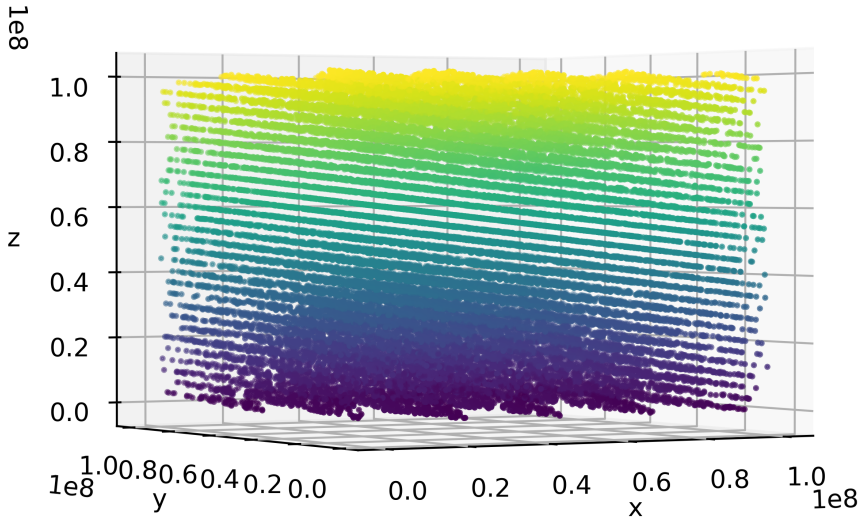
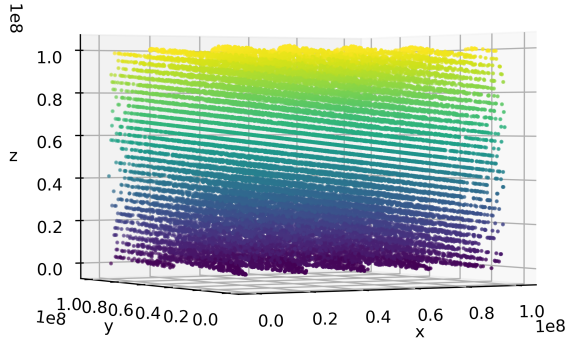
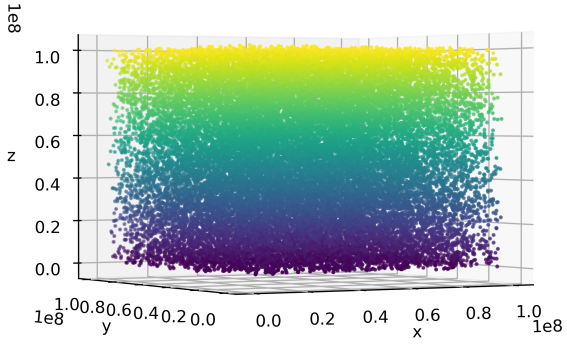
1. We can simply import the data from the "rand_points.txt" file and make a 3D plot in pyplot to see whether we observe planes or not. The same can be done for np.random.randint as well to check if we observe the same planes. The viewing angle of the 3D plot was adjusted to best show the planes, however as the value of a and b for $ax + by = z$ are not the same for each plane, a few planes merge together making it hard to discern a difference, as can be seen below. The viewing angle was kept the same for the numpy random numbers and for the default random number generator in the C standard library. We also make another plot for random numbers generated using this C standard library on my MacBook. No major changes needed to be applied to the script besides changing the name of the output file in "test_broken_libc.py". All the plots were made in the file "problem_1.py". The plots can be seen below.



(a) Random numbers from "rand_points.txt" (C standard library).



(b) Random numbers from C standard library produced on my MacBook.



(c) Random numbers generated using np.random.randint.

Figure 1: 3D plot of random numbers generated using different methods to see if we can make out any patterns.

From Figure 1a, we count the number of observable planes to be exactly 30. This is the same for the random numbers generated using the same library on my MacBook. However, the numpy generated random numbers in Figure 1c do not have any observable planes, indicating that this is a better random number generator.

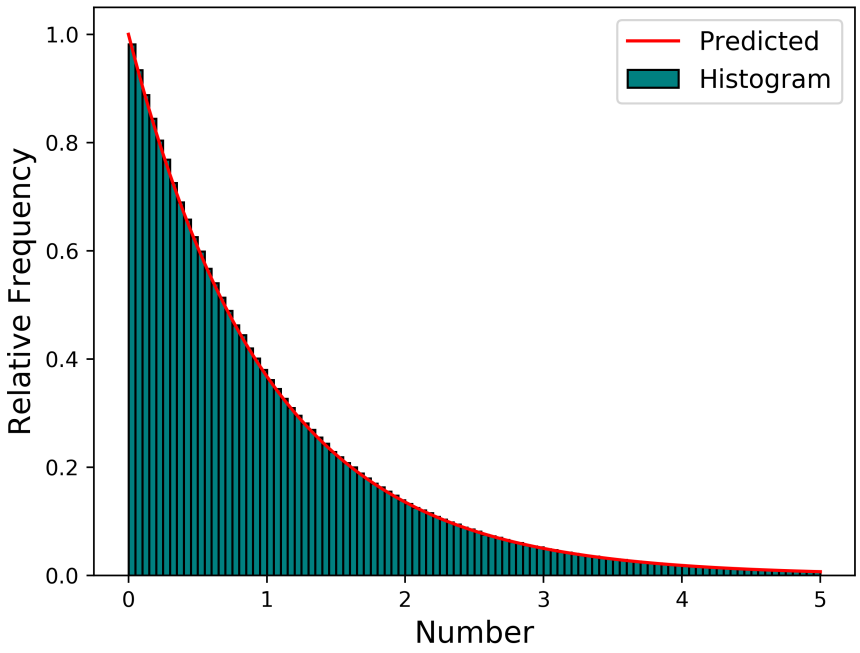
2. First, we shall look at how to randomly generate numbers for each distribution. We do so by calculating the inverse of the CDF for each distribution. Starting with the normalized Lorentzian, i.e. $1/\pi(1+x^2)$, and taking x to be our uniform generated random numbers, the CDF would be the integral of this from 0 up to y which is $\frac{1}{\pi}\arctan y$. Hence, we could generate random numbers by calculating the inverse CDF which gives us $\tan \pi x$. In order to better cover the range of tan and avoid the asymptotes we can substitute πx with $\pi(x-0.5)$ so that the input varies between $[-\pi/2, \pi/2]$. Therefore, our Lorentzian deviates can be generated as $\tan \pi(x-0.5)$. We can generate Gaussian deviates using the Box-Muller method with two random uniform deviates, θ_1 and θ_2. As we have already discussed this method in depth, we just state the result and generate the random numbers as $x = \sqrt{-2\ln\theta_1}\sin(2\pi\theta_2)$. Lastly, we can generate power law deviates, where α is our exponent, by integrating the normalised power law, $Cx^{-\alpha}$ (where $\alpha > 1$), from x_{min} to y to give $\frac{C}{1-\alpha}(y^{1-\alpha} - {x_{min}}^{1-\alpha})$. Note that C can only exist if we define our power law to exist where $x \geq x_{min}$ and $x_{min} \neq 0, -\infty, \infty$. Normalising and taking the inverse CDF, we can generate random numbers as $x_{min}(1-x)^{1/(1-\alpha)}$.

Now we decide which distributions we could use for the bounding distributions. The Lorentzian is an obvious candidate as it provides us with numbers inside our range for the exponential distribution (0 to ∞) and is always larger than the exponential. We can see this by writing out the Taylor series for our exponential: $e^{-x} = 1 - x + x^2/2 - x^3/6 + ...$ whereby if our random numbers are always between 0 and 1 (which they are), the leading order non-constant decay term (negative) is x, where as the leading order decay term for the Lorentzian is x^2. When x is between 0 and 1, $x^2 < x$ and therefore, the Lorentzian decays slower.

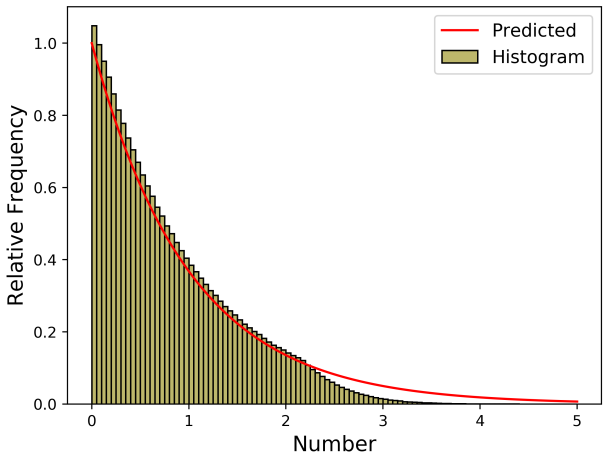
The Gaussian distribution is possible, however to be a bounding distribution it must always be greater than the distribution desired. At large x, the Gaussian will always be lower than the exponential distribution, regardless of the coefficient, and we can see this by calculating the point of intersection for a Gaussian: $Ae^{-x^2/2}$. Doing some algebra, we find that the point of intersection (at $x > 0$) is $x = 1 + \sqrt{1 + 2\ln A}$. As a result, the Gaussian cannot be used as a bounding distribution. However, we conduct the rejection method using a Gaussian $3e^{-0.5x^2}/\sqrt{2\pi}$ just for a proof of concept.

Lastly, we consider the power distribution between $[x_{min}, \infty]$. With suitable chosen α and x_{min}, the power distribution is always greater than the exponential. However, in order to accurately represent our exponential distribution, we would like to have $x_{min} = 0$ which we cannot have. Furthermore, only at $x_{min} \approx 0.9$ is the power distribution always greater than the exponential, so we cannot make x_{min} very small. This means that we cannot use the power distribution either. Therefore, lets consider the power distribution for another exponential centered at 1 with $x_{min} = 1$. However, even in this case in order for the power distribution to always be greater than the exponential, α must $= 1$, which was checked doing a quick plot on Desmos (a gif of which can be seen in the plots folder under "power_distribution.gif"). This cannot be used as $\alpha > 1$, so no matter what we choose, the power distribution cannot be used as a bounding distribution. Therefore we choose $\alpha = 1.1$ and just graph for the sake of analysis even though it cannot be used as a bounding distribution.

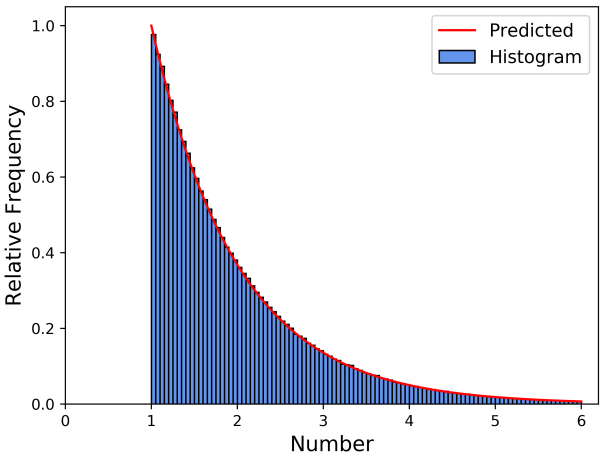
The method of rejections is explained in the python code in "problem_2.py". To quickly summarise, the method involves generating a uniformly distributed random number and if this number is less than the acceptance probability, characterised by dividing the exponential distribution by the respective probability distribution for the random numbers generated from that distribution, then we accept that number. In this way we can also calculate the ratio between the number of accepted numbers and the number of uniform deviates used to generate our distributions to see how efficient our generators are. Notably, our Lorentzian generator is already the most efficient it can possibly be as if we multiply the Lorentzian by a factor < 1, then the distribution is no longer bounding as the exponential is larger at $x = 0$. If we multiply by a factor > 1, the probability given by the rejection method ($(1 + x^2)e^{-x}$) becomes greater than 1 which is also not possible. We do not try and optimise the other distributions as they are not bounding distributions either ways. The resulting histograms can be seen below:



(a) Lorentzian Distribution: $1/(1+x^2)$.



(b) Gaussian distribution:$3e^{-0.5x^2}/\sqrt{2\pi}$.



(c) Power distribution: $x^{-1.5}$.

Figure 2: Histograms for an exponential distribution (e^{-x} and $e^{-(x-1)}$ for the power distribution) produced using the rejection method in Python.

The acceptance rate (number of uniform deviates for an exponential deviate) was determined to be $\approx 0.637, 0.310$ and 0.0998 for the Lorentzian, Gaussian and power distribution, respectively. Looking at Figure 2, we see that the Lorentzian and power distribution very accurately reflect their exponential distributions, even though the power distribution is not bounding as it is smaller than the exponential for only a very small range of x and hence, produces good results (even though the acceptance rate is much smaller). The Gaussian, as expected, is not quite accurate as at larger x, the Gaussian becomes smaller than the exponential and we see the histogram bars become lower for these values while at the same time being larger than the prediction for small x.

3. For a ratio-of-uniforms generator, we have the condition that $0 \leq u \leq \sqrt{p(v/u)}$ (where p represents the probability) or in our case since we are considering the exponential function: $0 \leq u \leq e^{-v/2u}$. We can determine the bounds on v using the method given in "Independent Sampling Methods" by Luca Martino et al. He states that if we define $c \leq v \leq d$ then:

$$c = -\sup_x \left(x\sqrt{p(x)} \right), \tag{1}$$

$$d = \sup_x \left(x\sqrt{p(x)} \right), \tag{2}$$

where $p(x)$ refers to the probability distribution (i.e. the exponential), $x = v/u$ and $\sup_x$ is the supremum of our set or the maximum value it can take. Therefore, we differentiate $x\sqrt{p(x)}$ and find its maximum value:

$$\frac{d}{dx}\left(x\sqrt{p(x)} \right) = \frac{d}{dx}\left(xe^{-x/2} \right) = -\frac{x}{2}e^{-x/2} + e^{-x/2} = 0, \tag{3}$$

$$e^{-x/2}\left(1 - \frac{x}{2}\right) = 0, \tag{4}$$

$$1 - \frac{x}{2} = 0, \quad (\text{as } \mathrm{e}^{-\mathrm{x}/2} \neq 0) \tag{5}$$

$$\therefore x = 2. \tag{6}$$

Now we can calculate the maximum value by plugging in 2 so $x\sqrt{e^{-x}} = 2e^{-1}$. Therefore,

$$c = -2e^{-1} \text{ and } d = 2e^{-1}, \tag{7}$$

$$\therefore v = [-2e^{-1}, 2e^{-1}]. \tag{8}$$

Now we write our script taking our values of u and v between these bounds (in "problem_3.py") and returning v/u. The numbers for v can be produced by subtracting 0.5 from our uniform deviates (so that we can get negative numbers, i.e. numbers that deviate between $[-0.5, 0.5]$) and then multiplying by $4e^{-1}$ so that the numbers vary between $[-2e^{-1}, 2e^{-1}]$. This is also the most efficient we can be without deviating from the exponential distribution or lowering our acceptance rate. The following histogram was produced as a result of this:

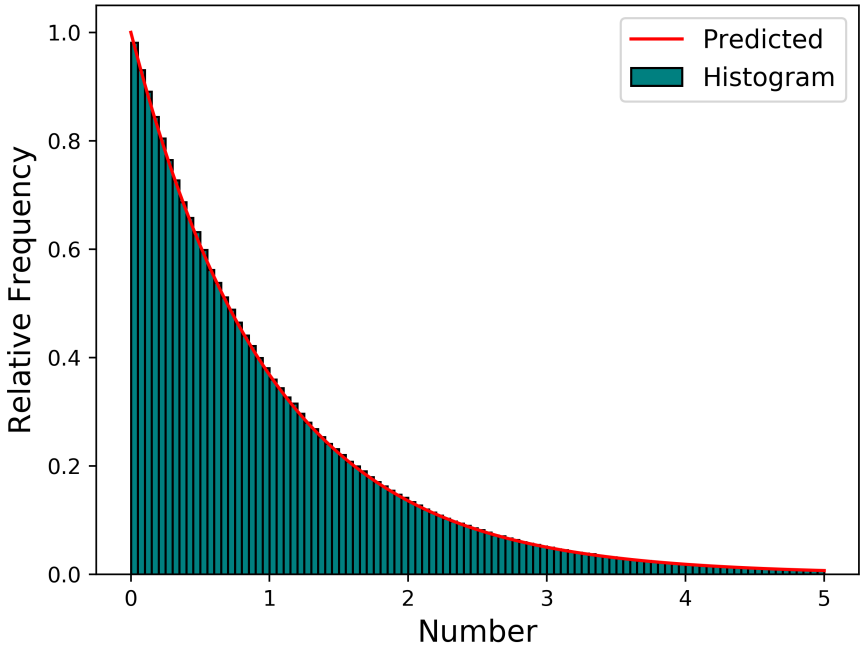


Figure 3: Normalised histogram of an exponential distribution produced using a ratio-of-uniform generator against the predicted distribution (e^{-x}).

Calculating the number of exponential deviates produced per uniform deviate (the acceptance rate), recognising that each test involves producing 2 uniform deviates, gives us a rate of 0.419977... $\approx$ 0.420. Looking at Figure 3 we see that the result is very accurate and follows the prediction almost exactly.