EINDHOVEN UNIVERSITY OF TECHNOLOGY

5LMB0 - MODEL PREDICTIVE CONTROL

Assignment 2 Report

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Preliminaries

A discrete-time state-space model of a buck converter system, sampled at $T_s = 2.5\ \mu s$, is given in the assignment as

$$x(k+1) = Ax(k) + Bu(k), \quad A = \begin{bmatrix} 0.9873 & 0.1124 \\ -0.0247 & 0.9886 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0282 \\ 0.4973 \end{bmatrix}, \tag{1}$$

where the state vector $x(k)$ consists of the capacitor voltage v_C and the inductor current i_L, respectively, and the input $u(k)$ is the transistor duty-cycle. The sets of constraints are defined as

$$\mathbb{X} := \{x \in \mathbb{R}^2 \ : \ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \leq x \leq \begin{bmatrix} 15 \\ 3 \end{bmatrix}\}, \tag{2a}$$

$$\mathbb{U} := \{u \in \mathbb{R} \ : \ 0 \leq u \leq 0.95\}. \tag{2b}$$

The steady-state to be reached is $x_{ss} = [10 \quad 1]^\top$ and $u_{ss} = 0.52$. Instead of controlling the state and input to these steady-state values, a shifted system is controlled to the origin in Task 1. The shifted system, with the shifted state $\bar{x}(k) := x(k) - x_{ss}$ and the shifted input $\bar{u}(k) := u(k) - u_{ss}$, has the same state-space matrices A and B as the original system. Moreover, the constraint set $(\bar{\mathbb{X}}, \bar{\mathbb{U}})$ of the shifted system is equal to that of the original system, but with the respective steady-state values subtracted. Therefore, the upper and lower bound on $\bar{x}(k)$ are $[5 \quad 2]^\top$ and $[-10 \quad -1]^\top$ respectively, and the upper and lower bound on $\bar{u}(k)$ are -0.43 and -0.52 respectively.

1 Task 1

1.1 Terminal cost weight and control law

Given is the following cost function for the shifted system

$$J(\bar{x}(k), \bar{U}_k) = \bar{x}_{N|k}^\top P \bar{x}_{N|k} + \sum_{i=0}^{N-1} \left(\bar{x}_{i|k}^\top Q \bar{x}_{i|k} + \bar{u}_{i|k}^\top R \bar{u}_{i|k} \right), \tag{3}$$

for which the prediction horizon $N = 10$, and cost matrices $Q = I \in \mathbb{R}^{2\times 2}$ and $R = 0.1$ are given. The terminal cost matrix P and the local control law K of the terminal set have to be computed, such that they are able to guarantee asymptotic stability of the closed-loop MPC system. According to slide set 4 of the course material, the origin of the closed-loop MPC system is asymptotically stable if Q and R are positive definite, and if P is positive definite and satisfies

$$(A + BK)^\top P (A + BK) - P \preceq -Q - K^\top R K, \tag{4}$$

where $K \in \mathbb{R}^{1\times 2}$ for which the spectral radius $\rho(A + BK) < 1$.

There are several ways to compute P and K. The chosen approach in this report, is solving a linear matrix inequality (LMI) with SeDuMi 1.3 and YALMIP as an interface. To rewrite (4) into a matrix of linear inequalities, the steps shown on slide 16 and 17 of set 4, are performed. Pre- and post-multiplying (4) with P^{-1} gives:

$$P^{-1}(A + BK)^\top P (A + BK) P^{-1} - P^{-1} P P^{-1} \preceq -P^{-1} Q P^{-1} - P^{-1} K^\top R K P^{-1}. \tag{5}$$

By constructing P as a symmetric positive definite matrix, the inverse matrix P^{-1} will also be symmetric positive definite. Therefore, the equation can be rearranged as follows

$$P^{-1} - (AP^{-1} + BKP^{-1})^\top P (AP^{-1} + BKP^{-1}) - P^{-1} Q P^{-1} - (KP^{-1})^\top R (KP^{-1}) \succeq 0. \tag{6}$$

Defining $O = P^{-1}$ and $Y = KP^{-1}$ allows (6) to be rewritten as follows

$$O - (AO + BY)^\top O^{-1} (AO + BY) - OQO - Y^\top R Y \succeq 0. \tag{7}$$

Rearranging this equation to a block matrix form results in

$$O - \begin{bmatrix}(AO+BY)^\top & O & Y^\top\end{bmatrix} \begin{bmatrix} O^{-1} & 0_{2\times2} & 0_{2\times1} \\ 0_{2\times2} & Q & 0_{2\times1} \\ 0_{1\times2} & 0_{1\times2} & R \end{bmatrix} \begin{bmatrix}(AO+BY) & O & Y\end{bmatrix} \succeq 0. \tag{8}$$

Finally Schur complement can be applied, this yields the following linear matrix inequality

$$\begin{bmatrix} O & (AO+BY)^\top & O & Y^\top \\ (AO+BY) & O & 0_{2\times2} & 0_{2\times1} \\ O & 0_{2\times2} & Q^{-1} & 0_{2\times1} \\ Y & 0_{1\times2} & 0_{1\times2} & R^{-1} \end{bmatrix} \succeq 0. \tag{9}$$

Solving the LMI with the SeDuMi 1.3 solver and YALMIP as an interface results in the following P and K values

$$P = \begin{bmatrix} 11.2043 & 1.1474 \\ 1.1474 & 1.7712 \end{bmatrix}, \quad K = \begin{bmatrix} -1.3749 & -1.6731 \end{bmatrix}.$$

From this can be concluded that (4) holds, for P is indeed a symmetric positive definite matrix, and the spectral radius $\rho(A+BK) = 0.8851 < 1$, i.e. all closed loop poles are within the unit circle and therefore stable.

1.2 Terminal set computation

Constraint matrices M, E and b for the MPC problem are constructed as was shown in the first course slide set. The terminal constraint matrices M_N and b_N are determined by the terminal set $\bar{\mathbb{X}}_T \subseteq \bar{\mathbb{X}}$, in order to guarantee recursive feasibility of the MPC problem. Slide set 5 of the course material shows that the terminal constraint set

$$\bar{\mathbb{X}}_T = \{\bar{x} \in \mathbb{R}^2 \ : \ M_N \bar{x} \leq b_N\}, \tag{10}$$

is constructed such that, for all states $\bar{x} \in \bar{\mathbb{X}}_T$ it holds that

$$\begin{aligned} (M + EK)\bar{x} &\leq b, \\ M_N(A+BK)\bar{x} &\leq b_N. \end{aligned}$$

Therefore, the terminal constraint set is both constraint admissible, due to $\bar{x} \in \bar{\mathbb{X}}_T$ implying $K\bar{x} \in \bar{\mathbb{U}}$, and invariant, due to $\bar{x} \in \bar{\mathbb{X}}_T$ implying $(A+BK)\bar{x} \in \bar{\mathbb{X}}_T$.

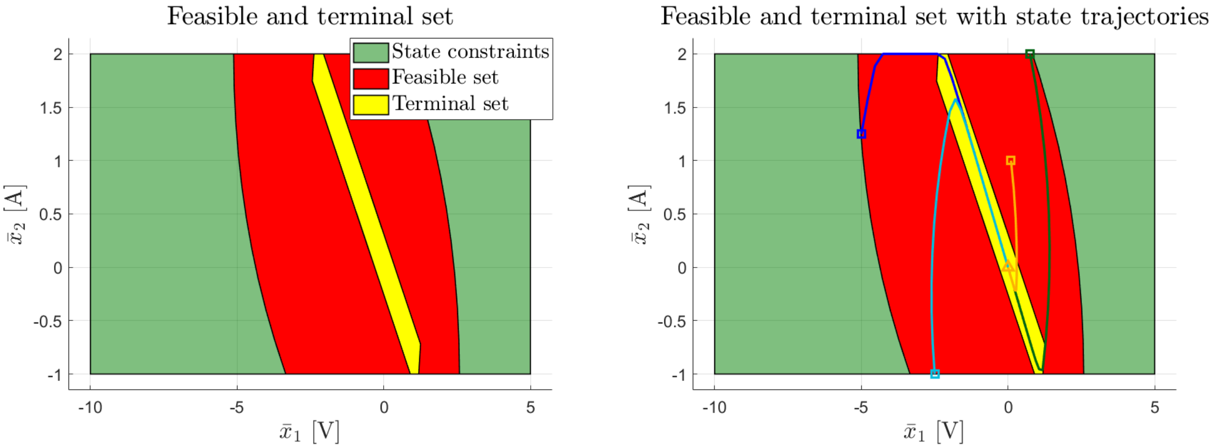
The set and the constraint matrices M_N and b_N are computed in MATLAB with help of the MPT3 toolbox. The following matrices are obtained

$$M_N = \begin{bmatrix} 0.7084 & -0.1566 \\ -0.7084 & 0.1566 \\ 1.3749 & 1.6731 \\ -1.3749 & -1.6731 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}, \quad b_N = \begin{bmatrix} 1 \\ 2 \\ 0.5200 \\ 0.4300 \\ 1 \\ 2 \end{bmatrix}.$$

Figure 1a depicts the terminal set and the feasible set of states for the given value of $N = 10$, in addition to the set of state constraints.

1.3 Simulation trajectories for the shifted system

Now that the terminal cost matrix P and the terminal constraint matrices M_N and b_N are known, the shifted buck converter system can be simulated. The MPC problem is solved in MATLAB following the methods depicted in the first course slide set. The system is simulated for several initial conditions, however these conditions have to be chosen carefully. Choosing initial conditions outside of the feasible set depicted in Figure 1a leads to an infeasible optimization problem. Simulation results for four different initial conditions are shown in Figure 2. All of the simulations manage to converge to zero. From these plots should be noted that all of the states stay strictly within the previously calculated feasible set. This can be seen clearly by looking at Figure 1b, where all of the four trajectories are visualized.



(a) *(b)*

Figure 1: (a) Terminal set and feasible set of states for $N = 10$. (b) Visualization of state trajectories.

2 Task 2

2.1 Adaptations on the MPC algorithm.

In Task 1, the next state at time step $k+1$, would be calculated as $\bar{x}(k+1) = A\bar{x}(k) + B\bar{u}(k)$. In simulation this is possible, however in reality, the next state would be measured instead. Since the measured state for the MPC controlled system would be the unshifted state $x(k+1)$, and the optimal input sequence $\bar{U}_k$ is computed for the shifted system, conversions have to be made. The actual system input $u(k)$ has to be calculated by adding the steady-state input u_{ss} to the shifted optimal input $\bar{u}(k)$. After applying the input and measuring the actual system state $x(k+1)$, the shifted state $\bar{x}(k+1)$ has to be computed by subtracting the steady-state vector x_{ss}. Then $\bar{x}(k+1)$ can be used in the next MPC loop. The loop structure is shown in Algorithm 1.

2.2 Simulation trajectories for the original system.

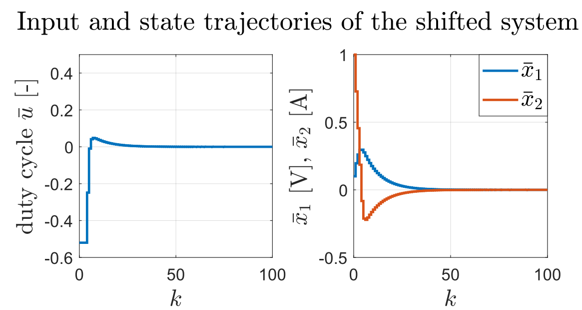
The original system is simulated for initial conditions $x(0) = \bar{x}(0) + x_{ss}$, where $\bar{x}(0)$ are the four different initial conditions given in Figure 2. However, one modification has to be made. The given steady-state vector $x_{ss} = [10 \quad 1]^\top$ is not accurate enough. Therefore $x_{ss} \neq Ax_{ss} + Bu_{ss}$, which results in numerical errors, rendering the modified MPC algorithm infeasible for some initial conditions. The actual steady-state vector x_{ss} is calculated as

$$x_{ss} = (I_{2\times 2} - A)^{-1} B u_{ss} = \begin{bmatrix} 10.00779169 \\ 1.000310982 \end{bmatrix}.$$

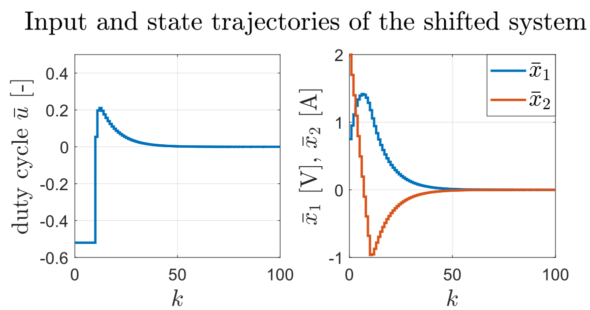
The resulting simulation plots, for the modified MPC algorithm, can be seen in Figure 3. As expected, the inputs and states indeed manage to converge to the steady-state values, whilst staying within their respective constraints.

Algorithm 1 Modified MPC algorithm

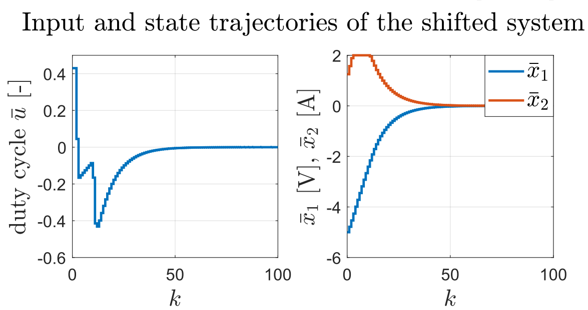
for k = 0 to ksim **do**
  $\bar{U}_k \leftarrow$ **quadprog**$(G, F\bar{x}(k), L, c + W\bar{x}(k))$
  $\bar{u}(k) = \bar{U}_k(1)$
  $u(k) = \bar{u}(k) + u_{ss}$
  $x(k+1) = Ax(k) + Bu(k)$
  $\bar{x}(k+1) = x(k+1) - x_{ss}$
end for



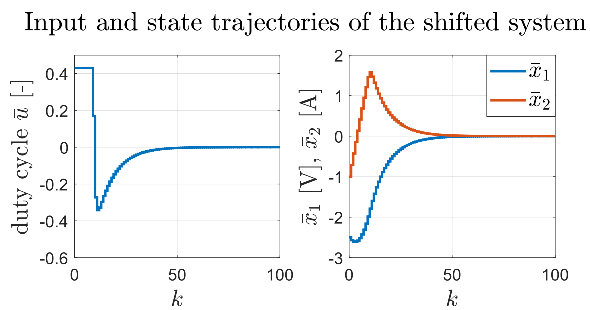
(a) For the provided initial condition $\bar{x}(0) = \begin{bmatrix} 0.1 & 1 \end{bmatrix}^\top$.



(b) For the initial condition $\bar{x}(0) = \begin{bmatrix} 0.75 & 2 \end{bmatrix}^\top$.

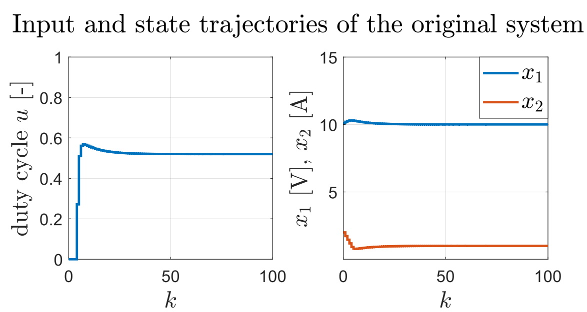


(c) For the initial condition $\bar{x}(0) = \begin{bmatrix} -5 & 1.25 \end{bmatrix}^\top$.

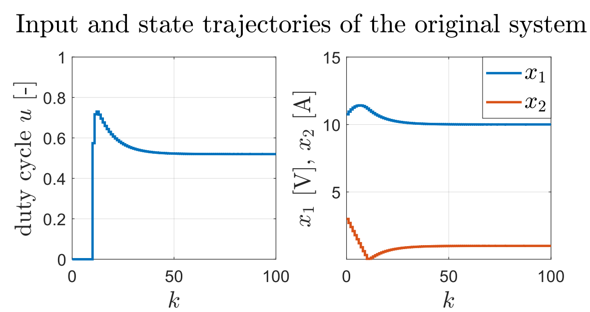


(d) For the initial condition $\bar{x}(0) = \begin{bmatrix} -2.5 & -1 \end{bmatrix}^\top$.

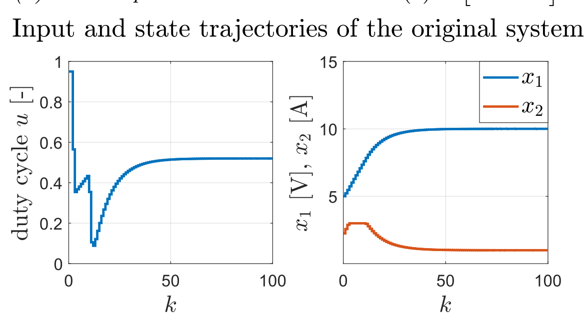
Figure 2: Simulations of the shifted system for several initial conditions.



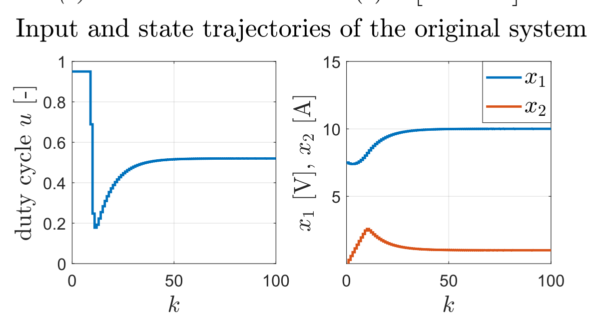
(a) For the provided initial condition $x(0) = \begin{bmatrix} 10.1 & 2 \end{bmatrix}^\top$.



(b) For the initial condition $x(0) = \begin{bmatrix} 10.75 & 3 \end{bmatrix}^\top$.



(c) For the initial condition $x(0) = \begin{bmatrix} 5 & 2.25 \end{bmatrix}^\top$.



(d) For the initial condition $x(0) = \begin{bmatrix} 7.5 & 0 \end{bmatrix}^\top$.

Figure 3: Simulations of the original and shifted system for several initial conditions.