Notes on monotonicity of $\alpha \mapsto D_{\alpha,z}$

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Assume that $z > 1/2$ and let $p = 2z$ and $q = \frac{2z-1}{2z}$ the dual parameter. Let $e := s(\varphi)$ and $f := s(\psi)$, and let $\sigma, \tau \in \mathcal{M}_*^+$ be such that $s(\sigma) = 1-e$, $s(\tau) = 1-f$. Put $\psi_0 := \psi+\tau$, $\varphi_0 = \varphi+\sigma$, then ψ_0, φ_0 are faithful positive normal functionals on $\mathcal{M}$ and we have $h_\varphi^\theta = eh_{\varphi_0}^\theta = h_{\varphi_0}^\theta e$ and $h_\psi^\theta = fh_{\psi_0}^\theta = h_{\psi_0}^\theta f$ for any $\theta > 0$. We will use the notations $L_L^p := L^p(\mathcal{M};\varphi_0)_L$, $L_R^p := L^p(\mathcal{M};\psi_0)_R$ and $L_\eta^p := C_\eta(L_L^p, L_R^p)$.

1 Remarks for the case $\alpha > 1$

1. Let $1 < \alpha \leq 2z$ and assume that $Q_{\alpha,z}(\psi\|\varphi) < \infty$, so that there exists a (unique) $y \in L^p(\mathcal{M})e$ such that $h_\psi^{\alpha/p} = yh_\varphi^{(\alpha-1)/p}$. Note that we may as well assume that $y \in fL^p(\mathcal{M})e$, so that
$$h_\psi = h_{\psi_0}^{\eta/q} y h_{\varphi_0}^{(1-\eta)/q} \in L_\eta^p,$$
where $\eta = (2z-\alpha)/(2z-1)$ and $Q_{\alpha,z}(\psi\|\varphi) = \|y\|_{2z}^{2z} = \|h_\psi\|_{p,\varphi_0,\psi_0,\eta}^p$. In this way, we may use interpolation theory directly, without first assuming that φ and ψ are faithful.

2. Note that we always have $h_\psi = h_{\psi_0}^{1/q} h_\psi^{1/p} \in L_R^p$ and $\|h_\psi\|_{L_R^p}^p = \|h_\psi^{1/p}\|_p^p = \psi(1)$. Assume that $1 < \alpha_1 \leq 2z$ is such that $Q_{\alpha_1,z}(\psi\|\varphi) < \infty$ and let $1 < \alpha < \alpha_1$. Then $\alpha = (1-\theta)\alpha_1 + \theta$ for some $\theta \in (0,1)$. Using the reiteration theorem as in [2], we obtain
$$Q_{\alpha,z}(\psi\|\varphi) \leq Q_{\alpha_1,z}(\psi\|\varphi)^{1-\theta}\psi(1)^\theta.$$
Taking the logarithm and noting that $\theta = (\alpha_1-\alpha)/(\alpha_1-1)$, we obtain directly that
$$D_{\alpha,z}(\psi\|\varphi) \leq D_{\alpha_1,z}(\psi\|\varphi).$$

3. Note that elements of the form $xh_{\varphi_0}^{1/q}$ with $x \in L_p(\mathcal{M})$ an analytic element with respect to $h_{\psi_0}^{it} \cdot h_{\varphi_0}^{-it}$ are contained in $L_L^p \cap L_R^p$. Since the analytic elements are dense in $L_p(\mathcal{M})$ [3, Lemma 10.4], it follows that $L_L^p \cap L_R^p$ is dense in both L_L^p and L_R^p. I am not sure if this implies that $L_L^p \cap L_R^p$ is dense in $L_{\eta_1}^p \cap L_{\eta_2}^p$, though, as required by the usual form of the reiteration theorem. In any case, the result by Cwikel can be used.

2 The case $\alpha \in (0,1)$

We will show that we can use complex interpolation and Kosaki L_p-spaces also in the case $\alpha \in (0,1)$ if $z > 1/2$. The proof is easier in the case $z \geq 1$, which we prove first.

Proposition 1. *Assume that $z \geq 1$. Then*

1. $\alpha \mapsto \log Q_{\alpha,z}(\psi\|\varphi)$ *is convex on* $(0,1)$
2. $\alpha \mapsto D_{\alpha,z}(\psi\|\varphi)$ *is monotone increasing on* $(0,1)$.

Proof. Put $\xi := h_\psi^{1/2} h_\varphi^{1/2} \in L_1(\mathcal{M})$. Let $\alpha \in (0,1)$ and put $\eta := \frac{z-\alpha}{2z-1}$, so that we have

$$0 \leq 1 - \frac{q}{2} = \frac{z-1}{2z-1} < \eta < \frac{z}{2z-1} = \frac{q}{2} \leq 1.$$

Then

$$\xi = h_\psi^{\frac{\eta}{q}}(h_\psi^{\frac{\alpha}{2z}} h_\varphi^{\frac{1-\alpha}{2z}})h_\varphi^{\frac{1-\eta}{q}} = h_{\psi_0}^{\frac{\eta}{q}}(h_\psi^{\frac{\alpha}{2z}} h_\varphi^{\frac{1-\alpha}{2z}})h_{\varphi_0}^{\frac{1-\eta}{q}} \in L^p_\eta$$

and $Q_{\alpha,z}(\psi\|\varphi) = \|\xi\|^p_{p,\psi_0,\varphi_0,\eta}$. The proof can be finished by reiteration theorem, similarly as in [2, Prop. 0.1] and the remark 2. in Section 1 above.

□

Now we turn to the case $1/2 < z < 1$. Note that a similar strategy as in the above proof works only for restricted values of α. We will need a bit more of the complex interpolation method. Let us denote $\Sigma := \Sigma(L^p_L, L^p_R) = L^p_L + L^p_R$ and let $\mathcal{F} := \mathcal{F}(L^p_L, L^p_R)$ be the set of functions $S := \{w \in \mathbb{C},\ \mathrm{Re}\,(w) \in [0,1]\} \to \Sigma$ that are

(i) bounded, continuous and analytic in the interior of S (with respect to the norm in Σ),

(ii) $f(it) \in L^p_L$, $f(1+it) \in L^p_R$, $t \in \mathbb{R}$,

(iii) the maps $t \mapsto f(it) \in L^p_L$ and $t \mapsto f(1+it) \in L^p_R$ are continuous and

$$\max\{\sup_t \|f(it)\|_{p,\varphi_0,L}, \sup_t \|f(1+it)\|_{p,\psi_0,R}\} < \infty.$$

We will use the following functions, defined on the strip S:

$$f(w) = h_\psi^{\frac{w}{q}+\frac{1-w}{p}} h_\varphi^{\frac{1-w}{q}+\frac{w}{p}}, \qquad w \in S. \tag{1}$$

Note that $f(w)$ is an element in $L_1(\mathcal{M})$. The next lemma shows that f has values in Σ.

Lemma 1. *We have $f \in \mathcal{F}$ and for each $\eta \in (0,1)$, we have*

$$\|f(\eta+it)\|^p_{p,\varphi_0,\psi_0,\eta} = Q_{1-\eta,z}(\psi\|\varphi).$$

Proof. For $\eta \in [0,1]$ we have

$$f(\eta+it) = h_\psi^{\frac{\eta}{q}} h_\psi^{i(\frac{1}{q}-\frac{1}{p})t} h_\psi^{\frac{1-\eta}{p}} h_\varphi^{\frac{\eta}{p}} h_\varphi^{i(\frac{1}{p}-\frac{1}{q})t} h_\varphi^{\frac{1-\eta}{q}} = h_{\psi_0}^{\frac{\eta}{q}}\big(h_{\psi_0}^{i(\frac{1}{q}-\frac{1}{p})t} h_\psi^{\frac{1-\eta}{p}} h_\varphi^{\frac{\eta}{p}} h_{\varphi_0}^{i(\frac{1}{p}-\frac{1}{q})t}\big)h_{\varphi_0}^{\frac{1-\eta}{q}}$$

By [3, Lemmas 10.1 and 10.2], $h_{\psi_0}^{it} \cdot h_{\varphi_0}^{-it}$ defines a strongly continuous group of isometries on $L_p(\mathcal{M})$ for every $1 \leq p \leq \infty$. This implies the property (iii) in the definition of $\mathcal{F}$. Also for $\eta \in (0,1)$, we see that $f(\eta+it) \in L^p_\eta$ and

$$\|f(\eta+it)\|^p_{p,\varphi_0,\psi_0,\eta} = \|h_{\psi_0}^{i(\frac{1}{q}-\frac{1}{p})t} h_\psi^{\frac{1-\eta}{p}} h_\varphi^{\frac{\eta}{p}} h_{\varphi_0}^{i(\frac{1}{p}-\frac{1}{q})t}\|^p_p = \|h_\psi^{\frac{1-\eta}{p}} h_\varphi^{\frac{\eta}{p}}\|^p_p = Q_{1-\eta,z}(\psi\|\varphi).$$

Since L^p_η for each η is continuously embedded in Σ, this implies that f is Σ-valued. Since by Hölder $\|h_\psi^{\frac{1-\eta}{p}} h_\varphi^{\frac{\eta}{p}}\|_p \le \psi(1)\varphi(1)$ for any η, f is also bounded. Note that as a function with values in $L_1(\mathcal{M})$, f is bounded, continuous on S and analytic in the interior. We now prove that the continuity and analyticity properties also hold in Σ (maybe this is already obvious, but I will give an argument similar to that in [1, Sec. 9.1,29.1] just for the case). Let $\mu_0(w,t)$ and $\mu_1(w,t)$ be the Poisson kernels associated with S. We then have

$$f(w) = \int_{\mathbb{R}} f(it)\mu_0(w,t)dt + \int f(1+it)\mu_1(w,t)dt.$$

The integrals are in $L_1(\mathcal{M})$, but since $t \mapsto f(it) \in L^p_L$ and $t \mapsto f(1+it) \in L^p_R$ are continuous and bounded in the respective norms, we see that the integrals also exist in Σ and since Σ is continuously embedded in $L_1(\mathcal{M})$, the above equality holds. This shows that $f : S \to \Sigma$ is continuous. Therefore, the expressions

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi - w} d\xi$$

for a suitable circle Γ around a point w in the interior of S are defined in Σ. Since f is analytic in $L_1(\mathcal{M})$, this expression is equal to $f(w)$, hence f is analytic in the interior of S.

□

Proposition 2. *Assume that* $1/2 < z < 1$. *Then*

1. $\alpha \mapsto \log Q_{\alpha,z}(\psi\|\varphi)$ *is convex on* $(0,1)$
2. $\alpha \mapsto D_{\alpha,z}(\psi\|\varphi)$ *is monotone increasing on* $(0,1)$.

Proof. Let $\alpha_1, \alpha_2 \in (0,1)$ and let $\alpha := (1-\theta)\alpha_1 + \theta\alpha_2$. Put $\eta_i = 1 - \alpha_i$, $i = 1,2$ so that $\eta := 1-\alpha = (1-\theta)\eta_1 + \theta\eta_2$. By the reiteration theorem, $L^p_\eta = C_\theta(L^p_{\eta_1}, L^p_{\eta_2})$. Let f be the function given by (1). Then $f_1 : w \mapsto f((1-w)\eta_1 + w\eta_2) \in \mathcal{F}(L^p_{\eta 1}, L^p_{\eta_2})$ and by usual arguments, we have

$$\|f(\eta)\|_{p,\varphi_0,\psi_0,\eta} = \|f_1(\theta)\|_{C_\theta(L^p_{\eta_1},L^p_{\eta_2})} \le (\sup_t \|f_1(it)\|_{L^p_{\eta_1}})^{1-\theta}(\sup_t \|f_1(1+it)\|_{L^p_{\eta_1}})^\theta.$$

Since $f_1(it) = f(\eta_1 + i(\eta_2 - \eta_1)t)$ and $f_1(1+it) = f(\eta_2 + i(\eta_2-\eta_1)t)$, we get from Lemma 1 that

$$Q_{1-\eta,z}(\psi\|\varphi) \le Q_{1-\eta_1,z}(\psi\|\varphi)^{1-\theta} Q_{1-\eta_2,z}(\psi,\varphi)^\theta.$$

This implies 1. Further, since $f(it) \in L^p_L$ and $\|f(it)\|^p_{L^p_L} = \psi(1)$, we obtain 2. similarly as in remark 2. of Section 1.

□

References

[1] A. Calderón. Intermediate spaces and interpolation, the complex method. *Studia Mathematica*, 24(2):113–190, 1964.

[2] F. Hiai, Monotonicity of $\alpha \mapsto D_{\alpha,z}$, (12/31/2023) notes.

[3] H. Kosaki. Applications of the complex interpolation method to a von Neumann algebra: Non-commutative L_p-spaces. *J. Funct. Anal.*, 56:26–78, 1984. doi:https://doi.org/10.1016/0022-1236(84)90025-9.