Mathematical Concepts 3

Exercise 1: Optimality in 2 dimensions

Let $f: \mathbb{R}^2 \to \mathbb{R}, (x_1, x_2) \mapsto -\cos(x_1^2 + x_2^2 + x_1 x_2)$

- (a) Create a contour plot of f in the range $[-2,2] \times [-2,2]$ with R.
- (b) Compute ∇f
- (c) Compute $\nabla^2 f$

Now, we define the restriction of f to $S_r = \{(x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 + x_1 x_2 < r\}$ with $r \in \mathbb{R}, r > 0$, i.e., $f_{|S_r|} : S_r \to \mathbb{R}, (x_1, x_2) \mapsto f(x_1, x_2)$.

- (d) Show that $f_{|S_{\overline{r}}|}$ with $\overline{r} = \pi/4$ is convex. *Hint:* The sum of a positive definite matrix and a positive semi-definite matrix is positive definite.
- (e) Find the local minimum \mathbf{x}^* of $f_{|S_{\overline{x}}|}$
- (f) Is \mathbf{x}^* a global minimum of f?

Exercise 2: Optimality in d dimensions

Let **X** be a d dimensional random vector and let **Y** be a one dimensional random vector with $Var(\mathbf{X}) = \Sigma_{\mathbf{X}, \mathbf{Y}} \in \mathbb{R}^{d \times 1}$. Further, let $f : \mathbb{R}^d \to \mathbb{R}, \mathbf{w} \mapsto Var(\mathbf{w}^\top \mathbf{X} - \mathbf{Y})$.

- (a) Show that f is convex.
- (b) Compute ∇f and $\nabla^2 f$
- (c) Under which condition exists a unique minimizer \mathbf{w}^* of f. Is this a global minimum? (if it exists)
- (d) Given the samples $(\mathbf{x}_i, y_i) \sim \mathbb{P}_{\mathbf{X}, \mathbf{Y}}$, under which condition is the least squares estimator a consistent estimator of \mathbf{w}^* in general?

¹This question is out of the scope of this lecture; however, it gives interesting insights into the entities we have computed.