Orbital Perturbations

Orbital Perturbations

Orbital Perturbations

- The Keplerian formulations we have developed do not account for the many other forces acting on spacecraft
- Generally, these forces are small and, as such, we treat them as orbital perturbations that slowly modify our Keplerian orbit

For a geocentric orbit, some examples of perturbations include:

- Atmospheric drag
- Solar radiation pressure
- Asphericity and non-uniformity of Earth's mass distribution
- Gravitational forces from other bodies (e.g., Moon, Sun)

Table 4.2 Magnitude of disturbing accelerations acting on a space vehicle whose area-to-mass ratio is *A/M*. Note that *A* is the projected area perpendicular to the direction of motion for air drag, and perpendicular to the Sun for radiation pressure

Source	Acceleration (m/s^2)	
	500 km	Geostationary orbit
Air drag*	$6 \times 10^{-5} A/M$	–
Radiation pressure	$4.7 \times 10^{-6} A/M$	$4.7 \times 10^{-6} A/M$
Sun (mean)	5.6×10^{-7}	3.5×10^{-6}
Moon (mean)	1.2×10^{-6}	7.3×10^{-6}
Jupiter (max.)	8.5×10^{-12}	5.2×10^{-11}

*Dependent on the level of solar activity

[Fortescue, Ch. 4]

Let's see how this changes our Equations of Motion

Orbital Perturbations

Equation of Motion

- For two-body motion of point masses m_1 and m_2, where $m_1 \gg m_2$, without perturbations we had:

$$\boxed{\ddot{\vec{\mathbf{r}}} = -\frac{\mu}{r^3}\vec{\mathbf{r}}} \quad \text{with initial conditions:} \quad \boxed{\vec{\mathbf{r}}(0) = \vec{\mathbf{r}}_0} \quad \boxed{\dot{\vec{\mathbf{r}}}(0) = \vec{\mathbf{v}}_0}$$

- Including the effects of perturbations on the two-body motion, we see the true equation of motion is:

$$\ddot{\vec{\mathbf{r}}} = -\frac{\mu}{r^3}\vec{\mathbf{r}} + \vec{\mathbf{f}}_p \quad \text{with initial conditions:} \quad \vec{\mathbf{r}}(0) = \vec{\mathbf{r}}_0 \quad \dot{\vec{\mathbf{r}}}(0) = \vec{\mathbf{v}}_0$$

$\vec{\mathbf{f}}_p$ is the perturbative acceleration (or specific force) due to the perturbing effects

- Two general approaches for dealing with perturbations: special perturbations and general perturbations

Many software, such as MATLAB, include packages with multiple ODE solvers

Orbital Perturbations

Numerical Integration – Fourth-Order Runge-Kutta

- A fairly good scheme for numerical integration is the Fourth-Order Runge-Kutta method:

$$\mathbf{x}(t_{k+1}) = \mathbf{x}(t_k) + \frac{1}{6}[\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4] \quad \text{where} \quad \begin{aligned} \mathbf{k}_1 &= h\mathbf{F}(\mathbf{x}_k, t_k) \\ \mathbf{k}_2 &= h\mathbf{F}\left(\mathbf{x}_k + \frac{1}{2}\mathbf{k}_1, t_k + \frac{1}{2}h\right) \\ \mathbf{k}_3 &= h\mathbf{F}\left(\mathbf{x}_k + \frac{1}{2}\mathbf{k}_2, t_k + \frac{1}{2}h\right) \\ \mathbf{k}_4 &= h\mathbf{F}(\mathbf{x}_k + \mathbf{k}_3, t_k + h) \end{aligned}$$

Cowell's Method

- Essentially a brute-force approach that requires rewriting the EoM as a system of first-order differential equations with all quantities represented in an inertial frame, $\mathcal{F}_I$:

$$\boxed{\ddot{\vec{\mathbf{r}}} = -\frac{\mu}{r^3}\vec{\mathbf{r}} + \vec{\mathbf{f}}_p} \quad \boxed{\vec{\mathbf{r}}(0) = \vec{\mathbf{r}}_0} \quad \boxed{\dot{\vec{\mathbf{r}}}(0) = \vec{\mathbf{v}}_0} \longrightarrow$$

$$\vec{\mathbf{r}} = \vec{\mathcal{F}}_I^{\mathrm{T}}\mathbf{r} \qquad \vec{\mathbf{r}}_0 = \vec{\mathcal{F}}_I^{\mathrm{T}}\mathbf{r}_0$$

$$\vec{\mathbf{v}} = \dot{\vec{\mathbf{r}}} = \vec{\mathcal{F}}_I^{\mathrm{T}}\mathbf{v} \qquad \vec{\mathbf{v}}_0 = \vec{\mathcal{F}}_I^{\mathrm{T}}\mathbf{v}_0$$

$$\vec{\mathbf{f}}_p = \vec{\mathcal{F}}_I^{\mathrm{T}}\mathbf{f}_p$$

$$\begin{bmatrix}\dot{\mathbf{r}} \\ \dot{\mathbf{v}}\end{bmatrix} = \begin{bmatrix}\mathbf{v} \\ -\dfrac{\mu}{(\mathbf{r}^{\mathrm{T}}\mathbf{r})^{3/2}}\mathbf{r} + \mathbf{f}_p\end{bmatrix}, \qquad \begin{bmatrix}\mathbf{r}(0) \\ \mathbf{v}(0)\end{bmatrix} = \begin{bmatrix}\mathbf{r}_0 \\ \mathbf{v}_0\end{bmatrix}$$

The equations are then directly integrated numerically

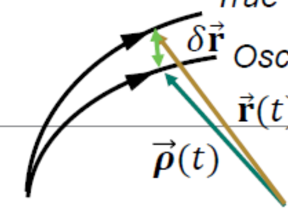
Advantages

- Straightforward (easy to program)
- Can handle any number of perturbations

Disadvantages

- Requires small time steps (making it slow and computationally expensive)
- Round-off errors accumulate rapidly, inaccurate long-term

Orbital Perturbations



Encke's Method

- More sophisticated than Cowell's method, but requires less computation
- Works by numerically integrating the deviation from the true (perturbed) orbit and a reference two-body orbit

Deviation from Osculating Orbit:

$$\delta\vec{\mathbf{r}} = \vec{\mathbf{r}} - \vec{\boldsymbol{\rho}}$$

1 *True (Perturbed) Orbit*

$$\boxed{\ddot{\vec{\mathbf{r}}} = -\frac{\mu}{r^3}\vec{\mathbf{r}} + \vec{\mathbf{f}}_p,} \quad \boxed{\vec{\mathbf{r}}(0) = \vec{\mathbf{r}}_0,} \quad \boxed{\dot{\vec{\mathbf{r}}}(0) = \vec{\mathbf{v}}_0}$$

2 *Reference Two-Body Orbit* ← called the <u>osculating orbit</u>

$$\ddot{\vec{\boldsymbol{\rho}}} = -\frac{\mu}{\rho^3}\vec{\boldsymbol{\rho}}, \quad \vec{\boldsymbol{\rho}}(0) = \vec{\mathbf{r}}_0, \quad \dot{\vec{\boldsymbol{\rho}}}(0) = \vec{\mathbf{v}}_0$$

- Difference between 1 and 2 :

$$\delta\ddot{\vec{\mathbf{r}}} = \ddot{\vec{\mathbf{r}}} - \ddot{\vec{\boldsymbol{\rho}}} = -\frac{\mu}{r^3}\vec{\mathbf{r}} + \vec{\mathbf{f}}_p - \left(-\frac{\mu}{\rho^3}\vec{\boldsymbol{\rho}}\right) = -\frac{\mu}{r^3}\vec{\mathbf{r}} + \frac{\mu}{\rho^3}\vec{\boldsymbol{\rho}} + \vec{\mathbf{f}}_p = -\mu\left[\frac{\vec{\mathbf{r}}}{r^3} - \frac{\vec{\boldsymbol{\rho}}}{\rho^3}\right] + \vec{\mathbf{f}}_p = -\mu\left[\frac{\vec{\mathbf{r}}}{r^3} - \frac{\vec{\mathbf{r}} - \delta\vec{\mathbf{r}}}{\rho^3}\right] + \vec{\mathbf{f}}_p$$

$$\delta\ddot{\vec{\mathbf{r}}} = -\frac{\mu}{\rho^3}\left[\delta\vec{\mathbf{r}} - \left(1 - \frac{\rho^3}{r^3}\right)\vec{\mathbf{r}}\right] + \vec{\mathbf{f}}_p$$

- Difference between the initial conditions of 1 and 2 :

$$\delta\vec{\mathbf{r}}(0) = \vec{\mathbf{0}}, \quad \delta\dot{\vec{\mathbf{r}}}(0) = \vec{\mathbf{0}},$$

where $\vec{\mathbf{r}} = \vec{\boldsymbol{\rho}} + \delta\vec{\mathbf{r}}$

Encke's method numerically integrates the deviation between the two orbits

However, we have an issue that the term $1 - \frac{\rho^3}{r^3}$, which is the difference between two almost equal quantities for small $\delta\vec{\mathbf{r}}$ → leads to loss of precision

Orbital Perturbations

converges rapidly for small q and avoids loss of precision when computing $1-\frac{\rho^3}{r^3}$ directly

Encke's Method

- To address the issue, we can use the following small variable: $2q = 1 - \dfrac{r^2}{\rho^2}$

  which we can rewrite in the following form: $1-\dfrac{\rho^3}{r^3} = 1-(1-2q)^{-\frac{3}{2}}$

- Expand with a Taylor series: $1-\dfrac{\rho^3}{r^3} = 1-\left(1+3q+\dfrac{3\times 5}{2!}q^2+\cdots\right) = -3q-\dfrac{3\times 5}{2!}q^2+\cdots$

- From the original definition of q and $\boxed{\delta\vec{\mathbf{r}} = \vec{\mathbf{r}} - \vec{\boldsymbol{\rho}}}$ we can find:

$$q = \frac{1}{2}\left(1-\frac{r^2}{\rho^2}\right) = \frac{1}{2}\left(\frac{\rho^2-r^2}{\rho^2}\right) = \frac{1}{2}\left(\frac{\vec{\boldsymbol{\rho}}\cdot\vec{\boldsymbol{\rho}}-\vec{\mathbf{r}}\cdot\vec{\mathbf{r}}}{\rho^2}\right) = \frac{1}{2}\left(\frac{\vec{\boldsymbol{\rho}}\cdot\vec{\boldsymbol{\rho}}-(\vec{\boldsymbol{\rho}}+\delta\vec{\mathbf{r}})\cdot(\vec{\boldsymbol{\rho}}+\delta\vec{\mathbf{r}})}{\rho^2}\right)$$

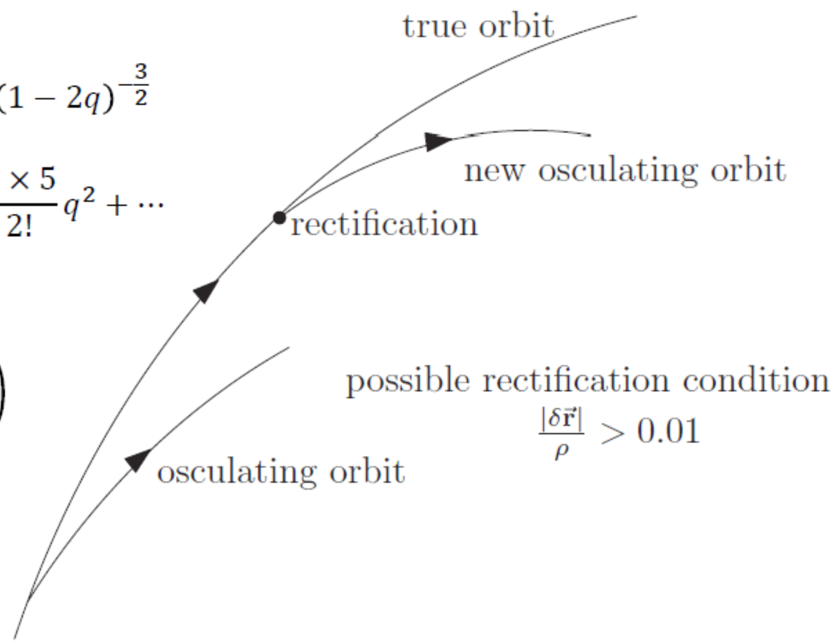
$$= -\frac{\delta\vec{\mathbf{r}}\cdot(\vec{\boldsymbol{\rho}}+\frac{\delta\vec{\mathbf{r}}}{2})}{\rho^2}$$

- Which for very small $\delta\vec{\mathbf{r}}$ compared to $\vec{\boldsymbol{\rho}}$:

$$q \approx -\frac{\vec{\boldsymbol{\rho}}\cdot\delta\vec{\mathbf{r}}}{\rho^2}$$

Advantages

- Reduces # of integration steps (increases time step)
- Faster than Cowell's method for equivalent accuracy



When $\delta\vec{\mathbf{r}}$ is no longer small compared to $\vec{\boldsymbol{\rho}}$, Encke's method requires rectification, i.e., a new osculating orbit is defined using the initial conditions of the true orbit at the time of rectification

Orbital Perturbations

N.B. Lagrange's planetary equations can be expressed w.r.t. change in variables other than time, e.g., true anomaly θ

General Perturbations

- Unlike special perturbations, general perturbations are valid for any set of initial conditions and are concerned with finding analytical expressions for the change in the orbital elements $\{a, e, i, \Omega, \omega, t\}$ w.r.t. time, i.e.,

$$\frac{da}{dt}, \quad \frac{de}{dt}, \quad \frac{di}{dt}, \quad \frac{d\Omega}{dt}, \quad \frac{d\omega}{dt}$$

- Generally, these differential equations of the orbital elements are often referred to as *Lagrange's Planetary Equations*
- The expressions we will present are due to Gauss, and, hence, are called **Gauss' Variational Equations (GVEs)**

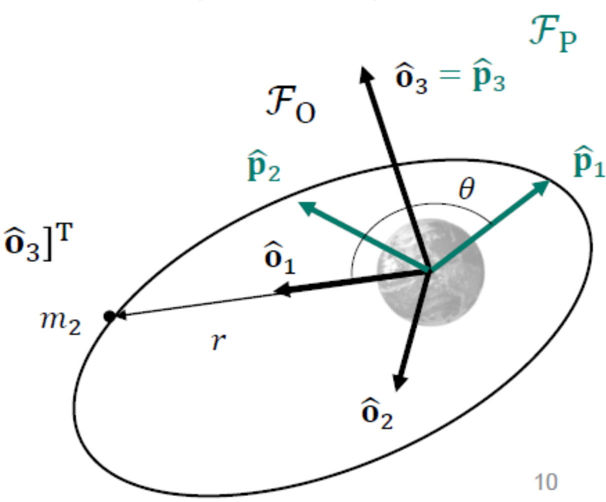
We will only derive one of these expressions, da/dt, but following a similar procedure they can all be obtained (see Ch. 7.7)

Gauss' Variational Equations (GVEs)

- Start with the perturbed two-body equation of motion: $\boxed{\ddot{\vec{\mathbf{r}}} = -\frac{\mu}{r^3}\vec{\mathbf{r}} + \vec{\mathbf{f}}_p}$
- We will use the orbiting frame, $\mathcal{F}_O$, which is a cylindrical coordinate system: $\vec{\mathcal{F}}_O = [\hat{\mathbf{o}}_1 \quad \hat{\mathbf{o}}_2 \quad \hat{\mathbf{o}}_3]^{\mathrm{T}}$

$$\boxed{\vec{\mathbf{r}} = \mathcal{F}_O^{\mathrm{T}}\begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}} \qquad \boxed{\vec{\mathbf{v}} = \mathcal{F}_O^{\mathrm{T}}\begin{bmatrix} \dot{r} \\ r\dot{\theta} \\ 0 \end{bmatrix}} \qquad \boxed{\vec{\mathbf{h}} = \mathcal{F}_O^{\mathrm{T}}\begin{bmatrix} 0 \\ 0 \\ r^2\dot{\theta} \end{bmatrix}} \qquad \vec{\mathbf{f}}_p = \mathcal{F}_O^{\mathrm{T}}\begin{bmatrix} f_r \\ f_\theta \\ f_z \end{bmatrix}$$

$$\vec{\mathbf{r}} = r\hat{\mathbf{o}}_1 \qquad \vec{\mathbf{v}} = \dot{r}\hat{\mathbf{o}}_1 + r\dot{\theta}\hat{\mathbf{o}}_2 \qquad \vec{\mathbf{h}} = r^2\dot{\theta}\hat{\mathbf{o}}_3 \qquad \vec{\mathbf{f}}_p = f_r\hat{\mathbf{o}}_1 + f_\theta\hat{\mathbf{o}}_2 + f_z\hat{\mathbf{o}}_3$$



Orbital Perturbations

Gauss' Variational Equations (GVEs)

- We are going to start with the orbital energy equation to find an expression for $\dfrac{da}{dt}$

Differentiate *Rearrange*

$$\boxed{\varepsilon = -\frac{\mu}{2a}} \qquad \frac{d\varepsilon}{dt} = \frac{da}{dt}\frac{\mu}{2a^2} \longrightarrow \frac{da}{dt} = \frac{2a^2}{\mu}\frac{d\varepsilon}{dt}$$

Substitute in $\boxed{\vec{\mathbf{v}} = \dot{r}\hat{\mathbf{o}}_1 + r\dot{\theta}\hat{\mathbf{o}}_2}$ $\boxed{\vec{\mathbf{f}}_p = f_r\hat{\mathbf{o}}_1 + f_\theta\hat{\mathbf{o}}_2 + f_z\hat{\mathbf{o}}_3}$

$$\boxed{\varepsilon = \frac{\vec{\mathbf{v}}\cdot\vec{\mathbf{v}}}{2} - \frac{\mu}{r}} \qquad \frac{d\varepsilon}{dt} = \vec{\mathbf{v}}\cdot\dot{\vec{\mathbf{v}}} + \dot{r}\frac{\mu}{r^2} = \vec{\mathbf{v}}\cdot\left(-\frac{\mu}{r^3}\vec{\mathbf{r}}\right) + \vec{\mathbf{v}}\cdot\vec{\mathbf{f}}_p + \mu\frac{\vec{\mathbf{r}}\cdot\vec{\mathbf{v}}}{r^3} = \vec{\mathbf{v}}\cdot\vec{\mathbf{f}}_p = [\dot{r}\hat{\mathbf{o}}_1 + r\dot{\theta}\hat{\mathbf{o}}_2]\cdot[f_r\hat{\mathbf{o}}_1 + f_\theta\hat{\mathbf{o}}_2 + f_z\hat{\mathbf{o}}_3] = \dot{r}f_r + r\dot{\theta}f_\theta$$

$$\boxed{\dot{r} = \frac{\vec{\mathbf{r}}\cdot\vec{\mathbf{v}}}{r}} \qquad \boxed{\ddot{\vec{\mathbf{r}}} = \dot{\vec{\mathbf{v}}} = -\frac{\mu}{r^3}\vec{\mathbf{r}} + \vec{\mathbf{f}}_p}$$

We already know $\dot{r}$ and $r\dot{\theta}$ from two-body motion:

$$\boxed{\dot{r} = \sqrt{\frac{\mu}{a(1-e^2)}}\,e\sin\theta} \qquad \boxed{r\dot{\theta} = \sqrt{\frac{\mu}{a(1-e^2)}}(1+e\cos\theta)}$$

This allows us to express it in terms of orbital elements and substitute into our other expression for $d\varepsilon/dt$, then rearrange for da/dt:

$$\frac{d\varepsilon}{dt} = \sqrt{\frac{\mu}{a(1-e^2)}}[e\sin\theta\, f_r + (1+e\cos\theta)f_\theta] \longrightarrow \frac{da}{dt} = \frac{2a^2}{\mu}\sqrt{\frac{\mu}{a(1-e^2)}}[e\sin\theta\, f_r + (1+e\cos\theta)f_\theta]$$

$$\boxed{\frac{da}{dt} = \frac{2a^2}{\sqrt{\mu a(1-e^2)}}[e\sin\theta\, f_r + (1+e\cos\theta)f_\theta]}$$

Orbital Perturbations

$$\vec{\mathbf{f}}_p = f_r\hat{\mathbf{o}}_1 + f_\theta\hat{\mathbf{o}}_2 + f_z\hat{\mathbf{o}}_3$$

Gauss' Variational Equations (GVEs)

- Following a similar approach, we can find the variational equations for all of the orbital elements, based on $\vec{\mathbf{f}}_p$

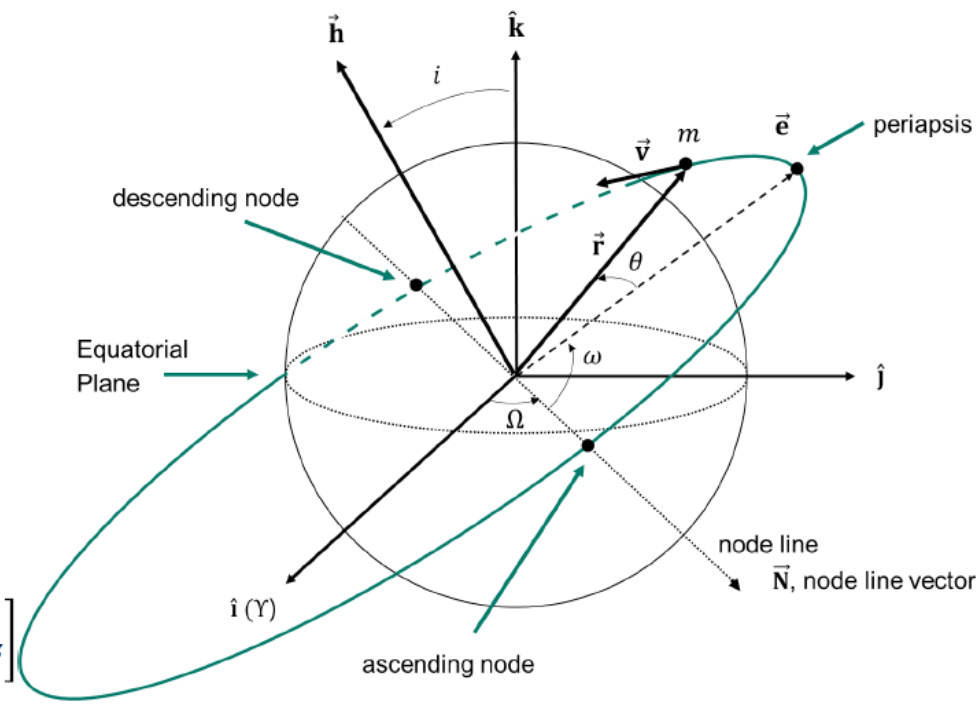
$$\frac{da}{dt} = \frac{2a^2}{\sqrt{\mu a(1-e^2)}}\left[e\sin\theta\, f_r + (1+e\cos\theta)f_\theta\right]$$

$$\frac{de}{dt} = \sqrt{\frac{a(1-e^2)}{\mu}}\left[\sin\theta\, f_r + \frac{2\cos\theta + e(1+\cos^2\theta)}{(1+e\cos\theta)}f_\theta\right]$$

$$\frac{di}{dt} = \sqrt{\frac{a(1-e^2)}{\mu}}\frac{\cos(\omega+\theta)}{1+e\cos\theta}f_z$$

$$\frac{d\Omega}{dt} = \sqrt{\frac{a(1-e^2)}{\mu}}\frac{\sin(\omega+\theta)}{\sin i\,(1+e\cos\theta)}f_z$$

$$\frac{d\omega}{dt} = \sqrt{\frac{a(1-e^2)}{\mu}}\left[-\frac{\cos\theta}{e}f_r + \frac{(2+e\cos\theta)\sin\theta}{e(1+e\cos\theta)}f_\theta - \frac{\sin(\omega+\theta)}{\tan i\,(1+e\cos\theta)}f_z\right]$$



Orbital Perturbations

For spacecraft, typically S is taken as the **frontal area** or cross-sectional area

Brief Overview of Common Orbital Perturbations

Atmospheric Drag

- For spacecraft in low Earth orbits, atmospheric effects are not negligible
- Dominant influences of drag are: orbit contraction and circularization

$$\vec{\mathbf{f}}_d = \frac{1}{2}\rho_\infty v_\infty^2 \frac{c_D S}{m}\left(-\frac{\vec{\mathbf{v}}}{v}\right)$$

$\vec{\mathbf{f}}_d$ = specific force due to drag

c_D = drag coefficient

S = reference area

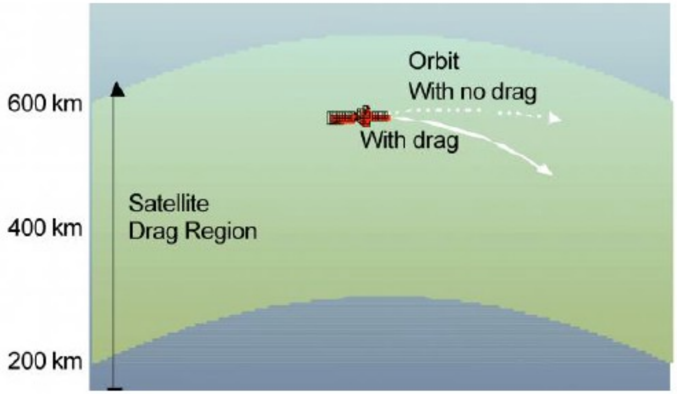
$\vec{\mathbf{v}}$ = s/c velocity vector

v = s/c speed, $v = |\vec{\mathbf{v}}|$

ρ_∞ = atmospheric density

v_∞ = speed relative to atmosphere

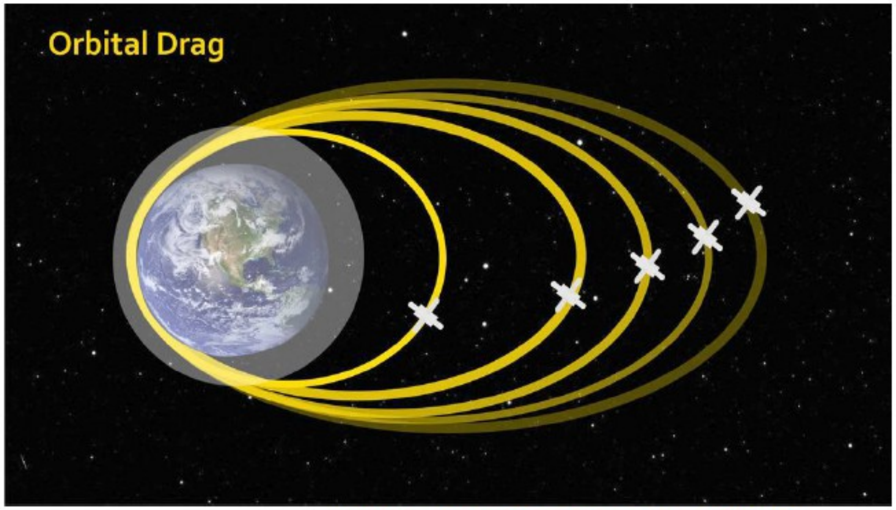
m = s/c mass



[NOAA, https://www.swpc.noaa.gov/impacts/satellite-drag]

What is the coefficient of drag for a spacecraft?

- Flow field does not have much intermolecular interaction (*molecules that interact with the s/c surface do not have further interactions with the flow field, e.g., no shock waves form*)
- Typically, $\boxed{c_D \sim 2.0}$ is used in calculations



[NASA, https://svs.gsfc.nasa.gov/12457]

Orbital Perturbations

Brief Overview of Common Orbital Perturbations

Solar Radiation Pressure (SRP)

- Small perturbation that acts on spacecraft due to solar radiation
- Solar radiation carries momentum that exerts a small but measurable pressure on a spacecraft
    - SRP produces a force on the spacecraft
    - SRP often produces a torque on the spacecraft
- Force acting on the spacecraft is dependent on the area-to-mass ratio, and is inversely proportional to the distance to the Sun

$$p_{\text{SRP}} = \frac{I_{\odot}}{c}$$

$I_{\odot}$ = solar irradiance

c = speed of light

Values for Earth

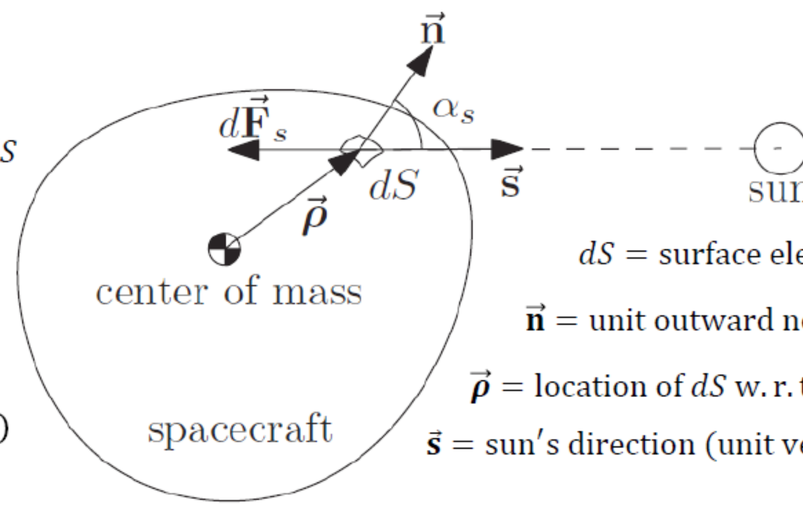
- Solar Irradiance at 1 AU (the *solar constant*): $G_{\text{SC}} = 1361\ \text{W/m}^2$
- Solar Radiation Pressure ($p_{\oplus}$): $p_{\oplus} = 4.5 \times 10^{-6}\ \text{Pa}$

Projected area of surface element in the sun's direction: $dA = \cos\alpha_s\, dS = \vec{\mathbf{n}} \cdot \vec{\mathbf{s}}\, dS$

Solar pressure force on dS is given by: $d\vec{\mathbf{F}}_s = -p_{\oplus} dA \vec{\mathbf{s}} = -p_{\oplus} \vec{\mathbf{n}} \cdot \vec{\mathbf{s}}\, dS \vec{\mathbf{s}}$

Total force due to SRP is given by: $\vec{\mathbf{F}}_s = \int_{S_{ws}} d\vec{\mathbf{F}}_s = -p_{\oplus} \vec{\mathbf{s}} \int_{S_{ws}} \vec{\mathbf{n}} \cdot \vec{\mathbf{s}}\, dS$

S_{ws} = wetted (lit) portion of the spacecraft surface (i.e., where $\vec{\mathbf{n}} \cdot \vec{\mathbf{s}} \geq 0$)



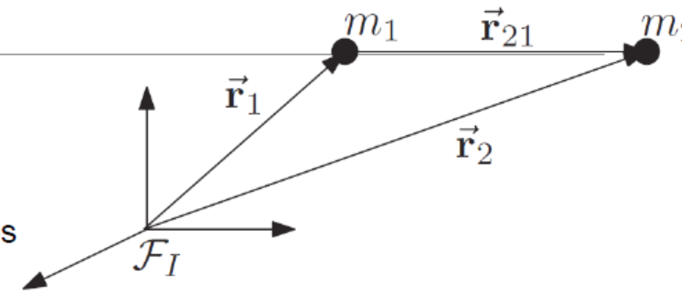
dS = surface element

$\vec{\mathbf{n}}$ = unit outward normal

$\vec{\boldsymbol{\rho}}$ = location of dS w.r.t CoM

$\vec{\mathbf{s}}$ = sun's direction (unit vector)

Orbital Perturbations



Brief Overview of Common Orbital Perturbations

Gravitational Perturbations due to Non-Spherical Primary Body

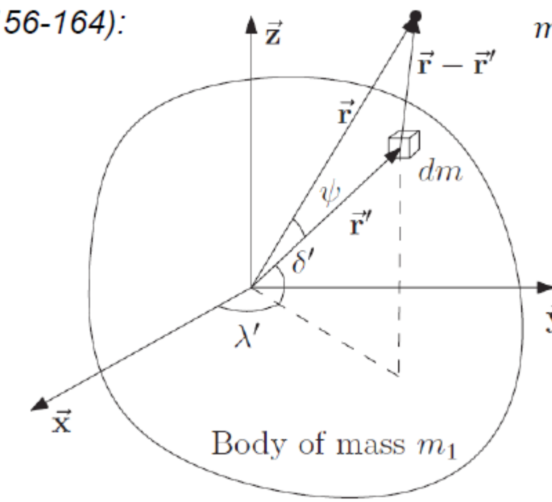
- Up until this point, we have considered the gravitational force based on point masses

$$\boxed{\vec{\mathbf{F}} = \frac{Gm_1m_2}{r^3}\vec{\mathbf{r}}} \qquad \boxed{\vec{\mathbf{f}} = \frac{Gm_1}{r^3}\vec{\mathbf{r}} = \frac{\mu}{r^3}\vec{\mathbf{r}}}$$

Force per unit mass can be obtained from the potential function: $\phi = \frac{Gm_1}{r}$

Extending to a series of point masses on m_2: $\phi = \sum_i \frac{Gm_i}{r_i}$

We will skip the derivation of the gravitational potential for an arbitrary body (see pp. 156-164):



$m_i \rightarrow dm = \rho dV$

$\sum_i \rightarrow \int_V$

- Gravitational potential due to a body is given by:

$$\phi(\vec{\mathbf{r}}) = \frac{Gm_1}{r} + \frac{G}{r}\sum_{n=2}^{\infty}\int_V \rho(\vec{\mathbf{r}})\left(\frac{r'}{r}\right)^n P_n(\cos\psi)dV$$

Two-body potential for a point mass

Perturbative force per unit mass

$$\phi_p(\vec{\mathbf{r}}) = \frac{G}{r}\sum_{n=2}^{\infty}\int_V \rho(\vec{\mathbf{r}})\left(\frac{r'}{r}\right)^n P_n(\cos\psi)dV$$

$P_n(x) =$ Legendre Polynomials

First three are:

$P_0(x) = 1$

$P_1(x) = x$

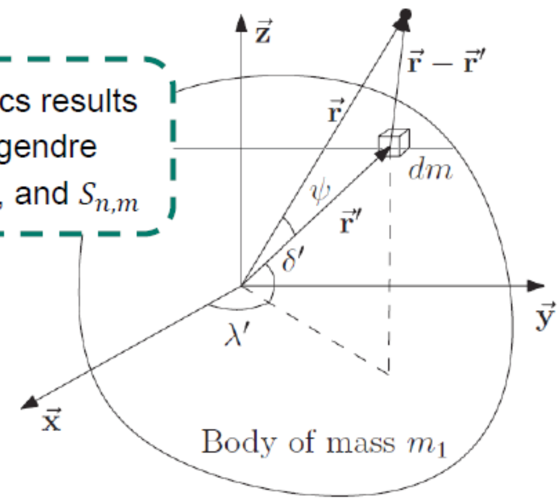
$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$

Generally given by:

$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$

Orbital Perturbations

An addition theorem for spherical harmonics results in the appearance of the associated Legendre functions $P_{n,m}$ and the coefficients J_n, $C_{n,m}$, and $S_{n,m}$



Brief Overview of Common Orbital Perturbations

Gravitational Perturbations due to Non-Spherical Primary Body

- Now, to evaluate the integrals $\cos\psi$ is represented in spherical coordinates and we get the common form of the **perturbing gravitational potential of the body**:

$$\phi_p(\vec{\mathbf{r}}) = \frac{Gm_1}{r}\left[-\sum_{n=2}^{\infty} J_n\left(\frac{R_e}{r}\right)^n P_n(\sin\delta) + \sum_{n=2}^{\infty}\sum_{m=1}^{n}\left(\frac{R_e}{r}\right)^n P_{n,m}(\sin\delta)\left[C_{n,m}\cos(m\lambda) + S_{n,m}\sin(m\lambda)\right]\right]$$

where R_e is some normalizing radius for the body m_1

J_n, $C_{n,m}$, and $S_{n,m}$ are coefficients:

$$J_n = -\frac{1}{R_e^n m_1}\int_V \rho(\vec{\mathbf{r}}')(r')^n P_n(\sin\delta')dV$$

$$C_{n,m} = \frac{1}{R_e^n m_1}2\frac{(n-m)!}{(n+m)!}\int_V \rho(\vec{\mathbf{r}}')(r')^n P_{n,m}(\sin\delta')\cos(m\lambda')\,dV$$

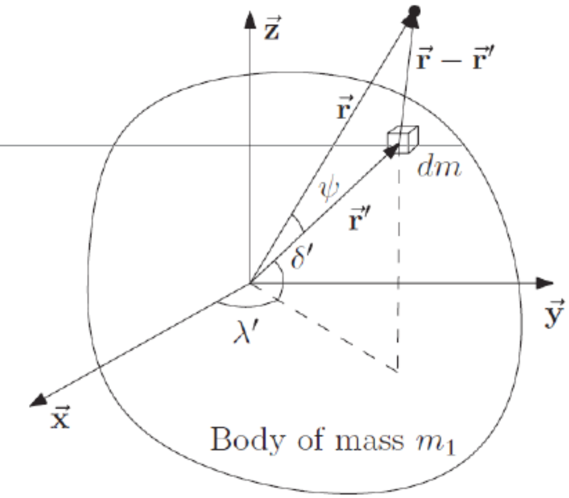
$$S_{n,m} = \frac{1}{R_e^n m_1}2\frac{(n-m)!}{(n+m)!}\int_V \rho(\vec{\mathbf{r}}')(r')^n P_{n,m}(\sin\delta')\sin(m\lambda')\,dV$$

Associated Legendre functions: $P_{n,m}(x) = (1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}P_n(x)$

In practice, coefficients J_n, $C_{n,m}$, and $S_{n,m}$ are **determined experimentally from satellite observations and can be obtained from tables**

Now, let's make some observations on this function

Orbital Perturbations



Brief Overview of Common Orbital Perturbations

Gravitational Perturbations due to Non-Spherical Primary Body

- Now, to evaluate the integrals $\cos\psi$ is represented in spherical coordinates and we get the common form of the **perturbing gravitational potential of the body**:

$$\phi_p(\vec{\mathbf{r}}) = \frac{Gm_1}{r}\left[-\sum_{n=2}^{\infty} J_n \left(\frac{R_e}{r}\right)^n P_n(\sin\delta) + \sum_{n=2}^{\infty}\sum_{m=1}^{n}\left(\frac{R_e}{r}\right)^n P_{n,m}(\sin\delta)\left[C_{n,m}\cos(m\lambda) + S_{n,m}\sin(m\lambda)\right]\right]$$

where R_e is some normalizing radius for the body m_1

$J_n \leftarrow$ zonal harmonic coefficients

$C_{n,m} \leftarrow$ tesseral harmonic coefficients

$S_{n,m} \leftarrow$ sectoral harmonic coefficients

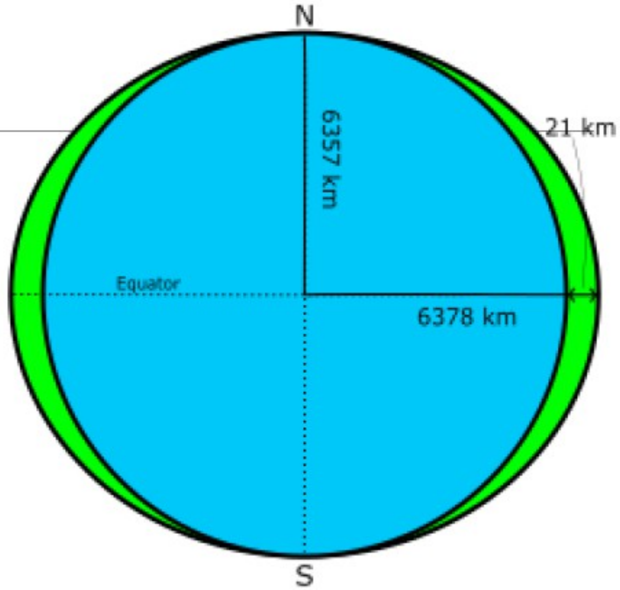
Notes:

- If the body is *rotationally symmetric* about $\vec{\mathbf{z}}$, $C_{n,m} = S_{n,m} = 0$

$$\phi_p(\vec{\mathbf{r}}) = -\frac{Gm_1}{r}\sum_{n=2}^{\infty} J_n \left(\frac{R_e}{r}\right)^n P_n(\sin\delta)$$

- A property of Legendre polynomials is that they satisfy the orthogonality property so that if the body is *spherically symmetric* the perturbing potential $\phi_p = 0$, and the resulting force per unit mass on m_2 is the same as for a point mass m_1 at the CoM

Orbital Perturbations



Now we will consider the effects of the oblateness of the Earth

- Earth is not a perfect sphere, it is an oblate spheroid

$$\text{oblateness} = \frac{\text{equatorial radius} - \text{polar radius}}{\text{equatorial radius}}$$

- For the Earth, the most dominant perturbing effect is the J_2 term, which is a result of the Earth's oblate shape (flattened at the poles)
- Perturbing potential including J_2 effects only:

$$\phi_p = -\frac{\mu_\oplus}{r} J_2 R_\oplus^2 \left(\frac{3}{2}\sin^2\delta - \frac{1}{2}\right)$$

Note, R_e for Earth is the equatorial radius $R_\oplus$

Perturbative Force Per Unit Mass Due to J_2

- For this we represent the potential in ECI coordinates and find:

$$\vec{\mathbf{f}}_p = \frac{3\mu_\oplus J_2 R_\oplus^2}{2r^5}\left[\left(5\frac{(\vec{\mathbf{r}}\cdot\hat{\mathbf{g}}_3)^2}{r^2} - 1\right)\vec{\mathbf{r}} - 2(\vec{\mathbf{r}}\cdot\hat{\mathbf{g}}_3)\hat{\mathbf{g}}_3\right]$$

Table 4.3 Magnitude of low-order *J*, *C* and *S* values for Earth

J_2	1082.6×10^{-6}	C_{21}	0	S_{21}	0
J_3	-2.53×10^{-6}	C_{22}	1.57×10^{-6}	S_{22}	-0.90×10^{-6}
J_4	-1.62×10^{-6}	C_{31}	2.19×10^{-6}	S_{31}	0.27×10^{-6}
J_5	-0.23×10^{-6}	C_{32}	0.31×10^{-6}	S_{32}	-0.21×10^{-6}
J_6	0.54×10^{-6}	C_{33}	0.10×10^{-6}	S_{33}	0.20×10^{-6}

[Fortescue, Ch. 4; Curtis, Ch.4; De Ruiter, Ch.7 ;AI Solutions, "J2 Perturbation" https://ai-solutions.com/_freeflyeruniversityguide/j2_perturbation.htm]

Orbital Perturbations

$$\boxed{\vec{\mathbf{f}}_p = \frac{3\mu_\oplus J_2 R_\oplus^2}{2r^5}\left[\left(5\frac{(\vec{\mathbf{r}}\cdot\hat{\mathbf{g}}_3)^2}{r^2} - 1\right)\vec{\mathbf{r}} - 2(\vec{\mathbf{r}}\cdot\hat{\mathbf{g}}_3)\hat{\mathbf{g}}_3\right]}$$

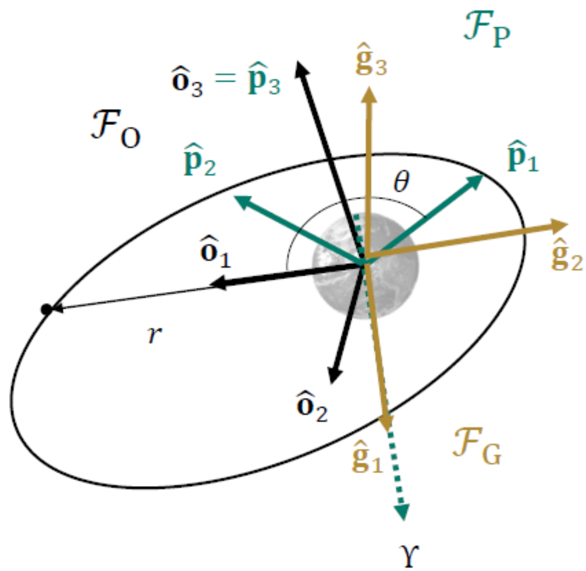
Effects of J_2 on the Orbital Elements

- We now have an expression of the perturbative force per unit mass due to J_2, we will use GVEs to determine its effect on OEs
- To do this, we need to express the perturbative force per unit mass in $\mathcal{F}_O$, which is a cylindrical coordinate system

We already know: $\boxed{\vec{\mathbf{r}} = r\hat{\mathbf{o}}_1}$

It can be shown that: $\hat{\mathbf{g}}_3 = \sin i \sin(\omega+\theta)\,\hat{\mathbf{o}}_1 + \sin i\cos(\omega+\theta)\,\hat{\mathbf{o}}_2 + \cos i\,\hat{\mathbf{o}}_3$

$$\vec{\mathbf{r}}\cdot\hat{\mathbf{g}}_3 = r\sin i \sin(\omega+\theta)$$



- So we can show $\vec{\mathbf{f}}_p$ in $\mathcal{F}_O$ directly:

$$\vec{\mathbf{f}}_p = \frac{3\mu_\oplus J_2 R_\oplus^2}{2r^5}\Bigg[\left(5\frac{(r\sin i\sin(\omega+\theta))^2}{r^2} - 1\right)r\hat{\mathbf{o}}_1 - 2(r\sin i\sin(\omega+\theta))(\sin i\sin(\omega+\theta)\,\hat{\mathbf{o}}_1 + \sin i\cos(\omega+\theta)\,\hat{\mathbf{o}}_2 + \cos i\,\hat{\mathbf{o}}_3)\Bigg]$$

where we can identify the following components:

$$f_r = \frac{3\mu_\oplus J_2 R_\oplus^2}{2r^4}(3\sin^2 i\sin^2(\omega+\theta) - 1) \qquad f_\theta = -\frac{3\mu_\oplus J_2 R_\oplus^2}{2r^4}\sin^2 i\sin^2\big(2(\omega+\theta)\big) \qquad f_z = -\frac{3\mu_\oplus J_2 R_\oplus^2}{2r^4}\sin 2i\sin(\omega+\theta)$$

Orbital Perturbations

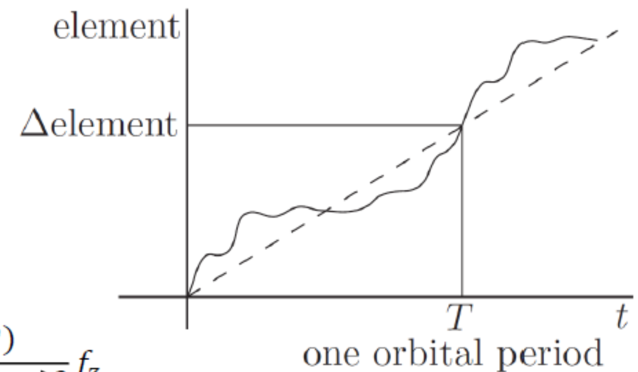
$$f_r = \frac{3\mu_\oplus J_2 R_\oplus^2}{2r^4}\left(3\sin^2 i \sin^2(\omega+\theta) - 1\right)$$

$$f_\theta = -\frac{3\mu_\oplus J_2 R_\oplus^2}{2r^4}\sin^2 i \sin^2\big(2(\omega+\theta)\big)$$

$$f_z = -\frac{3\mu_\oplus J_2 R_\oplus^2}{2r^4}\sin 2i \sin(\omega+\theta)$$

Effects of J_2 on the Orbital Elements

- We can now use the find the variation due to J_2 pertubations using the GVEs
- In general, perturbed orbital elements have *secular* and *period* variations
- Let's examine the secular variation in Ω



$$\frac{d\Omega}{dt} = \sqrt{\frac{a(1-e^2)}{\mu}}\frac{\sin(\omega+\theta)}{\sin i\,(1+e\cos\theta)}f_z$$

$$\dot{\theta} = \sqrt{\frac{\mu}{a^3}}\frac{(1+e\cos\theta)^2}{(1-e^2)^{3/2}}$$

We will find the variation in terms of the true anomaly θ:

$$\frac{d\Omega}{dt} = \frac{d\Omega}{d\theta}\dot{\theta} \longrightarrow \frac{d\Omega}{d\theta} = \frac{1}{\dot{\theta}}\frac{d\Omega}{dt}$$

Also, substituting in the two-body orbit equation for $\dot{\theta}$ we find: $\displaystyle \frac{d\Omega}{d\theta} = \frac{a^2(1-e^2)}{\mu}\frac{\sin(\omega+\theta)}{\sin i\,(1+e\cos\theta)^3}f_z$

We expect changes over an orbit in the elements to be small

Now, we can substitute in the polar equation of the orbit, the identity $2\sin i\cos i = \sin 2i$, and f_z to find:

$$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

$$\frac{d\Omega}{d\theta} = -\frac{3J_2R_\oplus^2}{a^2(1-e^2)^2}\cos i\sin^2(\omega+\theta)\,(1+e\cos\theta) \longrightarrow \Delta\Omega = \int_0^{\Delta\Omega} d\Omega = \int_0^{2\pi}\frac{d\Omega}{d\theta}d\theta = -\frac{3J_2R_\oplus^2}{a^2(1-e^2)^2}\int_0^{2\pi}\cos i\sin^2(\omega+\theta)\,(1+e\cos\theta)\,d\theta$$

To determine secular change in Ω, we look at the change over an orbit

Continued

Clarkson

Orbital Perturbations

$$\Delta\Omega = \int_0^{\Delta\Omega} d\Omega = \int_0^{2\pi} \frac{d\Omega}{d\theta} d\theta = -\frac{3J_2R_\oplus^2}{a^2(1-e^2)^2}\int_0^{2\pi} \cos i \sin^2(\omega+\theta)\,(1+e\cos\theta)\,d\theta$$

Effects of J_2 on the Orbital Elements

- By evaluating the integral and using trig. Identities, we obtain: $\Delta\Omega = \int_0^{\Delta\Omega} d\Omega = \int_0^{2\pi} \frac{d\Omega}{d\theta} d\theta = -\frac{3\pi J_2R_\oplus^2}{a^2(1-e^2)^2}\cos i$

$$\Delta T = 2\pi\sqrt{\frac{a^3}{\mu}}$$

- To obtain the secular (average) rate of change of Ω, denoted $\langle\dot{\Omega}\rangle$, we divide by the orbital period ΔT

$$\langle\dot{\Omega}\rangle = \frac{\Delta\Omega}{\Delta T} = -\frac{3J_2R_\oplus^2}{2(1-e^2)^2}\sqrt{\frac{\mu}{a^7}}\cos i$$

The secular change in Ω is called ***nodal regression***

$$\langle\dot{a}\rangle = 0 \qquad \langle\dot{e}\rangle = 0 \qquad \langle\dot{i}\rangle = 0$$

- Following the same process for the other orbital elements, we find:

$$\langle\dot{\omega}\rangle = \frac{3J_2R_\oplus^2}{4(1-e^2)^2}\sqrt{\frac{\mu}{a^7}}(5\cos^2 i - 1)$$

often called ***perigee advance***

Notes:

- Node line does not move for a polar orbit
- Regression changes direction if $i > 90°$
- Perigee advance direction is controlled by $(5\cos^2 i - 1)$
  - $\langle\dot{\omega}\rangle > 0$, if $0° \le i < 63.4°$ or $116.6° \le i < 180°$
  - $\langle\dot{\omega}\rangle < 0$, if $63.4° < i < 116.6°$
  - $\langle\dot{\omega}\rangle = 0$, if $i = 63.4°$ or $116.6°$ (apse line does not move)

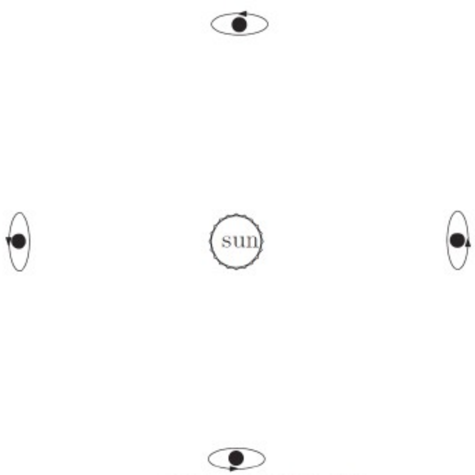
N.B. Oblateness of the Earth **only effects Ω and ω in the long-term**

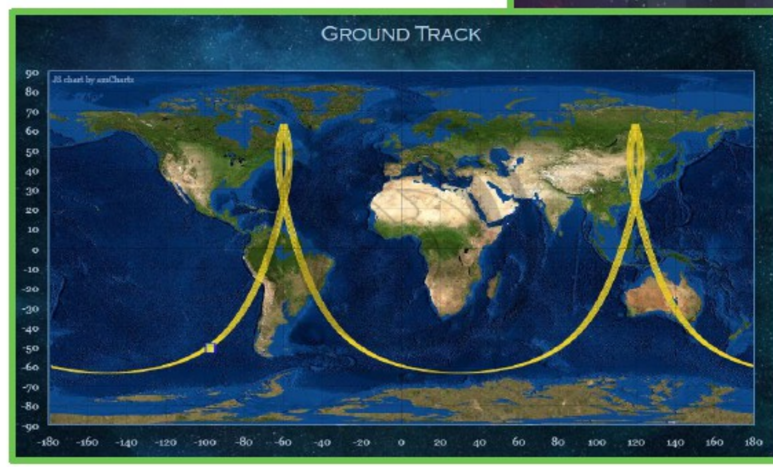
- Orbital plane rotates about the Earth's spin axis at an average rate of $\langle\dot{\Omega}\rangle$
- ω rotates about orbit normal at an average rate of $\langle\dot{\omega}\rangle$

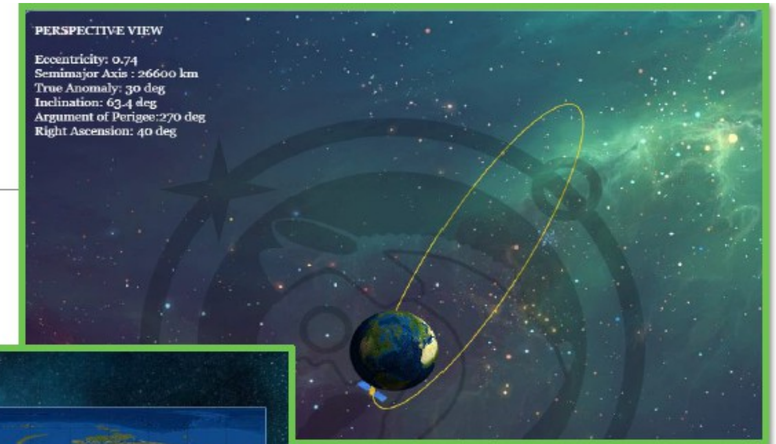
Orbital Perturbations

Special Types of Orbits

- Two orbit types leverage the secular variations of Ω and ω







Sun-Synchronous Orbits

- For a given a and e, we can choose i s.t. $\langle\dot{\Omega}\rangle = 360°/\text{year}$, i.e., the orbital plane rotates at the same rate as the Earth orbits around the sun

Molniya Orbits

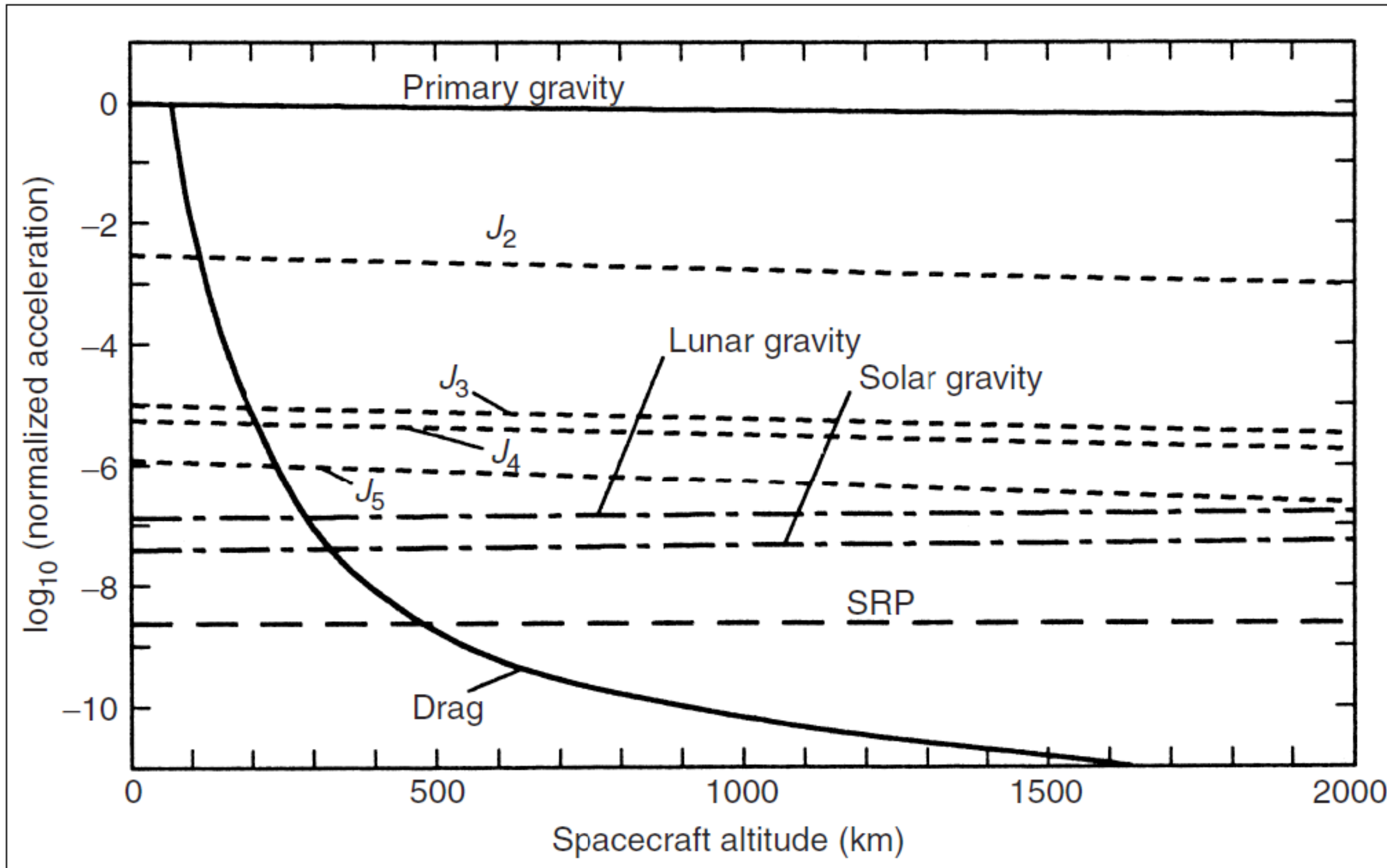
- By choosing $i = 63.4°$ or $116.6°$ the apse line does not move, and perigee does not advance ($\langle\dot{\omega}\rangle = 0$), i.e., a *frozen orbit*
- Highly eccentric, 12 h period, continuous coverage of high latitude regions, s/c spends majority of its time close to apogee

[NSSI, “Orbit Types”, https://www.youtube.com/watch?v=BvjlBpP4zU8]

Orbital Perturbations

Relative magnitude of main sources of perturbations acting on Earth-orbiting spacecraft (normalized to g) and using $A/m = 0.005\ \mathrm{m^2/kg}$

- Drag dominates at lower altitudes
- As altitude increases J_2 perturbations become the most significant effect
- SRP > Drag at around 600 km



[Fortescue, Ch. 4]

Low-Thrust Maneuvers

Low-Thrust Maneuvers

Low-thrust Transfers

- In contrast to maneuvers we have seen so far, we now consider **continuous low-thrust over a long period of time**
- We can easily derive equations that show this case using GVEs to model the low-thrust effects as perturbations to the OEs

We will use the GVEs for a, e and i:

$$\frac{da}{dt} = \frac{2a^2}{\sqrt{\mu a(1-e^2)}}[e\sin\theta\, f_r + (1+e\cos\theta)f_\theta]$$

$$\frac{de}{dt} = \sqrt{\frac{a(1-e^2)}{\mu}}\left[\sin\theta\, f_r + \frac{2\cos\theta + e(1+\cos^2\theta)}{(1+e\cos\theta)}f_\theta\right]$$

$$\frac{di}{dt} = \sqrt{\frac{a(1-e^2)}{\mu}}\frac{\cos(\omega+\theta)}{1+e\cos\theta}f_z$$

Given (ii) we can perform a Taylor series and drop the high-order terms, i.e., neglect $e^i f_j, i = 1,2,3,\dots, j = r,\theta,z$, terms

$$\frac{da}{dt} = 2\sqrt{\frac{a^3}{\mu}}f_\theta$$

$$\frac{de}{dt} = \sqrt{\frac{a}{\mu}}[\sin\theta\, f_r + 2\cos\theta\, f_\theta]$$

$$\frac{di}{dt} = \sqrt{\frac{a}{\mu}}\cos(\omega+\theta)\, f_z$$

Since thrust magnitude is const., we can express f in terms of steering angles

$$f = \sqrt{f_r^2 + f_\theta^2 + f_z^2}$$

<u>We will make some simplifying assumptions:</u>

(i) The magnitude of the applied thrust is small and constant

(ii) The eccentricity of the orbit throughout the maneuver remains small

(iii) The transfer time (T_m) is long relative to the orbital period (T)

Also, let us assume that we want to minimize transfer time (T_m)

Low-Thrust Maneuvers

Low-thrust Transfers

- In contrast to maneuvers we have seen so far, we now consider **continuous low-thrust over a long period of time**
- We can easily derive equations that show this case using GVEs to model the low-thrust effects as perturbations to the OEs

$$\frac{da}{dt} = 2\sqrt{\frac{a^3}{\mu}} f_\theta$$

$$\frac{de}{dt} = \sqrt{\frac{a}{\mu}} [\sin\theta\, f_r + 2\cos\theta\, f_\theta]$$

$$\frac{di}{dt} = \sqrt{\frac{a}{\mu}} \cos(\omega + \theta)\, f_z$$

We'll also assume $\omega = 0$ given our assumptions

Since thrust magnitude is const., we can express f in terms of steering angles

$$f = \sqrt{f_r^2 + f_\theta^2 + f_z^2}$$

$$f_r = f\cos\beta\sin\alpha\,, \qquad f_\theta = f\cos\beta\cos\alpha\,,$$

$$f_z = f\sin\beta$$

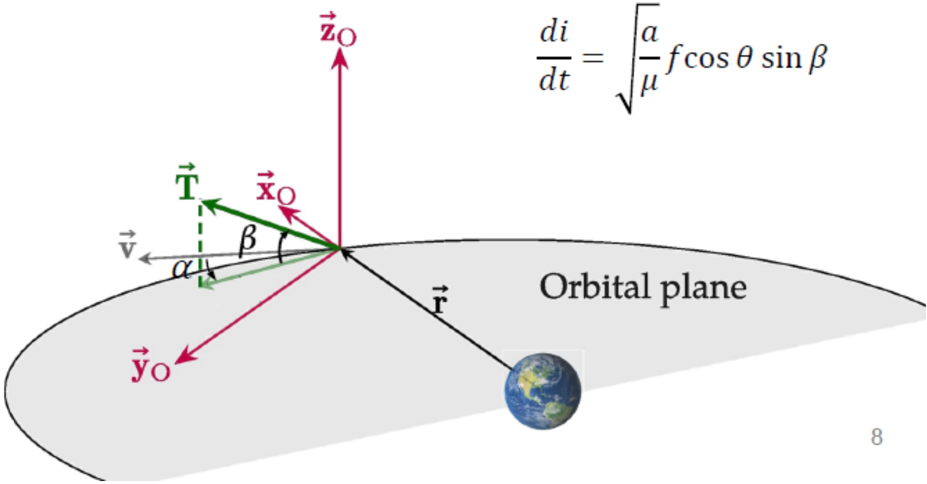
Two steering angles:

- α = angle between $\vec{\mathbf{v}}$ and thrust component in the orbital plane
- β = angle between the thrust vector and the orbital plane

$$\frac{da}{dt} = 2\sqrt{\frac{a^3}{\mu}} f\cos\beta\cos\alpha$$

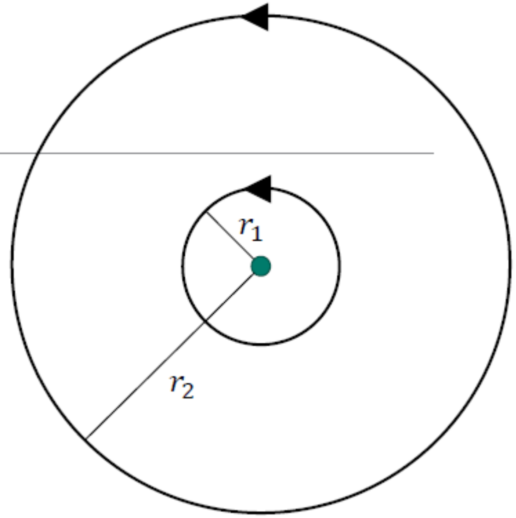
$$\frac{de}{dt} = \sqrt{\frac{a}{\mu}} f[\sin\theta\cos\beta\sin\alpha + 2\cos\theta\cos\beta\cos\alpha]$$

$$\frac{di}{dt} = \sqrt{\frac{a}{\mu}} f\cos\theta\sin\beta$$



Low-Thrust Maneuvers

$$\boxed{\frac{da}{dt} = 2\sqrt{\frac{a^3}{\mu}} f \cos\beta \cos\alpha}$$



Low-thrust Transfers

Coplanar Circle to Circle Transfers

Choose $\alpha = \beta = 0$ to maximize da/dt

- With our previous assumptions, in order to minimize T_{m} we will maximize da/dt by pointing the thrust along/against the velocity vector

$$\frac{1}{2}\sqrt{\frac{\mu}{a^3}}\frac{da}{dt} = f$$

This also leads to no inclination change (since thrust is in the plane)

- Velocity change (Δv) can be found by integration of both sides

$$\frac{1}{2}\int_{r_1}^{r_2} \sqrt{\frac{\mu}{a^3}} da = \int_0^{T_{\mathrm{m}}} f dt = \Delta v = f T_{\mathrm{m}}$$

$$\boxed{\Delta v = \sqrt{\frac{\mu}{r_1}} - \sqrt{\frac{\mu}{r_2}}}$$

To see the spiral nature of the transfer, we can substitute in: $dt = \sqrt{a^3/\mu}\, d\theta$

$$\frac{1}{2}\int_{r_1}^{a(t)} \frac{\mu}{a^3} da = \int_0^{\theta(t)} f d\theta$$

$$\boxed{f\theta(t) = \frac{\mu}{4}\left(\frac{1}{r_1^2} - \frac{1}{a^2(t)}\right)}$$

Evolution of a throughout the maneuver can be found by replacing the upper limit of the integral with $a(t)$

$$\boxed{\sqrt{\frac{\mu}{r_1}} - \sqrt{\frac{\mu}{a(t)}} = ft}$$

Low-Thrust Maneuvers

$$\frac{da}{dt} = 2\sqrt{\frac{a^3}{\mu}} f \cos\beta \cos\alpha$$

Low-thrust Transfers

Coplanar Circle to Circle Transfers

Choose $\alpha = \beta = 0$ to maximize da/dt

- With our previous assumptions, in order to minimize T_{m} we will maximize da/dt by pointing the thrust along/against the velocity vector

$$\frac{1}{2}\sqrt{\frac{\mu}{a^3}}\frac{da}{dt} = f$$

This also leads to no inclination change (since thrust is in the plane)

- Velocity change (Δv) can be found by integration of both sides

$$\frac{1}{2}\int_{r_1}^{r_2}\sqrt{\frac{\mu}{a^3}}da = \int_0^{T_{\mathrm{m}}} f dt = \Delta v = fT_{\mathrm{m}}$$

$$\Delta v = \sqrt{\frac{\mu}{r_1}} - \sqrt{\frac{\mu}{r_2}}$$

Evolution of a throughout the maneuver can be found by replacing the upper limit of the integral with $a(t)$

$$\sqrt{\frac{\mu}{r_1}} - \sqrt{\frac{\mu}{a(t)}} = ft$$

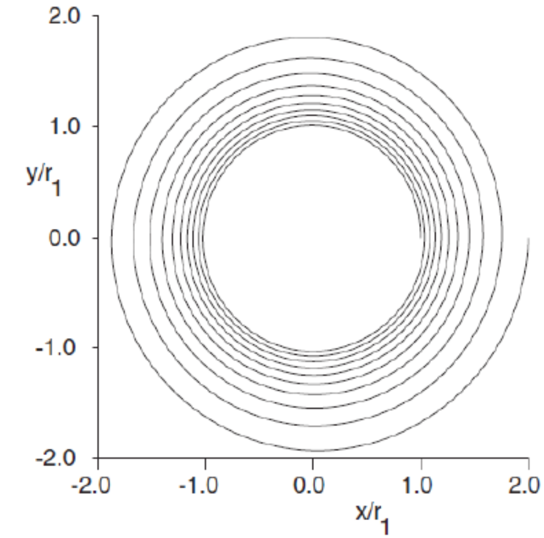


Figure 8.2 Circle-to-circle low thrust transfer for $r_2/r_1 = 2$ and a 10 revolution transfer

To see the spiral nature of the transfer, we can substitute in: $dt = \sqrt{a^3/\mu}d\theta$

$$\frac{1}{2}\int_{r_1}^{a(t)}\frac{\mu}{a^3}da = \int_0^{\theta(t)} f d\theta$$

$$f\theta(t) = \frac{\mu}{4}\left(\frac{1}{r_1^2} - \frac{1}{a^2(t)}\right)$$

Total number of revolutions: $[\theta(T_m)/2\pi]$

is found by setting $t = T_{\mathrm{m}}$ and $a(t) = a(T_{\mathrm{m}}) = r_2$

Clarkson

Low-Thrust Maneuvers

Low-thrust Transfers

Plane Change Maneuver

- With our previous assumptions, in order to minimize T_{m} we will maximize di/dt while setting $da/dt = de/dt = 0$

$$\boxed{\frac{da}{dt} = 2\sqrt{\frac{a^3}{\mu}} f \cos\beta \cos\alpha}$$

$$\boxed{\frac{de}{dt} = \sqrt{\frac{a}{\mu}} f[\sin\theta \cos\beta \sin\alpha + 2\cos\theta \cos\beta \cos\alpha]}$$

$$\boxed{\frac{di}{dt} = \sqrt{\frac{a}{\mu}} f \cos\theta \sin\beta}$$

Choose β to maximize di/dt

$$\sin\beta = \mathrm{sgn}(\cos\theta), \qquad \mathrm{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

$\mathrm{sgn}(\mathrm{x})$ *is the signum function*

- Setting $a = r_1$, $v_1 = \sqrt{\mu/r_1}$, and β according to the equation above: $\frac{di}{dt} = \frac{f}{v_1}|\cos\theta|$

Note, it turns out this is not a minimum time transfer (which requires varying β) from orbit to orbit, but it is close to optimal for small changes in inclination

Let us now relate the Δv for one orbit ($T_m = T$) to the change in inclination Δi,

Set $dt = T/(2\pi)d\theta$ and integrate from $t = 0$ to $t = T$:

$$\Delta i = \frac{fT}{2\pi v_1}\int_0^{2\pi} |\cos\theta|\, d\theta = \frac{2}{\pi}\frac{\Delta v}{v_1}$$

Low-Thrust Maneuvers

Quick Activity

Low-thrust Transfers

A satellite is in a prograde circular orbit about the Earth at an altitude of 500 km, and needs to be placed into a prograde circular orbit with an altitude of 16 000 km. If a low-thrust transfer is performed, calculate the total Δv and time of flight in years if the satellite exerts a constant specific thrust of $f = 6 \times 10^{-5}\ \mathrm{N/kg}$. Assume the radius of the Earth is $6371\ \mathrm{km}$ and $\mu_\oplus = 398\,600\ \mathrm{km^3/s^2}$.

Start by setting up your variables: $r_1 = 6371\ \mathrm{km} + 500\ \mathrm{km} = 6871\ \mathrm{km}$

$$r_2 = 6371\ \mathrm{km} + 16\,000\ \mathrm{km} = 22\,371\ \mathrm{km}$$

$$\boxed{\Delta v = \sqrt{\frac{\mu}{r_1}} - \sqrt{\frac{\mu}{r_2}}} \qquad \Delta v = \sqrt{\frac{\mu}{r_1}} - \sqrt{\frac{\mu}{r_2}} = \sqrt{\frac{(398\,600\ \mathrm{km^3/s^2})}{(6871\ \mathrm{km})}} - \sqrt{\frac{(398\,600\ \mathrm{km^3/s^2})}{(22\,371\ \mathrm{km})}} \qquad \boxed{\Delta v = 3.395\ \mathrm{km/s}}$$

Now, solve for the time of flight with the specific thrust of $f = 6 \times 10^{-5}\ \mathrm{N/kg}$.

$$\boxed{\Delta v = f T_\mathrm{m}} \qquad T_\mathrm{m} = \frac{\Delta v}{f} = \frac{3395\ \mathrm{m/s}}{6 \times 10^{-5}\ \mathrm{N/kg}} = 56\,583\,333\ \mathrm{s} \qquad \boxed{T_\mathrm{m} = 1.794\ \text{years}}$$

Low-Thrust Maneuvers

$\boxed{r_1 = 6871 \text{ km}}$ $\boxed{\Delta v = 3.395 \text{ km/s}}$

$\boxed{r_2 = 22\,371 \text{ km}}$ $\boxed{T_m = 1.794 \text{ years}}$

Quick Activity

Low-thrust Transfers

A satellite is in a prograde circular orbit about the Earth at an altitude of 500 km, and needs to be placed into a prograde circular orbit with an altitude of 16 000 km. If a low-thrust transfer is performed, calculate the total Δv and time of flight in years if the satellite exerts a constant specific thrust of $f = 6 \times 10^{-5}$ N/kg. Assume the radius of the Earth is 6371 km and $\mu_{\oplus} = 398\,600 \text{ km}^3/\text{s}^2$.

If on our next orbit, the thrust is then applied to increase the orbit's inclination, what would be the change in inclination after one orbit?

$$\boxed{\Delta i = \frac{2}{\pi}\frac{\Delta v}{v}}$$

$$\boxed{\Delta v = f T_m}$$

$$v_2 = \sqrt{\frac{\mu}{r_2}} = \sqrt{\frac{(398\,600 \text{ km}^3/\text{s}^2)}{(22\,371 \text{ km})}} = 4.22 \text{ km/s}$$

$$T = 2\pi\sqrt{\frac{r_2^3}{\mu}} = 2\pi\sqrt{\frac{(22\,371 \text{ km})^3}{(398\,600 \text{ km}^3/\text{s}^2)}} = 33\,300 \text{ s} = 9.25 \text{ h}$$

$$\Delta v = fT = \left(6 \times 10^{-5}\frac{\text{N}}{\text{kg}}\right)(33\,300 \text{ s}) = 1.998 \text{ m/s}$$

$$\Delta i = \frac{2}{\pi}\frac{\Delta v}{v} = \frac{2}{\pi}\frac{1.998 \text{ m/s}}{(4220 \text{ m/s})} = 3.014 \times 10^{-4} \text{ rad} \qquad \boxed{\Delta i = 3.014 \times 10^{-4} \text{ rad}}$$