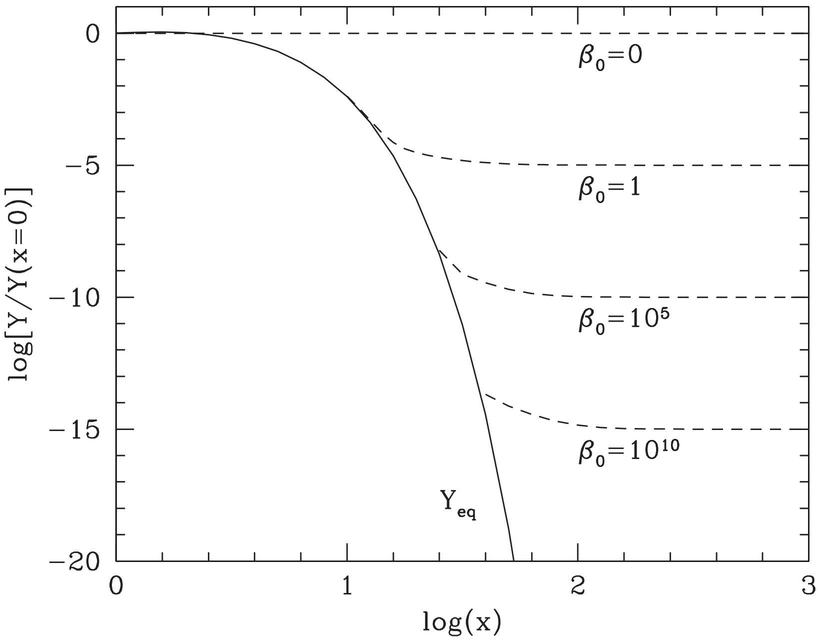

Fig. 3.5. The solution of Eq. (3.167) assuming a constant annihilation cross-section; $\beta = \beta_0$ (dashed curves). The solid curve shows the equilibrium abundance.

In what follows we present a simple, but relatively accurate, estimate of the relic abundances of various particle species. Rather than solving Eq. (3.167), which needs to be done numerically or by other approximate methods, we make the assumption that freeze-out occurs at a temperature $T_{\rm f}$, corresponding to $x_{\rm f}$, when $\Gamma/H = 1$, and that the relic abundance is simply given by $Y_i(x \to \infty) = Y_{i,\rm eq}(x_{\rm f})$. Using Eqs. (3.128) and (3.130) for $n_{i,\rm eq}$, and Eq. (3.146) for s, we have

$$Y_{i,\rm eq}(x) = \begin{cases} (45\zeta(3)/2\pi^4)[g_{i,\rm eff}/g_{*,s}(x)] & (x \ll 1) \\ (90/(2\pi)^{7/2})[g_i/g_{*,s}(x)]x^{3/2}e^{-x} & (x \gg 1), \end{cases} \tag{3.168}$$

where $g_{i,\rm eff} = g_i$ for bosons and $g_{i,\rm eff} = (3/4)g_i$ for fermions. The freeze-out temperature follows from $\Gamma(x_{\rm f}) = n_{i,\rm eq}(x_{\rm f})\beta(x_{\rm f}) = H(x_{\rm f})$. From Eq. (3.61) we have that in the radiation dominated era $H^2(t) = (8\pi G/3)\rho_{\rm r}(t)$. Substitution of Eq. (3.134) then gives

$$H(x) = \left(\frac{m_i m_{\rm Pl}}{x}\right)^2 \sqrt{\frac{4\pi^3 g_*(x)}{45}}, \tag{3.169}$$

where $m_{\rm Pl} = G^{-1/2}$ is the Planck mass in the natural units used here. Our definition of freeze-out then yields

$$x_{\rm f} = \sqrt{\frac{45}{\pi^7}}\frac{\zeta(3)}{2}\frac{g_{i,\rm eff}}{\sqrt{g_{*,s}(x_{\rm f})}}m_{\rm Pl}\,m_i\,\beta(x_{\rm f}) \quad (x_{\rm f} \ll 1)\,;$$

$$x_{\rm f}^{-1/2}e^{x_{\rm f}} = \sqrt{\frac{45}{32\pi^6}}\frac{g_i}{\sqrt{g_{*,s}(x_{\rm f})}}m_{\rm Pl}\,m_i\,\beta(x_{\rm f}) \qquad (x_{\rm f} \gg 1)\,. \tag{3.170}$$

Note that since $x_{\rm f}$ appears on both sides of these equations, they typically need to be solved numerically.

Let us first consider the case of hot relics that have remained relativistic to the present day, i.e. their rest mass $m_i \ll T_0 = 2.4 \times 10^{-4}\,$eV. Its energy density follows from Eq. (3.131), which can