Markov Representation of the Heath-Jarrow-Morton Model

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Abstract

The Heath-Jarrow-Morton (HJM) model of interest rates provides a fully general paradigm for the specification of interest rate term structure dynamics. However, there is no way to describe the dynamics of an arbitrary HJM model by means of a finite number of state variables while maintaining exact arbitrage freedom. This makes practical application of the model difficult.

This paper provides a construction of a class of arbitrage free term structure models that are Markovian in a finite number of state variables, consistent with an arbitrary initial term structure, and capable of representing almost any term structure of forward rate volatility with arbitrary accuracy (subject to standard technical conditions). Put another way, this paper shows how to approximate a large class of HJM models with an arbitrage-free Markov model to arbitrary accuracy. The approximation can be made as accurate as desired by using a sufficiently large number of state variables. In practice, accuracy at the level of statistical uncertainty for the term structure of volatility of the US Treasury market can be achieved with five state variables for the first principal component.

1 Introduction

The literature on interest rate dynamics is characterized by two distinct approaches to the description of arbitrage free processes. One type of model treats the short rate as a Markov process, and by introduction of a time dependent drift term fitted to the initial data, yields an arbitrage free dynamics with specified short rate volatility. (See, for example, Hull and White [1].) Except in special cases it is not possible to find an explicit expression for the term structure at future times in terms of the short rate, though it is fully determined by the dynamical assumptions. The term structure at $t > 0$ as a function of the short rate can only be found by numerical analysis for a particular realization of the model parameters.[1] In spite of this difficulty, the Markov property of the dynamics makes these models amenable to simple numerical methods for calculation of asset values, such as the binomial lattice (*e.g.*, the Black-Derman-Toy model [6]).

The second approach can be viewed as an effort to model the dynamics of the entire term structure, rather than just the short rate. The original model of this type is that of Ho and Lee [7], a one factor discrete time model with volatility independent both of the level of interest rates and of maturity. The initial term structure is taken as an input, and the model explicitly determines the complete term structure at all later times as a function of the short rate and the elapsed time. Continuous time representations of the same dynamics, and related mean reverting models have been given by Heath, Jarrow and Morton [8] and by Jamshidian [9]. Models of this general type are often called "whole-yield" models. Although it is not possible to do so generally, in this simple case the dynamics can be represented as a Markov process in the short rate. This model is therefore also of the type described in the preceding paragraph.

Heath, Jarrow and Morton (HJM) have developed a general framework for constructing whole-yield models. Their framework permits an arbitrary term structure of volatility and covariance of forward rates across maturities. Essentially any reasonable term structure dynamics can be modeled by the HJM technique. The primary practical significance of the HJM approach is the control it provides over the term structure of volatility, and that it gives one complete information about the forward curve at all times. This last point is particularly important for the application of HJM models to valuation of mortgage backed securities and derivatives such as index amortizing swaps, since the cashflow models for these securities typically require future par yields or swap rates as inputs. Short rate Markov

[1] Explicit expressions have been found in models where the drift term is chosen ad hoc [2,3] or based on a simple economic model [4] and does not fit an arbitrary initial term structure. A trick of Dybvig's [5] allows one to adapt these models to fit an initial term structure, at the cost of modifying the relationship between the volatility and the short rate.

models inevitably must make some more or less crude approximation to derive these values along a scenario, resulting in valuation errors.

The general HJM model can be viewed as a joint Markov process in an infinite number of forward rates, *i.e.*, the state space of the dynamics is infinite dimensional — even when there is only one random factor. This means that efficient numerical techniques for solving diffusion equations in a small number of dimensions are not applicable in the HJM framework. Alternatively, using a Monte Carlo or other simulation technique, one is forced to track a large number of points on the forward rate curve at each time step. For example, Monte Carlo valuation of a 30 year security with monthly cashflows requires following the evolution of 360 forward rates to precisely model all the relevant interest rates.[2] A further difficulty is that the HJM framework is expressed in terms of the dynamics of fixed maturity forward rates, while empirical observations of the term structure and valuation models are most naturally expressed in terms of constant *relative* maturity rates such as the short rate, one year rate, etc.

The model framework presented in this paper can be viewed as a reformulation of the HJM approach to overcome the difficulties enumerated above, while retaining the desirable properties of generality of dynamical assumptions and explicit representation of the term structure. The development of the model proceeds in two steps.

The first is to find a class of volatility functions that permit a description of the (HJM) term structure dynamics as a joint Markov process with a finite number of state variables. Using these forward rate volatility functions, it is possible to approximate any HJM model with *continuous* forward rate volatility functions uniformly to arbitrary accuracy, while maintaining exact arbitrage freedom. This is the main result of this paper. Because this framework is based on the HJM approach, models based on these volatility functions share the property that the entire forward rate curve is known at any time in terms of the finite set of state variables.

The second step is to further restrict the volatility functions to those that permit a natural expression of the dynamics in terms of constant relative maturity forward rates. This permits one to model interest rate processes having time-independent volatility functions as a Markov process.

As will be seen in section 3, the two most important factors in the dynamics of the US Treasury term structure can be well described by versions of this model with five state variables each. This work extends the model described in reference [10], which was also later reproduced independently in [11].

[2] Presumably one could make approximations using interpolation between a smaller number of modeled forward rates, but one then has to control the arbitrage errors that are introduced.

The next section provides a derivation of the model, giving first the most general model with a single stochastic noise factor, then specializing to a model whose term structure of volatility is given as a function of interest rates, time and relative maturity. As with the original HJM model, it is straightforward to extend this framework to incorporate multiple stochastic factors.

2 Term Structure Dynamics

The model construction starts from the general HJM expression for forward rate dynamics in a one factor context. It is then shown that when the forward rate volatility takes a particular form, the HJM dynamics can be expressed as a Markov process in terms of a finite number of state variables. Finally, it is observed that the restricted class of forward rate volatilities can in fact approximate any (reasonable) term struture of volatility to arbitrary accuracy.

Let $\sigma(t, T, \boldsymbol{\theta})$ be the time t volatility of the T-maturity default-free zero coupon bond. $\boldsymbol{\theta}$ is an arbitrary set of time t forward rates or derivable values (such as spot rates or par bond yields) on which σ may depend. According to HJM, the forward rate evolution is then described by the stochastic differential equation [8]

$$df(t,T) = \sigma_T(t,T,\boldsymbol{\theta})\,(\sigma(t,T,\boldsymbol{\theta}) - \lambda(t))\;dt + \sigma_T(t,T,\boldsymbol{\theta})\,dz(t) \tag{1}$$

where $\sigma_T(t, T, \boldsymbol{\theta}) \equiv \partial_T \sigma(t, T, \boldsymbol{\theta})$ is the forward rate volatility, dz is a Brownian increment and $\lambda(t)$ is the market price of interest rate risk, which may depend on t and $\boldsymbol{\theta}$, and in general also on events prior to time t, but not on the maturity T.

Equation (1) can be formally integrated to give

$$f(t,T) = f(0,T) + \int_0^t ds\,\sigma_T(s,T,\boldsymbol{\theta})\,(\sigma(s,T,\boldsymbol{\theta}) - \lambda(s)) + \int_0^t dz(s)\,\sigma_T(s,T,\boldsymbol{\theta}) \tag{2}$$

which gives the forward rate in terms of its initial value at time 0, the cumulative drift, and the cumulative stochastic fluctuations.

As first shown in [10] the path dependence of the integrals in equation (2) can be captured by two state variables if the volatility term structure is "separable" into a time and rate dependent factor and a maturity dependent factor: $\sigma_T(t, T, \boldsymbol{\theta}) = \alpha(T)\dfrac{\beta(t,\boldsymbol{\theta})}{\alpha(t)}$, or equivalently, $\sigma_T(t, T, \boldsymbol{\theta}) = \beta(t, \boldsymbol{\theta}) \exp\left(-\int_t^T du\,\kappa(u)\right)$ where $\kappa(u)$ is a deterministic function of time, and is the short rate mean reversion in the risk-neutral measure. In this case the short rate $r(t)$ and the cumulative quadratic variation $V(t) = \alpha^2(t)\int_0^t ds \dfrac{\beta^2(s,\boldsymbol{\theta})}{\alpha^2(s)}$ fully parameterize the evolution of the term structure. The forward rate volatility functions allowed by this model are rather restricted. For example, a volatility term structure with a persistently "humped" shape — rising at short maturity to a maximum, then decreasing for longer maturities — is not permitted in this model. (It is possible to get a *temporarily* humped shape through appropriate time dependence of either α or β. However the location of the maximum will move to shorter maturity over time, and eventually disappear.)

This paper extends the result in [10] to permit more complicated volatility term structures, at the cost of introducing additional state variables. For example, a persistently humped volatility term structure can be obtained with a five state-variable version of this model. The following proposition describes the construction of the model.

Proposition: *Assume that the forward rate volatility can be written as a sum of terms each of which is a product of a rate and time dependent function and a maturity dependent function:*

$$\sigma_T(t, T, \boldsymbol{\theta}) = \sum_{i=1}^{N} \alpha_i(T) \frac{\beta_i(t, \boldsymbol{\theta})}{\alpha_i(t)} \,. \tag{3}$$

Define $N(N+3)/2$ state variables x_i, $V_{ij} = V_{ji}$, $i, j = 1, \ldots, N$ by

$$\begin{aligned} x_i(t) &= \int_0^t dz(s)\, \frac{\alpha_i(t)}{\alpha_i(s)} \beta_i(s, \boldsymbol{\theta}) + \int_0^t ds\, \frac{\alpha_i(t)\beta_i(s, \boldsymbol{\theta})}{\alpha_i(s)} \left(\sum_{k=1}^{N} \frac{A_k(t) - A_k(s)}{\alpha_k(s)} \beta_k(s, \boldsymbol{\theta}) - \lambda(s) \right) \\ V_{ij}(t) = V_{ji}(t) &= \int_0^t ds\, \frac{\alpha_i(t)\alpha_j(t)}{\alpha_i(s)\alpha_j(s)} \beta_i(s, \boldsymbol{\theta}) \beta_j(s, \boldsymbol{\theta}) \,, \end{aligned} \tag{4}$$

where $A_k(t) = \int_0^t ds\, \alpha_k(s)$.

Then the forward rate equation (2) can be expressed as

$$\tilde{f}(t, T) \equiv f(t, T) - f(0, T) = \sum_j \frac{\alpha_j(T)}{\alpha_j(t)} \left[x_j(t) + \sum_{i=1}^{N} \frac{A_i(T) - A_i(t)}{\alpha_i(t)} V_{ij}(t) \right] \,. \tag{5}$$

The state variables x_i and V_{ij} form a Markov process with evolution equations

$$\begin{aligned} dx_i(t) &= \left(x_i(t)\, \partial_t \log \alpha_i(t) + \sum_{k=1}^{N} V_{ik}(t) - \lambda(t)\beta_i(t, \boldsymbol{\theta}) \right) dt + \beta_i(t, \boldsymbol{\theta}) dz \\ \partial_t V_{ij}(t) &= \beta_i(t, \boldsymbol{\theta})\beta_j(t, \boldsymbol{\theta}) + V_{ij}(t)\, \partial_t \log(\alpha_i(t)\alpha_j(t)) \,. \end{aligned} \tag{6}$$

The proposition follows by straightforward calculation.

The assumption, restricting the allowable forward rate volatilities, can be viewed as an approximation sequence with increasing N. A consequence of the Stone-Weierstrass theorem is that a continuous function of M variables can be uniformly approximated on a compact set by a polynomial in the M variables [12]. Therefore a sum of the form of equation (3) can uniformly approximate any continuous forward rate volatility for bounded t, T and $\boldsymbol{\theta}$. Since we are generally interested in models at finite times and maturities and with bounded interest rates, compactness is not a problem. (For example, we can take $t, T < 1000$ years, and $f(t,T) < 10^6$ as model boundaries that should satisfy anyone.)

The only apparently significant limitation entailed by the application of the Stone-Weierstrass theorem to this case is to continuous functions. One might reasonably want to model forward rate volatilities having, say, jump discontinuities. This is clearly possible for some volatilities having the form of equation (3). For example, the forward rate volatility $\sigma_T(t,T,\boldsymbol{\theta}) = \sigma_1$ for $t < t_1$, and $\sigma_T(t,T,\boldsymbol{\theta}) = \sigma_2$ otherwise can be represented by a single term in the sum. On the other hand, a forward rate volatility having a discontinuity as a function of relative time to maturity — *e.g.*, $\sigma_T(t,T,\boldsymbol{\theta}) = \sigma_1$ for $(T-t) < T_1$, and $\sigma_T(t,T,\boldsymbol{\theta}) = \sigma_2$ otherwise — cannot be uniformly approximated by the sum in (3). It is easy to imagine the first kind of discontinuity arising from, say, an expectation of change in interest rate regime due to a discrete event, such as the monetary union contemplated in Europe. The second type of discontinuity seems more difficult to motivate, but could conceivably arise in a market where there were restrictions on the issuance or trading of securities with maturities in some range. Even in this case, though, one could model the discontinuous volatility by a continuous approximation — just not uniformly.

The state variables specified in the definition are of two types. The N variables x_i have stochastic evolution equations; the $N(N+1)/2$ variables V_{ij} are instantaneously non-stochastic. It is possible to eliminate the x_i's in favor of a particular set of N forward rates $f(t, t+T_i)$ for some set of distinct relative maturities T_i, or alternatively, the short rate and $N-1$ derivatives of the forward curve at $T=t$. However, the dynamical equations and representation of the forward curve are much simpler when expressed in terms of x_i.

The stochastic dynamics leads to a diffusion equation for asset prices. Asset prices are given as expectations with respect to the risk-neutral probability measure by

$$P^A(t) = E_t\left[\int_t^\infty ds\, C_\omega^A(s) e^{-\int_t^s du\, r_\omega(u)}\right]$$

where $C_\omega^A(s)$ denotes the cashflow rate of the asset at time s along the path ω and $r_\omega(s)$ is the corresponding short rate. From equation (5) the short rate is given at any time by $r(t) = f(0,t) + \sum_{i=1}^{N} x_i(t)$. The present value P^A of an asset with cashflow rate $C^A(t,x,V)$

(which in general depends on the term structure) then obeys the Feynman-Kac equation

$$\begin{aligned}\Bigg[\partial_t + \frac{1}{2}\sum_{i,j=1}^{N} \beta_i\beta_j\partial^2_{x_i x_j} + \sum_{i=1}^{N}\left(x_i(\partial_t \log\alpha_i(t)) + \sum_{j=1}^{N} V_{ij}\right)\partial_{x_i} + \\ \sum_{i,j=1}^{N}\left(\beta_i\beta_j + V_{ij}(\partial_t \log\alpha_i\alpha_j)\right)\partial_{V_{ij}} - f(0,t) - \sum_{i=1}^{N} x_i\Bigg] P^A(t,x,V) + C^A(t,x,V) = 0\,.\end{aligned} \tag{7}$$

Needless to say, this equation will be intractable for all but the simplest cases. However, by direct integration of equation (5) one obtains the prices of zero coupon bonds, $P(t,T)$, which of course satisfy (7). The result is

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left(-\sum_{j=1}^{N}\frac{A_j(T)-A_j(t)}{\alpha_j(t)}x_j(t) - \sum_{i,j=1}^{N}\frac{(A_i(T)-A_i(t))(A_j(T)-A_j(t))}{2\alpha_i(t)\alpha_j(t)}V_{ij}\right). \tag{8}$$

The foregoing analysis yields a class of models in the HJM framework that are joint Markov processes in a relatively small number of state variables. However the volatility structure of these models is still expressed in terms of functions of fixed absolute maturity (the $\alpha_i(T)$'s).

In order to obtain processes described more naturally in terms of *relative* maturity volatility functions, assume that each of the forward rate volatility functions $\alpha_i(T)\beta_i(t,\boldsymbol{\theta})/\alpha_i(t)$ can be expressed as the product of a function of the time and the forward rate curve and a function of the relative maturity. Setting $\alpha_i(T)/\alpha_i(t) = g_i(T-t)$ immediately implies $g_i(T-t) = e^{-\kappa_i(T-t)}$ for some constants κ_i. The term structure of forward rate volatility is then

$$\sigma_T(t,T,\boldsymbol{\theta}) = \sum_{i=1}^{N} e^{-\kappa_i(T-t)}\beta_i(t,\boldsymbol{\theta})\ . \tag{9}$$

With this choice of forward rate volatility, the stochastic equations (6) simplify slightly: one obtains

$$\begin{aligned} dx_i(t) &= \left(-\kappa_i x_i(t) + \sum_{k=1}^{N} V_{ik}(t) - \lambda(t)\beta_i(t,\boldsymbol{\theta})\right)dt + \beta_j(t,\boldsymbol{\theta})dz \\ \partial_t V_{ij}(t) &= \beta_i(t,\boldsymbol{\theta})\beta_j(t,\boldsymbol{\theta}) - (\kappa_i+\kappa_j)V_{ij}(t)\ . \end{aligned} \tag{10}$$

The forward rate equation (5) now becomes

$$\tilde{f}(t,T) = \sum_{i=1}^{N} e^{-\kappa_i(T-t)}\left[x_i(t) + \sum_{j=1}^{N}\frac{1-e^{-\kappa_j(T-t)}}{\kappa_j}V_{ij}(t)\right]\ . \tag{11}$$

This model will be referred to as the "exponential model."

Application of the Stone-Weierstrass theorem implies that any $\sigma_T(t, T, \boldsymbol{\theta})$ jointly continuous in t, $T - t$ and $\boldsymbol{\theta}$ can be uniformly approximated by the exponential model. As before, forward rate volatilities with discontinuities in t (or $\boldsymbol{\theta}$) can also be uniformly approximated by this model.

It is a simple matter to extend the foregoing analysis to incorporate multiple stochastic factors. The multifactor generalization of the rate dynamics (1) to include M independent stochastic processes is

$$df(t,T) = \sum_{i=1}^{M} \sigma_T^{(i)}(t,T,\boldsymbol{\theta}) \left(\sigma^{(i)}(t,T,\boldsymbol{\theta}) - \lambda^{(i)}(t,\boldsymbol{\theta})\right) dt + \sigma_T^{(i)}(t,T,\boldsymbol{\theta})\, dz^{(i)}(t). \tag{12}$$

The integrated version of this becomes $f(t,T) = f(0,T) + \sum_i \tilde{f}^{(i)}(t,T)$ where

$$\tilde{f}^{(i)}(t,T) = \int_0^t ds\, \sigma_T^{(i)}(s,T,\boldsymbol{\theta}) \left(\sigma^{(i)}(s,T,\boldsymbol{\theta}) - \lambda^{(i)}(s,\boldsymbol{\theta})\right) + \int_0^t dz^{(i)}(s)\, \sigma_T^{(i)}(s,T,\boldsymbol{\theta})\ . \tag{13}$$

The remainder of the derivation proceeds as before for each $\tilde{f}^{(i)}(t,T)$, with the representation of $\sigma_T^{(i)}$ as a sum of terms as in equation (3).

3 Empirical Results

It is reasonable to ask how many terms in the sum (3) are required to accurately represent the observed term structure of volatility. In this section, I show that this is possible with $N = 2$ or 3 for the leading three factors of the US Treasury term structure, corresponding to an approximately parallel shift, a twist and a "flex" of the yield curve.

The factors were obtained by maximum likelihood analysis of the covariances of week to week changes in the US Treasury term structure, described in more detail below. The term structures were obtained by "bootstrapping" the benchmark on-the-run yields as reported in the Federal Reserve H15 series for the period January 1, 1983 through January 5 1995.[3] Each yield curve was specified in terms of bond equivalent yields of the 3 month, 6 month and 1 year bills, and interpolated yields for note/bond maturities of 2, 3, 5, 7, 10 and 30 years. For each yield curve, a zero coupon curve was derived by iteratively solving for a continuous piecewise linear spot rate function, with knots at the given maturity dates, such that a T-maturity bond with coupon equal to the maturity T benchmark yield was priced at par. The resulting spot curve was then sampled at the benchmark maturities. The differences between successive weekly spot curves were computed, after making a small adjustment for

[3] Because the data are weekly averages of daily values, the resulting covariance matrix is lower than the expected covariance matrix for a time series of weekly "snapshots" by a factor 2/3. Accordingly, the reported factors have been rescaled by a factor $\sqrt{3/2}$.

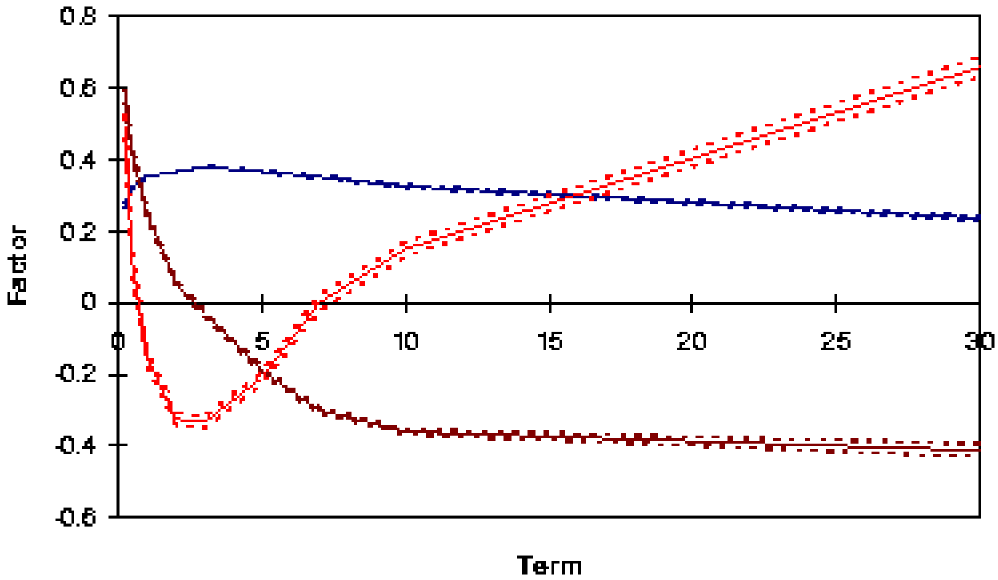

Figure 1 Largest principal components for US Treasury spot curve movements.

the "expected" component of the drift, as described below. Finally, the difference curves for the two weeks surrounding October 19, 1987 were excluded, resulting in a total of 623 difference curves.

The HJM forward rate stochastic equation (1) can be integrated to obtain the equation for spot rates. Taking T now to be the relative maturity (so $y(t,T)$ is the yield of the maturity $T+t$ zero coupon bond at time t), and making a discrete time approximation,

$$\begin{aligned} y(t+\Delta t, T) = & \frac{1}{T}\Bigg\{ \left(T y_T(t,T) + y(t,T) - y(t,t)\right)\Delta t + \\ & \sum_{i=1}^{N}\left(\frac{1}{2}{\sigma^{(i)}}^2(t,T,\boldsymbol{\theta})\Delta t - (\Delta z^{(i)}(t) + \lambda^{(i)}\Delta t)\sigma^{(i)}(t,T,\boldsymbol{\theta})\right)\Bigg\} + O(\Delta t^2)\,. \end{aligned} \tag{14}$$

Fitting the model means estimating the factors $\dfrac{\sigma^{(i)}(t,T,\boldsymbol{\theta})}{T}$ multiplying the $\Delta z^{(i)}$. The average drifts over the finite sample period of the driving random processes z_i are variables that cannot be disentangled from market prices of risk $\lambda^{(i)}$ whose rate dependence is the reciprocal of the corresponding volatility's. Whether the $\lambda^{(i)}$ are estimated based on some assumed form, or arbitrarily set to zero doesn't seem to significantly affect the estimation of the other model parameters. The estimation results reported here assumed $\lambda^{(i)} = 0$. The other systematic drift terms, due to the spot curve slope and to the quadratic variation were estimated and subtracted from the weekly differences. Both of these effects are tiny compared to the typical weekly fluctuations. The spot curve change, adjusted for these drift terms, will be denoted $\Delta y(t,T)$.

The estimation used here involved two steps. First, maximum likelihood estimation was applied to the model

$$\Delta y(t,T) = \frac{1}{T}\sum_{i=1}^{N}\left(\frac{r(t)}{r_0}\right)^{\gamma^{(i)}} \sigma^{(i)}\phi^{(i)}(T)\Delta z^{(i)}(t) \tag{15}$$

to estimate the scale factors $\sigma^{(i)}$, the exponents $\gamma^{(i)}$ and the "principal components" $\phi^{(i)}(T)$ which were constrained to be orthonormal. (Except for the interest rate dependence, this could be done by a simple eigenvector decomposition of the covariance matrix.) r_0 is a scaling constant, set arbitrarily to 0.05. The best fit exponents γ and scale parameters $\sigma^{(i)}$ for the three largest terms are given in table 1. The corresponding first three principal components are graphed in figure 1. In each case, the standard error estimates are based on a permutation bootstrap analysis with 100 resamplings [13]. In figure 1 the standard errors are shown as the dotted curves lying above and below the solid best-fit curves.

This functional form provides a parsimonious model of interest rate dependence of volatility, while permitting an unconstrained (non-parametric) volatility term structure.

Factor	Exponent ($\gamma^{(i)}$)	Scale ($\sigma^{(i)}$)
1	0.44 ± 0.08	2.91 ± 0.09
2	0.44 ± 0.09	1.06 ± 0.04
3	0.13 ± 0.10	0.60 ± 0.03

Table 1 Best fit to exponent and scale of power law dependence of term structure dynamics for the three largest factors. The units of the scale factors $\sigma^{(i)}$ are points per year.

After obtaining the best fit to this model, the functional form

$$\sum_{i=1}^{M(i)} \beta_i^{(i)} \frac{1-e^{-\kappa_i^{(i)}T}}{\kappa_i^{(i)}T} \tag{16}$$

was fit to the "principal components" $\phi^{(i)}(T)$ corresponding to the three largest values of $\sigma^{(i)}$. The number of exponential terms, $M(i)$ was chosen to be 2 for $i = 1, 2$ and 3 for $i = 3$. The fit was done using nonlinear least squares (with the boostrap estimates of the error on the ϕs used to scale the fitting accuracy).

The parameters of this fit are given in table 2, and the fitted curves are shown in figure 2, with the approximations replacing the best fit curves from step 1. The exponential model forms appear to fit quite well, though in every case they fall slightly outside the one-standard deviation error limits at some maturities.

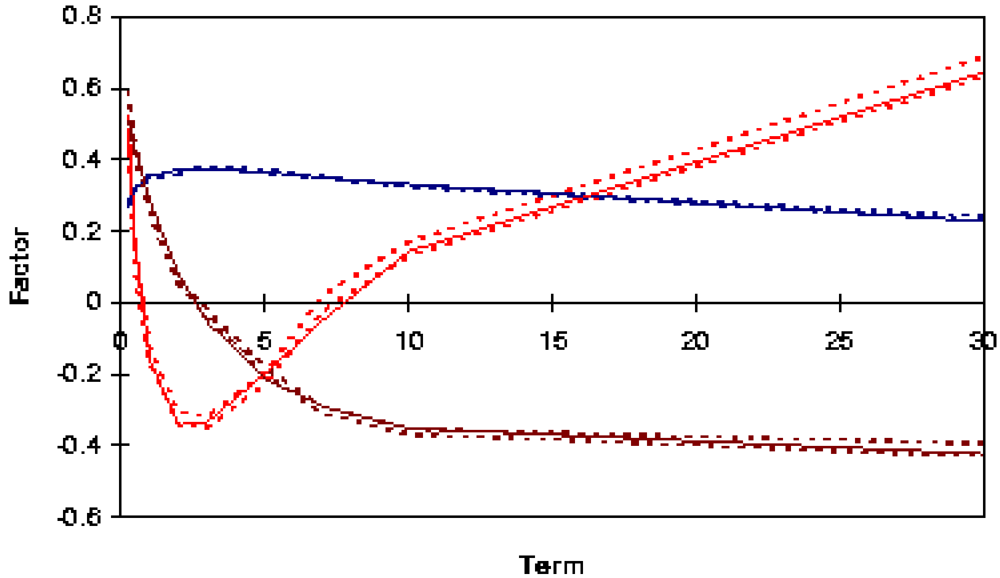

Figure 2 Exponential model fit to the leading principal components of figure 1.

	Term 1		Term 2		Term 3	
Factor	β_1	κ_1	β_2	κ_2	β_3	κ_3
1	0.422	0.046	−0.225	3.76	N/A	N/A
2	−0.528	0.007	1.135	0.715	N/A	N/A
3	1.622	0.020	−2.688	0.148	2.024	2.542

Table 2 Fit of three factor exponential model to empirical factors. The units of κ are inverse years.

A three factor implementation of the exponential model to the HJM model entails keeping track of 19 state variables. The first and second factors require 5 each, the third requires 9.

4 Conclusions

The exponential model provides a useful parameterization for the Heath-Jarrow-Morton approach to term structure dynamics, giving a Markov representation of the dynamics with a modest number of state variables. This framework permits, for example, the representation of a one factor model with an arbitrary short rate volatility function and exponential volatility term structure as a two state-variable Markov process. It also appears feasible to implement a numerical solution to the diffusion equation (7) for this case [14].

Assuming that the parameters of the fit to the largest factor are the most reliable, one can conclude that the long run behavior of interest rates is governed by a mean reversion strength of order 5% per year, or a timescale of about 20 years. This mean reversion is substantially smaller than some previously reported values, based for example on fits to yield curve shapes implied by the one factor Cox-Ingersoll-Ross model. An exception is the recent result of Chen and Scott based on fitting to a multi-factor CIR model: in a three factor fit, they find that the smallest mean reversion parameter is of order 0.5% per year, though with a fairly large error [15]. This corresponds more closely to the smallest values seen for the subleading factors. However, the data used here provide no information for maturities between 10 and 30 years (or beyond 30 years), so the best fit results are probably not terribly sensitive to the exact value of mean reversion timescales longer than about 20 years.

It is reassuring that the smallest value of $\kappa_i^{(i)}$ is positive for all three factors. A positive value implies that the volatility of very long forward rates approaches zero, and that the probability distribution of the future short rate has finite variance. If the smallest κ were zero or negative, forward rates would diverge and the short rate would grow arbitrarily large, since the corresponding terms in equation (10) then have exponentially growing behavior. (This is true even if the short rate volatility vanishes when the short rate exceeds some level.)

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