

# The Moore-Penrose Inverse of the Duplication Matrix

Ivan Jacob Agaloos Pesigan

The Moore-Penrose inverse of the duplication matrix  $\mathbf{D}_k$  is the  $\frac{1}{2}k(k+1) \times k^2$  matrix given by

$$\mathbf{D}_k^+ = (\mathbf{D}_k' \mathbf{D}_k)^{-1} \mathbf{D}_k' \quad (1)$$

where

$$\mathbf{D}_k^+ \text{vec}(\mathbf{A}) = \text{vech}(\mathbf{A}) \quad (\mathbf{A} = \mathbf{A}') \quad (2)$$

$\text{vec}(\cdot)$  is the vectorization of a matrix, and  $\text{vech}(\cdot)$  is the half-vectorization of a matrix.

## Examples

```
library(linearAlgebra)
```

```
A <- matrix(  
  data = c(  
    1.0, 0.5, 0.4,  
    0.5, 1.0, 0.6,  
    0.4, 0.6, 1.0  
  ),  
  ncol = 3  
)  
k <- dim(A)[1]
```

```
pinv_of_dcap(k)
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9]  
## [1,]    1  0.0  0.0  0.0    0  0.0  0.0  0.0    0  
## [2,]    0  0.5  0.0  0.5    0  0.0  0.0  0.0    0  
## [3,]    0  0.0  0.5  0.0    0  0.0  0.5  0.0    0  
## [4,]    0  0.0  0.0  0.0    1  0.0  0.0  0.0    0  
## [5,]    0  0.0  0.0  0.0    0  0.5  0.0  0.5    0  
## [6,]    0  0.0  0.0  0.0    0  0.0  0.0  0.0    1
```

```
pinv_of_dcap(k) %*% vec(A)
```

```
##      [,1]  
## [1,]  1.0  
## [2,]  0.5  
## [3,]  0.4  
## [4,]  1.0  
## [5,]  0.6  
## [6,]  1.0
```

```
all.equal(  
  c(pinv_of_dcap(k) %*% vec(A)),  
  vech(A)  
)  
  
## [1] TRUE
```

## Readings

See Magnus and Neudecker (2019) p. 56–57 and 444, Magnus and Neudecker (1980), and Abadir and Magnus (2005) ch. 11.

## References

- Abadir, K. M., & Magnus, J. R. (2005, August). *Matrix algebra*. Cambridge University Press. <https://doi.org/10.1017/cbo9780511810800>
- Magnus, J. R., & Neudecker, H. (1980). The elimination matrix: Some lemmas and applications. *SIAM Journal on Algebraic Discrete Methods*, 1(4), 422–449. <https://doi.org/10.1137/0601049>
- Magnus, J. R., & Neudecker, H. (2019, February). *Matrix differential calculus with applications in statistics and econometrics*. Wiley. <https://doi.org/10.1002/9781119541219>