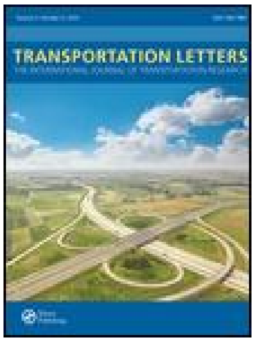

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Estimation of travel time reliability in large-scale networks

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Over the last two decades, travel time reliability has been increasingly used as a key performance indicator of transportation networks. Central to the assessment of travel time reliability is the estimation of the probability mass function (PMF) of total network travel time. This paper aims to present an efficient method to estimate the PMF of total network travel time using the universal generating function (UGF) method. Moreover, the paper proposes two truncation techniques to increase the computational efficiency of the UGF method. In order to assess the applicability of the method in practice, the method is tested on different networks. The results suggest that the method is computationally much more efficient than the standard crude Monte Carlo simulation technique at different confidence levels, and that it can be applied to real-world cases with a reasonable computation time.

Keywords: Travel time reliability, Large-scale transportation networks, Universal generating function (UGF), Monte Carlo simulation

1. Introduction

Travel time reliability is becoming a more critical and more relevant factor than "expected travel time" due to the substantial increase in traffic accidents in many urban areas (Loustau *et al.*, 2010). Total network travel time, also known as system-wide travel time (STT), has been recognized as one of the most prominent measures for appraisal of transportation network performance and design (Xu *et al.*, 2014).

Since travel time is the result of the interaction between traffic demand and supply, fluctuations in demand and supply disruptions give raise to uncertain travel times. It has been a common practice to model this uncertainty using probability theory and to treat travel time as a random variable. Earliest studies tended to focus on expected travel time as a primary cost to travelers without consideration of its variability. However, as evidenced by empirical research (Bogers *et al.*, 2008; van Lint *et al.*, 2008), not only is the distribution of travel time wide but also it is heavily skewed. As a result, last decade has witnessed an increasing number of studies that address measuring travel time variability (reliability) and modeling its effects on traveler's behavior (Bogers *et al.*, 2008; Brownstone and Small, 2005). Despite the importance of travel time reliability, there exists neither any unique definition of this concept nor any unique approach to measure it. Thus, several travel time reliability measures such as buffer index and planning time index have been proposed from different points of view and for different purposes. The reader is referred to a study by Pu (2011) for further details of these measures in which the analytic relationships between a number of reliability measures have been explored.

Whatever be the measure of assessment, central to the determination of the assessment of travel time reliability is the estimation of the probability distribution of STT. Estimation methods of the distribution of STT may be categorized into two general groups: (1) simulation-based methods (Chen *et al.*, 2007, 2010) and (2) numerical approximation methods (Clark and Watling, 2005; Ng and Waller, 2010; Ng *et al.*, 2011; Xu *et al.*, 2014). Although simulation-based methods are easy to implement and enjoy flexible frameworks, it has been proved that they are computationally demanding especially for large-scale networks (Xu *et al.*, 2014). To overcome this major drawback with simulation, several numerical approximation methods have been suggested. Clark and Watling (2005) developed a two-step procedure to estimate the probability distribution of total network travel time in the light of normal day-to-day variations in the travel demand matrix over a road traffic network. In the first step, the first four moments of the total network travel time are computed using an analytic method. Following the computation of these moments, a family of probability densities known as Johnson curves (Johnson, 1949) is fitted to these moments to construct the distribution of STT. The curve fitting procedure and in particular assumption about the type of probability distribution may be accompanied by some approximation errors and may increase the computational burden. To obviate the need for curve fitting procedure, Ng and Waller (2010) developed a computationally efficient methodology based on the fast Fourier transform (FFT) to construct the probability distribution function of total network travel time considering link capacity variations. They derived the theoretical bounds of approximation errors and presented methods

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to ensure the accuracy of approximations. Using this method requires to have complete information on the probability distribution of link capacity. However, the determination of this distribution is not easy in practice (Ng *et al*., 2011). In another study, Ng *et al*. (2011) proposed an alternative method to estimate upper bounds on the probability that STT exceeds a certain threshold. This method only requires the first N moments of total link travel times and their finite supports rather than their complete probability distributions. It is important to mention that the inclusion of higher order moments may significantly improve the accuracy of the approximations. This method is often appropriate for situations in which the probability distributions of total link travel time are unknown; otherwise, the adoption of methods such as FFT is recommended. Recently, seeking to introduce a new network-wide risk measure, Xu *et al*. (2014) suggested a method for the estimation of the probability distribution of STT assuming that the first four moments of STT are known a priori. It is apparent that computing the moments of STT is not a trivial task and it remarkably increases computational burden particularly for large-scale networks (Xu *et al*., 2014).

This paper aims to propose a computationally efficient and accurate method for estimating the probability distribution of STT, employing the universal generating function (UGF) method. This method obviates the need for efforts such as curve fitting and the determination of STT moments that can make the computations intractable in large networks. Furthermore, it uses the probability mass function (PMF) of total travel time instead of the PMF of capacity because it is easier in practice to determine the probability distribution of travel times than that of capacities.

The rest of this paper is organized as follows. The next section presents some notations used throughout the paper. This is followed by a section that describes briefly the universal generation function method. The definition of total network travel time, where link travel times are assumed to be stochastic, is then presented in Section 4. Also, the proposed method for estimation of the PMF of total network travel time is demonstrated in this section. The results of the implementation of the proposed method on a number of test networks are discussed in Section 5. Finally, Section 6 concludes the paper with a summary and suggestions for future research.

2. Notations

Some notations used in this paper are as follows:

n	number of elements of a multi-state system (MSS)
j	index of element, $j = 1, 2, \dots, n$
i	index of state
K_j	the number of states of element j
x_{ji}	performance rate of element j at state i
$\mathbf{x}_j$	set of possible performance rates of element j, $\mathbf{x}_j = \left\{x_{j1}, x_{j2}, \dots, x_{jk_j}\right\}$
$X_j(t)$	performance rate of element j at time instant t
p_{ji}	probability that element j is in state i
$\mathbf{p}_j$	vector of state probabilities for element j
z	a dummy variable
$u_{X_j}(z)$	z-transform (u-function) representing the performance distribution of element i
K	total number of states of an MSS
y_i	performance rate of an MSS in state $i \in \{1, 2, \dots, K\}$
$\mathbf{y}$	set of possible performance rates of an MSS, $\mathbf{y} = \left\{y_1, y_2, \dots, y_K\right\}$
$Y(t)$	performance rate of an MSS at time instant t
p_i	probability that an MSS is in state i
f	structure function of an MSS
Ψ_f	general UGF composition operator
$u_Y(z)$	u-function of an MSS
$\underset{+}{\otimes}$	composition operator over u-functions of elements connected in parallel
$\underset{\min}{\otimes}$	composition operator over u-functions of elements connected in series
A	set of all links of a network, $A = \{1, 2, \dots, m\}$
a	index of link, $a \in A$
t_a	random variable representing travel time on link a
$\tilde{t}_a$	travel time on link a given by the BPR volume-delay function
$\tilde{t}_a^f$	free-flow travel time on link a
v_a	traffic volume on link a
c_a	random variable representing capacity of link a
$\tilde{c}_a$	capacity of link a
STT	random variable representing total network travel time
T_a	total travel time experienced by all travelers traversing link a
$\boldsymbol{\tau}_a$	set of all values that T_a can take on, $\boldsymbol{\tau}_a = \{\tau_{ai}, 1 \le i \le K'_a\}$
$\mathbf{p}_a$	set of probabilities corresponding to different values of $\boldsymbol{\tau}_a$
$u_a(z)$	u-function of the travel time of link a
$u_S(z)$	u-function of STT
Γ	$\Gamma = \left\{T_a \mid a \in A\right\}$
Δ	length of discretization interval
Δ_a	length of discretization intervals of link a
S_a	number of discretization intervals (states) of link a
$S_{\max}$	maximum number of discretization intervals (states), $S_{\max} = \max\left\{S_a \mid a \in A\right\}$
Ω	confidence level of the total network travel time probability distribution
ω	confidence level of all links
$\bar{\omega}$	smallest confidence level that is greater than ω based on Equation 14
δ^{TRUN1}	first truncation operator
δ^{TRUN2}	second truncation operator

3. Universal generating function

In this section, we strive to briefly describe the UGF technique. To this end, we focus only on the essential concepts contributing to deep understanding of this paper. For more details about this technique, the reader is referred to a book by Levitin (2006).

The introduction of UGF technique can be traced back to a seminal work by Ushakov (1986). This technique has been proved to be remarkably efficient for reliability analysis in MSSs (Levitin, 2004).

A system performing its task with several distinct levels of efficiency (performance rates) is referred to as an MSS. An MSS usually comprises elements each of which has in turn different states ranging from perfect functioning to complete failure. Each state of the element is characterized by a performance rate and a probability of occurrence. Consider an MSS with n elements. Any element j, $1 \le j \le n$, has K_j states, each with a performance rate $x_{ji} \in \mathbf{x}_j$. The performance rate of element j at time instant t can be treated as a discrete random variable $X_j(t)$ such that $P\left(X_j(t) = x_{ji}\right) = p_{ji} \in \mathbf{p}_j$.

Definition 1. The z-transform (u-function) of the independent discrete random variable $X_j(t)$ is defined as a polynomial

$$u_{X_j}(z) = \sum_{i=1}^{K_j} p_{ji} z^{x_{ji}}. \tag{1}$$

The performance rate of an MSS can be calculated using the performance rates of its elements. Suppose an MSS has K distinct states and y_i is the performance rate of the MSS in state $i \in \{1, 2, \ldots, K\}$. As a result, the performance rate of the MSS at time t, $Y(t)$, can be viewed as a discrete random variable taking values in $\mathbf{y} = \{y_1, y_2, \ldots, y_K\}$ such that $P(Y(t) = y_i) = p_i$. Consider function $f{:}D \to \mathbf{y}$, where $D = \mathbf{x}_1 \times \ldots \times \mathbf{x}_j \ldots \times \mathbf{x}_n$. In the context of MMS, function $Y = f(X_1(t), X_2(t), \ldots, X_n(t))$ is called structure function. The z-transform of random variable $Y = f(X_1(t), X_2(t), \ldots, X_n(t))$ can be defined using universal generating operation Ψ_f as

$$u_Y(z) = \Psi_f\left\{u_{X_1}(z), u_{X_2}(z), \ldots, u_{X_n}(z)\right\} = \sum_{i_1=1}^{K_1}\sum_{i_2=1}^{K_2}\cdots\sum_{i_n=1}^{K_n}\left(p_{1i_1} \times p_{2i_2} \times \ldots \times p_{ni_n}\right) z^{f\left(x_{1i_1}, \ldots, x_{ni_n}\right)}. \quad (2)$$

After some algebraic manipulations, the above expression can be rewritten as the following polynomial:

$$u_Y(z) = \sum_{K=1}^{K} p_k z^{y_k} \quad (3)$$

The structure function represents the topology or the interaction among the elements of the MSS. Consider, for example, a flow transition system with two elements (pipes). Here, the performance rate of an element (a pipe) is defined as its capacity. As shown in Fig. 1*a*, the total performance rate of a pair of pipes connected in parallel is equal to the sum of the performance rates of the pipes. When the pipes are connected in series as depicted in Fig. 1*b*, the pipe with the lowest performance rate becomes the bottleneck of the system. As a result, the performance rate of the system is equal to the minimum of the performance rates of the pipes. On this basis, the u-function of a system composed of elements connected in parallel (parallel system) can be obtained as

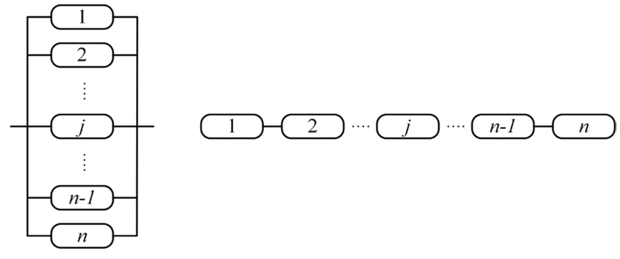
$$u_Y(z) = \underset{+}{\otimes}\left\{u_{X_1}(z), u_{X_2}(z), \ldots, u_{X_n}(z)\right\} = \sum_{i_1=1}^{K_1}\sum_{i_2=1}^{K_2}\cdots\sum_{i_n=1}^{K_n}\left(p_{1i_1} \times p_{2i_2} \times \ldots \times p_{ni_n}\right) z^{x_{1i_1}+x_{2i_2}+\ldots+x_{ni_n}}. \quad (4)$$

However, in cases where all elements of an MSS are arranged in series (series system) the u-function of the system is determined as

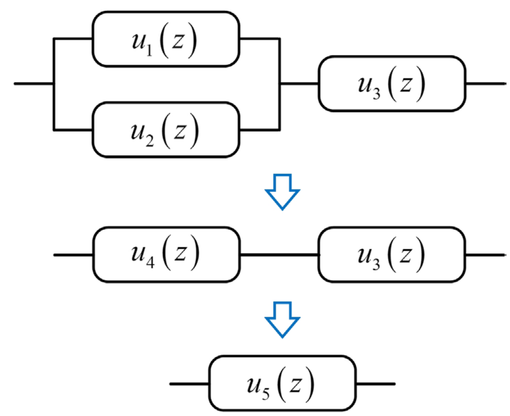
$$u_Y(z) = \underset{\min}{\otimes}\left\{u_{X_1}(z), u_{X_2}(z), \ldots, u_{X_n}(z)\right\} = \sum_{i_1=1}^{K_1}\sum_{i_2=1}^{K_2}\cdots\sum_{i_n=1}^{K_n}\left(p_{1i_1} \times p_{2i_2} \times \ldots \times p_{ni_n}\right) z^{\min\left\{x_{1i_1}+x_{2i_2}+\ldots+x_{ni_n}\right\}}. \quad (5)$$

In practice, in order to decrease computational burden, a recursive approach rather than direct use of Equation 2 is suggested as follows (Levitin, 2006):

(1) Find any pair of elements (j_1, j_2) connected in parallel or series in the MSS.
(2) Calculate the u-function of this pair using Equation 4 or 5.
(3) Replace the pair with a single element having the u-function calculated in the Step 2.
(4) If the MSS contains more than one element, return to Step 1,



1 Parallel and series flow transmission systems



2 A flow transmission system with the possibility of reducing computation effort

In other words, in many cases, the structure function of the entire MSS can be considered as the composition of the structure functions corresponding to some subsets of the system elements (MSS subsystems). The u-functions of the subsystems can be calculated separately and then the subsystems can be further treated as single equivalent elements (Levitin, 2006). Using the recursive algorithm, one can take advantage of the fact that some subsystems have the same performance rates in different states, which makes these states indistinguishable and decreases the total number of terms in the corresponding u-functions (Levitin, 2006). For example, consider a flow transmission system as shown in Fig. 2. The performance rates of elements and their corresponding probabilities are as follows:

$$\mathbf{x}_1 = \{1.5, 1, 0\},\ \mathbf{x}_2 = \{2, 1.5, 0\} \text{ and } \mathbf{x}_3 = \{4, 0\}$$

$$\mathbf{p}_1 = \{0.8, 0.1, 0.1\},\ \mathbf{p}_2 = \{0.7, 0.2, 0.1\} \text{ and } \mathbf{p}_3 = \{0.96, 0.4\}$$

Based on Equation 1, the u-functions of elements can be defined as follows:

$$u_1(z) = 0.8z^{1.5} + 0.1z^{1} + 0.1z^{0},$$

$$u_2(z) = 0.7z^{2} + 0.2z^{1.5} + 0.1z^{0},$$
$$u_3(z) = 0.96z^{4} + 0.04z^{0}.$$

The system shown in Figure 2 can be divided into two subsystems (i.e. A and B). The first subsystem A has two elements

Table 1 The analogy between an MSS and a transportation network

Reliability engineering	Transportation engineering
Multi-state system	Transportation network
Element	Link
Performance rate of an element	Total link travel time
Performance rate of an MSS	Total network travel time
State	State

connected in parallel. Thus, in accordance with Equation 4, the u-function of this subsystem can be obtained as

$$u_4(z) = u_1(z) \underset{+}{\otimes} u_2(z) = 0.56z^{3.5} \\ +0.23z^3 + 0.02z^{2.5} + 0.07z^2 + 0.1z^{1.5} \\ +0.01z^1 + 0.01z^0.$$

Sub-systems A and B are connected in series, therefore, the u-function of the system can be obtained using Equation 5 as

$$u_5(z) = u_3(z) \underset{\min}{\otimes} u_4(z) = 0.5376z^{3.5} \\ +0.2208z^3 + 0.0192z^{2.5} + 0.0672z^2 + 0.096z^{1.5} + 0.096z^1 + 0.0496z^0.$$

According to Equation 2, the total number of states for this system is equal to 18 (3 × 3 × 2). By employing the recursive method, however, the number of states for the system is reduced to seven because the states with the same performance rate are merged in this method.

It is worth mentioning that, based on Equation 2, the computational complexity of the UGF method is at worst

$$O\left(\prod_{j=1}^{n} K_j\right). \tag{6}$$

In most real cases, however, it can be drastically reduced due to the fact that states with the same performance rate are merged using the recursive approach.

4. Methodology

4.1. Problem statement

Let A denote the set of links in a transportation network. Assume that t_a is a random variable and represents travel time on link $a \in A$. The total travel time experienced by all travelers traversing link a can be defined as

$$T_a = t_a v_a, \tag{7}$$

where v_a represents the traffic volume on link a. In the rest of this paper, we use the term *total link travel time* to refer to the total time experienced by all travelers traversing a link.

Total network travel time denoted by STT is also a random variable defined as (Ng and Waller, 2010)

$$\text{STT} = \sum_{a \in A} T_a. \tag{8}$$

Based on these definitions, this paper seeks to estimate the probability distribution of STT, where T_as are assumed to be independent random variables. As pointed by Lo and Tung (2003), this assumption appears to be reasonable especially for relatively minor network disruptions such as traffic accidents and parking violations. It should be mentioned that the assumption has also been frequently used in the literature (Ng and Waller, 2010; Ng *et al.*, 2011; Rajabi-Bahaabadi *et al.*, 2015).

4.2. UGF in estimating STT's PMF

This section intends to show how a transportation network can be translated into an MMS for estimating the PMF of total network travel time. Table 1 summarizes the analogy between an MSS and a transportation network.

As can be seen from Table 1, to utilize the UGF method for estimation of the PMF of STT, the transportation network is assumed to represent an MSS. Then the performance of the MSS is the total network travel time, and network links represent the elements of the MSS.

As the probability distribution of total travel time of each link (T_a) is discrete and known, thus, the total travel time of each link $a \in A$ has a set of travel times $\boldsymbol{\tau}_a$ as follows:

$$\boldsymbol{\tau}_a = \{\tau_{ai},\ 1 \le i \le K'_a\}, \tag{9}$$

where τ_{ai} denotes the travel time of link a in state i, and K'_a represents the number of possible states for link a. Also, the probability of state i of link a is denoted by $p_{ai} = P(T_a = \tau_{ai})$ with a respective set of $\mathbf{p_a}$ where

$$\mathbf{p_a} = \{p_{ai}, \quad 1 \le i \le K_a\}. \tag{10}$$

Now, the u-function of the discrete random variable T_a can be written as

$$u_a(z) = \sum_{i=1}^{K'_a} p_{ai} z^{\tau_{ai}}. \tag{11}$$

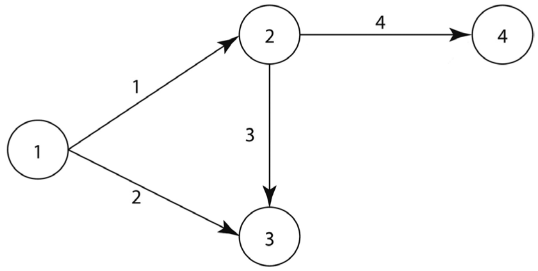
Regardless of network topology (i.e. the physical arrangement of links), as the total network travel time is equal to the sum of the total experienced travel time on its individual links (see Equation 8), the transportation network can be viewed as an MSS whose elements are connected in parallel. On this basis, the u-function of STT (i.e. $u_S(z)$) is calculated as

$$u_S(z) = \sum_{i_1=1}^{K'_1} \sum_{i_2=1}^{K'_2} \dots \sum_{i_n=1}^{K'_n} \left(p_{1i_1} \times p_{2i_2} \times \dots \times p_{mi_m}\right) z^{\tau_{1i_1}+\tau_{2i_2}+\dots+\tau_{mi_m}}. \tag{12}$$

As discussed in Section 3, to decrease computational burden, a recursive approach can be used rather than the direct use of Equation 12. To accomplish this approach, the following algorithmic steps can be carried out.

Algorithm 1. Total network travel time determination using UGF functions

1:	Input: $\Gamma = \{T_a \mid a \in A\}$ and $A = \{1, 2, \dots, m\}$
2:	Output: $u_S(z)$
3:	Begin
4:	Construct the u-function of all elements belonging to the set Γ using Equation 10
5:	$u'_0(z) \leftarrow u_1(z)$
6:	for a = 1:m−1
7:	$u'_a(z) = \underset{+}{\otimes} \{u'_{a-1}(z), u_{a+1}(z)\}$
8:	end
9:	$u_S(z) \leftarrow u'_{m-1}(z)$
10:	end



3 A schematic four-link network

For more clarification, consider the schematic four-link network of Fig. 3. The PMF of the total travel time of each link is presented in Table 2.

According to line 4 of Algorithm 1, the first task is to determine the u-function of each link as follows:

$$u_1(z) = 0.1z^5 + 0.9z^{10}$$

$$u_2(z) = 0.2z^1 + 0.8z^6$$

$$u_3(z) = z^8$$

$$u_4(z) = 0.2z^2 + 0.5z^7 + 0.3z^{10}$$

We then set $u'_0(z) = u_1(z)$ based on line 5 of Algorithm 1.

According to the first iteration of the *for* loop of lines 6–8, we can write

$$u'_1(z) = u'_0(z) \underset{+}{\otimes} u_2(z) = (0.1z^5 + 0.9z^{10}) \times (0.2z^1 + 0.8z^6)$$

$$= 0.02z^6 + 0.26z^{11} + 0.72z^{16}.$$

After executing the second and the third iterations of the *for* loop of lines 6–8, we will have

$$u'_2(z) = u'_1(z) \underset{+}{\otimes} u_3(z) = (0.02z^6 + 0.26z^{11} + 0.72z^{16}) \times (z^8)$$

$$= 0.02z^{14} + 0.26z^{19} + 0.72z^{24},$$

$$u'_3(z) = u'_2(z) \underset{+}{\otimes} u_4(z) = (0.02z^{14} + 0.26z^{19} + 0.72z^{24}) \times (0.2z^2 + 0.5z^7 + 0.3z^{10})$$

$$= 0.004z^{16} + 0.062z^{21} + 0.006z^{24} + 0.274z^{26} + 0.078z^{29} + 0.360z^{31} + 0.216z^{34}.$$

As there is no remaining link, based on line 9 of the algorithm, the u-function of the system is

$$u_S(z) = u'_3(z) = 0.004z^{16} + 0.062z^{21} + 0.006z^{24} + 0.274z^{26} + 0.078z^{29} + 0.360z^{31} + 0.216z^{34}.$$

Hence, the PMF of the total network travel time can be shown as the two following sets:

$$\boldsymbol{\tau}_{\mathrm{STT}} = \{16,\ 21,\ 24,\ 26,\ 29,\ 31,\ 34\}$$

$$\mathbf{p}_{\mathrm{STT}} = \{0.004,\ 0.062,\ 0.006,\ 0.274,\ 0.078,\ 0.360,\ 0.216\}$$

4.3. Discretization algorithm

In practice, the probability distribution function of link travel times are mostly presented in the form of continuous random variables, such as normal and log-normal (van Lint *et al*., 2008). (No doubt, the distribution should have a lower and an upper bound on the quantity of travel time, which is applicable by using truncated continuous distributions.) Hence, an important task would be to derive the PMFs (discrete distributions) from the original continuous ones. To this end, one possible way is to partition the travel time domain into identical infinitesimal intervals.

Let Δ_a and S_a denote the length of intervals and the number of states of link a, respectively. In order to observe the equity of travel time variation over all links, we suggest using the same interval length for all links, referred to as Δ; meaning that the wider the travel time distribution, the more the number of states should be assigned to the distribution. Thus, the link with the widest travel time distribution is given the bigger number of states, called $S_{\max}$. Obviously, the more accurate results will be obtained by increasing the value of parameter $S_{\max}$. On the negative, specifying a large $S_{\max}$ would result in more computation time. Therefore, a possible solution to determine a suitable quantity for $S_{\max}$ will be sensitivity analysis. By this solution, however, the following question will arise: Is there a unique suitable $S_{\max}$ for all problems disregarding link travel time PMFs, or it should be determined for each case specifically? Clearly, the determination of $S_{\max}$ on a case-by-case basis will be time-consuming. An alternative solution to even reduce the computation time would be to fix $S_{\max}$ for all cases and, then, use truncation techniques. In this paper, two truncation methods have been proposed that guarantee a given minimum allowable confidence level Ω for the resultant total network travel time probability distribution. These methods simply select the most probable states of the PMF of total travel time of each link so that the sum of the likelihood of all states of total network travel time equals Ω.

4.4. Truncation method 1

In this method, an identical confidence level ω is determined for all links as

$$\omega = \sqrt[m]{\Omega}, \tag{13}$$

where m is the number of links in the network. As seen, ω is determined so that the multiplication of the confidence level of

Table 2 Parameters of PMF of total link travel times

Link ID	1	2	3	4
Set of travel times	$t_1 = \{5, 10\}$	$t_2 = \{1, 6\}$	$t_3 = \{8\}$	$t_4 = \{2, 7, 10\}$
Set of probabilities	$p_1 = \{0.1, 0.9\}$	$p_2 = \{0.2, 0.8\}$	$p_3 = \{1\}$	$p_4 = \{0.2, 0.5, 0.3\}$

all links becomes Ω. To accomplish this, all states of the PMF of total travel time of each link are sorted descending with respect to their probabilities. Suppose that $\{1, 2, \ldots, K_a\}$ is the sorted set of all states of link a. Now, let $\bar{K}_a$ define the smallest number that the sum of the probability of all states smaller than/equal to it becomes greater than or equal to ω. In mathematical terms:

$$\sum_{i=1}^{\bar{K}_a} p_{ai} = \bar{\omega} \geq \omega, \tag{14}$$

where $\bar{\omega}$ is the smallest confidence level that is greater than ω.

Now, the first truncation operator over the u-function of each link a can be written as

$$\delta^{\text{TRUN1}}\left(u_a(z),\ \omega\right) = \frac{1}{\sum_{i=1}^{\bar{K}_a} p_{ai}} \sum_{k=1}^{\bar{K}_a} p_{ai} z^{\tau_{ai}}. \tag{15}$$

where $1/\sum_{i=1}^{\bar{K}_a} p_{ai}$ is used for rescaling the PMF to obtain a standard PMF, in which the sum of probability of all states becomes unity.

4.5. Truncation method 2

Since in truncation method 1 all links have states with total probability of $\bar{\omega} \geq \omega$, after the implementation of each iteration of the *for* loop of the Algorithm 1 for the ath link, it is expected to obtain a larger amount of $\bar{\omega}^a$ compared with what was intended, i.e. ω^a. Therefore, the u-function of the resultant PMF would contain a number of states with low probabilities. The purpose of Truncation Method 2 is to omit such unimportant states. To do so, suppose that after a iterations of the *for* loop of Algorithm 1, the u-function of $u'_a(z) = \sum_{i=1}^{K_V} p_{ai} z^{\tau_{ai}}$ has been obtained. As the confidence level until this stage of the algorithm should be equal to ω^a, the set of states $\{1, 2, \ldots, K_V\}$ can be sorted in descending order and the PMF can be truncated at confidence level of $\bar{\omega}_a (= \sum_{i=1}^{\bar{K}_V} p_{ai}) \geq \omega^Y (= \sqrt[(m-a)]{\Omega})$, where $\bar{K}_V$ is "the smallest state that the sum of the probability of all states smaller than/equal to it is greater than or equal to ω^a."

Therefore, in ath iteration of the *for* loop of Algorithm 1, the truncation operator can be written as follows:

$$\delta^{\text{TRUN2}}\left(u'_a(z),\ \bar{\omega}_a\right) = \frac{1}{\sum_{i=1}^{\bar{K}_V} p_{ai}} \sum_{i=1}^{\bar{K}_V} p_{ai} z^{\tau_{ai}}, \tag{16}$$

where $1/\sum_{i=1}^{\bar{K}_V} p_{ai}$ is a rescaling factor as discussed in case of Truncation Method 1.

5. Numerical examples

This section presents several numerical examples to assess the effectiveness of the proposed method in terms of computational efficiency and estimation accuracy. To this end, the method was coded in MATLAB software and all experiments were conducted on a laptop with an Intel 2.66 GHz CPU and 6 GB of memory.

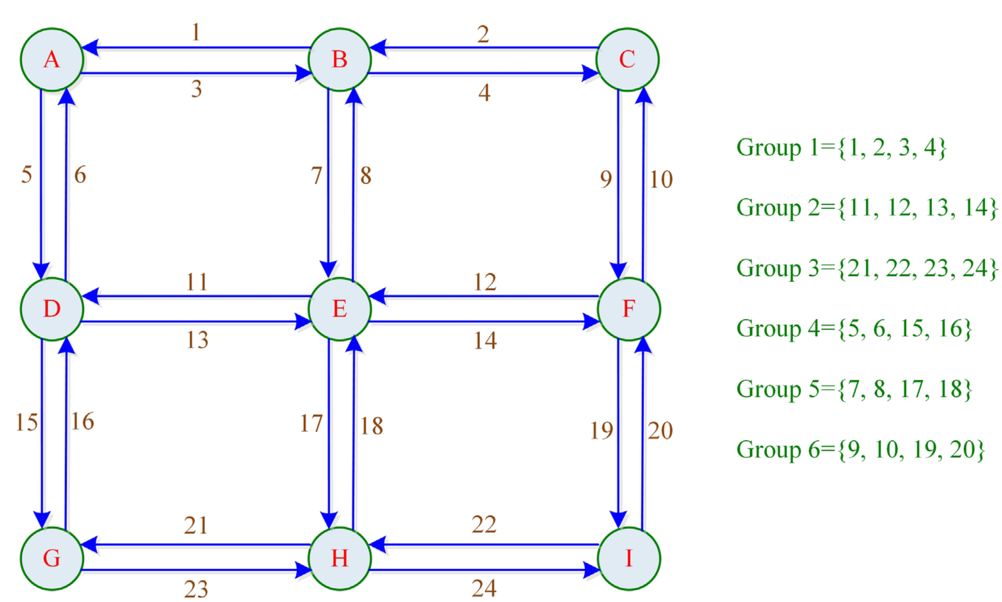
5.1. Example 1: A network with normally distributed link travel time

Based on normal distribution properties, the sum of several independent normal random variables (i.e. $Y = X_1 + X_2 + \ldots + X_n$, where $X_i \sim N(\mu_i, \sigma_i^2)$) is also normally distributed with the following parameters:

$$Y \sim N\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right) \tag{17}$$

Based on this characteristic, the normal distribution is a good benchmark for the examination of the precision of the proposed method.

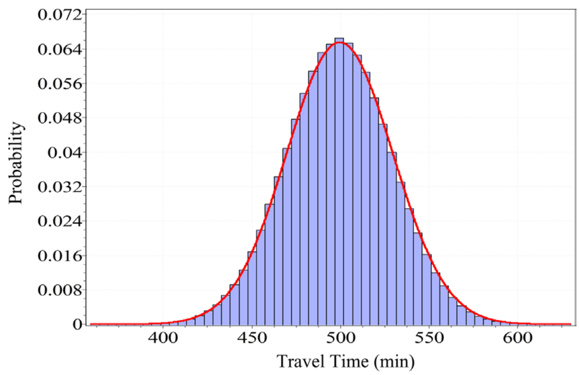
The network used in this example comprises 9 nodes and 24 links. The set of links is classified into six groups of four links as illustrated in Fig. 4. Each group of links follows a specified PMF symmetric around its mean. The mean and the standard



4 Test network for example 1 and example 2.

Table 3 Parameters of the probability distribution of link total travel time for each group

Group	Group 1	Group 2	Group 3	Group 4	Group 5	Group 6
Mean (μ)	499.63	1000.1	799.85	300.25	900.35	800.2
Standard deviation (σ)	29.993	30.003	99.905	19.928	99.952	39.748



5 PMF of total link travel time for group 1

deviation of the total link travel time of each group are included in Table 3. For the sake of brevity, only the PMF of total link travel time assigned to Group 1 and its corresponding normal distribution are depicted in Fig. 5.

As total link travel times are assumed to be normally distributed, fortunately, the probability distribution function of STT in this special case can be readily estimated. It follows that for this example, the STT will be a normal random variable with the following parameters:

$$\mu_{\mathrm{STT}} = 4 \times (499.63 + 1000.1 + 799.58 + 300.25 + 900.35 + 800.2) = 17200.44$$

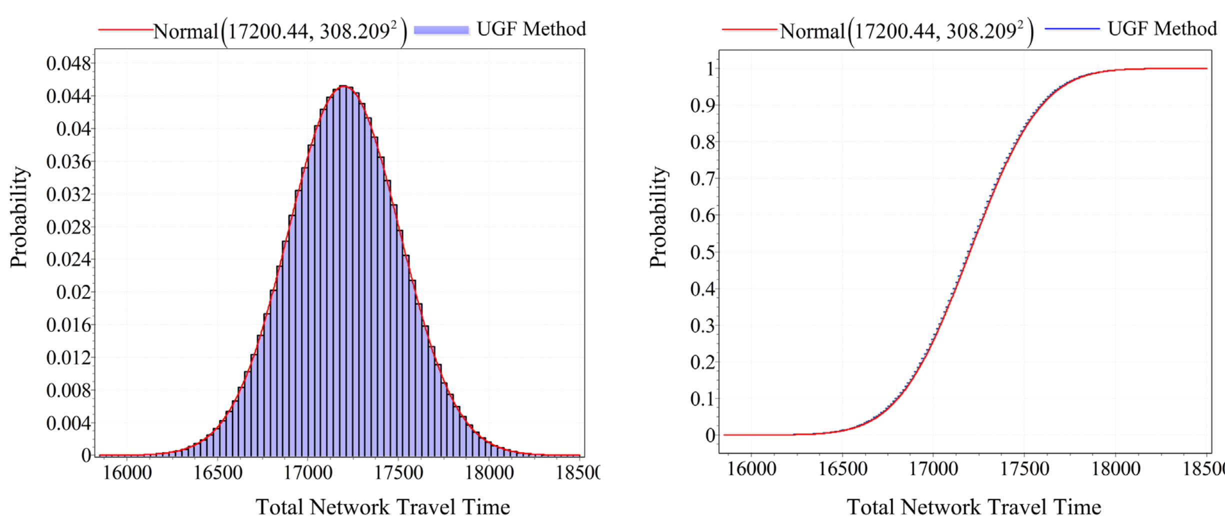
$$\sigma_{\mathrm{STT}} = \sqrt{4 \times \left(29.993^2 + 30.003^2 + 99.905^2 + 19.928^2 + 99.952^2 + 39.748^2\right)} = 308.209$$

The cumulative distribution function of STT has also been estimated using the UGF method. The PMF and the CDF of STT obtained from this method are shown in Fig. 6*a* and *b*, respectively. As can be seen in Fig. 6*b*, the PDF and CDF of $N(17200.44, 308.209^2)$ (the red curves) are well approximated by those of STT obtained from the UGF method. Furthermore, the results of Kolmogorov–Smirnov test show that the CDF of STT obtained from the UGF method matches the normal distribution $N(17200.44, 308.209^2)$ at the significance level of 0.01.

5.2. Example 2: A network with lognormally distributed link travel time

This section assesses the ability of the UGF method to estimate the PMF of STT, assuming that total link travel times follow the lognormal distribution as a case of skew distribution. Similar to the previous example, the set of links is classified into six groups of four links and each group of links has a specified PMF. The parameters of these distributions for each of the groups are summarized in Table 4. Figure 7 shows the PMF of link travel time assigned to Group 1 and its corresponding lognormal distribution as a case in point.

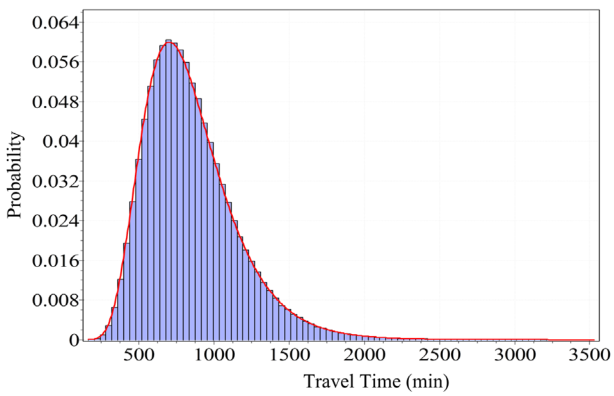
The cumulative distribution function of STT has been estimated using the UGF method. In order to examine the ability of the UGF method, a Monte Carlo simulation technique (with confidence level of 0.99 and relative error of 0.01) has been performed to estimate STT for this example. Figure 8*a* and *b,* respectively, compares the PMFs and the CDFs of STT obtained from the UGF method and the Monte Carlo simulation study, implying the fact that the UGF method can estimate STT with reasonable accuracy. Stated more explicitly, the two-sample



6 PMF and cumulative distribution function of STT

Table 4 Parameters of probability distribution of link travel time for each group

Group	Group 1	Group 2	Group 3	Group 4	Group 5	Group 6
Location parameter (μ)	6.677	6.908	5.702	6.213	6.796	6.684
Scale parameter (σ)	0.3524	0.1731	0.2575	0.2441	0.3324	0.2232



7 PMF of total link travel time for group 1

Kolmogorov–Smirnov test does not reject the null hypothesis that the PMFs obtained by simulation and UGF method are not different at the 0.05 significance level.

5.3. Example 3: A network with uniformly distributed link capacity

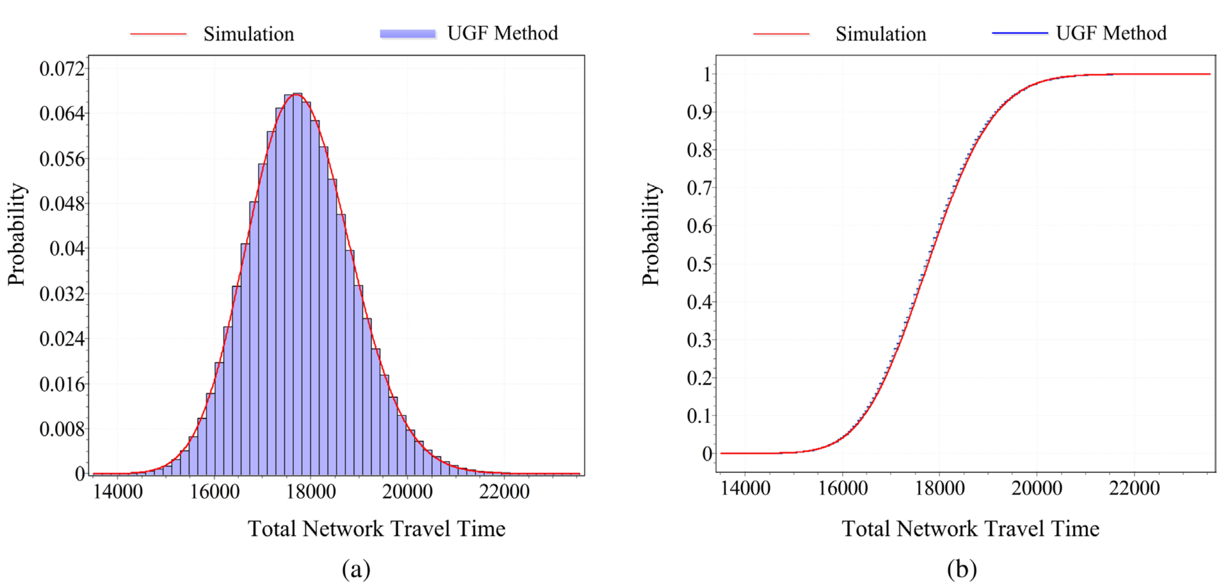
In this example, capacity is assumed to be the only source of uncertainty. For this purpose, the well-known Sioux Falls network as a medium-sized network is employed. This network consists of 24 nodes and 76 links as illustrated in Fig. 9. The network characteristics and demand matrix were derived from a study by Ferris *et al*. (1999). In order to determine the probability distribution function of total link travel time considering link capacity as a random variable, we adopted the method suggested in a study by Ng and Waller (2010) (see Appendix A). For the convenience of the reader, the PMFs of total link travel times are available at:

http://webpages.iust.ac.ir/mojtaba_rajabi/Data.htm

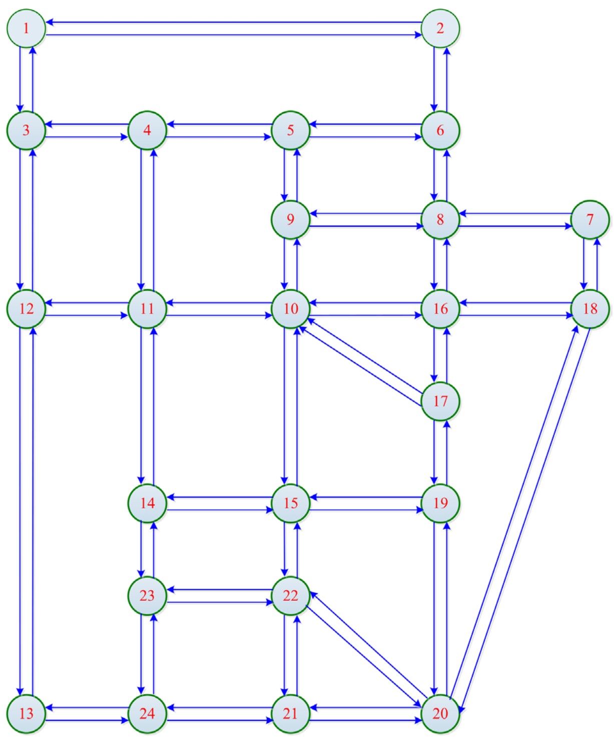
The comparison results of the estimation of STT using the UGF method and a Monte Carlo simulation technique for Sioux-Falls network are shown in Fig. 10. Based on the results of the two-sample Kolmogorov–Smirnov test, it can be concluded that the PMFs obtained by simulation and UGF method are not different at the 0.05 significance level. An important point to be noted is that the computation time of the UGF method and Monte Carlo simulation method (with confidence level of 0.99 and relative error of 0.01) are 0.53 and 1864 s, respectively.

As discussed earlier, when dealing with continuous distributions, an important point in utilizing the UGF method would be to determine a suitable quantity for S_{max}. To do so, we have examined a number of quantities ranging from 10 to 1000 (Fig. 11). As illustrated, after S_{max} reaches 100, the accuracy of PMF estimation does not alter much while the computation time increases dramatically due to the multiplicative property presented in Equation (6). Therefore, 100 can be a good quantity for S_{max}.

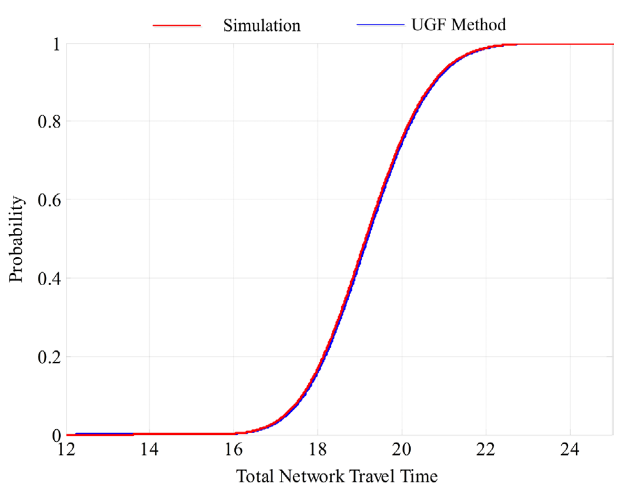
Figure 12 shows how the application of the proposed truncation methods can reduce the computation efforts in terms of sec of CPU time. As seen, only by truncating the PMF of each link and applying the UGF method (as in Truncation Method 1) the run time cannot be reduced much whereas by repetitive elimination of unimportant travel time states during the



8 PMF and cumulative distribution function of STT



9 Sioux-Falls network



10 CDF of STT for the Sioux-Falls network

implementation of Algorithm 1 (as in Truncation Method 2) a significant reduction would be obtained, especially for greater levels of confidence. To gain a better insight into the differences between the results of the different methods, the PMFs of STT obtained from different methods are illustrated in Fig. 13 for confidence level of 0.999.

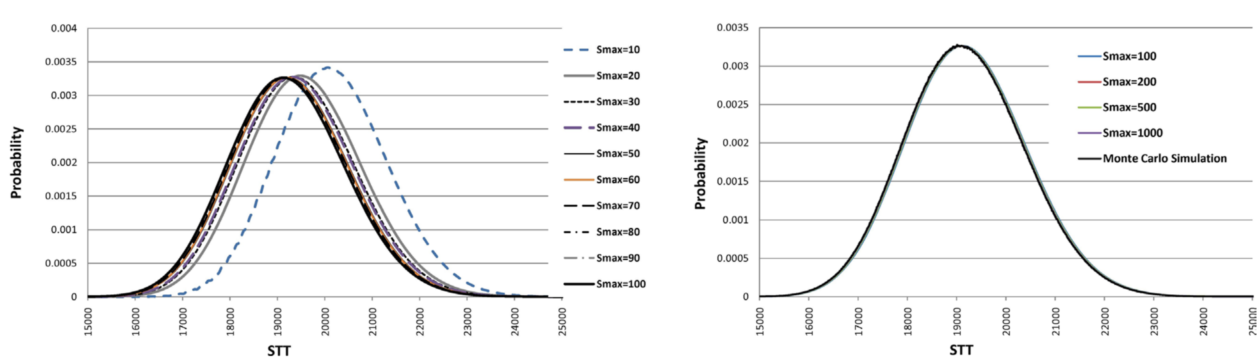
Different travel time reliability measures are commonly used in practice to determine the travel time variability of a route or a network (see Appendix B for more details). Table 5 compares several travel time reliability measures on the Sioux-Falls network obtained from Monte Carlo simulation, UGF method, Truncation Method 1 and Truncation Method 2. As can be seen from this table, the UGF method and the two truncation methods can estimate the travel time reliability measures accurately.

5.4. Example 4: Application of the method to a real large-scale network

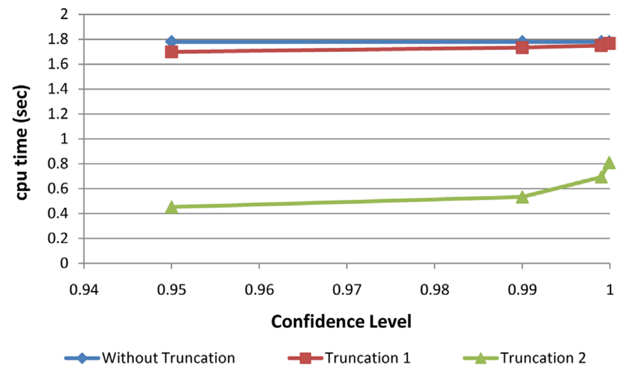
In order to investigate more characteristics of the application of the UGF method to large-scale networks, the network of Mashhad City, the second populous city in Iran, is employed. This network comprises 2765 links as depicted in Fig. 14. For the convenience of the reader, the PMFs are available at:

http://webpages.iust.ac.ir/mojtaba_rajabi/Data.htm

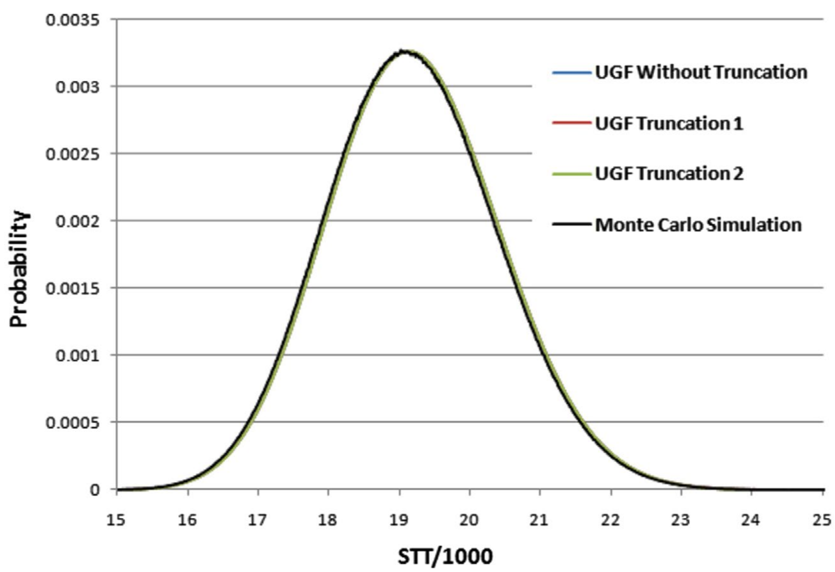
In this case, we have tended to evaluate the total network travel time uncertainty caused by traffic accidents. The total travel time experienced on each link is comprised of two parts: base travel time and delay. The first part, base travel time, is a constant quantity for each link, and can be simply calculated through multiplying accident-free travel time by link volume. The second part, total experienced delay, is a random variable. The PMF of the total experienced delay on each link is determined based on the capacity degradation caused by traffic accidents. Details of the determination of link delay PMFs, based on probabilistic traffic accident frequency and duration, cannot be discussed here due to page limitation, and more details can be found in Babaei (2013). It is obvious that traffic accidents can cause long queues on a link and therefore spilling back into some downstream links, and, hence, the assumption of delay independence between all links cannot be hold ideally. However, since making such assumption will result in a lower bound of total delays over the network, this would make a good tool to evaluate the minimum levels of travel time reliability and average delays.



11 The PMF of STT for the Sioux-Falls network with different values of S_{max}



12 CPU times of proposed truncation methods for different confidence levels



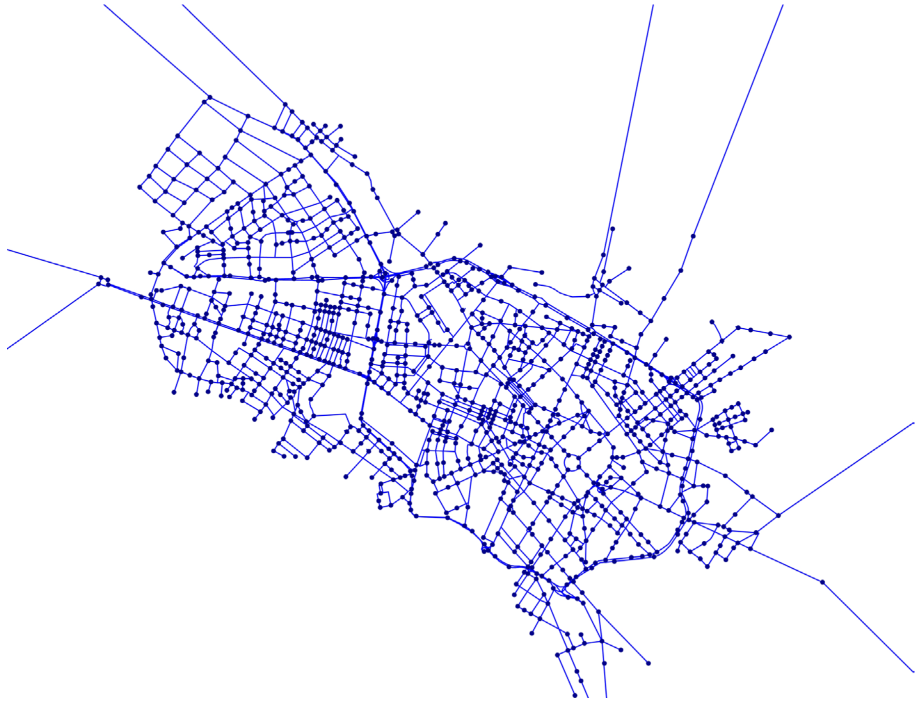
13 Comparison of PMF of STT obtained from different methods, Ω = 0.999and S_{max} = 100

Table 5 Comparison of travel time reliability measures

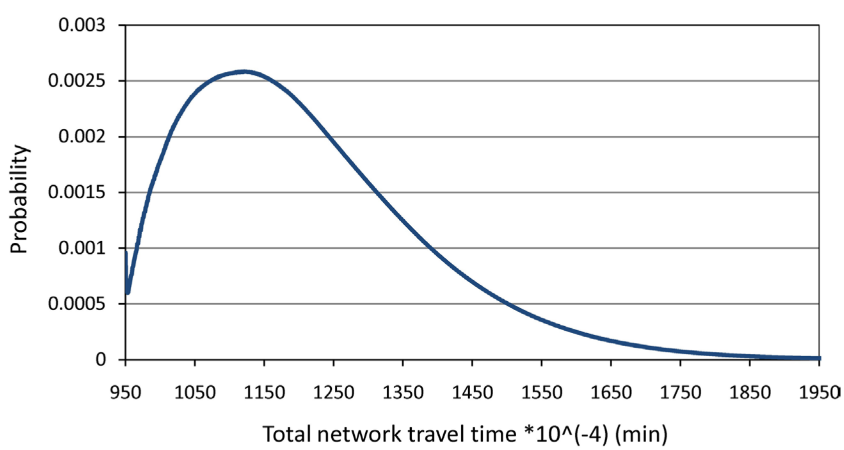
	COV	BT	PT	λ^{skew}	λ^{var}
Simulation	0.0634	0.1071	1.1854	1.0728	0.1635
UGF (without truncation)	0.0632	0.1067	1.1843	1.0662	0.1626
Truncation method 1	0.0632	0.1067	1.1843	1.0662	0.1626
Truncation method 2	0.0630	0.1068	1.1843	1.0662	0.1626

The average run time for this real-world case is 1.1248×10^6 CPU seconds using Monte Carlo simulation (with confidence level of 0.99 and relative error of 0.01). Applying the UGF method with $S_{max} = 200$, however, this is reduced to 14.8 CPU seconds. This reflects the fact that the UGF method has the ability to estimate the PMF of total STT within reasonable CPU time even in very large transportation networks. The results of the two-sample Kolmogorov–Smirnov test show that the PMFs obtained by simulation and UGF method are not different at the 0.05 significance level.

Figure 15 depicts the PMF of STT for Mashhad's network. The average total network travel time and the average total experienced delay for the whole network are approximately 12.18×10^6 and 2.67×10^6 min, respectively. This means that each of 517,000 passengers traveling during a one-hour off-peak is suffering 5 min delays due to only one source of recurrent events (i.e. traffic accidents), and this seems to be very large as compared with the average base travel time of 18.6 min.



14 Mashhad network



15 PMF of STT for Mashhad network

6. Conclusions

In this paper, the UGF has been adapted to estimate the probability distribution function of total network travel time. The method is easy to be implemented in practice since it requires only simple algebraic operations. The results included in the last section reveals the capability of this method in terms of accuracy and speed. For example, the application of the UGF method on a 76-link network has resulted in 0.53 s of CPU time, which is considerably smaller than the 1864 s of CPU time resulted from the Monte Carlo simulation method (with confidence level of 0.99 and relative error of 0.01). Furthermore, it has been shown that the use of the proposed Truncation Method 2, which repetitively eliminates unimportant travel time states during the implementation of Algorithm 1, can significantly reduce computation efforts while guaranteeing pre-specified confidence levels.

Since, in many cases, link travel times are subjected to simultaneous (correlated) fluctuations, an interesting topic for further research would be to extend the proposed method to the cases where travel times on adjacent links are spatially and/or temporally correlated.

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Appendix A. Determination of the probability density function of total experienced travel time of a link considering link capacity as a random variable.

Preposition 1. Assume that the travel time on link $a \in A$ is given by the widely used BPR volume-delay function as

$$\tilde{t}_a = \tilde{t}_a^f \left(1 + \alpha \left(\frac{v_a}{\tilde{c}_a}\right)^{\beta}\right).$$

By considering the capacity of link a as a random variable denoted by c_a, one can rewrite the BPR function as

$$t_a = \tilde{t}_a^f \left(1 + \alpha \left(\frac{v_a}{c_a}\right)^{\beta}\right).$$

Now let $f_{ca}(.)$ be the probability density function of the capacity of link a. Based on Ng and Waller (2010), the probability density function of the random variable $T_a = t_a v_a$ can be obtained as

$$f_a(x) = -f_{ca}(c^*)\frac{c^*}{\beta\left(v_a t_a^f - x\right)},$$

where

$$x > t_a^f v_a \quad \text{and} \quad c^* = \frac{v_a}{\left(\frac{1}{\alpha}\left(\frac{x}{v_a t_a^f} - 1\right)\right)^{1/\beta}}.$$

The proof of this preposition is straightforward using Theorem 7.1 of the book written by Ross (1997).

Appendix B. Travel time reliability measures

Let μ and σ be the mean and the standard deviation of STT, respectively. Also, let T_α denotes the αth percentile of STT. Different travel time reliability measures can be defined as follows:

$$\text{Coefficient of variation: COV} = \frac{\sigma}{\mu},$$

$$\text{Buffer time index: BI} = \frac{T_{95} - \mu}{\mu},$$

$$\text{Planning time index: PI} = \frac{T_{95}}{T_{15}},$$

$$\text{Width index: } \lambda^{\text{skew}} = \frac{T_{90} - T_{50}}{T_{50} - T_{10}},$$

$$\text{Skew index: } \lambda^{\text{var}} = \frac{T_{90} - T_{10}}{T_{50}}.$$