Half-Vectorization and the Duplication Matrix

Ivan Jacob Agaloos Pesigan

For a $k \times k$ symmetric matrix $\boldsymbol{A}$, there exists a unique $k^2 \times \frac{1}{2} k \left(k + 1 \right)$ matrix $\boldsymbol{D}_k$, called the duplication matrix, that transforms $\mathrm{vech} \left(\boldsymbol{A} \right)$ into $\mathrm{vec} \left(\boldsymbol{A} \right)$.

Thus,

$$\boldsymbol{D}_k \mathrm{vech} \left(\boldsymbol{A} \right) = \mathrm{vec} \left(\boldsymbol{A} \right) \quad \left(\boldsymbol{A} = \boldsymbol{A}^{\prime} \right) \tag{1}$$

and since $\boldsymbol{D}_k$ is full rank the half-vectorization is given by

$$\mathrm{vech} \left(\boldsymbol{A} \right) = \left(\boldsymbol{D}_k^{\prime} \boldsymbol{D}_k \right)^{-1} \boldsymbol{D}_k^{\prime} \mathrm{vec} \left(A \right) \quad \left(\boldsymbol{A} = \boldsymbol{A}^{\prime} \right) \tag{2}$$

$$\mathrm{vech} \left(\boldsymbol{A} \right) = \boldsymbol{D}_k^{+} \mathrm{vec} \left(A \right) \quad \left(\boldsymbol{A} = \boldsymbol{A}^{\prime} \right) \tag{3}$$

Examples

```
library(linearAlgebra)
```

```
A <- matrix(
  data = c(
    1.0, 0.5, 0.4,
    0.5, 1.0, 0.6,
    0.4, 0.6, 1.0
  ),
  ncol = 3
)
k <- dim(A)[1]
```

```
dcap(k)

##       [,1] [,2] [,3] [,4] [,5] [,6]
##  [1,]    1    0    0    0    0    0
##  [2,]    0    1    0    0    0    0
##  [3,]    0    0    1    0    0    0
##  [4,]    0    1    0    0    0    0
##  [5,]    0    0    0    1    0    0
##  [6,]    0    0    0    0    1    0
##  [7,]    0    0    1    0    0    0
##  [8,]    0    0    0    0    1    0
##  [9,]    0    0    0    0    0    1
```

```
dcap(k) %*% vech(A)

##       [,1]
##  [1,]  1.0
##  [2,]  0.5
##  [3,]  0.4
##  [4,]  0.5
##  [5,]  1.0
##  [6,]  0.6
##  [7,]  0.4
##  [8,]  0.6
##  [9,]  1.0
```

```r
all.equal(
  c(dcap(k) %*% vech(A)),
  vec(A)
)

## [1] TRUE
```

```r
pinv_of_dcap(k)

##      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9]
## [1,]    1  0.0  0.0  0.0    0  0.0  0.0  0.0    0
## [2,]    0  0.5  0.0  0.5    0  0.0  0.0  0.0    0
## [3,]    0  0.0  0.5  0.0    0  0.0  0.5  0.0    0
## [4,]    0  0.0  0.0  0.0    1  0.0  0.0  0.0    0
## [5,]    0  0.0  0.0  0.0    0  0.5  0.0  0.5    0
## [6,]    0  0.0  0.0  0.0    0  0.0  0.0  0.0    1
```

```r
pinv_of_dcap(k) %*% vec(A)

##      [,1]
## [1,]  1.0
## [2,]  0.5
## [3,]  0.4
## [4,]  1.0
## [5,]  0.6
## [6,]  1.0
```

```r
all.equal(
  c(pinv_of_dcap(k) %*% vec(A)),
  vech(A)
)

## [1] TRUE
```

Readings

See Magnus and Neudecker (2019) p. 56–57, Magnus and Neudecker (1980), and Abadir and Magnus (2005) ch. 11.

References

Abadir, K. M., & Magnus, J. R. (2005, August). *Matrix algebra*. Cambridge University Press. https://doi.org/10.1017/cbo9780511810800

Magnus, J. R., & Neudecker, H. (1980). The elimination matrix: Some lemmas and applications. *SIAM Journal on Algebraic Discrete Methods*, *1*(4), 422–449. https://doi.org/10.1137/0601049

Magnus, J. R., & Neudecker, H. (2019, February). *Matrix differential calculus with applications in statistics and econometrics*. Wiley. https://doi.org/10.1002/9781119541219