TERMIN 12 - zadaci za samostalan rad - rješenja

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Zadatak 1.
Izračunati

a) $\displaystyle\int \frac{dx}{\sqrt{4x-5x^2}}$, b) $\displaystyle\int \frac{dx}{1+\sqrt{x}}$, c) $\displaystyle\int \frac{dx}{(1+x)\sqrt{1-x}}$.

Rješenje

Vrijedi

a)

$$\begin{aligned}
I &= \int \frac{dx}{\sqrt{4x-5x^2}}\\
&= \int \frac{dx}{\sqrt{-\left(5x^2-4x\right)}}\\
&= \int \frac{dx}{\sqrt{-\left(\left(\sqrt{5}\cdot x\right)^2 - 2\cdot\left(\sqrt{5}\cdot x\right)\cdot\left(\frac{2}{\sqrt{5}}\right) + \left(\frac{2}{\sqrt{5}}\right)^2 - \left(\frac{2}{\sqrt{5}}\right)^2\right)}}\\
&= \int \frac{dx}{\left(\frac{2}{\sqrt{5}}\right)^2 - \left(\sqrt{5}\cdot x - \frac{2}{\sqrt{5}}\right)^2} \qquad \begin{cases} t = \sqrt{5}\cdot x - \frac{2}{\sqrt{5}}\\ dt = \sqrt{5}\,dx \ \Rightarrow \ dx = \frac{dt}{\sqrt{5}}\end{cases}\\
&= \int \frac{\frac{dt}{\sqrt{5}}}{\left(\frac{2}{\sqrt{5}}\right)^2 - t^2}\\
&= \frac{1}{\sqrt{5}}\cdot \arcsin\left(\frac{t}{\frac{2}{\sqrt{5}}}\right) + C\\
&= \frac{1}{\sqrt{5}}\cdot \arcsin\left(\frac{\sqrt{5}\cdot x - \frac{2}{\sqrt{5}}}{\frac{2}{\sqrt{5}}}\right) + C\\
&= \frac{1}{\sqrt{5}}\cdot \arcsin\left(\frac{\frac{5x-2}{\cancel{\sqrt{5}}}}{\frac{2}{\cancel{\sqrt{5}}}}\right) + C\\
&= \frac{1}{\sqrt{5}}\cdot \arcsin\left(\frac{5x-2}{2}\right) + C
\end{aligned}$$

b)

$$\begin{aligned}
I &= \int \frac{dx}{1+\sqrt{x}} \qquad \begin{cases} t = 1+\sqrt{x} \ \Rightarrow \ \sqrt{x} = t-1 \ \Rightarrow \ x = (t-1)^2\\ dx = 2\cdot(t-1)\ dt\end{cases}\\
&= \int \frac{2\cdot(t-1)\ dt}{t}\\
&= 2\cdot\left(\int dt - \int \frac{dt}{t}\right)\\
&= 2\cdot\left(t - \ln|t|\right) + C\\
&= 2\cdot\left(1+\sqrt{x} - \ln\left(1+\sqrt{x}\right)\right) + C
\end{aligned}$$

c)

$$\begin{aligned}
I &= \int \frac{dx}{(1+x)\sqrt{1-x}} \qquad \begin{cases} t = \sqrt{1-x} \ \Rightarrow \ t^2 = 1-x \ \Rightarrow \ x = 1-t^2\\ dx = -2t\,dt\end{cases}\\
&= \int \frac{-2\cancel{t}\,dt}{(1+1-t^2)\cdot\cancel{t}}\\
&= 2\cdot\int \frac{dt}{t^2-2}
\end{aligned} \tag{1}$$

Korištenjem metode neodređenih koeficijenata imamo da je

$$\begin{aligned}
&\frac{1}{t^2-2} = \frac{A}{t-\sqrt{2}} + \frac{B}{t+\sqrt{2}}\\
\Leftrightarrow \quad & 1 = A\left(t+\sqrt{2}\right) + B\left(t-\sqrt{2}\right)\\
\Leftrightarrow \quad & 1 = (A+B)\,t + \sqrt{2}\,(A-B)
\end{aligned}$$

odakle dobijamo sistem

$$\begin{cases} A + B = 0 \\ A - B = \frac{1}{\sqrt{2}} \end{cases}$$

čije je rješenje

$$A = \frac{1}{2\sqrt{2}}, \;\; B = -\frac{1}{2\sqrt{2}}.$$

Sada je integral (1) jednak:

$$\begin{aligned} I &= 2 \cdot \left(\int \frac{\frac{1}{2\sqrt{2}}}{t - \sqrt{2}} \, dt + \int - \frac{\frac{1}{2\sqrt{2}}}{t + \sqrt{2}} \, dt \right) \\ &= 2 \cdot \frac{1}{2\sqrt{2}} \cdot \left(\ln \left| t - \sqrt{2} \right| - \ln \left| t + \sqrt{2} \right| \right) + C \\ &= \frac{1}{\sqrt{2}} \cdot \left(\ln \left| \sqrt{1 - x} - \sqrt{2} \right| - \ln \left| \sqrt{1 - x} + \sqrt{2} \right| \right) + C. \end{aligned}$$

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Zadatak 2.

Izračunati

a) $\int \frac{\cos^5 x}{\sqrt[3]{\sin^8 x}}\,dx$, b) $\int \frac{dx}{\sin^4 x}$, c) $\int \operatorname{ctg}^3 x\,dx$

Rješenje

Vrijedi

a)

$$\begin{aligned}
I &= \int \frac{\cos^5 x}{\sqrt[3]{\sin^8 x}}\,dx \\
&= \int \frac{\left(\cos^2 x\right)^2}{(\sin x)^{\frac{8}{3}}}\,\cos x\,dx \\
&= \int \frac{\left(1-\sin^2 x\right)^2}{(\sin x)^{\frac{8}{3}}}\,\cos x\,dx \qquad \begin{cases} t=\sin x \\ dt=\cos x\,dx \end{cases} \\
&= \int \frac{\left(1-t^2\right)^2}{t^{\frac{8}{3}}}\,dt \\
&= \int \frac{1-2t^2+t^4}{t^{\frac{8}{3}}}\,dt \\
&= \int t^{-\frac{8}{3}}\,dt - 2\cdot\int t^{-\frac{2}{3}}\,dt + \int t^{\frac{4}{3}}\,dt \\
&= \frac{t^{-\frac{5}{3}}}{-\frac{5}{3}} - 2\cdot\frac{t^{\frac{1}{3}}}{\frac{1}{3}} + \frac{t^{\frac{7}{3}}}{\frac{7}{3}} + C \\
&= -\frac{3}{5}\cdot\sqrt[3]{\frac{1}{\sin^5 x}} - 6\cdot\sqrt[3]{\sin x} + \frac{3}{7}\cdot\sqrt[3]{\sin^7 x} + C
\end{aligned}$$

b)

$$\begin{aligned}
I &= \int \frac{dx}{\sin^4 x} \qquad \begin{cases} t=\operatorname{tg}\frac{x}{2} \\ \sin x=\frac{2t}{1+t^2} \\ dx=\frac{2\,dt}{1+t^2} \end{cases} \\
&= \int \frac{\frac{2\,dt}{1+t^2}}{\left(\frac{2t}{1+t^2}\right)^4} \\
&= \frac{1}{8}\cdot\int \frac{\left(1+t^2\right)^3}{t^4}\,dt \\
&= \frac{1}{8}\cdot\int \frac{1+3t^2+3t^4+t^6}{t^4}\,dt \\
&= \frac{1}{8}\cdot\left(\int t^{-4}\,dt + 3\cdot\int t^{-2}\,dt + 3\cdot\int dt + \int t^2\,dt\right) \\
&= \frac{1}{8}\cdot\left(\frac{t^{-3}}{-3} + 3\cdot\frac{t^{-1}}{-1} + 3t + \frac{t^3}{3}\right) + C \\
&= \frac{-\frac{1}{\operatorname{tg}^3\frac{x}{2}} - \frac{9}{\operatorname{tg}\frac{x}{2}} + 9\operatorname{tg}\frac{x}{2} + \operatorname{tg}^3\frac{x}{2}}{24} + C
\end{aligned}$$

c)

$$\begin{aligned}
I &= \int \operatorname{ctg}^3 x\,dx \\
&= \int \frac{dx}{\operatorname{tg}^3 x} \qquad \begin{cases} t=\operatorname{tg} x \\ dx=\frac{dt}{1+t^2} \end{cases} \\
&= \int \frac{dt}{t^3\cdot(1+t^2)}
\end{aligned} \tag{2}$$

Korištenjem metode neodređenih koeficijenata imamo da je

$$\begin{aligned}
&\frac{1}{t^3\cdot(1+t^2)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t^3} + \frac{Dt+E}{1+t^2} \\
\Leftrightarrow\quad & 1 = At^2\cdot\left(1+t^2\right) + Bt\cdot\left(1+t^2\right) + C\cdot\left(1+t^2\right) + (Dt+E)\,t^3 \\
\Leftrightarrow\quad & 1 = At^2 + At^4 + Bt + Bt^3 + C + Ct^2 + Dt^4 + Et^3 \\
\Leftrightarrow\quad & 1 = (A+D)\,t^4 + (B+E)\,t^3 + (A+C)\,t^2 + Bt + C
\end{aligned}$$

odakle dobijamo sistem

$$\begin{cases} A + D = 0 \\ B + E = 0 \\ A + C = 0 \\ B = 0 \\ C = 1 \end{cases}$$

čije je rješenje

$$A = -1, \;\; B = 0, \;\; C = 1, \;\; D = 1, \;\; E = 0.$$

Sada je integral (2) jednak:

$$\begin{aligned} I &= \int \frac{-1}{t}\,dt + \int \frac{1}{t^3}\,dt + \int \frac{t\,dt}{1+t^2} \\ &= -\ln|t| + \int t^{-3}\,dt + \frac{1}{2} \cdot \int \frac{2t\,dt}{1+t^2} \\ &= -\ln|t| + \frac{t^{-2}}{-2} + \frac{1}{2} \cdot \ln\left(1+t^2\right) + C \\ &= -\ln|\operatorname{tg} x| - \frac{1}{2\operatorname{tg}^2 x} + \frac{1}{2} \cdot \ln\left(1+\operatorname{tg}^2 x\right) + C. \end{aligned}$$

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Zadatak 3.

a) Izračunati površinu ograničenu lukom kosinusoide od $x = \frac{\pi}{6}$ do $x = \frac{5\pi}{6}$ i x-osom.

b) Izračunati površinu kruga poluprečnika r.

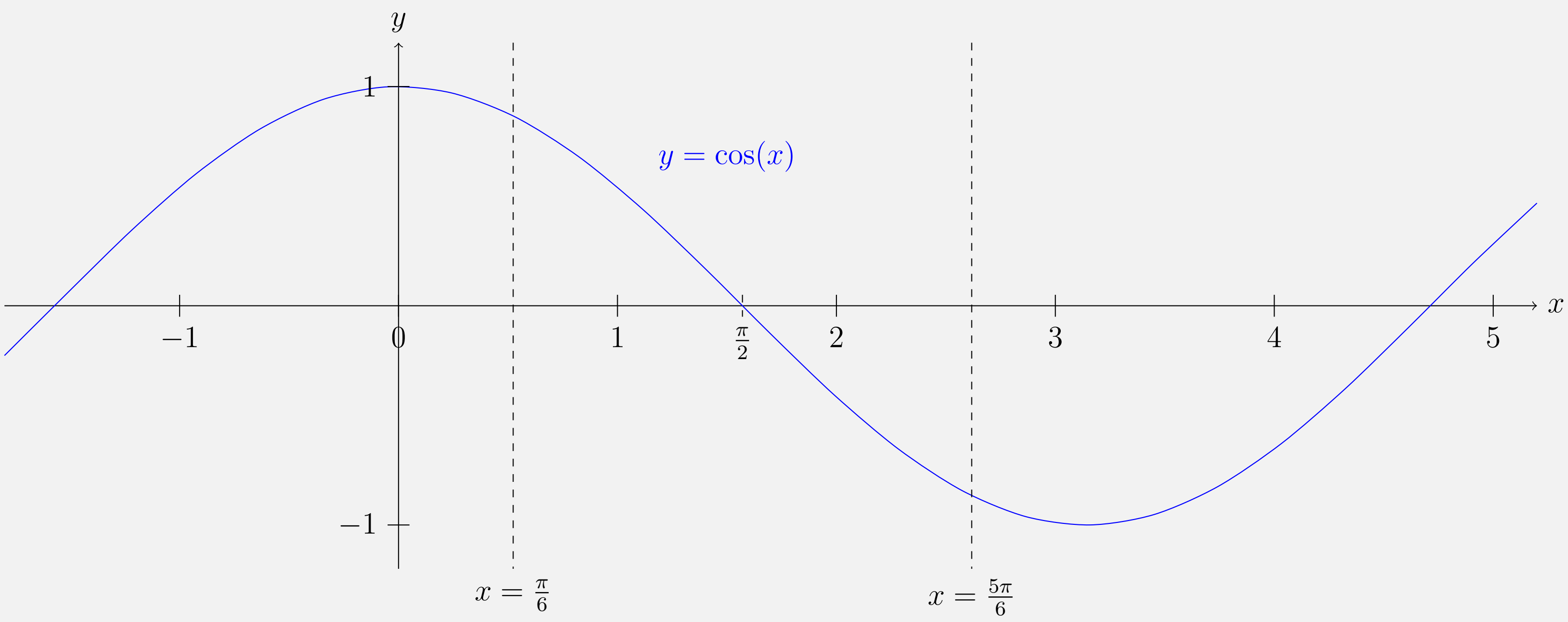
c) Odrediti površinu S ograničenu kubnom parabolom $y = x^3$ i pravom $y = 2x$.

Rješenje

Vrijedi

a) Ako površinu posmatramo kao fizičku veličinu, onda je ona strogo pozitivna te datu površinu posmatramo kao površinu između krive $y = \cos x$ i $y = 0$. Na dijelu između $x = \frac{\pi}{6}$ i $x = \frac{\pi}{2}$ je funkcija $y = \cos x$ veća, dok je između $x = \frac{\pi}{2}$ i $x = \frac{5\pi}{6}$ manja. Stoga, tražena površina je jednaka

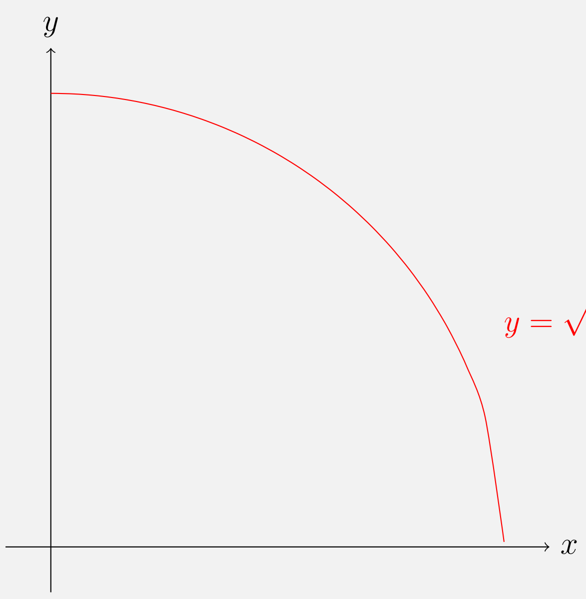
$$\begin{aligned} P &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\cos x - 0)\ dx + \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} (0 - \cos x)\ dx \\ &= \sin x\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} - \sin x\Big|_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \\ &= \left(\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{6}\right)\right) - \left(\sin\left(\frac{5\pi}{6}\right) - \sin\left(\frac{\pi}{2}\right)\right) \\ &= \left(1 - \frac{1}{2}\right) - \left(\frac{1}{2} - 1\right) \\ &= 1. \end{aligned}$$



b) Kako je jednačina kružnice

$$x^2 + y^2 = r^2$$

površinu kruga poluprečnika r možemo dobiti kao četvorostruku površinu ispod funkcije $f(x) = \sqrt{r^2 - x^2}$, ograničenu sa x-osom, između $x = 0$ i $x = r$.



Dakle,

$$P = 4 \cdot \int_0^r \sqrt{r^2 - x^2}\, dx \tag{3}$$

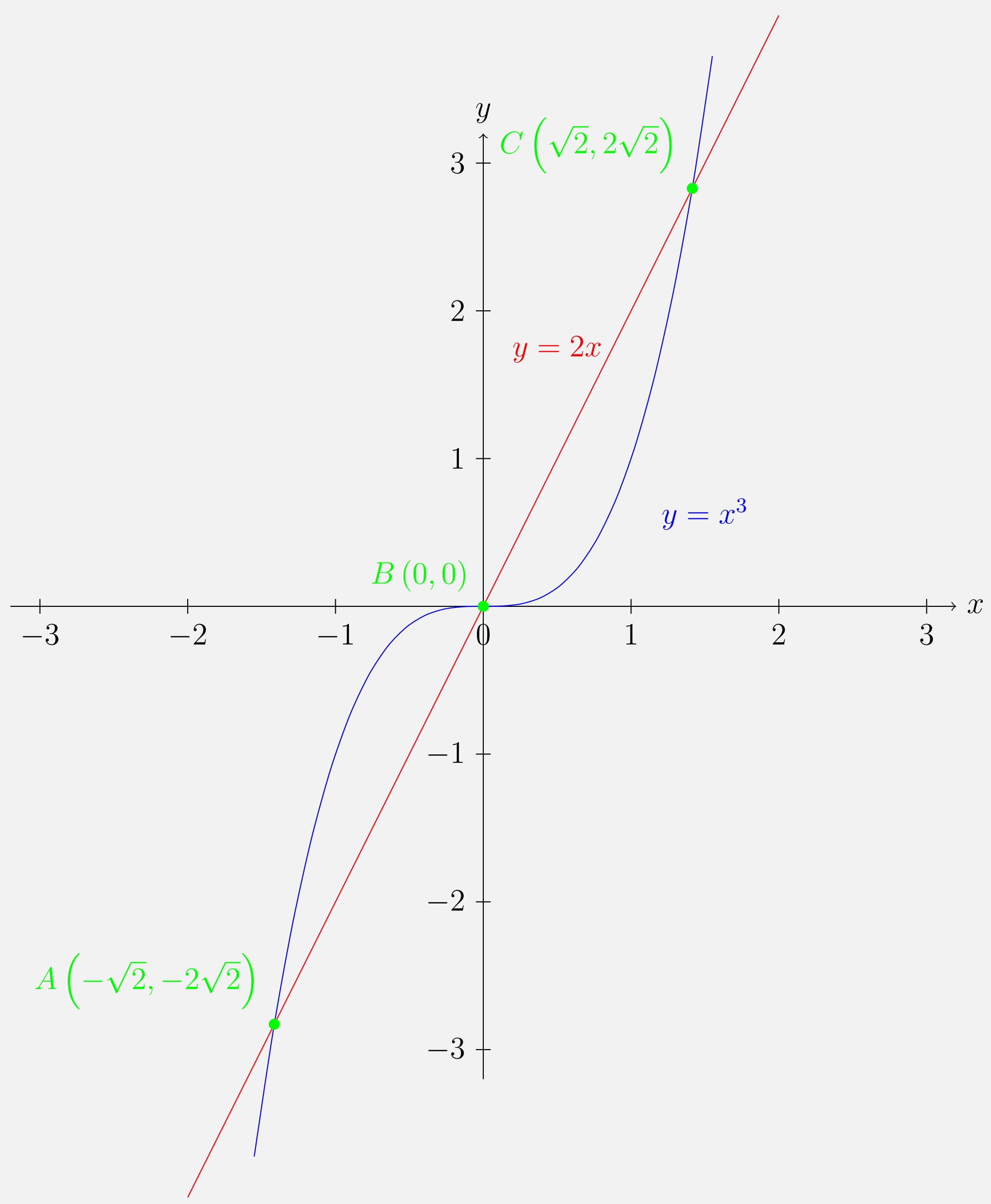
Odredimo prvo neodređeni integral:

$$
\begin{aligned}
I_n &= \int \sqrt{r^2 - x^2}\,dx \qquad \begin{cases} x = r\sin t \;\Rightarrow\; \sin t = \frac{x}{r} \;\Rightarrow\; t = \arcsin\left(\frac{x}{r}\right) \\ dx = r\cos t\,dt \end{cases} \\
&= \int \sqrt{r^2 - (r\sin t)^2} \cdot r\cos t\,dt \\
&= \int \sqrt{r^2 \cdot (1 - \sin^2 t)} \cdot r\cos t\,dt \\
&= \int |r| \cdot |\cos t| \cdot r\cos t\,dt \\
&= r^2 \cdot \int \cos^2 t\,dt \\
&= r^2 \cdot \int \frac{1 + \cos 2t}{2}\,dt \\
&= r^2 \cdot \left(\int \frac{dt}{2} + \frac{1}{2} \cdot \int \cos 2t\,dt\right) \\
&= r^2 \cdot \left(\frac{t}{2} + \frac{\sin 2t}{4}\right) + C \\
&= r^2 \cdot \left(\frac{\arcsin\left(\frac{x}{r}\right)}{2} + \frac{\sin\left(2 \cdot \arcsin\left(\frac{x}{r}\right)\right)}{4}\right) + C
\end{aligned}
$$

Vraćanjem u izraz (3) dobijamo da je površina kruga poluprečnika r jednaka

$$
\begin{aligned}
P &= 4 \cdot r^2 \cdot \left(\frac{\arcsin\left(\frac{x}{r}\right)}{2} + \frac{\sin\left(2 \cdot \arcsin\left(\frac{x}{r}\right)\right)}{4}\right)\Bigg|_0^r \\
&= 4r^2 \cdot \left(\left(\frac{\arcsin\left(\frac{r}{r}\right)}{2} + \frac{\sin\left(2 \cdot \arcsin\left(\frac{r}{r}\right)\right)}{4}\right) - \left(\frac{\arcsin\left(\frac{0}{r}\right)}{2} + \frac{\sin\left(2 \cdot \arcsin\left(\frac{0}{r}\right)\right)}{4}\right)\right) \\
&= 4r^2 \cdot \left(\left(\frac{\frac{\pi}{2}}{2} + \frac{\sin\left(2 \cdot \frac{\pi}{2}\right)}{4}\right) - \left(\cancel{\frac{0}{2}} + \cancel{\frac{\sin(2 \cdot 0)}{4}}\right)\right) \\
&= 4r^2 \cdot \left(\frac{\pi}{4} + \cancel{\frac{\sin \pi}{4}}\right) \\
&= 4r^2 \cdot \frac{\pi}{4} \\
&= r^2\pi.
\end{aligned}
$$

c) Skicirajmo funkcije $f(x) = x^3$ i $g(x) = 2x$.



Odredimo prvo presječne tačke funkcija $f(x) = x^3$ i $g(x) = 2x$:

$$\begin{aligned}
& x^3 = 2x \\
\Leftrightarrow \quad & x^3 - 2x = 0 \\
\Leftrightarrow \quad & x \cdot \left(x - \sqrt{2}\right) \cdot \left(x + \sqrt{2}\right) = 0 \\
\Leftrightarrow \quad & x_1 = -\sqrt{2},\;\; x_2 = 0,\;\; x_3 = \sqrt{2}.
\end{aligned}$$

Odavde dobijamo da su presječne tačke: $A\left(-\sqrt{2}, -2\sqrt{2}\right)$, $B\,(0,0)$ i $C\left(\sqrt{2}, 2\sqrt{2}\right)$.
Sada je površina između ovih funkcija jednaka:

$$\begin{aligned}
S &= \int_{-\sqrt{2}}^{0} \left(x^3 - 2x\right)\,dx + \int_{0}^{\sqrt{2}} \left(2x - x^3\right)\,dx \\
&= \left(\frac{x^4}{4} - x^2\right)\Bigg|_{-\sqrt{2}}^{0} + \left(x^2 - \frac{x^4}{4}\right)\Bigg|_{0}^{\sqrt{2}} \\
&= \cancel{\left(\frac{0^4}{4} - 0^2\right)} - \left(\frac{\left(-\sqrt{2}\right)^4}{4} - \left(-\sqrt{2}\right)^2\right) + \left(\left(\sqrt{2}\right)^2 - \frac{\left(\sqrt{2}\right)^4}{4}\right) - \cancel{\left(0^2 - \frac{0^4}{4}\right)} \\
&= -\left(\frac{4}{4} - 2\right) + \left(2 - \frac{4}{4}\right) \\
&= 2.
\end{aligned}$$

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Zadatak 4.
Ispitati konvergenciju integrala:

a) $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$, b) $\displaystyle\int_{1}^{+\infty} \sin x\, dx$, c) $\displaystyle\int_{1}^{+\infty} \frac{dx}{x\sqrt[3]{1+x^2}}$.

Rješenje

a) Neka je
$$f(x) = \frac{1}{1+x^2}.$$
Kako je
$$f(-x) = \frac{1}{1+(-x)^2} = f(x)$$
funkcija f je parna i vrijedi
$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = 2\cdot\int_{0}^{+\infty} \frac{dx}{1+x^2} = 2\cdot\left(\int_{0}^{1} \frac{dx}{1+x^2} + \int_{1}^{+\infty} \frac{dx}{1+x^2}\right) = 2\cdot\left(\operatorname{arctg} x\Big|_0^1 + \int_{1}^{+\infty} \frac{dx}{1+x^2}\right) = 2\cdot\left(\frac{\pi}{4} + \int_{1}^{+\infty} \frac{dx}{1+x^2}\right) \tag{4}$$
Sa druge strane, kako integral
$$\int_{1}^{+\infty} \frac{1}{x^\alpha}\, dx$$
konvergira za $\alpha > 1$, a divergira za $\alpha \leq 1$, imamo da integral
$$\int_{1}^{+\infty} \frac{dx}{x^2}$$
konvergira. Kako je
$$\begin{aligned} & 1+x^2 > x^2 \\ \Leftrightarrow \quad & \frac{1}{1+x^2} < \frac{1}{x^2} \\ \Leftrightarrow \quad & \int_{1}^{+\infty} \frac{dx}{1+x^2} < \int_{1}^{+\infty} \frac{dx}{x^2} \end{aligned}$$
integral
$$\int_{1}^{+\infty} \frac{dx}{1+x^2}$$
također konvergira, pa uvrštavanjem u izraz (4) zaključujemo da početni integral
$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$$
konvergira.

b) Kako
$$\lim_{b\to+\infty} \cos b$$
ne postoji, i kako je
$$\begin{aligned} I &= \int_{1}^{+\infty} \sin x\, dx \\ &= \lim_{b\to+\infty} \left(-\cos(x)\Big|_1^b\right) \\ &= \cos(1) - \lim_{b\to+\infty} \cos(b), \end{aligned}$$
početni integral divergira.

c) Slično kao i u dijelu zadatka pod a) vrijedi niz implikacija:
$$1+x^2 > x^2 \quad\Rightarrow\quad \sqrt[3]{1+x^2} > \sqrt[3]{x^2} \quad\Rightarrow\quad x\sqrt[3]{1+x^2} > x\sqrt[3]{x^2} \quad\Rightarrow\quad \frac{1}{x\sqrt[3]{1+x^2}} < \frac{1}{x\sqrt[3]{x^2}}.$$
Kako integral
$$\int_{1}^{+\infty} \frac{dx}{x\sqrt[3]{x^2}} = \int_{1}^{+\infty} \frac{dx}{x^{\frac{5}{3}}}$$
konvergira, konvergira i početni integral.

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Zadatak 5.
Izračunati

a) $\int_{-1}^{1} \frac{dx}{\sqrt[3]{x^2}}$, b) $\int_{0}^{\frac{\pi}{2}} \frac{2\sin x}{\cos^2 x}\, dx$, c) $\int_{-\infty}^{0} xe^x\, dx$

Rješenje

a) Podintegralna funkcija ima prekid u tački $x = 0$ te dati integral posmatramo kao nesvojstveni integral druge vrste:

$$\begin{aligned}
I &= \lim_{\epsilon\to 0^+}\left(\int_{-1}^{0-\epsilon} \frac{dx}{\sqrt[3]{x^2}}\right) + \lim_{\mu\to 0^+}\left(\int_{0+\mu}^{1} \frac{dx}{\sqrt[3]{x^2}}\right)\\
&= \lim_{\epsilon\to 0^+}\left(\int_{-1}^{0-\epsilon} x^{-\frac{2}{3}}\, dx\right) + \lim_{\mu\to 0^+}\left(\int_{0+\mu}^{1} x^{-\frac{2}{3}}\, dx\right)\\
&= \lim_{\epsilon\to 0^+}\left(\frac{x^{\frac{1}{3}}}{\frac{1}{3}}\right)\Bigg|_{-1}^{-\epsilon} + \lim_{\mu\to 0^+}\left(\frac{x^{\frac{1}{3}}}{\frac{1}{3}}\right)\Bigg|_{\mu}^{1}\\
&= 3\cdot\left(\lim_{\epsilon\to 0^+}\sqrt[3]{\epsilon} - \sqrt[3]{-1}\right) + 3\cdot\left(\sqrt[3]{1} - \lim_{\mu\to 0^+}\sqrt[3]{\mu}\right)\\
&= 3\cdot\big(0-(-1)\big) + 3\cdot(1-0)\\
&= 6.
\end{aligned}$$

b) Podintegralna funkcija ima prekid u tački $x = \frac{\pi}{2}$ te dati integral posmatramo kao nesvojstveni integral druge vrste:

$$I = \lim_{\epsilon\to 0^+}\left(\int_{0}^{\frac{\pi}{2}-\epsilon} \frac{2\sin x}{\cos^2 x}\, dx\right) \tag{5}$$

Riješimo sada neodređeni integral

$$\begin{aligned}
I_n &= \int \frac{2\sin x}{\cos^2 x}\, dx \qquad \begin{cases} t = \cos x \\ dt = -\sin x\, dx \end{cases}\\
&= \int \frac{-2\, dt}{t^2}\\
&= \frac{2}{t} + C\\
&= \frac{2}{\cos x} + C.
\end{aligned}$$

Uvrštavanjem u izraz (5) dobijamo

$$\begin{aligned}
I &= \lim_{\epsilon\to 0^+}\left(\frac{2}{\cos x}\right)\Bigg|_{0}^{\frac{\pi}{2}-\epsilon}\\
&= \lim_{\epsilon\to 0^+}\left(\frac{2}{\cos\left(\frac{\pi}{2}-\epsilon\right)}\right)^{+\infty} - \frac{2}{\cos(0)}\\
&= +\infty.
\end{aligned}$$

Dakle, integral I divergira.

c) Integral I je nesvojstveni integral prve vrste. Odredimo prvo neodređeni integral:

$$\begin{aligned}
I_n &= \int xe^x\, dx \qquad \begin{cases} u = x \\ du = dx \\ v = e^x \\ dv = e^x\, dx \end{cases}\\
&= xe^x - \int e^x\, dx\\
&= e^x(x-1) + C.
\end{aligned}$$

Sada je početni integral jednak

$$\begin{aligned}
I &= \big(e^x(x-1)\big)\Big|_{-\infty}^{0}\\
&= e^0\cdot(0-1) - \lim_{b\to-\infty}\big(e^b\cdot(b-1)\big)\\
&= -1 - \lim_{b\to-\infty}\frac{b-1}{e^{-b}}\\
&= -1 - \lim_{b\to-\infty}\frac{1}{-e^{-b}}^{\,0}\\
&= -1.
\end{aligned}$$

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Zadatak 6.
Ispitati konvergenciju i izračunati integral
$$\int_0^{+\infty} \frac{dx}{1+x^3}.$$

Rješenje

Kako je
$$\lim_{x\to+\infty} \frac{\frac{1}{1+x^3}}{\frac{1}{x^3}} = 1,$$
integrali $\int_0^{+\infty} \frac{dx}{1+x^3}$ i $\int_0^{+\infty} \frac{dx}{x^3}$ su ekvikonvergentni. Pošto integral $\int_0^{+\infty} \frac{dx}{x^3}$ konvergira, i početni integral konvergira.
Posmatrajmo prvo neodređeni integral
$$\begin{aligned} I_n &= \int \frac{dx}{1+x^3} \\ &= \int \frac{1}{(x+1)\left(x^2-x+1\right)}\,dx \end{aligned} \tag{6}$$

Koristeći metod neodređenih koeficijenata imamo:
$$\begin{aligned} & \frac{1}{(x+1)\left(x^2-x+1\right)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} \\ \Leftrightarrow \quad & 1 = A\left(x^2-x+1\right) + (Bx+C)(x+1) \\ \Leftrightarrow \quad & 1 = Ax^2 - Ax + A + Bx^2 + Cx + Bx + C \\ \Leftrightarrow \quad & 1 = (A+B)\,x^2 + (-A+B+C)\,x + (A+C) \end{aligned}$$

odakle dobijamo sistem
$$\begin{cases} A+B=0 \\ -A+B+C=0 \\ A+C=1 \end{cases}$$

čije je rješenje
$$A=\frac{1}{3},\ \ B=-\frac{1}{3},\ \ C=\frac{2}{3}$$

pa uvrštavanjem u izraz (6) dobijamo
$$\begin{aligned} I_n &= \int \frac{\frac{1}{3}}{x+1}\,dx + \int \frac{-\frac{1}{3}x+\frac{2}{3}}{x^2-x+1}\,dx \\ &= \frac{1}{3}\cdot \ln|x+1| - \frac{1}{3}\cdot \int \frac{x-2}{x^2-x+1}\,dx \\ &= \frac{1}{3}\cdot \ln|x+1| - \frac{1}{3}\cdot \int \frac{\frac{1}{2}\cdot(2x-1)-\frac{3}{2}}{x^2-x+1}\,dx \\ &= \frac{1}{3}\cdot \ln|x+1| - \frac{1}{6}\cdot \int \frac{(2x-1)\,dx}{x^2-x+1} + \frac{1}{3}\cdot\frac{3}{2}\cdot \int \frac{dx}{x^2-x+1} \\ &= \frac{1}{3}\cdot \ln|x+1| - \frac{1}{6}\cdot \ln\left|x^2-x+1\right| + \frac{1}{2}\cdot \int \frac{dx}{\left(x^2-2\cdot x\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2\right)-\left(\frac{1}{2}\right)^2+1} \\ &= \frac{1}{6}\cdot 2\cdot \ln|x+1| - \frac{1}{6}\cdot \ln\left|x^2-x+1\right| + \frac{1}{2}\cdot \int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}} \\ &= \frac{1}{6}\cdot \ln|x+1|^2 - \frac{1}{6}\cdot \ln\left|x^2-x+1\right| + \frac{1}{2}\cdot \int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} \qquad \begin{cases} t = x-\frac{1}{2} \\ dt = dx \end{cases} \\ &= \frac{1}{6}\cdot \ln\frac{|x+1|^2}{|x^2-x+1|} + \frac{1}{2}\cdot \int \frac{dt}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2} \\ &= \frac{1}{6}\cdot \ln\left|\frac{x^2+2x+1}{x^2-x+1}\right| + \frac{1}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\cdot \operatorname{arctg}\left(\frac{t}{\frac{\sqrt{3}}{2}}\right) + C \\ &= \frac{1}{6}\cdot \ln\left|\frac{x^2+2x+1}{x^2-x+1}\right| + \frac{1}{\sqrt{3}}\cdot \operatorname{arctg}\left(\frac{2t}{\sqrt{3}}\right) + C \\ &= \frac{1}{6}\cdot \ln\left|\frac{x^2+2x+1}{x^2-x+1}\right| + \frac{1}{\sqrt{3}}\cdot \operatorname{arctg}\left(\frac{2x-1}{\sqrt{3}}\right) + C. \end{aligned}$$

Vraćanjem u početni nesvojstveni integral dobijamo

$$
\begin{aligned}
I &= \lim_{b\to+\infty}\left(\frac{1}{6}\cdot\ln\left|\frac{x^2+2x+1}{x^2-x+1}\right| + \frac{1}{\sqrt{3}}\cdot\operatorname{arctg}\left(\frac{2x-1}{\sqrt{3}}\right)\right)\Bigg|_0^b \\
&= \lim_{b\to+\infty}\left(\frac{1}{6}\cdot\ln\left|\frac{b^2+2b+1}{b^2-b+1}\right| + \frac{1}{\sqrt{3}}\cdot\operatorname{arctg}\left(\frac{2b-1}{\sqrt{3}}\right)\right) - \left(\frac{1}{6}\cdot\ln\left|\frac{0^2+2\cdot 0+1}{0^2-0+1}\right| + \frac{1}{\sqrt{3}}\cdot\operatorname{arctg}\left(\frac{2\cdot 0-1}{\sqrt{3}}\right)\right) \\
&= \frac{1}{6}\cdot\lim_{b\to+\infty}\left(\ln\left|\frac{\cancel{b^2}\cdot\left(1+\overset{0}{\frac{2}{b}}+\overset{0}{\frac{1}{b^2}}\right)}{\cancel{b^2}\cdot\left(1-\overset{0}{\frac{1}{b}}+\overset{0}{\frac{1}{b^2}}\right)}\right|\right) + \frac{1}{\sqrt{3}}\cdot\lim_{b\to+\infty}\left(\operatorname{arctg}\left(\overset{+\infty}{\frac{2b-1}{\sqrt{3}}}\right)\right) - \left(\frac{1}{6}\cdot\ln|1| + \frac{1}{\sqrt{3}}\cdot\operatorname{arctg}\left(-\frac{1}{\sqrt{3}}\right)\right) \\
&= \cancel{\frac{1}{6}\cdot\ln(1)} + \frac{1}{\sqrt{3}}\cdot\frac{\pi}{2} - \cancel{\frac{1}{6}\cdot\ln(1)} - \frac{1}{\sqrt{3}}\cdot\left(-\frac{\pi}{6}\right) \\
&= \frac{\pi}{2\sqrt{3}} + \frac{\pi}{6\sqrt{3}} \\
&= \frac{3\pi+\pi}{6\sqrt{3}} \\
&= \frac{2\pi}{3\sqrt{3}}.
\end{aligned}
$$

Zadatak 7.
Izračunati
$$\int \frac{dx}{\sqrt[3]{\sin^5 x \cos x}}.$$

Rješenje

Vrijedi
$$\begin{aligned}
I &= \int \frac{dx}{\sqrt[3]{\sin^5 x \cos x}} \\
&= \int \frac{dx}{\sin^{\frac{5}{3}}(x) \cdot \cos^{\frac{1}{3}}(x) \cdot \frac{\cos^{\frac{5}{3}}(x)}{\cos^{\frac{5}{3}}(x)}} \\
&= \int \frac{dx}{\operatorname{tg}^{\frac{5}{3}}(x) \cdot \cos^2 x} \qquad \begin{cases} t = \operatorname{tg} x \\ dt = \frac{dx}{\cos^2 x} \end{cases} \\
&= \int \frac{dt}{t^{\frac{5}{3}}} \\
&= \int t^{-\frac{5}{3}}\, dt \\
&= \frac{t^{-\frac{2}{3}}}{-\frac{2}{3}} + C \\
&= -\frac{3}{2} \cdot \frac{1}{\sqrt[3]{t^2}} + C \\
&= \frac{-3}{2\sqrt[3]{\operatorname{tg}^2 x}} + C.
\end{aligned}$$

Zadatak 8.
Izračunati
$$\int \frac{dx}{\sqrt{\operatorname{tg} x}}.$$

Rješenje

Vrijedi

$$\begin{aligned}
I &= \int \frac{dx}{\sqrt{\operatorname{tg} x}} \qquad \begin{cases} t = \operatorname{tg} x \;\Rightarrow\; x = \operatorname{arctg} t \\ dx = \frac{dt}{1+t^2} \end{cases} \\
&= \int \frac{\frac{dt}{1+t^2}}{\sqrt{t}}] \\
&= \int \frac{dt}{(1+t^2)\sqrt{t}} \qquad \begin{cases} u = \sqrt{t} \;\Rightarrow\; t = u^2 \\ dt = 2u\,du \end{cases} \\
&= \int \frac{2\cancel{u}\,du}{(1+u^4)\cdot \cancel{u}} \\
&= \int \frac{2}{1+u^4}\,du
\end{aligned} \tag{7}$$

Integral (7) predstavlja integral racionalne funkcije. Funkciju u imeniocu je potrebno faktorisati nad poljem realnih brojeva. Kako je

$$\begin{aligned}
u^4+1 &= u^4+2u^2+1-2u^2 \\
&= \left(u^2+1\right)^2 - \left(\sqrt{2}u\right)^2 \\
&= \left(u^2-\sqrt{2}u+1\right)\cdot\left(u^2+\sqrt{2}u+1\right)
\end{aligned}$$

primjenom metode neodređenih koeficijenata dobijamo:

$$\begin{aligned}
&\frac{2}{\left(u^2-\sqrt{2}u+1\right)\cdot\left(u^2+\sqrt{2}u+1\right)} = \frac{Au+B}{u^2-\sqrt{2}u+1} + \frac{Cu+D}{u^2+\sqrt{2}u+1} \\
\Leftrightarrow\quad & 2 = (Au+B)\cdot\left(u^2+\sqrt{2}u+1\right) + (Cu+D)\cdot\left(u^2-\sqrt{2}u+1\right) \\
\Leftrightarrow\quad & 2 = Au^3+Bu^2+\sqrt{2}Au^2+\sqrt{2}Bu+Au+B+Cu^3+Du^2-\sqrt{2}Cu^2-\sqrt{2}Du+Cu+D \\
\Leftrightarrow\quad & 2 = (A+C)\,u^3 + \left(\sqrt{2}A+B-\sqrt{2}C+D\right)u^2 + \left(A+\sqrt{2}B+C-\sqrt{2}D\right)u + (B+D)
\end{aligned}$$

odakle dobijamo sistem

$$\begin{cases} A+C=0 \\ \sqrt{2}A+B-\sqrt{2}C+D=0 \\ A+\sqrt{2}B+C-\sqrt{2}D=0 \\ B+D=2 \end{cases}$$

čije je rješenje

$$A=-\frac{1}{\sqrt{2}},\;\; B=1,\;\; C=\frac{1}{\sqrt{2}},\;\; D=1.$$

Vraćanjem u izraz (7) dobijamo

$$\begin{aligned}
I &= \int \frac{-\frac{1}{\sqrt{2}}u+1}{u^2-\sqrt{2}u+1}\,du + \int \frac{\frac{1}{\sqrt{2}}u+1}{u^2+\sqrt{2}u+1}\,du \\
&= -\frac{1}{\sqrt{2}}\cdot\int \frac{u-\sqrt{2}}{u^2-\sqrt{2}u+1}\,du + \frac{1}{\sqrt{2}}\cdot\int\frac{u+\sqrt{2}}{u^2+\sqrt{2}u+1}\,du \\
&= -\frac{1}{\sqrt{2}}\cdot\int \frac{\frac{1}{2}\cdot\left(2u-\sqrt{2}\right)-\frac{\sqrt{2}}{2}}{u^2-\sqrt{2}u+1}\,du + \frac{1}{\sqrt{2}}\cdot\int\frac{\frac{1}{2}\cdot\left(2u+\sqrt{2}\right)+\frac{\sqrt{2}}{2}}{u^2+\sqrt{2}u+1}\,du \\
&= -\frac{1}{\sqrt{2}}\cdot\frac{1}{2}\cdot\int\frac{2u-\sqrt{2}}{u^2-\sqrt{2}u+1}\,du + \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{2}\cdot\int\frac{du}{u^2-\sqrt{2}u+1} + \frac{1}{\sqrt{2}}\cdot\frac{1}{2}\cdot\int\frac{2u+\sqrt{2}}{u^2+\sqrt{2}u+1}\,du + \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{2}\cdot\int\frac{du}{u^2+\sqrt{2}u+1} \\
&= -\frac{1}{2\sqrt{2}}\cdot\ln\left(u^2-\sqrt{2}u+1\right) + \frac{1}{2}\cdot\int\frac{du}{\left(u^2-2\cdot u\cdot\frac{\sqrt{2}}{2}+\left(\frac{\sqrt{2}}{2}\right)^2\right)-\left(\frac{\sqrt{2}}{2}\right)^2+1} + \\
&\quad \frac{1}{2\sqrt{2}}\cdot\ln\left(u^2+\sqrt{2}u+1\right) + \frac{1}{2}\cdot\int\frac{du}{\left(u^2+2\cdot u\cdot\frac{\sqrt{2}}{2}+\left(\frac{\sqrt{2}}{2}\right)^2\right)-\left(\frac{\sqrt{2}}{2}\right)^2+1} \\
&= -\frac{1}{2\sqrt{2}}\cdot\ln\left(u^2-\sqrt{2}u+1\right) + \frac{1}{2}\cdot\int\frac{du}{\left(u-\frac{\sqrt{2}}{2}\right)^2+\left(\frac{\sqrt{2}}{2}\right)^2} + \frac{1}{2\sqrt{2}}\cdot\ln\left(u^2+\sqrt{2}u+1\right) + \frac{1}{2}\cdot\int\frac{du}{\left(u+\frac{\sqrt{2}}{2}\right)^2+\left(\frac{\sqrt{2}}{2}\right)^2} \\
&= -\frac{1}{2\sqrt{2}}\cdot\ln\left(u^2-\sqrt{2}u+1\right) + \frac{1}{2}\cdot\frac{1}{\frac{\sqrt{2}}{2}}\cdot\operatorname{arctg}\left(\frac{u-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}\right) + \frac{1}{2\sqrt{2}}\cdot\ln\left(u^2+\sqrt{2}u+1\right) + \frac{1}{2}\cdot\frac{1}{\frac{\sqrt{2}}{2}}\cdot\operatorname{arctg}\left(\frac{u+\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}\right) + C \\
&= -\frac{1}{2\sqrt{2}}\cdot\ln\left(u^2-\sqrt{2}u+1\right) + \frac{1}{\sqrt{2}}\cdot\operatorname{arctg}\left(\sqrt{2}u-1\right) + \frac{1}{2\sqrt{2}}\cdot\ln\left(u^2+\sqrt{2}u+1\right) + \frac{1}{\sqrt{2}}\cdot\operatorname{arctg}\left(\sqrt{2}u+1\right) + C
\end{aligned}$$

Vraćanjem smjene $u = \sqrt{t} = \sqrt{\operatorname{tg} x}$ u prethodni izraz dobijamo da je konačno:

$$I = -\frac{1}{2\sqrt{2}} \cdot \ln\left(\operatorname{tg} x - \sqrt{2\operatorname{tg} x} + 1\right) + \frac{1}{\sqrt{2}} \cdot \operatorname{arctg}\left(\sqrt{2\operatorname{tg} x} - 1\right) + \frac{1}{2\sqrt{2}} \cdot \ln\left(\operatorname{tg} x + \sqrt{2\operatorname{tg} x} + 1\right) + \frac{1}{\sqrt{2}} \cdot \operatorname{arctg}\left(\sqrt{2\operatorname{tg} x} + 1\right) + C.$$

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Zadatak 9.
Izračunati

$$\int_1^{+\infty} \frac{x \ln x}{\left(1+x^2\right)^2}\, dx.$$

Rješenje

Prvo ćemo izračunati neodređeni integral

$$I_n = \int \frac{x \ln x}{\left(1+x^2\right)^2}\, dx. \tag{8}$$

Za rješavanje ovog integrala ćemo koristiti parcijalnu integraciju. Uzmimo $u = \ln x$. Tada je

$$dv = \frac{x}{\left(1+x^2\right)^2}\, dx$$

pa je

$$\begin{aligned} v &= \int \frac{x}{\left(1+x^2\right)^2}\, dx \qquad \begin{cases} t = 1+x^2 \\ dt = 2x\, dx \quad \Rightarrow \quad x\, dx = \frac{dt}{2} \end{cases} \\ &= \int \frac{\frac{dt}{2}}{t^2} \\ &= \frac{1}{2} \cdot \frac{t^{-1}}{-1} \\ &= -\frac{1}{2 \cdot (1+x^2)}. \end{aligned}$$

Sada je integral (8) jednak

$$\begin{aligned} I_n &= \int \frac{x \ln x}{\left(1+x^2\right)^2}\, dx \qquad \begin{cases} u = \ln x \\ du = \frac{dx}{x} \\ v = -\frac{1}{2\cdot\left(1+x^2\right)} \\ dv = \frac{x}{\left(1+x^2\right)^2}\, dx \end{cases} \\ &= -\frac{\ln x}{2 \cdot (1+x^2)} - \int -\frac{1}{2 \cdot (1+x^2)} \cdot \frac{dx}{x} \\ &= -\frac{\ln x}{2 \cdot (1+x^2)} + \frac{1}{2} \cdot \int \frac{dx}{x\,(x^2+1)}. \end{aligned} \tag{9}$$

Integral (9) predstavlja integral racionalne funkcije. Koristeći metod neodređenih koeficijenata imamo:

$$\begin{aligned} \frac{1}{x\,(x^2+1)} &= \frac{A}{x} + \frac{Bx+C}{x^2+1} \\ \Leftrightarrow \quad 1 &= A\left(x^2+1\right) + (Bx+C)\,x \\ \Leftrightarrow \quad 1 &= Ax^2 + A + Bx^2 + Cx \\ \Leftrightarrow \quad 1 &= (A+B)\,x^2 + Cx + A \end{aligned}$$

odakle dobijamo sistem

$$\begin{cases} A+B=0 \\ C=0 \\ A=1 \end{cases}$$

čije je rješenje

$$A=1, \;\; B=-1, \;\; C=0$$

pa uvrštavanjem u izraz (9) dobijamo

$$\begin{aligned} I_n &= -\frac{\ln x}{2 \cdot (1+x^2)} + \frac{1}{2} \cdot \left(\int \frac{1}{x}\, dx + \int \frac{-x}{1+x^2}\, dx \right) \\ &= \frac{1}{2} \cdot \left(-\frac{\ln x}{1+x^2} + \int \frac{dx}{x} - \frac{1}{2} \cdot \int \frac{x\, dx}{1+x^2} \right) \\ &= \frac{1}{2} \cdot \left(-\frac{\ln x}{1+x^2} + \ln|x| - \frac{1}{2} \cdot \ln\left|1+x^2\right| \right) + C \\ &= \frac{1}{2} \cdot \left(-\frac{\ln x}{1+x^2} + \frac{1}{2} \cdot \ln|x|^2 - \frac{1}{2} \cdot \ln\left|1+x^2\right| \right) + C \\ &= \frac{1}{2} \cdot \left(-\frac{\ln x}{1+x^2} + \frac{1}{2} \cdot \ln\left|\frac{x^2}{x^2+1}\right| \right) + C. \end{aligned}$$

Vraćanjem u početni nesvojstveni integral dobijamo

$$
\begin{aligned}
I &= \lim_{b\to+\infty}\left(\frac{1}{2}\cdot\left(-\frac{\ln x}{1+x^2}+\frac{1}{2}\cdot\ln\left|\frac{x^2}{x^2+1}\right|\right)\right)\Bigg|_1^b \\
&= \lim_{b\to+\infty}\left(\frac{1}{2}\cdot\left(-\frac{\ln b}{1+b^2}+\frac{1}{2}\cdot\ln\left|\frac{b^2}{b^2+1}\right|\right)\right)-\left(\frac{1}{2}\cdot\left(-\cancelto{0}{\frac{\ln 1}{1+1^2}}+\frac{1}{2}\cdot\ln\left|\frac{1^2}{1^2+1}\right|\right)\right) \\
&= -\frac{1}{2}\cdot\lim_{b\to+\infty}\frac{\ln b}{b^2+1}+\frac{1}{4}\cdot\lim_{b\to+\infty}\ln\left|\frac{b^2}{b^2+1}\right|-\frac{1}{4}\cdot\ln\left(\frac{1}{2}\right) \\
&= -\frac{1}{2}\cdot\lim_{b\to+\infty}\frac{\frac{1}{b}}{2b}+\frac{1}{4}\cdot\ln\left|\frac{\cancel{b^2}}{\cancel{b^2}\cdot\left(1+\cancelto{0}{\frac{1}{b^2}}\right)}\right|-\frac{1}{4}\cdot\ln\left(2^{-1}\right) \\
&= -\frac{1}{2}\cdot\lim_{b\to+\infty}\cancelto{0}{\frac{1}{2b^2}}+\frac{1}{4}\cdot\ln 1+\frac{1}{4}\cdot\ln 2 \\
&= \frac{\ln 2}{4}.
\end{aligned}
$$

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Zadatak 10.
U zavisnosti od vrijednosti realnog parametra p ispitati konvergenciju integrala

$$\int_1^2 \frac{dx}{x\ln^p x}.$$

Rješenje

Vrijedi

$$\begin{aligned}
I &= \int_1^2 \frac{dx}{x\ln^p x} \qquad \begin{cases} t = \ln x \\ dt = \frac{dx}{x} \end{cases} \\
&= \int_{\ln 1}^{\ln 2} \frac{dt}{t^p} \\
&= \lim_{\epsilon\to ^+} \int_{\epsilon}^{\ln 2} t^{-p}\, dt \\
&= \begin{cases} \lim\limits_{\epsilon\to 0^+} \left(\dfrac{\ln^{1-p}(2)}{1-p} - \dfrac{\epsilon^{1-p}}{1-p} \right), & p \neq 1 \\ \lim\limits_{\epsilon\to 0^+} \big(\ln(\ln 2) - \ln\epsilon \big), & p = 1. \end{cases}
\end{aligned}$$

Odavde vidimo da integral konvergira za $1-p>0$, odnosno $p<1$, a divergira za $p \geq 1$.