TERMIN 4 - zadaci za samostalan rad - rješenja

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Zadatak 1.
Dopuniti skup vektora $\{(1,1,1,2),(1,2,3,-3)\}$ do ortogonalne baze vektorskog prostora $\mathbb{R}^4$.

Rješenje

Neka je $V = Lin\{(1,1,1,2),(1,2,3,-3)\}$. Kako je prostor kolona matrice A ortogonalan na nula prostor matrice A^T, iz stepenaste forme

$$\left[\begin{array}{cc|cccc} \boxed{1} & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 1 & 0 \\ 2 & -3 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1\cdot(-2)+R_4]{\substack{R_1\cdot(-1)+R_2 \\ R_1\cdot(-1)+R_3}} \left[\begin{array}{cc|cccc} \boxed{1} & 1 & 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & -1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & -5 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2\cdot 5+R_4]{R_2\cdot(-2)+R_3} \left[\begin{array}{cc|cccc} \boxed{1} & 1 & 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & -7 & 5 & 0 & 1 \end{array}\right]$$

dobijamo $V^{\perp} = Lin\{(1,-2,1,0),(-7,5,0,1)\}$.
Da bismo skup $\{(1,1,1,2),(1,2,3,-3)\}$ doveli do ortogonalne baze prostora $\mathbb{R}^4$, potrebno je da bazni vektori prostora V i $V^{\perp}$ budu međusobno ortogonalni. Za ortogonalizaciju tih vektora iskoristićemo Gram-Šmitov postupak. Imamo da je:

$$\begin{aligned}
\overrightarrow{y_1} &= \overrightarrow{x_1}, \\
\overrightarrow{y_2} &= \overrightarrow{x_2} - \frac{(\overrightarrow{x_2},\overrightarrow{y_1})}{(\overrightarrow{y_1},\overrightarrow{y_1})}\cdot\overrightarrow{y_1}, \\
\overrightarrow{y_3} &= \overrightarrow{x_3}, \\
\overrightarrow{y_4} &= \overrightarrow{x_4} - \frac{(\overrightarrow{x_4},\overrightarrow{y_3})}{(\overrightarrow{y_3},\overrightarrow{y_3})}\cdot\overrightarrow{y_3},
\end{aligned}$$

odakle dobijamo

$$\begin{aligned}
\overrightarrow{y_1} &= (1,1,1,2), \\
\overrightarrow{y_2} &= (1,2,3,-3) - \frac{0}{7}\cdot(1,1,1,2) \\
&= (1,2,3,-3), \\
\overrightarrow{y_3} &= (1,-2,1,0), \\
\overrightarrow{y_4} &= (-7,5,0,1) - \frac{(-17)}{6}\cdot(1,-2,1,0) \\
&= \left(-\frac{25}{6},-\frac{4}{6},\frac{17}{6},1\right) \\
&= \frac{1}{6}\cdot(-25,-4,17,6).
\end{aligned}$$

Dakle, jedna od ortogonalnih baza prostora R^4 je

$$B_{\mathbb{R}^4} = \{(1,1,1,2),(1,2,3,-3),(1,-2,1,0),(-25,-4,17,6)\}.$$

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Zadatak 2.
Neka je

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 3 \\ -1 & 1 \\ 0 & 0 \\ -1 & 1 \end{bmatrix}.$$

Odrediti baze fundamentalnih potprostora matrice A pa provjeriti da li su ispunjeni odgovarajući uslovi njihove ortogonalnosti.

Rješenje

Iz proširene stepenaste forme matrice A:

$$\left[\begin{array}{cc|ccccc} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|ccccc} \boxed{1} & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1+R_5]{R_1+R_3} \left[\begin{array}{cc|ccccc} \boxed{1} & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 4 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2\cdot(-4)+R_5]{R_2\cdot(-4)+R_3} \left[\begin{array}{cc|ccccc} \boxed{1} & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -4 & 1 & 0 & 0 & 1 \end{array}\right]$$

dobijamo

$$B_{C(A)} = \left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}, \qquad B_{C\left(A^T\right)} = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^2, \qquad B_{N\left(A^T\right)} = \left\{ \begin{bmatrix} -4 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Kako je $\mathcal{A} : \mathbb{R}^2 \to \mathbb{R}^5$, zaključujemo da je

$$dim\left(C\left(A\right)\right) + dim\left(N\left(A\right)\right) = 2.$$

Kako je $dim\left(C\left(A\right)\right) = 2$, zaključujemo da je $dim\left(N\left(A\right)\right) = 0$ pa je

$$B_{N(A)} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}.$$

Neka je

$$\overrightarrow{x_1} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \overrightarrow{x_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \overrightarrow{x_3} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{i} \quad \overrightarrow{y_1} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \overrightarrow{y_2} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \overrightarrow{y_3} = \begin{bmatrix} -4 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \overrightarrow{y_4} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \overrightarrow{y_5} = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Kako vrijedi

$$\left(\overrightarrow{x_1}, \overrightarrow{x_3}\right) = 0 \quad \text{i} \quad \left(\overrightarrow{x_2}, \overrightarrow{x_3}\right) = 0$$

zaključujemo da je $C\left(A^T\right) \perp N\left(A\right)$.
Sa druge strane, kako vrijedi

$$\left(\overrightarrow{y_1}, \overrightarrow{y_3}\right) = 0, \quad \left(\overrightarrow{y_1}, \overrightarrow{y_4}\right) = 0, \quad \left(\overrightarrow{y_1}, \overrightarrow{y_5}\right) = 0, \quad \left(\overrightarrow{y_2}, \overrightarrow{y_3}\right) = 0, \quad \left(\overrightarrow{y_2}, \overrightarrow{y_4}\right) = 0 \text{ i } \left(\overrightarrow{y_2}, \overrightarrow{y_5}\right) = 0$$

zaključujemo da je $C\left(A\right) \perp N\left(A^T\right)$.
Ovim smo pokazali ortogonalnost fundamentalnih potprostora matrice A.

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Zadatak 3.
Neka je

$$A = \begin{bmatrix} 1 & -2 & a \\ -2 & 1 & b \\ -2 & -2 & c \end{bmatrix}.$$

Odrediti $a, b, c \in \mathbb{R}$ tako da matrica A ima ortogonalne kolone.

Rješenje

Kako su kolone $A_{\bullet 1}$ i $A_{\bullet 2}$ ortogonalne jer je $A_{\bullet 1}^T \cdot A_{\bullet 2} = 0$, matrica A će imati ortogonalne kolone ako vrijedi $A_{\bullet 1}^T \cdot A_{\bullet 3} = 0$ i $A_{\bullet 2}^T \cdot A_{\bullet 3} = 0$, tj. ako je

$$\begin{cases} a - 2b - 2c = 0 \\ -2a + b - 2c = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} a - 2b = 2c \\ -2a + b = 2c \end{cases}$$

čije je rješenje $a = -2c$ i $b = -2c$. Dakle, da bi matrica A imala ortogonalno rješenje, potrebno je da vrijedi

$$(a, b, c) = (-2c, -2c, c).$$

Odabiranjem proizvoljne vrijednosti parametra c dobijamo jedno rješenje sistema. Npr. za $c = -1$ dobijamo da je

$$(a, b, c) = (2, 2, -1).$$

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Zadatak 4.
Odrediti jednu ortonormiranu bazu prostora kolona matrice

$$A=\begin{bmatrix}1&3&8\\1&3&0\\1&-1&0\\1&-1&0\end{bmatrix}$$

a onda odrediti ortogonalnu dopunu od $C(A)$.

Rješenje

Iz proširene stepenaste forme matrice A:

$$\left[\begin{array}{ccc|cccc}\boxed{1}&3&8&1&0&0&0\\1&3&0&0&1&0&0\\1&-1&0&0&0&1&0\\1&-1&0&0&0&0&1\end{array}\right]\xrightarrow[R_1\cdot(-1)+R_4]{\substack{R_1\cdot(-1)+R_2\\R_1\cdot(-1)+R_3}}\left[\begin{array}{ccc|cccc}\boxed{1}&3&8&1&0&0&0\\0&0&-8&-1&1&0&0\\0&-4&-8&-1&0&1&0\\0&-4&-8&-1&0&0&1\end{array}\right]\xrightarrow{R_2\leftrightarrow R_3}\left[\begin{array}{ccc|cccc}\boxed{1}&3&8&1&0&0&0\\0&\boxed{-4}&-8&-1&0&1&0\\0&0&-8&-1&1&0&0\\0&-4&-8&-1&0&0&1\end{array}\right]$$

$$\xrightarrow{R_2\cdot(-1)+R_4}\left[\begin{array}{ccc|cccc}\boxed{1}&3&8&1&0&0&0\\0&\boxed{-4}&-8&-1&0&1&0\\0&0&\boxed{-8}&-1&1&0&0\\0&0&0&0&0&-1&1\end{array}\right]$$

dobijamo da je

$$C(A)=\{\overrightarrow{x_1},\overrightarrow{x_2},\overrightarrow{x_3}\}$$

pri čemu je

$$\overrightarrow{x_1}=\begin{bmatrix}1&1&1&1\end{bmatrix}^T,\ \overrightarrow{x_2}=\begin{bmatrix}3&3&-1&-1\end{bmatrix}^T,\ \overrightarrow{x_3}=\begin{bmatrix}8&0&0&0\end{bmatrix}^T.$$

Koristeći Gram-Šmitov postupak ortogonalizacije, ortogonalne vektore $\overrightarrow{y_1}$, $\overrightarrow{y_2}$ i $\overrightarrow{y_3}$ dobijamo iz formula:

$$\begin{aligned}\overrightarrow{y_1}&=\overrightarrow{x_1},\\ \overrightarrow{y_2}&=\overrightarrow{x_2}-\frac{(\overrightarrow{x_2},\overrightarrow{y_1})}{(\overrightarrow{y_1},\overrightarrow{y_1})}\cdot\overrightarrow{y_1},\\ \overrightarrow{y_3}&=\overrightarrow{x_3}-\frac{(\overrightarrow{x_3},\overrightarrow{y_1})}{(\overrightarrow{y_1},\overrightarrow{y_1})}\cdot\overrightarrow{y_1}-\frac{(\overrightarrow{x_3},\overrightarrow{y_2})}{(\overrightarrow{y_2},\overrightarrow{y_2})}\cdot\overrightarrow{y_2}.\end{aligned}$$

Računanjem dobijamo

$$\begin{aligned}\overrightarrow{y_1}&=\begin{bmatrix}1\\1\\1\\1\end{bmatrix},\\ \overrightarrow{y_2}&=\begin{bmatrix}3\\3\\-1\\-1\end{bmatrix}-\frac{4}{4}\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=\begin{bmatrix}2\\2\\-2\\-2\end{bmatrix},\\ \overrightarrow{y_3}&=\begin{bmatrix}8\\0\\0\\0\end{bmatrix}-\frac{8}{4}\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix}-\frac{16}{16}\cdot\begin{bmatrix}2\\2\\-2\\-2\end{bmatrix}=\begin{bmatrix}4\\-4\\0\\0\end{bmatrix}.\end{aligned}$$

Kako je

$$\|\overrightarrow{y_1}\|=2,\ \ \|\overrightarrow{y_2}\|=4\ \text{ i }\ \|\overrightarrow{y_3}\|=4\sqrt{2},$$

tražena ortonormirana baza prostora $C(A)$ je

$$B_{C(A)}=\left\{\frac{1}{2}\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix},\frac{1}{4}\cdot\begin{bmatrix}2\\2\\-2\\-2\end{bmatrix},\frac{1}{4\sqrt{2}}\cdot\begin{bmatrix}4\\-4\\0\\0\end{bmatrix}\right\}$$

odnosno

$$B_{C(A)}=\left\{\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\end{bmatrix},\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\end{bmatrix},\begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\\0\\0\end{bmatrix}\right\}.$$

Kako je $C(A)\perp N\left(A^T\right)$, iz stepenaste forme matrice A dobijamo da je ortogonalna dopuna prostora $C(A)$:

$$N\left(A^T\right)=Lin\left\{\begin{bmatrix}0\\0\\-1\\1\end{bmatrix}\right\}.$$

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Zadatak 5.

Da li postoji ortogonalna matrica $Q \in \mathcal{M}_3$ čija je prva kolona $\overrightarrow{q_1} = \begin{bmatrix} \frac{3}{5} & 0 & -\frac{4}{5} \end{bmatrix}^T$? Obrazložiti odgovor.

Rješenje

Neka su $\overrightarrow{q_1}, \overrightarrow{q_2}, \overrightarrow{q_3}$ kolone matrice Q. Da bi matrica Q bila ortogonalna, potrebno je da vrijedi

$$\overrightarrow{q_1} \perp \overrightarrow{q_2}, \ \ \overrightarrow{q_1} \perp \overrightarrow{q_3} \ \text{ i } \ \overrightarrow{q_2} \perp \overrightarrow{q_3}.$$

Neka je

$$\overrightarrow{q_2} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}.$$

Kako je $\overrightarrow{q_1} \perp \overrightarrow{q_2}$ imamo da je $\left(\overrightarrow{q_1}, \overrightarrow{q_2}\right) = 0$ odnosno

$$\frac{3}{5} \cdot x_2 + 0 \cdot y_2 - \frac{4}{5} \cdot z_2 = 0.$$

Odaberimo proizvoljan vektor $\overrightarrow{q_2}$ tako da vrijedi prethodna jednakost. Neka je

$$\overrightarrow{q_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Kako je $\overrightarrow{q_1} \perp \overrightarrow{q_3}$ i $\overrightarrow{q_2} \perp \overrightarrow{q_3}$, vrijedi $\left(\overrightarrow{q_1}, \overrightarrow{q_3}\right) = 0$ i $\left(\overrightarrow{q_2}, \overrightarrow{q_3}\right) = 0$. Ako je

$$\overrightarrow{q_3} = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$

imamo da je

$$\begin{aligned} \frac{3}{5} \cdot x_3 + 0 \cdot y_2 - \frac{4}{5} \cdot z_2 = 0 \\ 0 \cdot x_3 + 1 \cdot y_3 + 0 \cdot z_3 = 0. \end{aligned}$$

Iz prethodnog sistema dobijamo $y_3 = 0$ i $x_3 = \dfrac{4}{3} z_3$.
Izborom prozivoljne vrijednosti z_3 dobijamo vektor $\overrightarrow{q_3}$ koji je ortogonalan na vektore $\overrightarrow{q_1}$ i $\overrightarrow{q_2}$. Npr. za $z_3 = 1$ dobijamo

$$\overrightarrow{q_3} = \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}.$$

Konačno, matrica

$$Q = \begin{bmatrix} \frac{3}{5} & 0 & \frac{4}{3} \\ 0 & 1 & 0 \\ -\frac{4}{5} & 0 & 1 \end{bmatrix}$$

je jedna ortogonalna matrica čija je prva kolona $\overrightarrow{q_1}$. Važno je napomenuti da ovakvih matrica ima beskonačno mnogo, a gore navedena matrica Q je samo jedna od njih.

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Zadatak 6.

Neka je

$$V = Lin\left\{\begin{bmatrix}0\\2\\3\\4\end{bmatrix}, \begin{bmatrix}1\\3\\4\\5\end{bmatrix}\right\}.$$

Odrediti ortonormiranu bazu prostora $V^{\perp}$.

Rješenje

Ako bazne vektore prostora V predstavimo kao kolone matrice A, koristeći činjenicu da je $C(A) \perp N\left(A^T\right)$ iz stepenaste forme proširene matrice A

$$\left[\begin{array}{cc|cccc}0&1&1&0&0&0\\2&3&0&1&0&0\\3&4&0&0&1&0\\4&5&0&0&0&1\end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|cccc}\boxed{2}&3&0&1&0&0\\0&1&1&0&0&0\\3&4&0&0&1&0\\4&5&0&0&0&1\end{array}\right] \xrightarrow[R_1\cdot(-2)+R_4]{R_1\cdot\left(-\frac{3}{2}\right)+R_3} \left[\begin{array}{cc|cccc}\boxed{2}&3&0&1&0&0\\0&\boxed{1}&1&0&0&0\\0&-\frac{1}{2}&0&-\frac{3}{2}&1&0\\0&-1&0&-2&0&1\end{array}\right] \xrightarrow[R_2+R_4]{R_2\cdot\frac{1}{2}+R_3} \left[\begin{array}{cc|cccc}\boxed{2}&3&0&1&0&0\\0&\boxed{1}&1&0&0&0\\0&0&\frac{1}{2}&-\frac{3}{2}&1&0\\0&0&1&-2&0&1\end{array}\right]$$

dobijamo da je

$$V^{\perp} = Lin\left\{\begin{bmatrix}\frac{1}{2}\\-\frac{3}{2}\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-2\\0\\1\end{bmatrix}\right\}.$$

Da bismo odredili ortogonalnu bazu prostora $V^{\perp}$ koristimo Gram-Šmitov postupak ortogonalizacije:

$$\overrightarrow{y_1} = \begin{bmatrix}\frac{1}{2}\\-\frac{3}{2}\\1\\0\end{bmatrix},$$

$$\overrightarrow{y_2} = \begin{bmatrix}1\\-2\\0\\1\end{bmatrix} - \frac{\begin{bmatrix}1\\-2\\0\\1\end{bmatrix}^T \cdot \begin{bmatrix}\frac{1}{2}\\-\frac{3}{2}\\1\\0\end{bmatrix}}{\begin{bmatrix}\frac{1}{2}\\-\frac{3}{2}\\1\\0\end{bmatrix}^T \cdot \begin{bmatrix}\frac{1}{2}\\-\frac{3}{2}\\1\\0\end{bmatrix}} \cdot \begin{bmatrix}\frac{1}{2}\\-\frac{3}{2}\\1\\0\end{bmatrix} = \begin{bmatrix}1\\-2\\0\\1\end{bmatrix} - \frac{\cancel{7/2}}{\cancel{7/2}} \cdot \begin{bmatrix}\frac{1}{2}\\-\frac{3}{2}\\1\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\-1\\1\end{bmatrix}$$

Ortonormiranjem vektora $\overrightarrow{y_1}$ i $\overrightarrow{y_2}$ dobijamo ortonormiranu bazu prostora $V^{\perp}$:

$$\overrightarrow{q_1} = \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{3}{2}\right)^2 + 1^2 + 0^2}} \cdot \begin{bmatrix}\frac{1}{2}\\-\frac{3}{2}\\1\\0\end{bmatrix} = \frac{2}{\sqrt{14}} \cdot \begin{bmatrix}\frac{1}{2}\\-\frac{3}{2}\\1\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{14}}\\-\frac{3}{\sqrt{14}}\\\frac{2}{\sqrt{14}}\\0\end{bmatrix},$$

$$\overrightarrow{q_2} = \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + (-1)^2 + 1^2}} \cdot \begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\-1\\1\end{bmatrix} = \frac{2}{\sqrt{10}} \cdot \begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\-1\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{10}}\\-\frac{1}{\sqrt{10}}\\-\frac{2}{\sqrt{10}}\\\frac{2}{\sqrt{10}}\end{bmatrix}.$$

Dakle, ortonormirana baza prostora $V^{\perp}$ je

$$B_{V^{\perp}} = \left\{\begin{bmatrix}\frac{1}{\sqrt{14}}\\-\frac{3}{\sqrt{14}}\\\frac{2}{\sqrt{14}}\\0\end{bmatrix}, \begin{bmatrix}\frac{1}{\sqrt{10}}\\-\frac{1}{\sqrt{10}}\\-\frac{2}{\sqrt{10}}\\\frac{2}{\sqrt{10}}\end{bmatrix}\right\}.$$

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Zadatak 7.

Odrediti ortogonalnu projekciju i ortogonalnu komponentu vektora $\overrightarrow{x} = (4,-1,-3,4)$ na potprostor generisan vektorima $\overrightarrow{a} = (1,1,1,1)$, $\overrightarrow{b} = (1,2,2,-1)$ i $\overrightarrow{c} = (1,0,0,3)$.

Rješenje

Na početku, odredimo potprostor $V^{\perp}$ ortogonalan na potprostor V generisan vektorima $\overrightarrow{a}$, $\overrightarrow{b}$ i $\overrightarrow{c}$. Iz stepenaste forme proširene matrice

$$\left[\begin{array}{ccc|cccc} \boxed{1} & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1\cdot(-1)+R_4]{\substack{R_1\cdot(-1)+R_2 \\ R_1\cdot(-1)+R_3}} \left[\begin{array}{ccc|cccc} \boxed{1} & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 & 0 \\ 0 & -2 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2\cdot 2+R_4]{R_2\cdot(-1)+R_3} \left[\begin{array}{ccc|cccc} \boxed{1} & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 2 & 0 & 1 \end{array}\right]$$

vidimo da je vektor $\overrightarrow{c}$ linearno zavisan sa vektorima $\overrightarrow{a}$ i $\overrightarrow{b}$ pa je

$$B_V = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} \right\}$$

dok je

$$B_{V^{\perp}} = \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Vektor $\overrightarrow{x}$ moguće je razložiti kao zbir vektora $\overrightarrow{y} \in V$ i $\overrightarrow{z} \in V^{\perp}$.
Tada je $\overrightarrow{y}$ ortogonalna projekcija vektora $\overrightarrow{x}$ na potprostor V dok je $\overrightarrow{z}$ ortogonalna komponenta vektora $\overrightarrow{x}$. Iz jednačine

$$\overrightarrow{x} = \begin{bmatrix} 4 \\ -1 \\ -3 \\ 4 \end{bmatrix} = \alpha \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \beta \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} + \gamma \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \delta \cdot \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

dobijamo sistem

$$\begin{cases} \alpha + \beta - 3\delta = 4 \\ \alpha + 2\beta - \gamma + 2\delta = -1 \\ \alpha + 2\beta + \gamma = -3 \\ \alpha - \beta + \delta = 4 \end{cases}$$

čije je rješenje $\alpha = 3$, $\beta = -2$, $\gamma = -2$, $\delta = -1$. Odavde je

$$\overrightarrow{y} = \alpha \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \beta \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 5 \end{bmatrix}$$

$$\overrightarrow{z} = \gamma \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \delta \cdot \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix} = -2 \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \\ -1 \end{bmatrix}$$

pa je dakle vektor $\overrightarrow{y} = (1,-1,-1,5)$ ortogonalna projekcija vektora $\overrightarrow{x}$ na potprostor generisan vektorima $\overrightarrow{a}$, $\overrightarrow{b}$ i $\overrightarrow{c}$.
Vektor $\overrightarrow{z} = (3,0,-2,-1)$ je ortogonalna komponenta vektora $\overrightarrow{x}$.

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Zadatak 8.

Odrediti matricu ortogonalnog projektovanja na prostor $Lin\left\{\begin{bmatrix}1\\1\\-1\\0\end{bmatrix},\begin{bmatrix}1\\0\\1\\1\end{bmatrix}\right\}$.

Rješenje

Neka je vektorski prostor $V = Lin\left\{\begin{bmatrix}1\\1\\-1\\0\end{bmatrix},\begin{bmatrix}1\\0\\1\\1\end{bmatrix}\right\}$.

Korištenjem stepenaste forme proširene matrice A u kojoj se po kolonama nalaze bazni vektori vektorskog prostora V:

$$\left[\begin{array}{cc|cccc}\boxed{1}&1&1&0&0&0\\1&0&0&1&0&0\\-1&1&0&0&1&0\\0&1&0&0&0&1\end{array}\right]\xrightarrow[R_1+R_3]{R_1\cdot(-1)+R_2}\left[\begin{array}{cc|cccc}\boxed{1}&1&1&0&0&0\\0&\boxed{-1}&-1&1&0&0\\0&2&1&0&1&0\\0&1&0&0&0&1\end{array}\right]\xrightarrow[R_2+R_4]{R_2\cdot 2+R_3}\left[\begin{array}{cc|cccc}\boxed{1}&1&1&0&0&0\\0&\boxed{-1}&-1&1&0&0\\0&0&-1&2&1&0\\0&1&-1&1&0&1\end{array}\right]$$

dobijamo da je $V^\perp = Lin\left\{\begin{bmatrix}-1\\2\\1\\0\end{bmatrix},\begin{bmatrix}-1\\1\\0\\1\end{bmatrix}\right\}$. Kako je $V + V^\perp = \mathbb{R}^4$, skup

$$\mathcal{B}_N = \left\{\begin{bmatrix}1\\1\\-1\\0\end{bmatrix},\begin{bmatrix}1\\0\\1\\1\end{bmatrix},\begin{bmatrix}-1\\2\\1\\0\end{bmatrix},\begin{bmatrix}-1\\1\\0\\1\end{bmatrix}\right\}$$

je baza prostora $\mathbb{R}^4$. Po bazi $\mathcal{B}_N$, matrica ortogonalnog projektovanja na prostor V je

$$P_{\mathcal{B}_N} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}.$$

Kako je potrebno odrediti matricu ortogonalnog projektovanja na prostor V po standardnoj bazi $\mathcal{B}_S$ prostora $\mathbb{R}^4$ i kako je matrica prelaska

$$S_{\mathcal{B}_N\to\mathcal{B}_S} = \begin{bmatrix}1&1&-1&-1\\1&0&2&1\\-1&1&1&0\\0&1&0&1\end{bmatrix},$$

matricu projektovanja po standardnoj bazi $\mathcal{B}_S$ dobijamo kao:

$$\begin{aligned}P_{\mathcal{B}_S} &= S_{\mathcal{B}_N\to\mathcal{B}_S}\cdot P_{\mathcal{B}_N}\cdot S_{\mathcal{B}_S\to\mathcal{B}_N}\\&= \begin{bmatrix}1&1&-1&-1\\1&0&2&1\\-1&1&1&0\\0&1&0&1\end{bmatrix}\cdot\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\cdot\begin{bmatrix}1&1&-1&-1\\1&0&2&1\\-1&1&1&0\\0&1&0&1\end{bmatrix}^{-1}\\&= \begin{bmatrix}\frac{2}{3}&\frac{1}{3}&0&\frac{1}{3}\\\frac{1}{3}&\frac{1}{3}&-\frac{1}{3}&0\\0&-\frac{1}{3}&\frac{2}{3}&\frac{1}{3}\\\frac{1}{3}&0&\frac{1}{3}&\frac{1}{3}\end{bmatrix}.\end{aligned}$$

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Zadatak 9.
Data je matrica

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & -1 & -1 \end{bmatrix}.$$

Odrediti projekcije vektora $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^4$ na potprostore $N(A)$ i $C\left(A^T\right)$.

Rješenje

Iz stepenaste forme proširene matrice A^T

$$\left[\begin{array}{cccc|cccc} \boxed{1} & 1 & -1 & -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1\cdot(-1)+R_4]{\substack{R_1\cdot(-1)+R_2 \\ R_1\cdot(-1)+R_3}} \left[\begin{array}{cccc|cccc} \boxed{1} & 1 & -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & \boxed{-2} & 2 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -2 & 2 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2\cdot(-1)+R_4} \left[\begin{array}{cccc|cccc} \boxed{1} & 1 & -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & \boxed{-2} & 2 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right]$$

dobijamo da je

$$C\left(A^T\right) = Lin\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}\right\} \text{ i } N(A) = Lin\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right\}.$$

Vektor $\overrightarrow{x} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}$ moguće je razložiti kao zbir vektora $\overrightarrow{y} \in C\left(A^T\right)$ i $\overrightarrow{z} \in N(A)$.

Tada je $\overrightarrow{y}$ ortogonalna projekcija vektora $\overrightarrow{x}$ na potprostor $C\left(A^T\right)$ dok je $\overrightarrow{z}$ ortogonalna projekcija vektora $\overrightarrow{x}$ na potprostor $N(A)$.
Iz jednačine

$$\overrightarrow{x} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \alpha \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \beta \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} + \gamma \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \delta \cdot \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

dobijamo sistem

$$\begin{cases} \alpha + \beta - \gamma & = 1 \\ \alpha - \beta \quad - \delta & = 1 \\ \alpha + \beta + \gamma & = 2 \\ \alpha - \beta \quad + \delta & = 3 \end{cases}$$

čije je rješenje $\alpha = \dfrac{7}{4}$, $\beta = -\dfrac{1}{4}$, $\gamma = \dfrac{1}{2}$, $\delta = 1$. Odavde je

$$\overrightarrow{y} = \alpha \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \beta \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \frac{7}{4} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{4} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 2 \\ \frac{3}{2} \\ 2 \end{bmatrix}$$

$$\overrightarrow{z} = \gamma \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \delta \cdot \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

pa je dakle vektor $\overrightarrow{y} = \begin{bmatrix} \frac{3}{2} \\ 2 \\ \frac{3}{2} \\ 2 \end{bmatrix}$ projekcija vektora $\overrightarrow{x}$ na potprostor $C\left(A^T\right)$.

Vektor $\overrightarrow{z} = \begin{bmatrix} -\frac{1}{2} \\ -1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$ je projekcija vektora $\overrightarrow{x}$ na potprostor $N(A)$.

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Zadatak 10.
Neka je $\mathbb{R}_2(x)$ vektorski prostor realnih polinoma stepena ne većeg od 2 i neka je u tom vektorskom prostoru skalarni proizvod definisan sa

$$\big(p(x), q(x)\big) = \int_0^{+\infty} e^{-x} p(x)\, q(x)\; dx.$$

Odrediti ortonormiranu bazu tog prostora pomoću Gram-Šmitovog postupka ortogonalizacije polazeći od standardne baze tog prostora $\{1, x, x^2\}$.

Rješenje

Polazeći od polinoma $p_0(x) = 1$, $p_1(x) = x$ i $p_2(x) = x^2$, odredićemo ortogonalne polinome $q_0(x)$, $q_1(x)$ i $q_2(x)$ koristeći Gram-Šmitov postupak:

$$\begin{aligned}
q_0(x) &= p_0(x) = 1,\\
q_1(x) &= p_1(x) - \frac{\big(p_1(x), q_0(x)\big)}{\big(q_0(x), q_0(x)\big)} \cdot q_0(x),\\
q_2(x) &= p_2(x) - \frac{\big(p_2(x), q_0(x)\big)}{\big(q_0(x), q_0(x)\big)} \cdot q_0(x) - \frac{\big(p_2(x), q_1(x)\big)}{\big(q_1(x), q_1(x)\big)} \cdot q_1(x).
\end{aligned}$$

Koristeći definiciju skalarnog proizvoda dobijamo:

$$\big(q_0(x), q_0(x)\big) = \int_0^{+\infty} e^{-x} \cdot 1 \cdot 1\, dx = -e^{-x}\Big|_0^{+\infty} = -\lim_{x\to+\infty} e^{-x} - \big(-e^0\big) = 1$$

$$\big(p_1(x), q_0(x)\big) = \int_0^{+\infty} e^{-x} \cdot x \cdot 1\, dx \qquad \begin{cases} u = x \\ du = dx \\ v = -e^{-x} \\ dv = e^{-x}\, dx \end{cases}$$

$$= -xe^{-x}\Big|_0^{+\infty} - \int_0^{+\infty} -e^{-x}\, dx = -\left(\lim_{x\to+\infty} xe^{-x} - 0\cdot e^0\right) - e^{-x}\Big|_0^{+\infty} = -\lim_{x\to+\infty} \frac{x}{e^x} - \left(\lim_{x\to+\infty} e^{-x} - e^0\right) = 1$$

pa je

$$q_1(x) = x - \frac{1}{1} \cdot 1 = x - 1.$$

Dalje je

$$\big(p_2(x), q_0(x)\big) = \int_0^{+\infty} e^{-x} \cdot x^2 \cdot 1\, dx \qquad \begin{cases} u = x^2 \\ du = 2x\, dx \\ v = -e^{-x} \\ dv = e^{-x}\, dx \end{cases}$$

$$= -x^2e^{-x}\Big|_0^{+\infty} + \int_0^{+\infty} e^{-x} \cdot 2x\, dx = -\left(\lim_{x\to+\infty} x^2e^{-x} - 0^2\cdot e^0\right) + 2\cdot\big(p_1(x), q_0(x)\big) = -\lim_{x\to+\infty}\frac{x^2}{e^x} + 2 = 2$$

$$\begin{aligned}
\big(q_1(x), q_1(x)\big) &= \int_0^{+\infty} e^{-x} \cdot (x-1)^2\, dx = \int_0^{+\infty} e^{-x} \cdot \big(x^2 - 2x + 1\big)\, dx = \int_0^{+\infty} e^{-x}\cdot x^2\, dx - 2\cdot \int_0^{+\infty} e^{-x}\cdot x\, dx + \int_0^{+\infty} e^{-x}\, dx\\
&= \big(p_2(x), q_0(x)\big) - 2\cdot\big(p_1(x), q_0(x)\big) + \big(q_0(x), q_0(x)\big)\\
&= 2 - 2\cdot 1 + 1 = 1
\end{aligned}$$

$$\big(p_2(x), q_1(x)\big) = \int_0^{+\infty} e^{-x}\cdot x^2 \cdot (x-1)\, dx = \int_0^{+\infty} e^{-x}\cdot x^3\, dx - \int_0^{+\infty} e^{-x}\cdot x^2\, dx \qquad \begin{cases} u = x^3 \\ du = 3x^2\, dx \\ v = -e^{-x} \\ dv = e^{-x}\, dx \end{cases}$$

$$\begin{aligned}
&= -x^2e^{-x}\Big|_0^{+\infty} + \int_0^{+\infty} e^{-x}\cdot 3x^2\, dx - \int_0^{+\infty} e^{-x}\cdot x^2\, dx = -\left(\lim_{x\to+\infty}\frac{x^2}{e^x} - 0^2\cdot e^0\right) + 2\cdot\int_0^{+\infty} e^{-x}\cdot x^2\, dx\\
&= 2\cdot\big(p_2(x), q_0(x)\big) = 4
\end{aligned}$$

pa je

$$q_2(x) = x^2 - \frac{2}{1}\cdot 1 - \frac{4}{1}\cdot(x-1) = x^2 - 2 - 4x + 4 = x^2 - 4x + 2.$$

Dakle, ortogonalna baza prostora $\mathbb{R}_2(x)$ je

$$B_{\mathbb{R}_2(x)} = \big\{q_0(x), q_1(x), q_2(x)\big\} = \big\{1, x-1, x^2-4x+2\big\}.$$

Prethodnu bazu je potrebno ortonormirati.

Kako je

$$\begin{aligned}
\big(q_0(x), q_0(x)\big) &= 1\\
\big(q_1(x), q_1(x)\big) &= 1\\
\big(q_2(x), q_2(x)\big) &= \int_0^{+\infty} e^{-x}\cdot\left(x^2-4x+2\right)^2\,dx = \int_0^{+\infty} e^{-x}\cdot\left(x^4-8x^3+20x^2-16x+4\right)\,dx\\
&= \int_0^{+\infty} e^{-x}\cdot x^4\,dx - 8\cdot\int_0^{+\infty} e^{-x}\cdot x^3\,dx + 20\cdot\int_0^{+\infty} e^{-x}\cdot x^2\,dx - 16\cdot\int_0^{+\infty} e^{-x}\cdot x\,dx + 4\cdot\int_0^{+\infty} e^{-x}\,dx \qquad \begin{cases} u = x^4\\ du = 4x^3\,dx\\ v = -e^{-x}\\ dv = e^{-x}\,dx\end{cases}\\
&= -x^4e^{-x}\Big|_0^{+\infty} + \int_0^{+\infty} e^{-x}\cdot 4x^3\,dx - 8\cdot\int_0^{+\infty} e^{-x}\cdot x^3\,dx + 20\cdot\int_0^{+\infty} e^{-x}\cdot x^2\,dx - 16\cdot\int_0^{+\infty} e^{-x}\cdot x\,dx + 4\cdot\int_0^{+\infty} e^{-x}\,dx\\
&= -\left(\lim_{x\to+\infty}\frac{x^4}{e^x} - 0^4\cdot e^0\right) - 4\cdot\int_0^{+\infty} e^{-x}\cdot x^3\,dx + 20\cdot\big(p_2(x), q_0(x)\big) - 16\cdot\big(p_1(x), q_0(x)\big) + 4\cdot\big(q_0(x), q_0(x)\big) \qquad \begin{cases} u = x^3\\ du = 3x^2\,dx\\ v = -e^{-x}\\ dv = e^{-x}\,dx\end{cases}\\
&= -4\cdot\left(-x^3e^{-x}\Big|_0^{+\infty} + \int_0^{+\infty} e^{-x}\cdot 3x^2\,dx\right) + 20\cdot 2 - 16\cdot 1 + 4\cdot 1\\
&= 4\cdot\left(\lim_{x\to+\infty}\frac{x^3}{e^x} - 0^3\cdot e^0\right) - 12\cdot\int_0^{+\infty} e^{-x}\cdot x^2\,dx + 28\\
&= -12\cdot\big(p_2(x), q_0(x)\big) + 28\\
&= -12\cdot 2 + 28 = 4,
\end{aligned}$$

ortonormirana baza prostora $\mathbb{R}_2(x)$ je

$$\begin{aligned}
B_{\mathbb{R}_2(x)} &= \left\{\frac{1}{\|q_0(x)\|}\cdot q_0(x), \frac{1}{\|q_1(x)\|}\cdot q_1(x), \frac{1}{\|q_2(x)\|}\cdot q_2(x)\right\}\\
&= \left\{\frac{1}{\sqrt{\big(q_0(x), q_0(x)\big)}}\cdot q_0(x), \frac{1}{\sqrt{\big(q_1(x), q_1(x)\big)}}\cdot q_1(x), \frac{1}{\sqrt{\big(q_2(x), q_2(x)\big)}}\cdot q_2(x)\right\}\\
&= \left\{q_0(x), q_1(x), \frac{1}{2}\cdot q_2(x)\right\}\\
&= \left\{1, x-1, \frac{x^2-4x+2}{2}\right\}\\
&= \left\{1, x-1, \frac{x^2}{2}-2x+1\right\}.
\end{aligned}$$