MATH 240 - Assignment 2

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October 2015

(I worked with Ellen Chen)

1. Proof techniques
Solution 1:

(a) If x is an integer then $x^3 - x$ is divisible by 3.

Proof. (direct proof)

$$x^3 - x = x(x^2 - 1) = x(x+1)(x-1)$$
$$(x-1) \cdot x \cdot (x+1)$$

where x is $\in \mathbb{Z}$

$(x-1)$, x and $(x+1)$ are 3 consecutive integers. Hence, given that range of consecutive integers, there must be a factor that is divisible by 3.

eg) $\{1, 2, 3 \cdot 1\}$; $\{7, 8, 3 \cdot 3\}$ ...$\{x-1, x, x+1\}$

Thus, $x^3 - x$ is divisible by 3. □

(b) Let x and y be real numbers. If the product $x \cdot y$ is irrational then either x or y is an irrational number.

Proof. (contrapositive proof)

$\overline{p} \to \overline{q}$
If x and y are rational numbers, then the product $x \cdot y$ yields a rational number.

Assume x and y are rational numbers :

$$x = \frac{p_1}{q_1} \qquad y = \frac{p_2}{q_2}$$

where x $\in \mathbb{Q}$ and p_1, p_2, q_1, $q_2 \in \mathbb{Z}$ where $q_1 > 0$, $q_2 > 0$

$$x \cdot y = \frac{p_1}{q_1} \cdot \frac{p_2}{q_2} = \frac{p_1 \cdot p_2}{q_1 \cdot q_2}$$

where $p_1 \cdot p_2 \in \mathbb{Z}$ and $p_1 \cdot q_2 \in \mathbb{Z} > 0$

As the multiplication of two integers also yields an integer, it must follow that the numerator and denominator are both $\in \mathbb{Z}$. Hence, the result is a rational number as well. $\square$

2. Proofs by contradiction

Solution 1:

(a) There are no integers x and y such that $6x + 14y = 1$

Proof. (proof by contradiction)

Assuming that x, $y \in \mathbb{Z}$

$$6x + 14y = 1$$
$$2(3x + 7y) = 1$$
$$(2x + 7y) = c$$

for $c \in \mathbb{Z}$ as $x, y \in \mathbb{Z}$

$$\text{it follows that } 2 \cdot c = 1$$

which is a contradiction as a odd integer can't be even at the same time. $\square$

(b) Let x, y, $z \in \mathbb{Z}$. If $x^2 + y^2 = z^2$ then either x or y is even.

Proof. (proof by contradiction)

Assuming that x and y are both odd.

$$x = 2a + 1 \qquad y = 2b + 1$$

for a,$b \in \mathbb{Z}$

As x, y are both odd, it follows that x^2 and y^2 are also odd. Therefore, $x^2 + y^2$ is an even number being the sum of two odd numbers.

$$z^2 = x^2 + y^2$$

here z^2 is even

$$z^2 = (2k)^2 = 2 \cdot (2k^2) = x^2 + y^2$$
$$2(2k^2) = 2(2a^2 + 2b^2 + 2a + 2b + 1)$$

but $2a^2 + 2b^2 + 2a + 2b + 1$ is clearly odd

As $x^2+y^2 = z^2$ was shown to be both even and odd, this is a contradiction. $\square$

3. Proofs by induction
Solution 1:

(a) Prove by induction that $(1-\frac{1}{2^2})\cdot(1-\frac{1}{3^2})\cdot(1-\frac{1}{4^2})\cdots(1-\frac{1}{n^2})=\frac{n+1}{2n}$

Proof. (by induction)

$$\forall n \in \mathbb{Z} \geq 2$$

Base Case: $n = 2$

$$1-\frac{1}{2^2}=\frac{3}{4}$$

Assuming the following for $n \geq 2$:

$$\frac{n+1}{2n}=(1-\frac{1}{2^2})\cdot(1-\frac{1}{3^2})\cdots(1-\frac{1}{n^2})$$

Induction step: Proving for $n+1$

$$(1-\frac{1}{2^2})\cdot(1-\frac{1}{3^2})\cdots(1-\frac{1}{n^2})\cdot(1-\frac{1}{(n+1)^2})=\frac{(n+1)+1}{2(n+1)}$$

We know that

$$\frac{n+1}{2n}=(1-\frac{1}{2^2})\cdot(1-\frac{1}{3^2})\cdots(1-\frac{1}{n^2})$$

(from the induction hypothesis)

$$(1-\frac{1}{2^2})\cdot(1-\frac{1}{3^2})\cdots(1-\frac{1}{n^2})\cdot(1-\frac{1}{(n+1)^2})=(\frac{n+2}{2n+2})$$

$$(\frac{n+1}{2n})\cdot(1-\frac{1}{(n+1)^2})=(\frac{n+2}{2n+2})$$

$$\frac{n+1}{2n}\cdot(\frac{n^2+1+2n-1}{(n+1)^2})=(\frac{n+2}{2n+2})$$

$$\frac{n+1}{2n}\cdot(\frac{n^2+2n}{(n+1)^2})=(\frac{n+2}{2n+2})$$

$$(\frac{n+2}{2(n+1)})=(\frac{n+2}{2n+2})$$

□

(b) Prove by induction that, for any sets A_1, A_2, ..., A_n, De Morgan's Law generalises to : $\overline{A_1 \cup A_2 \cup ... \cup A_n}=\overline{A_1}\cap\overline{A_2}\cap ... \cap\overline{A_n}$

Proof. (by induction)

Base Case: For $n = 1$, it is trivial to show that:

$$\overline{A_1} = \overline{A_1}$$

For $n = 2$, he have that

$$\overline{A_1 \cup A_2} = \overline{A_1} \cap \overline{A_2}$$

which is the application of DeMorgan's law directly where $A = A_1$ and $B = A_2$.

Assuming that the assumption is true for $n \geq 1$ and for every sets $\{A_1, A_2, ..., A_n\}$.

Induction Step: Prove that the assumption still holds for $n + 1$ and for every set $\{A_1, A_2, ..., A_n, A_{n+1}\}$.

$$\overline{(A_1 \cup ... \cup A_{n+1})} = \overline{(A_1 \cup ... \cup A_n) \cup A_{n+1}}$$

which the application of DeMorgan's law identity with $A = A_1 \cup ... \cup A_n$ and $B = A_{n+1}$.

$$= \overline{(A_1 \cup ... \cup A_n)} \cap \overline{A_{n+1}}$$

$$= (\overline{A_1} \cap ... \cap \overline{A_n}) \cap \overline{A_{n+1}}$$

(by the induction hypothesis for $A_1, ..., A_n$)

$$= \overline{A_1} \cap ... \cap \overline{A_n} \cap \overline{A_{n+1}}$$

Thus, the assumption also holds for $n + 1$ for any sets. Hence, the proof is complete. □

4. Predicate Calculus

Solution 1:

(a) What is the negation of $\forall\ n \in \mathbb{N}$ (the remainder when n^2 is divided by 4 is either 0 or 1)

Negation : $\exists\ n \in \mathbb{N}$ (the remainder when n^2 is divised by 4 is different than 0 and 1)

(b) Let's consider the following :

$$\frac{n^2}{4} = (\frac{n}{2})^2$$

$\frac{n}{2}$ would either have a remainder of 0 or 1 if its even or not. Therefore, the original statement is true.

5. More predicate calculus
Solution 1:

(a) What is the negation of $\forall$ odd $m \in \mathbb{N}$ $\exists$ $n \in \mathbb{N}(m = n^2 - (n-1)^2)$

Negation : $\exists$ odd $m \in \mathbb{N}$ $\forall n \in \mathbb{N}(m = n^2 - (n-1)^2$

(b) Let m be odd.

$$\begin{aligned} m &= n^2 - (n-1)^2 \\ &= 2n - 1 \end{aligned}$$

which is obviously a odd number where there $\exists$ n such that it satisfies the equation.

This statement is true for all m. Hence, the original statement is true.

6. Axiomatic Systems

(a) Any two lines intersect in exactly one point.

By (A2) : if a line is defined using two distinct points, it must be a straight (linear) line.
By (A1) : if there are two straight line that are not parallel each one and other, they must intersect.

Therefore, it follows that every two lines intersect at a distinct point.

(b) There is at least one point.

By (A1) : there is at last always one line.
By (A2) : a line is defined using two distinct points.

Therefore, it follows that there must be a minimum of 2 points (a line) at each time for this axiomatic system to hold.

(c) No point lies on every line.

Assuming there is such point that lies on every line. By (A4) : It is possible to consider every such point as an intersection with an other line.
By (A2) : With these points, it is only possible to consider straight lines that intersect with a certain line.

Hence, it would be impossible to describe lines that are parallels if some points were to be on every other lines.