Problem Set 1

12—-2—2026

Part I (12 points)

At MIT the underlined problems must be done and turned in for grading. The "Others" are some suggested choices for more practice.

A listing like §1B: 2, 5b, 10 means do the indicated problems from supplementary Problems Section 1B.

1) Vector and Dot Product

§1A: 6, 7, 9, others 1, 4, 5, 8, 11

§1B: 2, 5b, 12, 13, others 3ab, 4, 5a, 10

2) Cross Product and Determinants

§1C: 2, 5a, others 1, 3, 6, 7

§1D: 2, 5, 7, others 1, 3, 4

Part II (15 points)

Problem 1 (5)
Find the dihedral angle between two faces of a regular tetrahedron.

Problem 2 (5: 2,3)

(a) Show that the polarization identity

$$\frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2\right) = u \cdot v$$

holds for any two n-vectors u and v.

(Use vector algebra, not components.)

(b) Given two non-zero vectors $\mathbf{u}$ and $\mathbf{v}$, give the formula for the unit vector which bisects the (smaller) angle between $\mathbf{u}$ and $\mathbf{v}$. (Use the notation $\hat{\mathbf{u}}$ for the unit vector in the $\mathbf{u}$-direction.)

Problem 3 (5:3.2) In this problem we examine tacking, which is the process sailboats use to travel against the wind. Sails are a familiar tool to harness the energy of the wind for transportation over the sea. Early ships had large fixed sails which would capture the wind blowing from behind to propel the ship forward. Even if the wind is blowing from behind at an (acute) angle the component of the wind vector perpendicular to the sail will push on the sail and hence on the boat. However, these early fixed sail ships had no way to go against the wind and had to rely on oarsmen if the wind was blowing in the wrong direction. A great advance that allowed boats to sail against the wind was the invention of movable sails in combination with a rudder and a keel. By carefully positioning the sail the boat can be made to sail into the wind –this process is called tacking As noted before, the component of the wind perpendicular to the sail pushes on the sail and, through it, the boat. The keel only allows the boat to move along its axis. (The rudder is used to turn the boat.) That is, for any force on the boat, only the component along the boat's axis actually pushes the boat. Described mathematically, the wind vector is first projected on the perpendicular to the sail to get the direction of the force on the sail. This resultant force is projected on the axis of the boat to find the direction the boat is being pushed. By orienting the sail correctly this double projection can result in a vector with a component pointing into the wind.

a) Let w1 be the projection of w onto the line ls. Show that w1 does not have a non zero component in the direction opposite w. (It is sufficient to show the projections on the sketch.) x b) Find the projection of w1 onto lB.

In this problem we examine tacking, which is the process sailboats use to travel.

In this picture, $\mathbf{w} = \vec{a}$ is the wind direction, the line L_s is perpendicular to the sail (with $0 \leq \alpha \leq \frac{\pi}{2}$), and the line L_b is along the boat's axis (with $0 \leq \beta < \frac{\pi}{2}$).

(b) What is the condition on α and β so that this projection has a component in the $-\hat{\mathbf{i}}$ direction?

(For warm-up you might try the specific case $\alpha = \frac{\pi}{3} = \beta$.)

Solutions

1) There are several ways to set up the tetrahedron for this problem. The simplest way is to inscribe it in the unit cube, so that it has vertices

$$(0,0,0),\ (1,1,0),\ (0,1,1),\ (1,0,1).$$

If you did not think of this, you could also set it up with one face defined by

$$(0,0,0),\ (1,0,0),\ \left(0, \frac{\sqrt{3}}{2}, 0\right)$$

and then figure out where the fourth vertex should be.

In this solution we use the simpler inscribed tetrahedron. We define the plane P_1 to contain

$$(0,0,0),\ (1,1,0),\ (1,0,1).$$

Taking the cross product, we see a normal to this plane is

$$\mathbf{n}_1 = \langle 1,0,1\rangle \times \langle 1,1,0\rangle = \langle -1,1,1\rangle.$$

Similarly, we define the plane P_2 to contain

$$(0,0,0),\ (1,1,0),\ (0,1,1).$$

A normal to this plane is

$$\mathbf{n}_2 = \langle 1,1,0\rangle \times \langle 0,1,1\rangle = \langle 1,-1,1\rangle.$$

The dihedral angle between the two faces is the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$.

The smaller angle θ between two planes is the smaller angle between their normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, with

$$\cos\theta = \left|\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right| = \frac{1}{3}.$$

(We use absolute value to ensure that we obtain the cosine of the smaller angle.) We conclude that

$$\theta = \cos^{-1}\left(\frac{1}{3}\right) \approx 1.23 \text{ rad} \approx 70.5^\circ.$$

(Any other tetrahedron will be similar (in the geometric sense) to this one, so it will have the same dihedral angle.)

2. (a) Calculating directly, we have

$$|\mathbf{u}+\mathbf{v}|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + 2\,\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v}.$$

Similarly,

$$|\mathbf{u}-\mathbf{v}|^2 = (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} - 2\,\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v}.$$

Subtracting these two equations gives

$$|\mathbf{u}+\mathbf{v}|^2 - |\mathbf{u}-\mathbf{v}|^2 = 4\,\mathbf{u}\cdot\mathbf{v},$$

as desired.

(b) Since $\mathbf{u}$ and $\mathbf{v}$ have the same length $\|\mathbf{u}\| = \|\mathbf{v}\| = 1$, $\mathbf{u}+\mathbf{v}$ bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

We make this a unit vector:
$$\frac{\mathbf{u}+\mathbf{v}}{|\mathbf{u}+\mathbf{v}|}.$$

3.(a) From the sketch with part (b), it is clear that no matter what the angle α, the vector $\mathbf{w}_2$ has a rightward component.

(b) In the sketch we have labeled the projection we want as $\mathbf{w}_2$. We see easily that

$$|\mathbf{w}_1| = a\cos\alpha \quad \text{and} \quad |\mathbf{w}_2| = |\mathbf{w}_1|\cos\beta = a\cos\beta\cos\alpha.$$

Thus,

$$\mathbf{w}_2 = a\cos\beta\cos\alpha\langle\cos(\alpha+\beta), \sin(\alpha+\beta)\rangle.$$

The component of $\mathbf{w}_2$ in the $\hat{\mathbf{i}}$ direction is

$$a\cos\alpha\cos\beta\cos(\alpha+\beta).$$

Since α and β are between 0 and $\frac{\pi}{2}$, this component is negative exactly when

$$\alpha+\beta > \frac{\pi}{2}.$$