

Question 3. Using vectors and the dot product, show the diagonals of a parallelogram have equal lengths if and only if it is a rectangle.

Solution.

We make use of the following properties of the dot product:

1. $\vec{v} \cdot \vec{v} = |\vec{v}|^2$
2. $\vec{v} \cdot \vec{w} = 0 \iff \vec{v} \perp \vec{w}$

Referring to the figure, let $ABCD$ be a parallelogram. Then

$$\vec{AB} = \vec{DC}.$$

We have

$$\vec{AC} = \vec{AB} + \vec{BC},$$

and

$$\vec{BD} = \vec{BC} + \vec{CD} = \vec{BC} - \vec{AB}.$$

Taking dot products,

$$\begin{aligned} |\vec{AC}|^2 &= \vec{AC} \cdot \vec{AC} = (\vec{AB} + \vec{BC}) \cdot (\vec{AB} + \vec{BC}) \\ &= |\vec{AB}|^2 + 2 \vec{AB} \cdot \vec{BC} + |\vec{BC}|^2. \end{aligned}$$

Similarly,

$$\begin{aligned} |\vec{BD}|^2 &= (\vec{BC} - \vec{AB}) \cdot (\vec{BC} - \vec{AB}) \\ &= |\vec{BC}|^2 - 2 \vec{AB} \cdot \vec{BC} + |\vec{AB}|^2. \end{aligned}$$

Hence,

$$|\vec{AC}| = |\vec{BD}| \iff \vec{AB} \cdot \vec{BC} = 0.$$

Therefore, the diagonals of a parallelogram are equal if and only if adjacent sides are perpendicular, that is, if and only if the parallelogram is a rectangle.

Let \vec{v} and \vec{w} be two vectors. If θ is the angle between them, then

$$\cos \theta = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}| |\vec{w}|}.$$

In our case,

$$\vec{v} \cdot \vec{w} = 2 + (-1) + 2 = 3.$$

Also,

$$|\vec{v}| = \sqrt{6}, \quad |\vec{w}| = \sqrt{6}.$$

Hence,

$$\cos \theta = \frac{3}{6} = \frac{1}{2}.$$

Therefore,

$$\sin \theta = \frac{\sqrt{3}}{2}.$$

Let PQR be a triangle. Given that

$$\vec{QP} = \langle a, 0, -2 \rangle, \quad \vec{QR} = \langle a, -2, 2 \rangle.$$

Then

$$\vec{QP} \cdot \vec{QR} = a^2 + 0 - 4 = a^2 - 4.$$

For $\vec{QP} \perp \vec{QR}$,

$$\vec{QP} \cdot \vec{QR} = 0.$$

Thus,

$$a^2 - 4 = 0, \quad a^2 = 4.$$

Hence,

$$a = 2 \quad \text{or} \quad a = -2.$$

Similarly,

$$\begin{aligned} |\vec{BD}|^2 &= |\vec{BC} - \vec{AB}|^2 = (\vec{BC} - \vec{AB}) \cdot (\vec{BC} - \vec{AB}) \\ &= |\vec{BC}|^2 - 2\vec{AB} \cdot \vec{BC} + |\vec{AB}|^2. \end{aligned}$$

Comparing the two results obtained above, we have

$$|\vec{AC}|^2 = |\vec{BD}|^2 \iff 4\vec{AB} \cdot \vec{BC} = 0.$$

This shows that the diagonals have equal lengths if and only if

$$\vec{AB} \perp \vec{BC},$$

that is, if and only if the sides of the parallelogram are orthogonal to each other.

Q.E.D.

Revision: Dot Product and Angles

(a) Find the angle between the vectors

$$\vec{u} = \hat{i} + \hat{j} + 2\hat{k} \text{ and } \vec{v} = 2\hat{i} - \hat{j} + \hat{k}.$$

(b) Let

$$P = (a, 1, -1), \quad Q = (0, 1, 1), \quad R = (a - 1, 3).$$

For what values of a is $\angle PQR$ a right angle?

Angle Between Two Vectors

Let \vec{v} and \vec{w} be two vectors. If θ is the angle between them, then

$$\cos \theta = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}| |\vec{w}|}.$$

Given Data

In our case,

$$\vec{v} \cdot \vec{w} = 2 + (-1) + 2 = 3$$

$$|\vec{v}| = \sqrt{6}, \quad |\vec{w}| = \sqrt{6}$$

Trigonometric Values

$$\cos \theta = \frac{3}{\sqrt{6} \cdot \sqrt{6}} = \frac{3}{6} = \frac{1}{2}$$

$$\sin \theta = \sqrt{1 - \cos^2 \theta} = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}$$

Vector Problem

Let the points be Q , P , and R .

$$\vec{QP} = \langle a, 0, -2 \rangle$$

$$\vec{QR} = \langle a, -2, 2 \rangle$$

Dot Product

$$\vec{QP} \cdot \vec{QR} = a^2 + (0)(-2) + (-2)(2)$$

$$= a^2 - 4$$

Condition

Since the vectors are perpendicular,

$$\vec{QP} \cdot \vec{QR} = 0$$

$$a^2 - 4 = 0$$

$$a^2 = 4$$

$$a = 2 \quad \text{or} \quad a = -2$$

Uses of the Dot Product

1. Find the angle between the vectors

$$\vec{A} = \hat{i} + 8\hat{j}, \quad \vec{B} = 4\hat{i} + \hat{j}$$

2. Take the points

$$P = (a, 1, -1), \quad Q = (0, 1, 1), \quad R = (a, -1, 3).$$

Find the value(s) of a so that $\angle PQR$ is a right angle.

3. Show that the diagonals of a parallelogram are perpendicular if and only if it is a rhombus, i.e., its four sides have equal length.

Solutions

1. Angle between the vectors

Using the dot product formula,

$$\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta$$

$$|\vec{A}| = \sqrt{1^2 + 8^2} = \sqrt{65}$$

$$|\vec{B}| = \sqrt{4^2 + 1^2} = \sqrt{17}$$

$$\vec{A} \cdot \vec{B} = (1)(4) + (8)(1) = 12$$

$$\cos \theta = \frac{12}{\sqrt{65}\sqrt{17}}$$

$$\theta = \cos^{-1} \left(\frac{12}{\sqrt{65}\sqrt{17}} \right)$$

Vectors (Compulsory)

$$\vec{QP} \cdot \vec{QR} = 0$$

$$a^2 = 4$$

$$a = \pm 2$$

$$\vec{A} + \vec{B} \quad \text{and} \quad \vec{A} - \vec{B}$$

$$(\vec{A} + \vec{B}) \cdot (\vec{A} - \vec{B}) = \vec{A} \cdot \vec{A} - \vec{B} \cdot \vec{B}$$

$$\vec{A} \cdot \vec{A} = \vec{B} \cdot \vec{B}$$

It is perpendicular if and only if two adjacent edges have equal length. In other words, the parallelogram is a rhombus.