Clase 04:

- Parámetro estimable.
- EIVMU
- Información de Fisher.
- Cota Mínima y eficiencia.
- Estadístico tipo U.

Estimador Puntual

$$T: \Lambda \to \Theta$$
$$x \to T(x) = \theta,$$

$$T: \Lambda \to \mathcal{P}$$
$$x \to T(x) = P_\theta \in \mathcal{P}.$$

$X_i \overset{iid}{\sim} N(\mu, \sigma^2) \quad i = 1, \dots, n \quad ; \quad \theta = (\mu, \sigma^2)$

$\Theta = \{(\mu, \sigma^2) : \mu \in \mathbb{R} \text{ y } \sigma^2 \in \mathbb{R}_+\} = \mathbb{R} \times \mathbb{R}_+$

$(5, 2) \to P_{(5,2)}.$

$$T_1 : \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}_+$$

$$T_1(x) = \underset{(\mu, \sigma^2) \in \Theta}{\arg\max} \, L(\mu, \sigma^2)$$

Exemplo de regresión lineal simple

Considerar una muestra $(y_1, x_1), \ldots, (y_n, x_n)$ del vector (Y, X) en que $Y_i | X_i = x_i \sim \text{Normal}(\mu_i, \sigma^2)$

$$\mu_i = \beta_0 + \beta_1 x_i, \quad \sigma^2 > 0$$ y el modelo

$$\boxed{Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i}, \quad \varepsilon_i \sim N(0, \sigma^2)$$

Por simplicidad, la muestra puede ser escrita

$$z = (y_1, \ldots, y_n, x_1, \ldots, x_n) \in \mathbb{R}^{2n}$$ y $\theta = (\beta_0, \beta_1) \in \mathbb{R}^2$.

$$T_2 : \mathbb{R}^{2n} \to \mathbb{R}^2$$

$$T_2(z) = \arg\min_{\beta_0, \beta_1} \left\{ \sum [y_i - (\beta_0 + \beta_1 x_i)] \right\}$$

Media posteriori

Modelo conjugado Poisson – Gamma

① $X_i \mid \theta = \theta_0 \sim \text{Poisson}(\theta_0)$

② $\theta \sim \text{Gamma}(\alpha, b)$; $\alpha, b > 0$

$$T_3 : \mathbb{N}^n \to \mathbb{R}_+$$

$$T_3(x) = E(\theta \mid x)$$

$$= \frac{\sum_{i=1}^{n} x_i + \alpha}{n + b}.$$

Medida frecuentista aleatoria

Exp: observar K muestras de tamaño n de de un misma variable X. ($X \sim$ Modelo(θ))

$$X_i^* = (X_{1i}, X_{2i}, \ldots, X_{ni}), \quad i = 1, 2, \ldots, K.$$

$X_1^*, X_2^*, \ldots, X_K^*$. Rep. de un mismo exp. K-veces.

$T(X_1^*), T(X_2^*), \ldots, T(X_K^*)$, los mismo para T'

K estimaciones para un mismo parámetro $g(\theta)$ [$e^{-\theta}$, $\log(\theta)$, θ^r, etc)

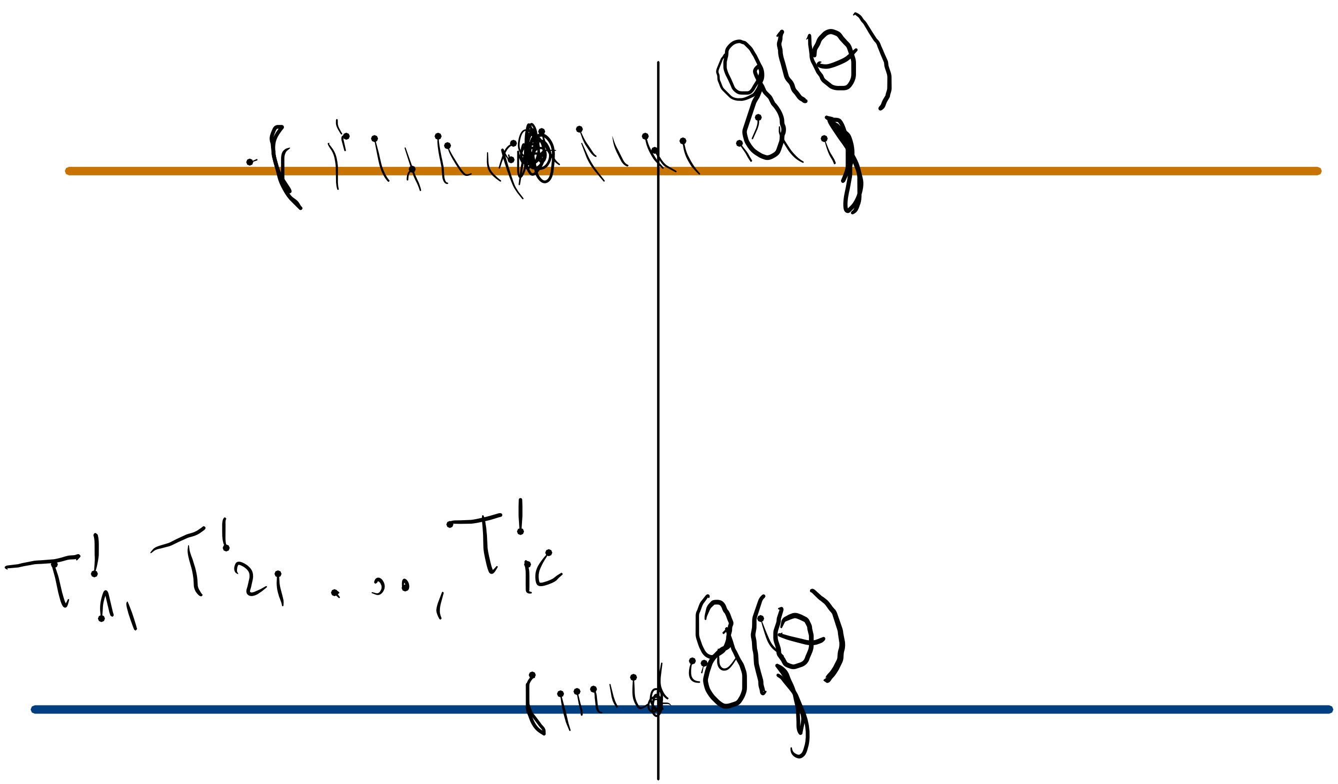
Evaluación frecuentista de $T(X)$

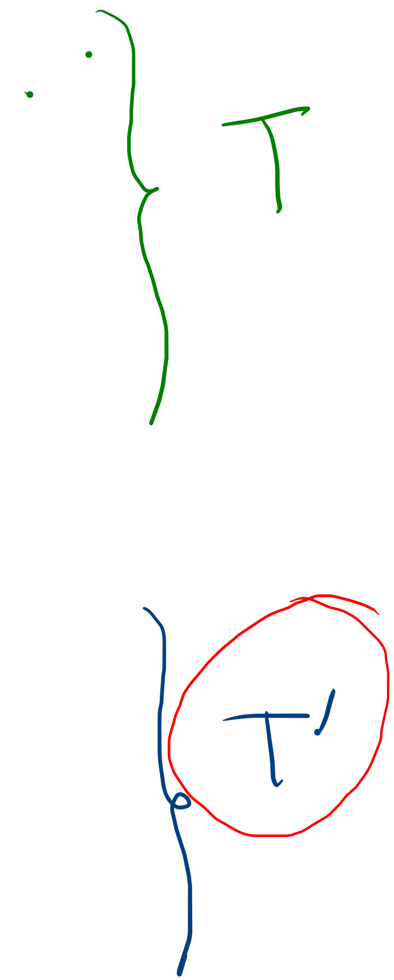
T y T' dos estimados

Estimador insesgado de varianza mínima uniforme.

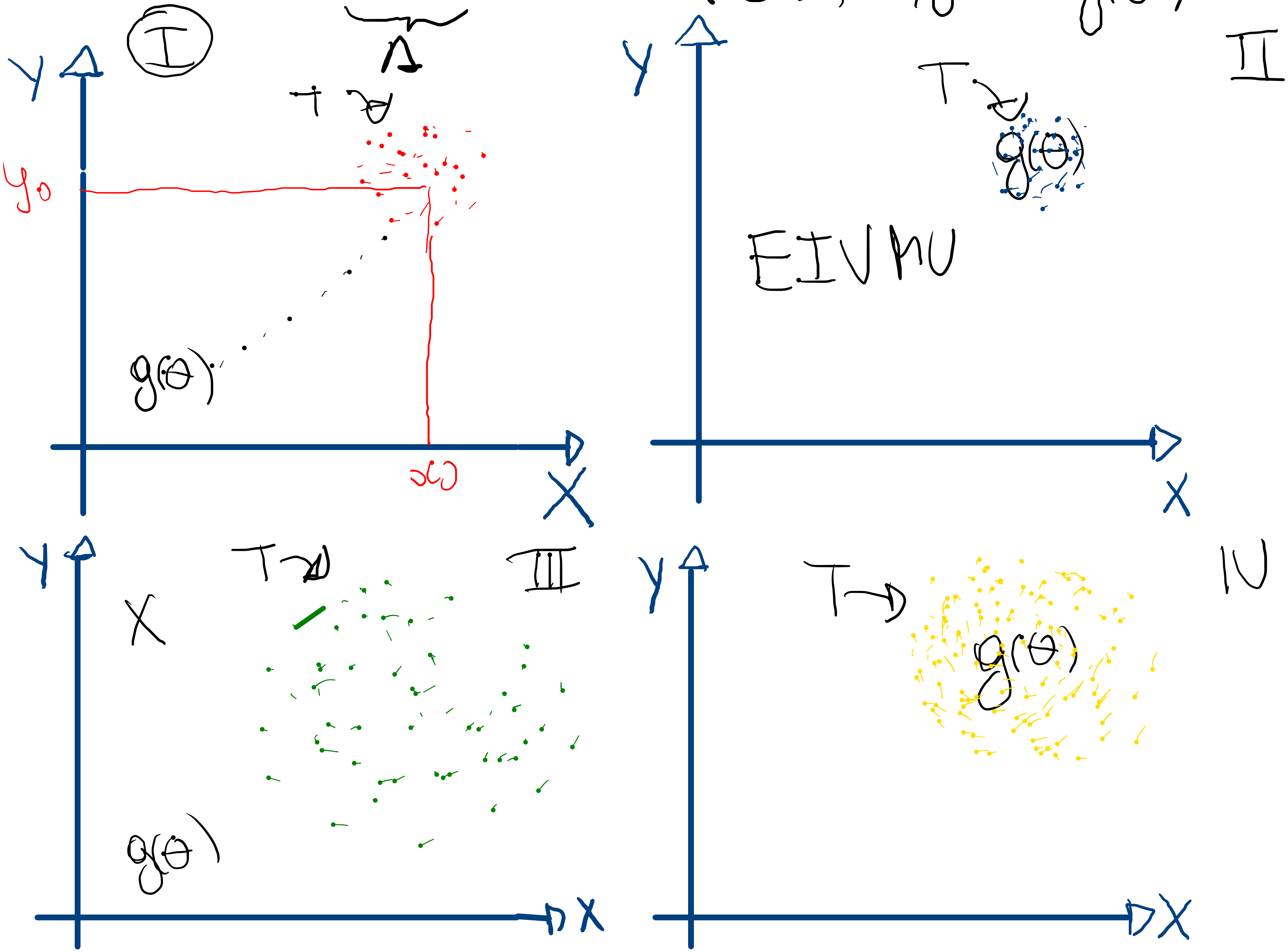
$T_1, T_2, \ldots, T_k$

$T'_1, T'_2, \ldots, T'_k$





$$T: \underbrace{\mathbb{R}\times\mathbb{R}}_{\Lambda} \to \mathbb{R}\ (\Theta);\ (x,y) \to g(\theta)$$



Caso especial: pérdida cuadrática y estimador insesgado

- $T: A \to \Theta$; $E(T(X)) = g(\theta)$ ①
- $L(\theta, T) = (\theta - T)^2$. ③
- $R_T(\theta) = E(L(\theta, T))$

$$= Var_\theta(T).$$

T^* ; D son los estimadores insesgados de $g(\theta)$

$$Var_\theta(T^*) \leq Var_\theta(T), \quad \forall T \in D, \forall \theta \in \Theta$$

T^* es un EIUMV.

Aplicación del teorema 4

$X_i \overset{iid}{\sim} U(0,\theta)$; $\theta > 0$; $f_\theta(x_i) = \frac{1}{\theta} \underset{(0,\theta)}{I}(x_i)$

$$f_\theta(x) = \prod_{i=1}^{n} f_\theta(x_i) = \theta^{-n} \prod_{i=1}^{n} \underset{(0,\theta)}{I}(x_i) \; ; 0 < x_{(1)} < x_{(2)} < \dots < x_{(n)} \leq \theta$$

$$= \underbrace{\theta^{-1} \underset{(0,\theta)}{I}(x_{(n)})} \times 1$$

$$= g_\theta(x_{(n)}) \cdot h(x)$$

$$h(x) = 1$$

$X_{(n)}$ es suficiente para $\theta \in \Theta$.

- $X_{(n)}$ es completo para $\theta \in \Theta$

- $X_{(n)} \sim g_\theta(x) = n\,\theta^{-n}\,x^{n-1}\,I_{(0,\theta)}(x)$

$$E(X_{(n)}) = \int_0^\theta x\, n\,\theta^{-n}\,x^{n-1}\,dx$$

$$= n\,\theta^{-n}\int_0^\theta x^{n}\,dx$$

$$= n\,\theta^{-n}\left[\frac{x^{n+1}}{n+1}\Big|_0^\theta\right]$$

$$= n\,\theta^{-n}\cdot\frac{\theta^{n+1}}{n+1} = \theta\cdot\frac{n}{n+1}$$

$$E\Big(\underbrace{\frac{n+1}{n}\,X_{(n)}}_{h(X_{(n)})}\Big) = \theta$$

$$h(X_{(n)}) = X_{(n)}\cdot\frac{n+1}{n}$$

|| es EIVMU para θ.

Ap. del Teorema 5.(i)

$X_i \overset{iid}{\sim} U(0,\theta)$; $\theta > 0$; $X_{(n)} \sim g_\theta(x) = n\theta^{-n} x^{n-1} I_{(0,\theta)}(x)$.

$$\mathcal{U} \doteq \{ U(X) : E(U(X)) = 0 \}$$

(i) $U(X_{(n)})$

$$E(U(X_{(n)})) = \int_0^\theta U(x)\, n\theta^{-n} x^{n-1}\, dx = 0$$

$$\Leftrightarrow \int_0^\theta U(x)\, x^{n-1}\, dx = 0$$

$$= \underbrace{\int_0^1 U(x)\, x^{n-1}\, dx}_{I_1} + \underbrace{\int_1^\theta U(x)\, x^{n-1}\, dx}_{I_2} = 0$$

$$I_2 = 0 \iff U(x) \equiv 0 \quad \forall x \geq 1 \qquad (C1)$$

$$\int_0^1 U(x)\, x^{n-1}\, dx = 0 \qquad (C2)$$

De la condición 1, tenemos que

$$E\big(h(X_{(n)})\, U(X_{(n)})\big) = \int_0^1 h(x)U(x)x^{n-1}\,dx + \overbrace{\cancel{\int_1^\theta h(x)U(x)x^{n-1}\,dx}}^{C1}_{0}$$

$$= \int_0^1 h(x)U(x)x^{n-1}\,dx, \quad \text{sujeto } (2)$$

Proponemos el siguiente estimador

$$h(x) = \begin{cases} c, & 0 \le x \le 1, \\ bx, & x > 1. \end{cases}$$

$h(X_{(n)})$: estimador.

$$E\left(h(X_{(n)})\,U(X_{(n)})\right) = \int_0^\theta h(x)\,U(x)\,x^{-n}\,dx$$

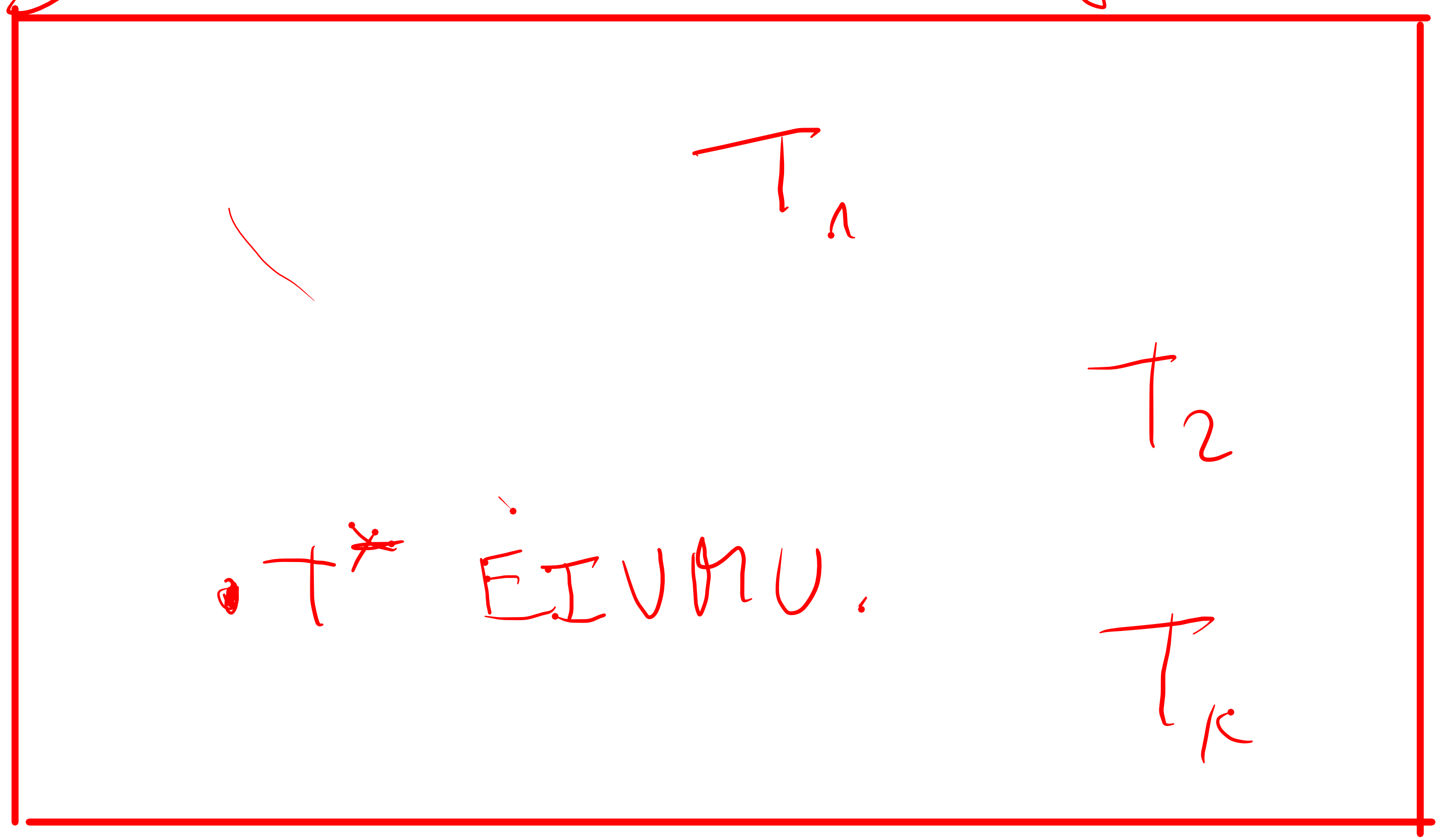
$U(x) \equiv 0, \ 1 < x \le \theta$

$$= c\int_0^1 \cancel{U(x)\,x^{-n}\,dx} + b\overbrace{\int_1^\theta x\,U(x)\,x^{-n}\,dx}$$

$$= 0 + b\int_1^\theta \cancel{U(x)\,x^{-n+1}}\,dx = 0.$$

0

D: estimadores insesgados



T^* puede no se eficiente.

Eje de las varianza

Función de log-verosimilitud

Sea $f_\theta(x)$ f.d.c $x=(x_1, x_2, \ldots, x_n)$ y $f_\theta(x_i)$ la i-ésima marginal.

$L(\theta)$; $L(\theta; x)$; $L(\theta | x)$; $L_x(\theta)$;

$$L_x : \Theta \to \mathbb{R}.$$

$$L_x(\theta) = \prod_{i=1}^{n} f_\theta(x_i).$$ F. de verosimilitud.

$X_i \overset{iid}{\sim} \text{Modelo}(\theta)$

$$\ell_x(\theta) = \ln\{L_x(\theta)\} = \sum_{i=1}^{n} \ln f_\theta(x_i),$$

$$\frac{\partial l_{x_i}(\theta)}{\partial \theta} = \sum_{i=1}^{n} \underbrace{\left\{ \frac{\partial \ln f_\theta(x_i)}{\partial \theta} \right\}}_{\text{Función escore.}}$$

$$\psi_\theta(x) = \left(\frac{\partial l_x(\theta)}{\partial \theta} \right)^2 ,$$ Función de X para un θ fijo.

$$E(\psi_\theta(x)) = E\left\{ \left(\frac{\partial l_x(\theta)}{\partial \theta} \right)^2 \right\} = IF_n(\theta)$$

Información de Fisher de la muestra $X_1, X_2, \dots, X_n$ sobre θ

$$IF_i(\theta) = E\left(\left(\frac{\partial l_{x_i}(\theta)}{\partial \theta}\right)^2\right) =$$

$$IF_n(\theta) = -E\left\{\frac{\partial^2}{\partial\theta^2} \ln f_\theta(x)\right\}, \quad n \geq 1$$ Restricto a la familia exp.

$$= -E\left\{\frac{\partial^2}{\partial\theta^2} \ln\left[\prod_{i=1}^n f_\theta(x_i)\right]\right\}, \quad n \geq 1$$ Independencia.

$$= -E\left\{\frac{\partial^2}{\partial\theta^2} \sum_{i=1}^n \ln f_\theta(x_i)\right\}$$

$$= -E\left\{\sum_{i=1}^n \frac{\partial^2}{\partial\theta^2} \ln f_\theta(x_i)\right\}$$

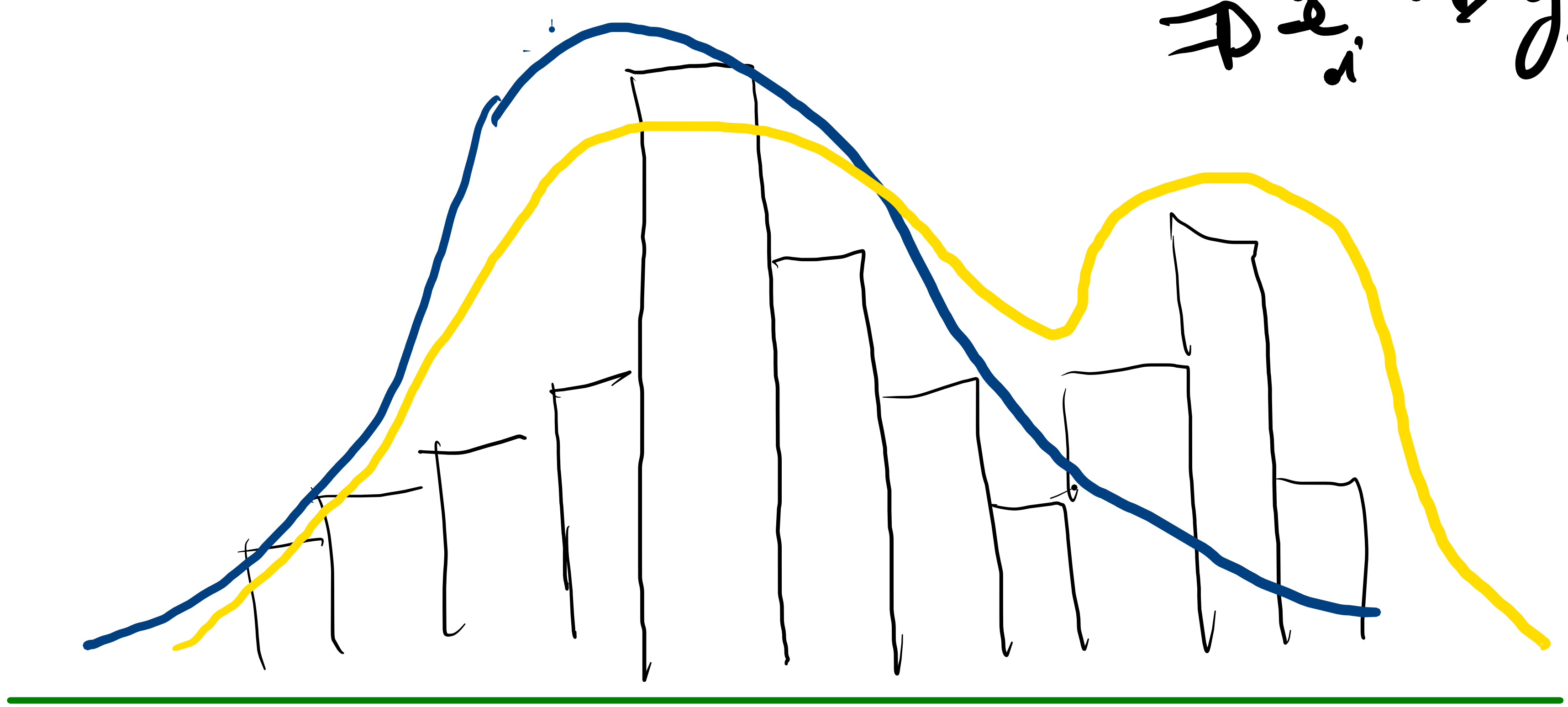
$$= \sum_{i=1}^n \underbrace{-E\left\{\frac{\partial^2}{\partial\theta^2} \ln f_\theta(x_i)\right\}}_{IF_i(\theta)}$$

Modelado paramétrico

$$\mu = g(\theta) = g\left(\beta_0 + \sum \beta_i x_i\right)$$

$\beta_0, \beta_i = 0$

$\Rightarrow x_i \rightarrow y_i$



$$\left(A, \mathcal{G}, \{P_F : F \in \mathcal{L}\}\right); \; F(x) = P(X \leq x)$$

- $\int x \, dF(x) < \infty$

- $\int |x|^2 \, dF(x) < \infty$.

$$IF_n(\theta) = \sum_{i=1}^{n} IF_i(\theta) , \qquad X_i$$

i.d. id. distribuido

$$= \sum_{i=1}^{n} IF_{i_0}(\theta) , \qquad f_\theta(x_j) = f_\theta(x_i)$$

$$i, j = 1, \ldots, n$$

$$= n\, IF_{i_0}(\theta) .$$

Supuesto i.i.d.

$$E_\theta(h(X)) = \int h(x) f_\theta(x) dx$$

$$\frac{\partial E_\theta(h(X))}{\partial \theta} = \frac{\partial}{\partial \theta} \int h(x) f_\theta(x) dx$$

Función escore e Información de Fisher

Información de Fisher caso independiente

Información de Fisher caso i.i.d