Greeks of Bermuda Swaptions: Financial Interpretation and Fast Computations

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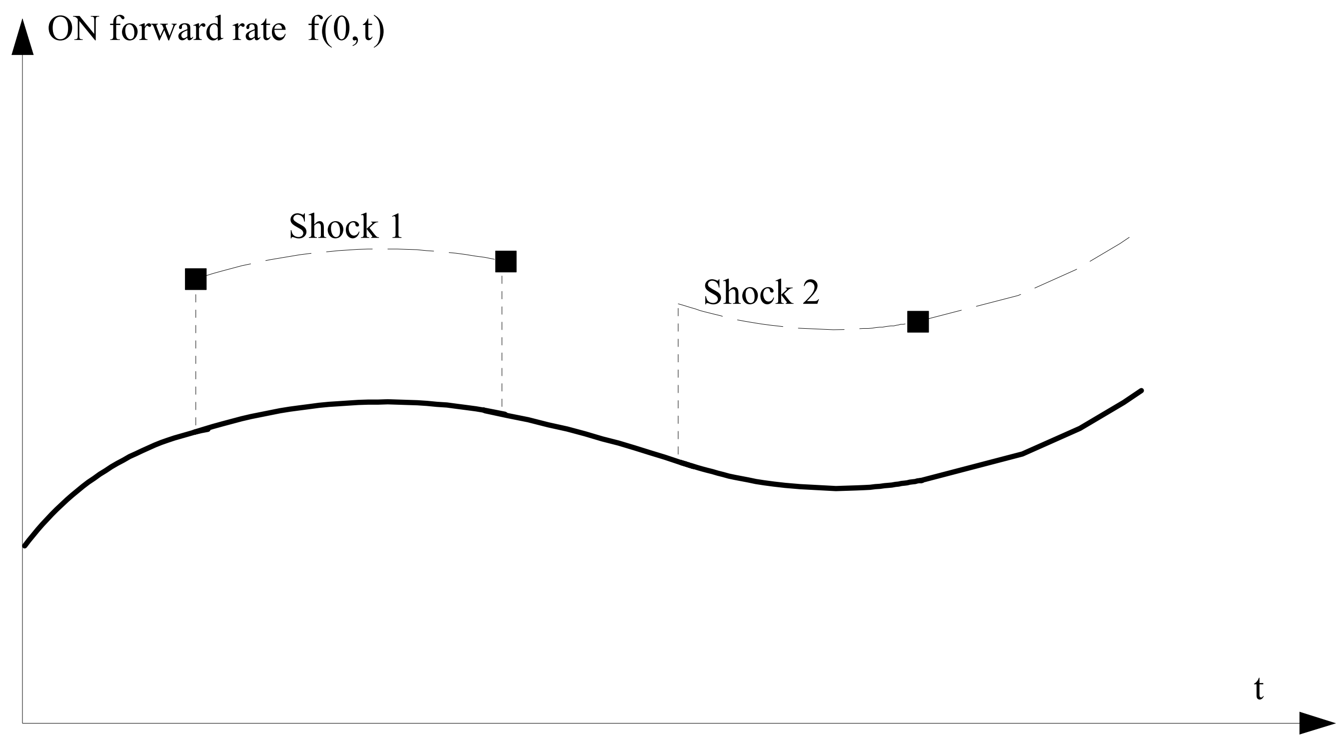
1 Goal

- Bermuda swaptions are American-style options on interest rate fixed-rate for floating-rate swaps.
- The most liquid of all interest rate exotics.
- Typically valued by numerically solving a PDE.
- Risk sensitivities – deltas, gammas, vegas – typically computed by numerical differentiation: “shock inputs and revalue the Bermudan”.
- Problems:
  - Slow: One PDE solution per Greek.
  - Inaccurate: Numerical noise is magnified by numerical differentiation.
- Is anything better possible?

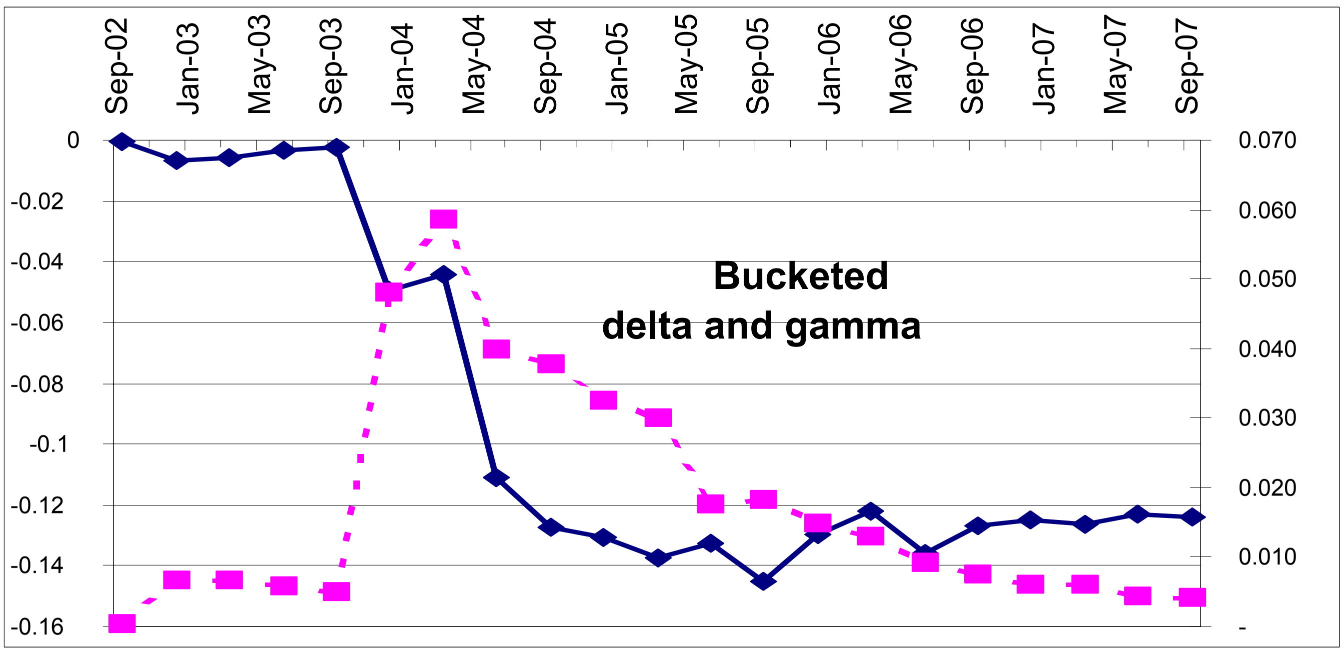
2 Deltas to shocks of initial interest rate curve

- Many equivalent parametrizations of the interest rate curve are possible.
- For the discussion on deltas, we parametrize it by instantaneous forward rates.
- Possible shocks:

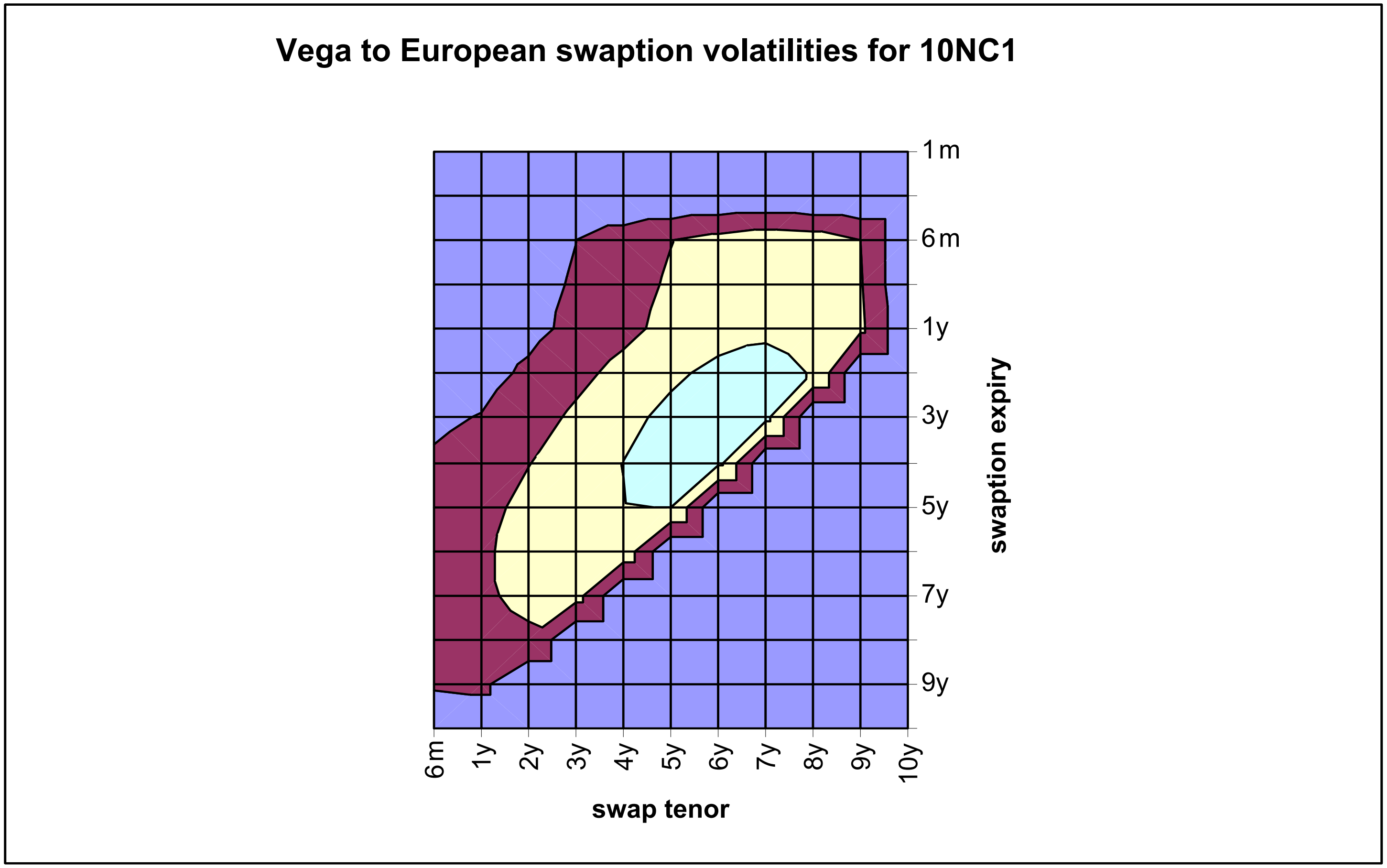


3 Curve sensitivities of Bermuda swaptions

- Bucketed deltas (solid line) – sensitivities of a Bermuda swaption to shocks $f(0,t) + \varepsilon 1_{\{T_i < t \le T_{i+1}\}}$ for a collection of dates T_i.
- Bucketed gammas (dotted line) – sensitivities of bucketed deltas to a parallel shift of the whole curve $f(0,\cdot) + \varepsilon$.



4 Volatility sensitivities of Bermuda swaptions



5 Are we stuck with bump-and-revalue method for deltas?

- Motivational example 1: Black's formula for options on forwards

$$F N\left(d_1\right) - K N\left(d_2\right).$$

  Delta is $N\left(d_1\right)$, computed along with the value.

- Motivational example 2: Black's formula on a PDE grid $\left(x_j, t_i.\right)$ If instrument value is $V\left(x=0, t=0\right)$ then its delta $\left(2 \cdot \Delta x\right)^{-1}\left[V(\Delta x, 0) - V(-\Delta x, 0)\right]$, computed at no extra computational cost.

- Motivational example 3: Monte-Carlo, integration by parts:

$$\begin{aligned} \frac{\partial}{\partial \theta} \mathbf{E}^{\theta}\left(V\left(\xi\right)\right) &= \frac{\partial}{\partial \theta} \int V\left(x\right) p_{\xi}\left(x; \theta\right) dx \\ &= \int V\left(x\right)\left(\frac{\partial}{\partial \theta} \log p_{\xi}\left(x; \theta\right)\right) p_{\xi}\left(x; \theta\right) dx \\ &= \mathbf{E}\left(V\left(\xi\right) w\left(\xi\right)\right). \end{aligned}$$

- Clearly in many cases, information needed for computing deltas is available during valuation.

6 Notations

- Instantaneous forward rates atr time t for forward period $[T, T+0]$ denoted by $f(t,T) = f(\omega, t, T)$.
- Zero coupon discount bonds:

$$P(t,T) = \exp\left(-\int_t^T f(t,s)\,ds\right).$$

- Discount (cc) rates

$$F(t,T) = -(T-t)^{-1}\log P(t,T) = (T-t)^{-1}\int_t^T f(t,s)\,ds.$$

- Money market account

$$B_T = \exp\left(\int_0^T f(t,t)\,ds\right) = \exp\left(\int_0^T r(t)\,ds\right).$$

7 Instruments

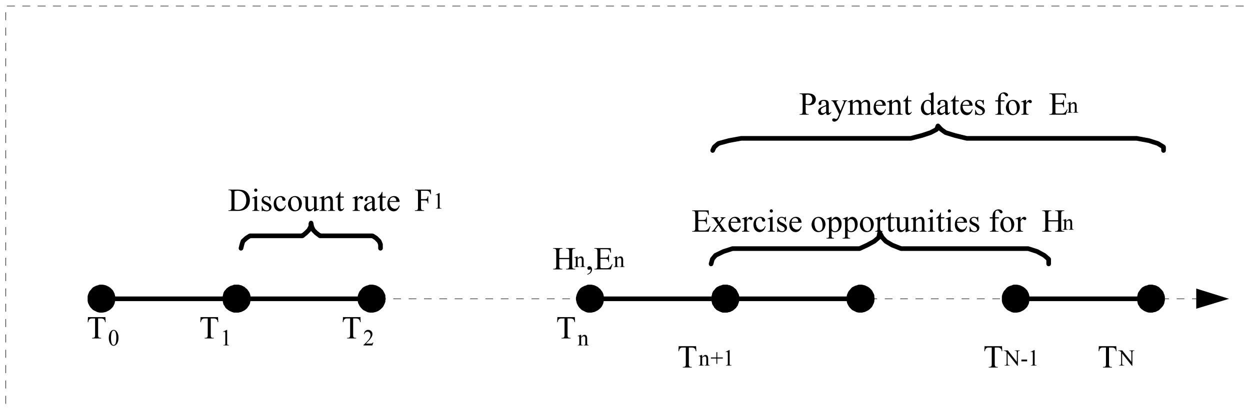
- Tenor structure: $0 = T_0 < T_1 < \ldots < T_N, \quad \tau_n = T_n - T_{n-1}$.
- The n-th swap starts at T_n. Fixed rate c, value at time t,

$$E_n(t) = P(t, T_n) - P(t, T_N) - c \sum_{k=n+1}^{N} P(t, T_k)\, \tau_k.$$

- European swaption with exercise date T_k: European option with payoff $\max\{E_k(T_k), 0\}$.
- Bermuda swaption H_0: Can exercise on any dates $T_1, \ldots, T_{N-1}$; if exercised on date T_k receive $E_k(T_k)$.
- Bermuda swaption H_n: Can exercise on dates $T_{n+1}, \ldots, T_{N-1}$ ("sub-Bermudan").

8 Backward recursion for valuation

- Timeline



- Main recursion: $n = N-2, \ldots, 0$,

$$\begin{aligned} H_n(T_n) &= e^{-\tau_{n+1} F_n(x)} \mathbf{E}_{T_n}^{T_{n+1}} \max\left\{ H_{n+1}(T_{n+1}), E_{n+1}(T_{n+1}) \right\}, \\ H_{N-1} &\equiv 0. \end{aligned}$$

- $\mathbf{E}_{T_n}(\cdot)$'s are usually computed on a PDE grid.

9 What inputs to shock?

- Deltas = sensitivity to interest rate curve shocks.
- What shocks to use? A shock to the interest rate curve affects many "moving parts".
- It is common to use a different "basis" for computing deltas.
- Look at slide 2. It is sometimes better to use shocks of "Shock 2" type:

  $$f(0,t) + \varepsilon 1_{\{t>T_i\}}$$

  and compute deltas to those shocks and convert back to deltas for shocks of type "Shock 1":

  $$f(0,t) + \varepsilon 1_{\{T_i<t\leq T_{i+1}\}},$$

  than compute the deltas to "Shock 2" type shocks directly.

10 Bright idea

- Let us shock $E_n(T_n)$'s directly and individually! They enter the formula for Bermudans explicitly, deltas must be the easiest to compute!
- What does it mean to shock $E_n(T_n)$ directly? Think Hull-White
- Short rate state $x(t)$, everything in terms of $x(t)$:

$$\begin{aligned} dx(t) &= (\theta(t) - ax(t))\, dt + \sigma(t)\, dW(t), \\ r(t) &= f(0,t) + x(t), \\ P(x(t), t, T) &= P(0,t,T)\exp(-b(t,T)\,x(t) + A(t,T)). \end{aligned}$$

- All exercise values are deterministic functions of $x(\cdot)$,

$$E_n(T_n) = E_n(x(T_n)).$$

- We will describe our method for HW, but it is applicable in a much broader setting.

11 Bermuda swaption valuation as a functional 1

- Define a set $C_0(\mathbb{R})$ of continuous functions that "do not grow too fast". Think of them as functions of the HW state variable $x(\cdot)$ and require $|f(x)| < Ke^{a|x|^{2-\varepsilon}}$, $\varepsilon > 0$.
- Think of the recursion $\quad n = N-1, \ldots, 1,$

$$\begin{aligned} H_{n-1}(x) &= e^{-\tau_n F_{n-1}(x)} \mathbf{E}^{T_n} \left(\max \{ H_n(x(T_n)), E_n(x(T_n)) \} \middle| \, x(T_{n-1}) = x \right), \\ H_{N-1} &\equiv 0. \end{aligned}$$

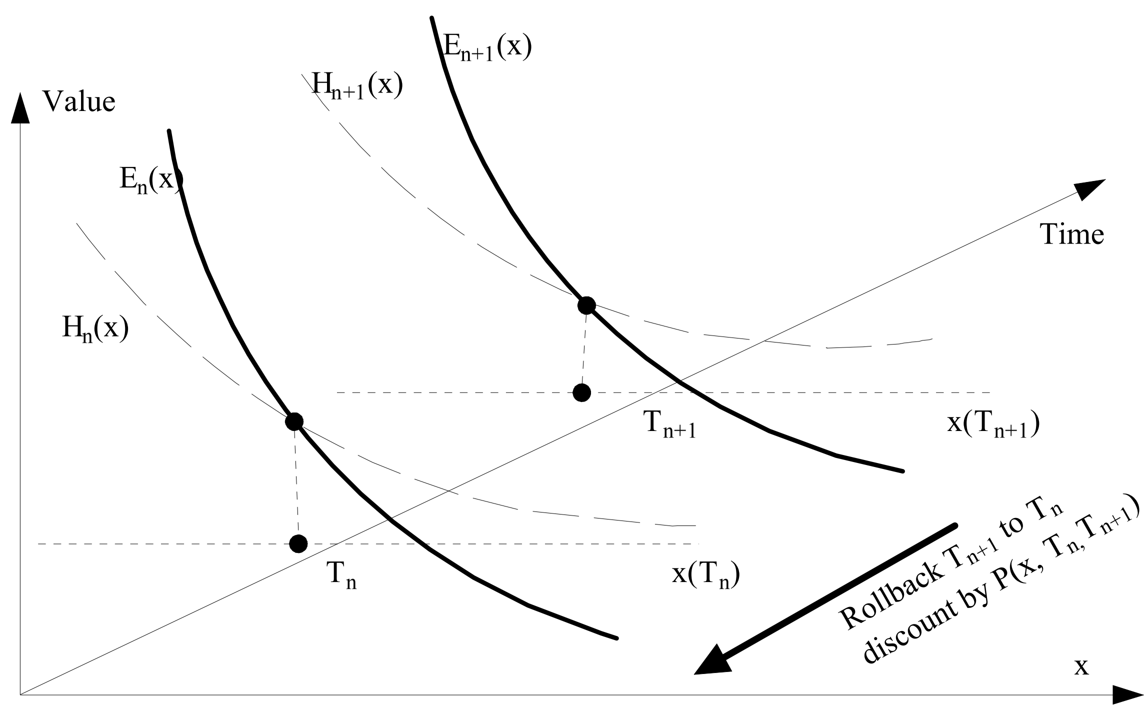
as a definition of functions $\{H_n(x)\}_{n=0}^{N-1}$ from arbitrary functions $E_n(x(T_n))$, $F_{n-1}(x) \in C_0(\mathbb{R})$. Treat $\mathbf{E}^{T_n}$ as an operator on $C_0(\mathbb{R})$:

$$f(x) \longmapsto \mathbf{E}^{T_n} \left(f(x(T_n)) \middle| \, x(T_{n-1}) = x \right).$$

- For any k we regard the Bermuda value H_k as a functional on $(C_0(\mathbb{R}))^{2(N-1)}$,

$$H_k : (C_0(\mathbb{R}))^{2(N-1)} \longrightarrow C_0(\mathbb{R}).$$

12 Bermuda swaption valuation as a functional 2



- Apply shocks to functions $E_n(x)$ and $F_{n-1}(x)$ directly!
- Compute deltas as sensitivities (as $\varepsilon \to 0$) to individual shocks

$$E_n(x) + \varepsilon D_n^e(x), \quad F_{n-1}(x) + \varepsilon D_{n-1}^f(x).$$

13 Model deltas

- Define two sets of model deltas, "underlying" deltas and "discount" deltas. Regard H_k as a functional $\left(C_0\left(\mathbb{R}\right)\right)^{2(N-1)} \longrightarrow C_0\left(\mathbb{R}\right)$, differentiate individual inputs in "directions" $D_n^e\left(x\right)$, $D_{n-1}^f\left(x\right)$,

$$\begin{aligned}\Delta_n^e H_k &= \frac{\partial}{\partial\varepsilon} H_k\left(\ldots, E_n\left(x\right) + \varepsilon D_n^e\left(x\right), \ldots\right)\Big|_{\varepsilon=0}, \\ \Delta_{n-1}^f H_k &= \frac{\partial}{\partial\varepsilon} H_k\left(\ldots, F_{n-1}\left(x\right) + \varepsilon D_{n-1}^f\left(x\right), \ldots\right)\Big|_{\varepsilon=0}.\end{aligned}$$

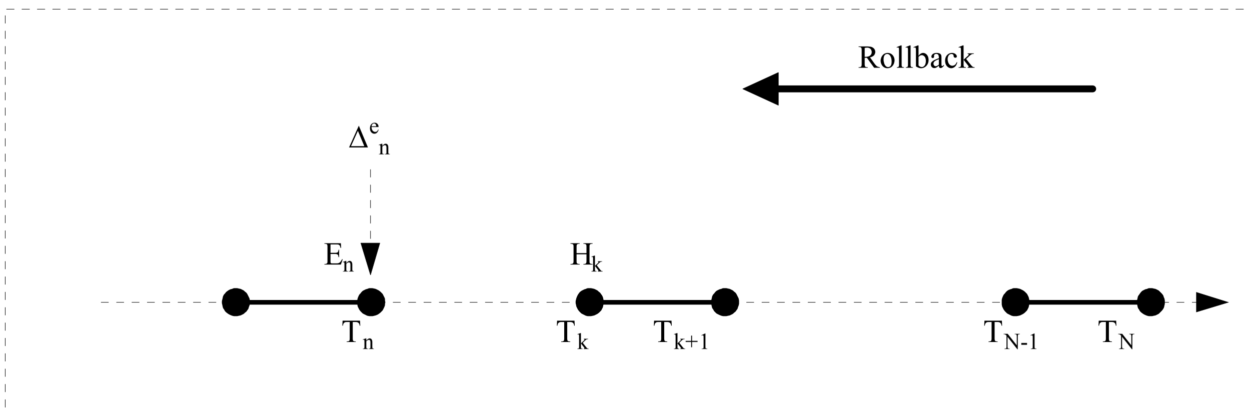
14 Recursion for model deltas 1

- Fix n. Let us try to compute $\Delta_n^e H_k$.
- Recall

$$H_k(x) = \exp\left(-\tau_{k+1} F_k(x)\right) \mathbf{E}_{T_k}^{T_{k+1}} \left(\max\left\{H_{k+1}, E_{k+1}\right\} \middle| \, x(T_k) = x\right).$$

- If $n < k+1$, then the computation of H_k will not be affected by a shock to $E_n(x)$ at all. So

$$\Delta_n^e H_k = 0.$$



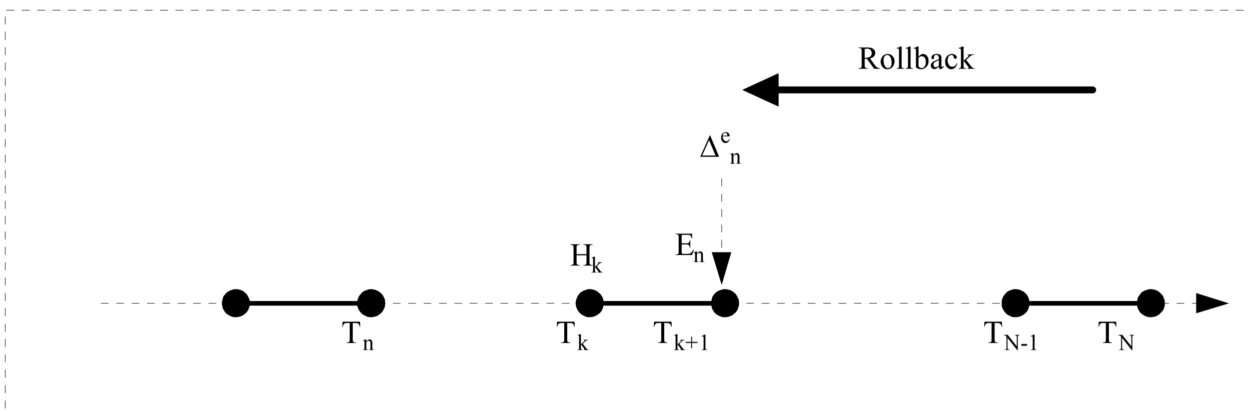
15 Recursion for model deltas 2

- Recall

$$H_k(x) = \exp(-\tau_{k+1} F_k(x))\, \mathbf{E}_{T_k}^{T_{k+1}} \left(\max\{H_{k+1}, E_{k+1}\} \middle| x(T_k) = x\right).$$

- If $n = k+1$, then H_{k+1} is unaffected by a shock to $E_n(x)$ but E_{k+1} obviously is affected. Formally differentiating the recursion above we get $(\max(x,k)' = 1_{\{x \geq k\}})$,

$$\begin{aligned}\Delta_n^e H_k &= \exp(-\tau_{k+1} F_k(x))\, \mathbf{E}_{T_k}^{T_{k+1}} \left(1_{\{E_{k+1} \geq H_{k+1}\}} \Delta_n^e E_{k+1} \middle| x(T_k) = x\right) \\ &= \exp(-\tau_{k+1} F_k(x))\, \mathbf{E}_{T_k}^{T_{k+1}} \left(1_{\{E_{k+1} \geq H_{k+1}\}} D_{k+1}^e \middle| x(T_k) = x\right).\end{aligned}$$



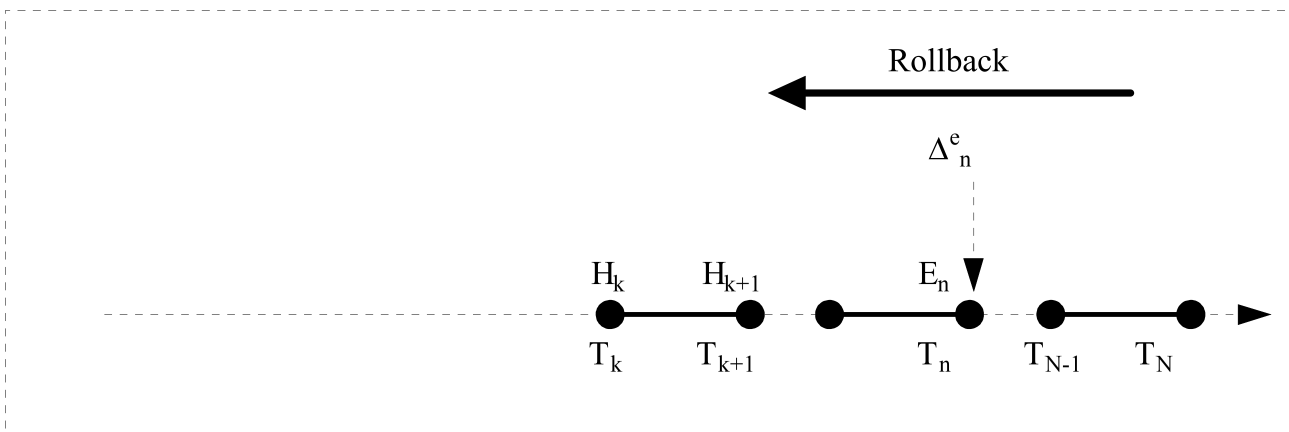
16 Recursion for model deltas 3

- Recall

$$H_k\left(x\right) = \exp\left(-\tau_{k+1} F_k\left(x\right)\right) \mathbf{E}_{T_k}^{T_{k+1}} \left(\max\left\{H_{k+1}, E_{k+1}\right\} \middle| \, x\left(T_k\right) = x\right).$$

- If $n > k+1$, then H_{k+1} is affected by a shock to $E_n\left(x\right)$ via recursive formulas but E_{k+1} is not (independent bumps to E_i). Thus

$$\begin{aligned}
\Delta_n^e H_k &= \exp\left(-\tau_{k+1} F_k\left(x\right)\right) \mathbf{E}_{T_k}^{T_{k+1}} \left(1_{\{E_{k+1} < H_{k+1}\}} \Delta_n^e H_{k+1} \middle| \, x\left(T_k\right) = x\right), \\
\Delta_n^e H_k &= B_{T_k} \mathbf{E}_{T_k} \left(B_{T_{k+1}}^{-1} 1_{\{E_{k+1} < H_{k+1}\}} \Delta_n^e H_{k+1} \middle| \, x\left(T_k\right) = x\right)
\end{aligned}$$



17 Lemma on differentiation

- Exchanging differentiation and expectation is justified by the following lemma

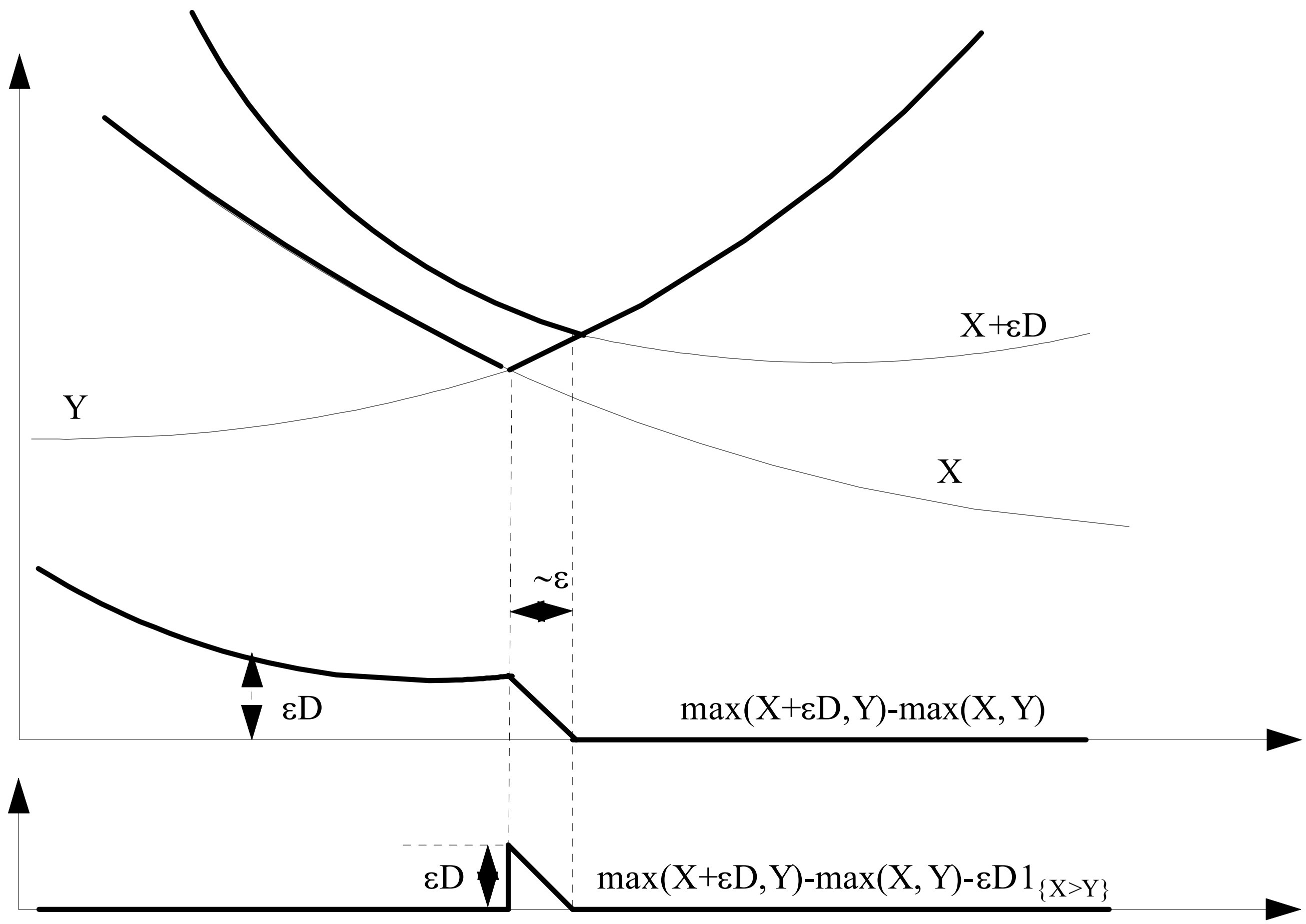
Lemma 1 : *Let X, Y and D be random variables such that*

$$\mathbf{E}\,|X| < \infty, \quad \mathbf{E}\,|Y| < \infty, \quad \mathbf{E}\,|D| < \infty.$$

If $\mathbf{P}\,(X = Y) = 0$ *then*

$$\lim_{\varepsilon \to 0} \varepsilon^{-1}\mathbf{E}\left(\max\left(X + \varepsilon D, Y\right) - \max\left(X, Y\right)\right) = \mathbf{E}\left(1_{\{X > Y\}} D\right).$$

18 Proof of the lemma



19 Unwrapping the recursion

- Our goal – deltas of H_0.
- Use recursive relations to "push deltas through".
- We have

$$\begin{aligned}
\Delta_n^e H_0 &= \mathbf{E}_0 \left(B_{T_1}^{-1} \cdot 1_{\{E_1 < H_1\}} \cdot \Delta_n^e H_1 \right) \\
&= \mathbf{E}_0 \left(B_{T_1}^{-1} \cdot 1_{\{E_1 < H_1\}} \cdot B_{T_1} \mathbf{E}_{T_1} B_{T_2}^{-1} \left(1_{\{E_2 < H_2\}} \cdot \Delta_n^e H_2 \right) \right) \\
&= \ldots \\
&= \mathbf{E}_0 \left(B_{T_{n-1}}^{-1} \cdot \prod_{i=1}^{n-1} 1_{\{E_i < H_i\}} \cdot \Delta_n^e H_{n-1} \right) \\
&= \mathbf{E}_0 \left(B_{T_n}^{-1} \cdot \prod_{i=1}^{n-1} 1_{\{E_i < H_i\}} \cdot 1_{\{E_n \geq H_n\}} \cdot D_n^e \right).
\end{aligned}$$

- This is our first main result.

20 Discount deltas

- Similar recursions and formulas hold for "discount" deltas $\Delta_n^f H_k$, model deltas of H_k with respect to shocks to discount rates $F_n(x)$.
- For the Bermudan H_0 we have

$$\Delta_n^f H_0 = -\tau_{n+1} \mathbf{E}_0 \left(B_{T_n}^{-1} \cdot \prod_{i=1}^{n-1} 1_{\{H_l > E_i\}} \cdot 1_{\{H_n > E_n\}} \times D_n^f \times H_n \right).$$

21 Optimal exercise time

- Our formula for deltas allows for a number of interesting interpretations of deltas.
- Define "**exercise**" **regions** for each exercise opportunity T_n,

$$R_n = \left\{x \in \mathbb{R} : H_n(x) \le E_n(x)\right\}, \quad 0 \le n \le N-1$$

and "**hold**" **regions**

$$R_n^c = \left\{x \in \mathbb{R} : H_n(x) > E_n(x)\right\}, \quad 0 \le n \le N-1.$$

- Define optimal exercise time (index)

$$\eta = \min\left\{n \ge 1 : x(T_n) \in R_n\right\}.$$

- Bermuda value via optimal exercise time:

$$H_0 = \sum_{m=1}^{N-1} \mathbf{E}_0\left(B_{T_m}^{-1} \cdot 1_{\{\eta=m\}} \cdot E_m(T_m)\right) = \mathbf{E}_0\left(B_{T_{\eta\wedge N}}^{-1} \cdot E_{\eta\wedge N}(T_{\eta\wedge N})\right).$$

22 Model deltas via optimal exercise time

- Recall

$$\Delta_n^e H_0 = \mathbf{E}_0 \left(B_{T_n}^{-1} \cdot \prod_{i=1}^{n-1} 1_{\{E_i < H_i\}} \cdot 1_{\{E_n \geq H_n\}} \cdot D_n^e \right).$$

- Note that

$$1_{\{\eta = n\}} = \prod_{i=1}^{n-1} 1_{\{E_i < H_i\}} \cdot 1_{\{E_n \geq H_n\}}.$$

- Thus, model deltas can also be expressed via optimal exercise time:

$$\Delta_n^e H_0 = \mathbf{E}_0 \left(B_{T_n}^{-1} \cdot 1_{\{\eta = n\}} \cdot D_n^e \right).$$

23 Model deltas as values of knock-outs

- Define T_n-knock-out instrument as a contingent claim that disappears ("knocks out") when $x(T_i)$ enters R_i for any $i = 1, \ldots, n-1$.
- A value of a T_n-knock-out with payoff ξ at time T_n is equal to

$$\mathbf{E}_0\left(B_{T_n}^{-1} \cdot 1_{\{\eta > n-1\}} \cdot \xi\right)$$

- Recall from previous slide

$$\Delta_n^e H_0 = \mathbf{E}_0\left(B_{T_n}^{-1} \cdot 1_{\{\eta = n\}} \cdot D_n^e\right) = \mathbf{E}_0\left(B_{T_n}^{-1} \cdot 1_{\{\eta > n-1\}} \cdot 1_{\{E_n \geq H_n\}} \cdot D_n^e\right).$$

- Thus $\Delta_n^e H_0$ is the price of a T_n-knock-out instrument with time-T_n payoff

$$1_{\{E_n \geq H_n\}} \cdot D_n^e.$$

- Carr in [Car01] established the intepretation of European option deltas as prices of contingent claims; we establish a similar interpretation for Bermudan swaptions, except the instruments are knock-outs.

24 The survival measure

- Value of a European option with payoff ξ at time T = integral of ξ with respect to state density at time T.
- Value of a knock-out option with payoff ξ at time T = integral of ξ with respect to the state "survival" density at time T.
- Define $\Psi(Y,t)$ for $Y \subset \mathbb{R}$ by

$$\Psi(Y,t) = \mathbf{E}_0\left(B_t^{-1} \cdot 1_{\left\{t \leq T_\eta\right\}} \cdot 1_{\{x(t) \in Y\}}\right).$$

"Discounted probability of $x(t) \in Y$ given that the state process $x(\cdot)$ did not knockout before t". Density $\psi(y,t)$ defined by

$$\Psi(Y,t) = \int_Y \psi(y,t)\, dy.$$

25 Model deltas via the survival density

- The formula for the n-th "undelrying" delta reads

$$\Delta_n^e H_0 = \mathbf{E}_0 \left(B_{T_n}^{-1} \cdot 1_{\left\{T_n \le T_\eta\right\}} \cdot 1_{\{E_n \ge H_n\}} \cdot D_n^e \right).$$

- The formula for the survival density reads

$$\psi\left(y; T_n\right) = \mathbf{E}_0 \left(B_{T_n}^{-1} \cdot 1_{\left\{T_n \le T_\eta\right\}} \cdot \delta_y \left\{x\left(T_n\right)\right\} \right).$$

- Using the formula of full probability we obtain from the two formulas that

$$\Delta_n^e H_0 = \int 1_{\{R_n\}}\left(y\right) \cdot D_n^e\left(y\right) \cdot \psi\left(y; T_n\right) \, dy.$$

- Consider
$$\Phi_{s,x}(Y,t) \triangleq B_s^{-1}\mathbf{E}_{s,x}\left(B_t^{-1}1_{\{x(t)\in Y\}}\right)$$
(the value of the contingent claim that pays $1_{\{x(t)\in Y\}}$ at time t, evaluated at time s conditioned on $x(s)=x$). We denote by $\phi_{s,x}(y;t)$ its density
$$\Phi_{s,x}(Y,t) = \int_Y \phi_{s,x}(y,t)\, dy.$$

- The density ϕ satisfies the forward Kolmogorov equation (Λ is the infinitesimal generator for $x(\cdot)$, Λ_y means it is applied to y-argument functions),
$$\frac{\partial}{\partial t}\phi_{s,x}(y,t) = \left(\Lambda_y^*\phi_{s,x}\right)(y,t) - r(t)\,\phi_{s,x}(y,t)\,.$$

27 Equation for the survival density

- For
$$T_n < t \le T_{n+1},$$
by conditioning on $x(T_n)$, we can obtain
$$\psi(y,t) = \mathbf{E}_0\left(B_{T_n}^{-1} \cdot 1_{\{\eta>n\}} \cdot \phi_{T_n, x(T_n)}(y,t)\right).$$

- Therefore, since both the expectation operator and Λ_y^* are linear,
$$\frac{\partial}{\partial t}\psi(y,t) = \left(\Lambda_y^*\psi\right)(y,t) - r(t)\,\psi(y,t). \tag{1}$$

- What happens when t "crosses over" T_{n+1}? Have special conditions (continuity)
$$\psi(y, T_{n+1}+0) = \psi(y, T_{n+1}) \times 1_{\left\{y \in R_{n+1}^c\right\}}(y). \tag{2}$$

- Algorithm: Start with $\psi(y,0) = \delta_0(y)$. Roll forward using the PDE (1). For $t = T_1$, apply the condition (2). Then roll forward using (1), and so on.

28 Fast computation of model deltas

- Remember formula for deltas

$$\Delta_n^e H_0 = \int 1_{\{R_n\}}(y) \cdot D_n^e(y) \cdot \psi(y; T_n)\, dy, \quad n = 1, \ldots, N-1.$$

- Remember formulas for the survival density

$$\begin{aligned} \frac{\partial}{\partial t}\psi(y,t) &= \left(\Lambda_y^* \psi\right)(y,t) - r(t)\,\psi(y,t), \quad T_n < t \leq T_{n+1}, \\ \psi(y, T_{n+1}+0) &= \psi(y, T_{n+1}) \times 1_{\{y \in R_{n+1}^c\}}(y). \end{aligned}$$

- Conclusion: At the expense of 1 PDE solution and $N-1$ integrations, we can get all “underlying” deltas $\Delta_n^e H_0$! (another $N-1$ integrations needed for “discount” deltas).
- Contrast that to the usual “shock-and-revalue” method that requires $N-1$ PDE solutions.
- See [And96] for forward-equation based algorithm for computing European option deltas.

29 Market deltas 1

- Really not that interested in the "model" deltas, i.e. deltas to shocks in $E_n(T_n)$, $F_n(T_n)$. Want deltas to shocks to the initial term curve.
- Let
$$f(0,t) \longmapsto f(0,t) + \varepsilon\theta(t),$$
be a shock to the initial curve. Denote by ∂_θ a derivative, of anything, with respect to that shock.
- This shock affects three things: exercise values at future times $E_n(T_n)$; discount rates at future times $F_n(T_n)$; and expectation operator $\mathbf{E}_{T_n}$ (NOT for Hull-White, but in general)
- Apply chain rule! Assume $\mathbf{E}_{T_n}$ is independent of the shock (can deal with general case, but it is a bit messier).

Market deltas 2

- Use

$$
\begin{aligned}
D_n^e(x) &= \partial_\theta E_n(x), \\
D_n^f(x) &= \partial_\theta F_n(x),
\end{aligned}
$$

as shocks to $E_n(x)$, $F_n(x)$ (“model” shocks). Can usually get them in closed form (definitely the case for HW model)

- Then

$$
\begin{aligned}
\partial_\theta H_0 &= \sum_{n=1}^{N-1} \int 1_{\{R_n\}}(x) \cdot \partial_\theta E_n(x) \cdot \psi(x; T_n)\, dx \\
&\quad - \sum_{n=0}^{N-2} \tau_{n+1} \int 1_{\{R_n^c\}}(x) \cdot \partial_\theta F_n(x) \cdot \psi(x; T_n)\, dx.
\end{aligned}
$$

- Market delta for any shock of initial term curve is a sum of integrals of known payoffs against the survival density.
- Multiple shocks θ – can use the same survival density ψ.

31 Do we need to compute all integrals?

- Can speed things up more by not computing all integrals. How? For example choose shocks θ_i such that $(\partial_i = \partial_{\theta_i})$

  $$\partial_i E_n \left(t = 0\right) = \delta_{\{n=i\}}$$

  (can always “rotate” to any other basis later).

- With such shocks it is likely that for $i \neq n$,

  $$\partial_i E_n \left(x, T_n\right) \ll \partial_n E_n \left(x, T_n\right).$$

  Can reduce the number of integrals by ignoring smaller contributions. Can implement an adaptive scheme that computes integrals to a certain accuracy.

- Bottom line: once we have the representation for deltas as above, we can do a lot of model-specific optimization.

- Analysis for vegas (sensitivity to volatilities) is more model specific, but can be performed along similar lines.
- Same general idea: identify volatility parameters internal to the model, bump them to compute model vegas, derive recursions for them, convert back to "market" vegas.
- Usually we can identify $N-1$ volatility parameters v_n, $n = 0, \ldots, N-2$, such that a shock to v_n does not affect $\mathbf{E}_{T_k} f\left(x\left(T_{k+1}\right)\right)$ for $k \neq n$.
- For HW:

$$\begin{aligned} \sigma\left(t\right) &= \sum 1_{t\in[T_n,T_{n+1}]} \times \sigma_n, \\ v_n &= \sigma_n. \end{aligned}$$

33 Vegas 2

- If ∇_n is the derivative with respect to a bump to v_n then (assume $n \neq 0$) as before, can differentiate through the recursive relation,

$$\begin{aligned} H_0 &= \mathbf{E}_0 B_{T_1}^{-1} \max\left(H_1, E_1\right), \\ \nabla_n H_0 &= \mathbf{E}_0 \left(B_{T_1}^{-1} \cdot 1_{\{H_1 > E_1\}} \cdot \nabla_n H_1\right) + \mathbf{E}_0 \left(B_{T_1}^{-1} \cdot 1_{\{H_1 \leq E_1\}} \cdot \nabla_n E_1\right) \\ &\approx \mathbf{E}_0 \left(B_{T_1}^{-1} \cdot 1_{\{H_1 > E_1\}} \cdot \nabla_n H_1\right). \end{aligned}$$

- Can iterate until get to H_n,

$$\nabla_n H_0 = \mathbf{E}_0 \left(B_{T_n}^{-1} \cdot \prod_{i=1}^{n} 1_{\{H_i > E_i\}} \cdot \nabla_n H_n \right).$$

- The same as for deltas, $\nabla_n H_0$ can be obtained by integrating $\nabla_n H_n\left(x\right)$ against the survival density $\psi\left(T_n, x\right)$ over the set $\left\{H_n\left(x\right) > E_n\left(x\right)\right\}$.

- The quantity $\nabla_n H_n$ has to be computed numerically: bump v_n and roll-back the payoff

$$\max\left(H_{n+1}\left(x\right), E_{n+1}\left(x\right)\right)$$

from T_{n+1} to T_n using the bumped volatility.

- Still saving a lot because the rollback only over one period $\left[T_n, T_{n+1}\right]$, not over $\left[0, T_{n+1}\right]$ as needed in the standard method.

35 Volatility correction for deltas

- In the section on deltas we assumed that the model's volatility does not depend on interest rates. Can relax it.
- If volatility depends on rates, then full delta = delta assuming the volatility is not affected by a rates shock + vega from the change in volatility.
- Need the same type of condition as for vegas: “locality”.
- For example can assume that the model's volatility is a deterministic function of $v_n\left(P\left(\cdot,\cdot\right)\right)$, $n=0,\ldots,N-2$, where each v_n is a function of rates, but shocks to v_n do not affect $\mathbf{E}_{T_k} f\left(x\left(T_{k+1}\right)\right)$ for $k \neq n$. Then the same approach as for vegas can be used.

- Valuation (lower bound) of Bermuda swaptions by Monte-Carlo is based on estimating the exercise and hold regions R_n and R_n^c in some optimization procedure and then valuing Bermuda swaption as a barrier swap using the formula

$$H_0 = \mathbf{E}_0 \left(B_{T_{\eta \wedge N}}^{-1} \cdot E_{\eta \wedge N} \left(T_{\eta \wedge N} \right) \right),$$

where $\eta = \eta(\omega)$ is now the index of the first time when a Monte-Carlo path ω enters the exercise region R_n.

- Recall that model deltas can be expressed in terms of η as well!

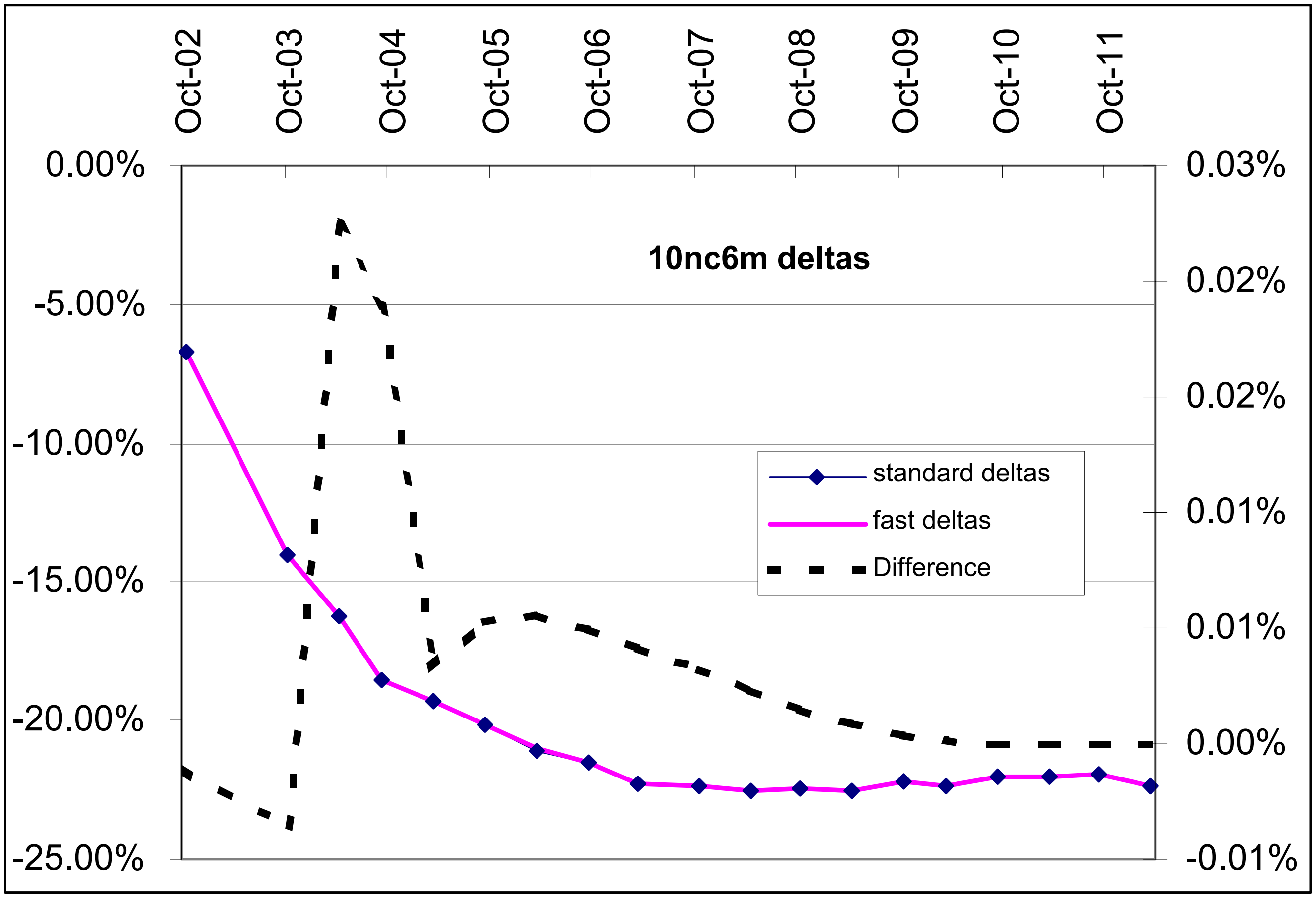
$$\Delta_n^e H_0 = \mathbf{E}_0 \left(B_{T_n}^{-1} \cdot 1_{\{\eta = n\}} \cdot D_n^e \right).$$

- Thus the deltas can be computed in the same simulation if D_n^e (and D_n^f) are known.

37 Applications to Monte-Carlo 2

- The future exercise values $E_n\left(T_n\right)$ and future discount rates $F_n\left(T_n\right)$ are "smooth" functions of the initial interest rate curve.
- For any shock θ to the initial curve, $\partial_\theta E_n\left(T_n\right), \partial_\theta F_n\left(T_n\right)$ can be computed by e.g. "pathwise differentiation" for each path in the same simulation.
- See [GZ99] for details on that. Our results for Bermuda deltas can be combined with theirs on computing $\partial_\theta E_n$, $\partial_\theta F_n$ in simulation to yield a viable delta computation scheme for Monte-Carlo.

Deltas for a 10y Bermudan 4.75% receiver with 1y no-call period as of October 1, 2002. Deltas on the left scale, difference on the right.



39 Conclusions 1

We developed an algorithm for fast computation of bucketed Greeks of Bermuda swaptions. The algorithm works like this:

- Shock inputs that are "natural" to each Bermuda swaption; define "model" deltas as sensitivities to those shocks;
- Derive recursive relations for "model" deltas;
- Use recursive relations to derive representations of "model" deltas as integrals with respect to the survival density;
- Derive a forward PDE for the survival density;
- Express "market" deltas in terms of "model" deltas using the chain rule;
- Compute "market" deltas as sums of integrals;
- Compute vegas and volatility adjustment to deltas the same way.

Much faster algorithm + many possibilities for further model-specific optimization + more accurate!

40 Conclusions 2

In addition to deriving a fast algorithm for Greeks computation, we developed

- Financial interpretations of deltas as prices of knockout contingent claims;
- Expressed deltas in terms of the optimal exercise time;
- Demonstrated how to use this representation to compute Bermuda swaption deltas in Monte-Carlo simulation.

41 Select reading

References

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