Classical Orbital Elements

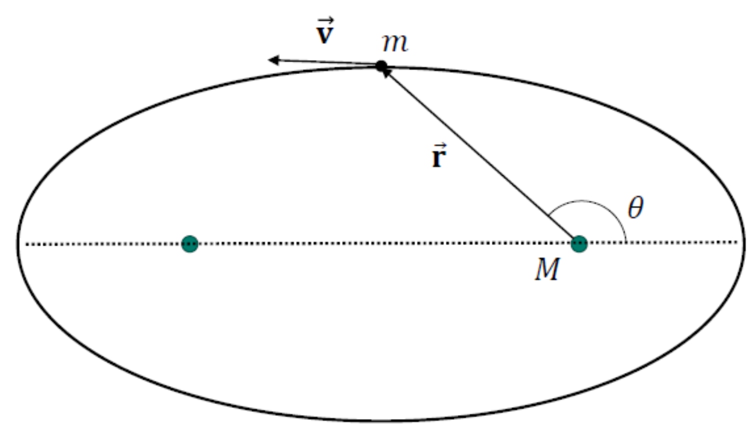
Orbit Description

- It is sufficient to describe an orbit in terms of the orbital state vectors (or Cartesian vectors), position, $\vec{\mathbf{r}}(t)$, and velocity, $\vec{\mathbf{v}}(t)$, and a given time, t

- We have found the equations of motion for the object: (a coupled system of second order ODES)

$$\dot{\vec{\mathbf{r}}} = \vec{\mathbf{v}}, \qquad \vec{\mathbf{r}}(0) = \vec{\mathbf{r}}_0$$
$$\dot{\vec{\mathbf{v}}} = -\frac{\mu}{r^3}\vec{\mathbf{r}}, \qquad \vec{\mathbf{v}}(0) = \vec{\mathbf{v}}_0$$

- For convenience, other sets of elements are used to describe orbits, let's look at the six: "classical orbital elements"



Clarkson

Classical Orbital Elements

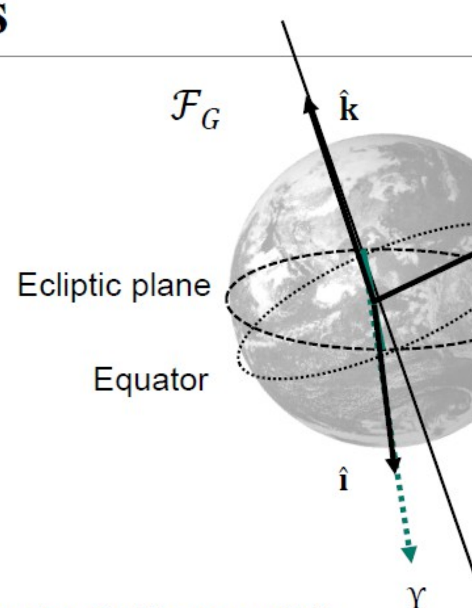
Review of Frames

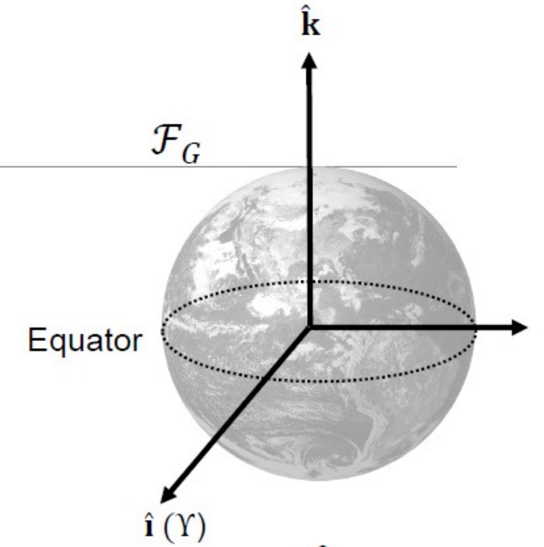
Geocentric-Equatorial Frame ($\mathcal{F}_G$)

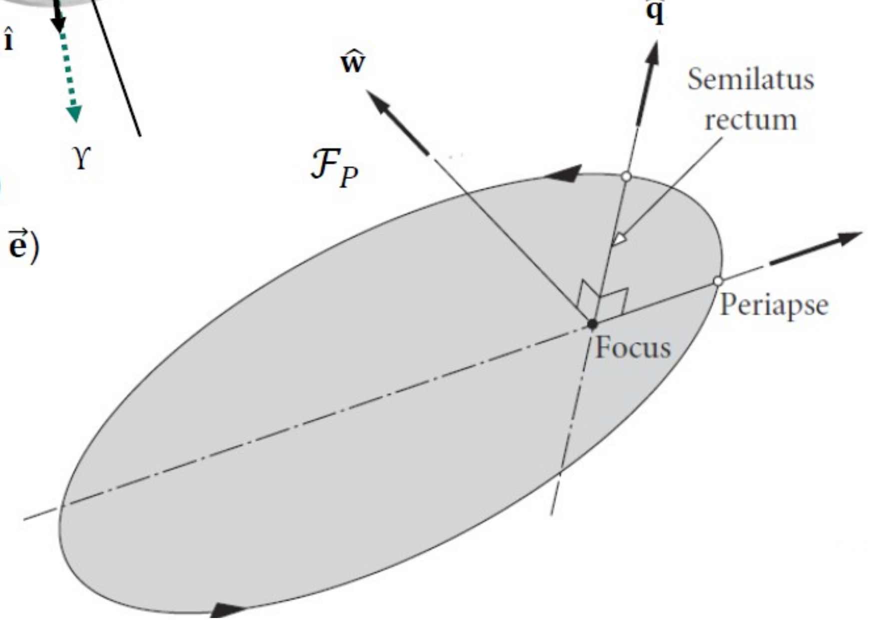
- Origin, O_G, at Earth's center of mass
- $\hat{\mathbf{g}}_1$ in the direction of the vernal equinox (Υ)
- $\hat{\mathbf{g}}_3$ towards Earth's north pole
- $\hat{\mathbf{g}}_2$ completes the right-hand rule

Perifocal Frame ($\mathcal{F}_P$)

- Origin, O_P, at the center of mass of the primary body (focus point)
- $\hat{\mathbf{p}}_1$ towards the orbit's periapsis (parallel to the eccentricity vector, $\vec{\mathbf{e}}$)
- $\hat{\mathbf{p}}_3$ normal to the orbit's plane (parallel to $\vec{\mathbf{h}}$)
- $\hat{\mathbf{p}}_2$ completes the right-hand rule (along p)







Classical Orbital Elements

The last three parameters also define a set of 3-1-3 Euler angles

Classical Orbital Elements (COEs)

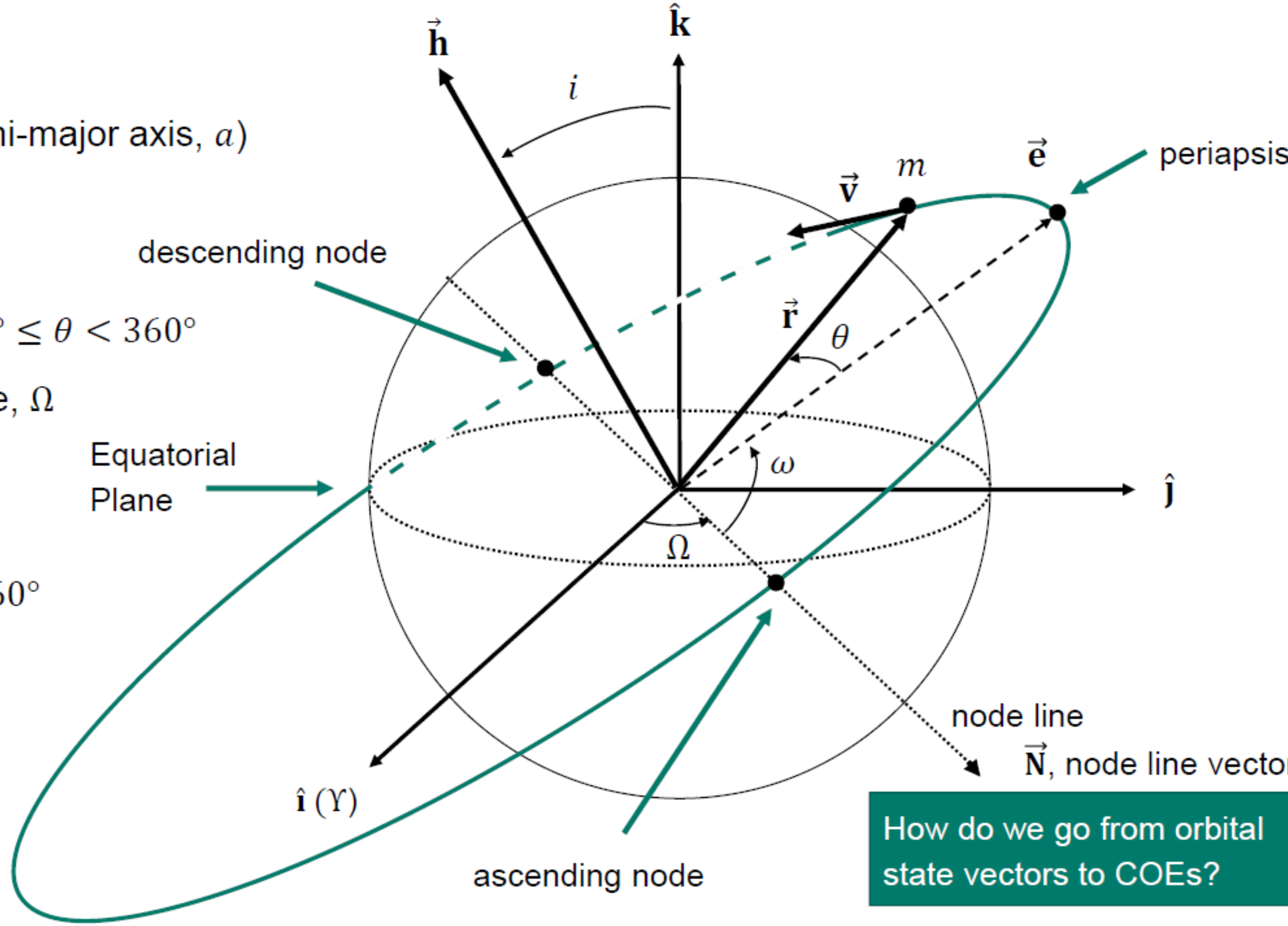
- Specific angular momentum, h (or semi-major axis, a)
- Eccentricity, e
- True anomaly, θ (also, sometime ν)
  (or sometimes epoch) $0° \leq \theta < 360°$
- Right ascension of the ascending node, Ω (RAAN) $0° \leq \Omega < 360°$
- Inclination, i $0° \leq i \leq 180°$
- Argument of perigee, ω $0° \leq \omega < 360°$

$i = 0°$ or $180°$ → Equatorial

$i < 90°$ → Prograde

$i = 90°$ → Polar

$i > 90°$ → Retrograde



How do we go from orbital state vectors to COEs?

Clarkson [De Ruiter, Ch. 3; Curtis, Ch. 4]

Classical Orbital Elements

Algorithm: COEs from State Vectors

$$\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$
$$\vec{\mathbf{v}} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$$

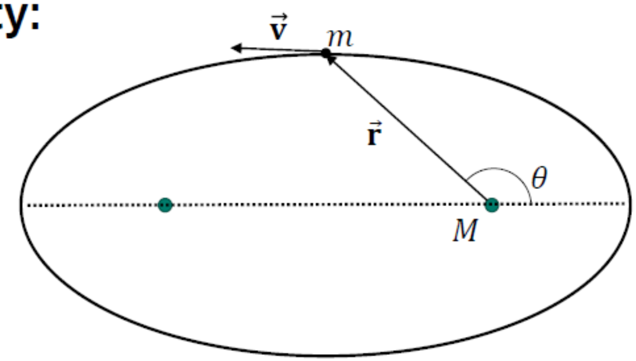
1. Calculate the distance:

$$r = \sqrt{\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}}$$
$$r = \sqrt{x^2 + y^2 + z^2}$$

2. Calculate the speed:

$$v = \sqrt{\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}}$$
$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

3. Calculate the radial velocity:



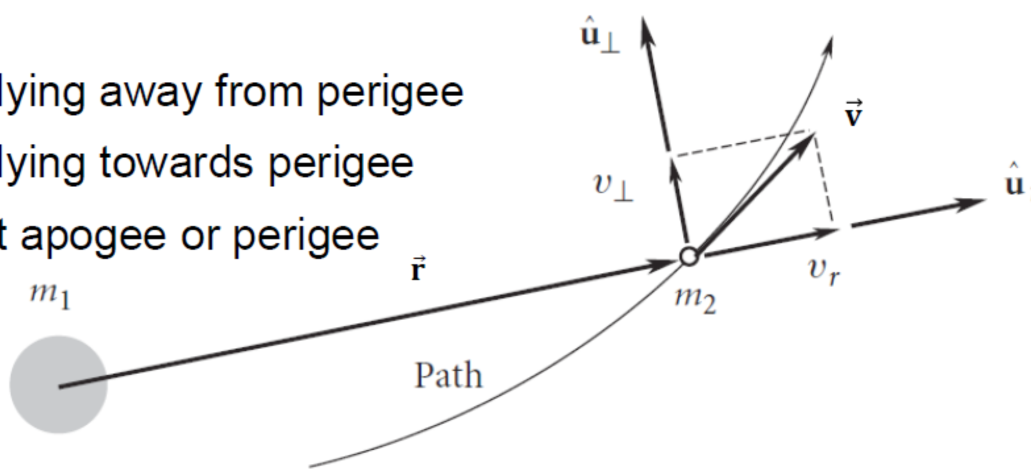
$$v_r = \frac{\vec{\mathbf{v}} \cdot \vec{\mathbf{r}}}{r}$$
$$v_r = \frac{(xv_x + yv_y + zv_z)}{r}$$

Note:

$v_r > 0$ ← Flying away from perigee

$v_r < 0$ ← Flying towards perigee

$v_r = 0$ ← At apogee or perigee



Classical Orbital Elements

Algorithm: COEs from State Vectors (cont.)

$$\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$
$$\vec{\mathbf{v}} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$$

4. Calculate the specific angular momentum:

$$\vec{\mathbf{h}} = \vec{\mathbf{r}} \times \vec{\mathbf{v}}$$

$$\vec{\mathbf{h}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x & y & z \\ v_x & v_y & v_z \end{vmatrix}$$

$$\vec{\mathbf{h}} = (yv_z - zv_y)\hat{\mathbf{i}} + (zv_x - xv_z)\hat{\mathbf{j}} + (xv_y - yv_x)\hat{\mathbf{k}}$$
$$\vec{\mathbf{h}} = h_x\hat{\mathbf{i}} + h_y\hat{\mathbf{j}} + h_z\hat{\mathbf{k}}$$

5. Calculate the magnitude of the specific angular momentum:

$$h = \sqrt{\vec{\mathbf{h}} \cdot \vec{\mathbf{h}}}$$

$$h = \sqrt{h_x^2 + h_y^2 + h_z^2} \leftarrow \underline{\text{First orbital element}}$$

Classical Orbital Elements

Algorithm: COEs from State Vectors (cont.)

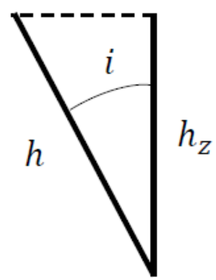
$$\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$
$$\vec{\mathbf{v}} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$$

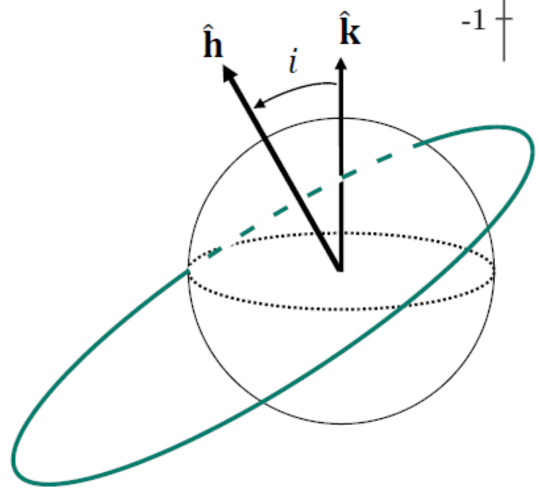
6. Calculate the inclination

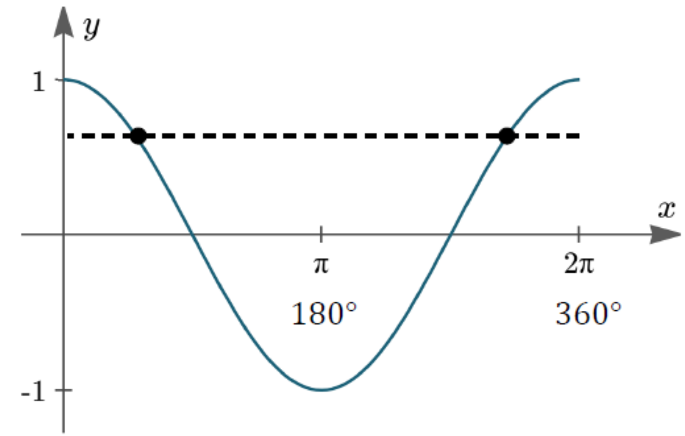
$$\vec{\mathbf{h}} = h_x\hat{\mathbf{i}} + h_y\hat{\mathbf{j}} + h_z\hat{\mathbf{k}}$$

$$\cos i = \frac{h_z}{h}$$

$$i = \cos^{-1}\left(\frac{h_z}{h}\right)$$ ← Second orbital element







Recall, $0° \leq i \leq 180°$

So, no quadrant ambiguity

$i = 0°$ or $180°$ → Equatorial

$i < 90°$ → Prograde

$i = 90°$ → Polar

$i > 90°$ → Retrograde

Classical Orbital Elements

Algorithm: COEs from State Vectors (cont.)

$$\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

$$\vec{\mathbf{v}} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$$

7. Calculate the node line vector:

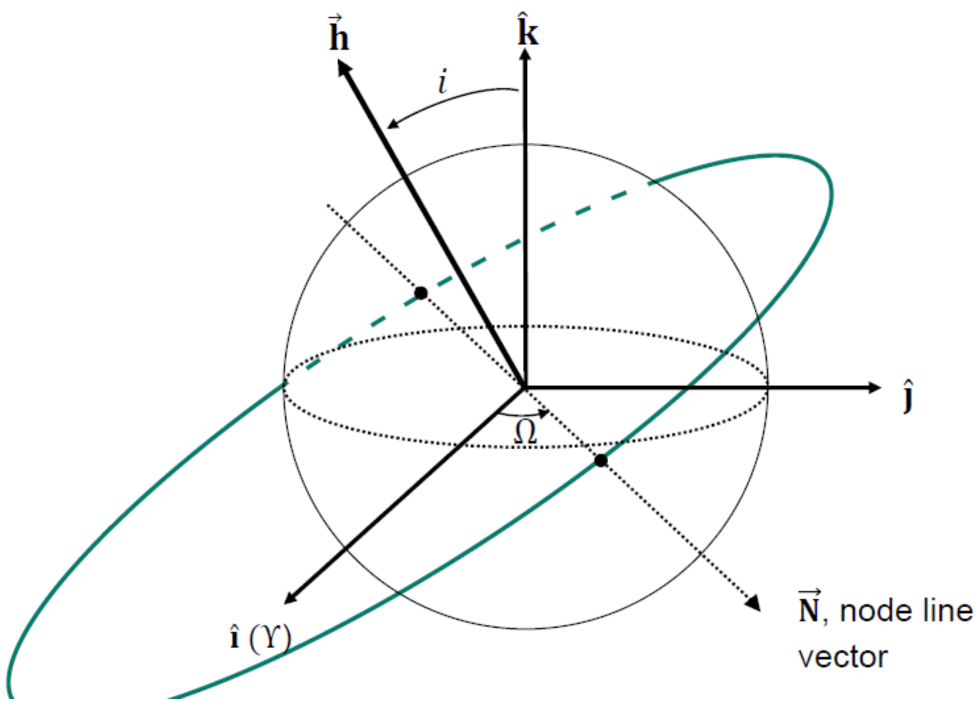
$$\vec{\mathbf{N}} = \hat{\mathbf{k}} \times \vec{\mathbf{h}}$$

$$\vec{\mathbf{N}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 1 \\ h_x & h_y & h_z \end{vmatrix}$$

$$\vec{\mathbf{N}} = \left(-h_y\right)\hat{\mathbf{i}} + (h_x)\hat{\mathbf{j}} + (0)\hat{\mathbf{k}} = N_x\hat{\mathbf{i}} + N_y\hat{\mathbf{j}}$$

8. Calculate the magnitude of the node line vector:

$$N = \sqrt{\vec{\mathbf{N}} \cdot \vec{\mathbf{N}}} = \sqrt{N_x^2 + N_y^2}$$



Classical Orbital Elements

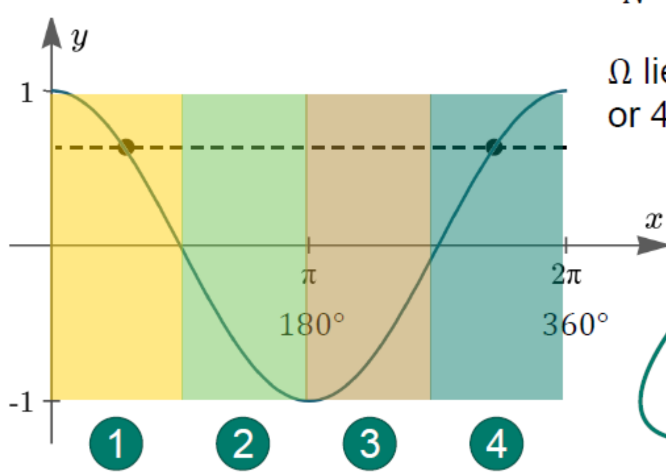
Algorithm: COEs from State Vectors (cont.)

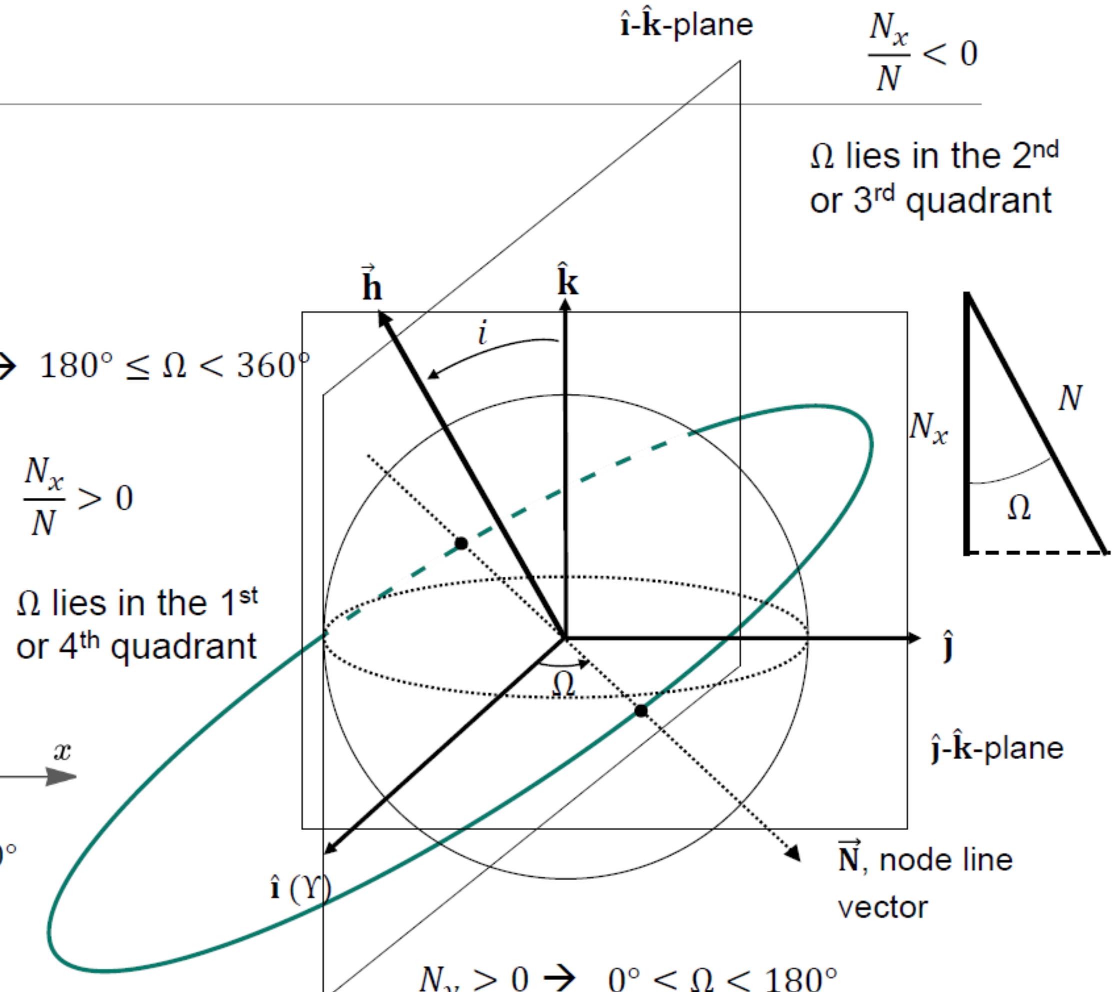
$$\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$
$$\vec{\mathbf{v}} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$$

9. Calculate the right ascension of the ascending node

$$\cos\Omega = \frac{N_x}{N}$$

$$\Omega = \cos^{-1}\left(\frac{N_x}{N}\right)$$





 [De Ruiter, Ch. 3; Curtis, Ch. 4; "Graphs of…", https://www.intmath.com/trigonometric-graphs/1-graphs-sine-cosine-amplitude.php , 2018]

Classical Orbital Elements

Algorithm: COEs from State Vectors (cont.)

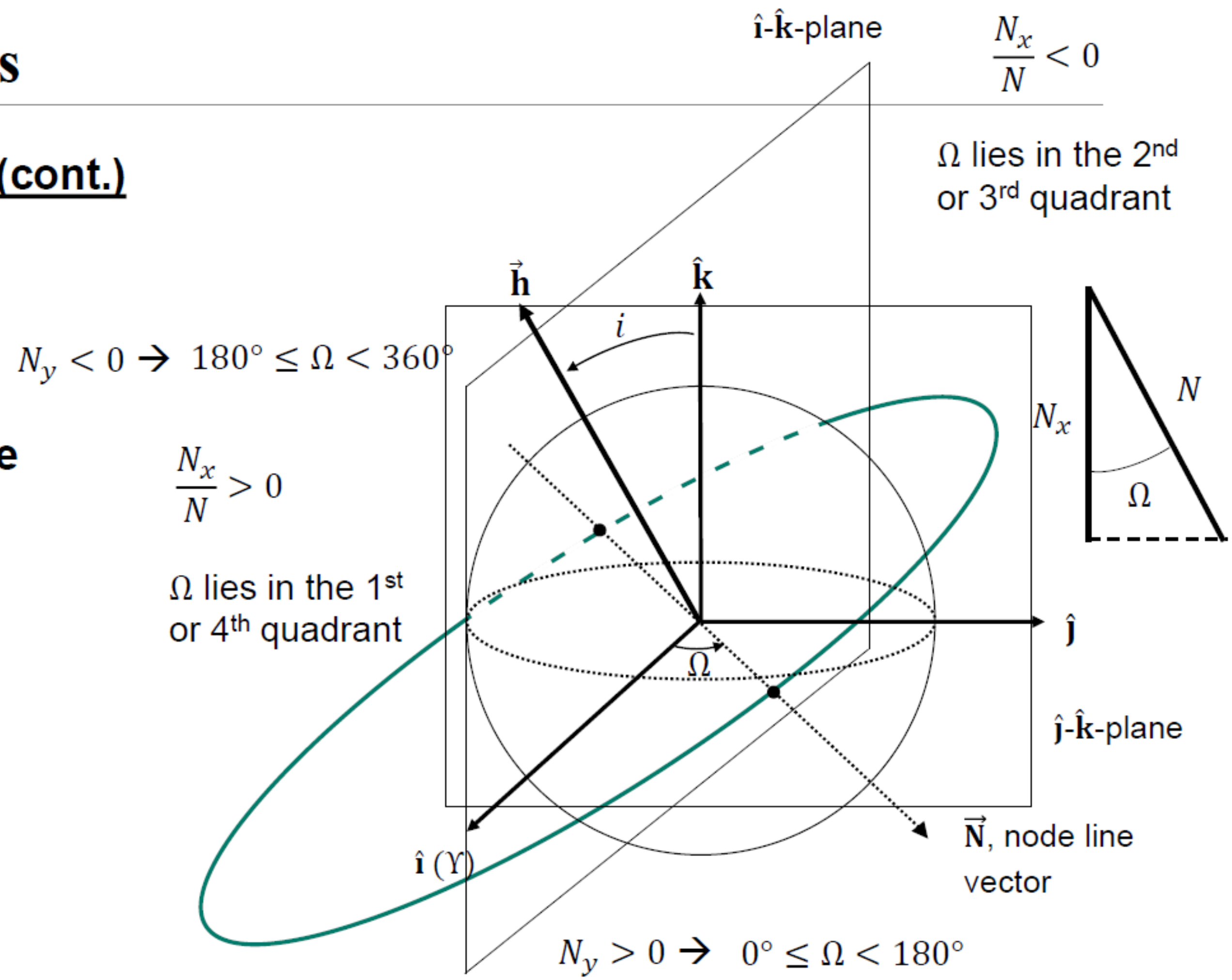
$$\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

$$\vec{\mathbf{v}} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$$

9. Calculate the right ascension of the ascending node

$$\Omega = \begin{cases} \cos^{-1}\left(\dfrac{N_x}{N}\right), & (N_y \geq 0) \\ 360° - \cos^{-1}\left(\dfrac{N_x}{N}\right) & (N_y < 0) \end{cases}$$

<u>Third orbital element</u>



Classical Orbital Elements

Algorithm: COEs from State Vectors (cont.)

$$\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$
$$\vec{\mathbf{v}} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$$

10. Calculate the eccentricity vector:

$$\vec{\mathbf{e}} = \frac{1}{\mu}\left[\vec{\mathbf{v}} \times \vec{\mathbf{h}} - \mu\frac{\vec{\mathbf{r}}}{r}\right] = \frac{1}{\mu}\left[\vec{\mathbf{v}} \times (\vec{\mathbf{r}} \times \vec{\mathbf{v}}) - \mu\frac{\vec{\mathbf{r}}}{r}\right]$$

$$\vec{\mathbf{e}} = \frac{1}{\mu}\left[\vec{\mathbf{v}} \times (\vec{\mathbf{r}} \times \vec{\mathbf{v}}) - \mu\frac{\vec{\mathbf{r}}}{r}\right] = \frac{1}{\mu}\left[\vec{\mathbf{r}}v^2 - \vec{\mathbf{v}}(\vec{\mathbf{r}} \cdot \vec{\mathbf{v}}) - \mu\frac{\vec{\mathbf{r}}}{r}\right]$$

$$\boxed{\vec{\mathbf{e}} = \frac{1}{\mu}\left[\left(v^2 - \frac{\mu}{r}\right)\vec{\mathbf{r}} - rv_r\vec{\mathbf{v}}\right]}$$

Vector Triple Product:
$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

$$v_r = \frac{\vec{\mathbf{v}} \cdot \vec{\mathbf{r}}}{r}$$

11. Calculate the eccentricity:

$$e = \sqrt{\vec{\mathbf{e}} \cdot \vec{\mathbf{e}}}$$

$$e = \sqrt{e_x^2 + e_y^2 + e_z^2}$$

Or, we can find in terms of scalars:

$$e = \frac{1}{\mu}\sqrt{(2\mu - rv^2)rv_r^2 + (\mu - rv^2)^2}$$

Fourth orbital element

Classical Orbital Elements

Algorithm: COEs from State Vectors (cont.)

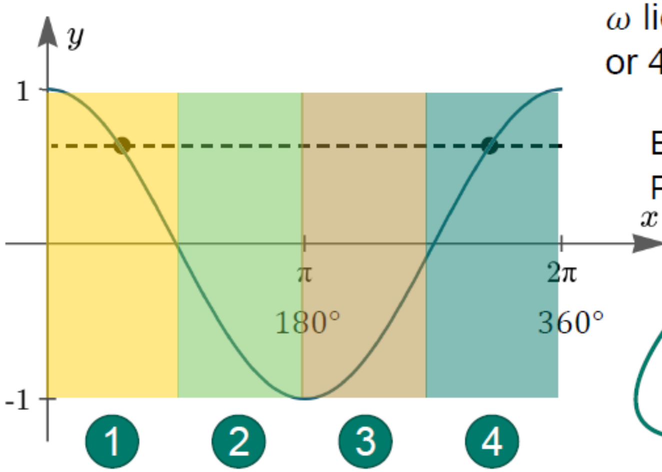
$$\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

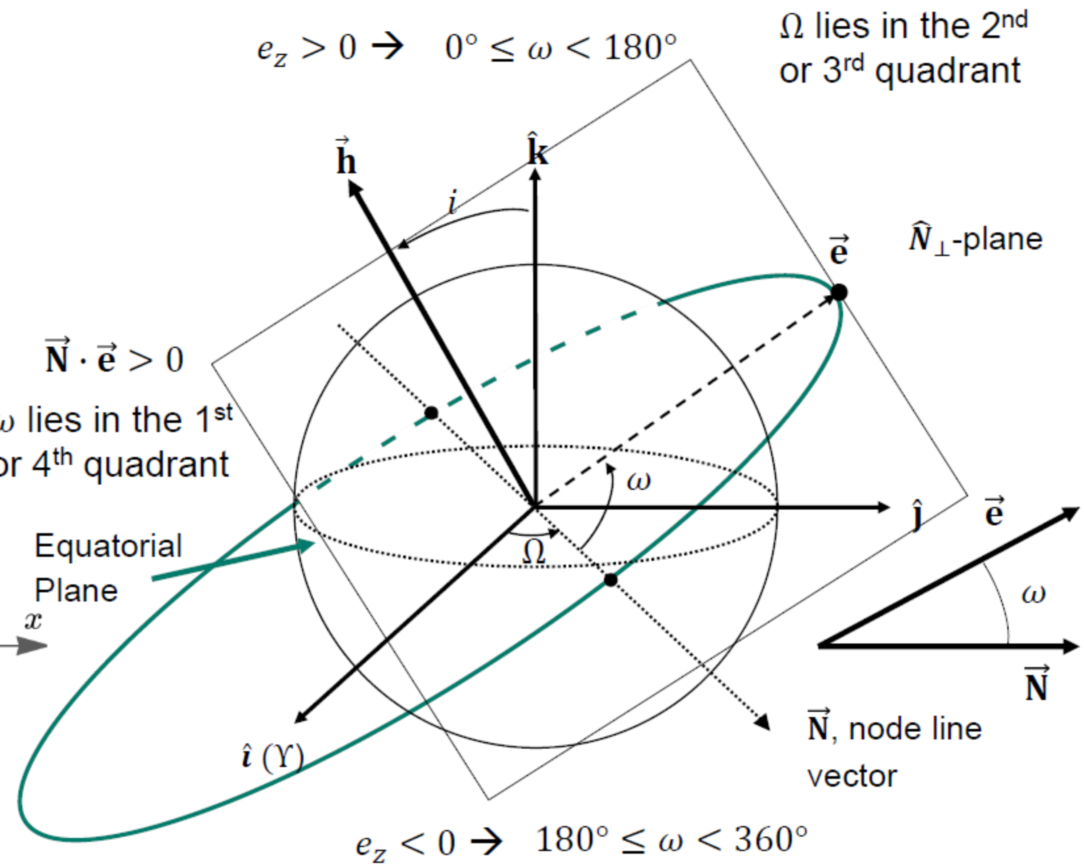
$$\vec{\mathbf{v}} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$$

12. Calculate the argument of perigee

$$\cos\omega = \frac{\vec{\mathbf{N}}\cdot\vec{\mathbf{e}}}{Ne}$$

$$\omega = \cos^{-1}\left(\frac{\vec{\mathbf{N}}\cdot\vec{\mathbf{e}}}{Ne}\right)$$





Clarkson [De Ruiter, Ch. 3; Curtis, Ch. 4; "Graphs of…", https://www.intmath.com/trigonometric-graphs/1-graphs-sine-cosine-amplitude.php , 2018]

Classical Orbital Elements

Algorithm: COEs from State Vectors (cont.)

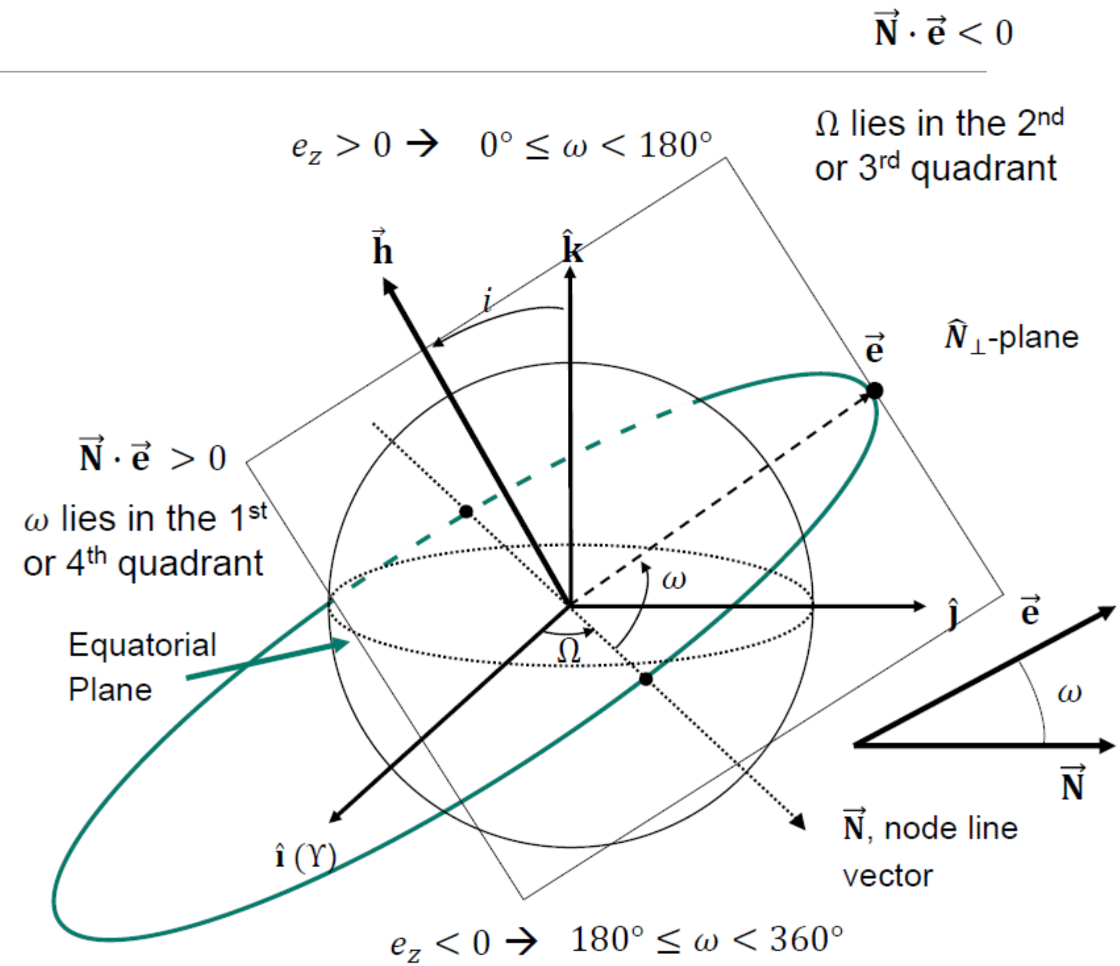
$\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

$\vec{\mathbf{v}} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$

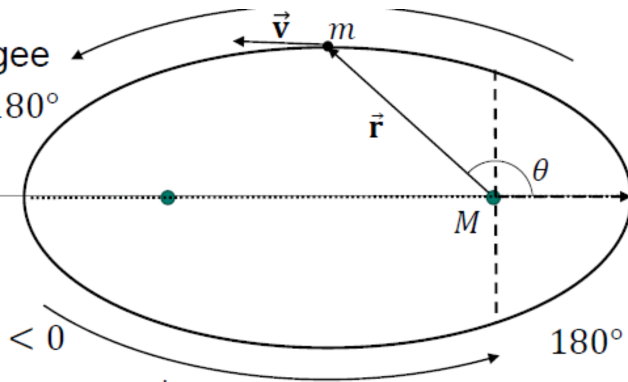
12. Calculate the argument of perigee

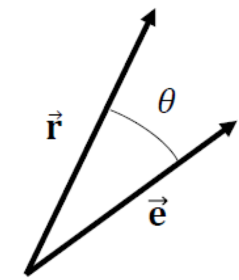
$$\omega = \begin{cases} \cos^{-1}\left(\dfrac{\vec{\mathbf{N}}\cdot\vec{\mathbf{e}}}{Ne}\right), & (e_z \geq 0) \\ 360° - \cos^{-1}\left(\dfrac{\vec{\mathbf{N}}\cdot\vec{\mathbf{e}}}{Ne}\right) & (e_z < 0) \end{cases}$$

Fifth orbital element



Classical Orbital Elements





Algorithm: COEs from State Vectors (cont.)

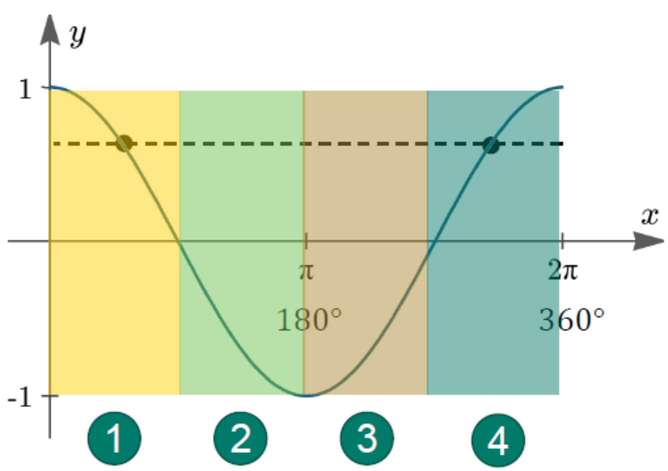
$$\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

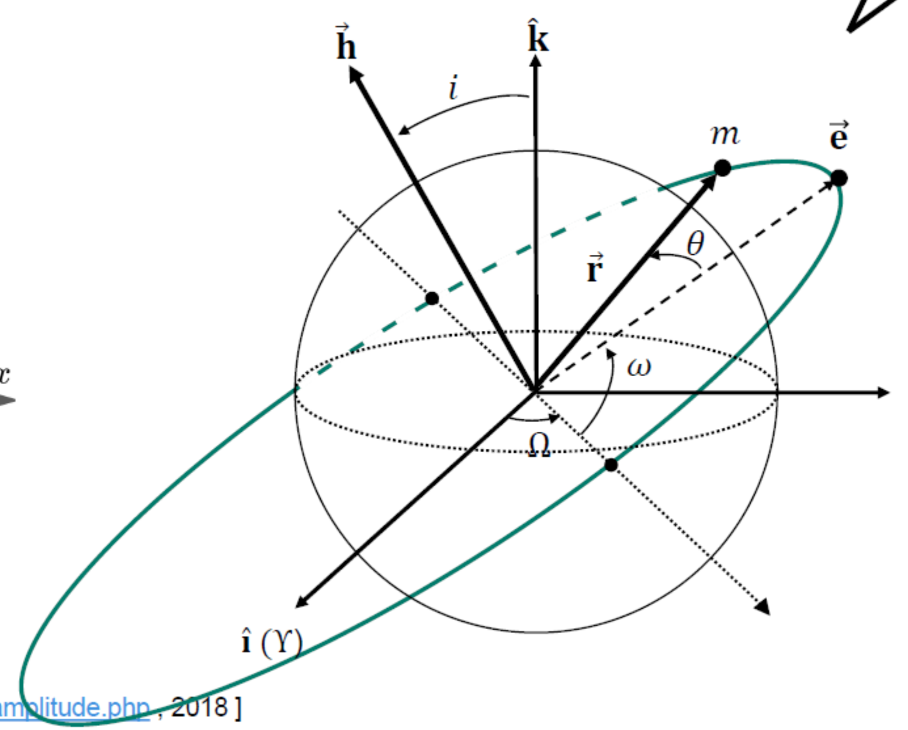
$$\vec{\mathbf{v}} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$$

13. Calculate the true anomaly

$$\cos\theta = \frac{\vec{\mathbf{e}} \cdot \vec{\mathbf{r}}}{er}$$

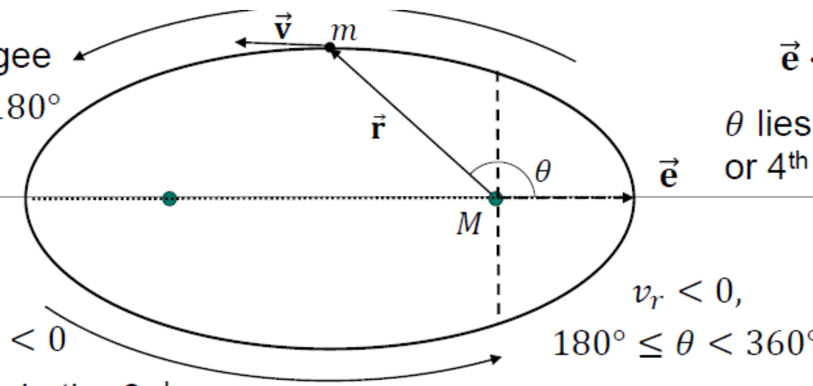
$$\theta = \cos^{-1}\left(\frac{\vec{\mathbf{e}} \cdot \vec{\mathbf{r}}}{er}\right)$$

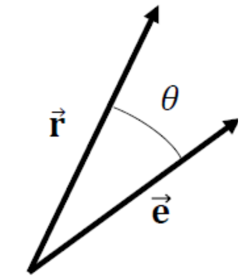




[De Ruiter, Ch. 3; Curtis, Ch. 4; "Graphs of…", https://www.intmath.com/trigonometric-graphs/1-graphs-sine-cosine-amplitude.php, 2018]

Classical Orbital Elements





Algorithm: COEs from State Vectors (cont.)

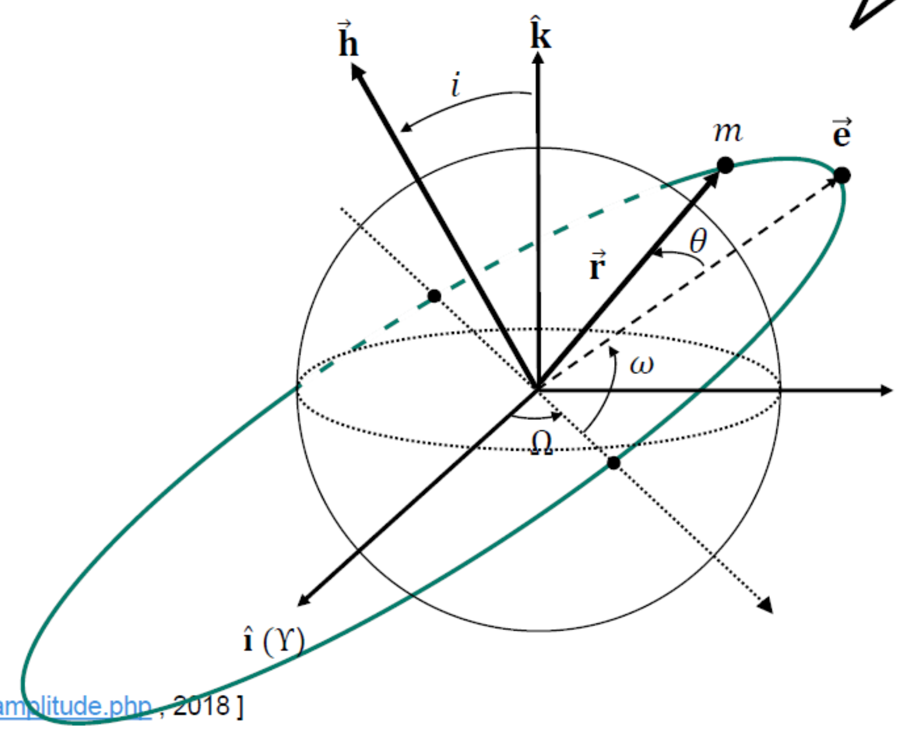
$$\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

$$\vec{\mathbf{v}} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$$

13. Calculate the true anomaly

$$\theta = \begin{cases} \cos^{-1}\left(\dfrac{\vec{\mathbf{e}} \cdot \vec{\mathbf{r}}}{er}\right), & (v_r \geq 0) \\ 360° - \cos^{-1}\left(\dfrac{\vec{\mathbf{e}} \cdot \vec{\mathbf{r}}}{er}\right) & (v_r < 0) \end{cases}$$

Sixth orbital element



 [De Ruiter, Ch. 3; Curtis, Ch. 4; "Graphs of…", https://www.intmath.com/trigonometric-graphs/1-graphs-sine-cosine-amplitude.php , 2018]

Classical Orbital Elements

Algorithm: COEs from State Vectors (cont.)

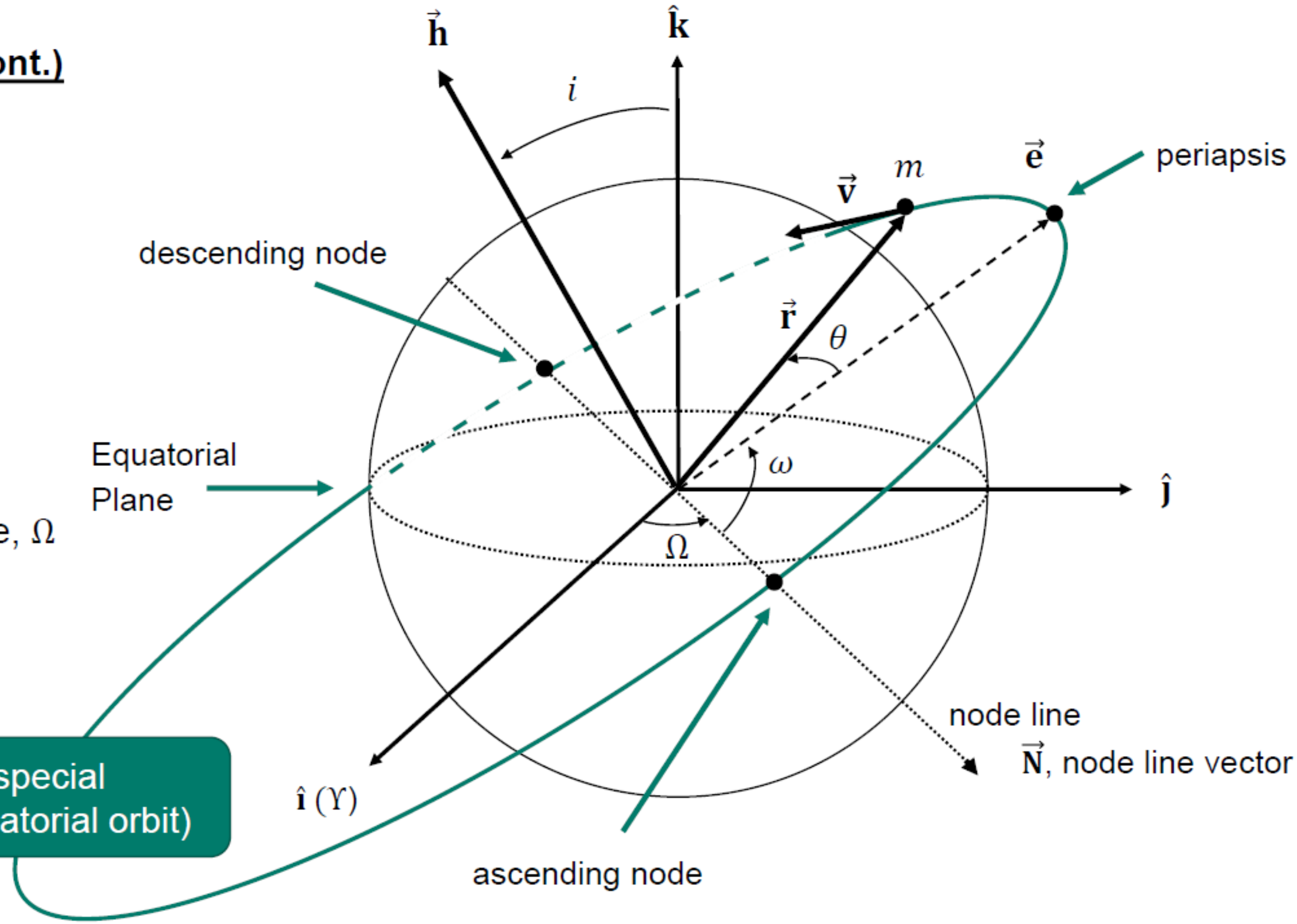
$\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

$\vec{\mathbf{v}} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$

Classical Orbital Elements (COEs)

- Specific angular momentum, h
- Eccentricity, e
- True anomaly, θ
- Right ascension of the ascending node, Ω
- Inclination, i
- Argument of perigee, ω

N.B. special cases exist that need special consideration (e.g., circular orbits or equatorial orbit)



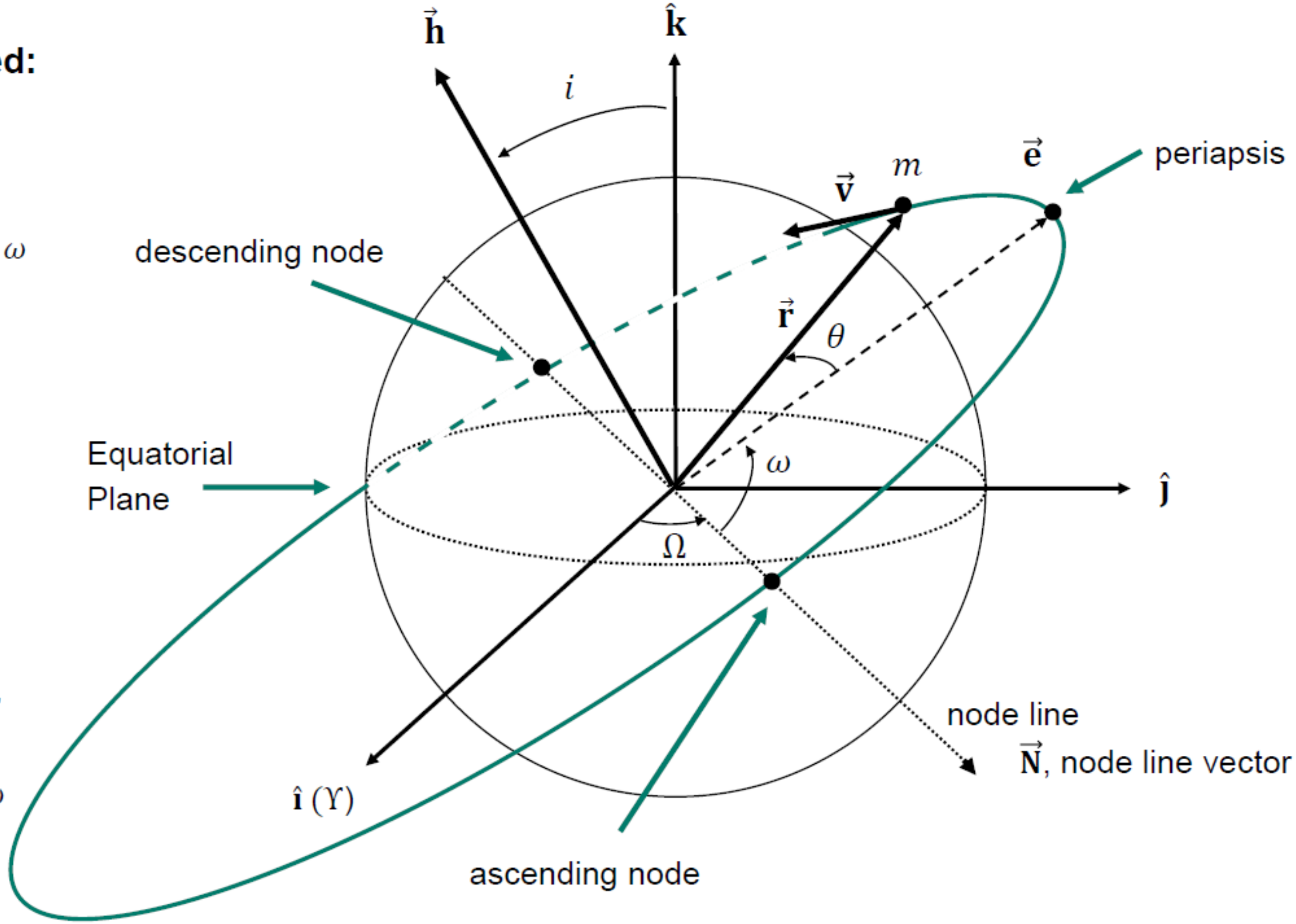
Clarkson

Classical Orbital Elements

N.B. special cases exist that need special consideration (e.g., circular orbits or equatorial orbit)

Cases where not all elements are defined:

- Circular orbit
  - *No periapsis, so ω is undefined*
  - Instead argument of latitude, $u \equiv \theta + \omega$ (angle from ascending node) is used
- Elliptical equatorial orbit
  - *No line of nodes, so Ω is undefined*
  - Instead longitude of periapsis, $\Pi \equiv \Omega + \omega$ (angle between periapsis and vernal equinox) is used
- Circular equatorial orbit
  - *No periapsis or node line, so ω and Ω are undefined, as is Π*
  - True longitude, $l = \theta + \Pi = \theta + \Omega + \omega$ (angle from the vernal equinox) may be used



Clarkson

COEs to State Vectors

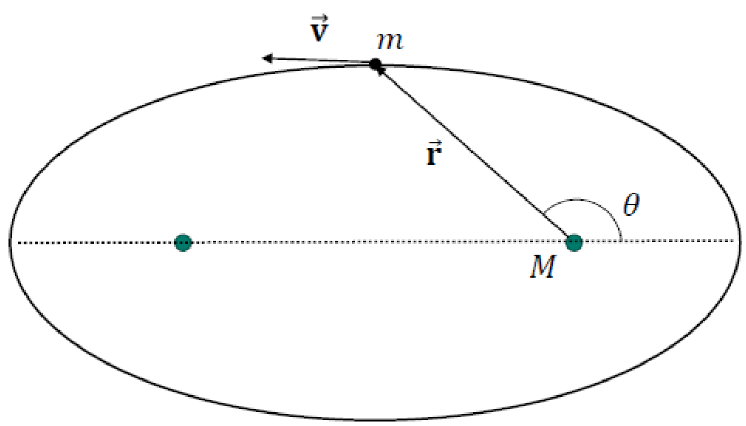
- As mentioned previously, it is sufficient to describe an orbit in terms of the orbital state vectors, position, $\vec{\mathbf{r}}(t)$, and velocity, $\vec{\mathbf{v}}(t)$ and a given time, t

- If we have the classical orbital elements, we can also solve for the state vectors

$$\dot{\vec{\mathbf{r}}} = \vec{\mathbf{v}}, \qquad \vec{\mathbf{r}}(0) = \vec{\mathbf{r}}_0$$

$$\dot{\vec{\mathbf{v}}} = -\frac{\mu}{r^3}\vec{\mathbf{r}}, \qquad \vec{\mathbf{v}}(0) = \vec{\mathbf{v}}_0$$

- The first step is recognizing the relationship between the perifocal frame and the ECI frame

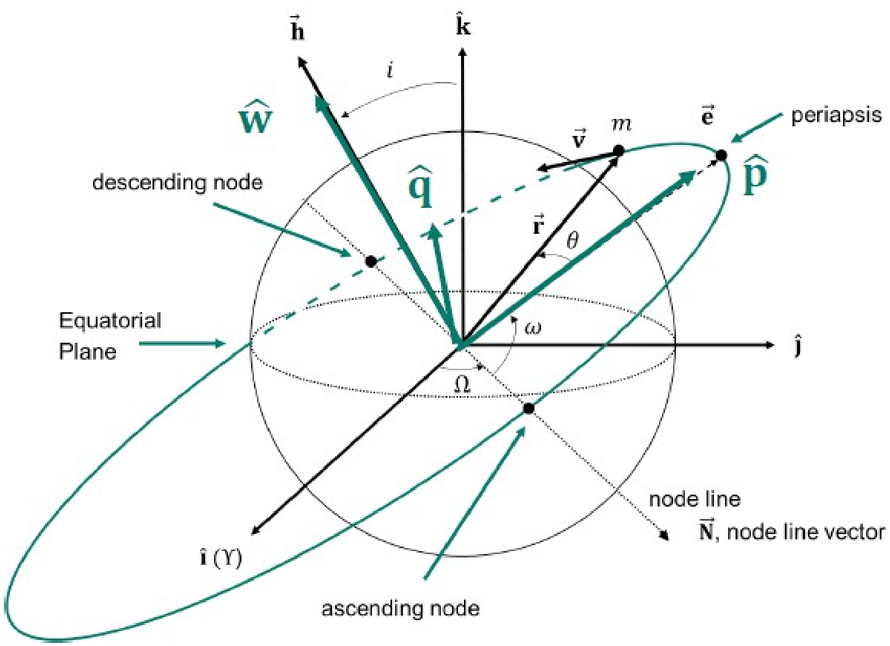


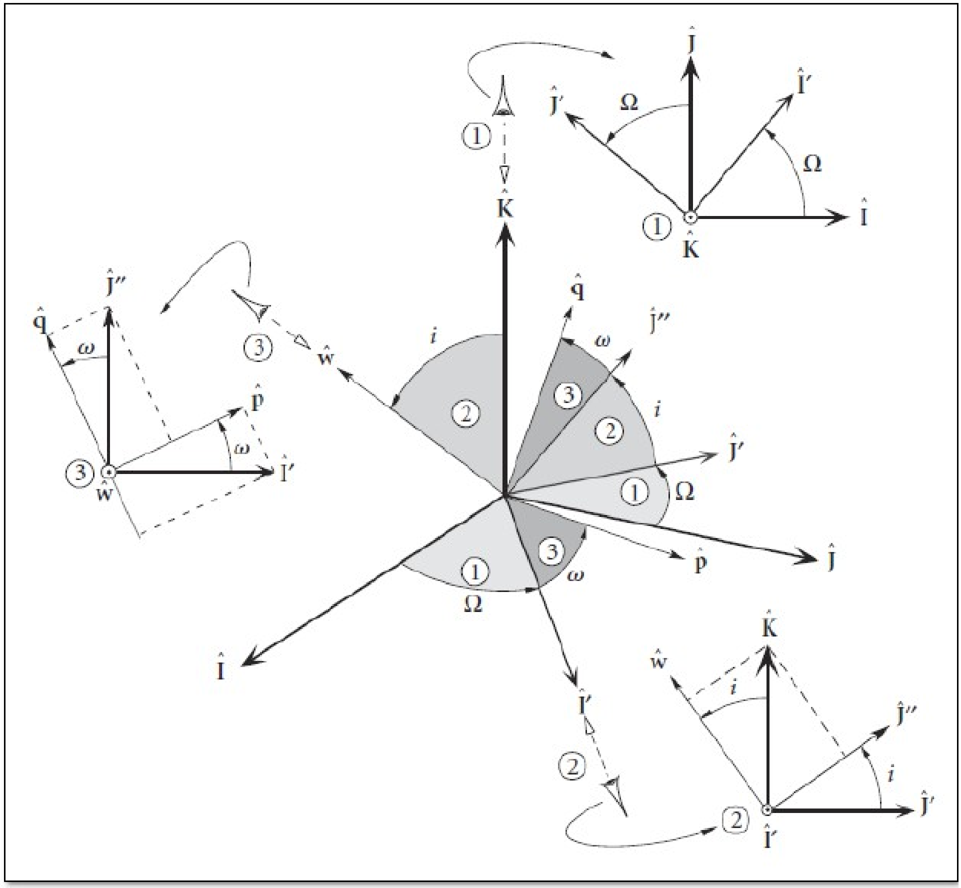
Clarkson

COEs to State Vectors

Transform between the **Geocentric-Equatorial Frame** ($\mathcal{F}_G$) to the **Perifocal Frame** ($\mathcal{F}_P$)

- Rotation matrix can be represented in terms of principal axis rotations and Euler angles





[Curtis, Ch. 4]

COEs to State Vectors

$\hat{\mathbf{K}} = \hat{\mathbf{K}}'$

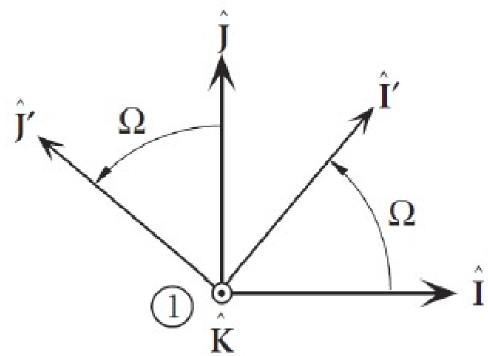
Transform between the **Geocentric-Equatorial Frame** ($\mathcal{F}_G$) to the **Perifocal Frame** ($\mathcal{F}_P$)

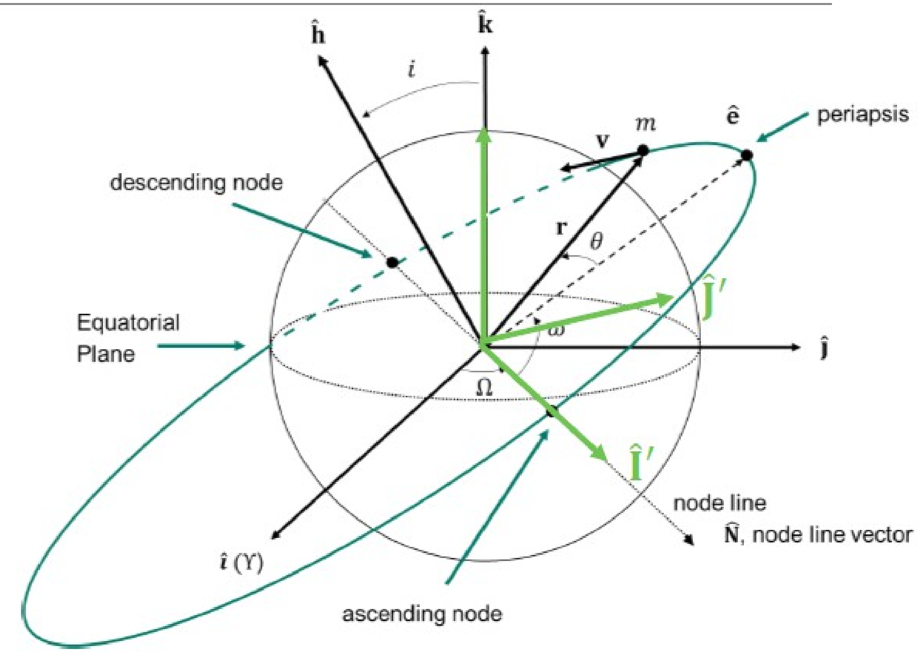
- Rotation matrix can be represented in terms of principal axis rotations and Euler angles

① Rotation about 3-axis ($\hat{\mathbf{K}}$)

$$\boxed{\mathbf{C}_3(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\mathbf{C}_3(\Omega) = \begin{bmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix}$$





COEs to State Vectors

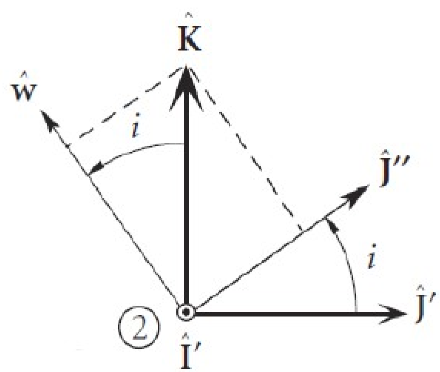
Transform between the **Geocentric-Equatorial Frame** ($\mathcal{F}_G$) to the **Perifocal Frame** ($\mathcal{F}_P$)

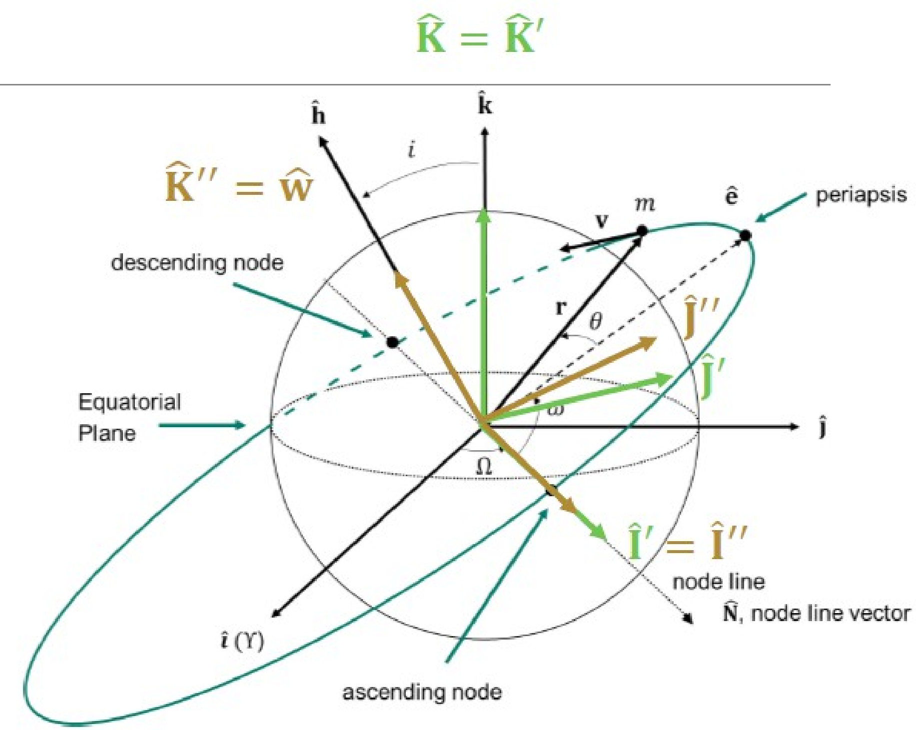
- Rotation matrix can be represented in terms of principal axis rotations and Euler angles

② Rotation about 1-axis ($\hat{\mathbf{I}}'$)

$$\boxed{\mathbf{C}_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}}$$

$$\mathbf{C}_1(i) \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{bmatrix}$$

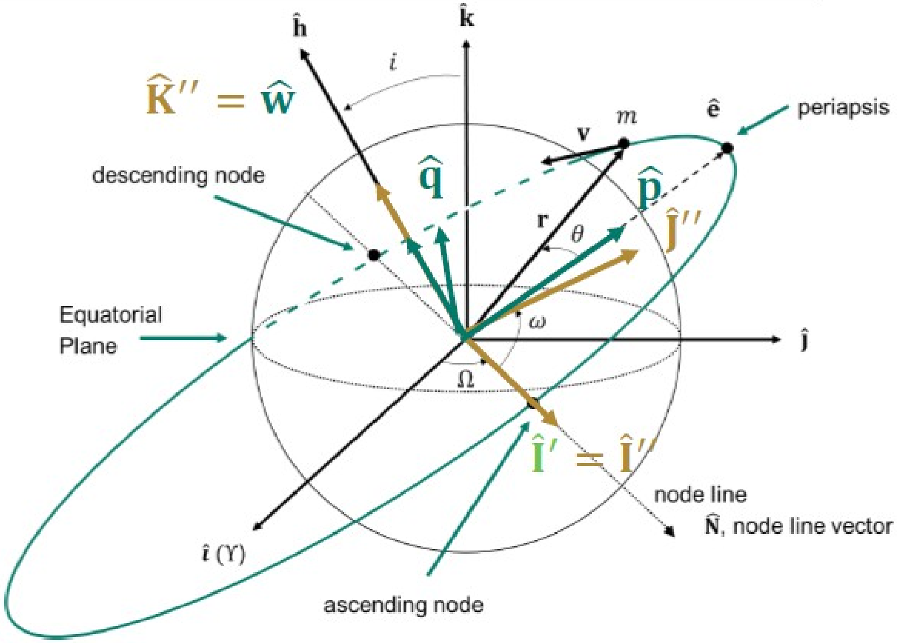




COEs to State Vectors

Transform between the **Geocentric-Equatorial Frame** ($\mathcal{F}_G$) to the **Perifocal Frame** ($\mathcal{F}_P$)

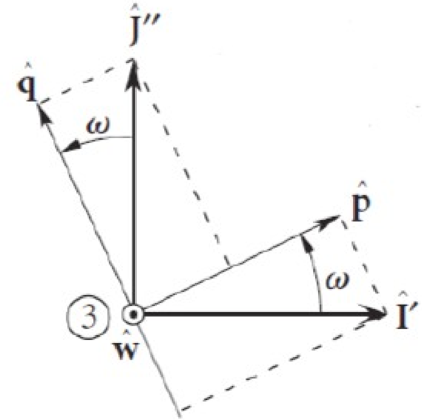
- Rotation matrix can be represented in terms of principal axis rotations and Euler angles



③ Rotation about 3-axis ($\hat{\mathbf{K}}$)

$$\boxed{\mathbf{C}_3(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\mathbf{C}_3(\omega) = \begin{bmatrix} \cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Clarkson

[Curtis, Ch. 4]

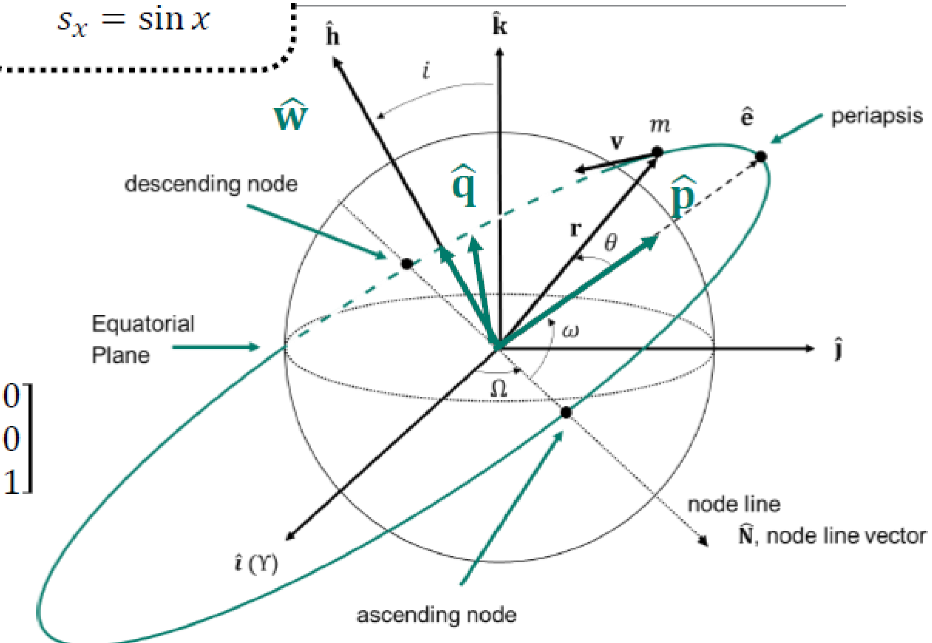
COEs to State Vectors

For simplicity:

$$c_x = \cos x$$
$$s_x = \sin x$$

Transform between the **Geocentric-Equatorial Frame** ($\mathcal{F}_G$) to the **Perifocal Frame** ($\mathcal{F}_P$)

- Rotation matrix can be represented in terms of principal axis rotations and Euler angles



$$\mathbf{C}_{PG} = \mathbf{C}_3(\omega)\mathbf{C}_1(i)\mathbf{C}_3(\Omega)$$

$$\mathbf{C}_{PG} = \begin{bmatrix} \cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{bmatrix} \begin{bmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{C}_{PG} = \begin{bmatrix} c_\Omega c_\omega - s_\Omega c_i s_\omega & s_\Omega c_\omega + c_\Omega c_i s_\omega & s_i s_\omega \\ -c_\Omega s_\omega - s_\Omega c_i c_\omega & -s_\Omega s_\omega + c_\Omega c_i c_\omega & s_i c_\omega \\ s_\Omega s_i & -c_\Omega s_i & c_i \end{bmatrix}$$

Remember!
$$\mathbf{C}_{BA} = \mathbf{C}_{AB}^{-1} = \mathbf{C}_{AB}^{\mathrm{T}}$$

$$\left.\begin{aligned} \mathbf{r}_G &= \mathbf{C}_{GP}\mathbf{r}_P \\ \mathbf{v}_G &= \mathbf{C}_{GP}\mathbf{v}_P \\ \vec{\boldsymbol{\mathcal{F}}}_G &= \mathbf{C}_{GP}\vec{\boldsymbol{\mathcal{F}}}_P \end{aligned}\right\}$$

What is $\mathbf{C}_{GP}$?

$$\mathbf{C}_{GP} = \mathbf{C}_{PG}^{\mathrm{T}}$$

$$\mathbf{C}_{GP} = \begin{bmatrix} c_\Omega c_\omega - s_\Omega c_i s_\omega & -c_\Omega s_\omega - s_\Omega c_i c_\omega & s_\Omega s_i \\ s_\Omega c_\omega + c_\Omega c_i s_\omega & -s_\Omega s_\omega + c_\Omega c_i c_\omega & -c_\Omega s_i \\ s_i s_\omega & s_i c_\omega & c_i \end{bmatrix}$$

[De Ruiter, Ch. 3; Curtis, Ch. 4]

COEs to State Vectors

$$\vec{\mathbf{r}} = \vec{\boldsymbol{\mathcal{F}}}_G^{\mathrm{T}} \mathbf{r}_G \qquad \vec{\boldsymbol{\mathcal{F}}}_G^{\mathrm{T}} = [\hat{\mathbf{i}} \quad \hat{\mathbf{j}} \quad \hat{\mathbf{k}}]$$
$$\vec{\mathbf{v}} = \vec{\boldsymbol{\mathcal{F}}}_G^{\mathrm{T}} \mathbf{v}_G$$

Now, given our orbital elements ($h, e, i, \Omega, \omega,$ and θ) we can compute the state vectors ($\mathbf{r}_G$ and $\mathbf{v}_G$) in the ECI frame:

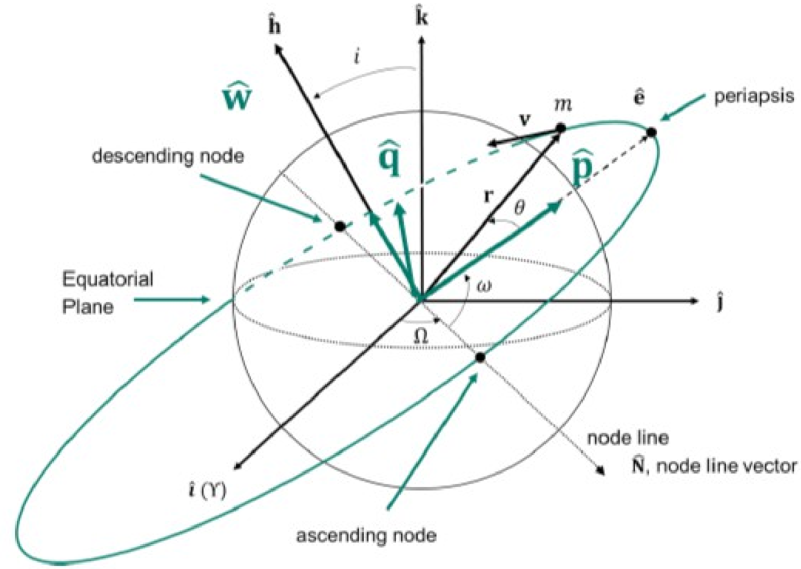
1. Find the position vector in $\mathcal{F}_P$: $\vec{\boldsymbol{\mathcal{F}}}_P^{\mathrm{T}} = [\hat{\mathbf{p}} \quad \hat{\mathbf{q}} \quad \hat{\mathbf{w}}]$

$$\vec{\mathbf{r}} = \vec{\boldsymbol{\mathcal{F}}}_P^{\mathrm{T}} \mathbf{r}_P$$
$$\mathbf{r}_P = [r\cos\theta \quad r\sin\theta \quad 0]^{\mathrm{T}}$$

$$\boxed{r = \frac{h^2/\mu}{1 + e\cos\theta}}$$

2. Find the velocity vector in $\mathcal{F}_P$:

$$\vec{\mathbf{v}} = \vec{\boldsymbol{\mathcal{F}}}_P^{\mathrm{T}} \mathbf{v}_P$$
$$\mathbf{v}_P = \left[-\frac{\mu}{h}\sin\theta \quad \frac{\mu}{h}(e + \cos\theta) \quad 0\right]^{\mathrm{T}}$$



3. Find the rotation matrix $\boldsymbol{C}_{GP}$:

$$\mathbf{C}_{GP} = \begin{bmatrix} c_\Omega c_\omega - s_\Omega c_i s_\omega & -c_\Omega s_\omega - s_\Omega c_i c_\omega & s_\Omega s_i \\ s_\Omega c_\omega + c_\Omega c_i s_\omega & -s_\Omega s_\omega + c_\Omega c_i c_\omega & -c_\Omega s_i \\ s_i s_\omega & s_i c_\omega & c_i \end{bmatrix}$$

4. Solve for $\mathbf{r}_G$ and $\mathbf{v}_G$:

$$\mathbf{r}_G = \mathbf{C}_{GP} \mathbf{r}_P$$
$$\mathbf{v}_G = \mathbf{C}_{GP} \mathbf{v}_P$$

[Curtis, Ch. 4]

COEs to State Vectors

Example

For a given earth orbit, the elements are: $h = 70{,}000\ \mathrm{km^2/s}, e = 0.74, i = 63.4°, \Omega = 40°, \omega = 270°$ and $\theta = 30°$. Find the state vectors ($\mathbf{r}_G$ and $\mathbf{v}_G$) in the ECI frame.

1. Find the position vector in $\mathcal{F}_P$:

$$\vec{\mathbf{r}} = \vec{\boldsymbol{\mathcal{F}}}_P^{\mathrm{T}} \mathbf{r}_P \qquad \boxed{r = \frac{h^2/\mu}{1 + e\cos\theta}} \qquad r = \frac{(70{,}000\ \mathrm{km^2/s})^2/(398{,}600\ \mathrm{km^3/s^2})}{1 + (0.74)\cos 30°} = 7492\ \mathrm{km}$$

$$\mathbf{r}_P = [r\cos\theta \quad r\sin\theta \quad 0]^{\mathrm{T}}$$

$$\mathbf{r}_P = [(7492\ \mathrm{km})\cos 30° \quad (7492\ \mathrm{km})\sin 30° \quad 0]^{\mathrm{T}}$$

$$\mathbf{r}_P = [6488\ \mathrm{km} \quad 3746\ \mathrm{km} \quad 0]^{\mathrm{T}}$$

2. Find the velocity vector in $\mathcal{F}_P$:

$$\vec{\mathbf{v}} = \vec{\boldsymbol{\mathcal{F}}}_P^{\mathrm{T}} \mathbf{v}_P$$

$$\mathbf{v}_P = \left[-\frac{\mu}{h}\sin\theta \quad \frac{\mu}{h}(e + \cos\theta) \quad 0\right]^{\mathrm{T}} = \left[-\frac{(398{,}600\ \mathrm{km^3/s^2})}{(70{,}000\ \mathrm{km^2/s})}\sin 30° \quad \frac{(398{,}600\ \mathrm{km^3/s^2})}{(70{,}000\ \mathrm{km^2/s})}((0.74) + \cos 30°) \quad 0\right]^{\mathrm{T}}$$

$$\mathbf{v}_P = [-2.847\ \mathrm{km/s} \quad 9.145\ \mathrm{km/s} \quad 0]^{\mathrm{T}}$$

Clarkson

[Curtis, Ch. 4]

COEs to State Vectors

Example

For a given Earth orbit, the elements are: $h = 70{,}000\ \mathrm{km}^2/s, e = 0.74, i = 63.4°, \Omega = 40°, \omega = 270°$ and $\theta = 30°$. Find the state vectors ($\mathbf{r}_G$ and $\mathbf{v}_G$) in the ECI frame.

3. Find the rotation matrix $\boldsymbol{C}_{GP}$:

$$\mathbf{C}_{GP} = \begin{bmatrix} c_\Omega c_\omega - s_\Omega c_i s_\omega & -c_\Omega s_\omega - s_\Omega c_i c_\omega & s_\Omega s_i \\ s_\Omega c_\omega + c_\Omega c_i s_\omega & -s_\Omega s_\omega + c_\Omega c_i c_\omega & -c_\Omega s_i \\ s_i s_\omega & s_i c_\omega & c_i \end{bmatrix}$$

$$\mathbf{C}_{GP} = \begin{bmatrix} c_{40°} c_{270°} - s_{40°} c_{63.4°} s_{270°} & -c_{40°} s_{270°} - s_{40°} c_{63.4°} c_{270°} & s_{40°} s_{63.4°} \\ s_{40°} c_{270°} + c_{40°} c_{63.4°} s_{270°} & -s_{40°} s_{270°} + c_{40°} c_{63.4°} c_{270°} & -c_{40°} s_{63.4°} \\ s_{63.4°} s_{270°} & s_{63.4°} c_{270°} & c_{63.4°} \end{bmatrix}$$

$$\mathbf{C}_{GP} = \begin{bmatrix} 0.2878 & 0.766 & 0.5748 \\ -0.343 & 0.6428 & -0.685 \\ -0.8942 & 0 & 0.4477 \end{bmatrix}$$

4. Solve for $\mathbf{r}_G$ and $\mathbf{v}_G$:

$$\mathbf{r}_G = \mathbf{C}_{GP}\mathbf{r}_P$$

$$\mathbf{r}_G = \begin{bmatrix} 0.2878 & 0.766 & 0.5748 \\ -0.343 & 0.6428 & -0.685 \\ -0.8942 & 0 & 0.4477 \end{bmatrix} \begin{bmatrix} 6488\ \mathrm{km} \\ 3746\ \mathrm{km} \\ 0 \end{bmatrix}$$

$$\boxed{\mathbf{r}_G = [4737\ \mathrm{km} \quad 182\ \mathrm{km} \quad -5802\ \mathrm{km}]^{\mathrm{T}}}$$

$$\mathbf{v}_G = \mathbf{C}_{GP}\mathbf{v}_P$$

$$\mathbf{v}_G = \begin{bmatrix} 0.2878 & 0.766 & 0.5748 \\ -0.343 & 0.6428 & -0.685 \\ -0.8942 & 0 & 0.4477 \end{bmatrix} \begin{bmatrix} -2.847\ \mathrm{km/s} \\ 9.145\ \mathrm{km/s} \\ 0 \end{bmatrix}$$

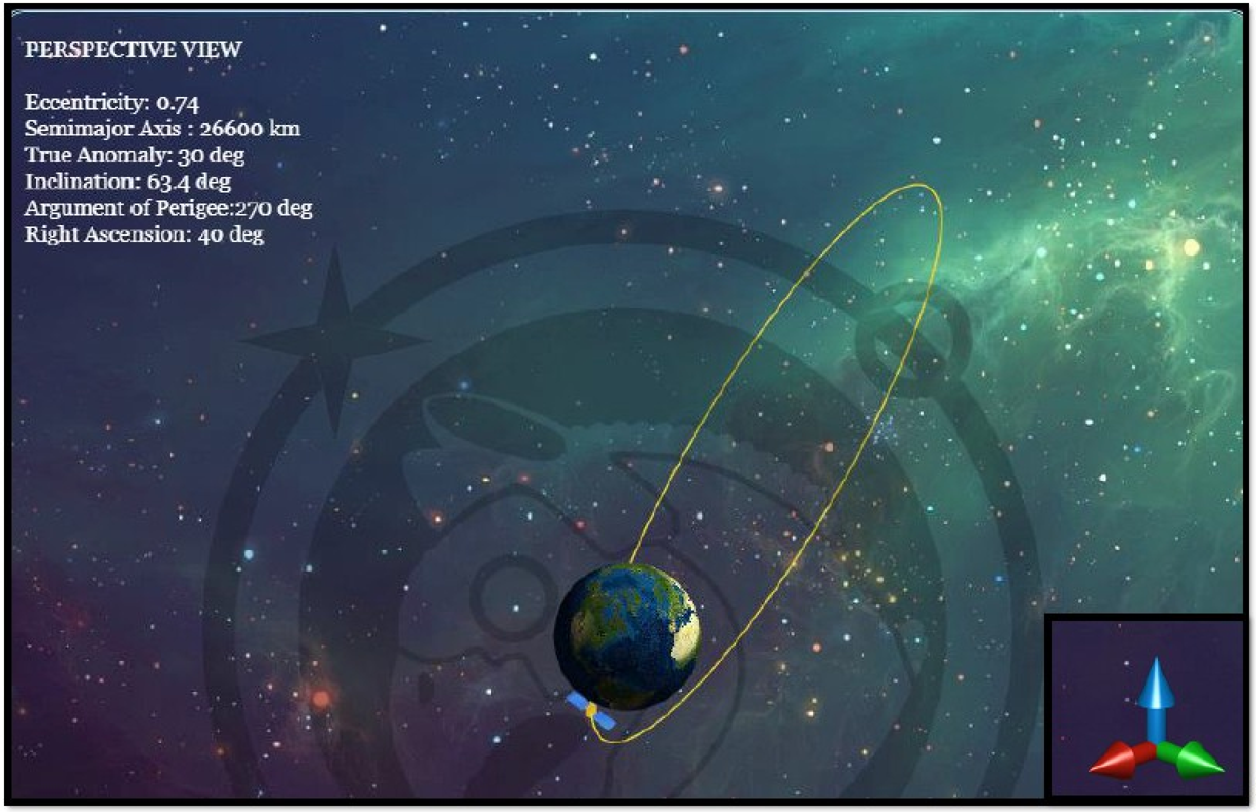
$$\boxed{\mathbf{v}_G = [6.186\ \mathrm{km/s} \quad 6.855\ \mathrm{km/s} \quad 2.546\ \mathrm{km/s}]^{\mathrm{T}}}$$

[Curtis, Ch. 4]

COEs to State Vectors

Example

For a given Earth orbit, the elements are: $h = 70{,}000\ \mathrm{km}^2/s, e = 0.74, i = 63.4°, \Omega = 40°, \omega = 270°$ and $\theta = 30°$. Find the state vectors ($\mathbf{r}_G$ and $\mathbf{v}_G$) in the ECI frame.



What does this orbit look like?

$r = 7492\ \mathrm{km}$

$r_\oplus = 6371\ \mathrm{km}$

Altitude: $1121\ \mathrm{km}$

Molniya Orbit

[HOMA, Online Space Orbit Simulator, http://en.homasim.com/orbitsimulation.php, 2020]

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