

Half-Vectorization and the Duplication Matrix

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For a $k \times k$ symmetric matrix \mathbf{A} , there exists a unique $k^2 \times \frac{1}{2}k(k+1)$ matrix \mathbf{D}_k , called the duplication matrix, that transforms $\text{vech}(\mathbf{A})$ into $\text{vec}(\mathbf{A})$.
Thus,

$$\mathbf{D}_k \text{vech}(\mathbf{A}) = \text{vec}(\mathbf{A}) \quad (\mathbf{A} = \mathbf{A}') \quad (1)$$

and since \mathbf{D}_k is full rank the half-vectorization is given by

$$\text{vech}(\mathbf{A}) = (\mathbf{D}_k' \mathbf{D}_k)^{-1} \mathbf{D}_k' \text{vec}(\mathbf{A}) \quad (\mathbf{A} = \mathbf{A}') \quad (2)$$

$$\text{vech}(\mathbf{A}) = \mathbf{D}_k^+ \text{vec}(\mathbf{A}) \quad (\mathbf{A} = \mathbf{A}') \quad (3)$$

Examples

```
library(linearAlgebra)
```

```
A <- matrix(  
  data = c(  
    1.0, 0.5, 0.4,  
    0.5, 1.0, 0.6,  
    0.4, 0.6, 1.0  
  ),  
  ncol = 3  
)  
k <- dim(A)[1]
```

```
dcap(k)
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6]  
## [1,]    1    0    0    0    0    0  
## [2,]    0    1    0    0    0    0  
## [3,]    0    0    1    0    0    0  
## [4,]    0    1    0    0    0    0  
## [5,]    0    0    0    1    0    0  
## [6,]    0    0    0    0    1    0  
## [7,]    0    0    1    0    0    0  
## [8,]    0    0    0    0    1    0  
## [9,]    0    0    0    0    0    1
```

```
dcap(k) %**% vech(A)
```

```
##      [,1]  
## [1,]  1.0  
## [2,]  0.5  
## [3,]  0.4  
## [4,]  0.5  
## [5,]  1.0  
## [6,]  0.6  
## [7,]  0.4  
## [8,]  0.6  
## [9,]  1.0
```

```
all.equal(
  c(dcap(k) %*% vech(A)),
  vec(A)
)

## [1] TRUE
```

```
pinv_of_dcap(k)

##      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9]
## [1,]    1  0.0  0.0  0.0    0  0.0  0.0  0.0    0
## [2,]    0  0.5  0.0  0.5    0  0.0  0.0  0.0    0
## [3,]    0  0.0  0.5  0.0    0  0.0  0.5  0.0    0
## [4,]    0  0.0  0.0  0.0    1  0.0  0.0  0.0    0
## [5,]    0  0.0  0.0  0.0    0  0.5  0.0  0.5    0
## [6,]    0  0.0  0.0  0.0    0  0.0  0.0  0.0    1
```

```
pinv_of_dcap(k) %*% vec(A)

##      [,1]
## [1,]  1.0
## [2,]  0.5
## [3,]  0.4
## [4,]  1.0
## [5,]  0.6
## [6,]  1.0
```

```
all.equal(
  c(pinv_of_dcap(k) %*% vec(A)),
  vech(A)
)

## [1] TRUE
```

Readings

See Magnus and Neudecker (2019) p. 56–57, Magnus and Neudecker (1980), and Abadir and Magnus (2005) ch. 11.

References

- Abadir, K. M., & Magnus, J. R. (2005, August). *Matrix algebra*. Cambridge University Press. <https://doi.org/10.1017/cbo9780511810800>
- Magnus, J. R., & Neudecker, H. (1980). The elimination matrix: Some lemmas and applications. *SIAM Journal on Algebraic Discrete Methods*, 1(4), 422–449. <https://doi.org/10.1137/0601049>
- Magnus, J. R., & Neudecker, H. (2019, February). *Matrix differential calculus with applications in statistics and econometrics*. Wiley. <https://doi.org/10.1002/9781119541219>