Ejercicio 2

2. Sea X una variable aleatoria con función de densidad

$$f_X(x) = kxI_{(0,1)}(x) + k(2-x)I_{[1,2)}(x)$$

a. Determine la constante k

$$\int_0^2 f_x(x)\,dx = k\int_0^1 x\,dx + k\int_1^2 (2-x)\,dx = 1$$

$$= k\left(\frac{1}{2}\right) + k\left(\frac{1}{2}\right) = 1 \Rightarrow k = 1 \quad \therefore f_x(x) = x\,I_{(0,1)}(x) + (2-x)\,I_{[1,2)}(x)$$

b. Obtener $F_X(x)$

$$x \le 0, \quad F_x(x) = 0$$

$$0 < x < 1, \quad F_x(x) = \int_0^x t\,dt = \frac{t^2}{2}\Big|_0^x = \frac{x^2}{2}$$

$$1 \le x < 2: \quad F_x(x) = \int_0^1 x\,dx + \int_1^x (2-t)\,dt = \frac{1}{2} + \left(2t - \frac{t^2}{2}\right)\Big|_1^x = \frac{1}{2} + \left(2x - \frac{x^2}{2}\right) - \left(2 - \frac{1}{2}\right) = 2x - \frac{x^2}{2} - 1$$

$$x \ge 2: \quad F_x(x) = 1$$

$$F_x(x) = \frac{x^2}{2} I_{(0,1)}(x) + \left(2x - \frac{x^2}{2} - 1\right) I_{[1,2)}(x) + I_{[2,\infty)}(x)$$

c. Hallar $E(X)$ y $V(X)$

$$f_X(x) = x\, I_{(0,1)}(x) + (2-x)\, I_{[1,2)}(x)$$

$$E(X) = \int_0^2 x\, f_X(x)\,dx = \int_0^1 x^2\,dx + \int_1^2 x(2-x)\,dx = \frac{1}{3} + \frac{2}{3} = 1$$

$$F_X(x) = \frac{x^2}{2}\, I_{(0,1)}(x) + \left(2x - \frac{x^2}{2} - 1\right) I_{[1,2)}(x) + I_{[2,\infty)}(x)$$

$$E(X) = \int_0^\infty 1 - F_X(x)\,dx = \int_0^1 \left(1 - \frac{x^2}{2}\right) dx + \int_1^2 \left(1 - 2x + \frac{x^2}{2} + 1\right) dx + \cancel{\int_2^\infty (1-1)\,dx} = \frac{5}{6} + \frac{1}{6} = 1$$

$$E(X^2) = \int_0^1 x^3\,dx + \int_1^2 x^2(2-x)\,dx = 0.25 + 0.9167 = 1.1667$$

$$V(X) = 1.1667 - 1^2 = 0.1667 = 1/6$$

d. Determinar el valor mediano de $X = me$

$$f_X(x) = x\, I_{(0,1)}(x) + (2-x)\, I_{[1,2)}(x)$$

$$F_X(x) = \frac{x^2}{2} I_{(0,1)}(x) + \left(2x - \frac{x^2}{2} - 1\right) I_{[1,2)}(x) + I_{[2,\infty)}(x$$

$$F_X(me) = 0.5$$

En $(0,1)$: $\frac{me^2}{2} = 0.5$

$me^2 = 1$

$\cancel{(me = -1)}$ o $me = 1$

En $[1,2)$: $2me - \frac{me^2}{2} - 1 = 0.5$

$$2me - \frac{me^2}{2} = 1.5$$

$$\frac{me^2}{2} - 2me + 1.5 = 0$$

$$me^2 - 4me + 3 = 0$$

$$(me-3)(me-1) = 0$$

$\cancel{me = 3}$ o $\boxed{me = 1}$ //

Ejercicio 3

3. Si una variable aleatoria X tiene la siguiente función de distribución

$$F_X(x) = (1 - \exp(-3x))I_{(0,\infty)}(x)$$

a. ¿Cuál es su función de densidad?

$$f_X(x) = F_X'(x) = -(-3)e^{-3x} = 3e^{-3x} I_{(0,\infty)}(x)$$

b. Calcular el percentil 60 de X

Sea r = percentil 60 de X $\Rightarrow F_X(r) = 0.6$

$$1 - e^{-3r} = 0.6$$

$$e^{-3r} = 0.4$$

$$-3r = \log(0.4)$$

$$r = -\frac{1}{3}\log(0.4) = 0.3054$$

c. ¿Cuál es su valor esperado?

$$f(x) = 3e^{-3x} I_{(0,\infty)}(x) \rightarrow E(x) = \int_0^\infty 3x e^{-3x} dx = 1/3$$

$$F_X(x) = (1 - \exp(-3x))I_{(0,\infty)}(x) \rightarrow E(x) = \int_0^\infty 1 - (1 - e^{-3x})\, dx = \int_0^\infty e^{-3x} dx = 1/3$$

d. Hallar su coeficiente de variación

$$E(x^2) = \int_0^\infty 3x^2 e^{-3x} dx = \frac{2}{9}$$

$$V(x) = \frac{2}{9} - \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$

$$\sigma_x = \sqrt{\frac{1}{9}} = \frac{1}{3}$$

$$CV_x = \frac{1/3}{1/3} \times 100\% = 100\%$$

Ejercicio 4

4. Un sensor ambiental detecta contaminación:

- Con probabilidad p no detecta nada y reporta $X = 0$
- Con probabilidad $1 - p$ detecta contaminación, y el nivel medido Y tiene densidad:

$$f_Y(y) = y \exp(-y) I_{(0,\infty)}(x)$$

$F_X(x)$

a. Identifique la función de distribución de X, así como $F^d(x)$, $F^{ac}(x)$, $f^d(x)$ y $f^{ac}(x)$

- Determinación de pesos : Peso discreto = $p = P(X=0)$

  Peso continuo = $1-p$

- Análisis intervalo a intervalo

$x < 0$: $F_X(x) = 0$ $\quad F^d(x) = 0$ $\quad F^{ac}(x) = 0$

$x = 0$: $F_X(x) = p\, F^d(x) + (1-p) F^{ac}(x) = p \times 1 + (1-p) \times 0 = p$

$F^d(x) = 1$, porque $X=0$ es el único punto discreto, $F^{ac}(x) = 0$

$x > 0$: $F_X(x) = F_X(y) = F_Y(y) = \int_0^y t e^{-t} dt = 1 - (y+1)e^{-y}$

↑ integración por partes

$F_X(x) = p \times 1 + (1-p)(1 - (x+1)e^{-x})$

$(1 - (x+1)e^{-x}) = p + (1-p) F^{ac}(x)$

$$\frac{(1-(x+1)e^{-x}) - p}{1-p} = F^{ac}(x)$$

$$F^{d}(x) = p\, I_{[0,\infty)}(x) \longrightarrow f^{d}$$

$$F^{ac}(x) = \left(1-(x+1)e^{-x}\right) I_{(0,\infty)}(x) \longrightarrow f^{ac}$$

$$F_X(x) = p\, I_{\{0\}}(x) + (1-p)\left(1-(x+1)e^{-x}\right) I_{(0,\infty)}(x)$$

- $f^{d}(x) = I_{\{0\}}(x)$ $\qquad P(X=0)=1$

- $f^{ac}(x) = F^{ac\,\prime}(x) = \dfrac{d}{dx}\left(1-(x+1)e^{-x}\right) = -(1)e^{-x} + \left(-(x+1)e^{-x}(-1)\right)$

$$= -e^{-x} + (x+1)e^{-x}$$

$$= x e^{-x} I_{(0,\infty)}(x)$$

Verificar que es densidad: $\displaystyle\int_0^{\infty} x e^{-x}\,dx = 1$

b. Calcule $P(X > t)$ para $p = 0.3$

$$F_X(x) = p\, I_{\{0\}}(x) + (1-p)\left(1-(x+1)e^{-x}\right) I_{(0,\infty)}(x)$$

$$P(X > t) = 1 - P(X \leq t) = 1 - F_X(t)$$

$$= (1-p)\, I_{\{0\}}(x) + \left(1-(1-p)\left(1-(t+1)e^{-t}\right)\right) I_{(0,\infty)}(x)$$

$$= 0.7\, I_{\{0\}}(x) + \left(1-0.3\left(1-(t+1)e^{-t}\right)\right) I_{(0,\infty)}(x)$$

c. Hallar $E(X)$ para $p = 0.9$ y $p = 0.2$

$$E(x) = \sum x f(x) + \int x f(x)\, dx$$

$$E(x) = p \times 0 \times 1 + (1-p) \times \int_0^\infty x \cdot x e^{-x}\, dx$$

$$= 0 + (1-p) \times 2 = 2(1-p)$$

Si $p = 0.9 \Rightarrow E(x) = 2 \times 0.1 = 0.2$

Si $p = 0.2 \Rightarrow E(x) = 2 \times 0.8 = 1.6$

d. Determinar $V(X)$ para $p = 0.9$ y $p = 0.2$

$$E(X^2) = p \times \underset{\underset{x^2}{\uparrow}}{0^2} \times \underset{\underset{f^d(x)}{\uparrow}}{1} + (1-p)\int_0^\infty \underset{\underset{x^2}{\uparrow}}{x^2} \cdot \underbrace{x e^{-x}}_{f^{ac}(x)}\,dx = 0 + (1-p)\,6 = 6\,(1-p)$$

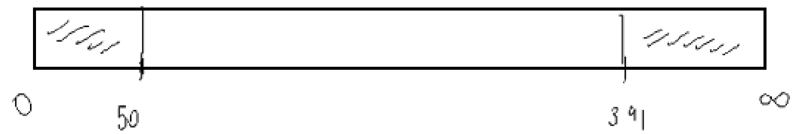
Si $p = 0.9 \Rightarrow$

$$E(X^2) = 6 \times 0.1 = 0.6$$
$$E(X) = 0.2$$
$$V(X) = 0.6 - 0.2^2 = 0.56$$

si $p = 0.2 \Rightarrow$

$$E(X^2) = 6 \times 0.8 = 4.8$$
$$E(X) = 1.6$$
$$V(X) = 4.8 - 1.6^2 = 2.24$$

$$f_X(x) = \frac{1}{100} \exp\left(-\frac{x}{100}\right) I_{(0,\infty)}(x)$$



4. Un sensor ambiental detecta contaminación:

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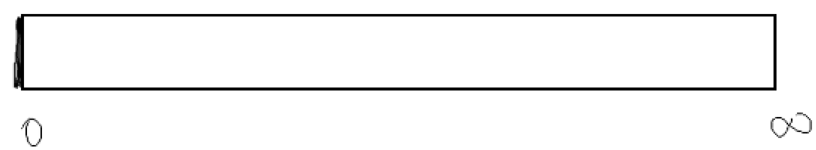
→ abs continuo



Ejercicio 5

5. Sea $Z \in \{1, 2\}$ una variable latente con $P(Z = 1) = \pi$. Condicionalmente a Z, la variable observada X tiene densidad:

$$\left.\begin{aligned} f_{X|Z=1}(x) &= \frac{x}{2} I_{(0,2)}(x) \\ f_{X|Z=2}(x) &= \frac{3}{4}(1 - x^2) I_{(-1,1)}(x) \end{aligned}\right\} \text{densidades condicionales}$$

a. Verifique que $f_{X|Z=1}$ y $f_{X|Z=2}$ son densidades válidas.

$$\int_0^2 \frac{x}{2}\,dx = 1 \quad ✓ \qquad \int_{-1}^{1} \frac{3}{4}(1-x^2)\,dx = 1 \quad ✓$$

b. Obtener la densidad marginal $f_X(x)$.

$$f_X(x) = \pi \frac{x}{2} I_{(0,2)}(x) + (1-\pi)\frac{3}{4}(1-x^2) I_{(-1,1)}(x) = \frac{3(1-\pi)}{4}(1-x^2) I_{(-1,0]}(x) + \left(\frac{\pi x}{2} + \frac{3(1-\pi)}{4}(1-x^2)\right) I_{(0,1)}(x) + \frac{\pi x}{2} I_{[1,2)}(x)$$

- $-1 < x \leq 0$: $\frac{3(1-\pi)}{4}(1-x^2)$
- $0 < x < 1$: $\frac{\pi x}{2} + \frac{3(1-\pi)}{4}(1-x^2)$
- $1 \leq x < 2$: $\frac{\pi x}{2}$

c. Obtener la función de distribución $F_X(x)$ y a partir de ella, $P(X < 0)$.

$$f(x) = \frac{3(1-\pi)}{4}(1-x^2)\, I_{(-1,0]}(x) + \left(\frac{\pi x}{2} + \frac{3(1-\pi)}{4}(1-x^2)\right) I_{(0,1)}(x) + \frac{\pi x}{2} I_{[1,2)}(x)$$

F_X:

$x \leq -1, \quad F_X(x) = 0$

$-1 < x \leq 0, \quad F_X(x) = \int_{-1}^{x} \frac{3(1-\pi)}{4}(1-t^2)\, dt = \ldots$

$0 < \cancel{x} < 1, \quad F_X(x) = \int_{-1}^{0} \frac{3(1-\pi)}{4}(1-x^2)\, dx + \int_{0}^{x} \frac{\pi}{2} t + \frac{3}{4}(1-\pi)(1-t^2)\, dt = \ldots$

$1 \leq x < 2, \quad F_X(x) = \int_{-1}^{0} \frac{3}{4}(1-\pi)(1-x^2)\, dx + \int_{0}^{1} \frac{\pi}{2} x + \frac{3}{4}(1-\pi)(1-x^2)\, dx + \int_{1}^{x} \frac{\pi}{2} t\, dt = \ldots$

$x \geq 2, \quad F_X(x) = 1$

d. Calcule $E(X|Z=1)$ Y $E(X|Z=2)$.

$$f_{X|Z=1}(x) = \frac{x}{2} I_{(0,2)}(x)$$

$$f_{X|Z=2}(x) = \frac{3}{4}(1-x^2) I_{(-1,1)}(x)$$

$$E(X|Z=1) = \int_0^2 \frac{x^2}{2}\,dx = \left.\frac{x^3}{6}\right|_0^2 = \frac{8}{6} = \frac{4}{3}$$

$$E(X|Z=2) = \int_{-1}^{1} \frac{3x(1-x^2)}{4}\,dx = \left.\left(\frac{3x^2}{8} - \frac{3x^4}{16}\right)\right|_{-1}^{1}$$

$$= \left(\frac{3}{8} - \frac{3}{16}\right) - \left(\frac{3}{8} - \frac{3}{16}\right) = 0$$

e. ¿Cuál es el valor de π si se sabe que $E(X) = 1$?

$$E(X) = \pi \times \frac{4}{3} + (1-\pi) \times 0 = 1$$

$$\pi \times \frac{4}{3} = 1$$

$$\pi = 0.75$$

f. Con el valor de π obtenido en la pregunta anterior, calcular y descomponer $V(X)$.

$$f_{X|Z=1}(x) = \frac{x}{2} I_{(0,2)}(x)$$

$$f_{X|Z=2}(x) = \frac{3}{4}(1-x^2) I_{(-1,1)}(x)$$

$$E(X \mid Z=1) = 4/3$$

$$E(X^2 \mid Z=1) = \int_0^2 x^2 \cdot \frac{x}{2}\, dx = \int_0^2 \frac{x^3}{2}\, dx = \frac{x^4}{8}\Big|_0^2 = \frac{2^4}{8} = 2$$

$$V(X \mid Z=1) = 2 - (4/3)^2 = \frac{2}{9}$$

$$E(X \mid Z=2) = 0$$

$$E(X^2 \mid Z=2) = \int_{-1}^{1} \frac{3}{4} x^2 (1-x^2)\, dx = 0.2$$

$$V(X \mid Z=2) = 0.2 - 0^2 = 0.2 = \frac{2}{10}$$

$$V(X) = \underbrace{\left[0.75 \times \frac{2}{9} + 0.25 \times \frac{2}{10}\right]}_{\text{intra}} + \underbrace{\left[0.75\left(4/3 - 1\right)^2 + 0.25\left(0 - 1\right)^2\right]}_{\text{inter}}$$

$$= 0.2167 + 0.3333 = 0.55$$