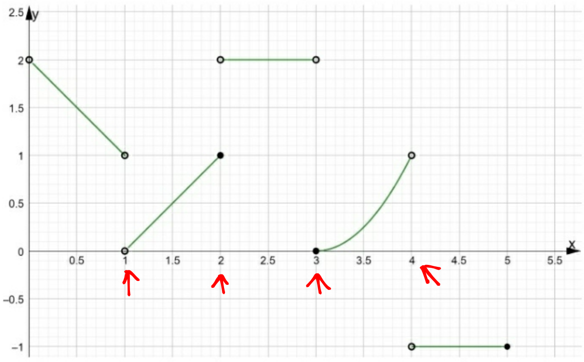
Teorema: $\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$

Ex01: Considere a função f, cujo gráfico é fornecido ao lado. Determine todos os valores de a tais que o limite $\lim_{x\to a} f(x)$ não existe.

Obs: nos extremos do intervalo não há limite central

VERIFIQUE!



Ex02: Os gráficos das funções f e g são mostrados a seguir. Utilizando as propriedades dos limites, encontre o limite indicado ou aponte a razão da não existência.

Gráfico de f

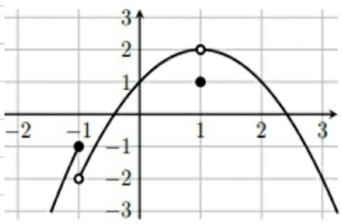
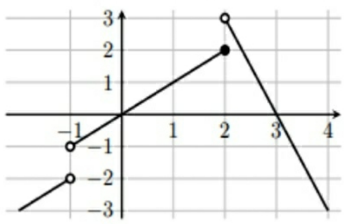


Gráfico de g



(a)	$\lim_{x\to 0}[f(x)+g(x)]$
(b)	$\lim_{x\to 1}[f(x)g(x)]$
(c)	$\lim_{x\to 2^+}[2f(x)+3g(x)]$
(d)	$\lim_{x\to 2^-}[x^2+g(x)]$
(e)	$\lim_{x\to 3}\left[\frac{g(x)}{f(x)}\right]$
(f)	$\lim_{x\to 3^+}\left[\frac{f(x)}{g(x)}\right]$

(a) $\lim_{x\to 0}[f(x)+g(x)] = \underbrace{\lim_{x\to 0} f(x)}_{1} + \underbrace{\lim_{x\to 0} g(x)}_{0} = 1$

(b) $\lim_{x\to 1} f(x)\cdot g(x) = \underbrace{\lim_{x\to 1} f(x)}_{2} \cdot \underbrace{\lim_{x\to 1} g(x)}_{1} = 2$

(c) $\lim_{x\to 2^+}[2f(x)+3g(x)] = 2\cdot\underbrace{\lim_{x\to 2^+} f(x)}_{1} + 3\cdot\underbrace{\lim_{x\to 2^+} g(x)}_{3} = 11$

(d) $\lim_{x\to 2^-}[x^2+g(x)] = \lim_{x\to 2^-} x^2 + \lim_{x\to 2^-} g(x) = \left(\underbrace{\lim_{x\to 2^-} x}_{4}\right)^2 + \underbrace{\lim_{x\to 2^-} g(x)}_{2} = 6$

(e) $\lim_{x\to 3}\left[\frac{g(x)}{f(x)}\right] = \dfrac{\underbrace{\lim_{x\to 3} g(x)}_{0}}{\underbrace{\lim_{x\to 3} f(x)}_{-2}} = \dfrac{0}{-2} = 0$

(f) $\lim_{x\to 3^+} \frac{f(x)}{g(x)} = \dfrac{\underbrace{\lim_{x\to 3^+} f(x)}_{-2}}{\underbrace{\lim_{x\to 3^+} g(x)}_{0}} = \dfrac{-2}{-0{,}00\ldots} = +\infty$

Ex03: Os gráficos das funções f e g são mostrados a seguir. Utilizando as propriedades dos limites, encontre o limite indicado ou aponte a razão da não existência.

Gráfico de f

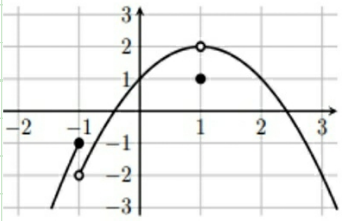
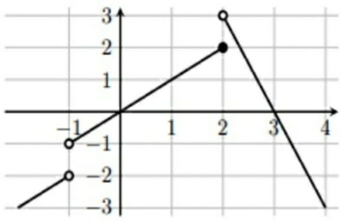


Gráfico de g



(g)	$\lim_{x\to2^-}\left[f(x)-g^2(x)\ln x\right]$
(h)	$\lim_{x\to1}\sqrt{1+f(x)+g(x)}$
(i)	$\lim_{x\to0}\sqrt[3]{\ln[f^2(x)]}$
(j)	$\lim_{x\to1}\log_3\left[g^4(x)+2f(x)\right]$
(k)	$\lim_{x\to0}\left[\operatorname{sen}^3\left(\frac{g(x)}{f(x)}\right)\right]$
(l)	$\lim_{x\to0}\left[\cos\left(\frac{f(x)}{g(x)}\right)\right]$ $= \nexists$

(g) $\lim_{x\to2^-}\left[f(x)-g^2(x).\ln x\right] = \lim_{x\to2^-} f(x) - \left(\lim_{x\to2^-} g(x)\right)^2 . \lim_{x\to2^-}\ln x =$

$= \underbrace{\lim_{x\to2^-} f(x)}_{1} - \left(\underbrace{\lim_{x\to2^-} g(x)}_{2}\right)^2 . \ln \underbrace{\lim_{x\to2^-} x}_{2} = 1+4\ln 2$

(h) $\lim_{x\to1}\sqrt{1+f(x)+g(x)} = \sqrt{\underbrace{\lim_{x\to1} 1}_{1} + \underbrace{\lim_{x\to1} f(x)}_{2} + \underbrace{\lim_{x\to1} g(x)}_{1}} = 2$

(i) $\lim_{x\to0}\sqrt[3]{\ln[f^2(x)]} = \sqrt[3]{\lim_{x\to0}\ln[f^2(x)]} =$

$= \sqrt[3]{\ln \lim_{x\to0}[f^2(x)]} = \sqrt[3]{\underbrace{\ln\left(\underbrace{\lim_{x\to0} f(x)}_{1}\right)^2}_{0}} = 0$

(j) $\lim_{x\to1}\log_3\left(g^4(x)+2f(x)\right) = \log_3\left(\left(\underbrace{\lim_{x\to1} g(x)}_{1}\right)^4 + 2\underbrace{\lim_{x\to1} f(x)}_{2}\right) =$

$= \log_3 5$

(k) $\lim_{x\to0}\left[\sin^3\left(\frac{g(x)}{f(x)}\right)\right] = \left(\lim_{x\to0}\left[\sin\left(\frac{g(x)}{f(x)}\right)\right]\right)^3 =$

$= \left(\sin\left(\frac{\cancel{\lim_{x\to0} g(x)}^{\,0}}{\underbrace{\lim_{x\to0} f(x)}_{1}}\right)\right)^3 = 0$

(i) $\lim_{x\to0}\left(\cos\left(\frac{\cancel{f(x)}^{\,1}}{\cancel{g(x)}_{\,0}}\right)\right) = \nexists$

Propriedade de Substituição Direta: Se f for uma função polinomial ou racional e a estiver no domínio de f, então

$$\boxed{\lim_{x \to a} f(x) = f(a)}$$

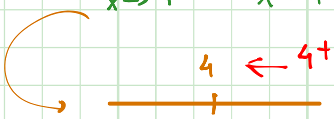
Pense um pouco!

Ex04: Calcule os limites:

a) $\displaystyle\lim_{x \to 3} \frac{x^2-4}{x-2}$	b) $\displaystyle\lim_{x \to 4^+} \frac{(6x-21)^3}{x-4}$	c) $\displaystyle\lim_{h \to 0} \frac{\sqrt{16+h}-4}{h}$

(a) $\displaystyle\lim_{x \to 3} \frac{x^2-4}{x-2} = \frac{(3)^2-4}{(3)-2} = 5$

(b) $\displaystyle\lim_{x \to 4^+} \frac{(6x-21)^3}{x-4} = \frac{(6\cdot 4-21)^3}{4{,}00\ldots 1-4} = \frac{27}{0{,}00\ldots} = \infty$

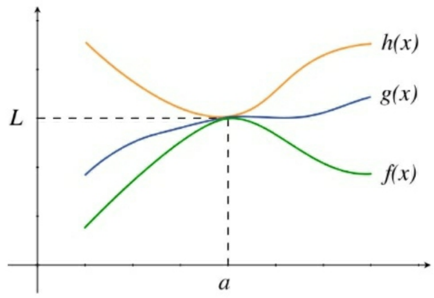


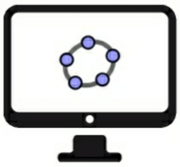
(c) $\displaystyle\lim_{h \to 0} \frac{\sqrt{16+h}-4}{h} = \frac{0}{0}$ ⇝ indetermin

$$\lim_{h \to 0} \frac{\sqrt{16+h}-4}{h} \cdot \frac{\sqrt{16+h}+4}{\sqrt{16+h}+4} = \lim_{h \to 0} \frac{(\cancel{16}+\cancel{h})-\cancel{16}}{\cancel{h}\left(\sqrt{16+h}+4\right)} = \frac{1}{8}$$

Se $f(x) \le g(x) \le h(x)$ quando $x \to a$ e

$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, então $\lim_{x\to a} g(x) = L$.





Ex05: Calcule os limites:

a) $\lim_{x\to 0} x^4 \cos\left(\frac{\pi}{x}\right)$

b) $\lim_{x\to 0^+} e^{-\frac{1}{x}} \operatorname{sen}\left(\frac{2\pi}{x}\right)$

(a) Sabemos:

$$-1 \le \cos(x) \le 1$$

$$-1 \le \cos\left(\frac{\pi}{x}\right) \le 1 \quad \to \text{Faço o argumento do cosseno } \pi/x;$$

$$-x^4 \le x^4 \cos\left(\frac{\pi}{x}\right) \le x^4 \quad \to \text{Multiplico por } x^4$$

$$-\lim_{x\to 0} x^4 \le \underbrace{\lim_{x\to 0} x^4 \cos\left(\frac{\pi}{x}\right)}_{\therefore\, = 0} \le \lim_{x\to 0} x^4 \quad \to \text{Aplico o limite}$$

$\to 0$ $\quad$ $\to 0$

(b) Sabemos:

$$-1 \le \sin(x) \le 1$$

$$-1 \le \sin\left(2\pi/x\right) \le 1$$

$$-e^{-1/x} \le e^{-1/x} \cdot \sin\left(2\pi/x\right) \le e^{-1/x}$$

$$-\lim_{x\to 0^+} e^{-1/x} \le \lim_{x\to 0^+} e^{-1/x} \sin\left(2\pi/x\right) \le \lim_{x\to 0^+} e^{-1/x}$$

$\to 0$ $\quad$ $\to 0$ $\quad$ $\to 0$

$$* \lim_{x\to 0^+} e^{-1/x} = e^{-1/0,000\ldots} = e^{-\infty} = \frac{1}{e^{\infty}} = 0$$