$f_{X,Y,Z,\dots}(x, y, z, \dots)$: dist. conjunta

$f_X(x)$: dist. marginales (U2) $\rightarrow$ $f_X(x) = \int_{R_Z}\int_{R_Y} f_{X,Y,Z}(x,y,z)\,dy\,dz$

$F_{X,Y}(x, y)$: dist. acumulada conjunta

$F_X(x)$: dist. acumulada marginal. $\rightarrow$ $F_X(x) = \lim_{y\to\infty} F_{X,Y}(x,y) = \int_{-\infty}^{x} f_X(u)\,du$

$f_{X,Y,Z,W}(x,y,z,w)$, $\quad f_{X,Y|Z,W}(x,y|z,w) = \dfrac{f_{X,Y,Z,W}(x,y,z,w)}{f_{Z,W}(z,w)}$

$P(A|B) = \dfrac{P(A\cap B)}{P(B)}$

$f_{Z,W}(z,w) = \int_{R_Y}\int_{R_X} f_{X,Y,Z,W}(x,y,z,w)\,dx\,dy$

X	Y	Z	$f(x,y,z)$
0	0	0	0.10
0	0	1	0.05
0	1	0	0.15
0	1	1	0.10
1	0	0	0.20
1	0	1	0.10
1	1	0	0.20
1	1	1	0.10

$$f(z|x,y) = \frac{f(x,y,z)}{f(x,y)}$$

$$f_{X,Y}(x,y) = \sum_z p(x,y,z)$$

$$\begin{cases} f_{X,Y}(0,0) = 0.10 + 0.05 = 0.15 \\ f_{X,Y}(0,1) = 0.15 + 0.10 = 0.25 \\ f_{X,Y}(1,0) = 0.20 + 0.10 = 0.30 \\ f_{X,Y}(1,1) = 0.20 + 0.10 = 0.30 \end{cases}$$

$$f(Z=0 \mid X=0, Y=0) = \frac{f(0,0,0)}{f_{X,Y}(0,0)} = \frac{0.10}{0.15} = 0.667$$

$$f(Z=1 \mid X=0, Y=0) = \frac{f(0,0,1)}{f_{X,Y}(0,0)} = \frac{0.05}{0.15} = 0.333$$

1

$$f(Z=0 \mid X=0, Y=1)$$

⋮

$$f_{X,Y,Z}(x,y,z) = \begin{cases} 6, & 0 < x < y < z < 1, \\ 0, & \text{en otro caso.} \end{cases}$$

$$f_Z(z) = \int_0^z \int_0^y 6\,dx\,dy = \int_0^z 6y\,dy = 3z^2\, I_{(0,1)}(z)$$

$$f_{X,Y|Z}(x,y \mid z) = \frac{6}{3z^2} = \frac{2}{z^2}, \quad 0 < x < y < \underset{\hookrightarrow \text{cte conocida}}{z} < 1$$

$$\int_0^z \int_0^y \frac{2}{z^2}\,dx\,dy = \int_0^z \frac{2}{z^2} \cdot x \Big|_0^y dy = \int_0^z \frac{2}{z^2} \cdot y\,dy = \frac{\cancel{2}}{z^2} \cdot \frac{y^2}{\cancel{2}} \Big|_0^z = \frac{z^2}{z^2} = 1 \Rightarrow \text{es una densidad}$$

Calcular $P(X < 0.15, Y > 0.25 | Z = 0.5)$

$$z = 0.5 \Rightarrow f(x,y \mid z=0.5) = \frac{2}{0.5^2} = 8, \quad 0 < x < y < 0.5$$

$$P(X < 0.15, Y > 0.25 \mid Z = 0.5) = \int_{0.25}^{0.5} \int_0^{0.15} 8\,dx\,dy = 0.3$$

$$X \rightarrow f_X(x) \begin{cases} \rightarrow E(X) \\ \rightarrow E(X^2) \\ \rightarrow V(X) \end{cases}$$

$$(X,Y) \rightarrow f_{X,Y}(x,y) \begin{cases} \rightarrow E(g(x,y)) \quad \text{p ej } E(XY),\ E(X+Y),\ E(-3X+4Y),\ \text{etc} \\ \rightarrow Cov(X,Y) \end{cases}$$

$$V(X) = E\left[(X-\mu_X)^2\right] \geq 0$$

$$Cov(X,Y) = E\left[(X-\mu_X)(Y-\mu_Y)\right] \in \mathbb{R} \begin{cases} < 0 \\ = 0 \\ > 0 \end{cases}$$

$$V(X) = E(X^2) - (E(X))^2$$
$$= E(X.X) - E(X)E(X)$$
$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

X = Tiempo de recorrido (h)

Y = Gasto calórico (kcal)

$E(X) = 2$ h

$E(Y) = 500$ kcal

$E(X)E(Y) = 1000$ hr.kcal

Ej: $Cov(XY) = 30$ hr.kcal

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\top}\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\right), \quad \mathbf{x} \in \mathbb{R}^{k}$$

Σ = matriz de varianzas y covarianzas

$$f_X(x) = \frac{1}{(2\pi)^{1/2}(\sigma^2)^{1/2}} \exp\left(-\frac{1}{2}(x-\mu) \times \frac{1}{\sigma^2} \times (x-\mu)\right), \quad x \in \mathbb{R}^{1}$$

σ^2

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N_3\left(\underbrace{\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}}_{E(\mathbf{X})}, \; \underbrace{\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & Cov(X_1,X_2) & Cov(X_1,X_3) \\ Cov(X_2,X_1) & \sigma_2^2 & Cov(X_2,X_3) \\ Cov(X_3,X_1) & Cov(X_3,X_2) & \sigma_3^2 \end{pmatrix}}_{Cov(\mathbf{X})}\right)$$

- Toda combinación lineal es normal:

$$\mathbf{a}^\top \mathbf{X} \sim \mathcal{N}(a^\top \mu,\ a^\top \Sigma a)$$

V. aleatoria:

$X \to Var(X) = 8$

$Y = 2X \to Var(Y) = 2^2 \cdot 8 = 32$

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_2\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & cov \\ cov & \sigma_2^2 \end{pmatrix} \right)$$

$$\begin{pmatrix} 1 & 5 \end{pmatrix} \begin{pmatrix} 10 & 2 \\ 2 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

$$Y = \begin{pmatrix} 1 & 5 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = X_1 + 5X_2 \sim N\left(\begin{pmatrix} 1 & 5 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \underset{1\times 2}{\begin{pmatrix} 1 & 5 \end{pmatrix}} \underset{2\times 2}{\Sigma} \underset{2\times 1}{\begin{pmatrix} 1 \\ 5 \end{pmatrix}} \right)$$

$$X_1 + 5X_2 \sim N\left(\mu_1 + 5\mu_2 ,\ \right)$$

- Cada subvector de $\mathbf{X}$ es Normal multivariado.

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} \sim N_4\left(\boldsymbol{\mu}_{4\times 1}, \Sigma_{4\times 4}\right)$$

$$\Rightarrow \quad \mathbf{Z} = \begin{pmatrix} X_1 \\ X_3 \end{pmatrix} \sim N_2\left(\boldsymbol{\mu}_{2\times 1}, \Sigma_{2\times 2}\right)$$

$$X_3 \sim N(\mu_3, \sigma_3^2)$$

- Si $\mathbf{Y} = A\mathbf{X} + b$, entonces

$$\mathbf{Y} \sim \mathcal{N}(A\mu + b,\ A\Sigma A^\top).$$

$X \to Var(X) = 8$

$Y = 2X + \underline{10} \to Var(Y) = 32$

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_2\left(\begin{pmatrix} 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 1 & 0.3 \\ 0.3 & 1.5 \end{pmatrix} \right)$$

$$\mathbf{Y} = \begin{pmatrix} 3 & 5 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + \begin{pmatrix} 4 \\ 10 \end{pmatrix} = \begin{pmatrix} 3X_1 + 5X_2 + 4 \\ -X_1 + 2X_2 + 10 \end{pmatrix} \sim N_2(\mu^*, \Sigma^*)$$

$$\mu^* = \begin{pmatrix} 3 & 5 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 7 \end{pmatrix} + \begin{pmatrix} 4 \\ 10 \end{pmatrix}$$

$$\Sigma^* = \begin{pmatrix} 3 & 5 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0.3 \\ 0.3 & 1.5 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 5 & 2 \end{pmatrix}$$

Distribución condicional

Particione el vector:

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix}, \qquad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}.$$

Entonces:

$$\mathbf{X}_1 \mid \mathbf{X}_2 = \mathbf{x}_2 \sim \mathcal{N}\left(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{x}_2 - \mu_2),\ \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\right)$$

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} \sim N_4\left(\mu = \begin{pmatrix} \underbrace{\begin{matrix} 10 \\ 5 \end{matrix}}_{\mu_1} \\ \underbrace{\begin{matrix} 0 \\ -2 \end{matrix}}_{\mu_2} \end{pmatrix},\ \Sigma = \begin{pmatrix} \overbrace{\begin{matrix} 4 & 1.2 \\ 1.2 & 3 \end{matrix}}^{\Sigma_{11}} & \overbrace{\begin{matrix} 0.5 & 0 \\ 0.8 & -0.3 \end{matrix}}^{\Sigma_{12}} \\ \underbrace{\begin{matrix} 0.5 & 0.8 \\ 0 & -0.3 \end{matrix}}_{\Sigma_{21}} & \underbrace{\begin{matrix} 2 & 0.6 \\ 0.6 & 1 \end{matrix}}_{\Sigma_{22}} \end{pmatrix} \right)$$

$$(X_1, X_2) \mid (X_3, X_4) \sim N_2\left(\underbrace{\begin{pmatrix} 10 \\ 5 \end{pmatrix} + \begin{pmatrix} 0.5 & 0 \\ 0.8 & -0.3 \end{pmatrix}\begin{pmatrix} 2 & 0.6 \\ 0.6 & 1 \end{pmatrix}^{-1}\left(\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} - \begin{pmatrix} 0 \\ -2 \end{pmatrix}\right)}_{\left(\ \right)_{2\times 1}},\ \underbrace{\begin{pmatrix} 4 & 1.2 \\ 1.2 & 3 \end{pmatrix} - \begin{pmatrix} 0.5 & 0 \\ 0.8 & -0.3 \end{pmatrix}\begin{pmatrix} 2 & 0.6 \\ 0.6 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 0.5 & 0.8 \\ 0 & -0.3 \end{pmatrix}}_{\left(\ \right)_{2\times 2}} \right)$$

$$\mathbf{x} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} \sim N_4\left(\boldsymbol{\mu} = \begin{pmatrix} 10 \\ 5 \\ 0 \\ -2 \end{pmatrix},\ \Sigma = \begin{array}{c c} & \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ & \begin{pmatrix} 4 & 1.2 & 0.5 & 0 \\ 1.2 & 3 & 0.8 & -0.3 \\ 0.5 & 0.8 & 2 & 0.6 \\ 0 & -0.3 & 0.6 & 1 \end{pmatrix} \end{array} \right) \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}$$

$$(X_1, X_4) \mid (X_2, X_3) \sim \ ?$$

$$\mathbf{x} = \begin{pmatrix} X_1 \\ X_4 \\ X_2 \\ X_3 \end{pmatrix} \sim N_4\left(\boldsymbol{\mu} = \begin{pmatrix} 10 \\ -2 \\ 5 \\ 0 \end{pmatrix},\ \Sigma = \begin{array}{c} \begin{array}{cccc} X_1 & X_4 & X_2 & X_3 \end{array} \\ \begin{pmatrix} 4 & 0 & 1.2 & 0.5 \\ 0 & 1 & -0.3 & 0.6 \\ 1.2 & -0.3 & 3 & 0.8 \\ 0.5 & 0.6 & 0.8 & 2 \end{pmatrix} \end{array} \right) \begin{array}{c} X_1 \\ X_4 \\ X_2 \\ X_3 \end{array}$$

$$\mathbf{X}_1 \mid \mathbf{X}_2 = \mathbf{x}_2 \sim \mathcal{N}\left(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{x}_2 - \mu_2),\ \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\right).$$

$P(X_1 < 1.5,\ X_2 < 2.5)$

```
pmvnorm(lower = c(-Inf, -Inf),
        upper = c(1.5, 2.5),
        mean = mu,
        sigma = Sigma)
```

```
[1] 0.5007897
```

$P(0 < x_1 < 1,\ X_2 > 2)$

```
> pmvnorm(lower = c(0, 2),
+         upper = c(1, Inf),
+         mean = mu,
+         sigma = Sigma)
[1] 0.1418144
```