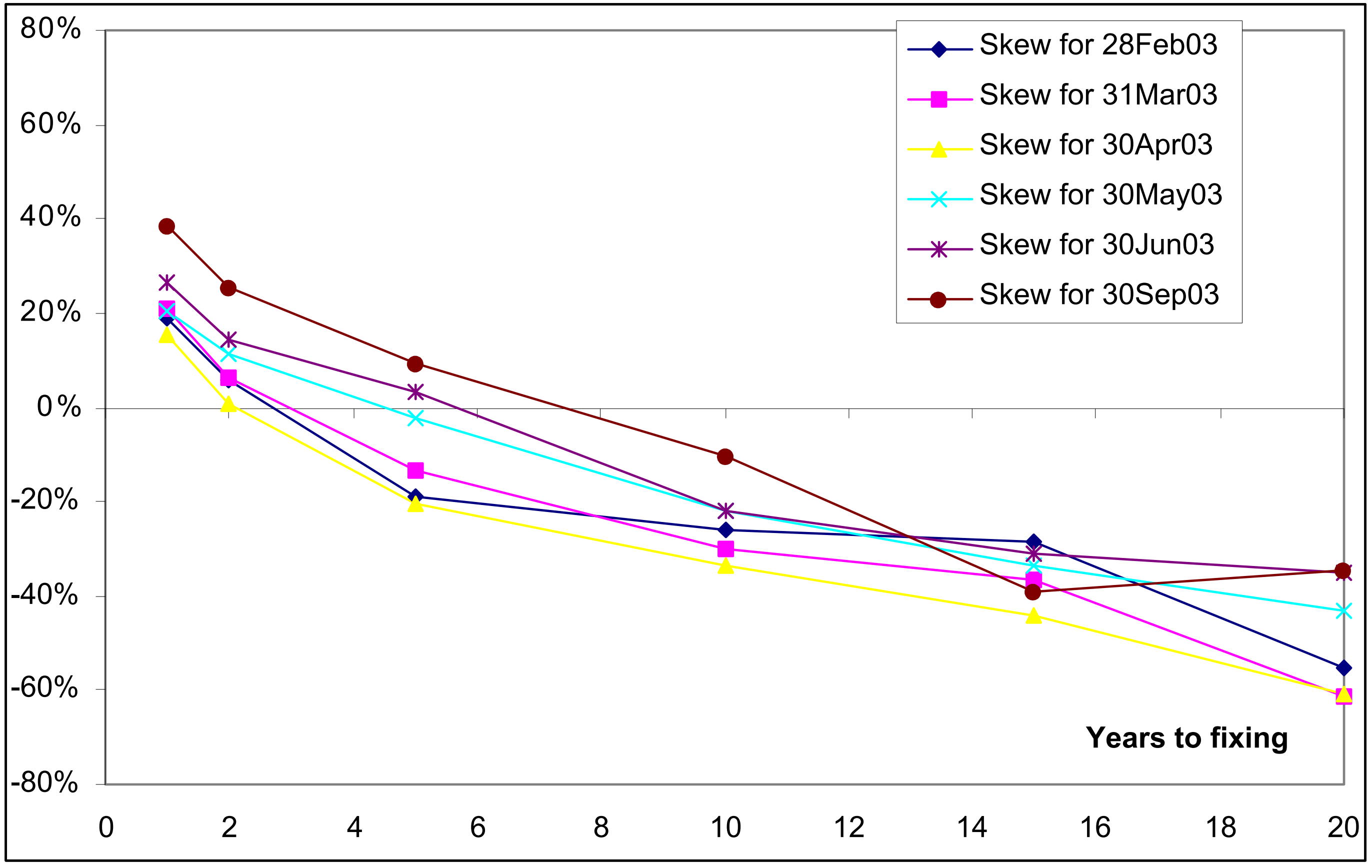


Figure 3:

29 Smile dynamics III

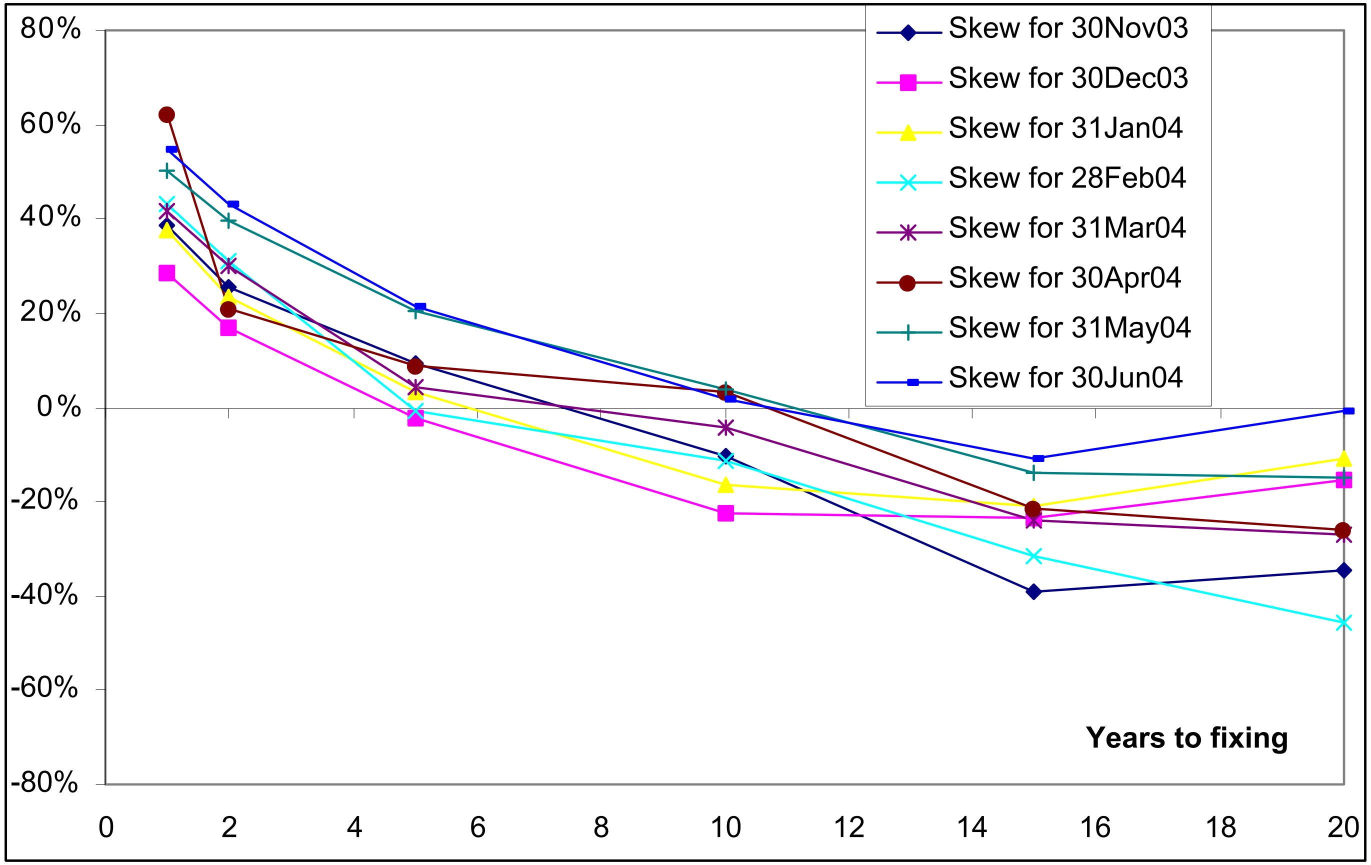


Figure 4:

Smile dynamics IV

- The shape of the skew versus $(T_n - t)$ stays remarkably consistent through time
- The level changes, but not by much – about 20%-30% over 1 year (not a lot for skew)
- Our model captures the effect of persistent skew shape (vs $(T_n - t)$) but not level change
- A better model? Perhaps

$$
\begin{aligned}
dL_n(t) &= \left(z_1(t)\,\beta(t;n)\,L_n(t) + (1 - z_1(t)\,\beta(t;n))\,L_n(0)\right)\sqrt{z(t)} \\
&\quad \times \sum_{k=1}^{K} \sigma_k(t;n)\; dW_k^{n+1}(t),
\end{aligned}
$$

where $z_1(t)$ another stochastic driver.

- This extension will capture all skew dynamics that we identified
- Is it worth it?

31 Conclusions

- The classic Stochastic Volatility forward Libor model is extended to account for variability of implied volatility skews across the swaption grid
- The extension allows for much closer recovery of implied volatility smiles across the whole swaption grid, providing a better platform for exotics pricing
- The average skew formula obtained can be of independent interest in other modeling contexts (equity, FX)
- Implied skew is time-homogeneous, a feature persistent through time
- The actual observed skew dynamics seems to roughly match the implied prediction (i.e. time-homogeneity) except for relatively minor skew-level changes

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