

# Parallel Double Greedy Submodular Maximization

Anonymous Author(s)

Affiliation

Address

email

## Abstract

Many machine learning problems can be formulated as maximization of submodular functions. Recently, [1] achieved a tight  $1/2$ -approximation for unconstrained submodular maximization using a double greedy algorithm. Unfortunately, the double greedy algorithm was developed and analyzed in the serial setting limiting our ability to leverage parallel hardware. In this work we propose and analyze two parallel extensions to the [1] double greedy algorithm. The first, *coordination-free* approach emphasizes speed at the cost of a weaker approximation guarantee. The second, *concurrency control* approach guarantees the same tight  $1/2$ -approximation, at the cost of additional coordination and reduced parallelism. We bound both the weaker approximation factor and the reduction in parallelism. We implement and evaluate both algorithms on multi-core hardware and billion edge graphs demonstrating both the scalability and tradeoffs of each approach.

## 1 Introduction

Motivate parallel machine learning from big data setting

Briefly mention two approaches (coordination-free, concurrency control)

In this paper, we focus on unconstrained non-monotone submodular maximization.. and say briefly why this problem is important

Contributions of the paper

1. We propose two parallel algorithms for unconstrained non-monotone submodular maximization, which allows one to choose between speed and stronger approximation guarantees.
2. We prove the serial equivalence of OCC to the sequential algorithm; we analytically bound the amount of coordination of the OCC algorithm for two example problems.
3. We provide approximation guarantees for hogwild; we analytically bound the expected loss in objective value for two example problems.
4. We demonstrate empirically using two synthetic and three real datasets that our parallel algorithms perform well in terms of both speed and objective values.

The bidirectional greedy algorithm [1] gives an approximation of  $E[F(A)] = 1/2f(OPT)$ , where  $A$  is the algorithm output, and  $OPT$  is an optimal solution.

The hogwild algorithm can give an approximation of  $E[F(A)] = \frac{1}{2}F(OPT) - \frac{1}{4}\sum_i E[\rho_i]$ , where  $\rho_i$  is the maximum difference in the marginal gain that may result from not knowing the full information when deciding whether to include or exclude element  $i$ .

The OCC algorithm [XP: for the lack of a better name] guarantees an outcome that is equivalent to a sequential run of the bidirectional greedy algorithm. Theoretical properties of the bidirectional

greedy algorithm immediately translates to the OCC algorithm – in particular, the OCC algorithm gives the same approximation factor of  $1/2$ . In contrast to the hogwild approach, OCC introduces more coordination and thus provides less concurrency.

## 2 Submodular maximization

What is submodular functions?

Why is maximization an important problem?

Describe the bidirectional greedy algorithm: The sequential bidirectional greedy [1] algorithm monotonically grows  $A^i$  and shrinks  $B^i$ .

## 3 Approaches for parallel learning

Two approaches that allow us to trade off speed with approximation guarantees.

### 3.1 Coordination free

Simply run everything in parallel. Optimized for speed, but does not necessarily provide the correct answer. Requires work to prove correctness.

### 3.2 Concurrency control

Ensures ‘serial equivalence’ – the outcome of the parallel algorithm is equivalent to some execution of the sequential algorithm. Locally, threads take actions that are guaranteed to be safe (i.e. preserves serial equivalence), and forces additional coordination only when they are unable to execute their action safely. Designed for correctness, but requires coordination that compromises speed. Work is only required to demonstrate that coordination is limited.

## 4 Hogwild Bidirectional Greedy Algorithm

[XP: Provide more intuition for what the OCC algorithm is doing.]

Algorithm 2<sup>1</sup> is the hogwild parallel bidirectional greedy unconstrained submodular maximization algorithm. We associate with each element  $e$  a time  $T_e$  at which Algorithm 2 line 8 is executed, and order the elements according to the times  $T_e$ . Let  $\iota(e)$  be the position of  $e$  in this ordering. This total ordering on elements also allows us to define monotonically non-decreasing sets  $A^i = \{e' : e' \in A, \iota(e') < i\}$ , and monotonically non-increasing sets  $B^i = A^i \cup \{e' : \iota(e') \geq i\}$ .

Note that in Algorithm 2, lines 6 and 7 may be executed in parallel with lines 10 and 11. Hence,  $\Delta_+^{\max}(e)$  and  $\Delta_-^{\max}(e)$  (lines 6 and 7) may be computed with different values of  $\hat{A}(e')$ . We denote by  $\hat{A}_e$  and  $\hat{B}_e$  respectively the vectors of  $\hat{A}$  and  $\hat{B}$  that are used in the computation of  $\Delta_+^{\max}(e)$  and  $\Delta_-^{\max}(e)$ . It immediately follows that

$$\begin{aligned}\Delta_+(e) &= F(A^{\iota(e)-1} \cup i) - F(A^{\iota(e)-1}), & \Delta_-(e) &= F(B^{\iota(e)-1} \setminus e) - F(B^{\iota(e)-1}) \\ \Delta_+^{\max}(e) &= F(\hat{A}_e \cup e) - F(\hat{A}_e), & \Delta_-^{\max}(e) &= F(\hat{B}_e \setminus e) - F(\hat{B}_e).\end{aligned}$$

**Lemma 4.1.** For any  $e \in V$ ,  $\hat{A}_e \subseteq A^{\iota(e)-1}$ ,  $\hat{B}_e \supseteq B^{\iota(e)-1}$ .

*Proof.* Consider any element  $e' \in V$ . If  $e' \in \hat{A}_e$ , it must be the case that the algorithm set  $\hat{A}(e')$  to 1 (line 10) before  $T_e$ , which implies  $\iota(e') < \iota(e)$ , and hence  $e' \in A^{\iota(e)-1}$ . So  $\hat{A}_e \subseteq A^{\iota(e)-1}$ . Similarly, if  $e' \notin \hat{B}_e$ , then the algorithm set  $\hat{B}(e')$  to 0 (line 11) before  $T_e$ , so  $\iota(e') < \iota(e)$ . Also,

<sup>1</sup>We present only the parallelized probabilistic versions of [1]. Both parallel algorithms can be easily extended to the deterministic version of [1]; the hogwild algorithm can also be extended to the multilinear version of [1].

**Algorithm 1: Serial submodular maximization**


---

```

1  $A^0 = \emptyset, B^0 = V$ 
2 for  $i = 1$  to  $n$  do
3    $\Delta_+(i) = F(A^{i-1} \cup i) - F(A^{i-1})$ 
4    $\Delta_-(i) = F(B^{i-1} \setminus i) - F(B^{i-1})$ 
5   Draw  $u_i \sim \text{Unif}(0, 1)$ 
6   if  $u_i < \frac{[\Delta_+(i)]_+}{[\Delta_+(i)]_+ + [\Delta_-(i)]_+}$  then
7      $A^i := A^{i-1} \cup i; B^i := B^{i-1}$ 
8   else
9      $A^i := A^{i-1}; B^i := B^{i-1} \setminus i$ 

```

---

**Algorithm 2: Hogwild bidirectional greedy**


---

```

1 for  $e \in V$  do  $\hat{A}(e) = 0, \hat{B}(e) = 1$ 
2
3 for  $p \in \{1, \dots, P\}$  do in parallel
4   while  $\exists$  element to process do
5      $e = \text{next element to process}$ 
6      $\Delta_+^{\max}(e) = F(\hat{A} \cup e) - F(\hat{A})$ 
7      $\Delta_-^{\max}(e) = F(\hat{B} \setminus e) - F(\hat{B})$ 
8     Draw  $u_e \sim \text{Unif}(0, 1)$ 
9     if  $u_e < \frac{[\Delta_+^{\max}(e)]_+}{[\Delta_+^{\max}(e)]_+ + [\Delta_-^{\max}(e)]_+}$  then
10        $\hat{A}(e) \leftarrow 1$ 
11     else  $\hat{B}(e) \leftarrow 0$ 

```

---

**Algorithm 3: Hogwild for separable sums**


---

```

1 for  $e \in V$  do  $\hat{A}(e) = 0$ 
2
3 for  $l = 1, \dots, L$  do  $\hat{\alpha}_l = 0, \hat{\beta}_l = \sum_{e \in S_l} w_l(e)$ 
4
5 for  $p \in \{1, \dots, P\}$  do in parallel
6   while  $\exists$  element to process do
7      $e = \text{next element to process}$ 
8      $\Delta_+^{\max}(e) = -\lambda v(e) + \sum_{S_l \ni e} g(\hat{\alpha}_l + w_l(e)) - g(\hat{\alpha}_l)$ 
9      $\Delta_-^{\max}(e) = +\lambda v(e) + \sum_{S_l \ni e} g(\hat{\beta}_l - w_l(e)) - g(\hat{\beta}_l)$ 
10    Draw  $u_e \sim \text{Unif}(0, 1)$ 
11    if  $u_e < \frac{[\Delta_+^{\max}(e)]_+}{[\Delta_+^{\max}(e)]_+ + [\Delta_-^{\max}(e)]_+}$  then
12       $\hat{A}(e) \leftarrow 1$ 
13    for  $l : e \in S_l$  do  $\hat{\alpha}_l \leftarrow \hat{\alpha}_l + w_l(e)$ 
14    else for  $l : e \in S_l$  do  $\hat{\beta}_l \leftarrow \hat{\beta}_l - w_l(e)$ 

```

---

**Algorithm 4: OCC bidirectional greedy**


---

```

1 for  $e \in V$  do  $\hat{A}(e) = \tilde{A}(e) = 0, \hat{B}(e) = \tilde{B}(e) = 1$ 
2
3 for  $i = 1, \dots, |V|$  do  $\text{result}(i) = 0$ 
4
5 for  $i = 1, \dots, |V|$  do  $\text{processed}(i) = \text{false}$ 
6
7  $\iota = 0$ 
8 for  $p \in \{1, \dots, P\}$  do in parallel
9   while  $\exists$  element to process do
10     $e = \text{next element to process}$ 
11     $\tilde{A}(e) \leftarrow 1$ 
12     $\tilde{B}(e) \leftarrow 0$ 
13     $i = \iota; \iota \leftarrow \iota + 1$ 
14     $\Delta_+^{\min}(e) = F(\tilde{A} \cup e) - F(\tilde{A})$ 
15     $\Delta_+^{\max}(e) = F(\hat{A} \cup e) - F(\hat{A})$ 
16     $\Delta_-^{\min}(e) = F(\tilde{B} \setminus e) - F(\tilde{B})$ 
17     $\Delta_-^{\max}(e) = F(\hat{B} \setminus e) - F(\hat{B})$ 
18    Draw  $u_e \sim \text{Unif}(0, 1)$ 
19    if  $u_e < \frac{[\Delta_+^{\min}(e)]_+}{[\Delta_+^{\min}(e)]_+ + [\Delta_-^{\max}(e)]_+}$  then
20       $\text{result}(i) \leftarrow 1$ 
21    else if  $u_e > \frac{[\Delta_+^{\max}(e)]_+}{[\Delta_+^{\max}(e)]_+ + [\Delta_-^{\min}(e)]_+}$  then
22       $\text{result}(i) \leftarrow -1$ 
23    wait until  $\forall j < i, \text{result}(j) \neq 0$ 
24    if  $\text{result}(i) = 0$  then  $\text{validate}(p, e, i)$ 
25
26    if  $\text{result}(i) = 1$  then
27       $\hat{A}(e) \leftarrow 1$ 
28       $\tilde{B}(e) \leftarrow 1$ 
29    else
30       $\tilde{A}(e) \leftarrow 0$ 
31       $\hat{B}(e) \leftarrow 0$ 
32     $\text{processed}(i) = \text{true}$ 

```

---

**Algorithm 5: validate( $p, e, i$ )**


---

```

1 wait until  $\forall j < i, \text{processed}(j) = \text{true}$ 
2  $\Delta_+^{\text{exact}}(e) = F(\hat{A} \cup e) - F(\hat{A})$ 
3  $\Delta_-^{\text{exact}}(e) = F(\hat{B} \setminus e) - F(\hat{B})$ 
4 if  $u_e < \frac{[\Delta_+^{\text{exact}}(e)]_+}{[\Delta_+^{\text{exact}}(e)]_+ + [\Delta_-^{\text{exact}}(e)]_+}$  then  $\text{result}(i) \leftarrow 1$ 
5
6 else  $\text{result}(i) \leftarrow -1$ 
7

```

---

$e' \notin A$  because the execution of line 11 excludes the execution of line 10. Therefore,  $e' \notin A^{\iota(e)-1}$ , and  $e' \notin B^{\iota(e)-1}$ . So  $\hat{B}_e \subseteq B^{\iota(e)-1}$ .  $\square$

**Corollary 4.2.** *Submodularity of  $F$  implies  $\Delta_+(e) \leq \Delta_+^{\max}(e)$ , and  $\Delta_-(e) \leq \Delta_-^{\max}(e)$ .*

#### 4.1 Hogwild for separable sums $F$

For some functions  $F$ , we can maintain sketches / statistics to aid the computation of  $\Delta_+^{\max}$ ,  $\Delta_-^{\max}$ , and obtain the bounds given in Corollary 4.2. In particular, we consider functions of the form  $F(X) = \sum_{l=1}^L g(\sum_{i \in X \cup S_l} w_l(i)) - \lambda \sum_{i \in X} v(i)$ , where  $S_l \subseteq V$  are (possibly overlapping) groups of elements in the ground set,  $g$  is a non-decreasing concave scalar function, and  $w_l(i)$  and  $v(i)$  are non-negative scalar weights. It is easy to see that  $F(X \cup e) - F(X) = \sum_{l: e \in S_l} [g(w_l(e) + \sum_{i \in X \cup S_l} w_l(i)) - g(\sum_{i \in X \cup S_l} w_l(i))] - \lambda v(e)$ . Define

$$\begin{aligned} \hat{\alpha}_l &= \sum_{j \in \hat{A} \cup S_l} w_l(j), & \hat{\alpha}_{l,e} &= \sum_{j \in \hat{A}_e \cup S_l} w_l(j), & \alpha_l^{\iota(e)-1} &= \sum_{j \in A^{\iota(e)-1} \cup S_l} w_l(j). \\ \hat{\beta}_l &= \sum_{j \in \hat{B} \cup S_l} w_l(j), & \hat{\beta}_{l,e} &= \sum_{j \in \hat{B}_e \cup S_l} w_l(j), & \beta_l^{\iota(e)-1} &= \sum_{j \in B^{\iota(e)-1} \cup S_l} w_l(j). \end{aligned}$$

Algorithm 3 updates  $\hat{\alpha}_l$  and  $\hat{\beta}_l$ , and computes  $\Delta_+^{\max}(e)$  and  $\Delta_-^{\max}(e)$  using  $\hat{\alpha}_{l,e}$  and  $\hat{\beta}_{l,e}$ . Following arguments analogous to that of Lemma 4.1, we can show:

**Lemma 4.3.** For each  $l$  and  $e \in V$ ,  $\hat{\alpha}_{l,e} \leq \alpha_l^{\iota(e)-1}$  and  $\hat{\beta}_{l,e} \geq \beta_l^{\iota(e)-1}$ .

**Corollary 4.4.** Concavity of  $g$  implies that  $\Delta$ 's computed by Algorithm 3 satisfy

$$\begin{aligned} \Delta_+^{\max}(e) &\geq \sum_{S_l \ni e} [g(\alpha_l^{\iota(e)-1} + w_l(e)) - g(\alpha_l^{\iota(e)-1})] - \lambda v(e) = \Delta_+(e), \\ \Delta_-^{\max}(e) &\geq \sum_{S_l \ni e} [g(\beta_l^{\iota(e)-1} - w_l(e)) - g(\beta_l^{\iota(e)-1})] + \lambda v(e) = \Delta_-(e), \end{aligned}$$

## 5 Optimistic Concurrency Control for Bidirectional Greedy Algorithm

[XP: Provide more intuition for what the OCC algorithm is doing.]

Algorithm 4<sup>2</sup> is the OCC bidirectional greedy algorithm. Unlike the hogwild algorithm, the OCC algorithm ensures serial equivalence by maintaining four sets  $\hat{A}$ ,  $\tilde{A}$ ,  $\hat{B}$ ,  $\tilde{B}$ , which serve as upper and lower bounds on  $A$  and  $B$ . Each thread can determine locally if a decision to include / exclude an element can be taken safely. Otherwise, the validation process (Algorithm 5) waits until it is certain about  $A$ ,  $B$  before proceeding.

The serialization order is given by  $\iota(e)$ , which is the value of  $\iota$  at line 13 of Algorithm 4. We define  $\hat{A}_e$ ,  $\hat{B}_e$ ,  $A^{\iota(e)-1}$ ,  $B^{\iota(e)-1}$  as before with the hogwild algorithm, and additionally let  $\tilde{A}_e$  and  $\tilde{B}_e$  be the vectors of  $\tilde{A}$  and  $\tilde{B}$  that are used in the computation of  $\Delta_+^{\min}(e)$  and  $\Delta_-^{\min}(e)$ . We will show that the outcome of Algorithm 4 is equivalent to the sequential algorithm executed with ordering  $\iota$ .

**Lemma 5.1.**  $\hat{A}_e \subseteq A^{\iota(e)-1} \subseteq \tilde{A}_e \setminus e$ , and  $\hat{B}_e \supseteq B^{\iota(e)-1} \supseteq \tilde{B}_e \cup e$ .

*Proof.* Clearly,  $e \in \tilde{B}_e \cup e$  but  $e \notin \tilde{A}_e \setminus e$ . By definition,  $e \in B^{\iota(e)-1}$  but  $e \notin A^{\iota(e)-1}$ . The OCC algorithm (Algorithm 4) only modifies  $\hat{A}(e)$  and  $\hat{B}(e)$  on Lines 27, 31 when processing  $e$ , which occurs after the computation of the  $\Delta$ 's on Lines 14 - 17, so  $e \in \hat{B}_e$  but  $e \notin \hat{A}_e$ .

Consider any  $e' \neq e$ . Suppose  $e' \in \hat{A}_e$ . This is only possible if we have processed Line 27, which implies (by Line 23) that  $\forall j < \iota(e')$ ,  $\text{result}(j) \neq 0$ . But at the time when  $\hat{A}_e$  is read (Lines 14 - 17), we have  $\text{result}(e) = 0$ , so it must be the case that  $\iota(e') < \iota(e)$ . Thus,  $e' \in A^{\iota(e)-1}$ .

Now suppose  $e' \in A^{\iota(e)-1}$ . By definition, this implies  $\iota(e') < \iota(e)$  and  $e \in A$ , which in turn implies  $\text{result}(\iota(e')) = 1$ . Hence, it must be the case that we have already set  $\tilde{A}(e') \leftarrow 1$  (by the ordering imposed by  $\iota$  on Line 13), but never execute  $\tilde{A}(e') \leftarrow 0$ , so  $e' \in \tilde{A}_e$ .

<sup>2</sup>The synchronization required by the OCC algorithm can be further reduced, for example, by replacing the operations from Line 23 onwards by a non-blocking function call.

An analogous argument shows  $e' \notin \hat{B}_e \implies e' \notin B^{\iota(e)-1} \implies e' \notin \tilde{B}_e \cup e$ .  $\square$

**Lemma 5.2.** *During the validation process for element  $e$ , on Lines 2, 3 of Algorithm 5, we have  $\hat{A} = A^{\iota(e)-1}$  and  $\hat{B} = B^{\iota(e)-1}$ .*

*Proof.* At Line 2, we have guaranteed that  $\forall j < \iota(e)$ ,  $\text{processed}(j) = \text{true}$ . Thus,  $\forall e' : \iota(e') < \iota(e)$ ,  $e' \in \hat{A} \iff e' \in A^{\iota(e)-1}$  and  $e' \in \hat{B} \iff e' \in B^{\iota(e)-1}$ . On the other hand,  $\text{result}(\iota(e)) = 0$ , so all  $e'$  such that  $\iota(e') > \iota(e)$  are blocked on Algorithm 4 Line 23, hence  $\hat{A}(e') = 0$  and  $\hat{B}(e') = 1$ . Thus,  $\hat{A} = A^{\iota(e)-1}$  and  $\hat{B} = B^{\iota(e)-1}$ .  $\square$

Algorithm 4 computes

$$\begin{aligned} \Delta_+^{\min}(e) &= F(\tilde{A}_e) - F(\tilde{A}_e \setminus e), & \Delta_+^{\max}(e) &= F(\hat{A}_e \cup e) - F(\hat{A}) \\ \Delta_-^{\min}(e) &= F(\tilde{B}_e) - F(\tilde{B}_e \cup e), & \Delta_-^{\max}(e) &= F(\hat{B}_e \setminus e) - F(\hat{B}). \end{aligned}$$

**Corollary 5.3.** *Submodularity of  $F$  implies that the  $\Delta$ 's computed by Algorithm 4 satisfy:  $\Delta_+^{\min}(e) \leq \Delta_+(e) = \Delta_+^{\text{exact}}(e) \leq \Delta_+^{\max}(e)$ , and  $\Delta_-^{\min}(e) \leq \Delta_-(e) = \Delta_-^{\text{exact}}(e) \leq \Delta_-^{\max}(e)$ .*

## 5.1 Separable sums $F$

Analogous to the hogwild algorithm, we maintain  $\hat{\alpha}_l, \hat{\beta}_l$  and additionally  $\tilde{\alpha}_l = \sum_{j \in \tilde{A} \cup S_l} w_l(j)$  and  $\tilde{\beta}_l = \sum_{j \in \tilde{B} \cup S_l} w_l(j)$ . It can be shown that  $\hat{\alpha}_{l,e} \leq \alpha^{\iota(e)-1} \leq \tilde{\alpha}_{l,e} - w_l(e)$  and  $\hat{\beta}_{l,e} \geq \beta^{\iota(e)-1} \geq \tilde{\beta}_{l,e} + w_l(e)$ , which then allows us to compute our bounds for  $\Delta$ 's.

## 6 Analysis of Algorithms

[XP: What is the purpose of this section? We want to say that we have a choice between a slower but serially equivalent algorithm and a faster algorithm which is not serially equivalent. Nevertheless, the slower algorithm is not too slow, and the approximation guarantee of the algorithm that is not serially equivalent is not too weak.]

### 6.1 Approximation of hogwild bidirectional greedy

Let  $F$  be submodular and non-negative. We assume for each  $e$ , there exists  $\rho_e \geq 0$  such that  $\Delta_+^{\max}(e) - \rho_e \leq \Delta_+(e)$ , and  $\Delta_-^{\max}(e) - \rho_e \leq \Delta_-(e)$ . This is possible, for example, by defining

$$\begin{aligned} \rho_e &:= \max\{\Delta_+^{\max}(e) - \Delta_+(e), \Delta_-^{\max}(e) - \Delta_-(e)\} \\ &\leq \max_{S, T \subseteq V} \{[F(S \cup e) - F(S)] - [F(S \cup T \cup e) - F(S \cup T)]\} \leq F(e)\kappa_F \end{aligned}$$

where  $\kappa_F$  is the total curvature of  $F$ . Summing over  $e$  then gives us  $\sum_e \rho_e \leq \kappa_F \sum_e F(e)$ .

**Theorem 6.1.** *Let  $F$  be a non-negative (monotone or non-monotone) submodular function. The hogwild bidirectional greedy algorithm solves the unconstrained problem  $\max_{A \subseteq V} F(A)$  with approximation  $E[F(A_{\text{hog}})] \geq \frac{1}{2}F^* - \frac{1}{4} \sum_{i=1}^n E[\rho_i]$ , where  $A_{\text{hog}}$  is the output of the algorithm,  $F^*$  is the optimal value, and  $\rho_i$  is a random variable such that  $\rho_i \geq \Delta_+^{\max}(i) - \Delta_+(i)$  and  $\rho_i \geq \Delta_-^{\max}(i) - \Delta_-(i)$ .*

We prove the theorem in Appendix A.

**Example: max graph cut.** Assuming bounded delay of  $\tau$  and edges with unit weight, we can bound  $\sum_i E[\rho_i] \leq 2\tau \frac{\#\text{edges}}{2N}$ . The approximation of the hogwild algorithm is then  $E[F(A^n)] \geq \frac{1}{2}F(\text{OPT}) - \tau \frac{\#\text{edges}}{2N}$ . In sparse graphs, the hogwild algorithm is off by a small additional term, which albeit grows linearly in  $\tau$ .

**Example: set cover.** Consider the simple set cover function,  $F(A) = \sum_{l=1}^L \min(1, |A \cap S_l|) - \lambda|A| = |\{l : A \cap S_l \neq \emptyset\}| - \lambda|A|$ , with  $0 < \lambda \leq 1$ . We assume that there is some bounded delay  $\tau$ . Suppose also the  $S_l$ 's form a partition, so each element  $e$  belongs to exactly one set. Then,  $\sum_e E[\rho_e] \geq \tau + L(1 - \lambda^\tau)$ , which is linear in  $\tau$  but independent of  $N$ .

## 6.2 Correctness of OCC

**Theorem 6.2.** *OCC bidirectional greedy is serially equivalent to bidirectional greedy.*

*Proof.* We will denote by  $A_{seq}^i, B_{seq}^i$  the sets generated by the sequential algorithm, reserving  $A^i, B^i$  for sets generated by the OCC algorithm. It suffices to show by induction that  $A_{seq}^i = A^i$  and  $B_{seq}^i = B^i$ . For the base case,  $A^0 = \emptyset = A_{seq}^0$ , and  $B^0 = V = B_{seq}^0$ . Consider any element  $e$ . The OCC algorithm includes  $e \in A$  iff  $u_e < [\Delta_+^{\min}(e)]_+ + [\Delta_-^{\max}(e)]_+^{-1}$  on Algorithm 4 Line 19 or  $u_e < [\Delta_+^{\text{exact}}(e)]_+ + [\Delta_-^{\text{exact}}(e)]_+^{-1}$  on Algorithm 5 Line 4. In both cases, Corollary 5.3 implies  $u_e < [\Delta_+(e)]_+ + [\Delta_-(e)]_+^{-1}$ . By induction,  $A^{\iota(e)-1} = A_{seq}^{\iota(e)-1}$  and  $B^{\iota(e)-1} = B_{seq}^{\iota(e)-1}$ , so the threshold is exactly that computed by the sequential algorithm. Hence, the OCC algorithm includes  $e \in A$  iff the sequential algorithm includes  $e \in A$ . (An analogous argument works for the case where  $e$  is excluded from  $B$ .) □

As an immediate consequence, theoretical properties of the sequential algorithm are preserved by the OCC algorithm, including the approximation guarantees:

**Theorem 6.3.** *Let  $F$  be a non-negative (monotone or non-monotone) submodular function. The OCC bidirectional greedy algorithm solves the unconstrained problem  $\max_{A \subseteq V} F(A)$  with approximation  $E[F(A_{OCC})] \geq \frac{1}{2}F^*$ , where  $A_{OCC}$  is the output of the algorithm, and  $F^*$  is the optimal value.*

## 6.3 Scalability of OCC

Whenever an element is validated, it needs to wait for all earlier elements to be processed, and it blocks all later elements from being processed. Each validation effectively constitutes a barrier to the parallel processing. Hence, the scalability of the OCC algorithm is dependent on the number of validated elements. Nevertheless, we show for a couple of example problems that the number of validated elements can be bounded. Full details of the bounds are given in Appendix B.

**Example: max graph cut.** Assume that there is a bounded delay  $\tau$ . The expected number of validated elements is upper bounded by  $\tau \frac{2\#edges}{N}$ .

**Example: set cover.** Assuming a bounded delay  $\tau$  and non-overlapping sets  $S_l$ 's, the expected number of validated elements is upper bounded by  $2\tau$ .

## 7 Evaluation

We implemented the parallel and sequential double greedy algorithms in Java / Scala. Experiments were conducted on Amazon EC2 using one cc2.8xlarge machine, up to 16 threads, for 10 iterations.

We measured the runtime and speedup (ratio of runtime on 1 thread to runtime on  $p$  threads). For the hogwild algorithm, we measured  $F(A_{hog}) - F(A_{seq})$ , the difference between the objective value on the sets returned by hogwild and the sequential algorithms. We verified the correctness of the OCC algorithm by comparing the sets returned by the OCC and sequential algorithms, and also measured the fraction of elements that are validated by OCC.

We tested our parallel algorithms on the max graph cut and set cover problems with two synthetic graphs and three real datasets (Table 1). Graphs were pre-processed to remove self-loops. We found that vertices were typically indexed such that vertices close to each other in the graph were also close in their indices. To reduce this dependency, we randomly permuted the ordering of vertices. For the max graph cut problem, we removed directions on edges to obtain undirected graphs. The set cover problem is reduced to a vertex cover on the directed graph.

Graph	# vertices	# edges	Description
Erdos-Renyi	20,000,000	$\approx 2 \times 10^6$	Each edge is included with probability $5 \times 10^{-6}$ .
ZigZag	25,000,000	2,025,000,000	Expander graph. The 81-regular zig-zag product between the Cayley graph on $\mathbb{Z}_{2500000}$ with generating set $\{\pm 1, \dots, \pm 5\}$ , and the complete graph $K_{10}$ .
Friendster	10,000,000	625,279,786	Subgraph of social network. [2]
Arabic-2005	22,744,080	631,153,669	2005 crawl of Arabic web sites [3, 4, 5].
UK-2005	39,459,925	921,345,078	2005 crawl of the .uk domain [3, 4, 5].
IT-2004	41,291,594	1,135,718,909	2004 crawl of the .it domain [3, 4, 5].

Table 1: Synthetic and real graphs used in the evaluation of our parallel algorithms.

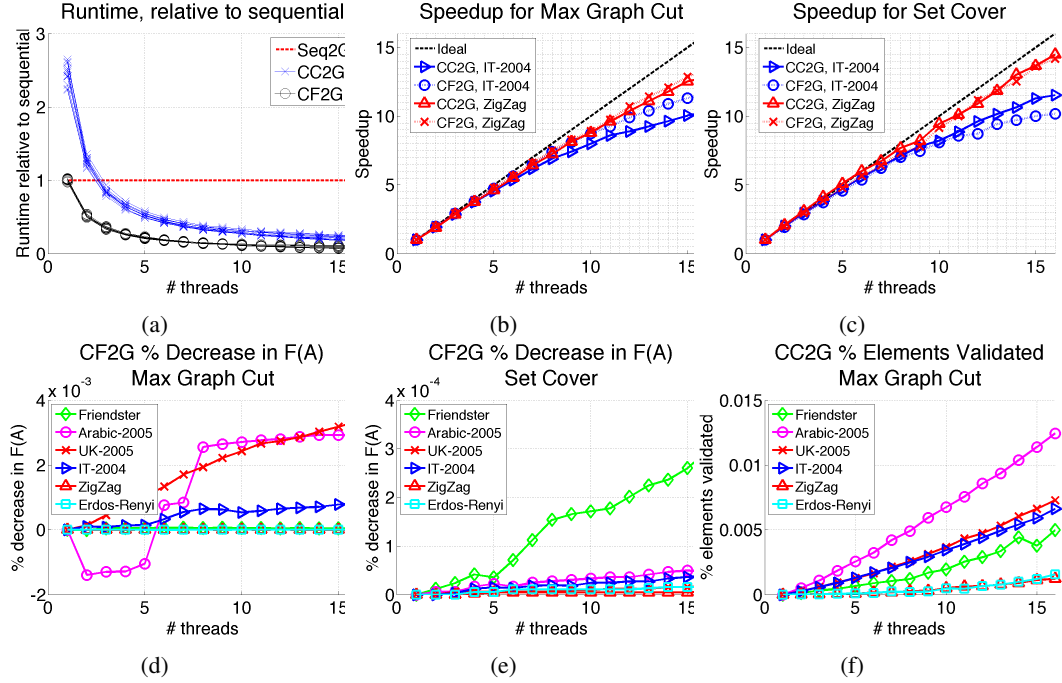


Figure 1: Experimental results. Fig 1a – runtime of the parallel algorithms as a ratio to that of the sequential algorithm. Each curve shows the runtime of a parallel algorithm on a particular graph for a particular function  $F$ . Fig 1b, 1c – speedup (ratio of runtime on one thread to that on  $p$  threads). Fig 1d, 1e – % difference between objective values of the sequential and hogwild algorithms, i.e.  $[F(A_{\text{hogwild}})/F(A_{\text{sequential}}) - 1] \times 100\%$ . Fig 1f – percentage of elements validated by the OCC algorithm on the max graph cut problem.

Due to space constraints, we only present part of our results in Figure 1, deferring full results to Appendix D. **Runtime, Speedup:** Both parallel algorithms are faster than the sequential algorithm with three or more threads, and show good speedup properties as more threads are added ( $\sim 10\times$  or more for all graphs and both functions). **Objective value:** The objective value of the hogwild algorithm decreases with the number of threads, but differs from the sequential objective value by less than 0.01%. **Validations:** The OCC algorithm validates more elements as threads are added, but less than 0.015% are validated with 16 threads, which has negligible effect on the runtime / speedup.

## 7.1 Adversarial ordering

To highlight the philosophical differences between the two parallel algorithms, we conducted experiments on a ring Cayley graph on  $\mathbb{Z}_{10^6}$  with generating set  $\{\pm 1, \dots, \pm 1000\}$ . The algorithms are presented with an adversarial ordering, with permutation, so vertices close in the ordering are adjacent to one another, and tend to be processed concurrently. This causes CF2G to make more mistakes, and CC2G to face more uncertainty.

As Figure 2 shows, CC2G sacrifices speed to ensure serial equivalence, eventually validating  $> 90\%$  of elements. On the other hand, CF2G focuses on speed, resulting in faster runtime, but delivering an objective value  $F(A_{CF2G})$  that is only 20% of  $F(A_{Seq2G})$ .<sup>3</sup>

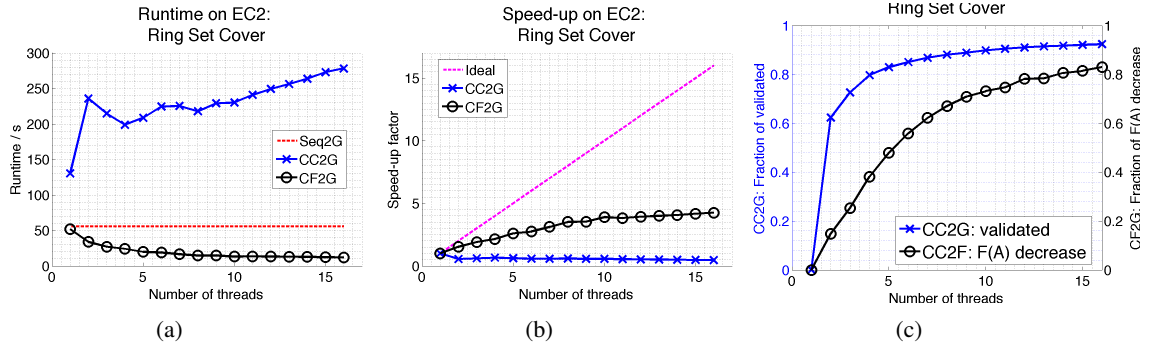


Figure 2: Experimental results for ring graph on set cover problem.

[XP: These experiments were conducted by using an atomic integer to select elements to process. We could instead use a partitioning scheme, which has 2 advantages. Firstly, there is less coordination – for CF2G, we essentially have no coordination. Secondly, when faced with an adversarial ordering, the partitioning scheme allows big jumps / re-orderings, which reduces the number of validations and *increases* the objective value of both parallel algorithms.]

In general, it is possible to avoid such adversarial ordering by randomly permuting the elements when pre-processing, or by using a different partitioning scheme, which we will discuss in a full paper.

## 8 Related Work

**Similar approach, different problem:** OCC DP-means [6]; Hogwild SGD [7]; Hogwild LDA [8] / parameter servers [9, 10]

**Similar problem, different approach:** distributed greedy submodular maximization for monotone functions [11]

## 9 Discussions

**Conclusion:** [XP: link back to intro, motivation]; we present two approaches to parallelizing unconstrained submodular maximization, which allows one to choose between speed and tight approximation guarantees.

**Future work:** constrained maximization, minimization; distributed setting, where communication costs and delays are higher, and function evaluations are challenging.

## References

- [1] Niv Buchbinder, Moran Feldman, Joseph (Seffi) Naor, and Roy Schwartz. A tight linear time (1/2)-approximation for unconstrained submodular maximization. In *Proceedings of the 2012 IEEE 53rd Annual Symposium on Foundations of Computer Science, FOCS '12*, pages 649–658, Washington, DC, USA, 2012. IEEE Computer Society. ISBN 978-0-7695-4874-6. doi: 10.1109/FOCS.2012.73. URL <http://dx.doi.org/10.1109/FOCS.2012.73>.
- [2] Jure Leskovec. Stanford network analysis project, 2011. URL <http://snap.stanford.edu/index.html>.

<sup>3</sup>For the set cover problem, both algorithms have to perform atomic updates to the statistics  $\alpha_l$  and  $\beta_l$ . Under the adversarial ordering, there are more concurrent atomic updates, and hence, CF2G does not achieve good speedup. It is possible to compute  $F$  directly, which would provide better speedup but slower runtimes.



432 [3] Paolo Boldi and Sebastiano Vigna. The WebGraph framework I: Compression techniques. In *Proc. of the*  
433 *Thirteenth International World Wide Web Conference (WWW 2004)*, pages 595–601, Manhattan, USA,  
434 2004. ACM Press.

435 [4] Paolo Boldi, Marco Rosa, Massimo Santini, and Sebastiano Vigna. Layered label propagation: A  
436 multiresolution coordinate-free ordering for compressing social networks. In *Proceedings of the 20th*  
437 *international conference on World Wide Web*. ACM Press, 2011.

438 [5] Paolo Boldi, Bruno Codenotti, Massimo Santini, and Sebastiano Vigna. Ubicrawler: A scalable fully  
439 distributed web crawler. *Software: Practice & Experience*, 34(8):711–726, 2004.

440 [6] Xinghao Pan, Joseph E Gonzalez, Stefanie Jegelka, Tamara Broderick, and Michael Jordan. Optimistic  
441 concurrency control for distributed unsupervised learning. In *Advances in Neural Information Processing*  
442 *Systems* 26. 2013.

443 [7] Benjamin Recht, Christopher Re, Stephen J. Wright, and Feng Niu. Hogwild: A lock-free approach to  
444 parallelizing stochastic gradient descent. In *Advances in Neural Information Processing Systems (NIPS)*  
445 24, pages 693–701, Granada, 2011.

446 [8] Amr Ahmed, Mohamed Aly, Joseph Gonzalez, Shravan Narayanamurthy, and Alexander J. Smola. Scalable  
447 inference in latent variable models. In *Proceedings of the 5th ACM International Conference on Web*  
448 *Search and Data Mining (WSDM)*, 2012.

449 [9] Mu Li, Li Zhou, Zichao Yang, Aaron Li, Fei Xia, David G Andersen, and Alexander Smola. Parameter  
450 server for distributed machine learning. In *Big Learn workshop, at NIPS*, Lake Tahoe, 2013.

451 [10] Qirong Ho, James Cipar, Henggang Cui, Seunghak Lee, Jin Kyu Kim, Phillip B. Gibbons, Garth A Gibson,  
452 Greg Ganger, and Eric Xing. More effective distributed ml via a stale synchronous parallel parameter  
453 server. In *Advances in Neural Information Processing Systems* 26. 2013.

454 [11] Baharan Mirzasoleiman, Amin Karbasi, Rik Sarkar, and Andreas Krause. Distributed submodular max-  
455 imization: Identifying representative elements in massive data. In *Advances in Neural Information*  
456 *Processing Systems* 26. 2013.

457

458

459

460

461

462

463

464

465

466

467

468

469

470

471

472

473

474

475

476

477

478

479

480

481

482

483

484

485

## A Proof of bound for hogwild

We follow the proof outline of [1].

Let  $OPT$  be an optimal solution to the problem. Define  $O^i := (OPT \cup A^i) \cap B^i$ . Note that  $O^i$  coincides with  $A^i$  and  $B^i$  on elements  $1, \dots, i$ , and  $O^i$  coincides with  $OPT$  on elements  $i+1, \dots, n$ . Hence,

$$\begin{aligned} O^i \setminus i+1 &\supseteq A^i \\ O^i \cup i+1 &\subseteq B^i. \end{aligned}$$

**Lemma A.1.** For every  $1 \leq i \leq n$ ,  $\Delta_+(i) + \Delta_-(i) \geq 0$ .

*Proof.* This is just Lemma II.1 of [1]. □

**Lemma A.2.** For every  $1 \leq i \leq n$ ,

$$E[F(O^{i-1}) - F(O^i)] \leq \frac{1}{2}E[f(A^i) - f(A^{i-1}) + f(B^i) - f(B^{i-1}) + \rho_i].$$

*Proof.* We follow the proof outline of [1]. First, note that it suffices to prove the inequality conditioned on knowing  $A^{i-1}$  and  $j$ , then applying the law of total expectation. Under this conditioning, we also know  $B^{i-1}$ ,  $O^{i-1}$ ,  $\Delta_+(i)$ ,  $\Delta_+^{\max}(i)$ ,  $\Delta_-(i)$ ,  $\Delta_-^{\max}(i)$ , and  $\rho_i$ .

We consider the following 9 cases.

**Case 1:**  $0 < \Delta_+(i) \leq \Delta_+^{\max}(i)$ ,  $0 \leq \Delta_-^{\max}(i)$ . Since both  $\Delta_+^{\max}(i) > 0$  and  $\Delta_-^{\max}(i) > 0$ , the probability of including  $i$  is just  $\Delta_+^{\max}(i)/(\Delta_+^{\max}(i) + \Delta_-^{\max}(i))$ , and the probability of excluding  $i$  is  $\Delta_-^{\max}(i)/(\Delta_+^{\max}(i) + \Delta_-^{\max}(i))$ .

$$\begin{aligned} E[F(A^i) - F(A^{i-1}) | A^{i-1}, j] &= \frac{\Delta_+^{\max}(i)}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)} (F(A^{i-1} \cup i) - F(A^{i-1})) \\ &= \frac{\Delta_+^{\max}(i)}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)} \Delta_+(i) \\ &\geq \frac{\Delta_+^{\max}(i)}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)} (\Delta_+^{\max}(i) - \rho_i) \\ E[F(B^i) - F(B^{i-1}) | A^{i-1}, j] &= \frac{\Delta_-^{\max}(i)}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)} (F(B^{i-1} \setminus i) - F(B^{i-1})) \\ &= \frac{\Delta_-^{\max}(i)}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)} \Delta_-(i) \\ &\geq \frac{\Delta_-^{\max}(i)}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)} (\Delta_-^{\max}(i) - \rho_i) \end{aligned}$$

$$\begin{aligned}
& E[F(O^{i-1}) - F(O^i)|A^{i-1}, j] \\
&= \frac{\Delta_+^{\max}(i)}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)} (F(O^{i-1}) - F(O^{i-1} \cup i)) \\
&\quad + \frac{\Delta_-^{\max}(i)}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)} (F(O^{i-1}) - F(O^{i-1} \setminus i)) \\
&= \begin{cases} \frac{\Delta_+^{\max}(i)}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)} (F(O^{i-1}) - F(O^{i-1} \cup i)) & \text{if } i \notin OPT \\ \frac{\Delta_-^{\max}(i)}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)} (F(O^{i-1}) - F(O^{i-1} \setminus i)) & \text{if } i \in OPT \end{cases} \\
&\leq \begin{cases} \frac{\Delta_+^{\max}(i)}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)} (F(B^{i-1} \setminus i) - F(B^{i-1})) & \text{if } i \notin OPT \\ \frac{\Delta_-^{\max}(i)}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)} (F(A^{i-1} \cup i) - F(A^{i-1})) & \text{if } i \in OPT \end{cases} \\
&= \begin{cases} \frac{\Delta_+^{\max}(i)}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)} \Delta_-(i) & \text{if } i \notin OPT \\ \frac{\Delta_-^{\max}(i)}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)} \Delta_+(i) & \text{if } i \in OPT \end{cases} \\
&\leq \begin{cases} \frac{\Delta_+^{\max}(i)}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)} \Delta_-^{\max}(i) & \text{if } i \notin OPT \\ \frac{\Delta_-^{\max}(i)}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)} \Delta_+^{\max}(i) & \text{if } i \in OPT \end{cases} \\
&= \frac{\Delta_+^{\max}(i) \Delta_-^{\max}(i)}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)}
\end{aligned}$$

where the first inequality is due to submodularity:  $O^{i-1} \setminus i \supseteq A^{i-1}$  and  $O^{i-1} \cup i \subseteq B^{i-1}$ .

Putting the above inequalities together:

$$\begin{aligned}
& E[F(O^{i-1}) - F(O^i)|A^{i-1}, j] - \frac{1}{2} E[f(A^i) - f(A^{i-1}) + f(B^i) - f(B^{i-1}) + \rho_i | A^{i-1}, j] \\
&\leq \frac{1/2}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)} \left[ 2\Delta_+^{\max}(i)\Delta_-^{\max}(i) - \Delta_-^{\max}(i)(\Delta_-^{\max}(i) - \rho_i) \right. \\
&\quad \left. - \Delta_+^{\max}(i)(\Delta_+^{\max}(i) - \rho_i) \right] - \frac{1}{2} \rho_i \\
&= \frac{1/2}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)} \left[ -(\Delta_+^{\max}(i) - \Delta_-^{\max}(i))^2 + \rho_i(\Delta_+^{\max}(i) + \Delta_-^{\max}(i)) \right] - \frac{1}{2} \rho_i \\
&\leq \frac{\frac{1}{2} \rho_i (\Delta_+^{\max}(i) + \Delta_-^{\max}(i))}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)} - \frac{1}{2} \rho_i \\
&= 0.
\end{aligned}$$

**Case 2:**  $0 < \Delta_+(i) \leq \Delta_+^{\max}(i)$ ,  $\Delta_-^{\max}(i) < 0$ . In this case, the algorithm always choses to include  $i$ , so  $A^i = A^{i-1} \cup i$ ,  $B^i = B^{i-1}$  and  $O^i = O^{i-1} \cup i$ :

$$\begin{aligned}
& E[F(A^i) - F(A^{i-1})|A^{i-1}, j] = F(A^{i-1} \cup i) - F(A^{i-1}) = \Delta_+(i) > 0 \\
& E[F(B^i) - F(B^{i-1})|A^{i-1}, j] = F(B^{i-1}) - F(B^{i-1}) = 0 \\
& E[F(O^{i-1}) - F(O^i)|A^{i-1}, j] = F(O^{i-1}) - F(O^{i-1} \cup i) \\
&\leq \begin{cases} 0 & \text{if } i \in OPT \\ F(B^{i-1} \setminus i) - F(B^{i-1}) & \text{if } i \notin OPT \end{cases} \\
&= \begin{cases} 0 & \text{if } i \in OPT \\ \Delta_-(i) & \text{if } i \notin OPT \end{cases} \\
&\leq 0 \\
&< \frac{1}{2} E[f(A^i) - f(A^{i-1}) + f(B^i) - f(B^{i-1}) + \rho_i | A^{i-1}, j]
\end{aligned}$$

where the first inequality is due to submodularity:  $O^{i-1} \cup i \subseteq B^{i-1}$ .

**Case 3:**  $\Delta_+(i) \leq 0 < \Delta_+^{\max}(i)$ ,  $0 < \Delta_-(i) < \Delta_-^{\max}(i)$ . Analogous to Case 1.

**Case 4:**  $\Delta_+(i) \leq 0 < \Delta_+^{\max}(i)$ ,  $\Delta_-(i) \leq 0$ . This is not possible, by Lemma A.1.

**Case 5:**  $\Delta_+(i) \leq \Delta_+^{\max}(i) \leq 0$ ,  $0 < \Delta_-(i) \leq \Delta_-^{\max}(i)$ . Analogous to Case 2.

**Case 6:**  $\Delta_+(i) \leq \Delta_+^{\max}(i) \leq 0$ ,  $\Delta_-(i) \leq 0$ . This is not possible, by Lemma A.1.

□

**[XP: Note]** If we weaken the assumption of  $\Delta_+(i) \leq \Delta_+^{\max}(i)$  to  $\Delta_+(i) \leq \Delta_+^{\max}(i) + \epsilon_i$ , then in Case 6 above, we can instead bound

$$\begin{aligned} E[F(O^{i-1}) - F(O^i) | A^{i-1}, j] &\leq \frac{\Delta_+^{\max}(i) \Delta_-^{\max}(i) + \epsilon \max(\Delta_+^{\max}(i), \Delta_-^{\max}(i))}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)} \\ &\leq \frac{\Delta_+^{\max}(i) \Delta_-^{\max}(i) + \epsilon(\Delta_+^{\max}(i) + \Delta_-^{\max}(i))}{\Delta_+^{\max}(i) + \Delta_-^{\max}(i)}. \end{aligned}$$

The bound of Lemma A.2 becomes

$$E[F(O^{i-1}) - F(O^i)] \leq \frac{1}{2} E[f(A^i) - f(A^{i-1}) + f(B^i) - f(B^{i-1}) + \rho_i + 2\epsilon_i],$$

and the bound of Theorem 6.1 becomes  $E[F(A)] \geq \frac{1}{2} F^* - \frac{1}{4} \sum_i E[\rho_i + 2\epsilon_i]$ .

We will now prove Theorem 6.1.

*Proof of Theorem 6.1.* Summing up the statement of Lemma A.2 for all  $i$  gives us a telescoping sum, which reduces to:

$$\begin{aligned} E[F(O^0) - F(O^n)] &\leq \frac{1}{2} E[F(A^n) - F(A^0) + F(B^n) - F(B^0)] + \frac{1}{2} \sum_{i=1}^n E[\rho_i] \\ &\leq \frac{1}{2} E[F(A^n) + F(B^n)] + \frac{1}{2} \sum_{i=1}^n E[\rho_i]. \end{aligned}$$

Note that  $O^0 = OPT$  and  $O^n = A^n = B^n$ , so  $E[F(A^n)] \geq \frac{1}{2} F(OPT) - \frac{1}{4} \sum_i E[\rho_i]$ . □

### A.1 Example: max graph cut

Let  $C^{ji} = \{j+1, \dots, i-1\}$ ,  $D^i = \{i+1, \dots, n\}$ . Denote  $\tilde{A}^j = V \setminus A^j \setminus C^{ji} \setminus D^i = \{1, \dots, j\} \setminus A^j$  be the elements up to  $j$  that are not included in  $A$ . Let  $w_i(S) = \sum_{j \in S, (i,j) \in E} w(i, j)$ . For the max graph cut function, it is easy to see that

$$\begin{aligned} \Delta_+ &\geq -w_i(A^j) - w_i(C^{ji}) + w_i(D^i) + w_i(\tilde{A}^j) \\ \Delta_+^{\max} &= -w_i(A^j) + w_i(C^{ji}) + w_i(D^i) + w_i(\tilde{A}^j) \\ \Delta_- &\geq +w_i(A^j) - w_i(C^{ji}) + w_i(D^i) - w_i(\tilde{A}^j) \\ \Delta_-^{\max} &= +w_i(A^j) + w_i(C^{ji}) + w_i(D^i) - w_i(\tilde{A}^j) \end{aligned}$$

Thus, we can set  $\rho_i = 2w_i(C^{ji})$ .

Suppose we have bounded delay  $\tau$ , so  $|C^{ji}| \leq \tau$ . Then  $w_i(C^{ji})$  has a hypergeometric distribution with mean  $\frac{\deg(i)}{N} \tau$ , and  $E[\rho_i] = 2\tau \frac{\deg(i)}{N}$ . The approximation of the hogwild algorithm is then  $E[F(A^n)] \geq \frac{1}{2} F(OPT) - \tau \frac{\#\text{edges}}{2N}$ . In sparse graphs, the hogwild algorithm is off by a small additional term, which albeit grows linearly in  $\tau$ .

## A.2 Example: set cover

[XP: For now, consider a toy problem, with (1) disjoint sets, (2) bounded delay, (3)  $\lambda \leq 1$ .]

Consider the simple set cover function, for  $\lambda < 1$ :

$$F(A) = \sum_{l=1}^L \min(1, |A \cap S_l|) - \lambda|A| = |\{l : A \cap S_l \neq \emptyset\}| - \lambda|A|.$$

We assume that there is some bounded delay  $\tau$ .

Suppose also the  $S_l$ 's form a partition, so each element  $e$  belongs to exactly one set. Let  $n_l$  denote  $|S_l|$  the size of  $S_l$ . Given any ordering  $\pi$ , let  $e_l^t$  be the  $t$ th element of  $S_l$  in the ordering, i.e.  $|\{e' : \pi(e') \leq \pi(e_l^t) \wedge e' \in S_l\}| = t$ .

For any  $e \in S_l$ , we get

$$\begin{aligned}\Delta_+(e) &= -\lambda + 1\{A^{\iota(e)-1} \cap S_l = \emptyset\} \\ \Delta_+^{\max}(e) &= -\lambda + 1\{\hat{A}_e \cap S_l = \emptyset\} \\ \Delta_-(e) &= +\lambda - 1\{B^{\iota(e)-1} \setminus e \cap S_l = \emptyset\} \\ \Delta_-^{\max}(e) &= +\lambda - 1\{\hat{B}_e \setminus e \cap S_l = \emptyset\}\end{aligned}$$

Let  $\eta$  be the position of the first element of  $S_l$  to be accepted, i.e.  $\eta = \min\{t : e_l^t \in A \cap S_l\}$ . (For convenience, we set  $\eta = n_l$  if  $A \cap S_l = \emptyset$ .) We first show that  $\eta$  is independent of  $\pi$ : for  $\eta < n_l$ ,

$$\begin{aligned}P(\eta|\pi) &= \frac{\Delta_+^{\max}(e_l^\eta)}{\Delta_+^{\max}(e_l^\eta) + \Delta_-^{\max}(e_l^\eta)} \prod_{t=1}^{\eta-1} \frac{\Delta_-^{\max}(e_l^t)}{\Delta_+^{\max}(e_l^t) + \Delta_-^{\max}(e_l^t)} \\ &= \frac{1-\lambda}{1-\lambda+\lambda} \prod_{t=1}^{\eta-1} \frac{\lambda}{1-\lambda+\lambda} \\ &= (1-\lambda)\lambda^{\eta-1},\end{aligned}$$

and  $P(\eta = n_l|\pi) = \lambda^{\eta-1}$ . [XP: This independence depends on the assumption of disjoint sets, which in turn allows us to decouple the randomness of the algorithm from the randomness of ordering in the below proof.]

Note that,  $\Delta_-^{\max}(e) - \Delta_-(e) = 1$  iff  $e = e_l^{n_l}$  is the last element of  $S_l$  in the ordering, there are no elements accepted up to  $\hat{B}_{e_l^{n_l}} \setminus e_l^{n_l}$ , and there is some element  $e'$  in  $\hat{B}_{e_l^{n_l}} \setminus e_l^{n_l}$  that is rejected and not in  $B^{\iota(e_l^{n_l})-1}$ . Denote by  $m_l \leq \min(\tau, n_l - 1)$  the number of elements before  $e_l^{n_l}$  that are inconsistent between  $\hat{B}_{e_l^{n_l}}$  and  $B^{\iota(e_l^{n_l})-1}$ . Then  $\mathbb{E}[\Delta_-^{\max}(e_l^{n_l}) - \Delta_-(e_l^{n_l})] = P(\Delta_-^{\max}(e_l^{n_l}) \neq \Delta_-(e_l^{n_l}))$  is

$$\lambda^{n_l-1-m_l}(1-\lambda^{m_l}) = \lambda^{n_l-1}(\lambda^{-m_l} - 1) \leq \lambda^{n_l-1}(\lambda^{-\min(\tau, n_l-1)} - 1) \leq 1 - \lambda^\tau.$$

If  $\lambda = 1$ ,  $\Delta_+^{\max}(e) \leq 0$ , so no elements before  $e_l^{n_l}$  will be accepted, and  $\Delta_-^{\max}(e_l^{n_l}) = \Delta_-(e_l^{n_l})$ .

On the other hand,  $\Delta_+^{\max}(e) - \Delta_+(e) = 1$  iff  $(A^{\iota(e)-1} \setminus \hat{A}_e) \cap S_l \neq \emptyset$ , that is, if an element has been accepted in  $A$  but not yet observed in  $\hat{A}_e$ . Since we assume a bounded delay, only the first  $\tau$  elements after the first acceptance of an  $e \in S_l$  may be affected.

$$\begin{aligned}\mathbb{E} \left[ \sum_{e \in S_l} \Delta_+^{\max}(e) - \Delta_+(e) \right] &= \mathbb{E}[\#\{e : e \in S_l \wedge e_l^\eta \in A^{\iota(e)-1} \wedge e_l^\eta \notin \hat{A}_e\}] \\ &= \mathbb{E}[\mathbb{E}[\#\{e : e \in S_l \wedge e_l^\eta \in A^{\iota(e)-1} \wedge e_l^\eta \notin \hat{A}_e\} \mid \eta = t, \pi(e_l^t) = k]] \\ &= \sum_{t=1}^{n_l} \sum_{k=t}^{N-n+t} P(\eta = t, \pi(e_l^t) = k) \mathbb{E}[\#\{e : e \in S_l \wedge e_l^\eta \in A^{\iota(e)-1} \wedge e_l^\eta \notin \hat{A}_e\} \mid \eta = t, \pi(e_l^t) = k] \\ &= \sum_{t=1}^{n_l} P(\eta = t) \sum_{k=t}^{N-n+t} P(\pi(e_l^t) = k) \mathbb{E}[\#\{e : e \in S_l \wedge e_l^\eta \in A^{\iota(e)-1} \wedge e_l^\eta \notin \hat{A}_e\} \mid \eta = t, \pi(e_l^t) = k].\end{aligned}$$

Under the assumption that every ordering  $\pi$  is equally likely, and a bounded delay  $\tau$ , conditioned on  $\eta = t, \pi(e_l^t) = k$ , the random variable  $\#\{e : e \in S_l \wedge e_l^\eta \in A^{t(e)-1} \wedge e_l^\eta \notin \hat{A}_e\}$  has hypergeometric distribution with mean  $\frac{n_l-t}{N-k}\tau$ . Also,  $P(\pi(e_l^t) = k) = \frac{n_l}{N} \binom{n-1}{t-1} \binom{N-n}{k-t} / \binom{N-1}{k-1}$ , so the above expression becomes

$$\begin{aligned}
& \mathbb{E} \left[ \sum_{e \in S_l} \Delta_+^{\max}(e) - \Delta_+(e) \right] \\
&= \sum_{t=1}^{n_l} P(\eta = t) \sum_{k=t}^{N-n+t} \frac{n_l}{N} \frac{\binom{n-1}{t-1} \binom{N-n}{k-t}}{\binom{N-1}{k-1}} \frac{n-t}{N-k} \tau \\
&= \frac{n_l}{N} \tau \sum_{t=1}^{n_l} P(\eta = t) \sum_{k=t}^{N-n+t} \frac{\binom{k-1}{t-1} \binom{N-k}{n-t}}{\binom{N-1}{n-1}} \frac{n-t}{N-k} \quad (\text{symmetry of hypergeometric}) \\
&= \frac{n_l}{N} \tau \sum_{t=1}^{n_l} \frac{P(\eta = t)}{\binom{N-1}{n-1}} \sum_{k=t}^{N-n+t} \binom{k-1}{t-1} \binom{N-k-1}{n-t-1} \\
&= \frac{n_l}{N} \tau \sum_{t=1}^{n_l} \frac{P(\eta = t)}{\binom{N-1}{n-1}} \binom{N-1}{n-1} \quad (\text{Lemma C.1, } a = N-2, b = n_l-2, j = 1) \\
&= \frac{n_l}{N} \tau \sum_{t=1}^{n_l} P(\eta = t) \\
&= \frac{n_l}{N} \tau.
\end{aligned}$$

Now if we define  $\rho_e = \Delta_+^{\max}(e) - \Delta_+(e) + \Delta_-^{\max}(e) - \Delta_-(e)$ , we get

$$\begin{aligned}
\mathbb{E} \left[ \sum_e \rho_e \right] &= \mathbb{E} \left[ \sum_e \Delta_+^{\max}(e) - \Delta_+(e) + \Delta_-^{\max}(e) - \Delta_-(e) \right] \\
&= \sum_l \mathbb{E} \left[ \sum_{e \in S_l} \Delta_+^{\max}(e) - \Delta_+(e) \right] + \mathbb{E} \left[ \sum_{e \in S_l} \Delta_-^{\max}(e) - \Delta_-(e) \right] \\
&\leq \tau \frac{\sum_l n_l}{N} + L(1 - \lambda^\tau) \\
&= \tau + L(1 - \lambda^\tau).
\end{aligned}$$

Note that  $\mathbb{E}[\sum_e \rho_e]$  does not depend on  $N$  and is linear in  $\tau$ . Also, if  $\tau = 0$  in the sequential case, we get  $\mathbb{E}[\sum_e \rho_e] \leq 0$ .

## B Upper bound on expected number of elements sent for validation

Let  $N$  be the number of elements, i.e. the cardinality of the ground set. Let  $P$  be the number of processors.

We assume that the total ordering assigns elements to processors in a round robin fashion. Thus, we assume  $C^{ji} = \{i - p + 1, \dots, i - 1\}$  has  $p - 1$  elements.

We call element  $i$  *dependent* on  $i'$  if  $\exists A, F(A \cup i) - F(A) \neq F(A \cup i' \cup i) - F(A \cup i')$  or  $\exists B, F(B \setminus i) - F(B) \neq F(B \cup i' \setminus i) - F(B \cup i')$ , i.e. the result of the transaction on  $i'$  will affect the computation of  $\Delta$ 's for  $i$ . For example, for the graph cut problem, every vertex is dependent on its neighbors; for the separable sums problem,  $i$  is dependent on  $\{i' : \exists S_i, i \in S_i, i' \in S_i\}$ .

Let  $n_i$  be the number of elements that  $i$  is dependent on.

Now, we note that if  $C^{ji}$  does not contain any elements on which  $i$  is dependent, then  $\Delta_+^{\max}(i) = \Delta_+(i) = \Delta_+^{\min}(i)$  and  $\Delta_-^{\max}(i) = \Delta_-(i) = \Delta_-^{\min}(i)$ , so  $i$  will not be validated (in either deterministic or probabilistic versions). Conversely, if  $i$  is validated, there must be some element  $i' \in C^{ji}$  such that  $i$  is dependent on  $i'$ .

$$\begin{aligned}
& E(\text{number of validated elements}) \\
&= \sum_i P(i \text{ validated}) \\
&\leq \sum_i P(\exists i' \in C^{ji}, i \text{ depends on } i') \\
&= \sum_i 1 - P(\forall i' \in C^{ji}, i \text{ does not depend on } i') \\
&= \sum_i 1 - \prod_{k=1}^{P-1} \frac{N - k - n_i}{N - k} \\
&= \sum_i 1 - \prod_{k=1}^{P-1} \left(1 - \frac{n_i}{N - k}\right) \\
&\leq \sum_i 1 - \left(1 - \sum_{k=1}^{P-1} \frac{n_i}{N - k}\right) \quad (\text{Weierstrass inequality}) \\
&= \left(\sum_i n_i\right) \left(\sum_{k=1}^{P-1} \frac{1}{N - k}\right) \\
&\leq \frac{P-1}{N-P+1} \sum_i n_i.
\end{aligned}$$

The key quantity in the above inequality is  $\sum_i n_i$ . Typically, we expect each element  $i$  to depend on a small fraction of the ground set. For example, in the graph cut problem,  $\sum_i n_i = 2|E|$  is twice the number of edges. If the graph is sparse with  $|E| \approx s|V| \log |V|$ , where  $0 \leq s \ll 1$  and  $P \ll N$ , then  $\frac{P-1}{N-P+1} \sum_i n_i \approx 2s(P-1) \log N$ , which grows sublinearly with  $N$ .

Note that the bound established above is generic to all algorithms that follow the basic transactional model we proposed (round-robin optimistic concurrency control), and is not specific to  $F$  or even submodular maximization. Thus, while our bounds provide a fundamental limit, additional knowledge of  $F$  can lead to better analyses on the algorithm's concurrency.

### B.1 Tighter general bound?

Define  $\rho_i = \max_{S \subseteq V} \{[F(S \cup i) - F(S)] - [F(S \cup C^{ji} \cup i) - F(S \cup C^{ji})]\} \leq F(i) - F(V) + F(V \setminus i)$

[XP: Is there theory along these lines?]

Then, we can bound

$$\begin{aligned}\Delta_+^{\min} &\leq \Delta_+^{\max} \leq \Delta_+^{\min} + \rho_i && (\text{choosing } S = A^j) \\ \Delta_-^{\min} &\leq \Delta_-^{\max} \leq \Delta_-^{\min} + \rho_i && (\text{choosing } S = A^j \cup D^i)\end{aligned}$$

Thus,

$$\begin{aligned}&E(\text{number of validated elements}) \\&= \sum_i P(i \text{ validated}) \\&= \sum_i P\left(\frac{\Delta_+^{\min}}{\Delta_+^{\min} + \Delta_-^{\max}} \leq u_i \leq \frac{\Delta_+^{\max}}{\Delta_+^{\max} + \Delta_-^{\min}}\right) \\&= \sum_i \frac{\Delta_+^{\max}}{\Delta_+^{\max} + \Delta_-^{\min}} - \frac{\Delta_+^{\min}}{\Delta_+^{\min} + \Delta_-^{\max}} \\&\leq \sum_i \frac{\Delta_+^{\min} + \rho_i}{\Delta_+^{\min} + \rho_i + \Delta_-^{\min}} - \frac{\Delta_+^{\min}}{\Delta_+^{\min} + \rho_i + \Delta_-^{\min}} \\&= \sum_i \frac{\rho_i}{\Delta_+^{\min} + \rho_i + \Delta_-^{\min}}\end{aligned}$$

## B.2 Upper bound for max graph cut

Denote  $\tilde{A}^j = V \setminus A^j \setminus C^{ji} \setminus D^i = \{1, \dots, j\} \setminus A^j$  be the elements up to  $j$  that are not included in  $A$ . Let  $w_i(S) = \sum_{j \in S, (i,j) \in E} w(i, j)$ . For the max graph cut function, it is easy to see that

$$\begin{aligned}\Delta_+^{\min} &= \max(0, -w_i(A^j) - w_i(C^{ji}) + w_i(D^i) + w_i(\tilde{A}^j)) \\ \Delta_+^{\max} &= \max(0, -w_i(A^j) + w_i(C^{ji}) + w_i(D^i) + w_i(\tilde{A}^j)) \\ \Delta_-^{\min} &= \max(0, +w_i(A^j) - w_i(C^{ji}) + w_i(D^i) - w_i(\tilde{A}^j)) \\ \Delta_-^{\max} &= \max(0, +w_i(A^j) + w_i(C^{ji}) + w_i(D^i) - w_i(\tilde{A}^j))\end{aligned}$$

Consider the following cases.

- $\Delta_+^{\max} = 0$ . Then  $\Delta_+^{\min} = 0$  and also

$$w_i(A^j) > w_i(C^{ji}) + w_i(D^i) + w_i(\tilde{A}^j) \implies w_i(A^j) + w_i(D^i) > w_i(C^{ji}) + w_i(\tilde{A}^j)$$

$$\text{so } \Delta_-^{\min} > 0 \text{ and } \Delta_-^{\max} > 0. \text{ Thus } \frac{\Delta_+^{\max}}{\Delta_+^{\max} + \Delta_-^{\min}} - \frac{\Delta_+^{\min}}{\Delta_+^{\min} + \Delta_-^{\max}} = 0 - 0 = 0.$$

- $\Delta_-^{\max} = 0$ . Then  $\Delta_-^{\min} = 0$  and also

$$w_i(\tilde{A}^j) > w_i(C^{ji}) + w_i(D^i) + w_i(A^j) \implies w_i(\tilde{A}^j) + w_i(D^i) > w_i(C^{ji}) + w_i(A^j)$$

$$\text{so } \Delta_+^{\min} > 0 \text{ and } \Delta_+^{\max} > 0. \text{ Thus } \frac{\Delta_+^{\max}}{\Delta_+^{\max} + \Delta_-^{\min}} - \frac{\Delta_+^{\min}}{\Delta_+^{\min} + \Delta_-^{\max}} = 1 - 1 = 0.$$



- $\Delta_+^{\max} > 0$  and  $\Delta_-^{\max} > 0$ . Then,

$$\begin{aligned}
& \frac{\Delta_+^{\max}}{\Delta_+^{\max} + \Delta_-^{\min}} - \frac{\Delta_+^{\min}}{\Delta_+^{\min} + \Delta_-^{\max}} \\
&= \frac{-w_i(A^j) + w_i(C^{ji}) + w_i(D^i) + w_i(\tilde{A}^j)}{-w_i(A^j) + w_i(C^{ji}) + w_i(D^i) + w_i(\tilde{A}^j) + \max(0, +w_i(A^j) - w_i(C^{ji}) + w_i(D^i) - w_i(\tilde{A}^j))} \\
&\quad - \frac{\max(0, -w_i(A^j) - w_i(C^{ji}) + w_i(D^i) + w_i(\tilde{A}^j))}{\max(0, -w_i(A^j) - w_i(C^{ji}) + w_i(D^i) + w_i(\tilde{A}^j)) + w_i(A^j) + w_i(C^{ji}) + w_i(D^i) - w_i(\tilde{A}^j)} \\
&= \min \left( 1, \frac{-w_i(A^j) + w_i(C^{ji}) + w_i(D^i) + w_i(\tilde{A}^j)}{2w_i(D^i)} \right) \\
&\quad - \max \left( 0, \frac{-w_i(A^j) - w_i(C^{ji}) + w_i(D^i) + w_i(\tilde{A}^j)}{2w_i(D^i)} \right) \\
&= \min \left( 1, \frac{w_i(C^{ji})}{w_i(D^i)} \right)
\end{aligned}$$

Thus,

$$E(\# \text{ of validated elements}) = \sum_i \frac{\Delta_+^{\max}}{\Delta_+^{\max} + \Delta_-^{\min}} - \frac{\Delta_+^{\min}}{\Delta_+^{\min} + \Delta_-^{\max}} \leq \sum_i \min \left( 1, \frac{w_i(C^{ji})}{w_i(D^i)} \right)$$

[XP: Not sure how to sum this over  $i$ .]

$$\sum_{\pi} \sum_i \min(1, w_i(C)/w(D^i)) \leq E(\sum_i w_i(C)) = c * \sum_i \deg(i)/n$$

### B.3 Upper bound for set cover

We make the same assumptions as before in the hogwild analysis, i.e. the sets  $S_l$  form a partition of  $V$ , there is a bounded delay  $\tau$ .

Observe that for any  $e \in S_l$ ,  $\Delta_-^{\min}(e) \neq \Delta_-^{\max}(e)$  if  $\hat{B}_e \setminus e \cap S_l \neq \emptyset$  and  $\tilde{B}_e \setminus e \cap S_l = \emptyset$ .

This is only possible if  $e_l^{n_l} \notin \tilde{B}_e$  and  $\tilde{B}_e \supset \hat{A}_e \cap S_l = \emptyset$ , that is  $\pi(e) \geq \pi(e_l^{n_l}) - \tau$  and  $\forall e' \in S_l, (\pi(e') < \pi(e_l^{n_l}) - \tau) \implies (e' \notin A)$ . The latter condition is achieved with probability  $\lambda^{n_l - m_l}$ , where  $m_l = \#\{e' : \pi(e') \geq \pi(e_l^{n_l}) - \tau\}$ . Thus,

$$\begin{aligned}
\mathbb{E} [\#\{e : \Delta_-^{\min}(e) \neq \Delta_-^{\max}(e)\}] &= \mathbb{E}[m_l \mathbf{1}(\forall e' \in S_l, (\pi(e') < \pi(e_l^{n_l}) - \tau) \implies (e' \notin A))] \\
&= \mathbb{E}[\mathbb{E}[m_l \mathbf{1}(\forall e' \in S_l, (\pi(e') < \pi(e_l^{n_l}) - \tau) \implies (e' \notin A)) | u_{1:N}]] \\
&= \mathbb{E}[m_l \mathbb{E}[\mathbf{1}(\forall e' \in S_l, (\pi(e') < \pi(e_l^{n_l}) - \tau) \implies (e' \notin A)) | u_{1:N}]] \\
&= \mathbb{E}[m_l \lambda^{n_l - m_l}] \\
&\leq \lambda^{(n_l - \tau) +} \mathbb{E}[m_l] \\
&= \lambda^{(n_l - \tau) +} \mathbb{E}[\mathbb{E}[m_l | \pi(e_l^{n_l}) = k]] \\
&= \lambda^{(n_l - \tau) +} \sum_{k=n_l}^N P(\pi(e_l^{n_l}) = k) \mathbb{E}[m_l | \pi(e_l^{n_l}) = k].
\end{aligned}$$

918 Conditioned on  $\pi(e_l^{n_l}) = k$ ,  $m_l$  is a hypergeometric random variable with mean  $\frac{n_l-1}{k-1}\tau$ . Also  
919  $P(\pi(e_l^{n_l}) = k) = \frac{n_l}{N} \binom{n_l-1}{0} \binom{N-n_l}{N-k} / \binom{N-1}{N-k}$ . The above expression is therefore  
920  
921  $\mathbb{E} [\#\{e : \Delta_-^{\min}(e) \neq \Delta_-^{\max}(e)\}]$   
922  
923  $= \lambda^{(n_l-\tau)+} \sum_{k=n_l}^N \frac{n_l}{N} \frac{\binom{n_l-1}{0} \binom{N-n_l}{N-k}}{\binom{N-1}{N-k}} \frac{n_l-1}{k-1} \tau$   
924  
925  
926  $= \lambda^{(n_l-\tau)+} \frac{n_l}{N} \tau \sum_{k=n_l}^N \frac{\binom{N-k}{0} \binom{k-1}{n_l-1}}{\binom{N-1}{n_l-1}} \frac{n_l-1}{k-1}$  (symmetry of hypergeometric)  
927  
928  
929  $= \lambda^{(n_l-\tau)+} \frac{n_l}{N} \frac{\tau}{\binom{N-1}{n_l-1}} \sum_{k=n_l}^N \binom{N-k}{0} \binom{k-2}{n_l-2}$   
930  
931  
932  $= \lambda^{(n_l-\tau)+} \frac{n_l}{N} \frac{\tau}{\binom{N-1}{n_l-1}} \binom{N-1}{n_l-1}$  (Lemma C.1,  $a = N-2$ ,  $b = n_l-2$ ,  $j = 2$ ,  $t = n_l$ )  
933  
934  $= \lambda^{(n_l-\tau)+} \frac{n_l}{N} \tau.$   
935  
936

937 Now we consider any element  $e \in S_l$  with  $\pi(e) < \pi(e_l^{n_l}) - \tau$  that is validated. (Note that  $e_l^{n_l} \in \hat{B}_e$   
938 and  $\tilde{B}_e$ , so  $\Delta_-^{\min}(e) = \Delta_-^{\max}(e) = \lambda$ .) It must be the case that  $\hat{A}_e \cap S_l = \emptyset$ , for otherwise  
939  $\Delta_+^{\min}(e) = \Delta_+^{\max}(e) = -\lambda$  and we do not need to validate. This implies that  $\Delta_+^{\max}(e) = 1 - \lambda \geq u_i$ .  
940 At validation, if  $A^{(e)-1} \cap S_l = \emptyset$ , we accept  $e$  into  $A$ . Otherwise,  $A^{(e)-1} \cap S_l \neq \emptyset$ , which implies  
941 that some other element  $e' \in S_l$  has been accepted. Thus, we conclude that every element  $e \in S_l$  that  
942 is validated must be within  $\tau$  of the first accepted element  $e_l^\eta$  in  $S_l$ . The expected number of such  
943 elements is exactly as we computed in the hogwild analysis:  $\frac{n_l}{N} \tau$ .  
944

945 Hence, the expected number of elements that we need to validate is upper bounded as

946 
$$\mathbb{E}[\#\text{validated}] \leq \sum_l (1 + \lambda^{(n_l-\tau)+}) \frac{n_l}{N} \tau$$
  
947  
948 
$$\leq \sum_l 2 \frac{n_l}{N} \tau$$
  
949  
950 
$$= 2\tau.$$
  
951  
952  
953  
954  
955  
956  
957  
958  
959  
960  
961  
962  
963  
964  
965  
966  
967  
968  
969  
970  
971

## C Lemma

**Lemma C.1.**  $\sum_{k=t}^{a-b+t} \binom{k-j}{t-j} \binom{a-k+j}{b-t+j} = \binom{a+1}{b+1}.$

*Proof.*

$$\begin{aligned}
& \sum_{k=t}^{a-b+t} \binom{k-j}{t-j} \binom{a-k+j}{b-t+j} \\
&= \sum_{k'=0}^{a-b} \binom{k'+t-j}{t-j} \binom{a-k'-t+j}{b-t+j} \\
&= \sum_{k'=0}^{a-b} \binom{k'+t-j}{k'} \binom{a-k'-t+j}{a-b-k'} \quad (\text{symmetry of binomial coeff.}) \\
&= (-1)^{a-b} \sum_{k'=0}^{a-b} \binom{-t+j-1}{k'} \binom{-b+t-j-1}{a-b-k'} \quad (\text{upper negation}) \\
&= (-1)^{a-b} \binom{-b-2}{a-b} \quad (\text{Chu-Vandermonde's identity}) \\
&= \binom{a+1}{a-b} \quad (\text{upper negation}) \\
&= \binom{a+1}{b+1} \quad (\text{symmetry of binomial coeff.})
\end{aligned}$$

□

1026 **D Full experiment results**

1027  
1028  
1029  
1030  
1031  
1032  
1033  
1034  
1035  
1036  
1037  
1038  
1039  
1040  
1041  
1042  
1043  
1044  
1045  
1046  
1047  
1048  
1049  
1050  
1051  
1052  
1053  
1054  
1055  
1056  
1057  
1058  
1059  
1060  
1061  
1062  
1063  
1064  
1065  
1066  
1067  
1068  
1069  
1070  
1071  
1072  
1073  
1074  
1075  
1076  
1077  
1078  
1079

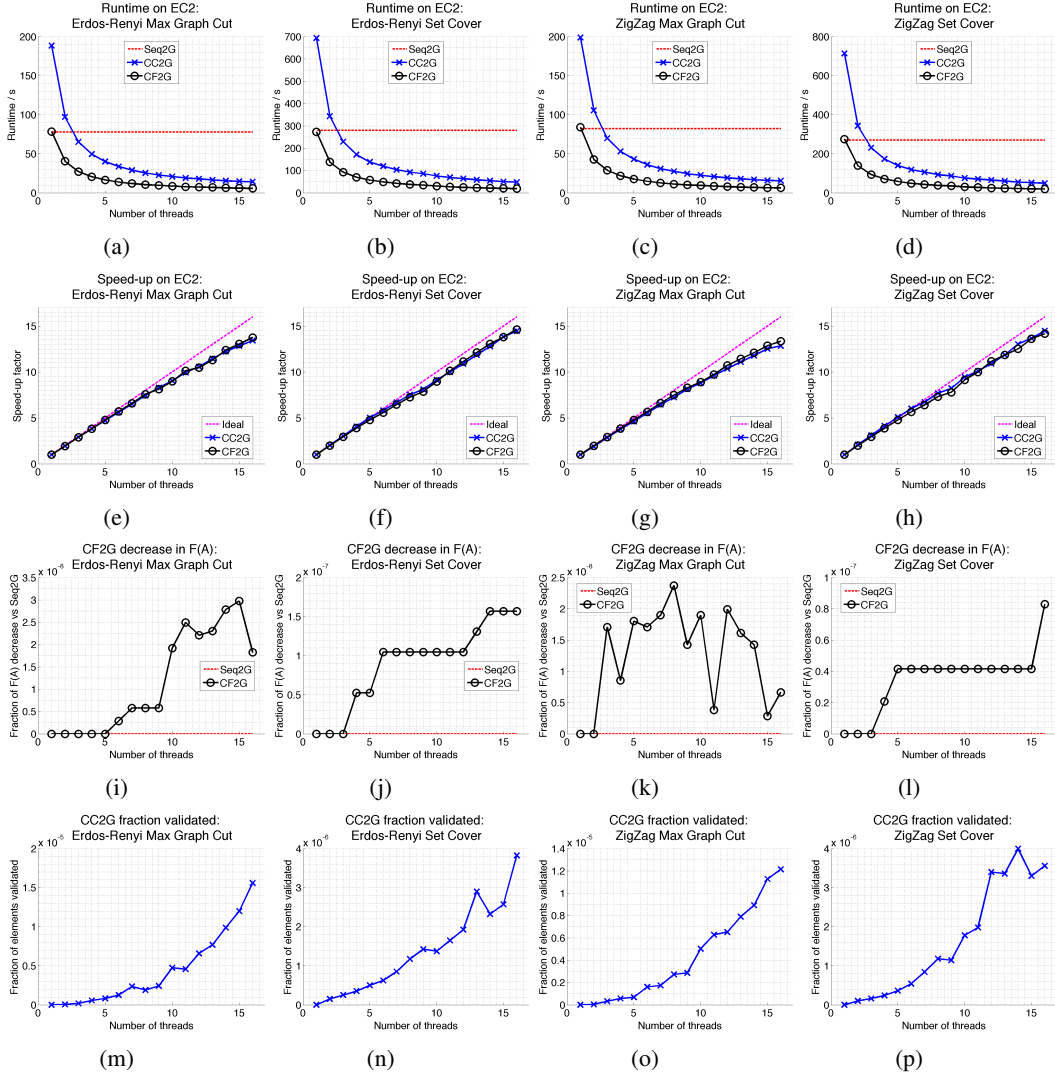


Figure 3: Experimental results on Erdos-Renyi and ZigZag synthetic graphs.

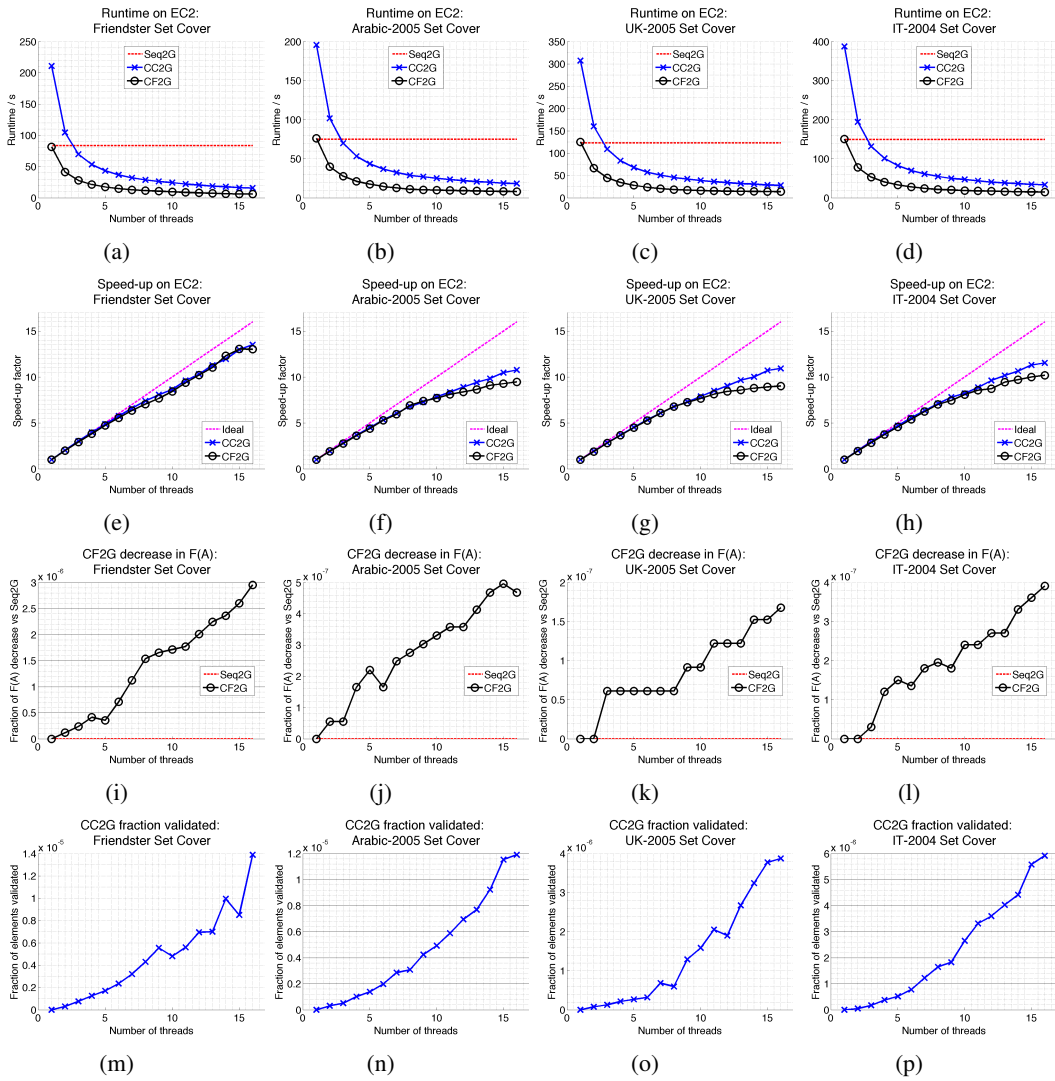


Figure 4: Set cover on 4 real graphs.

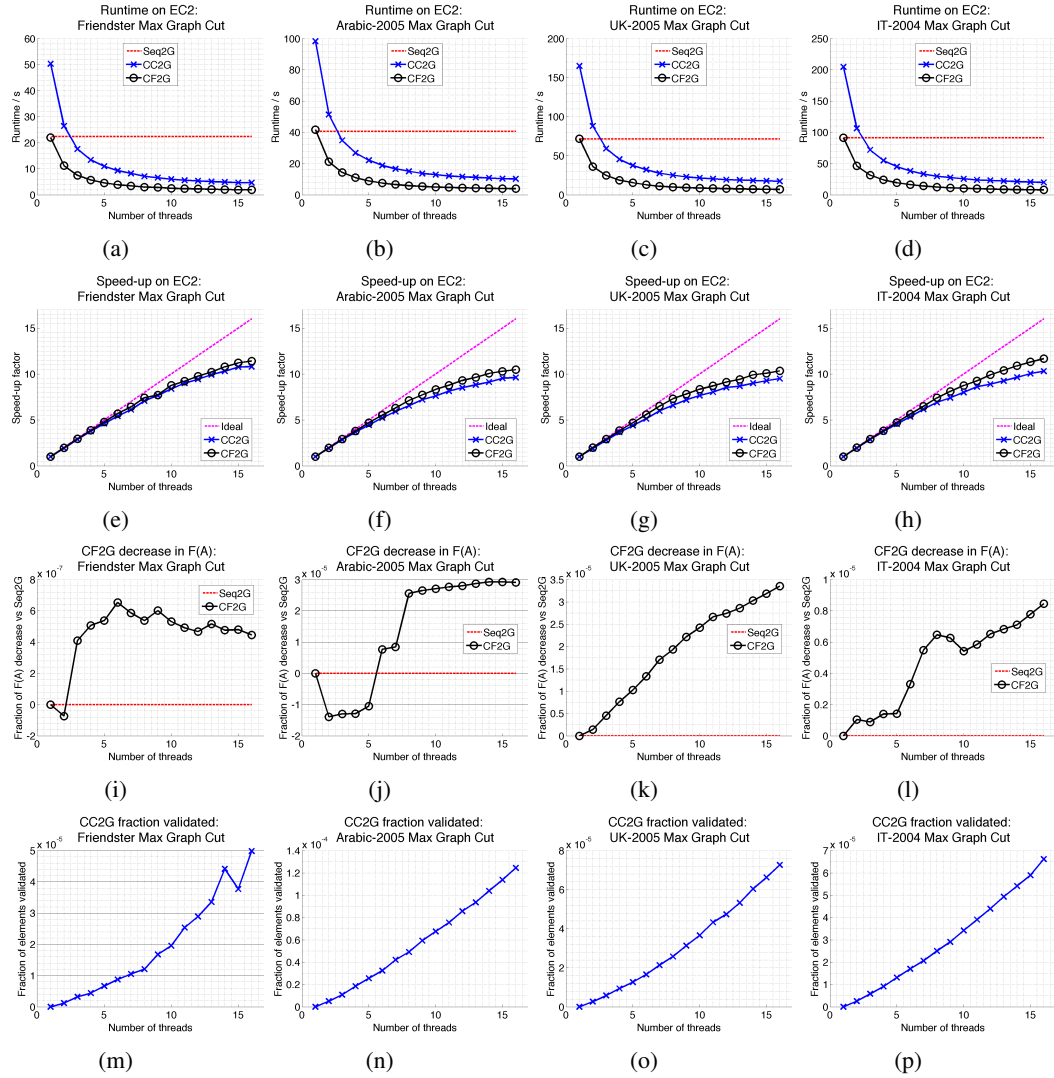


Figure 5: Max graph cut on 4 real graphs.

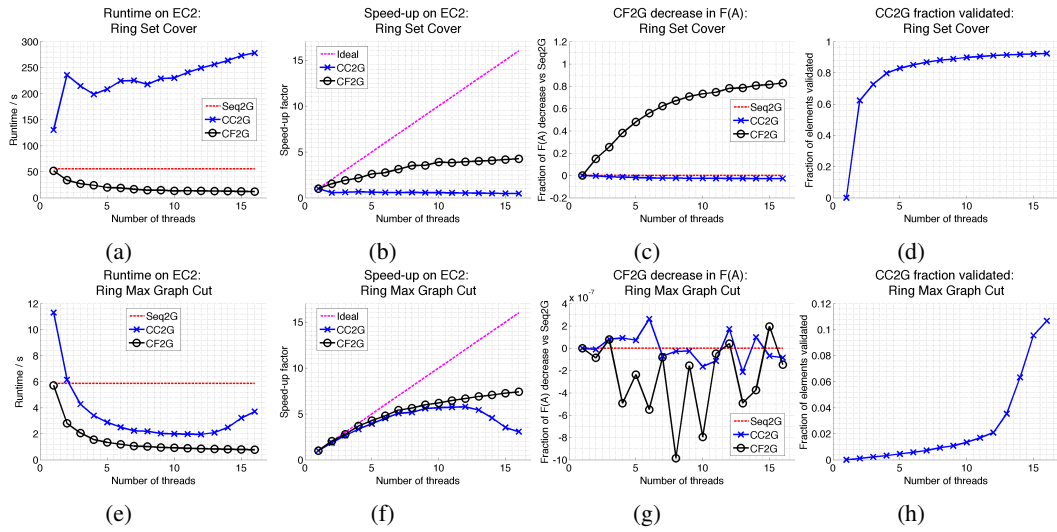


Figure 6: Experimental results for ring graph on set cover problem.

## E Full experiment results, partitioning with work stealing



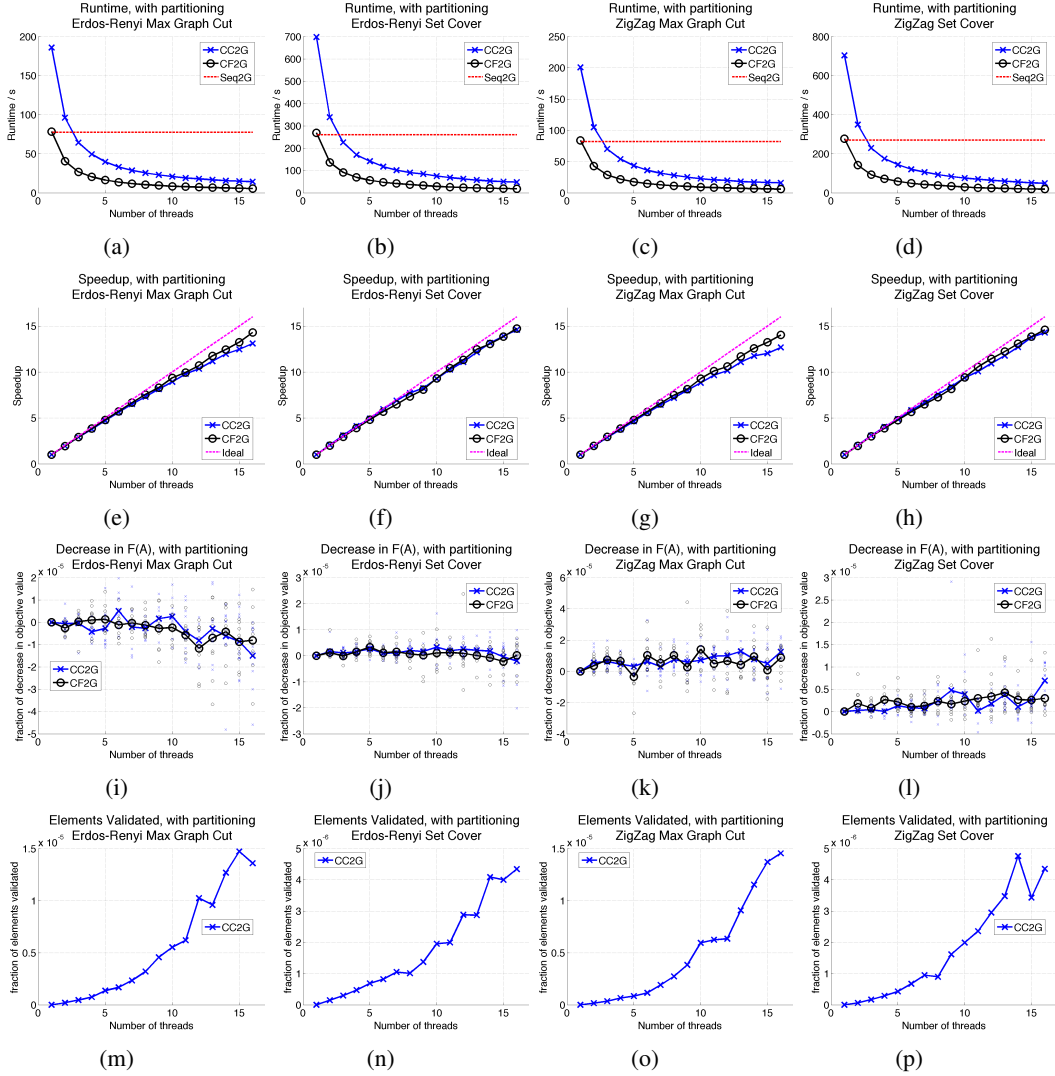


Figure 7: Experimental results (with partitioning) on Erdos-Renyi and ZigZag synthetic graphs.

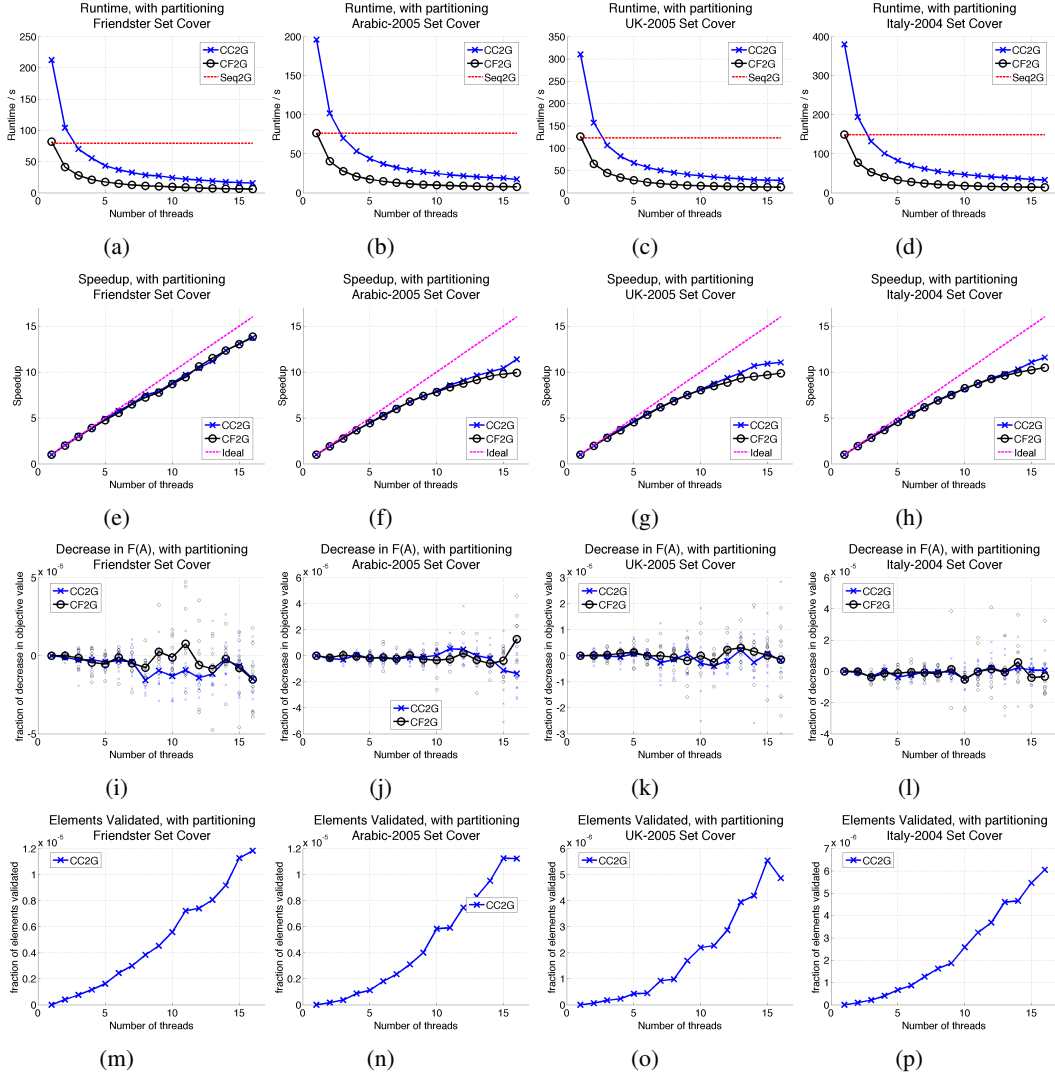


Figure 8: Set cover (with partitioning) on 4 real graphs.

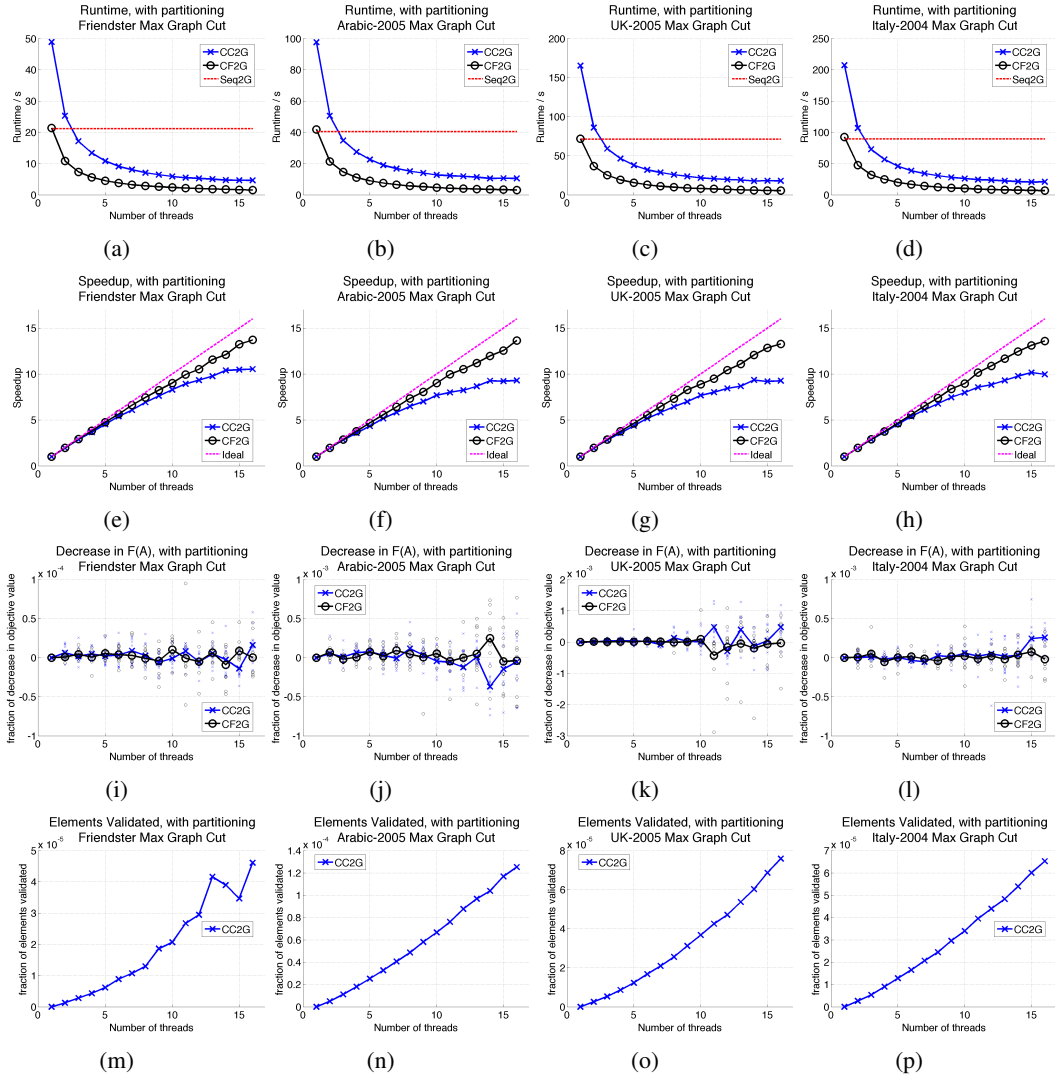


Figure 9: Max graph cut (with partitioning) on 4 real graphs.

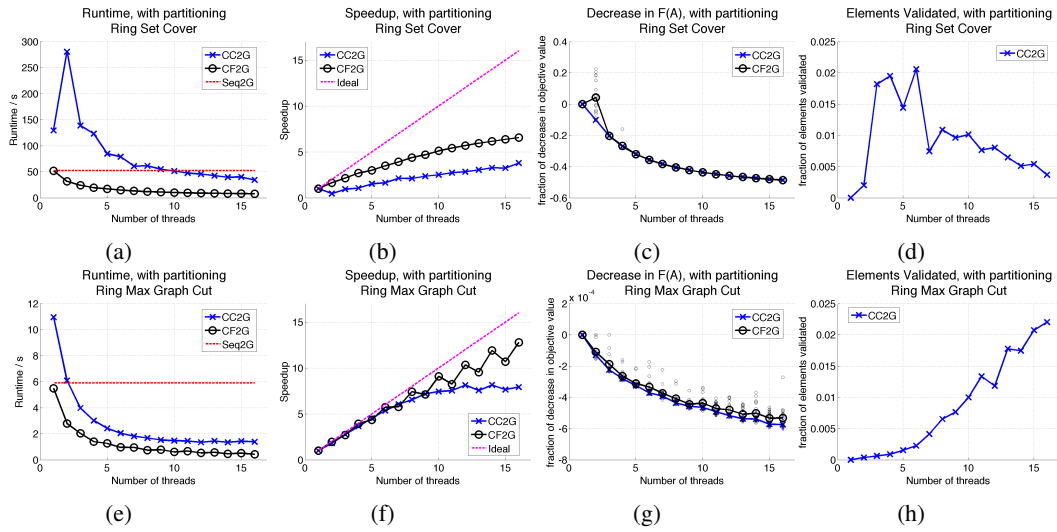


Figure 10: Experimental results (with partitioning) for ring graph on set cover problem.