The obstacle problem introduced is to find $u \in K$ such that

$$a(u, v-u) \geq (f, v-f)_\Omega \quad \forall v \in K, \tag{1}$$

where

$$K = \{v \in H_0^1(\Omega) | v \geq \phi a.e in \Omega.$$

Here K is a closed and convex admissible set of functions and

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v$$

and

$$(u,v)_\Omega = \int_\Omega uv.$$

Introduce the functional

$$E(u) = \frac{1}{2}a(u,u) - (f,u)_\Omega.$$

We can see that (1) is equivalent to finding

$$u = \text{argmin}_{v\in K} E(v). \tag{2}$$

To see this, consider the Gateaux derivative of E in the direction of v. We can see that

$$\begin{aligned}
&\lim_{t\to 0} \frac{E(u+t(v-u)) - E(u)}{t} && (3)\\
&= \lim_{t\to 0} \frac{1/2a(u+t(v-u), u+t(v-u)) - (f, u+t(v-u)) - 1/2a(u,u) + (f,u)}{t} && (4)\\
&= \lim_{t\to 0} \Bigg\{ 1/2\Bigg[a(u,u) + ta(v-u,u) + ta(v-u,u) + t^2 a(v-u, v-u)\Bigg] && (5)\\
&\quad - (f,u) - t(f, v-u) - 1/2a(u,u) + (f,u)\Bigg\}/t && (6)\\
&= \lim_{t\to 0} 1/2\Bigg(a(v-u,u) + a(v-u,u) + ta(v-u,v-u)\Bigg) - (f, v-u) && (7)\\
&= a(u, v-u) - (f, v-u). && (8)
\end{aligned}$$

Thus we see that requiring the directional derivative of E at u to be positive in every possible direction is equivalent to requiring (1).

It is not difficult to check that if $u \in H^2$ then u will satisfy

$$-\Delta u \geq f,\ u \geq \phi,\ (-\Delta u - f)(u - \phi) = 0, \text{a.e. in } \Omega. \tag{9}$$

The natural thing to do for a finite element method is to choose $K_h \subset K$, and define the numerical solution u_h to be

$$u_h = \text{argmin}_{v_h \in K_h} E(v_h).$$

For the linear finite element method we choose a mesh $\mathcal{T}$ and define

$$K_h^1 = \{v \in H_0^1(\Omega) | v(x_i) \geq \phi(x_i) \text{ at every node } x_i \text{ of } \mathcal{T} \text{ and } v|_T \in P_1(T) \forall T \in \mathcal{T}\}.$$

Suppose that $u_h, v_h \in K_h \subset K$ and that K has basis $\Phi = \{\phi_i\}_{i=1}^N$. We have in mind further that Φ is a typical finite element basis. That is, each ϕ_i is supported on only a few triangles T of the mesh $\mathcal{T}$ and there exists a set of nodes $X = \{x_i\}_{i=1}^N$ such that $\phi_i(x_j) = \delta_{ij}$.

Given a triangle $T \in \mathcal{T}$ let $X(T)$ be the nodes that lie in T.

Now we may express

$$\int_\Omega \nabla u_h \cdot \nabla v_h = \sum_{T \in \mathcal{T}} \int_T \nabla u_h \cdot \nabla v_h = \sum_{T \in \mathcal{T}} \int_T \nabla \left(\sum_{i=1}^N u_{h,i} \phi_i \right) \cdot \left(\nabla \sum_{j=1}^N v_{h,j} \phi_j \right) \tag{10}$$

$$= \sum_{T \in \mathcal{T}} \int_T \sum_{i=1}^N u_{h,i} \nabla \phi_i \cdot \sum_{j=1}^N v_{h,j} \nabla \phi_j. \tag{11}$$

Now assuming $\nabla \phi_i = 0$ if $i \notin X(T)$ we have

$$= \sum_{T \in \mathcal{T}} \int_T \sum_{i,j \in X(T)} u_{h,i} v_{h,j} \nabla \phi_i \cdot \nabla \phi_j = \sum_{T \in \mathcal{T}} \sum_{i,j \in X(T)} u_{h,i} v_{h_j} \int_T \nabla \phi_i \cdot \nabla \phi_j. \tag{12}$$

Assume all the normal reference element stuff. Then

$$\begin{aligned} \partial_p \left(\phi_i(x) \right) &= \partial_p \left(\hat{\phi}_i \circ T^{k^{-1}}(x) \right) \\ &= \partial_1 \hat{\phi}_i (T^{k^{-1}}(x)) \partial_p T_1^{k^{-1}}(x) + \partial_2 \hat{\phi}_2 (T^{k^{-1}}(x)) \partial_p T_2^{k^{-1}}(x). \end{aligned}$$

Now, for any function v defined on T, define

$$\vec{v} = [v(x_{l_1}), v(x_{l_2}) \ldots, v(x_{l_M})]$$

where M is the number of nodes lying in T (corresponding to the degree of the FEM method). Then

$$\partial_p v(x) = \sum_{i=1}^M v_i \partial_p \phi_i(x) = D_p(x) \vec{v}$$

where

$$\begin{aligned} D_p(x) &= [\partial_p \phi_1(x), \partial_p \phi_2(x), \ldots, \partial_p \phi_M(x)] \\ &= \left(\partial_p T^{k^{-1}} \right)^T [\nabla \hat{\phi}_1(x) | \nabla \hat{\phi}_2(x) | \ldots | \nabla \hat{\phi}_M(x)] \end{aligned}$$