Set Cardinality

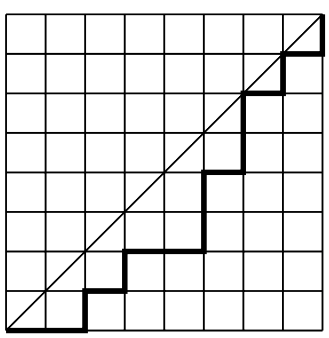
1. For each of the following claims, state whether it is true or false and then prove your assertion.

   - All finite sets have an *injection* to $\mathbb{N}$
   - All finite sets have a *surjection* to $\mathbb{N}$
   - If A is a countably infinite set (i.e., $|A| = |\mathbb{N}|$) and B is a also a countably infinite set (i.e., $|B| = |\mathbb{N}|$), then $A \cup B$ is also countable.
   - If A is countably infinite and B is uncountably infinite, then $A \cup B$ is countable.
   - If A is countably infinite and B is uncountably infinite, then $A \cap B$ is countable.

2. Consider a square grid with length and width n. The bottom left corner is considered position $(0, 0)$ and the upper right corner is position (n, n) *(*Note that the first item in the tuple is the square along the horizontal axis and the second element is the index along the vertical axis)*. You can see an example grid below.

   Our goal is to count the number of unique ways a robot starting at cell $(0, 0)$ can reach cell (n, n) by only moving up, down, left, right on the grid on each move. We would like you to do two things:

   1. In your own words, argue why the given set is infinite (as opposed to finite).
   2. Show that the number of unique ways the robot can reach position (n, n) is *countably infinite*. *Hint: Try showing that a superset of this one is countably infinite.*



3. Use a proof by diagonalization to show that the following set is uncountable:

   $F = \mathcal{P}(\mathbb{N})$

   In other words, prove that the power set of the natural numbers (the set of all subsets of the natural numbers) is uncountable.