Smooth Beamforming Vector Reconstruction for 802.11ax MIMO-OFDM via Manifold Optimization

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***Abstract*—Channel smoothing is a technique utilized to enhance channel estimation in communication systems. In orthogonal frequency division multiplexing (OFDM) systems, channel smoothing exploits the correlation or smoothness between channel responses in the frequency domain. However, its application to WiFi beamforming multiple-input multiple-output (MIMO) systems poses challenges due to the destructive nature of the beamforming vector provided by the receiver, resulting in a similarly destructive beamforming channel and rendering channel smoothing technology impractical. Consequently, we propose a scheme enabling the transmitter to reconstruct the destructive beamforming vector provided by the receiver, thereby facilitating the utilization of channel smoothing techniques. Initially, we derive the objective function and constraints for the smooth beamforming vector and subsequently simplify the function to render it differentiable, employing manifold optimization technique based on Barzilai-Borwein algorithm is used to solve the problem. Performance gains are validated through simulation.**

***Index Terms*—Channel smoothing, beamforming, Manifold optimization, Barzilai-Borwein**

I. INTRODUCTION

MULTIPLE-INPUT MULTIPE-OUTPUT Orthogonal Frequency Division Multiplexing (MIMO-OFDM) enhances system capacity and reliability by transmitting multiple independent data streams simultaneously and employing multiple antennas for transmission. In non-beamforming MIMO-OFDM channels, receiver-side channel smoothing stands as the conventional technique [1]. It effectively reduces channel estimation errors and diminishes the Packet Error Rate (PER) by dynamically selecting the order of the frequency domain filter based on the disparity vectors between adjacent subcarriers and the amplitude of each subcarrier.

Despite its efficacy in non-beamforming scenarios, channel smoothing poses challenges in beamforming MIMO-OFDM channels due to the rank-inverse effect [2]. In essence, while MIMO techniques allow for the simultaneous transmission of multiple data streams to bolster throughput and reliability in wireless communications, beamforming exacerbates this issue. MIMO systems exploit the diversity of signal propagation paths between transmitter and receiver by employing multiple antennas to augment channel capacity. Typically, Singular Value Decomposition (SVD) is utilized to decompose the channel matrix, yielding eigenvectors and eigenvalues that characterize the channel's spatial modes and their respective strengths. In Wi-Fi systems employing beamforming techniques, the receiver furnishes eigenvectors to the transmitter to guide the beamforming process. However, due to wireless channel fluctuations and interference, the order of eigenvectors may vary among different subcarriers, resulting in a phenomenon known as rank reversal effect. This effect leads to a hairpin shift in the order of eigenvectors, causing discontinuous or unstable variations in the beamforming channel across subcarriers. Consequently, such variations impede the accurate decoding and recovery of transmitted data at the receiving end, undermining the performance of beamforming. Hence, each WiFi frame within the IEEE 802.11n/ac/ax standards incorporates a signal in its preamble, directing the receiver to disable standard channel smoothing during the estimation of the beamformed signal [3] [4] [5].

In this paper, we will investigate the use of manifold optimization [6] to optimize the proposed optimization expression to obtain the rotation vectors to compute the smooth beamforming vectors used for destructive subcarriers through the transmitter rotation singular vectors. This technique is an improvement of the method in [7], where the optimization expression proposed in that paper has has poor performance for dealing with the case of continuous multiple subcarrier discontinuities. In short, this paper will use the results in [7] as the initial point of manifold optimization to get a rotation vector with better performance.

The paper is organized as follows: System model, Beamforming is described in Section II. Manifold optimization algorithms are explained in Section III. The simulation results illustrating the performance are given in section IV, and the conclusion is given in section V.

II. SYSTEM MODEL

A MIMO OFDM system has N receive antennas, M transmit antennas, and K subcarriers. The resulting channel estimate on the ith subcarrier can be expressed as

$$\widehat{\mathbf{H}}_i = \mathbf{H}_i + \mathbf{Z}_i \tag{1}$$

where $\mathbf{H}_i \in \mathbb{C}^{M\times N}$ is the MIMO OFDM channel and $\mathbf{Z}_i \in \mathbb{C}^{M\times N}$ is the additive noise.At this point is the channel probe frame, which is non-BFed, thus the FIR filter can be used at the receiver side to smooth the traveled channel [7], and the FIR filter window value is set to a fixed value of 5, leaving the design of the filter for future research.

After $\widehat{\mathbf{H}}_i$ passes through the FIR filter, we are able to obtain the channel estimate $\overline{\mathbf{H}}_i$ after channel smoothing, at which point we perform the SVD decomposition of $\overline{\mathbf{H}}_i$ to obtain the following equation,

$$\overline{\mathbf{H}}_i = \mathbf{U}_i \boldsymbol{\Lambda}_i \mathbf{V}_i^H \tag{2}$$

where $\mathbf{V}_i \in \mathbb{C}^{M \times N}$ is the right singular value vector after $\overline{\mathbf{H}}_i$ undergoes SVD decomposition. The quantized $\mathbf{V}_i$ will be packed into a CSI frame and sent to the AP, which will use the $\mathbf{V}_i$ calculation to generate beamforming vectors. The channel sounding protocol in 802.11ax comprises the AP's channel sounding frame and the STA's CSI report frame [8]. The AP employs a forced-zero precoder, and at the ith subcarrier, the beamforming matrix formed at the AP is referenced to Eq. in [7],

$$\mathbf{T}_i = \mathbf{S}_i(\mathbf{S}_i^H \mathbf{S}_i)^{-1} \mathbf{P}_i \tag{3}$$

where $\mathbf{S}_i = [\mathbf{v}_1^1, \mathbf{v}_2^1, ..., \mathbf{v}_{\underline{K}}^1]$ consists of a vector of maximal right singular values of $\overline{\mathbf{H}}_i$. $\mathbf{P}_i$ it is a diagonal matrix with power normalization factors on the diagonals. In the case of street estimation, after the signal passes through the beamforming matrix $\mathbf{T}_i$, the signal transmitted on the i subcarrier can be expressed as,

$$\mathbf{Y}_i = \mathbf{H}_i \mathbf{T}_i \mathbf{X}_i + \mathbf{Z}_i \tag{4}$$

where $\mathbf{Y}_i \in \mathbb{C}^{M \times 1}$ is the received signal vector, $\mathbf{X}_i \in \mathbb{C}^{N \times 1}$ is the signal vector being sent, $\mathbf{Z}_i \in \mathbb{C}^{M \times 1}$ is the additive noise vector. $\mathbf{H}_i \in \mathbb{C}^{M \times N}$ is the street estimation matrix, used to estimate the influence of the channel. STA can use the leading code of the MIMO data frame to estimate the beamforming channel denoted as,

$$\mathbf{h}_i = \mathbf{H}_i \mathbf{T}_i + \mathbf{z}_i \tag{5}$$

where channel estimation errors $\mathbf{z}_i$. Multiply the received signal vector $\mathbf{Y}_i$ with the beamforming matrix $\mathbf{h}_i$ to get the signal vector $\mathbf{X}_i$ after removing channel influence. The $\mathbf{X}_i = \mathbf{h}_i^{-1}\mathbf{Y}_i$. This step is essentially the interference elimination and signal enhancement of the received signal, making the received signal more clear and reliable.Due to the rank inversion effect, the resulting singular vectors of the channel matrix obtained after performing the SVD decomposition may be destructive at the subcarrier, which is carried to the AP by the CSI reporting frame, leading to the unsmoothness of the beamforming vectors $\mathbf{T}_i$, which in turn leads to the unsmoothness of $\mathbf{h}_i$, affecting the reception and demodulation of the signal. In the next section we will propose a solution at the AP side to address the nonsmoothness of $\mathbf{h}_i$ such that the default channel smoothing can be reapplied to $\mathbf{h}_i$ to improve the performance of the channel.

III. Promblem Formulation And Solution

In order to ensure the smoothness of the MIMO-OFDM channel, it is necessary to ensure the smoothness of the beam-forming vectors, which depends on the continuity of the singular vectors among the subcarriers. The optimization expression proposed in [7] has poor performance when dealing with discontinuity of successive multiple subcarriers. So we rewrite the optimization expression as

$$\begin{aligned} &\min_{\boldsymbol{Q}_i} \sum_{i=1}^{N} \left\| \boldsymbol{V}_i \boldsymbol{Q}_i - \boldsymbol{V}_{i+1} \boldsymbol{Q}_{i+1} \right\|_F^2 \\ &s.t. \quad \boldsymbol{Q}_i^H \boldsymbol{Q}_i = \mathbf{I}_{N_r}, \end{aligned} \tag{6}$$

We can rewrite either of the terms in (6) as

$$\begin{aligned} &\left\| \boldsymbol{V}_i \boldsymbol{Q}_i - \boldsymbol{V}_{i+1} \boldsymbol{Q}_{i+1} \right\|_F^2 \\ &= \mathbf{Tr}((\boldsymbol{V}_i \boldsymbol{Q}_i - \boldsymbol{V}_{i+1} \boldsymbol{Q}_{i+1})(\boldsymbol{V}_i \boldsymbol{Q}_i - \boldsymbol{V}_{i+1} \boldsymbol{Q}_{i+1})^H) \\ &= 2N_r - \mathbf{Tr}(\boldsymbol{Q}_i \boldsymbol{Q}_{i+1}^H \boldsymbol{V}_{i+1}^H \boldsymbol{V}_i + \boldsymbol{Q}_{i+1} \boldsymbol{Q}_i^H \boldsymbol{V}_i^H \boldsymbol{V}_{i+1}), \end{aligned} \tag{7}$$

Since N_r is known and computable, we can remove the term N_r from the simplified optimization expression, from which we obtain the simplified optimization expression (8), where $\boldsymbol{R}_i$ is equal to $\boldsymbol{V}_{i+1}^H \boldsymbol{V}_i$,

$$\begin{aligned} &\min_{\boldsymbol{Q}_i} \sum_{i=1}^{N-1} -\mathbf{Tr}(\boldsymbol{Q}_i \boldsymbol{Q}_{i+1}^H \boldsymbol{R}_i) - \mathbf{Tr}(\boldsymbol{Q}_{i+1} \boldsymbol{Q}_i^H \boldsymbol{R}_i^H) \\ &s.t. \quad \boldsymbol{Q}_i^H \boldsymbol{Q}_i = \mathbf{I}_{N_r}, \end{aligned} \tag{8}$$

For the sake of better expression in the following we will abbreviate (8) as

$$\begin{aligned} &\min_{\boldsymbol{Q}_i} f(\boldsymbol{Q}_i) \\ &s.t. \quad \boldsymbol{Q}_i^H \boldsymbol{Q}_i = \mathbf{I}_{N_r}, \end{aligned} \tag{9}$$

Formula (8) is the final expression we get after simplification, and we will use the $Barzilai - Borwein$(BB) algorithm [9] on the Riemannian manifold to solve this optimization expression. The (BB) method is categorized within the family of first-order optimization algorithms, which solely rely on the gradient of the cost function and do not incorporate second-order information such as the Hessian matrix. If we denote $g^{(k)} = \nabla f(x^{(k)})$ and $F^{(k)} = \nabla^2 f(x^{(k)})$, then a first-order method can be expressed as $x^{(k+1)} = x^{(k)} - \alpha_k g^{(k)}$, where the step size α_k can be fixed or obtained through line search. First-order methods are simple but have slow convergence rates. The Newton's method is given by $x^{(k+1)} = x^{(k)} - (F^{(k)})^{-1} g^{(k)}$, which has a faster convergence rate, albeit at the expense of computing the Hessian matrix. The Barzilai-Borwein (BB) method approximates $(F^{(k)})^{-1}$ using $\alpha_k g^{(k)}$. So in the Euclidean space, the basic idea of the BB method is to solve, for $k \geq 1$, the least-squares problem

$$\min_t \left\| s_k t - y_k \right\|_2, \tag{10}$$

where $s_k := x_{k+1} - x_k$ and $y_k := \nabla f(x_{k+1}) - \nabla f(x_k)$, assuming that $x_{k+1} \neq x_k$, we can obtain the unique solution by the least squares method $t = \frac{s_k^T y_k}{s_k^T s_k}$. When $s_k^T y_k > 0$, the BB step-length is chosen to be

$$\alpha_{k+1}^{BB} = \frac{s_k^T s_k}{s_k^T y_k} \tag{11}$$

Similar to the Euclidean case, the Riemannian BB method approximates the action of the Riemannian Hessian of f at a particular point by a scalar multiple of the identity matrix. At the $(k+1)$-th step, the Hessian acts as a linear map from $T_{x_{k+1}}M$ to $T_{x_{k+1}}M$. Instead of the difference $x_{k+1}-x_k$, we can consider the vector $\eta_k=-\alpha_k g_k$, which belongs to $T_{x_k}M$, and transport it to $T_{x_{k+1}}M$, resulting in [10],

$$s_k := \mathcal{T}_{\eta_k}(\eta_k) = \mathcal{T}_{x_k \longrightarrow x_{k+1}}(-\alpha_k g_k) = -\alpha_k \mathcal{T}_{x_k \longrightarrow x_{k+1}}(g_k) \tag{12}$$

Here, we assume the existence of a vector transport between x_k and x_{k+1}. To obtain y_k, we subtract two gradients lying in different tangent spaces. To maintain coherence with the manifold structure and operate on $T_{x_{k+1}}M$, this subtraction should occur after transporting g_k to $T_{x_{k+1}}M$, ensuring:

$$y_k := g_{k+1} - \mathcal{T}_{x_k \longrightarrow x_{k+1}}(g_k) = g_{k+1} + \frac{1}{\alpha_k}\mathcal{T}_{\eta_k}(\eta_k) \tag{13}$$

The Riemannian counterpart of the secant equation (10) can be written again as

$$s_k t = y_k, \tag{14}$$

The least-squares approximation with respect to the metric of $T_{x_{k+1}}\mathcal{M}$ yields $t=\frac{\langle s_k,y_k\rangle_{x_{k+1}}}{\langle s_k,s_k\rangle_{x_{k+1}}}$. Therefore the Riemannian BB step length has the form [10],

$$\alpha_{k+1}^{BB} = \frac{\langle s_k, s_k\rangle_{x_{k+1}}}{\langle s_k, y_k\rangle_{x_{k+1}}} \tag{15}$$

By the symmetry of (10), the Riemannian BB step is generated,

$$\alpha_{k+1}^{BB\,\prime} = \frac{\langle s_k, y_k\rangle_{x_{k+1}}}{\langle y_k, y_k\rangle_{x_{k+1}}} \tag{16}$$

The tensor-tensor product between $\mathcal{A}\in\mathbb{R}^{n\times p\times l}$ and $\mathcal{B}\in\mathbb{R}^{p\times m\times l}$ is defined as [11],

$$\mathcal{A} * \mathcal{B} = \mathrm{fold}(\mathrm{bcirc}(\mathcal{A})\cdot \mathrm{unfold}(\mathcal{B}) \in \mathbb{R}^{n\times m\times l} \tag{17}$$

The orthogonal projector operator from Euclidean space ε to $T_{\mathcal{Q}}\mathrm{St}(n,p,l)$ is,

$$\mathbf{P}_{\mathcal{Q}}(\mathcal{U}) = \mathcal{U} - \mathcal{Q} * \frac{1}{2}(\mathcal{Q}^{\mathrm{T}} * \mathcal{U} + \mathcal{U}^{\mathrm{T}} * \mathcal{Q}) \tag{18}$$

Moreover, we assume that the retraction R applied on the manifold is a $t-QR$ based retraction, denoted by [12],

$$R_{\mathcal{Q}}(\mathcal{V}) = qf(\mathcal{Q}+\mathcal{V}) \tag{19}$$

where $\mathcal{Q}\in\mathrm{St}(n,p,l), \mathcal{V}\in T_{\mathcal{Q}}\mathrm{St}(n,p,l)$, and $qf(\mathcal{A})$ denotes the $\mathcal{U}$ factor of the t-QR decomposition of $\mathcal{A}\in L^{-1}(\mathbb{C}_*^{n\times p\times l})$ as $\mathcal{A}=\mathcal{U}*\mathcal{R}$, where $\mathcal{U}\in\mathrm{St}(n,p,l)$ and $\mathcal{R}\in L^{-1}(\mathbb{C}_{upp+}^{p\times p\times l})$. The gradient of smooth function f defined on $\mathrm{St(n,p,l)}$ is,

$$\begin{aligned}\mathrm{grad}f(\mathcal{Q}) &= \mathbf{P}_{\mathcal{Q}}(\mathrm{grad}\bar{f}(\mathcal{Q})) \\ &= \mathrm{grad}\bar{f}(\mathcal{Q}) - \mathcal{Q} * \mathrm{sym}(\mathcal{Q}^T * \mathrm{grad}\bar{f}(\mathcal{Q}))\end{aligned} \tag{20}$$

where $\bar{f}$ defined on $\mathbb{R}^{n\times p\times l}$ coincides with f on $\mathrm{St}(n,p,l)$: $f=\bar{f}|_{\mathrm{St}(n,p,l)}$, and $\mathrm{sym}(\mathcal{A})=\frac{\mathcal{A}+\mathcal{A}^T}{2}$.
The vector transport on $\mathrm{St}(n,p,l)$ is,

$$\begin{aligned}\tau_{\eta_{\mathcal{Q}}}\xi_{\mathcal{Q}} &= \mathbf{P}_{R_{\mathcal{Q}}(\eta_{\mathcal{Q}})}\xi_{\mathcal{Q}} \\ &= \xi_{\mathcal{Q}} - \mathcal{Y} * \mathrm{sym}(\mathcal{Y}^{\mathrm{T}} * \xi_{\mathcal{Q}}) \in T_{\mathcal{Y}}\mathrm{St}(n,p,l)\end{aligned} \tag{21}$$

where $\mathcal{Y}=R_{\mathcal{Q}}(\eta_{\mathcal{Q}})$ is any retraction on $\mathrm{St}(n,p,l)$.

Next, we will take the derivative of Equation (8) to obtain an expression for the gradient in Euclidean space. It is clear that the expression should be a piecewise function,

$$\mathrm{grad}\bar{f}(Q_i) = \begin{cases} -2R_i^H Q_{i+1}, & i=1 \\ -2R_i^H Q_{i+1} - 2R_{i-1}Q_{i-1}, & 1<i<sub \\ -2R_{i-1}Q_{i-1}, & i=sub \end{cases} \tag{22}$$

where sub is the number of subcarriers.
Define the parameters for line search as follows: the step length reduction factor σ belonging to the interval $(0,1)$; the sufficient decrease parameter γ within $(0,1)$; an integer parameter M for nonmonotone line search where $M>0$; and upper and lower bounds for the step length, denoted as $\alpha_{\max}>\alpha_{\min}>0$. The retraction and vector transport operators have been defined in Equations (19) and (21). Set the initial values: the starting point Q_0 comes from the closed-form solution obtained in paper [7].

Algorithm 1 Transmit beamforming with band smoothing based on BB algorithm

Initialization: $f_0=f(Q_0)$ and $g_0=\nabla^{(\mathcal{R})}f(Q_0)$
for $k=0,1,2,...$ **do**
  find the smallest $h=0,1,2,...$ such that $f(R_{Q_k}(-\sigma^h\alpha_k^{BB}g_k)) \le \max_{1\le j\le min\{k+1,M\}} f_{k+1-j} - \gamma\sigma^h\alpha_k^{BB}\langle g_k,g_k\rangle_{Q_k}$
  and set $\alpha_k := \sigma^h\alpha_k^{BB}$.
  Compute $Q_{k+1}=R_{Q_k}(-\alpha_k g_k)$.
  Compute $f_{k+1}=f(Q_{k+1})$.
  Compute $g_{k+1}=\nabla^{(\mathcal{R})}f(Q_{k+1})$.
  Compute y_k and s_k as in (12) and (13), respectively.
  Set $\tau_{k+1}=\frac{\langle s_k,s_k\rangle_{Q_{k+1}}}{\langle s_k,y_k\rangle_{Q_{k+1}}}$.
  Set
  $\alpha_{k+1}^{BB} = \begin{cases}\min\{\alpha_{max},\max\{\alpha_{min},\tau_{k+1}\}\} & \text{if } \langle \mathrm{s_k},\mathrm{y_k}\rangle_{Q_{k+1}}>0, \\ \alpha_{max} & \text{otherwise}\end{cases}$
end for
for i=1 to sub **do**
  Compute $V_i^*=V_iQ_i^*$
end for

IV. Simulation Evalution

In the simulation, we consider the latest 802.11ax [13] standard for MIMO. The simulation bandwidth is 80MHz, which consists of 242 active subcarriers. There are 4 transmit and 4 receive antennas and there are 4 space-time streams. The APEP is 2000 bytes long. MCS7 ((64 QAM with rate 5/6 binary convolutional code(BCC)) was used. Binary convolutional coding is used. The delay Model is Model-D. There is no large-scale fading effect on the channel output, and the channel output is not normalized. The transmitting and receiving distance is 1 meter.Three schemes are evaluated.

* Conventional:Tx uses unsmoothed BF vectors, and Rx doesn't applies channel smoothing .

* Testing: Tx uses smoothed BF vectors byclosed-form solution [7], and Rx applies channel smoothing.
* Manifold: Tx uses smoothed BF vectors as Algorithm 1, and Rx applies channel smoothing.

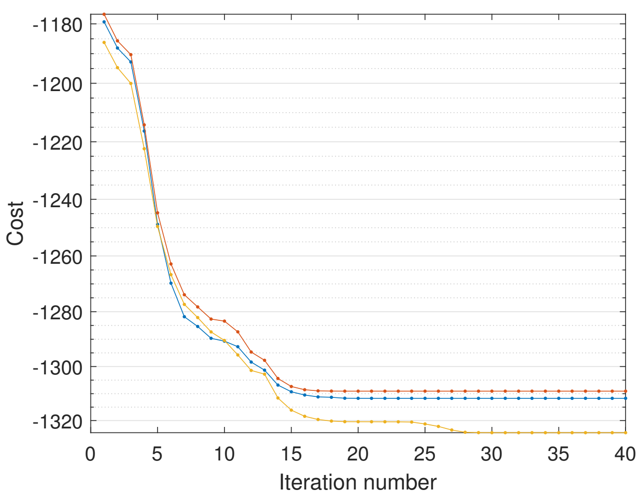

Fig. 1. Convergence of manifold optimization

Fig. 1 shows convergence of cost values(values of $f(\boldsymbol{Q}_i)$) with increasing number of iterations, the three lines in Fig. 1 show the iterative convergence of the manifold optimization algorithm at different SNR. We can see that the manifold optimization algorithm converges to around -1300 in about 20 iterations. The value before using manifold optimization is about -1180, and it can be concluded that there is roughly about 120 improvement in manifold optimization.
Fig. 2 and Fig. 3 shows the PER performances. In Fig. 2, for MCS7, our proposed scheme using streamer optimization outperforms the conventional scheme by about 1.8 db gain. For comparison purposes, we also simulated the performance in [7], and it can also be observed that the scheme using streamform optimization achieves about 0.3 dB gain over it. The reason for this is that our proposed objective function is able to take more into account the smoothing on all subcarriers, instead of focusing only on the unsmoothing between a certain neighboring subcarriers.To illustrate the versatility of the proposed method, we test the performance of the proposed method with different number of receive antennas and data streams. 4 transmit and receive antennas and 4 data streams are shown in Fig. 2 transmit antennas, 2 receive antennas and 2 data streams are shown in Fig. 3. In Fig. 3, we can see that the benefits of the proposed method are more significant. Our proposed scheme using streamer optimization outperforms the conventional scheme by about 8 dB gain, and versus the method in [7], a gain of 6 dB was obtained.

V. Conclusion

In this letter, we delved into the task of restoring the advantages of channel smoothing in beamformed MIMO-OFDM systems. Our examination centered on the 802.11ax

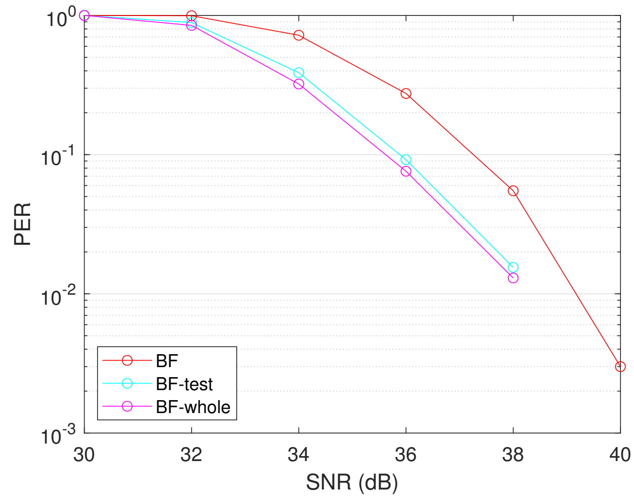

Fig. 2. PER performances of different schemes($M = N = 4$ antennas and $J = 4$ data streams)

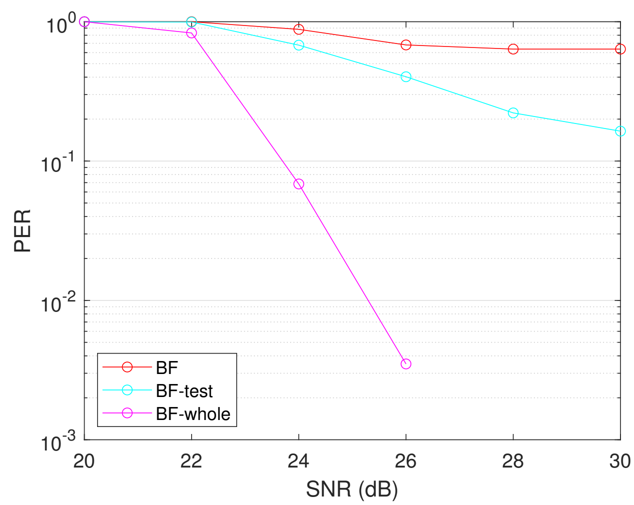

Fig. 3. PER performances of different schemes($M = 4, N = 2$ attennas and $J = 2$ data steams)

protocol, where the receiver furnishes feedback to the transmitter through the transmission of the singular vectors of the channel matrix for each subcarrier. Consequently, the transmitter utilizes this feedback to derive beamforming vectors for individual subcarriers.First we formulate an optimization problem to improve the smoothness of the singular vectors, and then we obtain the optimal solution to this problem using manifold optimization based on the Barzilai-Borwein algorithm. Following the application of beamforming vector smoothing at the transmitter, the receiver can reclaim the benefits of channel smoothing. Simulation results indicate a potential performance improvement of approximately 2 dB in PER, And the performance is better than the method in [7].

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