Complexity Theory

1. Prove that P closed under the star operator ($*$). (Hint: Given a language $L \in P$ and input a, use dynamic programming to build up solutions for each substring of a to see if $a \in L^*$)

2. *PSPACE* is the complexity class containing all problems that can be solved using a *Deterministic Turing Machine* that uses at most a polynomial amount of additional space (on the tape). Prove that $P \subseteq$ *PSPACE*. (*Hint: Think about a generic problem in P, and the DTM that decides it. Try to use more than a polynomial space with this machine given the allotted time.*)

3. You are working for *Shoogle* (A data company that definitely does not exist) and you are given the following task from your boss: *Shoogle* is interested in investing in their employees by offering a large amount of professional development. They have collected a series of *workshops* that each cover a subset of *topics*. For example, workshop 1 might cover machine learning and databases, workshop 2 might cover machine learning and data mining, and workshop 3 might cover advanced data structures (note that each workshop can cover multiple topics and there can be overlap across workshops). In addition, *Shoogle* wants to make sure that over the course of the professional development workshops, at least one topic in each sub-area of interest is represented. For example, they might want to make sure there is at least one AI workshop (machine learning or reinforcement learning or neural networks, etc.), at least one systems workshop (databases or cloud computing), etc.

   So...the problem is this: Given the availability of the professional development workshop, what topics each covers, and which areas need to be covered, can you devise a schedule over multiple weeks that ensures that at least one topic in every area is covered by at least one workshop?

   Let's formalize this a bit: The input contains multiple items:

   - T: A list of individual topics that can be covered (e.g., machine learning).
   - $W = \{w_1, w_2, ..., w_m | w_i \subseteq T\}$: A list of workshops, each of which is a subset of the topics T that will be covered in that workshop. There are m workshops total.
   - $S = \{s_1, s_2, ..., s_n | s_i \subseteq W\}$: The schedule availability for each week. Each s_i is the subset of workshops that can be scheduled in week i. There are n weeks.
   - $A = \{a_1, a_2, ..., a_p | a_i \subseteq T\}$: The subject areas that need to be covered. Each area is a list of Topics in that area.

   You want to output Yes if there is some schedule of workshops over the n weeks such that every area has at least one topic in T that is covered in at least one workshop.

   This problem is *NP-Complete*. For your reduction, use *3-SAT*.

4. You are working for *Amaphon* (an online realtor that definitely does not exist) and you are given the following task from your boss: Your users have a profile and may or may not have an in-store balance (i.e., a sum of money that can be directly spent on items on *Amaphon*). For example, maybe one user has $\$31.47$ in their account right now. Your team wants to build a new feature in which customers can select a type of product (e.g., Holiday gifts or clothes or electronics) and the site will automatically suggest a set of items they can purchase for which the total price is exactly their current balance. This means the customer gets a bunch of cool products AND they get to clear out their balance to exactly 0 all at once.

   To clarify this problem a bit, consider the following: You are given as input the users balance B, and a list of n products in their chosen category along with the prices of those products. Only the prices really matter, so let's call this list $P = \{p_1, p_2, ...p_n\}$ Your task is to find any subset of these prices that totals exactly B.

   You suspect this problem might be *NP-Complete*. Prove it! For your reduction, use *3-SAT*.