Ejemplo 8a

Supongamos una mezcla con dos mecanismos generadores:

- Si $Z = 1$, X toma valores solo en $[0, 1]$ con densidad $f_1(x) = 2xI_{[0,1]}(x)$
- Si $Z = 2$, X toma valores solo en $(1, 3]$ con densidad $f_2(x) = \frac{1}{2}I_{(1,3]}(x)$

Verificar que $f_1(x)$ y $f_2(x)$ son densidades

$$\int_0^1 2x\,dx = \frac{\cancel{2}x^2}{\cancel{2}}\Big|_0^1 = 1 \qquad \int_1^3 \frac{1}{2}\,dx = \frac{x}{2}\Big|_1^3 = 1$$

Ejemplo 8b

Suponiendo que $P(Z = 1) = \overset{\pi_1}{0.6}$ y $P(Z = 2) = \overset{\pi_2}{0.4}$, obtener la densidad mezcla.

$$f_X(x) = 0.6\,f_1(x) + 0.4\,f_2(x) = 1.2x\,I_{[0,1]}(x) + 0.2\,I_{(1,3]}(x)$$

$$\int_0^1 1.2x\,dx + \int_1^3 0.2\,dx = \frac{1.2x^2}{2}\Big|_0^1 + 0.2x\Big|_1^3 = 0.6\,(\underbrace{1^2 - 0^2}_{1}) + 0.2(3-1) = 0.6 + 0.4 = 1$$

Ejemplo 8c

Obtener la función de distribución acumulada.

$$f_1(x) = 2xI_{[0,1]}(x)$$
$$f_2(x) = \frac{1}{2}I_{(1,3]}(x)$$

$F_1(x)$: $x < 0$, $F_1(x) = 0$

$0 \leq x \leq 1$, $F_1(x) = \int_0^x 2t\,dt = \frac{\cancel{2}t^2}{\cancel{2}}\Big|_0^x = x^2$

$x > 1$, $F_1(x) = 1$

$F_2(x)$: $x \leq 1$, $F_2(x) = 0$

$1 < x \leq 3$, $F_2(x) = \int_1^x \frac{1}{2}\,dt = \frac{t}{2}\Big|_1^x = \frac{x-1}{2}$

$x > 3$, $F_2(x) = 1$

Entonces, construyendo $F_X(x)$:

$x < 0$, $F_1(x) = 0$ $F_2(x) = 0$ $F_X(x) = 0.6 \times 0 + 0.4 \times 0 = 0$

$0 \leq x \leq 1$, $F_1(x) = x^2$ $F_2(x) = 0$ $F_X(x) = 0.6\,x^2 + 0.4 \times 0 = 0.6x^2$

$1 < x \leq 3$, $F_1(x) = 1$ $F_2(x) = \frac{x-1}{2}$ $F_X(x) = 0.6 \times 1 + 0.4\left(\frac{x-1}{2}\right) = 0.6 + 0.2(x-1) = 0.2x + 0.4$

$x > 3$, $F_1(x) = 1$ $F_2(x) = 1$ $F_X(x) = 0.6 \times 1 + 0.4 \times 1 = 1$

$$F_X(x) = 0.6x^2\, I_{[0,1]}(x) + (0.2x + 0.4)\, I_{(1,3]}(x) + I_{(3,\infty)}(x)$$

Ejemplo 8d

Obtener el valor esperado de X.

$$f_1(x) = 2xI_{[0,1]}(x)$$
$$f_2(x) = \frac{1}{2}I_{(1,3]}(x)$$

$$E(X \mid Z = 1) = \int_0^1 x \cdot 2x \, dx = \int_0^1 2x^2 \, dx = \frac{2x^3}{3} \Big|_0^1 = \frac{2}{3}$$

$$E(X \mid Z = 2) = \int_1^3 x \cdot \frac{1}{2} \, dx = \int_1^3 \frac{x}{2} \, dx = \frac{x^2}{4} \Big|_1^3 = \frac{3^2 - 1^2}{4} = 2$$

$$E(X) = 0.6 \times \frac{2}{3} + 0.4 \times 2 = 0.4 + 0.8 = 1.2$$

Ejemplo 8e

Obtener la varianza total de X y su descomposición.

$$f_1(x) = 2x I_{[0,1]}(x)$$
$$f_2(x) = \frac{1}{2} I_{(1,3]}(x)$$

$$E(X \mid Z=1) = \frac{2}{3}$$

$$E(X^2 \mid Z=1) = \int_0^1 x^2 \cdot 2x \, dx = \int_0^1 2x^3 \, dx = \frac{2x^4}{4} \Big|_0^1 = \frac{1}{2}$$

$$V(X \mid Z=1) = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{2} - \frac{4}{9} = \frac{9-8}{18} = \frac{1}{18}$$

$$E(X \mid Z=2) = 2$$

$$E(X^2 \mid Z=2) = \int_1^3 x^2 \cdot \frac{1}{2} \, dx = \frac{x^3}{6} \Big|_1^3 = \frac{3^3 - 1^3}{6} = \frac{26}{6} = \frac{13}{3}$$

$$V(X \mid Z=2) = \frac{13}{3} - 2^2 = \frac{13}{3} - \frac{12}{3} = \frac{1}{3}$$

$$\sum_{i=1}^{2} \pi_i V(X \mid Z=i) = 0.6 \times 1/18 + 0.4 \times 1/3 = 1/6$$

$$\sum_{i=1}^{2} \pi_i \left(E(X \mid Z_i) - E(X)\right)^2$$

$$= 0.6\left(\frac{2}{3} - 1.2\right)^2 + 0.4\left(2 - 1.2\right)^2$$

$$= 0.4267$$

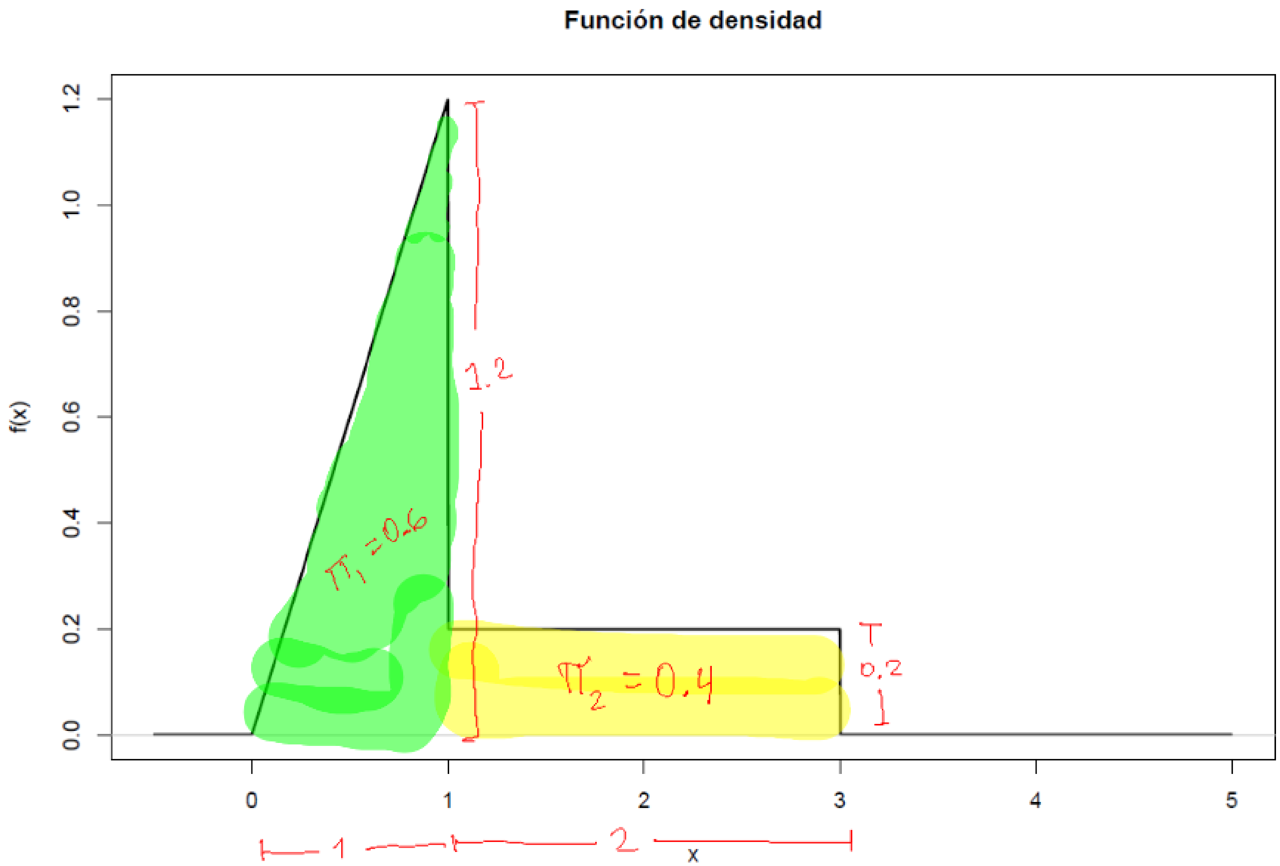
variab. intra

variab. inter

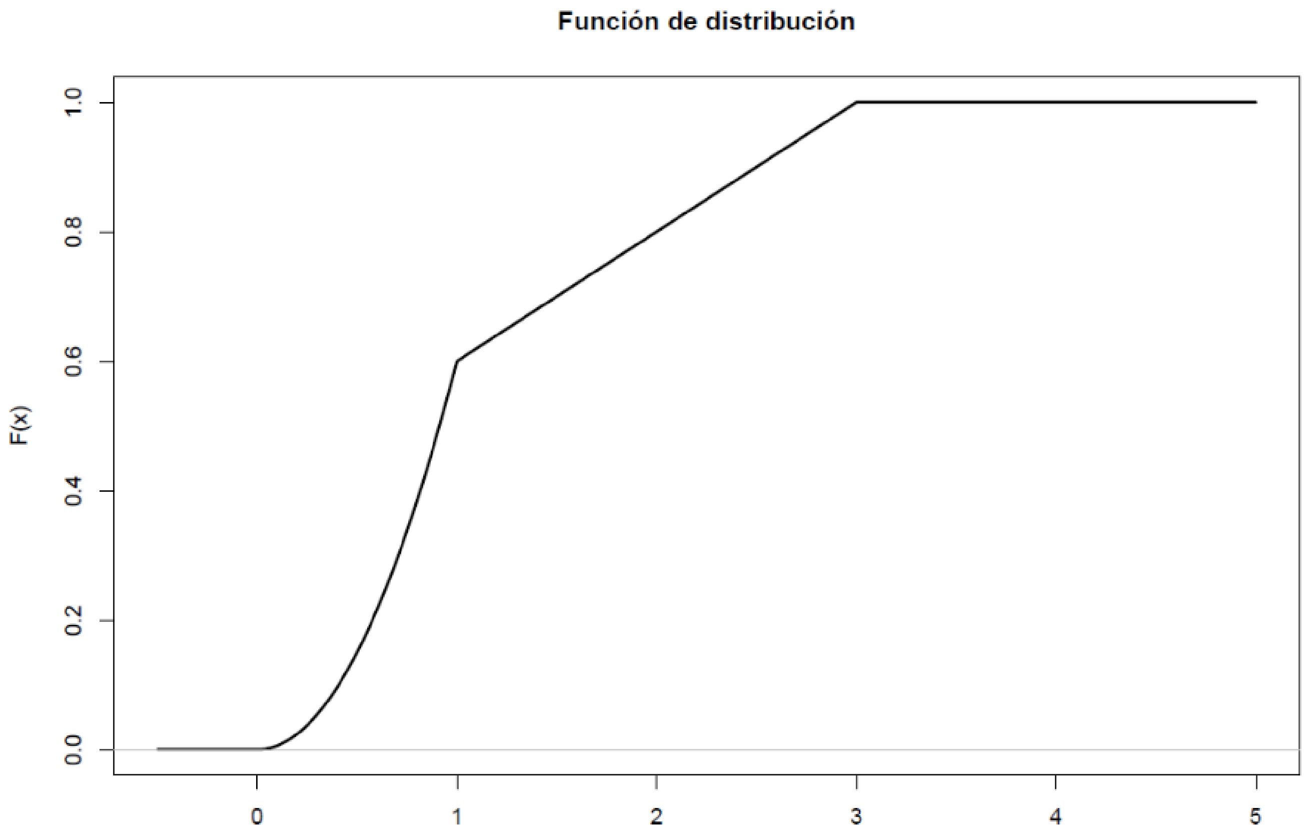
$$Var(X) = 0.1667 + 0.4267$$

$$= 0.5934$$

$$f_X(x) = 1.2x \, I_{[0,1]}(x) + 0.2 \, I_{(1,3]}(x)$$



$$F_X(x) = 0.6x^2 I_{[0,1]}(x) + (0.2x + 0.4) I_{(1,3]}(x) + I_{(3,\infty)}(x)$$



Ejemplo 9a

Supongamos una mezcla con dos mecanismos generadores:

- Si $Z = 1$, X toma valores en $\{0, 1, 2\}$ con función de probabilidad $f_1(X = x|Z = 1) = 0.2I_{\{0\}}(x) + 0.3I_{\{1\}}(x) + 0.5I_{\{2\}}(x)$
- Si $Z = 2$, X toma valores en $\{2, 3, 4\}$ con función de probabilidad $f_2(X = x|Z = 2) = 0.4I_{\{2\}}(x) + 0.3I_{\{3\}}(x) + 0.3I_{\{4\}}(x)$

Verificar que $f_1(X = x|Z = 1)$ y $f_2(X = x|Z = 2)$ son funciones de probabilidad.

$0.2 + 0.3 + 0.5 = 1$ $\qquad$ $0.4 + 0.3 + 0.3 = 1$

Ejemplo 9b

Suponiendo que $P(Z = 1) = 0.75$ y $P(Z = 2) = 0.25$, obtener la función de probabilidad mezcla.

$f_X(x) = 0.75 f_1(X = x|Z=1) + 0.25 f_2(X = x|Z = 1)$

$P(X=0) = 0.75 \times 0.2 + 0.25 \times 0 = 0.75 \times 0.20 = 0.15$

$P(X=1) = 0.75 \times 0.3 + 0.25 \times 0 = 0.75 \times 0.30 = 0.225$

$P(X=2) = 0.75 \times 0.5 + 0.25 \times 0.4 = 0.475$

$P(X=3) = 0.75 \times 0 + 0.25 \times 0.3 = 0.075$

$P(X=4) = 0.75 \times 0 + 0.25 \times 0.3 = 0.075$

$f_X(x) = 0.15 I_{\{0\}}(x) + 0.225 I_{\{1\}}(x) + 0.475 I_{\{2\}}(x) + 0.075 I_{\{3,4\}}(x)$

Ejemplo 9c

Obtener la función de distribución acumulada.

$$f_X(x) = 0.15\, I_{\{0\}}(x) + 0.225\, I_{\{1\}}(x) + 0.475\, I_{\{2\}}(x) + 0.075\, I_{\{3,4\}}(x)$$

$$\left.\begin{array}{ll} x < 0: & F_X(x) = 0 \\ 0 \leq x < 1: & F_X(x) = 0.15 \\ 1 \leq x < 2: & F_X(x) = 0.15 + 0.225 = 0.375 \\ 2 \leq x < 3: & F_X(x) = 0.15 + 0.225 + 0.475 = 0.85 \\ 3 \leq x < 4: & F_X(x) = 0.85 + 0.075 = 0.925 \\ x \geq 4: & F_X(x) = 1 \end{array}\right\}$$

$$F_X(x) = 0.15\, I_{[0,1)}(x) + 0.375\, I_{[1,2)}(x) + 0.85\, I_{[2,3)}(x) + 0.925\, I_{[3,4)}(x) + I_{[4,\infty)}(x)$$

Ejemplo 9d

Obtener el valor esperado de X.

$$f_1(X = x|Z = 1) = 0.2I_{\{0\}}(x) + 0.3I_{\{1\}}(x) + 0.5I_{\{2\}}(x)$$

$$f_2(X = x|Z = 2) = 0.4I_{\{2\}}(x) + 0.3I_{\{3\}}(x) + 0.3I_{\{4\}}(x)$$

$$\left.\begin{array}{l} E(X|Z=1) = 0.2\times 0 + 0.3\times 1 + 0.5\times 2 = 1.3 \\ E(X|Z=2) = 0.4\times 2 + 0.3\times 3 + 0.3\times 4 = 2.9 \end{array}\right\} E(X) = 0.75\times 1.3 + 0.25\times 2.9 = 1.7$$

$$f_X(x) = 0.15 I_{\{0\}}(x) + 0.225 I_{\{1\}}(x) + 0.475 I_{\{2\}}(x) + 0.075 I_{\{3,4\}}(x)$$

$$E(X) = 0.15\times 0 + 0.225\times 1 + 0.475\times 2 + 0.075\times 3 + 0.075\times 4 = 1.7$$

Ejemplo 9e

Obtener la varianza total de X y su descomposición.

$$f_1(X = x|Z = 1) = 0.2I_{\{0\}}(x) + 0.3I_{\{1\}}(x) + 0.5I_{\{2\}}(x)$$

$$f_2(X = x|Z = 2) = 0.4I_{\{2\}}(x) + 0.3I_{\{3\}}(x) + 0.3I_{\{4\}}(x)$$

$$E(X|Z=1) = 1.3$$

$$E(X^2|Z=1) = 0.2\times 0^2 + 0.3\times 1^2 + 0.5\times 2^2 = 2.3$$

$$V(X|Z=1) = 2.3 - 1.3^2 = 0.61$$

$$E(X|Z=2) = 2.9$$

$$E(X^2|Z=2) = 0.4\times 2^2 + 0.3\times 3^2 + 0.3\times 4^2 = 9.1$$

$$V(X|Z=2) = 9.1 - 2.9^2 = 0.69$$

$$\sum_{i=1}^{2} \pi_i V(X|Z=i) = 0.75\times 0.61 + 0.25\times 0.69 = 0.63$$ (Variabilidad dentro de los subpoblaciones)

$$\sum_{i=1}^{n} \pi_i (E(X|Z=i) - E(X))^2 = 0.75\times(1.3-1.7)^2 + 0.25(2.9-1.7)^2 = 0.48$$ (Variabilidad entre las subpoblaciones)

$$V(X) = 0.63 + 0.48 = 1.11$$

$$f_X(x) = 0.15\, I_{\{0\}}(x) + 0.225\, I_{\{1\}}(x) + 0.475\, I_{\{2\}}(x) + 0.075\, I_{\{3,4\}}(x)$$

$$E(X) = 0.15 \times 0 + 0.225 \times 1 + 0.475 \times 2 + 0.075 \times 3 + 0.075 \times 4 = 1.7$$

$$E(X^2) = 0.15 \times 0^2 + 0.225 \times 1^2 + 0.475 \times 2^2 + 0.075 \times 3^2 + 0.075 \times 4^2 = 4$$

$$V(X) = 4 - 1.7^2 = 1.11$$

1. Sea X una variable aleatoria con función de probabilidad:

$$f_X(x) = P(X = x) = \frac{c}{x(x+1)} I_{\{1,2,3,\dots\}}(x)$$

a. Determine la constante c

$$\sum_{x=1}^{\infty} \frac{c}{x(x+1)} = 1 \quad \Rightarrow \quad c \boxed{\sum_{x=1}^{\infty} \left(\frac{1}{x} - \frac{1}{x+1} \right)} = 1$$

Sea $S_n = \sum_{x=1}^{n} \left(\frac{1}{x} - \frac{1}{x+1} \right) = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \dots + \cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}} + \cancel{\frac{1}{n}} - \frac{1}{n+1} = 1 - \frac{1}{n+1}$

$$S_n = 1 - \frac{1}{n+1}$$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \left(1 - \frac{1}{n+1} \right) = 1$$

$$\Rightarrow \boxed{\sum_{x=1}^{\infty} \left(\frac{1}{x} - \frac{1}{x+1} \right) = 1} \quad \Rightarrow c = 1 \qquad f_X(x) = \frac{1}{x(x+1)} I_{\{1,2,3,\dots\}}(x)$$

b. Calcule $P(X \geq n)$

$$f_X(x) = \frac{1}{x(x+1)} I_{\{1,2,3,\dots\}}(x)$$

$$P(X \geq n) = \sum_{x=n}^{\infty} \frac{1}{x(x+1)}$$

$$\underset{(m>n)}{S_m} = \sum_{x=n}^{m} \left(\frac{1}{x} - \frac{1}{x+1} \right) = \frac{1}{n} - \cancel{\frac{1}{n+1}} + \cancel{\frac{1}{n+1}} - \cancel{\frac{1}{n+2}} + \dots + \cancel{\frac{1}{m}} - \frac{1}{m+1} = \frac{1}{n} - \frac{1}{m+1}$$

$$\lim_{m \to \infty} S_m = \lim_{m \to \infty} \left(\frac{1}{n} - \frac{1}{m+1} \right) = \frac{1}{n}$$

$$\Rightarrow P(X \geq n) = \frac{1}{n}$$

c. ¿Cuál es el valor de $E(X)$? No está definido

$$f_X(x) = \frac{1}{x(x+1)} I_{\{1,2,3,\dots\}}(x)$$

$$E(x) = \sum_{x=1}^{\infty} \cancel{x} \cdot \frac{1}{\cancel{x}(x+1)} = \sum_{x=1}^{\infty} \frac{1}{x+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots = \infty$$

d. ¿Cuál es el valor de la mediana de X?

$$f_X(1) = \frac{1}{1\times 2} = \frac{1}{2}$$

$$f_X(2) = \frac{1}{2\times 3} = \frac{1}{6} \qquad F_X(2) = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$

Sea m: mediana

$F(m) \geq 0.5$

$$P(X \leq m) = \sum_{x=1}^{m} \frac{1}{x(x+1)} = \sum_{x=1}^{m} \left(\frac{1}{x} - \frac{1}{x+1}\right) = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \dots + \cancel{\frac{1}{m}} - \frac{1}{m+1} = 1 - \frac{1}{m+1} = \frac{m+\cancel{1}-\cancel{1}}{m+1}$$

$$= \frac{m}{m+1}$$

$$\frac{m}{m+1} \geq 0.5$$

$m \geq 0.5m + 0.5$

$0.5m \geq 0.5 \Rightarrow m \geq 1 \Rightarrow \boxed{m=1}$ es la mediana ✓

e. Calcule $P(X \geq 8|X \geq 4) = \dfrac{P(X \geq 8)}{P(X \geq 4)}$

De la pregunta b sabemos que

$$P(X \geq n) = \frac{1}{n}$$

Entonces: $P(X \geq 8 \mid X \geq 4) = \dfrac{1/8}{1/4} = \dfrac{1}{2} = 0.5$