Context Free Languages

1. For each of the languages below, provide a context-free grammar that generates it (Note that some of these might also be regular languages, but we still want a grammar for each). For all parts, $\Sigma = \{0, 1\}$:

   - Strings that contain exactly two 1's OR exactly two 0's
   - Strings of even length that contain 1100 directly in the center (i.e., $w1100u \mid |w| = |u|$)
   - $ww^R uu^R \mid w \in \Sigma \wedge u \in \Sigma$

2. Draw PDAs for each of the languages in the previous exercise (note that you can draw a DFA / NFA if the language happens to be regular).

3. For this question, you will prove that context-free languages are NOT closed under intersection. Do this by showing the following:

   - **Part 1:** First, show that $A = \{a^m b^n c^n | m, n \geq 0\}$ is context-free by producing a context-free grammar that generates it.
   - **Part 2:** Do the same, but for language $B = \{a^n b^n c^m | m, n \geq 0\}$
   - **Part 3:** Lastly, find the intersection of these two sets and use the pumping lemma to show that the intersection language is not context-free.

4. Let us define a new operation using the $\diamond$ symbol as such: if A and B are languages, then $A \diamond B = \{xy | x \in A, y \in B, |x| = |y|\}$. Prove that if A and B are regular languages, then $A \diamond B$ must be a context-free language.

5. Suppose you have a context-free language C and a regular language R. Prove that $C \cap R$ is context-free.

6. Consider the language $A = \{w \mid w \in \{a, b, c\}^* \wedge F(w, a) = F(w, b) = F(w, c)\}$ where $F(w, a)$ counts the number of occurences of character a in string w. Prove that A is not context-free by using the result of the previous question (*Hint: Assume this language is a CFL and intersect it with a regular language of your choice!*)

7. Prove that the language A from the previous question is not context-free again, but this time do so by utilizing the *Pumping Lemma for Context-Free Languages*.