Clase 10

I	II	III
$H_0: \theta = \theta_0$	$H_0: \theta = \theta_0$	$H_0: \theta \leq \theta_0$
$H_1: \theta = \theta_1$	$H_1: \theta > \theta_0$	$H_1: \theta > \theta_0$
L.N-P	Corolario del L.N-P	Teorema 17

¿Que pasa Θ_0 y Θ_1 son arbitrarias o generales?

$H_0: \theta \in \Theta_0$

$H_1: \theta \in \Theta_1$ //

→ En el caso general, perdemos TUP.

→ Mantener el tamaño del test.

Razon de Verosimilitud para hipótesis generales.

$\underbrace{\text{Razon de Verosimilitud}}_{RV}$

$\Theta_0 \cup \Theta_1 = \Theta \; , \; \Theta_0 \cap \Theta_1 = \emptyset .$

$H_0 : \theta \in \Theta_0$

$H_1 : \theta \in \Theta_1$

- $\dim(\Theta) > \dim(\Theta_0)$.

- Resultado: $\left[\sup_{\theta \in \Theta_1} f_\theta(x) = \sup_{\theta \in \Theta} f_\theta(x) \right]$; $f_\theta(x) : \Theta \longrightarrow \mathbb{R}$.

$$\lambda(x) = \frac{\text{"sup sobre } H_1\text{"}}{\text{"sup sobre } H_0\text{"}} = \frac{\sup_{\theta \in \Theta_1} f_\theta(x)}{\sup_{\theta \in \Theta_0} f_\theta(x)} \begin{cases} \geq c > 1 \\ < 1 \end{cases}$$

$|\lambda'| \leq c$

RV Generalizada (RVG)

$$\lambda'(x) = \frac{\sup_{\theta \in \Theta_0} f_\theta(x)}{\sup_{\theta \in \Theta} f_\theta(x)} \neq \lambda(x)$$

$\Theta_0 \subset \Theta$

$$0 < \lambda' \leq 1$$

$$A_1 \subset \mathcal{X}, \quad (\mathcal{X}, \mathcal{G}, \{P_\theta : -\})$$

$$A_1 = \{x \in \mathcal{X} : \lambda(x) \geq c_\alpha\}$$

$$A_1' = \{x \in \mathcal{X} : \lambda'(x) \leq c_\alpha'\}$$

$$A_1 = A_1'.$$

Preferencia LF sobre Θ

$$\theta_1 \overset{LF}{\precsim} \theta_2 \iff f_{\theta_1}(x) \le f_{\theta_2}(x).$$

$$x \in A_1 \iff \lambda(x) \geq c_\alpha$$

$$\iff \frac{\sup_{H_1} f_\theta(x)}{\sup_{H_0} f_\theta(x)} \geq c_\alpha$$

$$\iff \frac{\sup_{\theta \in \Theta} f_\theta(x)}{\sup_{\theta \in \Theta_0} f_\theta(x)} \geq c_\alpha,$$ [condición]

$$\iff \frac{\sup_{\theta \in \Theta_0} f_\theta(x)}{\sup_{\theta \in \Theta} f_\theta(x)} \leq \frac{1}{c_\alpha} \; (c'_\alpha)$$

$$\iff \lambda'(x) \leq c'_\alpha \iff x \in A'_1. \quad //$$

Equivalencia en decisión

$$T(x) = \begin{cases} 1, & \lambda(x) > c_\alpha \\ 0, & c.c \end{cases} \iff T'(x) = \begin{cases} 1, & \lambda'(x) < \frac{1}{c_\alpha} \\ 0, & c.c \end{cases}$$

$c'_\alpha \sim$

Si $\lambda'(x) = \dfrac{f_{\hat{\theta}_0}(x)}{f_{\hat{\theta}}(x)}$, $\hat{\theta}_0$ es EMV sobre H_0
$\hat{\theta}$ es EMV sobre M

$$-2\log(\lambda'(x)) \xrightarrow{d} \chi^2_k.$$

Obs:

> Θ es un conjunto compacto ($\Theta \subset \mathbb{R}^k$)

> $f_\theta(x) : \Theta \to \mathbb{R}$ continua en Θ.

$$\sup_{\theta \in \Theta_i} f_\theta(x) = \max_{\theta \in \Theta_i} f_\theta(x) \quad // \quad i = 0, 1 . / \Theta .$$

$$\lambda'(x) = \frac{f_{\hat{\theta}_0}(x)}{f_{\hat{\theta}}(x)} .$$

Ejemplo: $X_1, \ldots, X_n$, $X_i \overset{iid}{\sim} \text{Normal}(\mu, \sigma^2)$

$H_0: \mu = \mu_0$ v/s $H_1: \mu \neq \mu_0$; $\theta = (\mu, \sigma^2)$

$\checkmark \; \Theta = \{(\mu, \sigma^2) : \mu \in \mathbb{R} \text{ y } \sigma^2 \in \mathbb{R}_+\} \subset \mathbb{R}^2$

$\checkmark \; \Theta_0 = \{(\mu_0, \sigma^2) : \sigma^2 \in \mathbb{R}_+\} \subset \mathbb{R}$

$\checkmark \; \Theta_1 = \{(\mu, \sigma^2) : \mu \neq \mu_0 \text{ y } \sigma^2 \in \mathbb{R}\}$

$\checkmark \; \dim(\Theta) = 2 > \dim(\Theta_0) = 1$

$\checkmark \; \Theta \subset \mathbb{R}^2$

$$-2 \log(\lambda'(x)) \xrightarrow{d} \chi^2_k, \quad k = 1$$

Un EMV de θ dado $X=x$.

$$\hat{\theta} = \hat{\theta}(x) = \arg\max_{\theta \in \Theta} f_\theta(x).$$

- El EMV de $\theta = (\mu, \sigma^2)$ sobre todo Θ es

$$\hat{\theta} = (\hat{\mu}, \hat{\sigma}^2) = \left(\bar{X}, n^{-1} \sum_i (X_i - \bar{X})^2\right)$$

- Sobre $H_0: \mu = \mu_0$ y $\sigma^2 \in \mathbb{R}_+$, el EMV sobre Θ_0

$$\hat{\theta}_0 = (\hat{\mu}_0, \hat{\sigma}^2_0) = \left(\mu_0, n^{-1} \sum_i (X_i - \mu_0)^2\right)$$

$x = (x_1, \dots, x_n)$

$$\lambda'(x) = \frac{f_{\hat{\theta}_0}(x)}{f_{\hat{\theta}}(x)} = \frac{\left(\frac{1}{\sqrt{2\pi\hat{\sigma}_0^2}}\right)^n \cdot \exp\left\{-\frac{1}{2\hat{\sigma}_0^2}\sum_i (x_i - \mu_0)^2\right\}}{\left(\frac{1}{\sqrt{2\pi\hat{\sigma}^2}}\right)^n \exp\left\{-\frac{1}{2\hat{\sigma}^2}\sum_i (x_i - \bar{x})^2\right\}}$$

$$= \left(\frac{\hat{\sigma}^2}{\hat{\sigma}_0^2}\right)^n \exp\left\{-\frac{1}{2\hat{\sigma}_0^2}\sum_i (x_i - \mu_0)^2 + \frac{1}{2\hat{\sigma}^2}\sum (x_i - \bar{x})^2\right\}$$

$$= \left(\frac{\hat{\sigma}^2}{\hat{\sigma}_0^2}\right)^n e^{\alpha},$$

Notemos que:

$$\hat{\sigma}_0^2 = n^{-1}\sum_i (x_i - \mu_0)^2$$

$$\hat{\sigma}^2 = n^{-1}\sum_i (x_i - \bar{x})^2$$

$$\partial = -\frac{\sum (x_i - \mu_0)^2}{2 \underbrace{\frac{\sum (x_i - \mu_0)^2}{n}}_{\hat{\sigma}_0^2}} + \frac{\sum (x_i - \bar{x})^2}{2 \frac{\sum (x_i - \bar{x})^2}{n}} = -\frac{n}{2} + \frac{n}{2} = 0$$

$$\Rightarrow$$

$$\lambda'(x) = \left(\frac{\hat{\sigma}^2}{\hat{\sigma}_0^2} \right)^{n/2} \leq c_\alpha$$

Usando el resultado

$$\sum (x_i - \mu_0)^2 = \sum (x_i - \bar{x})^2 + n(\bar{x} - \mu_0)^2$$

$$\lambda'(x) = \left(\frac{\cancel{n^{-1}} \sum (x_i - \bar{x})^2}{\cancel{n^{-1}} \left[\sum (x_i - \bar{x})^2 + n(\bar{x} - \mu_0)^2 \right]} \right)^{n/2} \cdot \left(\frac{1}{\sum (x_i - \bar{x})^2} \right)^{n/2}$$

$$= \left(\frac{1}{1 + \frac{n(\bar{x} - \mu_0)^2}{\sum (x_i - \bar{x})^2}} \right)^{n/2}$$

$$= \left(\frac{1}{1 + \frac{(\bar{x} - \mu_0)^2}{S^2}} \right)^{n/2} \qquad S^2 = \frac{\sum (x_i - \bar{x})^2}{n}$$

T' (TRVG)

$$\lambda'(x) \leq c'_\alpha$$

$$\Leftrightarrow \quad \frac{1}{\lambda'(x)} \geq \underbrace{\frac{1}{c'_\alpha}}_{K_\alpha}$$

$$\Leftrightarrow \left(1 + \frac{n(\bar{x}-\mu_0)^2}{s^2}\right)^{n/2} \geq K_\alpha$$

...

$$\underbrace{\sqrt{n}\,\frac{|\bar{x}-\mu_0|}{s}}_{\psi(x)} \geq \underbrace{\left(K_\alpha^{-2/n} - 1\right)(n-1)}_{K'_\alpha}$$

$$P(\lambda'(x) \leq c_\alpha)$$

$$\|$$

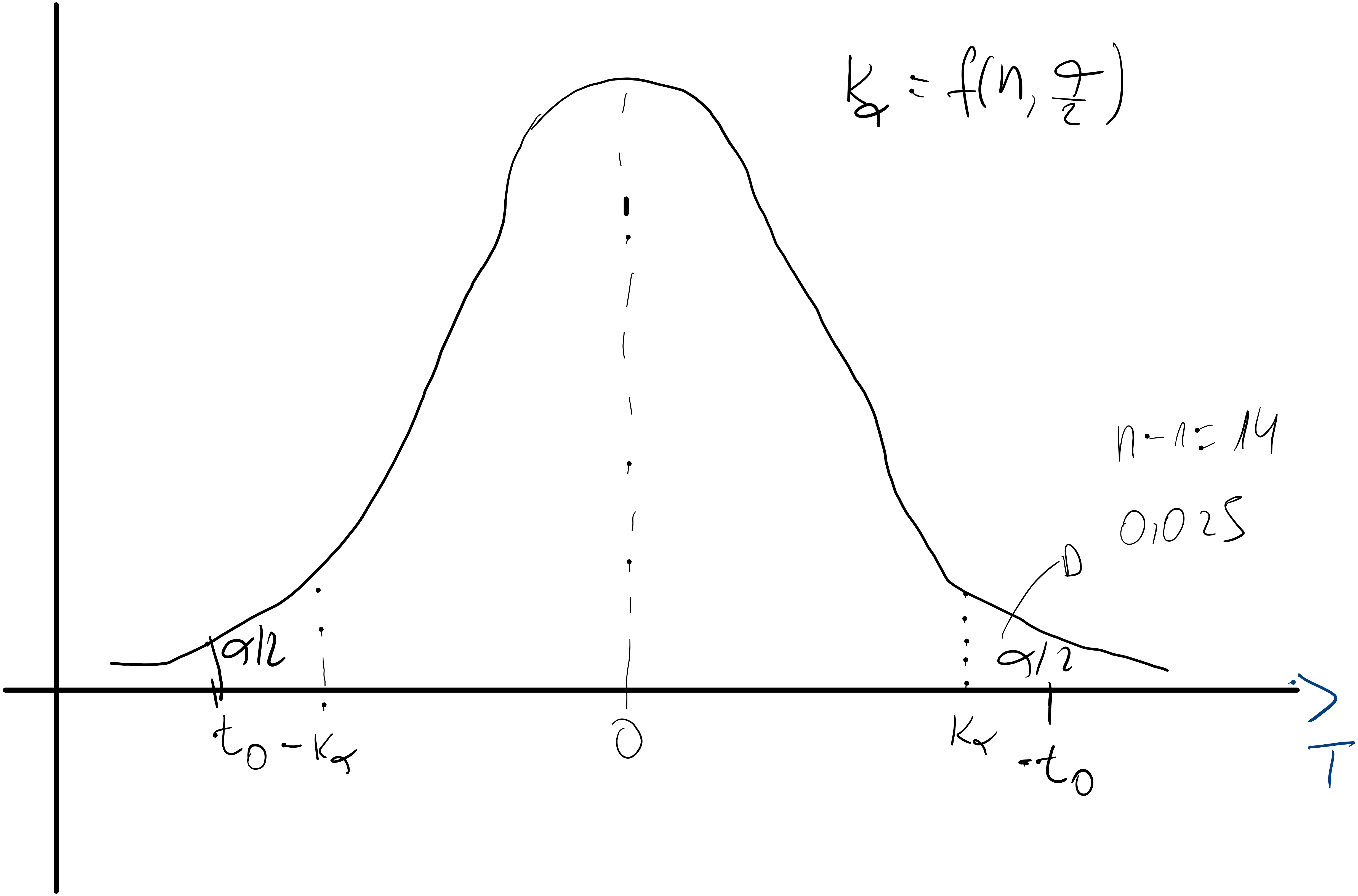
$$P\left(\sqrt{n}\,\frac{|\bar{x}-\mu_0|}{s} \geq K_\alpha\right)$$

$$T(x) = \begin{cases} 1, & \dfrac{\sqrt{n}\,|\bar{X} - \mu_0|}{S} \geq K'_\alpha \\ 0, & c.c \end{cases}$$

Se sabe que $\dfrac{\sqrt{n}(\bar{X} - \mu_0)}{S} \sim t_{n-1}$

Notemos que $|a| \geq b \iff a \leq -b$ or $a \geq b$

$$\frac{\sqrt{n}(\bar{X} - \mu_0)}{S} \leq -K'_\alpha \quad \text{or} \quad \frac{\sqrt{n}(\bar{X} - \mu_0)}{S} \geq K'_\alpha$$



$K_\alpha := f(n, \frac{\alpha}{2})$
$n-1 = 14$
0,025
$\alpha/2$
$\alpha/2$
D
t_0
$-K_\alpha$
0
K_α
$-t_0$
T

$$P\text{-valor}_{H_0} = P_{H_0}\left(\lambda(X) \geq \lambda(x_0) \right)$$

$$\vdots$$

$$= P\left(\frac{\sqrt{n}\,|\bar{X} - \mu_0|}{S} \geq \frac{\sqrt{n}\,|\bar{x} - \mu_0|}{s} \right)$$

$$= P\left\{ \frac{\sqrt{n}(\bar{X} - \mu_0)}{S} \leq -\frac{\sqrt{n}(\bar{x} - \mu_0)}{s} \right.$$

$$\left. \text{or } \frac{\sqrt{n}(\bar{X} - \mu)}{S} \geq \underbrace{\frac{\sqrt{n}(\bar{x} - \mu_0)}{s}}_{t_0} \right\}$$

$$= P\left(T_{n-1} \leq -t_0 \text{ or } T_{n-1} \geq t_0 \right)$$

Dimensiones de Θ y $B \subset \Theta$

Caso 1: $\Theta = \mathbb{R}$ y $H_0: \theta = \theta_0$ $(\theta \in \Theta_0 = \{\theta_0\})$

$\dim(\Theta) = 1$

$\xrightarrow{d} \chi^2_1.$

$\dim(\Theta_0) = 0.$

Caso 2: $\Theta = \mathbb{R} \times \mathbb{R}_+$ y $H_0: \mu = \mu_0$

$\Theta_0 = \{(\mu, \sigma^2) : \mu = \mu_0 \text{ y } \sigma^2 \in \mathbb{R}_+\}$ $(\mu, \sigma) \doteq \sigma^2$

$\dim(\Theta) = 2$

$\dim(\Theta_0) = 1$

Caso 3: MLG

$$E(Y_i) = \mu = \mu(\beta X_i)$$

$$\beta X_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2}, \quad \beta = (\beta_0, \beta_1, \beta_2)$$

$$\beta \in \Theta = \mathbb{R}^3$$

$$H_0: \beta_1 = 0 \qquad H_0: \beta_1 = 0 \wedge \beta_0, \beta_2 \in \mathbb{R}$$

$$H_1: \beta_1 \neq 0 \qquad H_1: \beta_1 \neq 0 \wedge \beta_0, \beta_2 \in \mathbb{R}.$$

$$\Theta_0 = \{(\beta_0, \beta_2) : \beta_0, \beta_2 \in \mathbb{R}\}, \quad \beta_1 = 0$$

$$\dim(\Theta) = 3 \qquad 3 - 2 = 1$$

$$\dim(\Theta_0) = 2 \qquad -2\log(\lambda'(x)) \xrightarrow{d} \chi^2_1$$