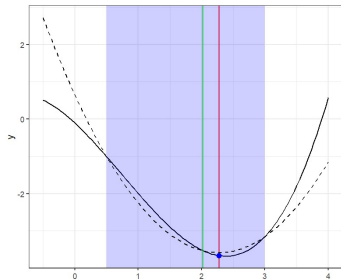# Optimization in Machine Learning

Univariate optimization: Brent's method
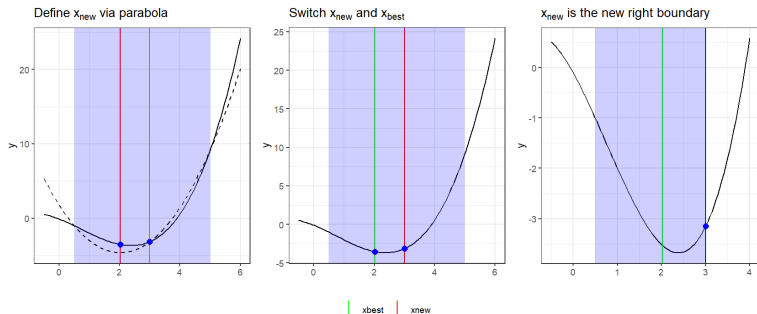

Learning goals

- Quadratic interpolation
- Brent's procedure

QUADRATIC INTERPOLATION

Similar to golden ratio procedure but select x^{new} differently: x^{new} as minimum of a parabola fitted through

$$(x^{\text{left}}, f^{\text{left}}), (x^{\text{best}}, f^{\text{best}}), (x^{\text{right}}, f^{\text{right}}).$$

Left: Fit parabola (dashed) and propose minimum (red) as new point. Middle: Switch / not switch with x^{best}. Right: New interval.

QUADRATIC INTERPOLATION

Similar to golden ratio procedure but select x^{new} differently: x^{new} as minimum of a parabola fitted through

$$(x^{\text{left}}, f^{\text{left}}), (x^{\text{best}}, f^{\text{best}}), (x^{\text{right}}, f^{\text{right}}).$$
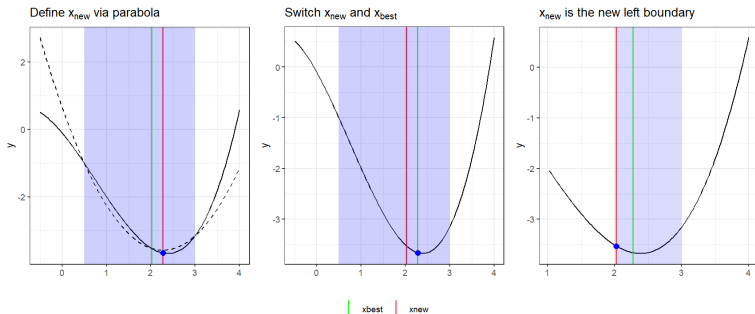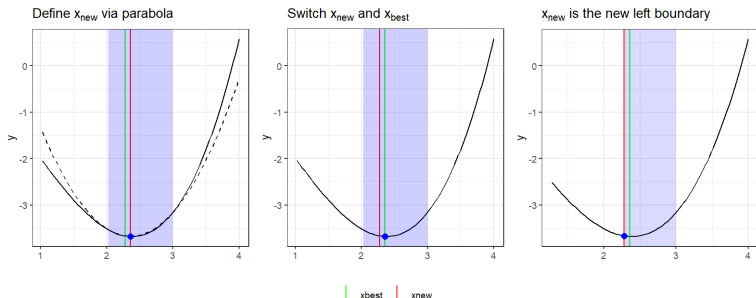

Left: Fit parabola (dashed) and propose minimum (red) as new point. Middle: Switch / not switch with x^{best}. Right: New interval.

QUADRATIC INTERPOLATION

Similar to golden ratio procedure but select x^{new} differently: x^{new} as minimum of a parabola fitted through

$$\left(x^{\text{left}}, f^{\text{left}}\right), \left(x^{\text{best}}, f^{\text{best}}\right), \left(x^{\text{right}}, f^{\text{right}}\right).$$

Left: Fit parabola (dashed) and propose minimum (red) as new point. Middle: Switch / not switch with x^{best}. Right: New interval.

QUADRATIC INTERPOLATION

Similar to golden ratio procedure but select x^{new} differently: x^{new} as minimum of a parabola fitted through

$$\left(x^{\text{left}}, f^{\text{left}}\right), \left(x^{\text{best}}, f^{\text{best}}\right), \left(x^{\text{right}}, f^{\text{right}}\right).$$
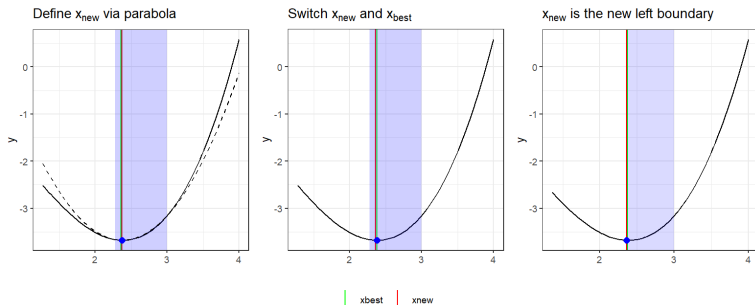

Left: Fit parabola (dashed) and propose minimum (red) as new point. Middle: Switch / not switch with x^{best}. Right: New interval.

QUADRATIC INTERPOLATION

Similar to golden ratio procedure but select x^{new} differently: x^{new} as minimum of a parabola fitted through

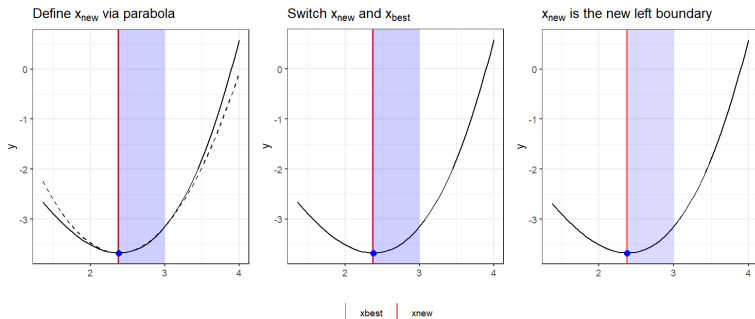$$(x^{left}, f^{left}), (x^{best}, f^{best}), (x^{right}, f^{right}).$$

Left: Fit parabola (dashed) and propose minimum (red) as new point. Middle: Switch / not switch with x^{best}. Right: New interval.

QUADRATIC INTERPOLATION COMMENTS

- Quadratic interpolation **not robust**. The following may happen:
  - Algorithm suggests the same x^{new} in each step,
  - x^{new} outside of search interval,
  - Parabola degenerates to line and no real minimum exists
- Algorithm must then abort, finding a global minimum is not guaranteed.

BRENT'S METHOD

- Brent proposed an algorithm (1973) that alternates between golden ratio search and quadratic interpolation as follows:
  - Quadratic interpolation step acceptable if: (i) x^{new} falls within $[x^{left}, x^{right}]$ (ii) x^{new} sufficiently far away from x^{best}
    (Heuristic: Less than half of movement of step before last)
  - Otherwise: Proposal via golden ratio
- Benefit: Fast convergence (quadratic interpolation), unstable steps (e.g. parabola degenerated) stabilized by golden ratio search
- Convergence guaranteed if the function *f* has a local minimum
- Used in R-function `optimize()`

EXAMPLE: MLE POISSON

- Poisson density: $f(k \mid \lambda) := \mathbb{P}(x = k) = \frac{\lambda^k \cdot \exp(-\lambda)}{k!}$
- Negative log-likelihood for n observations:

$$-\ell(\lambda, \mathcal{D}) = -\log \prod_{i=1}^{n} f\left(x^{(i)} \mid \lambda\right) = -\sum_{i=1}^{n} \log f\left(x^{(i)} \mid \lambda\right)$$

Method
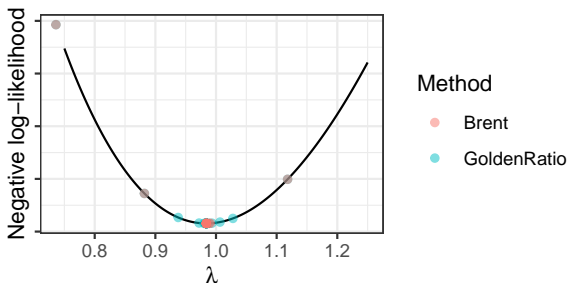
- Brent
- GoldenRatio

GR and Brent converge to minimum at $x^* \approx 1$.
But: GR needs ≈ 45 it., Brent only needs ≈ 15 it. for same tolerance.