Bicolimit Presentations of Type Theories

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Universal Algebra

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- excludes predicates from signatures;
- formulas = universally quantified equations of terms.

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Dihedral groups $D_n = \langle r, s \mid s^2 = 1, r^n = 1, srs = r^{-1} \rangle$
Categorically, we form a colimit (coequalizer):

$$\begin{tikzcd} F\{e_1, e_2, e_3\} \arrow[r, shift left, "{e_1,e_2,e_3 \mapsto 1,1,r^{-1}}"] \arrow[r, shift right, "{e_1,e_2,e_3 \mapsto s^2,r^n,srs}"'] & F\{s, r\} \arrow[r] & D_n \end{tikzcd}$$

Overview

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Models of $\mathcal{T}$... functors out of $\mathcal{T}$ preserving the structure

Pros = quite simple definition, functors have many nice properties
Cons = it is quite hard to present concrete examples of theories

We take inspiration from universal algebra and *present* examples of type theories giving their *presentation*.

Goal of this talk:
Show that we can construct examples of type theories via bicolimits of free type theories + explain how this interacts with semantics

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What needs to be captured?

Types (living in a context) $\Gamma \vdash A$ Type

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Contexts (can be extended) $\Gamma \vdash$

Natural Models of DTT

Definition (Representable Natural Transformation)

Let F, G be presheaves $\mathcal{C}^{op} \to \mathbf{Set}$. Then a natural transformation $\alpha \colon F \to G$ is called *representable* if, for every $\beta \colon よc \to G$, the pullback

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & F \arrow[d, "\alpha"] \\ よc \arrow[r, "\beta"] & G \end{tikzcd}$$

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is a representable presheaf.

Definition (Natural Model) [Awo18]

A *natural model* in a category $\mathcal{C}$ with a terminal object is a representable natural transformation $p\colon Tm \to Ty$.

The same as CwA, CwF.

Explanation of Natural Models

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the object representing the pullback of $A\colon よ\Gamma \to Ty$ along p is seen as the context extension $\Gamma.A$

Unit Types

A type theory has unit types if we have symbols $\mathbb{1}, \star$ together with the following rules:

$$\frac{}{\Gamma \vdash \mathbb{1}\ Ty}\ \mathbb{1}\text{-form} \qquad \frac{}{\Gamma \vdash \star : \mathbb{1}}\ \mathbb{1}\text{-intro}$$

$$\frac{\Gamma \vdash t : \mathbb{1}}{\Gamma \vdash t \equiv \star : \mathbb{1}.}\ \mathbb{1}\text{-}\eta$$

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Definition (Natural Models with Unit Types) [AN18]

A *natural model with unit types* is a natural model $p\colon Tm \to Ty \in \mathbf{Set}^{\mathcal{C}^{op}}$ together with maps $1 \xrightarrow{\mathbb{1}} Ty, 1 \xrightarrow{\star} Tm$ such that the following square is a pullback:

$$\begin{tikzcd} 1 \arrow[r, "\star"] \arrow[d, "id"'] \arrow[dr, phantom, "\lrcorner", very near start] & Tm \arrow[d, "p"] \\ 1 \arrow[r, "\mathbb{1}"] & Ty. \end{tikzcd}$$

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Can we have a parametric definition of semantics?

First we need a definition of type theory!

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Categories with Representable Maps

Definition (Category with Representable maps) [Uem21]

A *category with representable maps (CwR)* is a category $\mathcal{C}$ with finite limits and a class of *representable* maps $R \subseteq \mathcal{C}^{\rightarrow}$ that

- is closed under compositions and contains every isomorphism;
- is pullback-stable;
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Main example: $\mathbf{Set}^{\mathcal{C}^{op}}$ with representable natural transformations.

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- its objects represent judgement forms (Ty, $Tm, \ldots$);
- arrows are derivations;
- limits are used to create more complicated judgements ($\Gamma \vdash J_1\ \Gamma \vdash J_2$, empty judgement, $\ldots$);
- representable arrows are used to describe judgements that can appear in contexts and exponentials along those are used to bind variables (moving the judgements in contexts).

Functorial Semantics

Definition (Model of a CwR) [Uem21]

A *model of a CwR* $\mathcal{T}$ consists of a category $\mathcal{C}$ with a terminal object and a CwR functor $M\colon \mathcal{T} \to \mathbf{Set}^{\mathcal{C}^{op}}$.

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'Examples'

- Let NM be the CwR that is freely generated by $Tm \rightharpoonup Ty$. Then models of NM are natural models.

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- Let $NM_{\mathbb{1},\star,\eta}$ be the CwR freely generated by

$$\begin{tikzcd} 1 \arrow[r, "\star"] \arrow[d, "id"'] \arrow[dr, phantom, "\lrcorner", very near start] & Tm \arrow[d, "p"] \\ 1 \arrow[r, "\mathbb{1}"] & Ty. \end{tikzcd}$$

Then its models are natural models with unit types.

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What does it mean to freely generate? Is it always possible?

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2-Category of Type Theories

Definition (Rep) [Uemura in a private conversation]

We denote **Rep** the 2-category that has

- 0-cells. . . small CwRs;
- 1-cells. . . functors preserving all the CwR structure;
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Theorem (Type theories can be glued) [Bourke & J.]

Rep has all bicolimits.

Generators

Definition (Marked Category with Squares) [Bourke & J.]

A *marked category with squares* is a category $\mathcal{C}$ equipped with a class of arrows $M \subseteq \mathcal{C}^{\rightarrow}$ and a class of commutative squares $S \subseteq Sq(\mathcal{C})$ such that any square whose domain and codomain are isos is in S, and both arrows and squares are closed under composition.

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Definition ($\mathbf{Cat}_m$) [Bourke & J.]

We denote $\mathbf{Cat}_m$ the 2-category that has

- 0-cells. . . small marked categories with squares;
- 1-cells. . . functors preserving all the marking;
- 2-cells. . . natural transformations such that naturality square at a marked arrow is marked.

Free Generation

We have a forgetful 2-functor $U\colon \mathbf{Rep} \to \mathbf{Cat}_m$ sending $\mathcal{T}$ to $\mathcal{T}$ with representable maps and pullback squares.

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Theorem (Free generation is possible) [Bourke & J.]

U has a left biadjoint $F\colon \mathbf{Cat}_m \to \mathbf{Rep}$.

Interaction of Free Constructions and Semantics

We have a 2-functor $Mod\colon \mathbf{Rep}^{op} \to \mathbf{CAT}$ that sends a type theory to its category of models.

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Mod preserves all bipullbacks.

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We have also models of marked categories with squares:

Definition (Model of a Marked Category with Squares)

A *model of* $\mathcal{C} \in \mathbf{Cat}_m$ in a category $\mathcal{D}$ with a terminal object is a $\mathbf{CAT}_m$ functor $\mathcal{C} \to \mathbf{Set}^{\mathcal{D}^{op}}$ (where marked arrows are the representable natural transformations and squares are pullback squares).

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Theorem [Bourke & J.]

For every $\mathcal{C} \in \mathbf{Cat}_m$, we have $Mod(F\mathcal{C}) \simeq Mod(\mathcal{C})$.

Examples of Type Theories I

- Set $NM := F(Tm \overset{p}{\rightarrowtriangle} Ty)$, then $Mod(NM)$ is equivalent to natural models.

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- Set $NM := F(Tm \overset{p}{\rightarrowtriangle} Ty)$, then $Mod(NM)$ is equivalent to natural models.
- Models of the following bipushout in **Rep**

$$\begin{tikzcd}
F(a \quad b \xrightarrow{\alpha} c) \arrow[r, "(a \mapsto 1,\, \alpha \mapsto p)"] \arrow[d] & NM \arrow[d] \\
F(a \to b \to c) \arrow[r] & NM_{\mathbb{1},\star} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

are natural models with two maps $\star\colon 1 \to Tm$ and $\mathbb{1}\colon 1 \to Ty$ such that $p\star = \mathbb{1}$.

Examples of Type Theories II

- Let $\mathcal{C} \in \mathbf{Cat}_m$ be the free commutative square and $\mathcal{D}$ the free marked commutative square. Then models of the following bipushout in **Rep**

$$\begin{tikzcd} F\mathcal{C} \arrow[r, "f"] \arrow[d] & NM_{\mathbb{1},\star} \arrow[d] \\ F\mathcal{D} \arrow[r] & NM_{\mathbb{1},\star,\eta} \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd},$$

where f is the map choosing the square $\begin{tikzcd} 1 \arrow[r, "\star"] \arrow[d, "id"] & Tm \arrow[d, "p"] \\ 1 \arrow[r, "\mathbb{1}"] & Ty \end{tikzcd}$, are natural models with unit types.

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Thank you for your attention!

References

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