

Linear algebra for AI and ML

August 27

(Lecture #6)



$$Ax = b$$

$$A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m$$

- left invertibility
- right invertibility
- invertibility of matrix A

} → generalize the concept of inverse of a real number.

$$(a \neq 0)$$

(A has lin. indep. columns,) 

($A^T A$ is invertible) 

$B = (A^T A)^{-1} A^T$ ← pseudo inverse of A

$$\begin{bmatrix} n \times n \\ n \times m \end{bmatrix} \begin{bmatrix} n \times m \end{bmatrix} = \begin{bmatrix} B \end{bmatrix}_{n \times m}$$

$$A \in \mathbb{R}^{m \times n}$$

A has lin.
cols. $\begin{bmatrix} A \end{bmatrix}_{n \times m}$

$$\begin{bmatrix} A^T \\ n \times m \end{bmatrix} \begin{bmatrix} A \end{bmatrix}_{n \times m} = \begin{bmatrix} A^T A \\ n \times n \end{bmatrix}_{n \times n}$$

computations:

$$Ax = b$$

($A \in \mathbb{R}^{m \times m}$)

$b \in \mathbb{R}^m$

square matrix

Particular case:

U

$$\begin{bmatrix} u_{11} & u_{12} & \dots & u_{1m} \\ 0 & u_{22} & \dots & u_{2m} \\ \vdots & & \ddots & \\ 0 & 0 & \dots & u_{mm} \end{bmatrix} \xrightarrow{x = b}$$

upper triangular matrix

If $u_{ii} \neq 0$, for $i=1, 2, \dots, m$

$x_m = \frac{b_m}{u_{mm}}$

$$\begin{bmatrix} u_{11} & u_{12} & \dots & u_{1m} \\ 0 & u_{22} & \dots & u_{2m} \\ \vdots & & \ddots & \\ 0 & 0 & \dots & u_{mm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\begin{bmatrix} u_{mm} x_m \\ \downarrow \end{bmatrix} = \begin{bmatrix} \vdots \\ b_m \end{bmatrix}$$

$$u_{m-1,m-1} \quad x_{m-1} + u_{m-1,m} \underset{\uparrow}{x_m} = b_{m-1}$$

$$x_{m-1} = \frac{1}{u_{m-1,m-1}} [b_{m-1} - u_{m-1,m} x_m]$$

:

:

(backward substitution)

$$x = U^{-1} b$$

Given $Ax = b$

can we transform this system of linear equations in canonical form (upper triangular system or lower triangular system) provided that the solution of the transformed system does not change.

- - Gaussian elimination
- - LU decomposition

↔ QR decomposition

$$A = QR ; Ax = b \Rightarrow QRx = b \\ \Rightarrow Rx = Q^T b$$

$$\begin{array}{l} A = \overbrace{L U}^{\leftarrow} \\ = \left[\begin{array}{c|c} \cancel{x_1} & \cancel{x_2} \\ \hline \cancel{x_1} & \cancel{x_2} \end{array} \right] \left[\begin{array}{c|c} x_1 & x_2 \\ \hline 0 & 1 \end{array} \right] \end{array}$$

$$Ax = L(Ux) \neq Ly = b$$

" " " " "

- Cholesky decomposition
 - LU decomposition ← }
 - QR decomposition ← }
 - Eigenvalue, eigenvector decomposition
 - SVD
- ↑
Low rank approximations

LU decomposition:

$$A \in \mathbb{R}^{m \times m}$$

$$\underbrace{L_{m-1} \dots L_2 L_1}_L A = U$$

$$L^\top A = U$$
$$\Rightarrow A = LU$$

construct

$$L_{m-1}, L_{m-2}, \dots, L_2, L_1$$

s.t.

U: upper
triangular.

To prove: L is lower triangular.

setting: $L = \underbrace{L_1 L_2 \dots L_{m-1}}_{\begin{matrix} 1 & 1 & \dots & 1 \end{matrix}}$

$$\begin{array}{c}
 \left[\begin{array}{cccc} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array} \right] \xrightarrow{L_1} \left[\begin{array}{cccc} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{array} \right] \xrightarrow{L_2} \left[\begin{array}{cccc} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{array} \right] \xrightarrow{L_3} \\
 A \qquad L_1 A \qquad L_2 L_1 A
 \end{array}$$

$$\left[\begin{array}{cccc} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{array} \right] = U$$

$$\underbrace{L_3 L_2 L_1}_L A$$

$$A = LU = \underbrace{\begin{matrix} L_1 & L_2 & L_3 \end{matrix}}_L \underbrace{U}_{\text{U}}$$

Ex: $A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix} \in \mathbb{R}^{4 \times 4}$

$$L_1 A = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{array} \right] \left[\begin{array}{cccc} 2 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 \\ 7 & 9 & 5 & 0 \\ 7 & 9 & 8 & 1 \end{array} \right]$$

$\left\{ \begin{array}{l} R_2 - 2R_1 \\ R_3 - 4R_1 \\ R_4 - 3R_1 \end{array} \right\}$ on I

$\xrightarrow{\quad l_{21}, l_{31}, l_{41} \quad}$

$$= \left[\begin{array}{cccc} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 4 & 6 & 8 \end{array} \right]$$

$$L_2 L_1 A = \begin{bmatrix} 1 & & & \\ -3 & 1 & & \\ -4 & & 1 & \\ 7 & & & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 0 \\ & 1 & 1 & 1 \\ & 3 & 5 & 5 \\ & 4 & 6 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$\left\{ \begin{array}{l} R_3 - 3R_2 \\ R_4 - 4R_2 \end{array} \right\}$$

$$L_3 L_2 L_1 A = \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}}_{L_3} \underbrace{\begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}}_{L_2 L_1 A} = \underbrace{\begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}}_U$$

$$L_3 L_2 L_1 A = U$$

$$A = \underbrace{L_1 L_2 L_3}_{= L} U$$

To check:

i) L_1, L_2, L_3 are invertible

ii) Compute $L_1^{-1}, L_2^{-1}, L_3^{-1}$

iii) Show $L_1^{-1} L_2^{-1} L_3^{-1}$ is a lower triangular matrix.

H.W.

→

i) Since L_1, L_2, L_3 are lower triangular and
diagonals are all 1's ($\neq 0$) ; L_1, L_2, L_3
are invertible.

ii) Inverses of L_1, L_2, L_3

$$\left[\begin{array}{ccc|c} 1 & & & \\ -2 & 1 & & \\ -4 & & 1 & \\ -3 & & & 1 \end{array} \right] \xrightarrow{\text{Row operations}} = \left[\begin{array}{ccc|c} 1 & & & \\ 2 & 1 & & \\ 4 & & 1 & \\ 3 & & & 1 \end{array} \right]$$

L_1^{-1}

$$\left[\begin{array}{ccc|c} 1 & & & \\ -3 & 1 & & \\ -4 & & 1 & \\ -3 & & & 1 \end{array} \right] \xrightarrow{\text{Row operations}} = \left[\begin{array}{ccc|c} 1 & & & \\ 3 & 1 & & \\ 4 & & 1 & \\ 3 & & & 1 \end{array} \right]$$

L_2^{-1}

Similarly for L_3 also.

iii) To show $L_1^T L_2^T L_3^T$ is lower triangular

$$\begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 4 & 3 & 1 & \\ 3 & 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 3 & 1 & \\ & 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

L_1^T L_2^T L_3^T

$$= \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 4 & 3 & 1 & \\ 2 & 4 & 3 & 1 \end{bmatrix}$$

$L_1^T L_2^T L_3^T$

: lower
triangular

General case of $A \in \mathbb{R}^{m \times m}$:

Let y_k denote the k^{th} column of matrix A at the beginning of step k .

Choose L_k s.t.

$$y_k = \begin{bmatrix} y_{1,k} \\ \vdots \\ y_{k,k} \\ \boxed{y_{k+1,k}} \\ \vdots \\ \boxed{y_{m,k}} \end{bmatrix} \xrightarrow{L_k} L_k y_k = \begin{bmatrix} y_{1,k} \\ \vdots \\ \boxed{y_{k,k}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$l_{jk} = \frac{y_{jk}}{y_{kk}} \quad k < j \leq m$$

$$L_k = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & -L_{k+1,k} & & \ddots \\ & \vdots & & \\ & -L_{m,k} & & \end{bmatrix}_{m \times m}$$

\uparrow
 k^{th} column

Observation: L_k can be inverted by just changing the sign (negating the entries) below diagonal.

Define $\mathbf{l}_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ l_{k+1,k} \\ \vdots \\ l_{m,k} \end{bmatrix}_{m \times 1}$ k entries are zero.

$$\mathbf{e}_k = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}_{m \times 1} \quad \text{k^{th} position.}$$

Then check: $L_k = I - \mathbf{l}_k \mathbf{e}_k^T$

\mathbf{l}_k k^{th} position

\mathbf{e}_k^T k^{th} column

$\mathbf{l}_k \mathbf{e}_k^T = \begin{bmatrix} 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & \dots & -l_{k+1,k} & \ddots & 0 \\ \vdots & \vdots & \ddots & l_{m,k} & 0 & \ddots \end{bmatrix}_{m \times m}$

$\mathbf{l}_k \mathbf{e}_k^T$ k^{th} row

observe: $e_k^T l_k = 0$

$$\begin{bmatrix} 0 & 0 \dots & 1 & 0 \dots & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ l_{k+1,k} \\ \vdots \\ l_{m,k} \end{bmatrix} \begin{cases} k \text{ position} \\ m-k \text{ position} \end{cases} = 0$$

check:

$$(I - \cancel{l_k e_k^T}) \underbrace{(I + \cancel{l_k e_k^T})^{-1}}_{L_k} = I + \cancel{l_k e_k^T} - \cancel{l_k e_k^T} - \cancel{l_k e_k^T l_k e_k^T} = I$$

Given: $A \in \mathbb{R}^{m \times m}$, $b \in \mathbb{R}^m$
 A is invertible (A has full col. rank /

(previous)

\downarrow
 $\exists L, U$ s.t.

$$A = LU$$

$$Ax = b$$

$$LUx = b$$

"y"

cols of A are lin.
indep.)

First solve:

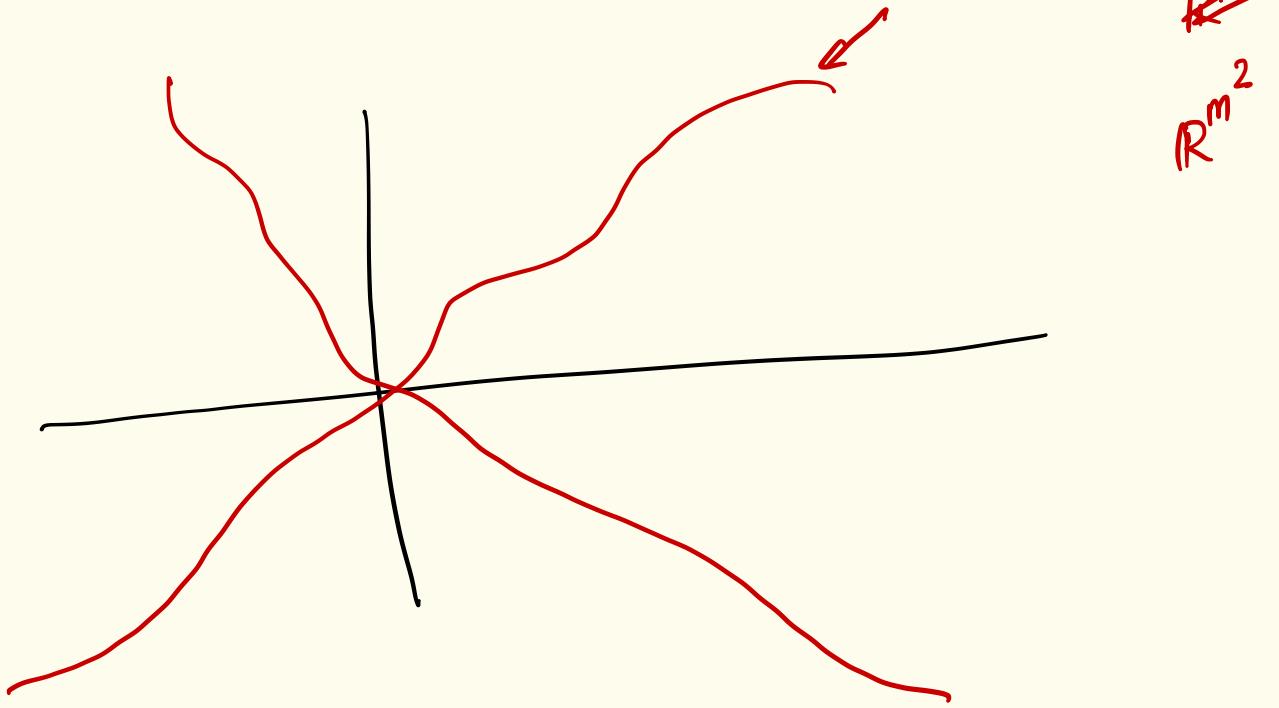
$$Ly = b$$

\downarrow by forward substitution

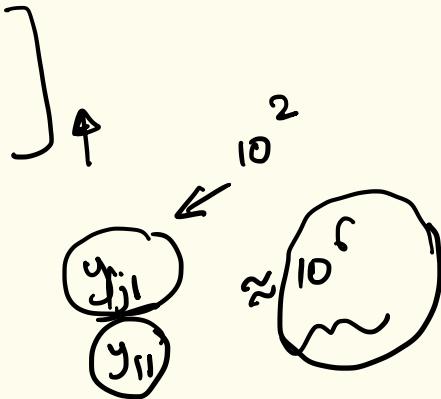
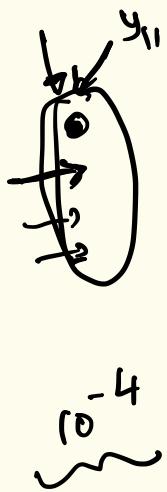
$$Ux = y$$

by backward substitution

Observation: Gaussian \equiv LU elimination



\mathbb{R}^{m^2}



LU

with
partial
pivoting

0

Numerical
linear
algebra
Trefethen
bau

Backward stability of computations

A. Wilkinson

D. Watkins : Matrix Computations

Computational
Linear
Algebra
Spring