# A multi-currency model with FX volatility skew

Vladimir Piterbarg
Barclays Capital

# 1 Long-dated cross-currency modeling

- PRDC – most widely traded and liquid cross-currency exotics.
  - Long-dated ($>$30 years) swaps with coupons that are options on the FX rate (a dollar-yen rate)
  - They are often Bermuda-style callable, or have knockout provisions, or (recently) TARNs
  - Exposed to the moves in the FX rate, and in the interest rates in both currencies
- Market-standard model:
  - Three factors (one per spot FX, domestic rates, foreign rates)
  - Gaussian for rates
  - Log-normal for the spot FX
  - PDE-based (low number of factors)
  - Forward FX is lognormal so closed-form calibration to European FX options

## 2 FX process

- Log-normal assumption for the FX rates
  - Technically very convenient
  - Little basis in reality
  - Significant volatility skew in FX
  - PRDCs often look like call spreads on FX rate, ie very sensitive to the slope of the implied volatility
- To quantify FX vol skew exposure, need a model that incorporates the skew
- Typical problem:
  - skew-inducing mechanism (local volatility, jumps, stochastic volatility) imposed on *instantaneous* spot FX process
  - To calibrate, need a *term* distribution of forward FX rate(s)
  - With stochastic interest rates, particularly difficult to link the two

## 3 Overview of the proposed model

- Model presented here:
  - Local volatility for the FX rate
  - Same number of factors as in the "standard" model, so can use the same PDE method
  - Focus on calibration to FX options
  - Procedure for almost instantaneous calibration to FX volatility smiles of all expiries
  - Based on skew averaging
  - Allows for stochastic volatility extensions

# 4 The notations

- Two interest rate processes in two currencies and a process for FX
- **P** be the domestic risk-neutral measure.
- $P_i(t,T)$, $r_i(t)$, $i = d, f$, the domestic and foreign zero-coupon discount bonds and short rates in domestic and foreign currencies
- $S(t)$ the spot FX rate, domestic per foreign
- The forward FX rate is denoted by $F(t,T)$,

$$F(t,T) = \frac{P_f(t,T)}{P_d(t,T)} S(t)$$

- $(W_d(t), W_f(t), W_S(t))$ is a Brownian motion under $\mathbf{P}$, correl

$$\begin{pmatrix} 1 & \rho_{df} & \rho_{dS} \\ \rho_{df} & 1 & \rho_{fS} \\ \rho_{dS} & \rho_{fS} & 1 \end{pmatrix}$$

- Bond volatility functions $\sigma_i(t,T)$, $i = d, f$, of special (Markovian) form

$$\sigma_i(t,T) = \sigma_i(t) \int_t^T e^{-\int_t^s \varkappa_i(u)du} ds, \quad i = d, f. \tag{1}$$

## 5 The model

- Under domestic risk-neutral measure,

$$
\begin{aligned}
dP_d(t,T)/P_d(t,T) &= r_d(t)\,dt + \sigma_d(t,T)\,dW_d(t)\,, \\
dP_f(t,T)/P_f(t,T) &= r_f(t)\,dt - \rho_{fS}\sigma_f(t,T)\,\gamma(t,S(t))\,dt + \sigma_f(t,T)\,dW_f(t)\,, \\
dS(t)/S(t) &= (r_d(t) - r_f(t))\,dt + \gamma(t,S(t))\,dW_S(t)\,,
\end{aligned}
\tag{2}
$$

- FX skew via the local volatility function $\gamma(t,x)$.
- The "standard" Gaussian framework ([DH97]) is recovered by choosing the function $\gamma(t,x)$ that is independent of $x$, $\gamma(t,x) = \gamma(t)$.
- Use a parametric form of the local volatility function for calibration stability,

$$\gamma(t,x) = \nu(t)\left(\frac{x}{L(t)}\right)^{\beta(t)-1}. \tag{3}$$

- $\nu(t)$ is the relative volatility function, $\beta(t)$ is a time-dependent constant elasticity of variance (CEV) parameter and $L(t)$ is a time-dependent scaling constant ("level").
- Should not use "U-shaped" local volatility (known problems with dynamics/hedging), and any "skew-type" parametrization will work just as well.

# 6 Valuation

- 3 state variables $(r_d(\cdot), r_f(\cdot), S(\cdot))$. All market quantities (bonds, etc) are functions of these
- If Let $V = V(t, r_d, r_f, S)$ is the value of a security (eg a PRDC) then

$$\begin{aligned} & V_t + (\theta_d(t) - \varkappa_d(t)\, r_d)\, V_{r_d} \\ & + \left(\theta_f(t) + \rho_{fS}\sigma_f(t)\,\gamma(t, S(t)) - \varkappa_f(t)\, r_d\right) V_{r_f} \\ & + (r_d - r_f)\, S V_S \\ & + \frac{1}{2}\sigma_d^2(t)\, V_{r_d r_d} + \frac{1}{2}\sigma_f^2(t)\, V_{r_f r_f} + \frac{1}{2}\gamma^2(t, S)\, S^2 V_{SS} \\ & + \rho_{df}\sigma_d(t)\,\sigma_f(t)\, V_{r_d r_f} - \rho_{dS}\sigma_d(t)\,\gamma(t, S)\, S V_{r_d S} - \rho_{fS}\sigma_f(t)\,\gamma(t, S)\, S V_{r_f S} \\ = \; & r_d V. \end{aligned}$$

- PDE in three space dimensions is most efficiently solved by utilizing a level-splitting scheme, such as an ADI scheme from [CS88]

# 7 Overview of calibration

- Before using the model for exotics, need to chose model parameters to match market-observable prices of related securities
- Interest rate volatility structures $(\sigma_d(t), \sigma_f(t), \varkappa_d(t), \varkappa_f(t))$ are typically chosen to match European swaption values in the respective currencies
- Correlation parameters $\rho_{ij}$, $i, j = d, f, S$, are typically chosen either by historical estimation, or from prices of "quantos"
- For PRDCs, most important is the calibration of $\gamma(t, x)$.
- Options on the FX rate are traded across a wide range of maturities and strikes; use for $\gamma(t, x)$ calibration. However
  - No specific FX strike or expiry "most relevant" for PRDCs
  - Cancellable/knockout PRDCs cannot be decomposed into simple FX options.
  - Hence, the volatility function $\gamma(t, x)$ needs to be calibrated to prices of all available FX options across maturities and strikes.

# 8 Forward FX rate

- Call option on the FX rate with strike $K$ and maturity $T$ pays $(S(T) - K)^+$ at time $T$, and its value at time 0 is equal to

$$c(T, K) = \mathbf{E}_0 \left( e^{-\int_0^T r_d(s)\, ds} (S(T) - K)^+ \right).$$

- Spot FX dynamics are complex. However, forward FX rate is a martingale under the corresponding domestic forward measure.
- Switching to the forward measure leads to decoupling of discounting from the expected value calculations,

$$c(T, K) = P_d(0, T)\, \mathbf{E}_0^T \left( (F(T, T) - K)^+ \right).$$

- Need the dynamics of forward FX rate in the model so we can derive FX option prices in the model

# 9 Forward FX rate, cont

- The dynamics of forward FX rate follows from Ito's lemma. Under domestic $T$-forward measure,

$$\begin{aligned} dF(t,T)/F(t,T) &= \sigma_f(t,T)\, dW_d^T(t) - \sigma_d(t,T)\, dW_d^T(t) \\ &\quad + \gamma(t, F(t,T)\, D(t,T))\, dW_S^T(t) . \end{aligned} \tag{4}$$

- Can represent with a single stochastic driver by computing quadratic variation,

$$\frac{dF(t,T)}{F(t,T)} = \Lambda(t, F(t,T)\, D(t,T))\, dW_F(t), \tag{5}$$

  where

$$\begin{aligned} \Lambda(t,x) &= \left(a(t) + b(t)\,\gamma(t,x) + \gamma^2(t,x)\right)^{1/2}, \\ a(t) &= (\sigma_f(t,T))^2 + (\sigma_d(t,T))^2 - 2\rho_{df}\sigma_f(t,T)\,\sigma_d(t,T), \\ b(t) &= 2\rho_{fS}\sigma_f(t,T) - 2\rho_{dS}\sigma_d(t,T), \end{aligned}$$

  and $D(t,T) \triangleq P_d(t,T)/P_f(t,T)$.

- If $\gamma(t,x)$ is a function of time $t$ only, then the $\Lambda(t, F(t,T)\, D(t,T)) = \Lambda(t)$ is also a deterministic function of time, and $F(T,T)$ is lognormally distributed.

## 10 Markovian representation for forward FX rate

- Need simpler dynamics of the forward FX rate to derive approximations to options
- SDE for $F(t,T)$

$$\frac{dF(t,T)}{F(t,T)} = \Lambda\left(t, F(t,T)\, D(t,T)\right) dW_F(t)$$

  not closed in $F$. Has an extra stochastic process $D(\cdot,T)$
- Approximate $D(t,T)$ with $D_0(t,T) \triangleq P_d(0,t,T)/P_f(0,t,T)$? Not accurate
- Can we find $\tilde{\Lambda}(t,x)$ such that, in the model

$$\frac{dF(t,T)}{F(t,T)} = \tilde{\Lambda}\left(t, F(t,T)\right) dW_F(t),$$

  the values of European options $\{c(t,T,K)\}$ for all $t, K$, *match exactly* the values of the same options in the original model?
- Main result (a-la Dupire): Yes we can, and

$$\tilde{\Lambda}^2(t,x) = \mathbf{E}_0^T\left(\Lambda^2\left(t, F(t,T)\, D(t,T)\right) \middle| \, F(t,T) = x\right).$$

## 11 Proof of Markovian representation

- Recall $c(t,T,K) = P_d(0,t)\,\mathbf{E}_0^T\left((F(t,T)-K)^+\right)$
- The local volatility $\tilde{\Lambda}(t,x)$ such that the values of European options in the model $\frac{dF(t,T)}{F(t,T)} = \tilde{\Lambda}(t, F(t,T))\, dW_F(t)$ match $\{c(,T,K)\}_{t,K}$ is given by the well-known result by Dupire (see [Dup94]),

$$\left(K\tilde{\Lambda}(t,K)\right)^2 = 2\frac{\partial\left(c(t,T,K)/P_d(0,t)\right)/\partial t}{\partial^2\left(c(t,T,K)/P_d(0,t)\right)/\partial K^2}. \tag{6}$$

- To compute the right-had side, we first write (delta-functions justified by Tanaka's formula, see [KS97])

$$d\left(F(t,T)-K\right)^+ = 1_{\{F(t,T)>K\}}dF(t,T) + \frac{1}{2}\delta_{\{F(t,T)=K\}}d\left\langle F(t,T)\right\rangle,$$

since $F(t,T)$ is a martingale under the domestic $T$-forward measure,

$$\mathbf{E}^T\left(F(t,T)-K\right)^+ - \left(F(0,T)-K\right)^+ = \frac{1}{2}\int_0^t \mathbf{E}^T\left(\delta_{\{F(t,T)=K\}}d\left\langle F(t,T)\right\rangle\right).$$

## 12 Proof of Markovian representation, cont

- Then

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{c(t,T,K)}{P_d(0,t)} &= \frac{\partial}{\partial t}\left(\mathbf{E}^T\left(F(t,T)-K\right)^+ - \left(F(0,T)-K\right)^+\right)\\
&= \frac{1}{2}\mathbf{E}^T\left(\delta_{\{F(t,T)=K\}} d\left\langle F(t,T)\right\rangle\right)\\
&= \frac{1}{2}\mathbf{E}^T\left(\delta_{\{F(t,T)=K\}}\right)\mathbf{E}^T\left(d\left\langle F(t,T)\right\rangle \middle| F(t,T)=K\right)\\
&= \frac{1}{2}\frac{\partial^2}{\partial K^2}\frac{c(t,T,K)}{P_d(0,t)}\mathbf{E}^T\left(d\left\langle F(t,T)\right\rangle \middle| F(t,T)=K\right)
\end{aligned}$$

- Since,

$$d\left\langle F(t,T)\right\rangle = F^2(t,T)\,\Lambda^2(t,F(t,T)\,D(t,T))\,dt,$$

we obtain

$$\frac{\partial}{\partial t}\frac{c(t,T,K)}{P_d(0,t)} = \frac{1}{2}\frac{\partial^2}{\partial K^2}\frac{c(t,T,K)}{P_d(0,t)}K^2\mathbf{E}^T\left(\Lambda^2(t,F(t,T)\,D(t,T))\middle| F(t,T)=K\right).$$

Comparing to (6), the result follows

## 13 Simplifying Markovian representation

- The result

$$\tilde{\Lambda}^2(t,x) = \mathbf{E}_0^T\left(\Lambda^2(t, F(t,T)\, D(t,T)) \middle| F(t,T) = x\right).$$

is very intuitive – the Markovian dynamics is defined by the diffusion coefficient that is the expected value of the original diffusion coefficient conditioned on the underlying.

- This is exact for European options and, hence, all derivatives with European-style payoffs.
- Need a way to compute (approximate) the conditional expected value
- Recall that

$$\Lambda(t,x) = \left(a(t) + b(t)\,\gamma(t,x) + \gamma^2(t,x)\right)^{1/2}.$$

# 14 Approximation

- The dynamics of $F(t,T)$ can be approximated by

$$\frac{dF(t,T)}{F(t,T)} = \hat{\Lambda}(t, F(t,T))\, dW_F(t), \tag{7}$$

where

$$\begin{aligned}
\hat{\Lambda}(t,x) &= \left(a(t) + b(t)\,\hat{\gamma}(t,x) + \hat{\gamma}^2(t,x)\right)^{1/2}, \\
\hat{\gamma}(t,x) &= \nu(t)\left(x\frac{D_0(t,T)}{L(t)}\right)^{\beta(t)-1}\left(1 + (\beta(t)-1)\, r(t)\left(\frac{x}{F(0,T)} - 1\right)\right), \\
r(t) &= \frac{\int_0^t \chi_{Z,F}(s)\, ds}{\int_0^t \chi_{F,F}(s)\, ds}, \\
\chi_{Z,F}(t) &= -a(t) - \frac{b(t)}{2}\gamma(t, F(0,T)\, D_0(t,T)), \\
\chi_{F,F}(t) &= a(t) + b(t)\,\gamma(t, F(0,T)\, D_0(t,T)) + \gamma^2(t, F(0,T)\, D_0(t,T)).
\end{aligned}$$

## 15 Comparison to a simpler approximation

- Recall from previous slide,

$$\hat{\gamma}(t,x) = \nu(t)\left(x\frac{D_0(t,T)}{L(t)}\right)^{\beta(t)-1}\left(1+(\beta(t)-1)\,r(t)\left(\frac{x}{F(0,T)}-1\right)\right).$$

- Had we used the approximation $D(t,T) \approx D_0(t,T) = P_d(0,t,T)/P_f(0,t,T)$, the corresponding formula would simply be

$$\hat{\gamma}_0(t,x) = \nu(t)\left(x\frac{D_0(t,T)}{L(t)}\right)^{\beta(t)-1}.$$

- Hence, accounting for the stochastic nature of $D(t,T)$ introduces an adjustment to the slope of the local volatility function $\hat{\gamma}(t,x)$ around the forward $x = F(0,T)$.

- The size of the correction depends on $r(t)$, which can be interpreted as a "regression coefficient" between $F(\cdot,T)$ and $D(\cdot,T)$

## 16 Deriving the approximation

- The approximation is based on the following result, that for any $c$,

$$\begin{aligned}\mathbf{E}^T\left(\left.(D(t,T))^c\right| F(t,T)=x\right) \approx (D_0(t,T))^c \\ \times\left(1+c\times\frac{\int_0^t \chi_{Z,F}(s)\,ds}{\int_0^t \chi_{F,F}(s)\,ds}\times\left(\frac{x}{F(0,T)}-1\right)\right),\end{aligned} \tag{8}$$

- This can be proven by approximating the joint dynamics of $D(t,T)$, $F(t,T)$ with Gaussian.
- Recall $\tilde{\Lambda}^2(t,x) = \mathbf{E}^T\left(\Lambda^2(t,F(t,T)\,D(t,T))\middle|\, F(t,T)=x\right)$.
- The approximation is obtained by applying (8) to the rhs of the above, once we recall that

$$\begin{aligned}\Lambda^2(t,F(t,T)\,D(t,T)) &= a(t) \\ &\quad + b(t)\,\gamma(t,F(t,T)\,D(t,T)) + \gamma^2(t,F(t,T)\,D(t,T)),\\ \gamma(t,x) &= \nu(t)\left(\frac{F(t,T)}{L(t)}\right)^{\beta(t)-1}(D(t,T))^{\beta(t)-1}.\end{aligned}$$

# 17 Next step

- We have derived an “autonomous” equation for $F(\cdot, T)$.
  - It is a one-dimensional SDE with the diffusion coefficient given by a local volatility function $\hat{\Lambda}(t, x)$.
  - Prices of options on $F(T, T)$ can in principle now be computed in a one-dimensional PDE
  - a significant reduction of effort from a three-dimensional one!
  - However, an even faster method is possible
- Know the slope of the local volatility function $(\hat{\Lambda}(t, x))$ for all $t$ at the forward, $x = F(0, T)$. Time-dependent slope.
- Can we relate it to the “total”, or “effective” (time-independent) slope of the local volatility function?
- Method of skew (or, more generally parameter) averaging.
- Time-dependent parameters are replaced with “effective”, time-constant ones, thus allowing to relate model and market parameters directly without actually performing any option calculations

- Let $X(t)$ be a stochastic process defined by

$$dX(t) = g(t, X(t))\, dW(t), \quad X(0) = x_0,$$

- Main result: the distribution of $X(T)$ is well-approximated by the distribution of $Y(T)$, where the stochastic process $Y(t)$ is defined by

$$dY(t) = \sigma(t)\, \bar{g}(Y(t))\, dW(t), \quad Y(0) = x_0,$$

and the functions $\sigma(t)$, $\bar{g}(y)$ are such that

$$\begin{aligned} \sigma(t) &= g(t, x_0), \quad \bar{g}(x_0) = 1, \\ \frac{\partial}{\partial x}\bar{g}(x)\Big|_{x=x_0} &= \int_0^T w(t) \frac{\frac{\partial}{\partial x} g(t, x)\big|_{x=x_0}}{g(t, X_0)} dt, \end{aligned}$$

- The weights $w(t)$ in the last equation are given by

$$w(t) = \frac{u(t)}{\int_0^T u(t)\, dt}, \quad u(t) = g^2(t, x_0) \int_0^t g^2(s, x_0)\, ds.$$

## 19 Idea of skew averaging proof

- Skew averaging in the small-slope limit
- Let $\varepsilon > 0$ be small (expansion parameter). Define

$$\begin{aligned} g_\varepsilon(t,x) &= g(t, x_0 + (x - x_0)\varepsilon)/\sigma(t), \\ \bar{g}_\varepsilon(x) &= \bar{g}(x_0 + (x - x_0)\varepsilon), \end{aligned}$$

- Define two families of diffusions indexed by $\varepsilon$,

$$\begin{aligned} dX_\varepsilon(t) &= g_\varepsilon(t, X_\varepsilon(t))\,\sigma(t)\;dW(t), \quad X_\varepsilon(0) = x_0, \\ dY_\varepsilon(t) &= \bar{g}_\varepsilon(Y_\varepsilon(t))\,\sigma(t)\;dW(t), \quad Y_\varepsilon(0) = x_0. \end{aligned}$$

- Define

$$q(\varepsilon) = \mathbf{E}\left(X_\varepsilon(T) - Y_\varepsilon(T)\right)^2.$$

- We look for conditions on $\bar{g}(\cdot)$ that minimize $q(\varepsilon)$ for small $\varepsilon$
- The condition ensures that options will all strikes are recovered as best as possible

## 20 Idea of skew averaging proof, cont

- The main result: Any function $\bar{g}$ that minimizes $q(\varepsilon)$ for small $\varepsilon$ satisfies the condition
$$\frac{\partial \bar{g}(x_0)}{\partial x} = \int_0^T \frac{\partial g(t, x_0)}{\partial x} w(t)\, dt,$$
where
$$w(t) = \frac{v^2(t)\,\sigma^2(t)}{\int_0^T v^2(t)\,\sigma^2(t)\, dt},$$
$$v^2(t) = \mathbf{E}\left((X_0(t) - x_0)^2\right).$$

- Comments:
  - "Total skew" $\frac{\partial \bar{g}(x_0)}{\partial x}$ is the average of "local skews" $\frac{\partial g(t,x_0)}{\partial x}$ with weights $w(t)$
  - Weights proportional to total variance, i.e. local slope further away matters more
  - Can get the same result under different criteria, i.e. robust

## 21 Applying skew averaging to the approximate forward FX dynamics

- Recall

$$\frac{dF(t,T)}{F(t,T)} = \hat{\Lambda}(t, F(t,T))\, dW_F(t)$$

- Use the skew averaging result with

$$\begin{aligned} g(t,x) &= x\hat{\Lambda}(t,x), \\ \bar{g}(x) &= \delta_F \frac{x}{F(0,T)} + (1-\delta_F), \\ x_0 &= F(0,T). \end{aligned}$$

## 22 Applying skew averaging to the approximate forward FX dynamics, the result

- Obtain the approximate dynamics

$$dF(t,T) = \hat{\Lambda}(t, F(0,T))\left(\delta_F F(t,T) + (1-\delta_F) F(0,T)\right) dW_F(t),$$

where

$$\delta_F = 1 + \int_0^T w(t) \frac{b(t)\eta(t) + 2\hat{\gamma}(t, F(0,T))\eta(t)}{2\hat{\Lambda}^2(t, F(0,T))} dt, \quad (9)$$

$$\eta(t) = \hat{\gamma}(t, F(0,T))(1 + r(t))(\beta(t) - 1),$$

$$w(t) = \frac{u(t)}{\int_0^T u(t)\, dt}, \quad u(t) = \hat{\Lambda}^2(t, F(0,T)) \int_0^t \hat{\Lambda}^2(s, F(0,T))\, ds.$$

## 23 Applying skew averaging to the approximate forward FX dynamics, cont

- In particular, $F(\cdot, T)$ follows a standard displaced-diffusion SDE with the skew parameter $\delta_F$. The value $c(T, K)$ of a call option on the FX rate with maturity $T$ and strike $K$ is equal to

$$c(T, K) = P_d(0, T)\, c_{\text{Black}}\left(\frac{F(0, T)}{\delta_F}, K + \frac{1 - \delta_F}{\delta_F} F(0, T), \sigma_F \delta_F, T\right), \tag{10}$$

$$\sigma_F = \left(\frac{1}{T} \int_0^T \hat{\Lambda}^2 (t, F(0, T))\, dt\right)^{1/2}, \tag{11}$$

where $c_{\text{Black}}(F, K, \sigma, T)$ is the Black formula

## Calibration

- Maturities $0 = T_0 < T_1 < \ldots < T_N$ are given, and the model (2) is to be calibrated to FX options with maturities $\{T_n\}_{n=1}^N$.
- Need to determine time-dependent functions $\nu(t)$ and $\beta(t)$. WLOG assumed piecewise-constant $\nu(t) = \sum_{n=1}^N \nu_n \cdot 1_{(T_{n-1},T_n]}(t)$, $\beta(t) = \sum_{n=1}^N \beta_n \cdot 1_{(T_{n-1},T_n]}(t)$
- Recall main result: an approximation to forward FX rates by a displaced-diffusion process.
- First, express market prices of FX options in terms of the displaced-diffusion model. For each maturity $T_n$, a *market* volatility $\sigma_n^*$ and a *market* skew parameter $\delta_n^*$ are determined

## 25 Calibration, cont

- Then, we need to

  Find model parameters $\{(\nu_n, \beta_n)\}_{n=1}^N$ such that "effective" volatility $\sigma_F = \sigma_F(T_n)$ and "effective" skew $\delta_F = \delta_F(T_n)$, computed for each expiry $T_n$ from $\{(\nu_k, \beta_k)\}_{k=1}^N$ according to the formulas (11), (9), match the market-implied values $\{(\sigma_n^*, \delta_n^*)\}_{n=1}^N$.

- Calculations can be organized so that the problem can be split into $N$ sequential problems, each one involving only a two-dimensional root search
- In particular, $\nu_n$, $\beta_n$ can be found from $\sigma_F(T_n)$, $\delta_F(T_n)$ and previously found $\{(\nu_k, \beta_k)\}_{k=1}^{n-1}$
- Fractions of a second on a computer. No need to price options during search– model and market parameters linked directly

## 26 Quality of fit

- FX volatility smiles in the model well-approximated by displaced-diffusion type smiles (skews, really)
- Can have sloped implied volatilities.. A significant improvement over flat Black volatilities in the “standard” 3-factor model
- Particularly important as PRDCs often very sensitive to the slope of the smile
- Allows to measure smile slope sensitivity intrinsically in the model
- But – may not be enough to match the market-observed volatility smile (that can be U-shaped)

## 27 Possible extensions

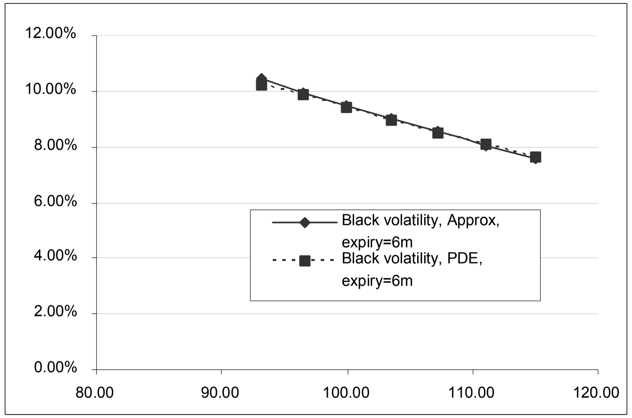
- Two approaches possible
  - Stay in the local volatility context, use more convex local volatility
    - Simple to extend
    - Dynamics questionable
  - Extend the idea to other smile-generating mechanisms
    - In particular stochastic volatility is relatively easy to incorporate since averaging methods have been developed for SV models (see [Pit05b], [Pit05c])
    - One more factor, so use Monte-Carlo?
    - Believable dynamics
    - Short-term smile could be too flat

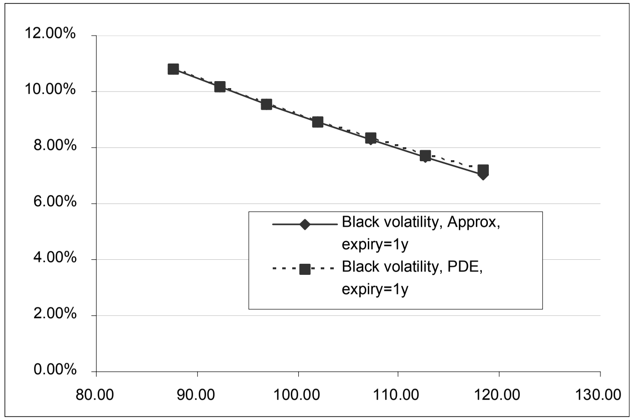
- Differences, in implied Black volatilities, between values of FX options computed using the PDE method and the approximation method. Strikes across rows

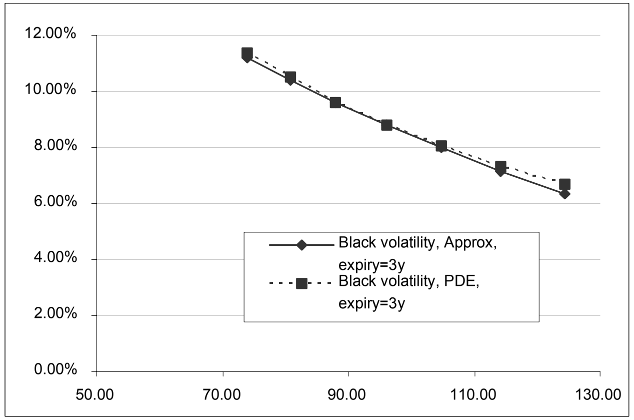
| Expiry | Error 1 | Error 2 | Error 3 | Error 4 | Error 5 | Error 6 | Error 7 |
|---|---|---|---|---|---|---|---|
| 6m | -0.26% | -0.16% | -0.12% | -0.10% | -0.07% | -0.02% | 0.05% |
| 1y | -0.06% | -0.06% | -0.07% | -0.06% | -0.03% | 0.03% | 0.14% |
| 3y | 0.14% | 0.07% | 0.00% | -0.02% | 0.01% | 0.11% | 0.28% |
| 5y | 0.17% | 0.11% | 0.04% | 0.00% | 0.03% | 0.12% | 0.30% |
| 7y | 0.21% | 0.16% | 0.07% | 0.02% | 0.03% | 0.12% | 0.29% |
| 10y | 0.19% | 0.19% | 0.11% | 0.04% | 0.03% | 0.08% | 0.20% |
| 15y | -0.01% | 0.08% | 0.07% | 0.01% | -0.04% | -0.08% | -0.09% |
| 20y | -0.39% | -0.17% | -0.08% | -0.05% | -0.06% | -0.08% | -0.11% |
| 25y | -0.62% | -0.31% | -0.15% | -0.06% | -0.03% | -0.02% | -0.02% |
| 30y | -0.82% | -0.42% | -0.18% | -0.06% | 0.02% | 0.05% | 0.06% |

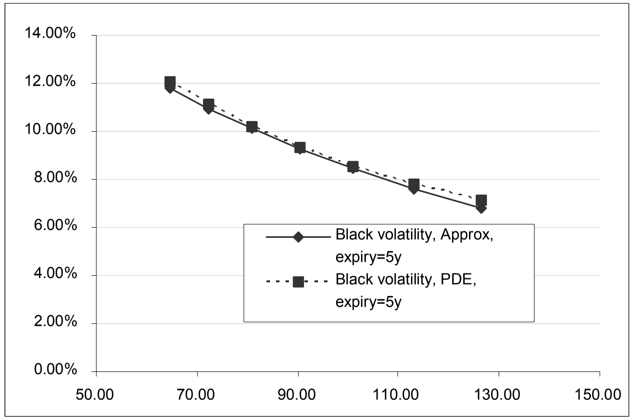
# 29 Test results, cont

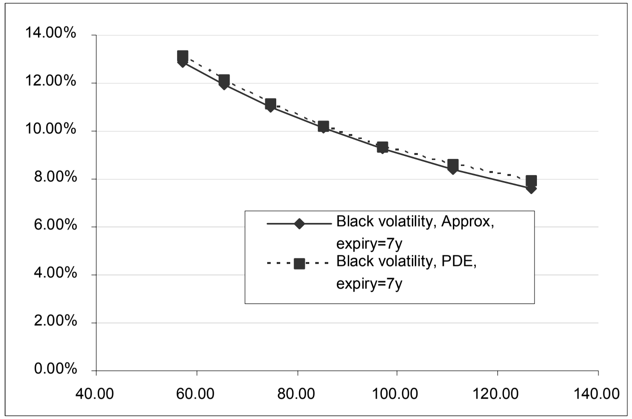
Same in Figures

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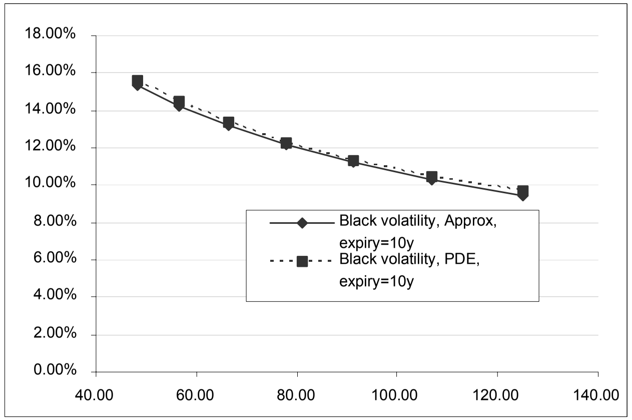
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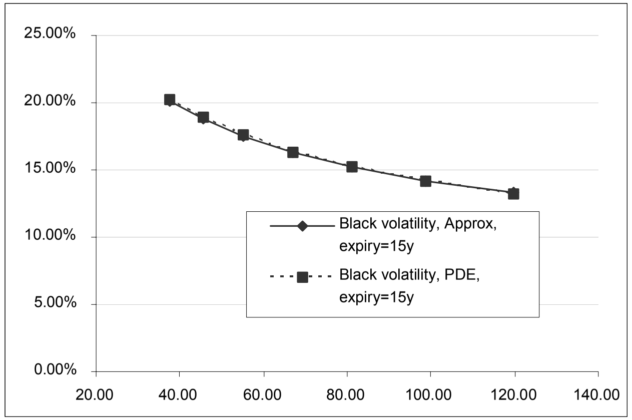
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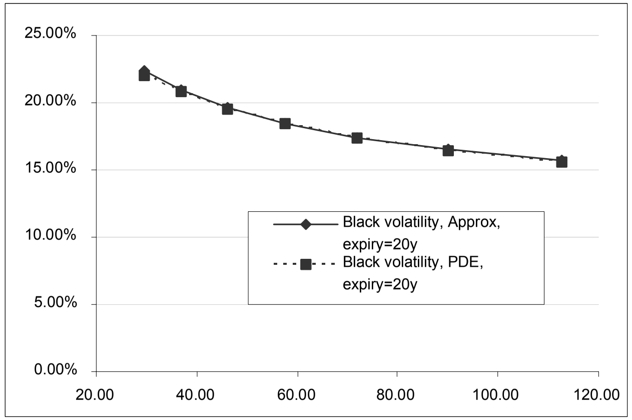
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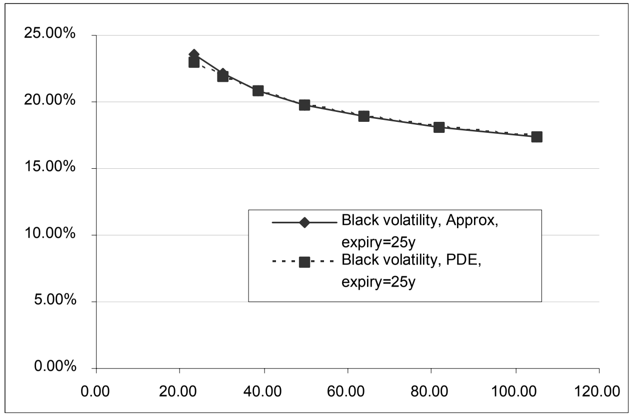
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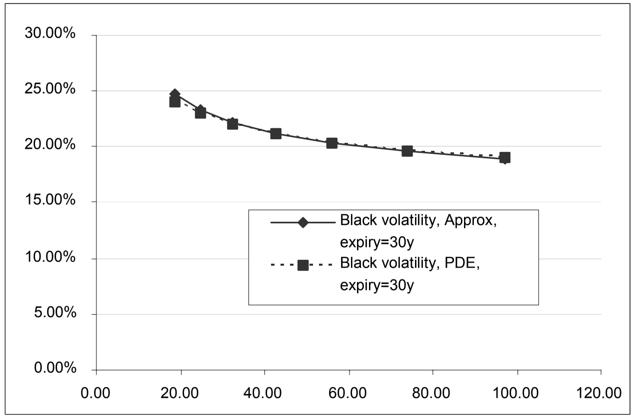
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Black volatility, Approx, expiry=30y
Black volatility, PDE, expiry=30y

- PRDC pays a coupon

$$C_n(S) = \min\left(\max\left(g_f \frac{S(T_n)}{s} - g_d, b_l\right), b_u\right), \quad n = 1, \ldots, N-1.$$

- $g_f$ and $g_d$ are called the *foreign* and the *domestic* coupons
- $b_l$ and $b_u$ are the *floor* and the *cap* on the payoff.
- The scaling factor $s$ is often called the *initial FX rate*. All the parameters can vary from coupon to coupon, ie depend on $n$, $n = 1, \ldots, N-1$.
- Bermuda-style callable, knockout on the FX rate, or a TARN
- We consider callable and knockout ones
- Three sets of parameters for each: low-leverage, medium-leverage, high-leverage.
  - Leverage determined by the combination of $g_f$, $g_d$.
  - Option notional $h = \frac{g_f}{s}$, option strike $k = \frac{s g_d}{g_f}$. Higher strikes + higher notional implies higher leverage (same expected coupon)

# 31 Results for PRDCs

- Results in percentage points of the notional

| Leverage | Low | Medium | High |
|---|---|---|---|
| **Trade details** | | | |
| Foreign coupon | 4.50% | 6.25% | 9.00% |
| Domestic coupon | 2.25% | 4.36% | 8.10% |
| Barrier | 110.00 | 120.00 | 130.00 |
| **PV, lognormal model** | | | |
| Underlying | -8.66 | -9.24 | -9.35 |
| Cancellable | 13.61 | 17.13 | 23.16 |
| Knockout | 4.16 | 8.08 | 14.12 |
| **PV, skew model** | | | |
| Underlying | -10.67 | -11.66 | -10.86 |
| Cancellable | 11.90 | 14.62 | 20.37 |
| Knockout | 1.52 | 2.89 | 6.32 |
| **Diff, skew - lognormal** | | | |
| Underlying | -2.01 | -2.43 | -1.52 |
| Cancellable | -1.71 | -2.51 | -2.79 |
| Knockout | -2.64 | -5.19 | -7.80 |

- Parameters chosen so the underlying swap has roughly the same value. Higher leverage – higher option(cancellable or knockout) value

## 32 Skew impact on the underlying

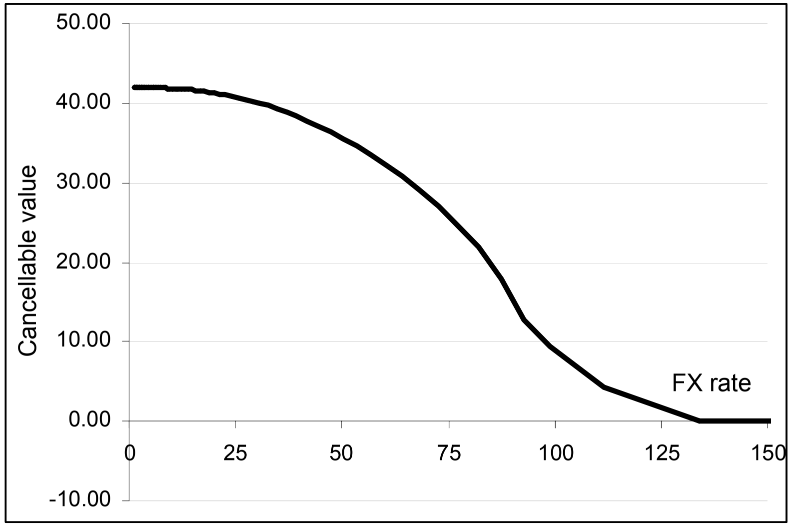
- PRDC swaps consist of (short) FX call options with low strikes ("low" means "weak dollar" for the analysis in this section).
- From figures, introduction of the skew *increases* the implied volatility of low-strike options
- The value of PRDCs are pushed *down* (for the issuer)
- The effect is the most pronounced for medium-leverage swaps. Why? The total effect is a combination of
  - the change in impled volatilities,
  - the level of sensitivities of the options to those changes.

## 33 Skew impact on the cancellables

- The (negative) impact can be seen uniformly increasing with the increased leverage.
- The effect is quite substantial, between $-1.70\%$ to $-2.80\%$.
- The impact is even more pronounced on knockouts, with the values changing by the amounts ranging from $-2.60\%$ to $-7.80\%$.
- These changes in values are comparable to typical profits booked by the issuer.
- Hence, not accounting for the FX skew can easily show a profit on a trade that was actually a loss.
- The conclusion: *accounting for the FX skew, for example in the way developed in this talk, is absolutely critical for proper pricing and risk-managing the PRDC book.*

## 34 Skew impact on the cancellables, cont

- To understand skew impact, look at the value of a cancellable PRDC at $T = 5y$ a function of the spot FX rate.



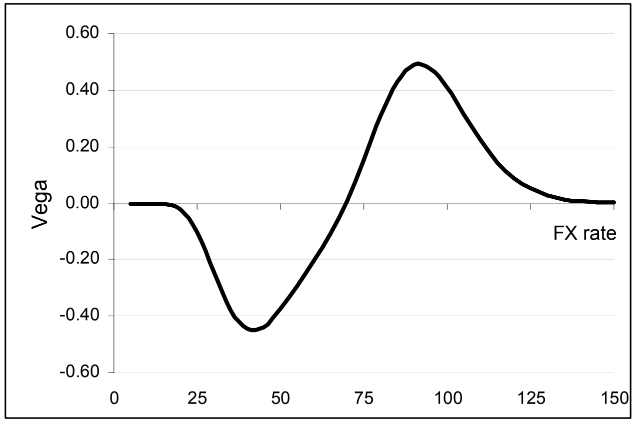

- It is optimal to cancel the swap if the FX rate is high enough
- The payoff is concave for $S <$forward, reflecting the negative convexity of short FX option positions, the PRDC coupons due to the issuer.
- For $S >$forward, the payoff is convex, reflecting the cancel option at higher strike.

## 35 Skew impact on the cancellables, cont

- Skew affects the value of a cancellable in two ways.
  - First, the higher volatility for lower strikes means that the “left”, concave, side of the payoff is valued *lower*.
  - The lower volatility for high strikes means that the “right”, convex side of the payoff is also valued *lower*.
  - Double wammy!
  - The profile of the cancellable PRDC is similar to a call spread
- For knockout PRDCs:
  - Effect is similar to that of a cancellable PRDC, as it goes to zero for high values of the FX rate.
  - Extra sensitivity comes from discontinuity at the boundary. Digital-like options are known to be primarily affected by the slope of the FX smile.

## 36 Skew impact on the cancellables, cont

- Another way to understand the exposures is via FX Vega profile (FX vega vs FX rate)




- Same conclusions:
  - as low-strike volatilities are increased, the value of the cancellable PRDC goes down
  - as high-strike volatilities are decreased, the value of the cancellable PRDC goes down

# 37 Conclusions

- The shape of the exposure of cancellable and knockout PRDCs to FX volatilities make it impossible to reproduce the impact of the FX volatility smile within a less-flexible model
  - It is often possible to introduce the smile effect into valuation by judiciously choosing the strikes to which to calibrate a no-smile model to
  - Here, there is no single strike to encapsulate the required dependence.
  - One cannot find the "effective" strike, as it is the slope of the FX volatility smile that matters.
- Hence, a model that explicitly incorporates the FX volatility smile into a multi-dimensional, cross-currency model suitable for valuation of power-reverse dual-currency securities is required
- We have developed such a model with nearly instantaneous calibration
- Extension of the standard three-factor log-normal model and uses the same valuation method
- Details in [Pit05a]

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