Parallel Double Greedy Submodular Maximization

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Abstract

Bidirectional greedy is a sequential algorithm that does not speed well to problems of large speed. We present 2 approaches to extend the bidirectional greedy algorithm to a parallel setting. The first, 'hogwild' approach emphasizes speed at the cost of worsening the approximation by an additive factor; the second approach guarantees the same approximation bound by sacrificing concurrency.

1 Introduction

The bidirectional greedy algorithm [1] gives an approximation of $E[F(A)] = 1/2 f(OPT)$, where A is the algorithm output, and OPT is an optimal solution.

The hogwild algorithm can give an approximation of $E[F(A)] = \frac{1}{2}F(OPT) - \frac{1}{4}\sum_i E[\rho_i]$, where ρ_i is the maximum difference in the marginal gain that may result from not knowing the full information when deciding whether to include or exclude element i.

The OCC algorithm [XP: for the lack of a better name] guarantees an outcome that is equivalent to a sequential run of the bidirectional greedy algorithm. Theoretical properties of the bidirectional greedy algorithm immediately translates to the OCC algorithm – in particular, the OCC algorithm gives the same approximation factor of $1/$. In contrast to the hogwild approach, OCC introduces more coordination and thus provides less concurrency.

2 Submodular maximization

The sequential bidirectional greedy [1] algorithm monotonically grows A^i and shrinks B^i.

3 Approaches for parallel learning

Two approaches that allow us to trade off speed with approximation guarantees.

3.1 Coordination free

Simply run everything in parallel. Optimized for speed, but does not necessarily provide the correct answer. Requires work to prove correctness.

3.2 Concurrency control

Ensures 'serial equivalence' – the outcome of the parallel algorithm is equivalent to some execution of the sequential algorithm. Locally, threads take actions that are guaranteed to be safe (i.e. preserves serial equivalence), and forces additional coordination only when they are unable to execute their action safely. Designed for correctness, but requires coordination that compromises speed. Work is only required to demonstrate that coordination is limited.

Algorithm 1: Serial submodular maximization

$A^0 = \emptyset$, $B^0 = V$
for $i = 1$ *to* n **do**
  $\Delta_+(i) = F(A^{i-1} \cup i) - F(A^{i-1})$
  $\Delta_-(i) = F(B^{i-1} \backslash i) - F(B^{i-1})$
  Draw $u_i \sim Unif(0,1)$
  if $u_i < \frac{[\Delta_+(i)]_+}{[\Delta_+(i)]_+ + [\Delta_-(i)]_+}$ **then**
    $A^i := A^{i-1} \cup i$; $B^i := B^{i-1}$
  else
    $A^i := A^{i-1}$; $B^i := B^{i-1} \backslash i$

Algorithm 2: Hogwild bidirectional greedy

1 **for** $e \in V$ **do** $\hat{A}(e) = 0$, $\hat{B}(e) = 1$
2 **for** $p \in \{1, \dots, P\}$ **do in parallel**
3   **while** $\exists$ *element to process* **do**
4     $e =$ next element to process
5     $\Delta_+^{\max}(e) = F(\hat{A} \cup e) - F(\hat{A})$
6     $\Delta_-^{\max}(e) = F(\hat{B} \backslash e) - F(\hat{B})$
7   Draw $u_e \sim Unif(0,1)$
8   **if** $u_e < \frac{[\Delta_+^{\max}(e)]_+}{[\Delta_+^{\min}(e)]_+ + [\Delta_-^{\max}(e)]_+}$ **then**
9       $\hat{A}(e) \leftarrow 1$
10   **else** $\hat{B}(e) \leftarrow 0$

Algorithm 3: Hogwild for separable sums

for $e \in V$ **do** $\hat{A}(e) = 0$
for $l = 1, \dots, L$ **do** $\hat{\alpha}_l = 0$, $\hat{\beta}_l = \sum_{e \in S_l} w_l(e)$
for $p \in \{1, \dots, P\}$ **do in parallel**
  while $\exists$ *element to process* **do**
    $e =$ next element to process
    $\Delta_+^{\max}(e) =$
    $-\lambda v(e) + \sum_{S_l \ni e} g(\hat{\alpha}_l + w_l(e)) - g(\hat{\alpha}_l)$
    $\Delta_-^{\max}(e) =$
    $+\lambda v(e) + \sum_{S_l \ni e} g(\hat{\beta}_l - w_l(e)) - g(\hat{\beta}_l)$
    Draw $u_e \sim Unif(0,1)$
    if $u_e < \frac{[\Delta_+^{\max}(e)]_+}{[\Delta_+^{\min}(e)]_+ + [\Delta_-^{\max}(e)]_+}$ **then**
      $\hat{A}(e) \leftarrow 1$
      for $l : e \in S_l$ **do** $\hat{\alpha}_l \leftarrow \hat{\alpha}_l + w_l(e)$
    else **for** $l : e \in S_l$ **do** $\hat{\beta}_l \leftarrow \hat{\beta}_l - w_l(e)$

Algorithm 4: OCC bidirectional greedy

for $e \in V$ **do** $\hat{A}(e) = \tilde{A}(e) = 0$, $\hat{B}(e) = \tilde{B}(e) = 1$
for $i = 1, \dots, |V|$ **do** result$(i) = 0$
for $i = 1, \dots, |V|$ **do** processed$(i) = false$
$\iota = 0$
for $p \in \{1, \dots, P\}$ **do in parallel**
  while $\exists$ *element to process* **do**
    $e =$ next element to process
    $\tilde{A}(e) \leftarrow 1$
    $\tilde{B}(e) \leftarrow 0$
    $i = \iota$; $\iota \leftarrow \iota + 1$
    $\Delta_+^{\min}(e) = F(\tilde{A} \cup e) - F(\tilde{A})$
    $\Delta_+^{\max}(e) = F(\hat{A} \cup e) - F(\hat{A})$
    $\Delta_-^{\min}(e) = F(\tilde{B} \backslash e) - F(\tilde{B})$
    $\Delta_-^{\max}(e) = F(\hat{B} \backslash e) - F(\hat{B})$
    Draw $u_e \sim Unif(0,1)$
    if $u_e < \frac{[\Delta_+^{\min}(e)]_+}{[\Delta_+^{\min}(e)]_+ + [\Delta_-^{\max}(e)]_+}$ **then**
      result$(i) \leftarrow 1$
    else if $u_e > \frac{[\Delta_+^{\max}(e)]_+}{[\Delta_+^{\max}(e)]_+ + [\Delta_-^{\min}(e)]_+}$ **then**
      result$(i) \leftarrow -1$
    wait until $\forall j < i$, result$(j) \neq 0$
    if *result*$(i) = 0$ **then** validate(p, e, i)
    if *result*$(i) = 1$ **then**
      $\hat{A}(e) \leftarrow 1$
      $\tilde{B}(e) \leftarrow 1$
    else
      $\tilde{A}(e) \leftarrow 0$
      $\hat{B}(e) \leftarrow 0$
    processed$(i) = true$

Algorithm 5: validate(p, e, i)

wait until $\forall j < i$, processed$(j) = true$
$\Delta_+(e) = F(\hat{A} \cup e) - F(\hat{A})$
$\Delta_-(e) = F(\hat{B} \backslash e) - F(\hat{B})$
if $u_e < \frac{[\Delta_+(e)]_+}{[\Delta_+(e)]_+ + [\Delta_-(e)]_+}$ **then** result$(i) \leftarrow 1$
else result$(i) \leftarrow -1$

4 Hogwild for arbitrary submodular F

Algorithm 2[1] is the hogwild parallel bidirectional greedy unconstrained submodular maximization algorithm. We associate with each element e a time T_e at which Algorithm 2 line 7 is executed, and order the elements according to the times T_e. Let $\iota(e)$ be the position of e in this ordering. This total ordering on elements also allows us to define sets A^i, B^i corresponding to that obtained by the serial algorithm; specifically, $A^i = \{e' : e' \in A, \iota(e') < i\}$ and $B^i = A^i \cup \{e' : \iota(e') \geq i\}$.

[1] We present only the parallelized probabilistic versions of [1]. Both parallel algorithms can be easily extended to the deterministic version of [1]; the hogwild algorithm can also be extended to the multilinear version of [1].

Note that in Algorithm 2, lines 5 and 6 may be executed in parallel with lines 9 and 10. Hence, $\Delta_+^{\max}(e)$ and $\Delta_-^{\max}(e)$ (lines 5 and 6) may be computed with different values of $\hat{A}(e')$. We denote by $\hat{A}_e$ and $\hat{B}_e$ respectively the vectors of $\hat{A}$ and $\hat{B}$ that are used in the computation of $\Delta_+^{\max}(e)$ and $\Delta_-^{\max}(e)$. It immediately follows that

$$\begin{aligned} \Delta_+(e) &= F(A^{\iota(e)-1} \cup i) - F(A^{\iota(e)-1}), & \Delta_-(e) &= F(B^{\iota(e)-1} \backslash e) - F(B^{\iota(e)-1}) \\ \Delta_+^{\max}(e) &= F(\hat{A}_e \cup e) - F(\hat{A}_e), & \Delta_-^{\max}(e) &= F(\hat{B}_e \backslash e) - F(\hat{B}_e). \end{aligned}$$

Lemma 4.1. *For any $e \in V$, $\hat{A}_e \subseteq A^{\iota(e)-1}$, $\hat{B}_e \supseteq B^{\iota(e)-1}$.*

Proof. Consider any element $e' \in V$. If $e' \in \hat{A}_e$, it must be the case that the algorithm set $\hat{A}(e')$ to 1 (line 9) before T_e, which implies $\iota(e') < \iota(e)$, and hence $e' \in A^{\iota(e)-1}$. So $\hat{A}_e \subseteq A^{\iota(e)-1}$. Similarly, if $e' \notin \hat{B}_e$, then the algorithm set $\hat{B}(e')$ to 0 (line 10) before T_e, so $\iota(e') < \iota(e)$. Also, $e' \notin A$ because the execution of line 10 excludes the execution of line 9. Therefore, $e' \notin A^{\iota(e)-1}$, and $e' \notin B^{\iota(e)-1}$. So $\hat{B}_e \subseteq B^{\iota(e)-1}$. □

Corollary 4.2. *Submodularity of F implies $\Delta_+(e) \leq \Delta_+^{\max}(e)$, and $\Delta_-(e) \leq \Delta_-^{\max}(e)$.*

4.1 Hogwild for separable sums F

For some functions F, we can maintain sketches / statistics to aid the computation of $\Delta_+^{\max}$, $\Delta_-^{\max}$, and obtain the bounds given in Corollary 4.2. In particular, we consider functions of the form $F(X) = \sum_{l=1}^{L} g\left(\sum_{i \in X \cup S_l} w_l(i)\right) - \lambda \sum_{i \in X} v(i)$, where $S_l \subseteq V$ are (possibly overlapping) groups of elements in the ground set, g is a non-decreasing concave scalar function, and $w_l(i)$ and $v(i)$ are non-negative scalar weights. It is easy to see that $F(X \cup e) - F(X) = \sum_{l: e \in S_l} \left[g\left(w_l(e) + \sum_{i \in X \cup S_l} w_l(i)\right) - g\left(\sum_{i \in X \cup S_l} w_l(i)\right)\right] - \lambda v(e)$. Define

$$\begin{aligned} \hat{\alpha}_l &= \sum_{j \in \hat{A} \cup S_l} w_l(j), & \hat{\alpha}_{l,e} &= \sum_{j \in \hat{A}_e \cup S_l} w_l(j), & \alpha_l^{\iota(e)-1} &= \sum_{j \in A^{\iota(e)-1} \cup S_l} w_l(j). \\ \hat{\beta}_l &= \sum_{j \in \hat{B} \cup S_l} w_l(j), & \hat{\beta}_{l,e} &= \sum_{j \in \hat{B}_e \cup S_l} w_l(j), & \beta_l^{\iota(e)-1} &= \sum_{j \in B^{\iota(e)-1} \cup S_l} w_l(j). \end{aligned}$$

Algorithm 3 updates $\hat{\alpha}_l$ and $\hat{\beta}_l$, and computes $\Delta_+^{\max}(e)$ and $\Delta_-^{\max}(e)$ using $\hat{\alpha}_{l,e}$ and $\hat{\beta}_{l,e}$. Following arguments analogous to that of Lemma 4.1, we can show:

Lemma 4.3. *For each l and $e \in V$, $\hat{\alpha}_{l,e} \leq \alpha_l^{\iota(e)-1}$ and $\hat{\beta}_{l,e} \geq \beta_l^{\iota(e)-1}$.*

Corollary 4.4. *Concavity of g implies that Algorithm 3 computes*

$$\begin{aligned} \Delta_+^{\max}(e) &\geq \sum_{S_l \ni e} \left[g(\alpha_l^{\iota(e)-1} + w_l(e)) - g(\alpha_l^{\iota(e)-1}) \right] - \lambda v(e) &&= \Delta_+(e), \\ \Delta_-^{\max}(e) &\geq \sum_{S_l \ni e} \left[g(\beta_l^{\iota(e)-1} - w_l(e)) - g(\beta_l^{\iota(e)-1}) \right] + \lambda v(e) &&= \Delta_-(e), \end{aligned}$$

5 Concurrency control

Algorithm 4 is the OCC bidirectional greedy algorithm. Unlike the hogwild algorithm, the OCC algorithm ensures serial equivalence by maintaining four sets $\hat{A}$, $\tilde{A}$, $\hat{B}$, $\tilde{B}$, which serve as upper and lower bounds on A and B. Each thread can determine locally if a decision to include / exclude an element can be taken safely. Otherwise, the validation process (Algorithm 5) waits until it is certain about A, B before proceeding.

The serialization order is given by $\iota(e)$, which is the value of ι at line 10 of Algorithm 4. We will show that the outcome of Algorithm 4 is equivalent to the sequential algorithm executed with ordering given by ι.

Lemma 5.1. *$\hat{A}_e \subseteq A^{\iota(e)-1} \subseteq \tilde{A}_e \backslash e$, and $\hat{B}_e \supseteq B^{\iota(e)-1} \supseteq \tilde{B}_e \backslash e$.*

Proof. There are 4 parts to this proof.

1. $e' \in \hat{A}_e \implies e' \in A^{\iota(e)-1}$.
2. $e' \in A^{\iota(e)-1} \implies e' \in \tilde{A}_e \backslash e$.
3. $e' \notin \hat{B}_e \implies e' \notin B^{\iota(e)-1}$.
4. $e' \notin B^{\iota(e)-1} \implies e' \notin \tilde{B}_e \backslash e$.

□

Algorithm 4 computes

$$\Delta_+^{\min}(e) = F(\tilde{A}_e) - F(\tilde{A}_e \backslash e), \qquad \Delta_+^{\max}(e) = F(\hat{A}_e \cup e) - F(\hat{A})$$
$$\Delta_-^{\min}(e) = F(\tilde{B}_e) - F(\tilde{B}_e \cup e), \qquad \Delta_-^{\max}(e) = F(\hat{B}_e \backslash e) - F(\hat{B}).$$

Corollary 5.2. *By submodularity, $\Delta_+^{\min}(e) \leq \Delta_+(e) \leq \Delta_+^{\max}(e)$, $\Delta_-^{\min}(e) \leq \Delta_-(e) \leq \Delta_-^{\max}(e)$.*

5.1 Separable sums F

Analogous to the hogwild algorithm, we maintain $\hat{\alpha}_l$, $\hat{\beta}_l$ and additionally $\tilde{\alpha}_l = \sum_{j \in \tilde{A} \cup S_l} w_l(j)$ and $\tilde{\beta}_l = \sum_{j \in \tilde{B} \cup S_l} w_l(j)$. It can be shown that $\hat{\alpha}_{l,e} \leq \alpha^{\iota(e)-1} \leq \tilde{\alpha}_{l,e} - w_l(e)$ and $\hat{\beta}_{l,e} \geq \beta^{\iota(e)-1} \geq \tilde{\beta}_{l,e} + w_l(e)$, which then allows us to compute our bounds for Δ's as $\Delta_+^{\max}(e) = \sum_{S_l \ni e} [g(\hat{\alpha}_{l,e} + w_l(e)) - g(\hat{\alpha}_{l,e})] - \lambda v(e)$, $\Delta_+^{\min}(e) = \sum_{S_l \ni e} [g(\tilde{\alpha}_{l,e}) - g(\tilde{\alpha}_{l,e} - w_l(e))] - \lambda v(e)$, $\Delta_-^{\max}(e) = \sum_{S_l \ni e} \left[g(\hat{\beta}_{l,e} - w_l(e)) - g(\hat{\beta}_{l,e})\right] + \lambda v(e)$, $\Delta_-^{\min}(e) = \sum_{S_l \ni e} \left[g(\tilde{\beta}_l^{\iota(e)-1}) - g(\tilde{\beta}_l^{\iota(e)-1} + w_l(e))\right] + \lambda v(e)$.

6 Analysis of algorithms

6.1 Approximation of hogwild bidirectional greedy

Theorem 6.1. *Let F be a non-negative (monotone or non-monotone) submodular function. The hogwild bidirectional greedy algorithm solves the unconstrained problem $\max_{A \subset V} F(A)$ with approximation*

$$E[F(A)] \geq \frac{1}{2} F^* - \frac{1}{4} \sum_{i=1}^{n} E[\rho_i],$$

where A is the output of the algorithm, F^* is the optimal value, and ρ_i is a random variable such that $\rho_i \geq \Delta_+^{\max}(i) - \Delta_+(i)$ and $\rho_i \geq \Delta_-^{\max}(i) - \Delta_-(i)$.

We prove the theorem in Appendix A.

Assumption

F is submodular and non-negative. We assume that we can bound

$$\Delta_+^{\max} - \rho_i \leq \Delta_+ \leq \Delta_+^{\max} \leq \Delta_+ + \rho_i, \qquad \Delta_-^{\max} - \rho_i \leq \Delta_- \leq \Delta_-^{\max} \leq \Delta_- + \rho_i$$

This is possible, for example, by defining

$$\begin{aligned} \rho_i \quad &:= \quad \max\{\Delta_+^{\max}(i) - \Delta_+(i), \Delta_-^{\max}(i) - \Delta_-(i)\} \\ &\leq \quad \max_{S,T \subseteq V} \{[F(S \cup i) - F(S)] - [F(S \cup T \cup i) - F(S \cup T)]\} \quad \leq \quad F(i)\kappa_F \end{aligned}$$

where S plays the role of A^j and T plays the role of $\{j+1,\ldots,i-1\}$, and κ_F is the total curvature of F. Summing over i then gives us $\sum_i \rho_i \leq \kappa_F \sum_i F(i)$.

[XP: Is there theory along these lines? Can we tighten this for non-monotone functions?]

Example: max graph cut. Assuming bounded delay of τ and edges with unit weight, we can bound $\sum_i E[\rho_i] \leq 2\tau\frac{\#\text{edges}}{2N}$ The approximation of the hogwild algorithm is then $E[F(A^n)] \geq= \frac{1}{2}F(OPT) - \tau\frac{\#\text{edges}}{2N}$. In sparse graphs, the hogwild algorithm is off by a small additional term, which albeit grows linearly in τ.

Example: set cover. Consider the simple set cover function, $F(A) = \sum_{l=1}^{L} \min(1, |A \cap S_l|) - \lambda|A| = |\{l : A \cap S_l \neq \emptyset\}| - \lambda|A|$, with $0 < \lambda \leq 1$. We assume that there is some bounded delay τ. Suppose also the S_l's form a partition, so each element e belongs to exactly one set. Then, $\sum_e E[\rho_e] \geq \tau + L(1-\lambda^\tau)$, which is linear in τ but independent of N.

6.2 Correctness of OCC

Theorem 6.2. *OCC bidirectional greedy is serially equivalent to bidirectional greedy.*

Proof. Outline: We first show that the sampling using $\Delta_+^{\min}$, $\Delta_+^{\max}$, $\Delta_-^{\min}$, $\Delta_-^{\max}$ is 'safe', i.e. is equivalently to sampling using Δ_+ and Δ_-. Secondly, we show the validation process is correct – specifically when the validation is executed, it is the case that $\hat{A} = A^{\iota(e)-1}$ and $\hat{B} = B^{\iota(e)-1}$. ☐

6.3 Scalability of OCC

We discuss the bound on the number of elements sent for validation in Appendix B

Example: max graph cut. The expected number of validated elements is upper bounded by $\tau\frac{2\#edges}{N}$.

Example: set cover. Under the same settings as for the hogwild analysis, the expected number of validated elements is upper bounded by 2τ.

7 Evaluation

We implemented the parallel and sequential double greedy algorithms in Java / Scala. Experiments were conducted on Amazon EC2 using one cc2.8xlarge machine, up to 16 threads, for 10 iterations.

We measured the runtime and speedup (ratio of runtime on 1 thread to runtime on p threads). For the hogwild algorithm, we measured $F(A_{hog}) - F(A_{seq})$, the difference between the objective value on the sets returned by hogwild and the sequential algorithms. We verified the correctness of the OCC algorithm by comparing the sets returned by the OCC and sequential algorithms, and also measured the fraction of elements that are validated by OCC.

Graph	# vertices	# edges	Description
Erdos-Renyi	20,000,000	$\approx 2 \times 10^6$	Each edge is included with probability 5×10^{-6}.
ZigZag	25,000,000	2,025,000,000	Expander graph. The 81-regular zig-zag product between the Cayley graph on $\mathbb{Z}_{2500000}$ with generating set $\{\pm1,\ldots,\pm5\}$, and the complete graph K_{10}.
Friendster	10,000,000	625,279,786	Subgraph of social network. [2]
Arabic-2005	22,744,080	631,153,669	2005 crawl of Arabic web sites [3, 4, 5].
UK-20005	39,459,925	921,345,078	2005 crawl of the .uk domain [3, 4, 5].

Table 1: Synthetic and real graphs used in the evaluation of our parallel algorithms.

We tested our parallel algorithms on the max graph cut and set cover problems with two synthetic graphs and three real datasets (Table 1). Graphs were pre-processed to remove self-loops. We found that vertices were typically indexed such that vertices close to each another in the graph were also close in their indices. To reduce this dependency, we randomly permuted the ordering of vertices.

For the max graph cut problem, we removed directions on edges to obtain undirected graphs. The set cover problem is reduced to a vertex cover on the directed graph.

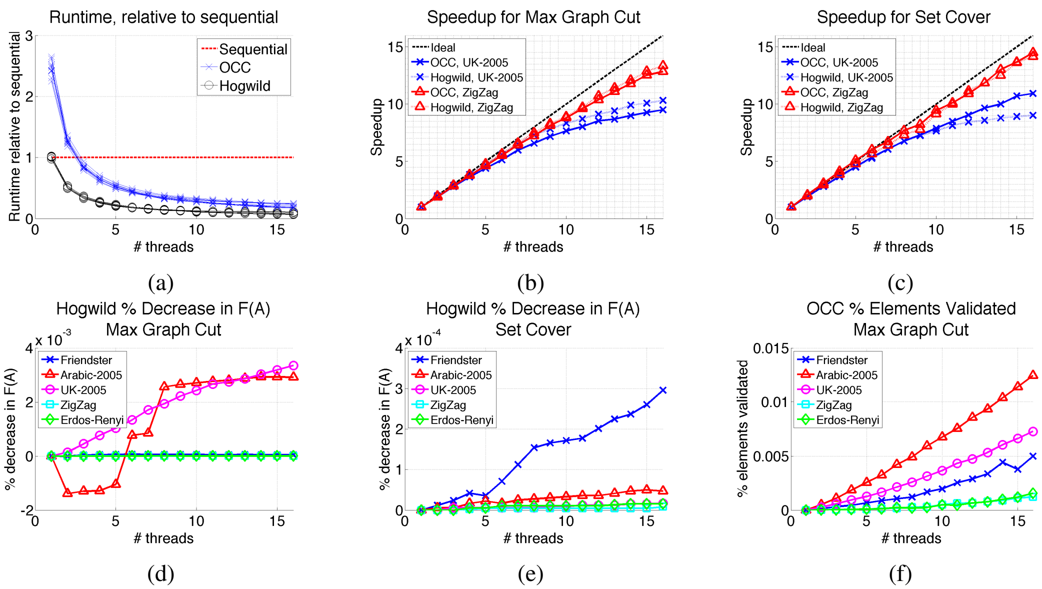

Figure 1: Experimental results. 1a – runtime of the parallel algorithms as a ratio to that of the sequential algorithm. Each curve shows the runtime of a parallel algorithm on a particular graph for a particular function F. 1b, 1c – speedup (ratio of runtime on one thread to that on p threads). 1d, 1e – % difference between objective values of the sequential and hogwild algorithms, i.e. $[F(A_{hogwild})/F(A_{sequential}) - 1] \times 100\%$. 1f – % of elements validated by the OCC algorithm on the max graph cut problem.

Due to space constraints, we only present part of our results in Figure 1, deferring full results to Appendix D. **Runtime, Speedup:** Both parallel algorithms are faster than the sequential algorithm with three or more threads, and show good speedup properties as more threads are added ($\sim$ 10x or more for all graphs and both functions). **Objective value:** The objective value of the hogwild algorithm decreases with the number of threads, but differs from the sequential objective value by less than 0.01%. **Validations:** The OCC algorithm validates more elements as threads are added, but less than 0.015% are validated with 16 threads, which has negligible effect on the runtime / speedup.

8 Discussions

Conclusion: [XP: link back to intro, motivation]; we present two approaches to parallelizing unconstrained submodular maximization, which allows one to choose between speed and tight approximation guarantees.

Future work: constrained maximization, minimization; distributed setting, where communication costs and delays are higher, and function evaluations are challenging.

References

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A Proof of bound for hogwild

We follow the proof outline of [1].

Let OPT be an optimal solution to the problem. Define $O^i := (OPT \cup A^i) \cap B^i$. Note that O^i coincides with A^i and B^i on elements $1, \ldots, i$, and O^i coincides with OPT on elements $i+1, \ldots, n$. Hence,

$$
\begin{aligned}
O^i \backslash i + 1 &\supseteq A^i \\
O^i \cup i + 1 &\subseteq B^i.
\end{aligned}
$$

Lemma A.1. *For every* $1 \leq i \leq n$, $\Delta_+(i) + \Delta_-(i) \geq 0$.

Proof. This is just Lemma II.1 of [1]. □

Lemma A.2. *For every* $1 \leq i \leq n$,

$$
E[F(O^{i-1}) - F(O^i)] \leq \frac{1}{2} E[f(A^i) - f(A^{i-1}) + f(B^i) - f(B^{i-1}) + \rho_i].
$$

Proof. We follow the proof outline of [1]. First, note that it suffices to prove the inequality conditioned on knowing A^{i-1} and j, then applying the law of total expectation. Under this conditioning, we also know B^{i-1}, O^{i-1}, $\Delta_+(i)$, $\Delta_+^{\max}(i)$, $\Delta_-(i)$, $\Delta_-^{\max}(i)$, and ρ_i.

We consider the following 9 cases.

Case 1: $0 < \Delta_+(i) \leq \Delta_+^{\max}(i)$, $0 \leq \Delta_-^{\max}(i)$. Since both $\Delta_+^{\max}(i) > 0$ and $\Delta_-^{\max}(i) > 0$, the probability of including i is just $\Delta_+^{\max}(i)/(\Delta_+^{\max}(i) + \Delta_-^{\max}(i))$, and the probability of excluding i is $\Delta_-^{\max}(i)/(\Delta_+^{\max}(i) + \Delta_-^{\max}(i))$.

$$
\begin{aligned}
E[F(A^i) - F(A^{i-1}) | A^{i-1}, j] &= \frac{\Delta_+^{\max}(i)}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)} (F(A^{i-1} \cup i) - F(A^{i-1})) \\
&= \frac{\Delta_+^{\max}(i)}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)} \Delta_+(i) \\
&\geq \frac{\Delta_+^{\max}(i)}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)} (\Delta_+^{\max}(i) - \rho_i) \\
E[F(B^i) - F(B^{i-1}) | A^{i-1}, j] &= \frac{\Delta_-^{\max}(i)}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)} (F(B^{i-1} \backslash i) - F(B^{i-1})) \\
&= \frac{\Delta_-^{\max}(i)}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)} \Delta_-(i) \\
&\geq \frac{\Delta_-^{\max}(i)}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)} (\Delta_-^{\max}(i) - \rho_i)
\end{aligned}
$$

$$
\begin{aligned}
&E[F(O^{i-1}) - F(O^i)|A^{i-1}, j] \\
=&\frac{\Delta_+^{\max}(i)}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)}(F(O^{i-1}) - F(O^{i-1} \cup i)) \\
&+ \frac{\Delta_-^{\max}(i)}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)}(F(O^{i-1}) - F(O^{i-1} \backslash i)) \\
=&\begin{cases} \frac{\Delta_+^{\max}(i)}{\Delta_+^{\max}(i)+\Delta_-^{\max}(i)}(F(O^{i-1}) - F(O^{i-1} \cup i)) & \text{if } i \notin OPT \\ \frac{\Delta_-^{\max}(i)}{\Delta_+^{\max}(i)+\Delta_-^{\max}(i)}(F(O^{i-1}) - F(O^{i-1} \backslash i)) & \text{if } i \in OPT \end{cases} \\
\leq&\begin{cases} \frac{\Delta_+^{\max}(i)}{\Delta_+^{\max}(i)+\Delta_-^{\max}(i)}(F(B^{i-1} \backslash i) - F(B^{i-1})) & \text{if } i \notin OPT \\ \frac{\Delta_-^{\max}(i)}{\Delta_+^{\max}(i)+\Delta_-^{\max}(i)}(F(A^{i-1} \cup i) - F(A^{i-1})) & \text{if } i \in OPT \end{cases} \\
=&\begin{cases} \frac{\Delta_+^{\max}(i)}{\Delta_+^{\max}(i)+\Delta_-^{\max}(i)}\Delta_-(i) & \text{if } i \notin OPT \\ \frac{\Delta_-^{\max}(i)}{\Delta_+^{\max}(i)+\Delta_-^{\max}(i)}\Delta_+(i) & \text{if } i \in OPT \end{cases} \\
\leq&\begin{cases} \frac{\Delta_+^{\max}(i)}{\Delta_+^{\max}(i)+\Delta_-^{\max}(i)}\Delta_-^{\max}(i) & \text{if } i \notin OPT \\ \frac{\Delta_-^{\max}(i)}{\Delta_+^{\max}(i)+\Delta_-^{\max}(i)}\Delta_+^{\max}(i) & \text{if } i \in OPT \end{cases} \\
=&\frac{\Delta_+^{\max}(i)\Delta_-^{\max}(i)}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)}
\end{aligned}
$$

where the first inequality is due to submodularity: $O^{i-1} \backslash i \supseteq A^{i-1}$ and $O^{i-1} \cup i \subseteq B^{i-1}$.

Putting the above inequalities together:

$$
\begin{aligned}
&E[F(O^{i-1}) - F(O^i)|A^{i-1}, j] - \frac{1}{2}E[f(A^i) - f(A^{i-1}) + f(B^i) - f(B^{i-1}) + \rho_i|A^{i-1}, j] \\
&\leq \frac{1/2}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)}\Big[2\Delta_+^{\max}(i)\Delta_-^{\max}(i) - \Delta_-^{\max}(i)(\Delta_-^{\max}(i) - \rho_i) \\
&\qquad - \Delta_+^{\max}(i)(\Delta_+^{\max}(i) - \rho_i)\Big] - \frac{1}{2}\rho_i \\
&= \frac{1/2}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)}\Big[- (\Delta_+^{\max}(i) - \Delta_-^{\max}(i))^2 + \rho_i(\Delta_+^{\max}(i) + \Delta_-^{\max}(i))\Big] - \frac{1}{2}\rho_i \\
&\leq \frac{\frac{1}{2}\rho_i(\Delta_+^{\max}(i) + \Delta_-^{\max}(i))}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)} - \frac{1}{2}\rho_i \\
&= 0.
\end{aligned}
$$

Case 2: $0 < \Delta_+(i) \leq \Delta_+^{\max}(i)$, $\Delta_-^{\max}(i) < 0$. In this case, the algorithm always choses to include i, so $A^i = A^{i-1} \cup i$, $B^i = B^{i-1}$ and $O^i = O^{i-1} \cup i$:

$$
\begin{aligned}
E[F(A^i) - F(A^{i-1})|A^{i-1}, j] &= F(A^{i-1} \cup i) - F(A^{i-1}) = \Delta_+(i) > 0 \\
E[F(B^i) - F(B^{i-1})|A^{i-1}, j] &= F(B^{i-1}) - F(B^{i-1}) = 0 \\
E[F(O^{i-1}) - F(O^i)|A^{i-1}, j] &= F(O^{i-1}) - F(O^{i-1} \cup i) \\
&\leq \begin{cases} 0 & \text{if } i \in OPT \\ F(B^{i-1} \backslash i) - F(B^{i-1}) & \text{if } i \notin OPT \end{cases} \\
&= \begin{cases} 0 & \text{if } i \in OPT \\ \Delta_-(i) & \text{if } i \notin OPT \end{cases} \\
&\leq 0 \\
&< \frac{1}{2}E[f(A^i) - f(A^{i-1}) + f(B^i) - f(B^{i-1}) + \rho_i|A^{i-1}, j]
\end{aligned}
$$

where the first inequality is due to submodularity: $O^{i-1} \cup i \subseteq B^{i-1}$.

Case 3: $\Delta_+(i) \le 0 < \Delta_+^{\max}(i)$, $0 < \Delta_-(i) < \Delta_-^{\max}(i)$. Analogous to Case 1.

Case 4: $\Delta_+(i) \le 0 < \Delta_+^{\max}(i)$, $\Delta_-(i) \le 0$. This is not possible, by Lemma A.1.

Case 5: $\Delta_+(i) \le \Delta_+^{\max}(i) \le 0$, $0 < \Delta_-(i) \le \Delta_-^{\max}(i)$. Analogous to Case 2.

Case 6: $\Delta_+(i) \le \Delta_+^{\max}(i) \le 0$, $\Delta_-(i) \le 0$. This is not possible, by Lemma A.1.

□

([XP: Note] If we weaken the assumption of $\Delta_+(i) \le \Delta_+^{\max}(i)$ to $\Delta_+(i) \le \Delta_+^{\max}(i) + \epsilon_i$, then in Case 6 above, we can instead bound

$$\begin{aligned} E[F(O^{i-1}) - F(O^i)|A^{i-1}, j] &\le \frac{\Delta_+^{\max}(i)\Delta_-^{\max}(i) + \epsilon \max(\Delta_+^{\max}(i), \Delta_-^{\max})}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)} \\ &\le \frac{\Delta_+^{\max}(i)\Delta_-^{\max}(i) + \epsilon(\Delta_+^{\max}(i) + \Delta_-^{\max})}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)}. \end{aligned}$$

The bound of Lemma A.2 becomes

$$E[F(O^{i-1}) - F(O^i)] \le \frac{1}{2}E[f(A^i) - f(A^{i-1}) + f(B^i) - f(B^{i-1}) + \rho_i + 2\epsilon_i],$$

and the bound of Theorem 6.1 becomes $E[F(A)] \ge \frac{1}{2}F^* - \frac{1}{4}\sum_i E[\rho_i + 2\epsilon_i]$.)

We will now prove Theorem 6.1.

Proof of Theorem 6.1. Summing up the statement of Lemma A.2 for all i gives us a telescoping sum, which reduces to:

$$\begin{aligned} E[F(O^0) - F(O^n)] &\le \frac{1}{2}E[F(A^n) - F(A^0) + F(B^n) - F(B^0)] + \frac{1}{2}\sum_{i=1}^n E[\rho_i] \\ &\le \frac{1}{2}E[F(A^n) + F(B^n)] + \frac{1}{2}\sum_{i=1}^n E[\rho_i]. \end{aligned}$$

Note that $O^0 = OPT$ and $O^n = A^n = B^n$, so $E[F(A^n)] \ge \frac{1}{2}F(OPT) - \frac{1}{4}\sum_i E[\rho_i]$. □

A.1 Example: max graph cut

Let $C^{ji} = \{j+1, \ldots, i-1\}$, $D^i = \{i+1, \ldots, n\}$. Denote $\tilde{A}^j = V \backslash A^j \backslash C^{ji} \backslash D^i = \{1, \ldots, j\} \backslash A^j$ be the elements up to j that are not included in A. Let $w_i(S) = \sum_{j \in S, (i,j) \in E} w(i,j)$. For the max graph cut function, it is easy to see that

$$\begin{aligned} \Delta_+ &\ge -w_i(A^j) - w_i(C^{ji}) + w_i(D^i) + w_i(\tilde{A}^j) \\ \Delta_+^{\max} &= -w_i(A^j) + w_i(C^{ji}) + w_i(D^i) + w_i(\tilde{A}^j) \\ \Delta_- &\ge +w_i(A^j) - w_i(C^{ji}) + w_i(D^i) - w_i(\tilde{A}^j) \\ \Delta_-^{\max} &= +w_i(A^j) + w_i(C^{ji}) + w_i(D^i) - w_i(\tilde{A}^j) \end{aligned}$$

Thus, we can set $\rho_i = 2w_i(C^{ji})$.

Suppose we have bounded delay τ, so $|C^{ji}| \le \tau$. Then $w_i(C^{ji})$ has a hypergeometric distribution with mean $\frac{\deg(i)}{N}\tau$, and $E[\rho_i] = 2\tau\frac{\deg(i)}{N}$. The approximation of the hogwild algorithm is then $E[F(A^n)] \ge \frac{1}{2}F(OPT) - \tau\frac{\#\text{edges}}{2N}$. In sparse graphs, the hogwild algorithm is off by a small additional term, which albeit grows linearly in τ.

A.2 Example: set cover

[XP: For now, consider a toy problem, with (1) disjoint sets, (2) bounded delay, (3) $\lambda \leq 1$.]

Consider the simple set cover function, for $\lambda < 1$:

$$F(A) = \sum_{l=1}^{L} \min(1, |A \cap S_l|) - \lambda|A| = |\{l : A \cap S_l \neq \emptyset\}| - \lambda|A|.$$

We assume that there is some bounded delay τ.

Suppose also the S_l's form a partition, so each element e belongs to exactly one set. Let n_l denote $|S_l|$ the size of S_l. Given any ordering π, let e_l^t be the tth element of S_l in the ordering, i.e. $|\{e' : \pi(e') \leq \pi(e_l^t) \wedge e' \in S_l\}| = t$.

For any $e \in S_l$, we get

$$\begin{aligned}
\Delta_+(e) &= -\lambda + 1\{A^{\iota(e)-1} \cap S_l = \emptyset\} \\
\Delta_+^{\max}(e) &= -\lambda + 1\{\hat{A}_e \cap S_l = \emptyset\} \\
\Delta_-(e) &= +\lambda - 1\{B^{\iota(e)-1} \backslash e \cap S_l = \emptyset\} \\
\Delta_-^{\max}(e) &= +\lambda - 1\{\hat{B}_e \backslash e \cap S_l = \emptyset\}
\end{aligned}$$

Let η be the position of the first element of S_l to be accepted, i.e. $\eta = \min\{t : e_l^t \in A \cap S_l\}$. (For convenience, we set $\eta = n_l$ if $A \cap S_l = \emptyset$.) We first show that η is independent of π: for $\eta < n_l$,

$$\begin{aligned}
P(\eta|\pi) &= \frac{\Delta_+^{\max}(e_l^\eta)}{\Delta_+^{\max}(e_l^\eta) + \Delta_-^{\max}(e_l^\eta)} \prod_{t=1}^{\eta-1} \frac{\Delta_-^{\max}(e_l^t)}{\Delta_+^{\max}(e_l^t) + \Delta_-^{\max}(e_l^t)} \\
&= \frac{1-\lambda}{1-\lambda+\lambda} \prod_{t=1}^{\eta-1} \frac{\lambda}{1-\lambda+\lambda} \\
&= (1-\lambda)\lambda^{\eta-1},
\end{aligned}$$

and $P(\eta = n_l|\pi) = \lambda^{\eta-1}$. [XP: This independence depends on the assumption of disjoint sets, which in turn allows us to decouple the randomness of the algorithm from the randomness of ordering in the below proof.]

Note that, $\Delta_-^{\max}(e) - \Delta_-(e) = 1$ iff $e = e_l^{n_l}$ is the last element of S_l in the ordering, there are no elements accepted up to $\hat{B}_{e_l^{n_l}} \backslash e_l^{n_l}$, and there is some element e' in $\hat{B}_{e_l^{n_l}} \backslash e_l^{n_l}$ that is rejected and not in $B^{\iota(e_l^{n_l})-1}$. Denote by $m_l \leq \min(\tau, n_l - 1)$ the number of elements before $e_l^{n_l}$ that are inconsistent between $\hat{B}_{e_l^{n_l}}$ and $B^{\iota(e_l^{n_l})-1}$. Then $\mathbb{E}[\Delta_-^{\max}(e_l^{n_l}) - \Delta_-(e_l^{n_l})] = P(\Delta_-^{\max}(e_l^{n_l}) \neq \Delta_-(e_l^{n_l}))$ is

$$\lambda^{n_l-1-m_l}(1-\lambda^{m_l}) \quad = \quad \lambda^{n_l-1}(\lambda^{-m_l}-1) \quad \leq \quad \lambda^{n_l-1}(\lambda^{-\min(\tau,n_l-1)}-1) \quad \leq \quad 1-\lambda^\tau.$$

If $\lambda = 1$, $\Delta_+^{\max}(e) \leq 0$, so no elements before $e_l^{n_l}$ will be accepted, and $\Delta_-^{\max}(e_l^{n_l}) = \Delta_-(e_l^{n_l})$.

On the other hand, $\Delta_+^{\max}(e) - \Delta_+(e) = 1$ iff $(A^{\iota(e)-1} \backslash \hat{A}_e) \cap S_l \neq \emptyset$, that is, if an element has been accepted in A but not yet observed in $\hat{A}_e$. Since we assume a bounded delay, only the first τ elements after the first acceptance of an $e \in S_l$ may be affected.

$$\begin{aligned}
&\mathbb{E}\left[\sum_{e \in S_l} \Delta_+^{\max}(e) - \Delta_+(e)\right] \\
&= \mathbb{E}[\#\{e : e \in S_l \wedge e_l^\eta \in A^{\iota(e)-1} \wedge e_l^\eta \notin \hat{A}_e\}] \\
&= \mathbb{E}[\mathbb{E}[\#\{e : e \in S_l \wedge e_l^\eta \in A^{\iota(e)-1} \wedge e_l^\eta \notin \hat{A}_e\} \mid \eta = t, \pi(e_l^t) = k]] \\
&= \sum_{t=1}^{n_l} \sum_{k=t}^{N-n+t} P(\eta = t, \pi(e_l^t) = k) \mathbb{E}[\#\{e : e \in S_l \wedge e_l^\eta \in A^{\iota(e)-1} \wedge e_l^\eta \notin \hat{A}_e\} \mid \eta = t, \pi(e_l^t) = k] \\
&= \sum_{t=1}^{n_l} P(\eta = t) \sum_{k=t}^{N-n+t} P(\pi(e_l^t) = k) \mathbb{E}[\#\{e : e \in S_l \wedge e_l^\eta \in A^{\iota(e)-1} \wedge e_l^\eta \notin \hat{A}_e\} \mid \eta = t, \pi(e_l^t) = k].
\end{aligned}$$

Under the assumption that every ordering π is equally likely, and a bounded delay τ, conditioned on $\eta = t, \pi(e_l^t) = k$, the random variable $\#\{e : e \in S_l \wedge e_l^\eta \in A^{\iota(e)-1} \wedge e_l^\eta \notin \hat{A}_e\}$ has hypergeometric distribution with mean $\frac{n_l - t}{N-k}\tau$. Also, $P(\pi(e_l^t) = k) = \frac{n_l}{N}\binom{n-1}{t-1}\binom{N-n}{k-t}/\binom{N-1}{k-1}$, so the above expression becomes

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{e \in S_l} \Delta_+^{\max}(e) - \Delta_+(e)\right] \\
&= \sum_{t=1}^{n_l} P(\eta = t) \sum_{k=t}^{N-n+t} \frac{n_l}{N} \frac{\binom{n-1}{t-1}\binom{N-n}{k-t}}{\binom{N-1}{k-1}} \frac{n-t}{N-k}\tau \\
&= \frac{n_l}{N}\tau \sum_{t=1}^{n_l} P(\eta = t) \sum_{k=t}^{N-n+t} \frac{\binom{k-1}{t-1}\binom{N-k}{n-t}}{\binom{N-1}{n-1}} \frac{n-t}{N-k} && \text{(symmetry of hypergeometric)} \\
&= \frac{n_l}{N}\tau \sum_{t=1}^{n_l} \frac{P(\eta = t)}{\binom{N-1}{n-1}} \sum_{k=t}^{N-n+t} \binom{k-1}{t-1}\binom{N-k-1}{n-t-1} \\
&= \frac{n_l}{N}\tau \sum_{t=1}^{n_l} \frac{P(\eta = t)}{\binom{N-1}{n-1}} \binom{N-1}{n-1} && \text{(Lemma C.1, } a = N-2, b = n_l - 2, j = 1\text{)} \\
&= \frac{n_l}{N}\tau \sum_{t=1}^{n_l} P(\eta = t) \\
&= \frac{n_l}{N}\tau.
\end{aligned}
$$

Now if we define $\rho_e = \Delta_+^{\max}(e) - \Delta_+(e) + \Delta_-^{\max}(e) - \Delta_-(e)$, we get

$$
\begin{aligned}
\mathbb{E}\left[\sum_e \rho_e\right] &= \mathbb{E}\left[\sum_e \Delta_+^{\max}(e) - \Delta_+(e) + \Delta_-^{\max}(e) - \Delta_-(e)\right] \\
&= \sum_l \mathbb{E}\left[\sum_{e \in S_l} \Delta_+^{\max}(e) - \Delta_+(e)\right] + \mathbb{E}\left[\sum_{e \in S_l} \Delta_-^{\max}(e) - \Delta_-(e)\right] \\
&\le \tau \frac{\sum_l n_l}{N} + L(1 - \lambda^\tau) \\
&= \tau + L(1 - \lambda^\tau).
\end{aligned}
$$

Note that $\mathbb{E}\left[\sum_e \rho_e\right]$ does not depend on N and is linear in τ. Also, if $\tau = 0$ in the sequential case, we get $\mathbb{E}\left[\sum_e \rho_e\right] \le 0$.

B Upper bound on expected number of elements sent for validation

Let N be the number of elements, i.e. the cardinality of the ground set. Let P be the number of processors.

We assume that the total ordering assigns elements to processors in a round robin fashion. Thus, we assume $C^{ji} = \{i - p + 1, \ldots, i - 1\}$ has $p - 1$ elements.

We call element i *dependent* on i' if $\exists A, F(A \cup i) - F(A) \neq F(A \cup i' \cup i) - F(A \cup i')$ or $\exists B, F(B \backslash i) - F(B) \neq F(B \cup i' \backslash i) - F(B \cup i')$, i.e. the result of the transaction on i' will affect the computation of Δ's for i. For example, for the graph cut problem, every vertex is dependent on its neighbors; for the separable sums problem, i is dependent on $\{i' : \exists S_l, i \in S_l, i' \in S_l\}$.

Let n_i be the number of elements that i is dependent on.

Now, we note that if C^{ji} does not contain any elements on which i is dependent, then $\Delta_+^{\text{max}}(i) = \Delta_+(i) = \Delta_+^{\text{min}}(i)$ and $\Delta_-^{\text{max}}(i) = \Delta_-(i) = \Delta_-^{\text{min}}(i)$, so i will not be validated (in either deterministic or probabilistic versions). Conversely, if i is validated, there must be some element $i' \in C^{ji}$ such that i is dependent on i'.

$$
\begin{aligned}
&E(\text{number of validated elements}) \\
=&\sum_i P(i \text{ validated}) \\
\leq&\sum_i P(\exists i' \in C^{ji}, i \text{ depends on } i') \\
=&\sum_i 1 - P(\forall i' \in C^{ji}, i \text{ does not depend on } i') \\
=&\sum_i 1 - \prod_{k=1}^{P-1} \frac{N - k - n_i}{N - k} \\
=&\sum_i 1 - \prod_{k=1}^{P-1} \left(1 - \frac{n_i}{N - k}\right) \\
\leq&\sum_i 1 - \left(1 - \sum_{k=1}^{P-1} \frac{n_i}{N - k}\right) && \text{(Weierstrass inequality)} \\
=&\left(\sum_i n_i\right)\left(\sum_{k=1}^{P-1} \frac{1}{N - k}\right) \\
\leq&\frac{P - 1}{N - P + 1} \sum_i n_i.
\end{aligned}
$$

The key quantity in the above inequality is $\sum_i n_i$. Typically, we expect each element i to depend on a small fraction of the ground set. For example, in the graph cut problem, $\sum_i n_i = 2|E|$ is twice the number of edges. If the graph is sparse with $|E| \approx s|V| \log |V|$, where $0 \leq s \ll 1$ and $P \ll N$, then $\frac{P-1}{N-P+1} \sum_i n_i \approx 2s(P - 1) \log N$, which grows sublinearly with N.

Note that the bound established above is generic to all algorithms that follow the basic transactional model we proposed (round-robin optimistic concurrency control), and is not specific to F or even submodular maximization. Thus, while our bounds provide a fundamental limit, additional knowledge of F can lead to better analyses on the algorithm's concurrency.

B.1 Tighter general bound?

Define $\rho_i = \max_{S \subseteq V} \{[F(S \cup i) - F(S)] - [F(S \cup C^{ji} \cup i) - F(S \cup C^{ji})]\} \leq F(i) - F(V) + F(V \backslash i)$

[XP: Is there theory along these lines?]

Then, we can bound

$$\Delta_+^{\min} \le \Delta_+^{\max} \le \Delta_+^{\min} + \rho_i \qquad \text{(choosing } S = A^j\text{)}$$
$$\Delta_-^{\min} \le \Delta_-^{\max} \le \Delta_-^{\min} + \rho_i \qquad \text{(choosing } S = A^j \cup D^i\text{)}$$

Thus,

$$\begin{aligned}
&E(\text{number of validated elements}) \\
&= \sum_i P(i \text{ validated}) \\
&= \sum_i P\left(\frac{\Delta_+^{\min}}{\Delta_+^{\min} + \Delta_-^{\max}} \le u_i \le \frac{\Delta_+^{\max}}{\Delta_+^{\max} + \Delta_-^{\min}}\right) \\
&= \sum_i \frac{\Delta_+^{\max}}{\Delta_+^{\max} + \Delta_-^{\min}} - \frac{\Delta_+^{\min}}{\Delta_+^{\min} + \Delta_-^{\max}} \\
&\le \sum_i \frac{\Delta_+^{\min} + \rho_i}{\Delta_+^{\min} + \rho_i + \Delta_-^{\min}} - \frac{\Delta_+^{\min}}{\Delta_+^{\min} + \rho_i + \Delta_-^{\min}} \\
&= \sum_i \frac{\rho_i}{\Delta_+^{\min} + \rho_i + \Delta_-^{\min}}
\end{aligned}$$

B.2 Upper bound for max graph cut

Denote $\tilde{A}^j = V \backslash A^j \backslash C^{ji} \backslash D^i = \{1, \ldots, j\} \backslash A^j$ be the elements up to j that are not included in A. Let $w_i(S) = \sum_{j \in S, (i,j) \in E} w(i,j)$. For the max graph cut function, it is easy to see that

$$\begin{aligned}
\Delta_+^{\min} &= \max(0, -w_i(A^j) - w_i(C^{ji}) + w_i(D^i) + w_i(\tilde{A}^j)) \\
\Delta_+^{\max} &= \max(0, -w_i(A^j) + w_i(C^{ji}) + w_i(D^i) + w_i(\tilde{A}^j)) \\
\Delta_-^{\min} &= \max(0, +w_i(A^j) - w_i(C^{ji}) + w_i(D^i) - w_i(\tilde{A}^j)) \\
\Delta_-^{\max} &= \max(0, +w_i(A^j) + w_i(C^{ji}) + w_i(D^i) - w_i(\tilde{A}^j))
\end{aligned}$$

Consider the following cases.

- $\Delta_+^{\max} = 0$. Then $\Delta_+^{\min} = 0$ and also

  $$w_i(A^j) > w_i(C^{ji}) + w_i(D^i) + w_i(\tilde{A}^j) \implies w_i(A^j) + w_i(D^i) > w_i(C^{ji}) + w_i(\tilde{A}^j)$$

  so $\Delta_-^{\min} > 0$ and $\Delta_-^{\max} > 0$. Thus $\frac{\Delta_+^{\max}}{\Delta_+^{\max} + \Delta_-^{\min}} - \frac{\Delta_+^{\min}}{\Delta_+^{\min} + \Delta_-^{\max}} = 0 - 0 = 0$.

- $\Delta_-^{\max} = 0$. Then $\Delta_-^{\min} = 0$ and also

  $$w_i(\tilde{A}^j) > w_i(C^{ji}) + w_i(D^i) + w_i(A^j) \implies w_i(\tilde{A}^j) + w_i(D^i) > w_i(C^{ji}) + w_i(A^j)$$

  so $\Delta_+^{\min} > 0$ and $\Delta_+^{\max} > 0$. Thus $\frac{\Delta_+^{\max}}{\Delta_+^{\max} + \Delta_-^{\min}} - \frac{\Delta_+^{\min}}{\Delta_+^{\min} + \Delta_-^{\max}} = 1 - 1 = 0$.

- $\Delta_+^{\max} > 0$ and $\Delta_-^{\max} > 0$. Then,

$$\begin{aligned}
&\frac{\Delta_+^{\max}}{\Delta_+^{\max} + \Delta_-^{\min}} - \frac{\Delta_+^{\min}}{\Delta_+^{\min} + \Delta_-^{\max}} \\
=& \frac{-w_i(A^j) + w_i(C^{ji}) + w_i(D^i) + w_i(\tilde{A}^j)}{-w_i(A^j) + w_i(C^{ji}) + w_i(D^i) + w_i(\tilde{A}^j) + \max(0, +w_i(A^j) - w_i(C^{ji}) + w_i(D^i) - w_i(\tilde{A}^j))} \\
&- \frac{\max(0, -w_i(A^j) - w_i(C^{ji}) + w_i(D^i) + w_i(\tilde{A}^j))}{\max(0, -w_i(A^j) - w_i(C^{ji}) + w_i(D^i) + w_i(\tilde{A}^j)) + w_i(A^j) + w_i(C^{ji}) + w_i(D^i) - w_i(\tilde{A}^j)} \\
=& \min\left(1, \frac{-w_i(A^j) + w_i(C^{ji}) + w_i(D^i) + w_i(\tilde{A}^j)}{2w_i(D^i)}\right) \\
&- \max\left(0, \frac{-w_i(A^j) - w_i(C^{ji}) + w_i(D^i) + w_i(\tilde{A}^j)}{2w_i(D^i)}\right) \\
=& \min\left(1, \frac{w_i(C^{ji})}{w_i(D^i)}\right)
\end{aligned}$$

Thus,

$$E(\text{\# of validated elements}) = \sum_i \frac{\Delta_+^{\max}}{\Delta_+^{\max} + \Delta_-^{\min}} - \frac{\Delta_+^{\min}}{\Delta_+^{\min} + \Delta_-^{\max}} \leq \sum_i \min\left(1, \frac{w_i(C^{ji})}{w_i(D^i)}\right)$$

[XP: Not sure how to sum this over i.]

$$\sum_\pi \sum_i \min(1, w_i(C)/w(D^i)) \leq E(\sum_i w_i(C)) = c * \sum_i deg(i)/n$$

B.3 Upper bound for set cover

We make the same assumptions as before in the hogwild analysis, i.e. the sets S_l form a partition of V, there is a bounded delay τ.

Observe that for any $e \in S_l$, $\Delta_-^{\min}(e) \neq \Delta_-^{\max}(e)$ if $\hat{B}_e \backslash e \cap S_l \neq \emptyset$ and $\tilde{B}_e \backslash e \cap S_l = \emptyset$.

This is only possible if $e_l^{n_l} \notin \tilde{B}_e$ and $\tilde{B}_e \supset \hat{A}_e \cap S_l = \emptyset$, that is $\pi(e) \geq \pi(e_l^{n_l}) - \tau$ and $\forall e' \in S_l, (\pi(e') < \pi(e_l^{n_l}) - \tau) \implies (e' \notin A)$. The latter condition is achieved with probability $\lambda^{n_l - m_l}$, where $m_l = \#\{e' : \pi(e') \geq \pi(e_l^{n_l}) - \tau\}$. Thus,

$$\begin{aligned}
\mathbb{E}\left[\#\{e : \Delta_-^{\min}(e) \neq \Delta_-^{\max}(e)\}\right] &= \mathbb{E}[m_l\, 1(\forall e' \in S_l, (\pi(e') < \pi(e_l^{n_l}) - \tau) \implies (e' \notin A))] \\
&= \mathbb{E}[\mathbb{E}[m_l\, 1(\forall e' \in S_l, (\pi(e') < \pi(e_l^{n_l}) - \tau) \implies (e' \notin A))|u_{1:N}]] \\
&= \mathbb{E}[m_l\, \mathbb{E}[1(\forall e' \in S_l, (\pi(e') < \pi(e_l^{n_l}) - \tau) \implies (e' \notin A))|u_{1:N}]] \\
&= \mathbb{E}[m_l \lambda^{n_l - m_l}] \\
&\leq \lambda^{(n_l - \tau)_+} \mathbb{E}[m_l] \\
&= \lambda^{(n_l - \tau)_+} \mathbb{E}[\mathbb{E}[m_l|\pi(e_l^{n_l}) = k]] \\
&= \lambda^{(n_l - \tau)_+} \sum_{k=n_l}^{N} P(\pi(e_l^{n_l}) = k) \mathbb{E}[m_l|\pi(e_l^{n_l}) = k]].
\end{aligned}$$

Conditioned on $\pi(e_l^{n_l}) = k$, m_l is a hypergeometric random variable with mean $\frac{n_l-1}{k-1}\tau$. Also $P(\pi(e_l^{n_l}) = k) = \frac{n_l}{N}\binom{n_l-1}{0}\binom{N-n_l}{N-k}/\binom{N-1}{N-k}$. The above expression is therefore

$$
\begin{aligned}
&\mathbb{E}\left[\#\{e : \Delta_-^{\min}(e) \neq \Delta_-^{\max}(e)\}\right] \\
&= \lambda^{(n_l-\tau)_+} \sum_{k=n_l}^{N} \frac{n_l}{N} \frac{\binom{n_l-1}{0}\binom{N-n_l}{N-k}}{\binom{N-1}{N-k}} \frac{n_l-1}{k-1}\tau \\
&= \lambda^{(n_l-\tau)_+} \frac{n_l}{N}\tau \sum_{k=n_l}^{N} \frac{\binom{N-k}{0}\binom{k-1}{n_l-1}}{\binom{N-1}{n_l-1}} \frac{n_l-1}{k-1} && \text{(symmetry of hypergeometric)} \\
&= \lambda^{(n_l-\tau)_+} \frac{n_l}{N} \frac{\tau}{\binom{N-1}{n_l-1}} \sum_{k=n_l}^{N} \binom{N-k}{0}\binom{k-2}{n_l-2} \\
&= \lambda^{(n_l-\tau)_+} \frac{n_l}{N} \frac{\tau}{\binom{N-1}{n_l-1}} \binom{N-1}{n_l-1} && \text{(Lemma C.1, } a = N-2, b = n_l-2, j = 2, t = n_l\text{)} \\
&= \lambda^{(n_l-\tau)_+} \frac{n_l}{N}\tau.
\end{aligned}
$$

Now we consider any element $e \in S_l$ with $\pi(e) < \pi(e_l^{n_l}) - \tau$ that is validated. (Note that $e_l^{n_l} \in \hat{B}_e$ and $\tilde{B}_e$, so $\Delta_-^{\min}(e) = \Delta_-^{\max}(e) = \lambda$.) It must be the case that $\hat{A}_e \cap S_l = \emptyset$, for otherwise $\Delta_+^{\min}(e) = \Delta_+^{\max}(e) = -\lambda$ and we do not need to validate. This implies that $\Delta_+^{\max}(e) = 1 - \lambda \geq u_i$. At validation, if $A^{\iota(e)-1} \cap S_l = \emptyset$, we accept e into A. Otherwise, $A^{\iota(e)-1} \cap S_l \neq \emptyset$, which implies that some other element $e' \in S_l$ has been accepted. Thus, we conclude that every element $e \in S_l$ that is validated must be within τ of the first accepted element $e_l^{\eta}\ inS_l$. The expected number of such elements is exactly as we computed in the hogwild analysis: $\frac{n_l}{N}\tau$.

Hence, the expected number of elements that we need to validate is upper bounded as

$$
\begin{aligned}
\mathbb{E}[\#\text{validated}] &\leq \sum_l (1 + \lambda^{(n_l-\tau)_+})\frac{n_l}{N}\tau \\
&\leq \sum_l 2\frac{n_l}{N}\tau \\
&= 2\tau.
\end{aligned}
$$

C Lemma

Lemma C.1. $\sum_{k=t}^{a-b+t} \binom{k-j}{t-j}\binom{a-k+j}{b-t+j} = \binom{a+1}{b+1}$.

Proof.

$$
\begin{aligned}
&\sum_{k=t}^{a-b+t} \binom{k-j}{t-j}\binom{a-k+j}{b-t+j} \\
&= \sum_{k'=0}^{a-b} \binom{k'+t-j}{t-j}\binom{a-k'-t+j}{b-t+j} \\
&= \sum_{k'=0}^{a-b} \binom{k'+t-j}{k'}\binom{a-k'-t+j}{a-b-k'} && \text{(symmetry of binomial coeff.)} \\
&= (-1)^{a-b} \sum_{k'=0}^{a-b} \binom{-t+j-1}{k'}\binom{-b+t-j-1}{a-b-k'} && \text{(upper negation)} \\
&= (-1)^{a-b} \binom{-b-2}{a-b} && \text{(Chu-Vandermonde's identity)} \\
&= \binom{a+1}{a-b} && \text{(upper negation)} \\
&= \binom{a+1}{b+1} && \text{(symmetry of binomial coeff.)}
\end{aligned}
$$

□

D Full experiment results

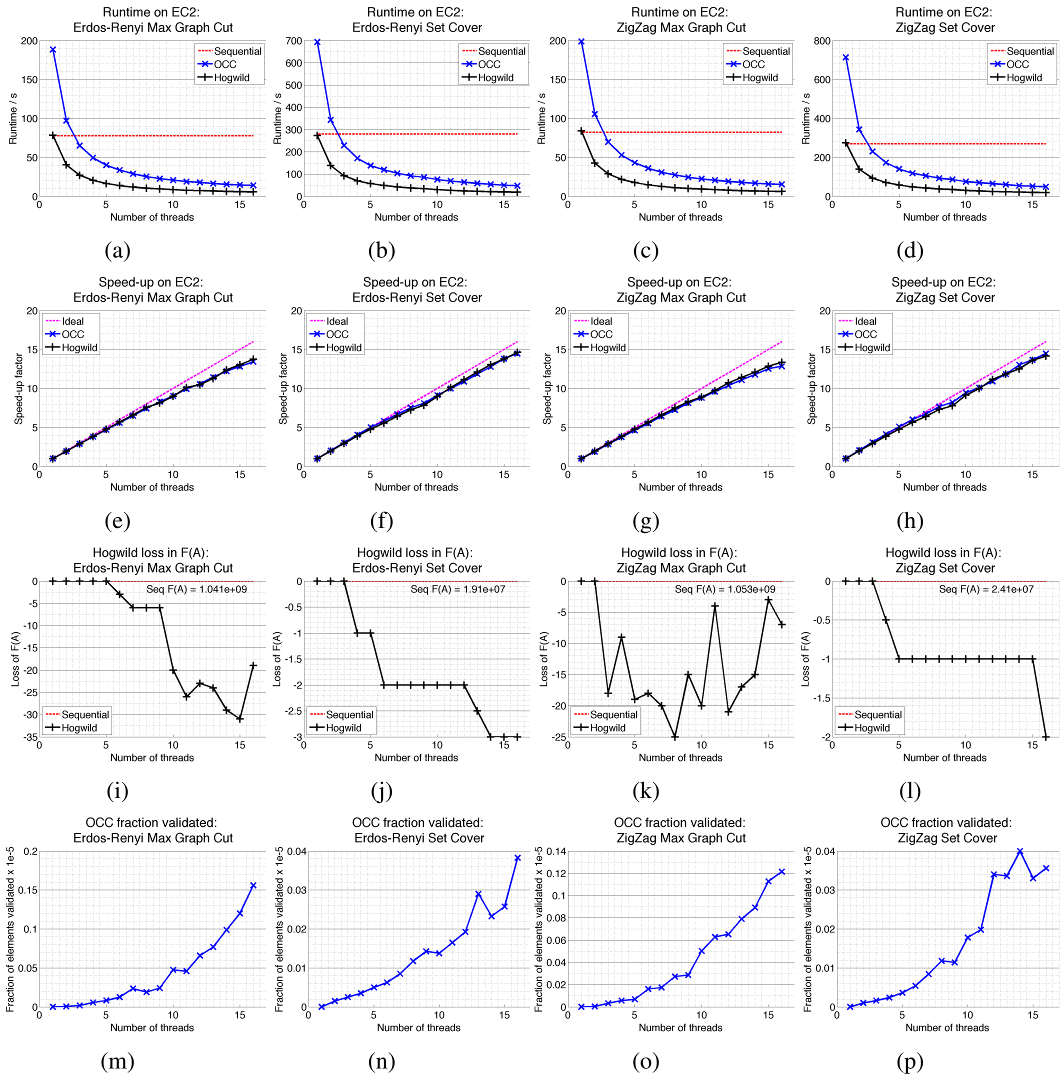
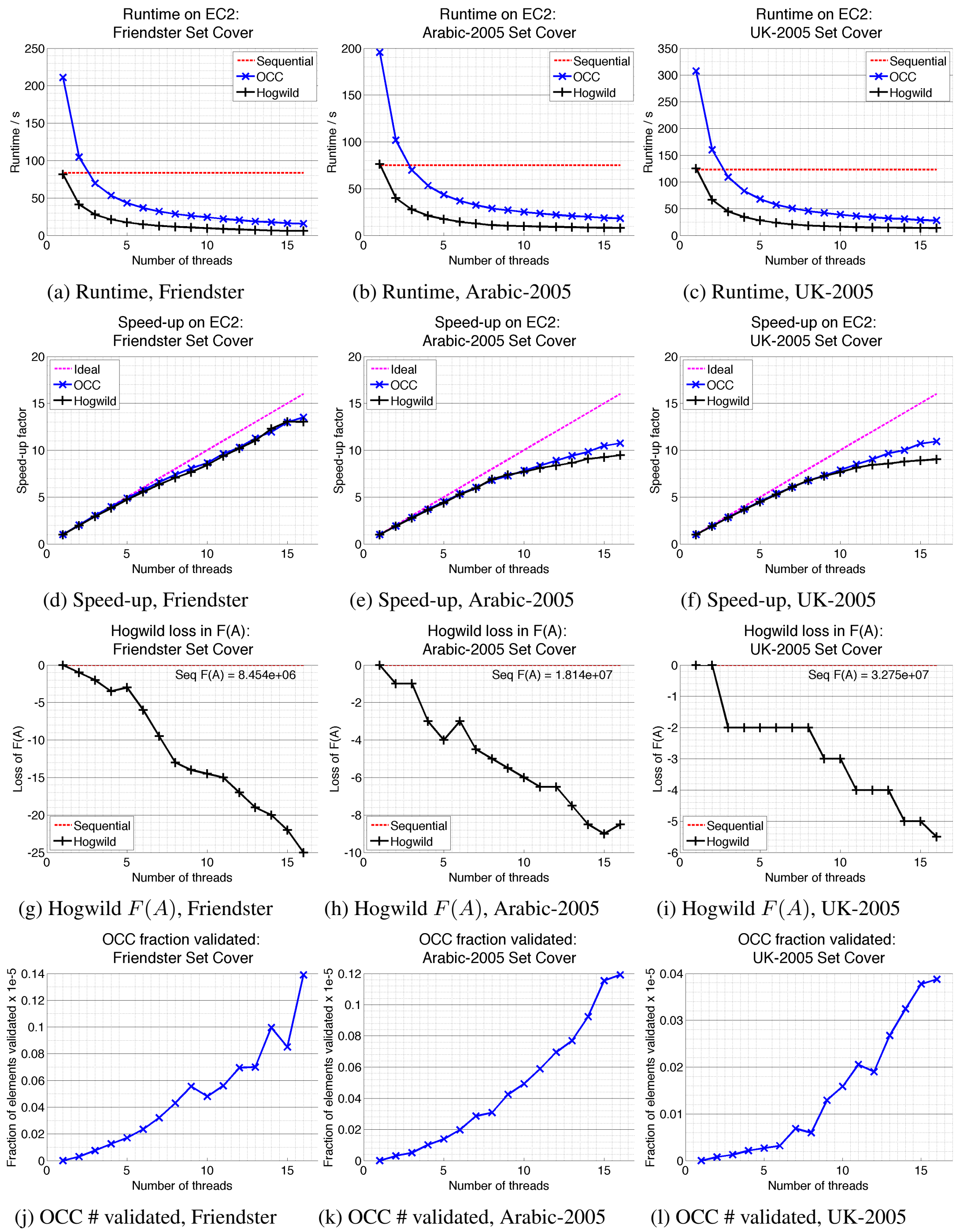


Figure 2: Experimental results on Erdos-Renyi and ZigZag synthetic graphs.



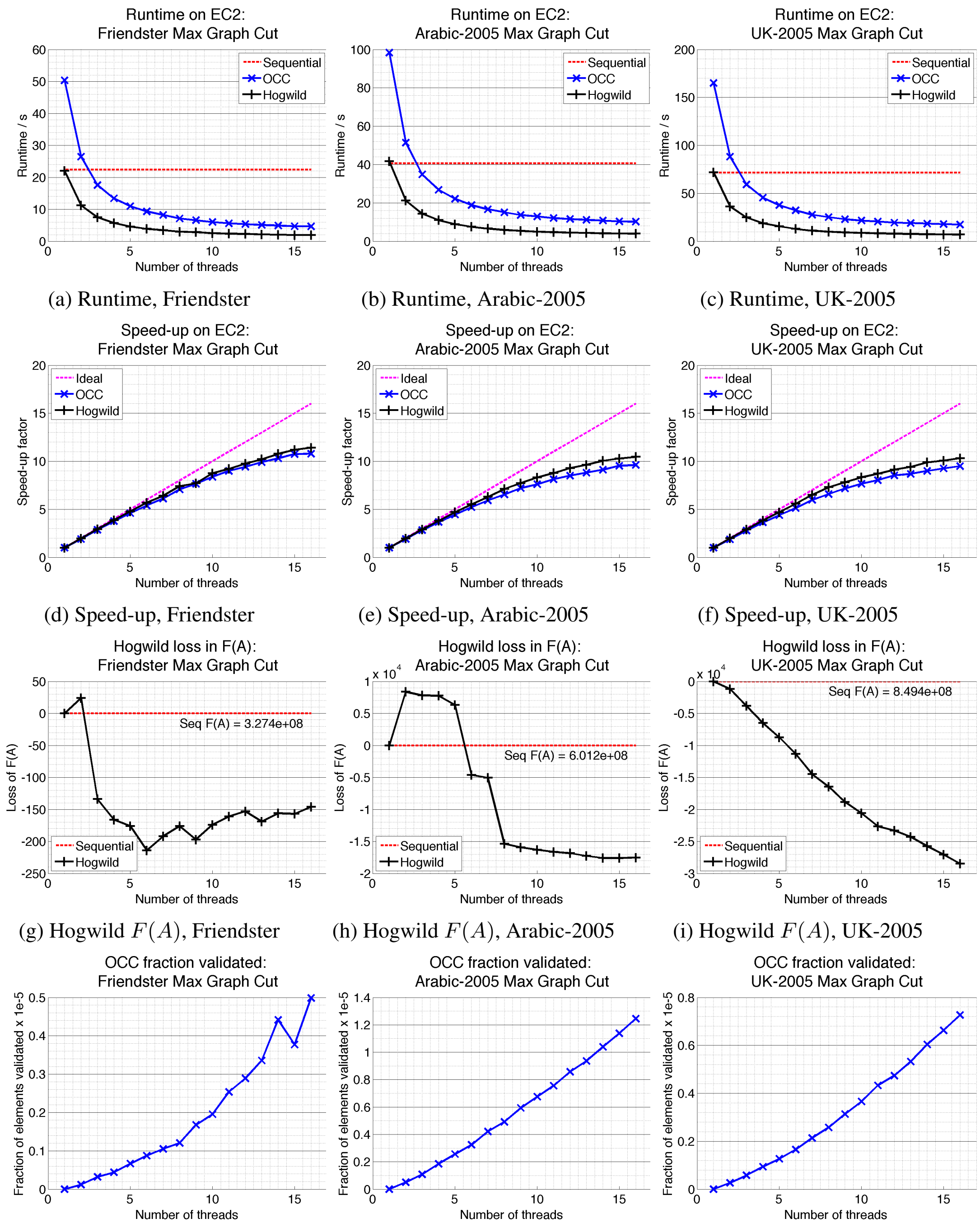
(a) Runtime, Friendster (b) Runtime, Arabic-2005 (c) Runtime, UK-2005

(d) Speed-up, Friendster (e) Speed-up, Arabic-2005 (f) Speed-up, UK-2005

(g) Hogwild $F(A)$, Friendster (h) Hogwild $F(A)$, Arabic-2005 (i) Hogwild $F(A)$, UK-2005

(j) OCC # validated, Friendster (k) OCC # validated, Arabic-2005 (l) OCC # validated, UK-2005

Figure 3: Set cover on 3 real graphs.



(a) Runtime, Friendster (b) Runtime, Arabic-2005 (c) Runtime, UK-2005

(d) Speed-up, Friendster (e) Speed-up, Arabic-2005 (f) Speed-up, UK-2005

(g) Hogwild $F(A)$, Friendster (h) Hogwild $F(A)$, Arabic-2005 (i) Hogwild $F(A)$, UK-2005

(j) OCC # validated, Friendster (k) OCC # validated, Arabic-2005 (l) OCC # validated, UK-2005

Figure 4: Max graph cut on 3 real graphs.