TERMIN 11 - zadaci za samostalan rad - rješenja

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Zadatak 1.
Izračunati

a) $\displaystyle\int \frac{\operatorname{arctg}\sqrt{x}}{\sqrt{x}} \cdot \frac{dx}{1+x}$, b) $\displaystyle\int \frac{1+\ln x}{x}\,dx$, c) $\displaystyle\int \frac{e^x+1}{e^x+x}\,dx$, d) $\displaystyle\int \frac{\operatorname{ctg} x}{\ln(\sin x)}\,dx$.

Rješenje

Vrijedi

a)

$$\begin{aligned}
I &= \int \frac{\operatorname{arctg}\sqrt{x}}{\sqrt{x}} \cdot \frac{dx}{1+x} \\
&= \int \frac{2\operatorname{arctg}\sqrt{x}}{1+x} \cdot \frac{dx}{2\sqrt{x}} \qquad \begin{cases} t = \sqrt{x} \\ dt = \frac{1}{2\sqrt{x}}\,dx \end{cases} \\
&= \int \frac{2\operatorname{arctg} t}{1+t^2}\,dt \qquad \begin{cases} u = \operatorname{arctg} t \\ du = \frac{1}{1+t^2}\,dt \end{cases} \\
&= \int 2u\,du \\
&= u^2 + C \\
&= (\operatorname{arctg} t)^2 + C \\
&= \operatorname{arctg}^2\left(\sqrt{x}\right) + C
\end{aligned}$$

b)

$$\begin{aligned}
I &= \int \frac{1+\ln x}{x}\,dx \\
&= \int \frac{1}{x}\,dx + \int \frac{\ln x}{x}\,dx \qquad \begin{cases} t = \ln x \\ dt = \frac{1}{x}\,dx \end{cases} \\
&= \ln|x| + \int t\,dt \\
&= \ln|x| + \frac{t^2}{2} + C \\
&= \ln|x| + \frac{\ln^2(x)}{2} + C
\end{aligned}$$

c)

$$\begin{aligned}
I &= \int \frac{e^x+1}{e^x+x}\,dx \qquad \begin{cases} t = e^x + x \\ dt = (e^x+1)\,dx \end{cases} \\
&= \int \frac{dt}{t} \\
&= \ln|t| + C \\
&= \ln|e^x + x| + C
\end{aligned}$$

d)

$$\begin{aligned}
I &= \int \frac{\operatorname{ctg} x}{\ln(\sin x)}\,dx \\
&= \int \frac{\cos x\,dx}{\sin x \cdot \ln(\sin x)} \qquad \begin{cases} t = \sin x \\ dt = \cos x\,dx \end{cases} \\
&= \int \frac{dt}{t \cdot \ln t} \qquad \begin{cases} u = \ln t \\ du = \frac{dt}{t} \end{cases} \\
&= \int \frac{du}{u} \\
&= \ln|u| + C \\
&= \ln|\ln t| + C \\
&= \ln\big|\ln(\sin x)\big| + C.
\end{aligned}$$

Zadatak 2.
Izračunati

a) $\displaystyle\int_0^4 x\sqrt{x^2+9}\,dx$, b) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{dx}{\cos^2 x\left(1+\operatorname{tg}^2 x\right)}$, c) $\displaystyle\int_{-2021}^{2021} \cos(2021\cdot x)\,dx$, d) $\displaystyle\int_1^4 \frac{\sqrt{x}}{1+\sqrt{x}}\,dx$.

Rješenje

Ako je funkcija f neprekidna na segmentu $[a,b]$ i F njena primitivna funkcija na segmentu $[a,b]$, tada vrijedi Njutn-Lajbnicova formula

$$\int_a^b f(x)\,dx = F(b)-F(a).$$

Kako je podintegralna funkcija neprekidna u granicama integrala, za svaki od narednih određenih integrala, prvo ćemo odrediti neodređeni integral a potom iskoristiti Njutn-Lajbnicovu formulu i odrediti rješenje.

a)

$$\begin{aligned} I_n &= \int x\sqrt{x^2+9}\,dx \qquad \begin{cases} t = x^2+9 \\ dt = 2x\,dx \ \Rightarrow \ x\,dx = \frac{dt}{2} \end{cases} \\ &= \int \sqrt{t}\cdot\frac{dt}{2} \\ &= \frac{1}{2}\cdot\int t^{\frac{1}{2}}\,dt \\ &= \frac{1}{2}\cdot\frac{t^{\frac{3}{2}}}{\frac{3}{2}}+C \\ &= \frac{1}{3}\cdot\sqrt{\left(x^2+9\right)^3}+C \end{aligned}$$

Odavde dobijamo da je vrijednost određenog integrala:

$$\begin{aligned} I_o &= \frac{1}{3}\cdot\sqrt{\left(4^2+9\right)^3}-\frac{1}{3}\cdot\sqrt{\left(0^2+9\right)^3} \\ &= \frac{1}{3}\cdot\left(5^3-3^3\right) \\ &= \frac{98}{3}. \end{aligned}$$

b)

$$\begin{aligned} I_n &= \int \frac{dx}{\cos^2 x\left(1+\operatorname{tg}^2 x\right)} \qquad \begin{cases} t = \operatorname{tg} x \\ dt = \frac{dx}{\cos^2 x} \end{cases} \\ &= \int \frac{dt}{1+t^2} \\ &= \operatorname{arctg} t + C \\ &= \operatorname{arctg}(\operatorname{tg} x) + C \\ &= x + C \end{aligned}$$

Odavde dobijamo da je vrijednost određenog integrala:

$$\begin{aligned} I_o &= \frac{\pi}{3}-\frac{\pi}{4} \\ &= \frac{\pi}{12}. \end{aligned}$$

c)

$$\begin{aligned} I_n &= \int \cos(2021\cdot x)\,dx \qquad \begin{cases} t = 2021x \\ dt = 2021\,dx \ \Rightarrow \ dx = \frac{dt}{2021} \end{cases} \\ &= \int \cos t\cdot\frac{dt}{2021} \\ &= \frac{1}{2021}\cdot\sin t + C \\ &= \frac{1}{2021}\cdot\sin(2021x) \end{aligned}$$

Odavde dobijamo da je vrijednost određenog integrala:

$$\begin{aligned} I_o &= \frac{1}{2021}\cdot\sin(2021\cdot 2021)-\frac{1}{2021}\cdot\sin\big(2021\cdot(-2021)\big) \\ &= \frac{2}{2021}\cdot\sin\left(2021^2\right). \end{aligned}$$

d)

$$
\begin{aligned}
I_n &= \int \frac{\sqrt{x}}{1+\sqrt{x}}\,dx \\
&= \int \frac{2x}{1+\sqrt{x}}\cdot\frac{dx}{2\sqrt{x}} \qquad \begin{cases} t = 1+\sqrt{x} \;\Rightarrow\; \sqrt{x} = t-1 \;\Rightarrow\; x = (t-1)^2 \\ dt = \frac{dx}{2\sqrt{x}} \end{cases} \\
&= \int \frac{2\cdot(t-1)^2}{t}\,dt \\
&= 2\cdot\int \frac{t^2-2t+1}{t}\,dt \\
&= 2\cdot\left(\int t\,dt + \int -2\,dt + \int \frac{dt}{t}\right) \\
&= 2\cdot\left(\frac{t^2}{2} - 2t + \ln|t|\right) + C \\
&= t^2 - 4t + \ln\left(t^2\right) + C \\
&= \left(1+\sqrt{x}\right)^2 - 4\cdot\left(1+\sqrt{x}\right) + \ln\left(1+\sqrt{x}\right)^2 + C
\end{aligned}
$$

Odavde dobijamo da je vrijednost određenog integrala:

$$
\begin{aligned}
I_o &= \left(\left(1+\sqrt{4}\right)^2 - 4\cdot\left(1+\sqrt{4}\right) + \ln\left(1+\sqrt{4}\right)^2\right) - \left(\left(1+\sqrt{1}\right)^2 - 4\cdot\left(1+\sqrt{1}\right) + \ln\left(1+\sqrt{1}\right)^2\right) \\
&= \left(9 - 12 + \ln(9)\right) - \left(4 - 8 + \ln(4)\right) \\
&= 1 + \ln\left(\frac{9}{4}\right).
\end{aligned}
$$

Zadatak 3.
Izračunati

a) $\displaystyle\int \frac{3x^2+4x}{x^2+x}\,dx$, b) $\displaystyle\int \frac{x^4+x^2+2x}{x^2+1}\,dx$, c) $\displaystyle\int \frac{x^2}{2x^2+x+1}\,dx$, d) $\displaystyle\int \frac{x^3+1}{x\,(1-x)^3}\,dx$.

Rješenje

Vrijedi

a)

$$\begin{aligned}
I &= \int \frac{3x^2+4x}{x^2+x}\,dx \\
&= \int \frac{\left(3x^2+3x\right)+x}{x^2+x}\,dx \\
&= \int \frac{3\cancel{\left(x^2+x\right)}}{\cancel{x^2+x}}\,dx + \int \frac{\cancel{x}}{\cancel{x}\,(x+1)}\,dx \\
&= 3x + \ln|x+1| + C
\end{aligned}$$

b)

$$\begin{aligned}
I &= \int \frac{x^4+x^2+2x}{x^2+1}\,dx \\
&= \int \frac{x^2\cdot\left(x^2+1\right)+2x}{x^2+1}\,dx \\
&= \int \frac{x^2\cdot\cancel{\left(x^2+1\right)}}{\cancel{x^2+1}}\,dx + \int \frac{2x\,dx}{x^2+1} \\
&= \frac{x^3}{3} + \int \frac{2x\,dx}{x^2+1} \qquad \begin{cases} t = x^2+1 \\ dt = 2x\,dx \end{cases} \\
&= \frac{x^3}{3} + \int \frac{dt}{t} \\
&= \frac{x^3}{3} + \ln|t| \\
&= \frac{x^3}{3} + \ln\left(x^2+1\right) + C
\end{aligned}$$

c)

$$\begin{aligned}
I &= \int \frac{x^2}{2x^2+x+1}\,dx \\
&= \int \frac{\left(x^2+\frac{x}{2}+\frac{1}{2}\right)-\left(\frac{x}{2}+\frac{1}{2}\right)}{2x^2+x+1}\,dx \\
&= \int \frac{\frac{1}{2}\cdot\cancel{\left(2x^2+x+1\right)}}{\cancel{2x^2+x+1}}\,dx - \frac{1}{2}\cdot\int \frac{x+1}{2x^2+x+1}\,dx \\
&= \frac{x}{2} - \frac{1}{2}\cdot\int \frac{\frac{1}{4}\cdot(4x+1)+\frac{3}{4}}{2x^2+x+1}\,dx \\
&= \frac{x}{2} - \frac{1}{8}\cdot\int \frac{(4x+1)\;dx}{2x^2+x+1} - \frac{3}{8}\cdot\int \frac{dx}{2x^2+x+1}
\end{aligned} \tag{1}$$

Dalje je

$$\begin{aligned}
I_1 &= \int \frac{(4x+1)\;dx}{2x^2+x+1} \qquad \begin{cases} t = 2x^2+x+1 \\ dt = (4x+1)\;dx \end{cases} \\
&= \int \frac{dt}{t} \\
&= \ln|t| + C_1 \\
&= \ln\left(2x^2+x+1\right) + C_1
\end{aligned}$$

i

$$\begin{aligned}
I_2 &= \int \frac{dx}{2x^2+x+1} \\
&= \frac{1}{2}\cdot\int \frac{dx}{x^2+\frac{1}{2}\cdot x+\frac{1}{2}} \\
&= \frac{1}{2}\cdot\int \frac{dx}{x^2+2\cdot x\cdot\frac{1}{4}+\left(\frac{1}{4}\right)^2-\left(\frac{1}{4}\right)^2+\frac{1}{2}} \\
&= \frac{1}{2}\cdot\int \frac{dx}{\left(x+\frac{1}{4}\right)^2+\frac{8-1}{16}} \qquad \begin{cases} t = x+\frac{1}{4} \\ dt = (4x+1)\;dx \end{cases}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2} \cdot \int \frac{dt}{t^2 + \left(\frac{\sqrt{7}}{4}\right)^2} \\
&= \frac{1}{2} \cdot \frac{1}{\frac{\sqrt{7}}{4}} \cdot \operatorname{arctg}\left(\frac{t}{\frac{\sqrt{7}}{4}}\right) + C_2 \\
&= \frac{2}{\sqrt{7}} \cdot \operatorname{arctg}\left(\frac{4 \cdot \left(x + \frac{1}{4}\right)}{\sqrt{7}}\right) + C_2 \\
&= \frac{2}{\sqrt{7}} \cdot \operatorname{arctg}\left(\frac{4x+1}{\sqrt{7}}\right) + C_2.
\end{aligned}
$$

Uvrštavanjem u izraz (1) dobijamo:

$$
\begin{aligned}
I &= \frac{x}{2} - \frac{1}{8} \cdot \ln\left(2x^2 + x + 1\right) - \frac{3}{8} \cdot \frac{2}{\sqrt{7}} \cdot \operatorname{arctg}\left(\frac{4x+1}{\sqrt{7}}\right) + C \\
&= \frac{x}{2} - \frac{1}{8} \cdot \ln\left(2x^2 + x + 1\right) - \frac{3}{4\sqrt{7}} \cdot \operatorname{arctg}\left(\frac{4x+1}{\sqrt{7}}\right) + C
\end{aligned}
$$

d) Korištenjem metode neodređenih koeficijenata imamo da je

$$
\begin{aligned}
& \frac{x^3+1}{x\,(1-x)^3} = \frac{-x^3-1}{x\,(x-1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3} \\
\Leftrightarrow \quad & -x^3 - 1 = A\,(x-1)^3 + Bx\,(x-1)^2 + Cx\,(x-1) + Dx \\
\Leftrightarrow \quad & -x^3 - 1 = A\left(x^3 - 3x^2 + 3x - 1\right) + Bx\left(x^2 - 2x + 1\right) + C\left(x^2 - x\right) + Dx \\
\Leftrightarrow \quad & -x^3 - 1 = Ax^3 - 3Ax^2 + 3Ax - A + Bx^3 - 2Bx^2 + Bx + Cx^2 - Cx + Dx \\
\Leftrightarrow \quad & -x^3 - 1 = (A+B)\,x^3 + (-3A - 2B + C)\,x^2 + (3A + B - C + D)\,x + (-A)
\end{aligned}
$$

odakle dobijamo sistem

$$
\left\{
\begin{aligned}
A + B &= -1 \\
-3A - 2B + C &= 0 \\
3A + B - C + D &= 0 \\
-A &= -1
\end{aligned}
\right.
$$

čije je rješenje

$$A = 1,\ \ B = -2,\ \ C = -1,\ \ D = -2.$$

Sada je početni integral jednak:

$$
\begin{aligned}
I &= \int \frac{1}{x}\,dx + \int \frac{-2}{x-1}\,dx + \int \frac{-1}{(x-1)^2}\,dx + \int \frac{-2}{(x-1)^3}\,dx \\
&= \ln|x| - 2\ln|x-1| + \frac{1}{x-1} - 2 \cdot \frac{(x-1)^{-2}}{-2} + C \\
&= \ln|x| - 2\ln|x-1| + \frac{1}{x-1} + \frac{1}{(x-1)^2} + C
\end{aligned}
$$

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Zadatak 4.
Izračunati

a) $\displaystyle\int \sqrt{1-x^2}\,dx$, b) $\displaystyle\int e^{x+\ln x}\,dx$, c) $\displaystyle\int x^2 \operatorname{arctg} x\,dx$, d) $\displaystyle\int x^3\cos x\,dx$.

Rješenje

U narednim zadacima ćemo koristiti parcijalnu integraciju:

$$\int u\,dv = uv - \int v\,du.$$

a)

$$\begin{aligned}
I = \int \sqrt{1-x^2}\,dx \qquad & \begin{cases} u = \sqrt{1-x^2} \\ du = \frac{1}{2\sqrt{1-x^2}}\cdot(-2x)\,dx = \frac{-x\,dx}{\sqrt{1-x^2}} \\ v = x \\ dv = dx \end{cases} \\
&= x\sqrt{1-x^2} + \int \frac{x^2\,dx}{\sqrt{1-x^2}} \\
&= x\sqrt{1-x^2} + \int \frac{x^2-1+1}{\sqrt{1-x^2}}\,dx \\
&= x\sqrt{1-x^2} + \int \frac{-\left(1-x^2\right)}{\sqrt{1-x^2}}\,dx + \int \frac{1}{\sqrt{1-x^2}}\,dx \\
&= x\sqrt{1-x^2} - \int \sqrt{1-x^2}\,dx + \arcsin x + C_1
\end{aligned}$$

Odavde je

$$\begin{aligned}
& I = x\sqrt{1-x^2} - I + \arcsin x + C_1 \\
\Leftrightarrow \quad & 2I = x\sqrt{1-x^2} + \arcsin x + C_1 \\
\Leftrightarrow \quad & I = \frac{x\sqrt{1-x^2} + \arcsin x}{2} + C.
\end{aligned}$$

Napomena: Zadatak se može uraditi i korištenjem smjene $x = \sin t$.

b)

$$\begin{aligned}
I &= \int e^{x+\ln x}\,dx \\
&= \int e^x \cdot e^{\ln x}\,dx \\
&= \int x\cdot e^x\,dx \qquad \begin{cases} u = x \\ du = dx \\ v = e^x \\ dv = e^x\,dx \end{cases} \\
&= xe^x - \int e^x\,dx \\
&= xe^x - e^x + C
\end{aligned}$$

c)

$$\begin{aligned}
I = \int x^2 \operatorname{arctg} x\,dx \qquad & \begin{cases} u = \operatorname{arctg} x \\ du = \frac{dx}{1+x^2} \\ v = \frac{x^3}{3} \\ dv = x^2\,dx \end{cases} \\
&= \frac{x^3}{3}\operatorname{arctg} x - \int \frac{x^3}{3\cdot(1+x^2)}\,dx \\
&= \frac{x^3}{3}\operatorname{arctg} x - \frac{1}{3}\cdot\int \frac{x^3+x-x}{1+x^2}\,dx \\
&= \frac{x^3}{3}\operatorname{arctg} x - \frac{1}{3}\cdot\int \frac{x\cdot\cancel{(x^2+1)}}{\cancel{x^2+1}}\,dx + \frac{1}{3}\cdot\int\frac{x\,dx}{1+x^2} \\
&= \frac{x^3}{3}\operatorname{arctg} x - \frac{1}{3}\cdot\int x\,dx + \frac{1}{3}\cdot\int \frac{2x\,dx}{2\cdot(1+x^2)} \qquad \begin{cases} t = 1+x^2 \\ dt = 2x\,dx \end{cases} \\
&= \frac{x^3}{3}\operatorname{arctg} x - \frac{1}{3}\cdot\frac{x^2}{2} + \frac{1}{6}\cdot\int\frac{dt}{t} \\
&= \frac{x^3}{3}\operatorname{arctg} x - \frac{x^2}{6} + \frac{\ln|t|}{6} + C \\
&= \frac{2x^3\operatorname{arctg} x - x^2 + \ln\left(1+x^2\right)}{6} + C.
\end{aligned}$$

d)

$$
\begin{aligned}
I &= \int x^3 \cos x \, dx && \begin{cases} u = x^3 \\ du = 3x^2 \, dx \\ v = \sin x \\ dv = \cos x \, dx \end{cases} \\
&= x^3 \sin x - \int 3x^2 \sin x \, dx && \begin{cases} u = 3x^2 \\ du = 6x \, dx \\ v = -\cos x \\ dv = \sin x \, dx \end{cases} \\
&= x^3 \sin x - \left(-3x^2 \cos x - \int -6x \cos x \, dx \right) \\
&= x^3 \sin x + 3x^2 \cos x - \int 6x \cos x \, dx && \begin{cases} u = 6x \\ du = 6 \, dx \\ v = \sin x \\ dv = \cos x \, dx \end{cases} \\
&= x^3 \sin x + 3x^2 \cos x - \left(6x \sin x - \int 6 \sin x \, dx \right) \\
&= x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + C.
\end{aligned}
$$

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Zadatak 5.

Izračunati

a) $\int \frac{x^2}{1+x^6}\,dx$, b) $\int \frac{dx}{\sin x}$, c) $\int \frac{x^5}{x^6-x^3-2}\,dx$, d) $\int \sin x \cdot \left(1-\cos^2 x\right)\,dx$.

Rješenje

a)

$$\begin{aligned} I &= \int \frac{x^2}{1+x^6}\,dx \\ &= \int \frac{3x^2\,dx}{3\cdot\left(1+(x^3)^2\right)} \qquad \begin{cases} t = x^3 \\ dt = 3x^2\,dx \end{cases} \\ &= \frac{1}{3}\cdot\int \frac{dt}{1+t^2} \\ &= \frac{1}{3}\operatorname{arctg} t + C \\ &= \frac{1}{3}\operatorname{arctg}\left(x^3\right) + C \end{aligned}$$

b)

$$\begin{aligned} I &= \int \frac{dx}{\sin x} \\ &= \int \frac{\sin x\,dx}{\sin^2 x} \\ &= -\int \frac{-\sin x\,dx}{1-\cos^2 x} \qquad \begin{cases} t = \cos x \\ dt = -\sin x\,dx \end{cases} \\ &= -\int \frac{dt}{1-t^2} \\ &= \int \frac{dt}{(t-1)(t+1)} \end{aligned} \tag{2}$$

Metodom neodređenih koeficijenata dobijamo:

$$\begin{aligned} & \frac{1}{(t-1)(t+1)} = \frac{A}{t-1} + \frac{B}{t+1} \\ \Leftrightarrow\quad & 1 = A(t+1) + B(t-1) \\ \Leftrightarrow\quad & 1 = (A+B)t + (A-B) \end{aligned}$$

odakle dobijamo sistem

$$\begin{cases} A+B=0 \\ A-B=1 \end{cases}$$

čije je rješenje

$$A = \frac{1}{2},\quad B = -\frac{1}{2}.$$

Vraćanjem u izraz (2) dobijamo

$$\begin{aligned} I &= \int \frac{\frac{1}{2}}{t-1}\,dt + \int \frac{-\frac{1}{2}}{t+1}\,dt \\ &= \frac{1}{2}\cdot\left(\ln|t-1| - \ln|t+1|\right) + C \\ &= \frac{1}{2}\cdot\ln\left|\frac{\cos x - 1}{\cos x + 1}\right| + C \\ &= \frac{1}{2}\cdot\ln\left(\frac{1-\cos x}{1+\cos x}\right) + C. \end{aligned}$$

c)

$$\begin{aligned} I &= \int \frac{x^5}{x^6-x^3-2}\,dx \\ &= \int \frac{x^3\cdot\frac{3x^2\,dx}{3}}{(x^3)^2 - x^3 - 2} \qquad \begin{cases} t = x^3 \\ dt = 3x^2\,dx \end{cases} \\ &= \frac{1}{3}\cdot\int \frac{t}{t^2-t-2}\,dt \end{aligned} \tag{3}$$

Metodom neodređenih koeficijenata dobijamo:

$$\begin{aligned} & \frac{t}{t^2-t-2} = \frac{t}{(t-2)(t+1)} = \frac{A}{t-2} + \frac{B}{t+1} \\ \Leftrightarrow \quad & t = A(t+1) + B(t-2) \\ \Leftrightarrow \quad & t = (A+B)\,t + (A-2B) \end{aligned}$$

odakle dobijamo sistem

$$\begin{cases} A + B = 1 \\ A - 2B = 0 \end{cases}$$

čije je rješenje

$$A = \frac{2}{3}, \;\; B = \frac{1}{3}.$$

Vraćanjem u izraz (3) dobijamo

$$\begin{aligned} I &= \frac{1}{3} \cdot \left(\int \frac{\frac{2}{3}}{t-2}\,dt + \int \frac{\frac{1}{3}}{t+1}\,dx \right) \\ &= \frac{1}{3} \cdot \left(\frac{2}{3} \ln|t-2| + \frac{1}{3} \ln|t+1| \right) + C \\ &= \frac{1}{9} \cdot \left(2 \ln\left|x^3-2\right| + \ln\left|x^3+1\right| \right) + C. \end{aligned}$$

d)

$$\begin{aligned} I &= \int \sin x \cdot \left(1-\cos^2 x\right)\,dx \\ &= \int \left(\cos^2 x - 1\right) \cdot (-\sin x\,dx) \qquad \begin{cases} t = \cos x \\ dt = -\sin x\,dx \end{cases} \\ &= \int \left(t^2-1\right)\,dt \\ &= \int t^2\,dt - \int dt \\ &= \frac{t^3}{3} - t + C \\ &= \frac{\cos^3 x}{3} - \cos x + C \end{aligned}$$

Zadatak 6.
Izračunati

$$\int \frac{x \cdot \sqrt[3]{2+x}}{x+\sqrt[3]{2+x}}\,dx.$$

Rješenje

Uzimanjem smjene $t=\sqrt[3]{2+x}$ dobijamo:

$$\begin{cases} t=\sqrt[3]{2+x} \quad \Rightarrow \quad t^3=2+x \quad \Rightarrow \quad x=t^3-2 \\ dx=3t^2\,dt \end{cases}$$

pa početni integral postaje

$$\begin{aligned} I &= \int \frac{\left(t^3-2\right)\cdot t}{t^3-2+t}\cdot 3t^2\,dt \\ &= 3\cdot\int \frac{t^6-2t^3}{t^3+t-2}\,dt. \end{aligned} \tag{4}$$

Integral (4) predstavlja integral racionalne funkcije pa nakon dijeljenja polinoma t^6-2t^3 polinomom t^3+t-2 dobijamo

$$\begin{array}{l} \left(t^6 \qquad -2t^3\right):\left(t^3+t-2\right)=t^3-t+\dfrac{t^2-2t}{t^3+t-2} \\ \underline{-t^6-t^4+2t^3} \\ \quad -t^4 \\ \quad \underline{\;t^4 \qquad +t^2-2t} \\ \qquad\qquad t^2-2t \end{array}$$

pa vraćanjem u izraz (4) dobijamo

$$\begin{aligned} I &= 3\cdot\int \frac{\left(t^3-t\right)\left(t^3+t-2\right)+\left(t^2-2t\right)}{t^3+t-2}\,dt \\ &= 3\cdot\left(\int \frac{\left(t^3-t\right)\cancel{\left(t^3+t-2\right)}}{\cancel{t^3+t-2}}\,dt+\int \frac{t^2-2t}{(t-1)\left(t^2+t+2\right)}\,dt\right) \\ &= 3\cdot\left(\int t^3\,dt-\int t\,dt+\int \frac{t^2-2t}{(t-1)\left(t^2+t+2\right)}\,dt\right) \\ &= 3\cdot\left(\frac{t^4}{4}-\frac{t^2}{2}+\int \frac{t^2-2t}{(t-1)\left(t^2+t+2\right)}\,dt\right). \end{aligned} \tag{5}$$

Koristeći metod neodređenih koeficijenata imamo:

$$\begin{aligned} & \frac{t^2-2t}{(t-1)\left(t^2+t+2\right)}=\frac{A}{t-1}+\frac{Bt+C}{t^2+t+2} \\ \Leftrightarrow \quad & t^2-2t=A\left(t^2+t+2\right)+(Bt+C)(t-1) \\ \Leftrightarrow \quad & t^2-2t=At^2+At+2A+Bt^2+Ct-Bt-C \\ \Leftrightarrow \quad & t^2-2t=(A+B)\,t^2+(A-B+C)\,t+(2A-C) \end{aligned}$$

odakle dobijamo sistem

$$\begin{cases} A+B=1 \\ A-B+C=-2 \\ 2A-C=0 \end{cases}$$

čije je rješenje

$$A=-\frac{1}{4},\;\; B=\frac{5}{4},\;\; C=-\frac{1}{2}$$

pa uvrštavanjem u izraz (5) dobijamo

$$\begin{aligned} I &= 3\cdot\left(\frac{t^4}{4}-\frac{t^2}{2}+\int \frac{-\frac{1}{4}}{t-1}\,dt+\int \frac{\frac{5}{4}t-\frac{1}{2}}{t^2+t+2}\,dt\right) \\ &= 3\cdot\left(\frac{t^4}{4}-\frac{t^2}{2}-\frac{1}{4}\cdot\ln|t-1|+\int \frac{\frac{5}{8}\cdot(2t+1)-\frac{5}{8}-\frac{1}{2}}{t^2+t+2}\,dt\right) \\ &= 3\cdot\left(\frac{t^4}{4}-\frac{t^2}{2}-\frac{1}{4}\cdot\ln|t-1|+\frac{5}{8}\cdot\int \frac{(2t+1)\;dt}{t^2+t+2}-\int \frac{\frac{5}{8}+\frac{1}{2}}{\left(t^2+2\cdot t\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2\right)-\left(\frac{1}{2}\right)^2+2}\,dt\right) \\ &= 3\cdot\left(\frac{t^4}{4}-\frac{t^2}{2}-\frac{1}{4}\cdot\ln|t-1|+\frac{5}{8}\cdot\ln\left|t^2+t+2\right|-\int \frac{\frac{9}{8}}{\left(t+\frac{1}{2}\right)^2+\frac{7}{4}}\,dt\right) \qquad \begin{cases} u=t+\frac{1}{2} \\ du=dt \end{cases} \end{aligned}$$

$$= 3 \cdot \left(\frac{t^4}{4} - \frac{t^2}{2} - \frac{1}{4} \cdot \ln|t-1| + \frac{5}{8} \cdot \ln\left|t^2+t+2\right| - \frac{9}{8} \cdot \int \frac{du}{u^2 + \left(\frac{\sqrt{7}}{2}\right)^2} \right)$$

$$= 3 \cdot \left(\frac{t^4}{4} - \frac{t^2}{2} - \frac{1}{4} \cdot \ln|t-1| + \frac{5}{8} \cdot \ln\left|t^2+t+2\right| - \frac{9}{8} \cdot \frac{1}{\frac{\sqrt{7}}{2}} \cdot \operatorname{arctg}\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{7}}{2}} \right) \right) + C$$

$$= 3 \cdot \left(\frac{t^4}{4} - \frac{t^2}{2} - \frac{1}{4} \cdot \ln|t-1| + \frac{5}{8} \cdot \ln\left|t^2+t+2\right| - \frac{9}{4\sqrt{7}} \cdot \operatorname{arctg}\left(\frac{2t+1}{\sqrt{7}} \right) \right) + C.$$

Nakon vraćanja početne smjene $t = \sqrt[3]{2+x} = (2+x)^{\frac{1}{3}}$ dobijamo

$$I = 3 \cdot \left(\frac{(2+x)^{\frac{4}{3}}}{4} - \frac{(2+x)^{\frac{2}{3}}}{2} - \frac{1}{4} \cdot \ln\left|(2+x)^{\frac{1}{3}} - 1\right| + \frac{5}{8} \cdot \ln\left|(2+x)^{\frac{2}{3}} + (2+x)^{\frac{1}{3}} + 2\right| - \frac{9}{4\sqrt{7}} \cdot \operatorname{arctg}\left(\frac{2 \cdot (2+x)^{\frac{1}{3}} + 1}{\sqrt{7}} \right) \right) + C.$$

Zadatak 7.
Izračunati
$$\int \sqrt{x^2+2x+2}\,dx.$$

Rješenje

Integral ćemo riješiti korištenjem prve Ojlerove smjene:

$$\begin{aligned}
& \sqrt{x^2+2x+2} = x+t \\
\Leftrightarrow \quad & x^2+2x+2 = (x+t)^2 \\
\Leftrightarrow \quad & \cancel{x^2}+2x+2 = \cancel{x^2}+2xt+t^2 \\
\Leftrightarrow \quad & 2x(1-t) = t^2-2 \\
\Leftrightarrow \quad & x = \frac{t^2-2}{2\cdot(1-t)} \\
\Rightarrow \quad & dx = \frac{\left(t^2-2\right)'\cdot\left(2\cdot(1-t)\right)-\left(t^2-2\right)\cdot\left(2\cdot(1-t)\right)'}{4\cdot(1-t)^2}\,dt \\
\Leftrightarrow \quad & dx = \frac{4t\cdot(1-t)+2\cdot\left(t^2-2\right)}{4\cdot(1-t)^2}\,dt \\
\Leftrightarrow \quad & dx = \frac{4t-4t^2+2t^2-4}{4\cdot(1-t)^2}\,dt \\
\Leftrightarrow \quad & dx = \frac{-t^2+2t-2}{2\cdot(1-t)^2}\,dt.
\end{aligned}$$

Sada je početni integral jednak

$$\begin{aligned}
I &= \int \sqrt{x^2+2x+2}\,dx \\
&= \int \left(\frac{t^2-2}{2\cdot(1-t)}+t\right)\cdot\frac{-t^2+2t-2}{2\cdot(1-t)^2}\,dt \\
&= \int \frac{t^2-2+2t\cdot(1-t)}{2\cdot(1-t)}\cdot\frac{-t^2+2t-2}{2\cdot(1-t)^2}\,dt \\
&= \int \frac{-t^2+2t-2}{2\cdot(1-t)}\cdot\frac{-t^2+2t-2}{2\cdot(1-t)^2}\,dt \\
&= \int \frac{-\left(t^2-2t+2\right)}{-2\cdot(t-1)}\cdot\frac{-\left(t^2-2t+2\right)}{2\cdot(t-1)^2}\,dt \\
&= -\frac{1}{4}\cdot\int \frac{\left(t^2-2t+2\right)^2}{(t-1)^3}\,dt \\
&= -\frac{1}{4}\cdot\int \frac{\left(t^2-2t\right)^2+2\cdot\left(t^2-2t\right)\cdot 2+2^2}{(t-1)^3}\,dt \\
&= -\frac{1}{4}\cdot\int \frac{t^4-4t^3+4t^2+4t^2-8t+4}{(t-1)^3}\,dt \\
&= -\frac{1}{4}\cdot\int \frac{t^4-4t^3+8t^2-8t+4}{(t-1)^3}\,dt \qquad (6)
\end{aligned}$$

Integral (6) predstavlja integral racionalne funkcije pa nakon dijeljenja polinoma $t^4-4t^3+8t^2-8t+4$ polinomom t^3-3t^2+3t-1 dobijamo

$$\begin{array}{l}
\left(\;t^4-4t^3+8t^2-8t+4\right):\left(t^3-3t^2+3t-1\right) = t-1+\dfrac{2t^2-4t+3}{t^3-3t^2+3t-1} \\
\underline{-\,t^4+3t^3-3t^2\;\;+t} \\
\quad -\,t^3+5t^2-7t+4 \\
\quad \underline{\;\;t^3-3t^2+3t-1} \\
\qquad\quad 2t^2-4t+3
\end{array}$$

pa vraćanjem u izraz (6) dobijamo

$$\begin{aligned}
I &= -\frac{1}{4}\cdot\int \frac{(t-1)\cdot(t-1)^3+\left(2t^2-4t+3\right)}{(t-1)^3}\,dt \\
&= -\frac{1}{4}\cdot\left(\int \frac{(t-1)\cdot\cancel{(t-1)^3}}{\cancel{(t-1)^3}}\,dt+\int \frac{2t^2-4t+3}{(t-1)^3}\,dt\right). \qquad (7)
\end{aligned}$$

Koristeći metod neodređenih koeficijenata imamo:

$$\begin{aligned}
& \frac{2t^2-4t+3}{(t-1)^3} = \frac{A}{t-1}+\frac{B}{(t-1)^2}+\frac{C}{(t-1)^3} \\
\Leftrightarrow \quad & 2t^2-4t+3 = A\cdot(t-1)^2+B\cdot(t-1)+C \\
\Leftrightarrow \quad & 2t^2-4t+3 = At^2-2At+A+Bt-B+C \\
\Leftrightarrow \quad & 2t^2-4t+3 = At^2+(-2A+B)\,t+(A-B+C)
\end{aligned}$$

odakle dobijamo sistem

$$\begin{cases} A = 2 \\ -2A + B = -4 \\ A - B + C = 3 \end{cases}$$

čije je rješenje

$$A = 2, \;\; B = 0, \;\; C = 1$$

pa uvrštavanjem u izraz (7) dobijamo

$$\begin{aligned} I &= -\frac{1}{4} \cdot \left(\int t \, dt - \int dt + \int \frac{2}{t-1} \, dt + \int \frac{1}{(t-1)^3} \, dt \right) \\ &= -\frac{1}{4} \cdot \left(\frac{t^2}{2} - t + 2 \cdot \ln|t-1| + \frac{(t-1)^{-2}}{-2} \right) + C \\ &= -\frac{t^2}{8} + \frac{t}{4} - \frac{\ln|t-1|}{2} + \frac{1}{8 \cdot (t-1)^2} + C \end{aligned}$$

Nakon vraćanja početne smjene $t = \sqrt{x^2 + 2x + 2} - x$ dobijamo

$$I = -\frac{\left(\sqrt{x^2+2x+2} - x\right)^2}{8} + \frac{\sqrt{x^2+2x+2} - x}{4} - \frac{\ln\left|\sqrt{x^2+2x+2} - x - 1\right|}{2} + \frac{1}{8 \cdot \left(\sqrt{x^2+2x+2} - x - 1\right)^2} + C.$$

★ ★ ★ ★

Zadatak 8.
Izračunati
$$\int \frac{x^3 \arccos x}{\sqrt{1-x^2}}\, dx.$$

Rješenje

Integral ćemo riješiti korištenjem parcijalne integracije. Uzmimo da je $u = \arccos x$. Tada je $dv = \frac{x^3}{\sqrt{1-x^2}}\, dx$ pa je

$$\begin{aligned}
v &= \int \frac{x^3}{\sqrt{1-x^2}}\, dx \\
&= \int -x^2 \cdot \frac{-x\, dx}{\sqrt{1-x^2}} \qquad \begin{cases} t = \sqrt{1-x^2} \;\Rightarrow\; t^2 = 1-x^2 \;\Rightarrow\; -x^2 = t^2 - 1 \\ dt = \frac{1}{2\cdot\sqrt{1-x^2}} \cdot (-2x)\, dx \;\Rightarrow\; dt = \frac{-x\, dx}{\sqrt{1-x^2}} \end{cases} \\
&= \int \left(t^2 - 1\right) dt \\
&= \frac{t^3}{3} - t + C \\
&= \frac{\left(\sqrt{1-x^2}\right)^3}{3} - \sqrt{1-x^2} + C.
\end{aligned}$$

Sada je

$$\begin{aligned}
I &= \int \frac{x^3 \arccos x}{\sqrt{1-x^2}}\, dx \qquad \begin{cases} u = \arccos x \\ du = -\frac{dx}{\sqrt{1-x^2}} \\ v = \frac{\left(\sqrt{1-x^2}\right)^3}{3} - \sqrt{1-x^2} \\ dv = \frac{x^3}{\sqrt{1-x^2}}\, dx \end{cases} \\
&= \left(\frac{\left(\sqrt{1-x^2}\right)^3}{3} - \sqrt{1-x^2}\right) \cdot \arccos x - \int \left(\frac{\left(\sqrt{1-x^2}\right)^3}{3} - \sqrt{1-x^2}\right) \cdot \left(-\frac{dx}{\sqrt{1-x^2}}\right) \\
&= \left(\frac{\left(\sqrt{1-x^2}\right)^3}{3} - \sqrt{1-x^2}\right) \cdot \arccos x + \int \frac{\left(\sqrt{1-x^2}\right)^3}{3} \cdot \frac{dx}{\sqrt{1-x^2}} - \int \sqrt{1-x^2} \cdot \frac{dx}{\sqrt{1-x^2}} \\
&= \left(\frac{\left(\sqrt{1-x^2}\right)^3}{3} - \sqrt{1-x^2}\right) \cdot \arccos x + \int \frac{1-x^2}{3}\, dx - \int dx \\
&= \left(\frac{\left(\sqrt{1-x^2}\right)^3}{3} - \sqrt{1-x^2}\right) \cdot \arccos x + \int \frac{dx}{3} - \frac{1}{3} \cdot \int x^2\, dx - x + C \\
&= \left(\frac{\left(\sqrt{1-x^2}\right)^3}{3} - \sqrt{1-x^2}\right) \cdot \arccos x + \frac{x}{3} - \frac{1}{3} \cdot \frac{x^3}{3} - x + C \\
&= \left(\frac{\left(\sqrt{1-x^2}\right)^3}{3} - \sqrt{1-x^2}\right) \cdot \arccos x - \frac{2x}{3} - \frac{x^3}{9} + C.
\end{aligned}$$

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Zadatak 9.
Izračunati
$$\int e^{\operatorname{arctg} x} \cdot \left(x^2+1\right)^{-\frac{3}{2}}\, dx.$$

Rješenje

Uzmimo smjenu
$$\begin{cases} x = \operatorname{tg} t \quad \Rightarrow \quad t = \operatorname{arctg} x \\ dx = \frac{dt}{\cos^2 t}. \end{cases}$$

Odavde je
$$\begin{aligned}
I &= \int e^{\operatorname{arctg}(\operatorname{tg} t)} \cdot \left((\operatorname{tg} t)^2 + 1\right)^{-\frac{3}{2}} \cdot \frac{dt}{\cos^2 t} \\
&= \int e^t \cdot \left(\frac{\cos^2 x + \sin^2 x}{\cos^2 x}\right)^{-\frac{3}{2}} \cdot \frac{dt}{\cos^2 t} \\
&= \int e^t \cdot \cos^3 x \cdot \frac{dt}{\cos^2 t} \\
&= \int e^t \cdot \cos t \, dt \qquad \begin{cases} u = \cos t \\ du = -\sin t \, dt \\ v = e^t \\ dv = e^t \, dt \end{cases} \\
&= e^t \cos t - \int -e^t \sin t \, dt \\
&= e^t \cos t + \int e^t \sin t \, dt \qquad \begin{cases} u = \sin t \\ du = \cos t \, dt \\ v = e^t \\ dv = e^t \, dt \end{cases} \\
&= e^t \cos t + e^t \sin t - \int e^t \cos t \, dt \\
&= e^t \cdot (\cos t + \sin t) - I,
\end{aligned}$$

odakle dobijamo
$$\begin{aligned}
& 2I = e^t \cdot (\cos t + \sin t) \\
\Leftrightarrow \qquad & I = \frac{e^t \cdot (\cos t + \sin t)}{2} + C
\end{aligned}$$

pa nakon vraćanja smjene $t = \operatorname{arctg} x$ dobijamo
$$I = \frac{e^{\operatorname{arctg} x} \cdot \left(\cos(\operatorname{arctg} x) + \sin(\operatorname{arctg} x)\right)}{2} + C.$$

★★★★★

Zadatak 10.
Izračunati

$$\int \frac{dx}{x^2 \cdot (x^2+1)^2}.$$

Rješenje

Korištenjem metode neodređenih koeficijenata imamo da je

$$\begin{aligned}
& \frac{1}{x^2 \cdot (x^2+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2} \\
\Leftrightarrow \quad & 1 = Ax \cdot \left(x^2+1\right)^2 + B \cdot \left(x^2+1\right)^2 + (Cx+D) \cdot x^2 \cdot \left(x^2+1\right) + (Ex+F) \cdot x^2 \\
\Leftrightarrow \quad & 1 = Ax \cdot \left(x^4+2x^2+1\right) + B \cdot \left(x^4+2x^2+1\right) + (Cx+D) \cdot \left(x^4+x^2\right) + Ex^3 + Fx^2 \\
\Leftrightarrow \quad & 1 = Ax^5 + 2Ax^3 + Ax + Bx^4 + 2Bx^2 + B + Cx^5 + Dx^4 + Cx^3 + Dx^2 + Ex^3 + Fx^2 \\
\Leftrightarrow \quad & 1 = (A+C)\,x^5 + (B+D)\,x^4 + (2A+C+E)\,x^3 + (2B+D+F)\,x^2 + Ax + B
\end{aligned}$$

odakle dobijamo sistem

$$\begin{cases} A+C=0 \\ B+D=0 \\ 2A+C+E=0 \\ 2B+D+F=0 \\ A=0 \\ B=1 \end{cases}$$

čije je rješenje

$$A=0,\ \ B=1,\ \ C=0,\ \ D=-1,\ \ E=0,\ \ F=-1.$$

Sada je početni integral jednak:

$$\begin{aligned}
I &= \int \frac{1}{x^2}\,dx + \int \frac{-dx}{x^2+1} + \int \frac{-dx}{(x^2+1)^2} \\
&= -\frac{1}{x} - \operatorname{arctg} x - \int \frac{dx}{(x^2+1)^2}
\end{aligned} \tag{8}$$

Riješimo integral

$$\begin{aligned}
I_1 &= \int \frac{dx}{(x^2+1)^2} \qquad \begin{cases} x = \operatorname{tg} t \quad \Rightarrow \quad t = \operatorname{arctg} x \\ dx = \frac{dt}{\cos^2 t} \end{cases} \\
&= \int \frac{\frac{dt}{\cos^2 t}}{\left(\operatorname{tg}^2 x + 1\right)^2} \\
&= \int \frac{\frac{dt}{\cos^2 t}}{\left(\frac{\cos^2 x + \sin^2 x}{\cos^2 x}\right)^2} \\
&= \int \cos^2 t\,dt \\
&= \int \frac{1+\cos 2t}{2}\,dt \\
&= \int \frac{dt}{2} + \frac{1}{2} \cdot \int \cos 2t\,dt \\
&= \frac{t}{2} + \frac{1}{2} \cdot \frac{\sin 2t}{2} + C_1 \\
&= \frac{\operatorname{arctg} x}{2} + \frac{\sin\left(2 \operatorname{arctg} x\right)}{4} + C_1.
\end{aligned}$$

Uvrštavanjem u izraz (8) dobijamo

$$\begin{aligned}
I &= -\frac{1}{x} - \operatorname{arctg} x - \left(\frac{\operatorname{arctg} x}{2} + \frac{\sin\left(2 \operatorname{arctg} x\right)}{4} + C_1\right) \\
&= -\frac{1}{x} - \frac{3 \operatorname{arctg} x}{2} - \frac{\sin\left(2 \operatorname{arctg} x\right)}{4} + C
\end{aligned}$$