ЧАС 5

1. Одредити Фуријеову трансформацију Хевисајдове ф-је и ф-је знака.

а) $u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$

$$\frac{du(t)}{dt} = \delta(t) \longleftrightarrow j\omega \cdot \mathcal{F}\{u(t)\}$$

$$\mathcal{F}\{u(t)\} = \frac{1}{j\omega} \cdot \mathcal{F}\left\{\frac{du(t)}{dt}\right\} = \frac{1}{j\omega} \mathcal{F}\{\delta(t)\} = \frac{1}{j\omega}$$

б) $sgn(t) = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases}$

$$\frac{sgn(t)}{2} = u(t) - \frac{1}{2} \Rightarrow \frac{d}{dt}\left\{\frac{sgn(t)}{2}\right\} = \frac{du(t)}{dt} = \delta(t)$$

$$\Rightarrow \mathcal{F}\left\{\frac{sgn(t)}{2}\right\} = \frac{1}{j\omega}$$

$\Rightarrow$ иста деривација $\rightarrow$ иста Фуријеова трансформација

$$u(t) = \frac{sgn(t)}{2} + \frac{1}{2}$$

$$\mathcal{F}\{u(t)\} = \mathcal{F}\left\{\frac{sgn(t)}{2} + \frac{1}{2}\right\} = \frac{1}{j\omega} + \pi\delta(\omega)$$

$\Rightarrow$ $\delta(t) \longleftrightarrow 1$

$1 \longleftrightarrow 2\pi\delta(\omega)$

$\Rightarrow$ За случај апериодичних сигнала, аутокорелациона функција и спектрална густина снаге чине Фуријеов трансформациони пар

2. Одредити Фуријеову трансформацију следећих сигнала:

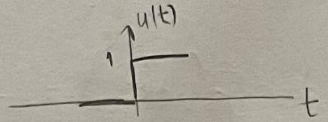
а) $x_a(t) = e^{-at} \cdot u(t)$

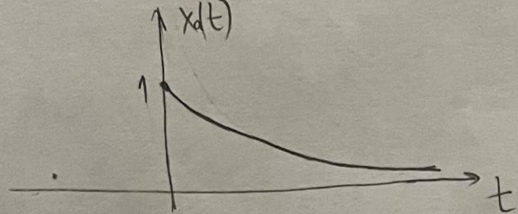
б) $x_b(t) = e^{at} \cdot u(-t)$

в) $x_c(t) = e^{-a\cdot|t|}$

г) $x(t) = sgn(t)$

а)



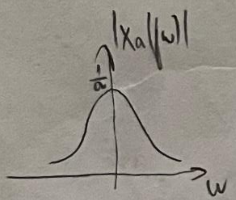


$$X_a(j\omega) = \int_{-\infty}^{+\infty} x_a(t) e^{-j\omega t} dt = \int_{0}^{+\infty} e^{-at} \cdot e^{-j\omega t} dt =$$

$$= \int_0^{+\infty} e^{-(a+j\omega)t} dt = \frac{-1}{a+j\omega} \cdot e^{-(a+j\omega)t}\Big|_0^{+\infty}$$

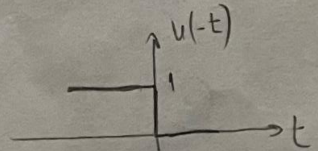
$$= -\frac{0-1}{a+j\omega} = \boxed{\frac{1}{a+j\omega}} \cdot \frac{a-j\omega}{a-j\omega} =$$

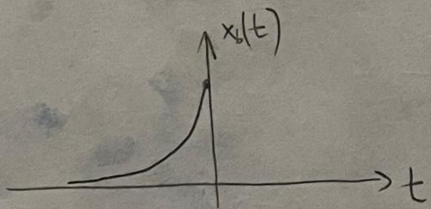
$$\boxed{= \frac{a}{a^2+\omega^2} - j \cdot \frac{\omega}{a^2+\omega^2}}$$



$$|X_a(j\omega)| = \left|\frac{a-j\omega}{a^2+\omega^2}\right| = \frac{1}{a^2+\omega^2} \cdot \sqrt{a^2+\omega^2} = \frac{1}{\sqrt{a^2+\omega^2}}$$

б)





$$X_b(j\omega) = \int_{-\infty}^{+\infty} x_b(t) e^{-j\omega t} dt = \int_{-\infty}^{0} e^{at} \cdot e^{-j\omega t} dt =$$

$$= \int_{-\infty}^{0} e^{(a-j\omega)t} dt = \frac{1}{a-j\omega} \cdot e^{(a-j\omega)t}\Big|_{-\infty}^{0} =$$

$$\boxed{= \frac{1}{a-j\omega}} \cdot \frac{a+j\omega}{a+j\omega} =$$

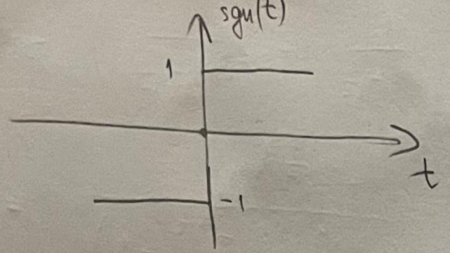
$$\boxed{= \frac{a}{a^2+\omega^2} + j\frac{\omega}{a^2+\omega^2}}$$

$$\boxed{\begin{array}{l} x_b(t) = x_a(-t) \\ \Rightarrow X_b(j\omega) = X_a(-j\omega) \end{array}}$$

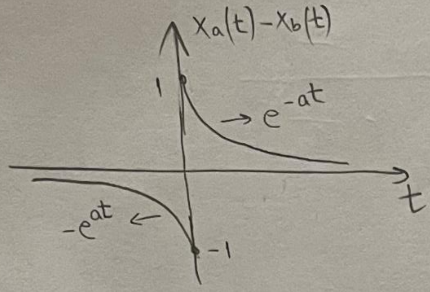
в) $x_c(t) = x_a(t) + x_b(t) \Rightarrow X_c(j\omega) = X_a(j\omega) + X_b(j\omega) = \frac{2a}{a^2+\omega^2} \in \mathbb{R}$

$X_c(j\omega) = X_c(\omega)$ – реална функција од ω

1)



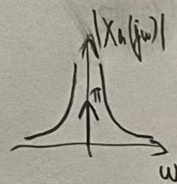
$\Rightarrow$ Није апсолутно интеграбилна $\Rightarrow$ ф. трансформација се не може одредити директно из дефиниционог интеграла



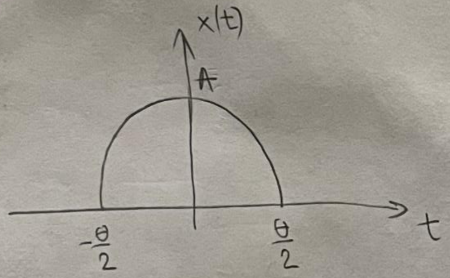
$$sgn(t) = \lim_{a\to 0}\left(x_a(t) - x_b(t)\right)$$

$$sgn(t) \longleftrightarrow \lim_{a\to 0}\left[\frac{1}{a+j\omega} - \frac{1}{a-j\omega}\right] = \frac{2}{j\omega}$$

$$\frac{sgn(t)}{2} = u(t) - \frac{1}{2} \Rightarrow \mathcal{F}\{u(t)\} = \frac{1}{j\omega} + \pi\delta(\omega) \qquad \mathcal{F}\{u(t)\} = \begin{cases} \pi\delta(\omega), & \omega = 0 \\ \frac{1}{j\omega}, & \omega \neq 0 \end{cases}$$



3. Одредити спектралну густину амплитуда сигнала приказаног на слици.



$$x(t) = A\cdot\cos(\beta t), \quad |t| \leq \frac{\theta}{2}$$

$\beta = ?$

$\cos(\beta t) = 0$

$\beta t = \frac{\pi}{2}$

$\beta\frac{\theta}{2} = \frac{\pi}{2} \Rightarrow \beta = \frac{\pi}{\theta}$

$$\Rightarrow x(t) = A\cdot\cos\left(\frac{\pi}{\theta}\cdot t\right), \quad |t| \leq \frac{\theta}{2}$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt = \int_{-\frac{\theta}{2}}^{\frac{\theta}{2}} A\cos\left(\frac{\pi}{\theta}t\right)\cdot e^{-j\omega t}dt = \frac{A}{2}\int_{-\frac{\theta}{2}}^{\frac{\theta}{2}}\left(e^{j\frac{\pi}{\theta}t} + e^{-j\frac{\pi}{\theta}t}\right)\cdot e^{-j\omega t}dt =$$

$$= \frac{A}{2}\left[\int_{-\frac{\theta}{2}}^{\frac{\theta}{2}} e^{-j\left(\omega-\frac{\pi}{\theta}\right)t}dt + \int_{-\frac{\theta}{2}}^{\frac{\theta}{2}} e^{-j\left(\omega+\frac{\pi}{\theta}\right)t}dt\right] =$$

$$= \frac{A}{2}\left[\frac{1}{-j\left(\omega-\frac{\pi}{\theta}\right)}\cdot\left(e^{-j\left(\omega-\frac{\pi}{\theta}\right)\cdot\frac{\theta}{2}} - e^{j\left(\omega-\frac{\pi}{\theta}\right)\cdot\frac{\theta}{2}}\right) - \frac{1}{j\left(\omega+\frac{\pi}{\theta}\right)}\left(e^{-j\left(\omega+\frac{\pi}{\theta}\right)\cdot\frac{\theta}{2}} - e^{j\left(\omega+\frac{\pi}{\theta}\right)\cdot\frac{\theta}{2}}\right)\right]$$

$$= A\cdot\left(\frac{\sin\left(\omega-\frac{\pi}{\theta}\right)\cdot\frac{\theta}{2}}{\omega-\frac{\pi}{\theta}} + \frac{\sin\left(\omega+\frac{\pi}{\theta}\right)\cdot\frac{\theta}{2}}{\omega+\frac{\pi}{\theta}}\right) = A\left(\frac{\sin\left(\frac{\omega\theta}{2}-\frac{\pi}{2}\right)}{\omega-\frac{\pi}{\theta}} + \frac{\sin\left(\frac{\omega\theta}{2}+\frac{\pi}{2}\right)}{\omega+\frac{\pi}{\theta}}\right) =$$

$$= A\left(\frac{-\cos\left(\frac{\omega\theta}{2}\right)}{\omega-\frac{\pi}{\theta}} + \frac{\cos\left(\frac{\omega\theta}{2}\right)}{\omega+\frac{\pi}{\theta}}\right) = A\cdot\cos\left(\frac{\omega\theta}{2}\right)\left(\frac{1}{\omega+\frac{\pi}{\theta}} - \frac{1}{\omega-\frac{\pi}{\theta}}\right) =$$

$$= A\cdot\cos\left(\frac{\omega\theta}{2}\right)\cdot\frac{\left(-2\frac{\pi}{\theta}\right)}{\omega^2-\left(\frac{\pi}{\theta}\right)^2} = A\cdot\cos\left(\frac{\omega\theta}{2}\right)\cdot\frac{\left(-2\frac{\pi}{\theta}\right)}{\left(\frac{\pi}{\theta}\right)^2\left(\left(\frac{\omega\theta}{\pi}\right)^2-1\right)} =$$

$$= \frac{2A\cdot\cancel{\frac{\pi}{\theta}}}{\left(\frac{\pi}{\theta}\right)^2}\cdot\frac{\cos\left(\frac{\omega\theta}{2}\right)}{1-\left(\frac{\omega\theta}{\pi}\right)^2} = \frac{2A\theta}{\pi}\cdot\frac{\cos\left(\frac{\omega\theta}{2}\right)}{1-\left(\frac{\omega\theta}{\pi}\right)^2}$$

Нуле: $\frac{\omega\theta}{2} = \frac{\pi}{2} + k\pi \quad , k\in\mathbb{Z}$

$$\omega = \frac{\pi}{\theta} + k\cdot\frac{2\pi}{\theta}$$

$$f = \frac{1}{2\theta} + k\cdot\frac{1}{\theta}$$

али за $k=0$ имамо лажну нулу!

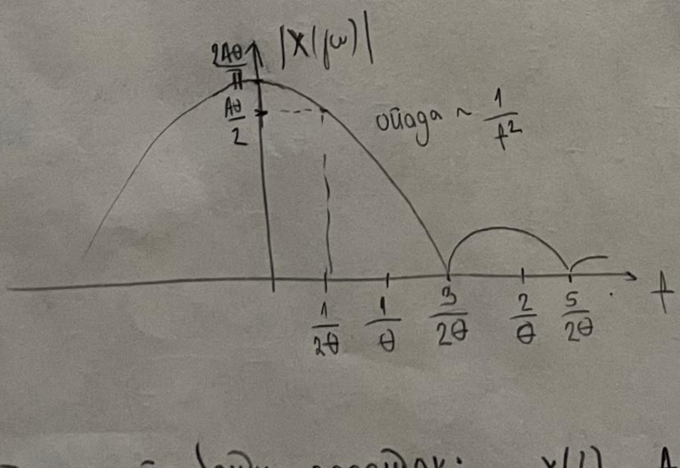
односно за $\omega = \frac{\pi}{\theta}$

$$\frac{\cos\left(\frac{\omega\theta}{2}\right)}{1-\left(\frac{\omega\theta}{\pi}\right)^2} = \frac{\cos\left(\frac{\pi}{2}\right)}{1-1} = \frac{0}{0} \;!$$

$$\lim_{\omega\to\frac{\pi}{\theta}} \frac{2A\theta}{\pi}\cdot\frac{\cos\left(\frac{\omega\theta}{2}\right)}{1-\left(\frac{\omega\theta}{\pi}\right)^2} \underset{\text{Л.П.}}{=} \lim_{\omega\to\frac{\pi}{\theta}} \frac{\cancel{2}A\cancel{\theta}}{\cancel{\pi}}\cdot\frac{\cancel{-}\sin\left(\frac{\omega\theta}{2}\right)\cdot\frac{\cancel{\theta}}{2}}{\cancel{-}\cancel{2}\cdot\frac{\omega\cancel{\theta}}{\cancel{\pi}}\cdot\frac{\cancel{\theta}}{\pi}} = \lim_{\omega\to\frac{\pi}{\theta}} \frac{A\cdot\pi}{2\omega}\cdot\sin\left(\frac{\omega\theta}{2}\right) =$$

$$\boxed{= \frac{A\theta}{2}}$$

$\Rightarrow$ Нуле: $\omega = \frac{\pi}{\theta} + k\cdot\frac{2\pi}{\theta}, \quad k\in\mathbb{Z}\setminus\{0\}$



* самостално пробати задатак: $x(t) = A\cdot\cos^2\left(\frac{\pi}{\theta}\cdot t\right), \quad |t|\le\frac{\theta}{2}$