Clase 5: Intervalos de Confianza y de credibilidad.

1 IC clásicos via M. pivotal.

2 IC asintóticos

3 I. credibilidad Bayesiano y ICDM.

4: I.C multimodal y Regiones de confianza.

E. puntal

$$T : \Delta \longrightarrow \Theta \longrightarrow P_{\hat{\mu}\sigma}$$

$$T(x) = \hat{\mu}(x) \approx \mu.$$

E. Intervalar:

$$(a, b) \subset \Theta.$$

$$T : \Delta \longrightarrow I_{\Theta}$$

$$T(x) = [a, b] \ni \mu$$

$$x = (x_1, x_2, \ldots, x_n) \quad \{P_{\mu\sigma^2} : \mu \in T(x)\} \subset \{P_{\mu\sigma^2} : (\mu, \sigma^2) \in \Theta\}$$

Tres intervalos

$\alpha \in (0,1)$

$1-\alpha \in (0,1)$

$95\%, \quad \alpha = 0{,}05$

$(1-\alpha) \cdot 100\%$

$[a, b]$: IC

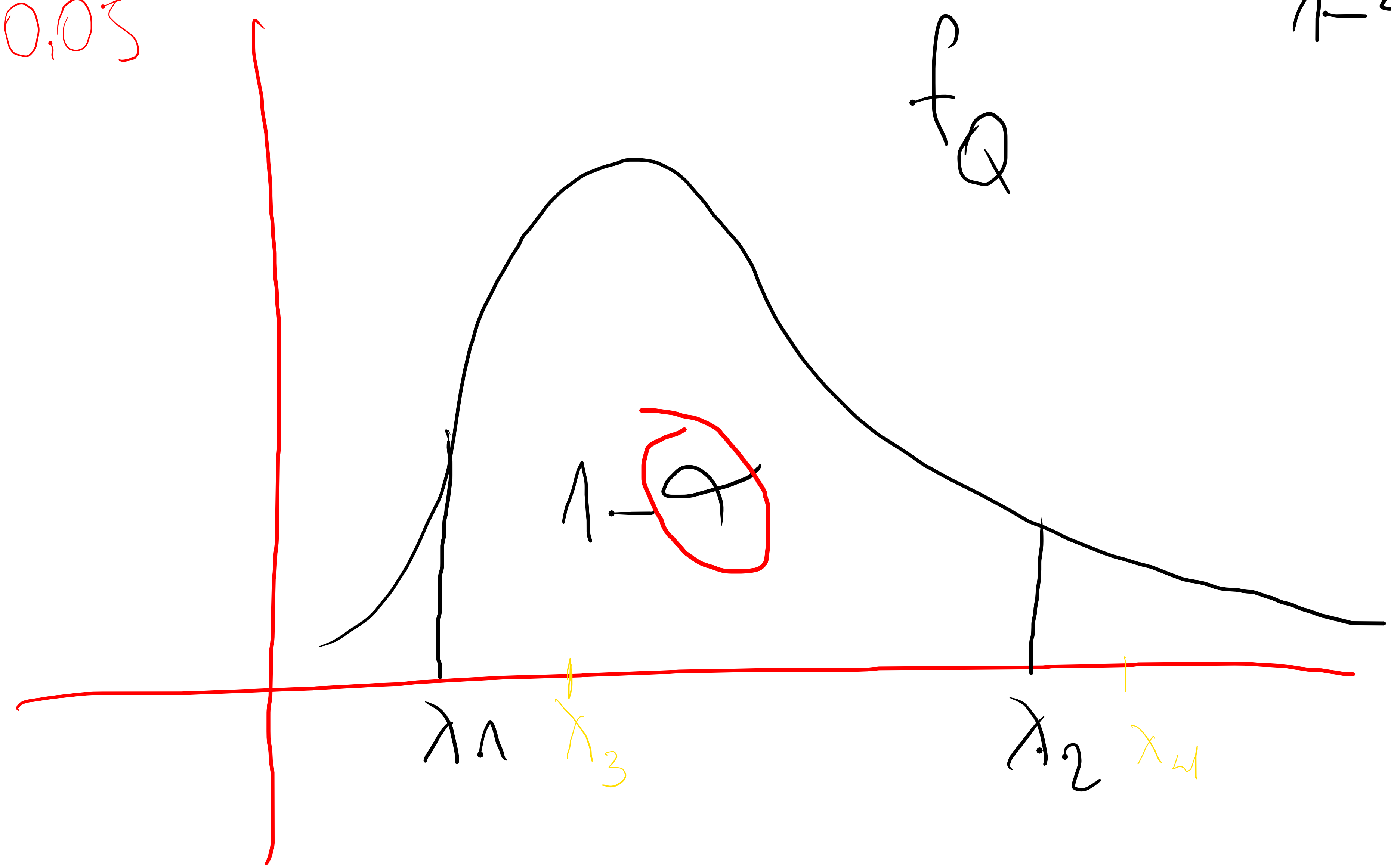
$(a, +\infty)$: Límite inferior

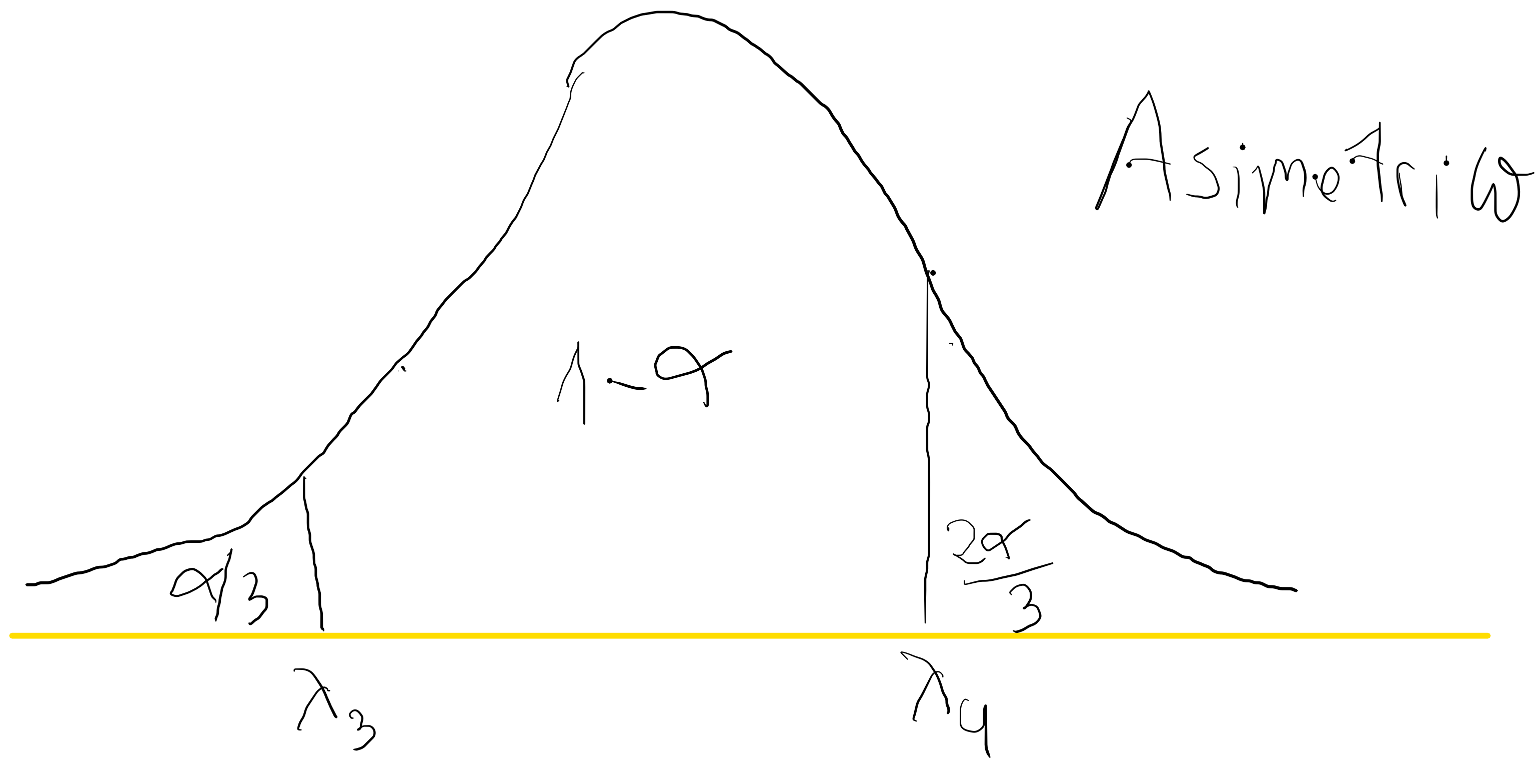
$(-\infty, b)$: Límite superior

Cantidad pivotal $Q(X;\theta) \sim$ Modelo ✓

$\alpha = 0{,}05$

$1-\alpha$





Asimetrico

simétrico

Asumir $X_i \overset{iid}{\sim} N(\mu, \sigma^2) \quad i = 1, \dots, n$

$$\bar{X}_n = n^{-1} \sum_{i=1}^{n} X_i \sim N(\mu, \sigma^2/n)$$

- $E(\bar{X}) = n^{-1} \sum_{i=1}^{n} E(X_i) = n^{-1} \sum_{i=1}^{n} \mu = \cancel{n^{-1}} \cancel{n} \mu = \mu$

- $Var(\bar{X}) = (n^{-1})^2 \sum_{i=1}^{n} Var(X_i)$, por ind.

$$= n^{-2} \sum_{i} \sigma^2 = n^{-2} n^1 \sigma^2 = n^{-1} \sigma^2 = \frac{\sigma^2}{n}$$

$$Q(X; \theta) = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$

- No depende de (μ, σ^2)
- Es el mismo para cada (μ, σ^2)

$X_1, \ldots, X_n$

Empíricos

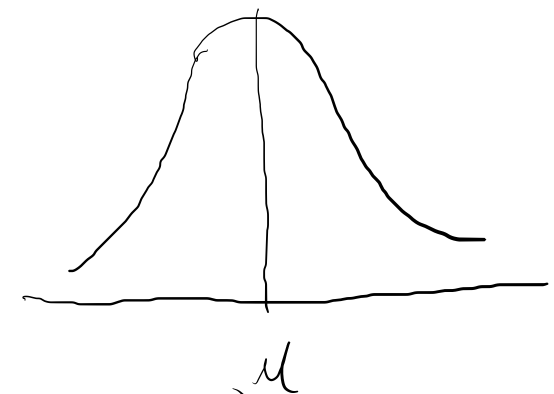
$\bar{x} = n^{-1} \sum_{i=1}^{n} x_i \in \mathbb{R}$

Promedio - media empírica.

$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$

Teoricamente

$N(\mu, \sigma^2)$

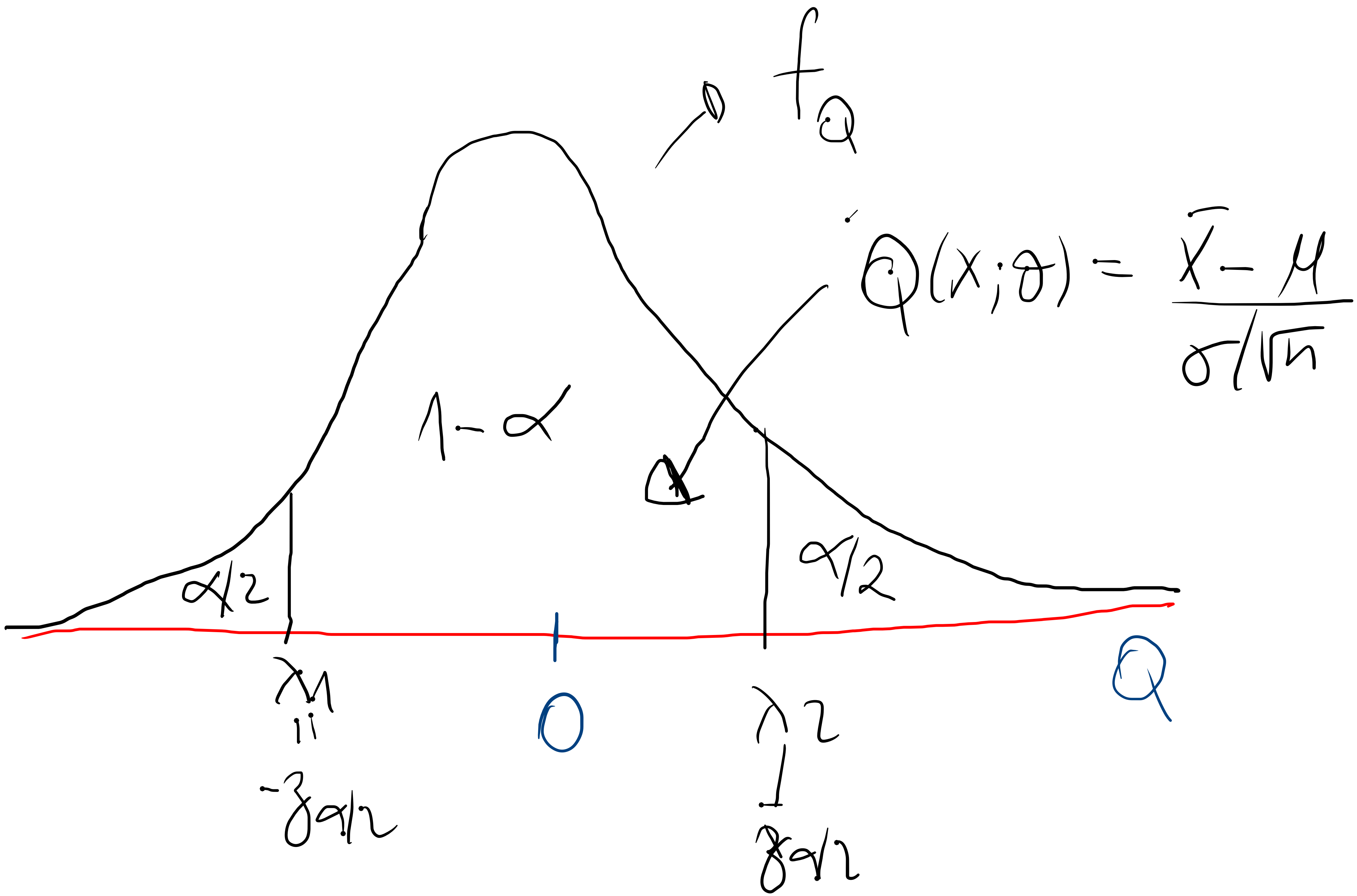


μ: media, esperanza, promedio, valor esperado del modelo estadístico para X.

Si $T(x)$ no es pivotal, entonces casi siempre

$$P_\theta(\lambda_1 \le T(x) \le \lambda_2) = g(\theta) = 1-\alpha, \quad \forall \theta \in \Theta$$

Clásico



f_Q
$Q(X;\theta) = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$
$1-\alpha$
$\alpha/2$
$\alpha/2$
λ_1
$\shortparallel$
$-z_{\alpha/2}$
0
λ_2
$\shortparallel$
$z_{\alpha/2}$
Q

$$P\left(-t_{\alpha/2} \leq \overbrace{\frac{\bar{X}-\mu}{S/\sqrt{n}}}^{t_{n-1}} \leq t_{\alpha/2}\right) = 1-\alpha, \ \forall (\mu,\sigma^2) \in \Theta$$

Variable Aleatoria $(\bar{X}, S)$

$$= P\left(-t_{\alpha/2} \cdot \frac{S}{\sqrt{n}} \leq \bar{X}-\mu \leq t_{\alpha/2} \cdot \frac{S}{\sqrt{n}}\right)$$

μ es constante

$$= P\left(-\bar{X}-t_{\alpha/2} \cdot \frac{S}{\sqrt{n}} \leq -\mu \leq -\bar{X}+t_{\alpha/2} \frac{S}{\sqrt{n}}\right)$$

$$= P\left(\underbrace{\bar{X}+t_{\alpha/2} \frac{S}{\sqrt{n}}}_{L.S} \geq \mu \geq \underbrace{\bar{X}-t_{\alpha/2} \cdot \frac{S}{\sqrt{n}}}_{L.I}\right), \ (a, b)$$

$t_{n-1, \alpha/2}$

$$P_{\Theta}\left(\bar{X}-t_{\alpha/2} \frac{S}{\sqrt{n}} \leq \mu \leq \bar{X}+t_{\alpha/2} \frac{S}{\sqrt{n}}\right) = 1-\alpha$$

$T(x) \subset \Theta$

$X_1^* = (X_{11}, \ldots, X_{1n})$

$\vdots$

$X_k^* = (X_{k1}, \ldots, X_{kn})$

$X_{j1} \overset{iid}{\sim} \text{Modelo}(\theta)$

$$I_{T_j}(\theta) = \begin{cases} 1, & \theta \in T(x) \\ 0, & c.c \end{cases} \quad ; \quad T_j = T(X_j^*), \; j = 1, \ldots, k.$$

$$P_k = k^{-1} \sum_{i=1}^{k} I_{T_j}(\theta) : \text{prop; frec. relativa.}$$

$$P_k \xrightarrow{k \to \infty} 1 - \alpha$$

$\cancel{\approx}$ $\cancel{=}$ | Converge.

X_i: altura en metros $i = 1, \ldots, 100$

$T(x) = [1{,}3\ ;\ 2] \ni \mu$ del 95%

Falso: $P_\theta(1{,}3 \leq \underset{\theta}{\mu} \leq 2) = 0{,}95$ $\otimes$

No es v.a.

X: discreto, $\Delta \subset \mathbb{N}$

$T(X) \subset \Theta$; $P_\theta(\theta \in T(X)) \overset{\approx}{\geq} 1-\alpha$.

n grande $(n \to \infty)$

$X_1, \dots, X_n \sim Bern(\theta)$ $i = 1, \dots, n$

$\bar{X} = n^{-1} \sum_i X_i \sim$ Modelo discreto

$Q_n(x;\theta) = \dfrac{\bar{X} - \theta}{\sigma/\sqrt{n}}$; $\sigma = \sqrt{Var(X_i)}$. → Pivotal asintótico

Sea P_n la medida de prob. de Q_n.

"$P_n(A) \approx P_Z(A)$, cuando n sea grande" $(n \to \infty)$

$$\mathrm{Var}(\bar{X}) \xrightarrow{n\to\infty} \frac{1}{IF_n(\hat{\theta})}, \quad \hat{\theta}: \text{es un estimador de máx. verosimilitud.}$$

$$Q_n(X;\theta) = \frac{\bar{X}-\theta}{\sqrt{IF_n(\hat{\theta})}} \xrightarrow{d} N(0,1)$$

$$P_n(A) \approx P_Z(A), \quad \forall A \in \mathcal{G}.$$

IC asintoticos

$$\frac{\hat{\theta} - \theta}{\sqrt{(IF_n(\hat{\theta}))^{-1}}} \xrightarrow[d]{} N(0,1)$$

$$F_n \approx \Phi(z) \qquad\qquad F_Z(z) = \Phi(z).$$

$q_1 \;,\; q_2$

$$\Phi\left(-z_{\alpha/2} \le \frac{\bar{X}-\theta}{\sqrt{\frac{\bar{X}(1-\bar{X})}{n}}} \le z_{\alpha/2}\right) \approx 1-\alpha$$

$X_i \overset{iid}{\sim} \text{Bernoulli}(\theta)$, $i = 1, \dots, n$.

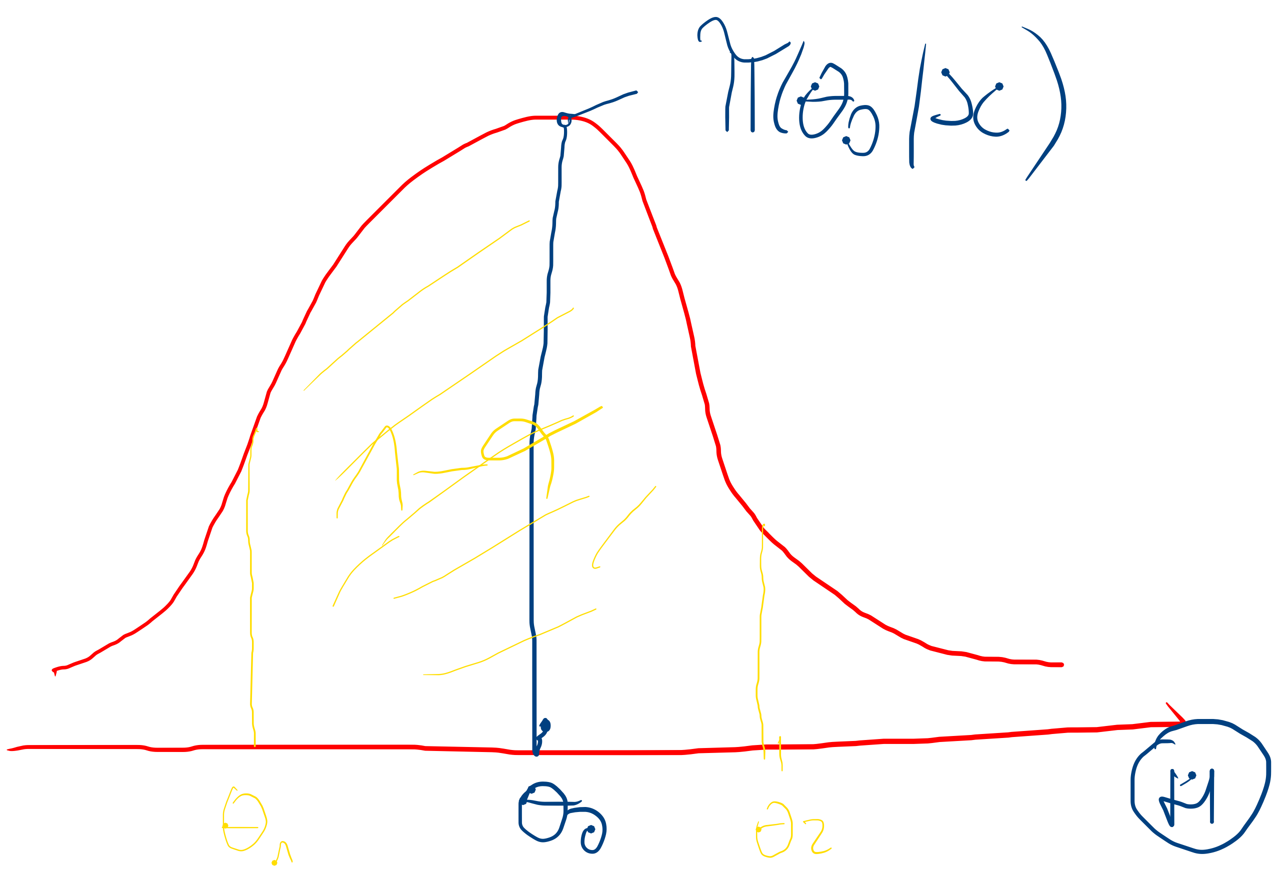
$\Phi : \Phi(z) = P(Z \le z)$, $Z \sim N(0,1)$.

Intervalos de Credibilidad Bayesiano (ICB)

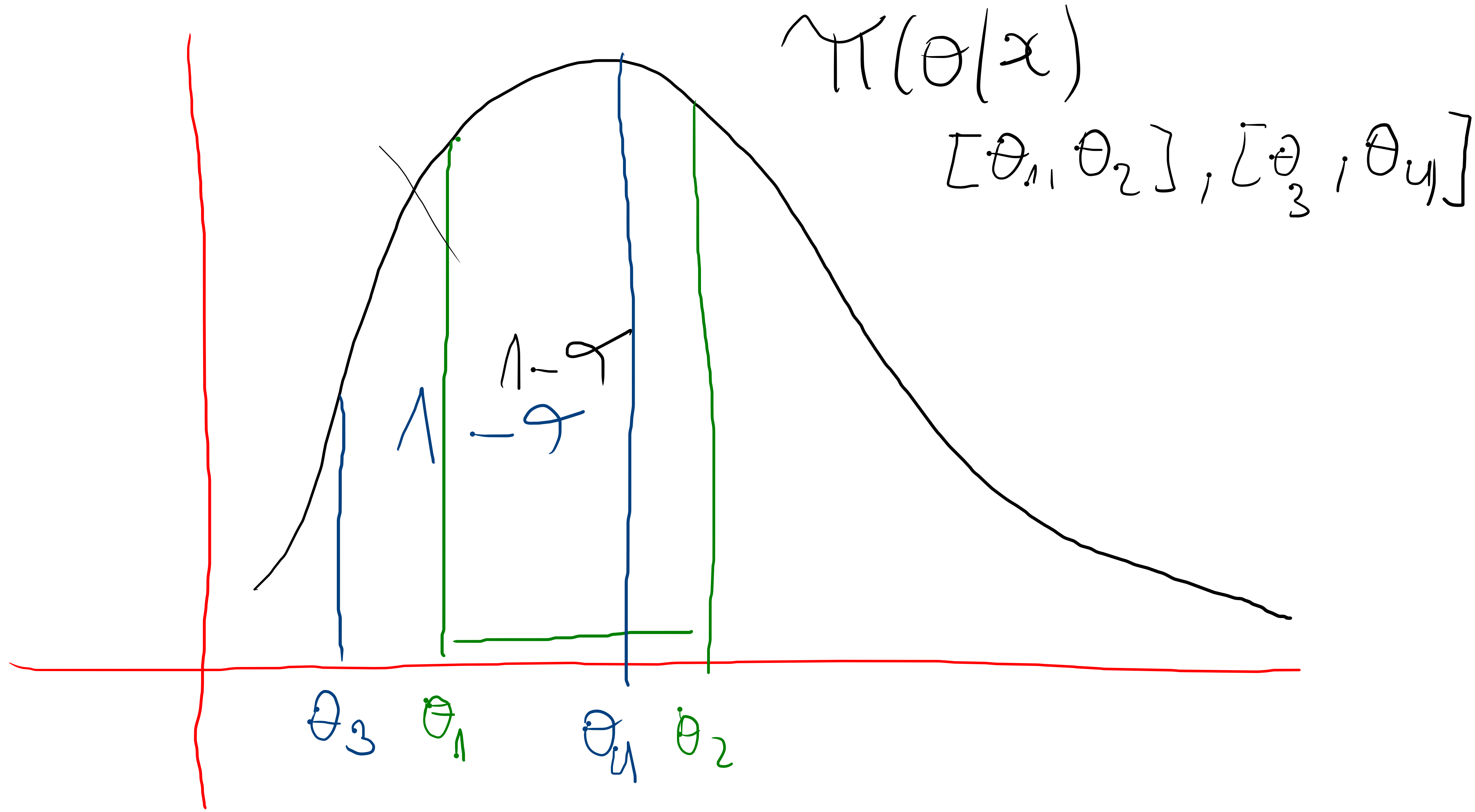
$$\pi(\theta) + f(x|\theta) = \pi(\theta|x)$$

$\pi(\theta)$ ↓ creencia del investigador.

$(X,\theta) \sim P_{X\theta}$



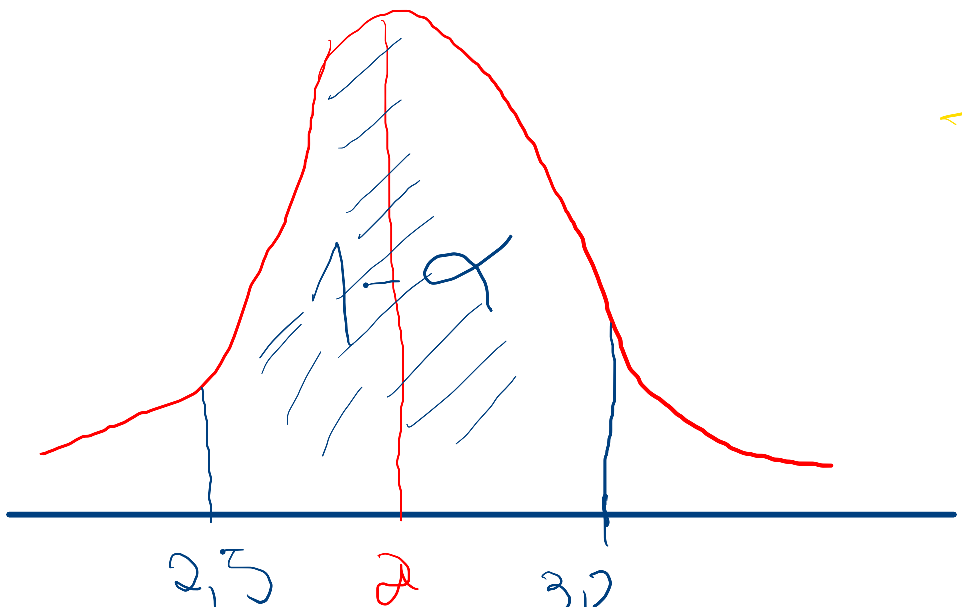
$ICB = [\theta_1, \theta_2]$ del $1-\alpha$



Caso Normal conjugado

$$\left.\begin{array}{l} X_i \mid \mu = \mu_0 \overset{iid}{\sim} Normal(\mu_0, 1) \\ \mu \sim Normal(a, b) \end{array}\right\} \mu \mid x \sim Normal(\mu_x, \sigma_x^2)$$

Verdadero: $P(2{,}5 \leq \mu \leq 3{,}2) = 1-\alpha$ ✓

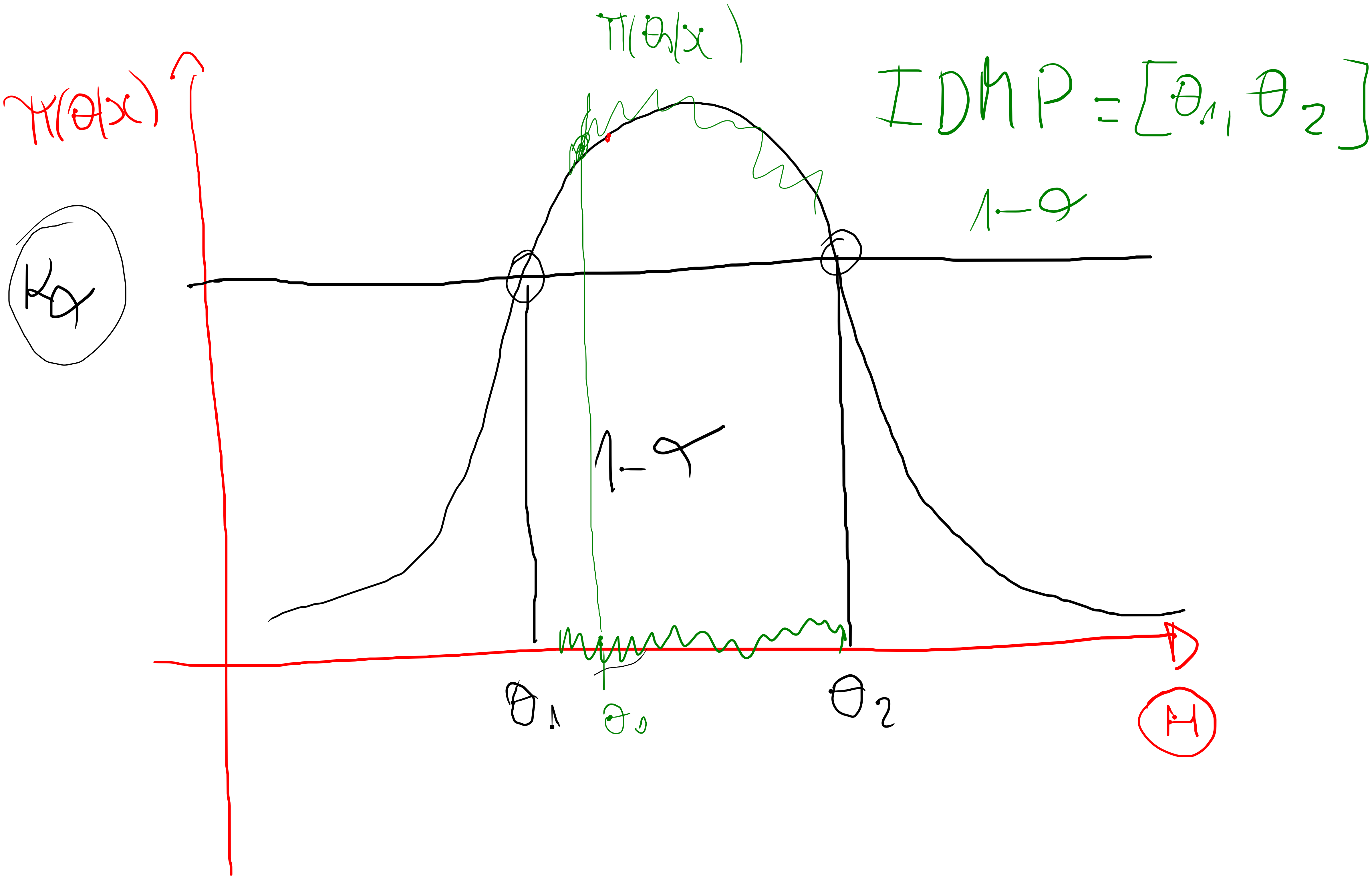


$T(x) = [2{,}5 ; 3{,}2]$

$\pi(\theta \mid x)$: difícil de encontrar en el caso no conjugado.

K_α :

$$\{\theta : \pi(\theta|x) > K_\alpha\} \subset \underset{\cup}{\overset{M}{}}\!IDMP \subset \{\theta : \pi(\theta|x) \geq K_\alpha\}$$



$$\mu | x \sim N\Big(\underbrace{\frac{\sum x_i + \mu_0}{n+1}}_{\text{media}} ; \underbrace{\frac{1}{n+1}}_{\text{varianza}} \Big)$$

$$Z = \frac{\mu - \frac{\sum x_i + \mu_0}{n+1}}{\sqrt{1/(n+1)}} \sim N(0,1)$$

$$P(t_1 \leq Z \leq t_2) = 0{,}95$$

...

Resumen

$$T_1(x) = [a_1, b_1] \ni \mu$$

Normal(μ, σ^2)

$$T_2(x) = [a_2, b_2] \ni \sigma^2$$

$$(\mu, \sigma^2) \in T_1(x) \times T_2(x) \subseteq \Theta.$$

$$\mathcal{P}_0 = \{ P_{\mu\sigma^2} : (\mu, \sigma^2) \in T_1 \times T_2 \} \subset \mathcal{P}.$$

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un modelo $T_3(x) = [a_3, b_3] \ni \theta$, prob. $1-\alpha$

Región de Θ donde mi creencia a posterior indica que está θ con prob igual a $1-\alpha$.

$$P_{\theta|x}(A) = \int_A \pi(\theta|x)\,dx,$$

Ex 1 : $X_i \overset{iid}{\sim} \text{Exp}(\theta)$, $f_\theta(x) = \theta e^{-x\theta}$.

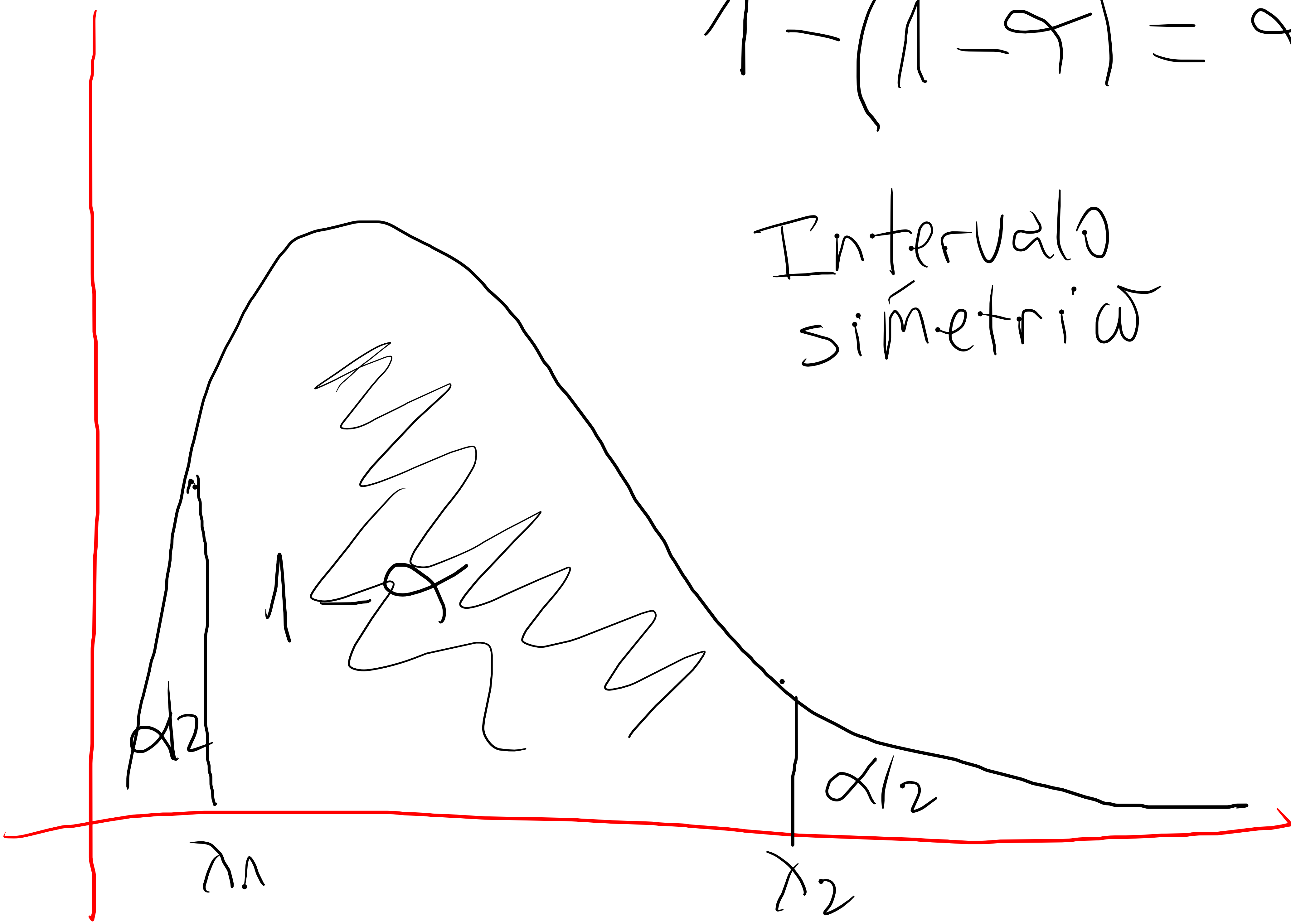
$$Q(X;\theta) = 2\theta \sum X_i \sim \chi^2_{2n}$$

λ_1, λ_2 : $P\Big(\underbrace{\lambda_1 \leq Q(X;\theta) \leq \lambda_2}_{\theta} \Big) = 1-\alpha$

$$P\left(\frac{\lambda_1}{2\sum X_i} \leq \theta \leq \frac{\lambda_2}{2\sum X_i} \right) = 1-\alpha$$

$$1-(1-\alpha)=\alpha$$

Intervalo simétrico



$$IC_{1-\alpha} = \left[\frac{q_1}{2\sum X_i} ; \frac{q_2}{2\sum X_i}\right] \subseteq \Theta$$

$$\{P_\theta : \theta \in IC_{1-\alpha}\} \subseteq P = \{P_\theta : \theta \in \Theta\}$$

Dos muestras independientes. Poblaciones normales

$X_1, X_2, \ldots, X_n, \quad X_i \sim \text{Normal}(\mu_1, \sigma^2)$

$Y_1, \ldots, Y_m, \quad Y_i \sim \text{Normal}(\mu_2, \sigma^2)$

$$\theta = \mu_1 - \mu_2$$

$$\bar{X} - \bar{Y} \sim \text{Normal}\left(\mu_1 - \mu_2, \sigma^2\left(\frac{1}{n} + \frac{1}{m}\right)\right) \checkmark$$

$$Q(X, Y; \theta) = \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{S_p\sqrt{\frac{1}{n} + \frac{1}{m}}} \sim t_{n+m+2}$$

$$IC_{1-\alpha} = \left[\overline{X} - \overline{Y} \pm t_{\alpha/2} \, S_p \sqrt{\frac{1}{n} + \frac{1}{m}} \right] \subseteq \Theta$$

$$\Downarrow$$

$$\mu_1 - \mu_2 \,.$$

IC assintotico

$X_1, X_2, \ldots, X_n$; $X_i \sim \text{Bernoulli}(\theta)$

$\hat{\theta} = \bar{X}$ y $IF_n(\theta) = \dfrac{n}{\theta(1-\theta)}$

$$Q = \frac{\hat{\theta} - \theta}{\sqrt{IF_n(\theta)^{-1}}} = \frac{\bar{X} - \theta}{\sqrt{\left(\dfrac{n}{\bar{X}(1-\bar{X})}\right)^{-1}}} = \frac{\bar{X} - \theta}{\sqrt{\dfrac{\bar{X}(1-\bar{X})}{n}}} \xrightarrow{d} N(0,1)$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$F_n \approx \Phi(z) \qquad\qquad \Phi(z)$$

q_1 q_2

$$P(\lambda_1 \leq Q(x;\theta) \leq \lambda_2) = 1-\alpha$$

$$\left[\bar{X} \pm Z_{\alpha/2}\sqrt{\frac{\bar{X}(1-\bar{X})}{n}}\right]$$

$\bar{X} \sim$ Discreto

ICB

$X_1, X_2, \dots, X_n \; ; \quad X_i \mid \mu \sim \text{Normal}(\mu, 1)$

$\mu \sim \text{Normal}(\mu_0, 1)$

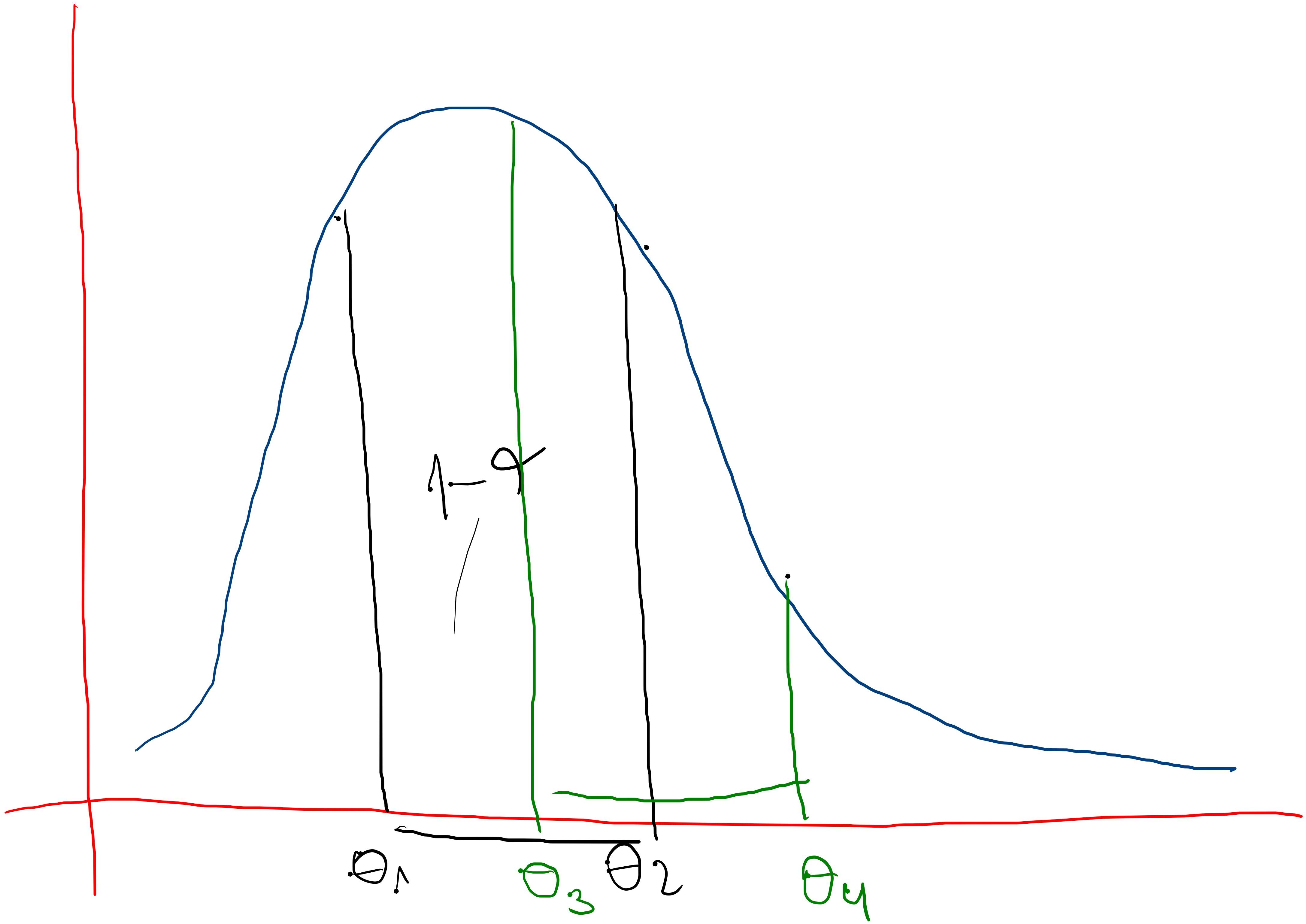
$$\mu \mid x \sim \text{Normal}\left(\overbrace{\frac{\sum x_i + \mu_0}{n+1}}^{\mu|x} \; ; \; \overbrace{\frac{1}{n+1}}^{\sigma^2|x} \right)$$

$1 - \alpha = 0{,}95$

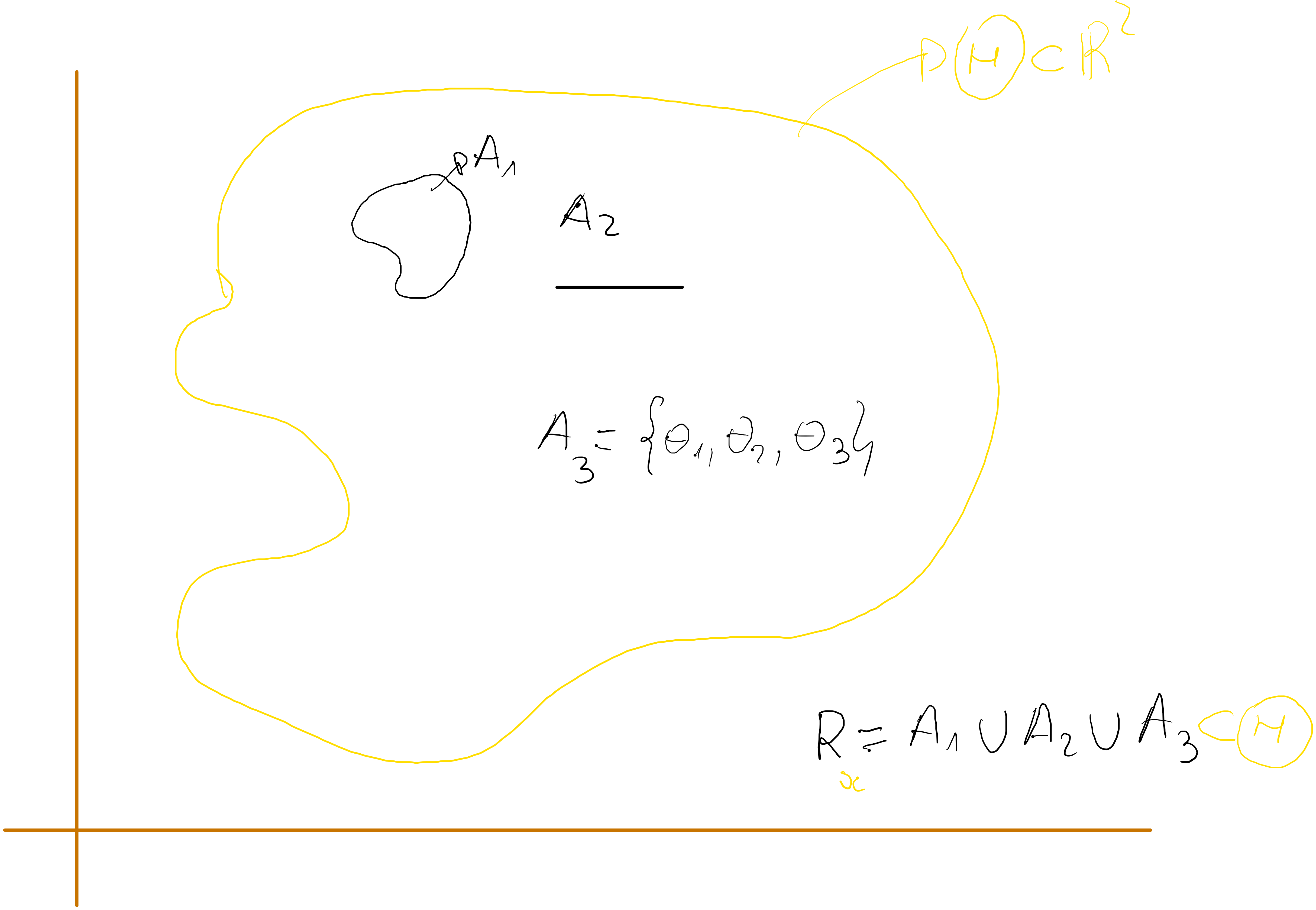
$$\frac{\mu \mid x - \dfrac{\sum x_i + \mu_0}{n+1}}{\sqrt{\dfrac{1}{n+1}}} \sim N(0,1)$$

$$t_i = \frac{\pm \dfrac{\sum x_i + \mu_0}{n+1}}{\sqrt{\dfrac{1}{n+1}}} = \pm 1{,}96 \,, \quad i = 1, 2$$

$$ICB_{1-\alpha} = \left[\frac{\sum x_i + \mu_0}{n+1} \pm 1{,}96 \sqrt{\frac{1}{n+1}} \right] \subseteq \Theta$$



1-α
θ1
θ3
θ2
θ4



$$A_3 = \{\theta_1, \theta_2, \theta_3\}$$

$$R_x = A_1 \cup A_2 \cup A_3 \subset H$$