# Notes on *-autonomous categories

Anna Jenčová

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## 1 Definitions of a *-autonomous category

(nLab) Let $(\mathcal{C}, \otimes, I)$ be a symmetric monoidal category.

### 1.1 Definition 1

(i) $(\mathcal{C}, \otimes, I)$ is closed; (the functor $- \otimes B$ has a right adjoint $[B, -]$. In fact, this defines a functor $[-,-] : C^{op} \times C \to C$ (internal hom) and the isomorphism

$$\mathcal{C}(A \otimes B, C) \simeq \mathcal{C}(A, [B, C])$$

is natural in all 3 variables $A, B, C$. This follows by Yoneda (nLab). The corresponding map $\mathcal{C}(A \otimes B, C) \ni f \mapsto \hat{f} \in \mathcal{C}(A, [B, C])$ is the transpose of $f$.)

(ii) there is a dualizing object $0 \in \mathcal{C}$, such that for every $A$ the transpose of $ev_{A,0}$ is an iso:

$$\mathcal{C}([A, 0] \otimes A, 0) \simeq \mathcal{C}(A, [[A, 0], 0]), \qquad ev_{A,0} \mapsto \hat{ev}_{A,0} \equiv j_A$$

Here $ev_{A,B} : [A, B] \otimes A \to B$ is the counit of the adjunction in 2. Note that we used symmetry.

### 1.2 Definition 2

(1) A symmetric monoidal category $(\mathcal{C}, \otimes, I)$;

(2) $(-)^* : \mathcal{C}^{op} \to \mathcal{C}$ full and faithful functor such that

$$\mathcal{C}(A \otimes B, C) \simeq \mathcal{C}(A, (B \otimes C^*)^*) \qquad \text{(natural in } A, B, C),$$

that is, $[B, C] \simeq (B \otimes C^*)^*$.

### 1.3 Equivalence

$1 \to 2$: Define the contravariant endofunctor: $A^* = [A, 0]$, then $j_A$ in (ii) is a natural isomorphism $id_{\mathcal{C}} \to (-)^{**}(?)$, so $(-)^*$ is full and faithful, all else is clear.

$2 \to 1$: Define internal hom as $[A, B] := (A \otimes B^*)^*$, then (i) holds. Define the dualizing object as $0 = I^*$

## 1.4 Compact closed categories

An object $A \in \mathcal{C}$ is dualizable if there is some $A^* \in \mathcal{C}$ (dual object) and morphisms $\cup_A : I \to A^* \otimes A$ and $\cap_A : A \otimes A^* \to I$ such that the snake identities hold:

$$(\cap_A \otimes A) \circ (A \otimes \cup_A) = A, \qquad (A^* \otimes \cap_A) \circ (\cup_A \otimes A^*) = A^*.$$

$\mathcal{C}$ is compact closed if every object is dualizable. Equivalently, $\mathcal{C}$ is *-autonomous and there is an iso

$$(A \otimes B)^* \simeq A^* \otimes B^*, \qquad \text{natural in } A, B.$$

**Lemma 1.** *Let $\mathcal{C}$ be *-autonomous and $I = I^*$. Let $A$ be an object in $\mathcal{C}$. Then $A$ is dualizable iff there is an iso*

$$(A \otimes B)^* \simeq A^* \otimes B^*, \qquad \textit{natural in } A, B.$$

*Proof.* Assume $A$ is such that there is the iso. Then using the *-autonomous structure, we have

$$\mathcal{C}(A, A) \simeq \mathcal{C}(I \otimes A, A) \simeq \mathcal{C}(I, (A \otimes A^*)^*) \simeq \mathcal{C}(I, A^* \otimes A)$$

so we may put $\cup_A \in \mathcal{C}(I, A^* \otimes A^*)$ as the morphism corresponding to the identity $A$. Similarly, we use

$$\mathcal{C}(A, A) \simeq \mathcal{C}(A, A \otimes I) \simeq \mathcal{C}(A, (A^* \otimes I)^*) \simeq \mathcal{C}(A \otimes A^*, I)$$

to define $\cap_A$. In fact, take the subcategory consisting of $I, A, A^*$ and all their tensor products as objects and all morphisms, then we should get a compact closed category, so that indeed $A$ is dualizable. Conversely, assume that $A$ is dualizable

$\square$

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# 2 Symmetric monoidal categories (SMC)

**Monoidal category:** A category $C$ equipped with

- A functor $\otimes : C \times C \to C$;
- unit object $I \in C$;
- associator: natural iso $(a \otimes b) \otimes c \xrightarrow{\alpha_{a,b,c}} a \otimes (b \otimes c)$;
- left unitor: natural iso $I \otimes a \xrightarrow{\lambda_a} a$;
- right unitor: natural iso $a \otimes I \xrightarrow{\rho_a} a$
- **symmetric** if there is a symmetry: natural iso $a \otimes b \xrightarrow{\sigma_{a,b}} b \otimes a$ such that $\sigma_{b,a} = \sigma_{a,b}^{-1}$,

satisfying triangle, pentagon (+ symmetry) diagrams.

We will always assume that $C$ is a SMC.

## 2.1 Closed SMC

A SMC $C$ is **closed** if:

for every $b \in C$, the endofunctor $- \otimes b$ has a right adjoint $[b, -]$ (internal hom).

What does this mean?

(1) For all $a, c \in C$, $C(a \otimes b, c) \simeq C(a, [b, c])$, naturally in $a, c$.

(2) unit $\eta^b_a : a \to [b, a \otimes b]$, counit: $\epsilon^b_a : [b, a] \otimes b \to a$, natural transformations, triangle identities

‘ Relation of the two:

- Let $i$ be the iso of (1):

$$\eta^b_a \in C(a, [b, a \otimes b]) \simeq C(a \otimes b, a \otimes b), \qquad \eta^b_a = i(id_{a \otimes b})$$
$$\epsilon^b_a \in C([b, a] \otimes b, a) \simeq C([b, a], [b, a]), \qquad \epsilon^b_a = i^{-1}(id_{[b,a]}).$$

- Conversely, from $\eta^b$, $\epsilon^b$ of (2), we define $i$ as

$$g \in C(a \otimes b, c), \qquad i(g) = [b, g] \circ \eta^b_a$$

  with inverse

$$h \in C(a, [b, c]), \quad i^{-1}(h) = \epsilon^b_c \circ (h \otimes b)$$

Equivalently: a SMC $C$ is closed if and only if for all $b, c \in C$, there is an object $[b, c]$ and an **evaluation map** $eval_{b,c} : [b, c] \otimes b \to c$ that has the following **universal property**: for all $a \in C$ and $f : a \otimes b \to c$ there is a unique $h : a \to [b, c]$ such that

$$f = eval_{b,c} \circ (h \otimes b).$$

The evaluation map is the counit $eval_{b,c} = \epsilon^b_c$ above.

Internal hom is a functor $[-, -] : C^{op} \times C \to C$ and the isomorphism in (1) is natural in all 3 variables $a, b, c$. This follows by Yoneda (nLab).

## 2.2 Compact SMC

A SMC is **compact** if each object $a \in C$ has a dual $a^* \in C$ such that there are maps $\cup_a : I \to a^* \otimes a$ and $\cap_a : a \otimes a^* \to I$ satisfying the snake identities

$$(\cap_a \otimes id_a) \circ (id_a \otimes \cup_a) = id_a, \qquad (id_{a^*} \otimes \cap_a) \circ (\cup_a \otimes id_{a^*}) = id_{a^*}.$$

The following are easily seen by the snake identities (pictures):

(1) $a^*$ is determined up to iso;

(1) $I^* \simeq I$, by the isomorphisms

$$\rho_{I^*} \circ \cup_I : I \to I^*, \qquad \cap_I \circ \lambda^{-1}_{I^*} : I^* \to I;$$

(2) $a^{**} \simeq a$, indeed, we may define $\cup_{a^*} : I \to a \otimes a^*$ and $\cap_{a^*} : a^* \otimes a$ as

$$\cup_{a^*} = \sigma_{a^*,a} \circ \cup_a, \qquad \cap_{a^*} = \cap_a \circ \sigma_{a^*,a},$$

so that $a$ is dual to $a^*$, and use (1);

(3) if we fix $a^*$ and $\cup_a$ ($\cap_a$), then $\cap_a$ ($\cup_a$) is uniquely determined;

(4) any assignment $a \mapsto a^*$ defines a functor $C \to C^{op}$ (if $f : a \to b$, we can use $\cup_a$ and $\cap_b$ to "bend the wires" to obtain a map $b^* \to a^*$, this is obviously functorial);

(5) $(a \otimes b)^* \simeq a^* \otimes b^*$, we can clearly put (using symmetry)

$$\cup_{a\otimes b} = \cup_a \otimes \cup_b, \qquad \cap_{a\otimes b} = \cap_a \otimes \cap_b$$

(5) $C$ is closed, with $[b, c] = b^* \otimes c$: the iso $i : C(a \otimes b, c) \simeq C(a, b^* \otimes c)$ can be obtain (roughly) as

$$i(g) = g \circ \cup_b, \qquad i^{-1}(h) = \cap_b \circ h,$$

the maps are inverses by the snake identities, naturality is obvious, since $i$ does nothing on $a$ or $c$. The unit and counit of the adjunction are given as

$$\eta^b_a = a \otimes \cup_b : a \to b^* \otimes a \otimes b, \qquad \epsilon^b_a = \cap_b \otimes a : b^* \otimes a \otimes b \to a.$$

(6) Can we state a theorem like: $C$ is compact if and only if for each $b \in C$ there is some $b^* \in C$ such that $b^* \otimes -$ is the right adjoint of $- \otimes b$ and ...? What should be the additional conditions?

# 3 Kleisli categories and monoidal monads

A **monad** on $C$ is a triple $(P, \eta, \mu)$, where:

- $P : C \to C$ is an endofunctor;
- $\eta : Id_C \to P$, $\mu : P^2 \to P$ are natural transformations satisfying some triangles and squares.

## 3.1 Kleisli categories

The **Kleisli category** $C_P$ has the same objects as $C$, with morphisms:

$$C_p(a, b) = C(a, P(b)),$$

the identity $id_a = \eta_a$ and for $f \in C_p(a, b)$, $g \in C_p(b, c)$, the composition is defined as

$$g \circ f := \mu_c \circ P(g) \circ f.$$

We have the following adjunction:

- the **left adjoint functor** $F_P : C \to C_P$ is defined as $a \mapsto a$ and for $f : a \to b$, we put $F_P(f) \in C_P(a, b) = C(a, P(b))$ as $\eta_b \circ f$;
- the **right adjoint functor** $G_P : C_P \to C$ is given as $a \mapsto P(a)$ and for $f \in C_P(a, b) = C(a, P(b))$ we put $G_P(f) \in C(P(a), P(b))$ as $G_P(f) = \mu_b \circ P(f)$.

This is indeed an adjunction, where the unit is given by $\eta$ and the counit is determined as $\epsilon_a = id_{P(a)} \in C_P(P(a), a)$.

## 3.2 Monoidal monads

A monad is **monoidal** [**?** ] if there are maps

$$\kappa_{a,b} : Pa \otimes Pb \to P(a \otimes b), \qquad a, b \in C,$$

natural in $a, b$ and such that

- $(P, \eta, \kappa)$ is a **monoidal functor**, that is, some diagrams involving $P$, $\alpha$, $\lambda$, $\rho$, $\kappa$ and $\eta$ commute;
- additional diagrams containing $\mu$ commutes;
- **symmetric**: additionally a diagram with $\sigma$ commutes.

A monoidal functor is **strict** if $\kappa$ is iso.

**Proposition 1.** *[**?** , Prop. 1.2.2] There is a bijective correspondence between:*

*(i) families of morphisms $\{\kappa_{a,b}\}$ such that $(P, \eta, \mu, \kappa)$ is a (symmetric) monoidal monad;*

*(ii) (symmetric) monoidal structures on $C_P$ such that the left adjoint functor $F_P : C \to C_P$ is strict monoidal.*

If $(P, \eta, \mu, \kappa)$ is a symmetric monoidal monad, we define the monoidal structure on $C_P$ as follows. The functor

$$\otimes_P : C_P \times C_P \to C_P$$

is given as as $a \otimes_P b = a \otimes b$ on objects, and for $f \in C_P(a, c) = C(a, P(c))$ and $g \in C_P(b, d) = C(b, p(d))$, we define $f \otimes_P g \in C_P(a \otimes_P b, c \otimes_P d) = C(a \otimes b, P(c \otimes d))$ as

$$f \otimes_P g := \kappa_{b,d} \circ (f \otimes g).$$

The associator and unitors and symmetry in $C_P$ can be defined from those in $C$ by composition with $\eta$.

# 4 Compact Kleisli categories

Assume that $C$ is a symmetric monoidal closed category. Assume that $(P, \eta, \mu, \kappa)$ is a monoidal monad, such that the category $C_p$ with corresponding monoidal structure is compact. We will study some consequences of this.

## 4.1 First consequences

It follows that each object $a \in C_P$ has a dual object $a^*$, such that there is an isomorphism

$$i : C_P(a \otimes_P b^*, c) \simeq C_P(a, b \otimes_P c),$$

which is natural (in $C_P$) in $a$ and $c$. This means that for any arrows $a \xrightarrow{f} a'$ and $a' \otimes b^* \xrightarrow{h'} c$ in $C_P$, we have

$$i(h' \circ_P (f \otimes_P id^P_{b^*})) = i(h') \circ_P f,$$

and similarly, gor $c \xrightarrow{g} c'$ and $a \otimes b^* \xrightarrow{h} c$ in $C_P$, we get

$$i(g \circ_P h) = (id_b^P \otimes_P g) \circ_P i(h).$$

By definition of the Kleisli category, $i$ is an isomorphism (that is, a bijection of sets)

$$i : C(a \otimes b^*, P(c)) \simeq C(a, P(b \otimes c)).$$

We would like to show that $i$ is natural in $a$ and $c$ also in the category $C$.

So let $f \in C(a, a')$ and let $\tilde{f} := \eta_{a'} \circ f \in C(a, P(a')) = C_P(a, a')$. Let $h' \in C(a' \otimes b^*, P(c))$, then

$$\begin{aligned} h' \circ_P (\tilde{f} \otimes_P b^*) &= \mu_c \circ P(h') \circ (\tilde{f} \otimes_P \eta_{b^*}) = \mu_c \circ P(h') \circ \kappa_{a',b^*} \circ (\eta_{a'} \otimes \eta_{b^*}) \circ (f \otimes b^*) \\ &= \mu_c \circ P(h') \circ \eta_{a' \otimes b^*} \circ (f \otimes b^*) = \mu_c \circ \eta_{P(c)} \circ h' \circ (f \otimes b^*) \\ &= h' \circ (f \otimes b^*), \end{aligned}$$

where we used that $\kappa_{a',b^*} \circ (\eta_{a'} \otimes \eta_{b^*}) = \eta_{a' \otimes b^*}$, naturality of $\eta$ and the triangle identity. Similarly, we get for any $\bar{h}' \in C(a, P(b \otimes c))$,

$$\bar{h}' \circ_P f = \mu_{b \otimes c} \circ P(\bar{h}') \circ \eta_{a'} \circ f = \bar{h}' \circ f,$$

in particular, putting these together, this implies

$$i(h' \circ (f \otimes b^*)) = i(h' \circ_P (\tilde{f} \otimes_P id_{b^*})) = i(h') \circ_P \tilde{f} = i(h') \circ f.$$

Naturality in $c$ is proved similarly.

It follows that there is an isomorphism

$$C(a, P(b \otimes c)) \simeq C(a \otimes b^*, P(c)) \simeq C(a, [b^*, P(c)]),$$

natural in $a$ and $c$. By Yoneda, we get the isomorphism

$$P(b \otimes c) \simeq [b^*, P(c)],$$

natural in $c$. Putting $c = I$, we obtain

$$P(b) \simeq P(b \otimes I) \simeq [b^*, P(I)].$$

## 4.2 A construction of a monoidal monad

Fix an object $p \in C$ and assume that

- there is a bijective map $a \mapsto a^*$ on objects;
- for each $a \in C$, there is a morphism $\theta_a \in C(a, [a^*, p])$;
- for each morphism $f \in C(a, [b^*, p])$ there is some $\hat{f} \in C([a^*, p], [b^*, p])$

such that

(i) $\hat{\theta}_a = id_{[a^*,p]}$;

(ii) for $f \in C(a,[b^*,p])$, $\hat{f} \circ \theta_a = f$;

(iii) for $f \in C(a,[b^*,p])$ and $g \in C(b,[c^*,p])$,

$$(\hat{g} \circ f)^\wedge = \hat{g} \circ \hat{f}. \tag{1}$$

From this data, we may define a monad $(P_p, \theta, \nu)$. Here the functor $P_p$ acts as $a \mapsto [a^*,p]$ on objects and for $f \in C(a,b)$, we define $P_p(f) \in C([a^*,p],[b^*,p])$ by

$$P_p(f) := (\theta_b \circ f)^\wedge.$$

Moreover, $\nu$ is defined as $\nu_a = \hat{id}_{[a^*,p]}$. The fact that this is a monad follows easily from the properties (i)-(iii).

To make it monoidal, we add family of maps

$$\kappa_{a,b} : [a^*,p] \otimes [b^*,p] \to [(a \otimes b)^*,p],$$

such that

(iv) for all $a,b \in C$,

$$\theta_{a\otimes b} = \kappa \circ (\theta_a \otimes \theta_b);$$

(v) for $f \in C(a,[b^*,p])$ and $g \in C(c,[d^*,p])$,

$$\kappa_{b,d} \circ (\hat{f} \otimes \hat{g}) = (\kappa_{b,d} \circ (f \otimes g))^\wedge \circ \kappa_{a,c}.$$

Then one can check that $\kappa_{a,b}$ are natural in $a,b$ and that

$$\nu_{a\otimes b} \circ P_p(\kappa_{a,b}) \circ \kappa_{[a^*,p],[b^*,p]} = \kappa_{a,b} \circ (\nu_a \otimes \nu_b).$$

We also need some properties with respect to $\alpha, \lambda, \rho$ and $\sigma$:

(vi) for all $a,b,c$,

$$\kappa_{a,b\otimes c} \circ (id_{[a^*,p]} \otimes \kappa_{b,c}) \circ \alpha_{[a^*,p],[b^*,p],[c^*,p]} = P_p(\alpha_{a,b,c}) \circ \kappa_{a\otimes b,c} \circ (\kappa_{a,b} \otimes id_{[c^*,p]})$$

(vii) for all $a$,

$$(\theta_a \circ \lambda_a)^\wedge \circ \kappa_{I,a} \circ (\theta_I \otimes id_{[a^*,p]}) = \lambda_{[a^*,p]}$$
$$(\theta_a \circ \rho_a)^\wedge \circ \kappa_{I,a} \circ (id_{[a^*,p]} \otimes \theta_I) = \rho_{[a^*,p]};$$

(viii) for all $a,b$,

$$(\theta_{b\otimes a} \circ \sigma_{a,b})^\wedge \circ \kappa_{a,b} = \kappa_{b,a} \circ \sigma_{[a^*,p],[b^*,p]}.$$

Then $(P_p, \theta, \nu, \kappa)$ is a monoidal monad, [**?** ].

## 4.3 The Kleisli category $C_p$

The Kleisli category $C_p := C_{P_p}$ has the same objects as $C$, with morphisms $C_p(a,b) = C(a,[b^*,p])$, the identity is $id^p_a = \theta_a$ and for $f \in C_p(a,b)$, $g \in C_p(b,c)$, the composition is fiven as

$$g \circ_p f = \hat{g} \circ f.$$

*Remark* 1. Let $j$ be the natural iso (in $C$):

$$j : C(a \otimes b, c) \simeq C(a,[b,c])$$

Note that $C_p(a,b)$ can be identified with $C(a \otimes b^*, p)$, with composition given by

$$j^{-1}(j(\psi)^\wedge \circ j(\varphi)), \qquad \varphi \in C(a \otimes b^*, p),\ \psi \in C(b \otimes c^*, p).$$

We equip $C_p$ with the tensor product $\otimes_p$ defined by $a \otimes_p b = a \otimes b$ on objects and $f \otimes_p g = \kappa \circ (f \otimes g)$ on morphisms. Then $(C_p, \otimes_p, I)$ is a symmetric monoidal category, with the natural isomorphisms $\alpha, \lambda, \rho, \sigma$ extended by $\theta$, that is, $\alpha^p := \theta \circ \alpha$, $\lambda^p := \theta \circ \lambda$, $\rho^p := \theta \circ \rho$, $\sigma^p := \theta \circ \sigma$.

## 4.4 When is $C_p$ closed?

We need to define the internal hom $b \overset{p}{\multimap} c$, such that $b \overset{p}{\multimap} -$ is the right adjoint of $b \otimes_p -$ in $C_p$. In fact, it is enough to specify $b \overset{p}{\multimap} c$ on objects and to find an iso

$$C_p(a \otimes_p b, c) \simeq C_p(a, b \overset{p}{\multimap} c)$$

natural in $a$. As for the isomorphism, we must have

$$C(a \otimes b, [c^*, p]) \simeq C(a, [(b \overset{p}{\multimap} c)^*, p])$$

Since $C$ is SMC, we have

$$C(a \otimes b, [c^*, p]) \simeq C((a \otimes b) \otimes c^*, p) \simeq C(a \otimes (b \otimes c^*), p) \simeq C(a, [b \otimes c^*, p])$$

and the isomorphisms are natural (in $C$) in all variables. This suggests to define $b \overset{p}{\multimap} c$ as the object such that $(b \overset{p}{\multimap} c)^* = b \otimes c^*$. Since $(-)^*$ is bijective, such an object exists and is unique.

As for naturality of the isomorphism, let us denote by $i$ the resulting isomorphism

$$i : C(a \otimes b, [c^*, p]) \simeq C(a, [b \otimes c^*, p])$$

Let $f \in C_p(a', a) = C(a', [a^*, p])$. Then naturality means that we require

$$i(h \circ_p (f \otimes_p id^p_b)) = i(h) \circ_p f = i(h)^\wedge \circ f.$$

on the left hand side we obtain

$$h \circ_p (f \otimes_p id^p_b) = \hat{h} \circ \kappa_{a,b} \circ (f \otimes \theta_b) = \hat{h} \circ s_{a,b} \circ (f \otimes b),$$

where $\hat{h} \circ s_{a,b} \in C([a^*, p] \otimes b, [c^*, p])$. By naturality in $C$, we see that

$$i(\hat{h} \circ s_{a,b} \circ (f \otimes b)) = i(\hat{h} \circ s_{a,b}) \circ f,$$

where $i(\hat{h} \circ s_{a,b}) \in C([a^*, p], [b \otimes c^*, p])$. It follows that we need to have

$$i(\hat{h} \circ s_{a,b}) = i(h)^\wedge. \tag{2}$$