Traffic Assignment Using a Density-Based Travel-Time Function for Intelligent Transportation Systems

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***Abstract*—This paper presents and shows why density-based travel-time function is consistent with the fundamental diagram from traffic theory and then reviews the applications of travel time in intelligent transportation systems. This paper presents a density-based travel-time function that does not have the ill-posedness that is present in flow-based travel-time functions. The classic steady-state traffic assignment is cast using this new travel-time function, and corresponding mathematical programming formulations are proposed. It is shown that the modified Beckman formulation based on the density-based travel-time function provides a unique solution for link flows and link densities where the Wardrop condition has nonunique values for arc traffic densities.**

***Index Terms*—Travel time, shockwaves, traffic assignment.**

I. Introduction

TRANSPORTATION systems are built on the expectation of providing mobility with safety. Travel time is the most suited variable of interest in developing mobility performance index. Travel time is used in many design and operations problems in transportation. For instance, it is used to perform traffic assignment ([1]), whether that is static assignment ([2]) or dynamic traffic assignment ([3]). Feedback control based dynamic traffic assignment is also based on travel time ([4]–[8]). Travel time is an essential element of transportation planning models ([9], [10]). Similarly, it is used to develop performance measures for before and after events studies. It is also used to monitor the performance of the system in real time, as well as for aggregate performance over a time period for decision making and policy considerations.

Federal Highway Administration (FHWA) report ([11]) presents different techniques for the collection of data related to travel time. Researchers have studied the estimation and prediction of travel time for many applications ([12]–[14]).

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Dynamic model for travel time using density was first presented in [15] which was followed by a continuum field model in [3].

This paper presents the static traffic assignment problem framed using this density based travel time function. By using the density based travel time function we are able to use the macroscopic traffic flow theory and also connect it to microscopic theory since we use the speed of a vehicle as the travel time function. Hence this new framework provides a unification of the traffic theory for traffic assignment problem.

Outline: The remainder of this paper is organized as follows. We review the travel time function used in practice in Section II, where in Section II-A we show its application in static traffic assignment. Section III develops the formula for travel time delay at signalized intersections using queuing theory. Finally Section IV provides the formulation of the static assignment problems as mathematical programming problems and provides analysis of those models. We provide some ITS applications of travel time function in Section V.

II. Flow Based Travel Time Function and Its Applications

Flow based travel time function used in steady state conditions in its generic form is given by Equation (1)

$$T(f) = t_f \phi\left(\frac{f}{C}\right). \tag{1}$$

In this formula, f is the traffic volume on the link, C is the capacity of the link given by the maximum flow possible on that link, and t_f is the time taken to traverse the link in free flow conditions, i.e., when the density is zero and consequently the vehicles have free flow (maximum) speed.

Bureau of Public Roads (BPR) (now FHWA) uses formula (2) for $\phi(\cdot)$ ([16]) which has a parameter α with a typical value of 1 and β which typically ranges between 2 and 12. The plot of the BPR travel time function with respect to flow is shown in Fig. 1.

$$\phi\left(\frac{f}{C}\right) = \left(1 + \beta\left(\frac{f}{C}\right)^{\alpha}\right). \tag{2}$$

A flow based steady state travel time function, $T(f)$, T being the travel time function, f being the flow, is designed to satisfy

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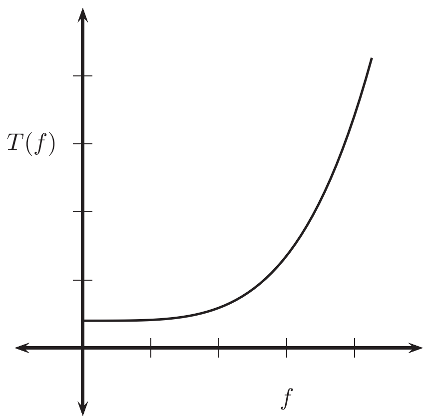

Fig. 1. BPR travel time function.

TABLE I
NOTATION FOR NETWORK

$\mathfrak{N}$	Set of Vertices
$\mathfrak{A}$	Set of Directed Edges
$\mathfrak{R}$	Set of Origin Nodes
$\mathfrak{S}$	Set of Destination Nodes
$\mathfrak{K}$	Set of Paths for O-D pair $r - s$, $r \in \mathfrak{R}$, $s \in \mathfrak{S}$
x_a	Flow on Edge $a \in \mathfrak{A}$
t_a	Travel time on Edge $a \in \mathfrak{A}$
f_k^{rs}	Flow on Path $k \in \mathfrak{K}$ between O-D pair $r - s$
c_k^{rs}	Travel time on Path $k \in \mathfrak{K}$ between O-D pair $r - s$
f_{rs}	O-D Trip Rate between O-D pair $r - s$
$\delta_{a,k}^{rs}$	$\delta_{a,k}^{rs} = 1$, if a is in path k between r and s, otherwise 0

certain properties ([17]), such as:

- **Second Order Continuously Differentiable**: $T(\cdot) \in C^2$
- **Positive**: $\forall\, f \geq 0, T(f) \geq 0$
- **Monotonic**: $f_1 \geq f_2 \Rightarrow T(f_1) \geq T(f_2)$
- **Strictly Monotonic Slope**: $T'' > 0$
- **Bounded Slope**: $\exists M > 0, T' \leq M$
- **Unique**: $T'(0) > 0$

where we have used the prime superscript to denote differentiation. In literature we can find different travel time functions related to these properties ([17]–[19]).

A. *Traffic Assignment Problem*

For the sake of illustration, using the notation and example from [2] we consider a digraph with four vertices and four directed edges. There are two origin nodes, 1 and 2 and node 4 is the destination node. In this example we have 1–4 and 2–4 as two O-D pairs. The variables are described in Table I.

Traffic assignment problem deals with assigning O-D flows to the corresponding paths. User-equilibrium formulation involves flow assignment in such a way as to make travel times the same on the used alternate routes, and the system optimum formulation involves minimizing the total travel time.

1) User-Equilibrium: Wardrop's principle [20] states that the travel times should be the same in alternate paths between any O-D pair except for unused routes travel time on which can not be less than that of the used paths. User equilibrium formulation is based on this principle and in terms of a mathematical programming is given below:

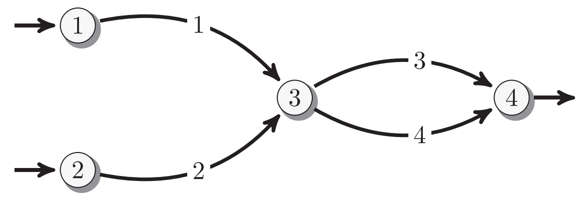

Fig. 2. Example network.

a) Mathematical programming for user-equilibrium: The mathematical programming associated with the user-equilibrium formulation ([2], [21]) is

$$\begin{aligned} \min\ z(x) &= \sum_a \int_0^{x_a} t_a(\omega) d\omega \\ \text{with} \sum_k f_k^{rs} &= f_{rs}\ \forall\, r, s \\ x_a &= \sum_r \sum_s \sum_k f_k^{rs} \delta_{a,k}^{rs} \\ f_k^{rs} &\geq 0\ \forall\, r, s. \end{aligned} \tag{3}$$

This user-equilibrium problem is the Beckmann transformation [1]. The travel time function from BPR that is typically used for this formulation is

$$t_a(x_a) = v_f \left(1 + 0.15 \left(\frac{x_a}{c_a}\right)^4\right). \tag{4}$$

2) System Optimal: The system optimal mathematic programming problem presented below involves solving for total travel time for all travelers (see [2], [21]).

a) Mathematical programming for system optimal: The mathematical programming problem statement for system optimal is presented below.

$$\begin{aligned} \min\ z(x) &= \sum_a x_a t_a(x_a) \\ \text{with} \sum_k f_k^{rs} &= f_{rs}\ \forall\, r, s \\ x_a &= \sum_r \sum_s \sum_k f_k^{rs} \delta_{a,k}^{rs} \\ f_k^{rs} &\geq 0\ \forall\, r, s. \end{aligned} \tag{5}$$

B. *Limitations of the Flow Based Travel Time Functions*

There are many speed density models that have been proposed for traffic (linear in Greenshield's model, [22], logarithmic in Greenberg model, [23], exponential in Underwood model, [24], piecewise linear in cell transmission model [25]), and many more such as Northwestern University model, Drew model, Pipes-Munjal model, and multi-regime models ([6], and references therein). Although we present Greenshield's model for our analysis, the steps can be modified for other models.

1) Fundamental Diagram: We use Greenshield's model that for a given free flow speed v_f and jam density ρ_m gives traffic speed as

$$v(\rho) = v_f \left(1 - \frac{\rho}{\rho_m}\right). \tag{6}$$

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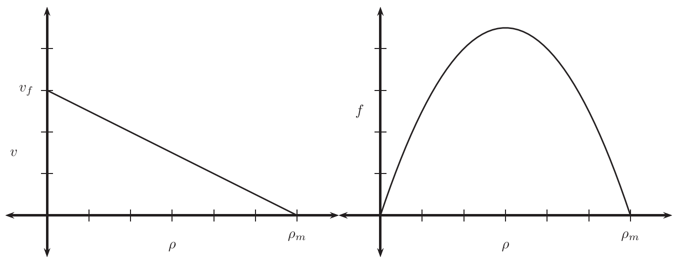

Fig. 3. Fundamental diagram.

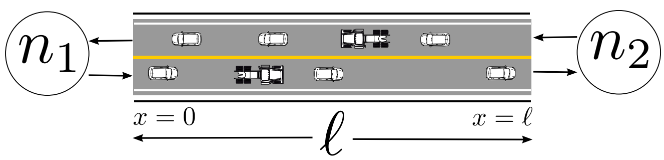

Fig. 4. Highway section.

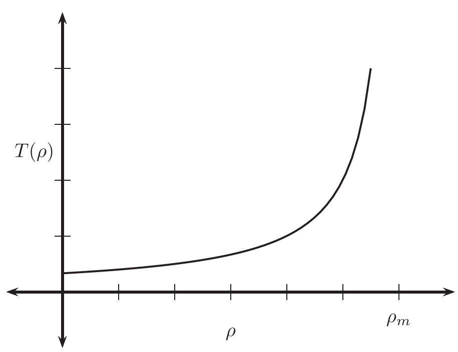

Fig. 5. Density based travel time function.

Traffic flow expression is

$$f(\rho) = v_f \rho \left(1 - \frac{\rho}{\rho_m}\right). \tag{7}$$

Fig. 3 shows the diagram.

Link capacity is the maximum flow on the link obtained as

$$C = \frac{1}{4} v_f \rho_m. \tag{8}$$

2) Travel Time Based on the Fundamental Diagram: Consider Fig. 4 showing a highway arc between nodes n_1 and node n_2, of length ℓ and a uniform traffic density on it being ρ_0).

The travel time on this arc given by the quotient of length and the speed is

$$T = \frac{\ell}{v_f \left(1 - \frac{\rho_0}{\rho_m}\right)}. \tag{9}$$

This travel time function based traffic density is plotted in Fig. 5.

The highway stretch is in equilibrium (steady state). The flow into the highway from node n_1 is equal to the flow out from the section to the node n_2, and is given by:

$$f = v_f \rho_0 \left(1 - \frac{\rho_0}{\rho_m}\right). \tag{10}$$

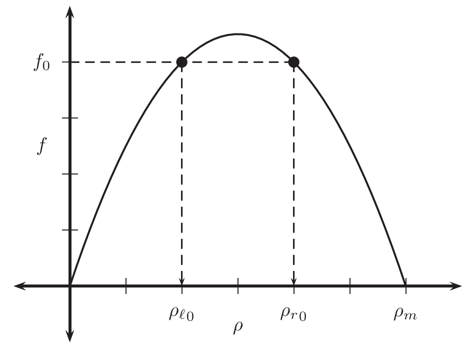

Fig. 6. Non-unique densities for the same flow.

TABLE II
INTERSECTION DELAY PARAMETERS

Parameter	**Meaning**
f_i	Rate of inflow at intersection
f_o	Rate of outflow from intersection
$F_i(t)$	Cumulative inflow at intersection
$F_o(t)$	Cumulative outflow at intersection
t_C	Cycle time
t_R	Red Time
t_G	Green Time
t_s	Saturated Flow Time
t_a	Time of arrival for a vehicle
t_d	Time of departure for a vehicle
t_D	Delay time for a vehicle ($t_D = t_d - t_a$)
$q(t)$	Queue length at time t ($q(t) = F_i(t) - F_o(t)$)
q_{av}	Average queue length per time at an intersection
t_{aD}	Average vehicle delay through an intersection
ℓ	Section length

For a given flow f_0 there are two corresponding densities $\rho_{\ell 0}$ and ρ_{r0} as shown in Fig. 6.

For that given flow there are two different density based travel times as shown in Equation (11)

$$T(\rho_{\ell 0}) = \frac{\ell}{v_f \left(1 - \frac{\rho_{\ell 0}}{\rho_m}\right)}, \quad T(\rho_{r0}) = \frac{\ell}{v_f \left(1 - \frac{\rho_{r0}}{\rho_m}\right)}. \tag{11}$$

A vehicle travels with a speed that is consistent with the traffic density, and hence its travel time depends on that density. Since flow gives two different densities and hence two different travel times, flow can not determine a unique travel time on a link.

III. AVERAGE TRAVEL TIME DELAY AND QUEUE LENGTH PER SIGNALIZED INTERSECTION

Travel time delay due to a signalized intersection can be calculated using Webster's formula ([26] and [27]). We will use Webster's uniform delay model since our problem statement deals with steady state conditions. This delay model is derived by considering the cumulative inflow of vehicles at the intersection as well as the cumulative outflow at the intersection.

The variables used in the analysis are shown in Table II.

Fig. 7 shows all these parameters, where the straight line plot of $F_i(t)$ is shown for a cycle, as well as the piecewise linear

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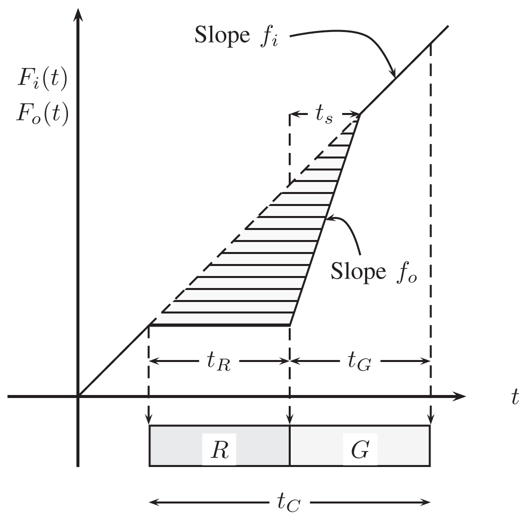
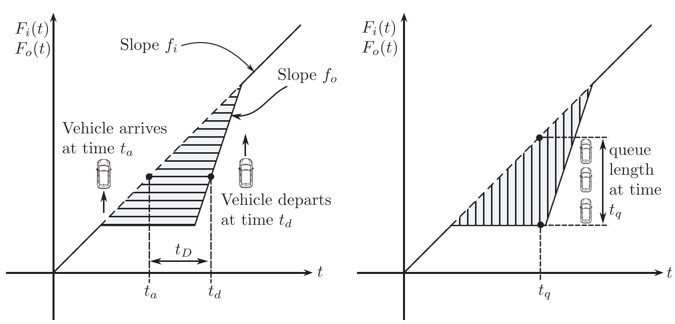


Fig. 7. Queuing at an intersection.



(a) (b)

Fig. 8. Vehicle delay and queue formation at an intersection.

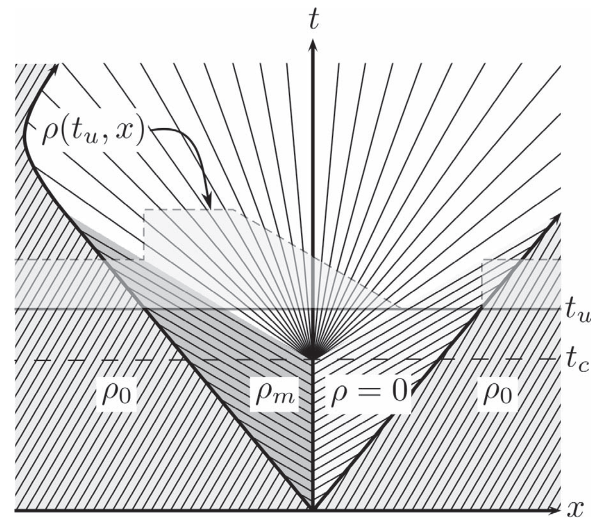


Fig. 9. Traffic characteristics.

plot of $F_o(t)$ is shown. The relationship between the cumulative flows and the flows are shown in Equation (12).

$$\frac{dF_i(t)}{dt} = f_i$$
$$\frac{dF_o(t)}{dt} = \begin{cases} 0, & \text{during red phase} \\ f_o, & \text{during saturation flow phase} \\ f_i, & \text{otherwise.} \end{cases} \quad (12)$$

Fig. 8(a) shows the arrival time for a vehicle obtained from $F_i(t_a)$. The departure time for the same vehicle satisfies $F_i(t_a) = F_o(t_d)$. Moreover, the delay time for the vehicle is given by $t_D = t_d - t_a$. Fig. 8(b) shows the queue length at some time t_q. Queue length is given by $q(t) = F_i(t) - F_o(t)$.

The number of vehicle arriving during one full cycle is $f_i t_C$. The aggregate delay for all these vehicles is the area of the triangle. The area of the triangle divided by the number of vehicles during a cycle is the average vehicle delay. By exploiting the equality $f_i(t_R + t_s) = f_o t_s$ we derive the average delay as

$$t_{aD} = \frac{t_R^2}{2t_C} \frac{f_o}{f_o - f_i}. \quad (13)$$

We can also obtain the average queue length as follows. This averaging will be over time as compared to the averaging over the number of vehicles that was performed for the delay calculations. The average queue length is calculated by averaging over the cycle time as shown in Equation (14).

$$q_{av} = \frac{1}{t_C} \int_0^{t_C} (F_i(t) - F_o(t))\, dt. \quad (14)$$

This is the area of the triangle in Fig. 7 divided by the cycle time. This area of the triangle is the same in Fig. 8(a) and (b), and hence is obtained from Equation (15). The average queue length is

$$q_{av} = f_i t_{aD}. \quad (15)$$

Substituting Equation (13) in Equation (15), we obtain the expression for the average queue length as

$$q_{av} = \frac{t_R^2}{2t_C} \frac{f_i f_o}{f_o - f_i}. \quad (16)$$

A. *Shock Based Average Travel Time Delay and Queue Length per Signalized Intersection*

Fig. 9 shows the shock analysis of the traffic density when the traffic light turns from green to red, and also from red to green. Before time $t = 0$ the traffic density is ρ_0 everywhere. At time $t = 0$ the signal at $x = 0$ turns red. We see immediately four wedge shaped regions emanating in positive time. The rightmost and the left most wedges have density ρ_0 and the other right one has density zero, and the left one has the traffic jam density (the queue) ρ_m. Shock speed is obtained using Rankine-Hugoniot condition ([28]), which gives $s = [f(\rho_m) - f(\rho_0)]/(\rho_m - \rho_0)$. The shock travels backwards with speed

$$s = -\frac{v_f \rho_0}{\rho_m - \rho_0} \left(1 - \frac{\rho_0}{\rho_m}\right). \quad (17)$$

Fig. 9 shows the red light being on till time $t = t_c$ at which time the light turns green. The inflow is a constant flow given by

$$f_i = v_f \rho_0 \left(1 - \frac{\rho_0}{\rho_m}\right). \quad (18)$$

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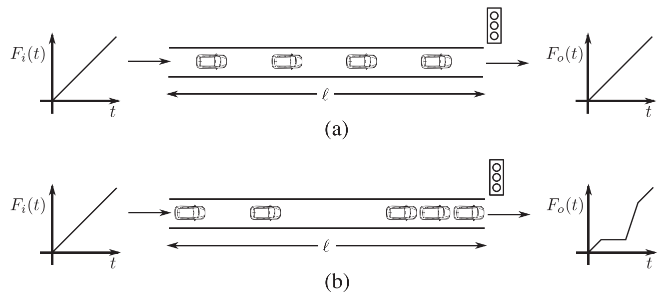

Fig. 10. Total vehicle delay. (a) Vehicle delay with uninterrupted flow. (b) Vehicle delay with interrupted flow.

The outflow is zero during the red phase. During the green phase the outflow is at the saturation level, and is given by

$$f_o = \frac{1}{4} v_f \rho_m. \tag{19}$$

The derivation of this saturation flow for this problem can be obtained from analysis ([29] and [30]).

Substituting Equation (18) and Equation (19) in Equation (13), the average travel time delay due to the signalized intersection is obtained as

$$t_{aD} = \frac{t_R^2}{2t_C} \frac{\rho_m^2}{\rho_m^2 - 4\rho_m \rho_0 + 4\rho_0^2}. \tag{20}$$

We can substitute Equation (18) and Equation (19) into Equation (16) to obtain the expression for the average queue length as

$$q_{av} = \frac{v_f t_R^2}{2t_C} \frac{\rho_0 \rho_m^2}{\rho_m^2 - 4\rho_m \rho_0 + 4\rho_0^2} \left(1 - \frac{\rho_0}{\rho_m}\right). \tag{21}$$

B. Travel Time Delay on a Link With Signalized Intersection at the End

In this subsection we calculate the average delay on a link that is terminated by a signalized intersection. Previously we calculated the travel time delay for a fixed length link, and also the average travel time delay through a signalized intersection. In order to calculate the average travel time through the link and the intersection, we need to calculate the travel time to reach the back end of the queue, and then the time to exit the front of the intersection. However, because of the queuing analysis using cumulative flows, the analysis becomes extremely simple as explained next.

If the link has green light all the time and is in steady state at density ρ_0, then the inflow rate and the outflow rate are both same as shown in Fig. 10(a). In that case the travel time to cross the link of length ℓ is given in Equation (9) and repeated here for convenience.

$$T_u = \frac{\ell}{v_f \left(1 - \frac{\rho_0}{\rho_m}\right)} \tag{22}$$

where we have used the subscript u to indicate uninterrupted flow. When we add a signalized intersection of a cycle time t_C such that the maximum queue length is less than ℓ and moreover, the entire queue is dissipated within one cycle, then the additional time delay for an average vehicle is given in Equation (20) and repeated here for convenience.

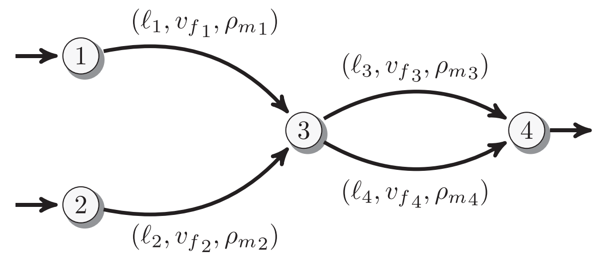

Fig. 11. Sample network with density based travel time.

$$t_{aD} = \frac{t_R^2}{2t_C} \frac{\rho_m^2}{\rho_m^2 - 4\rho_m \rho_0 + \rho_0^2}. \tag{23}$$

Hence, the total average travel time on a link is

$$T = \frac{\ell}{v_f \left(1 - \frac{\rho_0}{\rho_m}\right)} + \frac{t_R^2}{2t_C} \frac{\rho_m^2}{\rho_m^2 - 4\rho_m \rho_0 + \rho_0^2}. \tag{24}$$

Here, we are assuming that the density on the link remains constant in the steady state on all links. This is the assumption made for static traffic assignment as compared to the dynamic traffic assignment.

IV. Static Traffic Assignment Problem Formulation Using Density Based Travel Time Function

In this section we now build the density based traffic assignment problem. As shown in Section III, we can include all the signalized intersection on an arc, and then add the delay due to all the intersections on that arc on top of the link travel time to compute the overall travel time on an arc. Data on each arc can be represented by a list of parameters of the subcomponent arcs of that specific arc. Any subarc which is unsignalized will have its parameter data represented by (ℓ, v_f, ρ_m), and any signalized will have its parameter data represented by $(\ell, v_f, \rho_m, t_C, t_R)$. So, as an example, on a single arc, its data can be represented by

$$\mathfrak{a}_i = \Big((\ell_{i1}, v_{f_{i1}}, \rho_{mi1}, t_{Ci1}, t_{Ri1}) \\ (\ell_{i2}, v_{f_{i2}}, \rho_{mi2}), \ldots, \left(\ell_{i_n}, v_{f_{i_n}}, \rho_{m i_n}\right)\Big). \tag{25}$$

To illustrate the steps, we again use the sample network of Fig. 2 but we modify it with the new data that is needed to solve the density based problem. This is shown in Fig. 11. The digraph shows four nodes and four arcs. Nodes 1 and 2 are again the origin nodes and node 4 is the destination node. Hence, just like in the previous sample network, we again have two O-D pairs: 1–4 and 2–4. The data and parameters required in the flow based assignment were O-D flow matrix, and capacity on each link. This information is replaced in the density based assignment as shown in Table III. As we have

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TABLE III
REPLACEMENT IN THE DENSITY BASED ASSIGNMENT

Original	**Replacement**
Flow f	Density ρ
Capacity C	Arc Parameter List (ℓ, v_f, ρ_m)

TABLE IV
DENSITY RELATED ADDITIONAL NETWORK NOTATION

ρ_a	Density on arc $a \in \mathfrak{A}$
$f(\rho_a)$	Flow corresponding to ρ_a
ρ_{ak}^{rs}	Density on arc a due to path $k \in \mathfrak{K}$ between O-D pair $r - s$
fo_{rs}	O-D flow from r to s
n_I	Set of arcs entering node n
n_O	Set of arcs exiting node n

seen that flow alone can not fix the travel speed for vehicles. By specifying an O-D matrix of traffic densities, we are able to specify flow as well automatically, since flow is a function of traffic density. Moreover, the travel time now can be computed using the vehicle speed formula based on the density, which, instead of the capacity, utilize the free flow speed, jam density, and the arc length parameters, as well as cycle time parameters for the signalized intersections. For the sake of simplification, from now on we will consider only the link travel time delay instead of also having the signalized intersection delay for the analysis.

Density related additional notation needed for this section is shown in Table IV.

A. Flow Balance at Nodes

Although the new formulation is in terms of density as is necessitated by the travel time being a direct function of density rather than flow, the traffic inflow and outflow have to balance at nodes. To study the impact of this on the corresponding density values in equilibrium we study some cases next.

Consider a link having one inflow and one outflow as shown in Fig. 12. The parameters of the two links are shown in the figure. The flow balance at the node forces the inflow to be equal to the outflow giving us the following equation.

$$v_{f_1}\rho_1\left(1-\frac{\rho_1}{\rho_{m_1}}\right) = v_{f_2}\rho_2\left(1-\frac{\rho_2}{\rho_{m_2}}\right). \tag{26}$$

Given the OD flow by f_{13}, there are two different values of density ρ_1 that will satisfy that given flow. For each of those two values, there will be two values of ρ_2 as well. Hence there will be four different solutions matching the exact same flow conditions. However, the four conditions will have different travel times. The minimum density in each arc will produce the minimum travel time solution. If there were two alternate routes, then we could have same travel times in two routes but not necessarily with lowest possible travel times in them unless we force that condition.

In the case of having more than one route passing through the same two links, we require the route balance conditions also to match. Let us denote $f_{1,3}$ as the flow from origin 1 to destination 3. Let us divide this flow into two routes, f_1^{13} and f_2^{13}, for $\{k = 1, 2\}$. We have the route flow conservation condition

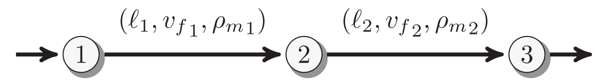

Fig. 12. Flow matching in series.

$$\sum_k f_k^{13} = f_{13}. \tag{27}$$

We use the notation f_{ak}^{rs} for the flow on the arc a corresponding to a route k of the origin destination pair rs. Now we can also define the density on an arc a corresponding to a route k of the origin destination pair rs using the corresponding flows as follows.

$$\rho_{ak}^{rs} = \frac{f_{ak}^{rs}}{f(\rho_a)}\rho_a. \tag{28}$$

This proves easily that

$$f_k^{rs} = f_{ak}^{rs}, \quad \text{if } \delta_{ak}^{rs} = 1. \tag{29}$$

For every node in a network, the sum of all the inflow to the node must equal the sum of all the outflow from the node. For instance, the flow balance for node 3 in Fig. 11 is as follows.

$$\sum_{i=1}^{2} v_{f_i}\rho_i\left(1-\frac{\rho_i}{\rho_{m_i}}\right) = \sum_{i=3}^{4} v_{f_i}\rho_i\left(1-\frac{\rho_i}{\rho_{m_i}}\right). \tag{30}$$

The node balance conditions also have conditions for balancing the route based flows similar to the case discussed above.

Now, we will consider the two main classical traffic assignment optimization problems as before. We will present the new density based formulations for the user-equilibrium and system optimum problems.

B. User-Equilibrium

User-equilibrium problem is based on Wardrop's principle [20]. However now, travel time is computed based on vehicle speed formula based on traffic density.

We will present this equilibrium condition as a necessary condition for the solution of a modified mathematical programming problem and provide the proof of this assertion.

1) Mathematical Programming Formulation:

Theorem 4.1: The user equilibrium problem is equivalent to the mathematical programming problem presented in Equation (31).

$$\min\ z\left(f_k^{rs}, \rho_a\right) = \sum_a \int_0^{f_a(f_k^{rs})} t_a(\omega)d\omega \tag{31}$$

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with the equality constraints:

$$\sum_k f_k^{rs} = f_{rs} \,\forall\, r, s \tag{32}$$

$$f_a\left(f_k^{rs}\right) = \sum_r \sum_s \sum_k f_{ak}^{rs} \delta_{a,k}^{rs} \tag{33}$$

$$\sum_r \sum_s \sum_k f_{ak}^{rs} \delta_{a,k}^{rs} = v_{f\,a} \rho_a \left(1 - \frac{\rho_a}{\rho_{m\,a}}\right) \tag{34}$$

$$f_{ak}^{rs} = f_k^{rs} \delta_{a,k}^{rs} \tag{35}$$

$$\sum_i v_{f\,i} \rho_i \left(1 - \frac{\rho_i}{\rho_{m\,i}}\right) = \sum_o v_{f\,o} \rho_o \left(1 - \frac{\rho_o}{\rho_{m\,o}}\right) \quad \forall\, i \in n_I,\ o \in n_O,\ n \in \mathfrak{N} \tag{36}$$

and the inequality constraint

$$\rho_a \geq 0 \,\forall\, a, \quad f_k^{rs} \geq 0 \,\forall\, k, r, s \tag{37}$$

$$\rho_{m\,a} \geq \rho_a \geq 0 \,\forall\, a. \tag{38}$$

The travel time function on the link $t_a(\rho_a)$ is a function of traffic density on the link and the link parameters as given by Equation (39).

$$t_a(\rho_a) = \frac{\ell_a}{v_{f\,a}\left(1 - \frac{\rho_a}{\rho_{m\,a}}\right)}. \tag{39}$$

Note that we can incorporate a different density based function that also takes into account the signalized and unsignalized intersection delays. We have already shown the formula for the signalized intersection delay, and we can also develop a queuing theory based formula for the unsignalized intersection delay.

Proof: The Kuhn–Tucker conditions for the mathematical programming problem given by Equation (31) can be obtained in terms of the Lagrangian given in Equation (40).

$$\begin{aligned}
&\mathfrak{L}(f, \rho, \lambda, \mu) \\
&= z\left[f_k^{rs}, \rho_a\right] + \sum_{rs} \lambda_{rs} \left(f_{rs} - \sum_k f_k^{rs}\right) \\
&\quad + \sum_a \mu_a \left(\sum_r \sum_s \sum_k f_k^{rs} \delta_{a,k}^{rs} - v_{f\,a} \rho_a \left(1 - \frac{\rho_a}{\rho_{m\,a}}\right)\right) \\
&\quad + \sum_n \gamma_n \left(\sum_i v_{f\,i} \rho_i \left(1 - \frac{\rho_i}{\rho_{m\,i}}\right) - \sum_o v_{f\,o} \rho_o \left(1 - \frac{\rho_o}{\rho_{m\,o}}\right)\right).
\end{aligned} \tag{40}$$

Here, λ_{rs} and μ_a are the Lagrangian multipliers. The relevant Kuhn–Tucker conditions $\forall\, k, r, s, a$ are:

$$\begin{aligned}
f_k^{rs} \frac{\partial \mathfrak{L}(f, \rho, \lambda, \mu)}{\partial f_k^{rs}} = 0, &\quad \frac{\partial \mathfrak{L}(f, \rho, \lambda, \mu)}{\partial f_k^{rs}} \geq 0 \\
\frac{\partial \mathfrak{L}(f, \rho, \lambda, \mu)}{\partial \lambda_{rs}} = 0, &\quad \frac{\partial \mathfrak{L}(f, \rho, \lambda, \mu)}{\partial \mu_a} = 0.
\end{aligned} \tag{41}$$

Applying these necessary conditions (41) to the mathematical program (31) we obtain the Wardrop conditions $\forall\, k, r, s$ as:

$$\begin{aligned}
f_k^{rs}\left(c_k^{rs} - \lambda_{rs}\right) = 0, &\quad c_k^{rs} - \lambda_{rs} \geq 0 \\
\sum_k f_k^{rs} = f_{rs}, &\quad f_k^{rs} \geq 0.
\end{aligned} \tag{42}$$

■

Because of inverse function theorem, each solution has a local injective mapping from density to flow except where $\partial f / \partial \rho = 0$. Hence locally we can consider the travel time to be a function of flow. We derive the local formula for travel time as a function flow next.

Traffic flow on an arc is a quadratic function of traffic density given by

$$f_a(\rho) = v_f \rho_a \left(1 - \frac{\rho_a}{\rho_m}\right). \tag{43}$$

The maximum flow f_m using the Greenshield formula is obtained at $\rho_m/2$ and is given by

$$f_m = \frac{v_f \rho_m}{4}. \tag{44}$$

We can solve for ρ_a in terms of f_a by solving the quadratic equation

$$\frac{v_f}{\rho_m} \rho_a^2 - v_f \rho_a + f_a = 0 \tag{45}$$

to yield, after using Equation (44) here,

$$\rho_a(f_a) = \frac{\rho_m}{2}\left(1 \pm \sqrt{1 - \frac{f_a}{f_m}}\right). \tag{46}$$

The travel time function can also be written in terms of the traffic flow locally (where $\partial f / \partial \rho \neq 0$) by utilizing Equation (46) as

$$t_a = \frac{\ell_a}{v_{f\,a}\left(1 - \frac{\rho_a}{\rho_{m\,a}}\right)} = \frac{\ell_a \rho_a}{f_a} = \frac{\ell_a}{f_a} \frac{\rho_m}{2}\left(1 \pm \sqrt{1 - \frac{f_a}{f_m}}\right). \tag{47}$$

We can rewrite this relationship as

$$t_a(x) = \frac{k}{x}\left(1 \pm \sqrt{1 - x}\right), \text{ with } x = \frac{f_a}{f_m}, \text{ and } k = \frac{\ell_a \rho_m}{2 f_m}. \tag{48}$$

Differentiating Equation (48) we obtain

$$\begin{aligned}
\frac{t_a\left(x(f_a)\right)}{df_a} &= \frac{t_a\left(x(f_a)\right)}{dx} \frac{dx}{df_a} \\
&= \left[-\frac{1}{x^2} \pm \left(\frac{1}{2\sqrt{1 - x}x} + \frac{\sqrt{1 - x}}{x^2}\right)\right] \frac{1}{f_m}.
\end{aligned} \tag{49}$$

The plots of Equations (48) for travel time as functions of flow and of its derivative in Equations (49) are shown in Fig. 13.

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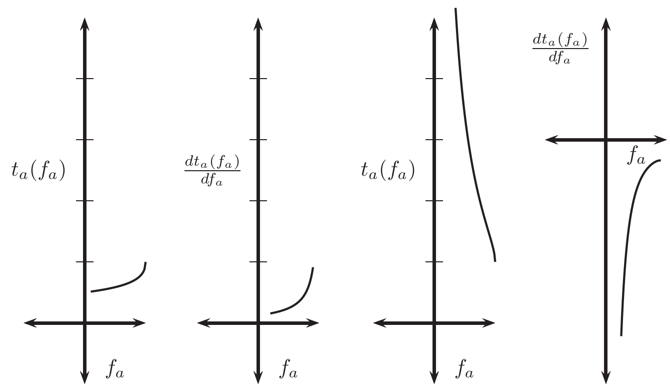

(a) (b) (c) (d)

Fig. 13. Travel time and derivatives plots.

The plots are drawn just to show the shapes of teh curve and the signs. These plots are not in true scale. The subfigure (a) shows the plot for the negative sign in Equation (48), plot (b) being its derivative. The subfigure (c) shows the plot for the positive sign in Equation (48), plot (d) being its derivative. The derivative has a positive sign in subfigure (b) as compared to the sign in subfigure (d).

The Hessian matrix with respect to arc flows f_a related to Equation (31) is a diagonal matrix with diagonal terms being the derivatives in Equation (49). This shows that in terms of arc flows there is a unique solution for the arc flows for the user equilibrium problem, which is obtained at flows corresponding to the uncongested densities, i.e., densities lower than the critical densities, i.e., densities leading to maximum flow. What this analysis shows is that the minimizing problem has a unique solution in terms of the arc flows and arc densities. However, the solution in terms of path flows and consequently densities on arcs that are assigned to various paths are not unique (see [2]).

Hence, we see that the density based travel time function retains the Wardrop condition as a necessary condition after modifying the mathematical programming problem to accomplish that. As we can see here though that with the density based travel time formulation we get a stronger result for the Beckman formulation. We obtain the unique arc densities also for the solution of the Beckman formulation, which we do not get by just solving the Wardrop condition.

C. System Optimal Solution

In this section we show how the system optimal solution can also be obtained as a solution of a new mathematical programming problem using the density based travel time function.

1) Mathematical Programming Formulation:

Theorem 4.2: The system optimal problem is equivalent to the mathematical programming problem presented in Equation (50).

$$\min\ z\left(f_k^{rs}, \rho_a\right) = \sum_a f\left(f_k^{rs}\right) t_a(\rho_a) \tag{50}$$

with the equality constraints:

$$\sum_k f_k^{rs} = f_{rs}\ \forall\, r, s \tag{51}$$

$$f\left(f_k^{rs}\right) = \sum_r \sum_s \sum_k f_{ak}^{rs} \delta_{a,k}^{rs} \tag{52}$$

$$\sum_r \sum_s \sum_k f_{ak}^{rs} \delta_{a,k}^{rs} = v_{f_a} \rho_a \left(1 - \frac{\rho_a}{\rho_{m_a}}\right) \tag{53}$$

$$f_{ak}^{rs} = f_k^{rs} \delta_{a,k}^{rs} \tag{54}$$

$$\sum_i v_{f_i} \rho_i \left(1 - \frac{\rho_i}{\rho_{m_i}}\right) = \sum_o v_{f_o} \rho_o \left(1 - \frac{\rho_o}{\rho_{m_o}}\right) \quad \forall\, i \in n_I, o \in n_O, n \in \mathfrak{N} \tag{55}$$

and the inequality constraint

$$\rho_a \geq 0\ \forall\, a, \quad f_k^{rs} \geq 0\ \forall\, k, r, s \tag{56}$$

$$\rho_{m_a} \geq \rho_a \geq 0\ \forall\, a. \tag{57}$$

The travel time function on the link $t_a(\rho_a)$ is given by Equation (39).

Proof: The Kuhn–Tucker conditions for the mathematical programming problem given by Equation (50) can be obtained in terms of the Lagrangian given in Equation (58).

$$\begin{aligned}
&\mathfrak{L}(f, \rho, \lambda, \mu) \\
&\quad = z\left[f_k^{rs}, \rho_a\right] + \sum_{rs} \lambda_{rs} \left(f_{rs} - \sum_k f_k^{rs}\right) \\
&\qquad + \sum_a \mu_a \left(\sum_r \sum_s \sum_k f_k^{rs} \delta_{a,k}^{rs} - v_{f_a} \rho_a \left(1 - \frac{\rho_a}{\rho_{m_a}}\right)\right) \\
&\qquad + \sum_n \gamma_n \left(\sum_i v_{f_i} \rho_i \left(1 - \frac{\rho_i}{\rho_{m_i}}\right) - \sum_o v_{f_o} \rho_o \left(1 - \frac{\rho_o}{\rho_{m_o}}\right)\right).
\end{aligned} \tag{58}$$

Here, λ_{rs} and μ_a are the Lagrangian multipliers. The relevant Kuhn–Tucker conditions $\forall\, k, r, s, a$ are:

$$\begin{aligned}
f_k^{rs} \frac{\partial \mathfrak{L}(f, \rho, \lambda, \mu)}{\partial f_k^{rs}} &= 0, \quad \frac{\partial \mathfrak{L}(f, \rho, \lambda, \mu)}{\partial f_k^{rs}} \geq 0 \\
\frac{\partial \mathfrak{L}(f, \rho, \lambda, \mu)}{\partial \lambda_{rs}} &= 0, \quad \frac{\partial \mathfrak{L}(f, \rho, \lambda, \mu)}{\partial \mu_a} = 0.
\end{aligned} \tag{59}$$

Applying Kuhn-Tucker conditions in this case we get $\forall\, k, r, s$:

$$\begin{aligned}
f_k^{rs}\left(\tilde{c}_k^{rs} - \lambda_{rs}\right) &= 0, \quad \tilde{c}_k^{rs} - \lambda_{rs} \geq 0 \\
\sum_k f_k^{rs} &= f_{rs}, \quad f_k^{rs} \geq 0.
\end{aligned} \tag{60}$$

Here, we have

$$\tilde{c}_k^{rs} = \sum_a \delta_{a,k}^{rs} \tilde{t}_a \tag{61}$$

where

$$\tilde{t}_a(\rho_a) = t_a(\rho_a) + f(\rho_a) \frac{dt_a(\rho_a)}{df(\rho_a)}. \tag{62}$$

■

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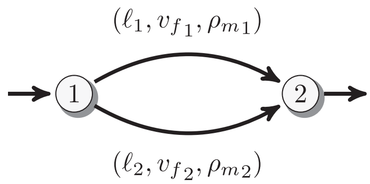

Fig. 14. Dynamic traffic routing.

The differentiation of the travel time function with respect to the traffic arc flow is the same as shown in Equation (49), which is valid because of the application of the inverse function theorem as was the case for the user equilibrium analysis.

V. I.T.S. Applications of Travel Time Function

There are many ITS applications where travel time function plays a pivotal role. Some such examples are presented below.

A. *Traffic Routing*

Traffic Routing is closely related to travel time as traffic should be routed based on that. Routing can be performed either as a user-equilibrium or a system optimum problem. Fig. 14 shows a sample network for this routing problem where the arc labels show the parameters such as the free flow speed v_{f_i}, jam density ρ_{m_i} and lengths of the arcs ℓ_i. The travel time is a function of the arc parameters and the traffic conditions on the arc. Traditionally, traffic routing in a static or steady state sense has been used as part of the four step planning ([31]) that includes (1) Trip generation, (2) Trip distribution, (3) Mode choice, and (4) Route assignment. Researchers have been developing dynamic routing algorithms ([4], [6]–[8]) which have become even more relevant due to the prevalence of smartphones with navigational capabilities.

1) User-Equilibrium: Routing involves splitting the incoming flow such as to node 1 in Fig. 14 into alternate routes, such as the two shown in the same figure, so as to equate the travel times on the two routes. This is the user-equilibrium traffic routing problem.

2) System-Optimum: Splitting the incoming flow into alternate routes so as to minimize the total travel times on the two routes is the system-optimum traffic routing problem.

B. *Ramp Metering*

Ramp metering involves controlling the inflow rate to affect the traffic conditions on the highway ([32]). The objective of the control law can be designed to minimize the travel time on the highway section as shown in Fig. 15. The figure shows the inflow to the highway section as $f(t)$ from upstream, and the controlling ramp flow as $r(t)$. The parameters of the highway section that dictate the travel time are shown as (v_f, ρ_m, ℓ).

C. *Signalized Intersection Control*

Signalized intersection control problems can also be designed using travel time as an objective function that should be minimized. For instance, in Fig. 16, we can design the phases of the two signalized intersections in order to minimize the sum of travel times from all origin nodes to the destination nodes.

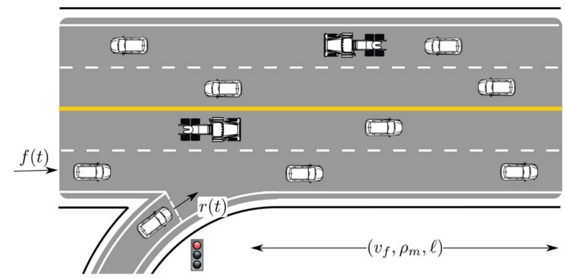

Fig. 15. Ramp metering.

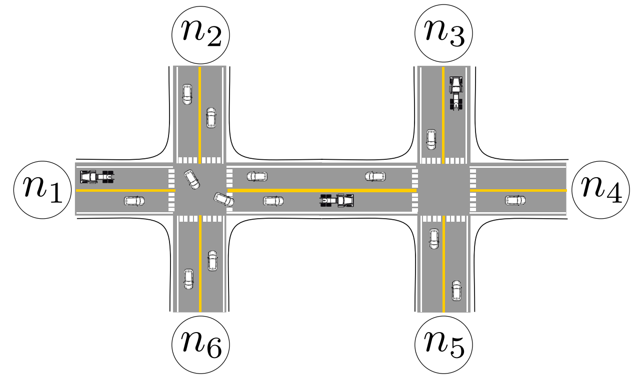

Fig. 16. Signalized intersection control.

D. *Traffic Network Control*

Coupled traffic network control based on travel time minimization for origin destination paths through the network can also be designed. The network can be composed of control actuators for traffic diversion, access (ramp) control, and signalized intersections.

VI. Conclusion

This paper presented a traffic density based travel time function as compared to flow based one and showed its applications in I.T.S. applications. This makes the traffic assignment problem framework consistent with the macroscopic and microscopic aspects of traffic theory. Travel time function that includes urban networks with signalized intersections was also developed. The paper then re-derived the formulation of static traffic assignment problems using the new travel time function, and also obtained their relationship to Wardrop condition for user equilibrium and marginal travel time equivalent for the system optimum problem. The paper showed that the Beckman formulation provides a unique solution for arc flows as well as arc densities, whereas the Wardrop condition alone has multiple arc density solutions. This travel time function is valid for sections where traffic density is uniform which is the case for steady state flows. The theory of travel time in a complete distributed fashion valid for time and space varying framework is the topic of a forthcoming paper by the authors ([33]).

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