CS 170 Homework 2

Due **Monday 2/5/2024, at 10:00 pm (grace period until 11:59pm)**

1 Study Group

List the names and SIDs of the members in your study group. If you have no collaborators, you must explicitly write "none".

Solution:

- Lakshya Nagal, SID: 3037935253
- Joyce Lu, SID: 3039208707
- Melad Mayaar, SID: 3038197632
- OH and Homework Party

2 Median

Recall the quickselect algorithm used for quickly finding the median of an array. Suppose you ran quickselect on the following array to find it's median.

12	78	13	1	97	45	48	26	85	100	78

1. Suppose you always pick the first element as the pivot. List the all the subarrays that quickselect recurses on, as well as the index k – the k smallest element of the current subarray which the algorithm returns.

   For example, quickselect begins on the entire array [12, 78, 13, 1, 97, 45, 48, 26, 85, 100, 78] and $k = 6$.

   Solution:

   1^{st} call

   S_R : $\boxed{78}\boxed{13}\boxed{97}\boxed{45}\boxed{48}\boxed{26}\boxed{85}\boxed{100}\boxed{78}$, $k = 6$

   2^{nd} call

   S_L : $\boxed{13}\boxed{45}\boxed{48}\boxed{26}$, $k = 4$

   3^{rd} call

   S_R : $\boxed{45}\boxed{48}\boxed{26}$, $k = 4$

   4^{th} call

   S_R : $\boxed{48}$, $k = 3$

   5^{th} call

   $k = 1$, return $\Rightarrow v = 48$

2. Specify which elements should be picked as pivots at each step in order to **maximize** the runtime of this algorithm. Write the numerical value of the elements, not their indices.
   Solution: The idea to maximize the runtime would be to chose the smallest or biggest elements as the pivot up to the median number and then pick the opposite of your inital choice (i.e. choose smallest elements as pivots then after median choose largest elements). One way would be picking 100, 97, 85, 78, 1, 12, 13, 26, 45, 48, as pivots.

3. Specify which elements should be picked as pivots at each step in order to **minimize** the runtime of this algorithm. Write the numerical value of the elements, not their indices.
   Solution: If we pick the median to be our initial pivot then, we return the median immediately after partitioning the input array. We call quickselect on the array above with $k = 6$ and our first pivot is $v = 48$...

$$\mathrm{S}_L : \boxed{12} \boxed{13} \boxed{1} \boxed{45} \boxed{26} \quad \mathrm{S}_v : \boxed{48} \quad \mathrm{S}_R : \boxed{78} \boxed{97} \boxed{85} \boxed{100} \boxed{78}$$

$$|S_L| < k \leq |S_L| + |S_v| \Rightarrow \textit{return } v = 48$$

3 Median of Medians

The QuickSelect(A, k) algorithm for finding the kth smallest element in an unsorted array A picks an arbitrary pivot, then partitions the array into three pieces: the elements less than the pivot, the elements equal to the pivot, and the elements that are greater than the pivot. It is then recursively called on the piece of the array that still contains the kth smallest element.

(a) Consider the array $A = [1, 2, \ldots, n]$ shuffled into some arbitrary order. What is the worst-case runtime of QuickSelect(A, $n/2$) in terms of n? Construct a sequence of 'bad' pivot choices that achieves this worst-case runtime.

Hint: refer to Q2.

Solution: A bad pivot would consist of the smallest or largest element of A such as the sequence $1, 2, 3, 4, \ldots$. This would yield the runtime:

$$n + (n-1) + (n-2) + \cdots + \frac{n}{2} = \Theta(n^2)$$

The above 'worst case' has a chance of occurring even with randomly-chosen pivots, so the worst-case time for QuickSelect is $\mathcal{O}(n^2)$, even though it achieves $\Theta(n)$ on average.

Based on QuickSelect, let's define a new algorithm DeterministicSelect that deterministically picks a consistently good pivot every time. This pivot-selection strategy is called 'Median of Medians', so that the worst-case runtime of DeterministicSelect(A, k) is $\mathcal{O}(n)$.

Median of Medians

1. Group the array into $\lfloor n/5 \rfloor$ groups of 5 elements each (ignore any leftover elements)
2. Find the median of each group of 5 elements (as each group has a constant 5 elements, finding each individual median is $\mathcal{O}(1)$)
3. Create a new array with only the $\lfloor n/5 \rfloor$ medians, and find the true median of this array using DeterministicSelect.
4. Return this median as the chosen pivot

(b) Let p be the pivot chosen by DETERMINISTICSELECT on A. Show that at least $3n/10$ elements in A are less than or equal to p, and that at least $3n/10$ elements are greater than or equal p.

Solution:

Example: $\mathbf{n = 15}$

A:

92	3	10	26	81	72	5	16	18	23	85	100	12	15	48

Split into $\lfloor \frac{n}{5} \rfloor$ groups: S_i for $0 < i \leq \lfloor \frac{n}{5} \rfloor$

S_1:

92	3	10	26	81

S_2:

72	5	16	18	23

S_3:

85	100	12	15	48

Find the median of S_i:

3	10	**26**	81	92	5	16	**18**	23	72	12	15	**48**	85	100

Create a new array, M_f, with the medians of S_i:

M_f:

26	18	48

Find the median of M_f:

M_f:

18	**26**	48

Let m denote the median of M_f, that is $m = \left\lfloor \frac{|M_f|}{2} \right\rfloor$.

M_f contains $\lfloor \frac{n}{5} \rfloor$ elements. To find m we need to retrieve the $\frac{\frac{n}{5}}{2} = \frac{n}{10}$ element. We know that every element that comes before m (there are $\frac{n}{10}$ elements) is less than or equal to the m. Since we intially grouped n elements into S_i subarrays with five elements in each then found the median of S_i, this implies that there are exactly two elements that are less than or equal to the median of S_i. Consequently every element from M_f that is less than or equal to m implicitly has two elements that are also less than or equal to m. Leaving us with the following equation:

$$\frac{n}{10}(2+1) = \frac{3n}{10}$$

where $\frac{n}{10}$ is the number of elements to the left of m, for which there are **2** elements in S_i that are less than or equal to m, as well as the elements to the left of m in M_f (**+1**). By symmetry, the argument for the number of elements that are greater than or equal to m would be the same and yield the same results as above.

(c) Show that the worst-case runtime of DeterministicSelect(A, k) using the 'Median of Medians' strategy is $\mathcal{O}(n)$. *Hint: Using the Master theorem will likely not work here. Find a recurrence relation for $T(n)$, and try to use induction to show that $T(n) \leq c \cdot n$ for some $c > 0$.*

Solution: Recurrence relation is $T(n) = T(n/5) + T(7n/10) + O(n)$

Now we want to show that $T(n) \leq cn, c > 0$

Base Case:

The case for small n is trivial, as shown in 3(b) with $n = 15$

Inductive Hypothesis:

$$T\left(\frac{n}{5}\right) \leq c\left(\frac{n}{5}\right)$$
$$T\left(\frac{7n}{10}\right) \leq c\left(\frac{7n}{10}\right)$$

Inductive Step:

$$\begin{aligned} T(n) &\leq c\left(\frac{n}{5}\right) + c\left(\frac{7n}{10}\right) + bn \\ cn &= c\left(\frac{n}{5}\right) + c\left(\frac{7n}{10}\right) + bn \\ c &= c\left(\frac{9}{10}\right) + b \\ c &= 10b \end{aligned}$$

Thus we have shown that $T(n) \leq c \cdot n$ for $c > 0$

4 The Resistance

We are playing a variant of The Resistance, a board game where there are n players, s of which are spies. In this variant, in every round, we choose a subset of players to go on a mission. A mission succeeds if the subset of the players does not contain a spy, but fails if at least one spy goes on the mission. After a mission completes, we only know its outcome and not which of the players on the mission were spies.

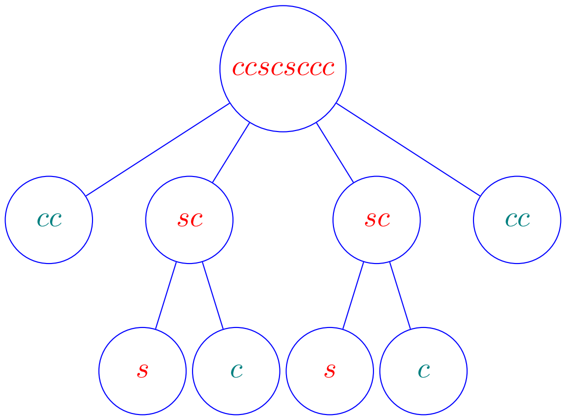
Come up with a strategy that identifies all the spies in $O(s\log(n/s))$ missions. **Describe your strategy and analyze the number of missions needed.**

Hint 1: consider evenly splitting the n players into x disjoint groups (containing $\approx n/x$ players each), and send each group on a mission. At most how many of these x missions can fail? What should you set x to be to ensure that you can reduce your problem by a factor of at least $1/2$?

Hint 2: it may help to try a small example like $n = 8$ and $s = 2$ by hand.

Solution: The strategy would be to send the players on a mission, if the mission fails, we split the n players evenly into $2s$ groups, and recurse on the groups stopping when the mission returns succesful.

Example: $\mathbf{n = 8, s = 2 = spy, c} \in \mathbf{n,}\ c \neq s$ **with** $\mathbf{x = 2s = 4}$



Since at every level at most s missions will fail all the way down to the the $l-1$ level (because at the last l level we know who is a spy and no further missions are needed), and additionallly the height of the tree is $l = \log_{2s} n$ then we have the following relationship:

$$\begin{aligned} s\cdot(l-1) &= s(\log_{2s} n - 1) \\ &= s(\log_{2s} n - \log_{2s} 2s) \\ &= s(\log_{2s}\left(\frac{n}{2s}\right) \\ &= s\log\left(\frac{n}{s}\right) \end{aligned}$$

Giving a total runtime of $O(s\log(n/s))$

5 Among Us

You are playing a party game with n other friends, who play either as imposters or crewmates. You do not know who is a crewmate and who is a imposter, but all your friends do. There are always more crewmates than there are imposters.

Your goal is to identify one player who is certain to be a crewmate.

Your allowed 'query' operation is as follows: you pick two players as partners. You ask each player if their partner is a crewmate or a imposter. When you do this, a crewmate must tell the truth about the identity of their partner, but a imposter doesn't have to (they may lie or tell the truth about their partner).

Your algorithm should work regardless of the behavior of the imposters.

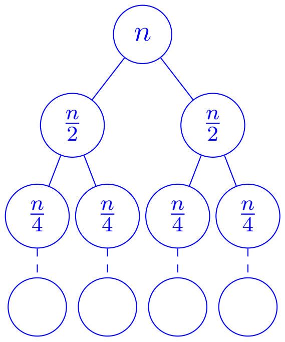
(a) For a given player x, devise an algorithm that returns whether or not x is a crewmate using $O(n)$ queries. Just an informal description of your test and a brief explanation of why it works is needed.

Solution: The simplest solution would be to take a vote from the $n-1$ players and go with the popular vote. This is guaranteed to expose player x since the majority of the $n-1$ players are crewmates who can only tell the truth. This would yield a time complexity of $O(n)$ since at most you have ask n.

(b) Show how to find a crewmate in $O(n \log n)$ queries (where one query is taking two players x and y and asking x to identify y and y to identify x). *Hint: Split the players into two groups, recurse on each group, and use part (a). What invariant must hold for at least one of the two groups?*

Give a 3-part solution.

Solution:



Algorithm Description:

We split the players into two groups and keep splitting until there are around 3 players in each group. At which point we can run the algorithm from part (a) and keep only the players that are voted as being crewmates by the majority of the group.

Proof of Correctness:

Inductive Hypothesis:

The algorithm above indentifies a crewmate amongst n players or less.

Base Case: n≤ 3

From the description of the game, we know there are more players than imposter, so the majority vote should suffice to find a crewmate.

Inductive Step:

From our inductive hypothesis we know that the algorithm works for groups with less than or equal to 3 players. Since we recursively keep splitting the n players into 2 groups until we are eventually left with a finite number of players in each group. From our base case we know that the algorithm in part (a) can identify a crewmate, so keep on those players.

Runtime Analysis:

The above algorithm gives the following recurrence relation: $T(n) = 2T(\frac{n}{2}) + O(n)$. From the tree abovewe can see our branching factor is 2. The total amount of work done is $\sum_{k=0}^{\log_2 n} 2^k \frac{n}{2^k} = n \sum_{k=0}^{\log_2 n} 1 = O(n \log n)$

6 Modular Fourier Transform

Fourier transforms (FT) have to deal with computations involving irrational numbers which can be tricky to implement in practice. Motivated by this, in this problem you will demonstrate how to do a Fourier transform in modular arithmetic, using modulo 5 as an example.

(a) There exists $\omega \in \{0, 1, 2, 3, 4\}$ such that $\{\omega^0, \omega^1, \omega^2, \omega^3\}$ the are 4^{th} roots of unity (modulo 5), i.e., solutions to $z^4 = 1 \pmod 5$. When doing the FT in modulo 5, this ω will serve a similar role to the primitive root of unity in our standard FT. Show that $\{1, 2, 3, 4\}$ are the 4^{th} roots of unity (modulo 5), with $\omega = 2$ as the primitive root. Also show that $1 + \omega + \omega^2 + \omega^3 = 0 \pmod 5$ for $\omega = 2$.

Solution:

$$\begin{aligned}
1^4 &= 1 \equiv 1 \pmod 5 \\
2^4 &= 16 \equiv 1 \pmod 5 \\
3^4 &= 81 \equiv 1 \pmod 5 \\
4^4 &= 256 \equiv 1 \pmod 5
\end{aligned}$$

Since $(1, 2, 3, 4)$ raised to the 4^{th} gives back 1, $(1, 2, 3, 4)$ are the 4^{th} roots of unity (modulo 5).

$$\begin{aligned}
2^0 &= 1 \equiv 1 \pmod 5 \\
2^1 &= 2 \equiv 2 \pmod 5 \\
2^2 &= 4 \equiv 4 \pmod 5 \\
2^3 &= 8 \equiv 3 \pmod 5 \\
2^4 &= 16 \equiv 1 \pmod 5 \\
2^5 &= 32 \equiv 2 \pmod 5 \\
2^6 &= 64 \equiv 4 \pmod 5 \\
2^7 &= 128 \equiv 3 \pmod 5 \\
&\vdots \\
1 + \omega + \omega^2 + \omega^3 &= 1 + 2 + 4 + 3 = 10 \equiv 0 \pmod 5
\end{aligned}$$

(b) Using the FFT, produce the transform of the sequence $(0, 2, 3, 0)$ modulo 5; that is, evaluate the polynomial $2x + 3x^2$ at $\{1, 2, 4, 3\}$ using the recursive FFT algorithm defined in class, but with $\omega = 2$ and in modulo 5 instead of with $\omega = i$ in the complex numbers. Note that all calculations should be performed modulo 5.

Hint: You can verify your calculation by evaluating the polynomial at $\{1, 2, 4, 3\}$ *using the slow method (i.e. DFT).*

Solution:

```
$FFT([0,2,3,2],[1,3,4,2])$
    $E_0(x) = 0 + 3x$
    $FFT([0,3],[1,4])$
        $E_1(x) = 0 \hookrightarrow 0$
        $O_1(x) = 3 \hookrightarrow 3$
        $\hookrightarrow P(1) = E_1(1^2) + 1 \cdot O_1(1^2) = 0 + 3 = 3 \equiv 3 \pmod 5$
        $\hookrightarrow P(4) = E_1(4^2) + 4 \cdot O_1(4^2) = 0 + 12 = 12 \equiv 2 \pmod 5$
    $O_0(x) = 2 + 0x$
    $FFT([2,0],[1,4])$
        $E_1(x) = 2 \hookrightarrow 2$
        $O_1(x) = 0 \hookrightarrow 0$
        $\hookrightarrow P(1) = E_1(1^2) + 1 \cdot O_1(1^2) = 2 + 0 = 2 \equiv 2 \pmod 5$
        $\hookrightarrow P(4) = E_1(4^2) + 4 \cdot O_1(4^2) = 2 + 0 = 2 \equiv 2 \pmod 5$
  $\hookrightarrow P(1) = E(1^2) + 1 \cdot O(1^2) = 3 + 2 = 5 \equiv 0 \pmod 5$
  $\hookrightarrow P(2) = E(2^2) + 2 \cdot O(2^2) = 2 + 4 = 6 \equiv 1 \pmod 5$
  $\hookrightarrow P(4) = E(4^2) + 4 \cdot O(4^2) = 3 + 8 = 11 \equiv 1 \pmod 5$
  $\hookrightarrow P(3) = E(3^2) + 3 \cdot O(3^2) = 2 + 6 = 8 \equiv 3 \pmod 5$
  $\Rightarrow$ return $\boxed{[0,1,1,3]}$
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(c) Now perform the inverse FFT on the sequence $(0, 1, 4, 0)$, also using the recursive algorithm. Recall that the inverse FFT is the same as the forward FFT, but using ω^{-1} instead of ω, and with an extra multiplication by 4^{-1} for normalization.

Solution:

Inverses: [1, 3, 4, 2]

```
$FFT([0,1,4,0],[1,3,4,2])$
    $E_0(x) = 0 + 4x$
    $FFT([0,4],[1,4])$
        $E_1(x) = 0 \hookrightarrow 0$
        $O_1(x) = 4 \hookrightarrow 4$
      $\hookrightarrow P(1) = E_1(1^2) + 1 \cdot O_1(1^2) = 0 + 4 = 4 \equiv 4 \pmod 5$
      $\hookrightarrow P(4) = E_1(4^2) + 4 \cdot O_1(4^2) = 0 + 16 = 16 \equiv 1 \pmod 5$
    $O_0(x) = 1 + 0x$
    $FFT([1,0],[1,4])$
        $E_1(x) = 1 \hookrightarrow 1$
        $O_1(x) = 0 \hookrightarrow 0$
      $\hookrightarrow P(1) = E_1(1^2) + 1 \cdot O_1(1^2) = 1 + 0 = 1 \equiv 1 \pmod 5$
      $\hookrightarrow P(4) = E_1(4^2) + 4 \cdot O_1(4^2) = 1 + 0 = 1 \equiv 1 \pmod 5$
  $\hookrightarrow P(1) = E(1^2) + 1 \cdot O(1^2) = 4 + 1 = 5 \equiv 0 \pmod 5$
  $\hookrightarrow P(3) = E(3^2) + 3 \cdot O(3^2) = 16 + 3 = 19 \equiv 4 \pmod 5$
  $\hookrightarrow P(4) = E(4^2) + 4 \cdot O(4^2) = 4 + 4 = 8 \equiv 3 \pmod 5$
  $\hookrightarrow P(2) = E(2^2) + 2 \cdot O(2^2) = 16 + 2 = 18 \equiv 3 \pmod 5$
  $\Rightarrow$ **return** $4 \cdot [0,4,3,3] = [0,16,12,12] = \boxed{[0,1,2,2] \pmod 5}$
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(d) Now show how to multiply the polynomials $2x + 3x^2$ and $3 - x$ using the FFT modulo 5. You may use the fact that the FFT of $(3, 4, 0, 0)$ modulo 5 is $(2, 1, 4, 0)$ without doing your own calculation.

Solution:

First we calculat ethe FFT of $2x + 3x^2$ and $3 - x$ (modulo 5) with the 4^{th} roots of unity from part (a). From part (b) we know that the FFT of $2x + 3x^2 = [0, 1, 1, 3]$. The FFT of $3 - x$ was given as $[2, 1, 4, 0]$. We know multiply the two vectors point wise resulting in $[0, 1, 4, 0]$. We then find the inverse of $0 + x + 4x^2 + 0x^3 \Rightarrow [0, 1, 4, 0]$ which we found in part (c) to get the final polynomial.