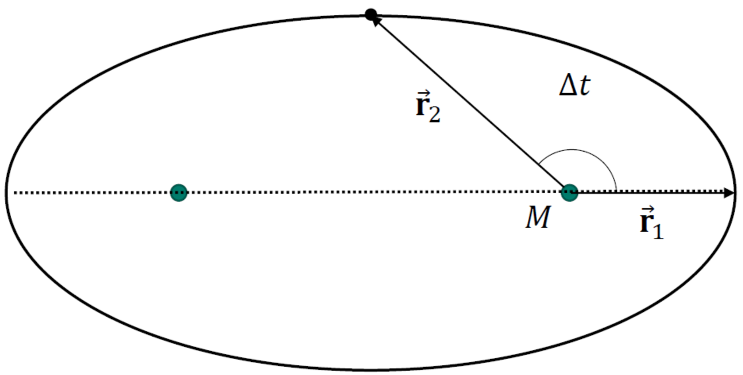
Orbit Determination

What is orbit determination?

- The process of estimating (or determining) the orbit of a body (e.g., spacecraft, asteroid, planet) in space
- Estimation is based on the type of observational measurements available
- There are many methods of orbit determination, and there are many very sophisticated methods based on statistical approaches
- When an orbit is determined from observational data, orbital propagation techniques are then used
- Orbits eventually diverge from their predicted paths
- *Chapter 4 presents three methods, we will start in this course with* **Lambert's Problem**



There are several solutions to Lambert's problem, we will focus on **Gauss' solution and restrict it to elliptical orbits with transfer angles less than 90°**

Lambert's Problem

- Orbit determination from **two position vectors** and **time**
- Very important problem that has been studied extensively
- First solved by Johann Heinrich Lambert (18th century)
- Applications in orbit transfer, rendezvous, and targeting

Orbit Determination

Lambert's Problem

- To start, let's recognize: (i) orbital motion is planar and (ii) the position ($\vec{\mathbf{r}}$) and velocity ($\vec{\mathbf{v}}$) vectors are not parallel
- This allows us to relate $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$ at one point in an orbit to another by the relations:

$$\left.\begin{aligned}\vec{\mathbf{r}}_2 &= F\vec{\mathbf{r}}_1 + G\vec{\mathbf{v}}_1 \\ \vec{\mathbf{v}}_2 &= F_t\vec{\mathbf{r}}_1 + G_t\vec{\mathbf{v}}_1\end{aligned}\right\}$$

Scalars F, G, F_t, and G_t are known as the **Lagrangian Coefficients**

We can obtain the velocity, $\vec{\mathbf{v}}_1$, if we know F and G

$$\vec{\mathbf{v}}_1 = \frac{\vec{\mathbf{r}}_2 - F\vec{\mathbf{r}}_1}{G}$$

To find F and G, let's start by expressing $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$ in the perifocal frame, $\mathcal{F}_p$

$$\vec{\boldsymbol{\mathcal{F}}}_{\mathrm{P}}^{\mathrm{T}} = [\hat{\mathbf{x}}_p \quad \hat{\mathbf{y}}_p \quad \hat{\mathbf{z}}_p]$$

$$\boxed{\begin{gathered}\vec{\mathbf{r}} = \vec{\boldsymbol{\mathcal{F}}}_{\mathrm{P}}^{\mathrm{T}}\mathbf{r}_{\mathrm{P}} \\ \mathbf{r}_{\mathrm{P}} = [r\cos\theta \quad r\sin\theta \quad 0]^{\mathrm{T}}\end{gathered}}$$

$$\boxed{\begin{gathered}\vec{\mathbf{v}} = \vec{\boldsymbol{\mathcal{F}}}_{\mathrm{P}}^{\mathrm{T}}\mathbf{v}_{\mathrm{P}} \\ \mathbf{v}_{\mathrm{P}} = \left[-\sqrt{\frac{\mu}{p}}\sin\theta \quad \sqrt{\frac{\mu}{p}}(e+\cos\theta) \quad 0\right]^{\mathrm{T}}\end{gathered}}$$

$$\begin{bmatrix}\vec{\mathbf{r}} \\ \vec{\mathbf{v}}\end{bmatrix} = \begin{bmatrix} r\cos\theta & r\sin\theta \\ -\sqrt{\frac{\mu}{p}}\sin\theta & \sqrt{\frac{\mu}{p}}(e+\cos\theta)\end{bmatrix}\begin{bmatrix}\hat{\mathbf{x}}_p \\ \hat{\mathbf{y}}_p\end{bmatrix}$$

$$\boxed{r = \frac{p}{1+e\cos\theta}}$$

Let's take the determinant and see if the matrix is invertible

$$\det\left(\begin{bmatrix} r\cos\theta & r\sin\theta \\ -\sqrt{\frac{\mu}{p}}\sin\theta & \sqrt{\frac{\mu}{p}}(e+\cos\theta)\end{bmatrix}\right) = r\sqrt{\frac{\mu}{p}}\cos\theta\,(e+\cos\theta) + r\sqrt{\frac{\mu}{p}}\sin^2\theta \;= r\sqrt{\frac{\mu}{p}}(1+e\cos\theta)$$

$$= \sqrt{\mu p} \quad = h$$

Since $h \neq 0$ for an orbit, the <u>matrix is invertible</u>

Orbit Determination

$$\boxed{\begin{bmatrix}\vec{\mathbf{r}}\\ \vec{\mathbf{v}}\end{bmatrix} = \begin{bmatrix} r\cos\theta & r\sin\theta \\ -\sqrt{\frac{\mu}{p}}\sin\theta & \sqrt{\frac{\mu}{p}}(e+\cos\theta)\end{bmatrix}\begin{bmatrix}\hat{\mathbf{x}}_p\\ \hat{\mathbf{y}}_p\end{bmatrix}}$$

Lambert's Problem

- We can now take the inverse: $\begin{bmatrix} r\cos\theta & r\sin\theta \\ -\sqrt{\frac{\mu}{p}}\sin\theta & \sqrt{\frac{\mu}{p}}(e+\cos\theta)\end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{p}(e+\cos\theta) & -\frac{r}{\sqrt{\mu p}}\sin\theta \\ \frac{1}{p}\sin\theta & \frac{r}{\sqrt{\mu p}}\cos\theta\end{bmatrix}$
- The unit vectors for the $\mathcal{F}_p$ can now be obtained from $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$

$$\begin{bmatrix}\hat{\mathbf{x}}_p\\ \hat{\mathbf{y}}_p\end{bmatrix} = \begin{bmatrix} \frac{1}{p}(e+\cos\theta) & -\frac{r}{\sqrt{\mu p}}\sin\theta \\ \frac{1}{p}\sin\theta & \frac{r}{\sqrt{\mu p}}\cos\theta\end{bmatrix}\begin{bmatrix}\vec{\mathbf{r}}\\ \vec{\mathbf{v}}\end{bmatrix}$$

So we can represent $\vec{\mathbf{r}}_1, \vec{\mathbf{v}}_1$ and $\vec{\mathbf{r}}_2, \vec{\mathbf{v}}_2$:

$$\begin{bmatrix}\hat{\mathbf{x}}_p\\ \hat{\mathbf{y}}_p\end{bmatrix} = \begin{bmatrix} \frac{1}{p}(e+\cos\theta_1) & -\frac{r_1}{\sqrt{\mu p}}\sin\theta_1 \\ \frac{1}{p}\sin\theta_1 & \frac{r_1}{\sqrt{\mu p}}\cos\theta_1\end{bmatrix}\begin{bmatrix}\vec{\mathbf{r}}_1\\ \vec{\mathbf{v}}_1\end{bmatrix} \quad (1)$$

$$\begin{bmatrix}\vec{\mathbf{r}}_2\\ \vec{\mathbf{v}}_2\end{bmatrix} = \begin{bmatrix} r_2\cos\theta_2 & r_2\sin\theta_2 \\ -\sqrt{\frac{\mu}{p}}\sin\theta_2 & \sqrt{\frac{\mu}{p}}(e+\cos\theta_2)\end{bmatrix}\begin{bmatrix}\hat{\mathbf{x}}_p\\ \hat{\mathbf{y}}_p\end{bmatrix} \quad (2)$$

- Substitute 1 into 2:

$$\begin{bmatrix}\vec{\mathbf{r}}_2\\ \vec{\mathbf{v}}_2\end{bmatrix} = \begin{bmatrix} r_2\cos\theta_2 & r_2\sin\theta_2 \\ -\sqrt{\frac{\mu}{p}}\sin\theta_2 & \sqrt{\frac{\mu}{p}}(e+\cos\theta_2)\end{bmatrix}\begin{bmatrix} \frac{1}{p}(e+\cos\theta_1) & -\frac{r_1}{\sqrt{\mu p}}\sin\theta_1 \\ \frac{1}{p}\sin\theta_1 & \frac{r_1}{\sqrt{\mu p}}\cos\theta_1\end{bmatrix}\begin{bmatrix}\vec{\mathbf{r}}_1\\ \vec{\mathbf{v}}_1\end{bmatrix} \longrightarrow$$

$$F = \frac{r_2}{p}(e\cos\theta_2 + \cos\theta_2\cos\theta_1 + \sin\theta_2\sin\theta_1)$$

$$G = \frac{r_1 r_2}{\sqrt{\mu p}}(\sin\theta_2\cos\theta_1 - \cos\theta_2\sin\theta_1)$$

Remember, we only need F and G

$$\boxed{\vec{\mathbf{r}}_2 = F\vec{\mathbf{r}}_1 + G\vec{\mathbf{v}}_1}$$

$$\boxed{\vec{\mathbf{v}}_2 = F_t\vec{\mathbf{r}}_1 + G_t\vec{\mathbf{v}}_1}$$

Using trig. identities and the polar eq. for the orbit (for $e\cos\theta_2$)

$$\boxed{r = \frac{p}{1+e\cos\theta}}$$

$$\boxed{F = 1 - \frac{r_2}{p}(1-\cos(\theta_2-\theta_1))}$$

$$\boxed{G = \frac{r_1 r_2}{\sqrt{\mu p}}\sin(\theta_2-\theta_1)}$$

Clarkson

8

$$F = 1 - \frac{r_2}{p}(1-\cos(\theta_2-\theta_1))$$

$$G = \frac{r_1 r_2}{\sqrt{\mu p}}\sin(\theta_2-\theta_1)$$

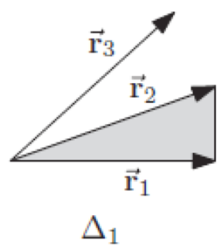
Orbit Determination

Lambert's Problem

- Looking closer at the Lagrangian Coefficients, we have: $r_1 = |\vec{\mathbf{r}}_1|, r_2 = |\vec{\mathbf{r}}_2|$, and $\theta_2 - \theta_1$ is just the angle between $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$
- So to determine the orbit, all we need is p ← Fortunately, we can find this by using $h = \sqrt{\mu p}$ and the sector-triangle area ratio η

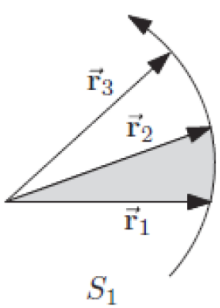
Sector-Triangle Area Ratio

- *We will try to describe this briefly (more details in de Ruiter, Ch. 4.2)*



Area of the triangle defined by $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$

$$\Delta_1 = \frac{1}{2}|\vec{\mathbf{r}}_1 \times \vec{\mathbf{r}}_2|$$



Area of the orbit sector defined by $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$

$$S_1 = \frac{h}{2}(t_2 - t_1)$$

(from Kepler's 2nd law)

Sector-triangle area ratio η

$$\eta_1 = \frac{S_1}{\Delta_1} = \frac{h(t_2-t_1)}{|\vec{\mathbf{r}}_1 \times \vec{\mathbf{r}}_2|}$$

Substitute in $\boxed{h = \sqrt{\mu p}}$ and rearrange:

$$p = \frac{\eta^2|\vec{\mathbf{r}}_1 \times \vec{\mathbf{r}}_2|^2}{\mu(t_2-t_1)^2}$$

So now, all we need to do is **find η given the transfer time**

The **sector-triangle ratio η for a two-body elliptical orbit between two vectors** can be solved for using a transcendental equation, which **must be solved iteratively**

$$\eta = 1 + \frac{m}{\eta^2}W\left(\frac{m}{\eta^2} - l\right)$$

where:

$$m = \frac{\mu(t_2-t_1)^2}{[2\sqrt{r_1 r_2}\cos((\theta_2-\theta_1)/2)]^3}$$

$$l = \frac{r_1 + r_2}{4\sqrt{r_1 r_2}\cos((\theta_2-\theta_1)/2)} - \frac{1}{2}$$

$$W(w) = \frac{2g - \sin 2g}{\sin^3 g}$$

$$g = 2\sin^{-1}\sqrt{w}$$

We can use an iterative procedure to compute (see Ch. 4.2 and 4.3)

Orbit Determination

<u>Summary of Gauss' Solution to Lambert's Problem</u> (elliptical orbit, transfer angle less than 90°)

1. Compute the sector-triangle area ratio

$$\boxed{\eta = 1 + \frac{m}{\eta^2} W\left(\frac{m}{\eta^2} - l\right)}$$

Using the secant method: $\eta_{i+1} = \eta_i - f(\eta_i)\dfrac{\eta_i - \eta_{i-1}}{f(\eta_i) - f(\eta_{i-1})}$

where: $f(x) = 1 - x + \dfrac{m}{x^2} W\left(\dfrac{m}{x^2} - l\right)$

With starting values: $\eta_1 = \eta_H + 0.1$, and $\eta_2 = \eta_H$

For questions in this course, it will be sufficient to approximate $\eta = \eta_H$

$$\eta_H = \frac{12}{22} + \frac{10}{22}\sqrt{1 + \frac{44}{9}\frac{m}{l + 5/6}}$$

where η_H is known as the **Hansen Approximation**

2. Compute the semi-parameter: $\boxed{p = \dfrac{\eta^2 |\vec{\mathbf{r}}_1 \times \vec{\mathbf{r}}_2|^2}{\mu (t_2 - t_1)^2}}$

3. Compute the Lagrangian Coefficients:

$$\boxed{F = 1 - \frac{r_2}{p}(1 - \cos(\theta_2 - \theta_1))} \qquad \boxed{G = \frac{r_1 r_2}{\sqrt{\mu p}} \sin(\theta_2 - \theta_1)}$$

4. Compute the velocity, $\vec{\mathbf{v}}_1$: $\boxed{\vec{\mathbf{v}}_1 = \dfrac{\vec{\mathbf{r}}_2 - F\vec{\mathbf{r}}_1}{G}}$

Now we have the state vectors $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{v}}_1$, which are **sufficient for determining the orbit**

The orbital elements can also be computed, if needed

Orbit Determination

Example Activity

$R_{earth} = 149.598023 \times 10^6$ km, $R_{mars} = 227.939186 \times 10^6$ km

$\mu_{earth} = 3.986 \times 10^5\ \mathrm{km}^3/s^2, \mu_{mars} = 4.305 \times 10^4\ \mathrm{km}^3/s^2, \mu_{sun} = 1.327144 \times 10^{11}\ \mathrm{km}^3/s^2$

It is desired to perform an interplanetary transfer from Earth to Mars. It is determined that a Hohmann transfer requires too much time. Assume that the Earth and Mars both possess coplanar circular orbits. At time $t = 0$, the Earth has true anomaly $\theta_E(0) = 0$,and Mars has true anomaly $\theta_M(0) = 30°$. The spacecraft is desired to arrive at Mars when Mars has a true anomaly $\theta_M = 45°$.

(a) Determine the time of flight of the transfer in days

(b) Determine the heliocentric velocity vector for the s/c upon departing Earth's SOI (assume $\eta = \eta_H$ for the sector-triangle area ratio)

For (a) solve for the orbital angular velocity of Mars, and then find the time of flight between $\theta_M(0)$ and $\theta_M(t)$

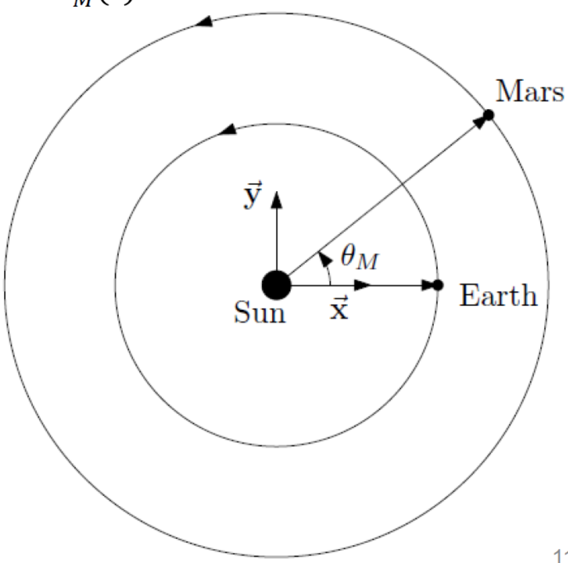
$$\boxed{n \equiv \frac{2\pi}{T} = \sqrt{\frac{\mu}{a^3}}} \qquad n_\mathrm{M} = \sqrt{\frac{\mu_S}{r_\mathrm{M}^3}} = \sqrt{\frac{132.7 \times 10^9\ \mathrm{km}^3/\mathrm{s}^2}{(227.9 \times 10^6\ \mathrm{km})^3}} = 1.059 \times 10^{-7}\ \mathrm{rad/s}$$

$$\boxed{\theta(t) - \theta(t_0) = n(t - t_0)}$$

$$t = \frac{\theta_M(t) - \theta_M(0)}{n_\mathrm{M}}$$

$$= \frac{45 \times \left(\frac{\pi}{180}\right) - 30\left(\frac{\pi}{180}\right)}{1.059 \times 10^{-7}}$$

$$= 2.4731 \times 10^6\ \mathrm{s} = 28.62\ \mathrm{days}$$



Example Activity

Orbit Determination

$R_{earth} = 149.598023 \times 10^6$ km, $R_{mars} = 227.939186 \times 10^6$ km

$\mu_{earth} = 3.986 \times 10^5\ \mathrm{km}^3/s^2, \mu_{mars} = 4.305 \times 10^4\ \mathrm{km}^3/s^2$, $\mu_{sun} = 1.327144 \times 10^{11}\ \mathrm{km}^3/s^2$

It is desired to perform an interplanetary transfer from Earth to Mars. It is determined that a Hohmann transfer requires too much time. Assume that the Earth and Mars both possess coplanar circular orbits. At time $t = 0$, the Earth has true anomaly $\theta_E(0) = 0$,and Mars has true anomaly $\theta_M(0) = 30°$. The spacecraft is desired to arrive at Mars when Mars has a true anomaly $\theta_M = 45°$.

(a) Determine the time of flight of the transfer in days

(b) Determine the heliocentric velocity vector for the s/c upon departing Earth's SOI (assume $\eta = \eta_H$ for the sector-triangle area ratio)

For (b) we need to solve Lambert's problem,

- Starting point is given by the position of the Earth at $t = 0$: $\vec{\mathbf{R}}_E = R_{earth}\vec{\mathbf{x}} = 149.598023 \times 10^6\ \vec{\mathbf{x}}$ km
- Final position of Mars at t: $\vec{\mathbf{R}}_M = R_{mars}\cos 45°\,\vec{\mathbf{x}} + R_{mars}\sin 45°\,\vec{\mathbf{y}}$

$$= 1.6118 \times 10^8\ \vec{\mathbf{x}} + 1.6118 \times 10^8\ \vec{\mathbf{y}}\ \mathrm{km}$$

- The angle of transfer: $\Delta\theta = 45°$

$$\boxed{\eta_H = \frac{12}{22} + \frac{10}{22}\sqrt{1 + \frac{44}{9}\frac{m}{l + 5/6}}}$$

$$\boxed{l = \frac{r_1 + r_2}{4\sqrt{r_1 r_2}\cos((\theta_2 - \theta_1)/2)} - \frac{1}{2}}$$

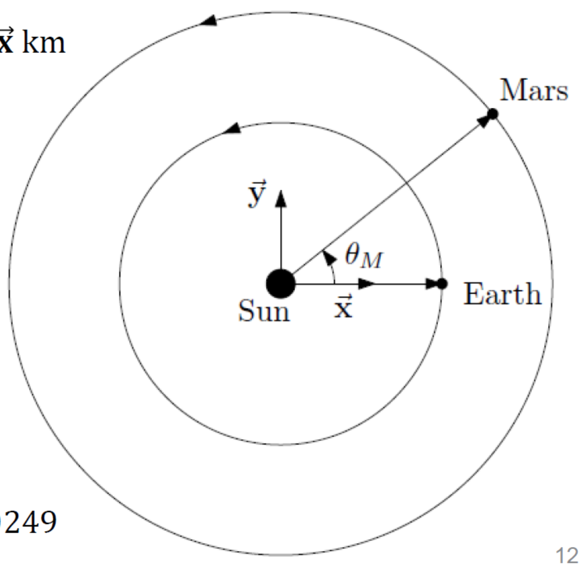
$$\boxed{m = \frac{\mu(t_2 - t_1)^2}{[2\sqrt{r_1 r_2}\cos((\theta_2 - \theta_1)/2)]^3}}$$

(1) Find the sector-triangle area ratio:

$$m = \frac{\mu_{sun} t^2}{[2\sqrt{R_{earth}R_{mars}}\cos(\Delta\theta/2)]^3} = 0.0204$$

$$l = \frac{R_{earth} + R_{mars}}{4\sqrt{R_{earth}R_{mars}}\cos(\Delta\theta/2)} - \frac{1}{2} = 0.0532$$

$$\eta_H = \frac{12}{22} + \frac{10}{22}\sqrt{1 + \frac{44}{9}\frac{m}{l + 5/6}} = 1.0249$$



Example Activity

Orbit Determination

$R_{earth} = 149.598023 \times 10^6$ km, $R_{mars} = 227.939186 \times 10^6$ km
$\mu_{earth} = 3.986 \times 10^5\ \mathrm{km}^3/s^2, \mu_{mars} = 4.305 \times 10^4\ \mathrm{km}^3/s^2\ , \mu_{sun} = 1.327144 \times 10^{11}\ \mathrm{km}^3/s^2$

It is desired to perform an interplanetary transfer from Earth to Mars. It is determined that a Hohmann transfer requires too much time. Assume that the Earth and Mars both possess coplanar circular orbits. At time $t = 0$, the Earth has true anomaly $\theta_E(0) = 0$,and Mars has true anomaly $\theta_M(0) = 30°$. The spacecraft is desired to arrive at Mars when Mars has a true anomaly $\theta_M = 45°$.

(a) Determine the time of flight of the transfer in days

(b) Determine the heliocentric velocity vector for the s/c upon departing Earth's SOI (assume $\eta = \eta_H$ for the sector-triangle area ratio)

For (b) we need to solve Lambert's problem,

$\eta_H = 1.0249$ $\quad \vec{\mathbf{R}}_E = 149.598023 \times 10^6\ \vec{\mathbf{x}}$ km

$\Delta\theta = 45°$ $\quad \vec{\mathbf{R}}_M = 1.6118 \times 10^8\ \vec{\mathbf{x}} + 1.6118 \times 10^8\ \vec{\mathbf{y}}$ km

$t = 2.4731 \times 10^6$ s $= 28.62$ days

(2) Compute the semiparameter:

$$\boxed{p = \frac{\eta^2 |\vec{\mathbf{r}}_1 \times \vec{\mathbf{r}}_2|^2}{\mu(t_2 - t_1)^2}} \qquad p = \frac{\eta^2 \left|\vec{\mathbf{R}}_E \times \vec{\mathbf{R}}_M\right|^2}{\mu_{sun} t^2} = 7.524 \times 10^8 \text{ km}$$

(3) Compute the Lagrangian Coefficients:

$$\boxed{F = 1 - \frac{r_2}{p}(1 - \cos(\theta_2 - \theta_1))} \qquad F = 1 - \frac{R_{mars}}{p}(1 - \cos\Delta\theta) = 0.9113$$

$$\boxed{G = \frac{r_1 r_2}{\sqrt{\mu p}} \sin(\theta_2 - \theta_1)} \qquad G = \frac{R_{earth} R_{mars}}{\sqrt{\mu_{sun} p}} \sin\Delta\theta = 2.413 \times 10^6$$

(4) Compute the velocity, $\vec{\mathbf{V}}$:

$$\boxed{\vec{\mathbf{v}}_1 = \frac{\vec{\mathbf{r}}_2 - F\vec{\mathbf{r}}_1}{G}}$$

$$\vec{\mathbf{V}} = \frac{\vec{\mathbf{R}}_M - F\vec{\mathbf{R}}_E}{G}$$

$$\vec{\mathbf{V}} = 10.3\ \vec{\mathbf{x}} + 66.8\ \vec{\mathbf{y}} \text{ km/s}$$