FTL FOR ONLINE LINEAR OPTIMIZATION

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- Let $\mathcal{A} = [-1, 1]$ and suppose that $z_t = \begin{cases} -\frac{1}{2}, & t = 1, \\ 1, & t \text{ is even}, \\ -1, & t \text{ is odd}. \end{cases}$

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- Another popular instantiation of the online learning problem is the online linear optimization problem, which is characterized by a linear loss function $\ell(a, z) = a^\top z$.
- Let $\mathcal{A} = [-1, 1]$ and suppose that $z_t = \begin{cases} -\frac{1}{2}, & t = 1, \\ 1, & t \text{ is even}, \\ -1, & t \text{ is odd}. \end{cases}$
- No matter how we choose the first action a_1^{FTL}, it will hold that FTL has a cumulative loss greater than (or equal) $T - 3/2$, while the best action in hindsight has a cumulative loss of $-1/2$.
- Thus, FTL's cumulative regret is at least $T - 1$, which is linearly growing in T.

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- Indeed, note that

$$a_{t+1}^{\mathrm{FTL}} = \operatorname*{arg\,min}_{a \in \mathcal{A}} \sum_{s=1}^{t} \ell(a, z_s) = \operatorname*{arg\,min}_{a \in [-1,1]} \sum_{s=1}^{t} z_s \, a = \begin{cases} -1, & \text{if } \sum_{s=1}^{t} z_s > 0, \\ 1, & \text{if } \sum_{s=1}^{t} z_s < 0, \\ \text{arbitrary}, & \text{if } \sum_{s=1}^{t} z_s = 0. \end{cases}$$

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$$= \begin{cases} +1, & \text{if } \sum_{s=1}^{t} z_s > 0, \\ -1, & \text{if } \sum_{s=1}^{t} z_s < 0, \\ \text{arbitrary}, & \text{if } \sum_{s=1}^{t} z_s = 0. \end{cases}$$

t	a_t^{FTL}	z_t	$\ell(a_t^{\mathrm{FTL}}, z_t)$	$\sum_{s=1}^{t} \ell(a_s^{\mathrm{FTL}}, z_s)$	$\sum_{s=1}^{t} z_s$
1	1	$-1/2$	$-1/2$	$-1/2$	$-1/2$

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1	1	$-1/2$	$-1/2$	$-1/2$	$-1/2$
2	1	1	1	$1 - 1/2$	$1/2$

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1	1	$-1/2$	$-1/2$	$-1/2$	$-1/2$
2	1	1	1	$1 - 1/2$	$1/2$
3	-1	-1	1	$2 - 1/2$	$-1/2$

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1	1	$-1/2$	$-1/2$	$-1/2$	$-1/2$
2	1	1	1	$1 - 1/2$	$1/2$
3	-1	-1	1	$2 - 1/2$	$-1/2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
T	$(-1)^T$	$(-1)^T$	1	$T - 1 - 1/2$	$(-1/2)^T$

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- Indeed, note that

$$a_{t+1}^{\mathrm{FTL}} = \operatorname*{arg\,min}_{a \in \mathcal{A}} \sum_{s=1}^{t} \ell(a, z_s) = \operatorname*{arg\,min}_{a \in [-1,1]} a \sum_{s=1}^{t} z_s = -\sum_{s=1}^{t} z_s$$

$$= \begin{cases} -1, & \text{if } \sum_{s=1}^{t} z_s > 0, \\ 1, & \text{if } \sum_{s=1}^{t} z_s < 0, \\ \text{arbitrary}, & \text{if } \sum_{s=1}^{t} z_s = 0. \end{cases}$$

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1	1	$-1/2$	$-1/2$	$-1/2$	$-1/2$
2	1	1	1	$1 - 1/2$	$1/2$
3	-1	-1	1	$2 - 1/2$	$-1/2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
T	$(-1)^T$	$(-1)^T$	1	$T - 1 - 1/2$	$(-1/2)^T$

- The best action has cumulative loss

$$\inf_{a \in \mathcal{A}} \sum_{s=1}^{T} \ell(a, z_s) = \inf_{a \in [-1,1]} a \underbrace{\sum_{s=1}^{T} z_s}_{=(-1/2)^T} = -1/2.$$