Mathematical Concepts 2

Exercise 1: Matrix Calculus

Let $\mathbf{x}, \mathbf{c} \in \mathbb{R}^d, \mathbf{u}, \mathbf{v} : \mathbb{R}^d \to \mathbb{R}^d, \mathbf{u}, \mathbf{v} \in \mathcal{C}^2, \mathbf{Y} : \mathbb{R}^d \to \mathbb{R}^{d \times d}$

(a) Compute $\frac{\partial \|\mathbf{x}-\mathbf{c}\|_2^2}{\partial \mathbf{x}}$

(b) Compute $\frac{\partial \|\mathbf{x}-\mathbf{c}\|_2}{\partial \mathbf{x}}$

(c) Compute $\frac{\partial \mathbf{u}^\top \mathbf{v}}{\partial \mathbf{x}}$

(d) Show that $\frac{\partial \mathbf{u}^\top \mathbf{Y}}{\partial \mathbf{x}} = \begin{pmatrix} \mathbf{u}^\top \frac{\partial \mathbf{y}_1}{\partial \mathbf{x}} + \mathbf{y}_1^\top \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \\ \vdots \\ \mathbf{u}^\top \frac{\partial \mathbf{y}_d}{\partial \mathbf{x}} + \mathbf{y}_d^\top \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \end{pmatrix}$ where $\mathbf{y}_i : \mathbb{R}^d \to \mathbb{R}^d, i = 1, \ldots, d$ are the column vectors of $\mathbf{Y}$.

(e) Compute $\frac{\partial^2 \mathbf{u}^\top \mathbf{v}}{\partial \mathbf{x} \partial \mathbf{x}^\top}$
Hint: use c), d)

Exercise 2: Optimality in 1d

Let $f : [-1, 2] \to \mathbb{R}, x \mapsto \exp(x^3 - 2x^2)$

(a) Compute f'

(b) Plot f and f' with `R`

(c) Find all possible candidates x^* for local maxima and minima.
Hint: exp is a strictly monotone function.

(d) Compute f''

(e) Determine if the candidates are local maxima, minima or neither.

(f) Find the global maximum and global minimum of f