Analytical formulas for pricing CMS products in the Libor Market Model with the stochastic volatility

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Abstract

In this paper, we develop a series of approximations for a fast analytical pricing of European constant maturity swap (CMS) products, such as CMS swaps, CMS caps/floors, and CMS spread options, for the LIBOR Market Model (LMM) with stochastic volatility. The derived formulas can also be used for model calibration to the market, including European swaptions and CMS products.

The first technical achievement of this work is related to the optimal calculation of the measure change. For single-rate CMS products, we have used the standard linear regression of the measure change, with optimally calculated coefficients. For the CMS spread options, where the linear procedure does not work, we propose a new effective *non-linear* measure change technique. The fit quality of the new results is confirmed numerically using Monte Carlo simulations.

The second technical advance of the article is a theoretical derivation of the generalized spread option price via two-dimensional Laplace transform presented in a closed form in terms of the complex Gamma-functions.

1 Introduction

A growing amount of CMS product transactions in the modern market has required extended analytical support. Many authors have reflected in their papers different theoretical and practical aspects of CMS product pricing. The main work has been related to pricing the most popular European CMS products, including both single-rate (CMS swap, CMS cap/floor) and double-rate products (CMS spread options).

We distinguish two primary directions in the research. The first one deals with the approximation of the CMS products for a concrete interest rate model. We can mention a work of Benhamou (2000), who approximated CMS swaps for the log-normal zero bond model using the Wiener Chaos technique, and also a more recent paper of Henrard (2007) approximating CMS swaps for Gaussian one-factor LMM (LIBOR Market Model) and HJM (Heath-Jarrow-Morton) models.

The second approach is related to CMS product pricing given market information for a yield curve and swaption implied volatilities. Namely, the market determines a distribution

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of the swap rates in so-called *swap measure*. This fact has been used by many researchers for the pricing approximations. Note, however, that CMS product prices are expressed explicitly via rates in a *forward measure*, different from the swap measure. Thus, dealing with the CMS European contracts in the swap measure has been faced with approximating the measure change (Radon-Nikodym derivative). Hagan (2003) and Pelsser (2000) proposed a linearization of the measure change in terms of the swap rate for the *single-rate* CMS products. This procedure, lacking interest rate model dynamics, was simplified due to heuristic, but numerically effective, considerations. Later, developing the previous ideas, Mercurio and Pallavicini (2006) made progress in static replication for CMS payments/options by European options using a special strike extrapolation.

The pricing of double-rate CMS products (CMS spread options) having more complicated analytics has been less covered by researchers. One can find corresponding references in Berrahoui (2004), where the author deals with the spread option approximation with a smile adjustment using a historical correlation between the rates. However, to our knowledge, no one has treated an accurate approximation of the CMS spread option for a rich interest rate model. This is one goal of our research.

In this paper, we present an effective analytical approximation for the CMS swap, CMS cap/floor, and CMS spread option for the LIBOR market model under stochastic volatility (LMM SV) (see Andersen-Andreasen (2002) and Piterbarg (2005)). The presence of liquid quotes for both European swaptions and CMS products suggests a simultaneous calibration to these instruments for consistent pricing of CMS-dependent exotics. One condition for effective calibration is the possibility of analytical pricing for underlying instruments—this was one motivation for the present work. Our choice of model, LMM SV, was determined by its rich dynamic properties, non-trivial correlation between the rates, and its standard position in the market.

However, during a market crisis, when the CMS products become less liquid and the exotic market falls into recession, the proposed approximation can be used for a fast pricing of the CMS products, given an LMM calibrated to European swaptions. The LMM, having a large parametric freedom, can be, in principal, calibrated to the whole swaption matrix. Thus, European CMS product evaluation with this model can provide a consistent pricing with whole-market information. Moreover, the CMS analytics deliver *arbitrage-free* prices, contrary to known static replication evaluations based on certain strike extrapolation assumptions for European swaptions.

The approximation of the CMS product pricings for the LMM SV involve several steps. First, we represent the forward CMS rates as known shifted-Heston processes using the Markovian projection elaborated by Piterbarg (2006) and Antonov and Misirpashaev (2006). Then, we refine or optimize the linearization technique of the measure change with respect to the standard heuristic methods. The linear approximation coefficients will not only depend on the yield curve but also on the model volatilities.

Whereas the measure-change linearization procedure works fine for the single-rate CMS products (CMS swaps and CMS caplets), it fails for the CMS spread option approximation. We will show that, for the pricing of such products, one needs an *non-linear extension* of the the measure-change process approximation.

After the appropriate measure choice, we will undertake the final step for the CMS spread option pricing, that is, evaluation of the average,

$$E\left[\left(C_1\, e^{y_1} - C_2\, e^{y_2} - K\right)^+\right],$$

where two stochastic variables, y_1 and y_2, have known characteristic functions and C_1, C_2, and K are some positive coefficients. This average is also known as a *generalized spread option* price. For Gaussian y_1 and y_2, there exists a numerical computational procedure involving one-dimensional integration for a general strike K, and a closed form (Margrabe (1978)) for the zero strike (see also a review of Carmona and Durrleman (2003)). However, for our case, where y_1 and y_2 are logarithms of two correlated Heston processes sharing the same stochastic volatility, one knows only their characteristic function, and the generalized spread option computation is more complicated. Dempster and Hong (2006) computed the price in terms of *three-dimensional* integration involving the characteristic function. In this work, we go further and represent the spread option as a *two-dimensional* integral. We have used the technique of a two-dimensional Laplace transform of the function $f(y_1, y_2) = (C_1\, e^{y_1} - C_2\, e^{y_2} - K)^+$, which we calculate explicitly in terms of Gamma functions. This extra dimension reduction in the numerical integration and known properties of the Gamma function permits more effective computation of the spread option.

The same result was developed simultaneously and independently by Hurd and Zhou (2009) using equivalent two-dimensional Fourier transform of the spread option pay-off function.

The text is organized as follows. Section 2 is devoted to the general methodology of the measure-change linearization. A review of the LMM SV model, and the Markovian projection of the swap rates for it, is provided in Section 3. The main results of the research concerning the single-rate CMS products (CMS swaps, CMS caplets) and the double-rate CMS products (the CMS spread options) can be found in Sections 4 and 5, respectively.

2 General methodology

Let $S(\tau)$ be some forward index fixed at time τ. We presume that the time origin is at zero and refer to t as the running time parameter, contrary to the fixed time parameter τ.

Consider a payment of $f(S(\tau))$ at time T for some function $f(x)$. In the forward T-measure associated with a zero bond $P(t,T)$ as numeraire, the present value of the payment is

$$V = P(0,T)E_T[f(S(\tau))]. \tag{1}$$

The above average, $E_T[f(S(\tau))]$, is our main computational goal.

In general, unless the distribution of the index is known in the forward T-measure, switching to S-martingale measure is more attractive from a computational point of view: Driftless processes can be easily approximated, for example, using the Markovian projection (MP) technique.

Suppose that there exists a martingale S-measure for the forward index process $S(t)$ with an expectation operator $E[\cdots]$ associated with numeraire $N(t)$ and having $W(t)$ as Brownian motion. Then, our payment price is

$$V = P(0,T)E_T[f(S(\tau))] = E\left[\frac{f(S(\tau))\, P(\tau,T)}{N(\tau)}\right].$$

Introduce a measure-change process (Radon-Nikodym derivative), martingale in the S-measure,

$$M(t) = \frac{P(t,T)}{N(t)\, P(0,T)}. \tag{2}$$

Then, the price in hand reduces to

$$V = P(0,T)\, E\left[f(S(\tau))\, M(\tau)\right]$$

or, in other words,

$$E_T[f(S(\tau))] = E\left[f(S(\tau))\, M(\tau)\right]. \tag{3}$$

Even, if we know the joint dynamics of $S(t)$ and $M(t)$ in the S-measure, a calculation of the average $E\left[f(S(\tau))\, M(\tau)\right]$ can be computationally tricky as it contains a bi-variate distribution[1]. Instead, one can apply an approximation involving integration over the single stochastic variable $S(\tau)$. Using the tower law of expectations,

$$E\left[f(S(\tau))\, M(\tau)\right] = E\left[f(S(\tau))\, E[M(\tau)|S(\tau)]\right], \tag{4}$$

we approximate the conditional expectation $E[M(\tau)|S(\tau)]$ as a linear function of $S(\tau)$,

$$E[M(\tau)|S(\tau)] \simeq A(\tau) + B(\tau)S(\tau). \tag{5}$$

The coefficients $A(\tau)$ and $B(\tau)$ can be calculated in the optimal way, minimizing

$$\chi^2(A(\tau), B(\tau)) = E\left[(M(\tau) - A(\tau) - B(\tau)S(\tau))^2\right]$$

for any given time t. Setting the derivatives $\frac{\partial \chi^2}{\partial A}$ and $\frac{\partial \chi^2}{\partial B}$ to zero, we obtain a linear system for optimal $A(\tau)$ and $B(\tau)$. Using martingale properties of the index process $S(t)$ and the measure change $M(t)$ under the S-measure, we can easily find the optimal values for $A(\tau)$ and $B(\tau)$,

$$B(\tau) = \frac{E[M(\tau)S(\tau)] - S_0}{Var[S(\tau)]} = \frac{E_T[S(\tau)] - S_0}{Var[S(\tau)]}, \tag{6}$$

$$A(\tau) = 1 - B(\tau)S_0, \tag{7}$$

where we denoted $S(0) = S_0$ and used $M(0) = 1$.

Finally, we approximate the average in the r.h.s. of (3), substituting the linearized conditional expectation (5) into (4) and using (7):

$$\begin{aligned} E_T[f(S(\tau))] &= E\left[f(S(\tau))\, M(\tau)\right] \\ &\simeq E\left[f(S(\tau))\right] + B(\tau)\, E\left[f(S(\tau))\, (S(\tau) - S_0)\right]. \end{aligned} \tag{8}$$

The averages underlying the optimal formula depend only on the marginal one-dimensional distribution of the index $S(t)$.

For a special case of $f(s) = s$, the approximation (8) becomes, by construction, exact:

$$E_T[S(\tau)] - S_0 = Var[S(\tau)]\, B(\tau).$$

Suppose now that we know, exactly or approximately, the dynamics of the processes $S(t)$ and $M(t)$ in the S-measure. For example, one can apply the Markovian projection technique.

[1] See the end of Section 4 where such direct calculi are treated.

Then, the expectation $E_T[f(S(\tau))]$ can be evaluated given the coefficient B from (6) and the expression (8). In Section 4, one can find other methods of calculating coefficient B, including heuristic model-independent methods.

The optimal formula (8) is justified if the linearization (5) is close to reality. This is indeed the fact for single-rate products where the index $S(t)$ is a CMS rate. However, for *spread* options, the index $S(t)$ is a difference between two CMS rates and, as we show below, the conditional expectation $E[M(\tau)|S(\tau)]$ has a V-shape which requires a *non-linear* approximation. The basis choice and computational details can be found in Section 5.

Note that a decent approximation can be obtained in the T-forward measure by means of a non-perturbative adjustment of the rate $S(\tau)$. Namely, let R_T (R_S) be a stochastic variable coinciding with $S(\tau)$ in the T-forward (respectively, S-) measure. Then, in terms of distributions, one can approximate

$$R_T \sim \frac{E_T[S(\tau)]}{S_0} R_S.$$

This leads to a simple pricing expression,

$$E_T[f(S(\tau))] \simeq E_S\left[f\left(\frac{E_T[S(\tau)]}{S_0}S(\tau)\right)\right]. \tag{9}$$

For options, this approximation gives quite satisfactory accuracy for close-to-ATM strikes but deteriorates on the edges.

3 LIBOR Market Model with stochastic volatility

In this section, we recall some facts about the LIBOR market model (LMM) with stochastic volatility (SV) and related formulas of the Markovian projection (MP).

Following the standard setup of LIBOR market models, we assume a set of maturities $0 = T_0 < T_1 < \cdots < T_N$ and define forward LIBOR rates, $L_n(t)$, as forward rates starting at T_n and ending at T_{n+1}. In terms of a zero-coupon bond with maturity T, $P(t,T)$, the LIBOR rates are given by

$$L_n(t) = \frac{1}{\delta_n}\left(\frac{P(t,T_n)}{P(t,T_{n+1})} - 1\right), \tag{10}$$

where δ_n is the day-count fraction from T_n to T_{n+1}. Each forward LIBOR process satisfies the SDE

$$dL_n = \cdots + (\beta_n(t)L_n + (1-\beta_n(t))l_n)\sigma_n(t)\cdot dW(t), \quad L_n(0) = l_n, \tag{11}$$

where $\beta_n(t)$ are scalar deterministic skew-functions, $\sigma_n(t)$ are vector F-component deterministic volatilities, and $W(t)$ is a vector F-component Brownian motion with independent elements, $E[dW_f(t)\, dW_{f'}] = 0$, for $f \neq f'$ and $E[dW_f^2(t)] = dt$.

Let process $S(t)$ be a function of the forward LIBOR rates without explicit time dependence and $S(0,\ldots,0) = 0$,

$$S(t) = S(L_1(t), \cdots, L_N(t)), \quad \text{with } S(0,\ldots,0) = 0. \tag{12}$$

This process can be, for example, a forward CMS rate. Then, one can apply the technique of Markovian projection, Piterbarg (2006) and Antonov-Misirpashaev (2006), to find the effective displaced diffusion $S^*(t)$ that optimally approximates the process $S(t)$ in its *martingale* measure,

$$dS^*(t) = (S^*(t)\beta(t) + (1-\beta(t))S_0)\,\sigma(t)\cdot dW(t), \qquad S^*(0) = S_0, \tag{13}$$

where $S_0 = S(l_1, \ldots, l_N)$ is the initial value of the process in hand. The optimal vector volatility $\sigma(t)$ and the skew parameter $\beta(t)$ can be expressed as

$$\sigma(t) = \sum_n \frac{l_n}{S_0}\frac{\partial S_0}{\partial l_n}\sigma_n(t) = \sum_n \frac{\partial(\ln S_0)}{\partial(\ln l_n)}\sigma_n(t) \tag{14}$$

and

$$\beta(t) = \frac{\sum_n \left(\frac{1}{2}\frac{\partial|\sigma(t)|^2}{\partial(\ln l_n)} + \frac{\partial(\ln S_0)}{\partial(\ln l_n)}\left(|\sigma(t)|^2 - (1-\beta_n(t))(\sigma(t)\cdot\sigma_n(t))\right)\right)\int_0^t(\sigma_n(\tau)\cdot\sigma(\tau))d\tau}{|\sigma(t)|^2\int_0^t|\sigma(\tau)|^2 d\tau}. \tag{15}$$

Introduce a stochastic volatility (SV) following the CIR process,

$$dz(t) = \theta(t)(1-z(t)) + \sqrt{z(t)}\gamma(t)\,dU(t), \quad z(0) = 1, \tag{16}$$

with mean-reversion $\theta(t)$, vol-of-vol $\gamma(t)$, and a scalar Brownian motion $U(t)$, independent of the vector Brownian motion $W(t)$ driving the LIBOR rates. Following Piterbarg (2005), one can generalize the LIBOR SDEs as

$$dL_n = \cdots + (\beta_n(t)L_n + (1-\beta_n(t))l_n)\sqrt{z(t)}\,\sigma_n(t)\cdot dW(t), \quad L_n(0) = l_n. \tag{17}$$

Due to independence of the Brownian motions underlying the SV and LIBOR rates, one can approximate the function $S(L_1, \ldots, L_N)$ as

$$dS^*(t) = (S^*(t)\beta(t) + (1-\beta(t))S^*(0))\sqrt{z(t)}\,\sigma(t)\cdot dW(t), \qquad S^*(0) = S_0, \tag{18}$$

with the same optimal parameters (14) and (15). Note also that this Brownian motion independence leaves the SV process unchanged under standard measures.

4 Single-rate CMS products

In this section, we consider the single-rate CMS products, such as the CMS swap and CMS cap.

Forward swap rate $S(t)$ starting at T_B and ending at T_E can be written as

$$S(t) = \frac{P(t,T_B) - P(t,T_E)}{\sum\limits_{i=B+1}^{E}\delta_{i-1}P(t,T_i)}. \tag{19}$$

Using the general approach from Section 2, we will calculate a single CMS payment price $P(0,T)E_T[S(\tau)]$ and a CMS caplet price $P(0,T)E_T[(S(\tau)-K)^+]$ for some strike K. As usual, we denote the CMS fixing date as τ and the CMS payment date as T.

We will refer to the denominator in the swap definition (19) as a *swap level*, or annuity, and denote it as

$$L_S(t) = \sum_{i=B+1}^{E} \delta_{i-1} P(t, T_i).$$

The martingale measure for the swap rate (19) is associated with the numeraire

$$N(t) = \frac{L_S(t)}{L_S(0)}.$$

The measure-change process (2) is

$$M(t) = \frac{L_S(0)}{P(0,T)} \frac{P(t,T)}{L_S(t)}. \tag{20}$$

In the LMM SV, the measure-change process is a certain function of LIBOR rates, $M(t) = M(L_1(t), \cdots, L_N(t))$. In order to apply the MP formulas (14) and (15) to the measure-change process, we need to reshape it to guarantee its zero value for zero LIBOR rates (12). For this, define

$$R(t) = M(t) - M_{zr},$$

where a zero rate value of the measure change is

$$M_{zr} = \frac{L_S(0)}{P(0,T)} \frac{1}{\sum\limits_{i=B+1}^{E} \delta_{i-1}}.$$

Thus, one can apply the MP formulas to the process $R(t)$,

$$R(t) = \frac{L_S(0)}{P(0,T) \sum\limits_{i=B+1}^{E} \delta_{i-1}} \frac{P(t,T) \sum\limits_{i=B+1}^{E} \delta_{i-1} - \sum\limits_{i=B+1}^{E} \delta_{i-1} P(t,T_i)}{\sum\limits_{i=B+1}^{E} \delta_{i-1} P(t,T_i)}.$$

In order to calculate the generalized payment price $E_T[f(S(\tau))]$ (8) and its optimal coefficient (6), one should approximate the rates $S(t)$ and $R(t)$. For this, we use the Markovian projection formulas (14) and (15), which requires calculation of the partial derivatives of the rate $S(t)$ over initial LIBOR values $\partial(\ln S_0)/\partial(\ln l_n)$ and $\partial^2(\ln S_0)/\partial(\ln l_n)\partial(\ln l_m)$. It can be done given the swap rate (19) expressed as a function of the initial LIBOR values $l_n = L_n(0)$,

$$S(0) = \frac{\prod\limits_{i=B}^{E-1} (1 + \delta_i l_i) - 1}{\sum\limits_{i=B+1}^{E} \delta_{i-1} \prod\limits_{j=i}^{E-1} (1 + \delta_j l_j)}. \tag{21}$$

The analogous formula can be written for the rate R using interpolation[2] if the payment date T does not fall to the LMM maturities T_i.

[2] One can choose the standard log-linear interpolation for the zero bonds.

In this way, we approximate our rates in the S-measure with[3]

$$dS(t) \simeq (S(t)\beta_S(t) + (1-\beta_S(t))S_0)\,\sqrt{z(t)}\,\sigma_S(t)\cdot dW(t), \tag{22}$$
$$dR(t) \simeq (R(t)\beta_R(t) + (1-\beta_R(t))R_0)\,\sqrt{z(t)}\,\sigma_R(t)\cdot dW(t), \tag{23}$$

where $S_0 = S(0)$ and $R_0 = R(0)$. To proceed with calculations for the fixing date τ, we should average skew parameters using Piterbarg (2005) formulas. For the averaged skew, we will use the same notations without the time parameter, $\beta_S(t) \to \beta_S$ and $\beta_R(t) \to \beta_R$. The SDEs above can be solved for $t = \tau$,

$$S(\tau) \simeq \frac{S_0\, e^{y_S(\tau)}}{\beta_S} - \frac{(1-\beta_S)S_0}{\beta_S}, \tag{24}$$
$$R(\tau) \simeq \frac{R_0\, e^{y_R(\tau)}}{\beta_R} - \frac{(1-\beta_R)R_0}{\beta_R}, \tag{25}$$

where processes $y_R(t)$ and $y_S(t)$ are defined as

$$y_S(\tau) = -\frac{1}{2}\beta_S^2 \int_0^\tau dt|\sigma_S(t)|^2 z(t) + \beta_S \int_0^\tau dW(t)\cdot\sigma_S(t)\sqrt{z(t)}, \tag{26}$$
$$y_R(\tau) = -\frac{1}{2}\beta_R^2 \int_0^\tau dt|\sigma_R(t)|^2 z(t) + \beta_R \int_0^\tau dW(t)\cdot\sigma_R(t)\sqrt{z(t)}. \tag{27}$$

The fact that the triple $\{y_S(t), y_R(t), z(t)\}$ is affine permits calculating the characteristic using a Riccati-like ODE (see Appendix B). Denote the characteristic function and moment generating function as ϕ and $\mathcal{K}$, respectively,

$$\phi_{SR}(t,\xi_S,\xi_R) = e^{\mathcal{K}_{SR}(t,\xi_S,\xi_R)} = E\left[e^{\xi_S\, y_S(t) + \xi_R\, y_R(t)}\right]$$

and

$$\phi_S(t,\xi) = e^{\mathcal{K}_S(t,\xi)} = E\left[e^{\xi\, y_S(t)}\right].$$

We are ready now to approximate the single CMS payment price and the CMS caplet. Indeed, the average $E_T[S(\tau)] = E[M(\tau)S(\tau)]$ can be obtained using the characteristic function $\phi_{SR}(t,\xi_S,\xi_R)$:

$$E[M(\tau)S(\tau)] - M_0\, S_0 = E[R(\tau)S(\tau)] - R_0\, S_0 = \frac{R_0\, S_0}{\beta_R\,\beta_S}\left(\phi_{SR}(\tau,1,1) - 1\right). \tag{28}$$

Thus, a single CMS payment price can be approximated as

$$P(0,T)E_T[S(\tau)] \simeq P(0,T)\,\frac{R_0\, S_0}{\beta_R\,\beta_S}\left(\phi_{SR}(\tau,1,1) - 1\right) + P(0,T)\, S_0. \tag{29}$$

The first term in the equality (29) represents the CMS payment convexity adjustment over the trivial value $P(0,T)\,S_0$. Note that the characteristic function underlying the above result is more than one if[4] $\sigma_S(t)\cdot\sigma_R(t) > 0$ for any time t. Positivity of the volatility dot product

[3]We omit asterisk superscripts in the approximation processes explicitly using "$\simeq$" symbols.

[4]We suppose also that the initial rate values S_0, R_0 and their skew parameters β_S, β_R are positive.

is quite natural, as far as correlation between the rates is, in general, positive. Thus, under these assumptions, the convexity adjustment (28) is positive.

For the CMS caplet pricing, we need to calculate the optimal coefficient (6) in order to use the general formula (8). Noticing that

$$Var[S(\tau)] = E[S^2(\tau)] - S_0^2 = \frac{S_0^2}{\beta_S^2}\left(\phi_S(\tau,2) - 1\right), \tag{30}$$

we obtain a formula for the optimal coefficient,

$$B(\tau) = \frac{E[M(\tau)S(\tau)] - S_0}{Var[S(\tau)]} \simeq \frac{R_0\beta_S\left(\phi_{SR}(\tau,1,1) - 1\right)}{S_0\beta_R\left(\phi_S(\tau,2) - 1\right)}. \tag{31}$$

One can find in the literature multiple references to heuristic linear approximation of the measure change process, which is model independent and based only on the knowledge of the yield curve. Hunt and Kennedy (2000) have proposed coefficient B to be equal to

$$B(\tau) \simeq \frac{R_0}{S_0}, \tag{32}$$

which coincides with (31) provided that volatilities and skews of the processes $S(t)$ and $R(t)$ are identical. Note that the model independent approximation of the coefficient $B(\tau)$ gives an alternative to the CMS payment approximation. Namely, the underlying average in the T-forward measure can be approximated as

$$E_T[S(\tau)] = S_0 + B(\tau)\,Var[S(\tau)],$$

where the variance is calculated using (30) and the optimal coefficient via formula (32). This way is less accurate than (29), but still can be used for short fixing dates.

Let us proceed now with calculations of CMS options. For a given strike K, a call option (caplet) price $\mathcal{C} = P(0,T)E_T[(S(\tau)-K)^+]$ can be computed using the general formula (8) given already-calculated optimal coefficient $B(\tau)$,

$$\frac{\mathcal{C}}{P(0,T)} = E_T[(S(\tau)-K)^+] \simeq E\left[(S(\tau)-K)^+\right] + B(\tau)\,E\left[(S(\tau)-K)^+\,(S(\tau)-S_0)\right]. \tag{33}$$

Thus, the T-forward measure option in the l.h.s. is expressed in terms of the call option $E\left[(S(\tau)-K)^+\right]$ and *parabolic*[5] option $E\left[(S(\tau)-K)^+\,S(\tau)\right]$, both in the swap-measure. Substituting formulas (24) into the above averages, one can calculate the underlying averages as

$$\begin{aligned} E\left[(S(\tau)-K)^+\right] &= \frac{S_0}{\beta_S} E\left[\left(e^{y_S(\tau)} - k\right)^+\right], \\ E\left[(S(\tau)-K)^+(S(\tau)-S_0)\right] &= \frac{S_0^2}{\beta_S^2}\left(E\left[\left(e^{y_S(\tau)} - k\right)^+ e^{y_S(\tau)}\right] - E\left[\left(e^{y_S(\tau)} - k\right)^+\right]\right), \end{aligned}$$

where effective strike $k = 1 + \frac{K\beta_S}{S_0} - \beta_S$. To compute the options, one can use the Laplace or Fourier transform covered in detail in Appendix A,

$$E\left[\left(e^{y_S(\tau)} - k\right)^+\right] = \frac{k}{2\pi i}\int_{C+}\frac{d\xi}{\xi(\xi-1)} e^{\mathcal{K}_S(\tau,\xi) - \xi\,\ln k}, \tag{34}$$

$$E\left[\left(e^{y_S(\tau)} - k\right)^+ e^{y_S(\tau)}\right] = \frac{k}{2\pi i}\int_{C+}\frac{d\xi}{\xi(\xi-1)} e^{\mathcal{K}_S(\tau,\xi+1) - \xi\,\ln k}, \tag{35}$$

[5] We will also refer to it as *para-option*.

(see formulas (59) and (60)). These integrals can be evaluated in a closed form for the non-SV case, but require numerical one-dimensional integration for the SV case.

As mentioned before, we can compute the CMS cap price exactly given the processes $S(t)$ and $R(t)$, i.e., without linearizing the measure-change process. Namely, the CMS cap formula

$$\frac{\mathcal{C}}{P(0,T)} = E_T[(S(\tau)-K)^+] = E[(S(\tau)-K)^+\, M(\tau)]$$

can be rewritten as

$$\begin{aligned} E[(S(\tau)-K)^+\, M(\tau)] &= \frac{S_0}{\beta_S} E\left[\left(e^{y_S(\tau)}-k\right)^+ (M_{zr}+R(\tau))\right] \\ &= \frac{S_0}{\beta_S}\left(1-\frac{R_0}{\beta_R}\right) E\left[\left(e^{y_S(\tau)}-k\right)^+\right] \\ &+ \frac{S_0\, R_0}{\beta_S\, \beta_R} E\left[\left(e^{y_S(\tau)}-k\right)^+ e^{y_R(\tau)}\right], \end{aligned}$$

where the non-trivial term can be calculated as

$$E\left[\left(e^{y_S(\tau)}-k\right)^+ e^{y_R(\tau)}\right] = \frac{k}{2\pi i}\int_{C^+}\frac{d\xi}{\xi(\xi-1)} e^{\mathcal{K}_{SR}(\tau,\xi,1)-\xi\,\ln k}.$$

This expression fits closely the approximation (33) as far as the main error comes from the approximation of the rates by the MP.

Note that the Heston model can contain diverging moments, see Andersen-Piterbarg (2005). Thus, the parabolic option price is valid only for a finite second moment.

For reference, we also present the CMS call option price in the forward T-measure. Starting from the general adjusted formula (9), we formally use the swap representation (24) and obtain

$$\frac{\mathcal{C}}{P(0,T)} = E_T[(S(\tau)-K)^+] \simeq E\left[\left(\frac{E_T[S(\tau)]}{S_0}S(\tau)-K\right)^+\right] = \frac{E_T[S(\tau)]}{\beta_S} E\left[\left(e^{y_S(\tau)}-k'\right)^+\right] \tag{36}$$

where effective strike $k' = 1+\frac{K\beta_S}{E_T[S(\tau)]}-\beta_S$. The CMS average $E_T[S(\tau)]$ is obtained via the approximation (29) and the underlying option price is calculated via the Laplace transform (34).

5 CMS spread option

We consider a CMS spread $S(t) = S_1(t) - S_2(t)$, where S_1 and S_2 are some forward CMS rates. A call option on the spread fixed at τ with a payment date T and a strike K has the price

$$\mathcal{C}_{sp} = P(0,T)E_T[(S_1(\tau)-S_2(\tau)-K)^+].$$

A martingale measure for the spread $S(t)$ exists for multi-factor LMM, but the corresponding numeraire process $N_S(t)$ is complicated to calculate. Denote the martingale spread-measure expectation operator as $E_S[\cdots]$ and associated Brownian motion as $W_S(t)$. One can apply the general formulas (8) without *explicit* knowledge of the numeraire. Indeed, we can approximate the rate process $S_i(t)$ in the spread-measure using the MP in the corresponding martingale measure,

$$dS_i(t) \simeq \mu(t)\,dt + (S_i(t)\beta_i(t) + (1-\beta_i(t))S_i(0))\,\sqrt{z(t)}\,\sigma_i(t){\cdot}dW_S(t), \quad i=1,2,$$

where the drift $\mu(t)$ reflects a measure change between the rate martingale measure and the spread one. Note that the drift is common for both processes, as far as the spread $S_1(t)-S_2(t)$ is martingale in the spread-measure. Thus, one can simply set the drift to zero before proceeding with calculations[6]:

$$dS_i(t) \simeq (S_i(t)\beta_i(t) + (1-\beta_i(t))S_i(0))\,\sqrt{z(t)}\,\sigma_i(t){\cdot}dW_S(t), \quad i=1,2. \tag{37}$$

Then, we calculate the optimal coefficient $B(\tau)$ in the T-forward measure,

$$B(\tau) = \frac{E_T[S(\tau)] - S_0}{\mathrm{Var}_S[S(\tau)]} = \frac{E_T[S_1(\tau)] - E_T[S_2(\tau)] - S_0}{\mathrm{Var}_S[S(\tau)]}, \tag{38}$$

where $S_0 = S(0) = S_1(0) - S_2(0)$ and the variance $\mathrm{Var}_S[S(\tau)] = E_S[S^2(\tau)] - S_0^2$. The averages $E_T[S_1(\tau)]$ are evaluated using (29) and the variance can be computed via the characteristic function, see (52). Unfortunately, this approximation fails to fit the numerical answer. A reason for that is that the numeraire *linearization*,

$$E_S\left[\frac{P(\tau,T)}{N_S(\tau)}\,|\,S(\tau)\right] \simeq A(\tau) + B(\tau)\,S(\tau), \tag{39}$$

is far from reality and leads to big errors. Before describing the improvement procedure, we will briefly comment on the T-forward measure calculus with simple, but effective, adjustments.

Indeed, as in the expression (9), we calculate formally the MP coefficients (14-15) for both rates in the T-forward measure, neglecting the drifts (like in their *martingale* measures),

$$dS_i^*(t) = (S_i^*(t)\beta_i(t) + (1-\beta_i(t))S_i^*(0))\,\sqrt{z(t)}\,\sigma_i(t){\cdot}dW_T(t), \quad i=1,2, \tag{40}$$

and adjust the average,

$$S_i(\tau) \simeq \frac{E_T[S_i(\tau)]}{S_i(0)} S_i^*(\tau),$$

using $E_T[S_i(\tau)]$ calculated in the previous section, see formula (29). This gives

$$\begin{aligned} \mathcal{C}_{sp} &= P(0,T)E_T[(S_1(\tau) - S_2(\tau) - K)^+] \\ &\simeq P(0,T)E_T\left[\left(\frac{E_T[S_1(\tau)]}{S_1(0)} S_1^*(\tau) - \frac{E_T[S_2(\tau)]}{S_2(0)} S_2^*(\tau) - K\right)^+\right]. \end{aligned} \tag{41}$$

Analytical methods for computing the above expectation are given in Section 6.2. Note that the adjusted T-measure option price formula (41) works quite effectively for the near-ATM

[6] This modification, $S_i(t) \to S_i(t) + \int_0^t \mu(\tau)\,d\tau$, will also influence the diffusion term, but the drift order, as a square of volatility, is out of the MP scheme precision. Note also that due to independence of the Brownian motion underlying the forward rates and the SV Brownian motion, the SV SDE remains the same (16) under the spread measure.

cases and degenerates for out-of-money strikes. Actually, by enlarging the linear basis (39) in the spread-measure, one can achieve a uniform fit for a large variety of strikes.

To visualize a typical form of the spread-measure numeraire conditional expectation,

$$E_S\left[\frac{P(\tau,T)}{N_S(\tau)}\,|\,S(\tau)\right],$$

we consider two forward LIBOR rates $L_n(t)=\frac{1}{\delta_n}\left(\frac{P(t,T_n)}{P(t,T_{n+1})}-1\right)$, $n=1,2$, having, for simplicity, log-normal diffusion terms without stochastic volatility,

$$\begin{aligned} dL_1(t) &= L_1(t)\,\sigma_1(t)\cdot dW_{T_2}(t),\\ dL_2(t) &= L_2(t)\,\sigma_2(t)\cdot\left(dW_{T_2}(t)+dt\,\sigma_2(t)\frac{\delta_2\,L_2(t)}{1+\delta_2\,L_2(t)}\right),\end{aligned}$$

where Brownian motion W_{T_2} corresponds to the T_2-forward measure associated with zero bond $P(t,T_2)$ as numeraire. Obviously, the first LIBOR is martingale in this measure. Writing an SDE for the spread process $S(t)=L_1(t)-L_2(t)$,

$$dS(t)=\left(L_1(t)\,\sigma_1(t)-L_2(t)\,\sigma_2(t)\right)\cdot dW_{T_2}(t)-L_2(t)\,|\sigma_2(t)|^2\,\frac{\delta_2\,L_2(t)}{1+\delta_2\,L_2(t)}\,dt\,, \qquad (42)$$

we can find the martingale spread measure with Brownian motion W_S that cancels the drift in the SDE above. Namely, let $dW_{T_2}(t)=dW_S(t)+\rho(t)\,dt$ for some unknown process $\rho(t)$. Then, the drift in (42) is zero if

$$\left(L_1(t)\,\sigma_1(t)-L_2(t)\,\sigma_2(t)\right)\cdot\rho(t)=|\sigma_2(t)|^2\,\frac{\delta_2\,L_2^2(t)}{1+\delta_2\,L_2(t)}.$$

This equation has a solution if $|L_1(t)\,\sigma_1(t)-L_2(t)\,\sigma_2(t)|\neq 0$ for allowed values of LIBOR rates (positive in our case). It is definitely possible for the non-degenerate multi-factor case, i.e., for non-collinear vectors $\sigma_1(t)$ and $\sigma_2(t)$. Then, a solution for the process ρ follows

$$\rho(t)=\left(L_1(t)\,\sigma_1(t)-L_2(t)\,\sigma_2(t)\right)\;\frac{\delta_2\,L_2^2(t)|\sigma_2(t)|^2}{(1+\delta_2\,L_2(t))|L_1(t)\,\sigma_1(t)-L_2(t)\,\sigma_2(t)|^2}.$$

Thus, in the spread-measure, the LIBOR evolution

$$\begin{aligned} dL_1(t) &= L_1(t)\,\sigma_1(t)\cdot\left(dW_S(t)+\rho(t)\,dt\right),\\ dL_2(t) &= L_2(t)\,\sigma_2(t)\cdot\left(dW_S(t)+dt\,\left(\rho(t)+\sigma_2(t)\frac{\delta_2\,L_2(t)}{1+\delta_2\,L_2(t)}\right)\right)\end{aligned}$$

guarantees the zero drift for the spread process. The numeraire can be obtained via the SDE on the Radon-Nikodym derivative $M(t)=\frac{P(t,T_2)}{N_S(t)}$,

$$dM(t)=-M(t)\,\rho(t)\cdot dW_S(t),$$

martingale in the spread measure. On the graph below, we calculate numerically the conditional expectation $E_S\left[\frac{1}{N_S(T_1)}\,|\,S(T_1)\right]$ and compare it with the linear regression.

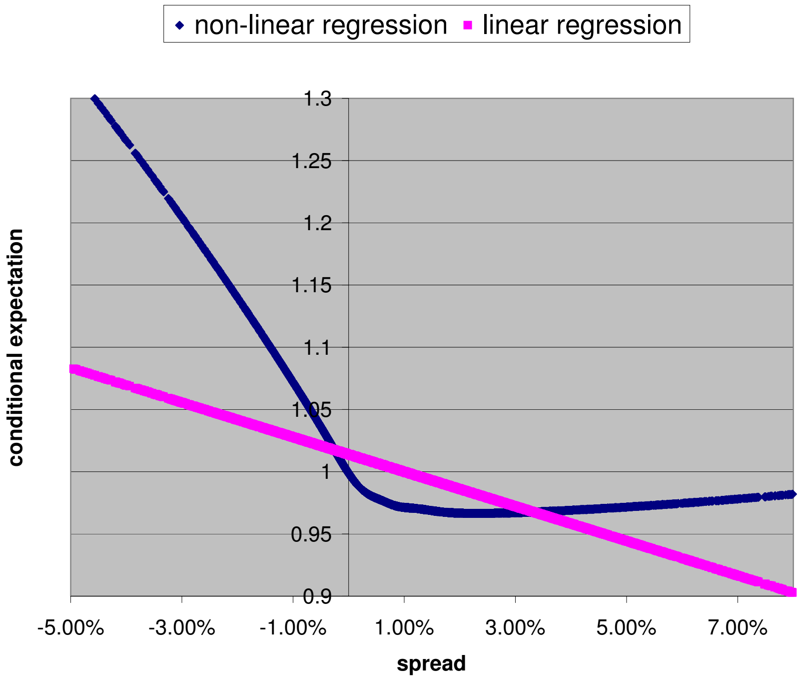

Figure 1: Conditional expectations $E_S\left[\frac{1}{N_S(T_1)} \,|\, S(T_1)\right]$ using non-linear regression (exact) and linear regression.

We have used time-independent volatilities $|\sigma_1(t)| = |\sigma_2(t)| = 25\%$ with 90% correlation, different initial values for the forward LIBOR rates, $L_1(0) = 5\%, L_2(0) = 4\%$, and times $T_1 = 10$, $T_2 = 11$, $T_3 = 12$.

It becomes clear now why the linear regression approximation was poor: The real conditional expectation, having a V-shape, is too far from linear. The "tick" has a singularity in the vicinity of the initial spread, $S(0) = L_1(0) - L_2(0) = 1\%$, which suggests modifying the linear basis, including the ATM option. Namely, we will try to approximate the measure change

$$M(\tau) = \frac{P(\tau, T)}{N_S(\tau)} \; \frac{1}{P(0,T)}$$

as

$$E_S\left[M(\tau) \,|\, S(\tau)\right] \simeq A(\tau) + B(\tau)\,\Delta S(\tau) + C(\tau)\,(\Delta S(\tau))^+, \tag{43}$$

where $\Delta S(\tau) = S(\tau) - S(0)$. The optimal coefficients $A(\tau)$, $B(\tau)$, and $C(\tau)$ can be found by minimizing

$$\chi^2(A(\tau), B(\tau), C(\tau)) = E_S\left[\left(M(\tau) - A(\tau) - B(\tau)\,\Delta S(\tau) - C(\tau)\,(\Delta S(\tau))^+\right)^2\right].$$

Setting the derivatives $\frac{\partial \chi^2}{\partial A}$, $\frac{\partial \chi^2}{\partial B}$, and $\frac{\partial \chi^2}{\partial C}$ to zero and omitting the time τ argument for better legibility, we obtain the linear system:

$$\begin{aligned}
E_S[M] &= A + C\,E_S[\Delta S^+],\\
E_S[M\Delta S] &= B\,E_S[\Delta S^2] + C\,E_S[(\Delta S^+)^2],\\
E_S[M\Delta S^+] &= A\,E_S[\Delta S^+] + B\,E_S[(\Delta S^+)^2] + C\,E_S[(\Delta S^+)^2].
\end{aligned}$$

The l.h.s. of the system above contains averages of the spread in the T-forward measure. Indeed, restoring temporally the time argument, we have

$$\begin{aligned}
E_S[M(\tau)] &= 1,\\
E_S[M(\tau)\Delta S(\tau)] &= E_T[\Delta S(\tau)],\\
E_S[M(\tau)(\Delta S(\tau))^+] &= E_T[(\Delta S(\tau))^+],
\end{aligned}$$

which permits rewriting the linear system as follows:

$$1 = A + C\,E_S[\Delta S^+], \tag{44}$$

$$E_T[\Delta S] = B\,E_S[\Delta S^2] + C\,E_S[(\Delta S^+)^2], \tag{45}$$

$$E_T[\Delta S^+] = A\,E_S[\Delta S^+] + B\,E_S[(\Delta S^+)^2] + C\,E_S[(\Delta S^+)^2]. \tag{46}$$

To find the optimal coefficients, one should be able to know both l.h.s. averages in the T-forward measure and the r.h.s ones in the spread-measure. The average $E_T[\Delta S(\tau)] = E_T[S_1(\tau)] - E_T[S_2(\tau)] - S_1(0) + S_2(0)$ can be approximated using the single-rate CMS product technique (29). The second average in the l.h.s., $E_T[\Delta S^+]$, is one of our computational goals, the ATM spread option price. But, as stated above and as will be confirmed numerically, the T-forward measure approximation (41) works well for near-ATM strikes. This permits applying the formula (41) for the average $E_T[\Delta S^+]$. The r.h.s. averages can be computed using the rate approximations $S_i(\tau)$ in the spread-measure (37). Further details on this can be found in Section 6.2.

Thus, having both l.h.s. and r.h.s. of the averages, one can easily solve the linear system (44-46) to obtain the measure change approximation (43), which permits calculating the averages $E_T[f(S(\tau))] = E_T[f(S_1(\tau) - S_2(\tau))]$ in the spread-measure,

$$E_T[f(S(\tau))] \simeq E_S[f(S(\tau))\,(A(\tau) + B(\tau)\,\Delta S(\tau) + C(\tau)\,(\Delta S(\tau))^+)], \tag{47}$$

approximating the rates in the spread-measure as (37).

In spite of the measure change *approximation* (43), the important averages are maintained *exactly* due to the optimal conditions (44-46): The approximate pricing formula (47) becomes exact for functions $f(S) = 1$, $f(S) = S$, and $f(S) = (S - S(0))^+$. This also implies the conservation of the put-call parity.

The spread call option approximation corresponding to $f(S) = (S - K)^+$ can be written as

$$E_T[(S(\tau) - K)^+] \simeq E_S[(S(\tau) - K)^+\,(A(\tau) + B(\tau)\,\Delta S(\tau) + C(\tau)\,(\Delta S(\tau))^+)]. \tag{48}$$

As mentioned above, the T-forward result (41) for the ATM strike $K = S(0)$ will be reproduced. Furthermore, the final spread-measure approximation (48) improves out-of-money behavior with respect to the T-measure approximation. One can use the advanced formula (48) for spread option pricing, given the model parameters, as well as using it as an *extrapolation* tool from ATM strikes to out-of-model ones.

Note, at the end, that the averages in the approximation (48) can be further simplified:

$$
\begin{aligned}
E_T[(S(\tau)-K)^+] &\simeq A(\tau)\,E_S[(S(\tau)-K)^+] \\
&+ B(\tau)\,E_S[(S(\tau)-K)^+\,\Delta S(\tau)] \\
&+ C(\tau)\,E_S[(S(\tau)-K)^+\,(S(\tau)-S(0))^+].
\end{aligned}
$$

Indeed, for two strike positions, $K > S_0$ and $K < S_0$, we have

$$
\begin{aligned}
E_T[(S(\tau)-K)^+] &\simeq A(\tau)\,E_S[(S(\tau)-K)^+] \\
&+ (B(\tau)+C(\tau))\,E_S[(S(\tau)-K)^+\,\Delta S(\tau)],
\end{aligned}
$$

for $K > S(0)$, and

$$
\begin{aligned}
E_T[(S(\tau)-K)^+] &\simeq A(\tau)\,E_S[(S(\tau)-K)^+] \\
&+ B(\tau)\,E_S[(S(\tau)-K)^+\,\Delta S(\tau)] \\
&+ C(\tau)\,E_S[(S(\tau)-S(0))^+\,(S(\tau)-K)],
\end{aligned}
$$

for $K < S(0)$.

6 Spread calculus

In this section, we address calculation techniques related to the spread between two rates, $S_1(t)$ and $S_2(t)$, approximated via the MP by (37),

$$dS_i(t) \simeq (S_i(t)\beta_i(t) + (1-\beta_i(t))S_i(0))\,\sqrt{z(t)}\,\sigma_i(t)\cdot dW_S(t), \quad i = 1, 2.$$

As in Section 4, we average the rate skew parameters for a fixed time $t = \tau$ using Piterbarg (2005) formulas and denote them by β_i. The replacement $\beta_i(t) \to \beta_i$ permits solving the rate SDEs, for the time τ,

$$S_i(\tau) \simeq \frac{S_i(0)\,e^{y_i}}{\beta_i} - \frac{(1-\beta_i)S_i(0)}{\beta_i}, \tag{49}$$

where stochastic variables y_i are defined as

$$y_i = -\frac{1}{2}\beta_i^2 \int_0^\tau dt|\sigma_i(t)|^2 z(t) + \beta_i \int_0^\tau dW_S(t)\cdot\sigma_i(t)\sqrt{z(t)}. \tag{50}$$

The characteristic function and the moment generation function of the variables,

$$\varphi(\xi_1, \xi_2) = e^{\mathcal{K}(\xi_1,\xi_2)} = E_S\left[e^{\xi_1\,y_1 + \xi_2\,y_2}\right], \tag{51}$$

can be calculated analytically, see Appendix B. Note that $\varphi(1,0) = E_S\left[e^{y_1}\right] = 1$ and $\varphi(0,1) = E_S\left[e^{y_2}\right] = 1$, which follows from the Ito formula.

In the subsections below, we will calculate spread $S(\tau) = S_1(\tau) - S_2(\tau)$ averages underlying the CMS spread pricing approximation: the spread variance, the spread option, and the spread para-option.

6.1 Spread variance

The spread variance, $\mathrm{Var}_S[S(\tau)]$, can be computed via the characteristic function (51). Namely, substituting the rate expressions (49) into the variance definition, we obtain

$$\begin{aligned}\mathrm{Var}_S[S(\tau)] &= \frac{S_1^2(0)}{\beta_1^2}\left(E_S\left[e^{2\,y_1}\right]-1\right)+\frac{S_2^2(0)}{\beta_2^2}\left(E_S\left[e^{2\,y_2}\right]-1\right)\\ &- 2\frac{S_1(0)\,S_2(0)}{\beta_1\,\beta_2}\left(E_S\left[e^{y_1+y_2}\right]-1\right).\end{aligned}$$

Substituting the underlying exponential averages expressed in terms of the characteristic function (51), we get

$$\mathrm{Var}_S[S(\tau)] = \frac{S_1^2(0)}{\beta_1^2}\left(\varphi(2,0)-1\right)+\frac{S_2^2(0)}{\beta_2^2}\left(\varphi(0,2)-1\right)-2\frac{S_1(0)\,S_2(0)}{\beta_1\,\beta_2}\left(\varphi(1,1)-1\right). \quad (52)$$

6.2 Spread option price

In this section, we present different techniques for spread option prices, including a new elegant result based on the two-dimensional Laplace (Fourier) transform.

Our goal is the evaluation of effective expression

$$\mathcal{C}_{sp} = E_S\left[\left(C_1\,e^{y_1}-C_2\,e^{y_2}-K\right)^+\right] \quad (53)$$

for some constant positive coefficients C_1, C_2, and strike K in terms of the characteristic function (51). This form of the spread option is equivalent to

$$E_S\left[(S(\tau)-k)^+\right] = E_S\left[(S_1(\tau)-S_2(\tau)-k)^+\right]$$

appearing in the final CMS spread option formula (48), provided that $C_i = \frac{S_i(0)}{\beta_i}$ and $K = k + \frac{(1-\beta_1)S_1(0)}{\beta_1} - \frac{(1-\beta_2)S_2(0)}{\beta_2}$. The effective form (53) can be also used for evaluation of the CMS spread option in the T-forward measure (41) under similar conditions.

We consider the strike K being positive. Otherwise, for negative strikes, we can use the put-call parity to revert to the positive effective strike situation,

$$E_S\left[\left(C_1\,e^{y_1}-C_2\,e^{y_2}-K\right)^+\right] = E_S\left[\left(C_2\,e^{y_2}-C_1\,e^{y_1}+K\right)^+\right]+C_1-C_2-K.$$

Thus, a put option can be transformed to the call option with the opposite sign strike via the index exchange $1 \leftrightarrow 2$.

Apart from a simple case of zero strike, which we will consider later, the exact integration of average (53) involves 2-dimensional complex integration containing the characteristic function (51). It can be performed in an effective manner using smart variable choices and control variates.

6.2.1 Zero strike case

For the zero strike, one can calculate the option price using one-dimensional integration for a general characteristic function and come up with a closed formula for the Gaussian case, also known as the Margrabe (1978) formula.

Indeed, transform the payoff for $K = 0$ as

$$(C_1\, e^{y_1} - C_2\, e^{y_2})^+ = C_1 e^{y_2} \left(e^{y_1 - y_2} - \frac{C_2}{C_1} \right)^+ .$$

Using Laplace transform of the maximum function (see Appendix A for more details), we obtain

$$(C_1\, e^{y_1} - C_2\, e^{y_2})^+ = C_2\, e^{y_2} \frac{1}{2\pi i} \int_{C_+} \frac{d\xi}{\xi(\xi - 1)} e^{\xi(y_1 - y_2) - \xi \ln\left(\frac{C_2}{C_1}\right)} .$$

Thus, after taking the expectation operator for $K = 0$, the spread option price becomes

$$\begin{aligned}
\mathcal{C}_{sp} &= E_S \left[(C_1\, e^{y_1} - C_2\, e^{y_2})^+ \right] \\
&= \frac{C_2}{2\pi i} \int_{C_+} \frac{d\xi}{\xi(\xi - 1)} E_S \left[e^{y_2 + \xi(y_1 - y_2)} \right] e^{-\xi \ln\left(\frac{C_2}{C_1}\right)} \\
&= \frac{C_2}{2\pi i} \int_{C_+} \frac{d\xi}{\xi(\xi - 1)} e^{\mathcal{K}(\xi, 1-\xi) - \xi \ln\left(\frac{C_2}{C_1}\right)} .
\end{aligned}$$

For the Gaussian case, the integral can be easily taken to restore the Margrabe (1978) formula.

6.2.2 Positive strike case

In this subsection, we consider a positive strike $K > 0$ and give a new spread option price formula based on the two-dimensional Laplace transform. As it will be proven in Appendix A, the spread option price can be represented in terms of a two-dimensional integral involving the double characteristic function,

$$\mathcal{C}_{sp} = \int_{C^{(1)}} d\xi_1 \int_{C^{(2)}} d\xi_2\, e^{\mathcal{K}(\xi_1, \xi_2)}\, L(\xi_1, \xi_2), \tag{54}$$

where the complex function $L(\xi_1, \xi_2)$ is defined as

$$L(\xi_1, \xi_2) = \frac{1}{(2\pi i)^2} \frac{C_1}{\xi_1(\xi_1 - 1)} \left(\frac{K}{C_1} \right)^{-(\xi_1 - 1)} \left(\frac{K}{C_2} \right)^{-\xi_2} \frac{\Gamma(-\xi_2)\, \Gamma(\xi_1 + \xi_2 - 1)}{\Gamma(\xi_1 - 1)}, \tag{55}$$

where $\Gamma(z)$ is the complex Gamma-function.

The contours in the integral (54) are parallel to the imaginary axis and chosen to satisfy the following conditions,

$$\Re\xi_2 < 0 \quad \text{and } \Re\xi_1 + \Re\xi_2 > 1.$$

Known asymptotics, series and effective quasi polynomial representations for Gamma-functions (see, for example, Press at al (2007)) make the numerical integration procedure of the final result quite attractive.

Note that the double-integral spread option formula (54-55) was developed simultaneously and independently by Hurd and Zhou (2009) using equivalent two-dimensional Fourier transform.

6.3 Spread para-option price

In this subsection, we briefly cover the spread para-option price.

The parabolic generalization of (53),

$$\mathcal{P}_{sp} = E_S\left[(C_1\, e^{y_1} - C_2\, e^{y_2})\,(C_1\, e^{y_1} - C_2\, e^{y_2} - K)^+\right], \tag{56}$$

is equivalent to

$$E_S\left[S(\tau)(S(\tau)-k)^+\right] = E_S\left[((S_1(\tau)-S_2(\tau))(S_1(\tau)-S_2(\tau)-k)^+\right]$$

under similar rescaling.

To evaluate the para-option (56), one can reuse the two-dimensional Laplace transform technique from the previous section. Below, we will comment on the pricing of

$$E_S\left[e^{y_1}\,(C_1\, e^{y_1} - C_2\, e^{y_2} - K)^+\right],$$

keeping in mind that its y_2 parity is calculated similarly. Then, the spread option expression (54) generalizes to the following para-option form,

$$E_S\left[e^{y_1}\,(C_1\, e^{y_1} - C_2\, e^{y_2} - K)^+\right] = \int_{C^{(1)}} d\xi_1 \int_{C^{(2)}} d\xi_2\, e^{\mathcal{K}(\xi_1+1,\xi_2)}\, L(\xi_1,\xi_2). \tag{57}$$

We see that the changes simply reduce to a shift of the first argument of the characteristic function.

7 Numerical experiments

For numerical comparison, we consider a 3-factor LMM setup with 20Y annual model tenor $T_n = nY$, see the model definition (17) and (16). Initial LIBOR rates $L_n(0)$ and their flat vector volatility modulus $|\sigma_n(t)| = |\sigma_n|$ are presented in the table below.

Index n	Dates T_n (years)	Initial Libors $L_n(0)$ (%)	Vols $\lvert\sigma_n\rvert$ (%)
0	1	3.34	39
1	2	3.43	38
2	3	3.52	37
3	4	3.61	36
4	5	3.70	35
5	6	3.78	34
6	7	3.87	33
7	8	3.96	32
8	9	4.05	31
9	10	4.14	30
10	11	4.23	29
11	12	4.32	28
12	13	4.41	27
13	14	4.50	26
14	15	3.43	25
15	16	3.38	24
16	17	3.32	23
17	18	3.27	22
18	19	3.21	21
19	20	3.16	20

This setup corresponds to the current highly volatile situation. We use typical flat skew values $\beta_n(t) = \beta_n = \frac{1}{2}$ and the following time-independent parameters for the stochastic volatility (16): vol-of-vol $\gamma(t) = \gamma = 1.3$ and mean-reversion $\theta(t) = \theta = 0.15$.

To set up correlations for the 3-Factor model, we define the full rank-correlation matrix

$$C_{ij} = \frac{\sigma_i \cdot \sigma_j}{|\sigma_i|\,|\sigma_j|} = e^{-0.1\,|T_i - T_j|}$$

and compress it to rank 3. This rank reduction procedure consists of choosing the three largest eigenvalues and zeroing the others with the subsequent diagonal normalization.

For our numerical experiments, we take two CMS rates with annual tenor for 2Y and 10Y, i.e., CMS2Y and CMS10Y, and check two fixing dates: 5Y and 10Y, supposing that the index payment date coincides with its fixing date.

We compare the CMS product analytical approximations (see table below) with the Monte Carlo simulations.

Product	Method	Formula
CMS payment	analytical approximation	(29)
CMS caplet	T-measure (forward measure approximation)	(36)
	S-measure (swap measure approximation)	(33)
CMS spread option	T-measure (forward measure approximation)	(41)
	S-measure (spread measure approximation)	(48)

7.1 CMS instruments numerical check: 5Y maturity

We consider here the fixing date τ equal to the payment date T set to 5Y. The forward rate processes are formally defined as

$$\text{CMS10 forward rate } \rightarrow S_1(t) = \frac{P(t,T_5) - P(t,T_{15})}{\sum\limits_{i=6}^{15} \delta_{i-1} P(t,T_i)},$$

$$\text{CMS2 forward rate } \rightarrow S_2(t) = \frac{P(t,T_5) - P(t,T_7)}{\sum\limits_{i=6}^{7} \delta_{i-1} P(t,T_i)}.$$

7.1.1 CMS10Y products for 5Y maturity

The 10 year rate has a forward value at the origin equal to 4.067 %. Its payment price and convexity adjustment are presented in the table below.

	Price	CA
Approx	3.808	37.2
MC	3.818	38.2

CMS10y caplets: Prices

Strikes	3.067	3.317	3.567	3.817	4.067	4.317	4.567	4.817	5.067
S-measure	154.2	139.7	126.2	114.0	103.0	93.3	84.8	77.4	70.9
T-measure	157.6	142.7	128.6	115.5	103.6	92.8	83.1	74.6	67.1
MC	155.4	140.8	127.4	115.1	104.2	94.4	85.9	78.5	72.0

CMS10Y caplets: Analytical approximation errors w.r.t. MC

Strikes	3.067	3.317	3.567	3.817	4.067	4.317	4.567	4.817	5.067
S-measure	-1.2	-1.2	-1.2	-1.1	-1.1	-1.1	-1.1	-1.1	-1.1
T-measure	2.2	1.8	1.2	0.4	-0.6	-1.7	-2.8	-3.9	-4.9

7.1.2 CMS2Y products for 5Y maturity

The 2 year rate has a forward value the origin equal to 3.739%. Its payment price and convexity adjustment are presented in the table below.

	Price	CA
Approx	3.286	12.7
MC	3.287	12.8

CMS2Y caplets: Prices

Strikes	2.739	2.989	3.239	3.489	3.739	3.989	4.239	4.489	4.739
S-measure	138.7	125.1	112.6	101.3	91.2	82.3	74.5	67.7	61.8
T-measure	139.7	125.9	113.1	101.5	91.1	81.8	73.6	66.4	60.1
MC	138.9	125.3	112.8	101.5	91.4	82.5	74.7	67.9	62.0

CMS2Y caplets: Analytical approximation errors w.r.t. MC

Strikes	2.739	2.989	3.239	3.489	3.739	3.989	4.239	4.489	4.739
S-measure	-0.2	-0.2	-0.2	-0.2	-0.2	-0.2	-0.2	-0.2	-0.2
T-measure	0.8	0.6	0.4	0.0	-0.4	-0.7	-1.1	-1.5	-1.8

7.1.3 CMS spread option for 5Y maturity

CMS10Y-CMS2Y spread options: Prices

Strikes	-0.672	-0.422	-0.172	0.078	0.328	0.578	0.828	1.078	1.328
S-measure	111.0	91.4	72.4	54.9	40.1	29.1	21.7	16.8	13.4
T-measure	114.0	93.9	74.6	56.4	40.1	27.0	17.9	12.2	8.7
MC	112.0	91.9	72.6	54.8	39.9	29.0	21.7	16.8	13.3

CMS10Y-CMS2Y spread options: Analytical approximation errors w.r.t. MC

Strikes	-0.672	-0.422	-0.172	0.078	0.328	0.578	0.828	1.078	1.328
S-measure	-1.0	-0.5	-0.2	0.1	0.2	0.1	0.0	0.0	0.1
T-measure	2.0	2.0	2.0	1.6	0.2	-2.0	-3.8	-4.6	-4.6

7.2 CMS instruments numerical check: 10Y maturity

We consider here the fixing date τ equal to the payment date T set to 10Y. The forward rate processes are formally defined as

$$\text{CMS10 forward rate } \rightarrow S_1(t) = \frac{P(t,T_{10}) - P(t,T_{20})}{\sum\limits_{i=11}^{20} \delta_{i-1} P(t,T_i)},$$

$$\text{CMS2 forward rate } \rightarrow S_2(t) = \frac{P(t,T_{10}) - P(t,T_{12})}{\sum\limits_{i=11}^{12} \delta_{i-1} P(t,T_i)}.$$

7.2.1 CMS10Y products for 10Y maturity

The 10 year rate has a forward value at the origin equal to 3.86377 %. Its payment price and convexity adjustment are presented in the table below.

	Price	CA
Approx	3.272	57.2
MC	3.262	56.3

CMS10y caplets: Prices

Strikes	2.864	3.114	3.364	3.614	3.864	4.114	4.364	4.614	4.864
S-measure	158.8	147.2	136.5	126.6	117.7	109.7	102.6	96.2	90.6
T-measure	162.4	150.1	138.3	127.3	117.0	107.4	98.6	90.5	83.2
MC	158.3	146.7	136.0	126.1	117.2	109.2	102.1	95.7	90.0

CMS10y caplets: Analytical approximation errors w.r.t. MC

Strikes	2.864	3.114	3.364	3.614	3.864	4.114	4.364	4.614	4.864
S-measure	0.4	0.5	0.5	0.5	0.5	0.5	0.5	0.6	0.6
T-measure	4.1	3.4	2.4	1.2	-0.3	-1.8	-3.5	-5.2	-6.8

7.2.2 CMS2Y products for 10Y maturity

The 2 year rate has a forward value the origin equal to 4.1843%. Its payment price and convexity adjustment are presented in the table below.

	Price	CA
Approx	3.166	24.3
MC	3.164	24.1

CMS2Y caplets: Prices

Strikes	3.184	3.434	3.684	3.934	4.184	4.434	4.684	4.934	5.184
S-measure	146.2	135.8	126.2	117.4	109.3	102.0	95.3	89.3	83.9
T-measure	147.3	136.6	126.5	117.0	108.3	100.3	93.0	86.4	80.3
MC	146.2	135.8	126.2	117.4	109.3	101.9	95.3	89.3	83.8

CMS2Y caplets: Analytical approximation errors w.r.t. MC

Strikes	3.184	3.434	3.684	3.934	4.184	4.434	4.684	4.934	5.184
S-measure	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
T-measure	1.2	0.7	0.2	-0.3	-1.0	-1.6	-2.3	-2.9	-3.5

7.2.3 CMS spread option for 10Y maturity

CMS10Y-CMS2Y spread options: Prices

Strikes	-1.321	-1.071	-0.821	-0.571	-0.321	-0.071	0.179	0.429	0.679
S-measure	105.0	89.1	73.4	58.4	44.9	33.2	24.2	17.6	13.0
T-measure	108.0	91.7	75.5	59.8	44.9	31.4	20.4	12.7	7.9
MC	104.0	87.9	72.1	57.3	44.4	33.9	26.2	20.6	16.7

CMS10Y-CMS2Y spread options: Analytical approximation errors w.r.t. MC

strikes	-1.321	-1.071	-0.821	-0.571	-0.321	-0.071	0.179	0.429	0.679
S-measure	1.0	1.2	1.3	1.1	0.5	-0.7	-2.0	-3.0	-3.7
T-measure	4.0	3.8	3.4	2.5	0.5	-2.5	-5.8	-7.9	-8.8

7.3 Observations

In the tables above, we have seen a good fit of the CMS product prices calculated using approximations with respect to Monte Carlo simulations. In particular, the simple T-forward measure adjusted result performs quite well for near-ATM options (the error does not exceed 1 bp). However, for far-from-ATM strikes, the T-forward measure approximation deteriorates up to $\sim$ 8.8 bps in the worst case. On the other hand, our main swap- or spread-measure method shows much more *homogeneous* fit (the biggest error is $\sim$ 3.8 bps).

For better visualization, we present below two graphs for CMS10Y caplets and CMS10Y-CMS2Y spread options (with 10Y maturity).

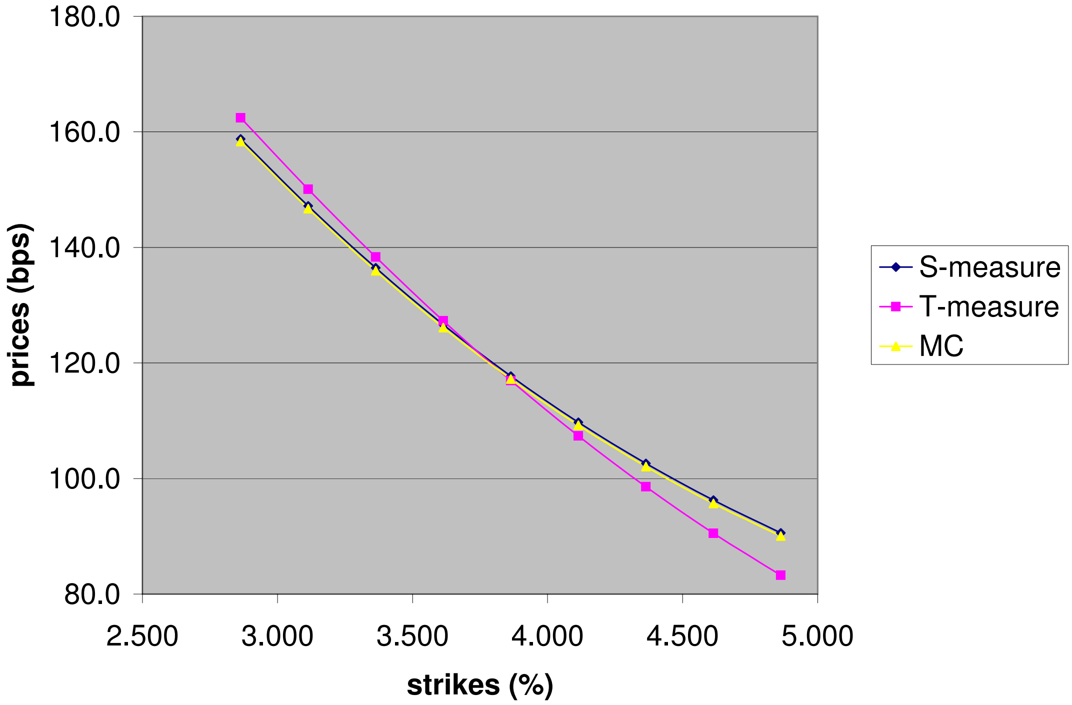

Figure 2: CMS10Y caplets for 10Y maturity.

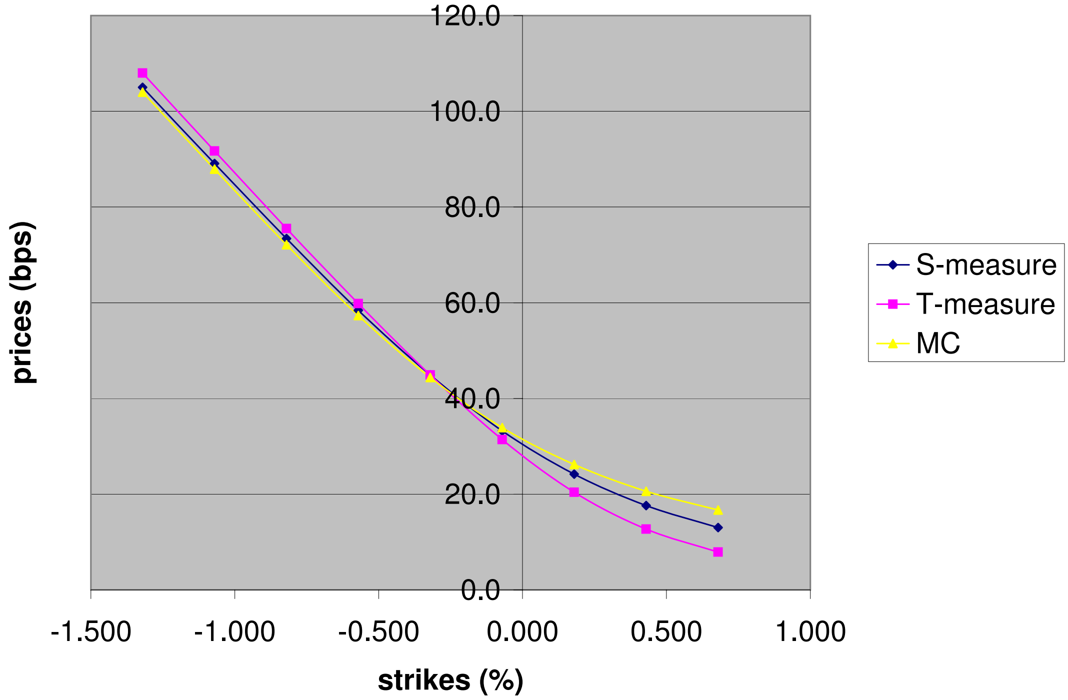

Figure 3: CMS10Y-CMS2Y spread option for 10Y maturity.

8 Conclusion

In this paper, we developed a series of approximations for fast analytical pricing of European CMS products for the LMM with stochastic volatility. The derived formulas can be used for model calibration to the market, including European swaptions and CMS products. This simultaneous calibration procedure is important for consistent pricing of CMS exotic instruments. However, during market crisis, when the CMS products become less liquid and the exotic market falls into recession, the proposed approximation can be used for a fast pricing of the CMS products given LMM calibrated to European swaptions. The LMM, having a rich parametric freedom, can be, in principle, calibrated to the whole swaption matrix. Thus, European CMS product evaluation with this model can provide a consistent pricing with whole-market information. Moreover, the CMS analytics deliver an arbitrage free price, contrary to known CMS evaluations based on certain strike extrapolation assumptions for European swaptions.

The main technical achievement of this work is related to the optimal calculation of the measure change. For single-rate CMS products, we have used the standard measure change, linear regression, with optimally calculated coefficients. For the CMS spread options, we have studied numerically the measure change regression to the spread and proposed to modify the effective *non-linear* regression basis. This new result was confirmed numerically.

We have also developed a new formula for the general spread option pricing given the double characteristic function via the two-dimensional Laplace transform.

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A Laplace transform technique for computation of averages

In this appendix, we recall the basics of the Laplace transform and its application to the calculation of averages given a characteristic function.

Consider first the one-dimensional case. Namely, suppose that our stochastic variable y has characteristic function

$$\varphi(\xi) = E\left[e^{\xi\, y}\right],$$

which exists in a certain regularity strip, $\alpha < \Re\xi < \beta$. We also define the moment generation function $\mathcal{K}$ as a logarithm of the characteristic function,

$$\varphi(\xi) = e^{\mathcal{K}(\xi)}.$$

In order to calculate an average,

$$E\left[f(y)\right],$$

using the characteristic function, one should apply the Laplace transform technique. Indeed, define the (double-sided) *Laplace transform*,

$$\mathfrak{L}[f](\xi) = \int_{-\infty}^{+\infty} dx f(x)\, e^{-\xi\, x},$$

which exists in a *region of convergence* for the parameter ξ. Under some technical conditions, the initial function can be restored via the *inverse Laplace transform*,

$$f(y) = \frac{1}{2\pi i}\int_C d\xi \mathfrak{L}[f](\xi) e^{\xi\, y},$$

where the contour C lies in the region of convergence.

Instead of the Laplace integral representation, one can use the Fourier one,

$$\mathfrak{F}[f](\omega) = \int_{-\infty}^{+\infty} dx f(x)\, e^{-i\,\omega\, x},$$

with

$$f(y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\xi \mathfrak{F}[f](\xi) e^{i\omega\, y}.$$

The Fourier transform is equivalent to the Laplace transform under $\xi = i\omega$ provided that the imaginary line belongs to the region of convergence. Otherwise, one should impose a regularization, applying, for example, the Fourier transform to $f(y)\, e^{-\alpha\, y}$, which makes the Fourier transform

$$\mathfrak{F}[f(y)\, e^{-\alpha\, y}](\omega) = \int_{-\infty}^{+\infty} dx f(x)\, e^{-(\alpha + i\,\omega)\, x}$$

converging. This regularized Fourier transform is equivalent to the Laplace integral representation, provided that the contour was shifted from the imaginary axis to a line $\Re\xi = \alpha$. Traditionally, practioners use the Fourier transform for option pricing, see, for example, Stein-Stein (1991), Heston (1993), Carr-Madan (1999). However, we prefer the Laplace transform to the Fourier one as being more formal for the regularization procedure.

As an important example, consider an option with pay-off $(e^y - k)^+$. It can be represented as

$$(e^y - k)^+ = \frac{k}{2\pi i}\int_{C^+} \frac{d\xi}{\xi(\xi-1)} e^{y\xi - \xi\,\ln k} \tag{58}$$

because the Laplace transform of the option pay-off has the form

$$\mathfrak{L}\left[(e^y - k)^+\right](\xi) = \int_{-\infty}^{+\infty} dx\,(e^x - k)^+\; e^{-\xi\,x} = \frac{k}{\xi(\xi-1)} e^{-\xi\,\ln k}.$$

The contour C in the integral (58) should lie in the region of convergence $\Re\xi > 1$ to guarantee existence of the integral underlying the Laplace transform.

To calculate the option price $E\left[(e^y - k)^+\right]$, one should take the expectation operator of both sides of the representation (58),

$$E\left[(e^y - k)^+\right] = \frac{k}{2\pi i}\int_{C^+} \frac{d\xi}{\xi(\xi-1)} E[e^{y\xi}]\,e^{-\xi\,\ln k} = \frac{k}{2\pi i}\int_{C^+} \frac{d\xi}{\xi(\xi-1)}\; e^{\mathcal{K}(\xi) - \xi\,\ln k}. \tag{59}$$

In the main body of this paper, we considered parabolic options, or para-options, with pay-off $e^y\,(e^y - k)^+$. Using the integral (58), one can easily evaluate the para-option price,

$$E\left[e^y(e^y - k)^+\right] = \frac{k}{2\pi i}\int_{C^+} \frac{d\xi}{\xi(\xi-1)} E[e^y\,e^{y\xi}]\,e^{-\xi\,\ln k} = \frac{k}{2\pi i}\int_{C^+} \frac{d\xi}{\xi(\xi-1)}\; e^{\mathcal{K}(\xi+1) - \xi\,\ln k}. \tag{60}$$

One can generalize the Laplace transform to multiple-dimension cases. As an application to spread options, we consider the 2D case for two stochastic variables y_1 and y_2 having the following characteristic function,

$$\varphi(\xi_1, \xi_2) = E\left[e^{\xi_1\,y_1 + \xi_2\,y_2}\right],$$

which exists in a certain regularity strip, $\alpha_1 < \Re\xi_1 < \beta_1$ and $\alpha_2 < \Re\xi_2 < \beta_2$. As for the 1D case, we define the moment generation,

$$\varphi(\xi_1, \xi_2) = e^{\mathcal{K}(\xi_1, \xi_2)}.$$

For a function with two arguments, $f(y_1, y_2)$, define the 2D Laplace transform,

$$\mathfrak{L}[f](\xi_1, \xi_2) = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 f(x_1, x_2) e^{-\xi_1\,x_1 - \xi_2\,x_2},$$

which exists for a region of convergence for the parameters ξ_1 and ξ_2. Under certain technical conditions, the initial function can be restored via the *inverse 2D Laplace transform*,

$$f(y_1, y_2) = \frac{1}{(2\pi i)^2}\int_{C^{(1)}} d\xi_1 \int_{C^{(2)}} d\xi_2\; \mathfrak{L}[f](\xi_1, \xi_2)\; e^{\xi_1\,y_1 + \xi_2\,y_2}, \tag{61}$$

where the contours $C^{(1)}$ and $C^{(2)}$ lie in the region of convergence.

Consider now a spread option with pay-off

$$\mathcal{O}(y_1, y_2) = (C_1\,e^{y_1} - C_2\,e^{y_2} - K)^+$$

for a positive strike K and notionals C_1 and C_2. The 2D Laplace transform of this function,

$$\mathfrak{L}[\mathcal{O}](\xi_1,\xi_2) = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \, (C_1\, e^{x_1} - C_2\, e^{x_2} - K)^+ \, e^{-\xi_1\, x_1 - \xi_2\, x_2},$$

can be presented in a closed form involving Gamma functions. First, let us determine its region of convergence. Existence of the integral over x_1 is equivalent to $\Re\xi_1 > 1$. In this case,

$$\mathfrak{L}[\mathcal{O}](\xi_1,\xi_2) = \frac{1}{(2\pi i)^2}\frac{C_1}{\xi_1(\xi_1-1)} \int_{-\infty}^{+\infty} dx_2 \; e^{-(\xi_1-1)\,\ln\left(\frac{C_2}{C_1}e^{x_2}+\frac{K}{C_1}\right)-\xi_2\,x_2}. \tag{62}$$

The $-\infty$ end convergence requires $\Re\xi_2 < 0$ and the $+\infty$ end gives $\Re\xi_1 + \Re\xi_2 > 1$, which implies our first contour requirement $\Re\xi_1 > 1$. Finally, the 2D Laplace transform exists if

$$\Re\xi_2 < 0 \quad \text{and } \Re\xi_1 + \Re\xi_2 > 1. \tag{63}$$

The integral underlying (62) can be represented in terms of the Beta function. Namely, making the integration variable change $u = e^{x_2}$,

$$\begin{aligned} I(\xi_1,\xi_2) \;&\equiv\; \int_{-\infty}^{+\infty} dx_2 \; e^{-(\xi_1-1)\,\ln\left(\frac{C_2}{C_1}e^{x_2}+\frac{K}{C_1}\right)-\xi_2\,x_2} && (64)\\ &=\; \left(\frac{K}{C_1}\right)^{-(\xi_1-1)} \int_0^{\infty} du\, \left(1+\frac{C_2}{K}\,u\right)^{-(\xi_1-1)} u^{-(\xi_2+1)}. && (65)\end{aligned}$$

After the scaling $v = \frac{C_2}{K}u$, we get

$$I(\xi_1,\xi_2) = \left(\frac{K}{C_1}\right)^{-(\xi_1-1)} \left(\frac{K}{C_2}\right)^{-\xi_2} \int_0^{\infty} dv\; \frac{v^{-(\xi_2+1)}}{(1+v)^{\xi_1-1}}.$$

Note that the last integral is the Beta function,

$$B(z,w) = \int_0^{\infty} dv\; \frac{v^{z-1}}{(1+v)^{z+w}} = \frac{\Gamma(z)\,\Gamma(w)}{\Gamma(z+w)}. \tag{66}$$

Finally, the Laplace transform (62) of the spread option pay-off can be written as

$$\mathfrak{L}[\mathcal{O}](\xi_1,\xi_2) = \frac{C_1}{\xi_1(\xi_1-1)}\left(\frac{K}{C_1}\right)^{-(\xi_1-1)} \left(\frac{K}{C_2}\right)^{-\xi_2} B(-\xi_2, \xi_1+\xi_2-1).$$

Using the inverse 2D Laplace form (61), we obtain the spread option payoff in the required form for the price calculation,

$$(C_1\, e^{y_1} - C_2\, e^{y_2} - K)^+ = \frac{1}{(2\pi i)^2}\int_{C^{(1)}} d\xi_1 \int_{C^{(2)}} d\xi_2 \; \mathfrak{L}[\mathcal{O}](\xi_1,\xi_2)\; e^{\xi_1\, y_1 + \xi_2\, y_2}.$$

Finally, taking the expectation of both sides of the above equality, we obtain the desired 2D integral containing the characteristic function,

$$E\left[(C_1\, e^{y_1} - C_2\, e^{y_2} - K)^+\right] = \frac{1}{(2\pi i)^2}\int_{C^{(1)}} d\xi_1 \int_{C^{(2)}} d\xi_2 \; \mathfrak{L}[\mathcal{O}](\xi_1,\xi_2)\; e^{\mathcal{K}(\xi_1,\xi_2)}. \tag{67}$$

The integration variables $\xi_1 \in C^{(i)}$ satisfy the conditions (63) and should lie inside the characteristic function regularity strip. Known asymptotics and series for Gamma functions makes the numerical integration procedure of the final result quite attractive.

B Characteristic function of two correlated Heston processes with independent stochastic volatility.

In this appendix, we develop a characteristic function analytical expression for correlated Heston processes. We start with a set of processes $S_i(t)$ following the Heston SDE,

$$S_i(t) = C_i \, e^{y_i(t)},$$

where processes $y_i(t)$ satisfy the standard log-Heston evolution

$$dy_i(t) = -\frac{1}{2}\, z(t)|\lambda_i(t)|^2 \, dt + \sqrt{z(t)}\lambda_i(t) \cdot dW(t), \quad y_i(0) = 0,$$

with the CIR process as common stochastic volatility

$$dz(t) = \theta(t)(1 - z(t)) + \sqrt{z(t)}\gamma(t) \, dU(t), \quad z(0) = 1,$$

where the scalar Brownian motion U is independent of the vector Brownian motions W.

We denote by $y(t)$ a vector process $y(t) = \{y_1(t), y_2(t), \cdots\}$ and notice that the system $\{y(t), z(t)\}$ is affine and its mutual characteristic function

$$\varphi(\xi, T) = E\left[e^{\xi \cdot y(T)}\right] = E\left[e^{\sum_i \xi_i \, y_i(T)}\right]$$

can be calculated explicitly for any complex vector $\xi = \{\xi_1, \xi_2, \cdots\}$. Indeed, consider a conditional expectation

$$u(t; z, y) = E\left[e^{\xi \cdot y(T)} \mid z(t) = z, y(t) = y\right],$$

which satisfies PDE (zero drift condition)

$$u'(t) \;-\; \frac{1}{2}\, z \sum_i |\lambda_i(t)|^2 \, \frac{\partial u}{\partial y_i} + \frac{1}{2}\, z \sum_{i,j} \lambda_i(t) \cdot \lambda_j(t) \, \frac{\partial^2 u}{\partial y_i \, \partial y_j} \tag{68}$$

$$+ \;\; \theta(t)(1 - z) \, \frac{\partial u}{\partial z} + \frac{1}{2}\gamma(t)^2 \, z \, \frac{\partial^2 u}{\partial z^2} = 0. \tag{69}$$

Looking for the solution in affine form,

$$u(t; z, y) = e^{A(t) + B(t)\, z + C(t) \cdot y},$$

for two scalar function $A(t)$ and $B(t)$ and a vector function $C(t)$, and imposing final conditions $A(T) = B(T) = 0$ and $C(T) = \xi$, we have

$$\begin{aligned} C'(t) &= 0, \\ B'(t) &= -\frac{1}{2}\gamma(t)^2 B^2(t) + \theta(t)\, B(t) - \frac{1}{2}\left(\sum_{i,j} \lambda_i(t) \cdot \lambda_j(t) \, \xi_i \, \xi_j - \sum_i |\lambda_i(t)|^2 \, \xi_i\right), \\ A'(t) &= -\theta(t)\, B(t). \end{aligned} \tag{70}$$

These ODEs can be resolved numerically or, for step-constant parameters, iteratively using the Riccati solution, to give the characteristic function value

$$\varphi(\xi, T) = e^{A(0) + B(0)}.$$

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