ARBITRAGES IN THE VOLATILITY SURFACE INTERPOLATION AND EXTRAPOLATION

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Abstract. The implied volatility surface is built from a discrete set of vanilla option quotes. To move from a discrete set to a continuous surface, interpolation and extrapolation are therefore needed in the expiry dimension as well as in the strike dimension. This paper will study the interpolation and extrapolation in the time-to-expiry dimension. In particular, we show here that a linear interpolation in variance, a very common way of interpolation, of two arbitrage-free slices of increasing variance can lead to an arbitrage in an interpolated slice. We also show that flat volatility extrapolation, another common usage, will also lead to arbitrages. To conclude, we present an alternative simple extrapolation.

1. Introduction

1.1. **The volatility surface.** There are many ways to build a volatility surface. They can be categorized as following:

- global parameterization in strike and time-to-expiry dimensions: this corresponds for example to surfaces built from a stochastic volatility model like Heston [Heston, 1993], as well as parameterizations where the solving of the parameters is done on the full surface where usually arbitrage free constraints are included like [Fengler and Hin, 2011].
- slice by slice parameterization: this is when slices are built expiry by expiry. The full surface is then built by interpolating those slices in the time to expiry dimension, usually along constant strike, or constant moneyness, or constant delta. Slices can be built from a stochastic volatility model, for example SABR [Hagan et al., 2002], or from a direct parameterization of the implied volatility, for example a spline [Kahalé, 2004], a quadratic [Dumas et al., 1998], or SVI [Gatheral and Lynch, 2004].

This work studies the interpolation in the time-to-expiry dimension for cases where the volatility surface is built slice by slice.

1.2. **Arbitrages.** Carr has shown that static arbitrage is avoided in a set of option prices if calendar spread and butterfly spread arbitrages are avoided [Carr and Madan, 2005].

In terms of undiscounted option prices, the calendar spread no arbitrage condition is just:

$$\frac{\partial C_0}{\partial t}(K,t) \geq 0 \tag{1}$$

where $C_0(K,t)$ is an undiscounted call option price of strike K and maturity t. This can be easily translated to standard option prices [Reiner, 2004, Fengler, 2009]:

$$\frac{\partial C}{\partial t} + (r-q)K\frac{\partial C}{\partial K} + qC(K,t) \geq 0 \tag{2}$$

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where $C(K,t)$ is a call option price of strike K and maturity t, r represents the risk free interest rate and q a dividend rate.

Butterfly spread arbitrage is avoided if (one just has to take the Butterfly spread value to the limit $\Delta K \to 0$):

$$\frac{\partial^2 C}{\partial K^2}(K,t) \geq 0 \tag{3}$$

Those can also be expressed directly in terms of variance. This stems from Gatheral derivation [Gatheral and Lynch, 2004] of local volatility $\sigma^{\star 2}(y,t)$ in terms of variance $w(y,t)$ where y is the forward moneyness $y = \log(\frac{K}{F(t)})$ with $F(t)$ the forward:

$$\sigma^{\star 2}(y,t) = \frac{\frac{\partial w}{\partial t}}{1 - \frac{y}{w}\frac{\partial w}{\partial y} + \frac{1}{4}\left(-\frac{1}{4} - \frac{1}{w} + \frac{y^2}{w^2}\right)\left(\frac{\partial w}{\partial y}\right)^2 + \frac{1}{2}\frac{\partial^2 w}{\partial y^2}} \tag{4}$$

Remember that the Dupire local volatility formula is [Dupire, 1994]:

$$\sigma^{\star 2}(K,t) = \frac{1}{2}\frac{\frac{\partial C_0}{\partial t}}{K^2 \frac{\partial^2 C_0}{\partial K^2}} \tag{5}$$

Calendar spread arbitrage is avoided if the numerator is positive, and Butterfly spread arbitrage is avoided if the denominator is positive. The relations can also be derived directly, without the local volatility analogy as in [Carr, 2004, Roper, 2010].

2. Linear interpolation in variance

It is well known that the surface resulting from linear interpolation in variance along the forward moneyness axis will be arbitrage free in terms of calendar spreads as long as the final variance is higher than the initial variance. This stems from the fact that the variance is monotonically increasing. In equation (4), the numerator will stay positive. But does it stay arbitrage free in terms of butterfly spreads?

2.1. **Arbitrage with linear interpolation.**

For a given moneyness y, let's call $w_0(y)$ the variance of the initial slice for the expiry $t = t_0$ and $w_1(y)$ the variance of the final slice for the expiry $t = t_1$. We assume that $w_0(y) \leq w_1(y)$. The linear interpolated variance along constant moneyness is $w(y,t) = \frac{t_1 - t}{t_1 - t_0}w_0(y) + \frac{t - t_0}{t_1 - t_0}w_1(y)$ for $t_0 \leq t \leq t_1$. To simplify the notation, we will make $t_0 = 0$ and $t_1 = 1$, we then have:

$$w(y,t) = (1-t)w_0(y) + tw_1(y) \tag{6}$$

If the denominator of Equation (4) is positive at $t = 0$ and $t = 1$, is it positive for every $t \in (0,1)$? Let h be the denominator of Equation (4)

$$h(y,t) = 1 - \frac{y}{w}\frac{\partial w}{\partial y} + \frac{1}{4}\left(-\frac{1}{4} - \frac{1}{w} + \frac{y^2}{w^2}\right)\left(\frac{\partial w}{\partial y}\right)^2 + \frac{1}{2}\frac{\partial^2 w}{\partial y^2} \tag{7}$$

In other words, we would like to know if the following proposition holds:

$$\forall y \in (-\infty, +\infty), h(y,0) \geq 0 \text{ and } h(y,1) \geq 0 \Rightarrow \forall t \in (0,1), h(y,t) \geq 0 \tag{8}$$

Table 1 shows several example of points with a given value, slope and curvature that verifies $h(y,0) \geq 0$ and $h(y,1) \geq 0$ but where at some time t, $h(y,t) < 0$.

Some look reasonable enough that they could stem from real volatility slices. For example the points with $w_0 = 0.114$ and $w_1 = 1.916$ would correspond to the surface plotted in Figure 1.

This shows that a-priori, one should not expect the linear interpolation to be arbitrage free. However this is only a local analysis. To really prove that linear

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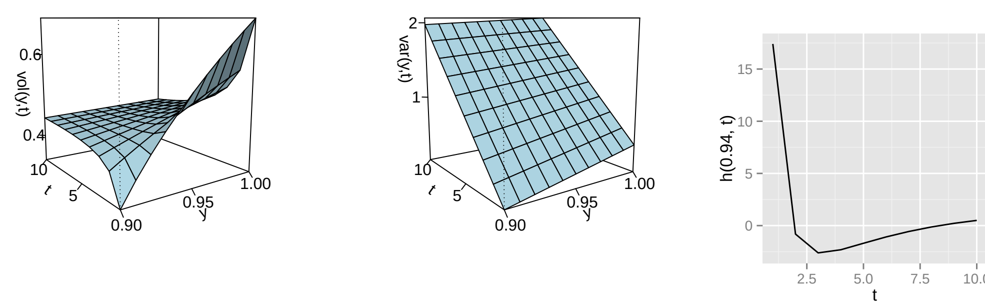
TABLE 1. Example of arbitrages due to linear interpolation

y	0.6	0.6	0.9	1	0.5
t	0.5	0.2	0.3	0.1	0.1
w_0	0.071	1.707	0.114	0.058	0.117
$\frac{\partial w_0}{\partial y}$	0.97	-3.552	3.566	0.789	0.974
$\frac{\partial^2 w_0}{\partial y^2}$	0	0	0	-8.451	1.931
w_1	0.265	3.767	1.976	4.116	0.568
$\frac{\partial w_1}{\partial y}$	0.308	-3.55	0.862	0.376	0.378
$\frac{\partial^2 w_1}{\partial y^2}$	0	0	0	-1.574	1.606
$h(y,0)$	6.229	0.002	142.3	26.451	0.048
$h(y,1)$	0.329	0.021	0.505	0.106	1.426
$h(y,t)$	-0.613	-0.018	-2.584	-4.18	-0.224

FIGURE 1. Arbitrage between two truncated slices of increasing variance using linear interpolation. The slices correspond to the $y = 0.9$ case in Table 1 with a time scaled between $t_0 = 1$ and $t_0 = 10$



(A) Volatility (B) Variance (C) $h(y = 0.94, t)$

interpolation breaks down, one would need to find two full slices that verify inequation $h(y,t) \geq 0$ for $y \in (-\infty, +\infty)$ and where the interpolation does not at some point.

2.2. **At-the-money interpolation.** Let f be the denominator of equation (4) at-the-money, that is when $y = 0$.

$$f(W) = 1 + \frac{1}{4}\left(-\frac{1}{4} - \frac{1}{w}\right)\left(\frac{\partial w}{\partial y}\right)^2 + \frac{1}{2}\frac{\partial^2 w}{\partial y^2} \text{ with } W = \begin{pmatrix} w \\ \frac{\partial w}{\partial y} \\ \frac{\partial^2 w}{\partial y^2} \end{pmatrix} \tag{9}$$

Let's look at g defined by

$$g(t) = f\left((1-t)W_0 + tW_1\right) - (1-t)f(W_0) - tf(W_1) \tag{10}$$

$$
\begin{aligned}
g(t) &= \frac{1}{16}\left((1-t)\left(\frac{\partial w_0}{\partial y}\right)^2 + t\left(\frac{\partial w_1}{\partial y}\right)^2 - (1-t)^2\left(\frac{\partial w_0}{\partial y}\right)^2 - t^2\left(\frac{\partial w_1}{\partial y}\right)^2 - 2t(1-t)\frac{\partial w_0}{\partial y}\frac{\partial w_1}{\partial y}\right)\\
&+ \frac{1}{4}\left(\frac{1-t}{w_0}\left(\frac{\partial w_0}{\partial y}\right)^2 + \frac{t}{w_1}\left(\frac{\partial w_1}{\partial y}\right)^2 - \frac{1}{(1-t)w_0 + tw_1}\left((1-t)\frac{\partial w_0}{\partial y} + t\frac{\partial w_1}{\partial y}\right)^2\right)\\
&= \frac{t(1-t)}{16}\left(\left(\frac{\partial w_0}{\partial y}\right)^2 + \left(\frac{\partial w_1}{\partial y}\right)^2 - 2\frac{\partial w_0}{\partial y}\frac{\partial w_1}{\partial y}\right)\\
&+ \frac{t(1-t)}{4((1-t)w_0 + tw_1)}\left(\left(\frac{\partial w_0}{\partial y}\right)^2\frac{w_1}{w_0} + \left(\frac{\partial w_1}{\partial y}\right)^2\frac{w_0}{w_1} - 2\frac{\partial w_0}{\partial y}\frac{\partial w_1}{\partial y}\right)\\
&= \frac{t(1-t)}{16}\left(\frac{\partial w_0}{\partial y} - \frac{\partial w_1}{\partial y}\right)^2 + \frac{t(1-t)}{4w_0w_1((1-t)w_0 + tw_1)}\left(\frac{\partial w_0}{\partial y}w_1 - \frac{\partial w_1}{\partial y}w_0\right)^2
\end{aligned}
$$

As $g(t) \geq 0$ for $t \in [0,1]$, f is concave. Furthermore, as $f(W_0) \geq 0$ and $f(W_1) \geq 0$, we have for $t \in [0,1]$, $f((1-t)W_0 + tW_1) \geq 0$, and therefore the no arbitrage property is preserved under linear interpolation at the money.

3. The Problem With Flat Extrapolation

In terms of implied volatility $\sigma(K,t)$, the total implied variance is $w(y) = \sigma(K,t)^2 t$ with $y = \log(\frac{K}{F(t)})$ expressing the log-moneyness, F the forward, K a strike, and t a maturity. We are interested in describing the volatility after the last slice at maturity t_n.

For $t \geq t_n$, the flat-extrapolated volatility will be:

$$
\begin{aligned}
\sigma(K,t) &= \sigma(K,t_n)\\
w(y,t) &= \sigma^2(K,t_n)\cdot t
\end{aligned}
$$

Let v defined as $v(y) = \sigma(F(t_n)e^y, t_n)^2$ for a given y and $t \geq T$. We have

$$
w(y,t) = v(y)\cdot t \tag{11}
$$

As $t \to \infty$, the extrapolated variance behaves like:

$$
\begin{aligned}
w(t) &= \mathcal{O}(t)\\
\frac{\partial w}{\partial y}(y,t) &= \mathcal{O}(t)\\
\frac{\partial^2 w}{\partial y^2}(y,t) &= \mathcal{O}(t)
\end{aligned}
$$

therefore,

$$
\begin{aligned}
1 - \frac{y}{w}\frac{\partial w}{\partial y} &= \mathcal{O}(1)\\
\left(\frac{1}{w} + \frac{y^2}{w^2}\right)\left(\frac{\partial w}{\partial y}\right)^2 &= \mathcal{O}(t)\\
\frac{1}{2}\frac{\partial^2 w}{\partial y^2} &= \mathcal{O}(t)
\end{aligned}
$$

and using equation (7):

$$
h(y,t) = -\frac{1}{16}v'(y)^2 t^2 + \mathcal{O}(t) \tag{12}
$$

When t grows, the h behaves like $-Rt^2$ where R is a positive function. Remembering that the butterfly spread no-arbitrage condition is equivalent to $h(y,t) \geq 0$, as soon as $v'(y) \neq 0$, there will be a butterfly spread arbitrage at one point in time if one uses the flat extrapolation in volatility. That is, if the last volatility slice is not flat, there will be a butterfly spread arbitrage in the flat extrapolation.

4. A Concrete Example

Let's choose a popular parameterization of the volatility surface: the SVI parameterization. The surface is parameterized expiry slice by expiry slice, and interpolated linearly in variance inside.

For a given expiry t, the SVI parameterization of the implied variance is [Gatheral, 2006]:

$$\sigma^2(K,t) = a + b\left[\rho(y-m) + \sqrt{(y-m)^2+\omega^2}\right] \tag{13}$$

Or in terms of total implied variance:

$$w(y,t) = \sigma^2(K,t)\cdot t = at + bt\left[\rho(y-m) + \sqrt{(y-m)^2+\omega^2}\right] \tag{14}$$

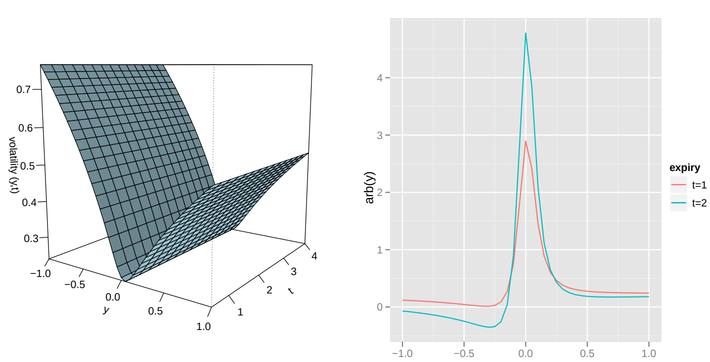
For given parameters a, b, ρ, m, y, ω and $t \geq t_n$, the flat-extrapolated volatility will be:

$$\begin{aligned}
\sigma(K,t) &= \sigma(K,t_n) \\
w(K,t) &= \sigma^2(K,t_n)\cdot t \\
&= at + bt\left[\rho(y-m) + \sqrt{(y-m)^2+\omega^2}\right]
\end{aligned}$$

The derivatives of the variance under SVI are:

$$\begin{aligned}
\frac{\partial w}{\partial y}(y,t) &= bt(\rho + \frac{y-m}{\sqrt{(y-m)^2+\omega^2}}) \\
\frac{\partial^2 w}{\partial y^2}(y,t) &= bt(\frac{1}{\sqrt{(y-m)^2+\omega^2}} - \frac{(y-m)^2}{((y-m)^2+\omega^2)^{\frac{3}{2}}})
\end{aligned}$$

Let's choose $t_n = 1$, $a = 0.02$, $b = 0.4$, $\omega = 0.1$, $\rho = -0.4$, $m = 0$. Those parameters comes from an actual fitted slice to the Nikkei Index. This slice verifies the no-arbitrage conditions, but the flat-extrapolated slice at $t = 2$ does not as the figure 2 shows.



(A) Volatility Surface. (B) $h(y,t)$

Figure 2. Flatly extrapolated SVI surface.

5. An Alternative Simple Extrapolation

5.1. **General Case.** Let's represent the last slice of the volatility surface by $\sigma(K,t) = \alpha_0 + \beta(K)$ where $\alpha_0 > 0$ is a base volatility level.

One simple fix is to keep the flat extrapolation only for the base level represented by α_0 so that the first derivative of the variance in y does not depend on the time to expiry. We will call this the α-extrapolation. The extrapolated total variance then becomes for $t \geq t_n$:

$$w(y,t) = \sigma^2(K,t_n) \cdot t = \alpha(t)t + \beta(K)t_n \tag{15}$$

with $\alpha(t_n) = \alpha_0$. In absence of arbitrage, we must have [Carr and Wu, 2002] when $t \to \infty$:

$$\frac{\partial \sigma}{\partial \ln(K)}(K,t) = \mathcal{O}(\frac{1}{t})$$

With the α-extrapolation, we have:

$$\begin{aligned}
\frac{\partial \sigma}{\partial \ln(K)}(K,t) &= \frac{\partial \sqrt{\frac{w}{t}}}{\partial \ln(K)}(t,K)\\
&= \frac{1}{2t}\frac{\partial w}{\partial y}\frac{1}{\sqrt{\frac{w}{t}}}\\
&= \frac{1}{2t}\frac{t_n}{\sqrt{\alpha(t) + \frac{\beta(K)t_n}{t}}}\frac{\partial \beta}{\partial y}\\
&= \mathcal{O}(\frac{1}{t})
\end{aligned}$$

when $\alpha(t) > 0$ and $\alpha(t) = \mathcal{O}(1)$.

How to choose α? A simple choice is the smallest volatility of the last slice. Ideally one would try to find a term structure for the at-the-money volatility using a simple time paramerization like for example $\alpha(t) = At^{-B}$ where A and B are fitted to the few last slices.

5.2. **SVI α-extrapolation.** In the case of SVI it is natural to choose $\alpha = a$, the extrapolated total variance then becomes:

$$w(y,t) = \sigma^2(K,t_n) \cdot t = at + bt_n\left[\rho(y-m) + \sqrt{(y-m)^2 + \omega^2}\right] \tag{16}$$

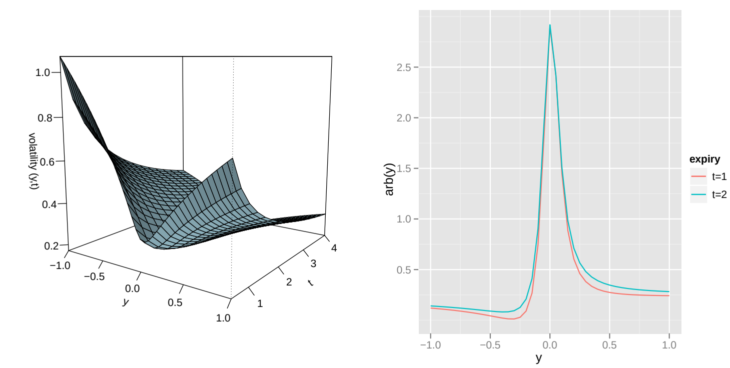
We suppose here that we always have $a \geq 0$, a desirable property for an SVI surface [Zeliade Systems, 2009]. With this constraint, the α-extrapolated total implied variance is monotonically increasing.

Figure 3 shows that the wings flatten with increasing expiry, as expected.

With SVI, our experience suggests that some markets exhibit a nearly constant a, while b decreases to 0 in an hyperbolic manner, very much like the simplest α-extrapolation. However, usually, the parameter m varies with time and this is not captured. Also some other markets exhibits varying a. A more refined way would be to extrapolate directly the SVI parameters using a term structure like [Gurrieri, 2011] and to ensure that the volatility still flattens in an appropriate manner.

6. Other Extrapolations

Another possibility for extrapolation is flat local volatility extrapolation. This is natural if local volatility pricing is used, and will result in a flattening implied volatility surface as long the the flat local volatility is small enough. But in reality there is no reason for it to be flat, and it would be better to extrapolate local



(A) Volatility Surface. (B) $h(y, t)$

FIGURE 3. SVI surface with α-extrapolation.

volatility from the last 2 slices via for example a Ae^{-Bt} parameterization with A and B positive parameters. This will guarantee absence of arbitrage. Unfortunately there is no simple way to get the implied volatility out of it.

The simplest is probably to just add a fictitious flat slice far away. One would then rely on linear interpolation between the fictitious slice and the previous slice for extrapolation. This is close to the constant alpha extrapolation in principle: the constant alpha extrapolation is like a flat slice at infinity. The problem with such an extrapolation is that a calendar spread arbitrage is guaranteed to happen as long as

$$\lim_{K \to +\infty} \sigma(K, T_n) = +\infty \tag{17}$$

where T_n is the last non extrapolated slice as there will be a strike K_0 where the total variance will decrease for $K > K_0$. This is a very common case in practice, so one has just to hope that this K_0 is never attained.

7. Conclusion

We have shown that linear interpolation preserves static arbitrage at-the-money. Counter-examples show that this is not true in general, locally. It is left to further research to prove the result globally: that is to find two full slices that have no arbitrage and where linear interpolation creates an arbitrage.

Regarding extrapolation in time, flat volatility extrapolation in time is not acceptable in general, and will always introduce butterfly spread arbitrages, except if the last slice is flat. We have proposed a better behaved simple new extrapolation, the α-extrapolation, that respects the flattening of the wings.

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