

Optimization Problems 1

**Exercise 1: Regression**

- (a) Show that ridge regression is a convex problem and compute its analytical solution (given the feature matrix  $\mathbf{X} \in \mathbb{R}^{n \times d}$  and the target vector  $\mathbf{y} \in \mathbb{R}^n$ ).
- (b) In Bayesian regression, we are interested in the posterior density  $p_{\boldsymbol{\theta}|\mathbf{X},\mathbf{y}}(\boldsymbol{\theta}) \propto p_{\mathbf{y}|\mathbf{X},\boldsymbol{\theta}}(\mathbf{y})p_{\boldsymbol{\theta}}(\boldsymbol{\theta})$ , where  $p_{\mathbf{y}|\mathbf{X},\boldsymbol{\theta}}$  is the likelihood and  $p_{\boldsymbol{\theta}}$  is the prior density. Assume the observations are i.i.d. with  $y_i|\mathbf{x}_i \sim \mathcal{N}(\mathbf{x}_i^\top \boldsymbol{\theta}, 1)$  and the parameters are also i.i.d. with  $\boldsymbol{\theta}_j \sim \mathcal{N}(0, \sigma_w^2)$ . Find the maximizer of the posterior density. What do you observe?
- (c) Find the prior density that would result in Lasso regression in b).

**Exercise 2: Classification**

- (a) In logistic regression, we model the conditional probability  $\mathbb{P}(y = 1|\mathbf{x}^{(i)}) = \frac{1}{1+\exp(-\boldsymbol{\theta}^\top \mathbf{x}^{(i)})}$  of the target  $y \in \{0, 1\}$  given a feature vector  $\mathbf{x}^{(i)}$ . From this it follows that  $\mathbb{P}(y = y^{(i)}|\mathbf{x}^{(i)}) = \mathbb{P}(y = 1|\mathbf{x}^{(i)})^{y^{(i)}}(1 - \mathbb{P}(y = 1|\mathbf{x}^{(i)}))^{1-y^{(i)}}$ . With this derive the empirical risk  $\mathcal{R}_{\text{emp}}$  as shown in the lecture following the maximum likelihood principle. (Assume the observations are independent)
- (b) Show that  $\mathcal{R}_{\text{emp}}$  of a) is convex.
- (c) Show that the first primal form of the linear SVM with soft constraints  $\min_{\boldsymbol{\theta}, \boldsymbol{\theta}_0, \zeta^{(i)}} \frac{1}{2} \|\boldsymbol{\theta}\|_2^2 + C \sum_{i=1}^n \zeta^{(i)}$  s.t.  $y^{(i)} (\boldsymbol{\theta}^\top \mathbf{x}^{(i)} + \boldsymbol{\theta}_0) \geq 1 - \zeta^{(i)} \quad \forall i \in \{1, \dots, n\}$  and  $\zeta^{(i)} \geq 0 \quad \forall i \in \{1, \dots, n\}$  and its second primal form  $\min_{\boldsymbol{\theta}, \boldsymbol{\theta}_0} \sum_{i=1}^n \max(1 - y^{(i)} (\boldsymbol{\theta}^\top \mathbf{x}^{(i)} + \boldsymbol{\theta}_0), 0) + \lambda \|\boldsymbol{\theta}\|_2^2$  are equivalent. What is the functional relationship between  $C$  and  $\lambda$ ?  
*Hint:* Try to insert the combined constraints into their associated objective.
- (d) Show that the second primal form of the linear SVM is a convex problem