Set Cardinality

1. Prove that the set of rational numbers $\mathbb{Q} = \frac{a}{b}$ where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$ is countable. *Hint: find a bijection between $\mathbb{Z} \times \mathbb{Z}^+$ and $\mathbb{N}$*

We will provide a bijection from $\mathbb{Z} \times \mathbb{Z}^+$ to $\mathbb{N}$. Note the following:

- let $f(n) = 0, -1, 1, -2, 3, ...$ be the bijection from class between $\mathbb{Z}$ and $\mathbb{N}$
- let $g(n) = 1, 2, 3, 4, ...$ be the bijection from class between $\mathbb{Z}^+$ and $\mathbb{N}$

Our listing then proceeds as follows:

- First list out all $f(n)$, $g(n)$ pairs that use only the first element from each list.
- Then list out all $f(n)$, $g(n)$ pairs that use the second element from at least one of $f(n)$ and $g(n)$
- Use the third elements from one of the lists, etc...

This is injective because each $f(n)$ is paired with each $g(n)$ exactly once and surjective because every rational is covered at some point.

2. Use a proof by diagonalization to show that the following set is uncountable:

$F = \{f : \mathbb{N} \to \mathbb{N} | (a > b) \to (f(a) > f(b))\}$

In other words, this is the set of all strictly increasing functions that map natural numbers to natural numbers. A function is strictly increasing if larger inputs are guaranteed to produce larger outputs.

For example, $f(x) = x^2$ is strictly increasing since if $a > b$, then $a^2 > b^2$. However, $f(x) = (x-5)^2$ is not strictly increasing since $1 < 2$ but $f(1) > f(2)$.

Easiest way is to turn this into the exact problem from class. For each increasing function $f(n)$, turn it into an infinite length binary number ($\{0,1\}^*$) as follows: Set bit i to 1 if $f(n)$ ever outputs i, 0 otherwise. Because the function is increasing, the 1 bits have a proper ordering (the first is $f(0)$, the second is $f(1)$. etc.). Thus, each function has a unique infinite bitstring that describes it.

Then, simply use the diagonalization from class to show the $\{0,1\}^\infty$ is uncountable.

3. Prove that every subset of the natural numbers $\mathbb{N}$ is countable.

Any subset of $\mathbb{N}$, A, satisfies the constraint $|A| \leq |\mathbb{N}|$, which is the definition of countable.

4. Prove the following claim: If A is a countably infinite set (i.e., $|A| = |\mathbb{N}|$) and B is a finite set (i.e., $|B| = n | n \in \mathbb{N}$), then $A \cup B$ is also countable.

Simply map the $n - c$ finite elements of B that are not also in A to the first $n - c$ elements in $\mathbb{N}$, then proceed to map elements in A to the rest according to the ordering of A.

5. Prove the following claim: If A is a countably infinite set (i.e., $|A| = |\mathbb{N}|$) and B is a also a countably infinite set (i.e., $|B| = |\mathbb{N}|$), then $A \cup B$ is also countable.

Simply alternate between elements in the ordering for A with elements in the ordering of B. Make sure to skip any element in B that is also a member of A.