Rényi divergences in quantum information theory

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What is a divergence?

- A "dissimilarity measure" on probability distributions:

  For probability distributions p, q

$$D(p\|q) \equiv \text{how different } p \text{ is from } q.$$

- A contrast functional:

$$D(p\|q) \geq 0, \qquad D(p\|q) = 0 \iff p = q.$$

- Not a metric (not necessarily symmetric)
- Other properties?

Rényi divergences

Axiomatic approach (A. Rényi, 1961):

There is a unique family of divergences $\{D_\alpha\}_{\alpha>0}$, satisfying certain postulates

$$D_\alpha(p\|q) = \frac{1}{\alpha-1}\log\left(\sum_k p_k^\alpha q_k^{1-\alpha}\right), \qquad 1 \neq \alpha > 0$$

$$D_1(p\|q) = \lim_{\alpha\to 1} D_\alpha(p\|q) = \sum_k p_k \log\left(\frac{p_k}{q_k}\right)$$

- Fundamental quantities in information theory
- For $\alpha = 1$ - Kullback-Leibler divergence (relative entropy, I-divergence)

A basic property: DPI and sufficient statistics

Data processing inequality: For a transformation

$T : \{1, \ldots, n\} \to \{1, \ldots, m\}$, with p^T, q^T induced distributions

$$D_\alpha(p^T \| q^T) \leq D_\alpha(p \| q)$$

- Any reasonable divergence should satisfy DPI!

Kullback-Leibler-Csiszár Theorem: If $\mathrm{supp}(p) \subseteq \mathrm{supp}(q)$, $\alpha > 1$

$D_\alpha(p^T \| q^T) = D_\alpha(p \| q) \iff T$ is a sufficient statistic for $\{p, q\}$:

- conditional expectations $E_p[\cdot | T] = E_q[\cdot | T]$
- T contains all information needed to distinguish p from q.

Quantum divergences

Quantum information theory:

- quantum states instead of probability measures
- simplest case: density matrices

$$\rho \in M_n(\mathbb{C}),\ \ \rho \geq 0,\ \ \mathrm{Tr}\,[\rho] = 1$$

- general case: normal states of a von Neumann algebra
  - covers most of interesting situations
  - powerful technical tools

Quantum divergences: dissimilarity measures for quantum states

Postulates for quantum divergences?

- Postulates similar to Rényi (Müller-Lennert et al, 2013)
- In the classical case (commuting density matrices) we get the unique family of Rényi divergences $\{D_\alpha\}_{\alpha>0}$
- In general quantum case: no unique solution

Quantum DPI

Quantum channel: a linear map $\Phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$

- completely positive: $\mathrm{id}_k : M_k(\mathbb{C}) \to M_k(\mathbb{C})$ identity map

$$\Phi \otimes \mathrm{id}_k \text{ is positive for any } k \geq 1$$

- trace-preserving: $\mathrm{Tr}\left[\Phi(\rho)\right] = \mathrm{Tr}\left[\rho\right]$

Equivalently: $\Phi \otimes \mathrm{id}_k$ maps states to states, for all k.

Data processing inequality for quantum divergences:

$$D(\Phi(\rho)\|\Phi(\sigma)) \leq D(\rho\|\sigma)$$

for any quantum channel Φ and any pair of states ρ, σ.

An important quantum divergence

Quantum relative entropy (Umegaki, 1962)

$$S(\rho\|\sigma) = \mathrm{Tr}\left[\rho\left(\log(\rho) - \log(\sigma)\right)\right]$$

- satisfies postulates, DPI (Lindblad, 1975)
- fundamental in quantum information theory
- operational interpretations: quantum communication, asymptotic hypothesis testing
- related to many important quantities
- entanglement measures, uncertainty relations

Quantum Rényi divergences

Petz-type (standard) quantum Rényi divergence: (Petz, 1985,1986)

$$D_\alpha(\rho\|\sigma) = \frac{1}{\alpha-1}\log \mathrm{Tr}\left[\rho^\alpha \sigma^{1-\alpha}\right], \qquad 1 \neq \alpha > 0$$

- satisfies postulates, DPI for $\alpha \in (0,2]$
- $\lim_{\alpha\to 1} D_\alpha(\rho\|\sigma) = S(\rho\|\sigma)$
- operational interpretation for $\alpha \in (0,1)$: (Audenaert et al., 2008, Nagaoka, 2006)

  asymptotic hypothesis testing (error exponents, direct part)

Quantum Rényi divergences

Minimal (sandwiched) quantum Rényi divergence: (Müller-Lennert et al, 2013, Wilde et al, 2014)

$$\tilde{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1} \log \mathrm{Tr}\left[\left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha\right], \qquad 1 \neq \alpha > 0$$

- satisfies postulates, DPI for $\alpha \in [1/2, \infty)$ (Frank and Lieb, 2013)
- $\lim_{\alpha \to 1} \tilde{D}_\alpha(\rho\|\sigma) = S(\rho\|\sigma)$
- operational interpretation for $\alpha > 1$: (Mosonyi and Ogawa, 2015)

  asymptotic hypothesis testing (error exponents, converse part)

Quantum Rényi divergences

$\alpha - z$-Rényi divergence: (Jaksic et al, 2011, Audenaert and Datta, 2015)

$$D_{\alpha,z}(\rho\|\sigma) = \frac{1}{\alpha - 1} \log \mathrm{Tr}\left[\left(\sigma^{\frac{1-\alpha}{2z}} \rho^{\frac{\alpha}{z}} \sigma^{\frac{1-\alpha}{2z}}\right)^{z}\right], \qquad 1 \neq \alpha > 0, z > 0$$

- satisfies postulates, DPI for (Zhang, 2020)

  $\alpha \in (0,1),\ \max\{\alpha, 1-\alpha\} \leq z$ or $\alpha > 1,\ \max\{\frac{\alpha}{2}, \alpha-1\} \leq z \leq \alpha$

- $\lim_{\alpha\to 1} D_{\alpha,z}(\rho\|\sigma) = S(\rho\|\sigma),\ z > 1$