1

$$S = 1 - \frac{1}{3\cdot 3} + \frac{1}{5\cdot 3^2} - \frac{1}{7\cdot 3^3} + \dots$$

$$= (-1)^0 \cdot \frac{1}{1\cdot 3^0} + (-1)^1 \cdot \frac{1}{3\cdot 3^1} + (-1)^2 \cdot \frac{1}{5\cdot 3^2} + (-1)^3 \cdot \frac{1}{7\cdot 3^3} + \dots$$

$$= \sum_{k=0}^{\infty} (-1)^k \cdot \frac{1}{(2k+1)\cdot 3^k}$$

$$= \sum_{k=0}^{\infty} (-1)^k \cdot \frac{1}{(2k+1)} \cdot \left(\frac{1}{3}\right)^k$$

$$= \sum_{k=0}^{\infty} (-1)^k \cdot \frac{1}{2k+1} \cdot \left(\frac{1}{\sqrt{3}}\right)^{2k} \quad \dots (1)$$

Posmatrajmo funkciju:

$$f(x) = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{1}{2k+1} \cdot x^{2k}$$

Ispitajmo da li $x = \frac{1}{\sqrt{3}}$ pripada domenu konvergencije.

$x_c = 0$,

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^n}{2n+1}}{\frac{(-1)^{n+1}}{2(n+1)+1}} \right| = \lim_{n\to\infty} \left| \frac{2n+3}{2n+1} \right| = 1$$

Kako je $\frac{1}{\sqrt{3}} \in (-1,1) \subseteq D$, red konvergira za $x = \frac{1}{\sqrt{3}}$.

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{2n+1} x^{2n}$$

$$= \frac{1}{x} \cdot \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{2n+1} x^{2n+1}$$

$$= \frac{1}{x} \cdot \int_0^x \left(\sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{2n+1} t^{2n+1} \right)' dt$$

$$= \frac{1}{x} \cdot \int_0^x \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{\cancel{2n+1}} \cdot \cancel{(2n+1)} \cdot t^{2n} \, dt$$

$$= \frac{1}{x} \cdot \int_0^x \sum_{n=0}^{\infty} (-1)^n \cdot (t^2)^n \, dt$$

$$= \frac{1}{x} \cdot \int_0^x \sum_{n=0}^{\infty} (-t^2)^n \, dt$$

$$= \frac{1}{x} \cdot \int_0^x \frac{1}{1+t^2} \, dt$$

$$= \frac{1}{x} \cdot \operatorname{arctg} t \Big|_0^x$$

$$= \frac{1}{x} \cdot \operatorname{arctg} x$$

Kako je $S = f\left(\frac{1}{\sqrt{3}}\right)$ vrijedi

$$S = \frac{1}{\frac{1}{\sqrt{3}}} \cdot \operatorname{arctg}\left(\frac{1}{\sqrt{3}}\right) = \sqrt{3} \cdot \frac{\pi}{6} \Rightarrow$$

$$\boxed{1 - \frac{1}{3 \cdot 3} + \frac{1}{5 \cdot 3^2} - \frac{1}{7 \cdot 3^3} + \ldots = \frac{\pi\sqrt{3}}{6}}$$

②

Koristeći identitet $1+2+\ldots+n = \frac{n(n+1)}{2}$ imamo:

$$S = \sum_{n=0}^{\infty} \frac{1+2+\ldots+n}{3^n} = \sum_{n=0}^{\infty} \frac{\frac{n(n+1)}{2}}{3^n} = \frac{1}{2} \cdot \sum_{n=0}^{\infty} n(n+1)\left(\frac{1}{3}\right)^n$$

Posmatrajmo funkciju

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{2} n \cdot (n+1) x^n$$

Ispitajmo da li $x = \frac{1}{3}$ pripada domenu konvergencije

$x_c = 0$,

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{\cancel{\frac{1}{2}} n \cancel{(n+1)}}{\cancel{\frac{1}{2}} \cancel{(n+1)}(n+2)}\right| = 1$$

Kako je $\frac{1}{3} \in (-1,1) \subseteq D$, red konvergira za $x = \frac{1}{3}$.

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{2} n(n+1) \cdot x^n$$

$$= \frac{1}{2}\left(\int_0^x \sum_{n=0}^{\infty} n(n+1)\, t^n\, dt\right)'$$

$$= \frac{1}{2} \cdot \left(\sum_{n=0}^{\infty} n\cancel{(n+1)} \frac{t^{n+1}}{\cancel{n+1}} \Big|_0^x \right)'$$

$$= \frac{1}{2} \cdot \left(\sum_{n=0}^{\infty} n \cdot x^{n+1}\right)'$$

$$= \frac{1}{2} \cdot \left(x^2 \cdot \sum_{n=0}^{\infty} n \cdot x^{n-1} \right)'$$

$$= \frac{1}{2} \cdot \left(x^2 \cdot \left(\int_0^x \sum_{n=0}^{\infty} n\, t^{n-1}\, dt \right)' \right)'$$

$$= \frac{1}{2} \cdot \left(x^2 \cdot \left(\sum_{n=0}^{\infty} \cancel{n} \cdot \frac{t^n}{\cancel{n}} \Big|_0^x \right)' \right)'$$

$$= \frac{1}{2} \cdot \left(x^2 \cdot \left(\sum_{n=0}^{\infty} x^n \right)' \right)'$$

$$= \frac{1}{2} \cdot \left(x^2 \cdot \left(\frac{1}{1-x} \right)' \right)'$$

$$= \frac{1}{2} \cdot \left(x^2 \cdot \frac{1}{(1-x)^2} \right)'$$

$$= \frac{1}{2} \cdot \frac{2x \cdot (1-x)^2 - x^2 \cdot 2(1-x) \cdot (-1)}{(1-x)^4}$$

$$= \frac{1}{2} \cdot \frac{\cancel{(1-x)} \cdot [2x(1-x) + 2x^2]}{(1-x)^4}$$

$$= \frac{1}{2} \cdot \frac{2x - \cancel{2x^2} + \cancel{2x^2}}{(1-x)^3}$$

$$= \frac{x}{(1-x)^3}$$

Kako je $S = f\left(\frac{1}{3}\right)$; vrijedi:

$$S = \frac{\frac{1}{3}}{\left(1-\frac{1}{3}\right)^3} = \frac{\frac{1}{3}}{\left(\frac{2}{3}\right)^3} = \frac{\frac{1}{3}}{\frac{8}{\cancel{27}_9}} \Rightarrow \boxed{\sum_{n=0}^{\infty} \frac{1+2+\ldots+n}{3^n} = \frac{9}{8}}$$

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$$S = \sum_{n=0}^{\infty} \frac{n^2+4n+1}{3^n} = \sum_{n=0}^{\infty} (n^2+4n+1)\cdot\left(\frac{1}{3}\right)^n$$

Posmatrajmo funkciju:

$$f(x) = \sum_{n=0}^{\infty} (n^2+4n+1)\cdot x^n$$

Ispitajmo da li $x=\frac{1}{3}$ pripada domenu konvergencije

$x_c = 0,$

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n^2+4n+1}{(n+1)^2+4(n+1)+1}\right| = \lim_{n\to\infty}\left|\frac{n^2+4n+1}{n^2+6n+6}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\cancel{n^2}\left(1+\cancel{\frac{4}{n}}^{0}+\cancel{\frac{1}{n^2}}^{0}\right)}{\cancel{n^2}\left(1+\cancel{\frac{6}{n}}^{0}+\cancel{\frac{6}{n^2}}^{0}\right)}\right| = 1$$

Kako je $\frac{1}{3} \in (-1,1) \subseteq D$, red konvergira za $x=\frac{1}{3}$.

Posmatrajmo funkciju

$$f(x) = \sum_{n=0}^{\infty} (n^2+4n+1)\, x^n$$

$$= \sum_{n=0}^{\infty} n^2x^n + 4\cdot\sum_{n=0}^{\infty} nx^n + \sum_{n=0}^{\infty} x^n$$

$$= x\cdot\sum_{n=0}^{\infty} n^2x^{n-1} + 4x\cdot\sum_{n=0}^{\infty} nx^{n-1} + \frac{1}{1-x}$$

$$= x \cdot \sum_{n=1}^{\infty} n^2 x^{n-1} + 4x \sum_{n=1}^{\infty} n x^{n-1} + \frac{1}{1-x}$$

$$= x \cdot \left(\int_0^x \sum_{n=1}^{\infty} n^2 t^{n-1} \, dt \right)' + 4x \cdot \left(\int_0^x \sum_{n=1}^{\infty} n t^{n-1} \, dt \right)' + \frac{1}{1-x}$$

$$= x \cdot \left(\sum_{n=1}^{\infty} n^2 \frac{t^n}{\cancel{n}} \Big|_0^x \right)' + 4x \cdot \left(\sum_{n=1}^{\infty} \cancel{n} \cdot \frac{t^n}{\cancel{n}} \Big|_0^x \right)' + \frac{1}{1-x}$$

$$= x \cdot \left(\sum_{n=1}^{\infty} n x^n \right)' + 4x \cdot \left(\sum_{n=1}^{\infty} x^n \right)' + \frac{1}{1-x}$$

$$= x \cdot \left(x \sum_{n=1}^{\infty} n x^{n-1} \right)' + 4x \cdot \left(\sum_{n=0}^{\infty} x^n - x^0 \right)' + \frac{1}{1-x}$$

$$= x \cdot \left(x \cdot \left(\int_0^x \sum_{n=1}^{\infty} n t^{n-1} \, dt \right)' \right)' + 4x \cdot \left(\frac{1}{1-x} - 1 \right)' + \frac{1}{1-x}$$

$$= x \cdot \left(x \cdot \left(\sum_{n=1}^{\infty} \cancel{n} \frac{t^n}{\cancel{n}} \Big|_0^x \right)' \right)' + 4x \cdot \frac{1}{(1-x)^2} + \frac{1}{1-x}$$

$$= x \cdot \left(x \cdot \left(\sum_{n=1}^{\infty} x^n \right)' \right)' + \frac{4x}{(1-x)^2} + \frac{1}{1-x}$$

$$= x \cdot \left(x \cdot \left(\sum_{n=0}^{\infty} x^n - x^0 \right)' \right)' + \frac{4x}{(1-x)^2} + \frac{1}{1-x}$$

$$= x \cdot \left(x \cdot \left(\frac{1}{1-x} - 1 \right)' \right)' + \frac{4x}{(1-x)^2} + \frac{1}{1-x}$$

$$= x \cdot \left(x \cdot \frac{1}{(1-x)^2} \right)' + \frac{4x}{(1-x)^2} + \frac{1}{1-x}$$

$$= x \cdot \frac{(1-x)^2 - x \cdot 2(1-x)(-1)}{(1-x)^4} + \frac{4x}{(1-x)^2} + \frac{1}{1-x}$$

$$= x \cdot \frac{\cancel{(1-x)} \cdot [1-x+2x]}{(1-x)^4} + \frac{4x}{(1-x)^2} + \frac{1}{1-x}$$

$$= \frac{x \cdot (1+x)}{(1-x)^3} + \frac{4x}{(1-x)^2} + \frac{1}{1-x}$$

$$= \frac{x(1+x) + 4x(1-x) + (1-x)^2}{(1-x)^3}$$

$$= \frac{x + x^2 + 4x - 4x^2 + 1 - 2x + x^2}{(1-x)^3}$$

$$= \frac{-2x^2 + 3x + 1}{(1-x)^3} \quad ;$$

Kako je $S = f\left(\frac{1}{3}\right)$ vrijedi

$$S = \frac{-2 \cdot \left(\frac{1}{3}\right)^2 + 3 \cdot \frac{1}{3} + 1}{\left(1 - \frac{1}{3}\right)^3} = \frac{-\frac{2}{9} + 1 + 1}{\left(\frac{2}{3}\right)^3} = \frac{\frac{-2+9+9}{9}}{\frac{8}{27}} = \frac{\frac{16}{9}}{\frac{8}{27}} = \frac{\cancel{27}^{3} \cdot \cancel{16}^{2}}{\cancel{9} \cdot \cancel{8}}$$

$$\boxed{S = \sum_{n=0}^{\infty} \frac{n^2 + 4n + 1}{3^n} = 6}$$

(4) $$f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{x^{2n+1}}{4n^2-1}$$

$x_c = 0$

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} \cdot \frac{1}{4n^2-1}}{(-1)^{n+2} \cdot \frac{1}{4(n+1)^2-1}} \right| = \lim_{n\to\infty} \left| \frac{4n^2+8n+3}{4n^2-1} \right| = 1$$

Za $x=1$ imamo:

$$S_1 = f(1) = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1^{2n+1}}{4n^2-1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2-1}$$

Lajbnicov kriterijum:

1° $\lim_{n\to\infty} \frac{1}{4n^2-1} = 0$ ✓

2° $a_{n+1} < a_n \Rightarrow \frac{1}{4(n+1)^2-1} < \frac{1}{4n^2-1} \Rightarrow 4(n+1)^2 \cancel{-1} > 4n^2 \cancel{-1}$

$\Rightarrow \cancel{4}(n+1)^2 > \cancel{4}n^2 \Rightarrow (n+1)^2 > n^2$ ✓

KVG

Za $x=-1$ imamo

$$S_{-1} = f(-1) = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{(-1)^{2n+1}}{4n^2-1} = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{-1}{4n^2-1}$$

$= -f(1) \Rightarrow$ KVG, jer red KVG za $x=1$

Dakle, domen konvergencije je $\boxed{D = [-1, 1]}$

Kako je $S_1 = f(1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2-1}$, zaključujemo da red konvergira za $x=1$ pa posmatramo funkciju:

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2-1} x^{2n+1}$$

$$= \int_0^x \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2-1} \cdot t^{2n+1} \right)' dt$$

$$= \int_0^x \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)\cancel{(2n+1)}} \cdot \cancel{(2n+1)}\, t^{2n}\, dt$$

$$= \int_0^x t \cdot \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} t^{2n-1}\, dt$$

$$= \int_0^x t \cdot \left(\int_0^t \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \cdot u^{2n-1} \right)' du \right) dt$$

$$= \int_0^x t \cdot \left(\int_0^t \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\cancel{2n-1}} \cdot \cancel{(2n-1)} \cdot u^{2n-2}\, du \right) dt$$

$$= \int_0^x t \cdot \left(\int_0^t \sum_{n=0}^{\infty} (-1)^{n+2} \cdot u^{2(n+1)-2}\, du \right) dt$$

$$= \int_0^x t \cdot \left(\int_0^t \sum_{n=0}^{\infty} (-1)^{n} \cdot u^{2n}\, du \right) dt$$

$$= \int_0^x t \cdot \left(\int_0^t \sum_{n=0}^{\infty} (-u^2)^n \, du \right) dt$$

$$= \int_0^x t \cdot \left(\int_0^t \frac{du}{1+u^2} \right) dt$$

$$= \int_0^x t \cdot \operatorname{arctg}(u) \Big|_0^t \, dt$$

$$= \int_0^x t \operatorname{arctg} t \, dt = \begin{cases} u = \operatorname{arctg} t & v = \frac{t^2}{2} \\ du = \frac{1}{1+t^2} dt & dv = t\,dt \end{cases}$$

$$= \frac{t^2}{2} \cdot \operatorname{arctg} t \Big|_0^x - \int_0^x \frac{\frac{t^2}{2}}{1+t^2} dt$$

$$= \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} \cdot \int_0^x \frac{t^2+1-1}{t^2+1} dt$$

$$= \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} \cdot \left[\int_0^x \frac{\cancel{t^2+1}}{\cancel{t^2+1}} dt - \int_0^x \frac{dt}{1+t^2} \right]$$

$$= \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} \cdot \left[t \Big|_0^x - \operatorname{arctg} t \Big|_0^x \right]$$

$$= \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} x + \frac{1}{2} \operatorname{arctg} x$$

$$= \frac{1}{2} \cdot \left((x^2+1) \operatorname{arctg} x - x \right)$$

Kako je $S_1 = f(1)$ vrijedi

$$S_1 = \frac{1}{2}\cdot\left(\left(1^2+1\right)\operatorname{arctg} 1 - 1\right) = \frac{1}{2}\cdot\left(2\cdot\frac{\pi}{4} - 1\right) = \frac{\frac{\pi}{2}-1}{2}$$

Dakle, $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2-1} = \frac{\pi}{4} - \frac{1}{2}}$

★ Da li smo do rješenja mogli doći na jednostavniji način, bez integracije 2 puta?

(5)

$$xy' = y\cdot\cos\left(\ln\left(\frac{y}{x}\right)\right) \quad ; \quad x \neq 0$$

$$\Leftrightarrow \quad y' = \frac{y}{x}\cdot\cos\left(\ln\left(\frac{y}{x}\right)\right)$$

HOMOGENA D.J. $\Rightarrow$ smjena $\frac{y}{x} = t \Rightarrow y = tx$

$y' = t'x + t \quad \Rightarrow$

$\ldots (1)$

$$t'x + t = t\cdot\cos(\ln(t))$$

$$\Leftrightarrow \quad t'x = t\cdot(\cos(\ln t) - 1)$$

$$\Leftrightarrow \quad \frac{dt}{dx}\cdot x = t\cdot(\cos(\ln t) - 1)$$

$$\Leftrightarrow \quad \frac{dt}{t\cdot(\cos(\ln t)-1)} = \frac{dx}{x}$$

$$(=) \int \frac{dt}{t\cdot(\cos(\ln t)-1)} = \int \frac{dx}{x}$$

smjena: $\ln t = u \Rightarrow \frac{dt}{t} = du$ ... (2)

$$\int \frac{du}{\cos u - 1} = \ln|x| + C_1$$

$$(=) \int \frac{1}{\cos u - 1} \cdot \frac{\cos u + 1}{\cos u + 1}\, du = \ln|x| + \ln|C_2|$$

$$\Leftrightarrow \int \frac{\cos u + 1}{\cos^2 u - 1}\, du = \ln|C_2 x|$$

$$(=) \underbrace{\int \frac{\cos u\, du}{-\sin^2 u}}_{I_1} + \underbrace{\int \frac{du}{-\sin^2 u}}_{I_2} = \ln|C_2 x|$$

$$I_1 = \int \frac{\cos u\, du}{-\sin^2 u} = \left\{ \begin{array}{l} w = \sin u \\ dw = \cos u\, du \end{array} \right.$$

$$= \int \frac{dw}{-w^2} = \int -w^{-2}\, dw = -\frac{w^{-1}}{(-1)} = \frac{1}{w} = \frac{1}{\sin u} + C_3$$

$$I_2 = \int \frac{du}{-\sin^2 u} = \operatorname{ctg} u + C_4$$

$$\frac{1}{\sin u} + \operatorname{ctg} u = \ln|c_2 x|$$

$$\Leftrightarrow \frac{1+\cos u}{\sin u} = \ln|c_2 x|$$

$$\overset{(2)}{\Rightarrow} \frac{1+\cos(\ln t)}{\sin(\ln t)} = \ln|c_2 x|$$

$$\overset{(1)}{\Rightarrow} \boxed{\frac{1+\cos(\ln(\frac{y}{x}))}{\sin(\ln(\frac{y}{x}))} = \ln|cx|}$$

⑥ $\quad xy' - y = \sqrt{x^2+y^2}$

1° Ako je $x=0$ diferencijalna jednačina postaje jednačina oblika:

$$-y = \sqrt{y^2} \Rightarrow -y = |y| \Rightarrow -\cancel{y} = \cancel{y}\cdot \operatorname{sgn}(y)$$

$$\Rightarrow \operatorname{sgn}(y) = -1 \Rightarrow y<0$$

Dakle, za $y<0$, rješenje početne jednačine je $x=0$.

2° Sada možemo pretpostaviti da je $x \neq 0$ pa je:

$$xy' - y = \sqrt{x^2+y^2}$$

$$\Leftrightarrow \quad xy' - y = \sqrt{x^2\left(1+\left(\frac{y}{x}\right)^2\right)}$$

$$\Leftrightarrow \quad xy' - y = |x| \cdot \sqrt{1+\left(\frac{y}{x}\right)^2} \quad /:x$$

$$\Leftrightarrow \quad y' - \frac{y}{x} = \frac{\cancel{x} \cdot \operatorname{sgn}(x)}{\cancel{x}} \cdot \sqrt{1+\left(\frac{y}{x}\right)^2}$$

HOMOGENA D.J. $\Rightarrow$ smjena $\frac{y}{x} = t \Rightarrow y = tx \Rightarrow$

$y' = t'x + t$

$\ldots(1)$

$$t'x + \cancel{t} - \cancel{t} = \operatorname{sgn}(x) \cdot \sqrt{1+t^2}$$

$$\Leftrightarrow \quad \frac{dt}{dx} \cdot x = \operatorname{sgn}(x) \cdot \sqrt{1+t^2}$$

$$\Leftrightarrow \quad \frac{dt}{\sqrt{1+t^2}} = \frac{\operatorname{sgn}(x) \cdot dx}{x}$$

$$\Leftrightarrow \quad \underbrace{\int \frac{dt}{\sqrt{1+t^2}}}_{I_1} = \underbrace{\int \frac{\operatorname{sgn}(x)\, dx}{x}}_{I_2} \qquad \ldots(2)$$

$$I_1 = \int \frac{dt}{\sqrt{1+t^2}} = \left\{ \begin{array}{l} t = \operatorname{tg} u \Rightarrow u = \operatorname{arctg}(t) \\ dt = \frac{du}{\cos^2 u} \end{array} \right.$$

$$I_1 = \int \frac{\frac{du}{\cos^2 u}}{\sqrt{\frac{\cos^2 u + \sin^2 u}{\cos^2 u}}} = \int \frac{\frac{du}{\cos^2 u}}{\sqrt{\frac{1}{\cos^2 u}}} = \int \frac{\frac{du}{\cos^2 u}}{\frac{1}{\cos u}}$$

$$= \int \frac{1}{\cos u}\, du = \int \frac{\cos u\, du}{\cos^2 u} = \int \frac{\cos u\, du}{1-\sin^2 u}$$

$$= \left\{ \begin{array}{l} \sin u = w \\ \cos u\, du = dw \end{array} \right.$$

$$= \int \frac{dw}{1-w^2} = \int \frac{-1}{(w-1)(w+1)}\, dw \qquad \ldots (3)$$

$$\frac{-1}{(w-1)(w+1)} = \frac{A}{w-1} + \frac{B}{w+1} \quad \Big/ \cdot (w-1)(w+1)$$

$$-1 = A(w+1) + B(w-1)$$

$$= (A+B)\,w + (A-B) \implies$$

$$\left. \begin{array}{l} A+B = 0 \\ A-B = -1 \end{array} \right\} +$$

$$2A = -1 \implies A = -\frac{1}{2}, \quad B = \frac{1}{2} \overset{(3)}{\implies}$$

$$\int \frac{dw}{1-w^2} = \int \frac{-\frac{1}{2}dw}{w-1} + \int \frac{\frac{1}{2}dw}{w+1} = -\frac{1}{2}\ln|w-1| + \frac{1}{2}\ln|w+1| + C$$

$$I_1 = \frac{1}{2}\ln\left|\frac{w+1}{w-1}\right| + C_1$$

$$= \frac{1}{2}\cdot\ln\left|\frac{\sin u + 1}{\sin u - 1}\right| + C_1$$

$$= \frac{1}{2}\cdot\ln\left(\frac{1+\sin u}{1-\sin u}\right) + C_1$$

$$= \frac{1}{2}\cdot\ln\left(\frac{1+\sin(\operatorname{arctg}(t))}{1-\sin(\operatorname{arctg}(t))}\right) + C_1$$

$$I_2 = \int \frac{\operatorname{sgn}(x)\cdot dx}{x} = \operatorname{sgn}(x)\cdot\int\frac{dx}{x} = \operatorname{sgn}(x)\cdot\ln|x| + C_2$$

$$\overset{(2)}{\Longrightarrow} \frac{1}{2}\cdot\ln\left(\frac{1+\sin(\operatorname{arctg}(t))}{1-\sin(\operatorname{arctg}(t))}\right) = \operatorname{sgn}(x)\cdot\ln|x| + C_3$$

$$\overset{(1)}{\Longrightarrow} \boxed{\frac{1}{2}\cdot\ln\left(\frac{1+\sin(\operatorname{arctg}(\frac{y}{x}))}{1-\sin(\operatorname{arctg}(\frac{y}{x}))}\right) = \operatorname{sgn}(x)\cdot\ln|x| + C}$$

★ Iskoristiti identitet $\sin(\operatorname{arctg}(x)) = \frac{x}{\operatorname{sgn}(x)\cdot\sqrt{x^2+1}}$ i pokušati redukovati izraz.

★★ Nacrtati familiju integralnih krivih u DESMOS-u. Šta se može zaključiti za rješenje $x=0,\ y<0$?

⑦ $y' \sin x \cos x = y + \cos x$

1° Ako je $\sin x = 0$, tada je $x = k\pi$, $k \in \mathbb{Z}$,
pa je $\cos x = 1$, za $x = 2k\pi$, $k \in \mathbb{Z}$, odnosno
$\cos x = -1$, za $x = (2k+1)\pi$, $k \in \mathbb{Z}$,

tj.

$$y' \cancel{\cdot 0 \cdot (\pm 1)} = y + (\pm 1) \Rightarrow y \pm 1 = 0 \Rightarrow$$

$$y = -1, \quad \text{za} \quad x = 2k\pi, \ k \in \mathbb{Z}$$

$$y = 1, \quad \text{za} \quad x = (2k+1)\pi, \ k \in \mathbb{Z}$$

2° Ako je $\cos x = 0$, tada je $x = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$
pa je:

$$y' \cancel{\sin x \cdot 0} = y + \cancel{0} \Rightarrow y = 0$$

$$y = 0, \quad \text{za} \quad x = \frac{\pi}{2} + k\pi, \ k \in \mathbb{Z}$$

3° Ako je $\sin x \cdot \cos x \neq 0$, tada je

$$y' = \frac{1}{\sin x \cdot \cos x} y + \frac{\cancel{\cos x}}{\sin x \cdot \cancel{\cos x}}$$

$$\Leftrightarrow \quad y' - \frac{1}{\sin x \cdot \cos x} y = \frac{1}{\sin x}$$

LINEARNA D.J.

$$y = e^{-\int -\frac{1}{\sin x \cdot \cos x}dx} \cdot \left(C + \int \frac{1}{\sin x} \cdot e^{\int -\frac{1}{\sin x \cdot \cos x}dx} dx \right) \quad \dots (1)$$

Neka je $I_1 = \int \frac{-1}{\sin x \cdot \cos x} dx$.

$$I_1 = -\int \frac{dx}{\sin x \cdot \cos x} = -\int \frac{\frac{dx}{\cos^2 x}}{\frac{\sin x \cdot \cancel{\cos x}}{\cos^{\cancel{2}} x}}$$

$$= -\int \frac{\frac{dx}{\cos^2 x}}{tg x} = \begin{cases} t = tg x \\ dt = \frac{dx}{\cos^2 x} \end{cases}$$

$$= -\int \frac{dt}{t} = -\ln|t| + C_1 = -\ln|tg x| + C_1 \overset{(1)}{\Longrightarrow}$$

$$y = e^{\ln|tg x|} \cdot \left(C + \int \frac{1}{\sin x} \cdot e^{-\ln|tg x|} dx \right)$$

$$y = tg x \cdot \left(C + \int \frac{1}{\sin x} \cdot \frac{1}{tg x} dx \right) \quad \dots (2)$$

Neka je $I_2 = \int \frac{1}{\sin x} \cdot \frac{1}{tg x} dx$

$$I_2 = \int \frac{1}{\sin x} \cdot \frac{1}{\frac{\sin x}{\cos x}} dx = \int \frac{\cos x \, dx}{\sin^2 x}$$

$$= \begin{cases} u = \sin x \\ du = \cos x \, dx \end{cases}$$

$$= \int \frac{du}{u^2} = -\frac{1}{u} + C_3 = -\frac{1}{\sin x} + C_3 \overset{(2)}{\Longrightarrow}$$

$$y = \operatorname{tg} x \cdot \left(C - \frac{1}{\sin x} \right) \Rightarrow$$

$$\boxed{y = C \cdot \operatorname{tg} x - \frac{1}{\cos x}}$$

★ singularno rješenje?

(8)

$$y' = \frac{-x-y-2}{2x+2y-1}$$ UOPŠTENA HOMOGENA D.J.

Kako je $\det = \begin{vmatrix} -1 & -1 \\ 2 & 2 \end{vmatrix} = 0$, uzimamo smjenu

$$-x-y = t \Rightarrow -1-y' = t' \Rightarrow y' = -t'-1 \quad \dots(1) \Rightarrow$$

$$-t'-1 = \frac{t-2}{-2t-1} \Rightarrow -t' = \frac{t-2}{-2t-1} + 1$$

$$-t' = \frac{t-2+(-2t-1)}{-2t-1} \Rightarrow -t' = \frac{-t-3}{-2t-1}$$

$$-t' = \frac{-(t+3)}{-(2t+1)} \Rightarrow -t' = \frac{t+3}{2t+1}$$

$$-\frac{dt}{dx} = \frac{t+3}{2t+1} \Rightarrow \frac{dt}{\frac{t+3}{2t+1}} = -dx$$

$$\int \frac{2t+1}{t+3}\,dt = \int -dx$$

$$\Leftrightarrow \int \frac{2(t+3)-5}{t+3}\,dt = -x + C_1$$

$$\Leftrightarrow \int 2\,dt - \int \frac{5\,dt}{t+3} = -x + C_1$$

$$\Leftrightarrow 2t - 5\ln|t+3| = -x + C_1$$

$$\overset{(1)}{\Rightarrow} 2(-x-y) - 5\ln|-x-y+3| = -x + C_1$$

$$\Leftrightarrow -2x - 2y - 5\ln|-(x+y-3)| = -x + C_1$$

$$\Leftrightarrow -(x + 2y + 5\ln|x+y-3|) = C_1 \Rightarrow$$

$$\boxed{x + 2y + 5\ln|x+y-3| = C}$$

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(9) $y \cdot (y - xy') = \sqrt{x^4 + y^4}$

$$\Leftrightarrow \quad y^2 - xy \cdot y' = \sqrt{x^4 \cdot \left(1 + \left(\frac{y}{x}\right)^4\right)}$$

$$\Leftrightarrow \quad y^2 - xy \cdot y' = x^2 \cdot \sqrt{1 + \left(\frac{y}{x}\right)^4}$$

1° Ako je $x = 0$, imamo:

$$y^2 - \cancel{0 \cdot y \cdot y'} = \sqrt{0^4 + y^4} \Rightarrow y^2 = \sqrt{y^4} \Rightarrow y^2 = y^2 \ \checkmark$$

Dakle, za $x = 0$ početna jednačina vrijedi za svako y.

2° Ako je $x \neq 0$, imamo:

$$y^2 - xy \cdot y' = x^2 \sqrt{1 + \left(\frac{y}{x}\right)^4} \quad / : x^2$$

$$\left(\frac{y}{x}\right)^2 - \frac{\cancel{x} y y'}{x^{\cancel{2}}} = \sqrt{1 + \left(\frac{y}{x}\right)^4}$$

HOMOGENA D.J. $\Rightarrow$ smjena: $\frac{y}{x} = t \Rightarrow y = tx$

$y' = t'x + t$

$\dots (1)$

$$t^2 - t \cdot (t'x + t) = \sqrt{1 + t^4}$$

$$\Leftrightarrow \quad \cancel{t^2} - txt' - \cancel{t^2} = \sqrt{1 + t^4}$$

$$\Leftrightarrow \quad -tx \cdot \frac{dt}{dx} = \sqrt{1 + t^4}$$

$$\Leftrightarrow \quad \frac{t\,dt}{\sqrt{1 + t^4}} = -\frac{dx}{x}$$

$$\int \frac{t\,dt}{\sqrt{1+t^4}} = \int -\frac{dx}{x} \qquad \ldots (2)$$

$$\underset{I_1}{\uparrow} \qquad\qquad \underset{I_2}{\uparrow}$$

$$I_1 = \int \frac{t\,dt}{\sqrt{1+t^4}} = \begin{cases} u = t^2 \\ du = 2t\,dt \Rightarrow \quad t\,dt = \frac{du}{2} \end{cases}$$

$$= \int \frac{\frac{du}{2}}{\sqrt{1+u^2}} = \frac{1}{2} \cdot \int \frac{du}{\sqrt{1+u^2}}$$

* Identičan integral je rađen u 6 zadatku pa ćemo iskoristiti gotovo rješenje:

$$I_1 = \frac{1}{2} \cdot \frac{1}{2} \cdot \ln\left(\frac{1+\sin(\operatorname{arctg}(u))}{1-\sin(\operatorname{arctg}(u))} \right) + C_1$$

ili tablični integral.

$$\int \frac{du}{\sqrt{1+u^2}} = \ln\left|\sqrt{1+u^2} + u\right| + C_1$$

do kog se dolazi koristenjem identiteta

$$\sin(\operatorname{arctg}(x)) = \frac{x}{\operatorname{sgn}(x) \cdot \sqrt{x^2+1}}.$$

$$I_2 = \int -\frac{dx}{x} = -\ln|x| + C_2 \quad \overset{(2)}{\Longrightarrow}$$

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$$\frac{1}{4}\ln\left|\sqrt{1+u^2}+u\right| + C_1 = -\ln|x| + C_2$$

$$\Leftrightarrow \quad \ln\sqrt[4]{\sqrt{1+u^2}+u} + \ln|x| = C_3$$

$$\Rightarrow \quad \ln\sqrt[4]{\sqrt{1+t^4}+t^2} + \ln|x| = C_3$$

$$\overset{(1)}{\Longrightarrow} \quad \ln\sqrt[4]{\sqrt{1+\left(\frac{y}{x}\right)^4}+\left(\frac{y}{x}\right)^2} + \ln\sqrt[4]{x^4} = C_3$$

$$\Leftrightarrow \quad \ln\sqrt[4]{x^4\cdot\left(\sqrt{\frac{x^4+y^4}{x^4}}+\frac{y^2}{x^2}\right)} = C_3$$

$$\Leftrightarrow \quad \ln\left(x^4\cdot\left(\frac{\sqrt{x^4+y^4}+y^2}{x^2}\right)\right)^{\frac{1}{4}} = C_3$$

$$\Leftrightarrow \quad \frac{1}{4}\cdot\ln\left(x^2\cdot\left(\sqrt{x^4+y^4}+y^2\right)\right) = C_3 \quad /\cdot 4 \; ; \qquad C_4 = 4C_3$$

$$\boxed{\ln\left(x^2\left(\sqrt{x^4+y^4}+y^2\right)\right) = C_4} \quad \Rightarrow$$

$$x^2\left(\sqrt{x^4+y^4}+y^2\right) = e^{C_4} = C \quad \Rightarrow$$

$$\boxed{x^2\left(\sqrt{x^4+y^4}+y^2\right) = C}$$

Rješenje $x=0$ nije singularno rješenje diferencijalne jednačine, jer se dobija iz opšteg rješenja uvrštavanjem $C=0$.

(10) $$y' = 2\sqrt{y-x} + 1$$

Data diferencijalna jednačina ne pripada grupi diferencijalnih jednačina sa razdvojenim promjenljivim, niti je homogena d.j. niti linearna d.j. što znači da se rješava smjenom.

$$y' - 1 = 2\sqrt{y-x}$$

smjena: $y - x = t \Rightarrow \quad y' - 1 = t' \Rightarrow$

$$t' = 2\sqrt{t} \Rightarrow$$

$$\frac{dt}{dx} = 2\sqrt{t} \Rightarrow$$

$$\frac{dt}{2\sqrt{t}} = dx \Rightarrow$$

$$\int \frac{dt}{2\sqrt{t}} = \int dx \Rightarrow$$

$$\sqrt{t} = x + C \Rightarrow$$

$$\boxed{\sqrt{y-x} = x + C}$$

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