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Source: `https://github.com/thomabir/astrophysics-ii`

Introduction | 1

1.1 Extragalactic sky

1.2 Cosmological model

1.3 Course overview

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1.5 Basic concepts

2 Cosmological Background

2.1 Cosmological Principle

The Cosmological principle states that the universe is homogeneous and isotropic on sufficiently large scales. This is a generalization of the Copernican principle, according to which there is no special place and no special direction in the universe.

The Cosmological principle is only valid for distances larger than a few hundred Mpc. Local perturbations about this uniform background will be described later.

2.2 Elements of General Relativity

Einstein field equations:

$$G_{\mu\nu} = \frac{8\pi G}{\rho} T_{\mu\nu}$$

Ideal fluid:

$$T_{\mu\nu} = \mathrm{diag}(\rho c^2, p, p, p)$$

2.3 FRW metric

The Friedmann-Robertson-Walker (FRW) metric is the metric of a homogeneous and isotropic universe:

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a(t)^2(\mathrm{d}\chi^2 + r(\chi)^2\,\mathrm{d}\Omega^2)$$

- χ: comoving radius
- $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$: solid angle element
- $a(t)$: scale factor
- $r(\chi) = f_K(\chi) = \begin{cases} \sin\chi & \text{closed case, positive curvature} \\ \chi & \text{flat case} \\ \sinh\chi & \text{open case, negative curvature} \end{cases}$

Hubble parameter

- Hubble parameter: $H := \dot{a}/a$
- today's value gets a subscript zero: H_0
- Because it is hard to measure H accurately, we write it as

  $$H_0 = 100h\frac{\mathrm{km}}{\mathrm{s\,Mpc}},$$

  where $h \approx 0.7$ is the dimensionless Hubble parameter.
- $H_0^{-1} \approx 10\,\mathrm{Gyr}$ is about the age of the universe
- $cH_0^{-1} \approx 4\,\mathrm{Gpc}$ is about the size of the observable universe

2.4 Friedmann equation

The Friedmann equations are derived by plugging the FRW metric into Einstein's equations:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2}$$
$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)$$

The critical density is defined as

$$\rho_{\text{crit}}(t) = \frac{3H(t)^2}{8\pi G}$$

Today, the critical density is about five hydrogen atoms per cubic metre, or one galaxy per Mpc^3.

Density parameters:

- The subscript i describes one component of the universe (i = radiation, dark matter, matter ...)
- density parameter: $\Omega_i(t) = \rho_i(t)/\rho_{\text{crit}}(t)$
- total energy density: $\rho(t) = \sum_i \rho_i(t)$
- total density parameter: $\Omega(t) = \rho(t)/\rho_{\text{crit}}(t)$
- curvature density parameter: $\Omega_{K,0} = 1 - \Omega_0 = -Kc^2/H_0^2 a_0^2$

With these definitions, the (first) Friedmann equation can be rewritten as

$$\frac{H}{H_0} = \sqrt{\frac{\rho}{\rho_{\text{crit},0}} + \Omega_{K,0}\left(\frac{a_0}{a}\right)^2}$$

2.5 Solutions

To solve the Friedmann equation, $\rho(t)$ or $\rho(a)$ need to be known. It can be calculated as

$$\rho = n\epsilon$$

where n is the particle number per unit volume and ϵ the energy per particle

- Relativistic matter. ϵ is constant with a, while $n \propto a^{-3}$. Thus $\rho \propto a^{-3}$.
- Radiation. $\epsilon = h\nu = hc/\lambda \propto a^{-1}$. Thus $\rho \propto a^{-4}$.
- Vacuum energy is constant in a

There is a generalization for general fluids:

- equation of state: $p = wpc^2$
- density: $\rho \propto a^{-3(1+w)}$
- $w = \begin{cases} 0 & \text{matter} \\ 1/3 & \text{radiation} \\ -1 & \text{vacuum energy} \end{cases}$

The results can be plugged into the Friedmann equation:

$$\begin{aligned}\frac{H}{H_0} &= \sqrt{\frac{\rho}{\rho_{\text{crit},0}} + \Omega_{K,0}\left(\frac{a_0}{a}\right)^2} \\ &= \sqrt{\Omega_{m,0}\left(\frac{a_0}{a}\right)^3 + \Omega_{r,0}\left(\frac{a_0}{a}\right)^4 + \Omega_{\Lambda,0} + \Omega_{K,0}\left(\frac{a_0}{a}\right)^2}\end{aligned}$$

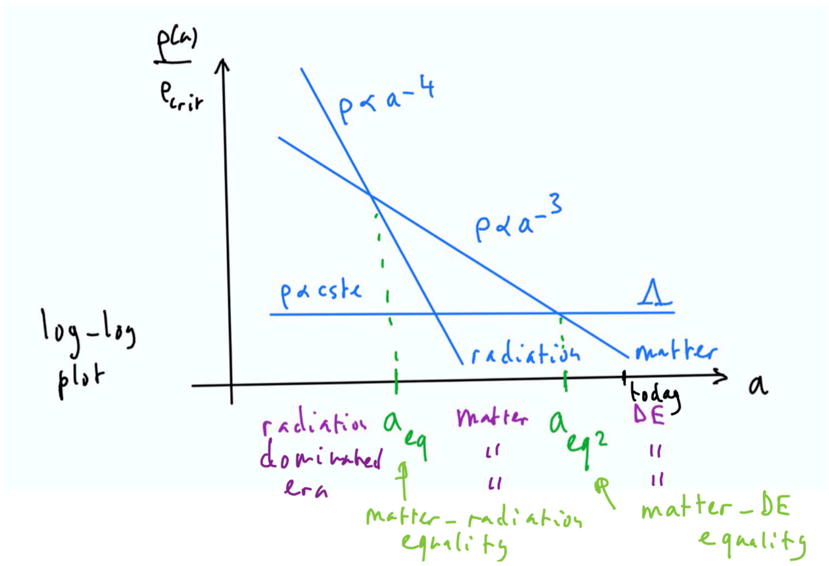


Figure 2.1: Domination of different components at different times

This is a differential equation with $\Omega_{i,0}$ as parameters.

The standard cosmological model:

- $\Omega_{m,0} \approx 0.3$
- $\Omega_{r,0} \approx 10^{-5}$
- $\Omega_{\Lambda,0} \approx 0.7$
- $\Omega_{K,0} \approx 0$
- $\Omega_0 \approx 1$
- $h \approx 0.7$

At different times, the universe is dominated by different components. Approximations:

- Matter dominated:

$$\frac{H}{H_0} = \sqrt{\Omega_{m,0}\left(\frac{a_0}{a}\right)^3} \implies a \propto t^{2/3}$$

- Radiation dominated:

$$\frac{H}{H_0} = \sqrt{\Omega_{r,0}\left(\frac{a_0}{a}\right)^4} \implies a \propto t^{1/2}$$

- Λ dominated:

$$\frac{H}{H_0} \propto \text{constant} \implies a \propto e^{Ht}$$

- General fluid ($w \neq -1$):

$$\rho \propto a^{-3(1+w)} \implies a \propto t^{\frac{2}{3(1+w)}}$$

2.6 Distances and times

2.6.1 Angular distance & Luminosity distance

The comoving distance χ and the proper distance $a\chi$ to a source are not directly observable. However, the angular size θ and the flux F of an object can be measured directly.

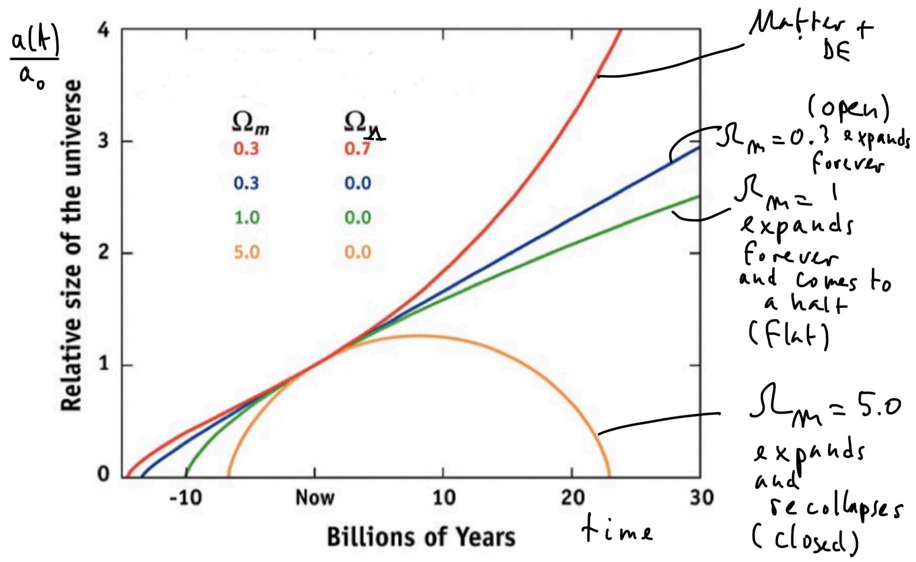


Figure 2.2: Evolution

Intrinsic properties of the source:

- its size D
- its luminosity L

Properties of space-time:

- the comoving distance to the source χ
- the proper distance to the source $a(t)\chi$

Measurable quantities for an observer:

- the angular size θ
- the flux F

In Euclidean space, the following relations hold:

$$\theta = \frac{D}{d} \qquad\qquad F = \frac{L}{4\pi d^2}$$

where d is the distance to the source. In FRW-space, we define the following:

- the angular-diameter distance d_A satisfies $\theta = D/d_A$. One can show $d_A = ar(\chi)$
- the luminosity distance d_L satisfies $F = L/4\pi d_L^2$. One can show that $d_L = r(\chi)/a$

2.6.2 Comoving radius

We measured the redshift of a photon that has travelled to us on a radial trajectory. How far away (in comoving distance) is the source?

$$\begin{aligned}
0 &= \mathrm{d}s^2 && \text{photon} \\
&= c\,\mathrm{d}t^2 - a(t)^2[\mathrm{d}\chi^2 + r(\chi)^2\,\mathrm{d}\Omega^2] && \text{FRW metric} \\
\implies c\,\mathrm{d}t &= a(t)\,\mathrm{d}\chi && \mathrm{d}\Omega^2 = 0 \text{ on a radial trajectory} \\
\implies \mathrm{d}\chi &= \frac{c\,\mathrm{d}t}{a(t)} && \\
&= \frac{c\,\mathrm{d}a}{a^2 H(a)} && H = \frac{\dot a}{a}, \text{ so } \mathrm{d}t = \frac{\mathrm{d}a}{aH(a)} \\
\implies \chi(a) &= c\int_a^{a_0} \frac{\mathrm{d}a'}{a'^2 H(a')} && \chi(a_0) = 0
\end{aligned}$$

$H(a)$ has to be obtained from the Friedmann equations. As a result, we will get $\chi(a, a_0)$. We can use $a/a_0 = 1/(1 + z)$ to get $\chi(z, a_0)$. Since a_0 can be defined arbitrarily (for example, $a_0 = 1$), we get $\chi(z)$.

2.6.3 Comoving Horizon

Suppose a (hypothetical, non-interacting) photon was emitted at the Big Bang. How far (in comoving distance) could it have travelled until now? This comoving distance is called the comoving horizon. We plug into the previous equation, with $a = 0$ at the start:

$$\chi(a) = c \int_0^{a_0} \frac{\mathrm{d}a'}{a'^2 H(a')}$$

2.6.4 Age of the Universe

How old is the universe?

$$\begin{aligned} t_0 &= \int_0^{t_0} \mathrm{d}t \\ &= \int_0^{a_0} \frac{\mathrm{d}a}{aH(a)} && H(a) = \dot{a}/a \end{aligned}$$

$H(a)$ is again found from the Friedmann equation. With the standard cosmology, $t_0 \approx$ 14 Gyr.

2.7 Thermal history

According to the Big Bang paradigm, the universe was once hot and dense, and now it expands and cools down. Today, it is far from thermal equilibrium, but it must have been in thermal equilibrium at some point in the past if it continuously expands.

A system is in thermal equilibrium if $\Gamma \gg H$

- Γ = interactions/time is the interaction rate
- $H = \dot{a}/a$ is the Hubble constant

Similarly, a system is in thermal equilibrium if $\tau_\Gamma \ll \tau_H$

- $\tau_\Gamma = 1/\Gamma$ is the characteristic timescale of interactions
- $\tau_H = 1/H$ is the characteristic timescale of expansion

We already know about H. Γ is defined as

$$\Gamma = n v \sigma$$

- n number density, particles/volume
- v velocity of particles
- σ scattering cross-section, has units of area.

At early times, $\Gamma \gg H$. Particles are in thermal equilibrium with the plasma and coupled to photons. This scenario will be treated in section 2.8.1.

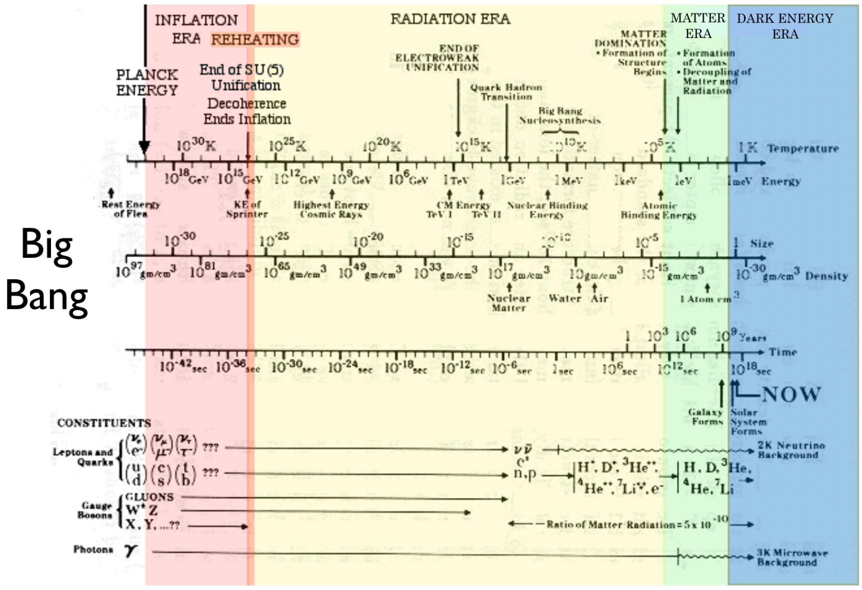
At later times $\Gamma \ll H$. Particles are not in thermal equilibrium and are decoupled from photons. See section 2.8.2

The decoupling or "freeze out" happens when $\Gamma \approx H$. This transition is described by the Boltzmann equation in section 2.8.3.

In table 2.1 and fig. 2.2, an overview of the thermal history of the universe is given.

Event	time	redshift	energy temp.
Inflation A phase of extremely rapid exponential expansion, caused by a phase transitions where the inflaton field emerged. Inflation explains properties of the universe which are difficult to account for without.	?	?	?
Baryogenesis Baryons (protons, neutrons) are formed from quarks. Weirdly, there are way more baryons formed than antibaryons. This is the matter-antimatter asymmetry.	?	?	?
QCD phase transition The universe has cooled sufficiently such that hadrons (baryons and mesons) can form.	10^{-5} s	10^{12}	200 MeV $3 \cdot 10^{12}$ K
Pions annihilate and decay, the only hadrons left are nucleons (protons and neutrons).	10^{-4} s		50 MeV 10^{12} K
Dark Matter freeze-out Dark Matter interacts very weakly with ordinary matter, so it decouples early on.	?	?	?
Electron-positron annihilation Electrons and positrons annihilate through $e^+ + e^- \to 2\gamma$. Since the number of charged particles decreases, neutrinos decouple.	4 s	$2 \cdot 10^9$	0.3 MeV $5 \cdot 10^9$ K
Big Bang nucleosynthesis Light nuclei such as D and He get synthesized. They are still ionized.	3 min	$4 \cdot 10^8$	0.08 MeV 10^9 K
Matter-radiation equality	$6 \cdot 10^4$ yr	3400	0.75 eV 8700 K
Recombination Formation of neutral atoms through $e^- + p^+ \to H + \gamma$	$2 \cdot 10^5$ yr	1200	0.34 eV 4000 K
Surface of last scattering The number density of charged particles has decreased enough for photons to decouple. These photons form the CMB.			
Reionization Stars form and re-ionize hydrogen.	$2 \cdot 10^8$ yr	20	4 meV 50 K
Dark Energy - Matter equality	9 Gyr	0.4	0.33 meV 3.8 K
Today	13.8 Gyr	0	0.24 eV 2.7 K

Table 2.1: Thermal history of the universe



Adapted from Kolb & Turner 1990

Figure 2.3: History of the Big Bang

2.8 Thermodynamics

To describe the evolution of the universe quantitatively, a few definitions are required:

- The probability that a particle is in a volume $\mathrm{d}^3x\,\mathrm{d}^3p$ at time t is given by the **(phase space) distribution function**:

  $$f(\mathbf{x},\mathbf{p},t)\,\mathrm{d}^3x\,\mathrm{d}^3p$$

  In a homogeneous and isotropic universe, we have $f(\mathbf{x},\mathbf{p},t) = f(p,t)$.
- The number of particles per unit volume is given by the **number density**:

  $$n(t) = 4\pi \int f(p,t)p^2\,\mathrm{d}p$$

- The energy per unit volume is given by the **energy density**:

  $$\rho(t) = 4\pi \int E(p)f(p,t)p^2\,\mathrm{d}p\,,$$

  with $E(p) = \sqrt{p^2+m^2}$.
- The **pressure** is given by

  $$P(t) = 4\pi c^2 \int \frac{p^2}{3E(p)} f(p,t)p^2\,\mathrm{d}p$$

  Since we work in natural units, we drop the c.

2.8.1 At early times ($\Gamma > H$)

The distribution function of particles in thermodynamic equilibrium is the Bose-Einstein or the Fermi-Dirac distribution:

$$f_{\mathrm{eq}}(p,t) = \frac{g}{(2\pi)^3}\left[\exp\left(\frac{E(p)-\mu}{T} \pm 1\right)\right]^{-1}$$

- $+$ is for fermions and $-$ for bosons
- g is a spin degeneracy factor. Examples: $g_\nu = 1$, $g_\gamma = 2$, $g_{\text{quark}} = 6$
- μ is the chemical potential, which is the response of a thermodynamics system to a change of particle number. Usually, $\mu = 0$ for our purposes.
- T is the temperature of the universe, which is time dependent.

For *non-relativistic particles*, $T \ll m$ and $E \approx m + p^2/2m$. Plugging this in yields

$$n = g\left(\frac{mT}{2\pi}\right)^{3/2} \exp\left(\frac{p-m}{T}\right) \qquad \rho = nm \qquad P = nT$$

For *relativistic particles*, $T \gg m$ and $E \approx p$. For both fermions and bosons, this yields

$$n = gT^3 \qquad \rho = gT^4 \qquad P = \frac{\rho}{3}$$

In the case of a relativistic gas, $T \propto a^{-1}$, so $n \propto a^{-3}$ and $\rho \propto a^{-4}$. This is what we have already seen in section 2.4.

2.8.2 At late times ($\Gamma < H$)

The transition between non-equilibrium and equilibrium takes place at the decoupling time of freeze-out time t_f. At $t > t_f$, the transition function is

$$f(p,t) = f\left(p\frac{a(t)}{a(t_f)}, t_f\right)$$

The shape of the function is “frozen in” at the freeze-out time t_f.

For a relativistic particle,

$$f(p,t) = \frac{g}{(2\pi)^3}\left[\exp\left(\frac{pa(t)}{T_f a(t_f)} \pm 1\right)\right]^{-1},$$

which is the same as for an equilibrium particle, but with $T_f \coloneqq T_f a(t_f)/a(t)$.

2.8.3 Boltzmann equation

The Boltzmann equation

$$\frac{\mathrm{d}f_i}{\mathrm{d}t} = c_i[f_i]$$

describes the time evolution of the distribution function. $c_i[f_i]$ is the collision term.

f only depends on p and t, so we can write

$$\frac{\mathrm{d}f_i}{\mathrm{d}t} = \frac{\partial f_i}{\partial t} + \frac{\partial f_i}{\partial p}\frac{\partial p}{\partial t}$$

The last term can be simplified. Because $p = p_0 a^{-1}$, $\mathrm{d}p/\mathrm{d}t = -p_0 a^{-1}\dot{a} = -pH$. The Boltzmann equation then becomes

$$\begin{aligned}
\frac{\partial f_i}{\partial t} - pH(t)\frac{\partial f_i}{\partial p} &= c_i[f_i] \\
\implies \frac{\partial}{\partial t}\int \mathrm{d}^3p\, f_i - H(t)\int \mathrm{d}^3p\, p\frac{\partial f_i}{\partial p} &= \int \mathrm{d}^3p\, c_i[f_i] \\
\implies \frac{\mathrm{d}n_i}{\mathrm{d}t} + 3H(t)n_i &= \int \mathrm{d}^3p\, c_i[f_i]
\end{aligned}$$

In the last line, we used partial integration.[1] If the system is collisionless, $c_i[f_i] = 0$, and the solution of the Boltzmann equation is $n_i \propto a^{-3}$. The $3H(t)n_i$ is called the Hubble drag term.

We now look at reactions of the type $i + j \leftrightarrow a + b$. The collision term is then of the form

$$c_i[f_i] = \alpha(T)n_a n_b - \beta(T)n_i n_j$$

- $\alpha(T)$ is the production rate
- $\beta(T)$ is the destruction rate

To simplify the equation, we make a few assumptions:

- a and b are in equilibrium with a general plasma at temperature T
- $n_i = n_j$ (this is the case for antiparticles)
- radiation era: $a \propto t^{1/2} \propto T^{-1}$

The equation is then

$$\frac{\mathrm{d}n_i}{\mathrm{d}t} + 3H(t)n_i = \beta(T)(n_{i,\mathrm{eq}}^2 - n_i^2)$$

To analyse the equation, we define

- $x = m_i/T$ is used as a time variable
- $y_i = n_i/S$, where S is the entropy
- $\Gamma(x) = n_{i,\mathrm{eq}}(x)\beta(x)$

The equation is then

$$\frac{x}{y_{i,\mathrm{eq}}}\frac{\mathrm{d}y_i}{\mathrm{d}x} = -\frac{\Gamma(x)}{H(x)}\left[\left(\frac{y_i}{y_{i,\mathrm{eq}}}\right)^2 - 1\right]$$

Initial conditions: $x \ll 1$ at early times, so $y_i = y_{i,\mathrm{eq}}$.

No analytical solution is known. A set of numerical solutions is shown in fig. 2.4

2.9 Particle relics

There are two types of relics:

- Hot relics freeze out when the particles are still relativistic. Since $x = m_i/T$, this means $x_f \ll 1$
- Cold relics freeze out when the particles are already non-relativistic, with $x_f \gg 1$

2.9.1 Hot relics

Hot relics are still relativistic today, so their rest mass is $m_i \ll T_0 = 2.4 \cdot 10^{-4}$ eV. An example would be massless neutrinos.

The solution of the Boltzmann equation is

$$\Omega_{i,0}h^2 = \frac{g_{i,\mathrm{eff}}}{2}\left[\frac{g_{*s}(x_0)}{g_{*s}(x_f)}\right]^{4/3}\Omega_{\gamma,0}h^2$$

The g's are degeneracy factors and satisfy $g_{*s}(x_0) \leq g_{*s}(x_f)$, and the photon density is $\Omega_{\gamma,0}h^2 = 2.5 \cdot 10^{-5}$. It follows that $\Omega_{i,0}$ is very small, which means that hot relic particles contribute very little to today's energy density.

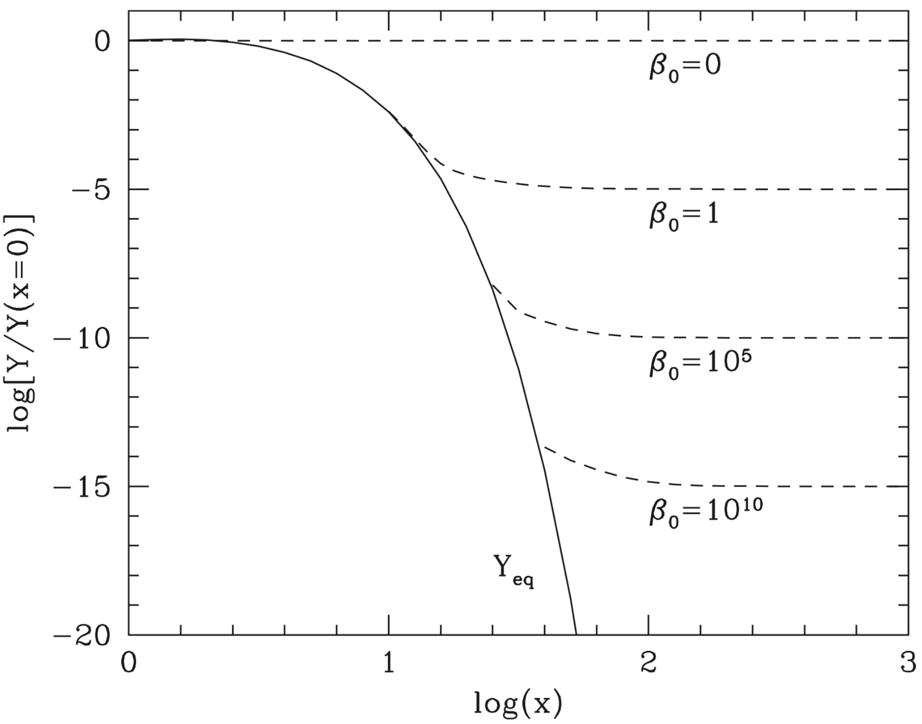


Figure 2.4: Solutions of the Boltzmann equation for different destruction rates β. We assume $\beta(T) = \beta_0$ is constant. The vertical axis is a proxy for abundance, and the horizontal axis for time. First, particles remain in equilibrium (solid line), then they decouple (dashed lines), leaving relic abundances. When β_0 is large, thermal equilibrium is maintained longer, so the relic abundance is lower.

2.9.2 WIMPs

Next we consider weakly interacting massive particles (WIMPs). Examples are massive neutrinos and stable, light supersymmetric particles. WIMPs can either be hot ($x_f \ll 1$) or cold ($x_f \gg 1$).

Hot WIMPs

The solution of the Boltzmann equation yields

$$\Omega_{i,0}h^2 = 7.64 \cdot 10^{-2} \left(\frac{g_{i,\mathrm{eff}}}{g_{*s}(x_f)} \right) \frac{m_i}{\mathrm{eV}}$$

Since we know that $\Omega_{i,0} < 1$, we can get a constraint $m_i < 100\,\mathrm{eV}$ on the mass of the hot WIMPs, which is possible for neutrinos. However, hot WIMPs are ruled out as dark matter by structure formation arguments.

Cold WIMPs

Boltzmann says

$$\Omega_{i,0}h^2 = \begin{cases} 1.8 \left(\frac{m_i}{\mathrm{GeV}}\right)^{-2} \left[1 + 0.17 \ln\left(\frac{m_i}{\mathrm{GeV}}\right)\right] & \text{if } m_i < 100\,\mathrm{GeV} \\ \left(\frac{m_i}{3\,\mathrm{TeV}}\right)^2 & \text{if } m_i > 100\,\mathrm{GeV} \end{cases}$$

To get $\Omega_{i,0} < 1$, we need m_i to be between 1.4 GeV and 3 TeV. Cold Wimps are good candidates for dark matter.

2.10 Primordial nucleosynthesis

This is the epoch where protons and neutrons first combined to nuclei (not atoms) heavier than hydrogen. Heavier atoms can also be synthesized in stars through nuclear reactions, which has to be differentiated in observations.

Initial conditions

Initially, the temperature is $T < 10^{13}$ K and the associated energy scale is $k_B T \approx$ 0.8 MeV. Protons and neutrons are in thermal equilibrium and interact weakly via the processes $p + e \leftrightarrow n + \nu_e$ and $n + \bar{e} \leftrightarrow p + \bar{\nu}_e$. Since the mass of the nucleons is around 940 MeV, they are already non-relativistic. The masses of neutrons and protons are slightly different, so their abundances after freeze out are different.

Nuclear reactions

Once the temperature drops below 1 MeV, which is the binding energy of a typical nucleus, the nuclei start forming. A host of nuclear reactions can occur then:

$$
\begin{aligned}
\mathrm{p} + \mathrm{n} &\rightleftharpoons \gamma + \mathrm{D} & \mathrm{D} + \mathrm{n} &\rightleftharpoons \gamma + {}^3\mathrm{H} \\
\mathrm{D} + \mathrm{D} &\rightleftharpoons \mathrm{p} + {}^3\mathrm{H} & \mathrm{D} + \mathrm{p} &\rightleftharpoons \gamma + {}^3\mathrm{He} \\
\mathrm{D} + \mathrm{D} &\rightleftharpoons \mathrm{n} + {}^3\mathrm{He} & {}^3\mathrm{H} + \mathrm{p} &\rightleftharpoons \mathrm{n} + {}^3\mathrm{He} \\
{}^3\mathrm{H} &\rightleftharpoons \mathrm{e} + \bar{\nu}_e + {}^3\mathrm{He} & {}^3\mathrm{H} + \mathrm{p} &\rightleftharpoons \gamma + {}^4\mathrm{He} \\
{}^3\mathrm{H} + \mathrm{D} &\rightleftharpoons \mathrm{n} + {}^4\mathrm{He} & {}^3\mathrm{He} + \mathrm{n} &\rightleftharpoons \gamma + {}^4\mathrm{He} \\
{}^3\mathrm{He} + \mathrm{D} &\rightleftharpoons \mathrm{p} + {}^4\mathrm{He} & 2\,{}^3\mathrm{He} &\rightleftharpoons 2\,\mathrm{p} + {}^4\mathrm{He} \\
{}^7\mathrm{Li} + \mathrm{p} &\rightleftharpoons 2\,{}^4\mathrm{He} & {}^4\mathrm{He} + {}^3\mathrm{H} &\rightleftharpoons \gamma + {}^7\mathrm{Li} \\
{}^4\mathrm{He} + {}^3\mathrm{He} &\rightleftharpoons \gamma + {}^7\mathrm{Be} & {}^7\mathrm{Be} + \mathrm{e} &\rightleftharpoons \nu_e + {}^7\mathrm{Li}
\end{aligned}
$$

For each of these reactions, there is a Boltzmann equation which describes the evolution of the number densities. This coupled system of equations can be solved numerically to find the abundances of nuclei. The solution for a few nuclei is shown in fig. 2.5.

2.11 Recombination

As the universe cools down to a temperature that is lower than the binding energy of hydrogen (13.6 eV), some hydrogen atoms start to form via the reaction $p + e \rightarrow H + \gamma$. This epoch is called recombination, even though electrons and protons combine for the first time. Other atoms also start forming, but we only care about hydrogen for now.s

The ionization fraction x_e is the ratio between the number density of electrons and the number density of baryons (protons, hydrogen):

$$x_e := \frac{n_e}{n_b}$$

For given initial conditions, the ionization fraction can be calculated as a function of z by solving the Boltzmann equation, see fig. 2.6

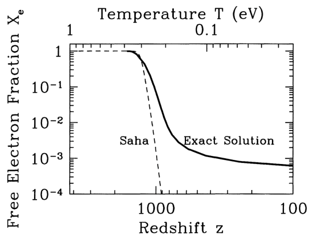


Figure 2.6: The ionization X_e as a function of redshift. Before recombination (at high redshift), the universe is fully ionized with $X_e = 1$. The Saha solution assumes thermal equilibrium, which is only valid until recombination. After the freeze-out, a residual ionization fraction ($X_e \approx 10^{-3}$)remains.

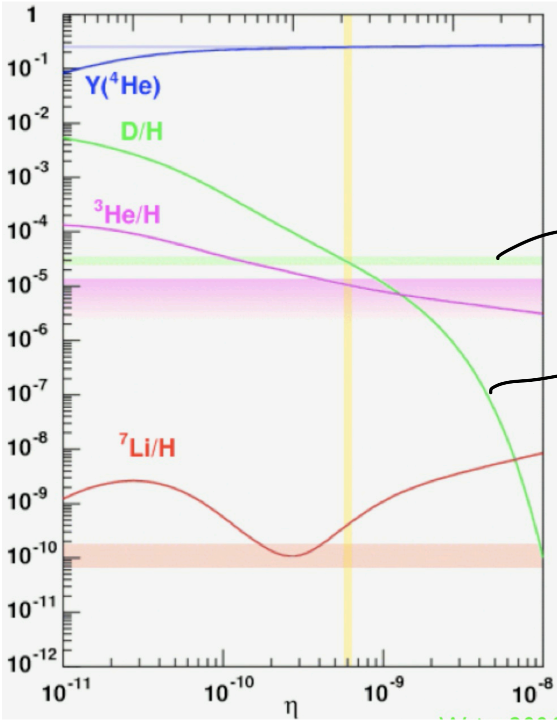


Figure 2.5: The abundances of D, ^{4}He, ^{3}He, and ^{7}Li, as a function of the photon to baryon ration η. The yellow vertical area indicates the measured value of η, while the shaded horizontal bars are the measured abundances. The modelled abundances agree well with the experiments, except for ^{7}Li, which is probably due to uncertainties in how much ^{7}Li is destroyed in stars.

2.11.1 Recombination

The time of recombination is defined as the time when there are ten times more baryons than electrons: $x_e(z_{\text{rec}}) = 0.1$. This happens at a temperature of $T_{\text{rec}} \approx 0.3\,\text{eV}$ and a redshift of $z \approx 1300$. Note that T_{rec} is smaller than the binding energy of hydrogen. This is because recombination is delayed by the high abundance of photons.

2.11.2 Decoupling

The interaction of electrons and photons (in the relevant energy regime) is described by Thomson scattering, with a scattering cross-section of $\sigma_T = 6.65 \cdot 10^{-25}\,\text{cm}^2$. The reaction rate is then (see section 2.7)

$$\Gamma_T = n_e \sigma_T c,$$

where n_e is the number density of electrons, and c the speed of photons. We know that strong coupling occurs as long as $\Gamma_T \gg H$, so we define the moment of decoupling such that

$$\Gamma_T(z_{\text{dec}}) = H(z_{\text{dec}})$$

This happens at $z \approx 1100$, $E \approx 0.26\,\text{eV}$, and $T \approx 3000\,\text{K}$, after the universe is 380 000 yr old. Note that $z_{\text{dec}} < z_{\text{rec}}$, so decoupling occurs soon after recombination.

After decoupling, the universe is transparent to photons. When an observer today stares into empty space, the photons he measures come from the surface of last scattering, where the photons interacted for the last time during decoupling. This is the cosmic microwave background (CMB), which has a temperature of

$$T_{\text{CMB}} = T_{\text{dec}} \frac{a_0}{a_{\text{dec}}} \approx 3\,\text{K}$$

Measurements of the CMB show a blackbody spectrum with remarkably deviations of $\Delta T / T \approx 10^{-5}$, as can be seen in fig. 2.7.

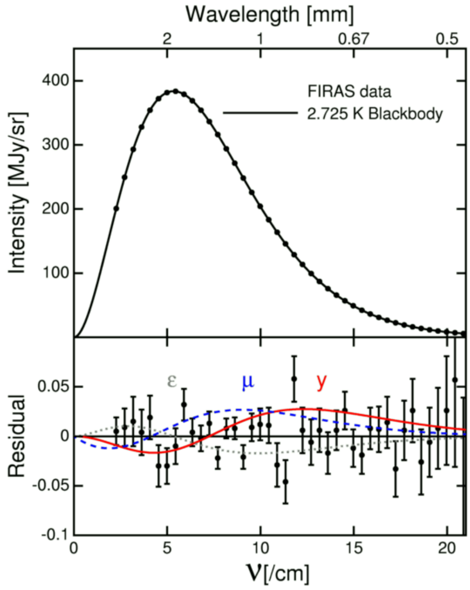


Figure 2.7: The measured spectrum of the cosmic microwave background and a fit with a blackbody spectrum. The residuals show an average deviation of only $\Delta T/T \approx 10^{-5}$.

2.12 ΛCDM model

The ΛCDM model is a refinement of the Big Bang model, and it is the standard model of cosmology. Three observational pillars justify this model:

- the expansion of the universe,
- the big bang nucleosynthesis, and
- the cosmic microwave background.

According to ΛCDM, the energy content of the universe today is as follows:

- Radiation are those particles that are still relativistic today:
  - photons: $T \approx 2.73\,\mathrm{K}$, $\Omega_{\gamma,0} = 2.5 \cdot 10^{-5}$
  - massless[2] neutrinos: $\Omega_{\nu,0} = 0.68\Omega_{\gamma,0}$

[2] Because of neutrino oscillations, we strongly suspect that neutrinos actually have mass, but it is very low.

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