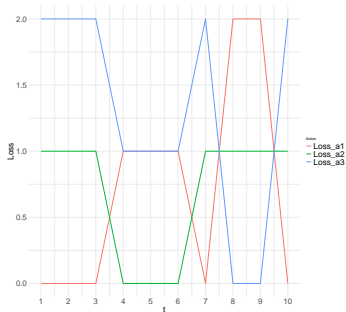


Advanced Machine Learning

Follow the leader on OLO problems



Learning goals

- Get to know online linear optimization (OLO) problems
- See that FTL might fail for these problems
- Understanding the root cause for FTL's flaw

FTL FOR ONLINE LINEAR OPTIMIZATION

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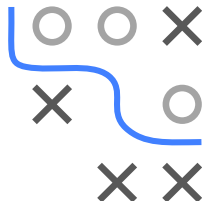


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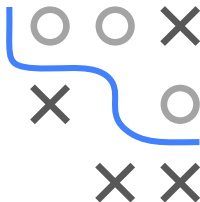
- No matter how we choose the first action a_1^{FTL} , it will hold that FTL has a cumulative loss greater than (or equal) $T - 3/2$, while the best action in hindsight has a cumulative loss of $-1/2$.
- Thus, FTL's cumulative regret is at least $T - 1$, which is linearly growing in T .



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- Indeed, note that

$$\begin{aligned} a_{t+1}^{\text{FTL}} &= \arg \min_{a \in \mathcal{A}} \sum_{s=1}^t L(a, z_s) = \arg \min_{a \in [-1, 1]} a \sum_{s=1}^t z_s \\ &= \begin{cases} -1, & \text{if } \sum_{s=1}^t z_s > 0, \\ 1, & \text{if } \sum_{s=1}^t z_s < 0, \\ \text{arbitrary}, & \text{if } \sum_{s=1}^t z_s = 0. \end{cases} \end{aligned}$$



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t	\hat{a}_t^{FTL}	z_t	$L(\hat{a}_t^{\text{FTL}}, z_t)$	$\sum_{s=1}^t L(\hat{a}_s^{\text{FTL}}, z_s)$	$\sum_{s=1}^t z_s$
1	1	-1/2	-1/2	-1/2	-1/2



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1	1	-1/2	-1/2	-1/2	-1/2
2	1	1	1	1 - 1/2	1/2

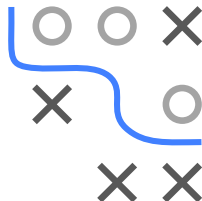


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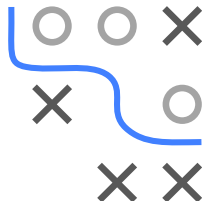


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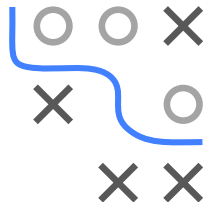
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2	1	1	1	1 - 1/2	1/2
3	-1	-1	1	2 - 1/2	-1/2
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
T	$(-1)^T$	$(-1)^T$	1	$T - 1 - 1/2$	$(-1/2)^T$



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\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
T	$(-1)^T$	$(-1)^T$	1	$T - 1 - 1/2$	$(-1/2)^T$

- The best action has cumulative loss

$$\inf_{a \in \mathcal{A}} \sum_{s=1}^T L(a, z_s) = \inf_{a \in [-1, 1]} a \underbrace{\sum_{s=1}^T z_s}_{=(-1/2)^T} = -1/2.$$

FTL FOR ONLINE LINEAR OPTIMIZATION

- Thus, we see: FTL can fail for **online linear optimization problems**, although it is well suited for **online quadratic optimization problems**!
- The reason is that the action selection of FTL is not stable enough (caused by the loss function), which is fine for **the latter problem**, but problematic for **the former**.
- One has to note that the online linear optimization problem example above, where FTL fails, is in fact an adversarial learning setting: The environmental data is generated in such a way that the FTL learner is fooled in each time step.

