TERMIN 2 - zadaci za samostalan rad - rješenja

★ K1 09.02.2022. ②

Zadatak 1.
Pokazati jednakost $e^{i\pi} + 1 = 0$.

Rješenje

Kako je $e^{i\phi} = \cos\phi + i\sin\phi$, vrijedi

$$e^{i\pi} + 1 = \cos\pi + \sin\pi + 1 = -1 + 0 + 1 = 0$$

što je trebalo dokazati.

Zadatak 2.
Riješiti jednačinu $z^4 + 1 + i = 0$.

Rješenje

Vrijedi:

$$\begin{aligned}
& z^4 + 1 + i = 0 \\
\Leftrightarrow\quad & z^4 = -1 - i \\
\Leftrightarrow\quad & z^4 = \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) \\
\Leftrightarrow\quad & z = \sqrt[8]{2}\left(\cos\frac{\frac{5\pi}{4}+2k\pi}{4} + i\sin\frac{\frac{5\pi}{4}+2k\pi}{4}\right),\ \ k\in\{0,1,2,3\} \\
\Leftrightarrow\quad & z_0 = \sqrt[8]{2}\left(\cos\frac{\frac{5\pi}{4}}{4} + i\sin\frac{\frac{5\pi}{4}}{4}\right) = \sqrt[8]{2}\cdot \operatorname{cis}\frac{5\pi}{16}, \\
& z_1 = \sqrt[8]{2}\left(\cos\frac{\frac{5\pi}{4}+2\pi}{4} + i\sin\frac{\frac{5\pi}{4}+2\pi}{4}\right) = \sqrt[8]{2}\cdot \operatorname{cis}\frac{13\pi}{16}, \\
& z_2 = \sqrt[8]{2}\left(\cos\frac{\frac{5\pi}{4}+4\pi}{4} + i\sin\frac{\frac{5\pi}{4}+4\pi}{4}\right) = \sqrt[8]{2}\cdot \operatorname{cis}\frac{21\pi}{16}, \\
& z_3 = \sqrt[8]{2}\left(\cos\frac{\frac{5\pi}{4}+6\pi}{4} + i\sin\frac{\frac{5\pi}{4}+6\pi}{4}\right) = \sqrt[8]{2}\cdot \operatorname{cis}\frac{29\pi}{16}.
\end{aligned}$$

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Zadatak 3.
Odrediti sva rješenja jednačine

$$3(z-i)^3 = \frac{1-3i}{1+i} - \frac{2i}{1-i}$$

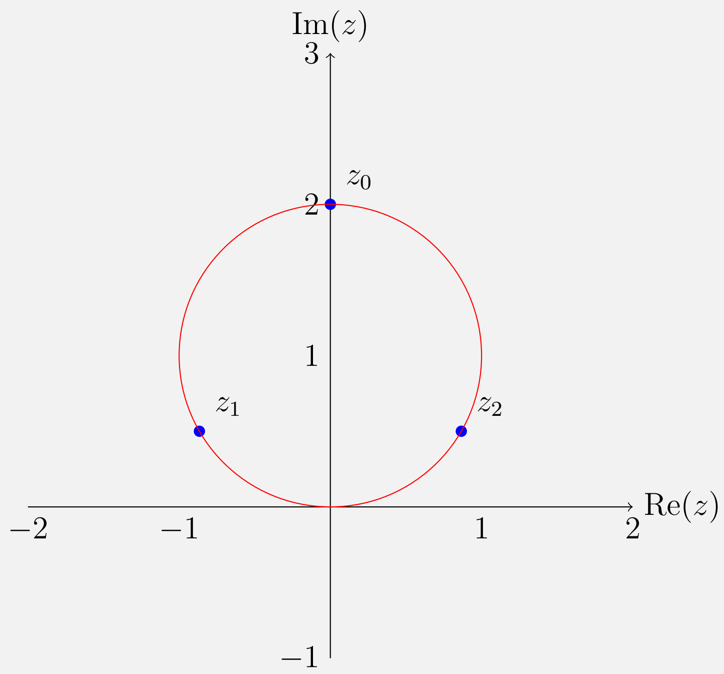
i predstaviti ih u kompleksnoj ravni.

Rješenje

Vrijedi:

$$\begin{aligned}
& 3(z-i)^3 = \frac{1-3i}{1+i} - \frac{2i}{1-i} \\
\Leftrightarrow \quad & 3(z-i)^3 = \frac{1-3i}{1+i}\cdot\frac{1-i}{1-i} - \frac{2i}{1-i}\cdot\frac{1+i}{1+i} \\
\Leftrightarrow \quad & 3(z-i)^3 = \frac{1-3i-i-3}{2} - \frac{2i-2}{2} \\
\Leftrightarrow \quad & 3(z-i)^3 = \frac{-2-4i-2i+2}{2} \\
\Leftrightarrow \quad & 3(z-i)^3 = -3i \\
\Leftrightarrow \quad & (z-i)^3 = -i \\
\Leftrightarrow \quad & z-i = \sqrt[3]{\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}} \\
\Leftrightarrow \quad & z = i + \cos\frac{\frac{3\pi}{2}+2k\pi}{3} + i\sin\frac{\frac{3\pi}{2}+2k\pi}{3}, \quad k\in\{0,1,2\} \\
\Leftrightarrow \quad & z_0 = i + \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i+0+i = 2i \\
& z_1 = i + \cos\frac{\frac{3\pi}{2}+2\pi}{3} + i\sin\frac{\frac{3\pi}{2}+2\pi}{3} = i+\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6} = i - \frac{\sqrt{3}}{2} - \frac{1}{2}i = -\frac{\sqrt{3}}{2} + \frac{1}{2}i \\
& z_2 = i + \cos\frac{\frac{3\pi}{2}+4\pi}{3} + i\sin\frac{\frac{3\pi}{2}+4\pi}{3} = i+\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6} = i + \frac{\sqrt{3}}{2} - \frac{1}{2}i = \frac{\sqrt{3}}{2} + \frac{1}{2}i
\end{aligned}$$

Na narednoj slici prikazana su rješenja početne jednačine z_0, z_1 i z_2 u kompleksnoj ravni.



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Zadatak 4.
U kompleksnoj ravni predstaviti sve kompleksne brojeve z koji zadovoljavaju uslov

1. $z = \overline{z} + 2i$,
2. $\arg z = \dfrac{\pi}{4}$.

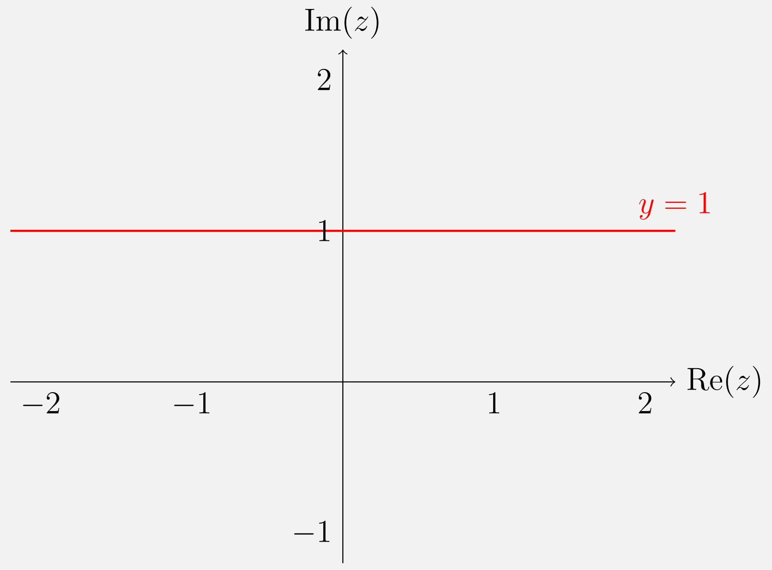
Da li postoji kompleksan broj z koji zadovoljava oba uslova? Ako postoji, odrediti ga i predstaviti ga u kompleksnoj ravni.

Rješenje

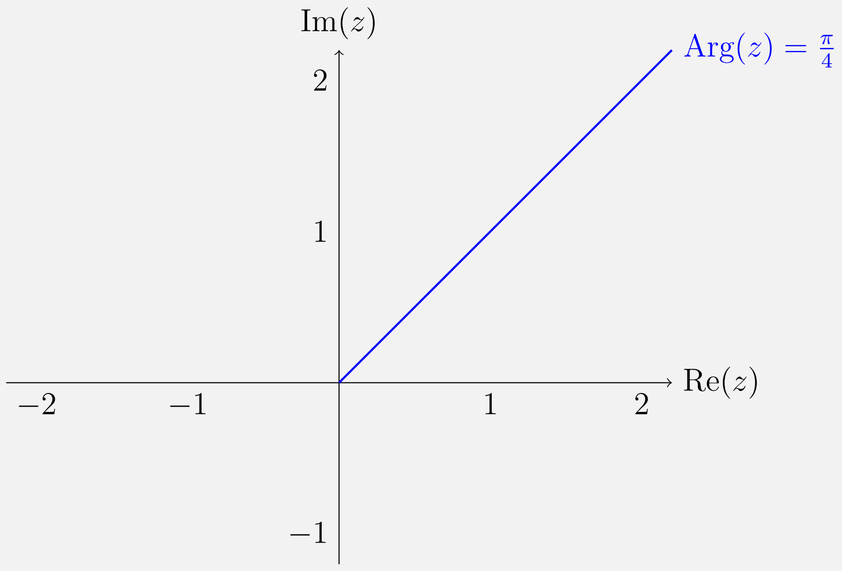
Neka je kompleksan broj $z = x + yi$. Iz prvog uslova imamo:

$$x + yi = x - yi + 2i \Rightarrow 2yi = 2i \Rightarrow y = 1.$$

Odavde zaključujemo da je z kompleksan broj čiji je imaginarni dio jednak 1 pa je skup svih kompleksnih brojeva z koji zadovoljavaju 1. uslov prikazan na narednoj slici:



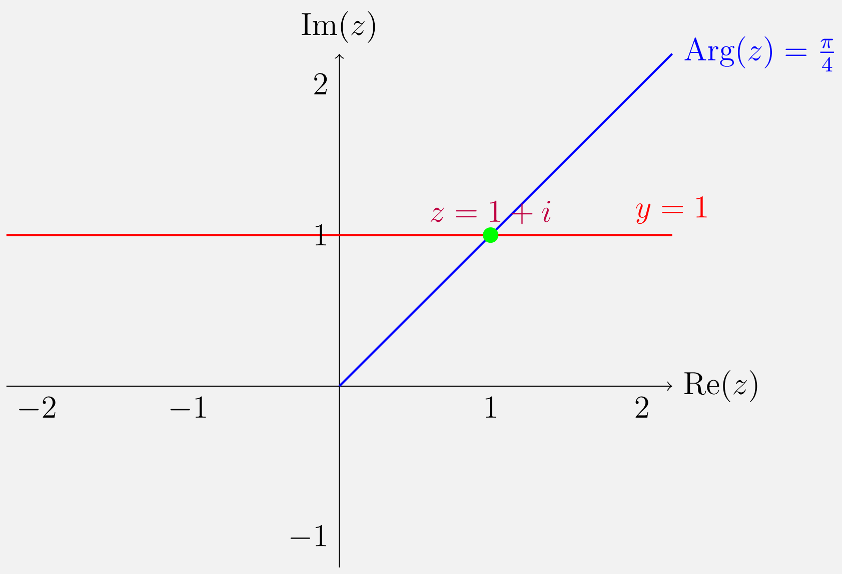
Na narednoj slici su prikazani svi kompleksni brojevi z za koje je ispunjen drugi uslov: $\arg z = \dfrac{\pi}{4}$.



Prilikom traženja kompleksnog broja $z = x + yi$ koji zadovoljava oba uslova koristimo uslov $\mathrm{tg}(\arg z) = \dfrac{\mathrm{Im} z}{\mathrm{Re} z}$ odakle je, nakon uvrštavanja $\arg z = \frac{\pi}{4}$ i $y = 1$:

$$\mathrm{tg}\frac{\pi}{4} = \frac{1}{x} \Rightarrow x = 1$$

pa je $z = 1 + i$.
Na narednoj slici prikazan je kompleksan broj z koji zadovoljava oba uslova.



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Zadatak 5.
Naći sva rješenja jednačine

$$z^3 = \left(\frac{8}{\sqrt{3}}\left(-\sqrt{3}+3i\right)\right)^{50}$$

u skupu kompleksnih brojeva.

Rješenje

Neka je $u = -\sqrt{3}+3i$. Vrijedi

$$|u| = \sqrt{\left(\sqrt{3}\right)^2 + 3^2} = \sqrt{12} = 2\sqrt{3}$$

pa je

$$u = 2\sqrt{3}\cdot\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = 2\sqrt{3}\cdot e^{i\frac{2\pi}{3}}.$$

Sada je

$$z^3 = \left(\frac{8}{\sqrt{3}}\cdot 2\sqrt{3}\cdot e^{i\frac{2\pi}{3}}\right)^{50} = 16^{50}\cdot e^{i\frac{100\pi}{3}} = \left(2^4\right)^{50}\cdot e^{i\cdot\left(16\cdot 2\pi+\frac{4\pi}{3}\right)} = 2^{200}\cdot\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)$$

odakle je na osnovu Muavrove formule

$$z = \sqrt[3]{2^{200}}\cdot\left(\cos\frac{\frac{4\pi}{3}+2k\pi}{3}+i\sin\frac{\frac{4\pi}{3}+2k\pi}{3}\right)$$

pa imamo tri rješenja:

$$z_0 = \sqrt[3]{2^{200}}\left(\cos\frac{4\pi}{9}+i\sin\frac{4\pi}{9}\right),$$
$$z_1 = \sqrt[3]{2^{200}}\left(\cos\frac{10\pi}{9}+i\sin\frac{10\pi}{9}\right),$$
$$z_2 = \sqrt[3]{2^{200}}\left(\cos\frac{16\pi}{9}+i\sin\frac{16\pi}{9}\right).$$

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Zadatak 6.
Izračunati $1-i-z$ ako je

$$z=\frac{(1-i)^{10}\cdot(\sqrt{3}+i)^5}{(-1-i\sqrt{3})^{10}}.$$

Rješenje

Neka je $u=1-i$, $v=\sqrt{3}+i$ i $w=-1-\sqrt{3}i$. Tada je

$$|u|=\sqrt{1^2+(-1)^2}=\sqrt{2}\Rightarrow u=\sqrt{2}\cdot\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)=\sqrt{2}\cdot e^{i\frac{7\pi}{4}},$$
$$|v|=\sqrt{\left(\sqrt{3}\right)^2+1^2}=2\Rightarrow v=2\cdot\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=2\cdot e^{i\frac{\pi}{6}},$$
$$|w|=\sqrt{(-1)^2+\left(-\sqrt{3}\right)^2}=2\Rightarrow w=2\cdot\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=2\cdot e^{i\frac{4\pi}{3}}.$$

Sada je

$$\begin{aligned}
z&=\frac{\left(\sqrt{2}\cdot e^{i\frac{7\pi}{4}}\right)^{10}\cdot\left(2\cdot e^{i\frac{\pi}{6}}\right)^5}{\left(2\cdot e^{i\frac{4\pi}{3}}\right)^{10}}\\
&=\frac{\cancel{2^5}\cdot e^{i\frac{70\pi}{4}}\cdot\cancel{2^5}\cdot e^{i\frac{5\pi}{6}}}{\cancel{2^{10}}\cdot e^{i\frac{40\pi}{3}}}\\
&=e^{i\pi\cdot\left(\frac{70}{4}+\frac{5}{6}-\frac{40}{3}\right)}\\
&=e^{i\frac{210+10-160}{12}\pi}\\
&=e^{i5\pi}\\
&=\cos 5\pi+i\sin 5\pi\\
&=\cos\pi+i\sin\pi\\
&=-1.
\end{aligned}$$

Odavde je

$$1-i-z=1-i-(-1)=2-i.$$

Zadatak 7.
Ako je

$$f(n) = \left(\frac{1+i}{\sqrt{2}}\right)^n + \left(\frac{1-i}{\sqrt{2}}\right)^n,$$

gdje je $n \in \mathbb{N}$, dokazati da je

$$f(n+4) + f(n) = 0.$$

Rješenje

Vrijedi

$$\begin{aligned}
f(n+4)+f(n) &= \left(\frac{1+i}{\sqrt{2}}\right)^{n+4} + \left(\frac{1-i}{\sqrt{2}}\right)^{n+4} + \left(\frac{1+i}{\sqrt{2}}\right)^n + \left(\frac{1-i}{\sqrt{2}}\right)^n \\
&= \left(\frac{1+i}{\sqrt{2}}\right)^n \cdot \left(\frac{1+i}{\sqrt{2}}\right)^4 + \left(\frac{1-i}{\sqrt{2}}\right)^n \cdot \left(\frac{1-i}{\sqrt{2}}\right)^4 + \left(\frac{1+i}{\sqrt{2}}\right)^n + \left(\frac{1-i}{\sqrt{2}}\right)^n \\
&= \left(\frac{1+i}{\sqrt{2}}\right)^n \cdot \frac{\left((1+i)^2\right)^2}{\left(\sqrt{2}\right)^4} + \left(\frac{1-i}{\sqrt{2}}\right)^n \cdot \frac{\left((1-i)^2\right)^2}{\left(\sqrt{2}\right)^4} + \left(\frac{1+i}{\sqrt{2}}\right)^n + \left(\frac{1-i}{\sqrt{2}}\right)^n \\
&= \left(\frac{1+i}{\sqrt{2}}\right)^n \cdot \frac{\left(\cancel{1}+2i+\cancel{i^2}\right)^2}{4} + \left(\frac{1-i}{\sqrt{2}}\right)^n \cdot \frac{\left(\cancel{1}-2i+\cancel{i^2}\right)^2}{4} + \left(\frac{1+i}{\sqrt{2}}\right)^n + \left(\frac{1-i}{\sqrt{2}}\right)^n \\
&= \left(\frac{1+i}{\sqrt{2}}\right)^n \cdot \frac{\cancel{4}i^2}{\cancel{4}} + \left(\frac{1-i}{\sqrt{2}}\right)^n \cdot \frac{\cancel{4}i^2}{\cancel{4}} + \left(\frac{1+i}{\sqrt{2}}\right)^n + \left(\frac{1-i}{\sqrt{2}}\right)^n \\
&= -\left(\frac{1+i}{\sqrt{2}}\right)^n - \left(\frac{1-i}{\sqrt{2}}\right)^n + \left(\frac{1+i}{\sqrt{2}}\right)^n + \left(\frac{1-i}{\sqrt{2}}\right)^n \\
&= 0,
\end{aligned}$$

što je trebalo dokazati.

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Zadatak 8.
Ako su $a, b \in \mathbb{C}$ takvi da je

$$|a| = |b| = 1 \quad \text{i} \quad ab \neq -1$$

dokazati da je

$$\frac{a+b}{1+ab} \in \mathbb{R}.$$

Rješenje

Neka je $a = x_1 + y_1 i$ i $b = x_2 + y_2 i$, $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Tada je, koristeći identitet $z\overline{z} = |z|^2$:

$$\begin{aligned}
\frac{a+b}{1+ab} &= \frac{a+b}{1+ab} \cdot \frac{1-\overline{a}\overline{b}}{1-\overline{a}\overline{b}} \\
&= \frac{a+b-a\overline{a}\overline{b}-\overline{a}b\overline{b}}{1+ab-\overline{a}\overline{b}-a\overline{a}b\overline{b}} \\
&= \frac{a+b-|a|^2\,\overline{b}-\overline{a}|b|^2}{1+ab-\overline{a}\overline{b}-|a|^2|b|^2} \\
&= \frac{a+b-\overline{b}-\overline{a}}{\cancel{1}+ab-\overline{a}\overline{b}-\cancel{1}} \\
&= \frac{(\cancel{x_1}+y_1 i)+(\cancel{x_2}+y_2 i)-(\cancel{x_2}-y_2 i)-(\cancel{x_1}-y_1 i)}{(x_1+y_1 i)(x_2+y_2 i)-(x_1-y_1 i)(x_2-y_2 i)} \\
&= \frac{2y_1 i + 2y_2 i}{\cancel{x_1x_2}+x_2y_1 i+x_1y_2 i-\cancel{y_1y_2}-(\cancel{x_1x_2}-x_2y_1 i-x_1y_2 i-\cancel{y_1y_2})} \\
&= \frac{2i(y_1+y_2)}{2x_1y_2 i+2x_2y_1 i} \\
&= \frac{\cancel{2i}(y_1+y_2)}{\cancel{2i}(x_1y_2+x_2y_1)} \\
&= \frac{y_1+y_2}{x_1y_2+x_2y_1},
\end{aligned}$$

što je realan broj, za $ab \neq -1$, čime je dokaz završen.

Zadatak 9.
Ako za $\varepsilon \in \mathbb{C}$ vrijedi $\varepsilon^{2n} = 1$, odrediti broj z ako je

$$z = 1 + \varepsilon + \varepsilon^2 + \cdots + \varepsilon^{n-1}.$$

Rješenje

Iz početnog uslova imamo

$$\varepsilon^{2n} - 1 = 0 \quad \Leftrightarrow \quad (\varepsilon^n - 1) \cdot (\varepsilon^n + 1) = 0$$

odakle razlikujemo dva slučaja:

1. slučaj: $\varepsilon^n - 1 = 0 \quad \Leftrightarrow \quad \varepsilon^n = 1$
   Odavde je $\varepsilon = \sqrt[n]{1} = \sqrt[n]{\cos 0 + i \sin 0}$ pa je na osnovu Muavrove formule

$$\varepsilon = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}, \quad k \in \{0, 1, 2, \ldots, n-1\}.$$

   Primijetimo da za $k = 0$ imamo $\varepsilon = \cos 0 + i \sin 0 = 1$, odnosno za svako $k \in \{1, 2, \ldots, n-1\}$ vrijedi $\varepsilon \neq 0$. Razlikujemo dva slučaja:

   (a) $\varepsilon = 1$
   U ovom slučaju imamo da je

$$z = 1 + 1 + 1^2 + \cdots + 1^{n-1} = \underbrace{1 + 1 + 1 + \cdots + 1}_{n} = n.$$

   (b) $\varepsilon \neq 1 \quad \Leftrightarrow \quad \varepsilon = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}, \quad k \in \{1, 2, \ldots, n-1\}$
   Ako broj $z = 1 + \varepsilon + \varepsilon^2 + \cdots + \varepsilon^{n-1}$ pomnožimo sa $1 - \varepsilon$, pri čemu je sada sigurno $1 - \varepsilon \neq 0$, imamo:

$$\begin{aligned}
& z \cdot (1 - \varepsilon) = \left(1 + \varepsilon + \varepsilon^2 + \cdots + \varepsilon^{n-1}\right) \cdot (1 - \varepsilon) \\
\Leftrightarrow \quad & z \cdot (1 - \varepsilon) = 1 + \cancel{\varepsilon} + \cancel{\varepsilon^2} + \cdots + \cancel{\varepsilon^{n-1}} - \cancel{\varepsilon} - \cancel{\varepsilon^2} - \cdots - \cancel{\varepsilon^{n-1}} - \varepsilon^n \\
\Leftrightarrow \quad & z = \frac{1 - \varepsilon^n}{1 - \varepsilon} \\
\Leftrightarrow \quad & z = \frac{1 - 1}{1 - \varepsilon} \\
\Leftrightarrow \quad & z = 0.
\end{aligned}$$

   (c) slučaj: $\varepsilon^n + 1 = 0 \quad \Leftrightarrow \quad \varepsilon^n = -1$
   Odavde je $\varepsilon = \sqrt[n]{-1} = \sqrt[n]{\cos \pi + i \sin \pi}$ pa je na osnovu Muavrove formule

$$\varepsilon = \cos \frac{\pi + 2k\pi}{n} + i \sin \frac{\pi + 2k\pi}{n}, \quad k \in \{0, 1, 2, \ldots, n-1\}.$$

   Sada je $\varepsilon \neq 1$ pa nakon množenja z sa $1 - \varepsilon$ dobijamo:

$$\begin{aligned}
& z = \frac{1 - \varepsilon^n}{1 - \varepsilon} \\
\Leftrightarrow \quad & z = \frac{1 - (-1)}{1 - \varepsilon} \\
\Leftrightarrow \quad & z = \frac{2}{1 - \left(\cos\left(\frac{(2k+1)\pi}{n}\right) + i \sin\left(\frac{(2k+1)\pi}{n}\right)\right)} \\
\Leftrightarrow \quad & z = \frac{2}{1 - \cos\left(\frac{(2k+1)\pi}{n}\right) - i \sin\left(\frac{(2k+1)\pi}{n}\right)} \\
\Leftrightarrow \quad & z = \frac{2}{2\sin^2\left(\frac{(2k+1)\pi}{2n}\right) - 2i \sin\left(\frac{(2k+1)\pi}{2n}\right) \cos\left(\frac{(2k+1)\pi}{2n}\right)} \\
\Leftrightarrow \quad & z = \frac{\cancel{2}}{\cancel{2} \sin\left(\frac{(2k+1)\pi}{2n}\right) \cdot \left(\sin\left(\frac{(2k+1)\pi}{2n}\right) - i \cos\left(\frac{(2k+1)\pi}{2n}\right)\right)} \\
\Leftrightarrow \quad & z = \frac{1}{\sin\left(\frac{(2k+1)\pi}{2n}\right) \cdot \left(\sin\left(\frac{(2k+1)\pi}{2n}\right) - i \cos\left(\frac{(2k+1)\pi}{2n}\right)\right)} \cdot \frac{\sin\left(\frac{(2k+1)\pi}{2n}\right) + i \cos\left(\frac{(2k+1)\pi}{2n}\right)}{\sin\left(\frac{(2k+1)\pi}{2n}\right) + i \cos\left(\frac{(2k+1)\pi}{2n}\right)} \\
\Leftrightarrow \quad & z = \frac{\sin\left(\frac{(2k+1)\pi}{2n}\right) + i \cos\left(\frac{(2k+1)\pi}{2n}\right)}{\sin\left(\frac{(2k+1)\pi}{2n}\right) \cdot \left(\sin^2\left(\frac{(2k+1)\pi}{2n}\right) + \cos^2\left(\frac{(2k+1)\pi}{2n}\right)\right)} \\
\Leftrightarrow \quad & z = 1 + i \operatorname{ctg}\left(\frac{(2k+1)\pi}{2n}\right).
\end{aligned}$$

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Zadatak 10.
Ako je $|a| = |b| = |c| = r$, pri čemu su $a, b, c \in \mathbb{C}$, dokazati jednakost

$$\left|\frac{ab + bc + ca}{a + b + c}\right| = r.$$

Rješenje

Neka je

$$\begin{aligned}
a &= |a|\, e^{i\phi_a} = re^{i\phi_a},\\
b &= |b|\, e^{i\phi_b} = re^{i\phi_b},\\
c &= |c|\, e^{i\phi_c} = re^{i\phi_c}.
\end{aligned}$$

Sada je

$$\begin{aligned}
\left|\frac{ab + bc + ca}{a + b + c}\right| &= \left|\frac{re^{i\phi_a} \cdot re^{i\phi_b} + re^{i\phi_b} \cdot re^{i\phi_c} + re^{i\phi_c} \cdot re^{i\phi_a}}{re^{i\phi_a} + re^{i\phi_b} + re^{i\phi_c}}\right|\\
&= \left|\frac{r^2 \cdot \left(e^{i(\phi_a+\phi_b)} + e^{i(\phi_b+\phi_c)} + e^{i(\phi_c+\phi_a)}\right)}{r \cdot \left(e^{i\phi_a} + e^{i\phi_b} + e^{i\phi_c}\right)}\right|\\
&= r \cdot \left|\frac{e^{i(\phi_a+\phi_b+\phi_c)} \cdot \left(e^{-i\phi_c} + e^{-i\phi_a} + e^{-i\phi_b}\right)}{e^{i\phi_a} + e^{i\phi_b} + e^{i\phi_c}}\right|\\
&= r \cdot \left|e^{i(\phi_a+\phi_b+\phi_c)}\right| \cdot \left|\frac{e^{-i\phi_a} + e^{-i\phi_b} + e^{-i\phi_c}}{e^{i\phi_a} + e^{i\phi_b} + e^{i\phi_c}}\right|\\
&= r \cdot \left|\cos\left(\phi_a + \phi_b + \phi_c\right) + i \sin\left(\phi_a + \phi_b + \phi_c\right)\right| \cdot \left|\frac{\cos\left(-\phi_a\right) + i\sin\left(-\phi_a\right) + \cos\left(-\phi_b\right) + i\sin\left(-\phi_b\right) + \cos\left(-\phi_c\right) + i\sin\left(-\phi_c\right)}{\cos\left(\phi_a\right) + i\sin\left(\phi_a\right) + \cos\left(\phi_b\right) + i\sin\left(\phi_b\right) + \cos\left(\phi_c\right) + i\sin\left(\phi_c\right)}\right|\\
&= r \cdot \sqrt{\cos^2\left(\phi_a + \phi_b + \phi_c\right) + \sin^2\left(\phi_a + \phi_b + \phi_c\right)} \cdot \left|\frac{\left(\cos\left(\phi_a\right) + \cos\left(\phi_b\right) + \cos\left(\phi_c\right)\right) - i\left(\sin\left(\phi_a\right) + \sin\left(\phi_b\right) + \sin\left(\phi_c\right)\right)}{\left(\cos\left(\phi_a\right) + \cos\left(\phi_b\right) + \cos\left(\phi_c\right)\right) + i\left(\sin\left(\phi_a\right) + \sin\left(\phi_b\right) + \sin\left(\phi_c\right)\right)}\right|\\
&= r \cdot \frac{\sqrt{\left(\cos\left(\phi_a\right) + \cos\left(\phi_b\right) + \cos\left(\phi_c\right)\right)^2 + \left(\sin\left(\phi_a\right) + \sin\left(\phi_b\right) + \sin\left(\phi_c\right)\right)^2}}{\sqrt{\left(\cos\left(\phi_a\right) + \cos\left(\phi_b\right) + \cos\left(\phi_c\right)\right)^2 + \left(\sin\left(\phi_a\right) + \sin\left(\phi_b\right) + \sin\left(\phi_c\right)\right)^2}}\\
&= r
\end{aligned}$$

što je trebalo dokazati.