**A QUADRATIC VOLATILITY CHEYETTE MODEL**


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**Abstract**

In this paper we present an extension of the one factor blended Cheyette model for pricing single currency exotics, allowing for a more adequate fit to the swaption volatility smile. We first present a general framework based on the HJM model and then make a separability assumption on the instantaneous forward rate volatility, thus enabling a representation of the discount curve in a finite number of Markovian state variables. We show a practical application of this family of models by analyzing calibration and pricing in the case of a quadratic volatility function. By doing so, we provide a novel and parsimonious specification of the Cheyette model. Then for calibration purposes, we develop fast and accurate approximations for European swaptions, based on standard projection and averaging techniques. We also improve the usual naïve mean state estimation by the use of Gaussian approximations. Last we present an efficient large step Monte-Carlo simulation for strongly path dependent exotics.



**Keywords:** HJM, Cheyette, Smile, Quadratic volatility, Averaging techniques.

## 1. Introduction

The Libor Market Model (LMM from now on) first introduced in Brace and Al (1997), either in its native lognormal specification or its extended stochastic volatility version as described in Piterbarg (2003) have now become standard tools in almost all financial institutions. Whether used as risk-management models or solely for benchmarking purposes, they boast substantial advantages such as intuitive primary variables, i.e. the forward rates, simplicity of calibration through analytical, respectively semi-analytical swaption pricing approximations and straightforward Monte-Carlo implementation. However they still exhibit shortcomings such as the absence of a Markovian representation of the discount curve, the computational cost which is quadratic in the number of dates and the lack of information on how to interpolate discount factors for non primary dates. Although a lot of effort has been put into tactically addressing these issues, it seems that a more strategic solution emerged with a new family of models, first introduced by Cheyette (1992) and Ritchken and Sankarasubramanian (1995). These models known as Quasi-Gaussian or Cheyette models start from the HJM specification of instantaneous forward rates dynamics and assume separability of volatility. From now on we will favor the terminology Cheyette models to describe such dynamics.

Under the Cheyette assumptions, there exists a closed form representation of discount bond prices in a small number of Markovian state variables at any arbitrary time and the constraint on the instantaneous volatility is loose enough so that a wide variety of interest rate smiles can be reproduced. However the above works don't provide insight into how fast and accurate calibration can be obtained. This question has been solved in the works of Andreasen (2001), in the case of the single factor blended Cheyette Model and Andreasen (2005) for multifactor blended stochastic volatility Cheyette models. The latter borrows ideas already used in the calibration of Stochastic Volatility LMM introduced in Piterbarg (2003), calculating volatility weights along the forward and optimally averaging parameters across time to derive efficient calibration routines. In the present work, instead, we suggest a parametric local volatility model of the instantaneous volatility. We place ourselves in a one factor Cheyette model and derive fast and accurate analytical formulas for pricing swaptions and present a fast second order accurate Monte-Carlo algorithm. This paper is organized as follows. In the second section we recall the HJM framework while in the third section we present the specifics of the Cheyette model. In the fourth section we choose to study a time dependent quadratic volatility version of the Cheyette model and present an accurate analytical approximation for swaption prices. In the fifth section we study the accuracy of this approximation. In the sixth section we present an efficient large step Monte-Carlo simulation which can be used for strongly path dependent structures.

## 2. Notations and a brief review of the HJM formalism

Here we recall the usual notations prevailing in Fixed Income modeling. We will denote the price at time $t$ of a pure discount bond paying one unit of domestic currency at time $T$ by $P(t,T)$. The instantaneous forward rate at time $t$ maturing at time $T$ will be denoted by $f(t,T)$. By construction, discount bond prices and instantaneous forward rates relate as follows:

$$P(t,T)=\exp\left(-\int_t^T f(t,s)\,ds\right)\Leftrightarrow f(t,T)=-\frac{\partial\ln\left(P(t,T)\right)}{\partial T}$$

The short rate prevailing at time $t$ will be denoted by $r(t)$. By definition $r(t)=f(t,t)$.

Assuming the existence of a risk-neutral measure, we posit the dynamics of discount bond prices under this measure as:

$$\frac{dP(t,T)}{P(t,T)}=r(t)\,dt-\sigma(t,T)\,dW(t)$$

where:

- $\sigma(t,T)$ is the volatility of discount bonds on which we make no particular assumptions other than the usual regularity properties
- $W$ is a standard one dimensional Brownian motion under the risk-neutral measure

Under these settings one can show easily that the dynamics of the instantaneous forward rates are completely determined and such that:

$$df(t,T)=\frac{\partial\sigma(t,T)}{\partial T}\sigma(t,T)\,dt+\frac{\partial\sigma(t,T)}{\partial T}dW(t)$$

We introduce the forward rate volatility $v(t,T)=\dfrac{\partial\sigma(t,T)}{\partial T}$ and rephrase the dynamics of forward rates as:

$$df(t,T)=v(t,T)\left(\int_t^T v(t,s)\,ds\right)dt+v(t,T)\,dW(t)$$

Here, we have summarized the HJM framework for instantaneous forward rates. In the next section we start to make additional assumptions on instantaneous volatilities to obtain the general one factor Cheyette model.

## 3. The Cheyette model

So far we treated of fairly well known general results within the HJM framework. We now impose more specific conditions on the volatility structure of forward rates. These assumptions can be found in Cheyette (1992) and can be summarized as such: we assume that the instantaneous volatility function is the product of a function which solely depends on the maturity and of a function of time and of a state vector $\theta$ which might be an arbitrary set of time $t$ forward rates and derivables values. Therefore under these assumptions we can write without loss of generality that there exist two functions $\alpha,\beta$ such that:

$$v(t,T) = \alpha(T)\frac{\beta(t,\theta)}{\alpha(t)}$$

Based on that volatility specification we deduce the following expression for instantaneous forward rates:

$$f(t,T) = f(0,T) + \int_0^t \frac{\alpha(T)}{\alpha(s)}\beta(s,\theta)\left(\int_s^T \frac{\alpha(u)}{\alpha(s)}\beta(s,\theta)du\right)ds + \int_0^t \frac{\alpha(T)}{\alpha(s)}\beta(s,\theta)\,dW(s)$$

which we can rewrite as:

$$f(t,T) = f(0,T) + \int_0^t \frac{\alpha(T)}{\alpha(s)}\beta(s,\theta)\big(A(T)-A(s)\big)\frac{\beta(s,\theta)}{\alpha(s)}ds + \int_0^t \frac{\alpha(T)}{\alpha(s)}\beta(s,\theta)\,dW(s)$$
$$A(t) = \int_0^t \alpha(u)du$$

Forward rates can be more conveniently expressed using the following set of Markovian processes :

$$f(t,T) = f(0,T) + \frac{\alpha(T)}{\alpha(t)}\left(x(t) + \frac{A(T)-A(t)}{\alpha(t)}y(t)\right)$$
$$x(t) = \int_0^t \frac{\alpha(t)}{\alpha(s)}\beta(s,\theta)\big(A(t)-A(s)\big)\frac{\beta(s,\theta)}{\alpha(s)}ds + \int_0^t \frac{\alpha(t)}{\alpha(s)}\beta(s,\theta)\,dW(s) \quad (3.1)$$
$$y(t) = \int_0^t \left(\frac{\alpha(t)}{\alpha(s)}\right)^2 \beta^2(s,\theta)\,ds$$

In the special case of the short rate we get:

$$r(t) = f(t,t) = f(0,t) + x(t)$$

Therefore the variable $x(t)$ can be interpreted as a centered version of the short rate. The variable $y(t)$ serves as an adjustment to guarantee repricing of the initial discount curve.

The dynamics of our state variables under the risk neutral measure are readily obtained:

$$
\begin{aligned}
dx(t) &= \left(\frac{\alpha'(t)}{\alpha(t)}x(t)+y(t)\right)dt+\beta(t,\theta)\,dW(t)\\
dy(t) &= \left(\beta^2(t,\theta)+2\frac{\alpha'(t)}{\alpha(t)}y(t)\right)dt\\
x(0) &= 0\\
y(0) &= 0
\end{aligned}
\tag{3.2}
$$

Recalling the definition of the discount bond prices we also derive the following reconstruction formula:

$$
P(t,T)=\frac{P(0,T)}{P(0,t)}\exp\left(-\frac{A(T)-A(t)}{\alpha(t)}x(t)-\frac{1}{2}\left(\frac{A(T)-A(t)}{\alpha(t)}\right)^2 y(t)\right) \tag{3.3}
$$

Furthermore we add the assumption that the volatility dependence on maturity should actually appear through a dependence on time to maturity, which can be modeled by assuming:

$$
\begin{aligned}
&\frac{\alpha(T)}{\alpha(t)}=\exp\left(-\kappa(T-t)\right)\\
&\kappa \geq 0
\end{aligned}
$$

Under this parameterisation the forward rate volatility becomes:

$$
v(t,T)=\exp\left(-\kappa(T-t)\right)\beta(t,\theta)
$$

With these assumptions we have built a model which enables us to express the whole discount curve as a function of a parsimonious set of Markovian variables, thereby making the dimensionality of many pricing problems much easier to handle.

Indeed, let us consider a European option which terminal payoff at time $T$ can be expressed as a function of discount bond prices. We denote the payoff function as $H$ and the price function at time $t$ of this option as $h$. Then by non arbitrage arguments it is well known that $h$ solves the following parabolic partial differential equation:

$$
\begin{aligned}
&\frac{\partial h}{\partial t}+\left(y(t)-\kappa x\right)\frac{\partial h}{\partial x}+\frac{1}{2}\beta(t,\theta)^2\frac{\partial^2 h}{\partial x^2}+\left(\beta(t,\theta)^2-2\kappa y\right)\frac{\partial h}{\partial y}=\left(x+f(0,t)\right)h\\
&h(T,x,y)=H(x,y)
\end{aligned}
\tag{3.4}
$$

This PDE can be efficiently solved in a two dimensional ADI scheme as described in Craig and Sneyd

(1988). Extension of (3.4) to Bermudan options through dynamic programming is straightforward.

We also note that the ubiquitous Hull and White model is just a particular case of the above framework with $\beta(t,\theta)=\sigma(t)$ where $\sigma$ is a deterministic function of time. Although the latter proves to be attractive for its analytical tractability, it is unable to reproduce the swaption market implied smile and its usefulness tends to be limited to hybrid models. Therefore, from now on, we wish to consider Cheyette models with more general volatility structures. In the next section we suggest a possible solution to this problem.

## 4. Quadratic Volatility Cheyette model

In this parameterization we postulate a quadratic volatility for the rate state variable so that the Cheyette dynamics can be written as follows:

$$\begin{aligned}
&dx(t)=\left(-\kappa x(t)+y(t)\right)dt+\beta\left(t,x(t)\right)dW(t)\\
&dy(t)=\left(-2\kappa y(t)+\beta^2\left(t,x(t)\right)\right)dt\\
&\beta(t,x)=a(t)x^2(t)+b(t)x(t)+c(t)\\
&x(0)=0\\
&y(0)=0
\end{aligned}\tag{4.1}$$

It is intuitively clear that the quadratic parameter $a$ will control the curvature of the smile, $b$ will mostly control its slope around the ATM forward whereas $c$ will control the ATM volatility level. We now need to come up with a fast and smooth calibration routine. To this intent we first establish approximate dynamics for the forward swap rate.

Let us first consider a swap agreement fixing at time $T_0$ and paying at time $(T_i)_{1\le i\le N}$.

Using Ito's lemma, it is clear that the exact dynamics of the underlying forward swap rate $S$ in the annuity measure are given by:

$$\begin{aligned}
&dS(t)=\eta\left(t,x(t),y(t)\right)dW^A(t)\\
&\eta\left(t,x(t),y(t)\right)=\frac{\partial g}{\partial x}\left(t,x(t),y(t)\right)\times\beta\left(t,x(t)\right)\\
&S(t)=g\left(t,x(t),y(t)\right)=\frac{P(t,T_0)-P(t,T_N)}{A(t)}\\
&A(t)=\sum_{i=1}^{N}P(t,T_i)\delta_i
\end{aligned}$$

where:

- $A$ is the annuity process
- $W^A$ is a standard Brownian motion under the annuity measure

- The annuity measure is the measure where the numeraire is the annuity process

A natural candidate for approximate swap rate dynamics should be a driftless quadratic volatility model for which there exist very accurate analytical approximations. So we postulate the following approximation:

$$
\begin{aligned}
&dS(t) \simeq \sigma(t, S(t))\, dW^A(t) \\
&\sigma(t, S) = \Sigma(t)\left(\alpha(t) S + (1-\alpha(t)) S(0) + \gamma(t)(S - S(0))^2\right)
\end{aligned}
\qquad (4.2)
$$

The coefficients of the quadratic form involved in (4.2) can be obtained by identifying derivatives of the exact and approximate volatilities along the forward and solving analytically the following system :

$$
\begin{aligned}
&\eta(t, \bar{x}(t)) = \sigma(t, (S(0))) \\
&\left.\frac{\partial \eta}{\partial x}\right|_{\substack{x=\bar{x}(t),\\ y=\bar{y}(t)}} = \left.\frac{\partial \sigma}{\partial x}\right|_{S=S(0)} \\
&\left.\frac{\partial^2 \eta}{\partial x^2}\right|_{\substack{x=\bar{x}(t),\\ y=\bar{y}(t)}} = \left.\frac{\partial^2 \sigma}{\partial x^2}\right|_{S=S(0)}
\end{aligned}
\qquad (4.3)
$$

where the mean state $\bar{x}(t) = E^A[x(t)], \bar{y}(t) = E^A[y(t)]$ is still to be determined. Using Girsanov's theorem it is easy to obtain the exact dynamics of the state vector under the annuity measure as:

$$
\begin{aligned}
&dx(t) = \left(-\kappa x(t) + y(t) + \frac{\partial ln(A(t))}{\partial x}(t)\beta^2(t, x(t))\right)dt + \beta(t, x)\, dW^A(t) \\
&dy(t) = \left(-2\kappa y(t) + \beta^2(t, x)\right)dt
\end{aligned}
$$

which yields the following implicit ODE:

$$
\begin{aligned}
&d\bar{x}(t) = \left(-\kappa \bar{x}(t) + \bar{y}(t) + E^A\left[\frac{\partial ln(A(t))}{\partial x}(t)\beta^2(t, x(t))\right]\right)dt \\
&d\bar{y}(t) = \left(-2\kappa \bar{y}(t) + E^A\left[\beta^2(t, x(t))\right]\right)dt
\end{aligned}
\qquad (4.4)
$$

ODE system (4.4) is difficult to solve in practice. To circumvent this difficulty practitioners generally neglect convexity and approximate expectation terms with values along the forward i.e.:

$$E^A\left[\frac{\partial ln(A(t))}{\partial x}(t)\beta^2(t,x(t))\right]\simeq\left.\frac{\partial ln(A(t))}{\partial x}\right|_{\substack{x=\bar{x}(t),\\ y=\bar{y}(t)}}\beta^2(t,\bar{x}(t))$$
$$E^A\left[\beta^2(t,x(t))\right]\simeq\beta^2(t,\bar{x}(t))$$

which transforms (4.4) into the explicit ODE system:

$$\begin{aligned} d\bar{x}(t)&=\left(-\kappa\bar{x}(t)+\bar{y}(t)+\left.\frac{\partial ln(A(t))}{\partial x}\right|_{\substack{x=\bar{x}(t),\\ y=\bar{y}(t)}}\beta^2(t,\bar{x}(t))\right)dt \\ d\bar{y}(t)&=\left(-2\kappa\bar{y}(t)+\beta^2(t,\bar{x}(t))\right)dt \end{aligned} \tag{4.5}$$

ODE (4.5) can be solved efficiently with a multidimensional Runge-Kutta solver. However although this choice of mean state yields very good accuracy in the case of a Blended Cheyette model, it does not perform very well in the quadratic case. Instead as demonstrated in Chibane and Law (2012), we favour a more accurate approximation, which draws from the Hull-White model structure for which moments of state variable $x$ can be expressed as simple functions of $y$. So under the Hull-White assumptions we get:

$$\begin{aligned} E\left[x^2(t)\right]&=\bar{y}(t)+\bar{x}^2(t) \\ E\left[x^3(t)\right]&=3\bar{y}(t)\bar{x}(t)+\bar{x}^3(t) \\ E\left[x^4(t)\right]&=3\bar{y}^2(t)+6\bar{y}(t)\bar{x}^2(t)+\bar{x}^4(t) \end{aligned} \tag{4.6}$$

So to compute the mean state in the quadratic volatility Cheyette we use approximation (4.6) and get

$$\begin{aligned} d\bar{x}(t)&=\left(-\kappa\bar{x}(t)+\bar{y}(t)+\left.\frac{\partial ln(A(t))}{\partial x}\right|_{\substack{x=\bar{x}(t),\\ y=\bar{y}(t)}}\bar{\beta}^2(t,\bar{x}(t),\bar{y}(t))\right)dt \\ d\bar{y}(t)&=\left(-2\kappa\bar{y}(t)+\bar{\beta}^2(t,\bar{x}(t),\bar{y}(t))\right)dt \\ \bar{\beta}^2(t,x,y)&=\beta^2(t,x)+6a^2(t)yx^2+6a(t)b(t)yx+3a^2(t)y^2+\left(b^2(t)+2a(t)c(t)\right)y \end{aligned} \tag{4.7}$$

Similarly, ODE (4.7) can be solved efficiently in a multidimensional Runge-Kutta scheme. Once the mean state $(\bar{x}(t),\bar{y}(t))$ has been determined for all times and projection (4.3) has been performed we are left with finding an efficient option analytical pricer for a time-dependent quadratic volatility model. Fortunately thanks to works by Andersen (2009) and Chibane (2011) it is possible to obtain very accurate averaging formulas which transform a time dependent quadratic volatility model onto a constant quadratic volatility model such that the smiles implied for a given expiry $T$ by the two models essentially coïncide. According to Chibane (2011) the optimal constant parameter model is given by:

$$
\begin{aligned}
&dS(t) \simeq \bar{\sigma}(S(t))\,dW^A(t)\\
&\bar{\sigma}(S(t)) = \bar{\Sigma}\left(\bar{\alpha}S + (1-\bar{\alpha})S(0) + \bar{\gamma}(S - S(0))^2\right)\\
&\bar{\Sigma}^2 = \frac{1}{T}\int_0^T \Sigma^2(t)\,dt\\
&\bar{\alpha} = 2\frac{\int_0^T \alpha(t)\Sigma^2(t)\int_0^t \Sigma^2(s)\,ds}{\bar{\Sigma}^4 T^2}\\
&\bar{\gamma} = 3\frac{\int_0^T \alpha(t)\Sigma^2(t)\left(\int_0^t \alpha(t)\Sigma^2(s)\int_0^s \Sigma^2(u)\,du\right)dt - \frac{\bar{\alpha}^2\bar{\Sigma}^6 T^3}{6} + 2\int_0^T \gamma(t)\Sigma^2(t)\left(\int_0^t \Sigma^2(s)\int_0^s \Sigma^2(u)\,du\right)dt}{\bar{\Sigma}^6 T^3}
\end{aligned}
\tag{4.8}
$$

However in case of highly varying curvature coefficient $\gamma$, the above spot volatility averaging is no longer accurate and requires adjustments in the spirit of Andersen (2009) as described below:

$$
\begin{aligned}
&\bar{\Sigma} = \Sigma_{naive} + \frac{\delta}{\sqrt{T}}\\
&\Sigma_{naive}^2 = \frac{1}{T}\int_0^T \Sigma^2(t)\,dt\\
&\delta = \frac{-\left(2 + \frac{1}{2}\bar{\gamma}_{adj}\Sigma_{naive}^2\right) + \sqrt{4 - 2\bar{\gamma}_{adj}\Sigma_{naive}^2 - \frac{1}{12}\bar{\gamma}_{adj}^2\Sigma_{naive}^2 + 4\bar{\gamma}_{adj}\Sigma_{naive}U(T)}}{\bar{\gamma}_{adj}\Sigma_{naive}}\\
&U(T) = \Sigma_{naive} - \frac{h_1(T)}{\Sigma_{naive}^3} + \frac{h_2(T)}{\Sigma_{naive}}\\
&h_1(T) = \frac{1}{2}\int_0^T \Sigma^2(t)\gamma_{adj}(t)v^4(t)\,dt\\
&h_2(T) = \frac{1}{2}\int_0^T \Sigma^2(t)\gamma_{adj}(t)v^2(t)\,dt\\
&\gamma_{adj}(t) = 2S(0)\gamma(t)\\
&\bar{\gamma}_{adj} = 2S(0)\bar{\gamma}\\
&v(t) = \sqrt{\frac{1}{s}\int_0^t \Sigma^2(s)\,ds}
\end{aligned}
\tag{4.9}
$$

We can feed the average parameters obtained from (4.9) into the quadratic volatility model option pricing formulas displayed in the appendix A. In practice we would calibrate our model to a set of coterminal swaptions. For example for a semi-annual 20Y Bermudan swaption we would ideally calibrate to the (6Mx19Y6M), (1Yx19Y),…, (19Y6Mx6M) coterminal swaptions where the first term in () represents a swaption expiry and the second represents the underlying swap tenor. In the next section we show how well our approximation compares to the 2D finite difference pricing underlying (3.4).

## 5. Numerical examples

We applied the calibration strategy described in section 4. to the USD swaption market as of 31st July 2012. In the graphs below we compared how well our approximation was doing in terms of fitting the volatility smile and how accurate it is compared to the 2D finite difference underlying pricing PDE (3.4). In practice the calibration process is basically instantaneous. Also the calibration approximation seems to be very accurate for a wide range of expiries and strikes and that the quadratic parametrisation of the Cheyette volatility is exhaustive enough to match this kind of smiles. For reference the finite difference scheme was run with a resolution of 40 steps per year, 400 $x$ states, 20 $y$ states and we used a mean reversion speed of $\kappa = 0.1\%$.

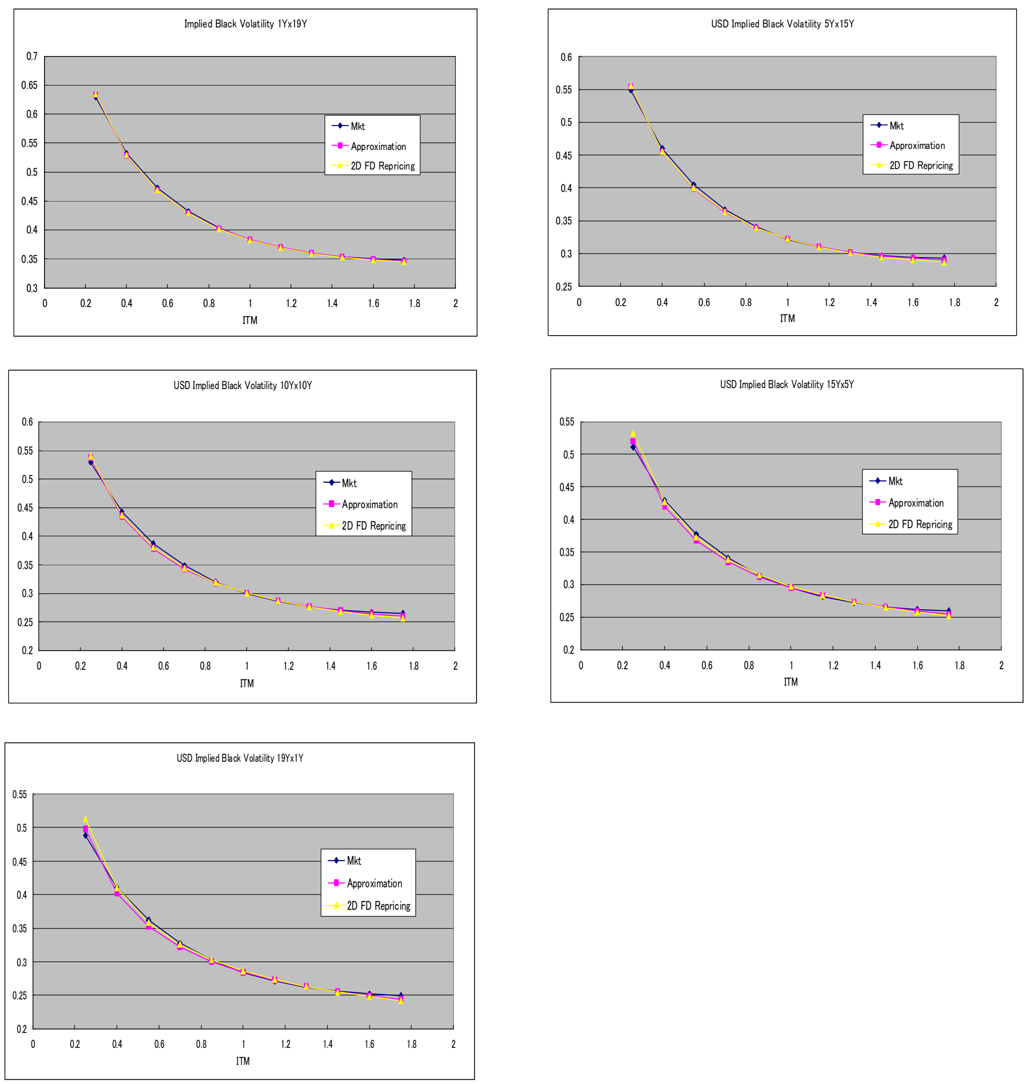


**Figure 5.1 Repricing Accuracy of the Quadratic Approximation**

## 6. Large Step Monte-Carlo

For strongly path dependent exotics, PDE (3.4) might not be adequate. Instead a Monte-Carlo simulation will be required for valuation purposes. Although a standard Euler scheme applied to the state variables $\left(x(t), y(t)\right)$ might give acceptable accuracy in some particular cases, it will in general require a lot of time steps because of the high state dependency of quadratic volatility function. Consequently, simulation schemes based on quasi-random numbers such as Sobol sequences will require a high number of paths to reach decent accuracy, thereby drastically hampering performance. Instead we suggest using a second order accurate scheme similar to the one introduced in Glasserman (2000). The adaptation of this scheme in the context of the one factor Cheyette model can be represented as:

$$
\begin{aligned}
x(t+\Delta t) \simeq\; & x(t) + \left(y(t) - \kappa x(t)\right)\Delta t + \beta\left(t, x(t)\right)\sqrt{\Delta t} Z_1 \\
& + \frac{1}{2}\beta^{(1)}\left(t, x(t)\right)\beta\left(t, x(t)\right)\Delta t\left(\left(Z_1\right)^2 - 1\right) \\
& - \kappa\beta\left(t, x(t)\right)\frac{(\Delta t)^{\frac{3}{2}}}{\sqrt{3}} Z_3 \\
& + \frac{1}{2}\beta\left(t, x(t)\right)\left(\beta^{(1)}\left(t, x(t)\right)^2 + \beta\left(t, x(t)\right)\beta^{(2)}\left(t, x(t)\right)\right)(\Delta t)^{\frac{3}{2}} Z_1\left(\frac{\left(Z_1\right)^2}{3} - 1\right) \\
& + \left(\beta^{(1)}\left(t, x(t)\right)\left(y(t) - \kappa x(t)\right) + \frac{1}{2}\beta\left(t, x(t)\right)^2 \beta^{(2)}\left(t, x(t)\right)\right)\frac{(\Delta t)^{\frac{3}{2}}}{\sqrt{3}} Z_2
\end{aligned}
$$

$$
y(t+\Delta t) \simeq y(t) + \left(\beta\left(t, x(t)\right)^2 - 2\kappa y(t)\right)\Delta t + 2\beta\left(t, x(t)\right)^2 \beta^{(1)}\left(t, x(t)\right)\frac{(\Delta t)^{\frac{3}{2}}}{\sqrt{3}} Z_3
$$

where $\left(Z_1, Z_2, Z_3\right)$ are standard normal random variables equipped with the following correlation structure:

$$
\begin{aligned}
& correl\left(Z_i, Z_j\right) = \rho_{i,j} \\
& \rho_{1,2} = \frac{\sqrt{3}}{2}, \quad \rho_{1,3} = \frac{\sqrt{3}}{2}, \quad \rho_{2,3} = \frac{1}{2}
\end{aligned}
$$

Below we display convergence results of the improved simulation scheme and of the Euler scheme for 1 year into 10 year USD swaptions. It is clear that the second order simulation is drastically more efficient than the Euler scheme.

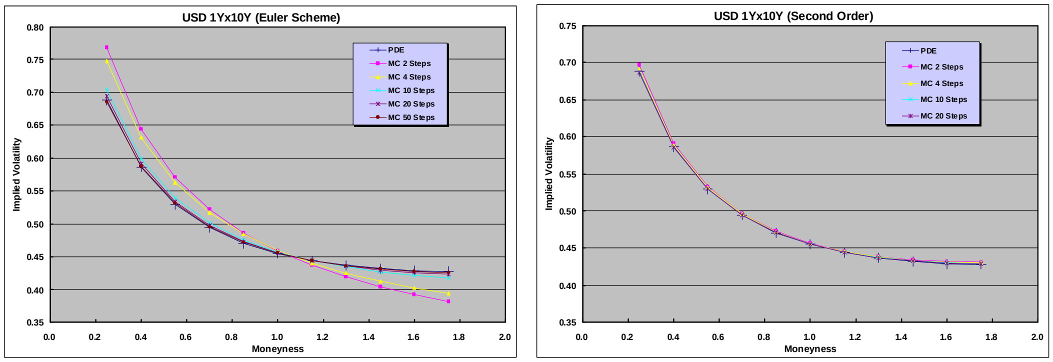


**Figure 6.1 Convergence of Monte-Carlo simulation**

## 7. Conclusion

In this work, we have adapted work started in Andreasen (2001) and Andreasen (2005) by making the Cheyette volatility parameterization a parsimonious function of the state variable which is able to fit a general swaption smile without the need of a stochastic volatility component. We suggested fast and accurate calibration procedures as well as a new and efficient Monte-Carlo simulation scheme. These features can be extended to the multi-factor Cheyette context in the spirit of Andreasen (2005) but this is left for further research.

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**Appendix A. Option pricing Formulas for the quadratic volatility model**

We consider the following constant parameter quadratic volatility model:

$$
\begin{aligned}
&dS(t) = \eta(S)\,dW(t)\\
&\eta(S) = \left(a\left(S(t)-S(0)\right)^2 + b\left(S(t)-S(0)\right)+c\right)\\
&a = 1
\end{aligned}
$$

We give option pricing formula for unit second order coefficients. Using standard time change argument, for arbitrary second order coefficient the formulas can be adapted by replacing the expiry parameter with $a^2T$ .There are different option pricing formulas depending on the root structure of the local volatility function for a strike *K* and expiry *T*. Call (resp. put) option prices can be obtain from prices and reciprocally by put-call parity.

## Table of option pricing formulas

| Volatility Root Structure | Type Of Option Price | Pricing Formula |
|---|---|---|
| Two distinct roots<br>$s_l < s_u < S(0)$ | Put | $P_K(0) = \frac{(K-s_u)(S(0)-s_l)}{s_u-s_l}N(x_u) - \frac{(K-s_u)(S(0)-s_u)}{s_u-s_l}N(y_u) - \frac{(K-s_l)(S(0)-s_u)}{s_u-s_l}N(x_l) + \frac{(K-s_l)(S(0)-s_l)}{s_u-s_l}N(y_l)$<br>$x_u = \frac{\ln X_1 + \frac{1}{2}T}{\sqrt{T}}$<br>$y_u = \frac{\ln X_2 + \frac{1}{2}T}{\sqrt{T}}$<br>$x_l = x_u - \sqrt{T}$<br>$y_l = y_u - \sqrt{T}$<br>$X_1 = \frac{(K-S_u)(S(0)-s_l)}{(K-s_l)(S(0)-s_u)}$<br>$X_2 = \frac{(K-s_u)(S(0)-s_u)}{(K-s_l)(S(0)-s_l)}$ |
| Two distinct roots<br>$S(0) < s_l < s_u$ | Call | $C_K(0) = \frac{(K_Z-s_u)(S_Z(0)-s_l)}{s_u-s_l}N(x_u) - \frac{(K_Z-s_u)(S_Z(0)-s_u)}{s_u-s_l}N(y_u) - \frac{(K_Z-s_l)(S_Z(0)-s_u)}{s_u-s_l}N(x_l) + \frac{(K_Z-s_l)(S_Z(0)-s_l)}{s_u-s_l}N(y_l)$<br>$x_u = \frac{\ln X_1 + \frac{1}{2}T}{\sqrt{T}}$<br>$y_u = \frac{\ln X_2 + \frac{1}{2}T}{\sqrt{T}}$<br>$x_l = x_u - \sqrt{T}$<br>$y_l = y_u - \sqrt{T}$<br>$X_1 = \frac{(K_Z-S_u)(S_Z(0)-s_l)}{(K_Z-s_l)(S_Z(0)-s_u)}$<br>$X_2 = \frac{(K_Z-s_u)(S_Z(0)-s_u)}{(K_Z-s_l)(S_Z(0)-s_l)}$<br>$S_Z(0) = s_l + s_u - S(0)$<br>$K_Z = s_l + s_u - K$ |

| Volatility Root Structure | Type Of Option Price | Pricing Formula |
|---|---|---|
| One single root $s > S(0)$ | Put | $P_K(0) = (S(0)-s)(K-s)\sqrt{T}\{d_+N(d_+)+n(d_+)-d_-N(d_-)-n(d_-)\}$ <br> $d_\pm = \dfrac{\pm\dfrac{1}{S(0)-s}-\dfrac{1}{K-s}}{\sqrt{T}}$ |
| One single root $s < S(0)$ | Call | $C_K(0) = (S_Z(0)-s)(K_Z-s)\sqrt{T}\{d_+N(d_+)+n(d_+)-d_-N(d_-)-n(d_-)\}$ <br> $d_\pm = \dfrac{\pm\dfrac{1}{S_Z(0)-s}-\dfrac{1}{K_Z-s}}{\sqrt{T}}$ <br> $S_Z(0) = s - S(0)$ <br> $K_Z(0) = s - K$ |
| Two single roots straddling the forward: $s_l < S(0) < s_u$ | Put | $P_K(0) = \dfrac{(K-s_u)(S(0)-s_l)}{s_u-s_l}(1-N(x_u)) - \dfrac{(K-s_l)(S(0)-s_u)}{s_u-s_l}N(x_l)$ <br> $x_u = \dfrac{\ln\left(\dfrac{\tilde{K}}{Y(0)}\right)+\dfrac{1}{2}T}{\sqrt{T}}$ <br> $x_l = \dfrac{-\ln\left(\dfrac{\tilde{K}}{Y(0)}\right)+\dfrac{1}{2}T}{\sqrt{T}}$ <br> $\tilde{K} = \dfrac{K-s_u}{K-s_l}$ <br> $Y(0) = \dfrac{S(0)-s_u}{S(0)-s_l}$ |

| Volatility Root Structure | Type Of Option Price | Pricing Formula |
|---|---|---|
| No real root. Two absorbing boundaries straddling the forward $L < S(0) < U$ | Put | $p_K(0) = \frac{1}{2}\frac{\exp\left(\frac{d^2T}{2}\right)}{\cos(dY(0))}\sum_{n=1}^{+\infty} a_n e^{-\lambda_n T}\left(\tilde{K}\left(I(z_l,\tilde{z},\alpha_+,\beta_-)-I(z_l,\tilde{z},\alpha_-,\beta_+)\right)-d\left(J(z_l,\tilde{z},\alpha_-,\beta_+)-J(z_l,\tilde{z},\alpha_+,\beta_-)\right)\right)$ <br> $+2(K-s_l)\frac{\cos(dY_l)}{\cos(dY(0))}\sum_{n\geq 1}\lambda_n b_n I_n$ <br> $I(z_l,\tilde{z},a,b)=\int_{z_l}^{\tilde{z}}\sin(az+b)\,dz$ <br> $I(z_l,\tilde{z},a,b)=\int_{z_l}^{\tilde{z}}\cos(az+b)\,dz$ <br> $a_n=\frac{2}{z_u-z_l}\sin\left(-\frac{n\pi z_l}{z_u-z_l}\right)$ <br> $\lambda_n=\frac{1}{2}\left(\frac{n\pi}{z_u-z_l}\right)^2$ <br> $\alpha_\pm = d\pm\frac{n\pi}{z_u-z_l}$ <br> $\beta_\pm = dY(0)\pm\frac{n\pi z_l}{z_u-z_l}$ <br> $I_{2n+1}=\int_0^T e^{\left(\frac{d^2}{2}-\lambda_{2n+1}\right)t}dt$ <br> $b_n=\frac{1}{n\pi}\sin\left(-\frac{n\pi z_l}{z_u-z_l}\right)$ <br> $Y_l=\frac{1}{d}\arctan\left(\frac{s_l+e}{d}\right)$ <br> $Y(0)=\frac{1}{d}\arctan\left(\frac{S(0)+e}{d}\right)$ <br> $\tilde{K}=\frac{K-s_u}{K-s_l}$ <br> $d=\sqrt{c^2-\frac{b^2}{4}}$ <br> $e=\frac{b}{2}$ |