Introduction to Functional Programming

Jeremy Gibbons

University of Oxford

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Part 0

Aims and objectives

0.1 Aims

- *functional programming* is *programming with values*:
  value-oriented programming
- no ‘actions’, no side-effects — a radical departure from ordinary (imperative or OO) programming
- surprisingly, it is a powerful (and fun!) paradigm
- better ways of gluing programs together:
  component-oriented programming
- ideas are applicable in ordinary programming languages too:
  aim to introduce you to the ideas, to improve your day-to-day programming
- (I don’t expect you all to become functional programmers)

0.2 Motivation

LISP is worth learning [because of] the profound enlightenment experience you will have when you finally get it. That experience will make you a better programmer for the rest of your days, even if you never actually use LISP itself a lot.

Eric S. Raymond, American computer programmer (1957–)
How to Become a Hacker
`www.catb.org/~esr/faqs/hacker-howto.html`

You can never understand one language until you understand at least two.

Ronald Searle, British artist (1920–2011)

0.3 Contents

1. value-oriented programming
2. lists and other algebraic datatypes
3. types, polymorphism, type classes
4. monads and applicative functors
5. laziness, infinite data structures, co-programming

Concurrently, John Hughes will be talking about *property-based testing* using *QuickCheck*, illustrating most of these ideas.

0.4 Expressions vs statements

- in ordinary programming languages the world is divided into a world of statements and a world of expressions
- statements:
  - x:=E, s1 ; s2, **while** b **do** s
  - execution order is important

    $i{:=}i+1\ ;\ a{:=}a*i \quad \neq \quad a{:=}a*i\ ;\ i{:=}i+1$

- expressions:
  - eg a+b∗c, a and not b
  - evaluation order is unimportant (assuming no side-effects): in (2∗a∗y+b) ∗ (2∗a∗y+c), evaluate either parenthesis first (or both together!)

0.4 Optimizations

- useful optimizations:
  - branch merging:

```
    if b then p else p end
=   p
```

  - common subexpression elimination:

```
    z := (2*a*y+b)*(2*a*y+c)
=   t := 2*a*y ; z := (t+b)*(t+c)
```

  - parallel execution: evaluate subexpressions concurrently
- most optimizations require *referential transparency*
  - all that matters about the expression is its value
  - follows from 'no side effects'
  - ...which follows from 'no :='
  - with assignments, side-effect-freeness is very hard to check

0.4 Brevity

- expressions are much shorter and simpler than the corresponding statements
- eg compare using expression:

```
z := (2*a*y+b)*(2*a*y+c)
```

with not using expressions:

```
ac := 2; ac *= a; ac *= y; ac += b; t := ac;
ac := 2; ac *= a; ac *= y; ac += c; ac *= t;
z := ac
```

- but in order to discard statements,
  the expression language must be extended
- functional programming is
  programming with an extended expression language

Part 1

Value-oriented programming

1.1 Programs

We'll be using the *GHCi* implementation of *Haskell*.
A *script* is essentially just a collection of definitions:

$$
\begin{array}{l}
\quad \text{-- compute the square of an integer} \\
\mathit{square} :: \mathit{Integer} \to \mathit{Integer} \\
\mathit{square}\ x = x \times x \\
\quad \text{-- smaller of two arguments} \\
\mathit{smaller} :: (\mathit{Integer}, \mathit{Integer}) \to \mathit{Integer} \\
\mathit{smaller}\ (x, y) = \mathbf{if}\ x \leqslant y\ \mathbf{then}\ x\ \mathbf{else}\ y
\end{array}
$$

Bigger *programs* are divided into *modules*.
A standalone program includes a 'main' expression.

1.1 REPL

Value-oriented programming accommodates a *read-eval-print loop* (REPL):

Prelude⟩ :l sample.hs
Ok, one module loaded.

Main⟩ 42
42

Main⟩ 6×7
42

Main⟩ $\mathit{square}\ 7 - \mathit{smaller}\ (3, 4) - \mathit{square}\ (\mathit{smaller}\ (2, 3))$
42

Main⟩ $\mathit{square}\ 3141592654$
9869604403666763716

(I typed the things in green.)

1.1 Layout

- elegant and unobtrusive syntax
- structure obtained by layout, not punctuation
- all definitions in same scope must start in the same column
- indentation from start of definition implies continuation

$$
\begin{array}{l}
\mathit{smaller} :: (\mathit{Integer}, \mathit{Integer}) \to \mathit{Integer} \\
\mathit{smaller}\ (x, y) \\
\quad = \mathbf{if} \\
\qquad\quad x \leqslant y \\
\qquad \mathbf{then} \\
\qquad\quad x \\
\qquad \mathbf{else} \\
\qquad\quad y
\end{array}
$$

- blank lines around code in literate script!
- use spaces, not tabs!

1.2 Evaluation

- interpreter evaluates expression by reducing to simplest possible form
- reduction is rewriting using meaning-preserving simplifications: *replacing equals by equals*

$$
\begin{array}{ll}
 & \mathit{square}\ (3+4) \\
\Rightarrow & \quad \{\ \text{definition of } +\ \} \\
 & \mathit{square}\ 7 \\
\Rightarrow & \quad \{\ \text{definition of } \mathit{square}\ \} \\
 & 7 \times 7 \\
\Rightarrow & \quad \{\ \text{definition of } \times\ \} \\
 & 49
\end{array}
$$

- expression 49 cannot be reduced any further: *normal form*
- *applicative order* evaluation: reduce arguments before expanding function definition (call by value, eager evaluation)

1.2 Alternative evaluation orders

- other evaluation orders are possible:

$$
\begin{array}{ll}
& \mathit{square}\ (3+4) \\
\Rightarrow & \quad \{ \text{definition of } \mathit{square} \} \\
& (3+4) \times (3+4) \\
\Rightarrow & \quad \{ \text{definition of } + \} \\
& 7 \times (3+4) \\
\Rightarrow & \quad \{ \text{definition of } + \} \\
& 7 \times 7 \\
\Rightarrow & \quad \{ \text{definition of } \times \} \\
& 49
\end{array}
$$

- final result is the same: if two evaluation orders terminate, both yield the same result (*confluence*)
- *normal order* evaluation: expand function definition before reducing arguments (call by need, lazy evaluation)

1.2 Non-terminating evaluations

- consider script

$$
\begin{array}{l}
\mathit{three} :: \mathit{Integer} \to \mathit{Integer} \\
\mathit{three}\ x = 3 \quad \text{-- NB argument unused} \\[4pt]
\mathit{infinity} :: \mathit{Integer} \\
\mathit{infinity} = 1 + \mathit{infinity}
\end{array}
$$

- two different evaluation orders:

$$
\begin{array}{ll}
 & \mathit{three}\ \mathit{infinity} \\
\Rightarrow & \quad \{\ \text{defn of } \mathit{infinity}\ \} \\
 & \mathit{three}\ (1 + \mathit{infinity}) \\
\Rightarrow & \quad \{\ \text{defn of } \mathit{infinity}\ \} \\
 & \mathit{three}\ (1 + (1 + \mathit{infinity})) \\
\Rightarrow & \ldots
\end{array}
\qquad
\begin{array}{ll}
 & \mathit{three}\ \mathit{infinity} \\
\Rightarrow & \quad \{\ \text{defn of } \mathit{three}\ \} \\
 & 3
\end{array}
$$

- not all evaluation orders terminate, even on the same expression; Haskell uses lazy evaluation

1.3 Functions

- naturally, FP is a matter of functions
- script defines *functions* (*square*, *smaller*)
- (script actually defines *values*; indeed, in FP functions are values)
- function transforms (one or more) arguments into result
- *deterministic*: same arguments always give same result
- may be *partial*: result may sometimes be undefined ($\perp$)
- eg cosine, square root; distance between two cities; compiler; text formatter; process controller

1.3 Function types

- *type declaration* in script specifies type of function
- eg $square :: Integer \rightarrow Integer$
- in general, $f :: A \rightarrow B$ indicates that function f takes arguments of type A and returns results of type B
- *apply* function to argument: $f\ x$
- sometimes parentheses are necessary: $square\ (3 + 4)$
  (function application is an operator, binding more tightly than $+$)
- be careful not to confuse the function f with the value $f\ x$

1.3 Lambda

- notation for anonymous functions
- eg $\lambda x \rightarrow x \times x$ as another way of writing *square*
- eg $\lambda a\ b \rightarrow a$ (which we'll call *const* later)
- ASCII '\' is nearest equivalent to Greek λ
- from Church's λ-calculus theory of computability (1941)

1.3 Declaration vs expression style

- Haskell is a committee language
- Haskell supports two different programming styles
- *declaration style*: using equations, patterns and expressions

$$
\begin{aligned}
&quad :: Integer \to Integer \\
&quad\ x = square\ x \times square\ x
\end{aligned}
$$

- *expression style*: emphasising the use of expressions

$$
\begin{aligned}
&quad :: Integer \to Integer \\
&quad = \lambda x \to square\ x \times square\ x
\end{aligned}
$$

- expression style is often more flexible
- experienced programmers use both simultaneously

1.3 Extensionality

- two functions are equal ($f = g$) if they give equal results for all arguments ($f\ x = g\ x$ for every x of the right type)
- eg these two functions are equal:

$$
\begin{aligned}
&\mathit{double}, \mathit{double}' :: \mathit{Integer} \rightarrow \mathit{Integer} \\
&\mathit{double}\ x\ \ = x + x \\
&\mathit{double}'\ x = 2 \times x
\end{aligned}
$$

- the important thing about a function is its mapping from arguments to results
- *intentional* properties (eg how a mapping is described) are irrelevant

1.3 Currying

- replace single structured argument by several simpler ones

$$
\begin{aligned}
&add :: (Integer, Integer) \to Integer\\
&add\ (x, y) = x + y
\end{aligned}
$$

$$
\begin{aligned}
&add' :: Integer \to (Integer \to Integer)\\
&add'\ x\ y = x + y
\end{aligned}
$$

- useful for reducing number of parentheses
- *add* takes a pair of *Integer*s and returns an *Integer*
- *add'* takes an *Integer* and returns a function of type $Integer \to Integer$
- eg $add'\ 3$ is a function; $(add'\ 3)\ 4$ reduces to 7
- can be written just $add'\ 3\ 4$ (see why shortly)

1.4 Operators

- functions with alphabetic names are *prefix*: $f\ 3\ 4$
- functions with symbolic names are *infix*: $3 + 4$
- make an alphabetic name infix by enclosing in backquotes: $17\ `\mathit{mod}`\ 10$
- make symbolic operator prefix (and curried) with parentheses: $(+)\ 3\ 4$
- thus, $\mathit{add'} = (+)$
- extend notion to include one argument too: *sectioning*
- eg $(1/)$ is the reciprocal function, (>0) is the positivity test

1.4 Associativity

- why operators at all? why not prefix notation?
- there is a problem of ambiguity:

$$x \otimes y \otimes z$$

—does this mean $(x \otimes y) \otimes z$ or $x \otimes (y \otimes z)$?

- sometimes it doesn't matter, eg addition

$$(x + y) + z = x + (y + z)$$

the operator $+$ is associative

1.4 Association

- some operators are not associative ($-$, $/$, $\uparrow$)
- to disambiguate without parentheses, operators may *associate* to the left or right
- eg subtraction associates to the left: $5 - 4 - 2 = -1$
- function application associates to the left: $f\ a\ b$ means $(f\ a)\ b$
- function type operator associates to the right:
  $\mathit{Integer} \to \mathit{Integer} \to \mathit{Integer}$ means $\mathit{Integer} \to (\mathit{Integer} \to \mathit{Integer})$

1.4 Precedence

- association does not help when operators are mixed

$$x \oplus y \otimes z$$

  what does this mean: $(x \oplus y) \otimes z$ or $x \oplus (y \otimes z)$?
- to disambiguate without parentheses, we use *precedence* (binding power)
- eg $\times$ has higher precedence (binds more tightly) than $+$

$$\mathbf{infixl}\ 7\ \times$$
$$\mathbf{infixl}\ 6\ +$$

- function application can be seen as an operator, and has the highest precedence, so $\mathit{square}\ 3 + 4 = 13$

1.4 Composition

- glue functions together with *function composition*
- defined as follows:

$$(\circ) :: (Integer \to Integer) \to (Integer \to Integer) \to (Integer \to Integer)$$
$$f \circ g = \lambda x \to f\,(g\,x)$$

- eg function $square \circ double$ takes 3 to 36
- equivalent definition: $(\circ)\ f\ g\ x = f\,(g\,x)$
- associative, so parentheses not needed in $f \circ g \circ h$
- (actually has a different type; explained later)

1.5 Definitions

- we've seen some simple definitions of functions so far
- can also define other kinds of values:

$$
\begin{array}{l}
\mathit{here} :: \mathit{String} \\
\mathit{here} = \texttt{"Oxford"}
\end{array}
$$

- all so far have had an identifier (and perhaps formal parameters) on the left, and an expression on the right
- other forms possible: conditional, pattern-matching and local definitions

1.5 Conditional definitions

- earlier definition of *smaller* used a *conditional expression*:

$$
\begin{aligned}
&smaller :: (Integer, Integer) \rightarrow Integer \\
&smaller\ (x, y) = \mathbf{if}\ x \leqslant y\ \mathbf{then}\ x\ \mathbf{else}\ y
\end{aligned}
$$

- could also use *guarded equations*:

$$
\begin{aligned}
&smaller :: (Integer, Integer) \rightarrow Integer \\
&smaller\ (x, y) \\
&\quad \mid x \leqslant y = x \\
&\quad \mid x > y = y
\end{aligned}
$$

- each *clause* has a *guard* and an *expression* separated by $=$
- last guard can be *otherwise* (synonym for *True*)
- especially convenient with three or more clauses
- *declaration style*: guard; *expression style*: **if** ... **then** ... **else**...

1.5 Pattern matching

- define function by several equations
- arguments on lhs not just variables, but *patterns*
- patterns may be *variables* or *constants* (or *constructors*, later)
- eg

$$
\begin{aligned}
&\mathit{day} :: \mathit{Integer} \to \mathit{String}\\
&\mathit{day}\ 1 = \texttt{"Saturday"}\\
&\mathit{day}\ 2 = \texttt{"Sunday"}\\
&\mathit{day}\ \_ = \texttt{"Weekday"}
\end{aligned}
$$

- also *wildcard pattern* _
- evaluate by reducing argument to normal form,
  then applying first matching equation
- result is $\perp$ if argument has no normal form, or no equation matches

1.5 Local definitions

- repeated subexpression can be captured in a *local definition*

$$
\begin{aligned}
&\mathit{qroots} :: (\mathit{Float}, \mathit{Float}, \mathit{Float}) \to (\mathit{Float}, \mathit{Float}) \\
&\mathit{qroots}\ (a, b, c) = ((-b - \mathit{sd}) \mathbin{/} (2 \times a), (-b + \mathit{sd}) \mathbin{/} (2 \times a)) \\
&\quad \mathbf{where}\ \mathit{sd} = \mathit{sqrt}\ (b \times b - 4 \times a \times c)
\end{aligned}
$$

- scope of ‘where’ clause extends over whole right-hand side
- multiple local definitions can be made:

$$
\begin{aligned}
&\mathit{demo} :: \mathit{Integer} \to \mathit{Integer} \to \mathit{Integer} \\
&\mathit{demo}\ x\ y = (a + 1) \times (b + 2) \\
&\quad \mathbf{where}\ a = x - y \\
&\quad \phantom{\mathbf{where}\ } b = x + y
\end{aligned}
$$

(nested scope, so layout rule applies here too:
all definitions must start in same column)

- in conjunction with guarded equations, the scope of a **where** clause covers all guard clauses

1.5 let-expressions

- a **where** clause is syntactically attached to an equation
- also: definitions local to an expression

$$
\begin{aligned}
&\mathit{demo} :: \mathit{Integer} \to \mathit{Integer} \to \mathit{Integer} \\
&\mathit{demo}\ x\ y = \mathbf{let}\ a = x - y \\
&\qquad\qquad\qquad\quad\ b = x + y \\
&\qquad\qquad\quad\ \mathbf{in}\ (a + 1) \times (b + 2)
\end{aligned}
$$

- *declaration style*: **where**; *expression style*: **let … in…**
- **let**-expressions are more flexible than **where** clauses

1.6 Functions as first-class citizens

- *functional programming* concerns functions (of course!)
- functions are first-class citizens of the language
- functions have all the rights of other types:
  - may be passed as arguments
  - may be returned as results
  - may be stored in data structures
  - etc
- functions that manipulate functions are *higher order*

Slogan: higher-order functions allow
new and better means of modularizing programs

1.7 Functions as arguments and results

- functions may be taken as arguments

$$
\begin{aligned}
&\mathit{applyToPair} :: (\mathit{Integer} \to \mathit{Integer}) \to (\mathit{Integer}, \mathit{Integer}) \to (\mathit{Integer}, \mathit{Integer}) \\
&\mathit{applyToPair}\ f\ (x, y) = (f\ x, f\ y)
\end{aligned}
$$

- functions may also be returned as results

$$
\begin{aligned}
&\mathit{addOrMul} :: \mathit{Bool} \to (\mathit{Integer} \to \mathit{Integer} \to \mathit{Integer}) \\
&\mathit{addOrMul}\ b = \mathbf{if}\ b\ \mathbf{then}\ (+)\ \mathbf{else}\ (\times)
\end{aligned}
$$

- those are rather artifical examples. . .
- partial application
- currying
- function composition (again)
- lots more, especially tomorrow: *parametrizable program schemes*

1.7 Partial application

- consider $add'\ x\ y = x + y$
- type $Integer \rightarrow Integer \rightarrow Integer$; takes two *Integer*s and returns an *Integer* (eg $add'\ 3\ 4 = 7$)
- another view: type $Integer \rightarrow (Integer \rightarrow Integer)$ (remember, $\rightarrow$ associates to the right); takes a single *Integer* and returns an $Integer \rightarrow Integer$ function (eg $add'\ 3$ is the *Integer*-transformer that adds three)
- need not apply function to all its arguments at once: *partial application*; result will then be a function, awaiting remaining arguments
- in fact, partial application is the norm; every function takes *exactly one argument*!
- sectioning ($(3+)$) is partial application of binary operators

1.7 Currying

- a function taking pair of arguments can be transformed into a function taking two successive arguments, and vice versa
- recall

$$
\begin{aligned}
&\mathit{add} :: (\mathit{Integer}, \mathit{Integer}) \to \mathit{Integer} \\
&\mathit{add}\ (x, y) = x + y
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{add'} :: \mathit{Integer} \to \mathit{Integer} \to \mathit{Integer} \\
&\mathit{add'}\ x\ y = x + y
\end{aligned}
$$

- $\mathit{add'}$ is called the *curried* version of add
- named after logician Haskell B. Curry (like the language), though actually due to Schönfinkel
- thus, pair-consuming functions are unnecessary

- transformations are implementable as higher-order operations

$$\begin{aligned}&\mathit{curry} :: ((a, b) \to c) \to (a \to b \to c)\\&\mathit{curry}\ f = \lambda a\ b \to f\ (a, b)\end{aligned}$$

$$\begin{aligned}&\mathit{uncurry} :: (a \to b \to c) \to ((a, b) \to c)\\&\mathit{uncurry}\ g = \lambda (a, b) \to g\ a\ b\end{aligned}$$

- eg $\mathit{add'} = \mathit{curry}\ \mathit{add}$
- a related higher-order operation: flip arguments of binary function (later: $\mathit{reverse} = \mathit{foldl}\ (\mathit{flip}\ (:))\ [\,]$)

$$\begin{aligned}&\mathit{flip} :: (a \to b \to c) \to (b \to a \to c)\\&\mathit{flip}\ f = \lambda b\ a \to f\ a\ b\end{aligned}$$

1.7 Function composition

- recall function composition

$$
\begin{aligned}
&(\circ) :: (\mathit{Integer} \to \mathit{Integer}) \to (\mathit{Integer} \to \mathit{Integer}) \to \mathit{Integer} \to \mathit{Integer} \\
&(f \circ g)\ x = f\ (g\ x)
\end{aligned}
$$

- takes two functions that 'meet in the middle' and an argument to one; returns the result from the other
- equivalently, type $(\mathit{Integer} \to \mathit{Integer}) \to (\mathit{Integer} \to \mathit{Integer}) \to (\mathit{Integer} \to \mathit{Integer})$
- takes two functions, glues them together to form a third

1.7 Repeated composition

- double application: eg $\mathit{twice}\ \mathit{square}\ 3 = 81$

$$\begin{aligned}&\mathit{twice} :: (\mathit{Integer} \to \mathit{Integer}) \to (\mathit{Integer} \to \mathit{Integer})\\&\mathit{twice}\ f = f \circ f\end{aligned}$$

- generalize: eg $\mathit{iter}\ 4\ (2\times)\ 1 = 2 \times 2 \times 2 \times 2 \times 1$

$$\begin{aligned}&\mathit{iter} :: \mathit{Int} \to (\mathit{Integer} \to \mathit{Integer}) \to (\mathit{Integer} \to \mathit{Integer})\\&\mathit{iter}\ 0\ f = \mathit{id}\\&\mathit{iter}\ n\ f = f \circ \mathit{iter}\ (n-1)\ f\end{aligned}$$

1.8 Reasoning about programs

- functional programs are just equations
- lazy semantics means that rules of ordinary algebra (substitution of equals for equals) apply
- given

  $$\mathit{three}\ x = 3$$

  can replace 3 anywhere by $\mathit{three}\ x$ for any suitably-typed expression x (even $x = 1\ \text{‘}\mathit{div}\text{‘}\ 0$)
- simple proofs by *equational reasoning*

1.9 Equational reasoning

- equations as definitions intended for evaluation
- ...but also useful for reasoning: proofs
- better than testing, because exhaustive
- eg given

$$swap\ (x, y) = (y, x)$$

we can show that *swap*ping twice is the identity:

$$\begin{array}{ll} & swap\ (swap\ (a, b)) \\ = & \quad \{ \text{definition of } swap \} \\ & swap\ (b, a) \\ = & \quad \{ \text{definition of } swap \} \\ & (a, b) \end{array}$$

- program text used as proof rules

1.9 Another simple example

- given

$$
\begin{aligned}
&\mathit{curry}\ f\ a\ b = f\ (a, b) \\
&\mathit{fst}\ (a, b) = a \\
&\mathit{const}\ a\ b = a
\end{aligned}
$$

prove that $\mathit{const} = \mathit{curry}\ \mathit{fst}$:

$$
\begin{array}{ll}
 & \mathit{curry}\ \mathit{fst}\ a\ b \\
= & \quad \{\ \text{definition of } \mathit{curry}\ \} \\
 & \mathit{fst}\ (a, b) \\
= & \quad \{\ \text{definition of } \mathit{fst}\ \} \\
 & a \\
= & \quad \{\ \text{definition of } \mathit{const}\ \} \\
 & \mathit{const}\ a\ b
\end{array}
$$