Part B matrices of equation Let

$$A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -7 & 3 \\ 1 & 0 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ -1 & 2 \end{bmatrix}$$

$$a)\ AB \qquad b)\ BA \qquad c)\ AC \qquad d)\ -AC$$

$$AB = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -7 & 3 \\ 1 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 17 & -42 & 38 \\ 4 & -7 & 11 \end{bmatrix}$$

b)

$$BA = \begin{bmatrix} 2 & -7 & 3 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$$

Since this is a 2×3 matrix multiplied by a 2×2 matrix,

$$BA \text{ is not defined.}$$

c)

$$BC = \begin{bmatrix} 2 & -7 & 3 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -15 & -17 \\ -3 & 7 \end{bmatrix}$$

d)

$$AC = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \\ -1 & 2 \end{bmatrix}$$

Since this is a 3×2 matrix, matrix multiplication is not defined.

$$AC \text{ is not defined.}$$

Section 10: Meaning of Matrix Multiplication

Matrices:

Often represent linear relations between variables.
Example: Change of coordinate system.

For example:

$$\begin{cases} U_1 = 2x_1 + 2x_2 + 3x_3 \\ U_2 = 2x_1 + 4x_2 + 5x_3 \\ U_3 = x_1 + x_2 + 2x_3 \end{cases}$$

Express using matrix multiplication:

$$\begin{bmatrix} 2 & 2 & 3 \\ 2 & 4 & 5 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

$$AX = U$$

Entries in the matrix product

The entries in the matrix product are obtained by the dot product between a row of A and a column of X.

$$(A \text{ is a } 3 \times 2 \text{ matrix}), \quad (X \text{ is a } 2 \times 1 \text{ column matrix})$$

Entries of AB

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 3 \\ 2 & 2 \end{pmatrix}$$

$$AB = \begin{pmatrix} 7 & 14 \\ 7 & 14 \\ 2 & 3 \end{pmatrix}$$

The width of A must equal the height of B.

If AB represents doing transformation B, then transformation A,

$$(AB)X = A(BX) \qquad \text{(Associativity)}$$

$$\text{Note: } AB \neq BA$$

Entries in the matrix product AB are obtained by taking the dot product between a row of A and a column of B.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 3 \\ 2 & 0 \\ 1 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 11 & 14 \\ 7 & 6 \\ 3 & 5 \end{bmatrix}$$

The width of A must equal the height of B.

$$(AB)X = A(BX) \quad \text{(Associativity)}$$

$$AB \neq BA$$

Identity Matrix

The identity matrix is a matrix that does nothing.

$$IX = X$$

$$I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

In general,

$$I_{n\times n} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Example: Rotation in the Plane

Rotation by 90° counterclockwise:

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Apply R to $\hat{\imath}$:

$$R\hat{\imath} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \hat{\jmath}$$

Apply R to $\hat{\jmath}$:

$$R\hat{\jmath} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} = -\hat{\imath}$$

Formula:

$$A^{-1} = \frac{1}{\det A}\,\text{adjugate}(A)$$

Steps on a 3×3 example

$$A = \begin{pmatrix} 2 & 3 & 3 \\ 2 & 4 & 5 \\ 1 & 1 & 2 \end{pmatrix}$$

1) Minors:

$$\begin{pmatrix} 3 & -1 & -2 \\ -1 & 3 & -1 \\ 3 & 4 & 2 \end{pmatrix}$$

2) Cofactors: (flip signs in a checkerboard pattern)

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 3 & 1 & -2 \\ -3 & 1 & 1 \\ 3 & -4 & 2 \end{pmatrix}$$

3) Transpose:
Switch rows and columns.

$$\begin{pmatrix} 3 & -3 & 3 \\ 1 & 1 & -4 \\ -2 & 1 & 2 \end{pmatrix} \qquad \text{Adjugate}$$

Divide by determinant of A:

$$\det A = \begin{vmatrix} 2 & 3 & 3 \\ 2 & 4 & 5 \\ 1 & 1 & 2 \end{vmatrix} = 3$$

$$A^{-1} = \frac{1}{3} \begin{pmatrix} 3 & -3 & 3 \\ 1 & 1 & -4 \\ -2 & 1 & 2 \end{pmatrix}$$