TERMIN 9 - zadaci za samostalan rad - rješenja

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Zadatak 1.
Izračunati prvi izvod funkcije:

a) $f(x) = \dfrac{\ln(3\sin x) + \cos x}{3^x}$,

b) $f(x) = \operatorname{arctg}\left(\sqrt{\dfrac{1-x}{1+x}}\right)$.

Rješenje

Vrijedi:

a)

$$\begin{aligned}
f'(x) &= \frac{\big(\ln(3\sin x) + \cos x\big)' \cdot 3^x - \big(\ln(3\sin x) + \cos x\big) \cdot (3^x)'}{(3^x)^2} \\
&= \frac{\left(\frac{1}{3\sin x} \cdot (3\sin x)' - \sin x\right) \cdot 3^x - \big(\ln(3\sin x) + \cos x\big) \cdot 3^x \cdot \ln 3}{3^{2x}} \\
&= \frac{3^x \cdot \left(\frac{3\cos x}{3\sin x} - \sin x - \big(\ln(3\sin x) + \cos x\big) \cdot \ln 3\right)}{3^{2x}} \\
&= \frac{\operatorname{ctg} x - \sin x - \big(\ln(3\sin x) + \cos x\big) \cdot \ln 3}{3^x}
\end{aligned}$$

b)

$$\begin{aligned}
f'(x) &= \frac{1}{1 + \left(\sqrt{\frac{1-x}{1+x}}\right)^2} \cdot \left(\sqrt{\frac{1-x}{1+x}}\right)' \\
&= \frac{1}{1 + \frac{1-x}{1+x}} \cdot \frac{1}{2 \cdot \sqrt{\frac{1-x}{1+x}}} \cdot \left(\frac{1-x}{1+x}\right)' \\
&= \frac{1}{\frac{1+x+1-x}{1+x}} \cdot \frac{1}{2 \cdot \sqrt{\frac{1-x}{1+x}}} \cdot \frac{(1-x)' \cdot (1+x) - (1-x) \cdot (1+x)'}{(1+x)^2} \\
&= \frac{1+x}{2} \cdot \frac{1}{2 \cdot \sqrt{\frac{1-x}{1+x}}} \cdot \frac{-(1+x) - (1-x)}{(1+x)^2} \\
&= \frac{1+x}{2} \cdot \frac{\sqrt{1+x}}{2 \cdot \sqrt{1-x}} \cdot \frac{-2}{(1+x)^2} \\
&= \frac{-1}{2 \cdot \sqrt{1-x} \cdot \sqrt{1+x}} \\
&= -\frac{1}{2\sqrt{1-x^2}}
\end{aligned}$$

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Zadatak 2.
Izračunati prvi i drugi izvod funkcije:

a) $f(x) = e^{e^2}$,

b) $f(x) = e^{x^2}$,

c) $f(x) = x^{e^2}$,

d) $f(x) = x^{x^2}$.

Rješenje

a) Kako je $f(x) = e^{e^2}$ konstanta, vrijedi

$$f'(x) = 0,$$

$$f''(x) = 0.$$

b) Vrijedi

$$\begin{aligned} f'(x) &= e^{x^2} \cdot \left(x^2\right)' \\ &= e^{x^2} \cdot 2x, \end{aligned}$$

$$\begin{aligned} f''(x) &= \left(e^{x^2} \cdot 2x\right)' \\ &= 2 \cdot \left(\left(e^{x^2}\right)' \cdot x + e^{x^2} \cdot (x)'\right) \\ &= 2 \cdot \left(e^{x^2} \cdot 2x \cdot x + e^{x^2}\right) \\ &= 2e^{x^2} \cdot \left(2x^2 + 1\right). \end{aligned}$$

c) Vrijedi

$$f'(x) = e^2 \cdot \left(x^{e^2-1}\right),$$

$$\begin{aligned} f''(x) &= \left(e^2 \cdot \left(x^{e^2-1}\right)\right)' \\ &= e^2 \cdot \left(e^2 - 1\right) \cdot x^{e^2-2}. \end{aligned}$$

d) Vrijedi

$$\begin{aligned} f'(x) &= \left(e^{\ln\left(x^{x^2}\right)}\right)' \\ &= \left(e^{x^2 \cdot \ln x}\right)' \\ &= e^{x^2 \ln x} \cdot \left(x^2 \cdot \ln x\right)' \\ &= e^{x^2 \ln x} \cdot \left(2x \cdot \ln x + x^2 \cdot \frac{1}{x}\right) \\ &= e^{x^2 \ln x} \cdot (2x \ln x + x), \end{aligned}$$

$$\begin{aligned} f''(x) &= \left(e^{x^2 \ln x} \cdot (2x \ln x + x)\right)' \\ &= \left(e^{x^2 \ln x}\right)' \cdot (2x \ln x + x) + e^{x^2 \ln x} \cdot (2x \ln x + x)' \\ &= e^{x^2 \ln x} \cdot (2x \ln x + x) \cdot (2x \ln x + x) + e^{x^2 \ln x} \cdot \left((2x)' \ln x + 2x \cdot (\ln x)' + 1\right) \\ &= e^{x^2 \ln x} \cdot (2x \ln x + x)^2 + e^{x^2 \ln x} \cdot \left(2 \ln x + 2x \cdot \frac{1}{x} + 1\right) \\ &= e^{x^2 \ln x} \cdot \left((2x \ln x + x)^2 + 2 \ln x + 3\right). \end{aligned}$$

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Zadatak 3.
Izračunati graničnu vrijednost

a) $\lim\limits_{x\to\frac{\pi}{4}} \dfrac{\ln(\operatorname{tg} x)}{\sin x - \cos x}$,

b) $\lim\limits_{x\to 0^+} \ln x \cdot \operatorname{tg} x$.

Rješenje

Vrijedi:

a)

$$
\begin{aligned}
L &= \lim_{x\to\frac{\pi}{4}} \frac{\ln(\operatorname{tg} x)}{\sin x - \cos x}\\
&= \lim_{x\to\frac{\pi}{4}} \frac{\big(\ln(\operatorname{tg} x)\big)'}{(\sin x - \cos x)'}\\
&= \lim_{x\to\frac{\pi}{4}} \frac{\frac{1}{\operatorname{tg} x}\cdot(\operatorname{tg} x)'}{\cos x - (-\sin x)}\\
&= \lim_{x\to\frac{\pi}{4}} \frac{\frac{1}{\operatorname{tg} x}\cdot\frac{1}{\cos^2 x}}{\cos x + \sin x}\\
&= \frac{\frac{1}{\operatorname{tg}\frac{\pi}{4}}\cdot\frac{1}{\cos^2\frac{\pi}{4}}}{\cos\frac{\pi}{4} + \sin\frac{\pi}{4}}\\
&= \frac{\frac{1}{\left(\frac{\sqrt{2}}{2}\right)^2}}{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}}\\
&= \frac{\frac{1}{\frac{1}{2}}}{2\cdot\frac{\sqrt{2}}{2}}\\
&= \frac{2}{\sqrt{2}}\\
&= \sqrt{2},
\end{aligned}
$$

b)

$$
\begin{aligned}
L &= \lim_{x\to 0^+} \ln x \cdot \operatorname{tg} x\\
&= \lim_{x\to 0^+} \frac{\ln x}{\operatorname{ctg} x}\\
&= \lim_{x\to 0^+} \frac{(\ln x)'}{(\operatorname{ctg} x)'}\\
&= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{\sin^2 x}}\\
&= \lim_{x\to 0^+} \frac{-\sin^2 x}{x}\\
&= \lim_{x\to 0^+} \left(\cancelto{1}{\frac{\sin x}{x}}\cdot\left(\cancelto{0}{-\sin x}\right)\right)\\
&= 0.
\end{aligned}
$$

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Zadatak 4.
Izračunati n-ti izvod funkcije:

a) $f(x) = \ln x$,

b) $f(x) = x^{n-1} \cdot \ln x$.

Rješenje

Vrijedi:

a) Vrijedi

$$\begin{aligned}
f'(x) &= \frac{1}{x} = x^{-1} \\
f''(x) &= (-1) \cdot x^{-2} \\
f'''(x) &= (-1) \cdot (-2) \cdot x^{-3} \\
f^{IV}(x) &= (-1) \cdot (-2) \cdot (-3) \cdot x^{-4} \\
&\vdots \\
f^{(n)}(x) &= (-1) \cdot (-2) \cdot (-3) \cdot \dots \cdot \big(-(n-1)\big) \cdot x^{-n} \\
&= (-1)^{n-1} \cdot (n-1)! \cdot x^{-n}.
\end{aligned}$$

Dokaz je moguće izvesti i primjenom principa matematičke indukcije jer vrijedi:

$$\begin{aligned}
f'(x) &= (-1)^{1-1} \cdot (1-1)! \cdot x^{-1} \\
&= \frac{1}{x}
\end{aligned}$$

i

$$\begin{aligned}
f^{(n+1)}(x) &= \left(f^{(n)}(x)\right)' \\
&= \left((-1)^{n-1} \cdot (n-1)! \cdot x^{-n}\right)' \\
&= (-1)^{n-1} \cdot (n-1)! \cdot (-n) \cdot x^{-(n+1)} \\
&= (-1)^{n} \cdot n! \cdot x^{-(n+1)}.
\end{aligned}$$

b) Kako je

$$(x^m)^{(n)} = m \cdot (m-1) \cdot (m-2) \dots (m-n+1)\, x^{m-n},$$

pri čemu je $(x^m)^{(n)} = 0$ za $n > m$, te koristeći Lajbnicovu formulu

$$(f \cdot g)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} \cdot g^{(n-k)}$$

i dio zadatka a) imamo da je

$$\begin{aligned}
\left(x^{n-1} \cdot \ln x\right)^{(n)} &= \sum_{k=0}^{n} \binom{n}{k} \cdot \left(x^{n-1}\right)^{(k)} \cdot (\ln x)^{(n-k)} \\
&= \sum_{k=0}^{n-1} \binom{n}{k} \cdot \Big(\big((n-1) \cdot (n-2) \dots (n-1-k+1)\big) \cdot x^{n-1-k}\Big) \cdot \Big((-1)^{n-k-1} \cdot (n-k-1)! \cdot x^{-(n-k)}\Big) \\
&= \sum_{k=0}^{n-1} \binom{n}{k} \cdot \big((n-1) \cdot (n-2) \dots (n-k) \cdot (n-k-1)!\big) \cdot (-1)^{n-k-1} \cdot x^{n-1-k} \cdot x^{-n+k} \\
&= \sum_{k=0}^{n-1} \binom{n}{k} \cdot (n-1)! \cdot (-1)^{n-1-k} \cdot x^{n-1-k-n+k} \\
&= (n-1)! \cdot \sum_{k=0}^{n-1} \binom{n}{k} \cdot (-1)^{n-1-k} \cdot x^{-1} \\
&= -(n-1)! \cdot \frac{1}{x} \cdot \sum_{k=0}^{n-1} \binom{n}{k} \cdot (-1)^{n-k} \\
&= -(n-1)! \cdot \frac{1}{x} \cdot \left(\left(\sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^{n-k}\right) - \binom{n}{n} \cdot (-1)^{n-n}\right) \\
&= \frac{-(n-1)!}{x} \cdot \left(\left(\sum_{k=0}^{n} \binom{n}{k} \cdot 1^k \cdot (-1)^{n-k}\right) - 1\right) \\
&= \frac{-(n-1)!}{x} \cdot \left(\big(1 + (-1)\big)^n - 1\right) \\
&= \frac{(n-1)!}{x}.
\end{aligned}$$

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Zadatak 5.
Izračunati graničnu vrijednost

a) $\lim\limits_{x\to 0} \dfrac{\ln(1+x)-x}{\operatorname{tg}^2 x}$,

b) $\lim\limits_{x\to 0} x^{\sin x}$.

Rješenje

Vrijedi:

a)

$$\begin{aligned}
L &= \lim_{x\to 0} \frac{\ln(1+x)-x}{\operatorname{tg}^2 x} \\
&= \lim_{x\to 0} \frac{\big(\ln(1+x)-x\big)'}{\big(\operatorname{tg}^2 x\big)'} \\
&= \lim_{x\to 0} \frac{\frac{1}{1+x}-1}{2\operatorname{tg} x\cdot(\operatorname{tg} x)'} \\
&= \lim_{x\to 0} \frac{\frac{1-(1+x)}{1+x}}{2\cdot\frac{\sin x}{\cos x}\cdot\frac{1}{\cos^2 x}} \\
&= \lim_{x\to 0} \frac{\frac{-x}{1+x}}{\frac{2\sin x}{\cos^3 x}} \\
&= -\frac{1}{2}\cdot\lim_{x\to 0} \frac{\cos^3 x\cdot x}{(1+x)\cdot\sin x} \\
&= -\frac{1}{2}\cdot\lim_{x\to 0} \frac{\cancelto{1}{\cos^3 x}}{\cancelto{1}{1+x}}\cdot\lim_{x\to 0}\frac{1}{\cancelto{1}{\frac{\sin x}{x}}} \\
&= -\frac{1}{2},
\end{aligned}$$

b)

$$\begin{aligned}
L &= \lim_{x\to 0} x^{\sin x} \\
&= \lim_{x\to 0} e^{\ln x^{\sin x}} \\
&= \lim_{x\to 0} e^{\sin x\cdot\ln x} \\
&= e^{\lim\limits_{x\to 0}\frac{\ln x}{\frac{1}{\sin x}}} \\
&= e^{\lim\limits_{x\to 0}\frac{(\ln x)'}{\left(\frac{1}{\sin x}\right)'}} \\
&= e^{\lim\limits_{x\to 0}\frac{\frac{1}{x}}{\frac{1'\cdot\sin x-1\cdot(\sin x)'}{\sin^2 x}}} \\
&= e^{-\lim\limits_{x\to 0}\frac{\sin^2 x}{x\cdot\cos x}} \\
&= e^{-\lim\limits_{x\to 0}\left(\cancelto{1}{\frac{\sin x}{x}}\cdot\cancelto{0}{\operatorname{tg} x}\right)} \\
&= e^0 \\
&= 1.
\end{aligned}$$

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Zadatak 6.
Odrediti parametar k tako da prava $y = kx + 1$ bude tangenta krive $y^2 = 4x$ i naći tačku dodira.

Rješenje

Kriva $y^2 = 4x$ se može podijeliti u dvije odvojene funkcije u zavisnosti od vrijednosti y.

1. $y = \sqrt{4x} = 2\sqrt{x},\ \ y \geq 0$
   Neka je $A(x_0, y_0)$ tačka dodira tangente $y = kx + 1$ i funkcije $y = 2\sqrt{x}$. Tada je $y_0 = 2\sqrt{x_0}$ pa kako tačka $A(x_0, 2\sqrt{x_0})$ pripada tangenti $y = kx + 1$, vrijedi:

$$2\sqrt{x_0} = kx_0 + 1 \tag{1}$$

   Sa druge strane, imamo da koeficijent pravca k tangente $y = kx + 1$ ima vrijednost prvog izvoda funkcije $y = 2\sqrt{x}$ u tački dodira $A(x_0, 2\sqrt{x_0})$ tj.

$$k = y'(x_0) = \cancel{2} \cdot \frac{1}{\cancel{2}\sqrt{x_0}} = \frac{1}{\sqrt{x_0}} \tag{2}$$

   pa uvrštavanjem u jednačinu (1) imamo

$$\begin{aligned} & 2\sqrt{x_0} = \frac{1}{\sqrt{x_0}} \cdot x_0 + 1 \\ \Leftrightarrow \quad & 2\sqrt{x_0} = \sqrt{x_0} + 1 \\ \Leftrightarrow \quad & \sqrt{x_0} = 1 \\ \Leftrightarrow \quad & x_0 = 1. \end{aligned}$$

   Odavde je tačka dodira $A(x_0, 2\sqrt{x_0})$ jednaka $A(1, 2)$, a parametar k je jednak $\frac{1}{\sqrt{x_0}} = 1$.

2. $y = -\sqrt{4x} = -2\sqrt{x},\ \ y < 0$
   Neka je sada $B(x_1, y_1)$ tačka dodira tangente $y = kx + 1$ i funkcije $y = -2\sqrt{x}$. Tada je $y_1 = 2\sqrt{x_1}$ pa kako tačka $B(x_1, -2\sqrt{x_1})$ pripada tangenti $y = kx + 1$, vrijedi:

$$-2\sqrt{x_1} = kx_1 + 1 \tag{3}$$

   Sa druge strane, imamo da koeficijent pravca k tangente $y = kx + 1$ ima vrijednost prvog izvoda funkcije $y = -2\sqrt{x}$ u tački dodira $B(x_1, -2\sqrt{x_1})$ tj.

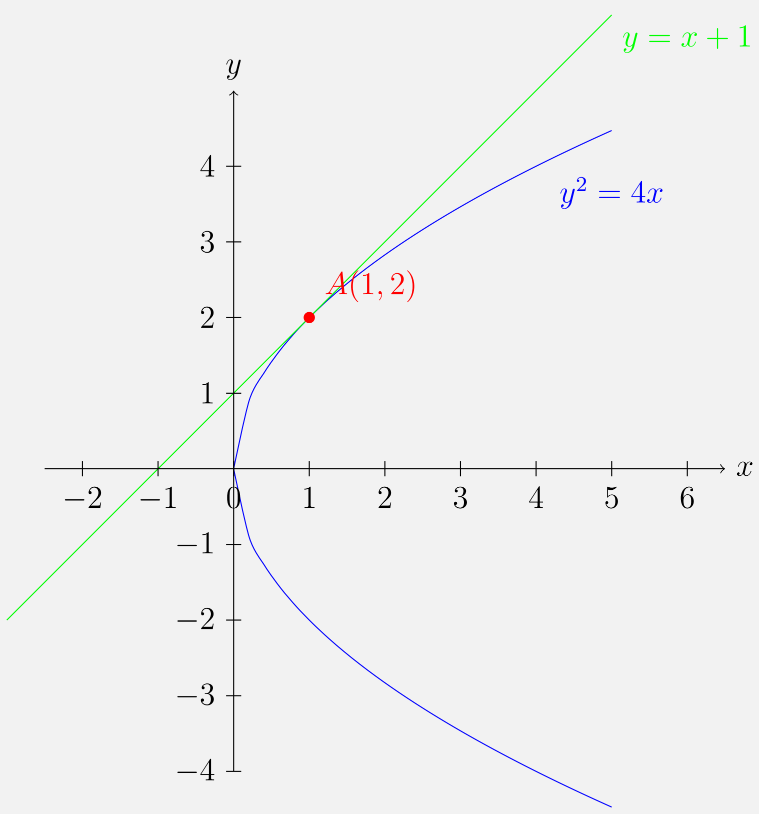
$$k = y'(x_1) = -\cancel{2} \cdot \frac{1}{\cancel{2}\sqrt{x_1}} = \frac{-1}{\sqrt{x_1}} \tag{4}$$

   pa uvrštavanjem u jednačinu (3) imamo

$$\begin{aligned} & -2\sqrt{x_1} = \frac{-1}{\sqrt{x_1}} \cdot x_1 + 1 \\ \Leftrightarrow \quad & -2\sqrt{x_1} = -\sqrt{x_1} + 1 \\ \Leftrightarrow \quad & \sqrt{x_1} = -1 \\ \Rightarrow \quad & x_1 \notin \mathbb{R} \end{aligned}$$

   Dakle, u ovom slučaju nemamo rješenje.

Prikaz krive $y^2 = 4x$ i tangente $y = x + 1$ sa tačkom dodira $A(1, 2)$ dat je na sljedećoj slici.



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Zadatak 7.

Za koju vrijednost parametra a kriva $y = \dfrac{ax - x^3}{x}$ siječe Ox osu pod uglom od $45°$?

Rješenje

Neka je $A(x_0, y_0)$, $x_0 \neq 0$, presječna tačka krive $y = \dfrac{ax - x^3}{x}$ i Ox ose. Tada je $y_0 = 0$ i

$$\begin{aligned} & ax_0 - x_0^3 = 0 \\ \Leftrightarrow \quad & x_0 \cdot \left(a - x_0^2\right) = 0 \\ \Leftrightarrow \quad & x_0^2 = a \\ \Leftrightarrow \quad & x_0 = \pm\sqrt{a}. \end{aligned}$$

Razlikujemo dvije mogućnosti.

1. $x_0 = \sqrt{a} \;\; \wedge \;\; y_0 = 0$
   Ugao pod kojim kriva $y = \dfrac{ax - x^3}{x}$ siječe Ox osu je zapravo ugao pod kojim tangenta t na krivu u tački $A(\sqrt{a}, 0)$ siječe Ox osu.
   Kako je $x \neq 0$, kriva $y = \dfrac{ax - x^3}{x}$ se može zapisati i kao $y = a - x^2, \;\; x \neq 0$.
   Koeficijent pravca k tangente t koja u tački $A(\sqrt{a}, 0)$ siječe Ox osu jednak je tangensu ugla koji tangenta t gradi sa Ox osom.
   Takođe sa druge strane, koeficijent pravca k jednak je prvom izvodu funkcije $y = a - x^2, \;\; x \neq 0$ u tački $x_0 = \sqrt{a}$, pa imamo:

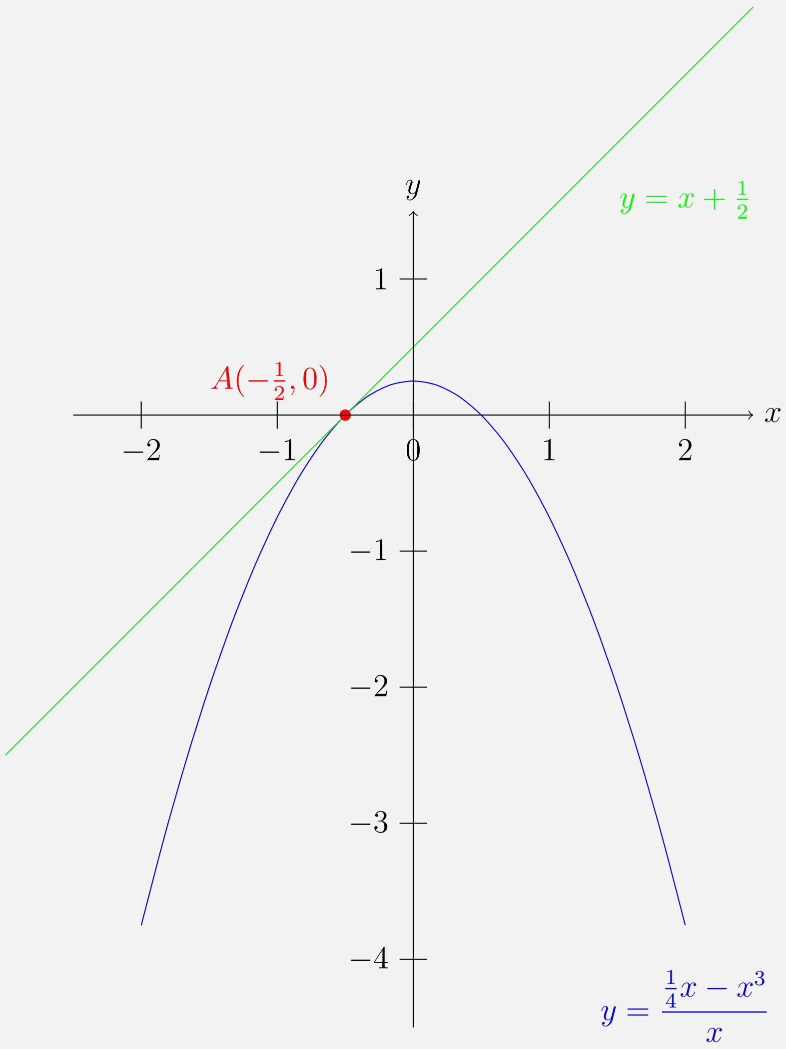
$$\begin{aligned} & k = \operatorname{tg} 45° = y'(\sqrt{a}) \\ \Leftrightarrow \quad & k = 1 = -2\sqrt{a} \\ \Leftrightarrow \quad & \sqrt{a} = -\frac{1}{2} \\ \Rightarrow \quad & a \notin \mathbb{R}. \end{aligned}$$

   U ovom slučaju nemamo rješenje.

2. $x_0 = -\sqrt{a} \;\; \wedge \;\; y_0 = 0$
   Slično kao i u prethodnom slučaju imamo da je

$$\begin{aligned} & k = \operatorname{tg} 45° = y'(-\sqrt{a}) \\ \Leftrightarrow \quad & k = 1 = 2\sqrt{a} \\ \Leftrightarrow \quad & \sqrt{a} = \frac{1}{2} \\ \Rightarrow \quad & a = \frac{1}{4}. \end{aligned}$$

Dakle $a = \dfrac{1}{4}$, a grafički prikaz funkcije $y = \dfrac{\frac{1}{4}x - x^3}{x}$ i ugla pod kojim ta kriva siječe Ox osu dat je na sljedećoj slici.



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Zadatak 8.
Izračunati graničnu vrijednost

$$\lim_{x\to+\infty} x^2 \cdot \ln\left(\frac{\frac{1}{x}}{\sin\left(\frac{1}{x}\right)}\right).$$

Rješenje

Uzmimo smjenu $t = \frac{1}{x}$. Sada imamo $t \to 0$ pa je početni limes jednak:

$$\begin{aligned}
L &= \lim_{t\to 0}\left(\frac{1}{t^2}\cdot\ln\left(\frac{t}{\sin t}\right)\right)\\
&= \lim_{t\to 0}\frac{\ln\left(\frac{t}{\sin t}\right)}{t^2}\\
&= \lim_{t\to 0}\frac{\left(\ln\left(\frac{t}{\sin t}\right)\right)'}{\left(t^2\right)'}\\
&= \lim_{t\to 0}\frac{\frac{1}{\frac{t}{\sin t}}\cdot\left(\frac{t}{\sin t}\right)'}{2t}\\
&= \lim_{t\to 0}\frac{\frac{\sin t}{t}\cdot\frac{\sin t - t\cos t}{\sin^2 t}}{2t}\\
&= \lim_{t\to 0}\frac{\frac{\sin t - t\cos t}{t\sin t}}{2t}\\
&= \frac{1}{2}\cdot\lim_{t\to 0}\frac{\left(\sin t - t\cos t\right)'}{\left(t^2\sin t\right)'}\\
&= \frac{1}{2}\cdot\lim_{t\to 0}\frac{\cos t - \left(t\cos t\right)'}{\sin t + t\cos t}\\
&= \frac{1}{2}\cdot\lim_{t\to 0}\frac{\cos t - \left(\cos t - t\sin t\right)}{2t\sin t + t^2\cos t}\\
&= \frac{1}{2}\cdot\lim_{t\to 0}\frac{t\sin t}{2t\sin t + t^2\cos t}\\
&= \frac{1}{2}\cdot\lim_{t\to 0}\frac{t\sin t}{t\cdot\left(2\sin t + t\cos t\right)}\\
&= \frac{1}{2}\cdot\lim_{t\to 0}\frac{\sin t}{2\sin t + t\cos t}\\
&= \frac{1}{2}\cdot\lim_{t\to 0}\frac{\left(\sin t\right)'}{\left(2\sin t + t\cos t\right)'}\\
&= \frac{1}{2}\cdot\lim_{t\to 0}\frac{\cancelto{1}{\cos t}}{2\cancelto{1}{\cos t} + \cancelto{1}{\cos t} - \cancelto{0}{t\sin t}}\\
&= \frac{1}{2}\cdot\frac{1}{2\cdot 1 + 1 - 0}\\
&= \frac{1}{6}.
\end{aligned}$$

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Zadatak 9.
Data je kriva $y = xe^{\frac{1}{x}}$. Naći jednačinu tangente krive u tački $x = \alpha$ kao i njen granični položaj kad $\alpha \to +\infty$.

Rješenje

Odredimo prvi izvod funkcije $y = xe^{\frac{1}{x}}$:

$$\begin{aligned} y' &= x' \cdot e^{\frac{1}{x}} + x \cdot \left(e^{\frac{1}{x}}\right)' \\ &= e^{\frac{1}{x}} + x \cdot e^{\frac{1}{x}} \cdot \left(\frac{1}{x}\right)' \\ &= e^{\frac{1}{x}} + x \cdot e^{\frac{1}{x}} \cdot \frac{-1}{x^2} \\ &= e^{\frac{1}{x}} \cdot \left(1 - \frac{1}{x}\right). \end{aligned}$$

Prvi izvod funkcije $y = xe^{\frac{1}{x}}$ u tački $x = \alpha$ odgovara koeficijentu pravca tangente t u toj tački. Dakle

$$k = e^{\frac{1}{\alpha}} \cdot \left(1 - \frac{1}{\alpha}\right).$$

Kako tangenta t prolazi kroz tačku $\left(\alpha, \alpha \cdot e^{\frac{1}{\alpha}}\right)$, koristeći jednačinu prave kroz jednu tačku dobijamo da je jednačina tangente t:

$$\begin{aligned} t: &\quad y - \alpha \cdot e^{\frac{1}{\alpha}} = e^{\frac{1}{\alpha}} \cdot \left(1 - \frac{1}{\alpha}\right) \cdot (x - \alpha) \\ t: &\quad y - \alpha \cdot e^{\frac{1}{\alpha}} = e^{\frac{1}{\alpha}} \cdot \left(1 - \frac{1}{\alpha}\right) \cdot x - e^{\frac{1}{\alpha}} \cdot \left(1 - \frac{1}{\alpha}\right) \cdot \alpha \\ t: &\quad y - \cancel{\alpha \cdot e^{\frac{1}{\alpha}}} = e^{\frac{1}{\alpha}} \cdot \left(1 - \frac{1}{\alpha}\right) \cdot x - \cancel{\alpha \cdot e^{\frac{1}{\alpha}}} + e^{\frac{1}{\alpha}} \\ t: &\quad y = e^{\frac{1}{\alpha}} \cdot \left(1 - \frac{1}{\alpha}\right) \cdot x + e^{\frac{1}{\alpha}}. \end{aligned}$$

Granični položaj tangente kad $\alpha \to +\infty$ dobijamo nakon što odredimo limes:

$$\lim_{\alpha \to +\infty} \cancelto{1}{e^{\frac{1}{\alpha}}} \cdot \left(1 - \cancelto{0}{\frac{1}{\alpha}}\right) \cdot x + \cancelto{1}{e^{\frac{1}{\alpha}}} = x + 1.$$

Dakle, granični položaj tangente t kad $\alpha \to +\infty$ je

$$y = x + 1.$$

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Zadatak 10.
Izračunati n-ti izvod funkcije

$$f(x)=\frac{1+x}{\sqrt{1-x}}.$$

Rješenje

Funkciju $f(x)$ ćemo zapisati u obliku pogodnom za diferenciranje. Kako je

$$\begin{aligned} f(x)&=\frac{2-(1-x)}{\sqrt{1-x}}\\ &=\frac{2}{\sqrt{1-x}}-\sqrt{1-x}\\ &=2\cdot(1-x)^{-\frac{1}{2}}-(1-x)^{\frac{1}{2}} \end{aligned} \tag{5}$$

Odredimo sada n-te izvode funkcija $g(x)=2\cdot(1-x)^{-\frac{1}{2}}$ i $h(x)=(1-x)^{\frac{1}{2}}$, a n-ti izvod funkcije $f(x)$ ćemo pronaći kao razliku dobijenih n-tih izvoda.
Vrijedi

$$\begin{aligned} g'(x)&=2\cdot\left(-\frac{1}{2}\right)\cdot(1-x)^{-\frac{1}{2}-1}\cdot(-1)\\ &=2\cdot\frac{1}{2}\cdot(1-x)^{-\frac{3}{2}}, \end{aligned}$$

$$\begin{aligned} g''(x)&=2\cdot\left(\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot(1-x)^{-\frac{3}{2}-1}\cdot(-1)\\ &=2\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot(1-x)^{-\frac{5}{2}},\\ &\vdots \end{aligned}$$

$$\begin{aligned} g^{(n)}(x)&=2\cdot\frac{1}{2}\cdot\frac{3}{2}\dots\frac{2n-1}{2}\cdot(1-x)^{-\frac{2n+1}{2}}\\ &=2\cdot\frac{(2n-1)!!}{2^n}\cdot(1-x)^{-\frac{2n+1}{2}}. \end{aligned}$$

Na sličan način dobijamo da je

$$\begin{aligned} h'(x)&=\left(\frac{1}{2}\right)\cdot(1-x)^{\frac{1}{2}-1}\cdot(-1)\\ &=-\frac{1}{2}\cdot(1-x)^{-\frac{1}{2}}, \end{aligned}$$

$$\begin{aligned} h''(x)&=-\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot(1-x)^{-\frac{1}{2}-1}\cdot(-1)\\ &=-\frac{1}{2}\cdot\frac{1}{2}\cdot(1-x)^{-\frac{3}{2}}, \end{aligned}$$

$$\begin{aligned} h'''(x)&=-\frac{1}{2}\cdot\left(\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot(1-x)^{-\frac{3}{2}-1}\cdot(-1)\\ &=-\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot(1-x)^{-\frac{5}{2}},\\ &\vdots \end{aligned}$$

$$\begin{aligned} h^{(n)}(x)&=-\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\dots\frac{2n-3}{2}\cdot(1-x)^{-\frac{2n-1}{2}}\\ &=-\frac{1}{2}\cdot\frac{(2n-3)!!}{2^{n-1}}\cdot(1-x)^{-\frac{2n-1}{2}}. \end{aligned}$$

Primijetimo da oznaka $(2n-1)!!$ predstavlja proizvod svih neparnih brojeva do $2n-1$. Dakle

$$(2n-1)!!=1\cdot3\cdot5\dots(2n-1).$$

Iz izraza (5) imamo da je:

$$\begin{aligned} f^{(n)}(x)&=g^{(n)}(x)-h^{(n)}(x)\\ &=2\cdot\frac{(2n-1)!!}{2^n}\cdot(1-x)^{-\frac{2n+1}{2}}-\left(-\frac{1}{2}\cdot\frac{(2n-3)!!}{2^{n-1}}\cdot(1-x)^{-\frac{2n-1}{2}}\right)\\ &=\frac{2\cdot(2n-1)\cdot(2n-3)!!}{2^n\cdot(1-x)^n\sqrt{1-x}}+\frac{(2n-3)!!\cdot(1-x)}{2^n\cdot(1-x)^n\sqrt{1-x}}\\ &=\frac{(2n-3)!!}{2^n\cdot(1-x)^n\sqrt{1-x}}\cdot\big(2\cdot(2n-1)+(1-x)\big),\quad x<1. \end{aligned}$$