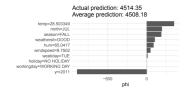
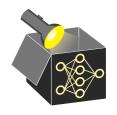
Interpretable Machine Learning

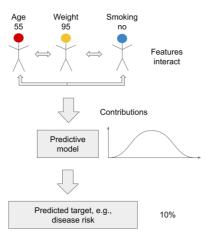
Shapley Values for Local Explanations



Learning goals

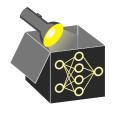
- See model predictions as a cooperative game
- Transfer the Shapley value concept from game theory to machine learning







• Game: Make prediction $\hat{f}(x_1, x_2, \dots, x_p)$ for a single observation **x**



- Game: Make prediction $\hat{f}(x_1, x_2, \dots, x_p)$ for a single observation **x**
- Players: Features x_j, j ∈ {1,...,p} which cooperate to produce a prediction
 → How can we make a prediction with a subset of features without changing the model?
 - \leadsto PD function: $\hat{f}_{S}(\mathbf{x}_{S}):=\int_{X_{-S}}\hat{f}(\mathbf{x}_{S},X_{-S})d\mathbb{P}_{X_{-S}}$ ("removing" by marginalizing over -S)



- Game: Make prediction $\hat{f}(x_1, x_2, \dots, x_p)$ for a single observation **x**
- Players: Features $x_j, j \in \{1, ..., p\}$ which cooperate to produce a prediction \rightarrow How can we make a prediction with a subset of features without changing the model?
 - \leadsto PD function: $\hat{f}_S(\mathbf{x}_S) := \int_{X_{-S}} \hat{f}(\mathbf{x}_S, X_{-S}) d\mathbb{P}_{X_{-S}}$ ("removing" by marginalizing over -S)
- Value function / payout of coalition $S \subseteq P$ for observation **x**:

$$v(S) = \hat{\mathit{f}}_{S}(\mathbf{x}_{S}) - \mathbb{E}_{\mathbf{x}}(\hat{\mathit{f}}(\mathbf{x})), \text{ where } \hat{\mathit{f}}_{S} : \mathcal{X}_{S} \mapsto \mathcal{Y}$$

 \rightarrow subtraction of $\mathbb{E}_{\mathbf{x}}(\hat{f}(\mathbf{x}))$ ensures that ν is a value function with $\nu(\emptyset) = 0$

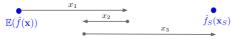
$$\mathbb{E}(\hat{f}(\mathbf{x})) \qquad \qquad x_1 \qquad \qquad \hat{f}_S(\mathbf{x}_S)$$



- Game: Make prediction $\hat{f}(x_1, x_2, \dots, x_p)$ for a single observation \mathbf{x}
- Players: Features $x_j, j \in \{1, ..., p\}$ which cooperate to produce a prediction \rightarrow How can we make a prediction with a subset of features without changing the model?
 - \leadsto PD function: $\hat{f}_S(\mathbf{x}_S) := \int_{X_{-S}} \hat{f}(\mathbf{x}_S, X_{-S}) d\mathbb{P}_{X_{-S}}$ ("removing" by marginalizing over -S)
- Value function / payout of coalition $S \subseteq P$ for observation **x**:

$$v(S) = \hat{\mathit{f}}_{S}(\mathbf{x}_{S}) - \mathbb{E}_{\mathbf{x}}(\hat{\mathit{f}}(\mathbf{x})), \text{ where } \hat{\mathit{f}}_{S} : \mathcal{X}_{S} \mapsto \mathcal{Y}$$

ightharpoonup subtraction of $\mathbb{E}_{\mathbf{x}}(\hat{f}(\mathbf{x}))$ ensures that v is a value function with $v(\emptyset)=0$

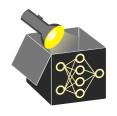


• Marginal contribution: $v(S \cup \{j\}) - v(S) = \hat{f}_{S \cup \{j\}}(\mathbf{x}_{S \cup \{j\}}) - \hat{f}_{S}(\mathbf{x}_{S})$ $\rightarrow \mathbb{E}_{\mathbf{x}}(\hat{f}(\mathbf{x}))$ cancels out due to the subtraction of value functions



Shapley value ϕ_i of feature *j* for observation **x** via **order definition**:

$$\phi_j(\mathbf{x}) = \frac{1}{|P|!} \sum_{\tau \in \Pi} \hat{f}_{S_j^{\tau} \cup \{j\}}(\mathbf{x}_{S_j^{\tau} \cup \{j\}}) - \hat{f}_{S_j^{\tau}}(\mathbf{x}_{S_j^{\tau}})$$
marginal contribution of feature j



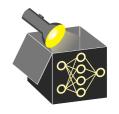
- Interpretation: Feature x_i contributed ϕ_i to difference between $\hat{f}(\mathbf{x})$ and average prediction
 - → Note: Marginal contributions and Shapley values can be negative
- For exact computation of $\phi_i(\mathbf{x})$, the PD function $\hat{f}_S(\mathbf{x}_S) = \frac{1}{n} \sum_{i=1}^n \hat{f}(\mathbf{x}_S, \mathbf{x}_{-S}^{(i)})$ for any set of features S can be used which yields

$$\phi_j(\mathbf{x}) = \frac{1}{|P|! \cdot n} \sum_{\tau \in \Pi} \sum_{i=1}^n \hat{f}(\mathbf{x}_{S_j^{\tau} \cup \{j\}}, \mathbf{x}_{-\{S_j^{\tau} \cup \{j\}\}}^{(i)}) - \hat{f}(\mathbf{x}_{S_j^{\tau}}, \mathbf{x}_{-S_j^{\tau}}^{(i)})$$

 \rightarrow Note: \hat{f}_S marginalizes over all other features -S using all observations $i=1,\ldots,n$

 Exact Shapley value computation is problematic for high-dimensional feature spaces

 \leadsto For 10 features, there are already |P|!= 10! \approx 3.6 million possible orders of features



- Exact Shapley value computation is problematic for high-dimensional feature spaces
 - ightharpoonup For 10 features, there are already |P|!= 10! pprox 3.6 million possible orders of features
- Additional problem due to estimation of the marginal prediction $\hat{f}_{S_j^\tau}$: Averaging over the entire data set for each coalition S_j^τ introduced by τ can be very expensive for large data sets



- Exact Shapley value computation is problematic for high-dimensional feature spaces
 - \leadsto For 10 features, there are already |P|!= 10! \approx 3.6 million possible orders of features
- Additional problem due to estimation of the marginal prediction $\hat{f}_{S_j^{\tau}}$: Averaging over the entire data set for each coalition S_j^{τ} introduced by τ can be very expensive for large data sets
- Solution to both problems is sampling: Instead of averaging over $|P|! \cdot n$ terms, we approximate it using a limited amount of M random samples of τ to build coalitions S_j^{τ}



- Exact Shapley value computation is problematic for high-dimensional feature spaces
 - ightharpoonup For 10 features, there are already $|P|!=10!\approx 3.6$ million possible orders of features
- Additional problem due to estimation of the marginal prediction $\hat{f}_{S_j^{\tau}}$: Averaging over the entire data set for each coalition S_j^{τ} introduced by τ can be very expensive for large data sets
- Solution to both problems is sampling: Instead of averaging over $|P|! \cdot n$ terms, we approximate it using a limited amount of M random samples of τ to build coalitions S_i^{τ}



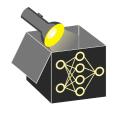
Estimation of ϕ_i for observation **x** of model \hat{t} fitted on data \mathcal{D} using sample size M:

1 For m = 1, ..., M do:



Estimation of ϕ_i for observation **x** of model \hat{t} fitted on data \mathcal{D} using sample size M:

- **1** For m = 1, ..., M do:
 - Select random order / perm. of feature indices $\tau = (\tau^{(1)}, \dots, \tau^{(p)}) \in \Pi$



Estimation of ϕ_i for observation **x** of model \hat{f} fitted on data \mathcal{D} using sample size M:

- **1** For m = 1, ..., M do:
 - Select random order / perm. of feature indices $\tau = (\tau^{(1)}, \dots, \tau^{(p)}) \in \Pi$
 - **2** Determine coalition $S_m := S_i^{\tau}$, i.e., the set of feat. before feat. j in order τ



Estimation of ϕ_i for observation **x** of model \hat{t} fitted on data \mathcal{D} using sample size M:

- **1** For m = 1, ..., M do:
 - Select random order / perm. of feature indices $\tau = (\tau^{(1)}, \dots, \tau^{(p)}) \in \Pi$
 - **2** Determine coalition $S_m := S_i^{\tau}$, i.e., the set of feat. before feat. j in order τ
 - **3** Select random data point $\mathbf{z}^{(m)} \in \mathcal{D}$



Estimation of ϕ_i for observation **x** of model \hat{f} fitted on data \mathcal{D} using sample size M:

- **1** For m = 1, ..., M do:
 - Select random order / perm. of feature indices $\tau = (\tau^{(1)}, \dots, \tau^{(p)}) \in \Pi$
 - Determine coalition $S_m := S_i^{\tau}$, i.e., the set of feat. before feat. j in order τ
 - Select random data point $\mathbf{z}^{(m)} \in \mathcal{D}$
 - Construct two artificial obs. by replacing feature values from **x** with $\mathbf{z}^{(m)}$:



Estimation of ϕ_i for observation **x** of model \hat{t} fitted on data \mathcal{D} using sample size M:

- **1** For m = 1, ..., M do:
 - Select random order / perm. of feature indices $\tau = (\tau^{(1)}, \dots, \tau^{(p)}) \in \Pi$
 - Determine coalition $S_m := S_i^{\tau}$, i.e., the set of feat. before feat. j in order τ
 - Select random data point $\mathbf{z}^{(m)} \in \mathcal{D}$

Construct two artificial obs. by replacing feature values from
$$\mathbf{x}$$
 with $\mathbf{z}^{(m)}$:

• $\mathbf{x}_{+j}^{(m)} = (\underbrace{X_{\tau^{(1)}}, \dots, X_{\tau^{(|S_m|-1)}}, X_j}_{\mathbf{x}_{S_m \cup \{j\}}}, \underbrace{Z_{\tau^{(|S_m|+1)}}^{(m)}, \dots, Z_{\tau^{(p)}}^{(m)}}_{\mathbf{z}_{-\{S_m \cup \{j\}\}}})$ takes features

 $S_m \cup \{j\}$ from **x**



Estimation of ϕ_i for observation **x** of model \hat{t} fitted on data \mathcal{D} using sample size M:

- **1** For m = 1, ..., M do:
 - Select random order / perm. of feature indices $\tau = (\tau^{(1)}, \dots, \tau^{(p)}) \in \Pi$
 - Determine coalition $S_m := S_i^{\tau}$, i.e., the set of feat. before feat. j in order τ
 - Select random data point $\mathbf{z}^{(m)} \in \mathcal{D}$

Construct two artificial obs. by replacing feature values from
$$\mathbf{x}$$
 with $\mathbf{z}^{(m)}$:

• $\mathbf{x}_{+j}^{(m)} = (\underbrace{x_{\tau^{(1)}, \dots, x_{\tau^{(|S_m|-1)}}, x_j}_{\mathbf{x}_{S_m \cup \{j\}}}, \underbrace{z_{\tau^{(|S_m|+1)}, \dots, z_{\tau^{(p)}}}^{(m)})$ takes features

 $S_m \cup \{i\}$ from **x**

$$\bullet \ \ \mathbf{x}_{-j}^{(m)} = (\underbrace{x_{\tau^{(1)}}, \dots, x_{\tau^{(|S_m|-1)}}}_{\mathbf{x}_{S_m}}, \underbrace{z_{j}^{(m)}, z_{\tau^{(|S_m|+1)}}^{(m)}, \dots, z_{\tau^{(p)}}^{(m)}}_{\mathbf{z}_{-S_m}^{(m)}}) \text{ takes features}$$

 S_m from **x**



Estimation of ϕ_i for observation **x** of model \hat{t} fitted on data \mathcal{D} using sample size M:

- **1** For m = 1, ..., M **do**:
 - Select random order / perm. of feature indices $\tau = (\tau^{(1)}, \dots, \tau^{(p)}) \in \Pi$
 - Determine coalition $S_m := S_i^{\tau}$, i.e., the set of feat. before feat. j in order τ
 - Select random data point $\mathbf{z}^{(m)} \in \mathcal{D}$

Construct two artificial obs. by replacing feature values from
$$\mathbf{x}$$
 with $\mathbf{z}^{(m)}$:

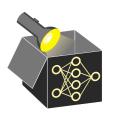
• $\mathbf{x}_{+j}^{(m)} = (\underbrace{x_{\tau^{(1)}, \dots, x_{\tau^{(|S_m|-1)}, x_j}}_{\mathbf{x}_{S_m \cup \{j\}}}, \underbrace{z_{\tau^{(|S_m|+1)}, \dots, z_{\tau^{(p)}}}^{(m)})$ takes features

 $S_m \cup \{i\}$ from **x**

$$\bullet \ \ \mathbf{x}_{-j}^{(m)} = (\underbrace{x_{\tau^{(1)}}, \dots, x_{\tau^{(|S_m|-1)}}}_{\mathbf{x}_{S_m}}, \underbrace{z_{j}^{(m)}, z_{\tau^{(|S_m|+1)}}^{(m)}, \dots, z_{\tau^{(p)}}^{(m)}}_{\mathbf{z}_{-S_m}^{(m)}}) \text{ takes features}$$

 S_m from **x**

Compute difference
$$\phi_j^m = \hat{f}(\mathbf{x}_{+j}^{(m)}) - \hat{f}(\mathbf{x}_{-j}^{(m)})$$
 $\leadsto \hat{f}_{S_m}(\mathbf{x}_{S_m})$ is approximated by $\hat{f}(\mathbf{x}_{-j}^{(m)})$ and $\hat{f}_{S_m \cup \{j\}}(\mathbf{x}_{S_m \cup \{j\}})$ by $\hat{f}(\mathbf{x}_{+j}^{(m)})$ over M iters.



Estimation of ϕ_i for observation **x** of model \hat{f} fitted on data \mathcal{D} using sample size M:

- **1** For m = 1, ..., M **do**:
 - Select random order / perm. of feature indices $\tau = (\tau^{(1)}, \dots, \tau^{(p)}) \in \Pi$
 - Determine coalition $S_m := S_i^{\tau}$, i.e., the set of feat. before feat. j in order τ
 - Select random data point $\mathbf{z}^{(m)} \in \mathcal{D}$

Construct two artificial obs. by replacing feature values from
$$\mathbf{x}$$
 with $\mathbf{z}^{(m)}$:

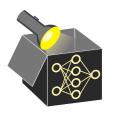
• $\mathbf{x}_{+j}^{(m)} = (\underbrace{x_{\tau^{(1)}, \dots, x_{\tau^{(|S_m|-1)}, x_j}}_{\mathbf{x}_{S_m \cup \{j\}}}, \underbrace{z_{\tau^{(|S_m|+1)}, \dots, z_{\tau^{(p)}}}^{(m)})$ takes features

 $S_m \cup \{i\}$ from **x**

$$\bullet \ \ \mathbf{x}_{-j}^{(m)} = (\underbrace{x_{\tau^{(1)}}, \dots, x_{\tau^{(|S_m|-1)}}}_{\mathbf{x}_{S_m}}, \underbrace{z_{j}^{(m)}, z_{\tau^{(|S_m|+1)}}^{(m)}, \dots, z_{\tau^{(p)}}^{(m)}}_{\mathbf{z}_{-S_m}^{(m)}}) \text{ takes features}$$

 S_m from **x**

- **6** Compute difference $\phi_i^m = \hat{f}(\mathbf{x}_{+i}^{(m)}) \hat{f}(\mathbf{x}_{-i}^{(m)})$ $\rightsquigarrow \hat{f}_{S_m}(\mathbf{x}_{S_m})$ is approximated by $\hat{f}(\mathbf{x}_{-i}^{(m)})$ and $\hat{f}_{S_m \cup \{i\}}(\mathbf{x}_{S_m \cup \{i\}})$ by $\hat{f}(\mathbf{x}_{+i}^{(m)})$ over M iters.
- 2 Compute Shapley value $\phi_i = \frac{1}{M} \sum_{m=1}^{M} \phi_i^m$



SHAPLEY VALUE APPROXIMATION - ILLUSTRATION

Definition

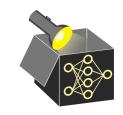
x: obs. of interest

 \mathbf{x} with feature values in S_m (other are replaced)

$$\phi_j(\mathbf{x}) = \frac{1}{M} \sum_{m=1}^{M} \left[\hat{f}(\mathbf{x}_{+j}^{(m)}) - \hat{f}(\mathbf{x}_{-j}^{(m)}) \right]$$

 \mathbf{x} with feature values in $S_m \cup \{j\}$

	Temperature	Humidity	Windspeed	Year
\boldsymbol{x}	10.66	56	11	2012
x_{+j}	10.66	56	$random: z_{windspeed}^{(m)}$	2012
x_{-j}	10.66	56	$random: z_{windspeed}^{(m)}$	$random: z_{year}^{(m)}$
			,	•
				7



SHAPLEY VALUE APPROXIMATION - ILLUSTRATION

Definition

Contribution of feature
$$j$$
 to coalition S_m
$$\phi_j(\mathbf{x}) = \frac{1}{M} \sum_{m=1}^{M} \left[\hat{f}(\mathbf{x}_{+j}{}^{(m)}) - \hat{f}(\mathbf{x}_{-j}{}^{(m)}) \right]$$
$$:= \Delta(j, S_m)$$



- $\Delta(j, S_m) = \hat{f}(\mathbf{x}_{+j}^{(m)}) \hat{f}(\mathbf{x}_{-j}^{(m)})$ is the marginal contribution of feature j to coalition S_m
- Here: Feature *year* contributes +700 bike rentals if it joins coalition $S_m = \{temp, hum\}$

	Temperature	Humidity	Windspeed	Year	Count	
\boldsymbol{x}	10.66	56	11	2012		
x_{+j}	10.66	56	$random: z_{windspeed}^{(m)}$	2012	5600	700
x_{-j}	10.66	56	$random: z_{windspeed}^{(m)}$	$random: z_{year}^{(m)}$	4900	700
			`	•	$\stackrel{}{\frown}$	$\Delta(j,S_m)$
				${\mathcal J}$	f	marginal contribution

SHAPLEY VALUE APPROXIMATION - ILLUSTRATION

Definition

$$\phi_j(\mathbf{x}) = \frac{1}{M} \sum_{j=1}^{M} \left[\hat{f}(\mathbf{x}_{+j}^{(m)}) - \hat{f}(\mathbf{x}_{-j}^{(m)}) \right]$$



- Compute marginal contribution of feature j towards the prediction across all randomly drawn feature coalitions S_1, \ldots, S_m
- Average all *M* marginal contributions of feature *j*
- Shapley value ϕ_j is the payout of feature j, i.e., how much feature year contributed to the overall prediction in bicycle counts of a specific observation \mathbf{x}

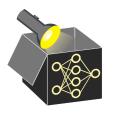
$$m=1$$
 2 M 300 $\Delta(j,S_m)$



We take the general axioms for Shapley Values and apply it to predictions:

• Efficiency: Shapley values add up to the (centered) prediction:

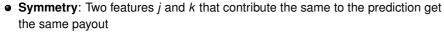
$$\sum_{j=1}^{p} \phi_j = \hat{f}(\mathbf{x}) - \mathbb{E}_{\mathbf{x}}(\hat{f}(X))$$



We take the general axioms for Shapley Values and apply it to predictions:

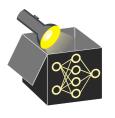
• Efficiency: Shapley values add up to the (centered) prediction:

$$\sum_{j=1}^{p} \phi_j = \hat{f}(\mathbf{x}) - \mathbb{E}_{\mathbf{x}}(\hat{f}(X))$$

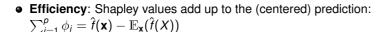


→ interaction effects between features are fairly divided

$$\hat{f}_{S \cup \{j\}}(\mathbf{x}_{S \cup \{j\}}) = \hat{f}_{S \cup \{k\}}(\mathbf{x}_{S \cup \{k\}})$$
 for all $S \subseteq P \setminus \{j, k\}$ then $\phi_j = \phi_k$



We take the general axioms for Shapley Values and apply it to predictions:





- **Symmetry**: Two features *j* and *k* that contribute the same to the prediction get the same payout
 - → interaction effects between features are fairly divided

$$\hat{f}_{S \cup \{j\}}(\mathbf{x}_{S \cup \{j\}}) = \hat{f}_{S \cup \{k\}}(\mathbf{x}_{S \cup \{k\}})$$
 for all $S \subseteq P \setminus \{j, k\}$ then $\phi_j = \phi_k$

 Dummy / Null Player: Shapley value of a feature that does not influence the prediction is zero → if a feature was not selected by the model (e.g., tree or LASSO), its Shapley value is zero

$$\hat{f}_{S\cup\{j\}}(\mathbf{x}_{S\cup\{j\}})=\hat{f}_S(\mathbf{x}_S)$$
 for all $S\subseteq P$ then $\phi_j=0$

We take the general axioms for Shapley Values and apply it to predictions:

• **Efficiency**: Shapley values add up to the (centered) prediction:

$$\sum_{j=1}^{p} \phi_j = \hat{f}(\mathbf{x}) - \mathbb{E}_{\mathbf{x}}(\hat{f}(X))$$

• **Symmetry**: Two features *j* and *k* that contribute the same to the prediction get the same payout

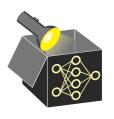
 $\stackrel{\leadsto}{\sim}$ interaction effects between features are fairly divided

$$\hat{f}_{S\cup\{j\}}(\mathbf{x}_{S\cup\{j\}}) = \hat{f}_{S\cup\{k\}}(\mathbf{x}_{S\cup\{k\}})$$
 for all $S \subseteq P \setminus \{j,k\}$ then $\phi_j = \phi_k$

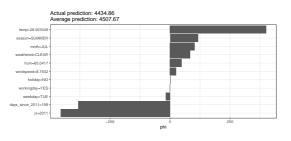
 Dummy / Null Player: Shapley value of a feature that does not influence the prediction is zero → if a feature was not selected by the model (e.g., tree or LASSO), its Shapley value is zero

$$\hat{f}_{S\cup\{j\}}(\mathbf{x}_{S\cup\{j\}})=\hat{f}_S(\mathbf{x}_S)$$
 for all $S\subseteq P$ then $\phi_j=0$

• Additivity: For a prediction with combined payouts, the payout is the sum of payouts: $\phi_j(v_1) + \phi_j(v_2) \leadsto$ Shapley values for model ensembles can be combined



BIKE SHARING DATASET





- Shapley values of observation i = 200 from the bike sharing data
- Difference between model prediction of this observation and the average prediction of the data is fairly distributed among the features (i.e., $4434-4507\approx-73$)
- Feature value temp = 28.5 has the most positive effect, with a contribution (increase of prediction) of about +400

ADVANTAGES AND DISADVANTAGES

Advantages:

- Solid theoretical foundation in game theory
- Prediction is fairly distributed among the feature values → easy to interpret for a user
- Contrastive explanations that compare the prediction with the average prediction

Disadvantages:

- Without sampling, Shapley values need a lot of computing time to inspect all possible coalitions
- Like many other IML methods, Shapley values suffer from the inclusion of unrealistic data observations when features are correlated

