Reticular Action Model (RAM) Notation Notes

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Contents

Chapter 1

Description

This is a collection of my notes on the Reticular Action Model (RAM) notation that accompanies the `ramR` package (Pesigan, 2021) in the `R` statistical environment (R Core Team, 2020). You can install the released version of `ramR` from GitHub with:

```
remotes::install_github("jeksterslab/ramR")
```

These notes are based on the following resources:

- Boker and McArdle (2005)
- McArdle and McDonald (1984)
- McArdle (2005)

See GitHub Pages for the html deployment.

Chapter 2

Reticular Action Model (RAM) Matrix Notation

2.1 Full Model

Definition 2.1.

$$\mathbf{v} = \mathbf{A}\mathbf{v} + \mathbf{u} \tag{2.1}$$

where

- $\mathbf{v}$ and $\mathbf{u}$ are $t \times 1$ vectors of random variables
- $\mathbf{A}$ is a $t \times t$ matrix of *directed* or *asymmetric* relationship from column variable v_j to row variable v_i
  - $\mathbf{A}$ represent the regression of each of the t variables $\mathbf{v}$ on the other $t-1$ variables
  - diagonal $a_{i,i}$ is zero
  - u_i represent the residual of v_i
  - if all regression coefficients on other variables are zero (i.e., i^{th} row of $\mathbf{A}$ consists of zeros), then the variable v_i is considered the same as its own residual u_i

Definition 2.2.

$$\mathbf{S} = \mathbb{E}\left\{\mathbf{u}\mathbf{u}'\right\}, \tag{2.2}$$

where

- $\mathbf{S}$ is a $t \times t$ matrix of *undirected* or *symmetric* relationship
  - the notation Ω is used in other sources for $\mathbf{S}$
- $\mathbb{E}$ is the expectation operator

Definition 2.3.

$$\mathbf{C} = \mathbb{E}\left\{\mathbf{v}\mathbf{v}'\right\}, \tag{2.3}$$

where

- $\mathbf{C}$ is a $t \times t$ variance-covariance matrix
  - the notation Σ is used in other sources for $\mathbf{C}$

Definition 2.4.

$$\mathbf{v} = \mathbf{A}\mathbf{v} + \mathbf{u}$$

can be rewritten as

$$\begin{aligned} \mathbf{v} - \mathbf{A}\mathbf{v} &= \mathbf{u} \\ \mathbf{u} &= \mathbf{v} - \mathbf{A}\mathbf{v} \\ \mathbf{u} &= \left(\mathbf{I} - \mathbf{A}\right)\mathbf{v} \end{aligned} \tag{2.4}$$

assuming that $(\mathbf{I} - \mathbf{A})$ is non-singular,

$$\mathbf{E} = \left(\mathbf{I} - \mathbf{A}\right)^{-1} \tag{2.5}$$

then

$$\begin{aligned} \mathbf{v} &= \left(\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{u} \\ &= \mathbf{E}\mathbf{u}. \end{aligned} \tag{2.6}$$

Using the definitions above, $\mathbf{S}$ and $\mathbf{C}$ are given by

$$\begin{aligned} \mathbf{S} &= \left(\mathbf{I} - \mathbf{A}\right)\mathbf{C}\left(\mathbf{I} - \mathbf{A}\right)^{-1} \\ &= \mathbf{E}^{-1}\mathbf{C}\left(\mathbf{E}^{-1}\right)^{\intercal} \end{aligned} \tag{2.7}$$

$$\begin{aligned} \mathbf{C} &= \left(\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{S}\left[\left(\mathbf{I} - \mathbf{A}\right)^{-1}\right]^{\intercal} \\ &= \mathbf{E}\mathbf{S}\mathbf{E}^{\intercal} \end{aligned} \tag{2.8}$$

2.2 Given vs. Hidden Variables

Definition 2.5.

$$\mathbf{v} = \begin{bmatrix} \mathbf{g}_{p \times 1} \\ \mathbf{h}_{q \times 1} \end{bmatrix} \tag{2.9}$$

$$t = p + q \tag{2.10}$$

- $\mathbf{g}$ may be considered observed, manifest or *given* variables
- $\mathbf{h}$ may be considered unobserved, latent, or *hidden* variables

Definition 2.6.

$$\mathbf{F} = \left[\mathbf{I}_{p \times p} : \mathbf{0}_{p \times q}\right] \tag{2.11}$$

- the $\mathbf{F}$ matrix acts as a *filter* to select the manifest variables out of the full set of manifest and latent variables

$$\mathbf{g} = \mathbf{F}\mathbf{v} \tag{2.12}$$

$$\begin{aligned} \mathbf{g} &= \mathbf{F}\left(\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{u} \\ &= \mathbf{F}\mathbf{E}\mathbf{u} \end{aligned} \tag{2.13}$$

Definition 2.7.

$$\mathbf{M} = \mathbb{E}\left\{\mathbf{g}\mathbf{g}^\top\right\} \tag{2.14}$$

$$\begin{aligned} \mathbf{M} &= \mathbf{F}\left(\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{S}\left[\left(\mathbf{I} - \mathbf{A}\right)^{-1}\right]^\top\mathbf{F}^\top \\ &= \mathbf{F}\mathbf{E}\mathbf{S}\mathbf{E}^\top\mathbf{F}^\top \\ &= \mathbf{F}\mathbf{C}\mathbf{F}^\top \end{aligned} \tag{2.15}$$

- when components of $\mathbf{v}$ are permuted, the columns of $\mathbf{F}$ can be correspondingly permuted
- the rows and columns of $\mathbf{C}$ that are filtered out by $\mathbf{F}$ contain useful information about the latent variable structure.

The equations above completely define RAM.

Chapter 3

Reticular Action Model (RAM) Path Diagram

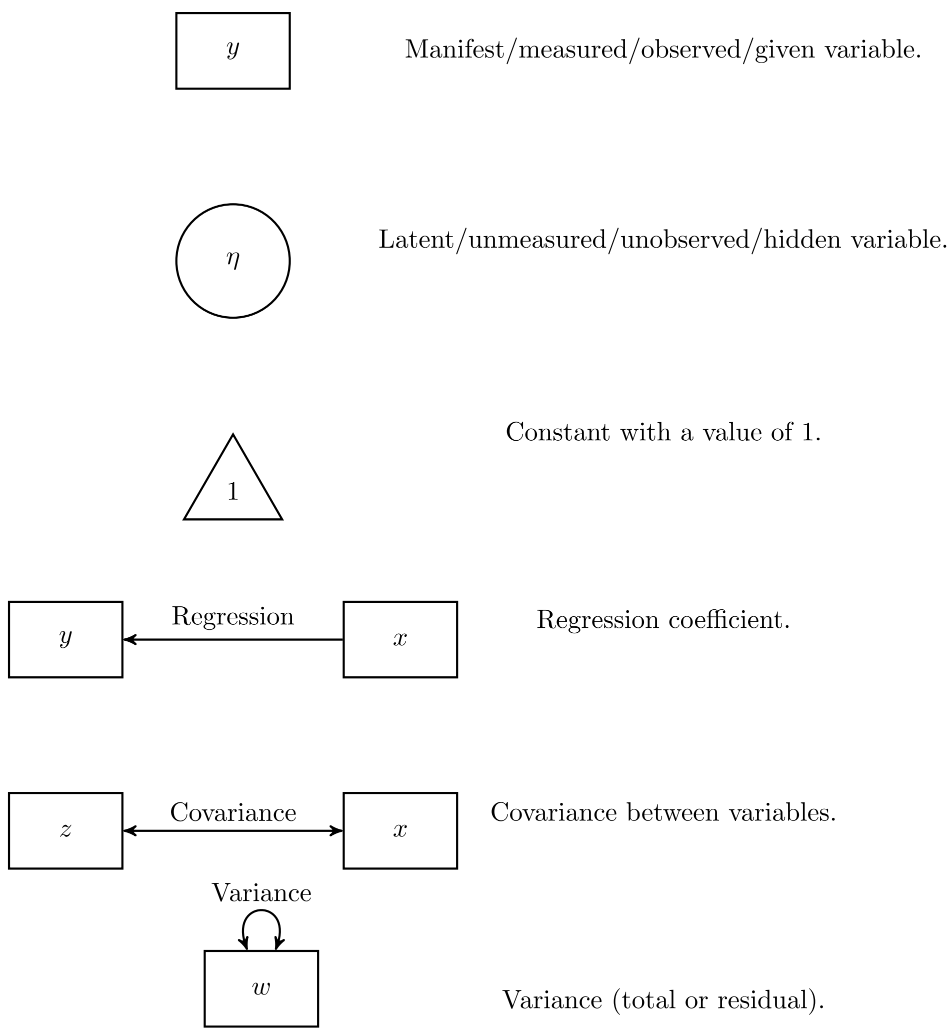
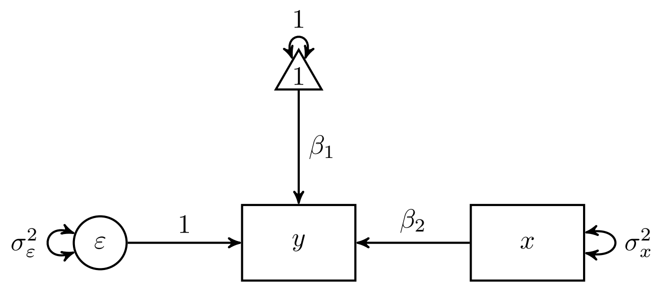


Figure 3.1: Path Diagram Elements



$$y = \alpha + \beta x + \varepsilon$$

Figure 3.2: Two-Variable Regression Model

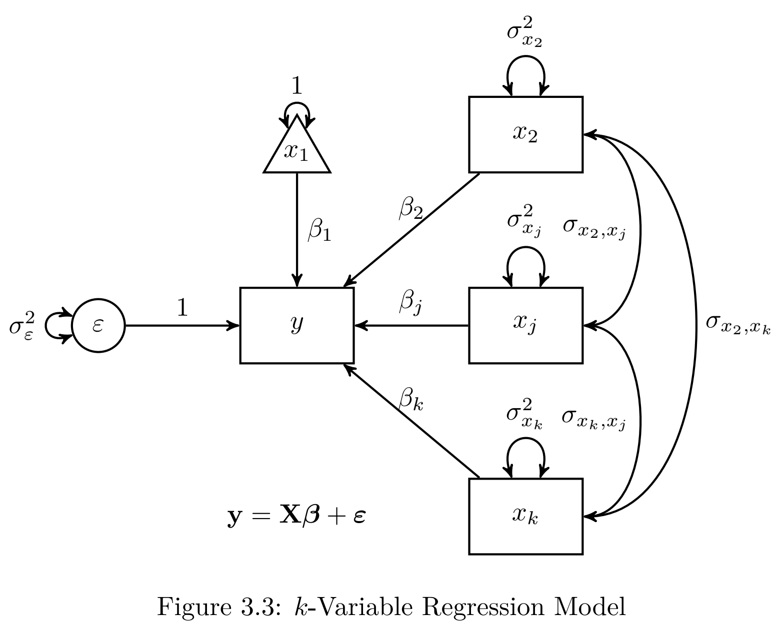
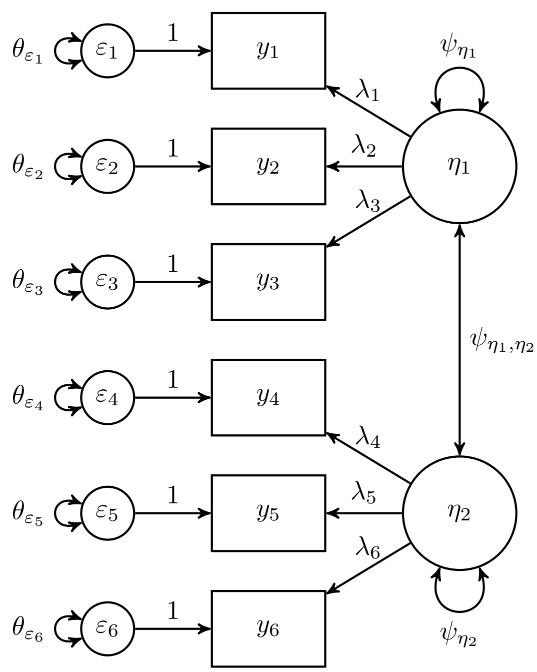


Figure 3.3: k-Variable Regression Model



$$\mathbf{y} = \boldsymbol{\Lambda}\boldsymbol{\eta} + \boldsymbol{\varepsilon}$$

Figure 3.4: Two-Factor Confirmatory Factor Analysis Model

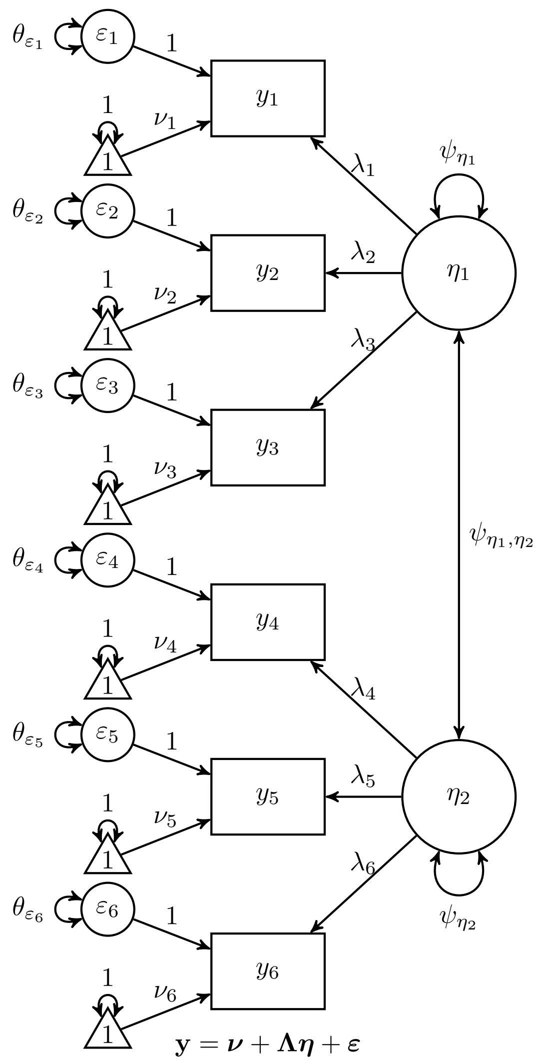
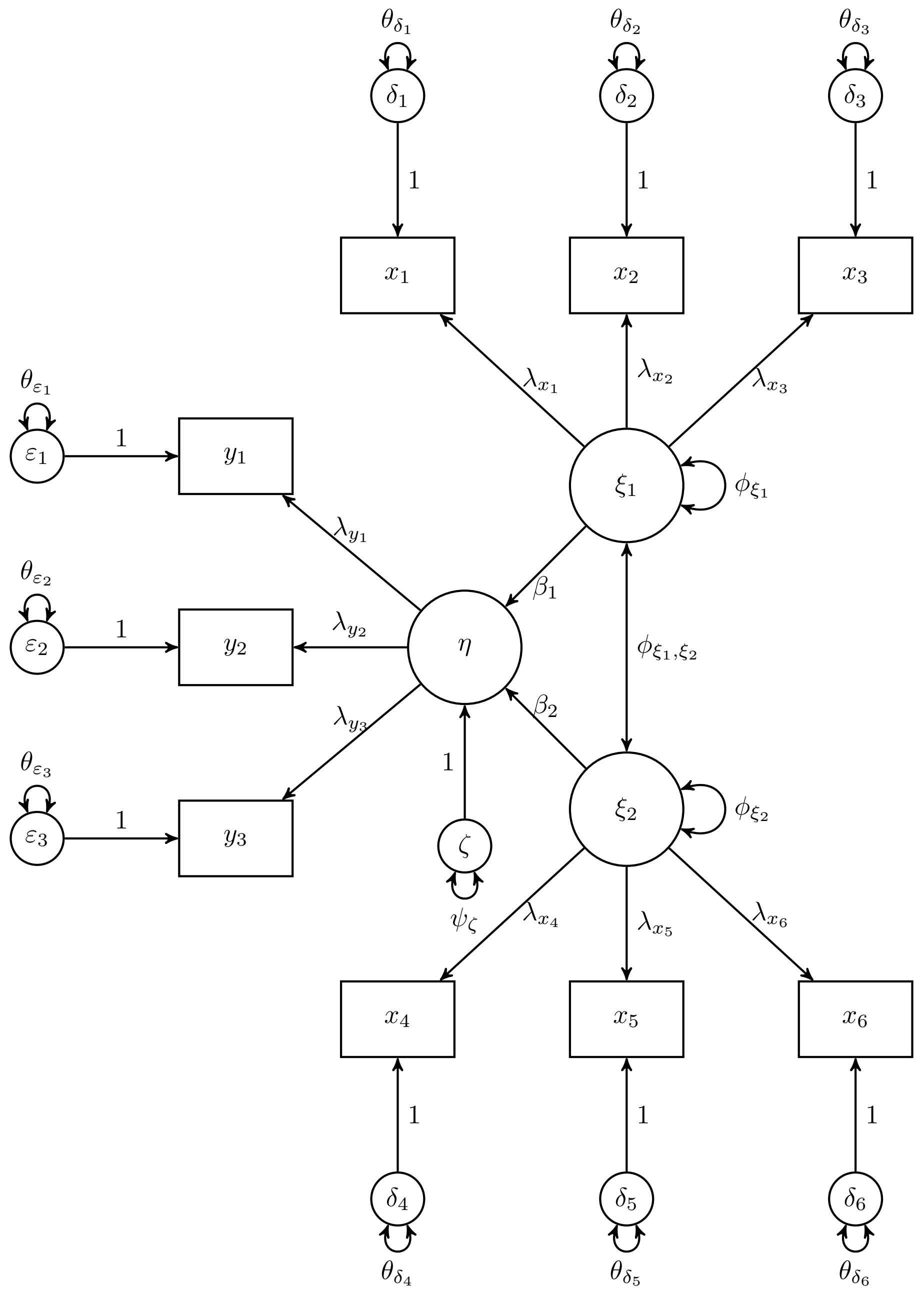


Figure 3.5: Two-Factor Confirmatory Factor Analysis Model with Mean Structure



$$\boldsymbol{\eta} = \mathbf{B}\boldsymbol{\eta} + \boldsymbol{\Gamma}\boldsymbol{\xi} + \boldsymbol{\zeta}, \mathbf{y} = \boldsymbol{\Lambda}_y\boldsymbol{\eta} + \boldsymbol{\varepsilon}, \mathbf{x} = \boldsymbol{\Lambda}_x\boldsymbol{\xi} + \boldsymbol{\delta}$$

Figure 3.6: Path Model with Latent Variables

Chapter 4

Student's t-test

In this section, the Student's t-test is presented as a structural equation model using the RAM notation. Let y be a continuous dependent variable, x be a dichotomous independent variable ($x = \{0, 1\}$), and ε be the stochastic error term with mean 0 and constant variance of σ^2_{ε} across the values of x. The associations of the variables are given by

$$y = \alpha + \beta x + \varepsilon$$

where

- α is the expected value of y when $x = 0$
- β is the unit change in y for unit change in x
- $\alpha + \beta$ is the expected value of y when $x = 1$

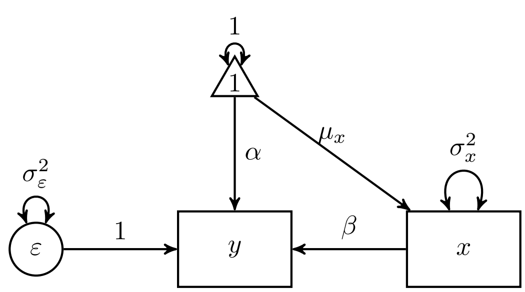


Figure 4.1: Student's t-test

4.1 Symbolic

Let $\{y, x, \varepsilon\}$ be the variables of interest.

$$\mathbf{A} = \begin{pmatrix} 0 & \beta & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_x^2 & 0 \\ 0 & 0 & \sigma_\varepsilon^2 \end{pmatrix}$$

$$\begin{aligned}
\mathbf{C} &= (\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{S}\left[(\mathbf{I} - \mathbf{A})^{-1}\right]^{\mathsf{T}} \\
&= \mathbf{E}\mathbf{S}\mathbf{E}^{\mathsf{T}} \\
&= \begin{pmatrix} 1 & \beta & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_x^2 & 0 \\ 0 & 0 & \sigma_\varepsilon^2 \end{pmatrix} \begin{pmatrix} 1 & \beta & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\mathsf{T}} \\
&= \begin{pmatrix} \sigma_x^2\beta^2 + \sigma_\varepsilon^2 & \beta\sigma_x^2 & \sigma_\varepsilon^2 \\ \sigma_x^2\beta & \sigma_x^2 & 0 \\ \sigma_\varepsilon^2 & 0 & \sigma_\varepsilon^2 \end{pmatrix}
\end{aligned}$$

$$\mathbf{F} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{aligned}
\mathbf{M} &= \mathbf{F}\,(\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{S}\left[(\mathbf{I} - \mathbf{A})^{-1}\right]^{\mathsf{T}}\mathbf{F}^{\mathsf{T}} \\
&= \mathbf{F}\mathbf{E}\mathbf{S}\mathbf{E}^{\mathsf{T}}\mathbf{F}^{\mathsf{T}} \\
&= \mathbf{F}\mathbf{C}\mathbf{F}^{\mathsf{T}} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \sigma_x^2\beta^2 + \sigma_\varepsilon^2 & \beta\sigma_x^2 & \sigma_\varepsilon^2 \\ \sigma_x^2\beta & \sigma_x^2 & 0 \\ \sigma_\varepsilon^2 & 0 & \sigma_\varepsilon^2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{\mathsf{T}} \\
&= \begin{pmatrix} \sigma_x^2\beta^2 + \sigma_\varepsilon^2 & \beta\sigma_x^2 \\ \sigma_x^2\beta & \sigma_x^2 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathbf{v} &= (\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{u} \\
&= \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & \beta & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]^{-1} \begin{pmatrix} \alpha \\ \mu_x \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \alpha + \beta\mu_x \\ \mu_x \\ 0 \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
\mathbf{u} &= (\mathbf{I} - \mathbf{A})\,\mathbf{v} \\
&= \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & \beta & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \begin{pmatrix} \alpha + \beta\mu_x \\ \mu_x \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \alpha + \beta\mu_x \\ \mu_x \\ 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{g} &= \mathbf{F}\,(\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{u} \\
&= \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & \beta & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right]^{-1} \begin{pmatrix} \alpha \\ \mu_x \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \alpha + \beta\mu_x \\ \mu_x \end{pmatrix}
\end{aligned}
$$

4.1.1 Using the `ramR` Package

```
A
```

```
##   y   x      e
## y "0" "beta" "1"
## x "0" "0"    "0"
## e "0" "0"    "0"
```

```
S
```

```
##   y   x            e
## y "0" "0"          "0"
## x "0" "sigma[x]^2" "0"
## e "0" "0"          "sigma[varepsilon]^2"
```

```
u
```

```
##   u
## y "alpha"
## x "mu[x]"
## e "0"
```

```
Filter
```

```
##   y x e
## y 1 0 0
## x 0 1 0
```

The covariance expectations can be symbolically derived using the `ramR::C()` function with `A` of class `yac.symbol`.

```
ramR::C(Ryacas::ysym(A), S)
```

```
## {{sigma[x]^2*beta^2+sigma[varepsilon]^2,                          beta*sigma[x]^2,                sig
##  {                       sigma[x]^2*beta,                              sigma[x]^2,
##  {                    sigma[varepsilon]^2,                                     0,                sig
```

$$\mathbf{C} = \begin{pmatrix} \sigma_x^2\beta^2 + \sigma_\varepsilon^2 & \beta\sigma_x^2 & \sigma_\varepsilon^2 \\ \sigma_x^2\beta & \sigma_x^2 & 0 \\ \sigma_\varepsilon^2 & 0 & \sigma_\varepsilon^2 \end{pmatrix}$$

The covariance expectations for the observed variables can be symbolically derived using the `ramR::M()` function with `A` of class `yac.symbol`.

```
ramR::M(Ryacas::ysym(A), S, Filter)
```

```
## {{sigma[x]^2*beta^2+sigma[varepsilon]^2,                          beta*sigma[x]^2},
##  {                       sigma[x]^2*beta,                              sigma[x]^2}}
```

$$\mathbf{M} = \begin{pmatrix} \sigma_x^2\beta^2 + \sigma_\varepsilon^2 & \beta\sigma_x^2 \\ \sigma_x^2\beta & \sigma_x^2 \end{pmatrix}$$

The mean expectations can be symbolically derived using the `ramR::v()` function with `A` of class `yac.symbol`.

```
ramR::v(Ryacas::ysym(A), u)
```

```
## {{alpha+beta*mu[x]},
##  {           mu[x]},
##  {               0}}
```

$$\mathbf{v} = \begin{pmatrix} \alpha + \beta\mu_x \\ \mu_x \\ 0 \end{pmatrix}$$

The mean expectations for the observed variables can be symbolically derived using the `ramR::g()` function with `A` of class `yac.symbol`.

```
ramR::g(Ryacas::ysym(A), u, Filter)
```

```
## {{alpha+beta*mu[x]},
##  {           mu[x]}}
```

$$\mathbf{g} = \begin{pmatrix} \alpha + \beta\mu_x \\ \mu_x \end{pmatrix}$$

4.2 Numerical Example

Let df be a random sample from a population with the following parameters

Parameter	$x = 0$	$x = 1$
Sample Size	500	500
$\mathbb{E}(y \mid x)$	0	1
$\mathrm{Var}(y \mid x)$	1	1

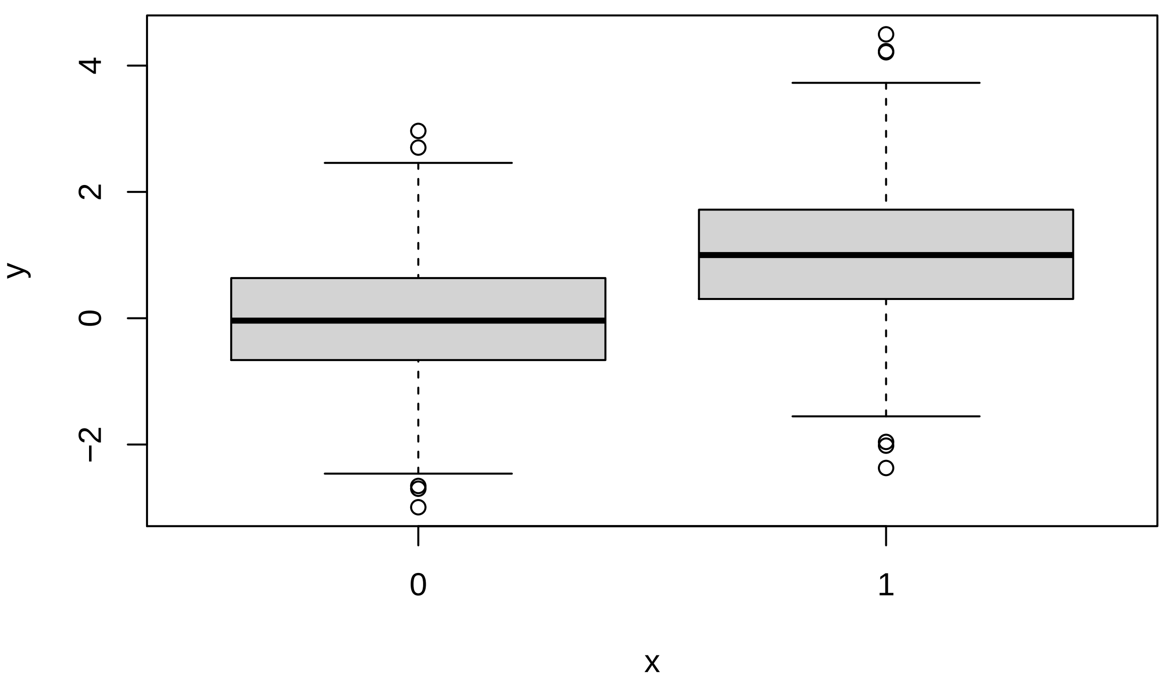
Parameter	Description	Value
α	$\mathbb{E}(y \mid x = 0)$	0
β	$\mathbb{E}(y \mid x = 1) - \mathbb{E}(y \mid x = 0)$	1

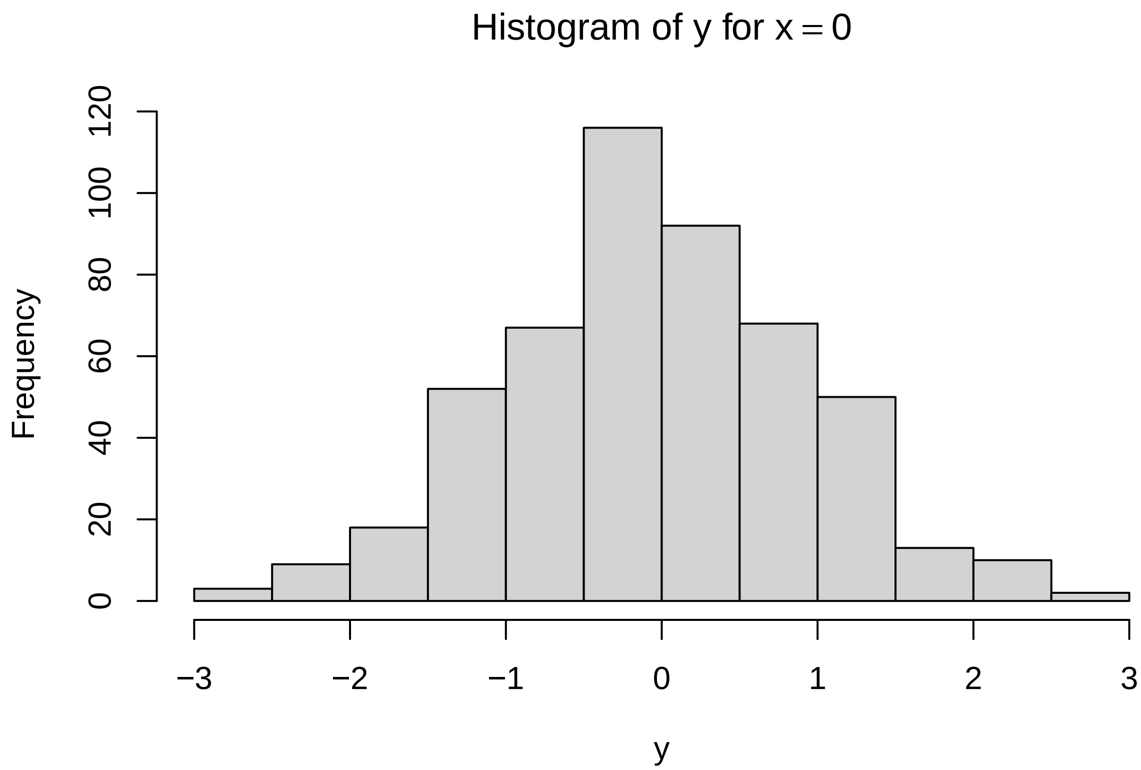
```
head(df)
```

```
##            y x
## 1  1.3709584 0
## 2 -0.5646982 0
## 3  0.3631284 0
## 4  0.6328626 0
## 5  0.4042683 0
## 6 -0.1061245 0
```

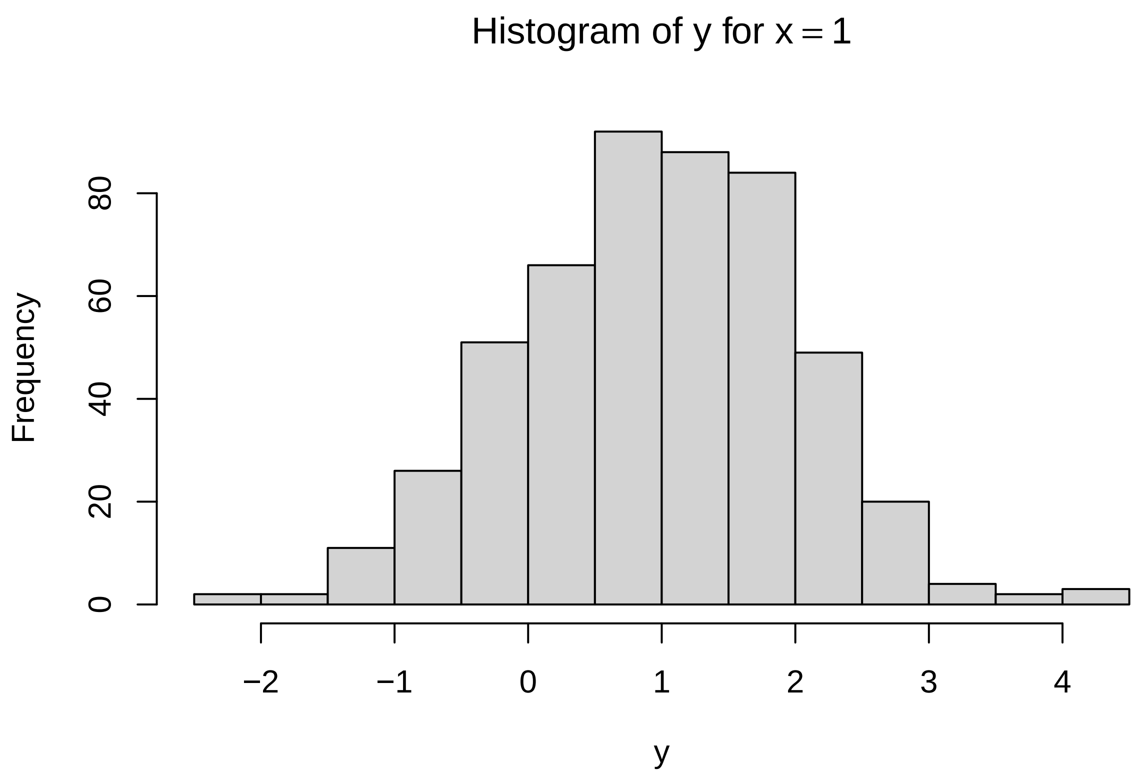
```
summary(df)
```

```
##        y                  x
##  Min.   :-2.9931   Min.   :0.0
##  1st Qu.:-0.2770   1st Qu.:0.0
##  Median : 0.4503   Median :0.5
##  Mean   : 0.4742   Mean   :0.5
##  3rd Qu.: 1.2492   3rd Qu.:1.0
##  Max.   : 4.4953   Max.   :1.0
```





Histogram of y for x = 0
Frequency
0
20
40
60
80
100
120
−3
−2
−1
0
1
2
3
y



Histogram of y for x = 1
Frequency
0
20
40
60
80
−2
−1
0
1
2
3
4
y

4.2.1 t-test

```r
t.test(y ~ x, data = df)
```

```
##
##  Welch Two Sample t-test
##
## data:  y by x
## t = -15.897, df = 994.36, p-value < 2.2e-16
## alternative hypothesis: true difference in means is not equal to 0
## 95 percent confidence interval:
##  -1.1329278 -0.8839594
## sample estimates:
## mean in group 0 mean in group 1
##     -0.03004622      0.97839737
```

4.2.2 Linear Regression

```r
summary(lm(y ~ x, data = df))
```

```
##
## Call:
## lm(formula = y ~ x, data = df)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -3.3501 -0.6517  0.0086  0.6858  3.5169
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) -0.03005    0.04486   -0.67    0.503
## x            1.00844    0.06344   15.90   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.003 on 998 degrees of freedom
## Multiple R-squared:  0.2021, Adjusted R-squared:  0.2013
## F-statistic: 252.7 on 1 and 998 DF,  p-value: < 2.2e-16
```

4.2.3 Structural Equation Modeling

4.2.3.1 lavaan (Rosseel, 2012)

```r
model <- "
  y ~ x
"
fit <- lavaan::sem(
  model,
  data = df,
  meanstructure = TRUE,
```

```
  fixed.x = FALSE
)
lavaan::summary(fit)
```

```
## lavaan 0.6-7 ended normally after 12 iterations
##
##   Estimator                                         ML
##   Optimization method                           NLMINB
##   Number of free parameters                          5
##
##   Number of observations                          1000
##
## Model Test User Model:
##
##   Test statistic                                 0.000
##   Degrees of freedom                                 0
##
## Parameter Estimates:
##
##   Standard errors                             Standard
##   Information                                 Expected
##   Information saturated (h1) model          Structured
##
## Regressions:
##                    Estimate  Std.Err  z-value  P(>|z|)
##   y ~
##     x                 1.008    0.063   15.913    0.000
##
## Intercepts:
##                    Estimate  Std.Err  z-value  P(>|z|)
##    .y                -0.030    0.045   -0.671    0.503
##     x                 0.500    0.016   31.623    0.000
##
## Variances:
##                    Estimate  Std.Err  z-value  P(>|z|)
##    .y                 1.004    0.045   22.361    0.000
##     x                 0.250    0.011   22.361    0.000
```

4.2.3.2 OpenMx (Boker et al., 2020)

RAM matrices can be used to specify models in `OpenMx`. Note, however, that the `u` vector in the RAM notation is `M` in the `OpenMx` notation.

```
mxData <- OpenMx::mxData(
  observed = df,
  type = "raw"
)
mxA <- OpenMx::mxMatrix(
  type = "Full",
  nrow = 3,
  ncol = 3,
  free = c(
```

```
    F, T, F,
    F, F, F,
    F, F, F
  ),
  values = c(
    0, 0.20, 1,
    0, 0, 0,
    0, 0, 0
  ),
  labels = c(
    NA, "beta", NA,
    NA, NA, NA,
    NA, NA, NA
  ),
  byrow = TRUE,
  name = "mxA"
)
mxS <- OpenMx::mxMatrix(
  type = "Symm",
  nrow = 3,
  ncol = 3,
  free = c(
    F, F, F,
    F, T, F,
    F, F, T
  ),
  values = c(
    0, 0, 0,
    0, 0.20, 0,
    0, 0, 0.20
  ),
  labels = c(
    NA, NA, NA,
    NA, "sigma2x", NA,
    NA, NA, "sigma2e"
  ),
  byrow = TRUE,
  name = "mxS"
)
mxM <- OpenMx::mxMatrix(
  type = "Full",
  nrow = 1,
  ncol = 3,
  free = c(
    T, T, F
  ),
  values = c(
    0.20,
    0.20,
    0
  ),
  labels = c(
    "alpha",
```

```
    "mux",
    NA
  ),
  byrow = TRUE,
  name = "mxM"
)
mxF <- OpenMx::mxMatrix(
  type = "Full",
  nrow = 2,
  ncol = 3,
  free = FALSE,
  values = c(
    1, 0, 0,
    0, 1, 0
  ),
  byrow = TRUE,
  name = "mxF"
)
expRAM <- OpenMx::mxExpectationRAM(
  A = "mxA",
  S = "mxS",
  F = "mxF",
  M = "mxM",
  dimnames = c(
    "y",
    "x",
    "e"
  )
)
objML <- OpenMx::mxFitFunctionML()
mxMod <- OpenMx::mxModel(
  name = "Student's t test",
  data = mxData,
  matrices = list(
    mxA,
    mxS,
    mxF,
    mxM
  ),
  expectation = expRAM,
  fitfunction = objML
)
fit <- OpenMx::mxRun(mxMod)
```

```
## Running Student's t test with 5 parameters
```

```
summary(fit)
```

```
## Summary of Student's t test
##
## free parameters:
##      name matrix row col    Estimate  Std.Error A
```

```
## 1    beta    mxA   1   2  1.00844356 0.06337369
## 2 sigma2x    mxS   2   2  0.25000000 0.01118034
## 3 sigma2e    mxS   3   3  1.00402596 0.04490152
## 4   alpha    mxM   1   y -0.03004621 0.04481202
## 5     mux    mxM   1   x  0.49999999 0.01581140
##
## Model Statistics:
##                |  Parameters  |  Degrees of Freedom  |  Fit (-2lnL units)
##        Model:             5                   1995             4293.478
##    Saturated:             5                   1995                   NA
## Independence:             4                   1996                   NA
## Number of observations/statistics: 1000/2000
##
## Information Criteria:
##       |  df Penalty  |  Parameters Penalty  |  Sample-Size Adjusted
## AIC:       303.4776              4303.478                 4303.538
## BIC:     -9487.4941              4328.016                 4312.136
## CFI: NA
## TLI: 1   (also known as NNFI)
## RMSEA:  0  [95% CI (NA, NA)]
## Prob(RMSEA <= 0.05): NA
## To get additional fit indices, see help(mxRefModels)
## timestamp: 2021-02-14 03:27:22
## Wall clock time: 0.04273891 secs
## optimizer:  SLSQP
## OpenMx version number: 2.18.1
## Need help?  See help(mxSummary)
```

4.2.4 Using the `ramR` Package

```
A
```

```
##   y        x e
## y 0 1.008444 1
## x 0 0.000000 0
## e 0 0.000000 0
```

```
S
```

```
##   y         x        e
## y 0 0.0000000 0.000000
## x 0 0.2502503 0.000000
## e 0 0.0000000 1.006038
```

```
u
```

```
##             u
## y -0.03004622
## x  0.50000000
## e  0.00000000
```

```
Filter
```

```
##   y x e
## y 1 0 0
## x 0 1 0
```

The covariance expectations can be numerically derived using the `ramR::C()` function.

```
ramR::C(A, S)
```

```
##           y         x        e
## y 1.2605321 0.2523633 1.006038
## x 0.2523633 0.2502503 0.000000
## e 1.0060380 0.0000000 1.006038
```

The covariance expectations for the observed variables can be numerically derived using the `ramR::M()` function.

```
ramR::M(A, S, Filter)
```

```
##           y         x
## y 1.2605321 0.2523633
## x 0.2523633 0.2502503
```

The mean expectations can be numerically derived using the `ramR::v()` function.

```
ramR::v(A, u)
```

```
##           v
## y 0.4741756
## x 0.5000000
## e 0.0000000
```

The mean expectations for the observed variables can be numerically derived using the `ramR::g()` function.

```
ramR::g(A, u, Filter)
```

```
##           g
## y 0.4741756
## x 0.5000000
```

4.3 Equations to RAM

The `ramR` package has a utility function to convert structural equations to RAM notation. The Student's *t*-test can be expressed in the following equations

```
eq <- "
  # LHS OPERATION RHS LABEL
  e     by        y   1
  y     on        x   beta
  e     with      e   sigma[varepsilon]^2
  x     with      x   sigma[x]^2
  y     on        1   alpha
  x     on        1   mu[x]
"
```

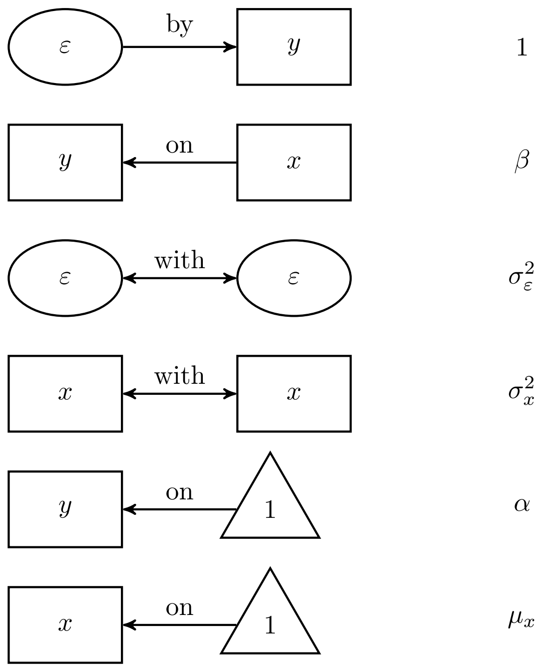


Figure 4.2: Student's t-test's Structural Equations

The error term is treated as a latent variable and defined with the operation `by`. Its value is constrained to 1. The regression of y on x is defined by operation `on`. It is labeled as `beta`. The variance of x and the error variance are defined using the operation `with`. These are labeled `sigma[x]^2` and `sigma[varepsilon]^2` respectively. The intercept and the mean of x are defined using the operation `on 1`. These are labeled `alpha` and `mu[x]` respectively.

The `ramR::Eq2RAM` converts the equations to RAM notation.

```
ramR::Eq2RAM(eq)
```

```
## $par.table
##   lhs   op rhs           par.label par.index
## 1   e   by   y                   1         1
## 2   y   on   x                beta        p1
## 3   e with   e sigma[varepsilon]^2        p2
## 4   x with   x          sigma[x]^2        p3
## 5   y   on   1               alpha        p4
## 6   x   on   1               mu[x]        p5
##
```

```
## $variables
## [1] "y" "x" "e"
##
## $g.variables
## [1] "y" "x"
##
## $h.variables
## [1] "e"
##
## $A
##   y   x      e
## y "0" "beta" "1"
## x "0" "0"    "0"
## e "0" "0"    "0"
##
## $S
##   y   x            e
## y "0" "0"          "0"
## x "0" "sigma[x]^2" "0"
## e "0" "0"          "sigma[varepsilon]^2"
##
## $u
##   u
## y "alpha"
## x "mu[x]"
## e "0"
##
## $Filter
##   y x e
## y 1 0 0
## x 0 1 0
```

4.4 Equations to Expectations

The ramR package has a utility function to convert structural equations to expectations both symbolically and numerically.

```
eq <- "
  # LHS OPERATION RHS LABEL
  e     by        y   1
  y     on        x   beta
  e     with      e   sigma[varepsilon]^2
  x     with      x   sigma[x]^2
  y     on        1   alpha
  x     on        1   mu[x]
"
```

```
ramR::Eq2Expectations(eq)
```

```
## $par.table
##   lhs   op rhs           par.label par.index
## 1   e   by   y                   1         1
```

```
## 2   y   on   x              beta        p1
## 3   e with   e sigma[varepsilon]^2        p2
## 4   x with   x          sigma[x]^2        p3
## 5   y   on   1               alpha        p4
## 6   x   on   1               mu[x]        p5
##
## $variables
## [1] "y" "x" "e"
##
## $g.variables
## [1] "y" "x"
##
## $h.variables
## [1] "e"
##
## $A
## {{   0, beta,    1},
##  {   0,    0,    0},
##  {   0,    0,    0}}
##
## $S
## {{                  0,                   0,                   0},
##  {                  0,          sigma[x]^2,                   0},
##  {                  0,                   0, sigma[varepsilon]^2}}
##
## $u
## {{alpha},
##  {mu[x]},
##  {    0}}
##
## $Filter
## {{1, 0, 0},
##  {0, 1, 0}}
##
## $v
## {{alpha+beta*mu[x]},
##  {           mu[x]},
##  {               0}}
##
## $g
## {{alpha+beta*mu[x]},
##  {           mu[x]}}
##
## $C
## {{sigma[x]^2*beta^2+sigma[varepsilon]^2,                       beta*sigma[x]^2,                   si
##  {                     sigma[x]^2*beta,                            sigma[x]^2,
##  {                 sigma[varepsilon]^2,                                     0,                   si
##
## $M
## {{sigma[x]^2*beta^2+sigma[varepsilon]^2,                       beta*sigma[x]^2},
##  {                     sigma[x]^2*beta,                            sigma[x]^2}}
```

```
eq <- "
  # LHS OPERATION RHS VALUE
```

```
  e     by        y   1.00
  y     on        x   1.00
  e     with      e   1.00
  x     with      x   0.25
  y     on        1   0.00
  x     on        1   0.50
"
```

```
ramR::Eq2Expectations(eq)
```

```
## $par.table
##   lhs   op rhs par.label par.index
## 1   e   by   y      1.00      1.00
## 2   y   on   x      1.00      1.00
## 3   e with   e      1.00      1.00
## 4   x with   x      0.25      0.25
## 5   y   on   1      0.00      0.00
## 6   x   on   1      0.50      0.50
##
## $variables
## [1] "y" "x" "e"
##
## $g.variables
## [1] "y" "x"
##
## $h.variables
## [1] "e"
##
## $A
##   y x e
## y 0 1 1
## x 0 0 0
## e 0 0 0
##
## $S
##   y    x e
## y 0 0.00 0
## x 0 0.25 0
## e 0 0.00 1
##
## $u
##     u
## y 0.0
## x 0.5
## e 0.0
##
## $Filter
##   y x e
## y 1 0 0
## x 0 1 0
##
## $v
##     v
```

```
## y 0.5
## x 0.5
## e 0.0
##
## $g
##     g
## y 0.5
## x 0.5
##
## $C
##      y    x e
## y 1.25 0.25 1
## x 0.25 0.25 0
## e 1.00 0.00 1
##
## $M
##      y    x
## y 1.25 0.25
## x 0.25 0.25
```

Chapter 5

One-Way Analysis of Variance

In this section, one-way analysis of variance is presented as a structural equation model using the RAM notation. Let y be a continuous dependent variable, x be a categorical independent variable with three levels ($x = \{0, 1, 2\}$). The dependent variable x can be dummy coded as

x	x_1	x_2
$x = 0$	0	0
$x = 1$	1	0
$x = 2$	0	1

ε is the stochastic error term with mean 0 and constant variance of σ^2_{ε} across the values of the regressors. The associations of the variables are given by

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$$

where

- β_0 is the expected value of y when $x = 0$
- β_1 is the unit change in y for unit change in x_1 while x_2 is constant
- β_2 is the unit change in y for unit change in x_2 while x_1 is constant
- $\beta_0 + \beta_1$ is the expected value of y when $x = 1$
- $\beta_0 + \beta_2$ is the expected value of y when $x = 2$

5.1 Symbolic

Let $\{y, x_1, x_2, \varepsilon\}$ be the variables of interest.

$$\mathbf{A} = \begin{pmatrix} 0 & \beta_1 & \beta_2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sigma^2_{x_1} & 0 & 0 \\ 0 & 0 & \sigma^2_{x_2} & 0 \\ 0 & 0 & 0 & \sigma^2_{\varepsilon} \end{pmatrix}$$

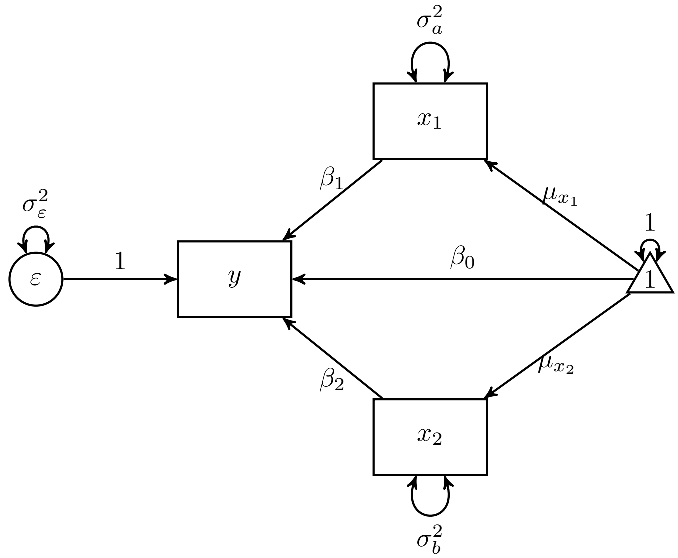


Figure 5.1: One-Way Analysis of Variance

$$
\begin{aligned}
\mathbf{C} &= (\mathbf{I}-\mathbf{A})^{-1}\,\mathbf{S}\left[(\mathbf{I}-\mathbf{A})^{-1}\right]^{\mathsf{T}} \\
&= \mathbf{E}\mathbf{S}\mathbf{E}^{\mathsf{T}} \\
&= \begin{pmatrix} 1 & \beta_1 & \beta_2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sigma_{x_1}^2 & 0 & 0 \\ 0 & 0 & \sigma_{x_2}^2 & 0 \\ 0 & 0 & 0 & \sigma_\varepsilon^2 \end{pmatrix}
\begin{pmatrix} 1 & \beta_1 & \beta_2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{\mathsf{T}} \\
&= \begin{pmatrix} \sigma_{x_1}^2\beta_1^2 + \sigma_{x_2}^2\beta_2^2 + \sigma_\varepsilon^2 & \beta_1\sigma_{x_1}^2 & \beta_2\sigma_{x_2}^2 & \sigma_\varepsilon^2 \\ \sigma_{x_1}^2\beta_1 & \sigma_{x_1}^2 & 0 & 0 \\ \sigma_{x_2}^2\beta_2 & 0 & \sigma_{x_2}^2 & 0 \\ \sigma_\varepsilon^2 & 0 & 0 & \sigma_\varepsilon^2 \end{pmatrix}
\end{aligned}
$$

$$
\mathbf{F} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}
$$

$$
\begin{aligned}
\mathbf{M} &= \mathbf{F}\left(\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{S}\left[\left(\mathbf{I}-\mathbf{A}\right)^{-1}\right]^{\mathsf{T}}\mathbf{F}^{\mathsf{T}} \\
&= \mathbf{FESE}^{\mathsf{T}}\mathbf{F}^{\mathsf{T}} \\
&= \mathbf{FCF}^{\mathsf{T}} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\begin{pmatrix} \sigma^2_{x_1}\beta_1^2 + \sigma^2_{x_2}\beta_2^2 + \sigma^2_{\varepsilon} & \beta_1\sigma^2_{x_1} & \beta_2\sigma^2_{x_2} & \sigma^2_{\varepsilon} \\ \sigma^2_{x_1}\beta_1 & \sigma^2_{x_1} & 0 & 0 \\ \sigma^2_{x_2}\beta_2 & 0 & \sigma^2_{x_2} & 0 \\ \sigma^2_{\varepsilon} & 0 & 0 & \sigma^2_{\varepsilon} \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{\mathsf{T}} \\
&= \begin{pmatrix} \sigma^2_{x_1}\beta_1^2 + \sigma^2_{x_2}\beta_2^2 + \sigma^2_{\varepsilon} & \beta_1\sigma^2_{x_1} & \beta_2\sigma^2_{x_2} \\ \sigma^2_{x_1}\beta_1 & \sigma^2_{x_1} & 0 \\ \sigma^2_{x_2}\beta_2 & 0 & \sigma^2_{x_2} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{v} &= \left(\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{u} \\
&= \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & \beta_1 & \beta_2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right]^{-1} \begin{pmatrix} \beta_0 \\ \mu_{x_1} \\ \mu_{x_2} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \beta_0 + \beta_1\mu_{x_1} + \beta_2\mu_{x_2} \\ \mu_{x_1} \\ \mu_{x_2} \\ 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{u} &= \left(\mathbf{I}-\mathbf{A}\right)\mathbf{v} \\
&= \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & \beta_1 & \beta_2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right] \begin{pmatrix} \beta_0 + \beta_1\mu_{x_1} + \beta_2\mu_{x_2} \\ \mu_{x_1} \\ \mu_{x_2} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \beta_0 + \beta_1\mu_{x_1} + \beta_2\mu_{x_2} \\ \mu_{x_1} \\ \mu_{x_2} \\ 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{g} &= \mathbf{F}\left(\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{u} \\
&= \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & \beta_1 & \beta_2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right]^{-1} \begin{pmatrix} \beta_0 \\ \mu_{x_1} \\ \mu_{x_2} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \beta_0 + \beta_1\mu_{x_1} + \beta_2\mu_{x_2} \\ \mu_{x_1} \\ \mu_{x_2} \end{pmatrix}
\end{aligned}
$$

5.1.1 Using the `ramR` Package

```
A
```

```
##    y   x1        x2        e
## y  "0" "beta[1]" "beta[2]" "1"
## x1 "0" "0"       "0"       "0"
## x2 "0" "0"       "0"       "0"
## e  "0" "0"       "0"       "0"
```

```
S
```

```
##    y   x1               x2               e
## y  "0" "0"              "0"              "0"
## x1 "0" "sigma[x[1]]^2" "0"              "0"
## x2 "0" "0"              "sigma[x[2]]^2" "0"
## e  "0" "0"              "0"              "sigma[varepsilon]^2"
```

```
u
```

```
##    u
## y  "beta[0]"
## x1 "mu[x[1]]"
## x2 "mu[x[2]]"
## e  "0"
```

```
Filter
```

```
##    y x1 x2 e
## y  1  0  0 0
## x1 0  1  0 0
## x2 0  0  1 0
```

The covariance expectations can be symbolically derived using the `ramR::C()` function with `A` of class `yac.symbol`.

```
ramR::C(Ryacas::ysym(A), S)
```

```
## {{sigma[x[1]]^2*beta[1]^2+sigma[x[2]]^2*beta[2]^2+sigma[varepsilon]^2,
##  {                                              sigma[x[1]]^2*beta[1],
##  {                                              sigma[x[2]]^2*beta[2],
##  {                                                sigma[varepsilon]^2,
```

$$
\mathbf{C} = \begin{pmatrix} \sigma_{x_1}^2\beta_1^2 + \sigma_{x_2}^2\beta_2^2 + \sigma_{\varepsilon}^2 & \beta_1\sigma_{x_1}^2 & \beta_2\sigma_{x_2}^2 & \sigma_{\varepsilon}^2 \\ \sigma_{x_1}^2\beta_1 & \sigma_{x_1}^2 & 0 & 0 \\ \sigma_{x_2}^2\beta_2 & 0 & \sigma_{x_2}^2 & 0 \\ \sigma_{\varepsilon}^2 & 0 & 0 & \sigma_{\varepsilon}^2 \end{pmatrix}
$$

The covariance expectations for the observed variables can be symbolically derived using the `ramR::M()` function with `A` of class `yac.symbol`.

```
ramR::M(Ryacas::ysym(A), S, Filter)
```

```
## {{sigma[x[1]]^2*beta[1]^2+sigma[x[2]]^2*beta[2]^2+sigma[varepsilon]^2,
##  {                                                sigma[x[1]]^2*beta[1],
##  {                                                sigma[x[2]]^2*beta[2],
```

$$\mathbf{M} = \begin{pmatrix} \sigma_{x_1}^2\beta_1^2 + \sigma_{x_2}^2\beta_2^2 + \sigma_{\varepsilon}^2 & \beta_1\sigma_{x_1}^2 & \beta_2\sigma_{x_2}^2 \\ \sigma_{x_1}^2\beta_1 & \sigma_{x_1}^2 & 0 \\ \sigma_{x_2}^2\beta_2 & 0 & \sigma_{x_2}^2 \end{pmatrix}$$

The mean expectations can be symbolically derived using the `ramR::v()` function with `A` of class `yac.symbol`.

```
ramR::v(Ryacas::ysym(A), u)
```

```
## {{beta[0]+beta[1]*mu[x[1]]+beta[2]*mu[x[2]]},
##  {                              mu[x[1]]},
##  {                              mu[x[2]]},
##  {                                     0}}
```

$$\mathbf{v} = \begin{pmatrix} \beta_0 + \beta_1\mu_{x_1} + \beta_2\mu_{x_2} \\ \mu_{x_1} \\ \mu_{x_2} \\ 0 \end{pmatrix}$$

The mean expectations for the observed variables can be symbolically derived using the `ramR::g()` function with `A` of class `yac.symbol`.

```
ramR::g(Ryacas::ysym(A), u, Filter)
```

```
## {{beta[0]+beta[1]*mu[x[1]]+beta[2]*mu[x[2]]},
##  {                              mu[x[1]]},
##  {                              mu[x[2]]}}
```

$$\mathbf{g} = \begin{pmatrix} \beta_0 + \beta_1\mu_{x_1} + \beta_2\mu_{x_2} \\ \mu_{x_1} \\ \mu_{x_2} \end{pmatrix}$$

5.2 Numerical Example

Let `df` be a random sample from a population with the following parameters

Parameter	$x = 0$	$x = 1$	$x = 2$
Sample Size	500	500	500
$\mathbb{E}(y \mid x)$	0	2	1
$\mathrm{Var}(y \mid x)$	1	1	1

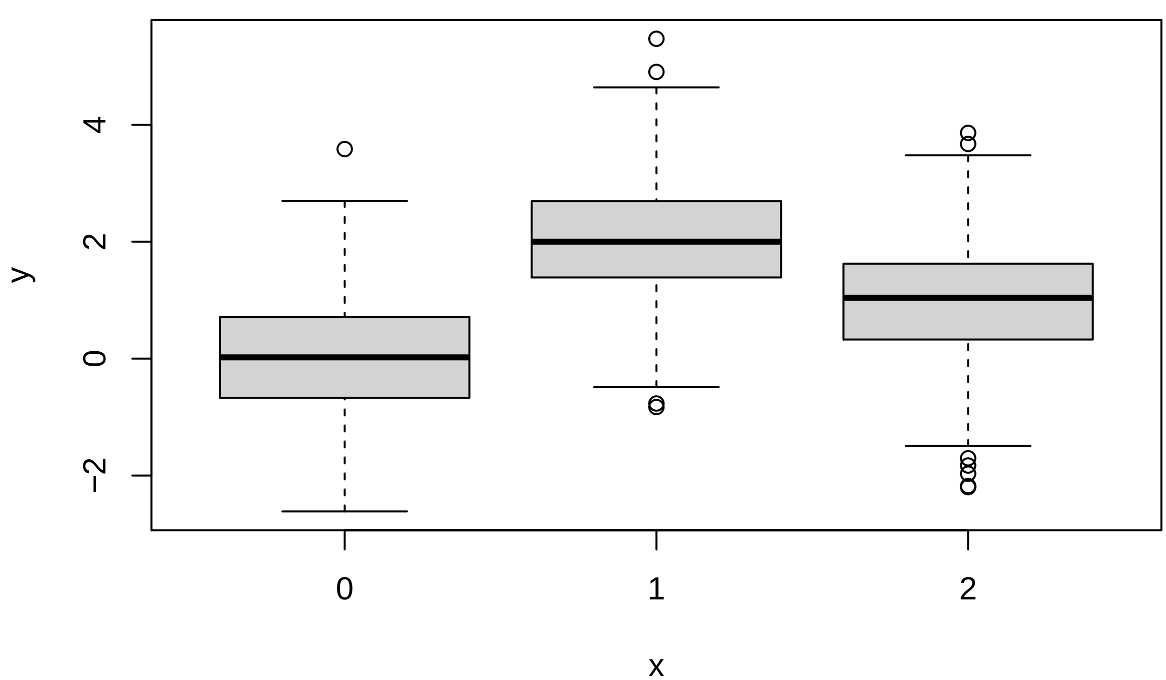
Parameter	Description	Value
β_0	$\mathbb{E}(y \mid x = 0)$	0
β_1	$\mathbb{E}(y \mid x = 1) - \mathbb{E}(y \mid x = 0)$	2
β_2	$\mathbb{E}(y \mid x = 2) - \mathbb{E}(y \mid x = 0)$	1

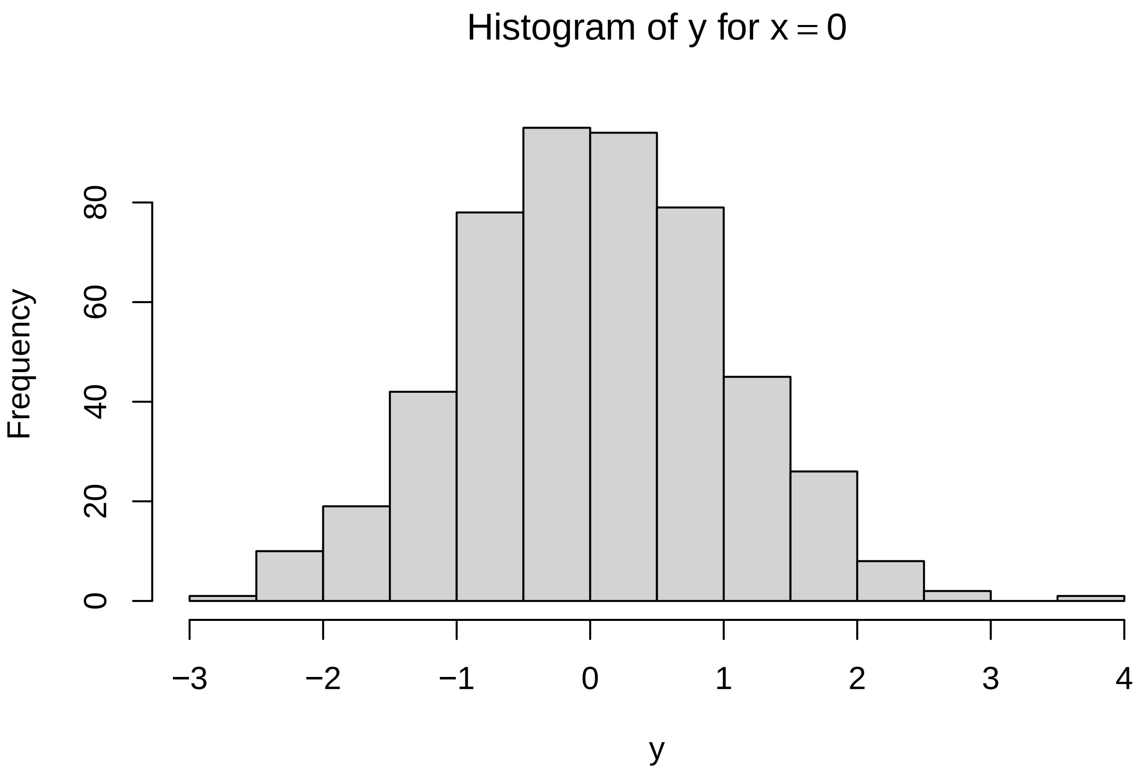
```
head(df)
```

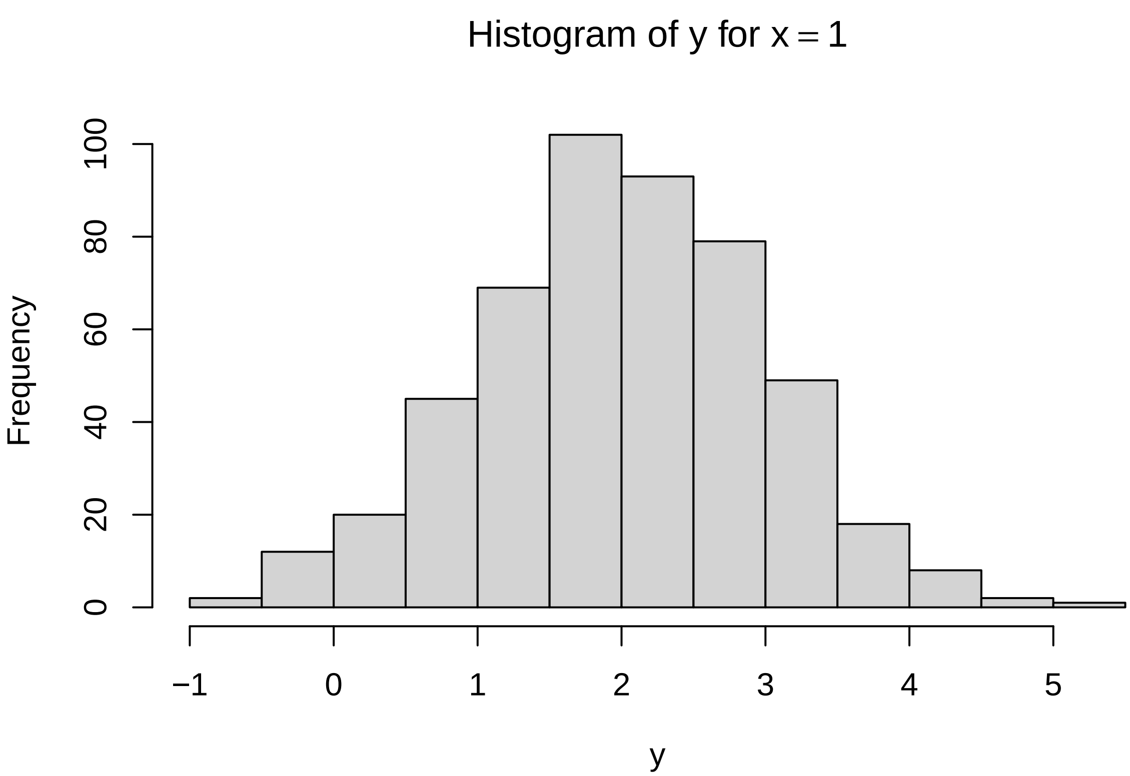
```
##            y x
## 1 -0.6013830 0
## 2 -0.1358161 0
## 3 -0.9872728 0
## 4  0.8319250 0
## 5 -0.7950595 0
## 6  0.3404646 0
```

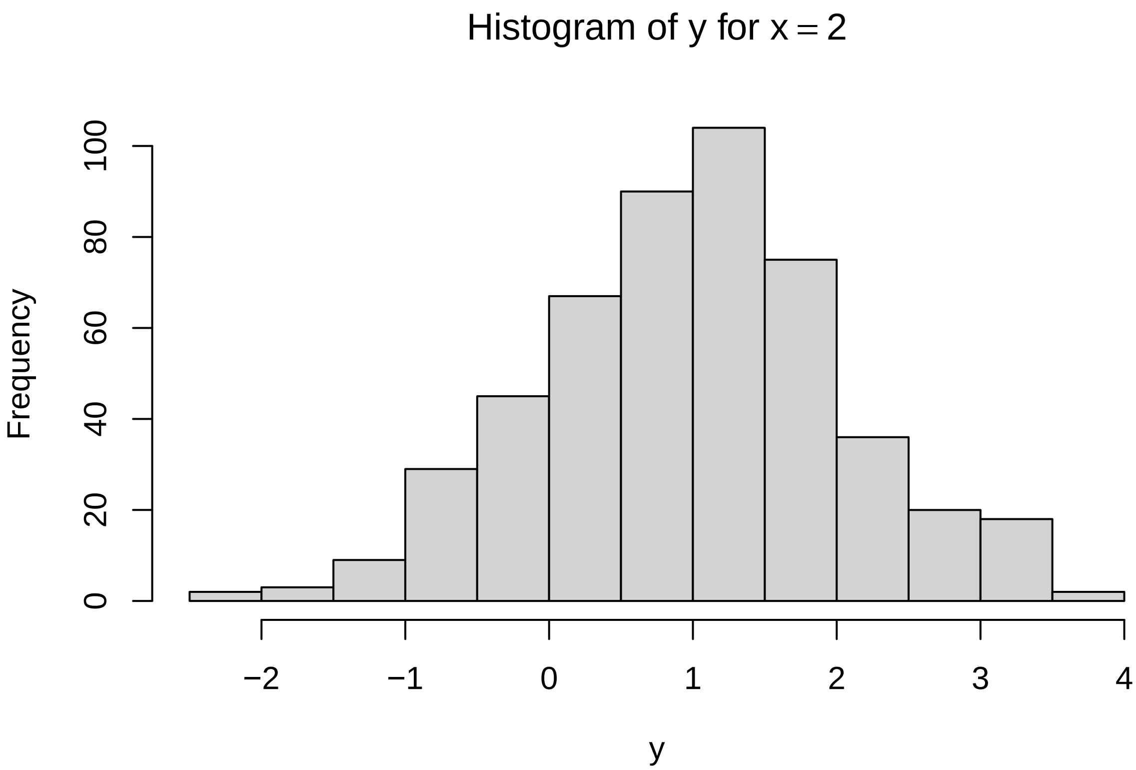
```
summary(df)
```

```
##        y             x
##  Min.   :-2.61364   0:500
##  1st Qu.: 0.08094   1:500
##  Median : 1.02617   2:500
##  Mean   : 1.00814
##  3rd Qu.: 1.90112
##  Max.   : 5.47091
```




Histogram of y for x = 0
Frequency
0
20
40
60
80
−3
−2
−1
0
1
2
3
4
y


Histogram of y for x = 1
Frequency
0
20
40
60
80
100
−1
0
1
2
3
4
5
y



5.2.1 One-Way Analysis of Variance

Make sure that x is of class `factor` for `lm` and `aov` to treat it as a categorical variable.

```
str(df)
```

```
## 'data.frame':    1500 obs. of  2 variables:
##  $ y: num  -0.601 -0.136 -0.987 0.832 -0.795 ...
##  $ x: Factor w/ 3 levels "0","1","2": 1 1 1 1 1 1 1 1 1 1 ...
```

```
summary(aov(y ~ x, data = df))
```

```
##               Df Sum Sq Mean Sq F value Pr(>F)
## x              2  983.8   491.9   471.4 <2e-16 ***
## Residuals   1497 1562.2     1.0
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

5.2.2 Linear Regression

```
summary(lm(y ~ x, data = df))
```

```
##
## Call:
```

```
## lm(formula = y ~ x, data = df)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -3.1792 -0.6469  0.0021  0.6751  3.5538
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  0.03083    0.04569   0.675      0.5
## x1           1.98309    0.06461  30.694   <2e-16 ***
## x2           0.94884    0.06461  14.686   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.022 on 1497 degrees of freedom
## Multiple R-squared:  0.3864, Adjusted R-squared:  0.3856
## F-statistic: 471.4 on 2 and 1497 DF,  p-value: < 2.2e-16
```

5.2.3 Structural Equation Modeling

We have to dummy code the data set first before fitting the model. The `model.matrix` function which is used to create a design matrix can be used to dummy code `x`. Make sure that `x` is a `factor`. The first column of the design matrix is a matrix of ones. Since we do not need this column, we can replace this column with the values of `y`. Make sure to name rename the first column as `lavaan` relies on the column names.

```
df_dummy <- model.matrix(y ~ x, data = df)
df_dummy[, 1] <- df$y
colnames(df_dummy)[1] <- "y"
head(df_dummy)
```

```
##            y x1 x2
## 1 -0.6013830  0  0
## 2 -0.1358161  0  0
## 3 -0.9872728  0  0
## 4  0.8319250  0  0
## 5 -0.7950595  0  0
## 6  0.3404646  0  0
```

5.2.3.1 lavaan (Rosseel, 2012)

```
model <- "
  y ~ x1 + x2
"
fit <- lavaan::sem(
  model,
  data = df_dummy,
  meanstructure = TRUE,
  fixed.x = FALSE
)
lavaan::summary(fit)
```

```
## lavaan 0.6-7 ended normally after 22 iterations
##
##   Estimator                                         ML
##   Optimization method                           NLMINB
##   Number of free parameters                          9
##
##   Number of observations                          1500
##
## Model Test User Model:
##
##   Test statistic                                 0.000
##   Degrees of freedom                                 0
##
## Parameter Estimates:
##
##   Standard errors                             Standard
##   Information                                 Expected
##   Information saturated (h1) model          Structured
##
## Regressions:
##                    Estimate  Std.Err  z-value  P(>|z|)
##   y ~
##     x1                1.983    0.065   30.725    0.000
##     x2                0.949    0.065   14.701    0.000
##
## Covariances:
##                    Estimate  Std.Err  z-value  P(>|z|)
##   x1 ~~
##     x2               -0.111    0.006  -17.321    0.000
##
## Intercepts:
##                    Estimate  Std.Err  z-value  P(>|z|)
##    .y                 0.031    0.046    0.676    0.499
##     x1                0.333    0.012   27.386    0.000
##     x2                0.333    0.012   27.386    0.000
##
## Variances:
##                    Estimate  Std.Err  z-value  P(>|z|)
##    .y                 1.041    0.038   27.386    0.000
##     x1                0.222    0.008   27.386    0.000
##     x2                0.222    0.008   27.386    0.000
```

5.2.3.2 OpenMx (Boker et al., 2020)

RAM matrices can be used to specify models in `OpenMx`. Note, however, that the `u` vector in the RAM notation is `M` in the `OpenMx` notation.

```
mxData <- OpenMx::mxData(
  observed = df_dummy,
  type = "raw"
)
mxA <- OpenMx::mxMatrix(
  type = "Full",
```

```
  nrow = 4,
  ncol = 4,
  free = c(
    F, T, T, F,
    F, F, F, F,
    F, F, F, F,
    F, F, F, F
  ),
  values = c(
    0, 0.20, 0.20, 1,
    0, 0, 0, 0,
    0, 0, 0, 0,
    0, 0, 0, 0
  ),
  labels = c(
    NA, "beta1", "beta2", NA,
    NA, NA, NA, NA,
    NA, NA, NA, NA,
    NA, NA, NA, NA
  ),
  byrow = TRUE,
  name = "mxA"
)
mxS <- OpenMx::mxMatrix(
  type = "Symm",
  nrow = 4,
  ncol = 4,
  free = c(
    F, F, F, F,
    F, T, F, F,
    F, F, T, F,
    F, F, F, T
  ),
  values = c(
    0,    0,    0,    0,
    0, 0.20,    0,    0,
    0,    0, 0.20,    0,
    0,    0,    0, 0.20
  ),
  labels = c(
    NA, NA, NA, NA,
    NA, "sigma2x1", NA, NA,
    NA, NA, "sigma2x2", NA,
    NA, NA, NA, "sigma2e"
  ),
  byrow = TRUE,
  name = "mxS"
)
mxM <- OpenMx::mxMatrix(
  type = "Full",
  nrow = 1,
  ncol = 4,
  free = c(
```

```
    T, T, T, F
  ),
  values = c(
    0.20,
    0.20,
    0.20,
    0
  ),
  labels = c(
    "beta0",
    "mux1",
    "mux2",
    NA
  ),
  byrow = TRUE,
  name = "mxM"
)
mxF <- OpenMx::mxMatrix(
  type = "Full",
  nrow = 3,
  ncol = 4,
  free = FALSE,
  values = c(
    1, 0, 0, 0,
    0, 1, 0, 0,
    0, 0, 1, 0
  ),
  byrow = TRUE,
  name = "mxF"
)
expRAM <- OpenMx::mxExpectationRAM(
  A = "mxA",
  S = "mxS",
  F = "mxF",
  M = "mxM",
  dimnames = c(
    "y",
    "x1",
    "x2",
    "e"
  )
)
objML <- OpenMx::mxFitFunctionML()
mxMod <- OpenMx::mxModel(
  name = "One Way Analysis of Variance",
  data = mxData,
  matrices = list(
    mxA,
    mxS,
    mxF,
    mxM
  ),
  expectation = expRAM,
```

```
  fitfunction = objML
)
fit <- OpenMx::mxRun(mxMod)
```

```
## Running One Way Analysis of Variance with 8 parameters
```

```
summary(fit)
```

```
## Summary of One Way Analysis of Variance
##
## free parameters:
##       name matrix row col   Estimate   Std.Error A
## 1    beta1    mxA   1   2 1.98308662 0.064543779
## 2    beta2    mxA   1   3 0.94883814 0.064543143
## 3 sigma2x1    mxS   2   2 0.22222230 0.008114416
## 4 sigma2x2    mxS   3   3 0.22222238 0.008114420
## 5  sigma2e    mxS   4   4 1.04147460 0.038029458
## 6    beta0    mxM   1   y 0.03083127 0.045639092
## 7     mux1    mxM   1  x1 0.33333343 0.012171613
## 8     mux2    mxM   1  x2 0.33333344 0.012171612
##
## Model Statistics:
##                |  Parameters  |  Degrees of Freedom  |  Fit (-2lnL units)
##        Model:             8                   4492              8319.17
##    Saturated:             9                   4491                   NA
## Independence:             6                   4494                   NA
## Number of observations/statistics: 1500/4500
##
## Information Criteria:
##       |  df Penalty  |  Parameters Penalty  |  Sample-Size Adjusted
## AIC:      -664.8302               8335.170                 8335.266
## BIC:    -24531.8162               8377.676                 8352.262
## To get additional fit indices, see help(mxRefModels)
## timestamp: 2021-02-14 03:27:24
## Wall clock time: 0.03658605 secs
## optimizer:  SLSQP
## OpenMx version number: 2.18.1
## Need help?  See help(mxSummary)
```

5.2.4 Using the ramR Package

```
A
```

```
##    y       x1        x2 e
## y  0 2.008444 0.9885797 1
## x1 0 0.000000 0.0000000 0
## x2 0 0.000000 0.0000000 0
## e  0 0.000000 0.0000000 0
```

```r
S
```

```
##    y        x1        x2         e
## y  0 0.0000000 0.0000000 0.0000000
## x1 0 0.2223705 0.0000000 0.0000000
## x2 0 0.0000000 0.2223705 0.0000000
## e  0 0.0000000 0.0000000 0.9823083
```

```r
u
```

```
##            u
## y  0.3333333
## x1 0.3333333
## x2 0.3333333
## e  0.0000000
```

```r
Filter
```

```
##    y x1 x2 e
## y  1  0  0 0
## x1 0  1  0 0
## x2 0  0  1 0
```

The covariance expectations can be numerically derived using the `ramR::C()` function.

```r
ramR::C(A, S)
```

```
##            y        x1        x2         e
## y  2.0966368 0.4466185 0.2198309 0.9823083
## x1 0.4466185 0.2223705 0.0000000 0.0000000
## x2 0.2198309 0.0000000 0.2223705 0.0000000
## e  0.9823083 0.0000000 0.0000000 0.9823083
```

The covariance expectations for the observed variables can be numerically derived using the `ramR::M()` function.

```r
ramR::M(A, S, Filter)
```

```
##            y        x1        x2
## y  2.0966368 0.4466185 0.2198309
## x1 0.4466185 0.2223705 0.0000000
## x2 0.2198309 0.0000000 0.2223705
```

The mean expectations can be numerically derived using the `ramR::v()` function.

```r
ramR::v(A, u)
```

```
##            v
## y  1.3323411
## x1 0.3333333
## x2 0.3333333
## e  0.0000000
```

The mean expectations for the observed variables can be numerically derived using the `ramR::v()` function.

```
ramR::g(A, u, Filter)
```

```
##            g
## y  1.3323411
## x1 0.3333333
## x2 0.3333333
```

5.3 Equations to RAM

The `ramR` package has a utility function to convert structural equations to RAM notation. One-way analysis of variance with three levels can be expressed in the following equations

```
eq <- "
  # VARIABLE1 OPERATION VARIABLE2 LABEL
  e           by        y         1
  y           on        x1        beta[1]
  y           on        x2        beta[2]
  e           with      e         sigma[varepsilon]^2
  x1          with      x1        sigma[x[1]]^2
  x2          with      x2        sigma[x[2]]^2
  y           on        1         beta[0]
  x1          on        1         mu[x[1]]
  x2          on        1         mu[x[2]]
"
```

The error term is treated as a latent variable and defined with the operation `by`. Its value is constrained to 1. The regression of y on x_1 and x_2 is defined by operation `on`. The coefficients are labeled as `beta[1]` and `beta[2]` respectively. The variance of x_1, x_2 and the error variance are defined using the operation `with`. These are labeled `sigma[x[1]]^2`, `sigma[x[2]]^2`, and `sigma[varepsilon]^2` respectively. The intercept and the mean of x_1 and x_2 are defined using the operation `on 1`. These are labeled `beta[0]`, `mu[x[1]]`, and `mu[x[2]]` respectively.

The `ramR::Eq2RAM` converts the equations to RAM notation.

```
ramR::Eq2RAM(eq)
```

```
## $par.table
##   lhs   op rhs           par.label par.index
## 1   e   by   y                   1         1
## 2   y   on  x1             beta[1]        p1
## 3   y   on  x2             beta[2]        p2
## 4   e with   e sigma[varepsilon]^2        p3
## 5  x1 with  x1       sigma[x[1]]^2        p4
## 6  x2 with  x2       sigma[x[2]]^2        p5
## 7   y   on   1             beta[0]        p6
## 8  x1   on   1            mu[x[1]]        p7
## 9  x2   on   1            mu[x[2]]        p8
##
## $variables
```

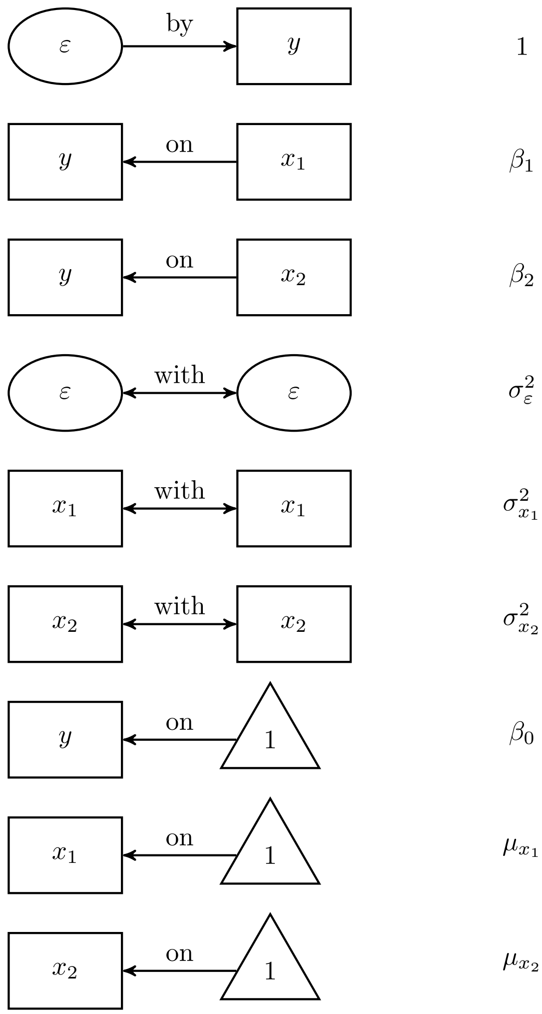


Figure 5.2: One-Way Analysis of Variance's Structural Equations

```
## [1] "y"  "x1" "x2" "e"
##
## $g.variables
## [1] "y"  "x1" "x2"
##
## $h.variables
## [1] "e"
##
## $A
##    y   x1        x2        e
## y  "0" "beta[1]" "beta[2]" "1"
## x1 "0" "0"       "0"       "0"
## x2 "0" "0"       "0"       "0"
## e  "0" "0"       "0"       "0"
##
## $S
##    y   x1              x2              e
## y  "0" "0"             "0"             "0"
## x1 "0" "sigma[x[1]]^2" "0"             "0"
## x2 "0" "0"             "sigma[x[2]]^2" "0"
## e  "0" "0"             "0"             "sigma[varepsilon]^2"
##
## $u
##    u
## y  "beta[0]"
## x1 "mu[x[1]]"
## x2 "mu[x[2]]"
## e  "0"
##
## $Filter
##    y x1 x2 e
## y  1  0  0 0
## x1 0  1  0 0
## x2 0  0  1 0
```

5.4 Equations to Expectations

The `ramR` package has a utility function to convert structural equations to expectations both symbolically and numerically.

```
eq <- "
  # VARIABLE1 OPERATION VARIABLE2 LABEL
  e         by        y         1
  y         on        x1        beta[1]
  y         on        x2        beta[2]
  e         with      e         sigma[varepsilon]^2
  x1        with      x1        sigma[x[1]]^2
  x2        with      x2        sigma[x[2]]^2
  y         on        1         beta[0]
  x1        on        1         mu[x[1]]
  x2        on        1         mu[x[2]]
"
```

```
ramR::Eq2Expectations(eq)
```

```
## $par.table
##   lhs   op rhs           par.label par.index
## 1   e   by   y                   1         1
## 2   y   on  x1             beta[1]        p1
## 3   y   on  x2             beta[2]        p2
## 4   e with   e sigma[varepsilon]^2        p3
## 5  x1 with  x1       sigma[x[1]]^2        p4
## 6  x2 with  x2       sigma[x[2]]^2        p5
## 7   y   on   1             beta[0]        p6
## 8  x1   on   1            mu[x[1]]        p7
## 9  x2   on   1            mu[x[2]]        p8
##
## $variables
## [1] "y"  "x1" "x2" "e"
##
## $g.variables
## [1] "y"  "x1" "x2"
##
## $h.variables
## [1] "e"
##
## $A
## {{      0, beta[1], beta[2],       1},
##  {      0,       0,       0,       0},
##  {      0,       0,       0,       0},
##  {      0,       0,       0,       0}}
##
## $S
## {{                  0,                   0,                   0,                   0},
##  {                  0,       sigma[x[1]]^2,                   0,                   0},
##  {                  0,                   0,       sigma[x[2]]^2,                   0},
##  {                  0,                   0,                   0, sigma[varepsilon]^2}}
##
## $u
## {{ beta[0]},
##  {mu[x[1]]},
##  {mu[x[2]]},
##  {       0}}
##
## $Filter
## {{1, 0, 0, 0},
##  {0, 1, 0, 0},
##  {0, 0, 1, 0}}
##
## $v
## {{beta[0]+beta[1]*mu[x[1]]+beta[2]*mu[x[2]]},
##  {                                mu[x[1]]},
##  {                                mu[x[2]]},
##  {                                       0}}
##
## $g
```

```
## {{beta[0]+beta[1]*mu[x[1]]+beta[2]*mu[x[2]]},
##  {                                mu[x[1]]},
##  {                                mu[x[2]]}}
##
## $C
## {{sigma[x[1]]^2*beta[1]^2+sigma[x[2]]^2*beta[2]^2+sigma[varepsilon]^2,
##  {                                              sigma[x[1]]^2*beta[1],
##  {                                              sigma[x[2]]^2*beta[2],
##  {                                                sigma[varepsilon]^2,
##
## $M
## {{sigma[x[1]]^2*beta[1]^2+sigma[x[2]]^2*beta[2]^2+sigma[varepsilon]^2,
##  {                                              sigma[x[1]]^2*beta[1],
##  {                                              sigma[x[2]]^2*beta[2],
```

```
eq <- "
  # VARIABLE1 OPERATION VARIABLE2 LABEL
  e           by        y         1
  y           on        x1        2
  y           on        x2        1
  e           with      e         1
  x1          with      x1        0.22222222222
  x2          with      x2        0.22222222222
  y           on        1         0
  x1          on        1         0.33333333333
  x2          on        1         0.33333333333
"
```

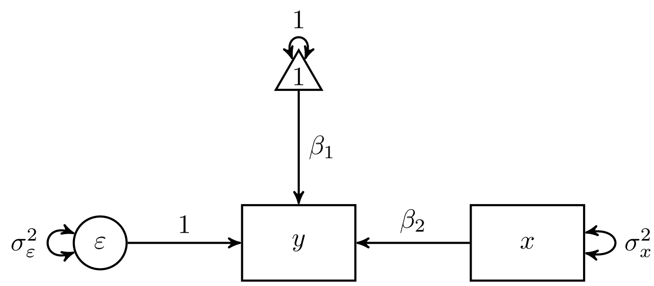
```
ramR::Eq2Expectations(eq)
```

```
## $par.table
##   lhs   op rhs par.label par.index
## 1   e   by   y 1.0000000 1.0000000
## 2   y   on  x1 2.0000000 2.0000000
## 3   y   on  x2 1.0000000 1.0000000
## 4   e with   e 1.0000000 1.0000000
## 5  x1 with  x1 0.2222222 0.2222222
## 6  x2 with  x2 0.2222222 0.2222222
## 7   y   on   1 0.0000000 0.0000000
## 8  x1   on   1 0.3333333 0.3333333
## 9  x2   on   1 0.3333333 0.3333333
##
## $variables
## [1] "y"  "x1" "x2" "e"
##
## $g.variables
## [1] "y"  "x1" "x2"
##
## $h.variables
## [1] "e"
##
## $A
##    y x1 x2 e
```

```
## y  0  2  1 1
## x1 0  0  0 0
## x2 0  0  0 0
## e  0  0  0 0
##
## $S
##    y        x1        x2 e
## y  0 0.0000000 0.0000000 0
## x1 0 0.2222222 0.0000000 0
## x2 0 0.0000000 0.2222222 0
## e  0 0.0000000 0.0000000 1
##
## $u
##            u
## y  0.0000000
## x1 0.3333333
## x2 0.3333333
## e  0.0000000
##
## $Filter
##    y x1 x2 e
## y  1  0  0 0
## x1 0  1  0 0
## x2 0  0  1 0
##
## $v
##            v
## y  1.0000000
## x1 0.3333333
## x2 0.3333333
## e  0.0000000
##
## $g
##            g
## y  1.0000000
## x1 0.3333333
## x2 0.3333333
##
## $C
##            y        x1        x2 e
## y  2.1111111 0.4444444 0.2222222 1
## x1 0.4444444 0.2222222 0.0000000 0
## x2 0.2222222 0.0000000 0.2222222 0
## e  1.0000000 0.0000000 0.0000000 1
##
## $M
##            y        x1        x2
## y  2.1111111 0.4444444 0.2222222
## x1 0.4444444 0.2222222 0.0000000
## x2 0.2222222 0.0000000 0.2222222
```

Chapter 6

Two-Variable Regression Model



$$y = \alpha + \beta x + \varepsilon$$

Figure 6.1: Two-Variable Regression Model

Chapter 7

k-Variable Regression Model

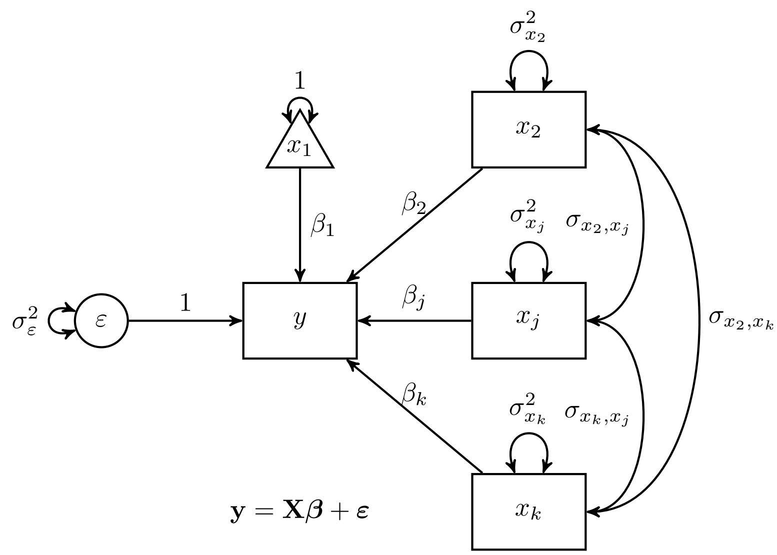


Figure 7.1: k-Variable Regression Model

Chapter 8

The Simple Mediation Model

Let y, m, x, ε_y, and ε_m be random variables whose associations are given by

$$y = \beta_0 + \beta_1 x + \beta_2 m + \varepsilon_y \tag{8.1}$$

$$m = \alpha_0 + \alpha_1 x + \varepsilon_m \tag{8.2}$$

or combined

$$\begin{aligned} y &= \beta_0 + \beta_1 x + \beta_2(\alpha_0 + \alpha_1 x + \varepsilon_m) + \varepsilon_y \\ &= \beta_0 + \beta_1 x + \beta_2 \alpha_0 + \alpha_1 \beta_2 x + \beta_2 \varepsilon_m + \varepsilon_y \end{aligned} \tag{8.3}$$

where

- β_1 is the path from x on y
- β_2 is the path from m to y
- α_1 is the path from x to m
- ε_y and ε_m are uncorrelated error terms with means of zero and constant variances of $\sigma^2_{\varepsilon_y}$ and $\sigma^2_{\varepsilon_m}$ respectively
- β_0 and α_0 are intercepts

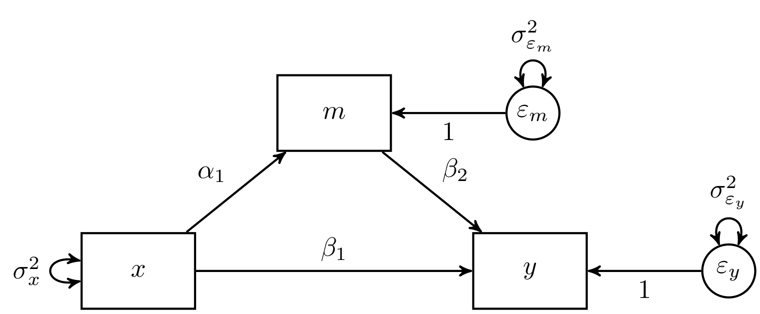


Figure 8.1: The Simple Mediation Model

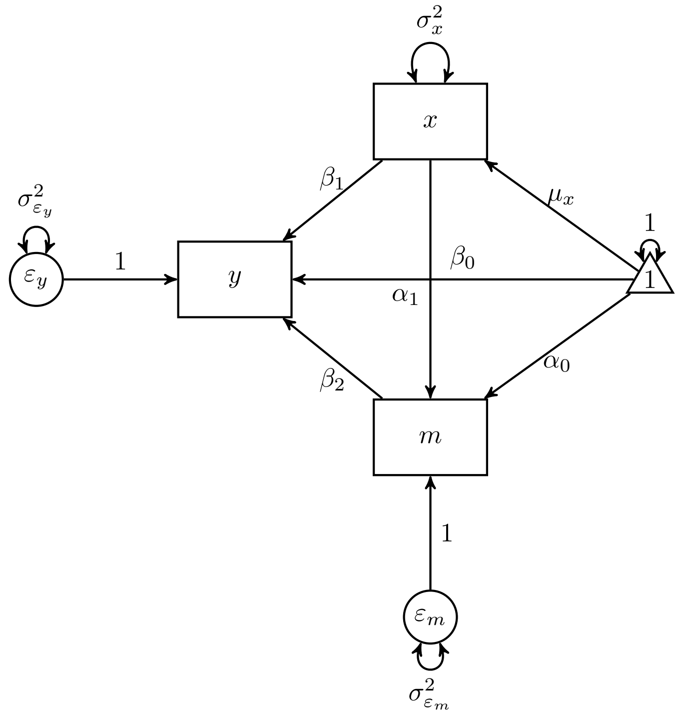


Figure 8.2: The Simple Mediation Model with Mean Structure

8.1 Symbolic

Let $\{y, m, x, \varepsilon_y, \varepsilon_m\}$ be the variables of interest.

$$\mathbf{A} = \begin{pmatrix} 0 & \beta_2 & \beta_1 & 1 & 0 \\ 0 & 0 & \alpha_1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma^2_x & 0 & 0 \\ 0 & 0 & 0 & \sigma^2_{\varepsilon_y} & 0 \\ 0 & 0 & 0 & 0 & \sigma^2_{\varepsilon_m} \end{pmatrix}$$

$$
\begin{aligned}
\mathbf{C} &= (\mathbf{I}-\mathbf{A})^{-1}\,\mathbf{S}\left[(\mathbf{I}-\mathbf{A})^{-1}\right]^{\mathsf{T}} \\
&= \mathbf{E}\mathbf{S}\mathbf{E}^{\mathsf{T}} \\
&= \begin{pmatrix} 1 & \beta_2 & \beta_1+\beta_2\alpha_1 & 1 & \beta_2 \\ 0 & 1 & \alpha_1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_x^2 & 0 & 0 \\ 0 & 0 & 0 & \sigma_{\varepsilon_y}^2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_{\varepsilon_m}^2 \end{pmatrix}
\begin{pmatrix} 1 & \beta_2 & \beta_1+\beta_2\alpha_1 & 1 & \beta_2 \\ 0 & 1 & \alpha_1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}^{\mathsf{T}} \\
&= \begin{pmatrix}
\sigma_x^2\beta_1^2 + 2\sigma_x^2\beta_1\beta_2\alpha_1 + \sigma_x^2\beta_2^2\alpha_1^2 + \beta_2^2\sigma_{\varepsilon_m}^2 + \sigma_{\varepsilon_y}^2 & \beta_1\alpha_1\sigma_x^2 + \beta_2\alpha_1^2\sigma_x^2 + \beta_2\sigma_{\varepsilon_m}^2 & \beta_1\sigma_x^2 + \beta_2\alpha_1\sigma_x^2 & \sigma_{\varepsilon_y}^2 & \beta_2\sigma_{\varepsilon_m}^2 \\
\alpha_1^2\sigma_x^2\beta_2 + \alpha_1\sigma_x^2\beta_1 + \beta_2\sigma_{\varepsilon_m}^2 & \sigma_x^2\alpha_1^2 + \sigma_{\varepsilon_m}^2 & \alpha_1\sigma_x^2 & 0 & \sigma_{\varepsilon_m}^2 \\
\sigma_x^2\beta_1 + \sigma_x^2\beta_2\alpha_1 & \sigma_x^2\alpha_1 & \sigma_x^2 & 0 & 0 \\
\sigma_{\varepsilon_y}^2 & 0 & 0 & \sigma_{\varepsilon_y}^2 & 0 \\
\sigma_{\varepsilon_m}^2\beta_2 & \sigma_{\varepsilon_m}^2 & 0 & 0 & \sigma_{\varepsilon_m}^2
\end{pmatrix}
\end{aligned}
$$

$$
\mathbf{F} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}
$$

$$
\begin{aligned}
\mathbf{M} &= \mathbf{F}\,(\mathbf{I}-\mathbf{A})^{-1}\,\mathbf{S}\left[(\mathbf{I}-\mathbf{A})^{-1}\right]^{\mathsf{T}}\mathbf{F}^{\mathsf{T}} \\
&= \mathbf{F}\mathbf{E}\mathbf{S}\mathbf{E}^{\mathsf{T}}\mathbf{F}^{\mathsf{T}} \\
&= \mathbf{F}\mathbf{C}\mathbf{F}^{\mathsf{T}} \\
&= \begin{pmatrix}
\sigma_x^2\beta_1^2 + 2\sigma_x^2\beta_1\beta_2\alpha_1 + \sigma_x^2\beta_2^2\alpha_1^2 + \beta_2^2\sigma_{\varepsilon_m}^2 + \sigma_{\varepsilon_y}^2 & \beta_1\alpha_1\sigma_x^2 + \beta_2\alpha_1^2\sigma_x^2 + \beta_2\sigma_{\varepsilon_m}^2 & \beta_1\sigma_x^2 + \beta_2\alpha_1\sigma_x^2 \\
\alpha_1^2\sigma_x^2\beta_2 + \alpha_1\sigma_x^2\beta_1 + \beta_2\sigma_{\varepsilon_m}^2 & \sigma_x^2\alpha_1^2 + \sigma_{\varepsilon_m}^2 & \alpha_1\sigma_x^2 \\
\sigma_x^2\beta_1 + \sigma_x^2\beta_2\alpha_1 & \sigma_x^2\alpha_1 & \sigma_x^2
\end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{v} &= (\mathbf{I}-\mathbf{A})^{-1}\,\mathbf{u} \\
&= \left[\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & \beta_2 & \beta_1 & 1 & 0 \\ 0 & 0 & \alpha_1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right]^{-1} \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \mu_x \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \beta_0 + \beta_2\alpha_0 + \beta_2\alpha_1\mu_x + \beta_1\mu_x \\ \alpha_0 + \alpha_1\mu_x \\ \mu_x \\ 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{u} &= \left(\mathbf{I} - \mathbf{A}\right) \mathbf{v} \\
&= \left[\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & \beta_2 & \beta_1 & 1 & 0 \\ 0 & 0 & \alpha_1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right] \begin{pmatrix} \beta_0 + \beta_1 \alpha_0 + \beta_1 \alpha_1 \mu_x + \beta_2 \mu_x \\ \alpha_0 + \alpha_1 \mu_x \\ \mu_x \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \beta_0 + \beta_1 \alpha_0 + \beta_1 \alpha_1 \mu_x + \mu_x \beta_2 \\ \alpha_0 + \alpha_1 \mu_x \\ \mu_x \\ 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{g} &= \mathbf{F} \left(\mathbf{I} - \mathbf{A}\right)^{-1} \mathbf{u} \\
&= \left[\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & \beta_2 & \beta_1 & 1 & 0 \\ 0 & 0 & \alpha_1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right]^{-1} \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \mu_x \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \beta_0 + \beta_2 \alpha_0 + \beta_2 \alpha_1 \mu_x + \beta_1 \mu_x \\ \alpha_0 + \alpha_1 \mu_x \\ \mu_x \end{pmatrix}
\end{aligned}
$$

8.1.1 Using the `ramR` Package

```
A
```

```
##    y   m        x          ey  em
## y  "0" "beta[2]" "beta[1]"  "1" "0"
## m  "0" "0"       "alpha[1]" "0" "1"
## x  "0" "0"       "0"        "0" "0"
## ey "0" "0"       "0"        "0" "0"
## em "0" "0"       "0"        "0" "0"
```

```
S
```

```
##    y   m   x            ey                         em
## y  "0" "0" "0"          "0"                        "0"
## m  "0" "0" "0"          "0"                        "0"
## x  "0" "0" "sigma[x]^2" "0"                        "0"
## ey "0" "0" "0"          "sigma[varepsilon[y]]^2" "0"
## em "0" "0" "0"          "0"                        "sigma[varepsilon[m]]^2"
```

```
u
```

```
##    u
## y  "beta[0]"
```

```
## m  "alpha[0]"
## x  "mu[x]"
## ey "0"
## em "0"
```

```
filter
```

```
##   y m x ey em
## y 1 0 0  0  0
## m 0 1 0  0  0
## x 0 0 1  0  0
```

The covariance expectations can be symbolically derived using the `ramR::C()` function with `A` of class `yac.symbol`.

```
ramR::C(Ryacas::ysym(A), S, simplify = TRUE)
```

```
## {{sigma[x]^2*beta[1]^2+2*sigma[x]^2*beta[1]*beta[2]*alpha[1]+sigma[x]^2*beta[2]^2*alpha[1]^2+beta[2]
##  {                                                             alpha[1]^2*sigma[x]^2*beta[2]+alpha[1]*s
##  {
##  {
##  {
```

$$\mathbf{C} = \begin{pmatrix} \sigma_x^2\beta_1^2 + 2\sigma_x^2\beta_1\beta_2\alpha_1 + \sigma_x^2\beta_2^2\alpha_1^2 + \beta_2^2\sigma_{\varepsilon_m}^2 + \sigma_{\varepsilon_y}^2 & \beta_1\alpha_1\sigma_x^2 + \beta_2\alpha_1^2\sigma_x^2 + \beta_2\sigma_{\varepsilon_m}^2 & \beta_1\sigma_x^2 + \beta_2\alpha_1\sigma_x^2 & \sigma_{\varepsilon_y}^2 & \beta_2\sigma_{\varepsilon_m}^2 \\ \alpha_1^2\sigma_x^2\beta_2 + \alpha_1\sigma_x^2\beta_1 + \beta_2\sigma_{\varepsilon_m}^2 & \sigma_x^2\alpha_1^2 + \sigma_{\varepsilon_m}^2 & \alpha_1\sigma_x^2 & 0 & \sigma_{\varepsilon_m}^2 \\ \sigma_x^2\beta_1 + \sigma_x^2\beta_2\alpha_1 & \sigma_x^2\alpha_1 & \sigma_x^2 & 0 & 0 \\ \sigma_{\varepsilon_y}^2 & 0 & 0 & \sigma_{\varepsilon_y}^2 & 0 \\ \sigma_{\varepsilon_m}^2\beta_2 & \sigma_{\varepsilon_m}^2 & 0 & 0 & \sigma_{\varepsilon_m}^2 \end{pmatrix}$$

The covariance expectations for the observed variables can be symbolically derived using the `ramR::M()` function with `A` of class `yac.symbol`.

```
ramR::M(Ryacas::ysym(A), S, filter, simplify = TRUE)
```

```
## {{sigma[x]^2*beta[1]^2+2*sigma[x]^2*beta[1]*beta[2]*alpha[1]+sigma[x]^2*beta[2]^2*alpha[1]^2+beta[2]
##  {                                                             alpha[1]^2*sigma[x]^2*beta[2]+alpha[1]*s
##  {
```

$$\mathbf{M} = \begin{pmatrix} \sigma_x^2\beta_1^2 + 2\sigma_x^2\beta_1\beta_2\alpha_1 + \sigma_x^2\beta_2^2\alpha_1^2 + \beta_2^2\sigma_{\varepsilon_m}^2 + \sigma_{\varepsilon_y}^2 & \beta_1\alpha_1\sigma_x^2 + \beta_2\alpha_1^2\sigma_x^2 + \beta_2\sigma_{\varepsilon_m}^2 & \beta_1\sigma_x^2 + \beta_2\alpha_1\sigma_x^2 \\ \alpha_1^2\sigma_x^2\beta_2 + \alpha_1\sigma_x^2\beta_1 + \beta_2\sigma_{\varepsilon_m}^2 & \sigma_x^2\alpha_1^2 + \sigma_{\varepsilon_m}^2 & \alpha_1\sigma_x^2 \\ \sigma_x^2\beta_1 + \sigma_x^2\beta_2\alpha_1 & \sigma_x^2\alpha_1 & \sigma_x^2 \end{pmatrix}$$

The mean expectations can be symbolically derived using the `ramR::v()` function with `A` of class `yac.symbol`.

```
ramR::v(Ryacas::ysym(A), u, simplify = TRUE)
```

```
## {{beta[0]+beta[2]*alpha[0]+beta[2]*alpha[1]*mu[x]+beta[1]*mu[x]},
##  {                                        alpha[0]+alpha[1]*mu[x]},
##  {                                                          mu[x]},
##  {                                                              0},
##  {                                                              0}}
```

$$\mathbf{v} = \begin{pmatrix} \beta_0 + \beta_2\alpha_0 + \beta_2\alpha_1\mu_x + \beta_1\mu_x \\ \alpha_0 + \alpha_1\mu_x \\ \mu_x \\ 0 \\ 0 \end{pmatrix}$$

The mean expectations for the observed variables can be symbolically derived using the `ramR::g()` function with `A` of class `yac.symbol`.

```
ramR::g(Ryacas::ysym(A), u, filter, simplify = TRUE)
```

```
## {{beta[0]+beta[2]*alpha[0]+beta[2]*alpha[1]*mu[x]+beta[1]*mu[x]},
##  {                                        alpha[0]+alpha[1]*mu[x]},
##  {                                                          mu[x]}}
```

$$\mathbf{g} = \begin{pmatrix} \beta_0 + \beta_2\alpha_0 + \beta_2\alpha_1\mu_x + \beta_1\mu_x \\ \alpha_0 + \alpha_1\mu_x \\ \mu_x \end{pmatrix}$$

8.2 Numerical Example

Let `df` be a random sample from a population with the following parameters

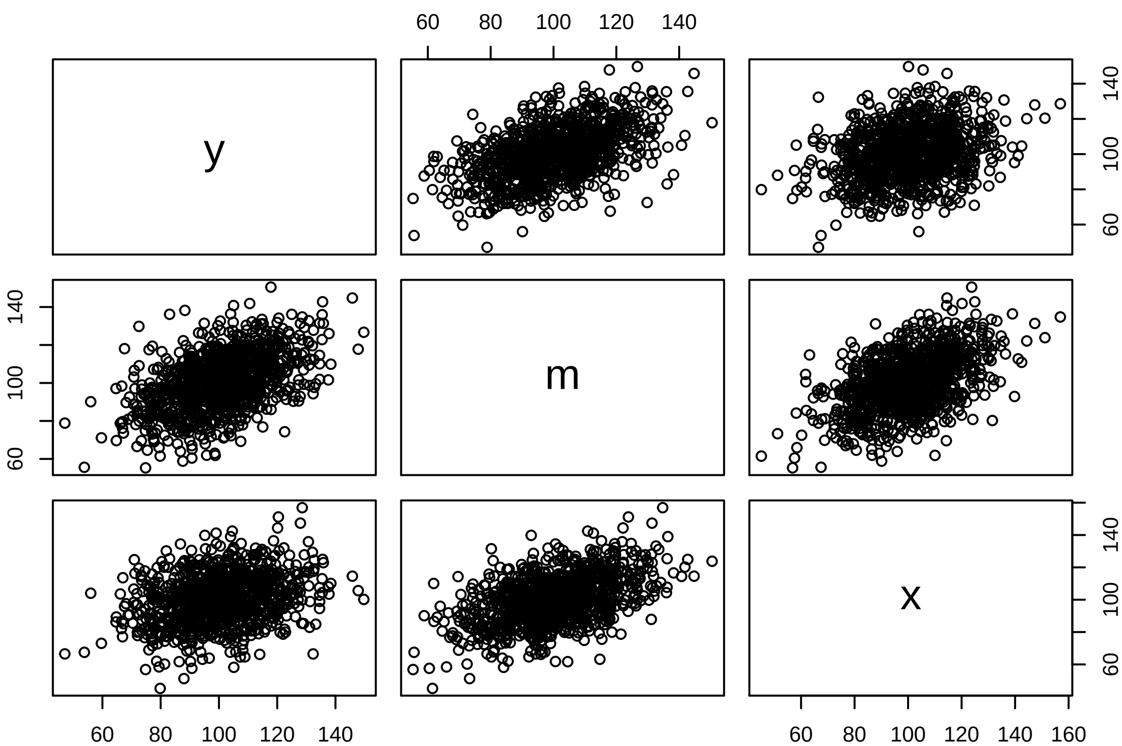
Parameter	Value
β_1	0
β_2	0.5
α_1	0.5
$\sigma^2_{\varepsilon_y}$	168.75
$\sigma^2_{\varepsilon_m}$	168.75
σ^2_x	225
β_0	50
α_0	50
μ_x	100

```
head(df)
```

```
##           y         m         x
## 1 107.32121  80.22215  64.60716
## 2 109.83608 109.42737 100.20734
## 3  97.41131 107.89265  79.57780
## 4  87.75538 106.43004  80.82994
## 5  80.77481 104.18033  99.07128
## 6  97.27327  98.70669 108.24974
```

```
summary(df)
```

```
##        y                m                x         
##  Min.   : 47.09   Min.   : 55.30   Min.   : 45.16  
##  1st Qu.: 90.02   1st Qu.: 90.39   1st Qu.: 90.09  
##  Median :100.34   Median : 99.84   Median :100.14  
##  Mean   :100.22   Mean   :100.36   Mean   :100.32  
##  3rd Qu.:110.30   3rd Qu.:110.40   3rd Qu.:110.29  
##  Max.   :149.73   Max.   :150.48   Max.   :156.91
```



8.2.1 Linear Regression

```
summary(lm(y ~ x + m, data = df))
```

```
## 
## Call:
## lm(formula = y ~ x + m, data = df)
## 
## Residuals:
##     Min      1Q  Median      3Q     Max 
## -42.413  -8.837  -0.045   8.804  39.148 
## 
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)    
## (Intercept) 48.47917    3.19559  15.171   <2e-16 ***
## x            0.04486    0.03101   1.447    0.148
```

```
## m            0.47072    0.03136  15.012   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 12.98 on 997 degrees of freedom
## Multiple R-squared:  0.2464, Adjusted R-squared:  0.2449
## F-statistic:   163 on 2 and 997 DF,  p-value: < 2.2e-16
```

```
summary(lm(m ~ x, data = df))
```

```
##
## Call:
## lm(formula = m ~ x, data = df)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -43.182  -9.426   0.280   8.705  38.824
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) 52.04456    2.77350   18.77   <2e-16 ***
## x            0.48163    0.02734   17.62   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 13.1 on 998 degrees of freedom
## Multiple R-squared:  0.2372, Adjusted R-squared:  0.2365
## F-statistic: 310.4 on 1 and 998 DF,  p-value: < 2.2e-16
```

8.2.2 Structural Equation Modeling

8.2.2.1 lavaan (Rosseel, 2012)

```
# Covariance Structure
model <- "
  y ~ beta1 * x + beta2 * m
  m ~ alpha1 * x
  indirect := alpha1 * beta2
"
# With Mean Structure
model <- "
  y ~ beta1 * x + beta2 * m
  m ~ alpha1 * x
  y ~~ sigma2ey * y
  m ~~ sigma2em * m
  x ~~ sigma2x * x
  y ~ beta0 * 1
  m ~ alpha0 * 1
  x ~ mux * 1
  indirect := alpha1 * beta2
"
```

```
fit <- lavaan::sem(
  model,
  data = df,
  meanstructure = TRUE,
  fixed.x = FALSE
)
lavaan::summary(fit)
```

```
## lavaan 0.6-7 ended normally after 28 iterations
##
##   Estimator                                         ML
##   Optimization method                           NLMINB
##   Number of free parameters                          9
##
##   Number of observations                          1000
##
## Model Test User Model:
##
##   Test statistic                                 0.000
##   Degrees of freedom                                 0
##
## Parameter Estimates:
##
##   Standard errors                             Standard
##   Information                                 Expected
##   Information saturated (h1) model          Structured
##
## Regressions:
##                    Estimate  Std.Err  z-value  P(>|z|)
##   y ~
##     x       (bet1)    0.045    0.031    1.449    0.147
##     m       (bet2)    0.471    0.031   15.034    0.000
##   m ~
##     x       (alp1)    0.482    0.027   17.636    0.000
##
## Intercepts:
##                    Estimate  Std.Err  z-value  P(>|z|)
##    .y       (bet0)   48.479    3.191   15.193    0.000
##    .m       (alp0)   52.045    2.771   18.784    0.000
##     x        (mux)  100.320    0.479  209.300    0.000
##
## Variances:
##                    Estimate  Std.Err  z-value  P(>|z|)
##    .y      (sgm2y)  167.964    7.512   22.361    0.000
##    .m      (sgm2m)  171.335    7.662   22.361    0.000
##     x      (sgm2x)  229.740   10.274   22.361    0.000
##
## Defined Parameters:
##                    Estimate  Std.Err  z-value  P(>|z|)
##     indirect          0.227    0.020   11.441    0.000
```

8.2.3 Using the `ramR` Package

```
A
```

```
##     y         m          x ey em
## y   0 0.4707226 0.04486201  1  0
## m   0 0.0000000 0.48162865  0  1
## x   0 0.0000000 0.00000000  0  0
## ey  0 0.0000000 0.00000000  0  0
## em  0 0.0000000 0.00000000  0  0
```

```
S
```

```
##     y m        x      ey       em
## y   0 0   0.0000   0.000   0.0000
## m   0 0   0.0000   0.000   0.0000
## x   0 0 229.9701   0.000   0.0000
## ey  0 0   0.0000 168.469   0.0000
## em  0 0   0.0000   0.000 171.6784
```

```
u
```

```
##             u
## y    48.47917
## m    52.04456
## x   100.31982
## ey    0.00000
## em    0.00000
```

```
filter
```

```
##   y m x ey em
## y 1 0 0  0  0
## m 0 1 0  0  0
## x 0 0 1  0  0
```

The covariance expectations can be numerically derived using the `ramR::C()` function.

```
ramR::C(A, S)
```

```
##             y        m         x      ey        em
## y   223.47047 110.8927  62.45425 168.469  80.81291
## m   110.89267 225.0237 110.76020   0.000 171.67843
## x    62.45425 110.7602 229.97013   0.000   0.00000
## ey  168.46896   0.0000   0.00000 168.469   0.00000
## em   80.81291 171.6784   0.00000   0.000 171.67843
```

The covariance expectations for the observed variables can be numerically derived using the `ramR::M()` function.

```
ramR::M(A, S, filter)
```

```
##           y        m         x
## y 223.47047 110.8927  62.45425
## m 110.89267 225.0237 110.76020
## x  62.45425 110.7602 229.97013
```

The mean expectations can be numerically derived using the `ramR::v()` function.

```
ramR::v(A, u)
```

```
##           v
## y  100.2221
## m  100.3615
## x  100.3198
## ey   0.0000
## em   0.0000
```

The mean expectations for the observed variables can be numerically derived using the `ramR::v()` function.

```
ramR::g(A, u, filter)
```

```
##          g
## y 100.2221
## m 100.3615
## x 100.3198
```

8.3 Equations to RAM

The `ramR` package has a utility function to convert structural equations to RAM notation. The simple mediation model can be expressed in the following equations

```
eq <- "
  # VARIABLE1 OPERATION VARIABLE2 LABEL
  ey          by        y         1
  em          by        m         1
  y           on        x         beta[1]
  y           on        m         beta[2]
  m           on        x         alpha[1]
  ey          with      ey        sigma[varepsilon[y]]^2
  em          with      em        sigma[varepsilon[m]]^2
  x           with      x         sigma[x]^2
  y           on        1         beta[0]
  m           on        1         alpha[0]
  x           on        1         mu[x]
"
```

The `ramR::Eq2RAM` converts the equations to RAM notation.

```
ramR::Eq2RAM(eq)
```

```
## $par.table
##    lhs   op rhs              par.label par.index
## 1   ey   by   y                      1         1
## 2   em   by   m                      1         1
## 3    y   on   x                beta[1]        p1
## 4    y   on   m                beta[2]        p2
## 5    m   on   x               alpha[1]        p3
## 6   ey with  ey sigma[varepsilon[y]]^2        p4
## 7   em with  em sigma[varepsilon[m]]^2        p5
## 8    x with   x             sigma[x]^2        p6
## 9    y   on   1                beta[0]        p7
## 10   m   on   1               alpha[0]        p8
## 11   x   on   1                  mu[x]        p9
##
## $variables
## [1] "y"  "m"  "x"  "ey" "em"
##
## $g.variables
## [1] "y" "m" "x"
##
## $h.variables
## [1] "ey" "em"
##
## $A
##    y   m         x          ey  em
## y  "0" "beta[2]" "beta[1]"  "1" "0"
## m  "0" "0"       "alpha[1]" "0" "1"
## x  "0" "0"       "0"        "0" "0"
## ey "0" "0"       "0"        "0" "0"
## em "0" "0"       "0"        "0" "0"
##
## $S
##    y   m   x            ey                       em
## y  "0" "0" "0"          "0"                      "0"
## m  "0" "0" "0"          "0"                      "0"
## x  "0" "0" "sigma[x]^2" "0"                      "0"
## ey "0" "0" "0"          "sigma[varepsilon[y]]^2" "0"
## em "0" "0" "0"          "0"                      "sigma[varepsilon[m]]^2"
##
## $u
##    u
## y  "beta[0]"
## m  "alpha[0]"
## x  "mu[x]"
## ey "0"
## em "0"
##
## $Filter
##   y m x ey em
## y 1 0 0  0  0
## m 0 1 0  0  0
```

```
## x 0 0 1  0  0
```

8.4 Equations to Expectations

The `ramR` package has a utility function to convert structural equations to expectations both symbolically and numerically.

```
eq <- "
  # VARIABLE1 OPERATION VARIABLE2 LABEL
  ey          by        y         1
  em          by        m         1
  y           on        x         beta[1]
  y           on        m         beta[2]
  m           on        x         alpha[1]
  ey          with      ey        sigma[varepsilon[y]]^2
  em          with      em        sigma[varepsilon[m]]^2
  x           with      x         sigma[x]^2
  y           on        1         beta[0]
  m           on        1         alpha[0]
  x           on        1         mu[x]
"
```

```
ramR::Eq2Expectations(eq)
```

```
## $par.table
##    lhs   op rhs              par.label par.index
## 1   ey   by   y                      1         1
## 2   em   by   m                      1         1
## 3    y   on   x                beta[1]        p1
## 4    y   on   m                beta[2]        p2
## 5    m   on   x               alpha[1]        p3
## 6   ey with  ey sigma[varepsilon[y]]^2        p4
## 7   em with  em sigma[varepsilon[m]]^2        p5
## 8    x with   x             sigma[x]^2        p6
## 9    y   on   1                beta[0]        p7
## 10   m   on   1               alpha[0]        p8
## 11   x   on   1                  mu[x]        p9
##
## $variables
## [1] "y"  "m"  "x"  "ey" "em"
##
## $g.variables
## [1] "y" "m" "x"
##
## $h.variables
## [1] "ey" "em"
##
## $A
## {{       0,  beta[2],  beta[1],        1,        0},
##  {       0,        0, alpha[1],        0,        1},
##  {       0,        0,        0,        0,        0},
##  {       0,        0,        0,        0,        0},
```

```
##  {       0,       0,       0,       0,       0}}
##
## $S
## {{                    0,                    0,                    0,                    0,
##  {                    0,                    0,                    0,                    0,
##  {                    0,                    0,           sigma[x]^2,                    0,
##  {                    0,                    0,                    0, sigma[varepsilon[y]]^2,
##  {                    0,                    0,                    0,                    0, sig
##
## $u
## {{ beta[0]},
##  {alpha[0]},
##  {   mu[x]},
##  {       0},
##  {       0}}
##
## $Filter
## {{1, 0, 0, 0, 0},
##  {0, 1, 0, 0, 0},
##  {0, 0, 1, 0, 0}}
##
## $v
## {{beta[0]+beta[2]*alpha[0]+(beta[1]+beta[2]*alpha[1])*mu[x]},
##  {                                  alpha[0]+alpha[1]*mu[x]},
##  {                                                    mu[x]},
##  {                                                        0},
##  {                                                        0}}
##
## $g
## {{beta[0]+beta[2]*alpha[0]+(beta[1]+beta[2]*alpha[1])*mu[x]},
##  {                                  alpha[0]+alpha[1]*mu[x]},
##  {                                                    mu[x]}}
##
## $C
## {{sigma[x]^2*(beta[1]+beta[2]*alpha[1])^2+sigma[varepsilon[y]]^2+sigma[varepsilon[m]]^2*beta[2]^2,
##  {                  alpha[1]*sigma[x]^2*(beta[1]+beta[2]*alpha[1])+sigma[varepsilon[m]]^2*beta[2],
##  {                                                        sigma[x]^2*(beta[1]+beta[2]*alpha[1]),
##  {                                                                       sigma[varepsilon[y]]^2,
##  {                                                                sigma[varepsilon[m]]^2*beta[2],
##
## $M
## {{sigma[x]^2*(beta[1]+beta[2]*alpha[1])^2+sigma[varepsilon[y]]^2+sigma[varepsilon[m]]^2*beta[2]^2,
##  {                  alpha[1]*sigma[x]^2*(beta[1]+beta[2]*alpha[1])+sigma[varepsilon[m]]^2*beta[2],
##  {                                                        sigma[x]^2*(beta[1]+beta[2]*alpha[1]),
```

```r
eq <- "
  # VARIABLE1 OPERATION VARIABLE2 LABEL
  ey          by        y         1
  em          by        m         1
  y           on        x         0.00
  y           on        m         0.50
  m           on        x         0.50
  ey          with      ey        168.75
  em          with      em        168.75
```

```
  x          with      x         225
  y          on        1         50
  m          on        1         50
  x          on        1         100
"
```

```
ramR::Eq2Expectations(eq)
```

```
## $par.table
##    lhs   op rhs par.label par.index
## 1   ey   by   y      1.00      1.00
## 2   em   by   m      1.00      1.00
## 3    y   on   x      0.00      0.00
## 4    y   on   m      0.50      0.50
## 5    m   on   x      0.50      0.50
## 6   ey with  ey    168.75    168.75
## 7   em with  em    168.75    168.75
## 8    x with   x    225.00    225.00
## 9    y   on   1     50.00     50.00
## 10   m   on   1     50.00     50.00
## 11   x   on   1    100.00    100.00
##
## $variables
## [1] "y"  "m"  "x"  "ey" "em"
##
## $g.variables
## [1] "y" "m" "x"
##
## $h.variables
## [1] "ey" "em"
##
## $A
##    y   m   x ey em
## y  0 0.5 0.0  1  0
## m  0 0.0 0.5  0  1
## x  0 0.0 0.0  0  0
## ey 0 0.0 0.0  0  0
## em 0 0.0 0.0  0  0
##
## $S
##    y m   x     ey     em
## y  0 0   0   0.00   0.00
## m  0 0   0   0.00   0.00
## x  0 0 225   0.00   0.00
## ey 0 0   0 168.75   0.00
## em 0 0   0   0.00 168.75
##
## $u
##      u
## y   50
## m   50
## x  100
## ey   0
```

```
## em   0
##
## $Filter
##   y m x ey em
## y 1 0 0  0  0
## m 0 1 0  0  0
## x 0 0 1  0  0
##
## $v
##      v
## y  100
## m  100
## x  100
## ey   0
## em   0
##
## $g
##     g
## y 100
## m 100
## x 100
##
## $C
##          y      m      x     ey      em
## y  225.000 112.50  56.25 168.75  84.375
## m  112.500 225.00 112.50   0.00 168.750
## x   56.250 112.50 225.00   0.00   0.000
## ey 168.750   0.00   0.00 168.75   0.000
## em  84.375 168.75   0.00   0.00 168.750
##
## $M
##        y     m      x
## y 225.00 112.5  56.25
## m 112.50 225.0 112.50
## x  56.25 112.5 225.00
```

Chapter 9

The Standardized Simple Mediation Model

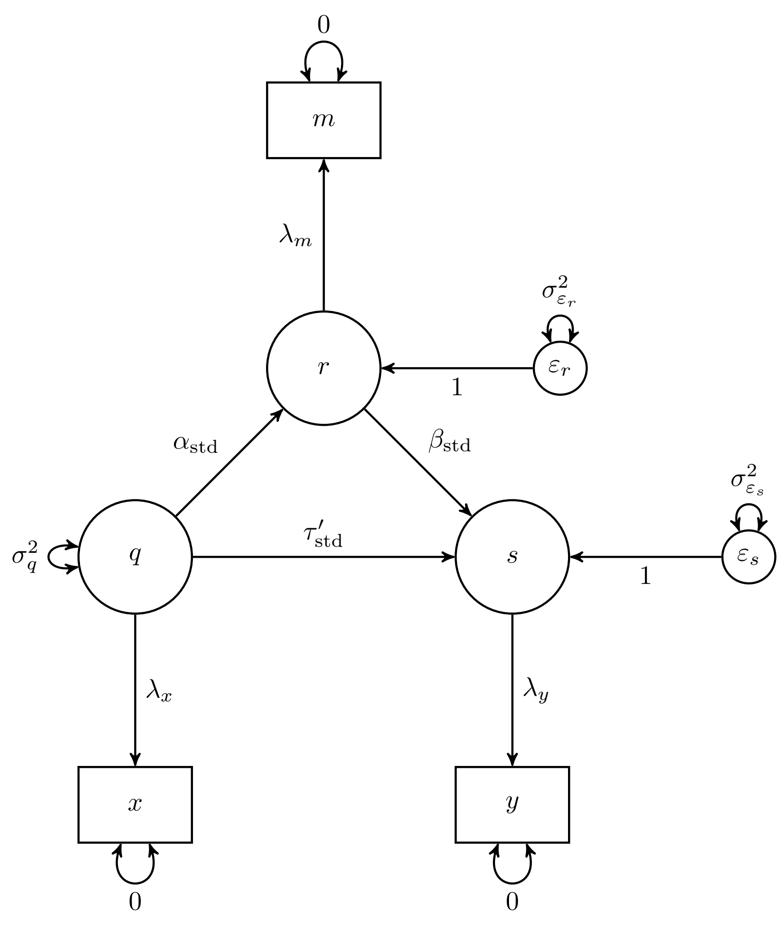


Figure 9.1: The Standardized Simple Mediation Model

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