Set Cardinality

1. Prove that the set of rational numbers $\mathbb{Q} = \frac{a}{b}$ where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$ is countable. *Hint: find a bijection between* $\mathbb{Z} \times \mathbb{Z}^+$ *and* $\mathbb{N}$

2. Use a proof by diagonalization to show that the following set is uncountable:

   $F = \{f : \mathbb{N} \rightarrow \mathbb{N} | (a > b) \rightarrow (f(a) > f(b))\}$

   In other words, this is the seet of all strictly increasing functions that map natural numbers to natural numbers. A function is strictly increasing if larger inputs are guaranteed to produce larger outputs.

   For example, $f(x) = x^2$ is strictly increasing since if $a > b$, then $a^2 > b^2$. However, $f(x) = (x-5)^2$ is not strictly increasing since $1 < 2$ but $f(1) > f(2)$.

3. Prove that every subset of the natural numbers $\mathbb{N}$ is countable.

4. Prove the following claim: If A is a countably infinite set (i.e., $|A| = |\mathbb{N}|$) and B is a finite set (i.e., $|B| = n | n \in \mathbb{N}$), then $A \cup B$ is also countable.

5. Prove the following claim: If A is a countably infinite set (i.e., $|A| = |\mathbb{N}|$) and B is a also a countably infinite set (i.e., $|B| = |\mathbb{N}|$), then $A \cup B$ is also countable.