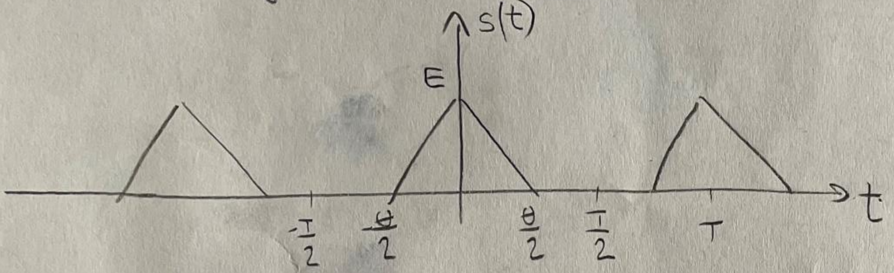
ЧАС 2 – 3

1.) Развити у Фуријеов ред поворку троугаоних импулса.



$$s(t)=\begin{cases}0, & -\frac{T}{2}\le t<-\frac{\theta}{2}\\ \frac{2E}{\theta}t+E, & -\frac{\theta}{2}\le t\le 0\\ -\frac{2E}{\theta}t+E, & 0\le t\le\frac{\theta}{2}\\ 0, & \frac{\theta}{2}<t\le\frac{T}{2}\end{cases}$$

$a_n \neq 0$, $b_n = 0$

$$s(t)=\frac{a_0}{2}+\sum_{n=1}^{+\infty} a_n\cdot\cos(n\omega_0 t)$$

$$a_n=\frac{2}{T}\int_T s(t)\cos(n\omega_0 t)dt=\frac{4}{T}\int_0^{\frac{\theta}{2}}\left(-\frac{2E}{\theta}t+E\right)\cdot\cos(n\omega_0 t)dt=$$

$$=\frac{4}{T}\left[\int_0^{\frac{\theta}{2}}-\frac{2E}{\theta}\cdot t\cdot\cos(n\omega_0 t)dt+\int_0^{\frac{\theta}{2}}E\cdot\cos(n\omega_0 t)dt\right]=$$

$$=\frac{4E}{T}\left[-\frac{2}{\theta}\int_0^{\frac{\theta}{2}}t\cdot\cos(n\omega_0 t)dt+\int_0^{\frac{\theta}{2}}\cos(n\omega_0 t)dt\right]$$

$$I_1=\int_0^{\frac{\theta}{2}}t\cdot\cos(n\omega_0 t)dt=\frac{t}{n\omega_0}\cdot\sin(n\omega_0 t)\Big|_0^{\frac{\theta}{2}}-\frac{1}{n\omega_0}\int_0^{\frac{\theta}{2}}\sin(n\omega_0 t)dt=$$

$$\left|\begin{array}{ll}u=t & dv=\cos(n\omega_0 t)dt\\ du=dt & v=\frac{1}{n\omega_0}\cdot\sin(n\omega_0 t)\end{array}\right|$$

$$=\frac{t}{n\omega_0}\cdot\sin(n\omega_0 t)\Big|_0^{\frac{\theta}{2}}+\frac{1}{(n\omega_0)^2}\cdot\cos(n\omega_0 t)\Big|_0^{\frac{\theta}{2}}=$$

$$=\frac{\theta}{2n\omega_0}\cdot\sin\left(n\omega_0\frac{\theta}{2}\right)+\frac{1}{(n\omega_0)^2}\cos\left(n\omega_0\frac{\theta}{2}\right)-\frac{1}{(n\omega_0)^2}$$

$$I_2=\int_0^{\frac{\theta}{2}}\cos(n\omega_0 t)dt=\frac{1}{n\omega_0}\cdot\sin(n\omega_0 t)\Big|_0^{\frac{\theta}{2}}=\frac{1}{n\omega_0}\cdot\sin\left(n\omega_0\frac{\theta}{2}\right)$$

$$a_n=\frac{4E}{T}\left[\cancel{\frac{-1}{n\omega_0}\sin\left(n\omega_0\frac{\theta}{2}\right)}-\frac{2}{\theta}\cdot\frac{1}{(n\omega_0)^2}\left(\cos\left(n\omega_0\frac{\theta}{2}\right)-1\right)+\cancel{\frac{1}{n\omega_0}\cdot\sin\left(n\omega_0\frac{\theta}{2}\right)}\right]=$$

$$=\frac{4E}{T}\cdot\frac{2}{\theta}\cdot\frac{1}{(n\omega_0)^2}\cdot\left(1-\cos\left(n\omega_0\frac{\theta}{2}\right)\right)\quad\Big/\cdot\frac{2}{2}$$

$$\boxed{\sin^2\frac{x}{2} = \frac{1-\cos x}{2}} \qquad \boxed{\frac{\sin^2(x)}{x^2} = \operatorname{sinc}^2(x)}$$

$$\Rightarrow a_n = \frac{4E}{T}\cdot\frac{4}{\theta}\cdot\frac{1}{(n\omega_0)^2}\cdot\frac{1-\cos\left(n\omega_0\frac{\theta}{2}\right)}{2} = \frac{16E}{\theta\cdot T}\cdot\frac{1}{(n\omega_0)^2}\cdot\sin^2\left(n\omega_0\frac{\theta}{4}\right) =$$

$$= \frac{16E\cdot\sin^2\left(n\frac{2\pi}{T}\cdot\frac{\theta}{4}\right)}{\theta\cdot \cancel{T}\cdot n^2\cdot\frac{4\pi^2}{T^{\cancel{2}}}} = \frac{4E\cdot\sin^2\left(n\pi\frac{L}{2}\right)}{n^2\pi^2\cdot L\cdot\frac{L}{L}\cdot\frac{4}{4}} = EL\cdot\frac{\sin^2\left(n\pi\frac{L}{2}\right)}{\left(n\pi\frac{L}{2}\right)^2} =$$

$$\boxed{= EL\cdot\operatorname{sinc}^2\left(n\frac{L}{2}\right)}$$

$$\boxed{F_n = \frac{EL}{2}\cdot\operatorname{sinc}^2\left(n\frac{L}{2}\right)}$$

Нуле:

$$\frac{n\pi L}{2} = k\pi$$

$$\frac{n\pi\theta}{2T} = k\pi$$

$$n\omega_0\frac{\theta}{4} = k\pi$$

$k=1: \Rightarrow n\omega_0 = \frac{4\pi}{\theta} \quad \left(nf_0 = \frac{2}{\theta}\right)$

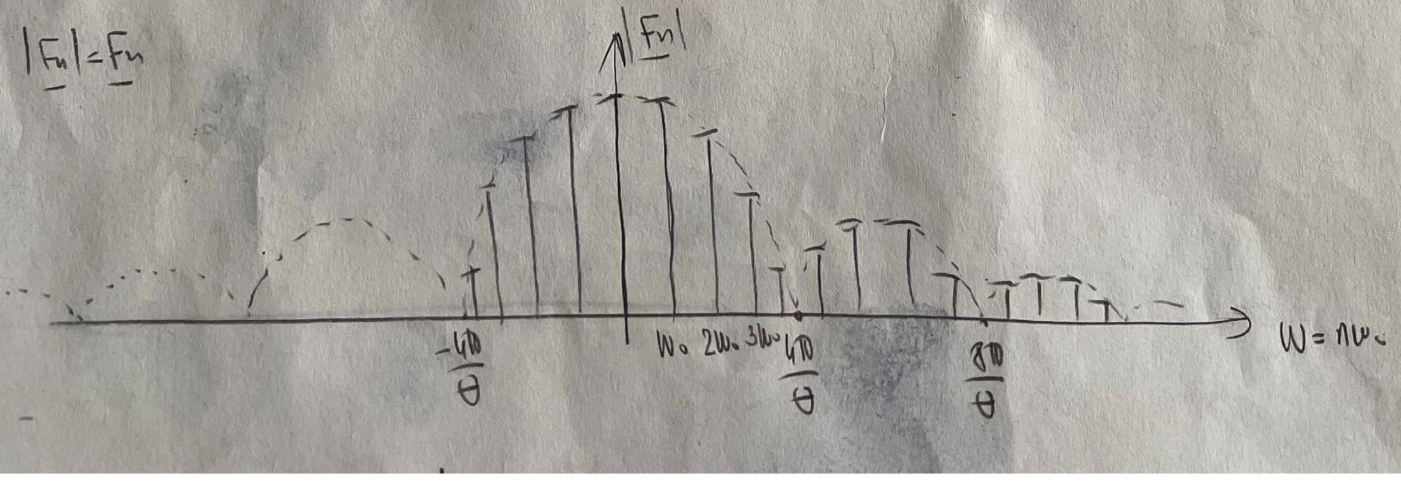
$k=2 \Rightarrow n\omega_0 = \frac{8\pi}{\theta} \quad \left(nf_0 = \frac{4}{\theta}\right)$

$$n\omega_0 = \omega = \frac{4\pi}{\theta}\cdot k = \frac{4\pi}{T}\cdot\frac{T}{\theta}\cdot k = \boxed{2\omega_0\cdot\frac{1}{L}\cdot k}$$

Број хармоника до прве нуле у спектру: $\frac{\frac{2}{\theta}}{f_0} = \frac{\frac{2}{\theta}}{\frac{1}{T}} = \frac{2T}{\theta} = \boxed{\frac{2}{L}}$

$n = \left\lfloor \frac{2}{L} \right\rfloor$ ↳ заокружено на доњи цели део $\left(\begin{array}{l}\lfloor 1{,}2\rfloor = 1\\ \lfloor 2{,}2\rfloor = 2\end{array}\right)$

$|F_n| = F_n$



$$P = \frac{1}{T}\int_T s^2(t)\,dt = \frac{2}{T}\int_0^{\frac{T}{2}} s^2(t)\,dt = \frac{2}{T}\int_0^{\frac{\theta}{2}}\left(-\frac{2E}{\theta}t + E\right)^2 dt =$$

$$= \frac{2}{T}\int_0^{\frac{\theta}{2}}\left(E^2 - \frac{4E^2}{\theta}t + \frac{4E^2}{\theta^2}t^2\right)dt =$$

$$= \frac{2}{T}\left(E^2\cdot\frac{\theta}{2} - \frac{\cancel{2}E^2}{\cancel{\theta}}\cdot\frac{\theta^{\cancel{2}}}{\cancel{4}} + \frac{\cancel{4}E^2}{3\cancel{\theta^2}}\cdot\frac{\cancel{\theta^3}}{\cancel{8}_2}\right) = \frac{2}{T}\left(E^2\frac{\theta}{2} - \frac{E^2\theta}{2} + \frac{E^2\theta}{6}\right) =$$

$$= \frac{E^2}{3}\cdot\frac{\theta}{T} = \boxed{\frac{E^2}{3}\cdot\alpha}$$

Hip. $E = 1$
$\alpha = \frac{1}{2}$ $\Rightarrow \boxed{P = \frac{1}{6}}\ [V^2] = 0{,}167\ [V^2]$

$N = \frac{2}{\alpha} = \underline{\underline{4}}$

$|\underline{F_0}| = \frac{1}{4}$

$|\underline{F_1}| = \frac{1}{\cancel{4}}\cdot\frac{\sin^2\left(\frac{\pi}{4}\right)}{\frac{\pi^2}{\cancel{16}_4}} = \frac{\frac{1}{\cancel{2}}}{\frac{\pi^2}{\cancel{4}_2}} = \frac{2}{\pi^2}$

$|\underline{F_2}| = \frac{1}{\cancel{4}}\cdot\frac{\sin^2\left(\frac{\pi}{2}\right)}{\frac{\pi^2}{\cancel{4}}} = \frac{1}{\pi^2}$

$|\underline{F_3}| = \frac{1}{\cancel{4}}\cdot\frac{\sin^2\left(\frac{3\pi}{4}\right)}{\frac{9\pi^2}{\cancel{16}_4}} = \frac{\frac{2}{\cancel{4}}}{\frac{9\pi^2}{\cancel{4}}} = \frac{2}{9\pi^2}$

$|\underline{F_4}| = \frac{1}{4}\cdot\frac{\sin^2(\pi)}{\pi^2} = 0$

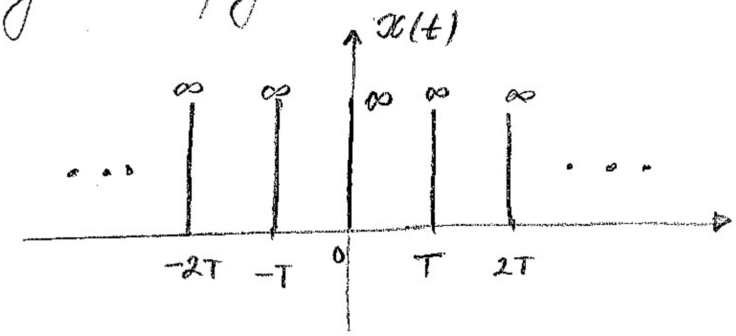
$$P' = \left(\frac{1}{4}\right)^2 + 2\cdot\left(\left(\frac{2}{\pi^2}\right)^2 + \left(\frac{1}{\pi^2}\right)^2 + \left(\frac{2}{9\pi^2}\right)^2\right) = \frac{1}{16} + 2\cdot\left(\frac{4}{\pi^4} + \frac{1}{\pi^4} + \frac{4}{81\pi^4}\right) =$$

$$= \frac{1}{16} + 2\frac{409}{81\pi^4} = \frac{1}{16} + \frac{818}{81\pi^4} = 0{,}166\ [V^2]$$

$\eta = \frac{P'}{P} \cong \boxed{99{,}6\ \%}$

③ Извршити спектралну анализу (одредити ампл. и фазни спектар) сигнала у виду периодичне делта ф-је.
2.T.

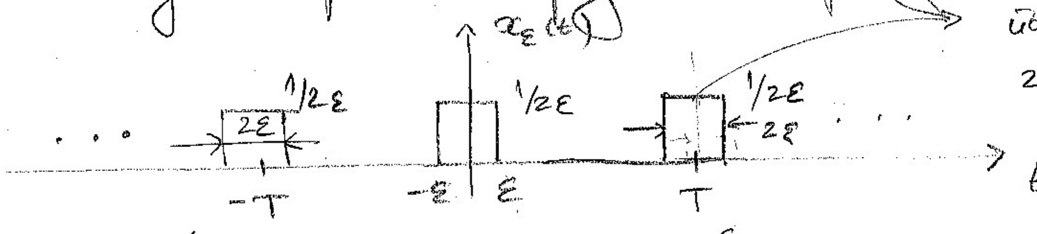
$$x(t) = \sum_{n=-\infty}^{+\infty} \delta(t-nT)$$



- периодична поворка Диракових импулса
- Површина импулса је 1
(Амплитуда бесконачно велика, а ширина импулса бесконачно мала)

периодичан сигнал $\longrightarrow$ Фуријеов ред $X_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jn\omega_0 t}\, dt$

Како? Подимо од поворке правоугаоних импулса кратког трајања периоде T



површина импулса $2\varepsilon \cdot 1/2\varepsilon = 1$!

$$X_n^{(\varepsilon)} = \frac{1}{T}\int_{-T/2}^{T/2} x_\varepsilon(t)\, e^{-jn\omega_0 t}\, dt = \frac{1}{T}\int_{-\varepsilon}^{\varepsilon} \frac{1}{2\varepsilon}\, e^{-jn\omega_0 t}\, dt =$$

$$= \frac{1}{2\varepsilon T}\cdot \left.\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right|_{-\varepsilon}^{\varepsilon} = \frac{1}{\varepsilon T n\omega_0}\cdot\frac{\sin(n\omega_0\varepsilon)}{1}$$

$$= \frac{1}{T}\,\frac{\sin(n\omega_0\varepsilon)}{(n\omega_0\varepsilon)} = \frac{1}{T}\,\mathrm{sinc}(n\omega_0\varepsilon)$$

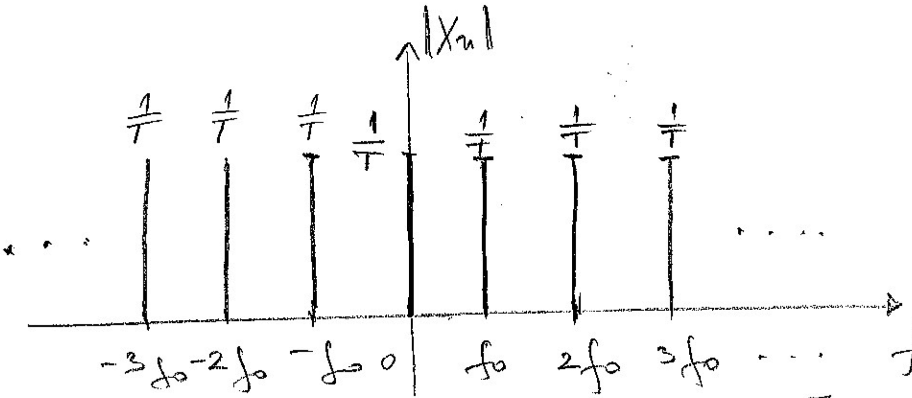
Када $\varepsilon \to 0 \Rightarrow x_\varepsilon(t) \longrightarrow x(t)$, $X_n^\varepsilon \to \frac{1}{T}$

Имамо да је $X_n = \lim_{\varepsilon\to 0} X_n^\varepsilon = \frac{1}{T}$

$$\boxed{x(t) = \sum_{n=-\infty}^{+\infty} \frac{1}{T}\, e^{jn\omega_0 t}}$$

$x(t)$ парна ф-ја спектар је реалан и парна ф-ја учестаности

$X_n = |X_n|$



$X_n = |X_n|$ $\quad n = 0, \pm 1, \pm 2, \dots$

$\theta_n = 0$

сви хармоници имају једнаке амплитуде а фазни помаци су нула

математички спектар

Како гласи физички спектар?
(само за позитивне фрекв.)

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t = \frac{a_0}{2} + \sum_{n=1}^{+\infty} C_n \cos(n\omega_0 t + \theta_n)$$

пошто је $x(t)$ парна ф-ја $\Rightarrow b_n = 0$

$C_n = 2 \cdot |X_n|$

$C_n = \frac{2}{T}$

$\theta_n = 0$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \cdot \cos n\omega_0 t \, dt \quad , \; n = 0, 1, 2, \dots$$

$$a_0 = \underbrace{\frac{2}{T} \int_{-T/2}^{T/2} x(t) \, dt}_{\text{површина испод импулса}} = \frac{2}{T}$$

$$a_n^{\varepsilon} = \frac{2}{T} \cdot \frac{1}{2\varepsilon} \cdot \frac{\sin n\omega_0 t}{n\omega_0} \Bigg|_{-\varepsilon}^{\varepsilon} = \frac{2}{T} \cdot \frac{1}{2\varepsilon} \cdot 2 \; \frac{\sin n\omega_0 t}{n\omega_0} \Bigg|_{0}^{\varepsilon}$$

$$= \frac{2}{T} \cdot \frac{\sin n\omega_0 \varepsilon}{n\omega_0 \varepsilon}$$

$$\varepsilon \to 0 \quad \Rightarrow a_n^{\varepsilon} \to \frac{2}{T} \quad , \; a_n = \frac{2}{T} \quad , \; n = 1, 2, \dots$$

$$\hat{x}(t) = \frac{1}{T} + \sum_{n=1}^{+\infty} \frac{2}{T} \cdot \cos n\omega_0 t = \frac{1}{T} \cdot \left(1 + 2 \cdot \cos \omega_0 t + 2 \cdot \cos 2\omega_0 t + \cdots \right)$$

Питати студенте