Linear algebra for AI & ML

Basis and dimension.

In $\mathbb{R}^n$, the maximal linearly independent collection contains n vectors.

A basis is a collection of n linearly independent vectors in $\mathbb{R}^n$.

Any vector $v \in \mathbb{R}^n$ can be written as a unique linear combination of basis vectors.

$B = \{v_1, \ldots, v_n\}$ is a basis of $\mathbb{R}^n$.

$v \in \mathbb{R}^n$

Consider the linear combination,

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n + \beta v = 0$$

Linear functions

$f : \mathbb{R}^n \rightarrow \mathbb{R}$ is called as a linear function if $\forall\, x, y \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$

$$\boxed{f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)} \quad : \text{ superposition principle}$$

$$\Updownarrow$$

$$\boxed{\begin{array}{l} f(\alpha x) = \alpha f(x) \\ f(x+y) = f(x) + f(y) \end{array}}$$

$$f\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1^2 + x_2^2 + \cdots + x_n^2 \quad \Leftarrow \text{not linear}$$

$$= |x_1| + |x_2| + \cdots + |x_n| \quad \Leftarrow \text{not linear}$$

$$= \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \quad \Leftarrow \text{linear for scalars } \alpha_i\text{'s} \in \mathbb{R}$$

Let $f: \mathbb{R}^n \to \mathbb{R}$ be a linear map.

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \qquad e_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{th} \text{ position.}$$

$$= x_1 e_1 + x_2 e_2 + \dots + x_n e_n$$

$$f(x) = f(x_1 e_1 + x_2 e_2 + \dots + x_n e_n)$$

$$= x_1 \underbrace{f(e_1)}_{\alpha_1} + x_2 \underbrace{f(e_2)}_{\alpha_2} + \dots + x_n \underbrace{f(e_n)}_{\alpha_n}$$

$$a = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} f(e_1) \\ \vdots \\ f(e_n) \end{bmatrix}$$

$f(x) = a^T x$ — inner product of $a \in \mathbb{R}^n$ with $x \in \mathbb{R}^n$

$= \langle a, x \rangle$

outer product $a x^T$

Affine maps:

$f: \mathbb{R}^n \to \mathbb{R}$ is an affine map if

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \quad \text{where } \alpha + \beta = 1$$

Every affine map is of the form

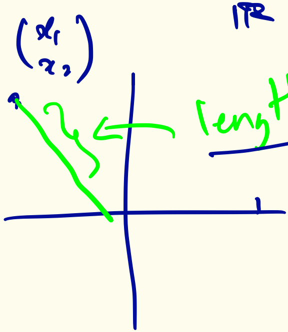
$$f(x) = a^T x + b \quad \text{where } a \in \mathbb{R}^n \text{ and } b \in \mathbb{R}$$

Norm of a vector

$$f\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix} = \sqrt{x_1^2+x_2^2+\cdots+x_n^2}$$

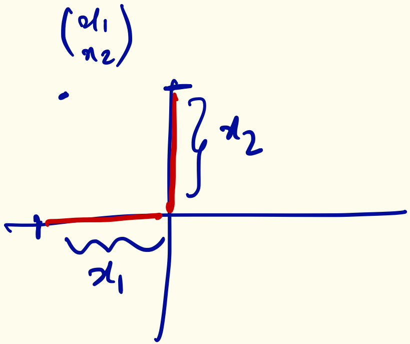
: 2-norm of $x\in\mathbb{R}^n$

$\mathbb{R}^2$



length = $\sqrt{x_1^2+x_2^2}$

$f(x)=\|x\|$



length $= |x_1| + |x_2|$

$\|\ \|: \mathbb{R}^n \longrightarrow \mathbb{R}$ is a norm if

i) $\|x\| \geq 0$ and $\|x\|=0 \iff x=0$

ii) $\|\alpha x\| = |\alpha|\,\|x\|$

iii) $\|x+y\| \leq \|x\| + \|y\|$

$$f(x) = \|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

- vector p-norm

where $p \geq 1$

Distance : consider $\| \ \| : \mathbb{R}^n \longrightarrow \mathbb{R}$

$\forall x, y \in \mathbb{R}^n$

$$d(x, y) = \|x - y\|$$