A STOCHASTIC VOLATILITY FORWARD LIBOR MODEL WITH A TERM STRUCTURE OF VOLATILITY SMILES

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ABSTRACT. Volatility smiles of European swaptions of various expiries and maturities typically have different slopes. This important feature of interest rate markets has not been incorporated in any of the practical interest rate models available to date. In this paper, we build a model that treats the swaption skew matrix as a market input and is calibrated to it. The model is constructed as an extension of a *Stochastic Volatility Forward Libor* model, with local volatility functions imposed upon forward Libor rates being time-dependent and Libor-rate specific. The focus of the paper is on deriving efficient European swaption approximation formulas that allow calibration of the model to all European swaptions across all expiries, maturities and strikes. The main conceptual contribution of the paper is its focus on recovering all available market volatility skew information across a full swaption grid within a consistent model. The model we develop has a potential to change the way skew calibration is approached, in the same way the introduction of the log-normal forward Libor model had changed the way volatility calibration is approached. The main technical contribution of the paper is a formula for the "effective" skew in a stochastic volatility model, a formula that relates a total amount of skew generated by the model over a given time period to the time-dependent slope of the instantaneous local volatility function. A new "effective" volatility approximation for stochastic volatility models with time-dependent volatility functions is also derived. The formulas we obtain are simple and intuitive; their applicability goes beyond interest rate modeling.

1. INTRODUCTION

The basic log-normal *Forward Libor* (also known as *Libor Market*, or *BGM*) model has proved to be an essential tool for pricing and risk-managing interest rate derivatives. It was the first model that was easy to calibrate to the grid of at-the-money swaptions volatilities across all swaption expiries and underlying swaps' maturities. This calibration ability is recognized as perhaps the most important requirement for a model to be successfully applied to such complicated interest rate derivatives as flexi-caps, chooser caps, Bermuda swaptions, and various callable Libor exotics (see e.g. [Pit03]).

While the log-normal forward Libor model has established a standard for incorporating all available at-the-money volatility information, it was less successful in recovering other essential characteristics of interest rate markets, particularly the volatility smile. In a log-normal forward Libor model, swaptions of the same expiry/maturity but of different strikes have the same implied Black volatilities, a feature of the model that is inconsistent with

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the market. Various extensions of the log-normal forward Libor model have been proposed. These extensions have been designed to incorporate the volatility smile effect in one form or another. Local volatility, jump-diffusion, and stochastic volatility extensions of a forward Libor model have all been introduced. Arguably, the most successful of these efforts has been the *Stochastic Volatility forward Libor* model, a model in which each forward Libor rate follows a process with a deterministic local volatility function, with a stochastic volatility perturbation uniformly applied to all Libor rates. The model, which we call the *FL-SV* model from now on, is presented in [ABR01] and [AA02].

In the FL-SV model, a time-independent local volatility function is applied to all forward Libor rates. It determines the slope of the volatility smile at-the-money (a parameter called a *skew* from now on). The stochastic volatility component, also the same for all forward Libor rates, is responsible for the curvature of the volatility smile. The higher the volatility of the stochastic variance process, the higher the curvature of the volatility smile.

Despite being a significant improvement over the log-normal FL model, the FL-SV model still suffers from certain limitations. The same local volatility function and the same stochastic volatility scaling are applied to each forward Libor rate. It results in the same general shape of the volatility smile (the same slope/curvature) being implied for swaptions of different expiries/maturities. The skews of different swaptions are, however, substantially different. Thus, the FL-SV model still falls short of matching the full term structure of volatility smiles across the whole swaption grid.

The importance of recovering all volatility smiles for all swaptions cannot be overemphasized. Structured interest rate derivatives, derivatives that are a natural application for FL models because of their complicated volatility dependence, are rarely, if ever, depend on at-the-money volatilities. They are typically structured with strikes that are far away from money, and typically depend on swaption volatilities with different expiries and maturities. To accurately value such an instrument, a model should be able to match all market volatilities for all strikes for many, if not all, swaption expiry/maturity combinations. As market sophistication grows and bid/ask spreads on structured derivatives tighten, such requirements become more and more imperative.

A model that can calibrate to market-implied skews for the whole swaption grid is needed. This paper introduces such a model, a model that aims to do to the skew structure modeling what a log-normal forward Libor model did for the volatility structure modeling. We build a model that takes a swaption grid of volatility skews as an external input. By allowing the model parameters responsible for inducing skews on swaptions to be time-dependent and Libor-rate specific, we are able to create a model that can recover all available market volatility smile information.

In the model developed in this paper, we keep the stochastic volatility process the same for all forward Libor rates, just like in the "classic" FL-SV model. Only the skews (the slopes of local volatility functions) are allowed to vary across different forward Libor rates and times. While extending the model even further with swaption-specific volatility smile curvatures is certainly possible, our testing indicates that this is unnecessary. It appears that curvatures of volatility smiles for different swaptions can be well-captured by a single stochastic volatility driver (this is in no small part helped by using a stochastic variance process that incorporates mean reversion, so the connection between curvatures of volatility smiles of different expiries can be controlled in the model). We construct a model that captures the effect of changing skews across the swaption grid, a model we call the *FL-TSS* model, for *Term Structure Of Skews*. Formal definition of the model is given by the equations (4.1), (4.3) below.

Valuation in any FL model is most typically done using a Monte-Carlo simulation. Extending the valuation algorithm to handle time-dependent and forward Libor rate-specific local volatility functions is a trivial exercise. Calibration, however, is a different matter. Obtaining accurate and fast approximations to the prices of European swaptions and caplets is a much more complicated problem. Such approximations are critical for calibration. This is the problem we solve in this paper. The first technical result (Theorem 5.1) shows that in the FL-TSS model, swap rates approximately satisfy SDEs of the same form as forward Libor rates. (Similar types of results are available for almost all types of forward Libor models, so little new ground is broken here.) Thus, only a single type of an SDE needs to be considered when deriving approximations to values of European caps and swaptions.

The main technical contributions of this paper are Theorems 6.1 and 6.4, with practical applications given by Corollary 6.3. These results relate time-dependent model volatilities and skews (for forward Libor or swap rates) to market-implied term ones. The result on skews (Theorem 6.1, Proposition 6.2, Corollary 6.3) is a particularly new and important one. It relates a time-dependent slope of a local-volatility function in a stochastic volatility model to the total amount of skew the model generates. The result is of an independent interest and can be applied in a broader context of equity, FX or commodity modeling. The averaging result on time-dependent volatilities in a stochastic volatility model (Theorem 6.4) is also new, even though the problem itself is not (see [ABR01], [SKSE02]). The procedure we use is more transparent than most others. We also believe that the approximation we derive perform better in the current high volatility-of-variance market environment than the previously available ones.

The averaging results obtained have a particularly suitable form for calibration applications, i.e. for the process of finding model volatilities and skews to match market-implied ones. The formulas relate market and model skews and volatilities *directly*, without the need to use formulas for European option valuation (either direct or inverse). This speeds up calibration considerably. We further simplify and speed up the problem of calibrating to the full set of market volatility information by treating skew calibration and volatility calibration separately. This separation also allows the model to escape the problem of overfitting, a problem the model might be perceived to be susceptible to because of the significantly increased number of parameters. Another potential problem, that of a potentially non-stationary evolution of the term structure of skews implied by the model, is addressed by adding strong time-homogeneity objectives to the skew calibration.

This paper is organized as follows. Section 2 is devoted to the review of relevant literature. A simple stochastic volatility model, a model used to concisely encode volatility smile information for each swaption, is reviewed in Section 3. The FL-TSS model is presented in Section 4. Approximate dynamics of swap rates in the FL-TSS model are derived in Section 5. Main technical results on the averaging of model parameters in a stochastic volatility model are presented in Section 6. Calibration is discussed in Section 7. Section 8 briefly discusses forward skew structures generated by the model. Test results are presented in Section 9. We conclude in Section 10. Proofs of technical results are presented the Appendix.

2. Literature review

First introduced in [BGM96] and [Jam97], forward Libor models are in the mainstream of interest rate modeling. For general information on the model, textbooks such as [BM01], [Reb02], [MR97] provide a good starting point.

Various extensions of forward Libor models that attempt to incorporate volatility smiles of interest rates have been proposed. Local volatility type extensions were pioneered in [AA00]. A stochastic volatility extension is proposed in [ABR01], and is further extended in [AA02]. A different approach to stochastic volatility forward Libor models is described in [Reb02]. Jump-diffusion forward Libor models are treated in [GM01], [GK99]. In [Sin02], a combination stochastic volatility/jump diffusion forward Libor model is advocated. While some approaches are more successful in fitting market-observed volatility smiles for a subset of swaptions that others, all previous authors share one thing in common – they do not attempt to calibrate volatility smiles for the whole swaption grid. With overly-parsimonious specifications of volatility-smile-generating mechanisms (be that stochastic volatility, jumps, or a combination of both) such goal would be unattainable anyway. In this paper we explicitly specify the calibration to the full swaption skew grid as a goal of modeling, and allow for a flexible (although tightly-controlled) specification of skew parameters to achieve the goal.

We present approximation formulas for European swaptions/caps that are based on parameter averaging results in stochastic volatility models with time-dependent parameters. Related and alternative approaches can be found in [ABR01], [SKSE02], [Zho03], and [AA02]. An excellent book [Lew00] must also be mentioned.

The calibration of the FL-TSS model consists of two sub-problems, a skew calibration and a volatility calibration. The problem of volatility calibration is virtually the same for our model as for other types of forward Libor models. Common techniques and approaches are available in the textbooks [BM01], [Reb02], as well as in the papers [Ped98], [SC00], [LM02], [Ale02], [Gat02], [Wu03], to name a few. Skew calibration discussed in this paper is similar in spirit to the volatility calibration, albeit, arguably, simpler.

Introduction of a non-time-homogeneous skew structure in a model raises questions of the model dynamics and its impact on pricing and hedging of interest rate derivatives. The issues that arise are similar to the issues that a non-time-homogeneous volatility structure would pose. The latter are well-covered in, for example, [Reb03].

3. Simple model for European swaptions

Let

$$\begin{aligned} 0 &= T_0 < T_1 < \cdots < T_N, \\ \tau_n &= T_n - T_{n-1}, \end{aligned}$$

be a tenor structure, i.e. e a sequence of approximately equally spaced dates. Let $S_{n,m}(t)$ be a swap rate for a swap with a first fixing date T_n and the last payment date T_m,

$$S_{n,m}(t) = \frac{P(t,T_n) - P(t,T_m)}{\sum_{i=n+1}^{m} \tau_i P(t,T_i)}.$$

By switching to an appropriate measure, a European swaption can be regarded as a European option on a relevant swap rate. For a European swaption on the swap rate $S_{n,m}$, its expiry is defined as T_n, and maturity as $T_m - T_n$. European swaptions of different strikes but the same expiry/maturity depend on the term distribution of the relevant swap rate, i.e. a distribution between the observation date and the rate fixing date. A model that specifies a term distribution of this swap rate only is termed a "simple model" (This is in contrast to a term structure model of interest rates that describes the evolution of the whole interest rate curve). An example of such a model is the Black model for European swaptions, where a lognormal distribution is prescribed for the rate.

It is by now pretty common to use some flavor of a stochastic volatility model to price European swaptions. For concreteness, we will use the following one (see [AA02]). A term distribution of a swap rate S is described by 3 parameters: the SV volatility λ, the skew parameter b and the volatility of variance parameter η. In terms of this parameters, a swap rate $X(t)$ is assumed to follow the following distribution (in the appropriate swap measure),

$$\begin{aligned} dz(t) &= \theta(z_0 - z(t))\, dt + \eta\sqrt{z(t)}\, dV(t), && (3.1)\\ dS(t) &= \lambda(bS(t) + (1-b)S(0))\sqrt{z(t)}\, dW(t), && (3.2)\\ z(0) &= z_0, \\ \langle dV, dW\rangle &= 0. \end{aligned}$$

Note that even though the equation describes the dynamics of a swap rate through time, only its terminal distribution is relevant for this application of the model.

The model is calibrated separately for each swap rate $S_{n,m}$, resulting in a grid of triples $\left\{(\lambda_{n,m}, b_{n.m}, \eta_{n,m})\right\}_{n,m=1}^{N}$ (only grid points with $n+m<N$ hold meaningful information). This grid (thereupon referred to as "the swaption grid") contains all available market information for all expiries/maturities/strikes. Note that this notion extends the concept of the swaption grid of ATM volatilities (an object extensively discussed in the context of log-normal forward Libor models) with information on each individual swaptions' skew.

Note that the SDE depends on another parameter, θ, the mean reversion of variance. The meaning of this parameter will be made clear later. For now we note that we regard it as a global parameter, the same for the whole swaption grid.

Let us briefly comment on the effects of different inputs on the volatility smile produced by the model. The parameter λ is responsible for the overall level of the volatility smile. It is related to, although not exactly equal to, an at-the-money Black swaption volatility. It is expressed in the same units as the ATM Black volatility (note that the two coincide when $b=1$, $\eta=0$). The parameter b is responsible for the slope of the volatility smile at-the-money. The parameter η controls the curvature of the smile.

An important advantage of the model (3.1)-(3.2) over other SV specifications in its tractability. swaptions can be valued almost instantaneously using transform-based methods. We refer to [Lew00] and [AA02] for details.

The model (3.1)-(3.2) is typically used for $b\in[0,1]$ only. The SDEs however are valid even for b outside of $[0,1]$. It is convenient to apply the model for b in the interval that is somewhat larger than $[0,1]$. We are particularly interested in being able to use negative values of b. The point $\hat{S}$ such that

$$b\hat{S} + (1-b)S(0) = 0$$

is a "graveyard" state; once $S(t)$ reaches it , ti stays at $\hat{S}$ point forever. This behavior is somewhat unrealistic; the advantages of being able to use negative values of b outweighed this concern.

The parameter θ, the mean reversion of variance, influences the speed with which the volatility smile flattens out as the swaption expiry increases. It appears that, for the appropriate choice of θ, all market smiles can be well matched with the same η used across the whole swaption grid. As of the time of writing (October 2003), such a value of θ appears to be in the $10\% - 15\%$ range. Our test results use $\eta_{n,m} \equiv \eta = 130\%$ and $\theta = 15\%$, see Section 9.

4. A model consistent with the swaption grid

4.1. **On the need for a consistent model.** To properly price and risk-manage exotic interest rate derivatives, a term structure model that incorporates all available swaption term distribution information for all swaptions is needed. For the problem of recovering at-the-money swaption volatilities for all swaptions, the standard log-normal forward Libor model is probably the best choice. Structured interest rate derivatives rarely, if ever, depend on at-the-money volatilities only. For that reason, using a log-normal specification is questionable, as the model does not incorporate volatility smile information. Probably the best attempt so far to include volatility smile information in a forward Libor model has appeared in [ABR01], [AA02]. Let us quickly review it here; we will use it as a base to build on.

4.2. **A Stochastic Volatility Forward Libor model.** We define by $P(t,T)$ the value, at time t, of a zero coupon bond paying \$1 at time T. We define spanning Libor rates

$$\begin{aligned} L_n(t) &\triangleq L(t,T_n,T_{n+1}) = \frac{P(t,T_n) - P(t,T_{n+1})}{\tau_{n+1} P(t,T_{n+1})}, \\ n &= 0,\dots,N-1. \end{aligned}$$

A stochastic variance process $z(t)$ is defined by the SDE (same as (3.1))

$$\begin{aligned} dz(t) &= \theta(z_0 - z(t))\,dt + \eta\sqrt{z(t)}\,dV(t), \\ z(0) &= z_0. \end{aligned} \tag{4.1}$$

Let

$$dW(t) = (dW_1(t),\dots,dW_K(t))$$

be a K-dimensional Brownian motion (under the measure $\mathbf{P}$) independent of dV. The Stochastic Volatility forward Libor (FL-SV) model is defined by the following dynamics imposed on the spanning forward Libor rates,

$$dL_n(t) = (\beta L_n(t) + (1-\beta) L_n(0))\sqrt{z(t)} \sum_{k=1}^{K} \sigma_k(t;n) \left(\sqrt{z(t)} \mu_k(t;n) + dW_k(t) \right), \quad n = 1,\dots,N-1. \tag{4.2}$$

Here

$$\left\{ (\sigma_k(t;n))_{k=1}^{K}, \quad t \ge 0 \right\}_{n=1}^{N-1}$$

is a collection of K-dimensional instantaneous caplet volatility functions, and $(\mu_k(t;n))_{k=1}^{K}$ are K-dimensional numeraire-specific drift that ensures lack of arbitrage within the model. Expressions for μ for different choices of the numeraire can be found in [ABR01].

In the model (4.1), (4.2), each swap rate $S(t) = S_{n,m}(t)$ can be shown to approximately satisfy the following SDE,

$$dS(t) = \sigma(t)(\beta S(t) + (1-\beta) S(0)) \sqrt{z(t)}\, dU(t)$$

for some Brownian motion $U(t)$ and some time-dependent volatility function $\sigma(t)$. Superficially it looks exactly the same as (3.2), but there is a critical difference. In the model (4.1)-(4.2), the parameters β and η are *universal* for all Libor rates L_n (and, by extension, for all swap rates $S_{n,m}$). In the simple model (3.1)-(3.2), a different parameter $b_{n,m}$ and $\eta_{n,m}$ is applied to each swap rate. While the assumption that all volatility of variance parameters $\eta_{n,m}$ are the same appears to be consistent with the market (for the appropriate choice of θ, the mean reversion of variance), it is not the case for the skew parameters $b_{n,m}$. This means that the simple stochastic volatility model (4.1)-(4.2) cannot reproduce all swaption volatility smiles for all combinations of expiry/maturity.

4.3. **A model with a term structure of skew.** As clear from the analysis in the previous section, the specification (4.2) is too rigid to account for variability in swaption skews across expiries/maturities. To remedy that we propose to extend the model with Libor-rate specific and time-dependent skews. We assume that a collection of time-dependent functions (*instantaneous forward Libor skews*)

$$\{\beta(t;n), \quad t \geq 0\}_{n=1}^{N-1},$$

is specified, and define the dynamics of spanning forward Libor rates by

$$\begin{aligned} dL_n(t) &= (\beta(t;n) L_n(t) + (1-\beta(t;n)) L_n(0)) \sqrt{z(t)} \\ &\quad \times \sum_{k=1}^{K} \sigma_k(t;n) \left(\sqrt{z(t)} \mu_k(t;n) + dW_k(t) \right), \\ n &= 1, \ldots, N-1. \end{aligned} \tag{4.3}$$

In particular, under the T_{n+1}-forward measure (the measure for which $P(\cdot, T_{n+1})$ is a numeraire), the rate L_n is a martingale and the equations for the rate $L_n(\cdot)$ can be written as

$$\begin{aligned} dL_n(t) &= (\beta(t;n) L_n(t) + (1-\beta(t;n)) L_n(0)) \sqrt{z(t)} \\ &\quad \times \sum_{k=1}^{K} \sigma_k(t;n)\, dW_k^{n+1}(t), \\ n &= 1, \ldots, N-1. \end{aligned} \tag{4.4}$$

We call the resulting model the *Term-Structure-of-Skew forward Libor* model (*FL-TSS* model).

Incorporating time-dependent skews into a Monte-Carlo valuation is a trivial exercise. This is not so for calibration. A successful calibration procedure requires availability of fast European option valuation methods. For stochastic volatility models with time-dependent local volatility functions, none have been developed so far. The main technical contribution of this paper is the derivation of fast and accurate approximations for European option values in this case.

In the context of forward Libor models, the first step is deriving an approximate SDE for a swap rate.

5. Swap rate approximation

In this section we derive approximate dynamics of swap rates in the FL-TSS model, the dynamics that will ultimately allow us to derive approximate pricing formulas for European swaptions.

Let $S_{n,m}(t)$ be a swap rate for a swap with a first fixing date T_n and the last payment date T_m,

$$S_{n,m}(t) = \frac{P(t, T_n) - P(t, T_m)}{\sum_{i=n+1}^{m} \tau_i P(t, T_i)}.$$

Under the swap measure $\mathbf{Q}^{n,m}$, the measure for which $\sum_{i=n+1}^{m} \tau_i P(t, T_i)$ is a numeraire, the swap rate $S_{n,m}(t)$ is a martingale. We denote the Brownian motions under this measure by $dW^{n,m}$.

Theorem 5.1. *In the TSS-FL model given by (4.1), (4.3), the approximate dynamics of the swap rate $S_{n,m}(t)$ under the swap measure $\mathbf{Q}^{n,m}$ are given by the SDE*

$$\begin{aligned} dS_{n,m}(t) &= (\beta(t; n, m) S_{n,m}(t) + (1 - \beta(t; n, m)) S_{n,m}(0)) \sqrt{z(t)} \\ &\quad \times \sum_{k=1}^{K} \sigma_k(t; n, m)\, dW_k^{n,m}(t), \end{aligned} \tag{5.1}$$

where

$$\sigma_k(t; n, m) = \sum_{i=n}^{m-1} q_i(n, m) \sigma_k(t; i), \tag{5.2}$$

$$\beta(t; n, m) = \sum_{i=n}^{m-1} p_i(n, m) \beta(t; i), \tag{5.3}$$

and

$$q_i(n, m) = \frac{L_i(0)}{S_{n,m}(0)} \frac{\partial S_{n,m}(0)}{\partial L_i(0)}, \tag{5.4}$$

$$p_i(n, m) = \frac{\sum_{k=1}^{K} \sigma_k(t; i) \sigma_k(t; n, m)}{\sum_k \sigma_k^2(t; n, m)}, \tag{5.5}$$

$$i = n, \ldots, m - 1.$$

The proof is given in Appendix A.

We note that the approximate SDE (5.1) for a swap rate $S_{n,m}$ is of the same form as the exact SDE (4.4) for a Libor rate L_n.

6. Relating time-dependent volatility and skew to constant ones

6.1. **Pricing European options via parameter averaging.** As derived in the previous section, each swap rate and each forward Libor rate in the FL-TSS model (each under its appropriate measure) follows an SDE of the form

$$dS(t) = \sigma(t) (\beta(t) S(t) + (1 - \beta(t)) S(0)) \sqrt{z(t)}\, dU(t), \tag{6.1}$$

where $U(t)$ is a Brownian motion, $\sigma(t)$ is a time-dependent instantaneous volatility of the rate S, and $\beta(t)$ is a time-dependent instantaneous skew of the rate S (both $\sigma(\cdot)$ and $\beta(\cdot)$ are related to model parameters via (5.2), (5.3)).

We denote the fixing date for rate $S(\cdot)$ (the same as the expiry date for the corresponding European swaption) by T.

Let us rewrite (3.2) using slightly different notations for the rate to distinguish it from (6.1). The simple model specifies that

$$d\bar{S}(t) = \lambda \left(b\bar{S}(t) + (1-b)\bar{S}(0)\right) \sqrt{z(t)}\, dU(t). \tag{6.2}$$

As explained in Section 3, the market-implied distribution of $\bar{S}(T)$ is encoded in the implied parameters λ, b and η, the time-independent parameters in the SDE (3.2). As explained above, the success of calibration hinges on the availability of a fast European option pricing formulas (or approximations) in the model (6.1). What would be even more advantageous, however, is to have formulas that relate time-dependent parameters $\sigma(t)$, $\beta(t)$ in (6.1) to the "effective", constant parameters λ and b in (6.2). If we could discover such relationships, then the calibration procedure can be set up in terms of the inputs $\{\sigma_k(t;n), \quad t \geq 0\}_{n,k}$ and $\{\beta(t;n), \quad t \geq 0\}_n$ and outputs $\left\{\left(\lambda_{n,m}, b_{n.m}, \eta_{n,m}\right)\right\}_{n,m}$ directly, without the need for costly direct/inverse option valuations during calibration's non-linear search.

The problem of finding an "effective" skew in a stochastic volatility model is new; we approach it first. The problem of finding an "effective" volatility has been studied in various contexts before (see [SKSE02], [Zho03], [ABR01]), and several solutions exist. Later in the paper, we give our own solution to the problem which, we believe, is more effective than the alternatives, and is easier to understand too.

6.2. **"Effective" skew.** The next theorem, the main technical contribution of this paper, derives an effective skew, a time-homogeneous local volatility function that approximates a time-dependent one. The idea of the result is to find a linear combination of time-dependent local volatility functions that provides the best approximation. We formulate it as a limit result, with practical applications derived later in Corollary 6.3.

Theorem 6.1. *Let us fix a time horizon $T > 0$. Let $f(t,x)$ be a local volatility function,*

$$f(\cdot,\cdot) \in C^1\left([0,T] \times \mathbb{R}, \mathbb{R}^+\right),$$

satisfying the usual growth requirements. Let $\sigma(t)$, $t \in [0,T]$ be a function of time only. Fix $x_0 \in \mathbb{R}$. For any $\varepsilon > 0$, define a re-scaled local volatility function

$$f_\varepsilon(t,x) = f\left(t\varepsilon^2, x_0 + (x - x_0)\varepsilon\right). \tag{6.3}$$

Without loss of generality we can assume that

$$f(t,x_0) \equiv 1, \quad t \in [0,T],$$

which implies

$$f_\varepsilon(t,x_0) \equiv 1, \quad t \in [0,T]. \tag{6.4}$$

Let $w(t)$, $t \in [0,T]$, be a weight function such that

$$\int_0^T w(t)\, dt = 1, \tag{6.5}$$

and let us define an averaged local volatility function

$$\bar{f}_{\varepsilon}^{2}(x)=\int_{0}^{T} f_{\varepsilon}^{2}(t,x)\,w(t)\,dt. \tag{6.6}$$

Define two families of diffusions indexed by ε,

$$\begin{aligned} dX_{\varepsilon}(t) &= f_{\varepsilon}(t,X_{\varepsilon}(t))\sqrt{z(t)}\sigma(t)\,dW(t),\\ dY_{\varepsilon}(t) &= \bar{f}_{\varepsilon}(Y_{\varepsilon}(t))\sqrt{z(t)}\,dW(t),\\ X_{\varepsilon}(0) &= x_0,\\ Y_{\varepsilon}(0) &= x_0, \end{aligned}$$

for $t\in[0,T]$, *where* $z(t)$ *is defined by (4.1). If the weights* $w(t)$ *are given by the expression*

$$\begin{aligned} w(t) &= \frac{v^{2}(t)\,\sigma^{2}(t)}{\int_{0}^{T}v^{2}(t)\,\sigma^{2}(t)\,dt}, \\ v^{2}(t) &= \mathbf{E}\left(z(t)\,(X_0(t)-x_0)^2\right), \end{aligned} \tag{6.7}$$

then

$$\int_{-\infty}^{\infty}\left(\mathbf{E}\,(Y_{\varepsilon}(T)-K)^{+}-\mathbf{E}\,(X_{\varepsilon}(T)-K)^{+}\right)dK=o\left(\varepsilon^{2}\right),\quad \varepsilon\to 0. \tag{6.8}$$

The theorem is proved in Appendix B.

Before applying the theorem to the problem of deriving an "effective" skew formula, let us comment on its substance. The theorem seeks to replace a time-dependent local volatility function $f_{\varepsilon}(t,x)$ with a time-homogeneous one $\bar{f}_{\varepsilon}(x)$, which is a weighted average (in t) of the original one, see (6.6). The weights are chosen to satisfy the condition (6.8). The expression on the left is an integral of the differences of European option prices obtained in the pre-averaged and averaged models. The criteria is natural because we are interested in approximating prices of European options for all strikes when replacing f with $\bar{f}$. We note that for any choice of $w(t)$, the expression on the left-hand side of (6.8) is of the order $O(\varepsilon^2)$. The only choice that gives a better approximation is (6.7).

Differentiating (6.6) with respect to x, evaluating the resulting equality at $x=x_0$, using (6.4), and setting $\varepsilon=1$, we obtain the following relation between the slopes of $\bar{f}$ and f,

$$\frac{\partial\bar{f}(x_0)}{\partial x}=\int_{0}^{T}\frac{\partial f(t,x_0)}{\partial x}w(t)\,dt. \tag{6.9}$$

Interestingly, a function that satisfies (6.9) solves a differently formulated (albeit related) "effective skew" problem[1]. We present it for completeness.

Proposition 6.2. *Let us fix a time horizon* $T>0$. *Let* $g(t,x)$ *be a time-dependent, and* $\bar{g}(x)$ *a time-independent local volatility functions,*

$$\begin{aligned} g(\cdot,\cdot) &\in C^{1}\left([0,T]\times\mathbb{R},\mathbb{R}^{+}\right),\\ \bar{g}(\cdot) &\in C^{1}\left(\mathbb{R},\mathbb{R}^{+}\right), \end{aligned}$$

[1]This observation, and the idea of the proof, was communicated to us by Leif Andersen.

satisfying the usual growth requirements. Let $\sigma(t)$, $t \in [0,T]$ be a function of time only. Fix $x_0 \in \mathbb{R}$. For any $\varepsilon > 0$, define re-scaled local volatility functions (note that time t is not rescaled, unlike (6.3)),

$$\begin{aligned} g_\varepsilon(t,x) &= g(t, x_0 + (x - x_0)\varepsilon), \\ \bar{g}_\varepsilon(x) &= \bar{g}(x_0 + (x - x_0)\varepsilon), \end{aligned}$$

assuming without loss of generality that

$$\begin{aligned} g(t,x_0) &\equiv 1, \quad t \in [0,T], \\ \bar{g}(x_0) &= 1. \end{aligned}$$

Define two families of diffusions indexed by ε,

$$\begin{aligned} &dX_\varepsilon(t) = g_\varepsilon(t, X_\varepsilon(t)) \sqrt{z(t)}\sigma(t)\, dW(t), \\ &dY_\varepsilon(t) = \bar{g}_\varepsilon(Y_\varepsilon(t)) \sqrt{z(t)}\, dW(t), \\ &X_\varepsilon(0) = x_0, \\ &Y_\varepsilon(0) = x_0, \end{aligned}$$

for $t \in [0,T]$, where $z(t)$ is defined by (4.1).

$$q(\varepsilon) = \mathbf{E}(X_\varepsilon(T) - Y_\varepsilon(T))^2. \tag{6.10}$$

It can be shown that

$$q(0) = q'(0) = 0.$$

Under the conditions above, any function $\bar{g}$ that minimizes $q''(0)$,

$$q''(0) \to \min, \tag{6.11}$$

satisfies the condition (compare to (6.9))

$$\frac{\partial \bar{g}(x_0)}{\partial x} = \int_0^T \frac{\partial g(t,x_0)}{\partial x} w(t)\, dt. \tag{6.12}$$

The condition (6.11) is related to (6.8) and provides an alternative meaning to the notion of approximating a time-inhomogeneous diffusion with a time-homogeneous one. The proof of the proposition is given in the Appendix.

Applying Theorem 6.1 to the problem of finding an "effective" skew for the equation (6.1) in the FL-TSS model, we obtain the following corollary.

Corollary 6.3. *The effective skew b for the equation*

$$dS(t) = \sigma(t)(\beta(t) S(t) + (1 - \beta(t)) S(0)) \sqrt{z(t)}\, dU(t)$$

over a time horizon $[0,T]$ is given by

$$b = \int_0^T \beta(t) w(t)\, dt, \tag{6.13}$$

where the weights $w(\cdot)$ are given by

$$w(t) = \frac{v^2(t)\,\sigma^2(t)}{\int_0^T v^2(t)\,\sigma^2(t)\,dt}, \tag{6.14}$$

$$v^2(t) = z_0^2 \int_0^t \sigma^2(s)\,ds + z_0\eta^2 e^{-\theta t} \int_0^t \sigma^2(s)\,\frac{e^{\theta s} - e^{-\theta s}}{2\theta}\,ds.$$

The proof of Corollary is given in Appendix D. The corollary solves the problem of finding the "effective" skew for forward Libor/swap rate evolutions in the FL-TSS model, allowing super-fast calibration procedures to be developed (more on that in Section 7).

Example 6.1. *Recall that $\beta(t)$ has the meaning of being a slope of the local volatility function at time t. The "effective" skew b is related to the slope of the implied volatility smile. So the slope of the term smile is equal to an average of instantaneous slopes, with weights $w(t)$. Under the assumption of no stochastic volatility, $\eta = 0$, using scaling $z_0 = 1$, and constant swaption volatility $\sigma(t) \equiv \sigma$, we obtain particularly simple formulas for the "effective" skew,*

$$v^2(t) = \sigma^2 t,$$

$$w(t) = \frac{t}{\int_0^T t\,dt} = \frac{t}{T^2/2},$$

so that

$$b = \frac{1}{T^2/2} \int_0^T t\beta(t)\,dt.$$

We test these approximations, both in the context of an FL-TSS model, and on a stand-alone basis, in Section 9. Test results indicate that the effective skew approximation is very accurate for realistic time-dependent volatility skew functions.

6.3. **"Effective" volatility.** The result of Corollary 6.3 shows how to approximate the SDE (6.1) with an SDE with a constant skew b (see (6.13)),

$$\begin{aligned} dS(t) &= \sigma(t)\,(bS(t) + (1-b)\,S(0))\,\sqrt{z(t)}\,dU(t), \\ S(0) &= S_0. \end{aligned} \tag{6.15}$$

The volatility function in (6.15) is still time-dependent, and the SDE has not been completely reduced to the form of (6.2). This step is taken next. The formal result on "effective" volatility is stated as Theorem 6.4 below.

The method we develop is based on representing a European option price as an integral of a known function against the distribution of the term stochastic variance. In particular, for an at-the-money ($K = S_0$) option,

$$\mathbf{E}\,(S(T) - S_0)^+ = \mathbf{E}\left(\mathbf{E}\left(\left.(S(T) - S_0)^+\right|\, z(\cdot)\right)\right). \tag{6.16}$$

Because the Brownian motion that drives $z(t)$ is independent of the Brownian motion that drives $S(t)$, the distribution of $S(T)$ (in the model (6.15)) conditioned on a particular path $\{z(t)\}_{t=0}^T$ is a shifted lognormal. Hence, the inside condition expectation in (6.16) can be evaluated easily to yield

$$\begin{aligned} \mathbf{E}\,(S(T) - S_0)^+ &= \mathbf{E}g\,(Z(T)), \\ Z(T) &= \int_0^T \sigma^2(t)\,z(t)\,dt, \end{aligned}$$

where g is a known function,

$$
\begin{aligned}
g(x) &= \frac{S_0}{b}\left(2\Phi\left(b\sqrt{x}/2\right)-1\right), \\
\Phi(y) &= \mathbf{P}(\xi<y), \\
\xi &\sim \mathcal{N}(0,1).
\end{aligned}
\tag{6.17}
$$

The problem of finding an "effective" variance can then be represented as finding such λ that

$$
\mathbf{E}g\left(\int_0^T \sigma^2(t)\,z(t)\,dt\right)=\mathbf{E}g\left(\lambda^2\int_0^T z(t)\,dt\right).
\tag{6.18}
$$

Neither of the expected values on both sides of (6.18) is easy to compute. The Laplace transform of $\int_0^T \sigma^2(t)\,z(t)\,dt$ in the model (6.15) is, however, easy to compute numerically (see [AA02] and the Appendix E for details). This observation suggests approximating $g(x)$ with a function of the form

$$
g(x)\approx a+be^{-cx}.
\tag{6.19}
$$

We choose the coefficients a, b, c from the local second-order fit condition around the mean of $\int_0^T \sigma^2(t)\,z(t)\,dt$,

$$
\begin{aligned}
g(\zeta) &= a+be^{-c\zeta}, \\
g'(\zeta) &= -bce^{-c\zeta}, \\
g''(\zeta) &= bc^2e^{-c\zeta}, \\
\zeta &= \mathbf{E}Z(T)=\int_0^T \sigma^2(t)\left(\mathbf{E}z(t)\right)dt \\
&= z_0\int_0^T \sigma^2(t)\,dt.
\end{aligned}
\tag{6.20}
$$

These imply (only c is important for the derivation below) that

$$
c=-\frac{g''(\zeta)}{g'(\zeta)}.
\tag{6.21}
$$

Having computed the coefficients a, b, c, the problem (6.18) is approximated with

$$
\begin{aligned}
a+b\mathbf{E}\exp\left(-c\int_0^T \sigma^2(t)\,z(t)\,dt\right) &= a+b\mathbf{E}\exp\left(-c\lambda^2\int_0^T z(t)\,dt\right), \\
\mathbf{E}\exp\left(-c\int_0^T \sigma^2(t)\,z(t)\,dt\right) &= \mathbf{E}\exp\left(-c\lambda^2\int_0^T z(t)\,dt\right).
\end{aligned}
\tag{6.22}
$$

These considerations define the effective volatility approximation, the statement that we present as a theorem.

Theorem 6.4. *Let us denote the Laplace transform of $Z(T)$ by*

$$
\varphi(\mu)\triangleq\mathbf{E}\exp\left(-\mu Z(T)\right),
$$

and the Laplace transform of the integral of $z(t)$ *by*

$$\varphi_0(\mu) \triangleq \mathbf{E}\exp\left(-\mu \int_0^T z(t)\, dt\right).$$

The second-order accurate "effective" volatility λ *is given as a solution to the equation*

$$\varphi_0\left(-\frac{g''(\zeta)}{g'(\zeta)}\lambda^2\right) = \varphi\left(-\frac{g''(\zeta)}{g'(\zeta)}\right), \tag{6.23}$$

where

$$\zeta = z_0 \int_0^T \sigma^2(t)\, dt$$

and the function $g(\cdot)$ *is given by 6.17.*

Proof. The homogenization problem (6.18) is replaced with (6.22), for which the expression (6.21) for c is used.

Remark 6.1. *The expression on the right-hand side of (6.23) is easily (and cheaply) computable, see Appendix E. The expression on the left-hand side (6.23) is available in closed form (see (E.3) in Appendix E). Thus, the equation (6.23) is trivial to solve for* λ *numerically. Because of the simplicity of functions involved, obtaining the solution is extremely fast.*

Remark 6.2. *The effective volatility* λ *as given by Theorem 6.4 is second-order accurate in the sense that the approximation (6.19) is second-order accurate with the choice of parameters in (6.20). We note that our method does not readily lend itself to obtaining higher-order approximations. This fact is of little relevance as the quality of the approximation is excellent as it is.*

Test results in Section 9 show that the approximation derived in Theorem 6.4 is very accurate for the values of distribution parameters typically observed in the market.

7. Calibration

The results presented in Section 5 (Theorem 5.1), Section 6.2 (Theorem 6.1) and Section 6.3 (Theorem 6.4) provide the connections between the model parameters $\{\sigma_k(t;n), \quad t \geq 0\}_{n,k}$ and $\{\beta(t;n), \quad t \geq 0\}_n$ for the model (4.1), (4.3), and the parameters $\{(\lambda_{n,m}, b_{n,m}, \eta)\}_{n,m}$ for the simple model (3.1), (3.2). Let us collect these formulas here. The time-dependent volatilities and skews for each swap rate are defined by

$$\begin{aligned}
\sigma_k(t;n,m) &= \sum_{i=n}^{m-1} q_i(n,m)\,\sigma_k(t;i), && (7.1)\\
\beta(t;n,m) &= \sum_{i=n}^{m-1} p_i(n,m)\,\beta(t;i), && (7.2)\\
n,m &= 1,\ldots,N-1.
\end{aligned}$$

where the weights $q_i(n,m)$, $p_i(n,m)$ are given by (5.4), (5.5).

The effective skew for each swap rate is defined by

$$\begin{aligned} b_{n,m} &= \int_0^{T_n} \beta(t;n,m)\, w_{n,m}(t)\; dt, \\ n,m &= 1,\ldots,N-1. \end{aligned} \tag{7.3}$$

with weights $w_{n,m}(t)$ given by (6.14) (note that the dependence of weights on the expiry/maturity is now explicitly reflected in the notations).

The effective volatility for each swap rate $\lambda_{n,m}$ is given as a solution to the equation

$$\varphi_0\left(-\frac{g''\left(z_0\int_0^{T_n}\sigma_{n,m}^2(t)\;dt\right)}{g'\left(z_0\int_0^{T_n}\sigma_{n,m}^2(t)\;dt\right)}\lambda_{n,m}^2\right) = \varphi\left(-\frac{g''\left(z_0\int_0^{T_n}\sigma_{n,m}^2(t)\;dt\right)}{g'\left(z_0\int_0^{T_n}\sigma_{n,m}^2(t)\;dt\right)}\right), \tag{7.4}$$

$$\begin{aligned} \sigma_{n,m}^2(t) &= \sum_{k=1}^{K}\sigma_k^2(t;n,m), \\ n,m &= 1,\ldots,N-1, \end{aligned} \tag{7.5}$$

with functions φ_0, φ defined in Theorem 6.4.

The purpose of model calibration is to obtain model parameters $\{\sigma_k(t;n)\,,\quad t\geq 0\}_{n,k}$ and $\{\beta(t;n)\,,\quad t\geq 0\}_n$ from the market-implied ones $\left\{\left(\lambda_{n,m}^*, b_{n,m}^*, \eta^*\right)\right\}_{n,m}$. This is typically done by regarding the formulas above as equations on the model parameters, recast in a least-squared-fit form

$$\begin{aligned} \sum_{n,m}\left(b_{n,m}-b_{n,m}^*\right)^2 &\to \min, \\ \sum_{n,m}\left(\lambda_{n,m}-\lambda_{n,m}^*\right)^2 &\to \min, \end{aligned}$$

where $b_{n,m}^*$, $\lambda_{n,m}^*$ are implied from the market, and $b_{n,m}$, $\lambda_{n,m}$ are computed from the model parameters via equations (7.3) and (7.4). Conditions on the regularity/time-homogeneity/smoothness of the model parameters are also typically imposed, most often in the similar least-squared-fit form . The model parameters are then found by numerical optimization.

Combining volatility and skew calibration in the same optimization is not the best course of action. Numerical properties of calibration algorithms are typically improved if the problem is split into a sequence of smaller subproblems. This typically reduces the dangers of overfitting, makes implied parameters more stable, and makes the procedure faster. In the problem at hand, two subproblems can be clearly defined. One is the fitting of the term structure of swaption skews $\{b_{n,m}\}_{n,m}$. The other is the fitting of the term structure of swaption volatilities$\{\lambda_{n,m}\}_{n,m}$.

At first glance, separating the two calibrations appears impossible. The equations for the skew fit, (7.3), involve model volatilities $\{\sigma_k(t;n)\,,\quad t\geq 0\}_{n,k}$ (through the dependence of the weights $\{w_{n,m}(t)\}$ on the latter). Similarly, the equations for swaption volatilities, (7.4), involve model skews $\{\beta(t;n)\,,\quad t\geq 0\}_n$. The dependence, however, is very mild. The particular form of the local volatility function in (4.3) (and (3.2)) was chosen with this issue in mind. Let us explain this important fact for (3.2). We note that for any value of the skew parameter b in (3.2), the value of the local volatility function at the money, i.e. at the point

$S = S_0$, is always

$$\lambda S_0,$$

the value that does not depend on b. The independence of the value of the local volatility function at-the-money from the skew parameter gives rise to an almost-independence of the skew parameter b from the implied parameter λ.

The discussion above justifies the following procedure for the full calibration of the model. It is carried out in 3 steps.

Step 1: The model's skew parameters $\{\beta(t;n), \quad t \geq 0\}_n$ are all set to the same value $\bar{\beta}$ (chosen, for example, to be the average of $b_{n,m}$ over the whole swaption grid). The model volatilities $\{\sigma_k(t;n), \quad t \geq 0\}_{n,k}$ are calibrated to $\{\lambda_{n,m}\}_{n,m}$ using equations (7.4) and regularity conditions imposed on the model volatilities;

Step 2: Using model volatilities $\{\sigma_k(t;n), \quad t \geq 0\}_{n,k}$ obtained on the previous step, the model skews $\{\beta(t;n), \quad t \geq 0\}_n$ are now calibrated to the swaption skews $\{b_{n,m}\}_{n,m}$ using equations (7.3) and regularity conditions imposed on the model skews;

Step 3: (Optional) Finally, the model volatilities $\{\sigma_k(t;n), \quad t \geq 0\}_{n,k}$ are re-calibrated to $\{\lambda_{n,m}\}_{n,m}$ using equations (7.4) (and regularity conditions) with $\{\beta(t;n), \quad t \geq 0\}_n$ calibrated on the previous step.

The algorithm is a form of a "relaxation" optimization in which nearly-orthogonal subproblems are solved sequentially, so that on each step the parameters that are not being solved for are taken from the previous step. As such, Steps 2 and 3 can be repeated multiple times. Our experience shows that one cycle, even without Step3, is enough to get a very good fit.

Steps 1 and 3 above constitute standard volatility calibration problems. They are extensively covered in the literature, and we do not contribute anything to the standard algorithm. Step 2 is of course new. Let us discuss it in more detail. With model volatilities $\{\sigma_k(t;n), \quad t \geq 0\}_{n,k}$ fixed on the previous step, all weight functions $\{w_{n,m}(\cdot)\}_{n,m}$ in (7.3) can be precomputed. The part of the objective function related to the recovery of market inputs then becomes

$$\begin{aligned} O_{fit}(\{\beta(\cdot;\cdot)\}) &= \sum_{n,m=1}^{N} O_{n,m}(\{\beta(\cdot;\cdot)\}), \\ O_{n,m}(\{\beta(\cdot;\cdot)\}) &= \left(\int_0^{T_n} \left(\sum_{i=n}^{m-1} \beta(t;i)\, p_i(n,m)\right) w_{n,m}(t)\, dt - b^*_{n,m}\right)^2. \end{aligned}$$

The term $O_{fit}(\{\beta(\cdot;\cdot)\})$ penalizes the deviation of the model skews from the market-implied ones, as required by the condition (7.3). To regularize the problem and to impart desired behavior on the model skews, we find it useful to add another term. We define it by

$$O_{reg}(\{\beta(\cdot;\cdot)\}) = \sum_{n,m} \left(\beta(T_n;m) - \beta(T_{n-1};m-1)\right)^2. \tag{7.6}$$

This term penalizes deviation of the term structure of model skews from a perfectly time-homogeneous one. Finally, the complete objective function is given by the weighted combination of the two,

$$O(\{\beta(\cdot;\cdot)\}) = O_{fit}(\{\beta(\cdot;\cdot)\}) + \alpha O_{reg}(\{\beta(\cdot;\cdot)\}),$$

with the weight α defining relative importance of time-homogeneity over market fit. The calibration procedure finds $\beta(\cdot;\cdot)$ that minimizes $O(\{\beta(\cdot;\cdot)\})$,

$$\beta(\cdot;\cdot) = \arg\min O(\{\beta(\cdot;\cdot)\}).$$

As a final note we observe that $O(\{\beta(\cdot;\cdot)\})$ is quadratic in $\{\beta(\cdot;\cdot)\}$, a fact that can be explored to make skew calibration almost instantaneous.

8. Forward skew structure

The classic FL-SV model has one theoretical advantage over the FL-TSS model: its skew structure is time-homogeneous (since it is constant). As the FL-TSS model aims to fit the market skew structure, time homogeneity is almost invariably lost. Time-homogeneity is widely regarded as an important property of any model. Time-dependent model skew structure in the FL-TSS model implies certain evolution of the term structure of swaption skews going forward (the forward skew, as it is sometimes known). Should the TSS-FL model imply wildly unrealistic implied evolution of the term structure of swaption skews, hedging parameters implied by the model, and ultimately prices of exotic interest rate derivatives, would be very suspect.

In the model we present, the degree of desired time-homogeneity is controlled by imposing the non-time homogeneity penalty (7.6) during the skew calibration step (see Section 7). In effect, the calibration tries to find the best fit to the current term structure of skew while retaining as much time-homogeneity in the skew structure as requested. This can be taken further by only considering time-homogeneous skew structures and finding the best fit among those. That might compromise the ability to match current market skew too much, however.

Graphs of instantaneous forward Libor skews in Figure 3 present a picture of how the model skews evolve over time in an FL-TSS model. See Section 9.2 for more details.

9. Test results

9.1. **Test of the effective skew formula.** In this section we show typical performance of the effective skew formula from Section 6.2 (see Theorem 6.1) For clarity, we assume no stochastic volatility ($\eta = 0$), and test the simple formula presented in Example 6.1. We value simple European options of different strikes, all with expiry in $T = 30$ (years), on a stock with spot price $S_0 = 100$, under zero interest rate and dividend rate conditions. We compute option values in the model

$$dS(t) = \sigma(\beta(t) S(t) + (1-\beta(t)) S_0)\ dW(t).$$

We use $\sigma = 10\%$ and $\beta(t) = t/T$ (constant volatility, skew linearly growing from 0 to 1). For each strike, three values are computed. The "true" value is computed using a PDE (Crank-Nicholson scheme). The effective skew approximation is computed using the closed form value from the shifted lognormal model

$$dS(t) = \sigma(bS(t) + (1-b) S_0)\ dW(t)$$

with b computed according to the effective skew formula in Example 6.1 ($b \approx 68\%$ in this case). Another approximation is computed using the closed form value from the same shifted lognormal model, but with b computed using a naive arithmetic average of $\beta(\cdot)$. The results are presented in Figure 1, where the values from the three methods are presented on the left scale, and errors of the two approximations (versus the "true" value) are shown on the right scale. All numbers are in implied Black volatilities. Option strike is given on the x-axis. The effective skew approximation works very well, even for these long-dated options with a time-dependent skew $\beta(\cdot)$ that varies across its full range (from 0% to 100%). Across a wide range of strikes, errors are within 0.10% Black volatility. The effective skew approximation works much better that the naive one where the arithmetic average of the skew is used. This demonstrates that not only our approximation formula is very accurate, but it is also relevant and non-trivial, in the sense that other ad-hoc averaging rules do not provide nearly as good accuracy.

9.2. **Skew calibration and forward skew tests.** We perform comprehensive tests of approximation formulas derived, by valuing a representative collection of swaptions using Monte-Carlo methods and the approximations. Before presenting the results, let us describe the market data used. We use stylized market data, both in an effort to highlight relevant conclusions, and to allow other researchers to reproduce/compare with our results. We use an interest rate curve with continuously-compounded zero rates flat at 5.00%. All market swaption volatilities ($\lambda_{n,m}$ in the notations of Section 3) are set to 15%, and all volatility of variance parameters ($\eta_{n,m}$ in the notations of Section 3) to 130%. Mean reversion of variance θ is set to 15%[2].

Swaption skews ($b_{n,m}$) are allowed to vary across the swaption grid. Market skews are given in Table 1 and Figure 2. Note the wide range of values that the skews were allowed to take. Note that all these values are roughly consistent with the Euro swaption market as of October 2003.

The classic FL-SV model is calibrated to this data, using the skew $\bar{b} = 20\%$, which is computed as an average of all market swaption skews. We calibrated the FL-TSS model to the actual grid of market swaption skews, given by Table 1 (and to swaption volatilities λ / volatilities of variance η as detailed above), as described in Section 7, without Step 3. The model dynamics are based on 6-month Libor rates, i.e. $\tau_n \equiv \tau = 0.5$ for all n. Each model has two factors.

The results of skew calibration are presented in Table 2 and Figure 3, where the term structure of model skews is presented (note that the calibrated volatility structure $\sigma_k(t;n)$ is not shown because the corresponding tables are large; they are available upon request). In formulas throughout the paper, model skews of instantaneous forward Libor rates $\beta(t;n)$ are indexed by a pair of time/Libor index (t,n). In the test results, we adopt a slightly different convention of parametrizing them by time/offset (both measured in years), and denoting them by $\beta^*(t,\mathcal{T})$. The connection between the two is very simple,

$$\beta(t;n) = \beta^*(t, \tau \times n - t).$$

The parameter $\mathcal{T}$ has the interpretation of the distance from time t to the fixing date of a Libor rate. Both Table 2 and Figure 3 show $\beta^*(t_i, \mathcal{T}_j)$ for knot points $(t_i, \mathcal{T}_j)$; other values $\beta^*(t,\mathcal{T})$ are computed using bilinear interpolation.

One of the advantages of parametrization $\beta^*(t,\mathcal{T})$ over $\beta(t;n)$ is that time-homogeneity, or lack thereof, is easier to discern with the former. Had the model skew structure been time-homogeneous, $\beta^*(t,\mathcal{T})$ would be a function of $\mathcal{T}$ only (all lines in Figure 3 would be

[2] We note that these values of volatility of variance/mean reversion of variance generate rather convex volatility smiles that do not flatten out significantly over long times (in accordance with what is observed in the market). Many approximation methods become inaccurate for these, rather extreme, values of parameters (in particular assymptotic expansions in [ABR01], as noted in [AA02]).

on top of each other). By examining Figure 3 we see that this is not the case. As calendar time t goes forward, the skews appear to go down, with the "middle portion" of each $\beta^* (t, \cdot)$ decaying faster that the front and back end. This is of course a consequence of calibrating the model's skew structure to the market. If the market skew structure is not time homogeneous, the model one cannot be, either. Despite the lack of time-homogeneity, the dynamics of the term structure of skews appears to be smooth and controlled, and not totally unreasonable, thus strengthening our belief in the merits of the proposed approach.

9.3. **Model comparison tests.** Tables 3, 4, 5, 6, 7, and Figures 4, 5, 6, 7, 8 present test results, across various models, for a collection of swaptions of different strikes, expiries and maturities. All numbers are in implied Black volatilities. The following models are considered.

- "Simple" – a simple stochastic volatility model for swaptions, as presented in Section 3.
- "FL-TSS" - a Term Structure of Skews forward Libor model, as defined in Section 4.3, equations (4.1), (4.2).
- "FL-SV" - a "classic" Stochastic Volatility forward Libor model, as defined in Section 4.3, equations (4.1), (4.3).

"MC" refers to Monte-Carlo results obtained using 524,288 ($= 2^{19}$) paths with 16 steps per year, and "Approx" to results obtained using the approximations developed in this paper.

Each table shows results for a particular strike offset = -2.00%, -1.00%, 0.00%, 1.00% and 2.00%. For example, a strike offset of 1.00% means that a strike used for each swaption in the table or figure is equal to the at-the-money forward swap rate for that swaption plus 1.00%.

Each swaption is identified by its expiry and maturity. For example, a 5y10y swaption has an expiry in 5 years; it gives the holder a right to exercise into a swap that will have maturity of 10 years on the date of exercise.

Actual model parameters used are described in details in Section 9.2. In particular, the table of market term skews for the simple model is given in Table 1 and Figure 2. The calibrated (model) instantaneous forward Libor skews for the FL-TSS model are given in Table 2 and Figure 3. For the FL-SV model, the global skew of 20.00% (the average of skews across the swaption grid) is used.

Each of the tables lists swaption expiry/maturity in the first row, values of swaptions in different models in the next five rows, Monte-Carlo standard errors in the next two rows, and differences between different models/methods in the last 6 rows. The figures present the same information in a graphical form. There are 4 sub-figures per each figure, (A), (B), (C), and (D).

- Sub-figure (A) shows Monte-Carlo standard errors for the FL-TSS and FL-SV models. This is useful to assess significance of various comparison results, as any difference of more that 3 standard deviations away from zero indicates significance of the result beyond random noise.
- Sub-figure (B) shows the differences between Monte-Carlo values and approximation values. This figure is the main indicator of the quality of approximations we developed in Theorems 5.1, 6.1 and 6.4.

- Sub-figure (C) shows the differences between simple model values (i.e. market values) and approximation values from the forward Libor models. These results mainly indicate the quality of the calibration procedure presented in Section 7.
- Sub-figure (D) shows the differences between simple model values (i.e. market values) and actual Monte-Carlo values for forward Libor models. Th results are a combination of the results from sub-figures (B) and (C). They provide information on the overall ability of the FL-TSS and the FL-SV models to recover market swaption prices.

Test results below are described using Figures rather than Tables, as the former provide an easier way to understand the results. The results are discussed in detail below. Overall, the following conclusions are reached,

- The quality of approximations developed in Theorems 5.1 (approximate swap rate dynamics), 6.1 (effective skew approximation), and 6.4 (effective volatility approximation) appear to be excellent across a wide range of strikes/expiries and maturities;
- The calibration procedure developed in Section 7 works well and results in an accurate calibration to at-the-money swaption volatilities, and to market skews for all swaption expiries/maturities;
- The FL-TSS model significantly outperforms the FL-SV model in its ability to recover market implied volatilities for non-at-the-money swaptions, and the extra complexity of the model is justified by its enhanced capabilities.

9.3.1. *Approximation and calibration quality for at-the-money swaptions.* The results for at-the-money swaptions are shown in Table 5 and Figure 6. The quality of approximations is excellent for at-the-money swaptions. The difference between Monte-Carlo and approximation values is within 0.15% Black volatility for all but one swaption (15y20y), for which it is an also-respectable 0.25% Black volatility. For the approximations we developed, the 15y20y swaption is probably the hardest of all in the whole swaption grid to handle, because of the significant convexity in the underlying swap rate induced by a long swap tenor of 20 years, compounded over a long time to expiration of 15 years.

Note that skew approximations matters little for at-the-money swaptions. The results for at-the-money swaptions (in particular sub-figure (B)) validate the approximation for the swap rate dynamics presented in Theorem 5.1, and the approximation results for the "effective" volatility in Theorem 6.4. (We comment that, even though the market swaption grid of $\lambda_{n,m}$ was chosen flat at 15% for our tests, forward Libor volatilities $\sigma_k(t;n)$ were not constant, so that the tests for Theorem 6.4 were still meaningful.)

The quality of calibration, as given by sub-figure (C), is also excellent for at-the-money swaption volatilities.

On the sub-figure (B), the quality of approximations appears to be slightly worse for short-expiry swaptions (1 year expiry) than for longer-expiry ones, which may seem somewhat odd. This result is most likely the result of a discretization error in the Monte-Carlo simulation being higher for shorter times because of the somewhat insufficient number of time steps (16 per year) used.

9.3.2. *Approximation and calibration quality for swaptions with positive strike offset.* The results for swaptions with strike offsets of 1% and 2% (in-the-money receiver swaptions) are shown in Tables 6, 7 and Figures 7, 8. The quality of approximations (sub-figures (B) for both figures) appears to still be excellent, with 1% strike offset results similar to at-the-money results, and 2% strike offset results slightly worse, but still mostly within 0.3 Black volatilities). As it is the case for the at-the-money swaptions, the 15y20y swaption has the worst approximation quality. Note that all three approximations (Theorems 5.1, 6.1, and 6.4) are used, so the test results validate all three approximations. Skew calibration (sub-figures (C), FL-TSS model) is also shown to be excellent.

9.3.3. *Approximation and calibration quality for swaptions with negative strike offset.* The results for swaptions with strike offsets of -1% and -2% (in-the-money payer swaptions) are shown in Tables 4, 3 and Figures 5, 4. They are generally in line with the results for positive strike offset swaptions (see previous section). We witness slight deterioration of approximation quality for negative strike offsets for longer-dated (25 years expiry) swaptions, especially for -2.00% strike offset for he FL-TSS model (sub-figures (B)). Also of interest are significantly higher standard errors for 1 year expiry swaptions for -2.00% strike offset. This is more the consequence of the chosen scale (implied Black volatilities), since these deep-in-the-money swaptions have very little volatility dependence.

9.3.4. *Recovery of market skews.* In tis section we highlight the differences between the FL-TSS and the FL-SV models in regards to their skew recovery capabilities. Recall that the "classic" FL-SV model has only one skew parameter applied to all forward Libor rates (which was set to 20% in our tests). The FL-TSS model developed in this paper, in contrast, has a term structure of skews that was calibrated to the market skews of the swaption grid. We focus on sub-figures (C) and (D) for Figures 4, 5, 6, 7, 8 (the difference between (C) and (D) is that (C) is based on approximation results and (D) is based on actual Monte-Carlo results; he difference between the two is in the quality of approximations). For at-the-money swaptions, there is no difference between the two models (as expected). For strikes in and out of the money, however, we see that extra capabilities of the FL-TSS model allow it to track market swaption prices (or implied Black volatilities) a lot closer. For strike offsets of $\pm 2.00\%$, the FL-SV model can be off by $1.00\% - 1.50\%$ Black volatility, and for strike offsets of $\pm 1.00\%$, the FL-SV model can be off by up to 0.50% Black volatility.

Overall, the FL-TSS model significantly outperforms the FL-SV model in its ability to recover market skew information. Extra complexity of the FL-TSS model is justified by its enhanced capabilities.

10. Conclusions

In this paper, we have extended the classic Stochastic Volatility forward Libor model to account for variability of implied volatility skews across the swaption grid. We have developed, for the first time, a viable, practical model that is capable of fully incorporating all available market volatility information (including swaption skews). New theoretical results on averaging of model parameters have been obtained (see Theorems 6.1, 6.4 and Corollary 6.3). The results facilitate fast, accurate and efficient pricing of, and calibration to, European swaptions and caps across all strikes, expiries and maturities. Test results indicate that the approximations that we derived are excellent, even for high volatility of variance/low mean reversion of variance parameters. They also show that the new model's ability to match market-implied volatility skews is materially better than that of the standard Stochastic Volatility forward Libor model, allowing for more accurate pricing/hedging of exotic interest rate derivatives.

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Appendix A. Proof of Theorem 5.1

Since the swap rate $S_{n,m}(t)$ is a martingale under the swap measure $\mathbf{Q}^{n,m}$, we can ignore dt terms when writing an SDE for $dS_{n,m}(t)$. We have,

(A.1)

$$\begin{aligned} dS_{n,m}(t) &= \sum_{i=n}^{m-1} \frac{\partial S_{n,m}(t)}{\partial L_i(t)} dL_i(t) \\ &= \sum_{i=n}^{m-1} \frac{\partial S_{n,m}(t)}{\partial L_i(t)} \left(\beta(t;i) L_i(t) + (1-\beta(t;i)) L_i(0)\right) \sqrt{z(t)} \sum_{k=1}^{K} \sigma_k(t;i)\, dW_k^{n,m}(t). \end{aligned}$$

We would like to write this equation in an "autonomous" form

(A.2)

$$dS_{n,m}(t) \approx \left(\beta(t;n,m) S_{n,m}(t) + \left(1-\beta(t;n,m)\right) S_{n,m}(0)\right) \sqrt{z(t)} \sum_{k=1}^{K} \sigma_k(t;n,m)\ dW_k^{n,m}(t),$$

where $\beta(t;n,m)$ is the time-dependent skew for the swap rate $S_{n,m}$, and $\sigma_k(\cdot;n,m)$ are factor volatilities for the swap rate. To get from (A.1) to (A.2), we impose two conditions. The first one is that the right-hand-sides should agree "along the forward path",

$$\begin{aligned} S_{n,m}(t) &= S_{n,m}(0), \\ L_i(t) &= L_i(0). \end{aligned}$$

Secondly, the slopes should agree along the forward path as well.

From the first condition we obtain

$$\sum_{i=n}^{m-1} L_i(0) \frac{\partial S_{n,m}(0)}{\partial L_i(0)} \left(\sum_{k=1}^{K} \sigma_k(t;i)\ dW_k^{n,m} \right) = S_{n,m}(0) \sum_{k=1}^{K} \sigma_k(t;n,m)\ dW_k^{n,m}(t).$$

Let us define

$$q_i = \frac{L_i(0)}{S_{n,m}(0)} \frac{\partial S_{n,m}(0)}{\partial L_i(0)}.$$

Then for each k, $k=1,\ldots,K$, the weights for the standard swaption volatility approximation follows,

$$\sigma_k(t;n,m) = \sum_{i=n}^{m-1} q_i \sigma_k(t;i). \tag{A.3}$$

Substituting (A.3) into (A.2) we obtain

(A.4)

$$dS_{n,m}(t) = \left(\beta(t;n,m) S_{n,m}(t) + \left(1-\beta(t;n,m)\right) S_{n,m}(0)\right) \sqrt{z(t)} \sum_{i=n}^{m-1} q_i \sum_{k=1}^{K} \sigma_k(t;i)\ dW_k^{n,m}(t).$$

Differentiating the right-hand-side of (A.4) with respect to L_j we obtain

(A.5)

$$\frac{\partial}{\partial L_j(t)} \left(\left(\beta(t;n,m) S_{n,m}(t) + \left(1-\beta(t;n,m)\right) S_{n,m}(0)\right) \sqrt{z(t)} \sum_{i=n}^{m-1} q_i \sum_{k=1}^{K} \sigma_k(t;i)\ dW_k^{n,m}(t) \right)$$

$$= \beta(t;n,m) \frac{\partial S_{n,m}(t)}{\partial L_j(t)} \sqrt{z(t)} \sum_{i=n}^{m-1} q_i \sum_{k=1}^{K} \sigma_k(t;i)\ dW_k^{n,m}(t).$$

Differentiating the right-hand side of (A.1) with respect to L_j we obtain,

(A.6)

$$\begin{aligned}
&\frac{\partial}{\partial L_j(t)} \sum_{i=n}^{m-1} \frac{\partial S_{n,m}(t)}{\partial L_i(t)} \left(\beta(t;i) L_i(t) + (1-\beta(t;i)) L_i(0)\right) \sqrt{z(t)} \sum_{k=1}^{K} \sigma_k(t;i)\ dW_k^{n,m}(t) \\
&= \sum_{i=n}^{m-1} \frac{\partial^2 S_{n,m}(t)}{\partial L_j(t)\,\partial L_i(t)} \left(\beta(t;i) L_i(t) + (1-\beta(t;i)) L_i(0)\right) \sqrt{z(t)} \sum_{k=1}^{K} \sigma_k(t;i)\ dW_k^{n,m}(t) \\
&+ \frac{\partial S_{n,m}(t)}{\partial L_j(t)} \beta(t;j) \sqrt{z(t)} \sum_{k=1}^{K} \sigma_k(t;j)\ dW_k^{n,m}(t).
\end{aligned}$$

Equating (A.5) and (A.6) along the forward path, and ignoring the second-order term we obtain,

$$\begin{aligned}
&\frac{\partial S_{n,m}(0)}{\partial L_j(0)} \beta(t;n,m) \sqrt{z(t)} \sum_{i=n}^{m-1} q_i \sum_{k=1}^{K} \sigma_k(t;i)\ dW_k^{n,m}(t) \\
= \ &\frac{\partial S_{n,m}(0)}{\partial L_j(0)} \beta(t;j) \sqrt{z(t)} \sum_{k=1}^{K} \sigma_k(t;j)\ dW_k^{n,m}(t), \quad j = 1, \ldots, N.
\end{aligned}$$

Cancelling $\frac{\partial S_{n,m}(0)}{\partial L_j(0)} \sqrt{z(t)}$ on both sides and equating diffusion coefficients for each factor Brownian motion $W_k^{n,m}$, and using (A.3) we obtain

$$\beta(t;n,m)\,\sigma_k(t;n,m) = \beta(t;j)\,\sigma_k(t;j), \quad j = 1, \ldots, N, \quad k = 1, \ldots, K.$$

These equations cannot be solved simultaneously for all j of course. We reformulate the problem in the least-squares sense: find $\beta(t;n,m)$ such that

$$\sum_{j,k} \left(\beta(t;n,m)\,\sigma_k(t;n,m) - \beta(t;j)\,\sigma_k(t;j)\right)^2 \to \min.$$

This problem can be easily solved to yield

$$\beta(t;n,m) = \sum_j \beta(t;j) \frac{\sum_k \sigma_k(t;j)\,\sigma_k(t;n,m)}{\sum_k \sigma_k^2(t;n,m)}.$$

The theorem is proved. As a final note, we observe that the coefficient

$$\frac{\sum_k \sigma_k(t;j)\,\sigma_k(t;n,m)}{\sum_k \sigma_k^2(t;n,m)}$$

can be regarded as a regression coefficient of $dS_{n,m}(t)$ on $dL_j(t)$.

Appendix B. Proof of Theorem 6.1

The stochastic variance process $z(t)$ is Markovian. We denote its infinitesimal generator by L^z,

$$\begin{aligned}
L^z &: C_b^2(\mathbb{R}) \to C_b(\mathbb{R}), \\
L^z &: \phi \longmapsto \theta(z_0 - z) \frac{\partial \phi}{\partial z} + \frac{1}{2}\eta^2 z \frac{\partial^2 \phi}{\partial z^2}.
\end{aligned}$$

We note that the process $X_0(t)$ ($\equiv Y_0(t)$) satisfies the following SDE,

$$\begin{aligned} dX_0(t) &= \sqrt{z(t)}\sigma(t)\, dW(t), \\ X_0(0) &= x_0. \end{aligned}$$

Let us denote the Markov semi-group of operators that corresponds to the process $(X_0(t), z(t))$ by $P_0(s,t)$, and the time-dependent infinitesimal generator by $L_0(t)$,

$$\begin{aligned} P_0(s,t) &: C_b\left(\mathbb{R}^2\right) \to C_b\left(\mathbb{R}^2\right), \\ P_0(s,t) &: \phi \longmapsto P_0(s,t)\phi, \\ \left[P_0(s,t)\phi\right](x,z) &= \mathbf{E}_s\left(\phi(X_0(t), z(t)) \middle| X_0(s) = x, z(s) = z\right), \\ L_0(t) &: C_b^2\left(\mathbb{R}^2\right) \to C_b\left(\mathbb{R}^2\right), \\ L_0(t) &: \phi \longmapsto \frac{1}{2}\sigma^2(t)\, z \frac{\partial^2\phi}{\partial x^2} + L^z. \end{aligned}$$

Let us denote the same for $(X_\varepsilon(\cdot), z(\cdot))$ and for $(Y_\varepsilon(\cdot), z(\cdot))$ by $P_\varepsilon^X(s,t)$, $L_\varepsilon^X(t)$ and $P_\varepsilon^Y(s,t)$, $L_\varepsilon^Y(t)$, respectively.

From the general operator semigroup theory it follows that

$$P_\varepsilon^Y(0,T) = P_\varepsilon^X(0,T) + \int_0^T P_\varepsilon^Y(0,t)\left(L_\varepsilon^Y(t) - L_\varepsilon^X(t)\right) P_\varepsilon^X(t,T)\, dt. \tag{B.1}$$

Expressed in terms of the Markovian semigroup, the value at time 0 of a European call option with strike K and time to maturity T under the process $(X_\varepsilon(\cdot), z(\cdot))$ is given by

$$\left\langle \delta_{x_0,z_0}, P_\varepsilon^X(0,T)\pi_K \right\rangle,$$

where the payoff π_K is defined by

$$\pi_K(x,z) = (x-K)^+.$$

The notation δ_{x_0,z_0} is used for the usual delta function at (x_0, z_0), and $\langle\cdot,\cdot\rangle$ for the scalar product of functions in $C(\mathbb{R}^2)$.

Similarly, the price of a the same call option under the diffusion $Y_\varepsilon(\cdot)$ is given by

$$\left\langle \delta_{x_0,z_0}, P_\varepsilon^Y(0,T)\pi_K \right\rangle.$$

Then (see (6.8)),

$$\begin{aligned} \Delta &\triangleq \int \left(\mathbf{E}(Y_\varepsilon(T) - K)^+ - \mathbf{E}(X_\varepsilon(T) - K)^+\right) dK \\ &= \int \left\langle \delta_{x_0,z_0}, \left(P_\varepsilon^Y(0,T) - P_\varepsilon^X(0,T)\right)\pi_K \right\rangle\, dK \\ &= \int_0^T \int \left\langle \delta_{x_0,z_0}, P_\varepsilon^Y(0,t)\left(L_\varepsilon^Y(t) - L_\varepsilon^X(t)\right) P_\varepsilon^X(t,T)\pi_K \right\rangle\, dK\, dt. \end{aligned}$$

It follows from the Taylor expansion of f_ε and $\bar{f}_\varepsilon$ around $x=0$ and (6.5) that the difference $L_\varepsilon^Y(t) - L_\varepsilon^X(t)$ is of the order $O(\varepsilon^2)$. Similarly, $L_\varepsilon^Y(t) - L_0(t)$ and $L_\varepsilon^X(t) - L_0(t)$ are of the order $O(\varepsilon)$. Therefore, $P_\varepsilon^Y(0,t) - P_0(0,t)$ and $P_\varepsilon^X(t,T) - P_0(t,T)$ are of the order

$O(\varepsilon)$. Hence, we can replace $P_\varepsilon^Y(0,t)$ and $P_\varepsilon^X(t,T)$ in the equation above with $P_0(0,t)$ and $P_0(t,T)$, respectively, so that

$$\begin{aligned}\Delta &= \Delta_1 + o\left(\varepsilon^2\right), \\ \Delta_1 &= \int_0^T \int_{-\infty}^{\infty} \left\langle \delta_{x_0,z_0}, P_0(0,t)\left(L_\varepsilon^Y(t) - L_\varepsilon^X(t)\right) P_0(t,T)\,\pi_K \right\rangle \, dK\, dt.\end{aligned}$$

To prove the theorem it is enough to show that $\Delta_1 = o(\varepsilon^2)$.

Note that for any function $\phi \in C_b(\mathbb{R}^2)$,

$$\langle \delta_{x_0,z_0}, P_0(0,t)\,\phi \rangle = \langle P^*(0,t)\,\delta_{x_0}, \phi \rangle,$$

by definition, where $P^*(0,t)$ is the conjugate Markov semi-group acting on measures on $C_b(\mathbb{R}^2)$. In particular, $P_0^*(0,t)\,\delta_{x_0,z_0}$ is the (unconditional) density $p(t,x,z)$ of the random variable $(X_0(t), z(t))$. Thus,

$$\begin{aligned}\Delta_1 &= \int_0^T \int_{-\infty}^{\infty} \int\int p(t,x,z)\left[\left(L_\varepsilon^Y(t) - L_\varepsilon^X(t)\right) P_0(t,T)\,\pi_K\right](x)\; dz\, dx\; dK\, dt \\ &= \int_0^T \int_{-\infty}^{\infty} \int\int p(t,x,z)\left[\left(\frac{1}{2}\sigma^2(t)\, z\left(\bar{f}_\varepsilon^2(x) - f_\varepsilon^2(t,x)\right)\frac{\partial^2}{\partial x^2}\right) P_0(t,T)\,\pi_K\right](x)\; dz\, dx\; dK\, dt,\end{aligned}$$

where in the last expression, the operator $L_\varepsilon^Y(t) - L_\varepsilon^X(t)$ is written out explicitly.

Clearly

$$[P_0(t,T)\,\pi_K](x,z)$$

is the value, at time t, of the call option with expiry T and strike K, if $(X_0(t), z(t)) = (x,z)$. Then

$$\frac{\partial^2}{\partial x^2}[P(t,T)\,\pi_K](x,z)$$

is the gamma $\gamma(x,z,K)$ (second derivative of the option with respect to the underlying). Hence

$$\begin{aligned}\Delta_1 &= \frac{1}{2}\int_0^T \int_{-\infty}^{\infty} \int\int zp(t,x,z)\left(\bar{f}_\varepsilon^2(x) - f_\varepsilon^2(t,x)\right)\sigma^2(t)\,\gamma(x,z,K)\; dz\, dx\; dK\, dt \\ &= \frac{1}{2}\int_0^T \int\int zp(t,x,z)\left(\bar{f}_\varepsilon^2(x) - f_\varepsilon^2(t,x)\right)\left(\int_{-\infty}^{\infty}\gamma(x,z,K)\; dK\right) dz\, dx\; dt.\end{aligned}$$

The distribution of the process $X_0(\cdot)$ is translation-invariant ($f_0(t,x) \equiv 1$), hence

$$\gamma(x,z,K) = \gamma(x-K,z)$$

and thus

$$\int_{-\infty}^{\infty}\gamma(x,z,K)\; dK = -\int_{-\infty}^{\infty}\gamma(x,z,K)\; dx = -1,$$

with the last equality following from the fact that $\gamma(x,z,K)$ is the derivative of the delta of the option with respect to x (spot). Therefore

$$\begin{aligned} \Delta_1 &= -\frac{1}{2}\int_0^T \left(\int\int zp(t,x,z)\left(\bar{f}_\varepsilon^2(x) - f_\varepsilon^2(t,x)\right) dz\,dx\right)\sigma^2(t)\,dt \\ &= -\frac{1}{2}\int_0^T \left(\int \hat{p}(t,x)\left(\bar{f}_\varepsilon^2(x) - f_\varepsilon^2(t,x)\right) dx\right)\sigma^2(t)\,dt, \\ \hat{p}(t,x) &\triangleq \int zp(t,x,z)\,dz. \end{aligned}$$

Using the fact that $f_\varepsilon(s,x_0) = 1$ for all $s \in [0,T]$, and expanding to the first order around (s,x_0),

$$f_\varepsilon(s,x) = 1 + \varepsilon\frac{\partial f(s\varepsilon^2,x_0)}{\partial x}(x-x_0) + O(\varepsilon).$$

Then

$$f_\varepsilon^2(s,x) = 1 + 2\varepsilon\frac{\partial f(s\varepsilon^2,x_0)}{\partial x}(x-x_0) + \varepsilon^2\left[\frac{\partial f(s\varepsilon^2,x_0)}{\partial x}\right]^2(x-x_0)^2 + o\left(\varepsilon^2\right).$$

Integrating this against an (arbitrary at this point) weight function $w(s)$, $s \in [0,T]$, and using (6.5) yields

$$\bar{f}_\varepsilon^2(x) = 1 + 2\varepsilon(x-x_0)\int_0^T \frac{\partial f(s\varepsilon^2,x_0)}{\partial x}w(s)\,ds + \varepsilon^2(x-x_0)^2\int_0^T\left[\frac{\partial f(s\varepsilon^2,x_0)}{\partial x}\right]^2 w(s)\,ds + o\left(\varepsilon^2\right).$$

Then

$$\begin{aligned} &\int \hat{p}(t,x)\left(\bar{f}_\varepsilon^2(x) - f_\varepsilon^2(t,x)\right) dx \\ &= 2\varepsilon\left(\frac{\partial f(s\varepsilon^2,x_0)}{\partial x} - \int_0^T \frac{\partial f(s\varepsilon^2,x_0)}{\partial x}w(s)\,ds\right)\int \hat{p}(t,x)(x-x_0)\,dx \\ &\quad +\varepsilon^2\left(\left[\frac{\partial f(s\varepsilon^2,x_0)}{\partial x}\right]^2 - \int_0^T\left[\frac{\partial f(s\varepsilon^2,x_0)}{\partial x}\right]^2 w(s)\,ds\right)\int \hat{p}(t,x)(x-x_0)^2\,dx \\ &\quad +o\left(\varepsilon^2\right). \end{aligned}$$

Since

$$\begin{aligned} \int p(t,x)(x-x_0)\,dx &= 0, \\ \int \hat{p}(t,x)(x-x_0)^2\,dx &= v^2(t), \end{aligned}$$

where

$$v^2(t) \triangleq \mathbf{E}\left(z(t)(X_0(t)-x_0)^2\right),$$

the expression simplifies to yield

$$\int \hat{p}(t,x)\left(\bar{f}_\varepsilon^2(x) - f_\varepsilon^2(t,x)\right)\,dx = \varepsilon^2 v^2(t)\left(\left[\frac{\partial f(t\varepsilon^2,x_0)}{\partial x}\right]^2 - \int_0^T \left[\frac{\partial f(s\varepsilon^2,x_0)}{\partial x}\right]^2 w(s)\,ds\right) + o\left(\varepsilon^2\right).$$

Thus,

$$\begin{aligned}
\Delta_1 &= \Delta_2 + o(\varepsilon), \\
\Delta_2 &= -\frac{\varepsilon^2}{2}\int_0^T \left(\left[\frac{\partial f(t\varepsilon^2,x_0)}{\partial x}\right]^2 - \int_0^T \left[\frac{\partial f(s\varepsilon^2,x_0)}{\partial x}\right]^2 w(s)\,ds\right) v^2(t)\,\sigma^2(t)\,dt \\
&= -\frac{\varepsilon^2}{2}\int_0^T \left[\frac{\partial f(t\varepsilon^2,x_0)}{\partial x}\right]^2 v^2(t)\,\sigma^2(t)\,dt + \frac{\varepsilon^2}{2}\left(\int_0^T \left[\frac{\partial f(t\varepsilon^2,x_0)}{\partial x}\right]^2 w(t)\,dt\right)\left(\int_0^T v^2(t)\,\sigma^2(t)\,dt\right)
\end{aligned}$$

To cancel Δ_2 we choose

$$w(t) = \frac{v^2(t)\,\sigma^2(t)}{\int_0^T v^2(t)\,\sigma^2(t)\,dt}, \quad t \in [0,T].$$

With the weights set according to this formula,

$$\begin{aligned}
\Delta_2 &= 0, \\
\Delta &= o(\varepsilon),
\end{aligned}$$

and the theorem is proved.

Appendix C. Proof of Proposition 6.2

Define

$$u(t) = z(t)\,\sigma^2(t).$$

Differentiating $q(\varepsilon)$ in ε we obtain,

$$q(\varepsilon) = \mathbf{E}\int_0^T \left(g_\varepsilon(t,X_\varepsilon(t)) - \bar{g}_\varepsilon(Y_\varepsilon(t))\right)^2 u(t)\,dt,$$

$$q'(\varepsilon) = \mathbf{E}\int_0^T \left(g_\varepsilon(t,X_\varepsilon(t)) - \bar{g}_\varepsilon(Y_\varepsilon(t))\right)\left(\frac{\partial}{\partial\varepsilon}g_\varepsilon(t,X_\varepsilon(t)) - \frac{\partial}{\partial\varepsilon}\bar{g}_\varepsilon(Y_\varepsilon(t))\right)u(t)\,dt, \tag{C.1}$$

$$\begin{aligned}
q''(\varepsilon) &= \mathbf{E}\int_0^T \left(\frac{\partial}{\partial\varepsilon}g_\varepsilon(t,X_\varepsilon(t)) - \frac{\partial}{\partial\varepsilon}\bar{g}_\varepsilon(Y_\varepsilon(t))\right)^2 u(t)\,dt \\
&\quad + \mathbf{E}\int_0^T \left(g_\varepsilon(t,X_\varepsilon(t)) - \bar{g}_\varepsilon(Y_\varepsilon(t))\right)\left(\frac{\partial^2}{\partial\varepsilon^2}g_\varepsilon(t,X_\varepsilon(t)) - \frac{\partial^2}{\partial\varepsilon^2}\bar{g}_\varepsilon(Y_\varepsilon(t))\right)u(t)\,dt.
\end{aligned}$$

As we put and $\varepsilon = 0$ in the equation above we note that the second term disappears. Hence,

$$q''(0) = \mathbf{E}\int_0^T \left(\left.\frac{\partial}{\partial\varepsilon}g_\varepsilon(t,X_\varepsilon(t))\right|_{\varepsilon=0} - \left.\frac{\partial}{\partial\varepsilon}\bar{g}_\varepsilon(Y_\varepsilon(t))\right|_{\varepsilon=0}\right)^2 u(t)\,dt.$$

Note that

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}g_\varepsilon(t,X_\varepsilon(t)) &= \left[\varepsilon\left(\frac{\partial}{\partial\varepsilon}X_\varepsilon\right)(t)+(X_\varepsilon(t)-x_0)\right]\frac{\partial g}{\partial x}(t,x_0+\varepsilon(X_\varepsilon(t)-x_0)),\\ \frac{\partial}{\partial\varepsilon}\bar{g}_\varepsilon(Y_\varepsilon(t)) &= \left[\varepsilon\left(\frac{\partial}{\partial\varepsilon}Y_\varepsilon\right)(t)+(Y_\varepsilon(t)-x_0)\right]\frac{\partial\bar{g}}{\partial x}(x_0+\varepsilon(Y_\varepsilon(t)-x_0)).\end{aligned}$$

In particular (recall that $Y_0(t)=X_0(t)$),

$$\begin{aligned}\left.\frac{\partial}{\partial\varepsilon}g_\varepsilon(t,X_\varepsilon(t))\right|_{\varepsilon=0} &= (X_0(t)-x_0)\frac{\partial g}{\partial x}(t,x_0),\\ \left.\frac{\partial}{\partial\varepsilon}\bar{g}_\varepsilon(Y_\varepsilon(t))\right|_{\varepsilon=0} &= (X_0(t)-x_0)\frac{\partial\bar{g}}{\partial x}(x_0).\end{aligned}$$

Thus,

$$\begin{aligned}q''(0) &= \int_0^T\mathbf{E}\left[(X_0(t)-x_0)^2u(t)\right]\left(\frac{\partial g}{\partial x}(t,x_0)-\frac{\partial\bar{g}}{\partial x}(x_0)\right)^2dt\\ &= \int_0^T v^2(t)\sigma^2(t)\left(\frac{\partial g}{\partial x}(t,x_0)-\frac{\partial\bar{g}}{\partial x}(x_0)\right)^2dt.\end{aligned}$$

where the definition of $v^2(t)$ (see (6.7)) was used in the last equality. Differentiating with respect to $\frac{\partial\bar{g}}{\partial x}(x_0)$ and setting the resulting derivative to zero we obtain the following necessary condition for the minimum of $q''(0)$,

$$\begin{aligned}\frac{\partial\bar{g}}{\partial x}(x_0) &= \frac{\int_0^T v^2(t)\sigma^2(t)\frac{\partial g}{\partial x}(t,x_0)\,dt}{\int_0^T v^2(t)\sigma^2(t)\,dt}\\ &= \int_0^T w(t)\frac{\partial g}{\partial x}(t,x_0)\,dt,\end{aligned}$$

as stated in the proposition.

Appendix D. Proof of Corollary 6.3

To prove the corollary, we set

$$\begin{aligned}f(t,x) &= \frac{1}{x_0}(\beta(t)x+(1-\beta(t))x_0),\\ x_0 &= S(0),\end{aligned}$$

and apply Theorem 6.1 with the pre-limiting value of the small parameter ε set to 1,

$$\varepsilon=1.$$

Note that

$$f_1(t,x)=f(t,x).$$

The "effective" skew function for the time horizon $[0,T]$ is given by

$$\bar{f}_1(x)=\frac{1}{x_0}\left(\int_0^T w(t)(\beta(t)x+(1-\beta(t))x_0)^2dt\right)^{1/2}.$$

We approximate

$$\bar{f}_1(x)$$

with a linear function that has the same value and the same slope as $\bar{f}_1(x)$ at $x = x_0$. We note that

$$\begin{aligned}
\bar{f}_1(x_0) &= 1, \\
\bar{f}_1'(x_0) &= \frac{1}{x_0} \left. \frac{2\int_0^T w(t)\beta(t)(\beta(t)x + (1-\beta(t))x_0)\,dt}{2\left(\int_0^T w(t)(\beta(t)x + (1-\beta(t))x_0)^2\,dt\right)^{1/2}} \right|_{x=x_0} \\
&= \frac{1}{x_0}\int_0^T w(t)\beta(t)\,dt.
\end{aligned}$$

Hence

$$\begin{aligned}
\bar{f}_1(x) &\approx \frac{1}{x_0}(bx + (1-b)x_0), \\
b &= \int_0^T w(t)\beta(t)\,dt,
\end{aligned}$$

as asserted by the corollary.

To compute $v^2(t)$, we recall its definition from Theorem 6.1,

$$\begin{aligned}
v^2(t) &= \mathbf{E}\left((X_0(t) - x_0)^2 z(t)\right), \\
dz(t) &= \theta(z_0 - z(t))\ dt + \eta\sqrt{z(t)}\,dW_0(t), \\
dX_0(t) &= \sqrt{z(t)}\sigma(t)\ dW(t), \\
X_0(0) &= x_0, \\
\langle dW_0, dW\rangle &= 0.
\end{aligned}$$

Conditioning on $z(t)$ and using conditional independence of X_0 and z we obtain,

$$\begin{aligned}
\mathbf{E}\left[(X_0(t) - x_0)^2 z(t)\right] &= \mathbf{E}\left[z(t)\mathbf{E}\left[(X_0(t) - x_0)^2\middle| z(\cdot)\right]\right] \\
&= \mathbf{E}\left[z(t)\int_0^t z(s)\sigma^2(s)\ ds\right] \\
&= \int_0^t \sigma^2(s)\mathbf{E}\left[z(t)z(s)\right]\ ds.
\end{aligned} \tag{D.1}$$

Clearly

$$z(t) - z_0 = e^{-\theta(t-s)}(z(s) - z_0) + noise,$$

so that

$$\begin{aligned}
\mathbf{E}[z(t)z(s)] &= z_0^2 + \mathbf{E}\left[e^{-\theta(t-s)}(z(s) - z_0)z(s)\right] \\
&= z_0^2 - e^{-\theta(t-s)}z_0^2 + e^{-\theta(t-s)}\mathbf{E}\left[z^2(s)\right] \\
&= e^{-\theta(t-s)}\mathbf{E}\left[z^2(s)\right] + z_0^2\left(1 - e^{-\theta(t-s)}\right).
\end{aligned}$$

Also we have from the equation on dz, that

$$dz^2(t) = 2z(t)\left[\theta(z_0 - z(t))\ dt + \eta\sqrt{z(t)}\,dW_0(t)\right] + \eta^2 z(t)\ dt.$$

Denoting

$$v_2(t) = \mathbf{E}\left[z^2(t)\right]$$

and taking expectations of both sides of the SDE for dz^2, using the fact that

$$\mathbf{E}[z(t)] = z_0,$$

we obtain an ODE for v_2,

$$\begin{aligned} v_2'(t) &= 2\theta z_0^2 - 2\theta v_2(t) + \eta^2 z_0, \\ v_2(0) &= z_0^2. \end{aligned}$$

This equation can be easily solved to obtain,

$$\begin{aligned} v_2(t) - v_2(0) &= z_0\varepsilon^2 \int_0^t e^{-2\theta(t-u)} du \\ &= z_0\varepsilon^2 \frac{1-e^{-2\theta t}}{2\theta}. \end{aligned}$$

Plugging it back into (D.1) yields

$$\begin{aligned} v^2(t) &= \mathbf{E}\left[(S(t)-x_0)^2 z(t)\right] = \int_0^t \sigma^2(s)\left[e^{-\theta(t-s)}v_2(s) + z_0^2\left(1-e^{-\theta(t-s)}\right)\right] ds \\ &= \int_0^t \sigma^2(s)\left(e^{-\theta(t-s)}z_0^2 + e^{-\theta(t-s)}z_0^2\eta^2\frac{1-e^{-2\theta s}}{2\theta} + z_0^2\left(1-e^{-\theta(t-s)}\right)\right) ds \\ &= \int_0^t \sigma^2(s)\left(z_0^2 + z_0^2\eta^2 e^{-\theta(t-s)}\frac{1-e^{-2\theta s}}{2\theta}\right) ds \\ &= z_0^2\int_0^t \sigma^2(s)\, ds + z_0\eta^2 e^{-\theta t}\int_0^t \sigma^2(s)\,\frac{e^{\theta s}-e^{-\theta s}}{2\theta}\, ds. \end{aligned}$$

Appendix E. Laplace transform of integrated stochastic variance

Recall the definitions

$$\begin{aligned} Z(T) &= \int_0^T \sigma^2(t)\, z(t)\, dt, \\ \varphi(\mu) &\triangleq \mathbf{E}\exp(-\mu Z(T)), \\ \varphi_0(\mu) &\triangleq \mathbf{E}\exp\left(-\mu\int_0^T z(t)\, dt\right), \end{aligned}$$

where the stochastic variance process $z(\cdot)$ follows (4.1). As explained in [AA02], the function $\varphi(\mu)$ can be represented as

$$\varphi(\mu) = \exp\left(A(0,T) - z_0 B(0,T)\right),$$

where the functions $A(t,T)$, $B(t,T)$ satisfy the Riccati system of ODEs

$$A'(t,T) - \theta z_0 B(t,T) = 0, \tag{E.1}$$

$$B'(t,T) - \theta B(t,T) - \frac{1}{2}\eta^2 B^2(t,T) + \mu\sigma^2(t) = 0, \tag{E.2}$$

with terminal conditions

$$\begin{aligned} A(T,T) &= 0, \\ B(T,T) &= 0. \end{aligned}$$

The system of ODEs is trivial to solve numerically.

The function $\varphi_0(\mu)$ satisfies the same system of equations with $\sigma(t) \equiv 1$. In this case, the equations can be solved explicitly, to yield

$$\begin{aligned} \varphi_0(\mu) &= \exp\left(A_0(0,T) - z_0 B_0(0,T)\right), \\ B_0(0,T) &= \frac{2\mu\left(1-e^{-\gamma T}\right)}{(\theta+\gamma)\left(1-e^{-\gamma T}\right)+2\gamma e^{-\gamma T}}, \\ A_0(0,T) &= \frac{2\theta z_0}{\eta^2}\log\left(\frac{2\gamma}{\theta+\gamma\left(1-e^{-\gamma T}\right)+2\gamma e^{-\gamma T}}\right) - 2\theta z_0 \frac{\mu}{\theta+\gamma} T, \\ \gamma &= \sqrt{\theta^2+2\eta^2\mu}. \end{aligned} \tag{E.3}$$

TABLE 1. A swaption grid of market skews. Rows are different swaption expiries, and columns are underlying swap maturities.

	1Y	**5Y**	**10Y**	**15Y**	**20Y**	**25y**	**30y**
1Y	37.6%	41.6%	46.6%	51.6%	56.6%	61.6%	66.6%
5Y	24.8%	28.8%	33.8%	38.8%	43.8%	48.8%	53.8%
10Y	11.0%	15.0%	20.0%	25.0%	30.0%	35.0%	40.0%
15y	-0.3%	3.8%	8.8%	13.8%	18.8%	23.8%	
20y	-9.0%	-5.0%	0.0%	5.0%	10.0%		
25y	-15.3%	-11.3%	-6.3%	-1.3%			
30y	-19.0%	-15.0%	-10.0%				

TABLE 2. Instanteneous forward Libor skews for the FL-TSS model, as calibrated from the table of market skews in Table 1. Values given for the knot points are defined by the first row/first column; bilinear interpolation used elsewhere. Rows are times t in years, columns are offsets from t in years. See page 18 for more details.

	$\mathcal{T}=1$	$\mathcal{T}=5$	$\mathcal{T}=10$	$\mathcal{T}=15$	$\mathcal{T}=20$	$\mathcal{T}=30$
$\mathbf{t=0.5}$	36.03%	44.69%	55.83%	69.58%	93.40%	100.0%
$\mathbf{t=4.5}$	13.29%	19.66%	30.50%	45.22%	70.37%	93.8%
$\mathbf{t=9.5}$	-10.36%	-6.69%	-3.45%	6.56%	30.89%	76.5%
$\mathbf{t=14.5}$	-26.96%	-32.54%	-40.53%	-28.91%	0.42%	
$\mathbf{t=19.5}$	-36.58%	-50.28%	-68.77%	-60.96%	-29.86%	
$\mathbf{t=24.5}$	-41.53%	-49.06%	-72.91%	-69.45%		
$\mathbf{t=29.5}$	-39.09%	-53.38%	-72.99%			

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TABLE 3. Test results, in implied Black volatilities, for swaptions with strike offset from at-the-money of -2.00%. See page 19 for more details.

Expiry/Maturity	**1y1y**	**1y10y**	**1y20y**	**5y1y**	**5y10y**	**5y20y**
Simple	21.50%	21.15%	20.76%	19.46%	19.15%	18.80%
FL-TSS (MC)	21.45%	21.33%	20.90%	19.42%	19.14%	18.65%
FL-TSS (Approx)	21.47%	21.14%	20.74%	19.45%	19.16%	18.76%
FL-SV (MC)	21.99%	22.13%	22.11%	19.55%	19.52%	19.41%
FL-SV (Approx)	22.12%	22.13%	22.14%	19.62%	19.61%	19.61%
FL-TSS std error	0.92%	0.94%	0.98%	0.11%	0.10%	0.09%
FL-SV std error	0.92%	0.94%	0.98%	0.11%	0.10%	0.09%
FL-TSS (MC) - Simple	-0.05%	0.18%	0.14%	-0.04%	-0.01%	-0.15%
FL-SV (MC) - Simple	0.49%	0.98%	1.34%	0.09%	0.37%	0.60%
FL-TSS (Approx) - Simple	-0.03%	-0.01%	-0.02%	0.00%	0.02%	-0.04%
FL-SV (Approx) - Simple	0.62%	0.98%	1.38%	0.16%	0.47%	0.80%
FL-TSS (MC) - FL-TSS (Appr)	-0.03%	0.19%	0.16%	-0.03%	-0.03%	-0.11%
FL-SV (MC) -FL- SV (Appr)	-0.13%	0.00%	-0.04%	-0.07%	-0.10%	-0.20%

Expiry/Maturity	**15y1y**	**15y5y**	**15y20y**	**25y1y**	**25y5y**
Simple	18.26%	18.10%	17.59%	18.61%	18.47%
FL-TSS (MC)	17.87%	17.80%	16.85%	18.07%	17.81%
FL-TSS (Approx)	18.30%	18.11%	17.55%	18.68%	18.48%
FL-SV (MC)	17.37%	17.37%	17.04%	17.26%	17.11%
FL-SV (Approx)	17.59%	17.54%	17.60%	17.41%	17.32%
FL-TSS std error	0.05%	0.04%	0.04%	0.04%	0.04%
FL-SV std error	0.05%	0.05%	0.04%	0.04%	0.04%
FL-TSS (MC) - Simple	-0.39%	-0.30%	-0.74%	-0.54%	-0.67%
FL-SV (MC) - Simple	-0.89%	-0.73%	-0.55%	-1.35%	-1.36%
FL-TSS (Approx) - Simple	0.04%	0.01%	-0.04%	0.07%	0.01%
FL-SV (Approx) - Simple	-0.67%	-0.56%	0.01%	-1.20%	-1.15%
FL-TSS (MC) - FL-TSS (Appr)	-0.43%	-0.32%	-0.70%	-0.60%	-0.68%
FL-SV (MC) -FL- SV (Appr)	-0.22%	-0.17%	-0.56%	-0.16%	-0.21%

TABLE 4. Test results, in implied Black volatilities, for swaptions with strike offset from at-the-money of -1.00%. See page 19 for more details.

Expiry/Maturity	**1y1y**	**1y10y**	**1y20y**	**5y1y**	**5y10y**	**5y20y**
Simple	17.02%	16.87%	16.71%	15.27%	15.13%	14.97%
FL-TSS (MC)	16.94%	16.78%	16.63%	15.26%	15.12%	14.89%
FL-TSS (Approx)	17.00%	16.86%	16.70%	15.27%	15.14%	14.95%
FL-SV (MC)	17.18%	17.16%	17.17%	15.34%	15.32%	15.24%
FL-SV (Approx)	17.28%	17.29%	17.30%	15.34%	15.34%	15.34%
FL-TSS std error	0.13%	0.13%	0.13%	0.06%	0.06%	0.05%
FL-SV std error	0.13%	0.13%	0.13%	0.06%	0.06%	0.05%
FL-TSS (MC) - Simple	-0.09%	-0.09%	-0.08%	-0.01%	0.00%	-0.08%
FL-SV (MC) - Simple	0.16%	0.29%	0.46%	0.08%	0.20%	0.27%
FL-TSS (Approx) - Simple	-0.02%	-0.01%	-0.01%	0.00%	0.01%	-0.02%
FL-SV (Approx) - Simple	0.26%	0.42%	0.60%	0.07%	0.22%	0.37%
FL-TSS (MC) - FL-TSS (Appr)	-0.06%	-0.08%	-0.07%	-0.01%	-0.01%	-0.06%
FL-SV (MC) -FL- SV (Appr)	-0.10%	-0.13%	-0.13%	0.00%	-0.02%	-0.10%

Expiry/Maturity	**15y1y**	**15y5y**	**15y20y**	**25y1y**	**25y5y**
Simple	14.89%	14.82%	14.58%	15.36%	15.30%
FL-TSS (MC)	14.68%	14.65%	14.16%	15.13%	14.96%
FL-TSS (Approx)	14.93%	14.82%	14.58%	15.42%	15.29%
FL-SV (MC)	14.48%	14.48%	14.23%	14.79%	14.67%
FL-SV (Approx)	14.59%	14.55%	14.60%	14.84%	14.76%
FL-TSS std error	0.04%	0.03%	0.03%	0.03%	0.03%
FL-SV std error	0.04%	0.04%	0.03%	0.03%	0.03%
FL-TSS (MC) - Simple	-0.21%	-0.16%	-0.42%	-0.22%	-0.34%
FL-SV (MC) - Simple	-0.41%	-0.34%	-0.35%	-0.56%	-0.62%
FL-TSS (Approx) - Simple	0.04%	0.00%	0.01%	0.06%	0.00%
FL-SV (Approx) - Simple	-0.30%	-0.27%	0.03%	-0.51%	-0.53%
FL-TSS (MC) - FL-TSS (Appr)	-0.25%	-0.17%	-0.42%	-0.28%	-0.34%
FL-SV (MC) -FL- SV (Appr)	-0.12%	-0.07%	-0.38%	-0.05%	-0.09%

TABLE 5. Test results, in implied Black volatilities, for at-the-money . See page 19 for more details.

Expiry/Maturity	**1y1y**	**1y10y**	**1y20y**	**5y1y**	**5y10y**	**5y20y**
Simple	14.17%	14.16%	14.15%	12.91%	12.90%	12.88%
FL-TSS (MC)	14.31%	14.29%	14.29%	13.00%	12.97%	12.91%
FL-TSS (Approx)	14.14%	14.15%	14.15%	12.91%	12.90%	12.88%
FL-SV (MC)	14.32%	14.31%	14.31%	13.00%	12.98%	12.90%
FL-SV (Approx)	14.16%	14.17%	14.19%	12.92%	12.92%	12.92%
FL-TSS std error	0.04%	0.03%	0.03%	0.04%	0.03%	0.03%
FL-SV std error	0.03%	0.03%	0.03%	0.04%	0.03%	0.03%
FL-TSS (MC) - Simple	0.14%	0.13%	0.14%	0.09%	0.07%	0.03%
FL-SV (MC) - Simple	0.15%	0.15%	0.17%	0.10%	0.08%	0.02%
FL-TSS (Approx) - Simple	-0.02%	-0.01%	0.00%	0.00%	0.01%	0.00%
FL-SV (Approx) - Simple	0.00%	0.02%	0.04%	0.01%	0.02%	0.04%
FL-TSS (MC) - FL-TSS (Appr)	0.16%	0.14%	0.14%	0.08%	0.07%	0.03%
FL-SV (MC) -FL- SV (Appr)	0.16%	0.14%	0.13%	0.09%	0.06%	-0.01%
Expiry/Maturity	**15y1y**	**15y5y**	**15y20y**	**25y1y**	**25y5y**	
Simple	12.92%	12.91%	12.89%	13.31%	13.31%	
FL-TSS (MC)	12.89%	12.90%	12.70%	13.34%	13.23%	
FL-TSS (Approx)	12.95%	12.91%	12.93%	13.36%	13.30%	
FL-SV (MC)	12.89%	12.88%	12.64%	13.33%	13.22%	
FL-SV (Approx)	12.92%	12.88%	12.93%	13.32%	13.25%	
FL-TSS std error	0.03%	0.03%	0.02%	0.03%	0.03%	
FL-SV std error	0.03%	0.03%	0.02%	0.03%	0.03%	
FL-TSS (MC) - Simple	-0.03%	-0.02%	-0.19%	0.03%	-0.08%	
FL-SV (MC) - Simple	-0.03%	-0.04%	-0.25%	0.02%	-0.09%	
FL-TSS (Approx) - Simple	0.03%	0.00%	0.04%	0.06%	-0.01%	
FL-SV (Approx) - Simple	0.00%	-0.03%	0.04%	0.02%	-0.06%	
FL-TSS (MC) - FL-TSS (Appr)	-0.06%	-0.01%	-0.23%	-0.02%	-0.07%	
FL-SV (MC) -FL- SV (Appr)	-0.03%	-0.01%	-0.29%	0.00%	-0.03%	

TABLE 6. Test results, in implied Black volatilities, for swaptions with strike offset from at-the-money of 1.00%. See page 19 for more details.

Expiry/Maturity	**1y1y**	**1y10y**	**1y20y**	**5y1y**	**5y10y**	**5y20y**
Simple	14.60%	14.69%	14.80%	12.84%	12.93%	13.02%
FL-TSS (MC)	14.50%	14.57%	14.67%	12.91%	12.97%	13.00%
FL-TSS (Approx)	14.58%	14.68%	14.81%	12.85%	12.93%	13.03%
FL-SV (MC)	14.33%	14.31%	14.30%	12.85%	12.83%	12.73%
FL-SV (Approx)	14.38%	14.39%	14.40%	12.80%	12.80%	12.79%
FL-TSS std error	0.03%	0.03%	0.03%	0.03%	0.03%	0.03%
FL-SV std error	0.03%	0.03%	0.02%	0.03%	0.03%	0.03%
FL-TSS (MC) - Simple	-0.10%	-0.13%	-0.13%	0.07%	0.04%	-0.02%
FL-SV (MC) - Simple	-0.26%	-0.38%	-0.50%	0.01%	-0.10%	-0.29%
FL-TSS (Approx) - Simple	-0.02%	-0.01%	0.01%	0.00%	0.00%	0.01%
FL-SV (Approx) - Simple	-0.21%	-0.30%	-0.40%	-0.04%	-0.13%	-0.22%
FL-TSS (MC) - FL-TSS (Appr)	-0.08%	-0.12%	-0.14%	0.07%	0.04%	-0.03%
FL-SV (MC) -FL- SV (Appr)	-0.05%	-0.08%	-0.10%	0.06%	0.03%	-0.07%
Expiry/Maturity	**15y1y**	**15y5y**	**15y20y**	**25y1y**	**25y5y**	
Simple	12.04%	12.09%	12.24%	12.07%	12.12%	
FL-TSS (MC)	12.15%	12.16%	12.14%	12.29%	12.23%	
FL-TSS (Approx)	12.07%	12.08%	12.31%	12.12%	12.10%	
FL-SV (MC)	12.29%	12.25%	11.98%	12.52%	12.42%	
FL-SV (Approx)	12.28%	12.25%	12.30%	12.51%	12.45%	
FL-TSS std error	0.03%	0.03%	0.02%	0.02%	0.02%	
FL-SV std error	0.03%	0.03%	0.02%	0.03%	0.03%	
FL-TSS (MC) - Simple	0.10%	0.07%	-0.10%	0.22%	0.11%	
FL-SV (MC) - Simple	0.25%	0.16%	-0.27%	0.45%	0.30%	
FL-TSS (Approx) - Simple	0.03%	-0.01%	0.07%	0.05%	-0.02%	
FL-SV (Approx) - Simple	0.24%	0.16%	0.05%	0.44%	0.32%	
FL-TSS (MC) - FL-TSS (Appr)	0.07%	0.08%	-0.17%	0.17%	0.13%	
FL-SV (MC) -FL- SV (Appr)	0.01%	0.00%	-0.32%	0.01%	-0.02%	

TABLE 7. Test results, in implied Black volatilities, for swaptions with strike offset from at-the-money of 2.00%. See page 19 for more details.

Expiry/Maturity	**1y1y**	**1y10y**	**1y20y**	**5y1y**	**5y10y**	**5y20y**
Simple	15.71%	15.89%	16.10%	13.62%	13.78%	13.95%
FL-TSS (MC)	15.45%	15.59%	15.75%	13.65%	13.77%	13.85%
FL-TSS (Approx)	15.69%	15.88%	16.11%	13.62%	13.78%	13.97%
FL-SV (MC)	15.14%	15.12%	15.09%	13.57%	13.54%	13.42%
FL-SV (Approx)	15.33%	15.34%	15.35%	13.53%	13.54%	13.53%
FL-TSS std error	0.04%	0.04%	0.03%	0.04%	0.03%	0.03%
FL-SV std error	0.04%	0.03%	0.03%	0.04%	0.03%	0.03%
FL-TSS (MC) - Simple	-0.26%	-0.30%	-0.35%	0.04%	0.00%	-0.10%
FL-SV (MC) - Simple	-0.57%	-0.77%	-1.00%	-0.05%	-0.24%	-0.54%
FL-TSS (Approx) - Simple	-0.02%	-0.01%	0.01%	0.01%	0.00%	0.02%
FL-SV (Approx) - Simple	-0.38%	-0.56%	-0.75%	-0.08%	-0.24%	-0.42%
FL-TSS (MC) - FL-TSS (Appr)	-0.24%	-0.29%	-0.36%	0.03%	0.00%	-0.12%
FL-SV (MC) -FL- SV (Appr)	-0.18%	-0.21%	-0.25%	0.04%	0.00%	-0.12%
Expiry/Maturity	**15y1y**	**15y5y**	**15y20y**	**25y1y**	**25y5y**	
Simple	11.81%	11.91%	12.19%	11.37%	11.46%	
FL-TSS (MC)	11.98%	12.01%	12.08%	11.72%	11.71%	
FL-TSS (Approx)	11.84%	11.89%	12.27%	11.41%	11.42%	
FL-SV (MC)	12.25%	12.19%	11.86%	12.12%	12.03%	
FL-SV (Approx)	12.24%	12.20%	12.25%	12.14%	12.08%	
FL-TSS std error	0.03%	0.03%	0.02%	0.02%	0.02%	
FL-SV std error	0.03%	0.03%	0.02%	0.03%	0.03%	
FL-TSS (MC) - Simple	0.17%	0.10%	-0.11%	0.36%	0.25%	
FL-SV (MC) - Simple	0.44%	0.28%	-0.33%	0.76%	0.58%	
FL-TSS (Approx) - Simple	0.03%	-0.01%	0.09%	0.05%	-0.03%	
FL-SV (Approx) - Simple	0.42%	0.30%	0.06%	0.77%	0.62%	
FL-TSS (MC) - FL-TSS (Appr)	0.14%	0.12%	-0.20%	0.31%	0.28%	
FL-SV (MC) -FL- SV (Appr)	0.02%	-0.02%	-0.39%	-0.02%	-0.05%	

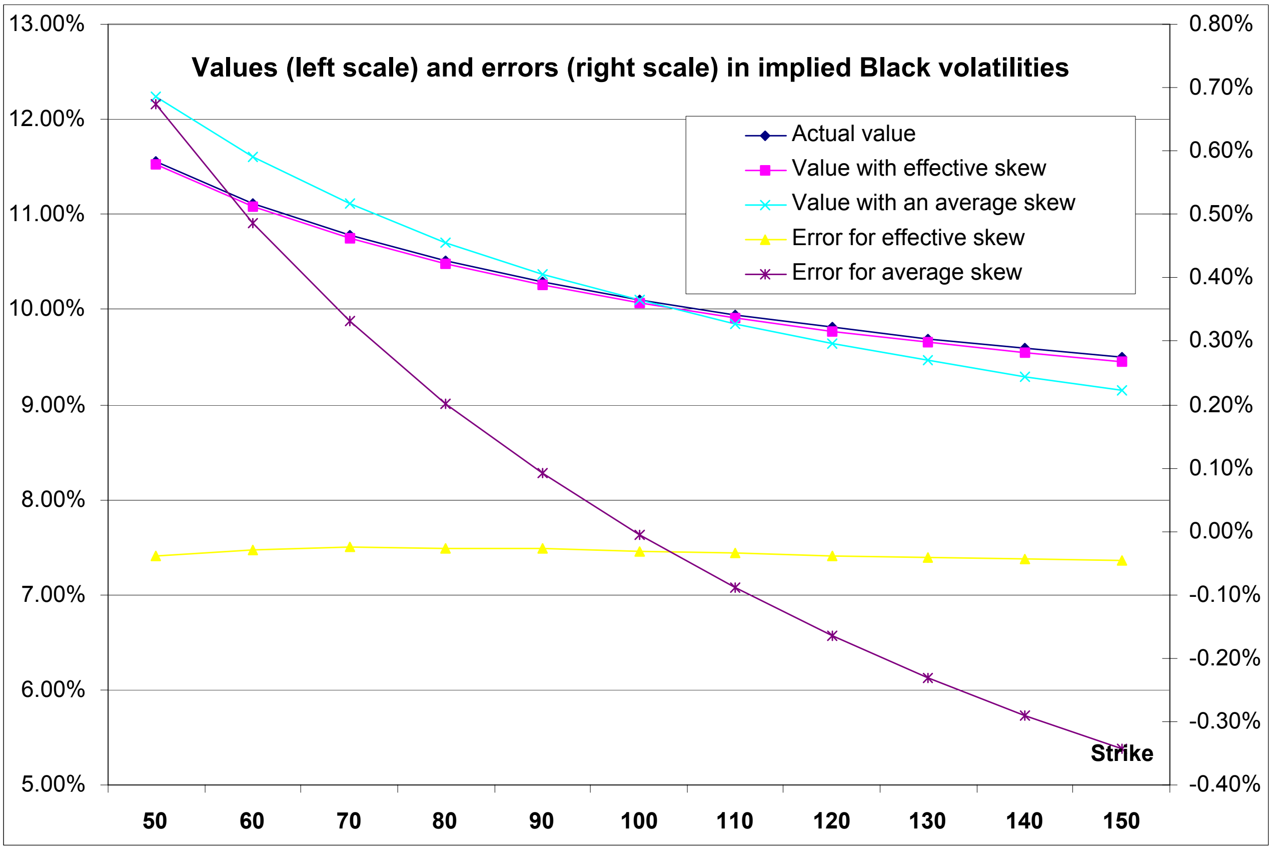

FIGURE 1. Test results for the effective skew formula in Section 6.2. European options with 30 years to expiry and with different strikes (x-axis) are valued in the model with time-dependent skew (the model skew grows linearly from 0.0 to 1.0 over 30 years). Actual value: computed using a PDE. Effective skew value: computed using the formula in Section 6.2. Average skew value: computed using a simple average of time-dependent skews. The three values are plotted on the left scale, differences between the effective and the simple average skew approximations and the actual values – on the right scale. All values in implied Black volatilitie.

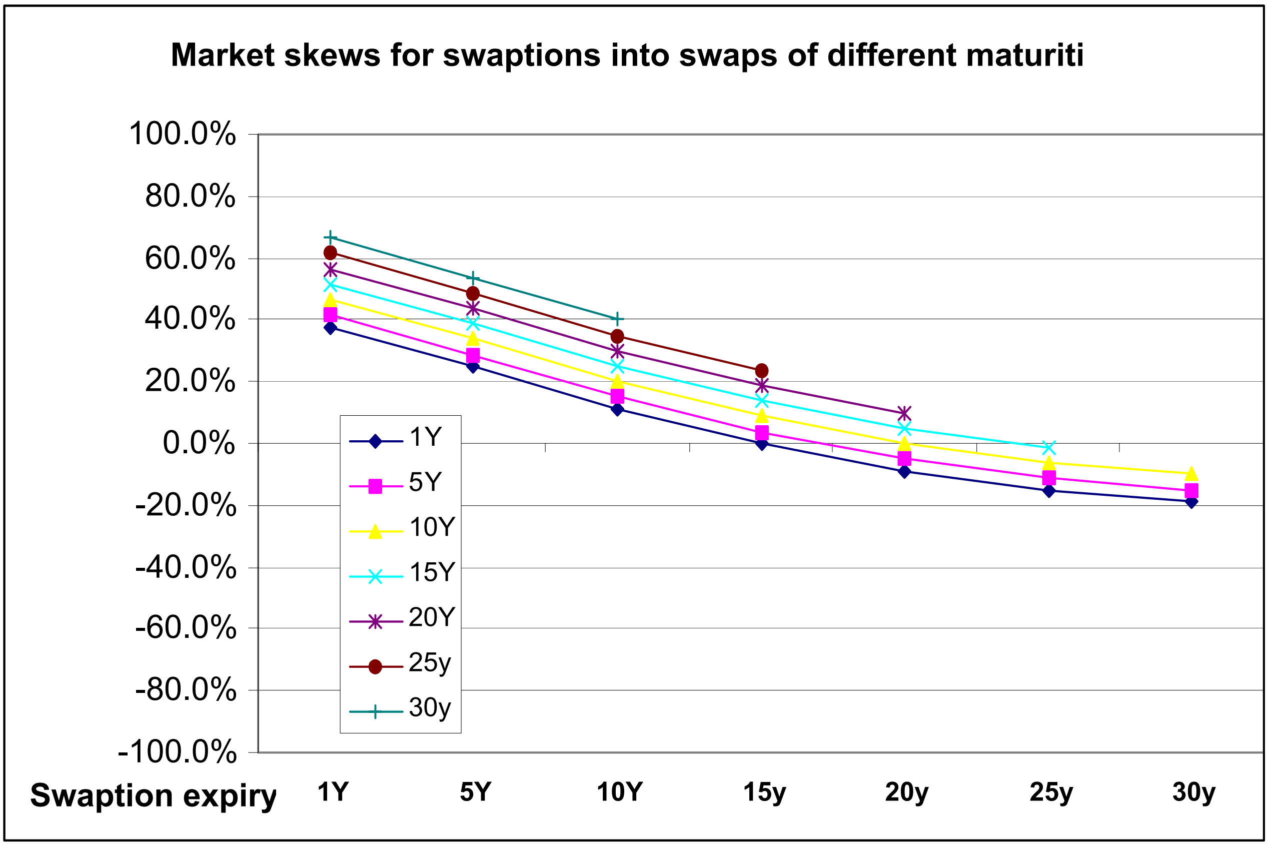

FIGURE 2. Results from Table 1 are plotted here. See the table for explanation.

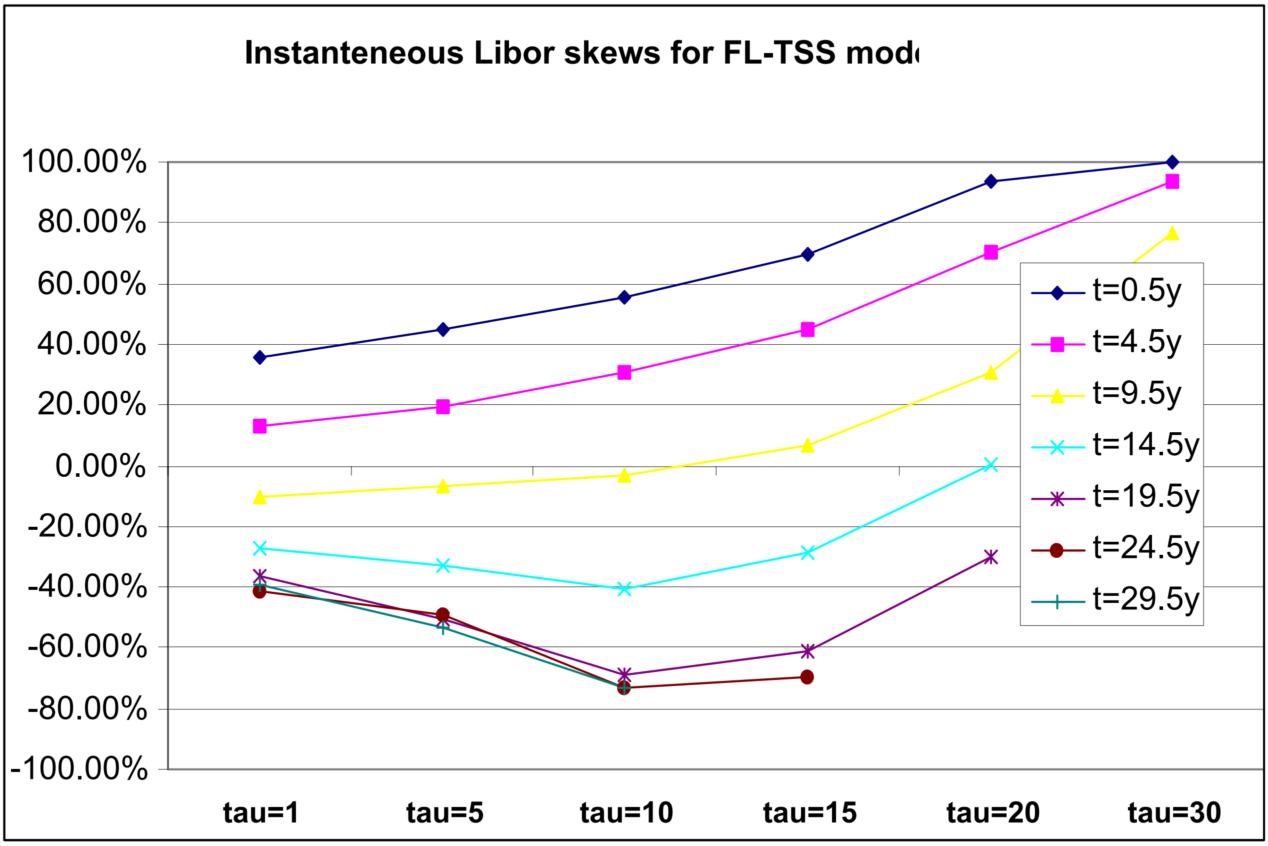

FIGURE 3. Results from Table 2 are plotted here. See the table for explanation.

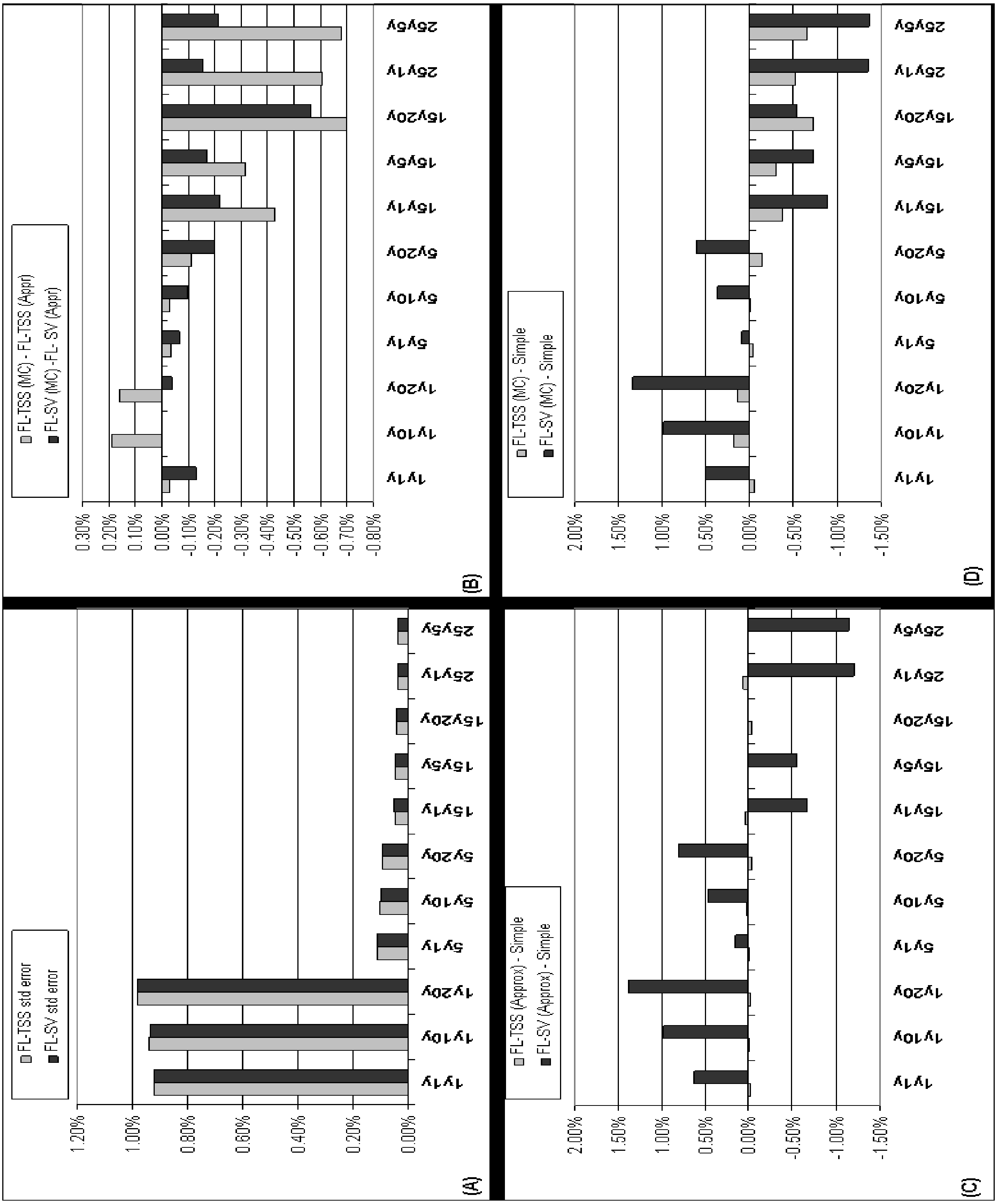

Figure 4. Test results, in implied Black volatilities, for swaptions with strike offset from at-the-money of -2.00%. See page 19 for more details.

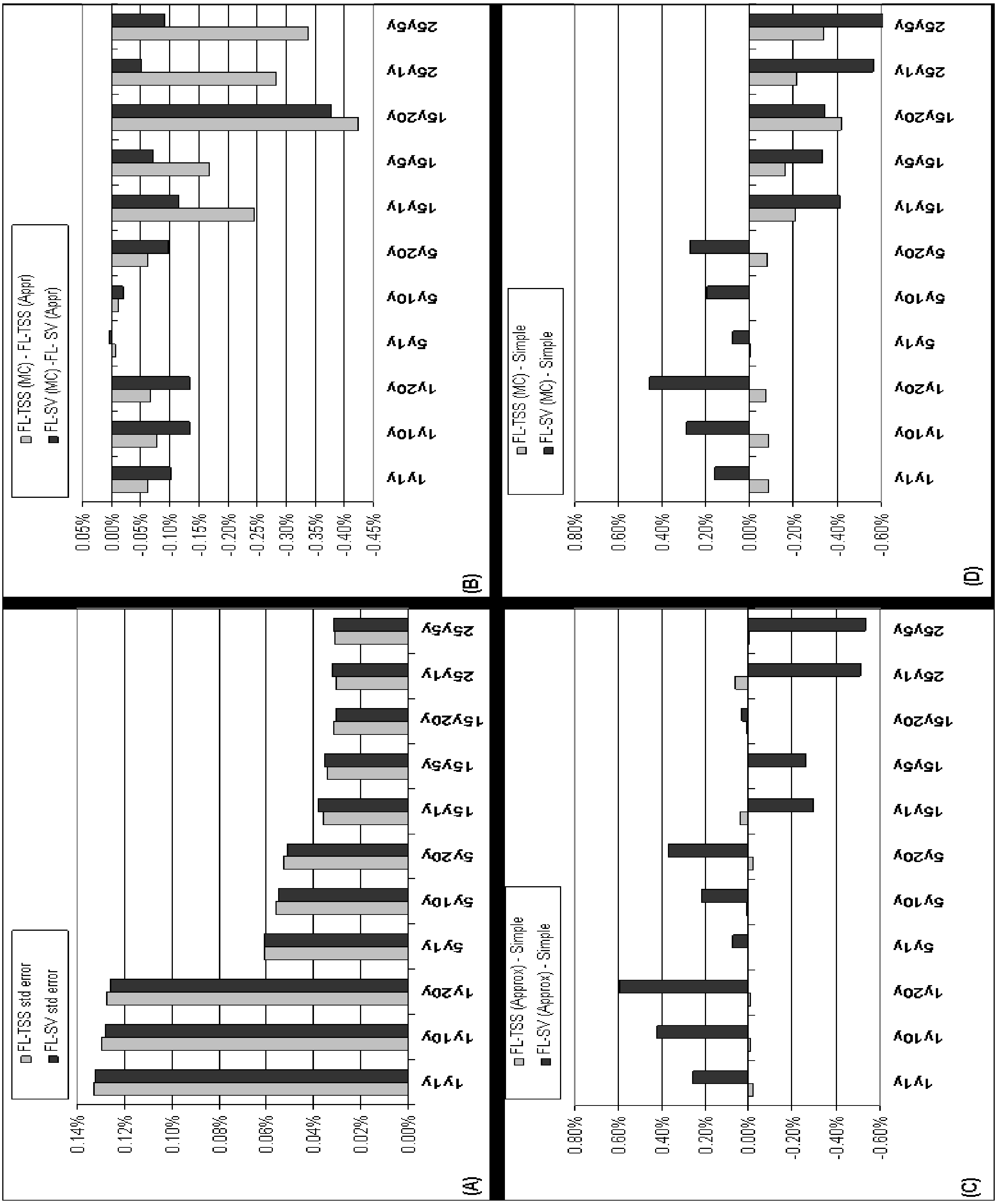

FIGURE 5. Test results, in implied Black volatilities, for swaptions with strike offset from at-the-money of -1.00%. See page 19 for more details.

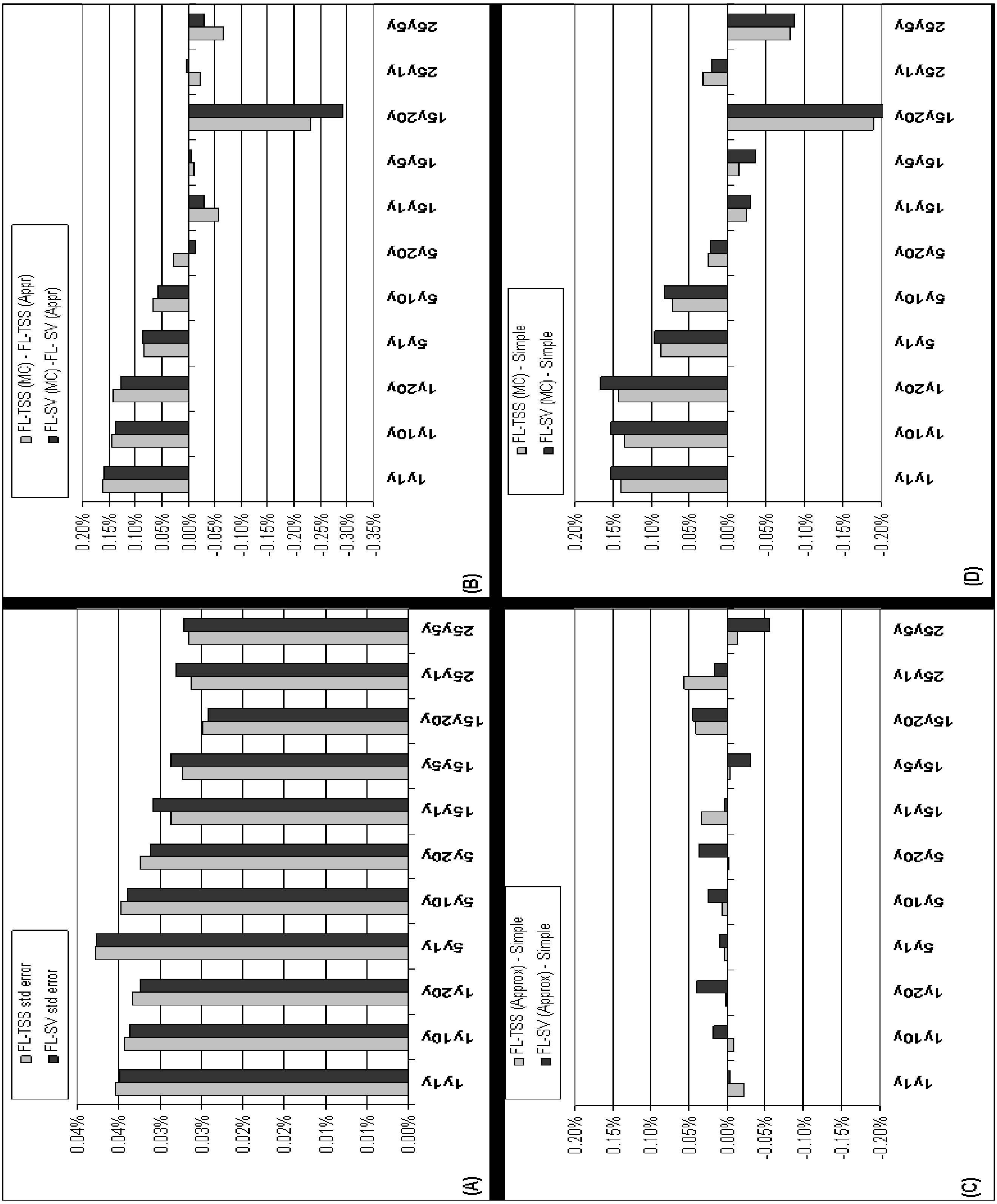

Figure 6. Test results, in implied Black volatilities, for at-the-money swaptions. See page 19 for more details.

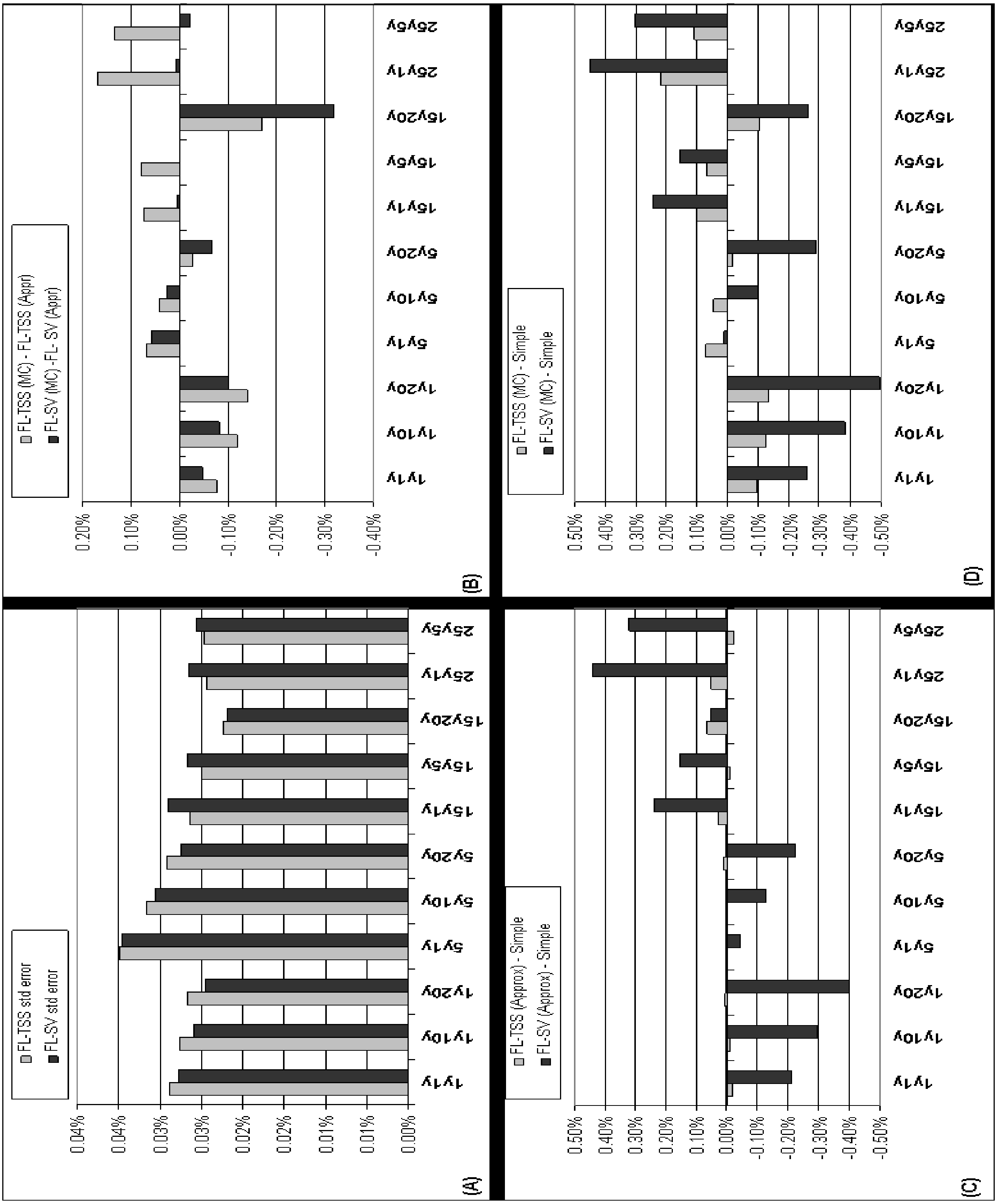

FIGURE 7. Test results, in implied Black volatilities, for swaptions with strike offset from at-the-money of 1.00%. See page 19 for more details.

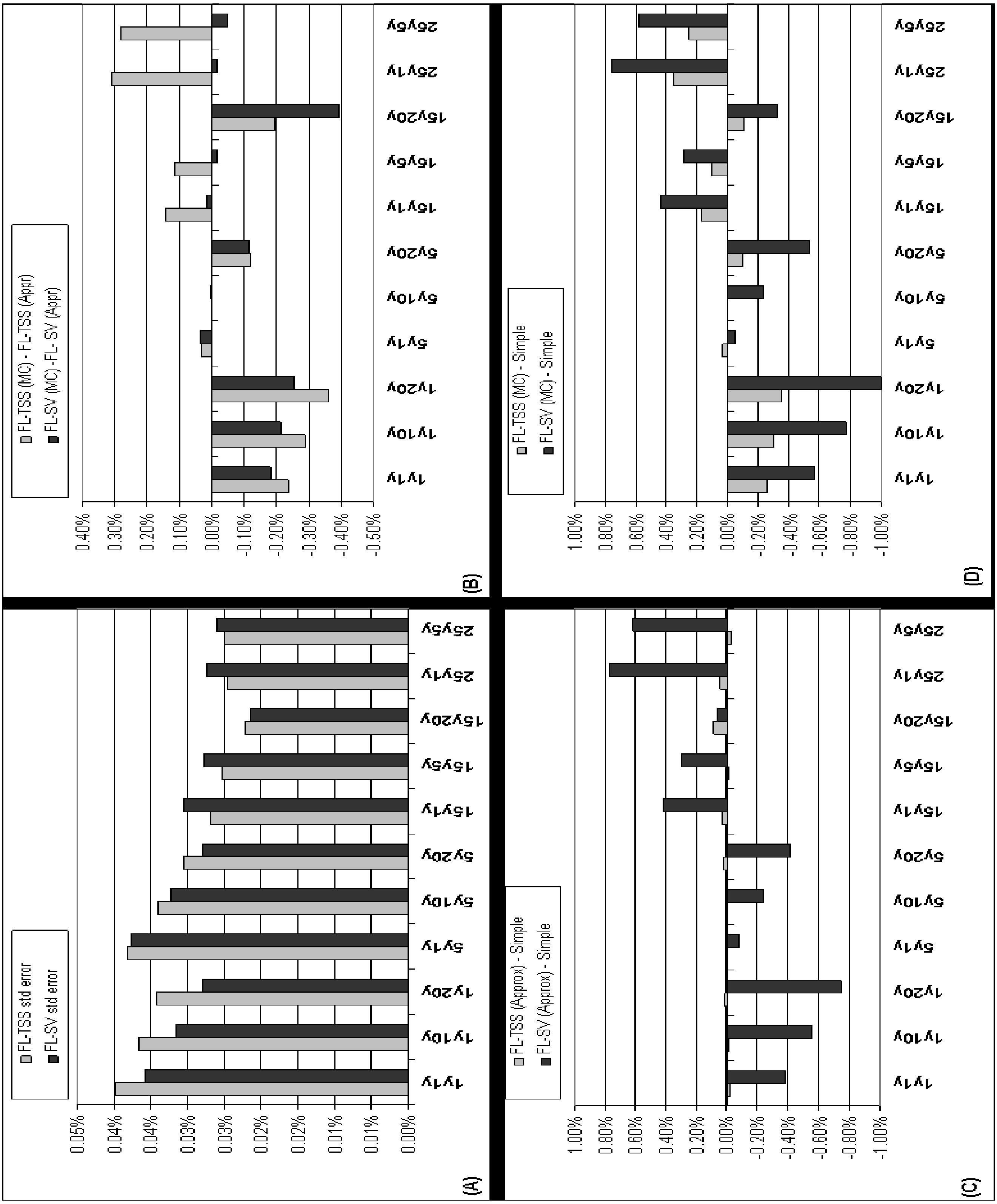

FIGURE 8. Test results, in implied Black volatilities, for swaptions with strike offset from at-the-money of 2.00%. See page 19 for more details.