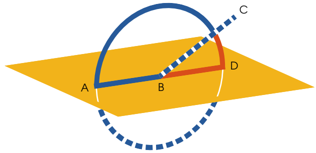
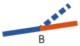


A PART *of a straight line cannot be in the plane of reference and a part in a plane more elevated.*

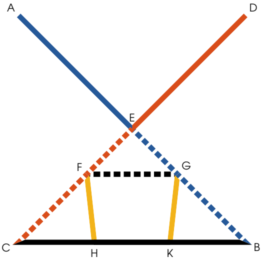
For, if possible, let a part  of the straight line  be in the plane of reference , and a part  be in a plane more elevated.

There will then be in  some straight line continuous with  in a straight line. Let it be  .

Therefore  is a common segment of the two straight lines  and  , which is impossible, inasmuch as, if we describe a circle  with centre  and radius  , then the diameters will cut off unequal circumferences  and  of the circle.

Therefore, a part of a straight line cannot be in the plane of reference and a part in a plane more elevated.

Q. E. D.



I*f two straight lines cut one another, they are in one plane, and every triangle is in one plane.*

For let the two straight lines and cut one another at the point .

I say that and are in one plane, and every triangle is in one plane.

Take the points and at random on and , let and be joined, and let and be drawn across.

I say first that the triangle lies in one plane.

For, if part of the triangle , either or , is in the plane of reference, and the rest in another, then a part also of one of the straight lines or is in the plane of reference, and a part in another.

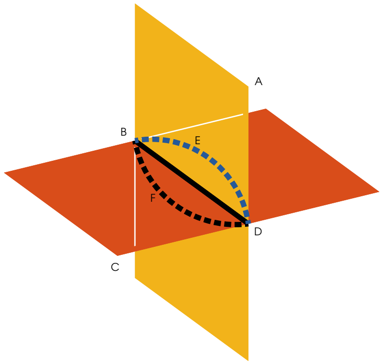
But, if the part of the triangle is in the plane of reference, and the rest in another, then a part also of both the straight lines and is in the plane of reference and a part in another, which was proved absurd. (pr. 1)

Therefore the triangle lies in one plane.

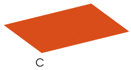
But, in whatever plane the triangle is, each of the straight lines and also is, and in whatever plane each of the straight lines  and  is,  and  also are. (pr. 1)

Therefore the straight lines  and  are in one plane; and every triangle is in one plane.

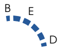
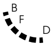
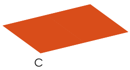
Q. E. D.

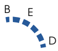
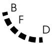


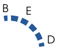
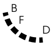
IF *two planes cut one another, their common section is a straight line.*

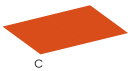
For let the two planes  and  cut one another, and let the line  be their common section.

I say that the line  is a straight line.

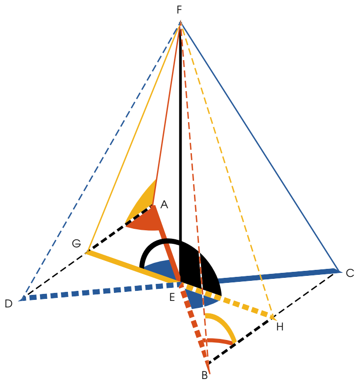
For, if not, join the straight line  from  to  in the plane , and the straight line  in the plane .

Then the two straight lines  and  have the same ends and clearly enclose an area, which is absurd.

∴  and  are not straight lines.

Similarly we can prove that neither will there be any other straight line joined from  to  except , the common section of the planes  and .

Q. E. D.



I*F a straight line be set up at right angles to two straight lines which cut one another, at their common point of section, it will also be at right angles to the plane passing through them.*

For let a straight line EF be set up at right angles to the two straight lines AB and CD at E, the point at which the lines cut one another.

I say that EF is also at right angles to the plane passing through AB and CD.

Cut off AE, EB, CE, and ED equal to one another. Draw any straight line GH across through E at random. Join AD and CB, and join FA, FG, FD, FC, FH, and FB from a point F taken at random on EF. (pr. 2 ??)

Now, since the two straight lines AE and ED equal the two straight lines CE and EB and contain equal angles, therefore the base AD equals the base CB, and the triangle DAE equals the triangle ECB, so that DAE = EBC. (pr. ?? ??)

But GAE = EBH, ∴ GAE and EBH are two triangles which have two angles equal to two angles respectively, and one side equal to one side, namely that adjacent to the equal angles, that is to say, AE = EB. Therefore they also have the remaining sides equals to the remaining sides, that is, GE = EH, and AG = BH. (pr. ?? ??)

And, since AE = EB, while FE is common and at right angles, therefore the base FA equals the base FB.

For the same reason, FC = FD. (pr. ??)

And, since AD = CB, and FA = FB, the two sides FA and AD equal the two sides FB and BC respectively, and the base FD was proved equal to the the base FC, therefore the angle FAD also equals the angle FCB. (pr. ??)

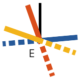
And since, again, AG was proved equal to BH, and further, FA = FB, the two sides FA and AG equal the two sides FB and BH, and the angle FAD was proved equal to the angle FCB, therefore the base FG = FH. (pr. ??)

Again, since GE was proved equal to EH, and EF is common, the two sides GE and EF equal the two sides HE and EF, and the base FG equals the base FH, ∴ FGE = FEH. (pr. ??)

Therefore each of the angles FGE and FEH is right.

∴ 

is at right angles to 

drawn at random through 

.

Similarly we can prove that 

also makes right angles with all the straight lines which meet it and are in the plane of reference.

But a straight line is at right angles to a plane when it makes right angles with all the straight lines which meet it and are in that same plane, therefore 

is at right angles to the plane of reference. (def. ??)

But the plane of reference is the plane through the straight lines 

and 

.

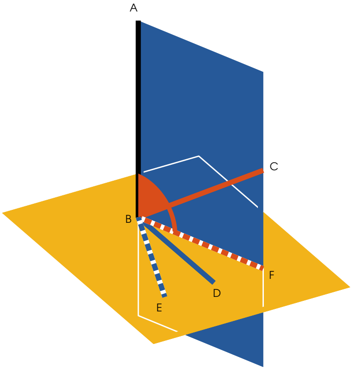
Therefore 

is at right angles to the plane through 

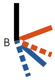
and 

.

Q. E. D.



If a straight line be set up at right angles to three straight lines which meet one another, at their common point of section, the three straight are in one plane.

For let a straight line AB be set up at right angles to the three straight lines BC, BD and BE at their point of meeting at .

I say that BC, BD, and BE are in one plane.

For suppose that they are not, but, if possible, let BD and BE be in the plane of reference  and BC in one more elevated. Produce the plane  through AB and BC. (pr. 3)

 intersects  in a straight line. Let the intersection be BF. Therefore the three straight lines AB, BC, and BF are in one plane , namely that drawn through AB and BC.

Now, since AB is at right angles to each of the straight lines BD and BE, therefore AB is also at right angles to the plane  through BD and BE. (pr. 4)

But the plane  through 

and 

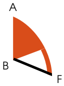
is the plane of reference, therefore 

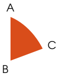
is at right angles to the plane of reference.

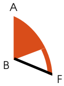
Thus 

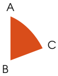
also makes right angles with all the straight lines which meet it and lie in the plane of reference. (def. ??)

But 

, which is the plane of reference, meets it, therefore the angle 

is right. And, by hypothesis, the angle 

is also right, therefore the angle 

= 

, and they lie in one plane, which is impossible.

Therefore the straight line 

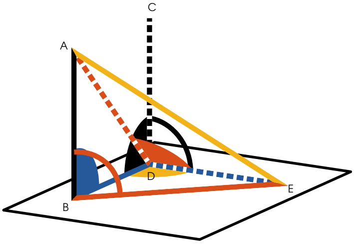
is not in a more elevated plane. Therefore the three straight lines 

, 

, and 

are in one plane.

Q. E. D.



IF *two straight lines be at right angles to the same plane, the straight lines are parallel.*

For let the two straight lines AB and CD be at right angles to the plane of reference.

I say that AB ∥ CD.

Let them meet the plane of reference at the points B and D.

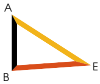
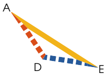
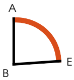
Join the straight line BD. Draw DE in ⊥ BD, and make DE = AB. (pr. ??, pr. ??)

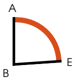
Now, since AB is at right angles to , it also makes right angles with all the straight lines which meet it and lie in the plane of reference. (def. ??)

But each of the straight lines BD and BE lies in and meets AB, therefore each of the angles ∠ABD and ∠ABE is right. For the same reason each of the angles ∠CDB and ∠CDE is also right.

And since in △ABD and △BDE AB = DE, and BD is common, therefore the two sides AB and BD equal the two sides ED and DB. And they include right angles ∠ABD and

, therefore the base  = . (pr. ??)

And, since in  and   =  while  = , the two sides  and  equal the two sides  and , and  is their common base, therefore the angle  equals the angle . (pr. ??)

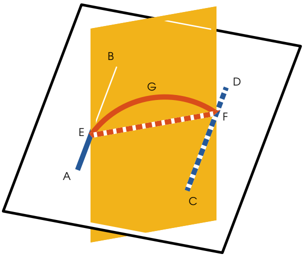
But the angle  is right, therefore the angle  is also right. Therefore  is at right angles to .

But it is also at right angles to each of the straight lines  and , therefore  is set up at right angles to the three straight lines , , and  at their intersection. Therefore the three straight lines , , and  lie in one plane. (pr. 5)

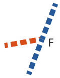
But in whatever plane  and  lie,  also lies, for every triangle lies in one plane. (pr. 2)

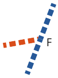
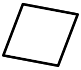
Therefore the straight lines , , and  are in one plane. And each of the angles  and  is right, therefore  is parallel to . (pr. ??)

Q. E. D.

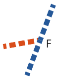
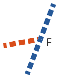


If two straight lines are parallel and points be taken at random on each of them, the straight line joining the points is in the same plane with the parallel straight lines.

Let  and  be two parallel straight lines, and let points  and  be taken at random on them respectively.

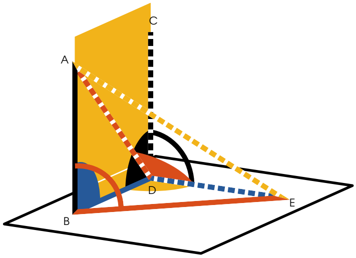
I say that the straight line joining the points  and  lies in the same plane  with the parallel straight lines.

For suppose it is not, but, if possible, let it be in a more elevated plane. Draw a plane  through . Its intersection with the plane of reference is a straight line. Let it be . (pr. 3)

∴ the two straight lines  and  enclose an area, which is impossible. ∴ the straight line joined from  to  is not in a plane more elevated. ∴ the straight line joined from  to  lies in the plane through the parallel straight lines  and .

∴, if two straight lines are parallel and points are taken at random on each of them, then the straight line joining the points is in the same plane with the parallel straight lines.

Q. E. D.



I*F two straight lines be parallel, and one of them be at right angles to any plane, then the remaining one will also be at right angles to the same plane.*

Let AB and CD be two parallel straight lines, and let one of them, AB, be at right angles to the plane of reference.

I say that the remaining one, CD, is also at right angles to the same plane.

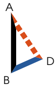
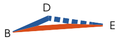
Let AB and CD meet the plane of reference at the points B and D. Join BD. Then AB, CD, and BD lie in one plane. (pr. 7)

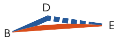
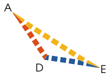
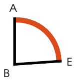
Draw DE in ⊥ BD, make DE = AB, and join BE, AE, and AD. (pr. ??, pr. ??)

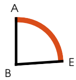
Now, since AB ⊥, therefore AB is also at right angles to all the straight lines which meet it and lie in. Therefore each of the angles ABD and ABE is right. (def. ??)

And, since the straight line BD falls on the parallels AB and CD, ∴ ABD + CDB.

But the angle ABD is right, therefore the angle CDB is also right. ∴ CD ⊥ BD. (pr. ??)

And since in  and  AB = DE, and BD is common, the two sides AB and BD equal the two sides ED and DB, and  = , for each is right, therefore the base AD = BE. (pr. ??)

And since in  and  AB = DE, and BE = AD, the two sides AB and BE equal the two sides ED and DA respectively, and AE is their common base, ∴  = . (pr. ??)

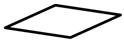
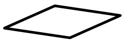
But the angle  is right, therefore the angle  is also right. ∴ ED ⊥ AD. But it is also ⊥ DB. ∴ ED is also at right angles to the plane through BD and DA. (pr. 4)

∴ ED also makes right angles with all the straight lines which meet it and lie in the plane through BD and DA. But DC lies in the plane  through BD and DA inasmuch as AB and BD lie in the plane  through BD and

DA, and DC also lies in the plane 

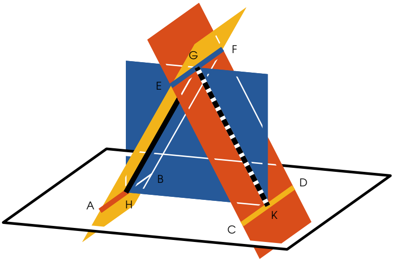
in which AB and BD lie.

∴ ED ⊥ DC, so that CD ⊥ DE. But CD ⊥ BD. ∴ CD is set up at right angles to the two straight lines DE and DB so that CD is also at right angles to the plane through DE and DB. (pr. 4)

But the plane through DE and DB is the plane of reference , ∴ CD ⊥ .

Therefore, if two straight lines are parallel, and one of them is at right angles to any plane, then the remaining one is also at right angles to the same

Q. E. D.



STRAIGHT *lines which are parallel to the same straight line and are not in the same plane with it are also parallel to one another.*

For let each of the straight lines AB, CD be parallel to EF, not being in the same plane  with it. I say that AB ∥ CD.

For let a point  be taken at random on EF, and from it let there be drawn GH, in the plane  through EF, AB, at right angles to EF, and GK in the plane  through FE, CD again at right angles to EF.

Now, since EF is at right angles to each of the straight lines GH, GK, ∴ EF ⊥ to the plane HK through GH, GK. (pr. 4)

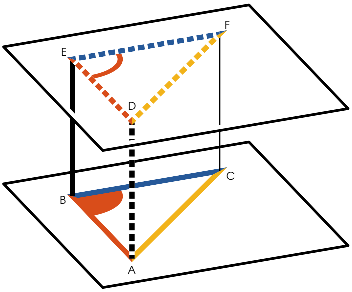
And EF ∥ AB, ∴ AB ⊥ HK. (pr. 8)

For the same reason CD ⊥ HK. ∴ each of the straight lines AB, CD ⊥ HK.

But, if two straight lines be at right angles to the same plane, the straight lines are parallel (pr. 6).

Therefore AB ∥ CD.

Q. E. D.



IF *two straight lines meeting one another be parallel to two straight lines meeting one another not in the same plane, they will contain equal angles.*

For let the two straight lines AB, BC meeting one another be parallel to the two straight lines DE, EF meeting one another, not in the same plane. I say that the angle BAC is equal to the angle EDF.

For let BA, BC, ED, EF be cut off equal to one another, and let AD, CF, BE, AC, DF be joined.

Now, since BA is equal and parallel to ED, therefore AD is also equal and parallel to BE. (pr. ??)

For the same reason CF is also equal and parallel to BE.

Therefore each of the straight lines AD, CF is equal and parallel to BE.

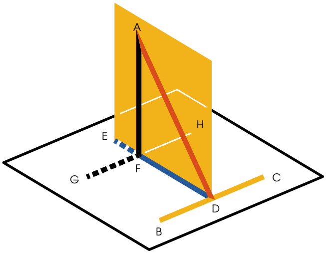
But straight lines which are parallel to the same straight line and are not in the same plane with it are parallel to one another. (pr. 9)

Therefore AD is parallel and equal to CF.

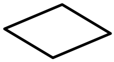
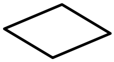
And AC, DF join them. Therefore AC is also equal and parallel to DF. (pr. ??)

Now, since in BAC and EDF AB = DE, BC = EF and AC = DF, ∴ BAC = EDF. (pr. ??)

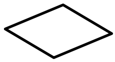
Q. E. D.

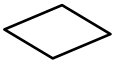


F ROM *a given elevated point to draw a straight line perpendicular to a given plane.*

Let A be the given elevated point, and the plane of reference the given plane . Thus it is required to draw from A a straight line perpendicular to .

Let any straight line BC be drawn, at random, in the plane of reference, and let AD be drawn from the point A perpendicular to BC. (pr. ??)

If then AD ⊥ , that which was enjoined will have been done.

But, if not, let DE be drawn from the point D at right angles to BC and in  (pr. ??), let AF be drawn from A perpendicular to DE (pr. ??), and let GH be drawn through the point F parallel to BC. (pr. ??)

Now, since BC is at right angles to each of the straight lines DA, DE, therefore BC is also at right angles to the plane  through ED, DA. (pr. 4)

And  is parallel to it; but, if two straight lines be parallel, and one of them be at right angles to any plane, the remaining one will also be at right angles to the same plane. (pr. 8)

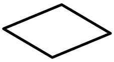
∴  ⊥ 

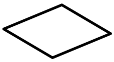
Therefore  is also at right angles to all the straight lines which meet it and are in  . (def. ??)

But  meets it and is in the plane through , ∴  ⊥ , so that  ⊥ .

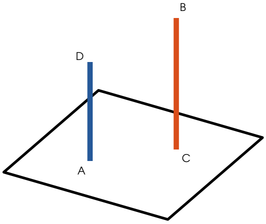
But  ⊥ , ∴  ⊥  and .

But, if a straight line be set up at right angles to two straight lines which cut one another, at the point of section, it will also be at right angles to the plane through them. (pr. 4)

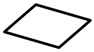
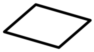
∴  ⊥  .

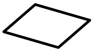
Therefore from the given elevated point  the straight line  has been drawn perpendicular to the plane of reference  .

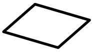
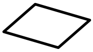
Q. E. F.

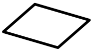


To set up a straight line at right angles to a given plane from a given point in it.

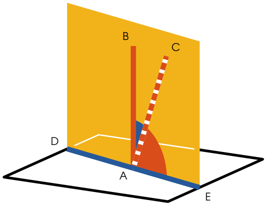
Let the plane of reference be the given plane  , and A the point in it. Thus it is required to set up from A a straight line at right angles to the  .

Let any elevated point B be conceived, from B let BC be drawn perpendicular to  (pr. 11), and through the point A let AD be drawn parallel to BC. (pr. ??)

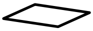
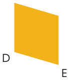
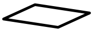
Then, since AD ∥ CB, while one of them, BC ⊥  , ∴ AD ⊥  . (pr. 8)

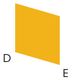
Therefore AD has been set up at right angles to the given plane  from the point A in it.

Q. E. F.



FROM *the same point two straight lines cannot be set up at right angles to the same plane on the same side.*

For, if possible, from the same point  let the two straight lines ,  be set up at right angles to the plane of reference  and on the same side, and let a plane  be drawn through , . It will then make, as section through  in , a straight line. (pr. 3)

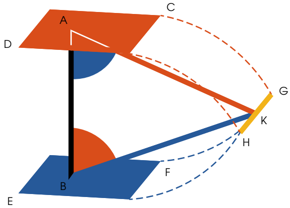
Let it make ; therefore the straight lines , ,  are in one plane .

And, since  ⊥ , it will also make right angles with all the straight lines which meet it and are in the plane of reference. (def. ??)

But  meets it and is in ; therefore the angle  is right.

For the same reason the angle  is also right; ∴  = . And they are in one plane: which is impossible.

Q. E. D.



PLANES *to which the same straight line is at right angles will be parallel.*

For let any straight line A B be at right angles to each of the planes D C, E F. I say that the planes are parallel.

For, if not, they will meet when produced.

Let them meet; they will then make, as common section, a straight line. (pr. 3)

Let them make G H. Let a point K be taken at random on G H, and let A K, B K be joined.

Now, since A B ⊥ E F, ∴ A B ⊥ B K, which is a straight line in E F produced. (def. ??)

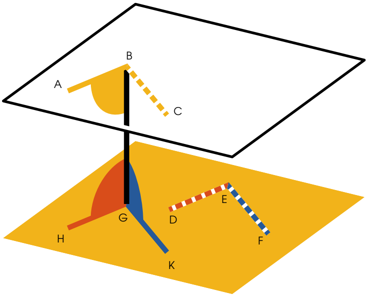
∴ A B K is right.

For the same reason B A K is also right.

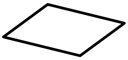
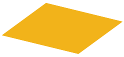
Thus, in the triangle A B K, A B K + B A K = two right angles, which is impossible. (pr. ??)

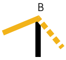
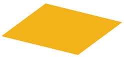
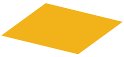
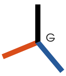
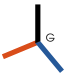
Therefore the planes D C, E F will not meet when produced. ∴ D C ∥ E F (def. ??)

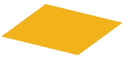
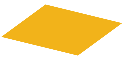
Q. E. D.

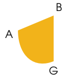
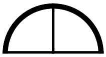


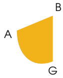
If two straight lines meeting one another be parallel to two straight lines meeting one another, not being in the same plane, the planes through them are parallel.

For let the two straight lines AB, BC meeting one another be parallel to the two straight lines DE, EF meeting one another, not being in the same plane. I say that the planes ,  produced through AB, BC and DE, EF will not meet one another.

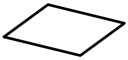
For let BG be drawn from the point  perpendicular to the . (pr. 11) And let it meet  at the point . Through  let be drawn GH ∥ ED, and GK ∥ EF. (pr. ??)

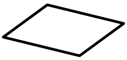
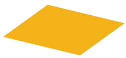
Now, since BG ⊥ , therefore it will also make right angles with all the straight lines which meet it and are in . (def. ??) ∴  =  = .

And, since  ∥  (pr. 9), ∴  +  = . (pr. ??)

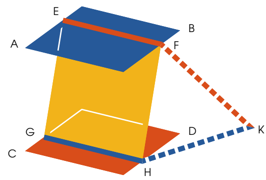
But  = , ∴  = , ∴  ⊥ .

For the same reason  ⊥ .

Since then the straight line  is set up at right angles to the two straight lines ,  which cut one another, ∴  ⊥ . (pr. 4)

But planes to which the same straight line is at right angles are parallel (pr. 14), ∴  ∥ 

Q. E. D.



IF *two parallel planes be cut by any plane, their common sections are parallel.*

For let the two parallel planes AB, CD be cut by the plane EFGH, and let EF, GH be their common sections. I say that EF ∥ GH.

For, if not, EF, GH will, when produced, meet either in the direction of F, H or of E, G.

Let them be produced, as in the direction of F, H, and let them, first, meet at K.

Now, since EK is in AB, therefore all the points on EK are also in AB. (pr. 1)

But K is one of the points on the straight line EK, therefore K is in AB.

For the same reason K is also in CD, therefore the planes AB, CD will meet when produced.

But they do not meet, ∵ they are parallel. (hyp.)

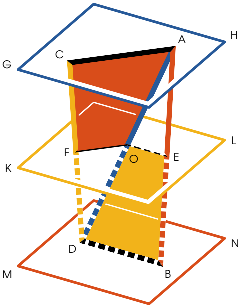
Therefore the straight lines EF, GH will not meet when produced in the direction of F, H.

Similarly we can prove that neither will the straight lines EF, GH meet when produced in the direction of E, G.

But straight lines which do not meet in either direction are parallel. (def. ??)

∴ EF ∥ GH.

Q. E. D.



I*f two straight lines be cut by parallel planes, they will be cut in the same ratios.*

For let the two straight lines AB, CD be cut by the parallel planes GH, KL, MN at the points A, E, B and C, F, D.

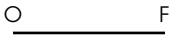
I say that, AE : EB :: CF : FD.

For let AC, BD, AD be joined, let AD meet the KL at the point O, and let EO, OF be joined.

Now, since the two parallel planes KL, MN are cut by the plane OEDB, their common sections EO, BD are parallel. (pr. 16)

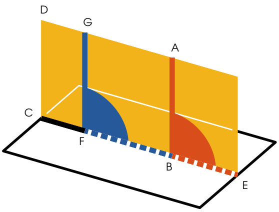
For the same reason, since the two GH, KL are cut by the plane CAFO, their common sections AC, OF are parallel. (pr. 16)

And, since EO ∥ BD, one of the sides of the triangle ADB, therefore, proportionally, AE : EB :: AO : OD. (pr. ??)

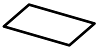
Again, since  ∥ , one of the sides of the triangle , proportionally,  :  ::  . (pr. ??)

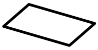
But it was also proved that,  :  ::  : , ∴  :  ::  : .

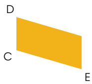
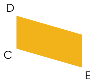
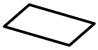
Q. E. D.

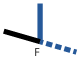
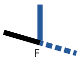
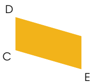


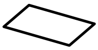
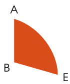
I*f two straight lines be cut by parallel planes, they will be cut in the same ratios.*

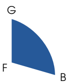
For let any straight line  be at right angles to the plane of reference  .

I say that all the planes through  are also at right angles to  .

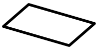
For let  be drawn through , let  be the common section of  and  .

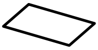
Let a point  be taken at random on , and from  let  ⊥  be drawn in  . (pr. ??)

Now, since  ⊥ ,  is also at right angles to all the straight lines which meet it and are in the plane of reference. (def. ??) So that it is also at right angles to ; ∴  = .

But  = , ∴  ∥ . (pr. ??)

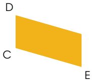
But 

⊥  , ∴ 

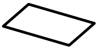
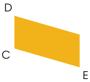
⊥  . (pr. 8)

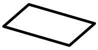
Now a plane is at right angles to a plane, when the straight lines drawn, in one of the planes, at right angles to the common section of the planes are at right angles to the remaining plane. (def. ??)

And 

, drawn in one of the planes 

at right angles to 

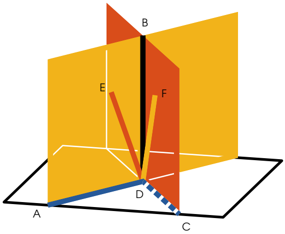
, the common section of the planes, was proved to be at right angles to  . ∴ 

⊥  .

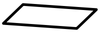
Similarly also it can be proved that all the planes through 

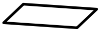
are at right angles to the plane of reference.

Q. E. D.



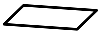
*I*F *two planes which cut one another be at right angles to any plane, their common section will also be at right angles to the same plane.*

For let the two planes  ,  be at right angles to the plane of reference  , and let  be their common section.

I say that  ⊥  .

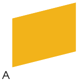
For suppose it is not, and from the point  let  be drawn in  at right angles to the straight line  , and  in  at right angles to  .

Now, since  ⊥  , and  has been drawn in the  at right angles to  , their common section, ∴  ⊥  . (def. ??)

Similarly we can prove that  ⊥  .

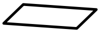
Therefore from the same point  two straight lines have been set up at right angles to the plane of reference on the same side: which is impossible. (pr. 13)

Therefore no straight line except the common section 

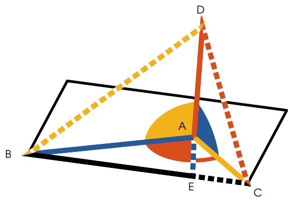
of 

, 

can be set up from the point 

at right angles to  .

Q. E. D.



If a solid angle be contained by three plane angles, any two, taken together in any manner, are greater than the remaining one.

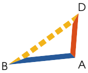
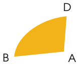
For let the solid angle at be contained by the three plane angles , , .

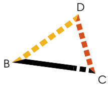
I say that any two of the angles , , , taken together in any manner, are greater than the remaining one.

If now = = , it is manifest that any two are greater than the remaining one.

But, if not, let be greater, and on the straight line , and at the point on it, let the angle be constructed, in the plane , equal to the angle .

Let be made equal to , and let , drawn across through the point , cut the straight lines , at the points , ; let , be joined.

Now, since in  and   = , and  is common, two sides are equal to two sides; and  = , ∴  = . (pr. ??)

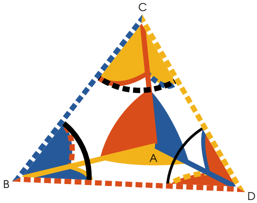
And, since in  the two sides ,  are greater than  (pr. ??), and of these  was proved equal to , therefore the remainder  is greater than the remainder .

Now, since in  and   = , and  is common, and the base  is greater than the base , ∴  > . (pr. ??)

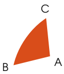
But  = , ∴ ,  are greater than .

Similarly we can prove that the remaining angles also, taken together two and two, are greater than the remaining one.

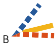
Q. E. D.

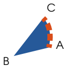
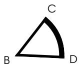


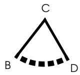
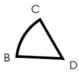
ANY *solid angle is contained by plane angles less than four right angles.*

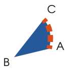
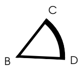
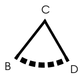
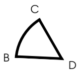
Let the angle at  be a solid angle contained by the plane angles , , .

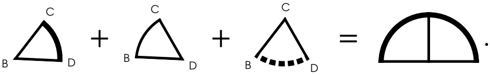
I say that the angles , ,  are less than four right angles.

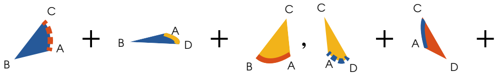
For let points , ,  be taken at random on the straight lines , ,  respectively, and let , ,  be joined.

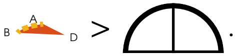
Now, since the solid angle at  is contained by the three plane angles , , , any two are greater than the remaining one. (pr. 20)

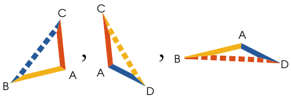
For the same reason  +  > , and  +  > .

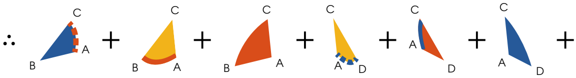
Therefore the six angles  +  + ,  +  +  >  +  + .

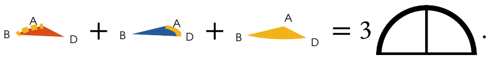
But .
(pr. ??)

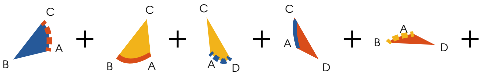
∴ 

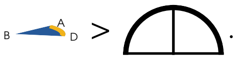
.

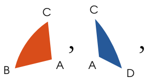
And, since the three angles of each of the triangles  are equal to two right angles,

∴ 

.

And 

.

Therefore the remaining three angles ,

 containing the solid angle  are less than four right angles.

Q. E. D.