Vector Lengths

Let the vectors be

$$A = (1,\, 2), \quad B = (1,\, -7), \quad C = (1,\, 2,\, 3).$$

Find the lengths of A, $A + B$, and C.

Solution

The length of a vector is given by the Pythagorean theorem.

Length of A

$$|A| = \sqrt{1^2 + 2^2} = \sqrt{5}.$$

Vector $A + B$

$$A + B = (1 + 1,\ 2 + (-7)) = (2,\, -5).$$

$$|A + B| = \sqrt{2^2 + (-5)^2} = \sqrt{29}.$$

Length of C

$$|C| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}.$$

Force in Vectors

The diagram shows two tugs pulling a barge. One tug pulls with a force of 20000 N at an angle of 40° above the horizontal. The other tug pulls with a force F at an angle of 35° below the horizontal.

What force should the other tug pull with to keep the barge moving straight to the right?

Solution

To keep the barge moving straight, the vertical components of the forces must cancel.

$$20000 \sin 40^\circ = F \sin 35^\circ$$

Solving for F:

$$F = \frac{20000 \sin 40^\circ}{\sin 35^\circ}$$

$$F \approx 22413.32 \text{ N}$$

Proofs Using Vectors

1. The median of a triangle is a vector from a vertex to the midpoint of the opposite side. Show that the sum of the medians of a triangle is zero.

Proof

Let A, B, and C be the position vectors of the vertices of a triangle.

The median of side AB is the vector from vertex C to the midpoint of AB. Let the midpoint of AB be P.

The position vector of P is

$$\vec{OP} = \frac{1}{2}(A + B).$$

Hence, the median from C is

$$\vec{CP} = \vec{OP} - \vec{OC} = \frac{1}{2}(A + B) - C.$$

Likewise, let Q be the midpoint of AC and R be the midpoint of BC.
The median from B is

$$\vec{BQ} = \frac{1}{2}(A + C) - B,$$

and the median from A is

$$\vec{AR} = \frac{1}{2}(B + C) - A.$$

Therefore, the sum of the medians is

$$\vec{CP}+\vec{BQ}+\vec{AR} = \left(\frac{1}{2}(A+B)-C\right)+\left(\frac{1}{2}(A+C)-B\right)+\left(\frac{1}{2}(B+C)-A\right).$$

Simplifying,

$$= \frac{1}{2}(2A+2B+2C)-(A+B+C)=0.$$

Hence, the sum of the medians of a triangle is zero.

Recitation video Centriod of a triangle

$$coordinate free proofs$$

$$Show that the three medians of a triangle interests in a point \frac{2}{3} of the way from each vertex$$

$$M is the midpoint of \vec{BC}, so$$

$$\vec{OM} = \frac{1}{2}(\vec{OB}+\vec{OC})$$

$$\vec{OP} = \vec{OA}+\frac{2}{3}(\vec{AM})$$

$$\vec{AM} = \vec{OM}-\vec{OC}$$

$$\vec{AM} = \frac{1}{2}\vec{OB}+\frac{1}{2}\vec{OC}-\vec{OA}$$

$$\vec{OP} = \vec{OA}+\frac{2}{3}(\frac{1}{2}\vec{OB}+\frac{1}{2}\vec{OC}-\vec{OA})$$

$$\vec{OP} = \frac{1}{3}\vec{OA}+\frac{1}{3}\vec{OB}+\frac{1}{2}\vec{OC}$$

Example on dot product

$$1a) Complete (1,2,-4)(2,3,5)$$

$$1b) Is the angle between two vectors acute, abtuse or right$$

$$\mathbf{1.}2) suppose B = (2,2,1) suppose also that B makes an angle of 30^\circ with A and AB = 6 Find |\vec{OP}|$$

$$\mathbf{1.}3) if AB = 0 What is the angle between A and B?$$

$$\mathbf{A} = (4, 2, -4), \quad \mathbf{B} = (2, 3, 5)$$

$$\mathbf{A} \cdot \mathbf{B} = 8 + 6 - 20 = -6$$

The angle between the vectors is obtuse.

$$|\mathbf{B}| = \sqrt{38}$$

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}| \cos\theta$$

$$\theta = \frac{\pi}{2} \text{ or } 90^\circ$$

Uses of Dot Product

1. Find the angle between the vectors

$$\vec{a} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}, \quad \vec{b} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$$

Solution:

$$\vec{a} \cdot \vec{b} = (1)(2) + (1)(-1) + (2)(1) = 2 - 1 + 2 = 3$$

Geometrically,

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$$

$$|\vec{a}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}, \quad |\vec{b}| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$$

$$3 = \sqrt{6} \cdot \sqrt{6} \cos\theta$$

$$3 = 6\cos\theta \quad \Rightarrow \quad \cos\theta = \frac{1}{2}$$

$$\therefore\ \theta = \frac{\pi}{3}$$

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2(a). Show that the vectors $(1, 3)$ and $(-2, 2)$ are orthogonal

$$(1,3) \cdot (-2,2) = (1)(-2) + (3)(2) = -2 + 6 = 0$$

$$\therefore \text{ the vectors are orthogonal.}$$

—

2(b). For what value of a are the vectors $(1, a)$ and $(2, 3)$ at right angles?

$$(1, a) \cdot (2, 3) = 0$$

$$2 + 3a = 0 \quad \Rightarrow \quad a = -\frac{2}{3}$$

—

2(c).

In the figure, the vectors $\vec{A}$ and $\vec{B}_1$ are orthogonal, as are $\vec{A}$ and $\vec{B}_2$. If all vectors have the same length and

$$\vec{A} = (a, a),$$

find the coordinates of $\vec{B}_1$ and $\vec{B}_2$.

Answer

Vectors are orthogonal if their dot product is zero. So, taking the dot product,

$$(1,3) \cdot (-2,3) = -2 + 6 = 4 \neq 0$$

$$\therefore \text{ the vectors are not orthogonal.}$$

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(b) Setting the dot product to zero and solving,

$$(1, a) \cdot (2, 3) = 0$$

$$2 + 3a = 0 \quad \Rightarrow \quad a = -\frac{2}{3}$$

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(c) $\vec{B}_1$ is $\vec{A}$ rotated 90° clockwise. We will show that

$$\vec{B}_1 = (a_2, -a_1)$$

It is easy to check that

$$|(a_2, -a_1)| = |\vec{A}| \quad \text{and} \quad (a_2, -a_1) \cdot \vec{A} = 0$$

The figure above shows that putting a negative sign on the a_1 term means the vector is turned clockwise from $\vec{A}$. Thus, $\vec{B}_1$ is $\vec{A}$ rotated 90° clockwise.

Similarly, for $\vec{B}_2$, we find

$$\vec{B}_2 = (-a_2, a_1)$$