~~AEE~~ 343

Problem Set 3

Problem #1

From Table A.1 for $M_e = 2.4$: $\boxed{A_e/A^* = 2.403}$,

$p_o/p_e = 14.62$ and $T_o/T_e = 2.152$.

Hence:

$$p_o = \frac{p_o}{p_e}\, p_e = 14.62\ (1\ \text{atm}) = \boxed{14.62\ \text{atm}}$$

$$T_o = \frac{T_o}{T_e}\, T_e = 2.152\ (519°R) = \boxed{1117°R}$$

Problem #2

Air flows steadily and isentropically from standard atmospheric conditions to a receiver pipe through a converging duct. The cross section area of the throat of the converging duct is 0.05 ft². Determine the mass flowrate through the duct if the receiver pressure is **(a)** 10 psia; **(b)** 5 psia. Sketch temperature–entropy diagrams for situations (a) and (b). Verify results obtained with values from the appropriate table in Appendix E with calculations involving ideal gas equations.

This problem is similar to Example 11.5

The mass flowrate is obtained at the throat with Eq. 11.40. Thus,

$$\dot{m} = \rho_{th} A_{th} V_{th} \qquad (1)$$

The throat density can be obtained with Eq. 11.60. Thus,

$$\rho_{th} = \rho_0 \left[\frac{1}{1 + \left(\frac{k-1}{2}\right) Ma_{th}^2} \right]^{\frac{1}{k-1}} \qquad (2)$$

To determine the throat Mach number we use Eq. 11.59. Thus,

$$Ma_{th} = \sqrt{\left(\frac{2}{k-1}\right) \left[\left(\frac{p_0}{p_{th}}\right)^{\frac{k-1}{k}} - 1 \right]} \qquad (3)$$

The critical throat pressure is obtained with Eq. 11.61. Thus,

$$p_{th}^* = p_0 \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}} = (14.7\ \text{psia}) \left(\frac{2}{1.40+1}\right)^{\frac{1.4}{1.4-1}} = 7.76\ \text{psia}$$

If the receiver pressure, p_{re}, is greater than or equal to p_{th}^*, then $p_{th} = p_{re}$ and the flow is not choked. If $p_{re} < p_{th}^*$, then $p_{th} = p_{th}^*$ and the flow is choked.

The velocity at the throat is obtained with Eqs. 11.36 and 11.46 combined to yield

$$V_{th} = Ma_{th} \sqrt{R T_{th} k} \qquad (4)$$

where T_{th} is obtained with Eq. 11.56. Thus,

$$T_{th} = \frac{T_0}{1 + \left(\frac{k-1}{2}\right) Ma_{th}^2} \qquad (5)$$

(con't)

(con't)

(a) For $p_{re} = 10\ psia > p_{th}^* = 7.76\ psia$, $p_{th} = 10\ psia$ and we use Eq. 3 to calculate the throat Mach number. Thus,

$$Ma_{th} = \sqrt{\left(\frac{2}{1.40-1}\right)\left[\left(\frac{14.7\ psia}{10\ psia}\right)^{\frac{1.40-1}{1.40}} - 1\right]} = 0.7628$$

From Eq. 2 we obtain

$$\rho_{th} = \left(2.38 \times 10^{-3}\ \frac{slug}{ft^3}\right)\left[\frac{1}{1+\left(\frac{1.40-1}{2}\right)(0.7628)^2}\right]^{\frac{1}{1.40-1}} = 1.807 \times 10^{-3}\ \frac{slug}{ft^3}$$

From Eq. 5 we get

$$T_{th} = \frac{519\,°R}{1+\left(\frac{1.40-1}{2}\right)(0.7628)^2} = 464.9\,°R$$

and with Eq. 4

$$V_{th} = (0.7628)\sqrt{\left(1716\ \frac{ft \cdot lb}{slug \cdot °R}\right)\frac{(1.40)(464.9\,°R)}{\left(1\ \frac{lb}{slug \cdot \frac{ft}{s^2}}\right)}} = 806.2\ \frac{ft}{s}$$

With Eq. 1 we obtain

$$\dot{m} = \left(1.807 \times 10^{-3}\ \frac{slug}{ft^3}\right)\left(0.05\ ft^2\right)\left(806.2\ \frac{ft}{s}\right) = \underline{\underline{0.0728}}\ \frac{slug}{s}$$

Alternatively, using Table E.1 with

$$\frac{p_{th}}{p_0} = \frac{10\ psia}{14.7\ psia} = 0.6803$$

The closest value of Ma_{th} is

$$Ma_{th} = 0.76$$

For $Ma_{th} = 0.76$, we get from Table E.1

$$T_{th} = (0.89644)\,T_0 = (0.89644)(519\,°R) = 465\,°R$$

Then with Eq. 4

$$V_{th} = 0.76\sqrt{\left(1716\ \frac{ft \cdot lb}{slug \cdot °R}\right)\frac{(1.40)(465\,°R)}{\left(1\ \frac{lb}{slug \cdot \frac{ft}{s^2}}\right)}} = 803\ \frac{ft}{s}$$

(con't)

(con't)

For $Ma_{th} = 0.76$ we get from Table E.1

$$\rho_{th} = 0.76086\,\rho_0 = (0.76086)\left(2.38\times10^{-3}\ \frac{slug}{ft^3}\right) = 1.81\times10^{-3}\ \frac{slug}{ft^3}$$

Now, with Eq. 1 we obtain

$$\dot{m} = \left(1.81\times10^{-3}\ \frac{slug}{ft^3}\right)\left(0.05\ ft^2\right)\left(803\ \frac{ft}{s}\right) = \underline{\underline{0.0727}}\ \frac{slug}{ft^3}$$

(b) For $p_{re} = 5\,psia < p^* = 7.76\,psia$, $p_{th} = 7.76\,psia$ and $Ma_{th} = 1.0$. From Eq. 2,

$$\rho_{th} = \left(2.38\times10^{-3}\ \frac{slug}{ft^3}\right)\left[\frac{1}{1+\left(\frac{1.40-1}{2}\right)}\right]^{\frac{1}{1.40-1}} = 1.509\times10^{-3}\ \frac{slug}{ft^3}$$

From Eq. 5 we obtain

$$T_{th} = \frac{519\,°R}{1+\left(\frac{1.40-1}{2}\right)} = 432.5\,°R$$

and with Eq. 4

$$V_{th} = \sqrt{\frac{\left(1716\ \frac{ft\cdot lb}{slug\cdot °R}\right)(1.40)(432.5\,°R)}{\left(1\ \frac{lb}{slug\cdot \frac{ft}{s^2}}\right)}} = 1019\ \frac{ft}{s}$$

With Eq. 1 we obtain

$$\dot{m} = \left(1.509\times10^{-3}\ \frac{slug}{ft^3}\right)\left(0.05\ ft^2\right)\left(1019\ \frac{ft}{s}\right) = 0.0769\ \frac{slug}{s}$$

Alternatively, from Table E.1 for $Ma = 1.0$

$$T_{th} = (0.83333)(519\,°R) = 432.5\,°R$$

and

$$\rho_{th} = (0.63394)\left(2.38\times10^{-3}\ \frac{slug}{ft^3}\right) = 1.509\times10^{-3}\ \frac{slug}{ft^3}$$

Then with Eq. 4

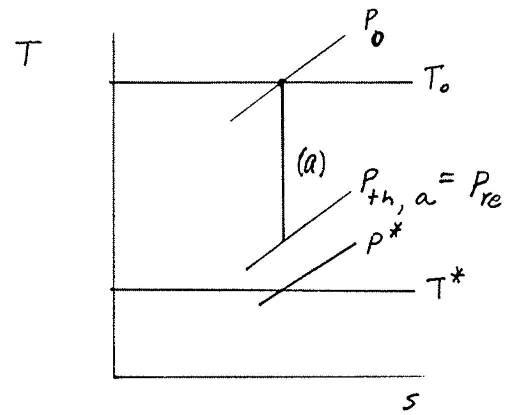
$$V_{th} = \sqrt{\frac{\left(1716\ \frac{ft\cdot lb}{slug\cdot °R}\right)(1.40)(432.5\,°R)}{\left(1\ \frac{lb}{slug\cdot \frac{ft}{s^2}}\right)}} = 1019\ ft/s$$

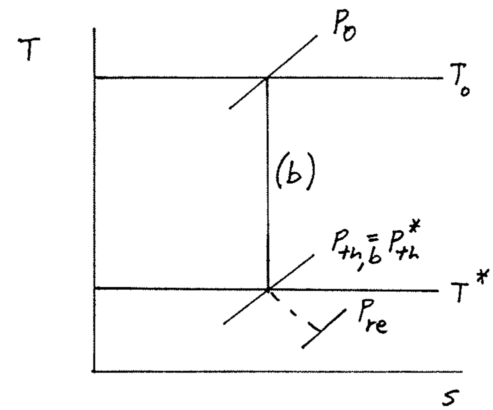
(con't)

(con't)

and with Eq. 1 we obtain

$$\dot{m} = \left(1.509 \times 10^{-3} \frac{slug}{ft^3}\right)\left(0.05\ ft^2\right)\left(1019\ \frac{ft}{s}\right) = 0.0769\ \frac{slug}{s}$$





Problem #3 and #4

An ideal gas enters subsonically and flows isentropically through a choked converging–diverging duct having a circular cross section area, A, that varies with axial distance from the throat, x, according to the formula

$$A = 0.1 + x^2$$

where A is in square feet and x is in feet. For this flow situation, sketch the side view of the duct and graph the variation of Mach number, static temperature to stagnation temperature ratio, T/T_0, and static pressure to stagnation pressure ratio, p/p_0, through the duct from $x = -1.0$ ft to $x = +1.0$ ft. Also show the possible fluid states at $x = -1.0$ ft, 0 ft, and $+1.0$ ft using temperature–entropy coordinates. Consider the gas as being **(a)** air; **(b*)** helium (use $0.051 \leq \text{Ma} \leq 5.193$).

This is like Example 11.8.

Since

$$A = \pi r^2$$

and

$$A = 0.1 + x^2$$

then

$$r = \frac{0.1 + x^2}{\pi} \qquad (1)$$

With Eq. 1 we can determine r values corresponding to values of x. The are summarized in the graph and tables

duct is choked,

$$A^* = 0.1 \text{ ft}^2$$

and

$$\frac{A}{A^*} = 1 + \frac{x^2}{0.1} \qquad (2)$$

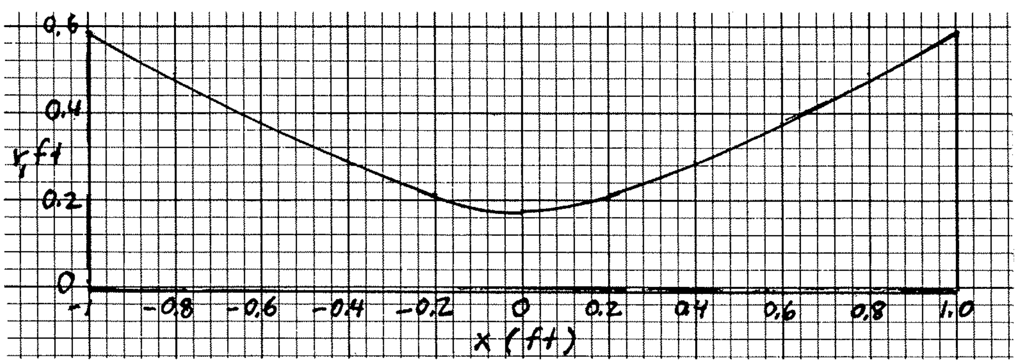
With Eq. 2 we can determine $\frac{A}{A^*}$ values corresponding to values of x. These $\frac{A}{A^*}$ values are tabulated

(a) For air, we can obtain the values of Ma, $\frac{T}{T_0}$, and $\frac{P}{P_0}$ corresponding to the closest values of $\frac{A}{A^*}$ in Table E.1. These values are summarized in the Table on the next page.

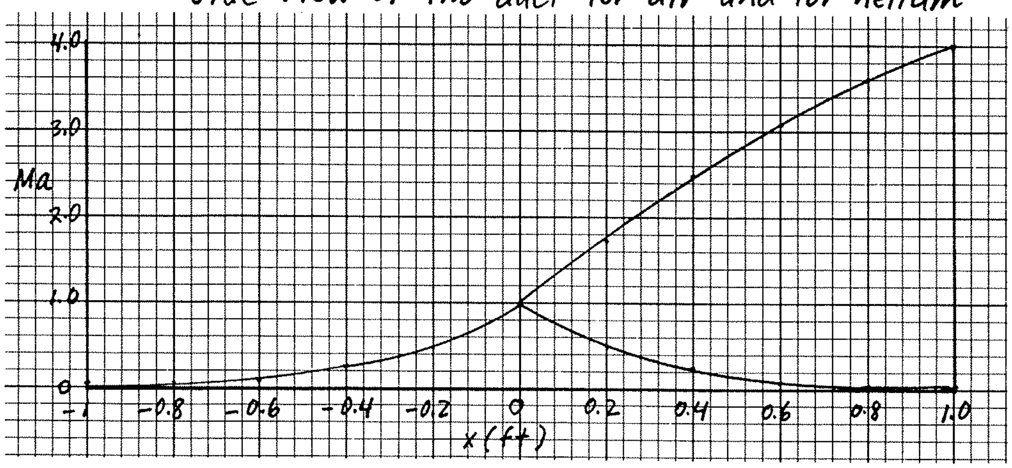
(con't)

(con't)

x (ft)	From Eq. 1 r (ft)	From Eq. 2 $\frac{A}{A^*}$	From Table E.1 $\frac{A}{A^*}$	Ma	$\frac{T}{T_0}$	$\frac{P}{P_0}$	state
			subsonic solution				
-1.0	0.592	11.0	11.592	0.05	0.9995	0.99825	a
-0.8	0.485	7.4	7.2616	0.08	0.99872	0.99553	
-0.6	0.383	4.6	4.4968	0.13	0.99664	0.98826	
-0.4	0.288	2.6	2.5968	0.23	0.98953	0.96383	
-0.2	0.211	1.4	1.4018	0.47	0.95769	0.85958	
0	0.178	1.0	1.0	1.0	0.83333	0.52828	b
0.2	0.211	1.4	1.4018	0.47	0.95769	0.85958	
0.4	0.288	2.6	2.5968	0.23	0.98953	0.96383	
0.6	0.383	4.6	4.4968	0.13	0.99664	0.98826	
0.8	0.485	7.4	7.2616	0.08	0.99872	0.99553	
1.0	0.592	11.0	11.592	0.05	0.9995	0.99825	c
			supersonic solution				
0.2	0.211	1.4	1.3967	1.76	0.61747	0.18499	
0.4	0.288	2.6	2.588	2.48	0.4484	0.06038	
0.6	0.383	4.6	4.6573	3.10	0.34223	0.02345	
0.8	0.485	7.4	7.4501	3.60	0.2784	0.01138	
1.0	0.592	11.0	10.719	4.00	0.2381	0.00658	d



Side view of the duct for air and for helium



Variation of Mach number for air

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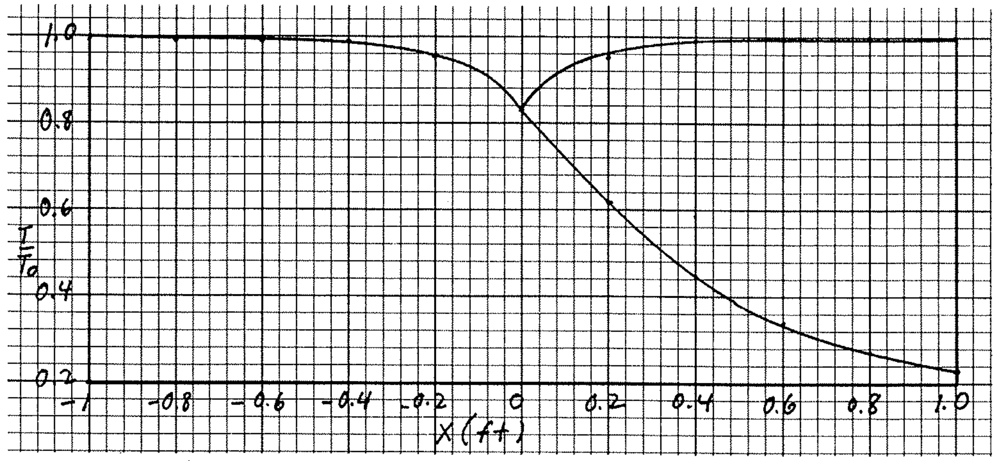
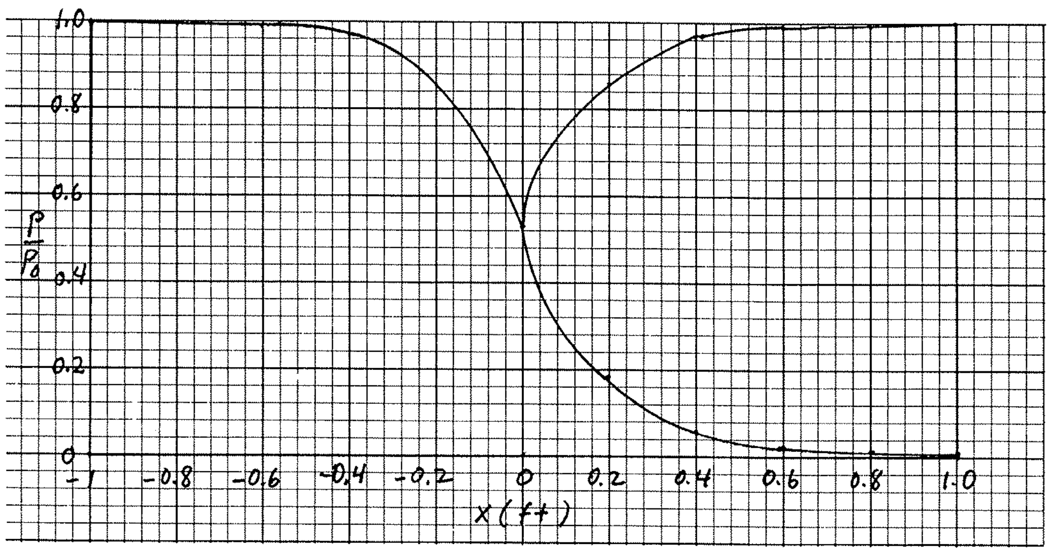


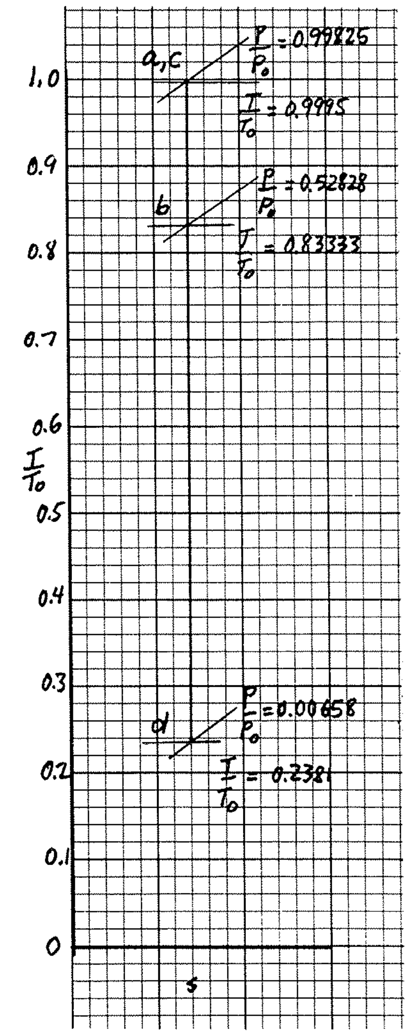
Variation of static temperature to stagnation temperature ratio for air



Variation of static pressure to stagnation pressure ratio for air

(con't)

(con't)



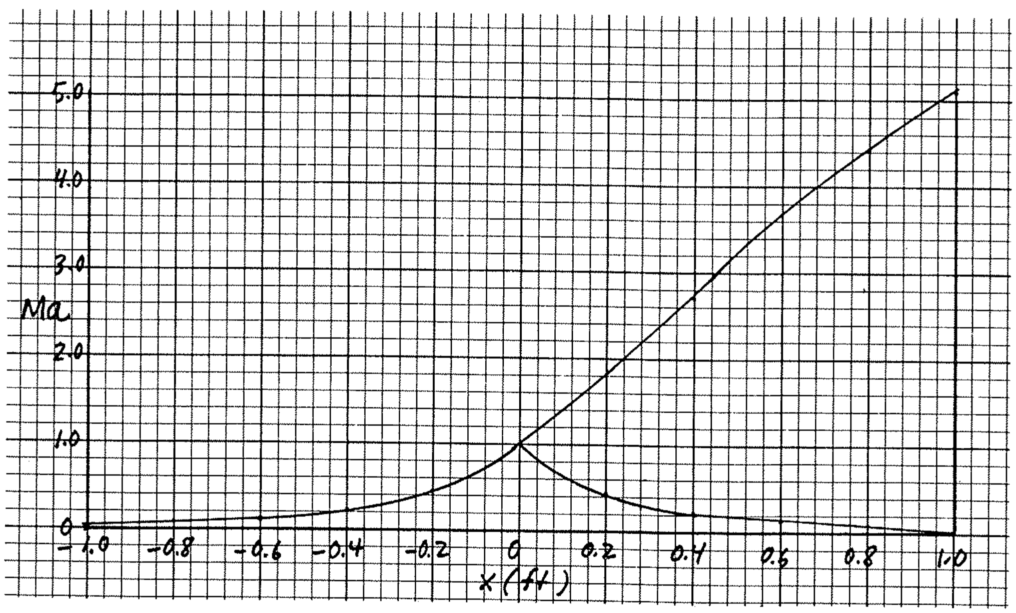
Temperature-entropy diagram for air

(b) For helium we enter program ISENTROP with $k=1.66$ and with Ma values within the range specified in the problem statement and obtain values of $\frac{A}{A^*}$ (Eq. 11.71), x (Eq. 2), $\frac{T}{T_0}$ (Eq. 11.56) and $\frac{p}{p_0}$ (Eq. 11.59). These values are tabulated and graphed on pages that follow.

(con't)

(con't)

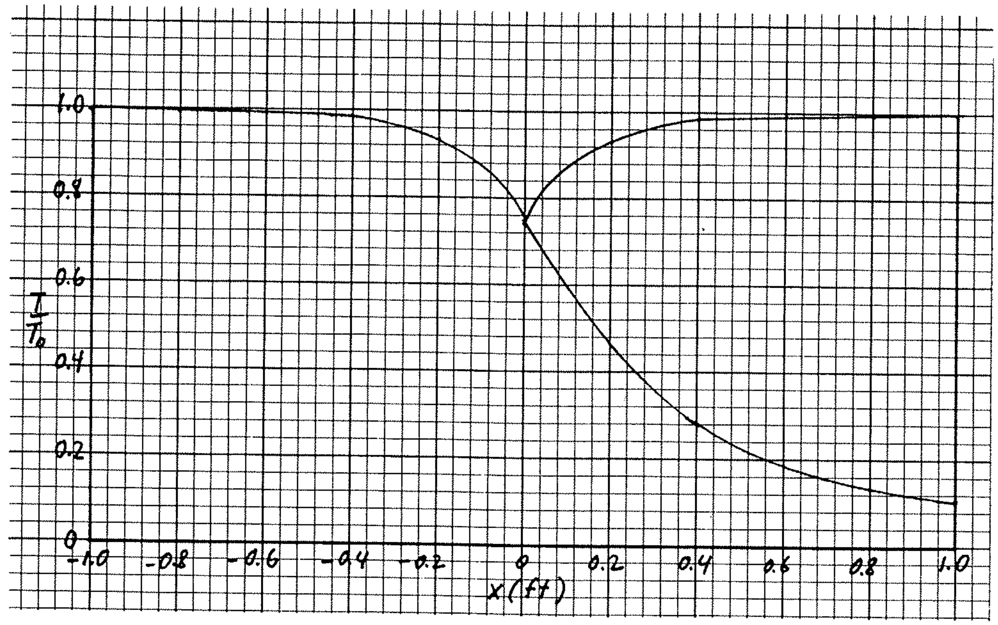
Ma	From $\frac{A}{A^*}$	program Eq. 2 $x(ft)$	ISENTROP with $k=1.66$ $\frac{T}{T_o}$	$\frac{P}{P_o}$	state
			subsonic solution		
0.051	11.06	±1.00	0.99914	0.99784	a, c
0.076	7.43	±0.80	0.99809	0.99522	
0.123	4.62	±0.60	0.99503	0.98755	
0.223	2.61	±0.40	0.98385	0.95989	
0.460	1.40	±0.20	0.93473	0.84386	
1.00	1.00	0	0.75188	0.48808	b
			supersonic solution		
1.855	1.40	0.20	0.46827	0.14833	
2.778	2.60	0.40	0.28195	0.04141	
3.647	4.60	0.60	0.18556	0.01446	
4.448	7.40	0.80	0.13282	0.00624	
5.193	11.0	1.00	0.10102	0.00313	d



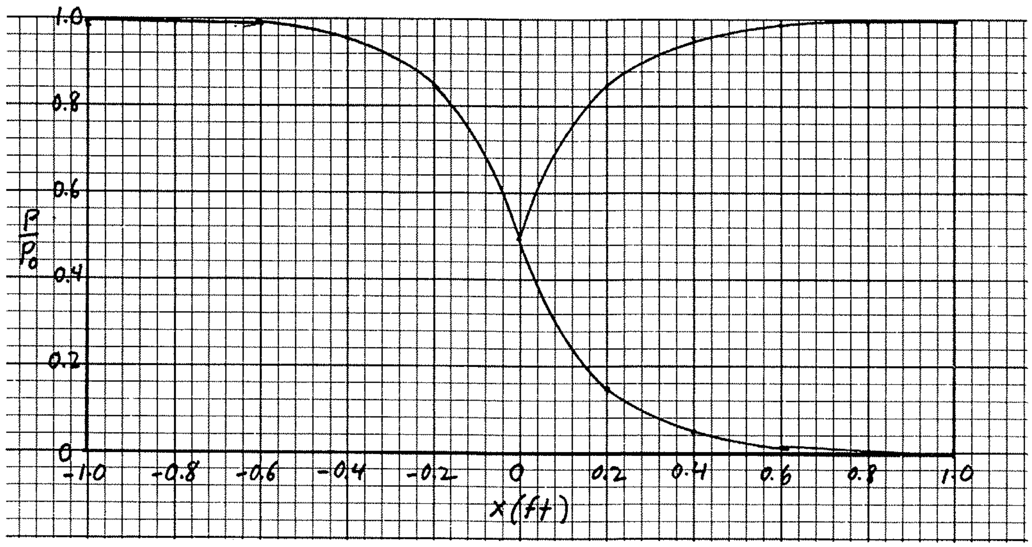
Variation of Mach number for helium

(con't)

(con't)



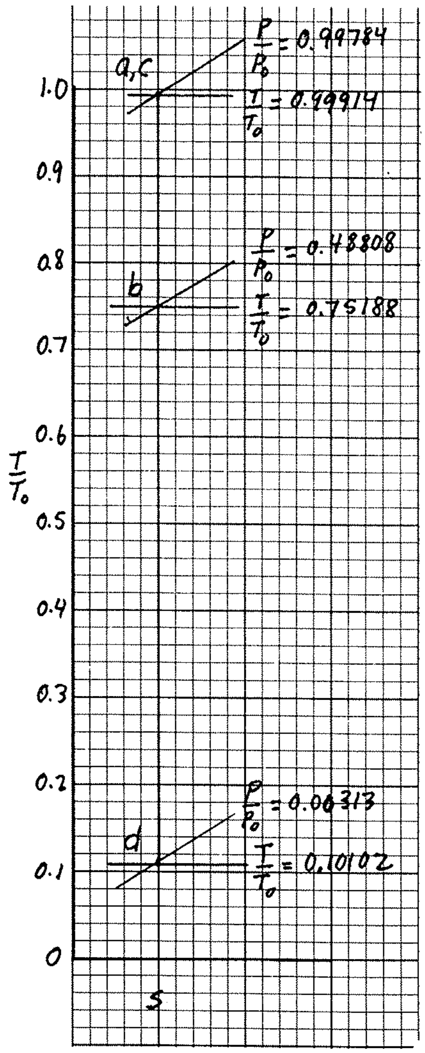
Variation of static temperature to stagnation temperature ratio for helium



Variation of static pressure to stagnation pressure ratio for helium

(con't)

(con't)



Temperature entropy diagram for helium

Problem 5 best to approach problem by determining conditions P_{e3}, P_{e5}, P_{e6}

Condition 3

at $\frac{A}{A^*} = 1.53 \xrightarrow[\text{subsonic}]{\text{App. A.1}} M_{e3} = 0.42$

$\frac{P_o}{P_{e3}} = 1.13$ $\boxed{P_{e3} = 0.885 \text{ atm}}$

Condition 6

at $\frac{A}{A^*} = 1.53 \xrightarrow[\text{supersonic}]{\text{App. A.1}} M_{e6} = 1.88$

$\frac{P_o}{P_{e6}} = 6.5$ $\boxed{P_{e6} = 0.154 \text{ atm}}$

Condition 5, normal shock at exit

$M_{e6} = 1.88 \longrightarrow M_{e5} = 0.6$ $\frac{P_{e5}}{P_{e6}} = 3.96$

$\frac{P_{e5}}{P_o} = \frac{P_{e5}}{P_{e6}} \cdot \frac{P_{e6}}{P_o} = 3.96 \frac{1}{6.5} = 0.61$ $\boxed{P_{e5} = 0.65 \text{ atm}}$

$\therefore$ (a) subsonic everywhere ($P_e = 0.94$ atm)

(b) condition 3 ($P_e = 0.886$ atm)

(c) shock inside diverging section ($P_e = 0.75$ atm)

(d) condition 6 ($P_e = 0.154$ atm)

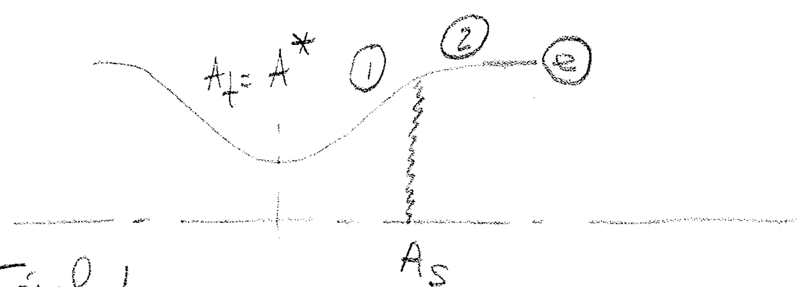
Part (c)

Method #1 : direct method using Eq (5.28)

$$M_e^2 = -\frac{1}{\gamma-1} + \sqrt{\frac{1}{(\gamma-1)^2} + \left(\frac{2}{\gamma-1}\right)\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}} \Bigg(\underbrace{\frac{P_{01}}{P_e}}_{\frac{1}{0.75}}\ \underbrace{\frac{A_t}{A_e}}_{\frac{1}{1.53}}\Bigg)^2}$$

With $\gamma = 1.4$, $\boxed{M_e = 0.486}$

Method #2 : trial an error solution



$$\frac{A_e}{A_t} = 1.53$$

Trial 1

Assume $\frac{A_s}{A_t} = \frac{A_s}{A^*} = 1.2$

App. A.1 (supersonic) @ $\frac{A_s}{A^*} = 1.2 \longrightarrow M_1 = 1.54$

App. A.2 @ $M_1 = 1.54 \longrightarrow M_2 = 0.687 \quad \frac{P_{02}}{P_{01}} = 0.917$

App. A.1 @ $M_2 = 0.687 \longrightarrow \frac{A_s}{A_2^*} = 1.1$

$$\frac{A_e}{A_2^*} = \frac{A_e}{A_1^*}\frac{A_1^*}{A_s}\frac{A_s}{A_2^*} = \frac{A_e}{A_t}\frac{A_t}{A_s}\frac{A_s}{A_2^*}$$

A_t A_t

$$= (1.53)\left(\frac{1}{1.2}\right)(1.1)$$

$$\frac{A_e}{A_2^*} = 1.4$$

App. A.1 (subsonic) @ $\frac{A_e}{A_2^*} = 1.4 \longrightarrow M_e = 0.47$

$$\frac{p_{oe}}{p_e} = \frac{p_{o2}}{p_e} = 1.16$$

$$p_e = \frac{p_e}{p_{o2}}\frac{p_{o2}}{p_{o1}}p_{o1} = \frac{1}{1.16}(0.917)(1) = 0.79\text{ atm}$$

$\therefore$ the assumed shock position $\frac{A_s}{A_t} = 1.2$ corresponds to $p_e = 0.79$ atm.

Since the actual exit pressure is lower the shock is further downstream at a larger area ratio A_s/A_t

Trial 2

Assume $\frac{A_s}{A_t} = \frac{A_s}{A^*} = 1.3$ (larger than previous guess)

App. A.1 (supersonic) @ $\frac{A_s}{A^*} = 1.3 \longrightarrow M_1 = 1.66$

App. A.2 @ $M_1 = 1.66 \longrightarrow M_2 = 0.65$

$\frac{P_{o2}}{P_{o1}} = 0.872$

App. A.1 @ $M_2 = 0.65 \longrightarrow \frac{A_s}{A_2^*} = 1.113$

$$\frac{A_e}{A_2^*} = \frac{A_e}{A_t}\frac{A_t}{A_s}\frac{A_s}{A_2^*} = (1.53)\left(\frac{1}{1.3}\right)(1.113)$$

$$\frac{A_e}{A_2^*} = 1.31$$

App. A.1 (subsonic) @ $\frac{A_e}{A_2^*} = 1.31 \longrightarrow M_e = 0.52$

$$\frac{P_{oe}}{P_e} = 1.2 = \frac{P_{o2}}{P_e}$$

$$P_e = \frac{P_e}{P_{o2}} \frac{P_{o2}}{P_{o1}} P_{o1} = \frac{1}{1.2}(0.872)(1) = 0.73 \text{atm}$$

$\therefore$ the assumed shock position $\frac{A_s}{A_t} = 1.3$ corresponds to $p_e = 0.73$ atm.

Since the actual exit pressure is higher, the shock is further upstream at a lower area ratio $\frac{A_s}{A_t}$

Trial 3

Assume $\frac{A_s}{A_t} = \frac{A_s}{A^*} = 1.27$

Note: Guess $\frac{A_s}{A_t}$ should be between 1.2 and 1.3

Since the solution is closer with the guessed value of 1.3, we choose $A_s/A_t = 1.27$

App A.1 @ $\frac{A_s}{A^*} = 1.27 \longrightarrow M_1 = 1.62$
(supersonic)

App. A.2 @ $M_1 = 1.62 \longrightarrow M_2 = 0.663$

$$\frac{P_{o2}}{P_{o1}} = 0.888$$

App A.1 @ $M_2 = 0.663 \longrightarrow \frac{A_s}{A_2^*} = 1.13$

$$\frac{A_e}{A_2^*} = \frac{A_e}{A_t}\frac{A_t}{A_s}\frac{A_s}{A_2^*} = (1.53)\frac{1}{1.27}(1.13)$$

$$\frac{A_e}{A_2^*} = 1.36$$

App A.1 @ $\frac{A_e}{A_2^*} = 1.36 \longrightarrow M_e = 0.49$
(subsonic)

$$\frac{P_{oe}}{P_e} = \frac{P_{o2}}{P_e} = 1.18$$

$$P_e = \frac{P_e}{P_{o2}}\frac{P_{o2}}{P_{o1}}P_{o1} = \frac{1}{1.18}(0.888)(1) = 0.75 \text{ atm}$$

(match found!)

Solution: $M_e = 0.49$ and $A_s/A_t = 1.27$