Chapter 2

Kinematics

2.1 Introduction

Kinematics is the description of the motion of *points*, *bodies*, and *systems of bodies*. It does only describe **how** things are moving, but **not why**. To describe the kinematics of a moving point, we will refer to position vectors, which are generically defined in $\mathbb{R}^3$, and their derivatives. For an extended body, we need to additionally take into account rotations $\phi \in SO(3)$ to completely define its configuration.

In the following, we will first discuss the basics of kinematics by describing the motion of points and single bodies before moving on to serial systems of bodies in section 2.8.

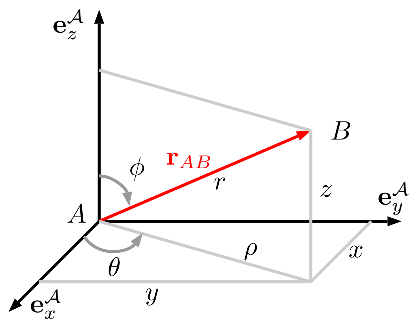


Figure 2.1: Representation of positions using Cartesian, cylindrical, or spherical coordinates.

2.2 Position

The position of a point B relative to point A can be written as

$$\mathbf{r}_{AB}. \tag{2.1}$$

For points in the three dimensional space, positions are represented by vectors $\mathbf{r} \in \mathbb{R}^3$. In order to numerically express the components of a vector, it is necessary to define a reference frame $\mathcal{A}$ and to express the vector in this frame:

$$_{\mathcal{A}}\mathbf{r}_{AB}. \tag{2.2}$$

The unit vectors $\left(\mathbf{e}_x^{\mathcal{A}}, \mathbf{e}_y^{\mathcal{A}}, \mathbf{e}_z^{\mathcal{A}}\right)$ of frame $\mathcal{A}$ form an ortho-normal basis of $\mathbb{R}^3$.

2.2.1 Representation of Positions

Representing a position in the three dimensional space requires three parameters.

Cartesian coordinates

The most common approach is to work with Cartesian coordinates and hence parameterize the position by

$$\boldsymbol{\chi}_{Pc} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{2.3}$$

which implies that a position vector is simply given by

$$_{\mathcal{A}}\mathbf{r} = x\mathbf{e}_x^{\mathcal{A}} + y\mathbf{e}_y^{\mathcal{A}} + z\mathbf{e}_z^{\mathcal{A}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{2.4}$$

Cylindrical coordinates

A second approach is to work with cylindrical coordinates

$$\chi_{Pz} = \begin{pmatrix} \rho \\ \theta \\ z \end{pmatrix}, \tag{2.5}$$

which implies that a position vector is given by

$${}_{\mathcal{A}}\mathbf{r} = \begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \\ z \end{pmatrix}. \tag{2.6}$$

Spherical coordinates

A third method is to use spherical coordinates

$$\chi_{Ps} = \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}, \tag{2.7}$$

which implies that a position vector is given by

$${}_{\mathcal{A}}\mathbf{r} = \begin{pmatrix} r\cos\theta\sin\phi \\ r\sin\theta\sin\phi \\ r\cos\phi \end{pmatrix}. \tag{2.8}$$

Note: All the previously introduced parameterizations require three parameters to describe a position in 3D space, meaning that they are at the same time also minimal representations. This will be different for rotations as shown in section 2.4.5. In most of the cases, people work with Cartesian coordinates due to the simple properties for vector calculus.

2.3 Linear Velocity

The velocity of point B relative to point A is given by

$$\dot{\mathbf{r}}_{AB}. \tag{2.9}$$

For three dimensional space, velocities are represented by vectors $\dot{\mathbf{r}} \in \mathbb{R}^3$. For performing vector algebra, the same rules as introduced for the positions need to hold.

2.3.1 Representation of Linear Velocities

There exists a linear mapping $\mathbf{E}_P(\chi)$ between velocities $\dot{\mathbf{r}}$ and the derivatives of the representation $\dot{\chi}_P$:

$$\dot{\chi}_P = \mathbf{E}_P(\chi_P)\,\dot{\mathbf{r}} \tag{2.10}$$
$$\dot{\mathbf{r}} = \mathbf{E}_P^{-1}(\chi_P)\,\dot{\chi}_P \tag{2.11}$$

Cartesian Coordinates

For Cartesian coordinates, the mapping is simply the identity:

$$\mathbf{E}_{Pc}\left(\boldsymbol{\chi}_{Pc}\right) = \mathbf{E}_{Pc}^{-1}\left(\boldsymbol{\chi}_{Pc}\right) = \mathbb{I} \tag{2.12}$$

Cylindrical Coordinates

For cylindrical coordinates we get

$$\dot{\mathbf{r}}\left(\boldsymbol{\chi}_{Pz}\right) = \begin{pmatrix} \dot{\rho}\cos\theta - \rho\dot{\theta}\sin\theta \\ \dot{\rho}\sin\theta + \rho\dot{\theta}\cos\theta \\ \dot{z} \end{pmatrix}, \tag{2.13}$$

which can be solved for

$$\dot{\boldsymbol{\chi}}_{Pz} = \begin{pmatrix} \dot{\rho} \\ \dot{\theta} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \dot{x}\cos\theta + \dot{y}\sin\theta \\ -\dot{x}\sin\theta/\rho + \dot{y}\cos\theta/\rho \\ \dot{z} \end{pmatrix} = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta/\rho & \cos\theta/\rho & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{E}_{Pz}} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}. \tag{2.14}$$

The inverse is

$$\mathbf{E}_{Pz}^{-1}\left(\boldsymbol{\chi}_{Pz}\right) = \frac{\partial \mathbf{r}\left(\boldsymbol{\chi}_{Pz}\right)}{\partial \boldsymbol{\chi}_{Pz}} = \begin{bmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.15}$$

Spherical Coordinates

Using the same approach for spherical coordinates results in:

$$\mathbf{E}_{Ps} = \begin{bmatrix} \cos\theta\sin\phi & \sin\phi\sin\theta & \cos\phi \\ -\sin\theta/(r\sin\phi) & \cos\theta/(r\sin\phi) & 0 \\ (\cos\phi\cos\theta)/r & (\cos\phi\sin\theta)/r & -\sin\phi/r \end{bmatrix}, \tag{2.16}$$

$$\mathbf{E}_{Ps}^{-1} = \begin{bmatrix} \cos\theta\sin\phi & -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ \sin\phi\sin\theta & r\cos\theta\sin\phi & r\cos\phi\sin\theta \\ \cos\phi & 0 & -r\sin\phi \end{bmatrix}. \tag{2.17}$$

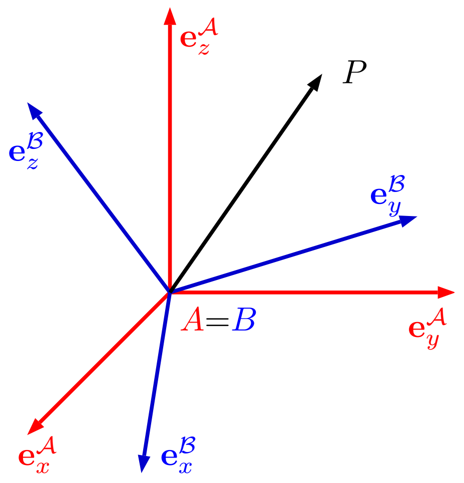


Figure 2.2: Generic 3D rotation between two frames $\mathcal{A}$ and $\mathcal{B}$.

2.4 Rotation

While the configuration of a point is fully described by a position, bodies additionally require a *rotation* to define their pose. As theoretical abstraction of rotations,

$$\phi_{\mathcal{AB}} \in SO(3) \tag{2.18}$$

is often used to indicate the orientation of body fixed frame $\mathcal{B}$ with respect to a reference frame $\mathcal{A}$. It is important to understand that, since $\phi_{\mathcal{AB}}$ lives in $SO(3)$, there is no numerical equivalent to a position such as "angular position". Instead, the orientation $\phi_{\mathcal{AB}}$ can be parametrized in several ways. For a better understanding, we will start by defining the mapping between coordinate frames by means of *rotation matrices* and then show how these relate to different parametrizations.

2.4.1 Rotation Matrices

Consider the situation depicted in Fig. 2.2 with a reference frame $\mathcal{A}$. The position vector of a point P which is fixed in this frame is written as

$${}_{\mathcal{A}}\mathbf{r}_{AP} = \begin{bmatrix} {}_{\mathcal{A}}r_{AP_x} \\ {}_{\mathcal{A}}r_{AP_y} \\ {}_{\mathcal{A}}r_{AP_z} \end{bmatrix}. \tag{2.19}$$

Consider now a reference frame $\mathcal{B}$ which is rotated w.r.t. $\mathcal{A}$. The origin B of frame $\mathcal{B}$ coincides with the origin A of frame $\mathcal{A}$. The position vector of point P, this time expressed in frame $\mathcal{B}$, is

$${}_{\mathcal{B}}\mathbf{r}_{AP} = \begin{bmatrix} {}_{\mathcal{B}}r_{AP_x} \\ {}_{\mathcal{B}}r_{AP_y} \\ {}_{\mathcal{B}}r_{AP_z} \end{bmatrix}. \tag{2.20}$$

By writing the unit vectors of $\mathcal{B}$ expressed in frame $\mathcal{A}$ as $\left({}_{\mathcal{A}}\mathbf{e}_x^{\mathcal{B}}, {}_{\mathcal{A}}\mathbf{e}_y^{\mathcal{B}}, {}_{\mathcal{A}}\mathbf{e}_z^{\mathcal{B}}\right)$, we can write the mapping between the two position vectors ${}_{\mathcal{A}}\mathbf{r}_{AP}$ and ${}_{\mathcal{B}}\mathbf{r}_{AP}$ as

$${}_{\mathcal{A}}\mathbf{r}_{AP} = {}_{\mathcal{A}}\mathbf{e}_x^{\mathcal{B}} \cdot {}_{\mathcal{B}}r_{AP_x} + {}_{\mathcal{A}}\mathbf{e}_y^{\mathcal{B}} \cdot {}_{\mathcal{B}}r_{AP_y} + {}_{\mathcal{A}}\mathbf{e}_z^{\mathcal{B}} \cdot {}_{\mathcal{B}}r_{AP_z}. \tag{2.21}$$

The mapping shown in (2.21) can be rewritten in compact form as

$$\begin{aligned} {}_{\mathcal{A}}\mathbf{r}_{AP} &= \begin{bmatrix} {}_{\mathcal{A}}\mathbf{e}_x^{\mathcal{B}} & {}_{\mathcal{A}}\mathbf{e}_y^{\mathcal{B}} & {}_{\mathcal{A}}\mathbf{e}_z^{\mathcal{B}} \end{bmatrix} \cdot {}_{\mathcal{B}}\mathbf{r}_{AP} \\ &= \mathbf{C}_{\mathcal{AB}} \cdot {}_{\mathcal{B}}\mathbf{r}_{AP}. \end{aligned} \tag{2.22}$$

The term $\mathbf{C}_{\mathcal{AB}}$ is a 3×3 matrix called *rotation matrix*. Since the columns of $\mathbf{C}_{\mathcal{AB}}$ are orthogonal unit vectors, $\mathbf{C}_{\mathcal{AB}}$ is orthogonal, meaning that

$$\mathbf{C}_{\mathcal{AB}}^T \cdot \mathbf{C}_{\mathcal{AB}} = \mathbb{I}_3 \tag{2.23}$$

A consequence of (2.23) is that $\mathbf{C}_{\mathcal{BA}} = \mathbf{C}_{\mathcal{AB}}^{-1} = \mathbf{C}_{\mathcal{AB}}^T$. The rotation matrix $\mathbf{C}_{\mathcal{AB}}$ belongs to the *special orthonormal group SO(3)*. This requires to apply a special type of algebra that is different from classical $\mathbb{R}^3$ vector algebra.

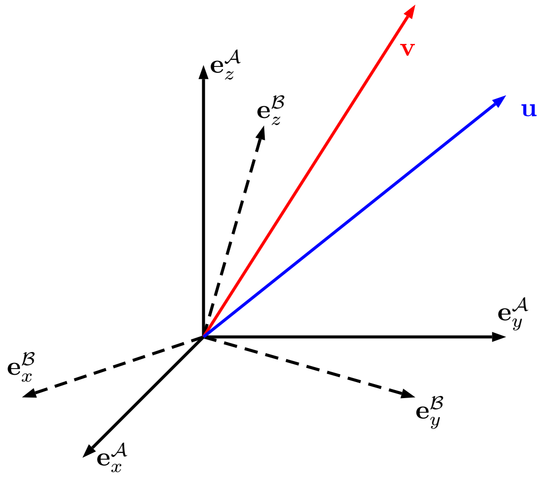


Figure 2.3: A *passive* rotation can be interpreted as the rotation of a coordinate frame, and an *active* rotation as the rotation of an object $\mathbf{u}$ which yields $\mathbf{v}$.

2.4.2 Active vs. Passive Rotation

Rotations can have two different interpretations, which lead to the definition of the so-called *active* and *passive* rotations.

Passive Rotation

Passive rotations, also known as rotation transformations, correspond to a mapping between coordinate frames as shown in (2.21). A passive rotation $\mathbf{C}_{\mathcal{AB}}$ maps the same object $\mathbf{u}$ from frame $\mathcal{B}$ to frame $\mathcal{A}$:

$$ {}_{\mathcal{A}}\mathbf{u} = \mathbf{C}_{\mathcal{AB}} \cdot {}_{\mathcal{B}}\mathbf{u} \tag{2.24}$$

Active Rotation

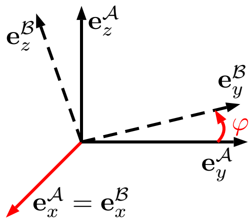
An active rotation, often indicated with a 3×3 matrix $\mathbf{R}$, is an *operator* that rotates a vector ${}_{\mathcal{A}}\mathbf{u}$ to a vector ${}_{\mathcal{A}}\mathbf{v}$ in the same reference frame $\mathcal{A}$:

$$ {}_{\mathcal{A}}\mathbf{v} = \mathbf{R} \cdot {}_{\mathcal{A}}\,\mathbf{u}. \tag{2.25}$$

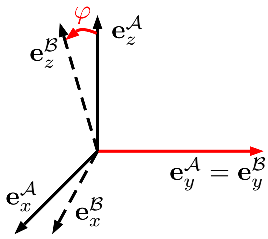
Active rotations are not very relevant in robot dynamics and are hence not used in the course of this lecture.

2.4.3 Elementary Rotations

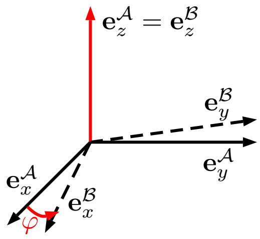
The most simple and at the same time most often appearing rotations are elementary rotations, i.e. rotations around one of the basis vectors $\mathbf{e}_x^{\mathcal{A}}$, $\mathbf{e}_y^{\mathcal{A}}$ or $\mathbf{e}_z^{\mathcal{A}}$. Given a rotation angle φ, the three elementary rotations are:



$$\mathbf{C}_{\mathcal{AB}} = \mathbf{C}_x(\varphi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix} \tag{2.26}$$



$$\mathbf{C}_{\mathcal{AB}} = \mathbf{C}_y(\varphi) = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{bmatrix} \tag{2.27}$$



$$\mathbf{C}_{\mathcal{AB}} = \mathbf{C}_z(\varphi) = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.28}$$

2.4.4 Composition of Rotations

Consider three reference frames $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. The coordinates of vector $\mathbf{u}$ can be mapped from $\mathcal{B}$ to $\mathcal{A}$ by writing

$$_{\mathcal{A}}\mathbf{u} = \mathbf{C}_{\mathcal{AB}} \cdot {_{\mathcal{B}}\mathbf{u}}. \tag{2.29}$$

We can also write

$$_{\mathcal{B}}\mathbf{u} = \mathbf{C}_{\mathcal{BC}} \cdot {_{\mathcal{C}}\mathbf{u}}. \tag{2.30}$$

By combining the last two equations, we can write

$$\begin{aligned} {_{\mathcal{A}}\mathbf{u}} &= \mathbf{C}_{\mathcal{AB}} \cdot (\mathbf{C}_{\mathcal{BC}} \cdot {_{\mathcal{C}}\mathbf{u}}) \\ &= \mathbf{C}_{\mathcal{AC}} \cdot {_{\mathcal{C}}\mathbf{u}}. \end{aligned} \tag{2.31}$$

The resulting rotation matrix $\mathbf{C}_{\mathcal{AC}} = \mathbf{C}_{\mathcal{AB}} \cdot \mathbf{C}_{\mathcal{BC}}$ can be interpreted as the rotation obtained by rotating frame $\mathcal{A}$ until it coincides with frame $\mathcal{B}$, and then rotating frame $\mathcal{B}$ until it coincides with frame $\mathcal{C}$.

2.4.5 Representation of Rotations

As discussed in the previous sections, generic rotations in the three-dimensional space are represented by 3×3 rotation matrices, i.e. by means of 9 parameters. These parameters, however, are not independent, but constrained by the orthogonality conditions shown in (2.23). Hence, only three independent parameters such as *Euler angles* are needed to obtain a minimal representation of rotations in space. Other non-minimal representations can be derived, namely the *angle-axis* and the *unit quaternion* representation. We will briefly discuss advantages and disadvantages of each parametrization, as well as derive the mapping from one implementation to the other. For a more detailed analysis of three dimensional rotations (which goes beyond the scope of this lecture), the reader is referred to [1].

Euler Angles

A rotation in space can be understood as a sequence of three elementary rotations defined in (2.26) to (2.28). To fully describe all possible orientations, two successive rotations should not be made around parallel axes. When the first and third rotations are made around the same axis, the parametrization is called *proper* Euler angles. When all three angles are different, we typically refer to *Tait–Bryan*, *Cardan* or *roll-pitch-yaw* angles. The latter ones are often used in robotics.

Note: Please note again that there applies a different type of algebra to rotations than what we know from typical position vectors. Hence, never add, subtract or simply multiply Euler angles, angle-axis, or quaternions.

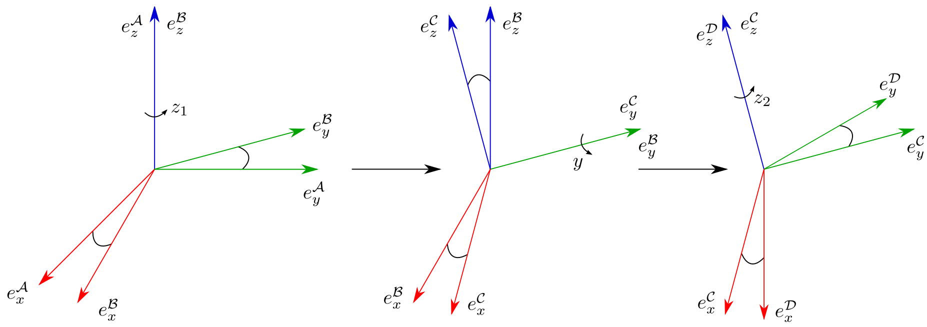


Figure 2.4: ZYZ Euler Angles as three successive rotations around z, y, and z axes. Rotation from $\mathcal{A}$-frame to $\mathcal{D}$-frame: (z–y'–z") – (yaw–pitch–yaw)

ZYZ Euler Angles ZYZ Euler Angles are also known as *proper* Euler Angles. The rotation angles can be collected in a parameter vector

$$\chi_{R,eulerZYZ} = \begin{pmatrix} z_1 \\ y \\ z_2 \end{pmatrix}. \tag{2.32}$$

The resulting rotation matrix is obtained by a concatenation of elementary rotations (see Fig.2.4) given by

$$\begin{aligned} \mathbf{C}_{\mathcal{AD}} &= \mathbf{C}_{\mathcal{AB}}(z_1)\mathbf{C}_{\mathcal{BC}}(y)\mathbf{C}_{\mathcal{CD}}(z_2) \quad \Rightarrow \quad {}_{\mathcal{A}}\mathbf{r} = \mathbf{C}_{\mathcal{AD}\mathcal{D}}\mathbf{r} \\ &= \begin{bmatrix} \cos z_1 & -\sin z_1 & 0 \\ \sin z_1 & \cos z_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos y & 0 & \sin y \\ 0 & 1 & 0 \\ -\sin y & 0 & \cos y \end{bmatrix} \begin{bmatrix} \cos z_2 & -\sin z_2 & 0 \\ \sin z_2 & \cos z_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c_y c_{z_1} c_{z_2} - s_{z_1} s_{z_2} & -c_{z_2} s_{z_1} - c_y c_{z_1} s_{z_2} & c_{z_1} s_y \\ c_{z_1} s_{z_2} + c_y c_{z_2} s_{z_1} & c_{z_1} c_{z_2} - c_y s_{z_1} s_{z_2} & s_y s_{z_1} \\ -c_{z_2} s_y & s_y s_{z_2} & c_y \end{bmatrix}. \end{aligned} \tag{2.33}$$

Analyzing (2.33) allows to find the solution of the inverse problem. Given a rotation matrix

$$\mathbf{C}_{\mathcal{AD}} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}, \tag{2.34}$$

the ZYZ Euler angles are given by

$$\chi_{R,eulerZYZ} = \begin{pmatrix} z_1 \\ y \\ z_2 \end{pmatrix} = \begin{pmatrix} atan2\left(c_{23}, c_{13}\right) \\ atan2\left(\sqrt{c_{13}^2 + c_{23}^2}, c_{33}\right) \\ atan2\left(c_{32}, -c_{31}\right) \end{pmatrix}. \tag{2.35}$$

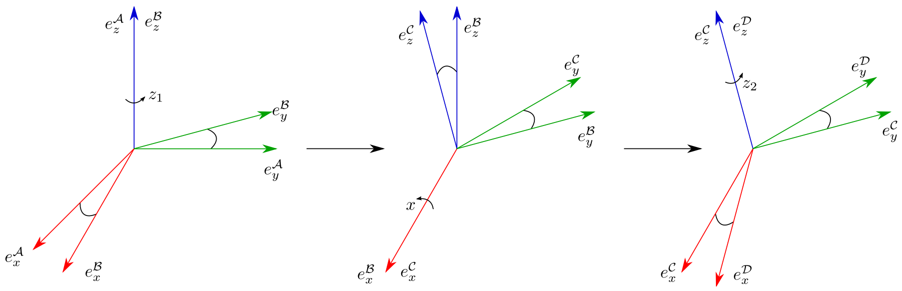


Figure 2.5: ZXZ Euler Angles as three successive rotations around z, x, and z axes. Rotation from $\mathcal{A}$-frame to $\mathcal{D}$-frame: (z–x'–z") – (yaw–roll–yaw)

ZXZ Euler Angles ZXZ Euler Angles are also known as *proper* Euler Angles. The rotation angles can be collected in a parameter vector

$$\boldsymbol{\chi}_{R,eulerZXZ} = \begin{pmatrix} z_1 \\ x \\ z_2 \end{pmatrix}. \tag{2.36}$$

The resulting rotation matrix is obtained by a concatenation of elementary rotations (Fig.2.5) given by

$$\begin{aligned}
\mathbf{C}_{\mathcal{AD}} &= \mathbf{C}_{\mathcal{AB}}(z_1)\mathbf{C}_{\mathcal{BC}}(x)\mathbf{C}_{\mathcal{CD}}(z_2) \quad \Rightarrow \quad {}_{\mathcal{A}}\mathbf{r} = \mathbf{C}_{\mathcal{AD}\,\mathcal{D}}\mathbf{r} \\
&= \begin{bmatrix} \cos z_1 & -\sin z_1 & 0 \\ \sin z_1 & \cos z_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos x & -\sin x \\ 0 & \sin x & \cos x \end{bmatrix} \begin{bmatrix} \cos z_2 & -\sin z_2 & 0 \\ \sin z_2 & \cos z_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_{z_1}c_{z_2} - c_x s_{z_1} s_{z_2} & -c_{z_1}s_{z_2} - c_x c_{z_2} s_{z_1} & s_x s_{z_1} \\ c_{z_2}s_{z_1} + c_x c_{z_1} s_{z_2} & c_x c_{z_1} c_{z_2} - s_{z_1} s_{z_2} & -c_{z_1} s_x \\ s_x s_{z_2} & c_{z_2} s_x & c_x \end{bmatrix}.
\end{aligned} \tag{2.37}$$

Analyzing (2.37) allows to find the solution of the inverse problem. Given a rotation matrix

$$\mathbf{C}_{\mathcal{AD}} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}, \tag{2.38}$$

the Euler Angles are given by

$$\boldsymbol{\chi}_{R,eulerZXZ} = \begin{pmatrix} z_1 \\ x \\ z_2 \end{pmatrix} = \begin{pmatrix} atan2\,(c_{13}, -c_{23}) \\ atan2\left(\sqrt{c_{13}^2 + c_{23}^2}, c_{33}\right) \\ atan2\,(c_{31}, c_{32}) \end{pmatrix}. \tag{2.39}$$

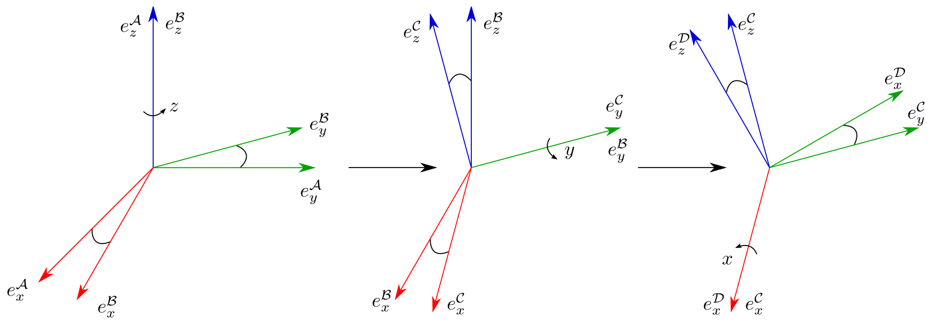


Figure 2.6: ZYX Euler Angles as three successive rotations around z, y, and x. Rotation from $\mathcal{A}$-frame to $\mathcal{D}$-frame: (z–y'–x") – (yaw–pitch–roll)

ZYX Euler Angles ZYX Euler Angles, also known as Tait-Bryan angles, are often used for flying vehicles and called yaw-pitch-roll. The rotation angles can be collected in a parameter vector

$$\chi_{R,eulerZYX} = \begin{pmatrix} z \\ y \\ x \end{pmatrix}. \tag{2.40}$$

The resulting rotation matrix is obtained by a concatenation of elementary rotations given by

$$\begin{aligned}
\mathbf{C}_{\mathcal{AD}} &= \mathbf{C}_{\mathcal{AB}}(z)\mathbf{C}_{\mathcal{BC}}(y)\mathbf{C}_{\mathcal{CD}}(x) \quad \Rightarrow \quad {}_{\mathcal{A}}\mathbf{r} = \mathbf{C}_{\mathcal{AD}}\,{}_{\mathcal{D}}\mathbf{r} \\
&= \begin{bmatrix} \cos z & -\sin z & 0 \\ \sin z & \cos z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos y & 0 & \sin y \\ 0 & 1 & 0 \\ -\sin y & 0 & \cos y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos x & -\sin x \\ 0 & \sin x & \cos x \end{bmatrix} \\
&= \begin{bmatrix} c_y c_z & c_z s_x s_y - c_x s_z & s_x s_z + c_x c_z s_y \\ c_y s_z & c_x c_z + s_x s_y s_z & c_x s_y s_z - c_z s_x \\ -s_y & c_y s_x & c_x c_y \end{bmatrix}.
\end{aligned} \tag{2.41}$$

Given a rotation matrix as in (2.38), the inverse solution is

$$\chi_{R,eulerZYX} = \begin{pmatrix} z \\ y \\ x \end{pmatrix} = \begin{pmatrix} atan2\left(c_{21}, c_{11}\right) \\ atan2\left(-c_{31}, \sqrt{c_{32}^2 + c_{33}^2}\right) \\ atan2\left(c_{32}, c_{33}\right) \end{pmatrix}. \tag{2.42}$$

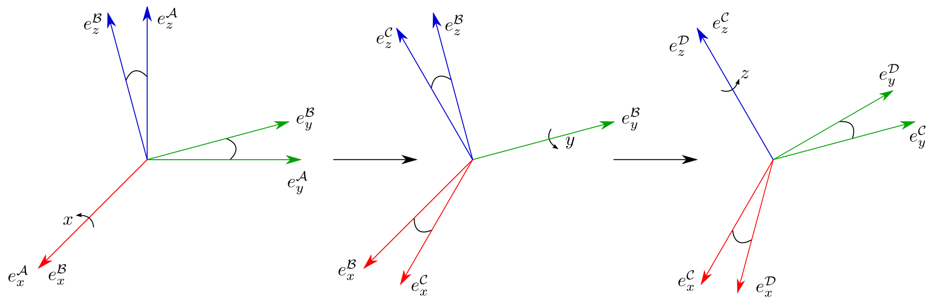


Figure 2.7: XYZ Euler Angles as three successive rotations around x, y, and z. Rotation from $\mathcal{A}$-frame to $\mathcal{D}$-frame: (x–y'–z") – (roll–pitch–yaw)

XYZ Euler Angles XYZ Euler Angles are also known as Cardan angles. The rotation angles can be collected in a parameter vector

$$\chi_{R,eulerXYZ} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{2.43}$$

The resulting rotation matrix is obtained by a concatenation of elementary rotations (see Fig.2.7) given by

$$\begin{aligned} \mathbf{C}_{\mathcal{AD}} &= \mathbf{C}_{\mathcal{AB}}(x)\mathbf{C}_{\mathcal{BC}}(y)\mathbf{C}_{\mathcal{CD}}(z) \quad \Rightarrow \quad {}_{\mathcal{A}}\mathbf{r} = \mathbf{C}_{\mathcal{AD}\,\mathcal{D}}\mathbf{r} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos x & -\sin x \\ 0 & \sin x & \cos x \end{bmatrix} \begin{bmatrix} \cos y & 0 & \sin y \\ 0 & 1 & 0 \\ -\sin y & 0 & \cos y \end{bmatrix} \begin{bmatrix} \cos z & -\sin z & 0 \\ \sin z & \cos z & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c_y c_z & -c_y s_z & s_y \\ c_x s_z + c_z s_x s_y & c_x c_z - s_x s_y s_z & -c_y s_x \\ s_x s_z - c_x c_z s_y & c_z s_x + c_x s_y s_z & c_x c_y \end{bmatrix}. \end{aligned} \tag{2.44}$$

Given a rotation matrix as in (2.38), the inverse solution is

$$\chi_{R,eulerXYZ} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} atan2\left(-c_{23}, c_{33}\right) \\ atan2\left(c_{13}, \sqrt{c_{11}^2 + c_{12}^2}\right) \\ atan2\left(-c_{12}, c_{11}\right) \end{pmatrix}. \tag{2.45}$$

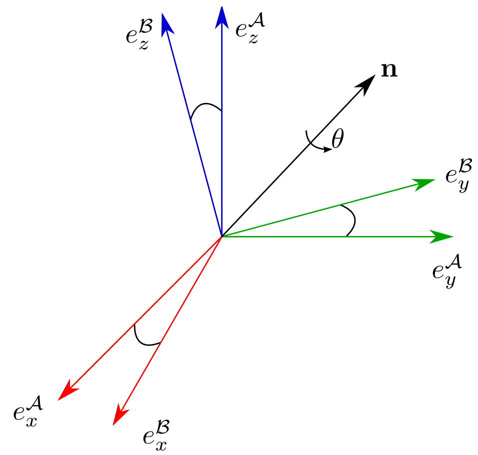


Figure 2.8: The angle-axis representation defines the orientation of two coordinate frames as a rotation of an angle θ around an axis $\mathbf{n}$.

Angle Axis

The *angle-axis* is a non-minimal implementation of rotations which is defined by an angle θ and an axis $\mathbf{n}$. The vector $\mathbf{n} \in \mathbb{R}^3$ defines the direction around which the rotation is made, while the scalar $\theta \in \mathbb{R}$ defines the rotation magnitude:

$$\boldsymbol{\chi}_{R,AngleAxis} = \begin{pmatrix} \theta \\ \mathbf{n} \end{pmatrix} \tag{2.46}$$

This representation features four parameters and the unitary length constraint $\|\mathbf{n}\| = 1$. It is possible to combine these two quantities to obtain a *rotation vector*, or *Euler vector*, defined as

$$\boldsymbol{\varphi} = \theta \cdot \mathbf{n} \in \mathbb{R}^3. \tag{2.47}$$

It is important to note that, although $\boldsymbol{\varphi}$ belongs to $\mathbb{R}^3$, the sum operation is in general non-commutative, i.e. $\boldsymbol{\varphi}_1 + \boldsymbol{\varphi}_2 \neq \boldsymbol{\varphi}_2 + \boldsymbol{\varphi}_1$.

With angle axis parameters $\boldsymbol{\varphi}_{\mathcal{AB}}$, the rotation matrix results to

$$\mathbf{C}_{\mathcal{AB}} = \begin{bmatrix} n_x^2(1-c_\theta)+c_\theta & n_x n_y(1-c_\theta)-n_z s_\theta & n_x n_z(1-c_\theta)+n_y s_\theta \\ n_x n_y(1-c_\theta)+n_z s_\theta & n_y^2(1-c_\theta)+c_\theta & n_y n_z(1-c_\theta)-n_x s_\theta \\ n_x n_z(1-c_\theta)-n_y s_\theta & n_y n_z(1-c_\theta)+n_x s_\theta & n_z^2(1-c_\theta)+c_\theta \end{bmatrix}. \tag{2.48}$$

Given a rotation matrix as in (2.38), the angle axis parameters are:

$$\theta = cos^{-1}\left(\frac{c_{11}+c_{22}+c_{33}-1}{2}\right), \tag{2.49}$$

$$\mathbf{n} = \frac{1}{2sin(\theta)}\begin{pmatrix} c_{32}-c_{23} \\ c_{13}-c_{31} \\ c_{21}-c_{12} \end{pmatrix}. \tag{2.50}$$

As it can be seen from (2.50), this representation encounters a problem for $\theta = 0$ and $\theta = \pi$ since $sin(\theta) = 0$, meaning that the rotation vector is not defined. For $\theta = 0$ it can have any direction, for $\theta = \pi$ it can point in two opposite directions.

Unit Quaternions

A non-minimal representation of rotations which does not suffer from the disadvantage encountered with the angle axis is provided by *unit quaternions*, also known as Euler parameters. Considering a rotational vector $\boldsymbol{\varphi} \in \mathbb{R}^3$, a unit quaternion $\boldsymbol{\xi}$ is defined by

$$\boldsymbol{\chi}_{R,quat} = \boldsymbol{\xi} = \begin{pmatrix} \xi_0 \\ \check{\boldsymbol{\xi}} \end{pmatrix} \in \mathbb{H}, \tag{2.51}$$

where

$$\begin{aligned} \xi_0 &= \cos\left(\frac{\|\boldsymbol{\varphi}\|}{2}\right) = \cos\left(\frac{\theta}{2}\right) \\ \check{\boldsymbol{\xi}} &= \sin\left(\frac{\|\boldsymbol{\varphi}\|}{2}\right)\frac{\boldsymbol{\varphi}}{\|\boldsymbol{\varphi}\|} = \sin\left(\frac{\theta}{2}\right)\mathbf{n} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}. \end{aligned} \tag{2.52}$$

The first parameter ξ_0 is called the real part of the quaternion, the latter $\check{\boldsymbol{\xi}}$ the imaginary part. The unit quaternion fulfills the constraint

$$\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2 = 1. \tag{2.53}$$

Similarly to the angle axis, the rotation matrix calculated from quaternions is

$$\begin{aligned} \mathbf{C}_{\mathcal{AD}} &= \mathbb{I}_{3\times 3} + 2\xi_0\left[\check{\boldsymbol{\xi}}\right]_\times + 2\left[\check{\boldsymbol{\xi}}\right]_\times^2 = \left(2\xi_0^2 - 1\right)\mathbb{I}_{3\times 3} + 2\xi_0\left[\check{\boldsymbol{\xi}}\right]_\times + 2\check{\boldsymbol{\xi}}\check{\boldsymbol{\xi}}^T \\ &= \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2q_1q_2 - 2q_0q_3 & 2q_0q_2 + 2q_1q_3 \\ 2q_0q_3 + 2q_1q_2 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2q_2q_3 - 2q_0q_1 \\ 2q_1q_3 - 2q_0q_2 & 2q_0q_1 + 2q_2q_3 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}. \end{aligned} \tag{2.54}$$

Given a rotation matrix as in (2.38), the corresponding quaternions are

$$\boldsymbol{\chi}_{R,quat} = \boldsymbol{\xi}_{\mathcal{AD}} = \tfrac{1}{2}\begin{pmatrix} \sqrt{c_{11} + c_{22} + c_{33} + 1} \\ sgn(c_{32} - c_{23})\sqrt{c_{11} - c_{22} - c_{33} + 1} \\ sgn(c_{13} - c_{31})\sqrt{c_{22} - c_{33} - c_{11} + 1} \\ sgn(c_{21} - c_{12})\sqrt{c_{33} - c_{11} - c_{22} + 1} \end{pmatrix}. \tag{2.55}$$

When working with quaternions, there exists a special algebra that allows to directly work with the quaternion paramterization (and not only the rotation matrices). The interested reader should have a look at [5], which provides a very complete, yet compact and well understandable introduction to quaternions. In short, the following rules are important. For inversion with $\boldsymbol{\xi}^{-1}$ parameterizing $\mathbf{C}^{-1} = \mathbf{C}^T$ it holds that:

$$\boldsymbol{\xi} = \begin{pmatrix} \xi \\ \check{\boldsymbol{\xi}} \end{pmatrix} \quad \overset{inverse}{\rightarrow} \quad \boldsymbol{\xi}^{-1} = \begin{pmatrix} \xi \\ -\check{\boldsymbol{\xi}} \end{pmatrix} \tag{2.56}$$

If $\boldsymbol{\xi}_{\mathcal{AB}}$ and $\boldsymbol{\xi}_{\mathcal{BC}}$ represent the quaternions corresponding to $\mathbf{C}_{\mathcal{AB}}$ and $\mathbf{C}_{\mathcal{BC}}$, their multiplication associated with $\mathbf{C}_{\mathcal{AC}} = \mathbf{C}_{\mathcal{AB}}\mathbf{C}_{\mathcal{BC}}$ is

$$\boldsymbol{\xi}_{\mathcal{AB}} \otimes \boldsymbol{\xi}_{\mathcal{BC}} = \begin{pmatrix} \xi_{0,\mathcal{AB}} \cdot \xi_{0,\mathcal{BC}} - \check{\boldsymbol{\xi}}_{\mathcal{AB}}^T \cdot \check{\boldsymbol{\xi}}_{\mathcal{BC}} \\ \xi_{0,\mathcal{AB}} \cdot \check{\boldsymbol{\xi}}_{\mathcal{BC}} + \xi_{0,\mathcal{BC}} \cdot \check{\boldsymbol{\xi}}_{\mathcal{AB}} + \left[\check{\boldsymbol{\xi}}_{\mathcal{AB}}\right]_\times \cdot \check{\boldsymbol{\xi}}_{\mathcal{BC}} \end{pmatrix} \tag{2.57}$$

$$= \begin{bmatrix} \xi_0 & -\xi_1 & -\xi_2 & -\xi_3 \\ \xi_1 & \xi_0 & -\xi_3 & \xi_2 \\ \xi_2 & \xi_3 & \xi_0 & -\xi_1 \\ \xi_3 & -\xi_2 & \xi_1 & \xi_0 \end{bmatrix}_{\mathcal{AB}} \begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}_{\mathcal{BC}} \tag{2.58}$$

2.5 Angular Velocity

Consider a frame $\mathcal{B}$ which is moving with respect to a fixed frame $\mathcal{A}$. The *angular velocity* ${}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}}$, which describes the rotational motion of $\mathcal{B}$ w.r.t. $\mathcal{A}$, is defined by the limit

$${}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}} = \lim_{\epsilon \to 0} \frac{\boldsymbol{\varphi}_{\mathcal{B}(t)\mathcal{B}(t+\epsilon)}}{\epsilon}. \tag{2.59}$$

As discussed in the last section, a rotational vector $\boldsymbol{\varphi}$ is, in general, not a proper vector. However, for $\epsilon \to 0$, the angular velocity is defined as the ratio of a proper vector $\boldsymbol{\varphi}_{\mathcal{B}(t)\mathcal{B}(t+\epsilon)}$ and a scalar. Hence, angular velocities can be summed according to the rules of vector summation. Consequently, the relative rotation of $\mathcal{A}$ w.r.t. $\mathcal{B}$ is

$$\boldsymbol{\omega}_{\mathcal{AB}} = -\boldsymbol{\omega}_{\mathcal{BA}}. \tag{2.60}$$

It can be shown that the relationship between the angular velocity vector ${}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}}$ and a time varying rotation matrix $\mathbf{C}_{\mathcal{AB}}(t)$ is defined by

$$[{}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}}]_{\times} = \dot{\mathbf{C}}_{\mathcal{AB}} \cdot \mathbf{C}_{\mathcal{AB}}^T, \tag{2.61}$$

with $[{}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}}]_{\times}$ being the skew symmetric matrix of ${}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}}$:

$$[{}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}}]_{\times} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}, \qquad {}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}. \tag{2.62}$$

Angular velocities can be transformed like other vectors:

$${}_{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{AB}} = \mathbf{C}_{\mathcal{BA}} \cdot {}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}}. \tag{2.63}$$

The corresponding cross-product matrix $[\boldsymbol{\omega}]_{\times}$ is transformed by

$$[{}_{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{AB}}]_{\times} = \mathbf{C}_{\mathcal{BA}} \cdot [{}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}}]_{\times} \cdot \mathbf{C}_{\mathcal{AB}}. \tag{2.64}$$

The angular velocity of consecutive coordinate systems is given by

$${}_{\mathcal{D}}\boldsymbol{\omega}_{\mathcal{AC}} = {}_{\mathcal{D}}\boldsymbol{\omega}_{\mathcal{AB}} + {}_{\mathcal{D}}\boldsymbol{\omega}_{\mathcal{BC}} \tag{2.65}$$

Similarly to the vector addition used before, it is important that all vectors are expressed in the same reference system $\mathcal{D}$.

Example 2.5.1: Angular velocity from rotation matrix

Determine the angular velocity of $\mathcal{B}$ with respect to $\mathcal{A}$ in case of an elementary rotation with $\alpha(t)$ around $\mathbf{e}_x^{\mathcal{A}}$ using the corresponding rotation matrix:

$$\mathbf{C}_{\mathcal{AB}}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha(t)) & \sin(\alpha(t)) \\ 0 & -\sin(\alpha(t)) & \cos(\alpha(t)) \end{bmatrix} \tag{2.66}$$

$$[{}_{\mathcal{A}}\omega_{\mathcal{AB}}]_\times = \dot{\mathbf{C}}_{\mathcal{AB}}\mathbf{C}_{\mathcal{AB}}^T \tag{2.67}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\dot{\alpha}\sin\alpha & \dot{\alpha}\cos\alpha \\ 0 & -\dot{\alpha}\cos\alpha & -\dot{\alpha}\sin\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \tag{2.68}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \dot{\alpha} \\ 0 & -\dot{\alpha} & 0 \end{bmatrix} \tag{2.69}$$

Un-skewing this matrix results in:

$${}_{\mathcal{A}}\omega_{\mathcal{AB}} = \begin{pmatrix} \dot{\alpha} \\ 0 \\ 0 \end{pmatrix}. \tag{2.70}$$

2.5.1 Time Derivatives of Rotation Parameterizations

As introduced in section 2.4.5, there exist different parameterizations for a rotation. Similarly to what we have seen for linear velocity, their derivatives can be mapped to angular velocity:

$${}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}} = \mathbf{E}_R(\boldsymbol{\chi}_R) \cdot \dot{\boldsymbol{\chi}}_R. \tag{2.71}$$

In the following, these mappings will be derived and discussed.

Time Derivatives of Euler Angles ZYX ⇔ Angular Velocity

Given a set of ZYX Euler angles $\boldsymbol{\chi}_{R,eulerZYX} = \begin{bmatrix} z & y & x \end{bmatrix}^T$ and their time derivatives $\dot{\boldsymbol{\chi}}_{R,eulerZYX} = \begin{bmatrix} \dot{z} & \dot{y} & \dot{x} \end{bmatrix}^T$, we wish to find the mapping $\mathbf{E}_{R,eulerZYX} = \mathbf{E}_R(\boldsymbol{\chi}_{R,eulerZYX}) \in \mathbb{R}^{3\times 3}$ that maps $\dot{\boldsymbol{\chi}}$ to ${}_{A}\boldsymbol{\omega}_{AB}$. The columns of $\mathbf{E}(\boldsymbol{\chi}_{R,eulerZYX})$ are the components of the unit vectors around which the angular velocities are applied expressed in a fixed frame $\mathcal{A}$. These are obtained by selecting the columns of a rotation matrix which is built up by successive elementary rotations specified by the Euler angle parametrization.

Starting from the reference frame $\mathcal{A}$, the first rotation will be an elementary rotation around ${}_{A}\mathbf{e}_z^{\mathcal{A}}$, which is simply given by

$${}_{\mathcal{A}}\mathbf{e}_z^{\mathcal{A}} = \mathbb{I}_{3\times 3}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{2.72}$$

After an elementary rotation around ${}_{A}\mathbf{e}_z^{\mathcal{A}}$, the y axis ${}_{\mathcal{A}}\mathbf{e}_y^{\mathcal{A}'}$ will be expressed by

$${}_{\mathcal{A}}\mathbf{e}_y^{\mathcal{A}'} = \mathbf{C}_{\mathcal{AA}'}(z) \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(z) & -\sin(z) & 0 \\ \sin(z) & \cos(z) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\sin(z) \\ \cos(z) \\ 0 \end{bmatrix}. \tag{2.73}$$

After an elementary rotation around ${}_{\mathcal{A}}\mathbf{e}_y^{\mathcal{A}'}$, the x axis ${}_{\mathcal{A}}\mathbf{e}_x^{\mathcal{A}''}$ will be expressed by

$$
\begin{aligned}
{}_{\mathcal{A}}\mathbf{e}_x^{\mathcal{A}''} &= \mathbf{C}_{\mathcal{A}\mathcal{A}'}(z) \cdot \mathbf{C}_{\mathcal{A}'\mathcal{A}''}(y) \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \cos(z) & -\sin(z) & 0 \\ \sin(z) & \cos(z) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(y) & 0 & \sin(y) \\ 0 & 1 & 0 \\ -\sin(y) & 0 & \cos(y) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \cos(y)\cos(z) \\ \cos(y)\sin(z) \\ -\sin(y) \end{bmatrix}.
\end{aligned} \tag{2.74}
$$

Finally, the mapping $\mathbf{E}(\boldsymbol{\chi}_R)$ will be computed as:

$$
\mathbf{E}_{R,eulerZYX} = \begin{bmatrix} {}_{\mathcal{A}}\mathbf{e}_z^{\mathcal{A}} & {}_{\mathcal{A}}\mathbf{e}_y^{\mathcal{A}'} & {}_{\mathcal{A}}\mathbf{e}_x^{\mathcal{A}''} \end{bmatrix} = \begin{bmatrix} 0 & -\sin(z) & \cos(y)\cos(z) \\ 0 & \cos(z) & \cos(y)\sin(z) \\ 1 & 0 & -\sin(y) \end{bmatrix}. \tag{2.75}
$$

It is easy to find that $det(\mathbf{E}_{R,eulerZYX}) = -\cos(y)$. The mapping then becomes singular when $y = \pi/2 + k\pi, \forall k \in \mathbb{Z}$. This means that although we can always describe an angular velocity using Euler Angle time derivatives, the inverse is not always possible. The inverse mapping is given by:

$$
\mathbf{E}_{R,eulerZYX}^{-1} = \begin{bmatrix} \frac{\cos(z)\sin(y)}{\cos(y)} & \frac{\sin(y)\sin(z)}{\cos(y)} & 1 \\ -\sin(z) & \cos(z) & 0 \\ \frac{\cos(z)}{\cos(y)} & \frac{\sin(z)}{\cos(y)} & 0 \end{bmatrix}. \tag{2.76}
$$

Time Derivatives of Euler Angles XYZ ⇔ Angular Velocity

Analog to the previous derivation, the projection matrix for XYZ Euler Angles and its inverse are

$$
\mathbf{E}_{R,eulerXYZ} = \begin{bmatrix} 1 & 0 & \sin(y) \\ 0 & \cos(x) & -\cos(y)\sin(x) \\ 0 & \sin(x) & \cos(x)\cos(y) \end{bmatrix}, \tag{2.77}
$$

$$
\mathbf{E}_{R,eulerXYZ}^{-1} = \begin{bmatrix} 1 & \frac{\sin(x)\sin(y)}{\cos(y)} & \frac{-\cos(x)\sin(y)}{\cos(y)} \\ 0 & \cos(x) & \sin(x) \\ 0 & \frac{-\sin(x)}{\cos(y)} & \frac{\cos(x)}{\cos(y)} \end{bmatrix}. \tag{2.78}
$$

Time Derivatives of Euler Angles ZYZ ⇔ Angular Velocity

The projection matrix for ZYZ Euler angles and its inverse are

$$\mathbf{E}_{R,eulerZYZ} = \begin{bmatrix} 0 & -\sin(z_1) & \cos(z_1)\sin(y) \\ 0 & \cos(z_1) & \sin(z_1)\sin(y) \\ 1 & 0 & \cos(y) \end{bmatrix}, \tag{2.79}$$

$$\mathbf{E}^{-1}_{R,eulerZYZ} = \begin{bmatrix} \frac{-\cos(y)\cos(z_1)}{\sin(y)} & \frac{-\cos(y)\sin(z_1)}{\sin(y)} & 1 \\ -\sin(z_1) & \cos(z_1) & 0 \\ \frac{\cos(z_1)}{\sin(y)} & \frac{\sin(z_1)}{\sin(y)} & 0 \end{bmatrix}. \tag{2.80}$$

Time Derivatives of Euler Angles ZXZ ⇔ Angular Velocity

The projection matrix for ZXZ Euler angles and its inverse are

$$\mathbf{E}_{R,eulerZXZ} = \begin{bmatrix} 0 & \cos(z_1) & \sin(z_1)\sin(x) \\ 0 & \sin(z_1) & -\cos(z_1)\sin(x) \\ 1 & 0 & \cos(x) \end{bmatrix}, \tag{2.81}$$

$$\mathbf{E}^{-1}_{R,eulerZXZ} = \begin{bmatrix} \frac{-\cos(x)\sin(z_1)}{\sin(x)} & \frac{\cos(x)\cos(z_1)}{\sin(x)} & 1 \\ \cos(z_1) & \sin(z_1) & 0 \\ \frac{\sin(z_1)}{\sin(x)} & \frac{-\cos(z_1)}{\sin(x)} & 0 \end{bmatrix}. \tag{2.82}$$

Time Derivative of Rotation Quaternion ⇔ Angular Velocity

For quaternions it can be shown that the following relations hold:

$$_{\mathcal{I}}\boldsymbol{\omega}_{\mathcal{IB}} = 2\mathbf{H}(\boldsymbol{\xi}_{\mathcal{IB}})\dot{\boldsymbol{\xi}}_{\mathcal{IB}} = \mathbf{E}_{R,quat}\dot{\boldsymbol{\chi}}_{R,quat} \tag{2.83}$$

$$\dot{\boldsymbol{\xi}}_{\mathcal{IB}} = \frac{1}{2}\mathbf{H}(\boldsymbol{\xi}_{\mathcal{IB}})^T {}_{\mathcal{I}}\boldsymbol{\omega}_{\mathcal{IB}} = \mathbf{E}^{-1}_{R,quat}\dot{\boldsymbol{\chi}}_{R,quat}, \tag{2.84}$$

with

$$\mathbf{H}(\boldsymbol{\xi}) = \begin{bmatrix} -\check{\boldsymbol{\xi}} & \left[\check{\boldsymbol{\xi}}\right]_\times + \xi_0 \mathbb{I}_{3\times 3} \end{bmatrix} \in \mathbb{R}^{3\times 4} \tag{2.85}$$

$$= \begin{bmatrix} -\xi_1 & \xi_0 & -\xi_3 & \xi_2 \\ -\xi_2 & \xi_3 & \xi_0 & -\xi_1 \\ -\xi_3 & -\xi_2 & \xi_1 & \xi_0 \end{bmatrix}. \tag{2.86}$$

Hence, the mapping matrix $\mathbf{E}_R$ and its inverse for a quaternion representation (2.51) are

$$\mathbf{E}_{R,quat} = 2\mathbf{H}(\boldsymbol{\xi}), \tag{2.87}$$

$$\mathbf{E}^{-1}_{R,quat} = \frac{1}{2}\mathbf{H}(\boldsymbol{\xi})^T. \tag{2.88}$$

Time Derivative of Angle Axis $\Leftrightarrow$ Angular Velocity

For angle axis it can be shown that the following relations hold:

$$_I\boldsymbol{\omega}_{IB} = \mathbf{n}\dot{\theta} + \dot{\mathbf{n}}\sin\theta + [\mathbf{n}]_\times \dot{\mathbf{n}}(1-\cos\theta) \tag{2.89}$$

$$\dot{\theta} = \mathbf{n}^T {}_I\boldsymbol{\omega}_{IB}, \quad \dot{\mathbf{n}} = \left(-\frac{1}{2}\frac{\sin\theta}{1-cos\theta}[\mathbf{n}]_\times^2 - \frac{1}{2}[\mathbf{n}]_\times\right) {}_I\boldsymbol{\omega}_{IB} \quad \forall\theta \in \mathbb{R}\backslash\{0\} \tag{2.90}$$

Hence, the mapping matrix $\mathbf{E}_R$ and its inverse for the angle axis (2.46) are

$$\mathbf{E}_{R,angleaxis} = \begin{bmatrix}\mathbf{n} & \sin\theta\mathbb{I}_{3\times3} + (1-\cos\theta)[\mathbf{n}]_\times\end{bmatrix} \tag{2.91}$$

$$\mathbf{E}^{-1}_{R,angleaxis} = \begin{bmatrix}\mathbf{n}^T \\ -\frac{1}{2}\frac{\sin\theta}{1-cos\theta}[\mathbf{n}]_\times^2 - \frac{1}{2}[\mathbf{n}]_\times\end{bmatrix} \tag{2.92}$$

Time Derivative of Rotation Vector $\Leftrightarrow$ Angular Velocity

For a rotation vector it can be shown that the following relations hold:

$$_I\boldsymbol{\omega}_{IB} = \left[\mathbb{I}_{3\times3} + [\boldsymbol{\varphi}]_\times\left(\frac{1-\cos\|\boldsymbol{\varphi}\|}{\|\boldsymbol{\varphi}\|^2}\right) + [\boldsymbol{\varphi}]_\times^2\left(\frac{\|\boldsymbol{\varphi}\| - \sin\|\boldsymbol{\varphi}\|}{\|\boldsymbol{\varphi}\|^3}\right)\right]\dot{\boldsymbol{\varphi}} \quad \forall\|\boldsymbol{\varphi}\| \in \mathbb{R}\backslash\{0\} \tag{2.93}$$

$$\dot{\boldsymbol{\varphi}} = \left[\mathbb{I}_{3\times3} - \frac{1}{2}[\boldsymbol{\varphi}]_\times + [\boldsymbol{\varphi}]_\times^2\frac{1}{\|\boldsymbol{\varphi}\|^2}\left(1 - \frac{\|\boldsymbol{\varphi}\|}{2}\frac{\sin\|\boldsymbol{\varphi}\|}{1-\cos\|\boldsymbol{\varphi}\|}\right)\right] {}_I\boldsymbol{\omega}_{IB} \quad \forall\|\boldsymbol{\varphi}\| \in \mathbb{R}\backslash\{0\} \tag{2.94}$$

Hence, the mapping matrix $\mathbf{E}_R$ and its inverse for the rotation vector (2.47) are

$$\mathbf{E}_{R,rotationvector} = \left[\mathbb{I}_{3\times3} + [\boldsymbol{\varphi}]_\times\left(\frac{1-\cos\|\boldsymbol{\varphi}\|}{\|\boldsymbol{\varphi}\|^2}\right) + [\boldsymbol{\varphi}]_\times^2\left(\frac{\|\boldsymbol{\varphi}\| - \sin\|\boldsymbol{\varphi}\|}{\|\boldsymbol{\varphi}\|^3}\right)\right] \tag{2.95}$$

$$\mathbf{E}^{-1}_{R,rotationvector} = \left[\mathbb{I}_{3\times3} - \frac{1}{2}[\boldsymbol{\varphi}]_\times + [\boldsymbol{\varphi}]_\times^2\frac{1}{\|\boldsymbol{\varphi}\|^2}\left(1 - \frac{\|\boldsymbol{\varphi}\|}{2}\frac{\sin\|\boldsymbol{\varphi}\|}{1-\cos\|\boldsymbol{\varphi}\|}\right)\right] \tag{2.96}$$

2.6 Transformation

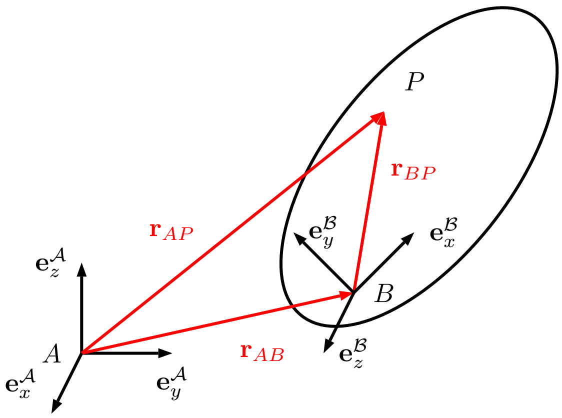


Figure 2.9: Single body with body fixed frame $\mathcal{B}$.

In the most general case, two reference frames have a position offset and relative rotation (see Fig. 2.9). As a result, a point P can be transformed from one frame to another using a homogeneous transformation matrix $\mathbf{T}$ which is a combined translation and rotation:

$$\mathbf{r}_{AP} = \mathbf{r}_{AB} + \mathbf{r}_{BP} \tag{2.97}$$

$$_{\mathcal{A}}\mathbf{r}_{AP} = {_{\mathcal{A}}\mathbf{r}_{AB}} + {_{\mathcal{A}}\mathbf{r}_{BP}} = {_{\mathcal{A}}\mathbf{r}_{AB}} + \mathbf{C}_{\mathcal{AB}} \cdot {_{\mathcal{B}}\mathbf{r}_{BP}} \tag{2.98}$$

$$\begin{pmatrix} _{\mathcal{A}}\mathbf{r}_{AP} \\ 1 \end{pmatrix} = \underbrace{\begin{bmatrix} \mathbf{C}_{\mathcal{AB}} & _{\mathcal{A}}\mathbf{r}_{AB} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}}_{\mathbf{T}_{\mathcal{AB}}} \begin{pmatrix} _{\mathcal{B}}\mathbf{r}_{BP} \\ 1 \end{pmatrix} \tag{2.99}$$

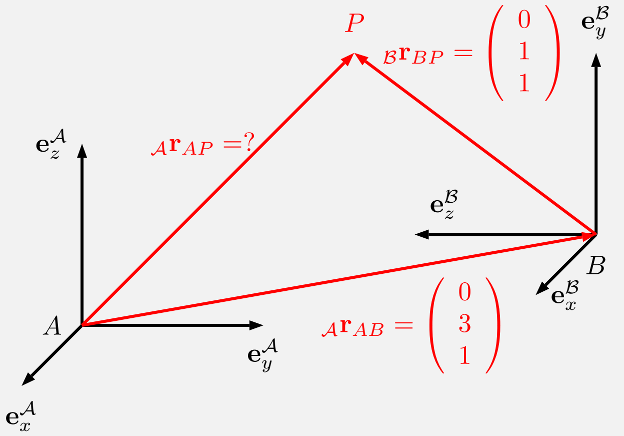
The inverse of the homogeneous transformation can be calculated as

$$\mathbf{T}_{\mathcal{AB}}^{-1} = \begin{bmatrix} \mathbf{C}_{\mathcal{AB}}^T & \overbrace{-\mathbf{C}_{\mathcal{AB}}^T {_{\mathcal{A}}\mathbf{r}_{AB}}}^{_{\mathcal{B}}\mathbf{r}_{BA}} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}. \tag{2.100}$$

Consecutive homogeneous transformations are given by

$$\mathbf{T}_{\mathcal{AC}} = \mathbf{T}_{\mathcal{AB}}\mathbf{T}_{\mathcal{BC}}. \tag{2.101}$$

Example 2.6.1: Homogeneous transformation



What is the homogeneous transformation matrix $\mathbf{T}_{\mathcal{AB}}$ and the position vector ${}_{\mathcal{A}}\mathbf{r}_{AP}$?

The homogeneous transformation matrix is

$$\mathbf{T}_{\mathcal{AB}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.102}$$

Correspondingly, the position vector can be calculated as:

$$\begin{pmatrix} {}_{\mathcal{A}}\mathbf{r}_{AP} \\ 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 2 \\ 1 \end{pmatrix}. \tag{2.103}$$

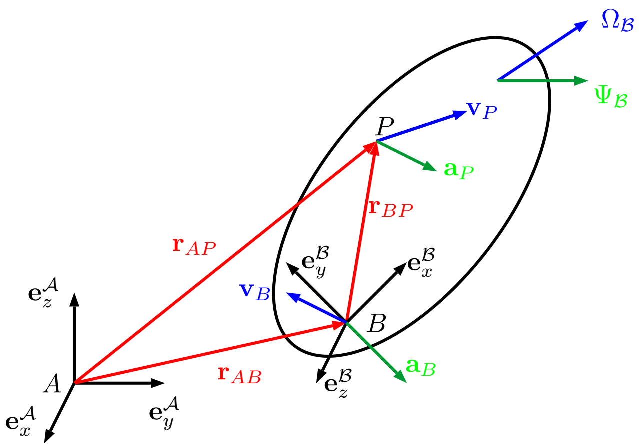


Figure 2.10: Rigid body with velocity and acceleration.

2.7 Velocity in Moving Bodies

Figure 2.10 depicts the velocities of a single body. Assuming $\mathcal{A}$ being in inertial fixed frame, we have:

- $\mathbf{v}_P$: the absolute velocity of P
- $\mathbf{a}_P = \dot{\mathbf{v}}_P$: the absolute acceleration of P
- $\boldsymbol{\Omega}_{\mathcal{B}} = \boldsymbol{\omega}_{\mathcal{AB}}$: (absolute) angular velocity of body $\mathcal{B}$
- $\boldsymbol{\Psi}_{\mathcal{B}} = \dot{\boldsymbol{\Omega}}_{\mathcal{B}}$: (absolute) angular acceleration of body $\mathcal{B}$

At this point it is important to understand the difference between the velocity, i.e. the absolute time variation of a position, expressed in a frame $\mathcal{C}$:

$$_{\mathcal{C}}\left(\dot{\mathbf{r}}_{AP}\right) =_{\mathcal{C}} \left(\frac{d}{dt}\mathbf{r}_{AP}\right) =_{\mathcal{C}} \mathbf{v}_{AP}, \tag{2.104}$$

and the time differentiation of the coordinates of a position vector:

$$\left(_{\mathcal{C}}\dot{\mathbf{r}}_{AP}\right) = \left(_{\mathcal{C}}\mathbf{r}_{AP}\right)^{\cdot} = \frac{d}{dt}\left(_{\mathcal{C}}\mathbf{r}_{AP}\right). \tag{2.105}$$

They are only equal in case $\mathcal{C}$ is selected as an inertial frame. Following the transformation introduced before, we can write the position of P as:

$$_{\mathcal{A}}\mathbf{r}_{AP} = {_{\mathcal{A}}\mathbf{r}_{AB}} + {_{\mathcal{A}}\mathbf{r}_{BP}} = {_{\mathcal{A}}\mathbf{r}_{AB}} + \mathbf{C}_{\mathcal{AB}} \cdot {_{\mathcal{B}}\mathbf{r}_{BP}}. \tag{2.106}$$

Differentiating with respect to time results in

$$_{\mathcal{A}}\dot{\mathbf{r}}_{AP} = {_{\mathcal{A}}\dot{\mathbf{r}}_{AB}} + \mathbf{C}_{\mathcal{AB}} \cdot {_{\mathcal{B}}\dot{\mathbf{r}}_{BP}} + \dot{\mathbf{C}}_{\mathcal{AB}} \cdot {_{\mathcal{B}}\mathbf{r}_{BP}} \tag{2.107}$$

Since P is a point on the rigid body $\mathcal{B}$, the relative velocity ${_{\mathcal{B}}\dot{\mathbf{r}}_{BP}} = 0$. Furthermore, from (2.61) it can be seen that $\dot{\mathbf{C}}_{\mathcal{AB}} = \left[{_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}}}\right]_{\times} \cdot \mathbf{C}_{\mathcal{AB}}$, yielding

$$_{\mathcal{A}}\dot{\mathbf{r}}_{AP} = {_{\mathcal{A}}\dot{\mathbf{r}}_{AB}} + \left[{_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}}}\right]_{\times} \cdot \mathbf{C}_{\mathcal{AB}} \cdot {_{\mathcal{B}}\mathbf{r}_{BP}} \tag{2.108}$$

$$= {_{\mathcal{A}}\dot{\mathbf{r}}_{AB}} + {_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}}} \times {_{\mathcal{A}}\mathbf{r}_{BP}} \tag{2.109}$$

This is the very famous *rigid body formulation* for velocities, also known as velocity composition rule. It can be reformulated to

$$\mathbf{v}_P = \mathbf{v}_B + \boldsymbol{\Omega} \times \mathbf{r}_{BP}. \tag{2.110}$$

Applying the same calculation rules for accelerations results in

$$\mathbf{a}_P = \mathbf{a}_B + \boldsymbol{\Psi} \times \mathbf{r}_{BP} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{BP}). \tag{2.111}$$

Some Notes on Vector Differentiation

Be careful with vector differentiation in moving coordinate systems. In particular, it is not true that the velocity generally equals the time derivative of the position:

$$\mathbf{v}_P \neq \dot{\mathbf{r}}_{AP} \tag{2.112}$$

Equality only holds if the differentiation is done in non-moving systems represented by $\mathcal{A}$:

$$_{\mathcal{A}}\mathbf{v}_P = {}_{\mathcal{A}}\dot{\mathbf{r}}_{AP} \tag{2.113}$$

In case a moving system $\mathcal{B}$ is used for representation, the *Euler differentiation rule* must be applied

$$
\begin{aligned}
_{\mathcal{B}}\mathbf{v}_P &= \mathbf{C}_{\mathcal{B}\mathcal{A}} \cdot \frac{d}{dt}\left(\mathbf{C}_{\mathcal{A}\mathcal{B}} \cdot {}_{\mathcal{B}}\mathbf{r}_{AP}\right) && (2.114)\\
&= \mathbf{C}_{\mathcal{B}\mathcal{A}} \cdot \Big(\mathbf{C}_{\mathcal{A}\mathcal{B}} \cdot {}_{\mathcal{B}}\dot{\mathbf{r}}_{AP} + \dot{\mathbf{C}}_{\mathcal{A}\mathcal{B}} \cdot {}_{\mathcal{B}}\mathbf{r}_{AP}\Big) && (2.115)\\
&= \mathbf{C}_{\mathcal{B}\mathcal{A}} \cdot \left(\mathbf{C}_{\mathcal{A}\mathcal{B}} \cdot {}_{\mathcal{B}}\dot{\mathbf{r}}_{AP} + [{}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{A}\mathcal{B}}]_\times \cdot \mathbf{C}_{\mathcal{A}\mathcal{B}} \cdot {}_{\mathcal{B}}\mathbf{r}_{AP}\right) && (2.116)\\
&= {}_{\mathcal{B}}\dot{\mathbf{r}}_{AP} + \mathbf{C}_{\mathcal{B}\mathcal{A}} \cdot [{}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{A}\mathcal{B}}]_\times \cdot \mathbf{C}_{\mathcal{A}\mathcal{B}} \cdot {}_{\mathcal{B}}\mathbf{r}_{AP} && (2.117)\\
&\overset{(2.64)}{=} {}_{\mathcal{B}}\dot{\mathbf{r}}_{AP} + {}_{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{A}\mathcal{B}} \times {}_{\mathcal{B}}\mathbf{r}_{AP} && (2.118)
\end{aligned}
$$

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