Combined Final Projects - Spring 2023

Dr. Aris Spanos* & Jonathan Gendron*

*Department of Economics, Virginia Polytechnic Institute and State University

May 1, 2023

1 Introduction

1.1 Topic

This final project is going to study LIBOR rates: $r1$ will be covered as a time series model in Section 2, and then $r1, r2, r3$ & $r4$ will be combined into a panel data model in Section 3.

1.2 Objective

The objective of this final project is to show how to correctly model LIBOR rates, both as a time series and a panel data model.

1.3 Brief Literature Review

LIBOR rates represent daily swap rates of maturities in years for the exchange of U.S. Treasury bond rates for London Inter Bank Offered Rate (i.e. LIBOR). One approach in the literature for LIBOR rates is to difference the data to ensure it's stationary, and then run a Simple Normal model (Lai & Xing, 2008). This model presumes the data is Normal, Independent and Identically distributed. There are many issues with this, especially since it's only covariates are a constant and a static mean over time which is quite questionable (see the table in Section 2.2 for more details on the assumptions).

An approach that is far more common in the literature for LIBOR rates are to use AR(p) or VAR(p) models, these allow us to model dependence over time (see the table in Section 2.4 for more details on the assumptions). The VAR(p) and ARMA(1,3) models have been used to model and forecast the undifferenced LIBOR-Federal Funds rate spread and found the forecasts followed similar patterns (but note that there is a margin of error in the forecasts graph) (Dbouk, Jamali, & Kryzanowski, 2015). LIBOR rate were also forecasted with an AR(1) model, but found no variation in mean for various rates, so factor dynamics were used instead (Bali, Heidari, & Wu, 2009). The issues faced in this glimpse of the literature are not surprising due to the issues with using AR(p) or VAR(p) models, they presume

Normality and can't account for departures in linearity, homoskedasticity or homogeneity (Poudyal & Spanos, 2022). We'll see this is an issue because like many time series data, the data instead exhibit a Student's t distribution. This is from the same elliptically symmetric family, but is more leptokurtic (Spanos, 2019). Note one study that detrended LIBOR, ran a VAR(p) (which accounts for dependence) and then found it exhibited a normal distribution when testing for normality (Erdogan & Dayan, n.d.). This doesn't mean it's necessarily normal though, this normality could have been imposed by the VAR(p) model since it presumes the data follows a normal distribution (Erdogan & Dayan, n.d.). The only way to definitively confirm if the data of that study is truly normal would be to obtain it and M-S test it.

2 Methodology - Time Series

In order to make a panel data model, misspecification (M-S) testing must occur for each of the cross sections to highlight any model violations. So after discussing data and the process of M-S testing, this section will then review how to model this as a time series model. Section 3 will then extend this into a panel data model.

2.1 Data

The data are LIBOR rates r1, r2, r3 & r4, these represent daily swap rates of maturities for 1, 2, 3 & 4 years for the exchange of U.S. Treasury bond rates for London Inter Bank Offered Rate (i.e. LIBOR) (Lai & Xing, 2008). The time frame for all variables is ($t = 1257$), and we are modeling 4 cross sections ($i = 4$, for each rate).

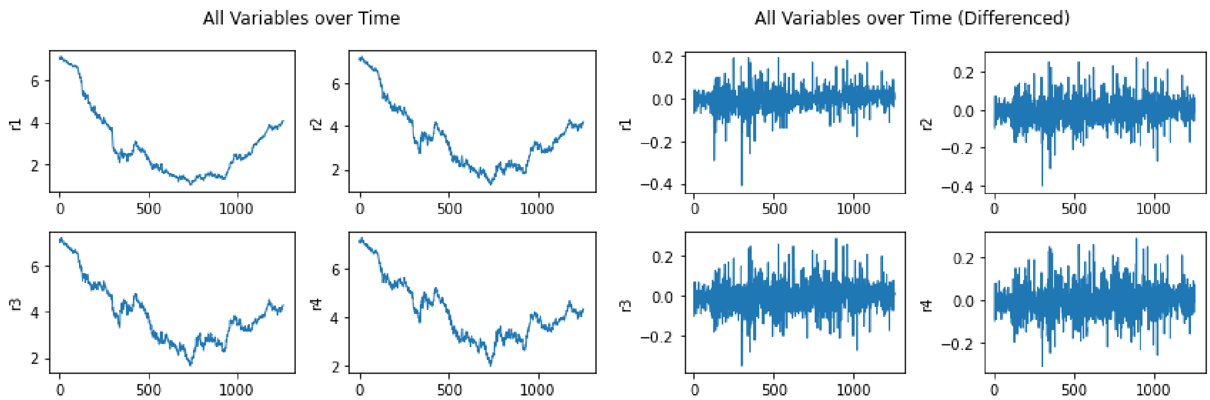

Figure 1

2.2 Initial Statistical Model

The core of the model will be the following Simple Normal Model estimation over time (t):

$$\Delta r1_t = \alpha_0 + \alpha_1 \mu + \varepsilon_t, t \in \mathbb{T}$$

Simple Normal Model
Statistical GM: $X_t = \mu + u_t, t \in \mathbb{N} = \{1, 2, ..., n, ...\}$
[1] Normality: $X_t \sim N(.,.), x_t \in \mathbb{R}, t \in \mathbb{N}$
[2] Constant mean: $E(X_t) = \mu, t \in \mathbb{N}$
[3] Constant variance: $Var(X_t) = \sigma^2, t \in \mathbb{N}$
[4] Independence: $\{X_t, t \in \mathbb{N}\}$ is an independent process

As we'll see in the next section though, M-S testing reveals that not all of the assumptions of this model hold. For example, heterogeneity is not even present! Accordingly, we'll respecify this into a more accurate model, this process starts with the first round of M-S testing (as seen in the next subsection).

2.3 M-S Testing - 1st Round - Revising the Initial Statistical Model

M-S testing runs auxiliary regressions on the first and second moments of the residual to see if there are any violations of the following core model assumptions:

- (1) Distribution
- (2) Linearity
- (3) Homoskedasticity
- (4) Independence
- (5) Homogeneity (i.e. t-invariance)

If any terms in the auxiliary regressions are found to be significant, this means the model exhibits that violation and needs to be adjusted. For instance, if a lag term was found significant, these would mean there is a form of dependence that must be accounted for in the model. Note that the first moment's auxiliary regression includes the original model specification and additional terms to catch violations while the second moment's auxiliary regression only includes additional terms to catch violations (excluding the original model specification). This is because the conditional variance isn't estimated with parameters, while the conditional mean is (Spanos, 2019).

To M-S test this data model, our first round of M-S testing tests for Independence [4] and t-invariance [5] as follows:

- (4) First order dependence: This is done by adding lag terms of both the independent and the dependent variables in the auxiliary regression of the first moment. We tested up to 4 lags.
- (4) Second order dependence: This is done by taking the squares of the above lag terms, but in the auxiliary regression of the second moment.
- (5) Mean heterogeneity: This is done by adding trend polynomials ($t_0^i, i = 1, 2, 3, 4$) into the auxiliary regression of the first moment. Note that $t_0 = \frac{2t-n-1}{n-1}$.
- (5) Variance heterogeneity: This is done by adding the above trend polynomials ($t_0^i, i = 1, 2, 3, 4$), but into the auxiliary regression of the second moment.

If any of these violations were found, these terms are added into the statistical model which is run again to regenerate the residuals for the auxiliary regressions to be used in the next round of M-S testing. M-S testing is an iterative process that only stops when there are no more departures found, which signals that the model is statistically adequate because all of the departures have been accounted for (Spanos, 2019).

Remember that any terms added to the respecified model are dropped from the second moment's auxiliary regression, but since this is the 1st round of M-S testing before respecification, the below auxiliary regressions don't yet reflect this. These adjusted auxiliary regressions will then be used for the second round of M-S testing. As for this first round of M-S testing, issues with trend and dependence were found in the 1st moment's auxiliary regression, so these will be added into the 2nd round of M-S testing. Then this updated statistical model will be ran again to regenerate the residuals, and if trend and dependence are not longer an issue, we'll look at linearity and homoskedasticity. Note that the $r1_{t-1}$ term was included because the dependent variable being differenced can impose unit roots which can drastically skew the results; it's significance highlights that it needs to be added to the respecified model. See Section 3.2 for the respecified auxiliary regressions that confirms the issues described above. The following auxiliary regressions were used in this 1st round of M-S testing, $\diamond$ denotes significant:

$$\hat{u}_t = \underset{(.008)}{.026}^{\diamond} - \underset{(.000002)}{.000006\mu}^{\diamond} - \underset{(.005)}{.017r1_{t-1}}^{\diamond} - \underset{(.009)}{.0075t_0} + \underset{(.026)}{.066t_0^2}^{\diamond} - \underset{(.009)}{.014t_0^3} + \underset{(.018)}{.014t_0^4} + \underset{(.028)}{.080\Delta r1_{t-1}}^{\diamond} + \underset{(.028)}{.051\Delta r1_{t-2}}$$

$$-\underset{(.028)}{.047\Delta r1_{t-3}} + \underset{(.028)}{.0054\Delta r1_{t-4}} + v_{1t}$$

$$\hat{u}_t^2 = \underset{(.000)}{.002}^{\diamond} - \underset{(.000)}{.0001r1_{t-1}^2} - \underset{(.001)}{.004t_0}^{\diamond} + \underset{(.003)}{.009t_0^2}^{\diamond} + \underset{(.002)}{.002t_0^3} - \underset{(.003)}{.006t_0^4} + \underset{(.028)}{.015\Delta r1_{t-1}^2} + \underset{(.028)}{.063\Delta r1_{t-2}^2}^{\diamond}$$

$$+\underset{(.028)}{.062\Delta r1_{t-3}^2}^{\diamond} - \underset{(.028)}{.027\Delta r1_{t-4}^2} + v_{2t}$$

2.4 M-S Testing - 2nd Round - Revising the Initial Statistical Model

The initial model updated to the following which accounts for trend and dependence in the 1st moment with it's added covariates:

$$\Delta r1_t = \gamma_0 + \gamma_1 t_0 + \gamma_2 t_0^2 + \gamma_3 \Delta r1_{t-1} + \gamma_4 \Delta r1_{t-2} + \gamma_5 \Delta r1_{t-3} + u_t, t \in \mathbb{T}$$

Note that μ is no longer included since there are other confounding factors that were found. More precisely, the presence of a trend term would interfere with the measurement of μ (because an average measurement assumes Independence and Identically Distributed (IID), which is clearly violated if trend or dependence are observed). Also note that this model now looks very similar to an AR model, but enhanced with the trend terms:

AR(p=3) Model
Statistical GM: $Y_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i Y_{t-i} + u_t, t \in \mathbb{N}$
[1] Normality: $(Y_1, Y_2, ..., Y_n) \sim N(.,.), t \in \mathbb{N}$
[2] Linearity: $E(Y_t \| \sigma(Y_{2t})) = \alpha_0 + \sum_{i=1}^{p} \alpha_i Y_{t-i}, t \in \mathbb{N}$
[3] Constant variance: $Var(Y_t \| \sigma(Y_{2t})) = \sigma_0^2, Y_{2t} = (Y_{t-1}, ..., Y_{t-p}), t \in \mathbb{N}$
[4] Markov(p): $\{Y_t, t \in \mathbb{N}\}$ is a Markov(p) process
[5] $\theta = (\alpha_0, \alpha_1, ..., \alpha_p, \sigma_0^2)$ constant over t

When testing the resulting residual it's plain to see that the distribution is still leptokurtic, and heteroskedasticity is present. This is where this round of M-S testing would test to confirm departures in linearity and homoskedasticity, but since these are characteristic of the Student's t distribution we can instead run our model as Student's t. If this doesn't work though, then we'd go back to here and do M-S testing of a different form. For reference, Figure 2 shows what the residuals look like after respecification:

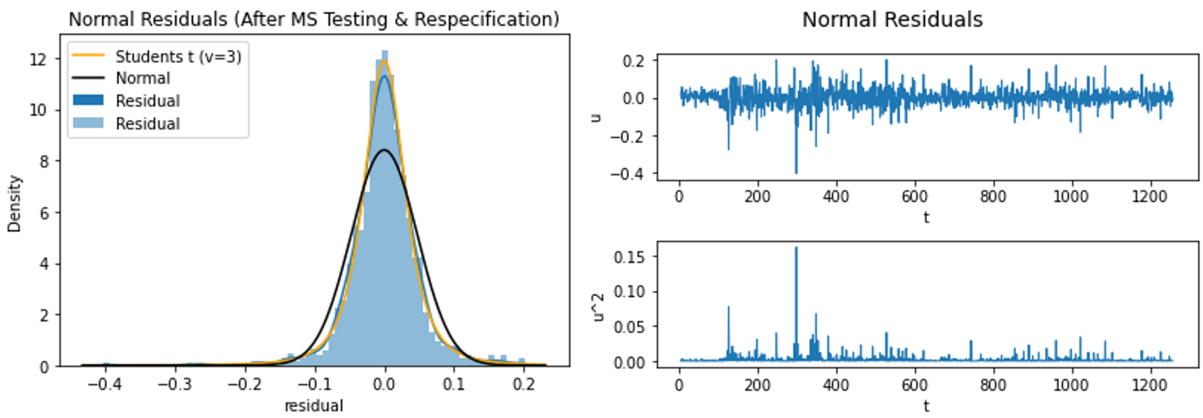

Figure 2: The model's current degrees of freedom (d.f.) are $v = 2$, but note that $v = 3$ is a nearly perfect fit to the current residual

2.5 M-S Testing - 3rd Round - Testing Student's t AR model

Since our residual exhibits a leptokurtic distribution and heteroskedasticity, we are going to now run our model as a Student's t AR model (Poudyal & Spanos, 2022).

Student's t AR(p=3) Model
Statistical GM: $Y_t = \beta_0 + \sum_{i=1}^{p} \beta_i Y_{t-i} + \varepsilon_t, t \in \mathbb{N}$
[1] Student's t: $D(Y_t \mid Y_{t-1}^0; \theta)$ is Student's t with $v + k$ d.f.
[2] Linearity: $E(Y_t \mid \sigma(Y_{t-1}^0)) = \beta_0 + \sum_{i=1}^{p} \beta_i Y_{t-i}$
[3] Heteroskedasticity: $Var(Y_t \mid \sigma(Y_{t-1}^0)) = \omega_t^2(Y_{t-1}^0)$ $\omega_t^2 = \frac{v}{v+t-3}\sigma_0^2(1 + \sum_{j=1}^{t-1}\sum_{i=1}^{t-1}\{\delta_{ij}[Y_{t-i} - \mu][Y_{t-j} - \mu]\})$ $\delta_{ij} = 0$ for $\lvert i-j \rvert > p, \delta_{ij} = \delta_{kl}$ for $\lvert i-j \rvert = \lvert k-l \rvert$
[4] Weak dependence: $\{Y_t, t \in \mathbb{N}\}$ is a Markov(p) process
[5] t-invariance: $\theta = (\beta_0, \beta_1, ..., \beta_p, \sigma_0^2, \mu, \delta_1, ..., \delta_p)$

To M-S test this model, we are going to use the following auxiliary regressions ($\diamond$ denotes significant):

$$\hat{u}_t = \underset{(.101)}{.051} - \underset{(7.689)}{3.152}\Delta\hat{r1}_t - \underset{(23.348)}{16.688}\Delta\hat{r1}_t^2 - \underset{(.089)}{.041} r1_{t-1} - \underset{(.064)}{.024} t_0 + \underset{(.446)}{.185} t_0^2 - \underset{(.010)}{.011} t_0^3 + \underset{(.018)}{.017} t_0^4 + \underset{(.243)}{.128}\hat{u}_{t-1} - \underset{(.145)}{.005}\hat{u}_{t-2}$$

$$-\underset{(.498)}{.199}\hat{u}_{t-3} - \underset{(.125)}{.048}\hat{u}_{t-4} + v_{1t}$$

$$\hat{u}_t^2 = \underset{(.000)}{.001}^{\diamond} + \underset{(.00008)}{.0001}\hat{\sigma}_t^2 - \underset{(3.378)}{2.206}\Delta\hat{r1}_t^2 - \underset{(.000)}{.00006} r1_{t-1}^2 - \underset{(.001)}{.004} t_0^{\diamond} + \underset{(.003)}{.006} t_0^{2\diamond} + \underset{(.002)}{.003} t_0^3 - \underset{(.003)}{.005} t_0^4$$

$$-\underset{(.0001)}{.00005}\hat{\sigma}_{t-1}^2 + \underset{(.0001)}{.0002}\hat{\sigma}_{t-2}^{2\diamond} - \underset{(.0001)}{.0001}\hat{\sigma}_{t-3}^2 + \underset{(.00008)}{.00007}\hat{\sigma}_{t-4}^2 + v_{2t}$$

Heterogeneity is still captured with the trend terms used with the Normal, but Student's t captures the following violations differently:

- (2 - Linearity) via the first moment's $\Delta\hat{r1}_t^2$
- (3 - Homoskedasticity) via the second moment's $\hat{\sigma}_t^2$ & $\Delta\hat{r1}_t^2$
- (3 - Dynamic Heteroskedasticity) via the second moment's $\hat{\sigma}_{t-i}^2$ s.t. $i = 1, 2, ..., 4$
- (4 - Markhov dependence) via the first moment's $\hat{u}_{t-i}$ s.t. $i = 1, 2, ..., 4$

In the 1st moment's auxiliary regression the following terms were dropped because they were found insignificant ($\hat{u}_{t-i}$ s.t. $i = 2, 3, 4$), and in the 2nd moment's auxiliary regression ($\hat{\sigma}_{t-i}^2$ s.t. $i = 3, 4$) was dropped for the same reason. After these were dropped, the following violations were found from the above M-S testing:

- (3 - Homoskedasticity) via $\hat{\sigma}_t^2$
- (3 - Dynamic Heteroskedasticity) via the second moment's $\hat{\sigma}_{t-2}^2$
- (5) Variance Heterogeneity (via t_0^4)

2.6 Results - Final Statistical Model After Respecification

Our final model will be the following Student's t AR model d.f. $v = 3$:

$$\Delta r1_t = \underset{(.003)}{.014}^{\diamond} - \underset{(.003)}{.009t_0}^{\diamond} + \underset{(.010)}{.063t_0^{2}}^{\diamond} - \underset{(.002)}{.012r1_{t-1}}^{\diamond} + \underset{(.019)}{.032\Delta r1_{t-1}} - \underset{(.022)}{.020\Delta r1_{t-2}} - \underset{(.031)}{.064\Delta r1_{t-3}}^{\diamond} + u_t, t \in \mathbb{T}$$

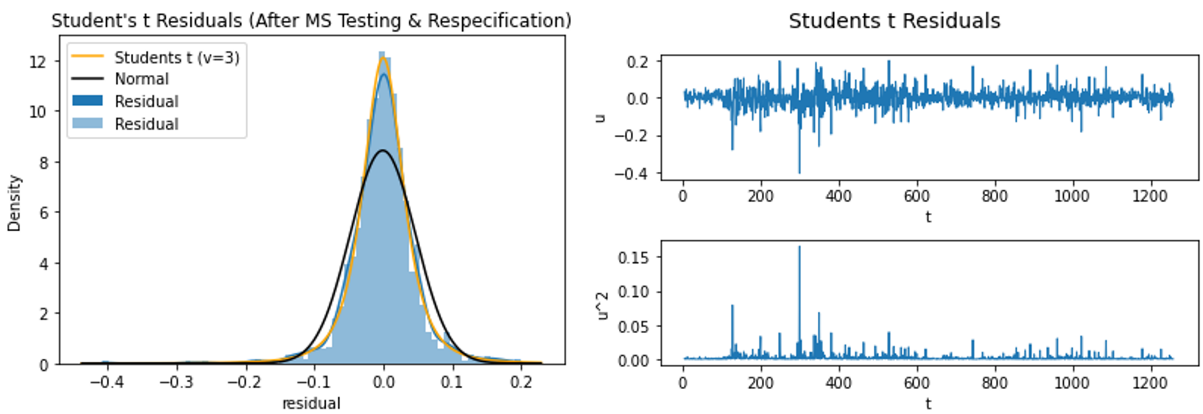

Figure 3: The residuals after this respecification

2.7 Results Compared to the Literature

The results found in Section 2.6 are quite different compared to the ARs of the literature, primarily because they include three additional parameters for trend (via t_0 & t_0^2) and the unit root (via $r1_{t-1}$) (Bali et al., 2009). Simple Normal models miss another three parameters (6 total) since the lag terms aren't included ($\Delta r1_{t-k}$ s.t. $k = 1, 2, 3$), so their performance is even worse since they include only the constant and the parameter for a static μ (Lai & Xing, 2008). Not surprisingly, models with the most appropriate independent variables will generate estimates that provide the most accurate inferences, while models that don't include enough independent variables will generate the least accurate inferences. For example, a one unit increase in t_0^2 won't impact either the Simple Normal, AR(p) or VAR(p) models since it isn't a typical parameter, but it is a parameter in the StAR(3) model and will cause $\Delta r1_t$ to increase respectively by 0.063. It is plain to see the StAR(3) will provide the most accurate inferences, while the other models will provide confounding inferences since they don't capture the full picture.

3 Methodology - Panel Data

A panel data model will be made in this section by first M-S testing each cross section (similar to how it was done above in Section 2). Then the respecified time series models will be combined into one panel data model.

3.1 M-S Testing the Cross Sections as Time Series Models - Initial Models

Just like before, each cross section will initially test the Simple Normal Model $\Delta rj_t = \alpha_0 + \alpha_1\mu + \varepsilon_t, t \in \mathbb{T}$ s.t. $j = 1, 2, 3, 4$

3.2 M-S Testing the Cross Sections - Revising the Initial Statistical Models

This subsection will simply show which violations were found for each rate (when analyzed as time series models). These were found with the same initial auxiliary regressions used in the 1st round of M-S testing in Section 2 ($j = 1, 2, 3, 4$). Remember that insignificant terms for trends or lags of differenced variables were dropped one by one to ensure these excess coefficients weren't masking other violations (since parameters and their significance can change when the parameterization changes):

$$\hat{u}j_t = \beta_0 + \alpha_0\mu + \beta_1 t_0 + \beta_2 t_0^2 + \beta_3 t_0^3 + \beta_4 t_0^4 + \beta_5\Delta rj_{t-1} + \beta_6\Delta rj_{t-2} + \beta_7\Delta rj_{t-3} + \beta_8\Delta rj_{t-4} + \beta_9 rj_{t-1} + v_{1t}$$

$$\hat{u}j_t^2 = \delta_0 + \delta_1 t_0 + \delta_2 t_0^2 + \delta_3 t_0^3 + \delta_4 t_0^4 + \delta_5\Delta rj_{t-1}^2 + \delta_6\Delta rj_{t-2}^2 + \delta_7\Delta rj_{t-3}^2 + \delta_8\Delta rj_{t-4}^2 + \delta_9 rj_{t-1}^2 + v_{2t}$$

1st Round: All rates (r1, r2, r3, r4) found violations in:

- (1) Non-Normality (should instead be Student's t distribution, see histogram of residual below)
- (2) Non-linearity (could be fixed by Student's t, see histogram of residual below)
- (3) Heteroskedasticity (could be fixed by Student's t, see histogram of residual below)
- (4) Dependence (first order, via Δrj_{t-3})
- (4) Dependence (second order, via Δrj^2_{t-3})
- (5) Mean heterogeneity (via t_0^2)
- Unit roots (via rj_{t-1})

For r1 & r2, the following violations were additionally found:

- (4) Dependence (second order, via Δrj^2_{t-2})
- (5) Heterogeneity
  - Mean (via t_0 & t_0^2)
  - Variance (via t_0^4)

For r1, the following violations were additionally found:

- (5) Dependence (second order, via $\Delta r1^2_{t-2}$)

For r3 & r4, the following violations were additionally found:

- (4) Dependence (second order, via rj^2_{t-1}). This highlights a potential issue of a unit root process in the variance.
- (5) Variance heterogeneity (via t_0)

2nd Round: Before going to Student's t, we can respecify dependence and heterogeneity as follows for each rate:

$$\Delta r2_t = \gamma_0 + \gamma_1 t_0 + \gamma_2 t_0^2 + \gamma_3 \Delta r2_{t-3} + \gamma_4 r2_{t-1} + u_t, t \in \mathbb{T}$$

$$\Delta r3_t = \gamma_0 + \gamma_1 t_0^2 + \gamma_2 \Delta r3_{t-3} + \gamma_3 r3_{t-1} + u_t, t \in \mathbb{T}$$

$$\Delta r4_t = \gamma_0 + \gamma_1 t_0^2 + \gamma_2 \Delta r4_{t-3} + \gamma_3 r4_{t-1} + u_t, t \in \mathbb{T}$$

When we run these in M-S testing again (without μ, see below), we'll see first order dependence and heterogeneity are accounted for now, but there are still some issues with second order dependence and heterogeneity. In addition, when looking at the histograms of the residuals, it is apparent that the need for using Student's t is still present.

r1 was as follows:

- $\hat{u}_t = \underset{(.012)}{.0009} - \underset{(.005)}{.0004 r1_{t-1}} - \underset{(.009)}{.0008 t_0} + \underset{(.021)}{.0015 t_0^2} + \underset{(.028)}{.008 \Delta r1_{t-1}} + \underset{(.028)}{.014 \Delta r1_{t-2}} + \underset{(.028)}{.0002 \Delta r1_{t-3}} + v_{2,1t}$
- $\hat{u}_t^2 = \underset{(.000)}{.002}^{\diamond} + \underset{(.00008)}{.0001 r1_{t-1}^2} + \underset{(.002)}{.002 t_0^3} - \underset{(.003)}{.006 t_0^4}^{\diamond} + \underset{(.028)}{.036 \Delta r1_{t-1}^2} + \underset{(.028)}{.086 \Delta r1_{t-2}^2}^{\diamond} + \underset{(.028)}{.087 \Delta r1_{t-3}^2}^{\diamond} + v_{2,2t}$

r2 was as follows:

- $\hat{u}_t = \underset{(.012)}{.0009} - \underset{(.005)}{.0004 r2_{t-1}} - \underset{(.009)}{.0008 t_0} + \underset{(.021)}{.0015 t_0^2} + \underset{(.028)}{.008 \Delta r2_{t-1}} + \underset{(.028)}{.014 \Delta r2_{t-2}} + \underset{(.028)}{.0002 \Delta r2_{t-3}} + v_{2,1t}$
- $\hat{u}_t^2 = \underset{(.001)}{.004}^{\diamond} + \underset{(.00008)}{.00004 r2_{t-1}^2} + \underset{(.002)}{.001 t_0^3} - \underset{(.003)}{.006 t_0^4}^{\diamond} + \underset{(.028)}{.033 \Delta r2_{t-1}^2} + \underset{(.028)}{.005 \Delta r2_{t-2}^2} + \underset{(.028)}{.072 \Delta r2_{t-3}^2}^{\diamond} + v_{2,2t}$

r3 was as follows:

- $\hat{u}_t = \underset{(.015)}{.024} - \underset{(.005)}{.008 r3_{t-1}} - \underset{(.008)}{.015 t_0} + \underset{(.018)}{.026 t_0^2} + \underset{(.028)}{.018 \Delta r3_{t-1}} + \underset{(.028)}{.007 \Delta r3_{t-2}} + \underset{(.028)}{.0006 \Delta r3_{t-3}} + v_{2,1t}$
- $\hat{u}_t^2 = \underset{(.001)}{.007}^{\diamond} - \underset{(.00003)}{.0002 r3_{t-1}^2}^{\diamond} - \underset{(.001)}{.003 t_0}^{\diamond} + \underset{(.028)}{.034 \Delta r3_{t-1}^2} - \underset{(.028)}{.008 \Delta r3_{t-2}^2} + \underset{(.028)}{.068 \Delta r3_{t-3}^2}^{\diamond} + v_{2,2t}$

r4 was as follows:

- $\hat{u}_t = \underset{(.017)}{.028} - \underset{(.005)}{.008 r4_{t-1}} - \underset{(.008)}{.015 t_0} + \underset{(.015)}{.022 t_0^2} + \underset{(.028)}{.021 \Delta r4_{t-1}} + \underset{(.028)}{.009 \Delta r4_{t-2}} + \underset{(.028)}{.005 \Delta r4_{t-3}} + v_{2,1t}$
- $\hat{u}_t^2 = \underset{(.001)}{.008}^{\diamond} - \underset{(.00003)}{.0002 r4_{t-1}^2}^{\diamond} + \underset{(.001)}{.003 t_0}^{\diamond} + \underset{(.028)}{.043 \Delta r4_{t-1}^2} - \underset{(.028)}{.012 \Delta r4_{t-2}^2} + \underset{(.028)}{.082 \Delta r4_{t-3}^2}^{\diamond} + v_{2,2t}$

All rates found violations in:

- (4) Second order dependence via ($\Delta r j_{t-3}^2$)
  - Note that r1 also saw this via $r1_{t-2}^2$

Additional violations were found for r1 & r2:

- (5) Variance heterogeneity via (t_0^4)

Additional violations were found for r3 & r4:

- (5) Variance heterogeneity via (t_0)
- Unit root via (rj^2_{t-1})

It is apparent majority of the dependence and heterogeneity is accounted for, but we can plainly see the resulting residuals are non-normal and better match a Student's t distribution. The resulting residuals for r2, r3 & r4 are below, see Section 2.4 for r1:

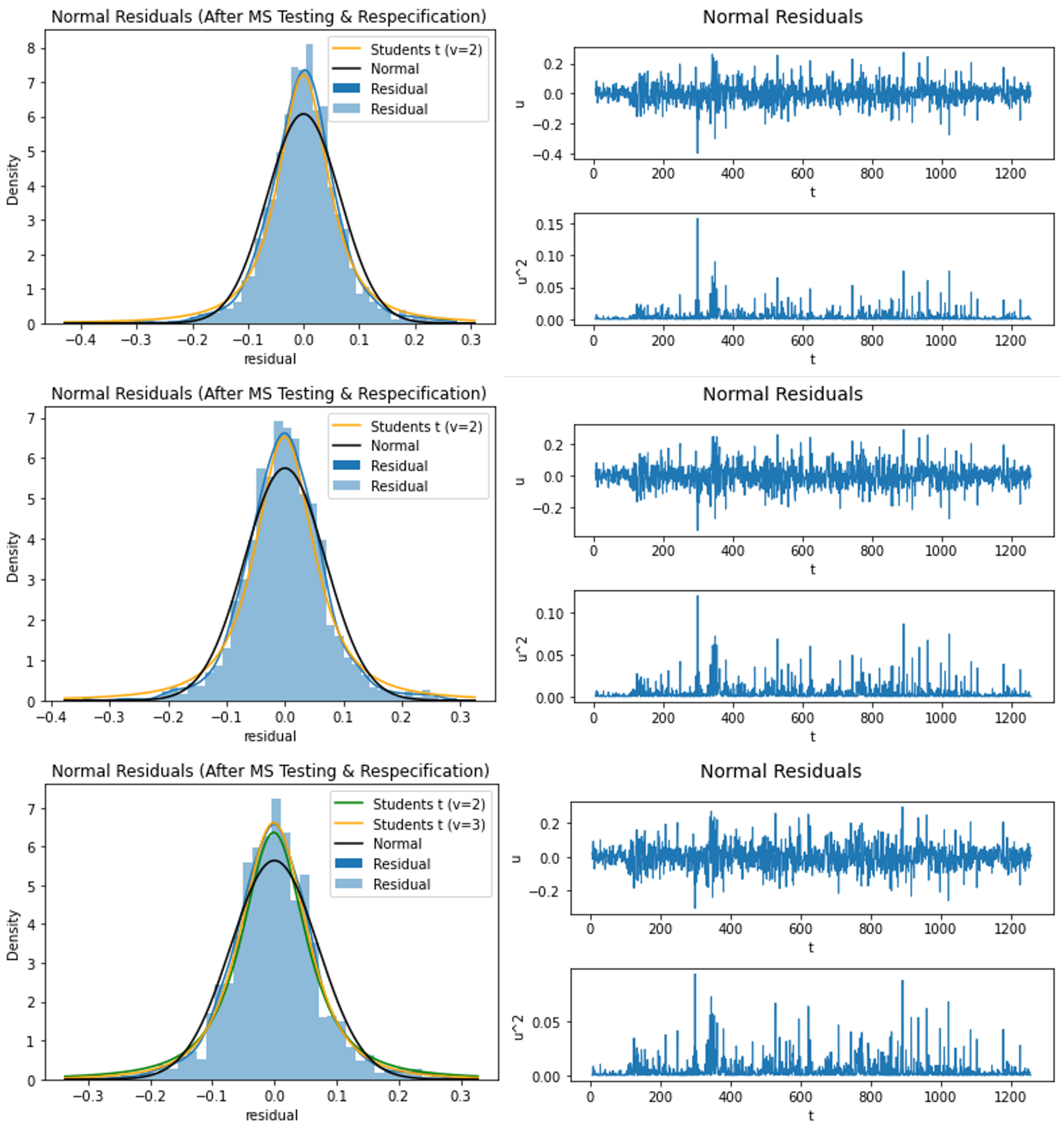

Figure 4

3.3 StAR Estimations

Before setting up the StVAR(p) model, we need to ensure that Student's t is the best fit for each cross section. We can see this is the case with the residuals of each cross section respectively for r2, r3 & r4 (see Section 2.6 for r1). Note that this used the same covariates of the respecified models of Section 3.3 above:

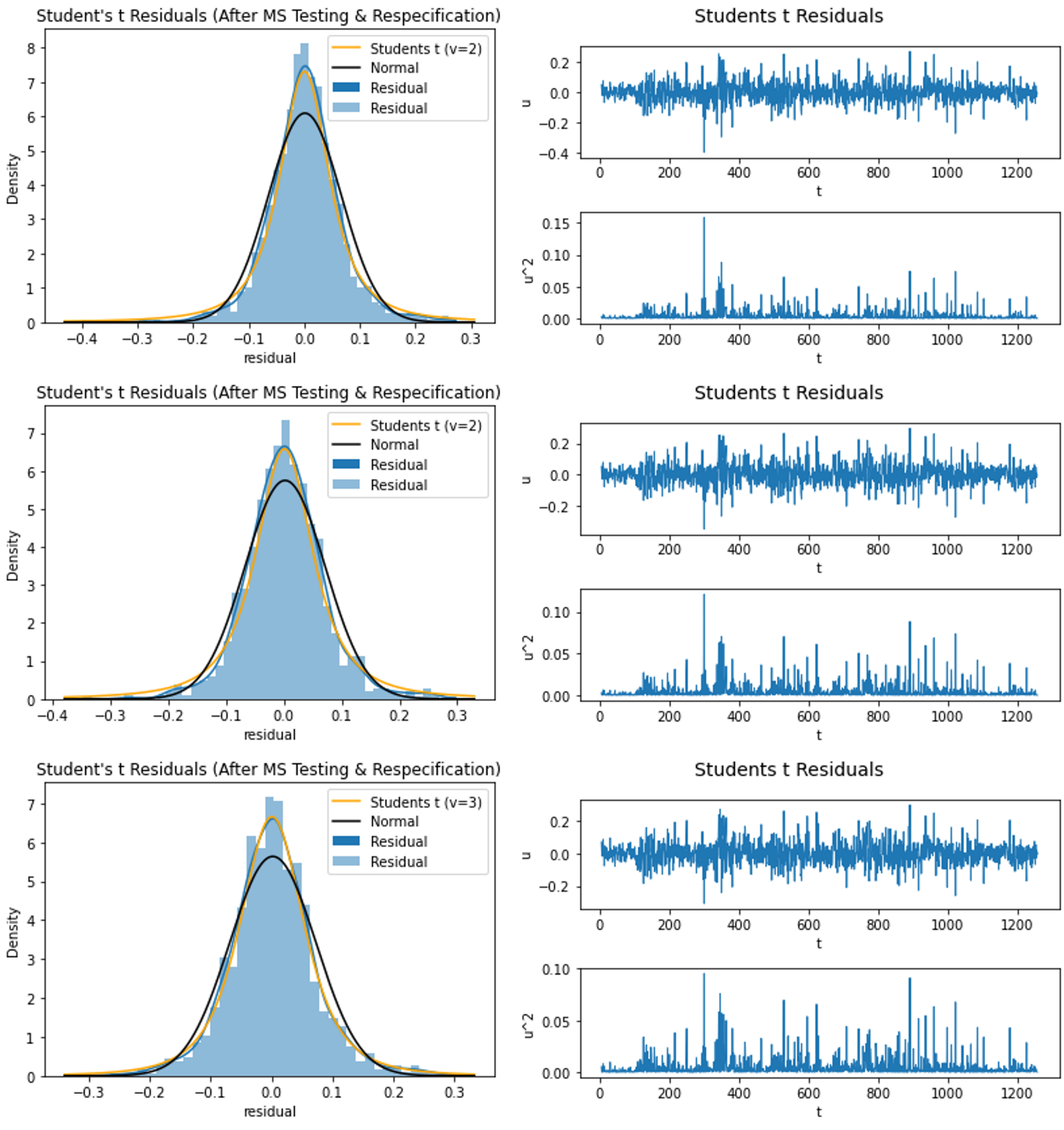

Figure 5

3.4 StVAR Estimation - Combining Time Series into Panel Data Model

This is where things get interesting, all four time series models (for each rate) are now going to be combined into one panel data model. Earlier we found the furthest lag was 3, and the largest trend polynomial was t_0^2, so we'll give our StVAR(p) 3 lags and trend polynomials t_0 & t_0^2. As for degrees of freedom, v was 2 for some variables but 3 for others, so we first tried $v = 3$. There were issues in computations of standard errors though due to missing values, so both v of 2 and 4 were tested in it's stead. We'll see later that $v = 4$ is a better fit overall compared to 2 (see Figure 6 below), so the StVAR(3) used $v = 4$. The next section will show M-S testing this model and then the final result.

Student's t VAR(p=3) Model
Statistical GM: $Z_t = a_0(t) + \sum_{i=1}^{p} A_i^T Z_{t-i} + u_t, t \in \mathbb{N}$
[1] Student's t: $(Z_t \vert Z_{t-1}^0), Z_{t-1}^0 = (Z_{t-1}, ..., Z_1)$
[2] Linearity: $E(Z_t \vert \sigma(Z_{t-1}^0)) = a_0(t) + \sum_{i=1}^{p} A_i^T Z_{t-i}$
[3] Heteroskedasticity: $Var(Z_t \vert \sigma(Z_{t-1}^0)) = q(Z_{t-1}^0)$ depends on Z_{t-1}^0 $q(Z_{t-1}^0) = (\frac{v}{v+kp-2})V[1 + \frac{1}{v}\sum_{i=1}^{p}(Z_{t-i} - \mu(t))Q_i^{-1}(Z_{t-i} - \mu(t))]$
[4] Markov(p): $\{Z_t, t \in \mathbb{N}\}$ is a Markov(p) process
[5] t-invariance: $\theta = (a_0, \mu, A_1, ..., A_p, V, Q_1, ..., Q_p)$ are constant over t

3.5 Results - Final Statistical Model

Our StVAR(3) with trends will look as follows:

$$\Delta r_t = \alpha_0 + \sum_{k=1}^{3} \alpha_{1k} \Delta r_{t-k} + \alpha_2 t_0 + \alpha_3 t_0^2 + \sum_{k=1}^{4} \alpha_{4k} r_{t-1} + u_t$$

$$\Delta r_t = \begin{bmatrix} \Delta r1_t \\ \Delta r2_t \\ \Delta r3_t \\ \Delta r4_t \end{bmatrix}, \alpha_0 = \begin{bmatrix} \alpha_{10} \\ \alpha_{20} \\ \alpha_{30} \\ \alpha_{40} \end{bmatrix}, \alpha_{1k} = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} & \alpha_{41} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} & \alpha_{42} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} & \alpha_{43} \end{bmatrix}, \Delta r_{t-k} = \begin{bmatrix} \Delta r1_{t-1} & \Delta r1_{t-2} & \Delta r1_{t-3} \\ \Delta r2_{t-1} & \Delta r2_{t-2} & \Delta r2_{t-3} \\ \Delta r3_{t-1} & \Delta r3_{t-2} & \Delta r3_{t-3} \\ \Delta r4_{t-1} & \Delta r4_{t-2} & \Delta r4_{t-3} \end{bmatrix}$$

$$, \alpha_2 = \begin{bmatrix} \alpha_{15} \\ \alpha_{25} \\ \alpha_{35} \\ \alpha_{45} \end{bmatrix}, \alpha_3 = \begin{bmatrix} \alpha_{16} \\ \alpha_{26} \\ \alpha_{36} \\ \alpha_{46} \end{bmatrix}, \alpha_{4k} = \begin{bmatrix} \alpha_{17} & \alpha_{27} & \alpha_{37} & \alpha_{47} \end{bmatrix}, r_{t-1} = \begin{bmatrix} r1_{t-1} \\ r2_{t-1} \\ r3_{t-1} \\ r4_{t-1} \end{bmatrix}, u_t = \begin{bmatrix} u_{1t} \\ u_{2t} \\ u_{3t} \\ u_{4t} \end{bmatrix}$$

	α_0	t_0	t_0^2	$r1_{t-1}$	$r2_{t-1}$	$r3_{t-1}$	$r4_{t-1}$	$\Delta r1_{t-1}$	$\Delta r2_{t-1}$	$\Delta r3_{t-1}$	$\Delta r4_{t-1}$
$\Delta r1_t$	$-.021^\diamond$ (.009)	$.019^\diamond$ (.000)	$.018^\diamond$ (.000)	$.127^\diamond$ (.012)	$-.572^\diamond$ (.045)	$.808^\diamond$ (.066)	$-.361^\diamond$ (.034)	$.919^\diamond$ (.070)	$-2.460^\diamond$ (.117)	$3.046^\diamond$ (.152)	$-1.204^\diamond$ (.093)
$\Delta r2_t$	$-.062^\diamond$ (.011)	$.033^\diamond$ (.003)	$.021$ (.011)	$.219^\diamond$ (.015)	$-.900^\diamond$ (.061)	$1.185^\diamond$ (.095)	$-.494^\diamond$ (.050)	$1.271^\diamond$ (.102)	$-3.382^\diamond$ (.174)	$3.903^\diamond$ (.221)	$-1.391^\diamond$ (.134)
$\Delta r3_t$	$-.061^\diamond$ (.011)	$.032^\diamond$ (.004)	$.022$ (.011)	$.226^\diamond$ (.015)	$-.930^\diamond$ (.062)	$1.222^\diamond$ (.096)	$-.500^\diamond$ (.051)	$1.408^\diamond$ (.108)	$-4.183^\diamond$ (.185)	$5.074^\diamond$ (.240)	$-1.846^\diamond$ (.145)
$\Delta r4_t$	$-.056^\diamond$ (.011)	$.031^\diamond$ (.004)	$.024^\diamond$ (.012)	$.250^\diamond$ (.016)	$-1.039^\diamond$ (.064)	$1.388^\diamond$ (.098)	$-.592^\diamond$ (.051)	$1.396^\diamond$ (.110)	$-4.377^\diamond$ (.186)	$5.539^\diamond$ (.249)	$-2.110^\diamond$ (.151)

	$\Delta r1_{t-2}$	$\Delta r2_{t-2}$	$\Delta r3_{t-2}$	$\Delta r4_{t-2}$	$\Delta r1_{t-3}$	$\Delta r2_{t-3}$	$\Delta r3_{t-3}$	$\Delta r4_{t-3}$
$\Delta r1_t$	$-.072$ (.092)	$.474^\diamond$ (.161)	$-.936^\diamond$ (.197)	$.551^\diamond$ (.120)	$-.099$ (.122)	$-.321$ (.187)	$.170$ (.241)	$.229$ (.147)
$\Delta r2_t$	$-.157$ (.130)	$.783^\diamond$ (.233)	$-1.526^\diamond$ (.287)	$.894^\diamond$ (.175)	$-.214$ (.168)	$-.337$ (.259)	$.141$ (.353)	$.346$ (.218)
$\Delta r3_t$	$-.176$ (.141)	$.745^\diamond$ (.254)	$-1.493^\diamond$ (.317)	$.924^\diamond$ (.189)	$-.207$ (.188)	$-.447$ (.295)	$.262$ (.385)	$.329$ (.236
$\Delta r4_t$	$-.207$ (.145)	$.871^\diamond$ (.260)	$-1.636^\diamond$ (.326)	$.977^\diamond$ (.194)	$-.251$ (.196)	$-.281$ (.304)	$.099$ (.400)	$.354$ (.243)

If we look at the residuals, we can see that every cross section matches the Student's t distribution close to degrees of freedom ($v = 4$), lags of 3, and trends of t_0 & t_0^2.

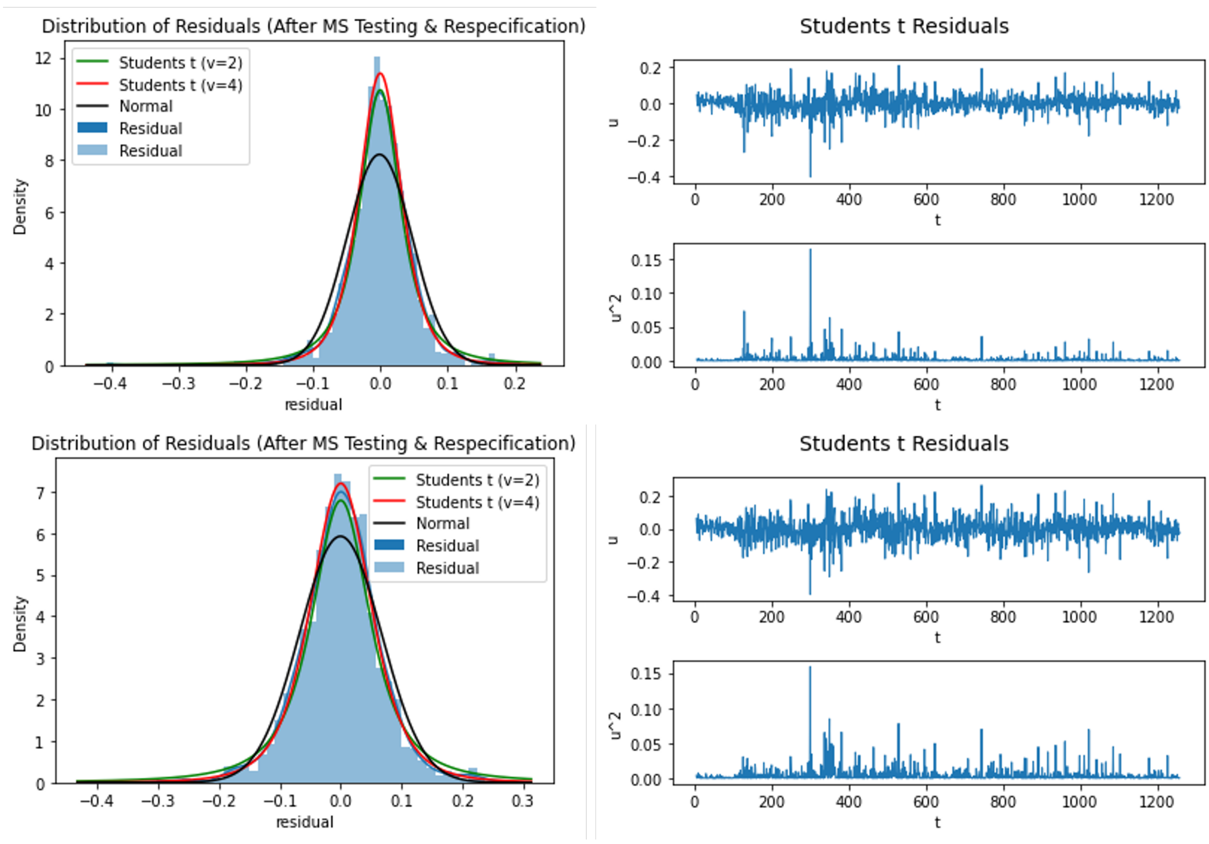

Figure 6: Respectively residuals for r1 & r2 from StVAR(3)

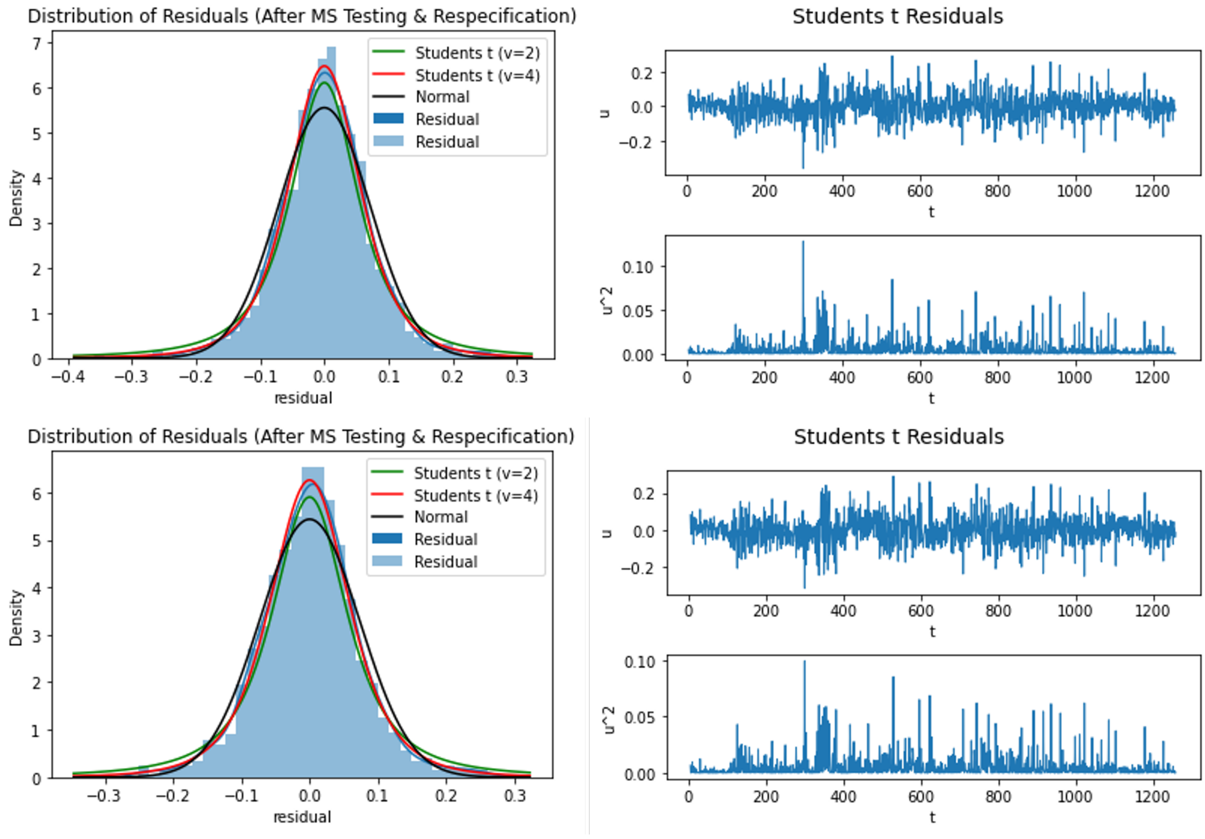

Figure 7: Respectively residuals for r3 & r4 from StVAR(3)

3.6 M-S Testing the Results

M-S testing occurs for the residuals of each equation in the system of the StVAR(3). So we'll test $\hat{u_{jt}}$ & $\hat{u^2_{jt}}$ s.t. $j = 1, 2, 3, 4$. This will look as follows (see Appendix for detailed auxiliary regression results):

$$\hat{u}_{jt} = \alpha_{j0} + \overbrace{\sum_{j=1}^{4}\sum_{k=1}^{4} \alpha_{jk}\Delta r_{jt-k} + \sum_{j=1}^{4}\sum_{i=1}^{2} \alpha_{ji+4} t_0^i + \sum_{j=1}^{4} \alpha_{j7} r_{jt-1}}^{OriginalSpecification} + \sum_{j=1}^{4}\sum_{i=3}^{4} \alpha_{ji+5} t_0^i + \sum_{i=1}^{3} \alpha_{ji+9}\hat{u}_{t-i} + \alpha_{j14}\hat{y} + \alpha_{j15}\hat{y}^2 + v_{j1t}$$

$$\hat{u^2_{jt}} = \alpha_{j0} + \sum_{i=1}^{4} \alpha_{ji} t_0^i + \sum_{i=1}^{4} \alpha_{ji+5}\hat{\sigma}^2_{t-i} + \alpha_{j10}\hat{\sigma}^2_t + \alpha_{j11}\Delta \hat{rj}_t^2 + v_{j2t}$$

Remember that for Student's t, heterogeneity is still captured with the trend terms, but dependence, linearity and heteroskedasticity are captured differently:

- (2) via the first moment's $\Delta \hat{rj}_t^2$
- (3 - Homoskedasticity) via the second moment's $\hat{\sigma}^2_t$ & $\Delta \hat{rj}_t^2$

- (3 - Dynamic Heteroskedasticity) via the second moment's $\hat{\sigma}^2_{t-i}$ s.t. $i = 1, 2, ..., 4$
- (4 - Markhov dependence) via the first moment's $\hat{u}_{t-i}$ s.t. $i = 1, 2, ..., 4$

The following violations were found for all rates:

- (4) Markhov dependence (via the first moment's u_{t-3})

The following violations were found for r2, r3 & r4:

- (5) Variance Heterogeneity (via t_0^4)

3.7 Results Compared to the Literature

The results found in Section 3.5 are quite different compared to the AR(p) and VAR(p) models of the literature, primarily because they include six additional parameters for trend (via t_0 & t_0^2) and the unit roots (via rj_{t-1}) (Bali et al., 2009; Dbouk et al., 2015; Erdogan & Dayan, n.d.). Simple Normal models are even worse, since they are missing three times the amount of missing parameters (18 total) because of all the lag terms not included (Δrj_{t-k} s.t. $k = 1, 2, 3$)(Lai & Xing, 2008). All that is accounted for in the Simple Normal model are two parameters: the constant and the static parameter for μ. As expected, similar models with many more (or less) independent variables will generate estimates that are different from each other, so the results of this paper will be systematically different leading to systematically different inferences. For example, a one unit increase in t_0 won't impact either the Simple Normal, AR(p) or VAR(p) models since it isn't a parameter, but it is a parameter in the StVAR(3) model and will cause each Δrj_t to change. It is plain to see the StVAR(3) will provide the most accurate inferences, while the other models will provide confounding inferences since they don't capture the full picture.

4 Concluding Remarks

This study has shown how there are key violations in normality, linearity, homoskedasticity, independence and homogeneity when modeling LIBOR rates with the typical Simple Normal model, the AR(p) model and the VAR(p) model. By using M-S testing, we started with the Simple Normal model, and found all of these violations. Even when respecifying into an AR(p) model with a trend, we still saw violations in normality, linearity and homoskedasticity. This is why we introduced the Student's t models, which worked incredibly well (as confirmed by M-S testing the final StVAR(3) model). The only piece that still needs to be accounted for is variance heterogeneity (via t_0^4 for r2, r3 & r4). The issues with the commonly used models highlights a crucial issue for majority of the finance literature. Similar models that presume Normality and/or aren't designed to catch departures from the core probabilistic assumptions are regularly used (linearity, homoskedasticity, independence and homogeneity). Moving forward, they should test their data to see if any of these departures are present, and if they are present, the model should be adjusted to handle this.

As we saw in Sections 2.7 & 3.6, the models in this paper provide very different results compared to the literature, which lead to completely different inferences. The time series models miss three parameters when using the traditional AR(p) models, while six parameters are missed when using the Simple Normal model. Panel data models are far worse, these miss six parameters when using the traditional VAR(p) models, while eighteen parameters are missed when using the Simple Normal model. These missing parameters don't pick up violations of the model assumptions, which will provide completely different inferences that don't capture the full picture.

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Appendix - StVAR Auxiliary Regressions

This section is organized by cross section, $\diamond$ denotes significance. Note that for all rates the following were found insignificant in early rounds of M-S testing, so they were dropped from the auxiliary regressions: t_0^3, t_0^4 & $\hat{u}_{t-4}$ from the 1st moment, and from the 2nd moment $\hat{\sigma}^2_{t-i}$ s.t. $i = 2, 3, 4$.

r1 was as follows:

- $\hat{u}_{1t} = \textit{Original Specification} - \underset{(.139)}{.712}\Delta\hat{r1}_t^{\diamond} + \underset{(4.392)}{.266}\Delta\hat{r1}_t^2 - \underset{(.030)}{.036}\hat{u}_{t-1} - \underset{(.029)}{.033}\hat{u}_{t-2} - \underset{(.029)}{.055}\hat{u}_{t-3}^{\diamond} + v_{1,1t}$
- $\hat{u}^2_{1t} = \underset{(.000)}{.002}^{\diamond} - \underset{(.001)}{.001}t_0^3 - \underset{(.001)}{.002}t_0^4 + \underset{(.00006)}{.00007}\hat{\sigma}^2_t + \underset{(.656)}{.504}\Delta\hat{r1}_t^2 + \underset{(.00006)}{.000045}\hat{\sigma}^2_{t-1} + v_{1,2t}$

r2 was as follows:

- $\hat{u}_{2t} = \textit{Original Specification} - \underset{(.120)}{.672}\Delta\hat{r2}_t^{\diamond} + \underset{(2.751)}{3.605}\Delta\hat{r2}_t^2 - \underset{(.029)}{.020}\hat{u}_{t-1} - \underset{(.029)}{.008}\hat{u}_{t-2} - \underset{(.029)}{.074}\hat{u}_{t-3}^{\diamond} + v_{2,1t}$
- $\hat{u}^2_{2t} = \underset{(.001)}{.005}^{\diamond} - \underset{(.001)}{.0002}t_0^3 - \underset{(.001)}{.005}t_0^{4\diamond} + \underset{(.00009)}{.00004}\hat{\sigma}^2_t + \underset{(.892)}{1.277}\Delta\hat{r2}_t^2 + \underset{(.00008)}{.00007}\hat{\sigma}^2_{t-1} + v_{2,2t}$

r3 was as follows:

- $\hat{u}_{3t} = \textit{Original Specification} - \underset{(.125)}{.790}\Delta\hat{r3}_t^{\diamond} + \underset{(2.835)}{4.629}\Delta\hat{r3}_t^2 - \underset{(.028)}{.012}\hat{u}_{t-1} - \underset{(.029)}{.019}\hat{u}_{t-2} - \underset{(.029)}{.083}\hat{u}_{t-3}^{\diamond} + v_{3,1t}$
- $\hat{u}^2_{3t} = \underset{(.001)}{.005}^{\diamond} + \underset{(.001)}{.00002}t_0^3 - \underset{(.001)}{.006}t_0^{4\diamond} + \underset{(.00009)}{.00007}\hat{\sigma}^2_t + \underset{(.919)}{1.506}\Delta\hat{r3}_t^2 + \underset{(.00008)}{.00007}\hat{\sigma}^2_{t-1} + v_{3,2t}$

r4 was as follows:

- $\hat{u}_{4t} = \textit{Original Specification} - \underset{(.131)}{.805}\Delta\hat{r4}_t^{\diamond} + \underset{(2.822)}{4.925}\Delta\hat{r4}_t^2 - \underset{(.029)}{.008}\hat{u}_{t-1} - \underset{(.029)}{.020}\hat{u}_{t-2} - \underset{(.029)}{.091}\hat{u}_{t-3}^{\diamond} + v_{4,1t}$
- $\hat{u}^2_{4t} = \underset{(.001)}{.005}^{\diamond} + \underset{(.001)}{.00005}t_0^3 - \underset{(.001)}{.006}t_0^{4\diamond} + \underset{(.00009)}{.00007}\hat{\sigma}^2_t + \underset{(.914)}{1.632}\Delta\hat{r4}_t^2 + \underset{(.00008)}{.00007}\hat{\sigma}^2_{t-1} + v_{4,2t}$