Linear algebra for AI and ML

September 1

(Lecture # 7)

$Ax = b$ $\quad A \in \mathbb{R}^{m\times n}$, $\quad b \in \mathbb{R}^n$

Q. $\exists\ x \in \mathbb{R}^n$ s.t. $Ax = b$

Assumption $m = n$ ✓

1) Existence & uniqueness ✗

2) How to compute x?? — LU / Gaussian elimination
— QR decomposition. Assumption $m = n$.

3) sensitivity analysis.

Orthogonal matrices:

Recall: Inner product / dot product of vector.

$x, y \in \mathbb{R}^n$; $\quad x^T y = \langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$

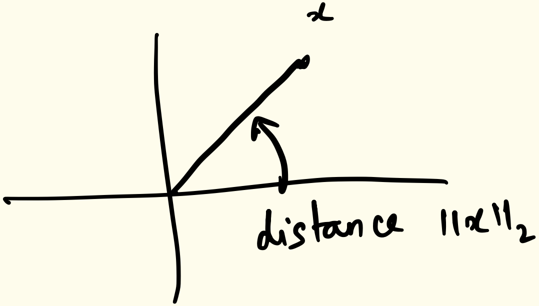
where $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ & $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

$\langle x, y\rangle = x^T y = y^T x$; (Bilinear fn x, y)

Norm: $\|x\|_2 = \sqrt{\langle x, x \rangle}$

$$= \sqrt{x^T x}$$

$$= \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} = \sqrt[2]{\sum_{i=1}^{n} x_i^2}$$



$$\|x - y\|_2 = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

$\downarrow$

usual distance formula, we studied in coordinate geometry.

$$\cos\theta = \frac{x^T y}{\|x\|_2 \|y\|_2}$$

Here θ is angle between the vectors x & y.

x & y orthogonal $\iff x^T y = 0$

Orthogonal matrix:

A matrix $Q \in \mathbb{R}^{n\times n}$ is called as orthogonal if $QQ^T = I$

$\Rightarrow$ Q^T is the inverse Q.

$$Q = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix} \in \mathbb{R}^{n\times n}$$

$QQ^T = I = Q^TQ$ $\Rightarrow$ columns of Q $\{q_1, \dots, q_n\}$ is an orthonormal set.

$\langle q_i, q_j \rangle = \delta_{ij}$

$i, j = 1, 2, \dots, n$

Properties of orthogonal matrices:

If $Q \in \mathbb{R}^{n \times n}$ is an orthogonal matrix, then for any $x, y \in \mathbb{R}^n$

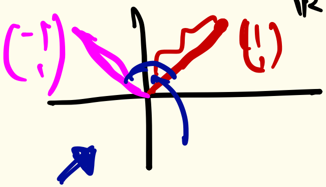
a) $\langle Qx, Qy \rangle = \langle x, y \rangle$

b) $\|Qx\|_2 = \|x\|_2$

pf: $\langle Qx, Qy \rangle = (Qx)^T Qy = x^T (Q^T Q) y = x^T y = \langle x, y \rangle$ ∎

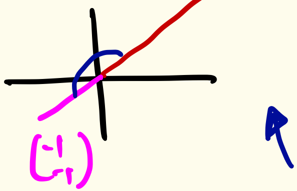
Think of matrix as a linear transformation $\begin{pmatrix}19\\17\end{pmatrix}$ from $\mathbb{R}^n \to \mathbb{R}^n$ $(n=2)$

$\mathbb{R}^2$ $\qquad\qquad$ $\mathbb{R}^2$

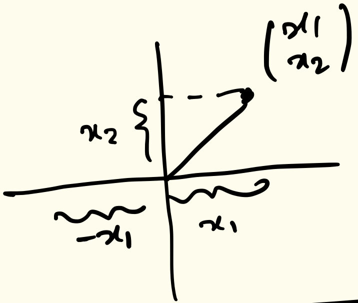


$\xrightarrow{A}$

$A = \begin{bmatrix} 10 & 9 \\ 9 & 8 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} =$



$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad Ax = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}$$



$\boxed{\text{Multiplication of a vector by an orthogonal matrix is backward stable.}}$

Suppose if we can write for $A \in \mathbb{R}^{n \times n}$

$$A = QR$$

where $Q \in \mathbb{R}^{n \times n}$ orthogonal

& $R \in \mathbb{R}^{n \times n}$ an upper triangular matrix.

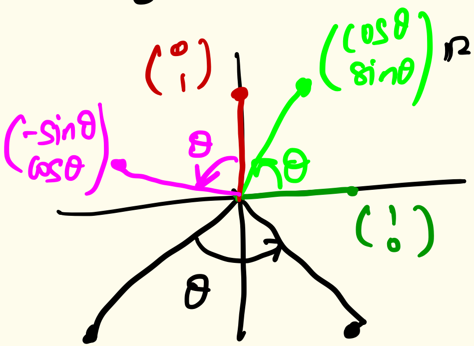
$$Ax = b \Rightarrow QRx = b \Rightarrow \underbrace{Rx}_{\text{backward substitution.}} = Q^T b$$

(*) Rotators : $(n=2)$

$$Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$$

Q is such that it rotates every vector in $\mathbb{R}^2$ in an anticlockwise direction by an angle θ.

It is just enough to study the action of Q on $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$.



$$Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}q_{11}\\q_{21}\end{bmatrix}$$

$$Q\begin{pmatrix}1\\0\end{pmatrix} = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}, \quad Q\begin{pmatrix}0\\1\end{pmatrix} = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix} \Rightarrow Q = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$$

clearly Q is orthogonal.

<u>Fact:</u> Rotators can be used to create zeros in a vector.

Let $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x$ s.t. $\boxed{x_2 \neq 0.}$

$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ be such that

$Q^T x = \begin{bmatrix} y \\ 0 \end{bmatrix}$ for $y \neq 0$

$$Q^T x = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \boxed{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \begin{bmatrix} (\cos\theta) x_1 + (\sin\theta) x_2 \\ \boxed{(-\sin\theta) x_1 + (\cos\theta) x_2} \end{bmatrix} = \boxed{\begin{bmatrix} y \\ \boxed{0} \end{bmatrix}}$$

$\sqrt{x_1^2 + x_2^2}$

$$\Leftrightarrow \quad x_1 \sin\theta = x_2 \cos\theta \quad \leftarrow (*)$$

$$\sin\theta = \frac{x_2}{\sqrt{x_1^2 + x_2^2}} \; ; \; \cos\theta = \frac{x_1}{\sqrt{x_1^2 + x_2^2}}$$

$$\boxed{\sin\theta = x_2 \; ; \; \cos\theta = x_1}$$

Given a matrix $A \in \mathbb{R}^{2\times 2}$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$\exists$ a rotator Q^T s.t.

$$Q^T A = Q^T \begin{bmatrix} \boxed{\begin{matrix} a_{11} \\ a_{21} \end{matrix}} & \begin{matrix} a_{12} \\ a_{22} \end{matrix} \end{bmatrix} = \underbrace{\begin{bmatrix} \boxed{\begin{matrix} * \\ 0 \end{matrix}} & \begin{matrix} * \\ * \end{matrix} \end{bmatrix}}_{R}$$

$$Q^T A = R$$

$$\Rightarrow \boxed{A = QR}$$

$n=3$

Rotator in $\mathbb{R}^3$:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longrightarrow \begin{bmatrix} y \\ 0 \\ 0 \end{bmatrix}$$

Hint: 2 - rotators.

$i=j=3$

$$\underbrace{\begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{Q_1^T} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y^* \\ 0 \\ x_3 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & & \cos\theta_2 \end{bmatrix}}_{Q_2^T} \begin{bmatrix} y^* \\ \boxed{0} \\ x_3 \end{bmatrix} = \begin{bmatrix} y \\ \boxed{0} \\ 0 \end{bmatrix}$$

$$Q_2^T Q_1^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y \\ 0 \\ 0 \end{bmatrix}$$