ONLINE CONVEX OPTIMIZATION

- One of the most relevant instantiations of the online learning problem is the problem of *online convex optimization* (OCO), which is characterized by a loss function $L : \mathcal{A} \times \mathcal{Z} \to \mathbb{R}$, which is convex w.r.t. the action, i.e., $a \mapsto L(a, z)$ is convex for any $z \in \mathcal{Z}$.

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- Note that both OLO and OQO belong to the class of online convex optimization problems:
  - *Online linear optimization (OLO) with convex action spaces:* $\ell(a, z) = a^\top z$ is a convex function in $a \in \mathcal{A}$, provided $\mathcal{A}$ is convex.
  - *Online quadratic optimization (OQO) with convex action spaces:* $\ell(a, z) = \frac{1}{2}\|a - z\|_2^2$ is a convex function in $a \in \mathcal{A}$, provided $\mathcal{A}$ is convex.

ONLINE GRADIENT DESCENT: MOTIVATION

- We have seen that the FTRL algorithm with the ℓ_2 norm regularization $\psi(a) = \frac{1}{2\eta}\,||a||_2^2$ achieves satisfactory results for online linear optimization (OLO) problems, that is, if $\ell(a, z) = L^{\text{lin}}(a, z) = a^\top z$, then we have
  - *Fast updates* — If $\mathcal{A} = \mathbb{R}^d$, then

$$a_{t+1}^{\mathrm{FTRL}} = a_t^{\mathrm{FTRL}} - \eta z_t, \quad t = 1, \dots, T;$$

  - *Regret bounds* — By an appropriate choice of η and some (mild) assumptions on $\mathcal{A}$ and $\mathcal{Z}$, we have

$$R_T^{\mathrm{FTRL}} = o(T).$$

ONLINE GRADIENT DESCENT: MOTIVATION

Apparently, the nice form of the loss function L^{lin} is responsible for the appealing properties of FTRL in this case. Indeed, since $\nabla_a L^{\mathrm{lin}}(a, z) = z$ note that the update rule can be written as

$$a_{t+1}^{\mathrm{FTRL}} = a_t^{\mathrm{FTRL}} - \eta z_t = a_t^{\mathrm{FTRL}} - \eta \nabla_a L^{\mathrm{lin}}(a_t^{\mathrm{FTRL}}, z_t).$$

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Interpretation: In each time step $t+1$, we are following the direction with the steepest decrease of the loss (represented by $-\nabla L^{\mathrm{lin}}(a_t^{\mathrm{FTRL}}, z_t)$) from the current "position" a_t^{FTRL} with the step size η.

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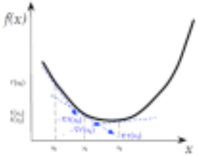

$\Rightarrow$ Gradient Descent.

ONLINE GRADIENT DESCENT: MOTIVATION

- **Question:** How to transfer this idea of the Gradient Descent for the update formula to other loss functions, while still preserving the regret bounds?
- **Solution (for convex losses):** Recall the equivalent characterization of convexity of differentiable convex functions:

$$
\begin{aligned}
f: S \to \mathbb{R} \text{ is convex } &\Leftrightarrow f(y) \geq f(x) + (y-x)^\top \nabla f(x) \text{ for any } x, y \in S \\
&\Leftrightarrow f(x) - f(y) \leq (x-y)^\top \nabla f(x) \text{ for any } x, y \in S.
\end{aligned}
$$

- This means if we are dealing with a loss function $L: \mathcal{A} \times \mathcal{Z} \to \mathbb{R}$, which is convex and differentiable in its first argument ($\mathcal{A}$ has also to be convex), then

$$
L(a, z) - L(\tilde{a}, z) \leq (a - \tilde{a})^\top \nabla_a L(a, z), \ \forall a, \tilde{a} \in \mathcal{A}, z \in \mathcal{Z}.
$$

ONLINE GRADIENT DESCENT: MOTIVATION

- **Reminder:** $(a,z) - (\bar{a},z) \leq (a - \bar{a})^\top \nabla_a (a,z) \; \forall a, \bar{a} \in \mathcal{A}, z \in \mathcal{Z}.$

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- Let $z_1, \ldots, z_T$ arbitrary environmental data and $a_1, \ldots, a_T$ be some arbitrary action sequence. Substitute $\tilde{z}_t := \nabla_a \ell(a_t, z_t)$ and note that

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$$R_T(\tilde{a}) = \sum_{t=1}^{T} L(a_t, z_t) - L(\tilde{a}, z_t) \leq \sum_{t=1}^{T} (a_t - \tilde{a})^\top \nabla_a L(a_t, z_t)$$

$$= \sum_{t=1}^{T} (a_t - \tilde{a})^\top \tilde{z}_t = \sum_{t=1}^{T} a_t^\top \tilde{z}_t - \tilde{a}^\top \tilde{z}_t = \sum_{t=1}^{T} L^{\text{lin}}(a_t, \tilde{z}_t) - L^{\text{lin}}(\tilde{a}, \tilde{z}_t).$$

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Conclusion: The regret of a learner with respect to a differentiable and convex loss function ℓ is bounded by the regret corresponding to an online linear optimization problem with environmental data $\tilde{z}_t = \nabla_a \ell(a_t, z_t)$.

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- **We know:** Online linear optimization problems can be tackled by means of the FTRL algorithm!

$\rightsquigarrow$ Incorporate the substitution $\tilde{z}_t = \nabla_a \ell(a_t, z_t)$ into the update formula of FTRL with squared L2-norm regularization.

ONLINE GRADIENT DESCENT: DEFINITION AND PROPERTIES

- The corresponding algorithm which chooses its action according to these considerations is called the *Online Gradient Descent* (OGD) algorithm with step size $\eta > 0$. It holds in particular,

$$a_{t+1}^{\mathrm{OGD}} = a_t^{\mathrm{OGD}} - \eta \nabla_a L(a_t^{\mathrm{OGD}}, z_t), \quad t = 1, \ldots, T. \tag{1}$$

(Technical side note: For this update formula we assume that $\mathcal{A} = \mathbb{R}^d$. Moreover, the first action a_1^{OGD} is arbitrary.)

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- We have the following connection between FTRL and OGD:
  - Let $\tilde{z}_t^{\mathrm{OGD}} := \nabla_a L(a_t^{\mathrm{OGD}}, z_t)$ for any $t = 1, \dots, T$.
  - The update formula for FTRL with L_2 norm regularization for the linear loss L^{lin} and the environmental data $\tilde{z}_t^{\mathrm{OGD}}$ is

$$a_{t+1}^{\mathrm{FTRL}} = a_t^{\mathrm{FTRL}} - \eta \tilde{z}_t^{\mathrm{OGD}} = a_t^{\mathrm{FTRL}} - \eta \nabla_a L(a_t^{\mathrm{OGD}}, z_t).$$

  - If we have that $a_1^{\mathrm{FTRL}} = a_1^{\mathrm{OGD}}$, then it iteratively follows that $a_{t+1}^{\mathrm{FTRL}} = a_{t+1}^{\mathrm{OGD}}$ for any $t = 1, \dots, T$ in this case.

ONLINE GRADIENT DESCENT: DEFINITION AND PROPERTIES

- With the deliberations above we can infer that

$$
\begin{aligned}
R_{T,L}^{\mathrm{OGD}}(\tilde{a} \mid (z_t)_t) &= \sum_{t=1}^{T} \left(L(a_t^{\mathrm{OGD}}, z_t) - L(\tilde{a}, z_t)\right) \\
&\leq \sum_{t=1}^{T} L^{\mathrm{lin}}(a_t^{\mathrm{OGD}}, \tilde{z}_t^{\mathrm{OGD}}) - L^{\mathrm{lin}}(\tilde{a}, \tilde{z}_t^{\mathrm{OGD}}) \\
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&= R_{T,L^{\mathrm{lin}}}^{\mathrm{FTRL}}(\tilde{a} \mid (\tilde{z}_t^{\mathrm{OGD}})_t),
\end{aligned}
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where we write in the subscripts of the regret the corresponding loss function and also include the corresponding environmental data as a second argument in order to emphasize the connections.

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\end{aligned}
$$

  where we write in the subscripts of the regret the corresponding loss function and also include the corresponding environmental data as a second argument in order to emphasize the connections.

- *Interpretation:* The regret of the FTRL algorithm (with L_2 norm regularization) for the online linear optimization problem (characterized by the linear loss L^{lin}) with environmental data $\tilde{z}_t^{\mathrm{OGD}}$ is an upper bound for the OGD algorithm for the online convex problem (characterized by a differentiable convex loss L) with the original environmental data z_t.

ONLINE GRADIENT DESCENT: REGRET

- Due to this connection we immediately obtain a similar decomposition of the regret upper bound into a bias term and a variance term as for the FTRL algorithm for OLO problems.
- **Corollary.** Using the OGD algorithm on any online convex optimization problem (with differentiable loss function) leads to a regret of OGD with respect to any action $\tilde{a} \in \mathcal{A}$ of

$$
\begin{aligned}
R_T^{\mathrm{OGD}}(\tilde{a}) &\leq \frac{1}{2\eta}\|\tilde{a}\|_2^2 + \eta \sum_{t=1}^{T} \left\| z_t^{\mathrm{OGD}} \right\|_2^2 \\
&= \frac{1}{2\eta}\|\tilde{a}\|_2^2 + \eta \sum_{t=1}^{T} \left\| \nabla_a \ell(a_t^{\mathrm{OGD}}, z_t) \right\|_2^2 .
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\end{aligned}
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- Note that the step size $\eta > 0$ of OGD has the same role as the regularization magnitude of FTRL: It should balance the trade-off between the bias- and the variance-term.

ONLINE GRADIENT DESCENT: REGRET

- As a consequence, we can also derive a similar order of the regret for the OGD algorithm on OCO problems as for the FTRL on OLO problems by imposing a slightly different assumption on the (new) “variance” term $\sum_{t=1}^{T} \|\nabla_a (a_t^{\mathrm{OGD}}, z_t)\|_2^2$.

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- **Corollary:** Suppose we use the OGD algorithm on an online convex optimization problem with a convex action space $\mathcal{A} \subset \mathbb{R}^d$ such that
  - $\sup_{\tilde{a} \in \mathcal{A}} \|\tilde{a}\|_2 \leq B$ for some finite constant $B > 0$
  - $\sup_{a \in \mathcal{A}, z \in \mathcal{Z}} \|\nabla_a \ell(a, z)\|_2 \leq V$ for some finite constant $V > 0$.

  Then, by choosing the step size η for OGD as $\eta = \frac{B}{V\sqrt{2T}}$ we get

$$R_T^{\mathrm{OGD}} \leq BV\sqrt{2T}.$$

REGRET LOWER BOUNDS FOR OCO

- **Theorem.** For any online learning algorithm there exists an online convex optimization problem characterized by a convex loss function L, a bounded (convex) action space $\mathcal{A} = [-B, B]^d$ and bounded gradients $\sup_{a \in \mathcal{A}, z \in \mathcal{Z}} ||\nabla_a L(a, z)||_2 \leq V$ for some finite constants $B, V > 0$, such that the algorithm incurs a regret of $\Omega(\sqrt{T})$ in the worst case.

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- Recall that under (almost) the same assumptions as the theorem above, we have $R_T^{\text{OGD}} \le BV\sqrt{2T}$.

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$\rightsquigarrow$ This result shows that the Online Gradient Descent is *optimal* regarding its order of its regret with respect to the time horizon T.