Introduction to Limit Theory

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Overview

- Notation: $\lim_{x\to a} f(x) = L$ indicates that values of $f(x)$ approach L as x approaches a.
- Intuition: closeness of x to a forces closeness of $f(x)$ to L.
- Use algebraic simplification first when limits yield indeterminate forms such as $0/0$.

Definition

For a function $f : \mathbb{R} \to \mathbb{R}$, we say $\lim_{x\to a} f(x) = L$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $0 < |x - a| < \delta$, it follows that $|f(x) - L| < \varepsilon$.

Example

Evaluate $\displaystyle\lim_{x\to 2} \frac{x^2 - 4}{x - 2}$.

Solution. Since $x^2 - 4 = (x - 2)(x + 2)$, for $x \neq 2$ we have $\frac{x^2-4}{x-2} = x + 2$. Hence the limit equals $\lim_{x\to 2}(x + 2) = 4$. (*removable discontinuity*)

Numerical verification (script)

```
import math


def f(x):
return (x**2 - 4) / (x - 2)


for h in [1e-1, 1e-2, 1e-3, 1e-4]:
print(f"x=2+h -> {f(2+h):.6f} x=2-h -> {f(2-h):.6f}")
# Expected output: both sequences approach 4
```

Listing 1: Two-sided evaluation near $x = 2$

Key takeaways

- The ε–δ definition formalizes the intuitive notion of approach.
- Simplification often eliminates indeterminate forms.
- A limit may exist even when f is undefined at the point.