1 Study Group

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2 Course Policies

(a)

- Midterm 1: Monday, 2/26/2024, 7:00 PM - 9:00 PM
- Midterm 2: Tuesday, 4/2/2024, 7:00 PM - 9:00 PM
- Final: Friday, 5/10/2024, 7:00 PM - 10:00 PM (Group 20)

(b) It is recommended that homework be finished by 10:00 PM.

(c) Nothing, since there are no exeptions after the deadline of 11:59 PM; however, you may use one of your two drops.

(d) Ed is the primary source of communication between students and staff.

(e) I have read and understood the course syllabus and policies.

3 Understanding Academic Dishonesty

(a) Not okay.

(b) Not okay.

(c) Not okay.

(d) Okay.

4 Math Potpourri

(a) Simplify the following expressions into a single logarithm (i.e. in the form $\log_a(b)$:

(i)

$$\frac{\ln(x)}{\ln(y)} = \frac{\log_e(x)}{\log_e(y)} = \boxed{\log_y(x)}$$

(ii)

$$\ln(x) + \ln(y) = \boxed{\ln(x \cdot y)}$$

(iii)

$$\ln(x) - \ln(y) = \boxed{\ln\left(\frac{x}{y}\right)}$$

(iv)

$$170\ln(x) = \boxed{\ln(x^{170})}$$

(b) Give a simple proof for each of the following identities:

(a)

$$\begin{aligned} x^{\log_{\frac{1}{x}}(y)} &= \frac{1}{\frac{1}{x^{\log_{\frac{1}{x}}(y)}}} \\ &= \frac{1}{\left(\frac{1}{x}\right)^{\log_{\frac{1}{x}}(y)}} \\ &= \frac{1}{y} \end{aligned}$$

(b)

$$\begin{aligned} \sum_{i=1}^{n} i &= 1 + 2 + 3 + \cdots + (n-1) + n \\ &= (1+n) + (2 + (n-1)) + (3 + (n-2)) + \cdots \\ &= (n+1) + (n+1) + (n+1) + \cdots \\ &= (n+1) \cdot \left(\frac{n}{2}\right) \\ &= \frac{n(n+1)}{2} \end{aligned}$$

(c)

If r = 1 then...

$$\sum_{k=0}^{n} ar^k = \sum_{k=0}^{n} a = a\sum_{k=0}^{n} 1 = a \cdot (n+1)$$

If r $\neq$ 1 then...

$$\text{Let } x_n = \sum_{k=0}^{n} ar^k$$

$$x_n = ar^0 + ar^1 + ar^2 + \cdots + ar^n$$

Multiply both sides by r

$$rx_n = ar^1 + ar^2 + ar^3 + \cdots + ar^{n+1}$$

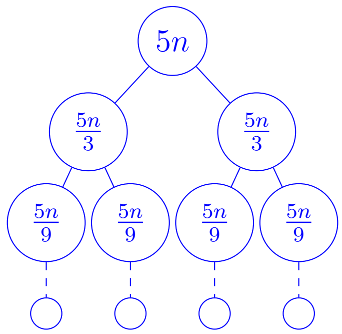
$$x_n - rx_n = ar^0 - ar^{n+1}$$

$$x_n(1-r) = a(1-r^{n+1})$$

$$\sum_{k=0}^{n} ar^k = \frac{a(1-r^{n+1})}{1-r}$$

5 Recurrence Relations

(a) $T(n) = 2T(n/3) + 5n$



Level 1: $5n$
Level 2: $2\frac{5n}{3}$
Level 3: $4\frac{5n}{9}$
$\vdots$

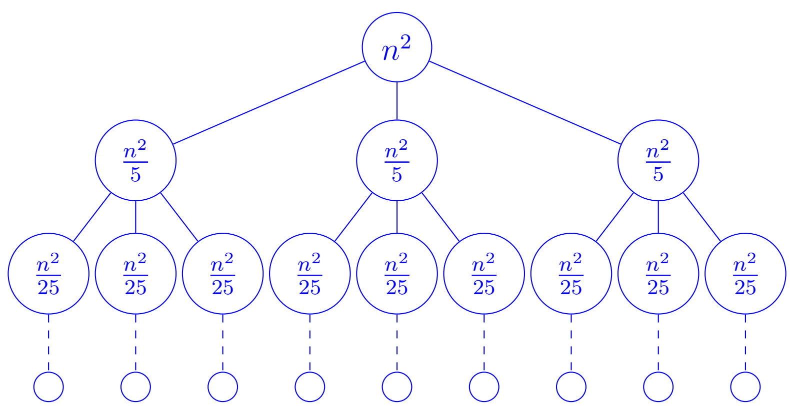
$$
\begin{aligned}
\sum_{k=0}^{\log_3 n} 2^k \cdot \frac{5n}{3^k} &= 5n \sum_{k=0}^{\log_3 n} \left(\frac{2}{3}\right)^k \leq \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k \\
&= 5n \left[\left(\frac{2}{3}\right)^0 + \left(\frac{2}{3}\right)^1 + \left(\frac{2}{3}\right)^2 + \cdots) \right] \\
&= 5n \left[\frac{1}{1 - \frac{2}{3}} \right] \\
&= 15n \\
&= \boxed{\Theta(n)}
\end{aligned}
$$

(b) $T(n) = 169T(n/170) + \Theta(n)$

There are 169 branches for every node. At the k^{th} level we do $\frac{169^k n}{170^k}$ work and there are $\log_{170} n$ levels in the tree.

$$\begin{aligned}
\sum_{k=0}^{\log_{170} n} \frac{169^k n}{170^k} &= n\left[\sum_{k=0}^{\log_{170} n}\left(\frac{169}{170}\right)^k\right] \leq n\left[\sum_{k=0}^{\infty}\left(\frac{169}{170}\right)^k\right] \\
&= n\left[\left(\frac{169}{170}\right)^0 + \left(\frac{169}{170}\right)^1 + \left(\frac{169}{170}\right)^2 + \cdots\right] \\
&= n\left[\frac{1}{1-\frac{169}{170}}\right] \\
&= \boxed{\Theta(n)}
\end{aligned}$$

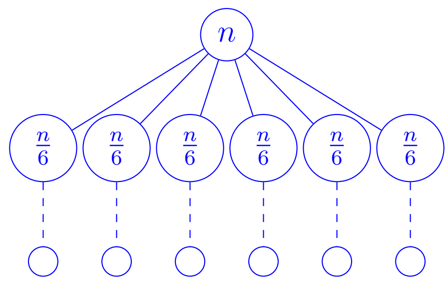
(c) An algorithm A takes $\Theta(n^2)$ time to partition the input into 5 sub-problems of size $n/5$. Describe the recurrence relation for the run-time $T(n)$ of $\mathcal{A}$ and find its asymptotic order of growth.



Giving the recurrence relation $\Rightarrow T(n) = 3T(n/5) + \Theta(n^2)$
Level 1: n^2 work, Level 2: $3\frac{n^2}{5}$ work, Level 3: $9\frac{n^2}{25}$ work, ...

$$\begin{aligned}
\sum_{k=0}^{\log_5(n)} \frac{3^k n^2}{5^k} &= n^2 \sum_{k=0}^{\log_5(n)}\left(\frac{3}{5}\right)^k \leq n^2\sum_{k=0}^{\infty}\left(\frac{3}{5}\right)^k \\
&= n^2\left[\left(\frac{3}{5}\right)^0 + \left(\frac{3}{5}\right)^1 + \left(\frac{3}{5}\right)^2 + \ldots\right] \\
&= n^2\left[\frac{1}{1-\frac{3}{5}}\right] \\
&= \boxed{\Theta\left(n^2\right)}
\end{aligned}$$

(d) $T(n) = 6T(n/6) + \Theta(n)$



At level k^{th} level we are doing $\frac{6^k n}{6^k}$ work and there are $\log_6 n$ levels...

$$\begin{aligned}\sum_{k=0}^{\log_6 n} \left(\frac{\cancel{6^k} n}{\cancel{6^k}}\right) &= n \sum_{k=0}^{\log_6 n} 1 \\ &= n \log_6(n) \\ &= \boxed{\Theta\left(n \log n\right)}\end{aligned}$$

(e) $T(n) = T(3n/5) + T(4n/5)$(We have $T(1) = 1$)

We can try finding an upper and lower bound for $T(n)$:

$$\text{UPPER: } T(n) \leq 2T(4n/5)$$
$$\text{LOWER: } T(n) \geq 2T(3n/5)$$

We can use Masters Theorem to solve the two realtions..

$$\begin{aligned}
\text{UPPER: } a = 2, b = \frac{5}{4}, d = 0 \\
\log_{\frac{5}{4}} 2 > d = 0 \\
\therefore \mathcal{O}(n^{\log_{\frac{5}{4}} 2}) \approx \mathcal{O}(n^{3.11}) \\
\text{LOWER: } a = 2, b = \frac{5}{3}, d = 0 \\
\log_{\frac{5}{3}} 2 > d = 0 \\
\therefore \Omega(n^{\log_{\frac{5}{3}} 2}) \approx \Omega(n^{1.36})
\end{aligned}$$

$$T(n) = an^b, T(1) = 1$$
$$T(1) = a1^b = a = 1$$

Plug in $n = an^b$ into $T(n)$ with $a = 1$

$$\begin{aligned}
n^b &= \left(\frac{3n}{5}\right)^b + \left(\frac{4n}{5}\right)^b \\
n^b &= \left(\frac{3^b}{5^b}\right) n^b + \left(\frac{4^b}{5^b}\right) n^b \\
1 &= \frac{3^b + 4^b}{5^b} \\
5^b &= 3^b + 4^b \\
b &= 2, \text{ since } 9 + 16 = 25 \\
&\boxed{\Theta(n^2)}
\end{aligned}$$

6 In Between Functions

(a) Try setting $f(n)$ to a polynomial of degree d, where d is a very large constant. So $f(n) = a_0 + a_1 n + a_2 n^2 + \cdots + a_d n^d$. For which values of k (if any) does f fail to satisfy (1)?

Since the last term dominates, in order for (1) to not be satisfied then $d \geq k$, and/or a would have to a large value such that $a_d n^d \geq n^k$.

(b) Now try setting $f(n)$ to a^n, for some constant $a > 1$ that's as small as possible while still satisfying (1) (e.g. 1.000001). For which values of c (if any) does f fail to satisfy (2)?

$$f(n) = a^n, a > 1$$
$$\text{Let } a = 2^b \Rightarrow b = log_2(a), f(n) = 2^{bn}$$

$$2^{bn} \geq 2^{cn}$$
$$b \geq c$$
$$log_2(a) \geq c$$

In order for (2) to not be satisfied, $b = log_2(a)$ would have to be greater than c.

(c) Find a function $D(n)$ such that setting $f(n) = O(nD(n))$ satisfies both (1) and (2). Give a proof that your answer satisfies both.

$$\text{Let } D(n) = log(n)$$
$$f(n) = O(n^{logn})$$

$$n^{\log n} > n^k$$
$$\log n > k \quad \textbf{(i)}$$

$$n^{\log n} < 2^{cn}$$
$$\log(n^{\log n}) < \log(2^{cn})$$
$$\log n \log n < cn \log 2$$
$$\log^2 n < cn \log 2 \quad \textbf{(ii)}$$

Since $\log n$ is greater than any constant k, **(i)** satisfies (1). Similarly $\log^2 n$ is less than $cn \log 2$ **(ii)** satisfying (2).