We consider the so-called obstacle problem of finding $u \in K$ such that for all $v \in K$,

$$a(u, v-u) \geq (f, v-u)_\Omega \tag{1}$$

where

$$K = \{v \in H_0^1 \,|\, v \geq \phi \text{ a.e. in } \Omega\}, \tag{2}$$

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v, \tag{3}$$

and

$$(f,v)_\Omega = \int_\Omega fv. \tag{4}$$

The adaptive numerical method described above is implemented in four steps:

1. Define an initial triangular mesh, $\mathcal{T}_1$, of Ω. Use the standard finite element method with linear basis to find an approximate solution, u_h^1, to (1).That is, $u_h^1 \in K^1$ satisfies

$$a(u_h^1, v_h - u_h^1) \geq (f, v_h - u_h^1)_\Omega \, \forall v_h \in K^1 \tag{5}$$

   where

$$\begin{aligned} K^1 = \{&v_h \in H_0^1(\Omega) \,|\, v_h(v_i) \geq \phi(v_i) \text{ at all vertices } v_i \text{ of } \mathcal{T}_1 \\ &\text{and } \forall K \in T_1,\, v_h|_K \in P_1(K)\}. \end{aligned}$$

2. Find the free boundary elements of $\mathcal{T}_1$. An element $K \in \mathcal{T}_1$ is said to be a free boundary element if there exist vertices v_1 and v_2 of K such that $u_h^1(v_1) = \phi(v_1)$ and $u_h^1(v_2) > \phi(v_2)$. Let the set of free boundary elements of $\mathcal{T}_1$ be called $\mathcal{T}_1^F$.

3. Create a new mesh of Ω, $\mathcal{T}_2$, by refining all the elements in $\mathcal{T}_1^F$. Denote the refinement of $\mathcal{T}_1^F$ as $\mathcal{T}_2^F$, let h_1^F be the maximal edge length in $\mathcal{T}_1^F$, let h_2^F be the maximal edge length in $\mathcal{T}_2^F$. We choose a *mesh refinement parameter* C, and ensure that we refine T_1^F enough times to respect the *mesh refinement criteria*

$$h_2^F \leq C \left(h_1^F\right)^{4/3}. \tag{6}$$

   In particular, if we use the simple red-green refinement approach for a two-dimensional mesh, we refine every element of $\mathcal{T}_1^F$ $\left\lceil \ln\left(C \left(h_1^F\right)^{1/3}\right)\right\rceil$ times.

   The elements $K \notin T_1^F$ are left unrefined, except in the case that they must be refined in order to avoid hanging nodes.

4. Use finite element method with quadratic basis on the mesh $\mathcal{T}_2$ to find a new approximate solution u_h^2 to (1). $u_h^2 \in K^2$ satisfies

$$a(u_h^2, v_h - u_h^2) \geq (f, v_h - u_h^2)_\Omega \, \forall v_h \in K^2 \tag{7}$$

where

$$K^2 = \{v_h \in H_0^1(\Omega) \,|\, v_h(v_i) \geq \phi(v_i) \text{ at all nodes } v_i \text{ of } \mathcal{T}_2 \\ \text{and } \forall K \in T_2,\, v_h|_K \in P_2(K)\}.$$

In this context a node of $\mathcal{T}_2$ is defined as any point that is a vertex of one of the triangles of $\mathcal{T}_2$ or is a midpoint of two such vertices.

1 Numerical Results

In this section we test the adaptive method on two test cases, comparing the method with the standard linear and quadratic finite element method, as well as an adaptive method in which we simply refine the free boundary mesh $\mathcal{T}_1^F$ one time rather than respecting the mesh refinement criteria (6). We shall refer to the adaptive method which does not respect (6) as the *non-respecting* adaptive method, and the adaptive method which does respect (6) as the adaptive method.

1.1 Example 1

Let the domain $\Omega = [-1.5, 1.5]^2$ and define

$$\phi(x) = 0 \tag{8}$$

$$f(x) = -2 \tag{9}$$

$$u_0(r) = \frac{r^2}{2} - \ln(r) - 1/2. \tag{10}$$

where $r = \sqrt{x_1^2 + x_2^2}$.

With these choices of parameters the exact solution to (1)–(2) is

$$u(r) = \begin{cases} 0 & \text{if } r < 1 \\ \frac{r^2}{2} - \ln(r) - 1/2 & \text{otherwise.} \end{cases} \tag{11}$$

In the numerical examples we choose the mesh refinement parameter $C = 10$.

In Figures 1–3 we compare the accuracy of each of the four numerical methods, as well as the time required to compute each solution. We can see that the adaptive method outperforms the other methods in both regards. As expected, the non-respecting adaptive method begins to lose accuracy compared the adaptive method as the mesh size becomes small.

We report the same results in tabular form in Tables 1 – 4. In addition, for each numerical method we approximate the order of convergence by comparing

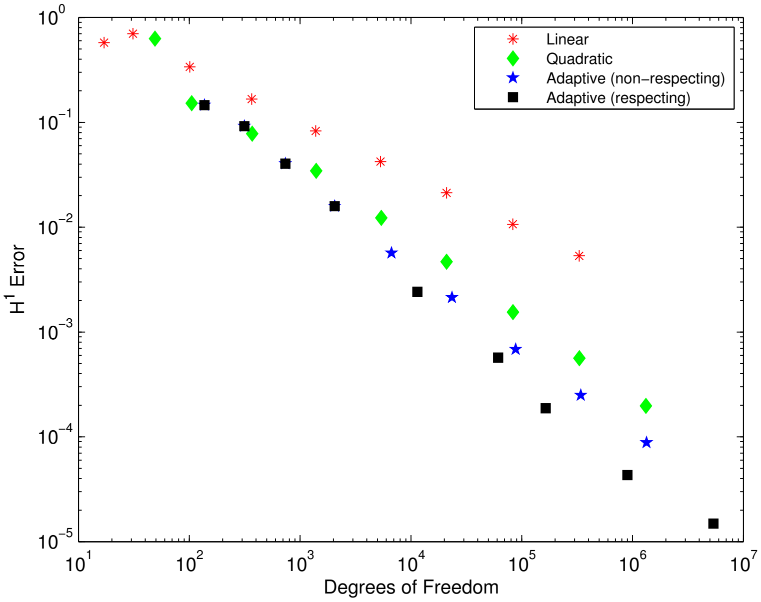

Figure 1: Degrees of freedom vs. H^1 error for Example 1.

the error achieved by the i^{th} approximation to the error achieved by the first approximation. In particular, let N_i denote the number of degrees of freedom corresponding to the i^{th} approximation. Let h_i denote the largest edge length from the mesh $\mathcal{T}_i^1$, and define E_i as the $H^1(\Omega)$ error achieved by approximation i. We estimate the order of convergence in terms of N for the method by $p \approx \frac{\log(E_i/E_1)}{\log(N_i/N_1)}$. We estimate the order of convergence in terms of h via $p \approx \frac{\log(E_i/E_1)}{\log(h_i/h_1)}$. As a second estimate of convergence order in terms of h we find the least-squares solution to the system of equations $\ln E_i = p \ln h_i + c$ and report the value of p, a second estimate of the convergence order in terms of N is computed similarly. In Tables 1 – 4 we can see each method tending toward its expected order of convergence.

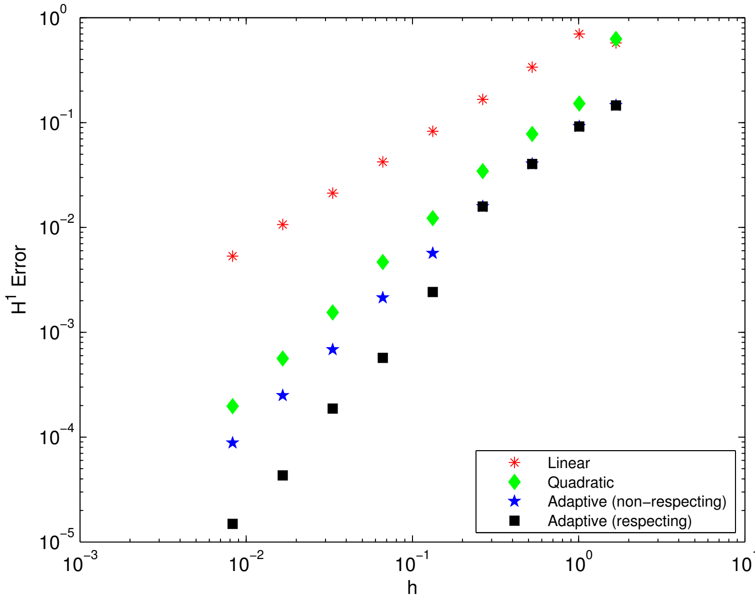

Figure 2: Mesh size vs. H^1 error for Example 1.

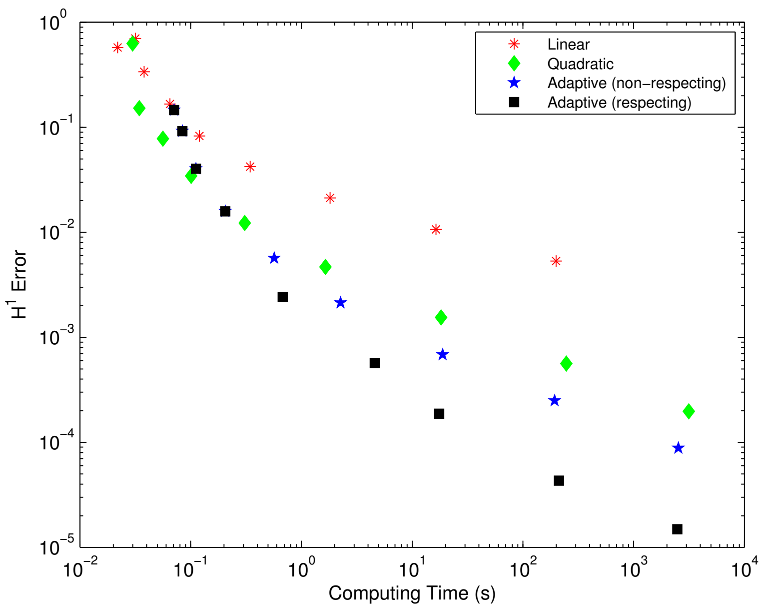

Figure 3: Solution time (s) vs. H^1 error for Example 1.

N	h	H^1 Error	N Conv. Rate	h Conv. Rate	Comp. Time (s)
17	1.677e+00	1.207e+00	-	-	0
31	1.008e+00	7.486e-01	-7.945e-01	9.372e-01	9.372e-01
101	5.252e-01	3.527e-01	-6.902e-01	1.059e+00	1.059e+00
366	2.650e-01	1.800e-01	-6.198e-01	1.031e+00	1.031e+00
1385	1.325e-01	9.085e-02	-5.878e-01	1.019e+00	1.019e+00
5308	6.629e-02	4.588e-02	-5.692e-01	1.012e+00	1.012e+00
20958	3.315e-02	2.292e-02	-5.569e-01	1.010e+00	1.010e+00
83087	1.657e-02	1.146e-02	-5.482e-01	1.009e+00	1.009e+00
330653	8.286e-03	5.749e-03	-5.414e-01	1.007e+00	1.007e+00
Best Fit			-5.308e-01	1.007e+00	

Table 1: Results for linear method in Example 1.

N	h	H^1 Error	N Conv. Rate	h Conv. Rate	Comp. Time (s)
49	1.677e+00	5.900e-01	-	-	0
105	1.008e+00	1.390e-01	-1.897e+00	2.839e+00	2.839e+00
369	5.252e-01	7.419e-02	-1.027e+00	1.786e+00	1.786e+00
1397	2.650e-01	2.064e-02	-1.001e+00	1.817e+00	1.817e+00
5409	1.325e-01	7.199e-03	-9.367e-01	1.736e+00	1.736e+00
20973	6.629e-02	2.639e-03	-8.928e-01	1.674e+00	1.674e+00
83317	3.315e-02	9.073e-04	-8.708e-01	1.651e+00	1.651e+00
331319	1.657e-02	3.206e-04	-8.524e-01	1.628e+00	1.628e+00
1320561	8.286e-03	1.122e-04	-8.398e-01	1.613e+00	1.613e+00
Best Fit			-8.037e-01	1.563e+00	

Table 2: Results for quadratic method in Example 1.

N	h	H^1 Error	N Conv. Rate	h Conv. Rate	Comp. Time (s)
137	1.677e+00	2.050e-01	-	-	0
320	1.008e+00	6.373e-02	-1.377e+00	2.295e+00	2.295e+00
815	5.252e-01	2.825e-02	-1.111e+00	1.707e+00	1.707e+00
2413	2.650e-01	8.183e-03	-1.123e+00	1.746e+00	1.746e+00
7185	1.325e-01	3.106e-03	-1.058e+00	1.651e+00	1.651e+00
24593	6.629e-02	1.190e-03	-9.921e-01	1.594e+00	1.594e+00
90277	3.315e-02	4.474e-04	-9.441e-01	1.562e+00	1.562e+00
345815	1.657e-02	1.512e-04	-9.207e-01	1.562e+00	1.562e+00
1349265	8.286e-03	5.192e-05	-9.006e-01	1.559e+00	1.559e+00
Best Fit			-8.796e-01	1.517e+00	

Table 3: Results for adaptive (non-respecting) method in Example 1.

N	h	H^1 Error	N Conv. Rate	h Conv. Rate	Comp. Time (s)
137	1.677e+00	2.050e-01	-	-	0
320	1.008e+00	6.373e-02	-1.377e+00	2.295e+00	2.295e+00
815	5.252e-01	2.825e-02	-1.111e+00	1.707e+00	1.707e+00
2413	2.650e-01	8.183e-03	-1.123e+00	1.746e+00	1.746e+00
13725	1.325e-01	1.481e-03	-1.070e+00	1.943e+00	1.943e+00
80125	6.629e-02	3.434e-04	-1.003e+00	1.979e+00	1.979e+00
196377	3.315e-02	1.099e-04	-1.036e+00	1.919e+00	1.919e+00
1147579	1.657e-02	2.388e-05	-1.003e+00	1.962e+00	1.962e+00
Best Fit			-9.897e-01	1.958e+00	

Table 4: Results for adaptive (respecting) method in Example 1.

1.2 Example 2

As in Example 1, let the domain be $\Omega = [-1.5, 1.5]^2$. Define the parameters

$$\phi(x) = 0, \tag{12}$$

$$f(r) = \begin{cases} \sqrt{1-r^2} & \text{if } r < 1 \\ 0 & \text{otherwise} \end{cases}, \tag{13}$$

$$r^* = 0.6979651482, \tag{14}$$

$$u_0(r) = -\left(r^*\right)^2 \frac{\ln(r/2)}{\sqrt{1-\left(r^*\right)^2}}. \tag{15}$$

where $r = \sqrt{x_1^2 + x_2^2}$.

With these choices of parameters the exact solution to (1)–(2) is

$$u(r) = \begin{cases} \sqrt{1-r^2} & \text{if } r < r^* \\ -\left(r^*\right)^2 \frac{\ln(r/2)}{\sqrt{1-(r^*)^2}} & \text{otherwise.} \end{cases} \tag{16}$$

Again we choose the mesh refinement parameter $C = 10$.

Figures 4 and 6 compare the error and solution time in each of the methods. Once again the adaptive method outperforms the other methods.

Tables 5– 8 provide the same data as the figures in tabular form, and attempt to estimate the convergence rate of each method. Again, the adaptive method converges at a rate near N^{-1}, as predicted.

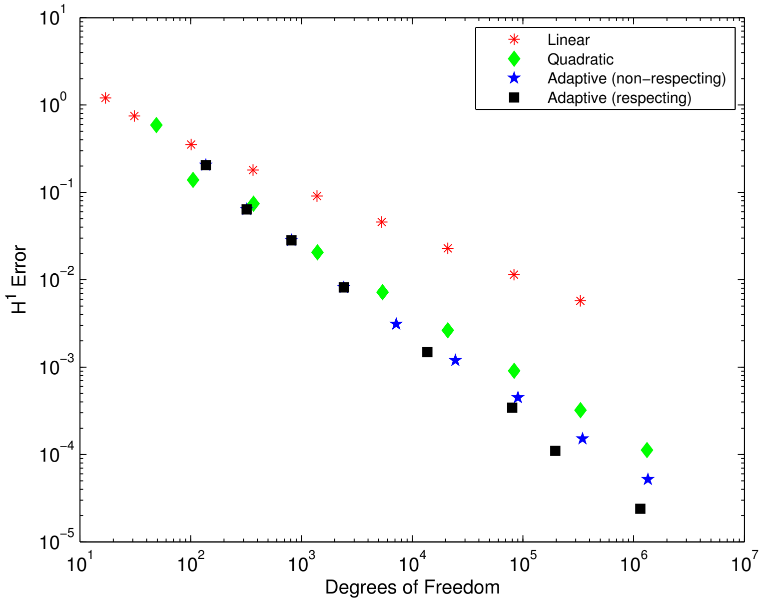

Figure 4: Degrees of freedom vs. H^1 error for Example 2.

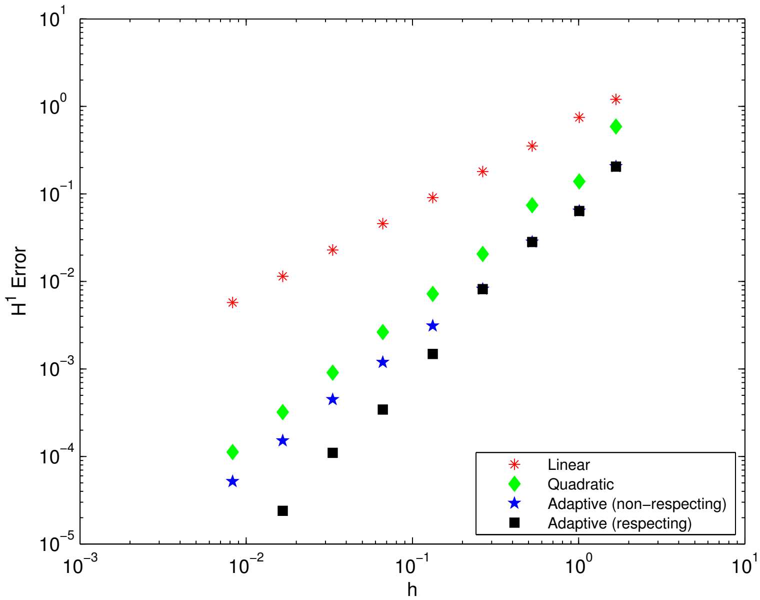

Figure 5: Mesh size vs. H^1 error for Example 2.

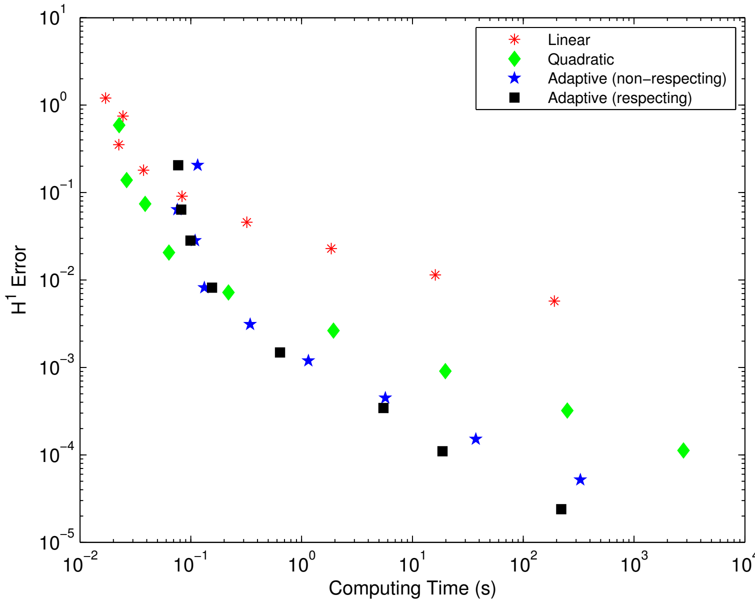

Figure 6: Solution time (s) vs. H^1 error for Example 2.

N	h	H^1 Error	N Conv. Rate	h Conv. Rate	Comp. Time (s)
17	1.677e+00	5.765e-01	-	-	0
31	1.008e+00	7.005e-01	3.244e-01	-3.827e-01	-3.827e-01
101	5.252e-01	3.386e-01	-2.985e-01	4.582e-01	4.582e-01
366	2.650e-01	1.666e-01	-4.043e-01	6.727e-01	6.727e-01
1385	1.325e-01	8.278e-02	-4.410e-01	7.646e-01	7.646e-01
5308	6.629e-02	4.216e-02	-4.553e-01	8.095e-01	8.095e-01
20958	3.315e-02	2.124e-02	-4.638e-01	8.412e-01	8.412e-01
83087	1.657e-02	1.065e-02	-4.699e-01	8.646e-01	8.646e-01
330653	8.286e-03	5.327e-03	-4.743e-01	8.821e-01	8.821e-01
Best Fit			-5.014e-01	9.470e-01	

Table 5: Results for linear method in Example 2

N	h	H^1 Error	N Conv. Rate	h Conv. Rate	Comp. Time (s)
49	1.677e+00	6.289e-01	-	-	0
105	1.008e+00	1.520e-01	-1.864e+00	2.789e+00	2.789e+00
369	5.252e-01	7.797e-02	-1.034e+00	1.798e+00	1.798e+00
1397	2.650e-01	3.447e-02	-8.668e-01	1.574e+00	1.574e+00
5409	1.325e-01	1.225e-02	-8.372e-01	1.552e+00	1.552e+00
20973	6.629e-02	4.679e-03	-8.088e-01	1.517e+00	1.517e+00
83317	3.315e-02	1.549e-03	-8.075e-01	1.531e+00	1.531e+00
331319	1.657e-02	5.615e-04	-7.961e-01	1.521e+00	1.521e+00
1320561	8.286e-03	1.973e-04	-7.908e-01	1.519e+00	1.519e+00
Best Fit			-7.477e-01	1.454e+00	

Table 6: Results for quadratic method in Example 2.

N	h	H^1 Error	N Conv. Rate	h Conv. Rate	Comp. Time (s)
137	1.677e+00	1.459e-01	-	-	0
314	1.008e+00	9.188e-02	-5.573e-01	9.076e-01	9.076e-01
737	5.252e-01	4.031e-02	-7.643e-01	1.108e+00	1.108e+00
2053	2.650e-01	1.582e-02	-8.207e-01	1.204e+00	1.204e+00
6693	1.325e-01	5.669e-03	-8.351e-01	1.280e+00	1.280e+00
23485	6.629e-02	2.136e-03	-8.211e-01	1.307e+00	1.307e+00
88285	3.315e-02	6.834e-04	-8.292e-01	1.367e+00	1.367e+00
341451	1.657e-02	2.500e-04	-8.144e-01	1.379e+00	1.379e+00
1340053	8.286e-03	8.805e-05	-8.067e-01	1.396e+00	1.396e+00
Best Fit			-8.239e-01	1.423e+00	

Table 7: Results for adaptive (non-respecting) method in Example 2.

N	h	H^1 Error	N Conv. Rate	h Conv. Rate	Comp. Time (s)
137	1.677e+00	1.459e-01	-	-	0
314	1.008e+00	9.188e-02	-5.573e-01	9.076e-01	9.076e-01
737	5.252e-01	4.031e-02	-7.643e-01	1.108e+00	1.108e+00
2053	2.650e-01	1.582e-02	-8.207e-01	1.204e+00	1.204e+00
11429	1.325e-01	2.423e-03	-9.263e-01	1.615e+00	1.615e+00
61305	6.629e-02	5.707e-04	-9.083e-01	1.716e+00	1.716e+00
164273	3.315e-02	1.874e-04	-9.391e-01	1.697e+00	1.697e+00
897539	1.657e-02	4.314e-05	-9.247e-01	1.760e+00	1.760e+00
5366497	8.286e-03	1.488e-05	-8.690e-01	1.731e+00	1.731e+00
Best Fit			-9.172e-01	1.828e+00	

Table 8: Results for adaptive (respecting) method in Example 2.