

# Resource quantification of the no-programming thm.

ans: BQP, UQP (Kubricki)

1. No-programming theorem, UQP
2. approximate (deterministic/probabilistic) UQP
3.  $\epsilon$ -UQP
4. Main result
5. Proof:  $\epsilon$ -UQP and embeddings  $S_i^d \rightarrow B(\mathcal{H})$
6. embeddings  $S_i^d \rightarrow B(\mathcal{H}_m)$  and type 2-constants (Kirsch spaces)

$$1) \text{ UQP } \psi \xrightarrow{\text{embed}} \boxed{V} \xrightarrow{\text{embed}} U\psi U^\dagger = \text{tr}_{\mathcal{H}_m} (V(\psi \otimes \phi_m) V^\dagger)$$

$\psi \in \mathcal{D}(\mathcal{H})$  (input)  
 $\phi_m \in \mathcal{D}(\mathcal{H}_m)$  (memory)

$$\dim(\mathcal{H}) = d$$

$$\dim(\mathcal{H}_m) = m$$

Def:  $V \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H}_m)$  is a UQP (universal programmable q-process)

if for  $\forall U \in \mathcal{U}(\mathcal{H}) \exists \phi_m \in \mathcal{D}(\mathcal{H}_m)$  s.t.

$$\forall \psi \in \mathcal{D}(\mathcal{H}), \text{tr}_m V(\psi \otimes \phi_m) V^\dagger = U\psi U^\dagger$$

(Nielsen & Chuang, 1997)

No-programming theorem:

$V \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H}_m)$  is a UQP, then  $m = \infty$ .

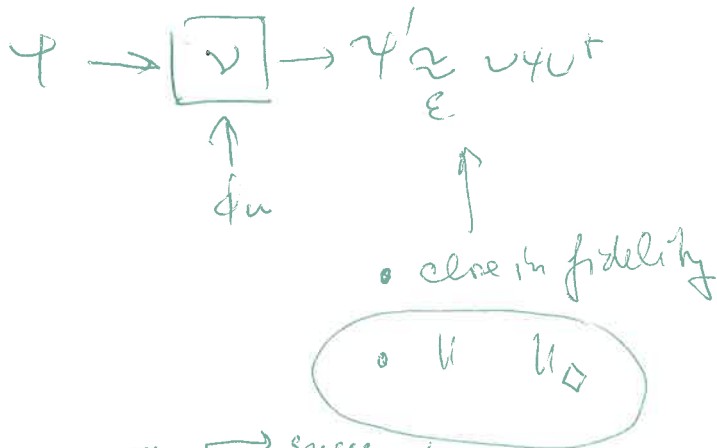


Q. What if  $m \geq 0$ ?

2. Approximate UQP, deterministic & Probabilistic

$$\|\psi\psi^\dagger - \text{tr}_{\mathcal{H}_B}[\psi(\psi^\dagger \phi_A)\psi]\| \leq \epsilon$$

deterministic:



probabilistic

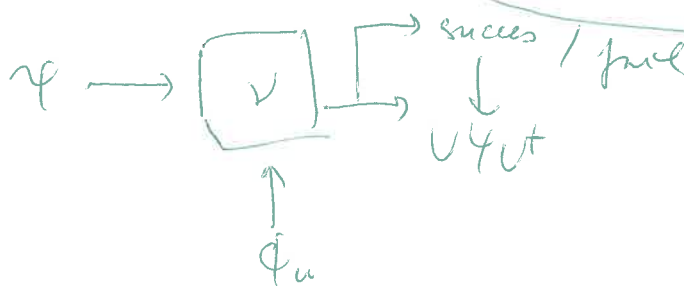


figure of merit: success probability

$$1 - \epsilon \leq p_{\text{succ}} < 1$$

→ maximizing

$m \rightarrow \epsilon$   
 $m \rightarrow d$   
 upper bounds

$$\epsilon \geq 0 \Rightarrow m \geq 0$$

$$\epsilon > 0 \Rightarrow$$

conditions of UQP:

(post-based teleportation)

$$m \sim \exp\left(\frac{d^2 \log d}{\epsilon}\right)$$

fix  $d$ ,  $m(\epsilon) = \left(\frac{1}{\epsilon}\right)^{d^2}$  (probabilistic)

(fidelity, etc.)

deterministic [this work]

lower bounds

$$m(d) \geq \frac{d^2}{\epsilon}$$

fix  $\epsilon$

$$\text{fix } d : m(\epsilon) \geq \left(\frac{1}{\epsilon}\right)^d$$

known:  $d_{1/\varepsilon}^2 \leq m(d_\varepsilon) \leq \exp(d_\varepsilon^2) \xrightarrow{PBT}$

main result: improve lower bounds:  $\exp[(1-\varepsilon)d^2] \leq$

3.  $\varepsilon$ -UPQPs:  $\gamma \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H}_m)$  is an  $\varepsilon$ -UPQP if  
 $\forall \psi \in \mathcal{U}(\mathcal{H}) \exists \phi_m \in \mathcal{D}(\mathcal{H}_m)$  s.t.

$$\frac{1}{2} \| \text{tr}_m \gamma(\cdot \otimes \phi_m) \gamma^\dagger - U \cdot U^\dagger \|_\diamond \leq \varepsilon$$

Comment: (i)  $\varepsilon$ -UPQPs are deterministic approx UPQPs

(ii) if  $\gamma$  is  $10\varepsilon$ -UPQP with  $\text{rank} \geq 1/\varepsilon$

then  $\gamma$  is also  $2\varepsilon$ -UPQP

→ both kinds

Thm 1  $\gamma \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H}_m)$  is an  $\varepsilon$ -UPQP, then  
 $m \geq 2 \frac{(1-\varepsilon)}{\varepsilon}$ , c more constant

5. Proof: First step: Understanding  $\gamma \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H}_m)$  as embeddings

$$\begin{array}{ccc} \mathcal{L}_1(\mathcal{H}) & \rightarrow & \mathcal{B}(\mathcal{H}_m) \\ \downarrow & & \nearrow \text{Bessel proc} \\ \{O_p \text{ on } \mathcal{H}_m \text{ with true norm}\} & & \end{array}$$

Second step: Understanding isometric embeddings  $\mathcal{L}_1(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}_m)$



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$\varepsilon$ -UPQPs and embeddings

$\gamma$  - unitary in  $\mathcal{U}(\mathcal{H} \otimes \mathcal{H}_m)$

$$\mathcal{H}_{dm} \simeq \mathcal{M}_d \otimes \mathcal{M}_m, \gamma = \sum_i A_i \otimes B_i$$

→ [www.uni.lodz.pl](http://www.uni.lodz.pl)

$$\phi_V: \sigma \mapsto \sum_i (\text{tr} \sigma A_i) B_i$$

$$S(\mathcal{X}) \rightarrow B(\mathcal{H}_m)$$

Fact:  $\|\phi_V: S_1(\mathcal{X}) \rightarrow B(\mathcal{H}_m)\| \leq \|V\|_{B(\mathcal{X} \otimes \mathcal{H})} = 1$

$\uparrow$  trace norm       $\downarrow$  op. norm  
 embedding map

Th. 2: If  $V$  is an  $\epsilon$ -UTQP, then  $\phi_V$  is an approx. isometry:

$$\forall \sigma \in S(\mathcal{X}) \quad (1-\epsilon) \leq \|\phi_V(\sigma)\|_{B(\mathcal{H}_m)} \leq \|\sigma\|_{S_1(\mathcal{X})}$$

$\Downarrow$

Rem:  $\|\phi_V\| \cdot \|\phi_V^{-1}\| \leq \frac{1}{(1-\epsilon)^{1/2}}$

$\epsilon = 0 \Rightarrow$  perfect UTQP  $\Rightarrow \phi_V$ -isometry  $\rightarrow$  not possible

Thm 1 Embeddings  $S_1(\mathcal{X}) \rightarrow B(\mathcal{H}_m)$  and type 2 constants

$V$   $\epsilon$ -UTQP  $\rightarrow \phi_V$  embedding (embedding, approx. isometry)

$$\phi_V: S_1(\mathcal{X}) \rightarrow \phi_V(S_1(\mathcal{X})) \subseteq B(\mathcal{H}_m)$$

• We look for a metric

$$\|\phi_V\| \|\phi_V^{-1}\| \leq \frac{1}{(1-\epsilon)^{1/2}}$$

(property of  $\rightarrow$  invariant ~~sets~~ of  $S_1(\mathcal{X}), B(\mathcal{H}_m)$ )  
 (property of  $\rightarrow$  invariant when  $\phi_V$  is applied)

Def: A normed space  $X$  is called of type 2 if for every finite sequence  $\{x_i\} \subset X$

$$\mathbb{E} \left\| \sum_i \varepsilon_i x_i \right\|_X^2 \leq T \left( \sum_i \|x_i\|^2 \right)^{1/2} \quad \begin{array}{l} \varepsilon_i \in \{\pm 1\} \\ \uparrow \\ \text{random variable} \end{array} \quad \text{for some constant } T$$

The inf. of constants  $T$  is called the type 2 constant of  $X \equiv T_2(X)$

Comments:  
 (i)  $X \equiv$  Hilbert space, then  $T_2(X) = 1$  (parallelogram law)

(ii) For  $\ell_1(X)$ :  $T_2(\ell_1(X)) \geq \sqrt{d}$

$$T_2(B(\ell_m)) \leq c \sqrt{\log m}$$

(iii) For subspaces:  $T_2(S) \leq T_2(X)$   
 $S \subset X$

(iv) If  $\phi: X \rightarrow Y$  linear isomorphism  
 $T_2(X) \leq \|\phi\| \|\phi^{-1}\| T_2(Y)$

Proof of Thm 1

$$T_2(\ell_1(X)) \leq \frac{1}{(1-\varepsilon)^{1/2}} T_2(B(\ell_m))$$

$$\sqrt{d} \leq$$

$$\leq \frac{1}{(1-\varepsilon)^{1/2}} \sqrt{\log m}$$

$$\Rightarrow m \geq 2^{\frac{(1-\varepsilon)m}{\varepsilon}}$$

$$\exp(\frac{d}{(1-\varepsilon)d}) \leq m \leq \exp(d^2 \log d)$$

OPTIMALITY in Thm 1 / Fix  $\varepsilon$ ,



Standard Q-teleportation is also an E-VPBP,  $(1-\epsilon)^2/g^2, m=d^2$

Rem the bound is true for E-PBP schemes which implement  
the diagonal unitaries