TERMIN 8 - zadaci za samostalan rad - rješenja

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Zadatak 1.
Izračunati graničnu vrijednost

a) $\lim\limits_{x\to 0}(1+\operatorname{tg} x)^{\frac{1}{3x}}$,

b) $\lim\limits_{x\to 1}\dfrac{\sqrt[3]{x+7}-2}{x-1}$.

Rješenje

Vrijedi:

a)

$$\begin{aligned}\lim_{x\to 0}(1+\operatorname{tg} x)^{\frac{1}{3x}} &= \lim_{x\to 0}(1+\operatorname{tg} x)^{\frac{1}{\operatorname{tg} x}\cdot\frac{\operatorname{tg} x}{3x}}\\ &= e^{\lim\limits_{x\to 0}\frac{\frac{\sin x}{\cos x}}{3x}}\\ &= e^{\lim\limits_{x\to 0}\xcancel{\frac{\sin x}{x}}^{1}\cdot\frac{1}{\xcancel{3\cos x}^{1}}}\\ &= e^{\frac{1}{3}},\end{aligned}$$

b)

$$\begin{aligned}\lim_{x\to 1}\frac{\sqrt[3]{x+7}-2}{x-1} &= \lim_{x\to 1}\left(\frac{\sqrt[3]{x+7}-2}{x-1}\cdot\frac{\sqrt[3]{(x+7)^2}+2\sqrt[3]{x+7}+2^2}{\sqrt[3]{(x+7)^2}+2\sqrt[3]{x+7}+2^2}\right)\\ &= \lim_{x\to 1}\left(\frac{\cancel{x-1}}{\cancel{(x-1)}\cdot\left(\xcancel{\sqrt[3]{(x+7)^2}}^{4}+2\xcancel{\sqrt[3]{x+7}}^{2}+2^2\right)}\right)\\ &= \frac{1}{12}.\end{aligned}$$

Zadatak 2.
Izračunati graničnu vrijednost:

a) $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x}}$,

b) $\lim\limits_{x\to 0}(1+\sin x)^{\operatorname{ctg} x}$.

Rješenje

Vrijedi:

a)

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x}} &= \lim_{x\to 0}\left(1+\left(\frac{\sin x}{x}-1\right)\right)^{\frac{1}{x}}\\
&= \lim_{x\to 0}\left(1+\frac{\sin x - x}{x}\right)^{\frac{x}{\sin x - x}\cdot\frac{\sin x - x}{x}\cdot\frac{1}{x}}\\
&= e^{\lim\limits_{x\to 0}\frac{\sin x - x}{x^2}}\\
&\overset{LP}{=} e^{\lim\limits_{x\to 0}\frac{\cos x - 1}{2x}}\\
&\overset{LP}{=} e^{\lim\limits_{x\to 0}\frac{-\cancelto{0}{\sin x}}{2}}\\
&= e^0\\
&= 1,
\end{aligned}$$

b)

$$\begin{aligned}
\lim_{x\to 0}(1+\sin x)^{\operatorname{ctg} x} &= \lim_{x\to 0}(1+\sin x)^{\frac{1}{\sin x}\cdot\sin x\cdot\operatorname{ctg} x}\\
&= e^{\lim\limits_{x\to 0}\cancel{\sin x}\cdot\frac{\cancelto{1}{\cos x}}{\cancel{\sin x}}}\\
&= e.
\end{aligned}$$

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Zadatak 3.

Izračunati graničnu vrijednost:

a) $\lim\limits_{x\to 0}(1-\cos x)\operatorname{ctg} x$,

b) $\lim\limits_{x\to 0}\dfrac{\ln\left(1+\sin^2 x\right)}{e^{x^2}-1}$.

Rješenje

Vrijedi:

a)

$$\begin{aligned}
\lim_{x\to 0}(1-\cos x)\operatorname{ctg} x &= \lim_{x\to 0}\left(2\sin^2\frac{x}{2}\cdot\frac{\cos x}{\sin x}\right)\\
&= \lim_{x\to 0}\left(2\sin^2\frac{x}{2}\cdot\frac{\cos x}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)\\
&= 0,
\end{aligned}$$

b)

$$\begin{aligned}
\lim_{x\to 0}\frac{\ln\left(1+\sin^2 x\right)}{e^{x^2}-1} &= \lim_{x\to 0}\left(\frac{\ln\left(1+\sin^2 x\right)}{\sin^2 x}\cdot\frac{\sin^2 x}{e^{x^2}-1}\right)\\
&= \lim_{x\to 0}\left(\frac{\ln\left(1+\sin^2 x\right)}{\sin^2 x}\cdot\frac{\sin^2 x}{x^2}\cdot\frac{x^2}{e^{x^2}-1}\right)\\
&= \lim_{x\to 0}\left(\cancelto{1}{\frac{\ln\left(1+\sin^2 x\right)}{\sin^2 x}}\cdot\cancelto{1}{\frac{\sin^2 x}{x^2}}\cdot\frac{1}{\cancelto{1}{\frac{e^{x^2}-1}{x^2}}}\right)\\
&= 1.
\end{aligned}$$

Zadatak 4.
Izračunati graničnu vrijednost

a) $\lim\limits_{x\to 0} \frac{1}{x} \ln\left(\sqrt{\frac{1+x}{1-x}}\right)$,

b) $\lim\limits_{x\to a} \frac{\sin x - \sin a}{x-a}$.

Rješenje

Vrijedi:

a)

$$\begin{aligned}
\lim_{x\to 0} \frac{1}{x} \ln\left(\sqrt{\frac{1+x}{1-x}}\right) &= \lim_{x\to 0}\left(\frac{1}{x}\cdot\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\right)\\
&= \frac{1}{2}\cdot\lim_{x\to 0}\left(\frac{1}{x}\cdot\big(\ln(1+x)-\ln(1-x)\big)\right)\\
&= \frac{1}{2}\cdot\lim_{x\to 0}\left(\cancelto{1}{\frac{\ln(1+x)}{x}}+\cancelto{1}{\frac{\ln\big(1+(-x)\big)}{-x}}\right)\\
&= \frac{1}{2}\cdot(1+1)\\
&= 1,
\end{aligned}$$

b)

$$\begin{aligned}
\lim_{x\to a}\frac{\sin x-\sin a}{x-a} &= \lim_{x\to a}\frac{\cancel{2}\cos\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)}{\cancel{2}\cdot\frac{x-a}{2}}\\
&= \lim_{x\to a}\left(\cos\left(\frac{x+a}{2}\right)\cdot\cancelto{1}{\frac{\sin\left(\frac{x-a}{2}\right)}{\frac{x-a}{2}}}\right)\\
&= \cos\left(\frac{a+a}{2}\right)\\
&= \cos a.
\end{aligned}$$

Zadatak 5.
Odrediti $a \in \mathbb{R}$ tako da funkcija

a) $f(x) = \begin{cases} \dfrac{1-\cos x}{x^2}, & x \neq 0 \\ a, & x = 0 \end{cases}$

b) $f(x) = \begin{cases} 5 - x^2, & x \leq -1 \\ x - a, & x > -1 \end{cases}$

bude neprekidna.

Rješenje

Da bi funkcija $f(x)$ bila neprekidna na svom domenu, potrebno je da budu ispunjeni sljedeći uslovi:

a)

$$
\begin{aligned}
& \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) \\
\Leftrightarrow \quad & \lim_{x\to 0^-} \frac{1-\cos x}{x^2} = \lim_{x\to 0^+} \frac{1-\cos x}{x^2} = a \\
\Leftrightarrow \quad & \lim_{x\to 0^-} \left(\frac{1-\cos x}{x^2} \cdot \frac{1+\cos x}{1+\cos x} \right) = \lim_{x\to 0^+} \left(\frac{1-\cos x}{x^2} \cdot \frac{1+\cos x}{1+\cos x} \right) = a \\
\Leftrightarrow \quad & \lim_{x\to 0^-} \frac{1-\cos^2 x}{x^2 \cdot (1+\cos x)} = \lim_{x\to 0^+} \frac{1-\cos^2 x}{x^2 \cdot (1+\cos x)} = a \\
\Leftrightarrow \quad & \lim_{x\to 0^-} \left(\cancelto{1}{\frac{\sin^2 x}{x^2}} \cdot \frac{1}{1+\cancelto{1}{\cos x}} \right) = \lim_{x\to 0^+} \left(\cancelto{1}{\frac{\sin^2 x}{x^2}} \cdot \frac{1}{1+\cancelto{1}{\cos x}} \right) = a \\
\Leftrightarrow \quad & \frac{1}{2} = \frac{1}{2} = a \\
\Leftrightarrow \quad & a = \frac{1}{2}.
\end{aligned}
$$

b)

$$
\begin{aligned}
& \lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) = f(-1) \\
\Leftrightarrow \quad & \lim_{x\to -1^-} \left(5 - x^2\right) = \lim_{x\to -1^+} (x-a) = 5 - (-1)^2 \\
\Leftrightarrow \quad & 5 - (-1)^2 = -1 - a = 5 - (-1)^2 \\
\Leftrightarrow \quad & 4 = -1 - a = 4 \\
\Leftrightarrow \quad & a = -5.
\end{aligned}
$$

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Zadatak 6.
Izračunati
$$\lim_{x\to 0}\frac{1+\sin x-\cos x}{1+\sin px-\cos px},\quad p\neq 0.$$

Rješenje

Vrijedi:

$$\begin{aligned}
\lim_{x\to 0}\frac{1+\sin x-\cos x}{1+\sin px-\cos px} &= \lim_{x\to 0}\frac{2\sin^2\left(\frac{x}{2}\right)+2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{2\sin^2\left(\frac{px}{2}\right)+2\sin\left(\frac{px}{2}\right)\cos\left(\frac{px}{2}\right)}\\
&= \lim_{x\to 0}\frac{\cancel{2}\sin\left(\frac{x}{2}\right)\cdot\left(\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)\right)}{\cancel{2}\sin\left(\frac{px}{2}\right)\cdot\left(\sin\left(\frac{px}{2}\right)+\cos\left(\frac{px}{2}\right)\right)}\\
&= \lim_{x\to 0}\frac{\cancelto{1}{\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}}\cdot\frac{\cancel{x}}{\cancel{2}}\cdot\left(\cancelto{0}{\sin\left(\frac{x}{2}\right)}+\cancelto{1}{\cos\left(\frac{x}{2}\right)}\right)}{\cancelto{1}{\frac{\sin\left(\frac{px}{2}\right)}{\frac{px}{2}}}\cdot\frac{p\cancel{x}}{\cancel{2}}\cdot\left(\cancelto{0}{\sin\left(\frac{px}{2}\right)}+\cancelto{1}{\cos\left(\frac{px}{2}\right)}\right)}\\
&= \frac{1}{p}.
\end{aligned}$$

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Zadatak 7.

Ako je poznato da je granična vrijednost $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\ln a$ izračunati

$$\lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}},\quad a,b>0.$$

Rješenje

Vrijedi:

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}} &= \lim_{x\to 0}\left(1+\left(\frac{a^x+b^x}{2}-1\right)\right)^{\frac{1}{x}}\\
&= \lim_{x\to 0}\left(1+\frac{a^x+b^x-2}{2}\right)^{\frac{2}{a^x+b^x-2}\cdot\frac{a^x+b^x-2}{2}\cdot\frac{1}{x}}\\
&= e^{\lim\limits_{x\to 0}\dfrac{a^x-1+b^x-1}{2x}}\\
&= e^{\frac{1}{2}\cdot\lim\limits_{x\to 0}\left(\dfrac{a^x-1}{x}+\dfrac{b^x-1}{x}\right)}\\
&= e^{\frac{1}{2}\cdot(\ln a+\ln b)}\\
&= e^{\frac{1}{2}\cdot\ln(ab)}\\
&= e^{\ln\left(\sqrt{ab}\right)}\\
&= \sqrt{ab}.
\end{aligned}$$

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Zadatak 8.
Odrediti $a, b \in \mathbb{R}$ tako da funkcija

$$f(x) = \begin{cases} \dfrac{\sin(ax)}{4x}, & x < 0 \\ b^2x^2 + b(x+2), & 0 \le x \le 2 \\ e^{\frac{1}{2-x}} - 1, & x > 2 \end{cases}$$

bude neprekidna za svako $x \in \mathbb{R}$.

Rješenje

Da bi funkcija $f(x)$ bila neprekidna na svom domenu, potrebno je da budu ispunjeni sljedeći uslovi:

$$\begin{aligned}
& \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) \quad \wedge \quad \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2) \\
\Leftrightarrow \quad & \lim_{x\to 0^-} \frac{\sin(ax)}{4x} = \lim_{x\to 0^+} \left(b^2x^2 + b(x+2)\right) = b^2\cdot 0^2 + b\cdot(0+2) \quad \wedge \quad \lim_{x\to 2^-} \left(b^2x^2 + b(x+2)\right) = \lim_{x\to 2+} \left(e^{\frac{1}{2-x}} - 1\right) = b^2\cdot 2^2 + b\cdot(2+2) \\
\Leftrightarrow \quad & \lim_{x\to 0^-} \frac{\sin(ax)}{ax\cdot\frac{4}{a}} = 2b = 2b \quad \wedge \quad 4b^2 + 4b = -1 = 4b^2 + 4b \\
\Leftrightarrow \quad & \frac{a}{4} = 2b \quad \wedge \quad 4b^2 + 4b = -1 \\
\Leftrightarrow \quad & a = 8b \quad \wedge \quad 4b^2 + 4b + 1 = 0 = 0 \\
\Leftrightarrow \quad & a = 8b \quad \wedge \quad (2b+1)^2 = 0 \\
\Leftrightarrow \quad & a = 8b \quad \wedge \quad b = -\frac{1}{2} \\
\Leftrightarrow \quad & a = -4 \quad \wedge \quad b = -\frac{1}{2}.
\end{aligned}$$

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Zadatak 9.
Odrediti konstante A, B i C tako da je

$$\lim_{x\to+\infty}\left(\sqrt{x^4+2x^3}-Ax^2-Bx-C\right)=0.$$

Rješenje

Vrijedi:

$$\begin{aligned}
& \lim_{x\to+\infty}\left(\sqrt{x^4+2x^3}-Ax^2-Bx-C\right)=0\\
\Leftrightarrow\quad & \lim_{x\to+\infty}\left(\sqrt{x^4+2x^3}-\left(Ax^2+Bx\right)\right)-\lim_{x\to+\infty}C=0\\
\Leftrightarrow\quad & \lim_{x\to+\infty}\left(\left(\sqrt{x^4+2x^3}-\left(Ax^2+Bx\right)\right)\cdot\frac{\sqrt{x^4+2x^3}+\left(Ax^2+Bx\right)}{\sqrt{x^4+2x^3}+\left(Ax^2+Bx\right)}\right)=C\\
\Leftrightarrow\quad & \lim_{x\to+\infty}\frac{x^4+2x^3-\left(Ax^2+Bx\right)^2}{\sqrt{x^4+2x^3}+\left(Ax^2+Bx\right)}=C\\
\Leftrightarrow\quad & \lim_{x\to+\infty}\frac{x^4+2x^3-\left(A^2x^4+2ABx^3+B^2x^2\right)}{\sqrt{x^4\cdot\left(1+\frac{2}{x}\right)}+x^2\left(A+\frac{B}{x}\right)}=C\\
\Leftrightarrow\quad & \lim_{x\to+\infty}\frac{x^4\cdot\left(1-A^2\right)+x^3\cdot\left(2-2AB\right)-B^2x^2}{x^2\cdot\left(\sqrt{1+\frac{2}{x}}+A+\frac{B}{x}\right)}=C.
\end{aligned}\tag{1}$$

Kako je vrijednost limesa (1) konačan realan broj C, zaključujemo da vrijedi

$$1-A^2=0 \tag{2}$$
$$2-2AB=0. \tag{3}$$

Iz jednačine (2) vidimo da je $A=\pm1$ pa imamo dvije mogućnosti.

1. $A=-1 \;\Rightarrow\; B=-1$
   Sada limes (1) postaje:

$$\lim_{x\to+\infty}\frac{-\cancel{x^2}}{\cancel{x^2}\cdot\left(\sqrt{1+\overset{0}{\cancel{\tfrac{2}{x}}}}-1-\overset{0}{\cancel{\tfrac{1}{x}}}\right)}=+\infty\neq C$$

   pa ovu mogućnost odbacujemo.

2. $A=1 \;\Rightarrow\; B=1$
   Sada limes (1) postaje:

$$\lim_{x\to+\infty}\frac{-\cancel{x^2}}{\cancel{x^2}\cdot\left(\sqrt{1+\overset{0}{\cancel{\tfrac{2}{x}}}}+1-\overset{0}{\cancel{\tfrac{1}{x}}}\right)}=-\frac{1}{2}$$

   odakle dobijamo rješenje: $A=1, B=1, C=-\dfrac{1}{2}$.

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Zadatak 10.

Ispitati neprekidnost funkcije

$$f(x) = \lim_{n\to+\infty} \frac{x + x^2 e^{nx}}{1 + e^{nx}}.$$

Rješenje

U zavisnosti od vrijednosti promjenljive x, razlikujemo tri mogućnosti.

1. $x > 0 \quad \Rightarrow \quad nx \to +\infty$
   Imamo da je

$$f(x) = \lim_{n\to+\infty} \frac{\cancel{e^{nx}} \cdot \left(x^2 + \cancelto{0}{\frac{x}{e^{nx}}}\right)}{\cancel{e^{nx}} \cdot \left(1 + \cancelto{0}{\frac{1}{e^{nx}}}\right)} = x^2.$$

2. $x < 0 \quad \Rightarrow \quad nx \to -\infty$
   Imamo da je

$$f(x) = \lim_{n\to+\infty} \frac{x + \cancelto{0}{x^2 e^{nx}}}{1 + \cancelto{0}{e^{nx}}} = x.$$

3. $x = 0 \quad \Rightarrow \quad nx = 0$
   Imamo da je

$$f(x) = \lim_{n\to+\infty} \frac{x + x^2 e^{nx}}{1 + e^{nx}} = \frac{x + x^2 e^0}{1 + e^0} = \frac{x + x^2}{2}.$$

   Za $x = 0$ imamo dakle da je

$$f(0) = \frac{0 + 0^2}{2} = 0.$$

Sada je

$$f(x) = \begin{cases} x, & x < 0 \\ 0, & x = 0 \\ x^2, & x > 0 \end{cases}.$$

Kako je

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) = 0,$$

zaključujemo da je $f(x)$ neprekidna funkcija na cijelom svom domenu.