The Moore-Penrose Inverse of the Duplication Matrix

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The Moore-Penrose inverse of the duplication matrix $\boldsymbol{D}_k$ is the $\frac{1}{2}k\left(k+1\right) \times k^2$ matrix given by

$$\boldsymbol{D}_k^+ = \left(\boldsymbol{D}_k' \boldsymbol{D}_k\right)^{-1} \boldsymbol{D}_k' \tag{1}$$

where

$$\boldsymbol{D}_k^+ \text{vec}\left(A\right) = \text{vech}\left(\boldsymbol{A}\right) \quad \left(\boldsymbol{A} = \boldsymbol{A}'\right) \tag{2}$$

$\text{vec}\left(\cdot\right)$ is the vectorization of a matrix, and $\text{vech}\left(\cdot\right)$ is the half-vectorization of a matrix.

Examples

```
library(linearAlgebra)
```

```
A <- matrix(
  data = c(
    1.0, 0.5, 0.4,
    0.5, 1.0, 0.6,
    0.4, 0.6, 1.0
  ),
  ncol = 3
)
k <- dim(A)[1]
```

```
pinv_of_dcap(k)
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9]
## [1,]    1  0.0  0.0  0.0    0  0.0  0.0  0.0    0
## [2,]    0  0.5  0.0  0.5    0  0.0  0.0  0.0    0
## [3,]    0  0.0  0.5  0.0    0  0.0  0.5  0.0    0
## [4,]    0  0.0  0.0  0.0    1  0.0  0.0  0.0    0
## [5,]    0  0.0  0.0  0.0    0  0.5  0.0  0.5    0
## [6,]    0  0.0  0.0  0.0    0  0.0  0.0  0.0    1
```

```
pinv_of_dcap(k) %*% vec(A)
```

```
##      [,1]
## [1,]  1.0
```

```
## [2,]  0.5
## [3,]  0.4
## [4,]  1.0
## [5,]  0.6
## [6,]  1.0
```

```
all.equal(
  c(pinv_of_dcap(k) %*% vec(A)),
  vech(A)
)

## [1] TRUE
```

Readings

See Magnus and Neudecker (2019) p. 56–57 and 444, Magnus and Neudecker (1980), and Abadir and Magnus (2005) ch. 11.

References

Abadir, K. M., & Magnus, J. R. (2005, August). *Matrix algebra*. Cambridge University Press. https://doi.org/10.1017/cbo9780511810800

Magnus, J. R., & Neudecker, H. (1980). The elimination matrix: Some lemmas and applications. *SIAM Journal on Algebraic Discrete Methods*, *1*(4), 422–449. https://doi.org/10.1137/0601049

Magnus, J. R., & Neudecker, H. (2019, February). *Matrix differential calculus with applications in statistics and econometrics*. Wiley. https://doi.org/10.1002/9781119541219