What is a computer? Proof Techniques

1. Suppose you have an alphabet Σ that has a specific length m (i.e., $|\Sigma| = m$). How long is the alphabet Σ^n? In other words, what is the value of $|\Sigma^n|$? Write out your formula and provide 1 or 2 sentences justifying the formula.

   $|\Sigma^n| = m^n$

2. If the set A has a elements, and the set B has b elements, how many elements are in $A \times B$? Explain your answer in a sentence or two.

   $a \times b$. For each element in A, it can be paired with each element in B.

3. Suppose that $\Sigma = \{0, 1\}$. Given the following intuitive descriptions of alphabets, provide a description of each of the same alphabets using formal, set notation.

   - Strings that begin with 011 or begin with 100.
   - Strings that contain four 1s in a row at least once.
   - Strings that have two 1s in a row somewhere in the string. However, the two ones are NOT the first two characters or the last two characters.

   There are several ways to do these. Here is one way:

   - $\{W = w_1 w_2 ... w_n | w_1 w_2 w_3 = 011 || w_1 w_2 w_3 = 100\}$
   - $\{W | W \in \{\Sigma^*\} \times \{1111\} \times \{\Sigma^*\}\}$
   - $\{W | W \in \{\Sigma^+\} \times \{11\} \times \{\Sigma^+\}\}$

4. Suppose I build a new computing machine that can be programmed to recognize a lot of different functions! It is called the *Flogrammable Device*. In order to program this machine, you can type out your program on a *tape*, but this type can only hold ten unique charactes. Each character is from the alphabet $\Sigma = \{a, b, c, 0, 1\}$ and any combination of these 10 characters is a valid program. You CANNOT have fewer than 10 characters or the code will not compile.

   Now suppose that you read online that somehow, there are only 10,000,000 important functions that need to be computed by the *Flogrammable Device* for it to cover all important functionality. Can we program each of the 10,000,000 functions on this machine? How do you know or not?

There are 5 character options and 10 characters of length, so $5^10 = 9,765,625 < 10,000,000$ unique programs. So every function cannot be implemented.

5. **Proof Practice:** Prove that every graph $G = (V, E)$ such that $|V| \geq 2$ contains two nodes with equal degrees.

Assume for the sake of contradiction that we have a graph G for which no two nodes have the same degree. This means that G has $|V|$ nodes and each has a unique degree. The possible degrees that each node can have span from 0 to $|V| - 1$, but notice that the one node that has degree exactly $|V| - 1$ connects to every other node, so it cannot be the case that any node has degree exactly 0. Thus, there are $|V|$ nodes and $|V| - 1$ (1 to $|V| - 1$) unique degrees each can have. By the pigeonhole principle, two nodes must have the same degree, contradicting our original assumption.