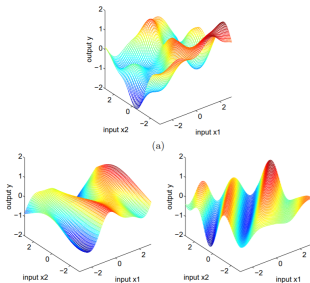


Introduction to Machine Learning

Covariance Functions for GPs



Learning goals

- Covariance functions encode key assumptions about the GP
- Know common covariance functions like squared exponential and Matérn

COVARIANCE FUNCTION OF A GP

The marginalization property of the Gaussian process implies that for any finite set of input values, the corresponding vector of function values is Gaussian:

$$\mathbf{f} = \left[f\left(\mathbf{x}^{(1)}\right), \dots, f\left(\mathbf{x}^{(n)}\right) \right] \sim \mathcal{N}(\mathbf{m}, \mathbf{K}),$$

- The covariance matrix \mathbf{K} is constructed based on the chosen inputs $\{\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}\}$.
- Entry \mathbf{K}_{ij} is computed by $k\left(\mathbf{x}^{(i)}, \mathbf{x}^{(j)}\right)$.
- Technically, for **every** choice of inputs $\{\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}\}$, \mathbf{K} needs to be positive semi-definite in order to be a valid covariance matrix.
- A function $k(., .)$ satisfying this property is called **positive definite**.



COVARIANCE FUNCTION OF A GP / 2

- Recall, the purpose of the covariance function is to control to which degree the following is fulfilled:

If two points $\mathbf{x}^{(i)}, \mathbf{x}^{(j)}$ are close in \mathcal{X} -space, their function values $f(\mathbf{x}^{(i)}), f(\mathbf{x}^{(j)})$ should be close (**correlated!**) in \mathcal{Y} -space.

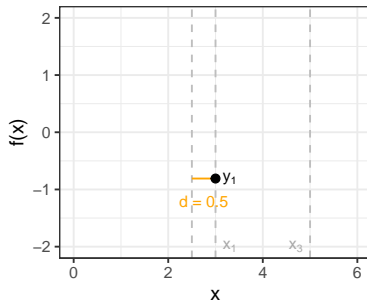
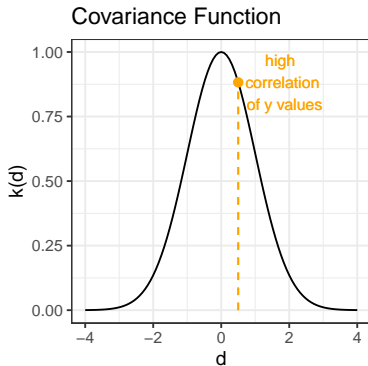
- Closeness of two points $\mathbf{x}^{(i)}, \mathbf{x}^{(j)}$ in input space \mathcal{X} is measured in terms of $\mathbf{d} = \mathbf{x}^{(i)} - \mathbf{x}^{(j)}$:

$$k(\mathbf{x}^{(i)}, \mathbf{x}^{(j)}) = k(\mathbf{d})$$



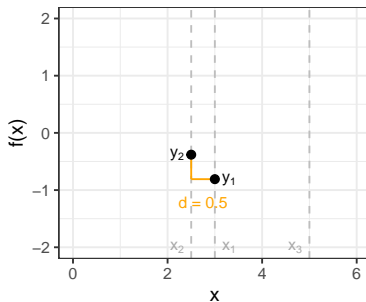
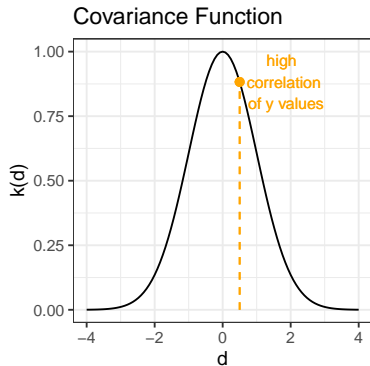
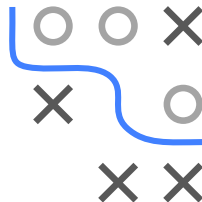
COVARIANCE FUNCTION OF A GP: EXAMPLE

- Let $f(\mathbf{x})$ be a GP with $k(\mathbf{x}, \mathbf{x}') = \exp(-\frac{1}{2}\|\mathbf{d}\|^2)$ with $\mathbf{d} = \mathbf{x} - \mathbf{x}'$.
- Consider two points $\mathbf{x}^{(1)} = 3$ and $\mathbf{x}^{(2)} = 2.5$.
- If you want to know how correlated their function values are, compute their correlation!



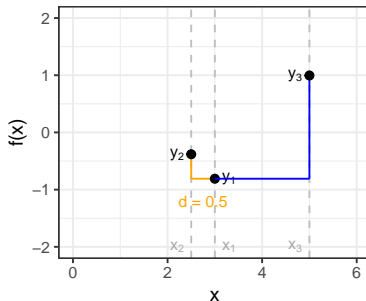
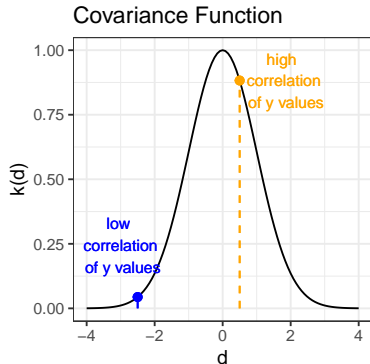
COVARIANCE FUNCTION OF A GP: EXAMPLE

- Assume we observed a value $y^{(1)} = -0.8$, the value of $y^{(2)}$ should be close under the assumption of the above Gaussian process.



COVARIANCE FUNCTION OF A GP: EXAMPLE

- Let us compare another point $\mathbf{x}^{(3)}$ to the point $\mathbf{x}^{(1)}$
- We again compute their correlation
- Their function values are not very much correlated; $y^{(1)}$ and $y^{(3)}$ might be far away from each other



COVARIANCE FUNCTIONS

There are three types of commonly used covariance functions:

- $k(., .)$ is called stationary if it is as a function of $\mathbf{d} = \mathbf{x} - \mathbf{x}'$, we write $k(\mathbf{d})$.

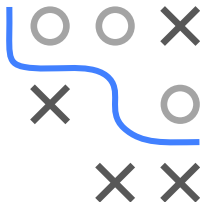
Stationarity is invariance to translations in the input space:

$$k(\mathbf{x}, \mathbf{x} + \mathbf{d}) = k(\mathbf{0}, \mathbf{d})$$

- $k(., .)$ is called isotropic if it is a function of $r = \|\mathbf{x} - \mathbf{x}'\|$, we write $k(r)$.

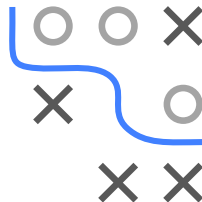
Isotropy is invariance to rotations of the input space and implies stationarity.

- $k(., .)$ is a dot product covariance function if k is a function of $\mathbf{x}^T \mathbf{x}'$



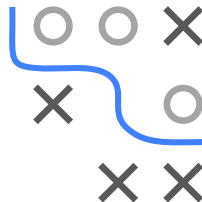
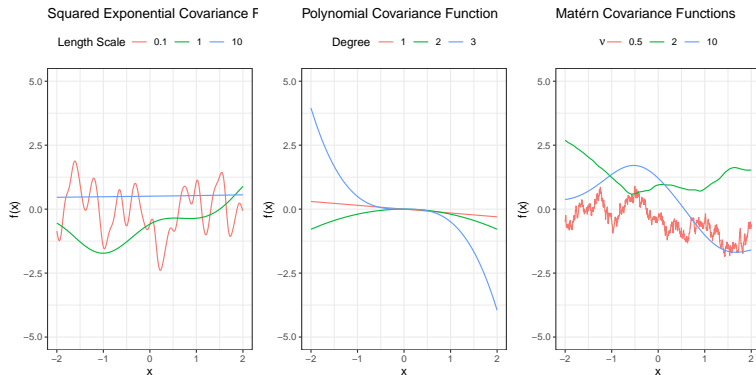
COMMONLY USED COVARIANCE FUNCTIONS

Name	$k(\mathbf{x}, \mathbf{x}')$
constant	σ_0^2
linear	$\sigma_0^2 + \mathbf{x}^T \mathbf{x}'$
polynomial	$(\sigma_0^2 + \mathbf{x}^T \mathbf{x}')^p$
squared exponential	$\exp\left(-\frac{\ \mathbf{x} - \mathbf{x}'\ ^2}{2\ell^2}\right)$
Matérn	$\frac{1}{2^\nu \Gamma(\nu)} \left(\frac{\sqrt{2\nu}}{\ell} \ \mathbf{x} - \mathbf{x}'\ \right)^\nu K_\nu\left(\frac{\sqrt{2\nu}}{\ell} \ \mathbf{x} - \mathbf{x}'\ \right)$
exponential	$\exp\left(-\frac{\ \mathbf{x} - \mathbf{x}'\ }{\ell}\right)$



$K_\nu(\cdot)$ is the modified Bessel function of the second kind.

COMMONLY USED COVARIANCE FUNCTIONS / 2



- Random functions drawn from Gaussian processes with a Squared Exponential Kernel (left), Polynomial Kernel (middle), and a Matérn Kernel (right, $\ell = 1$).
- The length-scale hyperparameter determines the “wiggleness” of the function.
- For Matérn, the ν parameter determines how differentiable the process is.

SQUARED EXPONENTIAL COVARIANCE FUNCTION

The squared exponential function is one of the most commonly used covariance functions.

$$k(\mathbf{x}, \mathbf{x}') = \exp\left(-\frac{\|\mathbf{x} - \mathbf{x}'\|^2}{2\ell^2}\right).$$

Properties:

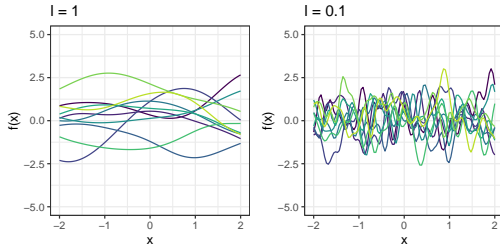
- It depends merely on the distance $r = \|\mathbf{x} - \mathbf{x}'\| \rightarrow$ isotropic and stationary.
- Infinitely differentiable \rightarrow sometimes deemed unrealistic for modeling most of the physical processes.



CHARACTERISTIC LENGTH-SCALE

$$k(\mathbf{x}, \mathbf{x}') = \exp \left(-\frac{1}{2\ell^2} \|\mathbf{x} - \mathbf{x}'\|^2 \right)$$

ℓ is called **characteristic length-scale**. Loosely speaking, the characteristic length-scale describes how far you need to move in input space for the function values to become uncorrelated. Higher ℓ induces smoother functions, lower ℓ induces more wiggly functions.



CHARACTERISTIC LENGTH-SCALE / 2

For $p \geq 2$ dimensions, the squared exponential can be parameterized:

$$k(\mathbf{x}, \mathbf{x}') = \exp \left(-\frac{1}{2} (\mathbf{x} - \mathbf{x}')^\top \mathbf{M} (\mathbf{x} - \mathbf{x}') \right)$$

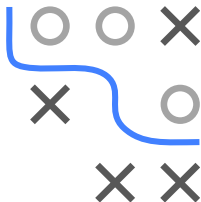
Possible choices for the matrix \mathbf{M} include

$$\mathbf{M}_1 = \ell^{-2} \mathbf{I} \quad \mathbf{M}_2 = \text{diag}(\ell)^{-2} \quad \mathbf{M}_3 = \Gamma \Gamma^\top + \text{diag}(\ell)^{-2}$$

where ℓ is a p -vector of positive values and Γ is a $p \times k$ matrix.

The 2nd (and most important) case can also be written as

$$k(\mathbf{d}) = \exp \left(-\frac{1}{2} \sum_{j=1}^p \frac{d_j^2}{l_j^2} \right)$$



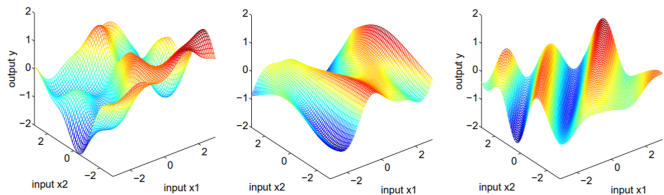
CHARACTERISTIC LENGTH-SCALE / 3

What is the benefit of having an individual hyperparameter ℓ_i for each dimension?

- The ℓ_1, \dots, ℓ_p hyperparameters play the role of **characteristic length-scales**.
- Loosely speaking, ℓ_i describes how far you need to move along axis i in input space for the function values to be uncorrelated.
- Such a covariance function implements **automatic relevance determination** (ARD), since the inverse of the length-scale ℓ_i determines the relevancy of input feature i to the regression.
- If ℓ_i is very large, the covariance will become almost independent of that input, effectively removing it from inference.
- If the features are on different scales, the data can be automatically **rescaled** by estimating ℓ_1, \dots, ℓ_p



CHARACTERISTIC LENGTH-SCALE / 4



For the first plot, we have chosen $\mathbf{M} = \mathbf{I}$: the function varies the same in all directions. The second plot is for $\mathbf{M} = \text{diag}(\ell)^{-2}$ and $\ell = (1, 3)$: The function varies less rapidly as a function of x_2 than x_1 as the length-scale for x_1 is less. In the third plot $\mathbf{M} = \Gamma\Gamma^\top + \text{diag}(\ell)^{-2}$ for $\Gamma = (1, -1)^\top$ and $\ell = (6, 6)^\top$. Here Γ gives the direction of the most rapid variation. (Image from Rasmussen & Williams, 2006)