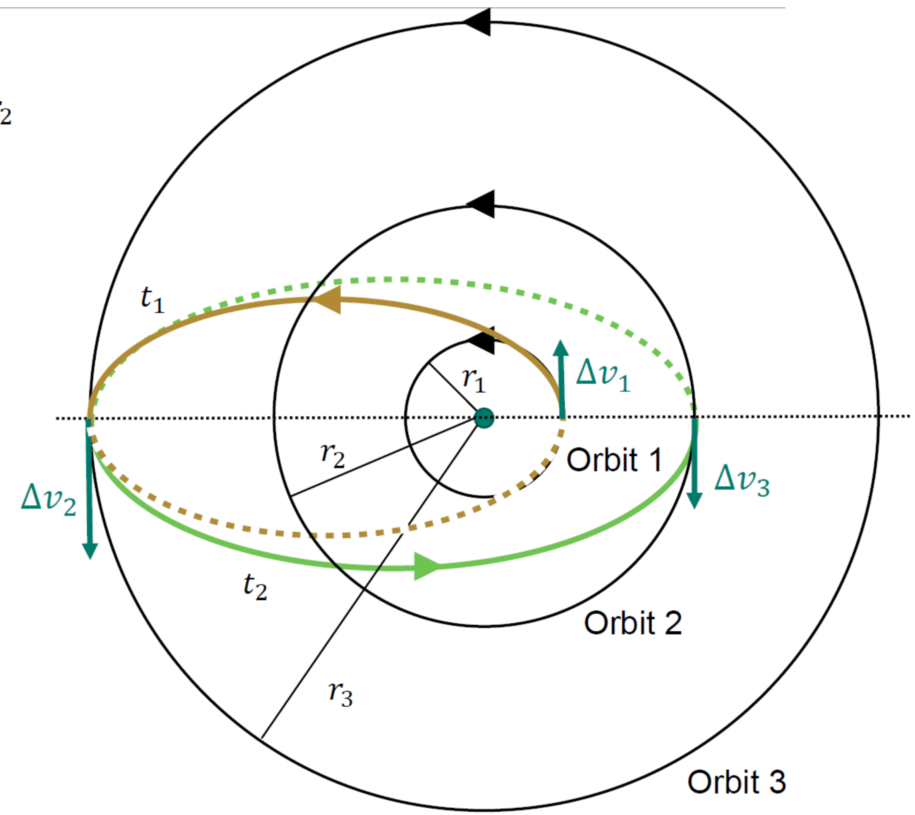
Orbital Maneuvers

Bi-Elliptic Transfer

- Triple impulse transfer for two concentric circular orbits of radius r_1 and r_2
- Both orbits have the same direction of motion
  i.e., both *prograde* or both *retrograde*
- Combines the idea of two Hohmann transfers given a larger third orbit
- Sometimes more fuel efficient than the Hohmann transfer

Consists of three impulses in sequence:

- **First impulse** → changes orbit to elliptic transfer orbit 1 that is tangent to the circular orbit 1 at periapsis
- **Second impulse** → applied at apoapsis of transfer orbit 1 and places s/c into transfer orbit 2 (tangent to orbit 3 at apoapsis)
- **Third impulse** → applied at periapsis of transfer orbit 2 and places the s/c into circular orbit 2



*Bi-elliptic transfer is minimum Δv transfer maneuver if $r_2/r_1 > 15.58$
(i.e., better than a Hohman transfer)

Clarkson

Orbital Maneuvers

Bi-Elliptic Transfer

- Semi-major axes of the transfer orbits: $a_{t1} = \frac{r_1 + r_3}{2}$ $a_{t2} = \frac{r_2 + r_3}{2}$

- Orbital speeds of initial and target orbits: $v_1 = \sqrt{\frac{\mu}{r_1}}$ $v_2 = \sqrt{\frac{\mu}{r_2}}$

- Speeds at the transfer points for Transfer Orbit 1:

periapsis:

$$v_{t1p} = \sqrt{\mu\left(\frac{2}{r_1} - \frac{1}{a_{t1}}\right)}$$

apoapsis:

$$v_{t1a} = \sqrt{\mu\left(\frac{2}{r_3} - \frac{1}{a_{t1}}\right)}$$

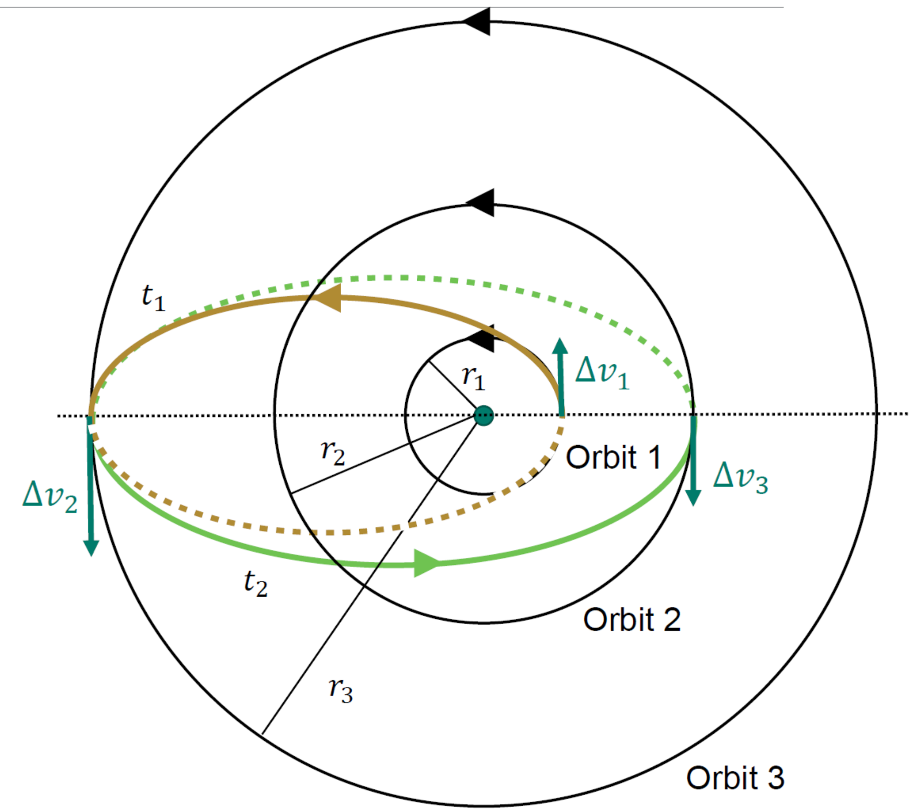
- Speeds at the transfer points for Transfer Orbit 2:

apoapsis:

$$v_{t2a} = \sqrt{\mu\left(\frac{2}{r_3} - \frac{1}{a_{t2}}\right)}$$

periapsis:

$$v_{t2p} = \sqrt{\mu\left(\frac{2}{r_2} - \frac{1}{a_{t2}}\right)}$$



Clarkson

Orbital Maneuvers

$$a_{t1} = \frac{r_1 + r_3}{2}$$

$$a_{t2} = \frac{r_2 + r_3}{2}$$

$$v_1 = \sqrt{\frac{\mu}{r_1}} \quad v_2 = \sqrt{\frac{\mu}{r_2}}$$

$$v_{t1p} = \sqrt{\mu\left(\frac{2}{r_1} - \frac{1}{a_{t1}}\right)} \quad v_{t1a} = \sqrt{\mu\left(\frac{2}{r_3} - \frac{1}{a_{t1}}\right)}$$

$$v_{t2a} = \sqrt{\mu\left(\frac{2}{r_3} - \frac{1}{a_{t2}}\right)} \quad v_{t2p} = \sqrt{\mu\left(\frac{2}{r_2} - \frac{1}{a_{t2}}\right)}$$

Bi-Elliptic Transfer

- Now, we'll consider the change in speed at each impulse

First Impulse

- Since $a_{t1} > r_1$, we have $v_{t1p} > v_1$, yielding:

$$\Delta v_1 = v_{t1p} - v_1 = \sqrt{\mu\left(\frac{2}{r_1} - \frac{1}{a_{t1}}\right)} - \sqrt{\frac{\mu}{r_1}}$$

Second Impulse

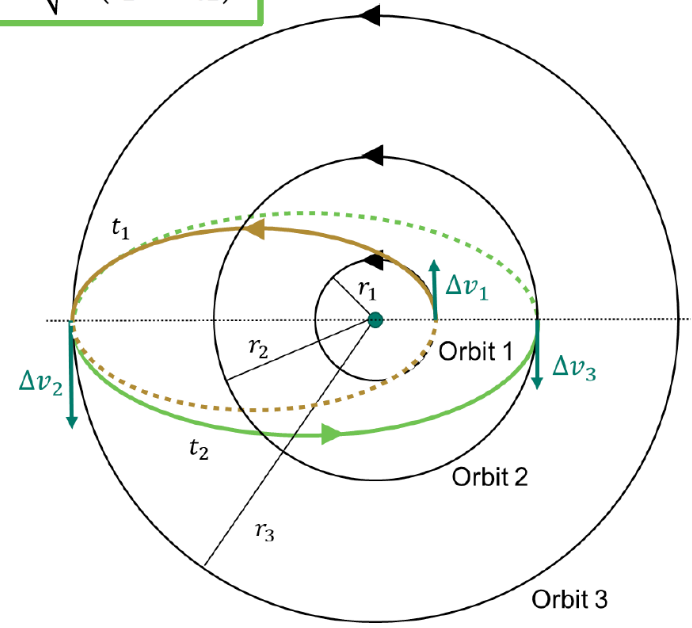
- Since $a_{t2} > a_{t1}$, we have $v_{t2a} > v_{t1a}$, yielding:

$$\Delta v_2 = v_{t2a} - v_{t1a} = \sqrt{\mu\left(\frac{2}{r_3} - \frac{1}{a_{t2}}\right)} - \sqrt{\mu\left(\frac{2}{r_3} - \frac{1}{a_{t1}}\right)}$$

Third Impulse

- Since $a_{t2} > r_2$, we have $v_{t2p} > v_2$, yielding:

$$\Delta v_3 = v_{t2p} - v_2 = \sqrt{\mu\left(\frac{2}{r_2} - \frac{1}{a_{t2}}\right)} - \sqrt{\frac{\mu}{r_2}}$$



Total Delta-v: $\Delta v = \Delta v_1 + \Delta v_2 + \Delta v_3$

$$\Delta v = \sqrt{\mu\left(\frac{2}{r_1} - \frac{1}{a_{t1}}\right)} - \sqrt{\frac{\mu}{r_1}} + \sqrt{\mu\left(\frac{2}{r_3} - \frac{1}{a_{t2}}\right)} - \sqrt{\mu\left(\frac{2}{r_3} - \frac{1}{a_{t1}}\right)} + \sqrt{\mu\left(\frac{2}{r_2} - \frac{1}{a_{t2}}\right)} - \sqrt{\frac{\mu}{r_2}}$$

Time of Flight (half the period of each transfer ellipse):

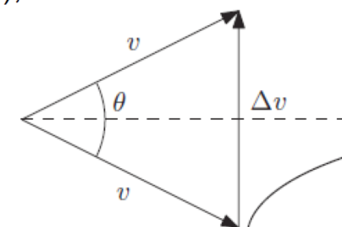
$$TOF = \pi\left(\sqrt{\frac{a_{t1}^3}{\mu}} + \sqrt{\frac{a_{t2}^3}{\mu}}\right)$$

Clarkson

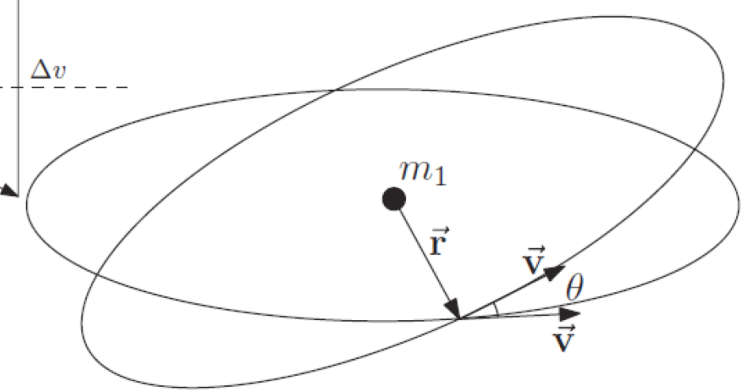
Orbital Maneuvers

Plane Change Maneuvers

- Two orbital elements associated with the plane of the orbit are: inclination, i, and right ascension of the ascending node, Ω
- In order to change these (without affecting size or shape of the orbit), we rotate $\vec{\mathbf{v}}$ about $\vec{\mathbf{r}}$



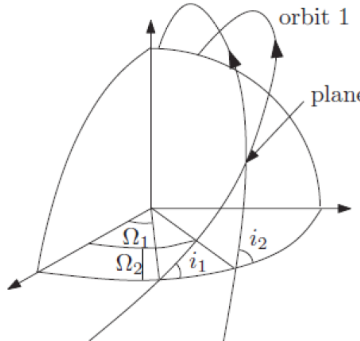
If plane change maneuver is performed when $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$ are perpendicular, and there is no change in orbital speed, then:



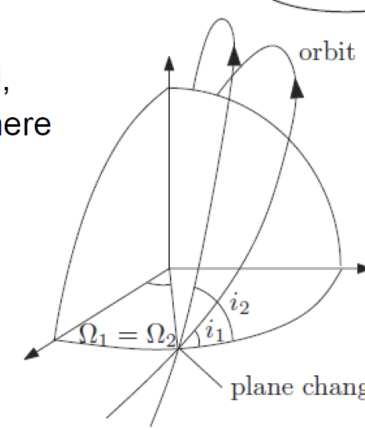
$$\Delta v = 2v \sin\frac{\theta}{2}$$

Plane change Δv can be *very* large, so should be performed at point of minimum orbital speed

Note, this will alter both i and Ω



If a pure inclination change is needed, then the plane change must occur where the orbit crosses the equatorial plane



N.B. this is for no change in orbital speed (but this concept can be applied to find combined maneuvers)

Clarkson

Orbital Maneuvers

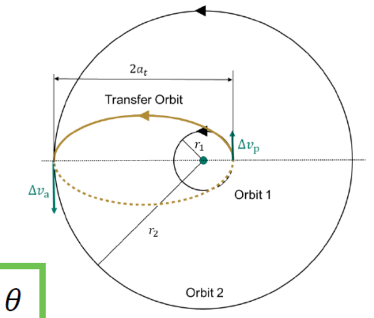
Combined Maneuvers

- In general, we can change combinations of shape, size, and orbital plane by combining the maneuvers previously discussed

Example

- Let's say we have a satellite that was inserted into a circular low-Earth orbit with a semi-major axis, a, and a non-zero inclination i, and we would like to place our satellite in a circular geosynchronous equatorial orbit (or *geostationary* orbit) with an altitude, h, what would be the total Δv of the combined maneuver?

$$\boxed{\Delta v = \left|\sqrt{\mu\left(\frac{2}{r_1}-\frac{1}{a_t}\right)} - \sqrt{\frac{\mu}{r_1}}\right| + \left|\sqrt{\frac{\mu}{r_2}} - \sqrt{\mu\left(\frac{2}{r_2}-\frac{1}{a_t}\right)}\right|}$$



First, what are we changing?

Orbit size (a) ← We can change between two co-planar circular orbits using a **Hohmann Transfer**

Orbit plane (i) ← We can combine a **plane change maneuver** with the Hohmann transfer at either the periapsis or apoapsis points

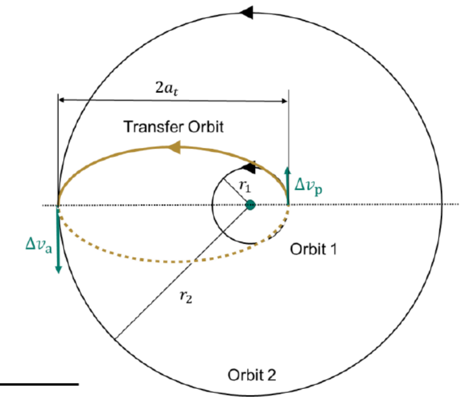
$$\boxed{\Delta v = 2v\sin\frac{\theta}{2}}$$

- Recall, we would like to do the plane change when the orbital velocity is lowest → so we choose the apoapsis point of the transfer ellipse

Orbital Maneuvers

Example (cont.)

- Let's say we have a satellite that was inserted into a circular low-Earth orbit with a semi-major axis, a, and a non-zero inclination i, and we would like to place our satellite in a circular geosynchronous equatorial orbit (or *geostationary* orbit) with an altitude, h, what would be the total Δv of the combined maneuver?



Let's set up our variables

Orbit 1: $a_1 = r_1 \qquad i_1$

Orbit 2: $h = r_2 - R_{\oplus} \qquad i_2$

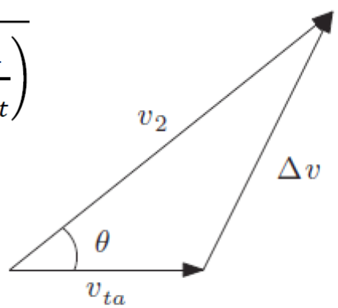
Let's find the semi-major axis of the transfer ellipse: $a_t = \dfrac{r_1 + r_2}{2}$

At periapsis: $\Delta v_p = v_{tp} - v_1 = \sqrt{\mu\left(\dfrac{2}{r_1} - \dfrac{1}{a_t}\right)} - \sqrt{\dfrac{\mu}{r_1}}$

At apoapsis (orbital speed change and plane change):

$$v_{ta} = \sqrt{\mu\left(\frac{2}{r_2} - \frac{1}{a_t}\right)}$$

$$v_2 = \sqrt{\frac{\mu}{r_2}}$$



$$\Delta v_a = \sqrt{v_{ta}^2 + v_2^2 - 2 v_{ta} v_2 \cos\theta}$$

Total Δv of the combined maneuver: $\Delta v = \Delta v_p + \Delta v_a$

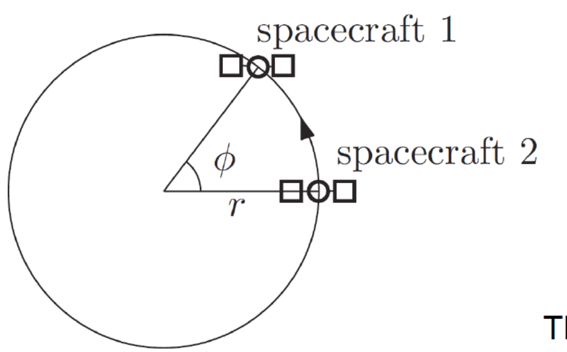
$$\Delta v = \sqrt{\mu\left(\frac{2}{r_1} - \frac{1}{a_t}\right)} - \sqrt{\frac{\mu}{r_1}} + \sqrt{v_{ta}^2 + v_2^2 - 2 v_{ta} v_2 \cos\theta}$$

...we could then substitute back in the given terms to solve

Orbital Maneuvers

Rendezvous

- Consider two spacecraft in a circular orbit, separated by a phase angle ϕ (assuming s/c 1 is ahead so that $\phi > 0$)



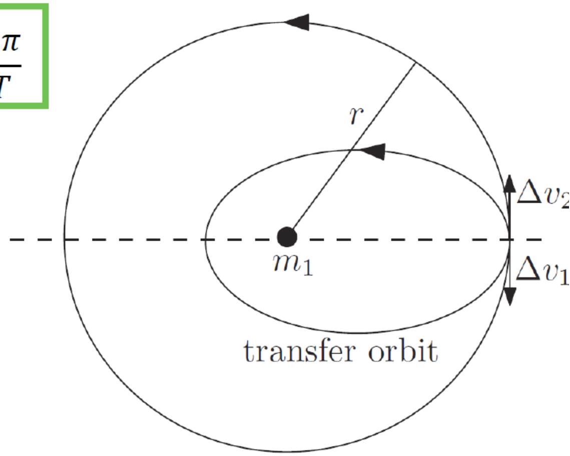
If we want s/c 2 to rendezvous with s/c 1 we must perform a **phasing maneuver**

- Let's start by recognizing that the angular rate is constant for a circular orbit, i.e., $\dot{\theta} = n$

$$\boxed{\theta(t) - \theta(t_0) = n(t - t_0)} \qquad \phi = n\Delta t \qquad \boxed{n = \frac{2\pi}{T}}$$

Time s/c 1 is ahead of s/c 2: $\Delta t_{12} = \dfrac{\phi T}{2\pi}$

This also means that the time for s/c 1 to be at the location of s/c 2 is given by: $\Delta t_{21} = T - \dfrac{\phi T}{2\pi}$



- So, if s/c 2 transfers into an elliptical orbit with period: $T_{trans} = T - \dfrac{\phi T}{2\pi}$ it will rendezvous with s/c 1 after one revolution of the transfer orbit

- The semi-major axis of the transfer orbit is obtained by substituting in the transfer orbit period:

$$\boxed{T_{trans} = 2\pi\sqrt{\frac{a_t^3}{\mu}}} \qquad a_t = \left[\mu\left(\frac{T_{trans}}{2\pi}\right)^2\right]^{\frac{1}{3}}$$

From our coplanar transfer equations we have:

$$\Delta v = \Delta v_1 + \Delta v_2 = 2\left|\sqrt{\frac{\mu}{r}} - \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a_t}\right)}\right|$$

Since we are returning to the same orbit, the magnitude of the burns are equal but in opposite directions, i.e., $\Delta v_1 = \Delta v_2$

Orbital Maneuvers

Is this maneuver feasible?

<u>No</u>, the transfer orbit would collide with Earth!

Quick Activity

A spacecraft is trying to dock with the International Space Station (ISS) and let's assume that they are both in the same circular orbit with an altitude of 420 km. If the spacecraft is separated from the ISS by a phase angle $\phi = \pi/4$, what is the Δv of the rendezvous maneuver? You are given: $\mu_\oplus = 398\,600\ \mathrm{km^3/s^2}$ and $R_\oplus = 6371$ km

Let's collect our variables

$\phi = \pi/4$

Orbit: $h = 420$ km

$h = r - R_\oplus$

$r = h + R_\oplus$

$r = 420\text{ km} + 6371\text{ km}$

$r = 6791$ km

We know we eventually want to solve this equation:

$$\Delta v = \Delta v_1 + \Delta v_2 = 2\left|\sqrt{\frac{\mu}{r}} - \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a_t}\right)}\right|$$

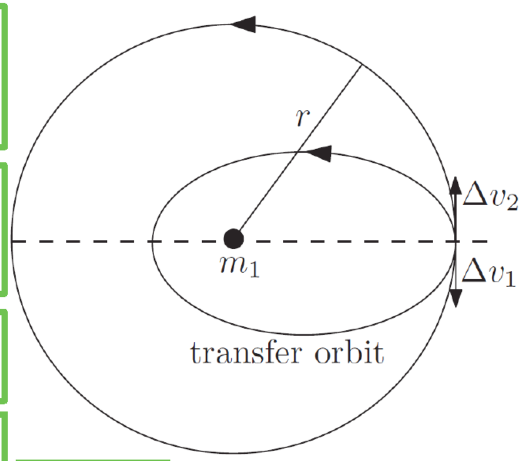
$$a_t = \left[\mu\left(\frac{T_{trans}}{2\pi}\right)^2\right]^{\frac{1}{3}}$$

$$T_{trans} = T - \frac{\phi T}{2\pi}$$

$$T = 2\pi\sqrt{\frac{a^3}{\mu}}$$

$$n = \frac{2\pi}{T}$$

$$\phi = n\Delta t$$



Find the period for the circular orbit:

$$T = 2\pi\sqrt{\frac{a^3}{\mu_\oplus}} = 2\pi\sqrt{\frac{r^3}{\mu_\oplus}} = 5569\text{ s}$$

Find the period of the transfer orbit:

$$T_{trans} = T - \frac{\phi T}{2\pi} = T\left(1 - \frac{\phi}{2\pi}\right) = T\left(\frac{7}{8}\right) = 4873\text{ s}$$

Find a_t:

$$a_t = \left[\mu\left(\frac{T_{trans}}{2\pi}\right)^2\right]^{\frac{1}{3}} = 6212\text{ km}$$

Solve for Δv:

$$\Delta v = \Delta v_1 + \Delta v_2 = 2\left|\sqrt{\frac{\mu}{r}} - \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a_t}\right)}\right| = 2\left|\sqrt{\frac{398\,600}{6791}} - \sqrt{398\,600\left(\frac{2}{6791} - \frac{1}{6212}\right)}\right| = 0.7315\text{ km/s}$$