$\Omega = \mathbb{R}$ $\quad P(\Omega) = 1$

$\Omega \xrightarrow{X} A \subset \mathbb{R} \to (-\infty, \infty)$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

```
> fu1 = function(x){x+2}
> integrate(fu1, -2, -1)$value
[1] 0.5
```

```
> fu2 = function(x){x**0}
> integrate(fu2, -1, 1)$value
[1] 2
```

```
> fu3 = function(x){(2-x)/3}
> integrate(fu3, 1, 2)$value
[1] 0.1666667
```

Ejemplo 6a

Hallar el valor de a ($\in \mathbb{R}$) para que $f_X(x)$ sea una función de densidad:

$$f_X(x) = \begin{cases} a(x+2), & -2 \le x < -1, \\ a, & -1 \le x < 1, \\ \dfrac{2-x}{3}, & 1 \le x \le 2, \\ 0, & \text{en otro caso.} \end{cases}$$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$$\int_{-\infty}^{-2} \cancel{0}\,dx + \int_{-2}^{-1} a(x+2)\,dx + \int_{-1}^{1} a\,dx + \int_{1}^{2} \frac{2-x}{3}\,dx + \int_{2}^{\infty} \cancel{0}\,dx = 1$$

$$a\int_{-2}^{-1} (x+2)\,dx + a\int_{-1}^{1} 1\,dx + \int_{1}^{2} \frac{2-x}{3}\,dx = 1$$

$$\frac{a}{2} + 2a + \frac{1}{6} = 1$$

$$\frac{5a}{2} = \frac{5}{6} \Rightarrow a = \frac{1}{3}$$

Ejemplo 6a

Hallar el valor de a para que $f_X(x)$ sea una función de densidad:

$$f_X(x) = \begin{cases} a(x+2), & -2 \le x < -1, \\ a, & -1 \le x < 1, \\ \dfrac{2-x}{3}, & 1 \le x \le 2, \\ 0, & \text{en otro caso.} \end{cases}$$

función de densidad

$$f(x) = \frac{1}{3}(x+2)\, I_{[-2,1)}(x) + \frac{1}{3} I_{[-1,1)}(x) + \frac{2-x}{3} I_{[1,2]}(x)$$

Ejemplo 6b

Calcular $P(\underbrace{X < 0}_{\text{disjuntos}} \cup \underbrace{X > 1.5}_{\text{disjuntos}} | X > -0.5) = P(X<0 \mid X>-0.5) + P(X>1.5 \mid X>-0.5)$

$$= \frac{P(-0.5<X<0)}{P(X>-0.5)} + \frac{P(X>1.5)}{P(X>-0.5)} = \frac{0.1667}{0.6667} + \frac{0.0417}{0.6667} = 0.3125$$

$$* \; P(-0.5<X<0) = \int_{-0.5}^{0} \frac{1}{3}\, dx = \frac{1}{6} = 0.1667$$

$$* \; P(X>1.5) = \int_{1.5}^{2} \frac{2-x}{3}\, dx = 0.0417$$

$$* \; P(X>-0.5) = \int_{-0.5}^{1} \frac{1}{3} dx + \int_{1}^{2} \frac{2-x}{3}\, dx = 0.5 + 0.1667 = 0.6667$$

Ejemplo 6c

Obtener la función de distribución de X

$x < -2$, $\quad F_X(x) = 0$

$-2 \le x < 1$, $\quad F_X(x) = \int_{-2}^{x} \frac{1}{3}(t+2)\,dt = \left(\frac{t^2}{6} + \frac{2}{3}t\right)\Big|_{-2}^{x} = \left(\frac{x^2}{6} + \frac{2x}{3}\right) - \underbrace{\left(\frac{4}{6} - \frac{4}{3}\right)}_{-\frac{2}{3}} = \frac{x^2}{6} + \frac{2}{3}x + \frac{2}{3}$

$-1 \le x < 1$, $\quad F_X(x) = \int_{-2}^{-1} \frac{1}{3}(x+2)\,dx + \int_{-1}^{x} \frac{1}{3}\,dt = \frac{1}{6} + \frac{t}{3}\Big|_{-1}^{x}$

$= \frac{1}{6} + \frac{x}{3} - \frac{-1}{3} = \frac{x}{3} + \frac{1}{6} + \frac{1}{3} = \frac{x}{3} + \frac{1}{2}$

$1 \le x \le 2$, $\quad F_X(x) = \int_{-2}^{-1} \frac{1}{3}(x+2)\,dx + \int_{-1}^{1} \frac{1}{3}\,dx + \int_{1}^{x} \frac{2-t}{3}\,dt = \frac{1}{6} + \frac{2}{3} + \left(\frac{2t}{3} - \frac{t^2}{6}\right)\Big|_{1}^{x}$

$= \frac{1}{6} + \cancel{\frac{2}{3}} + \left(\frac{2x}{3} - \frac{x^2}{6}\right) - \left(\cancel{\frac{2}{3}} - \frac{1}{6}\right)$

$= -\frac{x^2}{6} + \frac{2}{3}x + \frac{1}{3}$

$x > 2 \quad F_X(x) = 1$

Rspta: $F_X(x) = \left(\frac{x^2}{6} + \frac{2}{3}x + \frac{2}{3}\right) I_{[-2,-1)}(x) + \left(\frac{x}{3} + \frac{1}{2}\right) I_{[-1,1)}(x) + \left(-\frac{x^2}{6} + \frac{2}{3}x + \frac{1}{3}\right) I_{[1,2]}(x) + I_{(2,\infty)}(x)$ //

Valor esperado

$$\mu_X = E(X) = \int_{-\infty}^{\infty} x f(x)dx = \int_{0}^{\infty} (1 - F(x))dx - \int_{-\infty}^{0} F(x)dx$$

Ejemplo 6c

Calcular el valor esperado de X

función de densidad

$$f(x) = \frac{1}{3}(x+2)\, I_{[-2,1)}(x) + \frac{1}{3}\, I_{[-1,1)}(x) + \frac{2-x}{3}\, I_{[1,2]}(x)$$

$$\mu_x = E(x) = \int_{-2}^{-1} \frac{x}{3}(x+2)\, dx + \int_{-1}^{1} \frac{x}{3}\, dx + \int_{1}^{2} \frac{x(2-x)}{3}\, dx = -0.2222 + 0 + 0.2222 = 0$$

$$F_x(x) = \left(\frac{x^2}{6} + \frac{2}{3}x + \frac{2}{3}\right) I_{[-2,-1)}(x) + \left(\frac{x}{3} + \frac{1}{2}\right) I_{[-1,1)}(x) + \left(-\frac{x^2}{6} + \frac{2}{3}x + \frac{1}{3}\right) I_{[1,2]}(x) + I_{(2,\infty)}(x)$$

$$\mu_x = E(x) = \int_{0}^{1} \left(1 - \left(\frac{x}{3} + \frac{1}{2}\right)\right) dx + \int_{1}^{2} 1 - \left(-\frac{x^2}{6} + \frac{2}{3}x + \frac{1}{3}\right) dx + \int_{2}^{\infty} (1-1)\, dx - \int_{-\infty}^{-2} 0\, dx - \int_{-2}^{-1} \frac{x^2}{6} + \frac{2}{3}x + \frac{2}{3}\, dx - \int_{-1}^{0} \frac{x}{3} + \frac{1}{2}\, dx$$

$$= \cancel{0.3333} + \cancel{0.0556} + 0 - 0 - \cancel{0.0556} - \cancel{0.3333} = 0$$

Varianza

La varianza mide la dispersión de los valores de la variable aleatoria respecto a su esperanza matemática. Se calcula como:

$$\sigma_X^2 = Var(X) = \int_{-\infty}^{\infty} (x_i - \mu_X)^2 f_X(x_i)\,dx = E(X^2) - \underbrace{\mu_X^2}_{0}$$

Ejemplo 6d

Calcular la varianza de X

$$f(x) = \frac{1}{3}(x+2)\, I_{[-2,1)}(x) + \frac{1}{3} I_{[-1,1)}(x) + \frac{2-x}{3} I_{[1,2]}(x)$$

$$\sigma^2 = \underbrace{\int_{-\infty}^{\infty} x^2 f_X(x)\,dx}_{E(x^2)} = \cancel{\int_{-\infty}^{-2} x^2 \cdot 0\,dx} + \int_{-2}^{-1} \frac{x^2}{3}(x+2)\,dx + \int_{-1}^{1} \frac{x^2}{3}\,dx + \int_{1}^{2} \frac{x^2(2-x)}{3}\,dx + \cancel{\int_{2}^{\infty} x^2 \cdot 0\,dx}$$

$$\doteq 0.3056 + 0.2222 + 0.3056 = 0.8334 = E(x^2)$$

Dado que $\mu_X = E(X) = 0$, $E(X^2) = \sigma^2 = 0.8334$

Ejemplo 7a

Suponga que X representa a la variable aleatoria Concentración **real** (en $\mu g/m^3$) de dióxido de carbono (NO_2) por hora en una ciudad costera, cuya función de densidad es:

$$f_X(x) = \frac{1}{100}\exp(-\frac{x}{100})I_{(0,\infty)}(x)$$

Obtener la función de distribución de X

$F(x)$:

$x \leq 0$: $F_X(x) = 0$

$x > 0$: $F_X(x) = \int_0^x \frac{1}{100} e^{-\frac{t}{100}} dt = -\int_0^{-x/100} e^u du = -e^u \Big|_0^{-\frac{x}{100}} = -e^{-\frac{x}{100}} - (-\underbrace{e^0}_{1}) = 1 - e^{-\frac{x}{100}}$

$u = \frac{-t}{100}$ $\qquad t = 0 \rightarrow u = 0$

$du = \frac{-dt}{100}$ $\qquad t = x \rightarrow u = \frac{-x}{100}$

✓ $F_X(x) = (1 - e^{-x/100}) I_{(0,\infty)}(x)$

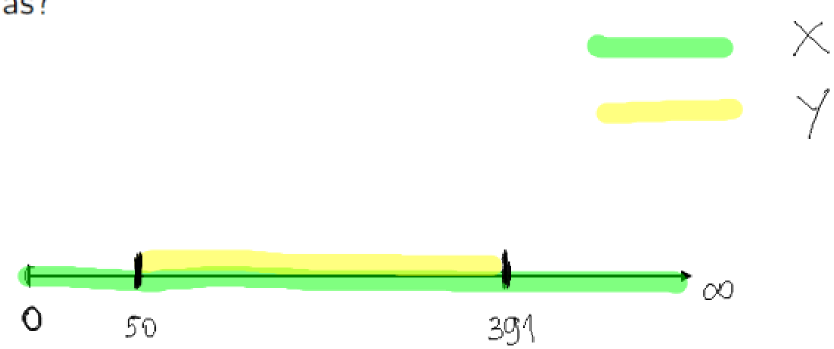
$\lim_{x \to \infty} F_X(x) = 1$

Ejemplo 7b

Sin embargo, esta concentración es **registrada** (Y) mediante un instrumento de medición cuyo límite inferior es de 50 $\mu g/m^3$ y el superior, 391 $\mu g/m^3$. Por lo tanto:

$$Y = 50I_{(0,50]}(x) + xI_{(50,391)}(x) + 391I_{[391,\infty)}(x)$$

Conocemos el comportamiento probabilístico de las concentraciones reales (X) pero, ¿cuál es el comportamiento probabilístico (Y) de las observadas?
Obtener $F_Y(y)$, $F_Y^d(y)$ y $F_Y^{ac}(y)$



$$\underbrace{\begin{matrix} x = 30 \\ x = 120 \\ x = 400 \end{matrix}}_{\text{Valor real}} \begin{matrix} \Rightarrow \\ \Rightarrow \\ \Rightarrow \end{matrix} \underbrace{\begin{matrix} Y = 50 \\ y = 120 \\ y = 391 \end{matrix}}_{\text{Valor registrado}}$$

$P(Y = 50) = P(X \leq 50) = F_X(50)$

$P(Y = 391) = P(X > 391) = 1 - P(X \leq 391) = 1 - F_X(391)$

$$\alpha = P(Y = 50) + P(Y = 391)$$

Intervalo $50 \leq y < 391$

$$F_Y(y) = P(Y \leq y) = P(X \leq y) = F_X(y) = 1 - \exp\left(-\frac{y}{100}\right)$$

$$F_Y^d(y) = \frac{\alpha_1}{\alpha_1 + \alpha_2} = \frac{1 - \exp(-50/100)}{1 - \exp(-50/100) + \exp(-391/100)} = \frac{0.394}{0.414} = 0.952$$

$$\downarrow \quad \frac{\alpha_1}{\alpha}$$

$$F_Y(y) = \alpha F_Y^d(y) + (1-\alpha)\boxed{F_Y^{ac}(y)} \; ?$$

$$1 - \exp\left(-\frac{y}{100}\right) = \cancel{\alpha} \times \frac{\alpha_1}{\cancel{\alpha}} + (1-\alpha) F_Y^{ac}(y)$$

$$\cancel{1} - \exp\left(-\frac{y}{100}\right) = \left(\cancel{1} - \exp\left(-\frac{50}{100}\right)\right) + \left(\exp\left(-\frac{50}{100}\right) - \exp\left(-\frac{391}{100}\right)\right) F_Y^{ac}(y)$$

$$\frac{\exp(-50/100) - \exp(-y/100)}{\exp(-50/100) - \exp(-391/100)} = F_Y^{ac}(y)$$

Intervalo $y < 50$

$$F_Y(y) = 0$$

$$F_Y^d(y) = 0$$

$$F_Y^{ac}(y) = 0$$

Se verifica que:

$$\alpha F_Y^d(y) + (1-\alpha) F_Y^{ac}(y) = F_Y(y)$$

Intervalo $50 \leq y < 391$

$$F_Y(y) = P(Y \leq y) = P(X \leq y) = F_X(y) = 1 - \exp\left(-\frac{y}{100}\right)$$

$$F_Y^d(y) = \frac{\alpha_1}{\alpha_1 + \alpha_2} = \frac{1 - \exp(-50/100)}{1 - \exp(-50/100) + \exp(-391/100)} = \frac{0.394}{0.414} = 0.952$$

$F_Y^{ac}(y)$ es obtenido mediante la igualdad $\alpha F_Y^d(y) + (1-\alpha) F_Y^{ac}(y) = F_Y(y)$

$$1 - \exp\left(-\frac{50}{100}\right) + \left[\exp\left(-\frac{50}{100}\right) - \exp\left(-\frac{391}{100}\right)\right] F_Y^{ac}(y) = 1 - \exp\left(-\frac{y}{100}\right)$$

Despejando:

$$F_Y^{ac}(y) = \frac{\exp\left(-\frac{50}{100}\right) - \exp\left(-\frac{y}{100}\right)}{\exp\left(-\frac{50}{100}\right) - \exp\left(-\frac{391}{100}\right)}$$

Intervalo $y \geq 391$

$$F_Y(y) = 1$$

$$F_Y^d(y) = 1$$

$$F_Y^{ac}(y) = 1$$

$$F_Y(y) = \left(1 - \exp\left(-\frac{y}{100}\right)\right) I_{[50,391)}(y) + I_{[391,\infty)}(y)$$

$$F_Y^d(y) = 0.952 I_{[50,391)}(y) + I_{[391,\infty)}(y)$$

$$F_Y^{ac}(y) = \frac{\exp\left(-\frac{50}{100}\right) - \exp\left(-\frac{y}{100}\right)}{\exp\left(-\frac{50}{100}\right) - \exp\left(-\frac{391}{100}\right)} I_{[50,391)}(y) + I_{[391,\infty)}(y)$$

Ejemplo 7c

Obtener $f_Y^d(y)$ y $f_Y^{ac}(y)$

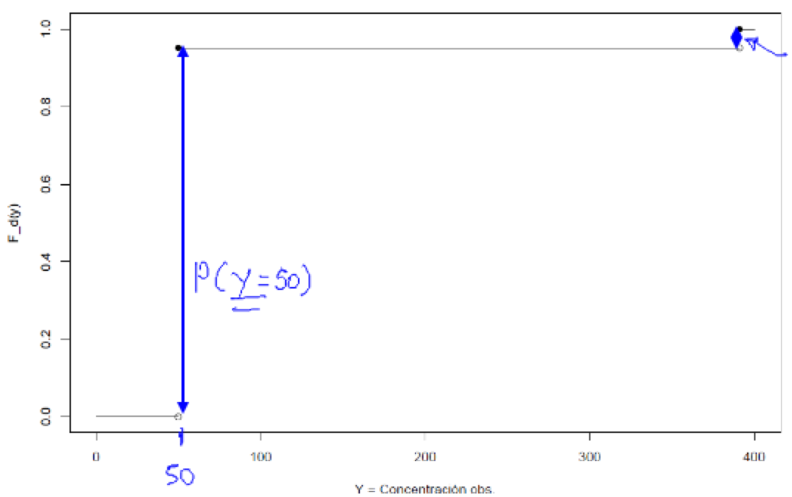
$f_Y^d(y)$, usaremos los pesos $\begin{cases} \alpha_1 = 0.394 \\ \alpha_2 = 0.02 \end{cases}$

$\alpha_1 + \alpha_2 = 0.414$

$$\cancel{f_Y^d(y) = \begin{cases} 0.394, & y = 50 \\ 0.02, & y = 391 \end{cases}}$$

$$f_Y^d(y) = \begin{cases} \dfrac{0.394}{0.414} = 0.952, & y = 50 \\ \dfrac{0.02}{0.414} = 0.048, & y = 391 \end{cases}$$ ✓

$$F_Y^d(y) = 0.952 I_{[50,391)}(y) + I_{[391,\infty)}(y)$$



$P(Y=391) = 1 - 0.952$
$= 0.048$

$$\boxed{f_Y^d(y) = 0.952 I_{\{50\}}(y) + 0.048 I_{\{391\}}(y)}$$

Para encontrar $f^{ac}_{(}y)$ utilizaremos el concepto de derivada de la f.d.a. absolutamente continua.

$$f_Y^{ac}(y) = \frac{dF_Y^{ac}(y)}{dy}$$

$$F_Y^{ac}(y) = \frac{\exp\left(-\frac{50}{100}\right) - \exp\left(-\frac{y}{100}\right)}{\exp\left(-\frac{50}{100}\right) - \exp\left(-\frac{391}{100}\right)} I_{[50,391)}(y) + I_{[391,\infty)}(y)$$

$$F_Y^{'ac}(y) = \frac{-\exp\left(-\frac{y}{100}\right) \times -\frac{1}{100}}{\exp\left(\frac{-50}{100}\right) - \exp\left(\frac{-391}{100}\right)} = \frac{\frac{1}{100}\exp\left(-\frac{y}{100}\right)}{\exp\left(-\frac{50}{100}\right) - \exp\left(-\frac{391}{100}\right)} I_{[50,391)}(y) = f_Y^{ac}(y)$$