Exercise 1

Course: *Numerical Solutions of Partial Differential Equations*
by Xiaozhou Li, Spring, 2019. `http://xiaozhouli.com/teaching/ns/index.html`

Exercise 1.1 *(Singular Perturbations Problem)*

(a) Use the method of undetermined coefficients to determine a pth-order accurate finite difference approximation to $u''(x)$ based on 3 general points,

$$u''(x) = c_0 u(x_0) + c_1 u(x_1) + c_2 u(x_2).$$

(b) Use this finite difference formula to solve a Perturbed Problem

$$\begin{aligned} \epsilon u''(x) - u'(x) &= f(x), \\ u(0) = 1, \quad u(1) &= 3, \end{aligned}$$

with $\epsilon = 0.01$ and $f(x) = -1$. What is the exact solution for this problem? If the grid is given by $0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = 1$, please design a grid such that the maximum error $\|E\|_\infty$ is smaller than 10^{-5} (require $m \le 100$). You should plot the finite difference solution and compared with the exact solution. What is the smallest value of m based on the given requirement?

Exercise 1.2 *(Poisson problem)*

Writing a script to solve the Poisson problem on a square $m \times m$ grid with $\Delta x = \Delta y = h$, using the 5-point Laplacian. It is set up to solve a test problem for which the exact solution is $u(x, y) = \exp(x + y/2)$, using Dirichlet boundary conditions and the right hand side $f(x, y) = 1.25 \exp(x + y/2)$.

(a) Test your script by performing a grid refinement study to verify that it is second order accurate.

(b) Modify the script so that it works on a rectangular domain $[a_x, b_x] \times [a_y, b_y]$, but still with $\Delta x = \Delta y = h$. Test your modified script on a non-square domain.

(c) Further modify the code to allow $\Delta x \neq \Delta y$ and test the modified script.

(d) Show that the 9-point Laplacian (3.17) has the truncation error derived in Section 3.5. **Hint:** To simplify the computation, note that the 9-point Laplacian can be written as the 5-point Laplacian (with known truncation error) plus a finite difference approximation that models $\frac{1}{6}h^2 u_{xxyy} + O(h^4)$.

(e) Modify your script to use the 9-point Laplacian (3.17) instead of the 5-point Laplacian, and to solve the linear system (3.18) where f_{ij} is given by (3.19). Perform a grid refinement study to verify that fourth order accuracy is achieved.