Vector Components

The dot product:

$$\vec{A} \cdot \vec{B} = \sum a_i b_i = |\vec{A}|\,|\vec{B}| \cos\theta$$

Applications

1. **Finding lengths and angles**

   From the dot product formula:

   $$\cos\theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}|\,|\vec{B}|}$$

2. **Detect perpendicular vectors**

   $$\vec{A} \perp \vec{B} \quad \Longleftrightarrow \quad \vec{A} \cdot \vec{B} = 0$$

3. **Component of $\vec{A}$ along direction $\hat{u}$ (unit vector)**

   Scalar component:

   $$|\vec{A}| \cos\theta = |\vec{A}|\,|\hat{u}| \cos\theta$$

   Since $|\hat{u}| = 1$:

   $$\vec{A} \cdot \hat{u}$$

Area

To find the area of a pentagon → easier to first find the area of a triangle.

Area of a Triangle Using Vectors

$$\text{Area} = \frac{1}{2}|\vec{A}|\,|\vec{B}| \sin\theta$$

We can find $\cos\theta$ first, then use:

$$\sin^2\theta + \cos^2\theta = 1$$

Using Rotation

Let $\vec{A}'$ be $\vec{A}$ rotated by 90°.

$$\theta' = \frac{\pi}{2} - \theta$$

$$\cos(\theta') = \sin(\theta)$$

Therefore,

$$|\vec{A}|\,|\vec{B}|\sin\theta = |\vec{A}|\,|\vec{B}|\cos(\theta') = \vec{A}' \cdot \vec{B}$$

Component Form

If

$$\vec{A} = (a_1, a_2) \quad \text{and} \quad \vec{B} = (b_1, b_2)$$

Then rotating $\vec{A}$ by 90° gives:

$$\vec{A}' = (-a_2, a_1)$$

Now,

$$\vec{A}' \cdot \vec{B} = (-a_2, a_1) \cdot (b_1, b_2)$$

$$= -a_2 b_1 + a_1 b_2$$

$$= a_1 b_2 - a_2 b_1$$

$$\therefore \text{Area} = \frac{1}{2}|a_1 b_2 - a_2 b_1|$$

Determinant and Area

$$\det(\vec{A}, \vec{B})$$

$$\det(\vec{A}, \vec{B}) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$\det(\vec{A}, \vec{B}) = a_1 b_2 - a_2 b_1$$

Geometric Interpretation

Area of the parallelogram formed by $\vec{A}$ and $\vec{B}$:

$$\text{Area} = |\vec{A}|\,|\vec{B}|\sin\theta = \left|\det(\vec{A},\vec{B})\right|$$

Area of the triangle:

$$\text{Area of } \triangle = \frac{1}{2}|\vec{A}|\,|\vec{B}|\sin\theta = \frac{1}{2}\left|\det(\vec{A},\vec{B})\right|$$

Determinant in Space

For three vectors $\vec{A},\vec{B},\vec{C}$:

$$\det(\vec{A},\vec{B},\vec{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Volume of the parallelepiped:

$$\text{Volume} = \left|\det(\vec{A},\vec{B},\vec{C})\right|$$

Determinant in Space

Let $\vec{A} = (a_1,a_2,a_3)$, $\vec{B} = (b_1,b_2,b_3)$, $\vec{C} = (c_1,c_2,c_3)$.

$$\det(\vec{A},\vec{B},\vec{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Expanding:

$$= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Geometrical Interpretation

$$\det(\vec{A},\vec{B},\vec{C}) = \pm\text{Volume of the parallelepiped}$$

$$\text{Volume} = \left|\det(\vec{A},\vec{B},\vec{C})\right|$$

Cross Product

Let $\vec{A}$ and $\vec{B}$ be vectors in $\mathbb{R}^3$.

$$\vec{A} \times \vec{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$