A two level algorithm for an obstacle problem

Abstract. Due to the inequality feature of the obstacle problem, the standard quadratic finite element method for solving the problem can only achieve an error bound of the form $\mathcal{O}(N^{-3/4+\epsilon})$ with N the total number of degrees of freedom, and $\epsilon > 0$ arbitrary. To achieve a better error bound, the key lies in how to capture the free boundary accurately. In this paper, we propose a two level algorithm for solving the obstacle problem. The first part of the algorithm is through the use of the linear elements on a quasi-uniform mesh. Then information on the approximate free boundary from the linear element solution is used in the construction of a quadratic finite element method. Under some reasonable assumptions, the numerical solution from the two level algorithm is shown to have a nearly optimal error bound of $\mathcal{O}(N^{-1+\epsilon})$, $\epsilon > 0$ arbitrary.

Keywords. Obstacle problem, variational inequality, free-boundary problem, error estimation

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1 Introduction

Many problems in physical and engineering sciences are modeled by partial differential equations. However, various more complex physical processes are described by variational inequalities (VIs). Variational inequalities form an important family of nonlinear problems arising in diverse application areas, for example, elastoplasticity, contact mechanics, heat control problems, options pricing problems in finance, Nash-equilibria in management science. Therefore, how to solve variational inequalities efficiently is very attractive to mathematicians, engineers and economists. Variational inequalities are closely related to free-boundary problems. The classical formulation of a variational inequality is usually expressed through the presence of an unknown region or boundary. So a variational inequality can be also viewed as a free-boundary problem. Moreover, many free-boundary problems can be reformulated as variational inequalities. The formulation of a variational inequality is advantageous over that of a free-boundary problem, especially for numerical solutions, since in a variational inequality there is no explicit involvement of an unknown region or boundary.

In this paper, we consider an obstacle problem, which is a representative elliptic variational inequality (EVI) of first kind ([5]). For more examples of EVIs, we refer the reader to the monograph [3]. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz boundary $\partial\Omega$.

An obstacle problem. Let $f \in L^2(\Omega)$, and $\psi \in H^2(\Omega)$ with $\psi \le 0$ on $\partial\Omega$. The obstacle problem is to find $u \in K$ such that

$$a(u, v-u) \ge (f, v-u)_\Omega \quad \forall\, v \in K, \tag{1.1}$$

where

$$K = \{v \in H_0^1(\Omega) : v \ge \psi \text{ a.e in } \Omega\} \tag{1.2}$$

is a closed and convex admissible set of the space $H_0^1(\Omega)$, and

$$\begin{aligned} a(u,v) &= \int_\Omega \nabla u \cdot \nabla v\, dx, \\ (f,v)_\Omega &= \int_\Omega f v\, dx. \end{aligned}$$

The obstacle problem has a unique solution ([3]). It arises in a variety of applications, such as the membrane deformation in elasticity theory, and the non-parametric minimal and capillary surfaces as geometrical problems. The elastic-plastic torsion problem and the cavitation problem in the theory of lubrication also can be regarded as obstacle type problems. If the solution has the regularity $u \in H^2(\Omega)$, then it satisfies the relations (see, e.g., [1])

$$-\triangle u \ge f, \quad u \ge \psi, \quad (-\triangle u - f)(u-\psi) = 0 \quad \text{a.e. in } \Omega. \tag{1.3}$$

Therefore, we have the following relations

$$\begin{aligned} -\triangle u &\ge f \quad \text{in } \Omega^0 = \{x \in \Omega : u(x) = \psi(x)\}, \\ -\triangle u &= f \quad \text{in } \Omega^+ = \{x \in \Omega : u(x) > \psi(x)\}. \end{aligned}$$

Regarding the solution regularity of the obstacle problem, Brezis proved the following result (see [6, 7]): if $\partial\Omega$ is smooth, $f \in L^\infty(\Omega) \cap BV(\Omega)$, and $\psi \in C^3(\bar{\Omega})$, then the solution of the problem (1.1) has the regularity $u \in W^{s,p}(\Omega)$ with $1 < p < \infty$ and $s < 2 + 1/p$.

The finite element method is the dominant numerical discretization method for variational inequalities. Optimal convergence order can be reached by the linear elements ([4, 5, 9]) under the regularity assumption $u \in H^2(\Omega)$. For the quadratic element solutions, an error bound $\mathcal{O}(h^{3/2-\epsilon})$, $\epsilon > 0$ arbitrary, is derived for $H^1(\Omega)$-norm in [8] under regularity assumption that $u \in W^{s,p}(\Omega)$ with $1 < p < \infty$ and $s < 2 + 1/p$. In terms of the total number of degrees of freedom N, the error bound for the linear element solution is $\mathcal{O}(N^{-1/2})$, whereas that for the quadratic element solution is $\mathcal{O}(N^{-3/4+\epsilon})$ for an arbitrarily small $\epsilon > 0$. For variational inequalities, higher order elements do not lead to higher order convergence. Therefore, it is common to use low order elements in solving variational inequalities. In this paper, we introduce a two level algorithm using both linear and quadratic elements to solve the obstacle problem such that the error bound is expected to be $\mathcal{O}(N^{-1+\epsilon})$ for an arbitrarily small $\epsilon > 0$. In the error analysis for the two level algorithm, we adopt the assumption that $u \in W^{s,p}(\Omega)$ with $1 < p < \infty$ and $s < 2 + 1/p$. Moreover, we assume that $u|_{\Omega^0} \in H^3(\Omega^0)$ and $u|_{\Omega^+} \in H^3(\Omega^+)$, where Ω^0 is the contact area and $\Omega^+ = \Omega \backslash \Omega^0$. This is a reasonable assumption. In the contact area, $u = \psi$, so $u|_{\Omega^0} \in H^3(\Omega^0)$ is just the assumption $\psi|_{\Omega^0} \in H^3(\Omega^0)$, which is implied by $\psi \in H^3(\Omega)$. $u|_{\Omega^+} \in H^3(\Omega^+)$ can be considered as the solution of an Poisson equation with the free-boundary as Dirichlet boundary condition. This error bound is proved under some assumption on the behavior of the numerical solution. The idea of the algorithm is outlined as follows. First, solve the obstacle problem with linear elements on a quasiuniform mesh $\mathcal{T}_h$, and identify free-boundary elements. Then refine the free-boundary elements into elements with mesh size $h_* = O(h^{4/3})$ to obtain a new mesh. Finally, we solve the obstacle problem on this new mesh with the quadratic elements.

The rest of the paper is organized as follows: In Section 2, we introduce the two level algorithm. In Section 3, we derive a priori error estimates for this algorithm. In Section 4, we present numerical examples to provide numerical evidence of the error bound.

2 A two level algorithm

We assume Ω is a polygonal domain. For a subdivision $\mathcal{T}_h$ of $\overline{\Omega}$ into triangles, let $h_T = \mathrm{diam}(T)$ and $h = \max\{h_T : T \in \mathcal{T}_h\}$. All the subdivisions, including the refined meshes, are constructed so that the minimal angle condition is satisfied.

We introduce the following two level quadratic finite element method for the obstacle problem.

1. Solve the obstacle problem on a quasi-uniform mesh $\mathcal{T}_h$ with the linear elements, that is, find $u_h \in K_h^1$ such that

$$a(u_h, v_h - u_h) \geq (f, v_h - u_h)_\Omega \quad \forall\, v_h \in K_h^1, \tag{2.1}$$

   where

$$K_h^1 = \{v_h \in V_h^1 :\ v_h(x) \geq \psi(x) \text{ at all vertices of } \mathcal{T}_h\}$$

   and

$$V_h^1 = \{v_h \in H^1(\Omega) :\ v_h|_T \in P_1(T)\ \forall\, T \in \mathcal{T}_h\}.$$

2. Identify the subset $\mathcal{T}_h^F \subset \mathcal{T}_h$ of free-boundary elements, i.e., the subset of the elements $T \in \mathcal{T}_h$ such that $T = T_1 \cup T_2$ with $|T_1| > 0$, $|T_2| > 0$, $u_h = \psi$ on T_1, and $u_h > \psi$ on T_2. Refine all the elements in $\mathcal{T}_h^F$ into new elements with mesh size $h_* = O(h^{4/3})$. Denote the new mesh by $\mathcal{T}_h^*$.

3. Solve the obstacle problem with quadratic elements over the new mesh $\mathcal{T}_h^*$, i.e., find $u_h^* \in K_h^2$ such that

$$a(u_h^*, v_h - u_h^*) \geq (f, v_h - u_h^*)_\Omega \quad \forall\, v_h \in K_h^2, \tag{2.2}$$

where

$$K_h^2 = \{v_h \in V_h^2 :\ v_h(m) \geq \psi(m) \text{ at all midpoints } m \text{ on element edges of } \mathcal{T}_h^*\}$$

and

$$V_h^2 = \{v_h \in H^1(\Omega) :\ v_h|_T \in P_2(T)\ \forall\, T \in \mathcal{T}_h^*\}.$$

In the iterative procedure for solving the problem (2.2), for the initial guess we use the interpolant of u_h in V_h^2.

In the next section, under the solution smoothness assumption $v \in V$, where

$$V = \{v \in W^{s,p}(\Omega),\ 1 < p < \infty, s < 2 + 1/p;\ v|_{\Omega^0} \in H^3(\Omega^0) \text{ and } v|_{\Omega^+} \in H^3(\Omega^+)\}, \tag{2.3}$$

and a natural assumption on the numerical solution, we will prove that for the two level algorithm, $\|u - u_h^*\|_1 \leq Ch^{2-\epsilon}$; or, in terms of the total number of degrees of freedom N, $\|u - u_h^*\|_1 \leq CN^{-1+\epsilon}$.

3 Error estimates

First recall the following boundedness and stability properties of the bilinear form $a(u, v)$.

Lemma 3.1

$$a(u, v) \leq C_b \|u\|_1 \|v\|_1 \quad \forall\, u, v \in H_0^1(\Omega), \tag{3.1}$$

$$a(v, v) \geq C_s \|v\|_1^2 \quad \forall\, v \in H_0^1(\Omega), \tag{3.2}$$

where C_b and C_s are positive constants independent of the mesh size.

The main purpose of the section is to bound the error for the two level method. We assume $u \in V$. Let us group the elements of $\mathcal{T}_h^*$ into three kinds:

$$\begin{aligned} \mathcal{T}_h^+ &= \{T \in \mathcal{T}_h^* : T \subset \Omega^+\}, \\ \mathcal{T}_h^0 &= \{T \in \mathcal{T}_h^* : T \subset \Omega^0\}, \\ \mathcal{T}_h^b &= \mathcal{T}_h^* \backslash (\mathcal{T}_h^+ \cup \mathcal{T}_h^0). \end{aligned}$$

We will assume

$$\bigcup_{T\in\mathcal{T}_h^b} T \subset \bigcup_{T\in\mathcal{T}_h^F} T, \tag{3.3}$$

where $\mathcal{T}_h^F$ was defined in Step 2 of the algorithm. This is a reasonable assumption if h is small enough, given the first order convergence of u_h to u in $H^1(\Omega)$. Write the error as

$$e = u - u_h^* = (u - u_I) + (u_I - u_h^*),$$

where $u_I \in V_h^2$ is the usual continuous piecewise quadratic polynomial interpolant. Then

$$\|u - u_I\|_1 \lesssim h^{2-\epsilon}, \tag{3.4}$$

where "$\lesssim \cdots$" stands for "$\leq C \cdots$", where C is a positive generic constant independent of h and other parameters, which may take different values at different appearances. In fact, because $u|_{\Omega^0} \in H^3(\Omega^0)$ and $u|_{\Omega^+} \in H^3(\Omega^+)$, for any $T \in \mathcal{T}_h^0 \cup \mathcal{T}_h^+$, we have

$$\|u - u_I\|_{1,T} \leq Ch^2|u|_{3,T},$$

and for any $T \in \mathcal{T}_h^b$, we have

$$\|u - u_I\|_{1,T} \leq Ch_*^{3/2-3\epsilon/4}|u|_{5/2-3\epsilon/4,T} \leq Ch^{2-\epsilon}|u|_{5/4-3\epsilon/4,T},$$

with $s = 5/2 - 3\epsilon/4 < 2 + 1/p$, $p = 2$ in (2.3).

Using a technique similar to that in [8, 9], we derive the following error estimate.

Theorem 3.2 *Let u and u_h^* be the solutions of* (1.1) *and* (2.2)*, respectively. Assume $u \in V$ defined in* (2.3)*, $\psi \in H^3(\Omega)$, $f \in H^1(\Omega) \cap L^\infty(\Omega)$, and* (3.3)*. Then for the two level method introduced in Section 2, we have*

$$\|u - u_h^*\|_1 \leq Ch^{2-\epsilon}, \tag{3.5}$$

for an arbitrary $\epsilon > 0$.

Proof. By the stability of the bilinear form $a(u, v)$, we have

$$C_s\|u_I - u_h^*\|_1^2 \leq a(u_I - u_h^*, u_I - u_h^*) \equiv A_1 + A_2, \tag{3.6}$$

where

$$A_1 = a(u_I - u, u_I - u_h^*),$$
$$A_2 = a(u - u_h^*, u_I - u_h^*).$$

By the boundedness of the bilinear form, the term A_1 is bounded by

$$A_1 \le C_b \|u_I - u\|_1 \|u_I - u_h^*\|_1 \le \frac{C_s}{2}\|u_I - u_h^*\|_1^2 + \frac{C_b^2}{2C_s}\|u_I - u\|_1^2. \tag{3.7}$$

To bound A_2, we first recall the relations

$$\begin{aligned} -\triangle u = f \quad & \text{in } \Omega^+ = \{x \in \Omega : u(x) > \psi(x)\}, \\ -\triangle u \ge f \quad & \text{in } \Omega^0 = \{x \in \Omega : u(x) = \psi(x)\}. \end{aligned}$$

Note that $u_I - u_h^* = 0$ on $\partial\Omega$. We have

$$a(u, u_I - u_h^*) = \int_\Omega \nabla u \cdot \nabla(u_I - u_h^*)\, dx = -\int_\Omega \Delta u(u_I - u_h^*)\, dx. \tag{3.8}$$

Let $v_h = u_I$ in (2.2),

$$a(u_h^*, u_I - u_h^*) \ge (f, u_I - u_h^*)_\Omega = \sum_{T\in\mathcal{T}_h} \int_T f(u_I - u_h^*)\, dx. \tag{3.9}$$

Combining (3.9) and (3.8), we obtain

$$A_2 = a(u - u_h^*, u_I - u_h^*) \le \sum_{T\in\mathcal{T}_h} \int_T -(\Delta u + f)(u_I - u_h^*)\, dx \equiv A_3 + A_4 + A_5, \tag{3.10}$$

where

$$\begin{aligned} A_3 &= \sum_{T\in\mathcal{T}_h^+} \int_T w(u_I - u_h^*)\, dx, \\ A_4 &= \sum_{T\in\mathcal{T}_h^0} \int_T w(u_I - u_h^*)\, dx, \\ A_5 &= \sum_{T\in\mathcal{T}_h^b} \int_T w(u_I - u_h^*)\, dx. \end{aligned}$$

Here, we denote $w := -\Delta u - f$. It is easy to see that $A_3 = 0$.

To estimate A_4, as in [8], we introduce

$$P_0^T v = \frac{1}{|T|}\int_T v\, dx, \qquad R_0^T v = v - P_0^T v.$$

Since $w \ge 0$, we get $P_0^T w \ge 0$. Due to $u_h^* \in K_h^2$, we have $u_h^*(m) \ge \psi(m)$ for all the midpoints m on the edges of the element T, implying

$$\int_T (\psi_I - u_h^*)\, dx = \frac{|T|}{3}\sum_{i=1}^{3}(\psi - u_h^*)(m_i) \le 0.$$

Then we get

$$\int_T w(\psi_I - u_h^*)\,dx \le \int_T R_0^T w(\psi_I - u_h^*)\,dx$$
$$= \int_T R_0^T w\, R_0^T(\psi_I - u_h^*)\,dx \le \|R_0^T w\|_{0,T}\|R_0^T(\psi_I - u_h^*)\|_{0,T}.$$

Note that $u = \psi$ in $T \in \mathcal{T}_h^0$ and $\psi \in H^3(\Omega)$; so

$$\int_T w(\psi_I - u_h^*)\,dx \le Ch^2|w|_{1,T}|\psi_I - u_h^*|_{1,T}$$
$$\le Ch^2|w|_{1,T}(|\psi_I - \psi|_{1,T} + |\psi - u|_{1,T} + |u - u_h^*|_{1,T}).$$

We then apply interpolation error estimates to get

$$\int_T w(\psi_I - u_h^*)\,dx \le Ch^2|w|_{1,T}(h^2|\psi|_{3,T} + |u - u_h^*|_{1,T}). \tag{3.11}$$

Hence,

$$A_4 = \sum_{T\in\mathcal{T}_h^0}\int_T w(\psi_I - u_h^*)\,dx \le Ch^2|w|_{1,\Omega}(h^2|\psi|_{3,\Omega} + \|u - u_h^*\|_1). \tag{3.12}$$

Consider the term

$$A_5 = \sum_{T\in\mathcal{T}_h^b}\int_T w(u_I - u + \psi - \psi_I)dx + \sum_{T\in\mathcal{T}_h^b}\int_T w(u-\psi)dx + \sum_{T\in\mathcal{T}_h^b}\int_T w(\psi_I - u_h^*)dx.$$

Since $w(u-\psi) = 0$ by (1.3), we can write

$$A_5 = A_{5,1} + A_{5,2},$$

where

$$A_{5,1} = \sum_{T\in\mathcal{T}_h^b}\int_T w[(u-\psi)_I - (u-\psi)]dx,$$
$$A_{5,2} = \sum_{T\in\mathcal{T}_h^b}\int_T w(\psi_I - u_h^*)dx.$$

Using a technique similar to that in [2], we can bound the term $A_{5,1}$ as follows:

$$A_{5,1} \lesssim h^{4-\epsilon}\|w\|_{L^\infty(\Omega)}\|u-\psi\|_{s,p,\Omega}. \tag{3.13}$$

Indeed, for any $v \in W^{s,p}(\Omega)$ with $1 < p < \infty$ and $s < 2 + 1/p$, by Cauchy-Schwarz inequality and the interpolation error estimate, we get

$$\|v_I - v\|_{0,1,T} = \int_T |v_I - v|\,dx \le |T|^{1-\frac{1}{p}}\|v_I - v\|_{0,p,T} \lesssim h_*^{s+2-\frac{2}{p}}\|v\|_{s,p,T},$$

and then,

$$\begin{aligned}\sum_{T\in\mathcal{T}_h^b}\|v_I-v\|_{0,1,T} &\lesssim h_*^{s+2-\frac{2}{p}}\sum_{T\in\mathcal{T}_h^b}\|v\|_{s,p,T}\\ &\lesssim h_*^{s+2-\frac{2}{p}}\left(\sum_{T\in\mathcal{T}_h^b}1\right)^{1-\frac{1}{p}}\left(\sum_{T\in\mathcal{T}_h^b}\|v\|_{s,p,T}^p\right)^{\frac{1}{p}}\\ &\lesssim h_*^s\|v\|_{s,p,\Omega}.\end{aligned}$$

Hence,

$$\begin{aligned}A_{5,1} &= \sum_{T\in\mathcal{T}_h^b}\int_T w[(u-\psi)_I-(u-\psi)]dx\\ &\le \|w\|_{L^\infty(\Omega)}\sum_{T\in\mathcal{T}_h^b}\|(u-\psi)_I-(u-\psi)\|_{0,1,T}\\ &\lesssim h_*^s\|w\|_{L^\infty(\Omega)}\|u-\psi\|_{s,p,\Omega}.\end{aligned}$$

Taking $p=1+\epsilon_1$, $s=2+\frac{1}{p}-\epsilon_2=3-3\epsilon/4$, with $\epsilon=\frac{4}{3}\left(\frac{\epsilon_1}{1+\epsilon_1}+\epsilon_2\right)$ in the above inequality, and note that $h_*=O(h^{4/3})$, we get the bound (3.13) for the term $A_{5,1}$.

Finally, let us bound the term $A_{5,2}$. We know that

$$\begin{aligned}\int_T w(\psi_I-u_h^*)\,dx &\le \int_T R_0^T w\,R_0^T(\psi_I-u_h^*)\,dx\\ &= \int_T R_0^T w\,[R_0^T(\psi_I-\psi)+R_0^T(\psi-u)+R_0^T(u-u_h^*)]\,dx. \qquad (3.14)\end{aligned}$$

By interpolation error estimate, for the first and third terms on the right hand side of (3.14), we have

$$\begin{aligned}\sum_{T\in\mathcal{T}_h^b}\int_T R_0^T w\,R_0^T(\psi_I-\psi)\,dx &\le \sum_{T\in\mathcal{T}_h^b}\|R_0^T w\|_{0,T}\|R_0^T(\psi_I-\psi)\|_{0,T}\\ &\le C\sum_{T\in\mathcal{T}_h^b}h_*^{3/2-3\epsilon/4}\|w\|_{1/2-3\epsilon/4,T}|\psi_I-\psi|_{1,T}\\ &\le C\sum_{T\in\mathcal{T}_h^b}h_*^{7/2-3\epsilon/4}\|w\|_{1/2-3\epsilon/4,T}|\psi|_{3,T}\\ &\le C\sum_{T\in\mathcal{T}_h^b}h^{\frac{14}{3}-\epsilon}\|w\|_{1/2-3\epsilon/4,T}|\psi|_{3,T}\\ &\le Ch^{\frac{14}{3}-\epsilon}\|w\|_{1/2-3\epsilon/4,\Omega}|\psi|_{3,\Omega},\end{aligned}$$

and

$$\begin{aligned}\sum_{T\in\mathcal{T}_h^b}\int_T R_0^T w\, R_0^T(u-u_h^*)\,dx &\le \sum_{T\in\mathcal{T}_h^b}\|R_0^T w\|_{0,T}\|R_0^T(u-u_h^*)\|_{0,T}\\ &\le C\sum_{T\in\mathcal{T}_h^b} h_*^{3/2-3\epsilon/4}\|w\|_{1/2-3\epsilon/4,T}|u-u_h^*|_{1,T}\\ &\le C\sum_{T\in\mathcal{T}_h^b} h^{2-\epsilon}\|w\|_{1/2-3\epsilon/4,T}|u-u_h^*|_{1,T}\\ &\le Ch^{2-\epsilon}\|w\|_{1/2-3\epsilon/4,\Omega}|u-u_h^*|_{1,\Omega}.\end{aligned}$$

Next we bound the second term on the right hand side of (3.14).

$$\begin{aligned}\int_T R_0^T w\, R_0^T(\psi-u)\,dx &\le \|R_0^T w\|_{0,1,T}\|R_0^T(\psi-u)\|_{0,\infty,T}\\ &\le |T|^{1-\frac{1}{p}}\|R_0^T w\|_{0,p,T}\|R_0^T(\psi-u)\|_{0,\infty,T}\\ &\le Ch_*^{3-\frac{1}{p}-\epsilon_1}\|w\|_{\frac{1}{p}-\epsilon_1,p,T}|\psi-u|_{1,\infty,T}.\end{aligned}$$

Now, we need to estimate $|\psi-u|_{1,\infty,T}$ for any $T\in\mathcal{T}_h^b$. From the assumption $\psi\in H^3(\Omega)$ and $u\in V$, we know that

$$\nabla(\psi-u)\in W^{1+1/t-\epsilon_2,t}(\Omega)\hookrightarrow C^{0,\alpha}(\Omega),\ \text{with}\ \alpha=1-1/t-\epsilon_2.$$

By the assumption (3.3), since $T\in\mathcal{T}_h^b$, there is a point $Q\in T$ such that $\nabla(\psi-u)(Q)=0$. Then for any $x\in T\in\mathcal{T}_h^b$, we have

$$\begin{aligned}|\nabla(\psi-u)(x)| &= |\nabla(\psi-u)(x)-\nabla(\psi-u)(Q)|\\ &\le C|x-Q|^\alpha\|\psi-u\|_{2+1/t-\epsilon_2,t,\Omega}\\ &\le Ch_*^\alpha\|\psi-u\|_{2+1/t-\epsilon_2,t,\Omega}.\end{aligned}$$

Thus,

$$|\psi-u|_{1,\infty,T}\le Ch_*^\alpha\|\psi-u\|_{2+1/t-\epsilon_2,t,\Omega}.$$

Then

$$\begin{aligned}\sum_{T\in\mathcal{T}_h^b}\int_T R_0^T w\, R_0^T(\psi-u)\,dx &\le Ch_*^{\alpha+3-\frac{1}{p}-\epsilon_1}\left(\sum_{T\in\mathcal{T}_h^b}\|w\|_{\frac{1}{p}-\epsilon_1,p,T}\right)\|\psi-u\|_{2+1/t-\epsilon_2,t,\Omega}\\ &\le Ch_*^{\alpha+3-\frac{1}{p}-\epsilon_1}\left(\sum_{T\in\mathcal{T}_h^b}1\right)^{1-\frac{1}{p}}\|w\|_{\frac{1}{p}-\epsilon_1,p,\Omega}\|\psi-u\|_{2+1/t-\epsilon_2,t,\Omega}\\ &\le Ch_*^{3-\epsilon}\|w\|_{\frac{1}{p}-\epsilon_1,p,\Omega}\|\psi-u\|_{2+1/t-\epsilon_2,t,\Omega}.\end{aligned}$$

where $\alpha = 1 - 1/t - \epsilon_2$, $\epsilon_3 = 1/t$, $p = 1 + \epsilon_1$, and $\epsilon = \frac{\epsilon_1(2+\epsilon_1)}{1+\epsilon_1} + \epsilon_2 + \epsilon_3 > 0$. Note that $h_* = O(h^{4/3})$, we obtain

$$\sum_{T \in \mathcal{T}_h^b} \int_T R_0^T w \, R_0^T(\psi - u)\, dx \le h^{4-\epsilon} \|w\|_{\frac{1}{p}-\epsilon_1,p,\Omega} \|\psi - u\|_{2+1/t-\epsilon_2,t,\Omega}.$$

The proof is completed then. ■

Suppose the free boundary is a "regular" curve. Then there are $\mathcal{O}(h^{-1})$ elements to be refined to smaller size, and the number of smaller size elements will be $\mathcal{O}(h^{-5/3})$. Thus, after the refinement, there are $\mathcal{O}(h^{-2})$ elements of the size h, and $\mathcal{O}(h^{-5/3})$ elements of the size h^*. So the total number of elements is $\mathcal{O}(h^{-2})$, which means that the total degrees of freedom do not increase significantly compared to the standard quadratic element on mesh $\mathcal{T}_h$. Let N be the number of degrees of freedom for the linear element space V_h^1, then the error bound (3.5) implies that

$$\|u - u_h^*\|_1 \le C\, N^{-1+\epsilon},$$

instead of obtaining convergence rate $\mathcal{O}(N^{-3/4+\epsilon})$ in [8]. This two level algorithm can also applied to other variational inequalities of the first kind.

4 Numerical examples

Consider the following 1-d example. The domain is $\Omega = [0, 1]$. The obstacle function $\psi = -10(x^2 - x + \frac{3}{16})$, and $f = 2$. The exact solution is

$$u(x,y) = \begin{cases} -x^2 + (10 - \dfrac{3\sqrt{30}}{2})x, & \text{if } 0 \le x \le \sqrt{30}/12, \\ -10(x^2 - x + \frac{3}{16}), & \text{if } \sqrt{30}/12 \le x \le 1 - \sqrt{30}/12, \\ -(1-x)^2 + (10 - \frac{3\sqrt{30}}{2})(1-x), & \text{if } 1 - \sqrt{30}/12 \le x \le 1 \end{cases}$$

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