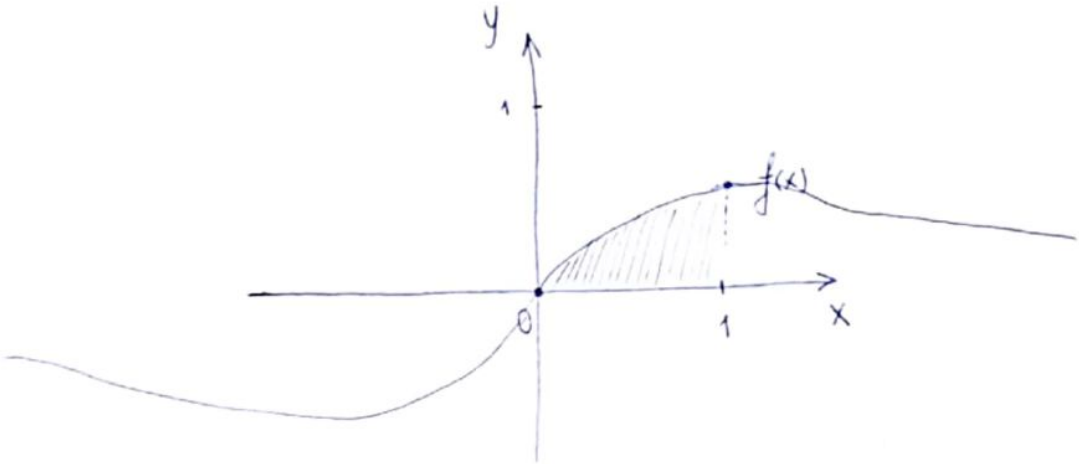
(11)



a) $P = \int_0^1 f(x)\,dx = \int_0^1 \frac{x}{1+x^2}\,dx$

Odredimo prvo neodređeni integral.

$I_n = \int \frac{x\,dx}{1+x^2} = \begin{cases} t = 1+x^2 \\ dt = 2x\,dx \Rightarrow x\,dx = \frac{dt}{2} \end{cases}$

$= \int \frac{\frac{dt}{2}}{t} = \frac{1}{2}\ln|t| + C = \frac{1}{2}\ln(1+x^2) + C$

Sada je vrijednost tražene površine:

$P = \frac{1}{2}\ln(1+x^2)\Big|_0^1 = \frac{1}{2}\ln 2 - \cancel{\frac{1}{2}\ln 1} = \boxed{\frac{\ln 2}{2}}$

b) Zapreminu tijela nastalog rotacijom date figure oko x-os dobijamo po formuli:

$V = \pi \cdot \int_0^1 f^2(x)\,dx = \pi \cdot \int_0^1 \frac{x^2}{(1+x^2)^2}\,dx$

Odredimo ponovo neodređeni integral

$I_n = \int \frac{x^2}{(1+x^2)^2}\,dx$

Dati integral predstavlja integral racionalne funkcije. Metodom neodređenih koeficijenata dobijamo:

$$\frac{x^2}{(1+x^2)^2} = \frac{Ax+B}{1+x^2} + \frac{Cx+D}{(1+x^2)^2} \Rightarrow$$

$$\begin{aligned} x^2 &= (Ax+B)\cdot(1+x^2) + Cx+D \\ &= Ax+B+Ax^3+Bx^2+Cx+D \\ &= Ax^3+Bx^2+(A+C)x+(B+D) \Rightarrow \end{aligned}$$

$$\begin{array}{llll} A & & & = 0 \\ & B & & = 1 \\ A & & +C & = 0 \\ & B & +D & = 0 \\ \hline \end{array}$$

$$A=0,\ B=1,\ C=0,\ D=-1$$

Sada je:

$$I_n = \int \frac{dx}{1+x^2} - \int \frac{dx}{(1+x^2)^2}$$

$$= \operatorname{arctg}(x) - I_2 \qquad \dots (1)$$

Integral

$$I_2 = \int \frac{dx}{(1+x^2)^2}$$

rješavamo korištenjem trigonometrijske smjene: $x = \operatorname{tg} t \Rightarrow t = \operatorname{arctg} x$, $dx = \frac{dt}{\cos^2 t}$

$$I_2 = \int \frac{\frac{dt}{\cos^2 t}}{(1+\operatorname{tg}^2 t)^2} = \int \frac{\frac{dt}{\cos^2 t}}{\left(\frac{\cos^2 t + \sin^2 t}{\cos^2 t}\right)^2} = \int \frac{\frac{dt}{\cos^2 t}}{\frac{1}{\cos^4 t}}$$

$$= \int \cos^2 t \, dt = \int \frac{1+\cos 2t}{2} dt = \int \frac{dt}{2} + \int \frac{\cos 2t \, dt}{2}$$

$$= \frac{t}{2} + \frac{1}{2} \cdot \frac{\sin 2t}{2} + C$$

Nakon vraćanja smjene $t = \operatorname{arctg} x$ dobijamo

$$I_2 = \frac{\operatorname{arctg} x}{2} + \frac{\sin(2\operatorname{arctg} x)}{4} + C$$

pa vraćanjem u izraz (1) dobijamo:

$$I_n = \operatorname{arctg} x - \left(\frac{\operatorname{arctg} x}{2} + \frac{\sin(2\operatorname{arctg} x)}{4} \right) + C$$

$$= \frac{\operatorname{arctg} x}{2} - \frac{\sin(2\operatorname{arctg} x)}{4} + C$$

pa je zapremina dobijenog tijela.

$$V = \pi \cdot \left[\frac{\operatorname{arctg} x}{2} - \frac{\sin(2\operatorname{arctg} x)}{4} \right] \Bigg|_0^1$$

$$= \pi \left[\left(\frac{\operatorname{arctg}(1)}{2} - \frac{\sin(2\operatorname{arctg}(1))}{4} \right) - \left(\frac{\operatorname{arctg}(0)}{2} - \frac{\sin(2\operatorname{arctg}(0))}{4} \right) \right]$$

$$= \pi \cdot \left[\frac{\frac{\pi}{4}}{2} - \frac{\sin(2 \cdot \frac{\pi}{4})}{4} - \cancel{\frac{0}{2}} + \cancel{\frac{\sin(2 \cdot 0)}{4}} \right]$$

$$= \pi \cdot \left(\frac{\pi}{8} - \frac{1}{4} \right)$$

$$= \frac{\pi \cdot (\pi - 2)}{8}$$

(12)

$$I = \int \frac{dx}{\cos x \cdot \sqrt[3]{\sin^2 x}}$$

$$= \int \frac{\cos x \, dx}{\cos^2 x \cdot \sqrt[3]{\sin^2 x}} = \begin{cases} t = \sin x \\ dt = \cos x \, dx \end{cases}$$

$$= \int \frac{dt}{(1-t^2)\cdot t^{\frac{2}{3}}} = \begin{cases} u = t^{\frac{1}{3}} \\ du = \frac{1}{3}\cdot t^{-\frac{2}{3}} dt \Rightarrow \frac{dt}{t^{\frac{2}{3}}} = 3du \end{cases}$$

$$= \int \frac{3du}{1-u^6}$$

$$= \int \frac{-3}{u^6-1} \, du$$

$$= \int \frac{-3}{(u^3-1)\cdot(u^3+1)} \, du$$

$$= \int \frac{-3}{(u-1)(u^2+u+1)\cdot(u+1)(u^2-u+1)} \, du$$

Primjenom metode neodređenih koeficijenata dobijamo:

$$\frac{-3}{(u-1)\cdot(u^2+u+1)\cdot(u+1)\cdot(u^2-u+1)} = \frac{A}{u-1} + \frac{Bu+C}{u^2+u+1} + \frac{D}{u+1} + \frac{Eu+F}{u^2-u+1}$$

$$-3 = A\cdot(u^2+u+1)\cdot(u^3+1) + (Bu+C)\cdot(u-1)\cdot(u^3+1) +$$
$$D\cdot(u^3-1)\cdot(u^2-u+1) + (Eu+F)\cdot(u^3-1)\cdot(u+1)$$

$$= A\cdot(u^5+u^4+u^3+u^2+u+1) + (Bu+C)\cdot(u^4-u^3+u-1) +$$
$$D\cdot(u^5-u^4+u^3-u^2+u-1) + (Eu+F)\cdot(u^4+u^3-u-1)$$

$$= Au^5+Au^4+Au^3+Au^2+Au+A+Bu^5+Cu^4-Bu^4-Cu^3$$
$$+Bu^2+Cu-Bu-C+Du^5-Du^4+Du^3-Du^2+Du-D+$$
$$Eu^5+Fu^4+Eu^4+Fu^3-Eu^2-Fu-Eu-F$$

$$= (A+B+D+E)u^5 + (A-B+C-D+E+F)u^4 +$$
$$(A-C+D+F)u^3 + (A+B-D-E)u^2 +$$
$$(A-B+C+D-E-F)u + (A-C-D-F)$$

$$\begin{array}{rrrrrrcrl}
A & +B & & +D & +E & & = & 0 & \\
A & -B & +C & -D & +E & +F & = & 0 & /\cdot(-1) \\
A & & -C & +D & & +F & = & 0 & \\
A & +B & & -D & -E & & = & 0 & \\
A & -B & +C & +D & -E & -F & = & 0 & \\
A & & -C & -D & & -F & = & -3 & \\
\hline
A & +B & & +D & +E & & = & 0 & /\cdot(-1) \\
A & +B & & -D & -E & & = & 0 & \\
 & B & -2C & +2D & -E & & = & 0 & \\
2A & -2B & +2C & & & & = & 0 & /:2 \\
2A & -B & & -2D & +E & & = & -3 & \\
\hline
\end{array}$$

$$\begin{array}{l} A - B + C = 0 \\ 2A + 2B = 0 \quad /:2 \\ \left.\begin{array}{l} A + 2B - 2C + 3D = 0 \\ A - 2B - 3D = -3 \end{array}\right\} + \end{array}$$

$$\begin{array}{l} A + B = 0 \\ \left.\begin{array}{l} A - B + C = 0 \quad /\cdot 2 \\ 2A - 2C = -3 \end{array}\right\} + \end{array}$$

$$\left.\begin{array}{l} A + B = 0 \quad /\cdot 2 \\ 4A - 2B = -3 \end{array}\right\} +$$

$$6A = -3 \Rightarrow$$

$$A = -\frac{1}{2},\quad B = \frac{1}{2},\quad C = 1,\quad D = \frac{1}{2},\quad E = -\frac{1}{2},\quad F = 1$$

Sada dobijamo:

$$I = \int \frac{-\frac{1}{2}}{u-1}\,du + \int \frac{\frac{1}{2}u + 1}{u^2+u+1}\,du + \int \frac{\frac{1}{2}}{u+1}\,du + \int \frac{-\frac{1}{2}u + 1}{u^2-u+1}\,du$$

$$= -\frac{1}{2}\ln|u-1| + \int \frac{\frac{1}{4}\cdot(2u+1) + \frac{3}{4}}{u^2+u+1}\,du + \frac{1}{2}\ln|u+1| + \int \frac{-\frac{1}{4}\cdot(2u-1) + \frac{3}{4}}{u^2-u+1}\,du$$

$$= -\frac{1}{2}\ln|u-1| + \frac{1}{4}\cdot\ln(u^2+u+1) + \frac{3}{4}\int \frac{du}{(u+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} - \frac{1}{4}\ln(u^2-u+1)$$

$$+ \frac{3}{4}\cdot\int \frac{du}{(u-\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} + \frac{1}{2}\ln|u+1|$$

$$I = -\frac{1}{2}\ln|u-1| + \frac{1}{4}\ln(u^2+u+1) + \frac{3}{4}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\operatorname{arctg}\left(\frac{u+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

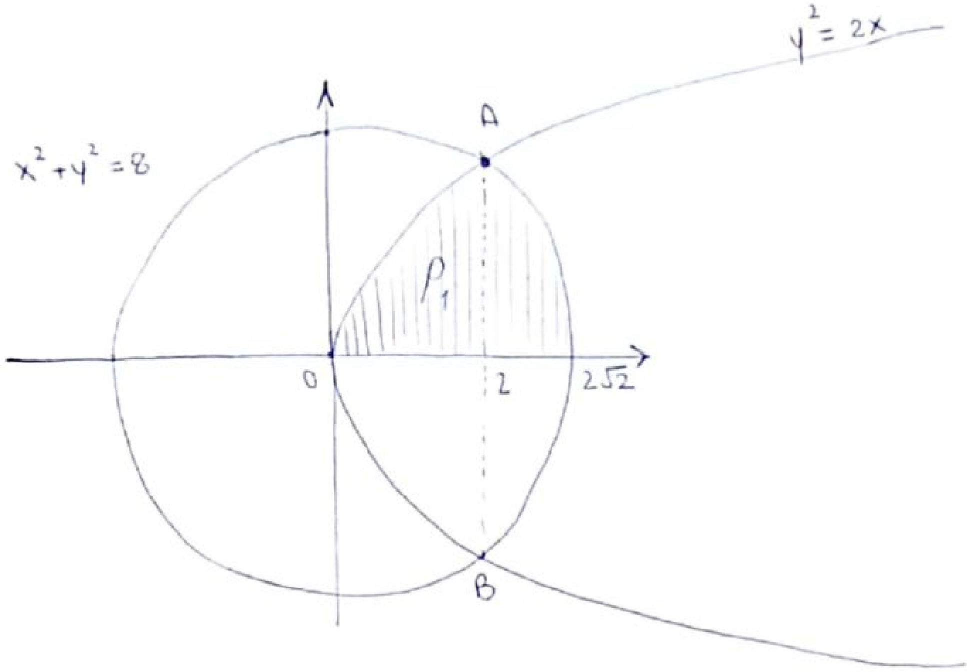
$$+\frac{1}{2}\ln|u+1| - \frac{1}{4}\ln(u^2-u+1) + \frac{3}{4}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\operatorname{arctg}\left(\frac{u-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

$$= \frac{1}{2}\ln\left|\frac{u+1}{u-1}\right| + \frac{1}{4}\ln\left|\frac{u^2+u+1}{u^2-u+1}\right| + \frac{\sqrt{3}}{2}\left(\operatorname{arctg}\left(\frac{2u+1}{\sqrt{3}}\right) + \operatorname{arctg}\left(\frac{2u-1}{\sqrt{3}}\right)\right) + C$$

Nakon vraćanja smjene $u = t^{\frac{1}{3}} = \sin^{\frac{1}{3}}x$ dobijamo

$$\boxed{I = \frac{1}{2}\ln\left|\frac{1+\sqrt[3]{\sin x}}{1-\sqrt[3]{\sin x}}\right| + \frac{1}{4}\ln\left|\frac{\sqrt[3]{\sin^2 x}+\sqrt[3]{\sin x}+1}{\sqrt[3]{\sin^2 x}-\sqrt[3]{\sin x}+1}\right| + \frac{\sqrt{3}}{2}\left(\operatorname{arctg}\left(\frac{2\sqrt[3]{\sin x}+1}{\sqrt{3}}\right) + \operatorname{arctg}\left(\frac{2\sqrt[3]{\sin x}-1}{\sqrt{3}}\right)\right) + C}$$

13



Odnos u kom parabola $y^2 = 2x$ dijeli površinu kruga $x^2+y^2=8$ jednak je odnosu površine P_1 koju grade ove dvije krive omeđene x-osom i površine polukruga P_2.

Imajući u vidu da je poluprečnik kruga, tj. kružnice, $x^2+y^2=8$ jednak $\sqrt{8} = 2\sqrt{2}$, imamo da je

$$P_2 = \frac{1}{2} \cdot (\sqrt{8})^2 \pi = 4\pi$$

Da bismo odredili površinu P_1, potrebno je da odredimo presječne tačke krivih $y^2 = 2x$ i $x^2+y^2=8$.

Kako se posmatrane površi, odnosno krive, nalaze iznad x-ose, imamo da je jednačina parabole $y = \sqrt{2x}$, a kružnice $y = \sqrt{8-x^2}$.

Tačku presjeka ovih funkcija (A) dobijamo izjednačavanjem ovih jednačina:

$$\sqrt{2x} = \sqrt{8-x^2} \Rightarrow \qquad 2x = 8 - x^2 \Rightarrow$$

$$x^2 + 2x - 8 = 0$$

$$x_{1,2} = \frac{-2 \pm \sqrt{4-4\cdot 1\cdot(-8)}}{2} = \frac{-2 \pm 6}{2}$$

$$x_1 = -4 < 0 \quad \times$$

$$x_2 = 2 \quad \checkmark$$

Površinu P_1 sada dobijamo kao

$$P_1 = \int_0^2 \sqrt{2x}\, dx + \int_2^{2\sqrt{2}} \sqrt{8-x^2}\, dx \qquad \ldots(2)$$

Rješavanjem neodređenih integrala dobijamo:

$$I_1 = \int \sqrt{2x}\, dx = \int \sqrt{2}\, x^{\frac{1}{2}}\, dx = \sqrt{2} \cdot \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$= \frac{2\sqrt{2}}{3} \sqrt{x^3} + C$$

$$I_2 = \int \sqrt{8-x^2}\, dx = \begin{cases} x = \sqrt{8}\, \sin t \Rightarrow t = \arcsin\left(\frac{x}{\sqrt{8}}\right) \\ dx = \sqrt{8}\, \cos t\, dt \end{cases}$$

$$= \int \sqrt{8 - 8\sin^2 t} \cdot \sqrt{8}\, \cos t\, dt$$

$$= \int \sqrt{8} \cdot \sqrt{1-\sin^2 t}\, \sqrt{8} \cdot \cos t\, dt$$

$$= 8 \cdot \int \cos^2 t\, dt$$

$$= 8 \cdot \int \frac{1+\cos 2t}{2}\, dt$$

$$= 4 \cdot \left(\int dt + \int \cos 2t \, dt \right)$$

$$= 4 \cdot \left(t + \frac{\sin 2t}{2} \right) + C$$

$$= 4 \cdot \arcsin\left(\frac{x}{\sqrt{8}}\right) + 2 \sin\left(2 \arcsin\left(\frac{x}{\sqrt{8}}\right)\right) + C$$

Vraćanjem u izraz (2) dobijamo:

$$P_1 = \frac{2\sqrt{2}}{3} \sqrt{x^3} \Big|_0^2 + \left(4 \arcsin\left(\frac{x}{\sqrt{8}}\right) + 2\sin\left(2\arcsin\left(\frac{x}{\sqrt{8}}\right)\right)\right) \Big|_2^{2\sqrt{2}}$$

$$= \frac{2\sqrt{2}}{3} \cdot \left(\sqrt{2^3} - \cancel{\sqrt{0^3}}\right) + 4\arcsin\left(\frac{\cancel{2\sqrt{2}}}{\cancel{\sqrt{8}}}\right) + 2\sin\left(2\arcsin\left(\frac{2\sqrt{2}}{\sqrt{8}}\right)\right)$$

$$- \left(4 \arcsin\left(\frac{\cancel{2}}{2\sqrt{2}}\right) + 2\sin\left(2\arcsin\left(\frac{2}{\sqrt{8}}\right)\right)\right)$$

$$= \frac{2 \cdot 4}{3} + 4 \cdot \frac{\pi}{2} + 2\sin\left(\cancel{2} \cdot \frac{\pi}{\cancel{2}}\right) - \left(\cancel{4} \cdot \frac{\pi}{\cancel{4}} + 2\sin\left(2 \cdot \frac{\pi}{4}\right)\right)$$

$$= \frac{8}{3} + 2\pi + \cancel{2 \cdot 0} - (\pi + 2)$$

$$= \frac{8}{3} + 2\pi - \pi - 2 \quad \Rightarrow$$

$$P_1 = \pi + \frac{2}{3}$$

Konačno, traženi odnos Q je:

$$Q = \frac{P_1}{P_2} = \frac{\pi + \frac{2}{3}}{4\pi} = \frac{1}{4} + \frac{1}{6\pi} = \boxed{\frac{3\pi + 2}{12\pi}}$$

(14)

Odredimo prvo neodređeni integral

$$I_n = \int \frac{dx}{x\sqrt{1+x^5+x^{10}}} = \int \frac{x^4 dx}{x^5 \cdot \sqrt{1+x^5+x^{10}}} = \begin{cases} t = x^5 \\ dt = 5x^4 dx \end{cases}$$

$$= \frac{1}{5} \cdot \int \frac{dt}{t \cdot \sqrt{1+t+t^2}} \qquad \dots (3)$$

Ovaj integral dalje rješavamo korištenjem Ojlerove smjene:

$$\sqrt{1+t+t^2} = t+u \quad /^2 \qquad \Rightarrow \quad u = \sqrt{1+t+t^2} - t$$

$$1+t+\cancel{t^2} = \cancel{t^2} + 2tu + u^2$$

$$t \cdot (1-2u) = u^2 - 1$$

$$t = \frac{u^2-1}{1-2u}$$

$$dt = \frac{(u^2-1)'(1-2u) - (u^2-1)\cdot(1-2u)'}{(1-2u)^2}\, du$$

$$= \frac{2u(1-2u) + 2(u^2-1)}{(1-2u)^2}\, du$$

$$= \frac{2\cdot(-u^2+u-1)}{(1-2u)^2}\, du$$

Sada integral (3) postaje:

$$I_n = \frac{1}{5} \int \frac{\frac{2(-u^2+u-1)}{(1-2u)^2}\, du}{\frac{u^2-1}{1-2u} \cdot \left(\frac{u^2-1}{1-2u}+u\right)}$$

$$= \frac{2}{5} \int \frac{\frac{-u^2+u-1}{\cancel{(1-2u)^2}}\, du}{\frac{u^2-1}{\cancel{1-2u}} \cdot \frac{u^2-1+u-2u^2}{\cancel{1-2u}}}$$

$$= \frac{2}{5} \cdot \int \frac{\cancel{(-u^2+u-1)}\, du}{(u^2-1) \cdot \cancel{(-u^2+u-1)}}$$

$$= \frac{2}{5} \cdot \int \frac{du}{(u-1)(u+1)}$$

Posljednji integral je integral racionalne funkcije i metodom neodređenih koeficijenata dobijamo

$$\frac{1}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1} \quad / \ (u-1)(u+1)$$

$$1 = A(u+1) + B(u-1)$$
$$= (A+B)u + (A-B) \Rightarrow$$

$$\left.\begin{array}{l} A+B = 0 \\ A-B = 1 \end{array}\right\} \Rightarrow$$
$$\overline{A = \frac{1}{2},\ B = -\frac{1}{2}}$$

pa je naš neodređeni integral jednak:

$$I_n = \frac{2}{5} \cdot \left[\int \frac{\frac{1}{2}}{u-1}\, du + \int \frac{-\frac{1}{2}}{u+1}\, du \right]$$

$$= \frac{2}{5} \cdot \left(\frac{1}{2} \ln|u-1| - \frac{1}{2} \ln|u+1| \right) + C$$

$$= \frac{1}{5} \cdot \ln \left| \frac{u-1}{u+1} \right| + C$$

Nakon vraćanja smjene : $u = \sqrt{1+t+t^2} - t$

$$= \sqrt{1+x^5+x^{10}} - x^5$$

dobijamo da je početni neodređeni integral:

$$I_n = \frac{1}{5} \cdot \ln \left| \frac{\sqrt{1+x^5+x^{10}} - x^5 - 1}{\sqrt{1+x^5+x^{10}} - x^5 + 1} \right| + C$$

Sada je vrijednost početnog nesvojstvenog integrala:

$$I = \lim_{b \to \infty} \left(\frac{1}{5} \cdot \ln \left| \frac{\sqrt{1+b^5+b^{10}} - b^5 - 1}{\sqrt{1+b^5+b^{10}} - b^5 + 1} \right| \right) - \frac{1}{5} \cdot \ln \left| \frac{\sqrt{1+1^5+1^{10}} - 1^5 - 1}{\sqrt{1+1^5+1^{10}} - 1^5 + 1} \right|$$

$$\ldots (4)$$

Odredimo sada graničnu vrijednost:

$$L = \lim_{b \to \infty} \left(\frac{1}{5} \ln \left| \frac{\sqrt{1+b^5+b^{10}} - b^5 - 1}{\sqrt{1+b^5+b^{10}} - b^5 + 1} \right| \right)$$

$$= \frac{1}{5} \cdot \ln \left| \lim_{b \to \infty} \frac{\left(\sqrt{1+b^5+b^{10}} - b^5\right) \cdot \frac{\sqrt{1+b^5+b^{10}}+b^5}{\sqrt{1+b^5+b^{10}}+b^5} - 1}{\left(\sqrt{1+b^5+b^{10}} - b^5\right) \cdot \frac{\sqrt{1+b^5+b^{10}}+b^5}{\sqrt{1+b^5+b^{10}}+b^5} + 1} \right|$$

$$= \frac{1}{5} \cdot \ln \left| \lim_{b \to \infty} \frac{\frac{1+b^5+\cancel{b^{10}}-\cancel{b^{10}}}{\sqrt{1+b^5+b^{10}}+b^5} - 1}{\frac{1+b^5+\cancel{b^{10}}-\cancel{b^{10}}}{\sqrt{1+b^5+b^{10}}+b^5} + 1} \right|$$

$$= \frac{1}{5} \cdot \ln \left| \lim_{b \to \infty} \frac{\frac{1+\cancel{b^5}-\sqrt{1+b^5+b^{10}}-\cancel{b^5}}{\cancel{\sqrt{1+b^5+b^{10}}+b^5}}}{\frac{1+b^5+\sqrt{1+b^5+b^{10}}+b^5}{\cancel{\sqrt{1+b^5+b^{10}}+b^5}}} \right|$$

$$= \frac{1}{5} \cdot \ln \left| \lim_{b \to \infty} \frac{1 - \sqrt{b^{10} \cdot \left(1+\frac{1}{b^5}+\frac{1}{b^{10}}\right)}}{1+2b^5+\sqrt{b^{10}\left(1+\frac{1}{b^5}+\frac{1}{b^{10}}\right)}} \right|$$

$$= \frac{1}{5} \cdot \ln \left| \lim_{b \to \infty} \frac{\cancel{b^5} \cdot \left(\frac{1}{b^5} - \sqrt{1+\frac{1}{b^5}+\frac{1}{b^{10}}}\right)}{\cancel{b^5}\left(\frac{1}{b^5}+2+\sqrt{1+\frac{1}{b^5}+\frac{1}{b^{10}}}\right)} \right|$$

$$= \frac{1}{5} \cdot \ln \left| \frac{-1}{2+1} \right|$$

$$= \frac{1}{5} \ln \left(\frac{1}{3} \right)$$

pa nakon uvrštavanja u (4) dobijamo

$$I = \frac{1}{5} \ln\left(\frac{1}{3}\right) - \frac{1}{5} \ln\left(\left|\frac{\sqrt{3}-2}{\sqrt{3}}\right|\right) = \frac{\ln\left(\frac{1}{3}\right) - \ln\left(\frac{2-\sqrt{3}}{\sqrt{3}}\right)}{5}$$

$$= \frac{1}{5} \cdot \ln\left(\frac{\frac{1}{3}}{\frac{2-\sqrt{3}}{\sqrt{3}}}\right) = \frac{1}{5} \ln\left(\frac{1}{\sqrt{3}\,(2-\sqrt{3})}\right) = \boxed{-\frac{1}{5} \ln\left(2\sqrt{3}-3\right)}$$

(15) Odredimo prvo neodređeni integral.

$$I_n = \int x^n e^{-x}\, dx = \begin{cases} u = x^n & v = -e^{-x} \\ du = n\, x^{n-1} dx & dv = e^{-x} dx \end{cases}$$

$$= -x^n e^{-x} + n \cdot \int x^{n-1} e^{-x} dx = \begin{cases} u = x^{n-1} & v = -e^{-x} \\ du = (n-1)\, x^{n-2} dx & dv = e^{-x} dx \end{cases}$$

$$= -x^n e^{-x} + n\left(-x^{n-1} e^{-x} + (n-1) \cdot \int x^{n-2} e^{-x} dx\right)$$

$$= -x^n e^{-x} - n x^{n-1} e^{-x} + n(n-1) \cdot I_{n-2}$$

Uočavamo rekurzivnu relaciju

$$I_n = -x^n e^{-x} + n \cdot I_{n-1}$$

pa ako integral $I_{n-2} = \int x^{n-2} e^{-x} dx$ nastavimo rješavati parcijalnom integracijom $(n-2)$ puta dobijamo:

$$I_n = -x^n e^{-x} - n\, x^{n-1} e^{-x} - n(n-1)\, x^{n-2} e^{-x} - n(n-1)(n-2)\, x^{n-3} e^{-x}$$

$$- \dots - n(n-1)(n-2) \dots (3) \cdot (2) \cdot x^1 e^{-x} - n!\, e^{-x}$$

$$= -\frac{x^n + n x^{n-1} + n(n-1)\, x^{n-2} + \dots + n(n-1)(n-2) \dots 3 \cdot 2\, x + n!}{e^x}$$

Sada vrijednost nesvojstvenog integrala dobijamo kao.

$$I = \left(-\frac{x^n + n x^{n-1} + n(n-1)\, x^{n-2} + \dots + n(n-1)(n-2) \dots 3 \cdot 2\, x + n!}{}\right)\Bigg|_0^{+\infty}$$

$$= \lim_{b \to +\infty} \left(- \frac{b^n + n b^{n-1} + n(n-1) b^{n-2} + \dots + (n(n-1)(n-2) \cdots 2) b + n!}{e^b} \right.$$

$$\left. + \frac{\cancel{0^n} + \cancel{n 0^{n-1}} + n(n-1) 0^{n-2} + \dots + n(n-1)(n-2) \cdots \cancel{2 \cdot 0} + n!}{\underbrace{e^0}_{=1}} \right)$$

Kako je eksponencijalna funkcija mnogo brže rastuća u odnosu na polinomsku imamo da je vrijednost prvog limesa jednaka 0 pa je vrijednost nesvojstvenog integrala

$$I = \int_0^{+\infty} x^n e^{-x} \, dx = n!$$

Sada je

$$\lim_{n \to +\infty} \left(\frac{\pi}{2n!} \cdot \int_0^{+\infty} x^n e^{-x} \, dx \right) =$$

$$= \lim_{n \to \infty} \left(\frac{\pi}{2\cancel{n!}} \cdot \cancel{n!} \right)$$

$$= \boxed{\frac{\pi}{2}}$$