Contents

Abstract

In this thesis we construct the unbounded Kasparov product in the real setting, based on the work in the complex setting of [MR16]. We recreate the results therein, and where the real and complex settings diverge, we prove analogous results in the real setting. Thereby we create the framework needed for the general lifting construction of the unbounded Kasparov product in [MR16, Section 4]. To develop a feel for the theory of real C^* algebras, we give the classification of real graded continuous trace from [Ek14]. We also give an introduction to KK-theoretic duality, introducing the notion of non-commutative Poincare duality and propose classes implementing Poincare duality for the non-commutative 2-torus in the real setting.

Introduction

There has been renewed interest in real C^* algebras and their topological invariants owing to their new-found applications in condensed matter physics, [BKR16], where the hope is to use the unbounded Kasparov product to put various topological invariants stemming from physical considerations on a firm mathematical footing. Non-commutative differential geometry might be the natural setting in which to study many condensed matter systems, as the dynamics of condensed matter systems may often be described by unbounded non-commuting derivations. The success of this approach is evident in the description of the Integer Quantum Hall Effect, where the only consistent theoretical model is that of [BvESB94].
The study of real C^* algebras has a different flavor than the study of complex C^* algebras. The difference may essentially be ascribed to the superfically trivial fact that -1 and 1 are not homotopic through unitaries in the reals, so the group of unitaries of $\mathbb{R}$ is $\mathbb{Z}_2$ There is also the essential difference that there are three different real normed division algebras, in contrast to the single complex one. The continuous algebras provide us with a class of algebras which is particulary well-suited to illustrate the pecularities of real C^* algebras. Continuous trace algebras also serve to illustrate the advantages of the bundle theoretic viewpoint, where the ideas of classical geometry may be brought to bear on non-commutative problems. The classification of continuous trace algebras via cohomological means was completed in the 1980's, as summarized in [RW98]. The classification was expanded in [Ek14] to real continuous trace algebras with groupoid actions. In this thesis we reproduce the classification of Real Graded continuous trace algebras from [Ek14], showing that the Brauer group of a space (T, τ) with involution classifies the continuous trace algebras with primitive ideal space (T, τ) up to stable isomorphism.
One of the most important unifying themes across the thesis is the use of the bundle-theoretic viewpoint as enabling an analysis of C^* algebraic properties, both directly for the continuous trace algebras and in the use of the local-global principle in constructing the unbounded Kasparov product. This viewpoint also has computational perspectives in condensed matter physics, as outlined in [Pro17].
Within the past few years there has been substantial progress in the unbounded approach to KK-theory, beginning with the PhD Thesis of Bram Mesland as elucidated in [Mes09]. and [KL13]. This development has hinged on several technical developments drawing on ideas from ordinary differential geometry. To work with differential structures the appro-

priate setting is found to be the categories of operator algebras and modules, serving as non-commutative generalization of differential algebras and bundles. In [Mes09] a crucial ingredient in the construction of the unbounded Kasparov product is found to be the non-commutative analogue of a connection, serving to define a version of $1 \otimes \mathscr{D}_2$ which is well-defined on the interior tensor product. Then to provide a suitable differential structure to accommodate the construction of connections requires a non-commutative analogue of a frame. This analogue is provided by the introduction of projective operator modules, which come with a canonical frame. The final technical tool in the construction is the Local-Global principle, developed separately in [Pie06] and [KL12] which is used to show that the operator implementing the unbounded Kasparov product is self-adjoint and regular. The framework in which these ideas come together is the notion of an unbounded $\mathcal{A} - \mathcal{B}$-correspondence, which turns out to be the essential ingredient in the construction of the unbounded Kasparov product. It was shown in [MR16] that using the tools we have just mentioned, one can recover the entirety of complex KK-theory in the unbounded setting. In the thesis we take these results and generalize them to incorporate the real setting, showing that we can also construct the unbounded Kasparov product for real algebras. In order to do so, we show a number of results concerning unbounded multipliers and approximate units. These are essential ingredients in [MR16, Section 4] for showing that given any two composable Kasparov modules, we may lift them to composable unbounded modules. With our results, all the proofs go through smoothly in the real case. This ends the line of inquiry investigating unbounded KK-theory as vehicle for capturing bounded KK-theory. What we show is that unbounded KK-theory completely captures bounded KK-theory in both the real and complex cases, but it does not lend itself to immediate extension to all cases of interest due to the reliance on bounded approximate units in our differential algebras. This opens the door to the program begun in [Kaa16], [Kaa15] and [Mes09] of building an abelian group from the unbounded KK-theory. The work of [Kaa15] and [Kaa16] seems to be promising in that it contains both an equivalence relation and a direct sum, but the theory is still not fully formed. An interesting further development in real KK-theory, of particular interest to the ongoing investigation of physical consequences thereof, is showing KK-theoretic Poincare duality for the real non-commutative torus. We have worked on this, but not been able to come to any useful results.

In summary, the thesis illustrates how ideas from commutative geometry may be brought to bear on seemingly intractable problems in non-commutative geometry, as well as hinting at how the operator theoretic viewpoint leads to new insights in commutative differential geometry. The thesis also illustrates the added flexibility stemming from working with unbounded operators.

1 A brief introduction to real C^* algebras

1.1 Definitions and basic properties

In order to define unbounded KK-theory in the real setting, it is necessary for us to define graded real C^* and explore some of their representation theory. In this section we shall give such a basic exposition, for a more in-depth review of the theory of real C^* algebras, we refer to [GG82].

Definition 1.1. A real C^* -algebra A is a Banach *-algebra A over $\mathbb{R}$ *-isomorphic to a subset of $B(H)$ for a real Hilbert space H.

Remark 1.2. A C^* algebra or Hilbert space is real if it has $\mathbb{R}$ as scalar field.

As in the complex case, it can be shown that the concrete definition is equivalent to a purely algebraic definition:

Theorem 1.3. A real Banach *-algebra is a real C^* algebra if and only if it satisfies:

1. $\|x^*x\| = \|x\|^2$
2. $1 + x^*x$ is invertible in A^+, the unitization of A,.

For a proof, see [GG82].

Remark 1.4. The assumption that $1 + x^*x$ is invertible is necessary and does not follow from the other axioms. For instance we could consider the complex numbers as a real C^* - algebra with trivial involution, and thereby we would have $0 < \|1 + x^*x\| = \left\|1 + i^2\right\| = 0$. Henceforth all C^* -algebras considered will be real, unless stated otherwise.

The motivating geometric example for the construction of real C^* is viewing them as non-commutative generalizations of Atiyah's real spaces.

Definition 1.5 (Atiyah's Real spaces)**.** A Real space is a pair (X, τ) consisting of a locally compact Hausdorff space X with a continuous involution τ. The set of continuous Real functions from (X_1, τ_1) to (X_2, τ_2) is defined as

$$C_R(X_1, X_2) = \{f \in C(X_1, X_2) \mid f(\tau_1(x)) = \tau_2(f(x))\}$$

When considering morphisms sets between real spaces, we shall always suppress the R and often suppress the explicit involutions.

Given a real space we can define a real C^* algebra.

Definition 1.6 (C^* version of Real spaces)**.** Let (X, τ) be a Real space. Define the algebra:

$$C_0(X, \tau) = \{f \in C(X, \mathbb{C}) | f(\tau(x)) = \overline{f(x)}\}$$

Example 1.7. 1. Given a Real space, the algebra $C_0(X, \tau)$ is a real C^* -algebra, with $(f)^*(x) = \overline{f(x)}$.

2. The field $\mathbb{R}$ is itself a real C^* -algebra, equipped with the star $x^* = x$. The complex numbers are a real C^* algebra, with involution $z^* = \overline{z}$. The quarternions $\mathbb{H}$ are a real C^* algebra, with involution defined on generators as $i^* = -i, j^* - j, k^* = -k$ and $1^* = 1$.

   The algebras $M_n(\mathbb{R}$, $M_n(\mathbb{C})$ and $M_n(\mathbb{H})$ are all real C^* -algebras, with the *-operation stemming from the entrywise-*-operation composed with the transpose.

3. There is a real version of the multiplier algebra, which may be defined exactly as in the complex case, viewing A as a Hilbert C^* -module over itself, i.e. $\langle a, a' \rangle = a^* a$. Then we define the multiplier algebra of A as the adjointables on A, $M(A) = L(A)$.

As in the complex case, we have a direct classification of the finite dimensional real C^* -algebras

Theorem 1.8. Every finite dimensional real C^* -algebra is a direct sum of matrix algebras over the quarternions, the reals and the complex numbers.

Thus we naturally also have a notion of real AF and UHF algebras but with a slightly more complicated classification theory, [Gio88].

Definition 1.9. Given a real C^* algebra A, define $A_{\mathbb{C}} = A \otimes_{\mathbb{R}} \mathbb{C}$. This is a complex C^* algebra with *-operation given by:

$$(a + ib)^* = a^* - ib^*$$

and norm

$$\|a + ib\|^2_{A_{\mathbb{C}}} = \|a^* a + b^* b\|_A$$

The continuous functional calculus is an essential element in the toolkit of operator algebras, and we have a version of it for real C^* -algebras:

Definition 1.10. Suppose A is a real C^* algebra. Define the spectrum of an element $a \in A$ to be the spectrum of a in $A_{\mathbb{C}}$.

We can define the state space and irreducible representations of real C^* algebras analogous to the complex case:

Definition 1.11. Let A be a real C^* algebra.

1. A state on A is a real positive linear functional $\rho : A \to \mathbb{R}$ with $\|\rho\| = 1$. A state which cannot be written as a convex combination of other states is a pure state.

2. A representation of A C^* algebra is a *-homomorphism $A \to B(H)$, where H is a real Hilbert space. A representation is irreducible if $\pi(A)$ has no invariant subspaces except for H.

3. Let $\hat{A}$ denote the space of unitary equivalence classes of irreducible representations of A.

4. Define $\mathcal{I}(A)$ as the set of ideals of A. For $I \in \mathcal{I}$, define $U(I) = \{J \in \mathcal{I} | J \setminus I \neq \emptyset\}$. This is a subbasis for a topology on $\mathcal{I}$ and gives rise to a topology on the space of representations through the map $\pi \mapsto \ker \pi$. The subspace topology on $\hat{A}$ is the Jacobson topology.

Remark 1.12. Let A be a real C^* algebra. By the GNS-construction every state corresponds to a representation of A. Likewise, the pure states correspond to irreducible representations. For a proof of this, see [GG82].

One of the nice features of real C^* algebras is that we have additional structure on $\hat{A}$.

Lemma 1.13. Let π be an irreducible representation of a real C^* -algebra A on a real Hilbert space H. Then the commutant of $\pi(A)$ is $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

Proof. As $\pi(A)'$ is a real C^* -algebra, it suffices to show that it is a real division algebra since the only real division algebras are $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$, [UW60]. Let $p \in \pi(A)'$ be a spectral projection of a self-adjoint element $x \in \pi(A)'$, this exists by [MV97, Remark 20.18]. Then pH and $(1-p)H$ are both invariant subspaces for $\pi(A)$. As π was assumed irreducible, either $p = 1$ or $1 - p = 1$. As all non-trivial spectral projections agree the spectrum of x must be a one-point set, ie. an eigenvalue. Therefore x is of the form $x = \lambda \cdot 1$, for $\lambda \in \mathbb{R}$. If $y \in \pi(A)'$, then $y^*y = \lambda \cdot 1$ for $\lambda \in \mathbb{R}$, and likewise for yy^*. The spectra of yy^* and y^*y coincide except possibly for the set $\{0\}$ it follows $yy^* = y^*y = \lambda \cdot 1$ and as such either $y = 0$, or y is invertible with inverse $y^{-1} = \|y\|^{-2} y^*$. This implies that $\pi(A)'$ is a division algebra as desired, showing the theorem. □

Definition 1.14. Let π be an irreducible representation of a real C^* algebra A.

1. π is of real type if $\pi(A)' \cong \mathbb{R}$,
2. π is of complex type if $\pi(A)' \cong \mathbb{C}$,
3. π is of quartenionic type if $\pi(A)' \cong \mathbb{H}$,

Definition 1.15. A Real C^* -algebra is a pair (A, τ) where A is a complex C^* -algebra and τ is a complex-linear involution such that:

$$\tau(ab) = \tau(b)\tau(a)$$
$$\tau(a^*) = \tau(a)^*$$

Such a map is a *-anti-automorphism.

Remark 1.16. The categories of real and Real C^* algebras are equivalent via. the following construction: Given a real C^* algebra A with complexification $A_{\mathbb{C}}$, define the anti-linear *-automorphism:

$$\sigma : A_{\mathbb{C}} \to \mathcal{A}_{\mathbb{C}}$$
$$\sigma(a + ib) = a - ib$$

From this, define the *-anti-automorphism τ:

$$\tau(a+ib) = (\sigma(a+ib))^* = a^* + ib^*$$

Then it is easy to see that $(A_{\mathbb{C}}, \tau)$ defines a Real C^* algebra. Conversely, given a Real C^* algebra (A, τ), define:

$$\begin{aligned}
&\sigma : A \to A \\
&\sigma(a) = (\tau(a))^* \\
&A_{real} = \{a \in A : \sigma(a) = a\}
\end{aligned}$$

Then the algebra A_{real} is a real C^* algebra, and the two constructions are inverses of each other. The construction shows that a Real C^* algebra (A, τ) is equivalent to (A, σ) where A is a complex C^* algebra and σ is an anti-linear involution commuting with the adjoint.

Definition 1.17. A Real Hilbert space is a pair (H, σ), where H is a complex Hilbert space and σ is anti-linear involution on H.

Remark 1.18. The categories of real and Real Hilbert spaces are equivalent via. the following construction: Given a real Hilbert space H, define (H, σ) as $H \otimes \mathbb{C}$ with involution $\sigma(\xi \otimes z) = \xi \otimes \overline{z}$. Conversely, given a Real Hilbert space (H, σ) define $H_{real} = \{\xi \in H \mid \sigma(\xi) = \xi\}$.

The definition of Real Hilbert spaces allows us to define the Real version of $B(H)$

Definition 1.19. Suppose (H, σ) is a Real Hilbert space. Define $B(H)$ the *-anti-automorphism θ on $B((H, \sigma))$ for $x \in B(H)$ as:

$$\theta(x) = \sigma(x^*)\sigma$$

Definition 1.20. Suppose (A, τ) is a Real C^* algebra, and $\pi : A \to B(H)$ is a complex *-representation of A on a Real Hilbert space. Then $B(H)$ comes with a *-anti-automorphism θ, see Definition 1.19. If π satisfies

$$\pi(\tau(a)) = \theta(\pi(a))$$

π is a Real representation.

Definition 1.21. Given a *-algebra A over k, define the set A^{op} as the set A. We equip A^{op} with the structure of a Real algebra. Let $a^{op}, b^{op} \in A^{op}$ and define the operations:

$$\begin{aligned}
&(ab)^{op} = b^{op}a^{op} \\
&\lambda a^{op} = (\lambda a)^{op} \\
&(a^{op})^* = (a^*)^{op} \\
&a^{op} + b^{op} = (a+b)^{op} \\
&\tau(a^{op}) = (\tau(a))^{op}
\end{aligned}$$

We can see that a Real C^* algebra (A,τ) satisfies $A \cong A^{op}$ as complex algebras, with the isomorphism implemented by τ. Given a representation of (A,τ) we define a representation of the opposite algebra.

Definition 1.22. Let π be a representation of a Real C^* algebra (A,τ) on (H,σ). Define $\pi^{op} : A^{op} \to B(H)^{op}$:

$$\pi^{op}(a^{op}) = \pi(\tau(a))$$

An interesting property of Real C^* -algebras is that τ allows us to define an involution on the space of irreducible representations of a Real algebra.

Proposition 1.23. If (A,τ) is a Real C^* -algebra, τ induces an involution on $\hat{A}$.

Proof. Let π be an irreducible *-representation of a Real algebra $(A,\tau) \to (H,\sigma)$. Define the map $\pi^{op} : A^{op} \to B(H)^{op}$ as in Definition 1.22. We have the *-anti-automorphism θ on $B(H)$, $\theta : B(H)^{op} \to B(H)$ from Definition 1.19. This allows us to define a new representation $A \to B(H)$:

$$\theta_*(\pi) = \theta\pi^{op}(\tau))$$

We wish to show the map θ_* is an involution on $\hat{A}$. To this end consider $\theta_*(\theta_*(\pi))$. Unravelling this expression we get:

$$\begin{aligned}\theta_*(\theta_*(\pi)) &= \theta_*(\theta(\pi^{op}(\tau))) \\ &= \theta(\theta(\pi(\tau(\tau)))) \\ &= \pi\end{aligned}$$

Hence θ_* is an involution on $\hat{A}$. □

Theorem 1.24 (Gelfand-Naimark)**.** The category of real commutative C^* -algebras with non-degenerate *-homomorphisms as the morphisms is equivalent to the category of locally compact spaces with proper continuous maps as the morphisms. Given a commutative real C^* algebra A we define the space X of $\mathbb{R}$-linear characters of A, so $A \cong C_0(X,\tau)$ where τ is the involution on the character space of A, defined in Proposition 1.23.

Remark 1.25. The space X is homeomorphic to the space of $\mathbb{C}$-linear characters of $A_{\mathbb{C}}$.

These statements combine to give a real version of the continuous functional calculus.

Proposition 1.26. Assume A is a real C^* algebra. Let $a \in A$ be a normal element in A. Then the algebra $C^*(\{x\})$ is a commutative algebra real C^* algebra, by Theorem 1.24 $C^*(\{x\}) \cong C_0(Spec(x),\tau$. Thus $f(x) \in C^*(\{x\})$ for $f \in C_0(Spec(x),\tau)$.

For a proof of these results see [GG82]. The three possible commutants of an irreducible representation is reflected in the involution on the irreducible representations, as encapsulated in the following theorem.

Theorem 1.27. Let π be an irreducible representation of a real C^* algebra A on a real Hilbert space H. Let $(A_{\mathbb{C}}, \tau)$ be the complexification of A with the involution τ defined as in Remark 1.16.

1. If π is of real type, the Real representation $\pi_{\mathbb{C}} : A_{\mathbb{C}} \to B(H \otimes \mathbb{C})$ of $A_{\mathbb{C}}$ induced by π is fixed under the involution on $\hat{A}_{\mathbb{C}}$ defined in Proposition 1.23.

2. Suppose that $\pi(A)' \cong \mathbb{C}$. This gives H the structure of a complex Hilbert space, H acquires two different structures of a complex Hilbert space: One by multiplication and one by multiplication with the conjugate.

   The representation π is complex-linear with respect to both of these structures. These structures give two representations $\pi'_{\mathbb{C}}, \pi''_{\mathbb{C}}$ of $A_{\mathbb{C}}$ which are inequivalent and interchanged by the involution in Proposition 1.23.

3. Assume that $\pi(A)' \cong \mathbb{H}$, and view H as a quartenionic Hilbert space, H^h through the action of $\pi(A)'$. The complexified representation $\pi_{\mathbb{C}}$ is fixed under the involution in Proposition 1.23.

We let σ denote the complex conjugation in the proof of this theorem, and let θ be the *-antiautomorphism on $B(H)$ as defined in Definition 1.19.

Proof. 1. Assume that π is of real type, the commutant of π is $\mathbb{R}$ and thus the commutant of the complexification $\pi_{\mathbb{C}}$ is $\mathbb{C}$. To see that $\pi_{\mathbb{C}}$ is fixed by the involution, consider the following calculation for $a \in A$ where ^{t} denotes the transpose:

$$\theta_*(\pi_{\mathbb{C}})(a) = \tau(\pi_{\mathbb{C}}^{\mathrm{op}}(\theta(a))) = \tau(\pi_{\mathbb{C}}^{\mathrm{op}}(a^*)) = \tau(\pi(a)^t) = (\pi(a)^t)^t$$

   This shows that $\pi_{\mathbb{C}}$ is fixed.

2. Consider the case where π is of complex type, then it suffices to show that $\theta_*(\pi_{\mathbb{C}})$ and $\pi_{\mathbb{C}}$ are inequivalent representations. We may thus assume for contradiction that they are equivalent. Viewing π as an irreducible representation on a complex Hilbert space H^c, we may expand it to an irreducible complex representation of $A_{\mathbb{C}}$, which is equivalent to $\theta_*(\pi^c)$. If we have the element $a + ib \in A_{\mathbb{C}}$ then $\sigma(a + ib) = a - ib, \theta(a + ib) = a^* + ib^*$. Thus, by our assumption $\theta_*(\pi^c)(a+ib) = \tau(\pi_{\mathbb{C}}^{\mathrm{op}})(a^* + ib^*) = \overline{\pi(a)} + \overline{\pi(b)}$, where $\bar{\cdot}$ denotes the complex conjugation.

   We have a canonical identification of $H_{\mathbb{C}} \cong H^c \oplus \overline{H^c}$, implying that the complexification of π may be canonically identified with $\pi^c \oplus \theta_*(\pi^c)$ by our previous calculations. For this to be equivalent to $\pi^c \oplus \pi^c$, would require the commutant of its image to be isomorphic to $M_2(\mathbb{C})$. However, the commutant of $\pi_{\mathbb{C}}$ must be isomorphic to $\pi(A)' \otimes \mathbb{C}$, leading to the desired contradiction.

3. The last case we need to consider is the case where π is of quartenionic type. In this case the commutant of $\pi_{\mathbb{C}}$ commutant is $\mathbb{H} \otimes \mathbb{C} \cong M_2(\mathbb{C})$, with the isomorphism given

by mapping

$$
\begin{aligned}
i \otimes i &\mapsto e_{12} + e_{21} \\
j \otimes i &\mapsto e_{11} - e_{22} \\
k \otimes i &\mapsto -ie_{12} + ie_{21}
\end{aligned}
$$

That $\pi_{\mathbb{C}}(A_{\mathbb{C}})' \cong M_2(\mathbb{C})$ implies that $\pi_{\mathbb{C}} : A \to H \otimes \mathbb{C}$ is equivalent to $\tilde{\pi}_{\mathbb{C}} \oplus \tilde{\pi}_{\mathbb{C}} : A \to V \oplus V$, where V is a complex Hilbert space and $\tilde{\pi}_{\mathbb{C}}$ is irreducible on V. This reduces our problem to showing that $\tilde{\pi}_{\mathbb{C}} \oplus \tilde{\pi}_{\mathbb{C}}$ is fixed by the involution. To see this, start by noting that the involution is $\sigma \oplus \sigma$. We note that the projections onto e_{11} and e_{22} corresponds to $\frac{1}{2}(1+i)$ and $\frac{1}{2}(1-i)$, under the identification $\mathbb{C} \cong \mathbb{H} \otimes M_2(\mathbb{C})$. Thus in a slight abuse of notation:

$$
\pi_{\mathbb{C}} = \frac{1}{2}(1 \otimes 1 + j \otimes i)\tilde{\pi}_{\mathbb{C}} + \frac{1}{2}(1 \otimes 1 - j \otimes i)\tilde{\pi}_{\mathbb{C}}
$$

It is clear that $\sigma \oplus \sigma$ interchanges these, showing that $\pi_{\mathbb{C}}$ is fixed under the involution up to unitary equivalence.

□

Before we proceed, we need to introduce the notion of a graded C^* algebra and the graded commutator as well as the graded tensor product.

Definition 1.28. 1. Suppose that Γ is a discrete group and A is a k-algebra. The algebra A is Γ-graded if there is a decomposition of A into k-vector spaces $A^{(g)}$ such that:

$$
\begin{aligned}
A &= \bigoplus_{g \in \Gamma} A^{(g)} \\
A^{(g)} A^{(g')} &\subset A^{(gg')}
\end{aligned}
$$

In the case where Γ is $\mathbb{Z}_2$, the grading can equivalently be given by an automorphism $\gamma : A \to A$ such that $\gamma(A^{(0)}) = A^{(0)}$, and $\gamma(A^{(1)}) = -A^{(1)}$. From now on we will restrict ourselves to the case $\Gamma = \mathbb{Z}_2$.

2. Suppose A is a graded C^* algebra. An element $x \in A^{(i)}$ is homogeneous, and every element in A may be written as a finite sum of homogeneous elements. Given a homogeneous element x, we denote its degree as $\deg x$.

3. The grading is called inner if the grading automorphism, is implemented by a self-adjoint unitary $g \in M(A)$ such that $\gamma(a) = gag^* = (-1)^n a$ for every $a \in A^{(n)}$. This operator is known as the grading operator. In case the grading is not implemented by a self-adjoint unitary in $M(A)$, we shall say that the grading is outer.

4. A *-subalgebra is called a graded *-subalgebra if it is invariant under the grading operator.

5. An ungraded C^* -algebra A can be viewed as a graded algebra by setting $A^{(1)} = 0$.

6. Let B be a graded C^* algebra. A Hilbert B-module E_B is graded if there is a decomposition of E_B such that:

$$\begin{aligned}
&E_B = E_B^{(0)} \oplus E_B^{(1)} \\
&B^{(i)} E_B^{(j)} \subset E_B^{(j+i)} \\
&\langle , \rangle_B : E_B^{(i)} \times E_B^{(j)} \to B^{(i+j)}
\end{aligned}$$

As in the C^* -algebraic case, the grading can equivalently be given by an automorphism $\gamma : E_B \to E_B$ such that $\gamma(E_B^{(0)}) = E_B^{(0)}$, and $\gamma(E_B^{(1)}) = -E_B^{(1)}$. From now on we will restrict ourselves to the case $\Gamma = \mathbb{Z}_2$.

7. Let B be a graded C^* algebra and let E_B, F_B be graded Hilbert modules. Define $L(E_B, F_B)$:

$$\begin{aligned}
L^{(i)}(E_B, F_B) &= \bigoplus_{k,j \in \mathbb{Z}_2,\ k+j=i} L(E_B^{(k)}, F_B^{(j)}) \\
L(E_B, F_B) &= L^{(0)}(E_B, F_B) \oplus L^{(1)}(E_B, F_B)
\end{aligned}$$

Definition 1.29. Given two graded C^* -algebras A and B a *-homomorphism $\varphi : A \to B$, φ is graded if $\varphi(A^{(i)}) \subset B^{(i)}$.

Definition 1.30. Let x, y be homogeneous elements. We define the graded commutator as

$$[x, y] = xy - (-1)^{\deg(x)\deg(y)} yx$$

Proposition 1.31. The graded commutator satisfies the following relations on homogeneous elements.

1. $[x, y] + (-1)^{\deg x \deg y}[y, x] = 0$
2. $[x, yz] = [x, y]z - (-1)^{\deg x \deg y} y[x, z]$
3. $(-1)^{\deg x \deg z}[[x, y], z] + (-1)^{\deg x \deg y}[[y, z], x] + (-1)^{\deg y \deg z}[[z, x], y] = 0$

Proof. We shall only show the second equality as the calculations become progressively longer and the nature of the relevant calculations is illustrated in this calculation. Let x, y and z be homogeneous elements, and remark that $yz \in A^{(\deg y + \deg z)}$. Thus we get

$$\begin{aligned}
xyz - (-1)^{\deg x(\deg y + \deg z)} yzx &= xyz - (-1)^{\deg x(\deg y+\deg z)} y((-1)^{\deg x \deg z} xz + [z, x]) \\
&= [x, y]z - (-1)^{\deg x(\deg y+\deg z)} y[x, z]
\end{aligned}$$

as desired. ∎

Definition 1.32. Given graded C^* -algebras A and B, we may form their algebraic tensor product $A \odot B$. We equip it with the grading defined on homogeneous elements as: $\deg(x \otimes y) = \deg(x) + \deg(y)$. We define the product on homogeneous elements by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{\deg b_1 \deg a_2}(a_1 a_2 \otimes b_1 b_2)$$

and we extend it by linearity to the entirety of the algebraic tensor product. Suppose π, ρ be graded faithful representations of A and B on H and K respectively, where H and K are graded Hilbert spaces. We define $\pi \otimes \rho$ on $B(H \otimes K)$. Completing in the norm derived from this representation gives us the minimal graded tensor product of A and B. The proof of the fact that this is well-defined is essentially the same as for ungraded C^* algebras, and may be found in [Kas80b].

All tensor products in this thesis are minimal.

Definition 1.33. Suppose that A and B are graded C^* algebras, and consider their tensor product $A \otimes B$. Let E_A and E_B be graded Hilbert modules over A and B. Consider the algebraic tensor product $E_A \odot E_B$, with the grading defined on homogeneous elements as: $\deg(x \otimes y) = \deg(x) + \deg(y)$. This is an $A \otimes B$ module through the action of $a \otimes b \in A \otimes B$ on $x \otimes y \in E_A \odot E_B$ as:

$$(x \otimes y)(a \otimes b) = (-1)^{\deg y \deg a}(xa \otimes yb)$$

We define the exterior tensor product $E_A \otimes E_B$ as the completion of $E_A \odot E_B$ with respect to the inner product:

$$\langle (x_1 \otimes x_2), (y_1 \otimes y_2) \rangle = (-1)^{\deg x_2 (\deg x_1 + \deg x_2)} \langle x_1, y_1 \rangle \otimes \langle x_2, y_2 \rangle$$

Definition 1.34. Let A and B be graded C^* -algebras, and let E_A and E_B be A and B-Hilbert modules respectively. Assume that we have a representation $\pi : A \to L(E)$, and define the sesquilinear map:

$$\begin{aligned} &\langle \cdot, \cdot \rangle_{int} : E_A \odot E_B \to B \\ &\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle_{int} = \langle y_1, \pi(\langle x_1, x_2 \rangle_A) y_2 \rangle_B \end{aligned}$$

Define the nullspace of the semi-norm $\| \langle \cdot, \cdot \rangle_{int} \|$:

$$\mathcal{N} = \{ z \in E_A \odot E_B : \langle z, z \rangle_{int} = 0 \}$$

We define the interior tensor product:

$$E_A \otimes_\pi E_B = \overline{(E_A \odot E_B)/\mathcal{N}}^{\| \langle , \rangle_{int} \|}$$

This is a Hilbert B-module with inner product $\langle , \rangle_{int}$. We shall suppress the subscript $_{int}$ from the notation from now on. Often we will write $\otimes_A$ instead of π when the representation is unambigious.

For a proof that these tensor-products are well-defined, we refer to [Lan95]. Throughout the thesis all tensor products will be graded, unless stated otherwise . To bring back the concreteness of what we are working with, we have some examples of real C^* algebras:

Example 1.35. 1. We define the rotation algebras as $C(S^1, \mathbb{R}) \rtimes \mathbb{Z}$ with $\mathbb{Z}$ acting by rotation by $n\theta$. If $\mathbb{Z}$ is acting by rotation with an irrational angle, we get the real irrational algebra.

2. We may also consider a commutative torus algebra, namely $C(S^1, \mathbb{R}) \otimes C(S^1, \tau_0)$, with τ_0 the complex conjugation on the circle. One should notice that this is just of one three different choices we could have taken for a real algebra reflecting the torus, showing once again the wide range of choices afforded by choice of involution. This is contrast to the complex case, where we get $C(S^1) \otimes C(S^1)$ no matter what, leading to a different geometry as reflected in the K-theory of these algebras.

3. Consider the space $\ell^2(\mathbb{N}, \mathbb{R})$ and the operator $Se_k = e_{k+1}$. We define $\mathcal{T} = C^*(\{S\})$. This fits into an extension as in the complex case, see [Sch93, Chapter 1]:

$$0 \longrightarrow \mathbb{K} \longrightarrow \mathcal{T} \longrightarrow C(S_1, \tau_0) \longrightarrow 0$$

as in the complex case.

4. We construct higher-dimensional non-commutative analogues of the Toeplitz algebra which also fit into extensions. Let $S_1, \ldots, S_n$ be a family of unilateral shifts with orthogonal ranges. As in the complex case this gives rise to the extension, see [Sch93, Chapter 1]:

$$0 \longrightarrow \mathbb{K}(H) \longrightarrow C^*(\{S_1, \ldots, S_n\}) \longrightarrow O_n \longrightarrow 0$$

This defines O_n, the n'th real Cuntz algebra.

5. The prototypical example of graded real C^* -algebras is given by the Clifford algebras $Cl_{p,q}$, as in the appendix on Clifford algebras.

6. We define the real suspension and anti-suspension as the tensor product with:

$$C_0(\mathbb{R}, \mathrm{id})$$
$$C_0(\mathbb{R}, -\mathrm{id})$$

For K-theory the algebras encoding the suspension are also essential and provide good examples of how different involutions on the underlying topological space X induce entirely different C^* -algebras. For further examples, and an introduction to the construction of crossed products in the real case, we refer to [Sch93, Ch. 1]. We end our brief exposition of the fundamental theory of real C^* algebras with a theorem which will allow us to switch seamlessly between the complex and the real case. For notation and results regarding unbounded operators on Hilbert modules, we refer to [Lan95, Chapter 9, Chapter 10].

Theorem 1.36. Let B be real C^* -algebra, and let E_B be a Hilbert B-module.

1. The operator $\mathscr{D} : \mathrm{Dom}\,\mathscr{D} \to E_B$ is self-adjoint and regular if and only if $\mathscr{D} \otimes 1$ is self-adjoint and regular as an operator $\mathrm{Dom}\,\mathscr{D} \otimes \mathbb{C} \to E_{B\otimes\mathbb{C}} \cong E_B \otimes \mathbb{C}$.

2. Let $\mathscr{D}$ be self-adjoint and regular on E_B. Then the operator $(\mathscr{D}^2 + 1)^{-1}$ is compact if and only if $(\mathscr{D} + i)^{-1}$ and $(\mathscr{D} - i)^{-1}$ are compact.

3. The map sending $T \mapsto T \otimes 1$: is an isometry $L(E_B) \otimes \mathbb{C} \to L(E_B \otimes \mathbb{C})$.

Proof. Remark that all tensor products in this proof are ungraded.

1. Start by noting that $E_B \otimes_B B \cong E_B$, thus $E_B \otimes \mathbb{C} \cong E_B \otimes_B B \otimes \mathbb{C} \cong E_{B\otimes\mathbb{C}}$.

   Assume that $\mathscr{D}$ is self-adjoint and regular as a map $\operatorname{Dom}\mathscr{D} \to E_B$. Then consider the map $\mathscr{D} \otimes 1 : (\operatorname{Dom}\mathscr{D}) \otimes \mathbb{C}$. As we have assumed that the the mapping $1 + \mathscr{D}^*\mathscr{D}$ is surjective, it follows that the mapping $1 \otimes 1 + \mathscr{D}^*\mathscr{D} \otimes 1$ is also surjective, giving regularity. To see self-adjointness, consider any element $x \in \operatorname{Dom}((\mathscr{D} \otimes 1)^*)$. This element must satisfy that $\langle x, (\mathscr{D} \otimes 1)y\rangle$ is well-defined for all $y \in \operatorname{Dom}\mathscr{D}\otimes\mathbb{C}$. However, x must be of the form $\xi_1 \otimes 1+ \xi_2 \otimes i$, and we may assume $y = \eta \otimes 1$, for $\eta \in \operatorname{Dom}\mathscr{D}$, as the domain is a linear space. Thus we have

   $$\langle x, (\mathscr{D} \otimes 1)y\rangle = \langle \xi_1 \otimes 1, (\mathscr{D} \otimes 1)\eta_1\rangle - 1 \otimes i \, \langle \xi_2 \otimes 1, (\mathscr{D} \otimes 1)(\eta_1 \otimes 1)\rangle$$

   This gives an element of $B \oplus iB$ and in order for this element to be well-defined, both terms must lie in B. Thus we get that $\xi_i \in \operatorname{Dom}\mathscr{D}^*$, and by self-adjointness of $\mathscr{D}$ they must lie in $\operatorname{Dom}\mathscr{D}$.

   Conversely, assume that $\mathscr{D}\otimes 1 : \operatorname{Dom}\mathscr{D}\otimes\mathbb{C} \to E_B\otimes\mathbb{C}$ is self-adjoint and regular. Start by assuming $\mathscr{D}\otimes 1$ is self-adjoint, and consider $\mathscr{D}\otimes 1$ as a map $E_B \oplus iE_B \to E_B \oplus iE_B$, ie $\mathscr{D} \otimes 1 = \mathscr{D} \oplus \mathscr{D}$. Then self-adjointness implies that $(\mathscr{D} \oplus \mathscr{D})^* = \mathscr{D} \oplus \mathscr{D}$. Therefore $\mathscr{D}^* = \mathscr{D}$, and thus $\mathscr{D}$ is self-adjoint.

   For regularity, let $x \in E_B$, then there is an element $\xi \otimes z \in E_B \otimes \mathbb{C}$ such that $((\mathscr{D}^*\mathscr{D} + 1) \otimes 1)\xi \otimes z = x \otimes 1$. Then

   $$\begin{aligned} ((\mathscr{D}^*\mathscr{D} + 1) \otimes 1)(\xi \otimes \overline{z} + \xi \otimes z) &= x \otimes 1 \\ ((\mathscr{D}^*\mathscr{D} + 1) \otimes 1)\xi' \otimes 1 &= x \otimes 1 \end{aligned}$$

   Thus $(\mathscr{D}^*\mathscr{D} + 1)\xi' = x$, showing regularity.

2. It is clear by the $C_0(\mathbb{R})$-functional calculus for self-adjoint regular operators that if $(\mathscr{D}\otimes 1\pm 1\otimes i)^{-1}$ are compact, then $(\mathscr{D}^2\otimes 1+1\otimes 1)^{-1} = (\mathscr{D}\otimes 1\pm 1\otimes i)^{-1}(\mathscr{D}\otimes 1\mp 1\otimes i)^{-1}$ is compact. For the converse, assume that $(\mathscr{D}^2+1)^{-1}$ is compact. Consider the operator

   $$\mathscr{D}(\mathscr{D}^2 + 1)^{-1/2}$$

   This is bounded, as it is the bounded transform of a self-adjoint regular operator, see [Lan95, Chapter 9]. By the functional calculus, $(\mathscr{D}^2 + 1)^{-1/2}$ is also compact, and $(\mathscr{D} - i)^{-1} = (i + \mathscr{D})(\mathscr{D}^2 + 1)^{-1}$. Applying the $C_0(\mathbb{R})$-functional for self-adjoint regular operators, we get:

$$\begin{aligned}
&(\mathscr{D}-i)^{-1}\\
&=\mathscr{D}(\mathscr{D}^2+1)^{-1}+i(\mathscr{D}^2+1)^{-1}\\
&=\underbrace{\mathscr{D}(\mathscr{D}^2+1)^{-1/2}}_{bounded}\underbrace{(\mathscr{D}^2+1)^{-1/2}}_{compact}+i\underbrace{(\mathscr{D}^2+1)^{-1}}_{compact}\\
&=\underbrace{(\mathscr{D}(\mathscr{D}^2+1)^{-1/2}+i(\mathscr{D}^2+1)^{-1/2})}_{bounded}\underbrace{(\mathscr{D}^2+1)^{-1/2}}_{compact}
\end{aligned}$$

Thus $(\mathscr{D}-i)^{-1}$ is compact. Similarly, $(\mathscr{D}+i)^{-1}$ is compact.

3. Clear.

□

Henceforth we shall usually suppress the tensor product when considering complexifications and let it be implicit in the notation as to whether we are working with a complexified algebra or not. We now turn our attention to a brief exposition of the theory of real continuous trace algebras, which highlights some of the essential differences between real and complex C^* -algebras.

2 Real K and KK-theory

2.1 Basic properties

We can now proceed to consider the K-theory as well as its corresponding K-homology of real C^* -algebras in their joint guise of Kasparovs KK-theory. There are several reasons for considering unbounded real KK-theory. Unbounded real KK-theory can detect orientation, and can detect differences between algebras which are equal in the complex case, eg. $\mathbb{H}$ and $M_2(\mathbb{R})$ which have isomorphic complexifications but are clearly different as real algebras. We start by defining KKO-theory.

Definition 2.1. Let A and B be separable real C^* algebras. We define the Kasparov (A, B) cycles $E(A, B)$ as the triples $E = (E, \pi, F)$, where:

1. E is a countably generated $\mathbb{Z}_2$ graded real Hilbert B module
2. The map π is a graded real *-homomorphism $A \to L(E)$.
3. The operators $[\pi(A), F]$, $\pi(a)(F^2 - 1)$ and $(F^2 - 1)\pi(a)$ are in $K(E)$ for all a in A.

A triple (E, π, F) is degenerate if all three operators in the third requirement are identically zero for every a in A. We can endow $E(A, B)$ with a binary operation through the direct sum:

$$(E_1, \pi_1, F_1) \oplus (E_2, \pi_2, F_2) = (E_1 \oplus E_2, \pi_1 \oplus \pi_2, F_1 \oplus F_2)$$

In order to reduce the unwieldy collection of cycles to a group, we define an equivalence relation akin to the one used to define K-homology.

Definition 2.2. Two cycles $E_1 = (E, \pi, F')$ and (E, π, F) are operator homotopic if there is a strictly continuous family F_t giving rise to a path of cycles (E, π, F_t) where $F_0 = F, F_1 = F'$. We define the equivalence relation $\sim_{oh}$ as the equivalence relation stemming from addition of degenerate modules and operator homotopy.

Definition 2.3. Define $KKO(A, B)$ as the group $E(A, B)/\sim_{oh}$. Define the higher KKO groups by the formula

$$K_{p,q}K^{r,s}O(A, B) = KKO(A\hat{\otimes}Cl_{p,q}, B\hat{\otimes}Cl_{r,s})$$

For details and proofs, see [Kas80a], where the the equivalence between formal and topological Bott periodicity is explored. We summarize some of the properties of operator KKO theory in the theorem below, for a proof see [Kas80a] or [Sch93].

Theorem 2.4. 1. Operator KO_*-theory is a stable covariant 8-periodic functor from the category of real graded C^* algebras to the category of abelian groups. The functor KO_* takes short exact sequences of real graded C^* -algebras to a 24-term cyclic exact sequence of abelian groups.

2. Operator KO^*-homology is a stable 8-periodic contravariant functor from the category of real C^* algebras to the category of abelian groups. If we have a short exact sequence with a completely positive splitting, KO^* applied to the sequence gives rise to a 24-term cyclic exact sequence.

3. The functors KO_* and KO^* are combined in the stable bifunctor KKO, where $KKO(A,\mathbb{R}) \cong KO^0(A)$ and $KKO(\mathbb{R},A) \cong KO_0(A)$. We summarize some of the most important properties of K^*K_*O relating to formal Bott periodicity

$$\begin{aligned} &KO_n(A) \cong KO_0(A \otimes Cl_{0,n}) \\ &KO(A) \cong KKO(\mathbb{R},A) \\ &KKO(\mathbb{R},Cl_{p,q} \otimes A) \cong KKO(Cl_{q,p},A) \end{aligned}$$

The functor K^*K_*O also satisfies topological Bott-periodicity, ie. it is 8-periodic under taking suspensions.

Lemma 2.5. Let A be a real C^* algebra then the isomorphism between $K_0(A)$ and $KO(\mathbb{R},A)$ is given by: $[p] \mapsto [(\pi_{\mathbb{R}}, pA^n, 0)]$ for $[p] \in KO_0(A)$.

Proof. We start by showing the mapping is well-defined. Let $p \sim_h q$ be unitarily equivalent projections in $\mathbb{K} \otimes A$, with the equivalence implemented by some unitary U. Then $U(pA^n)U^* = qA^m$, so the mapping is well-defined. To see the mapping is injective, assume that $[(\pi, pA^n, 0)]$ is a trivial cycle, implying that $0 - Id_{pA^n} = 0$, so $p = 0$. Given an arbitrary cycle $[(\pi, E, F)]$ with F self-adjoint, consider $[(\pi, \ker(F), 0)]$ and $[(\pi, \ker(F)^\perp, F)]$. These are both well-defined as the range of F is closed, it being Fredholm, and by [**?**, Theorem 3.2], $\ker(F)$ is a complemented submodule. To show surjectivity, it suffices to show that the second cycle is degenerate. This follows from a swindle argument as F is a linear isomorphism in the second cycle, and all commutants with π are zero. Additivity is clear, showing the desired. □

We use the periodicity of KKO-theory to derive the following standard results on operator KO-theory of the reals, [Sch93, Section 1].

Example 2.6. The K-theory of the reals is:

$$KO_*(\mathbb{R}) = \begin{cases} 0 & \mathbb{Z} \\ 1 & \mathbb{Z}_2 \\ 2 & \mathbb{Z}_2 \\ 4 & \mathbb{Z} \end{cases}$$

By formal Bott periodicity in KKO-theory,

$$KO^n(\mathbb{R}) \cong KKO(Cl_{0,n} \otimes \mathbb{R}, \mathbb{R}) \cong KO_{-n}(\mathbb{R}),$$

allowing us to derive the following table:

$$KO_*(\mathbb{R}) = \begin{cases} KO_0 & \mathbb{Z} \\ KO_{-1} & \mathbb{Z}_2 \\ KO_{-2} & \mathbb{Z}_2 \\ KO_{-4} & \mathbb{Z} \end{cases}$$

As described in the section on real C^* -algebras, we have both suspensions and anti-suspensions in real bivariant K-theory, which are related to the Clifford algebras via. the following lemma.

Lemma 2.7. We have the isomorphisms.

$$KKO(A \otimes Cl_{0,n}, B) \cong KK(A \otimes C_0(\mathbb{R}^n, \mathrm{id}), B)$$
$$KKO(A \otimes Cl_{n,0}, B) \cong KK(A \otimes C_0(\mathbb{R}^n, -\mathrm{id}), B)$$

Thus we have the isomorphisms $KO^n(A) = KO(C_0(\mathbb{R}^n) \otimes A)$, and $KO^{-n}(A) \cong KO(A \otimes C_0(\mathbb{R}, -\mathrm{id}))$.

For a proof, see [Kas80a] We may immediately use these results to calculate the K-theory of the algebra $C(S^1, \mathrm{id})$, giving a flavor of the theory, as well as characterizing how tensoring with $C(S^1, \tau)$ affects the K-theory of an algebra.

Theorem 2.8. The KO-theory of $C(S^1, \mathrm{id})$ is given by the following table.

$$KO_n(C(S^1, \mathrm{id})) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_2 & n = 0 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & n = 1 \\ \mathbb{Z}_2 & n = 2 \\ \mathbb{Z} & n = 3, 4, 7 \\ 0 & n = 5, 6 \end{cases}$$

For any real C^* -algebra A we have the isomorphism

$$KO_n(A \otimes C(S^1, \tau_0)) \cong KO_n(A) \oplus KO_{n-1}(A)$$

This allows us to compute the K-theory of $C(S^1, \tau_0)$.

Proof. Consider the split short exact sequence, with the split begin given by $x \mapsto (z \mapsto x)$ for $x \in \mathbb{R}$.

$$0 \longrightarrow C_0(\mathbb{R}, \mathrm{id}) \longrightarrow C(S^1, \mathrm{id}) \longrightarrow \mathbb{R} \longrightarrow 0$$

Thus the 24-periodic exact sequence in K-theory reduces to the split short exact sequence

$$0 \longrightarrow KO_{n+1}(\mathbb{R}) \longrightarrow KO(C(S^1, \mathrm{id})) \longrightarrow KO_n(\mathbb{R}) \longrightarrow 0$$

giving the desired table. To show the second part of the lemma, consider the split short exact sequence

$$0 \longrightarrow A \otimes C_0(i\mathbb{R}) \longrightarrow A \otimes C(S^1, \tau) \longrightarrow A \longrightarrow 0$$

with the split being given by the map $a \mapsto a \otimes 1$. We can infer that the 24-periodic long-exact sequence reduces to the split short exact sequence

$$0 \longrightarrow KO_{n-1}(A) \longrightarrow KO_n(A \otimes C(S^1, \tau)) \longrightarrow KO_n(A) \longrightarrow 0$$

giving $KO_n(A \otimes C(S^1, \tau)) \cong KO_{n-1}(A) \oplus KO_n(A)$ as desired. $\square$

The benefit of real K-theory over complex K-theory is exhibited in these two cases, where we can see a much finer structure on the K-theory of the real circle.
In recent years real K-theory has been applied in the realm of topological insulators in order to, once again, explain global behaviors by the presence of these torsion classes. This has also explictly used the construction of the unbounded Kasparov product, [BKR16], in a form which we shall recreate at a later point in the thesis.

2.2 K-homology of $\mathbb{R}$

The standard picture of KK-theory works very well in purely algebraic context, but in practice the operators that encode the geometry, commutative or otherwise, of a given space are naturally defined as unbounded regular operators. To remedy this shortcoming, it is often preferable to work with unbounded KK-theory which is defined from suitable unbounded Kasparov cycles. We start by working in the bounded case, and then argue that for suitable operators unbounded operators, the bounded results essentially also hold.

Definition 2.9. Let H be a real Hilbert space with the structure of a $Cl_{0,k}$-module, and let $F(H, H)$ denote the odd Fredholm operators on H. Define F_k as those elements of $F(H, H)$ which are $Cl_{0,k}$-linear and self-adjoint.

We define the Clifford index of operators lying in F_k, referring to Theorem A.5 for the notation on $\hat{A}_k$ and $\hat{\mathfrak{M}}_k$.

Definition 2.10. Let T be an operator in $F_k(H, H)$. Define the Clifford index of T as:

$$\mathrm{Ind}_k(T) = [\ker T] \in \hat{A}_k \cong KO_k(\mathbb{R})$$

This is well-defined due to the Atiyah-Bott-Shapiro isomorphism, Theorem A.5.

In order to justify calling this construction an index, we check that Ind_0 recovers the usual Fredholm index.

Example 2.11. We see $Cl_0 = \mathbb{R}$ and $Cl_{0,1} = \mathbb{C}$. A $\mathbb{Z}_2$ graded Cl_0-module is simply a graded $\mathbb{R}$-vector space, $V_0 \oplus V_1$. We observe that $[V \oplus 0] = -[0 \oplus V]$ in $\hat{A}_0$ since $V \oplus V \cong V \otimes \mathbb{C}$ may be extended to a graded $Cl_{0,1}$-module. Given $T \in F_k(H, H)$ we get $\mathrm{Ind}_0(T) = [\ker T_0 \oplus \ker T_1] = [\ker T_0 \oplus 0] - [\ker T_1 \oplus 0]$, recovering the Fredholm Index of T_0.

We wish to show that the index map is well-defined as a map from K-homology, so we need it to be homotopy invariant and a group homomorphism.

Theorem 2.12. The Clifford index is constant on connected components of F_k, i.e. it is homotopy-invariant.

Proof. Let $T \in F_k$, as 0 is an isolated point in the spectrum we may assume that the non-zero spectrum of T lies outside of $[-2, 2]$. Pick a neighborhood U of T in F_k such that for all $S \in U$ we have that $\sigma(S^2) \subset [0, 1/2) \cup (1, \infty)$ and $\left\|T^2 - S^2\right\| \leq 1/2$. Fix $S \in U$ and let W be the range of the spectral projection of S^2 onto $[0, 1/2]$. Consider also the orthogonal projection $P : H \to \ker T$. Then we claim that $p : W \cong \ker T$. To do this, pick $v \in W \in (\ker T)^\perp$. We have that the equality

$$\left\langle (T^2 - S^2)v, v\right\rangle \geq \left(2 - \frac{1}{2}\right) \|v\|^2 \geq \|v\|^2$$

As $\left\|T^2 - S^2\right\| < \frac{1}{2}$, we get that $\|v\| < 1/2$, and as such we get that that $v = 0$. In order to see surjectivity, pick $v \in W^\perp \cap \ker T$ and observe that $\left\langle (S^2 - T^2)v, v\right\rangle \geq \|v\|^2$ and as such is 0. We see that W is a $\mathbb{Z}_2\mathbb{Z}$ graded submodule of H, as well as splitting as $W = \ker S \oplus (\ker(S)^\perp \cap W)$. The projection P onto the kernel of T also preserves the graded module structure. Defining $V = (\ker(S)^\perp \cap W)$ we get the equivalence

$$\ker T \cong \ker S \oplus V$$

In order to see that the class is the same, we need to give V the structure of Cl_{k+1} module in order to map it to A_k. Let $S_V = S|_V$. This is a symmetric $\mathbb{Z}_2$ $Cl_{0,k}$-linear graded map. Thus the operator $J = (S_V^2)^{-1/2} S_V$ is as well. We see that $J^2 = Id$. Decomposing $V = V_0 \oplus V_1$, $J = e_{21}J_0 + e_{12}J_1$. The graded endomorphism $\tilde{J} = -e_{21}J_0 + e_{12}J_1$ squares to -1, and as such makes V into a Cl_{k+1}-module as desired. Thus $\operatorname{Ind}_k T = \operatorname{Ind}_k S$ as desired. □

We refer to the following reuslt: [ML89, Page 217].

Proposition 2.13. Given self-adjoint elliptic $Cl_{0,k}$-linear (pseudo)-differential operator $\mathscr{D}$, the class $[\ker(\mathscr{D})]$ in $\hat{A}_k$ equals $[\ker(\mathscr{D}/(1 + \mathscr{D}^2)^{-1/2})]$.

To construct cycles representing the generators of the real K-homology groups presenting the flavor of the theory, it suffices to construct unbounded elliptic operators with kernels corresponding to the generators of the groups $\hat{A}_k$. Alternatively, we could use the construction in Lemma 2.5.

Theorem 2.14. The following cycles are generators of the non-trivial real K-homology groups of the reals.

1. The group $KO^0(\mathbb{R})$ is generated by the cycle

$$E = \left(\mathbb{R}, \ell^2(\mathbb{N}, \mathbb{R}) \oplus \ell^2(\mathbb{N}, \mathbb{R}), \begin{pmatrix} 0 & D \\ D^* & 0 \end{pmatrix}\right)$$

Where the operator D is defined on the core spanned by the standard basis as:

$$\begin{aligned} D &: \ell^2(\mathbb{N},\mathbb{R}) \to \ell^2(\mathbb{N},\mathbb{R}) \\ e_k &\mapsto k e_{k+1} \end{aligned}$$

The adjoint of D also has the standard basis as core, where it is defined as

$$D^*(e_k) = \begin{cases} 0 & k = 1 \\ \frac{1}{k-1} e_{k-1} & k \geq 2 \end{cases}$$

2. The group $KO^{-1}(\mathbb{R})$ is generated by

$$\big(Cl_{0,1}, L^2(S^1,\mathbb{C}), \gamma_1 d_\theta\big)$$

where θ is the angular direction on the circle, and γ_1 is the generator of $Cl_{0,1} \cong \mathbb{C}$.

3. The group $KO^{-2}(\mathbb{R})$ is generated by

$$(Cl_{0,2}, L^2(S^1 \times S^1, \mathbb{H}), \gamma_1\partial_{\theta_1} + \gamma_2\partial_{\theta_2})$$

where θ_i are the angular directions on the torus and γ_1, γ_2 are the generators of $Cl_{0,2} \cong \mathbb{H}$.

4. The group $KO^{-4}(\mathbb{R})$ is generated by:

$$(Cl_{0,4}, (\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})) \otimes_{\mathbb{R}} (\mathbb{H} \oplus \mathbb{H}), \mathscr{D})$$

Where the representation of $Cl_{0,4}$ is either of the two irreducible graded representations.

Proof. We start by noting that for every n the cycles are given by self-adjoint elliptic $Cl_{0,n}$-linear operators. To start, we wish to show that:

$$\mathrm{Ind}_n(T) : KO^n(\mathbb{R}) \to \hat{A}_k$$

is a homomorphism. We start by checking that it is well-defined on equivalence classes. Invariance under unitary isomorphism is clear, and by Theorem 2.12 the map $\mathrm{Ind}_n(T)$ is invariant under operator homotopy. Thus it is well-defined as a function $KO^*(\mathbb{R}) \to \hat{A}_*$ To see that Ind_n is a homomorphism, assume that (π, F, H) is a degenerate cycle, so it is $Cl_{0,k}$-linear. Furthermore, degeneracy implies that $F^2 = 1$ and $F = F^*$. Therefore F is a self-adjoint unitary and and thus has trivial kernel. Hence $\mathrm{Ind}_n(F) = 0$. Additivity is clear, hence can infer that $\mathrm{Ind}_n : KO^n \to \hat{A}_n$ is homomorphism. As we know what the left-hand side is by Bott periodicity, we only need to show that our cycles in $KO^n(\mathbb{R})$ have the appropriate index.

1. Define the operator

$$\begin{aligned} D &: \ell^2(\mathbb{N},\mathbb{R}) \to \ell^2(\mathbb{N},\mathbb{R}) \\ e_k &\mapsto k e_{k+1} \end{aligned}$$

The operator clearly has the standard basis as core of its minimal closure, and is thereby densely defined. The adjoint of D likewise has the standard as core, where it is given by:

$$D^*(e_k) = \begin{cases} 0 & k=1 \\ \frac{1}{k-1} e_{k-1} & k \geq 2 \end{cases}$$

Consider the cycle

$$E = \left(\mathbb{R}, \ell^2(\mathbb{N},\mathbb{R}) \oplus \ell^2(\mathbb{N},\mathbb{R}), \begin{pmatrix} 0 & D \\ D^* & 0 \end{pmatrix} \right)$$

By Example 2.11 $\mathrm{Ind}_0 \left(\begin{pmatrix} 0 & D \\ D^* & 0 \end{pmatrix} \right)$ is the Fredholm Index of D, which is readily seen to be 1. Thus E generates $KO^0(\mathbb{R})$.

2. Consider the operator $\mathscr{D} = \gamma_1 \otimes d_\theta$, in the Fourier basis we can write this on the basis vectors as:

$$\begin{aligned} \mathscr{D} &: \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \\ e_k &\mapsto k e_k \end{aligned}$$

Remark that the considerations above on domains can be reused for this operator. We wish to determine the Clifford index of this operator and thus we take $[\ker(\mathscr{D})] \in \hat{A}_1 \cong \mathbb{Z}_2$. The kernel consists of the constant $Cl_{0,1}$-functions, equipped with the canonical grading. Thus the kernel is $Cl_{0,1}$ viewed as a module over itself. This is is the generator of $\hat{\mathfrak{M}}_1$, and thus its image in the quotient $\hat{\mathfrak{M}}_1 \to \hat{A}_1$ is the generator of $\hat{A}_1$.

3. Consider the operator $\mathscr{D} = \gamma_1 \partial_{\theta_1} + \gamma_2 \partial_{\theta_2}$ on $L^2(S^1 \times S^1, \mathbb{H})$. If we consider $\mathscr{D}$ as an operator in the Fourier basis, we can write it as

$$\begin{aligned} \mathscr{D}(k_j e_j, c_n e_n) &= (-\gamma_1\gamma_1 j k_j e_j, -\gamma_2\gamma_1 n c_n e_n) \\ &= (j k_j e_j, \gamma_1\gamma_2 n c_n e_n) \end{aligned}$$

where $k_j, c_n \in \mathbb{H}$. Thus $\mathscr{D} f = 0$ implies $k_j = c_n = 0$ for all non-zero j, n. Therefore $f = (k_0, c_0)$ and is thereby a constant $\mathbb{H}$-valued function. We see that the kernel is isomorphic to the constant $\mathbb{H}$-valued functions and as such is the generator of $\hat{\mathfrak{M}}_2$, and thereby the generator of $\hat{A}_2$.

4. Let $\pi : M_2(\mathbb{H}) \to \mathbb{H} \otimes_{\mathbb{H}\oplus\mathbb{H}} M_2(\mathbb{H}) \cong \mathbb{H} \oplus \mathbb{H}$ denote either of the two graded irreducible representations of $M_2(\mathbb{H})$, see Proposition A.3. Define the map $\mathscr{D} : \ell^2(\mathbb{Z}) \otimes_{\mathbb{R}} (\mathbb{H}\oplus\mathbb{H}) \to$

$\ell^2(\mathbb{Z}) \otimes_{\mathbb{R}} (\mathbb{H} \oplus \mathbb{H})$ as:

$$e_k \otimes (h_1, h_2) \mapsto ke_k \otimes (h_1, h_2)$$

The kernel of this map is isomorphic to $\mathbb{H} \oplus \mathbb{H}$, which is the generator of $\hat{A}_4$ by construction. Thus the cycle

$$(Cl_{0,4}, (\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})) \otimes_{\mathbb{R}} (\mathbb{H} \oplus \mathbb{H}), \mathscr{D})$$

generates $KO^{-4}(\mathbb{R})$.

□

The above calculations can be seen to be a specific case of the KO^*-generalization of the analytic index of Clifford-linear operators from spin geometry.

Remark 2.15. Let $x \in KO^n(A)$ and $y \in KO_m(A)$. Then we may take the Kasparov product $x \otimes y$, to get an element in $KKO(Cl_{0,n}, Cl_{0,m}) \cong KKO(Cl_{n,m}, \mathbb{R})$. This element is uniquely determined up to operator homotopy by its Clifford index.

We will give a concrete formula for the calculation of this product in a later section, by applying the unbounded Kasparov product.

3 Construction of the unbounded Kasparov product over real C^* -algebras

The development of unbounded KK-theory is foreshadowed in Kasparov's original paper [Kas80a] where all Bott elements are bounded transforms of naturally defined unbounded cycles. The study of unbounded KK-theory begins with the paper by Baaj and Julg, [BJ83] where they define unbounded KK-cycles:

Definition 3.1 (Unbounded KK-cycles)**.** Let A and B be real or complex C^* -algebras. Define the set $\Psi(A, B)$ of unbounded Kasparov $A{-}B$ cycles as the set of quadruples $(E, \pi, \mathcal{A}, \mathscr{D})$ where E is a graded Hilbert B-module and $\mathcal{A}$ is a dense subalgebra of A, $\pi : A \to L(E)$ is a *-homomorphism, and $\mathscr{D}$ is a degree one unbounded self-adjoint regular densely defined operator on E, satisfying the conditions:

1. $(1 + \mathscr{D}^2)^{-1}$ is locally compact, in the sense that $\pi(a)(1 + \mathscr{D}^2)^{-1}$ and $(1 + \mathscr{D}^2)^{-1}\pi(a)$ are compact for all $a \in A$

2. The domain of $\mathscr{D}$ is preserved under the action of $\mathcal{A}$, ie. for every $a \in \mathcal{A}$ and $\xi \in \mathrm{Dom}(\mathscr{D})$, $\pi(a)\xi$ is in $\mathrm{Dom}(\mathscr{D})$.

3. For all $a \in \mathcal{A}$, the operator $[\mathscr{D}, \pi(a)]$ extends uniquely to an operator in $L(E)$.

As an example, consider the standard differentation operator on $L^2(\mathbb{R})$, and $\mathcal{A}$ as $C_0^\infty(\mathbb{R})$. We recognize the cycles defining the real K-homology of $\mathbb{R}$ as unbounded Kasparov cycles.

Remark 3.2. The commutators and gradings on A and E_B are related:

$$\begin{aligned}
&[\mathscr{D}, \pi(a)] = \mathscr{D}\pi(a) - \pi(\gamma(a))\mathscr{D} \\
&[\mathscr{D}, \pi(a)]^* = -[\mathscr{D}, \pi(\gamma_A(a^*))] \\
&\pi(\gamma_A(a)) = \gamma_{E_B}\pi(a)\gamma_{E_B}
\end{aligned}$$

The utility of unbounded Kasparov modules was immediate, as it allowed [BJ83] to write the product operator in the exterior Kasparov product simply as

$$D_1 \otimes 1 + 1 \otimes D_2$$

It was an open problem for many years to construct an unbounded version of the interior Kasparov product, but since the work of [Mes09] this problem has come ever closer to being solved in full generality through the work of [Kaa15], [BMvS13] and various others. In this section, we shall give the hitherto most general construction of the unbounded interior Kasparov product by expanding the results [MR16] to the real setting. We generally follow their methods closely, only occasionally adapting them to the real case when necessary.

As it will turn out the natural setting for the unbounded Kasparov product is the category of operator spaces, not C^* -algebras so we need to introduce the framework of operator spaces and algebras, in particular complete differential algebras.

3.1 Operator spaces and algebras

Definition 3.3. 1. An operator algebra $\mathcal{A}$ is a closed subalgebra of a C^* algebra B. As we may represent B isometrically on $B(H)$, we may assume $\mathcal{A} \subset B(H)$.

2. An operator *-algebra is an operator algebra $\mathcal{A} \subset B(H)$ with a completely bounded involution $^* : \mathcal{A} \to \mathcal{A}$. This involution will in general not coincide with the involution on B.

3. An operator space X is a closed subspace of a C^* -algebra.

4. Let $\mathcal{A}$ be an operator algebra, and X be an operator space. If there is a continuous (left or right)-action $\mathcal{A} \times X \to X$, X is a (left or right)-$\mathcal{A}$-module.

5. Let A and B be C^* -algebras. Suppose that $X \subset A$ and $Y \subset B$ are operator spaces, and let $\psi : X \to Y$ be a linear map. There are unique norms $\|\cdot\|_n$ on $M_n(X)$ and $M_n(Y)$ stemming from the unique norms on $M_n(A)$ and $M_n(B)$ respectively. Define $\varphi_n = \varphi \otimes 1_n : M_n(X) \to M_n(Y)$. If

$$\sup_{n\in\mathbb{N}} \sup_{\|a\|_n \leq 1} \|\varphi_n(a)\|_n < \infty$$

Then φ is completely bounded, with

$$\|\varphi\|_{cb} = \sup_{n\in\mathbb{N}} \sup_{\|a\|_n \leq 1} \|\varphi_n(a)\|_n .$$

Likewise, φ is completely isometric if φ_n is an isometry for all n, and completely contractive if φ_n is a contraction for all n.

Remark 3.4. There are two C^* algebras associated to a given operator algebra:

1. $C^*(\mathcal{A}) \subset A$, the smallest subalgebra of A containing $\mathcal{A}$.

2. A: The C^* -closure of A, which stems from viewing $\mathcal{A}$ as a Banach *-algebra and completing in the norm defined from the spectral radius of a^*a.

Definition 3.5. Given an unbounded Kasparov module $(\mathcal{A}, E_B, \mathscr{D})$ with grading γ define the algebra $\mathcal{A}_D = \{a \in A | a \operatorname{Dom}(\mathscr{D}) \subset \operatorname{Dom}(\mathscr{D}), [\mathscr{D}, a] \in L(E_B)\}$. We may equip this with the structure of an operator $^*-$algebra via. the representation

$$\pi_{\mathscr{D}} : \mathcal{A} \to L((E_B) \oplus E_B)$$
$$a \mapsto \begin{pmatrix} \pi(a) & 0 \\ [\mathscr{D}, \pi(a)] & \pi(\gamma(a)) \end{pmatrix}$$

with the *-operation given by $\pi_{\mathscr{D}}(a)^* = \pi(\gamma(a))$. Remark that this operator extends to a bounded operator for every a, as $[\mathscr{D}, \pi(a)]$ extends to a bounded operator. We shall always assume $\mathcal{A}_{\mathscr{D}}$ to be equipped with the topology coming from $\|\pi_{\mathscr{D}}(a)\|_{L(E_B \oplus E_B)}$.

Assumption 3.6. The representation π will be assumed to be faithful throughout the thesis, as else we could consider the representation

$$a \mapsto a \oplus \pi_{\mathscr{D}}(a) \in A \oplus L_B(E_B \oplus E_B)$$

which does not change the results, but clutters the calculations.

As motivation for the definition of the Lipschitz representation, remark that if $a \in C_0^1(\mathbb{R})$ and $\mathscr{D}$ is the derivative on $\mathbb{R}$, $[\pi(a), \mathscr{D}] = a'$. In general, one should think of the commutators with $\mathscr{D}$ as derivatives with respect to $\mathscr{D}$. An important generalization of the Lipschitz algebra is the general notion of a differentiable algebra, which we will define later.

Definition 3.7. We define the Lipschitz algebra associated to a self-adjoint regular operator $\mathscr{D}$:

$$\mathrm{Lip}(\mathscr{D}) = \{T \in L(E_B) \mid T\,\mathrm{Dom}(\mathscr{D}^*) \subset \mathrm{Dom}(\mathscr{D}), [\mathscr{D}^*, T] \in L(E_B)\}$$

We can use this to define the general framework of differentiable algebras, which are algebras acting where the operators behave like differentiable functions.

Definition 3.8. We define a differentiable algebra $\mathcal{A}$ as a separable operator *-subalgebra of $\mathrm{Lip}(\mathscr{D})$ closed in the topology stemming from $\pi_{\mathscr{D}}$. Projecting onto the first coordinate of $\pi_{\mathscr{D}}(\mathcal{A})$, the C^* closure of $\mathcal{A}$ algebra coincides with the closure of $\mathcal{A}$ as a subalgebra of $L(E_B)$, thereby giving a subalgebra of A.

We need to define a tensor product on the category of operator spaces, as we are working in the setting of operator spaces rather than Hilbert C^* -modules.

Definition 3.9. Given two operator spaces X, Y we may define their algebraic tensor product $X \odot Y$. Define the Haagerup norm for $z \in M_n(X \odot Y)$

$$\|z\|^2 = \inf \left\{ \left\| \sum_{i=1}^m x_i x_i^* \right\| \left\| \sum_{i=1}^m y_i^* y_i \right\| : z = \sum_{i=1}^m x_i \otimes y_i \right\}$$

We shall denote the Haagerup norm as $\|\cdot\|_{\widetilde{\otimes}}$. Let $\mathcal{A}$ be an operator algebra. If X is a left and Y is a right operator module, define the Haagerup module tensor product:

$$X \widetilde{\otimes}_{\mathcal{A}} Y$$

as the quotient of $X \widetilde{\otimes} Y$ by the closed linear spans of expressions of the form $x \otimes ay - xa \otimes y$. The Haagerup tensor product serves to make multiplication $X \widetilde{\otimes} \mathcal{A} \to X$ continuous.

Remark 3.10. In case E_B and F_C are Hilbert modules, with a representation $\pi : B \to F_C$, we have the cb. isomorphism $E_B \widetilde{\otimes}_B F_C \cong E_B \otimes_B F_C$, see [BP91].

To see this is well-defined, we refer to [BP91]. Likewise, we would like to define the notion of inner product operator modules.

Definition 3.11. An inner product operator module $\mathcal{E}$ is a normed right operator module over an operator *-algebra $\mathcal{B}$ with a sesquilinear pairing $\mathcal{E} \times \mathcal{E} \to \mathcal{B}$ such that

$$\begin{aligned}
&\langle e_1, e_2 b\rangle = \langle e_1, e_2\rangle\, b \\
&\langle e_1, e_2\rangle^* = \langle e_2, e_1\rangle \\
&\langle e, e\rangle \geq 0 \text{ in } B \\
&\langle e, e\rangle = 0 \Leftrightarrow e = 0
\end{aligned}$$

Further we require that $\langle\cdot,\cdot\rangle$ satisfies a weak version of the Cauchy-Schwarz inequality for $C > 0$:

$$\|\langle e_1, e_2\rangle\| \leq C \, \|e_1\|_{\mathcal{E}} \, \|e_2\|_{\mathcal{E}}$$

for all matrix norms.

Remark 3.12. We do not require $\mathcal{E}$ to be complete in the inner product $\langle,\rangle_{\mathcal{E}}$, and in general the topology induced from the $\mathcal{B}$-valued inner product will differ from the norm topology on $\mathcal{E}$.

The differentiable-algebraic analogue of having a differentiable increasing net converging to the identity is:

Definition 3.13. Let $\mathcal{A}$ be an operator algebra, then a bounded approximate unit for $\mathcal{A}$ is a net $(u_\lambda)_{\lambda\in I}$ such that $\sup_{\lambda\in I} \|u_\lambda\| < \infty$ and $\lim_{\lambda\in I} \|u_\lambda a - a\| = \lim_{\lambda\in I} \|a u_\lambda - a\| = 0$. We say that it is commutative if $u_\lambda u_\mu = u_\mu u_\lambda$. The approximate unit is said to be sequential if the index set is the natural numbers with the usual ordering.

As we are working over operator algebras and operator spaces, we slightly need to expand the notion of a representation so we may analyze $\mathcal{A}$ via. its representation theory as for C^* -algebras.

Definition 3.14. Let H be a real graded Hilbert space. A completely bounded (cb) representation of an operator algebra $\mathcal{A}$ is a completely bounded homomorphism $\pi : \mathcal{A} \to B(H)$. A representation is non-degenerate if $\pi(A)H$ is dense in H.

One would expect that approximate units of an operator algebra converge strongly to idempotents under completely bounded representations, and showing this result is our next goal.

Lemma 3.15. Let $\pi : \mathcal{A} \to B(H)$ be a completely bounded representation. Then

$$H = \overline{\pi(\mathcal{A})H} \oplus (\pi(\mathcal{A})H)^\perp = \overline{\pi(\mathcal{A})^* H} \oplus (\pi(\mathcal{A})^* H)^\perp$$

Defining $\mathrm{Nil}(\pi(\mathcal{A})) = \{h \in H : \pi(a)h = 0, \quad \forall a \in A\}$, there is the equality $\mathrm{Nil}(\pi(\mathcal{A})) = \overline{(\pi(\mathcal{A})^*)}^\perp$.

Proof. Let $a \in \mathcal{A}$ and $\xi, \eta \in H$. Consider the identity:

$$\langle \pi(a)\xi, \eta\rangle = \langle \xi, \pi(a)^*\eta\rangle$$

Thus if $\xi \in (\pi(\mathcal{A})^* H)^\perp$, $\pi(a)\xi = 0$ for every a. This in turn implies that $\xi \notin \overline{\pi(\mathcal{A})H}$, hence $\xi \in \overline{\pi(\mathcal{A})H}$. Hence $(\pi(\mathcal{A})^* H) = \pi(\mathcal{A})H$. This shows the first identity.
For the identity on $\mathrm{Nil}(\pi(A))$, let $h \in \mathrm{Nil}(\pi(\mathcal{A})), v \in H$ and $a \in \mathcal{A}$. Then

$$\langle h, \pi(a)^* v\rangle = \langle \pi(a)h, v\rangle = 0$$

So $\mathrm{Nil}(\pi(\mathcal{A})) \subset \overline{\pi(A)^* H}^\perp$. For the converse, let $h \in \overline{\pi(A)^* H}^\perp$, $v \in H$, and $a \in \mathcal{A}$. Then

$$\langle \pi(a)h, v\rangle = \langle h, \pi(a)^* v\rangle = 0$$

hence $\pi(a)h = 0$, and $h \in \mathrm{Nil}(\pi(\mathcal{A}))$. □

With this minor remark out of the way, we can show the claimed strong convergence.

Proposition 3.16. Let $\mathcal{A}$ be an operator algebra with bounded approximate unit $(u_\lambda)_{\lambda\in I}$ and a completely bounded representation $\pi : \mathcal{A} \to B(H)$. Then $\pi(u_\lambda)$ converges in the strong operator topology to an idempotent $q \in B(H)$ satisfying the following

1. $q\pi(a) = \pi(a)q = \pi(a)$
2. $qH = \overline{\pi(\mathcal{A})H}$.
3. $(1-q)H = \mathrm{Nil}(\pi(\mathcal{A}))$.
4. $\|q\| \le \|\pi\| \sup_{\lambda\in I} \|u_\lambda\|$

Proof. Define the projection $q : H \to \overline{\pi(\mathcal{A})H}$ and the projection $p_* : H \to \overline{\pi(\mathcal{A})^* H}$. We may define the self-adjoint operator $t = p + (1-p_*)$. We wish to show that t is injective with dense image. To see that it is injective, let $x \in ker(t)$. Then $px = (p_* - 1)x$, which implies that $px \in \overline{\pi(\mathcal{A})H} \cap \overline{\pi(\mathcal{A})H}^\perp = \{0\}$. Thus $px = 0$, and thereby we get we get that $x = (1-p)x = p_* x$, and $x \in \mathrm{Nil}(\pi(\mathcal{A})^*) \cap \overline{\pi(\mathcal{A})^* H} = \{0\}$. By self-adjointness $\overline{((p + (1-p_*)))H} = \ker(p + (1-p_*))^\perp = H$, giving that t has dense image.
Thus $\overline{\pi(\mathcal{A})H} + \mathrm{Nil}\,\pi(\mathcal{A})$ is dense. Pick an arbitrary $\xi \in H$ and $\varepsilon > 0$. Then there exist $\eta_0 \in \overline{\pi(\mathcal{A})H}$ and $\eta_1 \in \mathrm{Nil}\,\pi(\mathcal{A})$ such that $\|\xi - (\eta_0 + \eta_1)\| \le \varepsilon/4C$ for $C = \sup_{\lambda\in I} \|\pi(u_\lambda)\|$. Pick $\lambda \in I$ such that for all $\mu > \lambda$, $\|\pi(u_\lambda - u_\mu)\eta_0\| < \varepsilon/2$. Then we may perform the following estimate to show that $\pi(u_\lambda)$ is strongly Cauchy.

$$\begin{aligned}
&\|\pi(u_\lambda - u_\mu)\xi\| \\
&\le \|\pi(u_\lambda - u_\mu)(\eta_0 + \eta_1)\| + \|\pi(u_\lambda - u_\mu)(\xi - (\eta_0 + \eta_1))\| \\
&\le \|\pi(u_\lambda - u_\mu)\eta_0\| + \|\pi(u_\lambda - u_\mu)\| \, \|\xi - (\eta_0 + \eta_1)\| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2}
\end{aligned}$$

Therefore we may define the limit q of $\pi(u_\lambda)$ on $\pi(\mathcal{A})H$ as the strong topology is complete on bounded sets. We see that q commutes with the representation, and that q is an idempotent from the construction. As it is an idempotent it has closed range. Thus $\mathrm{Im}(q) = \overline{\pi(\mathcal{A})H}$. If we consider $(1-q)$, we see that $(1-q)\pi(a) = 0$, thus $\mathrm{Im}(1-q) = \mathrm{Nil}(\pi(\mathcal{A}))$. To see that the norm of q is bounded as claimed, consider the following

$$\|q\| = \sup_{h, \|h\|=1} \left\| \lim_{\lambda} \pi(u_\lambda) h \right\| \leq \|\pi\| \sup \|u_\lambda\| .$$

□

If we wish to drop the assumption of our approximate unit being bounded, it is necessary to find a different method of proof. Fundamentally, we are only working on non-commutative analogues of complete manifolds, as expanded upon in [MR16, Section 2] and in order to generalize to non-complete manifolds with symmetric operators, we need an entirely new framework as introduced in for example [Kaa15]. As a geometric corollary of the previous propositions, we get that $H \cong \overline{\pi(\mathcal{A})H} + \overline{\pi(\mathcal{A})^* H}$. We gather up a plethora of results illustrating the geometric control imposed by the presence of a bounded approximate unit in the following theorems.

Theorem 3.17. Let $\mathcal{A}$ be an operator algebra with a bounded approximate unit and an essential cb. representation $\pi : \mathcal{A} \to B(H)$. The norm on $M_n(\mathcal{A})$ is equivalent to the norm $\|\cdot\|_{op,n}$, $\|a\|_{op,n} = \sup_{\|b\|_n \leq 1} \|ab\|_n$ where $\|\cdot\|_n$ is the matrix norm.

Proof. We start by noting that $(u_\lambda)_{\lambda \in \Lambda}$ gives rise to bounded approximate units $1_n \cdot u_\lambda$ on $M_n(\mathcal{A})$, and as such we may simply run the argument for u_λ. It is clear that $\|a\|_{op,n} \leq \|a\|$. Letting u be a bounded approximate unit, we clearly have $\frac{1}{c} \|u_\lambda\| \leq 1$ for some fixed c. For all $\varepsilon > 0$ there exists a λ such that $\|b - bu_\lambda\| < \varepsilon$. This gives us the following string of inequalities

$$\frac{1}{c}(\|b\|_{op,n} - \varepsilon) < \frac{1}{c} \|b\| - \|b - bu_\lambda\| \leq \frac{1}{c} \|bu_\lambda\| \leq \|b\|_{op,n}$$

Showing the desired. □

Definition 3.18. Let $T : \mathcal{A} \to \mathcal{A}$ where $\mathcal{A}$ is an operator algebra. Define $\|T\|_{op} = \sup_{n \in \mathbb{N}} \|T \otimes 1_n\|_{op,n}$.

An operator $T : \mathcal{A} \to \mathcal{A}$ is cb. if and only $\|T\|_{op} < \infty$. Thus, by Theorem 3.17 we can use $\|\cdot\|_{op}$ to define the completely bounded version of the strict topology on an operator algebra:

Definition 3.19. Let $\mathcal{A}$ be an operator algebra with a bounded approximate unit. We define the multiplier algebra of $\mathcal{A}$ as the strict closure of $\mathcal{A}$, ie. by

$$M(\mathcal{A}) = \{T : \mathcal{A} \to \mathcal{A} \mid \exists (b_\lambda)_{\lambda \in \Lambda} \subset \mathcal{A}, \lim_{\lambda \in \Lambda} \|b_\lambda a - Ta\|_{op} = \lim_{\lambda \in \Lambda} \|ab_\lambda - Ta\|_{op} = 0, \forall a \in A\}$$

Where we define $\|T\| = \|T\|_{op}$.

Theorem 3.20.
1. Let $\mathcal{A}$ be an operator algebra with a bounded approximate unit and an essential cb. representation $\pi : \mathcal{A} \to B(H)$. The cb. representation extends to a representation of the multiplier algebra of $\mathcal{A}$, such that $\pi(1) = 1$.
2. With the further assumtion that π is a cb. isormorphism in addition to being a cb. representation:

(a) The strict closure of $\mathcal{A}$, ie. the multiplier algebra of $\mathcal{A}$ is cb-isomorphic to the idealiser of $\pi(\mathcal{A})$.

(b) Every element in $M(\mathcal{A})$ is the strict limit of a bounded net in $\mathcal{A}$.

(c) Closed ideals of $\mathcal{A}$ descend to closed ideals of $M(\mathcal{A})$.

3. Assume that $\pi : \mathcal{A} \to B(H)$ is a cb. representation of an operator with bounded approximate unit. Then π extends to a representation $M(\mathcal{A}) \to B(H)$ such that $\pi(1)$ is an idempotent and $\overline{\pi(\mathcal{A})H} = \pi(1)H$ and $(1 - \pi(1))H = \mathrm{Nil}(\pi(\mathcal{A}))$.

Proof. 1. We have assumed that $H = \overline{\pi(\mathcal{A})H}$, so for all $h \in H$ $u_\lambda h$ converges to h. Picking $b \in M(\mathcal{A})$ and utilizing that $M(\mathcal{A})$ is the strict closure of $\mathcal{A}$, we get $\sup_\lambda \|bu_\lambda\| < \infty$ and that $(bu_\lambda a)_{\lambda\in\Lambda}$ is norm-Cauchy in $\mathcal{A}$ for all $a \in \mathcal{A}$. Defining

$$\pi(b)\pi(a)h = \lim_{\lambda\in\Lambda} \pi(bu_\lambda a)h$$

we see that we get a Cauchy net. Therefore $\pi(bu_\lambda)$ converges for every $h \in \pi(\mathcal{A})H$. As this is a dense subspace and the net $\pi(bu_\lambda)$ is uniformly bounded, we may infer that the net is strongly Cauchy on H. Thus $h \mapsto \lim_{\lambda\in\Lambda} \pi(bu_\lambda)h$ gives us a bounded operator on H. By definition we have $\pi(ab) = \pi(a)\pi(b)$ for $a \in \mathcal{A}, b \in M(\mathcal{A})$. Letting $a, b \in M(\mathcal{A})$ we have the following

$$\pi(a)\pi(b)h = \pi(a) \lim_{\lambda\in\Lambda} \pi(bu_\lambda)h = \lim_{\lambda\in\Lambda} \pi(abu_\lambda)h$$

Where we have used that $bu_\lambda \in \mathcal{A}$, so that the extension defines a homomorphism. To show uniqueness of this extension, we remark that for all $a \in \mathcal{A}$ and $b \in M(\mathcal{A})$ we have that $bu_\lambda a \to ba$ in $\mathcal{A}$, it follows by essentiality that this extension is unique.

2. By what we have just shown, π extends to representation of $M(\mathcal{A})$. Let $T \in \pi(M(\mathcal{A}))$, and let $(b_\lambda)_{\lambda\in I} \subset \mathcal{A}$ be a net satisfying

$$\lim_{\lambda\in I} \|b_\lambda a - Ta\| = \lim_{\lambda\in I} \|ab - aT\| = 0$$

where we have suppressed π. It follows that $T\pi(a) \in \pi(\mathcal{A}), \pi(a)T \in \pi(\mathcal{A})$. Thus $T \in \pi(M(\mathcal{A}))$ idealizes $\pi(\mathcal{A})$. For the other inclusion, let $T \in B(H)$ such that $T\pi(\mathcal{A}) \subset \mathcal{A}, \pi(\mathcal{A})T \subset \mathcal{A}$. Considering the net $T\pi(u_\lambda)$, we get

$$\begin{aligned}
&\|T(\pi(u_\lambda a)) - T\pi(a)\| \leq \|T\| \, \|\pi(u_\lambda a - a)\| \to 0 \\
&\|\pi(a)T\pi(u_\lambda) - \pi(a)T\| \to 0
\end{aligned}$$

As π is assumed to be a cb. isomorphism and essential, it follows that $T = \pi(b)$ for some $b \in M(\mathcal{A})$. To see that T is the strict limit of a bounded net, we simply apply Theorem 3.17. To see that if $J \subset \mathcal{A}$ is a closed ideal in $\mathcal{A}$, it will also be a closed ideal in $M(\mathcal{A})$, consider $T \in M(\mathcal{A})$ and $b \in J$. The net $u_\lambda Tb$ will converge to Tb in norm, however $u_\lambda T \in \mathcal{A}$ so the net actually lies in J. As J is closed $Tb \in J$ and likewise for bT.

3. We have the cb. isomorphism $H \cong qH \oplus (1-q)H$ where q is given as in Proposition 3.16. Since π is essential on qH and 0 on $(1-q)H$, by the first part of the theorem we get a representation $\pi : M(\mathcal{A}) \to B(qH)$ which is zero $(1-q)H$, thereby giving the desired representation. By construction, $\pi(1) = q$, showing the desired.

□

In case $\mathcal{A}$ is a differentiable algebra, we can characterize the multiplier algebra of $\mathcal{A}$ in a more concrete fashion. The idea is that if we think of $\mathcal{A}$ as $C_0^1(M)$ on a complete manifold, then $M(\mathcal{A}) = C_b^1(M)$.

Proposition 3.21. Let $\mathscr{D} : \mathrm{Dom}(\mathscr{D}) \to E_B$ be self-adjoint and regular and $\mathcal{A} \subset \mathrm{Lip}(\mathscr{D})$ be closed with bounded approximate unit, i.e. a differentiable algebra with bounded approximate unit. If the representation of A is essential, then the multiplier algebra $M(\mathcal{A})$ is cb-isomorphic to:

$$\tilde{M}(\mathcal{A}) = \{T \in M(A) : T\,\mathrm{Dom}(\mathscr{D}) \subset \mathrm{Dom}(\mathscr{D}), T\mathcal{A}, \mathcal{A}T \subset \mathcal{A}, [\mathscr{D}, T] \in L(E_B)\}$$

Further, the spectrum of an operator is preserved under this mapping.

Proof. The algebra $\tilde{M}$ is clearly a subalgebra of $M(A)$ and included in the idealizer of $\pi_{\mathscr{D}}(\mathcal{A})$ in $L_B(E \oplus E)$. To see the other inclusion, start by considering

$$T\pi_{\mathscr{D}}(a) = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} a & 0 \\ [\mathscr{D}, a] & \gamma(a) \end{pmatrix} = \begin{pmatrix} T_{11}a + T_{12}[\mathscr{D}, a] & T_{12}\gamma(a) \\ T_{21}a + T_{22}[\mathscr{D}, a] & T_{22}\gamma(a) \end{pmatrix}$$

For this to be an element of $\pi_{\mathscr{D}}(\mathcal{A})$, we must have $T_{12}\gamma(a) = 0$, so $T_{12} = 0$ by essentiality of A. This implies that $\gamma(T_{11}a) = T_{22}\gamma(a)$, so $\gamma(T_{11}) = T_{22}$ by essentiality. As $\mathcal{A}$ is essential, we see that T_{11} must preserve the domain of $\mathscr{D}$. We can infer that T_{21} must satisfy the equation

$$T_{21}a + \gamma(T)[\mathscr{D}, a] = [\mathscr{D}, T_{11}a]$$

By essentiality we again infer that $T_{21} = [\mathscr{D}, T_{11}]$, and as such is bounded. This shows that the algebra $\tilde{M}(\mathcal{A})$ we have defined also contains the idealizer of $\pi_{\mathscr{D}}(\mathcal{A})$ as desired, giving the desired equality.

For the second part of the theorem, note that $\overline{AE_B} = E_B$ so $\pi_{\mathscr{D}}(1) = 1$ through Theorem 3.20. Recycling the arguments of the first and second parts of Theorem 3.20 then gives that $M(\mathcal{A})$ maps to $L_B(E_B)$ and that the idealizer of $\pi_{\mathscr{D}}(\mathcal{A})$ equals $\pi_{\mathscr{D}}(M(\mathcal{A}))$. By our equivalent characterizations of the norm on $M(\mathcal{A})$ in Theorem 3.17 we get the desired cb-isomorphism. The spectral invariance follows from a result of Mesland, [Mes09, Theorem B.3], which follows through in the real case as the spectra are identical. □

Definition 3.22. Let $\mathscr{D} : \mathrm{Dom}(\mathscr{D}) \to E_B$ be a self-adjoint and regular, and $\mathcal{A} \subset \mathrm{Lip}(\mathscr{D})$ be a differentiable algebra with bounded approximate unit, with $\mathcal{A}$ a dense subset of the C^* -algebra A. If the representation of A is essential we define the unitization of $\mathcal{A}$, $\mathcal{A}^+ \subset M(\mathcal{A})$ as the algebra generated by $\mathcal{A}$ and $\pi_{\mathscr{D}}(1)$.

One can show that the unitization exists in general, this is done in [Mey01].

Assumption 3.23. From now on we shall assume that all representations appearing in Kasparov modules are essential.

Remark 3.24. That the representations featured in the Kasparov modules is essential is not unreasonable, as one may show that any Kasparov module is equivalent to one where the representation is essential,[Kas80a].

Definition 3.25. Let $\mathcal{A}$ be a differentiable algebra with a dense right ideal $\operatorname{Dom} c$. A linear operator $c : \operatorname{Dom} c \to \mathcal{A}$ is an unbounded multiplier if $c(ab) = (ca)b$ for $b, c \in \operatorname{Dom} c$. If in addition c satisfies:

1. c is closed.
2. It is formally symmetric in the inner product $\langle a, b\rangle = a^*b$, i.e. $(ca)^*b = a^*(cb)$
3. The operators $c^2 + 1$ is surjective and $(c^2 + 1)^{-1}$ is a bounded multiplier for $\mathcal{A}$.

c is an unbounded self-adjoint multiplier. Further, we say that c is positive if $(ca)^*a \geq 0$ in A.

Lemma 3.26. Let $c : \operatorname{Dom} c \to \mathcal{A}$ be an unbounded multiplier, then the operator $(c^2 - \lambda) : \operatorname{Dom} c \to \mathcal{A}$ is bijective for all positive λ, and $(c^2 - \lambda)^{-1} \in M(\mathcal{A})$. If c is positive then $c - \lambda$ is bijective, and $(c - \lambda)^{-1} \in M(\mathcal{A})$ for all positive λ.

Proof. The operator $(c^2 + 1)^{-1}$ is bounded, hence $c^2 - 1$ is bijective in the C^*-closure A of $\mathcal{A}$. This lets us conclude that the operator $g = (c^2 - \lambda)(c^2 - 1)^{-1}$ is invertible in $M(A)$, giving that $c^2 - \lambda$ is bijective on $\mathcal{A}$. Thus by the spectral invariance of Proposition 3.21 we conclude that $(c^2 - \lambda)(c^2 - 1)^{-1}$ is invertible in $M(\mathcal{A})$, thus $c^2 - \lambda$ is invertible giving that $c^2 - \lambda : \operatorname{Dom} c \to \mathcal{A}$ is bijective in $\mathcal{A}$ as well. The operator g satisfies the equation $g^{-1} = (c^2 - 1)^{-1}(c^2 - \lambda) = 1 - (\lambda - 1)(c^2 - \lambda)^{-1}$ in $M(A)$. As 1 and g^{-1} are both in $M(\mathcal{A})$, $(c^2 - \lambda)^{-1} \in M(\mathcal{A})$. The positive case is proved in an entirely analogous manner. □

It turns out the existence of a bounded approximate unit is enough to ensure a symmetric multiplier is also closed.

Lemma 3.27. For a differentiable algebra with a bounded approximate unit it is sufficient for a multiplier $c : \operatorname{Dom} c \to \mathcal{A}$ to satisfy that $(ca)^*b = a^*(cb)$ for every $a, b \in \operatorname{Dom} c$ for it to be closable.

Proof. As $\mathcal{A}$ has a bounded approximate unit, the norm on $\mathcal{A}$ can be equivalently characterized as $\|a\| = \|a\|_{op}$. Letting a_n be a sequence converging to zero in $\operatorname{Dom} c$, and $ca_n \to b$. By symmetry of c and the identity $\|(ca_n)\| = \|(ca_n)^*\|$, we have

$$b^*a = \lim_{n\to\infty} (ca_n)^*a = \lim_{n\to\infty} a_n^*(ca) = 0$$

Thus $\|b^*\|_{op} = 0$, implying $\|b^*\| = 0$ as desired. □

It follows that if $(c^2 + 1)^{-1}$ is densely defined and bounded, the closure of c is a self-adjoint unbounded multiplier with domain $(c^2 + 1)^{-1}\mathcal{A}$. We define a complete multiplier as:

Definition 3.28. Let $(\mathcal{A}, E_B, \mathscr{D})$ be an unbounded Kasparov module and c self-adjoint multiplier of $\mathcal{A}$, c is complete if

1. $(c^2+1)^{-1} \in \mathcal{A}$
2. $\mathrm{Im}((\mathscr{D}^2+1)^{-1/2}(c^2+1)^{-1/2}) = \mathrm{Im}((c^2+1)^{-1/2}(\mathscr{D}^2+1)^{-1/2})$
3. The operator $[\mathscr{D}, c]$ is bounded on $\mathrm{Im}((c^2+1)^{-1/2}(\mathscr{D}^2+1)^{-1})$.

These sets are natural to consider when working with unbounded operators, as they are the natural domains for $c\mathscr{D}$ and $\mathscr{D}c$. We can now prove the essential theorem to expanding the expanded technical theorem of [MR16] to the real setting, relating approximate units, positive self-adjoint multipliers and and the differential operator $\mathscr{D}$. This is one of the results where the proof requires the most modification to fit into the real setting. Before we proceed with the proof, we recall the result of [Mes09, Theorem 4.24] that an operator subalgebra of the Lipschitz algebra is closed under the holomorphic functional calculus. This result allows us to use the holomorphic functional calculus freely, implicitly passing back and forth between the C^*-closure A of $\mathcal{A}$ and $\mathcal{A}$ itself.

Theorem 3.29. Let $\mathscr{D} : \mathrm{Dom}(\mathscr{D}) \to E_B$ be a self-adjoint and regular operator, and $\mathcal{A} \subset \mathrm{Lip}(\mathscr{D})$ such that $AE_B = E_B$. Then the following are equivalent

1. There is an approximate commutative unit $(u_n)_{n\in\mathbb{N}}$ for $\mathcal{A}$ such that $\|[u_n, \mathscr{D}]\| \to 0$
2. There exists a positive self-adjoint multiplier c for $\mathcal{A}$.
3. We have a strictly positive element $h \in \mathcal{A}$ such that $\mathrm{Im}((\mathscr{D}^2+1)^{-1/2}h = \mathrm{Im}(h(\mathscr{D}^2+1)^{-1/2}$, along with a constant c such that $i[\mathscr{D}, h] \leq ch^2$ in $A \otimes \mathbb{C}$.

Proof. 1 $\Rightarrow$ 2 Let $\varepsilon < 1$. Pick a countable subset with dense linear span $(a_i)_{i\in\mathbb{N}} \subset \mathcal{A}$, without loss of generality we may assume that $\|(u_{n+1}-u_n)a_i\| \leq \varepsilon^{2n}$,as well as $\|[\mathscr{D}, u_n]\| \leq \varepsilon^{2n}$. Define $d_n = u_{n+1} - u_n$ and define the candidate self-adjoint multiplier

$$c = \sum_{n=1}^{\infty} \varepsilon^{-n} d_n$$

This is densely defined, as may be readily verified for any fixed a_i, with $i < k < l$, and $c_j = \sum_{n=1}^{j} \varepsilon^{-n} d_n$. The estimates:

$$\begin{aligned}\left\|\sum_{n=k}^{l} \varepsilon^{-n} d_n a_i\right\| &\leq \sum_{n=k}^{l} \varepsilon^{-n} \|(u_{n+1}-u_n)a_i\| \\ &\leq \sum_{n=k}^{l} \varepsilon^{n}\end{aligned}$$

show that $c_k a_i$ is a Cauchy sequence, so $ca_i \in \mathcal{A}$. The operator c is clearly symmetric, so it suffices for us to show that $(c^2+1)^{-1/2}$ is densely defined and bounded. Consider the increasing sequence

$$\begin{aligned}\|[\mathscr{D}, c_k]\| &= \left\|\sum_{n=1}^{k} \varepsilon^{-n}([\mathscr{D}, u_{n+1}] - [\mathscr{D}, u_n]\right\| \leq \sum_{n=1}^{k} \varepsilon^{-n} \left(\|[\mathscr{D}, u_n]\| + \|[\mathscr{D}, u_n]\|\right) \\ &\leq 2\sum_{n=1}^{k} \varepsilon^{n}\end{aligned}$$

from which it follows that $\|[\mathscr{D}, c]\| = \sup_k \|[\mathscr{D}, c_k]\| < \infty$. We can now proceed with our calculations through the holomorphic functional calculus

$$\begin{aligned}&\sup_k \left\|[\mathscr{D}, (c_k^2+1)^{-1/2}\right\| = \sup_k \left\|\int_{\mathrm{Re}z=1/2} (z^2+1)^{-1/2}[\mathscr{D}, (z-c_k)^{-1}]dz\right\| \\ &\sup_k \left\|\int_{\mathrm{Re}z=1/2} (z^2+1)^{-1/2}(z-c_k)^{-1}[\mathscr{D}, c_k](z-c_k)^{-1}dz\right\| \\ &\leq \sup_k \left\|\int_{\mathrm{Re}z=1/2} (z^2+1)^{-1/2}\left\|(z-c_k)^{-1}\right\|^2 dz\right\| \|[\mathscr{D}, c_k]\| \\ &\leq \sup_k \left\|\int_{\mathrm{Re}z=1/2} (z^2+1)^{-1/2}\left\|(z-c_1)^{-1}\right\|^2 dz\right\| \|[\mathscr{D}, c]\| < \infty\end{aligned}$$

This shows that the sequence $(c_k^2+1)^{-1/2}$ is bounded, so we need only check that it is strictly Cauchy for the limit to exist.

$$\begin{aligned}&\left\|((c_l^2+1)^{-1/2} - (c_m^2+1)^{-1/2})a_i\right\| \\ &\leq \left\|\left(1+\sum_{n=1}^{l}\varepsilon^{2n}d_n^2\right)^{-1/2} - \left(1+\sum_{n=1}^{l}\varepsilon^{2n}d_n^2 + \sum_{k=l+1}^{m}\varepsilon^{2k}d_k^2\right)^{-1/2}\right\| \|a_i\|\end{aligned}$$

Picking l sufficently large, we see that this expression may be bounded by ε independent of m, giving that the sequence is strictly Cauchy. Thus we have the existence of $(c^2+1)^{-1/2} \in M(\mathcal{A})$, and we have shown that c is a positive self-adjoint unbounded multiplier. We need to check the remainder of the criteria of c being a complete multiplier. At first we verify the equality

$$\mathrm{Im}((c_k^2+1)^{-1/2}(\mathscr{D}^2+1)^{-1/2}) = \mathrm{Im}((\mathscr{D}^2+1)^{-1/2}(c_k^2+1)^{-1/2})$$

As $(c^2+1)^{-1/2}$ is the strict limit of $(c_k^2+1)^{-1/2}$, we may write

$$y = \lim_k (c_k^2+1)^{-1/2}(\mathscr{D}^2+1)^{-1/2}x$$

for every element y in the image of $(c^2+1)^{-1/2}$. We may then write up the following identity

$$
\begin{aligned}
&(c_k^2+1)^{-1/2}(\mathscr{D}^2+1)^{-1/2}x \\
&= -(\mathscr{D}^2+1)^{-1/2}(c_k^2+1)^{-1/2} \\
&+ (\mathscr{D}^2+1)^{-1/2}(c_k^2+1)^{-1/2}[(\mathscr{D}^2+1)^{1/2},(c_k^2+1)^{1/2}](c_k^2+1)^{-1/2}(\mathscr{D}^2+1)^{-1/2}
\end{aligned}
$$

We now need to show that the sequence $[(\mathscr{D}^2+1)^{1/2},(c_k^2+1)^{1/2}]$ is uniformly bounded in operator norm. To do this, consider the following calculations

$$
\begin{aligned}
&[(\mathscr{D}^2+1)^{1/2},(c_k^2+1)^{1/2}] \\
&= \int_{z=1/2+it}(z^2+1)^{1/2}[(\mathscr{D}^2+1)^{1/2},(c_k-z)^{-1}]dz \\
&= \int_{z=1/2+it}(z^2+1)^{1/2}(c_k-z)^{-1}[(\mathscr{D}^2+1)^{1/2},c_k](c_k-z)^{-1}dz \\
&= \int_{w=i(t^2+1)+t}\int_{z=1/2+it}(z^2+1)^{1/2}(w^2+1)^{1/2}(c_k-z)^{-1}(\mathscr{D}-w)^{-1}[\mathscr{D},c_k](\mathscr{D}-w)^{-1}(c_k-z)^{-1}dzdw
\end{aligned}
$$

This sequence is clearly uniformly bounded in operator norm, so we may perform the limiting procedure for c_k as follows

$$
\begin{aligned}
&\lim_k (c_k^2+1)^{-1/2}[\mathscr{D},c_k](c_k^2+1)^{-1/2}(\mathscr{D}^2+1)^{-1/2}x \\
&= (c^2+1)^{-1/2}[\mathscr{D},c](c^2+1)^{-1/2}(\mathscr{D}^2+1)^{-1/2}x
\end{aligned}
$$

thereby showing the desired. The other inclusion is entirely analogous. To see that $[\mathscr{D},c]$ is bounded on $\mathrm{Im}((c^2+1)^{-1}(\mathscr{D}^2+1)^{-1})$ we simply remark that it is the strong limit of the uniformly bounded family of operators $([c_k,\mathscr{D}])_{k\in\mathbb{N}}$. Finally, we need to show that $(c^2+1)^{-1/2}\in\mathcal{A}$. We start by showing that it is an element of A, then proceeding to show that it actually lies in $\mathcal{A}$. To see this is the case, consider the commutative subalgebra $B=C_0(X)$ of A generated by $(u_n)_{n\in\mathbb{N}}$. We utilize that every unbounded multiplier is specified by its Gelfand transform, see eg. [Woo79, Theorem 2.1,Theorem 2.3].Fixing $0<t<1$ define the family of sets $X_n=\{x\in X|u_n(x)\geq t\}$.

Let $x \in X, k \in \mathbb{N}$ and consider $m \geq k$. Then we may calculate as follows for $x \in X_k$

$$\begin{aligned}
\sum_{n=0}^{\infty} \varepsilon^{-n} d_n(x) &\geq \sum_{n=k}^{\infty} \varepsilon^{-n} d_n(x) \\
&= \sum_{n=k}^{m} \varepsilon^{-n} d_n(x) + \sum_{j=m+1}^{\infty} \varepsilon^{-j} d_j(x) \\
&\geq \sum_{n=k}^{\infty} \varepsilon^{-k} d_n(x) + \sum_{j=m+1}^{\infty} \varepsilon^{-j} d_j(x) \\
&= \varepsilon^{-k}(u_{m+1} - u_k)(x) + \sum_{j=m+1}^{\infty} \varepsilon^{-j} d_j(x) \\
&\geq \varepsilon^{-k}(u_{m+1} - t)(x) + \sum_{j=m+1}^{\infty} \varepsilon^{-j} d_j(x)
\end{aligned}$$

As u_n is an approximate unit and $\sum_{j=m+1}^{\infty} \varepsilon^{-j} d_j(x)$ converges pointwise to zero, we get the estimate

$$\sum_{n \in \mathbb{N}} \varepsilon^{-n} d_n(x) \geq (1-t)\varepsilon^{-k}$$

To see that $(c^2+1)^{-1/2}$ actually lies in $\mathcal{A}$, note that $u_n(c^2+1)^{-1/2}$ converges to $(c^2+1)^{-1/2}$ in A-norm. We have that $[\mathscr{D}, u_n]$ is bounded and $[\mathscr{D}, u_n]$ goes to zero, so we may derive:

$$\begin{aligned}
[\mathscr{D}, u_n(c^2+1)^{-1/2}] &= u_n[\mathscr{D}, (c^2+1)^{-1/2}] + [\mathscr{D}, u_n](c^2+1)^{-1/2} \\
&= \int_{z=(/2+ti} u_n(z^2+1)^{-1/2}(c+z)^{-1}[c, \mathscr{D}](c+z)^{-1} dz - [\mathscr{D}, u_n](c^2+1)^{-1/2} \\
&= u_n \int_{z=1/2+ti} (z^2+1)^{-1/2}(c+z)^{-1}[c, \mathscr{D}](c+z)^{-1} dz - [\mathscr{D}, u_n](c^2+1)^{-1/2}
\end{aligned}$$

which converges to $[\mathscr{D}, (c^2+1)^{-1/2}]$ as desired. As such $\pi_{\mathscr{D}}(u_n(c^2+1)^{-1/2})$ converges to $\pi_{\mathscr{D}}((c^2+1)^{-1/2})$. As all $u_n(c^2+1)^{-1/2}$ lie in $\mathcal{A}$, we may infer that $(c^2+1)^{-1/2} \in \mathcal{A}$.

$2 \Rightarrow 1$ To see the claim, consider $f_n(x) = \exp(-x/n)$. Then, using the standard identities for derivatives of operator-valued functions, we get the equalities

$$\begin{aligned}
[\mathscr{D}, f_n(c)]y &= \int_0^1 \partial_s(\exp(-c(1-s)/n)\mathscr{D}\exp(-cs/n)y) ds \\
&= -\frac{1}{n}\int_0^1 \exp(-c(1-s)/n)[\mathscr{D}, c]\exp(-cs/n) y ds
\end{aligned}$$

As everything is bounded, we may extend the equality to the entirety of E_B, as well as getting the inequality $\|[\mathscr{D}, f_n(c)]\| \leq \frac{1}{n}\|[\mathscr{D}, c]\|$ To see that we actually have an approximate unit, note that $(c^2+1)^{-1/2}\mathcal{A}$ is dense, thereby $(c^2+1)^{-1/2}$ generates an essential ideal wherein $f_n(c)$ is clearly an approximate unit.

$2 \Leftrightarrow 3$ Pick an unbounded multiplier c on $\mathcal{A}$ and remark that $h = (1+c^2)^{-1/2}$ has dense range and is positive. If $h \in \mathcal{A}$ is positive with dense range, define the operator $c = h^{-1}$, which is densely defined on $\mathrm{Im}(h)$. We can infer the domain relation from the identity

$$(h^{-2}+1)^{-1/2} = h(1+h^2)^{-1/2}$$

and $1+h^2$ is invertible, so $(1+h^2)^{-1/2}$ is a bijection on $\mathrm{Dom}(\mathscr{D}) = \mathrm{Im}((\mathscr{D}^2+1)^{-1/2})$. Then we have the equalities

$$\begin{aligned}&\mathrm{Im}(h(h^2+1)^{-1/2}(\mathscr{D}^2+1)^{-1/2}) = \mathrm{Im}(h(1+\mathscr{D}^2)^{-1/2})\\ &= \mathrm{Im}((\mathscr{D}^2+1)^{-1/2})h = \mathrm{Im}((\mathscr{D}^2+1)^{-1/2}h(1+h^2)^{-1/2})\end{aligned}$$

Finally, from the assumption that $i[\mathscr{D}, h] \leq ch^2$ for some $c \in \mathbb{R}^+$, it can be inferred for $e \in \mathrm{Im}(h(\mathscr{D}^2+1)^{-1/2}h)$ that

$$\begin{aligned}&\left\langle i[\mathscr{D}, h^{-1}]e, e\right\rangle_B = -i\left\langle h^{-1}i[\mathscr{D}, h]h^{-1}e, e\right\rangle_B\\ &= \left\langle i[\mathscr{D}, h]h^{-1}e, h^{-1}e\right\rangle_B \leq C\left\langle he, h^{-1}e\right\rangle_B = C\left\langle e, e\right\rangle_B\end{aligned}$$

Then by taking a sequence $hy_n \to y \in E_B$ we can infer boundedness on the entirety of $\mathrm{Im}(h(\mathscr{D}^2+1)^{-1/2})$

□

Remark 3.30. This theorem allows us to assume that every approximate unit for a differentiable algebra is even, self-adjoint commutative and increasing.

This completes our study of differential algebras as independent objects, and we will henceforth be considering these in the context of constructing the unbounded Kasparov product.

3.2 Connections

In this section we establish the differential framework needed for the construction of the unbounded Kasparov product, the essential ingredient in the construction of the unbounded Kasparov product in [Mes09]. The most important part of the framework is notion of a projector operator module. This allows the construction of a connection, serving to make the naive product $1 \otimes \mathscr{D}_2$ well-defined on the interior tensor product. The modification is much more direct than in the bounded case, as it is based on the idea of the covariant derivative for a vector bundle. As such it is instructive to think of the product $1 \otimes_\nabla \mathscr{D}_2$ exactly as a covariant derivative with $\mathscr{D}_2$ being the standard derivative and ∇ encoding the derivative on the bundle. That is, we think of $E_B \otimes_B F_C$ as the tensor product of two vector bundles, with E_B serving the role of the tangent bundle.

Definition 3.31. Let $\mathcal{B}$ be an operator algebra with a bounded approximate unit u_λ. Then as for C^* modules, define the standard right $\mathcal{B}$-module $H_\mathcal{B} = H\widetilde{\otimes}\mathcal{B}$.

Assumption 3.32. Let B and C be graded σ-unital C^* -algebras. For the remainder of this section, we fix an unbounded essential (B, C)-module $(\mathcal{B}, F_C, \mathscr{D})$ where $\mathcal{B}$ is assumed to have a bounded approximate unit, with $\mathcal{B}^+$ denoting the unitization of $\mathcal{B}$. We define the corresponding differential representation of $\mathcal{B}$ as

$$\pi_{\mathscr{D}}(b) = \begin{pmatrix} b & 0 \\ [\mathscr{D}, b] & \gamma(b) \end{pmatrix} \in L_C(F_C \oplus F_C)$$

We see that the grading operator Γ on $H_{\mathcal{B}^+}$ can be written as $\Gamma(b_i)_{i\in\hat{\mathbb{Z}}} = (\text{sign}(i)\gamma(b_i))_{i\in\hat{\mathbb{Z}}}$. Defining the self-adjoint unitary

$$\begin{aligned} &\varepsilon : H_{\mathcal{B}^+} \to H_{\mathcal{B}^+} \\ &\varepsilon((b_i)_{i\in\hat{\mathbb{Z}}}) = (\text{sign}(i)b_i)_{i\in\hat{\mathbb{Z}}} \end{aligned}$$

the grading operator on $H_{\mathcal{B}^+}$ may be given as $\varepsilon\,\text{diag}(\gamma_{\mathcal{B}^+})$, a factorization which we can already see simplifies our constructions as we may represent $H_{\mathcal{B}^+}$ as an operator $\mathcal{B}^+$ module via. the representation

$$\begin{aligned} (b_i)_{i\in\mathbb{Z}\setminus\{0\}} \mapsto &\begin{pmatrix} b_i & 0 \\ \text{sign}(i)[\mathscr{D}, b_i]_{\mathcal{B}^+} & \Gamma(b_i) \end{pmatrix}_{i\in\hat{\mathbb{Z}}} \\ = &\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} \begin{pmatrix} b_i & 0 \\ [\mathscr{D}, b_i]_{\mathcal{B}^+} & \gamma(b_i) \end{pmatrix}_{i\in\hat{\mathbb{Z}}} \end{aligned}$$

An immediate question which one must ask is then whether the inner product on $H_{\mathcal{B}^+}$ derived from H_{B^+} is actually $\mathcal{B}^+$-valued. This is true as shown in [Mes09], and we state it here without proof.

Proposition 3.33. The standard B-valued inner product on $H_{\mathcal{B}}$, ie.

$$\langle x, y\rangle = \sum_{i\in\hat{\mathbb{Z}}} x_i^* y_i$$

takes values in $\mathcal{B}^+$.

As the class of a Kasparov module is invariant under compact perturbations, we need a good operator-algebraic notion of the compact operators on $H_{\mathcal{B}^+}$. Likewise, we would like an analogue of the adjointable operators on $H_{\mathcal{B}^+}$.

Definition 3.34. We define the compacts on $H_{\mathcal{B}^+}$ as $\mathbb{K}(H_{\mathcal{B}^+}) = \mathbb{K}\widetilde{\otimes}\mathcal{B}^+$, we define $L(H_{\mathcal{B}^+})$ as the subset of completely bounded maps $T : H_{\mathcal{B}^+} \to H_{\mathcal{B}^+}$ that have an adjoint with respect to the standard inner product.

Working only in the context of the standard module turns out to be too inflexible in applications, as shown for instance in [BMvS13] where the concept of an unbounded projection was introduced in order to be able to handle a C^* -module stemming from the non-commutative Hopf fibration. Recalling the Serre-Swan theorem, one may be tempted to think of projective operator modules as generalized vector bundles, and it is exactly this we shall use as the base point of our intuition.

Definition 3.35. Let $\mathcal{B}$ be an operator *-algebra. A projective operator module $\mathcal{E}$ over $\mathcal{B}$ is a $\mathcal{B}$-module which is completely isometrically unitarily isomorphic to $p\,\mathrm{Dom}(p)$ for some even projection on $H_{\mathcal{B}^+}$ with the additional requirement that $(e_i)_{i\in\hat{\mathbb{Z}}} \subset \mathrm{Dom}(p)$

We can characterize which modules are of this form with the aid of the concept of a frame, the *-module analogue of a frame in a vector bundle ie. a collection of sections giving a basis for every fiber. As for vector bundles, our definition of a section is local in the sense that it is a coordinate-wise definition, but as we shall shortly show having a frame has global consequences for a module.

Definition 3.36. Let E_B be a C^* module over B. The sequence $(x_i)_{i\in\hat{\mathbb{Z}}} \subset E_B$ is a frame for E_B if:

1. $\gamma_{E_B}(x_i) = \mathrm{sign}_i\, x_i$.
2. The sequence of finite rank operators

$$\chi_n = \sum_{1\leq |i|\leq n} |x_i\rangle\langle x_i|$$

is an approximate unit for the finite-rank operators, with $\|\chi_n\| \leq 1$. $(x_i)_{i\in\hat{\mathbb{Z}}}$

We shall refer to χ_n as the canonical approximate identity corresponding to the frame.

The mental model of χ_n is the strongly convergent symmetric projection in $B(\ell^2(\hat{\mathbb{Z}}))$
To illustrate the analogy with the frame of a vector bundle, we have the following theorem showing that a $\mathcal{B}$-module has a frame with uniformly bounded correlations if and only if the corresponding C^* module is the completion of a projective $\mathcal{B}$ module.

Proposition 3.37. Let $\mathcal{B}$ be a differentiable algebra with E_B a graded C^* module over B. Then E_B is the completion of a projective operator module $\mathcal{E}_{\mathcal{B}}$ if and only if there is a frame $(x_i)_{i\in\hat{\mathbb{Z}}}$ such that $(\langle x_i, x_j\rangle)_{i\in\hat{\mathbb{Z}}}$ has finite norm in $H_{\mathcal{B}^+}$ for each j.

Proof. Start by assuming that $\mathcal{E}_{\mathcal{B}}$ is projective, then $\mathcal{E}_{\mathcal{B}} \subset H_{B^+}$ and therefore we may define the frame $x_i = pe_i$ and consider the inner product

$$\left\langle pe_i, \sum_{1\leq |j|\leq n} pe_n \langle pe_n, pe_j\rangle \right\rangle = \left\langle pe_i, \sum_{1\leq |j|\leq n} e_n \langle e_n, pe_j\rangle \right\rangle$$

This converges for $n\to\infty$ since pe_i and all pe_j are in $H_{\mathcal{B}^+}$. This implies that $|pe_i\rangle\langle pe_k|$ is a frame as χ_n is a column finite approximate unit for the finite rank operators. To show the converse, we wish to show that the operator p induced on $H_{\mathcal{B}^+}$ by the matrix $(\langle x_i, x_j\rangle)_{ij\in\hat{\mathbb{Z}}}$ is a projection. Start by observing that p has the following domain

$$\mathrm{Dom}(p) = \left\{ (b_i)_{i\in\hat{\mathbb{Z}}} : \forall j \in \hat{\mathbb{Z}} \lim_{k\to\infty} \left(\sum_{1\leq |i|\leq k} \langle x_i, x_j\rangle\, b_j \right) \in \mathcal{B} \right\}$$

We see that p is densely defined as all e_i lie in $\mathrm{Dom}(p)$ by column finiteness. In order to see that p is closed, let q_i be the projection onto the submodule spanned by e_i. As we have assumed that p is column finite, $q_i p \in L(H_{\mathcal{B}^+})$. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $H_{\mathcal{B}^+}$ converging to z and assume that $pz_n \to h$. Then $q_i p z_n \to q_i h$. By continuity we also get that $q_i p z_n \to q_i z$ for all i. This implies that $q_i p z = q_i h$ and that $pz = h \in H_{\mathcal{B}^+}$, so p is closed. We wish to show that p is self-adjoint, so pick $z \in \mathrm{Dom}(p^*)$, ie. $\langle pw, z\rangle = \langle w, x\rangle$ for all $w \in \mathrm{Dom}(p)$. As the basis vectors e_i are in the domain of p and using the definition of the adjoint, we may compute:

$$\begin{aligned}\lim_{n\in\mathbb{N}} \sum_{1\le|i|\le n} q_i pzn &= \lim_{n\to\infty} \sum_{1\le|i|\le n} e_i \langle e_i, q_i pz\rangle \\ &= \lim_{n\in\mathbb{N}} \sum_{1\le|i|\le n} e_i \langle pe_i, z\rangle \\ &= \lim_{n\in\mathbb{N}} \sum_{1\le|i|\le n} e_i \langle e_i, x\rangle \\ &= x = p^* z\end{aligned}$$

This implies that $pz = x$, giving the desired. Finalizing the proof, we may define $\mathcal{E}_{\mathcal{B}}$ by

$$\mathcal{E}_{\mathcal{B}} = \{e \in E_B : (\langle x_i, e\rangle)_{i\in\hat{\mathbb{Z}}} \in \mathrm{Dom}(p)\}$$

We clearly have that $x_i \in \mathcal{E}_{\mathcal{B}}$, and that $\mathcal{E}_{\mathcal{B}}$ is closed in $H_{\mathcal{B}^+}$ follows from the observation that a convergent net in $\mathcal{E}_{\mathcal{B}}$ will also be convergent in E_B thus must of the form $(\langle x_i, e\rangle)_{i\in\hat{\mathbb{Z}}}$. ☐

Definition 3.38. Let $\mathcal{E}_{\mathcal{B}}$ be a projective C^* -module. Define the canonical column-finite frame associated to $\mathcal{E}_{\mathcal{B}}$ as the frame in Proposition 3.37.

We have now constructed our geometric setup, showing that our modules with frames behave as generalized vector bundles. As a next logical step, we wish to construct the analogue of the differentiable sections of our bundle. For this purpose we start by defining the set of universal 1-forms, from which we shall eventually construct our analogue of the differentiable sections.

Definition 3.39. We define the universal 1-forms $\Omega^1(B, \mathcal{B})$ as the kernel of the map $B\widetilde{\otimes}\mathcal{B} \to B$ given by $b_1 \otimes b_2 \mapsto \gamma(b_1)b_2$ when $B, \mathcal{B}$ are unital. In the non-unital case, we consider $\Omega(B^+, \mathcal{B}^+)$ such that the universal derivation:

$$\begin{aligned} db &: B \to \Omega^1(B^+, \mathcal{B}^+) \\ b &\mapsto 1 \otimes b + \gamma(b) \otimes 1 \end{aligned}$$

is well-defined. Associated to this we have the universal short exact sequence where m is the multiplication map:

$$0 \longrightarrow \mathcal{E}_B \widetilde{\otimes}_{\mathcal{B}^+} \Omega^1(B^+, \mathcal{B}^+) \longrightarrow E_B \widetilde{\otimes}_{\mathcal{B}} \mathcal{B}^+ \xrightarrow{m} E_B \longrightarrow 0$$

A split of this is a map $s : \mathcal{E}_{\mathcal{B}} \to E_B \widetilde{\otimes} \mathcal{B}^+$ such that $m(s) = \iota_{\mathcal{E}_{\mathcal{B}}}$, where $\iota_{\mathcal{E}_{\mathcal{B}}}$ is the inclusion of $\mathcal{E}_{\mathcal{B}}$ into E_B, [BMvS13, Proposition 2.22].

We are interested in splittings the universal exact sequence for the universal 1-forms, as we may use such a split to construct an analogue the covariant derivative based on the universal derivative, which we then use to construct an explicit form for a covariant derivation associated to an operator $\mathscr{D}$. We may construct a split using our column finite frames.

Lemma 3.40. Let $(x_i)_{i\in\hat{\mathbb{Z}}}$ be a column finite frame defining a projective $\mathcal{B}$ submodule $\mathcal{E}_{\mathcal{B}} \subset E_B$. Then the map

$$\begin{aligned} &s : \mathcal{E}_{\mathcal{B}} \to E \widetilde{\otimes} \mathcal{B}^+ \\ &e \mapsto \sum_{i\in\hat{\mathbb{Z}}} \gamma(x_i) \otimes \langle x_i, e\rangle\,, \qquad \text{for } e \in \mathcal{E}_{\mathcal{B}} \end{aligned}$$

defines a contractive $\mathcal{B}^+$-linear split of the universal exact sequence.

Proof. We start by checking that s is well-defined. Let $\varepsilon > 0$ and $e \in \mathcal{E}_{\mathcal{B}}$ and pick $n, m \in \mathbb{N}$ such that

$$\left\| \sum_{n\le|i|\le m} \pi_{\mathscr{D}}(\langle x_i, e\rangle)^* \pi_{\mathscr{D}}(\langle x_i, e\rangle) \right\|_{L(E_B\oplus E_B)} < \varepsilon$$

which we may do as $e \in \mathcal{E}_{\mathcal{B}}$. We can perform the following estimate

$$\begin{aligned} &\left\| \sum_{n\le|i|\le m} \gamma(x_i) \otimes \langle x_i, e\rangle \right\|^2_{\widetilde{\otimes}} \\ &\le \left\| \sum_{n\le|i|\le m} |x_i\rangle\langle x_i| \right\|_{\mathbb{K}(E_B)} \left\| \sum_{n\le|i|\le m} \pi_{\mathscr{D}}(\langle x_i, e\rangle)^* \pi_{\mathscr{D}}(\langle x_i, e\rangle) \right\|_{\mathcal{B}^+} \\ &\le \left\| \sum_{n\le|i|\le m} \pi_{\mathscr{D}}(\langle x_i, e\rangle)^* \pi_{\mathscr{D}}(\langle x_i, e\rangle) \right\|_{L(E_B\oplus E_B)} < \varepsilon \end{aligned}$$

Thus the partial sums from the definition of s give rise to a Cauchy sequence in $\widetilde{\otimes}$-norm. To show continuity, consider the following estimate

$$\begin{aligned} &\|s(e)\|^2_{\widetilde{\otimes}} \\ &\le \lim_{k\to\infty} \left\| \sum_{1\le|i|\le k} |x_i\rangle\langle x_i| \right\|_{\mathbb{K}(E_B)} \left\| \sum_{1\le|i|\le k} \pi_{\mathscr{D}}(\langle x_i, e\rangle)^* \pi_{\mathscr{D}}(\langle x_i, e\rangle) \right\|_{\mathcal{B}} \\ &\le \|e\|^2_{\mathcal{E}_{\mathcal{B}}} \end{aligned}$$

This shows that our split is well-defined and contractive as desired. □

We can now use this split to define our connection.

Definition 3.41. Let $\mathcal{E}_\mathcal{B}$ be a projective operator module, then any completely bounded linear operator $\nabla : \mathcal{E}_\mathcal{B} \to E\widetilde{\otimes}\Omega^1(B^+, \mathcal{B}^+)$ satisfying the Leibniz rule for the universal derivation d

$$\nabla(eb) = \nabla(e)b + \gamma(e) \otimes db$$

is a connection.

The motivating example for this definition is the following

Example 3.42. Given a splitting s we may define a connection associated to this split.

$$\nabla_s(e) = s(e) - \gamma(e) \otimes 1$$

For the module $H_{\mathcal{B}^+}$ we may define the connection $(\varepsilon d)((b_i)_{i\in\hat{\mathbb{Z}}}) = (\varepsilon((db_i)))_{i\in\hat{\mathbb{Z}}}$.

Before proceeding, we define an isometry v implementing the stablization map $\mathcal{E}_\mathcal{B} \to H_{\mathcal{B}^+}$, providing a framework into which to place the split s. Given a column finite frame $(x_i)_{i\in\hat{\mathbb{Z}}}$ and a projective module $\mathcal{E}_\mathcal{B}$ we get an induced stabilization map, $v : \mathcal{E}_\mathcal{B} \to H_{\mathcal{B}^+}$ extending to E_B and H_{B^+}.

Definition 3.43. For a projective operator module $\mathcal{E}_\mathcal{B}$ define the isometry v implementing the stabilization $\mathcal{E}_\mathcal{B} \to H_{\mathcal{B}^+}$.

$$v(e) = (\langle e, x_i\rangle)_{i\in\hat{\mathbb{Z}}}$$

The adjoint v^* of v is

$$v^*((b_i)_{i\in\hat{\mathbb{Z}}}) = \sum_{i\in\hat{\mathbb{Z}}} x_i b_i$$

Define the associated projection $p = vv^*$.

In order to understand the utility of the example given above, we have the following lemma illuminating the concrete form of the covariant derivative in terms of the frame.

Lemma 3.44. The operator v is an even isometry, and the connection associated to the splitting may be characterized through v as follows $\nabla_s = v^*\varepsilon dv$.

Proof. We have

$$\begin{aligned}
\nabla_s(e) &= s(e) - \gamma(e) \otimes 1 \\
&= \sum_{i\in\hat{\mathbb{Z}}} \gamma(x_i) \otimes \langle x_i, e\rangle - \gamma(e) \otimes 1 \\
&= \sum_{i\in\hat{\mathbb{Z}}} (\gamma(x_i) \otimes 1)(1 \otimes \langle x_i, e\rangle - \gamma(\langle x_i, e\rangle) \otimes 1) \\
&= \sum_{i\in\hat{\mathbb{Z}}} \gamma(x_i) \otimes d(\langle x_i, e\rangle)
\end{aligned}$$

By the calculations we have just performed, we see that $\nabla_s = v^* \circ \varepsilon d \circ v$. $\square$

As we desired that our framework should also work for unbounded, ie. non-regular projections, we need to consider a concrete derivation constructed from $\mathscr{D}$ which we may use to construct a suitable connection.

Definition 3.45. Define the derivation $\delta_{\mathscr{D}}$ as $[\mathscr{D}, \cdot]$. We define the set of $\delta_{\mathscr{D}}$-1-forms as

$$\Omega^1_{\mathscr{D}} = \overline{\{\pi(b_i)[\mathscr{D}, b_i'] : b_i, b_i' \in \mathcal{B}\}} \subset L_C(F_C)$$

By the universality of $\Omega^1(B^+, \mathcal{B}^+)$ there is a map

$$j_{\mathscr{D}} : \Omega^1(B^+, \mathcal{B}^+) \to \Omega^1_{\mathscr{D}} \qquad\qquad db \mapsto [\mathscr{D}, \pi(b)]$$

Thus we get a connection $\nabla_{\mathscr{D}} = (1 \otimes j_{\mathscr{D}}) \circ (\nabla_s)$.

We might be in the situation that the space of differentiable elements in $\mathcal{E}_{\mathcal{B}} \widetilde{\otimes}_{\mathcal{B}^+} F_C$ with respect to ∇ is not complete, and as such we need to enlarge our module to remedy this malady. We start by considering the free case, and use this as our reference.

Definition 3.46. We define the space

$$H_{\Omega^1_{\mathscr{D}}} = H_{\mathcal{B}^+} \widetilde{\otimes}_{\mathcal{B}^+} \Omega^1_{\mathscr{D}}$$

consisting of sequences $(\omega)_{j \in \hat{\mathbb{Z}}}$ such that the sum $\sum_{j \in \hat{\mathbb{Z}}} \omega_j^* \omega_j$ converges in $L_C(F_C)$

This allows us to define our operator module $\mathcal{E}^\nabla$ which can see both the action of $\mathscr{D}$ on $\mathcal{B}$ and F_C.

Definition 3.47. Given a column-finite frame $(x_i)_{i \in \hat{\mathbb{Z}}}$ for $\mathcal{E}_{\mathcal{B}}$ define the space $\mathcal{E}^\nabla_{\mathcal{B}} \subset E_B$:

$$\mathcal{E}^\nabla_{\mathcal{B}} = \left\{ e \in E_B : \lim_{n \to \infty} \left(\sum_{1 \le |k| \le n} \langle x_i, \gamma(x_k) \rangle [\mathscr{D}, \langle x_k, e \rangle] \right)_{i \in \hat{\mathbb{Z}}} \in H_{\Omega^1_{\mathscr{D}}} \right\}$$

This is an operator $\mathcal{B}$ module in the representation:

$$\pi_\nabla(e) = \begin{pmatrix} v(e) & 0 \\ vv^* \varepsilon [\mathscr{D}, v(e)] & v(\gamma(e)) \end{pmatrix} \in \bigoplus_{i \in \hat{\mathbb{Z}}} L_C(F_C \oplus F_C), \quad \|e\|_{\mathcal{E}^\nabla_{\mathcal{B}}} = \|\pi_\nabla(e)\|$$

where $\gamma = \operatorname{diag}(\gamma_{\mathcal{B}^+})$. We shall use the notation $\mathscr{D}_\varepsilon = \varepsilon \operatorname{diag}(\mathscr{D})$ on $H_{B^+} \widetilde{\otimes}_{B^+} F_C$.

Remark 3.48. The two closed graded derivations defined below

$$[\operatorname{diag}(\mathscr{D}), T]_\gamma = \operatorname{diag}(\mathscr{D}) T - \gamma T \gamma \operatorname{diag} \mathscr{D} [\mathscr{D}_\varepsilon, T]_\Gamma = \mathscr{D}_\varepsilon T - \Gamma T \Gamma \mathscr{D}_\varepsilon$$

are related as $[\mathscr{D}_\varepsilon, T]_\Gamma = [\operatorname{diag} \mathscr{D}, \varepsilon T]_\gamma = \varepsilon [\operatorname{diag}(\mathscr{D}), T]_\gamma$. Thus they have the same domain.

One should also remark that with the gradings defined on $\mathcal{E}_{\mathcal{B}}^{\nabla}$ as before, we have the identity:

$$\begin{aligned}\left(\sum_{1\leq|k|\leq n}\langle x_i,x_k\rangle\left[\mathscr{D},\langle x_k,e\rangle\right]\right)^* &= \sum_{1\leq|k|\leq n}-[\mathscr{D},\gamma(\langle e,x_k\rangle)]\left\langle\gamma(x_k),x_i\right\rangle\\ &= \sum_{1\leq|k|\leq n}\gamma([\mathscr{D},\langle e,x_k\rangle]\left\langle x_k,\gamma(x_i)\right\rangle)\end{aligned}$$

So that the sequence of row vectors

$$\left(\sum_{1\leq|k|\leq n}[\mathscr{D},\langle e,x_k\rangle]\left\langle x_k,\gamma(x_i)\right\rangle\right)^t_{i\in\hat{\mathbb{Z}}} \tag{1}$$

converges.

As we shall see in the following lemma, the module $\mathcal{E}_{\mathcal{B}}^{\nabla}$ is isomorphic to $\mathcal{E}_{\mathcal{B}}$ in the case where $\mathcal{E}_{\mathcal{B}}$ is projective, reinforcing the analogy of projective $\mathcal{B}$-modules as being unbounded C^1-vector bundles in a suitable sense. We also show that $\mathcal{E}_{\mathcal{B}}$ is an inner product $\mathcal{B}$-module, allowing us to use our results on these.

Theorem 3.49. The operator module $\mathcal{E}_{\mathcal{B}}^{\nabla}$ has the following properties.

1. The inner product $\mathcal{E}_{\mathcal{B}}\times\mathcal{E}_{\mathcal{B}}\to\mathcal{B}$ extends to $\mathcal{E}_{\mathcal{B}}^{\nabla}\times\mathcal{E}_{\mathcal{B}}^{\nabla}\to\mathcal{B}$.

2. For every $e\in\mathcal{E}_{\mathcal{B}}^{\nabla}$ the operator $e^*:\mathcal{E}_{\mathcal{B}}^{\nabla}:\to\mathcal{B}$ given as $e^*(f)=\langle e,f\rangle$ is completely bounded and adjointable, with adjoint given as $(e^*)^*(b)=eb$, satisfying the estimate $\|e^*\|_{cb}\leq 2\,\|e\|_{\mathcal{E}_{\mathcal{B}}^{\nabla}}$.

3. For every projection $p:\mathrm{Dom}(p)\to H_{\mathcal{B}^+}$ such $\mathcal{E}_{\mathcal{B}}=p\,\mathrm{Dom}(p)$ is a projective module, there is a completely contractive dense inclusion $\iota:\mathcal{E}_{\mathcal{B}}\to\mathcal{E}_{\mathcal{B}}^{\nabla}$, if $p\in L(H_{\mathcal{B}^+})$, ι is a *cb*-isomorphism.

Proof. 1. Let $f,e\in\mathcal{E}_{\mathcal{B}}^{\nabla}$ we wish to show that $\langle e,f\rangle$ lies in $\mathcal{B}$. Let $(x_i)_{i\in\hat{\mathbb{Z}}}$ be the canonical frame for $\mathcal{E}_{\mathcal{B}}$. Consider the series of column vectors

$$\sum_{j\in\hat{\mathbb{Z}}}\left(\langle x_i,\gamma(x_j)\rangle\left[\mathscr{D},\langle x_j,e\rangle\right]\right)_{i\in\hat{\mathbb{Z}}}$$

which for $e\in\mathcal{E}^{\nabla}$ is norm-convergent in $H_{\mathcal{B}^+}\widetilde{\otimes}_{\mathcal{B}^+}F_C$ by definition of $\mathcal{E}_{\mathcal{B}}^{\nabla}$. Then we may consider the partial sums

$$\left[\mathscr{D},\sum_{1\leq|j|\leq n}\langle e,x_j\rangle\left\langle x_j,f\right\rangle\right]=\sum_{1\leq|j|\leq n}\gamma(\langle e,x_j\rangle)[\mathscr{D},\langle x_j,f\rangle]+[\mathscr{D},\langle e,x_j\rangle]\left\langle x_j,f\right\rangle$$

In order to see that both terms on the right hand side are convergent, consider the following where we use the pairing between row and column vectors, ie. the standard inner product product, we get

$$\begin{aligned}\left\| \sum_{1\le|j|\le n} \gamma(\langle e, x_j\rangle)[\mathscr{D}, \langle x_j, f\rangle] \right\| &= \left\| \sum_{1\le|j|\le n} \sum_{i\in\hat{\mathbb{Z}}} \langle \gamma(e), x_i\rangle \langle x_i, \gamma(x_j)\rangle [\mathscr{D}, \langle x_j, f\rangle] \right\| \\ &= \left\| \sum_{1\le|j|\le n} (\langle \gamma(e), x_i\rangle)^*_{i\in\hat{\mathbb{Z}}} \cdot (\langle x_i, \gamma(x_j)\rangle [\mathscr{D}, \langle x_j, f\rangle])_{i\in\hat{\mathbb{Z}}} \right\| \\ &\le \|e\|_{E_B} \left\| \sum_{1\le|j|\le n} (\langle x_i, \gamma(x_j)\rangle [\mathscr{D}, \langle x_j, f\rangle])_{i\in\hat{\mathbb{Z}}} \right\|\end{aligned}$$

by our initial considerations this is finite. Thus we have shown the desired since $\delta_{\mathscr{D}}$ is a closed derivation and x_i is a frame for $\mathcal{E}_{\mathcal{B}}$, so $\sum_{i\in\hat{\mathbb{Z}}}(\langle e, x_i\rangle)\langle x_i, f\rangle$ converges to $\langle e, f\rangle$.

2. We have the following equalities

$$\begin{aligned}\begin{pmatrix} \langle e, f\rangle & 0 \\ [\mathscr{D}, \langle e, f\rangle] & \gamma(\langle e, f\rangle)\end{pmatrix} &= \sum_{i\in\hat{\mathbb{Z}}} \begin{pmatrix} \langle e, x_i\rangle & 0 \\ 0 & \langle \gamma(e), x_i\rangle\end{pmatrix} \begin{pmatrix} \langle x_i, f\rangle & 0 \\ \sum_{j\in\hat{\mathbb{Z}}} \langle x_i, \gamma(x_j)\rangle & [\mathscr{D}, \langle x_i, f\rangle]\end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 \\ \sum_{j\in\hat{\mathbb{Z}}}[\mathscr{D}, \langle e, x_j\rangle] & \langle x_j, \gamma(x_i)\rangle\end{pmatrix} \begin{pmatrix} \langle \gamma(x_i), f\rangle & 0 \\ 0 & \langle x_i, f\rangle\end{pmatrix}\end{aligned}$$

All series converge by Equation (1) and the definition via. limits of $\mathcal{E}_{\mathcal{B}}^{\nabla}$. These equalities also hold for matrices of elements in $\mathcal{E}_{\mathcal{B}}^{\nabla}$, giving rise to the estimate

$$\|e^*(f)\|_{\mathcal{B}} \le \|e\|_E \|f\|_{\mathcal{E}_{\mathcal{B}}^{\nabla}} + \|e\|_{\mathcal{E}_{\mathcal{B}}^{\nabla}} + \|f\|_E \le 2\|e\|_{\mathcal{E}_{\mathcal{B}}^{\nabla}} \|f\|_{\mathcal{E}_{\mathcal{B}}^{\nabla}}$$

so we infer

$$\|e^*\|_{cb} \le 2\|e\|_{\mathcal{E}^{\nabla}}$$

Which proves the claim.

3. To see the final point we start by using that v is a partial isometry

$$\begin{aligned}&\left\| \begin{pmatrix} v(e) & 0 \\ vv^*\varepsilon[\mathscr{D}, v(e)] & \gamma(v(e))\end{pmatrix} \right\| \\ &= \left\| \begin{pmatrix} p & 0 \\ 0 & p\end{pmatrix} \begin{pmatrix} v(e) & 0 \\ \varepsilon[\mathscr{D}, v(e)] & v(\gamma(e))\end{pmatrix} \right\| \\ &\le \left\| \begin{pmatrix} v(e) & 0 \\ \varepsilon[\mathscr{D}, v(e)] & v(\gamma(e))\end{pmatrix} \right\|\end{aligned}$$

which shows the first of the statement. Recall that Γ is the grading operator on $H_{\mathcal{B}^+}$

and that as p is even, we have $\Gamma p = p\Gamma$. We may also deduce the following relations

$$\begin{aligned} p\varepsilon &= p\gamma\varepsilon\gamma \\ &= p\Gamma\gamma = \Gamma p\gamma = \varepsilon\gamma p\gamma \\ &= \varepsilon\gamma(p) \\ [\mathscr{D}_\varepsilon, p]_\Gamma v(e) &= [\mathrm{diag}(\mathscr{D}), \varepsilon p]_\gamma v(e) \\ &= [\mathscr{D}, \varepsilon v(e)] - \varepsilon\gamma p\gamma[\mathscr{D}, v(e)] = [\mathscr{D}, \varepsilon v(e)] - p\varepsilon[\mathscr{D}, v(e)] \end{aligned}$$

These two considerations taken together allow us to write the following equality

$$\begin{pmatrix} v(e) & 0 \\ vv^*\varepsilon[\mathscr{D}, v(e)] & \gamma(v(e)) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -[\mathscr{D}_\varepsilon, p]_\Gamma & 1 \end{pmatrix} \begin{pmatrix} v(e) & 0 \\ \varepsilon[\mathscr{D}, v(e)] & v(\gamma(e)) \end{pmatrix}$$

By boundedness of the commutator in the matrix $\begin{pmatrix} 1 & 0 \\ -[\mathscr{D}_\varepsilon, p] & 1] \end{pmatrix}$, it is invertible. This gives the desired inverse.

□

In order to construct the unbounded Kasparov product, we need some analogues of the compact operators and the adjointables. Further, we need to show that these have the desired differential structure. Doing this construction and proving that the algebras have the appropriate structure for our purposes is the goal of the next couple of pages.

Definition 3.50. We may view $\mathcal{E}_\mathcal{B}$ as a proper submodule of $\mathcal{E}_\mathcal{B}^\nabla$. We consider the finite rank operators $Fin(\mathcal{E}_\mathcal{B})$ as an algebra of operators on $\mathcal{E}_\mathcal{B}^\nabla$ via. the following representation, for $K \in Fin(\mathcal{E}_\mathcal{B})$.

$$\pi_\nabla(K) = \begin{pmatrix} vKv^* & 0 \\ p[\mathscr{D}_\varepsilon, vKv^*]p & vKv^* \end{pmatrix}$$

where P comes from a defining column frame, and v is the isometry $\mathcal{E}_\mathcal{B} \to H_{B^+}$.

Lemma 3.51. The representation above is well-defined on the finite rank operators.

Proof. For $e, f \in \mathcal{E}_\mathcal{B}$ we consider the column and row vectors

$$\begin{aligned} v|e\rangle &= v(e) = (\langle x_i, e\rangle)_{i\in\hat{\mathbb{Z}}} \\ \langle f|v^* &= v(f)^* = (\langle f, x_i\rangle)^t_{i\in\hat{\mathbb{Z}}} \end{aligned}$$

where ^{t} is the transpose. These are elements of H_{B^+} and $H^t_{B^+}$ respectively. Therefore the rank one operator given as $|e\rangle\langle f|$ satisfies that

$$[\mathscr{D}_\varepsilon, v|e\rangle\langle f|v^*]$$

is a bounded matrix. This shows that the representation is well-defined. □

This allows us to define the compact operators $\mathbb{K}(\mathcal{E}_{\mathcal{B}}^{\nabla})$ as the closure of $\pi_{\nabla}(Fin_{\mathcal{B}}(\mathcal{E}_{\mathcal{B}}))$ in operator space norm. We are now in a situation where we may show existence of suitable approximate units.

Lemma 3.52. Let $(\mathcal{B}, F_C, \mathscr{D})$ be a defining Kasparov module for $\mathcal{B}$ and $\mathcal{E}_{\mathcal{B}}$ a projective operator module. Let $(u_n)_{n\in\mathbb{N}}$ be the canonical approximate unit associated to the column finite frame $(x_i)_{i\in\hat{\mathbb{Z}}}$, and let $K \in Fin_{\mathcal{B}}(\mathcal{E}_{\mathcal{B}})$. Then $[\mathscr{D}_\varepsilon, vKv^*]$ extends to a bounded adjointable operator in $L_C(H_{B^+}\widetilde{\otimes}_{B^+}F_C)$ and

$$vKv^*\,\mathrm{Dom}(\mathscr{D}_\varepsilon) \subset \mathrm{Dom}(\mathscr{D}_\varepsilon) \tag{2}$$

$$\lim_{n\to\infty} v\chi_n v^*[\mathscr{D}_\varepsilon, vKv^*] = vv^*[\mathscr{D}_\varepsilon, vKv^*] \tag{3}$$

$$\lim_{n\to\infty} [\mathscr{D}_\varepsilon, vKv^*]v\chi_n v^* = [\mathscr{D}_\varepsilon, vKv^*]vv^* \tag{4}$$

where all limits are in operator norm.

Proof. By linearity and continuity it is sufficient to show this for rank one operators, ie. for $K = |e\rangle\langle f|$. Then

$$vKv^* = (\langle x_i, e\rangle \langle f, x_i\rangle)_{ij\in\hat{\mathbb{Z}}} \in \mathbb{K}\widetilde{\otimes}\mathcal{B}$$

and thus lies in the domain of $[\mathscr{D}_\varepsilon, \cdot]$. We handle only one of the limits, as the proof of the other equality is identical in form. Here we tacitly use the identities for $\mathscr{D}_\varepsilon$ established in Remark 3.48.

$$\begin{aligned}
&\lim_{n\to\infty} v\chi_n v^*[\mathscr{D}_\varepsilon, vKv^*] = \lim_{n\to\infty}\left(\sum_{1\le|k|\le n} \langle x_i, \gamma(x_k)\rangle [\mathscr{D}, \langle x_k, e\rangle \langle f, x_j\rangle]\right)_{i,j\in\hat{\mathbb{Z}}} \\
&= \lim_{n\to\infty}\left(\sum_{1\le|k|\le n} \langle x_i, \gamma(x_k)\rangle \gamma(\langle x_k, e\rangle)[\mathscr{D}, \langle f, x_j\rangle] + \langle x_i, \gamma(x_k)\rangle [\mathscr{D}, \langle x_k, e\rangle] \langle f, x_j\rangle\right)_{i,j\in\hat{\mathbb{Z}}} \\
&= (\langle x_i, \gamma(e)\rangle [\mathscr{D}, \langle f, x_j\rangle)_{i,j\in\hat{\mathbb{Z}}} + \lim_{n\to\infty}\left(\sum_{1\le|k|\le n} \langle x_i, \gamma(x_k)\rangle [\mathscr{D}, \langle x_k, e\rangle] \langle f, x_j\rangle\right)_{i,j\in\hat{\mathbb{Z}}}
\end{aligned}$$

The first term is well-defined as $f \in \mathcal{E}_{\mathcal{B}}$ and the second is well-defined as $e \in \mathcal{E}_{\mathcal{B}} \subset \mathcal{E}_{\mathcal{B}}^{\nabla}$ □

We may now define the non-commutative notion of completeness of our non-commutative analogues of vector bundles.

Definition 3.53. As usual, we let $(\mathcal{B}, F_C, \mathscr{D})$ be an unbounded Kasparov module and let $\mathcal{E}_{\mathcal{B}}$ be a projective operator module with canonical approximate unit $(\chi_n)_{n\in\mathbb{N}}$. If there is an approximate unit $(u_n)_{n\in\mathbb{N}} \subset \mathrm{conv}\{\chi_n : n \in \mathbb{N}\}$ such that for all $K \in \mathbb{K}(E_B)$ the sequence

$$p[\mathscr{D}_\varepsilon, vu_nv^*]p$$

converges to zero strictly, $\mathcal{E}_{\mathcal{B}}$ is a complete projective operator module. Note this is a sequence of operators $H_{B^+}\widetilde{\otimes}_{B^+}F_C \to H_{B^+}\widetilde{\otimes}_{B^+}F_C$. In light of **??** this implies that $u_n \to p$.

Lemma 3.54. Let $\mathcal{E}_{\mathcal{B}}$ be a complete projective operator module over $\mathcal{B}$. Then $\mathbb{K}(\mathcal{E}_{\mathcal{B}}^{\nabla})$ has a bounded approximate unit consisting of elements in $\mathrm{conv}(\chi_n)$.

Proof. Let $\chi_n = \sum_{1\leq|i|\leq n}|x_i\rangle\langle x_i|$ be the canonical approximate unit associated to a column finite frame. Let $(u_n)_{n\in\mathbb{N}}$ be an approximate unit as in the assumptions. For each $x \in H_{B^+}\widetilde{\otimes}_{B^+}F$ the sequence $p[\mathscr{D}, vu_nv^*]px$ converges, implying that $\sup_{n\in\mathbb{N}} \|p[\mathscr{D}_\varepsilon, vu_nv^*]px\| < \infty$ hence $\sup_{n\in\mathbb{N}} \|p[\mathscr{D}_\varepsilon, vu_nv^*]p\| < \infty$ It follows from the uniform boundedness principle that $\sup_n \|\pi_\nabla(u_n)\| < \infty$. Picking $K \in Fin_{\mathcal{B}}(\mathcal{E}_{\mathcal{B}})$ and using that vKv^* is domain preserving for $\mathscr{D}_\varepsilon$, we calculate:

$$p[\mathscr{D}_\varepsilon, vu_nKv^*]p = p[\mathscr{D}_\varepsilon, vu_nv^*]vKv^* + vu_nv^*[\mathscr{D}_\varepsilon, vKv^*]p$$

The final term converges to $p[\mathscr{D}_\varepsilon, vKv^*]p$. By continuity and unfirom boundedness, it follows $\pi_\nabla(u_nK) \to \pi_\nabla(K)$ for all $K \in \mathbb{K}(\mathcal{E}_{\mathcal{B}}^{\nabla})$. ☐

Proposition 3.55. Let $\mathcal{E}_{\mathcal{B}} = p\,\mathrm{Dom}(p)$ be a projective operator module with defining column finite frame $(x_i)_{i\in\hat{\mathbb{Z}}}$, and canonical approximate unit χ_n. Then either of the conditions:

1. There exists an approximate unit $u_n \in \mathrm{conv}\{\chi_n : n \in \mathbb{N}\}$ for $\mathbb{K}(E_B)$ such that $p[\mathscr{D}_\varepsilon, u_n]p \to 0$ in norm on $H_{\mathcal{B}^+}\widetilde{\otimes}_{\mathcal{B}^+}F_C$.

2. The projection p is a countable direct sum of finite even projections $p_k \in M_{2m_k}(\mathcal{B}^+)$.

3. The projection p lies in $L(H_{\mathcal{B}^+})$.

is sufficient to infer completeness of the module.

Proof. 1. As norm convergence implies strict convergence, we get completeness immediately.

2. As such it suffices to show that the second condition implies the first. Given a countable family of finite projections with $[\mathscr{D}_\varepsilon, p_i]$ bounded, it holds that $p_i[\mathscr{D}_\varepsilon, p_i]p_i = 0$, and we have

$$p_k = \sum_{1\leq|i|\leq m_k} |pe_i^k\rangle\langle pe_i^k|$$

We may identify $\bigoplus_{k=0}^{\infty}(\mathcal{B}^+)^{2m_k}$ with $H_{\mathcal{B}^+}$ and define $p = \oplus_{i=1}^{\infty}p_i$, thus getting the approximate unit $u_n = \oplus_{i=1}^{n}p_i$. As p_i is defined explicitly we see that u_n is a subsequence of the approximate unit associated to the frame (pe_k^i). Hence u_n lies in the convex hull of the canonical approximate unit. To finalize the argument, observe that $p[\mathscr{D}, u_n]p = \sum_{i=1}^{n} p[\mathscr{D}, p_i]p = \sum_{i=1}^{n} p_i[\mathscr{D}, p_i]p_i = 0$.

3. In order to see that the third condition implies completeness of the module, we remark that $p \in L_{\mathcal{B}^+}(H_{\mathcal{B}^+})$ if and only if $p \otimes Id_F$ preserves the domain of $\mathscr{D}_\varepsilon$ and the commutator $[\mathscr{D}_\varepsilon, p \otimes Id_F]$ is adjointable. Letting q_k be the increasing family of symmetric projections onto $e_i, 1 \leq |i| \leq k$. Define $x_i = pe_i$ and the approximate unit

$\chi_n = \sum_{i=1}^n |x_i\rangle\langle x_i|$. We see that for all $y = px$ we have $\chi_n y = pq_n py = pq_n y$. As such we can calculate as below

$$p[\mathscr{D}_\varepsilon, \chi_n]p = p[\mathscr{D}_\varepsilon, pq_n]p = p[\mathscr{D}_\varepsilon, p]q_n p$$

As q_n converges strongly to the identity and the commutator is bounded, we see that $p[\mathscr{D}_\varepsilon, \chi_n]px \to p[\mathscr{D}_\varepsilon, p]px = 0$.

□

This establishes the second leg of the tripod on which the construction of the Kasparov product rests, and we can now construct the third leg based on the method of localization.

Assumption 3.56. From here $\mathcal{E}_\mathcal{B}$ is a projective operator module. We shall be working with the fixed Kasparov module $(\mathcal{B}, F_C, \mathscr{D})$.

Definition 3.57. Define the operator

$$(1 \otimes_\nabla \mathscr{D})(e \otimes f) = \gamma(e) \otimes \mathscr{D}(f) + \nabla_\mathscr{D}(e) f$$

on elementary tensors. We shall consider minimal closure of $1 \otimes_\nabla \mathscr{D}$ on $\mathcal{E} \otimes_\mathcal{B} \operatorname{Dom} \mathscr{D}$.

This operator shall turn out take the on the role of covariant derivative on the interior tensor product, where $\nabla_\mathscr{D}$ serves to symmetrize the operator. We may define the stabilized version of $1 \otimes_\nabla \mathscr{D}$ operator, which is more amenable to calculations.

Definition 3.58. Recall that $p = vv^*$ and define $\operatorname{Dom}(\partial) = v \operatorname{Dom}(1 \otimes_\nabla \mathscr{D}) \oplus (1 - p) H_\mathcal{B} \widetilde{\otimes}_{B^+} F_C$ with the operator ∂ defined as

$$\partial = v(1 \otimes_\nabla \mathscr{D})v^*$$
$$\partial(vy + (1-p)z) = v(1 \otimes_\nabla \mathscr{D})y$$

We may thus denote it as $v(1 \otimes_\nabla \mathscr{D})v^*$.

This transformation preserves the geometric information of $1 \otimes_\nabla \mathscr{D}$, as evidenced by the proposition below.

Lemma 3.59. The operator ∂ is self-adjoint and regular if and only if $1 \otimes_\nabla \mathscr{D}$ is self-adjoint and regular.

Proof. A closed densely defined symmetric operator T is self-adjoint and regular if and only $T \pm i : \operatorname{Dom} T \to E$ have dense range. Assume that $(1 \otimes_\nabla \mathscr{D}) \pm i$ both have dense range. Picking $x = vy + (1-p)z \in \operatorname{Dom} 1 \otimes_\nabla \mathscr{D}$ we get

$$(\partial \pm i)x = v(1 \otimes_\nabla \mathscr{D} \pm i) \pm i(1-p)z$$
$$(1 \otimes_\nabla \pm i)y = v^*(\partial \pm i)x$$

As $\operatorname{Ran} v$ and $\operatorname{Ran}(1-p)$ are orthogonal, it can be seen that $(\partial \pm i)$ has dense range in $H_{B^+} \widetilde{\otimes}_{B^+} F_C$ if and only if $(1 \otimes_\nabla \mathscr{D}) \pm i$ has dense range in $(E_B \widetilde{\otimes}_B F_C$ □

This reduces our problem to showing self-adjointness and regularity of $1 \otimes_\nabla \mathscr{D}$ to considering ∂. As B is represented essentially on F_C we have the canonical isomorphism

$$H_{B^+} \widetilde{\otimes}_{B^+} F_C \cong \bigoplus_{i \in \hat{\mathbb{Z}}} F_C$$

Thus $\mathscr{D}_\varepsilon$ is equivalent to $1 \otimes_d \varepsilon \mathscr{D}$ where εd is the trivial connection. We can now show that the operator $1 \otimes_\nabla \mathscr{D}$ is well-defined on the interior tensor product through the following lemma giving an explicit formula on elementary tensors. More importantly, we can evantualyl use the lemma to show that ∂ is self-adjoint and regular by the continuity of the map g defined therein.

Lemma 3.60. Let $(\mathcal{B}, F_C, \mathscr{D})$ be the essential unbounded Kasparov module defining $\mathcal{B}$, and let $\mathcal{E}_\mathcal{B} \subset \mathcal{E}_\mathcal{B}^\nabla$ be a graded complete projective module with defining frame $(x_i)_{i \in \hat{\mathbb{Z}}}$. We may express $1 \otimes_\nabla \mathscr{D}$ on elementary tensors $e \otimes f \in \mathcal{E} \otimes_{\mathcal{B}^+} \operatorname{Dom} \mathscr{D}$

$$\begin{aligned} \gamma(e) \otimes \mathscr{D} f + \nabla(e) f &= \sum_{i \in \hat{\mathbb{Z}}} x_i \otimes \langle x_i, \gamma(e) \rangle \mathscr{D} f + \gamma(x_i) \otimes [\mathscr{D}, \langle x_i, e \rangle] f & (5) \\ &= \sum_{i \in \hat{\mathbb{Z}}} x_i \otimes \langle x_i, \gamma(e) \rangle \mathscr{D} f + \sum_{i,j \in \hat{\mathbb{Z}}} x_i \otimes \langle x_i, \gamma(x_j) \rangle [\mathscr{D}, \langle x_j, e \rangle] f & (6) \\ &= \sum_{i \in \hat{\mathbb{Z}}} x_i \otimes \mathscr{D} \langle \gamma(x_i), e \rangle f & (7) \end{aligned}$$

In particular, this entails that $1 \otimes_\nabla \mathscr{D} = v^* \partial v = v^* \mathscr{D}_\varepsilon v$ on $\mathcal{E} \otimes_\mathcal{B} \operatorname{Dom} \mathscr{D}$ and that $\partial = p \mathscr{D}_\varepsilon p$ on $v\mathcal{E} \otimes_{\mathcal{B}^+} \operatorname{Dom} \mathscr{D}$. The map

$$\begin{aligned} g : \mathcal{E}_\mathcal{B}^\nabla \widetilde{\otimes}_{\mathcal{B}^+} G(\mathscr{D}) &\to G(1 \otimes_\nabla \mathscr{D}) \\ e \otimes \begin{pmatrix} f \\ \mathscr{D} f \end{pmatrix} &\mapsto \begin{pmatrix} e \otimes f \\ (1 \otimes_\nabla \mathscr{D})(e \otimes f) \end{pmatrix} \end{aligned}$$

is a completely bounded operator with dense range. Thus $1 \otimes_\nabla \mathscr{D}$ is continuous in graph norm, allowing us to expand the result by continuity.

Proof. Our first goal will be to show that the sum in Equation (6) is convergent, so that g is well-defined. The first term converges as χ_n is an approximate unit by assumption. To see the second term converges, let $z \in \mathcal{E}_\mathcal{B}^\nabla \widetilde{\otimes}_{\mathcal{B}^+} G(\mathscr{D})$. Let $\sum_{k \in \hat{\mathbb{Z}}} e_k \otimes f_k \in \mathcal{E}_\mathcal{B}^\nabla \odot_{\mathcal{B}^+} G(\mathscr{D})$, be a representation of z. Then consider the estimate below, where we repeatedly use the canonical approximate unit stemming from the defining frame.

$$\begin{aligned} & \left\| \sum_{i,j,k \in \hat{\mathbb{Z}}} x_i \otimes \langle x_i, \gamma(x_j) \rangle [\mathscr{D}, \langle x_j, e_k \rangle] f_k \right\|_{\widetilde{\otimes}}^2 & (8) \\ \leq & \left\| \sum_{i \in \hat{\mathbb{Z}}} |x_i\rangle\langle x_i| \right\|_{\mathbb{K}(E_B)} \left\| \left(\sum_{j,k \in \hat{\mathbb{Z}}} \langle x_i, \gamma(x_j) \rangle [\mathscr{D}, \langle x_j, e_k \rangle] f_k \right)_{i \in \hat{\mathbb{Z}}} \right\|_{E_B \oplus E_B}^2 & (9) \end{aligned}$$

We wish to estimate Equation (9), to this end note that

$$\left\langle \sum_{j,k\in\hat{\mathbb{Z}}} \langle x_i, \gamma(x_j)\rangle\, [\mathscr{D}, \langle x_j, e_k\rangle] f_k, \sum_{j,k\in\hat{\mathbb{Z}}} \langle x_i, \gamma(x_j)\rangle\, [\mathscr{D}, \langle x_j, e_k\rangle] f_k \right\rangle = \left\langle (f_k)_{k\in\hat{\mathbb{Z}}}, \pi_\nabla((e_k)_{k\in\hat{\mathbb{Z}}})\pi_\nabla((e_k)_{k\in\hat{\mathbb{Z}}})^*(f_k)_{k\in\hat{\mathbb{Z}}}\right\rangle$$

Thus, by complete boundedness of the representation π_∇ we can continue our estimates as

$$\begin{aligned}
&\left\| \left(\sum_{j,k\in\hat{\mathbb{Z}}} \langle x_i, \gamma(x_j)\rangle\, [\mathscr{D}, \langle x_j, e_k\rangle] f_k \right)_{i\in\hat{\mathbb{Z}}} \right\|^2_{E_B\oplus E_B} \\
&\leq \left\| \sum_{k\in\hat{\mathbb{Z}}} \pi_\nabla(e_k)\pi_\nabla(e_k)^* \right\|_{L(F_C\oplus F_C)} \left\| \sum_{k\in\hat{\mathbb{Z}}} \langle f_k, f_k\rangle \right\|_{E_B} \\
&\leq \left\| \sum_{k\in\hat{\mathbb{Z}}} \pi_\nabla(e_k)\pi_\nabla(e_k)^* \right\|_{L(F_C\oplus F_C)} \left\| \sum_{k\in\hat{\mathbb{Z}}} \left\langle \begin{pmatrix} f_k \\ \mathscr{D} f_k \end{pmatrix}, \begin{pmatrix} f_k \\ \mathscr{D} f_k \end{pmatrix} \right\rangle \right\|_{E_B\oplus E_B}
\end{aligned}$$

which shows that Equation (6) and the following sums are convergent. In order to show continuity of g, we still need to control $e\otimes f \mapsto \gamma(e)\otimes\mathscr{D}f$. We start by recalling the result, that $E\widetilde{\otimes}_B F$ is isometrically isomorphic to $E\otimes_B F$ if E, F are Hilbert modules, [Ble97]. For this purpose, we have the following estimates, where again e_k and f_k are non-zero only for finitely many k.

$$\begin{aligned}
\left\| \sum_{k\in\hat{\mathbb{Z}}} \gamma(e_k)\otimes\mathscr{D} f_k \right\|^2 &\leq \left\| \sum_{k\in\hat{\mathbb{Z}}} |\gamma(e_k)\rangle\langle\gamma(e_k)| \right\|_{\mathbb{K}(E)} \left\| \sum_{k\in\hat{\mathbb{Z}}} \langle \mathscr{D} f_k, \mathscr{D} f_k\rangle \right\| \\
&\leq \left\| \sum_{k\in\hat{\mathbb{Z}}} \pi_\nabla(e_k)\pi_\nabla(e_k)^* \right\| \left\| \sum_{k\in\hat{\mathbb{Z}}} \left\langle \begin{pmatrix} f_k \\ \mathscr{D} f_k \end{pmatrix}, \begin{pmatrix} f_k \\ \mathscr{D} f_k \end{pmatrix} \right\rangle \right\|
\end{aligned}$$

Drawing the above estimates together, we get the estimate:

$$\begin{aligned}
\left\| g\left(\sum_{k\in\hat{\mathbb{Z}}} e_k\otimes \begin{pmatrix} f_k \\ \mathscr{D} f_k \end{pmatrix} \right) \right\|_{G(1\otimes_\nabla\mathscr{D})} &\leq 2 \left\| \sum_{k\in\hat{\mathbb{Z}}} \pi_\nabla(e_k)\pi_\nabla(e_k)^* \right\| \left\| \sum_{k\in\hat{\mathbb{Z}}} \left\langle \begin{pmatrix} f_k \\ \mathscr{D} f_k \end{pmatrix}, \begin{pmatrix} f_k \\ \mathscr{D} f_k \end{pmatrix} \right\rangle \right\| \\
&\leq 2 \left\| \sum_{k\in\hat{\mathbb{Z}}} e_k e_k^* \right\|_{\mathcal{E}^\nabla_{\mathcal{B}^+}} \|f_k^* f_k\|_{G(\mathscr{D})}
\end{aligned}$$

recall that the Haagerup norm is defined as the infimum we have shown for an arbitrary representation $\sum_{k\in\hat{\mathbb{Z}}} e_k \otimes \begin{pmatrix} f_k \\ \mathscr{D} f_k \end{pmatrix}$ of z that

$$\left\| g\left(\sum_{k\in\hat{\mathbb{Z}}} e_k \otimes \begin{pmatrix} f_k \\ \mathscr{D} f_k \end{pmatrix}\right)\right\| \leq 2 \left\|\sum_{k\in\hat{\mathbb{Z}}} e_k e_k^*\right\|_{\mathcal{E}^{\nabla}_{\mathcal{B}+}} \|f_k^* f_k\|_{G(\mathscr{D})}$$

Thus taking the infimum over representations of z, we get:

$$\begin{aligned}\|g(z)\| &\leq 2\inf\left(\left\|\sum_{k\in\hat{\mathbb{Z}}} e_k e_k^*\right\|_{\mathcal{E}^{\nabla}_{\mathcal{B}+}} \|f_k^* f_k\|_{G(\mathscr{D})} \mid z = \sum_{k\in\hat{\mathbb{Z}}} e_k \otimes \begin{pmatrix} f_k \\ \mathscr{D} f_k \end{pmatrix}\right) \\ &\leq 2\,\|z\|_{\widetilde{\otimes}}\end{aligned}$$

as desired. □

As promised the lemma shows well-definedness of $1 \otimes_\nabla \mathscr{D}$ on the balanced tensor product, if we write out the representation explicitly:

$$\begin{aligned}(1 \otimes_\nabla \mathscr{D})(eb \otimes f) &= \sum_{i\in\hat{\mathbb{Z}}} x_i \otimes \mathscr{D}\pi(\langle\gamma(x_i), eb\rangle) f \\ &= \sum_{i\in\hat{\mathbb{Z}}} x_i \otimes \mathscr{D}\pi(\langle\gamma(x_i), e\rangle)\pi(b) f \\ &= (1 \otimes_\nabla \mathscr{D})(e \otimes \pi(b) f)\end{aligned}$$

The complete boundedness of g will come in useful later, as the operator taking the graph of $\mathscr{D}$ to the graph of $1 \otimes_\nabla \mathscr{D}$ is now known to be continuous with respect to the operator space norm.

Lemma 3.61. Let $\mathcal{E}_\mathcal{B}$ be a projective operator module with a column finite frame $(x_i)_{i\in\hat{\mathbb{Z}}}$ and $R \in \mathrm{conv}\{\chi_n : n \in \mathbb{N}\}$. Then R satisfies:

1. $vRv^* : \mathrm{Dom}(\mathscr{D}_\varepsilon) \to \mathrm{Dom}(\partial)$.

2. $vRv^* : \mathrm{Dom}(\partial)^* \to \mathrm{Dom}(\mathscr{D}_\varepsilon)$.

3. If $\mathcal{E}_\mathcal{B}$ is complete, $vRv^* : \mathrm{Dom}(\partial)^* \to \mathrm{Dom}(\mathscr{D}_\varepsilon) \cap \mathrm{Dom}(\partial) \subset \mathrm{Dom}\,\partial$

Proof. 1. It suffices to consider $R = \chi_n$, as then the result follows by linearity. For the first part of the statement, we consider the family of adjointable operators $(H_B \widetilde{\otimes}_B F)^2 \to (vE\widetilde{\otimes}_B F)^2$.

$$\pi^p_{\mathscr{D}}(\chi_k) = \begin{pmatrix} v\chi_k v^* & 0 \\ p[\mathscr{D}_\varepsilon, v\chi_k v^*] & v\chi_k v^* \end{pmatrix}$$

Letting $x = h \otimes f \in H_{\mathcal{B}} \otimes_{\mathcal{B}} \operatorname{Dom} \mathscr{D} \subset \operatorname{Dom}(\mathscr{D}_\varepsilon)$ we get the following identity

$$v\chi_k v^*(x) = \sum_{1\leq |i| \leq k} vx_i \otimes \langle x_i, v^*(h)\rangle f = \sum_{1\leq |i|\leq k} vx_i \otimes \langle vx_i, h\rangle f$$

As both $v(x_i)$ and h lie in $H_{\mathcal{B}+}$ we get that the inner product $\langle vx_i, h\rangle$ takes values $\mathcal{B}$ as well, thereby allowing us to conclude that $\langle vx_i, h\rangle f$ lies in the domain of $\mathscr{D}$. This shows that $v\chi_k v^*(x)$ lies in $v\mathcal{E} \otimes_{\mathcal{B}} \operatorname{Dom} \mathscr{D} \subset \operatorname{Dom}(\partial)$. Applying the explicit formula we have derived for $(1 \otimes_\nabla \mathscr{D})(e \otimes f)$ we arrive at the following expression:

$$\pi^p_{\mathscr{D}}(\chi_k) \begin{pmatrix} x \\ \mathscr{D}_\varepsilon x \end{pmatrix} = \begin{pmatrix} v\chi_k v^* x \\ p(\mathscr{D}_\varepsilon) v\chi_k v^* x \end{pmatrix} = \begin{pmatrix} v\chi_k v^* x \\ \partial v\chi_k v^* x \end{pmatrix}$$

This implies that $H_{\mathcal{B}} \otimes_{\mathcal{B}} \operatorname{Dom} \mathscr{D}$ contains the direct sum $\bigoplus_{i\in\hat{\mathbb{Z}}} \operatorname{Dom} \mathscr{D}$ and as such is a core for $\mathscr{D}_\varepsilon$. Thereby we have that every element in the family $\pi^p_{\mathscr{D}}(\chi_k)$ maps a dense subspace of $G(\mathscr{D}_\varepsilon)$ to $G(\partial)$, showing the first part of the lemma.

2. To show the second part of the lemma, consider $\pi^p_{\mathscr{D}}(\chi_k)^*$. By what we have just shown, this is a map $G(\partial)^\perp \to G(\mathscr{D}_\varepsilon)^\perp$. Defining the unitary U as

$$\begin{aligned} U &: (E_B \widetilde{\otimes}_{B^+} F_C)^2 \to (E_B \widetilde{\otimes}_{B^+} F_C) \\ &(x, y) \mapsto (-y, x) \end{aligned}$$

we have the standard equalities $G(\partial)^\perp = UG(\partial^*)$ and $G(\mathscr{D}_\varepsilon)^\perp = UG(\mathscr{D}_\varepsilon)$. This allows us to compute the following for every $x \in \operatorname{Dom}(\partial)^*$.

$$\begin{aligned} \pi^p_{\mathscr{D}}(\chi_k)^* \begin{pmatrix} -\partial^* x \\ x \end{pmatrix} &= \begin{pmatrix} v\chi_k v^* & -[\mathscr{D}_\varepsilon, v\chi_k v^*]p \\ 0 v\chi_k v^* & \end{pmatrix} \begin{pmatrix} -\partial^* x \\ x \end{pmatrix} \\ &\begin{pmatrix} -v\chi_k \partial^* x - [\mathscr{D}_\varepsilon, v\chi_k v^*]x \\ v\chi_k v^* x \end{pmatrix} \end{aligned}$$

Thus $v\chi_k v^*$ lies in the dmoain of $\mathscr{D}_\varepsilon$ as long as x is in the domain of ∂^*.

3. To show that vRv^* has the claimed properties in case that $\mathcal{E}_{\mathcal{B}}$ is complete, by Part (2) of the lemma it suffices to show that the range of vRv^* is in $\operatorname{Dom}(\partial)$. It again suffices to consider $v\chi_n v^*$. By completeness of $\mathcal{E}_{\mathcal{B}}$ we may pick an approximate unit $(u_n) \subset \operatorname{conv}(\chi_n)$, and let $x \in \operatorname{Dom}(\partial)^*$. By Part (1) of the lemma, we have $vu_n v^* v\chi v^* x \in \operatorname{Dom}(\partial)$, and the following norm limit in $H_{\mathcal{B}+} \widetilde{\otimes} F_C$.

$$\lim_{n\to\infty} vu_n v^* v\chi_n v^* x = v\chi_n v^* x$$

Thus, by applying the second part of the lemma and this limit we get that the operator

$$p[\mathscr{D}_\varepsilon, vu_k kv^*]p$$

is well-defined and bounded on the space $v\mathcal{E} \otimes_{\mathcal{B}+} \operatorname{Dom} \mathscr{D}$. Thus it extends by continuity to the entirety of $H_{\mathcal{B}+}\widetilde{\otimes}_{B^+}F_C$. The identity

$$(\partial vu_kv^* - vu_kv^*\mathscr{D}_\varepsilon)p = p[\mathscr{D}_\varepsilon, vu_kv^*]p \quad \text{for } x \in \mathrm{Dom}(\partial)\cap\mathrm{Dom}(\mathscr{D}_\varepsilon)\cap(pH_{\mathcal{B}^+}\widetilde{\otimes}F_C)$$

together with the continuity of $p[\mathscr{D}_\varepsilon, vu_kv^*]p$ gives that $(\partial vu_kv^* - vu_kv^*\mathscr{D}_\varepsilon)p$ is bounded. As we also have the strict convergences

$$\begin{aligned} p[\mathscr{D}_\varepsilon, vu_kv^*]p &\to 0 \\ vu_kv^* &\to p \end{aligned}$$

we may deduce that the following limit exists

$$\begin{aligned} \lim_{k\to\infty} \partial vu_kv^*v\chi_nv^*x &= \lim_{k\to\infty} vu_kv^*\mathscr{D}_\varepsilon\chi_nv^*x + \partial vu_kv^*v\chi_nv^*x - vu_kv^*\mathscr{D}_\varepsilon\chi_nv^*x \\ &= \lim_{k\to\infty} vu_kv^*\mathscr{D}_\varepsilon\chi_nv^*x + p[\mathscr{D}_\varepsilon, vu_kv^*]pv\chi_nv^*x \end{aligned}$$

By closedness of ∂, we get that $v\chi_nv^*$ lies in $\mathrm{Dom}(\partial)$.

□

This ends our study of connections and projective $\mathcal{B}$-modules as objects in themselves, as we are now sufficiently equipped to use them to show existence of the unbounded Kasparov product. We proceed with showing that $(1\otimes_\nabla \mathscr{D})$ is a self-adjoint and regular operator and that $\mathbb{K}(\mathcal{E}_{\mathcal{B}}^\nabla)$ is a differentiable algebra.

3.3 Localization and sums of self-adjoint operators

In this section we return to the local viewpoint of the section on continuous trace algebras, where we view Hilbert C-modules as bundles of Hilbert spaces over the state space of C, through the formalism of localizations. We start by using the formalism to show that the covariant version of $\mathscr{D}_2$ is actually self-adjoint and regular. Using this result and modifying the techniques of [KL12] to the graded case, we show that the sum of weakly anti-commuting operators is locally self-adjoint, and thereby globally self-adjoint and regular.
The method which lets us study regular self-adjoint operators through local considerations is the results of [Pie06] and [KL12]: An operator is self-adjoint and regular if and only if the operator on each Hilbert localized space is itself a self-adjoint operator. Though the result is stated in the complex case, it readily carries over to the real case by Theorem 1.36.

Definition 3.62. Given a state φ on B we can construct the localization E_B^φ of E_B with respect to this state via. the (pre)-inner product $\varphi(\langle x, y\rangle)$, where we take the quotient by the nullifier of the inner product and complete as usual. Alternatively, given a cyclic representation π, we can consider (π, H_π, ξ_pi) and define $E_B^\pi = E_B \otimes_\pi H_\pi$. These two definitions are equivalent. Denote the embedding map ι_φ.

Definition 3.63. Given a regular self-adjoint operator T on a Hilbert B-module E_B as well as a representation π of B on the Hilbert space H_π we define the localization T_0^π on $\mathscr{D}(T)\otimes_\pi H_\pi \subset E_B\otimes_\pi H_\pi$ via. the formula $T_0^\pi(x\otimes h) = Tx\otimes h$. The closure of this operator is denoted T^π and is called the localization of T with respect to π.

Theorem 3.64 (The Local-Global Principle)**.** A closed densely defined operator T is self-adjoint and regular if and only if all localizations are self-adjoint.

As we are working in the setting of differential algebras rather than Hilbert modules, we need to show that the representation associated with the localization is a morphism in the category of spaces.

Proposition 3.65. Given an unbounded Kasparov module and a state φ we get a completely contractive map $\pi_{\mathscr{D}^\varphi} : \mathcal{B} \to \mathrm{Lip}(\mathscr{D}^\varphi)$ by localizing the map $\pi_{\mathscr{D}}$.

Proof. By definition, $\iota_\varphi(\mathrm{Dom}\,\mathscr{D})$ is a core for $\mathscr{D}^\varphi$. For every $b \in \mathcal{B}$ and $f \in \mathrm{Dom}\,\mathscr{D}$ we have $\pi_\varphi(b)\iota_\varphi(f) = \iota_\varphi(bf) \in \iota_\varphi(\mathrm{Dom}\,\mathscr{D})$. Thereby $\pi_\varphi(b)$ preserves the core $\iota_\varphi(\mathscr{D})$ for $\mathscr{D}^\varphi$. For the commutator we calculate:

$$\begin{aligned}
\|[\mathscr{D}^\varphi, \pi_\varphi(b)]\iota_\varphi(f)\|^2 &= \|\iota_\varphi([\mathscr{D}, b]f)\| \\
&= \varphi(\langle [\mathscr{D}, b]f, [\mathscr{D}, b]f\rangle) \leq \|[\mathscr{D}, b]\|^2 \varphi(\langle f, f\rangle) \\
&= \|[\mathscr{D}, b]\|^2 \|\iota_\varphi(f)\|^2
\end{aligned}$$

Giving boundedness of the commutator on the core. Therefore $\pi_\varphi([\mathscr{D}, b])$ is well-defined and equals $[\mathscr{D}^\varphi, \pi_\varphi(b)]$, so we write

$$\pi_{\mathscr{D}^\varphi}(b) = \pi_\varphi(\pi_{\mathscr{D}}(b))$$

giving complete contractivity of the map $\pi_{\mathscr{D}}(b) \mapsto \pi_{\mathscr{D}^\varphi}(b)$, showing that it is a morphism in the category of operator spaces. We define $\mathcal{B}^\varphi$ as the completion of $\mathcal{B}$ in the norm induced by $\pi_{\mathscr{D}^\varphi}$ and as such we may also define the localized module $\mathcal{E}_{\mathcal{B}^\varphi}$ over $\mathcal{B}^\varphi$ through the mapping $H_{\mathcal{B}^+} \to H_{\mathcal{B}^+,\varphi}$. ☐

We proceed to directly apply the results for localizations, showing that all regularity properties pass to localizations, thereby setting the stage for the proof of the self-adjointness and regularity of the covariant product operator.

Lemma 3.66. Let $\mathcal{E}_{\mathcal{B}}$ be a complete projective operator module for $(\mathcal{B}, F_C, \mathscr{D})$ with column finite frame $(x_i)_{i\in\hat{\mathbb{Z}}}$. Then the localized module $\mathcal{E}_{\mathcal{B}}^\varphi$ is a complete projective module for the localized Kasparov module and $(1 \otimes_\nabla \mathscr{D}^\varphi) = (1 \otimes_\nabla \mathscr{D})^\varphi$. Thus we can infer:

1. We have the mapping $v\pi_\varphi(\chi_n)v^* : \mathrm{Dom}(\partial^\varphi)^* \to \mathrm{Dom}\,\partial^\varphi$.
2. There is an approximate in $\mathrm{conv}(\chi_n)$ such that $p[\mathscr{D}_\varepsilon^\varphi, v\pi_\varphi(u_n)v^*]p$ converges to 0 in the *-strong sense on the Hilbert space defined as $H_{B^+}\widetilde{\otimes}_{B^+}F^\varphi$.

Proof. We need to check that the defining frame of $\mathcal{E}_{\mathcal{B}}$ passes down and the accompanying approximate unit in the convex hull of the canonical approximate unit. The column finiteness of the localized frame is immediate from the previous proposition, since it shows that $\left\|\pi_{\mathscr{D}^\varphi}(\langle x_i, e\rangle)_{i\in\hat{\mathbb{Z}}}\right\| \leq \left\|\pi_{\mathscr{D}}(\langle x_i, e\rangle)_{i\in\hat{\mathbb{Z}}}\right\|$. The sequence $p[\mathscr{D}_\varepsilon, vu_nv^*]p$ is uniformly bounded and converges strictly 0, hence the localized sequence does so as well. This shows that $\mathcal{E}_{\mathcal{B}^\varphi}$ is a complete projective module for $(\mathcal{B}^\varphi, F^\varphi, \mathscr{D}^\varphi)$. Consider the operator $(1 \otimes_\nabla \mathscr{D}^\varphi)$, which is defined on the core $(\mathcal{E} \otimes_{\mathcal{B}^+} \mathrm{Dom}\,\mathscr{D}^\varphi)$, while the operator $(1 \otimes_\nabla \mathscr{D})^\varphi$ is defined on

$\iota_\varphi(\mathrm{Dom}(1 \otimes_\nabla \mathscr{D}))$. Thus we would like to determine a common core for $(1 \otimes_\nabla \mathscr{D})^\varphi$ and $1 \otimes_\nabla \mathscr{D}^\varphi$. Our candidate is the space: $V = \iota_\varphi(\mathcal{E} \otimes_{\mathcal{B}} \mathrm{Dom}\,\mathscr{D})$. It is clear that this is a core for $(1 \otimes_\nabla \mathscr{D})^\varphi$, since $\mathcal{E}_{\mathcal{B}} \otimes_{\mathcal{B}} \mathrm{Dom}\,\mathscr{D}$ is a core for $1 \otimes_\nabla \mathscr{D}$. In order to show that is is also a core for $1 \otimes_\nabla \mathscr{D}^\varphi$, we unravel the definition of the operator on $e \otimes f_k \in \iota_\varphi(\mathcal{E}_{\mathcal{B}} \otimes_{\mathcal{B}} \mathrm{Dom}\,\mathscr{D})$, where f_k is a sequence converging to $f \in \mathrm{Dom}\,\mathscr{D}^\varphi$ in graph norm.

$$\begin{aligned}1 \otimes_\nabla \mathscr{D}^\varphi(e \otimes f_k) &= \gamma(e) \otimes \mathscr{D}^\varphi f_k + \nabla_{\mathscr{D}^\varphi}(e) f_k \\ &= \gamma(e) \otimes \mathscr{D}^\varphi f_k + \sum_{i \in \hat{\mathbb{Z}}} \gamma(x_i) \otimes [\mathscr{D}^\varphi, \langle x_i, e\rangle] f_k\end{aligned}$$

The first term will clearly converge to $\gamma(e) \otimes \mathscr{D}^\varphi f$, by definition of the graph norm. To show convergence of the second term, we apply the norm estimates stemming from the Haagerup norm to show that it is a Cauchy sequence. We have implicitly used that the localization map is contractive to perform these estimates.

$$\begin{aligned}\left\| \sum_{i \in \hat{\mathbb{Z}}} \gamma(x_i) \otimes [\mathscr{D}^\varphi, \langle x_i, e\rangle](f_k - f_l) \right\|_{\widetilde{\otimes}} &\leq \left\| \sum_{i \in \hat{\mathbb{Z}}} |x_i\rangle\langle x_i| \right\|_{\mathbb{K}(E_B)} \|[\mathscr{D}^\varphi, \langle x_i, e\rangle](f_k - f_l)\| \\ &\leq \|[\mathscr{D}, \langle x_i, e\rangle]\|^2 \, \|(f_k - f_l)\|^2\end{aligned}$$

The norm of the first term is finite, as e is assumed to lie in $\mathcal{E}_{\mathcal{B}}$. This tells us that we may approximate any element $y \in \mathcal{E} \widetilde{\otimes}_{\mathcal{B}} \mathrm{Dom}\,\mathscr{D}^\varphi$ by elements of V in graph norm of $1 \otimes_\nabla \mathscr{D}^\varphi$. This implies that the closure of $(1 \otimes_\nabla \mathscr{D})^\varphi$ over V contains the defining domain of $1 \otimes_\nabla \mathscr{D}^\varphi$, showing that V is a common core for the operators. Since they coincide on the core, we may infer that

$$1 \otimes_\nabla \mathscr{D}^\varphi = (1 \otimes_\nabla \mathscr{D})^\varphi$$

as desired. To see the first claim, simply apply Lemma 3.61 to the defining frame of localized module $\mathcal{E}_{\mathcal{B}^\varphi}$. □

Theorem 3.67. Letting $\mathcal{E}_{\mathcal{B}}$ be a complete projective operator module for $(\mathcal{B}, F_C, \mathscr{D})$. Then $(1 \otimes_\nabla \mathscr{D})$ is self-adjoint and regular.

Proof. We have reduced our problem to showing that for every state φ on C, the operator ∂^φ is self-adjoint and regular on the Hilbert space $(E_B \otimes_B F_C)^\varphi \cong E_B \otimes_B F_C^\varphi$. Letting $(u_n)_{n \in \mathbb{N}} \subset \mathrm{conv}\{\chi_n : n \in \mathbb{N}\}$ be an approximate unit for the complete projective operator module associated to $(\mathcal{B}, F_C, \mathscr{D})$, then by Lemma 3.66:

$$\begin{aligned}&v\pi_\varphi(u_n)v^* : \mathrm{Dom}(\partial^\varphi)^* \to \mathrm{Dom}\,\partial^\varphi \\ &p[\mathscr{D}_\varepsilon^\varphi, v\pi_\varphi(u_n)v^*]p \to 0, \quad {}^* \text{ strongly on } H_{\mathcal{B}^+} \widetilde{\otimes}_{B^+} F^\varphi.\end{aligned}$$

We may also apply Lemma 3.60 to conclude that $\partial^\varphi x = p\mathscr{D}_\varepsilon^\varphi p x$ on the core $H_{\mathcal{B}^+} \widetilde{\otimes}_{\mathcal{B}^+} \mathrm{Dom}\,\mathscr{D}^\varphi$

of ∂^φ. Drawing these two results together, let $x \in H_{\mathcal{B}^+}\widetilde{\otimes}_{\mathcal{B}^+} \operatorname{Dom} \mathscr{D}^\varphi$:

$$\begin{aligned}[\partial^\varphi, vu_kv^*]x &= \partial^\varphi vu_kv^*x - vu_kv^*\partial^\varphi x \\ &= p\mathscr{D}_\varepsilon^\varphi pvu_kv^*x - vu_kv^*p\mathscr{D}_\varepsilon^\varphi px \\ &= p[\mathscr{D}_\varepsilon^\varphi, vu_kv^*]px \to 0\end{aligned}$$

which converges to zero in norm. By the uniform boundedness of $p[\mathscr{D}_\varepsilon^\varphi, vu_kv^*]p$ the result may be expanded to the entirety of $H_\mathcal{B}\widetilde{\otimes}_\mathcal{B}F_C^\varphi$. Since the closure of $[\partial^\varphi, vu_kv^*]$ is equal to the closure of $[(\partial^\varphi)^*, vu_kv^*]$, it may be inferred that the latter converges strictly to zero on $H_{\mathcal{B}^+}\widetilde{\otimes}_{\mathcal{B}^+}F_C$. As such, for $y \in \operatorname{Dom}(\partial^\varphi)^*$, we get $vu_kv^*y \in \operatorname{Dom}\partial^\varphi$ through application of Lemma 3.66, and $vu_kv^*y \to y$. We conclude that $\partial^\varphi vu_kv^*y$ is convergent to $\partial^\varphi y$ as below since

$$\begin{aligned}(\partial^\varphi)^*y &= \lim_{k\to\infty} vu_kv^*(\partial^\varphi)^*y = \lim_{k\to\infty} \partial^\varphi vu_kv^*y - [(\partial^\varphi)^*, vu_kv^*]y \\ &= \lim_{k\to\infty} \partial^\varphi vu_kv^*y\end{aligned}$$

Thus $\operatorname{Dom}\partial^\varphi$ is a core for the operator $(\partial^\varphi)^*$. This lets us conclude that ∂^φ is self-adjoint since it is closed and symmetric, and thus, by the Local-Global principle, it is self-adjoint and regular, and finally by Lemma 3.59, that $1 \otimes_\nabla \mathscr{D}$ is self-adjoint and regular. □

Definition 3.68. The algebra of adjointable operators on $\mathcal{E}^\nabla$ is the idealiser of $\pi_\nabla(\mathbb{K}(\mathcal{E}^\nabla))$ inside the algebra $L_C((pH_{\mathcal{B}^+}\widetilde{\otimes}_{B^+}F)^2)$. We shall denote it by $L_\mathcal{B}(\mathcal{E}^\nabla)$.

To see that our definitions for differentiable modules interact in the same fashion as for Hilbert modules, we have the following proposition.

Proposition 3.69. If $\mathcal{E}_\mathcal{B}$ is a complete projective module then $L_\mathcal{B}(\mathcal{E}^\nabla)$ is an operator *-algebra, which is isometrically isomorphic to $M(\mathbb{K}(\mathcal{E}^\nabla))$. This algebra coincides with a closed subalgebra of the Lipschitz algebra of $(1 \otimes_\nabla \mathscr{D})$, and as such $\mathbb{K}(\mathcal{E}^\nabla)$ is a differentiable algebra.

Proof. The operator $1 \otimes_\nabla \mathscr{D}$ is self-adjoint and regular, so for finite rank K we have the following equality

$$[1 \otimes_\nabla \mathscr{D}, vKv^*] = p[\mathscr{D}_\varepsilon, vKv^*]p$$

Let $T \in L_\mathcal{B}(\mathcal{E}^\nabla)$ then there is a sequence T_n satisfying that T_nK is finite rank for K finite rank, and that T_nK and $p[\mathscr{D}_\varepsilon, vT_nKv^*]p$ are both convergent with T_nK to TK. By definition of ∂ we have the equality

$$p[\mathscr{D}_\varepsilon, vT_nKv^*]p = [\partial, vT_nKv^*]$$

Giving that

$$\partial(vT_nKv^*x) = [\partial, vT_nKv^*] + vT_nKv^*x$$

is convergent for every $x \in \operatorname{Dom} \partial$. Hence TK preserves the domain of ∂ for every K, and $vFin_B(\mathcal{E}_{\mathcal{B}})v^* \operatorname{Dom} \partial$ is dense in $p \operatorname{Dom} \partial$ in graph norm. It follows that T preserves the core of ∂ and on this core the operator

$$[\partial, vTv^*]vKv^*x = [\partial, vTKv^*]x - v\gamma(T)v^*[\partial, vKv^*]x$$

is bounded. This implies that vTv^* is in $\operatorname{Lip}(\partial)$, or equivalently in $T \in \operatorname{Lip}(1 \otimes_\nabla \mathscr{D})$ as desired. The remaining part may be shown as in Proposition 3.21. □

Definition 3.70. Let E be a graded complex C^* module. Assume two odd self-adjoint regular operators S and T on E satisfy

1. There is a core X for T such that $(S \pm i\lambda)^{-1}X \subset \operatorname{Dom} T$.
2. We have the inclusions $T(S \pm i\lambda)^{-1}X \subset \operatorname{Dom} S$.
3. The operator $[S, T](S \pm i\lambda)^{-1}$ is bounded on X.

for all $\lambda > 0$. Then S and T weakly anti-commute.

We have the following result

Lemma 3.71. If S and T weakly anticommute then $(S \pm \lambda i)^{-1}$ preserves the domain of T, and the commutator $[S, T](S \pm \lambda i)^{-1}$ is bounded on $\operatorname{Dom} T$. Hence:

$$S((S - \lambda i)^{-1} \operatorname{Dom} T) \subset \operatorname{Dom} T$$
$$T(\operatorname{Im}(S - \lambda i) \operatorname{Dom} T) \subset \operatorname{Dom} s$$

Thus the commutator is defined on $\operatorname{Im}(S \pm \lambda i)^{-1}(T \pm ic)^{-1}$.

Proof. We may expand the commutator:

$$[T, (S + i\lambda)^{-1}]x = (S \mp \lambda i)^{-1}[S, T](S \pm i\lambda)^{-1}x$$

This operator is bounded by definition which implies that $(S \pm i\lambda)^{-1}$ preserves the domain of T by [FMR14, Proposition 2.1]. Given $x \in \operatorname{Dom} T$ we may pick a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ converging to x, satisfying $Tx_n \to Tx$ as X is a core for T. We then calculate:

$$T(S - i\lambda)x_n = -(S + i\lambda)^{-1}Tx_n + (S + i\lambda)^{-1}[S, T](S - i\lambda)^{-1}x_n$$

As the resolvent preserves the domain, we may take the limit of the above sequence to get $T(S - i\lambda)x = -(S + i\lambda)^{-1}Tx + (S + i\lambda)^{-1}[S, T](S - i\lambda)^{-1}x$. As this sequence is convergent, the limit is $T(S - i\lambda)^{-1}x \in \operatorname{Dom} S$. In order to see that we have the inclusion $S((S - i\lambda)^{-1} \operatorname{Dom} T) \subset \operatorname{Dom} T$ consider the equality

$$S(S - i\lambda)^{-1}(T - i\mu)^{-1} = (T - i\mu)^{-1} + i\lambda(S - i\lambda)^{-1}(T - i\mu)^{-1}$$

□

We can draw all our work together into a unified framework into which we place the construction of the unbounded Kasparov product, namely the unbounded correspondence.

Definition 3.72. Given an unbounded Kasparov module $(\mathcal{B}, F_C, \mathscr{D}_2)$ with bounded approximate unit for $\mathcal{B}$, an $\mathcal{A}-\mathcal{B}$ correspondence for $(\mathcal{B}, F_C, \mathscr{D}_2)$ is a quadruple $(\mathcal{A}, \mathcal{E}_\mathcal{B}, \mathscr{D}_1, \nabla)$ such that:

1. The module $\mathcal{E}_\mathcal{B}$ is a projective $\mathcal{B}$ operator module.
2. The algebra $\mathcal{A}$ is a *-algebra satisfying $\mathcal{A} \subset L_\mathcal{B}(\mathcal{E}^\nabla_\mathcal{B}) \cap \operatorname{Lip}(\mathscr{D}_1)$.
3. The operator $\mathscr{D}_1$ is self-adjoint regular operator satisfying that $(\mathscr{D}_1^2+1)^{-1/2} \in L_\mathcal{B}(\mathcal{E}^\nabla_\mathcal{B})$ with $a(\mathscr{D}_1^2+1)^{-1/2} \in \mathbb{K}_\mathcal{B}(\mathcal{E}^\nabla_\mathcal{B})$ for all $a \in \mathcal{A}$.
4. The operator $\nabla : \mathcal{E}^\nabla_\mathcal{B} \to E\widetilde{\otimes}_B\Omega^1_\mathscr{D}$ is a connection such that $\nabla((\mathscr{D}_1^2+1)^{-1/2}\varepsilon) \subset \operatorname{Dom}(\mathscr{D}_1 \otimes 1)$,and the operator $[\nabla, \mathscr{D}_1](\mathscr{D}_1^2+1)^{-1/2} : \mathcal{E}^\nabla_\mathcal{B} \to E_B\widetilde{\otimes}_{B^+}\Omega^1_{\mathscr{D}_2}$ is completely bounded.

such a correspondence is strongly complete if there exists an approximate unit for A such that $[\mathscr{D}_1, u_n] \to 0$ and $[1 \otimes_\nabla \mathscr{D}_2, u_n \otimes Id_{F_C}] \to 0$ in norm.

To see that the notion of an unbounded correspondence is the correct one, we show that $D_1 = \mathscr{D}_1 \otimes 1$ and $D_2 = 1 \otimes_\nabla D_2$ weakly anticommute. One should remark that weakly anticommuting is defined only on the complexifications, as the property is only required to prove self-adjointness and regularity of the sum operator.

Lemma 3.73. Let $(\mathcal{A}, \mathcal{E}_\mathcal{B}, \mathscr{D}_1, \nabla)$ be an $\mathcal{A}-\mathcal{B}$ correspondence. Then the self-adjoint regular operators $D_1 = \mathscr{D}_1 \otimes 1$ and $D_2 = 1 \otimes_\nabla \mathscr{D}_2$ weakly anticommute, with core $X = \mathcal{E}^\nabla_\mathcal{B}\widetilde{\otimes}_\mathcal{B}\mathscr{D}_2$.

Proof. By Lemma 3.60 the map $g : \mathcal{E}^\nabla_\mathcal{B}\widetilde{\otimes}_{\mathcal{B}^+}G(\mathscr{D}_2) \to G(1 \otimes_\nabla \mathscr{D}_2)$ given as

$$e \otimes \begin{pmatrix} f \\ \mathscr{D}_2 f \end{pmatrix} \mapsto \begin{pmatrix} e \otimes f \\ 1 \otimes_\nabla \mathscr{D}_2(e \otimes f) \end{pmatrix}$$

has dense range. Thus $X = \mathcal{E}^\nabla_\mathcal{B} \otimes_\mathcal{B} \operatorname{Dom} \mathscr{D}_2$ is a core for D_2. As

$$(\mathscr{D}_1^2+1)^{-1/2} : \mathcal{E}^\nabla_\mathcal{B} \to \mathcal{E}^\nabla_\mathcal{B},$$

so will $(D_1 \pm i)^{-1}$ and thereby we get the first part of the criterion for being weakly anticommuting. To check the second part, remark that by definition

$$D_1((D_2+i)^{-1}(D_2-i)^{-1})^{1/2}X \subset \operatorname{Dom} D_2,$$

and thus $D_1(D_2 \pm i)^{-1}X \subset \operatorname{Dom} D_2$. On X we may calculate the graded commutator

$$[D_1, (D_2 \pm i)^{-1}] = (D_2 \mp i)^{-1}[D_2, D_1](D_2 \pm i)^{-1}$$

which can be seen to be bounded from:

$$\begin{aligned}[D_2, D_1](e \otimes f) &= (D_1 \otimes 1)(\gamma(e) \otimes \mathscr{D}_2 f + \nabla(e) f) + \gamma(D_1 e) \otimes \mathscr{D}_2 f + \gamma(D_1 e) \otimes f + \nabla(D_2 e) \otimes f \\ &= [\nabla, D_2](e) f\end{aligned}$$

By definition $[\nabla, D_1](D_1 \pm i)^{-1}$ is bounded, finishing the claim. $\square$

We can now show local compactness of the resolvents.

Lemma 3.74. Assume we are in the setup of Lemma 3.73, then the operators $(D_2^2+1)^{-1/2}K$, and $K(D_2^2+1)^{-1/2}$ are compact for $K \in \mathbb{K}(E_B)$.

Proof. For every $e \in \mathcal{E}$ the following series is norm-convergent.

$$\sum_{i\in\hat{\mathbb{Z}}}[\mathscr{D}_{2,\varepsilon}, \langle x_i, e\rangle]^*[\mathscr{D}_{2,\varepsilon}, \langle x_i, e\rangle]$$

Thus we may conclude that the operator given as $D_2|e\rangle - |\gamma(e)\rangle\mathscr{D}_2$ acting on f as:

$$\begin{aligned}\sum_{i\in\hat{\mathbb{Z}}} x_i \otimes \mathscr{D}_2 \langle x_i, e\rangle - \gamma(e)\otimes \mathscr{D}_2 f &= \sum_{i\in\hat{\mathbb{Z}}} x_i \otimes (\mathscr{D}_{2,\varepsilon}\langle x_i, e\rangle - \langle x_i, \gamma(e)\rangle \mathscr{D}_{2,\varepsilon})f \\ &= \sum_{i\in\hat{\mathbb{Z}}} x_i \otimes [\mathscr{D}_{2,\varepsilon}, \langle x_i, e\rangle]f\end{aligned}$$

is bounded. Let $(u_n)_{n\in\mathbb{N}}$ be an approximate unit for B and consider the operator $|e\rangle(\mathscr{D}_2^2 + 1)^{-1/2} : F_C \to E_B\widetilde{\otimes}_B F_C$. We have the equality:

$$\begin{aligned}|e\rangle(\mathscr{D}_2^2+1)^{-1/2} &= \lim_{n\to\infty}|eu_n\rangle(\mathscr{D}_2^2+1)^{-1/2} \\ &= \lim_{n\to\infty}|e\rangle u_n(\mathscr{D}_2^2+1)^{-1/2}\end{aligned}$$

Giving that our operator is the norm limit of compact operators, and thus is compact by the assumption of locally compact resolvent of $\mathscr{D}_2$. To expand this to all compacts, consider the calculation below

$$\begin{aligned}&((1\otimes_\nabla \mathscr{D}_2)\pm i)^{-1}|e_1\rangle\langle e_2| \\ &= |\gamma(e_1)\rangle(\mathscr{D}_2)\pm i)^{-1}\langle e_2| \\ &+ ((1\otimes_\nabla \mathscr{D}_2)\pm i)^{-1}(|e_1\rangle\mathscr{D}_2 - (1\otimes_\nabla \mathscr{D}_2)|\gamma(e_1)\rangle \pm i|e_1-\gamma(e_1)\rangle)(\mathscr{D}_2\pm i)^{-1}|e_2\rangle\end{aligned}$$

Thus the operator $(1\otimes_\nabla \mathscr{D}_2 \pm i)^{-1}K$ is compact for every $K \in \mathbb{K}(E)\otimes 1$, so we have shown the desired by Theorem 1.36. $\square$

After this brief aside demonstrating the consequences of the existence of an unbounded correspondence for whether two unbounded self-adjoint operators weakly anti-commute, we proceed by showing the applications of weak anti-commutativity. In particular, we shall show that if two self-adjoint regular operators weakly anti-commute, their sum is self-adjoint and regular.

Lemma 3.75. Let W be a self-adjoint regular operator on E. Let $(f_n)_{n\in\mathbb{N}}$ be a uniformly bounded sequence of functions, converging uniformly on compact subsets of $\mathbb{R}$ to $f \in C_0(\mathbb{R})$. Then $f_n(W)$ converges strongly to $f(W)$.

Proof. Let $x \in \mathscr{D}(W)$. Define $\varphi(t) = (t+i)^{-1}$, and $y = (W+i)x$. This gives the identity $x = \varphi(W)y$, as the regularity of W implies that the function is well-defined. As φ vanishes at infinity, $f_n\varphi \to f$ uniformly, not just on compact subsets. Thus

$$f_n(W)x = ((f_n\varphi)(W))y \to f(W)x$$

Therefore $f_n(W)$ converges strongly to $f(W)$ on a dense subset of E. As the sequence is uniformly bounded, we get strong convergence globally. ☐

Define the family of operators $Y_\mu = [S,T](S-i\mu)^{-1}$, for $\mu \in \mathbb{R}$. This family is adjointable as the domain of Y_μ includes the dense submodule $(S-i)^{-1}(E)$. Thus we get

$$Y_\mu^*\xi = -(S+i\mu)^{-1}[S,T]\xi$$

for every $\xi \in \mathrm{Dom}([S,T])$.

Lemma 3.76. The sequence of operators

$$R_n = -\frac{i}{n}\left(\frac{-i}{n}S+1\right)^{-1}[S,T]\left(\frac{i}{n}S-1\right)^{-1}$$

converges strongly to the zero operator. Furthermore, R_n is adjointable and the adjoint converges to the zero operator as well.

Proof. Write R_n in the form

$$R_n = \left(\frac{i}{n}S+1\right)^{-1}Y_{-1}(S-i)(S-in)^{-1}$$

Every factor is adjointable, so R_n is adjointable. By Lemma 3.75 $(S-i)(S-in)^{-1}$ converges strongly to 0. Further, $\left(\frac{i}{n}S+1\right)^{-1}X_{-1}$ is a uniformly bounded sequence. This proves the desired, as the proof for the adjoint proceeds in an analogous fashion. ☐

In order to control the sum of S and T it is necessary to control their commutators.

Lemma 3.77. Let S and T be self-adjoint regular weakly anti-commuting operators, then there exists a $C > 0$ such that:

$$\langle [S,T]\xi,\xi\rangle \leq \frac{1}{2}\langle S\xi,S\xi\rangle + C\langle \xi,\xi\rangle$$

Proof. Taking S and T's self-adjointness into account and the assumption that they weakly anticommute, we see that the form $\langle [S,T]\xi,\xi\rangle$ is self-adjoint for every $\xi \in \mathrm{Dom}([S,T])$. Then by

$$\begin{aligned}
2\langle [S,T]\xi,\xi\rangle &= -(\left\langle [S,T]\mu\xi,\mu^{-1}\xi\right\rangle + \left\langle \mu^{-1}\xi,[S,T]\mu\xi\right\rangle \\
&\leq \mu^2\langle [S,T]\xi,[S,T]\xi\rangle + \mu^{-2}\langle\xi,\xi\rangle \\
&\leq \mu^2||[S,T](S+\mu i)^{-1}||^2\langle (S+\mu i)\xi,(S+\mu i)\xi\rangle - \mu^{-2}\langle\xi,\xi\rangle \\
&\leq \mu^2||X_{-1}||^2\langle S\xi,S\xi\rangle + (\mu^2(||X_{-1}+\mu^{-2}||)\langle\xi,\xi\rangle
\end{aligned}$$

we see that choosing $\mu = \frac{1}{\|Y_{-1}\|}$ shows the desired. □

After having achieved control over the commutator of self-adjoint regular weakly anti-commuting operators, we can show we can control the graph norm of the sum via. the sum of the graph norms.

Lemma 3.78. Let S and T be self-adjoint regular weakly anti-commuting operators. There is a constant $C > 0$ such that for every $\xi \in \operatorname{Dom}(S) \cap \operatorname{Dom}(T)$.

$$\langle (S+T)\xi, (S+T)\xi \rangle \geq \frac{1}{2}||S\xi||^2 + ||T\xi||^2 - C||\xi||^2$$

Proof. We start by showing the inequality on $\operatorname{Dom}([S,T])$. Here we can apply Lemma 3.77 to get:

$$\begin{aligned}\langle (S+T)\xi, (S+T)\xi \rangle &= \langle S\xi, S\xi \rangle + \langle T\xi, T\xi \rangle + \langle [S,T]\xi, \xi \rangle \\ &\geq \frac{1}{2}||S\xi||^2 + ||T\xi||^2 - C||\xi||^2\end{aligned}$$

Consider $\xi \in \operatorname{Dom}(S) \cap \operatorname{Dom}(T)$. Define the sequence

$$\xi_n = \left(\frac{i}{n}S + 1\right)^{-1} \xi \in \operatorname{Dom}([S,T])$$

We have the convergence $\xi_n \to \xi$ and $S\xi_n \to S\xi$ by Lemma 3.75. Therefore the only thing left to show is that $T\xi_n \to T\xi$. We calculate:

$$\begin{aligned}T\xi_n &= T\left(\frac{i}{n}S + 1\right)^{-1} \xi \\ &= \left(\frac{i}{n}S + 1\right)^{-1} T\xi + \frac{i}{n}\left(\frac{i}{n}S + 1\right)^{-1} [S,T] \left(\frac{i}{n}S - 1\right)^{-1} \xi \\ &= \left(\frac{i}{n}S + 1\right)^{-1} T\xi + R_n\xi\end{aligned}$$

By the result in Lemma 3.76 this implies that $T\xi_n \to T\xi$ as desired. □

We can now show the result that the sum of weakly anticommuting self-adjoint regular operators is again self-adjoint.

Theorem 3.79. Assume that the self-adjoint regular operators S and T weakly anti-commute. Then the operator $S+T$ with $\operatorname{Dom}(S+T) = \operatorname{Dom}(S) \cap \operatorname{Dom}(T)$ is self-adjoint. Further, $S+T$ has core $\operatorname{Im}((S \pm i)^{-1} \operatorname{Dom} T) = \operatorname{Im}((S \pm \lambda i)^{-1}(T \pm i)^{-1})$.

Proof. We shall proceed by first showing that $S+T$ is closed and symmetric, using this to show self-adjointness. By Lemma 3.78 we get that the convergence of a sequence in the graph norm of $S+T$ is equivalent to the sequence converging in the graph norms stemming from both operators. Therefore closedness of S and T gives closedness of $S+T$. In order to show that $(S+T)^* = S+T$, we need only consider one inclusion since symmetry is immediate, namely

$$\mathrm{Dom}((S+T)^*) \subset \mathrm{Dom}(S) \cap \mathrm{Dom}(T)$$

To do this, consider an element $\xi \in \mathscr{D}((S+T)^*)$. Consider the sequence

$$\xi_n = \left(-\frac{i}{n}S+1\right)^{-1}\xi$$

which will lie in $\mathrm{Dom}(S)$, and is norm-convergent to ξ. As all operators concerned are closed, we need only show that $\xi_n \in \mathrm{Dom}(S) \cap \mathrm{Dom}(T)$ and $(S+T)^*\xi_n = (S+T)\xi_n$ is convergent. To see $\xi_n \in \mathrm{Dom}(S) \cap \mathrm{Dom}(T)$, let $\eta \in \mathrm{Dom}(T) \cap \mathrm{Dom}(S)$. Then we can perform the following calculations

$$\begin{aligned}
\langle \xi_n, T\eta \rangle &= \left\langle \left(\frac{-i}{n}S+1\right)^{-1}\xi, T\eta \right\rangle \\
&= \left\langle \xi, T\left(\frac{i}{n}S+1\right)^{-1}\eta \right\rangle + \left\langle \xi, \frac{i}{n}\left(\frac{i}{n}S+1\right)^{-1}[S,T]\left(\frac{i}{n}S-1\right)^{-1}\eta \right\rangle \\
&= \left\langle \xi, (S+T)\left(\frac{i}{n}S+1\right)^{-1}\eta \right\rangle + \left\langle \xi, S\left(\frac{i}{n}S+1\right)^{-1}\eta \right\rangle + \langle R_n^*\xi, \eta \rangle \\
&= \left\langle \left(\frac{i}{n}S+1\right)^{-1}(S+T)^*\xi, \eta \right\rangle + \langle S\xi_n, \eta \rangle + \langle R_n^*\xi, \eta \rangle
\end{aligned}$$

By self-adjointness of T we may now conclude $\xi \in \mathrm{Dom}(T)$, as well as

$$T\xi_n = \left(-\frac{i}{n}S+1\right)^{-1}(S+T)^*\xi - S\xi_n + R_n^*\xi$$

Now we need only show that $(S+T)\xi_n$ converges in E. By the expression for $T\xi_n$ we can see that

$$(S+T)\xi_n = \left(\frac{-i}{n}S+1\right)^{-1}(S+T)\xi + iR_n^*\xi$$

which converges in E as R_n^* converges strongly to 0. Thus $((S+T)^*\xi_n)_{n\in\mathbb{N}}$ converges, and we have shown the desired result on $\mathrm{Dom}(S+T)$ and also that the domain has the claimed core. □

We now take a brief detour back to localizations, in order to show that the localization procedure preserves weak anticommutativity and sums. We first state a lemma without proof.

Lemma 3.80. Let S and T be self-adjoint regular weakly anti-commuting operators. For any state φ the operators S^φ and T^φ weakly anti-commute, with core $\mathcal{E}^\varphi = \iota_\varphi(\mathrm{Dom}(T))$.

The only remaining hurdle is showing that the process of localization is finitely additive.

Lemma 3.81. The localization of a sum is equal to the sum of the localizations. That is,

$$S^\varphi + T^\varphi = (S+T)^\varphi$$

for any state φ.

Proof. By definition we have $\mathscr{D}(S^\varphi + T^\varphi) = \mathscr{D}(S^\varphi) \cap \mathscr{D}(T^\varphi)$. The inclusion $(S+T)^\varphi \subset S^\varphi + T^\varphi$ is immediate as $(S+T)_0^\varphi \subset S_0^\varphi + T_0^\varphi$, and $S^\varphi + T^\varphi$ is closed. To show the other inclusion, we start by showing that

$$(S^\varphi + i\mu)^{-1}(\mathscr{D}(T^\varphi)) \subset \mathscr{D}((S+T)^\varphi)$$

Pick $\xi \in \mathscr{D}(T^\varphi)$. Then there is a sequence $\eta_n \subset \mathrm{Dom}(T)$ such that $\iota_\varphi(\eta_n)$ converges to ξ and $\iota_\varphi(T\eta_n)$ converges to $T^\varphi(\xi)$. Applying our assumption that S and T weakly anticommute we get the following

$$(S^\varphi + i\mu)^{-1}(\iota_\varphi(\eta_n)) = \iota_\varphi((S+i\mu)^{-1}\eta_n) \in \iota_\varphi(\mathrm{Dom}(S) \cap \mathrm{Dom}(T)) \subset \mathscr{D}((S+iT)^\varphi)$$

By continuity we may infer that $(S^\varphi + i\mu)(\iota_\varphi(\eta_n))$ converges to $(S^\varphi + i\mu)^{-1}\xi$. Thus it suffices to show that $(S+T)^\varphi(S^\varphi + i\mu)^{-1}(\eta_n)$ converges in the norm of E, which may be done with an argument analogous to the one in Theorem 3.79. To finalize the proof of the inclusion, let $\xi \in \mathscr{D}(S^\varphi + T^\varphi)$. Define the sequence $\xi_n = \left(\frac{i}{n}S^\varphi + 1\right)^{-1}\xi$. As in Lemma 3.78 we have that ξ_n converges to ξ, and $(S^\varphi + T^\varphi)\xi_n$ converges to $(S^\varphi + T^\varphi)\xi$ in E^φ. This shows the desired, as $\xi_n \in \mathscr{D}((S+T)^\varphi)$. $\square$

Drawing all our work together, we have finally reached the point where we can show that the sum of regular self-adjoint normal operators is once again self-adjoint, thereby the last piece of the puzzle showing the existence of the unbounded Kasparov product.

Theorem 3.82. Let S and T be weakly anticommuting self-adjoint regular operators. Then the sum operator

$$Z = S + T$$

is self-adjoint and regular.

Proof. Let φ be a state on B. By the Local-Global principle it suffices to show that Z^φ is self-adjoint. By Lemma 3.81 we get

$$Z^\varphi = S^\varphi + T^\varphi$$

Thus $(Z^\varphi)^* = (S^\varphi + T^\varphi)^*$. By Lemma 3.80 S^φ and T^φ weakly anti-commute and by Theorem 3.79 we get that $S^\varphi + T^\varphi$ is again self-adjoint. Thus

$$(Z^\varphi)^* = (S^\varphi + T^\varphi)^* = (S+T)^\varphi = S^\varphi + T^\varphi$$

which is exactly Z^φ, as desired. $\square$

3.4 The Kasparov Product

We may now show that we have constructed an unbounded version of the Kasparov product, applicable in both real and complex cases. We start by stating a result by Kucerovsky, [Kuc97, Theorem 13].

Theorem 3.83 (Kucerovsky's criterion)**.** Assume that we have unbounded Kasparov modules $x = (E_B, \pi_1, \mathscr{D}_1) \in \Psi(A, B)$ and $y = (E_C, \pi_2, \mathscr{D}_2) \in \Psi(B, C)$. Let $W \subset \pi(\mathcal{A})E_B$ be a dense subset. For every $e \in W$ define the operator:

$$\begin{aligned} T_e &: E_B \to E_B \otimes_B E_C \\ f &\mapsto e \otimes f \end{aligned}$$

The module $z = (E_B \otimes_B E_C, \pi_1 \otimes 1, D) \in \Psi(B, C)$ represents the Kasparov product of x and y if the following conditions are satisfied:

1. For $e \in W$ the commutator

$$\left[\begin{pmatrix} D & 0 \\ 0 & \mathscr{D}_2 \end{pmatrix}, \begin{pmatrix} 0 & T_e \\ T_e^* & 0 \end{pmatrix}\right]$$

   extends to a bounded operator.

2. $\mathrm{Dom}(D) \subset \mathrm{Dom}(D_1 \otimes 1)$.

3. There exists $C \in \mathbb{R}$ such that

$$\langle (\mathscr{D}_1 \otimes 1)x, Dx\rangle + \langle Dx, (\mathscr{D}_1 \otimes 1)x\rangle \geq C \langle x, x\rangle$$

   for every $x \in \mathrm{Dom}(D)$.

Bringing together the results we have shown on connections, localization and differential algebras we can show that we have constructed the unbounded Kasparov module representing the product of two composable cycles. The proof is a mixture of the proof given in [KL13] and [MR16], where we have chosen to forego the spectral approach of [MR16] and instead work with the localizations of [KL13] in order to make the results more self-contained.

Theorem 3.84. Let $(\mathcal{B}, F_C, \mathscr{D}_2)$ be an unbounded Kasparov $B{-}C$ module and let $(\mathcal{A}, \mathcal{E}_\mathcal{B}, \mathscr{D}_1, \nabla)$ be an $\mathcal{A} - \mathcal{B}$ correspondence for $(\mathcal{B}, F_C, \mathscr{D}_2)$. Then the module $(\mathcal{A}, (E_B \widetilde{\otimes}_{B^+} F_C, \mathscr{D}_1 \otimes 1 + 1 \otimes_\nabla \mathscr{D}_2)$ is an an unbounded Kasparov module representing the product of $(\mathcal{A}, E_B, \mathscr{D}_1)$ and $(\mathcal{B}, F_C, \mathscr{D}_1)$.

Proof. The proof is broken up into two stages. We start by showing that we have an unbounded Kasparov module, and then we check the requirements for Kucerovsky's criterion. By Theorem 3.67 the operators $\mathscr{D}_1 \otimes 1$ and $(1 \otimes_\nabla \mathscr{D}_2)$ are both self-adjoint and regular. Appealing to Lemma 3.73 we get that they weakly anti-commute, and by Theorem 3.82 we thus get that the sum is self-adjoint and regular.
In order to see that our operator has locally compact resolvent, we follow the method of [KL13]. Let $(u_n)_{n\in\mathbb{N}}$ be an approximate unit for $\mathbb{K}(E_B)$.

Consider the commutative diagram of operator modules, where ι_* denotes the inclusion operator, as implemented by multiplication with the resolvent.

$$\begin{tikzcd}
\mathrm{Dom}(\mathscr{D}_1 \otimes 1 + 1 \otimes_\nabla \mathscr{D}_2) \arrow[r, "\iota_1"] \arrow[d, "\iota_3"] \arrow[dr, "\iota"] & \mathrm{Dom}(1 \otimes_\nabla \mathscr{D}_2) \arrow[d, "\iota_2"] \\
\mathrm{Dom}(\mathscr{D}_1 \otimes 1) \arrow[r, "\iota_4"] & (E_B \widetilde{\otimes}_B F_C)
\end{tikzcd}$$

We need to show that $\pi(a) \circ \iota : \mathrm{Dom}(\mathscr{D}_1 \otimes 1 + 1 \otimes_\nabla \mathscr{D}_2)$ is compact, where $\pi(a)$ acts by multiplication on the first component of the tensor product. By commutativity of the diagram we get

$$(u_n \otimes 1) \circ \pi(a) \circ \iota = (u_n \otimes 1) \circ \pi(a) \circ \iota_2 \circ \iota_1$$

The operator $(u_m \otimes 1) \circ \pi(a) \circ \iota_1 \circ \iota_2$ is compact by Lemma 3.74, implying that

$$(u_m \otimes 1) \circ \pi(a) \circ \iota : \mathcal{E}_{\mathcal{B}}^{\nabla} \widetilde{\otimes}_{\mathcal{B}^+} \mathrm{Dom}\, \mathscr{D}_2 \to E_B \widetilde{\otimes}_{B^+} F_C$$

is compact for all n. Going back to the diagram, we have the identity

$$(u_m \otimes 1) \circ \pi(a) \circ \iota = (u_m \otimes 1) \circ \pi(a) \circ \iota_4 \circ \iota_3.$$

when viewed as operators

$$\mathcal{E}_{\mathcal{B}}^{\nabla} \widetilde{\otimes}_{\mathcal{B}^+} \mathrm{Dom}\, \mathscr{D}_2 \to E_B \widetilde{\otimes}_{B^+} F_C$$

As we are working over a correspondence, we have $(\pi(a) \circ \iota_4) = \pi(a) \circ (\mathscr{D}_1^2 \otimes 1 + 1)^{-1/2} = K \otimes 1$ where $K \in \mathbb{K}(\mathcal{E}_{\mathcal{B}}^{\nabla}) \subset \mathbb{K}(E_B)$. Hence the sequence $((u_m \otimes 1) \circ \pi(a) \circ \iota_4)_{m \in \mathbb{N}} \subset L(\mathcal{E}_{\mathcal{B}}^{\nabla} \widetilde{\otimes}_{\mathcal{B}^+} \mathrm{Dom}\, \mathscr{D}_2, E_B \widetilde{\otimes}_{B^+} F_C)$ converges to the bounded operator $(\pi(a) \circ \iota_4) \in L(\mathrm{Dom}(\mathscr{D}_1 \otimes 1 + 1 \otimes_\nabla \mathscr{D}_2), E_B \otimes_B F_C)$ in operator norm. This shows that $\pi(a) \circ \iota_4$ is compact, hence $\pi(a) \circ \iota$ is compact. Consequently,

$$(\mathcal{A}, E_B \otimes_B F_C, \mathscr{D}_1 \otimes 1 + 1 \otimes_\nabla \mathscr{D}_2)$$

is an unbounded Kasparov $A - C$ module.

In order to see that it is the product of the two modules as claimed, we need to invoke Kucerovsky's criterion, see Theorem 3.83.

Define $W = \mathrm{Dom}(1 \otimes_\nabla \mathscr{D}_2)$, and define the operators T_e, and T_e^* as in Theorem 3.83. Define the operator Q,

$$Q = \left[\begin{pmatrix} \mathscr{D}_1 \otimes 1 + 1 \otimes_\nabla \mathscr{D}_2 & 0 \\ 0 & \mathscr{D}_2 \end{pmatrix}, \begin{pmatrix} 0 & T_e \\ T_e^* & 0 \end{pmatrix} \right]$$

the boundedness of which we need to check. Calculating, for $(e' \otimes f', f) \in \mathrm{Dom}(\mathscr{D}_1 \otimes +1 +$

$1 \otimes_\nabla \mathscr{D}_2) \oplus \mathrm{Dom}(\mathscr{D}_1)$:

$$Q\begin{pmatrix} e' \otimes f' \\ f \end{pmatrix} = \begin{pmatrix} \mathscr{D}_1 e \otimes f + (-1)^{\partial e} \nabla_{\mathscr{D}_2}(e) f \\ \langle e, \mathscr{D}_1 e' \rangle f + [\mathscr{D}_2, \langle e, e' \rangle] f + (-1)^{\partial(e')} \langle e, \nabla_{\mathscr{D}_2}(e') \rangle f \end{pmatrix}$$
$$= \begin{pmatrix} \mathscr{D}_1 e \otimes f + (-1)^{\partial e} \nabla(e) f \\ \langle \mathscr{D}_1 e, e' \rangle f + \langle \nabla(e), e' \rangle f \end{pmatrix}.$$

For fixed $e \in W$ we see that Q extends to a bounded operator as desired.
The remaining item we need to check is the semi-boundedness condition. Define $s = \mathscr{D}_1 \otimes 1, t = 1 \otimes_\nabla \mathscr{D}_2$. We consider s and t on their common core $V = (s + \lambda i)^{-1} \mathrm{Dom}(t) \mathrm{Ran}(s + \lambda i)^{-1}(t + \lambda i)^{-1}$, see Lemma 3.71. Here we can rewrite the commutator in Kucerovsky's criterion from a quadratic form to an operator expression, recalling that $[s, s] = 2s^2$:

$$\langle [\mathscr{D}_1 \otimes 1 + 1 \otimes_\nabla \mathscr{D}_2, \mathscr{D}_1 \otimes 1] x, x \rangle = 2 \langle sx, sx \rangle + \langle [s, t] x, x \rangle \geq - \|[s, t]\| \langle x, x \rangle$$

To establish semi-boundedness from below of $2 \langle sx, sx \rangle + \langle [s, t] x, x \rangle$, remark that $2 \langle sx, sx \rangle \geq 0$ and by Lemma 3.71 the commutator $[s, t]$ is bounded on V. Thus:

$$2 \langle sx, sx \rangle + \langle [s, t] x, x \rangle \geq - \|[s, t]\| \langle x, x \rangle \qquad \text{for } x \in V$$

By Theorem 3.79 V is a core for $s + t$, allowing us to conclude the semi-boundedness on the entirety of $\mathrm{Dom}\, s + t$. Thus our module represents the Kasparov product as claimed. $\square$

One may then inspect the proof of the lifting theorems in [MR16] for the Kasparov product, and see that these all readily go through in the case with no caveats or modifications necessary, and as such any Kasparov product can, in theory at least, be calculated in the unbounded theory. As remarked throughout the litterature, [Mes09],[Kaa15],[Kaa16],[BMvS13], a great loss of geometric data occurs when passing to KK-theory as it cannot see differential structures. As such it would be beneficial to develop a theory which encapsulates the geometric data, and this is is exactly what is attempted in [Kaa16]. It is however an open problem to develop an appropriate product for this theory, which also works for non-complete manifolds, eg. the half-open interval. This theory is also different in flavor from ordinary unbounded KK-theory, as the usual cycles are replaced with their generalizations, called modular cycles, thereby precluding usage of Kucerovsky's criterion to show that we recover the Kasparov product and leading to the necessity of much greater technical sophistication to show that we reocver the usual product. Along with the further work in developing an unbounded version of KK-theory encapsulating geometric data, it would also be interesting to see if one could use the methods of unbounded KK-theory to find concrete representatives of K-homology classes in the cases where we know we have Poincare duality in KK-theory. Doing this would in particular be interesting in the real case, as the cycles might be found to have some physical significance as encoding the geometry of the non-commutative space in question. Showing Poincare duality in the real case is, however, a hard problem as will be discussed in the next section.

A A brief introduction to Clifford Algebras

We wish to classify the Clifford modules, and from this construct a group $\hat{A}_*$, which it turns out it is isomorphic to the KO-theory of a point, and thereby directly to the K-homology of a point. Thus the Clifford modules which arise here will eventually turn out to relate directly to the kernels of suitable Dirac operators. We start by defining the Clifford algebras

Definition A.1. We define the real Clifford algebras as follows

$$Cl_{p,q} = \mathrm{span}_{\mathbb{R}}\{\gamma_1, \ldots, \gamma_p, \rho_1, \ldots, \rho_q | (\gamma_i)^2 = 1, \gamma_i^* = \gamma_i, (\rho_i)^2 = -1, (\rho_i)^* = -\rho_i\}$$

with $x_j x_i = -x_j x_i$ where x_j, x_i are distinct generators.

In analogy with the complex situation, we have the result that we may build the higher Clifford algebras from the lower Clifford algebras.

Lemma A.2. We have the isomorphism

$$Cl_{p,q} \hat{\otimes} Cl_{p',q'} \cong Cl_{p+p',q+q'}$$

We start by briefly recalling the classification of Clifford algebras, as given in the following table for $Cl_{0,i}, Cl_{i,0}$.

k	$Cl_{0,i}$	$Cl_{i,0}$
1	$\mathbb{C}$	$\mathbb{R} \oplus \mathbb{R}$
2	$\mathbb{H}$	$M_2(\mathbb{R})$
3	$\mathbb{H} \oplus \mathbb{H}$	$M_2(\mathbb{C})$
4	$M_2(\mathbb{H})$	$M_2(\mathbb{H})$
5	$M_4(\mathbb{C}))$	$M_2(\mathbb{H}) \oplus M_2(\mathbb{H})$
6	$M_8(\mathbb{R})$	$M_8(\mathbb{C})$
7	$M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$	$M_8(\mathbb{C}) \oplus M_8(\mathbb{C})$
8	$M_{16}(\mathbb{R})$	$M_{16}(\mathbb{R})$

Proposition A.3. The categories of ungraded Cl_{n-1}-modules and graded Cl_n modules are equivalent, via. the identification of $W^{(0)} \oplus W^{(1)} \mapsto W^{(0)}$ and $W^{(0)} \mapsto W^{(0)} \otimes_{Cl_n^0} Cl_n$

Thus by the classification of Clifford algebras and the result above we get

Definition A.4. Define $\hat{\mathfrak{M}}_k$ as the group generated by unitary equivalence classes of irreducible representations of $Cl_{0,k}$.

We now state the Atiyah-Bott-Shapiro(ABS) theorem.

Theorem A.5. Consider the map $\iota : \mathbb{R}^n \to \mathbb{R}^{n+1}$, this induces a morphism $\iota_* : Cl_n \to Cl_{n+1}$. Thus by restriction, we get a morphism $\iota^* : \hat{\mathfrak{M}}_{n+1} \to \hat{\mathfrak{M}}_n$. Defining the groups $\hat{A}_n = \hat{\mathfrak{M}}_n / \iota^*(\hat{\mathfrak{M}}_{n+1})$, we have an isomorphism $\hat{A}_n \cong KO^{-n}(\{*\}) \cong KO_n(\mathbb{R})$.

We have the accompanying table, in which we also have the groups $\hat{M}_n$ included. We note that the groups are generated by the unique irreducible representations of the simple algebras.

k	$\hat{A}_k$	$\hat{\mathfrak{M}}_k$
0	$\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{Z}_2$	$\mathbb{Z}$
2	$\mathbb{Z}_2$	$\mathbb{Z}$
3	0	$\mathbb{Z}$
4	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$
5	0	$\mathbb{Z}$
6	0	$\mathbb{Z}$
7	0	$\mathbb{Z}$

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