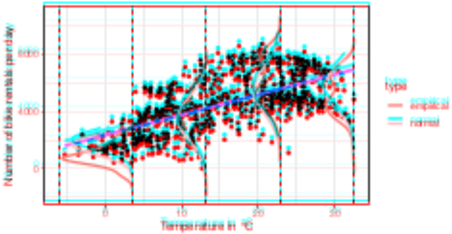
GENERALIZED LINEAR MODEL (GLM)

▸ Nelder and Wedderburn 1972

Problem: Target variable given feat. not always normally dist. ⇝ LM not suitable

- Target is binary (e.g., disease classification)
  ⇝ Bernoulli / Binomial distribution
- Target is count variable
  (e.g., number of sold products)
  ⇝ Poisson distribution
- Time until an event occurs
  (e.g., time until death)
  ⇝ Gamma distribution



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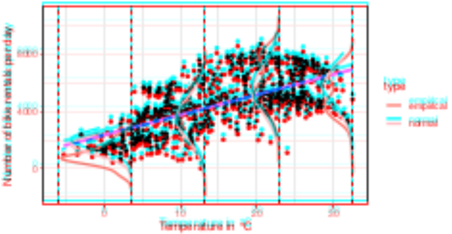

Solution: GLMs - extend LMs by allowing other distributions from exponential family

$$g(\mathbb{E}(y \mid \mathbf{x})) = \mathbf{x}^\top \theta \Leftrightarrow \mathbb{E}(y \mid \mathbf{x}) = g^{-1}(\mathbf{x}^\top \theta)$$

- Link function g links linear predictor $\mathbf{x}^\top \theta$ to expectation of distribution of $y \mid \mathbf{x}$
  $\leadsto$ LM is special case: Gaussian distribution for $y \mid \mathbf{x}$ with g as identity function
- Link function g and distribution need to be specified
- High-order and interaction effects can be manually added as in LMs
- Note: Interpretation of weights depend on link function and distribution

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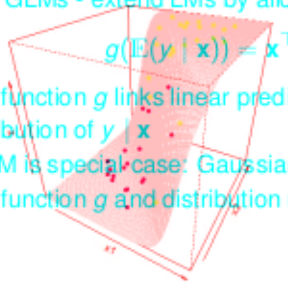
GLM - LOGISTIC REGRESSION

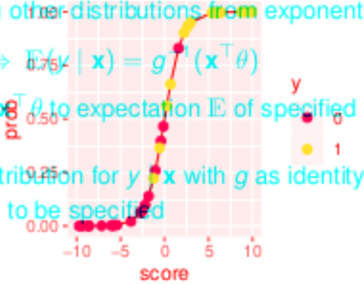
- Logistic regression ≙ GLM with Bernoulli distribution and logit link function:

$$g(x) = \log\left(\frac{x}{1-x}\right) \Rightarrow g^{-1}(x) = \frac{1}{1+\exp(-x)}$$

- Models probabilities for binary classification by

$$\pi(\mathbf{x}) = \mathbb{E}(y \mid \mathbf{x}) = P(y=1) = g^{-1}(\mathbf{x}^\top\theta) = \frac{1}{1+\exp(-\mathbf{x}^\top\theta)}$$





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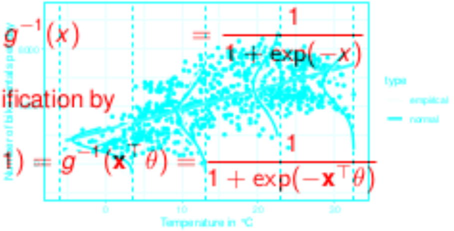

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- Typically, we set the threshold to 0.5 to predict classes, e.g.,
  - Class 1 if $\pi(\mathbf{x}) > 0.5$
  - Class 0 if $\pi(\mathbf{x}) \leq 0.5$

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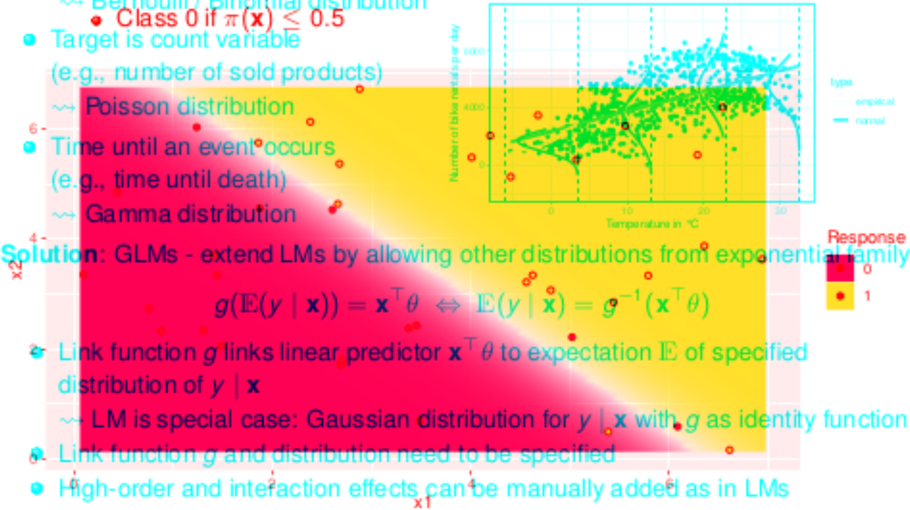

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GLM - LOGISTIC REGRESSION - INTERPRETATION

- **Recall:** Odds is ratio of two probabilities, odds ratio compares ratio of two odds
- Weights θ_j are interpreted linear as in LM (but w.r.t. log-odds)
  $\rightsquigarrow$ difficult to comprehend

$$\textit{log-odds} = \log\left(\frac{\pi(\mathbf{x})}{1-\pi(\mathbf{x})}\right) = \log\left(\frac{P(y=1)}{P(y=0)}\right) = \theta_0 + \theta_1 x_1 + \ldots + \theta_p x_p$$

Interpretation:
Changing x_j by one unit, changes log-odds of class 1 compared to class 0 by θ_j

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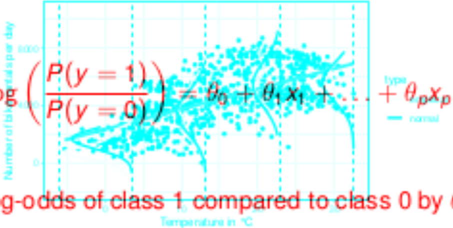

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- Odds for class 1 vs. class 0: $\textit{odds} = \frac{\pi(\mathbf{x})}{1-\pi(\mathbf{x})} = \exp(\theta_0 + \theta_1 x_1 + \ldots + \theta_p x_p)$
- Instead of interpreting changes w.r.t. log-odds, *odds ratio* is more common

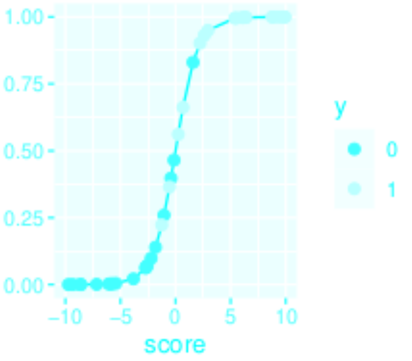
$$= \frac{\textit{odds}_{x_j+1}}{\textit{odds}} = \frac{\exp(\theta_0 + \theta_1 x_1 + \ldots + \theta_j(x_j+1) + \ldots + \theta_p x_p)}{\exp(\theta_0 + \theta_1 x_1 + \ldots + \theta_j x_j + \ldots + \theta_p x_p)} = \exp(\theta_j)$$

Interpretation: Changing x_j by one unit, changes the **odds ratio** for class 1 (compared to class 0) by the **factor** $\exp(\theta_j)$

GLM - LOGISTIC REGRESSION - EXAMPLE

- Create a binary target variable for bike rental data:
  - Class 1: "high number of bike rentals" $>$ 70% quantile (i.e., $\text{cnt} > 5531$)
  - Class 0: "low to medium number of bike rentals" (i.e., $\text{cnt} \leq 5531$)
- Fit a logistic regression model (GLM with Bernoulli distribution and logit link)

	Weights	SE	p-value
(Intercept)	-8.52	1.21	0.00
seasonSPRING	1.74	0.60	0.00
seasonSUMMER	-0.86	0.77	0.26
seasonFALL	-0.64	0.55	0.25
temp	0.29	0.04	0.00
hum	-0.06	0.01	0.00
windspeed	-0.09	0.03	0.00
days_since_2011	0.02	0.00	0.00



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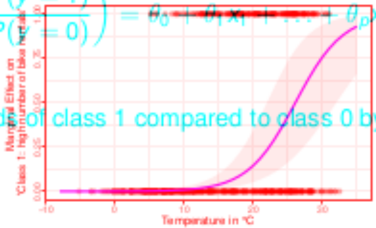

Interpretation

- If temp increases by $1°C$, odds ratio for class 1 increases by factor $\exp(0.29) = 1.34$ compared to class 0, c.p. ($\triangleq$ "high number of bike rentals" now 1.34 times more likely)

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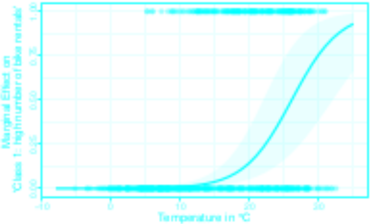

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