Exercise 1: Bayesian Linear Model

In the Bayesian linear model, we assume that the data follows the following law:

$$y = f(\mathbf{x}) + \epsilon = \boldsymbol{\theta}^T \mathbf{x} + \epsilon,$$

where $\varepsilon \sim \mathcal{N}(0, \sigma^2)$ and independent of **x**. On the data-level this corresponds to

$$y^{(i)} = f\left(\mathbf{x}^{(i)}\right) + \epsilon^{(i)} = \boldsymbol{\theta}^T \mathbf{x}^{(i)} + \epsilon^{(i)}, \text{ for } i \in \{1, \dots, n\}$$

where $\epsilon^{(i)} \sim \mathcal{N}(0, \sigma^2)$ are iid and all independent of the $\mathbf{x}^{(i)}$'s. In the Bayesian perspective it is assumed that the parameter vector $\boldsymbol{\theta}$ is stochastic and follows a distribution.

Assume we are interested in the so-called maximum a posteriori estimate of θ , which is defined by

$$\hat{\boldsymbol{\theta}} = \arg \max_{\boldsymbol{\theta}} p(\boldsymbol{\theta}|\mathbf{X}, \mathbf{y}).$$

(a) Show that if we choose a uniform distribution over the parameter vectors $\boldsymbol{\theta}$ as the prior belief, i.e.,

$$q(\boldsymbol{\theta}) \propto 1$$
,

then the maximum a posteriori estimate coincides with the empirical risk minimizer for the L2-loss (over the linear models).

(b) Show that if we choose a Gaussian distribution over the parameter vectors $\boldsymbol{\theta}$ as the prior belief, i.e.,

$$q(\boldsymbol{\theta}) \propto \exp \left[-\frac{1}{2\tau^2} \boldsymbol{\theta}^\top \boldsymbol{\theta} \right], \qquad \tau > 0,$$

then the maximum a posteriori estimate coincides for a specific choice of τ with the regularized empirical risk minimizer for the L2-loss with L2 penalty (over the linear models), i.e., the Ridge regression.

(c) Show that if we choose a Laplace distribution over the parameter vectors $\boldsymbol{\theta}$ as the prior belief, i.e.,

$$q(\boldsymbol{\theta}) \propto \exp\left[-\frac{\sum_{i=1}^{p} |\boldsymbol{\theta}_i|}{\tau}\right], \quad \tau > 0,$$

then the maximum a posteriori estimate coincides for a specific choice of τ with the regularized empirical risk minimizer for the L2-loss with L1 penalty (over the linear models), i.e., the Lasso regression.

Exercise 2: Gaussian Posterior Process

Assume your data follows the following law:

$$y = f + \varepsilon, \quad \varepsilon \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I}),$$

with $f = f(x) \in \mathbb{R}^n$ being a realization of a Gaussian process (GP), for which we a priori assume

$$f(\mathbf{x}) \sim \mathcal{GP}(m(\mathbf{x}), k(\mathbf{x}, \mathbf{x}')).$$

 \boldsymbol{x} here only consists of 1 feature that is observed for n data points.

- (a) Derive / define the prior distribution of f.
- (b) Derive the posterior distribution f|y.
- (c) Derive the posterior predictive distribution $y_*|x_*, \boldsymbol{x}, \boldsymbol{y}$ for a new sample x_* from the same data-generating process.
- (d) Implement the GP with squared exponential kernel, zero mean function and $\ell = 1$ from scratch for n = 2 observations (y, x). Do this as efficiently as possible by explicitly calculating all expensive computations by hand. Do the same for the posterior predictive distribution of y_* . Test your implementation using simulated data.