Linear algebra for AI & ML

September - 15

- Fundamentals of Matrix Computations
  D. Watkins

- Matrix computations
  Golub, Van Loan

$Ax = b$; $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$

find x s.t. $Ax = b$

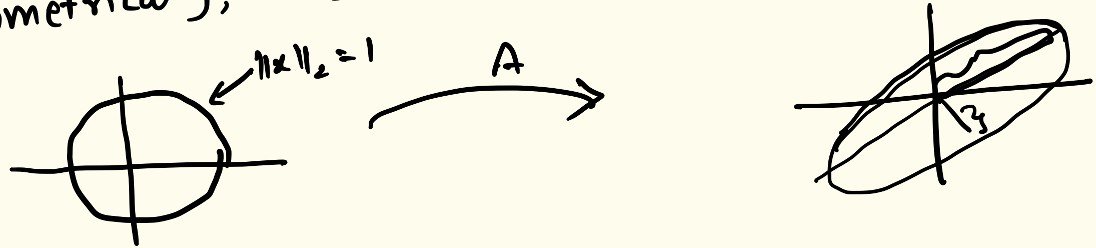
- Sensitivity analysis:

$A(x+\delta x) = b + \delta b$

Allowing perturbations in b only.

$$\boxed{\frac{\|\delta x\|_2}{\|x\|_2} \le \underbrace{\|A\|_2 \|A^{-1}\|_2}_{\kappa_2(A)} \frac{\|\delta b\|_2}{\|b\|_2}}$$

Geometrically, $\kappa_2(A)$.



$$\text{maxmag}(A) = \max_{\|x\|_2 = 1} \|Ax\|_2$$

$$\text{minmag}(A) = \min_{\|x\|_2 = 1} \|Ax\|_2$$

orthogonal matrices : $\text{maxmag}(Q) = 1$; $\text{minmag}(Q) = 1$

unit circle $\longrightarrow$ unit circle

$$k_2(A) = \frac{\text{maxmag}(A)}{\text{minmag}(A)}$$ ← want to prove !!

$$k_2(A) = \|A\|_2 \|A^{-1}\|_2$$ ← we know!!

Lemma: A is non-singular matrix.

$$\text{maxmag}(A) = \frac{1}{\text{minmag}(A^{-1})} \quad \text{and} \quad \boxed{\text{maxmag}(A^{-1}) = \frac{1}{\text{minmag}(A)}}$$

Pf: A is nonsingular (invertible)

$\Rightarrow \quad Ax = 0 \Leftrightarrow x = 0$

Also, A is invertible $\Rightarrow$ A is one-one.

$Ax = y \quad \Rightarrow \quad x = A^{-1}y$

$$\text{maxmag}(A) = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

$$= \max_{y \neq 0} \frac{\|y\|_2}{\|A^{-1}y\|_2} = \frac{1}{\min\limits_{y \neq 0} \frac{\|A^{-1}y\|_2}{\|y\|_2}}$$

$$= \frac{1}{\text{minmag}(A^{-1})}$$

$$k_2(A) = \|A\|_2 \, \|A^{-1}\|_2$$

$$= \text{maxmag}(A) \cdot \text{maxmag}(A^{-1})$$

$$\boxed{k_2(A) = \frac{\text{maxmag}(A)}{\text{minmag}(A)}}$$

Q: $\boxed{Ax = b}$ $\qquad$ $A(x + \delta x) = b + \delta b$

If $\text{cond}(A)$ is very large. (ill-conditioned matrix)

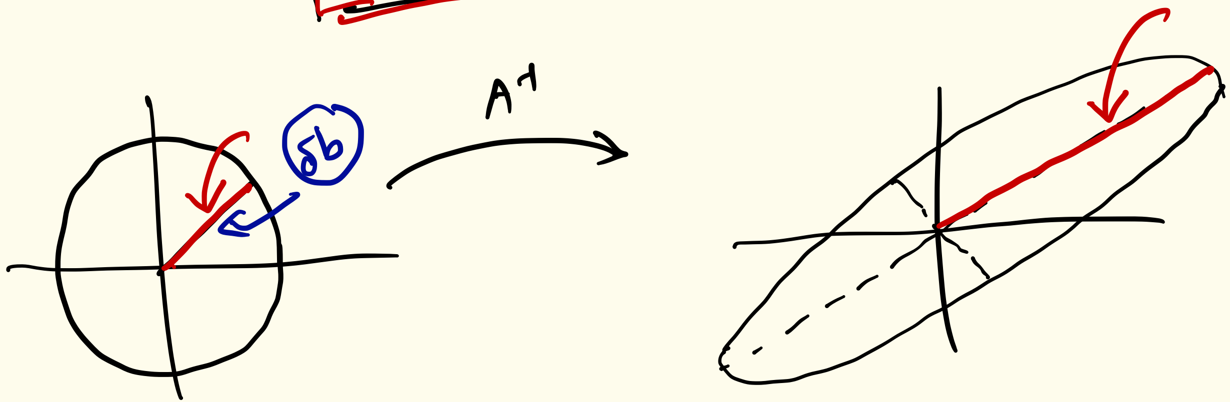
$\dfrac{\|\delta b\|_2}{\|b\|_2}$ is small.

$\dfrac{\|\delta x\|_2}{\|x\|_2}$ may be small.
may be large.

$$\cancel{Ax} + A\delta x = \cancel{b} + \delta b$$

$$A\delta x = \delta b$$

$$\Rightarrow \boxed{\delta x = A^{-1}\delta b}$$



$$Ax = b$$

$$X\beta = Y$$

$$\beta = \boxed{(X^T X)^{-1}}\, X^T Y$$

Another interpretation of ill-conditioned matrices:

$A \in \mathbb{R}^{n \times n}$

Let A be NOT invertible.

$\Rightarrow$ columns of A are linearly dependent.

$\Rightarrow$ $\exists\ x \in \mathbb{R}^n$; $x \neq 0$ such that

$$Ax = 0$$

Without loss of generality, assume $\|x\|_2 = 1$

$\Rightarrow$ $\text{minmag}(A) = 0$

$$\text{minmag}(A) = \min_{\|x\|_2 = 1} \|Ax\|_2$$

Let A be matrix which is ill-conditioned.

$$k_2(A) >> 1$$

$$\| $$

$$\frac{\text{maxmag}(A)}{\text{minmag}(A)} >> 1$$

WLOG assume that $\text{maxmag}(A) = \|A\|_2 = 1$

$$k_2(A) >> 1$$

$$\Rightarrow \frac{1}{\text{minmag}(A)} >> 1$$

$$\frac{1}{0.001}$$

$$\Rightarrow \text{minmag}(A) << 1$$

there exists a vector $x \in \mathbb{R}^n$; $\|x\|_2 = 1$

s.t.

$\|Ax\|_2$ is "very very small".

$\left\| \frac{A}{\|A\|_2} \right\|_2 = 1$

$\|Ax\|_2$ is "almost zero".

$\Rightarrow$ Ax is "almost zero".

$\Rightarrow$ columns of A are "almost linearly dependent".

$\Rightarrow$ matrix A is "almost non-invertible"
"almost singular".

$$\|Qx\|_2 = \|x\|_2 \qquad \forall\, x \in \mathbb{R}^n$$

$$\text{maxmag}(Q) = \max_{x \neq 0} \frac{\|Qx\|_2}{\|x\|_2} = 1$$

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \qquad \alpha > 0,\ \beta > 0$$

$$\alpha, \beta \in (0, 1)$$