DIFFUSION PROCESSES

Definition of a Diffusion Process

- A Markov process consists of three parts: a drift (deterministic), a random process and a jump process.
- A diffusion process is a Markov process that has continuous sample paths (trajectories). Thus, it is a Markov process with no jumps.
- A diffusion process can be defined by specifying its first two moments:

Definition 1. *A Markov process X_t with transition probability $P(\Gamma, t|x, s)$ is called a* **diffusion process** *if the following conditions are satisfied.*

i) (Continuity). For every x and every $\varepsilon > 0$

$$\int_{|x-y|>\varepsilon} P(dy, t|x, s) = o(t-s) \tag{1}$$

uniformly over $s < t$.

ii) (Definition of drift coefficient). There exists a function $a(x, s)$ such that for every x and every $\varepsilon > 0$

$$\int_{|y-x|\leqslant\varepsilon} (y-x)P(dy, t|x, s) = a(x, s)(s-t) + o(s-t). \tag{2}$$

uniformly over $s < t$.

iii) (Definition of diffusion coefficient). There exists a function $b(x,s)$ such that for every x and every $\varepsilon > 0$

$$\int_{|y-x|\leqslant\varepsilon} (y-x)^2 P(dy, t|x,s) = b(x,s)(s-t) + o(s-t). \quad (3)$$

uniformly over $s < t$.

- Under additional regularity assumptions we can show that a diffusion process is completely determined from its first two moments.

Theorem 1. *(Kolmogorov) Let* $f(x) \in C_b(\mathbb{R})$ *and assume that*

$$u(x,s) := \int f(y) P(dy, t|x,s) \in C_b^2(\mathbb{R}).$$

Assume furthermore that the functions $a(x,s)$, $b(x,s)$ *are continuous in both* x *and* s*. Then* $u(x,s) \in C^{2,1}(\mathbb{R} \times \mathbb{R}^+)$ *and it solves the* **final value problem**

$$-\frac{\partial u}{\partial s} = a(x,s)\frac{\partial u}{\partial x} + \frac{1}{2} b(x,s) \frac{\partial^2 u}{\partial x^2}, \quad \lim_{s \to t} u(s,x) = f(x). \tag{4}$$

- Assume that the transition function has a density:

$$P(\Gamma, t|x, s) = \int_\Gamma p(s, x, t, y)\, dy.$$

- Under some regularity assumptions we can derive the Fokker–Planck (forward Kolmogorov) equation.

Theorem 2. *(Kolmogorov) Assume that conditions* (1), (2), (3) *are satisfied and that* $p(\cdot, \cdot, t, y)$, $a(t, y), b(t, y) \in C^{2,1}(\mathbb{R} \times \mathbb{R}^+)$. *Then the transition probability density satisfies the equation*

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial y}\left(a(t, y)p\right) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(b(t, y)p\right), \quad \lim_{t \to s} p(s, x, t, y) = \delta(x - y). \tag{5}$$

Remark 1. *Assume that initial distribution of* X_t *is* $\rho_0(x)$ *and set* $s = 0$ *(the initial time) in* (5)*. Define*

$$p(t, y) := \int p(0, x, t, y)\, dx.$$

Integrating the forward Kolmogorov equation (5) *with respect to* x *we obtain the Fokker-Plank equation for* $p(t, y)$

$$\frac{\partial p(t, y)}{\partial t} = -\frac{\partial}{\partial y}\left(a(t, y)p(t, y)\right) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(b(t, y)p(t, y)\right).$$

The Fokker-Planck Equation in Arbitrary Dimensions

- the drift and diffusion coefficients of a diffusion process on $\mathbb{R}^2$ are defined as:

$$\lim_{t \to s} \frac{1}{t-s} \int_{|y-x|<\varepsilon} (y-x) P(s,x,t,dy) = \mathbf{a}(s,x)$$

- and

$$\lim_{t \to s} \frac{1}{t-s} \int_{|y-x|<\varepsilon} (y-x) \otimes (y-x) P(s,x,t,dy) = \mathbf{b}(s,x).$$

- The drift $a(s,x)$ is a d-dimensional vector field and the diffusion is a $d \times d$ symmetric matrix.
- The generator of a d dimensional diffusion process is

$$\begin{aligned} \mathcal{L} &= a(s,x) \cdot \nabla + \frac{1}{2} b(s,x) : \nabla\nabla \\ &= \sum_{j=1}^{d} a_j \frac{\partial}{\partial x_j} + \frac{1}{2} \sum_{i,j=1}^{d} b_{ij} \frac{\partial^2}{\partial x_j^2}. \end{aligned}$$

In Definition 1 we had to truncate the domain of integration since we didn't know whether the first and second moments exist. If we assume that there exists a $\delta > 0$ such that

$$\lim_{t \to s} \frac{1}{t-s} \int_{\mathbb{R}^d} |y-x|^{2+\delta} P(s,x,t,dy) = 0, \tag{6}$$

then we can extend the integration over the whole $\mathbb{R}^d$ and use expectations in the definition of the drift and the diffusion coefficient. Indeed, ,let $k = 0, 1, 2$ and notice that

$$\begin{aligned}
\int_{|y-x|>\varepsilon} |y-x|^k P(s,x,t,dy) &= \int_{|y-x|>\varepsilon} |y-x|^{2+\delta} |y-x|^{k-(2+\delta)} P(s,x,t,dy) \\
&\leqslant \frac{1}{\varepsilon^{2+\delta-k}} \int_{|y-x|>\varepsilon} |y-x|^{2+\delta} P(s,x,t,dy) \\
&\leqslant \frac{1}{\varepsilon^{2+\delta-k}} \int_{\mathbb{R}^d} |y-x|^{2+\delta} P(s,x,t,dy).
\end{aligned}$$

Using this estimate together with (6) we conclude that:

$$\lim_{t\to s}\frac{1}{t-s}\int_{|y-x|>\varepsilon}|y-x|^k P(s,x,t,dy)=0, \quad k=0,1,2.$$

This implies that assumption (6) is sufficient for the sample paths to be continuous ($k=0$) and for the replacement of the truncated integrals in (2) and (3) by integrals over $\mathbb{R}^d$ ($k=1$ and $k=2$, respectively). The definitions of the drift and diffusion coefficients become:

$$\lim_{t\to s}\mathbb{E}\left(\frac{X_t-X_s}{t-s}\Big|X_s=x\right)=\mathbf{a}(x,s) \tag{7}$$

and

$$\lim_{t\to s}\mathbb{E}\left(\frac{(X_t-X_s)\otimes(X_t-X_s)}{t-s}\Big|X_s=x\right)=\mathbf{b}(x,s) \tag{8}$$

Notice also that the continuity condition can be written in the form

$$\mathbb{P}\left(|X_t-X_s|\geqslant\varepsilon|X_s=x\right)=o(t-s).$$

Now it becomes clear that this condition implies that the probability of large changes in X_t over short time intervals is small. Notice, on the other hand, that the above condition implies that the sample paths of a diffusion process **are not differentiable**: if they where, then the right hand side of the above equation would have to be 0 when $t - s \ll 1$. The sample paths of a diffusion process have the regularity of Brownian paths. A Markovian process **cannot be** differentiable: we can define the derivative of a sample paths only with processes for which the past and future are not statistically independent when conditioned on the present.

Let us denote the expectation conditioned on $X_s = x$ by $\mathbb{E}^{s,x}$. Notice that the definitions of the drift and diffusion coefficients (7) and (8) can be written in the form

$$\mathbb{E}^{s,x}(X_t - X_s) = \mathbf{a}(x, s)(t - s) + o(t - s).$$

and

$$\mathbb{E}^{s,x}\Big((X_t - X_s) \otimes (X_t - X_s)\Big) = \mathbf{b}(x, s)(t - s) + o(t - s).$$

Consequently, the drift coefficient defines the **mean velocity vector** for the stochastic process X_t, whereas the diffusion coefficient (tensor) is a measure of the local magnitude of fluctuations of $X_t - X_s$ about the mean value. hence, we can write locally:

$$X_t - X_s \approx \mathbf{a}(s, X_s)(t - s) + \sigma(s, X_s)\,\xi_t,$$

where $\mathbf{b} = \sigma\sigma^{\mathbf{T}}$ and ξ_t is a mean zero Gaussian process with

$$E^{s,x}(\xi_t \otimes \xi_s) = (t - s)I.$$

Since we have that

$$W_t - W_s \sim \mathcal{N}(0, (t-s)I),$$

we conclude that we can write locally:

$$\Delta X_t \approx \mathbf{a}(s, X_s)\Delta t + \sigma(s, X_s)\Delta W_t.$$

Or, replacing the differences by differentials:

$$dX_t = \mathbf{a}(t, X_t)dt + \sigma(t, X_t)dW_t.$$

Hence, the sample paths of a diffusion process are governed by a **stochastic differential equation** (SDE).

- the proof of Theorem 1 is based on the Chapman-Kolmogorov equation and the use of a Taylor series expansion for $u(x,s) := \int f(y) P(dy, t|x, s)$:

$$u(z,s) - u(x,s) = \frac{\partial u(s,x)}{\partial x}(z-x) + \frac{1}{2}\frac{\partial^2 u(s,x)}{\partial x^2}(z-x)^2(1+r_\varepsilon), \quad |z-x| \leqslant \varepsilon.$$

- Using this and the CK equation we can obtain a formula for the finite difference

$$\frac{u(s_2,x) - u(s_1,x)}{s_2 - s_1}.$$

- We then obtain the backward Kolmogorov equation by passing to limit $s_2 \to s_1$ and using the definition of the drift and diffusion coefficients.