Transformation formulas for fractional Brownian motion

Céline Jost[*]

Department of Mathematics and Statistics, P.O. Box 68 (Gustaf Hällströmin katu 2b), 00014 University of Helsinki, Finland

Abstract

We derive a Molchan–Golosov-type integral transform which changes fractional Brownian motion of arbitrary Hurst index K into fractional Brownian motion of index H. Integration is carried out over $[0, t]$, $t > 0$. The formula is derived in the time domain. Based on this transform, we construct a prelimit which converges in $L^2(\mathbb{P})$-sense to an analogous, already known Mandelbrot–Van Ness-type integral transform, where integration is over $(-\infty, t]$, $t > 0$.
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1. Introduction

The *fractional Brownian motion with Hurst index* $H \in (0, 1)$, or *H-fBm*, is the continuous, centered Gaussian process $(B_t^H)_{t \in \mathbb{R}}$ with $B_0^H = 0$, a.s., and

$$\mathrm{Cov}_{\mathbb{P}}(B_s^H, B_t^H) = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |t - s|^{2H}), \quad s, t \in \mathbb{R}.$$

For $H = \frac{1}{2}$, fractional Brownian motion is standard Brownian motion (sBm) and denoted by W. One of the most important properties of fBm is the stationarity of its increments. For $H > \frac{1}{2}$,

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[*] Tel.: +358 9 19151510; fax: +358 9 19151400.
E-mail address: celine.jost@helsinki.fi.

increments are positively correlated and form a long-range dependent sequence, whereas for $H < \frac{1}{2}$, increments are negatively correlated and this sequence exhibits short-range dependence. Furthermore, fBm is H-self-similar, meaning that $(B^H_{at})_{t\in\mathbb{R}} \overset{d}{=} (a^H B^H_t)_{t\in\mathbb{R}}$ for every $a > 0$. FBm is interesting from a theoretical point of view, since it is fairly simple but neither Markov nor semimartingale. The latter fact, for example, makes integration with respect to it challenging. FBm was first mentioned and studied by Kolmogorov in 1940 under the name of *Wiener spiral* (see [6]). The modern name *fractional Brownian motion* was proposed by Mandelbrot and Van Ness in 1968, when they described fBm by a Wiener integral process of a *fractional integral kernel*, namely

$$B^H_t = \frac{1}{C'(H)} \int_{\mathbb{R}} \left((t-s)^{H-\frac{1}{2}}_+ - (-s)^{H-\frac{1}{2}}_+ \right) d\tilde{W}_s, \quad \text{a.s., } t \in \mathbb{R}, \tag{1.1}$$

for some sBm $(\tilde{W}_t)_{t\in\mathbb{R}}$ (see [7]). Here $x^a_+ := x^a \cdot 1_{(0,\infty)}(x)$,

$$C'(H) := \left(\int_0^\infty \left((1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}} \right)^2 ds + \frac{1}{2H} \right)^{\frac{1}{2}} = \frac{\Gamma(H+\frac{1}{2})}{(\Gamma(2H+1)\sin(\pi H))^{\frac{1}{2}}}$$

and Γ is the gamma function. In 1969, Molchan and Golosov represented fBm for positive t alternatively by

$$B^H_t = \frac{C(H)}{\Gamma(H+\frac{1}{2})} \int_0^t (t-s)^{H-\frac{1}{2}} F\left(\frac{1}{2} - H, H - \frac{1}{2}, H + \frac{1}{2}, \frac{s-t}{s} \right) dW_s, \quad \text{a.s.,} \tag{1.2}$$

for some (different) sBm $(W_t)_{t\in[0,\infty)}$, where F is Gauss' hypergeometric function and

$$C(H) := \left(\frac{2H\Gamma(H+\frac{1}{2})\Gamma(\frac{3}{2}-H)}{\Gamma(2-2H)} \right)^{\frac{1}{2}} \tag{1.3}$$

(see [8]). The kernel in (1.2) is a weighted fractional integral. Expressing it in terms of F is due to Decreusefond and Üstünel in 1999 (see [3]). Proofs for the real line transform (1.1) and the positive line transform (1.2) can also be found in [15], p. 320–325, and [9], respectively. Nowadays, these moving averages are called *time domain representations* of fBm. Given B^H, the sBms $\tilde{W}$ and W in (1.1) and (1.2) are unique (up to modification). FBm has been studied recently in connection to applications in finance and telecommunications. The filtrations generated by $(B^H_t)_{t\in[0,\infty)}$ and $(W_t)_{t\in[0,\infty)}$ do coincide, whereas this is not the case for the natural filtrations of $(B^H_t)_{t\in\mathbb{R}}$ and $(\tilde{W}_t)_{t\in\mathbb{R}}$. Therefore, in financial modelling, (1.2) seems a priori more convenient than (1.1) for representing fBm in terms of sBm. In 2002, Pipiras and Taqqu generalized the real line transform in the sense that, for any $K \in (0, 1)$, there exists a unique K-fBm $\tilde{B}^K$ such that, for all $t \in \mathbb{R}$, there holds

$$B^H_t = \frac{C'(K)\Gamma(H+\frac{1}{2})}{\Gamma(K+\frac{1}{2})C'(H)\Gamma(H-K+1)} \int_{\mathbb{R}} \left((t-s)^{H-K}_+ - (-s)^{H-K}_+ \right) d\tilde{B}^K_s, \quad \text{a.s.} \tag{1.4}$$

(see [11], Theorem 1.1). The integral in (1.4) is a *fractional Wiener integral*.

The first aim of this work is to prove a generalization of (1.2). Namely, we show that, for given $K \in (0, 1)$, there exists a unique $(B_t^K)_{t \in [0, \infty)}$ such that, for all $t \in [0, \infty)$, we have a.s. that

$$
\begin{aligned}
B_t^H &= \frac{C(H)C(K)^{-1}}{\Gamma(H - K + 1)} \\
&\quad \times \int_0^t (t - s)^{H-K} F\left(1 - K - H, H - K, 1 + H - K, \frac{s - t}{s}\right) dB_s^K.
\end{aligned}
\tag{1.5}
$$

A similar approach as in the proof of (1.4) in [11] is also used here. However, due to the more complicated kernel, the proof is technically more involved. Indeed, we have to compose not just simple fractional integrals, but weighted ones. For this purpose, we study the connection between weighted fractional integrals and some special functions. These results can be generally helpful when dealing with fBm. Second, we show that the real line transform (1.4) can be obtained alternatively by constructing an approximating sequence of fBms, which is based on the positive line transform (1.5).

This article is organized as follows. In Section 2, we explain the special functions involved in transformation (1.5). Section 3 is dedicated to the fractional calculus connected to fractional Wiener integration. In Section 4, we review the construction of the fractional Wiener integral in the time domain. The ingredients to derive formula (1.4) (as in [11]) will be repeated briefly within Sections 3 and 4. Then in Section 5, we prove (1.5) and show that (1.4) appears as a consequence.

2. Some special functions

The proof of (1.5) relies heavily on the understanding of some special Euler integrals and their respective extensions. We review their definitions, clarify how they relate to each other, and state well-known properties relevant for us.

The *gamma function* of $\alpha > 0$ is defined by

$$
\Gamma(\alpha) := \int_0^\infty \exp(-u)u^{\alpha-1} \, du.
$$

By partial integration, we obtain $\Gamma(\alpha + 1) = \alpha \Gamma(\alpha)$. This recursion is used to extend Γ to all $\alpha \in \mathcal{A} := \mathbb{R} \setminus -\mathbb{N}_0$. We have $\Gamma(1) = 1$. Moreover,

$$
\frac{1}{\Gamma(\beta)} := \lim_{\substack{\alpha \to \beta \\ \alpha \in \mathcal{A}}} \frac{1}{\Gamma(\alpha)} = 0, \quad \beta \in -\mathbb{N}_0.
\tag{2.1}
$$

The *beta function* of $\alpha, \beta > 0$ is defined by

$$
B(\alpha, \beta) := \int_0^1 (1 - v)^{\alpha-1} v^{\beta-1} dv.
$$

For $\beta > 1$, we have that $B(\alpha, \beta - 1) = \frac{\alpha+\beta-1}{\beta-1} B(\alpha, \beta)$. This recursion and the symmetry relation $B(\alpha, \beta) = B(\beta, \alpha)$ are used to extend B to all $\alpha, \beta \in \mathcal{A}$. B can be expressed in terms of Γ via (see [4], p. 9)

$$
B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}, \quad \alpha, \beta \in \mathcal{A}.
\tag{2.2}
$$

The *Pochhammer symbol* of $a \in \mathbb{R}$ is defined by

$$(a)_0 := 1 \quad \text{and} \quad (a)_k := a \cdot (a+1) \cdots \cdots (a+k-1), \quad k \in \mathbb{N}.$$

We have

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}, \quad a+k \in \mathcal{A}. \tag{2.3}$$

Moreover,

$$\binom{n}{k} = \frac{(-1)^k (-n)_k}{k!}, \quad k, n \in \mathbb{N}_0, n \geq k. \tag{2.4}$$

The *Gauss hypergeometric function* of parameters a, b, c and variable $z \in \mathbb{R}$ is defined by the formal power series

$$F(a, b, c, z) := {}_2F_1(a, b, c, z) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}.$$

We assume that $c \in \mathcal{A}$ for this to make sense. If $|z| < 1$ or $|z| = 1$ and $c - b - a > 0$, then the series converges absolutely. If, furthermore, $c > b > 0$ for $z \in [-1, 1)$ and $b > 0$ for $z = 1$, then it can be represented by the Euler integral

$$F(a, b, c, z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 v^{b-1}(1-v)^{c-b-1}(1-vz)^{-a} \, \mathrm{d}v \tag{2.5}$$

(see [4], p. 59). If $c > b > 0$, then the expression on the right-hand side of (2.5) is well-defined for all $z \in (-\infty, 1)$, and is therefore used as an extended definition of F. In order to extend F for fixed $z \in (-\infty, 1]$ to more general parameters, we consider Gauss' relations for neighbor functions. *Neighbors* of $F(a, b, c, z)$ are functions of type $F(a \pm 1, b, c, z)$, $F(a, b \pm 1, c, z)$ or $F(a, b, c \pm 1, z)$. For any two neighbors $F_1(z)$, $F_2(z)$ of $F(a, b, c, z)$, one has a linear relation of type

$$A(z)F(a, b, c, z) + A_1(z)F_1(z) + A_2(z)F_2(z) = 0,$$

where A, A_1 and A_2 are first-degree polynomials (see [1], p. 558). Based on these relations, we extend F for $z \in (-\infty, 1)$ to all $a, b, c \in \mathbb{R}$ such that $c \in \mathcal{A}$, and for $z = 1$ to all parameters that satisfy $c, c - b - a \in \mathcal{A}$. Important properties of F are the symmetry $F(a, b, c, z) = F(b, a, c, z)$, the reduction formula

$$F(0, b, c, z) = F(a, b, c, 0) = 1 \tag{2.6}$$

and the linear transformation formula (see [1], p. 559)

$$F(a, b, c, z) = (1 - z)^{-a} F\left(a, c - b, c, \frac{z}{z-1}\right), \quad z < 1. \tag{2.7}$$

F is smooth in every parameter and in z. We have (see [1], p. 557)

$$\frac{\mathrm{d}^n}{\mathrm{d}^n z}(z^{c-1} F(a, b, c, z)) = (c-n)_n z^{c-n-1} F(a, b, c-n, z), \quad n \in \mathbb{N}. \tag{2.8}$$

Moreover, for $c \in -\mathbb{N}_0$ we have (see [1], p. 556)

$$
\frac{F(a, b, c, z)}{\Gamma(c)} := \lim_{\substack{x \to c \\ x \in \mathcal{A}}} \frac{F(a, b, x, z)}{\Gamma(x)}
$$

$$
= \frac{(a)_{1-c}(b)_{1-c}}{\Gamma(2-c)} z^{1-c} F(a + 1 - c, b + 1 - c, 2 - c, z). \tag{2.9}
$$

The *third hypergeometric function* is one of Appell's four generalizations of $F(a, b, c, z)$ to two variables. It is defined by the formal series

$$
F_3(a, a', b, b', c, z, z') := \sum_{k,l=0}^{\infty} \frac{(a)_k (a')_l (b)_k (b')_l}{(c)_{k+l}} \frac{z^k}{k!} \frac{z'^l}{l!}.
$$

We assume that $c \in \mathcal{A}$ so that the summands are well-defined. If $|z|, |z'| < 1$, then the series converges absolutely; see [2], p. 74. If, additionally, $b, b', c - b - b' > 0$, then it has the double Euler integral representation (see [2], p. 77)

$$
F_3(a, a', b, b', c, z, z') = \frac{\Gamma(c)}{\Gamma(b)\Gamma(b')\Gamma(c - b - b')}
$$

$$
\times \int_0^1 \int_0^{1-v} u^{b-1} v^{b'-1} (1 - u - v)^{c-b-b'-1}
$$

$$
\times (1 - uz)^{-a} (1 - vz')^{-a'} \, du \, dv. \tag{2.10}
$$

The right-hand side of (2.10) is well-defined for all $z, z' < 1$ and is therefore used as extension of F_3. By substituting $w := \frac{u}{1-v}$ and using (2.5) we obtain

$$
F_3(a, a', b, b', c, z, z') = \frac{\Gamma(c)}{\Gamma(b')\Gamma(c - b')} \cdot \int_0^1 v^{b'-1} (1 - vz')^{-a'}
$$

$$
\times (1 - v)^{c-b'-1} F(a, b, c - b', (1 - v)z) \, dv. \tag{2.11}
$$

Also, we have the symmetry relations

$$
F_3(a, a', b, b', c, z, z') = F_3(a', a, b', b, c, z', z) = F_3(a', b, b', a, c, z', z). \tag{2.12}
$$

By using (2.11) and (2.12), one can derive neighbor relations for F_3, where here it is clear what we mean by neighbors. Using these, one can extend F_3 for $z, z' \in (-\infty, 1)$ to all parameters such that $c \in \mathcal{A}$. We have the reduction formulas

$$
F_3(a, 0, b, b', c, z, z') = F(a, b, c, z) \tag{2.13}
$$

and (see [2], p. 79–80)

$$
F_3\left(a, a', b, b', a + a', \frac{s-t}{s}, \frac{t-s}{t}\right) = \left(\frac{t}{s}\right)^{b'} F\left(a, b + b', a + a', \frac{s-t}{s}\right). \tag{2.14}
$$

F_3 is smooth in all arguments. In particular, we have the differentiation formula

$$
\frac{d}{dz}\left((1 - z)^{-a'} z^{c-1} F_3\left(a, a', b, b', c, z, \frac{z}{z-1}\right)\right)
$$

$$
= (c - 1) z^{c-2} (1 - z)^{-a'} F_3\left(a, a', b, b', c - 1, z, \frac{z}{z-1}\right)
$$

$$+ a' z^{c-1} (1 - z)^{-a'-1} F_3 \left(a, a' + 1, b, b', c, z, \frac{z}{z - 1} \right). \tag{2.15}$$

By combining neighbor relations linearly, we obtain the relation

$$\begin{aligned}
c F_3(a, a', b, b', c, z, z') &= c F_3(a, a', b, b' + 1, c, z, z') \\
&\quad - z' a' F_3(a, a' + 1, b, b' + 1, c + 1, z, z').
\end{aligned} \tag{2.16}$$

Both (2.15) and (2.16) are easily verified by using series expansions.

Remark 2.1. All four Euler integrals Γ, B, F and F_3 are analytic when considered as functions of complex arguments. The extensions that we defined above are the restrictions of the unique analytic extensions to real arguments.

3. Fractional calculus

Next, we review basic Riemann–Liouville fractional calculus and its connection to fBm. A general source on fractional calculus is [14]. See also [10] for details on fractional calculus in the context of fBm. We show that the integrand in (1.2) is a power weighted fractional integral of an indicator. Furthermore, we derive a composition formula for power weighted fractional integrals. After that, we will briefly repeat the fractional calculus for the real line case.

Definition 3.1. Let $\alpha > 0$. The *(right-sided) Riemann–Liouville fractional integral operator of order* α over the real line is defined by

$$(\mathcal{I}_-^\alpha f)(s) := \frac{1}{\Gamma(\alpha)} \int_s^\infty f(u)(u - s)^{\alpha-1} \, du, \quad s \in \mathbb{R}.$$

Let $T > 0$. The *(right-sided) Riemann–Liouville fractional integral operator of order* α over $[0, T]$ is defined by

$$(\mathcal{I}_{T-}^\alpha f)(s) := (\mathcal{I}_-^\alpha f 1_{[0,T)})(s), \quad s \geq 0.$$

Fractional calculus is a generalization of usual calculus: for $n \in \mathbb{N}$ and f suitably integrable, there holds $(\mathcal{I}_-^n f)(s) = \int_s^\infty \int_{s_{n-1}}^\infty \ldots \int_{s_1}^\infty f(u) \, du \, ds_1 \ldots ds_{n-1}$. Clearly, $\mathcal{I}_-^1 f$ exists for $f \in L^1(\mathbb{R})$. Furthermore, if $\alpha \in (0, 1)$, $p \in [1, \frac{1}{\alpha})$ and $f \in L^p(\mathbb{R})$, then $\mathcal{I}_-^\alpha f$ is well-defined (see [14], p. 94). Note that $\mathcal{I}_-^\alpha f$ depends on f only via $f 1_{[0,T]}$. So $\mathcal{I}_{T-}^\alpha$ in fact operates on functions $f : [0, T] \to \mathbb{R}$. If $f \in L^1[0, T]$, then $\mathcal{I}_{T-}^\alpha f \in L^1[0, T]$ for all $\alpha > 0$ (see [14], p. 48). Using (2.2) yields the composition formulas

$$\mathcal{I}_-^\alpha \mathcal{I}_-^\beta f = \mathcal{I}_-^{\alpha+\beta} f, \qquad f \in L^1(\mathbb{R}), \mathcal{I}_-^\beta f \in L^1(\mathbb{R}), \alpha, \beta \in (0, 1], \tag{3.1}$$

and

$$\mathcal{I}_{T-}^\alpha \mathcal{I}_{T-}^\beta f = \mathcal{I}_{T-}^{\alpha+\beta} f, \quad f \in L^1[0, T], \alpha, \beta > 0. \tag{3.2}$$

3.1. Extension and composition of fractional integrals with power weights

For $t \in (0, T]$, we have

$$(\mathcal{I}_{T-}^\alpha f 1_{[0,t)})(s) = (\mathcal{I}_{t-}^\alpha f)(s) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_s^t f(u)(u - s)^{\alpha-1} \, du & \text{if } s \in [0, t) \\ 0 & \text{if } s \geq t. \end{cases}$$

Assuming that $f \in L^1[s, t]$ and $\alpha > 0$ is vital to ensure the existence of the integral. However, by using extended Euler integrals, we are able to define $\mathcal{I}_{t-}^{\alpha} f$ for certain parametric, non-integrable functions f on the one hand and for negative α on the other hand. Therefore, fix $s \in (0, t)$ and consider first functions of type

$$f_{\gamma}(u) := u^{\gamma}, \quad \gamma \in \mathbb{R}.$$

Clearly, $(\mathcal{I}_{t-}^{\alpha} f_{\gamma})(s)$ is well-defined for $\alpha > 0$. Using (2.5) and (2.7), we obtain

$$(\mathcal{I}_{t-}^{\alpha} f_{\gamma})(s) = \frac{1}{\Gamma(1+\alpha)} (t-s)^{\alpha} s^{\gamma} F\left(-\gamma, \alpha, 1+\alpha, \frac{s-t}{s}\right) \tag{3.3}$$

$$= \frac{1}{\Gamma(1+\alpha)} (t-s)^{\alpha} t^{\gamma} F\left(-\gamma, 1, 1+\alpha, \frac{t-s}{t}\right). \tag{3.4}$$

For $\alpha \leq 0$, define the (improper) fractional integral $(\mathcal{I}_{t-}^{\alpha} f_{\gamma})(s)$ by the right-hand side of (3.3) or (3.4). In particular, for $\alpha \in -\mathbb{N}$ we have, by combining (3.3), (2.9) and (2.6), that $(\mathcal{I}_{t-}^{\alpha} f_{\gamma})(s) = (-\gamma)_{-\alpha} s^{\gamma+\alpha}$. Furthermore, (2.6) implies

$$(\mathcal{I}_{t-}^{\alpha} 1)(s) = \frac{1}{\Gamma(1+\alpha)} (t-s)^{\alpha}, \quad \alpha \in \mathbb{R}. \tag{3.5}$$

Next, we consider more general functions of type

$$f_{t,\gamma}^{\beta}(u) := u^{\gamma} (\mathcal{I}_{t-}^{\beta} 1)(u), \quad \gamma, \beta \in \mathbb{R}.$$

The integral $(\mathcal{I}_{t-}^{\alpha} f_{t,\gamma}^{\beta})(s)$ converges if and only if $\alpha, \beta + 1 > 0$. By (2.5), we have

$$(\mathcal{I}_{t-}^{\alpha} f_{t,\gamma}^{\beta})(s) = \frac{s^{\gamma} (t-s)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} F\left(-\gamma, \alpha, \alpha+\beta+1, \frac{s-t}{s}\right). \tag{3.6}$$

For ($\alpha \leq 0$ or $\beta + 1 \leq 0$) and $\beta \notin -\mathbb{N}$, we define the (improper) fractional integral $(\mathcal{I}_{t-}^{\alpha} f_{t,\gamma}^{\beta})(s)$ by the right-hand side of (3.6). Let $\beta \in -\mathbb{N}$. By using (3.5) and (2.1), we have $(\mathcal{I}_{t-}^{\alpha} f_{t,\gamma}^{\beta})(s) = 0$ for $\alpha > 0$. For $\alpha \leq 0$, we thus set $(\mathcal{I}_{t-}^{\alpha} f_{t,\gamma}^{\beta})(s) := 0$. Finally, we consider functions of type

$$f_{t,\gamma,\lambda}^{\beta}(u) := u^{\gamma} (\mathcal{I}_{t-}^{\beta} \cdot^{\lambda})(u), \quad \gamma, \beta, \lambda \in \mathbb{R}.$$

$f_{t,\gamma,\lambda}^{\beta} \in L^1[s, t]$ if and only if $\beta + 1 > 0$. Thus $(\mathcal{I}_{t-}^{\alpha} f_{t,\gamma,\lambda}^{\beta})(s)$ converges if and only if $\alpha, \beta + 1 > 0$. By using (3.4) and (2.11), we obtain

$$(\mathcal{I}_{t-}^{\alpha} f_{t,\gamma,\lambda}^{\beta})(s) = \frac{t^{\lambda} s^{\gamma} (t-s)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} F_3\left(-\lambda, -\gamma, 1, \alpha, \alpha+\beta+1, \frac{t-s}{t}, \frac{s-t}{s}\right). \tag{3.7}$$

For ($\alpha \leq 0$ or $\beta + 1 \leq 0$) and $\beta, \alpha + \beta \notin -\mathbb{N}$, we define the (improper) fractional integral $(\mathcal{I}_{t-}^{\alpha} f_{t,\gamma,\lambda}^{\beta})(s)$ by the right-hand side of (3.7). By combining (2.13) and (3.4), we obtain on the one hand

$$(\mathcal{I}_{t-}^{0} f_{t,\gamma,\lambda}^{\beta})(s) = f_{t,\gamma,\lambda}^{\beta}(s), \quad \beta \notin -\mathbb{N}, \tag{3.8}$$

and on the other hand

$$(\mathcal{I}_{t-}^{\alpha} \mathcal{I}_{t-}^{\beta} f_{\lambda})(s) = (\mathcal{I}_{t-}^{\alpha+\beta} f_{\lambda})(s) = (\mathcal{I}_{t-}^{\beta} \mathcal{I}_{t-}^{\alpha} f_{\lambda})(s), \quad \alpha, \beta, \alpha+\beta \notin -\mathbb{N}. \tag{3.9}$$

We can now easily state and prove a reduction formula for the composition of suitably power weighted fractional integrals:

Lemma 3.2 (*Composition Formula*). *Let* $\nu, \alpha, \beta \in \mathbb{R}$ *with* $\beta, \alpha + \beta \notin -\mathbb{N}$. *Then*

$$s^{\beta}(\mathcal{I}_{t-}^{\alpha} \cdot^{-\beta-\alpha} \mathcal{I}_{t-}^{\beta} \cdot^{\alpha+\nu})(s) = (\mathcal{I}_{t-}^{\alpha+\beta} \cdot^{\nu})(s), \quad s \in (0, t).$$

Proof. Combine (3.7), (2.12), (2.14) and (3.3). □

Note that, for parameters $\alpha, \beta + 1 > 0$, i.e. when the fractional integral on the left-hand side converges, the claim coincides with (10.44) on p. 192 in [14].

3.2. Fractional derivatives

In Section 3.1, we obtained a definition for fractional integrals of negative order of certain parametric functions in a very natural way. Observing (3.8) and (3.9) suggests that $\mathcal{I}_{t-}^{-\alpha}$ inverts the operator $\mathcal{I}_{t-}^{\alpha}$, i.e. $\mathcal{I}_{t-}^{-\alpha} f$ is indeed a kind of derivative of f. A fractional derivative for arbitrary functions is defined by the formal solution in g of *Abel's integral equation* $\mathcal{I}_{T-}^{\alpha} g = f$ (see [14], p. 29–30):

Definition 3.3. Let $\alpha \in (0, 1)$. The *(right-sided) Riemann–Liouville fractional derivative operator of order* α over the real line is defined by

$$(\mathcal{D}_{-}^{\alpha} f)(s) := \frac{-\mathrm{d}}{\mathrm{d}s}(\mathcal{I}_{-}^{1-\alpha} f)(s) = \frac{1}{\Gamma(1-\alpha)} \frac{-\mathrm{d}}{\mathrm{d}s} \int_{s}^{\infty} f(u)(u-s)^{-\alpha} \, \mathrm{d}u, \quad s \in \mathbb{R}.$$

Moreover,

$$\mathcal{D}_{-}^{0} f := f. \tag{3.10}$$

For $\alpha = [\alpha] + \{\alpha\}$, where $[\alpha]$ and $\{\alpha\}$ correspond to integer and fractional parts of α, define

$$(\mathcal{D}_{-}^{\alpha} f)(s) := (-1)^{[\alpha]} \frac{\mathrm{d}^{[\alpha]}}{\mathrm{d}s^{[\alpha]}}(\mathcal{D}_{-}^{\{\alpha\}} f)(s), \quad s \in \mathbb{R}.$$

The *(right-sided) Riemann–Liouville fractional derivative operator of order* $\alpha \geq 0$ over $[0, T]$ is defined by

$$(\mathcal{D}_{T-}^{\alpha} f)(s) := (\mathcal{D}_{-}^{\alpha} f 1_{[0,T)})(s), \quad s > 0.$$

For $t \in (0, T]$, there holds $\mathcal{D}_{T-}^{\alpha} f 1_{[0,t)} = \mathcal{D}_{t-}^{\alpha} f$. The fractional derivative $(\mathcal{D}_{t-}^{\alpha} f_{t,\gamma,\lambda}^{\beta})(s)$, $s \in (0, t)$, is (properly or improperly) defined for $\beta, 1 - \{\alpha\} + \beta \notin -\mathbb{N}$ via (3.7). The following shows that fractional derivatives and fractional integrals of negative order coincide.

Lemma 3.4. *Let* $\alpha \geq 0$. *For* $\gamma, \lambda \in \mathbb{R}$, $\beta, 1 - \{\alpha\} + \beta, \beta - \alpha \notin -\mathbb{N}$, *we have*

$$(\mathcal{D}_{t-}^{\alpha} f_{t,\gamma,\lambda}^{\beta})(s) = (\mathcal{I}_{t-}^{-\alpha} f_{t,\gamma,\lambda}^{\beta})(s), \quad s \in (0, t).$$

Proof. For $\alpha = 0$, the claim is true by (3.8) and (3.10). For $\alpha \in \mathbb{N}$ we have, by using (2.8), then (2.4) combined with (2.3) and the series representations of F and F_3, and finally (3.7) that

$$(\mathcal{D}^\alpha_{t-} f^\beta_{t,\gamma,\lambda})(s) = \frac{t^{\lambda+\beta}(-1)^\alpha}{\Gamma(\beta+1)} \frac{\mathrm{d}^\alpha}{\mathrm{d}s^\alpha} \left(s^\gamma \left(\frac{t-s}{t} \right)^\beta F \left(-\lambda, 1, 1+\beta, \frac{t-s}{t} \right) \right)$$

$$= \frac{t^{\lambda+\beta}(-1)^\alpha}{\Gamma(\beta+1)} \sum_{k=0}^{\alpha} \binom{\alpha}{k} (-\gamma)_k (-1)^k s^{\gamma-k} (1+\beta-\alpha+k)_{\alpha-k}$$

$$\cdot \left(\frac{t-s}{t} \right)^{\beta-\alpha+k} F \left(-\lambda, 1, 1+\beta-\alpha+k, \frac{t-s}{t} \right) \frac{(-1)^{\alpha-k}}{t^{\alpha-k}}$$

$$= \frac{t^\lambda (t-s)^{\beta-\alpha} s^\gamma}{\Gamma(\beta+1-\alpha)} F_3 \left(-\alpha, -\lambda, -\gamma, 1, 1+\beta-\alpha, \frac{s-t}{s}, \frac{t-s}{t} \right)$$

$$= (\mathcal{I}^{-\alpha}_{t-} f^\beta_{t,\gamma,\lambda})(s).$$

Using (2.15) and (2.16) yields, for $\alpha \in (0, 1)$,

$$(\mathcal{D}^\alpha_{t-} f^\beta_{t,\gamma,\lambda})(s) = \frac{-\mathrm{d}}{\mathrm{d}s} (\mathcal{I}^{1-\alpha}_{t-} f^\beta_{t,\gamma,\lambda})(s)$$

$$= \frac{t^{\lambda+\gamma+1-\alpha+\beta}}{\Gamma(2-\alpha+\beta)} \frac{-\mathrm{d}}{\mathrm{d}s} \left(\left(\frac{s}{t} \right)^\gamma \left(\frac{t-s}{t} \right)^{1-\alpha+\beta} F_3 \left(-\lambda, -\gamma, 1, 1-\alpha, 2-\alpha+\beta, \frac{t-s}{t}, \frac{s-t}{s} \right) \right)$$

$$= \frac{t^{\lambda-\alpha+\beta+\gamma}}{\Gamma(2-\alpha+\beta)} \left((1-\alpha+\beta) \left(\frac{t-s}{t} \right)^{-\alpha+\beta} \left(\frac{s}{t} \right)^\gamma F_3 \left(-\lambda, -\gamma, 1, 1-\alpha, 1-\alpha+\beta, \frac{t-s}{t}, \frac{s-t}{s} \right) \right.$$

$$\left. + \left(\frac{t-s}{t} \right)^{1-\alpha+\beta} \left(\frac{s}{t} \right)^{\gamma-1} (-\gamma) F_3 \left(-\lambda, -\gamma+1, 1, 1-\alpha, 2-\alpha+\beta, \frac{t-s}{t}, \frac{s-t}{s} \right) \right)$$

$$= \frac{t^\lambda s^\gamma (t-s)^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)} F_3 \left(-\lambda, -\gamma, 1, -\alpha, \beta-\alpha+1, \frac{t-s}{t}, \frac{s-t}{s} \right) = (\mathcal{I}^{-\alpha}_{t-} f^\beta_{t,\gamma,\lambda})(s).$$

By iterating, we obtain the claim for all α with $\{\alpha\} > 0$. $\square$

In view of Lemma 3.4, it is consistent to set

$$\mathcal{I}^{-\alpha}_{T-} := \mathcal{D}^\alpha_{T-}, \quad \alpha \geq 0.$$

General, nonparametric functions in the domain of $\mathcal{D}^\alpha_-$ must be suitably integrable and smooth. From (3.1), we obtain

$$\mathcal{D}^\alpha_- \mathcal{I}^\beta_- f = \mathcal{I}^{\beta-\alpha}_- f, \quad f \in L^1(\mathbb{R}), \mathcal{I}^\beta_- f \in L^1(\mathbb{R}), 0 < \alpha \leq \beta \leq 1. \tag{3.11}$$

Furthermore, from (3.2) it follows that

$$\mathcal{D}^\alpha_{T-} \mathcal{I}^\beta_{T-} f = \mathcal{I}^{\beta-\alpha}_{T-} f, \quad f \in L^1[0, T], 0 < \alpha \leq \beta. \tag{3.12}$$

As for ordinary calculus, the fractional operators $\mathcal{D}^\alpha_-$ and $\mathcal{I}^\beta_-$ do not commute. For details on their composition, see [14], subparagraphs 2.6 and 2.7.

3.3. The connection to fBm

For $H \in (0, 1)$, define weighted fractional integral operators by

$$(\mathbf{K}^H f)(s) := (\mathbf{K}^H_T f)(s) := C(H) s^{\frac{1}{2}-H} (\mathcal{I}^{H-\frac{1}{2}}_{T-} \cdot^{H-\frac{1}{2}} f)(s), \quad s \in (0, T),$$

and

$$(\mathbf{K}^{H,*} f)(s) := (\mathbf{K}_T^{H,*} f)(s) := C(H)^{-1} s^{\frac{1}{2}-H} (\mathcal{I}_{T-}^{\frac{1}{2}-H} \cdot^{H-\frac{1}{2}} f)(s), \quad s \in (0, T),$$

where $C(H)$ is the constant defined in (1.3). Note that $\mathbf{K}^H$ ($\mathbf{K}^{H,*}$) is a true weighted integral operator, i.e. of positive order, if and only if $H > \frac{1}{2}$ ($H < \frac{1}{2}$). $\mathbf{K}^{\frac{1}{2}}$ and $\mathbf{K}^{\frac{1}{2},*}$ are identity operators. By (3.3), we have

$$(\mathbf{K}^H 1_{[0,t)})(s) = \frac{C(H)}{\Gamma(H + \frac{1}{2})} (t-s)^{H-\frac{1}{2}} F\left(\frac{1}{2} - H, H - \frac{1}{2}, H + \frac{1}{2}, \frac{s-t}{s}\right) 1_{(0,t)}(s).$$

So for $t \in [0, T]$, (1.2) can be written as

$$B_t^H = \int_0^T (\mathbf{K}^H 1_{[0,t)})(s) \, \mathrm{d}W_s, \quad \text{a.s.} \tag{3.13}$$

Lemma 3.5. *Let $H, K \in (0, 1)$. Then*

$$(\mathbf{K}^{K,*} \mathbf{K}^H 1_{[0,t)})(s) = \frac{C(H)}{C(K)} s^{1-H-K} (\mathcal{I}_{t-}^{H-K} \cdot^{H+K-1})(s), \quad s \in (0, T). \tag{3.14}$$

Moreover,

$$(\mathbf{K}^K \mathbf{K}^{K,*} \mathbf{K}^H 1_{[0,t)})(s) = (\mathbf{K}^H 1_{[0,t)})(s), \quad s \in (0, T). \tag{3.15}$$

Proof. Let $s \in (0, t)$. By setting $\alpha := \frac{1}{2} - K$, $\beta := H - \frac{1}{2}$ and $\nu := H + K - 1$ in Lemma 3.2, it follows that

$$\begin{aligned}(\mathbf{K}^{K,*} \mathbf{K}^H 1_{[0,t)})(s) &= \frac{C(H)}{C(K)} s^{\frac{1}{2}-K} (\mathcal{I}_{t-}^{\frac{1}{2}-K} \cdot^{K-\frac{1}{2}} \cdot^{\frac{1}{2}-H} \mathcal{I}_{t-}^{H-\frac{1}{2}} \cdot^{H-\frac{1}{2}})(s) \\ &= \frac{C(H)}{C(K)} s^{1-H-K} (\mathcal{I}_{t-}^{H-K} \cdot^{H+K-1})(s).\end{aligned}$$

Using this and furthermore setting $\alpha := K - \frac{1}{2}$, $\beta := H - K$ and $\nu := H - \frac{1}{2}$ in Lemma 3.2 implies that

$$\begin{aligned}(\mathbf{K}^K \mathbf{K}^{K,*} \mathbf{K}^H 1_{[0,t)})(s) &= C(H) s^{\frac{1}{2}-K} (\mathcal{I}_{t-}^{K-\frac{1}{2}} \cdot^{K-\frac{1}{2}} \cdot^{1-H-K} \mathcal{I}_{t-}^{H-K} \cdot^{H+K-1})(s) \\ &= C(H) s^{\frac{1}{2}-H} (\mathcal{I}_{t-}^{H-\frac{1}{2}} \cdot^{H-\frac{1}{2}})(s) = (\mathbf{K}^H 1_{[0,t)})(s). \quad \square\end{aligned}$$

Remark 3.6. By setting $H = K$ in (3.14) and $H = \frac{1}{2}$ in (3.15), we obtain that $\mathbf{K}^{K,*} \mathbf{K}^K 1_{[0,t)} = 1_{[0,t)} = \mathbf{K}^K \mathbf{K}^{K,*} 1_{[0,t)}$. Moreover, (3.12) implies, for general $f \in L^2[0, T]$, that $\mathbf{K}^{K,*} \mathbf{K}^K f = f$ if $K > \frac{1}{2}$ and $\mathbf{K}^K \mathbf{K}^{K,*} f = f$ if $K < \frac{1}{2}$.

3.4. Fractional calculus for the real line analogue

Let $t \in \mathbb{R} \setminus \{0\}$ and set $1_{[0,t)} := -1_{[t,0)}$ for $t < 0$. For $\alpha > 0$, we have

$$(\mathcal{I}_-^\alpha 1_{[0,t)})(s) = \frac{1}{\Gamma(\alpha + 1)} ((t-s)^\alpha 1_{(-\infty,t)}(s) - (-s)^\alpha 1_{(-\infty,0)}(s)). \tag{3.16}$$

For $\alpha \le 0$, define the (improper) fractional integral $(\mathcal{I}_-^\alpha 1_{[0,t)})(s)$ by the right-hand side of (3.16). If $\alpha, \beta + 1 > 0$, then (2.2) implies that

$$(\mathcal{I}_-^\alpha \mathcal{I}_-^\beta 1_{[0,t)})(s) = (\mathcal{I}_-^{\alpha+\beta} 1_{[0,t)})(s), \quad s \in \mathbb{R}. \tag{3.17}$$

For $\alpha \le 0$ and $\beta + 1 > 0$, define the (improper) integral $(\mathcal{I}_-^\alpha \mathcal{I}_-^\beta 1_{[0,t)})(s)$ by the right-hand side of (3.17).

When dealing with fractional Wiener integrals, we need to invert $g = \mathcal{I}_-^\alpha f$ for $\alpha \in (0, \frac{1}{2})$ and $f \in L^2(\mathbb{R})$. However, (3.11) is in general not true for such f. Therefore, $\mathcal{D}_-^\alpha$ must be modified to a more convenient form (see [14], p. 109):

Definition 3.7. Let $\alpha \in (0, 1)$. The *(right-sided) Marchaud fractional derivative operator of order α* over the real line is defined by

$$(\mathbf{D}_-^\alpha f)(s) := \lim_{\epsilon \searrow 0} (\mathbf{D}_{-,\epsilon}^\alpha f)(s), \quad \text{a.e. } s \in \mathbb{R},$$

where

$$(\mathbf{D}_{-,\epsilon}^\alpha f)(s) := \frac{\alpha}{\Gamma(1-\alpha)} \int_\epsilon^\infty (f(s) - f(u+s))u^{-\alpha-1}\, \mathrm{d}u.$$

Furthermore,

$$\mathbf{D}_-^0 f := f.$$

The following is a consequence of Theorem 6.1, p. 125 in [14].

Lemma 3.8. *Let $\alpha \in (0, \frac{1}{2})$ and $f \in L^2(\mathbb{R})$. Then $\mathbf{D}_-^\alpha \mathcal{I}_-^\alpha f = f$.*

If f is piecewise continuously differentiable with $\mathrm{supp}(f') \subseteq (-\infty, y)$ for some $y \in \mathbb{R}$ and so that $\mathcal{D}_-^\alpha f$ exists, then $\mathbf{D}_-^\alpha f = \mathcal{D}_-^\alpha f$. In particular,

$$\mathbf{D}_-^\alpha \mathcal{I}_-^\beta 1_{[0,t)} = \mathcal{D}_-^\alpha \mathcal{I}_-^\beta 1_{[0,t)}, \quad \alpha \in [0, 1), \beta > -1. \tag{3.18}$$

Lemma 3.9. *For $\alpha \in [0, 1)$ and $\beta > -1$, we have that $\mathbf{D}_-^\alpha \mathcal{I}_-^\beta 1_{[0,t)} = \mathcal{I}_-^{-\alpha} \mathcal{I}_-^\beta 1_{[0,t)}$.*

Proof. For $\alpha = 0$, this is trivial. Let $\alpha \in (0, 1)$. Clearly, $\mathcal{D}_-^\alpha \mathcal{I}_-^\beta 1_{[0,t)} = \mathcal{I}_-^{\beta-\alpha} 1_{[0,t)}$. The claim follows from (3.18). $\quad\square$

In view of Lemma 3.9, we may set

$$\mathcal{I}_-^{-\alpha} := \mathbf{D}_-^\alpha, \quad \alpha \in [0, 1).$$

Hence (1.1) is equivalent to

$$B_t^H = \frac{\Gamma(H+\frac{1}{2})}{C'(H)} \int_\mathbb{R} (\mathcal{I}_-^{H-\frac{1}{2}} 1_{[0,t)})(s)\mathrm{d}\tilde{W}_s, \quad \text{a.s., } t \in \mathbb{R}. \tag{3.19}$$

4. Fractional Wiener integrals

First, we review — by combining time domain representation (3.13) with the standard Wiener integral — how one can identify a space of deterministic functions f for which the expression $\int_0^T f(s)\, \mathrm{d}B_s^H$ is well-defined. Second, we present the basic notation for integration over the

real line. For details on fractional Wiener integration over $[0, T]$ and $\mathbb{R}$, see [12] and [13], respectively.

4.1. Fractional Wiener integrals over $[0, T]$

Let $\mathcal{E}_T$ denote the space of elementary functions on $[0, T]$, i.e. functions of type $f(s) := \sum_{i=1}^m a_i 1_{[0, t_i)}(s)$, where $t_1, \ldots, t_m \in [0, T]$ and $a_1, \ldots, a_m \in \mathbb{R}$. Furthermore, let $\mathcal{H}_T(B^H)$ denote the completion in $L^2(\mathbb{P})$ of the linear space generated by the variables B_t^H, $t \in [0, T]$. We define the *fractional Wiener integral* of elementary f with respect to B^H in the natural way by the linear injection

$$
I_T^H : \mathcal{E}_T \to \mathcal{H}_T(B^H)
$$

$$
f \mapsto I_T^H(f) := \int_0^T f(s)\, \mathrm{d}B_s^H := \sum_{i=1}^m a_i B_{t_i}^H.
$$

Formula (3.13) and the linearity of $\mathbf{K}^H$ yield

$$
I_T^H(f) = \int_0^T f(s)\, \mathrm{d}B_s^H = \int_0^T (\mathbf{K}^H f)(s)\, \mathrm{d}W_s, \quad \text{a.s.,} \ f \in \mathcal{E}_T. \tag{4.1}
$$

In view of (4.1) and the classical Wiener–Itô isometry, for $H > \frac{1}{2}$ it is natural to define a linear space by

$$
\Lambda_T(H) := \left\{ f : [0, T] \to \mathbb{R} \mid \cdot^{H-\frac{1}{2}} f(\cdot) \in L^1[0, T] \text{ and } \int_0^T (\mathbf{K}^H f)(s)^2\, \mathrm{d}s < \infty \right\}.
$$

For $H < \frac{1}{2}$, we set

$$
\Lambda_T(H) := \{ f : [0, T] \to \mathbb{R} \mid \exists \phi_f \in L^2[0, T] \text{ such that } f = \mathbf{K}^{H,*}\phi_f \}.
$$

$\mathcal{E}_T$ is dense in $\Lambda_T(H)$ with respect to the scalar product defined by $\langle f, g \rangle_{\Lambda_T(H)} := (\mathbf{K}^H f, \mathbf{K}^H g)_{L^2[0,T]}$. We define the *(time domain) fractional Wiener integral* of $f \in \Lambda_T(H)$ with respect to B^H by

$$
I_T^H(f) := \int_0^T f(s)\, \mathrm{d}B_s^H := L^2(\mathbb{P}) - \lim_{n \to \infty} \int_0^T f_n(s)\, \mathrm{d}B_s^H = \int_0^T (\mathbf{K}^H f)(s)\, \mathrm{d}W_s,
$$

where $(f_n)_{n \in \mathbb{N}} \subseteq \mathcal{E}_T$ is an approximating sequence for f. By construction, $\int_0^T f(s)\, \mathrm{d}B_s^H$ is centered and Gaussian. $\Lambda_T(H)$ is complete if and only if $H < \frac{1}{2}$. So the distance preserving map I_T^H is an isometry if and only if $H < \frac{1}{2}$.

Example 4.1. An important fractional Wiener integral process is the autoregressive representation of fBm, which shows how to transform the complicated fBm into the simpler sBm. By (3.3), for all $t \in [0, T]$ we have a.s. that

$$
W_t = \int_0^T (\mathbf{K}^H \mathbf{K}^{H,*} 1_{[0,t)})(s)\, \mathrm{d}W_s = \int_0^T (\mathbf{K}^{H,*} 1_{[0,t)})(s)\, \mathrm{d}B_s^H
$$

$$
= \int_0^t \frac{(t-s)^{\frac{1}{2}-H}}{C(H)\Gamma(\frac{3}{2}-H)} F\left(\frac{1}{2} - H, \frac{1}{2} - H, \frac{3}{2} - H, \frac{s-t}{s} \right) \mathrm{d}B_s^H. \tag{4.2}
$$

Representation (1.2) and its reciprocal (4.2) together imply that $\mathbb{F}^{B^H} = \mathbb{F}^W$, where $\mathbb{F}^{B^H}$ is the completed filtration generated by $(B_t^H)_{t\in[0,T]}$. Furthermore, $\mathcal{H}_t(B^H) = \mathcal{H}_t(W)$, $t \in [0, T]$.

4.2. Fractional Wiener integrals over $\mathbb{R}$

The construction of the fractional Wiener integral over $\mathbb{R}$ is based on (3.19) and follows an analogous approximation procedure as in Section 4.1. For $H > \frac{1}{2}$, the linear space of integrands is given by

$$\Lambda(H) := \left\{ f \in L^1(\mathbb{R}) \mid \int_{\mathbb{R}} (\mathcal{I}_-^{H-\frac{1}{2}} f)(s)^2 \, \mathrm{d}s < \infty \right\},$$

whereas, for $H < \frac{1}{2}$, we have

$$\Lambda(H) := \{ f : \mathbb{R} \to \mathbb{R} \mid \exists \phi_f \in L^2(\mathbb{R}) \text{ such that } f = \mathcal{I}_-^{\frac{1}{2}-H} \phi_f \}.$$

The scalar product on $\Lambda(H)$ is defined by

$$\langle f, g \rangle_{\Lambda(H)} := \left(\frac{\Gamma(H+\frac{1}{2})}{C'(H)} \right)^2 (\mathcal{I}_-^{H-\frac{1}{2}} f, \mathcal{I}_-^{H-\frac{1}{2}} g)_{L^2(\mathbb{R})}.$$

The *(time domain) fractional Wiener integral* of $f \in \Lambda(H)$ with respect to B^H is defined by

$$I^H(f) := \int_{\mathbb{R}} f(s) \, \mathrm{d}B_s^H := \frac{\Gamma(H+\frac{1}{2})}{C'(H)} \int_{\mathbb{R}} (\mathcal{I}_-^{H-\frac{1}{2}} f)(s) \, \mathrm{d}\tilde{W}_s.$$

Example 4.2. By using (3.17) and (3.16), we obtain the autoregressive representation of fBm on the real line. For all $t \in \mathbb{R}$, we have a.s.

$$\tilde{W}_t = \frac{C'(H)}{\Gamma(H+\frac{1}{2})\Gamma(\frac{3}{2}-H)} \int_{\mathbb{R}} \left((t-s)_+^{\frac{1}{2}-H} - (-s)_+^{\frac{1}{2}-H} \right) \mathrm{d}B_s^H. \tag{4.3}$$

Let $H \neq \frac{1}{2}$. For $t < 0$, the interval of integration in (1.4) and in its inversion formula (4.3) is not $(-\infty, t]$, but $(-\infty, 0]$. Thus $\mathbb{G}^{B^H} \neq \mathbb{G}^{\tilde{W}}$, where $\mathbb{G}^{B^H}$ and $\mathbb{G}^{\tilde{W}}$ denote the completed natural filtrations of $(B_t^H)_{t\in\mathbb{R}}$ and $(\tilde{W}_t)_{t\in\mathbb{R}}$, respectively.

5. The transformation formula

We are now able to state and prove the main result.

Theorem 5.1. *Let $(B_t^H)_{t\in[0,T]}$ be an H-fBm and let $K \in (0, 1)$. Then there exists a unique (up to modification) K-fBm $(B_t^K)_{t\in[0,T]}$ such that, for all $t \in [0, T]$, we have a.s.*

$$B_t^H = C(H)C(K)^{-1} \int_0^T s^{1-H-K} (\mathcal{I}_{t-}^{H-K} \cdot^{H+K-1})(s) \, \mathrm{d}B_s^K$$

$$= C(K, H) \int_0^t (t-s)^{H-K} F\left(1-K-H, H-K, 1+H-K, \frac{s-t}{s} \right) \mathrm{d}B_s^K.$$

Here

$$C(K, H) := \frac{C(H)C(K)^{-1}}{\Gamma(H - K + 1)}.$$

Moreover, $\mathbb{F}^{B^H} = \mathbb{F}^{B^K}$ *and* $\mathcal{H}_t(B^H) = \mathcal{H}_t(B^K)$, $t \in [0, T]$.

Proof. Let $(W_t)_{t \in [0,T]}$ be the unique sBm that satisfies

$$B_t^H = \int_0^T (\mathbf{K}^H 1_{[0,t)})(s) \, dW_s, \quad \text{a.s., } t \in [0, T].$$

There exists a unique K-fBm $(B_t^K)_{t \in [0,T]}$ such that

$$W_t = \int_0^T (\mathbf{K}^{K,*} 1_{[0,t)})(s) \, dB_s^K, \quad \text{a.s., } t \in [0, T].$$

In fact, this is the K-fBm given by

$$B_t^K := \int_0^T (\mathbf{K}^K 1_{[0,t)})(s) \, dW_s = \int_0^T (\mathbf{K}^{H,*} \mathbf{K}^K 1_{[0,t)})(s) \, dB_s^H, \quad t \in [0, T], \tag{5.1}$$

where the equality follows from (3.15). By Example 4.1, we have $\mathbb{F}^{B^H} = \mathbb{F}^W = \mathbb{F}^{B^K}$ and $\mathcal{H}_t(B^H) = \mathcal{H}_t(W) = \mathcal{H}_t(B^K)$, $t \in [0, T]$. Using (3.15) implies, for all $t \in [0, T]$, that

$$B_t^H = \int_0^T (\mathbf{K}^K \mathbf{K}^{K,*} \mathbf{K}^H 1_{[0,t)})(s) \, dW_s = \int_0^T (\mathbf{K}^{K,*} \mathbf{K}^H 1_{[0,t)})(s) \, dB_s^K, \quad \text{a.s.}$$

We obtain the desired expressions by using (3.14) and (3.3).   □

In particular, we obtain an easy formula for the transformation of an fBm with positively correlated increments into an fBm with negatively correlated increments, and vice versa:

Corollary 5.2. *Let* $(B_t^H)_{t \in [0,T]}$ *be an* H-*fBm. Then there exists a unique (up to modification)* $(1 - H)$-*fBm* $(B_t^{1-H})_{t \in [0,T]}$ *such that*

$$B_t^H = \left(\frac{2H}{\Gamma(2H)\Gamma(3 - 2H)} \right)^{\frac{1}{2}} \int_0^t (t - s)^{2H-1} \, dB_s^{1-H}, \quad a.s., t \in [0, T].$$

Proof. Set $K := 1 - H$ in Theorem 5.1 and use (2.6).   □

Remark 5.3. In Theorem 5.1 and Corollary 5.2, the finite interval $[0, T]$ can be replaced by $[0, \infty)$. This is due to the fact that the kernel $(\mathbf{K}^{K,*} \mathbf{K}^H 1_{[0,t)})(s)$ and its reciprocal $(\mathbf{K}^{H,*} \mathbf{K}^K 1_{[0,t)})(s)$ — hence in particular B^K defined in (5.1) — do not depend on T.

Remark 5.4. The Malliavin derivative of B_t^H with respect to B^K is given by $(D^{B^K} B_t^H)(\omega) = C(K, H)(t - \cdot)^{H-K} F(1 - K - H, H - K, 1 + H - K, \frac{-t}{\cdot}) 1_{[0,t)}(\cdot)$.

Remark 5.5. The *Riemann–Liouville process with index* $H > 0$ is defined by

$$U_t^H := \sqrt{2H} \int_0^t (t - s)^{H - \frac{1}{2}} \, dW_s, \quad t \in [0, \infty).$$

We have $\mathrm{Cov}_{\mathbb{P}}(U_s^H, U_t^H) = \frac{2H}{H+\frac{1}{2}} s^{H+\frac{1}{2}} t^{H-\frac{1}{2}} F\left(\frac{1}{2} - H, 1, H + \frac{3}{2}, \frac{s}{t}\right)$ for $s \leq t$. U^H was already mentioned in [7] as an alternative form of fBm. U^H is H-self-similar, however, contrary to fBm, its increments are not stationary. By (3.5), we have $U_t^H = \sqrt{2H}\Gamma(H + \frac{1}{2})$ $\int_0^T (I_{T-}^{H-\frac{1}{2}} 1_{[0,t)})(s) \, \mathrm{d}W_s$, $t \in [0, T]$, so it is clear how one can define a fractional Wiener integral with respect to U^H. It is easy to show that, for the Riemann–Liouville process $U_t^K := \sqrt{2K} \int_0^t (t - s)^{K-\frac{1}{2}} \, \mathrm{d}W_s$, there holds

$$U_t^H = \frac{\sqrt{H}\Gamma(H + \frac{1}{2})}{\sqrt{K}\Gamma(K + \frac{1}{2})\Gamma(H - K + 1)} \int_0^t (t - s)^{H-K} \, \mathrm{d}U_s^K, \quad \text{a.s., } t \in [0, \infty).$$

By Corollary 5.2, the process B^H is hence obtained from B^{1-H} by the same transformation (modulo a constant) as the process U^H from U^{1-H}.

Remark 5.6. The *fundamental martingale* of B^H is defined by

$$M_t^H := \frac{\sqrt{2 - 2H}}{C(H)\Gamma\left(\frac{3}{2} - H\right)} \int_0^t s^{\frac{1}{2}-H}(t - s)^{\frac{1}{2}-H} \, \mathrm{d}B_s^H = \sqrt{2 - 2H} \int_0^t s^{\frac{1}{2}-H} \, \mathrm{d}W_s.$$

M^H is $(1 - H)$-self-similar. If B^K is the K-fBm in Theorem 5.1, then

$$M_t^H = \sqrt{\frac{1 - H}{1 - K}} \int_0^t s^{K-H} \, \mathrm{d}M_s^K, \quad \text{a.s., } t \in [0, \infty).$$

The Mandelbrot–Van Ness-type representation (1.4), where $\tilde{B}^K$ is the K-fBm defined by

$$\tilde{B}_t^K := \frac{\Gamma(K + \frac{1}{2})}{C'(K)} \int_{\mathbb{R}} (\mathcal{I}_-^{K-\frac{1}{2}} 1_{[0,t)})(s) \, \mathrm{d}\tilde{W}_s, \quad t \in \mathbb{R},$$

is obtained by combining (3.19) with composition formula (3.17), and then using (3.16). This is shown in detail in [11]. However, (1.4) can also be obtained directly from Theorem 5.1. In fact, (1.4) emerges as a boundary case of a suitable time-shifted Molchan–Golosov-type transform. In order to see this, let $(B_t^K)_{t\in\mathbb{R}}$ be a K-fBm. By Theorem 5.1 and Remark 5.3, the process defined by

$$B_t^H := C(K, H) \int_0^t (t - u)^{H-K} \hat{F}\left(\frac{u - t}{u}\right) \mathrm{d}B_u^K, \quad t \in [0, \infty),$$

where, for convenience, we set

$$\hat{F}(z) := F(1 - K - H, H - K, 1 + H - K, z),$$

is an H-fBm. B^K has stationary increments and so, for every $s > 0$, the process

$$B_t^{H,s} := C(K, H) \int_0^t (t - u)^{H-K} \hat{F}\left(\frac{u - t}{u}\right) \mathrm{d}B_{u-s}^K, \quad t \in [0, \infty), \tag{5.2}$$

is an H-fBm. So the increments of $B^{H,s}$ are stationary. By substituting $v := u - s$ in (5.2), we obtain that the time-shifted process

$$
\begin{aligned}
Z_t^{H,s} &:= B_{t+s}^{H,s} - B_s^{H,s} \\
&= C(K, H) \left(\int_{-s}^{t} (t - v)^{H-K} \hat{F}\left(\frac{v - t}{v + s}\right) dB_v^K - \int_{-s}^{0} (-v)^{H-K} \hat{F}\left(\frac{v}{v + s}\right) dB_v^K \right)
\end{aligned}
$$
(5.3)

is an H-fBm on $[-s, \infty)$. By using (2.6), the pointwise limit of the integrands as $s \to \infty$ suggests the following:

Corollary 5.7. *Let $(B_t^K)_{t\in\mathbb{R}}$ be a K-fBm. Then the process*

$$
Z_t^{H,\infty} := C(K, H) \int_{\mathbb{R}} \left((t - v)^{H-K} 1_{(-\infty, t)}(v) - (-v)^{H-K} 1_{(-\infty, 0)}(v) \right) dB_v^K
$$

is an H-fBm on the real line.

In order to conclude that the centered Gaussian process $Z^{H,\infty}$ is indeed a fractional Brownian motion, it can be shown that, for every $K \geq \frac{1}{2}$ and $t \in \mathbb{R}$, there exist constants $C_1 = C_1(K, H, t)$ and $s_1 = s_1(t) > 0$ such that

$$
E[Z_t^{H,s} - Z_t^{H,\infty}]^2 \leq C_1 s^{2H-2}, \quad s > s_1.
$$
(5.4)

In the same way, for $K < \frac{1}{2}$ and $t \in \mathbb{R}$, there exist constants $C_2 = C_2(K, H, t)$, $C_3 = C_3(K, H, t)$ and $s_2 = s_2(t) > 0$ with

$$
E[Z_t^{H,s} - Z_t^{H,\infty}]^2 \leq C_2 s^{2H-2} + C_3 s^{2K-2}, \quad s > s_2.
$$
(5.5)

An extensive technical proof for (5.4) and (5.5) is given in [5]. From (5.4) and (5.5), it follows that, for every $t \in \mathbb{R}$, we have $\lim_{s\to\infty} E[Z_t^{H,s} - Z_t^{H,\infty}]^2 = 0$. Hence, for all $t, t' \in \mathbb{R}$, we have $E[Z_t^{H,\infty} \cdot Z_{t'}^{H,\infty}] = \lim_{s\to\infty} E[Z_t^{H,s} \cdot Z_{t'}^{H,s}] = \frac{1}{2}\left(|t|^{2H} + |t'|^{2H} - |t' - t|^{2H}\right)$. This proves Corollary 5.7. Remark that, by combining a functional equation and the duplication formula for Γ (see [4], (6) on p. 3 and p. 5), we obtain $\frac{C(H)}{\Gamma(H+\frac{1}{2})} = C'(H)^{-1}$. So the constant $C(K, H)$ in Corollary 5.7 indeed coincides with the constant in (1.4).

Remark 5.8. In order to invert (5.3), define a K-fBm by $Z_t^{K,-s} := B_{t-s}^K - B_{-s}^K$, $t \in [0, \infty)$. Let $\tilde{F}(z) := F(1 - K - H, K - H, 1 + K - H, z)$. We have $B_t^{H,s} = C(K, H) \int_0^t (t - u)^{H-K} \hat{F}(\frac{u-t}{u}) dZ_u^{K,-s}$, $t \in [0, \infty)$, and hence $Z_t^{K,-s} = C(H, K) \int_0^t (t-u)^{K-H} \tilde{F}\left(\frac{u-t}{u}\right) dB_u^{H,s}$, $t \in [0, \infty)$. Thus $B_t^K = Z_{t+s}^{K,-s} - Z_s^{K,-s} = C(H, K)(\int_{-s}^t (t - v)^{K-H} \tilde{F}(\frac{v-t}{v+s}) dZ_v^{H,s} - \int_{-s}^0 (-v)^{K-H} \tilde{F}(\frac{v}{v+s}) dZ_v^{H,s})$, $t \in [-s, \infty)$.

Remark 5.9. Note that $L^2(\mathbb{P}) - \lim_{s\to\infty} B_{t+s}^{H,s}$ and $L^2(\mathbb{P}) - \lim_{s\to\infty} B_s^{H,s}$ do not exist separately. Indeed, the integrals $C(K, H) \int_{\mathbb{R}} (t - v)^{H-K} 1_{(-\infty, t)}(v) dB_v^K$ and $C(K, H) \int_{\mathbb{R}} (-v)^{H-K} 1_{(-\infty, 0)}(v) dB_v^K$, respectively, are not well-defined.

Remark 5.10. In the case $K = 1 - H$, the Mandelbrot–Van Ness-type integral in Corollary 5.7 can be truncated in $-s$ without changing the covariance. In fact, $Z_t^{H,s} = (\frac{2H}{\Gamma(2H)\Gamma(3-2H)})^{\frac{1}{2}} \int_{\mathbb{R}} ((t - v)^{2H-1} 1_{(-s,t)}(v) - (-v)^{2H-1} 1_{(-s,0)}(v)) dB_v^{1-H}$ is an H-fBm on $[-s, \infty)$.

Acknowledgements

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