Linear algebra for AI & ML

Linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if $\forall\, x, y \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

__Ex:__ '0' map

__Ex:__ $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$ if $m \le n$

__Ex:__ $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \stackrel{T}{\longmapsto} \begin{pmatrix} |x_1| \\ \vdots \\ |x_n| \end{pmatrix}$

$T : \mathbb{R}^n \to \mathbb{R}^n$

__Ex:__ zero-padding.

__Ex:__ $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \stackrel{T}{\longmapsto} \begin{pmatrix} x_1 \\ \frac{x_1 + x_2}{2} \\ x_2 \\ \frac{x_2 + x_3}{2} \\ \vdots \\ x_n \end{pmatrix}$

$T : \mathbb{R}^n \to \mathbb{R}^{2n-1}$

Let $e_1, e_2, \dots, e_n \in \mathbb{R}^n$ be the std. basis vectors.

$T: \mathbb{R}^n \to \mathbb{R}^m$ linear.

$T(e_1), T(e_2), \dots, T(e_n) \in \mathbb{R}^m$

Take any $x \in \mathbb{R}^n$, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$T(x) = T(x_1 e_1 + x_2 e_2 + \dots + x_n e_n)$$

$$= x_1 T(e_1) + x_2 T(e_2) + \dots + x_n T(e_n)$$

$$= \begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$m \times n$ matrix

the matrix of linear transformation.

Matrix vector multiplication:

$a_i^T = [a_{i1} \; a_{i2} \cdots a_{in}]$

$\uparrow$ i^{th} row of A

$A \in \mathbb{R}^{m \times n}, \quad x \in \mathbb{R}^n$

$$Ax = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} a_1^T x \\ a_2^T x \\ \vdots \\ a_m^T x \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} x_2 + \cdots + \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} x_n$$

Range of T

$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad \leftarrow$ linear

$R(T)$ = range of $T = \{ T(x) \mid x \in \mathbb{R}^n \} \subseteq \mathbb{R}^m$

Q: Is $R(T)$ a subspace??

$T(x) = \{ x_1 T(e_1) + x_2 T(e_2) + \cdots + x_n T(e_n) \mid x_1, \ldots, x_n \in \mathbb{R} \}$

$= \text{span}\{ T(e_1), T(e_2), \ldots, T(e_n) \}$

Ex: $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$T(x) = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$