

THE MARRIAGE PROBLEM.*

By PAUL R. HALMOS and HERBERT E. VAUGHAN.

In a recent issue of this journal Weyl¹ proved a combinatorial lemma which was apparently considered first by P. Hall.² Subsequently Everett and Whaples³ published another proof and a generalization of the same lemma. Their proof of the generalization appears to duplicate the usual proof of Tychonoff's theorem.⁴ The purpose of this note is to simplify the presentation by employing the statement rather than the proof of that result. At the same time we present a somewhat simpler proof of the original Hall lemma.

Suppose that each of a (possibly infinite) set of boys is acquainted with a finite set of girls. Under what conditions is it possible for each boy to marry one of his acquaintances? It is clearly necessary that every finite set of k boys be, collectively, acquainted with at least k girls; the Everett-Whaples result is that this condition is also sufficient.

We treat first the case (considered by Hall) in which the number of boys is finite, say n , and proceed by induction. For $n = 1$ the result is trivial. If $n > 1$ and if it happens that every set of k boys, $1 \leq k < n$, has at least $k + 1$ acquaintances, then an arbitrary one of the boys may marry any one of his acquaintances and refer the others to the induction hypothesis. If, on the other hand, some group of k boys, $1 \leq k < n$, has exactly k acquaintances, then this set of k may be married off by induction and, we assert, the remaining $n - k$ boys satisfy the necessary condition with respect to the as yet unmarried girls. Indeed if $1 \leq h \leq n - k$, and if some set of h bachelors were to know fewer than h spinsters, then this set of h bachelors together with the k married men would have known fewer than $k + h$ girls. An

* Received June 6, 1949.

¹ H. Weyl, "Almost periodic invariant vector sets in a metric vector space," *American Journal of Mathematics*, vol. 71 (1949), pp. 178-205.

² P. Hall, "On representation of subsets," *Journal of the London Mathematical Society*, vol. 10 (1935), pp. 26-30.

³ C. J. Everett and G. Whaples, "Representations of sequences of sets," *American Journal of Mathematics*, vol. 71 (1949), pp. 287-293. Cf. also M. Hall, "Distinct representatives of subsets," *Bulletin of the American Mathematical Society*, vol. 54 (1948), pp. 922-926.

⁴ C. Chevalley and O. Frink, Jr., "Bicomactness of Cartesian products," *Bulletin of the American Mathematical Society*, vol. 47 (1941), pp. 612-614.

application of the induction hypothesis to the $n-k$ bachelors concludes the proof in the finite case.

If the set B of boys is infinite, consider for each b in B the set $G(b)$ of his acquaintances, topologized by the discrete topology, so that $G(b)$ is a compact Hausdorff space. Write G for the topological Cartesian product of all $G(b)$; by Tychonoff's theorem G is compact. If $\{b_1, \dots, b_n\}$ is any finite set of boys, consider the set H of all those elements $g = g(b)$ of G for which $g(b_i) \neq g(b_j)$ whenever $b_i \neq b_j$, $i, j = 1, \dots, n$. The set H is a closed subset of G and, by the result for the finite case, H is not empty. Since a finite union of finite sets is finite, it follows that the class of all sets such as H has the finite intersection property and, consequently, has a non empty intersection. Since an element $g = g(b)$ in this intersection is such that $g(b') \neq g(b'')$ whenever $b' \neq b''$, the proof is complete.

It is perhaps worth remarking that this theorem furnishes the solution of the celebrated problem of the monks.⁵ Without entering into the history of this well-known problem, we state it and its solution in the language of the preceding discussion. A necessary and sufficient condition that each boy b may establish a harem consisting of $n(b)$ of his acquaintances, $n(b) = 1, 2, 3, \dots$, is that, for every finite subset B_0 of B , the total number of acquaintances of the members of B_0 be at least equal to $\sum n(b)$, where the summation runs over every b in B_0 . The proof of this seemingly more general assertion may be based on the device of replacing each b in B by $n(b)$ replicas seeking conventional marriages, with the understanding that each replica of b is acquainted with exactly the same girls as b . Since the stated restriction on the function n implies that the replicas satisfy the Hall condition, an application of the Everett-Whaples theorem yields the desired result.

UNIVERSITY OF CHICAGO

AND

UNIVERSITY OF ILLINOIS.

⁵ H. Balzac, *Les Cent Contes Drôlatiques*, IV, 9: *Des moines et novices*, Paris (1849).