① $y^{(5)} - 6y^{(4)} + 9y^{(3)} = 0$

$\Rightarrow \lambda^5 - 6\lambda^4 + 9\lambda^3 = 0$

$\Leftrightarrow \lambda^3 \cdot (\lambda^2 - 6\lambda + 9) = 0$

$\Leftrightarrow \lambda^3 \cdot (\lambda - 3)^2 = 0$

$\lambda_1 = \lambda_2 = \lambda_3 = 0, \quad \lambda_4 = \lambda_5 = 3$

$$\boxed{y = C_1 + C_2 x + C_3 x^2 + C_4 e^{3x} + C_5 x e^{3x}}$$

② $y^{(4)} - 4y^{(3)} + 8y'' - 8y' + 4y = 0$

$\Rightarrow \lambda^4 - 4\lambda^3 + 8\lambda^2 - 8\lambda + 4 = 0$

$\Leftrightarrow \lambda^4 - 2\lambda^3 + 2\lambda^2 - 2\lambda^3 + 6\lambda^2 - 8\lambda + 4 = 0$

$\Leftrightarrow \lambda^2 \cdot (\lambda^2 - 2\lambda + 2) - 2\lambda^3 + 4\lambda^2 - 4\lambda + 2\lambda^2 - 4\lambda + 4 = 0$

$\Leftrightarrow \lambda^2 \cdot (\lambda^2 - 2\lambda + 2) - 2\lambda(\lambda^2 - 2\lambda + 2) + 2(\lambda^2 - 2\lambda + 2) = 0$

$\Leftrightarrow (\lambda^2 - 2\lambda + 2) \cdot (\lambda^2 - 2\lambda + 2) = 0$

$\Leftrightarrow (\lambda^2 - 2\lambda + 2)^2 = 0$

$$\lambda_{1/2} = \frac{2 \pm \sqrt{4 - 4 \cdot 1 \cdot 2}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

$\lambda_1 = 1 + i, \quad \lambda_2 = 1 - i,$

$\lambda_3 = 1 + i, \quad \lambda_4 = 1 - i$

$$y = e^x \cdot (C_1 \cos x + C_2 \sin x) + xe^x (C_3 \cos x + C_4 \sin x)$$

$$\boxed{y = C_1 e^x \cos x + C_2 e^x \sin x + C_3 x e^x \cos x + C_4 x e^x \sin x}$$

(3) $$y'' - 3y' + 2y = 4x + e^x \cos x$$

Homogeno rješenje: $y'' - 3y' + 2y = 0 \Rightarrow$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$\Leftrightarrow (\lambda - 1)(\lambda - 2) = 0$$

$$\Leftrightarrow \lambda_1 = 1, \ \lambda_2 = 2 \Rightarrow$$

$$\boxed{y_h = C_1 e^x + C_2 e^{2x}}$$

1. partikularno rješenje:

$$p_1(x) = 4x$$

$$p(x) = e^{\alpha x}(P_{n_1}(x) \cos \beta x + P_{n_2}(x) \sin \beta x)$$

$$\left.\begin{array}{lll} \alpha = 0, & P_{n_1}(x) = 4x, & n_1 = 1 \\ \beta = 0, & P_{n_2}(x) = 0, & n_2 = 0 \end{array}\right\} \Rightarrow s = 1$$

$$\alpha + \beta i = 0 \neq \lambda_i \Rightarrow k = 0$$

$$y_{p_1} = (Ax + B)$$

$$y_{p_1}' = A$$

$$y_{p_1}'' = 0$$

$$y_{p1}'' - 3y_{p1}' + 2y_{p1} = 4x$$

$$\Rightarrow \ 0 - 3A + 2(Ax+B) = 4x$$

$$\Leftrightarrow \ -3A + 2Ax + 2B = 4x \ \Rightarrow$$

$$\begin{array}{l} 2A \qquad = 4 \\ -3A + 2B = 0 \\ \hline A = 2, \ B = 3 \end{array} \ \Rightarrow \ \boxed{y_{p1} = 2x+3}$$

2. partikularno rješenje:

$$r_2(x) = e^x \cos x$$

$$r(x) = e^{\alpha x} \cdot (P_{n_1}(x) \cos \beta x + P_{n_2}(x) \sin \beta x)$$

$$\left.\begin{array}{lll} \alpha = 1, & P_{n_1}(x) = 1, & n_1 = 0 \\ \beta = 1, & P_{n_2}(x) = 0, & n_2 = 0 \end{array}\right\} \Rightarrow s = 0$$

$$\alpha + \beta i = 1 + i \neq \lambda_i \ \Rightarrow \ k = 0$$

$$y_{p2} = e^x \cdot (C \cdot \cos x + D \sin x)$$

$$y_{p2}' = e^x \cdot (C \cdot \cos x + D \sin x) + e^x \cdot (-C \sin x + D \cos x)$$

$$= e^x \cdot ((C+D) \cos x + (-C+D) \sin x)$$

$$y_{p2}'' = e^x \cdot ((C+D) \cos x + (-C+D) \sin x) +$$

$$e^x \cdot (-(C+D) \sin x + (-C+D) \cos x)$$

$$= e^x \cdot ((\cancel{C} + D - \cancel{C} + D) \cos x + (-C + \cancel{D} - C - \cancel{D}) \sin x)$$

$$= e^x \cdot (2D \cos x - 2C \sin x)$$

$$y_{p2}'' - 3y_{p2}' + 2y_{p2} = e^x \cos x$$

$$\Leftrightarrow \quad e^x(2D\cos x - 2C\sin x) - 3e^x((C+D)\cos x + (-C+D)\sin x)$$
$$+ 2e^x(C\cos x + D\sin x) = e^x \cos x$$

$$\Leftrightarrow \quad \cancel{e^x} \cdot \left[(2D - 3\cdot(C+D) + 2C)\cos x + (-2C - 3\cdot(-C+D) + 2D)\sin x\right] = \cancel{e^x}\cos x$$

$$\Leftrightarrow \quad (2D - 3C - 3D + 2C)\cos x + (-2C + 3C - 3D + 2D)\sin x = \cos x$$

$$\Leftrightarrow \quad \left.\begin{aligned} -C - D &= 1 \\ C - D &= 0 \end{aligned}\right\} +$$

$$-2D = 1 \Rightarrow \quad D = \frac{-1}{2}, \quad C = -\frac{1}{2} \quad \Rightarrow$$

$$\boxed{y_{p2} = e^x \cdot \left(-\frac{1}{2}\cos x - \frac{1}{2}\sin x\right)}$$

Opste rjesenje: $y = y_h + y_{p1} + y_{p2}$

$$\boxed{y = C_1 e^x + C_2 e^{2x} + 2x + 3 - \frac{e^x}{2}(\cos x + \sin x)}$$

④ $$y'' - y' = (2x-1)e^{-x}$$

Homogeno rješenje: $y'' - y' = 0 \Rightarrow$

$$\lambda^2 - \lambda = 0$$

$$\Leftrightarrow \lambda(\lambda - 1) = 0$$

$$\Leftrightarrow \lambda_1 = 0, \ \lambda_2 = 1 \Rightarrow$$

$$\boxed{y_h = C_1 + C_2 e^x}$$

Partikularno rješenje:

$$r(x) = e^{-x} \cdot (2x-1)$$

$$r(x) = e^{\alpha x} \left(P_{n_1}(x) \cos\beta x + P_{n_2}(x) \sin\beta x \right)$$

$$\left.\begin{matrix} \alpha = -1, & P_{n_1}(x) = 2x-1, & n_1 = 1 \\ \beta = 0, & P_{n_2}(x) = 0, & n_2 = 0 \end{matrix}\right\} \Rightarrow s = 1$$

$$\alpha + \beta i = -1 \neq \lambda_i \Rightarrow k = 0$$

$$y_{p_1} = e^{-x} \cdot (Ax + B)$$

$$y_{p_1}' = -e^{-x} (Ax + B) + e^{-x} \cdot A$$

$$= e^{-x} \cdot (-Ax - B + A)$$

$$y_{p_1}'' = -e^{-x} (-Ax - B + A) + e^{-x} \cdot (-A)$$

$$= e^{-x} \cdot (Ax + B - A - A)$$

$$y_{p1}'' - y_{p1}' = e^{-x}\cdot(2x-1)$$

$$\Leftrightarrow \quad e^{-x}\cdot(Ax+B-2A) - e^{-x}\cdot(-Ax-B+A) = e^{-x}(2x-1)$$

$$\Leftrightarrow \quad \cancel{e^{-x}}\cdot(Ax+B-2A+Ax+B-A) = \cancel{e^{-x}}(2x-1)$$

$$\Leftrightarrow \quad 2Ax + (-3A+2B) = 2x-1$$

$$\Leftrightarrow \quad \begin{array}{l} 2A = 2 \\ -3A+2B = -1 \\ \hline A=1,\ B=1 \end{array} \Rightarrow$$

$$\boxed{y_{p1} = e^{-x}\cdot(x+1)}$$

Opste rješenje: $y = y_h + y_{p1}$

$$\boxed{y = C_1 + C_2e^{x} + e^{-x}(x+1)}$$

5 $\quad y^{(4)} + y'' = 6x$

Homogeno rješenje: $y^{(4)} + y'' = 0 \Rightarrow$

$$\lambda^4 + \lambda^2 = 0 \Rightarrow$$

$$\lambda^2\cdot(\lambda^2+1) = 0$$

$$\lambda_1 = \lambda_2 = 0,\ \lambda_3 = i,\ \lambda_4 = -i$$

$$\boxed{y_h = C_1 + C_2 x + C_3 \cos x + C_4 \sin x}$$

Partikularno rješenje:

$$r(x) = 6x$$

$$r(x) = e^{\alpha x} \cdot \left(P_{n1}(x) \cos \beta x + P_{n2}(x) \sin \beta x \right)$$

$$\left.\begin{array}{lll} \alpha = 0, & P_{n1}(x) = 6x, & n_1 = 1 \\ \beta = 0, & P_{n2}(x) = 0, & n_2 = 0 \end{array}\right\} \Rightarrow s = 1$$

$$\alpha + \beta i = 0 = \lambda_1 = \lambda_2 \Rightarrow k = 2$$

$$y_p = x^2 (Ax + B) = Ax^3 + Bx^2 \Rightarrow$$

$$y_p' = 3Ax^2 + 2Bx,$$

$$y_p'' = 6Ax + 2B,$$

$$y_p^{(3)} = 6A,$$

$$y_p^{(4)} = 0, \quad \Rightarrow$$

$$y_p^{(4)} + y_p'' = 6x \Rightarrow$$

$$0 + 6Ax + 2B = 6x \Rightarrow$$

$$\begin{array}{r} 6Ax = 6x \\ 2B = 0 \\ \hline A = 1, \ B = 0 \end{array} \Rightarrow$$

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$$\boxed{y_p = x^3}$$

Opšte rješenje: $y = y_h + y_p \Rightarrow$

$$\boxed{y = C_1 + C_2 x + C_3 \cos x + C_4 \sin x + x^3}$$

6

$$y'' + 3y' - 4y = e^{-4x} + xe^{-x}$$

Homogeno rješenje: $y'' + 3y' - 4y = 0 \Rightarrow$

$\lambda^2 + 3\lambda - 4 = 0 \Rightarrow$

$(\lambda + 4)(\lambda - 1) = 0 \Rightarrow$

$\lambda_1 = -4, \quad \lambda_2 = 1 \Rightarrow$

$$\boxed{y_h = C_1 e^{-4x} + C_2 e^{x}}$$

1. partikularno rješenje:

$r_1(x) = e^{-4x}$

$r(x) = e^{\alpha x} \cdot (P_{n1}(x) \cos \beta x + P_{n2}(x) \sin \beta x)$

$$\left.\begin{matrix} \alpha = -4, & P_{n1}(x) = 1, & n_1 = 0 \\ \beta = 0, & P_{n2}(x) = 0, & n_2 = 0 \end{matrix}\right\} \Rightarrow s = 0$$

$\alpha + \beta i = -4 = \lambda_1 \Rightarrow k = 1 \Rightarrow$

$$y_{p_1} = x \cdot e^{-4x} \cdot A = Axe^{-4x}$$

$$y_{p1}' = A\cdot\left(e^{-4x} + x\cdot e^{-4x}\cdot(-4)\right)$$
$$= A\cdot(1-4x)\,e^{-4x}$$

$$y_p'' = A\cdot\left(-4e^{-4x} + (1-4x)\cdot e^{-4x}\,(-4)\right)$$
$$= A\cdot\left(-4 + (1-4x)\cdot(-4)\right)e^{-4x}$$
$$= A\,(-4-4+16x)\,e^{-4x}$$

$$y_p'' + 3y_p' - 4y_p = e^{-4x} \Rightarrow$$

$$A\cdot(-8+16x)\,e^{-4x} + 3A\cdot(1-4x)e^{-4x} - 4Axe^{-4x} = e^{-4x}$$

$$\Rightarrow A(-8+\cancel{16x}+3-\cancel{12x}-\cancel{4x})\,\cancel{e^{-4x}} = \cancel{e^{-4x}}$$

$$\Rightarrow \quad -5A = 1 \Rightarrow A = -\frac{1}{5} \Rightarrow$$

$$\boxed{y_{p1} = -\frac{1}{5}xe^{-4x}}$$

2. partikularno rješenje

$$r_2(x) = xe^{-x}$$
$$r(x) = e^{\alpha x}\cdot\left(P_{n1}(x)\cos\beta x + P_{n2}(x)\sin\beta x\right)$$

$$\left.\begin{array}{lll} \alpha = -1, & P_{n1}(x) = x, & n_1 = 1 \\ \beta = 0, & P_{n2}(x) = 0, & n_2 = 0 \end{array}\right\} \Rightarrow s = 1$$

$$\alpha + \beta i = -1 \neq \lambda_i \Rightarrow k = 0$$

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$$y_{p2} = e^{-x} \cdot (Bx + C) \Rightarrow$$

$$y_{p2}' = -e^{-x} \cdot (Bx + C) + e^{-x} \cdot B$$

$$= (-Bx + B - C) e^{-x},$$

$$y_{p2}'' = (-B) \cdot e^{-x} + (-Bx + B - C) \cdot e^{-x} \cdot (-1)$$

$$= (Bx - 2B + C) e^{-x} \Rightarrow$$

$$y_{p2}'' + 3y_{p2}' - 4y_{p2} = xe^{-x} \Rightarrow$$

$$(Bx - 2B + C)e^{-x} + 3(-Bx + B - C)e^{-x} - 4(Bx + C)e^{-x}$$
$$= xe^{-x}$$

$$\Leftrightarrow (Bx - 2B + C - 3Bx + 3B - 3C - 4Bx - 4C)\cancel{e^{-x}} = x\cancel{e^{-x}}$$

$$-6Bx + B - 6C = x \Rightarrow$$

$$-6B = 1$$
$$B - 6C = 0 \Rightarrow C = \frac{B}{6}$$

$$B = -\frac{1}{6}, \quad C = -\frac{1}{36} \Rightarrow$$

$$\boxed{y_{p2} = \left(-\frac{1}{6}x - \frac{1}{36}\right)e^{-x}}$$

Opste rješenje: $y = y_h + y_{p1} + y_{p2} \Rightarrow$

$$\boxed{y = C_1 e^{-4x} + C_2 e^{x} - \frac{1}{5}xe^{-4x} - \left(\frac{1}{6}x + \frac{1}{36}\right)e^{-x}}$$

7 $\quad y'' + y' = x - \sin 2x$

Homogeno rješenje: $\quad y'' + y' = 0 \Rightarrow$

$\lambda^2 + \lambda = 0 \Rightarrow$

$\lambda(\lambda + 1) = 0 \Rightarrow$

$\lambda_1 = 0, \quad \lambda_2 = -1$

$$\boxed{y_h = C_1 + C_2 e^{-x}}$$

1. partikularno rješenje:

$p_1(x) = x$

$r(x) = e^{\alpha x} \cdot (P_{n_1}(x) \cos \beta x + P_{n_2}(x) \sin \beta x)$

$$\left.\begin{array}{lll} \alpha = 0, & P_{n_1}(x) = x, & n_1 = 1 \\ \beta = 0, & P_{n_2}(x) = 0, & n_2 = 0 \end{array}\right\} \Rightarrow s = 1$$

$\alpha + \beta i = 0 = \lambda_1 \Rightarrow k = 1 \Rightarrow$

$y_{p_1} = x \cdot (Ax + B) = Ax^2 + Bx$

$y_{p_1}' = 2Ax + B$

$y_{p_1}'' = 2A$

$y_{p_1}'' + y_{p_1}' = x \Rightarrow$

$2A + 2Ax + B = x \Rightarrow$

$$\begin{array}{l} 2A = 1 \\ 2A + B = 0 \\ \hline A = \frac{1}{2}, \quad B = -1 \end{array}$$

$$\boxed{y_{p1} = \frac{1}{2}x^2 - x}$$

2. partikularno rješenje:

$r_2(x) = -\sin 2x$

$r(x) = e^{\alpha x} \cdot (P_{n1}(x) \cos \beta x + P_{n2}(x) \sin \beta x)$

$$\left.\begin{matrix} \alpha = 0, & P_{n1}(x) = 0, & n_1 = 0 \\ \beta = 2, & P_{n2}(x) = -1, & n_2 = 0 \end{matrix}\right\} \Rightarrow s = 0$$

$\alpha + \beta i = 2i \neq \lambda_i \Rightarrow k = 0$

$y_{p2}(x) = C \cos 2x + D \sin 2x$

$y_{p2}' = C \cdot (-\sin 2x) \cdot 2 + D (\cos 2x) \cdot 2$

$= -2C \sin 2x + 2D \cos 2x$

$y_{p2}'' = -2C \cos(2x) \cdot 2 + 2D(-\sin 2x) \cdot 2$

$= -4C \cos(2x) - 4D \sin(2x)$

$y_{p2}'' + y_{p2}' = -\sin 2x \Rightarrow$

$-4C \cos 2x - 4D \sin 2x - 2C \sin 2x + 2D \cos 2x = -\sin 2x$

$\Leftrightarrow (-4C + 2D) \cos 2x + (-2C - 4D) \sin 2x = -\sin 2x$

$$\Rightarrow \left.\begin{matrix} -4C + 2D = 0 & / \cdot 2 \\ -2C - 4D = -1 & \end{matrix}\right\} +$$

$$-10C = -1 \Rightarrow$$

$$C = \frac{1}{10}, \quad D = \frac{2}{10} \Rightarrow$$

$$\boxed{y_{p2} = \frac{1}{10}\cos 2x + \frac{1}{5}\sin 2x}$$

Opšte rješenje: $y = y_h + y_{p1} + y_{p2} \Rightarrow$

$$\boxed{y = C_1 + C_2 e^{-x} + \frac{1}{2}x^2 - x + \frac{1}{10}\cos 2x + \frac{1}{5}\sin 2x}$$

Partikularno rješenje: $y(0) = 2, \; y'(0) = 1$

$$y(0) = 2 \Rightarrow \quad 2 = C_1 + C_2 e^0 + \cancel{\frac{1}{2}0^2} - \cancel{0} + \frac{1}{10}\cos(2\cdot 0) + \cancel{\frac{1}{5}\sin(2\cdot 0)}$$

$$\Rightarrow \quad C_1 + C_2 + \frac{1}{10} = 2 \quad \ldots (1)$$

$$y' = -C_2 e^{-x} + \frac{1}{2}\cdot 2x - 1 + \frac{1}{10}\cdot(-\sin 2x)\cdot 2 + \frac{1}{5}\cdot\cos(2x)\cdot 2$$

$$= -C_2 e^{-x} + x - 1 - \frac{1}{5}\sin 2x + \frac{2}{5}\cos(2x)$$

$$y'(0) = 1 \Rightarrow \quad 1 = -C_2 e^{-0} + \cancel{0} - 1 - \cancel{\frac{1}{5}\sin(2\cdot 0)} + \frac{2}{5}\cos(2\cdot 0)$$

$$\Rightarrow \quad -C_2 - 1 + \frac{2}{5} = 1 \quad \ldots (2)$$

Iz jednačina (1) i (2) imamo:

$$\left.\begin{aligned} C_1 + C_2 &= \frac{19}{10} \\ -C_2 &= \frac{16}{10} \end{aligned}\right\} +$$

$$C_1 = \frac{35}{10} = \frac{7}{2},$$

$$C_2 = -\frac{16}{10} = -\frac{8}{5}$$

$$\boxed{y_p = \frac{7}{2} - \frac{8}{5}e^{-x} + \frac{1}{2}x^2 - x + \frac{1}{10}\cos 2x + \frac{1}{5}\sin 2x}$$

(8) $y'' - 4y' + 3y = \sin^2 x$

Koristeći formulu $\cos 2x = \cos^2 x - \sin^2 x$

$$= 1 - \sin^2 x - \sin^2 x$$

dobijamo: $\cos 2x = 1 - 2\sin^2 x \Rightarrow$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

pa je početna jednačina ekvivalentna sa:

$$y'' - 4y' + 3y = \frac{1}{2} - \frac{1}{2}\cos 2x$$

- Homogeno rješenje: $y'' - 4y' + 3y = 0$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 1)\cdot(\lambda - 3) = 0$$

$$\lambda_1 = 1, \quad \lambda_2 = 3 \Rightarrow$$

$$\boxed{y_h = C_1 e^{x} + C_2 e^{3x}}$$

1. partikularno rješenje:

$$r_1(x) = \frac{1}{2}$$

$$r(x) = e^{\alpha x}\left(P_{n_1}(x)\cos\beta x + P_{n_2}(x)\sin\beta x\right)$$

$$\left.\begin{array}{lll} \alpha = 0, & P_{n_1}(x) = \frac{1}{2}, & n_1 = 0, \\ \beta = 0, & P_{n_2}(x) = 0, & n_2 = 0, \end{array}\right\} \Rightarrow s = 0$$

$$\alpha + i\beta = 0 \neq \lambda_i \Rightarrow k = 0$$

$$y_{p1} = A \Rightarrow$$

$$y_{p1}' = 0,$$

$$y_{p1}'' = 0$$

$$y_{p1}'' - 4y_{p1}' + 3y_{p1} = \frac{1}{2}$$

$$\Leftrightarrow \cancel{0} - \cancel{4\,0} + 3A = \frac{1}{2} \Rightarrow A = \frac{1}{6} \Rightarrow$$

$$\boxed{y_{p1} = \frac{1}{6}}$$

2. partikularno rješenje:

$$r_2(x) = -\frac{1}{2}\cos 2x$$

$$r(x) = e^{\alpha x}\cdot\left(P_{n_1}(x)\cos\beta x + P_{n_2}(x)\sin\beta x\right)$$

$$\left.\begin{array}{lll} \alpha = 0, & P_{n_1}(x) = -\frac{1}{2}, & n_1 = 0, \\ \beta = 2, & P_{n_2}(x) = 0, & n_2 = 0, \end{array}\right\} \Rightarrow s = 0$$

$$\alpha + i\beta = 2i \neq \lambda_i \Rightarrow k = 0 \Rightarrow$$

$$y_{p2} = B\cos 2x + C\sin 2x$$

$$y_{p2}' = -2B\sin 2x + 2C\cos 2x$$

$$y_{p2}'' = -4B\cos 2x - 4C\sin 2x \quad \Rightarrow$$

$$y_{p2}'' - 4y_{p2}' + 3y_{p2} = -\frac{1}{2}\cos 2x$$

$$\Rightarrow \quad -4B\cos 2x - 4C\sin 2x - 4\cdot(-2B\sin 2x + 2C\cos 2x)$$
$$+ 3(B\cos 2x + C\sin 2x) = -\frac{1}{2}\cos 2x$$

$$\Rightarrow \quad \cos 2x\cdot(-4B - 8C + 3B) +$$
$$\sin 2x\cdot(-4C + 8B + 3C) = -\frac{1}{2}\cos 2x$$

$$\Rightarrow \quad \left.\begin{aligned} -B - 8C &= -\frac{1}{2} \quad /\cdot 8 \\ 8B - C &= 0 \end{aligned}\right\} +$$

$$-65C = -4 \Rightarrow$$

$$C = \frac{4}{65}, \quad B = \frac{1}{130} \quad \Rightarrow$$

$$\boxed{y_{p2} = \frac{1}{130}\cos 2x + \frac{4}{65}\sin 2x}$$

Opste rjesenje: $\quad y = y_h + y_{p1} + y_{p2}$

$$\boxed{y = C_1 e^{x} + C_2 e^{3x} + \frac{1}{6} + \frac{1}{130}\cos 2x + \frac{4}{65}\sin 2x}$$

⑨ $$y'' - 2y' + y = \frac{e^x}{x}$$

- Homogeno rješenje: $y'' - 2y' + y = 0 \Rightarrow$

$\lambda^2 - 2\lambda + 1 = 0 \Rightarrow$

$(\lambda - 1)^2 = 0 \Rightarrow$

$\lambda_1 = \lambda_2 = 1 \Rightarrow$

$$\boxed{y_h = C_1 e^x + C_2 x e^x}$$

- Metod varijacije konstanti:

$$y = C_1(x) e^x + C_2(x) x e^x$$

$$C_1'(x) e^x + C_2'(x) x e^x = 0$$
$$C_1'(x)(e^x)' + C_2'(x)(xe^x)' = \frac{e^x}{x}$$

$$\left.\begin{aligned} C_1'(x) e^x + C_2'(x) x e^x &= 0 \quad / \cdot (-1) \\ C_1'(x) e^x + C_2'(x) \cdot (e^x + xe^x) &= \frac{e^x}{x} \end{aligned}\right\} +$$

$$C_2'(x) \cdot \left[\cancel{-xe^x} + e^x + \cancel{xe^x}\right] = \frac{e^x}{x}$$

$$C_2'(x) \cdot \cancel{e^x} = \frac{\cancel{e^x}}{x}$$

$$C_2'(x) = \frac{1}{x} \Rightarrow$$

$$\boxed{C_2(x) = \int \frac{1}{x}\,dx = \ln|x| + C_{21}}$$

Uvrštavanjem $C_2'(x) = \frac{1}{x}$ u jednacinu:

$$C_1'(x)\, e^x + C_2'(x)\, x e^x = 0$$

dobijamo

$$C_1'(x)\, e^x + \frac{1}{\cancel{x}} \cancel{x} e^x = 0$$

$$\Leftrightarrow \quad e^x \cdot \left(C_1'(x) + 1 \right) = 0 \quad \Rightarrow$$

$$C_1'(x) = -1 \quad \Rightarrow$$

$$\boxed{C_1(x) = \int -1\, dx = -x + C_1}$$

Sada je:

$$y = (-x + C_1)\, e^x + (\ln|x| + C_2)\, x e^x$$

$$= -x e^x + C_1 e^x + \ln|x|\, x e^x + C_2 x e^x$$

$$\boxed{y = C_1 e^x + C_2 x e^x + x e^x (\ln|x| - 1)}$$

★ NAPOMENA:

Kako je $C_2 x e^x - x e^x = (C_2 - 1) x e^x = C x e^x$, rješenje možemo zapisati i kao

$$y = C_1 e^x + C x e^x + x e^x \ln|x| .$$

(10) $y'' + 2y' + 5y = e^{-x}(\cos^2 x + \operatorname{tg} x)$

- Homogeno rješenje: $y'' + 2y' + 5y = 0 \Rightarrow$

$\lambda^2 + 2\lambda + 5 = 0 \Rightarrow$

$\lambda_{1,2} = \frac{-2 \pm \sqrt{4-4\cdot 5}}{2} = \frac{-2 \pm 4i}{2}$

$\lambda_{1,2} = -1 \pm 2i \Rightarrow$

$$\boxed{y_h = e^{-x} \cdot (C_1 \cos 2x + C_2 \sin 2x)}$$

- Metod varijacije konstanti:

$y = C_1(x) e^{-x} \cos 2x + C_2(x) e^{-x} \sin 2x$

$C_1'(x)\, e^{-x} \cos 2x + C_2'(x)\, e^{-x} \sin 2x = 0$

$C_1'(x) \cdot (e^{-x} \cos 2x)' + C_2'(x)\, (e^{-x} \sin 2x)' = e^{-x}(\cos^2 x + \operatorname{tg} x)$

$C_1'(x)\, e^{-x} \cos 2x + C_2'(x)\, e^{-x} \sin 2x = 0$

$C_1'(x) \cdot (-e^{-x} \cos 2x - 2e^{-x} \sin 2x) +$

$C_2'(x) \cdot (-e^{-x} \sin 2x + 2e^{-x} \cos 2x) = e^{-x}(\cos^2 x + \operatorname{tg} x)$

$(C_1'(x) \cos 2x + C_2'(x) \sin 2x)\, \cancel{e^{-x}} = 0$

$(C_1'(x) \cdot (-\cos 2x - 2\sin 2x) + C_2'(x)\, (2\cos 2x - \sin 2x))\, \cancel{e^{-x}}$

$= \cancel{e^{-x}}(\cos^2 x + \operatorname{tg} x)$

$$C_1'(x)\cdot \cos 2x \quad + C_2'(x)\sin 2x = 0 \quad \ldots(1)$$

$$C_1'(x)\cdot(-\cos 2x - 2\sin 2x) \quad + C_2'(x)\cdot(2\cos 2x - \sin 2x) = \cos^2 x + \operatorname{tg} x \quad \ldots(2)$$

$(1)\cdot(\cos 2x + 2\sin 2x)$;

$(2)\cdot \cos 2x$

$$\left.\begin{aligned} &C_1'(x)\ \cos 2x(\cos 2x + 2\sin 2x) + C_2'(x)\cos 2x \sin 2x = 0 \\ -&C_1'(x)\ \cos 2x(\cos 2x + 2\sin 2x) + C_2'(x)\cos 2x\cdot(2\cos 2x - \sin 2x) \\ &= \cos 2x(\cos^2 x + \operatorname{tg} x) \end{aligned}\right\} +$$

$$C_2'(x)\cdot\left[\cos 2x \sin 2x + \cos 2x\cdot(2\cos 2x - \sin 2x)\right] = \cos x\cdot(\cos^2 x + \operatorname{tg} x)$$

$$\Leftrightarrow\ C_2'(x)\cdot \cancel{\cos 2x}\cdot(\cancel{\sin 2x} + 2\cos 2x - \cancel{\sin 2x}) = \cancel{\cos 2x}\cdot(\cos^2 x + \operatorname{tg} x)$$

$$C_2'(x) = \frac{\cos^2 x + \frac{\sin x}{\cos x}}{2\cos 2x} =$$

$$C_2(x) = \int \frac{\cos^2 x + \operatorname{tg} x}{2\cdot(\cos^2 x - \sin^2 x)}\,dx$$

$$= \frac{1}{2}\cdot\int \frac{\cancel{\cos^2 x}\cdot\left(1 + \frac{\operatorname{tg} x}{\cos^2 x}\right)}{\cancel{\cos^2 x}\cdot\left(1 - \frac{\sin^2 x}{\cos^2 x}\right)}\cdot dx$$

$$= \frac{1}{2}\cdot\int \frac{1 + \operatorname{tg} x\cdot\frac{1}{\cos^2 x}}{1 - \operatorname{tg}^2 x}\,dx$$

$$= \frac{1}{2}\cdot\Bigg[\underbrace{\int \frac{dx}{1 - \operatorname{tg}^2 x}}_{I_1} + \underbrace{\int \frac{\operatorname{tg} x}{1 - \operatorname{tg}^2 x}\cdot\frac{dx}{\cos^2 x}}_{I_2}\Bigg]$$

$$I_1 = \int \frac{dx}{1-tg^2x} = \int \frac{\cos^2 x}{1-tg^2x} \cdot \frac{dx}{\cos^2 x}$$

$$= \int \frac{1}{\frac{1-tg^2x}{\cos^2 x}} \cdot \frac{dx}{\cos^2 x} = \int \frac{\frac{dx}{\cos^2 x}}{(1-tg^2x)\cdot \frac{\sin^2 x+\cos^2 x}{\cos^2 x}}$$

$$= \int \frac{\frac{dx}{\cos^2 x}}{(1-tg^2x)\cdot(tg^2x+1)} = \left\{ \begin{array}{l} t = tg\,x \\ dt = \frac{dx}{\cos^2 x} \end{array} \right.$$

$$= \int \frac{dt}{(1-t^2)(t^2+1)} = \int \frac{dt}{(1-t)(1+t)(t^2+1)}$$

$$= \int \frac{-1}{(t-1)(t+1)\,(t^2+1)}\,dt$$

$$\frac{-1}{(t-1)(t+1)(t^2+1)} = \frac{A}{t-1} + \frac{B}{t+1} + \frac{Ct+D}{t^2+1}$$

$$\Leftrightarrow\ -1 = A(t+1)(t^2+1) + B(t-1)(t^2+1) + (Ct+D)(t^2-1)$$

$$= A(t^3+t^2+t+1) + B(t^3-t^2+t-1) + Ct^3+Dt^2-Ct-D$$

$$= t^3\cdot(A+B+C) + t^2(A-B+D) + t(A+B-C) +$$

$$A-B-D \quad \Rightarrow$$

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$$\left.\begin{array}{l} A + B + C \quad\quad = 0 \\ A - B \quad\quad + D = 0 \\ A + B - C \quad\quad = 0 \\ A - B \quad\quad - D = -1 \end{array}\right\} +$$

$$\left.\begin{array}{l} A + B + C \quad = 0 \\ A + B - C \quad = 0 \end{array}\right\} +$$

$$2A - 2B \quad = -1$$

$$\left.\begin{array}{l} 2A + 2B \quad = 0 \\ 2A - 2B \quad = -1 \end{array}\right\} +$$

$$4A \quad = -1 \Rightarrow$$

$$A = -\frac{1}{4}, \quad B = \frac{1}{4}, \quad C = 0, \quad D = \frac{1}{2} \Rightarrow$$

$$I_1 = \int \frac{-\frac{1}{4}}{t-1}\,dt + \int \frac{\frac{1}{4}}{t+1}\,dt + \int \frac{\frac{1}{2}}{t^2+1}\,dt$$

$$= -\frac{1}{4}\ln|t-1| + \frac{1}{4}\ln|t+1| + \frac{1}{2}\operatorname{arctg}(t) + C$$

$$= \frac{1}{4}\ln\left|\frac{t+1}{t-1}\right| + \frac{1}{2}\operatorname{arctg}(t) + C$$

$$= \frac{1}{4}\ln\left|\frac{\operatorname{tg}x+1}{\operatorname{tg}x-1}\right| + \frac{1}{2}\operatorname{arctg}(\operatorname{tg}x) + C$$

$$= \frac{1}{4}\ln\left|\frac{\operatorname{tg}x+1}{\operatorname{tg}x-1}\right| + \frac{1}{2}x + C$$

$$I_2 = \int \frac{tg\,x}{1-tg^2x} \cdot \frac{dx}{\cos^2 x} = \begin{cases} t = tg\,x \\ dt = \frac{dx}{\cos^2 x} \end{cases}$$

$$= \int \frac{t}{1-t^2}\,dt = -\frac{1}{2} \cdot \int \frac{-2t\,dt}{1-t^2} = \begin{cases} u = 1-t^2 \\ du = -2t\,dt \end{cases}$$

$$= -\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\ln|u| = -\frac{1}{2}\ln|1-t^2| + C$$

$$= -\frac{1}{2}\ln|1-tg^2x| + C \qquad \Rightarrow$$

$$C_2(x) = \frac{1}{2}\cdot\left[\frac{1}{4}\ln\left|\frac{tgx+1}{tgx-1}\right| + \frac{x}{2} - \frac{1}{2}\ln|1-tg^2x|\right] + C_2$$

$$= \frac{1}{2}\cdot\frac{1}{4}\cdot\left(\ln\left|\frac{tgx+1}{tgx-1}\right| - 2\ln|(1-tgx)(1+tgx)| + 2x\right) + C_2$$

$$\boxed{C_2(x) = \frac{1}{8}\ln\left|\frac{tgx+1}{tgx-1}\right| - \frac{1}{4}\ln|1-tg^2x| + \frac{x}{4} + C_2}$$

Uvrštavanjem $C_2'(x) = \dfrac{\cos^2 x + tgx}{2\cos 2x}$ u jednačinu (1)

dobijamo:

$$C_1'(x) = -\frac{\cos^2 x + tgx}{2\cos^2 2x} \cdot \sin 2x$$

$$C_1(x) = -\int \frac{\cos^2 x + tgx}{\cancel{2}\cdot(\cos^2 x - \sin^2 x)^2} \cdot \cancel{2}\sin x\cos x\,dx$$

★ Ovaj integral se ponovo rješava smjenom $t = \operatorname{tg} x$, ali je dosta naporan pa ću zapisati samo konačni rezultat:

$$\boxed{C_1(x) = \frac{3\ln|\operatorname{tg}x+1| - \ln|\operatorname{tg}x-1| - 2\ln\left|\frac{1}{\cos x}\right|}{8} + \frac{1}{8(\operatorname{tg}x+1)} + \frac{3}{8(\operatorname{tg}x-1)} + C_1}$$

Opšte rješenje je:

$$y = e^{-x}\left(C_1(x)\cos(2x) + C_2(x)\sin(2x)\right).$$