A two level algorithm for an obstacle problem

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Abstract. Due to the inequality feature of the obstacle problem, the standard quadratic finite element method for solving the problem can only achieve an error bound of the form $\mathcal{O}(N^{-3/4+\epsilon})$, N being the total number of degrees of freedom, and $\epsilon > 0$ arbitrary. To achieve a better error bound, the key lies in how to capture the free boundary accurately. In this paper, we propose a two level algorithm for solving the obstacle problem. The first part of the algorithm is through the use of the linear elements on a quasi-uniform mesh. Then information on the approximate free boundary from the linear element solution is used in the construction of a quadratic finite element method. Under some reasonable assumptions, the numerical solution from the two level algorithm is shown to have a nearly optimal error bound of $\mathcal{O}(N^{-1+\epsilon})$, $\epsilon > 0$ arbitrary.

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1 Introduction

Many problems in physical and engineering sciences are modeled by partial differential equations. However, various more complex physical processes are described by variational inequalities (VIs). Variational inequalities form an important family of nonlinear problems arising in diverse application areas, for example, elastoplasticity, contact mechanics, heat control problems, options pricing problems in finance, Nash-equilibria in management science. Therefore, how to solve variational inequalities efficiently is very attractive to mathematicians, engineers and economists. Variational inequalities of the first kind are closely related to free-boundary problems. The classical formulation of a variational inequality is usually expressed through the presence of an unknown region or boundary. So a variational inequality can be also viewed as a free-boundary problem. Moreover, many free-boundary

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problems can be reformulated as variational inequalities. The formulation of a variational inequality is advantageous over that of a free-boundary problem, especially for numerical solutions, since in a variational inequality there is no explicit involvement of an unknown region or boundary.

In this paper, we consider an obstacle problem, which is a representative elliptic variational inequality (EVI) of first kind ([5]). For more examples of EVIs, we refer the reader to the monograph [3]. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz boundary $\partial\Omega$.

An obstacle problem. Let $f \in L^2(\Omega)$, and $\psi \in H^2(\Omega)$ with $\psi \leq 0$ on $\partial\Omega$. The obstacle problem is to find $u \in K$ such that

$$a(u, v-u) \geq (f, v-u)_\Omega \quad \forall\, v \in K, \tag{1.1}$$

where

$$K = \{v \in H_0^1(\Omega) : v \geq \psi \text{ a.e in } \Omega\} \tag{1.2}$$

is a closed and convex admissible set of the space $H_0^1(\Omega)$, and

$$\begin{aligned} a(u,v) &= \int_\Omega \nabla u \cdot \nabla v \, dx, \\ (f,v)_\Omega &= \int_\Omega f v \, dx.q \end{aligned}$$

The obstacle problem has a unique solution ([3]). It arises in a variety of applications, such as the membrane deformation in elasticity theory, and the non-parametric minimal and capillary surfaces as geometrical problems. The elastic-plastic torsion problem and the cavitation problem in the theory of lubrication also can be regarded as obstacle type problems. If the solution has the regularity $u \in H^2(\Omega)$, then it satisfies the relations (see, e.g., [1])

$$-\triangle u \geq f, \quad u \geq \psi, \quad (-\triangle u - f)(u - \psi) = 0 \quad \text{a.e. in } \Omega. \tag{1.3}$$

Therefore, we have the following relations

$$\begin{aligned} -\triangle u \geq f \quad & \text{in } \Omega^0 = \{x \in \Omega : u(x) = \psi(x)\}, \\ -\triangle u = f \quad & \text{in } \Omega^+ = \{x \in \Omega : u(x) > \psi(x)\}. \end{aligned}$$

Regarding the solution regularity of the obstacle problem, Brezis proved the following result (see [6, 7]): if $\partial\Omega$ is smooth, $f \in L^\infty(\Omega) \cap BV(\Omega)$, and $\psi \in C^3(\bar{\Omega})$, then the solution of the problem (1.1) has the regularity $u \in W^{s,p}(\Omega)$ with $1 < p < \infty$ and $s < 2 + 1/p$.

The finite element method is the dominant numerical discretization method for variational inequalities. Optimal convergence order can be reached by the linear elements ([4, 5, 9, 10, 11]) under the regularity assumption $u \in H^2(\Omega)$. For the quadratic element solutions, an error bound $\mathcal{O}(h^{3/2-\epsilon})$, $\epsilon > 0$ arbitrary, is derived for $H^1(\Omega)$-norm in [8] under regularity assumption that $u \in W^{s,p}(\Omega)$ with $1 < p < \infty$ and $s < 2 + 1/p$. In terms of the total number of degrees of freedom N, the error bound for the linear element solution is $\mathcal{O}(N^{-1/2})$, whereas that for the quadratic element solution is $\mathcal{O}(N^{-3/4+\epsilon})$ for an arbitrarily small $\epsilon > 0$. For variational inequalities, higher order elements do not lead to higher order convergence. Therefore, it is common to use low order elements in solving variational inequalities. In this paper, we introduce a two level algorithm using both linear and quadratic elements to solve the obstacle problem such that the error bound is expected to be $\mathcal{O}(N^{-1+\epsilon})$ for an arbitrarily small $\epsilon > 0$. In the error analysis for the two level algorithm, we adopt the assumption that $u \in W^{s,p}(\Omega)$ with $1 < p < \infty$ and $s < 2 + 1/p$. Moreover, we assume that $u|_{\Omega^0} \in H^3(\Omega^0)$ and $u|_{\Omega^+} \in H^3(\Omega^+)$, where Ω^0 is the contact area and $\Omega^+ = \Omega\backslash\Omega^0$. This is a reasonable assumption. In the contact area, $u = \psi$, so $u|_{\Omega^0} \in H^3(\Omega^0)$ is just the assumption $\psi|_{\Omega^0} \in H^3(\Omega^0)$, which is implied by $\psi \in H^3(\Omega)$. $u|_{\Omega^+} \in H^3(\Omega^+)$ can be considered as the solution of an Poisson equation with the free-boundary as Dirichlet boundary condition. The error bound is proved under some assumption on the behavior of the numerical solution. The idea of the algorithm is outlined as follows. First, solve the obstacle problem with linear elements on a quasiuniform mesh $\mathcal{T}_h$, and identify free-boundary elements. Then refine the free-boundary elements into elements with mesh size $h_* = O(h^{4/3})$ to obtain a new mesh. Finally, we solve the obstacle problem on this new mesh with the quadratic elements.

The rest of the paper is organized as follows: In Section 2, we introduce the two level algorithm. In Section 3, we derive a priori error estimates for this algorithm. In Section 4, we present numerical examples to provide numerical evidence of the error bound.

2 A two level algorithm

We assume Ω is a polygonal domain. For a subdivision $\mathcal{T}_h$ of $\overline{\Omega}$ into triangles, let $h_T = \mathrm{diam}(T)$ and $h = \max\{h_T : T \in \mathcal{T}_h\}$. All the subdivisions, including the refined meshes, are constructed so that the minimal angle condition is satisfied.

We introduce the following two level quadratic finite element method for the obstacle problem.

1 Solve the obstacle problem on a quasi-uniform mesh $\mathcal{T}_h$ with the linear elements, that is, find $u_h \in K_h^1$ such that

$$a(u_h, v_h - u_h) \geq (f, v_h - u_h)_\Omega \quad \forall\, v_h \in K_h^1, \tag{2.1}$$

where

$$K_h^1 = \{v_h \in V_h^1 : \; v_h(x) \geq \psi(x) \text{ at all vertices of } \mathcal{T}_h\}$$

and

$$V_h^1 = \{v_h \in H^1(\Omega):\ v_h|_T \in P_1(T)\ \forall T \in \mathcal{T}_h\}.$$

2 Identify the subset $\mathcal{T}_h^F \subset \mathcal{T}_h$ of free-boundary elements, i.e., the subset of the elements $T \in \mathcal{T}_h$ such that $T = T_1 \cup T_2$ with $|T_1| > 0$, $|T_2| > 0$, $u_h = \psi$ on T_1, and $u_h > \psi$ on T_2. Refine all the elements in $\mathcal{T}_h^F$ into new elements with mesh size $h_* = O(h^{4/3})$. Denote the new mesh by $\mathcal{T}_h^*$.

1. Solve the obstacle problem with quadratic elements over the new mesh $\mathcal{T}_h^*$, i.e., find $u_h^* \in K_h^2$ such that

$$a(u_h^*, v_h - u_h^*) \geq (f, v_h - u_h^*)_\Omega \quad \forall\, v_h \in K_h^2, \tag{2.2}$$

where

$$K_h^2 = \{v_h \in V_h^2:\ v_h(m) \geq \psi(m) \text{ at all midpoints } m \text{ on element edges of } \mathcal{T}_h^*\}$$

and

$$V_h^2 = \{v_h \in H^1(\Omega):\ v_h|_T \in P_2(T)\ \forall T \in \mathcal{T}_h^*\}.$$

In the iterative procedure for solving the problem (2.2), for the initial guess we use the interpolant of u_h in V_h^2.

In the next section, under the solution smoothness assumption $v \in V$, where

$$V = \{v \in W^{s,p}(\Omega):\ v|_{\Omega^0} \in H^3(\Omega^0) \text{ and } v|_{\Omega^+} \in H^3(\Omega^+)\} \tag{2.3}$$

with $1 < p < \infty$ and $s < 2 + 1/p$, and a natural assumption on the numerical solution, we will prove that for the two level algorithm, $\|u - u_h^*\|_1 \leq Ch^{2-\epsilon}$; or, in terms of the total number of degrees of freedom N, $\|u - u_h^*\|_1 \leq CN^{-1+\epsilon}$.

3 Error estimates

First recall the following boundedness and stability properties of the bilinear form $a(u, v)$.

Lemma 3.1

$$a(u, v) \leq C_b \|u\|_1 \|v\|_1 \quad \forall\, u, v \in H_0^1(\Omega), \tag{3.1}$$

$$a(v, v) \geq C_s \|v\|_1^2 \quad \forall\, v \in H_0^1(\Omega), \tag{3.2}$$

where C_b and C_s are positive constants independent of the mesh size.

The main purpose of the section is to bound the error for the two level method. We assume $u \in V$. Let us group the elements of $\mathcal{T}_h^*$ into three kinds:

$$\begin{aligned}\mathcal{T}_h^+ &= \{T \in \mathcal{T}_h^* : T \subset \Omega^+\},\\ \mathcal{T}_h^0 &= \{T \in \mathcal{T}_h^* : T \subset \Omega^0\},\\ \mathcal{T}_h^b &= \mathcal{T}_h^* \backslash (\mathcal{T}_h^+ \cup \mathcal{T}_h^0).\end{aligned}$$

We will assume

$$\bigcup_{T \in \mathcal{T}_h^b} T \subset \bigcup_{T \in \mathcal{T}_h^F} T, \tag{3.3}$$

where $\mathcal{T}_h^F$ was defined in Step 2 of the algorithm. This is a reasonable assumption if h is small enough, given the first order convergence of u_h to u in $H^1(\Omega)$. Write the error as

$$e = u - u_h^* = (u - u_I) + (u_I - u_h^*),$$

where $u_I \in V_h^2$ is the usual continuous piecewise quadratic polynomial interpolant. Then

$$\|u - u_I\|_1 \lesssim h^{2-\epsilon}, \tag{3.4}$$

where "$\lesssim \cdots$" stands for "$\leq C \cdots$", and C is a positive generic constant independent of h and other parameters, which may take different values at different appearances. In fact, because $u|_{\Omega^0} \in H^3(\Omega^0)$ and $u|_{\Omega^+} \in H^3(\Omega^+)$, for any $T \in \mathcal{T}_h^0 \cup \mathcal{T}_h^+$, we have

$$\|u - u_I\|_{1,T} \leq Ch^2 |u|_{3,T},$$

and for any $T \in \mathcal{T}_h^b$, we have

$$\|u - u_I\|_{1,T} \leq Ch_*^{3/2-3\epsilon/4} |u|_{5/2-3\epsilon/4,T} \leq Ch^{2-\epsilon} |u|_{5/2-3\epsilon/4,T},$$

with $s = 5/2 - 3\epsilon/4 < 2 + 1/p$, $p = 2$ in (2.3).

Using a technique similar to that in [8, 9], we derive the following error estimate.

Theorem 3.2 *Let u and u_h^* be the solutions of* (1.1) *and* (2.2)*, respectively. Assume $u \in V$ defined in* (2.3)*, $\psi \in H^3(\Omega)$, $f \in H^1(\Omega) \cap L^\infty(\Omega)$, and* (3.3)*. Then for the two level method introduced in Section 2, we have*

$$\|u - u_h^*\|_1 \leq Ch^{2-\epsilon}, \tag{3.5}$$

for an arbitrary $\epsilon > 0$.

Proof. By the stability of the bilinear form $a(u, v)$, we have

$$C_s \|u_I - u_h^*\|_1^2 \le a(u_I - u_h^*, u_I - u_h^*) \equiv A_1 + A_2, \tag{3.6}$$

where

$$\begin{aligned} A_1 &= a(u_I - u, u_I - u_h^*), \\ A_2 &= a(u - u_h^*, u_I - u_h^*). \end{aligned}$$

By the boundedness of the bilinear form, the term A_1 is bounded by

$$A_1 \le C_b \|u_I - u\|_1 \|u_I - u_h^*\|_1 \le \frac{C_s}{2} \|u_I - u_h^*\|_1^2 + \frac{C_b^2}{2C_s} \|u_I - u\|_1^2. \tag{3.7}$$

To bound A_2, we first recall the relations

$$\begin{aligned} -\triangle u = f \quad &\text{in } \Omega^+ = \{x \in \Omega : u(x) > \psi(x)\}, \\ -\triangle u \ge f \quad &\text{in } \Omega^0 = \{x \in \Omega : u(x) = \psi(x)\}. \end{aligned}$$

Note that $u_I - u_h^* = 0$ on $\partial\Omega$. We have

$$a(u, u_I - u_h^*) = \int_\Omega \nabla u \cdot \nabla(u_I - u_h^*)\, dx = -\int_\Omega \Delta u (u_I - u_h^*)\, dx. \tag{3.8}$$

Let $v_h = u_I$ in (2.2),

$$a(u_h^*, u_I - u_h^*) \ge (f, u_I - u_h^*)_\Omega = \sum_{T \in \mathcal{T}_h} \int_T f(u_I - u_h^*)\, dx. \tag{3.9}$$

Combining (3.9) and (3.8), we obtain

$$A_2 = a(u - u_h^*, u_I - u_h^*) \le \sum_{T \in \mathcal{T}_h} \int_T -(\Delta u + f)(u_I - u_h^*)\, dx \equiv A_3 + A_4 + A_5, \tag{3.10}$$

where

$$\begin{aligned} A_3 &= \sum_{T \in \mathcal{T}_h^+} \int_T w(u_I - u_h^*)\, dx, \\ A_4 &= \sum_{T \in \mathcal{T}_h^0} \int_T w(u_I - u_h^*)\, dx, \\ A_5 &= \sum_{T \in \mathcal{T}_h^b} \int_T w(u_I - u_h^*)\, dx. \end{aligned}$$

Here, we denote $w := -\Delta u - f$. It is easy to see that $A_3 = 0$.

To estimate A_4, as in [8], we introduce

$$P_0^T v = \frac{1}{|T|}\int_T v\,dx, \qquad R_0^T v = v - P_0^T v.$$

Since $w \geq 0$, we get $P_0^T w \geq 0$. Due to $u_h^* \in K_h^2$, we have $u_h^*(m) \geq \psi(m)$ for all the midpoints m on the edges of the element T, implying

$$\int_T (\psi_I - u_h^*)\,dx = \frac{|T|}{3}\sum_{i=1}^{3}(\psi - u_h^*)(m_i) \leq 0.$$

Then we get

$$\begin{aligned}\int_T w(\psi_I - u_h^*)\,dx &\leq \int_T R_0^T w(\psi_I - u_h^*)\,dx \\ &= \int_T R_0^T w\, R_0^T(\psi_I - u_h^*)\,dx \leq \|R_0^T w\|_{0,T}\|R_0^T(\psi_I - u_h^*)\|_{0,T}.\end{aligned}$$

Note that $u = \psi$ in $T \in \mathcal{T}_h^0$ and $\psi \in H^3(\Omega)$; so

$$\begin{aligned}\int_T w(\psi_I - u_h^*)\,dx &\leq Ch^2|w|_{1,T}|\psi_I - u_h^*|_{1,T} \\ &\leq Ch^2|w|_{1,T}(|\psi_I - \psi|_{1,T} + |\psi - u|_{1,T} + |u - u_h^*|_{1,T}).\end{aligned}$$

We then apply interpolation error estimates to get

$$\int_T w(\psi_I - u_h^*)\,dx \leq Ch^2|w|_{1,T}(h^2|\psi|_{3,T} + |u - u_h^*|_{1,T}). \tag{3.11}$$

Hence,

$$A_4 = \sum_{T\in\mathcal{T}_h^0}\int_T w(\psi_I - u_h^*)\,dx \leq Ch^2|w|_{1,\Omega}(h^2|\psi|_{3,\Omega} + \|u - u_h^*\|_1). \tag{3.12}$$

Consider the term

$$A_5 = \sum_{T\in\mathcal{T}_h^b}\int_T w(u_I - u + \psi - \psi_I)dx + \sum_{T\in\mathcal{T}_h^b}\int_T w(u - \psi)dx + \sum_{T\in\mathcal{T}_h^b}\int_T w(\psi_I - u_h^*)dx.$$

Since $w(u - \psi) = 0$ by (1.3), we can write

$$A_5 = A_{5,1} + A_{5,2},$$

where

$$A_{5,1} = \sum_{T\in\mathcal{T}_h^b} \int_T w[(u-\psi)_I - (u-\psi)]dx,$$

$$A_{5,2} = \sum_{T\in\mathcal{T}_h^b} \int_T w(\psi_I - u_h^*)dx.$$

Using a technique similar to that in [2], we can bound the term $A_{5,1}$ as follows:

$$A_{5,1} \lesssim h^{4-\epsilon}\|w\|_{L^\infty(\Omega)}\|u-\psi\|_{s,p,\Omega}. \tag{3.13}$$

Indeed, for any $v \in W^{s,p}(\Omega)$ with $1 < p < \infty$ and $s < 2 + 1/p$, by the Cauchy-Schwarz inequality and the interpolation error estimate, we get

$$\|v_I - v\|_{0,1,T} = \int_T |v_I - v|\, dx \le |T|^{1-\frac{1}{p}}\|v_I - v\|_{0,p,T} \lesssim h_*^{s+2-\frac{2}{p}}\|v\|_{s,p,T},$$

and then,

$$\begin{aligned}\sum_{T\in\mathcal{T}_h^b}\|v_I - v\|_{0,1,T} &\lesssim h_*^{s+2-\frac{2}{p}}\sum_{T\in\mathcal{T}_h^b}\|v\|_{s,p,T}\\ &\lesssim h_*^{s+2-\frac{2}{p}}\left(\sum_{T\in\mathcal{T}_h^b}1\right)^{1-\frac{1}{p}}\left(\sum_{T\in\mathcal{T}_h^b}\|v\|_{s,p,T}^p\right)^{\frac{1}{p}}\\ &\lesssim h_*^s\|v\|_{s,p,\Omega}.\end{aligned}$$

Hence,

$$\begin{aligned}A_{5,1} &= \sum_{T\in\mathcal{T}_h^b}\int_T w[(u-\psi)_I - (u-\psi)]dx\\ &\le \|w\|_{L^\infty(\Omega)}\sum_{T\in\mathcal{T}_h^b}\|(u-\psi)_I - (u-\psi)\|_{0,1,T}\\ &\lesssim h_*^s\|w\|_{L^\infty(\Omega)}\|u-\psi\|_{s,p,\Omega}.\end{aligned}$$

Taking $p = 1+\epsilon_1$, $s = 2 + \frac{1}{p} - \epsilon_2 = 3 - 3\epsilon/4$, with $\epsilon = \frac{4}{3}\left(\frac{\epsilon_1}{1+\epsilon_1} + \epsilon_2\right)$ in the above inequality, and note that $h_* = O(h^{4/3})$, we get the bound (3.13) for the term $A_{5,1}$.

Finally, let us bound the term $A_{5,2}$. We know that

$$\begin{aligned}\int_T w(\psi_I - u_h^*)\, dx &\le \int_T R_0^T w\, R_0^T(\psi_I - u_h^*)\, dx\\ &= \int_T R_0^T w\, [R_0^T(\psi_I - \psi) + R_0^T(\psi - u) + R_0^T(u - u_h^*)]\, dx. \end{aligned}\tag{3.14}$$

By interpolation error estimate, for the first and third terms on the right hand side of (3.14), we have

$$\begin{aligned}
\sum_{T\in\mathcal{T}_h^b}\int_T R_0^T w\, R_0^T(\psi_I-\psi)\,dx &\le \sum_{T\in\mathcal{T}_h^b}\|R_0^T w\|_{0,T}\|R_0^T(\psi_I-\psi)\|_{0,T}\\
&\le C\sum_{T\in\mathcal{T}_h^b} h_*^{3/2-3\epsilon/4}\|w\|_{1/2-3\epsilon/4,T}|\psi_I-\psi|_{1,T}\\
&\le C\sum_{T\in\mathcal{T}_h^b} h_*^{7/2-3\epsilon/4}\|w\|_{1/2-3\epsilon/4,T}|\psi|_{3,T}\\
&\le C\sum_{T\in\mathcal{T}_h^b} h^{\frac{14}{3}-\epsilon}\|w\|_{1/2-3\epsilon/4,T}|\psi|_{3,T}\\
&\le Ch^{\frac{14}{3}-\epsilon}\|w\|_{1/2-3\epsilon/4,\Omega}|\psi|_{3,\Omega},
\end{aligned}$$

and

$$\begin{aligned}
\sum_{T\in\mathcal{T}_h^b}\int_T R_0^T w\, R_0^T(u-u_h^*)\,dx &\le \sum_{T\in\mathcal{T}_h^b}\|R_0^T w\|_{0,T}\|R_0^T(u-u_h^*)\|_{0,T}\\
&\le C\sum_{T\in\mathcal{T}_h^b} h_*^{3/2-3\epsilon/4}\|w\|_{1/2-3\epsilon/4,T}|u-u_h^*|_{1,T}\\
&\le C\sum_{T\in\mathcal{T}_h^b} h^{2-\epsilon}\|w\|_{1/2-3\epsilon/4,T}|u-u_h^*|_{1,T}\\
&\le Ch^{2-\epsilon}\|w\|_{1/2-3\epsilon/4,\Omega}|u-u_h^*|_{1,\Omega}.
\end{aligned}$$

Next we bound the second term on the right hand side of (3.14).

$$\begin{aligned}
\int_T R_0^T w\, R_0^T(\psi-u)\,dx &\le \|R_0^T w\|_{0,1,T}\|R_0^T(\psi-u)\|_{0,\infty,T}\\
&\le |T|^{1-\frac{1}{p}}\|R_0^T w\|_{0,p,T}\|R_0^T(\psi-u)\|_{0,\infty,T}\\
&\le Ch_*^{3-\frac{1}{p}-\epsilon_1}\|w\|_{\frac{1}{p}-\epsilon_1,p,T}|\psi-u|_{1,\infty,T}.
\end{aligned}$$

Now, we need to estimate $|\psi-u|_{1,\infty,T}$ for any $T\in\mathcal{T}_h^b$. From the assumption $\psi\in H^3(\Omega)$ and $u\in V$, we know that

$$\nabla(\psi-u)\in W^{1+1/t-\epsilon_2,t}(\Omega)\hookrightarrow C^{0,\alpha}(\Omega),\ \text{with}\ \alpha=1-1/t-\epsilon_2.$$

By the assumption (3.3), since $T\in\mathcal{T}_h^b$, there is a point $Q\in T$ such that $\nabla(\psi-u)(Q)=0$. Then for any $x\in T\in\mathcal{T}_h^b$, we have

$$\begin{aligned}
|\nabla(\psi-u)(x)| &= |\nabla(\psi-u)(x)-\nabla(\psi-u)(Q)|\\
&\le C|x-Q|^\alpha\|\psi-u\|_{2+1/t-\epsilon_2,t,\Omega}\\
&\le Ch_*^\alpha\|\psi-u\|_{2+1/t-\epsilon_2,t,\Omega}.
\end{aligned}$$

Thus,

$$|\psi - u|_{1,\infty,T} \leq Ch_*^{\alpha} \|\psi - u\|_{2+1/t-\epsilon_2,t,\Omega}.$$

Then

$$\begin{aligned}
\sum_{T\in\mathcal{T}_h^b} \int_T R_0^T w\, R_0^T(\psi - u)\, dx &\leq Ch_*^{\alpha+3-\frac{1}{p}-\epsilon_1} \left(\sum_{T\in\mathcal{T}_h^b} \|w\|_{\frac{1}{p}-\epsilon_1,p,T} \right) \|\psi - u\|_{2+1/t-\epsilon_2,t,\Omega} \\
&\leq Ch_*^{\alpha+3-\frac{1}{p}-\epsilon_1} \left(\sum_{T\in\mathcal{T}_h^b} 1 \right)^{1-\frac{1}{p}} \|w\|_{\frac{1}{p}-\epsilon_1,p,\Omega} \|\psi - u\|_{2+1/t-\epsilon_2,t,\Omega} \\
&\leq Ch_*^{3-\epsilon} \|w\|_{\frac{1}{p}-\epsilon_1,p,\Omega} \|\psi - u\|_{2+1/t-\epsilon_2,t,\Omega},
\end{aligned}$$

where $\alpha = 1 - 1/t - \epsilon_2$, $\epsilon_3 = 1/t$, $p = 1 + \epsilon_1$, and $\epsilon = \frac{\epsilon_1(2+\epsilon_1)}{1+\epsilon_1} + \epsilon_2 + \epsilon_3 > 0$. Note that $h_* = O(h^{4/3})$, we obtain

$$\sum_{T\in\mathcal{T}_h^b} \int_T R_0^T w\, R_0^T(\psi - u)\, dx \leq h^{4-\epsilon} \|w\|_{\frac{1}{p}-\epsilon_1,p,\Omega} \|\psi - u\|_{2+1/t-\epsilon_2,t,\Omega}.$$

The proof is completed then. ■

Suppose the free boundary is a "regular" curve. Then there are $\mathcal{O}(h^{-1})$ elements to be refined to smaller size, and the number of smaller size elements will be $\mathcal{O}(h^{-5/3})$. Thus, after the refinement, there are $\mathcal{O}(h^{-2})$ elements of the size h, and $\mathcal{O}(h^{-5/3})$ elements of the size h^*. So the total number of elements is $\mathcal{O}(h^{-2})$, which means that the total degrees of freedom do not increase significantly compared to the standard quadratic element on mesh $\mathcal{T}_h$. Let N be the number of degrees of freedom for the linear element space V_h^1. Then the error bound (3.5) implies that

$$\|u - u_h^*\|_1 \leq C\, N^{-1+\epsilon},$$

instead of obtaining convergence rate $\mathcal{O}(N^{-3/4+\epsilon})$ in [8]. This two level algorithm can also applied to other variational inequalities of the first kind.

4 Numerical examples

In this section, we give two numerical examples to test the two level algorithm described in Section 2.

To implement the second step in the algorithm, we first need to find the free boundary elements of $\mathcal{T}_h$. An element $T \in \mathcal{T}_h$ is said to be a free boundary element if there exist vertices v_1 and v_2 of T such that $u_h(v_1) = \psi(v_1)$ and $u_h(v_2) > \psi(v_2)$. Let the set of free boundary elements of $\mathcal{T}_h$ be denoted by $\mathcal{T}_h^F$. Then, create a new mesh of Ω, $\mathcal{T}_h^*$, by refining all the elements in $\mathcal{T}_h^F$. Denote the refinement of $\mathcal{T}_h^F$ as $\mathcal{T}_*^F$. Let h^F and h_*^F be the maximal edge lengths in $\mathcal{T}_h^F$ and $\mathcal{T}_*^F$, respectively. We choose a *mesh refinement parameter* C, and ensure that we refine T_h^F enough times to respect the *mesh refinement criterion*

$$h_*^F \leq C \left(h^F\right)^{4/3}. \tag{4.1}$$

In particular, if we use the simple red-green refinement approach for a two-dimensional mesh, we refine every element of $\mathcal{T}_h^F$ by $\max\left(1, \left\lceil -\log_2\left(C\left(h^F\right)^{1/3}\right)\right\rceil\right)$ times. The elements $T \notin T_h^F$ are left unrefined, except in the case that they must be refined in order to avoid hanging nodes.

We test the two level algorithm on two examples, comparing the method with the standard linear and quadratic finite element method, as well as an algorithm in which we simply refine the free boundary mesh $\mathcal{T}_h^F$ one time rather than respecting the mesh refinement criterion (4.1). We shall refer to the two level algorithm which does not respect (4.1) as the *non-respecting* refinement method, and the two level algorithm which does respect (4.1) as the refinement method.

4.1 Example 1

Consider the obstacle problem (1.1)–(1.2) in the domain $\Omega = (-1.5, 1.5)^2$ with a constant right hand side term $f(x) = -2$ and the obstacle function $\psi(x) = 0$. Given the boundary condition $u_0(r) = \frac{r^2}{2} - \ln(r) - 1/2$, the exact solution to (1.1)–(1.2) is

$$u(r) = \begin{cases} 0 & \text{if } r < 1, \\ \frac{r^2}{2} - \ln(r) - 1/2 & \text{otherwise.} \end{cases}$$

In the numerical examples we choose the mesh refinement parameter $C = 10$. In Figures **??**–**??** we compare the accuracy of each of the four numerical methods, as well as the time required to compute each solution. We can see that the two level method outperforms the two standard methods in both regards. As expected, the non-respecting two level method does not converge as rapidly as the two level method.

We report the same results in tabular form in Tables 1–4. In addition, for each method we compute the numerical convergence orders. In particular, let N_i denote the number of degrees of freedom corresponding to the i^{th} approximation. Let h_i denote the largest edge length from the mesh $\mathcal{T}_h^i$, and define E_i as the $H^1(\Omega)$ error achieved by approximation i. The convergence rate in the term of N refers to α in $O(N^{-\alpha})$, and we estimate the order of

N	h	H^1 Error	N Conv. Rate	h Conv. Rate	Comp. Time (s)
17	1.677e+00	1.207e+00	-	-	1.702e-02
31	1.008e+00	7.486e-01	7.945e-01	9.372e-01	2.441e-02
101	5.252e-01	3.527e-01	6.371e-01	1.155e+00	2.242e-02
366	2.650e-01	1.800e-01	5.225e-01	9.835e-01	3.748e-02
1385	1.325e-01	9.085e-02	5.138e-01	9.868e-01	8.349e-02
5308	6.629e-02	4.588e-02	5.085e-01	9.862e-01	3.210e-01
20958	3.315e-02	2.292e-02	5.055e-01	1.002e+00	1.850e+00
83087	1.657e-02	1.146e-02	5.029e-01	9.993e-01	1.612e+01
330653	8.286e-03	5.749e-03	4.997e-01	9.956e-01	1.912e+02

Table 1: Results for linear method in Example 1.

N	h	H^1 Error	N Conv. Rate	h Conv. Rate	Comp. Time (s)
49	1.677e+00	5.900e-01	-	-	2.254e-02
105	1.008e+00	1.390e-01	1.897e+00	2.839e+00	2.635e-02
369	5.252e-01	7.419e-02	4.995e-01	9.632e-01	3.873e-02
1397	2.650e-01	2.064e-02	9.611e-01	1.871e+00	6.350e-02
5409	1.325e-01	7.199e-03	7.780e-01	1.520e+00	2.188e-01
20973	6.629e-02	2.639e-03	7.406e-01	1.449e+00	1.941e+00
83317	3.315e-02	9.073e-04	7.739e-01	1.540e+00	1.984e+01
331319	1.657e-02	3.206e-04	7.536e-01	1.501e+00	2.498e+02
1320561	8.286e-03	1.122e-04	7.590e-01	1.514e+00	2.791e+03

Table 2: Results for quadratic method in Example 1.

N	h	H^1 Error	N Conv. Rate	h Conv. Rate	Comp. Time (s)
137	1.677e+00	2.050e-01	-	-	1.153e-01
320	1.008e+00	6.373e-02	1.377e+00	2.295e+00	7.576e-02
815	5.252e-01	2.825e-02	8.701e-01	1.248e+00	1.096e-01
2413	2.650e-01	8.183e-03	1.142e+00	1.812e+00	1.325e-01
7185	1.325e-01	3.106e-03	8.877e-01	1.398e+00	3.440e-01
24593	6.629e-02	1.190e-03	7.798e-01	1.385e+00	1.151e+00
90277	3.315e-02	4.474e-04	7.523e-01	1.411e+00	5.704e+00
345815	1.657e-02	1.512e-04	8.077e-01	1.565e+00	3.733e+01
1349265	8.286e-03	5.192e-05	7.851e-01	1.542e+00	3.279e+02

Table 3: Results for refinement (non-respecting) method in Example 1.

N	h	H^1 Error	N Conv. Rate	h Conv. Rate	Comp. Time (s)
137	1.677e+00	2.050e-01	-	-	7.699e-02
320	1.008e+00	6.373e-02	1.377e+00	2.295e+00	8.188e-02
815	5.252e-01	2.825e-02	8.701e-01	1.248e+00	9.927e-02
2413	2.650e-01	8.183e-03	1.142e+00	1.812e+00	1.558e-01
13725	1.325e-01	1.481e-03	9.834e-01	2.467e+00	6.389e-01
80125	6.629e-02	3.434e-04	8.283e-01	2.110e+00	5.474e+00
196377	3.315e-02	1.099e-04	1.271e+00	1.644e+00	1.871e+01
1147579	1.657e-02	2.388e-05	8.646e-01	2.202e+00	2.205e+02

Table 4: Results for refinement (respecting) method in Example 1.

convergence in terms of N for the methods by $\alpha \approx -\frac{\log(E_i/E_{i-1})}{\log(N_i/N_{i-1})}$. We estimate the order of convergence in terms of h via $p \approx \frac{\log(E_i/E_{i-1})}{\log(h_i/h_{i-1})}$.

In Tables 1–4 we can see each method tending toward its expected order of convergence.

4.2 Example 2

As in Example 1, let the domain be $\Omega = [-2,2]^2$. Define $f(x) = 0$ and obstacle function as

$$\psi(r) = \begin{cases} \sqrt{1-r^2} & \text{if } r < 1, \\ 0 & \text{otherwise} \end{cases}.$$

The boundary condition is given as

$$u_0(r) = -\left(r^*\right)^2 \frac{\ln(r/2)}{\sqrt{1-\left(r^*\right)^2}},$$

where $r = \sqrt{x_1^2 + x_2^2}$ and $r^* = 0.6979651482$. Then, the exact solution to (1.1)–(1.2) is

$$u(r) = \begin{cases} \sqrt{1-r^2} & \text{if } r < r^*, \\ -\left(r^*\right)^2 \frac{\ln(r/2)}{\sqrt{1-\left(r^*\right)^2}} & \text{otherwise.} \end{cases}$$

Again we choose the mesh refinement parameter $C = 10$. Figures **??** and **??** compare the error and solution time in each of the methods. Once again the refinement method outperforms the other methods.

Tables 5–8 provide the same data as the figures in tabular form, and attempt to estimate the convergence rate of each method. Again, the two level method converges at a rate near N^{-1}, as predicted.

N	h	H^1 Error	N Conv. Rate	h Conv. Rate	Comp. Time (s)
17	1.677e+00	5.765e-01	-	-	2.194e-02
31	1.008e+00	7.005e-01	-3.244e-01	-3.827e-01	3.161e-02
101	5.252e-01	3.386e-01	6.154e-01	1.115e+00	3.797e-02
366	2.650e-01	1.666e-01	5.508e-01	1.037e+00	6.497e-02
1385	1.325e-01	8.278e-02	5.257e-01	1.010e+00	1.203e-01
5308	6.629e-02	4.216e-02	5.022e-01	9.740e-01	3.451e-01
20958	3.315e-02	2.124e-02	4.991e-01	9.889e-01	1.814e+00
83087	1.657e-02	1.065e-02	5.016e-01	9.968e-01	1.641e+01
330653	8.286e-03	5.327e-03	5.013e-01	9.990e-01	1.992e+02

Table 5: Results for linear method in Example 2.

N	h	H^1 Error	N Conv. Rate	h Conv. Rate	Comp. Time (s)
49	1.677e+00	6.289e-01	-	-	2.993e-02
105	1.008e+00	1.520e-01	1.864e+00	2.789e+00	3.435e-02
369	5.252e-01	7.797e-02	5.310e-01	1.024e+00	5.610e-02
1397	2.650e-01	3.447e-02	6.132e-01	1.194e+00	1.011e-01
5409	1.325e-01	1.225e-02	7.640e-01	1.493e+00	3.084e-01
20973	6.629e-02	4.679e-03	7.103e-01	1.389e+00	1.648e+00
83317	3.315e-02	1.549e-03	8.015e-01	1.595e+00	1.824e+01
331319	1.657e-02	5.615e-04	7.351e-01	1.464e+00	2.462e+02
1320561	8.286e-03	1.973e-04	7.565e-01	1.509e+00	3.153e+03

Table 6: Results for quadratic method in Exampl 2.

N	h	H^1 Error	N Conv. Rate	h Conv. Rate	Comp. Time (s)
137	1.677e+00	1.459e-01	-	-	7.092e-02
314	1.008e+00	9.188e-02	5.573e-01	9.076e-01	8.227e-02
737	5.252e-01	4.031e-02	9.655e-01	1.264e+00	1.015e-01
2053	2.650e-01	1.582e-02	9.133e-01	1.368e+00	1.793e-01
6693	1.325e-01	5.669e-03	8.682e-01	1.481e+00	4.070e-01
23485	6.629e-02	2.136e-03	7.777e-01	1.409e+00	1.356e+00
88285	3.315e-02	6.834e-04	8.605e-01	1.644e+00	6.685e+00
341451	1.657e-02	2.500e-04	7.436e-01	1.451e+00	4.174e+01
1340053	8.286e-03	8.805e-05	7.632e-01	1.505e+00	3.056e+02

Table 7: Results for refinement (non-respecting) method in Example 2.

N	h	H^1 Error	N Conv. Rate	h Conv. Rate	Comp. Time (s)
137	1.677e+00	1.459e-01	-	-	7.092e-02
314	1.008e+00	9.188e-02	5.573e-01	9.076e-01	8.428e-02
737	5.252e-01	4.031e-02	9.655e-01	1.264e+00	1.117e-01
2053	2.650e-01	1.582e-02	9.133e-01	1.368e+00	2.051e-01
11429	1.325e-01	2.423e-03	1.093e+00	2.708e+00	6.784e-01
61305	6.629e-02	5.707e-04	8.607e-01	2.087e+00	4.591e+00
164273	3.315e-02	1.874e-04	1.130e+00	1.607e+00	1.752e+01
897539	1.657e-02	4.314e-05	8.648e-01	2.119e+00	2.117e+02

Table 8: Results for refinement (respecting) method in Example 2.

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