Multivariate Optimization 1

Exercise 1: Gradient Descent

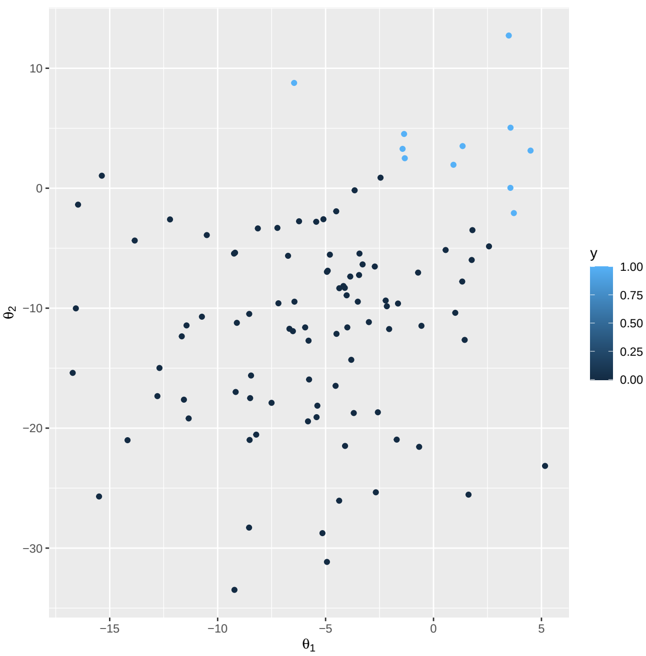
You are given the following data situation:

```
library(ggplot2)

set.seed(314)
n <- 100
X = cbind(rnorm(n, -5, 5),
  rnorm(n, -10, 10))
X_design = cbind(1, X)

z <- 2*X[,1] + 3*X[,2]
pr <- 1/(1+exp(-z))
y <- as.integer(pr > 0.5)
df <- data.frame(X = X, y = y)

ggplot(df) +
  geom_point(aes(x = X.1, y = X.2, color=y)) +
  xlab(expression(theta[1])) +
  ylab(expression(theta[2]))
```



In the following we want to estimate a logistic regression without intercept via gradient descent[1].

(a) The data situation is called complete separation, i.e., the classes can be perfectly classified with a linear classifier. Show that in this situation if $\tilde{\boldsymbol{\theta}}$ perfectly classifies the data then:
$\mathcal{R}_{\text{emp}}(\tilde{\boldsymbol{\theta}}) \geq \mathcal{R}_{\text{emp}}(\alpha\tilde{\boldsymbol{\theta}})$ with $\alpha > 1$.

(b) Visualize $\mathcal{R}_{\text{emp}}$ in $[-1, 4] \times [-1, 4]$.

[1]We chose this algorithm for educational purposes; in practice, we typically use second order algorithms.

(c) Find the gradient of $\mathcal{R}_{\text{emp}}$ for arbitrary $\boldsymbol{\theta}$.

(d) Solve the logistic regression via gradient descent. Use step width $\alpha = 0.01$, starting point $\boldsymbol{\theta}^{[0]} = (0, 0)^\top$ and train for 500 steps. Repeat this with $\alpha = 0.02$. Explain your observation.
Hint: a)

(e) Repeat d) but add an L2 penalization term (with $\lambda = 1$) to the objective. What do you observe now?

(f) Visualize the regularized $\mathcal{R}_{\text{emp}}$ in $[-1, 4] \times [-1, 4]$.

(g) Repeat e) but with backtracking. Set $\gamma = 0.9$ and $\tau = 0.5$