CS 170 Homework 8

Due **3/18/2024, at 10:00 pm (grace period until 11:59pm)**

1 Study Group

List the names and SIDs of the members in your study group. If you have no collaborators, you must explicitly write "none".

Solution: I worked on this homework with the following collaborators:

- Lakshya Nagal, SID: 3037935253

2 Faster Longest Increasing Subsequence

Recall the dynamic programming algorithm for LIS from lecture. It has the recurrence,

$$L[i] = \max_{j<i:A[j]<A[i]} L[j] + 1,$$

where $L[i]$ is the length of the longest increasing subsequence that includes and ends at $A[i]$. Using DP to compute all the $L[i]$'s takes $O(N^2)$ time, where N is the length of the array A. In this problem, we will see how to reformulate the problem so that we can use binary search to obtain a $O(N \log N)$ time algorithm.

Consider the following subproblem definition:

$M_i[j]$ = the smallest element that ends any subsequence of length j for $A[1 \dots i]$.

where M_i is 1-indexed.

We can set $M_i[k] = \infty$ if no increasing subsequence of length k exists in $A[1 \dots i]$.

(a) Given the following array of length 10, compute the values of M_8. Recall that M_8 only considers the elements $A[1 \dots 8]$. What is the length of the LIS of $A[1 \dots 8]$, and what is the last element of the LIS?

5	3	7	4	1	2	5	7	8	3

Solution:

$$M_8[1] = [5], [3], [7], [4], [1], [2], [5], [7] \to 1$$
$$M_8[2] = [5,7], [3,7], [3,4], [4,5], [4,7], [1,2], [1,5], [1,7], [2,5], [2,7] \to 2$$
$$M_8[3] = [3,4,5], [3,4,7], [3,5,7], [4,5,7], [1,2,5], [1,5,7], [2,5,7] \to 5$$
$$M_8[4] = [3,4,5,7], [1,2,5,7] \to 7$$
$$M_8[5] = M_8[6] = M_8[7] = M_8[8] = \infty$$

$M_8 = [1, 2, 5, 7, \infty, \infty, \infty, \infty]$

The longest subsequence is 4 and the last element is 7.

(b) Show that M_i is a strictly increasing array, i.e. that $M_i[j] < M_i[j+1]$ for all $j = 1, \ldots, N-1$.

Hint: Suppose there exists some j such that $M_i[j] \geq M_i[j+1]$, and show that this implies a contradiction.

Solution: Suppose there is some j such that $M_i[j] = e_j, M_i[j+1] = e_{j+1}$ where $e_j \geq e_{j+1}$. By definition e_j, e_{j+1} are the smallest elements that end any subsequence of length j and $j+1$ respectively. Because e_j is the smallest element in any subsequence of length j, then this implies that for any subsequence of length $j+1$, e_j occurs before e_{j+1} in the array, meaning that $e_j < e_{j+1}$, contradicting our initial assumption.

(c) Given the same array as part (a), compute M_{10}.

Solution:

$$M_{10}[1] = [5],[3],[7],[4],[1],[2],[5],[7],[8],[3] \rightarrow 1$$

$$M_{10}[2] = [5,7],[5,8],[3,7],[3,4],[3,5],[3,8],[7,8],[4,5],[4,7],[4,8],[1,2],[1,5],[1,7],[1,8],[1,3],[7,8] \rightarrow 2$$

$$M_{10}[3] = [5,7,8],[3,7,8],[3,4,5],[3,4,7],[3,4,8],[4,5,7],[4,5,8],[4,7,8],[1,2,5],[1,2,7],[1,2,8],[1,5,7],[1,7,8],[1,2,3],[2,5,7],[2,5,8],[2,7,8],[5,7,8] \rightarrow 3$$

$$M_{10}[4] = [3,5,7,8],[3,4,5,1],[3,4,5,8],[3,5,7,8],[4,5,7,8],[1,2,5,7],[1,2,5,8],[1,2,7,8],[1,5,7,8],[2,5,7,8] \rightarrow 7$$

$$M_{10}[5] = [1,2,5,7,8],[3,4,5,7,8] \rightarrow 8$$

$$M_{10}[6] = M_{10}[7] = M_{10}[8] = M_{10}[9] = M_{10}[10] = \infty$$

$$M_{10} = [1,2,3,7,8,\infty,\infty,\infty,\infty,\infty]$$

(d) Let j be the smallest index such that $M_i[j] \geq A[i+1]$. Prove that the length of the LIS ending on $A[i+1]$ is j.

Hint: Use the result from (b) to show that there exists a length j increasing subsequence ending on $A[i+1]$, and that there exist no longer increasing subsequence.

Solution: Given that $M_i[j] \geq A[i+1]$ we know that there exist an increasing subsequence of length $j-1$ ending at some element where $M_i[j-1] < A[i+1]$. If we include the $A[i+1]$ element to this subsequence we end up with an increasing subsequence of length j ending at A[i+1]. Showing that indeed the length of the LIS ending at $A[i+1]$ is j.

(e) Now show that only one element differs between M_i and M_{i+1}. Recall that M_i only accounts for $A[1 \ldots i]$, so we are trying to prove M_{i+1} can be computed for $A[1 \ldots i+1]$ by taking M_i and modifying one element.

Solution: When we transition from M_i to M_{i+1}, we are essentially including the element $A[i+1]$ to the set. Concluding that with the addition of $A[i+1]$, M_i and M_{i+1} differ by at most one element.

(f) Now combining the previous subparts, write pseudocode that finds the longest increasing subsequence of an array A in $O(N \log N)$ time.

Hint: a naive implementation using the 2D subproblem $M_i[j]$ would still yield a runtime of $O(N^2)$. To achieve the $O(N \log N)$ runtime, you only need to store a single 1D array M. Then, efficiently update M by using previous subparts.

Solution:

```
def longest_increasing_subsequence(A):
    n = len(A)
    M = [0] * (n+1)
    M[0] = 0
    parents = [0] * (n+1)
    l = 0

    for i in range(n):
        lo, hi = 0, l
        while lo <= hi:
            mid = (lo + hi) // 2
            if M[mid] < A[i]:
                low = mid+1
            else:
                hi = mid-1
        M[lo] = A[i]
        parents[i] = M[lo-1]

    lis = [0] * (l)
    k = M[l]
    for i in range(0, n, -1):
        if A[i] = k:
            lis[l] = A[i]
            l -= 1
        k = parent[i]
    return lis
```

- We iterate over every element of A and run binary search to find the largest index j such that $M[j] \leq A[i]$. Then we update M and parents (holds the indices of the elements that are part of the LIS).
- Starting at the last index of A, we trace back through parents to reconstruct LIS.

3 Max Independent Set Again

You are given a connected tree T with n nodes and a designated root r, where every vertex v has a weight $W[v]$. A set of nodes S is a k-independent set of T if $|S| = k$ and no two nodes in S have an edge between them in T. The weight of such a set is given by adding up the weights of all the nodes in S, i.e.

$$W(S) = \sum_{v \in S} W[v].$$

Given an integer $k \leq n$, your task is to find the maximum possible weight of any k-independent set of T. We will first tackle the problem in the special case that T is a binary tree, and then generalize our solution to a general tree T.

(a) Assume that T is a binary tree, i.e. every node has at most 2 children. Describe an $O(nk^2)$ algorithm that solves this special case, and analyze its runtime. Proof of correctness and space complexity analysis are not required.

Solution: Let $I_1(v, s)$ be the maximum weight of the independent set of size s with its root at v. Let $I_2(u, s)$ be the maximum weight of the independent set of size s rooted at $u \in children(v)$. Lets consider the case where v is included in the set. If v is included in the set, then we have the following recurrence:

$$I_1(v, s) = max\{I_2(v_L, s') + I_2(v_R, s - s')\}$$

This recurrence skips over the children of v since it was chosen and we consider new subproblems where s' is the size the new subset given by all possible sizes less than s. The next recurrence arises from skipping over v, in this case we consider $u \in children(v)$:

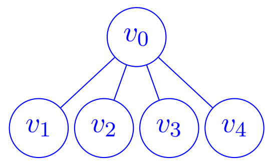
$$I_2(u, s) = max \begin{cases} I_1(u, s), \text{if we skip over } u \\ max\{W[u] + I_1(u_L, s') + I_1(u_R, s - s')\}, \text{if we choose } u \end{cases}$$

For both recurrences, subscripts R and L denote the right and left children of a vertex. For our base cases we have $I_1(v, 0) = 0$ and $I_2(u, 0) = 0$

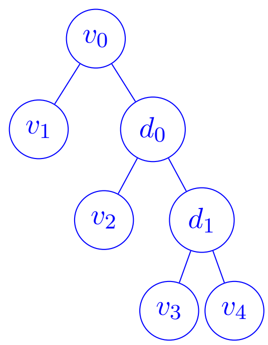
Runtime: There are $O(nk)$ subproblems each bounded by $O(k)$ to solve, for a total of $O(nk^2)$

(b) Now, consider any arbitrary tree T, with no restrictions on the number of children per node. Describe how we can add up to $O(n)$ "dummy" nodes (i.e. nodes with weight 0) to T, as well as some edges, to convert it into a binary tree T_b.

Solution: Consider the following tree:



Ignoring the left most child, we can add dummy nodes as follows:



If $|children(v_0)| > 2$, then we can take the right most two children and add dummy node d, as the parent of the two children, then add this dummy node as a child of v_0 Where $W[d] = 0$. We recursively do this until $|children(v_0)| = 2$. Adding up to $O(n)$ dummy nodes.

(c) Describe an $O(nk^2)$ algorithm to solve the general case (i.e. when T is any arbitrary tree), and analyze its runtime. Proof of correctness and space complexity analysis are not required.

Hint: there exists two ways (known to us) to solve this. One way is to combine parts (a) and (b), and then modify the recurrence to account for the dummy nodes. The other way involves 3D dynamic programming, in which you directly extend your recurrence from part (a) to iterate across vertices' children. We recommend the first way as it may be easier to conceptualize, but in the end it is up to you!

Solution: We begin by turning T, into a binary tree T'as seen in (b) which gives us the runtime $O(n)$, we then proceed to use the algorithm from part (a) to solve the problem, using T'. The recurrence from part (a) has to be updated to account for the dummy nodes. we can do this by adding a new case to our recurrences, that states that if we are at a dummynode, we skip it and consider the children of the dummy node, using the same recurrence as from part (a).

4 Canonical Form LP

Recall that any linear program can be reduced to a more constrained *canonical form* where all variables are non–negative, the constraints are given by $\leq$ inequalities, and the objective is the maximization of a cost function.
More formally, our variables are x_i. Our objective is $\max c^\top x = \max \sum_i c_i x_i$ for some constants c_i. The jth constraint is $\sum_i a_{ij} x_i \leq b_j$ for some constants a_{ij}, b_j. Finally, we also have the constraints $x_i \geq 0$.
An example canonical form LP:

$$\text{maximize } 5x_1 + 3x_2$$

$$\text{subject to } \begin{cases} x_1 + x_2 - x_3 \leq 1 \\ -(x_1 + x_2 - x_3) \leq -1 \\ -x_1 + 2x_2 + x_4 \leq 0 \\ -(-x_1 + 2x_2 + x_4) \leq 5 \\ x_1, x_2, x_3, x_4 \geq 0 \end{cases}$$

For each of the subparts below, describe how we should modify it to so that it satisfies canonical form. If it is impossible to do so, justify your reasoning.

Note that the subparts are independent of one another. Also, you may assume that variables are non-negative unless otherwise specified.

(a) Min Objective: $\min \sum_i c_i x_i$
Solution: To turn the maximization problem into a minimization, just multiply the coefficients of the objective function by -1.

(b) Lower Bound on Variable: $x_1 \geq b_1$
Solution: In this situation we can multiply both sides by -1. $-x_1 \leq -b_1$.

(c) Bounded Variable: $b_1 \leq x_1 \leq b_2$
Solution: We can re-write $b_1 \leq x_1 \leq b_2$ into $x_1 \leq b_2$ and $x_1 \geq b_1$ then re-write $x_1 \geq b_1$ into $-x_1 \leq -b_1$, so we have $b_1 \leq x_1 \leq b_2 \Leftrightarrow x_1 \leq b_2, -x_1 \leq -b_1$.

(d) Equality Constraint: $x_2 = b_2$
Solution: We can re-write $x_2 = b_2$ into $x_2 \geq b_2$ and $x_2 \leq b_2$ and re-write $x_2 \geq b_2$ into $-x_2 \leq -b_2$ since $x_2 = b_2 \Leftrightarrow x_2 \leq b_2, -x_2 \leq -b_2$.

(e) More Equality Constraint: $x_1 + x_2 + x_3 = b_3$
Solution: We can re-write as:

$$x_1 + x_2 + x_3 \le b_3$$
$$x_1 + x_2 + x_3 \ge b_3 \Leftrightarrow -x_1 - x_2 - x_3 \le -b_3$$

(f) Absolute Value Constraint: $|x_1 + x_2| \le b_2$ where $x_1, x_2 \in \mathbb{R}$
Solution: We can introduce new variables $y_1 = x_1 + x_2$ and $y_2 = -x_1 - x_2 \rightarrow -y_2 = x_1 + x_2$ and then re-write as:

$$y_1 \le b_2$$
$$-y_2 \le b_2$$

(g) Another Absolute Value Constraint: $|x_1 + x_2| \ge b_2$ where $x_1, x_2 \in \mathbb{R}$
Solution: Similar as above we introduce two non-negative variables x^+, x^- and replace x wherever it occurs by $x^+ - x^-$. So we have $x^+ - x^- \ge b_2 \Leftrightarrow -x^+ + x^- \le b_2$.

(h) Min Max Objective: $\min \max(x_1, x_2, x_3, x_4)$ *Hint: use a dummy variable!*
Solution:‘' Let $z = \max(x_1, x_2, x_3, x_4)$

$$\begin{aligned} \textbf{maximize: } & -z \\ \textbf{Subject to: } & z \ge x_1 \\ & z \ge x_2 \\ & z \ge x_3 \\ & z \ge x_4 \end{aligned}$$

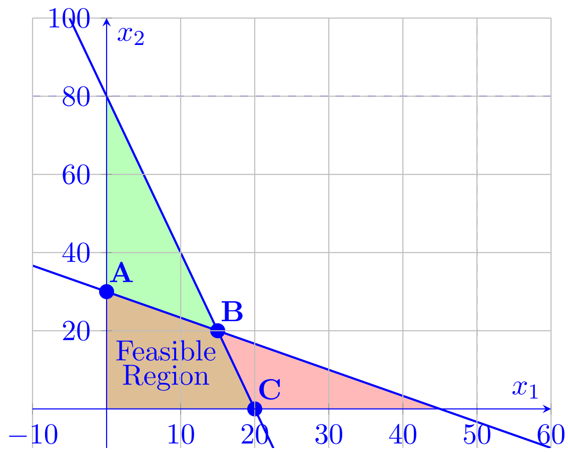
5 Baker

You are a baker who sells batches of brownies and cookies (unfortunately no brookies... for now). Each brownie batch takes 4 kilograms of chocolate and 2 eggs to make; each cookie batch takes 1 kilogram of chocolate and 3 eggs to make. You have 80 kilograms of chocolate and 90 eggs. You make a profit of 60 dollars per brownie batch you sell and 30 dollars per cookie batch you sell, and want to figure out how many batches of brownies and cookies to produce to maximize your profits.

(a) Formulate this problem as a linear programming problem; in other words, write a linear program (in canonical form) whose solution gives you the answer to this problem. Draw the feasible region, and find the solution using Simplex.

Solution: Let x_1 be the number of brownie batches, and x_2 to the number of cookie batches. Additionally it is given that we make \$60/batch of brownies and \$30/batch of cookies which is what we want to maximize. Two other inequalities arise from the fact that we have limited ingredients.

$$\begin{array}{rll} \textbf{maximize:} & 60x_1 + 30x_2 & \\ \textbf{Subject to:} & 4x_1 + x_2 \le 80 & (1) \\ & 2x_1 + 3x_2 \le 90 & (2) \\ & x_1 \ge 0 & (3) \\ & x_2 \ge 0 & (4) \end{array}$$



A: (0, 30), **B**: (15, 20), **C**: (20, 0)

After running simplex we arrive at the vertex given by equations (1) and (2) (from the re-written LP found during Simplex) since they satisfy the property $\forall_i c_i < 0$. After solving the system given by (1) and (2) from the original LP above, we get the optimal point $\mathbf{B} = (15, 20)$, which gives us the solution $(60)(15) + (30)(20) = \$1500$ coming from baking 15 batches of brownies and 20 batches of cookies.

(b) Suppose instead that the profit per brownie batch is P dollars and the profit per cookie batch remains at 30 dollars. For each vertex you listed in the previous part, give the range of P values for which that vertex is the optimal solution.
Solution: We have $Px_1 + 30x_2$ and the points **A**: (0, 30), **B**: (15, 20), **C**: (20, 0).

$$\begin{aligned}
&\text{Point } \mathbf{A} : (0, 30) \\
&\quad P(0) + 30(30) \geq 15P + 30(20) \\
&\quad P(0) + 30(30) \geq 20P + 30(0) \\
&\qquad \boxed{P \leq 20} \\
&\text{Point } \mathbf{B} : (15, 20) \\
&\quad 15P + 30(20) \geq P(0) + 30(30) \\
&\quad 15P + 30(20) \geq 20P + 0 \\
&\qquad \boxed{20 \leq P \leq 120} \\
&\text{Point } \mathbf{C} : (20, 0) \\
&\quad P(20) + 30(0) \geq P(0) + 30(30) \\
&\quad P(20) + 30(0) \geq 15P + 600 \\
&\qquad \boxed{P \geq 120}
\end{aligned}$$