Turbo Charging the Cheyette Model

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Abstract

We describe a number of ideas for speeding up and improving the performance of the Cheyette (1992) model with particular attention to the pricing of Bermudan swaptions. Specifically, we present

i. a better skew specification,
ii. a more efficient numerical method for pricing,
iii. closed-form approximations for calibration of the model,
iv. and a new idea for calibrating the speed of mean-reversion.

Notation and Terminology

We let $P(t,T)$ denote the time t price of a zero-coupon bond that matures at time T. Let

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T}$$

be the continuously compounded instantaneous forward rate, and let $\sigma(t,T)$ denote the (absolute) volatility of $f(t,T)$. The short rate is given by

$$r(t) = f(t,t) = -\frac{\partial \ln P(t,T)}{\partial T}\Big|_{T=t}$$

Further, we will fix a maturity grid

$$0 = t_0 < t_1 < \ldots$$

with day-count factors $\delta_i = t_i - t_{i-1}$, on which we will consider the discrete forward libor rates

$$F_k(t) = \frac{1}{\delta_{k+1}}\left[\frac{P(t,t_k)}{P(t,t_{k+1})} - 1\right]$$

Similarly we will let

$$S_{k,n}(t) = \frac{P(t,t_k) - P(t,t_n)}{\sum_{i=k+1}^{n} \delta_i P(t,t_i)}$$

be the time t forward par swap rate for the swap that runs over the times $\{t_k, \ldots, t_n\}$.

A European swaption on this swap rate that can be exercised at time t_k is by market convention called a $t_k\,x\,\left(t_n - t_k\right)$ *European swaption*.

A Bermudan swaption that gives the right to enter into a swap with fixed maturity t_n at the times $\left\{t_k,\ldots,t_{n-1}\right\}$ to will be termed a t_n*-no-call-*t_k Bermudan swaption, or in brief $t_n\,\mathrm{nc}\,t_k$. For this swaption we will let the swap rates

$$\left\{S_{k,n}, S_{k+1,n}, \ldots, S_{n-1,n}\right\}$$

be termed the *core* swap rates, and associated European swaptions will be called *core swaptions*.

European swaptions are often priced in the CEV model in which it is assumed that[1]

$$dS_{k,n}\left(t\right) = \lambda_{k,n}\left(t\right) S_{k,n}\left(t\right)^{\alpha}\, dZ\left(t\right) + O\left(dt\right) \tag{0}$$

where Z is a Brownian motion, $\lambda_{k,n}\left(\cdot\right)$ is a time-dependent function, and α is a positive non-negative constant. We note that $\alpha = 1$ corresponds to the Black model.

The Cheyette Model

The so-called Cheyette (1992) model[2] is essentially a specification of the volatility structure of the continuously compounded forward rates in the general Heath, Jarrow, and Morton (1992) (HJM) model that allows for Markovian representation of the dynamics of the yield curve. If the yield curve is driven by one Brownian motion a Cheyette specification of the forward rate volatilities will in general result in a model that is Markovian in two state variables – one being locally deterministic.

For the one-factor case the Cheyette specification of the forward rate volatility is

$$\sigma\left(t,T\right) = g\left(T\right) h\left(t\right) \tag{1}$$

where $g\left(\cdot\right)$ is a time-dependent positive function and h is a general non-negative process.

The standard HJM result is that

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[1] For European swaption pricing formulas for this model, see Andersen and Andreasen (2000).

[2] Independently, Jamshidian (1991), Babbs (1993), and Ritchken and Sankarasubrahmanyam (1993) investigated the same model specification as Cheyette. Jamshidian (1991) terms this type of volatility structure “quasi-gaussian”.

$$df(t,T)=\sigma(t,T)\left[\int_t^T \sigma(t,s)\,ds\,dt + dW(t)\right]$$

where W is a Brownian motion under the risk-neutral measure. Using the Cheyette specification of the volatility structure (1) we obtain

$$\begin{aligned} f(t,T) &= f(0,t)+g(T)\int_0^t h(u)^2 \int_u^T g(s)\,ds\,du + g(T)\int_0^t h(u)\,dW(u) \\ &= f(0,t)+\frac{g(T)}{g(t)}\left[x(t)+y(t)g(t)^{-1}\int_t^T g(s)\,ds\right] \end{aligned} \tag{2a}$$

where

$$\begin{aligned} x(t) &= g(t)\int_0^t h(u)^2 \int_u^t g(y)\,dy\,du + g(t)\int_0^t h(u)\,dW(u) \\ y(t) &= g(t)^2 \int_0^t h(u)^2\,du \end{aligned} \tag{2b}$$

Since

$$\begin{aligned} dx(t) &= \left(\frac{g'(t)}{g(t)}x(t)+y(t)\right)dt + g(t)h(t)\,dW(t) \\ dy(t) &= \left(g(t)^2 h(t)^2 + 2\frac{g'(t)}{g(t)}y(t)\right)dt \end{aligned} \tag{3}$$

If we introduce

$$\begin{aligned} \eta(t) &= \eta(t,x(t),y(t)) \equiv g(t)h(t,x(t),y(t)) \\ \kappa(t) &\equiv -g'(t)/g(t) \end{aligned} \tag{4}$$

we can write evolution of the state variables as

$$\begin{aligned} dx(t) &= (-\kappa(t)x(t)+y(t))\,dt + \eta(t)\,dW(t) \\ dy(t) &= \left(\eta(t)^2 - 2\kappa(t)y(t)\right)dt \end{aligned} \tag{5}$$

We see that that if $\eta=\eta(t,x(t),y(t))$ then (5) forms a Markov system, and from (2) that (x,y) are sufficient to describe any point on the yield curve. I.e. the evolution of the yield curve is Markovian.

It is clear (2a-b) with the restriction that $h(t)=h(t,x(t),y(t))$ also leads to a Markovian representation. The representation (5), however, leads to a more natural formulation of the model.

The pricing PDE associated with (5) and the condition $\eta=\eta(t,x(t),y(t))$ is

$$0=\frac{\partial}{\partial t}V+\left[-r+(-\kappa x+y)\frac{\partial}{\partial x}+\tfrac{1}{2}\eta^2\frac{\partial^2}{\partial x^2}+(\eta^2-2\kappa y)\frac{\partial}{\partial y}\right]V \tag{5}$$

with

$$r(t)=f(0,t)+x(t)$$

and the bond reconstruction formula

$$\begin{aligned}P(t,T)&=\frac{P(0,T)}{P(0,t)}\exp\left(-\left[\int_t^T e^{-\int_t^u \kappa(s)ds}du\right]x(t)-\frac{1}{2}\left[\int_t^T e^{-\int_t^u \kappa(s)ds}du\right]^2 y(t)\right)\\&\equiv\frac{P(0,T)}{P(0,t)}\exp\left(-G(t,T)x(t)-\frac{1}{2}G(t,T)^2 y(t)\right)\end{aligned} \tag{6}$$

The Volatility Skew

The standard implementation of the model uses

$$\eta(t)=\eta(t,r(t))$$

and particularly the CEV form

$$\eta(t)\propto r(t)^\alpha$$

is a popular choice. However, using a CEV skew for the short rate results in an implied volatility skew for the swap rates that is very far from that of the CEV pricing formula for European swaptions. Typically, the skew flattens dramatically as the tenor and maturity of the swap is increased. This means that when using the CEV-Cheyette model one has to use a significantly lower CEV power than what we see in the market for European swaptions. To circumvent this we are going to let the skew specification depend on the model's application. That is, we allow the user to specify a strip of swap rates that the model's volatility depends on, i.e.

$$\eta(t)=\eta\left(t_i,S_{k_i,n_i}(t)\right)\quad ,t\in(t_{i-1},t_i],\,i\le k_i<n_i$$

for $i = 1, 2, \ldots, n$. As an example, consider the case when we wish to price a Bermudan swaption on a swap with with fixed final maturity t_n that we can exercise into at the dates $\{t_k, \ldots, t_{n-1}\}$. For this Bermudan swaption a natural CEV specification is

$$\eta(t) = \begin{cases} \lambda(t) S_{k,n}(t)^{\alpha} & t < t_k \\ \lambda(t) S_{i,n}(t)^{\alpha} & t \geq t_k \ , t_{i-1} \leq t < t_i \end{cases} \tag{7}$$

where the function $\lambda(\cdot)$ is chosen so that the model hits the prices of the core swaptions.

The Auto-Correlation Structure

The auto correlation structure in the Cheyette model can be approximated by

$$\begin{aligned} &\operatorname{corr}\left[S_{i,n}(t_i), S_{j,n}(t_j)\right] \\ &\approx \operatorname{corr}\left[x(t_i), x(t_j)\right] \\ &\approx e^{-\int_{t_i}^{t_j}\kappa(u)du} \sqrt{\frac{\operatorname{var}\left[x(t_i)\right]}{\operatorname{var}\left[x(t_j)\right]}} \\ &\approx e^{-\int_{t_i}^{t_j}\kappa(u)du} \sqrt{\frac{\operatorname{var}\left[S_{i,n}(t_i)\right]}{\operatorname{var}\left[S_{j,n}(t_j)\right]}} \left[\frac{\partial S_{i,n}(t_i)/\partial x}{\partial S_{j,n}(t_j)/\partial x}\Big|_{x=y=0}\right]^{-1} \end{aligned} \tag{8}$$

Here we note that the middle term is determined by the implied volatilities of the European core swaptions and that

$$\frac{\partial S_i^n(u)}{\partial x}\Big|_{x(u)=y(u)=0} = -\frac{P(0,t_i)G(t,t_i) - P(0,t_n)G(t,t_n)}{\sum_{k=i+1}^{n}\delta_k P(0,t_k)} + S_i^n(0)\frac{\sum_{k=i+1}^{n}\delta_k P(0,t_k)G(t,t_k)}{\sum_{k=i+1}^{n}\delta_k P(0,t_k)}$$

Hence, the term-correlation structure in (8) is entirely determined by the level of mean-reversion.

The Variance Structure

We note that in the model the par swap rate evolves according to

$$dS_{k,n}(t) = \frac{\partial S_{k,n}}{\partial x}(t)\eta(t)dW^{(k,n)}(t)$$

where $W^{(k,n)}$ is a Brownian motion under the martingale measure with the annuity

$$\sum_{j=k+1}^{n} \delta_j P\left(\cdot, t_j\right)$$

as numeraire, and

$$\frac{\partial S_{k,n}}{\partial x}(t) = S_{k,n}(t)\left[\frac{-P\left(t,t_k\right)G\left(t,t_k\right)+P\left(t,t_n\right)G\left(t,t_n\right)}{P\left(t,t_k\right)-P\left(t,t_n\right)} + \frac{\sum_{j=k+1}^{n}\delta_j P\left(t,t_j\right)G\left(t,t_j\right)}{\sum_{j=k+1}^{n}\delta_j P\left(t,t_j\right)}\right]$$

This leads to the approximation

$$\operatorname{var}^{(k,n)}\left[S_{k,n}\left(t_k\right)\right] = \int_0^{t_k}\left[\frac{\partial S_{k,n}}{\partial x}(u)\,\eta(u)\right]^2_{x=0;y=0} du \tag{9}$$

Grid Solution

Numerical solution of the PDE (5) is not straightforward. The reason is that the lack diffusion in the second variable makes numerical solution of the equation prone to spurious oscillations and standard ADI finite-difference schemes and particularly explicit schemes will perform very poorly on this problem. Here we apply the Crank-Sneyd (1988) ADI scheme with a 5-point dicretisation in the second dimension. Relative to explicit schemes, this scheme has the usual benefits: second order accuracy in the time-domain and uniform stability in the von Neuman sense. Relative to the standard ADI schemes the scheme has two main advantages. First, the internal damping factor is higher for his scheme than standard ADI schemes and this reduces any tendencies to spurious oscillations. Second, the accuracy of the 5-point discretisation allows us to get away with relatively few points in the second dimension – approximately 10 for our case.

After rewriting the pricing PDE (5) as

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}V + \left[-r + \left(-\kappa x + y\right)\frac{\partial}{\partial x} + \frac{1}{2}\eta^2\frac{\partial^2}{\partial x^2}\right]V + \left[\left(\eta^2 - 2\kappa y\right)\frac{\partial}{\partial y}\right]V \\ &\equiv \frac{\partial}{\partial t}V + D_x V + D_y V \end{aligned}$$

our finite-difference scheme can be represented as

$$\begin{aligned} \left[\frac{1}{\Delta t} - \frac{1}{2}D_x\right]U(t) &= \left[\frac{1}{\Delta t} + \frac{1}{2}D_x + D_y\right]V(t+\Delta t) \\ \left[\frac{1}{\Delta t} - \frac{1}{2}D_y\right]V(t) &= \frac{1}{\Delta t}U(t) - \frac{1}{2}D_y V(t+\Delta t) \end{aligned} \tag{10}$$

where the operator D_x is approximated by the usual tri-diagonal matrix discretisation, and D_y is dicretised by a 5-point discretisation. This means that the first equation in (10) can be solved using the usual tri-diagonal matrix inversion, whereas the second equation in (10) has to be solved using a band diagonal solver. In both cases we use the standard algorithms of Press et al (1988).

Implementation and Calibration

Concerning the volatility specification, the model can be switched between the specification (7) and an equivalent form in continuously compounded yields. The model is easily extended to other functional forms than the CEV specification and/or to dependence of other yields.

The approximating formula (8) is used in conjunction with the term correlation structure of a calibrated Libor market model (see Andersen and Andreasen (1998)) to determine the level of mean-reversion. For Bermudan swaptions we use the term correlation structure of the core swap rates to calibrate the mean-reversion. We prefer to use a single constant for the mean-reversion rather than a full term structure. The mean-reversion can be calibrated in a split of a second. On the Libor market model side we use the approximation

$$\operatorname{corr}\left[S_{i,n}(t_i),S_{j,n}(t_j)\right]\approx\frac{\int_0^{t_i}dS_{i,n}(u)\cdot dS_{j,n}(u)}{\sqrt{\int_0^{t_i}dS_{i,n}(u)^2\cdot\int_0^{t_j}dS_{j,n}(u)^2}}\Big|_{F_k(u)=F_k(0)\,\forall u,k}\quad ,i\le j$$

Formula (9) is used to bootstrap the volatility structure, $\lambda(\cdot)$, in the model. For Bermudan swaptions we calibrate to the core swaptions. So if we consider a Bermudan swaption with the right to enter into a t_n maturity swap at one of the dates $\{t_k,\ldots,t_{n-1}\}$, and let $v_{p,q}$ be the initial implied Black volatility for an at-the-money European swaption over the interval $\{t_p,\ldots,t_q\}$ we choose $\{\lambda_i\}_{i=k,\ldots,n-1}$ so that

$$\sum_{i=k}^{j}\lambda_i^2 S_{i,n}(0)^{2\alpha}\int_{t_{i-1}}^{t_i}\left[\frac{\partial S_{j,n}}{\partial x}(u)\right]_{x=0;y=0}^2 du=\left(v_{j,n}\right)^2 S_{j,n}(0)t_j\quad ,(\text{here: }t_{k-1}\equiv 0)\quad (12)$$

for all $j=k,\ldots,n-1$.

The fact that we can bootstrap the volatility structure of the model makes the calibration procedure very fast. For example, calibration of the model to the ATM core swaptions for a 1nc30 Bermudan can be done in less than 0.05 seconds on a PC.

Numerical Tests

We first test the accuracy of the proposed algorithm for an actual Bermudan swaption. In Table 1 below we illustrate the typical numerical convergence for the price of a 30nc1 annual Bermudan swaption. Market conditions correspond to EUR at April 20, 2001 and the chosen model is the log-normal with zero mean-reversion, which is normally a difficult case.

Table 1: The Numerical Accuracy of the Finite Difference Scheme

grid	price	cpu [s]
25x50x5	5.4440%	0.33
30x60x6	5.3650%	0.39
40x80x8	5.3449%	0.67
50x100x10	5.3462%	1.28
60x120x12	5.3480%	1.97
70x140x14	5.3456%	2.66
80x160x16	5.3497%	3.48
90x180x18	5.3491%	4.35
100x200x20	5.3498%	5.47

Table 1 reports the numerical prices of Bermudan 30nc1 Receiver swaption for different sizes of the finite difference grid together with the associated CPU times. Here 50x100x10 refers to 50 time-steps, 100 x-steps and 10 y-steps. The listed CPU times refer to the computation time on a 1GHz Pentium PC. Market data are EUR as of April 20, 2001.

Table 1 shows that accurate numerical pricing can be achieved on a grid of size of only 40x80x8 grid points, and that this takes less than 1 second on a typical PC.

Andreasen, Cloke, and Piterbarg (2001, Cheyette-8) do a detailed investigation of the accuracy of the approximation (10) used in the calibration and the CEV specification. Table 2 reports an example of these results in the case when the model is calibrated to the at-the-money core swaptions of a 20nc1 Bermudan. In all cases the mid rows represent at-the-money. Market data are EUR as of April 20, 2001. We see that in all cases the implied volatilities are very close for the at-the-money case, whereas the agreement with the simple CEV model (0) for out-of-the-money swaptions is good but not perfect.

Table 2: Accuracy of Core Swaption Calibration and Skew Representation

5y x 15y	cev=0.0		cev=0.5		cev=1.0	
Strike	cheyette	simple model	cheyette	simple model	cheyette	simple model
4.21%	9.97%	10.23%	9.12%	9.28%	8.43%	8.38%
4.56%	9.68%	9.85%	8.91%	9.09%	8.25%	8.38%
4.91%	9.34%	9.51%	8.80%	8.92%	8.30%	8.38%
5.26%	9.07%	9.19%	8.68%	8.77%	8.31%	8.38%
5.61%	8.81%	8.91%	8.56%	8.63%	8.32%	8.38%
5.96%	8.60%	8.65%	8.47%	8.50%	8.35%	8.38%
6.31%	8.39%	8.40%	8.38%	8.38%	8.38%	8.38%
6.66%	8.19%	8.18%	8.29%	8.27%	8.40%	8.38%
7.01%	8.03%	7.97%	8.25%	8.16%	8.43%	8.38%
7.36%	7.86%	7.77%	8.18%	8.06%	8.47%	8.38%
7.71%	7.71%	7.59%	8.09%	7.98%	8.48%	8.38%
8.06%	7.62%	7.42%	8.05%	7.89%	8.52%	8.38%
8.41%	7.45%	7.25%	7.97%	7.81%	8.53%	8.38%
10y x 10y	cev=0.0		cev=0.5		cev=1.0	
Strike	cheyette	simple model	cheyette	simple model	cheyette	simple model
3.56%	10.40%	10.68%	9.07%	9.20%	8.21%	7.91%
4.06%	9.86%	10.05%	8.69%	8.92%	7.73%	7.91%
4.56%	9.34%	9.52%	8.52%	8.67%	7.78%	7.91%
5.06%	8.93%	9.06%	8.35%	8.45%	7.83%	7.91%
5.56%	8.56%	8.65%	8.18%	8.26%	7.83%	7.91%
6.06%	8.23%	8.29%	8.04%	8.09%	7.87%	7.91%
6.56%	7.95%	7.97%	7.92%	7.93%	7.91%	7.91%
7.06%	7.69%	7.68%	7.81%	7.78%	7.95%	7.91%
7.56%	7.46%	7.42%	7.72%	7.65%	7.98%	7.91%
8.06%	7.25%	7.17%	7.62%	7.53%	8.01%	7.91%
8.56%	7.04%	6.95%	7.52%	7.41%	8.03%	7.91%
9.06%	6.92%	6.75%	7.47%	7.31%	8.07%	7.91%
9.56%	6.72%	6.56%	7.38%	7.20%	8.14%	7.91%
15y x 5y	cev=0.0		cev=0.5		cev=1.0	
Strike	cheyette	simple model	cheyette	simple model	cheyette	simple model
3.64%	10.41%	10.60%	8.99%	9.11%	8.00%	7.83%
4.14%	9.94%	9.99%	8.65%	8.83%	7.66%	7.83%
4.64%	9.33%	9.47%	8.47%	8.59%	7.72%	7.83%
5.14%	8.91%	9.01%	8.28%	8.38%	7.72%	7.83%
5.64%	8.56%	8.61%	8.15%	8.19%	7.79%	7.83%
6.14%	8.22%	8.26%	8.00%	8.02%	7.81%	7.83%
6.64%	7.91%	7.94%	7.86%	7.87%	7.83%	7.83%
7.14%	7.68%	7.65%	7.77%	7.72%	7.89%	7.83%
7.64%	7.41%	7.39%	7.70%	7.59%	7.94%	7.83%
8.14%	7.23%	7.15%	7.58%	7.47%	7.96%	7.83%
8.64%	6.99%	6.94%	7.46%	7.36%	7.96%	7.83%
9.14%	6.84%	6.73%	7.39%	7.25%	7.98%	7.83%
9.64%	6.65%	6.54%	7.30%	7.15%	8.07%	7.83%

Table 2 reports the implied Black volatilities for the Cheyette model for different maturity, tenors, strikes, and CEV coefficients for the case when the model is calibrated to the core swaptions of a 20nc1 Bermudan.

With respect to pricing Bermudans and particular calibrating the mean-reversion, Andreasen, Cloke, and Piterbarg (2001) do wide variety of detailed tests of the consistency of the Cheyette model with the BGM. Below we give an example of one of these tests. Table 3 reports the values of a number of different Bermudan swaptions in a 1 factor BGM model and a Cheyette model where the mean-reversion is calibrated to the BGM model using the term-correlation methodology. Market data are EUR as of April 20, 2001.

Table 3: Bermudan swaption values in Cheyette and BGM 1F.

Contract	Bgm 1f, Value	std err	Cheyette
BermParswap10y0.045P1+1	7.64%	0.03%	7.62%
BermParswap10y0.055P1+1	3.91%	0.03%	3.79%
BermParswap10y0.055R1+1	2.38%	0.02%	2.36%
BermParswap10y0.065R1+1	7.04%	0.03%	7.02%
BermParswap20y0.05P1+1	12.63%	0.05%	12.52%
BermParswap20y0.06P1+1	7.06%	0.04%	6.82%
BermParswap20y0.06R1+1	4.49%	0.03%	4.51%
BermParswap20y0.07R1+1	12.35%	0.05%	12.34%
BermParswap30y0.06R1+1	5.23%	0.04%	5.38%
BermParswap30y0.065R10+1	5.04%	0.04%	5.03%
BermAccswap30y0.055R1+1	6.75%	0.07%	6.73%
BermAccswap30y0.065R1+1	20.05%	0.13%	20.07%
BermAccswap30y0.075R1+1	48.04%	0.17%	48.20%
BermAccswap30y0.06R5+5	10.17%	0.10%	10.31%
BermAccswap30y0.06R10+5	9.47%	0.10%	9.59%
BermAccswap30y0.07R5+5	27.74%	0.18%	27.86%
BermAccswap30y0.07R10+5	24.22%	0.19%	24.35%
BermAccswap40y0.06R1+1	20.63%	0.16%	20.87%
BermAccswap40y0.07R1+1	52.77%	0.25%	52.99%
BermAccswap40y0.08R1+1	113.25%	0.30%	113.41%
BermAccswap40y0.065R5+5	31.21%	0.22%	31.51%
BermAccswap40y0.065R10+5	29.82%	0.23%	30.13%
BermAccswap40y0.075R5+5	73.00%	0.37%	73.13%
BermAccswap40y0.075R10+5	67.56%	0.41%	67.74%

Table 3 shows the value of a portfolio of Bermudan swaptions in the BGM 1 factor model and the Cheyette model where the mean-reversion is calibrated to the BGM model using the term correlation methodology. Both models are log-normal. The reported standard error is the simulation error of the BGM model price

In table 3 the difference of the Cheyette and the BGM model is in all cases within one standard error of the Monte-Carlo simulation used in the BGM which illustrates the accuracy of the term correlation methodology.

References

Andersen, L. and J. Andreasen (1998): "Volatility Skews and Extensions of the LIBOR Market Model." *Applied Mathematical Finance* **7**, No 1, 1-32.

Andersen, L. and J. Andreasen (2000): "Factor Dependence of Bermudan Swaption Prices: Fact or Fiction?" Working paper, GRFP. Forthcoming in *Journal of Financial Economics*.

Andreasen, J. (2000): "Split Schemes for Numerical Solution of PDEs." Research note, GRFP.

Andreasen, J., P. Cloke, and V. Piterbarg (2001): *Test of Bank of America Bermuda Swaption Framework*. Research notes, Bank of America.

Babbs, S. (1993):"'Generalised Vasicek' Models of the Term Structure." *Applied Stochastic Models and Data Analysis*, Vol **1**, 49-62.

Cheyette, O. (1992): "Markov Representation of the Heath-Jarrow-Morton Model." Working paper, BARRA.

Craig, I. and A. Sneyd (1988): "An Alternating Implicit Scheme for Parabolic Equations with Mixed Derivatives". *Comput. Math. Applic.* **16**, No 4, 341-350.

Forsyth, P., K. Vetzal, and P. Zvan (1999): "Convergence of Lattice and PDE methods for Asian Options." *Proceedings of the 9th Annual Derivatives Securities Conference*, Boston.

Heath, D., Jarrow, R. and A. Morton (1992): "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation." *Econometrica* **60**, 77-106.

Jamshidian, F. (1991): "Bond and Option Evaluation in the Gaussian Interest Rate Model." *Research in Finance* **9**, 131-710.

Morton, K. and D. Mayers (1994): *Numerical Solution of Partial Differential Equations*. Cambridge University Press.

Press, W., S. Teukolsky, W. Vetterling, and B. Flannery (1988): *Numerical Recipes in C*. Cambridge University Press.

Ritchken, P. and Sankarasubrahmanyam (1993): "On Finite State Markovian Representations of the Term Structure." Working paper, Department of Finance, University of Southern California.