https://slds-lmu.github.io/website\_optimization/

## Mathematical Concepts 1

## Exercise 1: Gradient

Consider the bivariate function  $f: \mathbb{R}^2 \to \mathbb{R}, (x_1, x_2) \mapsto x_1^2 + 0.5x_2^2 + x_1x_2$ .

- (a) Show that f is smooth (as defined in the lecture).
- (b) Find the direction of greatest increase of f at  $\mathbf{x} = (1, 1)$ .
- (c) Find the direction of greatest decrease of f at  $\mathbf{x} = (1, 1)$ .
- (d) Find a direction in which f does not instantly change at  $\mathbf{x} = (1, 1)$ .
- (e) Assume there exists a differentiable parametrization of a curve  $\tilde{\mathbf{x}}: \mathbb{R} \to \mathbb{R}^2, t \mapsto \tilde{\mathbf{x}}(t)$  such that  $\forall t \in \mathbb{R}: f(\tilde{\mathbf{x}}(t)) = f(1,1)$ . Show that at each point of the curve  $\tilde{\mathbf{x}}$  the tangent line  $\frac{\partial \tilde{\mathbf{x}}}{\partial t}$  is perpendicular to the gradient  $\nabla f(\tilde{\mathbf{x}})$ .
- (f) Interpret (d), (e) geometrically

## Exercise 2: Convexity

Consider two convex functions  $f, g : \mathbb{R} \to \mathbb{R}$ .

- (a) Show that  $f + g : \mathbb{R} \to \mathbb{R}, x \mapsto f(x) + g(x)$  is convex.
- (b) Now, assume that g is additionally non-decreasing, i.e.,  $g(y) \ge g(x) \ \forall x \in \mathbb{R}, \forall y \in \mathbb{R}$  with y > x. Show that  $g \circ f$  is convex.

## Exercise 3: Taylor polynomials

Consider the bivariate function  $f: \mathbb{R}^2 \to \mathbb{R}, (x_1, x_2) \mapsto \exp(\pi \cdot x_1) - \sin(\pi \cdot x_2) + \pi \cdot x_1 \cdot x_2$ 

- (a) Compute the gradient of f for an arbitrary  $\mathbf{x}$ .
- (b) Compute the Hessian of f for an arbitrary  $\mathbf{x}$ .
- (c) State the first order taylor polynomial  $T_{1,\mathbf{a}}(\mathbf{x})$  expanded around the point  $\mathbf{a} = (0,1)$ .
- (d) State the second order taylor polynomial  $T_{2,\mathbf{a}}(\mathbf{x})$  expanded around the point  $\mathbf{a}=(0,1)$ .
- (e) Determine if  $T_{2,\mathbf{a}}$  is a convex function.