Quantum data hiding with continuous variable systems

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Introduction. Quantum state discrimination, which consists in identifying by means of a measurement an unknown quantum state, is one of the most fundamental tasks studied by quantum information science. Indeed, it captures the essential nature of scientific hypothesis testing in the quantum realm. At the same time, its conceptual simplicity allows for an in-depth mathematical analysis. The systematic study of the problem was initiated by Holevo [1–3] and Helstrom [4, 5], who solved the binary case, featuring only two states ρ and σ: the lowest error probability in discrimination is given by $P_e(\rho, \sigma; p) = \frac{1}{2}\left(1 - \|p\rho - (1-p)\sigma\|_1\right)$, where $\|X\|_1 \coloneqq \operatorname{Tr}|X|$ is the trace norm, and $p, 1-p$ are the a priori probabilities of ρ, σ, respectively. The investigation of quantum state discrimination in the asymptotic Stein setting has revealed compelling operational interpretations for the quantum relative entropy [6–8] and for the regularised relative entropy of entanglement [9–13]. Also the symmetric Chernoff setting has been studied in detail [14–16]. Recently, quantum *channel* discrimination [17, 18] has also become an important research topic [19–22], and its investigation either in the asymptotic [23–26] or in the energy-constrained [23, 27–29] setting has just witnessed notable progress.

A common assumption that is made when designing discrimination protocols is that any quantum measurement can in principle be carried out. This assumption, which has the advantage of leading to significant mathematical simplifications, does not take into account technological hurdles, which are likely to be a major concern on near-term quantum devices [30, 31], nor more fundamental constraints that appear e.g. in multi-partite settings, where only local quantum operations assisted by classical communication (LOCC) are available [32]. In general, let $\mathscr{M}$ be the class of available quantum measurements. The lowest probability of error in discriminating ρ and σ by employing measurements from $\mathscr{M}$ is denoted by $P_e^{\mathscr{M}}$, where we omitted the dependence from the 'scheme' $(\rho, \sigma; p)$.

For certain sets $\mathscr{M}$, a phenomenon called *quantum data hiding* against $\mathscr{M}$ can emerge [33–43]. Namely, it is possible for two states to be perfectly distinguishable when all measurements are available, yet to be almost indistinguishable when only measurements from $\mathscr{M}$ are employed. Informally, this means that $P_e \approx 0$ while $P_e^{\mathscr{M}} \approx 1/2$ for a certain scheme $(\rho, \sigma; p)$. This genuinely quantum behaviour can be exploited to hide a bit of information to somebody who can access the system only through measurements in $\mathscr{M}$ – hence the name.

The original construction provides a striking example of this phenomenon when $\mathscr{M} = \mathrm{LOCC}$ is the set of LOCC-implementable measurements on a finite-dimensional bipartite quantum system [33, 34]. For a scheme of the form $(\omega_S, \omega_A; 1/2)$, where ω_S, ω_A are the two extremal Werner states [44], it holds that $P_e = 0$, due to orthogonality, while it can be shown that $P_e^{\mathrm{LOCC}} = \frac{d-1}{2(d+1)}$ [33, 34, 37]. Thus, the greater d, the more 'efficient' this scheme becomes, in the sense that P_e^{LOCC} approaches $1/2$ while P_e stays equal to 0.

Discrimination of quantum states or channels in the presence of locality constraints has been studied in several different contexts [21, 42, 45–52], but always with the assumption of finite dimension. However, many systems of physical and technological interest are intrinsically infinite-dimensional. Among these, continuous variable (CV) quantum systems [53–58], which model e.g. electromagnetic modes travelling along an optical fibre, will likely play a major role in the future of quantum technologies, with applications ranging from quantum computation [59–64] to quantum communication [65].

In the case of a CV quantum system, another restricted set of measurements — identified by technological constraints — comprises all Gaussian operations assisted by classical computation (GOCC) [66]. The theoretical study of state discrimination with GOCC has been pioneered by Takeoka and Sasaki [66] and continued by Sabapathy and Winter, who have provided the first example of quantum data hiding against GOCC, achieved through a randomised construction that requires asymptotically many optical modes [67–70]. Let us remark in passing that it is only recently that experiments have been able to achieve a better performance than GOCC at coherent state discrimination [71–75].

In this contribution, we investigate the phenomenon of data hiding in the CV setting, thus filling a striking gap in the existing literature. In particular, we study the maximum efficiency of data hiding against LOCC for energy-constrained CV states, and we provide a simple, single-mode example of two states achieving data hiding against GOCC. Along the way, we develop a technique that allows us to state a quantitative, rigorous bound on the maximum disturbance introduced by the Braunstein–Kimble CV quantum teleportation protocol [76, 77].

Continuous variable teleportation. The Braunstein–Kimble teleportation protocol is a fundamental primitive in CV quantum information [76, 77]. It allows to teleport an m-mode system A to a distant location B, using only local operations, in particular homodyne detections, classical communication, and consuming as a resource m two-mode squeezed vacuum states $|\psi(r)\rangle$. For finite values of the squeezing parameter $r \in \mathbb{R}$ and non-ideal detection efficiency $\eta \in (0,1]$, the output state is not teleported perfectly, but is subjected to a noisy channel. The whole process is described by the transformation [77, Eq. (8)]

$$\rho_{RA} \otimes \left(\psi(r)^{\otimes m}\right)_{A'B} \longmapsto \widetilde{\rho}_{RB} = \left(I^R \otimes \mathcal{N}_{\lambda(r,\eta)}^{A\to B}\right)(\rho_{RA}), \qquad \lambda(r,\eta) := e^{-2r} + \frac{1-\eta^2}{\eta^2}. \tag{1}$$

where R is an arbitrary reference system, and the *Gaussian noise channel* $\mathcal{N}_\lambda^{A\to B}$ is defined by the Wigner function transformation $W_\rho \mapsto W_{\mathcal{N}_\lambda(\rho)} := W_\rho \star G_\lambda$, where $G_\lambda(z) := \frac{e^{-\|z\|^2/\lambda}}{\pi^m \lambda^m}$ and $\star$ denotes convolution.

For every fixed ρ_{RA}, the protocol approximates a perfect teleportation, in the limit of $r \to \infty$ and $\eta \to 1^-$. However, it was argued in [78, Section II.B] that such a convergence is strong but not uniform [79, Appendix A], implying that the values of r and η required to achieve a prescribed accuracy will depend on the input state. Since this dependence is not well understood, and moreover a precise description of the state is often not experimentally available, it is important to have an estimate of the accuracy that is based only on few physically relevant parameters. Our first result goes precisely in this direction. In the following statement, for an m-mode CV system we denote with a_j the annihilation operator corresponding to the j^{th} mode, and with $N := \sum_j a_j^\dagger a_j$ the total photon number Hamiltonian.

Theorem 1. *Let A, A', B be m-mode systems, and let R be an arbitrary quantum system. Fix an energy threshold $E > 0$, and consider a state ρ_{RA} such that $\operatorname{Tr}\rho_A N_A - \|z\|^2 \leq E$, where N_A is the photon number operator, and $\|z\|^2 := \sum_j |\operatorname{Tr}\rho_A a_j|^2$. Then, the error introduced by Braunstein–Kimble teleportation of ρ_{RA} over $(\psi(r)^{\otimes m})_{A'B}$ with detection efficiency $\eta \in (0,1]$ satisfies that*

$$\|\widetilde{\rho}_{RB} - \rho_{RB}\|_1 \leq \frac{2\,\Gamma\!\left(m+\frac{1}{2}\right)}{(m-1)!}\left(\sqrt{E} + \sqrt{E+1}\right)\sqrt{\lambda(r,\eta)}\,, \tag{2}$$

where $\widetilde{\rho}_{RB}$ and $\lambda(r,\eta)$ are given by (1).

The importance of Theorem 1 lies in the fact that it can be used to determine the values of r and η needed to reach a certain accuracy in the Braunstein–Kimble protocol. This is likely to be essential in designing CV quantum circuits with prescribed error tolerance where teleportation is an important primitive.

Data hiding against LOCC. We now come to the problem of data hiding against LOCCs in CV systems. Since these are infinite-dimensional, the results of [38, 41] do not apply, and indeed data hiding schemes are known to exist [33, 34, 41]. However, our intuition dictates that states of finite energy, that live in an 'effectively finite-dimensional' sector of the Hilbert space, should have their data hiding performance somehow limited. We now make this intuition quantitative.

Theorem 2. *Let A be a single-mode CV system, and let B be a generic quantum system. For a scheme $(\rho_{AB}, \sigma_{AB}; p)$ over the bipartite system AB with the property that $\operatorname{Tr}[\rho_A N_A]$, $\operatorname{Tr}[\sigma_A N_A] \leq E$, it holds that*

$$1 - 2P_e^{\text{LOCC}} \geq \frac{2\,(1-2P_e)^3}{27\pi\left(\sqrt{E}+\sqrt{E+1}\right)^2}\,. \tag{3}$$

The scaling with E is tight.

The inequality (3) answers our question: if $P_e \approx 0$, i.e. the two states are almost perfectly distinguishable with global measurements, then P_e^{LOCC} cannot be too close to $1/2$ unless the average energy of the two states is very large. In our paper we extend the above result to the multi-mode case [80, Theorem S7]. The statement is analogous, though it features a complicated dependence on the number of modes that makes it not very tight when said number is large. The proof of Theorem 2 rests on a generalisation of the 'teleportation argument' from [41, Theorem 16], which in turn relies on Theorem 1 in a crucial way.

Data hiding against GOCC. A CV quantum measurement belongs to the GOCC class if it can be implemented by (i) adding ancillary modes in the vacuum, (ii) applying Gaussian unitaries, and (iii) making (partial) homodyne detections, with the stipulation that these operations can be applied sequentially, and that each of them can depend on previous measurement outcomes [66] (cf. [81, SM, Definition S6]).

The operational importance of GOCCs stems primarily from the relative simplicity of their experimental realisation. In spite of this, their performance in state discrimination tasks is surprisingly good in many situations of practical interest. In the paper, we analyse two such tasks in great detail:

- First, we look at the discrimination of equiprobable single-mode thermal states with different average photon numbers (equivalently, different temperatures). Here, a thermal state with average photon number ν is given by $\tau_\nu := \frac{1}{\nu+1}\sum_{k=0}^{\infty}\left(\frac{\nu}{\nu+1}\right)^k |k\rangle\!\langle k|$. It turns out that

$$1 - 2P_e^{\mathrm{GOCC}} \geq \frac{1}{e}\left(1 - 2P_e\right) \tag{4}$$

  for all possible values of the two average photon numbers, with the constant $1/e$ being tight. Hence, no data hiding against GOCC is possible with thermal states.

- We then turn to 'less classical' states, namely, two consecutive Fock states $|n\rangle\!\langle n|$ and $|n+1\rangle\!\langle n+1|$. These are obviously orthogonal, so $P_e = 0$. However, discriminating them requires a photon counter, i.e. a non-GOCC measurement. Contrary to what intuition may tell us, however, there is also a Gaussian measurement that allows to retain a constant discrimination power even in the limit $n \to \infty$. By employing a homodyne detection, we are able to prove rigorously that

$$1 - 2P_e^{\mathrm{GOCC}} \geq c = c\left(1 - 2P_e\right) \tag{5}$$

  for some constant $c > 0$ and all $n \in \mathbb{N}$. Numerical evidence suggests that we can choose $c = 4/\pi^2$.

These two examples confirm that a reasonable accuracy in state discrimination can be achieved with simple Gaussian measurements for several cases of physical interest. While we know that data hiding against GOCC does appear when the number of modes is asymptotically large [67], the reader may wonder whether it exists at all for single-mode systems. We now exhibit an explicit example showing that this is indeed the case.

Example 3 (Even and odd thermal states)**.** For $\lambda \in [0,1)$, define the states

$$\omega_\lambda^+ := (1-\lambda^2)\sum_{n=0}^{\infty}\lambda^{2n}\,|2n\rangle\!\langle 2n|\ , \qquad \omega_\lambda^- := (1-\lambda^2)\sum_{n=0}^{\infty}\lambda^{2n}\,|2n+1\rangle\!\langle 2n+1|\ , \tag{6}$$

and consider the task of discriminating the equiprobable states ω_λ^+ and ω_λ^-. Note that this task can be simulated with a simple experimental procedure starting from a two-mode squeezed vacuum state $|\psi(r)\rangle_{AB}$. By measuring the photon number *parity* on system B [82–85], corresponding to the two-outcome POVM $\{\sum_{n\text{ even}}|n\rangle\!\langle n|_B\, ,\ \sum_{n\text{ odd}}|n\rangle\!\langle n|_B\}$, we are left with the states ω_λ^+ (even) or ω_λ^- (odd), where $\lambda = \tanh(r)$.

Now, since $\omega_\lambda^\pm$ are orthogonal, they are perfectly distinguishable (e.g. by means of a photon counter), and therefore $P_e = 0$. On the other hand, we prove that

$$1 - 2P_e^{\mathrm{GOCC}} \leq 2\left(\frac{1+\lambda}{1-\lambda}\right)^{-\frac{1+\lambda^2}{2\lambda}} \xrightarrow[\lambda\to 1^-]{} 0 \tag{7}$$

The proof of (7) relies on the simple yet effective upper bound on the 'distinguishability norm' associated with GOCC measurements put forth by Sabapathy and Winter [67].

Since $\lim_{\lambda\to 1^-} P_e^{\mathrm{GOCC}} = 1/2$ whereas $P_e = 0$ for all values of λ, we have found an example of data hiding against GOCC. Note that in order to make this scheme effective we need to increase the average energy of the two states. This is likely to be a universal feature of GOCC data hiding schemes [67], as it was of LOCC data hiding schemes (Theorem 2).

Note. The construction of our GOCC data hiding scheme builds on the results by Sabapathy and Winter [67], which appeared on the arXiv on the same day as our paper. In their work, these authors flesh out the general theory of GOCC state discrimination and provide an alternative construction of a GOCC data hiding scheme. This is of a different nature than ours: it employs convex mixtures of coherent states only, but has the drawback of requiring asymptotically many modes. Because of this, our two proposals do not overlap but rather complement each other; depending on the platform, one may be easier to realise or to break than the other, or vice versa. Also, the work by Sabapathy and Winter focuses extensively on data hiding against GOCC, and does not treat the case of data hiding against LOCC, which takes up the first part of this contribution.

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