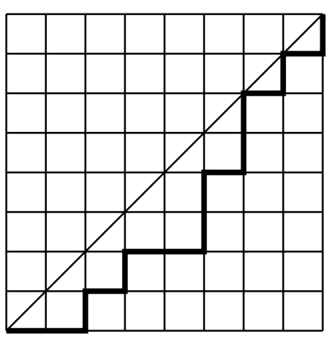
Set Cardinality

1. For each of the following claims, state whether it is true or false and then prove your assertion.

   - All finite sets have an *injection* to $\mathbb{N}$
   - All finite sets have a *surjection* to $\mathbb{N}$
   - If A is a countably infinite set (i.e., $|A| = |\mathbb{N}|$) and B is a also a countably infinite set (i.e., $|B| = |\mathbb{N}|$), then $A \cup B$ is also countable.
   - If A is countably infinite and B is uncountably infinite, then $A \cup B$ is countable.
   - If A is countably infinite and B is uncountably infinite, then $A \cap B$ is countable.

2. Consider a square grid with length and width n. The bottom left corner is considered position $(0, 0)$ and the upper right corner is position (n, n) *(*Note that the first item in the tuple is the square along the horizontal axis and the second element is the index along the vertical axis)*. You can see an example grid below.

   Our goal is to count the number of unique ways a robot starting at cell $(0, 0)$ can reach cell (n, n) by only moving up or to the right on each individual move. You need to do two things: 1) Prove that the number of unique moves is equal to the number of binary bitstrings of length $2n$ that contain exactly n 1's by describing a bijection between these two sets. Then, 2) Declare how many unique moves this is as a function of n.



3. Use a proof by diagonalization to show that the following set is uncountable:

   $F = \{f : \mathbb{N} \to \mathbb{N} | (a > b) \to (f(a) > f(b))\}$

   In other words, this is the set of all strictly increasing functions that map natural numbers to natural numbers. A function is strictly increasing if larger inputs are guaranteed to produce larger outputs.

For example, $f(x) = x^2$ is strictly increasing since if $a > b$, then $a^2 > b^2$. However, $f(x) = (x-5)^2$ is not strictly increasing since $1 < 2$ but $f(1) > f(2)$.