Student Name: | **Student No:**

AE470/ME570 – Assignment 4 (Solutions)

Spring 2023

Grade: / 10

Above not to be filled by student

Student Name: | **Student No:**

Important Information

The following constraints are provided to help students be successful in this Assignment[1]:

- **Must** use the provided assignment sheets (hand-written or typed solutions are both acceptable)
- **Must** include student name and number on all pages
- **Must** include page numbers on all pages
- **Must** be submitted as a single PDF file
- **Must** include solutions with the correct units
- **Must** include calculations or motivation for solutions
- **Must** be presented clearly

Students should check their answers for possible calculation errors. In addition to correctness of the solutions, clarity and organization in the presentation of solutions will be considered in the grading.

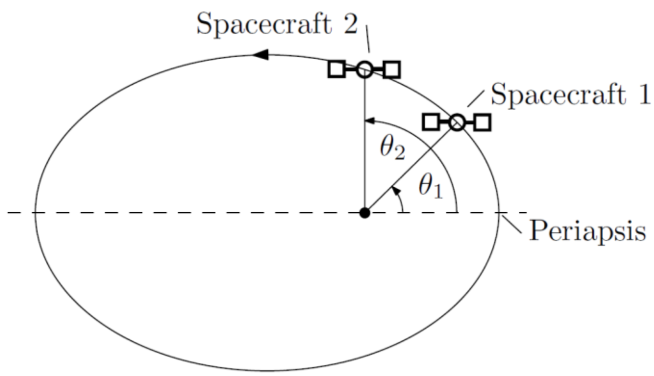
Due Date and Time	Percent of Final Grade	Submission Type
Apr. 7, 2023 at 11:59 p.m.[2]	10%	Individual

[1] Unlike in industry, or in other academic contexts, violating a constraint on this assignment will not result in the assignment being rejected (e.g., not being assessed or evaluated and instead considered not to have been submitted). Instead an appropriate penalty will be applied to the final evaluation.

[2] As per the course syllabus, late submissions will generally not be accepted (except for reasons of illness, etc.).

Problem 1 (25%)

Roscosmos has placed a space debris capturing satellite in the same orbit about the Earth as a large defunct satellite. The orbit is specified by: $e = 0.2$ and $a = 10{,}000\ \text{km}$, and at the present time the capturing satellite and the defunct satellite have true anomalies: $\theta_1 = 45°$ and $\theta_2 = 90°$, respectively. Determine the Δv that the capturing satellite must apply at periapsis to catch the defunct satellite in a single tangential maneuver. Note Earth's gravitational parameter: $\mu_{\oplus} = 3.986 \times 10^5\ \text{km}^3/\text{s}^2$.



Solution 1

To solve this problem, we utilize a similar procedure as in lecture (also described in Sec. 5.5 of the textbook), but for an elliptical orbit. Since we have an elliptical orbit, we must first compute the eccentric anomalies to find the current mean anomalies:

$$E_1 = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta_1}{2}\right) = 0.6523\ \text{rad}$$

$$E_2 = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta_2}{2}\right) = 1.3694\ \text{rad}$$

3%

Next, we use Kepler's equation to compute the mean anomalies:

$$M_1 = E_1 - e\sin E_1 = 0.5309\ \text{rad}$$

$$M_2 = E_2 - e\sin E_2 = 1.1735\ \text{rad}$$

3%

Now that we have the mean anomalies, we can see that spacecraft 2 (defunct satellite) is ahead of spacecraft 1 (capturing satellite) as follows:

$$\Delta M = M_2 - M_1 = 0.6426\ \text{rad}$$

1%

Next, we compute the mean orbital angular rate:

$$\eta = \sqrt{\frac{\mu_\oplus}{a^3}} = 6.3135 \times 10^4 \text{ rad/s}$$

2%

So, spacecraft 2 is ahead of spacecraft 1 by:

$$\frac{\Delta M}{n} = 1017.9 \text{ s}$$

2%

The orbital period is given by:

$$T = \frac{2\pi}{n} = 9952 \text{ s}$$

2%

As such, for the capturing satellite (spacecraft 1) to rendezvous with the defunct satellite (spacecraft 2) using a single maneuver applied at periapsis, it must move into an orbit with a period:

$$T_{trans} = T - \frac{\Delta M}{n} = 8934.2 \text{ s}$$

2%

So that the capturing satellite and defunct satellite will both arrive at periapsis after one revolution of the capturing satellite around the transfer orbit.

The semi-major axis of the transfer orbit is given by:

$$a_{trans} = \left[\mu_\oplus \left(\frac{T_{trans}}{2\pi}\right)\right]^{1/3} = 9306 \text{ km}$$

2%

The radius of perigee (same for both orbits):

$$r_p = a(1 - e) = 8000 \text{ km}$$

2%

Using the vis-viva equation, the orbital speeds at perigee for both the current orbit and the transfer orbit are as follows:

$$v_p = \sqrt{\mu_\oplus \left(\frac{2}{r_p} - \frac{1}{a}\right)} = 7.7324 \text{ km/s}$$

and

$$v_{p,trans} = \sqrt{\mu_\oplus \left(\frac{2}{r_p} - \frac{1}{a_{trans}}\right)} = 7.5377 \text{ km/s}$$

4%

Since $v_{p,trans} < v_p$, a tangential speed reduction is required of magnitude:

$$\Delta v = v_p - v_{p,trans} = 0.1947 \text{ km/s}$$

2%

Problem 2 (50%)

As part of a concept study for NASA's CubeSat UV Experiment (CUVE), which is a proposed mission to study atmospheric processes at Venus, you have been asked to analyze the interplanetary Hohmann transfer from Earth to Venus. You may assume that Earth and Venus are in circular, coplanar orbits.

The mothership carrying CUVE will start in a circular parking orbit about the Earth with a radius of $100{,}000\ \mathrm{km}$ and it will release CUVE when it arrives in a circular capture orbit about Venus with a radius of $50{,}000\ \mathrm{km}$.

The following parameters are given:

	Sun	**Earth**	**Venus**
Gravitational parameter μ $[\mathrm{km^3/s^2}]$	1.327×10^{11}	3.986×10^5	3.257×10^5
Mean orbital radius about the Sun [km]	-	149.6×10^6	108.2×10^6

Find the following parameters for the transfer:

(a) Semimajor axis of the transfer orbit
(b) Time of flight of the transfer
(c) Time in days of the required departure from Earth, assuming Earth, Venus and the Sun lie along the same line at $t = 0$ (on the same side of the Sun)
(d) Required hyperbolic excess speed upon exiting the Earth's sphere of influence and required hyperbolic excess speed upon entering Venus' sphere of influence
(e) Location and magnitude of the delta-v required for Earth departure
(f) Required arrival asymptote offset ($-b$) and magnitude of the delta-v required for Venus capture
(g) Total delta-v for the trip

Solution 2

(a) Semimajor axis of the transfer orbit is given by:

$$a_t = \frac{R_{Earth} + R_{venus}}{2} = 128.9 \times 10^6\ \mathrm{km}$$

2%

(b) The time of flight of the transfer is half the period of the transfer orbit:

$$T_{ev} = \pi \sqrt{\frac{a_t^3}{\mu_{sun}}} = 1.262 \times 10^7 \mathrm{s} = 146.1\ \mathrm{days}$$

2%

(c) Since we have assumed that Earth, Venus, and the Sun lie along the same line at $t = 0$, then the initial phase between them is as follows:

$$\phi(0) = 0\ \mathrm{rad} = 0°$$

2%

The following function can then find the mean orbital motions of Venus and Earth to solve for the required phase of departure:

$$n_{venus} = \sqrt{\frac{\mu_{sun}}{R_{venus}^3}} = 3.237 \times 10^{-7} \text{ rad/s}$$

$$n_{Earth} = \sqrt{\frac{\mu_{sun}}{R_{Earth}^3}} = 1.991 \times 10^{-7} \text{ rad/s}$$

3%

From this, we can find the required phase for departure:

$$\phi_d = \pi - n_{venus}T_{ev} = -0.9434 \text{ rad} = -54.05°$$

2%

Since $\phi_d < \phi(0)$, but $(n_{venus} - n_{earth}) > 0$, this means $\phi(t)$ is increasing, and hence, we will need to wait until the following required phase for departure:

$$\phi_d = -0.9430 \text{ rad} + 2\pi = 5.340 \text{ rad} = 305.9°$$

2%

The required time for departure can then be found by rearranging the following equation:

$$\phi_d = \phi(0) + (n_{venus} - n_{earth})t$$

$$t = \frac{\phi_d - \phi_0}{n_{venus} - n_{earth}} = 4.286 \times 10^7 \text{ s} = 496.1 \text{ days}$$

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(d) The planetary orbital speeds are:

$$V_{venus} = \sqrt{\frac{\mu_{sun}}{R_{venus}}} = 35.02 \text{ km/s}$$

$$V_{earth} = \sqrt{\frac{\mu_{sun}}{R_{earth}}} = 29.78 \text{ km/s}$$

3%

The heliocentric orbital speed for the transfer orbit at departure is given by:

$$V_{dep} = \sqrt{\mu_{sun}\left(\frac{2}{R_{earth}} - \frac{1}{a_t}\right)} = 27.29 \text{ km/s}$$

1.5%

The heliocentric orbital speed for the transfer orbit at arrival is given by:

$$V_{arr} = \sqrt{\mu_{sun}\left(\frac{2}{R_{venus}} - \frac{1}{a_t}\right)} = 37.73 \text{ km/s}$$

1.5%

With these equations, we can find the hyperbolic excess speed upon exiting the Earth's sphere of influence and the hyperbolic excess speed upon entering Venus' sphere of influence:

$$v_{\infty,dep} = V_{Earth} - V_{dep} = 2.496 \text{ km/s}$$

$$v_{\infty,arr} = V_{arr} - V_{venus} = 2.707 \text{ km/s}$$

*Note, since $V_{earth} > V_{dep}$, the spacecraft must depart in the direction opposite to Earth's velocity vector.

4%

(e) Using a tangential impulse to depart the circular parking orbit about the Earth, impulse location becomes the perigee of the departure hyperbola, with radius equal to that of the parking orbit, hence:

$$\frac{v_{p,dep}^2}{2} - \frac{\mu_{earth}}{r_{park}} = \frac{v_{\infty,dep}^2}{2}$$

Rearranging, we find the orbital speed at perigee of the departure hyperbola:

$$v_{p,dep} = \sqrt{v_{\infty,dep}^2 + \frac{2\mu_{earth}}{r_{park}}} = 3.768 \text{ km/s}$$

2%

The orbital speed of the circular parking orbit is given by:

$$v_{park} = \sqrt{\frac{\mu_{earth}}{r_{park}}} = 1.997 \text{ km/s}$$

2%

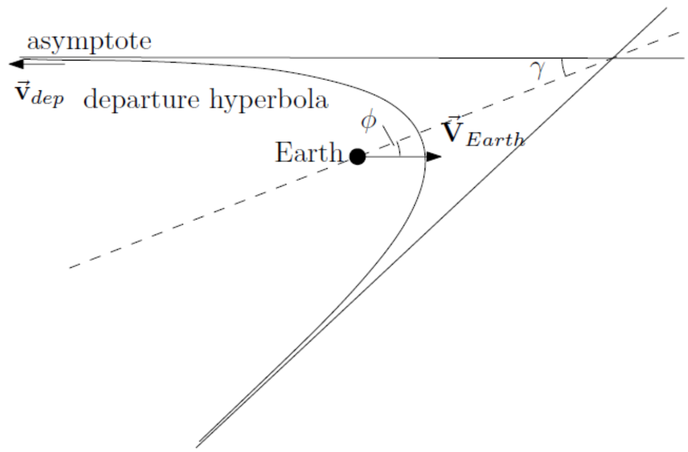
The required magnitude of the velocity change is then:

$$\Delta v_{dep} = v_{p,dep} - v_{park} = 1.772 \text{ km/s}$$

2%

Since $V_{dep} < V_{earth}$, the spacecraft must depart Earth's SOI in the opposite direction as the Earth's velocity vector, and the required phase of the tangential impulse is given by:

$$\phi = \gamma = \cos^{-1}\left(\frac{1}{e_{dep}}\right), \text{ where } e_{dep} \text{ is the eccentricity of the departure hyperbola}$$



To find e_{dep}, we must first compute the following:

$$a_{dep} = -\frac{\mu_{earth}}{2\varepsilon_{dep}}, \qquad \text{where } \varepsilon_{dep} = \frac{v_{\infty,dep}^2}{2} \text{ is the orbital energy of the departure orbit}$$

We then find:

$$a_{dep} = -\frac{\mu_{earth}}{v_{\infty,dep}^2} = -6.398 \times 10^4 \text{ km}$$

2%

Next, we rearrange the radius of perigee for the departure hyperbola to find the eccentricity:

$$r_{park} = a_{dep}(1 - e_{dep})$$

$$e_{dep} = 1 - \frac{r_{park}}{a_{dep}} = 2.563$$

2%

The location of (or the required phase of) the tangential impulse is then:

$$\phi_{dep} = \cos^{-1}\left(\frac{1}{e_{dep}}\right) = 1.170 \text{ rad} = 67.04°$$

2%

(f) Start by determining a_{arr} for the arrival hyperbola:

$$a_{arr} = -\frac{\mu_{venus}}{v_{\infty,arr}^2} = -44.44 \times 10^3 \text{ km}$$

2%

We need to transfer the spacecraft into a circular capture orbit of radius, r_{cap}, using a tangential maneuver, so we find the radius of periapsis of the hyperbolic arrival orbit must be r_{cap} and we can rearrange the following formula to solve for b:

$$r_{p2} = a_2 \left[1 - \sqrt{1 + \left(\frac{b_2}{a_2}\right)^2}\right]$$

$$b = a_{arr} \sqrt{\left(1 - \frac{r_{cap}}{a_{arr}}\right)^2 - 1} = -83.33 \times 10^3 \text{ km}$$

2%

This gives us the required arrival hyperbola asymptote offset:

$$-b = 83.33 \times 10^3 \text{ km}$$

1%

We can then find the orbital speed at periapsis of the arrival hyperbola:

$$v_{p,arr} = \sqrt{v_{\infty,arr}^2 + \frac{2\mu_{venus}}{r_{cap}}} = 4.512 \text{ km/s}$$

2%

And the orbital speed of the circular capture orbit is as follows:

$$v_{cap} = \sqrt{\frac{\mu_{venus}}{r_{cap}}} = 2.552 \text{ km/s}$$

2%

This allows us to find the magnitude of the delta-v required for Venus capture:

$$\Delta v_{arr} = v_{p,arr} - v_{cap} = 1.960 \text{ km/s}$$

2%

(g) The total Δv for the trip is given by:

$$\Delta v = \Delta v_{dep} + \Delta v_{arr} = 3.732 \text{ km/s}$$

2%

Problem 3 (25%)

The United States Space Surveillance Network made two observations of a satellite in orbit about the Earth using ground-based radar, with 740.6 s between each observation. The position vectors obtained at each time are as follows:

$$\vec{\mathbf{r}}_1 = \vec{\boldsymbol{\mathcal{F}}}_G^{\mathrm{T}} \begin{bmatrix} 3467.3 \\ 3467.3 \\ 4903.5 \end{bmatrix} \text{ km}, \qquad \vec{\mathbf{r}}_2 = \vec{\boldsymbol{\mathcal{F}}}_G^{\mathrm{T}} \begin{bmatrix} 0 \\ 0 \\ 7425.0 \end{bmatrix} \text{ km}.$$

Note that Earth's gravitational parameter is $\mu_\oplus = 3.986 \times 10^5 \text{ km}^3/\text{s}^2$ and you may assume $\eta = \eta_H$ as the sector-triangle ratio.

(a) What is the orbital period of the satellite?
(b) What is the eccentricity of the satellite's orbit?

Solution 3

(a) The orbital period is given by:

$$T = 2\pi \sqrt{\frac{a^3}{\mu_\oplus}}$$

So, we must find the semi-major axis a, using the orbital energy ε:

$$a = -\frac{\mu_\oplus}{2\varepsilon}$$

Orbital energy is by definition:

$$\varepsilon = \frac{\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}}{2} - \frac{\mu_\oplus}{|\vec{\mathbf{r}}|}$$

We shall evaluate this at time t_1 and since we know $\vec{\mathbf{r}}_1$, we just need to find $\vec{\mathbf{v}}_1$. Since we have $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$, and the time of flight $t_2 - t_1 = 740.6$ s, we can solve Lambert's problem to find $\vec{\mathbf{v}}_1$:

1%

$$\vec{\mathbf{v}}_1 = \frac{\vec{\mathbf{r}}_2 - F\vec{\mathbf{r}}_1}{G},$$

where

$$F = 1 - \frac{r_2}{p}[1 - \cos(\theta_2 - \theta_1)], \qquad G = \frac{r_1 r_2}{\sqrt{\mu p}} \sin(\theta_2 - \theta_1)$$

where $\theta_2 - \theta_1$ is the angle between $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$, and $r_1 = |\vec{\mathbf{r}}_1|$ and $r_2 = |\vec{\mathbf{r}}_2|$, i.e.,

$$r_1 = 6934.6 \text{ km}, \qquad r_2 = 7425 \text{ km}.$$

2%

We can also find $\theta_2 - \theta_1$ as follows:

$$\theta_2 - \theta_1 = \cos^{-1}\left(\frac{\vec{\mathbf{r}}_1 \cdot \vec{\mathbf{r}}_2}{r_1 r_2}\right) = 0.7854 \text{ rad} = 45°$$

2%

Now, in order to find the semi-latus rectum, p, we need the sector-triangle ratio η. However, we know that we can approximate $\eta = \eta_H$, as follows:

$$\eta_H = \frac{12}{22} + \frac{10}{22}\sqrt{1 + \frac{44}{9}\frac{m}{l + 5/6}}\,,$$

where

$$m = \frac{\mu_\oplus (t_2 - t_1)^2}{\left[2\sqrt{r_1 r_2}\cos\left((\theta_2 - \theta_1)/2\right)\right]^3} = 0.093796$$

$$l = \frac{r_1 + r_2}{\left(4\sqrt{r_1 r_2}\cos\left((\theta_2 - \theta_1)/2\right)\right)} - \frac{1}{2} = 0.041512$$

So, we obtain:

$$\eta_H = 1.1066$$

4%

We can now find p:

$$p = \frac{\eta^2 |\vec{\mathbf{r}}_1 \times \vec{\mathbf{r}}_2|^2}{\mu_\oplus (t_2 - t_1)^2} = 7425.2 \text{ km}$$

2%

Using r_1, r_2, p, θ_1 and θ_2 we can now solve for the Lagrangian coefficients:

$$F = 1 - \frac{r_2}{p}[1 - \cos(\theta_2 - \theta_1)] = 0.70712, \qquad G = \frac{r_1 r_2}{\sqrt{\mu p}}\sin(\theta_2 - \theta_1) = 669.24$$

4%

And then compute $\vec{\mathbf{v}}_1$:

$$\vec{\mathbf{v}}_1 = \frac{\vec{\mathbf{r}}_2 - F\vec{\mathbf{r}}_1}{G} = \vec{\boldsymbol{\mathcal{F}}}_G^{\mathrm{T}} \begin{bmatrix} -3.6636 \\ -3.6636 \\ 5.9136 \end{bmatrix} \text{ km/s}$$

2%

Now, we can find the orbital energy:

$$\varepsilon = \frac{\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}}{2} - \frac{\mu_\oplus}{|\vec{\mathbf{r}}|} = -26.573 \text{ km}^2/\text{s}^2$$

2%

And the semi-major axis is:

$$a = -\frac{\mu_\oplus}{2\varepsilon} = 7500.4 \text{ km}$$

2%

This allows us to finally compute the orbital period:

$$T = 2\pi\sqrt{\frac{a^3}{\mu_\oplus}} = 6464.4 \text{ s}$$

2%

(b) The eccentricity can now be found quite simply by using the semi-latus rectum and the semi-major axis, i.e.,

$$p = a(1 - e^2)$$

Which rearranges to give:

$$e = \sqrt{1 - \frac{p}{a}} = 0.1$$

2%