Reticular Action Model (RAM) Notation Notes

Ivan Jacob Agaloos Pesigan

2021-01-23

Contents

Chapter 1

Description

This is a collection of my notes on the Reticular Action Model (RAM) notation that accompanies the `ramR` package (Pesigan, 2021) in the `R` statistical environment (R Core Team, 2020). You can install the released version of `ramR` from GitHub with:

```
remotes::install_github("jeksterslab/ramR")
```

These notes are based on the following resources:

- Boker and McArdle (2005)
- McArdle and McDonald (1984)
- McArdle (2005)

See GitHub Pages for the html deployment.

Chapter 2

Reticular Action Model (RAM) Matrix Notation

2.1 Full Model

Definition 2.1.

$$\mathbf{v} = \mathbf{A}\mathbf{v} + \mathbf{u} \tag{2.1}$$

where

- $\mathbf{v}$ and $\mathbf{u}$ are $t \times 1$ vectors of random variables
- $\mathbf{A}$ is a $t \times t$ matrix of *directed* or *asymmetric* relationship from column variable v_j to row variable v_i
  - $\mathbf{A}$ represent the regression of each of the t variables $\mathbf{v}$ on the other $t-1$ variables
  - diagonal $a_{i,i}$ is zero
  - u_i represent the residual of v_i
  - if all regression coefficients on other variables are zero, then the variable v_i is considered the same as its own residual u_i

Definition 2.2.

$$\mathbf{S} = \mathbb{E}\left\{\mathbf{u}\mathbf{u}'\right\}, \tag{2.2}$$

where

- $\mathbf{S}$ is a $t \times t$ matrix of *undirected* or *symmetric* relationship
  - the notation Ω is used in other sources for $\mathbf{S}$
- $\mathbb{E}$ is the expectation operator

Definition 2.3.

$$\mathbf{C} = \mathbb{E}\left\{\mathbf{v}\mathbf{v}'\right\}, \tag{2.3}$$

where

- $\mathbf{C}$ is a $t \times t$ variance-covariance matrix
  - the notation Σ is used in other sources for $\mathbf{C}$

Definition 2.4.

$$\mathbf{v} = \mathbf{A}\mathbf{v} + \mathbf{u}$$

can be rewritten as

$$\begin{aligned} \mathbf{v} - \mathbf{A}\mathbf{v} &= \mathbf{u} \\ \mathbf{u} &= \mathbf{v} - \mathbf{A}\mathbf{v} \\ \mathbf{u} &= (\mathbf{I} - \mathbf{A})\,\mathbf{v} \end{aligned} \tag{2.4}$$

assuming that $(\mathbf{I} - \mathbf{A})$ is non-singular,

$$\mathbf{E} = (\mathbf{I} - \mathbf{A})^{-1} \tag{2.5}$$

then

$$\begin{aligned} \mathbf{v} &= (\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{u} \\ &= \mathbf{E}\mathbf{u}. \end{aligned} \tag{2.6}$$

Using the definitions above, $\mathbf{S}$ and $\mathbf{C}$ are given by

$$\begin{aligned} \mathbf{S} &= (\mathbf{I} - \mathbf{A})\,\mathbf{C}\,(\mathbf{I} - \mathbf{A})^{-1} \\ &= \mathbf{E}^{-1}\mathbf{C}\left(\mathbf{E}^{-1}\right)^{\mathsf{T}} \end{aligned} \tag{2.7}$$

$$\begin{aligned} \mathbf{C} &= (\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{S}\left[(\mathbf{I} - \mathbf{A})^{-1}\right]^{\mathsf{T}} \\ &= \mathbf{E}\mathbf{S}\mathbf{E}^{\mathsf{T}} \end{aligned} \tag{2.8}$$

2.2 Observed/Manifest/Given Variables vs. Unobserved/Latent/Hidden Variables

Definition 2.5.

$$\mathbf{v} = \begin{bmatrix} \mathbf{g}_{p\times 1} \\ \mathbf{h}_{q\times 1} \end{bmatrix} \tag{2.9}$$

$$t = p + q \tag{2.10}$$

- $\mathbf{g}$ may be considered observed, manifest or *given* variables
- $\mathbf{h}$ may be considered unobserved, latent, or *hidden* variables

Definition 2.6.

$$\mathbf{F} = \left[\mathbf{I}_{p\times p} : \mathbf{0}_{p\times q}\right] \tag{2.11}$$

- the $\mathbf{F}$ matrix acts as a *filter* to select the manifest variables out of the full set of manifest and latent variables

$$\mathbf{g} = \mathbf{F}\mathbf{v} \tag{2.12}$$

$$\begin{aligned} \mathbf{g} &= \mathbf{F}\left(\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{u} \\ &= \mathbf{F}\mathbf{E}\mathbf{u} \end{aligned} \tag{2.13}$$

Definition 2.7.

$$\mathbf{M} = \mathbb{E}\left\{\mathbf{g}\mathbf{g}^{\mathsf{T}}\right\} \tag{2.14}$$

$$\begin{aligned} \mathbf{M} &= \mathbf{F}\left(\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{S}\left[\left(\mathbf{I} - \mathbf{A}\right)^{-1}\right]^{\mathsf{T}}\mathbf{F}^{\mathsf{T}} \\ &= \mathbf{F}\mathbf{E}\mathbf{S}\mathbf{E}^{\mathsf{T}}\mathbf{F}^{\mathsf{T}} \\ &= \mathbf{F}\mathbf{C}\mathbf{F}^{\mathsf{T}} \end{aligned} \tag{2.15}$$

- when components of $\mathbf{v}$ are permuted, the columns of $\mathbf{F}$ can be correspondingly permuted
- the rows and columns of $\mathbf{C}$ that are filtered out by $\mathbf{F}$ contain useful information about the latent variable structure.

The equations above completely define RAM.

Chapter 3

Reticular Action Model (RAM) Path Diagram

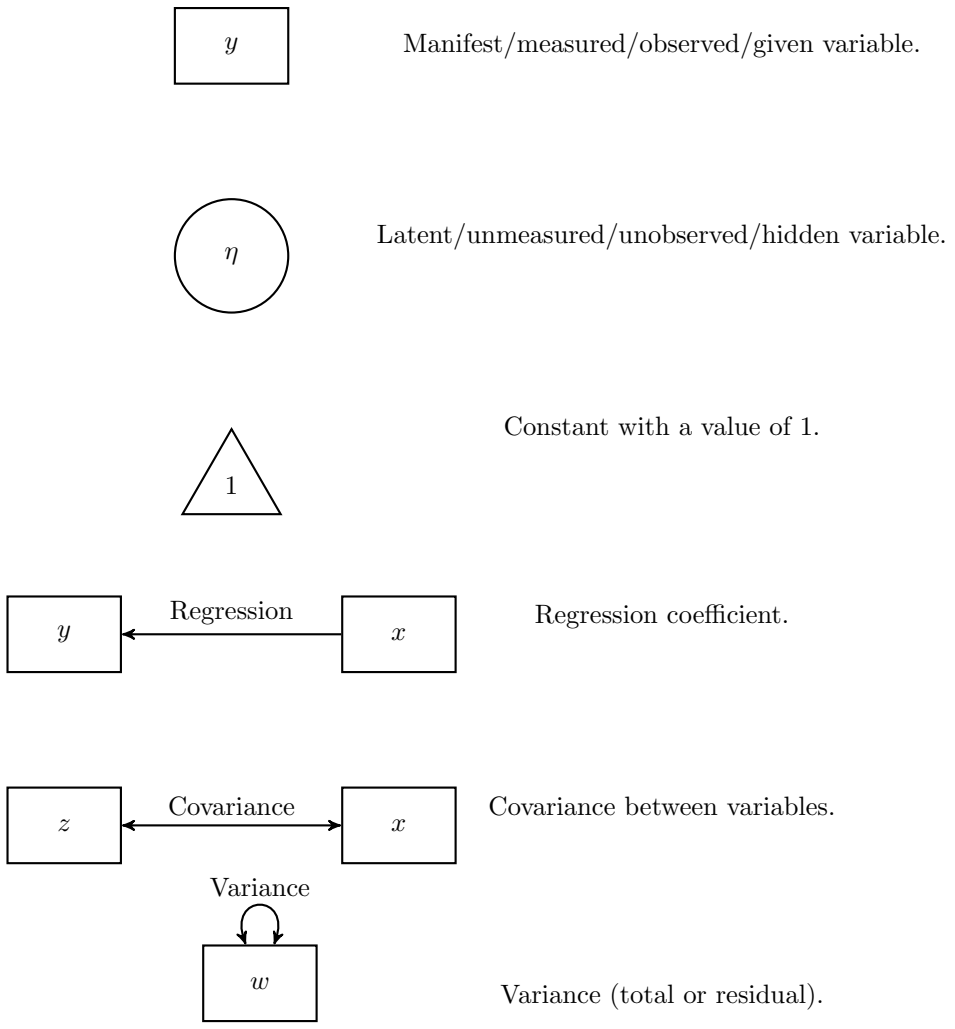
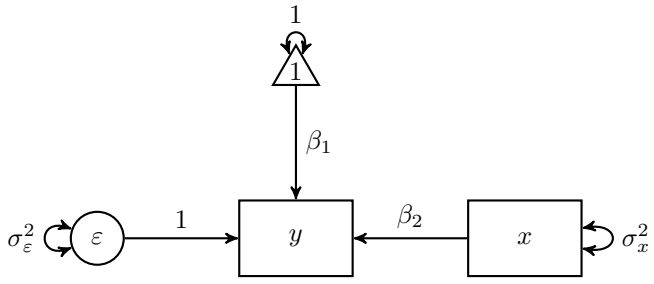


Figure 3.1: Path Diagram Elements



$$y = \alpha + \beta x + \varepsilon$$

Figure 3.2: Two-Variable Regression Model

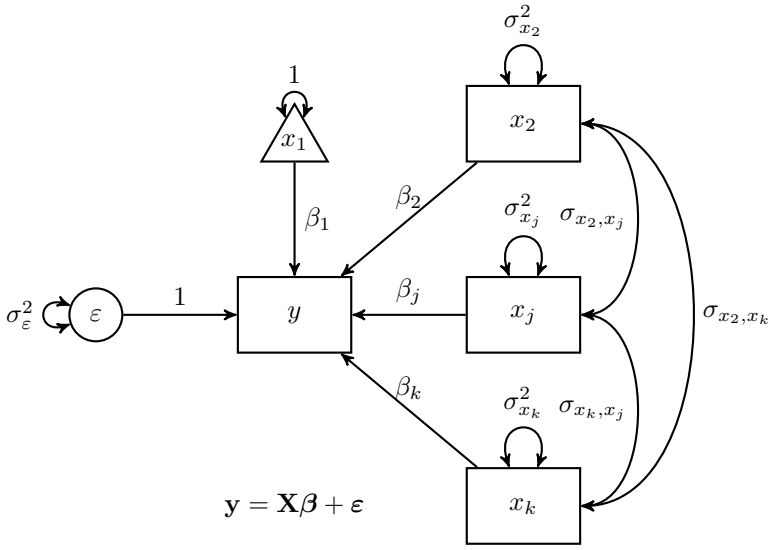
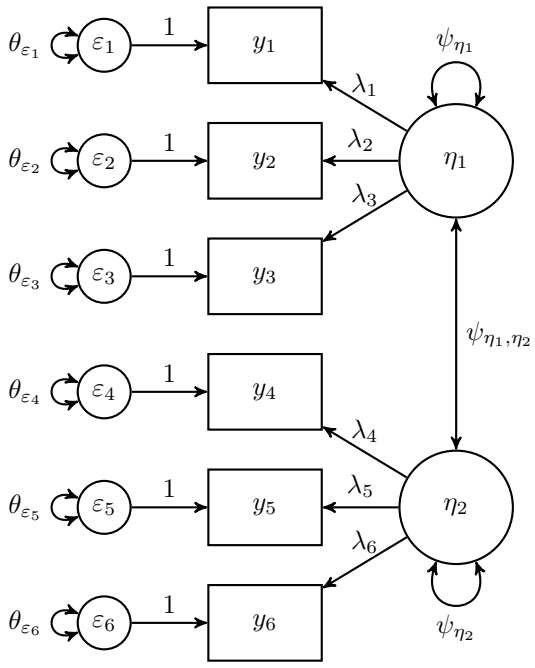


Figure 3.3: k-Variable Regression Model



$$\mathbf{y} = \boldsymbol{\Lambda\eta} + \boldsymbol{\varepsilon}$$

Figure 3.4: Two-Factor Confirmatory Factor Analysis Model

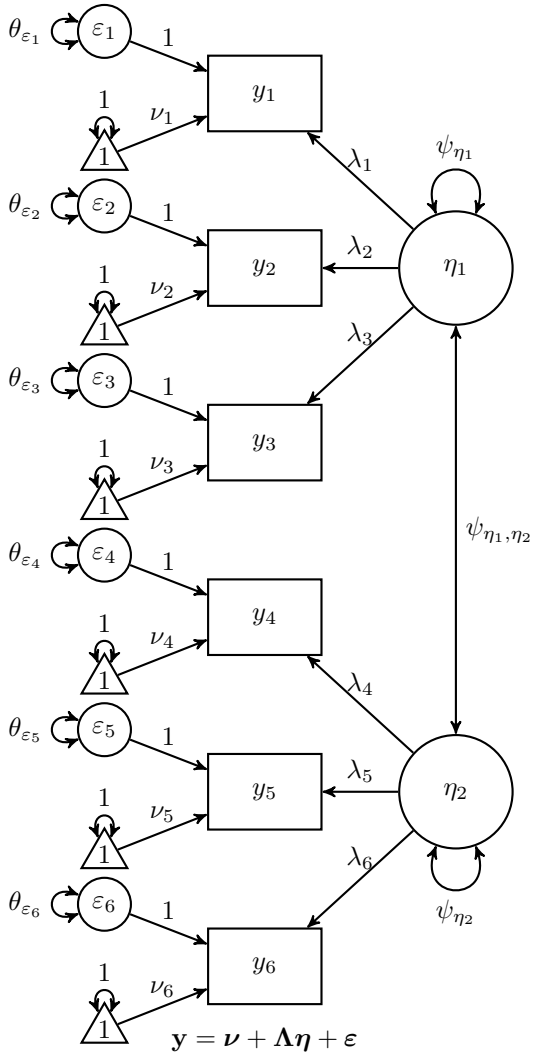
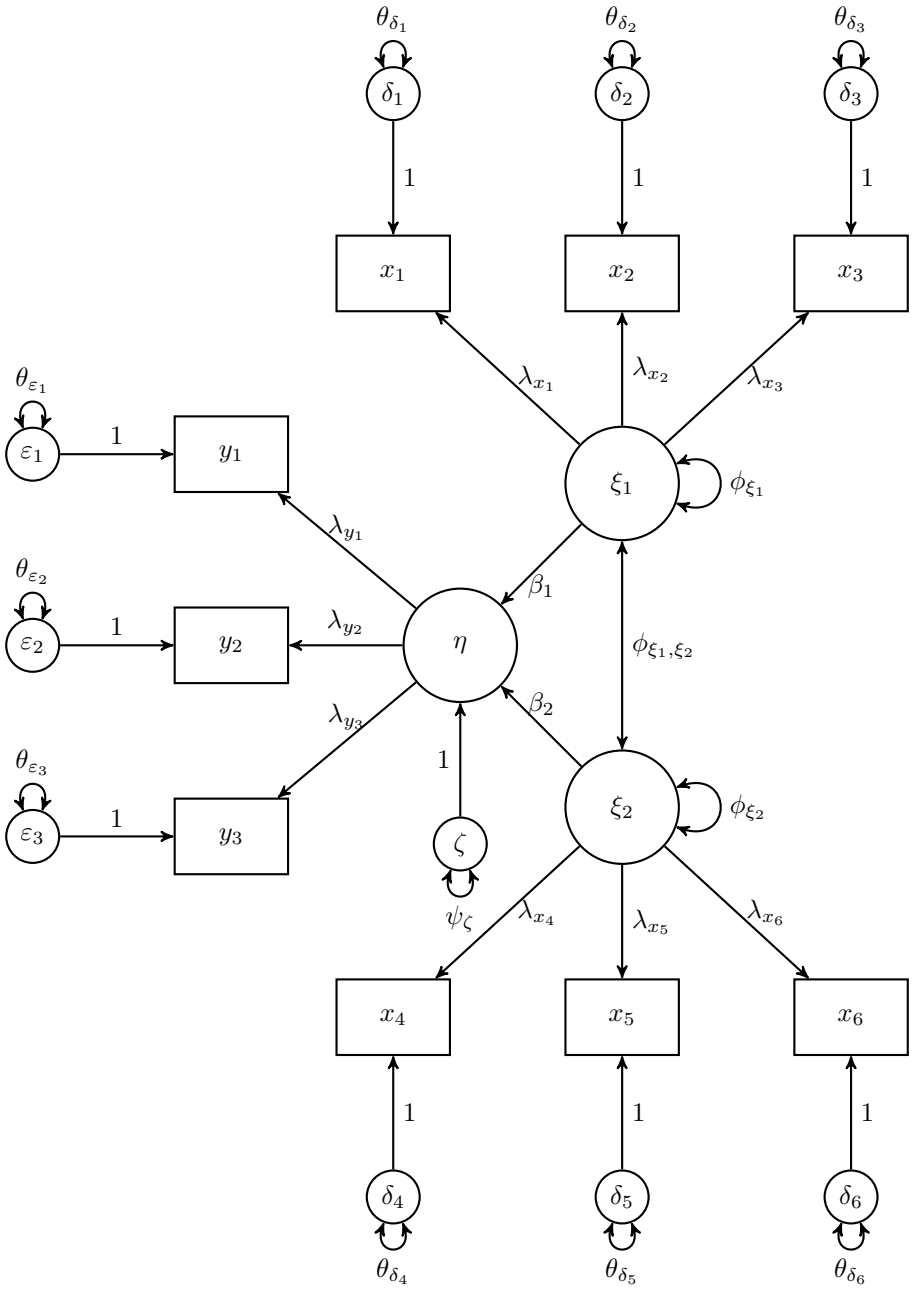


Figure 3.5: Two-Factor Confirmatory Factor Analysis Model with Mean Structure



$$\boldsymbol{\eta} = \mathbf{B}\boldsymbol{\eta} + \boldsymbol{\Gamma}\boldsymbol{\xi} + \boldsymbol{\zeta}, \mathbf{y} = \boldsymbol{\Lambda}_y\boldsymbol{\eta} + \boldsymbol{\varepsilon}, \mathbf{x} = \boldsymbol{\Lambda}_x\boldsymbol{\xi} + \boldsymbol{\delta}$$

Figure 3.6: Path Model with Latent Variables

Chapter 4

Student's t-test

In this section, the Student's t-test is presented as a structural equation model using the RAM notation. Let y be a continuous dependent variable, x be a dichotomous independent variable ($x = \{0, 1\}$), and ε be the stochastic error term with mean 0 and constant variance of σ^2_ε across the values of x. The associations of the variables are given by

$$y = \alpha + \beta x + \varepsilon$$

where

- α is the expected value of y when $x = 0$
- β is the unit change in y for unit change in x
- $\alpha + \beta$ is the expected value of y when $x = 1$

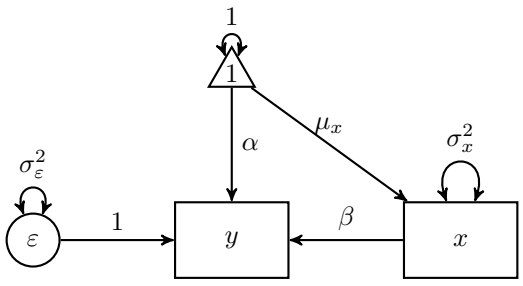


Figure 4.1: Student's t-test

4.1 Symbolic

Let $\{y, x, \varepsilon\}$ be the variables of interest.

$$\mathbf{A} = \begin{pmatrix} 0 & \beta & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_x^2 & 0 \\ 0 & 0 & \sigma_\varepsilon^2 \end{pmatrix}$$

$$\begin{aligned}
\mathbf{C} &= (\mathbf{I} - \mathbf{A})^{-1} \mathbf{S} \left[(\mathbf{I} - \mathbf{A})^{-1} \right]^\mathsf{T} \\
&= \mathbf{E}\mathbf{S}\mathbf{E}^\mathsf{T} \\
&= \begin{pmatrix} 1 & \beta & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_x^2 & 0 \\ 0 & 0 & \sigma_\varepsilon^2 \end{pmatrix} \begin{pmatrix} 1 & \beta & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^\mathsf{T} \\
&= \begin{pmatrix} \sigma_x^2\beta^2 + \sigma_\varepsilon^2 & \beta\sigma_x^2 & \sigma_\varepsilon^2 \\ \sigma_x^2\beta & \sigma_x^2 & 0 \\ \sigma_\varepsilon^2 & 0 & \sigma_\varepsilon^2 \end{pmatrix}
\end{aligned}$$

$$\mathbf{F} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{aligned}
\mathbf{M} &= \mathbf{F} (\mathbf{I} - \mathbf{A})^{-1} \mathbf{S} \left[(\mathbf{I} - \mathbf{A})^{-1} \right]^\mathsf{T} \mathbf{F}^\mathsf{T} \\
&= \mathbf{F}\mathbf{E}\mathbf{S}\mathbf{E}^\mathsf{T}\mathbf{F}^\mathsf{T} \\
&= \mathbf{F}\mathbf{C}\mathbf{F}^\mathsf{T} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \sigma_x^2\beta^2 + \sigma_\varepsilon^2 & \beta\sigma_x^2 & \sigma_\varepsilon^2 \\ \sigma_x^2\beta & \sigma_x^2 & 0 \\ \sigma_\varepsilon^2 & 0 & \sigma_\varepsilon^2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^\mathsf{T} \\
&= \begin{pmatrix} \sigma_x^2\beta^2 + \sigma_\varepsilon^2 & \beta\sigma_x^2 \\ \sigma_x^2\beta & \sigma_x^2 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathbf{v} &= (\mathbf{I} - \mathbf{A})^{-1} \mathbf{u} \\
&= \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & \beta & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right]^{-1} \begin{pmatrix} \alpha \\ \mu_x \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \alpha + \beta\mu_x \\ \mu_x \\ 0 \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
\mathbf{u} &= (\mathbf{I} - \mathbf{A})\,\mathbf{v} \\
&= \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & \beta & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \begin{pmatrix} \alpha + \beta\mu_x \\ \mu_x \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \alpha \\ \mu_x \\ 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{g} &= \mathbf{F}\,(\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{u} \\
&= \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & \beta & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right]^{-1} \begin{pmatrix} \alpha \\ \mu_x \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \alpha + \beta\mu_x \\ \mu_x \end{pmatrix}
\end{aligned}
$$

4.1.1 Using the `ramR` Package

```
A
```

```
##   y   x      e
## y "0" "beta" "1"
## x "0" "0"    "0"
## e "0" "0"    "0"
```

```
S
```

```
##   y   x            e
## y "0" "0"          "0"
## x "0" "sigma[x]^2" "0"
## e "0" "0"          "sigma[varepsilon]^2"
```

```
u
```

```
##   u
## y "alpha"
## x "mu[x]"
## e "0"
```

```
filter
```

```
##   y x e
## y 1 0 0
## x 0 1 0
```

The covariance expectations can be symbolically derived using the `ramR::C_sym()` function.

```
ramR::C_sym(A, S)
```

```
## {{sigma[x]^2*beta^2+sigma[varepsilon]^2,                    beta*sigma[x]^2,                    sig
##  {                      sigma[x]^2*beta,                         sigma[x]^2,
##  {                   sigma[varepsilon]^2,                                  0,                    sig
```

$$\mathbf{C} = \begin{pmatrix} \sigma_x^2\beta^2 + \sigma_\varepsilon^2 & \beta\sigma_x^2 & \sigma_\varepsilon^2 \\ \sigma_x^2\beta & \sigma_x^2 & 0 \\ \sigma_\varepsilon^2 & 0 & \sigma_\varepsilon^2 \end{pmatrix}$$

The covariance expectations for the observed variables can be symbolically derived using the `ramR::M_sym()` function.

```
ramR::M_sym(A, S, filter)
```

```
## {{sigma[x]^2*beta^2+sigma[varepsilon]^2,                    beta*sigma[x]^2},
##  {                      sigma[x]^2*beta,                         sigma[x]^2}}
```

$$\mathbf{M} = \begin{pmatrix} \sigma_x^2\beta^2 + \sigma_\varepsilon^2 & \beta\sigma_x^2 \\ \sigma_x^2\beta & \sigma_x^2 \end{pmatrix}$$

The mean expectations can be symbolically derived using the `ramR::v_sym()` function.

```
ramR::v_sym(A, u)
```

```
## {{alpha+beta*mu[x]},
##  {           mu[x]},
##  {               0}}
```

$$\mathbf{v} = \begin{pmatrix} \alpha + \beta\mu_x \\ \mu_x \\ 0 \end{pmatrix}$$

The mean expectations for the observed variables can be symbolically derived using the `ramR::g_sym()` function.

```
ramR::g_sym(A, u, filter)
```

```
## {{alpha+beta*mu[x]},
##  {           mu[x]}}
```

$$\mathbf{g} = \begin{pmatrix} \alpha + \beta\mu_x \\ \mu_x \end{pmatrix}$$

4.2 Numerical Example

Let `df` be a random sample with the following parameters

Parameter	$x = 0$	$x = 1$
Sample Size	500	500
μ	0	1
σ^2	1	1

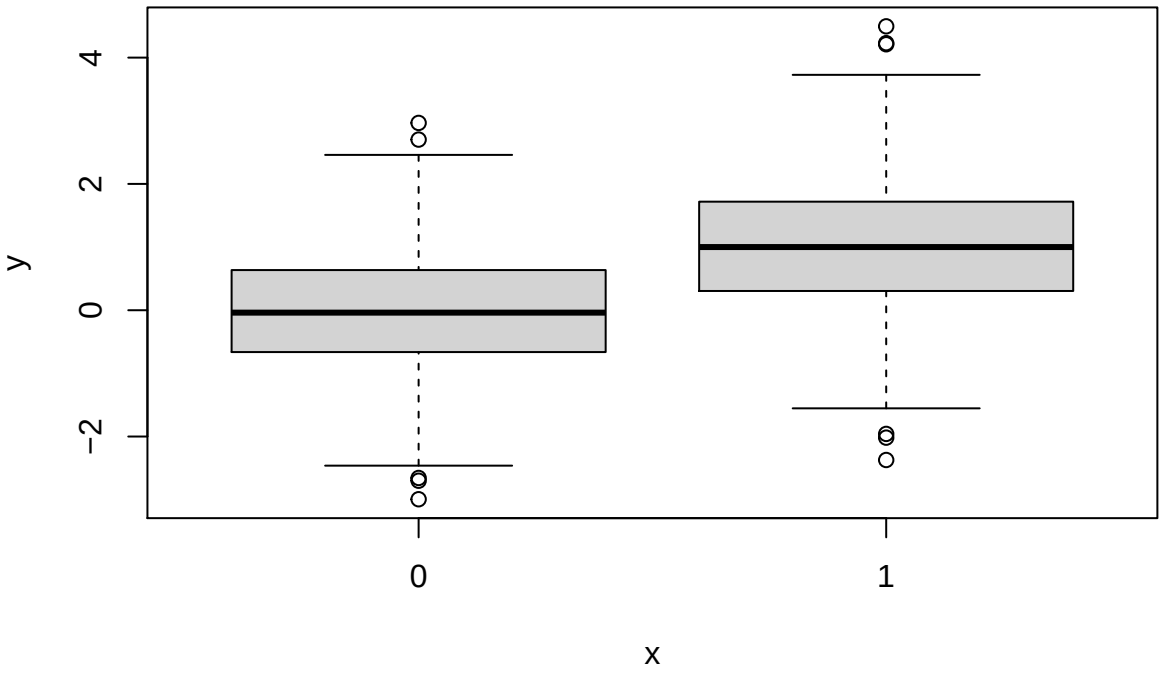
Parameter	Description	Estimate
α	Mean of $x = 0$.	0
β	Mean of $x = 1$ minus $x = 0$.	1

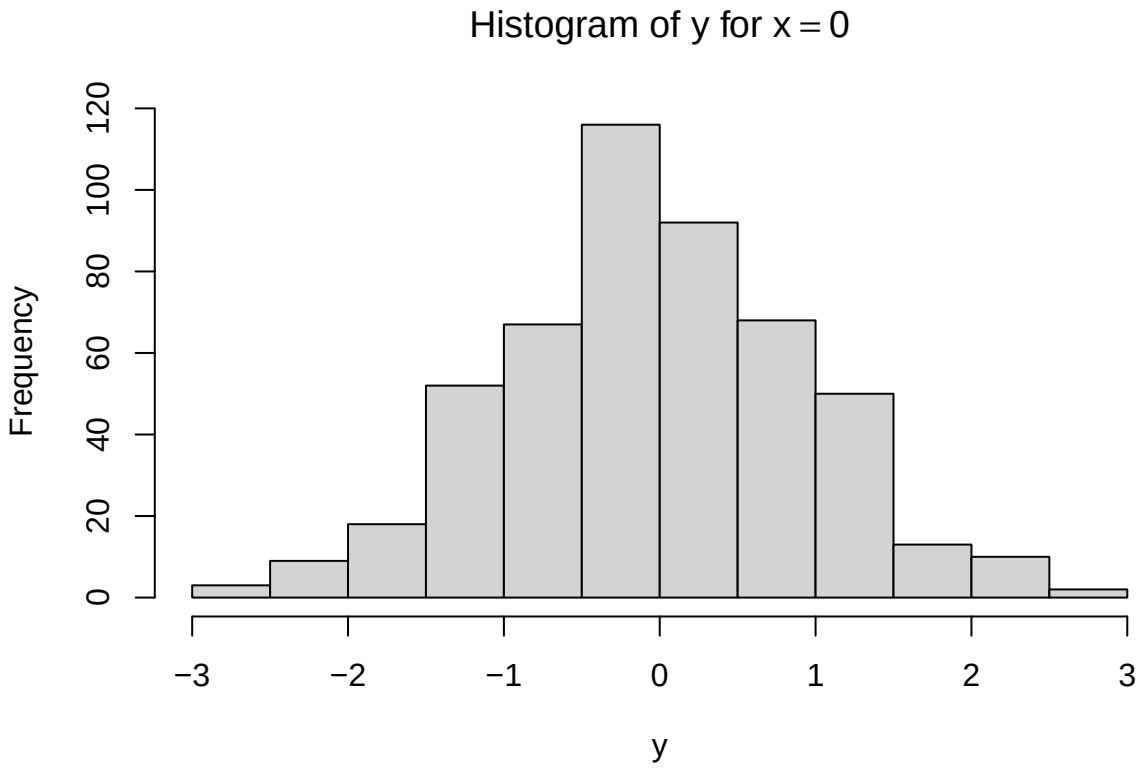
```
head(df)
```

```
##            y x
## 1  1.3709584 0
## 2 -0.5646982 0
## 3  0.3631284 0
## 4  0.6328626 0
## 5  0.4042683 0
## 6 -0.1061245 0
```

```
summary(df)
```

```
##        y                 x
##  Min.   :-2.9931   Min.   :0.0
##  1st Qu.:-0.2770   1st Qu.:0.0
##  Median : 0.4503   Median :0.5
##  Mean   : 0.4742   Mean   :0.5
##  3rd Qu.: 1.2492   3rd Qu.:1.0
##  Max.   : 4.4953   Max.   :1.0
```




Histogram of y for x = 0
Frequency
0
20
40
60
80
100
120
−3
−2
−1
0
1
2
3
y


Histogram of y for x = 1
Frequency
0
20
40
60
80
−2
−1
0
1
2
3
4
y

4.2.1 t-test

```r
t.test(y ~ x, data = df)
```

```
##
##  Welch Two Sample t-test
##
## data:  y by x
## t = -15.897, df = 994.36, p-value < 2.2e-16
## alternative hypothesis: true difference in means is not equal to 0
## 95 percent confidence interval:
##  -1.1329278 -0.8839594
## sample estimates:
## mean in group 0 mean in group 1
##     -0.03004622      0.97839737
```

4.2.2 Linear Regression

```r
summary(lm(y ~ x, data = df))
```

```
##
## Call:
## lm(formula = y ~ x, data = df)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -3.3501 -0.6517  0.0086  0.6858  3.5169
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) -0.03005    0.04486   -0.67    0.503
## x            1.00844    0.06344   15.90   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.003 on 998 degrees of freedom
## Multiple R-squared:  0.2021, Adjusted R-squared:  0.2013
## F-statistic: 252.7 on 1 and 998 DF,  p-value: < 2.2e-16
```

4.2.3 Structural Equation Modeling

4.2.3.1 lavaan (Rosseel, 2012)

```r
model <- "
  y ~ x
"
fit <- lavaan::sem(
  model,
  data = df,
  meanstructure = TRUE,
```

```
  fixed.x = FALSE
)
lavaan::summary(fit)
```

```
## lavaan 0.6-7 ended normally after 12 iterations
##
##   Estimator                                         ML
##   Optimization method                           NLMINB
##   Number of free parameters                          5
##
##   Number of observations                          1000
##
## Model Test User Model:
##
##   Test statistic                                 0.000
##   Degrees of freedom                                 0
##
## Parameter Estimates:
##
##   Standard errors                             Standard
##   Information                                 Expected
##   Information saturated (h1) model          Structured
##
## Regressions:
##                    Estimate  Std.Err  z-value  P(>|z|)
##   y ~
##     x                 1.008    0.063   15.913    0.000
##
## Intercepts:
##                    Estimate  Std.Err  z-value  P(>|z|)
##    .y                -0.030    0.045   -0.671    0.503
##     x                 0.500    0.016   31.623    0.000
##
## Variances:
##                    Estimate  Std.Err  z-value  P(>|z|)
##    .y                 1.004    0.045   22.361    0.000
##     x                 0.250    0.011   22.361    0.000
```

4.2.3.2 OpenMx (Boker et al., 2020)

RAM matrices can be used to specify models in `OpenMx`. Note, however, that the `u` vector in the RAM notation is `M` in the `OpenMx` notation.

```
mxData <- OpenMx::mxData(
  observed = df,
  type = "raw"
)
mxA <- OpenMx::mxMatrix(
  type = "Full",
  nrow = 3,
  ncol = 3,
  free = c(
```

```
    F, T, F,
    F, F, F,
    F, F, F
  ),
  values = c(
    0, 0.20, 1,
    0, 0, 0,
    0, 0, 0
  ),
  labels = c(
    NA, "beta", NA,
    NA, NA, NA,
    NA, NA, NA
  ),
  byrow = TRUE,
  name = "mxA"
)
mxS <- OpenMx::mxMatrix(
  type = "Symm",
  nrow = 3,
  ncol = 3,
  free = c(
    F, F, F,
    F, T, F,
    F, F, T
  ),
  values = c(
    0, 0, 0,
    0, 0.20, 0,
    0, 0, 0.20
  ),
  labels = c(
    NA, NA, NA,
    NA, "sigma2x", NA,
    NA, NA, "sigma2e"
  ),
  byrow = TRUE,
  name = "mxS"
)
mxM <- OpenMx::mxMatrix(
  type = "Full",
  nrow = 1,
  ncol = 3,
  free = c(
    T, T, F
  ),
  values = c(
    0.20,
    0.20,
    0
  ),
  labels = c(
    "alpha",
```

```
    "mux",
    NA
  ),
  byrow = TRUE,
  name = "mxM"
)
mxF <- OpenMx::mxMatrix(
  type = "Full",
  nrow = 2,
  ncol = 3,
  free = FALSE,
  values = c(
    1, 0, 0,
    0, 1, 0
  ),
  byrow = TRUE,
  name = "mxF"
)
expRAM <- OpenMx::mxExpectationRAM(
  A = "mxA",
  S = "mxS",
  F = "mxF",
  M = "mxM",
  dimnames = c(
    "y",
    "x",
    "e"
  )
)
objML <- OpenMx::mxFitFunctionML()
mxMod <- OpenMx::mxModel(
  name = "Student's t test",
  data = mxData,
  matrices = list(
    mxA,
    mxS,
    mxF,
    mxM
  ),
  expectation = expRAM,
  fitfunction = objML
)
fit <- OpenMx::mxRun(mxMod)
```

```
## Running Student's t test with 5 parameters
```

```
summary(fit)
```

```
## Summary of Student's t test
##
## free parameters:
##      name matrix row col    Estimate  Std.Error A
```

```
## 1    beta    mxA   1   2  1.00844356 0.06337369
## 2 sigma2x    mxS   2   2  0.25000000 0.01118034
## 3 sigma2e    mxS   3   3  1.00402596 0.04490152
## 4   alpha    mxM   1   y -0.03004621 0.04481202
## 5     mux    mxM   1   x  0.49999999 0.01581140
##
## Model Statistics:
##                |  Parameters  |  Degrees of Freedom  |  Fit (-2lnL units)
##        Model:              5                  1995              4293.478
##    Saturated:              5                  1995                    NA
## Independence:              4                  1996                    NA
## Number of observations/statistics: 1000/2000
##
## Information Criteria:
##       |  df Penalty  |  Parameters Penalty  |  Sample-Size Adjusted
## AIC:       303.4776               4303.478                4303.538
## BIC:     -9487.4941               4328.016                4312.136
## CFI: NA
## TLI: 1   (also known as NNFI)
## RMSEA:  0  [95% CI (NA, NA)]
## Prob(RMSEA <= 0.05): NA
## To get additional fit indices, see help(mxRefModels)
## timestamp: 2021-01-23 23:47:24
## Wall clock time: 0.05397487 secs
## optimizer:  SLSQP
## OpenMx version number: 2.18.1
## Need help?  See help(mxSummary)
```

4.2.4 Using the ramR Package

```
A
```

```
##   y        x e
## y 0 1.008444 1
## x 0 0.000000 0
## e 0 0.000000 0
```

```
S
```

```
##   y         x        e
## y 0 0.0000000 0.000000
## x 0 0.2502503 0.000000
## e 0 0.0000000 1.006038
```

```
u
```

```
##             u
## y -0.03004622
## x  0.50000000
## e  0.00000000
```

```
filter
```

```
##   y x e
## y 1 0 0
## x 0 1 0
```

The covariance expectations can be numerically derived using the `ramR::C_num()` function.

```
ramR::C_num(A, S)
```

```
##           y         x        e
## y 1.2605321 0.2523633 1.006038
## x 0.2523633 0.2502503 0.000000
## e 1.0060380 0.0000000 1.006038
```

The covariance expectations for the observed variables can be numerically derived using the `ramR::M_num()` function.

```
ramR::M_num(A, S, filter)
```

```
##           y         x
## y 1.2605321 0.2523633
## x 0.2523633 0.2502503
```

The mean expectations can be numerically derived using the `ramR::v_num()` function.

```
ramR::v_num(A, u)
```

```
##           v
## y 0.4741756
## x 0.5000000
## e 0.0000000
```

The mean expectations for the observed variables can be numerically derived using the `ramR::v_num()` function.

```
ramR::g_num(A, u, filter)
```

```
##           g
## y 0.4741756
## x 0.5000000
```

4.3 Equations to RAM

The `ramR` package has a utility function to convert structural equations to RAM notation. The Student's t-test can be expressed in the following equations

```
eq <- "
  # VARIABLE1 OPERATION VARIABLE2 LABEL
  e           by        y         1
  y           on        x         beta
  e           with      e         sigma[varepsilon]^2
  x           with      x         sigma[x]^2
  y           on        1         alpha
  x           on        1         mu[x]
"
```

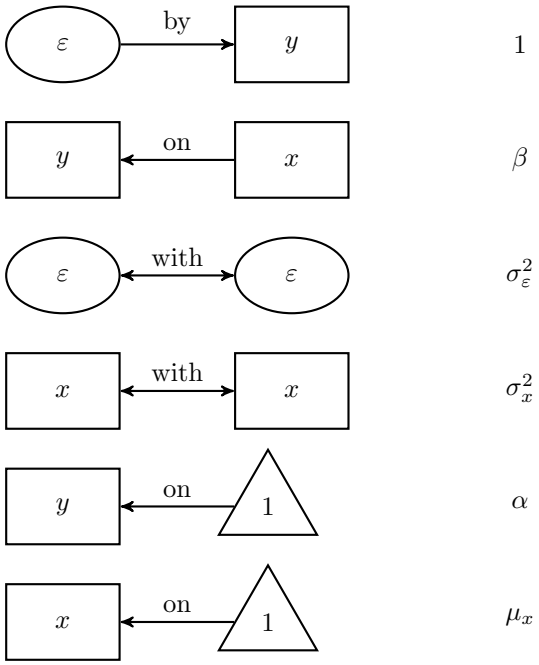


Figure 4.2: Student's t-test's Structural Equations

The error term is treated as a latent variable and defined with the operation `by`. Its value is constrained to 1. The regression of y on x is defined by operation `on`. It is labeled as `beta`. The variance of x and the error variance are defined using the operation `with`. These are labeled `sigma[x]^2` and `sigma[varepsilon]^2` respectively. The intercept and the mean of x are defined using the operation `on 1`. These are labeled `alpha` and `mu[x]` respectively.

The `ramR::eq2ram` converts the equations to RAM notation.

```
ramR::eq2ram(eq)
```

```
## $eq
##   var1   op var2               label
## 1    e   by    y                   1
## 2    y   on    x                beta
## 3    e with    e sigma[varepsilon]^2
## 4    x with    x          sigma[x]^2
## 5    y   on    1               alpha
## 6    x   on    1               mu[x]
##
```

```
## $variables
## [1] "y" "x" "e"
##
## $A
##   y   x      e
## y "0" "beta" "1"
## x "0" "0"    "0"
## e "0" "0"    "0"
##
## $S
##   y   x            e
## y "0" "0"          "0"
## x "0" "sigma[x]^2" "0"
## e "0" "0"          "sigma[varepsilon]^2"
##
## $filter
##   y x e
## y 1 0 0
## x 0 1 0
##
## $u
##   u
## y "alpha"
## x "mu[x]"
## e "0"
```

4.4 Equations to Expectations

The `ramR` package has a utility function to convert structural equations to expectations both symbolically and numerically.

```
eq <- "
  # VARIABLE1 OPERATION VARIABLE2 LABEL
  e           by        y         1
  y           on        x         beta
  e           with      e         sigma[varepsilon]^2
  x           with      x         sigma[x]^2
  y           on        1         alpha
  x           on        1         mu[x]
"
```

```
ramR::eq2exp_sym(eq)
```

```
## $variables
## [1] "y" "x" "e"
##
## $A
## {{   0, beta,    1},
##  {   0,    0,    0},
##  {   0,    0,    0}}
##
## $S
```

```
## {{                  0,                   0,                   0},
##  {                  0,          sigma[x]^2,                   0},
##  {                  0,                   0, sigma[varepsilon]^2}}
##
## $u
## {{alpha},
##  {mu[x]},
##  {    0}}
##
## $filter
## {{1, 0, 0},
##  {0, 1, 0}}
##
## $v
## {{alpha+beta*mu[x]},
##  {           mu[x]},
##  {               0}}
##
## $g
## {{alpha+beta*mu[x]},
##  {           mu[x]}}
##
## $C
## {{sigma[x]^2*beta^2+sigma[varepsilon]^2,                      beta*sigma[x]^2,                  si
##  {                      sigma[x]^2*beta,                           sigma[x]^2,
##  {                  sigma[varepsilon]^2,                                    0,                  si
##
## $M
## {{sigma[x]^2*beta^2+sigma[varepsilon]^2,                      beta*sigma[x]^2},
##  {                      sigma[x]^2*beta,                           sigma[x]^2}}
```

```
eq <- "
  # VARIABLE1 OPERATION VARIABLE2 VALUE
  e           by        y         1.00
  y           on        x         1.00
  e           with      e         1.00
  x           with      x         0.25
  y           on        1         0.00
  x           on        1         0.50
"
```

```
ramR::eq2exp_num(eq)
```

```
## $variables
## [1] "y" "x" "e"
##
## $A
##   y x e
## y 0 1 1
## x 0 0 0
## e 0 0 0
##
## $S
```

```
##   y    x e
## y 0 0.00 0
## x 0 0.25 0
## e 0 0.00 1
##
## $u
##     u
## y 0.0
## x 0.5
## e 0.0
##
## $filter
##   y x e
## y 1 0 0
## x 0 1 0
##
## $v
##     v
## y 0.5
## x 0.5
## e 0.0
##
## $g
##     g
## y 0.5
## x 0.5
##
## $C
##      y    x e
## y 1.25 0.25 1
## x 0.25 0.25 0
## e 1.00 0.00 1
##
## $M
##      y    x
## y 1.25 0.25
## x 0.25 0.25
```

Chapter 5

One-Way Analysis of Variance

In this section, one-way analysis of variance is presented as a structural equation model using the RAM notation. Let y be a continuous dependent variable, x be a categorical independent variable with three levels ($x = \{0, 1, 2\}$). The dependent variable x can be dummy coded as

x	x_1	x_2
$x = 0$	0	0
$x = 1$	1	0
$x = 2$	0	1

The associations of the variables are given by

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$$

where

- β_0 is the expected value of y when $x = 0$
- β_1 is the unit change in y for unit change in x_1 while x_2 is constant
- β_2 is the unit change in y for unit change in x_2 while x_1 is constant
- $\beta_0 + \beta_1$ is the expected value of y when $x = 1$
- $\beta_0 + \beta_2$ is the expected value of y when $x = 2$

5.1 Symbolic

Let $\{y, x_1, x_2, \varepsilon\}$ be the variables of interest.

$$\mathbf{A} = \begin{pmatrix} 0 & \beta_1 & \beta_2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sigma^2_{x_1} & 0 & 0 \\ 0 & 0 & \sigma^2_{x_2} & 0 \\ 0 & 0 & 0 & \sigma^2_{\varepsilon} \end{pmatrix}$$

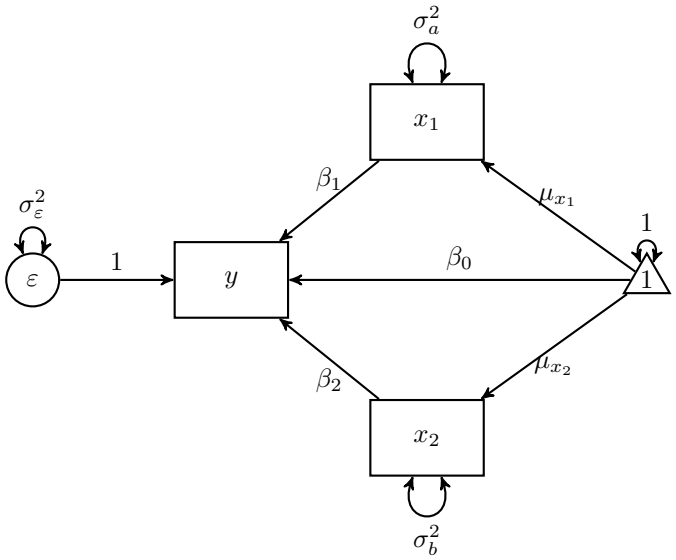


Figure 5.1: One-Way Analysis of Variance

$$\mathbf{C} = (\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{S}\left[(\mathbf{I} - \mathbf{A})^{-1}\right]^{\mathsf{T}}$$

$$= \mathbf{ESE}^{\mathsf{T}}$$

$$= \begin{pmatrix} 1 & \beta_1 & \beta_2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sigma_{x_1}^2 & 0 & 0 \\ 0 & 0 & \sigma_{x_2}^2 & 0 \\ 0 & 0 & 0 & \sigma_\varepsilon^2 \end{pmatrix} \begin{pmatrix} 1 & \beta_1 & \beta_2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{\mathsf{T}}$$

$$= \begin{pmatrix} \sigma_{x_1}^2\beta_1^2 + \sigma_{x_2}^2\beta_2^2 + \sigma_\varepsilon^2 & \beta_1\sigma_{x_1}^2 & \beta_2\sigma_{x_2}^2 & \sigma_\varepsilon^2 \\ \sigma_{x_1}^2\beta_1 & \sigma_{x_1}^2 & 0 & 0 \\ \sigma_{x_2}^2\beta_2 & 0 & \sigma_{x_2}^2 & 0 \\ \sigma_\varepsilon^2 & 0 & 0 & \sigma_\varepsilon^2 \end{pmatrix}$$

$$\mathbf{F} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$
\begin{aligned}
\mathbf{M} &= \mathbf{F}\left(\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{S}\left[\left(\mathbf{I}-\mathbf{A}\right)^{-1}\right]^{\mathsf{T}}\mathbf{F}^{\mathsf{T}} \\
&= \mathbf{FESE}^{\mathsf{T}}\mathbf{F}^{\mathsf{T}} \\
&= \mathbf{FCF}^{\mathsf{T}} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\begin{pmatrix} \sigma^2_{x_1}\beta_1^2+\sigma^2_{x_2}\beta_2^2+\sigma^2_\varepsilon & \beta_1\sigma^2_{x_1} & \beta_2\sigma^2_{x_2} & \sigma^2_\varepsilon \\ \sigma^2_{x_1}\beta_1 & \sigma^2_{x_1} & 0 & 0 \\ \sigma^2_{x_2}\beta_2 & 0 & \sigma^2_{x_2} & 0 \\ \sigma^2_\varepsilon & 0 & 0 & \sigma^2_\varepsilon \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{\mathsf{T}} \\
&= \begin{pmatrix} \sigma^2_{x_1}\beta_1^2+\sigma^2_{x_2}\beta_2^2+\sigma^2_\varepsilon & \beta_1\sigma^2_{x_1} & \beta_2\sigma^2_{x_2} \\ \sigma^2_{x_1}\beta_1 & \sigma^2_{x_1} & 0 \\ \sigma^2_{x_2}\beta_2 & 0 & \sigma^2_{x_2} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{v} &= \left(\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{u} \\
&= \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & \beta_1 & \beta_2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right]^{-1}
\begin{pmatrix} \beta_0 \\ \mu_{x_1} \\ \mu_{x_2} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \beta_0+\beta_1\mu_{x_1}+\beta_2\mu_{x_2} \\ \mu_{x_1} \\ \mu_{x_2} \\ 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{u} &= \left(\mathbf{I}-\mathbf{A}\right)\mathbf{v} \\
&= \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & \beta_1 & \beta_2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right]
\begin{pmatrix} \beta_0+\beta_1\mu_{x_1}+\beta_2\mu_{x_2} \\ \mu_{x_1} \\ \mu_{x_2} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \beta_0 \\ \mu_{x_1} \\ \mu_{x_2} \\ 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{g} &= \mathbf{F}\left(\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{u} \\
&= \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & \beta_1 & \beta_2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right]^{-1}
\begin{pmatrix} \beta_0 \\ \mu_{x_1} \\ \mu_{x_2} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \beta_0+\beta_1\mu_{x_1}+\beta_2\mu_{x_2} \\ \mu_{x_1} \\ \mu_{x_2} \end{pmatrix}
\end{aligned}
$$

5.1.1 Using the ramR Package

```
A
```

```
##    y   x1        x2        e
## y  "0" "beta[1]" "beta[2]" "1"
## x1 "0" "0"       "0"       "0"
## x2 "0" "0"       "0"       "0"
## e  "0" "0"       "0"       "0"
```

```
S
```

```
##    y   x1            x2            e
## y  "0" "0"           "0"           "0"
## x1 "0" "sigma[x1]^2" "0"           "0"
## x2 "0" "0"           "sigma[x2]^2" "0"
## e  "0" "0"           "0"           "sigma[varepsilon]^2"
```

```
u
```

```
##    u
## y  "beta[0]"
## x1 "mu[x1]"
## x2 "mu[x2]"
## e  "0"
```

```
filter
```

```
##    y x1 x2 e
## y  1  0  0 0
## x1 0  1  0 0
## x2 0  0  1 0
```

The covariance expectations can be symbolically derived using the `ramR::C_sym()` function.

```
ramR::C_sym(A, S)
```

```
## {{sigma[x1]^2*beta[1]^2+sigma[x2]^2*beta[2]^2+sigma[varepsilon]^2,
##  {                                            sigma[x1]^2*beta[1],
##  {                                            sigma[x2]^2*beta[2],
##  {                                            sigma[varepsilon]^2,
```

$$\mathbf{C} = \begin{pmatrix} \sigma^2_{x_1}\beta_1^2 + \sigma^2_{x_2}\beta_2^2 + \sigma^2_\varepsilon & \beta_1\sigma^2_{x_1} & \beta_2\sigma^2_{x_2} & \sigma^2_\varepsilon \\ \sigma^2_{x_1}\beta_1 & \sigma^2_{x_1} & 0 & 0 \\ \sigma^2_{x_2}\beta_2 & 0 & \sigma^2_{x_2} & 0 \\ \sigma^2_\varepsilon & 0 & 0 & \sigma^2_\varepsilon \end{pmatrix}$$

The covariance expectations for the observed variables can be symbolically derived using the `ramR::M_sym()` function.

```
ramR::M_sym(A, S, filter)
```

```
## {{sigma[x1]^2*beta[1]^2+sigma[x2]^2*beta[2]^2+sigma[varepsilon]^2,
##  {                                            sigma[x1]^2*beta[1],
##  {                                            sigma[x2]^2*beta[2],
```

$$\mathbf{M} = \begin{pmatrix} \sigma_{x_1}^2\beta_1^2 + \sigma_{x_2}^2\beta_2^2 + \sigma_\varepsilon^2 & \beta_1\sigma_{x_1}^2 & \beta_2\sigma_{x_2}^2 \\ \sigma_{x_1}^2\beta_1 & \sigma_{x_1}^2 & 0 \\ \sigma_{x_2}^2\beta_2 & 0 & \sigma_{x_2}^2 \end{pmatrix}$$

The mean expectations can be symbolically derived using the `ramR::v_sym()` function.

```
ramR::v_sym(A, u)
```

```
## {{beta[0]+beta[1]*mu[x1]+beta[2]*mu[x2]},
##  {                                mu[x1]},
##  {                                mu[x2]},
##  {                                     0}}
```

$$\mathbf{v} = \begin{pmatrix} \beta_0 + \beta_1\mu_{x_1} + \beta_2\mu_{x_2} \\ \mu_{x_1} \\ \mu_{x_2} \\ 0 \end{pmatrix}$$

The mean expectations for the observed variables can be symbolically derived using the `ramR::g_sym()` function.

```
ramR::g_sym(A, u, filter)
```

```
## {{beta[0]+beta[1]*mu[x1]+beta[2]*mu[x2]},
##  {                                mu[x1]},
##  {                                mu[x2]}}
```

$$\mathbf{g} = \begin{pmatrix} \beta_0 + \beta_1\mu_{x_1} + \beta_2\mu_{x_2} \\ \mu_{x_1} \\ \mu_{x_2} \end{pmatrix}$$

5.2 Numerical Example

Let `df` be a random sample with the following parameters

Parameter	$x = 0$	$x = 1$	$x = 2$
Sample Size	500	500	500
μ	0	2	1
σ^2	1	1	1

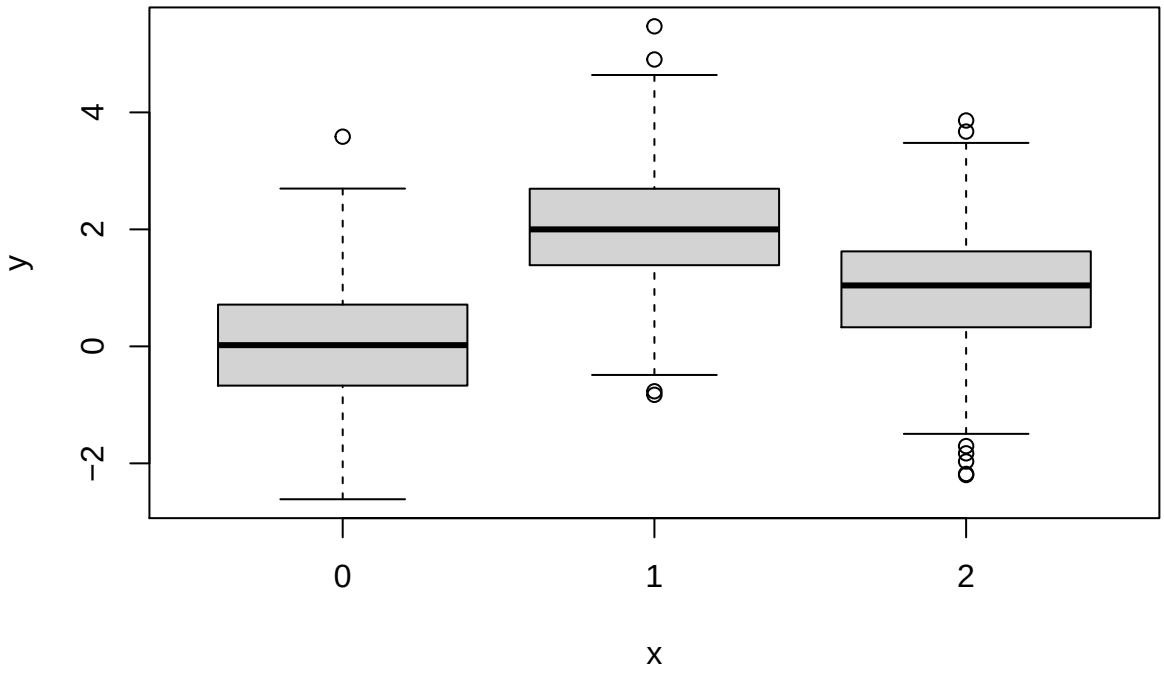
Parameter	Description	Estimate
β_0	Mean of $x = 0$.	0
β_1	Mean of $x = 1$ minus $x = 0$.	2
β_2	Mean of $x = 2$ minus $x = 0$.	1

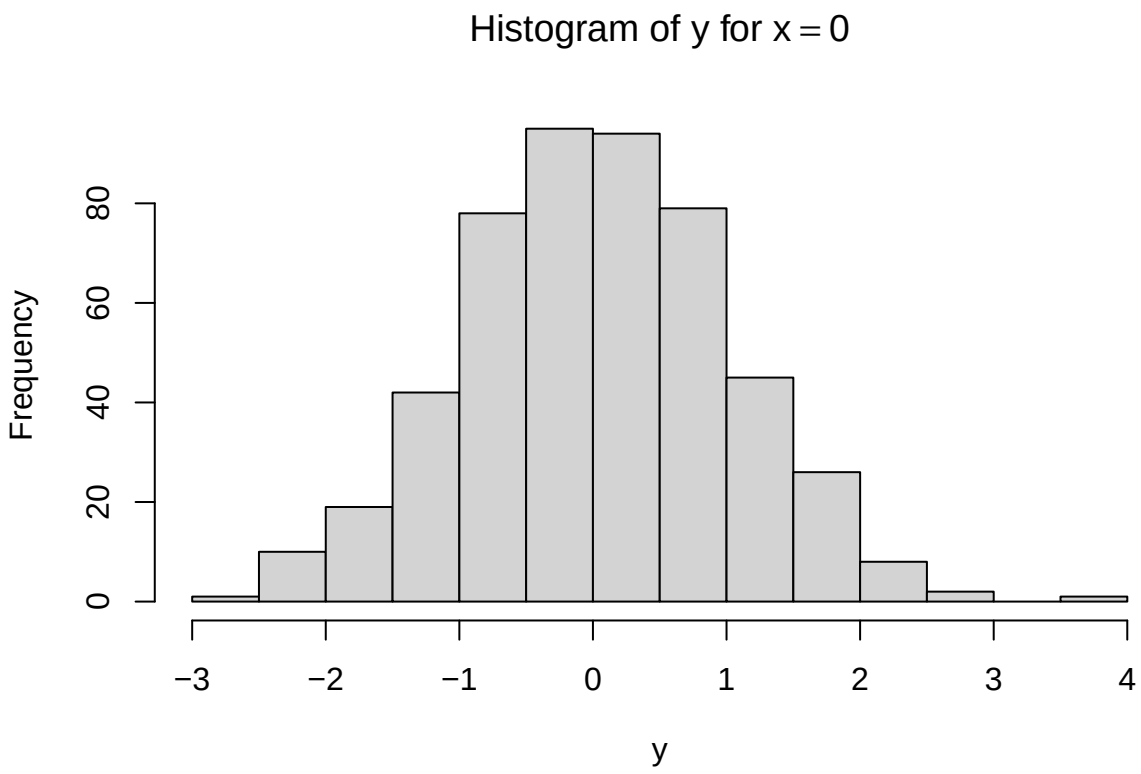
```
head(df)
```

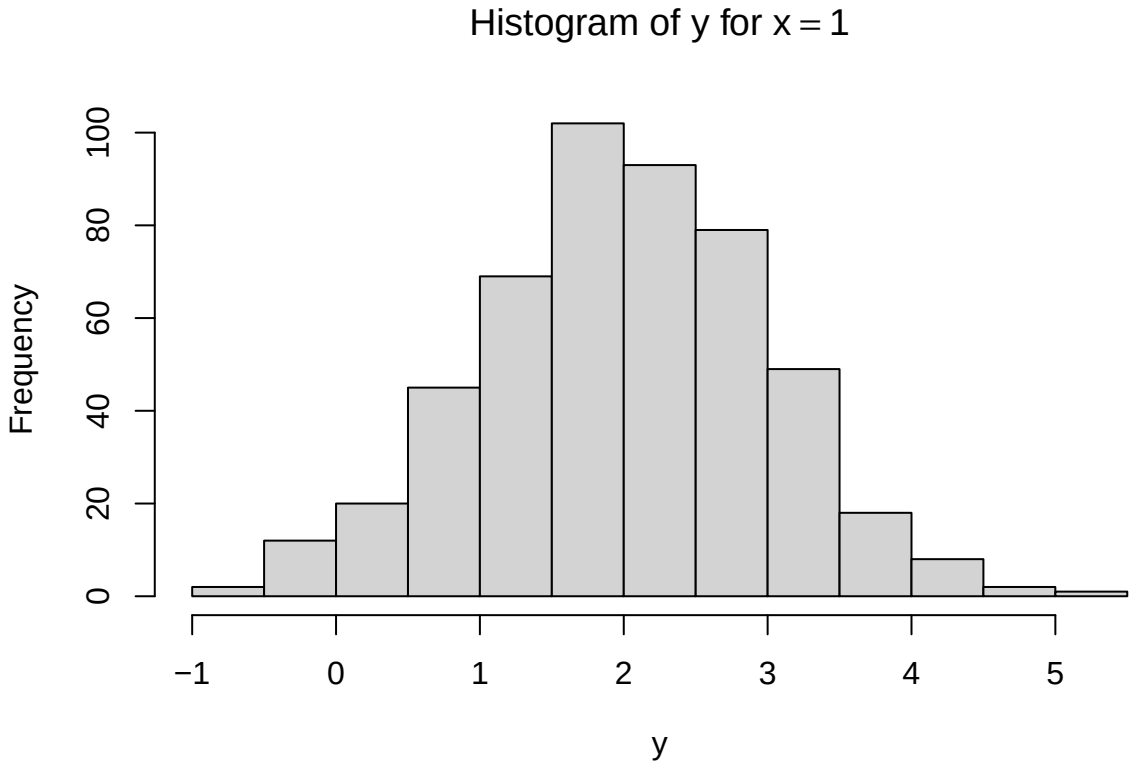
```
##           y x
## 1 -0.6013830 0
## 2 -0.1358161 0
## 3 -0.9872728 0
## 4  0.8319250 0
## 5 -0.7950595 0
## 6  0.3404646 0
```

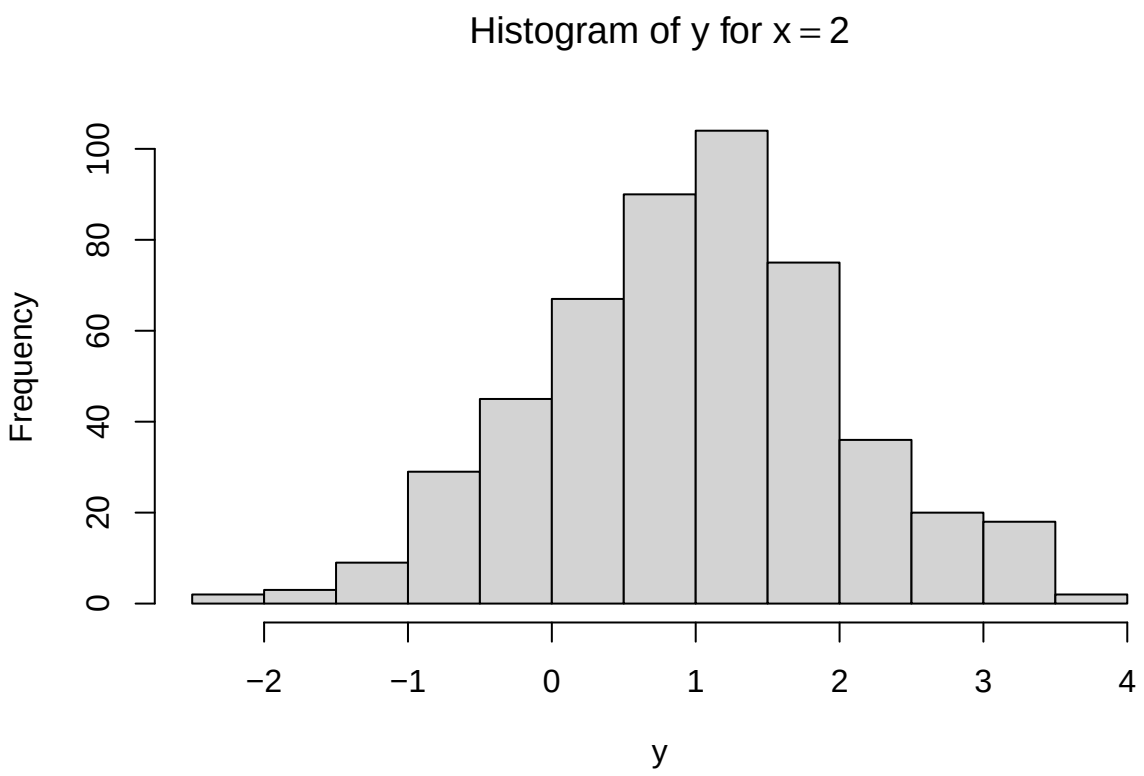
```
summary(df)
```

```
##        y           x
##  Min.   :-2.61364   0:500
##  1st Qu.: 0.08094   1:500
##  Median : 1.02617   2:500
##  Mean   : 1.00814
##  3rd Qu.: 1.90112
##  Max.   : 5.47091
```




Histogram of y for x = 0
Frequency
0
20
40
60
80
−3
−2
−1
0
1
2
3
4
y


Histogram of y for x = 1
Frequency
0
20
40
60
80
100
−1
0
1
2
3
4
5
y



5.2.1 One-Way Analysis of Variance

Make sure that x is of class `factor` for `lm` and `aov` to treat it as a categorical variable.

```
str(df)
```

```
## 'data.frame':    1500 obs. of  2 variables:
##  $ y: num  -0.601 -0.136 -0.987 0.832 -0.795 ...
##  $ x: Factor w/ 3 levels "0","1","2": 1 1 1 1 1 1 1 1 1 1 ...
```

```
summary(aov(y ~ x, data = df))
```

```
##               Df Sum Sq Mean Sq F value Pr(>F)
## x              2  983.8   491.9   471.4 <2e-16 ***
## Residuals   1497 1562.2     1.0
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

5.2.2 Linear Regression

```
summary(lm(y ~ x, data = df))
```

```
##
## Call:
```

```
## lm(formula = y ~ x, data = df)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -3.1792 -0.6469  0.0021  0.6751  3.5538
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  0.03083    0.04569   0.675      0.5
## x1           1.98309    0.06461  30.694   <2e-16 ***
## x2           0.94884    0.06461  14.686   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.022 on 1497 degrees of freedom
## Multiple R-squared:  0.3864, Adjusted R-squared:  0.3856
## F-statistic: 471.4 on 2 and 1497 DF,  p-value: < 2.2e-16
```

5.2.3 Structural Equation Modeling

We have to dummy code the data set first before fitting the model. The `model.matrix` function which is used to create a design matrix can be used to dummy code `x`. Make sure that `x` is a `factor`. The first column of the design matrix is a matrix of ones. Since we do not need this column, we can replace this column with the values of `y`. Make sure to name rename the first column as `lavaan` relies on the column names.

```
df_dummy <- model.matrix(y ~ x, data = df)
df_dummy[, 1] <- df$y
colnames(df_dummy)[1] <- "y"
head(df_dummy)
```

```
##            y x1 x2
## 1 -0.6013830  0  0
## 2 -0.1358161  0  0
## 3 -0.9872728  0  0
## 4  0.8319250  0  0
## 5 -0.7950595  0  0
## 6  0.3404646  0  0
```

5.2.3.1 lavaan (Rosseel, 2012)

```
model <- "
  y ~ x1 + x2
"
fit <- lavaan::sem(
  model,
  data = df_dummy,
  meanstructure = TRUE,
  fixed.x = FALSE
)
lavaan::summary(fit)
```

```
## lavaan 0.6-7 ended normally after 22 iterations
##
##   Estimator                                         ML
##   Optimization method                           NLMINB
##   Number of free parameters                          9
##
##   Number of observations                          1500
##
## Model Test User Model:
##
##   Test statistic                                 0.000
##   Degrees of freedom                                 0
##
## Parameter Estimates:
##
##   Standard errors                             Standard
##   Information                                 Expected
##   Information saturated (h1) model          Structured
##
## Regressions:
##                    Estimate  Std.Err  z-value  P(>|z|)
##   y ~
##     x1                1.983    0.065   30.725    0.000
##     x2                0.949    0.065   14.701    0.000
##
## Covariances:
##                    Estimate  Std.Err  z-value  P(>|z|)
##   x1 ~~
##     x2               -0.111    0.006  -17.321    0.000
##
## Intercepts:
##                    Estimate  Std.Err  z-value  P(>|z|)
##    .y                 0.031    0.046    0.676    0.499
##     x1                0.333    0.012   27.386    0.000
##     x2                0.333    0.012   27.386    0.000
##
## Variances:
##                    Estimate  Std.Err  z-value  P(>|z|)
##    .y                 1.041    0.038   27.386    0.000
##     x1                0.222    0.008   27.386    0.000
##     x2                0.222    0.008   27.386    0.000
```

5.2.3.2 OpenMx (Boker et al., 2020)

RAM matrices can be used to specify models in `OpenMx`. Note, however, that the `u` vector in the RAM notation is `M` in the `OpenMx` notation.

```
mxData <- OpenMx::mxData(
  observed = df_dummy,
  type = "raw"
)
mxA <- OpenMx::mxMatrix(
  type = "Full",
```

```
  nrow = 4,
  ncol = 4,
  free = c(
    F, T, T, F,
    F, F, F, F,
    F, F, F, F,
    F, F, F, F
  ),
  values = c(
    0, 0.20, 0.20, 1,
    0, 0, 0, 0,
    0, 0, 0, 0,
    0, 0, 0, 0
  ),
  labels = c(
    NA, "beta1", "beta2", NA,
    NA, NA, NA, NA,
    NA, NA, NA, NA,
    NA, NA, NA, NA
  ),
  byrow = TRUE,
  name = "mxA"
)
mxS <- OpenMx::mxMatrix(
  type = "Symm",
  nrow = 4,
  ncol = 4,
  free = c(
    F, F, F, F,
    F, T, F, F,
    F, F, T, F,
    F, F, F, T
  ),
  values = c(
    0,    0,    0,    0,
    0, 0.20,    0,    0,
    0,    0, 0.20,    0,
    0,    0,    0, 0.20
  ),
  labels = c(
    NA, NA, NA, NA,
    NA, "sigma2x1", NA, NA,
    NA, NA, "sigma2x2", NA,
    NA, NA, NA, "sigma2e"
  ),
  byrow = TRUE,
  name = "mxS"
)
mxM <- OpenMx::mxMatrix(
  type = "Full",
  nrow = 1,
  ncol = 4,
  free = c(
```

```
    T, T, T, F
  ),
  values = c(
    0.20,
    0.20,
    0.20,
    0
  ),
  labels = c(
    "beta0",
    "mux1",
    "mux2",
    NA
  ),
  byrow = TRUE,
  name = "mxM"
)
mxF <- OpenMx::mxMatrix(
  type = "Full",
  nrow = 3,
  ncol = 4,
  free = FALSE,
  values = c(
    1, 0, 0, 0,
    0, 1, 0, 0,
    0, 0, 1, 0
  ),
  byrow = TRUE,
  name = "mxF"
)
expRAM <- OpenMx::mxExpectationRAM(
  A = "mxA",
  S = "mxS",
  F = "mxF",
  M = "mxM",
  dimnames = c(
    "y",
    "x1",
    "x2",
    "e"
  )
)
objML <- OpenMx::mxFitFunctionML()
mxMod <- OpenMx::mxModel(
  name = "One Way Analysis of Variance",
  data = mxData,
  matrices = list(
    mxA,
    mxS,
    mxF,
    mxM
  ),
  expectation = expRAM,
```

```
  fitfunction = objML
)
fit <- OpenMx::mxRun(mxMod)
```

```
## Running One Way Analysis of Variance with 8 parameters
```

```
summary(fit)
```

```
## Summary of One Way Analysis of Variance
##
## free parameters:
##       name matrix row col   Estimate   Std.Error A
## 1    beta1    mxA   1   2 1.98308662 0.064543779
## 2    beta2    mxA   1   3 0.94883814 0.064543143
## 3 sigma2x1    mxS   2   2 0.22222230 0.008114416
## 4 sigma2x2    mxS   3   3 0.22222238 0.008114420
## 5  sigma2e    mxS   4   4 1.04147460 0.038029458
## 6    beta0    mxM   1   y 0.03083127 0.045639092
## 7     mux1    mxM   1  x1 0.33333343 0.012171613
## 8     mux2    mxM   1  x2 0.33333344 0.012171612
##
## Model Statistics:
##                |  Parameters  |  Degrees of Freedom  |  Fit (-2lnL units)
##        Model:              8                  4492               8319.17
##    Saturated:              9                  4491                    NA
## Independence:              6                  4494                    NA
## Number of observations/statistics: 1500/4500
##
## Information Criteria:
##       |  df Penalty  |  Parameters Penalty  |  Sample-Size Adjusted
## AIC:      -664.8302              8335.170                8335.266
## BIC:    -24531.8162              8377.676                8352.262
## To get additional fit indices, see help(mxRefModels)
## timestamp: 2021-01-23 23:47:26
## Wall clock time: 0.03291965 secs
## optimizer:  SLSQP
## OpenMx version number: 2.18.1
## Need help?  See help(mxSummary)
```

5.2.4 Using the ramR Package

```
A
```

```
##    y       x1        x2 e
## y  0 2.008444 0.9885797 1
## x1 0 0.000000 0.0000000 0
## x2 0 0.000000 0.0000000 0
## e  0 0.000000 0.0000000 0
```

```
S
```

```
##    y        x1        x2         e
## y  0 0.0000000 0.0000000 0.0000000
## x1 0 0.2223705 0.0000000 0.0000000
## x2 0 0.0000000 0.2223705 0.0000000
## e  0 0.0000000 0.0000000 0.9823083
```

```
u
```

```
##            u
## y  0.3333333
## x1 0.3333333
## x2 0.3333333
## e  0.0000000
```

```
filter
```

```
##    y x1 x2 e
## y  1  0  0 0
## x1 0  1  0 0
## x2 0  0  1 0
```

The covariance expectations can be numerically derived using the `ramR::C_num()` function.

```
ramR::C_num(A, S)
```

```
##            y        x1        x2         e
## y  2.0966368 0.4466185 0.2198309 0.9823083
## x1 0.4466185 0.2223705 0.0000000 0.0000000
## x2 0.2198309 0.0000000 0.2223705 0.0000000
## e  0.9823083 0.0000000 0.0000000 0.9823083
```

The covariance expectations for the observed variables can be numerically derived using the `ramR::M_num()` function.

```
ramR::M_num(A, S, filter)
```

```
##            y        x1        x2
## y  2.0966368 0.4466185 0.2198309
## x1 0.4466185 0.2223705 0.0000000
## x2 0.2198309 0.0000000 0.2223705
```

The mean expectations can be numerically derived using the `ramR::v_num()` function.

```
ramR::v_num(A, u)
```

```
##            v
## y  1.3323411
## x1 0.3333333
## x2 0.3333333
## e  0.0000000
```

The mean expectations for the observed variables can be numerically derived using the `ramR::v_num()` function.

```
ramR::g_num(A, u, filter)
```

```
##            g
## y  1.3323411
## x1 0.3333333
## x2 0.3333333
```

5.3 Equations to RAM

The `ramR` package has a utility function to convert structural equations to RAM notation. One-way analysis of variance with three levels can be expressed in the following equations

```
eq <- "
  # VARIABLE1 OPERATION VARIABLE2 LABEL
  e           by        y         1
  y           on        x1        beta1
  y           on        x2        beta2
  e           with      e         sigma[varepsilon]^2
  x1          with      x1        sigma[x1]^2
  x2          with      x2        sigma[x2]^2
  y           on        1         beta0
  x1          on        1         mu[x1]
  x2          on        1         mu[x2]
"
```

The error term is treated as a latent variable and defined with the operation `by`. Its value is constrained to 1. The regression of y on x_1 and x_2 is defined by operation `on`. The coefficients are labeled as `beta[1]` and `beta[2]` respectively. The variance of x_1, x_2 and the error variance are defined using the operation `with`. These are labeled `sigma[x1]^2`, `sigma[x2]^2`, and `sigma[varepsilon]^2` respectively. The intercept and the mean of x_1 and x_2 are defined using the operation `on 1`. These are labeled `beta[0]`, `mu[x1]`, and `mu[x2]` respectively.

The `ramR::eq2ram` converts the equations to RAM notation.

```
ramR::eq2ram(eq)
```

```
## $eq
##   var1   op var2               label
## 1    e   by    y                   1
## 2    y   on   x1               beta1
## 3    y   on   x2               beta2
## 4    e with    e sigma[varepsilon]^2
## 5   x1 with   x1         sigma[x1]^2
## 6   x2 with   x2         sigma[x2]^2
## 7    y   on    1               beta0
## 8   x1   on    1              mu[x1]
## 9   x2   on    1              mu[x2]
##
```

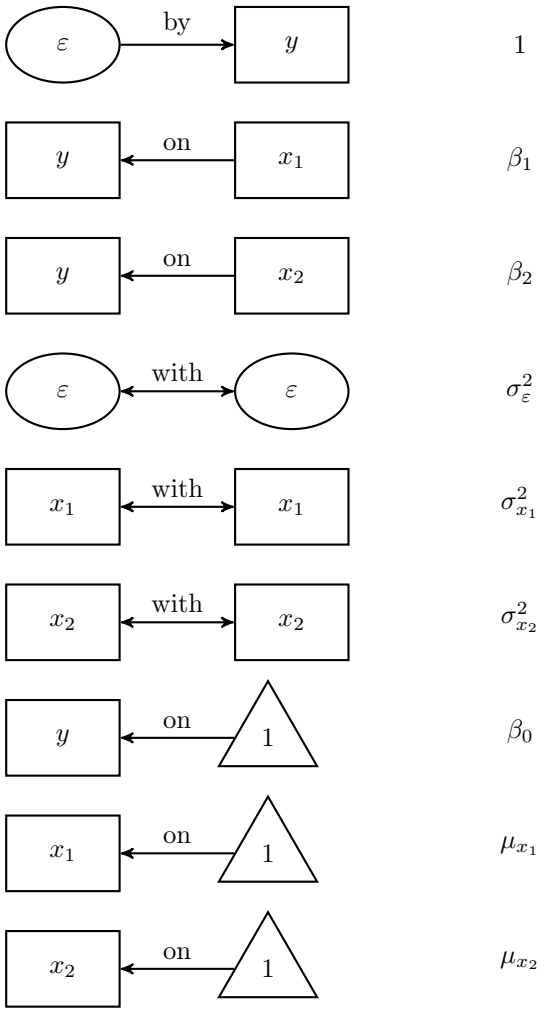


Figure 5.2: One-Way Analysis of Variance's Structural Equations

```
## $variables
## [1] "y"  "x1" "x2" "e"
##
## $A
##    y   x1      x2      e
## y  "0" "beta1" "beta2" "1"
## x1 "0" "0"     "0"     "0"
## x2 "0" "0"     "0"     "0"
## e  "0" "0"     "0"     "0"
##
## $S
##    y   x1            x2            e
## y  "0" "0"           "0"           "0"
## x1 "0" "sigma[x1]^2" "0"           "0"
## x2 "0" "0"           "sigma[x2]^2" "0"
## e  "0" "0"           "0"           "sigma[varepsilon]^2"
##
## $filter
##    y x1 x2 e
## y  1  0  0 0
## x1 0  1  0 0
## x2 0  0  1 0
##
## $u
##    u
## y  "beta0"
## x1 "mu[x1]"
## x2 "mu[x2]"
## e  "0"
```

5.4 Equations to Expectations

The `ramR` package has a utility function to convert structural equations to expectations both symbolically and numerically.

```
eq <- "
  # VARIABLE1 OPERATION VARIABLE2 LABEL
  e           by        y         1
  y           on        x1        beta1
  y           on        x2        beta2
  e           with      e         sigma[varepsilon]^2
  x1          with      x1        sigma[x1]^2
  x2          with      x2        sigma[x2]^2
  y           on        1         beta0
  x1          on        1         mu[x1]
  x2          on        1         mu[x2]
"
```

```
ramR::eq2exp_sym(eq)
```

```
## $variables
## [1] "y"  "x1" "x2" "e"
```

```
##
## $A
## {{    0, beta1, beta2,     1},
##  {    0,     0,     0,     0},
##  {    0,     0,     0,     0},
##  {    0,     0,     0,     0}}
##
## $S
## {{                  0,                   0,                   0,                     0},
##  {                  0,         sigma[x1]^2,                   0,                     0},
##  {                  0,                   0,         sigma[x2]^2,                     0},
##  {                  0,                   0,                   0, sigma[varepsilon]^2}}
##
## $u
## {{ beta0},
##  {mu[x1]},
##  {mu[x2]},
##  {     0}}
##
## $filter
## {{1, 0, 0, 0},
##  {0, 1, 0, 0},
##  {0, 0, 1, 0}}
##
## $v
## {{beta0+beta1*mu[x1]+beta2*mu[x2]},
##  {                         mu[x1]},
##  {                         mu[x2]},
##  {                              0}}
##
## $g
## {{beta0+beta1*mu[x1]+beta2*mu[x2]},
##  {                         mu[x1]},
##  {                         mu[x2]}}
##
## $C
## {{sigma[x1]^2*beta1^2+sigma[x2]^2*beta2^2+sigma[varepsilon]^2,
##  {                                          sigma[x1]^2*beta1,
##  {                                          sigma[x2]^2*beta2,
##  {                                        sigma[varepsilon]^2,
##
## $M
## {{sigma[x1]^2*beta1^2+sigma[x2]^2*beta2^2+sigma[varepsilon]^2,
##  {                                          sigma[x1]^2*beta1,
##  {                                          sigma[x2]^2*beta2,
```

```
eq <- "
  # VARIABLE1 OPERATION VARIABLE2 LABEL
  e           by        y         1
  y           on        x1        2
  y           on        x2        1
  e           with      e         1
  x1          with      x1        0.22222222222
  x2          with      x2        0.22222222222
```

```
  y           on        1         0
  x1          on        1         0.33333333333
  x2          on        1         0.33333333333
"
```

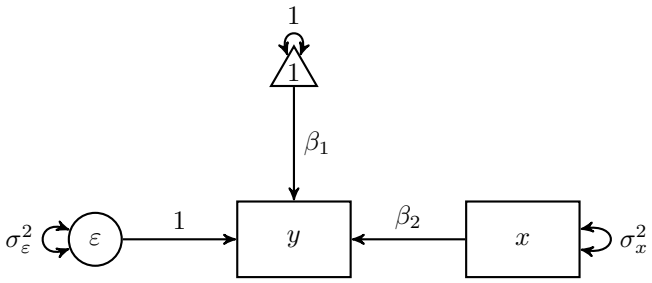
```
ramR::eq2exp_num(eq)
```

```
## $variables
## [1] "y"  "x1" "x2" "e"
##
## $A
##    y x1 x2 e
## y  0  2  1 1
## x1 0  0  0 0
## x2 0  0  0 0
## e  0  0  0 0
##
## $S
##    y        x1        x2 e
## y  0 0.0000000 0.0000000 0
## x1 0 0.2222222 0.0000000 0
## x2 0 0.0000000 0.2222222 0
## e  0 0.0000000 0.0000000 1
##
## $u
##            u
## y  0.0000000
## x1 0.3333333
## x2 0.3333333
## e  0.0000000
##
## $filter
##    y x1 x2 e
## y  1  0  0 0
## x1 0  1  0 0
## x2 0  0  1 0
##
## $v
##            v
## y  1.0000000
## x1 0.3333333
## x2 0.3333333
## e  0.0000000
##
## $g
##            g
## y  1.0000000
## x1 0.3333333
## x2 0.3333333
##
## $C
##            y        x1        x2 e
## y  2.1111111 0.4444444 0.2222222 1
```

```
## x1 0.4444444 0.2222222 0.0000000 0
## x2 0.2222222 0.0000000 0.2222222 0
## e  1.0000000 0.0000000 0.0000000 1
##
## $M
##            y        x1        x2
## y  2.1111111 0.4444444 0.2222222
## x1 0.4444444 0.2222222 0.0000000
## x2 0.2222222 0.0000000 0.2222222
```

Chapter 6

Two-Variable Regression Model



$$y = \alpha + \beta x + \varepsilon$$

Figure 6.1: Two-Variable Regression Model

Chapter 7

k-Variable Regression Model

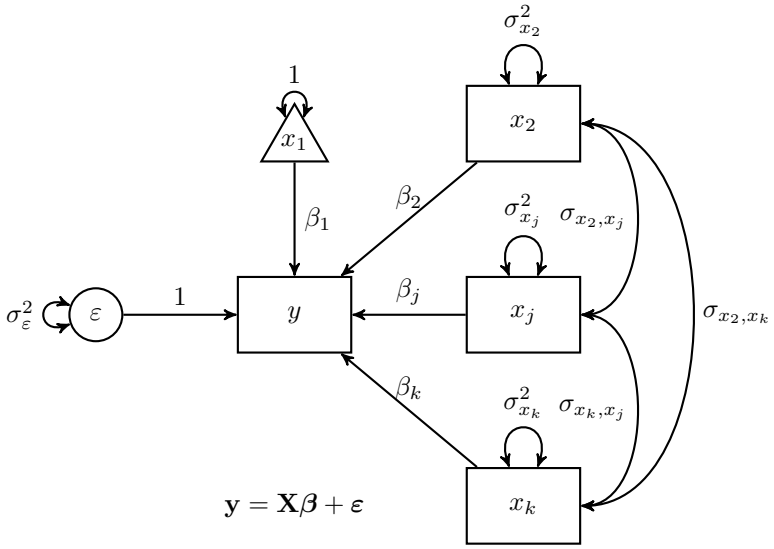


Figure 7.1: k-Variable Regression Model

Chapter 8

The Simple Mediation Model

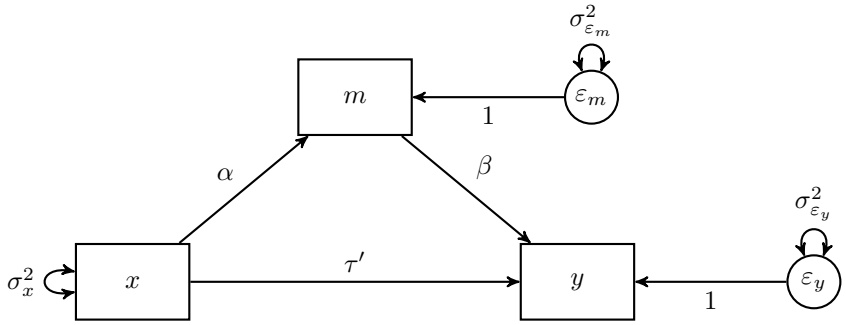


Figure 8.1: The Simple Mediation Model

Chapter 9

The Standardized Simple Mediation Model

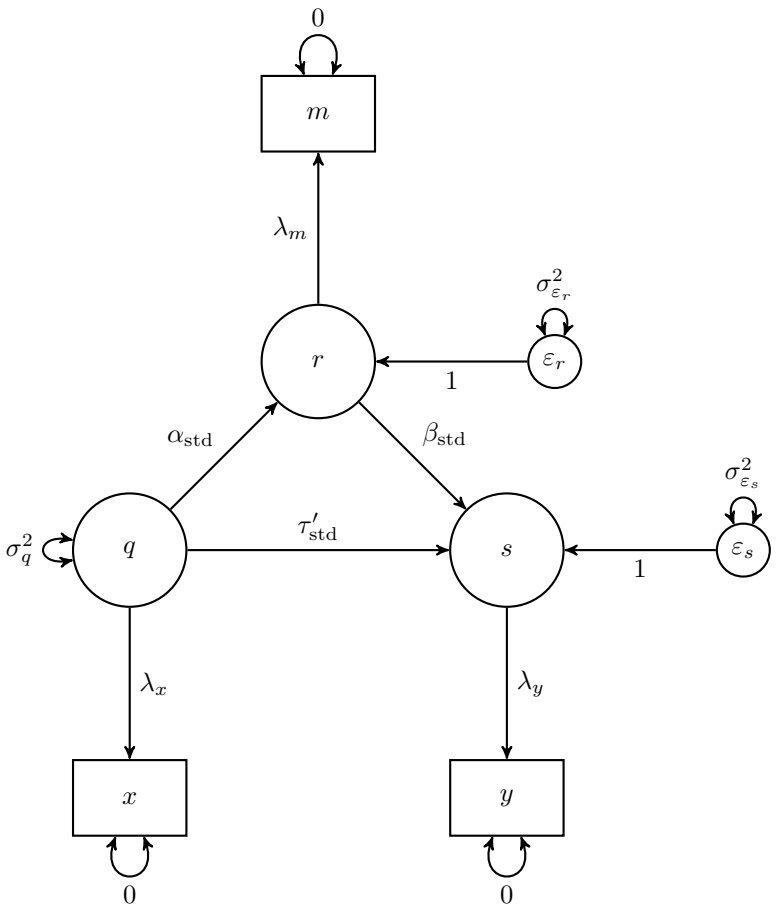


Figure 9.1: The Standardized Simple Mediation Model

Bibliography

Boker, S. M. and McArdle, J. J. (2005). Path analysis and path diagrams. In Everitt, B. S. and Howell, D. C., editors, *Encyclopedia of Statistics in Behavioral Science*, pages 1529–1531. John Wiley & Sons, Ltd, Chichester, UK.

Boker, S. M., Neale, M. C., Maes, H. H., Wilde, M. J., Spiegel, M., Brick, T. R., Estabrook, R., Bates, T. C., Mehta, P., von Oertzen, T., Gore, R. J., Hunter, M. D., Hackett, D. C., Karch, J., Brandmaier, A. M., Pritikin, J. N., Zahery, M., Kirkpatrick, R. M., Wang, Y., Goodrich, B., Driver, C., Massachusetts Institute of Technology, Johnson, S. G., Association for Computing Machinery, Kraft, D., Wilhelm, S., Medland, S., Falk, C. F., Keller, M., Manjunath B G, The Regents of the University of California, Ingber, L., Shao Voon, W., Palacios, J., Yang, J., Guennebaud, G., and Niesen, J. (2020). *OpenMx 2.18.1 User Guide*.

McArdle, J. J. (2005). The development of the RAM rules for latent variable structural equation modeling. In Maydeu-Olivares, A. and McArdle, J. J., editors, *Contemporary psychometrics: A festschrift for Roderick P. McDonald*, Multivariate applications book series, pages 225–273. Lawrence Erlbaum Associates, Mahwah, NJ.

McArdle, J. J. and McDonald, R. P. (1984). Some algebraic properties of the reticular action model for moment structures. *British Journal of Mathematical and Statistical Psychology*, 37(2):234–251.

Pesigan, I. J. A. (2021). *ramR: Reticular Action Model (RAM) Notation*. R package version 0.9.0.

R Core Team (2020). *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria.

Rosseel, Y. (2012). lavaan: An R package for structural equation modeling. *Journal of Statistical Software*, 48(2):1–36.