Linear Algebra for AI & ML
Assignment 2

NAME :- SHREYAS JENA
ROLL :- 20CS30049

1

Given :- $M = (1-p)A + pB$, where p = damping factor.

P.T. :- (i) M ~~has only +ve~~ remains a column stochastic matrix.
(ii) M has only +ve entries.

Proof :- (i) Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \vdots \\ a_{31} & & & \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}$; $(a_{ij} \geq 0)$ $(1 \leq i, j \leq n)$

where, each column sums to 1.

Now, $M = (1-p)A + pB$

$$= \begin{array}{cccc} ① & ② & & ⓝ \\ \end{array}$$

$$= \begin{bmatrix} (1-p)a_{11} + \frac{p}{n} & (1-p)a_{12} + p/n & \cdots & (1-p)a_{1n} + p/n \\ (1-p)a_{21} + p/n & & & \vdots \\ \vdots & & & \\ (1-p)a_{n1} + p/n & \cdots & & \end{bmatrix}$$

Consider sum of elements in i^{th} column :-

$$S_i = \left[(1-p)a_{1i} + p/n\right] + \left[(1-p)a_{2i} + p/n\right] + \cdots$$

$$= (1-p)\left(\sum_{j=1}^{n} a_{ji}\right) + (p/\cancel{n}) * \cancel{n}$$

Since, each column of A sums to 1, $\sum_{j=1}^{n} a_{ji} = 1$.

$$\therefore S_i = (1-p)(1) + (p)$$
$$= 1$$

$\therefore$ Sum of elements in i^{th} column of M is still 1.

$\therefore$ M remains a column stochastic matrix

(Proved)

(ii) Consider,

$(i, j)^{th}$ element of M

$$= M_{ij} = (1-p)\, a_{ij} + p/n$$

Now, $(1-p)\, a_{ij} + p/n \geq p/n$ (As, $a_{ij} \geq 0$, and $(1-p) > 0$)

> 0 (As, $p > 0$)

$$\boxed{\therefore\ M_{ij} > 0 \quad \forall\ 1 \leq i, j \leq n}$$

$\boxed{\therefore\ \text{M contains only +ve entries}}$ (Proved)

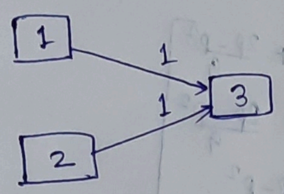
---

2 Redo computations for PageRank with transition matrix A replaced by M, for :-

(i) Dangling nodes,

(ii) Disconnected components.

(i) Dangling nodes :-



$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad B = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

$$M = (1-p)A + pB$$

$$= \begin{bmatrix} p/3 & p/3 & p/3 \\ p/3 & p/3 & p/3 \\ (1-p)+p/3 & (1-p)+p/3 & p/3 \end{bmatrix} = \begin{bmatrix} p/3 & p/3 & p/3 \\ p/3 & p/3 & p/3 \\ 1-2p/3 & 1-2p/3 & p/3 \end{bmatrix}.$$

Now,

let $x_0 = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$;

$$x_1 = M x_0 = \begin{bmatrix} P/3 & P/3 & P/3 \\ P/3 & P/3 & P/3 \\ 1-2P/3 & 1-2P/3 & P/3 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} P/3 \\ P/3 \\ \frac{2-P}{3} \end{bmatrix}$$

$$x_2 = M x_1 = \begin{bmatrix} P/3 & P/3 & P/3 \\ P/3 & P/3 & P/3 \\ \frac{3-2P}{3} & \frac{3-2P}{3} & P/3 \end{bmatrix} \begin{bmatrix} P/3 \\ P/3 \\ \frac{2-P}{3} \end{bmatrix}$$

$$= \begin{bmatrix} P^2/9 + P^2/9 + \frac{2P-P^2}{9} \\ P^2/9 + P^2/9 + \frac{2P-P^2}{9} \\ \frac{2(3P-2P^2) + 2P-P^2}{9} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2P+P^2}{9} \\ \frac{2P+P^2}{9} \\ \frac{8P-5P^2}{9} \end{bmatrix} = \begin{bmatrix} \frac{P(2+P)}{9} \\ \frac{P(2+P)}{9} \\ \frac{P(8-5P)}{9} \end{bmatrix}.$$

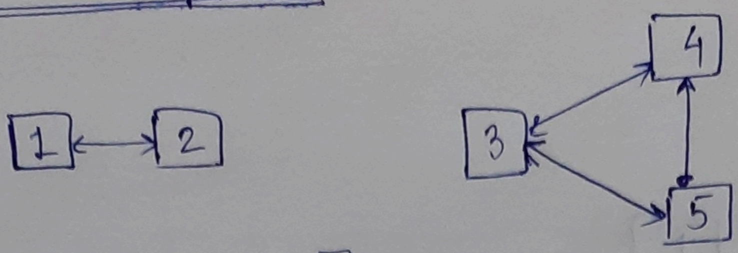
Now, $\frac{P(2+P)}{9} > 0$ (as $P > 0$)

$\frac{P(8-5P)}{9} > 0$, (as $P > 0$ and $P < 1 \Rightarrow 8-5P > 0$)

$\boxed{\therefore \ x_2 \neq 0}$ (As, was the case when we used A instead of M)

$\therefore$ Using M solves the issue of zero importance computed for some of the webpages. (Ans.)

(ii) <u>Disconnected components</u> :-



$$A = \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 & 0 \end{array}\right]$$

$$M = (1-p)A + pB \quad , \text{ where } B = \frac{1}{5}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \left[\begin{array}{cc|ccc} \frac{p}{5} & 1-\frac{4p}{5} & p/5 & p/5 & \cancel{p/5}\ p/5 \\ 1-\frac{4p}{5} & p/5 & p/5 & p/5 & \cancel{p/5}\ p/5 \\ \hline p/5 & p/5 & p/5 & \left(\frac{1-p}{2}+\frac{p}{5}\right) & \left(\frac{1-p}{2}+p/5\right) \\ p/5 & p/5 & \left(\frac{1-p}{2}+p/5\right) & p/5 & \left(\frac{1-p}{2}+p/5\right) \\ p/5 & p/5 & \left(\frac{1-p}{2}+\frac{p}{5}\right) & \left(\frac{1-p}{2}+\frac{p}{5}\right) & p/5 \end{array}\right]$$

Setting $p = 0.15$, we get :-

$$M = 0.85A + 0.15B$$

$$= \begin{bmatrix} 0.03 & 0.88 & 0.03 & 0.03 & 0.03 \\ 0.88 & 0.03 & 0.03 & 0.03 & 0.03 \\ 0.03 & 0.03 & 0.03 & 0.455 & 0.455 \\ 0.03 & 0.03 & 0.455 & 0.03 & 0.455 \\ 0.03 & 0.03 & 0.455 & 0.455 & 0.03 \end{bmatrix}$$

For the eigenvalue $\lambda = 1$,

Solving for $MX = X$ gives :-

$X = 0.2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ as the unique solution

$\therefore$ Problem of multiple eigenvectors ~~corresponding to $\lambda = 1$~~ for the same PageRank matrix M is resolved.

(Proved)