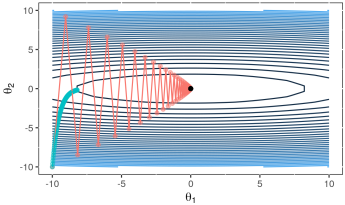
Optimization in Machine Learning

First order methods: Step size and optimality



Learning goals

- Impact of step size
- Fixed vs. adaptive
- Exact line search
- Armijo rule and backtracking

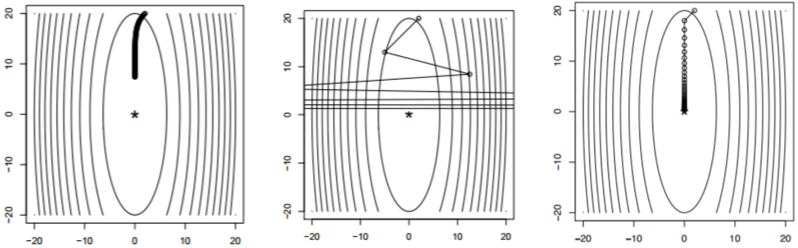
CONTROLLING STEP SIZE: FIXED & ADAPTIVE

Iteration t: Choose not only descent direction $\mathbf{d}^{[t]}$, but also step size $\alpha^{[t]}$

First approach: **Fixed** step size $\alpha^{[t]} = \alpha > 0$

- If α too small, procedure may converge very slowly (left)
- If α too large, procedure may not converge $\rightarrow$ "jumps" around optimum (middle)

Adaptive step size $\alpha^{[t]}$ can provide better convergence (right)



Steps of line searches for $f(\boldsymbol{x}) = 10x_1^2 + x_2^2/2$

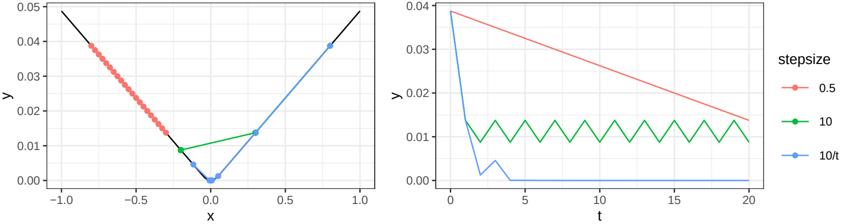
STEP SIZE CONTROL: DIMINISHING STEP SIZE

How can we adaptively control step size?

A natural way of selecting $\alpha^{[t]}$ is to decrease its value over time

Example: GD on

$$f(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } |x| \leq \delta, \\ \delta \cdot (|x| - 1/2 \cdot \delta) & \text{otherwise.} \end{cases}$$



GD with small constant (**red**), large constant (**green**), and diminishing (**blue**) step size

STEP SIZE CONTROL: EXACT LINE SEARCH

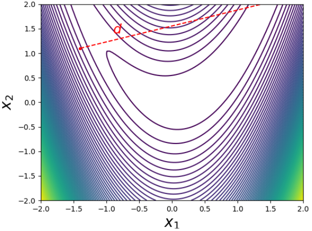
Use **optimal** step size in each iteration:

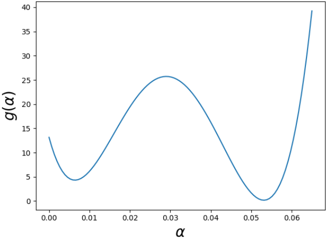
$$\alpha^{[t]} = \arg\min_{\alpha \in \mathbb{R}_{\geq 0}} g(\alpha) = \arg\min_{\alpha \in \mathbb{R}_{\geq 0}} f(\boldsymbol{x}^{[t]} + \alpha \mathbf{d}^{[t]})$$

Need to solve a **univariate** optimization problem in each iteration
$\Rightarrow$ univariate optimization methods

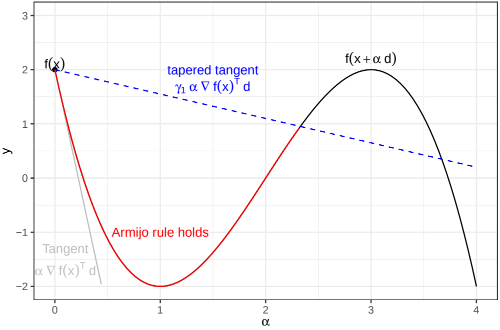
Problem: Expensive, prone to poorly conditioned problems

But: No need for *optimal* step size. Only need a step size that is “good enough”.
Reason: Effort may not pay off, but in some cases slows down performance.





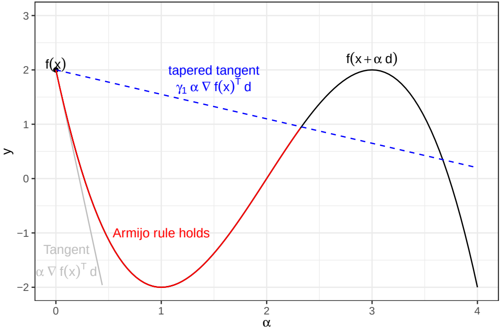
ARMIJO RULE



Inexact line search: Minimize objective “sufficiently” without computing optimal step size exactly

Common condition to guarantee “sufficient” decrease: **Armijo rule**

ARMIJO RULE

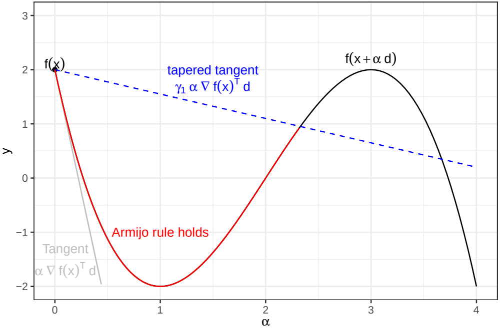


Fix $\gamma_1 \in (0, 1)$. α satisfies **Armijo rule** in **x** for descent direction **d** if

$$f(\mathbf{x} + \alpha\mathbf{d}) \leq f(\mathbf{x}) + \gamma_1 \alpha \nabla f(\mathbf{x})^\top \mathbf{d}.$$

Note: $\nabla f(\mathbf{x})^\top \mathbf{d} < 0$ (**d** *descent* dir.) $\implies f(\mathbf{x} + \alpha\mathbf{d}) < f(\mathbf{x})$.

ARMIJO RULE



Feasibility: For descent direction **d** and $\gamma_1 \in (0, 1)$, there exists $\alpha > 0$ fulfilling Armijo rule. In many cases, Armijo rule guarantees local convergence of GD and is therefore frequently used.

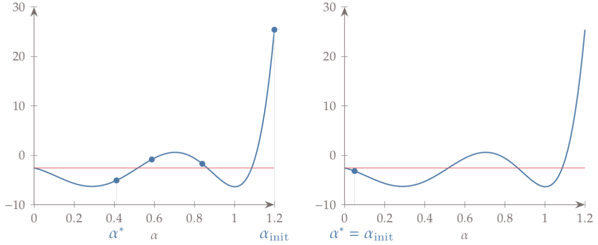
BACKTRACKING LINE SEARCH

Procedure to meet the Armijo rule: **Backtracking** line search

Idea: Decrease α until Armijo rule is met

Algorithm Backtracking line search

1: Choose initial step size $\alpha = \alpha_{\text{init}}$, $0 < \gamma_1 < 1$ and $0 < \tau < 1$
2: **while** $f(\boldsymbol{x} + \alpha \mathbf{d}) > f(\boldsymbol{x}) + \gamma_1 \alpha \nabla f(\boldsymbol{x})^\top \mathbf{d}$ **do**
3: Decrease α: $\alpha \leftarrow \tau \cdot \alpha$
4: **end while**



(Source: Martins and Ning. *Engineering Design Optimization*, 2021.)

WOLFE CONDITIONS

Backtracking is simple and shows good performance in practice

But: Two undesirable scenarios

1. Initial step size α_{init} is too large $\Rightarrow$ need multiple evaluations of f
2. Step size is too small with highly negative slopes

Solution for small step sizes:

- Fix γ_2 with $0 < \gamma_1 < \gamma_2 < 1$.
- α satisfies **sufficient curvature condition** in $\mathbf{x}$ for $\mathbf{d}$ if

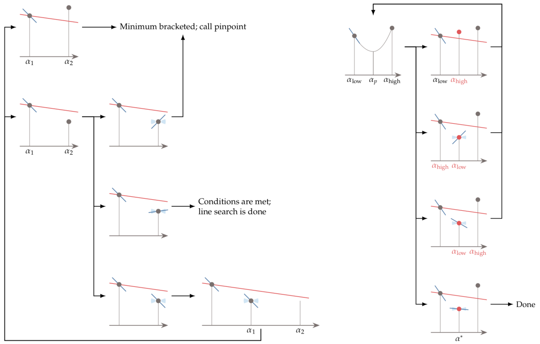
$$|\nabla f(\mathbf{x} + \alpha \mathbf{d})^\top \mathbf{d}| \leq \gamma_2 |\nabla f(\mathbf{x})^\top \mathbf{d}|.$$

Armijo rule + sufficient curvature condition = **Wolfe conditions**

WOLFE CONDITIONS

Algorithm for finding a Wolfe point (point satisfying Wolfe conditions):

1. **Bracketing:** Find interval containing Wolfe point
2. **Pinpointing:** Find Wolfe point in interval from bracketing



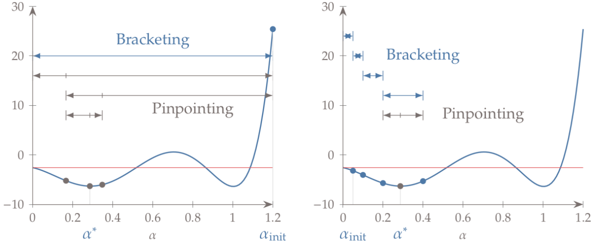
Left: Bracketing. **Right:** Pinpointing.

(Source: Martins and Ning. *EDO*, 2021.)

BRACKETING & PINPOINTING

Example:

- Large initial step size results in quick bracketing but multiple pinpointing steps (**left**).
- Small initial step size results in multiple bracketing steps but quick pinpointing (**right**).



Source: Martins and Ning. *EDO*, 2021.

GRADIENT DESCENT AND OPTIMALITY

- GD is a greedy algorithm: locally optimal moves
- If $\mathcal{R}(\theta)$ is **convex** and **differentiable**, and its **gradient** is **Lipschitz continuous**, GD is guaranteed to converge to the global minimum for small enough step size.