Reference Frames

Reference Frames

- Up until now, we have considered physical vectors without coordinate systems or reference frames; however, we must define **reference frames** in order to perform computations with vectors

- Reference frames describe the **position** and **orientation** of an object in space (as well as its kinematics and dynamics), through a set of *basis* vectors (*linearly independent subset that spans the vector space*)

- Typically, reference frames (denoted with an $\mathcal{F}$) are defined by a set of dextral (right-handed) and orthonormal (mutually-perpendicular unit) unit vectors, i.e.,

Dextral

$$\hat{\mathbf{x}}_1 \times \hat{\mathbf{y}}_1 = \hat{\mathbf{z}}_1$$

$$\hat{\mathbf{y}}_1 \times \hat{\mathbf{z}}_1 = \hat{\mathbf{x}}_1$$

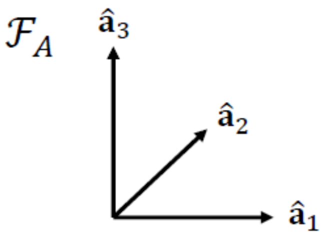
$$\hat{\mathbf{z}}_1 \times \hat{\mathbf{x}}_1 = \hat{\mathbf{y}}_1$$

Orthonormal

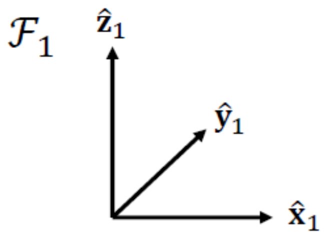
$$\hat{\mathbf{x}}_1 \cdot \hat{\mathbf{x}}_1 = \hat{\mathbf{y}}_1 \cdot \hat{\mathbf{y}}_1 = \hat{\mathbf{z}}_1 \cdot \hat{\mathbf{z}}_1 = 1$$

$$\hat{\mathbf{x}}_1 \cdot \hat{\mathbf{y}}_1 = \hat{\mathbf{x}}_1 \cdot \hat{\mathbf{z}}_1 = \hat{\mathbf{y}}_1 \cdot \hat{\mathbf{z}}_1 = 0$$

1-2-3 subscript notation for cartesian coordinate system



x-y-z variable notation for cartesian coordinate system



Reference Frames

What is an **inertial** reference frame?

- A reference frame is an inertial frame of reference if Newton's laws hold.

Newton's First Law: *"Every object persists in its state of rest or uniform motion in a straight line unless it is compelled to change that state by forces impressed on it."*

- A property of an inertial frame is that any frame that is stationary or moving with constant velocity (but *not* rotating) with respect to an inertial frame is also inertial (since acceleration with respect to those frames is the same)

In orbital mechanics there are many important reference frames, we will define a few common frames:

- Heliocentric-Ecliptic frame
- Geocentric-Equatorial or Earth-Centred Inertial (ECI) frame
- Perifocal frame
- Orbiting frame
- Body-fixed frame

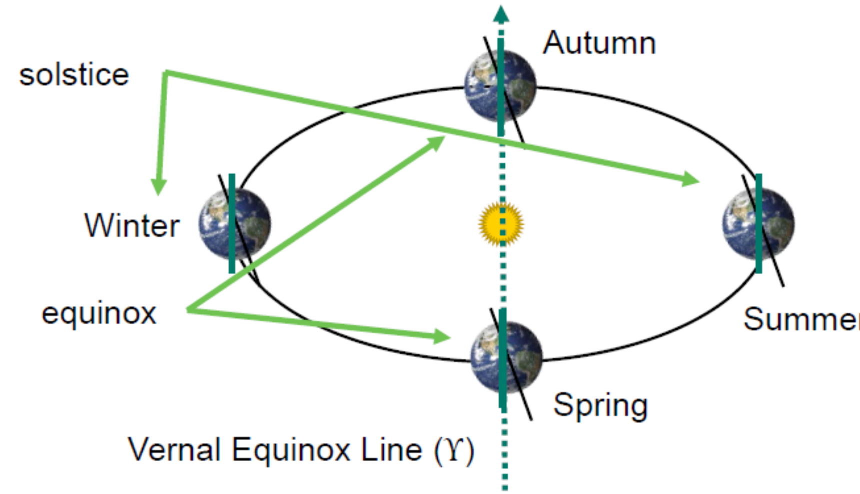
 [NASA, https://www.grc.nasa.gov/www/k-12/airplane/newton1g.html, 2015; Vatankhahghadim, 2019]

Reference Frames

Equinox: numbers of hours of daylight and darkness are equal
Solstice: longest or shortest period of daylight

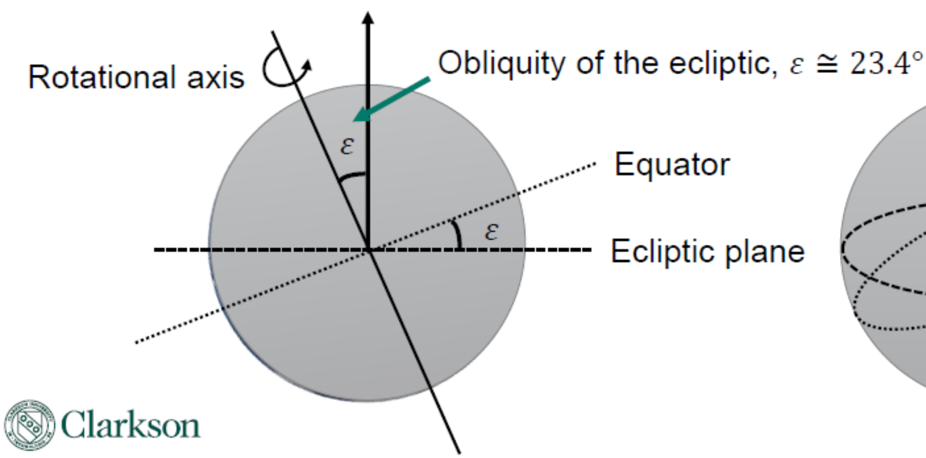
Heliocentric-Ecliptic Frame ($\mathcal{F}_{\mathrm{H}}$)

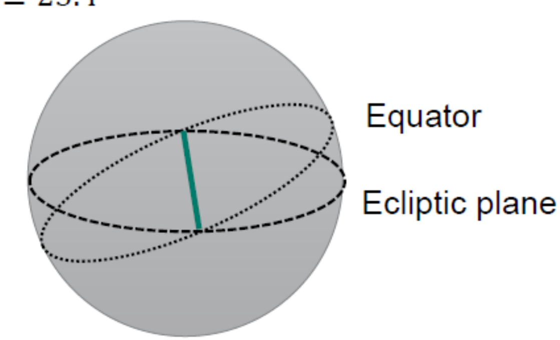
- Origin, O_{H}, at Sun's center of mass
- $\hat{\mathbf{h}}_1$ in the direction of the vernal equinox (Υ)
- $\hat{\mathbf{h}}_3$ normal to the ecliptic plane
- $\hat{\mathbf{h}}_2$ completes the right-hand rule

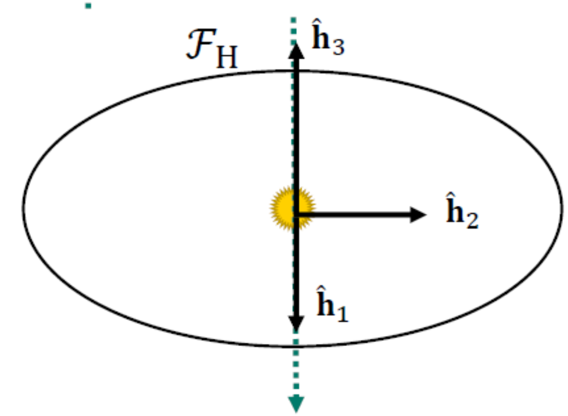


What is the **ecliptic**?

- The plane of Earth's orbit around the Sun







Clarkson

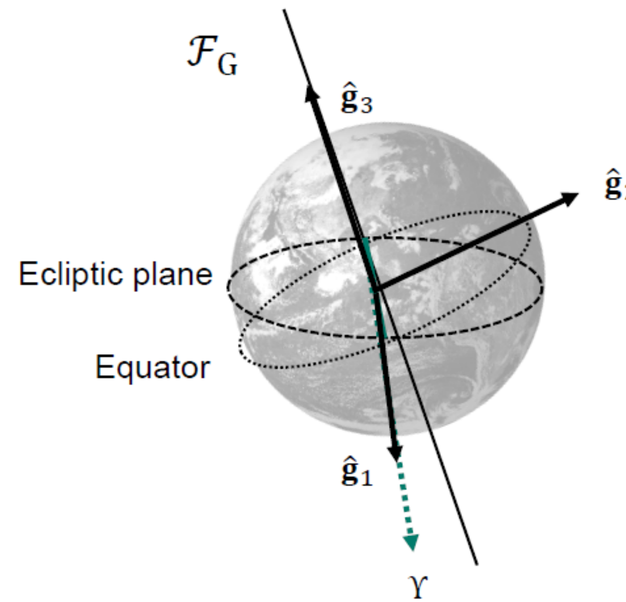
Reference Frames

Technically, **ECI is not inertial**, since Earth's center of mass accelerates in its orbit; however, the movement w.r.t. the Sun is slow enough to be **considered inertial for all practical purposes**

Geocentric-Equatorial or Earth-Centred Inertial (ECI) frame ($\mathcal{F}_{\mathrm{G}}$)

- Origin, O_{G}, at Earth's center of mass
- $\hat{\mathbf{g}}_1$ in the direction of the vernal equinox (Υ)
- $\hat{\mathbf{g}}_3$ towards Earth's north pole
- $\hat{\mathbf{g}}_2$ completes the right-hand rule

- The plane completed by $\hat{\mathbf{g}}_1$ and $\hat{\mathbf{g}}_2$ is on the equator
- Often, ECI frame will be represented using $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$



The Earth-Centred Earth-Fixed (ECEF) frame ($\mathcal{F}_{\mathrm{E}}$)

- Similar to $\mathcal{F}_{\mathrm{G}}$, but the 1-axis rotates with Earth (i.e., rotating frame)

 [De Ruiter, Ch. 2-3; Vatankhahghadim, 2019]

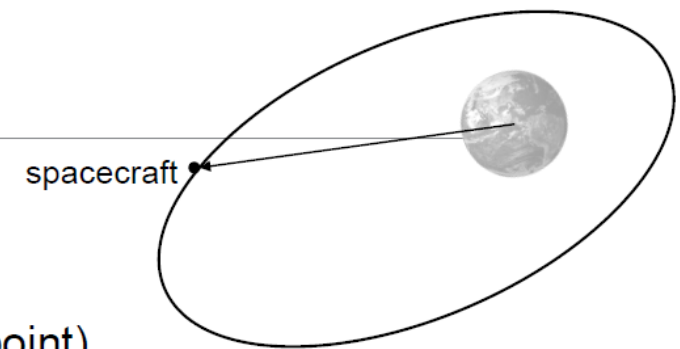
Reference Frames

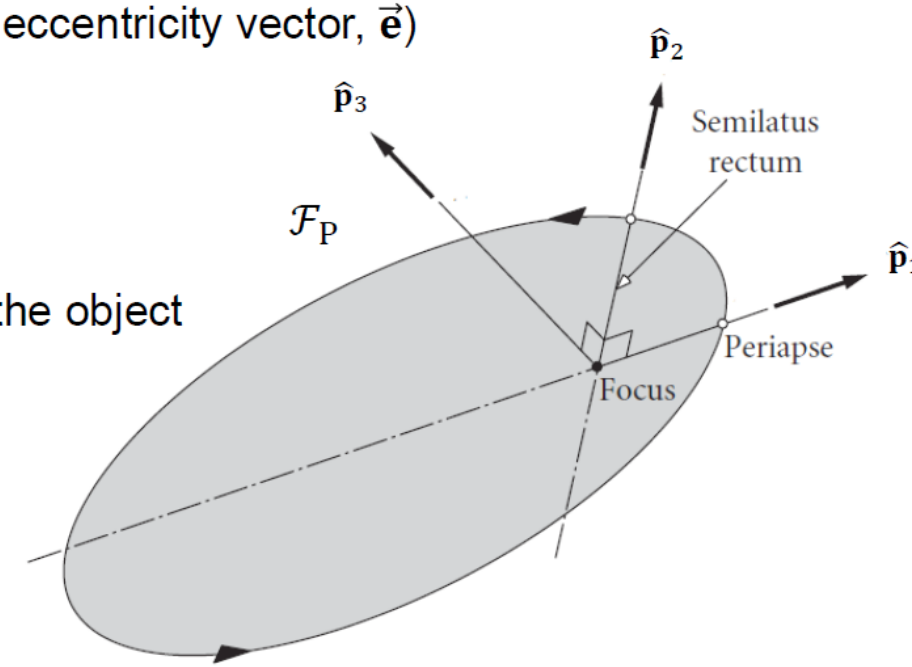
Perifocal Frame ($\mathcal{F}_\mathrm{P}$)

- Defined for an arbitrary orbit
- Origin, O_P, at the center of mass of the primary body (focus point)
- $\hat{\mathbf{p}}_1$ towards the orbit's periapsis (parallel to the eccentricity vector, $\vec{\mathbf{e}}$)
- $\hat{\mathbf{p}}_3$ normal to the orbit's plane (parallel to $\vec{\mathbf{h}}$)
- $\hat{\mathbf{p}}_2$ completes the right-hand rule (along p)

- The axes $\hat{\mathbf{p}}_1$ and $\hat{\mathbf{p}}_2$ are in the orbital plane of the object

- Often, $\mathcal{F}_\mathrm{P}$ will be represented using $\hat{\mathbf{p}}, \hat{\mathbf{q}}$ and $\hat{\mathbf{w}}$





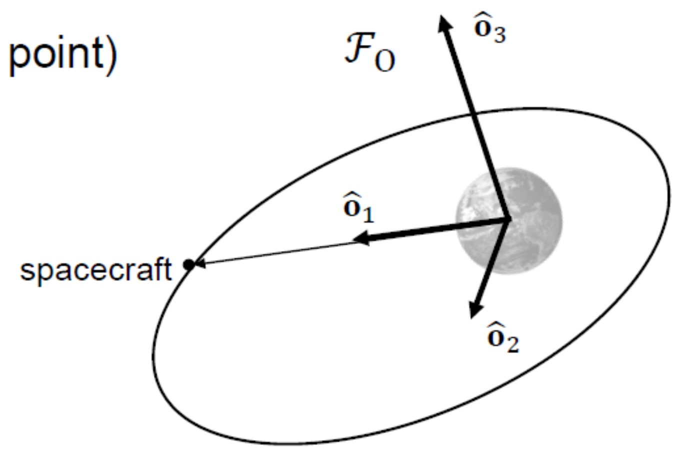
Reference Frames

Orbiting Frame ($\mathcal{F}_{\mathrm{O}}$)

- Defined for an arbitrary orbit
- Origin, O_{O}, at the center of mass of the primary body (focus point)
- $\hat{\mathbf{o}}_1$ towards the orbiting body
- $\hat{\mathbf{o}}_3$ normal to the orbit's plane (parallel to $\vec{\mathbf{h}}$)
- $\hat{\mathbf{o}}_2$ completes the right-hand rule

- The axes $\hat{\mathbf{o}}_1$ and $\hat{\mathbf{o}}_2$ are in the orbital plane of the object

- Similar to $\mathcal{F}_{\mathrm{P}}$, but the 1-axis rotates with orbiting body (i.e., rotating frame)

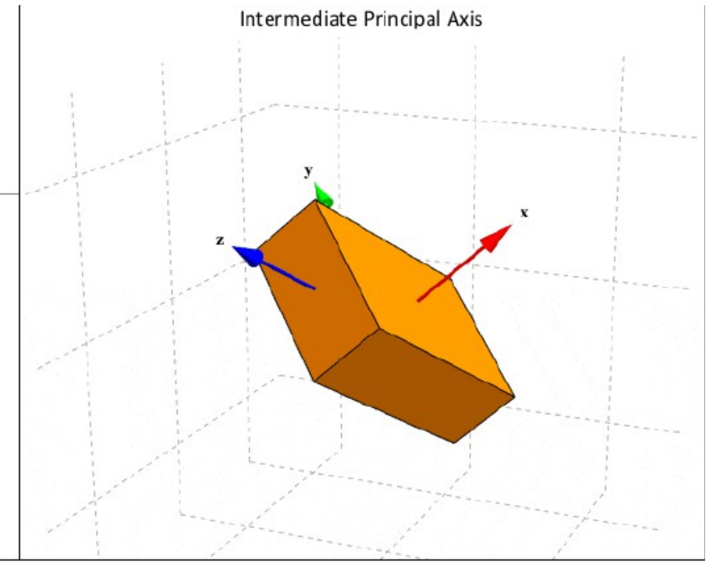


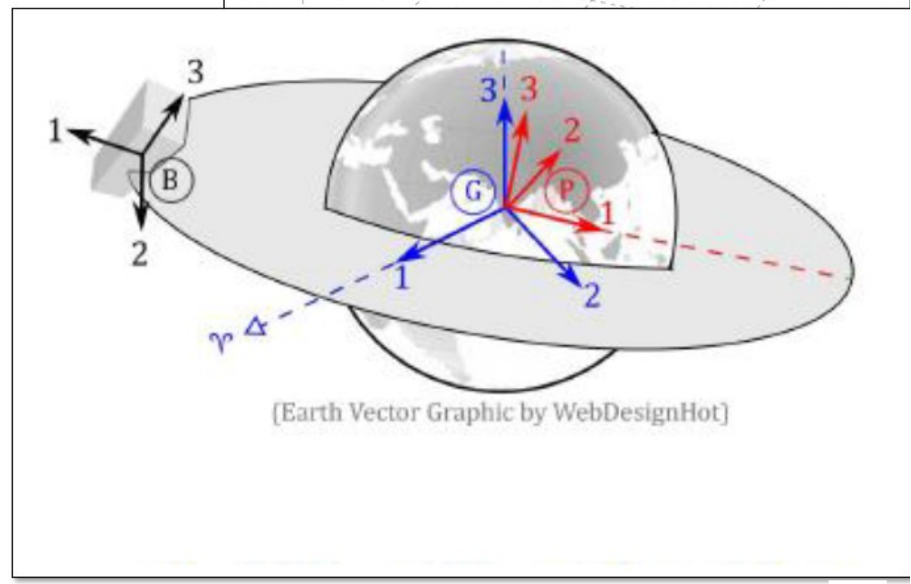
Reference Frames

Body-Fixed Frame ($\mathcal{F}_\mathrm{B}$)

- Defined for an arbitrary body (e.g., spacecraft)
- Origin, O_B, at the center of mass of the body
- $\hat{\mathbf{b}}_1$, $\hat{\mathbf{b}}_2$, $\hat{\mathbf{b}}_3$ are selected such that they point towards a fixed point on the body

- For example, along the spacecraft's principal axes that form a frame in which the moment of inertia matrix, $\mathbf{I}$, is diagonal





[Fouriest Series, https://fouriestseries.tumblr.com/post/91685028535/rotational-stability 2014; Vatankhahghadim, 2019]

Rotation Matrices

Generally, spacecraft dynamics uses multiple reference frames

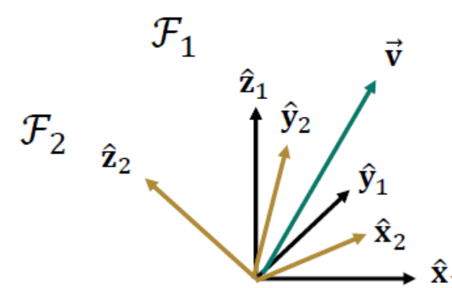
- Usually, want the orientation of a spacecraft
- For spacecraft, we call the orientation the *attitude*

As a result, we need to know how to do the following:

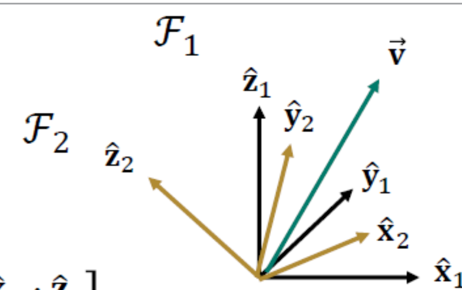
1. Describe the orientation of one reference frame with respect to another
2. Transform coordinates of a vector from one reference frame to another

We've seen that the same vector can be represented in multiple frames:

$$\vec{\mathbf{v}} = \vec{\boldsymbol{\mathcal{F}}}_1^{\mathrm{T}} \mathbf{v}_1 = \vec{\boldsymbol{\mathcal{F}}}_2^{\mathrm{T}} \mathbf{v}_2$$



Rotation Matrices



Now, let's transform the coordinates of vector $\vec{\mathbf{v}}$ from one frame to another

- Represent $\vec{\mathbf{v}}$ in $\mathcal{F}_2$ and $\mathcal{F}_1$

$$\vec{\mathbf{v}} = \vec{\boldsymbol{\mathcal{F}}}_2^{\mathrm{T}}\mathbf{v}_2 = \vec{\boldsymbol{\mathcal{F}}}_1^{\mathrm{T}}\mathbf{v}_1$$

$$\cancelto{1}{\vec{\boldsymbol{\mathcal{F}}}_2 \cdot \vec{\boldsymbol{\mathcal{F}}}_2^{\mathrm{T}}}\mathbf{v}_2 = \vec{\boldsymbol{\mathcal{F}}}_2 \cdot \vec{\boldsymbol{\mathcal{F}}}_1^{\mathrm{T}}\mathbf{v}_1 \;\;\Rightarrow \mathbf{v}_2 = \vec{\boldsymbol{\mathcal{F}}}_2 \cdot \vec{\boldsymbol{\mathcal{F}}}_1^{\mathrm{T}}\mathbf{v}_1 = \begin{bmatrix} \hat{\mathbf{x}}_2 \\ \hat{\mathbf{y}}_2 \\ \hat{\mathbf{z}}_2 \end{bmatrix} [\hat{\mathbf{x}}_1 \quad \hat{\mathbf{y}}_1 \quad \hat{\mathbf{z}}_1]\,\mathbf{v}_1 = \begin{bmatrix} \hat{\mathbf{x}}_2 \cdot \hat{\mathbf{x}}_1 & \hat{\mathbf{x}}_2 \cdot \hat{\mathbf{y}}_1 & \hat{\mathbf{x}}_2 \cdot \hat{\mathbf{z}}_1 \\ \hat{\mathbf{y}}_2 \cdot \hat{\mathbf{x}}_1 & \hat{\mathbf{y}}_2 \cdot \hat{\mathbf{y}}_1 & \hat{\mathbf{y}}_2 \cdot \hat{\mathbf{z}}_1 \\ \hat{\mathbf{z}}_2 \cdot \hat{\mathbf{x}}_1 & \hat{\mathbf{z}}_2 \cdot \hat{\mathbf{y}}_1 & \hat{\mathbf{z}}_2 \cdot \hat{\mathbf{z}}_1 \end{bmatrix} \mathbf{v}_1$$

$$\mathbf{v}_2 = \mathbf{C}_{21}\mathbf{v}_1$$

Definition: A *rotation matrix* describes the transformation from one frame ($\mathcal{F}_1$) to another ($\mathcal{F}_2$), and is defined as:

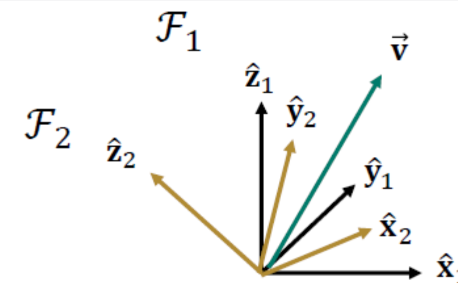
$$\mathbf{C}_{21} \equiv \vec{\boldsymbol{\mathcal{F}}}_2 \cdot \vec{\boldsymbol{\mathcal{F}}}_1^{\mathrm{T}} = \begin{bmatrix} \hat{\mathbf{x}}_2 \cdot \hat{\mathbf{x}}_1 & \hat{\mathbf{x}}_2 \cdot \hat{\mathbf{y}}_1 & \hat{\mathbf{x}}_2 \cdot \hat{\mathbf{z}}_1 \\ \hat{\mathbf{y}}_2 \cdot \hat{\mathbf{x}}_1 & \hat{\mathbf{y}}_2 \cdot \hat{\mathbf{y}}_1 & \hat{\mathbf{y}}_2 \cdot \hat{\mathbf{z}}_1 \\ \hat{\mathbf{z}}_2 \cdot \hat{\mathbf{x}}_1 & \hat{\mathbf{z}}_2 \cdot \hat{\mathbf{y}}_1 & \hat{\mathbf{z}}_2 \cdot \hat{\mathbf{z}}_1 \end{bmatrix} = \begin{bmatrix} \cos\theta_{11} & \cos\theta_{12} & \cos\theta_{13} \\ \cos\theta_{21} & \cos\theta_{22} & \cos\theta_{23} \\ \cos\theta_{31} & \cos\theta_{32} & \cos\theta_{33} \end{bmatrix}$$

← Also called a "**direction cosine matrix**"

(i and j represent the 1-2-3 axes for $\mathcal{F}_2$ and $\mathcal{F}_1$, respectively)

Rotation Matrices

Additional notes about the rotation matrix:



$$\mathbf{C}_{21} \equiv \vec{\mathcal{F}}_2 \cdot \vec{\mathcal{F}}_1^{\mathrm{T}} = \begin{bmatrix} \hat{\mathbf{x}}_2 \cdot \hat{\mathbf{x}}_1 & \hat{\mathbf{x}}_2 \cdot \hat{\mathbf{y}}_1 & \hat{\mathbf{x}}_2 \cdot \hat{\mathbf{z}}_1 \\ \hat{\mathbf{y}}_2 \cdot \hat{\mathbf{x}}_1 & \hat{\mathbf{y}}_2 \cdot \hat{\mathbf{y}}_1 & \hat{\mathbf{y}}_2 \cdot \hat{\mathbf{z}}_1 \\ \hat{\mathbf{z}}_2 \cdot \hat{\mathbf{x}}_1 & \hat{\mathbf{z}}_2 \cdot \hat{\mathbf{y}}_1 & \hat{\mathbf{z}}_2 \cdot \hat{\mathbf{z}}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{1,2} & \mathbf{y}_{1,2} & \mathbf{z}_{1,2} \end{bmatrix}$$

$$\mathbf{x}_{1,2} \qquad \mathbf{y}_{1,2} \qquad \mathbf{z}_{1,2}$$

- So, the rotation matrix is simply the basis vectors of $\mathcal{F}_1$ expressed in $\mathcal{F}_2$

Recall:

$$\mathbf{x}_{1,2} = \begin{bmatrix} \hat{\mathbf{x}}_1 \cdot \hat{\mathbf{x}}_2 \\ \hat{\mathbf{x}}_1 \cdot \hat{\mathbf{y}}_2 \\ \hat{\mathbf{x}}_1 \cdot \hat{\mathbf{z}}_2 \end{bmatrix}, \qquad \mathbf{y}_{1,2} = \begin{bmatrix} \hat{\mathbf{y}}_1 \cdot \hat{\mathbf{x}}_2 \\ \hat{\mathbf{y}}_1 \cdot \hat{\mathbf{y}}_2 \\ \hat{\mathbf{y}}_1 \cdot \hat{\mathbf{z}}_2 \end{bmatrix}, \qquad \mathbf{z}_{1,2} = \begin{bmatrix} \hat{\mathbf{z}}_1 \cdot \hat{\mathbf{x}}_2 \\ \hat{\mathbf{z}}_1 \cdot \hat{\mathbf{y}}_2 \\ \hat{\mathbf{z}}_1 \cdot \hat{\mathbf{z}}_2 \end{bmatrix}$$

The rotation matrix also has a number of **useful properties**:

- It is invertible, i.e., $\mathbf{C}_{21}^{-1}$
- It is orthogonal, i.e., its inverse is equal to its transpose, $\mathbf{C}_{21}^{-1} = \mathbf{C}_{21}^{\mathrm{T}}$

$$\mathbf{C}_{12} = \mathbf{C}_{21}^{-1} = \mathbf{C}_{21}^{\mathrm{T}}$$

$$\mathbf{C}_{21}\mathbf{C}_{21}^{-1} = \mathbf{C}_{21}\mathbf{C}_{21}^{\mathrm{T}} = \mathbf{1}$$

[for proofs see de Ruiter, p.13-14]

Given all of this, we can now say:

$$\mathbf{v}_1 = \mathbf{C}_{12}\mathbf{v}_2, \qquad \mathbf{v}_2 = \mathbf{C}_{21}\mathbf{v}_1$$

and,

$$\vec{\mathcal{F}}_1 = \mathbf{C}_{12}\vec{\mathcal{F}}_2 \qquad \vec{\mathcal{F}}_1^{\mathrm{T}} = \vec{\mathcal{F}}_2^{\mathrm{T}}\mathbf{C}_{21}$$

Rotation Matrices

Multiple Reference Frames

Consider three reference frames: $\vec{\mathbf{v}} = \vec{\boldsymbol{\mathcal{F}}}_{\mathrm{A}}^{\mathrm{T}}\mathbf{v}_{\mathrm{A}} = \vec{\boldsymbol{\mathcal{F}}}_{\mathrm{B}}^{\mathrm{T}}\mathbf{v}_{\mathrm{B}} = \vec{\boldsymbol{\mathcal{F}}}_{\mathrm{C}}^{\mathrm{T}}\mathbf{v}_{\mathrm{C}}$

$\mathbf{v}_{\mathrm{C}} = \vec{\boldsymbol{\mathcal{F}}}_{\mathrm{C}} \cdot \vec{\boldsymbol{\mathcal{F}}}_{\mathrm{B}}^{\mathrm{T}}\mathbf{v}_{\mathrm{B}} = \mathbf{C}_{\mathrm{CB}}\mathbf{v}_{\mathrm{B}}$

$\mathbf{v}_{\mathrm{B}} = \vec{\boldsymbol{\mathcal{F}}}_{\mathrm{B}} \cdot \vec{\boldsymbol{\mathcal{F}}}_{\mathrm{A}}^{\mathrm{T}}\mathbf{v}_{\mathrm{A}} = \mathbf{C}_{\mathrm{BA}}\mathbf{v}_{\mathrm{A}}$

$\mathbf{v}_{\mathrm{C}} = \vec{\boldsymbol{\mathcal{F}}}_{\mathrm{C}} \cdot \vec{\boldsymbol{\mathcal{F}}}_{\mathrm{A}}^{\mathrm{T}}\mathbf{v}_{\mathrm{A}} = \mathbf{C}_{\mathrm{CA}}\mathbf{v}_{\mathrm{A}}$

We can combine to find: $\mathbf{v}_{\mathrm{C}} = \mathbf{C}_{\mathrm{CB}}\mathbf{C}_{\mathrm{BA}}\mathbf{v}_{\mathrm{A}}$

Successive rotations of multiple frames can simply be represented by multiplication of the rotation matrices involved:

$$\mathbf{C}_{\mathrm{CA}} = \mathbf{C}_{\mathrm{CB}}\mathbf{C}_{\mathrm{BA}}$$

Principal rotation matrices

Rotation Matrices

Principal Rotations

Principal rotation matrices describe the change of frame obtained by rotating a frame about only one of its coordinate axes

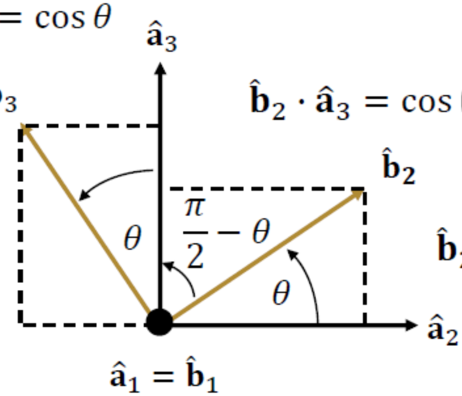
$$\mathbf{C}_1 \equiv \mathbf{C}_{\mathrm{BA}_1} = \vec{\mathcal{F}}_{\mathrm{B}} \cdot \vec{\mathcal{F}}_{\mathrm{A}}^{\mathrm{T}} = \begin{bmatrix} \cancelto{1}{\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{a}}_1} & \cancelto{0}{\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{a}}_2} & \cancelto{0}{\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{a}}_3} \\ \cancelto{0}{\hat{\mathbf{b}}_2 \cdot \hat{\mathbf{a}}_1} & \hat{\mathbf{b}}_2 \cdot \hat{\mathbf{a}}_2 & \hat{\mathbf{b}}_2 \cdot \hat{\mathbf{a}}_3 \\ \cancelto{0}{\hat{\mathbf{b}}_3 \cdot \hat{\mathbf{a}}_1} & \hat{\mathbf{b}}_3 \cdot \hat{\mathbf{a}}_2 & \hat{\mathbf{b}}_3 \cdot \hat{\mathbf{a}}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

$$\hat{\mathbf{b}}_3 \cdot \hat{\mathbf{a}}_3 = \cos\theta$$

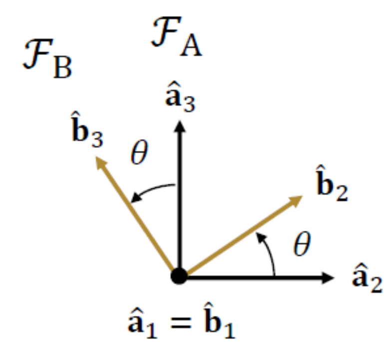
$$\hat{\mathbf{b}}_2 \cdot \hat{\mathbf{a}}_3 = \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\hat{\mathbf{b}}_3 \cdot \hat{\mathbf{a}}_2 = \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$$

$$\hat{\mathbf{b}}_2 \cdot \hat{\mathbf{a}}_2 = \cos\theta$$



For example, if $\mathcal{F}_{\mathrm{B}}$ is found by rotating $\mathcal{F}_{\mathrm{A}}$ about its 1-axis



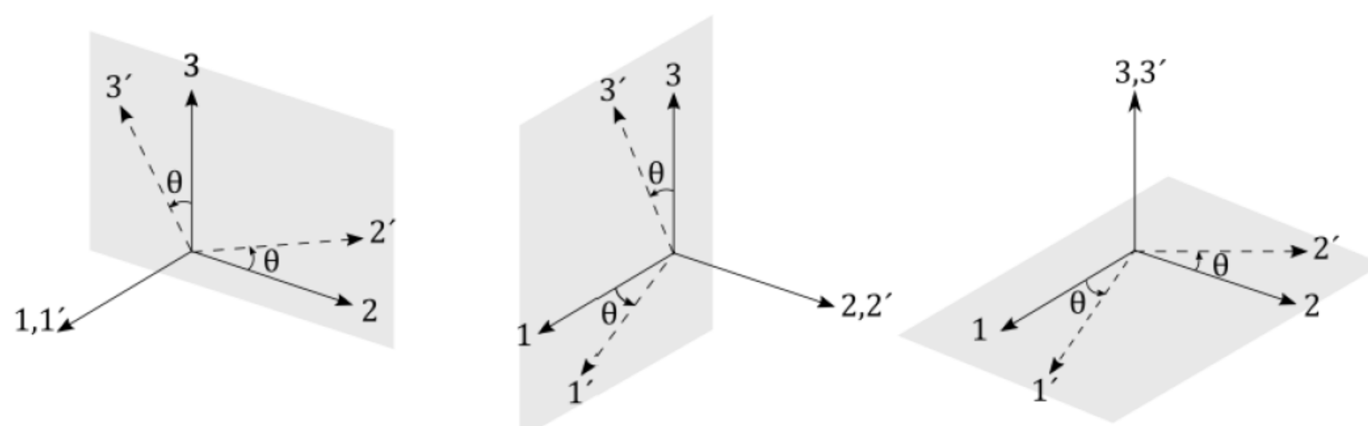
Clarkson

Rotation Matrices

Principal Rotations

Principal rotation matrices describe the change of frames obtained by rotating a frame about only one of its coordinate axes

$$\mathbf{C}_1(\theta) \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \quad \mathbf{C}_2(\theta) \equiv \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \quad \mathbf{C}_3(\theta) \equiv \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Clarkson [Curtis, Ch. 4]

Rotation Matrices

General Rotations

Let's now consider representations of general rotations, let's start with Euler's Theorem (obtained in 1775):

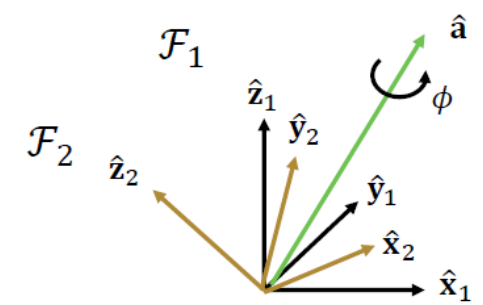
> **Euler's Theorem:** *The most general motion of a rigid body with one point fixed is a rotation about an axis through that point.*

Euler Axis-Angle $(\hat{\mathbf{a}}, \phi)$

Consider our two frames again, $\mathcal{F}_1$ and $\mathcal{F}_2$, where $\mathcal{F}_2$ can be obtained by a single rotation (ϕ) about some unit vector, which we will denote $\hat{\mathbf{a}} = \vec{\boldsymbol{\mathcal{F}}}_1^{\mathrm{T}}\mathbf{a}$

$$\mathbf{a} = [a_1 \quad a_2 \quad a_3]^{\mathrm{T}}$$

$$\mathbf{a}^{\mathrm{T}}\mathbf{a} = a_1^2 + a_2^2 + a_3^2 \equiv 1$$



We state, without proof, the rotation matrix is given by:

$$\mathbf{C}_{21} = \cos\phi\,\mathbf{1} + (1-\cos\phi)\mathbf{a}\mathbf{a}^{\mathrm{T}} - \sin\phi\,\mathbf{a}^{\times}$$

Note that the coordinates of $\mathbf{a}$ can be expressed in any reference frame, since $\mathbf{C}_{21}\mathbf{a} = \mathbf{a}$

[for proofs see de Ruiter, p.15-20]

Rotation Matrices

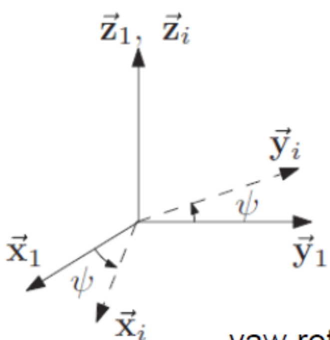
Euler Angles (ϕ, θ, ψ)

- One of the most common sets of parameters used to describe rotations, and uses successive principal rotations
- In three-dimensional space, 3 degrees of freedom are required to describe a general rotation

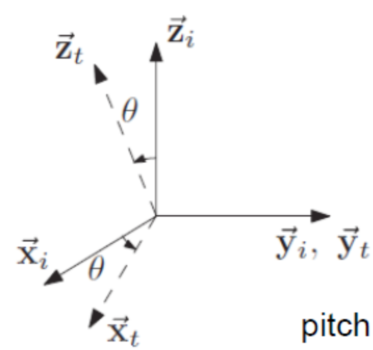
Euler Angles: set of three angles used in conjunction with 3 principal rotation matrices to describe orientation of one reference frame with respect to another

- There are many possible sequences, for example, consider $\mathcal{F}_1$ and $\mathcal{F}_2$ using a 3-2-1 attitude sequence:

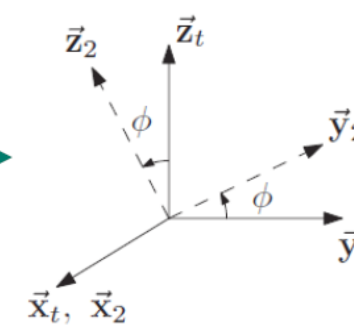
1. Rotation ψ about the 3-axis ($\hat{\mathbf{z}}_1 = \hat{\mathbf{z}}_i$)
2. Rotation θ about the 2-axis ($\hat{\mathbf{y}}_i = \hat{\mathbf{y}}_t$)
3. Rotation ϕ about the 1-axis ($\hat{\mathbf{x}}_t = \hat{\mathbf{x}}_2$)



yaw rotation



pitch rotation



roll rotation

Given these principal rotations, the rotation matrix from $\mathcal{F}_1$ and $\mathcal{F}_2$ is naturally given by: $\mathbf{C}_{21}(\phi, \theta, \psi) = \mathbf{C}_1(\phi)\mathbf{C}_2(\theta)\mathbf{C}_3(\psi)$

Clarkson

Rotation Matrices

Euler Angles (cont.)

Expanding on the rotation matrix from before, we see:

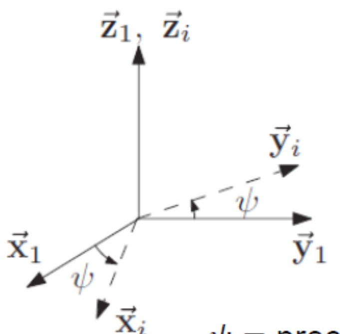
$$\mathbf{C}_{21}(\phi,\theta,\psi) = \mathbf{C}_1(\phi)\mathbf{C}_2(\theta)\mathbf{C}_3(\psi)$$

**where* $s_i = \sin i$ and $c_j = \cos j$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_\theta c_\psi & c_\theta s_\psi & -s_\theta \\ s_\phi s_\theta c_\psi - c_\phi s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi c_\theta \\ c_\phi s_\theta c_\psi + s_\phi s_\psi & c_\phi s_\theta s_\phi - s_\phi c_\psi & c_\phi c_\theta \end{bmatrix}$$

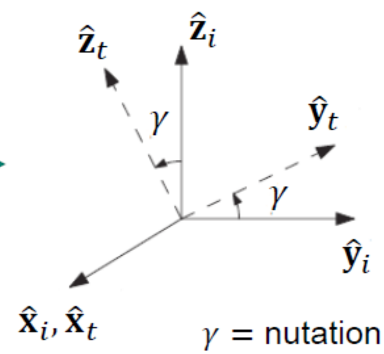
Another common sequence is 3-1-3, i.e.,

1. Rotation ψ about the 3-axis ($\hat{\mathbf{z}}_1 = \hat{\mathbf{z}}_i$)



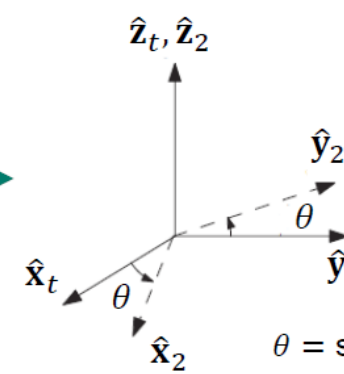
ψ = precession angle

2. Rotation γ about the 1-axis ($\hat{\mathbf{x}}_i = \hat{\mathbf{x}}_t$)



γ = nutation angle

3. Rotation θ about the 3-axis ($\hat{\mathbf{z}}_t = \hat{\mathbf{z}}_2$)



θ = spin angle

Rotation Matrices

Euler Angles (cont.)

Expanding on the rotation matrix from before, we see:

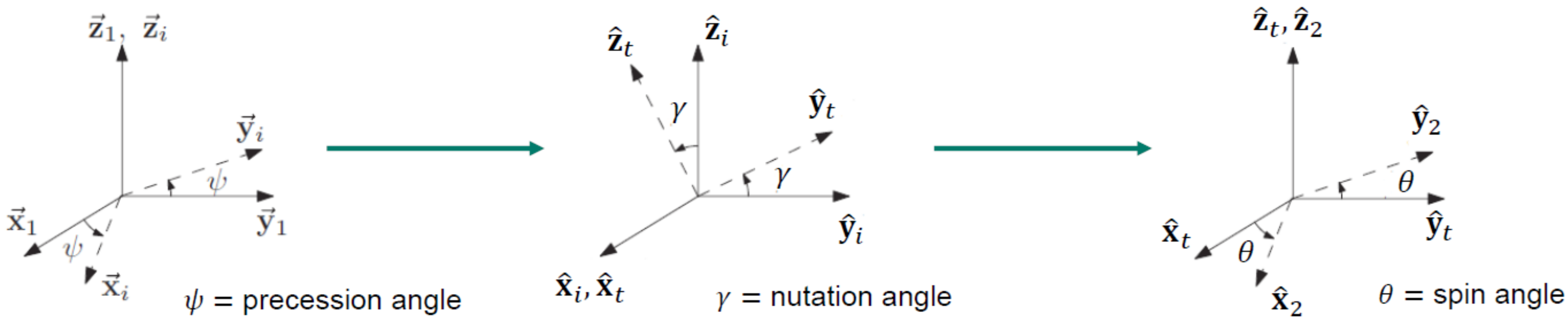
$$\mathbf{C}_{21}(\phi,\theta,\psi) = \mathbf{C}_1(\phi)\mathbf{C}_2(\theta)\mathbf{C}_3(\psi)$$

**where* $s_i = \sin i$ and $c_j = \cos j$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_\theta c_\psi & c_\theta s_\psi & -s_\theta \\ s_\phi s_\theta c_\psi - c_\phi s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi c_\theta \\ c_\phi s_\theta c_\psi + s_\phi s_\psi & c_\phi s_\theta s_\phi - s_\phi c_\psi & c_\phi c_\theta \end{bmatrix}$$

Another common sequence is 3-1-3, i.e.,

1. Rotation ψ about the 3-axis ($\hat{\mathbf{z}}_1 = \hat{\mathbf{z}}_i$)
2. Rotation γ about the 1-axis ($\hat{\mathbf{x}}_i = \hat{\mathbf{x}}_t$)
3. Rotation θ about the 3-axis ($\hat{\mathbf{z}}_t = \hat{\mathbf{z}}_2$)



ψ = precession angle

γ = nutation angle

θ = spin angle

Rotation Matrices

Euler Angles (cont.)

The 3-1-3 sequence results in the following rotation matrix: $\mathbf{C}_{21}(\theta, \gamma, \psi) = \mathbf{C}_3(\theta)\mathbf{C}_1(\gamma)\mathbf{C}_3(\psi)$

$$= \begin{bmatrix} c_\theta c_\psi - s_\theta c_\gamma s_\psi & s_\psi c_\theta + c_\gamma s_\theta c_\psi & s_\gamma s_\theta \\ -c_\psi s_\theta - c_\theta c_\gamma s_\psi & -s_\psi s_\theta + c_\theta c_\gamma c_\psi & s_\gamma c_\theta \\ s_\psi s_\gamma & -s_\gamma c_\psi & c_\gamma \end{bmatrix}$$

In short, any sequence 1,2,3-axis rotations of Euler angles $(\theta_1, \theta_2, \theta_3)$ can uniquely determine a rotation matrix:

$\mathbf{C} = \mathbf{C}_\gamma(\theta_3)\mathbf{C}_\beta(\theta_2)\mathbf{C}_\alpha(\theta_1)$, as long as the sequence of rotation axes, α, β, and γ, satisfy $\alpha \neq \beta$ and $\beta \neq \gamma$

- Note, it is important to specify the order of rotations, since **rotations do not commute**

N.B. Singularities can occur at certain angles,
e.g. if $\theta_2 = 0$ in a 3-1-3 combination

$$\mathbf{C}_{21} = \mathbf{C}_3(\theta_3)\mathbf{C}_3(\theta_1) = \mathbf{C}_3(\theta_3 + \theta_1),$$

which implies the first and third rotations collapse into one, and the attitude cannot be uniquely determined

Clarkson