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Spaces for the assessment of knowledge*

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The information regarding a particular field of knowledge is conceptualized as a large, specified set of questions (or problems). The *knowledge state* of an individual with respect to that domain is formalized as the subset of all the questions that this individual is capable of solving. A particularly appealing postulate on the family of all possible knowledge states is that it is closed under arbitrary unions. A family of sets satisfying this condition is called a *knowledge space*. Generalizing a theorem of Birkhoff on partial orders, we show that knowledge spaces are in a one-to-one correspondence with AND/OR graphs of a particular kind. Two types of economical representations of knowledge spaces are analysed: bases, and Hasse systems, a concept generalizing that of a Hasse diagram of a partial order. The structures analysed here provide the foundation for later work on algorithmic procedures for the assessment of knowledge.

1. Introduction: structuring a body of knowledge

In the foreseeable future, computers will take over a substantial part of the teacher's role, from kindergarten to college or even beyond. For the time being, however, even the most elaborate "computer assisted instruction" (CAI) systems would fare poorly, it must be granted, in a comparison with a competent teacher engaged in a one-to-one interaction with a student.

An important skill of a human teacher, yet lacking in the computor tutor, is the capability of efficiently assessing a student's knowledge. The procedure used to make such an assessment is typically implicit. The teacher will ask one question, then another, chosen as a function of the student's response to the first one, then still another. Quite rapidly, after only a few questions, a picture of the student's "knowledge state" will emerge. The teacher may not be able to justify the details of a particular sequence of questions asked to a student, and the assessment may not be very precise. Nevertheless, it cannot be denied that a sophisticated procedure is operating. This remarkable skill of the human teacher is a candidate for a simulation by a computer. A routine capable of providing an accurate initial assessment of a student knowledge state, and a constant updating of this assessment will have a wide applicability. For example, it will certainly be a crucial component of future CAI systems.

It is reasonable to ask whether knowledge could be assessed by adapting standard psychometric tests. Interactive versions of these tests have been worked out, under the

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label "tailored testing". (For a short introduction, see Lord, 1974.) Though such a method can render important services, the information that it provides is, in some respects at least, quite inferior to that available to a trained examiner after, say, a 45 min oral interrogation. The final result of the test is often a single number, a poor summary of the interaction that took place and of the "knowledge state" of the student.†

In this paper, we discuss a set theoretical representation of the knowledge states of individuals concerning a body of information. An effort will be made to have a general formalization, applicable in principle to most, if not all, fields. At a mature stage, this theory will provide a precise language in which efficient assessment procedures can be formulated and analysed, and more generally, issues and controversies about matters of knowledge can be translated and discussed.

A body of knowledge will be formalized as a set X of "notions". For the time being, a "notion" can be identified with a question or a problem. (A more abstract viewpoint will be taken later on.) An example is given below, which will often be used in the sequel. It involves a miniature body of knowledge with five notions labelled (a) to (e), taken from elementary probability and combinatorics:

(a) Let p be the probability of drawing a red ball in some urn. What is the probability of observing at least one red ball in a random sample of n balls, if the sampling is done with replacement? Give the formula.
(b) What is the probability of the joint realization of n independent events, each of which has a probability equal to p?
(c) Give the formula for the binomial coefficient $\binom{n}{k}$. Perform the computation for $n=7$ and $k=5$.
(d) In the experiment of Problem (a), what is the probability of observing exactly k red balls? Give the formula, and perform the computation for $n=5$ and $k=3$.
(e) Let $P(\mathrm{A})$ be the probability of an event A in a probability space. What is the probability that A is not realized in one trial?

Consider the problem of assessing, without asking redundant questions, the knowledge of individuals in a set X of notions. One of the first formalizations that comes to mind leads to the introduction of a "surmise" relation $S \subseteq X \times X$, with the following interpretation of the notation ySx: from observing a correct response to question x, it can be surmised that a correct response would also be given to question y; thus, this question should not be asked. More precisely, it is tempting to postulate that the relation S is actually a quasi order (reflexive, transitive) on X. In the case of the set $X=\{a, b, c, d, e\}$, for example, it may seem reasonable to assume that the surmise relation S is the partial order (reflexive, transitive, antisymmetric) represented by the Hasse diagram of Fig. 1. According to this representation, there are eleven possible states of knowledge, which are contained in the family

$$\begin{aligned}\kappa = \{&\varnothing, \{e\}, \{c\}, \{e, c\}, \{e, c, d\}, \{e, b\}, \{e, b, c\},\\ &\{e, b, d, c\}, \{e, b, a\}, \{e, b, a, c\}, \{e, c, d, b, a\}\}.\end{aligned} \tag{1.1}$$

The state $\{e, b, a\}$ could be identified, for example, by first proposing Problem c and observing that the student fails to solve it, and next verifying that b, and then a, are

† In multidimensional versions of these methods, the state of an individual is represented as a real vector. However, even this more general model may give a distorted representation of an individual knowledge state, which may result in costly erroneous decisions.

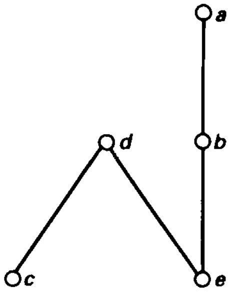


FIG. 1. Hasse diagram of the partial order (surmise) relation S in the case of the set $X = \{a, b, c, d, e\}$.

solved. This state is thus identified by three questions. A number of assessment procedures could be constructed along these lines.

We must point out that the surmise relation may capture more than the logical dependence between the notions. Writing "ySx" may mean that notion y logically precedes notion x, or more loosely, that the mastery of y can be surmised from that of x. In general, observing a correct response to x indicates that the knowledge state of the individual includes the set

$$Sx = \{y \mid ySx\}$$

of all the (surmised) antecedents of x.

The idea of formalizing the structure of a body of knowledge by a quasi order has an appealing simplicity, and is certainly applicable in some cases (as, for example, in the system developed by Misha Pavel at New York University, for the assessment of the mastery of UNIX‡). As a general model, however, it is too strong. A serious difficulty lies in the assumption that any notion has a unique set of antecedents. This is not realistic. An example can be given in the case of the set $\{a, b, c, d, e\}$ considered above. The set

$$Sa = \{x \mid xSa\} = \{e, b\}$$

contains all the antecedents of a. A straightforward method of solving Problem a is to use successively notions e, b and e again and compute

$$1-(1-p)^n. \tag{1.2}$$

There is another way of solving this problem. In Parzen's (1960) probability textbook, these five notions are introduced in the order

$$e < c < d < a < b. \tag{1.3}$$

A student of Parzen may solve Problem a through a special case of the binomial distribution, that is, by using successively c, d, and e and computing

$$1-\binom{n}{0}(1-p)^n. \tag{1.4}$$

This may strike the reader as an unlikely method. However, as indicated by the order (1.3), a student using Parzen's text is introduced to the concept of independence only

‡ UNIX is a trademark of Bell Laboratories.

after being given the formula for the binomial distribution, in the context of sampling with replacement. There may thus be a moment in the student's learning at which formula (1.4) may be the only method available. This suggests that Problem a has two possible sets of antecedents:

$$Sa = \{e, b\}$$

and

$$S'a = \{e, c, d\}.$$

In other words, the knowledge state of a student solving Problem a must include either Sa or $S'a$. The family κ should thus be augmented by at least one state, forming the family

$$\kappa' = \kappa \cup \{e, c, d, a\}. \tag{1.5}$$

However, κ' cannot be specified by a quasi order. Observe that the family κ is closed under union and intersection. This is an instance of a general situation, which we describe in Theorem 1.1.

1.1. THEOREM (BIRKHOFF, 1937)

For any set X, the formula

$$yQx \quad \text{iff} \quad (x \in A \text{ implies } y \in A, \text{ for all } A \in \varphi) \tag{1.6}$$

defines a one-to-one mapping r of the set of all families φ of subsets of X closed under intersection and union, to the set of all quasi orders Q on X.

Thus, a family of states is completely specified by a quasi order exactly when it is closed under union and intersection. The family κ' of formula (1.5) does not satisfy this condition since it does not contain the state

$$\{e, a\} = \{e, c, d, a\} \cap \{e, b, a\}.$$

Using formula (1.6), this family generates the partial order $r(\kappa')$ defined by the Hasse diagram in Fig. 2. However, other families are also consistent with $r(\kappa')$, such as that obtained by "closing" κ' for union and intersection.

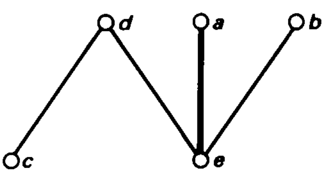


FIG. 2. Hasse diagram of the partial order $r(\kappa')$, with κ' defined by formula (1.5).

The situation in which a notion has several sets of antecedents is by no means exceptional. For another example, consider the problem of solving a system of linear equations. If a solution is provided by a student, we can infer either that the student is conversant with the notion of a determinant, and is familiar with Cramer's method; or is capable of manipulating equations, as in Gauss's method; or knows how to subtract, multiply and compute inverses of matrices. We venture that this situation is in fact typical of many fields.

The assumption of a family of states closed under union and intersection, which via Birkhoff's Theorem is tantamount to assuming a unique set of antecedents for each notion, is thus too strong and will be abandoned.

We shall develop a theory based on a family of states containing at least $\emptyset$ and X, the set of notions, and closed under union. This assumption may be motivated, for example, by considering the knowledge state of a student having several teachers. Such a student may end up with a knowledge state containing exactly those notions mastered by at least one of the teachers. We are not excluding other cases. Obviously, there may be a student who has mastered notions beyond the knowledge state of any teacher, or who has failed to learn a notion in a knowledge state of a particular teacher. *Some* students, however, may eventually learn all those and only those notions in the knowledge states of all the teachers. Our assumption accounts for this possibility.

Interestingly, Birkhoff's Theorem generalizes. Our investigation encounters here a subfield of artificial intelligence called "problem solving." A fundamental tool in there is the so-called "AND/OR graph" (see e.g. Nilsson, 1971). The generalization of Birkhoff's Theorem is that every family of states closed under union is in a one-to-one correspondence with a particular AND/OR graph, satisfying certain properties. This result is stated precisely in section 3. An attempt is made in section 4 to formulate useful, stronger or weaker forms of the theory. In section 5, we investigate the problem of providing a convenient summary of the information contained in a knowledge structure (in the sense, for example, that a Hasse diagram is a convenient summary of a partial order).

The theory presented in this paper is deterministic. It is consistent with an assessment procedure in which an individual state of knowledge is obtained by a sequence of binary decisions, with no explicit provisions made for possible errors committed at any stage. (An example of such a procedure is discussed by Degreef, Doignon, Ducamp & Falmagne, 1985). This is a limitation. For real-life applications it seems reasonable to postulate the existence of a probability distribution on the set of knowledge states, and to develop appropriate stochastic procedures. This approach will be taken in subsequent publications.

In general, our line of research must be regarded as complementary to that sometimes reported under the label "intelligent tutoring systems" (ITS, see, for example, Sleeman & Brown, 1982). We do not try to generate a model explaining the details of the failures or successes of a particular student attempting to solve a particular question. As made clear in this introduction, our long-term goal is to construct an efficient procedure for obtaining a student knowledge state, within a relatively coarse, but vast framework. Ultimately, such a procedure may evolve to be the core of a more adaptive system, the articulation of which may link various ITSs.

2. Fundamental concepts

2.1. DEFINITION

A *knowledge structure* is a pair (X, κ) consisting of a set X of *questions*, and a family κ of subsets of X called *states*. For any question x, the set of all states containing x

will be denoted by κ_x and the set $[x]$ of all questions belonging to the same states as x will be called a *notion*; thus,

$$[x]=\{y|\kappa_x=\kappa_y\}.$$

It is clear that the set of all notions is a partition of the set of questions. A knowledge structure is said to be *discriminating* iff each of its notions contains exactly one question.

A knowledge structure (X, κ) is called a *(knowledge) space* if the following two conditions are satisfied:

[K1] The set X and the empty set $\varnothing$ are states;

[K2] Every union of states is a state.

Occasionally, we shall refer to κ as a *space* or a *knowledge structure on* a set X, to mean that (X, κ) is a space or a knowledge structure, respectively. When X is not specified, it should be assumed that it is the union of the family κ. A space is said to be *quasi ordinal* iff

[K3] Every intersection of states is a state.

A discriminating, quasi ordinal space is a *partially ordinal* space. These terms will be justified in a moment. A knowledge structure satisfying [K3] will sometimes be called *$\cap$-stable.*

In our introductory section, two organizing principles were considered for a body of knowledge. One is that of a binary relation S, with the following interpretation:

$$ySx$$

means that if a student solves question x, it can be surmised that question y can also be solved by that student. The other organizing principle is that of a family of states. Our main goal in this section is to spell out precisely the correspondence between these two points of view. This will lead us to another formulation of Birkhoff's Theorem which, as far as we know, is due to Monjardet (1970). Some preliminary definitions are needed.

2.2. DEFINITION

Let X be a set (of questions), and let S be a binary relation on X. A set $K \subseteq X$ is called a *state of* (X, S) iff

$$ySx,\, x\in K \quad \text{imply}\; y\in K \tag{2.1}$$

for any x, $y\in X$. Thus, to any pair (X, S) corresponds a knowledge structure (X, κ), the states in κ being exactly those states of (X, S) defined by condition (2.1). We shall say in this situation that (X, κ) is the knowlege structure *derived from* (X, S), or more simply *derived from* the relation S.

A pair (X, S) is a *quasi ordered set*, or equivalently, S is a *quasi order on* X, iff S is a reflexive and transitive relation on X. A quasi ordered set (X, S) is a *partially ordered set* iff S is antisymmetric on X. In such a case, we also say that S is a *partial order on* X (cf. e.g. Roberts, 1979). The following result is immediate.

2.3. PROPOSITION

A knowledge structure derived from any relation is a quasi ordinal space.

It turns out that any quasi ordinal space can be derived from some quasi ordered set. This is a consequence of a result of Birkhoff (1937) (see also Monjardet, 1970), which is stated in Theorem 2.8.

2.4. DEFINITION

Let (Y, Q) be a quasi ordered set, and let h be a mapping of Y into Y, satisfying for all $x, y \in Y$, the three conditions:

(i) xQy implies $h(x)Qh(y)$;
(ii) $xQh(x)$;
(iii) $h^2(x) = h(x)$.

Then h is a *closure operator on* (Y, Q). Any $x \in Y$ is called *closed* iff $h(x) = x$.

Several cases of closures operators will be important in the sequel.

2.5. EXAMPLES

1. Consider the set $\dot{R}$ of all binary relations on a set X, ordered by inclusion. Let $R \to t(R)$ be a mapping of $\dot{R}$ into $\dot{R}$, defined by

$$t(R) = \bigcup_{n=0}^{\infty} R^n,$$

in which R^0 is the identity on X, $R^1 = R$, and for $n \geq 1$, R^{n+1} denotes the nth relative product of R with itself. Thus, $t(R)$ is the smallest reflexive and transitive relation including R. Then t is a closure operator, which will be referred to as the *transitive closure* of $\dot{R}$. The closed elements are the binary relations which are quasi orders.

2. Let (X, κ) be a $\cap$-stable knowledge structure. For any $A \subseteq X$, let $\kappa(A)$ be the smallest state of (X, κ) including A. It is easy to check that the mapping $\kappa: 2^X \to 2^X$ is a closure operator on $(2^X, \subseteq)$, the closed elements of 2^X being precisely the states of κ (which justifies the double usage of the symbol κ). This example can be regarded as a generalization of Example 1, and also of the two examples below.

3. Let $\dot{\kappa}$ be the set of all knowledge structures on a set X. Notice that any intersection of knowledge spaces in $\dot{\kappa}$ is a knowledge space. For any $\kappa \in \dot{\kappa}$, let $s(\kappa)$ be the smallest space including κ. Then, s is a closure operator on $(\dot{\kappa}, \subseteq)$, and the closed elements are the spaces. The mapping s will be called the *spatial closure* on $(\dot{\kappa}, \subseteq)$.

4. With $\dot{\kappa}$ as above, we associate to each $\kappa \in \dot{\kappa}$ the smallest quasi ordinal space $q(\kappa)$ including κ. The closure operator q will be referred to as the *quasi ordinal closure* on $(\dot{\kappa}, \subseteq)$. The closed elements are the quasi ordinal spaces.

Our next definition involves a slight generalization of a standard concept (Birkhoff, 1967).

2.6. DEFINITION

Let (Y, P) and (Z, Q) be two quasi ordered sets, and let (f, g) be a pair of mappings

$$f: Y \to Z, \qquad g: Z \to Y.$$

Then (f, g) is a *Galois connection* iff the following six conditions hold for all $y, y' \in Y$ and $z, z' \in Z$:

(i) yPy' and $y'Py$ imply $f(y) = f(y')$;
(ii) zQz' and $z'Qz$ imply $g(z) = g(z')$;
(iii) yPy' implies $f(y)Q^{-1}f(y')$;
(iv) zQz' implies $g(z)P^{-1}g(z')$;
(v) $yP(g \circ f)(y)$;
(vi) $zQ(f \circ g)(z)$.

The following facts are either well known, or easily verified:

2.7. PROPOSITION

Let (Y, P), (Z, Q), f and g be as Definition 2.6. Then:

(i) *$g \circ f$ and $f \circ g$ are closure operators, respectively, on (Y, P), (Z, Q);*

(ii) *there is at most one closed element in every equivalence class of (Y, P) (respectively (Z, Q));*

(iii) *the set Y_0 of all the closed elements of Y (respectively Z_0, Z) is partially ordered by $P_0 = P \cap (Y_0 \times Y_0)$ (respectively, $Q_0 = Q \cap (Z_0 \times Z_0)$;*

(iv) $f(Y) = Z_0$; $g(Z) = Y_0$;

(v) *the restriction $f_0^{-1} f_0$ of f on Y_0 is an anti-isomorphism between (Y_0, P_0) and (Z_0, Q_0); moreover $f_0^{-1} = g_0$, where g_0 is the restriction of g on Z_0.*

Birkhoff's Theorem can be formulated as a case of Proposition 2.7 in which the sets Y and Z are, respectively, the set of all knowledge structures and the set of all binary relations on a set X. The interest of this formulation (Monjardet, 1970) is that it clarifies the special role played by the quasi orders, and the correspondence between these concepts in the framework of the general correspondence between knowledge structures and binary relations.

2.8. THEOREM

Let $\dot{\kappa}$ be the set of all knowledge structures on a set X and let $\dot{R}$ be the set of all binary relations on X, both ordered by inclusion. Let $\kappa \to r(\kappa)$ be a mapping of $\dot{\kappa}$ into $\dot{R}$, defined by

$$x' r(\kappa) x \quad \text{iff} \quad \kappa_x \subseteq \kappa_{x'}$$

for all $x, x' \in X$. For any $R \in \dot{R}$ let $k(R)$ be the knowledge structure derived from R in the sense of Definition 2.2. Thus, $R \to k(R)$ is a mapping of $\dot{R}$ into $\dot{\kappa}$. Then, the pair (r, k) is a Galois connection. Moreover, $q = k \circ r$ is the quasi ordinal closure on $\dot{\kappa}$, and $t = r \circ k$ is the transitive closure on $\dot{R}$ [cf. Examples 2.5 (4) and (1)]. The closed sets are respectively the quasi ordinal spaces in $\dot{\kappa}$, and the quasi orders in $\dot{R}$.

In this terminology, Theorem 1.1 becomes:

2.9. COROLLARY

The Galois connection (r, k) induces a one-to-one correspondence between the quasi (respectively, partial) orders and the quasi (respectively, partially) ordinal spaces on the set X.

3. Surmise systems

The assumption that the knowledge structure is a quasi ordinal space is an interesting and useful special case. In general, however, the condition of $\cap$-stability is not realistic, and should be abandoned. As argued in our introductory section, the major difficulty lies in the result that, in a quasi ordinal space, to any question corresponds a unique set of "antecedent" questions: if a correct response to some question x is observed, it can be surmised that all the questions in a specified set Sx can also be answered correctly. It seems more reasonable to suppose that a correct response to x is consistent with several sets of antecedent questions. In artificial intelligence, it is customary to formalize such a situation by an "AND/OR graph". Let us illustrate this concept in the example of section 1, involving five questions a, b, c, d and e in elementary probability. Our analysis of this example led us to surmise that if question a is solved, then either all three questions in $\{e, c, d\}$ would also be solved, or possibly the two questions in $\{e, b\}$. This inference can be coded into the logical formula

$$a \Rightarrow (e \wedge c \wedge d) \vee (e \wedge b).$$

The other non-trivial formulae summarizing this example are

$$d \Rightarrow c \wedge e$$

and

$$b \Rightarrow e.$$

These three formulae represent the information contained in the knowledge space κ' of equation (1.5). In an AND/OR graph representation of this situation, each of the five questions is a vertex. Some of these vertices are OR-vertices, others are AND-vertices. This results in the AND/OR graph of Fig. 3.

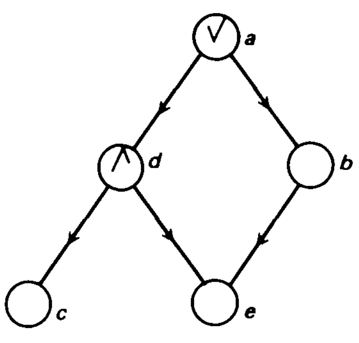


FIG. 3. AND/OR graph representation of the knowledge space κ' of equation (1.5).

It turns out that any AND/OR graph can be represented as a knowledge space (Proposition 3.8).

In some cases, using the concept of an AND/OR graph may force the addition of a number of fictitious vertices. Suppose, for example, that only the first of the three above formulae is available, namely:

$$a \Rightarrow (e \wedge c \wedge d) \vee (e \wedge b).$$

To code this formula as an AND/OR graph we would add two elements to the ground set, say α and β, corresponding, respectively, to $(e \wedge c \wedge d)$ and $(e \wedge b)$. This would lead to the AND/OR graph pictured in Fig. 4.

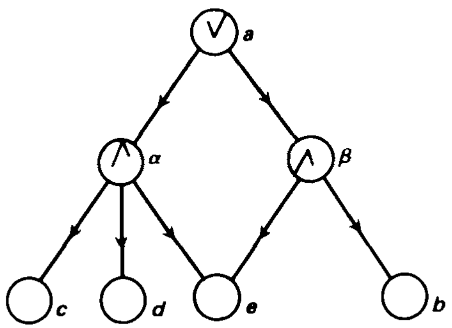


FIG. 4. AND/OR graph representation of the formula $a \Rightarrow (e \wedge c \wedge d) \vee (e \wedge b)$.

We find this machinery needlessly heavy. In the next definition, we introduce the equivalent concept of a *(disjunctive) surmise system.* The idea is that to each question x corresponds a collection $\sigma(x)$ of subsets $c_j(x)$ of questions, called the *clauses for* x. If one observes a correct response to question x, then the responses to all the questions in one of its clauses must be known. In terms of this concept, the AND/OR graph of Fig. 3 could be translated as follows:

$$\begin{aligned} \sigma(a) &= \{\{e, c, d\}, \{e, b\}\}, \\ \sigma(b) &= \{\{e\}\}, \\ \sigma(c) &= \{\varnothing\}, \\ \sigma(d) &= \{\{e, c\}\}, \\ \sigma(e) &= \{\varnothing\}. \end{aligned} \tag{3.1}$$

Other translations are possible. We could have, for example,

$$\sigma(e) = \{\varnothing, \{e\}\}$$

or

$$\sigma(e) = \{\{e\}\}.$$

Indeed, the interpretation of each of the three possibilities for $\sigma(e)$ is the same: from the single observation of a correct response to question e, nothing can be surmised but that fact. In the spirit of Birkhoff's Theorem, we shall investigate the correspondence between knowledge spaces and a particular class of (disjunctive) surmise systems.

3.1. DEFINITION

A pair (X, σ) is a *(disjunctive) surmise system* iff X is a set and $x \to \sigma(x)$ is a function which associates to each element $x \in X$ a family $\sigma(x)$ of subsets of X called the *clauses* *for* x. The function σ is called a *surmise mapping on* X. A *state of* a surmise system (X, σ) is any set $Y \subseteq X$ satisfying

$$\text{for all } y \in Y, \text{ there exists } C \in \sigma(y) \text{ such that } C \subseteq Y.$$

The set $l(\sigma)$ of all the states of (X, σ) is the knowledge structure on X *derived from* (X, σ), or equivalently, *derived from* the surmise mapping σ.

3.2. PROPOSITION

The knowledge structure derived from a surmise system is a space.

Note that such a space is not necessarily $\cap$-stable. The space derived from the surmise system defined in equations (3.1) is exactly that defined by equations (1.1) and (1.5), and we have seen that the $\cap$-stability was not satisfied.

We proceed to construct a Galois connection between the set $\dot{\kappa}$ of all knowledge structures on a set X, and the set $\dot{\sigma}$ of all surmise systems on X. The next definition introduces a quasi order on $\dot{\sigma}$.

3.3. DEFINITION

Let $\dot{\sigma}$ be the set of all surmise mappings on a set X. For any σ, $\sigma' \in \dot{\sigma}$, we write

$$\sigma' \ll \sigma$$

if for all $x \in X$ and $C \in \sigma(x)$, there is $C' \in \sigma'(x)$ such that $C' \subseteq C$.

Clearly, $\ll$ is a quasi order (but not generally a partial order), on $\dot{\sigma}$. We shall denote by

$$\approx \; = \; \ll \cap \ll^{-1}$$

the equivalence relation induced by $\ll$ on $\dot{\sigma}$.

The following concept will also be useful. For any $\varphi \subseteq 2^X$, we denote by $\hat{\varphi}$ the subfamily of φ containing all the sets in φ which are minimal for the inclusion. (Obviously, $\hat{\varphi}$ may be empty.) For example, taking $X_1 = \{x, y, z, w\}$ and $X_2 = \mathbb{R}$.

$$\varphi_1 = \{\{x, y, z\}, \{y, z\}, \{z, w\}, \{w\}\},$$

$$\varphi_2 = \{(-\infty, x) | x \in \mathbb{R}\}$$

yield, respectively,

$$\hat{\varphi}_1 = \{\{y, z\}, \{w\}\},$$

$$\hat{\varphi}_2 = \varnothing.$$

To any surmise mapping σ on X, we can associate a surmise mapping $\hat{\sigma}$ such that for all $x \in X$, $\hat{\sigma}(x)$ contains all the clauses for x which are minimal for the inclusion. Notice that we have then

$$\hat{\sigma} \gg \sigma.$$

3.4. REMARKS

1. With $X = \mathbb{R}$ and $\sigma(x) = \{(-\infty, y) | y < x\}$, we have $\hat{\sigma}(x) = \varnothing$. Hence $\hat{\sigma}$ is not equivalent to σ.

2. Two equivalent surmise systems have the same states, but the converse does not hold.

3. Any binary relation $R \subseteq X \times X$ may be regarded as a surmise system of a particular kind. Indeed, with $\sigma_R(x) = \{\{y | yRx\}\}$, the function $R \to \sigma_R$ is an injective mapping of $\dot{R}$ into $\dot{\sigma}$, satisfying

$$R \subseteq R' \quad \text{iff} \quad \sigma_R \ll \sigma_{R'}.$$

Thus, (X, σ_R) is a surmise system, the states of which coincide with the states of (X, R), as defined in section 2.2. Moreover, as a consequence of Birkhoff's Theorem, then the states of (X, σ_R) form a quasi ordinal space.

We now turn to our generalization of Birkhoff's Theorem. The particular surmise systems corresponding to knowledge spaces are introduced in the next definition.

3.5. DEFINITION

A surmise system (X, σ) is said to be *space-like* if the following three axioms are satisfied. For all $x \in X$ and $C, C' \in \sigma(x)$:

[R] $x \in C$ if $C \in \sigma(x)$;
(*each clause for* x *contains* x);

[T] for all $y \in C$ there exists $C'' \in \sigma(y)$ such that $C'' \subseteq C$;
(*each clause for* x *is a state*);

[I] ($C \subseteq C'$ or $C' \subseteq C$) implies $C = C'$;
(*any two clauses for* x *are pairwise incomparable with respect to inclusion*).

In the particular case in which a surmise system is essentially a relation (cf. section 3.4, point 3), the first two axioms correspond, respectively, to the reflexivity and transitivity of that relation. This justifies the letters chosen to denote these two axioms.

3.6. PROPOSITION

Let $(\dot{\sigma}, \ll)$ be the quasi order defined in section 3.3. Then:

(i) *in each equivalence class, there is at most one surmise system satisfying Axiom* [*I*];
(ii) *$\hat{\sigma}$ satisfies* [*I*], *for any surmise mapping* σ;
(iii) *if the set* X *is finite, then $\hat{\sigma} \approx \sigma$ for any surmise mapping* σ.

Proof: (i) Let (X, σ_1) and (X, σ_2) be two equivalent surmise systems satisfying Axiom [I]. For each $x \in X$ and each clause $C_1 \in \sigma_1(x)$, there is thus a clause $C_2 \in \sigma_2$ included in C_1. There must also be a clause $C_1' \in \sigma_1(x)$ included in C_2. We obtain $C_1' \subseteq C_1$, which implies $C_1' = C_1$, and thus $C_1 = C_2$. We conclude that any clause for x in (X, σ_1) is also a clause for x in (X, σ_2). By symmetry, we derive $\sigma_1 = \sigma_2$, establishing the required uniqueness.

Conditions (ii) and (iii) are immediate.

The main result of this section follows. The reader not interested in the strong formulation in terms of the Galois connection should skip to Corollary 3.8, and then to the comments after section 3.9.

3.7. THEOREM

Let $(\dot{\kappa}, \subseteq)$ be the partially ordered set of all knowledge structures on a finite set X; let $(\dot{\sigma}, \ll)$ be the quasi ordered set of all surmise systems on X (cf. 3.3.). Define two functions

$$\kappa \to s(\kappa) \in \dot{\sigma},$$
$$\sigma \to l(\sigma) \in \dot{\kappa},$$

respectively on $\dot{\kappa}$, $\dot{\sigma}$, as follows. For any knowledge structure $\kappa \in \dot{\kappa}$, let $s(\kappa)$ be the surmise mapping on X such that $\sigma(x)$ is the set of all minimal states of κ containing x; thus, in the notations of sections 2.1. and 3.1, $\sigma(x) = \hat{\kappa}_x$. For any surmise mapping $\sigma \in \dot{\sigma}$ let $l(\sigma)$ be the knowledge structure on X derived from (X, σ) in the sense of Definition 3.1. Then, the pair of mappings (s, l) is a Galois connection, the closed sets being respectively the knowledge spaces, and the space-like surmise systems.

Proof: We have to show that conditions (i)-(vi) of Definition 2.6, are satisfied, with $(Y, P) = (\dot{\kappa}, \subseteq)$, $(Z, Q) = (\dot{\sigma}, \ll)$, $f = s$ and $g = l$. We establish successively (iv), (ii) and (vi), and leave the three remaining conditions to the reader.

Suppose that

$$\sigma' \ll \sigma$$

and let K be a state of σ. Thus, for all $x \in K$, there exists $C \in \sigma(x)$ such that $C \subseteq K$. By definition of $\ll$, there is some $C' \in \sigma(x)$ such that $C' \subseteq C$. This implies that every state of σ is a state of σ'. That is,

$$l(\sigma) \subseteq l(\sigma')$$

and by symmetry

$$\sigma' \approx \sigma \quad \text{implies } l(\sigma') = l(\sigma).$$

Thus, (iv) and (ii) hold. To prove (vi), we observe that any clause $C \in [(s \circ l)(\sigma)](x)$ is, by definition, a minimal state of $l(\sigma)$ containing x. The clause C is thus a state of (X, σ) containing x, which implies that C includes some $C' \in \sigma(x)$. Thus, any clause $C \in [(s \circ l)(\sigma)](x)$ includes some clause $C' \in \sigma(x)$, which yields $\sigma \ll (s \circ l)(\sigma)$, the desired relation.

It remains to describe the closed sets. By Proposition 2.7 (iv), they define the ranges of s and l. For any $\kappa \in \dot{\kappa}$, $s(\kappa)$ satisfies Axioms [R] and [I] as an intermediate consequence of the definition. To prove that Axiom [T] also holds, we consider a clause $C \in [s(\kappa)](x)$ and take some $y \in C$. By definition of $s(\kappa)$, C is a state of κ, which implies, by the finiteness of X, that C includes a minimal state F of κ containing y. But then, F is necessarily a clause for y in $s(\kappa)$. This means that C is a state of $s(\kappa)$. in other words, Axiom [T] holds for $s(\kappa)$.

Conversely, assume that (X, σ) is a space-like surmise system. We shall establish that σ necessarily belongs to the range of s by proving that $\sigma = (s \circ l)(\sigma)$. Since (s, l) is a Galois connection, we already know that $\sigma \ll (s \circ l)(\sigma)$. On the other hand, if σ satisfies [T], then any clause $C \in \sigma(x)$ is a state, for any question x. This means that, for each of its elements y, C includes a clause for y in $(s \circ l)(\sigma)$. Thus, C is a state in $(X, (s \circ l)(\sigma))$, and one derives $(s \circ l)(\sigma) \ll \sigma$. Hence, σ and $(s \circ l)(\sigma)$ are equivalent. Moreover, each of these two surmise systems satisfies Axiom [I] (σ by assumption, and $(s \circ l)(\sigma)$ since it is in the range of s; see above). By Proposition 3.6 (i), we must have $\sigma = (s \circ l)(\sigma)$.

Finally, let us show that the range of l exactly consists of the knowledge spaces. Proposition 3.2 asserts that all the knowledge structures in the range of l are spaces. Conversely, let κ be a space. In view of the Galois connection, we have $\kappa \subseteq (l \circ s)(\kappa)$, and it is sufficient to show the opposite inclusion. Take $F \in (l \circ s)(\kappa)$. Since it is a state of $s(\kappa)$, we can select for each $x \in F$ a clause C_x for x in $s(\kappa)$, included in F. By definition of $s(\kappa)$, the clause C_x contains x and belongs to κ. Hence,

$$F = \bigcup_{x \in F} C_x.$$

Since κ is a space, $F \in \kappa$.

3.8. COROLLARY

There is a one-to-one correspondence between the knowledge spaces on a finite set and the space-like surmise systems on that set. Moreover, this correspondence reverses the quasi orders given by inclusion and $\ll$.

3.9. COROLLARY

In the finite case, the collection of all space-like surmise systems, ordered by $\ll$, *is a lattice anti-isomorphic to the lattice of the knowledge spaces ordered by inclusion.*

Corollary 3.8 is analogous to Corollary 2.9, stating the existence of a one-to-one correspondence between the quasi orders and the quasi ordinal spaces. It establishes a link between two superficially quite different formalizations: knowledge spaces on the one hand, and a class of structures akin to AND/OR graphs on the other hand. This result prompted us to investigate surmise systems more closely, regarding them as structures generalizing quasi orders. In particular, Corollaries 3.8 and 3.9 stress the importance of the class of surmise systems satisfying the Axioms [I], [R] and [T]. Notice that, as indicated, the space-like surmise systems in which, in addition, there is a unique clause for each question are essentially the quasi orders. Moreover, the closure operator on $(\dot{\sigma}, \ll)$ associates to each surmise system a space-like surmise system. This operator is the direct generalization of the transitive closure of a binary relation.

We mention that the existence of a surmise system corresponding to a knowledge space can be inferred from some work by Flament (1976). These results, however, were formulated in a context and using a terminology very different from ours.

4. Weakening and strengthening of the axioms

The concept of a knowledge space occupies a central place in our developments. Axiom [K2] in particular (the stability under union), seems to be a reasonable compromise between two somewhat conflicting objectives: constructing a system which is realistic enough to capture the complexity of empirical situations, and simple enough to be capable of practical applications.

Nevertheless, objections can be made. For example, Axiom [K2] may be criticized on the grounds that it may add "useless" states, that is, states which are never encountered in practice. Consider the case of two students exchanging information on a certain topic, and let K and K' be their respective states of knowledge, at the beginning of their interaction. It is certainly possible that both students may end up in a state containing not only all the questions in $K \cup K'$, but also some other questions, which the students would have derived spontaneously, so to speak, from the knowledge of all the elements in $K \cup K'$. Moreover, it is conceivable that such derivations are "automatic". That is, they always take place, no matter who the students are. This would mean of course that the state $K \cup K'$ could never occur empirically. Having this state in the knowledge structure may be regarded as needlessly complicating the picture.

At this state, we do not know how frequent this "automatic learning" may be, or even if it happens at all. In any event, our discussion suggests the following weakening of Axiom [K2]:

[K2′] For any subfamily of states φ, there exists a unique minimal state $K \in \kappa$ such that $\cup \varphi \subseteq K$.

This axiom requires that κ is a join-semilattice with respect to inclusion.

However, Axioms [K2] and [K2′] are both vulnerable to another counterexample. Consider the case of two world-class mathematicians, each of whom is an expert in a specialized subfield. It is certainly unrealistic to assume the existence, in general, of a mathematician whose knowledge equals or exceeds that of both of these experts.

The rest of this section is devoted to a discussion of various ways of strengthening the axioms defining a knowledge space. Consider the knowledge structure given in the example below.

4.1. EXAMPLES

$$\kappa = \{\varnothing, \{a\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\}\}.$$

Thus, κ is discriminating. We shall actually assume that the letters a, b, c and d represent notions. Suppose that some individual is in state $\{a\}$, and wishes to learn notion d. Since there is no "intermediate" state between $\{a\}$ and $\{a, b, d\}$ or $\{a, c, d\}$, this can only be done by mastering, simultaneously, either b and d, or c and d. This problem does not arise in the knowledge structure

$$\kappa' = \kappa \cup \{a, b\}.$$

In κ', the individual may progress from the initial state $\{a\}$ to a state containing d by steps involving only one new notion at a time. Notice that neither κ nor κ' are partially ordinal spaces. The concept illustrated by these two examples may be at the source of some pedagogical difficulties and deserves some attention.

Convention: In the sequel we shall often assume that the sets considered are finite. Any definition or result based on this assumption will be marked by the † symbol.

4.2. DEFINITION†

A knowledge structure (X, κ) is called *n-learnable* if for any state $K \in \kappa$ and any question $x \notin K$, there exists some positive integer m, and a chain of states $K_1 = K \subseteq K_2 \subseteq \cdots \subseteq K_m$, such that:

(i) $x \in K_m$;
(ii) $|K_{i+1} - K_i| \leq n$, for $1 \leq i \leq m-1$.

The *learnstep number* of a knowledge structure (X, κ) is the smallest positive integer n such that κ is n-learnable. The learnstep number of κ will be denoted by $lst(\kappa)$.

The learnstep number may reflect in part the (maximal) number of questions in some notion. (We recall that a notion has been defined as a class of equivalent questions; cf. Definition 2.1). It is thus important to sharply distinguish between these two parameters of a knowledge structure. We shall call the *plurality number* of a knowledge structure (X, κ), the largest number q such that $|[x]| = q$ for some $x \in X$. The plurality number of κ will be denoted by $plu(\kappa)$.

We omit the proof of the next Proposition.

4.3. PROPOSITION†

In a knowledge structure κ we always have

$$plu(\kappa) \leq lst(\kappa).$$

If κ is a quasi ordinal space, then $\mathrm{plu}(\kappa) = \mathrm{lst}(\kappa)$, *but the converse does not hold. In particular, the learnstep number of a partially ordinal space is equal to one.*

(But not conversely, as demonstrated by the knowledge structure κ' of Example 4.1.) Thus, we could strengthen the axioms defining a space by specifying that the learnstep number is equal to one, a requirement which might make good pedagogical sense (for a discriminating system; that is, after having formed the co-sets).

We shall also examine a possible additional axiom for space-like surmise systems. It will be seen to imply a learnstep number equal to one. Consider the surmise mapping defined in the example below.

4.4. EXAMPLE

Let σ be a surmise mapping on the set $\{a, b, c\}$, defined by the three equations

$$\sigma(a) = \{\{a, b\}, \{a, c\}\},$$

$$\sigma(b) = \{\{a, b\}, \{b, c\}\},$$

$$\sigma(c) = \{\{b, c\}, \{a, c\}\}.$$

It is easy to check that Axioms [I], [R] and [T] are satisfied. Thus, $(\{a, b, c\}, \sigma)$ is space-like. However, "cycles" can be observed. For example:

There is a clause for a containing b, namely, $\{a, b\}$;
There is a clause for b containing c, namely, $\{b, c\}$;
There is a clause for c containing a, namely, $\{a, c\}$.

The next definition formalizes this concept.

4.5. DEFINITION

The *precedence relation* R_σ of a surmise system (X, σ) is defined by

$$xR_\sigma y \quad \text{iff} \quad \text{there exists } C \in \sigma(y), \text{ such that } x \in C.$$

Whenever possible without ambiguity, we shall abbreviate R_σ as R. A surmise system is called *acyclic* iff its precedence relation is acyclic. It is clear that R is reflexive if Axiom [R] is satisfied, and in that case R is not acyclic.

The following fact will be useful.

4.6. PROPOSITION†

If (X, σ) is an acyclic surmise system, then $(X, (s \circ l)(\sigma)$ is also acyclic, with (s, l) the Galois connection defined in Theorem 3.7.

Proof: We shall use a contradiction. Let S be the precedence relation of $(s \circ l)(\sigma)$ and suppose that $x_1 S x_2$, $x_2 S x_3, \ldots, s_n S x_1$. Thus, by definition, there is a clause $C_i \in [(s \circ l)(\sigma)](x_{i+1})$ that contains x_i (with a cyclic index $i = 1, 2, \ldots, n$). Let R be the precedence relation of (X, σ). The proof proceeds by establishing the existence, for each index i, of questions $y_1^i, y_2^i, \ldots, y_{k_i}^i$ in X such that

$$x_1 R y_1^1, y_1^1 R y_2^1, \ldots, y_{k_1-1}^1 R x_2,$$

$$x_2 R y_1^2, y_1^2 R y_2^2, \ldots, y_{k_2-1}^2 R x_3,$$

$$\ldots$$

$$\ldots$$

$$x_n R y_1^n, y_1^n R y_2^n, \ldots, y_{k_n-1}^n R x_1.$$

This is a cycle of R, and the result follows. By the definition of $(s \circ l)(\sigma)$, the clause $C_i \in [(s \circ l)(\sigma)](x_{i+1})$ is a minimal state of $l(\sigma)$ among the states containing x_{i+1}. Thus, $C_i - \{x_i\}$ is not a state of $l(\sigma)$. In turn, this means that C_i is a state of σ, while $C_i - \{x_i\}$ is not. This necessarily implies the existence of some question $y_1^i \in (C_i - \{x_i\})$, such that no clause for y_1^i is included in $C_i - \{x_i\}$. Since $y_1^i \in C_i$ and C_i is a state, there exists a clause C_i for y_1^i included in C_i. We must have $x_i \in C_i$, since otherwise $C_1^i \subseteq (C_i - \{x_i\})$. If $y_1^i = x_{i+1}$, then $x_i R x_{i+1}$. Otherwise, $C_i - \{x_i, y_1^i\}$ is not a state of σ, but contains $\{x_{i+1}\}$. Using again the fact that C_i is a state of σ, we can assert the existence of y_i^2 such that either y_1^i belongs to a clause for y_2^i in σ, or x_i belongs to such a clause. In the second case, we replace y_1^i by y_2^i. This process may be pursued until, for some index k_i, $y_{k_i}^i = x_{i+1}$. Indeed, we are constructing a decreasing family of finite sets

$$D_j^i = C_i - \{y_1^i, y_2^i, \ldots, y_j^i\}$$

all of which contain x_{i+1}. Since this construction may be carried out for all indices i, $1 \leq i \leq n+1$, with $x_{n+1} = x_1$, the result follows.

4.7. REMARKS†

An examination of this proof suggests a construction of $(s \circ l)(\sigma)$ from σ which generalizes the classical construction of the transitive closure of a binary relation. Nevertheless, the example below shows that the precedence relation of $(X, (s \circ l)(\sigma))$ does not always coincide with the transitive closure of the precedence relation of (X, σ). The same example proves that the converse of Proposition 4.6 does not hold.

4.8. EXAMPLE

Let σ_1 be a surmise mapping on $X = \{a, b, c, d\}$ defined by

$$\sigma_1(a) = \{\{a\}\},$$

$$\sigma_1(b) = \{\{a, b\}\},$$

$$\sigma_1(c) = \{\{a, c\}\},$$

$$\sigma_1(d) = \{X\}.$$

The precedence relation of σ_1 is the partial order with Hasse diagram as shown in Fig. 5.

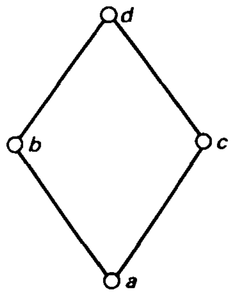


FIG. 5. Hasse diagram of the precedence relation of σ_1 in Example 4.8.

Next, consider the surmise mapping on X:

$$\sigma_2(a) = \{\{a\}\},$$
$$\sigma_2(b) = \{\{a, b\}, \{b, d\}\},$$
$$\sigma_2(c) = \{\{a, c\}\},$$
$$\sigma_2(d) = \{X\}.$$

It is easily checked that $(s \circ l)(\sigma_2) = \sigma_1$. However, the pair (d, b), which belongs to the precedence relation of σ_2 is not in the partial order. Moreover, (X, σ_1) is acyclic, while (X, σ_2) is not.

4.9. PROPOSITION

If a surmise mapping σ is acyclic, then its derived knowledge structure $l(\sigma)$ is discriminating. In general, the converse does not hold. If, however, σ has a unique clause for each question, then it is acyclic iff $l(\sigma)$ is discriminating. (In such a case, $l(\sigma)$ is thus a partially ordered space.)

Proof: Let σ be some acyclic surmise mapping, and let $\kappa = l(\sigma)$ and R_σ be its derived knowledge structure and precedence relation, respectively. In view of Proposition 4.6, we may assume that σ is space-like. [Indeed, σ and $(s \circ l)(\sigma)$ have the same derived knowledge structure.] Suppose that κ is not discriminating. Thus, there are a, b such that

$$(\forall K \in \kappa)(a \in K \Leftrightarrow b \in K). \tag{4.1}$$

Since κ is space-like, any clause for a is a state containing a. Using condition (4.1), this state also contains b, which yields

$$bR_\sigma a.$$

Similarly, any clause for b is a state containing a, and thus

$$aR_\sigma b$$

establishing that σ is not acyclic.

To see that the converse does not hold, take the surmise system of Example 4.4, which is not acyclic. Its derived knowledge structure is

$$\kappa = \{\emptyset, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\},$$

which is discriminating. The special case is clear.

We end this section by establishing a connection between acyclicity and the learnstep number.

4.10. PROPOSITION†

The knowledge structure (X, κ) derived from an acyclic surmise system (X, σ) has learnstep number equal to one.

Proof: Consider the precedence relation R of (X, σ), which by assumption is an acyclic relation. It is well known that such a relation can be embedded in a simple order on X; that is, we can list the elements of X as $x_1, x_2, \dots, x_t$ with $x_i R x_j$ implying $i<j$. Given any state K and any question x, the chain of subsets

$$K, K \cup \{x_1\}, K \cup \{x_1, x_2\}, \dots, K \cup \{x_1, x_2, \dots, x_t\}$$

includes a chain starting with K and ending with a subset containing x. Since two consecutive sets of the chain differ by at most one element, it only remains to show that for any $i = 1, 2, \dots, t$, the set

$$K_i = K \cup \{x_1, x_2, \dots, x_i\}$$

is a state. This is clear, since for any given $y \in K_i$, either some clause for y is included in K (if $y \in K$), or it is included in $\{x_1, x_2, \dots, x_i\}$ (if $y \in K_i - K$, because any clause for y is formed by elements preceding y).

5. Bases and Hasse systems

Knowledge structures encountered in practical applications may have a large number of states, and it is reasonable to consider the problem of describing such structures economically (for instance, in order to store the information in a computer's memory). In the case of a knowledge space, only some of the states must be specified, the remaining ones being generated by taking unions.

5.1. DEFINITION

Let (X, κ) be a knowledge structure. The *span* of a subfamily φ of κ is the family φ' of all sets which are unions of some states in φ. (In general, the elements of φ' need not be states.) We shall then say that φ' is *spanned by* φ. A *basis* for a knowledge space (X, κ) is a minimal family β of states spanning κ. By convention, the empty set is the union of the empty subfamily of β. Thus the empty set never belongs to a basis.

5.2. PROPOSITION

Let β be a basis for a knowledge space (X, κ). Then $\beta \subseteq \varphi$ for any subfamily φ or κ spanning κ.

5.3. COROLLARY

A knowledge space admits at most one basis.

5.4. DEFINITION

An *atom* at x in a knowledge structure (X, κ) is a minimal state containing x.

5.5. EXAMPLE

A knowledge space (X, κ) is quasi ordinal (or $\cap$-stable) iff there is exactly one atom for any question x, which is specified by $\cap \kappa_x$.

5.6. PROPOSITION

A state A is an atom in a knowledge space (X, κ) iff for any subfamily $\varphi \subseteq \kappa$, we have $A \in \rho$ whenever $A = \cup \varphi$.

5.7. PROPOSITION†

A finite knowledge space has a unique basis which is formed by all its atoms.

An infinite version of this proposition holds under an additional assumption on the knowledge space (the so-called finitary property of axiomatic convexity). We refer the interested reader to Hammer (1963) and Cohn (1965).

Returning to the considerations of economy motivating this section, we could also envisage a description of a knowledge space (X, κ) through one of the surmise systems on X whose states precisely form the family κ. In the finite case, we know by our fundamental Theorem 3.7 that this is possible with only one system satisfying axioms [I], [R], [T]. But this system is rather heavy: the number of clauses may be very large. For instance, if the space is also $\cap$-stable and discriminating, this approach leads to the corresponding partial order. In this case, a more sparing description is available, namely, the corresponding Hasse diagram. This suggests generalizing the concept of a Hasse diagram into one applicable to surmise systems (X, κ). A new axiom for surmise systems is needed.

5.8. AXIOM [M]

For each question x and each clause C for x, there is an atom at x that includes C.

5.9. PROPOSITION†

In a space-like surmise system, any clause is an atom. In particular, Axiom [M] *holds.*

5.10. DEFINITION

Let (X, σ) and (X, σ') be two surmise systems, and suppose that (X, σ')

1. has the same states as (X, σ);
2. satisfies Axiom [M];
3. is minimal with respect to $\ll$ among all those surmise systems satisfying 1 and 2.

Then (X, σ') is a *Hasse system* of (X, σ). The *Hasse systems* of a knowledge space (X, κ) are the Hasse systems of $s(\kappa)$.

5.11. REMARK†

Any surmise system (X, σ) has a Hasse system. Indeed, $(s \circ l)(\sigma)$ satisfies Axiom [M] and has the same states as σ.

5.12. EXAMPLE†

The Hasse diagram (X, H) of a partial order (X, P) is a Hasse system (regarding H and P as surmise systems with a unique clause for each question).

In fact:

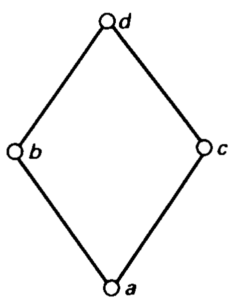
5.13. PROPOSITION†

A partial order admits a unique Hasse system, which is its Hasse diagram.

Proof: Let (X, σ) be a Hasse system of a partial order (X, P). We have to show that, for each x, $\sigma(x)$ contains exactly one clause, which is formed by the elements covered by x in the partial order. If $C \in \sigma(x)$, Axiom [M] implies cPx for all $c \in C$. Notice that $x \notin C$, by the minimality with respect to $\ll$, and that $\{x\} \cup \kappa(C)$ is a state, where $\kappa(C)$ is the smallest state containing C. Hence each question y covered by x must belong to $\{x\} \cup \kappa(C)$, and thus to $\kappa(C)$. Moreover, we must have $y \in C$ otherwise we would have $c \in C$ with yPc, cPx and $y \neq c$, $c \neq x$, contradicting the fact that y is covered by x. It follows that each clause for x contains all elements covered by x. Using the minimality of σ, we conclude that each clause for x is formed by the elements covered by x.

5.14. REMARKS

1. Axiom [M] cannot be dropped from Definition 5.10, without invalidating Proposition 5.13, as shown by the following example. Take $X = \{a, b, c, d\}$ and consider the partial order (X, P) pictured below.



We denote by δ the surmise mapping of the Hasse diagram of P, that is

$$\delta(a) = \{\varnothing\}$$

$$\delta(b) = \{\{a\}\}$$

$$\delta(c) = \{\{a\}\}$$

$$\delta(d) = \{\{b, c\}\}$$

We also introduce the surmise system (X, σ), defined by

$$\sigma(a) = \{\varnothing\}$$

$$\sigma(b) = \{\{a\}, \{c\}\}$$

$$\sigma(c) = \{\{a\}\}$$

$$\sigma(d) = \{\{b, c\}\}$$

Then δ and σ have the same states and satisfy $\sigma \ll \delta$ but not $\delta \ll \sigma$. It is thus clear that there is a minimal surmise system (X, θ) with $\theta \neq \delta$ but having the same states as δ. Notice that $\{c\}$ appears as a very unnatural clause for b and violates Axiom [M].

2. A quasi order has more than one Hasse system whenever it has an equivalence class with at least two elements and its underlying set has more than two elements.

It would be of interest to have a simple characterization of the knowledge spaces having a unique Hasse system. Notice that a nonacyclic space can have this property (see e.g. Example 4.4).

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