Some characterizations of reversibility of quantum channels

Anna Jenčová

Mathematical Institute, Slovak Academy of Sciences, Bratislava, Slovakia

Rényi Institute, Budapest, 2023

Reversible (sufficient) quantum channels

Let $\mathcal{S}$ be a set of quantum states, Φ a quantum channel.

We say that Φ is reversible (sufficient) with respect to $\mathcal{S}$ if there exists some channel Ψ (recovery channel) such that

$$\Psi \circ \Phi(\rho) = \rho, \qquad \rho \in \mathcal{S}.$$

Reference: Denes Petz's papers

The setting and assumptions

$B(\mathcal{H})$ - operators on a finite dimensional Hilbert space $\mathcal{H}$

- A set of states

$$\mathcal{S} \subset \{\rho \in B(\mathcal{H}),\ \rho \geq 0,\ \mathrm{Tr}\ \rho = 1\}$$

- A channel $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$, completely positive and trace preserving

Assumptions:

There is a faithful (full rank) state $\sigma \in \mathcal{S}$, its image $\Phi(\sigma) \in B(\mathcal{K})$ is also faithful.

Preservation of the relative entropy

The relative entropy: for states ρ, σ

$$D(\rho\|\sigma) = \begin{cases} \mathrm{Tr}\,[\rho(\log(\rho) - \log(\sigma))], & \mathrm{supp}(\rho) \leq \mathrm{supp}(\sigma) \\ \\ \infty, & \text{otherwise.} \end{cases}$$

- Data processing inequality: for a channel Φ

$$D(\Phi(\rho)\|\Phi(\sigma)) \leq D(\rho\|\sigma),$$

- If $D(\rho\|\sigma) < \infty$, then reversibility is equivalent to

$$D(\Phi(\rho)\|\Phi(\sigma)) = D(\rho\|\sigma), \qquad \forall \rho \in \mathcal{S}.$$

Petz

Universal recovery map

The Petz dual of Φ with respect to σ

$$\Phi_\sigma(\cdot) = \sigma^{1/2}\Phi^*(\Phi(\sigma)^{-1/2} \cdot \Phi(\sigma)^{-1/2})\sigma^{1/2}$$

- Φ_σ is a channel $B(\mathcal{K}) \to B(\mathcal{H})$ such that

$$\Phi_\sigma \circ \Phi(\sigma) = \sigma$$

- Φ is reversible with respect to $\mathcal{S}$ if and only if

$$\Phi_\sigma \circ \Phi(\rho) = \rho, \qquad \forall \rho \in \mathcal{S}$$

Petz

Semigroup of channels preserving $\mathcal{S}$

How to describe all channels reversible with respect to $\mathcal{S}$?

Let us consider the set of channels

$$\mathcal{C}_{\mathcal{S}} := \{\Theta : B(\mathcal{H}) \to B(\mathcal{H}),\ \Theta(\rho) = \rho,\ \forall \rho \in \mathcal{S}\}$$

- convex and compact semigroup (closed under composition)
- has a faithful fixed state: $\sigma \in \mathcal{S}$.

By the mean ergodic theorem, there is some $\mathcal{E}_{\mathcal{S}} \in \mathcal{C}_{\mathcal{S}}$ such that

$$\mathcal{E}_{\mathcal{S}} \circ \Theta = \Theta \circ \mathcal{E}_{\mathcal{S}} = \mathcal{E}_{\mathcal{S}}, \qquad \forall \Theta \in \mathcal{C}_{\mathcal{S}}.$$

We see that such $\mathcal{E}_{\mathcal{S}}$ is unique and

$$\mathcal{E}_{\mathcal{S}}^2 = \mathcal{E}_{\mathcal{S}}, \qquad \mathcal{E}_{\mathcal{S}}(\rho) = \rho, \quad \forall \rho \in \mathcal{S}.$$

The minimal sufficient subalgebra

The adjoint $\mathcal{E}_{\mathcal{S}}^*$ is a faithful conditional expectation

$$\Longrightarrow$$

its range is a subalgebra $\mathcal{M}_{\mathcal{S}} := \mathcal{E}_{\mathcal{S}}^*(B(\mathcal{H}))$.

$\mathcal{M}_{\mathcal{S}}$ is the minimal sufficient subalgebra with respect to $\mathcal{S}$:

- $\rho \mapsto \rho|_{\mathcal{M}_{\mathcal{S}}}$ is a sufficient channel
- $\mathcal{M}_{\mathcal{S}}$ is contained in any subalgebra with this property.

The range of a conditional expectation

Let $\mathcal{E} : B(\mathcal{H}) \to B(\mathcal{H})$ be such that $\mathcal{E}^*$ is a conditional expectation.

There is a decomposition $\mathcal{H} \equiv \oplus_n \mathcal{H}_n^L \otimes \mathcal{H}_n^R$ such that

$$\mathcal{E}^*(B(\mathcal{H})) \equiv \bigoplus_n B(\mathcal{H}_n^L) \otimes I_{\mathcal{H}_n^R}$$
$$\mathcal{E}(B(\mathcal{H})) \equiv \bigoplus_n B(\mathcal{H}_n^L) \otimes \omega_n$$

for some fixed states $\omega_n \in B(\mathcal{H}_n^R)$.

The Koashi-Imoto decomposition

Applying this to $\mathcal{E}_{\mathcal{S}}$, we obtain

$$\mathcal{M}_{\mathcal{S}} \equiv \bigoplus_n B(\mathcal{H}_n^{\mathcal{S},L}) \otimes I_{\mathcal{H}_n^{\mathcal{S},R}}$$

$$\rho \equiv \bigoplus_n \lambda_n(\rho)\rho_n \otimes \sigma_n, \qquad \rho \in \mathcal{S},$$

- $\lambda_n(\rho)$ is a probability dsitribution (classical part of $\mathcal{S}$)
- $\rho_n \in B(\mathcal{H}_n^{\mathcal{S},L})$ are states (depending on ρ)
- $\sigma_n \in B(\mathcal{H}_n^{\mathcal{S},R})$ are fixed states.

Koashi-Imoto, Hayden, etc., Luczak, Kuramochi

Generators of $\mathcal{M}_{\mathcal{S}}$

The minimal sufficient subalgebra is generated by:

- Connes cocycles:

$$\rho^{it}\sigma^{-it}, \qquad \rho \in \mathcal{S},\ t \in \mathbb{R}.$$

- Radon Nikodym derivatives:

$$\sigma^{it}(\sigma^{-1/2}\rho\sigma^{-1/2})\sigma^{-it}, \qquad \rho \in \mathcal{S},\ t \in \mathbb{R}.$$

Reversible channels with respect to $\mathcal{S}$

Assume that Φ is reversible.

- Let Ψ be a recovery channel, then $\Psi \circ \Phi \in \mathcal{C}_{\mathcal{S}}$, so that

$$\mathcal{E}_{\mathcal{S}} \circ (\Psi \circ \Phi) = (\Psi \circ \Phi) \circ \mathcal{E}_{\mathcal{S}} = \mathcal{E}_{\mathcal{S}}.$$

- Note that $\mathcal{E}_{\mathcal{S}} \circ \Psi$ is also a recovery channel, so we may assume

$$\mathcal{E}_{\mathcal{S}} \circ \Psi = \Psi, \qquad \Psi \circ \Phi = \mathcal{E}_{\mathcal{S}}.$$

- We then have $\Phi \circ \Psi = \mathcal{E}_{\mathcal{S}_0}$, where

$$\mathcal{S}_0 := \{\Phi(\rho),\ \rho \in \mathcal{S}\}.$$

Reversible channels with respect to $\mathcal{S}$

A channel $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ is reversible with respect to $\mathcal{S}$ iff

$$\Phi^*|_{\mathcal{M}_{\mathcal{S}_0}} : \mathcal{M}_{\mathcal{S}_0} \xrightarrow{iso} \mathcal{M}_{\mathcal{S}}.$$

Equivalently, there is

- a decomposition $\mathcal{K} \equiv \oplus_n \mathcal{K}_n^L \otimes \mathcal{K}_n^R$
- unitaries $U_n : \mathcal{H}_n^{\mathcal{S},L} \to \mathcal{K}_n^L$
- channels $\Phi_n : B(\mathcal{H}_n^{\mathcal{S},R}) \to B(\mathcal{K}_n^R)$

such that

$$\Phi\Big|_{B(\mathcal{H}_n^{\mathcal{S},L} \otimes \mathcal{H}_n^{\mathcal{S},R})} \equiv U_n^* \cdot U_n \otimes \Phi_n.$$

Reversible channels with respect to $\mathcal{S}$

Further conditions for reversibility: preserving the generators

- Connes cocycles

$$\Phi^*(\Phi(\rho)^{it}\Phi(\sigma)^{-it}) = \rho^{it}\sigma^{-it}, \qquad \rho \in \mathcal{S},\ t \in \mathbb{R};$$

- Radon-Nikodym derivatives:

$$\Phi^*(\Phi(\sigma)^{-1/2}\Phi(\rho)\Phi(\sigma)^{-1/2}) = \sigma^{1/2}\rho\sigma^{1/2}, \qquad \rho \in \mathcal{S};$$

- Petz dual

$$\Phi_\sigma \circ \Phi(\rho) = \rho, \qquad \rho \in \mathcal{S}.$$

Conditions on $\mathcal{S}$

Given a channel Φ, what are the conditions for states in $\mathcal{S}$?

We fix a faithful state $\sigma \in \mathcal{S}$. Then we must have

$$\mathcal{S} \subset \mathrm{Fix}(\Phi_\sigma \circ \Phi) := \{\rho,\ \Phi_\sigma \circ \Phi(\rho) = \rho\}.$$

Put

$$\mathcal{F} := \lim_n \frac{1}{n} \sum_{k=1}^{n} (\Phi_\sigma \circ \Phi)^k,$$

then $\mathcal{F}^*$ is a conditional expectation and

$$\mathcal{F}(B(\mathcal{H})) = \mathrm{Fix}(\Phi_\sigma \circ \Phi).$$

Conditions on $\mathcal{S}$

There is

- a decomposition $\mathcal{H} \equiv \oplus_n \mathcal{H}_n^{\Phi,\sigma,L} \otimes \mathcal{H}_n^{\Phi,\sigma,R}$
- and states $\omega_n \in B(\mathcal{H}_n^{\Phi,\sigma,R})$

such that Φ is reversible with respect to $\mathcal{S}$ if and only if all $\rho \in \mathcal{S}$ have the form

$$\rho \equiv \bigoplus_n \mu_n(\rho)\rho_n \otimes \omega_n$$

for some probability distribution $\mu(\rho)$ and states $\rho_n \in B(\mathcal{H}^{\Phi,\sigma,L})$.

Preservation of standard Rényi relative entropies

The standard (Petz-type) Rényi relative entropies, $\alpha > 0$:

$$D_\alpha(\rho\|\sigma) = \begin{cases} \dfrac{1}{\alpha-1} \log \mathrm{Tr}\ \rho^\alpha \sigma^{1-\alpha} & \alpha \neq 1 \\ \mathrm{Tr}\ \rho(\log\rho - \log\sigma), & \alpha = 1. \end{cases}$$

- satisfy data processing inequality for $\alpha \in (0,2]$.

Φ is sufficient with respect to $\mathcal{S}$ if and only if

$$D_\alpha(\Phi(\rho)\|\Phi(\sigma)) = D_\alpha(\rho\|\sigma), \qquad \forall \rho \in \mathcal{S}$$

for some $\alpha \in (0,2)$.
Petz, PetzJA, HMPB, HM,H

Preservation of sandwiched Rényi relative entropies

The sandwiched (minimal) Rényi relative entropies, $\alpha > 0$:

$$\tilde{D}_\alpha(\rho\|\sigma) = \begin{cases} \frac{1}{\alpha - 1} \log \mathrm{Tr} \left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha & \alpha \neq 1 \\ \mathrm{Tr}\ \rho(\log\rho - \log\sigma), & \alpha = 1. \end{cases}$$

- satisfy data processing inequality for $\alpha \in [1/2, \infty]$

Φ is sufficient with respect to $\mathcal{S}$ if and only if

$$\tilde{D}_\alpha(\Phi(\rho)\|\Phi(\sigma)) = \tilde{D}_\alpha(\rho\|\sigma), \qquad \forall \rho \in \mathcal{S}$$

for some $\alpha \in (1/2, \infty)$. JA, JA

Sandwiched Rényi relative entropy

We look at

$$\tilde{Q}_\alpha(\rho\|\sigma) := \mathrm{Tr}\,\left(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}}\right)^{\alpha},$$

so that

$$\tilde{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha-1}\log\tilde{Q}_\alpha(\rho\|\sigma)$$

- For $\alpha > 1$: interpolation L_p-norms
- For $\alpha \in (1/2, 1)$: a variational formula, relation to case $\alpha > 1$
- The case $\alpha = 1$ (relative entropy): solved by Petz

An interpolation L_p-norm with respect to a state

Let us define a norm in $B(\mathcal{H})$, for $\alpha \geq 1$:

$$\|X\|_{\alpha,\sigma} = \left(\mathrm{Tr}\ |\sigma^{\frac{1-\alpha}{2\alpha}} X \sigma^{\frac{1-\alpha}{2\alpha}}|^{\alpha}\right)^{\frac{1}{\alpha}}$$

We have for any state ρ:

$$\tilde{Q}_\alpha(\rho\|\sigma) = \|\rho\|_{\alpha,\sigma}^{\alpha}$$

The norm can be obtained by complex interpolation between

$$\|X\|_{1,\sigma} = \mathrm{Tr}\ |X| = \|X\|_1, \qquad \|X\|_{\infty,\sigma} = \|\sigma^{-\frac{1}{2}} X \sigma^{-\frac{1}{2}}\|$$

Hadamard three lines theorem

For any function on $S = \{z \in \mathbb{C},\ \mathrm{Re}(z) \in [0,1]\}$,

$$f : S \to B(\mathcal{H}), \quad \text{continuous, analytic in } \mathrm{int}(S)$$

- we have for any $\alpha > 1$,

$$\|f(1/\alpha)\|_{\alpha,\sigma} \leq \max_{t\in\mathbb{R}} \|f(it)\|_{\infty,\sigma} \max_{t\in\mathbb{R}} \|f(1+it)\|_1$$

- If equality holds for some $\alpha > 1$, then it holds for all

Hadamard three lines theorem

For any $\rho \geq 0$ and α, we define a function

$$f_{\rho,\alpha}(z) = \|\rho\|_{\alpha,\sigma}^{1-z\alpha} \sigma^{\frac{1-z}{2}} \left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}} \right)^{z\alpha} \sigma^{\frac{1-z}{2}}, \qquad z \in S$$

- $f_{\rho,\alpha}(1/\alpha) = \rho$,
- The equality in Hadamard three lines theorem is attained:

$$\|f_{\rho,\alpha}(1/\alpha)\|_{\alpha,\sigma} = \max_{t\in\mathbb{R}} \|f_{\alpha,\sigma}(it)\|_{\infty,\sigma} \max_{t\in\mathbb{R}} \|f_{\alpha,\sigma}(1+it)\|_1$$

Positive trace preserving maps are contractions

Let $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ be a positive trace preserving linear map:

- For $\alpha = 1$,

$$\|\Phi(X)\|_1 \leq \|X\|_1, \qquad X \in B(\mathcal{H})$$

- For $\alpha = \infty$,

$$\|\Phi(X)\|_{\infty,\Phi(\sigma)} = \|\Phi_\sigma^*(\sigma^{-1/2} X \sigma^{-1/2})\|_\infty \leq \|X\|_{\infty,\sigma}$$

- For $\alpha > 1$, by Riesz-Thorin (complex interpolation)

$$\|\Phi(X)\|_{\alpha,\Phi(\sigma)} \leq \|X\|_{\alpha,\sigma}, \qquad X \in B(\mathcal{H}).$$

Beigi

The case $\alpha = 2$

Let $\alpha = 2$.

- $\| \cdot \|_{s,\sigma}$ is a Hilbert space norm, with the inner product

$$\langle X, Y \rangle_\sigma = \mathrm{Tr}\ X^* \sigma^{1/2} Y \sigma^{1/2}$$

- For a positive trace preserving map $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$,

$$\langle X, \Phi(Y) \rangle_{\Phi(\sigma)} = \langle \Phi_\sigma(X), Y \rangle_\sigma, \qquad X \in B(\mathcal{K}),\ Y \in B(\mathcal{H})$$

- Since Φ is a contraction,

$$\|\Phi(Y)\|_{2,\Phi(\sigma)} = \|Y\|_{2,\sigma} \iff \Phi_\sigma \circ \Phi(Y) = Y.$$

Preservation and reversibility

Let Φ be a channel and assume that for some $\alpha > 1$,

$$\|\Phi(\rho)\|_{\alpha,\Phi(\sigma)} = \|\rho\|_{\alpha,\sigma}\left(\iff \tilde{D}_\alpha(\Phi(\rho)\|\Phi(\sigma)) = \tilde{D}_\alpha(\rho\|\sigma)\right)$$

For $\alpha = 2$, we get

$$\|\Phi(\rho)\|_{2,\Phi(\sigma)} = \|\rho\|_{2,\sigma} \iff \Phi_\sigma \circ \Phi(\rho) = \rho$$

so that Φ is reversible.

Preservation and reversibility

For $\alpha = \bar{\alpha} > 1$:

$$f(z) = f_{\rho,\bar{\alpha}}(z) = \|\rho\|_{\bar{\alpha},\sigma}^{1-z\bar{\alpha}} \sigma^{\frac{1-z}{2}} \left(\sigma^{\frac{1-\bar{\alpha}}{2\bar{\alpha}}} \rho \sigma^{\frac{1-\bar{\alpha}}{2\bar{\alpha}}} \right)^{z\bar{\alpha}} \sigma^{\frac{1-z}{2}}, \qquad z \in S$$

Then

$$\begin{aligned} \|\rho\|_{\bar{\alpha},\sigma} = \|f(1/\bar{\alpha})\|_{\bar{\alpha},\sigma} &= \|\Phi(f(1/\bar{\alpha}))\|_{\bar{\alpha},\Phi(\sigma)} \\ &\leq \max_{t\in\mathbb{R}} \|\Phi(f(it))\|_{\infty,\sigma} \max_{t\in\mathbb{R}} \|\Phi(f(1+it))\|_1 \\ &\leq \max_{t\in\mathbb{R}} \|f(it)\|_{\infty,\sigma} \max_{t\in\mathbb{R}} \|f(1+it)\|_1 = \|\rho\|_{\bar{\alpha},\sigma} \end{aligned}$$

We have equalities, for any $\alpha > 1$. This implies

$$\|\Phi(f(1/\alpha))\|_{\alpha,\Phi(\sigma)} = \|f(1/\alpha)\|_{\alpha,\sigma}, \qquad \alpha > 1.$$

Preservation and reversibility

We obtain

$$\|\Phi(\tau)\|_{2,\Phi(\sigma)} = \|\tau\|_{2,\sigma}, \text{ so that } \Phi_\sigma \circ \Phi(\tau) = \tau,$$

for

$$\tau := f(1/2) = \sigma^{\frac{1}{4}} \left(\sigma^{\frac{1-\bar{\alpha}}{2\bar{\alpha}}} \rho \sigma^{\frac{1-\bar{\alpha}}{2\bar{\alpha}}} \right)^{\frac{\bar{\alpha}}{2}} \sigma^{\frac{1}{4}}.$$

We know that $\Phi_\sigma \circ \Phi(\rho) = \rho$ iff ρ is of the form

$$\rho \equiv \bigoplus_n \rho_n \otimes \omega_n \qquad \text{(with fixed faithful states } \omega_n\text{)}$$

Since $\Phi_\sigma \circ \Phi(\sigma) = \sigma$ and $\Phi_\sigma \circ \Phi(\tau) = \tau$, this must be true.

A variational formula for $\alpha \in [1/2, 1)$

For $\alpha \in [1/2, 1)$, we have

$$\tilde{Q}_\alpha(\rho\|\sigma) = \inf_{X \in B(\mathcal{H})^{++}} \alpha \operatorname{Tr} \rho X + (1-\alpha) \operatorname{Tr} \left(\sigma^{\frac{1-\alpha}{2\alpha}} X^{-1} \sigma^{\frac{1-\alpha}{2\alpha}} \right)^{\frac{\alpha}{1-\alpha}}$$

With $\gamma := \frac{\alpha}{1-\alpha} > 1$, this can be written as

$$\tilde{Q}_\alpha(\rho\|\sigma) = \inf_{X \in B(\mathcal{H})^{++}} \alpha \operatorname{Tr} \rho X + (1-\alpha) \tilde{Q}_\gamma(\sigma^{1/2} X^{-1} \sigma^{1/2} \| \sigma).$$

If ρ is also faithful, attained at the unique element

$$\bar{X} = \sigma^{\frac{1}{2\gamma}} (\sigma^{\frac{1}{2\gamma}} \rho \sigma^{\frac{1}{2\gamma}})^{\alpha-1} \sigma^{\frac{1}{2\gamma}}.$$

Frank Lieb, Hiai

Positive trace preserving maps

Let $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ be a positive trace preserving map,

$$\tilde{Q}_\alpha(\rho\|\sigma) = \mathrm{Tr}\,\left(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha.$$

For $Y \in B(\mathcal{K})^{++}$, we have

$$\begin{aligned}\tilde{Q}_\gamma(\sigma^{1/2}\Phi^*(Y)^{-1}\sigma^{1/2}\|\sigma) &\leq \tilde{Q}_\gamma(\sigma^{1/2}\Phi^*(Y^{-1})\sigma^{1/2}\|\sigma)\\ &= \tilde{Q}_\gamma(\Phi_\sigma(\Phi(\sigma)^{1/2}Y^{-1}\Phi(\sigma)^{1/2})\|\sigma)\\ &\leq \tilde{Q}_\gamma(\Phi(\sigma)^{1/2}Y^{-1}\Phi(\sigma)^{1/2}\|\Phi(\sigma))\end{aligned}$$

We used the Choi inequality $\Phi^*(Y)^{-1} \leq \Phi^*(Y^{-1})$, definition of Φ_σ and monotonicity of $\tilde{Q}_\gamma$, $\gamma > 1$.

Positive trace preserving maps

We get, for $Y \in B(\mathcal{K})^{++}$,

$$\begin{aligned}\tilde{Q}_\alpha(\rho\|\sigma) &\le \alpha \operatorname{Tr} \rho\Phi^*(Y) + (1-\alpha)\tilde{Q}_\gamma(\sigma^{1/2}\Phi^*(Y)^{-1}\sigma^{1/2}\|\sigma) \\ &\le \alpha \operatorname{Tr} \Phi(\rho)Y + (1-\alpha)\tilde{Q}_\gamma(\Phi(\sigma)^{1/2}Y^{-1}\Phi(\sigma)^{1/2}\|\Phi(\sigma))\end{aligned}$$

Taking the inf,

$$\tilde{Q}_\alpha(\rho\|\sigma) \le \tilde{Q}_\alpha(\Phi(\rho)\|\Phi(\sigma)),$$

so that

$$\tilde{D}_\alpha(\rho\|\sigma) \ge \tilde{D}_\alpha(\Phi(\rho)\|\Phi(\sigma)).$$

Preservation and reversibility

Let $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ be a channel such that

$$\tilde{Q}_\alpha(\rho\|\sigma) = \tilde{Q}_\alpha(\Phi(\rho)\|\Phi(\sigma)).$$

If ρ is faithful, then the infima in the variational formulas are attained at unique $\bar{X} \in B(\mathcal{H})^{++}$ resp. $\bar{Y} \in B(\mathcal{K})$ and

$$\bar{X} = \Phi^*(\bar{Y}).$$

We also infer that

$$\begin{aligned}
\tilde{Q}_\gamma(\sigma^{1/2}\bar{X}^{-1}\sigma^{1/2}\|\sigma) &= \tilde{Q}_\gamma(\sigma^{1/2}\Phi^*(\bar{Y}^{-1})\sigma^{1/2}\|\sigma) \\
&= \tilde{Q}_\gamma(\Phi_\sigma(\Phi(\sigma)^{1/2}\bar{Y}^{-1}\Phi(\sigma)^{1/2})\|\sigma) \\
&= \tilde{Q}_\gamma(\Phi(\sigma)^{1/2}\bar{Y}^{-1}\Phi(\sigma)^{1/2}\|\Phi(\sigma))
\end{aligned}$$

Preservation and reversibility

Put

$$\mu = \sigma^{1/2}\bar{X}^{-1}\sigma^{1/2}, \qquad \nu = \Phi(\sigma)^{1/2}\bar{Y}^{-1}\Phi(\sigma)^{1/2}$$

Then

$$\Phi_\sigma(\nu) = \mu, \qquad \tilde{Q}_\gamma(\nu\|\Phi(\sigma)) = \tilde{Q}_\gamma(\mu\|\sigma) = \tilde{Q}_\gamma(\Phi_\sigma(\nu)\|\Phi_\sigma(\Phi(\sigma)))$$

By the results for $\gamma > 1$, $\Phi \circ \Phi_\sigma(\nu) = \nu$, so that

$$\Phi_\sigma \circ \Phi(\mu) = \Phi_\sigma \circ \Phi \circ \Phi_\sigma(\nu) = \Phi_\sigma(\nu) = \mu.$$

From

$$\mu = \sigma^{\frac{\gamma-1}{2\gamma}}\left(\sigma^{\frac{1}{2\gamma}}\rho\sigma^{\frac{1}{2\gamma}}\right)^{1-\alpha}\sigma^{\frac{\gamma-1}{2\gamma}},$$

we get $\Phi_\sigma \circ \Phi(\rho) = \rho$ as before.

Quantum hypothesis testing

Suppose $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ are given, one of them is the true state:

- we test the hypothesis $H_0 = \sigma$ against $H_1 = \rho$
- a test: an effect $0 \leq T \leq I$,

$$\mathrm{Tr}\,[T\omega] - \text{probability of rejecting } H_0 \text{ in the state } \omega$$

- error probabilities:

$$\alpha(T) = \mathrm{Tr}\,[\sigma T], \qquad \beta(T) = \mathrm{Tr}\,[\rho(I - T)]$$

- Bayes error probabilities for $\lambda \in [0, 1]$:

$$P_e(\lambda, \sigma, \rho, T) := \lambda\alpha(T) + (1 - \lambda)\beta(T)$$

Quantum Neyman-Pearson lemma

Put $P_{s,\pm} := \mathrm{supp}((\rho - s\sigma)_\pm)$, $P_{s,0} := I - P_{s,+} - P_{s,-}$.

A test T is Bayes optimal for $\lambda \in (0,1)$ if and only if

$$T = P_{s,+} + X, \quad 0 \le X \le P_{s,0}, \qquad s = \frac{\lambda}{1-\lambda}$$

and then

$$\begin{aligned} P_e(\lambda,\sigma,\rho) &:= \min_{0\le T\le I} P_e(\lambda,\sigma,\rho,T) \\ &= (1-\lambda)(1 - \mathrm{Tr}\left[(\rho - s\sigma)_+\right]) \\ &= (1-\lambda)(s - \mathrm{Tr}\left[(\rho - s\sigma)_-\right]) \\ &= \frac{1}{2}(1 - (1-\lambda)\|\rho - s\sigma\|_1). \end{aligned}$$

Data processing inequalities

We clearly have for any quantum channel Φ and $\lambda \in [0,1]$:

$$P_e(\lambda, \Phi(\sigma), \Phi(\rho)) \geq P_e(\lambda, \sigma, \rho),$$

or equivalently, for any $s \in \mathbb{R}$:

$$\|\Phi(\rho) - s\Phi(\sigma)\|_1 \leq \|\rho - s\sigma\|_1;$$

$$\mathrm{Tr}\left[(\Phi(\rho) - s\Phi(\sigma))_+\right] \leq \mathrm{Tr}\left[(\rho - s\sigma)_+\right];$$

$$\mathrm{Tr}\left[(\Phi(\rho) - s\Phi(\sigma))_-\right] \leq \mathrm{Tr}\left[(\rho - s\sigma)_-\right].$$

Equality in DPI

The following are equivalent:

- $P_e(\lambda, \Phi(\sigma), \Phi(\rho)) = P_e(\lambda, \sigma, \rho)$, $\lambda \in [0, 1]$;
- $\|\Phi(\rho) - s\Phi(\sigma)\|_1 = \|\rho - s\sigma\|_1$, $s \in \mathbb{R}$;
- $\mathrm{Tr}\left[(\Phi(\rho) - s\Phi(\sigma))_+\right] = \mathrm{Tr}\left[(\rho - s\sigma)_+\right]$, $s \in \mathbb{R}$;
- $\mathrm{Tr}\left[(\Phi(\rho) - s\Phi(\sigma))_-\right] = \mathrm{Tr}\left[(\rho - s\sigma)_-\right]$, $s \in \mathbb{R}$;
- $\Phi^*(Q_{s,+}) = P_{s,+}$, $s \in \mathbb{R}$;
- $\Phi^*(Q_{s,-}) = P_{s,-}$, $s \in \mathbb{R}$.

$$\big(\ Q_{s,\pm} = \mathrm{supp}((\Phi(\rho) - s\Phi(\sigma))_\pm)\big)$$

Can we get recoverability?

An integral formula for relative entropy

For any pair if states ρ, σ:

$$D(\rho\|\sigma) = \int_{-\infty}^{\infty} \frac{dt}{|t|(1-t)^2} \operatorname{Tr}\left[((1-t)\rho + t\sigma)_-\right]$$

For $\lambda \geq 0$ such that $\sigma \leq \rho \leq \lambda\sigma$:

$$D(\rho\|\sigma) = \int_0^{\lambda} \frac{ds}{s} \operatorname{Tr}\left[(\rho - s\sigma)_-\right] + \log(\lambda) + 1 - \lambda$$

If such λ does not exist, both sides are ∞.
(Frenkel, arxiv:2208.12194)