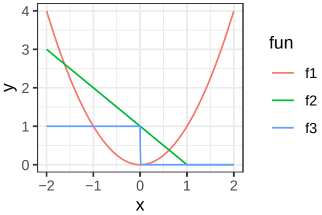
Optimization

Quadratic forms II



Learning goals

- Properites of Quadratic functions
-

PROPERTIES OF QUADRATIC FUNCTIONS

Univariate case

In 1-variable calculus, the second derivative ist just a single value $q''(x) = 2 \cdot a$. Basic properties of q can easily be read off:

- $q''(x) > 0$ implies convex, $q''(x) < 0$ implies concave.
- High absolute values mean high curvature, low absolute values mean a low curvature.

Now: Multivariate case

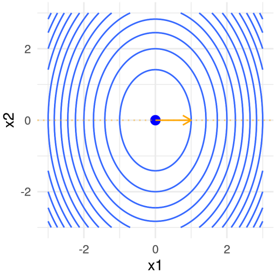
For a multivariate quadratic function, the second derivative is a symmetric matrix of values $\mathbf{H}$ (called the Hessian).

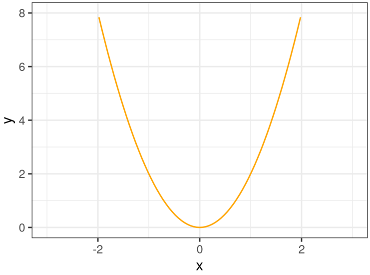
We will see how $\mathbf{H}$ encodes the basic properties of the function q.

PROPERTIES OF QUADRATIC FUNCTIONS (DIAG)

Example 1: A function composed of two univariate quadratic terms

$$
\begin{aligned}
q(\mathbf{x}) &= \mathbf{x}^\top \mathbf{A}\mathbf{x} = \mathbf{x}^\top \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 2 \cdot x_1^2 + x_2^2 \\
\text{with } \nabla q(\mathbf{x}) &= 2 \cdot \mathbf{A} \cdot \mathbf{x} = 4 \cdot x_1 + 2 \cdot x_2, \quad \boldsymbol{H} = 2 \cdot \mathbf{A} = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}
\end{aligned}
$$





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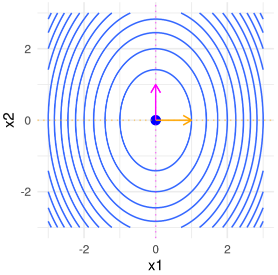

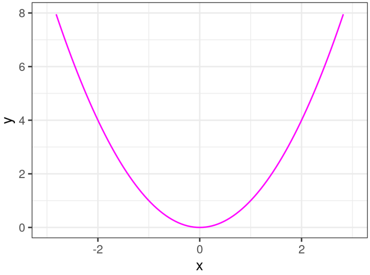

q has a high positive curvature of 4 in the direction of x_1, and a lower (positive) curvature of 2 in direction of x_2.

PROPERTIES OF QUADRATIC FUNCTIONS (DIAG)

Takeaway I:

- The Hessian encodes the curvature of the function
- If the Hessian **H** is diagonal, the diagonal elements encode the curvature of the function:
  - The i-th diagonal element gives us the curvature in the direction of $\boldsymbol{v} = \boldsymbol{e}_i$ because

$$\boldsymbol{v}^\top \boldsymbol{H}\boldsymbol{v} = \boldsymbol{e}_i^\top \boldsymbol{H}\boldsymbol{e}_i = h_{ii}.$$

  - The curvature in an arbitrary direction $\boldsymbol{v} \in \mathbb{R}^d$, $\|\boldsymbol{v}\| = 1$, is

$$\boldsymbol{v}^\top \mathbf{H}\boldsymbol{v} = h_{11}v_1^2 + h_{22}v_2^2 + ... + h_{dd}v_d^2.$$

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  $$\boldsymbol{v}^\top \mathbf{H}\boldsymbol{v} = h_{11}v_1^2 + h_{22}v_2^2 + \ldots + h_{dd}v_d^2.$$
- In the general case we look at the **eigenspectrum** of $\mathbf{H}$

Note: For diagonal matrices it is very easy to determine the eigenspectrum: The diagonal elements are the **eigenvalues**, and unit vectors are the **eigenvectors**.

$$\mathbf{H}\boldsymbol{e}_1 = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} = 4 \cdot \boldsymbol{e}_1; \qquad \mathbf{H}\boldsymbol{e}_2 = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} = 2 \cdot \boldsymbol{e}_2$$

PROPERTIES OF QUADRATIC FUNCTIONS

Example 2:

$$
\begin{aligned}
q(\mathbf{x}) &= \mathbf{x}^\top \mathbf{A}\mathbf{x} = \mathbf{x}^\top \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \mathbf{x}, \\
\text{with } \nabla q(\mathbf{x}) &= 2 \cdot \mathbf{A} \cdot \mathbf{x}, \qquad \nabla^2 q(\mathbf{x}) = \boldsymbol{H} = 2 \cdot \mathbf{A} = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}
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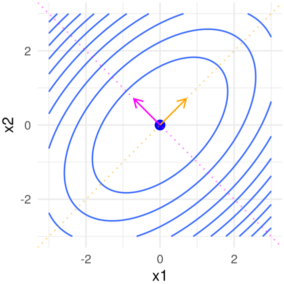
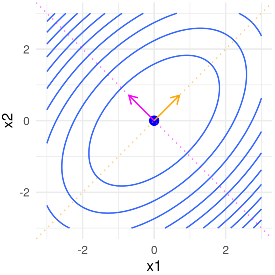


In the general case, the curvature is determined by the Eigenspectrum of $\boldsymbol{H}$.

PROPERTIES OF QUADRATIC FUNCTIONS

Takeaway II:

- Geometrically, directions of highest / lowest curvature are along the main axes of the ellipses representing the contour lines of q.
- Mathematically, the direction with the highest (lowest) curvature is the direction of the eigenvector $\boldsymbol{v}_{\text{max}}$ ($\boldsymbol{v}_{\text{min}}$) that belongs to the largest (smallest) eigenvalue λ_{max} (λ_{min}) of $\boldsymbol{H}$.



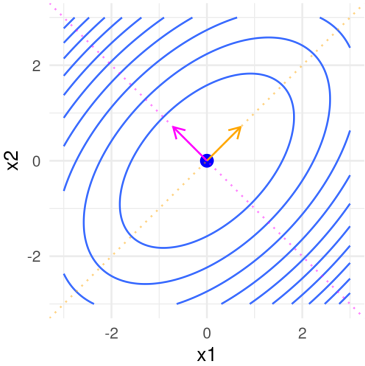
The eigenvectors and eigenvalues of $\boldsymbol{H} = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}$ are:

$$\boldsymbol{v}_{\text{min}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \lambda_{\text{min}} = 2$$

$$\boldsymbol{v}_{\text{max}} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad \lambda_{\text{max}} = 3$$

PROPERTIES OF QUADRATIC FUNCTIONS

The direction $\boldsymbol{v}_{\text{max}}$ is also the direction in which the function increases fastest.

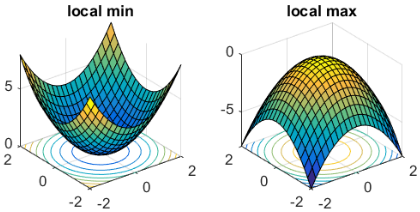


"Walking" the same distance along $\boldsymbol{v}_{\text{max}}$ (magenta) makes us pass more level curves than walking along any other direction.

PROPERTIES OF QUADRATIC FUNCTIONS

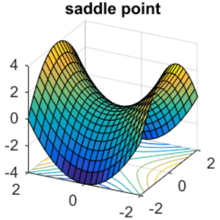
If we know the eigenspectrum of $\mathbf{A}$, i.e. the set of its eigenvalues $\{\lambda_1, \lambda_2, ..., \lambda_d\}$, we also know the eigenspectrum of $\boldsymbol{H} = 2 \cdot \mathbf{A}$. The following basic properties of q can be read off from this eigenspectrum:

- If **all** eigenvalues of the $\boldsymbol{H}$ are > 0 (we call $\boldsymbol{H}$ positive definite):
  - the function q is convex,
  - there is a unique global minimum.
- If **all** eigenvalues of the $\boldsymbol{H}$ are < 0 (we call $\boldsymbol{H}$ negative definite):
  - the function q is concave,
  - there is a unique global maximum.



PROPERTIES OF QUADRATIC FUNCTIONS

- If there are both positive and negative eigenvalues (we call $\boldsymbol{H}$ indefinite):
  - the function q is neither concave nor convex,
  - there is a saddle point.



PROPERTIES OF QUADRATIC FUNCTIONS

Example: Sketch the following function

$$q(\mathbf{x}) \quad = \quad \mathbf{x}^\top \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \mathbf{x}$$

Step 1: Compute the Hessian

$$\boldsymbol{H} \quad = \quad 2 \cdot \mathbf{A} = \begin{pmatrix} -2 & -2 \\ -2 & 2 \end{pmatrix}$$

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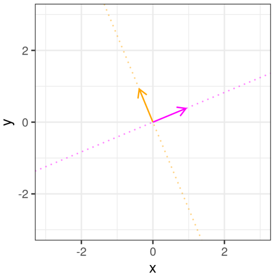
Step 2: Compute eigenvectors / -values:

$$\mathbf{v}_1 = \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix}, \qquad \lambda_1 = 2\sqrt{2}$$

$$\mathbf{v}_2 = \begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix}, \qquad \lambda_2 = -2\sqrt{2}.$$



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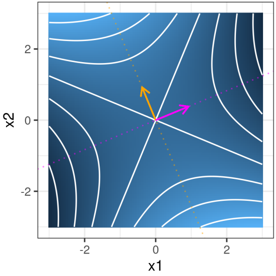

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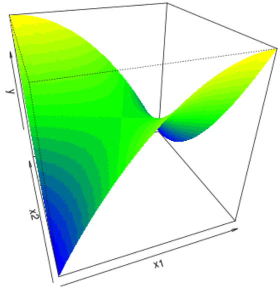
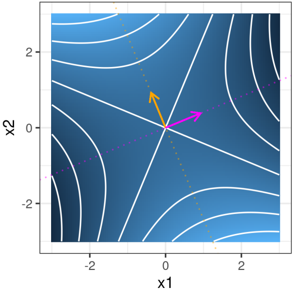
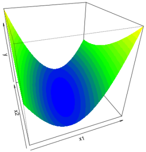


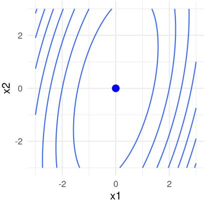


EIGENSPECTRUM AND CONDITION

The condition of squared full-rank matrix can be read off from its eigenspectrum: it is $\kappa(\mathbf{A}) = \frac{|\lambda_{\max}|}{|\lambda_{\min}|}$, with $\lambda_{\max}$ being the largest eigenvalue and $\lambda_{\min}$ being the lowest eigenvalue (in absolute terms). A high condition means:

- The absolute value of the biggest eigenvalue $\lambda_{\max}$ is much larger than the absolute value of the lowest eigenvalue $\lambda_{\min}$.
- The curvature in the direction of minimum curvature ($\mathbf{v}_{\max}$) is much lower than the one in the direction of maximum curvature ($\mathbf{v}_{\min}$).
- We will see later, that optimization algorithms like gradient descent will have difficulties optimizing such functions.

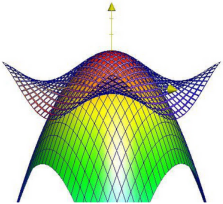




INTERPRETATION OF GENERAL FUNCTIONS

Every function can be locally approximated by a quadratic function via Taylor approximation:

$$f(x) \approx f(\tilde{\boldsymbol{x}}) + \nabla f(\tilde{\boldsymbol{x}})^\top (\mathbf{x} - \tilde{\boldsymbol{x}}) + \frac{1}{2}(\mathbf{x} - \tilde{\boldsymbol{x}})^\top \nabla^2 f(\tilde{\boldsymbol{x}})(\mathbf{x} - \tilde{\boldsymbol{x}})$$



f is shown as the hollow grid and its second-order approximation at $(0, 0)$ as a continuous surface. Source: `daniloroccatano.blog`.

By a basic geometric understanding of quadratic functions, we are able to understand a function's local geometry by looking at the Hessian $\nabla^2 f(\tilde{\boldsymbol{x}})$.