Bicolimit Presentations of Type Theories

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Overview

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Type theories = small categories with some structure
Semantics of $\mathcal{T}$ . . . functors out of $\mathcal{T}$ preserving the structure

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We take inspiration from universal algebra – to *present* an algebra, we can give its *presentation* (generators + relations). Categorically – colimit of free algebras.

Goal of this talk:
Show that we can construct examples via bicolimits of free type theories + explain how this interacts with semantics

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What needs to be captured?

Types (living in a context)

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Terms (living in a context and a type)

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Types (living in a context)

Terms (living in a context and a type)

Contexts (can be extended)

Natural Models of DTT

Definition (Representable Natural Transformation) [Algebraic geometers]

Let F, G be presheaves over a category $\mathcal{C}$. Then a natural transformation $\alpha\colon F \to G$ is called *representable* if, for every $\beta\colon よ c \to G$, the pullback

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F \arrow[d, "\alpha"] \\ よ c \arrow[r, "\beta"] & G \end{tikzcd}$$

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Definition (Natural Model) [Awodey/Fiore]

A *natural model* in a category $\mathcal{C}$ with a terminal object is a representable natural transformation $p\colon Tm \to Ty$.

THE SAME AS CwA

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the object representing the pullback of $A\colon よ\Gamma \to Ty$ along p is seen as the context extension $\Gamma.A$

Unit Types

A type theory has unit types if we have symbols $\mathbb{1}, \star$ together with the following rules:

$$\frac{}{\Gamma \vdash \mathbb{1}\ \mathit{Ty}}\ \mathbb{1}\text{-form} \qquad\qquad \frac{}{\Gamma \vdash \star : \mathbb{1}}\ \mathbb{1}\text{-intro}$$

$$\frac{\Gamma \vdash t : \mathbb{1}}{\Gamma \vdash t \equiv \star : \mathbb{1}.}\ \mathbb{1}\text{-}\eta$$

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Definition (Natural Models with Unit Types) [Folklore?]

A *natural model with unit types* is a natural model $p\colon \mathit{Tm} \to \mathit{Ty} \in \mathbf{Set}^{\mathcal{C}^{op}}$ together with maps $1 \xrightarrow{\mathbb{1}} \mathit{Ty}, 1 \xrightarrow{\star} \mathit{Tm}$ such that the following square is a pullback:

$$\begin{tikzcd} 1 \arrow[r, "\star"] \arrow[d, "id"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathit{Tm} \arrow[d, "p"] \\ 1 \arrow[r, "\mathbb{1}"] & \mathit{Ty}. \end{tikzcd}$$

Other Versions of DTT?

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Can we have a parametric definition of semantics?

First we need a definition of type theory!

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Categories with Representable Maps

Definition (Category with Representable maps) [Uemura]

A *category with representable maps (CwR)* is a category $\mathcal{C}$ with finite limits and a class of *representable* maps $R \subseteq \mathcal{C}^{\to}$ that are

- closed under compositions and contains every isomorphism;
- pullback-stable;
- exponentiable.

Maps in R will be denoted by $\rightarrowtriangle$.

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Main example: $\mathbf{Set}^{\mathcal{C}^{op}}$ with representable natural transformations.

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- its objects represent judgement forms (Ty, $Tm, \ldots$);
- arrows are derivations;
- limits are used to create more complicated judgements ($\Gamma \vdash J_1 \; \Gamma \vdash J_2$, empty judgement, $\ldots$);
- representable arrows are used to describe judgements that can appear in contexts and exponentials along those are used to bind variables (moving the judgements in contexts):

$$\frac{\Gamma \vdash A \; Ty \qquad \Gamma \vdash t : B}{\Gamma \vdash t : B} \, \pi_2 \qquad\qquad \frac{\Gamma, x : B \vdash A(x) \; Ty}{\Gamma \vdash B \; Ty.} \, \prod\nolimits_p(\pi_2)$$

Functorial Semantics

Definition (Model of a CwR) [Uemura]

A *model of a CwR* $\mathcal{T}$ consists of a category $\mathcal{C}$ with a terminal object and a CwR functor $M \colon \mathcal{T} \to \mathbf{Set}^{\mathcal{C}^{op}}$.

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What does it mean to freely generate? Is it always possible?

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2-Category of Type Theories

Definition (**Rep**)

We denote **Rep** the 2-category that has

- 0-cells. . . small CwRs;
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- 2-cells. . . natural transformations such that naturality square at a representable arrow is a pullback.

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Type theories can be glued!

Generators

Definition (Marked Category with Squares)

A *marked category with squares* is a category $\mathcal{C}$ equipped with a class of arrows $M \subseteq \mathcal{C}^{\rightarrow}$ and a clasee of commutative squares $S \subseteq Sq(\mathcal{C})$ such that any square whose domain and codomain are isos is in S, and both arrows and squares are closed under composition.

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Definition ($\mathbf{Cat}_m$)

We denote $\mathbf{Cat}_m$ the 2-category that has

- 0-cells... small marked categories with squares;
- 1-cells... functors preserving all the marking;
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Theorem ($\mathbf{Cat}_m$ is nice)

$\mathbf{Cat}_m$ is an accessible 2-category with all 2-limits and 2-colimits.

Free Generation

We have a forgetful 2-functor $U \colon \mathbf{Rep} \to \mathbf{Cat}_m$ sending $\mathcal{T}$ to $\mathcal{T}$ with representable maps and pullback squares.

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Theorem (U is nice)

U preserves directed colimits and flexible limits.

Corollary (U is nicer) [Bourke, Lack, Vokřínek]

U has a left biadjoint $F\colon \mathbf{Cat}_m \to \mathbf{Rep}$.

Interaction of Free Constructions and Semantics

We have a(n undefined) 2-functor $Mod \colon \mathbf{Rep}^{op} \to \mathbf{CAT}/\mathbf{CatT}$ that sends a type theory to its category of models (**CatT** are categories with a terminal object).

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Mod preserves all $(2,1)$-bilimits.

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We have also models of marked categories with squares:

Definition (Model of a Marked Category with Squares)

A *model of* $\mathcal{C} \in \mathbf{Cat}_m$ in a category $\mathcal{D}$ with a terminal object is a $\mathbf{CAT}_m$ functor $\mathcal{C} \to \mathbf{Set}^{\mathcal{D}^{op}}$ (where marked arrows are the representable natural transformations and squares are pullback squares).

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Theorem

For every $\mathcal{C} \in \mathbf{Cat}_m$, we have $Mod(F\mathcal{C}) \simeq Mod(\mathcal{C})$.

Examples of Type Theories I

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- Set $NM := F(Tm \xrightarrow{p} Ty)$, then $Mod(NM)$ is equivalent to natural models.
- Models of the following bipushout in **Rep**

$$\begin{tikzcd}
F(a \quad b \to c) \arrow[r, "{(1, p)}"] \arrow[d] & NM \arrow[d] \\
F(a \to b \to c) \arrow[r] \arrow[ur, phantom, "\ulcorner", near end] & NM_{\mathbb{1},\star}
\end{tikzcd}$$

are natural models with two maps $\star\colon 1 \to Tm$ and $\colon 1 \to Ty$ such that $p\star = \mathbb{1}$.

Examples of Type Theories II

- Let $\mathcal{C} \in \mathbf{Cat}_m$ be the free commutative square and $\mathcal{D}$ the free marked commutative square. Then models of the following bipushout in **Rep**

$$
\begin{tikzcd}
F\mathcal{C} \arrow[r, "f"] \arrow[d] & NM_{\mathbb{1},\star} \arrow[d] \\
F\mathcal{D} \arrow[r] & NM_{\mathbb{1},\star,\eta} \arrow[ul, phantom, "\ulcorner", very near start]
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where f is the map choosing the square $\begin{tikzcd} 1 \arrow[r, "\star"] \arrow[d, "id"] & Tm \arrow[d, "p"] \\ 1 \arrow[r, "\mathbb{1}"] & Ty \end{tikzcd}$, are natural models with unit types.

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Thank you for your attention!