(11) $$I = \int \frac{dx}{x^5 - x^2} = \int \frac{dx}{x^2 \cdot (x^3-1)} = \int \frac{dx}{x^2 \cdot (x-1)(x^2+x+1)}$$

Metod neodređenih koeficijenata.

$$\frac{1}{x^2 \cdot (x-1) \cdot (x^2+x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{Dx+E}{x^2+x+1}$$

$$\Leftrightarrow \quad 1 = Ax \cdot (x-1)(x^2+x+1) + B(x-1) \cdot (x^2+x+1) + Cx^2 \cdot (x^2+x+1) + (Dx+E) \cdot x^2 \cdot (x-1)$$

$$= Ax \cdot (x^3-1) + B(x^3-1) + C \cdot (x^4+x^3+x^2) + (Dx+E) \cdot (x^3-x^2)$$

$$= Ax^4 - Ax + Bx^3 - B + Cx^4 + Cx^3 + Cx^2 + Dx^4 + Ex^3 - Dx^3 - Ex^2$$

$$= x^4 \cdot (A + C + D) + x^3 \cdot (B + C - D + E) + x^2 \cdot (C - E) + x \cdot (-A) - B \Rightarrow$$

$$\begin{aligned}
A \quad\quad + C + D \quad\quad &= 0 \\
B + C - D + E &= 0 \\
C \quad\quad - E &= 0 \\
-A \quad\quad\quad\quad &= 0 \\
-B \quad\quad\quad &= 1
\end{aligned}$$

$$A = 0, \quad B = -1, \quad C = \frac{1}{3}, \quad D = -\frac{1}{3}, \quad E = \frac{1}{3}$$

$$I = \int \frac{-1}{x^2}\,dx + \int \frac{\frac{1}{3}}{x-1}\,dx + \int \frac{-\frac{1}{3}x+\frac{1}{3}}{x^2+x+1}\,dx$$

$$= \frac{1}{x} + \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3}\cdot\int \frac{x-1}{x^2+x+1}\,dx$$

$$= \frac{1}{x} + \frac{1}{3}\ln|x-1| - \frac{1}{3}\cdot\int \frac{\frac{1}{2}\cdot(2x+1)-\frac{3}{2}}{x^2+x+1}\,dx$$

$$= \frac{1}{x} + \frac{1}{3}\ln|x-1| - \frac{1}{3}\cdot\left[\frac{1}{2}\int \frac{(2x+1)dx}{x^2+x+1} - \frac{3}{2}\int \frac{dx}{x^2+x+1}\right]$$

$$= \frac{1}{x} + \frac{1}{3}\ln|x-1| - \frac{1}{6}\cdot\ln|x^2+x+1| + \frac{1}{2}\cdot\int \frac{dx}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$

$$= \frac{1}{x} + \frac{1}{3}\ln|x-1| - \frac{1}{6}\cdot\ln|x^2+x+1| + \frac{1}{\cancel{2}}\frac{1}{\frac{\sqrt{3}}{\cancel{2}}}\operatorname{arctg}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

$$\boxed{I = \frac{1}{x} + \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln|x^2+x+1| + \frac{1}{\sqrt{3}}\cdot\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C}$$

(12)

$$I = \int \frac{(9\cos x - 10 - 3\cos^2 x)\cdot \sin x\, dx}{(\cos x + 2)\cdot(\cos^2 x - 4\cos x + 8)} = \begin{cases} t = \cos x \\ dt = -\sin x\, dx \end{cases}$$

$$= \int \frac{(9t - 10 - 3t^2)\cdot(-dt)}{(t+2)\cdot(t^2 - 4t + 8)}$$

$$= \int \frac{3t^2 - 9t + 10}{(t+2)\cdot(t^2-4t+8)}\, dt$$

Metod neodređenih koeficijenata:

$$\frac{3t^2 - 9t + 10}{(t+2)\cdot(t^2-4t+8)} = \frac{A}{t+2} + \frac{Bt + C}{t^2 - 4t + 8}$$

$$\Leftrightarrow \quad 3t^2 - 9t + 10 = A(t^2 - 4t + 8) + (Bt + C)\cdot(t+2)$$

$$= At^2 - 4At + 8A + Bt^2 + Ct + 2Bt + 2C$$

$$= t^2(A+B) + t\cdot(-4A + 2B + C) + (8A + 2C)$$

$$\begin{aligned} A + B \quad\quad &= 3 \\ -4A + 2B + C &= -9 \quad / (-2) \\ 8A \quad\quad + 2C &= 10 \\ \hline A + B &= 3 \quad / \cdot 4 \\ 16A - 4B &= 28 \\ \hline 20A &= 40 \Rightarrow \end{aligned}$$

$$A = 2, \quad B = 1, \quad C = -3$$

$$I = \int \frac{2}{t+2}\,dt + \int \frac{t-3}{t^2-4t+8}\,dt$$

$$= 2\ln|t+2| + \int \frac{\frac{1}{2}\cdot(2t-4)-1}{t^2-4t+8}\,dt$$

$$= 2\ln|t+2| + \frac{1}{2}\int \frac{(2t-4)\,dt}{t^2-4t+8} - \int \frac{dt}{t^2-4t+8}$$

$$= 2\ln|t+2| + \frac{1}{2}\ln|t^2-4t+8| - \int \frac{dt}{(t-2)^2+2^2}$$

$$= 2\ln|t+2| + \frac{1}{2}\ln|t^2-4t+8| - \frac{1}{2}\operatorname{arctg}\left(\frac{t-2}{2}\right) + C$$

$$\boxed{I = 2\ln|2+\cos x| + \frac{1}{2}\ln|\cos^2 x-4\cos x+8| - \frac{1}{2}\operatorname{arctg}\left(\frac{\cos x-2}{2}\right)+C}$$

(13)

$$I = \int \frac{x + \sqrt[3]{x^2} + \sqrt[6]{x}}{x \cdot (1 + \sqrt[3]{x})} \, dx = \begin{cases} t = x^{\frac{1}{6}} \Rightarrow x = t^6 \\ dx = 6t^5 \, dt \end{cases}$$

$$= \int \frac{t^6 + t^4 + t}{t^6 \cdot (1 + t^2)} \cdot 6t^5 \, dt$$

$$= 6 \cdot \int \frac{\cancel{t} \cdot (t^5 + t^3 + 1) \cdot \cancel{t^5}}{\cancel{t^6} \cdot (t^2 + 1)} \, dt$$

$$= 6 \cdot \int \frac{t^5 + t^3 + 1}{t^2 + 1} \, dt$$

$$= 6 \cdot \int \frac{t^3 (t^2 + 1) + 1}{t^2 + 1} \, dt$$

$$= 6 \cdot \left[\int \frac{t^3 \cdot \cancel{(t^2 + 1)}}{\cancel{t^2 + 1}} \, dt + \int \frac{dt}{t^2 + 1} \right]$$

$$= 6 \cdot \left[\frac{t^4}{4} + \operatorname{arctg}(t) \right] + C$$

$$= \frac{3}{2} \left(x^{\frac{1}{6}} \right)^4 + 6 \operatorname{arctg}\left(x^{\frac{1}{6}} \right) + C$$

$$\boxed{I = \frac{3}{2} \cdot \sqrt[3]{x^2} + 6 \operatorname{arctg}\left(\sqrt[6]{x} \right) + C}$$

(14)

$$I = \int \frac{dx}{x \cdot \sqrt{4x^2+4x+3}}$$

★ Integrali oblika

$$\int R\left(x, \sqrt{ax^2+bx+c}\right) dx \quad , \qquad ax^2+bx+c \geq 0 \ \wedge \ a \neq 0$$

se svode na integrale racionalnih funkcija pomoću Ojlerovih smjena:

1° ako je $a>0$ smjena je $\sqrt{ax^2+bx+c} = x\sqrt{a} + t$

2° ako je $c>0$ smjena je $\sqrt{ax^2+bx+c} = xt + \sqrt{c}$

3° inače, ako je $a<0$ i $c \leq 0$ smjena je

$$\sqrt{ax^2+bx+c} = \sqrt{a(x-x_1)(x-x_2)} = t \cdot (x-x_1) \text{ ili } = t(x-x_2)$$

ako su $x_1, x_2 \in \mathbb{R}$.

Biramo prvu Ojlerovu smjenu:

$$\sqrt{4x^2+4x+3} = 2x+t \quad \Rightarrow$$

$$\cancel{4x^2}+4x+3 = \cancel{4x^2}+4xt+t^2 \quad \Rightarrow$$

$$4x \cdot (1-t) = t^2-3 \quad \Rightarrow$$

$$x = \frac{t^2-3}{4 \cdot (1-t)}$$

$$dx = \frac{2t \cdot 4(1-t) - (t^2-3) \cdot 4 \cdot (-1)}{16 \cdot (1-t)^2} dt = \frac{8t - 8t^2 + 4t^2 - 12}{16 \cdot (t-1)^2} dt$$

$$= \frac{-4 \cdot (t^2-2t+3)}{16 \cdot (t-1)^2} dt = \frac{t^2-2t+3}{-4 \cdot (t-1)^2} dt$$

Sada je:

$$I = \int \frac{\frac{t^2-2t+3}{-4\cdot(t-1)^2}}{\frac{t^2-3}{4\cdot(1-t)}\cdot\left(2\cdot\frac{t^2-3}{4\cdot(1-t)}+t\right)}\,dt$$

$$= \int \frac{\frac{t^2-2t+3}{\cancel{-4}\cdot(t-1)^{\cancel{2}}}}{\frac{t^2-3}{\cancel{-4}\cdot(\cancel{t-1})}\cdot\frac{t^2-3+2t(1-t)}{2\cdot(1-t)}}\,dt$$

$$= \int \frac{\frac{t^2-2t+3}{t-1}}{(t^2-3)\cdot\frac{-t^2+2t-3}{2\cdot(1-t)}}\,dt$$

$$= \int \frac{\frac{\cancel{t^2-2t+3}}{\cancel{t-1}}}{(t^2-3)\cdot\frac{-\cancel{(t^2-2t+3)}}{-2(\cancel{t-1})}}\,dt$$

$$= \int \frac{2\,dt}{t^2-3}$$

$$= 2\cdot\int\frac{dt}{t^2-(\sqrt{3})^2}$$

$$= \cancel{2}\cdot\frac{1}{\cancel{2}\sqrt{3}}\cdot\ln\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right| + C$$

$$\boxed{I = \frac{1}{\sqrt{3}}\cdot\ln\left|\frac{\sqrt{4x^2+4x+3}-2x-\sqrt{3}}{\sqrt{4x^2+4x+3}-2x+\sqrt{3}}\right| + C}$$

(15) $$I = \int \frac{x-1}{(x^2+2x)\cdot\sqrt{x^2+2x}}\,dx$$

1° Ojlerova smjena:

$$\sqrt{x^2+2x} = x+t \quad /^2$$

$$\cancel{x^2}+2x = \cancel{x^2}+2xt+t^2$$

$$2x\cdot(1-t) = t^2 \quad \Rightarrow \quad x = \frac{t^2}{2(1-t)}$$

$$dx = \frac{2t\cdot 2(1-t) - t^2\cdot 2\cdot(-1)}{4(1-t)^2}\,dt = \frac{4t-4t^2+2t^2}{4\cdot(t-1)^2}\,dt$$

$$dx = \frac{-t^2+2t}{2\cdot(t-1)^2}\,dt$$

Sada je:

$$I = \int \frac{\frac{t^2}{2\cdot(1-t)} - 1}{\left(\frac{t^2}{2(1-t)}+t\right)^3}\cdot\frac{-t^2+2t}{2(t-1)^2}\,dt$$

$$= \int \frac{\frac{t^2-2(1-t)}{2\cdot(1-t)}}{\left(\frac{t^2+2t(1-t)}{2(1-t)}\right)^3}\cdot\frac{-t^2+2t}{2\cdot(t-1)^2}\,dt$$

$$= \int \frac{\frac{t^2+2t-2}{2\cdot\cancel{(1-t)}}}{\frac{(-t^2+2t)^{\cancel{3}\,2}}{8\cdot\cancel{(1-t)^3}}}\cdot\frac{\cancel{-t^2+2t}}{2\cdot\cancel{(1-t)^2}}\,dt$$

$$= 2\int \frac{t^2+2t-2}{(-t^2+2t)^2}\,dt$$

$$= 2\cdot \int \frac{t^2+2t-2}{(t^2-2t)^2}\,dt$$

$$= 2\cdot \int \frac{t^2+2t-2}{t^2\cdot(t-2)^2}\,dt$$

Metod neodređenih koeficijenata:

$$\frac{t^2+2t-2}{t^2\cdot(t-2)^2} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t-2} + \frac{D}{(t-2)^2}$$

$$\Leftrightarrow\ t^2+2t-2 = At\cdot(t-2)^2 + B\cdot(t-2)^2 + Ct^2(t-2) + Dt^2$$

$$= At\cdot(t^2-4t+4) + B\cdot(t^2-4t+4) + C(t^3-2t^2) + Dt^2$$

$$= At^3 - 4At^2 + 4At + Bt^2 - 4Bt + 4B + Ct^3 - 2Ct^2 + Dt^2$$

$$= (A+C)t^3 + (-4A+B-2C+D)t^2 + (4A-4B)t + 4B$$

$$\begin{aligned} A \qquad\quad + C \qquad\quad &= 0 \\ -4A + B - 2C + D &= 1 \\ 4A - 4B \qquad\qquad &= 2 \\ 4B \qquad\qquad &= -2 \end{aligned}$$

$$A = 0,\quad B = -\frac{1}{2},\quad C = 0,\quad D = \frac{3}{2}$$

$$I = 2 \cdot \left[\int \frac{-\frac{1}{2}}{t^2}\, dt + \int \frac{\frac{3}{2}}{(t-2)^2}\, dt \right]$$

$$= 2 \cdot \left[-\frac{1}{2} \cdot \left(-\frac{1}{t} \right) + \frac{3}{2} \cdot \left(-\frac{1}{t-2} \right) \right] + C$$

$$= 2 \cdot \left(\frac{1}{2t} - \frac{3}{2(t-2)} \right) + C$$

$$= \frac{1}{t} - \frac{3}{t-2} + C$$

Nakon vraćanja smjene : $t = \sqrt{x^2+2x} - x$ dobijamo

$$I = \frac{1}{\sqrt{x^2+2x} - x} + \frac{-3}{\sqrt{x^2+2x} - x - 2} + C$$