TERMIN 7 - zadaci za samostalan rad - rješenja

★ K2 18.06.2021. ①

Zadatak 1.
Izračunati graničnu vrijednost

$$\lim_{n\to+\infty} \frac{n\cdot e^{\frac{1}{n}}}{\sqrt{n^2+1}}.$$

Rješenje

Kako je

$$\lim_{n\to+\infty} e^{\overset{0}{\cancel{\frac{1}{n}}}} = 1,$$

imamo da je

$$\lim_{n\to+\infty} \frac{n\cdot e^{\frac{1}{n}}}{\sqrt{n^2+1}} = \lim_{n\to+\infty} \frac{\cancel{n}}{\cancel{n}\cdot\sqrt{1+\underset{0}{\cancel{\frac{1}{n^2}}}}} \cdot \lim_{n\to+\infty} \overset{1}{\cancel{e^{\frac{1}{n}}}}$$

$$= 1.$$

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Zadatak 2.
Izračunati graničnu vrijednost

a) $\lim\limits_{n\to+\infty} \sqrt[n]{2023^n + 2024^n}$,

b) $\lim\limits_{n\to+\infty} \dfrac{2022^n + 2023^n}{2024^n}$.

Rješenje

Koristeći limes

$$\lim_{n\to+\infty} q^n = \begin{cases} 0, & |q| < 1, \\ 1, & q = 1, \\ +\infty, & q > 1 \end{cases}$$

imamo da vrijedi:

a)

$$\begin{aligned} \lim_{n\to+\infty} \sqrt[n]{2023^n + 2024^n} &= \lim_{n\to+\infty} \sqrt[n]{2024^n \cdot \left(\cancelto{0}{\left(\frac{2023}{2024}\right)^n} + 1\right)} \\ &= 2024, \end{aligned}$$

b)

$$\begin{aligned} \lim_{n\to+\infty} \frac{2022^n + 2023^n}{2024^n} &= \lim_{n\to+\infty} \left(\cancelto{0}{\left(\frac{2023}{2024}\right)^n} + \cancelto{0}{\left(\frac{2023}{2024}\right)^n}\right) \\ &= 0. \end{aligned}$$

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Zadatak 3.

Izračunati graničnu vrijednost

$$\lim_{n\to+\infty}\left[\left(1+\frac{2}{n}\right)^n\cdot\frac{n^2+3}{(2n+1)(2n-1)}\right].$$

Rješenje

Vrijedi

$$\begin{aligned}
L &= \lim_{n\to+\infty}\left[\left(1+\frac{2}{n}\right)^n\cdot\frac{n^2+3}{(2n+1)(2n-1)}\right]\\
&= \lim_{n\to+\infty}\left[\left(1+\frac{1}{\frac{n}{2}}\right)^{\frac{n}{2}\cdot\frac{2}{n}\cdot n}\cdot\frac{n^2+3}{4n^2-1}\right]\\
&= \lim_{n\to+\infty}\left[\left(\left(1+\frac{1}{\frac{n}{2}}\right)^{\frac{n}{2}}\right)^2\cdot\frac{\cancel{n^2}\cdot\left(1+\cancelto{0}{\frac{3}{n^2}}\right)}{\cancel{n^2}\cdot\left(4-\cancelto{0}{\frac{1}{n^2}}\right)}\right]\\
&= \frac{e^2}{4}.
\end{aligned}$$

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Zadatak 4.
Izračunati graničnu vrijednost

a) $\lim\limits_{n\to+\infty}\left(\dfrac{\sqrt{n}+2}{\sqrt{n}-1}\right)^{\sqrt{n}}$,

b) $\lim\limits_{n\to+\infty}\left(\dfrac{n^2+n+1}{n^2-n+1}\right)^{n}$.

Rješenje

Koristeći limes

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = e$$

imamo da vrijedi:

a)

$$\begin{aligned}
\lim_{n\to+\infty}\left(\frac{\sqrt{n}+2}{\sqrt{n}-1}\right)^{\sqrt{n}} &= \lim_{n\to+\infty}\left(\frac{\sqrt{n}-1+3}{\sqrt{n}-1}\right)^{\sqrt{n}}\\
&= \lim_{n\to+\infty}\left(1+\frac{3}{\sqrt{n}-1}\right)^{\sqrt{n}}\\
&= \lim_{n\to+\infty}\left(1+\frac{1}{\frac{\sqrt{n}-1}{3}}\right)^{\frac{\sqrt{n}-1}{3}\cdot\frac{3}{\sqrt{n}-1}\cdot\sqrt{n}}\\
&= e^{\lim\limits_{n\to+\infty}\dfrac{3\sqrt{n}}{\sqrt{n}-1}}\\
&= e^{\lim\limits_{n\to+\infty}\dfrac{3\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}\left(1-\frac{1}{\sqrt{n}}\right)}}\\
&= e^3,
\end{aligned}$$

b)

$$\begin{aligned}
\lim_{n\to+\infty}\left(\frac{n^2+n+1}{n^2-n+1}\right)^{n} &= \lim_{n\to+\infty}\left(\frac{n^2-n+1+2n}{n^2-n+1}\right)^{n}\\
&= \lim_{n\to+\infty}\left(1+\frac{2n}{n^2-n+1}\right)^{n}\\
&= \lim_{n\to+\infty}\left(1+\frac{1}{\frac{n^2-n+1}{2n}}\right)^{\frac{n^2-n+1}{2n}\cdot\frac{2n}{n^2-n+1}\cdot n}\\
&= e^{\lim\limits_{n\to+\infty}\dfrac{2n^2}{n^2-n+1}}\\
&= e^{\lim\limits_{n\to+\infty}\dfrac{2\cancel{n^2}}{\cancel{n^2}\cdot\left(1-\frac{1}{n}+\frac{1}{n^2}\right)}}\\
&= e^2.
\end{aligned}$$

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Zadatak 5.
Izračunati graničnu vrijednost

$$\lim_{n\to+\infty} \frac{\sqrt{\frac{1}{2}} + \sqrt{\frac{3}{5}} + \sqrt{\frac{5}{10}} + \cdots + \sqrt{\frac{2n-1}{n^2+1}}}{\sqrt{n}}.$$

Rješenje

Kako vrijedi

1. $\lim\limits_{n\to+\infty} \sqrt{n} = +\infty$ i
2. $\sqrt{n+1} > \sqrt{n}$

koristeći Štolcovu teoremu dobijamo

$$\begin{aligned}
L &= \lim_{n\to+\infty} \frac{\sqrt{\frac{1}{2}} + \sqrt{\frac{3}{5}} + \sqrt{\frac{5}{10}} + \cdots + \sqrt{\frac{2n-1}{n^2+1}}}{\sqrt{n}} \\
&= \lim_{n\to+\infty} \frac{\left(\cancel{\sqrt{\frac{1}{2}}} + \cancel{\sqrt{\frac{3}{5}}} + \cancel{\sqrt{\frac{5}{10}}} + \cdots + \cancel{\sqrt{\frac{2n-1}{n^2+1}}} + \sqrt{\frac{2\cdot(n+1)-1}{(n+1)^2+1}}\right) - \left(\cancel{\sqrt{\frac{1}{2}}} + \cancel{\sqrt{\frac{3}{5}}} + \cancel{\sqrt{\frac{5}{10}}} + \cdots + \cancel{\sqrt{\frac{2n-1}{n^2+1}}}\right)}{\sqrt{n+1} - \sqrt{n}} \\
&= \lim_{n\to+\infty} \frac{\sqrt{\frac{2n+1}{n^2+2n+2}}}{\sqrt{n+1} - \sqrt{n}} \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \\
&= \lim_{n\to+\infty} \frac{\sqrt{2n+1} \cdot \left(\sqrt{n+1} + \sqrt{n}\right)}{\sqrt{n^2+2n+2}} \\
&= \lim_{n\to+\infty} \frac{\cancel{\sqrt{n}} \cdot \left(\sqrt{2 + \cancelto{0}{\frac{1}{n}}}\right) \cdot \cancel{\sqrt{n}} \cdot \left(\sqrt{1 + \cancelto{0}{\frac{1}{n}}} + 1\right)}{\cancel{\sqrt{n^2}} \cdot \sqrt{1 + \cancelto{0}{\frac{2}{n}} + \cancelto{0}{\frac{2}{n^2}}}} \\
&= \frac{\sqrt{2} \cdot \left(\sqrt{1} + 1\right)}{\sqrt{1}} \\
&= 2\sqrt{2}.
\end{aligned}$$

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Zadatak 6.
Izračunati graničnu vrijednost
$$\lim_{n\to+\infty} \frac{\ln(n!)}{n}.$$

Rješenje

Kako vrijedi

1. $\lim\limits_{n\to+\infty} n = +\infty$ i
2. $n+1 > n$

koristeći Štolcovu teoremu dobijamo

$$\begin{aligned}
L &= \lim_{n\to+\infty} \frac{\ln(n!)}{n} \\
&= \lim_{n\to+\infty} \frac{\ln(n+1)! - \ln(n)!}{n+1-n} \\
&= \lim_{n\to+\infty} \ln\left(\frac{(n+1)!}{n!}\right) \\
&= \lim_{n\to+\infty} \ln\left(\frac{(n+1)\cancel{n!}}{\cancel{n!}}\right) \\
&= \lim_{n\to+\infty} \ln(n+1) \\
&= +\infty.
\end{aligned}$$

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Zadatak 7.
Izračunati

$$\lim_{n\to+\infty}\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{2}+\sqrt{4}}+\frac{1}{\sqrt{4}+\sqrt{6}}+\cdots+\frac{1}{\sqrt{2n}+\sqrt{2n+2}}\right).$$

Rješenje

Prvi način:
Kako vrijedi

1. $\lim\limits_{n\to+\infty}\sqrt{n}=+\infty$ i
2. $\sqrt{n+1}>\sqrt{n}$

koristeći Štolcovu teoremu dobijamo

$$\begin{aligned}
L&=\lim_{n\to+\infty}\frac{\dfrac{1}{\sqrt{2}+\sqrt{4}}+\dfrac{1}{\sqrt{4}+\sqrt{6}}+\cdots+\dfrac{1}{\sqrt{2n}+\sqrt{2n+2}}}{\sqrt{n}}\\
&=\lim_{n\to+\infty}\frac{\left(\cancel{\dfrac{1}{\sqrt{2}+\sqrt{4}}}+\cancel{\dfrac{1}{\sqrt{4}+\sqrt{6}}}+\cdots+\cancel{\dfrac{1}{\sqrt{2n}+\sqrt{2n+2}}}+\dfrac{1}{\sqrt{2\left(n+1\right)}+\sqrt{2\left(n+1\right)+2}}\right)-\left(\cancel{\dfrac{1}{\sqrt{2}+\sqrt{4}}}+\cancel{\dfrac{1}{\sqrt{4}+\sqrt{6}}}+\cdots+\cancel{\dfrac{1}{\sqrt{2n}+\sqrt{2n+2}}}\right)}{\sqrt{n+1}-\sqrt{n}}\\
&=\lim_{n\to+\infty}\frac{\dfrac{1}{\sqrt{2\left(n+1\right)}+\sqrt{2\left(n+1\right)+2}}}{\sqrt{n+1}-\sqrt{n}}\cdot\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\\
&=\lim_{n\to+\infty}\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{2n+2}+\sqrt{2n+4}}\\
&=\lim_{n\to+\infty}\frac{\cancel{\sqrt{n}}\cdot\left(\sqrt{1+\cancelto{0}{\frac{1}{n}}}+1\right)}{\cancel{\sqrt{n}}\cdot\left(\sqrt{2+\cancelto{0}{\frac{2}{n}}}+\sqrt{2+\cancelto{0}{\frac{4}{n}}}\right)}\\
&=\frac{\sqrt{1}+1}{\sqrt{2}+\sqrt{2}}\\
&=\frac{2}{2\sqrt{2}}\\
&=\frac{1}{\sqrt{2}}.
\end{aligned}$$

Drugi način:
Vrijedi:

$$\begin{aligned}
L&=\lim_{n\to+\infty}\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{2}+\sqrt{4}}+\frac{1}{\sqrt{4}+\sqrt{6}}+\cdots+\frac{1}{\sqrt{2n}+\sqrt{2n+2}}\right)\\
&=\lim_{n\to+\infty}\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{4}+\sqrt{2}}\cdot\frac{\sqrt{4}-\sqrt{2}}{\sqrt{4}-\sqrt{2}}+\frac{1}{\sqrt{6}+\sqrt{4}}\cdot\frac{\sqrt{6}-\sqrt{4}}{\sqrt{6}-\sqrt{4}}+\cdots+\frac{1}{\sqrt{2n+2}+\sqrt{2n}}\cdot\frac{\sqrt{2n+2}-\sqrt{2n}}{\sqrt{2n+2}-\sqrt{2n}}\right)\\
&=\lim_{n\to+\infty}\frac{1}{\sqrt{n}}\cdot\left(\frac{\sqrt{4}-\sqrt{2}}{2}+\frac{\sqrt{6}-\sqrt{4}}{2}+\cdots+\frac{\sqrt{2n+2}-\sqrt{2n}}{2}\right)\\
&=\frac{1}{2\sqrt{n}}\cdot\left(\cancel{\sqrt{4}}-\sqrt{2}+\cancel{\sqrt{6}}-\cancel{\sqrt{4}}+\cdots+\sqrt{2n+2}-\cancel{\sqrt{2n}}\right)\\
&=\lim_{n\to+\infty}\frac{\sqrt{2n+2}-\sqrt{2}}{2\sqrt{n}}\cdot\frac{\sqrt{2n+2}+\sqrt{2}}{\sqrt{2n+2}+\sqrt{2}}\\
&=\lim_{n\to+\infty}\frac{2n}{2\sqrt{n}\left(\sqrt{2n+2}+\sqrt{2}\right)}\\
&=\lim_{n\to+\infty}\frac{\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}\cdot\left(\sqrt{2+\cancelto{0}{\frac{2}{n}}}+\sqrt{\cancelto{0}{\frac{2}{n}}}\right)}\\
&=\frac{1}{\sqrt{2}}.
\end{aligned}$$

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Zadatak 8.
Izračunati

$$\lim_{n\to+\infty}\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right).$$

Rješenje

Vrijedi:

$$\begin{aligned}
L &= \lim_{n\to+\infty}\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right)\\
&= \lim_{n\to+\infty}\frac{2^2-1}{2^2}\cdot\frac{3^2-1}{3^2}\cdot\frac{4^2-1}{4^2}\cdots\frac{(n-1)^2-1}{(n-1)^2}\frac{n^2-1}{n^2}\\
&= \lim_{n\to+\infty}\frac{(2-1)\cdot(2+1)}{2\cdot 2}\cdot\frac{(3-1)\cdot(3+1)}{3\cdot 3}\cdot\frac{(4-1)\cdot(4+1)}{4\cdot 4}\cdots\frac{(n-1-1)\cdot(n-1+1)}{(n-1)\cdot(n-1)}\cdot\frac{(n-1)\cdot(n+1)}{n\cdot n}\\
&= \lim_{n\to+\infty}\frac{1\cdot\cancel{3}}{2\cdot\cancel{2}}\cdot\frac{\cancel{2}\cdot\cancel{4}}{\cancel{3}\cdot\cancel{3}}\cdot\frac{\cancel{3}\cdot\cancel{5}}{\cancel{4}\cdot\cancel{4}}\cdots\frac{\cancel{(n-2)}\cdot\cancel{n}}{\cancel{(n-1)}\cdot\cancel{(n-1)}}\cdot\frac{\cancel{(n-1)}\cdot(n+1)}{\cancel{n}\cdot n}\\
&= \lim_{n\to+\infty}\frac{n+1}{2n}\\
&= \lim_{n\to+\infty}\frac{\cancel{n}\cdot\left(1+\cancel{\frac{1}{n}}^{\,0}\right)}{2\cancel{n}}\\
&= \frac{1}{2}.
\end{aligned}$$

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Zadatak 9.
Ispitati da li je niz (a_n) definisan sa

$$a_{n+1} = \frac{6a_n + 6}{a_n + 7}, \ \ 0 < a_1 < 2$$

konvergentan. Ukoliko jeste, odrediti mu graničnu vrijednost.

Rješenje

Ispitajmo ograničenost i monotonost.

1. *Ograničenost*:
   Koristeći princip matematičke indukcije pokažimo da vrijedi $0 < a_n < 2$.
   Baza indukcije vrijedi jer iz postavke zadatka imamo da je $0 < a_1 < 2$.
   Pretpostavimo da vrijedi $0 < a_n < 2$ (indukcijska pretpostavka) i pokažimo da vrijedi $0 < a_{n+1} < 2$ (indukcijski korak). Dakle, potrebno je pokazati da vrijedi

$$\begin{aligned} & 0 < \frac{6a_n + 6}{a_n + 7} < 2 \\ \Leftrightarrow \quad & \frac{6a_n + 6}{a_n + 7} > 0 \ \wedge \ \frac{6a_n + 6}{a_n + 7} - 2 < 0 \\ \Leftrightarrow \quad & \frac{6\left(a_n + 1\right)}{a_n + 7} > 0 \ \wedge \ \frac{6a_n + 6 - 2a_n - 14}{a_n + 7} < 0 \\ \Leftrightarrow \quad & \frac{a_n + 1}{a_n + 7} > 0 \ \wedge \ \frac{4\left(a_n - 2\right)}{a_n + 7} < 0. \end{aligned}$$

   Kako je $0 < a_n < 2$, imamo da je $a_n + 1 > 0$ i $a_n + 7 > 0$ pa je $\frac{a_n + 1}{a_n + 7} > 0$. Sa druge strane je $a_n - 2 < 0$ i $a_n + 7 > 0$ pa je $\frac{a_n - 2}{a_n + 7} < 0$.
   Ovim smo pokazali da vrijedi $0 < a_n < 2$.

2. *Monotonost*:
   Posmatrajmo razliku $a_{n+1} - a_n$. Imamo da je

$$\begin{aligned} a_{n+1} - a_n &= \frac{6a_n + 6}{a_n + 7} - a_n \\ &= \frac{6a_n + 6 - a_n^2 - 7a_n}{a_n + 7} \\ &= \frac{-a_n^2 - a_n + 6}{a_n + 7} \\ &= \frac{-\left(a_n^2 + a_n - 6\right)}{a_n + 7} \\ &= \frac{-\left(a_n - 2\right)\left(a_n + 3\right)}{a_n + 7} \\ &= \frac{\left(2 - a_n\right)\left(a_n + 3\right)}{a_n + 7}. \end{aligned}$$

   Kako je niz $\{a_n\}$ ograničen, odnosno kako je vrijedi $a_n < 2$ i $a_n > 0$, vrijedi $2 - a_n > 0$, $a_n + 3 > 0$ i $a_n + 7 > 0$, odnosno vrijedi

$$\frac{\left(2 - a_n\right)\left(a_n + 3\right)}{a_n + 7} > 0,$$

   pa je $a_{n+1} - a_n > 0$, tj. $a_{n+1} > a_n$ na osnovu čega zaključujemo da je niz $\{a_n\}$ monotono rastući.

Kako je $\{a_n\}$ monotono rastući i ograničen odozgo sa $M = 2$, zaključujemo da je konvergentan.
Da bismo odredili graničnu vrijednost niza $\{a_n\}$, koristimo da vrijedi

$$\lim_{n \to +\infty} a_n = \lim_{n \to +\infty} a_{n+1} = L.$$

Imamo da je:

$$\begin{aligned} & \lim_{n \to +\infty} a_{n+1} = \lim_{n \to +\infty} \frac{6a_n + 6}{a_n + 7} \\ \Leftrightarrow \quad & L = \frac{6L + 6}{L + 7} \\ \Leftrightarrow \quad & \frac{L\left(L + 7\right) - \left(6L + 6\right)}{L + 7} = 0 \\ \Leftrightarrow \quad & \frac{L^2 + 7L - 6L - 6}{L + 7} = 0 \\ \Leftrightarrow \quad & \left(L - 2\right)\left(L + 3\right) = 0 \\ \Leftrightarrow \quad & L = 2 \ \vee \ L = -3. \end{aligned}$$

Kako je niz ograničen odozdo sa $N = 0$, rješenje $L = -3$ odbacujemo, pa zaključujemo da je granična vrijednost niza $\{a_n\}$ jednaka 2.

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Zadatak 10.

a) Ako je $\lim\limits_{n\to+\infty} a_n = L$, $L > 0$, tada vrijedi

$$\lim_{n\to+\infty} \sqrt[n]{a_1 a_2 \dots a_n} = L.$$

Dokazati.

b) Naći

$$\lim_{n\to+\infty} \sqrt[n]{(1+1)^1 \cdot \left(1+\frac{1}{2}\right)^2 \dots \left(1+\frac{1}{n}\right)^n}.$$

Rješenje

a) Neka je $\lim\limits_{n\to+\infty} \sqrt[n]{a_1 a_2 \dots a_n} = A$. Potrebno je dokazati da vrijedi $A = L$. ln-ovanjem izraza dobijamo

$$\begin{aligned} & \lim_{n\to+\infty} \ln \sqrt[n]{a_1 a_2 \dots a_n} = \ln(A) \\ \Leftrightarrow \quad & \lim_{n\to+\infty} \frac{\ln(a_1) + \ln(a_2) + \dots + \ln(a_n)}{n} = \ln(A). \end{aligned}$$

Kako vrijedi

(a) $\lim\limits_{n\to+\infty} n = +\infty$ i

(b) $n + 1 > n$

koristeći Štolcovu teoremu dobijamo

$$\begin{aligned} & \lim_{n\to+\infty} \frac{\ln(a_1) + \ln(a_2) + \dots + \ln(a_n)}{n} = \ln(A) \\ \Leftrightarrow \quad & \lim_{n\to+\infty} \frac{\big(\cancel{\ln(a_1)} + \cancel{\ln(a_2)} + \dots + \cancel{\ln(a_n)} + \ln(a_{n+1})\big) - \big(\cancel{\ln(a_1)} + \cancel{\ln(a_2)} + \dots + \cancel{\ln(a_n)}\big)}{\cancel{n} + 1 - \cancel{n}} = \ln(A) \\ \Leftrightarrow \quad & \lim_{n\to+\infty} \ln(a_{n+1}) = \ln(A). \end{aligned}$$

Kako je $\lim\limits_{n\to+\infty} a_{n+1} = \lim\limits_{n\to+\infty} a_n = L$, imamo da je

$$\ln(A) = \ln(L) \quad \Rightarrow \quad L = A,$$

čime je dokaz završen.

b) Na osnovu dijela zadatka a) zaključujemo da je

$$\lim_{n\to+\infty} \sqrt[n]{(1+1)^1 \cdot \left(1+\frac{1}{2}\right)^2 \dots \left(1+\frac{1}{n}\right)^n} = \lim_{n\to+\infty} \left(1+\frac{1}{n}\right)^n = e.$$