TERMIN 2 - zadaci za samostalan rad - rješenja

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Zadatak 1.

Za svaki od narednih linearnih operatora odrediti matricu u odnosu na standardnu bazu, kao i odgovarajući prostor slika i jezgra:

a) $\mathcal{O} : U \to V$ je operator koji svaki vektor $u \in U$ preslikava u $\overrightarrow{0_V}$, pri čemu je $dim\,(U) = m$ i $dim\,(V) = n$.

b) $\mathcal{R} : \mathbb{R}^3 \to \mathbb{R}^3$ je linearni operator koji vrši refleksiju svih vektora u prostoru u odnosu na xy ravan.

c) $\mathcal{A} : \mathbb{R}^3 \to \mathbb{R}^2$ je operator definisan sa $\mathcal{A}\,(a, b, c) = (2a - b + c, a + 2b - 3c)$.

Rješenje

a) Kako je

$$\mathcal{O}\left(\overrightarrow{e_1}\right) = \mathcal{O}\left(\overrightarrow{e_2}\right) = \cdots = \mathcal{O}\left(\overrightarrow{e_n}\right) = \overrightarrow{0_V}$$

matrica linearnog operatora $\mathcal{O}$ je

$$O = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{n\times m}$$

Kako je $rank\,(O) = 0$, zaključujemo da je $Im\,(O) = \overrightarrow{0_V} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1}$. Pošto je

$$dim\,\big(Im\,(O)\big) + dim\,\big(Ker\,(O)\big) = dim\,(U)$$

dobijamo da je $dim\,\big(Ker\,(O)\big) = dim\,(U) = m$, pa je $Ker\,(O) = U$.

b) Kako je

$$\mathcal{R}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \mathcal{R}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{i} \quad \mathcal{R}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix},$$

matrica linearnog operatora $\mathcal{R}$ je

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Kako je $rank\,(R) = 3$, zaključujemo da je $Im\,(R) = \mathbb{R}^3$. Pošto je

$$dim\,\big(Im\,(R)\big) + dim\,\big(Ker\,(R)\big) = dim\,\big(\mathbb{R}^3\big) = 3$$

dobijamo da je $dim\,\big(Ker\,(R)\big) = 0$, pa je $Ker\,(R) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$.

c) Kako je

$$\mathcal{A}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ \mathcal{A}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \text{i} \quad \mathcal{A}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -3 \end{bmatrix},$$

matrica linearnog operatora $\mathcal{A}$ je

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \end{bmatrix}.$$

Kako je $Ker\,(\mathcal{A}) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid \mathcal{A}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$, dobijamo sistem:

$$\begin{cases} 2x_1 - \ \ x_2 + \ \ x_3 = 0 \\ \ \ x_1 + 2x_2 - 3x_3 = 0 \end{cases} \Rightarrow \ 7x_1 - x_2 = 0$$

čije je rješenje $x_2 = 7x_1$ i $x_3 = 5x_1$ pa je

$$Ker\,(\mathcal{A}) = \left\{ \begin{bmatrix} x_1 \\ 7x_1 \\ 5x_1 \end{bmatrix} \mid x_1 \in \mathbb{R} \right\} = Lin \left\{ \begin{bmatrix} 1 \\ 7 \\ 5 \end{bmatrix} \right\}.$$

Kako je $dim\,\big(Ker\,(\mathcal{A})\big) = 1$, zaključujemo da je $dim\,\big(Im\,(\mathcal{A})\big) = 2$, pa je $Im\,(\mathcal{A}) = \mathbb{R}^2$.

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Zadatak 2.
Dato je linearno preslikavanje $\mathcal{A}:\mathbb{R}^2\to\mathbb{R}^2$ sa

$$\mathcal{A}(1,1)=(1,1) \quad \text{i} \quad \mathcal{A}(1,-2)=(1,4).$$

Odrediti matricu preslikavanja $\mathcal{A}$ u odnosu na standardnu bazu.

Rješenje

Predstavimo bazne vektore standardne baze prostora $\mathbb{R}^2$ kao linearnu kombinaciju vektora $(1,1)$ i $(1,-2)$:

$$\begin{aligned}(1,0)&=\alpha\cdot(1,1)+\beta\cdot(1,-2)\\(0,1)&=\gamma\cdot(1,1)+\delta\cdot(1,-2).\end{aligned}$$

Dobijamo sisteme

$$\begin{cases}\alpha+\beta=1\\ \alpha-2\beta=0\end{cases}$$

i

$$\begin{cases}\gamma+\delta=0\\ \gamma-2\delta=1\end{cases}$$

čija su rješenja $\alpha=\frac{2}{3}$, $\beta=\frac{1}{3}$, $\gamma=\frac{1}{3}$, $\delta=-\frac{1}{3}$.
Sada je

$$\begin{aligned}\mathcal{A}(1,0)&=\mathcal{A}\left(\frac{2}{3}\cdot(1,1)+\frac{1}{3}\cdot(1,-2)\right)\\&=\frac{2}{3}\cdot\mathcal{A}(1,1)+\frac{1}{3}\cdot\mathcal{A}(1,-2)\\&=\frac{2}{3}\cdot(1,1)+\frac{1}{3}\cdot(1,4)\\&=\left(\frac{2}{3}+\frac{1}{3},\frac{2}{3}+\frac{4}{3}\right)\\&=(1,2)\\&=1\cdot(1,0)+2\cdot(0,1).\end{aligned}$$

i

$$\begin{aligned}\mathcal{A}(0,1)&=\mathcal{A}\left(\frac{1}{3}\cdot(1,1)-\frac{1}{3}\cdot(1,-2)\right)\\&=\frac{1}{3}\cdot\mathcal{A}(1,1)-\frac{1}{3}\cdot\mathcal{A}(1,-2)\\&=\frac{1}{3}\cdot(1,1)-\frac{1}{3}\cdot(1,4)\\&=\left(\frac{1}{3}-\frac{1}{3},\frac{1}{3}-\frac{4}{3}\right)\\&=(0,-1)\\&=0\cdot(1,0)-1\cdot(0,1).\end{aligned}$$

Odavde konačno dobijamo matricu linearnog operatora $\mathcal{A}$ u odnosu na standardnu bazu

$$A=\begin{bmatrix}1&0\\2&-1\end{bmatrix}.$$

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Zadatak 3.

Neka je V prostor svih matrica $A \in \mathcal{M}_2$ čije jezgro sadrži vektor $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Odrediti bazu i dimenziju prostora V.

Rješenje

Kako je

$$\begin{aligned}
V &= \left\{ A \in \mathcal{M}_2 \;\middle|\; A \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \\
&= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\;\; a,b,c,d \in \mathbb{R} \right\} \\
&= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; \begin{bmatrix} a+2b \\ c+2d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\;\; a,b,c,d \in \mathbb{R} \right\} \\
&= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a+2b=0 \;\; \wedge \;\; c+2d=0,\;\; a,b,c,d \in \mathbb{R} \right\} \\
&= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a=-2b \;\; \wedge \;\; c=-2d,\;\; a,b,c,d \in \mathbb{R} \right\} \\
&= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a+2b=0 \;\; \wedge \;\; c+2d=0,\;\; a,b,c,d \in \mathbb{R} \right\} \\
&= \left\{ \begin{bmatrix} -2b & b \\ -2d & d \end{bmatrix},\;\; b,d \in \mathbb{R} \right\} \\
&= \left\{ b \cdot \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} + d \cdot \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix},\;\; b,d \in \mathbb{R} \right\} \\
&= Lin \left\{ \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix} \right\}.
\end{aligned}$$

Odavde dobijamo da je

$$B_V = \left\{ \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix} \right\}$$

i $dim\,(V) = 2$.

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Zadatak 4.
Ispitati da li postoji matrica A takva da je

$$Im(A) = Lin\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}\right) \quad \text{i} \quad Ker(A) = Lin\left(\begin{bmatrix}1\\2\\3\\4\end{bmatrix}\right).$$

Ako postoji, odrediti jednu takvu matricu.

Rješenje

Kako je $Ker(A) = Lin\left(\begin{bmatrix}1\\2\\3\\4\end{bmatrix}\right)$ i $Im(A) = Lin\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}\right)$ zaključujemo da je $\mathcal{A} : \mathbb{R}^4 \to \mathbb{R}^3$. Sa druge strane imamo da je

$$dim\left(Ker\left(A\right)\right) + dim\left(Im\left(A\right)\right) = dim\left(\mathbb{R}^4\right).$$

Pošto je $dim\left(Ker\left(A\right)\right) = 1$, $dim\left(Im\left(A\right)\right) = 2$, a $dim\left(\mathbb{R}^4\right) = 4$, prethodna jednakost ne vrijedi ni za jednu matricu A linearnog operatora $\mathcal{A}$.

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Zadatak 5.
Neka je dat linearni operator $\mathcal{A} : \mathbb{R}^4 \to \mathbb{R}^3$ definisan sa

$$\mathcal{A}(x, y, z, t) = (x - 3y + z + 2t, x - y + 2t, -x - 3y + 2z - 2t).$$

Odrediti bazu i dimenziju slike i jezgra linearnog operatora $\mathcal{A}$.

Rješenje

Matrica linearnog operatora $\mathcal{A}$ je

$$A = \begin{bmatrix} 1 & -3 & 1 & 2 \\ 1 & -1 & 0 & 2 \\ -1 & -3 & 2 & -2 \end{bmatrix}$$

Odredimo stepenastu formu matrice A:

$$\begin{bmatrix} \boxed{1} & -3 & 1 & 2 \\ 1 & -1 & 0 & 2 \\ -1 & -3 & 2 & -2 \end{bmatrix} \xrightarrow[R_1+R_3]{R_1\cdot(-1)+R_2} \begin{bmatrix} \boxed{1} & -3 & 1 & 2 \\ 0 & \boxed{2} & -1 & 0 \\ 0 & -6 & 3 & 0 \end{bmatrix} \xrightarrow{R_2\cdot 3+R_3} \begin{bmatrix} \boxed{1} & -3 & 1 & 2 \\ 0 & \boxed{2} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Iz stepenaste forme dobijamo da je

$$C(A) = Lin\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ -3 \end{bmatrix} \right\}$$

dok $N(A)$ dobijamo kao rješenje sistema:

$$\begin{cases} x - 3y + z + 2t = 0 \\ \quad 2y - z \qquad\; = 0 \end{cases}.$$

Iz druge jednačine dobijamo $z = 2y$, pa uvrštavanjem u prvu jednačinu dobijamo $x = y - 2t$, pa je

$$N(A) = \left\{ \begin{bmatrix} y - 2t \\ y \\ 2y \\ t \end{bmatrix} \right\} = \left\{ y \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} = Lin\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Kako je $Im(\mathcal{A}) = C(A)$ i $Ker(\mathcal{A}) = N(A)$, zaključujemo da je

$$B_{Im(\mathcal{A})} = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ -3 \end{bmatrix} \right\}$$

i

$$B_{Ker(\mathcal{A})} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Sada je jasno da je $rank(\mathcal{A}) = dim\big(Im(\mathcal{A})\big) = 2$ i $def(\mathcal{A}) = dim\big(Ker(\mathcal{A})\big) = 2$.

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Zadatak 6.
Neka je $\mathcal{A} : \mathbb{R}^2 \to \mathbb{R}^3$ linearni operator definisan sa

$$\mathcal{A}(x, y) = (x - 2y, 2x + y, x + y).$$

Ako su S i T standardne baze prostora $\mathbb{R}^2$ i $\mathbb{R}^3$ redom i

$$S' = \{(1, -1), (0, 1)\}$$

i

$$T' = \{(1, 1, 0), (0, 1, 1), (1, -1, 1)\}$$

odrediti

a) $[\mathcal{A}]_{S,T}$

b) $[\mathcal{A}]_{S,T'}$

c) $[\mathcal{A}]_{S',T}$

d) $[\mathcal{A}]_{S',T'}$.

Rješenje

a) Kako je

$$\begin{aligned}\mathcal{A}(1, 0) &= (1, 2, 1) = 1 \cdot (1, 0, 0) + 2 \cdot (0, 1, 0) + 1 \cdot (0, 0, 1),\\ \mathcal{A}(0, 1) &= (-2, 1, 1) = -2 \cdot (1, 0, 0) + 1 \cdot (0, 1, 0) + 1 \cdot (0, 0, 1),\end{aligned}$$

imamo da je

$$[\mathcal{A}]_{S,T} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

b) Ako stavimo

$$\begin{aligned}\mathcal{A}(1, 0) &= (1, 2, 1) = \alpha_1 \cdot (1, 1, 0) + \beta_1 \cdot (0, 1, 1) + \gamma_1 \cdot (1, -1, 1),\\ \mathcal{A}(0, 1) &= (-2, 1, 1) = \alpha_2 \cdot (1, 1, 0) + \beta_2 \cdot (0, 1, 1) + \gamma_2 \cdot (1, -1, 1),\end{aligned}$$

dobijamo sisteme

$$\begin{cases} \alpha_1 \qquad + \gamma_1 = 1 \\ \alpha_1 + \beta_1 - \gamma_1 = 2 \\ \qquad \beta_1 + \gamma_1 = 1 \end{cases} \quad \text{i} \quad \begin{cases} \alpha_2 \qquad + \gamma_2 = -2 \\ \alpha_2 + \beta_2 - \gamma_2 = 1 \\ \qquad \beta_2 + \gamma_2 = 1 \end{cases}$$

čija su rješenja $\alpha_1 = 1$, $\beta_1 = 1$, $\gamma_1 = 0$ i $\alpha_2 = -\frac{4}{3}$, $\beta_2 = \frac{5}{3}$, $\gamma_2 = -\frac{2}{3}$, pa je

$$[\mathcal{A}]_{S,T'} = \begin{bmatrix} 1 & -\frac{4}{3} \\ 1 & \frac{5}{3} \\ 0 & -\frac{2}{3} \end{bmatrix}.$$

c) Kako je

$$\begin{aligned}\mathcal{A}(1, -1) &= (3, 1, 0) = 3 \cdot (1, 0, 0) + 1 \cdot (0, 1, 0) + 0 \cdot (0, 0, 1),\\ \mathcal{A}(0, 1) &= (-2, 1, 1) = -2 \cdot (1, 0, 0) + 1 \cdot (0, 1, 0) + 1 \cdot (0, 0, 1),\end{aligned}$$

imamo da je

$$[\mathcal{A}]_{S',T} = \begin{bmatrix} 3 & -2 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

d) Ako stavimo

$$\begin{aligned}\mathcal{A}(1, -1) &= (3, 1, 0) = \alpha_1 \cdot (1, 1, 0) + \beta_1 \cdot (0, 1, 1) + \gamma_1 \cdot (1, -1, 1),\\ \mathcal{A}(0, 1) &= (-2, 1, 1) = \alpha_2 \cdot (1, 1, 0) + \beta_2 \cdot (0, 1, 1) + \gamma_2 \cdot (1, -1, 1),\end{aligned}$$

dobijamo sisteme

$$\begin{cases} \alpha_1 \qquad + \gamma_1 = 3 \\ \alpha_1 + \beta_1 - \gamma_1 = 1 \\ \qquad \beta_1 + \gamma_1 = 0 \end{cases} \quad \text{i} \quad \begin{cases} \alpha_2 \qquad + \gamma_2 = -2 \\ \alpha_2 + \beta_2 - \gamma_2 = 1 \\ \qquad \beta_2 + \gamma_2 = 1 \end{cases}$$

čija su rješenja $\alpha_1 = \frac{7}{3}$, $\beta_1 = -\frac{2}{3}$, $\gamma_1 = \frac{2}{3}$ i $\alpha_2 = -\frac{4}{3}$, $\beta_2 = \frac{5}{3}$, $\gamma_2 = -\frac{2}{3}$, pa je

$$[\mathcal{A}]_{S,T'} = \begin{bmatrix} \frac{7}{3} & -\frac{4}{3} \\ -\frac{2}{3} & \frac{5}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix}.$$

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Zadatak 7.
Neka je

$$A=\begin{bmatrix}0&1&2&3&4\\0&1&2&4&6\\0&0&0&1&2\end{bmatrix}.$$

Odrediti baze fundamentalnih potprostora matrice A.

Rješenje

Odredimo stepenastu formu matrice A:

$$\begin{bmatrix}0&\boxed{1}&2&3&4\\0&1&2&4&6\\0&0&0&1&2\end{bmatrix}\xrightarrow{R_1\cdot(-1)+R_2}\begin{bmatrix}0&\boxed{1}&2&3&4\\0&0&0&\boxed{1}&2\\0&0&0&1&2\end{bmatrix}\xrightarrow{R_2\cdot(-1)+R_3}\begin{bmatrix}0&\boxed{1}&2&3&4\\0&0&0&\boxed{1}&2\\0&0&0&0&0\end{bmatrix}.$$

Iz stepenaste forme dobijamo da je

$$B_{C(A)}=\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}3\\4\\1\end{bmatrix}\right\},$$

dok je

$$B_{C\left(A^T\right)}=B_{R(A)}=\left\{\begin{bmatrix}0\\1\\2\\3\\4\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\\2\end{bmatrix}\right\}.$$

Potprostor $N\left(A\right)$ dobijamo kao rješenje sistema:

$$\begin{cases}y+2z+3t+4u=0\\ \qquad\qquad\quad t+2u=0\end{cases}.$$

Iz druge jednačine dobijamo $t=-2u$, pa uvrštavanjem u prvu jednačinu dobijamo $y=-2z+2u$, pa je

$$N\left(A\right)=\left\{\begin{bmatrix}x\\-2z+2u\\z\\-2u\\u\end{bmatrix}\right\}=\left\{x\cdot\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}+z\cdot\begin{bmatrix}0\\-2\\1\\0\\0\end{bmatrix}+u\cdot\begin{bmatrix}0\\2\\0\\-2\\1\end{bmatrix}\right\}=Lin\left\{\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\-2\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\2\\0\\-2\\1\end{bmatrix}\right\}.$$

Odavde zaključujemo da je skup

$$B_{N(A)}=\left\{\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\-2\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\2\\0\\-2\\1\end{bmatrix}\right\}$$

baza vektorskog prostora $N\left(A\right)$.
Za određivanje baze potprostora $N\left(A^T\right)$ posmatrajmo stepenastu formu matrice A^T:

$$\begin{bmatrix}0&0&0\\\boxed{1}&1&0\\2&2&0\\3&4&1\\4&6&2\end{bmatrix}\xrightarrow[R_2\cdot(-4)+R_5]{\substack{R_2\cdot(-2)+R_3\\R_2\cdot(-3)+R_4}}\begin{bmatrix}0&0&0\\\boxed{1}&1&0\\0&0&0\\0&\boxed{1}&1\\0&2&2\end{bmatrix}\xrightarrow{R_4\cdot(-2)+R_5}\begin{bmatrix}0&0&0\\\boxed{1}&1&0\\0&0&0\\0&\boxed{1}&1\\0&0&0\end{bmatrix}\xrightarrow{R_2\leftrightarrow R_1}\begin{bmatrix}\boxed{1}&1&0\\0&0&0\\0&0&0\\0&\boxed{1}&1\\0&0&0\end{bmatrix}\xrightarrow{R_4\leftrightarrow R_2}\begin{bmatrix}\boxed{1}&1&0\\0&\boxed{1}&1\\0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

odakle dobijamo sistem $A^T\cdot\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$, odnosno

$$\begin{cases}x+y\qquad=0\\ \qquad y+z=0\end{cases}.$$

Iz prve jednačine dobijamo $x=-y$, a iz druge jednačine dobijamo $z=-y$ pa je

$$N\left(A^T\right)=\left\{\begin{bmatrix}-y\\y\\-y\end{bmatrix}\right\}=\left\{y\cdot\begin{bmatrix}-1\\1\\-1\end{bmatrix}\right\}=Lin\left\{\begin{bmatrix}-1\\1\\-1\end{bmatrix}\right\},$$

odakle dobijamo da je baza potprostora $N\left(A^T\right)$

$$B_{N\left(A^T\right)}=\left\{\begin{bmatrix}-1\\1\\-1\end{bmatrix}.\right\}$$

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Zadatak 8.
Neka je

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 & -1 & 1 & 1 & -2 \\ -3 & -6 & 2 & -7 & 7 & 0 & -6 & 3 \\ 1 & 2 & 2 & 5 & 3 & 3 & -1 & 0 \\ 2 & 4 & 0 & 6 & -2 & 1 & 3 & 0 \end{bmatrix}.$$

a) Odrediti $def\left(A^T\right)$.

b) Odrediti $rank\,(A)$.

c) Da li kolone $A_{\bullet 4}$, $A_{\bullet 5}$, $A_{\bullet 6}$, $A_{\bullet 7}$ čine bazu prostora $\mathbb{R}^4$?

Rješenje

Na osnovu dimenzija matrice A, zaključujemo da je ona reprezentacija linearnog operatora $\mathcal{A} : U \to V$, pri čemu vrijedi $dim\,(U) = 8$ i $dim\,(V) = 4$. Odredimo stepenastu formu matrice A:

$$\begin{bmatrix} \boxed{1} & 2 & 0 & 3 & -1 & 1 & 1 & -2 \\ -3 & -6 & 2 & -7 & 7 & 0 & -6 & 3 \\ 1 & 2 & 2 & 5 & 3 & 3 & -1 & 0 \\ 2 & 4 & 0 & 6 & -2 & 1 & 3 & 0 \end{bmatrix} \xrightarrow[R_1\cdot(-2)+R_4]{\substack{R_1\cdot 3+R_2 \\ R_1\cdot(-1)+R_3}} \begin{bmatrix} \boxed{1} & 2 & 0 & 3 & -1 & 1 & 1 & -2 \\ 0 & 0 & \boxed{2} & 2 & 4 & 3 & -3 & -3 \\ 0 & 0 & 2 & 2 & 4 & 2 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 4 \end{bmatrix} \xrightarrow{R_2\cdot(-1)+R_3} \begin{bmatrix} \boxed{1} & 2 & 0 & 3 & -1 & 1 & 1 & -2 \\ 0 & 0 & \boxed{2} & 2 & 4 & 3 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 & \boxed{-1} & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow{R_3\cdot(-1)+R_4} \begin{bmatrix} \boxed{1} & 2 & 0 & 3 & -1 & 1 & 1 & -2 \\ 0 & 0 & \boxed{2} & 2 & 4 & 3 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 & \boxed{-1} & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{-1} \end{bmatrix}.$$

a) Iz stepenaste forme matrice A vidimo da je

$$dim\,(C(A)) = dim\left(C\left(A^T\right)\right) = 4.$$

Kako je

$$dim\left(C\left(A^T\right)\right) + dim\left(N\left(A^T\right)\right) = dim\,(V)$$

zaključujemo da je $dim\left(N\left(A^T\right)\right) = 4 - 4 = 0$, pa je dakle $def\left(A^T\right) = 0$.

b) Kako je $rank\,(A) = dim\,(C\,(A))$, zaključujemo da je $rank\,(A) = 4$.

c) Da bi kolone $A_{\bullet 4}$, $A_{\bullet 5}$, $A_{\bullet 6}$, $A_{\bullet 7}$ činile bazu prostora $\mathbb{R}^4$, potrebno je da budu linearno nezavisne. Ispitajmo njihovu linearnu nezavisnost korištenjem stepenaste forme:

$$\begin{bmatrix} \boxed{3} & -1 & 1 & 1 \\ -7 & 7 & 0 & -6 \\ 5 & 3 & 3 & -1 \\ 6 & -2 & 1 & 3 \end{bmatrix} \xrightarrow[R_1\cdot(-2)+R_4]{\substack{R_1\cdot\frac{7}{3}+R_2 \\ R_1\cdot\left(-\frac{5}{3}\right)+R_3}} \begin{bmatrix} \boxed{3} & -1 & 1 & 1 \\ 0 & \boxed{\frac{14}{3}} & \frac{7}{3} & -\frac{11}{3} \\ 0 & \frac{14}{3} & \frac{4}{3} & -\frac{8}{3} \\ 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_2\cdot(-1)+R_3} \begin{bmatrix} \boxed{3} & -1 & 1 & 1 \\ 0 & \boxed{\frac{14}{3}} & \frac{7}{3} & -\frac{11}{3} \\ 0 & 0 & \boxed{-1} & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_3\cdot(-1)+R_4} \begin{bmatrix} \boxed{3} & -1 & 1 & 1 \\ 0 & \boxed{\frac{14}{3}} & \frac{7}{3} & -\frac{11}{3} \\ 0 & 0 & \boxed{-1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Na osnovu stepenaste forme prethodne matrice, vidimo da se kolona $A_{\bullet 7}$ može predstaviti kao linearna kombinacija kolona $A_{\bullet 4}$, $A_{\bullet 5}$ i $A_{\bullet 6}$ pa samim tim kolone $A_{\bullet 4}$, $A_{\bullet 5}$, $A_{\bullet 6}$, $A_{\bullet 7}$ nisu linearno nezavisne i ne čine bazu prostora $\mathbb{R}^4$.

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Zadatak 9.
Dato je linearno preslikavanje $\mathcal{A} : \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2(\mathbb{R})$ definisano sa

$$\mathcal{A}(X) = \begin{bmatrix} 1 & -3 \\ 0 & -3 \end{bmatrix} X + X \begin{bmatrix} -1 & 0 \\ 1 & 3 \end{bmatrix}.$$

Odrediti baze jezgra $Ker(\mathcal{A})$ i slike $Im(\mathcal{A})$.

Rješenje

Odredimo matricu linearnog operatora $\mathcal{A}$ u odnosu na standardnu bazu prostora $\mathcal{M}_2(\mathbb{R})$. Kako je

$$\mathcal{A}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & -3 \\ 0 & -3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\mathcal{A}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & -3 \\ 0 & -3 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix},$$

$$\mathcal{A}\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & -3 \\ 0 & -3 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ -3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ -4 & 0 \end{bmatrix},$$

$$\mathcal{A}\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & -3 \\ 0 & -3 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 0 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix},$$

uzimajući da su $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ i $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ bazni vektori prostora $\mathcal{M}_2(\mathbb{R})$, dobijamo da je matrica A linearnog operatora $\mathcal{A}$:

$$A = \begin{bmatrix} 0 & 1 & -3 & 0 \\ 0 & 4 & 0 & -3 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Odredimo stepenastu formu matrice A:

$$\begin{bmatrix} 0 & \boxed{1} & -3 & 0 \\ 0 & 4 & 0 & -3 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1\cdot(-4)+R_2} \begin{bmatrix} 0 & \boxed{1} & -3 & 0 \\ 0 & 0 & \boxed{12} & -3 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2\cdot\frac{1}{3}+R_3} \begin{bmatrix} 0 & \boxed{1} & -3 & 0 \\ 0 & 0 & \boxed{12} & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Iz stepenaste forme matrice A dobijamo da bazu prostora slika operatora $\mathcal{A}$ čine druga i treća kolona matrice A, odnosno

$$B_{Im(\mathcal{A})} = \left\{ \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -3 & 0 \\ -4 & 0 \end{bmatrix} \right\}.$$

Iz stepenaste forme matrice A dobijamo da je

$$N(A) = \left\{ \begin{bmatrix} x & y \\ z & t \end{bmatrix} \mid \begin{bmatrix} 0 & \boxed{1} & -3 & 0 \\ 0 & 0 & \boxed{12} & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

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$$\begin{cases} y - 3z = 0 \\ 12z - 3t = 0 \end{cases}.$$

Iz prethodnog sistema imamo da je $y = 3z$ i $t = 4z$ pa je sada

$$N(A) = \left\{ \begin{bmatrix} x & 3z \\ z & 4z \end{bmatrix} = x \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + z \cdot \begin{bmatrix} 0 & 3 \\ 1 & 4 \end{bmatrix} \right\}.$$

Sada zaključujemo da je

$$B_{Ker(\mathcal{A})} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 1 & 4 \end{bmatrix} \right\}.$$

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Zadatak 10.
Dato je preslikavanje $\mathcal{T}: P_3 \to \mathcal{M}_2(\mathbb{R})$ sa

$$\mathcal{T}\left(a_3x^3 + a_2x^2 + a_1x + a_0\right) = \begin{bmatrix} a_0 + a_3 - a_2 & a_0 + 2a_1 - a_2 \\ a_3 & a_0 - a_2 \end{bmatrix}.$$

a) Dokazati da je $\mathcal{T}$ linearni operator.

b) Odrediti matricu preslikavanja $\mathcal{T}$.

c) Odrediti jezgro, sliku, defekt i rang preslikavanja $\mathcal{T}$.

Rješenje

a) Neka su $\overrightarrow{P_1(x)} = a_3x^3 + a_2x^2 + a_1x + a_0$ i $\overrightarrow{P_2(x)} = b_3x^3 + b_2x^2 + b_1x + b_0$ vektori iz vektorskog prostora P_3. Da bismo pokazali da je $\mathcal{T}$ linearan operator, dovoljno je da pokažemo da za proizvoljne realne skalare α i β vrijedi

$$\mathcal{T}\left(\alpha \cdot \overrightarrow{P_1(x)} + \beta \cdot \overrightarrow{P_2(x)}\right) = \alpha \cdot \mathcal{T}\left(\overrightarrow{P_1(x)}\right) + \beta \cdot \mathcal{T}\left(\overrightarrow{P_2(x)}\right).$$

Kako je

$$\begin{aligned}
\mathcal{T}\left(\alpha \cdot \overrightarrow{P_1(x)} + \beta \cdot \overrightarrow{P_2(x)}\right) &= \mathcal{T}\left(\alpha \cdot \left(a_3x^3 + a_2x^2 + a_1x + a_0\right) + \beta \cdot \left(b_3x^3 + b_2x^2 + b_1x + b_0\right)\right) \\
&= \mathcal{T}\left(\left(\alpha a_3 + \beta b_3\right)x^3 + \left(\alpha a_2 + \beta b_2\right)x^2 + \left(\alpha a_1 + \beta b_1\right)x + \left(\alpha a_0 + \beta b_0\right)\right) \\
&= \begin{bmatrix} (\alpha a_0 + \beta b_0) + (\alpha a_3 + \beta b_3) - (\alpha a_2 + \beta b_2) & (\alpha a_0 + \beta b_0) + 2\left(\alpha a_1 + \beta b_1\right) - (\alpha a_2 + \beta b_2) \\ (\alpha a_3 + \beta b_3) & (\alpha a_0 + \beta b_0) - (\alpha a_2 + \beta b_2) \end{bmatrix} \\
&= \begin{bmatrix} \alpha \cdot (a_0 + a_3 - a_2) + \beta \cdot (b_0 + b_3 - b_2) & \alpha \cdot (a_0 + 2a_1 - a_2) + \beta \cdot (b_0 + 2b_1 - b_2) \\ \alpha \cdot a_3 + \beta \cdot b_3 & \alpha \cdot (a_0 - a_2) + \beta \cdot (b_0 - b_2) \end{bmatrix} \\
&= \alpha \cdot \begin{bmatrix} a_0 + a_3 - a_2 & a_0 + 2a_1 - a_2 \\ a_3 & a_0 - a_2 \end{bmatrix} + \beta \cdot \begin{bmatrix} b_0 + b_3 - b_2 & b_0 + 2b_1 - b_2 \\ b_3 & b_0 - b_2 \end{bmatrix} \\
&= \alpha \cdot \mathcal{T}\left(\overrightarrow{P_1(x)}\right) + \beta \cdot \mathcal{T}\left(\overrightarrow{P_2(x)}\right),
\end{aligned}$$

zaključujemo da je $\mathcal{T}$ linearni operator.

b) Da bismo odredili matricu operatora $\mathcal{T}$, odredićemo slike baznih vektora standardne baze prostora P_3 u odnosu na standardnu bazu prostora $\mathcal{M}_2(\mathbb{R})$. Kako je

$$\begin{aligned}
\mathcal{T}\left(x^3\right) &= \mathcal{T}\left(1 \cdot x^3 + 0 \cdot x^2 + 0 \cdot x + 0\right) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \\
\mathcal{T}\left(x^2\right) &= \mathcal{T}\left(0 \cdot x^3 + 1 \cdot x^2 + 0 \cdot x + 0\right) = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} = -1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \\
\mathcal{T}(x) &= \mathcal{T}\left(0 \cdot x^3 + 0 \cdot x^2 + 1 \cdot x + 0\right) = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \\
\mathcal{T}(1) &= \mathcal{T}\left(0 \cdot x^3 + 0 \cdot x^2 + 0 \cdot x + 1\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},
\end{aligned}$$

dobijamo da je matrica linearnog operatora $\mathcal{T}$:

$$T = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & -1 & 2 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

c) Odredimo stepenastu formu matrice T:

$$\begin{bmatrix} \boxed{1} & -1 & 0 & 1 \\ 0 & -1 & 2 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1\cdot(-1)+R_3} \begin{bmatrix} \boxed{1} & -1 & 0 & 1 \\ 0 & \boxed{-1} & 2 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \xrightarrow[R_2\cdot(-1)+R_4]{R_2+R_3} \begin{bmatrix} \boxed{1} & -1 & 0 & 1 \\ 0 & \boxed{-1} & 2 & 1 \\ 0 & 0 & \boxed{2} & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix} \xrightarrow{R_3+R_4} \begin{bmatrix} \boxed{1} & -1 & 0 & 1 \\ 0 & \boxed{-1} & 2 & 1 \\ 0 & 0 & \boxed{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Iz stepenaste forme vidimo da je

$$C(T) = Lin\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}\right\}.$$

Odavde zaključujemo da je

$$Im(\mathcal{T}) = Lin\left\{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\right\}$$

i $rank(\mathcal{T}) = 3$.

Sa druge strane, imamo da je

$$N(T) = \left\{ \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \mid \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

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$$\begin{cases} x - y \phantom{{}+2z} + t = 0 \\ - y + 2z + t = 0 \\ \phantom{x - y +{}} 2z \phantom{{}+t} = 0 \end{cases}.$$

Iz treće jednačine dobijamo $z = 0$, pa nakon uvrštavanja u drugu jednačinu dobijamo $y = t$, što dalje implicira $x = 0$. Sada je

$$N(T) = \left\{ \begin{bmatrix} 0 \\ t \\ 0 \\ t \end{bmatrix} \right\} = \left\{ t \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\} = Lin \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}. \right\}$$

Kako je $Ker(\mathcal{T}) \subseteq P_3$ i kako je

$$B_{P_3} = \left\{ x^3, x^2, x, 1 \right\},$$

zaključujemo da je

$$\begin{aligned} Ker(\mathcal{T}) &= Lin \left\{ 0 \cdot x^3 + 1 \cdot x^2 + 0 \cdot x + 1 \cdot 1 \right\} \\ &= Lin \left\{ x^2 + 1 \right\}. \end{aligned}$$

Lako zaključujemo i da je $def(\mathcal{T}) = 1$.