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Theoretical Note

How to Build a Knowledge Space by Querying an Expert*

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A particular field of knowledge is conceptualized as a set of problems (or questions). A person's *knowledge state* in this domain is formalized as the subset of problems this person is capable of solving. When the family of all knowledge states is closed under union, it is called a *knowledge space*. Doignon and Falmagne (1985) established a 1–1 correspondence between knowledge spaces and a class of *surmise systems*, a slight variant of AND/OR graphs. Here we rather obtain a 1–1 correspondence with a well defined class of quasi orders on the collection of all subsets of problems. The resulting approach to knowledge spaces helps to build such spaces for particular domains. We describe a procedure which relies on the answers of an expert to a carefully chosen sequence of information requests.

1. INTRODUCTION

A particular field of knowledge can be conceptualized as comprising a possibly large, but specified set of *notions*. The *knowledge state* of an individual in this domain can then be formalized as the subset of notions she/he has mastered. Here, a notion can be identified with a question or problem, or, rather, an equivalence class of questions or problems, testing just that notion. Doignon and Falmagne (1985) described the motivation and investigated the algebraic foundation of this approach in some detail. A number of knowledge assessment procedures based on this formalization have been developed (Falmagne and Doignon, 1988a; Falmagne and Doignon, 1988b; Falmagne, 1990; Degreef, Doignon, Ducamp, & Falmagne,

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1986). These procedures all start from a fixed knowledge structure of the domain, where a *knowledge structure* is defined as the collection of all possible knowledge states. (While any knowledge state is a subset of notions, in general not any such subset is a possible state. For instance, in the field of arithmetic, mastering long division implies mastering of subtraction, so a subset of notions (problems) containing long division but not containing subtraction is not a possible knowledge state. It is indeed this kind of restrictions that lends "structure" to a field.)

In this paper we are concerned with the problem of how to build such knowledge structures for particular domains. For this, we would not want to rely on our own, restricted knowledge, but we would rather consult an expert in the field. A straightforward approach results from presenting to the expert each of all subsets of problems and asking whether it constitutes a possible state. This seems a rather demanding task for the expert. Presented with a subset of problems, he has to decide how plausible it is that a subject has mastered this subset *and no other problem.* This means that the expert always has to consider a complete response pattern on the total set of problems, even if the presented subset has only a few elements. Besides, this approach is quite unmanageable in practice because of the sheer number of subsets.

Without any a priori assumption on the family of possible knowledge states, the above approach is the only one conceivable. If we assume a knowledge structure is such that any intersection and any union of knowledge states are again knowledge states—we call such a structure *closed* under union and intersection—, then, by a theorem of Birkhoff (1937, see Theorem 3.1 here), it can be equivalently well specified by a quasi order on the set of problems. Such a representation of knowledge structures by quasi orders would be well suited for eliciting the relevant information from experts (as we will argue in the next section). However, Doignon and Falmagne (1985), who recalled Birkhoff's result in this context, argue convincingly that the assumption of closure under intersection is not a realistic one for knowledge structures in practice. Assuming only closure under union, they showed that a representation by surmise systems (a variant of AND/OR graphs), instead of quasi orders, is possible. This result of Doignon and Falmagne, together with some other motivation they present, explains the central role played in the theory by knowledge structures closed under union; these are called *knowledge spaces.*

Accepting the assumption of closure under union, we can now reformulate our problem as that of designing a procedure for building a knowledge space for a particular domain by querying experts. Unfortunately, the representation by surmise systems is not very promising in this respect. The point of this paper, then, is to derive an alternative representation for knowledge spaces that is fitting our purpose. Again, quasi orders will enter the picture, but this time they will be relations on the power set of the set of problems. This representation is at the basis of a procedure that translates the responses of an expert to a set of queries of a specified form into a corresponding knowledge space. The principles of such a procedure are illustrated on a small example. A real-life, large scale application is to be reported elsewhere (Kambouri, Koppen, Villano, & Falmagne, 1989); it is based on the algorithm presented in Koppen (1989) as a result of a deeper elaboration of these principles. We also note here that similar theoretical work was done independently by Müller (1990).

The paper is organized as follows. In the next section we illustrate the quasi order representation (Birkhoff's theorem) and its limitation, and we describe more precisely the kind of procedure we are going to develop. In Section 3, we formally state the classical result of Birkhoff (1937) and we introduce the key concept of a *Galois connection*. Following Monjardet (1970), Birkhoff's result is derived from a Galois connection between the collection of knowledge structures and the collection of binary relations on the set of problems. Section 4 contains our main theoretical results. We establish a more general Galois connection between knowledge structures and relations on the power set of the set of problems. Theorem 4.5 can be seen as another extension of Birkhoff's result. This leads to the characterization of knowledge spaces as a well defined kind of quasi orders on this power set (Corollary 4.6). In Section 5, the surmise systems of Doignon and Falmagne (1985) are put in this context. The final section describes how the theoretical results of Section 4 give rise to an algorithm for obtaining a knowledge space from the expert's answers and shows the principles of such an algorithm by running the procedure on the small example presented in Section 2.

2. Background

Let us recall here a small academic example with five problems a, b, c, d, e, presented in Doignon and Falmagne (1985). The content of the five problems (drawn from the field of elementary probability) was examined and in a first analysis the following family $\mathscr{K}'$ of possible knowledge states was obtained ($\varnothing$ denotes the empty set):

$$\mathscr{K}' = \{\varnothing, \{c\}, \{e\}, \{b, e\}, \{c, e\}, \{a, b, e\}, \{b, c, e\}, \{c, d, e\}, \{a, b, c, e\}, \{b, c, d, e\}, \{a, b, c, d, e\}\}.$$

Notice that this family $\mathscr{K}'$ is closed under both union and intersection. By an old theorem of Birkhoff (1937, Theorem 3.1 here), this means that the family $\mathscr{K}'$ can be represented by a quasi order (a reflexive, transitive relation) $\mathscr{P}'$ on the set of problems. The correspondence between $\mathscr{P}'$ and $\mathscr{K}'$ is such that a pair of problems (x, y) is in $\mathscr{P}'$ if and only if any state of $\mathscr{K}'$ containing y also contains x. Thus $\mathscr{P}'$ can be thought of as a *surmise relation*: if a student solves y correctly and the pair (x, y) is in $\mathscr{P}'$, we can surmise that this student has also mastered x. In our case, $\mathscr{P}'$ is the partial order sketched in Fig. 1. Note that the above construction is reversible; that is, the family $\mathscr{K}'$ can be fully recovered from $\mathscr{P}'$.

This representation by quasi orders suggests an alternative approach to the problem of obtaining a knowledge structure by querying an expert. Instead of asking the expert directly for the knowledge structure, we ask him to specify the

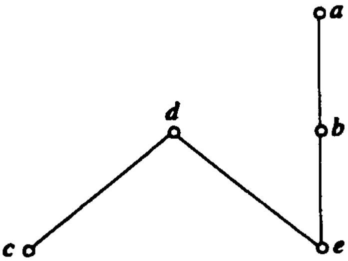

FIG. 1. Hasse diagram of the surmise relation $\mathscr{P}'$ corresponding to the knowledge structure $\mathscr{K}'$.

quasi order. That is, the expert is presented with a pair of problems and is asked whether solving the first one would imply solving the second one. This task of the expert seems much simpler than deciding directly about the "state-hood" of a subset: now he has at each instance only two problems to consider and he can ignore the rest. The maximum number of questions he will have to answer (i.e., the number of pairs) is only quadratic in the number of problems. Besides, using the transitivity of the quasi order we can make inferences and thus save information requests. Finally, quasi orders and the corresponding knowledge structures are related in such a way that inclusions are reversed. Thus, when we stop the process halfway and end up with a quasi order that is a part of the true quasi order, the corresponding knowledge structure will include the true knowledge structure. There may be some "nuisance" states left (states that will never be assigned to any students and that only act to slow down the assessment procedure), but at least no vital states are missing. So this procedure of obtaining a knowledge structure by asking an expert about a quasi order is clearly an attractive option. It is not available, however, for building a general knowledge space, since representability by a quasi order requires the extra assumption of closure under intersection.

To continue our example, in a second analysis of the five problems Doignon and Falmagne (1985) argued that a case could be made for including one more subset of problems, the set $\{a, c, d, e\}$, as a possible state. Thus we obtain the knowledge structure

$$\mathscr{K} = \mathscr{K}' \cup \{\{a, c, d, e\}\}.$$

It can easily be checked that $\mathscr{K}$ is still closed under union, but it is no longer closed under intersection (since $\{a, b, c, e\} \cap \{a, c, d, e\}$ is not a state). This means that $\mathscr{K}$ is a knowledge space which cannot be represented by a quasi order on the problems. For instance, we have to remove from $\mathscr{P}'$ the arrow from b to a. (It is no longer true that knowing a implies knowing b.) This leads to the partial order $\mathscr{P}$ of Fig. 2. Now the state $\{a, c, d, e\}$ has indeed been added, but some additional states have been introduced ($\{a, e\}$ and $\{a, c, e\}$) that do *not* appear in the knowledge structure $\mathscr{K}$.

Generalizing Birkhoff's theorem, Doignon and Falmagne (1985) showed that

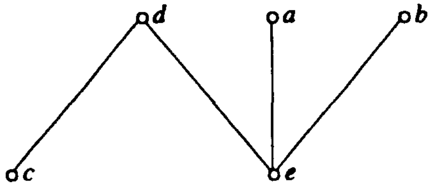

FIG. 2. Hasse diagram of the surmise relation $\mathscr{P}$ that "contains" the knowledge structure $\mathscr{K}$.

knowledge structures closed under union are in 1–1 correspondence with so-called *surmise systems*. The precise statement of this correspondence will be recalled in Section 5, but the essential point here is that this characterization does not seem to lend itself to an efficient procedure for obtaining a knowledge space from an expert.

In this paper an alternative representation for knowledge spaces is obtained by means of well-defined relations on the power set of the set of notions; in particular, subsets of notions are quasi ordered.[1] The derived representation leads to a procedure for uncovering a knowledge space that is very much like the quasi order procedure for a knowledge structure closed under union and intersection. In fact, it is an extension of this procedure. We start with presenting to the expert pairs of problems, querying whether a student that fails the first one would—in principle—also fail the second problem. This is in effect the same as we did in the procedure sketched above, and if we were willing to assume closure under intersection we could stop the process after this block of questions. Assuming only closure under union, however, we need more information and we continue with a second block of questions. Here the expert is presented a pair of problems together with a single problem and he is to indicate whether a student failing both problems of the pair would also fail the single problem. Next, in a third block, the query is whether failing a triple of problems would imply failing another, single problem. And so on. In general, the information asked from the expert is whether failing a particular subset of problems implies failing another specified problem.

This procedure saves—to a great extent—the advantages of the previously described procedure for the case where we assume closure under intersection. Of course we have lost the quadratic bound on the maximal number of questions, but because the encoding of the knowledge space is again in terms of quasi orders, we can use transitivity to draw inferences, both from positive and negative answers of the expert, thus reducing the information to be requested explicitly. (It will appear that we have in fact more inferences than just from transitivity.) Also, the queries to the expert are still as "local" as possible. In the first block he has, at each instance, to consider two problems only, in the second block three problems (a pair and a single problem), in the third block four, etc. Clearly things get bad toward the end, but because the inferences work all the way down, the number of questions to be asked in the later blocks can be greatly reduced. Still, if it happens that for

[1] We owe this basic idea to Jean-Claude Falmagne.

practical reasons we have to stop the procedure after, say, the second or third block of questions, and the encoding quasi order is only partly known, then, again, we have the advantage that, going from the quasi order to the encoded knowledge space, the inclusions are reversed. The obtained quasi order that is a part of the true quasi order translates into a knowledge space that includes the true knowledge space. So again we cannot lose vital states by interrupting the procedure.

3. Preliminary Definitions and Results

Let X be some fixed set; in our interpretation it is the collection of problems in some field of knowledge. In the present and next sections we do *not* assume X to be finite, although in practical applications this will be the case. Any knowledge structure $\mathscr{K}$ on X, that is, any family $\mathscr{K}$ of subsets of X, induces a relation $\mathscr{Q}$ on X by the definition

$$x\mathscr{Q}y \qquad \text{iff} \qquad (\text{for all } K \in \mathscr{K} : y \in K \text{ implies } x \in K).$$

Our interpretation of $x\mathscr{Q}y$ is that from the mastery of problem y it can be surmised that the same student also masters problem x. Denoting by $\mathscr{K}_x$ the subfamily of $\mathscr{K}$ consisting of the sets that contain the element x, we can write this equivalence more compactly as

$$x\mathscr{Q}y \qquad \text{iff} \qquad \mathscr{K}_x \supseteq \mathscr{K}_y. \tag{3.1}$$

It is immediate that the resulting relation $\mathscr{Q}$ is a quasi order (that is, it is reflexive and transitive). Conversely, any relation $\mathscr{Q}$ on X yields a knowledge structure $\mathscr{K}$ on X by setting

$$K \in \mathscr{K} \qquad \text{iff} \qquad (\text{for all } (x, y) \in \mathscr{Q} : y \in K \text{ implies } x \in K). \tag{3.2}$$

It is easy to check that the resulting knowledge structure $\mathscr{K}$ is closed both under union and intersection. The above discussion is summarized in the following classical result.

3.1. Theorem (Birkhoff, 1937). *The formulae* (3.1) *and* (3.2) *define a* 1–1 *correspondence between knowledge structures $\mathscr{K}$ on X that are closed under both union and intersection, and quasi orders $\mathscr{Q}$ on X.*

As shown in Monjardet (1970), this result can be considered as a corollary to the fact that the mappings (3.1) and (3.2) define a *Galois connection* between knowledge structures on X and relations on X (both collections ordered by inclusion).

3.2. Definition (Birkhoff, 1967). Let $(Y, \sqsubseteq)$ and $(Z, \sqsubseteq)$ be two partially ordered sets. A pair of mappings $f: Y \to Z$ and $g: Z \to Y$ defines a *Galois connection* between $(Y, \sqsubseteq)$ and $(Z, \sqsubseteq)$ iff, for $y, y' \in Y$ and $z, z' \in Z$,

(i) $y \sqsubseteq y'$ implies $f(y') \sqsubseteq f(y)$, and $z \sqsubseteq z'$ implies $g(z') \sqsubseteq g(z)$;
(ii) $y \sqsubseteq (g \circ f)(y)$, and $z \sqsubseteq (f \circ g)(z)$.

To state results in this context, the notion of a closure on a partially ordered set is useful.

3.3. Definition (Birkhoff, 1967). Let $(Y, \sqsubseteq)$ be a partially ordered set. A mapping $h\colon Y \to Y$ is a *closure* on $(Y, \sqsubseteq)$ when, for $y, y' \in Y$,

(i) $y \sqsubseteq h(y)$;
(ii) $y \sqsubseteq y'$ implies $h(y) \sqsubseteq h(y')$;
(iii) $(h \circ h)(y) = h(y)$.

Any $y \in Y$ such that $y = h(y)$ is called a *closed* element (under h).

3.4. Theorem (Birkhoff, 1967). *If the mappings $f\colon Y \to Z$ and $g\colon Z \to Y$ define a Galois connection between $(Y, \sqsubseteq)$ and $(Z, \sqsubseteq)$, then they induce a 1–1 correspondence between $g(Z)$ and $f(Y)$. More specifically, $f \circ g$ and $g \circ f$ are closures on $(Y, \sqsubseteq)$ and $(Z, \sqsubseteq)$ with the closed elements collected in $g(Z)$ and $f(Y)$, respectively; the restrictions of f and g to $g(Z)$ and $f(Y)$, respectively, are inverse order-reversing isomorphisms.*

As an example, (3.1) and (3.2) define a Galois connection leading to the 1–1 correspondence in Theorem 3.1. Here, the closure of a knowledge structure $\mathscr{K}$ on X is the smallest family of subsets of X containing $\mathscr{K}$ that is closed under union and intersection. For the relations on X, "closure" here means the reflexive transitive closure, mapping any relation $\mathscr{R}$ to the smallest quasi order containing $\mathscr{R}$.

Knowledge spaces, being closed only under union, cannot be captured by quasi orders on X. In the next section we will show, by establishing another Galois connection, that they are still in a 1–1 correspondence with a class of quasi orders, but we have to move one level up and consider quasi orders on the power set of X.

4. A Galois Connection for Knowledge Spaces

Let us first introduce some more notation. We denote by Ω the collection of all knowledge structures on X, so

$$\Omega = 2^{(2^X)},$$

the power set of the power set of X. Clearly, Ω is ordered by inclusion:

$$\mathscr{K} \subseteq \mathscr{K}' \quad \text{iff} \quad (\text{for all } K \in 2^X\colon K \in \mathscr{K} \text{ implies } K \in \mathscr{K}').$$

We denote by Ψ the collection of all binary relations on the power set of X:

$$\Psi = 2^{(2^X \times 2^X)}.$$

This collection is again ordered by inclusion:

$$\mathscr{R} \subseteq \mathscr{R}' \quad \text{iff} \quad (\text{for all } A, B \in 2^X\colon A\ \mathscr{R}\ B \text{ implies } A\ \mathscr{R}'\ B).$$

We proceed to construct a Galois connection (r, k) between the two partially ordered sets $(\Omega, \subseteq)$ and $(\Psi, \subseteq)$. With any knowledge structure $\mathscr{K}$ on X we can associate a relation $r(\mathscr{K})$ on the power set of X by the definition

$$A\ r(\mathscr{K})\ B \quad \text{iff} \quad (\text{for all } K \in \mathscr{K}\colon B \cap K \neq \varnothing \text{ implies } A \cap K \neq \varnothing).$$

With the notation $\mathscr{K}_A$ for the subcollection of sets in $\mathscr{K}$ that "meet" the subset A of X,

$$\mathscr{K}_A = \{K \in \mathscr{K} : A \cap K \neq \varnothing\},$$

this can be written as

$$A\ r(\mathscr{K})\ B \quad \text{iff} \quad \mathscr{K}_A \supseteq \mathscr{K}_B.$$

The empirical interpretation of $A\ r(\mathscr{K})\ B$ is that if a student masters some question in B, he also masters some question in A; or, equivalently, if he does not master any question in A, he also does not master any question in B. Notice the similarity with the corresponding definition (3.1) in the Birkhoff case. The next proposition follows easily from the definition of $\mathscr{K}_A$.

4.1. PROPOSITION. *For the mapping $r\colon \Omega \to \Psi$ defined, for $A, B \in 2^X$, by*

$$A\ r(\mathscr{K})\ B \quad \text{iff} \quad \mathscr{K}_A \supseteq \mathscr{K}_B$$

we have, if $\mathscr{K}, \mathscr{K}' \in \Omega$:

(i) *$r(\mathscr{K})$ extends $\supseteq$ on 2^X (that is, $A \supseteq B$ implies $A\ r(\mathscr{K})\ B$);*

(ii) *$r(\mathscr{K})$ is transitive (and thus, by (i), a quasi order);*

(iii) *if $A\ r(\mathscr{K})\ B_i$ for all i in some index set I, then $A\ r(\mathscr{K})(\bigcup_{i \in I} B_i)$;*

(iv) *if $\mathscr{K} \subseteq \mathscr{K}'$ then $r(\mathscr{K}) \supseteq r(\mathscr{K}')$;*

(v) *if $\mathscr{K}^c$ is the closure under union of $\mathscr{K}$, then $r(\mathscr{K}) = r(\mathscr{K}^c)$.*

Relations on 2^X that have the properties (i) to (iii) of Proposition 4.1 will play an important role in the sequel, so we investigate these more closely:

4.2. LEMMA. *For a transitive extension $\mathscr{P}$ of $\supseteq$ on 2^X, the following conditions are equivalent:*

(i) *$A\ \mathscr{P}\ B_i$ for all i in some index set I implies $A\ \mathscr{P}\ (\bigcup_{i \in I} B_i)$;*

(ii) *Any $A\mathscr{P} = \{Z \in 2^X : A\ \mathscr{P}\ Z\}$ has a maximum (for the inclusion) element A^*;*

(iii) *$A\ \mathscr{P}\ B$ iff (for all $x \in B$: $A\ \mathscr{P}\ \{x\}$).*

These conditions imply

(iv) $A \mathscr{P} B$ *implies* $A \mathscr{P} (B \cup A)$

and in case X *is finite*, (iv) *is equivalent to* (i)–(iii).

A transitive extension of $\supseteq$ on 2^X for which these conditions hold is called an *entail relation* for X. (Note that an entail relation for X is a relation on 2^X.) One has $A \mathscr{P} B$ iff $A^* \supseteq B$, with A^* as in (ii).

Proof. That (iii) implies (i) is immediate, as is the implication from (i) to (ii) and (iv) if we note that $A \mathscr{P} A$. The "only if" part of (iii) is given ($\mathscr{P}$ extends $\supseteq$) and the "if" part follows from (ii): If $A \mathscr{P} \{x\}$ for all $x \in B$, then, using the definition in (ii), $A^* \supseteq \{x\}$ for all $x \in B$, which implies $A^* \supseteq B$. So we have $A \mathscr{P} A^* \supseteq B$, thus $A \mathscr{P} B$. To see that (iv) implies (i) in the finite case it suffices to prove that $A \mathscr{P} B$ and $A \mathscr{P} C$ imply $A \mathscr{P} (B \cup C)$. From $(A \cup B) \supseteq A$ we have $(A \cup B) \mathscr{P} A$. By (iv) and $A \mathscr{P} C$, we also have $A \mathscr{P} (A \cup C)$. There follows $(A \cup B) \mathscr{P} (A \cup C)$. Applying (iv), we get $(A \cup B) \mathscr{P} (A \cup B \cup C)$. Since by (iv) $A \mathscr{P} (A \cup B)$, and $(A \cup B \cup C) \supseteq (B \cup C)$, we finally derive $A \mathscr{P} (B \cup C)$. ∎

Note that (iv) is not equivalent to (iii) in general. For a counterexample, take X infinite and define $\mathscr{P}$ by $A \mathscr{P} B$ iff $B \setminus A$ is finite.

Having defined the mapping $r: \Omega \to \Psi$, we now introduce a mapping that goes the other way. Again, the definition will be reminiscent of the corresponding mapping (3.2) in the Birkhoff case:

4.3. Proposition. *For the mapping* $k: \Psi \to \Omega$ *defined by*

$$K \in k(\mathscr{R}) \quad \textit{iff} \quad (\textit{for all } (A, B) \in \mathscr{R}: B \cap K \neq \varnothing \textit{ implies } A \cap K \neq \varnothing)$$

we have, if $\mathscr{R}, \mathscr{R}' \in \Psi$:

(i) $k(\mathscr{R})$ *is closed under union*;

(ii) *if* $\mathscr{R} \subseteq \mathscr{R}'$ *then* $k(\mathscr{R}) \supseteq k(\mathscr{R}')$.

The following lemma, giving an alternative characterization of the mapping k for the case of an entail relation, will be very useful.

4.4. Lemma. *If* $\mathscr{P} \in \Psi$ *is an entail relation for* X, *then, with* k *as in Proposition* 4.3,

$$K \in k(\mathscr{P}) \quad \textit{iff} \quad (\textit{for all } Z \in 2^X: (X \setminus K) \mathscr{P} Z \textit{ implies } (X \setminus K) \supseteq Z).$$

Proof. If $K \in k(\mathscr{P})$, then for any $A, B \in 2^X$ such that $A \mathscr{P} B$ and $K \cap A = \varnothing$ we have $K \cap B = \varnothing$. Putting $A = X \setminus K$ it follows that $(X \setminus K) \mathscr{P} B$ implies $(X \setminus K) \supseteq B$ for all $B \in 2^X$. Conversely, let K be such that $(X \setminus K) \mathscr{P} Z$ implies $(X \setminus K) \supseteq Z$ for all $Z \in 2^X$ and suppose for some $A, B \in 2^X$ we have $A \mathscr{P} B$ and $K \cap A = \varnothing$. This means $(X \setminus K) \supseteq A \mathscr{P} B$, thus $(X \setminus K) \mathscr{P} B$ and by our assumption $(X \setminus K) \supseteq B$. So, $K \cap B = \varnothing$ and $K \in k(\mathscr{P})$. ∎

Now we are ready to formulate our main result:

4.5. THEOREM. *The pair* (r, k), *where* r *is as in Proposition* 4.1 *and* k *as in Proposition* 4.3, *is a Galois connection between the partially ordered sets* $(\Omega, \subseteq)$ *and* $(\Psi, \subseteq)$. *The closed elements in* Ω *are the knowledge spaces and the closed elements in* Ψ *are the entail relations for* X.

Proof. The inclusion $\mathscr{R} \subseteq (r \circ k)(\mathscr{R})$ can be checked easily from the definitions of the mappings r and k. So in view of Propositions 4.1(iv) and 4.3(ii) the Galois connection is established if we can show that $\mathscr{K} \subseteq (k \circ r)(\mathscr{K})$ for $\mathscr{K} \in \Omega$. Since by Proposition 4.1(i) to (iii) any $r(\mathscr{K})$ is an entail relation, we can use Lemma 4.4 to see that

$$K \in (k \circ r)(\mathscr{K}) \qquad \text{iff} \qquad (\text{for all } Z \in 2^X\text{: } \mathscr{K}_{X \setminus K} \supseteq \mathscr{K}_Z \text{ implies } X \setminus K \supseteq Z).$$

For any $K, Z \in 2^X$, the inclusion $\mathscr{K}_{X \setminus K} \supseteq \mathscr{K}_Z$ implies $K \notin \mathscr{K}_Z$, so clearly $K \in \mathscr{K}$ together with this inclusion implies $X \setminus K \supseteq Z$. Consequently $\mathscr{K} \subseteq (k \circ r)(\mathscr{K})$.

Now let us determine the closed elements. If $\mathscr{K}$ is a space and $K \in 2^X$ we can, above, take $Z = X \setminus K_0$, where K_0 is the unique maximal element of $\mathscr{K}$ included in K. Then clearly $\mathscr{K}_{X \setminus K} \supseteq \mathscr{K}_{X \setminus K_0}$ and thus $K \in (k \circ r)(\mathscr{K})$ implies $K \subseteq K_0$, that is, $K \in \mathscr{K}$. So if $\mathscr{K}$ is a space we have $\mathscr{K} = (k \circ r)(\mathscr{K})$. This shows that any space is in the image of k and since Proposition 4.3(i) says that any element of $k(\Psi)$ is a space, we see that the knowledge spaces constitute the closed elements in Ω.

The closed elements in Ψ are the images by r and by Proposition 4.1(i) to (iii) any element of $r(\Omega)$ is an entail relation for X. We will show the converse by establishing $\mathscr{R} = (r \circ k)(\mathscr{R})$ for such a relation $\mathscr{R}$. One inclusion being checked above, it remains to show that $\mathscr{R} \supseteq (r \circ k)(\mathscr{R})$. So, suppose $A\ (r \circ k)(\mathscr{R})\ B$ and let A^* be the maximum element of $A\,\mathscr{R}$ (Lemma 4.2(ii)). Clearly, for all $Z \in 2^X$, $A^*\,\mathscr{R}\,Z$ implies $A\,\mathscr{R}\,A^*\,\mathscr{R}\,Z$ and thus $A^* \supseteq Z$. By Lemma 4.4, this means $X \setminus A^* \in k(\mathscr{R})$. From $A^* \supseteq A$ it follows that $X \setminus A^* \notin k(\mathscr{R})_A$, which, in view of $A\ (r \circ k)(\mathscr{R})\ B$, implies $(X \setminus A^*) \cap B = \varnothing$. This gives $A\,\mathscr{R}\,A^* \supseteq B$, thus $A\,\mathscr{R}\,B$. ∎

4.6. COROLLARY. *The mappings* r *and* k *induce a* 1–1 *correspondence between the collections of knowledge spaces on* X *and entail relations for* X.

Remark. We point out here that, independently, a very similar 1–1 correspondence was recently obtained by Burigana (1988). He did not derive it from a Galois connection, but formulated it directly in terms of closures (see below for this approach). Curiously enough, Burigana's motivation differs completely from ours: his paper is devoted to the study of regularity in sequences of stimuli.

4.7. *An Alternative Formulation.* The 1–1 correspondence induced by the Galois connection (r, k) can be described alternatively in terms of closures (cf. Definition 3.3). Observe first that any knowledge space $\mathscr{K}$ defines a closure $h_{\mathscr{K}}$ on $(2^X, \subseteq)$ by letting $h_{\mathscr{K}}(A)$ denote the maximum element $Z \in 2^X$ such that $\mathscr{K}_A \supseteq \mathscr{K}_Z$. (For a space, this is well-defined and it follows that $X \setminus h_{\mathscr{K}}(A)$ must be a state of

$\mathscr{K}$, in fact the largest state included in $X \backslash A$.) With our interpretation, $h_{\mathscr{K}}(A)$ collects the problems that we can infer a person will fail if we know this person fails all problems in A. On the other hand, for any entail relation $\mathscr{P}$ for X the mapping $A \to A^* = \bigcup A\,\mathscr{P}$ (see Lemma 4.2(ii)) yields a closure $h_{\mathscr{P}}$ on $(2^X, \subseteq)$. Now a space $\mathscr{K}$ and an entail relation $\mathscr{P}$ are paired in the 1–1 correspondence of Corollary 4.6 ($r(\mathscr{K}) = \mathscr{P}$ and $k(\mathscr{P}) = \mathscr{K}$) if and only if the induced closures are the same (iff $h_{\mathscr{K}} = h_{\mathscr{P}}$).

The mappings $\mathscr{K} \to h_{\mathscr{K}}$ and $\mathscr{P} \to h_{\mathscr{P}}$ are bijections. More specifically, for an arbitrary closure h on $(2^X, \subseteq)$ we find the knowledge space $\mathscr{K}$ such that $h = h_{\mathscr{K}}$ as

$$\mathscr{K} = \{X \backslash h(A) : A \in 2^X\},$$

and the entail relation $\mathscr{P}$ such that $h = h_{\mathscr{P}}$ is, for $A, B \in 2^X$, defined by

$$A\,\mathscr{P}\,B \qquad \text{iff} \qquad h(A) \supseteq B.$$

It may be checked that the defining properties of a closure ensure that the thus defined family $\mathscr{K}$ is indeed a space and the relation $\mathscr{P}$ indeed an entail relation for X.

4.8. *Birkhoff revisited.* By Lemma 4.2(iii) we know that for an entail relation $\mathscr{P}$ for X we do not lose any information by restricting its range to the singleton elements of 2^X. If, in addition, $\mathscr{P}$ is such that for $A \in 2^X$ and $y \in X$

$$A\,\mathscr{P}\,\{y\} \qquad \text{iff} \qquad (\text{for some } x \in A : \{x\}\,\mathscr{P}\,\{y\}),$$

we can also restrict the domain of $\mathscr{P}$ to the singleton sets and we have for $A, B \in 2^X$

$$A\,\mathscr{P}\,B \qquad \text{iff} \qquad (\text{for any } y \in B \text{ there is } x \in A \text{ such that } \{x\}\,\mathscr{P}\,\{y\}). \tag{*}$$

It is easy to check (with Lemma 4.4, for instance) that for such a $\mathscr{P}$ the knowledge space $k(\mathscr{P})$ is closed under intersection. Conversely, if a knowledge space $\mathscr{K}$ is closed under intersection, then the relation $\mathscr{P} = r(\mathscr{K})$ satisfies (*). For, if $A\,r(\mathscr{K})\,B$ and $y \in B$, we have $A\,r(\mathscr{K})\,\{y\}$, meaning that any state containing y contains an element of A. In particular the state $\bigcap \mathscr{K}_{\{y\}}$ contains some $x \in A$; consequently this x appears in any element of $\mathscr{K}_{\{y\}}$ and we obtain $\mathscr{K}_{\{x\}} \supseteq \mathscr{K}_{\{y\}}$. Thus the "only if" part of (*) is established; the "if" part poses no problems.

Since r and k are inverse mappings when restricted to knowledge spaces and entail relations for X, respectively, we obtain in this way a 1–1 correspondence between knowledge structures closed under union as well as intersection and quasi orders on 2^X satisfying (*). By identifying singleton sets with their element, the latter collection is in a natural 1–1 correspondence with the collection of quasi orders on X, which gives us Birkhoff's result.

5. The Relation with Surmise Systems

In this section we will assume that the set X is finite. Under that restriction Doignon and Falmagne (1985) obtained a different generalization of Birkhoff's result. They noted that a knowledge structure's being closed under intersection forces a notion to have one unique set of prerequisites, which is too restrictive. So, dropping the requirement of closure under intersection, Doignon and Falmagne (1985) were led to the definition of a *surmise system*, in which to each notion is assigned a collection of subsets of notions, representing the possible sets of prerequisites for that notion. Again by establishing a Galois connection, they derived a 1–1 correspondence between the collection of knowledge spaces and the collection of *space-like* surmise systems on X:

5.1. Definition. A *surmise system* on X is a mapping $\sigma: X \to \Omega$. The elements of $\sigma(x)$ are called the *clauses* for x. The *states* of σ are the sets $Z \in 2^X$ that contain a clause for each element:

$$\text{for all } x \in Z \text{ there is } C \in \sigma(x) \text{ such that } C \subseteq Z.$$

The surmise system σ is called *space-like* if each clause for x is a state containing x and the clauses for x are pairwise incomparable (with respect to inclusion).

5.2. Theorem (Doignon and Falmagne, 1985). *The collection of knowledge spaces on X is in 1–1 correspondence with the collection of space-like surmise systems on X. In this correspondence, the clauses for $x \in X$ in the space-like surmise system constitute the minimal states in the knowledge space that contain x.*

Since knowledge spaces are in 1–1 correspondence with both entail relations for X and space-like surmise systems, obviously the last two collections must be in 1–1 correspondence. We will investigate here how each entail relation for X induces a space-like surmise system.

If $\mathscr{P}$ is a relation on 2^X we denote by $\bar{\mathscr{P}}$ the complement of $\mathscr{P}$:

$$\bar{\mathscr{P}} = (2^X \times 2^X) \setminus \mathscr{P}.$$

From Lemma 4.2(iii) we know that in case $\mathscr{P}$ is an entail relation, it is uniquely determined by the sets $\bar{\mathscr{P}}\,\{x\}$, $x \in X$. To simplify notation we will in the sequel write $A\,\bar{\mathscr{P}}\,x$ for $A\,\bar{\mathscr{P}}\,\{x\}$ and also $A\,\mathscr{P}\,x$ for $A\,\mathscr{P}\,\{x\}$. So

$$\bar{\mathscr{P}}\,x = \{Z \in 2^X : \text{not } Z\,\mathscr{P}\,\{x\}\}.$$

We have for $A\,\mathscr{P}\,B$ the interpretation that if a subject has mastered a notion in B, he must also have mastered a notion in A. Thus $A\,\bar{\mathscr{P}}\,x$ means that it is still possible to know x without knowing anything in A; or, in terms of surmise systems, there is still a clause for x contained in $X \setminus A$. With this interpretation, the following theorem is not really surprising.

5.3. THEOREM. *For an entail relation $\mathscr{P}$ for X, define a mapping $s(\mathscr{P})\colon X \to \Omega$ by assigning to each $x \in X$ the collection of complements of maximal elements of $\bar{\mathscr{P}}\, x$. That is, $C \in s(\mathscr{P})\,(x)$ iff $(X\backslash C)\,\bar{\mathscr{P}}\, x$ and no strict superset of $X\backslash C$ has this property. Then $s(\mathscr{P})$ is a space-like surmise system and the states of $s(\mathscr{P})$ coincide with the states of the knowledge space $k(\mathscr{P})$.*

Proof. The clauses for any $x \in X$ are pairwise incomparable by definition and each clause C for x contains x since $(X\backslash C)\,\bar{\mathscr{P}}\, x$ implies $(X\backslash C) \not\supseteq \{x\}$. We have proved that $s(\mathscr{P})$ is space-like if we can show that any clause for x is a state. So suppose $C \in s(\mathscr{P})\,(x)$ and $x' \in C$. Since $X\backslash C$ is maximal in $\bar{\mathscr{P}}\, x$, it must be the case that $((X\backslash C) \cup \{x'\})\,\mathscr{P}\, x$. In view of this, the assumption $(X\backslash C)\,\mathscr{P}\, x'$ would lead to $(X\backslash C)\,\mathscr{P}\,((X\backslash C) \cup \{x'\})\,\mathscr{P}\, x$, contradicting $(X\backslash C)\,\bar{\mathscr{P}}\, x$. Thus, $(X\backslash C)\,\bar{\mathscr{P}}\, x'$ and, using finiteness of X, there is some maximal $A \supseteq (X\backslash C)$ such that $A\,\bar{\mathscr{P}}\, x'$. In other words, $(X\backslash A) \in s(\mathscr{P})\,(x')$ and $(X\backslash A) \subseteq C$.

For the states of $k(\mathscr{P})$ we use the characterization of Lemma 4.4. To show that any state of $s(\mathscr{P})$ is a state of $k(\mathscr{P})$ we take $K, Z \in 2^X$ such that $(X\backslash K)\,\mathscr{P}\, Z$ and for all $x \in K$ there is $C_x \subseteq K$ with $(X\backslash C_x)\,\bar{\mathscr{P}}\, x$. The proof that then $(X\backslash K) \supseteq Z$ follows by contradiction: if $x \in K \cap Z$ we have $(X\backslash K) \cup Z \supseteq \{x\}$ and, since $(X\backslash K)\,\mathscr{P}\, Z$ implies $(X\backslash K)\,\mathscr{P}\,((X\backslash K) \cup Z)$, we derive

$$(X\backslash C_x) \supseteq (X\backslash K)\,\mathscr{P}\,((X\backslash K) \cup Z) \supseteq \{x\},$$

contradicting $(X\backslash C_x)\,\bar{\mathscr{P}}\, x$. On the other hand, if K is a state of $k(\mathscr{P})$, Lemma 4.4 yields $(X\backslash K)\,\bar{\mathscr{P}}\, x$ for each $x \in K$. Using finiteness of X we may conclude that then for any $x \in K$ there must be a minimal $C_x \subseteq K$ such that $(X\backslash C_x)\,\bar{\mathscr{P}}\, x$. This means that K is a state of $s(\mathscr{P})$. ∎

6. THE CONSTRUCTION OF THE KNOWLEDGE SPACE

The theoretical results of Section 4 find a useful practical application in the problem of obtaining the knowledge structure of a particular domain by querying an expert in that field. We will sketch here a straightforward way of doing this that derives directly from the results of Section 4, and we illustrate the procedure on the five problem example of Section 2. For applications of practical importance (i.e., for a larger problem set) we need a more sophisticated version. Such a practicable procedure, based on an elaboration of the same principles, is described elsewhere (Koppen, 1989).

The basic idea is not to ask the expert directly for the states in the knowledge structure, but rather to ask for the corresponding entail relation for the set of notions. So, in principle, we present the expert with two subsets A and B of notions, and we ask whether or not it is safe to conclude that if a subject fails all problems in A (has not mastered any notion in A), then the same subject will fail all problems in B (has not mastered any notion in B). When this conclusion is valid, the pair (A, B) is in the relation, otherwise it is not.

Knowing that the relation we are looking for is an entail relation for X, we certainly need not offer the expert all pairs of subsets of notions. (We put ourselves in the comfortable position that our expert is perfectly reliable.) For one thing, from Lemma 4.2(iii) we know that we need only consider singleton sets in the range of such an entail relation. That is, we have to ask the expert only questions of the kind: If a subject fails all problems in A, does this imply that he will fail problem x? This would be a silly question to ask when x is an element of A, which reflects the fact that an entail relation extends the superset relation. In order to deal only with singletons in the range of our relation, we need the following easy Lemma, which characterizes such restrictions of entail relations. (The statement of this Lemma is in terms of X as the restricted range, identifying the singleton $\{x\}$ with the element x.)

6.1. LEMMA. *A relation $\hat{\mathscr{P}}$ included in $2^X \times X$ is the restriction of an entail relation for X iff it satisfies the following two conditions:*

(i) *$\hat{\mathscr{P}}$ includes the reverse membership relation $\ni$;*

(ii) *$[(B \hat{\mathscr{P}} z$ for all $z \in Z)$ & $Z \hat{\mathscr{P}} y]$ implies $B \hat{\mathscr{P}} y$.*

The lemma follows, since $\mathscr{P}$ defined by

$$A \mathscr{P} B \quad \text{iff} \quad A \hat{\mathscr{P}} b \text{ for all } b \in B$$

defines an entail relation for X if and only if $\hat{\mathscr{P}}$ satisfies the conditions in Lemma 6.1.

As a consequence of Lemma 6.1, the expert will be queried only about pairs of the form A, x. Moreover, not all such pairs need to be presented because inferences can be drawn from his responses, both from positive responses (validating the implication) and from negative responses (denying it). Let us, for notational convenience, introduce the shorthand A_x for the set $A \cup \{x\}$ with $A \in 2^X$ and $x \in X$. According to condition (ii) of Lemma 6.1, then, any positive inference that *directly* involves a new observation $A \mathscr{P} x$ must be of one of the two forms (a) or (b):

(a) We had already established $B \mathscr{P} a$ for all a in A, and then from $A \mathscr{P} x$ we infer $B \mathscr{P} x$ (thus $y = x$ and $Z = A$ in 6.1(ii)).

(b) We had already established $A_x \mathscr{P} y$, and then $A \mathscr{P} x$ leads to $A \mathscr{P} y$ (thus $B = A$ and $Z = A_x$ in 6.1(ii)).

The two cases (a) and (b) can be combined in one equivalent inference rule:

$$A \mathscr{P} x \quad \text{implies} \quad B \mathscr{P} y \qquad \text{whenever} \qquad B \mathscr{P} A \quad \text{and} \quad A_x \mathscr{P} y. \tag{6.1}$$

Indeed, (a) and (b) correspond to the special cases $y = x$ and $B = A$, respectively, while conversely any inference $B \mathscr{P} y$ obtained by (6.1) also follows by first inferring $A \mathscr{P} y$ (case (b)) and next $B \mathscr{P} y$ (case (a)). Note that in this discussion we only considered *direct* inferences from $A \mathscr{P} x$, that is, inferences in which A and/or x

appear in the conditions of the rule. We find *all* positive inferences by applying rule (6.1) iteratively to these direct inferences, and so on. In general, repeated application can yield new inferences.

A positive response can also lead to *negative* inferences according to similar rules; for instance,

$$A \mathscr{P} x \quad \text{implies} \quad B \bar{\mathscr{P}} y \qquad \text{whenever} \qquad A_x \mathscr{P} B \quad \text{and} \quad A \bar{\mathscr{P}} y \tag{6.2}$$

and

$$A \mathscr{P} x \quad \text{implies} \quad B \bar{\mathscr{P}} y \qquad \text{whenever} \qquad B \bar{\mathscr{P}} x \quad \text{and} \quad B_y \mathscr{P} A. \tag{6.3}$$

Both rules follow because the opposite assumption, $B \mathscr{P} y$, would lead to a contradiction. Note that, by Lemma 4.2(iv), the observation $A \mathscr{P} x$ is equivalent to $A \mathscr{P} A_x$. Thus, in the case of (6.2), $B \mathscr{P} y$ would give us $A \mathscr{P} A_x \mathscr{P} B \mathscr{P} y$, contradicting $A \bar{\mathscr{P}} y$. Similarly, with (6.3) we would have $B \mathscr{P} B_y \mathscr{P} A \mathscr{P} x$, contradicting $B \bar{\mathscr{P}} x$. When we get a negative response $A \bar{\mathscr{P}} x$ from the expert, we can only obtain negative inferences. We can formulate the rule

$$A \bar{\mathscr{P}} x \quad \text{implies} \quad B \bar{\mathscr{P}} y \qquad \text{whenever} \qquad A \mathscr{P} B \quad \text{and} \quad B_y \mathscr{P} x, \tag{6.4}$$

since $B \mathscr{P} y$ under the conditions of (6.4) would imply $A \mathscr{P} x$.

It must be noted that rules (6.2) to (6.4) do not necessarily find all possible negative inferences, even when rule (6.4) is applied iteratively to the obtained negative inferences. Rule (6.4), for instance, is a special case of the more general rule

$$A \bar{\mathscr{P}} x \quad \text{implies} \quad B \bar{\mathscr{P}} C \qquad \text{whenever} \qquad A \mathscr{P} B \quad \text{and} \quad (B \cup C) \mathscr{P} x.$$

But the inference $B \bar{\mathscr{P}} C$ cannot be processed in the restricted range version of $\mathscr{P}$. It means that for some $y \in C$ we must have $B \bar{\mathscr{P}} y$, but unless for all but one element z of C we had already established $B \mathscr{P} z$, we do not know which y to pick. (Of course, if we had already $B \bar{\mathscr{P}} y$ for some $y \in C$, the inference tells us nothing new.) So in general we would have to save all such inferences $B \bar{\mathscr{P}} C$ and after each new inference we would have to check whether any saved inference can now be consummated. This appears to be rather heavy. In practice we will use only the rules (6.2) to (6.4) that apply directly to the singleton set range of $\mathscr{P}$. Below will be indicated how we deal with their being incomplete.

Using the implications (6.1) really means that we get to our ultimate relation $\mathscr{P}$ via a number of intermediate relations, all of which are entail relations. In the absence of any information we start with a relation $\mathscr{P}_0$ which is set to the superset relation on the power set of notions. If, at time t, the response of the expert is negative, we do not change the relation; that is, in this case $\mathscr{P}_t = \mathscr{P}_{t-1}$. If, however, the response is positive, we add the newly established pair to the relation $\mathscr{P}_{t-1}$. This will result in a relation that, in general, is no longer an entail relation. But using the inferences of rule (6.1) we add in fact all pairs that are necessary to turn the new relation into such an entail relation. Thus we obtain $\mathscr{P}_t$ as the smallest entail relation for X that contains both $\mathscr{P}_{t-1}$ and the pair that induced the positive response at time t.

In order for this procedure to terminate (and in order to avoid asking unnecessary questions) we have to keep track of the pairs known not to belong to $\mathscr{P}$. At any time t these are collected in the relation $\mathscr{N}_t$. Since initially any pair can be in $\mathscr{P}$, we start with $\mathscr{N}_0 = \varnothing$. The inference rules (6.2), (6.3), and (6.4) show that either a positive or a negative response at time t may imply the exclusion from $\mathscr{P}$ of a number of new pairs, and $\mathscr{N}_t$ is obtained from $\mathscr{N}_{t-1}$ by adding these pairs. At time t the question to be presented to the expert is picked out of the pairs that are neither in $\mathscr{P}_{t-1}$ nor in $\mathscr{N}_{t-1}$. The complication arising from the fact that these rules do not necessarily find all excluded pairs is resolved by first computing the positive inferences of a positive answer, before actually posing the question to the expert. In the (relatively rare) case that we would observe that a positive answer would lead to a contradiction (a positive inference that is included in $\mathscr{N}_{t-1}$), we conclude that the chosen pair is actually excluded from $\mathscr{P}$. So we add it (together with the negative inferences drawn from it by (6.4)) to $\mathscr{N}_{t-1}$ and we choose another question. (The extra work of exploring the possible inferences of a candidate question is a sensible thing to do anyway. It can direct us in choosing a "most informative" question at time t.)

Thus we obtain an increasing chain $\{\mathscr{N}_t\}$ and at any time $\mathscr{N}_t \cap \mathscr{P}_t = \varnothing$. Since obviously with each observation either $\mathscr{N}_t$ or $\mathscr{P}_t$ (or both) increases, there must come a time s such that $\mathscr{N}_s \cup \mathscr{P}_s = 2^X \times 2^X$. (This, too, can be checked considering the restricted range only.) Then the process stops and we may conclude $\mathscr{P} = \mathscr{P}_s$. So we have constructed the relation $\mathscr{P}$ as the maximum element of an increasing chain $\{\mathscr{P}_t\}$ of entail relations for X. By the results of Section 4, then, the definition

$$\mathscr{K}_t = k(\mathscr{P}_t)$$

induces a decreasing chain $\{\mathscr{K}_t\}$ of knowledge spaces and the whole procedure can be summarized in the following scheme:

$$\begin{array}{rcccccccccc}
\varnothing = & \mathscr{N}_0 \subseteq & \mathscr{N}_1 \subseteq & \mathscr{N}_2 \subseteq & \cdots \subseteq & \mathscr{N}_t \subseteq & \cdots \subseteq & \mathscr{N}_s = \bar{\mathscr{P}}_s \\
\text{“}\supseteq\text{”} = & \mathscr{P}_0 \subseteq & \mathscr{P}_1 \subseteq & \mathscr{P}_2 \subseteq & \cdots \subseteq & \mathscr{P}_t \subseteq & \cdots \subseteq & \mathscr{P}_s = \mathscr{P} \\
k\colon & \downarrow & \downarrow & \downarrow & & \downarrow & & \downarrow \\
2^X = & \mathscr{K}_0 \supseteq & \mathscr{K}_1 \supseteq & \mathscr{K}_2 \supseteq & \cdots \supseteq & \mathscr{K}_t \supseteq & \cdots \supseteq & \mathscr{K}_s = \mathscr{K}
\end{array}$$

So at each moment t of the construction process we have a knowledge space $\mathscr{K}_t$ that is a conservative approximation of the final knowledge space $\mathscr{K}$; conservative in the sense that any $\mathscr{K}_t$ contains all states of $\mathscr{K}$. As mentioned in Section 2, this fact is of some practical significance. It means that when we want the knowledge space for future use in knowledge assessment procedures, we may interrupt the construction process at any time practical considerations lead us to do so, and we will end up with a knowledge space that possibly contains some "nuisance" states, but in which at least no vital states are missing.

In particular, we can start the construction process with the singleton subsets ("if a subject fails x, is it safe to conclude that he will also fail y?") and stop when all such questions have been asked. If $\mathscr{P}_t$ is the obtained relation at this point, then, by construction, $\mathscr{P}_t$ satisfies condition (*) of 4.8 and clearly any addition to $\mathscr{P}_t$ will invalidate the "only if" part of this equivalence. So the corresponding knowledge space $\mathscr{K}_t$ is closed under intersection (as it should be, since until now we have, in effect, asked questions regarding the quasi order on X, so we are still in the Birkhoff case) and it is the smallest such space. Thus, stopping the process at this point leaves us with the closure under intersection of the final knowledge space $\mathscr{K}$.

6.2. Example. Let us illustrate this procedure with the example from Section 2, where $X = \{a, b, c, d, e\}$. To simplify notation we will in the sequel denote subsets of X as strings without surrounding braces and separators between the elements. So abc denotes the subset $\{a, b\ c\}$. In this way we lose the distinction between a singleton set and its element, but the correct interpretation will be clear from context. Furthermore, pairs of subsets are given in a dot notation: $abc.de$ represents the pair of subsets $(\{a, b, c\}, \{d, e\})$.

Now suppose we want to recover the knowledge space

$$\mathscr{K} = \{\emptyset, c, e, be, ce, abe, bce, cde, abce, acde, bcde, X\},$$

given in Section 2, by gradually constructing the corresponding entail relation for X from the expert's responses. We have to ask the expert questions of the kind: does failing all in Z imply failing x, where Z runs through the subsets of X and x through X. The order of these questions is in principle arbitrary, but it makes sense to start with the simpler ones. We will adopt here a very straightforward rule for choosing the next question to ask. We order the subsets by increasing cardinality and within classes of equal cardinality we choose, arbitrarily, the lexicographic ordering; this ordering is also used for the elements of X. Thus we obtain an ordering of the pairs (Z, x) by letting (Z, x) precede (Z', x') iff Z precedes Z' or $Z = Z'$ and x precedes x'. We choose for the next question the first undecided pair in this ordering.

Using this design in questioning a perfectly reliable expert we would get the results gathered in Table 1. Here and in the sequel we show results for the relations $\mathscr{P}_t$ and $\mathscr{N}_t$ only restricted to the singleton set range. We start with a relation $\mathscr{P}_0$ set equal to the superset relation on the power set of X and the corresponding knowledge space $\mathscr{K}_0$ equal to the power set. In principle $\mathscr{N}_0 = \emptyset$ (any pair of subsets can be in the relation $\mathscr{P}$) and we would start inquiring about the pairs $\emptyset.x$, for $x \in X$. But a moment's thought will reveal that $\emptyset\, \mathscr{P}\, x$ means precisely that x does not appear at all in the knowledge space. We will save five questions here by assuming we know the specification of the set of problems X is right in the sense that $X = \bigcup \mathscr{K}$. On this assumption we can then "infer" $\emptyset\, \mathscr{N}\, x$ for all $x \in X$, giving us our initial $\mathscr{N}_0$. So our real first question to the expert is Does failing a imply failing b? As we can check from the above specification, $\mathscr{K}$ has states containing

TABLE 1

Observed Responses of the Expert, Based on the Space $\mathcal{K}$ of Example 6.1, and Inferences, Yielding Successive Approximations of the Entail Relation $\mathcal{P}$ and the Corresponding Knowledge Space $\mathcal{K}$

	At Start:	$\mathcal{N}_0 = \{\varnothing . x : x \in X\}$	$\mathcal{P}_0 = \supseteq$	$\mathcal{K}_0 = 2^X$
t	Observed	Adding to $\mathcal{N}$	Adding to $\mathcal{P}$	Deleting from $\mathcal{K}$
1 to 10	$a \mathcal{N} x,\ x \neq a$ $b \mathcal{N} x,\ x \neq b$ $c \mathcal{N} a,\ c \mathcal{N} b$	$a.b\ a.c\ a.d\ a.e$ $b.a\ b.c\ b.d\ b.e$ $c.a\ c.b$		
11	$c \mathcal{P} d$	$d.a\ \ d.b\ \ cd.a$ $cd.b$	$c.d\ \ ac.d\ \ bc.d$ $ce.d\ \ abc.d\ \ ace.d$ $bce.d\ \ abce.d$	$abde\ \ bde\ \ ade$ $abd\ \ de\ \ bd$ $ad\ \ d$
12	$c \mathcal{N} e$	$c.e\ \ d.e\ \ cd.e$		
13	$d \mathcal{N} c$	$d.c$		
14	$e \mathcal{P} a$		$e.a\ \ be.a\ \ ce.a$ $de.a\ \ bce.a\ \ bde.a$ $cde.a\ \ bcde.a$	$abcd\ \ acd\ \ abc$ $ac\ \ ab\ \ a$
15	$e \mathcal{P} b$		$e.b\ \ ae.b\ \ ce.b$ $de.b\ \ ace.b\ \ ade.b$ $cde.b\ \ acde.b$	$bcd\ \ bc\ \ b$
16	$e \mathcal{N} c$	$e.c\ \ ab.c\ \ ae.c$ $be.c\ \ abe.c$		
17	$e \mathcal{P} d$	$ad.c\ \ bd.c\ \ de.c$ $abd.c\ \ ade.c$ $bde.c\ \ abde.c$	$e.d\ \ ae.d\ \ be.d$ $abe.d$	cd
18	$ab \mathcal{N} d$	$ab.d\ \ ab.e$		
19	$ac \mathcal{N} b$	$ac.b\ \ ac.e\ \ ad.b$ $ad.e\ \ acd.b\ \ acd.e$		
20	$bc \mathcal{P} a$		$bc.a\ \ bcd.a$	ae
21	$bc \mathcal{N} e$	$bc.e\ \ bd.e\ \ abc.e$ $abd.e\ \ bcd.e\ \ abcd.e$		
22	$bd \mathcal{P} a$		$bd.a$	ace

b and not containing a (be, for instance), so we will get a negative response from our expert. The same is true for the next 9 questions, up to and including the question Does failing c imply failing b? Since we cannot infer anything from negative responses in the absence of any positive response, we can only add these pairs of subsets to the relation $\mathcal{N}_0$ to obtain $\mathcal{N}_{10}$, while still $\mathcal{P}_{10} = \mathcal{P}_0$ and $\mathcal{K}_{10} = \mathcal{K}_0$.

At $t = 11$ we get our first success: since any state of $\mathcal{K}$ containing d also contains

c, we get a positive response from our expert for the pair $c.d$. As we can see in Table 1, this gives us some extra inferences. The negative inference $d \mathcal{N} a$, for instance, follows, by transitivity of $\mathcal{P}$, from $c \mathcal{N} a$ and $c \mathcal{P} d$, and since $c \mathcal{P} d$ and the known $c \mathcal{P} c$ imply $c \mathcal{P} cd$, we similarly obtain $cd \mathcal{N} b$ from $c \mathcal{N} b$, and so on. These are all simple instances of rule (6.2). The positive inferences all follow from a special case of rule (6.1). With $c \mathcal{P} d$ we have $C \supseteq c \mathcal{P} d$ for any set C containing c and this adds to $\mathcal{P}_{11}$ all pairs $C.d$ where C contains c but not d. Adding pairs to $\mathcal{P}_{11}$ means deleting elements from $\mathcal{K}_{11}$ and the correspondence is via Lemma 4.4. Adding $c.d$ to $\mathcal{P}$, for instance, means that it is no longer true that $c \mathcal{P} Z$ implies $c \supseteq Z$ for all $Z \in 2^X$, so according to Lemma 4.4 we have to drop $X \backslash c = abde$ from $\mathcal{K}_{11}$. In the same way $ace \mathcal{P} d$ forces us to discard bd from $\mathcal{K}_{11}$ and so on.

The situation after the first positive response at $t = 11$ is depicted in Table 2 in the column with that heading. Here the problem of finding the relation $\mathcal{P}$ is represented as filling out a matrix, the rows of which are indexed by the subsets of X and the columns by the elements of X. In this and in the next columns of Table 2 boldface **p** and **n** entries represent positive, respectively negative inferences made from the expert's positive response to the latest question asked; italic p (n) denotes earlier positive (negative) inferences and roman p (n) denotes previously observed positive (negative) responses from the expert. A "$\ni$" entry corresponds to row–column pairs that are in $\mathcal{P}$ because the subset of the row contains the column element, while a dot indicates that the corresponding pair is still undecided. In a subcolumn is checked which rows contribute states to the current knowledge space; according to Lemma 4.4 the *complement* of a row indexing subset is a state as long as there is no positive (p, p or **p**) entry in that row. The checkmark is in boldface when the corresponding row has been completed, meaning that the complementary set is bound to be a state of the final knowledge space.

Returning to Table 1, we see that at times $t = 12$ and $t = 13$ we get negative responses, giving some negative inferences, and the next positive response is obtained at $t = 14$. From this we get a number of positive inferences and consequently a number of states can be discarded, all of this in the same way as we have seen above. In this way the process continues, until at $t = 22$ the relation $\mathcal{P}$ is completely known. With $\mathcal{K}_{22}$ we have indeed reconstructed the knowledge space $\mathcal{K}$ we used for deducing the expert's responses. In Table 2 we show the situations after each newly obtained positive response and consequent change in $\mathcal{K}_t$. The inquisitive reader may want to check that at each moment t the knowledge structure $\mathcal{K}_t$ is closed under union; that for $t = 0, ..., 17$ $\mathcal{K}_t$ is also closed under intersection, while this no longer holds from $t = 20$; and finally that indeed $\mathcal{K}_{17}$ is the closure under intersection of $\mathcal{K}_{22}$. The corresponding relation $\mathcal{P}_{17}$, considered as a partial order on X, equals exactly the partial order $\mathcal{P}$ we found in Section 2 (see Fig. 2) as our best try for a quasi order representation of $\mathcal{K}$.

This example illustrates the correspondence between knowledge spaces and entail relations, and it shows in principle how we can obtain the space by questioning an expert about the entail relation. However, it will also be clear from this example

TABLE 2

Situations after Each Positive Response of the Expert (see Table 1)

	$t=11$		$t=14$		$t=15$		$t=17$		$t=20$		$t=22$	
	a b c d e	$\mathscr{K}$	*a b c d e*	$\mathscr{K}$	*a b c d e*	$\mathscr{K}$	*a b c d e*	$\mathscr{K}$	*a b c d e*	$\mathscr{K}$	*a b c d e*	$\mathscr{K}$
∅	*n n n n n*	**√**	*n n n n n*	**√**	*n n n n n*	**√**	*n n n n n*	**√**	*n n n n n*	**√**	*n n n n n*	**√**
a	∋ n n n n	**√**	∋ n n n n	**√**	∋ n n n n	**√**	∋ n n n n	**√**	∋ n n n n	**√**	∋ n n n n	**√**
b	n ∋ n n n	**√**	n ∋ n n n	**√**	n ∋ n n n	**√**	n ∋ n n n	**√**	n ∋ n n n	**√**	n ∋ n n n	**√**
c	n n ∋ **p** ·		n n ∋ p n		n n ∋ p n		n n ∋ p n		n n ∋ p n		n n ∋ p n	
d	**n n** · ∋ ·	√	*n n* n ∋ *n*	**√**	*n n* n ∋ *n*	**√**	*n n* n ∋ *n*	**√**	*n n* n ∋ *n*	**√**	*n n* n ∋ *n*	**√**
e	· · · · ∋	√	**p** · · · ∋		p **p** · · ∋		p p n **p** ∋		p p n p ∋		p p n p ∋	
ab	∋ ∋ · · ·	√	∋ ∋ · · ·	√	∋ ∋ · · ·	√	∋ ∋ *n* · ·	√	∋ ∋ *n* n *n*	**√**	∋ ∋ *n* n *n*	**√**
ac	∋ · ∋ **p** ·		∋ · ∋ *p* ·		∋ · ∋ *p* ·		∋ · ∋ *p* ·		∋ n ∋ *p n*		∋ n ∋ *p n*	
ad	∋ · · ∋ ·	√	∋ · · ∋ ·	√	∋ · · ∋ ·	√	∋ · **n** ∋ ·	√	∋ *n n* ∋ *n*	**√**	∋ *n n* ∋ *n*	**√**
ae	∋ · · · ∋	√	∋ · · · ∋	√	∋ **p** · · ∋		∋ *p n* **p** ∋		∋ *p n p* ∋		∋ *p n p* ∋	
bc	· ∋ ∋ **p** ·		· ∋ ∋ *p* ·		· ∋ ∋ *p* ·		· ∋ ∋ *p* ·		**p** ∋ ∋ *p* ·		p ∋ ∋ *p* **n**	
bd	· ∋ · ∋ ·	√	· ∋ · ∋ ·	√	· ∋ · ∋ ·	√	· ∋ **n** ∋ ·	√	· ∋ *n* ∋ ·	√	**p** ∋ *n* ∋ *n*	
be	· ∋ · · ∋	√	**p** ∋ · · ∋		*p* ∋ · · ∋		*p* ∋ *n* **p** ∋		*p* ∋ *n p* ∋		*p* ∋ *n p* ∋	
cd	**n n** ∋ ∋ ·	√	*n n* ∋ ∋ *n*	**√**	*n n* ∋ ∋ *n*	**√**	*n n* ∋ ∋ *n*	**√**	*n n* ∋ ∋ *n*	**√**	*n n* ∋ ∋ *n*	**√**
ce	· · ∋ **p** ∋		**p** · ∋ *p* ∋		*p* **p** ∋ *p* ∋		*p p* ∋ *p* ∋		*p p* ∋ *p* ∋		*p p* ∋ *p* ∋	
de	· · · ∋ ∋	√	**p** · · ∋ ∋		*p* **p** · ∋ ∋		*p p* **n** ∋ ∋		*p p n* ∋ ∋		*p p n* ∋ ∋	
abc	∋ ∋ ∋ **p** ·		∋ ∋ ∋ *p* ·		∋ ∋ ∋ *p* ·		∋ ∋ ∋ *p* ·		∋ ∋ ∋ *p* ·		∋ ∋ ∋ *p n*	
abd	∋ ∋ · ∋ ·	√	∋ ∋ · ∋ ·	√	∋ ∋ · ∋ ·	√	∋ ∋ **n** ∋ ·	√	∋ ∋ *n* ∋ ·	√	∋ ∋ *n* ∋ *n*	**√**
abe	∋ ∋ · · ∋	√	∋ ∋ · · ∋	√	∋ ∋ · · ∋	√	∋ ∋ *n* **p** ∋		∋ ∋ *n p* ∋		∋ ∋ *n p* ∋	
acd	∋ · ∋ ∋ ·	√	∋ · ∋ ∋ ·	√	∋ · ∋ ∋ ·	√	∋ · ∋ ∋ ·	√	∋ *n* ∋ ∋ *n*	**√**	∋ *n* ∋ ∋ *n*	**√**
ace	∋ · ∋ **p** ∋		∋ · ∋ *p* ∋		∋ **p** ∋ *p* ∋		∋ *p* ∋ *p* ∋		∋ *p* ∋ *p* ∋		∋ *p* ∋ *p* ∋	
ade	∋ · · ∋ ∋	√	∋ · · ∋ ∋	√	∋ **p** · ∋ ∋		∋ *p* **n** ∋ ∋		∋ *p n* ∋ ∋		∋ *p n* ∋ ∋	
bcd	· ∋ ∋ ∋ ·	√	· ∋ ∋ ∋ ·	√	· ∋ ∋ ∋ ·	√	· ∋ ∋ ∋ ·	√	**p** ∋ ∋ ∋ ·		*p* ∋ ∋ ∋ *n*	
bce	· ∋ ∋ **p** ∋		**p** ∋ ∋ *p* ∋		*p* ∋ ∋ *p* ∋		*p* ∋ ∋ *p* ∋		*p* ∋ ∋ *p* ∋		*p* ∋ ∋ *p* ∋	
bde	· ∋ · ∋ ∋	√	**p** ∋ · ∋ ∋		*p* ∋ · ∋ ∋		*p* ∋ **n** ∋ ∋		*p* ∋ *n* ∋ ∋		*p* ∋ *n* ∋ ∋	
cde	· · ∋ ∋ ∋	√	**p** · ∋ ∋ ∋		*p* **p** ∋ ∋ ∋		*p p* ∋ ∋ ∋		*p p* ∋ ∋ ∋		*p p* ∋ ∋ ∋	
abcd	∋ ∋ ∋ ∋ ·	√	∋ ∋ ∋ ∋ ·	√	∋ ∋ ∋ ∋ ·	√	∋ ∋ ∋ ∋ ·	√	∋ ∋ ∋ ∋ ·	√	∋ ∋ ∋ ∋ *n*	**√**
abce	∋ ∋ ∋ **p** ∋		∋ ∋ ∋ *p* ∋		∋ ∋ ∋ *p* ∋		∋ ∋ ∋ *p* ∋		∋ ∋ ∋ *p* ∋		∋ ∋ ∋ *p* ∋	
abde	∋ ∋ · ∋ ∋	√	∋ ∋ · ∋ ∋	√	∋ ∋ · ∋ ∋	√	∋ ∋ **n** ∋ ∋	**√**	∋ ∋ *n* ∋ ∋	**√**	∋ ∋ *n* ∋ ∋	**√**
acde	∋ · ∋ ∋ ∋	√	∋ · ∋ ∋ ∋	√	∋ **p** ∋ ∋ ∋		∋ *p* ∋ ∋ ∋		∋ *p* ∋ ∋ ∋		∋ *p* ∋ ∋ ∋	
bcde	· ∋ ∋ ∋ ∋	√	**p** ∋ ∋ ∋ ∋		*p* ∋ ∋ ∋ ∋		*p* ∋ ∋ ∋ ∋		*p* ∋ ∋ ∋ ∋		*p* ∋ ∋ ∋ ∋	
X	∋ ∋ ∋ ∋ ∋	**√**	∋ ∋ ∋ ∋ ∋	**√**	∋ ∋ ∋ ∋ ∋	**√**	∋ ∋ ∋ ∋ ∋	**√**	∋ ∋ ∋ ∋ ∋	**√**	∋ ∋ ∋ ∋ ∋	**√**

Note. Indicated are positive and negative inferences from the latest response (**p** and **n**), earlier inferences (*p* and *n*) and earlier observed responses (p and n). The "∋" means that the row subset contains the column element, a dot that the corresponding pair is still undecided. The complement of a subset indexing a checked row is in the current knowledge space; if the check is in boldface it is known to be in the final space.

that, as such, the procedure would not be practicable, even for a very moderate number of problems in X. For instance, for a 20 problem set the equivalent of Table 2 would consist of over one million rows. The size of this table simply doubles with each additional problem. On closer inspection, however, it appears that, generally, many rows are redundant: the complete information on entail relation and knowledge space is contained in a subtable of considerably smaller size. An essential part of the algorithm presented in Koppen (1989) deals precisely with the issue of constructing just this minimal subtable, dynamically, in the course of questioning the expert. (The minimal subtable depends on the obtained responses.) The algorithm of Koppen (1989) cannot avoid the theoretical—but in practice uninteresting—worst case where all subsets are states (the expert will give only negative responses and the minimal subtable is the complete table), but it has proved to be applicable to the real-life situation of a set of 50 problems in U.S. high school mathematics. The results of this application will be reported elsewhere (Kambouri *et al.*, 1989). Let us here just mention that actually constructed minimal subtables in this case were on the order of 2000 rows, while the naive "Table 2 version" would contain well over 10^{15} rows. This gives an idea of the reduction that can be obtained once we go beyond the straightforward procedure described in this section for illustrative purposes.

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