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Source: `https://github.com/thomabir/astrophysics-ii`

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2.1 Cosmological Principle

The Cosmological principle states that the universe is homogeneous and isotropic on sufficiently large scales. This is a generalization of the Copernican principle, according to which there is no special place and no special direction in the universe.

The Cosmological principle is only valid for distances larger than a few hundred Mpc. Local perturbations about this uniform background will be described later.

2.2 Elements of General Relativity

Einstein field equations:

$$G_{\mu\nu} = \frac{8\pi G}{\rho} T_{\mu\nu}$$

Ideal fluid:

$$T_{\mu\nu} = \mathrm{diag}(\rho c^2, p, p, p)$$

2.3 FRW metric

The Friedmann-Robertson-Walker (FRW) metric is the metric of a homogeneous and isotropic universe:

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a(t)^2(\mathrm{d}\chi^2 + r(\chi)^2\,\mathrm{d}\Omega^2)$$

- χ: comoving radius
- $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$: solid angle element
- $a(t)$: scale factor
- $r(\chi) = f_K(\chi) = \begin{cases} \sin\chi & \text{closed case, positive curvature} \\ \chi & \text{flat case} \\ \sinh\chi & \text{open case, negative curvature} \end{cases}$

Hubble parameter

- Hubble parameter: $H := \dot{a}/a$
- today's value gets a subscript zero: H_0
- Because it is hard to measure H accurately, we write it as

  $$H_0 = 100h\frac{\mathrm{km}}{\mathrm{s\,Mpc}},$$

  where $h \approx 0.7$ is the dimensionless Hubble parameter.
- $H_0^{-1} \approx 10\,\mathrm{Gyr}$ is about the age of the universe
- $cH_0^{-1} \approx 4\,\mathrm{Gpc}$ is about the size of the observable universe

2.4 Friedmann equation

The Friedmann equations are derived by plugging the FRW metric into Einstein's equations:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2}$$
$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)$$

The critical density is defined as

$$\rho_{\text{crit}}(t) = \frac{3H(t)^2}{8\pi G}$$

Today, the critical density is about five hydrogen atoms per cubic metre, or one galaxy per Mpc^3.

Density parameters:

- The subscript i describes one component of the universe (i = radiation, dark matter, matter ...)
- density parameter: $\Omega_i(t) = \rho_i(t)/\rho_{\text{crit}}(t)$
- total energy density: $\rho(t) = \sum_i \rho_i(t)$
- total density parameter: $\Omega(t) = \rho(t)/\rho_{\text{crit}}(t)$
- curvature density parameter: $\Omega_{K,0} = 1 - \Omega_0 = -Kc^2/H_0^2 a_0^2$

With these definitions, the (first) Friedmann equation can be rewritten as

$$\frac{H}{H_0} = \sqrt{\frac{\rho}{\rho_{\text{crit},0}} + \Omega_{K,0}\left(\frac{a_0}{a}\right)^2}$$

2.5 Solutions

To solve the Friedmann equation, $\rho(t)$ or $\rho(a)$ need to be known. It can be calculated as

$$\rho = n\epsilon$$

where n is the particle number per unit volume and ϵ the energy per particle

- Relativistic matter. ϵ is constant with a, while $n \propto a^{-3}$. Thus $\rho \propto a^{-3}$.
- Radiation. $\epsilon = h\nu = hc/\lambda \propto a^{-1}$. Thus $\rho \propto a^{-4}$.
- Vacuum energy is constant in a

There is a generalization for general fluids:

- equation of state: $p = w\rho c^2$
- density: $\rho \propto a^{-3(1+w)}$
- $w = \begin{cases} 0 & \text{matter} \\ 1/3 & \text{radiation} \\ -1 & \text{vacuum energy} \end{cases}$

The results can be plugged into the Friedmann equation:

$$\begin{aligned}\frac{H}{H_0} &= \sqrt{\frac{\rho}{\rho_{\text{crit},0}} + \Omega_{K,0}\left(\frac{a_0}{a}\right)^2} \\ &= \sqrt{\Omega_{m,0}\left(\frac{a_0}{a}\right)^3 + \Omega_{r,0}\left(\frac{a_0}{a}\right)^4 + \Omega_{\Lambda,0} + \Omega_{K,0}\left(\frac{a_0}{a}\right)^2}\end{aligned}$$

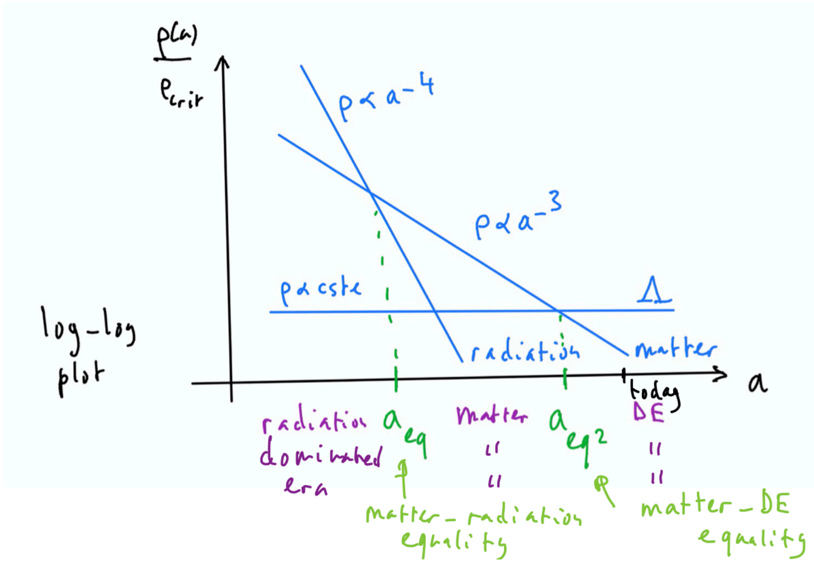


Figure 2.1: Domination of different components at different times

This is a differential equation with $\Omega_{i,0}$ as parameters.

The standard cosmological model:

- $\Omega_{m,0} \approx 0.3$
- $\Omega_{r,0} \approx 10^{-5}$
- $\Omega_{\Lambda,0} \approx 0.7$
- $\Omega_{K,0} \approx 0$
- $\Omega_0 \approx 1$
- $h \approx 0.7$

At different times, the universe is dominated by different components. Approximations:

- Matter dominated:
$$\frac{H}{H_0} = \sqrt{\Omega_{m,0}\left(\frac{a_0}{a}\right)^3} \implies a \propto t^{2/3}$$
- Radiation dominated:
$$\frac{H}{H_0} = \sqrt{\Omega_{r,0}\left(\frac{a_0}{a}\right)^4} \implies a \propto t^{1/2}$$
- Λ dominated:
$$\frac{H}{H_0} \propto \text{constant} \implies a \propto e^{Ht}$$
- General fluid ($w \neq -1$):
$$\rho \propto a^{-3(1+w)} \implies a \propto t^{\frac{2}{3(1+w)}}$$

2.6 Distances and times

2.6.1 Angular distance & Luminosity distance

The comoving distance χ and the proper distance $a\chi$ to a source are not directly observable. However, the angular size θ and the flux F of an object can be measured directly.

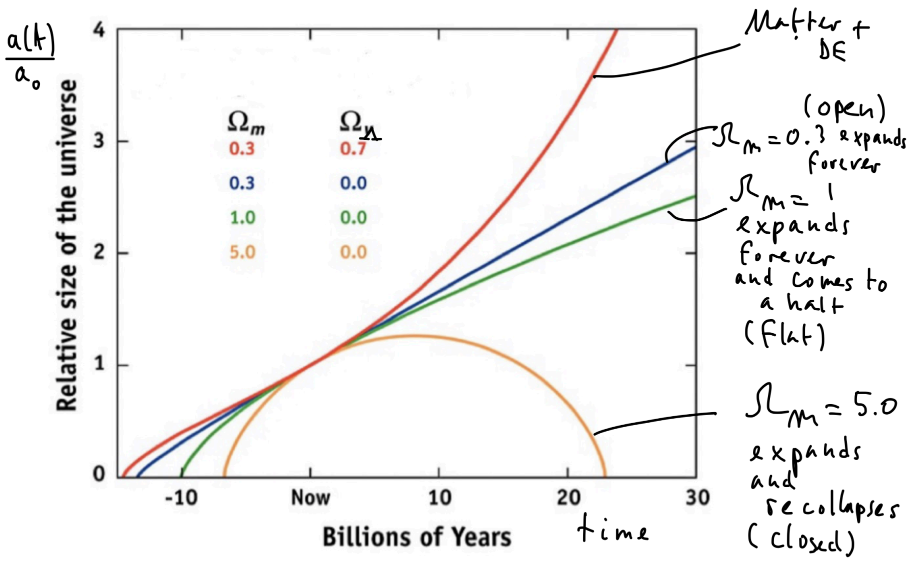


Figure 2.2: Evolution

Intrinsic properties of the source:

- its size D
- its luminosity L

Properties of space-time:

- the comoving distance to the source χ
- the proper distance to the source $a(t)\chi$

Measurable quantities for an observer:

- the angular size θ
- the flux F

In Euclidean space, the following relations hold:

$$\theta = \frac{D}{d} \qquad\qquad F = \frac{L}{4\pi d^2}$$

where d is the distance to the source. In FRW-space, we define the following:

- the angular-diameter distance d_A satisfies $\theta = D/d_A$. One can show $d_A = ar(\chi)$
- the luminosity distance d_L satisfies $F = L/4\pi d_L^2$. One can show that $d_L = r(\chi)/a$

2.6.2 Comoving radius

We measured the redshift of a photon that has travelled to us on a radial trajectory. How far away (in comoving distance) is the source?

$$\begin{aligned}
0 &= \mathrm{d}s^2 && \text{photon} \\
&= c\,\mathrm{d}t^2 - a(t)^2[\mathrm{d}\chi^2 + r(\chi)^2\,\mathrm{d}\Omega^2] && \text{FRW metric} \\
\implies c\,\mathrm{d}t &= a(t)\,\mathrm{d}\chi && \mathrm{d}\Omega^2 = 0 \text{ on a radial trajectory} \\
\implies \mathrm{d}\chi &= \frac{c\,\mathrm{d}t}{a(t)} && \\
&= \frac{c\,\mathrm{d}a}{a^2 H(a)} && H = \frac{\dot{a}}{a}, \text{ so } \mathrm{d}t = \frac{\mathrm{d}a}{aH(a)} \\
\implies \chi(a) &= c\int_a^{a_0} \frac{\mathrm{d}a'}{a'^2 H(a')} && \chi(a_0) = 0
\end{aligned}$$

$H(a)$ has to be obtained from the Friedmann equations. As a result, we will get $\chi(a, a_0)$. We can use $a/a_0 = 1/(1+z)$ to get $\chi(z, a_0)$. Since a_0 can be defined arbitrarily (for example, $a_0 = 1$), we get $\chi(z)$.

2.6.3 Comoving Horizon

Suppose a (hypothetical, non-interacting) photon was emitted at the Big Bang. How far (in comoving distance) could it have travelled until now? This comoving distance is called the comoving horizon. We plug into the previous equation, with $a = 0$ at the start:

$$\chi(a) = c\int_0^{a_0} \frac{\mathrm{d}a'}{a'^2 H(a')}$$

2.6.4 Age of the Universe

How old is the universe?

$$\begin{aligned} t_0 &= \int_0^{t_0} \mathrm{d}t \\ &= \int_0^{a_0} \frac{\mathrm{d}a}{aH(a)} \qquad\qquad H(a) = \dot{a}/a \end{aligned}$$

$H(a)$ is again found from the Friedmann equation. With the standard cosmology, $t_0 \approx$ 14 Gyr.

2.7 Thermal history

According to the Big Bang paradigm, the universe was once hot and dense, and now it expands and cools down. Today, it is far from thermal equilibrium, but it must have been in thermal equilibrium at some point in the past if it continuously expands.

A system is in thermal equilibrium if $\Gamma \gg H$

- Γ = interactions/time is the interaction rate
- $H = \dot{a}/a$ is the Hubble constant

Similarly, a system is in thermal equilibrium if $\tau_\Gamma \ll \tau_H$

- $\tau_\Gamma = 1/\Gamma$ is the characteristic timescale of interactions
- $\tau_H = 1/H$ is the characteristic timescale of expansion

We already know about H. Γ is defined as

$$\Gamma = n v \sigma$$

- n number density, particles/volume
- v velocity of particles
- σ scattering cross-section, has units of area.

At early times, $\Gamma \gg H$. Particles are in thermal equilibrium with the plasma and coupled to photons. This scenario will be treated in section 2.8.1.

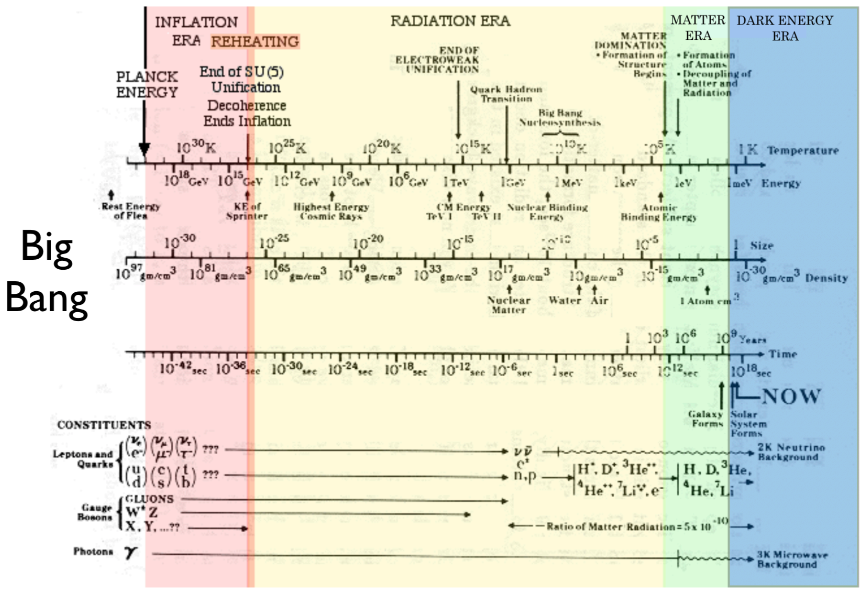
At later times $\Gamma \ll H$. Particles are not in thermal equilibrium and are decoupled from photons. See section 2.8.2

The decoupling or "freeze out" happens when $\Gamma \approx H$. This transition is described by the Boltzmann equation in section 2.8.3.

In table 2.1 and fig. 2.2, an overview of the thermal history of the universe is given.

Event	time	redshift	energy temp.
Inflation A phase of extremely rapid exponential expansion, caused by a phase transitions where the inflaton field emerged. Inflation explains properties of the universe which are difficult to account for without.	?	?	?
Baryogenesis Baryons (protons, neutrons) are formed from quarks. Weirdly, there are way more baryons formed than antibaryons. This is the matter-antimatter asymmetry.	?	?	?
QCD phase transition The universe has cooled sufficiently such that hadrons (baryons and mesons) can form.	10^{-5} s	10^{12}	200 MeV $3 \cdot 10^{12}$ K
Pions annihilate and decay, the only hadrons left are nucleons (protons and neutrons).	10^{-4} s		50 MeV 10^{12} K
Dark Matter freeze-out Dark Matter interacts very weakly with ordinary matter, so it decouples early on.	?	?	?
Electron-positron annihilation Electrons and positrons annihilate through $e^+ + e^- \rightarrow 2\gamma$. Since the number of charged particles decreases, neutrinos decouple.	4 s	$2 \cdot 10^9$	0.3 MeV $5 \cdot 10^9$ K
Big Bang nucleosynthesis Light nuclei such as D and He get synthesized. They are still ionized.	3 min	$4 \cdot 10^8$	0.08 MeV 10^9 K
Matter-radiation equality	$6 \cdot 10^4$ yr	3400	0.75 eV 8700 K
Recombination Formation of neutral atoms through $e^- + p^+ \rightarrow H + \gamma$	$2 \cdot 10^5$ yr	1200	0.34 eV 4000 K
Surface of last scattering The number density of charged particles has decreased enough for photons to decouple. These photons form the CMB.			
Reionization Stars form and re-ionize hydrogen.	$2 \cdot 10^8$ yr	20	4 meV 50 K
Dark Energy - Matter equality	9 Gyr	0.4	0.33 meV 3.8 K
Today	13.8 Gyr	0	0.24 eV 2.7 K

Table 2.1: Thermal history of the universe



Adapted from Kolb & Turner 1990

Figure 2.3: History of the Big Bang

2.8 Thermodynamics

To describe the evolution of the universe quantitatively, a few definitions are required:

- The probability that a particle is in a volume $\mathrm{d}^3x\,\mathrm{d}^3p$ at time t is given by the **(phase space) distribution function**:

  $$f(\boldsymbol{x}, \boldsymbol{p}, t)\,\mathrm{d}^3x\,\mathrm{d}^3p$$

  In a homogeneous and isotropic universe, we have $f(\boldsymbol{x}, \boldsymbol{p}, t) = f(p, t)$.
- The number of particles per unit volume is given by the **number density**:

  $$n(t) = 4\pi \int f(p,t)p^2\,\mathrm{d}p$$

- The energy per unit volume is given by the **energy density**:

  $$\rho(t) = 4\pi \int E(p)f(p,t)p^2\,\mathrm{d}p\,,$$

  with $E(p) = \sqrt{p^2 + m^2}$.
- The **pressure** is given by

  $$P(t) = 4\pi c^2 \int \frac{p^2}{3E(p)} f(p,t)p^2\,\mathrm{d}p$$

  Since we work in natural units, we drop the c.

2.8.1 At early times ($\Gamma > H$)

The distribution function of particles in thermodynamic equilibrium is the Bose-Einstein or the Fermi-Dirac distribution:

$$f_{\mathrm{eq}}(p,t) = \frac{g}{(2\pi)^3}\left[\exp\left(\frac{E(p) - \mu}{T} \pm 1\right)\right]^{-1}$$

- $+$ is for fermions and $-$ for bosons
- g is a spin degeneracy factor. Examples: $g_\nu = 1$, $g_\gamma = 2$, $g_{\text{quark}} = 6$
- μ is the chemical potential, which is the response of a thermodynamics system to a change of particle number. Usually, $\mu = 0$ for our purposes.
- T is the temperature of the universe, which is time dependent.

For *non-relativistic particles*, $T \ll m$ and $E \approx m + p^2/2m$. Plugging this in yields

$$n = g\left(\frac{mT}{2\pi}\right)^{3/2} \exp\left(\frac{p-m}{T}\right) \qquad \rho = nm \qquad P = nT$$

For *relativistic particles*, $T \gg m$ and $E \approx p$. For both fermions and bosons, this yields

$$n = gT^3 \qquad \rho = gT^4 \qquad P = \frac{\rho}{3}$$

In the case of a relativistic gas, $T \propto a^{-1}$, so $n \propto a^{-3}$ and $\rho \propto a^{-4}$. This is what we have already seen in section 2.4.

2.8.2 At late times ($\Gamma < H$)

The transition between non-equilibrium and equilibrium takes place at the decoupling time of freeze-out time t_f. At $t > t_f$, the transition function is

$$f(p,t) = f\left(p\frac{a(t)}{a(t_f)}, t_f\right)$$

The shape of the function is "frozen in" at the freeze-out time t_f.

For a relativistic particle,

$$f(p,t) = \frac{g}{(2\pi)^3}\left[\exp\left(\frac{pa(t)}{T_f a(t_f)} \pm 1\right)\right]^{-1},$$

which is the same as for an equilibrium particle, but with $T_f \coloneqq T_f a(t_f)/a(t)$.

2.8.3 Boltzmann equation

The Boltzmann equation

$$\frac{\mathrm{d}f_i}{\mathrm{d}t} = c_i[f_i]$$

describes the time evolution of the distribution function. $c_i[f_i]$ is the collision term.

f only depends on p and t, so we can write

$$\frac{\mathrm{d}f_i}{\mathrm{d}t} = \frac{\partial f_i}{\partial t} + \frac{\partial f_i}{\partial p}\frac{\partial p}{\partial t}$$

The last term can be simplified. Because $p = p_0 a^{-1}$, $\mathrm{d}p/\mathrm{d}t = -p_0 a^{-1}\dot{a} = -pH$. The Boltzmann equation then becomes

$$\begin{aligned}
\frac{\partial f_i}{\partial t} - pH(t)\frac{\partial f_i}{\partial p} &= c_i[f_i] \\
\implies \frac{\partial}{\partial t}\int \mathrm{d}^3p\, f_i - H(t)\int \mathrm{d}^3p\, p\frac{\partial f_i}{\partial p} &= \int \mathrm{d}^3p\, c_i[f_i] \\
\implies \frac{\mathrm{d}n_i}{\mathrm{d}t} + 3H(t)n_i &= \int \mathrm{d}^3p\, c_i[f_i]
\end{aligned}$$

In the last line, we used partial integration.[1] If the system is collisionless, $c_i[f_i] = 0$, and the solution of the Boltzmann equation is $n_i \propto a^{-3}$. The $3H(t)n_i$ is called the Hubble drag term.

We now look at reactions of the type $i + j \leftrightarrow a + b$. The collision term is then of the form

$$c_i[f_i] = \alpha(T)n_a n_b - \beta(T)n_i n_j$$

- $\alpha(T)$ is the production rate
- $\beta(T)$ is the destruction rate

To simplify the equation, we make a few assumptions:

- a and b are in equilibrium with a general plasma at temperature T
- $n_i = n_j$ (this is the case for antiparticles)
- radiation era: $a \propto t^{1/2} \propto T^{-1}$

The equation is then

$$\frac{\mathrm{d}n_i}{\mathrm{d}t} + 3H(t)n_i = \beta(T)(n_{i,\mathrm{eq}}^2 - n_i^2)$$

To analyse the equation, we define

- $x = m_i/T$ is used as a time variable
- $y_i = n_i/S$, where S is the entropy
- $\Gamma(x) = n_{i,\mathrm{eq}}(x)\beta(x)$

The equation is then

$$\frac{x}{y_{i,\mathrm{eq}}}\frac{\mathrm{d}y_i}{\mathrm{d}x} = -\frac{\Gamma(x)}{H(x)}\left[\left(\frac{y_i}{y_{i,\mathrm{eq}}}\right)^2 - 1\right]$$

Initial conditions: $x \ll 1$ at early times, so $y_i = y_{i,\mathrm{eq}}$.

No analytical solution is known. A set of numerical solutions is shown in fig. 2.4

2.9 Particle relics

There are two types of relics:

- Hot relics freeze out when the particles are still relativistic. Since $x = m_i/T$, this means $x_f \ll 1$
- Cold relics freeze out when the particles are already non-relativistic, with $x_f \gg 1$

2.9.1 Hot relics

Hot relics are still relativistic today, so their rest mass is $m_i \ll T_0 = 2.4 \cdot 10^{-4}$ eV. An example would be massless neutrinos.

The solution of the Boltzmann equation is

$$\Omega_{i,0}h^2 = \frac{g_{i,\mathrm{eff}}}{2}\left[\frac{g_{*s}(x_0)}{g_{*s}(x_f)}\right]^{4/3}\Omega_{\gamma,0}h^2$$

The g's are degeneracy factors and satisfy $g_{*s}(x_0) \leq g_{*s}(x_f)$, and the photon density is $\Omega_{\gamma,0}h^2 = 2.5 \cdot 10^{-5}$. It follows that $\Omega_{i,0}$ is very small, which means that hot relic particles contribute very little to today's energy density.

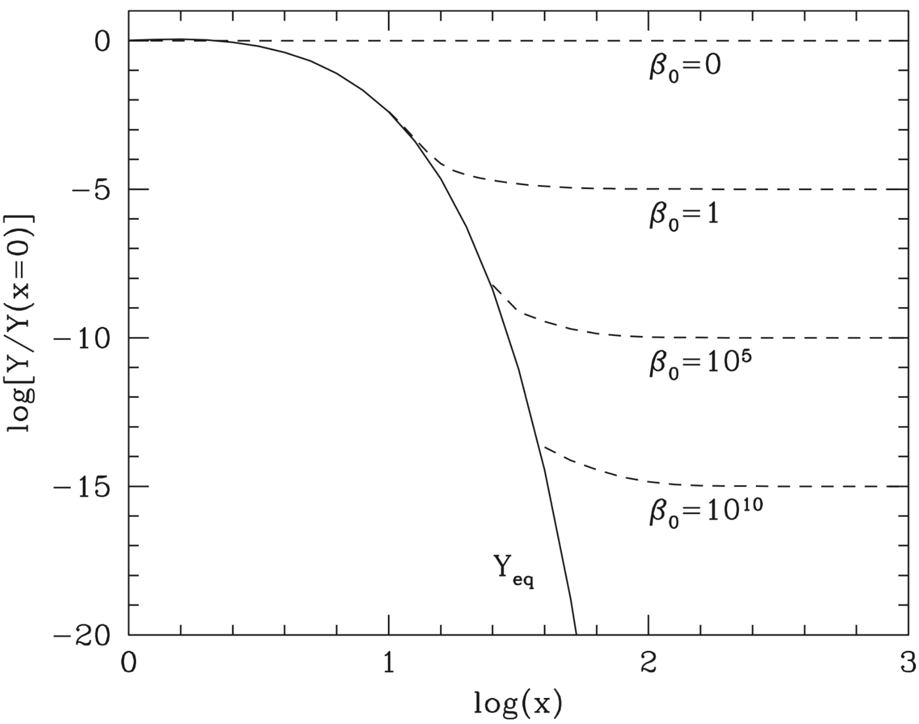


Figure 2.4: Solutions of the Boltzmann equation for different destruction rates β. We assume $\beta(T) = \beta_0$ is constant. The vertical axis is a proxy for abundance, and the horizontal axis for time. First, particles remain in equilibrium (solid line), then they decouple (dashed lines), leaving relic abundances. When β_0 is large, thermal equilibrium is maintained longer, so the relic abundance is lower.

2.9.2 WIMPs

Next we consider weakly interacting massive particles (WIMPs). Examples are massive neutrinos and stable, light supersymmetric particles. WIMPs can either be hot ($x_f \ll 1$) or cold ($x_f \gg 1$).

Hot WIMPs

The solution of the Boltzmann equation yields

$$\Omega_{i,0}h^2 = 7.64 \cdot 10^{-2} \left(\frac{g_{i,\text{eff}}}{g_{*s}(x_f)} \right) \frac{m_i}{\text{eV}}$$

Since we know that $\Omega_{i,0} < 1$, we can get a constraint $m_i < 100\,\text{eV}$ on the mass of the hot WIMPs, which is possible for neutrinos. However, hot WIMPs are ruled out as dark matter by structure formation arguments.

Cold WIMPs

Boltzmann says

$$\Omega_{i,0}h^2 = \begin{cases} 1.8 \left(\frac{m_i}{\text{GeV}} \right)^{-2} \left[1 + 0.17 \ln \left(\frac{m_i}{\text{GeV}} \right) \right] & \text{if } m_i < 100\,\text{GeV} \\ \left(\frac{m_i}{3\,\text{TeV}} \right)^2 & \text{if } m_i > 100\,\text{GeV} \end{cases}$$

To get $\Omega_{i,0} < 1$, we need m_i to be between 1.4 GeV and 3 TeV. Cold Wimps are good candidates for dark matter.

2.10 Primordial nucleosynthesis

This is the epoch where protons and neutrons first combined to nuclei (not atoms) heavier than hydrogen. Heavier atoms can also be synthesized in stars through nuclear reactions, which has to be differentiated in observations.

Initial conditions

Initially, the temperature is $T < 10^{13}\,\mathrm{K}$ and the associated energy scale is $k_B T \approx 0.8\,\mathrm{MeV}$. Protons and neutrons are in thermal equilibrium and interact weakly via the processes $p + e \leftrightarrow n + \nu_e$ and $n + \bar{e} \leftrightarrow p + \bar{\nu}_e$. Since the mass of the nucleons is around 940 MeV, they are already non-relativistic. The masses of neutrons and protons are slightly different, so their abundances after freeze out are different.

Nuclear reactions

Once the temperature drops below 1 MeV, which is the binding energy of a typical nucleus, the nuclei start forming. A host of nuclear reactions can occur then:

$$
\begin{aligned}
\mathrm{p} + \mathrm{n} &\rightleftharpoons \gamma + \mathrm{D} & \mathrm{D} + \mathrm{n} &\rightleftharpoons \gamma + {}^3\mathrm{H} \\
\mathrm{D} + \mathrm{D} &\rightleftharpoons \mathrm{p} + {}^3\mathrm{H} & \mathrm{D} + \mathrm{p} &\rightleftharpoons \gamma + {}^3\mathrm{He} \\
\mathrm{D} + \mathrm{D} &\rightleftharpoons \mathrm{n} + {}^3\mathrm{He} & {}^3\mathrm{H} + \mathrm{p} &\rightleftharpoons \mathrm{n} + {}^3\mathrm{He} \\
{}^3\mathrm{H} &\rightleftharpoons \mathrm{e} + \bar{\nu}_e + {}^3\mathrm{He} & {}^3\mathrm{H} + \mathrm{p} &\rightleftharpoons \gamma + {}^4\mathrm{He} \\
{}^3\mathrm{H} + \mathrm{D} &\rightleftharpoons \mathrm{n} + {}^4\mathrm{He} & {}^3\mathrm{He} + \mathrm{n} &\rightleftharpoons \gamma + {}^4\mathrm{He} \\
{}^3\mathrm{He} + \mathrm{D} &\rightleftharpoons \mathrm{p} + {}^4\mathrm{He} & 2\,{}^3\mathrm{He} &\rightleftharpoons 2\,\mathrm{p} + {}^4\mathrm{He} \\
{}^7\mathrm{Li} + \mathrm{p} &\rightleftharpoons 2\,{}^4\mathrm{He} & {}^4\mathrm{He} + {}^3\mathrm{H} &\rightleftharpoons \gamma + {}^7\mathrm{Li} \\
{}^4\mathrm{He} + {}^3\mathrm{He} &\rightleftharpoons \gamma + {}^7\mathrm{Be} & {}^7\mathrm{Be} + \mathrm{e} &\rightleftharpoons \nu_e + {}^7\mathrm{Li}
\end{aligned}
$$

For each of these reactions, there is a Boltzmann equation which describes the evolution of the number densities. This coupled system of equations can be solved numerically to find the abundances of nuclei. The solution for a few nuclei is shown in fig. 2.5.

2.11 Recombination

As the universe cools down to a temperature that is lower than the binding energy of hydrogen (13.6 eV), some hydrogen atoms start to form via the reaction $p + e \to H + \gamma$. This epoch is called recombination, even though electrons and protons combine for the first time. Other atoms also start forming, but we only care about hydrogen for now.s

The ionization fraction x_e is the ratio between the number density of electrons and the number density of baryons (protons, hydrogen):

$$x_e := \frac{n_e}{n_b}$$

For given initial conditions, the ionization fraction can be calculated as a function of z by solving the Boltzmann equation, see fig. 2.6

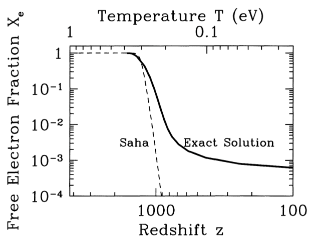


Figure 2.6: The ionization X_e as a function of redshift. Before recombination (at high redshift), the universe is fully ionized with $X_e = 1$. The Saha solution assumes thermal equilibrium, which is only valid until recombination. After the freeze-out, a residual ionization fraction ($X_e \approx 10^{-3}$)remains.

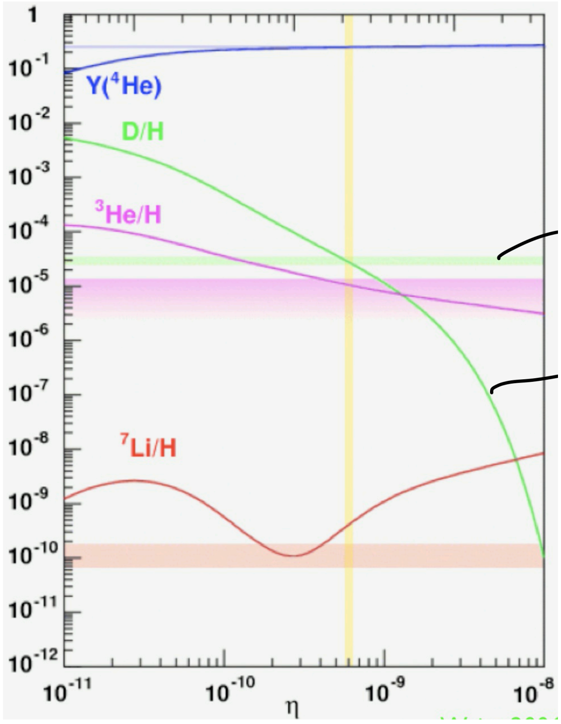


Figure 2.5: The abundances of D, ^{4}He, ^{3}He, and ^{7}Li, as a function of the photon to baryon ration η. The yellow vertical area indicates the measured value of η, while the shaded horizontal bars are the measured abundances. The modelled abundances agree well with the experiments, except for ^{7}Li, which is probably due to uncertainties in how much ^{7}Li is destroyed in stars.

2.11.1 Recombination

The time of recombination is defined as the time when there are ten times more baryons than electrons: $x_e(z_{\text{rec}}) = 0.1$. This happens at a temperature of $T_{\text{rec}} \approx 0.3\,\text{eV}$ and a redshift of $z \approx 1300$. Note that T_{rec} is smaller than the binding energy of hydrogen. This is because recombination is delayed by the high abundance of photons.

2.11.2 Decoupling

The interaction of electrons and photons (in the relevant energy regime) is described by Thomson scattering, with a scattering cross-section of $\sigma_T = 6.65 \cdot 10^{-25}\,\text{cm}^2$. The reaction rate is then (see section 2.7)

$$\Gamma_T = n_e \sigma_T c,$$

where n_e is the number density of electrons, and c the speed of photons. We know that strong coupling occurs as long as $\Gamma_T \gg H$, so we define the moment of decoupling such that

$$\Gamma_T(z_{\text{dec}}) = H(z_{\text{dec}})$$

This happens at $z \approx 1100$, $E \approx 0.26\,\text{eV}$, and $T \approx 3000\,\text{K}$, after the universe is 380 000 yr old. Note that $z_{\text{dec}} < z_{\text{rec}}$, so decoupling occurs soon after recombination.

After decoupling, the universe is transparent to photons. When an observer today stares into empty space, the photons he measures come from the surface of last scattering, where the photons interacted for the last time during decoupling. This is the cosmic microwave background (CMB), which has a temperature of

$$T_{\text{CMB}} = T_{\text{dec}} \frac{a_0}{a_{\text{dec}}} \approx 3\,\text{K}$$

Measurements of the CMB show a blackbody spectrum with remarkably deviations of $\Delta T/T \approx 10^{-5}$, as can be seen in fig. 2.7.

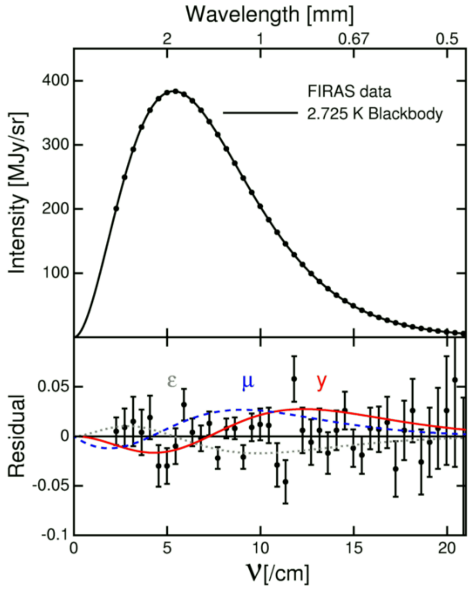


Figure 2.7: The measured spectrum of the cosmic microwave background and a fit with a blackbody spectrum. The residuals show an average deviation of only $\Delta T/T \approx 10^{-5}$.

2.12 ΛCDM model

The ΛCDM model is a refinement of the Big Bang model, and it is the standard model of cosmology. Three observational pillars justify this model:

- the expansion of the universe,
- the big bang nucleosynthesis, and
- the cosmic microwave background.

According to ΛCDM, the energy content of the universe today is made up of radiation, matter, dark matter, and dark energy. Their contributions today are:

- Photons and neutrinos: $\Omega_{\gamma,0} \approx \Omega_{\nu,0}/0.68 \approx 2.5 \cdot 10^{-5} h^{-2}$
- Baryons: $\Omega_{b,0} \approx 0.05$
- Dark Matter: $\Omega_{dm,0} \approx 0.25$
- Dark Energy: $\Omega_\Lambda \approx 0.7$

We now look at the constituents in more detail.

Radiation

As radiation, we classify those particles that are still relativistic today. Accordingly, they need to have very low mass.

- Most photons are part of the CMB and haven a temperature $T \approx 2.73\,\mathrm{K}$. They contribute only very little to the energy density: $\Omega_{\gamma,0} = 2.5 \cdot 10^{-5}$.
- Massless[2] neutrinos contribute about the same as photons: $\Omega_{\nu,0} = 0.68\Omega_{\gamma,0} \approx 1.7 \cdot 10^{-5}$.

[2] Because of neutrino oscillations, we strongly suspect that neutrinos actually have mass, but it is very low.

Baryons

Baryons from the Big Bang nucleosynthesis contribute $\Omega_b \approx 0.05$. They mostly occur in two forms:

- Even though only ten percent of baryons are found in stars, they are the largest fraction of visible matter.

- The rest of the baryons is in the form of various phases, such as cold gas, warm gas, and hot gas, both in the intergalactic and the interstellar medium.

Dark Matter

Dark Matter contributes $\Omega_{\text{dm}} \approx 0.25$. We know of its existence through its gravitational effects. There is evidence for dark matter in galaxies, galaxy clusters, large scale structure, and many others.

Dark matter is mostly non-baryonic, because baryons are constrained to $\Omega_b \approx 0.05$ by Big Bang nucleosynthesis. Also, the structure formation constraints below rule out baryons as dark matter candidates.

- Dark matter is cold (non-relativistic). It is thus called Cold Dark Matter (CDM)
- It interacts very weakly, so we can approximate it as non-collisional.

A good candidate for dark matter are particles beyond the standard model, such as WIMPs.

Massive Compact Halo Objects (MACHOs), such as black holes, are not good candidates, since objects with masses from 10^{-6} to $15\,M_\odot$ are ruled out.

Dark Energy

Dark Energy contributes $\Omega_{\text{de}} \approx 0.7$. It is needed to get a flat geometry for the universe, and to explain the recent acceleration of the expansion of the universe.

From the first Friedmann equation, we know that a constant energy density results in a scale factor $a \propto e^{Ht}$ that is exponentially growing. Consider an effective fluid with equation of state $p_{\text{DE}} = w\rho_{\text{DE}}c^2$. We demand $\ddot{a} > 0$, and the second Friedmann equation then yields $w < -1/3$.

There are a few possible candidates:

- The cosmological constant Λ corresponds to $w = -1$, independent of time.
- A model called *quintessence*, which is a dynamical scalar field, would give rise to another form of energy. Since it is dynamical, you can map it to an effective fluid, with a w that varies in time.
- There could be a theory of gravity that can improve upon or replace general relativity.

Current constraints demand $w = -1$ with an uncertainty of about 5 %. The cosmological constant is thus a model that is consistent with observations, and it is part of the ΛCDM model.

Summary

In total, $\Omega_0 \approx 1$, so the universe has a flat geometry. The exact nature of dark energy and dark matter are some of the most pressing questions in fundamental physics.

There are other ingredients to ΛCDM, such as the model of gravity and the choice of initial conditions. A process called inflation is also added, as we will see later.

The ΛCMD model is very successful to fit current observations, but there are some tensions that are starting to emerge with more detailed measurements from cosmological probes.

2.13 Problems with ΛCDM

The ΛCDM model without inflation has a few intrinsic fundamental problems.

2.13.1 The horizon problem

We have defined the particle horizon

$$\chi_h(t) = \int_0^t \frac{c\,\mathrm{d}t'}{a(t')},$$

which is the maximal comoving distance that a photon could have travelled from the big bang ($t = 0$) until time t. Let's look at the size of the horizon at decoupling time, where photons scattered for the last time. This corresponds to a redshift $z_{\mathrm{dec}} \approx 1100$. The integral evaluates to

$$\chi_h(t_{\mathrm{dec}}) \approx 180 h^{-1}\,\mathrm{Mpc}$$

The corresponding CMB angular scale is

$$\theta_h = \frac{\chi_h(t_{\mathrm{dec}})}{r(\chi_{\mathrm{dec}})} \approx 1.8\,\mathrm{deg},$$

where r is the comoving angular diameter distance. This is unexpected, because the CMB is extremely homogeneous across the whole sky, even though the photons from different areas of the sky could not have been in causal contact when they scattered! Something must be wrong here.

2.13.2 Flatness problem

We have observationally determined $\Omega_0 \approx 1$, so we have a flat geometry. Let's look at how Ω varies with time. For this, we can use the Friemann equation:

$$H(a)^2 = \frac{8\pi G}{3}\rho(a) - \frac{Kc^2}{a^2}$$

We look at the deviation of Ω from 1:

$$\begin{aligned}
\frac{1-\Omega(a)}{\Omega(a)} &= \Omega(a)^{-1} - 1 \\
&= -\frac{3Kc^2}{8\pi G\rho(a)a^2} \\
&\propto \begin{cases} a^2 & \text{in the radiation era } \rho \propto a^{-4} \\ a & \text{in the matter era } \rho \propto a^{-3} \end{cases}
\end{aligned}$$

Consider time t_i in the radiation era. Then

$$\begin{aligned}
\frac{\Omega_i^{-1} - 1}{\Omega_0^{-1} - 1} &= \frac{\Omega_i^{-1} - 1}{\Omega_{\mathrm{eq}}^{-1} - 1}\frac{\Omega_{\mathrm{eq}}^{-1} - 1}{\Omega_0^{-1} - 1} \\
&= \left(\frac{a_i}{a_{\mathrm{eq}}}\right)^2 \left(\frac{a_{\mathrm{eq}}}{a_0}\right) \\
&= \left(\frac{T_{\mathrm{eq}}}{T_i}\right)^2 \frac{T_0}{T_{\mathrm{eq}}} && T \propto a^{-1} \text{ for a relativistic gas}
\end{aligned}$$

We choose the Planck time as our starting point:

$$
\begin{aligned}
t_{\text{init}} &= t_{\text{Planck}} := \left(\frac{\hbar G}{c^5}\right)^{1/2} \\
T_{\text{Planck}} &= 10^{32}\,\text{K} \\
T_0 &= 3\,\text{K} \\
T_{\text{eq}} &= 10^4\,\text{K}
\end{aligned}
$$

Then

$$
\frac{\Omega_{\text{Planck}}^{-1} - 1}{\Omega_0^{-1} - 1} \approx 10^{-60}
$$

This means that any deviation of Ω from 1 at the Planck time has been amplified by 60 orders of magnitude until today. This is also called the fine tuning problem, which requires $\Omega = 1$ with extreme accuracy at early times. What is the physical explanation of this?

2.13.3 Monopole problem

Consider $T \approx 10^{14}$ GeV, which was the case during the grand unification (GUT) epoch, where the fundamental forces were unified. At these temperatures, there is a phase transition. Before the transition, the forces are unified, and during the transition, there is a spontaneous symmetry breaking. A prediction for topological defects that happen during this transition requires that there should be magnetic monopoles with an enormous density $\Omega_{\text{monopole},0} \approx 10^{11}$. If you still exist, this scenario is obviously ruled out by observations.

2.13.4 Structure formation problem

Today, even though the cosmological principle holds for large distance scales (a few 100 Mpc), we still see many structures on various smaller scales in the universe. What were the “seeds”, the initial perturbations that are responsible for the creation of these structures?

2.13.5 Initial condition problem

All of these problems can be resolved by postulating very special and finely tuned initial conditions. These could arise from physics in the quantum gravity era. The benefit of the inflation model is that all these problems could be resolved without having to resort to quantum gravity.

2.14 Inflation

Inflation is a possible solution for the disparities that we have seen in the previous section. We will tackle them one by one from here.

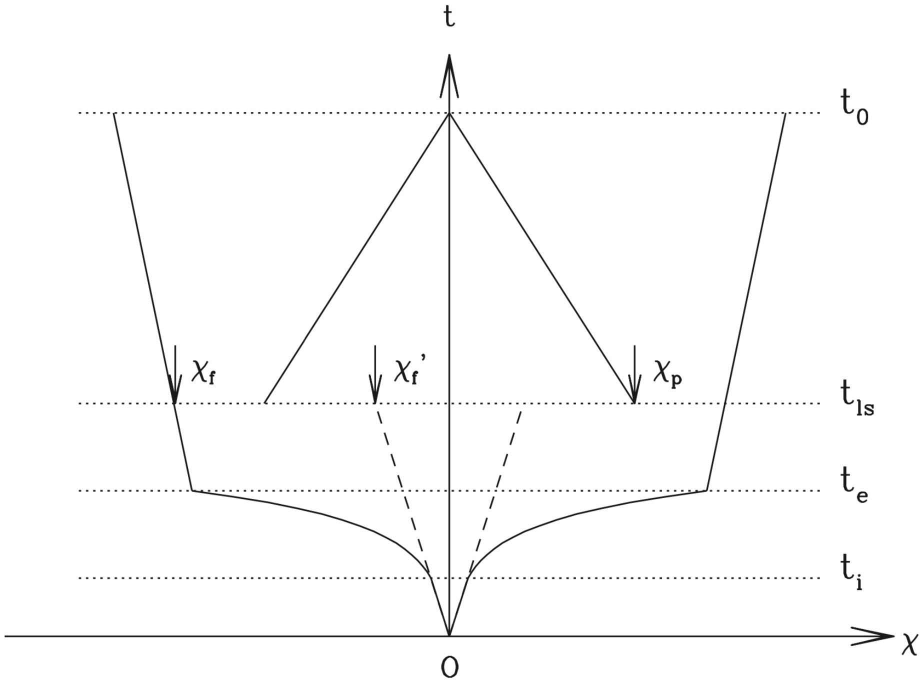


Figure 2.8: The light cone structure in an inflationary universe. Without inflation, the forward light cone (dashed line) would be smaller than our past light cone, χ_p, at the last scattering surface, resulting in causality problems. With a period of inflation from t_i to t_e, however, the forward light cone can be larger than the past light cone at t_{ls}.

2.14.1 Horizon problem

We define two things two reformulate the horizon problem:

- The size of the forward light cone:
$$\chi_f = \int_0^{t_{\text{ls}}} \frac{c\,\mathrm{d}t'}{a(t')}$$
- The size of the past light cone:
$$\chi_p = \int_{t_{\text{ls}}}^{t_0} \frac{c\,\mathrm{d}t'}{a(t')}$$
This is the maximum comoving distance that photons could have travelled since last scattering.

The horizon problem can be stated mathematically as $\chi_f' < \chi_p$: Regions which are observable today to have similar temperatures were apparently not in causal contact at last scattering. A solution is to increase χ_f' through a period of accelerated expansion called inflation. A sketch of this is shown in fig. 2.8.

Let t_i and t_e be the start and end time of inflation, with $\Delta t = t_e - t_i$. We know

$$\chi_f = \chi_h(t_{\text{ls}}) = \int_0^{t_{\text{ls}}} \frac{c\,\mathrm{d}t'}{a(t')}$$

For vacuum energy, we know that $\rho_{\text{vac}}(a)$ is constant. We assume that, at early times, there was some kind of vacuum energy which gives us

$$a \propto e^{Ht} \text{ with } H = \sqrt{8\pi G\rho_{\text{vac}}/3} = \text{constant},$$

exactly like our derivations for the cosmological constant. The contribution to χ_f during inflation is then

$$\begin{aligned}\chi_f(t_i, t_e) &= \int_{t_i}^{t_e} \frac{c\,\mathrm{d}t'}{a(t')} \\ &\propto \frac{1}{Ha(t_e)}(e^{H\Delta t} - 1)\end{aligned}$$

We see that χ_f grows exponentially during inflation. For how long does inflation have to last to solve our problems? We need $\Delta t > 60H^{-1}$, or $e^{H\Delta t} < 10^{25}$, if we assume $t_e \approx t_{\mathrm{GUT}}$. We thus require $a_e/a_i > e^{60}$, or in other words, we need 60 "e-foldings".

2.14.2 Flatness problem

One can show that

$$\frac{\Omega^{-1}(t_e) - 1}{\Omega^{-1}(t_i) - 1} = \frac{a(t_i)}{a(t_e)} < 10^{-52}$$

assuming 60 e-foldings. As a result, any curvature that was originally there gets flattened by inflation.

2.14.3 Monopole problem

The monopoles are diluted by the expansion during inflation by a factor

$$\left(\frac{a_e}{a_i}\right)^3 \approx 10^{78},$$

so the monopole density after inflation is practically zero.

2.14.4 Structure formation problem

As we will see later, inflation provides a mechanism to generate primordial fluctuations. Microscopic quantum fluctuations are turned into macroscopic classical fluctuations by the rapid expansion.

2.14.5 Initial condition problem

Inflation avoids having to set finely tuned initial conditions, since they will be diluted by the expansion. Quantum gravity is not required to solve this problem.

2.14.6 Realization

Inflation requires vacuum energy, which is realized as a scalar field $\phi(\boldsymbol{x}, t)$, called the *inflaton*. The potential of the scalar field is sketched in fig. 2.9.

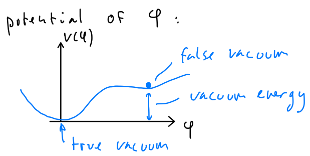


Figure 2.9: The potential of the inflaton field. During inflation, the field moves from its initial false vacuum state to a true vacuum state, which generates vaccum energy.

During inflation, ϕ is in a false vacuum state at a local minimum or a flat part of the potential. During the end of inflation, ϕ reaches the true vacuum state, which has a lower potential energy. This generates vacuum energy, which in turn drives accelerated expansion.

We impose a *slow roll condition*, which impedes the field from changing too quickly:

$$\dot{\phi}^2 \ll V(\phi)$$

There are several models, some of which are old inflation, new inflation, and chaotic inflation. There are many other models proposed, indicating that inflation is still an ongoing field of research.

Linear Perturbations | 3

We now drop the assumption that the universe is homogeneous and isotropic. In this chapter, we analyse small perturbations about the background, which can be handled with linear perturbation theory. These perturbations will lead to the structures that we observe today.

We are going to make a few approximations:

- The perturbations are sufficiently small to be treated by linear perturbation theory.
- We ignore relativistic effects and use a Newtonian approximation.

There are several ways to model this structure.

3.1 Ideal Fluid

First, we model the content of the universe as an expanding, self-gravitating, ideal fluid in the Newtonian approximation. We consider the following parameters:

- density $\rho(\boldsymbol{x}, t)$
- pressure $p(\boldsymbol{x}, t)$
- velocity $\boldsymbol{u}(\boldsymbol{x}, t) = \dot{\boldsymbol{r}}$
- gravitational potential $\phi(\boldsymbol{x}, t)$

The time evolution of these parameters is described by the *fluid equations*:

$$\begin{aligned}
&\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\boldsymbol{\nabla}_r \cdot \boldsymbol{u} && \text{continuity equation, conservation of mass} \\
&\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\frac{\boldsymbol{\nabla}_r p}{\rho} - \boldsymbol{\nabla}_r\phi && \text{acceleration = pressure force + gravitational force} \\
&\nabla_r^2\phi = 4\pi G\rho && \text{Poisson equation}
\end{aligned}$$

We define the *convective derivative*

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}_r,$$

which is the time derivative as one moves along fluid elements.

Assumptions

Let's consider the case where the fluid is expanding. In order to describe this, we can write the position vector as $\boldsymbol{r} = a(t)\boldsymbol{x}$, where $\boldsymbol{x}$ is a comoving coordinate. In the Newtonian treatment, we assume that the scale factor is a known input for the calculation. Then the (total) velocity is

$$\boldsymbol{u} = \dot{a}(t)\boldsymbol{x} + \boldsymbol{v},$$

where the first term is the Hubble expansion, and $\boldsymbol{v} = a\dot{\boldsymbol{x}}$ is the peculiar velocity.

We write the density as

$$\rho(\boldsymbol{x},t) = \bar{\rho}(t)[1+\delta(\boldsymbol{x},t)],$$

where $\bar{\rho}(t) \propto a^{-3}$ is the mean background density, and $\delta(\boldsymbol{x},t)$ is a small density perturbation.

We can plug these assumptions and definitions into the fluid equations to get

$$\begin{aligned}
\frac{\partial\delta}{\partial t} + \frac{1}{a}\boldsymbol{\nabla}\cdot[(1+\delta)\boldsymbol{v}] = 0\\
\frac{\partial\boldsymbol{v}}{\partial t} + \frac{\dot{a}}{a}\boldsymbol{v} + \frac{1}{a}[\boldsymbol{v}\cdot\boldsymbol{\nabla}]\boldsymbol{v} = -\frac{\boldsymbol{\nabla}\Phi}{a} - \frac{\boldsymbol{\nabla}p}{a\bar{\rho}(1+\delta)}\\
\nabla^2\Phi = 4\pi G\bar{\rho}a^2\delta
\end{aligned}$$

where

$$\Phi = \phi + \frac{1}{2}a\ddot{a}x^2$$

and $\boldsymbol{\nabla} = \boldsymbol{\nabla}_{\boldsymbol{x}}$.

Thermodynamics

We have three equations in four unknowns, so we still need the equation of state, which relates pressure to other variables, such as density and entropy.

The first law of thermodynamics states that internal energy can either be added by the means of heat or work:

$$\mathrm{d}U = \mathrm{d}Q + \mathrm{d}W$$

We can use $\mathrm{d}W = -p\,\mathrm{d}V$ and $\mathrm{d}Q = T\,\mathrm{d}S$. From now on, we assume we are dealing with an ideal gas. The equation of state for an ideal gas is

$$\begin{aligned}
p &= nk_BT\\
&= \frac{\rho}{\mu m_p}k_BT
\end{aligned}$$

where μ is the mean molecular weight and $n = N/V$ is the number density. The internal energy is

$$\begin{aligned}
U &= \frac{3}{2}Nk_BT\\
&= \frac{3}{2}\frac{\rho}{\mu m_p}Vk_BT
\end{aligned}$$

Then

$$\frac{\boldsymbol{\nabla}p}{\bar{\rho}} = \frac{1}{\bar{\rho}}\left[\left(\frac{\partial p}{\partial\rho}\right)_S\boldsymbol{\nabla}\rho + \left(\frac{\partial p}{\partial S}\right)_\rho\boldsymbol{\nabla}S\right]$$

Let $c_s^2 = (\partial p/\partial\rho)_S$ be the adiabatic sound speed squared. Then we get

$$\left(\frac{\partial p}{\partial S}\right)_\rho = \frac{2}{3}\rho T$$

and thus

$$\frac{\boldsymbol{\nabla}p}{\bar{\rho}} = c_s^2\boldsymbol{\nabla}\delta + \frac{2}{3}(1+\delta)T\boldsymbol{\nabla}S$$

The Euler equation can then be rewritten as

$$\frac{\partial\boldsymbol{v}}{\partial t} + \frac{\dot{a}}{a}\boldsymbol{v} + \frac{1}{a}(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = -\frac{\boldsymbol{\nabla}\Phi}{a} - \frac{c_s^2}{a}\frac{\boldsymbol{\nabla}\delta}{1+\delta} - \frac{2T}{3a}\boldsymbol{\nabla}S$$

Solving the fluid equation

For small density perturbations δ and small peculiar velocities $\boldsymbol{v}$, we can drop terms that are quadratic in these variables, such as $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$. We get the linearized fluid equations:

$$\frac{\partial\delta}{\partial t}+\frac{1}{a}\nabla\cdot\boldsymbol{v}=0$$
$$\frac{\partial\boldsymbol{v}}{\partial t}+\frac{\dot a}{a}\boldsymbol{v}=-\frac{\nabla\Phi}{a}-\frac{c_s^2}{a}\nabla\delta-\frac{2\bar T}{3a}\nabla S$$

We can combine them to get a single equation. First, take the derivative of the continuity equation. Then, use the Euler equation and the Poisson equation to get the following:

$$\frac{\partial^2\delta}{\partial t^2}+2\frac{\dot a}{a}\frac{\partial\delta}{\partial t}=4\pi G\bar\rho\delta+\frac{c_s^2}{a^2}\nabla^2\delta+\frac{2}{3}\frac{\bar T}{a^2}\nabla^2 S$$

This is a second order differential equation. We take the Fourier transform:

$$\delta(\boldsymbol{x},t)=\sum_{\boldsymbol{k}}\delta_{\boldsymbol{k}}\exp(i\boldsymbol{k}\cdot\boldsymbol{x})$$
$$\delta_{\boldsymbol{k}}(t)=\frac{1}{V}\int\delta(\boldsymbol{x},t)\exp(-i\boldsymbol{k}\cdot\boldsymbol{x})\,\mathrm{d}^3x\,,$$

where V is the volume of a sufficiently large box. The fluid equation then becomes[3]

[3] $\nabla\to i\boldsymbol{k}$ and $\nabla^2\to -k^2$

$$\frac{\mathrm{d}^2\delta_{\boldsymbol{k}}}{\mathrm{d}t^2}+2\frac{\dot a}{a}\frac{\mathrm{d}\delta_{\boldsymbol{k}}}{\mathrm{d}t}=\left[4\pi G\bar\rho-\frac{k^2c_s^2}{a^2}\right]\delta_{\boldsymbol{k}}-\frac{2}{3}k^2\frac{\bar T}{a^2}S_{\boldsymbol{k}}$$

The Poisson equation in Fourier space is

$$-k^2\Phi_{\boldsymbol{k}}=4\pi G\bar\rho a^2\delta_{\boldsymbol{k}}$$

The fluid equation can now be solved for every $\boldsymbol{k}$-mode independently.

We need initial conditions to solve the fluid equation. There are two pressure terms, $\delta_{\boldsymbol{k}}$ and $S_{\boldsymbol{k}}$. We distinguish two types of initial conditions:

- $\delta\neq 0$, $\delta S=0$: isentropic or adiabatic or curvature perturbations
- $\delta=0$, $\delta S\neq 0$: isocurvature perturbations

Adiabatic perturbations are naturally generated by inflation, so we only consider those and set $\delta S=0$.

The fluid equation can now be simplified further:

$$\frac{\mathrm{d}^2\delta_{\boldsymbol{k}}}{\mathrm{d}t^2}+2\frac{\dot a}{a}\frac{\mathrm{d}\delta_{\boldsymbol{k}}}{\mathrm{d}t}=\left[4\pi G\bar\rho-\frac{k^2c_s^2}{a^2}\right]\delta_{\boldsymbol{k}}$$

Consider the case where the expansion of the fluid can be ignored, so a is a constant. Then $\dot a=0$, so a further term drops out:

$$\frac{\mathrm{d}^2\delta_{\boldsymbol{k}}}{\mathrm{d}t^2}=\left[4\pi G\bar\rho-\frac{k^2c_s^2}{a^2}\right]\delta_{\boldsymbol{k}}$$
$$=-\omega^2\delta_{\boldsymbol{k}}$$

This is simply the equation of motion of a harmonic oscillator. We can rewrite

$$\omega^2=\left(\frac{c_s}{a}\right)^2[k^2-k_J^2]$$

where k_J is the JEANS wavenumber

$$k_J = \frac{a}{c_s}\sqrt{4\pi G\bar{\rho}}.$$

One can also define the Jeans wavelength

$$\lambda_J = \frac{2\pi a}{k_J},$$

and the Jeans mass

$$M_J = \frac{\pi}{6}\bar{\rho}\lambda_J^3,$$

which is the mass of a sphere with a radius of $\lambda_J/2$ and density $\bar{\rho}$.

Solutions

We first consider the case where $k > k_J$, or equivalently $\lambda < \lambda_J$, or $M < M_J$. Then $\omega^2 > 0$, and the equation of motion gives us an oscillatory solution:

$$\delta_{\boldsymbol{k}} \propto \exp(\pm i\omega t) \qquad \omega \in \mathbb{R}$$

This means that density perturbations don't grow, but just oscillate.

When $k < k_J$, the solution is

$$\delta_{\boldsymbol{k}} \propto \exp(\pm \alpha t) \qquad \alpha \in \mathbb{R},$$

which indicates exponentially decaying or growing modes. The growing modes lead to the growth of structure, which is called gravitational or Jeans instability. Once the perturbations become large, our assumptions are not valid any more, and we have to switch to non-linear perturbation theory.

Jeans mass

The Jeans mass is

$$\begin{aligned} M_J &= \frac{\pi}{6}\bar{\rho}\lambda_J^3 \\ &= \frac{\pi^{5/2}}{6}\frac{c_s^3}{G^{3/2}\bar{\rho}^{1/2}} \end{aligned}$$

Before recombination, photons and baryons are tightly coupled, and they act as a single fluid. We get $M_J \approx 10^{16}(\Omega_{b,0}h^2)^{-2}M_{\odot}$, which means that no baryonic perturbations smaller than a supercluster can grow before recombination.

After recombination, the baryons and the photons are decoupled. They have much smaller pressure, and $M_J \approx 10^5(\Omega_{b,0}h^2)^{-1/2}M_{\odot}$. Perturbations with masses larger than a globular cluster can grow.

We also have to take into account the expansion of the universe, which we neglected up to now. Furthermore, these results do not consider dark matter, but only baryons.

3.2 Collisionless gas

We now consider a collisionless gas, such as dark matter. In this case, there is no reason to impose thermodynamic equilibrium, not even locally. As a result, we cannot describe the gas with the thermodynamic variables we used before, since they are only defined in (at least local) thermodynamic equilibrium. We thus need to take one step back and consider distribution functions $f(\boldsymbol{x}, \boldsymbol{p}, t)$, whose dynamics are given by the collisionless Boltzmann equation,

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial p^i}\frac{\partial p^i}{\partial t} + \frac{\partial f}{\partial x^i}\frac{\partial x^i}{\partial t} \\ &= 0.\end{aligned}$$

As before, we can take the moments of the distribution functions:

$$\langle Q\rangle = \frac{1}{n}\int \mathrm{d}^3 p\, Q f(\boldsymbol{x}, \boldsymbol{p}, t)$$

with the comoving number density $n(\boldsymbol{x}, t) = \int \mathrm{d}^3 p\, f(\boldsymbol{x}, \boldsymbol{p}, t)$. Some moments are:

- The density $\rho(\boldsymbol{x}, t)$ can be obtained with $Q = m$, where m is the mass of a particle.
- The bulk velocity $\langle v_i\rangle$ can be obtained with $Q = v_i = p_i/(mn)$

We take moments of the Boltzmann equation. One can show that one gets the same equations as the fluid equations: the continuity equation and the Euler equation, with the pressure gradient term replaced as follows:

$$\frac{\nabla_i p}{\rho} \to \sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left[(1+\delta)\sigma_{ij}^2\right]$$

where

$$\sigma_{ij} = \langle v_i v_j\rangle - \langle v_i\rangle\langle v_j\rangle$$

The stress tensor is defined as $\rho\sigma_{ij}^2$.

If the stress tensor is small, we get the fluid equations with $\nabla p = 0$ and $c_s^2 = 0$, so a pressureless fluid.

On small scales, the stress tensor can be important. The large random velocities of the particles can make these particles diffuse and dampen the perturbations. This phenomenon is called *free streaming*. This is not important for cold dark matter, but it is important for hot dark matter.

3.3 Solutions

3.3.1 Collisionless gas

We assume adiabatic perturbations, neglect free streaming, and only consider cold dark matter. In the ideal fluid equations, we can set $p = 0$ and $c_s^2 = 0$. We get

$$\frac{\mathrm{d}^2\delta_{\boldsymbol{k}}}{\mathrm{d}t^2} + 2\frac{\dot a}{a}\frac{\mathrm{d}\delta_{\boldsymbol{k}}}{\mathrm{d}t} = 4\pi\bar\rho_m\delta_{\boldsymbol{k}}$$

We consider solutions for an expanding universe and find the following solutions (homework):

- The solution $\delta_- \propto H(t)$ is decaying, since H becomes smaller over time.

- The growing solution is

$$\delta_+ \propto H(t) \int_0^t \frac{\mathrm{d}t'}{a(t)^2 H(t')},$$

which is a result of gravitational instability. This is similar to what we found before, but this time we did not neglect expansion. We define $D(z) \propto \delta_+$ the linear growth factor.

Examples

- Matter dominated case ($\Omega_0 = \Omega_{m,0} = 1$): $a \propto t^{2/3}$ and $H \propto t^{-1}$. This yields

$$D \propto t^{2/3} \propto a$$

- OCDM ($\Omega_{m,0} < 1$, $\Omega_{\Lambda,0} = 0$) and ΛCDM ($\Omega_{m,0} + \Omega_{\Lambda,0} = 1$): We get an analytical solution where D grows slower than a. The faster expansion from ΛCDM slows down the growth of structure.
TODO Diagram
On the left is the OCDM case, on the right the ΛCDM case.
- In the general case, there is no analytical solution. The growth factor usually grows slower than the exponential growth we found in the non-expanding gravitational instability case. In other words, the universe expansion slows down with the growth of structure.

Let's look at how the gravitational potential grows, because we will need that result later. The Poisson equation in Fourier space is

$$-k^2 \Phi_{\boldsymbol{k}} = 4\pi G \bar{\rho} \delta_{\boldsymbol{k}},$$

which can be rearranged to

$$\begin{aligned} \Phi_{\boldsymbol{k}} &\propto a^2 \bar{\rho}_m \delta_{\boldsymbol{k}} \\ &\propto a^2 a^{-3} D(a) \\ &\propto \frac{D(a)}{a} \end{aligned}$$

This means that $\Phi_{\boldsymbol{k}}$ is constant in the matter dominated case, and $\Phi_{\boldsymbol{k}}$ decays in the OCDM and ΛCDM cases at late times.

3.3.2 Two non-relativistic components

We consider collisionless dark matter and collisionfull baryons. We assume that the pressureless dark matter dominates the matter density, so $\rho_{\text{tot}} \approx \rho_{\text{dm}}$.

There are now two differential equations:

$$\begin{aligned} \frac{\mathrm{d}^2 \delta_{\mathrm{dm},\boldsymbol{k}}}{\mathrm{d}t^2} + 2\frac{\dot{a}}{a}\frac{\mathrm{d}\delta_{\mathrm{dm},\boldsymbol{k}}}{\mathrm{d}t} &= 4\pi G \bar{\rho}_m \delta_{\mathrm{dm},\boldsymbol{k}} \\ \frac{\mathrm{d}^2 \delta_{\mathrm{b},\boldsymbol{k}}}{\mathrm{d}t^2} + 2\frac{\dot{a}}{a}\frac{\mathrm{d}\delta_{\mathrm{b},\boldsymbol{k}}}{\mathrm{d}t} + \frac{k^2 c_s^2}{a^2}\delta_{\mathrm{b},\boldsymbol{k}} &= 4\pi G \bar{\rho}_m \delta_{\mathrm{dm},\boldsymbol{k}} \end{aligned}$$

The first equation is the same as in the previous section, because the low-density baryons don't influence the dark matter much. To get analytical solutions, we assume that we are in the matter dominated case, and that $c_s^2 a$ is constant. The latter is equivalent to the condition that the baryons are a specific kind of polytropic fluid, which obeys $p \propto \rho^{4/3}$.

4 Nonlinear Perturbations

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