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GEOMETRY OF STATE SPACES OF OPERATOR ALGEBRAS

ERIK M. ALFSEN AND FREDERIC W. SHULTZ

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Preface

In this book we give a complete geometric description of state spaces of operator algebras, Jordan as well as associative. That is, we give axiomatic characterizations of those convex sets that are state spaces of C*-algebras and von Neumann algebras, together with such characterizations for the normed Jordan algebras called JB-algebras and JBW-algebras. These non-associative algebras generalize C*-algebras and von Neumann algebras respectively, and the characterization of their state spaces is not only of interest in itself, but is also an important intermediate step towards the characterization of the state spaces of the associative algebras.

This book gives a complete and updated presentation of the characterization theorems of [10], [11] and [71]. Our previous book *State spaces of operator algebras: basic theory, orientations and C*-products*, referenced as [AS] in the sequel, gives an account of the necessary prerequisites on C*-algebras and von Neumann algebras, as well as a discussion of the key notion of orientations of state spaces. For the convenience of the reader, we have summarized these prerequisites in an appendix which contains all relevant definitions and results (listed as (A1), (A2), ...), with reference back to [AS] for proofs, so that this book is self-contained.

Characterizing the state spaces of operator algebras among all convex sets is equivalent to characterizing the algebras (or their self-adjoint parts) among all ordered linear spaces. The problem was raised in this form in the early 1950s, but remained open for several decades. A brief survey of the history of the subject is given in the preface to [AS], and at the end of each chapter of the present book there are historical notes with references to the original papers.

Our axioms for state spaces are primarily of geometric character, but many of them also can be interpreted in terms of physics. As in our previous book, we have included a series of remarks discussing such interpretations.

The book is divided into three parts. Part I (Chapters 1 through 6) concerns Jordan algebras and their state spaces. Part II (Chapters 7 and 8) develops spectral theory for affine functions on convex sets. Part III (Chapters 9, 10 and 11) gives the axiomatic characterization of operator algebra state spaces, and explains how the algebras can be reconstructed from their state spaces.

We imagine that the backgrounds and interests of potential readers vary quite a bit, and we have tried to make the book adaptable for different groups of readers. Parts I and II are independent of each other, except for a few motivating examples. Part III makes use of the results and concepts from Parts I and II, but those interested in a short route to the state space characterization theorems may skim Parts I and II, and then proceed to Part III, referring back to the first two parts when necessary. We now will give a brief description of the content of each part.

Part I serves as an introduction for novices to the theory of JB-algebras and JBW-algebras, and of their state spaces. No previous knowledge of Jordan algebras is assumed. The reader familiar with C*-algebras and von Neumann algebras may want to keep in mind that the self-adjoint part of a C*-algebra is a JB-algebra, and the self-adjoint part of a von Neumann algebra is a JBW-algebra. Many results in the Jordan context are natural (if non-trivial) generalizations of known theorems on C*-algebras or von Neumann algebras, although there are sometimes significant differences in results and/or in proofs. To stress the similarities, we often have given a reference to the analogous result for C*-algebras or von Neumann algebras in [AS]. We have given full proofs of most results, but some of the more complex proofs we have referred to [67].

In quantum mechanics, observables usually are taken to be self-adjoint operators on a Hilbert space. Jordan algebras were originally introduced as a mathematical model of quantum mechanics (cf. [75]), motivated by the observation that the square of an observable has a physical interpretation, which the product of two observables will not have (unless they are simultaneously measurable). Since the Jordan product $a \circ b = \frac{1}{2}(ab + ba)$ can be expressed in terms of squares by $a \circ b = \frac{1}{4}((a+b)^2 - (a-b)^2)$, the set of observables should be closed under Jordan multiplication, but not necessarily under associative multiplication.

A JB-algebra is a real Jordan algebra which is also a Banach algebra under a norm with properties similar to those of C*-algebras. Not every JB-algebra A admits a representation as a Jordan algebra of self-adjoint operators on a complex Hilbert space, but there is a Jordan ideal J (described in terms of the exceptional Jordan algebra $H_3(\mathbf{O})$ of Hermitian matrices with entries in the octonians) which cannot be so represented, but for which A/J has a faithful representation on a Hilbert space. A JB-algebra which is monotone complete and has a separating set of normal states is called a JBW-algebra.

Chapters 1 through 4 contain standard algebraic background for JB-algebras and JBW-algebras, with a few results that are more specialized for later use. Chapter 5 investigates properties of JB-algebra state spaces, and JBW-algebra normal state spaces. Chapter 6 contains the theory of dynamical correspondences, which was first presented in [12], and has not appeared before in book form. A dynamical correspondence on a JB-algebra is a map which assigns to each element an order derivation, which is a generator of a one-parameter group of affine automorphisms of the state space (and also of the normal state space in the JBW-algebra case). The definition of a dynamical correspondence is motivated by quantum mechanics. Elements of a JB-algebra are thought of as representing observables. But in quantum mechanics physical variables play a dual role, as observables and as generators of one-parameter groups, and a dynamical correspondence gives the elements of a JB-algebra such a double identity. The main theorem on such correspondences says that a JB-algebra is the self-adjoint part of a C*-algebra (and a JBW-algebra is the self-adjoint part of a von Neumann algebra) iff it admits a dynamical correspondence, which will then provide the Lie part of the associative product. We also discuss the connection of dynamical correspondences with the concept of orientation used by Connes in his characterization of σ-finite von Neumann algebras via the natural cone $P_\xi^\natural$ of Tomita–Takesaki theory [36], and with the notion of orientation of state spaces introduced by the authors and discussed in detail in [AS], where it is shown that orientations of the state space are in 1-1 correspondence with C*-products.

Part II starts out with convex sets K which are at first very general, but later will be specialized to state spaces of algebras (Jordan or associative). In the case of the state space of a C*-algebra (or JB-algebra), the self-adjoint elements act as affine functions on the state space. A key feature of such algebras is that these elements admit a functional calculus, which, on the algebraic side, reflects a key property of the subalgebra generated by a single element, and, on the physical side, represents the application of a function to the outcome of an experiment. Thus a natural starting point in finding properties that characterize such state spaces is to give conditions that lead to a spectral theory and functional calculus for a space A of affine functions on K. This is the main focus of Part II.

This spectral theory is based on axioms involving concepts which we will now briefly describe. A key concept is that of an (abstract) *compression* $P : A \to A$ which generalizes a (concrete) compression $P : a \mapsto pap$ by a projection p in a C*-algebra (or a JB-algebra). Such maps are often thought of as representing the filters in the measurement process of quantum mechanics. Geometrically they correspond, in a 1-1 fashion, to a class of faces of K, called *projective faces* since they are determined by projections in the motivating examples. (In [7] and [9] compressions were called *P-projections*). Two positive affine functions $b, c \in A$ are said to be *orthogonal* if they live on a pair of *complementary* projective faces; then $a = b - c$ is said to be an *orthogonal decomposition* of a. The concept of an orthogonal decomposition plays an important role in the establishment of the functional calculus.

Part III contains the main result which characterizes state spaces of C*-algebras among all compact convex sets. A preliminary result gives a similar characterization of state spaces of JB-algebras, which is of interest in itself. The first step in the construction of an algebra from a compact convex set K is the spectral functional calculus for the space $A = A(K)$ of continuous affine functions on K. This step makes use of two assumptions. The first is the existence of orthogonal decompositions in $A(K)$. The other is the existence of "many" projective faces; more specifically, that the projective faces include all *norm exposed faces*, that is, all those faces which are zero-sets of positive bounded affine functions.

The functional calculus provides a candidate for a Jordan product defined in terms of squares, as explained above. This product need not be distributive, but this is achieved by the crucial *Hilbert ball axiom*, by which the face generated by each pair of extreme points of K is a norm exposed Hilbert ball (i.e., is affinely isomorphic to the closed unit ball of an arbitrary finite or infinite dimensional Hilbert space). Together with a technical assumption (which may be regarded as a non-commutative generalization of the decomposition of a probability measure in a discrete and a continuous part), these assumptions make A the self-adjoint part of a JB-algebra whose state space is K.

There is also an alternative axiom which serves the same purpose. That axiom is less geometrical, but has has a bearing on physical applications in that it involves three *pure state properties*, of which the crucial one describes the symmetry of transition probabilities in quantum mechanics. Thus, our characterization of JB-algebra state spaces is as follows.

Theorem *A compact convex set K is affinely homeomorphic to the state space of a JB-algebra (with the w^*-topology) iff K satisfies the conditions:*

(i) *Every $a \in A(K)$ admits a decomposition $a = b - c$ with $b, c \in A(K)^+$ and $b \perp c$.*
(ii) *Every norm exposed face of K is projective.*
(iii) *The σ-convex hull of the extreme points of K is a split face of K and K has the Hilbert ball property.*

Here the condition (iii) can be replaced by the alternative condition

(iii′) *K has the pure state properties.*

The state spaces of C*-algebras are characterized by adding more assumptions to those for state spaces of JB-algebras. The Hilbert ball property is strengthened to the *3-ball property*, by which the Hilbert balls in question are of dimension 0, 1 or 3.

If a JB-algebra A with state space K has the 3-ball property, then each extreme point of K generates a split face of K that is affinely isomorphic to the normal state space of $\mathcal{B}(H)$ for a suitable complex Hilbert space H. Each such affine isomorphism induces an irreducible representation of A as a Jordan algebra of self-adjoint operators on H, in analogy with the standard GNS-construction. However, different affine isomorphisms can give unitarily inequivalent representations, so in general, for each pure state there are two associated irreducible representations, up to unitary equivalence.

It turns out that the 3-ball property is not sufficient to insure that K is a C*-algebra state space. At this point we need to impose our our final condition. This is the axiom of *orientability*, which we will now briefly explain.

The key problem stems from the fact that a C*-algebra (unlike a JB-algebra) is not completely determined by its state space (as can be seen from the simple fact that a C*-algebra and its opposite algebra have the same state space). Thus we cannot recover a C*-algebra from the mere knowledge of the compact topology and the affine geometry of its state space. Some more structure is needed, and the clue to this additional structure already can be found by looking at the simplest non-commutative C*-algebra, the algebra $M_2(\mathbf{C})$ of 2×2-matrices. Its state space is a three dimensional Euclidean ball, which has two possible orientations, and those two determine the two possible C*-products on $M_2(\mathbf{C})$: the standard one, and its opposite product. In the general case, we need the concept of a *global orientation* of a compact convex set K with the 3-ball property. Loosely speaking, a global orientation of K is a "continuous choice of orientation" of all those faces of K that are 3-balls. Now K is said to be *orientable* if it admits a global orientation, and it is shown that such a global orientation of a compact convex set, with the other properties stated above, will determine a choice of the irreducible "GNS-type" representations such that the direct sum of these representations maps $A(K)$ Jordan-isomorphically onto the self-adjoint part of a C*-algebra with state space K. Thus our main result is the following.

Theorem *A compact convex set K is affinely homeomorphic to the state space of a C*-algebra (with the w^*-topology) iff K satisfies the following properties:*

(i) *Every $a \in A(K)$ admits a decomposition $a = b - c$ with $b, c \in A(K)^+$ and $b \perp c$.*
(ii) *Every norm exposed face of K is projective.*
(iii) *The σ-convex hull of the extreme points of K is a split face of K, and K has the 3-ball property.*
(iv) *K is orientable.*

If K satisfies these conditions, then $A(K)$ admits a unique Jordan product making it a JB-algebra with the given state space Furthermore, associative products on $A(K) + iA(K)$ that make it a C*-algebra with state space K will be in 1-1 correspondence with orientations of K.

Part III also contains characterizations of the normal state spaces of JBW-algebras and von Neumann algebras among all convex sets. One key property of these normal state spaces is that they are *spectral*, which means that their bounded affine functions admit unique orthogonal decompositions, and that their norm exposed faces are projective. In this context, there may be no extreme points, so the remaining axioms for JB-algebra and C*-algebra state spaces are not relevant here. Instead, the normal state spaces of JBW-algebras are shown to be those spectral convex sets that are *elliptic* (in that certain distinguished plane subsets are affinely isomorphic to circular disks). The normal state spaces of von Neumann algebras are shown to be those spectral convex sets that are elliptic and enjoy a generalized version of the 3-ball property. (There is no orientability axiom needed in this case.)

In this part of the book, it is also explained how the state spaces of C*-algebras and the normal state spaces of von Neumann algebras can be characterized among the corresponding state spaces for JB-algebras and JBW-algebras by assuming the existence of a dynamical correspondence, which will give the Lie part of the associative product (as proved in Part I). This result does not involve a construction from local geometric data like the theorems above, but is of interest because of its relationship to physics.

A brief description of the content of each chapter is given in the opening paragraph of that chapter.

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Erik M. Alfsen
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Contents

PART I

Jordan Algebras and Their State Spaces

1 JB-algebras

This chapter starts with the definition and basic properties of JB-algebras. We show that quotients of JB-algebras are again JB-algebras, and that Jordan isomorphisms between JB-algebras are isometries. Next we investigate properties of compressions (maps $a \mapsto \{pbp\}$ for p a projection); these will play a key role in our axiomatic investigations in Part II. This is followed by a discussion of operator commutativity (the Jordan analog of the associative notion of commutativity). The chapter ends with basic facts about order derivations on JB-algebras. (In Chapter 6, we will discuss order derivations in more detail.)

Introduction

The motivating example for a JB-algebra is a JC-algebra, i.e., a norm closed real linear space of self-adjoint operators on a Hilbert space, closed under the symmetrized product

$$a \circ b = \frac{1}{2}(ab + ba). \tag{1.1}$$

This product is bilinear and commutative, but not associative. A weakened form of the associative law holds for this product, and that leads to the following abstraction.

1.1. Definition. A *Jordan algebra* over $\mathbf{R}$ is a vector space A over $\mathbf{R}$ equipped with a commutative bilinear product $\circ$ that satisfies the identity

$$(a^2 \circ b) \circ a = a^2 \circ (b \circ a) \quad \text{for all } a, b \in A. \tag{1.2}$$

If $\mathcal{A}$ is any associative algebra over $\mathbf{R}$, then $\mathcal{A}$ becomes a Jordan algebra when equipped with the product (1.1), as does any subspace closed under $\circ$. If a Jordan algebra is isomorphic to such a subspace, it is said to be *special*; if it cannot be so imbedded it is *exceptional*. If $\mathcal{A}$ is a C*-algebra, then $A = \mathcal{A}_{\mathrm{sa}}$ is the primary example of a Jordan algebra that motivates our work.

In spite of the non-associative nature of the Jordan product, calculations in a special Jordan algebra are straightforward because of the associative algebra in which the Jordan algebra lives. There is then a procedure for verifying that the corresponding results hold in general Jordan algebras. This depends on a remarkable result called MacDonald's theorem that says roughly that most Jordan identities valid for special Jordan algebras are valid for all Jordan algebras. (We will discuss Macdonald's theorem in more detail when we first make use of it.)

In any Jordan algebra we define powers recursively: $x^1 = x$ and $x^n = x^{n-1} \circ x$. *A priori* we are in the difficult position of not knowing that the associative law holds for powers of a single element, i.e., that a Jordan algebra is power associative, cf. (A 45). We now set out to remedy this situation.

Note that the following special case of power associativity follows at once from commutativity and the Jordan identity (1.2) by taking $a = b = x$:

$$x(xx^2) = x^2x^2, \tag{1.3}$$

where for the moment we have indicated multiplication by juxtaposition. We are going to show that this implies power associativity for commutative algebras over $\mathbf{R}$, and thus that Jordan algebras are power associative. We start by finding an identity that is linear in each variable and is equivalent to (1.3). (This is a standard process, called "linearization" of the identity.)

1.2. Lemma. *Let $\mathcal{A}$ be an algebra over $\mathbf{R}$ with a product that is commutative and bilinear, but not necessarily associative. Then (1.3) is equivalent to*

$$\sum x_i(x_j(x_kx_l)) = \sum(x_ix_j)(x_kx_l) \tag{1.4}$$

where the sums are over distinct i, j, k, l with $1 \leq i, j, k, l \leq 4$.

Proof. Assume (1.3) holds. Replacing x by $\lambda_1x_1 + \lambda_2x_2 + \lambda_3x_3 + \lambda_4x_4$ (for real scalars $\lambda_1, \lambda_2, \lambda_3, \lambda_4$) gives

$$\sum \lambda_i\lambda_j\lambda_k\lambda_l\, x_i(x_j(x_kx_l)) = \sum \lambda_i\lambda_j\lambda_k\lambda_l\,(x_ix_j)(x_kx_l) \tag{1.5}$$

where the sums are over all i, j, k, l with $1 \leq i, j, k, l \leq 4$. Each side in (1.5) is a quartic polynomial in the variables $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ with coefficients in $\mathcal{A}$. Let $\phi : \mathcal{A} \to \mathbf{R}$ be a linear functional. Then applying ϕ to both sides of (1.5) leads to two polynomials with real coefficients that are equal for all values of the variables $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. It follows that the coefficients of corresponding terms must be equal. Thus in particular, equating coefficients of the $\lambda_1\lambda_2\lambda_3\lambda_4$ term on each side gives

$$\phi\Big(\sum x_i(x_j(x_kx_l))\Big) = \phi\Big(\sum(x_ix_j)(x_kx_l)\Big)$$

where the sum is now over all distinct i, j, k, l. Since linear functionals separate elements of $\mathcal{A}$, (1.4) follows.

Conversely, if (1.4) holds, substituting $x = x_1 = x_2 = x_3 = x_4$ gives (1.3). □

1.3. Proposition. *Let A be a (not necessarily associative) commutative algebra over* $\mathbf{R}$. *Then A is power associative iff it satisfies the identity $x(xx^2) = x^2x^2$ or the linearized version (1.4).*

Proof. The proof relies on Lemma 1.2 and a lengthy induction argument. To avoid interrupting the flow of ideas, we have relegated this proof to the appendix of this chapter. □

1.4. Corollary. *Every Jordan algebra over* $\mathbf{R}$ *is power associative.*

Proof. As remarked above, (1.3) is an immediate consequence of the Jordan identity. The corollary then follows from Proposition 1.3. (A proof can be given that avoids Proposition 1.3 (cf. [67, Lemma 2.4.5]), but we need Proposition 1.3 for other purposes anyway.) □

For complex Banach *-algebras, by the Gelfand–Neumark theorem (A 64) the norm axiom $\|a^*a\| = \|a\|^2$ characterizes norm closed *-algebras of operators on Hilbert space. It is natural to single out a class of normed Jordan algebras that act like Jordan algebras of self-adjoint operators on Hilbert space.

1.5. Definition. A *JB-algebra* is a Jordan algebra A over $\mathbf{R}$ with identity element 1 equipped with a complete norm satisfying the following requirements for $a, b \in A$:

$$\|a \circ b\| \leq \|a\|\|b\|, \tag{1.6}$$

$$\|a^2\| = \|a\|^2, \tag{1.7}$$

$$\|a^2\| \leq \|a^2 + b^2\|. \tag{1.8}$$

JB-algebras can also be defined without requiring a unit (multiplicative identity). A unit can always be adjoined, but this turns out to be non-trivial to show, cf. [67, Thm. 3.3.9].

1.6. Example. The self-adjoint part of a C*-algebra (A 57) is a JB-algebra with respect to the Jordan product $a \circ b = \frac{1}{2}(ab + ba)$. (We will make the convention that all C*-algebras mentioned have an identity.) The norm properties (1.6) and (1.7) follow from the definition of a C*-algebra. By (A 58) and (A 52), a C*-algebra satisfies the property (1.8).

Any norm closed Jordan subalgebra of $\mathcal{B}(H)_{\mathrm{sa}}$ is easily seen to be a JB-algebra. (The defining norm properties follow from the corresponding properties for $\mathcal{B}(H)_{\mathrm{sa}}$.)

1.7. Definition. A JC-algebra is a JB-algebra A that is isomorphic to a norm closed Jordan subalgebra of $\mathcal{B}(H)_{\mathrm{sa}}$. If A is a norm closed Jordan subalgebra of $\mathcal{B}(H)_{\mathrm{sa}}$, then we may also refer to A as a *concrete JC-algebra*, or as a *JC-algebra acting on* H. We will make the convention that our JC-algebras have an identity.

We will see in Proposition 1.35 that an isomorphism of JB-algebras is necessarily an isometry, so in Definition 1.7 we could replace "isomorphic" by "isometrically isomorphic".

These are the motivating examples for JB-algebras, but JB-algebras are slightly more general. Even in finite dimensions, not every JB-algebra is a JC-algebra. There is essentially just one exception: the algebra $H_3(\mathbf{O})$ of Hermitian 3×3 matrices with entries in the octonions (Cayley numbers). More precisely, a finite dimensional JB-algebra is a JC-algebra iff it does not contain $H_3(\mathbf{O})$ as a direct summand. In infinite dimensions, each JB-algebra A admits an exceptional ideal J (closely related to $H_3(\mathbf{O})$) such that A/J is a JC-algebra. We will discuss these results in more detail in Chapters 3 and 4.

We will see below (Lemma 1.10) that every JB-algebra not only is a Banach space, but also is an ordered linear space, with the positive cone being the set of squares. With this ordering every JB-algebra becomes an *order unit space*, whose definition we now review.

1.8. Definition. An *order unit space* is an ordered normed linear space A over $\mathbf{R}$ with a closed positive cone and an element e, satisfying

$$\|a\| = \inf\{\lambda > 0 \mid -\lambda e \le a \le \lambda e\}.$$

The element e is called the *distinguished order unit.*

Order unit spaces were investigated by Kadison [76], who called them *function systems*, since they are precisely the spaces that can be imbedded as subspaces of $C_{\mathbf{R}}(X)$ for a compact Hausdorff space X. For more background, see (A 12) and the succeeding results in the Appendix.

Remark. Jordan algebras have been proposed as a model for quantum mechanics. One reason for this is that, typically, observables are taken to be self-adjoint operators on a Hilbert space, and the space of such operators is closed under the Jordan product but not the associative product. One is then led to abstract axioms for a Jordan algebra. However, the fundamental Jordan axiom (1.2) has no obvious physical interpretation. It has therefore been proposed by von Neumann [98] and by Iochum and Loupias [69, 70] to replace (1.2) by power associativity. This leads to the notion of an order unit algebra (called a *Banach-power-associative algebra* in [69, 70]), which we now define. We will see that JB-algebras are precisely the commutative order unit algebras, cf. Lemma 1.10 and Theorem 2.49.

1.9. Definition. A *normed algebra* is a (not necessarily associative) algebra A with a norm such that $\|ab\| \leq \|a\| \, \|b\|$ for all $a, b \in A$. An order unit space A (A 14) which is also a power associative complete normed algebra (for the order unit norm) and satisfies

(i) the distinguished order unit 1 is a multiplicative identity,
(ii) $a^2 \in A^+$ for each $a \in A$,

will be called an *order unit algebra.*

For more background on order unit algebras, see (A 46) and the succeeding results in the Appendix.

1.10. Lemma. *Every JB-algebra A is a commutative order unit algebra with the given norm and positive cone $A^2 = \{a^2 \mid a \in A\}$.*

Proof. A has an identity, and is power associative (Corollary 1.4). Then the norm properties (1.6), (1.7), (1.8) imply that A is a commutative order unit algebra with the given norm and with a positive cone consisting of the squares in A (A 52). $\square$

Hereafter we will assume each JB-algebra A is given the order for which the positive cone is A^2. In Chapter 2 we will prove the converse of Lemma 1.10: that every commutative order unit algebra is a JB-algebra.

1.11. Theorem. *If A is a JB-algebra, then A is a norm complete order unit space with the identity 1 as distinguished order unit and with order unit norm coinciding with the given one. Furthermore, for each $a \in A$*

$$-1 \leq a \leq 1 \quad \Rightarrow \quad 0 \leq a^2 \leq 1. \tag{1.9}$$

Conversely, if A is a complete order unit space equipped with a Jordan product for which the distinguished order unit is the identity and satisfies (1.9), then A is a JB-algebra with the order unit norm.

Proof. Assume that A is a JB-algebra. By Lemma 1.10, A is a commutative order unit algebra, with the given norm, with a positive cone consisting of the squares, and with the distinguished order unit being the identity of A. By (A 51) the implication (1.9) is valid for commutative order unit algebras, and thus for A.

Conversely, let A be a Jordan algebra over $\mathbf{R}$, so that A is a commutative, power associative algebra with identity (Corollary 1.4). If A is also a complete order unit space for which the identity is the distinguished order unit and satisfies (1.9), then A is an order unit algebra (A 51). The norm in a commutative order unit algebra satisfies the properties (1.6), (1.7), (1.8) (A 52), so A is a JB-algebra. $\square$

Combining Theorem 1.11 and the definition of the order unit norm (A 14), we have the following in any JB-algebra:

$$\|a\| = \inf\{\lambda > 0 \mid -\lambda 1 \leq a \leq \lambda 1\}, \tag{1.10}$$

and the related inequalities (cf. (A 13)):

$$-\|a\| 1 \leq a \leq \|a\| 1. \tag{1.11}$$

Note that by (1.11) a JB-algebra is positively generated, i.e., $A = A^+ - A^+$, since $a = (a + \|a\|1) - \|a\|1$.

For each element a in a JB-algebra A we let $C(a, 1)$ denote the norm closed subalgebra generated by a and 1. By Corollary 1.4 the Jordan subalgebra generated by a single element and 1 is associative, and by (1.6) multiplication is jointly continuous, so $C(a, 1)$ is also associative.

1.12. Proposition. *If A is a JB-algebra and B is a norm closed associative subalgebra containing 1, in particular if $B = C(a, 1)$ for $a \in A$, then B is isometrically (order- and algebra-) isomorphic to $C_{\mathbf{R}}(X)$ for some compact Hausdorff space X.*

Proof. This follows from the fact that B is then an associative JB-algebra, that a JB-algebra is a commutative order unit algebra (Lemma 1.10), and that a commutative associative order unit algebra is isomorphic to some $C_{\mathbf{R}}(X)$ (A 48). □

We will frequently make use of Proposition 1.12, referring to its consequences as following from *spectral theory.* We give one application, proving that the norm condition (1.8) is not redundant. Let A be the algebra of all continuous functions on the closed unit ball in $\mathbf{C}$, holomorphic in the interior, and real on the real axis, with the supremum norm. Then it is easily verified that A is a commutative Banach algebra satisfying (1.6) and (1.7). If A satisfied (1.8), then A would be an associative JB-algebra, so by spectral theory $1 + a^2$ would be invertible for all $a \in A$. On the other hand, if $a \in A$ is the identity function on the unit disk, then $1 + a^2$ is the function $z \mapsto 1 + z^2$, which takes the value 0 at i, and thus is not invertible. Therefore (1.8) cannot hold for A.

In any Jordan algebra, we define the triple product $\{abc\}$ by

$$\{abc\} = (a \circ b) \circ c + (b \circ c) \circ a - (a \circ c) \circ b. \tag{1.12}$$

Note that $\{abc\}$ is linear in each factor, and $\{abc\} = \{cba\}$. In a special Jordan algebra, $\{abc\} = \frac{1}{2}(abc + cba)$ (where the right side involves the ordinary associative product).

The special case of the triple product $\{abc\}$ with $a = c$ will play an important role in the sequel. We will define $U_a : A \to A$ by

$$U_a b = \{aba\} = 2a \circ (a \circ b) - a^2 \circ b. \tag{1.13}$$

The maps U_a are generally much more convenient to work with than the Jordan multiplication $b \mapsto a \circ b$. For a special Jordan algebra (e.g., a JC-algebra), we have $U_a(b) = aba$.

The following two identities are valid in any Jordan algebra, and show that the Jordan triple product $\{aba\}$ acts very much like the associative product aba, which is a great aid in algebraic calculations:

$$\{\{aba\}c\{aba\}\} = \{a\{b\{aca\}b\}a\}, \tag{1.14}$$
$$\{bab\}^2 = \{b\{ab^2a\}b\}. \tag{1.15}$$

Note that (1.15) can be rephrased in terms of the maps U_a as

$$U_{\{aba\}} = U_a U_b U_a. \tag{1.16}$$

For special Jordan algebras $\{aba\} = aba$, so (1.14) and (1.15) follow immediately. It now follows from Macdonald's Theorem (discussed below) that these identities hold for all Jordan algebras.

1.13. Theorem (Macdonald). *Any polynomial Jordan identity involving three or fewer variables (and possibly involving the identity 1) which is of degree at most 1 in one of the variables and which holds for all special Jordan algebras will hold for all Jordan algebras.*

Proof. The interested reader can find a discussion with a complete proof in [72, p. 41] or [67, Thm. 2.4.13]. □

The following result can be derived from MacDonald's Theorem, and is sometimes more convenient to apply. Note that it implies that calculations in any Jordan algebra involving just two elements can be carried out by assuming that the Jordan algebra is a subalgebra of an associative algebra with respect to the product $a \circ b = \frac{1}{2}(ab + ba)$.

1.14. Theorem (Shirshov–Cohn). *Any Jordan algebra generated by two elements (and* 1*) is special.*

Proof. See [67, Thm. 2.4.14]. □

Our next goal is to prove the crucial property that the maps U_a are positive, i.e., $U_a(A^+) \subset A^+$. First we need to develop some properties of inverses.

1.15. Definition. An element a of a JB-algebra A is *invertible* if it is invertible in the Banach algebra $C(a,1)$. Its inverse in this subalgebra is then denoted a^{-1} and is called the inverse of a.

The next few results relate this notion to other concepts of invertibility.

1.16. Lemma. *Let $\mathcal{A}$ be an associative algebra, and consider the associated Jordan product $a \circ b = \frac{1}{2}(ab + ba)$. Then for a, b in $\mathcal{A}$, $ab = ba = 1$ iff*

$$a \circ b = 1 \quad \textit{and} \quad a^2 \circ b = a. \tag{1.17}$$

Proof. If $ab = ba = 1$, then (1.17) follows easily. Conversely, suppose a and b satisfy (1.17). Then

$$ab + ba = 2 \tag{1.18}$$

and

$$a^2 b + ba^2 = 2a \tag{1.19}$$

Multiplying (1.18) on the left by a and then on the right by a gives

$$a^2 b + aba = 2a = aba + ba^2$$

which implies

$$a^2 b = ba^2. \tag{1.20}$$

Substituting (1.20) into (1.19) gives

$$a^2 b = a = ba^2. \tag{1.21}$$

Multiplying the first of the equalities of (1.21) on the left by b and the second on the right by b gives $ba^2 b = ba$ and $ab = ba^2 b$. Hence by (1.18), $ab = ba = 1$. □

If elements a, b in a Jordan algebra satisfy (1.17) we will say that a and b are *Jordan invertible*, and that b is the *Jordan inverse* of a. We will now show that in the context of JB-algebras this is the same as invertibility as defined in Definition 1.15. (It will follow that the Jordan inverse of an element is unique.)

1.17. Proposition. *For elements a, b in a JB-algebra A, the following are equivalent:*

(i) *b is the inverse of a in the Banach algebra $C(a,1)$,*
(ii) *b is the Jordan inverse of a in A.*

Proof. (i) $\Rightarrow$ (ii). This follows from the associativity of the Jordan product for elements of $C(a,1)$.

(ii) $\Rightarrow$ (i). Let b be the Jordan inverse of a in A, and let B_0 be the Jordan subalgebra of A generated by a, b, and 1. By the Shirshov–Cohn Theorem (Theorem 1.14) B_0 is special, so we may assume that B_0 is a Jordan subalgebra of an associative algebra $\mathcal{C}$ with $x \circ y = \frac{1}{2}(xy + yx)$ for x, y in B_0. By Lemma 1.16, $ab = ba = 1$. Thus the associative subalgebra of $\mathcal{C}$ generated by a and b is commutative, and so the induced Jordan product on this subalgebra coincides with the associative product. By norm continuity of the Jordan product on A, the norm closure B of B_0 in A is also associative. Thus both Jordan and associative notions of inverse coincide in B, and by Proposition 1.12 there is an isomorphism ψ from B onto $C_{\mathbf{R}}(X)$ for a compact Hausdorff space X. Since a is invertible in B, then $\psi(a)$ will be invertible in $C_{\mathbf{R}}(X)$, and so will have compact range Y not containing zero. By the Stone–Weierstrass theorem we can uniformly approximate the function $\lambda \mapsto 1/\lambda$ on Y by polynomials. It follows that the inverse of $\psi(a)$ in $C_{\mathbf{R}}(X)$ is a norm limit of polynomials in $\psi(a)$ and 1, and thus that the inverse of a in B (namely b) is a norm limit of polynomials in a and 1. Thus b must lie in $C(a,1)$. $\square$

1.18. Definition. If a is an element of a JB-algebra A, the *spectrum of a*, denoted $\mathrm{sp}(a)$, is the set of all $\lambda \in \mathbf{R}$ such that $\lambda 1 - a$ is not invertible.

1.19. Corollary. *For each element a in a JB-algebra A, $\mathrm{sp}(a)$ is compact and non-empty, and there is a unique isomorphism from $C_{\mathbf{R}}(\mathrm{sp}(a))$ onto $C(a,1)$ taking the identity function on $\mathrm{sp}(a)$ to a. This isomorphism is also an isometric order isomorphism.*

Proof. We will proceed by constructing the inverse of the desired isomorphism. Let $b \mapsto \widehat{b}$ be an isometric isomorphism from $C(a,1)$ onto $C_{\mathbf{R}}(X)$ for a compact Hausdorff space X (cf. Proposition 1.12). An isomorphism preserves invertibility, so $\mathrm{sp}(a) = \mathrm{sp}(\widehat{a})$. By (A 55), $\mathrm{sp}(\widehat{a})$ is the range of $\widehat{a}$, so $\widehat{a}$ is a continuous map of X onto $\mathrm{sp}(a)$; in particular $\mathrm{sp}(a)$ is compact and non-empty. If $\widehat{a}(x) = \widehat{a}(y)$ then $\widehat{a}^n(x) = \widehat{a}^n(y)$ for $n = 1, 2, \ldots$. By density of polynomials in a and 1 in $C(a,1)$ and the fact that $b \mapsto \widehat{b}$ is an isometric isomorphism, polynomials in $\widehat{a}$ and 1 are dense in $C_{\mathbf{R}}(\mathrm{sp}(a))$, and so we conclude that $x = y$. Thus $\widehat{a}$ is a homeomorphism from X onto $\mathrm{sp}(a)$. The composition $b \mapsto \widehat{b} \circ \widehat{a}^{-1}$ is an isomorphism from $C(a,1)$ onto $C_R(\mathrm{sp}(a))$, which sends a to the identity map on $\mathrm{sp}(a)$. Call this map Φ.

If $\Psi : C(a,1) \to C_{\mathbf{R}}(\mathrm{sp}(a))$ is any isomorphism, then Ψ takes 1 to 1 and squares to squares, so must be a unital order isomorphism. By (1.10) Ψ is an isometry. Since polynomials in a are dense in $C(a,1)$, Ψ is uniquely determined by its value on a, and so if Ψ takes a to the identity map on $\mathrm{sp}(a)$, then $\Psi = \Phi$.

Finally, Φ^{-1} is the desired unique isomorphism from $C_{\mathbf{R}}(\mathrm{sp}(a))$ onto $C(a,1)$ taking the identity map on $\mathrm{sp}(a)$ to a. □

1.20. Definition. Let a be an element of a JB-algebra A. The *continuous functional calculus* is the unique isomorphism $f \mapsto f(a)$ from $C_{\mathbf{R}}(\mathrm{sp}(a))$ onto $C(a,1)$ taking the identity function on $\mathrm{sp}(a)$ to a.

1.21. Proposition. *Let a be an element of a JB-algebra A. The continuous functional calculus is an isometric order isomorphism from $C_{\mathbf{R}}(\mathrm{sp}(a))$ onto $C(a,1)$, and $f(a)$ has its usual meaning when f is a polynomial. Furthermore*

$$\mathrm{sp}(f(a)) = f(\mathrm{sp}(a)) \quad \textit{for } f \in C_{\mathbf{R}}(\mathrm{sp}(a)), \textit{ and} \tag{1.22}$$

$$(f \circ g)(a) = f(g(a)) \quad \textit{for } g \in C_{\mathbf{R}}(\mathrm{sp}(a)) \textit{ and } f \in C_{\mathbf{R}}(\mathrm{sp}(g(a))). \tag{1.23}$$

If $f \in C_{\mathbf{R}}(sp(a))$ and $f(0) = 0$, then $f(a)$ is in the (non-unital) norm closed subalgebra $C(a)$ generated by a.

Proof. $f \mapsto f(a)$ is an isometric order isomorphism by Corollary 1.19. If id denotes the identity map on $\mathrm{sp}(a)$, and p is the polynomial $p(\lambda) = \sum_i \alpha_i \lambda^i$, then

$$p(a) = \left(\sum_i \alpha_i (id)^i\right)(a) = \sum_i \alpha_i a^i,$$

where $(id)^i$ denotes pointwise multiplication of functions, so $p(a)$ has its usual meaning.

A unital isomorphism of algebras preserves invertibility and thus spectrum. Therefore $\mathrm{sp}(f) = \mathrm{sp}(f(a))$. By (A 55), $\mathrm{sp}(f)$ equals $f(\mathrm{sp}(a))$ and so (1.22) follows.

The equality (1.23) is clear for polynomials f, and then follows for arbitrary $f \in C_R(\mathrm{sp}(g))$ by continuity of the functional calculus and the Stone–Weierstrass theorem, since $f \mapsto f(a)$ is isometric.

If $f \in C_{\mathbf{R}}(\mathrm{sp}(a))$ and $f(0) = 0$, then f can be uniformly approximated on $\mathrm{sp}(a)$ by polynomials p_n without constant term. Now $p_n(a) \in C(a)$ for $n = 1, 2, \ldots$; hence also $f(a) \in C(a)$. □

1.22. Corollary. *If a is an element of a JB-algebra A, then $a \geq 0$ iff $\mathrm{sp}(a) \subset [0, \infty)$, and $||a|| = \sup\{|\lambda| \mid \lambda \in \mathrm{sp}(a)\}$.*

Proof. Since the functional calculus is an order isomorphism, then $a \geq 0$ iff $id \geq 0$, where id is the identity function on $\mathrm{sp}(a)$. But id is a square iff $\mathrm{sp}(a) \subset [0, \infty)$, which proves the first statement of the corollary. The second follows in similar fashion from the fact that the functional calculus is isometric. □

Note that if A is a JB-algebra, and B is a Jordan subalgebra containing the identity of A, and B is also a JB-algebra itself for the inherited norm, then *a priori* we have two notions of spectrum and order in B: those that come from its status as a JB-algebra, and those inherited from A. However, for a in B, we have $C(a, 1) \subset B$, so the spectrum of a is the same whether calculated in A or in B. Then by Corollary 1.22, $a \geq 0$ is equivalent for a viewed as a member of A or of B, so the order on B as a JB-algebra is that inherited from A.

1.23. Lemma. *Let A be a JB-algebra and $a \in A$. Then a is an invertible element iff U_a has a bounded inverse, and in this case the inverse map is $U_{a^{-1}}$.*

Proof. If a is invertible, let $c \in C(a, 1)$ be the inverse of a. Then by the associativity of $C(a, 1)$ we have $\{ac^2a\} = 1$. Now by (1.16) we have

$$U_a U_{c^2} U_a = U_{\{ac^2a\}} = I,$$

and thus U_a is invertible. To calculate its inverse, we first observe that since a^{-1} is the inverse of a in the associative subalgebra $C(a, 1)$, then $\{aa^{-1}a\} = a$. Thus by (1.16)

$$U_a = U_{\{aa^{-1}a\}} = U_a U_{a^{-1}} U_a.$$

Now multiplying by the inverse of U_a on both sides shows $U_{a^{-1}}$ is the inverse of U_a.

If a is not invertible, then by spectral theory there are elements $\{b_n\}$ in $C(a, 1)$ which have norm 1 and which satisfy $\|U_a b_n\| \to 0$. If U_a had a bounded inverse, applying it to the sequence $\{U_a b_n\}$ would show $\{b_n\} \to 0$, a contradiction to $\|b_n\| = 1$. □

The assumption that the inverse of U_a is bounded is unnecessary; a purely algebraic proof that a is invertible iff U_a is invertible can be found in [72, p. 52].

1.24. Lemma. *If a and b are invertible elements of a JB-algebra, then $\{aba\}$ is invertible with inverse $\{a^{-1}b^{-1}a^{-1}\}$.*

Proof. By Lemma 1.23, if a and b are invertible, then the maps U_a and U_b have bounded inverses, and thus by (1.16) so does $U_{\{aba\}} = U_a U_b U_a$. By

Lemma 1.23 it follows that $\{aba\}$ is invertible. For any invertible element x, calculating in $C(x,1)$ we find $U_{x^{-1}}x = x^{-1}$, so applying Lemma 1.23 and (1.16) gives

$$\begin{aligned}\{aba\}^{-1} = U_{\{aba\}^{-1}}\{aba\} = U_{\{aba\}}{}^{-1}\{aba\} &= (U_aU_bU_a)^{-1}\{aba\} \\ &= U_{a^{-1}}U_{b^{-1}}U_{a^{-1}}U_ab = \{a^{-1}b^{-1}a^{-1}\},\end{aligned}$$

which completes the proof. □

1.25. Theorem. *Let A be a JB-algebra. For each $a \in A$ the map U_a is positive, and has norm $\|a\|^2$.*

Proof. Let A^{-1} denote the set of invertible elements of A, and let $A_0 = A^{-1} \cap A^+$ be the positive invertible elements. By spectral theory, if $b \geq 0$, then b is invertible iff $b \geq \lambda 1$ for some $\lambda > 0$. It follows that A_0 is convex and open. By (1.11),

$$a \geq 0 \iff \|\,\|a\|\,1 - a\,\| \leq \|a\|,$$

so A^+ is closed. Therefore $A_0 = A^{-1} \cap A^+$ is both open and closed in A^{-1}. Note also that A_0 is dense in A^+, since if $a \geq 0$ and $\epsilon > 0$, then $a + \epsilon 1 \in A_0$.

We first show that U_a is positive for invertible a. Note that $(a^{-1})^2$ is in A_0 by spectral theory. By Lemma 1.24, $U_a(A^{-1}) \subset A^{-1}$, so in particular $U_a(A_0) \subset A^{-1}$. We next show that $1 \in U_a(A_0)$. To verify this, note that $1 = U_a((a^{-1})^2)$ (calculate in $C(a,1)$). Since $U_a(A_0)$ is convex, it is a connected subset of A^{-1}. Since A_0 is closed and open in A^{-1} and meets $U_a(A_0)$ (e.g., $1 \in U_a(A_0) \cap A_0$), then $U_a(A_0) \subset A_0$. By continuity of U_a and density of A_0 in A^+, we conclude that $U_a(A^+) \subset A^+$, and so U_a is positive.

Now let $a \in A$ be arbitrary, and let $b \in A_0$. By spectral theory there is an invertible c such that $c^2 = b$. Then by the identity (1.15)

$$U_cU_ab = \{c\{ac^2a\}c\} = \{cac\}^2,$$

and so by the first part of this proof

$$U_ab = U_{c^{-1}}(U_cU_ab) = U_{c^{-1}}\{cac\}^2 \geq 0.$$

Thus $U_a(A_0) \subset A^+$, and by continuity U_a is positive.

Finally, for any positive map T on A, it follows from (A 15) that $\|T\| = \|T1\|$. Applying this to U_a we conclude $\|U_a\| = \|U_a1\| = \|a^2\| = \|a\|^2$. □

1.26. Lemma. *If a, b are positive elements of a JB-algebra A, then (1.24) and (1.25) are equivalent and imply (1.26):*

$$\{aba\} = 0, \tag{1.24}$$

$$\{bab\} = 0, \tag{1.25}$$

$$a \circ b = 0. \tag{1.26}$$

Proof. (1.24) $\Leftrightarrow$ (1.25) Note that by spectral theory

$$b \geq 0 \quad \Rightarrow \quad b^2 \leq \|b\| b. \tag{1.27}$$

If (1.24) holds, then applying U_a to the inequality at the right side of (1.27) gives

$$0 \leq \{ab^2a\} \leq \|b\|\{aba\} = 0. \tag{1.28}$$

Then by the identity (1.15), $\{bab\}^2 = \{b\{ab^2a\}b\} = 0$, and (1.25) follows. Exchanging the roles of a and b shows that (1.25) implies (1.24).

Finally, we show that (1.24) and (1.25) together imply (1.26). The identity

$$(a \circ b)^2 = \tfrac{1}{4}\big(2a \circ \{bab\} + \{ab^2a\} + \{ba^2b\}\big) \tag{1.29}$$

is readily verified in any special Jordan algebra and thus holds for all Jordan algebras by MacDonald's theorem (Theorem 1.13). If $\{aba\} = 0$ and $\{bab\} = 0$, then by (1.28) we conclude that $\{ab^2a\} = 0$, and similarly $\{ba^2b\} = 0$. Now $a \circ b = 0$ follows from the identity (1.29). $\square$

1.27. Definition. Let A be a JB-algebra. Positive elements $a, b \in A$ are *orthogonal* if $\{aba\} = \{bab\} = 0$. We will write $a \perp b$ when a, b are orthogonal.

1.28. Proposition. *Let A be a JB-algebra. Each $a \in A$ can be expressed uniquely as a difference of positive orthogonal elements:*

$$a = a^+ - a^-, \quad \textit{where } a^+ \geq 0,\ a^- \geq 0, \textit{ and } a^+ \perp a^-.$$

Both a^+ and a^- will be in the norm closed subalgebra $C(a, 1)$ generated by a and 1.

Proof. The existence of such a decomposition follows at once from spectral theory. To prove uniqueness, we first show that orthogonal positive elements generate an associative subalgebra. Let $b, c \in A$ be orthogonal positive elements and let $C(b, c, 1)$ denote the norm closed Jordan subalgebra generated by b, c, and 1. We are going to show this algebra is associative by showing that

$$b^n \circ c^m = 0 \text{ for all } n, m \geq 1. \tag{1.30}$$

We first establish the implication

$$0 \leq x, y \in A \quad \text{and} \quad x \perp y \quad \Rightarrow \quad x^n \perp y \ \text{ for all } n \geq 1. \tag{1.31}$$

Let $0 \leq x, y \in A$. We may assume without loss of generality that $\|x\| \leq 1$ and $\|y\| \leq 1$. Then by spectral theory $x^n \leq x$ for all positive integers n, and so $0 \leq \{yx^ny\} \leq \{yxy\} = 0$. By Lemma 1.26, y and x^n are orthogonal. Thus the implication (1.31) follows.

Let m and n be positive integers. Applying (1.31) with $x = b$, and $y = c$, we conclude that $b^n \perp c$. Applying (1.31) again with $y = b^n$, $x = c$, and with m in place of n, gives $c^m \perp b^n$ for $m, n \geq 1$. By Lemma 1.26, this implies (1.30), which completes the proof that $C(b, c, 1)$ is associative.

Now let $a = b - c$ be any decomposition of a into orthogonal positive parts, and let $a = a^+ - a^-$ be the orthogonal decomposition of a within $C(a, 1)$ (which exists by spectral theory). By Proposition 1.12 we have $C(b, c, 1) \cong C_{\mathbf{R}}(Y)$ for some compact Hausdorff space Y. Note that $a^+, a^- \in C(a, 1) \subset C(b, c, 1)$. By the uniqueness of orthogonal decompositions in $C_{\mathbf{R}}(Y)$ we must have $b = a^+$ and $c = a^-$. $\square$

We will call the decomposition $a = a^+ - a^-$ in Proposition 1.28 the *orthogonal decomposition* of a.

Quotients of JB-algebras

An *ideal* of a Jordan algebra is a linear subspace closed under multiplication by elements from the algebra. We are going to prove that the quotient of a JB-algebra by a norm closed ideal is again a JB-algebra. We will imitate the standard proof of the analogous fact for C*-algebras, suitably "Jordanized". We begin by constructing an approximate identity.

1.29. Definition. Let J be a norm closed ideal in a JB-algebra A. An increasing net $\{v_\alpha\}$ in J is an *increasing approximate identity* for J if $0 \leq v_\alpha \leq 1$ for all α and $\lim \|v_\alpha \circ b - b\| = 0$ for all b in J.

1.30. Lemma. *If a, b are elements of a JB-algebra, then*

$$\|\{ab^2a\}\| = \|\{ba^2b\}\|. \tag{1.32}$$

If $a \geq 0$, then

$$\|a \circ b\|^2 \leq \|a\| \, \|\{bab\}\|. \tag{1.33}$$

Proof. We start with the identity

$$\{a\{b\{ba^2b\}b\}a\} = \{ab^2a\}^2$$

(which follows from Macdonald's theorem (Theorem 1.13)). Since the norm in A is the order unit norm, $\{ba^2b\} \leq \|\{ba^2b\}\|1$, and combining this with the last identity gives

$$\|\{ab^2a\}\|^2 = \|\{ab^2a\}^2\| = \|\{a\{b\{ba^2b\}b\}a\}\| \leq \|\{ba^2b\}\| \ \|\{ab^2a\}\|.$$

If $\{ab^2a\} \neq 0$, then dividing by $\|\{ab^2a\}\|$ gives

$$\|\{ab^2a\}\| \leq \|\{ba^2b\}\|,$$

and if $\{ab^2a\} = 0$ then this inequality is trivial. The opposite inequality follows by symmetry, so (1.32) holds.

To prove (1.33), using the identity (1.29) and (1.32) (and the JB-axioms (1.6), (1.7)) we have

$$\begin{aligned} \|(a \circ b)\|^2 = \|(a \circ b)^2\| &= \tfrac{1}{4}\|2a \circ \{bab\} + \{ab^2a\} + \{ba^2b\}\| \\ &\leq \tfrac{1}{2}\|a \circ \{bab\}\| + \tfrac{1}{2}\|\{ba^2b\}\| \\ &\leq \tfrac{1}{2}\|a\| \ \|\{bab\}\| + \tfrac{1}{2}\|\{ba^2b\}\|. \end{aligned} \tag{1.34}$$

Since $a \geq 0$, then by spectral theory $a^2 \leq \|a\|a$, so

$$\|\{ba^2b\}\| \leq \|a\| \ \|\{bab\}\|.$$

Combining this with (1.34) gives (1.33). □

1.31. Lemma. *Let a, b be invertible elements of a JB-algebra A. If $0 \leq a \leq b$, then $b^{-1} \leq a^{-1}$.*

Proof. Denote by $b^{-1/2}$ the positive square root of b^{-1}, which exists by spectral theory. Applying $U_{b^{-1/2}}$ to $a \leq b$ gives $\{b^{-1/2}ab^{-1/2}\} \leq 1$. By spectral theory and Lemma 1.24,

$$1 \leq \{b^{-1/2}ab^{-1/2}\}^{-1} = \{b^{1/2}a^{-1}b^{1/2}).$$

Applying $U_{b^{-1/2}} = (U_{b^{1/2}})^{-1}$ gives $b^{-1} \leq a^{-1}$. □

1.32. Lemma. *Every norm closed ideal J in a JB-algebra A contains an increasing approximate identity.*

Proof. Let $U = \{a \in J^+ \mid \|a\| < 1\}$. We are going to show that U is directed upwards. Define $f : [0,1) \to [0,\infty)$ and $g : [0,\infty) \to [0,1)$ by

$$f(t) = t(1-t)^{-1}, \qquad g(t) = t(1+t)^{-1}.$$

Note that the compositions $f\circ g$ and $g\circ f$ are the identity on their respective domains. By the functional calculus, since $f(0) = 0$ and $g(0) = 0$, we may view f as a map from U into J^+ and g as a map from J^+ into U (cf. Proposition 1.21). By (1.23) these maps are inverses. Since $f(t) = -1 + (1-t)^{-1}$ and $g(t) = 1 - (1+t)^{-1}$, Lemma 1.31 implies that f and g preserve order, and thus are (nonlinear) order isomorphisms. It follows that U is directed upwards since J^+ is.

Now we show that U is an approximate identity. We will show that $\lim_{u\in U} \|a - a\circ u\| = 0$ for $a \in J$. We may assume that $\|a\| \le 1$. For each $n = 1, 2, \ldots$ define $h_n(t) = 1 - 1/n$ for $|t| \ge 1/n$, $h(0) = 0$, and define h to be linear on $[0, 1/n]$ and on $[-1/n, 0]$. Then $\|h_n(a)\| \le 1$, and $h_n(a) \in C(a)$ (Proposition 1.21), so $h_n(a)$ is in U. Since $t^2(1 - h_n(t)) \le 1/n$ for $t \in [-1, 1]$, by spectral theory

$$\|a^2 - \{ah_n(a)a\}\| = \|a^2 \circ (1 - h_n(a))\| \le 1/n.$$

Now if $u \in U$ and $u \ge h_n(a)$ then

$$\{ah_n(a)a\} \le \{aua\} \le a^2,$$

so

$$\|a^2 - \{aua\}\| \le \|a^2 - \{ah_n(a)a\}\| \le 1/n.$$

Thus $\lim_{u\in U} \|a^2 - \{aua\}\| = 0$. Finally, by Lemma 1.30,

$$\begin{aligned}\|a - a\circ u\|^2 &= \|a\circ(1-u)\|^2 \\ &\le \|1-u\|\,\|\{a(1-u)a\}\| \\ &= \|1-u\|\,\|a^2 - \{uau\}\| \to 0,\end{aligned}$$

which completes the proof. □

If J is a norm closed ideal in a JB-algebra A, then the quotient space A/J is a Jordan algebra and a Banach space with the usual quotient norm

$$\|a + J\| = \inf_{b\in J} \|a + b\|.$$

(We will show in Lemma 1.34 that A/J is a JB-algebra.)

1.33. Lemma. *Let J be a norm closed ideal in a JB-algebra A. If $\{v_\alpha\}$ is an increasing approximate identity for J then*

(i) $\lim_\alpha \|\{(1 - v_\alpha)b(1 - v_\alpha)\}\| = 0$ *for all $b \in J$,*

(ii) $\|a+J\| = \lim_\alpha \|a - v_\alpha \circ a\| = \lim_\alpha \|\{(1-v_\alpha)a(1-v_\alpha)\}\|$ *for all* $a \in A$.

Proof. (i) We first establish

$$\lim_\alpha \|\{(1-v_\alpha)x^2(1-v_\alpha)\}\| = 0 \quad \text{for } x \in J. \tag{1.35}$$

By (1.32), and the fact that $(1-v_\alpha)^2 \le 1-v_\alpha$ (which follows from spectral theory), we have

$$\begin{aligned}\lim_\alpha \|\{(1-v_\alpha)x^2(1-v_\alpha)\}\| &= \lim_\alpha \|\{x(1-v_\alpha)^2 x\}\| \\ &\le \lim_\alpha \|\{x(1-v_\alpha)x\}\| \\ &= \lim_\alpha \|2x \circ (x \circ (1-v_\alpha)) - x^2 \circ (1-v_\alpha)\| = 0\end{aligned}$$

where we have used the definition (1.13) of the triple product in terms of Jordan multiplication and the definition of an approximate identity.

We next observe that

$$0 \le b \in J \quad \Rightarrow \quad b^{1/2} \in J, \tag{1.36}$$

and

$$b \in J \quad \Rightarrow \quad b^+ \in J \text{ and } b^- \in J. \tag{1.37}$$

These follow from the fact that we can uniformly approximate the functions $\lambda \mapsto \sqrt{\lambda}$ and $\lambda \mapsto |\lambda|$ by polynomials with constant term zero on compact subsets of $\mathbf{R}$ (which is a consequence of the Stone–Weierstrass theorem.) Now combining (1.35) and (1.36) gives (i) for $0 \le b \in J$. Applying (1.37) gives (i) for all $b \in J$.

(ii) For arbitrary b in J and a in A, $v_\alpha \circ b \to b$ so

$$\begin{aligned}\limsup_\alpha \|a - v_\alpha \circ a\| &= \limsup_\alpha \|a - v_\alpha \circ a + b - v_\alpha \circ b\| \\ &= \limsup_\alpha \|(1-v_\alpha) \circ (a+b)\| \\ &\le \|a+b\|.\end{aligned}$$

Thus

$$\limsup_\alpha \|a - v_\alpha \circ a\| \le \inf_{b \in J} \|a+b\| = \|a+J\|. \tag{1.38}$$

On the other hand

$$\|a+J\| \le \liminf_\alpha \|a - v_\alpha \circ a\|. \tag{1.39}$$

Combining (1.38) and (1.39) gives the first equality in (ii).

To prove the second inequality, first note that by linearity of the triple product in each factor, for arbitrary $a \in A$,

$$\{(1-v_\alpha)a(1-v_\alpha)\} = \{v_\alpha a v_\alpha\} - 2v_\alpha \circ a + a \ \in a + J$$

so

$$\|a + J\| \leq \liminf_\alpha \|\{(1-v_\alpha)a(1-v_\alpha)\}\|. \tag{1.40}$$

For the opposite inequality, by (i) and Lemma 1.26, for each b in J we have

$$\begin{aligned}\limsup_\alpha \|\{(1-v_\alpha)a(1-v_\alpha)\}\| &= \limsup_\alpha \|\{(1-v_\alpha)(a+b)(1-v_\alpha)\}\| \\ &\leq \limsup_\alpha \|(1-v_\alpha)^2\| \ \|a+b\| \\ &\leq \|a+b\|.\end{aligned}$$

Thus

$$\limsup_\alpha \|\{(1-v_\alpha)a(1-v_\alpha)\}\| \leq \inf_{b\in J} \|a+b\| = \|a+J\|. \tag{1.41}$$

Combining (1.40) and (1.41) gives the second equality in (ii). □

1.34. Lemma. *If A is a JB-algebra and J is a norm closed ideal, then A/J is a JB-algebra with the quotient norm.*

Proof. We first show

$$\|(a+J)\circ(b+J)\| \leq \|a+J\| \ \|b+J\|. \tag{1.42}$$

We have

$$\begin{aligned}\|a+J\| \ \|b+J\| &= \inf_{x\in J} \|a+x\| \inf_{y\in J} \|b+y\| \\ &\geq \inf_{x\in J} \inf_{y\in J} \|(a+x)(b+y)\| \\ &\geq \inf_{z\in J} \|(ab+z\| = \|ab+J\|\end{aligned}$$

which implies (1.42).

Let $\{v_\alpha\}$ be an increasing approximate unit for J. We next show

$$\|(a+J)^2\| = \|a+J\|^2. \tag{1.43}$$

We have from Lemma 1.33 and (1.15)

$$\begin{aligned}
\|a+J\|^2 &= \lim_\alpha \|\{(1-v_\alpha)a(1-v_\alpha)\}\|^2 \\
&= \lim_\alpha \|(\{(1-v_\alpha)a(1-v_\alpha)\})^2\| \\
&= \lim_\alpha \|\{(1-v_\alpha)\,\{a(1-v_\alpha)^2 a\}\,(1-v_\alpha)\}\| \\
&\le \lim_\alpha \|\{(1-v_\alpha)\,\{a(\|1-v_\alpha\|^2 1)a\}\,(1-v_\alpha)\}\| \\
&\le \lim_\alpha \|\{(1-v_\alpha)a^2(1-v_\alpha)\}\| \\
&= \|a^2+J\|.
\end{aligned}$$

The opposite inequality follows at once from (1.42), so we have established (1.43). Finally, we will prove

$$\|a^2+J\| \le \|a^2+b^2+J\|. \tag{1.44}$$

Applying Lemma 1.33 gives

$$\begin{aligned}
\|a^2+J\| &= \lim_\alpha \|\{(1-v_\alpha)a^2(1-v_\alpha)\}\| \\
&\le \lim_\alpha \|\{(1-v_\alpha)(a^2+b^2)(1-v_\alpha)\}\| \\
&= \|a^2+b^2+J\|
\end{aligned}$$

which establishes (1.44). □

1.35. Proposition. *Let A and B be JB-algebras and $\pi : A \to B$ a homomorphism such that $\pi(1) = 1$. Then $\pi(A)$ is a JB-algebra, $\|\pi\| \le 1$, and if π is 1-1, then π is an isometry from A into B.*

Proof. Assume first that π is injective. We first show for $a \in A$

$$a \ge 0 \iff \pi(a) \ge 0. \tag{1.45}$$

One direction is clear since π takes squares to squares. Suppose now that $\pi(a) \ge 0$. Let $a = x - y$ be the orthogonal decomposition of a into positive and negative parts (see Proposition 1.28). Then $\pi(a) = \pi(x) - \pi(y)$ is an orthogonal decomposition of $\pi(a)$. Since $\pi(a) \ge 0$, then by the uniqueness of the orthogonal decomposition $\pi(y) = 0$. Since π is injective, then $y = 0$, so $a \ge 0$. This establishes (1.45).

It follows that for each positive scalar λ,

$$-\lambda 1 \le \pi(a) \le \lambda 1 \iff -\lambda 1 \le a \le \lambda 1,$$

so $\|\pi(a)\| = \|a\|$. Thus π is an isometry. It follows in this case that $\pi(A)$ is complete and thus is closed in B, so in particular $\pi(A)$ is a JB-subalgebra of B.

In the general case, we factor π through A/J. By Lemma 1.34, A/J is a JB-algebra. By definition the quotient map $A \to A/J$ has norm at most 1, and so by the first paragraph the composition has norm at most 1, and the image is a JB-algebra. □

The requirement in Proposition 1.35 that π is unital (i.e., $\pi(1) = 1$) is not necessary. Let $p = \pi(1)$. Then $p^2 = p$ and we will see in Proposition 1.43 that $\{pBp\}$ is a JB-algebra with identity p. Thus we can apply Proposition 1.35 to $\pi : A \to \{pBp\}$.

Projections and compressions in JB-algebras

1.36. Definition. Let A be a JB-algebra. A *projection* is an element p of A satisfying $p^2 = p$. A *compression* on A is a map U_p for a projection p.

We will now develop some properties of compressions. Let p be a projection in a JB-algebra A. By the definition (1.13) of the triple product, using $p^2 = p$:

$$U_p a = \{pap\} = 2p \circ (p \circ a) - p \circ a. \tag{1.46}$$

1.37. Lemma. *Let a be a positive element and let p be a projection in a JB-algebra A. Then $\{pap\} = 0$ iff $p \circ a = 0$.*

Proof. If $\{pap\} = 0$, then Lemma 1.26 implies $p \circ a = 0$. Conversely, if $p \circ a = 0$, then (1.46) implies $\{pap\} = 0$. □

If p is a projection in a JB-algebra A, then we denote the projection $1 - p$ by p'. We record for future use:

$$p \circ a = \tfrac{1}{2}(a + \{pap\} - \{p'ap'\}), \tag{1.47}$$

which follows directly from (1.46).

1.38. Proposition. *Let p be a projection in a JB-algebra A. Then $\|U_p\| \le 1$, and*

$$U_p^2 = U_p \quad \textit{and} \quad U_p U_{1-p} = 0. \tag{1.48}$$

If $a \in A$ is positive, then

$$U_p a = 0 \quad \textit{iff} \quad U_{1-p} a = a. \tag{1.49}$$

Proof. We can assume $p \neq 0$. Since $\|p\| = \|p^2\| = \|p\|^2$, then $\|p\| = 1$. Thus by Theorem 1.25, $\|U_p\| = \|p\|^2 = 1$.

The identity

$$\{b\{bab\}b\} = \{b^2ab^2\} \tag{1.50}$$

follows from MacDonald's theorem, as does

$$\{(1-b)\{bab\}(1-b)\} = \{(b-b^2)a(b-b^2)\}.$$

Substituting p for b gives (1.48).

Now let $0 \leq a \in A$. If $U_{1-p}a = a$, then applying U_p to both sides gives $U_p a = 0$. Conversely, if $U_p a = 0$, then by Lemma 1.37, $p \circ a = 0$. Letting $p' = 1 - p$, we have $p' \circ a = a$. Combining this and (1.46) gives

$$U_{p'}a = 2p' \circ (p' \circ a) - p' \circ a = 2p' \circ a - p' \circ a = a,$$

which completes the proof of (1.49). □

If a, b are elements of a JB-algebra A, we write $[a, b]$ for the associated order interval, i.e., $[a, b] = \{x \in A \mid a \leq x \leq b\}$. If a is any positive element of A, face(a) denotes the face of A^+ generated by a, i.e.,

$$\text{face}(a) = \{y \in A \mid 0 \leq y \leq \lambda a \text{ for some } \lambda \geq 0\}.$$

1.39. Lemma. *Let A be a JB-algebra. Every projection $p \in A$ satisfies*

$$\text{face}(p) = \text{im}^+ U_p, \quad \textit{and} \quad \text{face}(p) \cap [0,1] = [0, p]. \tag{1.51}$$

Proof. If $a \in \text{face}(p)$ then $0 \leq a \leq \lambda p$ for some $\lambda \in \mathbf{R}$. Then $U_{p'}a = 0$, so by (1.49) $a = U_p a \in \text{im}^+ U_p$. Thus $\text{face}(p) \subset \text{im}^+ U_p$. Now suppose $a \in \text{im}^+ U_p$. Then $a = U_p a \leq U_p(\|a\| 1) = \|a\|\, p$, so $a \in \text{face}(p)$. Thus $\text{face}(p) = \text{im}^+ U_p$.

To prove the second statement of (1.51), let $a \in \text{face}(p) \cap [0,1]$. Then $a \in \text{im}\, U_p$, so $a = U_p a \leq U_p 1 = p$. Thus $\text{face}(p) \cap [0,1] \subset [0,p]$. The opposite inclusion is evident. □

1.40. Proposition. *The extreme points of the positive part $[0,1]$ of the unit ball of a JB-algebra A are the projections.*

Proof. Let a be an extreme point of $[0,1]$. By spectral theory, $a - a^2$ and a^2 are in $[0,1]$. Since

$$a = \tfrac{1}{2}((a - a^2) + a^2)$$

it follows that $a = a^2$.

Conversely, let p be a projection in A, and suppose

$$p = \lambda b + (1 - \lambda)c \quad \text{with} \quad b, c \in [0, 1] \tag{1.52}$$

and $0 < \lambda < 1$. Then b and c are in face$(p) \cap [0,1] = [0,p]$ and so $b \leq p$ and $c \leq p$ (Lemma 1.39). Therefore by (1.52) $b = c = p$, and thus p is an extreme point of $[0, 1]$. □

We will say a positive linear functional on a JB-algebra A with value 1 at 1 is a *state*, and will refer to the set of all states as the *state space* of A. This is consistent with the notion of state for order unit spaces discussed in [AS], cf. (A 17).

Each positive linear functional is (norm) continuous, with norm equal to its value at 1 (cf. (1.11)). We denote the space of all continuous linear functionals on A by A^*. We observe for future reference that for each state σ on a JB-algebra A, the map $(a, b) \mapsto \sigma(a \circ b)$ is a symmetric positive semi-definite bilinear form, and thus we have the Cauchy–Schwarz inequality

$$|\sigma(a \circ b)| \leq \sigma(a^2)^{1/2}\sigma(b^2)^{1/2}, \tag{1.53}$$

and (taking $b = 1$)

$$|\sigma(a)| \leq \sigma(a^2)^{1/2}. \tag{1.54}$$

We now explore some properties of compressions related to states. If $T : A \to A$ is a continuous linear operator, then $T^* : A^* \to A^*$ denotes the adjoint map, defined by $(T^*\sigma)(a) = \sigma(Ta)$ for $a \in A$ and $\sigma \in \mathcal{A}^*$.

1.41. Proposition. *Let p be a projection in a JB-algebra A, let $p' = 1 - p$, and let σ be a state on A. Then $\|U_p^*\| \leq 1$, and*

$$\sigma(p) = 1 \quad \Leftrightarrow \quad U_p^*\sigma = \sigma, \tag{1.55}$$

$$U_p^*\sigma = 0 \quad \Leftrightarrow \quad U_{p'}^*\sigma = \sigma, \tag{1.56}$$

$$\|U_p^*\sigma\| = 1 \quad \Leftrightarrow \quad U_p^*\sigma = \sigma. \tag{1.57}$$

Proof. By Proposition 1.38, $\|U_p\| \leq 1$, which implies that $\|U_p^*\| \leq 1$. We next establish the implication

$$\sigma(p') = 0 \quad \Rightarrow \quad U_p^*\sigma = \sigma. \tag{1.58}$$

If $\sigma(p') = 0$, by the Cauchy–Schwarz inequality (1.53), $\sigma(p' \circ b) = 0$ for all $b \in A$. Since $p' = 1 - p$, $\sigma(p \circ b) = \sigma(b)$. Applying this fact several times (with $p \circ a$ in place of b and then with a in place of b), and using the formula (1.46) for U_p, we find

$$\sigma(U_p a) = \sigma(2p \circ (p \circ a)) - \sigma(p \circ a) = \sigma(2p \circ a) - \sigma(a) = \sigma(a),$$

which proves the implication (1.58).

We now prove (1.55). If $\sigma(p) = 1$, then $\sigma(p') = 0$, so $U_p^* \sigma = \sigma$ follows from (1.58). Conversely, if $U_p^* \sigma = \sigma$, then applying both sides to 1 gives $\sigma(p) = 1$.

To establish (1.56), note that $U_p^* \sigma = 0$ implies $\sigma(p) = 0$, and so by (1.58) (with p' in place of p) we conclude that $U_{p'}^* \sigma = \sigma$. Conversely, if $U_{p'}^* \sigma = \sigma$, then by (1.48) we find $U_p^* \sigma = U_p^* U_{p'}^* \sigma = (U_{p'} U_p)^* \sigma = 0$.

To prove (1.57), note that as observed above the norm of the positive functional $U_p^* \sigma$ is its value at 1. Thus $\|U_p^* \sigma\| = 1$ implies $(U_p^* \sigma)(1) = 1$, and so $\sigma(p) = 1$. Now apply (1.55) to get $U_p^* \sigma = \sigma$. The converse implication is trivial. □

Property (1.57) has a physical interpretation ("neutrality" of the measurement process) which will be discussed in Part II of this book.

1.42. Definition. Let A be a JB-algebra. A *JB-subalgebra* of A is a norm closed linear subspace, closed under the Jordan product of A.

1.43. Proposition. *If p is a projection in a JB-algebra A, then $\{pAp\} = U_p(A)$ is a JB-subalgebra with identity p.*

Proof. By the Jordan identity (1.15), $\{pAp\}$ is closed under squaring, and since

$$a \circ b = \tfrac{1}{2}[(a+b)^2 - a^2 - b^2], \tag{1.59}$$

then $\{pAp\}$ is closed under Jordan multiplication. Since $U_p^2 = U_p$ it follows that $\{pAp\} = \{a \in A \mid U_p a = a\}$. Now continuity of U_p implies that $\{pAp\}$ is norm closed. For $a \in \{pAp\}$, $U_p a = a$ and $U_{1-p} a = 0$ by Proposition 1.38, and so by (1.47) $p \circ a = a$. Thus p is an identity for $\{pAp\}$. □

1.44. Definition. If p is a projection in a JB-algebra A, then A_p denotes the JB-subalgebra $\{pAp\}$.

1.45. Lemma. *If p is a projection in a JB-algebra A, then $a \circ b = 0$ for $a \in A_p$ and $b \in A_{p'}$.*

Proof. Since $A = A^+ - A^+$ (Proposition 1.28), by the positivity of U_p and $U_{p'}$ it follows that A_p and $A_{p'}$ are positively generated. Thus to establish the lemma we can assume $0 \leq a$ and $0 \leq b$. We will show $\{aba\} = 0$; by Lemma 1.26 this will imply $a \circ b = 0$.

Since $0 \leq b \leq \|b\|1$, applying $U_{p'}$ gives $0 \leq b \leq \|b\|p'$. Then by positivity of the map U_a,

$$0 \leq \{aba\} \leq \|b\|\{ap'a\}.$$

Since p is the identity of the Jordan algebra A_p, then $p' \circ a = 0$. From the definition of the map U_a in (1.13), $\{ap'a\} = 0$, which implies $\{aba\} = 0$. □

Operator commutativity

We want to describe the Jordan analog of elements commuting in an associative algebra.

1.46. Definition. Let A be a JB-algebra. T_a denotes Jordan multiplication by an element a. Elements a and b are said to *operator commute* if $T_a T_b = T_b T_a$.

It is easily verified that in a special Jordan algebra, elements that commute in the surrounding associative algebra will operator commute in the Jordan algebra.

1.47. Proposition. *Let A be a JB-algebra, let $a \in A$, and let p be a projection in A. Then the following are equivalent:*

(i) *a and p operator commute,*
(ii) $T_p a = U_p a$,
(iii) $(U_p + U_{p'})a = a$,
(iv) *p and a are contained in an associative subalgebra.*

Proof. (i) $\Rightarrow$ (ii) Assume a and p operator commute, so that T_a commutes with T_p. By the definition (1.13), $U_p = 2T_p^2 - T_p$, so T_a commutes with U_p. Thus

$$U_p a = U_p T_a 1 = T_a U_p 1 = T_a p = T_a T_p 1 = T_p T_a 1 = T_p a.$$

(ii) $\Rightarrow$ (iii) If $T_p a = U_p a$, then by the expression (1.47) for T_p

$$\tfrac{1}{2}(I + U_p - U_{p'})a = U_p a.$$

This rearranged gives (iii).

(iii) $\Rightarrow$ (i) The defining Jordan identity $a^2 \circ (b \circ a) = (a^2 \circ b) \circ a$ can be restated in the form $[T_{a^2}, T_a] = 0$, where the brackets denote the commutator. This is equivalent to the following identity, called the *linearized*

Jordan axiom:

$$[T_d, T_{b\circ c}] + [T_b, T_{c\circ d}] + [T_c, T_{d\circ b}] = 0.$$

To derive this, in the identity $[T_{a^2}, T_a] = 0$ replace a by $b + \lambda_1 c + \lambda_2 d$ and equate the coefficients of $\lambda_1\lambda_2$ in the resulting polynomials. For more details, see the analogous argument in the proof of Lemma 1.2.

If we take $b = c = p$ in the linearized Jordan axiom, this becomes

$$[T_d, T_p] + 2[T_p, T_{p\circ d}] = 0. \tag{1.60}$$

Note that by (1.60),

$$p \circ d = d \quad \Rightarrow \quad [T_d, T_p] = 0, \tag{1.61}$$

i.e., if $p \circ d = d$, then p and d operator commute.

Assume (iii). Let $r = U_p a$ and $s = U_{p'} a$. Then $a = r + s$, with $r \in A_p$ and $s \in A_{p'}$. Since p is the identity for A_p and p' for $A_{p'}$ (Proposition 1.43), then $p \circ r = r$ and $p' \circ s = s$. By (1.61), p operator commutes with r, and p' operator commutes with s. Since $p = 1 - p'$, then p also operator commutes with s, so p operator commutes with $a = r + s$, proving (i).

(iii) $\Leftrightarrow$ (iv) By Proposition 1.43, p is the identity for the JB-subalgebra A_p. Hence p and $U_p a$ generate an associative subalgebra of A_p. Similarly p' and $U_{p'} a$ generate an associative subalgebra of $A_{p'}$. For $x \in A_p$ and $y \in A_{p'}$ we have $x \circ y = 0$ (Lemma 1.45). It follows that p, $U_p a$, p', and $U_{p'} a$ generate an associative subalgebra B. By (iii), a and p are contained in the associative subalgebra B.

Conversely, if p and a are contained in an associative subalgebra, adjoining 1 and taking the norm closure gives a subalgebra isomorphic to $C_{\mathbf{R}}(X)$ (cf. Proposition 1.12). Then (iii) follows at once. □

1.48. Lemma. *Let a be an element of a JB-algebra A and p a projection in A. Then*

$$a \geq 0 \text{ operator commutes with } p \quad \Leftrightarrow \quad U_p a \leq a. \tag{1.62}$$

Proof. If $a \geq 0$ and p is a projection which operator commutes with a, then $U_p a + U_{p'} a = a$, so $U_p a \leq a$. Conversely, if $a \geq 0$ and $U_p a \leq a$, then $a - U_p a$ is a positive element such that $U_p(a - U_p a) = 0$. By (1.49), $U_{p'}(a - U_p a) = (a - U_p a)$. Thus $U_p a + U_{p'} a = a$ and so p and a operator commute by Proposition 1.47. □

We now observe that operator commutativity and ordinary commutativity coincide in the context of Jordan operator algebras. Recall that a JC-algebra is a norm closed Jordan subalgebra of $\mathcal{B}(H)_{\mathrm{sa}}$.

1.49. Proposition. *If a, b are elements of a JC-algebra A, then a, b commute iff a, b operator commute.*

Proof. Let $A \subset \mathcal{B}(H)_{\mathrm{sa}}$ be a JC-algebra. If a and b commute, it is straightforward to verify from the definition of the Jordan product that $a \circ (b \circ x) = b \circ (a \circ x)$ for all x, i.e., that a and b operator commute. Conversely, suppose that a and b operator commute. Let T_a denote Jordan multiplication by a on $\mathcal{B}(H)_{\mathrm{sa}}$. Then one easily verifies the following identity relating commutators of operators and elements:

$$[T_a, T_b](x) = \tfrac{1}{4}\big[[a, b], x\big].$$

Thus if a and b operator commute, then $[a, b]$ commutes with every element in A, and thus also with every element of the C*-subalgebra $\mathcal{A}$ of $\mathcal{B}(H)$ generated by A. Thus $[a, b]$ is central in $\mathcal{A}$. Let π be an irreducible representation of $\mathcal{A}$. Then $\pi([a, b]) = [\pi(a), \pi(b)]$ is central in $\pi(\mathcal{A})$, and thus is in the commutant of $\pi(\mathcal{A})$. By irreducibility, the commutant equals $\mathbf{C}1$ (A 80), so $\pi([a, b])$ is a multiple of the identity. But it is well known that the identity is not equal to the commutator of bounded operators ([80, 3.2.9]), so we must have $\pi([a, b]) = 0$. Since this is true for every irreducible representation of $\mathcal{A}$, and irreducible representations of $\mathcal{A}$ separate elements of $\mathcal{A}$ (A 82), then $[a, b] = 0$, and so a and b commute. (An alternative conclusion to this proof can also be based on the Kleinecke–Shirokov theorem [63, p. 128], which says that a commutator of two elements of a Banach algebra that commutes with one of the elements is quasi-nilpotent, i.e., its spectral radius is zero. Since $i[a, b]$ is self-adjoint, if it is quasi-nilpotent it must be zero.) □

1.50. Corollary. *If $\mathcal{A}$ is a C*-algebra, then the JB-algebra $\mathcal{A}_{\mathrm{sa}}$ is associative iff $\mathcal{A}$ is abelian.*

Proof. If $\mathcal{A}$ is abelian, then the Jordan product coincides with the ordinary product and thus is associative. If $\mathcal{A}_{\mathrm{sa}}$ is associative, then by associativity and commutativity of the Jordan product

$$a \circ (b \circ c) = (b \circ c) \circ a = b \circ (c \circ a) = b \circ (a \circ c)$$

for all $a, b, c \in \mathcal{A}_{\mathrm{sa}}$. Thus $T_a T_b = T_b T_a$ for all $a, b \in \mathcal{A}_{\mathrm{sa}}$, i.e., all elements of $\mathcal{A}_{\mathrm{sa}}$ operator commute. By Proposition 1.49 all elements of $\mathcal{A}_{\mathrm{sa}}$ commute. It follows that $\mathcal{A}$ is abelian. □

1.51. Definition. The *center* of a JB-algebra A consists of all elements that operator commute with every element of A. An element $z \in A$ is *central* if it is in the center. We will usually denote the center of A by $Z(A)$ or just by Z.

1.52. Proposition. *The center Z of a JB-algebra A is an associative JB-subalgebra containing the identity* 1.

Proof. To see that Z is closed under the Jordan product, let z_1 and z_2 be in Z and $x \in A$. Then

$$T_{z_1 \circ z_2} x = (z_1 \circ z_2) \circ x = x \circ (z_1 \circ z_2) = T_x(T_{z_1} T_{z_2} 1) = T_{z_1} T_{z_2} T_x 1 = T_{z_1} T_{z_2} x,$$

so $T_{z_1 \circ z_2} = T_{z_1} T_{z_2}$. Since T_{z_1} and T_{z_2} commute with every T_x, then so does their product $T_{z_1 \circ z_2}$. Hence $z_1 \circ z_2$ operator commutes with every element of A, and thus is in the center.

Finally, Z is associative because for z_1, z_2, and z_3 in Z:

$$z_1 \circ (z_2 \circ z_3) = T_{z_1} T_{z_2} T_{z_3} 1 = T_{z_3} T_{z_1} T_{z_2} 1 = z_3 \circ (z_1 \circ z_2) = (z_1 \circ z_2) \circ z_3.$$

By continuity of Jordan multiplication, Z is norm closed. □

For z in the center of a JB-algebra A and $a \in A$, we will write za or az instead of $z \circ a$. (As motivation, note that in a Jordan operator algebra context, by Proposition 1.49, z and a will commute so the Jordan product $z \circ a$ will coincide with the usual associative product za.) We then have for a and b in A and a central element z,

$$z(a \circ b) = (za) \circ b = a \circ (zb), \tag{1.63}$$

and

$$zU_a b = U_a(zb). \tag{1.64}$$

If a is an arbitrary element of A, and z_1, z_2 are central elements, then by (1.63)

$$z_1(z_2 a) = (z_1 z_2) a.$$

We will omit the parentheses and just write $z_1 z_2 a$.

If c is a central projection, then from the definition of the triple product together with (1.63)

$$U_c a = ca, \tag{1.65}$$

so $a \mapsto ca$ is a positive map. By (1.63) we also have

$$c(a \circ b) = (ca) \circ (cb). \tag{1.66}$$

Thus $U_c : a \mapsto ca$ is a Jordan homomorphism.

Recall that the center of a C*-algebra $\mathcal{A}$ consists of those elements in $\mathcal{A}$ that commute with all elements in $\mathcal{A}$.

1.53. Corollary. *If $\mathcal{A}$ is a C^*-algebra, then the center of the JB-algebra $\mathcal{A}_{\mathrm{sa}}$ consists of the self-adjoint elements of the center of $\mathcal{A}$.*

Proof. Since the self-adjoint elements span $\mathcal{A}$, an element $z = z^*$ is in the center of $\mathcal{A}$ iff z commutes with all self-adjoint elements of $\mathcal{A}$. By Proposition 1.49, this occurs iff z operator commutes with all elements in the JB-algebra $\mathcal{A}_{\mathrm{sa}}$, i.e., iff z is in the center of the JB-algebra $\mathcal{A}_{\mathrm{sa}}$. □

Order derivations on JB-algebras

We will now discuss an order-theoretic concept of derivation which was introduced by Connes [36]. We defined it in the general context of ordered Banach spaces in [AS, Chpt. 1], but for the reader's convenience we will give the definition here for JB-algebras. We begin with some motivating remarks.

Derivations occur in many different contexts. What is common for the various notions of a derivation δ is the fact that δ generates a one-parameter group of maps $e^{t\delta}$ which preserve the structure under study. In many algebraic contexts where the relevant structure is a bilinear product, this is equivalent to the Leibniz rule, which is often taken as the definition of a derivation. When the relevant structure is order-theoretic, the Leibniz rule is not meaningful, and so we are led to the following definition.

1.54. Definition. A bounded linear operator δ on a JB-algebra A is called an *order derivation* if $e^{t\delta}(A^+) \subset A^+$ for all $t \in \mathbf{R}$, or which is equivalent, if $\{e^{t\delta}\}_{t\in\mathbf{R}}$ is a one-parameter group of order automorphisms.

1.55. Proposition. *A bounded linear operator δ on a JB-algebra A is an order derivation iff the following implication holds for all $a \in A^+$ and $\rho \in (A^*)^+$:*

$$\rho(a) = 0 \ \Rightarrow\ \rho(\delta a) = 0. \tag{1.67}$$

Proof. This is an immediate consequence of (A 67). □

If A is the self-adjoint part of a C*-algebra $\mathcal{A}$, then for each $d \in \mathcal{A}$ we define a linear operator δ_d on A by

$$\delta_d(x) = \tfrac{1}{2}(dx + xd^*) \tag{1.68}$$

for $x \in A$. Note that $\delta_d(x) = (\delta_d(x))^*$. If A is the self-adjoint part of a von Neumann algebra $\mathcal{M}$, then by (A 183) the order derivations are exactly the maps δ_d for $d \in \mathcal{M}$.

Recall that we denote the Jordan multiplication determined by an element b of a JB-algebra A by T_b.

1.56. Lemma. *If A is a JB-algebra, then for each $b \in A$, T_b is an order derivation.*

Proof. Suppose $\rho \in (A^*)^+$, $a \in A^+$, and $\rho(a) = 0$. By the Cauchy–Schwarz inequality for JB-algebras (1.53),

$$\|\rho(T_b a)\|^2 = \|\rho(b \circ a)\|^2 \leq \rho(b^2)\rho(a^2). \tag{1.69}$$

By spectral theory, $a^2 \leq \|a\|a$. Now it follows from (1.69) that

$$\|\rho(T_b a)\|^2 \leq \rho(b^2)\|a\|\rho(a) = 0.$$

By Proposition 1.55, T_b is an order derivation. □

1.57. Definition. An order derivation δ on a JB-algebra A is *self-adjoint* if $\delta = T_a$ for some $a \in A$, and it is *skew-adjoint* (or just *skew*) if $\delta(1) = 0$.

We will see in the next chapter that the skew order derivations on a JB-algebra are exactly the *Jordan derivations*, i.e., the bounded linear operators δ which satisfy the Leibniz rule

$$\delta(a \circ b) = (\delta a) \circ b + a \circ (\delta b). \tag{1.70}$$

Note that not all order derivations on a JB-algebra satisfy the Leibniz rule. In fact, (1.70) implies $\delta(1) = 0$, and this is not true for any self-adjoint order derivation T_a with $a \neq 0$.

1.58. Proposition. *Let A be a JB-algebra with state space K and let δ be an order derivation on A. For every $t \in \mathbf{R}$, let $\alpha_t = e^{t\delta}$ and let α_t^* be the dual map defined on A^* by $(\alpha_t^* \rho) = \rho(\alpha_t(a))$ for $\rho \in A^*$ and $a \in A$. Then the following are equivalent:*

(i) *δ is skew,*
(ii) *$\alpha_t(1) = 1$ for all t,*
(iii) *$\alpha_t^*(K) \subset K$ for all t.*

Proof. (i)$\Rightarrow$(ii) Use the exponential series for $e^{t\delta}$.

(ii)$\Leftrightarrow$(iii) This follows immediately from the definition of states and of an order derivation.

(ii)$\Rightarrow$(i) This follows by differentiating the equality $\exp(t\delta)1 = 1$ at $t = 0$. □

Note that (iii) of Proposition 1.58 says that an order derivation δ of a JB-algebra A with state space K is skew iff the members of the dual one-parameter group $t \mapsto \exp(t\delta^*)$ are affine automorphisms of K.

1.59. Proposition. *The set $D(A)$ of order derivations of a JB-algebra A is a norm closed real linear space closed under the Lie product $[\delta_1, \delta_2] = \delta_1\delta_2 - \delta_2\delta_1$.*

Proof. By (A 68), the set of order derivations is a real linear space, which is closed under the Lie product. It is norm closed by Proposition 1.55. □

1.60. Proposition. *Every order derivation δ on a JB-algebra A can be decomposed uniquely as the sum of a self-adjoint and a skew derivation, namely $\delta = T_a + \delta'$ where $a = \delta(1)$ and $\delta' = \delta - T_a$.*

Proof. By the definition above, $\delta'(1) = \delta(1) - T_a(1) = a - a \circ 1 = 0$, so $\delta = T_a + \delta'$ is a decomposition of the desired type. If $\delta = T_b + \delta''$ is another such decomposition, then evaluation at 1 gives $\delta(1) = b \circ 1 = b$, so $b = a$. Therefore the decomposition is unique. □

We will investigate skew order derivations further in the next chapter on JBW-algebras. Later, when we study conditions that guarantee that a JB-algebra is the self-adjoint part of a C*-algebra, a key role will be played by skew order derivations.

Appendix: Proof of Proposition 1.3

Proposition 1.3. *Let A be a (not necessarily associative) commutative algebra over $\mathbf{R}$. Then A is power associative iff it satisfies the identity $x(xx^2) = x^2x^2$ or the linearized version (1.4).*

Proof. Assume A satisfies $x(xx^2) = x^2x^2$. To prove A is power associative, we follow the proof in [3]. Recall that powers of x are defined recursively: $x^1 = x$ and $x^{n+1} = x^n x$. We must prove that

$$x^j x^k = x^{j+k}. \tag{1.71}$$

We will prove this by induction on $n = j + k$. By the definition of powers and (1.3) this holds for $n \leq 4$. Suppose that it holds for $j + k < n$ where $n \geq 5$. Substituting $x_1 = x^\alpha$, $x_2 = x^\beta$, $x_3 = x^\gamma$, $x_4 = x^\delta$ into (1.4) where $\alpha + \beta + \gamma + \delta = n$ and $1 < \alpha, \beta, \gamma, \delta$ gives

$$\begin{aligned} &6(x^\alpha x^{n-\alpha} + x^\beta x^{n-\beta} + x^\gamma x^{n-\gamma} + x^\delta x^{n-\delta}) \\ &\qquad = 8(x^{\alpha+\beta}x^{\gamma+\delta} + x^{\alpha+\gamma}x^{\beta+\delta} + x^{\alpha+\delta}x^{\beta+\gamma}). \end{aligned} \tag{1.72}$$

Substituting $\alpha = \beta = \gamma = 1$, $\delta = n - 3$ into (1.72) and simplifying gives

$$x^3 x^{n-3} = 4x^2 x^{n-2} - 3x^n \text{ for } n \geq 5. \tag{1.73}$$

For $n = 5$ this reduces to

(1.74) $$x^3 x^2 = x^5.$$

Thus (1.71) holds for $j + k \leq 5$ and so we will assume hereafter that $n \geq 6$.

Substituting $\alpha = 2$, $\beta = \gamma = 1$, $\delta = n - 4$ in (1.72) gives

(1.75) $$3x^4 x^{n-4} = 8x^3 x^{n-3} + x^2 x^{n-4} - 6x^n.$$

Substituting (1.73) into (1.75) gives

(1.76) $$x^4 x^{n-4} = 11x^2 x^{n-2} - 10x^n.$$

Next substitute $\alpha = \beta = 2$, $\gamma = 1$, $\delta = n - 5$ in (1.72); this gives

(1.77) $$3x^5 x^{n-5} = 4x^4 x^{n-4} + 8x^3 x^{n-3} - 6x^2 x^{n-2} - 3x^n.$$

Substituting $\alpha = 3$, $\beta = \gamma = 1$, $\delta = n - 5$ in (1.72) gives

(1.78) $$3x^5 x^{n-5} = 8x^4 x^{n-4} - 3x^3 x^{n-3} + 4x^2 x^{n-2} - 6x^n.$$

Combining (1.77) and (1.78) gives

(1.79) $$4x^4 x^{n-4} = 11x^3 x^{n-3} - 10x^2 x^{n-2} + 3x^n.$$

Now substituting (1.73) and (1.76) into (1.79) gives

(1.80) $$x^2 x^{n-2} = x^n.$$

Now we complete the proof that (1.71) holds for $j + k = n$ by establishing

(1.81) $$x^k x^{n-k} = x^n \text{ for } 1 \leq k < n$$

by a separate induction on k for this fixed n. Note that by (1.80), the identity (1.81) holds for $k = 1, 2, n-2, n-1$. Suppose that it holds for all $k \leq p + 1$ with $p \leq n - 3$. Then substituting $\alpha = p$, $\beta = \gamma = 1$, $\delta = n - p - 2$ in (1.72) we get

(1.82) $$3x^{p+2} x^{n-p-2} = 8x^{p+1} x^{n-p-1} - 3x^p x^{n-p} + 4x^2 x^{n-2} - 6x^n.$$

By our latest induction hypothesis we conclude that the right side of (1.82) is equal to $3x^n$, and thus that $x^{p+2} x^{n-p-2} = x^n$. By induction, we conclude that (1.81) holds for all $k < n$. This also completes our original induction and thus finishes the proof. □

Notes

The finite dimensional "formally real" Jordan algebras (which are the same as finite dimensional JB-algebras, cf. [67, Cor. 3.1.7 and Cor. 3.3.8]), were introduced and classified by Jordan, von Neumann, and Wigner [75]. A Jordan algebra over $\mathbf{R}$ is *formally real* if

$$a_1^2 + a_2^2 + \cdots + a_n^2 = 0 \quad \Rightarrow \quad a_1 = a_2 = \cdots = a_n = 0.$$

Concrete Jordan algebras of self-adjoint operators on a Hilbert space (JC-algebras and JW-algebras) were studied by Effros and Størmer in [48], by Størmer in [124, 125, 126], and by Topping in [128, 129]. The definition of a JB-algebra was first given by Alfsen, Shultz, and Størmer [8], in a paper which gave a Jordan analog of the Gelfand–Naimark theorem (A 57), cf. Theorem 4.19.

The norm axioms that define a JB-algebra are a natural combination of the axioms for a C*-algebra (specialized to self-adjoint elements) together with the additional condition $\|a^2\| \leq \|a^2 + b^2\|$. This last condition is similar to one used by Segal [114] in a paper on axioms for quantum mechanics, and also appears in Arens' [18] abstract characterization of $C_{\mathbf{R}}(X)$. This can be thought of as a way of saying that elements should behave like self-adjoint operators (or like real numbers). In finite dimensions the same role is played by the "formally real" condition of [75]. However, this is not strong enough in infinite dimensions, as illustrated by the subalgebra of the disk algebra consisting of functions that are real-valued on the real axis. (See the discussion following Proposition 1.12). We have defined our JB-algebras to include an identity. If there is no identity, it is always possible to adjoin an identity, and to extend the product and norm to get a JB-algebra, as was shown in [25] and in [120].

Most of the results in this chapter can be found in [8]. We have generally followed the presentation there, except for perhaps a more order-theoretic flavor. We have made use of the notion of an order unit algebra (Definition 1.9), which was studied in [AS], and which was introduced by Iochum and Loupias under the name "Banach power-associative algebra" [69, 70]. The proof that Jordan algebras are power associative relies on Proposition 1.3 (proved in the appendix to this chapter), which is due to Albert [3]. We have followed the proofs in [67] for the construction of an approximate identity (Lemmas 1.30, 1.31, 1.32). The fact that in a JC-algebra, operator commutativity is the same as ordinary commutativity (Proposition 1.49) can be found in [66].

Order derivations were first defined by Connes [36]. The characterization of order derivations in Proposition 1.55 was first proved by Connes in the context of self-dual cones, then generalized to a wider class of ordered Banach spaces by Evans and Hanche-Olsen [50]. Similarly, the result that order derivations form a Lie algebra (Proposition 1.59) is in [36] and [50]. A more complete discussion of order derivations is given in [AS, Chpts. 1 and 6]. The notion of self-adjoint and skew order derivations was introduced by Alfsen and Shultz in [12], and is further discussed in Chapter 6.

There are important connections of JB-algebras with infinite dimensional holomorphy. For example, there is a 1-1 correspondence between symmetric tube domains and JB-algebras. For a brief discussion, see [67, pp. 92-93]. A more complete discussion can be found in the survey article of Kaup [82].

JB-algebras are in 1-1 correspondence with JB*-algebras. A JB*-algebra is a complex Jordan algebra with involution and complete norm such that $\|x \circ y\| \leq \|x\| \, \|y\|$, $\|x^*\| = \|x\|$, and $\|\{xx^*x\}\| = \|x\|^3$. The self-adjoint part of a JB*-algebra is a JB-algebra, cf. [67, Prop. 3.8.2], and the complexification of a JB-algebra is a JB*-algebra, a result due to J. D. M. Wright [137].

The authoritative work on JB-algebras is the book of Hanche-Olsen and Størmer [67]. Another good source of information is Upmeier's survey [130]. For general Jordan algebras, some sources are [33], [72], and [113].

2 JBW-algebras

JBW-algebras are the Jordan analogs of von Neumann algebras (also known as W*-algebras), and include the self-adjoint part of von Neumann algebras as a special case. In this chapter we will develop basic facts about JBW-algebras. We begin with the definition and the relevant topologies. Then we introduce an abstract notion of range projection, and a spectral theorem for JBW-algebras, derived from the spectral theorem for monotone complete $C_{\mathbf{R}}(X)$ (A 39). We then study the lattice of projections of a JBW-algebra, and establish a 1-1 correspondence of projections and σ-weakly closed hereditary subalgebras, and a correspondence of σ-weakly closed ideals with central projections. We show that the bidual of a JB-algebra is a JBW-algebra. Then we use the bidual to prove that JB-algebras are the same as commutative order unit algebras, to show that a unital order isomorphism of a JB-algebra is a Jordan isomorphism, and to show that skew order derivations are in fact Jordan derivations. We prove that every JBW-algebra has a unique predual consisting of the normal linear functionals. Then we develop some basic facts about JW-algebras (σ-weakly closed subalgebras of $\mathcal{B}(H)_{\mathrm{sa}}$), and we finish this chapter with an order-theoretic characterization of the maps $a \mapsto \{pap\}$ for projections p.

Introduction

2.1. Definition. An ordered Banach space A is *monotone complete* if each increasing net $\{b_\alpha\}$ which is bounded above has a least upper bound b in A. (We write $b_\alpha \nearrow b$ for such a net.) A bounded linear functional σ on a monotone complete space A is *normal* if whenever $b_\alpha \nearrow b$, then $\sigma(b_\alpha) \to \sigma(b)$.

2.2. Definition. A *JBW-algebra* is a JB-algebra that is monotone complete and admits a separating set of normal states.

In [AS], normal positive linear functionals on a von Neumann algebra were defined first (A 90), and then general normal linear functionals were defined to be linear combinations of normal positive linear functionals. We will see later (cf. Proposition 2.52) that every normal linear functional on a JBW-algebra is the difference of positive normal linear functionals, so the definitions of normal linear functionals on von Neumann algebras and on JBW-algebras are consistent.

We will call the set of normal states on a JBW-algebra M the *normal state space*, and denote it by K. We will let V denote the linear span of K in M^*.

The definition above is motivated by Kadison's intrinsic characterization of von Neumann algebras as those C*-algebras whose self-adjoint parts are monotone complete and admit a separating set of normal states (A 95). Thus the self-adjoint part of a von Neumann algebra with the usual Jordan product is an example of a JBW-algebra.

2.3. Definition. Let M be a JBW-algebra, K its normal state space, and V the linear span of K. We will call the topology on M defined by the duality of M and V the *σ-weak* topology. Thus the σ-weak topology is defined by the set of seminorms $a \mapsto |\rho(a)|$ for $\rho \in K$. The *σ-strong* topology on M is defined by the set of seminorms $a \mapsto \rho(a^2)^{1/2}$ for $\rho \in K$. Thus $a_\alpha \to a$ σ-strongly if $\rho((a_\alpha - a)^2) \to 0$ for all $\rho \in K$.

For the special case of the self-adjoint parts of von Neumann algebras, the σ-weak and σ-strong topologies as defined above coincide with the same terms in the von Neumann algebra context, cf. (A 93). What we have called the σ-weak and σ-strong topologies on a JBW-algebra are referred to as the weak and strong topologies respectively in [67].

Recall that for an element a of a JB-algebra A, T_a denotes Jordan multiplication and U_a is the map $U_a : b \mapsto \{aba\}$ (cf. (1.13)). If A^* is the dual space of A, and $T : A \to A$ is a bounded linear map, then $T^* : A^* \to A^*$ is defined by $(T^*\sigma)(b) = \sigma(Tb)$ for $b \in A$ and $\sigma \in A^*$. In the context of JBW-algebras, we will now show that U_a^* and T_a^* map V into V.

2.4. Proposition. *Jordan multiplication on a JBW-algebra M is σ-strongly and σ-weakly continuous in each variable separately, and is jointly σ-strongly continuous on bounded subsets. For each $a \in M$, U_a is σ-strongly and σ-weakly continuous, and T_a^* and U_a^* map V into V, and map normal linear functionals to normal linear functionals.*

Proof. We first prove σ-weak continuity of U_a and T_a. Note that if a is invertible, then by Lemma 1.23 and Theorem 1.25 the map U_a is an order isomorphism. Thus if a is invertible, then for each normal state σ on M, $b \mapsto \sigma(U_a b)$ is positive and normal, so $U_a^*\sigma$ is a multiple of a normal state. Thus U_a^* maps V into V when a is invertible.

For arbitrary a in M, choose $\lambda > \|a\|$. Then by spectral theory $\lambda 1 - a$ and $\lambda 1 + a$ are invertible, and from the linearity of the triple product (1.12) in each factor we have

$$U_a = \tfrac{1}{2}(U_{\lambda 1 + a} + U_{\lambda 1 - a} - 2\lambda^2 I),$$

where I is the identity map on A. It follows that U_a^* maps V into V for every a, and so U_a is σ-weakly continuous.

The identity relating the maps T_a and U_a,

$$T_a = \tfrac{1}{2}(U_{1+a} - U_a - I), \tag{2.1}$$

also follows from the linearity of the triple product in each factor. This implies that each map T_a is σ-weakly continuous, and that T_a^* maps V into V. The arguments just given, with trivial changes, also show that U_a^* and T_a^* take normal linear functionals to normal linear functionals.

To prove σ-strong continuity of U_a, suppose that $b_\alpha \to 0$ σ-strongly and fix a in M. Then by definition ${b_\alpha}^2 \to 0$ σ-weakly, so by the first part of this proof $\{a{b_\alpha}^2 a\} \to 0$ σ-weakly. Then using the identity (1.15) and $a^2 \le \|a^2\|1$ we have

$$\{ab_\alpha a\}^2 = \{a\{b_\alpha a^2 b_\alpha\}a\} \le \|a^2\|\{a{b_\alpha}^2 a\},$$

which proves that $\{ab_\alpha a\}^2 \to 0$ σ-weakly, and so $\{ab_\alpha a\} \to 0$ σ-strongly. Thus U_a is σ-strongly continuous, and by (2.1) so is T_a.

Finally, we prove multiplication is jointly σ-strongly continuous on bounded subsets. Note that from spectral theory, $a^4 \le \|a^2\|a^2$ (cf. (1.27)). Thus if $\{a_\alpha\}$ is a bounded net converging σ-strongly to zero, and σ is a normal state, then

$$\sigma(a_\alpha^4) \le \|a_\alpha^2\|\sigma(a_\alpha^2) \to 0.$$

Therefore the squaring map is σ-strongly continuous at zero on bounded sets. From $a \circ b = \frac{1}{2}((a+b)^2 - a^2 - b^2)$ it follows that multiplication is jointly σ-strongly continuous on bounded sets at zero. Finally, if $\{a_\alpha\}$ and $\{b_\alpha\}$ are bounded nets with $a_\alpha \to a$ and $b_\alpha \to b$ σ-strongly, then it follows from joint continuity of multiplication on bounded sets at zero and separate continuity (proven above), together with the equality

$$a \circ b - a_\alpha \circ b_\alpha = (a - a_\alpha) \circ b + (a_\alpha - a) \circ (b - b_\alpha) + a \circ (b - b_\alpha),$$

that $a_\alpha \circ b_\alpha \to a \circ b$ σ-strongly. □

2.5. Proposition. *Let M be a JBW-algebra. Then*

(i) *σ-strong convergence implies σ-weak convergence.*
(ii) *A bounded monotone net converges σ-strongly.*
(iii) *A monotone net of projections converges σ-strongly to a projection.*

Proof. (i) Let $\{a_\alpha\}$ be a net in M which converges σ-strongly to a. By the Cauchy–Schwarz inequality (1.53), for each normal state σ,

$$|\sigma(a_\alpha - a)| \le \sigma(1)\sigma\big((a_\alpha - a)^2\big)^{1/2},$$

and so $a_\alpha \to a$ σ-weakly.

(ii) If $\{a_\alpha\}$ is an increasing net with least upper bound a, then for normal states σ by (1.27) we have

$$\sigma((a-a_\alpha)^2) \leq \|a-a_\alpha\|\sigma(a-a_\alpha) \to 0.$$

Thus $a_\alpha \to a$ σ-strongly.

(iii) By part (ii), a monotone net converges σ-strongly. Now σ-strong joint continuity of multiplication on bounded subsets (Proposition 2.4) implies that the σ-strong limit of projections will be a projection. □

It also follows from joint σ-strong continuity of multiplication on bounded subsets and Proposition 2.5 that for a monotone net of elements $\{a_\alpha\}$ and any element b in a JBW-algebra we have

$$a_\alpha \nearrow a \quad \Rightarrow \quad U_{a_\alpha} b \to U_a b \ (\sigma\text{-strongly}), \tag{2.2}$$

$$a_\alpha \searrow a \quad \Rightarrow \quad U_{a_\alpha} b \to U_a b \ (\sigma\text{-strongly}). \tag{2.3}$$

2.6. Corollary. *Every σ-weakly continuous linear functional σ on a JBW-algebra M is normal.*

Proof. If $a_\alpha \nearrow a$, then by Proposition 2.5, $a_\alpha \to a$ σ-weakly, so $\sigma(a_\alpha) \to \sigma(a)$. Thus σ is normal. □

Later (in Proposition 2.52) we will prove that normal linear functionals are differences of positive normal linear functionals, from which the converse of Corollary 2.6 will follow (Corollary 2.56).

2.7. Proposition. *Let B be a σ-weakly closed Jordan subalgebra of a JBW-algebra M. Then B has an identity and is a JBW-algebra.*

Proof. If $\{b_\alpha\}$ is an ascending net in B that is bounded above, let $b \in M$ be its supremum in M. By Proposition 2.5, $b_\alpha \to b$ σ-weakly, so $b \in B$. Thus b is the supremum of $\{b_\alpha\}$ in B, which proves that B is monotone complete. Since norm convergence implies σ-weak convergence, B is norm closed as well as σ-weakly closed. We next show B has an identity. Let $\widetilde{B}$ be the linear span of B and 1. Then $\widetilde{B}$ is a norm closed subalgebra of M, and thus is a JB-algebra containing B as a norm closed ideal. By Lemma 1.32, B contains an increasing approximate identity $\{v_\alpha\}$. Let v be the least upper bound of this increasing net in M (and then also in B). Then $v_\alpha \circ a \to a$ in norm, while $v_\alpha \circ a \to v \circ a$ σ-weakly. Thus $v \circ a = a$, i.e., v is an identity for B. Finally, the normal states on M restrict to multiples of normal states on B, and so B has a separating set of normal states. □

2.8. Definition. If M is a JBW-algebra, a *JBW-subalgebra* is a σ-weakly closed Jordan subalgebra N of M. (Note that by Proposition 2.7 N will itself be a JBW-algebra.)

Recall that if p is a projection, then $U_p^2 = U_p$, and $U_p U_{p'} = U_{p'} U_p = 0$ (Proposition 1.38).

2.9. Proposition. *If p is a projection in a JBW-algebra M, then $M_p = U_p M$ is a JBW-subalgebra of M with identity p.*

Proof. By Proposition 1.43, M_p is a JB-subalgebra of M with identity p. It follows that $M_p = \{a \in M \mid U_p a = a\}$. Now σ-weak continuity of U_p implies that M_p is σ-weakly closed. □

2.10. Corollary. *If p is a projection in a JBW-algebra M, then $M_p + M_{p'} = \operatorname{im}(U_p + U_{p'})$ is a JBW-subalgebra of M.*

Proof. Since $M_p \circ M_{p'} = \{0\}$ (Lemma 1.45), then $M_p + M_{p'}$ is a Jordan subalgebra of M. Thus

$$M_p + M_{p'} = \operatorname{im} U_p + \operatorname{im} U_{p'} = \{a \in M \mid (U_p + U_{p'})a = a\},$$

so this subspace is σ-weakly closed as claimed. □

If a is an element in a JBW-algebra M, we will denote by $W(a,1)$ the σ-weakly closed subalgebra generated by a and 1.

2.11. Proposition. *If B is an associative Jordan subalgebra of a JBW-algebra M, then the σ-weak closure of B is an associative JBW-algebra which is isomorphic to a monotone complete $C_{\mathbf{R}}(X)$. In particular, this holds for $W(a,1)$.*

Proof. By separate σ-weak continuity of multiplication, the σ-weak closure of B is a σ-weakly closed subalgebra of M, and thus by Proposition 2.7 is a JBW-algebra. A simple argument using separate continuity of Jordan multiplication shows that the σ-weak closure of B is associative. By Proposition 1.12, an associative JBW-algebra is isomorphic to $C_{\mathbf{R}}(X)$ for some compact Hausdorff space X. Since a JBW-algebra is monotone complete, so is $C_{\mathbf{R}}(X)$. □

2.12. Corollary. *If a projection p in a JBW-algebra M operator commutes with an element a, then p operator commutes with every element in $W(a,1)$.*

Proof. By Proposition 1.47, p operator commutes with an element x iff x is in $M_p + M_{p'}$. If p operator commutes with a, then by Corollary

2.10, $M_p + M_{p'}$ is a σ-weakly closed Jordan subalgebra of M containing a and 1. It therefore contains $W(a,1)$. Thus for every $x \in W(a,1)$, x is in $M_p + M_{p'}$, and so p operator commutes with x. $\square$

Orthogonality and range projections

If E is a subset of a set X, then χ_E will denote the characteristic function of E, i.e., $\chi_E(x) = 1$ for $x \in X$ and $\chi_E(x) = 0$ for $x \in X \setminus E$. If p is a projection in a JB-algebra A, recall from Lemma 1.39:

$$\mathrm{face}(p) = \mathrm{im}^+ U_p.$$

We will use this frequently hereafter without further reference.

Recall also from Proposition 1.28 that each $a \in A$ has a unique orthogonal decomposition $a = a^+ - a^-$ where a^+, a^- are in A^+.

2.13. Proposition. *For every element a of a JBW-algebra M there is a unique projection p in $\overline{W(a,1) \cap \mathrm{face}(a^+)}$ (σ-weak closure) such that $U_p a \geq a$ and $U_p a \geq 0$. Furthermore, p is the smallest projection in M such that $U_p a^+ = a^+$.*

Proof. By Proposition 2.11, we can identify $W(a,1)$ with a monotone complete $C_{\mathbf{R}}(X)$. Let $E = \overline{\{x \in X \mid a(x) > 0\}}$. By (A 38), E is closed and open, $a \geq 0$ on E, $a \leq 0$ on $X \setminus E$, and $\chi_E a^+ = a^+$. Let $p = \chi_E$; then $p \in C_{\mathbf{R}}(X)$ and $p^2 = p$, so p is a projection in $C_{\mathbf{R}}(X)$ such that $U_p a^+ = a^+$.

Since $a \geq 0$ on E, then $pap \geq 0$, so $U_p a \geq 0$. Since $a \leq 0$ on $X \setminus E$, then $pap = \chi_E a \geq a$, so $U_p a \geq a$. Furthermore, by (A 38) there is an increasing sequence $\{a_n\}$ in $W(a,1) \cap \mathrm{face}(a^+)$ with supremum p (in $W(a,1)$). Let b be the supremum of $\{a_n\}$ in M. Then $a_n \to b$ σ-weakly by Proposition 2.5, and so b is in $W(a,1)$. Thus $b = p$ and $a_n \to p$ σ-weakly, which proves that $p \in \overline{W(a,1) \cap \mathrm{face}(a^+)}$.

To prove uniqueness, let q be any projection in $\overline{W(a,1) \cap \mathrm{face}(a^+)}$ such that $U_q a \geq a$ and $U_q a \geq 0$. Let $q = \chi_F$ for $F \subset X$, and set $G = X \setminus F$. Then

$$\chi_G a = (1 - \chi_F)a = a - \chi_F a = a - U_q a \leq 0$$

and

$$\chi_F a = U_q a \geq 0,$$

so

$$a \geq 0 \text{ on } F \quad \text{and} \quad a \leq 0 \text{ on } G.$$

By (A 38)(i), $E \subset F$, and so $p \leq q$. On the other hand, $U_p a^+ = a^+$, so $a^+ \in \text{face}(p)$. Thus

$$q \in \overline{\text{face}(a^+)} \subset \overline{\text{face}(p)} \subset \operatorname{im} U_p.$$

Then $q = U_p q \leq U_p 1 = p$, so $q \leq p$. Therefore $p = q$, proving uniqueness.

Finally, let q be any projection in M such that $U_q a^+ = a^+$. Then $a^+ \in \operatorname{im}^+ U_q = \text{face}(q)$, so

$$p \in \overline{\text{face}(a^+)} \subset \overline{\text{face}(q)} \subset \operatorname{im} U_q$$

and as above $p \leq q$ follows. Thus p is the least projection such that $U_p a^+ = a^+$. □

The conditions $U_p a \geq a$ and $U_p a \geq 0$ above will appear again in our axiomatic treatment of spectral theory in Part II.

2.14. Definition. Let a be a positive element of a JBW-algebra. The smallest projection p such that $U_p a = a$ is called the *range projection* of a and is denoted $r(a)$.

For later reference, we collect some elementary facts about $r(a)$.

2.15. Proposition. *Let a be a positive element of a JBW-algebra M with normal state space K. Then the range projection $r(a)$ satisfies the following properties.*

(2.4) $$r(a) \in W(a,1) \cap \overline{\text{face}(a)}.$$

(2.5) $$a \leq \|a\|\, r(a).$$

(2.6) $$\textit{If } \sigma \in K, \textit{ then } \quad \sigma(a) = 0 \iff \sigma(r(a)) = 0.$$

(2.7) $$a \leq b \quad \Rightarrow \quad r(a) \leq r(b).$$

(2.8) $$r(a) \leq p = p^2 \quad \Rightarrow \quad U_p a = a.$$

Proof. Property (2.4) follows from the definition of $r(a)$ and Proposition 2.13. Since $a \leq \|a\|\,1$, applying $U_{r(a)}$ to both sides gives (2.5). Property (2.6) follows from (2.4) and (2.5). Next, suppose $a \leq b$. By (2.5), $a \leq b \leq \|b\| r(b)$, so $a \in \text{face}(r(b)) = \operatorname{im}^+ U_{r(b)}$. Thus $U_{r(b)} a = a$, so $r(a) \leq r(b)$, which establishes (2.7). Finally, if p is a projection such that $r(a) \leq p$, then $a \in \text{face}(r(a)) \subset \text{face}(p)$, so $U_p a = a$. □

Let M be a JBW-algebra, and let W be a σ-weakly closed subalgebra of a JBW-algebra M containing the identity of M. If $0 \leq a \in W$, then $W(a,1) \subset W$, so by (2.4) $r(a) \in W$. Thus $r(a)$ will also be the least projection p in W such that $U_p a = a$, i.e., the notions of range projection in W and in M coincide.

Recall that positive elements a, b are orthogonal if $\{aba\} = 0$, and this is equivalent to $\{bab\} = 0$ by Lemma 1.26. We now characterize this notion in terms of range projections.

2.16. Proposition. *Let a, b be positive elements of a JBW-algebra M. Then $a \perp b$ iff $r(a) \perp r(b)$.*

Proof. If $r(a) \perp r(b)$, then by definition $U_{r(a)}r(b) = 0$, so using (2.5) gives

$$0 \leq U_{r(a)}b \leq \|b\| U_{r(a)}r(b) = 0.$$

Thus $U_{r(a)}b = 0$, which is equivalent to $U_b r(a) = 0$ (Lemma 1.26). Hence

$$0 \leq U_b a \leq \|a\| U_b r(a) = 0,$$

so $U_b a = 0$, i.e., $a \perp b$.

If $a \perp b$, then $U_a b = 0$ implies that U_a annihilates face(b), and therefore $U_a r(b) = 0$ by (2.4). Thus $a \perp b$ implies $a \perp r(b)$, and the same implication applied to the pair $r(b)$, a shows $r(b) \perp r(a)$. □

Now observe that if a and b are positive elements of a JBW-algebra such that $a \perp b$, then by Proposition 2.16, $r(a) \perp r(b)$, so $r(a) \perp b$, i.e., $U_r(a)b = 0$. This implies $U_{r(a)'}b = b$ (where $r(a)' = 1 - r(a)$), cf. equation (1.49). We record this observation for later reference:

$$(2.9) \qquad 0 \leq a, b \quad \text{and} \quad a \perp b \quad \Rightarrow \quad U_{r(a)}b = 0 \quad \text{and} \quad U_{r(a)'}b = b.$$

2.17. Corollary. *The normal states on a JBW-algebra M determine the order and the norm on M, i.e., for each $a \in M$,*

$$(2.10) \qquad a \geq 0 \iff \sigma(a) \geq 0 \textit{ for all } \sigma \in K, \textit{ and}$$

$$(2.11) \qquad \|a\| = \sup\{\, |\sigma(a)| \mid \sigma \in K \,\}.$$

Proof. Let $a \in M$ and suppose that $\sigma(a) \geq 0$ for all normal states σ. Let $a = a^+ - a^-$ be the orthogonal decomposition of a (Proposition 1.28). Suppose that a^- is not zero. Since by the definition of a JBW-algebra the normal states separate points, there is a normal state ω such that $\omega(r(a^-)) \neq 0$. By (2.6) $\omega(a^-) \neq 0$. Let $\tau = U^*_{r(a^-)}\omega$. Then $\tau(1) = (U^*_{r(a^-)}\omega)(1) = \omega(r(a^-)) \neq 0$, so τ is a non-zero positive σ-weakly continuous linear functional on M. By Corollary 2.6, τ is a multiple of a normal state. However, $U_{r(a^-)}(a^+) = 0$ (cf. (2.9)), so

$$\tau(a) = \omega(U_{r(a^-)}(a^+ - a^-)) = -\omega(a^-) < 0.$$

This contradicts our assumption that all normal states were positive on a, and thus we conclude that $a^- = 0$, and therefore $a \geq 0$. This proves (2.10).

We will now prove (2.11). Since the norm is the order unit norm (cf. (1.10)), and the normal states determine the order, then

$$|\sigma(a)| \leq 1 \text{ for all } \sigma \in K \iff -1 \leq a \leq 1 \iff \|a\| \leq 1,$$

which completes the proof. □

Note that by Corollary 2.17, M and V are in separating order and norm duality (A 21), and the positive cone M^+ is σ-weakly closed. Thus for each projection p in M, $\overline{\text{face}(p)} = \text{im}^+ U_p = \text{im}\, U_p \cap M^+$ is σ-weakly closed. If $0 \leq a$, then $r(a) \in \overline{\text{face}(a)}$ (cf. (2.4)) and $a \in \text{face}(r(a))$, so

$$\text{face}(r(a)) = \overline{\text{face}(a)} \quad (\sigma\text{-weak closure}). \tag{2.12}$$

2.18. Proposition. *Let p, q be projections in a JBW-algebra M. The following are equivalent.*

(i) $p \perp q$,
(ii) $p \circ q = 0$,
(iii) $p \leq q'$ *(where $q' = 1 - q$)*,
(iv) $p + q \leq 1$,
(v) $U_p U_q = 0$.

Proof. (i) $\Rightarrow$ (ii) This is part of Lemma 1.26.

(ii) $\Rightarrow$ (iii) If $p \circ q = 0$, then $(p+q)^2 = p + q$, so $p + q$ is a projection. Then $p + q \leq 1$ so $p \leq 1 - q$.

(iii) $\Leftrightarrow$ (iv) This is clear.

(iii) $\Rightarrow$ (v) If $p \leq 1 - q$, then $q \leq 1 - p$ so $U_p q \leq U_p(1-p) = 0$. Then for all $a \geq 0$,

$$U_p U_q a \leq U_p U_q(\|a\| 1) = \|a\| U_p q = 0.$$

Since every element of M is a difference of positive elements, $U_p U_q = 0$ follows.

(v) $\Rightarrow$ (i) If $U_p U_q = 0$, then $U_p q = 0$, which by definition means $p \perp q$. □

We will frequently make use of the equivalence of (i) and (iii) above without reference.

Spectral resolutions

2.19. Proposition. *If M is a JBW-algebra, a is any element of M, and p is a projection in M, then (i) and (ii) are equivalent:*

(i) $U_p a \geq a$ *and* $U_p a \geq 0$,

(ii) $U_p a \geq 0$, $U_{p'} a \leq 0$, *and p operator commutes with a.*

For each a in M, $r(a^+)$ is the smallest projection p satisfying (i) and (ii)

Proof. (i) $\Rightarrow$ (ii) Assume p satisfies (i). Then $U_p a - a \geq 0$. Since $U_p(U_p a - a) = 0$, we can use Proposition 1.38 (with $U_p a - a$ in place of a in (1.49)) to conclude that

$$U_{p'}(U_p a - a) = U_p a - a.$$

Hence $-U_{p'} a = U_p a - a$. From this we conclude that $a = U_p a + U_{p'} a$, so p operator commutes with a (Proposition 1.47). Also $U_{p'} a = a - U_p a \leq 0$, so (ii) holds.

(ii) $\Rightarrow$ (i) Suppose (ii) holds. Then $U_p a = a - U_{p'} a \geq a$, so (i) holds.

Finally, by Proposition 2.13, $p = r(a^+)$ satisfies (i). Let q be any projection satisfying (i) and (ii). Then $U_q a$ is positive and is fixed by U_q, so $r(U_q a) \leq q$ and similarly $r(-U_{q'} a) \leq q'$. Thus $U_q a$ and $-U_{q'} a$ are orthogonal (Proposition 2.16). Since a operator commutes with q, then $a = U_q a - (-U_{q'} a)$ (Proposition 1.47), so this equation must coincide with the unique orthogonal decomposition of a (Proposition 1.28). Thus $a^+ = U_q(a)$, so $r(a^+) \leq q$. □

We are now ready to state the spectral theorem. For each finite increasing sequence $\gamma = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ with $\lambda_0 < -\|a\|$ and $\lambda_n > \|a\|$, define

$$\|\gamma\| = \max_i \{\lambda_i - \lambda_{i-1}\}.$$

2.20. Theorem. (Spectral Theorem) *Let M be a JBW-algebra and let $a \in M$. Then there is a unique family $\{e_\lambda\}_{\lambda \in \mathbf{R}}$ of projections such that*

(i) *each e_λ operator commutes with a,*

(ii) $U_{e_\lambda} a \leq \lambda e_\lambda$, *and* $U'_{e_\lambda} a \geq \lambda e'_\lambda$,

(iii) $e_\lambda = 0$ *for* $\lambda < -\|a\|$, *and* $e_\lambda = 1$ *for* $\lambda > \|a\|$,

(iv) $e_\lambda \leq e_\mu$ *for* $\lambda < \mu$,

(v) $\bigwedge_{\mu > \lambda} e_\mu = e_\lambda$ *(greatest lower bound in M).*

The family $\{e_\lambda\}$ is given by $e_\lambda = 1 - r((a - \lambda 1)^+)$, and is contained in $W(a, 1)$. For each finite increasing sequence $\gamma = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ with

$\lambda_0 < -\|a\|$ *and* $\lambda_n > \|a\|$, *define*

$$s_\gamma = \sum_{i=1}^{n} \lambda_i (e_{\lambda_i} - e_{\lambda_{i-1}}). \tag{2.13}$$

Then

$$\lim_{\|\gamma\| \to 0} \|s_\gamma - a\| = 0. \tag{2.14}$$

Proof. We will apply the spectral theorem for monotone complete $C_{\mathbf{R}}(X)$ (A 39) to $W(a, 1)$. Note that by Proposition 2.5, increasing nets in M converge σ-weakly to their least upper bounds, so these least upper bounds are the same whether interpreted in $W(a, 1)$ or in M, and thus (v) is unambiguous. Furthermore, by the remarks following (2.4), the range projection of $(a - \lambda 1)^+$ calculated in $W(a, 1)$ and in M coincide, so $r((a - \lambda 1)^+)$ coincides with $r((a - \lambda 1)^+)$ as defined in (A 39).

Thus by (A 39), the family $\{e_\lambda\}$ defined in the theorem above satisfies (ii) through (v), and is the unique family in $W(a, 1)$ with this property. Since each e_λ is in the associative subalgebra $W(a, 1)$, each e_λ operator commutes with a (Proposition 1.47), which proves (i).

Finally, we prove uniqueness. Suppose that $\{e_\lambda\}$ is any family of projections in M satisfying (i) through (v). If $\lambda < \mu$, then $e_\lambda \leq e_\mu$, so $e_\lambda \in \text{face}(e_\mu)$; therefore $U_{e_\mu} e_\lambda = e_\lambda$. Thus e_μ and e_λ operator commute by (1.62). Then a and all of $\{e_\lambda\}$ mutually operator commute. From the definition of operator commutativity, it follows that a together with $\{e_\lambda\}$ generate an associative subalgebra, whose σ-weak closure is then isomorphic to a monotone complete $C_{\mathbf{R}}(X)$ (Proposition 2.11). Then by the uniqueness statement in (A 39) we must have $e_\lambda = 1 - r((a - \lambda)^+)$ which completes the proof of uniqueness in M. □

We will see in Part II (cf. Theorem 8.64) that a purely order-theoretic proof of the spectral theorem is possible, which relies solely on the following property established in Proposition 2.19:

For each $a \in M$, *there is a least projection* p *such that* $U_p a \geq a$ *and* $U_p a \geq 0$.

2.21. Definition. The family $\{e_\lambda\}$ in Theorem 2.20 is called the *spectral resolution* for a, and will be denoted by $\{e_\lambda^a\}$ when there is need to indicate the associated element a. We will call the e_λ's the *spectral projections* of a.

If we define the Stieltjes integral of a with respect to $\{e_\lambda\}$ as the norm limit of the approximating Riemann sums s_γ in (2.13), then we have shown

$$a = \int \lambda \, de_\lambda .$$

We note for later reference

$$e_0^a = 1 - r(a^+). \tag{2.15}$$

The lattices of projections and compressions

We let $\mathcal{P}(M)$ (or just $\mathcal{P}$) denote the set of projections of a JBW-algebra M, with the order inherited from M.

2.22. Lemma. *If p, q are projections in a JBW-algebra M, then $r(p+q)$ is the least upper bound of p and q in the set $\mathcal{P}$ of projections of M.*

Proof. Let $r = r(p+q)$. Then $p + q \in \operatorname{im}^+ U_r = \operatorname{face}(r)$, so p and q are in $\operatorname{face}(r) \cap [0,1] = [0,r]$ (using (1.51)), and thus $p \le r$ and $q \le r$. On the other hand, if e is any projection such that $e \ge p$ and $e \ge q$, then p and q and then $p+q$ are in $\operatorname{face}(e) = \operatorname{im}^+ U_e$, and so by the minimality of the range projection, $r \le e$. Thus r is the least upper bound of p and q in $\mathcal{P}$. □

Since $p \mapsto p' = 1 - p$ is an order reversing bijection, it follows from Lemma 2.22 that $\mathcal{P}$ is a lattice. For projections $p, q \in \mathcal{P}$, the lattice supremum is denoted by $p \vee q$ and the lattice infimum by $p \wedge q$.

2.23. Corollary. *If $p_1, \ldots, p_n$ are orthogonal projections in a JBW-algebra M, then $p_1 \vee \cdots \vee p_n = p_1 + \cdots + p_n$.*

Proof. By Lemma 2.22, $p_1 \vee p_2 = r(p_1 + p_2) = p_1 + p_2$. For $i > 2$, $p_1 \le p_i'$ and $p_2 \le p_i'$, so $p_1 + p_2 = p_1 \vee p_2 \le p_i'$. Thus $p_1 + p_2$, p_3, $\ldots$, p_n are orthogonal. Now the proof can be completed by induction. □

For the reader's convenience we recall the definition of a complete orthomodular lattice.

2.24. Definition. A lattice L is *complete* if every subset has a least upper bound. L is *orthomodular* if there is a map $p \mapsto p'$ that satisfies

(i) $p'' = p$,
(ii) $p \le q$ implies $p' \ge q'$,
(iii) $p \vee p' = 1$ and $p \wedge p' = 0$,
(iv) If $p \le q$, then $q = p \vee (q \wedge p')$.

The map $p \mapsto p'$ is called the *orthocomplementation.*

Note that (i) and (ii) imply that $p \mapsto p'$ is an order reversing bijection, so transforms greatest lower bounds to least upper bounds and vice versa. Thus if (i) and (ii) hold, then

$$(p \vee q)' = p' \wedge q' \quad \text{and} \quad (p \wedge q)' = p' \vee q'.$$

2.25. Proposition. *The set $\mathcal{P}$ of projections of a JBW-algebra is a complete orthomodular lattice, with orthocomplementation $p \mapsto p' = 1 - p$. If $\{p_\alpha\}$ is an increasing net of projections, then it converges σ-strongly to its least upper bound in the lattice $\mathcal{P}$.*

Proof. We will verify the properties (i) through (iv) of Definition 2.24. Properties (i) and (ii) are obvious.

(iii) Since $p \perp p'$, then $p \vee p' = p + p' = 1$ (Corollary 2.23). Then $p \wedge p' = (p' \vee p)' = 1' = 0$, which proves (iii).

(iv) Suppose $p \le q$. Then $p \le (q')'$ implies $p \perp q'$, so

$$q \wedge p' = (q' \vee p)' = (q' + p)' = 1 - (q' + p) = q - p. \tag{2.16}$$

Since $q \wedge p' \le p'$, then $q \wedge p'$ and p are orthogonal, so (2.16) gives

$$p \vee (q \wedge p') = p + (q \wedge p') = p + q - p = q.$$

Finally, the set of finite suprema of any set of projections forms an increasing net of projections (indexed by themselves). Any increasing net of projections converges σ-strongly to a projection p (Proposition 2.5), which is necessarily the least upper bound of this net in M and thus also in $\mathcal{P}$. Thus $\mathcal{P}$ is a complete lattice. □

The following was established during the proof above and will be used frequently:

$$p \le q \quad \Rightarrow \quad q \wedge p' = q - p. \tag{2.17}$$

(For motivation, note that if this is applied to the lattice of subsets of a set, it becomes the statement that if $E \subset F$, then $F \setminus E = F \cap E'$, where E' is the set complement of E.)

If p and q are elements of an orthomodular lattice, we say p and q are *orthogonal* if $p \leq q'$ (or equivalently, if $q \leq p'$). Note that for the lattice of projections in a JBW-algebra, this is equivalent to our previously defined notion of orthogonality for positive elements of a JBW-algebra, cf. Proposition 2.18.

We next investigate properties of the maps U_p (continuing the study begun in Chapter 1). We will generally be interested in the duality of M and the space $V = \operatorname{lin} K$. Recall that M and V are in separating order and norm duality (Corollary 2.17). By Proposition 2.4 each map U_p^* maps V into V. In what follows we will view U_p^* as acting on V (rather than M^*) and thus interpret $\operatorname{im} U_p^*$ to mean $U_p^*(V)$. (In other words, we interpret U_p^* as the adjoint map with respect to the duality of M and V.)

Note that $U_p \geq 0$ implies $U_p^* \geq 0$, and $\|U_p\| \leq 1$ implies $\|U_p^*\| \leq 1$. We observe that since $M = M^+ - M^+$ and $V = V^+ - V^+$, and since the maps U_p and U_p^* are positive, then

$$\operatorname{im} U_p = \operatorname{im}^+ U_p - \operatorname{im}^+ U_p \quad \text{and} \quad \operatorname{im} U_p^* = \operatorname{im}^+ U_p^* - \operatorname{im}^+ U_p^*. \tag{2.18}$$

2.26. Proposition. *If p, q are projections in a JBW-algebra M, then the following are equivalent:*

(i) $p \leq q$,
(ii) $\operatorname{im}^+ U_p \subset \operatorname{im}^+ U_q$,
(iii) $\operatorname{im} U_p \subset \operatorname{im} U_q$,
(iv) $U_q U_p = U_p$,
(v) $\operatorname{im}^+ U_p^* \subset \operatorname{im}^+ U_q^*$,
(vi) $\operatorname{im} U_p^* \subset \operatorname{im} U_q^*$,
(vii) $U_p U_q = U_p$.

Proof. (i) $\Rightarrow$ (ii) If $p \leq q$, then $\operatorname{face}(p) \subset \operatorname{face}(q)$, so

$$\operatorname{im}^+ U_p = \operatorname{face}(p) \subset \operatorname{face}(q) = \operatorname{im}^+ U_q.$$

(ii) $\Rightarrow$ (iii) This follows from (2.18).

(iii) $\Rightarrow$ (iv) This follows from $U_q^2 = U_q$.

(iv) $\Rightarrow$ (v) If $U_q U_p = U_p$, then $U_p^* = U_p^* U_q^*$. Let $\sigma \in \operatorname{im}^+ U_p^*$. Since $\|U_p^*\| \leq 1$ and $\|U_q^*\| \leq 1$, then

$$\|\sigma\| = \|U_p^* \sigma\| = \|U_p^* U_q^* \sigma\| \leq \|U_q^* \sigma\| \leq \|\sigma\|,$$

so $\|U_q^* \sigma\| = \|\sigma\|$. By "neutrality" of compressions (1.57), this implies $U_q^* \sigma = \sigma$. Thus $\operatorname{im}^+ U_p^* \subset \operatorname{im}^+ U_q^*$.

(v) $\Rightarrow$ (vi) This follows from (2.18).

(vi) $\Rightarrow$ (vii) If $\operatorname{im} U_p^* \subset \operatorname{im} U_q^*$, then $U_q^* U_p^* = U_p^*$ and so $U_p U_q = U_p$.

(vii) $\Rightarrow$ (i) If $U_p U_q = U_p$, then $U_p q' = 0$. Thus by (1.49), $U_{p'} q' = q'$. Then $q' = U_{p'} q' \leq U_{p'} 1 = p'$, which implies $p \leq q$. $\square$

2.27. Corollary. *A projection p operator commutes with an element a of a JBW-algebra iff p operator commutes with every spectral projection of a.*

Proof. Recall that the spectral projections of a are contained in $W(a, 1)$ (cf. Theorem 2.20). Now let p be a projection which operator commutes with a. Then p operator commutes with everything in $W(a, 1)$ by Corollary 2.12, and so p operator commutes with every e_λ.

Conversely, suppose p is an arbitrary projection that operator commutes with every spectral projection of a. By continuity of Jordan multiplication, p will operator commute with norm limits of linear combinations of the spectral projections, and thus with a by the spectral theorem. □

2.28. Proposition. *If p and q are projections in a JBW-algebra M, then p and q operator commute iff $U_p U_q = U_q U_p$, and then $U_p U_q = U_{p \wedge q}$ and $p \wedge q = p \circ q$.*

Proof. By the definition (1.13), $U_p = 2T_p^2 - T_p$, where $T_p a = p \circ a$ for $a \in M$. Therefore, if p and q operator commute, then U_p and U_q commute. Conversely, if U_p and U_q commute, then

$$U_p q = U_p U_q 1 = U_q U_p 1 = U_q p \leq U_q 1 = q.$$

By (1.62) p and q operator commute.

Assuming p and q operator commute, we next show $U_p U_q = U_{p \wedge q}$. Let $a \in M^+$, and $b = U_p U_q a = U_q U_p a$. Then $U_p b = b$ and $U_q b = b$, so by the definition of the range projection (Definition 2.14), $r(b) \leq p$ and $r(b) \leq q$. Therefore $r(b) \leq p \wedge q$. Thus $b \in \operatorname{face}(r(b)) \subset \operatorname{face}(p \wedge q) = \operatorname{im}^+ U_{p \wedge q}$, so $U_{p \wedge q} b = b$. Then substituting $b = U_p U_q a$ gives

$$U_{p \wedge q} U_p U_q a = U_p U_q a.$$

Since $p \wedge q \leq p$ and $p \wedge q \leq q$, applying Proposition 2.26 ((i) $\Rightarrow$ (vii)) to this equation gives

$$U_{p \wedge q} a = U_p U_q a$$

for all $a \in M^+$. Since such elements span M, then $U_p U_q = U_{p \wedge q}$ follows.

Finally, suppose p and q operator commute. By Proposition 1.47 $U_p q = T_p q$, so

$$p \circ q = T_p q = U_p(U_q 1) = U_{p \wedge q} 1 = p \wedge q.$$

This completes the proof. □

We note for future reference the following implications for projections p and q:

$$p \le q \quad \Rightarrow \quad p \text{ operator commutes with } q. \tag{2.19}$$

$$p \perp q \quad \Rightarrow \quad p \text{ operator commutes with } q. \tag{2.20}$$

To verify (2.19), observe that $p \le q$ implies $p \in \text{face}(q)$, so $U_q p = p$. Then q operator commutes with p by (1.62). If $p \perp q$, then $p \le q'$, so p and $q' = 1 - q$ operator commute by (2.19). Thus p and q operator commute.

2.29. Definition. Elements p and q of an orthomodular lattice are *compatible* if there exist orthogonal elements r, s, and t such that $p = r \vee s$ and $q = s \vee t$.

Note that projections p and q in a JBW-algebra M are compatible as elements of the lattice of projections of M iff there are projections r and s (necessarily orthogonal) such that $p = r + s$ and $q = s + t$ (since $r + s = r \vee s$ and $s + t = s \vee t$, cf. Corollary 2.23). Thus the next result shows that projections are compatible iff they operator commute.

2.30. Proposition. *Let p, q be projections in a JBW-algebra M. Then p and q operator commute iff there are mutually orthogonal projections r, s, and t such that*

$$p = r + s \quad \textit{and} \quad q = s + t. \tag{2.21}$$

These projections are unique, with $r = p \wedge q'$, $s = p \wedge q$, and $t = q \wedge p'$.

Proof. Assume (2.21) holds. Since r, s, and t are orthogonal, they operator commute with each other (cf. (2.20)), and thus $p = r + s$ and $q = s + t$ operator commute. Furthermore,

$$s = (r + s) \circ (s + t) = p \circ q = p \wedge q,$$
$$r = p - s = p - p \circ q = p \circ (1 - q) = p \wedge q',$$

and similarly $t = q \wedge p'$, so orthogonal projections satisfying (2.21) are uniquely determined.

Conversely, if p and q operator commute, let $r = p - p \circ q$, $s = p \circ q$, and $t = q - p \circ q$. Then $r + s = p$, and $s + t = q$. By Proposition 2.28, $r = p \circ (1 - q) = p \wedge q'$, $s = p \wedge q$, and $t = q \wedge p'$, so all three are projections. Finally, $r + s + t = p + q \wedge p' \le p + p' = 1$, so r, s, t are mutually orthogonal, and this completes the proof. $\square$

The following formula for the range projection will be very useful later.

2.31. Proposition. *Let M be a JBW-algebra, p any projection in M, and $a \geq 0$ in M. Then*

$$r(U_p a) = (r(a) \vee p') \wedge p = p - (p \wedge r(a)'). \tag{2.22}$$

Proof. Below we will repeatedly use

$$e \leq f \quad \Rightarrow \quad f - e = f \wedge e' \tag{2.23}$$

for projections e and f, which is just (2.17) restated. Note that the rightmost equality in (2.22) follows from (2.23) by taking $f = p$ and $e = p \wedge r(a)'$.

By (2.19) p' operator commutes with $p' \vee r(a)$, and thus so does p. Therefore by Proposition 2.28, $U_{p' \vee r(a)}$ and U_p commute. Since $p' \vee r(a) \geq r(a)$, then $U_{p' \vee r(a)} a = a$ (cf. (2.8)), so

$$U_{p' \vee r(a)} U_p a = U_p (U_{p' \vee r(a)} a) = U_p a.$$

By the definition of the range projection (Definition 2.14), this implies

$$r(U_p a) \leq p' \vee r(a).$$

Again by the definition, $r(U_p a) \leq p$, so

$$r(U_p a) \leq (p' \vee r(a)) \wedge p. \tag{2.24}$$

We now prove the reverse inequality. We will make repeated use of the fact that for a projection e and $b \geq 0$,

$$U_e b = 0 \iff U_{e'} b = b$$

(cf. (1.49)). Let $w = r(U_p a)$. Then $w \leq p$, so w, and then w', operator commutes with p. By definition, $U_w(U_p a) = U_p a$, so $U_{w'}(U_p a) = 0$. Therefore $U_{w' \wedge p} a = U_{w'} U_p a = 0$, which implies $U_{(w' \wedge p)'} a = a$. Thus $U_{w \vee p'} a = a$, which implies $w \vee p' \geq r(a)$. Now $w \vee p' \geq r(a) \vee p'$ follows. Since $w \leq p = (p')'$, then w is orthogonal to p', so we have

$$w + p' = w \vee p' \geq r(a) \vee p'.$$

This implies

$$w \geq (r(a) \vee p') - p' = (r(a) \vee p') \wedge p,$$

where the last equality follows from (2.23) with $e = p'$ and $f = r(a) \vee p'$. This is the opposite inequality to (2.24), and thus completes the proof. □

σ-weakly closed hereditary subalgebras

A Jordan subalgebra B of a JB-algebra A is *hereditary* if ($a \in A$ and $b \in B$ with $0 \leq a \leq b$) implies $a \in B$. Note that this just says B^+ is a face of A^+. Since $\operatorname{im}^+ U_p = \operatorname{face}(p)$, then $\operatorname{im} U_p$ for a projection p is an example of a hereditary subalgebra .

2.32. Proposition. *Let M be a JBW-algebra. Then there are 1-1 order preserving correspondences of projections of M, σ-weakly closed faces of M^+, and σ-weakly closed hereditary subalgebras of M, given by* $p \leftrightarrow \operatorname{face}(p) \leftrightarrow \operatorname{im} U_p = \{pMp\}$.

Proof. Since $\operatorname{face}(p) \cap [0,1] = [0,p]$ (cf. Lemma 1.39), then $p \leftrightarrow \operatorname{face}(p)$ is a 1-1 order preserving correspondence of projections and the faces they generate. We will show these faces are precisely the σ-weakly closed faces of M^+. Since $\operatorname{face}(p) = \operatorname{im}^+ U_p = U_p \cap M^+$, then $\operatorname{face}(p)$ is σ-weakly closed. Conversely, let H be any σ-weakly closed face of M^+. Let $H_0 = H \cap [0,1]$. We have

$$a \in H \quad \Rightarrow \quad r(a) \in H_0,$$

since $r(a) \in \overline{\operatorname{face}(a)} \subset H$ (cf. (2.4)). If $a, b \in H_0$, then $r(a+b) \in H_0$ and $a \leq r(a) \leq r(a+b)$ (and similarly $b \leq r(a+b)$) by (2.5) and (2.7). Thus H_0 is directed upwards. Since M is monotone complete, the net consisting of all elements of H_0 (indexed by themselves) must converge σ-strongly to an element p, which is then the largest member of H_0. Since $p \leq r(p) \in H_0$, then p is a projection. Thus $H_0 = [0,p]$ and so $H = \operatorname{face}(p)$. Thus we have shown that $p \leftrightarrow \operatorname{face}(p)$ is a 1-1 correspondence of projections and σ-weakly closed faces of M^+.

Next we show that $p \leftrightarrow \operatorname{im} U_p$ is a 1-1 correspondence of projections and σ-weakly closed hereditary subalgebras. Since $\operatorname{im}^+ U_p \cap [0,1] = \operatorname{face}(p) \cap [0,1] = [0,p]$, the correspondence $p \leftrightarrow \operatorname{im} U_p$ is bijective. By the remarks preceding this proposition, $\operatorname{im} U_p$ is a σ-weakly closed hereditary subalgebra. Thus what remains is to show that every σ-weakly closed hereditary subalgebra has the form $\operatorname{im} U_p$ for some projection p.

Let B be a σ-weakly closed hereditary subalgebra of M. By Proposition 2.7, B is a JB-algebra, so $B = B^+ - B^+$ (cf. (1.11)). By the definition of the term hereditary, $B^+ = B \cap M^+$ is a σ-weakly closed face of M^+. By the preceding paragraphs, $B^+ = \operatorname{im}^+ U_p$ for some projection p. Thus $B = B^+ - B^+ = \operatorname{im}^+ U_p - \operatorname{im}^+ U_p = \operatorname{im} U_p$, which completes the proof. □

In Chapter 5 we will prove a corresponding result for the normal state space K, by showing that every norm closed face of K is of the form $\operatorname{im}^+ U_p^* \cap K = \{\sigma \in K \mid \sigma(p) = 1\}$ for a projection p.

Center of a JBW-algebra

Recall that the center of a JB-algebra A is an associative JB-subalgebra consisting of all elements that operator commute with every element of A (Proposition 1.52).

2.33. Definition. An element s of a JB-algebra A which satisfies $s^2 = 1$ is called a *symmetry.*

2.34. Proposition. *If s is a symmetry in a JB-algebra A, then the map U_s is a Jordan automorphism of A, and U_s^2 is the identity map on A.*

Proof. For $b \in A$,

$$(U_s b)^2 = \{sbs\}^2 = \{s\{bs^2b\}s\} = \{sb^2s\} = U_s(b^2),$$

where we have used the identity (1.15). Thus U_s preserves squares, and therefore preserves the Jordan product (cf. (1.59)), so U_s is a Jordan homomorphism. By the identity (1.16) the map U_s^2 is the identity map, since $U_s^2 = U_s U_1 U_s = U_{\{s1s\}} = 1$. Thus U_s is its own inverse, and therefore is a Jordan isomorphism. □

Note that if s is a symmetry, then $p = \frac{1}{2}(s+1)$ is a projection such that $s = 2p - 1 = p - p'$ (where $p' = 1 - p$). Similarly, if p is a projection then $s = 2p - 1$ is a symmetry. There is the following useful relationship between U_s and U_p:

$$U_s = 2U_p + 2U_{p'} - I. \tag{2.25}$$

(Here I is the identity map.) To verify this, note that linearity of the triple product in each factor gives each of the following two equations:

$$U_s a = \{(p-p')a(p-p')\} = \{pap\} + \{p'ap'\} - \{pap'\} - \{p'ap\}.$$
$$a = U_1 a = U_{p+p'} a = \{(p+p')a(p+p')\} = \{pap\} + \{p'ap'\} + \{pap'\} + \{p'ap\}.$$

Adding these two equations gives (2.25).

Note also that $s = 2p - 1$ operator commutes with an element a iff p does, which in turn occurs iff $(U_p + U_{p'})a = a$ (Proposition 1.47). By (2.25) this is equivalent to $U_s a = a$. Thus if s is a symmetry, then

$$s \text{ operator commutes with } a \text{ iff } U_s a = a. \tag{2.26}$$

Hence if an element a is central, then $U_s a = a$ for all symmetries s. The converse is also true, leading to the following convenient characterization of the center in terms of symmetries.

2.35. Lemma. *An element a of a JBW-algebra M is central iff $U_s a = a$ for all symmetries s in M.*

Proof. By Corollary 2.27, an element a is central iff it operator commutes with every projection in M. Since the symmetries are exactly the elements $2p - 1$ for projections p, this in turn is equivalent to a operator commuting with all symmetries. By (2.26) this is equivalent to $U_s a = a$ for every symmetry s of M. □

2.36. Proposition. *The center Z of a JBW-algebra M is an associative JBW-subalgebra.*

Proof. By Proposition 1.52, Z is an associative JB-subalgebra. Since for each symmetry s, the map U_s is σ-weakly continuous, then by Lemma 2.35, the center of M is σ-weakly closed, and thus is an associative JBW-subalgebra of M. □

2.37. Lemma. *Let p be a projection in a JBW-algebra M. Then there is a smallest central projection $c(p) \geq p$.*

Proof. Let $X = \{\text{central projections } c \mid c \geq p\}$. Note that if c_1, c_2 are in X, then their product coincides with their greatest lower bound among projections (Proposition 2.28), so is again in X. Thus X is directed downwards. Let $c(p)$ be the greatest lower bound of X. Then by Proposition 2.5, $c(p)$ is a σ-weak limit of elements of X and is a projection. It is central since the center of M is σ-weakly closed. □

2.38. Definition. Let p be a projection in a JBW-algebra M. The smallest central projection $c(p) \geq p$ is called the *central cover* of p.

2.39. Proposition. *A σ-weakly closed subspace J of a JBW-algebra M is a Jordan ideal iff it has the form $J = cM$ for a (necessarily unique) central projection c.*

Proof. Let J be a σ-weakly closed ideal of M. Then J is a σ-weakly closed subalgebra of M, so by Proposition 2.7 has an identity c. For any element a in M, $c \circ a$ is in J so $c \circ (c \circ a) = (c \circ a)$. It follows that $U_c a = 2c \circ (c \circ a) - c \circ a = c \circ a = T_c a$, so c is central by Proposition 1.47. Then $cM \subset J$ and $J = cJ \subset cM$, so $J = cM$. □

We say a JBW-algebra A is the *(algebraic) direct sum* of subalgebras M_1 and M_2 (and write $M = M_1 \oplus M_2$) if M is the direct sum of M_1 and M_2 as linear spaces, and if $b_1 \circ b_2 = 0$ for all $b_1 \in M_1$ and $b_2 \in M_2$.

2.40. Proposition. *Let M be a JBW-algebra. If $M = M_1 \oplus M_2$, then M_1 and M_2 are σ-weakly closed Jordan ideals of the form cM and $c'M$ for a central projection c, where $c' = 1 - c$.*

Proof. It is straightforward to check that M_1 and M_2 are Jordan ideals of M. Since $M_1 = \{x \in M \mid x \circ M_2 = 0\}$, then M_1 (and similarly M_2) is σ-weakly closed. Finally, let c_1 and c_2 be central projections such that $M_1 = c_1 M$ and $M_2 = c_2 M$. Then c_1 is an identity for M_1 and c_2 for M_2, so $c_1 + c_2$ is an identity for M. Thus $c_1 + c_2 = 1$. □

2.41. Proposition. *Let M be a JBW-algebra and c a central projection. Then $M = cM \oplus c'M$, and $T_c = U_c$ is a homomorphism from M onto cM with kernel $c'M$.*

Proof. If c is a central projection, then by Lemma 1.45, $cM + c'M = M_c + M_{c'}$ is an algebraic direct sum, equal to M since $m = cm + c'm$ for each $m \in M$. By (1.66), $m \mapsto cm$ is a homomorphism. Since $cm = 0$ iff $c'm = m$, then the kernel of $m \mapsto cm$ is $c'M$. □

2.42. Definition. If $\{M_\alpha\}$ is a collection of JBW-algebras, then we define $\bigoplus_\alpha M_\alpha$ to consist of the elements (b_α) of the Cartesian product such that $\sup_\alpha \|b_\alpha\| < \infty$. We define a product by coordinatewise multiplication, and use the supremum norm. It is straightforward to verify that $\bigoplus_\alpha M_\alpha$ is a JB-algebra, and is monotone complete with a separating set of normal states, i.e., is a JBW-algebra.

Let $\{b_\alpha\}$ be a net in a JBW-algebra. We define *partial sums* to be sums of the form $b_F = \sum_{\alpha \in F} b_\alpha$, where F is a finite set of indices. We order such finite subsets F by inclusion, and define $\sum_\alpha b_\alpha$ to be the σ-strong limit of the partial sums (if this limit exists).

2.43. Lemma. *If $\{b_\alpha\}$ is a net of orthogonal positive elements in a JBW-algebra M, and if $\sup_\alpha \|b_\alpha\| < \infty$, then $\sum_\alpha b_\alpha$ exists and $\|\sum_\alpha b_\alpha\| = \sup_\alpha \|b_\alpha\|$.*

Proof. The partial sums form an increasing net, so it suffices to prove it is bounded above. For each finite set F of indices, by (2.5)

$$\sum_{\alpha \in F} b_\alpha \le \sum_{\alpha \in F} \|b_\alpha\| r(b_\alpha). \tag{2.27}$$

Since the elements $\{b_\alpha\}_{\alpha \in F}$ are orthogonal, so are their range projections, so

$$\sum_{\alpha \in F} \|b_\alpha\|\, r(b_\alpha) \le (\max_\alpha \|b_\alpha\|) \sum_{\alpha \in F} r(b_\alpha) \le (\max_\alpha \|b_\alpha\|)\, 1, \tag{2.28}$$

(where the last inequality follows from Corollary 2.23). It follows that the partial sums are bounded above, so converge σ-strongly to their least upper bound. Furthermore, (2.27) and (2.28) imply that $\|\sum_\alpha b_\alpha\| \leq \sup_\alpha \|b_\alpha\|$, and the opposite inequality is clear, so the proof is complete. □

In particular, for any orthogonal collection $\{p_\alpha\}$ of projections, the sum $\sum_\alpha p_\alpha$ converges, and if each p_α is central, then the sum will also be central.

2.44. Proposition. *If $\{c_\alpha\}$ is a maximal collection of orthogonal central projections in a JBW-algebra M, then $\sum_\alpha c_\alpha = 1$, and M is isomorphic to $\bigoplus_\alpha c_\alpha M$ via $a \mapsto (c_\alpha a)$. (We write $M = \bigoplus_\alpha c_\alpha M$.)*

Proof. By maximality of the collection $\{c_\alpha\}$, $\sum_\alpha c_\alpha = 1$. Therefore the map $a \mapsto (c_\alpha a)$ is clearly injective. To see that this map is surjective, suppose $b_\alpha \in c_\alpha M$ for each α, with $\sup_\alpha \|b_\alpha\| < \infty$. For each α, let $b_\alpha = {b_\alpha}^+ - {b_\alpha}^-$ be the orthogonal decomposition (Proposition 1.28). By spectral theory, $\|b_\alpha\| = \max(\|{b_\alpha}^+\|, \|{b_\alpha}^-\|)$, so $\sup_\alpha \|{b_\alpha}^+\| < \infty$ and $\sup_\alpha \|{b_\alpha}^-\| < \infty$. Thus $\sum_\alpha {b_\alpha}^+$ and $\sum_\alpha {b_\alpha}^-$ converge σ-strongly (Lemma 2.43). Hence $\sum_\alpha b_\alpha = \sum_\alpha ({b_\alpha}^+ - {b_\alpha}^-)$ converges σ-strongly to an element b such that $c_\alpha b = b_\alpha$ for each α. □

The bidual of a JB-algebra

For C*-algebras, the fact that the bidual comes with a natural product making it a von Neumann algebra (A 101) is a very useful tool, and we are now going to establish the analogous result for JB-algebras. We are going to prove the related result that the bidual of a commutative order unit algebra is an order unit algebra, and this will allow us to prove both that the bidual of a JB-algebra is a JBW-algebra, and that every commutative order unit algebra is in fact a JB-algebra.

We need first to review the notion of a base norm space, which is used in the next proof and which will play an important role in the rest of this book.

2.45. Definition. An ordered normed vector space V with a generating cone V^+ is said to be a *base norm space* if V^+ has a base K located on a hyperplane H $(0 \notin H)$ such that the closed unit ball of V is $\operatorname{co}(K \cup -K)$. The convex set K is called the *distinguished base* of V.

The dual space of every order unit space A is a base norm space, with distinguished base the state space of A, and we can recover A from K as the space $A(K)$ of w*-continuous affine functions on K, cf. (A 19) and (A 20). Thus the dual space of a JB-algebra A is an order unit space, with the state space of A as distinguished base. The dual of a base norm space with distinguished base K is an order unit space (A 19), and this dual space can be identified with $A_b(K)$, cf. (A 19) and (A 11). We will see later (Corollary 2.60) that the predual of a JBW-algebra is a base norm space, with the normal state space as distinguished base.

2.46. Theorem. *Let A be a commutative order unit algebra. Then A^{**} is an order unit algebra for a unique product that extends the original product on A and which is separately w^*-continuous. Each state on A extends uniquely to a w^*-continuous state (necessarily normal) on A^{**}. If K is the state space of A, then A^{**} can be identified as an order unit space with the space $A_b(K)$ of bounded affine functions on K. Thus A^{**} is monotone complete with a separating set of normal states.*

Proof. For each state σ on A define a symmetric bilinear form on A by $(a \,|\, b) = \sigma(ab)$. Since squares in A are positive, this form is positive semi-definite, so the Cauchy–Schwarz inequality holds. Define $N = \{a \in A \mid (a \,|\, a) = 0\}$. By the Cauchy–Schwarz inequality,

$$N = \{a \in A \mid (a \,|\, b) = 0 \text{ for all } b \in A\}.$$

Therefore N is a subspace of A. It follows that $(a + N \mid b + N) = (a \mid b)$ is well defined, symmetric, and positive semi-definite, so A/N is a pre-Hilbert space. Let H_σ denote its completion as a Hilbert space, and let $\phi_\sigma : A \to A/N \subset H_\sigma$ be the quotient map. Note that

$$\|\phi_\sigma(a)\|^2 = (a + N \mid a + N) = (a \,|\, a) = \sigma(a^2) \leq \|a^2\| \leq \|a\|^2,$$

where the last inequality follows from the definition of an order unit algebra (Definition 1.9). Thus

$$\|\phi_\sigma\| \leq 1.$$

Note that for $a, b \in A$ we have

$$\sigma(ab) = (\phi_\sigma(a) \mid \phi_\sigma(b)). \tag{2.29}$$

We identify H_σ with H_σ^{**} and let ϕ_σ^{**} be the bidual map from A^{**} into $H_\sigma^{**} = H_\sigma$. Since $\|\phi_\sigma\| \leq 1$, then

$$\|\phi_\sigma^{**}\| \leq 1. \tag{2.30}$$

For a, b in A^{**} define a function $a * b$ on the state space K of A by

$$\langle a * b, \sigma \rangle = (\phi_\sigma^{**}(a) \mid \phi_\sigma^{**}(b)). \tag{2.31}$$

Note that ϕ_σ^{**} is continuous as a map from the w*-topology on A^{**} to the weak topology on H_σ. It follows that $(a,b) \mapsto \langle a * b, \sigma\rangle$ is separately w*-continuous. Note also that for $a \in A$, viewing A as a subspace of A^{**}, we have $\phi_\sigma^{**}(a) = \phi_\sigma(a)$. Thus if $a, b \in A$, then by (2.31) and (2.29), $a * b$ coincides as a functional on K with the given product ab in A.

Since A is an order unit space, then A^* is a base norm space with distinguished base K, and A^{**} is an order unit space which can be identified with $A_b(K)$. (See the remarks preceding this theorem.) It follows that A^{**} is monotone complete with a separating set of normal states, since $A_b(K)$ evidently has these properties. Each $\sigma \in K$ is an element of A^*, so acts by evaluation as a w*-continuous functional on A^{**}. By density of A in A^{**}, this is the unique extension of σ to a w*-continuous functional on A^{**}. If $a_\alpha \nearrow a$ in $A^{**} = A_b(K)$, then a is the pointwise limit of a_α on K, and thus $a_\alpha \to a$ in the w*-topology. Thus every w*-continuous linear functional is normal.

Next let $a, b \in A^{**}$, and let $\{a_\alpha\}$ and $\{b_\beta\}$ be nets in A converging in the w*-topology to a and b respectively. Then for $\sigma \in K$,

$$\langle a * b, \sigma\rangle = \lim_\alpha \lim_\beta \langle a_\alpha * a_\beta, \sigma\rangle.$$

Thus $a * b$ is a pointwise limit of affine functions on K, and so must itself be affine. Furthermore, for $\sigma \in K$,

$$|\langle a * b, \sigma\rangle| = |(\phi_\sigma^{**}(a) \mid \phi_\sigma^{**}(b))| \le \|\phi_\sigma^{**}(a)\| \, \|\phi_\sigma^{**}(b)\| \le \|a\| \, \|b\|.$$

Thus $a * b$ is a bounded affine function on K, with

$$\sup_{\sigma \in K} |\langle a * b, \sigma\rangle| \le \|a\| \, \|b\|. \tag{2.32}$$

Therefore $a*b$ can be identified with an element of A^{**}. The map $(a,b) \mapsto a * b$ is our candidate for the extension of the given product on A to a product on A^{**}. Separate w*-continuity of this product and w*-density of A in A^{**} imply that this product is commutative and bilinear.

Since A is power associative, in particular the identity $x(xx^2) = x^2x^2$ is valid for all $x \in A$. Therefore the equivalent linearized version (1.4) holds in A. By w*-continuity of this product on A^{**} and w*-density of A in A^{**}, the identity (1.4) holds in A^{**}, and then by Proposition 1.3, A^{**} is power associative.

To show that A^{**} is an order unit algebra, by (A 51) there only remains to prove the implication

$$-1 \le a \le 1 \quad \Rightarrow \quad 0 \le a^2 \le 1.$$

Let $a \in A^{**}$ and assume $-1 \le a \le 1$. Then by (2.32), $\|a * a\| \le 1$. On the other hand, by (2.31), $a * a \ge 0$. By the definition of the order unit norm, we conclude that $0 \leq a * a \leq 1$. This completes the proof that A^{**} is a commutative order unit algebra.

Finally, by density of A in A^{**}, the product on A^{**} is uniquely determined by the product on A and by the requirement of separate w*-continuity. □

The proof above can easily be modified to show that the bidual of a JB-algebra is a JBW-algebra, but this will also follow when we show that order unit algebras and JB-algebras are the same. We remark that the clever proof above is due to Hanche-Olsen [64] (stated there for JB-algebras). Note that although the proof begins by using states to construct a Hilbert space, as in the GNS-construction, the remainder of the proof is fundamentally different. In particular, no representation of A as operators acting on a Hilbert space is involved. Indeed, we will see in Chapter 4 that there are JB-algebras for which no non-zero representation on a Hilbert space exists.

2.47. Lemma. *Let A be a commutative order unit algebra. Then the set of finite linear combinations of orthogonal projections is norm dense in A^{**}.*

Proof. By Theorem 2.46, the hypotheses of (A 53) are met, so this is an immediate consequence. □

The proof of the following lemma is elementary (after all, one has very little to work with) but fairly lengthy, so we will just give a reference for the proof. In the Jordan context, this lemma states that orthogonal projections operator commute, a result that we already know (cf. (2.20)).

2.48. Lemma. *Let A be a commutative power associative algebra over $\mathbf{R}$. If p and q are idempotents in A such that $pq = 0$, then*

$$p(qx) = q(px) \quad \textit{for all } x \in A.$$

Proof. [113, Lemma 5.2] □

Recall that every JB-algebra is a commutative order unit algebra (Lemma 1.10). Now we can prove the converse result, which was first proved by Iochum and Loupias [69, 70].

2.49. Theorem. *If A is a commutative order unit algebra, then A is a JB-algebra.*

Proof. By (A 52), the norm conditions that define a JB-algebra are satisfied by commutative order unit algebras. Thus we only need to prove the defining Jordan identity

$$(a^2 \circ b) \circ a = a^2 \circ (b \circ a). \tag{2.33}$$

Suppose first that a is a finite linear combination of orthogonal projections: $a = \sum_i \lambda_i p_i$, and $b \in A$. Then

$$(a^2 \circ b) \circ a = \sum_i (\lambda_i^2 p_i \circ b) \circ \Big(\sum_j \lambda_j p_j\Big) = \sum_{i,j} \lambda_i^2 \lambda_j (p_i \circ b) \circ p_j.$$

Similarly

$$a^2 \circ (b \circ a) = \Big(\sum_i \lambda_i^2 p_i\Big) \circ \Big(b \circ \sum_j \lambda_j p_j\Big) = \sum_{i,j} \lambda_i^2 \lambda_j p_i \circ (b \circ p_j).$$

Now (2.33) follows from commutativity and Lemma 2.48. For general a, we imbed A in the order unit algebra A^{**} and use the fact that in A^{**} every element can be approximated in norm by linear combinations of orthogonal projections (Lemma 2.47). Then (2.33) follows from norm continuity of multiplication on order unit algebras. □

2.50. Corollary. *Let A be a JB-algebra. Then its second dual A^{**} is a JBW-algebra for a unique product that extends the original product in A and which is separately w^*-continuous. Each state on A extends uniquely to a w^*-continuous state (necessarily normal) on A^{**}. If K is the state space of A, then A^{**} can be identified as an order unit space with $A_b(K)$.*

Proof. This follows from the analogous result for commutative order unit algebras (Theorem 2.46) and the fact that JB-algebras are the same as commutative order unit algebras (Theorem 2.49 and Lemma 1.10). □

We will often view A as a space of functionals on A^* and thus identify A with a subspace of A^{**}. Passing to the bidual is a very useful technique for proving certain results about JB-algebras, as we will now see.

The predual of a JBW-algebra

We will say a Banach space M is a *Banach dual space* if there is a Banach space M_* such that M is isometrically isomorphic to $(M_*)^*$. The space M_* is then called a *predual* of M. Note that elements of any predual can be viewed as functionals on M, and thus M_* can be isometrically imbedded in M^*.

2.51. Lemma. *Let M be a JBW-algebra, and N the subspace of M^* consisting of the normal linear functionals. Let J be the annihilator of N in M^{**}. Then N is norm closed, and there is a central projection c in M^{**} such that $J = (1-c)M^{**}$, and $N = U_c^*(M^*)$.*

Proof. By Corollary 2.50 each state on M has a unique extension to a normal state on M^{**}, which is then σ-weakly continuous (by the definition of the σ-weak topology). Furthermore, since M is an order unit space, then M^* is a base norm space (A 19), so every element of M^* is a linear combination of states (A 26). It follows that every element of M^* acts as a σ-weakly continuous linear functional on M^{**}.

Thus J is σ-weakly closed in M^{**}, as well as w*-closed. If $a \in M$, then by Proposition 2.4, T_a^* maps N into N. Since the multiplication on M^{**} is separately w*-continuous (Corollary 2.50), one readily verifies that T_a^{**} is multiplication by a on M^{**}. It follows that the annihilator J of N in M^{**} is invariant under multiplication by elements of M. By w*-density of M in M^{**}, J is invariant under multiplication by elements of M^{**}, and thus is a σ-weakly closed ideal of M^{**}.

By Proposition 2.39, J has the form $(1-c)M^{**} = U_{1-c}M^{**}$ for a central projection c. Since an increasing net bounded above is eventually bounded, a simple argument with the triangle inequality shows that a norm limit of normal functionals is normal. (See the proof of [AS, Lemma 2.85] for a similar argument.) Thus N is a norm closed subspace of M^*. By the bipolar theorem, N is the annihilator of J. Working in the duality of M^* and M^{**}, N is the annihilator in M^* of $\operatorname{im} U_{1-c}$, and thus equals the kernel of U_{1-c}^* on M^*. Since $U_{1-c} = I - U_c$, and U_c is idempotent, then the kernel of U_{1-c}^* equals the image of U_c^*, which completes the proof. □

2.52. Proposition. *Let M be a JBW-algebra. Every normal linear functional on M is the difference of positive normal linear functionals, and the normal state space of M is a split face of the state space of M.*

Proof. Since M^* is a base norm space (A 19), and thus is positively generated (A 26), then with the notation of Lemma 2.51, $N = U_c^*(M^*)$ is positively generated. Hence every normal linear functional on M is the difference of positive normal linear functionals. Since U_c^* and U_{1-c}^* are positive idempotent maps on M^* with sum equal to the identity map, it is straightforward to verify that the normal state space of M is a direct convex summand of the state space of M, i.e., a split face. □

If B is a subspace of a JB-algebra A, we say B is *monotone closed* in A if $\{b_\alpha\} \subset B$ and $b_\alpha \nearrow a \in A$ implies $a \in B$.

2.53. Lemma. *Let A be a JB-algebra. Then the smallest monotone closed subspace of A^{**} containing A is all of A^{**}.*

Proof. [67, Thm. 4.4.10] □

2.54. Lemma. *Let M be a JBW-algebra canonically imbedded in M^{**}, and $c \in M^{**}$ the central projection of Lemma 2.51. Then $a \mapsto ca$ is an isomorphism from M onto cM^{**}.*

Proof. By Proposition 2.41, $U_c : M^{**} \to cM^{**}$ is a Jordan homomorphism with kernel $(1-c)M^{**}$. By the choice of c, the kernel is the annihilator of the normal functionals on M, and so contains no element of M. Thus this map restricted to M is a Jordan isomorphism onto the image cM. We are going to show that $cM = cM^{**}$.

First we show that cM is monotone closed in cM^{**}. Suppose that $\{a_\alpha\} \subset M$ and $ca_\alpha \nearrow b$ in cM^{**}. Since U_c is an isomorphism on M, then $\{a_\alpha\}$ is an increasing net. Let a be the supremum of $\{a_\alpha\}$ in M. Then $\sigma(a_\alpha) \to \sigma(a)$ for all normal states σ on M. If σ is any state on M, then by Lemma 2.51, $U_c^*\sigma$ is normal on M, so $(U_c^*\sigma)(a_\alpha) \to (U_c^*\sigma)(a)$. Thus $\sigma(ca_\alpha) \to \sigma(ca)$ for all states σ on M, and so $ca_\alpha \to ca$ weak*. On the other hand, by Corollary 2.50 the w*-continuous extension of each state on M is a normal functional on M^{**}, so $\sigma(ca_\alpha) \to \sigma(b)$ for each state σ on M. We conclude that $b = ca \in cM$, which completes the proof that cM is monotone closed in cM^{**}.

Now let $M_0 = \{b \in M^{**} \mid U_c b \in U_c(M)\}$. If $\{b_\alpha\}$ is an increasing net in M_0 with least upper bound $b \in M^{**}$, then by σ-weak continuity and positivity of the map U_c, the net $\{U_c b_\alpha\}$ increases and converges σ-weakly to $U_c b$. Thus $U_c b_\alpha \nearrow U_c b$. Since cM is monotone closed in cM^{**}, then $U_c b \in U_c M$, proving that M_0 is monotone closed in M^{**}. By Lemma 2.53, M_0 must be all of M^{**}, and thus $U_c(M^{**}) = U_c(M)$. □

2.55. Theorem. *A JB-algebra M is a JBW-algebra iff it is a Banach dual space. In that case the predual M_* is unique and consists of the normal linear functionals on M.*

Proof. Assume M admits a predual M_*. Then the (closed) unit ball $[-1,1]$ of M is w*-compact, and $x \to \frac{1}{2}(x+1)$ is a homeomorphism of $[-1,1]$ onto $[0,1]$, so the positive part of the unit ball is w*-compact and thus w*-closed. It follows from the Krein–Šmulian theorem (A 34) that the positive cone M^+ is w*-closed. By the bipolar theorem, the positive elements of M_* determine the order on M, and in particular separate points of M. If we let $V = (M_*)^+ - (M_*)^+$, then the identity map on the unit ball of M is continuous from the w*-topology to the $\sigma(M,V)$ topology, and so by w*-compactness of the unit ball is a homeomorphism. Thus the w* and $\sigma(M,V)$ topologies coincide on the unit ball of M.

Now suppose that $\{a_\alpha\}$ is a bounded increasing net in M. Then for each positive σ in M_* the net $\{\sigma(a_\alpha)\}$ must converge. Therefore the net $\{a_\alpha\}$ is $\sigma(M,V)$ Cauchy. Since the unit ball is $\sigma(M,V)$ compact and therefore complete, this net has a $\sigma(M,V)$ limit a. Since the $\sigma(M,V)$ and w*-topologies coincide on the unit ball, $a_\alpha \to a$ weak*. As shown above, the positive functionals in M_* determine the order on M, so $a_\alpha \nearrow a$. Thus we have shown that M is monotone complete and that each functional in M_* is normal. We showed above that the positive functionals in M_* separate points, and therefore M admits a separating set of normal states, and so by definition is a JBW-algebra.

Conversely, suppose M is a JBW-algebra, and let N be the space of normal functionals. Let J be the annihilator of N in M^{**}, and c the central projection such that $J = U_{1-c}(M^{**})$ (Lemma 2.51). By Lemma 2.54, U_c is an isomorphism (and thus an isometry: cf. Proposition 1.35) from M onto $U_c(M^{**})$. From the fact that U_c is idempotent and $\|U_c\| \leq 1$, one readily verifies that $U_c(M^{**})$ is isometrically isomorphic to the dual space of $U_c^*(M^*) = N$. Thus U_c is an isometry of M onto the Banach dual of N. By Lemma 2.51, N is a closed subspace of M^* and thus is a Banach space.

Finally, let V be any Banach space that is a predual of M. By the first part of this proof, each element of V is a normal functional on M. Since V is complete, then V is a norm closed subset of the space N of normal functionals on M. Since V separates elements of M, by the Hahn–Banach theorem we must have $V = N$. □

2.56. Corollary. *The normal, w*-continuous, and σ-weakly continuous linear functionals on a JBW-algebra coincide. The σ-weak and w*-topologies coincide, and thus the unit ball of M is σ-weakly compact.*

Proof. By Theorem 2.55, the normal linear functionals are the elements of the unique predual that define the w*-topology, and thus coincide with the w*-continuous linear functionals. By Proposition 2.5, bounded monotone nets converge σ-weakly, so σ-weakly continuous linear functionals are normal. By the definition of the σ-weak topology, every positive normal linear functional is σ-weakly continuous. By Proposition 2.52 every normal linear functional is the difference of positive normal functionals. Thus every normal functional is σ-weakly continuous. Finally, the σ-weak topology is defined as convergence on positive normal linear functionals, which is then the same as convergence with respect to all normal linear functionals. The latter topology is precisely the w*-topology. □

2.57. Corollary. *If A is a JB-algebra, then A is σ-weakly dense in the JBW-algebra A^{**}.*

Proof. By the bipolar theorem, A is w*-dense in A^{**}. By Corollary 2.56 it follows that A is σ-weakly dense in A^{**}. □

2.58. Proposition. *If ρ is a normal functional on a JBW-algebra M, then there is a unique pair σ, τ of positive normal linear functionals* *such that*

$$\rho = \sigma - \tau \quad \textit{and} \quad \|\rho\| = \|\sigma\| + \|\tau\|. \tag{2.34}$$

Furthermore, there is a projection q in M such that

$$U_q^*\sigma = \sigma \quad \textit{and} \quad U_q^*\tau = 0. \tag{2.35}$$

Proof. Let ρ be a normal linear functional on M with $\|\rho\| = 1$. Recall that M is the dual of the space of normal linear functionals (Theorem 2.55). Let $b \in M$ be an element of the w*-compact unit ball M_1 on which ρ achieves its maximum 1. The set of such elements b is a w*-compact face of the unit ball of M, and so by the Krein–Milman theorem, we can choose b to be an extreme point of M_1. Since $a \mapsto 2a - 1$ is an affine homeomorphism of the positive unit ball $[0,1]$ of M onto $M_1 = [-1,1]$, it follows from Proposition 1.40 that the extreme points of the unit ball of M have the form $2q - 1 = q - q'$ for projections q (where $q' = 1 - q$). Now ρ can also be viewed as an element of M^*, and M^* is a base norm space (A 19). It follows from the definition of a base norm space that there exist positive functionals σ and τ such that $\rho = \sigma - \tau$ and $1 = \|\rho\| = \|\sigma\| + \|\tau\|$, cf. (A 26). We are going to show that σ and τ are normal, and that this decomposition is unique. Now

$$1 = \rho(b) = (\sigma - \tau)(q - q') = \sigma(q) + \tau(q') - \sigma(q') - \tau(q)$$

$$\leq \sigma(q) + \tau(q') \leq \|\sigma\| + \|\tau\| = 1.$$

It follows that $\sigma(q) = \|\sigma\|$ and $\tau(q') = \|\tau\|$, and so by Proposition 1.41, $U_q^*\sigma = \sigma$ and $U_q^*\tau = 0$. Then $U_q^*\rho = \sigma$. Since U_q is σ-weakly continuous (Proposition 2.4), and monotone nets converge σ-weakly (Proposition 2.5), then $\sigma = U_q^*\rho$ is normal, and so is $\tau = \sigma - \rho$. This completes the proof that the decomposition (2.34) exists. The construction of q depended only on ρ and not on the choice of σ and τ, so the equation $U_q^*\rho = \sigma$ implies uniqueness of the decomposition. □

Recall that a von Neumann algebra $\mathcal{M}$ also has a unique predual (A 95). The following describes the relationship of the predual of $\mathcal{M}$ and the predual of the JBW-algebra $\mathcal{M}_{\mathrm{sa}}$.

2.59. Proposition. *Let $\mathcal{M}$ be a von Neumann algebra with predual $\mathcal{M}_*$, and let $M = \mathcal{M}_{\mathrm{sa}}$, with predual M_*. Then $M_* + iM_* = \mathcal{M}_*$, and M_* has the norm inherited from $\mathcal{M}_*$.*

Proof. By definition, a positive normal linear functional on $\mathcal{M}$ is also normal on M, and thus is in M_*. The predual of $\mathcal{M}$ consists of the normal linear functionals (A 95), and every normal linear functional on $\mathcal{M}$ is a linear combination of positive normal linear functionals (cf. (A 90) and the remarks after Definition 2.2). Thus $\mathcal{M}_* = M_* + iM_*$.

Now let $\sigma \in M_*$. We need to show that the norm of σ as a linear functional on M coincides with its norm as a complex linear functional on $\mathcal{M}$. Write $\|\sigma\|_M$ and $\|\sigma\|_{\mathcal{M}}$ for these respective norms. If $\sigma \geq 0$, then $\|\sigma\|_M = \sigma(1) = \|\sigma\|_{\mathcal{M}}$, cf. (A 16) and (A 71). For arbitrary $\sigma \in M_*$, write $\sigma = \sigma^+ - \sigma^-$, where $\sigma^+, \sigma^- \in M_*^+$, and $\|\sigma\|_M = \|\sigma^+\|_M + \|\sigma^-\|_M$ (cf. Prop. 2.58). Then

$$\|\sigma\|_M \leq \|\sigma\|_{\mathcal{M}} \leq \|\sigma^+\|_{\mathcal{M}} + \|\sigma^-\|_{\mathcal{M}} = \|\sigma^+\|_M + \|\sigma^-\|_M = \|\sigma\|_M.$$

Therefore $\|\sigma\|_{\mathcal{M}} = \|\sigma\|_M$. □

Recall in Chapter 1 we showed that a JB-algebra A is an order unit space, and thus its state space is the distinguished base of the base norm space A^*.

2.60. Corollary. *The predual of a JBW-algebra M is a base norm space (V, K) where the normal state space K is the distinguished base. Thus M can be identified as an order unit space with $A_b(K)$.*

Proof. We need to prove $(M_*)_1 = \mathrm{co}(K \cup -K)$. If $\rho \in (M_*)_1$, choose a decomposition $\rho = \sigma - \tau$ as in equation (2.34). Then

$$\rho = \sigma - \tau = \|\sigma\|(\|\sigma\|^{-1}\sigma) + \|\tau\|(-\|\tau\|^{-1}\tau) + (1 - \|\rho\|)\, 0 \in \mathrm{co}(K \cup -K).$$

The second statement of the corollary follows from the fact that the dual of a base norm space (V, K) is $A_b(K)$ (A 11). □

Recall from (A 24) that positive linear functionals σ, τ in a base norm space are *orthogonal* if $\|\sigma - \tau\| = \|\sigma\| + \|\tau\|$. We will therefore refer to (2.34) as the *orthogonal decomposition* of ρ.

2.61. Corollary. *If A is a JB-algebra, then each state on A extends to a unique normal state on A^{**}, and the extension map is a bijection from the state space of A onto the normal state space of A^{**}.*

Proof. By Corollary 2.50, every state on A extends to a normal linear functional on A^{**}. Every normal state on A^{**} is w*-continuous by Corollary 2.56, so the normal extension is unique. Evidently two distinct states cannot have the same normal extension, so the extension process is one-to-one, and is surjective since every normal state on A^{**} is the extension of its restriction to A. □

2.62. Proposition. *Let M be a JBW-algebra with normal state space K, let p be a projection in M, and $M_p = U_p(M)$. The restriction map is an order preserving isometry from $U_p^*(M_*)$ onto $(M_p)_*$, and gives an affine isomorphism from $F_p = \{\sigma \in K \mid \sigma(p) = 1\}$ onto the normal state space of M_p. The inverse of the restriction map is the map $\sigma \mapsto \sigma \circ U_p$.*

Proof. Define $\psi : (M_p)_* \to M_*$ by

$$(\psi(\sigma))(x) = \sigma(U_p x).$$

It is readily verified that ψ is an isometry from $(M_p)_*$ onto $U_p^*(M_*)$, whose inverse is the restriction map. If σ is a normal state on M_p, then $\psi(\sigma)(p) = \sigma(p) = 1$, so $\psi(\sigma) \in F_p$. If $\omega \in F_p$, then $U_p^*\omega = \omega$ (cf. (1.55)), so $\psi(\omega|M_p) = \omega$. Thus ψ is an affine isomorphism of the normal state space of M_p onto F_p. □

2.63. Definition. Let M_1, M_2 be JBW-algebras. A positive map π from M_1 to M_2 is *normal* if whenever $a_\alpha \nearrow a$, then $\pi(a_\alpha) \nearrow \pi(a)$.

Note that any order isomorphism between JBW-algebras is normal, and then also any Jordan isomorphism.

2.64. Proposition. *Let M_1, M_2 be JBW-algebras. A positive map T from M_1 to M_2 is normal iff it is σ-weakly continuous.*

Proof. Since T is a positive map between order unit spaces, it is norm continuous (A 15). Observe that T is normal iff the dual map $T^* : M_2^* \to M_1^*$ sends normal states to normal functionals. In fact, if $a_\alpha \nearrow a$ in M_1, then $Ta_\alpha \nearrow Ta$ iff $\rho(Ta_\alpha) \nearrow \rho(Ta)$ for all normal states ρ on M_2, i.e , iff $T^*\rho$ is normal for all normal states ρ. Since the σ-weak topology is by definition the same as pointwise convergence on normal states, it follows by a similar argument that T is σ-weakly continuous iff T^* takes normal states to normal functionals. Thus normality for T is equivalent to σ-weak continuity. □

2.65. Theorem. *Let A be a JB-algebra and $\pi : A \to M$ a homomorphism into a JBW-algebra M. Then there is a unique normal homomorphism $\widetilde{\pi} : A^{**} \to M$ extending π.*

Proof. Note that $\pi^{**} : A^{**} \to M^{**}$ is by definition w*-continuous, so π^{**} is normal. By w*-density of A in A^{**}, π^{**} is a homomorphism. Recall from Lemma 2.54 that there is a central projection c in M^{**} such that U_c is an isomorphism from M onto $U_c(M^{**})$. Let π_0 be the inverse of this isomorphism, i.e., $\pi_0(U_c a) = a$. Finally, define $\widetilde{\pi}$ to be the composition $\widetilde{\pi} = \pi_0 \circ U_c \circ \pi^{**}$. Since the isomorphism π_0 is necessarily normal, then $\widetilde{\pi}$ is normal, and it extends π. Finally, uniqueness follows from the σ-weak density of A in A^{**}. □

2.66. Proposition. *Let M and N be JBW-algebras and $\pi : M \to N$ a normal homomorphism. Then $\pi(M)$ is a JBW-subalgebra of N.*

Proof. The kernel of π is a σ-weakly closed ideal of M, so by Proposition 2.39 has the form $U_{1-c}M$ for some central projection c in M. Then π is an isomorphism from U_cM onto $\pi(M)$. By Proposition 2.9, U_cM is a JBW-algebra, so the same is true of the isomorphic algebra $\pi(M)$.

Finally, the unit ball of M is σ-weakly compact (Corollary 2.56). Since $\|U_c\| \leq 1$ and U_c is idempotent, then U_c maps the unit ball of M onto that of U_cM, and the latter is mapped onto the unit ball of $\pi(M)$ by the isomorphism π. Thus by σ-weak continuity of π, the unit ball of $\pi(M)$ is σ-weakly compact, and then w*-compact in N. By the Krein–Šmulian theorem, $\pi(M)$ is w*-closed, and then σ-weakly closed. Thus it is a JBW-subalgebra of N. $\square$

2.67. Proposition. *Let A be a JB-algebra and M a JBW-algebra, and $\pi : A \to M$ a homomorphism. Then $\tilde{\pi}(A^{**}) = \overline{\pi(A)}$ (σ-weak closure.)*

Proof. By normality of $\tilde{\pi}$ and Proposition 2.66, $\tilde{\pi}(A^{**})$ is a JBW-subalgebra of M, and thus contains $\overline{\pi(A)}$. The opposite inclusion follows from the fact that A is σ-weakly dense in A^{**} (Corollary 2.57), and so by σ-weak continuity $\tilde{\pi}(A^{**}) \subset \overline{\pi(A)}$. $\square$

2.68. Proposition. *Let M be a JBW-algebra. Then the σ-weakly continuous linear functionals and the σ-strongly continuous linear functionals coincide. Thus the σ-weakly closed and σ-strongly closed convex subsets coincide.*

Proof. By Proposition 2.5, σ-weakly continuous linear functionals are σ-strongly continuous. If τ is σ-strongly continuous, since monotone nets converge σ-strongly (Proposition 2.5), then τ is normal, and therefore σ-weakly continuous (Corollary 2.56). $\square$

We can now derive the very useful Jordan analog of the Kaplansky density theorem for von Neumann algebras.

2.69. Proposition. (Kaplansky density theorem for JB-algebras) If M is a JBW-algebra and A is a JB-subalgebra that is σ-strongly dense in M, then the unit ball of A is σ-strongly dense in the unit ball of M. In particular, the unit ball of A is σ-strongly dense in the unit ball of A^{**}.

Proof. By Theorem 2.65, there is a normal extension $\pi : A^{**} \to M$ of the inclusion map of A in M. By Proposition 2.66, $\pi(A^{**})$ is σ-weakly closed in M. Since A is σ-strongly dense in M, then by Proposition 2.5, A is σ-weakly dense in M. Thus $\pi(A^{**}) \supset \pi(A) = A$ is σ-weakly dense in

M, and so $\pi(A^{**}) = M$. The kernel of π will be a σ-weakly closed ideal in A^{**}, and so is of the form $(1-c)A^{**}$ for a central projection c in A^{**}. Then π will be an isomorphism from cA^{**} onto M. Thus π takes the unit ball of cA^{**} onto the unit ball of M. It follows that π takes the unit ball of A^{**} onto the unit ball of M.

The unit ball of A is σ-weakly dense in the unit ball of A^{**} by the bipolar theorem. Thus by σ-weak continuity of π, the unit ball of $A = \pi(A)$ is σ-weakly dense in the unit ball of M. By Proposition 2.68, since the unit ball of A is convex, its σ-weak and σ-strong closures coincide, so it is σ-strongly dense in the unit ball of M.

Finally, since A is convex and σ-weakly dense in A^{**}, it is also σ-strongly dense. Now the second statement in the proposition follows from the first. □

JW-algebras

Recall that a JC-algebra is a norm closed Jordan subalgebra of $\mathcal{B}(H)_{\mathrm{sa}}$ (Definition 1.7). $\mathcal{B}(H)_{\mathrm{sa}}$ is monotone complete (A 76) and has a separating set of normal states (e.g., the vector states $\omega_\xi : a \mapsto (a\xi \mid \xi)$), so is a JBW-algebra. In [AS, Chpt. 2] the σ-weak topology was defined in terms of specific linear functionals on $\mathcal{B}(H)$ (A 84), which were shown to be precisely the normal linear functionals (A 92), so the term σ-weak in [AS, Chpt. 2] coincides with our usage here for $\mathcal{B}(H)_{\mathrm{sa}}$ viewed as a JBW-algebra.

2.70. Definition. A *JW-algebra* is a JB-algebra that is isomorphic to a σ-weakly closed Jordan subalgebra of $\mathcal{B}(H)_{\mathrm{sa}}$. A *concrete JW-algebra* is a σ-weakly closed Jordan subalgebra of $\mathcal{B}(H)_{\mathrm{sa}}$.

2.71. Example. Of course, $\mathcal{B}(H)_{\mathrm{sa}}$ is one example of a JW-algebra. Any concrete von Neumann algebra is by definition weakly closed (A 89), and thus also σ-weakly closed (A 88), and so the self-adjoint part of a von Neumann algebra is also a JW-algebra.

2.72. Lemma. *Every JW-algebra is a JBW-algebra. Furthermore, if M is a concrete JW-algebra, then the σ-weak topology on M coincides with that inherited from $\mathcal{B}(H)$.*

Proof. Let $B = \mathcal{B}(H)_{\mathrm{sa}}$. As was just discussed, B is a JBW-algebra, and thus so is any σ-weakly closed Jordan subalgebra, by Proposition 2.7. Let N_0 be the annihilator of M in B_* (the predual of B). By the bipolar theorem, since M is σ-weakly $= \sigma(B, B_*)$ closed, then it is equal to the annihilator of N_0. Each $a \in M$ induces a map $\widehat{a} : B_*/N_0 \to \mathbf{C}$ by $\widehat{a}(\rho + N_0) = \rho(a)$. By (A 32) $a \mapsto \widehat{a}$ is an isometric isomorphism from

M onto $(B_*/N_0)^*$. It follows that the elements of the predual of M are precisely the restrictions of the elements of the predual of B, and thus the two σ-weak topologies coincide on M. □

2.73. Lemma. *Let A and B be Banach spaces and $\pi : A \to B$ a linear isometry from A into B. Then π^{**} is an isometry from A^{**} onto a w*-closed subspace of B^{**}.*

Proof. We first show that π^* maps the closed unit ball of B^* onto the closed unit ball of A^*. For $\sigma \in (B^*)_1$ we have

$$\begin{aligned}\|\pi^*\sigma\| &= \sup\{(\pi^*\sigma)(a) \mid a \in A_1\} \\ &= \sup\{\sigma(\pi(a)) \mid a \in A_1\} \leq \|\sigma\|\|\pi(a)\| \leq 1,\end{aligned}$$

so π^* maps the unit ball of B^* into the unit ball of A^{**}. Let τ be in the unit ball of A^*. Then $\pi(a) \mapsto \tau(a)$ is a linear functional of norm at most 1 on $\pi(A)$. By the Hahn–Banach theorem we can extend this to a linear functional σ on B of norm at most 1, and then $\pi^*\sigma = \tau$, which proves that π^* maps the unit ball of B^* onto the unit ball of A^*.

We next show that π^{**} is an isometry from A^{**} into B^{**}. Let $a \in A^{**}$; then

$$\begin{aligned}\|\pi^{**}(a)\| &= \sup\{\sigma(\pi^{**}(a)) \mid \sigma \in (B^*)_1\} \\ &= \sup\{\pi^*(\sigma)(a) \mid \sigma \in (B^*)_1\} \\ &= \sup\{\tau(a) \mid \tau \in (A^*)_1\} = \|a\|.\end{aligned}$$

Finally, note that π^{**} is w*-continuous and the unit ball of A^{**} is w*-compact. Since π^{**} is an isometry, the unit ball of $\pi^{**}(A^{**})$ is the image of the unit ball of A^{**}, and so is w*-compact. By the Krein–Šmulian theorem, $\pi^{**}(A^{**})$ is w*-closed in B^{**}. □

2.74. Proposition. *If A and B are JB-algebras and $\pi : A \to B$ is a unital Jordan isomorphism from A into B, then π^{**} will be a Jordan isomorphism from A^{**} onto a σ-weakly closed subalgebra of B^{**}.*

Proof. Recall that the w*-and σ-weak topologies on a JBW-algebra coincide (Corollary 2.56). Now w*-density of A in A^{**}, w*-continuity of π^{**}, and separate w* $=$ σ-weak continuity of multiplication imply that π^{**} is a Jordan homomorphism. Since unital Jordan isomorphisms are isometries (Proposition 1.35), then π is an isometry. Hence by Lemma 2.73, π^{**} is an isometry onto a w*-closed subspace of B^{**}. Thus π^{**} is a Jordan isomorphism from A^{**} onto a σ-weakly closed subalgebra of B^{**}. □

Let B be a Banach space, and as usual identify B with a subspace of B^{**}. If A is a closed subspace of B, then we may also identify A with a subspace of B^{**}. If $\pi : A \to B$ is the inclusion map, then by Lemma 2.73, π^{**} is an isometry from A^{**} into B^{**}, and we will identify A^{**} with $\pi^{**}(A) \subset B^{**}$.

2.75. Lemma. *Let B be a Banach space and A a closed subspace. Let $\bar{A}$ be the w*-closure of A in B^{**}. Then $A^{**} = \bar{A}$, and*

$$\bar{A} \cap B = A.$$

Proof. Let $\pi : A \to B$ be the inclusion map. Since π^{**} is w*-continuous and A is w*-dense in A^{**}, it follows that $\pi^{**}(A)$ is w*-dense in $\pi^{**}(A^{**})$. We identify the former with A and the latter with A^{**} according to the remarks preceding this lemma. Then A^{**} is w*-closed in B^{**}, so $\bar{A} = A^{**}$ follows.

Now let V be the annihilator of A in B^*, and let V° be the annihilator of V in B^{**}. Then by the bipolar theorem (A 31) for the duality of B^* and B^{**}, $\bar{A} = V^\circ$. On the other hand, since norm closed subspaces of a Banach space are weakly closed, applying the bipolar theorem to the duality of B and B^*, we conclude that $V^\circ \cap B = A$. Combining these two equalities gives the result in the lemma. □

2.76. Lemma. *If $\mathcal{A}$ is a C*-algebra with state space K, then $(\mathcal{A}_{\mathrm{sa}})^{**}$ and $(\mathcal{A}^{**})_{\mathrm{sa}}$ can be identified as JBW-algebras.*

Proof. By (A 102) and Corollary 2.50, both $(\mathcal{A}^{**})_{\mathrm{sa}}$ and $(\mathcal{A}_{\mathrm{sa}})^{**}$ are order isomorphic to $A_b(K)$. The Jordan product on each is determined by the Jordan product on $\mathcal{A}_{\mathrm{sa}}$ and by separate w*-continuity of this product. The w*-topologies are determined by pointwise convergence on K, so coincide, and therefore the Jordan products do as well. □

2.77. Proposition. *If A is a JC-algebra, then A^{**} is a JW-algebra.*

Proof. Let π be the inclusion map from A into $\mathcal{B}(H)_{\mathrm{sa}}$. Then by Proposition 2.74, π^{**} is a Jordan isomorphism from A^{**} onto a σ-weakly closed Jordan subalgebra of $(\mathcal{B}(H)_{\mathrm{sa}})^{**}$. By Lemma 2.76 the latter can be identified with the self-adjoint part of the enveloping von Neumann algebra $\mathcal{B}(H)^{**}$ (A 101) of $\mathcal{B}(H)$. Thus A^{**} is a JW-algebra. □

2.78. Corollary. *If A is a JC-algebra and also a JBW-algebra, then A is a JW-algebra.*

Proof. Since A is a JC-algebra, A^{**} is a JW-algebra (Proposition 2.77). Since A is a JBW-algebra, there is an isomorphism from A onto a direct summand of A^{**} (Lemma 2.54). A direct summand of a JW-algebra is a JW-algebra. □

Order automorphisms of a JB-algebra

2.79. Definition. A *unital order automorphism* of a JB-algebra is a bijective linear map ϕ such that $\phi(1) = 1$, and such that both ϕ and ϕ^{-1} preserve order.

2.80. Theorem. *A unital order automorphism of a JB-algebra A is a Jordan algebra automorphism*

Proof. Let ϕ be an order automorphism of A. Recall (from Theorem 1.11) that A has the order unit norm, i.e.,

$$\|a\| = \inf\{\lambda \in \mathbf{R} \mid -\lambda 1 \leq a \leq \lambda 1\}.$$

Thus ϕ is an isometry. Then the dual map $\phi^* : A^* \to A^*$ is also an order isomorphism and an isometry, as is the bidual map $\phi^{**} : A^{**} \to A^{**}$. It suffices to prove that ϕ^{**} is a Jordan algebra isomorphism, and thus it is enough to prove the statement in the theorem when the algebra A is a JBW-algebra.

Since the projections in A are precisely the extreme points of the order interval $[0,1]$ (Proposition 1.40), it follows that ϕ maps projections to projections. Since projections p, q are orthogonal iff $p+q \leq 1$ (Proposition 2.18), then ϕ preserves orthogonality of projections. If $\sum_i \lambda_i p_i$ is any linear combination of orthogonal projections, then

$$\begin{aligned}\phi\left(\left(\sum_i \lambda_i p_i\right)^2\right) &= \phi\left(\sum_i \lambda_i^2 p_i\right) = \sum_i \lambda_i^2 \phi(p_i) \\ &= \left(\sum_i \lambda_i \phi(p_i)\right)^2 = \left(\phi\left(\sum_i \lambda_i p_i\right)\right)^2.\end{aligned}$$

By the spectral theorem (Theorem 2.20) every element of A is a norm limit of linear combinations of orthogonal projections, so by norm continuity of ϕ we have shown $\phi(a^2) = (\phi(a))^2$ for all $a \in A$. Since $a \circ b = \frac{1}{2}(a+b)^2 - a^2 - b^2)$, it follows that ϕ preserves the Jordan product. $\square$

We are now able to relate skew order derivations and Jordan derivations. (See Definition 1.57 and equation (1.70).)

2.81. Lemma. *Let A be a JB-algebra with state space K and let δ be an order derivation on A. For every $t \in \mathbf{R}$ let $\alpha_t = e^{t\delta}$. The following are equivalent:*

(i) *δ is skew.*

(ii) α_t *is a Jordan automorphism for all* t.
(iii) δ *is a Jordan derivation.*

Proof. (i) $\Rightarrow$ (ii) For each $t \in \mathbf{R}$, $\alpha_t(1) = 1$ (Proposition 1.58), so each α_t is a unital order isomorphism. Therefore each α_t is also a Jordan automorphism (Theorem 2.80).

(ii) $\Rightarrow$ (iii) Since $\alpha_t = e^{t\delta}$ is a Jordan automorphism, then

$$\begin{aligned}\delta(a \circ b) &= \lim_{t \to 0} t^{-1}(e^{t\delta}(a \circ b) - a \circ b)\\ &= \lim_{t \to 0} t^{-1}((e^{t\delta}a) \circ (e^{t\delta}b) - a \circ b)\\ &= \lim_{t \to 0} t^{-1}(e^{t\delta}a) \circ (e^{t\delta}b - b) + \lim_{t \to 0} t^{-1}(e^{t\delta}a - a) \circ b\\ &= a \circ (\delta b) + (\delta a) \circ b,\end{aligned}$$

so Leibniz' rule (1.70) holds and thus δ is a Jordan derivation.

(iii) $\Rightarrow$ (i) By Leibniz' rule (1.70), $\delta(1) = \delta(1 \circ 1) = 2\delta(1)$. Hence $\delta(1) = 0$. □

In general a Jordan automorphism need not fix the center. However, if it lies on the one-parameter group generated by a skew order derivation, then it must.

2.82. Proposition. *If* δ *is a skew order derivation on a JB-algebra* A *and* $z \in Z(A)$, *then*

(i) $e^{t\delta}z = z$ *for all* $t \in \mathbf{R}$, *and*
(ii) $\delta z = 0$.

Proof. By Corollary 2.50 the bidual A^{**} is a JBW-algebra. Since δ is skew, by Lemma 2.81, $e^{t\delta}$ is a Jordan automorphism. By σ-weak continuity of multiplication in each variable separately, the bidual map $(e^{t\delta})^{**}$ is also a Jordan automorphism. Furthermore, again by σ-weak continuity of multiplication, the center of A will be contained in the center of A^{**}, so it suffices to prove the lemma for the special case where A is a JBW-algebra. Recall that the center of a JBW-algebra is an associative JBW-algebra (Proposition 2.36). Then by the spectral theorem (Theorem 2.20) applied to the center, it is enough to prove the lemma for the case where z is a central projection. Since $e^{t\delta}$ is a Jordan automorphism, then $e^{t\delta}z$ is also a central projection for $t \in \mathbf{R}$. Thus $e^{t\delta}z - z$ is the difference of two central projections. Since the center is associative, it is isomorphic to $C_{\mathbf{R}}(X)$ for a compact Hausdorff space X, so the difference of two central projections has norm either zero or 1. Thus $\|e^{t\delta}z - z\|$ is either zero or 1. Since $\|e^{t\delta}z - z\|$ is a continuous function of t which is zero when $t = 0$, it must be zero for all t. This proves (i).

Now also $\delta z = \lim_{t \to 0} t^{-1}(e^{t\delta}z - z) = 0$, which proves (ii). □

Abstract characterization of compressions

We can now give an order-theoretic characterization of the maps U_p on JBW-algebras. Let M be a JBW-algebra. We will say that a positive projection (idempotent linear map) $P : M \to M$ is *normalized* if either it is the zero map or else has norm 1. Two positive projections P, Q on an ordered linear space are said to be *complementary* (and Q is said to be a complement of P and vice versa) if

$$\ker^+ Q = \operatorname{im}^+ P, \quad \ker^+ P = \operatorname{im}^+ Q,$$

cf. (A 4). Also we will say that a positive projection $P : M \to M$ is *complemented* if there exists a positive projection $Q : M \to M$ such that P, Q are complementary, and we will say that a σ-weakly continuous positive projection $P : M \to M$ is *bicomplemented* if there exists a σ-weakly continuous positive projection $Q : M \to M$ such that P, Q are complementary and the dual projections $P^*, Q^* : M_* \to M_*$ are also complementary. Note that if p is a projection in a JBW-algebra M, then by Proposition 1.38 and Proposition 1.41, U_p and U_{1-p} are bicomplementary normalized positive projections. We now show that these properties characterize compressions on JBW-algebras.

2.83. Theorem. *Let M be a JBW-algebra and let $P : M \to M$ be a σ-weakly continuous normalized positive projection. There exists a projection $p \in M$ such that $P = U_p$ iff P is bicomplemented; in this case p is unique: $p = P1$, and the complement P' of P is also unique: $P' = U_{1-p}$.*

Proof. As observed above, U_p and $U_{p'}$ are bicomplementary. Now let P, P' be any bicomplementary normalized positive projections on M. We will first establish the following properties:

(2.36) $$P1 + P'1 = 1,$$

(2.37) $$P1 \quad \text{is a projection,}$$

(2.38) $$\operatorname{im}^+(P) = \operatorname{face}(P1).$$

To verify these, first note that $P1 \le 1$ (since $\|P\| \le 1$), and $P(1 - P1) = 0$, so $P'(1 - P1) = 1 - P1$, which implies (2.36). To verify (2.37), by Proposition 1.40 it suffices to show $P1$ is an extreme point of $[0, 1]$. Suppose that

(2.39) $$P1 = \lambda a + (1 - \lambda)b,$$

where $0 < \lambda < 1$ and $a, b \in [0, 1]$. Applying P' to both sides shows $P'a = P'b = 0$ and thus $a = Pa \le P1$ and $b = Pb \le P1$. By (2.39) it follows that $a = b = P1$, so $P1$ is an extreme point of $[0,1]$, which verifies (2.37). Finally, suppose that $a \in \text{face}(P1)$. Then $P'a = 0$ so $Pa = a$, i.e., $a \in \text{im}^+(P)$. Now we prove the opposite inclusion. If a is an arbitrary element in $\text{im}^+(P)$, then applying P to $0 \le a \le \|a\|1$ gives $0 \le a = Pa \le \|a\|P1$, so $a \in \text{face}(P1)$, which completes the proof of (2.38).

Now let $p = P1$. We will complete the proof of the theorem by showing $P = U_p$ and $P' = U_{1-p}$. By (2.38) and (1.51), $\text{im}^+P = \text{im}^+U_p$. Since $\text{im}\,P$ and $\text{im}\,U_p$ are positively generated, then $\text{im}\,P = \text{im}\,U_p$. By (2.36) we also have $P'1 = 1 - P1 = 1 - p = U_{1-p}1$, so applying (2.38) with P' in place of P we conclude $\text{im}\,P' = \text{im}\,U_{1-p}$. Taking annihilators in M_*, we have

$$\ker P'^* = \ker U_{1-p}^*. \tag{2.40}$$

By hypothesis P^* and P'^* are complementary, and so are U_p^* and U_{1-p}^* by Proposition 1.41. Thus (2.40) implies

$$\text{im}^+P^* = \ker{}^+P'^* = \ker{}^+U_{1-p}^* = \text{im}^+U_p^*.$$

From this $\text{im}\,P^* = \text{im}\,U_p^*$ follows. Taking annihilators gives $\ker P = \ker U_p$. Thus the idempotent maps P and U_p have the same image and kernel, and so must be equal. A similar argument shows $U_{1-p} = P'$, and this completes the proof. □

We remark that the same theorem holds for a JB-algebra A with the dual maps P^*, P'^* living on A^*. The proof is the same except for the context being the duality of A and A^* instead of M and M_*.

Notes

Normed infinite dimensional Jordan algebras first appeared in the work of von Neumann [98]. Motivated by axiomatic quantum mechanics, he studied what are essentially JBW-algebras, though defined in quite different language. On physical grounds, he assumed power associativity instead of the Jordan identity (1.2). He proved the key fact that orthogonal projections operator commute (Lemma 2.48), and then showed that power associativity implies the Jordan identity. In modern language, he showed that the type I JBW-factors are not essentially different than those found in finite dimensions in [75]. Infinite dimensional Jordan algebras also appear briefly in Segal's paper [114] on axiomatic quantum mechanics.

Topping [128] generalized much of the structure theory for von Neumann algebras to JW-algebras. Størmer studied JW-algebras and the von Neumann algebras they generate in [125, 126]. Janssen [73, 74] studied a class of infinite dimensional Jordan algebras with a trace.

Monotone closed JB-algebras with a separating set of normal states, i.e., JBW-algebras (Definition 2.2), were used as a key tool in the proof of the Gelfand–Naimark type theorem in [8]. Most of the basic working tools for JBW-algebras described in Chapter 2 appear in [8]. The key technical result that the monotone closed subspace generated by a JB-algebra A in A^{**} is all of A^{**} (Lemma 2.53) we have just referred to [67], where the proof follows the lines of Pedersen's proof of the analogous result for C*-algebras.

JBW-algebras were first formally defined by Shultz [116], who showed that a JB-algebra is a Banach dual space iff it is monotone complete with a separating set of normal states (Theorem 2.55), and that a concrete JC-algebra admits a faithful weakly-closed representation (i.e., is a JW-algebra) iff it is a JBW-algebra (Corollary 2.78). This equivalence generalizes the abstract characterizations of von Neumann algebras due to Kadison [78] and to Sakai [110], cf. (A 95).

Edwards in [42, 43] studied quadratic ideals of JB-algebras, i.e., those subspaces J closed under the map $a \mapsto \{b_1 a b_2\}$ for all b_1, b_2 in the algebra. He showed that norm closed quadratic ideals are the same as the hereditary JB-subalgebras, and established the correspondence of σ-weakly closed hereditary subalgebras of JBW-algebras with projections in the algebra (Proposition 2.32).

The result that the bidual of a JB-algebra is also a JB-algebra (Corollary 2.50) is in [116], but we have used the much simpler proof due to Hanche-Olsen [64]. Hanche-Olsen's result is for JB-algebras, but we have first used essentially the same proof to show that the bidual of a commutative order unit algebra is an order unit algebra (Theorem 2.46). This is a key step in showing that commutative order unit algebras are the same as JB-algebras (Theorem 2.49), a result due to Iochum and Loupias [69, 70], and then Corollary 2.50 follows immediately.

The fact that projection lattices are orthomodular goes back to Loomis [93] for von Neumann algebras, in his lattice-theoretic axiomatization of dimension theory, and is proven for JW-algebras in [128] and for JBW-algebras, cf. Proposition 2.25, in [8]. The standard reference for orthomodular lattices is the book of Kalmbach [81]. Orthomodular lattices play an important role in the "quantum logic" approach to the foundations of quantum mechanics. This is discussed in more detail in the notes to Chapter 3, and in the remarks after Corollary 8.18.

The proof that unital order automorphisms of JB-algebras are Jordan automorphisms (Theorem 2.80) is based on the proof of Kadison's similar result for C*-algebras [77].

3 Structure of JBW-algebras

This chapter will begin by describing the notion of equivalence of projections in JBW-algebras. (This generalizes unitary equivalence rather than Murray–von Neumann equivalence of projections in von Neumann algebras.) We will define the notion of type I and type I_n JBW-algebras, and describe the classification of type I_n JBW-factors. (For the sake of brevity, most of these results will be stated without proof, with references given to [67].) Finally, we will investigate atomic JBW-algebras, in which every projection dominates a minimal projection. We will show that every JBW-algebra admits a decomposition into atomic and non-atomic parts, and that the atomic part is the direct sum of type I factors.

Equivalence of projections

Recall that projections p and q in a von Neumann algebra are (Murray–von Neumann) equivalent if there is an element v such that $v^*v = p$ and $vv^* = q$, cf. (A 161). Since Jordan multiplication is commutative, it is not obvious how to generalize this notion to JBW-algebras. However, Topping [128] showed that a satisfactory theory of equivalence can be built by defining projections p and q to be equivalent if there is a finite sequence $s_1, s_2, \ldots, s_n$ of symmetries such that $s_1 s_2 \ldots s_n p s_n \ldots s_2 s_1 = q$. Note that for von Neumann algebras this implies unitary equivalence, and indeed is the same as unitary equivalence (A 179).

3.1. Definition. Projections p and q in a JBW-algebra M are *equivalent* if there is a finite sequence $s_1, s_2, \ldots, s_n$ of symmetries such that $\{s_1\{s_2 \ldots \{s_n p s_n\} \ldots s_2\} s_1\} = q$. In this case we write $p \sim q$.

We will often make use of the following stronger property.

3.2. Definition. Projections p and q in a JBW-algebra M are *exchanged by the symmetry s* if $\{sps\} = q$. (This relation is symmetric, since $U_s^2 = 1$, but is not necessarily transitive.) We will say p and q are *exchangeable* if there exists a symmetry that exchanges them.

The following class of algebras will play a key role in our analysis of the structure of JBW-algebras.

3.3. Definition. Let B be a real associative algebra with identity and with involution $*$. Then $M_n(B)$ denotes the *-algebra of $n \times n$ matrices with entries in B, equipped with the involution $(a_{ij}) \mapsto (a_{ji}^*)$. We let $H_n(B)$ denote the set of Hermitian $(a_{ij}^* = a_{ji})$ matrices in $M_n(B)$ with the Jordan product $a \circ b = \frac{1}{2}(ab + ba)$ for $a, b \in H_n(B)$.

Let B be as above, and let $\{e_{ij}\}$ be the standard matrix units in $M_n(B)$. Note that for each i and j, $s_{ij} = e_{ij} + e_{ji} + (1 - e_{ii} - e_{jj})$ is a symmetry that exchanges e_{ii} and e_{jj}. Thus 1 can be written as the sum of n projections pairwise exchangeable by symmetries. The *Jordan coordinatization theorem* (Theorem 3.27) says roughly that the converse is true: given a Jordan algebra whose identity is the sum of n exchangeable projections $(4 \leq n < \infty)$, then the Jordan algebra is isomorphic to $H_n(B)$ for some associative *-algebra B. For JBW-algebras, we will see that this implies that the algebra is in fact a JW-algebra. Thus it will be very useful to be able to show that the identity can be written as a sum of at least four exchangeable projections.

3.4. Definition. Let M be a JBW-algebra. A *partial symmetry* is an element of M whose square is a projection. We define exchange of projections by partial symmetries in the same way as for symmetries (Definition 3.2). If e is a projection and $s^2 = e$, then we say s is an *e-symmetry.*

Note that if $s^2 = e$ where e is a projection, then by spectral theory s has spectrum in $\{0, 1\}$, so $s^3 = s$. Thus $s \circ e = s \circ s^2 = s^3 = s$, and therefore s is a symmetry in the JBW-algebra $M_e = \{eMe\}$.

3.5. Lemma. *Let e be a projection in a JBW-algebra M, and let p, q be projections in M_e. Then there is a symmetry exchanging p and q iff there is an e-symmetry exchanging p and q.*

Proof. Let e be any projection such that M_e contains p and q, and let s be any e-symmetry exchanging p and q. (Note that by the remarks preceding this lemma, $s \in M_e$.) Let $t = s + 1 - e$. Since e is the identity of M_e, then $(1 - e) \circ x = 0$ for all $x \in M_e$. It follows that $t^2 = 1$, and that $t \circ x = s \circ x$ for all $x \in M_e$. Thus by the definition (1.13) of U_t,

$$U_t p = 2t \circ (t \circ p) - t^2 \circ p = 2s \circ (s \circ p) - p.$$

Then

$$U_s p = 2s \circ (s \circ p) - s^2 \circ p = 2s \circ (s \circ p) - p = U_t p.$$

By hypothesis $U_s p = q$, so $U_t p = q$. Thus t is a symmetry that exchanges p and q.

Now let t be any symmetry that exchanges p and q, and let $f = p \vee q$. Define $s = \{ftf\}$. Recall that U_t is an automorphism of M (Proposition 2.34) and so U_t takes $f = p \vee q$ to $(U_t p) \vee (U_t q) = q \vee p = f$. By the Jordan identity (1.15)

$$s^2 = \{ftf\}^2 = \{f\{tf^2t\}f\} = \{f\{tft\}f\} = \{fff\} = f,$$

so s is an f-symmetry. Since p and q are in face$(f) = \mathrm{im}^+ U_f$, by the identity (1.16)

$$U_s p = U_{\{ftf\}} p = U_f U_t U_f p = q.$$

Thus s is a $(p \vee q)$-symmetry that exchanges p and q.

By assumption M_e contains p and q. By the previous paragraph, there is a $(p \vee q)$-symmetry s that exchanges p and q. By the first paragraph of this proof, applied to the pair $(M_e, M_{p\vee q})$ in place of (M, M_e), we can extend the $(p \vee q)$-symmetry s to an e-symmetry that exchanges p and q. □

In the following, we make use of the general triple product $\{abc\}$ defined previously (cf. (1.12)):

$$\{xay\} = (x \circ a) \circ y + (a \circ y) \circ x - (x \circ y) \circ a. \tag{3.1}$$

Note that this product is symmetric in x and y, i.e., $\{xay\} = \{yax\}$. By linearity of the triple product in each factor,

$$\{(x+y)a(x+y)\} = U_x a + U_y a + 2\{xay\},$$

so

$$\{xay\} = \tfrac{1}{2}(U_{x+y} - U_x - U_y)a. \tag{3.2}$$

Recall that if p is a projection in a JB-algebra and $t = p - p'$, where $p' = 1 - p$, then t is a symmetry, and each symmetry arises in this way from a unique projection. Note also that the equivalence of (iii), (iv), (vi) in Lemma 3.6 below generalizes the similar equivalence for von Neumann algebras in (A 188).

3.6. Lemma. *Let A be a JB-algebra, and let p be a projection in A. Let $p' = 1 - p$, $t = p - p'$, $a \in A$. Then the following are equivalent:*

(i) $U_p a = U_{p'} a = 0$,
(ii) $p \circ a = p' \circ a = \frac{1}{2} a$,
(iii) $t \circ a = 0$,
(iv) $U_t a = -a$,
(v) $a \in \{pAp'\}$.

If $a = q - q'$ where q is a projection, the above are equivalent to

(vi) $U_t q = q'$.

Proof. (i) $\Rightarrow$ (ii) By equation (1.47)

$$p \circ a = \tfrac{1}{2}(I + U_p - U_{p'})a,$$

so (i) implies that $p \circ a = \frac{1}{2}a$. Then $p' \circ a = (1-p) \circ a = \frac{1}{2}a$, so (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii) Trivial, since $t = p - p'$.

(iii) $\Rightarrow$ (iv) By the definition of U_t,

$$U_t a = 2t \circ (t \circ a) - t^2 \circ a = 2t \circ (t \circ a) - a,$$

so $t \circ a = 0$ implies (iv).

(iv) $\Rightarrow$ (i) By equation (2.25), $U_t = 2U_p + 2U_{p'} - I$. If (iv) holds, then

$$2U_p a + 2U_{p'} a - a = -a,$$

so $U_p a + U_{p'} a = 0$. Applying U_p and $U_{p'}$ to this equation gives (i).

(i) $\Rightarrow$ (v) By (3.2),

$$\{pap'\} = \tfrac{1}{2}(I - U_p - U_{p'})a. \tag{3.3}$$

Then (i) implies $a = 2\{pap'\} \in \{pAp'\}$.

(v) $\Rightarrow$ (i) If $a \in \{pAp'\}$, then by (3.3), for some $b \in A$, we have $a = \frac{1}{2}(I - U_p - U_{p'})b$. Applying U_p and $U_{p'}$ to both sides gives (i).

(iv) $\Rightarrow$ (vi) Let q be a projection and $a = q - q'$. If (iv) holds, then $U_t(q - q') = q' - q$. Adding to this the equation $U_t 1 = 1$ in the form $U_t(q + q') = q + q'$ gives $U_t(2q) = 2q'$, which implies (vi).

(vi) $\Rightarrow$ (iv) If $U_t q = q'$, since $U_t^2 = I$, then $U_t q' = q$. Now $U_t(q - q') = q' - q$ follows. □

3.7. Definition. Orthogonal projections p, q in a JB-algebra are *strongly connected* if there exists a $(p+q)$-symmetry s in $\{pMq\}$.

3.8. Lemma. *Orthogonal projections p, q in a JBW-algebra M are strongly connected iff they are exchangeable by a symmetry.*

Proof. We first show that

$$\{pMq\} = \{p(M_{p+q})q\}. \tag{3.4}$$

If $x \in \{pMq\}$, then by (3.2) there exists $b \in M$ such that

$$x = \tfrac{1}{2}(U_{p+q} - U_p - U_q)b.$$

Then since $U_pU_{p+q} = U_p$ and $U_qU_{p+q} = U_q$ (cf. Proposition 2.26),

$$x = \tfrac{1}{2}(U_{p+q} - U_p - U_q)(U_{p+q}b) = \{p(U_{p+q}b)q\} \in \{pM_{p+q}q\}.$$

This proves the left side of (3.4) is contained in the right, and the opposite inclusion is clear, so we have established (3.4). Thus p and q are strongly connected in M iff they are strongly connected in M_{p+q}. By Lemma 3.5, p and q are exchangeable by a symmetry in M iff they are exchangeable in M_{p+q}. Thus by working in M_{p+q}, without loss of generality we may suppose that $p + q = 1$, so that $q = p'$.

Suppose that s is a symmetry, and let $t = p - p'$. Applying the equivalence of (iii) and (v) in Lemma 3.6 we have

$$s \in \{pMp'\} \iff t \circ s = 0. \tag{3.5}$$

By the equivalence of (iii) and (vi) in Lemma 3.6 (with s in place of t and p in place of q)

$$s \circ t = 0 \iff U_s p = p'. \tag{3.6}$$

Combining (3.5) and (3.6), we conclude that p and p' are strongly connected iff they are exchanged by a symmetry, which completes the proof of the lemma. □

Comparison theory

In this section we state the basic results on equivalence of projections, just giving a reference for most proofs. Recall that the central cover $c(p)$ of a projection p is the least central projection that dominates p, cf. Definition 2.38.

3.9. Lemma. *Equivalent projections in a JBW-algebra M have the same central cover.*

Proof. By [67, 5.1.4], if p is a projection in M with central cover $c(p)$, then $c(p)$ is the least upper bound in the lattice of projections of all projections equivalent to p. The lemma follows. □

One virtue of Murray–von Neumann equivalence of projections is countable additivity (A 163). As remarked before, in the von Neumann algebra context the Jordan notion of equivalence is the same as unitary equivalence, so it is easy to see that countable additivity fails. However, we have the following substitute.

3.10. Proposition. *Let $\{p_\alpha\}$ and $\{q_\alpha\}$ be orthogonal collections of projections with $\sum_\alpha (p_\alpha + q_\alpha) \leq 1$. If p_α and q_α are exchanged by a symmetry for each α, then $p = \sum_\alpha p_\alpha$ and $q = \sum_\alpha q_\alpha$ are exchanged by a symmetry.*

Proof. [67, Lemma 5.2.9]. □

3.11. Corollary. *If M is a JBW-algebra in which 1 is the sum of infinitely many exchangeable projections, then for any $n < \infty$, 1 is the sum of n exchangeable projections.*

Proof. We may divide the set of exchangeable projections into n equinumerous subsets, and form the sum of each. The resulting n projections can be exchanged by symmetries by Proposition 3.10. □

The following is an analog of the comparison theorem (A 164) for projections in a von Neumann algebra, and directly generalizes (A 180).

3.12. Theorem. (Jordan comparison theorem) Let p and q be projections in a JBW-algebra M. Then there is a central projection c such that cp is exchangeable by a symmetry with a subprojection of cq, and $c'q$ is exchangeable by a symmetry with a subprojection of $c'p$.

Proof. [67, Thm. 5.2.13]. □

The following result relates the centers of M and M_p. Its proof relies on the Jordan comparison theorem.

3.13. Proposition. *Let p be a projection in a JBW-algebra M, and let Z be the center of M. Then the center of M_p is Zp. Furthermore,*

(i) *The map $z \mapsto zp$ (or equivalently, the map U_p) is an isomorphism from $c(p)Z$ onto the center Zp of M_p.*
(ii) *If e is a central projection in M_p, then $e = c(e)p$.*

Proof. The fact that the center of M_p is Zp can be found in [67, Prop. 5.2.17]. The statement (ii) is part of the proof of [67, Prop. 5.2 17]. Finally, to show the map $z \mapsto zp$ is 1-1 on $c(p)Z$, we show that its kernel is trivial. Let w be any central projection in $c(p)Z$ such that $wp = 0$, and let $w' = 1 - w$. Then $w'p = p$, so $w' \geq c(p)$. Since $w \in c(p)Z$, then $w \leq c(p) \leq w'$, which implies $w = 0$. □

Global structure of type I JBW-algebras

JBW-algebras can be classified into types I, II, III (cf. [67, Thm. 5.1.5]) in a manner similar to that for von Neumann algebras. However, we will just be interested in a decomposition into type I and non-type I summands (soon to be defined), and the further decomposition of type I into type I_n summands for $n = 0, 1, 2, \ldots, \infty$.

We review the situation for type I von Neumann algebras, cf. (A 168). The type I factors are the algebras $\mathcal{B}(H)$ where H is a finite or infinite dimensional Hilbert space (cf. [80, Thm. 6.6.1]). The dimension of H is the same as the cardinality of any set of minimal projections with sum the identity. More generally one defines a von Neumann algebra to be of type I_n if the identity is the sum of n equivalent abelian projections. Every type I von Neumann algebra is the direct sum of algebras of type I_n for various cardinal numbers n. We will see that the situation is quite similar for JBW-algebras, except that there is a greater variety of type I factors (especially of types I_2 and I_3).

3.14. Definition. A projection p in a JBW-algebra M is *abelian* if the algebra $M_p = \{pMp\}$ is associative.

Recall that T_a denotes Jordan multiplication by a. For elements a, b, c of a Jordan algebra,

$$a \circ (b \circ c) = (a \circ b) \circ c \iff T_a T_c b = T_c T_a b,$$

so a Jordan algebra is associative iff all elements operator commute (cf. Definition 1.46). Hence a projection p in a JBW-algebra M is abelian iff all elements of M_p operator commute. Thus p is abelian iff every element of M_p is central in M_p. If M is a concrete JW-algebra, then it follows from Proposition 1.49 that p is abelian iff M_p consists of mutually commuting elements. Thus this generalizes the notion of abelian projections in von Neumann algebras, cf. (A 165).

3.15. Definition. A JBW-algebra is *type I* if it contains an abelian projection with central cover 1.

3.16. Lemma. *Let M be a JBW-algebra. Then there is a unique decomposition $M = M_1 \oplus M_2$ where M_1 is of type I and M_2 has no summand of type I.*

Proof. [67, Thm. 5.1.5]. □

As discussed after Definition 3.3, one of our goals will be to describe circumstances when the identity can be written as the sum of (ideally at least four) equivalent projections. In JBW-algebras without type I direct summands we have the following result.

3.17. Proposition. *Let M be a JBW-algebra with no type I direct summand. Then there are four projections with sum 1 such that each pair is exchangeable by a symmetry.*

Proof. [67, Prop. 5.2.15]. □

The following result makes specific one way in which type I JBW-algebras have "many" abelian projections.

3.18. Lemma. *If q is any projection in a type I JBW-algebra M, then there is an abelian projection p with $p \leq q$ and $c(p) = c(q)$.*

Proof. [67, Lemma 5.3.2]. □

Recall that equivalent projections have the same central cover (Lemma 3.9). The following strong converse is true for abelian projections.

3.19. Lemma. *If p and q are abelian projections in a JBW-algebra with the same central cover, then there is a symmetry exchanging p and q.*

Proof. [67, 5.3.2]. □

3.20. Corollary. *Equivalent abelian projections in a JBW-algebra are exchangeable by a symmetry.*

Proof. By Lemma 3.9, equivalent projections have the same central cover. By Lemma 3.19, abelian projections with the same central cover are exchangeable. □

3.21. Definition. Let M be a JBW-algebra and n a cardinal number. Then M is of *type I_n* if the identity of M is the sum of n equivalent abelian projections. (If n is an infinite cardinal, such a sum converges σ-strongly, cf. Lemma 2.43.) M is of *type I_∞* if $M = \bigoplus_\alpha M_\alpha$ where each M_α is of type I_{n_α} for n_α an infinite cardinal.

If M is of type I_n, and $\{p_\alpha\}$ is a collection of equivalent abelian projections with sum 1, then the central covers $c(p_\alpha)$ coincide, and must dominate every p_α. Therefore this common central cover must equal 1. Conversely, if $\{p_\alpha\}$ is a collection of abelian projections with sum 1 and with $c(p_\alpha) = 1$ for all α, then these projections are equivalent (Lemma 3.19). Thus in the definition above we could replace the requirement that the projections be equivalent by the requirement that their central covers are all equal to 1. (This is the form of the definition of type I_n in [67, 5.3.3].) Note that in the definition above of a JBW-algebra of type I_∞, we are not assured that the identity is a sum of equivalent abelian projections. Finally, observe that M is of type I_1 iff the identity 1 is an abelian projection, i.e., iff M is associative.

3.22. Lemma. *Let M be a JBW-algebra of type I_n with $n < \infty$. Then n is uniquely determined by M, and any set of orthogonal non-zero projections with the same central cover has cardinality at most n.*

Proof. [67, Lemma 5.3.4]. □

3.23. Theorem. *Let M be a JBW-algebra of type I. Then there is a unique decomposition*

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_\infty,$$

where each M_n is either $\{0\}$ or of type I_n.

Proof. [67, Thm. 5.3.5]. □

3.24. Proposition. *If M is a JBW-algebra of type I_∞, then the identity can be expressed as the sum of four exchangeable projections.*

Proof. Write $M = \bigoplus_\alpha M_\alpha$ where M_α is of type I_{n_α} for n_α an infinite cardinal. Fix α, and write the identity of M_α as a sum of an infinite collection of equivalent abelian projections. These projections are exchangeable (Corollary 3.20), so by Corollary 3.11 we can write the identity of M_α as a sum of four projections e_α, f_α, g_α, h_α exchanged by symmetries. Repeat this construction for each α, and let $e = \sum_\alpha e_\alpha$, $f = \sum_\alpha f_\alpha$, $g = \sum_\alpha g_\alpha$, and $h = \sum_\alpha h_\alpha$. Then e, f, g, h are projections pairwise exchangeable by symmetries (Proposition 3.10), and $1 = e + f + g + h$, which completes the proof. □

Classification of type I JBW-factors

3.25. Definition. A JBW-algebra is a *JBW-factor* if its center consists of scalar multiples of the identity.

Since the center of a JBW-algebra is a JBW-subalgebra (Proposition 2.36), by the spectral theorem it is generated by its central projections. Thus a JBW-algebra M is a factor iff the only central projections are 0 and 1.

Our main purpose in this section is to develop the classification of type I JBW-factors. We will see that for $n \geq 4$ the classification of JBW-factors of type I_n is analogous to that for von Neumann algebras (except that the Hilbert space can be real, complex, or quaternionic), but that there are new possibilities for types I_3 and I_2.

The key to the classification of type I factors is the Jordan coordinatization theorem, which describes circumstances under which a Jordan algebra is isomorphic to one of the form $H_n(B)$ (see Definition 3.3). For $n = 3$, we have to allow algebras B which are not quite associative.

3.26. Definition. An algebra B is *alternative* if it satisfies the identities

$$a^2 b = a(ab) \quad \text{and} \quad ba^2 = (ba)a.$$

3.27. Theorem. *(Jordan coordinatization theorem) Let M be a Jordan algebra over $\mathbf{R}$ with identity 1. Assume that 1 is the sum of n idempotents $p_1, \ldots, p_n$ $(3 \leq n < \infty)$ with each pair p_i, p_j exchangeable by symmetries. Then there is a $*$-algebra B over $\mathbf{R}$ and a Jordan isomorphism of M onto $H_n(B)$ carrying p_i to the matrix unit e_{ii} for each i. Any such algebra B is alternative, and will be associative if $n \geq 4$.*

Proof. [67, Thm. 2.8.9 and Thm. 2.7.6]. (The statement of the Jordan coordinatization theorem in [67, Thm. 2.8.9] assumes that the projections are "strongly connected", which by Lemma 3.8 is the same as the projections being exchanged by a symmetry.) □

We will explain the idea of the proof of Theorem 3.27. Suppose that $M \cong H_n(B)$ with $n = 3$. The key is the observation for 3×3 matrices that the associative product is to the following extent determined by the Jordan product:

$$\begin{pmatrix} 0 & ab & 0 \\ b^* a^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 2 \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ a^* & 0 & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & b^* \\ 0 & b & 0 \end{pmatrix}. \tag{3.7}$$

We can identify the algebra B as a vector space with the matrices of the form

$$\begin{pmatrix} 0 & a & 0 \\ a^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

i.e., with the subspace $\{e_{11} M e_{22}\}$. Then we can use (3.7) to recover the associative multiplication from the Jordan product. However, for this purpose we need to be able to recover $ae_{13} + a^* e_{31}$ from $ae_{12} + a^* e_{21}$ and $b^* e_{23} + be_{32}$ from $be_{12} + b^* e_{21}$. We can use the symmetries $s_{23} = e_{11} + e_{23} + e_{32}$ and $s_{13} = e_{22} + e_{13} + e_{31}$, since

$$ae_{13} + a^* e_{31} = U_{s_{23}}(ae_{12} + a^* e_{21}),$$

and

$$b^* e_{23} + be_{32} = U_{s_{13}}(be_{12} + b^* e_{21}).$$

Thus for $x, y \in B = \{e_{11}Me_{22}\}$ the equation (3.7) becomes

$$xy = 2(U_{s_{23}}x) \circ (U_{s_{13}}y).$$

Similarly, letting $s_{12} = e_{12} + e_{21} + e_{33}$, since

$$U_{s_{12}}(ae_{12} + a^*e_{21}) = a^*e_{12} + ae_{21},$$

then we can recover the involution $*$ on $B = \{e_{11}Me_{22}\}$ by

$$x^* = U_{s_{12}}x.$$

Now for a Jordan algebra satisfying the hypotheses of the Jordan coordinatization theorem, one imitates this process, replacing the $\{e_{ii}\}$ by the given projections $\{p_i\}$, and the $\{s_{ij}\}$ by the given symmetries that exchange the projections. Finally, the Jordan axiom (1.2) is used to show that B is associative (if $n \geq 4$), and alternative if $n = 3$.

Note that this construction depends on the choice of the projections and the symmetries exchanging them, and so the algebra B may not be uniquely determined by M.

3.28. Definition. A non-zero projection p in a JBW-algebra M is *minimal* or an *atom* if $q \leq p$ for a non-zero projection q implies $q = p$.

3.29. Lemma. *A projection p in a JBW-algebra M is minimal iff $M_p = \mathbf{R}p$.*

Proof. Every projection in M_p is dominated by p, so if p is an atom, then the only non-zero projection in the JBW-algebra M_p is p. Then by the spectral theorem (Theorem 2.20), $M_p = \mathbf{R}p$. Conversely, since $\text{face}(p) = \text{im}^+U_p = M_p$, every projection under p is in M_p, so if $M_p = \mathbf{R}p$ then there are no projections between 0 and p, i.e., p is an atom. □

3.30. Lemma. *A non-zero projection in a JBW-factor is abelian iff it is minimal.*

Proof. If p is minimal, then $M_p = \mathbf{R}p$ is associative, so by definition p is abelian. If p is abelian, and q is a projection with $q \leq p$, then q is central in M_p (see the remarks following Definition 3.14). Then by Proposition 3.13(ii), $q = c(q)p$, and $c(q)$ must be 0 or 1 since M is a factor. Thus p is minimal. □

3.31. Definition. $\mathbf{H}$ will denote the quaternions, and $\mathbf{O}$ will denote the octonions (or Cayley numbers) with their usual involutions. We note that $\mathbf{H}$ is four dimensional over $\mathbf{R}$, and $\mathbf{O}$ is eight dimensional. The self-adjoint elements are just real multiples of the identity, and the skew adjoint elements are spanned by the elements b such that $b^2 = -1$. $\mathbf{H}$ is associative but $\mathbf{O}$ is only alternative. See [67, proof of Thm. 2.2.6].

3.32. Theorem. *Let M be a JBW-factor of type I_n for $3 \leq n < \infty$. Then $M \cong H_n(B)$ where B is *-isomorphic to* **R**, **C**, **H** *or* **O**. *In particular, M is finite dimensional. If $n \geq 4$, then B is *-isomorphic to* **R**, **C**, *or* **H**.

Proof. Let $p_1, \ldots, p_n$ be abelian projections with sum 1 with each pair exchangeable by symmetries. Choose a *-algebra B over $\mathbf{R}$ as in Theorem 3.27. (Note that if $n = 3$, then B might not be associative but rather only alternative.) Let $\{e_{ij}\}$ be the canonical matrix units of $M_n(B)$. Since $e_{11} = p_1$ is abelian, and M is a factor, then e_{11} is minimal (Lemma 3.30). Thus by Lemma 3.29, $\{e_{11}Me_{11}\} = \mathbf{R}e_{11}$, and $\{e_{11}Me_{11}\} = \{e_{11}H_n(B)e_{11}\} = B_{\mathrm{sa}}e_{11}$, so

$$B_{\mathrm{sa}} = \mathbf{R}1. \tag{3.8}$$

We are going to show that B is an (alternative) division algebra over $\mathbf{R}$, and is quadratic (i.e., every element $b \in B$ satisfies a quadratic equation over $\mathbf{R}$). Then we will make use of the result of Albert [4] that such an algebra must be isomorphic to $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$, or $\mathbf{O}$, cf. [67, Thm. 2.2.6]. Let $b \in B$. Since $B_{\mathrm{sa}} = \mathbf{R}1$, we can write $\frac{1}{2}(b + b^*) = \lambda 1$ for $\lambda \in \mathbf{R}$. Then $b - \lambda 1 = b - \frac{1}{2}(b + b^*) = \frac{1}{2}(b - b^*)$ is skew-adjoint, so its square is self-adjoint. Therefore $(b - \lambda 1)^2 = \alpha 1$ for some $\alpha \in \mathbf{R}$. Expanding we conclude that b satisfies a quadratic equation of the form

$$b^2 + \beta b + \gamma 1 = 0$$

for some scalars β, γ. Next we show that B is a division algebra, i.e., that each element of B is invertible. Let b be a non-zero element of B. If above $\gamma \neq 0$, then $b(b + \beta 1) = -\gamma 1$ implies that b is invertible. If $\gamma = 0$, then $b^2 = -\beta b$, so by the alternative law (Definition 3.26)

$$b(bb^*) = b^2 b^* = -\beta bb^*$$

and

$$(b^*b)b = b^*b^2 = b^*(-\beta b) = -\beta b^*b.$$

If either of the scalars b^*b or bb^* is non-zero, then we can conclude that $b = -\beta 1$, where $\beta \neq 0$, and so again we have shown b is invertible. However, both b^*b and bb^* cannot be zero, since if they were, then the fact that $be_{12} + b^*e_{21} \in H_n(B) = M$, and

$$(be_{12} + b^*e_{21})^2 = bb^*e_{11} + b^*be_{22},$$

would force $b = 0$. This completes the proof that B is a division algebra.

We conclude that B equals one of $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$, or $\mathbf{O}$ as algebras over $\mathbf{R}$. Finally, we check that the involution on B is the standard one. Note that B is spanned as a vector space over $\mathbf{R}$ by 1 and the elements $\jmath$ such that $\jmath^2 = -1$, so the involution is determined by its action on such elements. We are going to show $\jmath^* = -\jmath$. By (3.8) there is a non-zero real number δ such that $\jmath^*\jmath = \delta 1$. By the alternative law we have $(\jmath^*\jmath)\jmath = \jmath^*(\jmath^2)$, so $\delta\jmath = -\jmath^*$. Then $\jmath^* = -\delta\jmath$, so

$$\jmath = \jmath^{**} = (-\delta\jmath)^* = (-\delta)(-\delta\jmath) = \delta^2\jmath,$$

which implies that $\delta = \pm 1$. If $\delta = -1$, then $\jmath^* = \jmath$, so $\jmath \in B_{\mathrm{sa}} = \mathbf{R}1$, which contradicts $\jmath^2 = -1$. Thus $\delta = 1$, so we have shown $\jmath^* = -\jmath$. This completes the proof that there is a unique involution on B, and so the involution is the standard one. □

Recall that finite dimensional formally real Jordan algebras are the same as finite dimensional JB-algebras. (See the Notes to Chapter 1.) Each of the algebras listed in Theorem 3.32 is formally real [58, Prop. 2.92], and thus is a JBW-algebra. It is straightforward to check that the center is $\mathbf{R}1$ in each case, so each is a JBW-factor of type I. The analogous classification of finite dimensional formally real algebras is due to Jordan, von Neumann, and Wigner [66].

The methods described so far leave out JBW-factors of type I_2, which we will now discuss.

3.33. Definition. Let N be a real Hilbert space of dimension at least 2, and let $\mathbf{R}1$ denote a one dimensional real Hilbert space with unit vector 1. Let $M = N \oplus \mathbf{R}1$ (vector space direct sum). Give M the product $\circ$ defined by

$$(a + \alpha 1) \circ (b + \beta 1) = \beta a + \alpha b + ((a \mid b) + \alpha\beta)1.$$

Give M the norm $\|a + \lambda 1\| = \|a\|_2 + |\lambda|$ (where $\|\cdot\|_2$ denotes the Hilbert space norm). Then M is called a *spin factor.*

We will see in Proposition 3.37 that a spin factor is a JBW-factor, and so we will immediately apply the terminology of JB-algebras (e.g., we will refer to an element s of a spin factor as a symmetry if $s^2 = 1$.)

3.34. Lemma. *Let M be a spin factor, and write $M = N \oplus \mathbf{R}1$ as in Definition 3.33. Then N consists of all multiples of symmetries in M other than ± 1.*

Proof. From the definition of the product on N, each unit vector in N is a symmetry, and thus each element of N is a multiple of a symmetry.

Conversely, suppose $s = n + \lambda 1$ is a symmetry with $n \in N$. Then $s^2 = 1 = (n + \lambda 1)^2 = (n \mid n)1 + 2\lambda n + \lambda^2 1$. Then we must have $2\lambda n = 0$. Either $n = 0$ (which forces $\lambda^2 = 1$ and thus $s = \lambda 1 = \pm 1$), or else $\lambda = 0$, which implies $s = n \in N$. □

3.35. Definition. Let M be a JB-algebra. Elements a, b in M are *Jordan orthogonal* if $a \circ b = 0$.

Note that in a concrete JW-algebra, elements are Jordan orthogonal iff they anti-commute.

In a spin factor $M = N \oplus \mathbf{R}1$, each unit vector in N is a symmetry, and such symmetries are Jordan orthogonal iff they are orthogonal in the Hilbert space N. Thus an orthonormal basis for N is the same as a maximal set of Jordan orthogonal symmetries in N.

3.36. Example. $H_2(\mathbf{R}) = M_2(\mathbf{R})_{\mathrm{sa}}$ and $H_2(\mathbf{C}) = M_2(\mathbf{C})_{\mathrm{sa}}$ are spin factors of dimension 3 and 4 respectively. To see this, in each case let N be the trace zero elements, and for $x, y \in N$ define

$$(x \mid y) = \tau(x \circ y),$$

where τ is the normalized trace (i.e., $\tau(1) = 1$), and $\circ$ is the usual Jordan product of matrices. Note that the set of Pauli spin matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \imath \\ -\imath & 0 \end{pmatrix} \tag{3.9}$$

are symmetries in $M_2(\mathbf{C})_{\mathrm{sa}}$ that are both Jordan orthogonal and orthogonal unit vectors with respect to the tracial inner product. They span the trace zero matrices, so that it is easy to see the Jordan product on $M_2(\mathbf{R})_{\mathrm{sa}}$ and on $M_2(\mathbf{C})_{\mathrm{sa}}$ coincides with that for a spin factor. For the quaternions we can append to the list of Pauli spin matrices two more orthogonal symmetries $\jmath e_{12} - \jmath e_{21}$ and $k e_{12} - k e_{21}$ and then verify in similar fashion that with the usual Jordan product $H_2(\mathbf{H})$ is a spin factor. Similarly we can show that $H_2(\mathbf{O})$ is a spin factor. The fact that the usual norm on these algebras matches that defined for spin factors follows easily from the spectral theorem.

3.37. Proposition. *Every spin factor is a JBW-factor of type I_2, and every JBW-factor of type I_2 is a spin factor.*

Proof. Let $M = N \oplus \mathbf{R}1$ be a spin factor as defined above. We will first show M is a JB-algebra. Note that all of the axioms for the Jordan product and the norm in a JB-algebra refer just to two elements.

Furthermore, since by definition $N \circ N \subset \mathbf{R}1$, if a and b are arbitrary elements of M, then $\mathbf{R}1 + \mathbf{R}a + \mathbf{R}b$ is a subalgebra, which we can write in the form $H \oplus R1$ where H is a 2-dimensional subspace of N. It then suffices to show that $H \oplus R1$ is a JB-algebra. Note that $H \oplus \mathbf{R}1$ is itself a 3-dimensional spin factor. However, we saw above that $M_2(\mathbf{R})_{\mathrm{sa}}$ is a 3-dimensional spin factor, and it is clear that two spin factors of the same dimension are isomorphic. Since $M_2(\mathbf{R})_{\mathrm{sa}}$ is a JB-algebra, then so is the isomorphic algebra $H \oplus R1$, and this completes the proof that every spin factor is a JB-algebra, except for proving norm completeness.

The norm on M is equivalent to the norm on the Hilbert space direct sum $N \oplus \mathbf{R}1$. In fact

$$\|\lambda 1 \oplus n\|_2 = \left(|\lambda|^2 + \|n\|_2{}^2\right)^{1/2} \leq |\lambda| + \|n\|_2 = \|\lambda 1 + n\|,$$

and by the Cauchy–Schwarz inequality for $\mathbf{R}^2$,

$$\|\lambda 1 + n\| = |\lambda| + \|n\|_2 \leq (1^2 + 1^2)^{1/2}\left(|\lambda|^2 + \|n\|_2{}^2\right)^{1/2} = 2^{1/2}\|\lambda 1 \oplus n\|_2.$$

Thus

$$\|\lambda 1 \oplus n\|_2 \leq \|\lambda 1 + n\| \leq 2^{1/2}\|\lambda 1 \oplus n\|_2\,, \tag{3.10}$$

so M is complete and reflexive. Thus M is a JB-algebra and a Banach dual space, so is a JBW-algebra. To see that M is a factor, it suffices to show that the only central symmetries are ± 1 (so that the only central projections are 0 and 1.) Let t be any symmetry other than ± 1. Then from Lemma 3.34, t must be in N. Let s be a unit vector (and thus a symmetry) in N orthogonal to t. Then s is Jordan orthogonal to t, so $U_s t = 2s \circ (s \circ t) - s^2 \circ t = -t$. Thus by Lemma 2.35, t is not central.

Finally, we show that M is of type I_2. Let s be any symmetry in M. Then $s^2 = 1$ implies that the spectrum of s is contained in $\{0, 1\}$. Since every element of M has the form $\alpha 1 + \beta s$ where s is a symmetry, it follows that each element of M has a spectrum containing at most two numbers. This implies that any set of orthogonal non-zero projections in M has cardinality at most 2. Thus M contains minimal ($=$ abelian) projections, and so M is a type I factor. By Theorem 3.23, M is of type I_n for some n. Since M does not contain three orthogonal projections, then $n \leq 2$. Since 1 is not abelian (indeed, we showed above that there are non-central elements in M), then M is not of type I_1, so then M is of type I_2.

Conversely, let M be any JBW-factor of type I_2. By Lemma 3.22, any orthogonal set of non-zero projections has cardinality at most 2. Thus every projection other than 0 or 1 is a minimal projection, and so every spectral resolution (cf. Definition 2.21) contains at most one projection other than 0 and 1. It follows that the spectrum of every element has cardinality at most 2.

Now let N be the linear span of the symmetries in M other than ± 1. We are going to prove that

$$N \circ N \subset \mathbf{R}1. \tag{3.11}$$

Let s, t be symmetries in N. We will show $s \circ t \in \mathbf{R}1$, which will prove (3.11). Let p be the projection such that $s = p - p'$. By equation (1.47) we have

$$s \circ t = (p - p') \circ t = (U_p - U_{p'})t.$$

Since every projection other than 0 and 1 is minimal in M, then the range of U_p is $\mathbf{R}p$ and that of $U_{p'}$ is $\mathbf{R}p'$, so

$$s \circ t \in \mathbf{R}p + \mathbf{R}p' = \mathbf{R}1 + \mathbf{R}s. \tag{3.12}$$

On the other hand, the same argument applies to t in place of s, so

$$s \circ t = t \circ s \in \mathbf{R}1 + \mathbf{R}t. \tag{3.13}$$

If $s \circ t \in \mathbf{R}1$, we have proven our claim. Otherwise, by (3.13), $t \circ s = \lambda_1 1 + \lambda_2 t$ for scalars λ_1, λ_2 with $\lambda_2 \neq 0$. Then by (3.12), $\lambda_1 1 + \lambda_2 t \in \mathbf{R}1 + \mathbf{R}s$, so t lies in $\mathbf{R}1 + \mathbf{R}s$, say $t = \alpha 1 + \beta s$ for scalars α, β. Then $t^2 = 1$ implies that α or β must be zero. Since by assumption t is not a multiple of 1, then t must be a multiple of s, and so $s \circ t \in \mathbf{R}1$, which completes the proof of (3.11).

We next show that $M = N \oplus \mathbf{R}1$. If x is any element of M, with spectral decomposition $x = \alpha p + \beta p'$, then

$$x = \tfrac{1}{2}(\alpha + \beta)1 + \tfrac{1}{2}(\alpha - \beta)(p - p'),$$

so x is a linear combination of 1 and a symmetry other than ± 1. Thus $M = N + \mathbf{R}1$. This sum is direct, since if $1 = \sum \lambda_i s_i$ for a finite number of symmetries $s_1, \ldots, s_n$ other than ± 1, and scalars $\lambda_1, \ldots, \lambda_n$, then multiplying by any s_j shows that s_j is a multiple of 1, a contradiction.

Since $N \circ N \subset \mathbf{R}1$, then every element of N is a multiple of a symmetry. We define a symmetric bilinear form on N by

$$x \circ y = (x \mid y)1 \quad \text{for } x,\ y \text{ in } N.$$

Since every element of N is a multiple of a symmetry, it follows that this is positive definite, and thus equips N with an inner product. Furthermore, for s a symmetry in N and λ a scalar we have

$$(\lambda s \mid \lambda s) = \lambda^2 = \|\lambda s\|^2$$

so the inner product norm coincides with the given JBW-norm on N.

Define a linear functional τ on M by $\tau(\lambda 1 + x) = \lambda$. Then it is easy to check that τ is positive on squares and $\tau(1) = 1$, so τ is a state. Its kernel is N, so N is norm closed (and thus complete) for the JBW-norm. Since the inner product norm coincides with the JBW-norm, it follows that N is a Hilbert space. If N is zero dimensional, then $M = \mathbf{R}1$ is not of type I_2. If N is one dimensional, say $N = \mathbf{R}s$ for a symmetry s, then one verifies that s is central so M is not a factor. Thus N is of dimension at least 2.

From the fact that each member of N is a multiple of a symmetry, it follows at once that the Jordan product in M coincides with that defined for spin factors. If s is a symmetry not ± 1, one easily verifies from spectral theory that

$$\|\lambda 1 + \alpha s\| = |\lambda| + |\alpha| = |\lambda| + \|\alpha s\| = |\lambda| + \|\alpha s\|_2,$$

so that the JBW-norm coincides with the spin factor norm. This completes the proof that M is a spin factor. $\square$

We record the following fact that was established in the proof above.

3.38. Proposition. *Every spin factor is a reflexive Banach space, and thus every state on a spin factor is a normal state.*

Proof. We showed in the second paragraph of the proof of Proposition 3.37 that spin factors are reflexive. It follows that their dual and predual spaces coincide, so every state is normal (Theorem 2.55). $\square$

We have now described the type I_n factors for $1 \leq n < \infty$. Not surprisingly, the type I_∞ factors are the infinite dimensional versions of $H_n(B)$ for $B = \mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$, i.e., the self-adjoint bounded linear operators on real, complex, or quaternionic Hilbert space. Here is a summary of type I JBW-factors.

3.39. Theorem. *The type I JBW-factors can be divided into the following classes (up to isomorphism):*

(i) *the symmetric bounded operators on a real Hilbert space,*
(ii) *the self-adjoint bounded operators on a complex Hilbert space,*
(iii) *the self-adjoint bounded operators on a quaternionic Hilbert space,*
(iv) *the spin factors,*
(v) *the exceptional algebra $H_3(\mathbf{O})$ of 3×3 Hermitian matrices over the Cayley numbers.*

Moreover, these classes are mutually disjoint, with the exceptions that the matrix algebras $M_2(\mathbf{R})_{\mathrm{sa}}$, $M_2(\mathbf{C})_{\mathrm{sa}}$, $M_2(\mathbf{H})$ are all spin factors, and (i), (ii), (iii) coincide for Hilbert spaces of dimension 1.

Proof. Each type I factor is of type I_n for some n with $1 \leq n \leq \infty$ by Theorem 3.23. The factors of type I_2 are the spin factors by Proposition

3.37, and the factors of type I_n for $3 \leq n < \infty$ are described in Theorem 3.32. In a JBW-factor of type I_1, the identity 1 is abelian, so every element is central. For a factor, this is possible only if the factor is $\mathbf{R}1$. Finally, the fact that JBW-factors of type I_∞ are the bounded self-adjoint operators on a real, complex, or quaternionic Hilbert space can be found in [67, Thm. 7.5.11].

Finally, the disjointness of the isomorphism classes (i) – (v) (with the exceptions noted) follows by considering orthogonal minimal projections p and q and M_{p+q}. In the five cases this subalgebra will be isomorphic to $M_2(\mathbf{R})_{\mathrm{sa}}$, $M_2(\mathbf{C})_{\mathrm{sa}}$, $M_2(\mathbf{H})_{\mathrm{sa}}$, M, and $M_2(\mathbf{O})$. □

Atomic JBW-algebras

Recall that a minimal non-zero element of a lattice is called an *atom*. In the context of JBW-algebras, the relevant lattice will be the lattice of projections, so an atom will then be a minimal non-zero projection.

3.40. Definition. A JBW-algebra M is *atomic* if every non-zero projection dominates an atom.

Note that by a simple Zorn's lemma argument, in an atomic JBW-algebra every non-zero projection is the least upper bound of an orthogonal collection of atoms.

3.41. Lemma. *If q is an atom in a JBW-algebra M and p is any projection, then $U_p q$ is a multiple of an atom.*

Proof. Since q is minimal, then $U_q(M) = \mathbf{R}q$ (Lemma 3.29). Thus for any element a in M, by the identity (1.16)

$$U_{\{pqp\}}a = U_p U_q U_p a \in U_p(\mathbf{R}q) = \mathbf{R}U_p q = \mathbf{R}\{pqp\}. \tag{3.14}$$

Taking $a = 1$, we see that $\{pqp\}^2$ is a multiple of $\{pqp\}$, so $\{pqp\}$ is a multiple of a projection w. By (3.14), $U_w(M) = \mathbf{R}w$, so by Lemma 3.29 w is an atom or zero. Thus $U_p q = \{pqp\}$ is a multiple of an atom. □

3.42. Lemma. *Let M be a JBW-algebra, and $\mathcal{P}$ the lattice of projections in M. The least upper bound in $\mathcal{P}$ of the atoms in M is a central projection z such that zM is atomic and $(1-z)M$ contains no atoms. This decomposition of M into atomic and non-atomic parts is unique.*

Proof. By Proposition 2.25, the lattice of projections in M is complete, so the least upper bound z of the atoms exists. Let s be any symmetry in M. Then U_s is a Jordan isomorphism of M by Proposition 2.34. Therefore

U_s induces an isomorphism of the lattice of projections, and in particular gives a bijection of the set of atoms onto itself. Thus

$$\begin{aligned} U_s z &= U_s\big(\bigvee\{q \mid q \text{ is an atom}\}\big) \\ &= \bigvee\{U_s q \mid q \text{ is an atom}\} \\ &= \bigvee\{\text{atoms in } M\} = z. \end{aligned}$$

By Lemma 2.35, z is central.

To show zM is atomic, let p be a non-zero projection in zM. If $U_p q = 0$ for every atom q in M, then p would be orthogonal to every atom in M. By Proposition 2.18, p' would dominate every atom and therefore dominate z, so p would be orthogonal to z. Since p is not zero and $p \leq z$, this is impossible. Thus there is an atom q such that $U_p q$ is non-zero, and by Lemma 3.41 $U_p q$ is a multiple of an atom. Then $\|U_p q\|^{-1} U_p q$ is an atom under p, which completes the proof that zM is atomic.

By construction, z dominates all atoms, so $z'M$ contains no atoms. Finally, if $M = M_1 \oplus M_2$ with M_1 atomic and M_2 containing no atoms, then $M_1 = cM$ and $M_2 = c'M$ for a central projection c (Proposition 2.40). Since $M_1 = cM$ is atomic, then c must be the least upper bound of atoms by the remark following Definition 3.40. For every atom q, $c'q$ is a projection under q, and so $c'q$ must equal either zero or q. Since $c'M$ contains no atoms, then $c'q = 0$, and so $q = cq \leq c$. Therefore every atom is under c, and thus c is the least upper bound of the atoms in M, i.e., $c = z$. This establishes the uniqueness of the decomposition of M into atomic and non-atomic parts. □

3.43. Lemma. *If p is an atom in a JBW-algebra M, then the central cover $c(p)$ is a minimal projection in the center of M.*

Proof. Let $w \leq c(p)$ be a central projection. Then wp is a projection under p. Since p is an atom, $wp = p$ or $wp = 0$. In the first case, by the definition of the central cover, $w \geq c(p)$, and then $w = c(p)$. In the second case, $(c(p) - w)p = p$, so $c(p) - w \geq c(p)$. Thus $w = 0$. Therefore $c(p)$ is minimal among central projections of M. □

3.44. Proposition. *A JBW-factor M is type I iff it contains an atom.*

Proof. If M is type I, then by definition it contains an abelian projection, and an abelian projection in a factor is an atom (Lemma 3.30). Conversely, suppose that M contains an atom p. Then p is abelian, so its central cover must be 1 (since a factor contains no central projections other than 0 or 1). Thus M is of type I. □

3.45. Proposition. *A JBW-algebra M is atomic iff it is the direct sum of type I factors.*

Proof. Suppose first that M is atomic. Let $\{c_\alpha\}$ be a maximal collection of orthogonal minimal elements of the center of M. We claim $\sum_\alpha c_\alpha = 1$. If not, then by atomicity of M there is an atom $p \leq 1 - \sum_\alpha c_\alpha$. Then $c(p) \leq 1 - \sum_\alpha c_\alpha$, so $c(p)$ is a minimal element of the center of M (Lemma 3.43) orthogonal to all c_α. This contradicts the maximality of the collection $\{c_\alpha\}$, so we have shown $\sum_\alpha c_\alpha = 1$.

Let $M_\alpha = c_\alpha M$. Then M is the direct sum of the JBW-subalgebras M_α. Any central projection in $c_\alpha M$ is central in M, so by minimality of c_α in the center of M, we conclude that M_α is a factor. Note that by atomicity of M, each c_α dominates an atom, say p_α. Thus by Proposition 3.44, M_α is of type I.

Conversely, suppose that M is the direct sum of type I factors. By Lemma 3.18 every projection in a type I JBW-algebra dominates an abelian projection, and by Lemma 3.30, every abelian projection in a factor is an atom. Thus every type I factor is atomic. It follows that every direct sum of type I factors is atomic. □

We are going to develop the theory of dimension for projections in atomic JBW-algebras. The next two results are useful technical tools for that purpose.

3.46. Lemma. *Let p, q be projections in a JBW-algebra. Then there exists a symmetry $s \in M$ such that*

$$U_s(\{pqp\}) = \{qpq\}.$$

Proof. (Sketch) Let $a = p + q - 1$. Let s be a symmetry in $W(a, 1)$ such that $s \circ a = |a|$. Then by a Jordanized version of the argument in the proof of [AS, Lemma 6.46], one shows that s has the desired property. See [67, Lemma 5.2.1] for details. □

3.47. Lemma. *Let p, q be projections in a JBW-algebra. There exists a symmetry $s \in M$ such that*

$$U_s(p \vee q - p) = q - (p \wedge q).$$

Proof. By Lemma 3.46 there is a symmetry s such that U_s exchanges $\{p'qp'\}$ and $\{qp'q\}$. Since U_s is a Jordan isomorphism, it will exchange the associated range projections:

$$U_s(r(\{p'qp'\}) = r(\{qp'q\}). \tag{3.15}$$

From Proposition 2.31 and equation (2.17)

$$r(\{p'qp'\} = (p \vee q) \wedge p' = (p \vee q) - p \tag{3.16}$$

and

$$r(\{qp'q\}) = (q' \vee p') \wedge q = q \wedge (q \wedge p)' = q - (q \wedge p). \tag{3.17}$$

The lemma follows from (3.15), (3.16), and (3.17). □

The following result will not be needed in the sequel, but seems of independent interest. Let L be a lattice and $a, b \in L$. An ordered pair (a, b) is a *modular pair* (written $(a, b)M$) if for all x in L with $a \wedge b \leq x \leq b$ there holds $x = (x \vee a) \wedge b$. A lattice is *semimodular* if the relation M is symmetric, i.e., if $(a, b)M$ implies $(b, a)M$. Topping [128] showed that the projection lattice of a von Neumann algebra is semimodular.

3.48. Corollary. *The lattice of projections of a JBW-algebra M is semimodular.*

Proof. By Proposition 2.25, the lattice of projections in M is orthomodular. By [96, Cor. 36.14 and Thm. 29.8] in order to prove this lattice is semimodular it suffices to show that whenever p and q satisfy $p \vee q = 1$ and $p \wedge q = 0$, then there is a lattice isomorphism that takes p to q' and q to p'. If s is the symmetry in Lemma 3.47, then U_s induces a lattice isomorphism that takes $(p \vee q) - p = 1 - p = p'$ to $q - (p \wedge q) = q$. Then U_s also takes $p = 1 - p'$ to $q' = 1 - q$. Furthermore, $U_s q' = U_s^2 p = p$, so $U_s q = p'$. Thus U_s is the desired lattice isomorphism. □

We will now discuss the notion of dimension in the lattice of projections of a JBW-algebra. We will say a projection in an atomic JBW-algebra M is *finite* if it is the least upper bound of a finite set of atoms. The minimum number of atoms whose least upper bound is p is called the *dimension* of p, and is denoted $\dim(p)$. (For general background on dimension in atomic lattices, see (A 41) and the succeeding results in the Appendix.)

3.49. Definition. If p, q are projections in a JBW-algebra with $p \leq q$, we say q *covers* p if $q - p$ is an atom. (This is equivalent to saying that $p \leq q$ and that there is no projection strictly between p and q.)

3.50. Lemma. *(The covering property) Let p and u be projections in a JBW-algebra M with u an atom. Then either $u \leq p$ or else $p \vee u$ covers p.*

Proof. By Lemma 3.47, there is a symmetry s such that U_s exchanges $(p \vee u) - p$ with $u - (u \wedge p)$. Since u is an atom, then $u - (u \wedge p)$ is either equal to u or is zero. Since U_s takes atoms to atoms, then $(p \vee u) - p$ is either an atom or zero. In the former case, $(p \vee u)$ covers p. In the latter case, $p = p \vee u$ so $u \leq p$. $\square$

3.51. Proposition. *Let p be a finite projection in an atomic JBW-algebra M. Then p can be expressed as a finite sum of atoms, and the cardinality of any set of atoms with sum p is $\dim(p)$. Furthermore, every projection $q < p$ is finite with $\dim(q) < \dim(p)$.*

Proof. The result stated in the proposition follows from the fact that the projection lattice in a JBW-algebra has the covering property (Lemma 3.50) and from the general lattice-theoretic properties of dimension (A 44). (If $q \leq p$, (A 44) only states that $\dim(q) \leq \dim(p)$. However, if $q \leq p$ and $\dim(q) = \dim(p)$, then $q = p$, or else $p - q$ could be expressed as a finite sum of atoms, leading to $p = q + (p - q)$ being expressed as the sum of more than $\dim(p)$ atoms.) $\square$

3.52. Definition. If M is a JBW-algebra, then M_f denotes the linear span of the atoms in M.

3.53. Lemma. *Let M be a JBW-algebra. Then each element $x \in M_f$ can be expressed as a linear combination of orthogonal atoms, $x = \sum_i \lambda_i p_i$.*

Proof. Note that all atoms by definition are in the atomic part of M, so the same is true of their linear span. Thus without loss of generality we may assume M is atomic. Let x be a linear combination of atoms $u_1, \ldots, u_k$. Let $p = u_1 \vee \cdots \vee u_k$. Note that $x \in M_p$. We will prove the lemma by induction on $\dim(p)$.

If $\dim(p) = 1$, then p is an atom and $M_p = \mathbf{R}p$, so the result is clear. Suppose that the lemma holds for $\dim(p) \leq n - 1$. To carry out the induction, suppose that $\dim(p) = n$ and that $x \in M_p$. If x is a scalar multiple of p, by Proposition 3.51, p (and then x) is a finite sum of orthogonal atoms, so the result holds. Otherwise, there is some $\lambda \in \mathbf{R}$ such that $x - \lambda p \not\geq 0$ and $x - \lambda p \not\leq 0$. By Theorem 2.20 (i) and (ii) applied to M_p, there is a projection q in M_p such that

$$x = U_q x + U_{p-q} x \tag{3.18}$$

and

$$U_q(x - \lambda p) \geq 0 \quad \text{and} \quad U_{p-q}(x - \lambda p) \leq 0.$$

If $q = p$, since $x \in M_p$ then $x - \lambda p = U_p(x - \lambda p) = U_q(x - \lambda p) \geq 0$, contrary to our choice of λ. Similarly $p - q \neq p$. Thus by Proposition 3.51, q and $p - q$ are projections with dimension strictly smaller than $\dim(p)$.

By our induction hypothesis, we can express $U_q x$ as a linear combination of orthogonal atoms in M_q and $U_{p-q}x$ as a linear combination of orthogonal atoms in M_{p-q}. Each atom in M_q is orthogonal to each atom in M_{p-q}, so by (3.18) we are done. □

Notes

Structure theory for JBW-algebras is similar to that for von Neumann algebras, but differs in some important ways. In each case, it depends on comparison of projections. However, there is no good analog of Murray–von Neumann equivalence in JBW-algebras, so the definition of equivalent projections relies on exchanging projections by products of symmetries (Definition 3.1). In the context of von Neumann algebras, this is the same as unitary equivalence (A 179). Topping [128] made a thorough study of this notion of equivalence in the context of JW-algebras, and was able to carry over to JW-algebras many of the standard results on comparison of projections and structure of von Neumann algebras. Some of these results were then carried over to the context of JBW-algebras (especially JBW-factors) in [8], and were treated more systematically in [67]. The description of the center of M_p in Proposition 3.13 appears in [44].

For arbitrary Jordan algebras over **R** there is the classical coordinatization theorem (Theorem 3.27), cf. [72], which implies that JBW-algebras in which there are n exchangeable projections with sum 1 can be represented as $n \times n$ hermitian matrices with entries in a *-algebra B. In the type I factor case, this algebra can be taken to be the reals, complexes, quaternions, or octonions (if $n = 3$), cf. Theorems 3.32 and 3.39. To show this, we have relied on the result of Albert [4] that these are precisely the alternative division algebras over **R** such that each element satisfies a quadratic equation. Alternatively, we could have relied on the classification of finite dimensional formally real Jordan algebras in [75]. The infinite dimensional type I factor classification can be found explicitly in [67], and appeared first in [125].

Spin factors were first studied by Topping [128, 129]. The result that the spin factors are exactly the JBW-factors of type I_2 (Proposition 3.37) is due to Størmer [125]. Concretely represented atomic JW-algebras were studied in [128], where the JW-algebra version of the results in the last section of this chapter is proven.

The semimodularity of the projection lattice in a concretely represented JW-algebra is in [128], and is proved for JBW-algebras (Corollary 3.48) in [10]. The covering property (Definition 3.49), and the definition of dimension for atomic lattices with the covering property, is a lattice-theoretic abstraction of dimension for subspaces of a finite dimensional vector space, and is due to Maclaren [95].

Just as Jordan algebras originated in investigations of the foundations of quantum mechanics, there is also a lattice-theoretic approach to these foundations, usually called the "quantum logic" approach, which originated with the paper [29] of Birkhoff and von Neumann. Here the elements of the lattice are thought of as propositions, and one hopes to impose physically meaningful axioms that will lead to the lattice being isomorphic to the lattice of projections of $\mathcal{B}(H)$, or to more general lattices such as the lattice of projections of a von Neumann algebra or JBW-algebra.

If M is a type I_n JBW-factor with $3 \leq n < \infty$, the lattice of projections of M will form a projective geometry. For $n = 3$, this will be a projective plane, with the one dimensional projections being the points of the projective plane, and the two dimensional projections being the lines. If $n \geq 4$, by the fundamental theorem of projective geometry there will be a division ring D and a finite dimensional vector space V over D such that the lattice of projections of M will be isomorphic to the lattice of subspaces of V. If $n = 3$, such a division ring D will exist iff the projective geometry is Desarguesian, cf. [131, Thm. 2.16]. The projection lattice of $H_3(\mathbf{O})$ gives an example of a non-Desarguesian projective plane.

A projective geometry is the same as an irreducible modular complemented lattice of finite rank. Von Neumann's continuous geometries [99] are one generalization of this to infinite rank, and the lattice of projections of a type II_1 von Neumann factor will be a continuous geometry. However, in general, projection lattices of von Neumann algebras or JBW-algebas (even of $\mathcal{B}(H)$ for H infinite dimensional) are not modular. Orthomodularity (and semimodularity) are weakened forms of modularity that are satisfied by these projection lattices, as we have seen. For a more detailed discussion of the quantum logic approach, see the books of Beltrametti and Cassinelli [28], Varadarajan [131], and Piron [103].

A more complete discussion of the role of the octonions in a variety of fields of mathematics, and of $H_3(\mathbf{O})$ in particular, can be found in the survey article of Baez [24]. Some physical applications of the octonions and the associated exceptional Jordan algebra $H_3(\mathbf{O})$ are discussed in the papers of Corrigan and Hollowood [38], and Sierra [119].

4 Representations of JB-algebras

In this chapter we will discuss representations of JB- and JBW-algebras as Jordan algebras of self-adjoint operators on a Hilbert space. We will see that there is one crucial difference compared to the situation for C*-algebras: not every JB-algebra admits such a concrete representation. The "Gelfand–Naimark" type theorem (Theorem 4.19) states that there is a certain exceptional ideal, and modulo that ideal every JB-algebra admits a concrete representation, i.e., is a JC-algebra.

The first section will make use of the structure theory developed in the last chapter to determine which JBW-factors admit concrete representations (i.e., are JW-algebras). The next section is devoted to the Gelfand–Naimark type theorem for JB-algebras mentioned above. We then examine the relationship between a JC-algebra A and the real *-algebra it generates. We will see that for many JC-algebras, no matter what the representation, the algebra is the self-adjoint part of the real *-algebra it generates. (We will call this "universal reversibility".) In the penultimate section, we study the relationship between a JC-algebra and the C*-algebra it generates. Here the situation depends very much on the particular representation: every non-associative JC-algebra admits representations where it is not the self-adjoint part of the C*-algebra it generates. However, there is a particular representation which is quite useful, in which the generated C*-algebra is called the universal C*-algebra. This has the property that Jordan homomorphisms of the JC-algebra into any C*-algebra extend to *-homomorphisms on the universal C*-algebra.

The final section introduces Cartesian triples in JBW-algebras. Such triples played a central role in our investigations of orientations for von Neumann algebras, cf. (A 197), and will also play a role in our characterization of normal state spaces of von Neumann algebras in a later chapter.

Representations of JBW-factors as JW-algebras

In this section we will show that every JBW-factor except $H_3(\mathbf{O})$ is a JW-algebra, and that $H_3(\mathbf{O})$ cannot be so represented.

4.1. Theorem. *Every spin factor is a JW-algebra.*

Proof. We first construct representations for finite dimensional spin factors from tensor products of 2×2 matrix algebras. Fix $n \geq 1$. Let σ_1, σ_2, σ_3 be Jordan orthogonal symmetries in $M_2 = M_2(\mathbf{C})$, e.g., the

Pauli spin matrices (cf. (3.9)). For $1 \le j \le n$ define elements s_j in $M_2 \otimes \cdots \otimes M_2 \cong M_{2^n}$ by

$$s_{2j-1} = \sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1 \otimes \sigma_2 \otimes 1 \otimes \cdots \otimes 1$$

and

$$s_{2j} = \sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1 \otimes \sigma_3 \otimes 1 \otimes \cdots \otimes 1$$

(where there are $j-1$ factors of σ_1 and $n-j$ factors of 1). Note that for any $x_1, \ldots, x_n, y_1, \ldots, y_n$ in M_2,

$$(x_1 \otimes x_2 \otimes \cdots \otimes x_n) \circ (y_1 \otimes y_2 \otimes \cdots \otimes y_n) = (x_1 \circ y_1) \otimes \cdots \otimes (x_n \circ y_n)$$

as long as x_i and y_i commute for all except possibly one index i. (Evaluate the left-hand side making use of the fact that $x_j y_j = y_j x_j = x_j \circ y_j$ when x_j and y_j commute.) It follows that $\{s_1, \ldots, s_{2n}\}$ are Jordan orthogonal symmetries (cf. Definition 3.35). Let S be the real linear span of 1 and $s_1, \ldots, s_{2n}$. Let $M = N \oplus \mathbf{R}1$ be a $(2n+1)$-dimensional spin factor, and let $w_1, \ldots, w_{2n}$ be an orthonormal basis of N. Define $\Theta : M \to S$ by $\Theta(1) = 1$, $\Theta(w_i) = s_i$ for $1 \le i \le 2n$. Then Θ is a Jordan isomorphism from M onto $S \subset (M_{2^n})_{\rm sa} \cong \mathcal{B}(H)_{\rm sa}$, where H is a 2^n-dimensional complex Hilbert space, which proves that M is a JC-algebra. For $n \ge 2$, the $(2n)$-dimensional spin factor is a subalgebra of the $(2n+1)$-dimensional spin factor, so we have shown that all finite dimensional spin factors are JC-algebras.

Let S be the $(2n+1)$-dimensional spin factor constructed above, and $\mathcal{A} \subset M_{2^n}$ the C*-algebra generated by S. If $\mathcal{B}$ is any C*-algebra and $\pi : S \to \mathcal{B}_{\rm sa}$ a unital Jordan homomorphism, we are going to show that π extends to a *-homomorphism $\psi : \mathcal{A} \to \mathcal{B}$. (This says that $\mathcal{A}$ is the "universal C*-algebra" of S, a concept to be investigated in more detail later in this chapter, cf. Definition 4.37.)

Consider for each $j = 1, 2, \ldots, n$ the elements of $\mathcal{A}$,

$$\begin{aligned} t_j &= i s_{2j-1} s_{2j} = 1 \otimes \cdots \otimes 1 \otimes \sigma_1 \otimes 1 \otimes \cdots \otimes 1, \\ u_j &= t_1 t_2 \cdots t_{j-1} s_{2j-1} = 1 \otimes \cdots \otimes 1 \otimes \sigma_2 \otimes 1 \otimes \cdots \otimes 1, \\ v_j &= t_1 t_2 \cdots t_{j-1} s_{2j} = 1 \otimes \cdots \otimes 1 \otimes \sigma_3 \otimes 1 \otimes \cdots \otimes 1. \end{aligned}$$

Since $1, \sigma_1, \sigma_2, \sigma_3$ span $M_2 = M_2(\mathbf{C})$, it follows that $1, t_j, u_j, v_j$ span $\mathbf{C}1 \otimes \cdots \otimes M_2 \otimes \cdots \otimes \mathbf{C}1$, so $\mathcal{A} = M_{2^n}$. Furthermore, $\mathcal{A}$ is the linear span (over $\mathbf{C}$) of 1 and all finite products of the elements $s_1, s_2, \ldots, s_n$. Since these elements are Jordan orthogonal (i.e., anticommuting) symmetries, each such product either equals 1 or can be written in the form $s_{i_1} s_{i_2} \cdots s_{i_k}$ where $i_1 < i_2 < \cdots < i_k$ and $\{i_1, \ldots, i_k\}$ is any subset of $\{1, \ldots, 2n\}$. Together with the identity element 1 there are 2^{2n} such products. Since

M_{2^n} is 2^{2n}-dimensional over $\mathbf{C}$, it follows that these products together with 1 form a basis for M_{2^n}. Now we define $\psi : \mathcal{A} \to \mathcal{B}$ by

$$\psi\left(\alpha_0 1 + \sum \alpha_{i_1 \ldots i_k} s_{i_1} s_{i_2} \cdots s_{i_k}\right) = \alpha_0 1 + \sum \alpha_{i_1 \ldots i_k} \pi(s_{i_1})\pi(s_{i_2}) \cdots \pi(s_{i_k})$$

where the sum is over all k-tuples $i_1 < i_2 < \cdots < i_k$ for $1 \leq k \leq n$. Since $\pi : S \to \mathcal{B}_{\mathrm{sa}}$ is a unital Jordan homomorphism, it follows that $\pi(s_1)$, ..., $\pi(s_{2n})$ are Jordan orthogonal symmetries in $\mathcal{B}$, and thus anticommute. It is then straightforward to verify that ψ is a *-homomorphism that extends π.

Now we are ready for the infinite dimensional case. If M is an infinite dimensional spin factor, let $\{s_\alpha\}_{\alpha \in I}$ be a maximal set of Jordan orthogonal symmetries in $M \setminus \{1, -1\}$. For each finite subset F of the index set I, let M_F be the spin factor spanned by 1 and $\{s_\alpha \mid \alpha \in F\}$, and $\mathcal{A}_F$ the associated C*-algebra constructed above. Then the collection $\{M_F\}$ is directed upwards by inclusion. Let M_0 be the union of the subalgebras M_F for all finite subsets F, and let $\mathcal{A}_0$ be the algebraic inductive limit of the associated C*-algebras. (Note that any inclusion $F_1 \subset F_2$ of finite subsets of I extends to a Jordan isomorphism of M_{F_1} into M_{F_2}, and then to a *-homomorphism of $\mathcal{A}_{F_1}$ into $\mathcal{A}_{F_2}$.) Then $\mathcal{A}_0$ is a normed *-algebra whose completion $\mathcal{A}$ is a C*-algebra. (The defining norm properties for a C*-algebra hold in $\mathcal{A}_0$, and thus also in $\mathcal{A}$.) The imbedding of each M_F is isometric (Proposition 1.35), and thus extends to an isometric isomorphism from the union M_0 into $\mathcal{A}$. Since M_0 is dense in M, this isomorphism extends by continuity to an isometric isomorphism from M into $\mathcal{A}$. Thus by the Gelfand–Naimark theorem for C*-algebras (A 64), M is a JC-algebra. Since M is by hypothesis also a JBW-algebra, then M is a JW-algebra (Corollary 2.78). □

4.2. Lemma. *Let B be a real associative *-algebra for which $A = H_n(B)$ is equipped with a norm making it a JB-algebra. If $n \geq 2$ and $a \in M_n(B)$, then*

$$a^* a \leq 0 \quad \Rightarrow \quad a = 0.$$

Proof. We first show for $a \in M_n(B)$,

$$a^* a = 0 \quad \Rightarrow \quad aa^* = 0. \tag{4.1}$$

If $a^* a = 0$, then the spectrum of $a^* a$ in $M_n(B)$ is $\{0\}$, so by (A 54) the spectrum of aa^* in the associative algebra $M_n(B)$ is $\{0\}$. We will show the spectrum of aa^* in the JB-algebra A is $\{0\}$, which by Corollary 1.22 will imply $aa^* = 0$. Let $\lambda \in \mathbf{R}$ with $\lambda \neq 0$. Then $aa^* - \lambda 1$ is invertible in $M_n(B)$, and by uniqueness of inverses, its inverse is self-adjoint. By Lemma 1.16, $aa^* - \lambda 1$ has a Jordan inverse in A. Since this is true for all non-zero λ, then the spectrum of aa^* in A is $\{0\}$, which completes the proof of (4.1).

Now let $a \in M_n(B)$ and suppose that $a^*a \leq 0$. Let $\{e_{ij}\}$ be the standard system of matrix units in $M_n(B)$, and write $a = \sum_{i,j} b_{ij} e_{ij}$ with $b_{ij} \in B$ for $i, j = 1, \ldots, n$. For each $b \in B$, each i and any $j \neq i$, we have

$$b^*be_{ii} = e_{ii}(b^*e_{ij} + be_{ji})^2 e_{ii} \geq 0. \tag{4.2}$$

The inequality $a^*a \leq 0$ implies

$$0 \geq e_{ii}a^*ae_{ii} = \sum_k b_{ki}^* b_{ki} e_{ii}.$$

By (4.2) we conclude that $b_{ki}^* b_{ki} e_{ii} = 0$ for each i, k. It follows that $(b_{ki}e_{ii})^*(b_{ki}e_{ii}) = 0$, so by (4.1) $b_{ki}e_{ii} = 0$. By the uniqueness of the representation of elements of $M_n(B)$ with respect to the system of matrix units $\{e_{ij}\}$, we conclude that each b_{ki} is zero, so $a = 0$. □

4.3. Lemma. *If M is a JBW-algebra such that M is isomorphic to $H_n(B)$ for a real associative *-algebra B with $n \geq 2$, then M is a JW-algebra.*

Proof. We identify M with $H_n(B)$. We first show

$$x \in M_n(B) \quad \Rightarrow \quad x^*x \geq 0. \tag{4.3}$$

The proof we will give for (4.3) is similar to the standard proof of the positivity of x^*x for elements x in a C*-algebra, cf. (A 56), but differs in some essential ways since we are not working in a complex algebra. Let $x^*x = a - b$ be the orthogonal decomposition of x^*x in $C(x^*x, 1)$ (cf. Proposition 1.28), so that a, b are two positive elements of $C(x^*x, 1)$ such that $a \circ b = 0$. We are going to show $b = 0$, so that $x^*x = a \geq 0$ will follow. By the definition of $\{bab\}$ (cf. equation (1.13)) and the associativity of $C(x^*x, 1)$,

$$\{bab\} = 2b \circ (b \circ a) - (b \circ b) \circ a = -(b \circ b) \circ a = -b \circ (b \circ a) = 0.$$

Since the Jordan triple product $\{bab\}$ in $M_n(B)$ coincides with the associative product bab, we conclude that $bab = 0$.

Calculating in the associative *-algebra $M_n(B)$, we now find that

$$(xb)^*(xb) = bx^*xb = b(a - b)b = -b^3 \leq 0. \tag{4.4}$$

By Lemma 4.2, $xb = 0$, so by (4.4) $b^3 = 0$. By spectral theory, $b = 0$, which completes the proof of (4.3).

By spectral theory, every element of M^+ has the form a^2 for some $a \in M$. Thus by (4.3) we have $M^+ = \{x^*x \mid x \in M_n(B)\}$. Hence $M_n(B)$ is a *-algebra whose self-adjoint part is a complete order unit space with the positive cone $\{x^*x \mid x \in M_n(B)\}$. By (A 65), there is a representation π of the *-algebra $M_n(B)$ on a complex Hilbert space H such that π is an isometry on $M = H_n(B)$. Thus M is a JC-algebra, and therefore by Corollary 2.78 is a JW-algebra. □

4.4. Lemma. *If M is a JBW-algebra in which the identity is the sum of n projections ($4 \leq n \leq \infty$) exchangeable pairwise by symmetries, then M is a JW-algebra.*

Proof. We can assume $4 \leq n < \infty$, for if $n = \infty$, then the identity of M can be written as the sum of four exchangeable projections (Corollary 3.11). Then by Theorem 3.27, M is isomorphic to the self-adjoint part of $M_n(B)$ where B is an associative $*$-algebra over $\mathbf{R}$. Now the result follows from Lemma 4.3. □

Recall that $\mathbf{O}$ stands for the octonions or Cayley numbers.

4.5. Theorem. *Every JBW-factor other than $H_3(\mathbf{O})$ is a JW-algebra.*

Proof. Let M be a JBW-factor. If M is not of type I, then by Proposition 3.17, there are four exchangeable projections with sum 1. By Lemma 4.4, M is a JW-algebra. If M is of type I, then by Theorem 3.23 M is of type I_n where $1 \leq n \leq \infty$. By the definition of type I_n, if $n < \infty$, then 1 is the sum of n exchangeable projections, and if $n = \infty$ then 1 is the sum of infinitely many exchangeable projections. If $4 \leq n \leq \infty$, then by Lemma 4.4 M is a JW-algebra. If $n = 1$, then $M = \mathbf{R}1$ (Theorem 3.39), and so M is a JW-algebra. If $n = 2$, then by Proposition 3.37, M is a spin factor, and by Theorem 4.1 is then a JW-algebra. If $n = 3$, then by Theorem 3.32, $M \cong H_3(B)$ where B is the reals, complexes, quaternions, or octonions. In the first three cases, B is a real associative $*$-algebra, and so by Lemma 4.3, M is a JW-algebra □

Recall from Chapter 1 (cf. remarks following Definition 1.1) that a Jordan algebra A is *special* if there is an associative algebra $\mathcal{A}$ such that A is isomorphic to a Jordan subalgebra of $\mathcal{A}$ with the product $a \circ b = \frac{1}{2}(ab + ba)$. It is *exceptional* if it is not special.

4.6. Theorem. *$H_3(\mathbf{O})$ is exceptional.*

Proof. This is a classic result due to Albert [2]. The idea of the proof is as follows. If M were special, say $M \subset \mathcal{A}$ where $\mathcal{A}$ is an associative algebra, one can arrange that $\mathcal{A}$ is a *-algebra with $M \subset \mathcal{A}_{\mathrm{sa}}$, and use the three exchangeable projections in M to construct a 3×3 system of matrix units for $\mathcal{A}$. Then there is an associative *-algebra B and an isomorphism from $\mathcal{A}$ onto $M_n(B)$ taking the matrix units in $\mathcal{A}$ onto the matrix units of $M_n(B)$ and carrying $M = H_3(\mathbf{O})$ onto $H_n(B)$. Now (3.7) gives an isomorphism from $\mathbf{O}$ onto B, so $\mathbf{O}$ must be associative. Since the octonions are not associative, this is a contradiction. See [67, Cor. 2.8.5] for the details. □

Representations of JB-algebras as Jordan operator algebras

By Theorem 4.6 some JB-algebras admit no faithful representation on a Hilbert space, so the concept of a JB-algebra is strictly more general than that of a JC-algebra. But we will see in Theorem 4.19 that an arbitrary JB-algebra has an ideal (referred to as the *exceptional ideal*) such that the quotient is a JC-algebra. For this purpose we will show (in Lemma 4.14) that every JB-algebra admits a separating set of (abstract) factor representations as defined below. (We will reserve the term *concrete representation* for a Jordan homomorphism of a JB-algebra into $\mathcal{B}(H)_{\mathrm{sa}}$.)

4.7. Definition. Let A be a JB-algebra. A *factor representation* of A is a unital homomorphism π from A into a JBW-algebra such that the σ-weak closure of $\pi(A)$ is a JBW-factor. It is said to be a *JW-factor representation* if the σ-weak closure of $\pi(A)$ is a JW-factor.

By Theorem 4.5 a factor representation π of a JB-algebra A is a JW-factor representation unless the σ-weak closure of $\pi(A)$ (and then $\pi(A)$ itself) is isomorphic to $H_3(\mathbf{O})$. Thus a factor representation π of A is a JW-factor representation except when $\pi(A)$ is isomorphic to $H_3(\mathbf{O})$.

4.8. Lemma. *If M is a JBW-algebra of type I_n, and π is a factor representation, then $N = \overline{\pi(M)}$ (σ-weak closure) is a type I_n JBW-factor.*

Proof. Let $p_1, \ldots, p_n$ be abelian exchangeable projections in M with sum 1. Then by density of $\pi(M)$ in N, $\pi(p_1), \ldots, \pi(p_n)$ are also abelian projections in N with sum 1. If p_i and p_j are exchanged by a symmetry s, then $\pi(s)$ is a symmetry that exchanges $\pi(p_i)$ and $\pi(p_j)$. Thus N is of type I_n. □

4.9. Lemma. *If σ is a normal state on a JBW-algebra M, then there is a smallest central projection c with value 1 on σ.*

Proof. By taking complements, it is equivalent to show that there is a greatest central projection with value 0 on σ. We show the set of all such projections is directed upwards. Assume $\sigma(c_1) = \sigma(c_2) = 0$. Since $c_1c_2 \le c_1$, then $\sigma(c_1c_2) = 0$, hence

$$\sigma(c_1 \vee c_2) = \sigma(c_1 + c_2 - c_1c_2) = \sigma(c_1) + \sigma(c_2) - \sigma(c_1c_2) = 0.$$

Since σ is normal, it has value 0 on the least upper bound of such projections, which is itself a projection by Proposition 2.5(iii) and is central by Proposition 2.36. □

4.10. Definition. Let σ be a normal state on a JBW-algebra M. The least central projection with value 1 on σ is called the *central carrier* of σ, and is denoted $c(\sigma)$.

4.11. Definition. Let A be a JB-algebra, and σ a state on A. $\pi_\sigma : A \to c(\sigma)A^{**}$ is defined to be the composition of the natural injection of A into A^{**} followed by the homomorphism $U_{c(\sigma)}$ mapping A^{**} onto $c(\sigma)A^{**}$.

4.12. Definition. A state on a JB-algebra is *pure* if it is not a convex combination of two distinct states.

Pure states on JB-algebras generalize pure states on C*-algebras (A 60), and are a special case of pure states on general order unit spaces (A 17).

4.13. Definition. A state σ on a JB-algebra A is a *factor state* if its central carrier $c(\sigma)$ is a minimal projection in the center of A^{**}.

Note that σ is a factor state iff $c(\sigma)A^{**}$ is a JBW-factor, and thus iff π_σ is a factor representation. We will study the relationship of states and representations more thoroughly later in this book, after we have discussed the basic properties of Jordan state spaces. For now, we just remark that π_σ as defined above is an abstract analog of the GNS-representation associated with a state on a C*-algebra (cf. (A 63)).

4.14. Lemma. *If σ is a pure state on a JB-algebra A, then σ is a factor state, so π_σ is a factor representation. These factor representations π_σ separate the elements of A.*

Proof. Let σ be a pure state on A. Since A is σ-weakly dense in A^{**}, then $\pi_\sigma(A)$ is σ-weakly dense in $c(\sigma)A^{**}$. Suppose $0 < w < c(\sigma)$ is a central projection in $c(\sigma)A^{**}$. Define $\lambda = \sigma(w)$ and note that

$$\sigma(c(\sigma) - w) = 1 - \sigma(w) = 1 - \lambda.$$

By the definition of the central carrier, neither λ nor $1-\lambda$ equals 1, so $0<\lambda<1$. Note that $\|U_w^*\sigma\| = (U_w^*\sigma)(1) = \sigma(U_w 1) = \sigma(w) = \lambda$. Similarly, $\|U_{1-w}^*\sigma\| = \sigma(1-w) = 1-\lambda$. Now define

$$\sigma_1 = \|U_w^*\sigma\|^{-1}U_w^*\sigma = \lambda^{-1}U_w^*\sigma$$

and

$$\sigma_2 = \|U_{1-w}^*\sigma\|^{-1}U_{1-w}^*\sigma = (1-\lambda)^{-1}U_{1-w}^*\sigma.$$

Then σ_1 and σ_2 are distinct states, and since $U_w + U_{1-w}$ is the identity map on A^{**},

$$\lambda\sigma_1 + (1-\lambda)\sigma_2 = U_w^*\sigma + U_{1-w}^*\sigma = \sigma.$$

This contradicts the assumption that σ is pure. Thus no such central projection w exists. Then we've shown that $c(\sigma)$ is minimal among central projections in A^{**}, and hence that $c(\sigma)A^{**}$ is a factor.

Finally, let $c = c(\sigma)$. Then $\sigma(c) = 1$, so (1.55) implies that $U_c^*\sigma = \sigma$. If $\pi_\sigma(a) = ca = 0$, then

$$\sigma(a) = (U_c^*\sigma)(a) = \sigma(ca) = 0.$$

Thus $\pi_\sigma(a) = 0$ implies $\sigma(a) = 0$. By the Krein–Milman theorem the pure states separate points of A, so the associated factor representations π_σ separate points of A. □

4.15. Definition. Let A be a JB-algebra, and J a norm closed Jordan ideal. Then J is *purely exceptional* if every factor representation of J has image isomorphic to $H_3(\mathbf{O})$.

4.16. Lemma. *Let A be a JB-algebra, and J a norm closed Jordan ideal. Then every homomorphism π from J into a JBW-algebra M extends to a homomorphism from A into M.*

Proof. If $1 \in J$, then $J = A$, and there is nothing to prove, so we may assume $1 \notin J$. Let $B = J + R1 \subset A$. Then B is a JB-algebra. We also denote by $\pi : B \to M$ the unique extension of π to a unital homomorphism from B into M. By Theorem 2.65, there is a unique normal homomorphism $\widetilde{\pi}$ from B^{**} into M extending $\pi : B \to M$. By Lemma 2.75 we can identify B^{**} with the σ-weak closure $\bar{B}$ of B in A^{**}, and J^{**} with the σ-weak closure $\bar{J}$ of J in A^{**}. Thus $\widetilde{\pi}$ restricts to a normal homomorphism from $\bar{J}$ into M, extending $\pi : J \to M$. Since J is an ideal in A, then $\bar{J}$ is a σ-weakly closed ideal in A^{**}, so by Proposition 2.39 there is a central projection c in A^{**} such that $\bar{J} = cA^{**}$. Then $\widetilde{\pi} \circ U_c$ (restricted to A) is the desired extension. □

4.17. Lemma. *If J is a norm closed ideal in a JC-algebra A, then A/J is a JC-algebra.*

Proof. Let $\bar{J}$ be the σ-weak closure of J in A^{**}. By Lemma 2.75, $\bar{J} \cap A = J$, so the map $a + J \mapsto a + \bar{J}$ is an isomorphism from A/J into $A^{**}/\bar{J}$. We will be done if we show the latter is a JC-algebra. By Lemma 2.77, A^{**} is a JW-algebra, and by Proposition 2.39 there is a central projection c such that $\bar{J} = cA^{**}$. Thus U_{1-c} is an isomorphism of $A^{**}/\bar{J}$ onto $(1-c)A^{**}$. A direct summand of a JW-algebra is again a JW-algebra, so $A^{**}/\bar{J}$ is a JW-algebra, which completes the proof. □

4.18. Definition. If $\{\pi_\alpha\}$ is a family of concrete representations of a JB-algebra A with $\pi_\alpha : A \to \mathcal{B}(H_\alpha)_{\mathrm{sa}}$, then we define the direct sum $\bigoplus_\alpha \pi_\alpha$ by

$$\Big(\bigoplus_\alpha \pi_\alpha\Big)(a) = \bigoplus_\alpha \pi_\alpha(a) \in \bigoplus_\alpha \mathcal{B}(H_\alpha)_{\mathrm{sa}} \subset \mathcal{B}\Big(\bigoplus_\alpha H_\alpha\Big)_{\mathrm{sa}}.$$

The following is a "Gelfand–Naimark" type theorem for JB-algebras.

4.19. Theorem. *Let A be a JB-algebra. Then there is a unique ideal J such that A/J is a JC-algebra and J is purely exceptional.*

Proof. Let $\mathcal{J}$ be the set of all ideals J that are kernels of JW-factor representations of A, and let J be the intersection of all ideals in $\mathcal{J}$. For each $K \in \mathcal{J}$, let π_K be a JW-factor representation of A with kernel K. Let π be the direct sum of the representations π_K for $K \in \mathcal{J}$. Then the kernel of π is J, and by construction $\pi(A) \cong A/J$ is a JC-algebra.

To prove that J is purely exceptional, let π be a factor representation of J and let $M = \overline{\pi(J)}$. By Lemma 4.16, π extends to a homomorphism (also denoted π) from A into M. If M is a JW-factor, then π is a JW-factor representation of A, so its kernel contains J. Then $\pi(J) = \{0\}$ which contradicts π being a factor representation of J. Thus M is not a JW-factor, so by Theorem 4.5, $M \cong H_3(\mathbf{O})$. Thus J is purely exceptional.

To prove uniqueness, suppose J_1 is another norm closed Jordan ideal of A with the same properties as J. Then J_1 is annihilated by every factor representation of A onto a JW-factor, so by the definition of J, $J_1 \subset J$. By hypothesis, A/J_1 is a JC-algebra. Let π be a factor representation of A/J_1 and let $M = \overline{\pi(A/J_1)}$. By Lemma 4.17, $\pi(A/J_1)$ is a JC-algebra, and so cannot be dense in the finite dimensional factor $H_3(\mathbf{O})$. Thus by Theorem 4.5, M is a JW-factor. Each such representation pulls back to a JW-factor representation of A, which must then kill J. Thus each factor representation of A/J_1 kills $J/J_1 \subset A/J_1$. By Lemma 4.14 the factor representations separate points of A/J_1, so $J/J_1 = \{0\}$, i.e., $J \subset J_1$. □

4.20. Corollary. *Let A be a JB-algebra (respectively JBW-algebra). If A admits no factor representation onto $H_3(\mathbf{O})$, then A is a JC-algebra (respectively, JW-algebra).*

Proof. Assume A admits no factor representation onto $H_3(\mathbf{O})$, and let J be the purely exceptional ideal of Theorem 4.19. If J were not $\{0\}$, then by definition J would admit a factor representation onto $H_3(\mathbf{O})$. By Lemma 4.16 this would extend to a factor representation of A ontc $H_3(\mathbf{O})$, contrary to our assumption. Thus $J = \{0\}$. By Theorem 4.19, A/J is a JC-algebra, and thus so is A. Finally, if A is a JBW-algebra as well as a JC-algebra, then by Corollary 2.78 it is a JW-algebra. □

For JBW-algebras, we will see that the relevant ideal is σ-weakly closed.

4.21. Lemma. *Let M be a JBW-algebra of type I_3. Then M admits a unique decomposition $M = M_{\mathbf{R}} \oplus M_{\mathbf{C}} \oplus M_{\mathbf{H}} \oplus M_{\mathbf{O}}$ where all factor representations of $M_{\mathbf{R}}$, $M_{\mathbf{C}}$, $M_{\mathbf{H}}$, $M_{\mathbf{O}}$ are onto $H_3(\mathbf{R})$, $H_3(\mathbf{C})$, $H_3(\mathbf{H})$, $H_3(\mathbf{O})$ respectively.*

Proof. [67, Thm. 6.4.1]. □

4.22. Lemma. *Every JBW-algebra M of type I_2 is a JW-algebra.*

Proof. By Lemma 4.8, for every factor representation π of M, $\overline{\pi(M)}$ is a type I_2 factor. Every I_2 factor is a JW-algebra by Theorem 4.5. By Lemma 4.14, M admits a separating set of factor representations, each of which can then be taken to be a concrete representation. The direct sum of this separating set of factor representations then gives a representation of M as a JC-algebra. By Corollary 2.78, M is a JW-algebra. □

4.23. Theorem. *Let M be a JBW-algebra. Then M admits a unique decomposition $M = M_{sp} \oplus M_{exc}$ with M_{sp} a JW-algebra and M_{exc} a purely exceptional JBW-algebra of type I_3.*

Proof. By Lemma 3.16 and Theorem 3.23 we can write $M = M_1 \oplus M_2 \oplus M_3 \oplus \cdots \oplus M_\infty \oplus N$ where M_n is either $\{0\}$ or of type I_n for $1 \leq n \leq \infty$ and N has no type I direct summand. By Lemma 4.4, M_n for $4 \leq n < \infty$ is a JW-algebra. By definition, M_∞ is the direct sum of JBW-algebras whose identities are the sum of infinitely many exchangeable projections. By Lemma 4.4 each such direct summand is a JW-algebra, and thus so is M_∞. By Proposition 3.17, the identity in N is the sum of four exchangeable projections, so by Lemma 4.4, N is also a JW-algebra. M_1 is associative, and so is isomorphic to $C_{\mathbf{R}}(X)$. The latter is the self-adjoint part of $C_{\mathbf{C}}(X)$, which is a monotone complete C*-algebra with a separating set of normal states. By (A 95), $C_{\mathbf{C}}(X)$ is a von Neumann algebra, so

$M_1 = C_{\mathbf{R}}(X)$ is a JW-algebra. M_2 is a JW-algebra by Lemma 4.22. The remaining case is the I_3 summand M_3, which is a sum of JW-algebras and purely exceptional summands by Lemma 4.21. The direct sum of all the direct summands that are JW-algebras is again a JW-algebra, which completes the proof. □

Reversibility

We next discuss the relationship between a JC-algebra and the real $*$-algebra it generates. Recall that one motivating example for Jordan algebras is the self-adjoint part of an associative *-algebra. One might expect that a JC-algebra A would equal the self-adjoint part of the real $*$-algebra it generates. This is true iff A is *reversible* as defined below.

4.24. Definition. A Jordan subalgebra A of an associative $*$-algebra is *reversible* if for all $n > 0$

$$a_1, a_2, \ldots, a_n \in A \quad \Rightarrow \quad a_1 a_2 \cdots a_n + a_n \cdots a_2 a_1 \in A. \tag{4.5}$$

Note that the symmetric product in (4.5) is just twice the ordinary Jordan product if $n = 2$, so (4.5) holds for $n = 2$. The Jordan triple product in a special Jordan algebra satisfies

$$a_1 a_2 a_3 + a_3 a_2 a_1 = 2\{a_1 a_2 a_3\}, \tag{4.6}$$

so (4.5) always holds for $n = 3$. However, (4.5) can fail for $n = 4$ (cf. the remark after Proposition 4.31).

In principle, reversibility for a JC-algebra depends upon the representation, and in fact we will see that for certain spin factors reversibility does indeed vary with the representation. On the other hand, we will show that the self-adjoint parts of C*-algebras are reversible in every concrete representation.

If $A \subset \mathcal{B}(H)_{\mathrm{sa}}$ is a concrete JC-algebra, we will let $R_0(A)$ denote the real subalgebra generated by A in $\mathcal{B}(H)$. (Note that $R_0(A)$ is closed under adjoints, so coincides with the *-algebra generated by A.) We denote by $R(A)$ the norm closure of $R_0(A)$, and $\overline{R(A)}$ the σ-weak closure. Clearly A is reversible iff $A = R_0(A)_{\mathrm{sa}}$. Furthermore:

4.25. Lemma. *If A is a reversible JC-subalgebra of $\mathcal{B}(H)_{\mathrm{sa}}$, then $A = R(A)_{\mathrm{sa}}$. If in addition A is σ-weakly closed, then $A = (\overline{R(A)})_{\mathrm{sa}}$ (σ-weak closure).*

Proof. Let $x \in R(A)_{\mathrm{sa}}$. Then there is a sequence $\{x_n\}$ in $R_0(A)$ converging in norm to x. For $y_n = \frac{1}{2}(x_n + x_n^*)$ we have $y_n \in R_0(A)_{\mathrm{sa}} = A$ and $y_n \to \frac{1}{2}(x + x^*) = x$. Since by hypothesis A is norm closed, it follows that $x \in A$. The same argument applies if the norm topology is replaced by the σ-weak topology. □

We will now see that the existence of sufficiently many exchangeable projections with sum 1 is sufficient to guarantee reversibility. It is easy to verify that $H_n(B)$ is a reversible Jordan subalgebra of $M_n(B)$, and this will be the key to proving reversibility for a wide class of JW-algebras. We will use exchangeable projections with sum 1 to construct a system of matrix units and thus to show that $R_0(M) = M_n(B)$ and that $M = H_n(B)$. We recall the definition and the following well known result relating matrix units and matrix algebras.

4.26. Definition. Let R be an associative algebra over the reals with involution $*$. A set $\{e_{ij} \mid 1 \leq i, j \leq n\}$ of elements of R is a *self-adjoint system of $n \times n$ matrix units* if

(i) $e_{ij}^* = e_{ji}$ for all i, j,
(ii) $e_{11} + \cdots + e_{nn} = 1$,
(ii) $e_{ij}e_{km} = \delta_{jk}e_{im}$ for all i, j, k, m (where δ_{jk} is 1 if $j = k$ and 0 otherwise.)

4.27. Lemma. *Let R be an associative $*$-algebra containing a self-adjoint system of $n \times n$ matrix units $\{e_{ij}\}$. Let B be the relative commutant of $\{e_{ij}\}$. Then each element x of R admits a unique representation*

$$x = \sum_{i,j} b_{ij}e_{ij} \tag{4.7}$$

with $b_{ij} \in B$, and R is $*$-isomorphic to $M_n(B)$.

Proof. [67, Lemma 2.8.2]. See also (A 169). □

4.28. Lemma. *Let M be a JW-algebra acting on a Hilbert space H. Assume that the identity is a sum of n projections $p_1, p_2, \ldots, p_n$ (with $2 \leq n < \infty$) such that for each i, p_i and p_1 are exchanged by a symmetry s_i. (We include the case $i = 1$, where we can take $s_1 = 1$.) Define $e_{ij} = s_i p_1 s_j$ for $1 \leq i, j \leq n$. Then $\{e_{ij}\}$ is a self-adjoint system of $n \times n$ matrix units for the associative *-algebra generated by M in $\mathcal{B}(H)$.*

Proof. For all i, j, k we have $(e_{ij})^* = e_{ji}$ and

$$e_{ij}e_{jk} = s_i p_1 s_j s_j p_1 s_k = s_i p_1 s_k = e_{ik}.$$

Note that $e_{ii} = s_i p_1 s_i = p_i$ by our hypothesis, so $e_{jj}e_{kk} = 0$ for $j \neq k$. Thus for $j \neq k$,

$$e_{ij}e_{km} = e_{ij}e_{jj}e_{kk}e_{km} = 0,$$

which completes the proof. □

We will now establish the main result of this section. Under the hypotheses of Theorem 4.29, we will see that there is a close relationship between JW-algebras M acting on a Hilbert space H and the real algebra $R_0(M)$ generated by M. In particular, $M = R_0(M)_{sa}$, i.e., M is reversible. Furthermore, we will see in Corollary 4.38 that the norm closed real algebra $R(M)$ generated by M is determined up to *-isomorphism by M. A hint of why this might be expected is contained in the proof of Theorem 4.29. It is shown that $R_0(M)$ is isomorphic to $M_n(B)$ for an associative *-algebra B, via an isomorphism carrying M onto the set $H_n(B)$ of hermitian matrices with entries in B. We can identify B with any of the subspaces $\{be_{ij} + b^*e_{ji} \mid b \in B\}$ of $H_n(B)$ for $i \neq j$, and then can recover the associative multiplication on B from the Jordan product of elements of these subspaces, cf. (3.7), and also (4.8) below. If this sounds familiar to the reader, it should: the proof of the next result contains a proof of the Jordan coordinatization theorem (Theorem 3.27) for the case of special Jordan algebras.

4.29. Theorem. *Let M be a JW-algebra acting on a Hilbert space H. Assume that the identity is a sum of n pairwise exchangeable projections (with $3 \leq n \leq \infty$). Then M is reversible.*

Proof. By Corollay 3.11, we can assume $n < \infty$. Set $R = R_0(M)$. Thus R is the real subalgebra generated by M in $\mathcal{B}(H)$, and by the remark preceding Lemma 4.25, R coincides with the *-algebra generated by M. Let $\{e_{ij}\}$ be the set of $n \times n$ matrix units described in Lemma 4.28. Let B be the relative commutant of $\{e_{ij}\}$ in R; then by Lemma 4.27, $R = M_n(B)$ and (4.7) holds. We will identify $M_n(B)$ with the sums of the form (4.7). Under this identification, M is a Jordan subalgebra of $H_n(B)$ that generates $M_n(B)$ as a real associative algebra. We are going to show that M consists exactly of the hermitian matrices in $M_n(B)$, i.e., $M = H_n(B)$, which as remarked after Lemma 4.25, will show that M is reversible.

For $1 \leq i, j \leq n$ with $i \neq j$ define

$$B_{ij} = \{b \in B \mid be_{ij} + b^*e_{ji} \in M\}.$$

We are going to show that these subspaces all coincide, and that in fact they all equal B, i.e., $B_{ij} = B$ for all $i \neq j$.

Note that with the notation of Lemma 4.28,

$$e_{ij} + e_{ji} = s_i p_1 s_j + s_j p_1 s_i = 2\{s_i p_1 s_j\} \in M.$$

Now note that for distinct i, j, k, if $b_1 \in B_{ij}$ and $b_2 \in B_{jk}$, then

$$2(b_1 e_{ij} + b_1^* e_{ji}) \circ (b_2 e_{jk} + b_2^* e_{kj}) = b_1 b_2 e_{ik} + b_2^* b_1^* e_{ki}. \tag{4.8}$$

This equation is the key to recovering the associative multiplication from the Jordan product. (Note that taking $n = 3$, $i = 1$, $j = 3$, $k = 2$, and setting $a = b_1$, $b = b_2$ gives the equation (3.7) which was the key to the Jordan coordinatization theorem.) By (4.8)

$$B_{ij}B_{jk} \subset B_{ik} \quad \text{for } i, j, k \text{ distinct.} \tag{4.9}$$

Taking $b_2 = 1$ in (4.8) shows that

$$B_{ij} \subset B_{ik} \quad \text{for } i, j, k \text{ distinct.}$$

Interchanging the roles of the indices j, k shows that

$$B_{ij} = B_{ik} \quad \text{for } i, j, k \text{ distinct.} \tag{4.10}$$

Furthermore, if $b \in B_{ij}$, then

$$b^* e_{ij} + b e_{ji} = (e_{ij} + e_{ji})(b e_{ij} + b^* e_{ji})(e_{ij} + e_{ji}) \in M,$$

so

$$B_{ij}^* = B_{ij}. \tag{4.11}$$

On the other hand, by the definition of B_{ij} we have $B_{ij}^* = B_{ji}$, so (4.11) implies that $B_{ij} = B_{ji}$. Thus for all $i \neq 1$, $i \neq j$ we have

$$B_{12} = B_{1i} = B_{i1} = B_{ij}. \tag{4.12}$$

We also have $B_{12} = B_{1j}$ for all $j \neq 1$ by (4.10), and combining this with (4.12) shows $B_{ij} = B_{12}$ for all $i \neq j$. Furthermore, by (4.9) and (4.11), B_{12} is a $*$-subalgebra of B. (Here we've used $n \geq 3$.)

We next prove that $B_{12} = B$. Let $M_n(B_{12})$ denote the subalgebra of $R = M_n(B)$ generated by the set of matrix units $\{e_{ij}\}$ and by B_{12}. We will show this contains M, so that $M_n(B_{12})$ must coincide with the $*$-algebra $R = M_n(B)$ generated by M. Let $x \in M$, and write $x = \sum_{i,j} b_{ij} e_{ij}$ as in (4.7). Then $x = x^*$ implies that $b_{ij}^* = b_{ji}$ for each i, j. If $i \neq j$, then

$$b_{ij} e_{ij} + b_{ij}^* e_{ji} = b_{ij} e_{ij} + b_{ji} e_{ji} = e_{ii} x e_{jj} + e_{jj} x e_{ii} = 2\{e_{ii} x e_{jj}\} \in M,$$

so $b_{ij} \in B_{ij} = B_{12}$. For each i we have $b_{ii}^* = b_{ii}$ and

$$b_{ii} e_{ii} = e_{ii} x e_{ii} \in M,$$

and for any $j \neq i$,

$$b_{ii} e_{ij} + b_{ii}^* e_{ji} = b_{ii} e_{ij} + b_{ii} e_{ji} = 2(b_{ii} e_{ii}) \circ (e_{ij} + e_{ji}) \in M.$$

Thus $b_{ii} \in B_{ij} = B_{12}$. Therefore all coefficients $\{b_{ij}\}$ of x are in B_{12}, which completes the proof that M is contained in $M_n(B_{12})$. Thus $M_n(B_{12}) = M_n(B) = R$, from which $B_{12} = B$ follows as claimed.

Finally, we show $M = H_n(B)$. Let $x \in H_n(B)$. Write $x = \sum_{i,j} b_{ij} e_{ij}$ with each b_{ij} in B. Note that since $x = x^*$, then $b_{ji} = b_{ij}^*$ for all i, j. Then, by the definition of B_{ij} and the fact that $B = B_{ij}$, for $i \neq j$, each term $b_{ij}e_{ij} + b_{ji}e_{ji} = b_{ij}e_{ij} + b_{ij}^* e_{ji}$ is in M. Also for each i since $b_{ii} \in B = B_{ij}$ and $b_{ii}^* = b_{ii}$, then choosing any $j \neq i$,

$$b_{ii}(e_{ii} + e_{jj}) = (e_{ij} + e_{ji}) \circ (b_{ii}e_{ij} + b_{ii}e_{ji}) \in M.$$

Then $b_{ii}e_{ii} = e_{ii}(b_{ii}(e_{ii} + e_{jj}))e_{ii} \in M$. Thus each $x \in H_n(B)$ is a sum of terms in M, so is in M. Hence $M = H_n(B)$, and so M is reversible. □

4.30. Corollary. *Let $M \subset \mathcal{B}(H)_{\mathrm{sa}}$ be a concrete JW-algebra. Then M is reversible iff the I_2 direct summand of M is reversible.*

Proof. According to Lemma 3.16 and Theorem 3.23, we can write

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_\infty \oplus M_0$$

where M_n is of type I_n for $1 \leq i \leq \infty$ and M_0 has no type I direct summand. We first observe that M is reversible iff each of these direct summands is reversible. Indeed, if M_0, M_1, ..., M_∞ are reversible, let $a_1, \ldots, a_n \in M$. Let c_i $(0 \leq i \leq \infty)$ be the central projections in M such that $c_iM = M_i$. Then $c_ia_j \in M_i$ for each i and j, so by reversibility of M_i, $b_i = c_i(a_1a_2\cdots a_n + a_na_{n-1}\cdots a_1) \in M_i$. Now $\sum_i b_i$ converges σ-strongly to $a_1a_2\cdots a_n + a_na_{n-1}\cdots a_1$, so the latter is also in M. Thus M is reversible. The converse is clear.

We now show that each summand except M_2 is reversible, from which we can conclude that M is reversible iff M_2 is reversible. In all of the summands except M_1 and M_2, we can write the identity as the sum of $n \geq 3$ projections pairwise exchanged by symmetries, so by Theorem 4.29 each summand other than M_1 and M_2 is reversible. By definition, the identity of M_1 is abelian so as remarked after Definition 3.14, all elements of M_1 commute. Thus the Jordan product coincides with the associative product in M_1, and so M_1 is reversible, and we are done. □

4.31. Proposition. *Every spin factor M of dimension ≤ 4 is reversible in every faithful representation as a JC-algebra.*

Proof. Write $M = \mathbf{R}1 \oplus N$ as in Definition 3.33. Then N is of dimension 2 or 3, and each element of N is a multiple of a symmetry (Lemma 3.34). Let S be a maximal Jordan orthogonal set of symmetries in N, i.e., an orthonormal basis for N. Then S contains at most three symmetries, and every element of M is a linear combination of the symmetries in S and the identity 1. Assume M is faithfully represented as a Jordan algebra of self-adjoint operators on a Hilbert space. We need to show that if $a_1, \dots, a_n$ are elements of M, then $a_1 a_2 \dots a_n + a_n \dots a_2 a_1$ is in M. By linearity it suffices to prove this for $a_1 \dots a_n \in S$. Since Jordan orthogonal symmetries anticommute, we can rearrange the order of the factors (possibly multiplying the result by -1 in the process). Using the fact that the square of each symmetry is the identity, we can then reduce to the case where $a_1 \dots a_n$ are distinct (and then $n \leq 3$.) Thus

$$\tfrac{1}{2}(a_1 a_2 \dots a_n + a_n \dots a_2 a_1)$$

either coincides with the Jordan product or the Jordan triple product (see equation (4.6)), so this product is again in M. Thus M is reversible. □

Remark. The five dimensional spin factor is reversible in no representation. In fact, the implication (4.5) fails in every representation. The six dimensional spin factor admits both reversible and non-reversible representations. Finally, spin factors of dimension 7 or more are again reversible in no representation. We will not need this information; a proof of these remarks can be found in [67, Thm. 6.2.5].

We also note that the adjective "faithful" in Proposition 4.31 is redundant. It is straightforward to verify that spin factors are simple, i.e., have no proper closed ideals, and therefore every representation is faithful.

4.32. Definition. A JC-algebra A is *universally reversible* if $\pi(A)$ is reversible for every faithful representation $\pi : A \to \mathcal{B}(H)_{\mathrm{sa}}$.

By Proposition 4.31, every spin factor of dimension at most 4 is universally reversible. We next show that the same is true of the self-adjoint part of C*-algebras. To prove this, we will work with the biduals of both a C*-algebra $\mathcal{A}$ and its self-adjoint part $\mathcal{A}_{\mathrm{sa}}$.

4.33. Lemma. *Let A be a JC-algebra. If A^{**} is reversible in every representation as a concrete JW-algebra, then A is universally reversible.*

Proof. Let A be a concretely represented JC-algebra, and let $\mathcal{B}$ be the C*-algebra generated by A. Let $\widetilde{\mathcal{B}}$ be the enveloping von Neumann algebra of $\mathcal{B}$, which can be identified with $\mathcal{B}^{**}$ (A 101). We therefore view A as a JB-subalgebra of $\widetilde{\mathcal{B}}_{\mathrm{sa}}$. By Lemma 2.75, we can identify $\bar{A}$ with $A^{**} \subset (\mathcal{B}_{\mathrm{sa}})^{**} = (\mathcal{B}^{**})_{\mathrm{sa}} = \widetilde{\mathcal{B}}_{\mathrm{sa}}$, and we have $\bar{A} \cap \mathcal{B}_{\mathrm{sa}} = A$. (The identification of $(\mathcal{B}_{\mathrm{sa}})^{**}$ with $(\mathcal{B}^{**})_{\mathrm{sa}}$ follows from Lemma 2.76). Then for $a_1, \dots, a_n \in A$, by the reversibility of the concrete JW-algebra $A^{**} = \bar{A} \subset \widetilde{\mathcal{B}}$, we have

$$a_1 a_2 \cdots a_n + a_n \cdots a_2 a_1 \in \mathcal{B}_{\mathrm{sa}} \cap \bar{A} = A,$$

which completes the proof that A is reversible in every concrete representation as a JC-algebra. □

4.34. Proposition. *The self-adjoint part of every C*-algebra is universally reversible.*

Proof. Let A be the self-adjoint part of a C*-algebra, faithfully represented as a concrete JC-algebra acting on a Hilbert space H. (Note that we do not assume that $A+iA$ will be a C*-algebra in this representation!)

Assume first that A is isomorphic to the self-adjoint part of a von Neumann algebra $\mathcal{M}$ of type I_2. Then $\mathcal{M}$ is *-isomorphic to the 2×2 matrices with entries in the center $\mathcal{Z}$ of $\mathcal{M}$ (cf. (A 170) or [67, Lemma 7.4.5]). Define s_1, s_2, s_3 to be the Pauli spin matrices (cf. (3.9)), and $s_0 = 1$. Then s_1, s_2, s_3 are Jordan orthogonal symmetries, and every element $a \in A = M_2(\mathcal{Z})_{\mathrm{sa}}$ admits a unique representation $a = \sum_i z_i s_i$, where $z_0, \ldots, z_3$ are in the center Z of A. Indeed, since $\mathcal{Z} = Z + iZ$ (Corollary 1.53) we can write each element a of A in the form

$$a = \begin{pmatrix} w_1 & w_2 + iw_3 \\ w_2 - iw_3 & w_4 \end{pmatrix} = \tfrac{1}{2}(w_1 + w_4)s_0 + \tfrac{1}{2}(w_1 - w_4)s_1 + w_2 s_2 + w_3 s_3$$

where w_1, w_2, w_3, w_4 are in $\mathcal{Z}_{\mathrm{sa}} = Z$. Since all elements of A are linear combinations of elements of the form zs for $z \in Z$ and $s \in \{s_0, s_1, s_2, s_3\}$, to prove reversibility it suffices to show

$$\begin{aligned}(z_1 s_{i_1})(z_2 s_{i_2}) \cdots (z_n s_{i_n}) &+ (z_n s_{i_n}) \cdots (z_2 s_{i_2})(z_1 s_{i_1}) \\ &= z_1 z_2 \cdots z_n (s_{i_1} s_{i_2} \cdots s_{i_n} + s_{i_n} \cdots s_{i_2} s_{i_1})\end{aligned}$$

is in A for all choices of z_i in Z and all choices of $i_1, \cdots i_n$ in $\{0, 1, 2, 3\}$. Here $(s_{i_1} s_{i_2} \cdots s_{i_n} + s_{i_n} \cdots s_{i_2} s_{i_1})$ is in A by the universal reversibility of the spin factor given by the real linear span of $\{s_0, s_1, s_2, s_3\}$ (Proposition 4.31), and universal reversibility of A follows.

Next, suppose that A is isomorphic to the self-adjoint part of a von Neumann algebra, and that A is represented as a concrete JW-algebra. Write $A = A_2 \oplus B$ where A_2 is a JW-algebra of type I_2 and B has no type I_2 direct summand. Note that there is a central projection c such that $A_2 = cA$, so A_2 is also the self-adjoint part of a von Neumann algebra, which will also be of type I_2 (cf. (A 167)). By Corollary 4.30, B is reversible, and by the first part of this proof A_2 is reversible, so A is a reversible JW-algebra.

Finally, let A be isomorphic to the self-adjoint part of a C*-algebra $\mathcal{A}$. Then A^{**} is isomorphic to the self-adjoint part of the von Neumann algebra $\mathcal{A}^{**}$ (Lemma 2.76), and thus is reversible in every faithful representation as a JW-algebra by the preceding paragraph. Now the proposition follows from Lemma 4.33. □

The universal C*-algebra of a JB-algebra

We will see that the norm closed real *-algebra generated by a universally reversible JC-algebra in a faithful representation is independent of the representation up to *-isomorphism. However, the C*-algebras generated by faithful representations of a JC-algebra will generally depend upon the representation unless the algebra is abelian. A very useful tool in investigating these generated C*-algebras is the universal object described in Proposition 4.36.

4.35. Lemma. *Let A be a JB-algebra. Then there is a Hilbert space H of dimension large enough that for every Jordan homomorphism $\phi : A \to \mathcal{A}_{\mathrm{sa}}$ where $\mathcal{A}$ is a C*-algebra, the C*-subalgebra $\mathcal{A}_\phi$ generated by $\phi(A)$ can be *-isomorphically imbedded in $\mathcal{B}(H)$.*

Proof. The cardinality of $\phi(A)$ is at most equal to the cardinality of A. Thus there is an upper bound, independent of the choice of ϕ, on the cardinality of the *-algebra generated by ϕ in $\mathcal{A}$. Then there is a uniform upper bound on the cardinality of the set of all Cauchy sequences in this *-algebra, and hence on its closure $\mathcal{A}_\phi$.

Hence there is a uniform bound on the cardinality of the state space of $\mathcal{A}_\phi$, and of the dimension of the Hilbert space for each GNS-representation associated with a state on $\mathcal{A}_\phi$ (A 63). This gives a bound m on the dimension of the universal representation of $\mathcal{A}_\phi$ (A 83), independent of ϕ. Now let H be an m-dimensional Hilbert space. □

4.36. Proposition. *Let A be a JB-algebra. Then there is a pair $(\mathcal{A}, \pi)$ consisting of a C*-algebra $\mathcal{A}$ and a Jordan homomorphism π from A into $\mathcal{A}_{\mathrm{sa}}$ with the following universal properties:*

(i) *The C*-algebra generated by $\pi(A)$ is $\mathcal{A}$.*
(ii) *For every pair $(\mathcal{B}, \pi_1)$ consisting of a C*-algebra $\mathcal{B}$ and a Jordan homomorphism π_1 from A into $\mathcal{B}_{\mathrm{sa}}$, there is a *-homomorphism $\pi_2 : \mathcal{A} \to \mathcal{B}$ such that $\pi_1 = \pi_2 \circ \pi$.*

This pair $(\mathcal{A}, \pi)$ is unique up to isomorphism.

$$\begin{tikzcd}
\mathcal{A} \arrow[dr, "\pi_2"] & \\
A \arrow[u, "\pi"] \arrow[r, "\pi_1"'] & \mathcal{B}
\end{tikzcd}$$

Proof. Let H be as in Lemma 4.35, let $\{\pi_\alpha\}_{\alpha\in\mathcal{I}}$ be the set of all Jordan homomorphisms from A into $\mathcal{B}(H)_{\mathrm{sa}}$ (organized into a family with index set $\mathcal{I}$), and consider the direct sum $\pi = \bigoplus_{\alpha \in \mathcal{I}} \pi_\alpha$. Thus π is a Jordan homomorphism from A into $\mathcal{B}(\widetilde{H})_{\mathrm{sa}}$ where $\widetilde{H} = \bigoplus_{\alpha \in \mathcal{I}} H_\alpha$ with $H_\alpha = H$ for all $\alpha \in \mathcal{I}$. Let $\mathcal{A}$ be the C*-algebra generated by $\pi(A)$ in $\mathcal{B}(\widetilde{H})$. Then (i) is satisfied. We now show that (ii) is satisfied.

Let $(\mathcal{B}, \pi_1)$ be as in (ii). By replacing the C*-algebra $\mathcal{B}$ by the subalgebra generated by $\pi_1(A)$ in $\mathcal{B}$, we may assume without loss of generality that $\mathcal{B}$ is generated by $\pi_1(A)$. Then by the key property of H, we may assume $\mathcal{B} \subset \mathcal{B}(H)$, so that π_1 is a *-isomorphism from A into $\mathcal{B}(H)_{\mathrm{sa}}$. Thus $\pi_1 = \pi_\beta$ for some $\beta \in \mathcal{I}$.

Let ψ_β be the projection onto the β-coordinate in the direct sum $\bigoplus_{\alpha \in \mathcal{I}} \mathcal{B}(H_\alpha)$ (i.e., $\psi_\beta(\bigoplus_{\alpha \in \mathcal{I}} x_\alpha) = x_\beta$). Then for each $a \in A$,

$$\psi_\beta(\pi(a)) = \pi_\beta(a) = \pi_1(a).$$

Now let $\pi_2 : \mathcal{A} \to \mathcal{B}$ be the *-homomorphism obtained by restricting ψ_β to the C*-subalgebra $\mathcal{A}$ of $\mathcal{B}(\widetilde{H})$. Then $\pi_1 = \pi_2 \circ \pi$, which proves (ii).

Finally, if $(\mathcal{A}', \pi')$ is a second pair with the same property, then applying (ii) gives a *-homomorphism ϕ from $\mathcal{A}$ onto $\mathcal{A}'$ carrying π to π', and a *-homomorphism ϕ' from $\mathcal{A}'$ onto $\mathcal{A}$ carrying π' to π. The composition $\phi' \circ \phi$ agrees with the identity map on the image of A in $\mathcal{A}$, and so must equal the identity map on $\mathcal{A}$ by (i). Similarly $\phi \circ \phi'$ is the identity map on $\mathcal{A}'$. Thus ϕ is a *-isomorphism, and carries π to π', which completes the proof of uniqueness. □

Note that the map π_2 guaranteed in (ii) is unique since any two such factorizations agree on the image of A, which by (i) generates $\mathcal{A}$.

4.37. Definition. Let A be a JB-algebra. The pair $(\mathcal{A}, \pi)$ in Proposition 4.36 is called the *universal C*-algebra for A* and we will denote it by $(C_u^*(A), \pi)$.

Remark. If A is a JC-algebra, then π is faithful, so A can be identified with a Jordan subalgebra of $C_u^*(A)$. See [67, §7.1] for an alternative construction of $(C_u^*(A), \pi)$.

4.38. Corollary. *Let A be a universally reversible JC-algebra. Then the norm closed real *-algebra $R(A)$ generated by A in a faithful representation is determined by A up to *-isomorphism.*

Proof. Let $(\pi, \mathcal{A})$ be the universal C*-algebra for A, and let R be the norm closed real *-subalgebra of $\mathcal{A}$ generated by $\pi(A)$. Suppose $A \subset \mathcal{B}(H)$ is faithfully represented as a concrete JC-algebra. Let ψ be the *-homomorphism from $\mathcal{A}$ into $\mathcal{B}(H)$ such that $\psi(\pi(a)) = a$ for all $a \in A$. Then ψ maps R onto a dense subalgebra of $R(A)$. We next verify that ψ is isometric on R. For $b \in R$, by universal reversibility of A and Lemma

4.25 we have $b^*b \in R(\pi(A))_{\mathrm{sa}} = \pi(\mathcal{A})$. Since ψ restricted to $\pi(A)$ is a Jordan homomorphism and is 1-1, it is isometric on $\pi(A)$ (Proposition 1.35), and thus we have

$$\|\psi(b)\|^2 = \|\psi(b)^*\psi(b)\| = \|\psi(b^*b)\| = \|b^*b\| = \|b\|^2.$$

Thus ψ is isometric on R, and $\psi(R) = R(A)$ follows. □

4.39. Corollary. *A JC-algebra A is universally reversible iff its image in its universal C*-algebra is reversible.*

Proof. Suppose $A \subset \mathcal{B}(H)_{\mathrm{sa}}$ is a concrete representation, $(\pi, C_u^*(A))$ the universal C*-algebra for A, and that $\pi(A)$ is reversible in $C_u^*(A)$. By the universal property of $C_u^*(A)$, there is a *-homomorphism $\psi : C_u^*(A) \to \mathcal{B}(H)$ such that $\psi(\pi(a)) = a$ for all $a \in A$. If $a_1, a_2, \ldots, a_n$ are in A, then

$$a_1a_2\cdots a_n + a_n\cdots a_2a_1 = \psi(\pi(a_1)\pi(a_2)\cdots\pi(a_n) + \pi(a_n)\cdots\pi(a_2)\pi(a_1)).$$

Now by reversibilty of $\pi(A)$, $a_1a_2\cdots a_n + a_n\cdots a_2a_1 \in \psi(\pi(A)) = A$. Thus reversibility of A in $C_u^*(A)$ implies universal reversibility; the converse is trivial. □

If $\mathcal{A}$ is a C*-algebra, then we denote by $\mathcal{A}^{op}$ the opposite C*-algebra. $\mathcal{A}^{op}$ as a set coincides with $\mathcal{A}$, and is equipped with the same involution and norm. If $a \mapsto a^{op}$ denotes the identity map viewed as a map from $\mathcal{A}$ to $\mathcal{A}^{op}$, then the product on $\mathcal{A}^{op}$ is given by $a^{op}b^{op} = (ba)^{op}$. Thus $a \mapsto a^{op}$ is a *-anti-isomorphism from $\mathcal{A}$ onto $\mathcal{A}^{op}$.

4.40. Proposition. *Let A be a JB-algebra. Then there is a unique *-anti-automorphism Φ of $C_u^*(A)$ of period two that fixes all points in the image of A in $C_u^*(A)$. (We will refer to Φ as the canonical *-anti-automorphism associated with $C_u^*(A)$.)*

Proof. Let $\mathcal{A} = C_u^*(A)$, and let $\gamma : \mathcal{A} \to \mathcal{A}^{op}$ be the identity map, i.e., $\gamma(a) = a^{op}$. Let $\pi : A \to \mathcal{A}$ be the canonical map from A into its universal C*-algebra. By the universal property defining $\mathcal{A}$, the Jordan homomorphism $a \mapsto \gamma(\pi(a))$ lifts to a *-homomorphism ψ from $\mathcal{A}$ onto $\mathcal{A}^{op}$ such that $\psi(\pi(a)) = \gamma(\pi(a))$ for all $a \in A$. Now define $\Phi : \mathcal{A} \to \mathcal{A}$ by $\Phi = \gamma^{-1} \circ \psi$.

$$\begin{tikzcd}
 & \mathcal{A} \arrow[d, "\Phi"] \arrow[dr, "\psi"] & \\
A \arrow[ur, "\pi"] \arrow[r, "\pi"] & \mathcal{A} \arrow[r, "\gamma"] & \mathcal{A}^{op}
\end{tikzcd}$$

Then Φ is a *-anti-homomorphism of $\mathcal{A}$ into $\mathcal{A}$ that fixes $\pi(A)$. Thus $\Phi \circ \Phi$ is a *-homomorphism of $\mathcal{A}$ into itself that fixes $\pi(A)$. Since $\pi(A)$ generates $\mathcal{A}$ as a C*-algebra, it follows that $\Phi \circ \Phi$ is the identity map on $\mathcal{A}$. In particular, Φ is a *-anti-automorphism of period two of $\mathcal{A}$. If Φ' is another *-anti-automorphism of $\mathcal{A}$ fixing $\pi(A)$, then as above $\Phi^{-1} \circ \Phi'$ must be the identity map, so $\Phi = \Phi'$. □

4.41. Lemma. *Let $\mathcal{B}$ be a C*-algebra, and let A be a unital JB-subalgebra of $\mathcal{B}_{\mathrm{sa}}$ that generates $\mathcal{B}$ as a C*-algebra. Let Φ be a *-anti-automorphism of order 2 of $\mathcal{B}$ such that $\Phi(a) = a$ for all $a \in A$. Let $R(A)$ denote the norm closed real subalgebra of $\mathcal{B}$ generated by A. Then the following relations hold:*

(i) $R(A) = \{x \in \mathcal{B} \mid \Phi(x) = x^*\}$.
(ii) $R(A) \cap \imath R(A) = \{0\}$.
(iii) $\mathcal{B} = R(A) + \imath R(A)$.
(iv) *For x, y in $R(A)$ we have $\Phi(x + \imath y) = x^* + \imath y^*$.*

If $\mathcal{B}$ is a von Neumann algebra, and A is a JBW-subalgebra such that the von Neumann subalgebra of $\mathcal{B}$ generated by A is all of $\mathcal{B}$, then (i) through (iv) hold for the σ-weak closure $\overline{R(A)}$ in place of $R(A)$.

Proof. (i) Note that Φ restricted to $\mathcal{B}_{\mathrm{sa}}$ is a Jordan automorphism, so Φ is an isometry on $\mathcal{B}_{\mathrm{sa}}$ by Proposition 1.35. Then for arbitrary $b \in \mathcal{B}$, using the property $\|x^*x\| = \|x\|^2$ for elements x of a C*-algebra, we have

$$\|\Phi(b)\|^2 = \|\Phi(b)\Phi(b)^*\| = \|\Phi(b^*b)\| = \|b^*b\| = \|b\|^2,$$

so Φ is an isometry of $\mathcal{B}$ onto itself. Since each $a \in A$ is fixed by Φ, and since $x \mapsto \Phi(x)^*$ is a norm continuous (real-linear) *-automorphism of $\mathcal{B}$, it follows that

$$R(A) \subset \{x \in B \mid \Phi(x) = x^*\}. \tag{4.13}$$

To prove the opposite inclusion, let $b \in \mathcal{B}$ with $\Phi(b) = b^*$. Note that since A generates $\mathcal{B}$ as a C*-algebra, then $\mathcal{B}$ is the norm closure of $R(A)+\imath R(A)$. Thus there is a sequence $\{x_n + \imath y_n\}$ converging in norm to b with x_n and y_n in $R(A)$ for each n. By assumption $\Phi(b) = b^*$, so by (4.13)

$$x_n - \imath y_n = \Phi(x_n)^* - \imath\Phi(y_n)^* = (\Phi(x_n + \imath y_n))^* \to \Phi(b)^* = b.$$

Since $x_n + \imath y_n \to b$ and $x_n - \imath y_n \to b$, adding gives $x_n \to b$ and so $b \in R(A)$. This completes the proof of (i).

(ii) For $x \in R(A)$, by (4.13), $\Phi(ix) = \imath\Phi(x) = ix^* = -(\imath x)^*$. If also $ix \in R(A)$, then again applying (4.13), we have $\Phi(ix) = (\imath x)^*$, so $\imath x$ is not in $R(A)$ unless $x = 0$, which proves (ii).

(iii) Each $b \in \mathcal{B}$ can be written as $b = b_1 + i b_2$, where

$$b_1 = \tfrac{1}{2}(b + \Phi(b^*)) \quad \text{and} \quad b_2 = -\tfrac{i}{2}(b - \Phi(b^*)).$$

Since Φ has order 2, one readily verifies that $\Phi(b_j) = b_j^*$ for $j = 1, 2$. Thus by (i) both b_1 and b_2 are in $R(A)$, proving (iii).

(iv) This follows immediately from (4.13).

Finally, assume that $\mathcal{B}$ is a von Neumann algebra, and that A is a JBW-subalgebra that generates $\mathcal{B}$ as a von Neumann algebra. Then Φ restricted to $\mathcal{B}_{\text{sa}}$ is an order isomorphism, so is σ-weakly continuous there. Since Φ commutes with adjoints and is complex linear, by σ-weak continuity of the adjoint map it follows that Φ is σ-weakly continuous on all of $\mathcal{B}$. Now the proof of (i) – (iv) given above goes through with norm convergence replaced by σ-weak convergence throughout. □

4.42. Corollary. *Let A be a universally reversible JC-algebra, identified with its canonical image in $C_u^*(A)$. If Φ is the *-anti-automorphism of period 2 associated with $C_u^*(A)$, then A is the set $C_u^*(A)_{\text{sa}}^{\Phi}$ of self-adjoint fixed points of Φ.*

Proof. By assumption, A is fixed by Φ. By Lemma 4.41, any self-adjoint fixed point of Φ is in $R(A)_{\text{sa}}$, and by Lemma 4.25, $R(A)_{\text{sa}} = A$. □

4.43. Definition. Let M be a JBW-algebra. A pair $(\mathcal{M}, \pi)$ consisting of a von Neumann algebra $\mathcal{M}$ and normal Jordan homomorphism $\pi : M \to \mathcal{M}_{\text{sa}}$ is a *universal von Neumann algebra of M* if the following properties hold:

(i) $\pi(M)$ generates $\mathcal{M}$ as a von Neumann algebra.

(ii) If $\mathcal{N}$ is a von Neumann algebra and $\pi_1 : M \to \mathcal{N}_{\text{sa}}$ is a normal Jordan homomorphism, then there is a normal *-homomorphism π_2 from $\mathcal{M}$ into $\mathcal{N}$ such that $\pi_2 \circ \pi = \pi_1$.

We will now show that each JBW-algebra M has a unique universal von Neumann algebra, and we will generally denote this universal von Neumann algebra by $(W_u^*(M), \pi)$.

4.44. Lemma. *Let A be a JB-algebra, and $(C_u^*(A), \pi)$ its universal C^*-algebra. Then $(C_u^*(A)^{**}, \pi^{**})$ is a universal von Neumann algebra of A^{**}.*

Proof. Let $\widetilde{\phi}$ be a normal Jordan homomorphism from A^{**} into $\mathcal{M}_{\text{sa}}$ for a von Neumann algebra $\mathcal{M}$. Let ϕ be the restriction of $\widetilde{\phi}$ to A. Let $\psi : C_u^*(A) \to \mathcal{M}$ be the unique *-homomorphism that extends ϕ, and

$\widetilde{\psi} : (C_u^*(A))^{**} \to \mathcal{M}$ the unique extension to a normal *-homomorphism (A 103). Let $\widetilde{\pi} = \pi^{**}$.

$$\begin{tikzcd}
A^{**} \arrow[rr, "\widetilde{\pi}"] \arrow[dr, "\widetilde{\phi}"] & & C_u^*(A)^{**} \arrow[dl, "\widetilde{\psi}"'] \\
& \mathcal{M} & \\
A \arrow[uu] \arrow[ur, "\phi"] \arrow[rr, "\pi"] & & C_u^*(A) \arrow[uu] \arrow[ul, "\psi"']
\end{tikzcd}$$

Since $\psi \circ \pi = \phi$, it follows by σ-weak continuity that $\widetilde{\psi} \circ \widetilde{\pi} = \widetilde{\phi}$. This provides the desired factorization of $\widetilde{\phi}$, so the pair $(C_u^*(A)^{**}, \widetilde{\pi})$ has the necessary universal property. Since $\pi(A)$ generates $C_u^*(A)$ as a C*-algebra, then the von Neumann subalgebra of $C_u^*(A)^{**}$ generated by $\widetilde{\pi}(A^{**}) = \pi^{**}(A^{**})$ contains $C_u^*(A)$, and so by σ-weak density equals $C_u^*(A)^{**}$. □

4.45. Proposition. *Let M be a JBW-algebra. Then there exists a unique universal von Neumann algebra $(W_u^*(M), \pi)$, and a unique *-anti-isomorphism Φ of $W_u^*(M)$ of period 2 fixing each element of $\pi(M)$. If M is a JW-algebra and its image in $W_u^*(M)$ is reversible, then the set of self-adjoint fixed points of Φ is $\pi(M)$.*

Proof. It follows easily from the requirements (i), (ii) in Definition 4.43 that if M has a universal von Neumann algebra $(\mathcal{M}, \pi)$, then $(\mathcal{M}, \pi)$ is unique up to isomorphism. From Lemma 4.44, M^{**} admits a universal von Neumann algebra. It follows easily that every direct summand of M^{**} has the same property. In fact, let c be a central projection in M^{**} and let $(\mathcal{M}, \pi)$ be the universal von Neumann algebra for M^{**}. If $\mathcal{N}$ is a von Neumann algebra and $\pi_1 : cM^{**} \to \mathcal{N}_{\mathrm{sa}}$ is a normal Jordan homomorphism, then $\pi_1 \circ U_c : M^{**} \to \mathcal{N}_{\mathrm{sa}}$ is a normal Jordan homomorphism, so there is a normal *-homomorphism $\psi : \mathcal{M} \to \mathcal{N}$ such that $\psi \circ \pi = \pi_1 \circ U_c$. Let π_0 be the restriction of π to cM^{**}. Then $\psi \circ \pi_0 = \pi_1$.

Since c is central in M^{**}, then $\pi(c)$ is central in the JBW-algebra $\pi(M^{**})$, so commutes with all elements of $\pi(M^{**})$ (Proposition 1.49). Since the von Neumann algebra generated by $\pi(M^{**})$ is $\mathcal{M}$, then $\pi(c)$ is a central projection in $\mathcal{M}$. Define $\mathcal{M}_0 = \pi(c)\mathcal{M}$. Then $(\mathcal{M}_0, \pi_0)$ satisfies the defining properties of the universal von Neumann algebra for cM^{**}. By Lemma 2.54, M is isomorphic to a direct summand of M^{**}, so M possesses a universal von Neumann algebra.

Finally, the existence of Φ with the specified properties follows from Lemma 4.25 by essentially the same argument as in the proof of Proposition 4.40 and Corollary 4.42. □

For the case of a universally reversible JC-algebra, we can describe $C_u^*(A)$ in more concrete terms. The next several results lead to that description.

4.46. Lemma. *Let A be a universally reversible JC-algebra. Let $\mathcal{B}$ be a C^*-algebra, π_0 a faithful Jordan homomorphism from A into $\mathcal{B}_{\mathrm{sa}}$ such that the C^*-subalgebra of $\mathcal{B}$ generated by $\pi_0(A)$ is all of $\mathcal{B}$. If there exists a *-anti-isomorphism $\Phi_0 : \mathcal{B} \to \mathcal{B}$ of period 2 whose set of self-adjoint fixed points is $\pi_0(A)$, then $(\mathcal{B}, \pi_0)$ is the universal C^*-algebra of A.*

Proof. By the universal property of $(C_u^*(A), \pi)$, there is a *-homomorphism $\psi : C_u^*(A) \to \mathcal{B}$ such that $\pi_0 = \psi \circ \pi$. Let Φ be the *-anti-automorphism of period 2 associated with $C_u^*(A)$. Note that $\Phi_0 \circ \psi \circ \Phi$ is a *-homomorphism from $C_u^*(A)$ into $\mathcal{B}$. Since Φ leaves $\pi(A)$ pointwise fixed, then $\Phi \circ \pi = \pi$. Applying ψ to this equation gives

$$\psi \circ \Phi \circ \pi = \psi \circ \pi = \pi_0.$$

Since by assumption Φ_0 leaves $\pi_0(A)$ pointwise fixed, then $\Phi_0 \circ \pi_0 = \pi_0$, so when we apply Φ_0 to the equation above, we get $(\Phi_0 \circ \psi \circ \Phi) \circ \pi = \pi_0$.

$$\begin{tikzcd}
C_u^*(A) \arrow[r, "\Phi"] \arrow[dr, "\psi"] & C_u^*(A) \arrow[dr, "\psi"] & \\
A \arrow[u, "\pi"] \arrow[r, "\pi_0"] & \mathcal{B} & \mathcal{B} \arrow[l, "\Phi_0"']
\end{tikzcd}$$

By the uniqueness of ψ, we must have

$$\psi = \Phi_0 \circ \psi \circ \Phi.$$

Since Φ is of period 2, then $\Phi^{-1} = \Phi$, so the equation above implies

$$\psi \circ \Phi = \Phi_0 \circ \psi. \tag{4.14}$$

Let J be the kernel of ψ. By (4.14), $\Phi(J) \subset J$. Since Φ has period 2, we conclude $\Phi(J) = J$. Let $0 \leq j \in J_{\mathrm{sa}}$. Then $j + \Phi(j)$ is in J, is self-adjoint, positive, and is fixed by Φ. By Corollary 4.42, $j + \Phi(j)$ is in $\pi(A)$. Since ψ must be faithful on $\pi(A)$, then $j + \Phi(j) = 0$. Since $j \geq 0$, then $\Phi(j) \geq 0$ and so $j + \Phi(j) = 0$ implies $j = 0$. Thus $j^*j = 0$ for all $j \in J$. Therefore $J = \{0\}$, and so ψ is faithful. Since the C*-algebra generated by $\pi_0(A)$ is $\mathcal{B}$ and the image of a C*-algebra under a *-homomorphism is a C*-algebra (A 72), ψ is surjective. Thus ψ is a $*$-isomorphism from $C_u^*(A)$ onto $\mathcal{B}$ which carries π to π_0. It follows that $(\mathcal{B}, \pi_0)$ is the universal C*-algebra for A. $\square$

4.47. Proposition. *Let A be a universally reversible JC-subalgebra of $\mathcal{B}(H)_{\mathrm{sa}}$. Define $\pi_0 : A \to \mathcal{B}(H) \oplus \mathcal{B}(H)^{op}$ by $\pi_0(a) = a \oplus a^{op}$. Let $\mathcal{B}$ be the C*-subalgebra generated by $\pi_0(A)$. Then the pair $(\mathcal{B}, \pi_0)$ is the universal C*-algebra for A. The associated $*$-anti-automorphism Φ of period 2 of $\mathcal{B}$ is the map $\Phi(a \oplus b^{op}) = b \oplus a^{op}$.*

Proof. Let $\widetilde{\Phi}$ denote the map $a \oplus b^{op} \mapsto b \oplus a^{op}$ on $\mathcal{B}(H) \oplus \mathcal{B}(H)^{op}$. Note that $\widetilde{\Phi}$ is a *-anti-isomorphism of $\mathcal{B}(H) \oplus \mathcal{B}(H)^{op}$ of period 2 whose set of self-adjoint fixed points is $\{x \oplus x^{op} \mid x \in \mathcal{B}(H)_{\mathrm{sa}}\}$. Since $\widetilde{\Phi}$ is isometric and fixes $\pi_0(A)$, it leaves invariant the C*-algebra $\mathcal{B}$ generated by $\pi_0(A)$. By Lemma 4.41, the set of self-adjoint fixed points of $\widetilde{\Phi}$ on $\mathcal{B}$ is $R(\pi_0(A))_{\mathrm{sa}}$ where $R(\pi_0(A))$ is the norm closed real *-subalgebra of $\mathcal{B}$ generated by $\pi_0(A)$. Since A is by assumption universally reversible, by Lemma 4.25, $R(\pi_0(A))_{\mathrm{sa}} = \pi_0(A)$, so the set of self-adjoint fixed points of $\widetilde{\Phi}$ on $\mathcal{B}$ is $\pi_0(A)$. Now the desired result follows from Lemma 4.46. □

Now we will describe the universal C* and von Neumann algebras for those Jordan algebras that are actually self-adjoint parts of C* or von Neumann algebras. If A is the self-adjoint part of a C*-algebra $\mathcal{A}$, by Proposition 4.34, A is universally reversible. Therefore by Proposition 4.47, $C^*_u(A)$ is the C*-subalgebra of $\mathcal{A} \oplus \mathcal{A}^{op}$ generated by $\{a \oplus a^{op} \mid a \in A\}$. If $\mathcal{A}$ has no abelian part (i.e., has no one dimensional representations) then the universal algebra is actually equal to $\mathcal{A} \oplus \mathcal{A}^{op}$, and there is an analogous result for von Neumann algebras. We will now state these results more formally, but will not need them in the sequel.

4.48. Proposition. *If $\mathcal{A}$ is a unital C*-algebra with no one dimensional representations, then $C^*_u(\mathcal{A}_{\mathrm{sa}}) = \mathcal{A} \oplus \mathcal{A}^{op}$ with respect to the imbedding $a \to a \oplus a^{op}$.*

Proof. [67, 7.4.15]. □

4.49. Lemma. *Let $\mathcal{M}$ be a von Neumann algebra with no abelian direct summand. Then the universal von Neumann algebra of $\mathcal{M}_{\mathrm{sa}}$ is $\mathcal{M} \oplus \mathcal{M}^{op}$ with respect to the imbedding $a \to a \oplus a^{op}$.*

Proof. [67, Thm. 7.4.7]. □

Cartesian triples

We now introduce the notion of a Cartesian triple in a JBW-algebra, which generalizes the same notion in a von Neumann algebra, first introduced and studied in [AS, Chpt. 6]. In any von Neumann algebra containing a set of 2×2 matrix units, the Pauli spin matrices (cf. (3.9)) provide an example of a Cartesian triple. Our purpose here is to prove that the existence of a Cartesian triple of symmetries in a JBW-algebra is sufficient to guarantee that a JBW-algebra is isomorphic to the self-adjoint part of a von Neumann algebra containing 2×2 matrix units.

Recall that if e is a projection in a JB-algebra, an e-symmetry is an element s satisfying $s^2 = e$. By spectral theory applied to the Jordan subalgebra C(s,1), $e \circ s = s$, so $s \in M_e$. Then there are orthogonal projections p, q such that $s = p - q$, in fact $p = \frac{1}{2}(s + e)$ and $q = e - p$.

4.50. Definition. Let M be a JBW-algebra, and e a projection in M. A *Cartesian triple of e-symmetries* is an ordered triple (r, s, t) of mutually Jordan-orthogonal e-symmetries such that

$$U_r U_s U_t(x) = x \quad \text{for all } x \in M_e. \tag{4.15}$$

When there is no need to specify the projection e, we will say that (r, s, t) is a *Cartesian triple of partial symmetries.* When $e = 1$, we will say that (r, s, t) is a *Cartesian triple of symmetries.*

Remarks. Suppose (r, s, t) is a Cartesian triple of e-symmetries, and that $e = 1$. Then multiplying (4.15) on the left by U_r gives $U_r = U_s U_t$. Since each of U_r, U_s, U_t is its own inverse, then taking inverses gives $U_r = U_t U_s$. Thus U_s and U_t commute. Similar arguments show that any pair of U_r, U_s, U_t commute. It follows that (4.15) holds with any order for U_r, U_s, U_t. For arbitrary e, the same conclusion follows by applying the previous conclusion to U_r, U_s, U_t acting on the JB-algebra M_e.

We note also that there are triples of Jordan-orthogonal symmetries for which (4.15) fails. For example, if r, s, t, w are Jordan-orthogonal symmetries in a spin factor, then (4.15) fails since $U_r U_s U_t w = -w$, cf. Lemma 3.6.

4.51. Lemma. *Let M be a spin factor. Then M contains a non-zero Cartesian triple of symmetries iff M is four dimensional.*

Proof. Write $M = \mathbf{R}1 \oplus N$ as in Definition 3.33. Here N contains all symmetries other than ± 1. If dim $M = 4$, let r, s, t be an orthonormal basis of N. Then r, s, t are Jordan-orthogonal symmetries. Representing M as a JW-algebra (cf. Theorem 4.1), r, s, t anticommute, so one easily verifies that $U_r U_s U_t$ is the identity on M.

Now suppose that M is of unknown dimension, and admits a Cartesian triple r, s, t of symmetries. Recall that N is equipped with an inner product such that $a \circ b = (a \,|\, b)1$ for all $a, b \in N$. Suppose the dimension of M were greater than 4. Then N would have dimension greater than 3, so there would be a non-zero element w Jordan orthogonal to r, s, t. Then by the definition of a Cartesian triple and Lemma 3.6,

$$w = U_r U_s U_t w \quad \text{and} \quad U_r U_s U_t w = -w,$$

so $w = 0$, a contradiction. Thus we conclude M is at most four dimensional. On the other hand, r, s, t will be an orthonormal subset of N, so N is at least three dimensional and thus M is at least four dimensional. Thus M has dimension 4. □

4.52. Lemma. *Let M be the exceptional JBW-factor $H_3(\mathbf{O})$. Then M contains no non-zero Cartesian triple of partial symmetries.*

Proof. Let e be a non-zero projection in M and (r,s,t) a Cartesian triple of e-symmetries. We first show that $e \neq 1$ and that M_e is four dimensional.

Note that the center of M_e is $\mathbf{R}e$ by Proposition 3.13, so M_e is a JBW-factor. Write $r = p - q$ where p and q are orthogonal projections with sum e. By Lemma 3.6 applied in the JBW-algebra M_e, U_s exchanges p and q, so p and q are equivalent projections. We claim both are abelian projections in M_e. Suppose one were not abelian, say p. Since U_s is a Jordan automorphism of M_e, then q would also be non-abelian, and thus neither p nor q would be minimal (Lemma 3.30). This would imply that there are at least four mutually orthogonal non-zero projections in M, which by Lemma 3.22 would contradict M being a factor of type I_3. Thus e is the sum of the equivalent abelian projections p and q, and so by definition M_e is a factor of type I_2. This implies that $e \neq 1$, else $M = M_e$ would be both an I_2 factor and an I_3 factor, contradicting Lemma 3.22. Thus M_e is a spin factor containing a Cartesian triple of symmetries. By Lemma 4.51, M_e is four dimensional.

On the other hand, let $\{e_{ij}\}$ be the standard system of 3×3 matrix units for $H_3(\mathbf{O})$. Then $\{e_{33} H_3(\mathbf{O}) e_{33}\} = \mathbf{R}e_{33}$, so e_{33} is a minimal projection (Lemma 3.29). Since p, q, $1-e$ are orthogonal and non-zero, by Lemma 3.22, $1-e$ must be minimal. Minimal projections in a JBW-factor are exchangeable (Lemma 3.30 and Lemma 3.19), so there is a symmetry w exchanging $1-e$ and e_{33}. Then w also exchanges e and $1-e_{33} = e_{11}+e_{22}$. Thus U_w will be an isomorphism from M_e onto $M_{e_{11}+e_{22}} \cong H_2(\mathbf{O})$. It follows that the latter must be four dimensional. However, since $\mathbf{O}$ is of dimension 8 over $\mathbf{R}$, this is impossible. Thus we conclude that M contains no non-zero Cartesian triple of partial symmetries. □

4.53. Lemma. *If a JBW-algebra M contains a Cartesian triple of symmetries, then M is a JW-algebra, and is reversible in every faithful representation as a concrete JW-algebra.*

Proof. Let (r,s,t) be a Cartesian triple in M. We first show that M is a JW-algebra. Suppose there exists a factor representation π from M onto

$M_0 = H_3(\mathbf{O})$. Then $(\pi(r), \pi(s), \pi(t))$ is a Cartesian triple of symmetries in M_0. This is impossible by Lemma 4.52. Thus M admits no factor representation onto the exceptional factor $H_3(\mathbf{O})$. By Corollary 4.20, M is a JW-algebra.

Assume M is faithfully represented as a JW-subalgebra of $\mathcal{B}(H)_{\mathrm{sa}}$. By Corollary 4.30 it suffices to show that the I_2 summand of M is reversible. Let c be the central projection such that cM is the I_2 summand of M. We may assume $c \neq 0$ (for otherwise there is nothing to prove). Since multiplying by a central projection is a Jordan homomorphism, (cr, cs, ct) is a Cartesian triple in cM. Thus by replacing M with cM, it suffices to prove that a JW-algebra M of type I_2 containing a Cartesian triple of symmetries is reversible.

Let $s_0 = 1$ and let $(s_1, s_2, s_3) = (r, s, t)$ be the given Cartesian triple of symmetries in M. Let $x \in M$ and define

$$z_0 = (((x \circ s_1) \circ s_1) \circ s_2) \circ s_2), \tag{4.16}$$

$$z_i = (x - z_0) \circ s_i \quad \text{for } i = 1, 2, 3. \tag{4.17}$$

We are going to show that z_0, z_1, z_2, z_3 are central and that

$$x = \sum_i z_i s_i. \tag{4.18}$$

Let π be a factor representation of M and define $M_0 = \overline{\pi(M)}$ (σ-weak closure). Since M is of type I_2, by Lemma 4.8, M_0 is a type I_2 factor, i.e., a spin factor. Let $w_i = \pi(s_i)$ for $i = 0, 1, 2, 3$. Then (w_1, w_2, w_3) will be a Cartesian triple of symmetries in M_0. By Lemma 4.51, M_0 is 4-dimensional, so $\{w_0, w_1, w_2, w_3\}$ will be a basis for M_0. Thus we can write

$$\pi(x) = \sum_{i=0}^{3} \lambda_i w_i \tag{4.19}$$

for suitable scalars $\lambda_0, \ldots, \lambda_3$. One easily verifies that

$$\lambda_0 1 = (((\pi(x) \circ w_1) \circ w_1) \circ w_2) \circ w_2 \tag{4.20}$$

and

$$\lambda_i 1 = (\pi(x) - \lambda_0 1) \circ w_i \quad \text{for } i = 1, 2, 3. \tag{4.21}$$

Combining (4.16), (4.17), (4.20), (4.21) gives

$$\pi(z_i) = \lambda_i 1 \quad \text{for } i = 0, 1, 2, 3. \tag{4.22}$$

Thus for each i, each $a \in M$, and each factor representation π, the element $\pi(z_i)$ operator commutes with $\pi(a)$. Since factor representations separate points (Lemma 4.14), it follows that $z_0, \ldots, z_3$ are central in M. Furthermore, from (4.19) and (4.22) it follows that

$$\pi(x) = \pi\Big(\sum_i z_i s_i\Big)$$

for all factor representations π, from which (4.18) follows.

Finally, if $a_1, \ldots, a_n$ are in M, then we can write

$$a_k = \sum_{j=0}^{3} z_{jk} s_j$$

where each z_{jk} is central in M. Then $a_1 a_2 \cdots a_n + a_n \cdots a_2 a_1$ is a sum of terms of the form $z_{i_1} z_{i_2} \cdots z_{i_n}(s_{i_1} s_{i_2} \cdots s_{i_n} + s_{i_n} \cdots s_{i_2} s_{i_1})$ where each z_{i_j} is central and each s_{i_j} is in $\{s_1, s_2, s_3\}$. Since the symmetries s_i, s_j anticommute for $i \neq j$, each sum $(s_{i_1} s_{i_2} \cdots s_{i_n} + s_{i_n} \cdots s_{i_2} s_{i_1})$ can be rewritten in a similar form with $n \leq 3$. Such sums coincide either with a Jordan product of elements of M or a Jordan triple product, and thus are in M. Reversibility of M follows. □

We finish this section with some technical results on Cartesian triples that we will make frequent use of in later chapters.

4.54. Lemma. *Let $M \subset \mathcal{B}(H)_{\mathrm{sa}}$ be a JW-algebra, and let $R_0(M)$ be the real algebra generated by M. If (r, s, t) is a Cartesian triple of symmetries in M, then $j = rst$ is a central skew-adjoint element of $R_0(M)$ satisfying $j^2 = -1$.*

Proof. By definition, r, s, t are Jordan orthogonal symmetries, and thus anticommute. This implies that $j = rst$ is skew-adjoint and satisfies $j^2 = -1$. Furthermore, by assumption the symmetries satisfy $U_r U_s U_t = 1$ on M. Since conjugation by a symmetry is an algebraic automorphism of $\mathcal{B}(H)$, then $U_r U_s U_t = 1$ also holds on the real algebra $R_0(M)$ generated by M. Thus

$$rstxtsr = x \quad \text{for all } x \in R_0(M).$$

This implies that $rstx = xrst$, and so $j = rst$ is central in $R_0(M)$. □

4.55. Lemma. *Let $M \subset \mathcal{B}(H)_{\mathrm{sa}}$ be a reversible JW-algebra. If the real algebra $R_0(M)$ generated by M contains a skew-adjoint element j which is central in $R_0(M)$ and satisfies $j^2 = -1$, then $R_0(M) = M \oplus jM$*

with the inherited product, involution, and norm, and with $\jmath$ as complex unit, becomes a von Neumann algebra with self-adjoint part M. Thus M is isomorphic to the self-adjoint part of a von Neumann algebra.

Proof. We first show that

$$R_0(M) = M \oplus \jmath M \tag{4.23}$$

(direct sum as subspaces). If $x \in R_0(M)$, then

$$x = \tfrac{1}{2}(x + x^*) - \tfrac{1}{2}\jmath(\jmath(x - x^*)). \tag{4.24}$$

Since $R_0(M)$ is closed under the adjoint operation and M is reversible, then $x + x^* \in R_0(M)_{\mathrm{sa}} = M$ by reversibility. Furthermore, since $\jmath$ is central in $R_0(M)$ and skew-adjoint, then $\jmath(x - x^*) \in R_0(M)_{\mathrm{sa}} = M$. Thus $R_0(M) = M + \jmath M$. On the other hand, $M \cap \jmath M = \{0\}$, since each element of M is self-adjoint and each element of $\jmath M$ is skew-adjoint. This completes the proof of (4.23).

Since M is norm closed, it follows from (4.24) that $R_0(M) = M \oplus \jmath M$ is also norm closed. Since $\jmath$ is central in $R_0(A)$, with $\jmath^* = -1$ and $\jmath^* = -\jmath$, then we can view $R_0(M)$ as a complex *-algebra with imaginary unit scalar $\jmath$, and with the involution inherited from $\mathcal{B}(H)$. The norm inherited from $\mathcal{B}(H)$ is a norm on $R_0(M)$ viewed as a complex linear space, since for $x \in R_0(M)$,

$$\begin{aligned} \|(\alpha 1 + \beta \jmath)x\|^2 &= \|((\alpha 1 + \beta \jmath)x)^*(\alpha 1 + \beta \jmath)x\| \\ &= \|(\alpha^2 + \beta^2)x^*x\| \\ &= (\alpha^2 + \beta^2)\|x\|^2. \end{aligned}$$

Thus $R_0(M) = M \oplus \jmath M$ becomes a C*-algebra, with $(R_0(M))_{\mathrm{sa}} = M$. Since M is monotone complete with a separating set of normal states, then so is the C*-algebra $R_0(M)$, so the latter is a von Neumann algebra (A 95). □

Remark. In the context of the last theorem, it is not necessarily the case that $M + \imath M$ is a von Neumann algebra for the product inherited from $\mathcal{B}(H)$. For example, suppose that $M = M_2(\mathbf{C})_{\mathrm{sa}}$, and that $\mathcal{A}$ is the universal C*-algebra associated with the JB-algebra M (Definition 4.37). Identify M with its canonical image in $\mathcal{A}_{\mathrm{sa}}$. Then $\mathcal{A}$ is the direct sum of two copies of $M_2(\mathbf{C})$ (Proposition 4.48), and by definition the C*-subalgebra of $\mathcal{A}$ generated by M is all of $\mathcal{A}$. By finite dimensionality, $\mathcal{A}$ will be the complex *-algebra generated by M. If $M + iM$ were closed under the multiplication inherited from $\mathcal{A}$, then the complex *-algebra generated by M would be $M + iM$, so then $M + iM = \mathcal{A}$. Since $M + \imath M$

is 8-dimensional (as a real vector space), and $\mathcal{A}$ is 16-dimensional, this is impossible. Similarly, if $R_0(M)$ were closed under multiplication by $\imath$, then $R_0(M)$ would be the complex *-algebra generated by M, so $M + \jmath M = R_0(M) = \mathcal{A}$. As before, by counting dimensions this is shown to be impossible.

We will now discuss products $(x, y) \mapsto x{\star}y$ on $M + \imath M$ making $M + \imath M$ into a von Neumann algebra with involution $a + \imath b \mapsto a - \imath b$. We will say such a product is *Jordan compatible* if the induced Jordan product matches the given one on M, i.e., $a \circ b = \frac{1}{2}(a \star b + b \star a)$ for $a, b \in M$.

4.56. Theorem. *If a JBW-algebra M contains a Cartesian triple of symmetries (r, s, t), then M is isomorphic to the self-adjoint part of a von Neumann algebra. In particular, $\mathcal{M} = M + \imath M$ can be equipped with a unique von Neumann algebra product compatible with the given Jordan product and satisfying $\imath rst = 1$. For this product, the complex linear span of $1, r, s, t$ is a *-subalgebra *-isomorphic to $M_2(\mathbf{C})$, and thus $\mathcal{M} \cong M_2(C) \otimes \mathcal{B}$ for a von Neumann algebra $\mathcal{B}$.*

Proof. By Lemma 4.53, we may assume M is faithfully represented as a reversible JW-algebra, say $M \subset \mathcal{B}(H)_{\mathrm{sa}}$. Let $R_0(M)$ be the real *-algebra generated by M in $\mathcal{B}(H)$. Now by Lemma 4.54 the element $\jmath = -rst$ is central in $R_0(M)$ and satisfies $\jmath^* = -\jmath$, $\jmath^2 = -1$. By Lemma 4.55, $R_0(M) = M \oplus \jmath M$ becomes a von Neumann algebra for the complex unit $\jmath$ and the inherited norm, multiplication, and involution.

Now note that the identity map on M extends to a complex linear bijection from $M + \jmath M$ to $M + \imath M$ (with the former viewed as a complex space with complex unit $\jmath$) taking $a + \jmath b$ to $a + \imath b$ for $a, b \in M$. Thus we can use this to carry the product and norm from $M + \jmath M$ to $M + \imath M$, giving $M + \imath M$ an associative Jordan compatible product for which $M + \imath M$ becomes a von Neumann algebra. (In fact, this map is an isometry for the norms on $M + \jmath M$ and $M + \imath M$ inherited from $\mathcal{B}(H)$, but we will not need this result.) This product on $M + \imath M$ will not necessarily coincide with the associative product defined in $\mathcal{B}(H)$, though the Jordan products will agree. (See the remark preceding this lemma.) Since $1 = \jmath rst$ in $M \oplus \jmath M$, then for the corresponding product on $M + \imath M$, we have $\imath rst = 1$.

Now let $\mathcal{M}$ be any von Neumann algebra containing a Cartesian triple of symmetries r, s, t such that $\imath rst = 1$. Then the product of any two of the three symmetries is a multiple of the third, so the complex linear span of $1, r, s, t$ is a four dimensional complex *-subalgebra $\mathcal{C}_0$ of $\mathcal{M}$, which is easily seen to be isomorphic to $M_2(\mathbf{C})$. (In fact, the multiplication on $\mathcal{C}_0$ is determined by the fact that r, s, t are Jordan orthogonal, and thus are anticommuting symmetries. As we have observed before, the Pauli spin matrices form a Cartesian triple, and thus are also anticommuting symmetries. It is then straightforward to check that the map taking 1 to

1 and r, s, t to the three Pauli spin matrices (in the order listed in (3.9)) extends to a *-isomorphism from $\mathcal{C}_0$ onto $M_2(\mathbf{C})$.) Thus $\mathcal{M}$ contains a system of self-adjoint 2×2 matrix units $\{e_{ij}\}$, so it follows that $\mathcal{M} \cong M_2(C) \otimes \mathcal{B} \cong M_2(\mathcal{B})$, for $\mathcal{B}$ equal to the relative commutant of the set of matrix units $\{e_{ij}\}$ in $\mathcal{M}$ (Lemma 4.27). If $\mathcal{M}$ is represented as a weakly closed *-subalgebra of $\mathcal{B}(H)$, then the same will be true of $\mathcal{B}$, so $\mathcal{B}$ is a von Neumann algebra.

Now let $\star$ be a second Jordan compatible product on $\mathcal{M}$ such that $ir \star s \star t = 1$. We will show $\star$ coincides with the given product. We first sketch the idea. The Jordan product on $\mathcal{M}_{\mathrm{sa}}$ extends uniquely to a complex linear Jordan product on $\mathcal{M}$. As seen above, we can represent elements of $\mathcal{M}$ as 2×2 matrices. Then the associative product is determined by the Jordan product in the following sense:

$$(4.25) \qquad \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} xy & b \\ 0 & yx \end{pmatrix}.$$

(Note that a similar calculation (3.7) was behind the Jordan coordinatization theorem (Theorem 3.27). In that case, lacking $\mathcal{M} = \mathcal{M}_{\mathrm{sa}} + i\mathcal{M}_{\mathrm{sa}}$, the Jordan product on the self-adjoint part of the algebra did not determine the Jordan product on the whole algebra, and so we needed 3×3 matrix units instead of 2×2.)

Now we give the details of the proof. For each associative product on $\mathcal{M}$, we can create 2×2 matrix units from linear combinations of $1, r, s, t$, as shown above, and so can represent $\mathcal{M}$ as the algebra of 2×2 matrices over an algebra $\mathcal{B}$, where $\mathcal{B}$ is the relative commutant of r, s, t in $\mathcal{M}$. Since r, s, t are symmetries, any element x in $\mathcal{M}$ element commutes with each of r, s, t iff $U_r x = U_s x = U_t x = x$. Since these are Jordan expressions, the relative commutants for the two products coincide. Thus we can represent $\mathcal{M}$ with each product as 2×2 matrices over a fixed subset $\mathcal{B}$. The two products on $\mathcal{M}$ will then coincide iff the two products coincide when restricted to $\mathcal{B}$. Since the two products must induce the same Jordan product on $\mathcal{M}$, the equality of the two products then follows from (4.25). $\square$

By the bicommutant theorem, an irreducible von Neumann algebra $\mathcal{M} \subset \mathcal{B}(H)$ must be equal to $\mathcal{B}(H)$. For JW-algebras, this is not necessarily the case, since for example $M_n(\mathbf{R})$ acts irreducibly on $\mathbf{C}^n$. However, if we add the assumption that M is isomorphic to the self-adjoint part of $\mathcal{B}(H)$ the analogous conclusion holds, as we will show below (cf. Proposition 4.59). The remaining results will also be useful in our characterization of normal state spaces of von Neumann algebras.

4.57. Lemma. *Let M be a reversible JW-subalgebra of $\mathcal{B}(H)_{\mathrm{sa}}$. Suppose that there exists a self-adjoint system $\{e_{ij}\}$ of $n \times n$ matrix units* *in $\mathcal{B}(H)$ (with $1 < n < \infty$) such that $e_{ij} + e_{ji} \in M$ for all i, j. Suppose further that there is a partial symmetry t such that $e_{11} - e_{22}$, $e_{12} + e_{21}$, t form a Cartesian triple of partial symmetries in M. Then there is an element j in the center of the real algebra $R_0(M)$ generated by M such that $j^* = -j$ and $j^2 = -1$.*

Proof. Note that $e_{ij} = e_{ii}(e_{ij} + e_{ji}) \in R_0(M)$ for all $i \neq j$, so $\{e_{ij}\}$ is a self-adjoint system of matrix units for $R_0(M)$. By Lemma 4.27 we can identify $R_0(M)$ with $M_n(\mathcal{B})$ where $\mathcal{B}$ is the relative commutant of these matrix units in $R_0(M)$. By assumption M is reversible, so $M = R_0(M)_{\text{sa}} = H_n(\mathcal{B})$.

Define $r = e_{11} - e_{22}$ and $s = e_{12} + e_{21}$. Then by assumption there is an $(e_{11} + e_{22})$-symmetry t such that r, s, t form a Cartesian triple of $(e_{11} + e_{22})$-symmetries. Define $j_0 = rst$, and

$$M_{12} = (e_{11} + e_{22})M(e_{11} + e_{22}).$$

By Lemma 4.54 applied to M_{12}, j_0 satisfies $j_0^2 = -(e_{11} + e_{22})$ and $j_0^* = -j_0$, and j_0 is central in $R_0(M_{12})$. Since j_0 is in $M_n(\mathcal{B})$ and $j_0 = j_0(e_{11} + e_{22}) = (e_{11} + e_{22})j_0$, then j_0 admits a unique representation

$$j_0 = \sum_{i,j=1}^{2} b_{ij} e_{ij} \tag{4.26}$$

with each b_{ij} in $\mathcal{B}$. We will show that there is a central element z in $\mathcal{B}$ such that $j_0 = z(e_{11}+e_{22})$, and that $j = \sum_{i=1}^{n} z e_{ii}$ will have the properties specified in the lemma.

Since j_0 is central in $R_0(M_{12})$, it commutes with e_{11}, so

$$j_0 e_{11} = b_{21}e_{21} + b_{11}e_{11} = e_{11}j_0 = b_{12}e_{12} + b_{11}e_{11}.$$

By the uniqueness of the representation (4.26), $b_{21} = b_{12} = 0$. Since j_0 commutes with $e_{12}+e_{21}$, a similar calculation shows $b_{11} = b_{22}$. Thus there is an element $z \in \mathcal{B}$ such that

$$j_0 = ze_{11} + ze_{22} = z(e_{11} + e_{22}).$$

Since $j_0^* = -j_0$ and $j_0^2 = -(e_{11}+e_{22})$, then the element z satisfies $z^* = -z$ and $z^2 = -1$. We are going to show z is central in $\mathcal{B}$. Let $b \in \mathcal{B}$, and let $x = be_{12}+b^*e_{21}$. Then $x \in R_0(M)_{\text{sa}} = M$, and $(e_{11}+e_{22})x(e_{11}+e_{22}) = x$, so $x \in (e_{11} + e_{22})M(e_{11} + e_{22})$. In particular, j_0 must commute with x. Then

$$j_0 x = z(e_{11} + e_{22})(be_{12} + b^*e_{21}) = xj_0 = (be_{12} + b^*e_{21})z(e_{11} + e_{22})$$

which gives

$$zbe_{12} + zb^* e_{21} = bze_{12} + b^* ze_{21}.$$

This implies $zb = bz$, so z is central in $\mathcal{B}$ as claimed.

Finally, define $\jmath = \sum ze_{ii}$. Then $\jmath$ is clearly central in $M_n(\mathcal{B}) = R_0(M)$, and satisfies $\jmath^2 = -1$, $\jmath^* = -\jmath$. □

4.58. Lemma. *Let M be a JBW-algebra isomorphic to $\mathcal{B}(H)_{\mathrm{sa}}$. If e is a projection in M and r and s are Jordan orthogonal e-symmetries, then there exists an e-symmetry t in M such that (r, s, t) is a Cartesian triple.*

Proof. It suffices to prove the result for $M = \mathcal{B}(H)_{\mathrm{sa}}$. Define $t = \imath rs$. Since r and s anticommute, one finds that t is an e-symmetry that is Jordan orthogonal to r and s. Furthermore, $rst = -\imath e$ and then $tsr = (rst)^* = (-\imath e)^* = \imath e$. Therefore for $x \in e\mathcal{B}(H)e$,

$$U_r U_s U_t x = rstxtsr = -\imath ex(\imath e) = x,$$

so (r, s, t) is a Cartesian triple of e-symmetries in M. (See also (A 192)). □

We say a concrete JW-algebra $M \subset \mathcal{B}(H)_{\mathrm{sa}}$ is *irreducible* if there are no proper closed subspaces of H invariant under M.

4.59. Proposition. *If M is a JBW-algebra isomorphic to $B(K)_{\mathrm{sa}}$ for a complex Hilbert space K, and M is represented as an irreducible concrete JW-algebra in $\mathcal{B}(H)_{\mathrm{sa}}$, then $M = \mathcal{B}(H)_{\mathrm{sa}}$.*

Proof. The case where K is one dimensional is obvious, and so we assume that $\dim K > 1$. We first show that there is an element $\jmath$ in the center of the real algebra $R_0(M)$ generated by M such that $\jmath^* = -\jmath$ and $\jmath^2 = -1$. We are going to apply Lemma 4.57, so we show that $M \cong B(K)_{\mathrm{sa}}$ satisfies the hypotheses of that lemma. (Note that there is a subtlety here: associative computations are taking place in $\mathcal{B}(H)$ and not $\mathcal{B}(K)$.) We can write $1 = p_1 + \cdots + p_n$ where $p_1, p_2, \ldots, p_n$ are projections exchangeable by symmetries in M and $2 \le n < \infty$. For $\imath = 1, 2, \ldots, n$ let s_i be a symmetry in M that exchanges p_1 and p_i. Define

$$e_{ij} = s_i p_1 s_j \in \mathcal{B}(H) \quad \text{for } 1 \le i, j \le n.$$

By Lemma 4.28, $\{e_{ij}\}$ is a system of matrix units for $R_0(M) \subset \mathcal{B}(H)$, and $e_{ij} + e_{ji} = s_i p_1 s_j + s_j p_1 s_i \in M$.

Since $M \cong B(K)_{\mathrm{sa}}$, and $r = e_{11} - e_{22}$ and $s = e_{12} + e_{21}$ are Jordan orthogonal $(e_{11} + e_{22})$-symmetries, by Lemma 4.58 there is an $(e_{11} + e_{22})$-symmetry t in M such that (r, s, t) is a Cartesian triple. The existence of $\jmath$ as described above now follows from Lemma 4.57.

Since M is irreducible, the von Neumann algebra it generates is irreducible, and so has commutant $\mathbf{C}1$ in $\mathcal{B}(H)$ (A 80). Thus by the bicommutant theorem (A 88), the von Neumann algebra generated by M is all of $\mathcal{B}(H)$, so $\jmath$ is also central in $\mathcal{B}(H)$. Therefore $\jmath$ must be a scalar multiple of the identity. The only possibilities are $\jmath = \pm\imath 1$. It follows that $R_0(M)$ contains $\imath 1$. Thus $R_0(M)$ is a complex *-subalgebra of $\mathcal{B}(H)$. Hence $\overline{R_0(M)}$ is the von Neumann algebra generated by M, which as remarked above must be all of $\mathcal{B}(H)$. The self-adjoint part of a C*-algebra is reversible in every concrete representation (Proposition 4.34), so M is reversible. Hence $M = \overline{R_0(M)}_{\mathrm{sa}} = \mathcal{B}(H)_{\mathrm{sa}}$ by Lemma 4.25. □

Notes

The result that spin factors are JW-algebras (Theorem 4.1) is due to Topping [129], but the proof we have given is taken from that in [67]. The fact that $H_3(\mathbf{O})$ is exceptional (Theorem 4.6) is a classic result of Albert [2]. The "Gelfand–Naimark" type theorem for JB-algebras (Theorem 4.19) appeared in [8]. We have taken the decomposition of JBW-algebras of type I_3 (Lemma 4.21) from [67]; this result is due to Stacey [121]. The decomposition of an arbitrary JBW-algebra into JW and exceptional pieces (Theorem 4.23) appeared first in [116], but we have followed the proof in [67].

Reversibility in JC-algebras and JW-algebras was investigated by Størmer [124, 125, 126], and by Hanche-Olsen [66]. Those concrete JC-algebras that are the self-adjoint part of the C*-algebra they generate are characterized in [124]. The key result that a concrete JW-algebra is reversible iff its I_2 summand is reversible (Corollary 4.30) is in [125] and in [67, Thm. 5.3.10]. Our proof of the reversibility of a JBW-algebra whose identity is the sum of at least three exchangeable projections (Theorem 4.29) is essentially a proof of the Jordan coordinatization theorem (Theorem 3.27) for the case of special Jordan algebras, and is based on the proof in [67, Thm. 2.8.3].

The concept of the universal C*-algebra for a JB-algebra (Definition 4.37) first appeared in [11], but the construction of the universal C*-algebra in Proposition 4.36 is new. The universal C*-algebra was studied further in [66], where the results in Corollary 4.39, Corollary 4.42, and Proposition 4.48 can be found. The corresponding notion of a universal von Neumann algebra (Definition 4.43) comes from [67], as does the description of the universal von Neumann algebra for the self-adjoint part of a von Neumann algebra (Lemma 4.49).

Cartesian triples were introduced in the characterization of normal state spaces of von Neumann algebras by Iochum and Shultz [71], although not with that name, and the results of the final section of this chapter mostly appear in [71]. Cartesian triples played an important role in the treatment of orientations of von Neumann algebras and their normal state spaces, which was first announced in [13], and presented with complete proofs in [AS, Chpt. 7].

5 State Spaces of Jordan Algebras

In this chapter we will discuss properties of the normal state space of JBW-algebras. Since every JB-algebra state space is also the normal state space of a JBW-algebra (Corollary 2.61), these properties also apply to JB-algebra state spaces.

We will first define the notion of the carrier projection of a set of normal states, and the concept of orthogonality of states. Then we will characterize those maps that are the dual maps to Jordan homomorphisms. Next we will discuss traces on JB-algebras. These will play a key role in the proof of the main result of this chapter: the fact that there is a 1-1 correspondence between projections in a JBW-algebra and the norm closed faces of the normal state space. For most types of JBW-algebras we reduce this result to the corresponding result for norm closed faces of normal state spaces of von Neumann algebras. For JBW-algebras of type I_2 or I_3, we need a different argument based on traces.

We then study some properties of the state space that concern the pure states (i.e., the extreme points of the state space). One of these is the Hilbert ball property, which says that faces generated by pairs of pure states are affinely isomorphic to Hilbert balls. This is closely related to the physically interesting property that is called "symmetry of transition probabilities". These results also play a role in our proof of the fact that every normal state on an atomic JBW-algebra is a countable convex sum of orthogonal extreme points. This is then used to show that atomic JBW-algebras admit a notion of trace class elements that allows us to generalize the well known description of normal states on $\mathcal{B}(H)$ in terms of the trace and "density matrices." We finish by describing two properties related to faces and affine automorphisms of the normal state space: "symmetry" and "ellipticity".

Many of the aforementioned properties will reappear in later chapters in our abstract characterizations of those convex sets that are affinely isomorphic to Jordan state spaces or C*-algebras.

Carrier projections

5.1. Lemma. *Let M be a JBW-algebra, and F a non-empty set of positive normal functionals on M. Then there is a smallest projection p such that $\sigma(p') = 0$ (equivalently, $\sigma(p) = \|\sigma\|$) for all σ in F.*

Proof. Let

$$S = \{p \in M \mid p = p^2,\ \sigma(p') = 0 \text{ for all } \sigma \in F\}.$$

Note that for projections p, q and $\sigma \in F$,

$$\begin{aligned}
(5.1) \qquad \sigma(p') = \sigma(q') = 0 \quad &\Leftrightarrow \quad \sigma(p' + q') = 0 \\
&\Rightarrow \quad \sigma(r(p' + q')) = 0 \quad \text{(by equation (2.6))} \\
&\Rightarrow \quad \sigma(p' \vee q') = 0 \quad \text{(by Lemma 2.22)} \\
&\Rightarrow \quad \sigma((p \wedge q)') = 0.
\end{aligned}$$

Thus if p and q are in S, by (5.1) $p \wedge q$ is in S, so S is directed downwards. By Proposition 2.5, $\{q \in S\}$ converges σ-weakly to a projection p. Then $p \in S$ and evidently p is the minimal element of S. $\square$

The definition below generalizes the notion of the carrier projection of a normal state on a von Neumann algebra (A 106).

5.2. Definition. Let M be a JBW-algebra, and F a non-empty set of positive normal functionals on M. The smallest projection p such that $\sigma(p) = \|\sigma\|$ (equivalently, $\sigma(p') = 0$) for all σ in F is called the *support projection* or *carrier projection* of F, and is denoted carrier(F). For a single positive normal functional σ we will refer to the carrier projection of the set $\{\sigma\}$ as the carrier projection of σ.

If F is any set of normal positive functionals on a JBW-algebra M, then we have

$$(5.2) \qquad \bigvee \{\, \text{carrier}(\omega) \mid \omega \in F \,\} = \text{carrier}(F).$$

(The projection on the left takes the value 1 on every element of F, so dominates carrier(F). On the other hand, carrier(F) has the value $\|\omega\|$ on each $\omega \in F$, so dominates carrier(ω) for all $\omega \in F$.) Note also that by Lemma 5.1 and the equivalence (1.57), the carrier projection of a set F of normal positive functionals on a JBW-algebra M is the minimal projection p such that the image of U_p^* contains F.

The following definition is a special case of the notion of orthogonality in a base norm space (A 24).

5.3. Definition. Positive normal linear functionals σ, τ on a JBW-algebra M are *orthogonal* if

$$\|\sigma - \tau\| = \|\sigma\| + \|\tau\|.$$

We then write $\sigma \perp \tau$.

Each normal linear functional ρ on a JBW-algebra admits a unique decomposition $\rho = \sigma - \tau$ as a difference of positive normal functionals with $\sigma \perp \tau$ (Proposition 2.58). We then write ρ^+ in place of σ and ρ^- in place of τ, and refer to

$$\rho = \rho^+ - \rho^-$$

as the *orthogonal decomposition* of ρ.

5.4. Lemma. *Positive normal linear functionals σ, τ on a JBW-algebra M are orthogonal iff their carrier projections are orthogonal.*

Proof. If $\sigma \perp \tau$, let $\rho = \sigma - \tau$. Then this is the unique orthogonal decomposition of ρ, so by Proposition 2.58 there is a projection p such that

$$U_p^*\sigma = \sigma, \quad U_p^*\tau = 0.$$

Therefore $U_{p'}^*\tau = \tau$ (cf. (1.56)). Thus the carrier projections of σ and τ are under p and p' respectively (by the remarks following the definition of the carrier projection (Definition 5.2)), and so are orthogonal.

Conversely, suppose that p and q are the carrier projections of σ and τ, and that $p \perp q$. Then $\tau(p) \le \tau(q') = 0$, so $\tau(p) = 0$; similarly $\sigma(q) = 0$. Thus

$$(\sigma - \tau)(p - q) = \sigma(p) + \tau(q) = \|\sigma\| + \|\tau\|.$$

Since $\|p - q\| \le 1$, then

$$\|\sigma - \tau\| \ge \|\sigma\| + \|\tau\|.$$

The opposite inequality follows from the triangle inequality, so equality holds. It follows that $\sigma \perp \tau$. □

Note that as a consequence, for positive normal functionals σ and τ on a JBW-algebra

$$\sigma \perp \tau \quad \Leftrightarrow \quad U_{\text{carrier}(\sigma)}^*\tau = 0. \tag{5.3}$$

Indeed, if σ, τ have carrier projections p and q respectively, then

$$\sigma \perp \tau \quad \Rightarrow \quad p \perp q \quad \Rightarrow \quad U_p^*\tau = U_p^*U_q^*\tau = 0,$$

where the last equality follows from Proposition 2.18. On the other hand,

$$U_p^*\tau = 0 \quad \Rightarrow \quad U_{p'}^*\tau = \tau \quad \Rightarrow \quad p' \ge \text{carrier}(\tau) = q \quad \Rightarrow \quad p \perp q,$$

and so (5.3) follows.

5.5. Lemma. *Let ρ, σ_1 and σ_2 be positive normal functionals on a JBW-algebra M. If ρ is orthogonal to σ_1 and to σ_2, then ρ is orthogonal to $\lambda_1\sigma_1 + \lambda_2\sigma_2$ for all positive scalars λ_1, λ_2.*

Proof. If ρ is orthogonal to σ_1 and to σ_2, and p is the carrier projection of ρ, then by (5.3)

$$U_p^*(\lambda_1\sigma_1 + \lambda_2\sigma_2) = \lambda_1 U_p^*\sigma_1 + \lambda_2 U_p^*\sigma_2 = 0,$$

so ρ is orthogonal to $\lambda_1\sigma_1 + \lambda_2\sigma_2$ by (5.3). □

5.6. Definition. If M is a JBW-algebra with normal state space K, and p is a projection in M, then we define the norm closed face F_p of K by

$$F_p = \{\, \sigma \in K \mid \sigma(p) = 1 \,\}.$$

We say F_p is the *face of K associated with p*. A face of the form F_p we call a *projective face.*

One of the key results of this chapter will show that every norm closed face of the normal state space K of a JBW-algebra is projective (Theorem 5.32).

We note that by Proposition 1.41,

$$F_p = (\operatorname{im} U_p^*) \cap K = (\ker U_{p'}^*) \cap K. \tag{5.4}$$

Since the predual of M is positively generated (cf. Corollary 2.60), it follows that for each element a of a JBW-algebra and each projection p, we have $a = 0$ on F_p iff $a = 0$ on $\operatorname{im} U_p^*$. Thus for $a \in M$,

$$a = 0 \text{ on } F_p \iff U_p a = 0. \tag{5.5}$$

5.7. Proposition. *Let a, b be positive elements of a JBW-algebra M. Then $a \perp b$ iff there exists a projection p such that $a = 0$ on F_p and $b = 0$ on $F_{p'}$.*

Proof. Suppose that $a \perp b$. Let $p = r(b)$. (Recall that $r(b)$ denotes the range projection of b, cf. Definition 2.14.) Since positive elements are orthogonal iff their range projections are orthogonal (Proposition 2.16), then $r(a) \perp r(b) = p$, and so $r(a) \leq p'$. Now $p = 1$ on F_p implies that $p' = 0$ on F_p. Since $r(a) \leq p'$, then $r(a) = 0$ on F_p. By (2.6) this implies that $a = 0$ on F_p. Similarly since $r(b) = p = 0$ on $F_{p'}$, then $b = 0$ on $F_{p'}$.

Conversely, suppose that a and b are positive elements of M, and that there is a projection p such that $a = 0$ on F_p and $b = 0$ on $F_{p'}$. Then

$U_p a = 0$ (by (5.5)), so by (1.49) $U_{p'} a = a$. Then by definition $r(a) \le p'$. Similarly, $b = 0$ on $F_{p'}$ implies that $r(b) \le p$, and so $r(a) \perp r(b)$. This completes the proof that $a \perp b$. □

The lattice of projective faces

Recall that a JBW-algebra M can be identified as an order unit space with the space $A_b(K)$ of bounded affine functions on K, where K is the normal state space of M (cf. Corollary 2.60). Viewing elements of M as functionals on K, we have the following result, which shows that a projection p is determined by its associated face F_p.

5.8. Proposition. *If p is a projection in a JBW-algebra M, then*

$$p = \inf \{\, b \in M \mid 0 \le b \le 1 \text{ and } b = 1 \text{ on } F_p \,\}. \tag{5.6}$$

Proof. To verify this, let $b \in M$ with $0 \le b \le 1$ and $b = 1$ on F_p. Then $0 \le 1 - b \le 1$ and $1 - b = 0$ on F_p. Thus $U_p(1-b) = 0$ (cf. (5.5)), and so $U_{p'}(1-b) = 1 - b$. Therefore

$$1 - b = U_{p'}(1-b) \le U_{p'} 1 = p' = 1 - p,$$

which implies $b \ge p$. This proves (5.6). □

It follows from (5.6) that the complementary face $F_{p'}$ is determined by F_p. Note also that (5.6) implies

$$\operatorname{carrier}(F_p) = p, \tag{5.7}$$

and thus $p \mapsto F_p$ is a bijection from the set of projections of a JBW-algebra M to the set of projective faces of the normal state space of M.

5.9. Definition. Let M be a JBW-algebra with normal state space K. We will write $\mathcal{F}$ for the set of projective faces of K, ordered by inclusion. We define the map $F \mapsto F'$ on $\mathcal{F}$ by $(F_p)' = F_{p'}$ for each projection p, and call F' the *complementary face* of F.

5.10. Proposition. *Let M be a JBW-algebra with normal state space K. The set $\mathcal{F}$ of projective faces of K forms a complete orthomodular lattice, with $F \wedge G = F \cap G$, and with orthocomplementation $F \mapsto F'$. The map $p \mapsto F_p$ is a lattice ortho-isomorphism from the projection lattice of M onto $\mathcal{F}$. The inverse map takes a projective face to its carrier projection.*

Proof. The map $p \mapsto F_p$ is evidently order preserving, as is the inverse map $F \mapsto \operatorname{carrier}(F)$. Thus $\mathcal{P}$ and $\mathcal{F}$ are order isomorphic.

Since the lattice of projections is a complete orthomodular lattice with orthocomplementation $p \mapsto p'$ (Proposition 2.25), it follows that $\mathcal{F}$ is an orthomodular lattice with orthocomplementation $F_p \mapsto F_{p'}$. Now $F_p = \{\sigma \in K \mid \sigma(p') = 0\}$ and (5.1) imply that $F_p \cap F_q = F_{p \wedge q}$. Thus the greatest lower bound of projective faces in $\mathcal{F}$ is their intersection. □

Dual maps of Jordan homomorphisms

5.11. Lemma. *Let M be a JBW-algebra, and let p be a projection in M with associated face $F = F_p$. Then p is the unique positive bounded affine functional on the normal state space K of M with value 1 on F and value 0 on $F' = F_{p'}$.*

Proof. Recall that we can identify M with the set of bounded affine functions on K (Corollary 2.60). Let $b \in M^+$ satisfy $b = 1$ on F and $b = 0$ on F'. Then $U_{p'}b = 0$, so by the complementarity of U_p and $U_{p'}$ we have $U_p b = b$. We next show $b \leq 1$. If $\sigma \in K$, since $\|U_p^*\| \leq 1$ (Proposition 1.41), then $U_p^*\sigma = \lambda\omega$ for some $\omega \in F = \operatorname{im} U_p^* \cap K$ and $\lambda \in \mathbf{R}$ with $0 \leq \lambda \leq 1$. Then for all $\sigma \in K$,

$$\sigma(b) = \sigma(U_p b) = (U_p^*\sigma)(b) = \lambda\omega(b) = \lambda \leq 1,$$

since $b = 1$ on F. Thus $b \leq 1$ follows. Then

$$b = U_p b \leq U_p 1 = p.$$

Since $1 - b$ is 0 on F_p and 1 on $F_{p'}$, the same argument shows $1 - b \leq p' = 1 - p$, so $p \leq b$. Thus $b = p$. □

5.12. Lemma. *Let A be a JB-algebra with state space K. Then the map $a \mapsto a|K$ is an isomorphism from the order unit space $(A, 1)$ onto the order unit space $A(K)$ of w^*-continuous affine functions on K with the pointwise ordering and norm, with the constant function 1 as distinguished order unit.*

Proof. The fact that $(A, 1)$ is a complete order unit space was established in Theorem 1.11. The rest of the lemma follows from the corresponding statement for complete order unit spaces: (A 20). □

5.13. Lemma. *If M_1, M_2 are JBW-algebras with normal state spaces K_1, K_2, then $T \mapsto T^*|K_2$ is a 1-1 correspondence of positive unital σ-weakly continuous maps $T : M_1 \to M_2$ and affine maps from K_2 into K_1. Similarly, if A_1, A_2 are JB-algebras with state spaces K_1, K_2, then*

$T \mapsto T^*|K_2$ is a 1-1 correspondence of positive unital maps $T : A_1 \to A_2$ and affine w^*-continuous maps from K_2 into K_1.

Proof. Assume first that K_1, K_2 are the normal state spaces of the JBW-algebras M_1, M_2 respectively. If $\omega \in K_2$, then $T^*\omega = \omega \circ T$ is σ-weakly continuous, and thus is normal (Corollary 2.6). Thus T^* maps K_2 into K_1. If T_1 and T_2 are bounded linear maps from M_1 into M_2 such that T_1^* and T_2^* agree on K_2, then $T_1 a$ and $T_2 a$ agree on all normal states of M_2, so we must have $T_1 = T_2$. Thus $T \mapsto T^*|K_2$ is 1–1.

Now let $\phi : K_2 \to K_1$ be an affine map. Then ϕ extends uniquely to a linear map from the linear span of K_2 into the linear span of K_1. By Corollary 2.60, the unit ball of the preduals $(M_2)_*$, $(M_1)_*$ are $\mathrm{co}(K_2 \cup (-K_2))$ and $\mathrm{co}(K_1 \cup (-K_1))$ respectively, so the extension of ϕ has norm at most one. We also denote this extension by ϕ, and let $T = \phi^* : M_1 \to M_2$. Then T is a positive unital σ-weakly continuous map, and T^* restricted to K_2 equals ϕ. This shows that $T \mapsto T^*|K_2$ is surjective, which completes the proof of the first statement of the lemma.

Assume next that K_1, K_2 are the state spaces of JB-algebras A_1, A_2 respectively. If $T : A_1 \to A_2$ is a unital positive map, then T^* is a w*-continuous affine map from K_2 into K_1. Since states separate elements of a JB-algebra, then $T \mapsto T^*|K_2$ is injective. To see that this map is surjective, let ϕ be a w*-continuous affine map from K_2 into K_1. Identify A_i with $A(K_i)$ for $i = 1, 2$ (cf. Lemma 5.12). Define $T : A_1 \to A_2$ to be the map given by $Ta = a \circ \phi$. Then T^* restricted to K_2 coincides with ϕ. Thus $T \mapsto T^*|K_2$ is bijective. □

5.14. Lemma. *Let M_1, M_2 be JBW-algebras with normal state spaces K_1, K_2. Let $\phi : K_2 \to K_1$ be an affine map, and $T : M_1 \to M_2$ the positive unital σ-weakly continuous map such that $\phi = T^*|K_2$. Let F_1 and F_2 be projective faces of K_1 and K_2 respectively, with associated projections p_1 and p_2. Then these are equivalent:*

(i) $Tp_1 = p_2$,
(ii) $\phi^{-1}(F_1) = F_2$ *and* $\phi^{-1}(F_1') = F_2'$.

If ϕ is surjective, then (i) and (ii) are also equivalent to

(iii) $\phi(F_2) = F_1$ *and* $\phi(F_2') = F_1'$.

Proof. (i) $\Leftrightarrow$ (ii) Viewing elements of M_1 and M_2 as bounded affine functions on their normal state spaces, using $Tp_1 = p_1 \circ (T^*) = p_1 \circ \phi$, and applying Lemma 5.11, we have

$$\begin{aligned} Tp_1 = p_2 &\iff (Tp_1)^{-1}(1) = F_2 \quad \text{and} \quad (Tp_1)^{-1}(0) = F_2' \\ &\iff \phi^{-1}(p_1^{-1}(1)) = F_2 \quad \text{and} \quad \phi^{-1}(p_1^{-1}(0)) = F_2' \\ &\iff \phi^{-1}(F_1) = F_2 \quad \text{and} \quad \phi^{-1}(F_1') = F_2'. \end{aligned}$$

(iii) $\Rightarrow$ (i) Assume (iii) holds, and let $b = Tp_1$. Then $b \geq 0$ and $b = p_1 \circ \phi$ on K_2, so by (iii) b is a positive element of M_2 with value 1 on F_2 and value 0 on F_2'. By Lemma 5.11, $b = p_2$, so $Tp_1 = p_2$.

(ii) $\Rightarrow$ (iii) Assume ϕ is surjective, and that (ii) holds. Then applying ϕ to each side of the equations in (ii) gives (iii). □

5.15. Lemma. *Let M_1 and M_2 be JBW-algebras with normal state spaces K_1 and K_2, and let $\phi : K_2 \to K_1$ be an affine map. Let $T : M_1 \to M_2$ be the unital positive σ-weakly continuous map such that $T^*|K_2 = \phi$. Then the following are equivalent:*

(i) *T is a unital Jordan homomorphism from M_1 into M_2,*
(ii) *ϕ^{-1} preserves complements of projective faces,*
(iii) *ϕ^{-1} as a map from the lattice of projective faces of K_1 into the lattice of projective faces of K_2 preserves lattice operations and complements.*

Proof. Note first that if p is a projection in M_1, and q is the range projection of Tp', then

$$\phi^{-1}(F_p) = \phi^{-1}(p'^{-1}(0)) = (p' \circ \phi)^{-1}(0) = (Tp')^{-1}(0) = q^{-1}(0) = F_{q'},$$

where the penultimate equality follows from the fact that positive elements of M_2 and their range projections have the same annihilators in K_2 (cf. (2.6)). Thus the inverse image of a projective face is projective.

(i) $\Rightarrow$ (ii) Let F_1 be any projective face of K_1 and let $p_1 \in M_1$ be the associated projection. If (i) holds, let $p_2 = Tp_1 \in M_2$, and let F_2 be the corresponding projective face of K_2. By Lemma 5.14, $\phi^{-1}(F_1) = F_2$, and $\phi^{-1}(F_1') = F_2'$, so $\phi^{-1}(F_1') = (\phi^{-1}(F_1))'$. Thus ϕ^{-1} preserves complements.

(ii) $\Rightarrow$ (i) We first show T maps projections to projections, preserving orthogonality of projections. Let $p_1 \in M_1$ be a projection with associated face $F_1 \subset K_1$. Let $F_2 = \phi^{-1}(F_1)$; by the remarks in the first paragraph of this proof, F_2 is a projective face of K_2. Let $p_2 \in M_2$ be the associated projection. By (ii), $\phi^{-1}(F_1') = F_2'$, so by Lemma 5.14, $Tp_1 = p_2$. Thus T maps projections to projections and preserves complements of projections (since $T1 = 1$). Now suppose p and q are orthogonal projections. Since T is a positive map, then $p \leq q'$ implies $Tp \leq T(q') = (Tq)'$, so $Tp \perp Tq$. Thus T also preserves orthogonality of projections.

It follows that T preserves squares of elements with finite spectral decompositions. Since T is a positive unital map and M has the order unit norm, T is norm continuous (A 15). By continuity and the spectral theorem (Theorem 2.20), T preserves squares of all elements. It follows that T is a Jordan homomorphism.

(ii) $\Leftrightarrow$ (iii) The map ϕ^{-1} preserves intersections and thus lattice greatest lower bounds (cf. Proposition 5.10). Since complementation is order reversing, if ϕ^{-1} preserves complements, it will then necessarily also preserve least upper bounds. The reverse implication (iii) implies (ii) is trivial. □

5.16. Proposition. *If M_1, M_2 are JBW-algebras with normal state spaces K_1, K_2, then $T \mapsto T^*|K_2$ is a 1-1 correspondence of Jordan isomorphisms from M_1 onto M_2 and affine isomorphisms from K_2 onto K_1. Similarly, if A_1 and A_2 are JB-algebras with state spaces K_1, K_2, then the map $T \mapsto T^*|K_2$ is a 1-1 correspondence of Jordan isomorphisms from A_1 onto A_2 and affine homeomorphisms (for the w^*-topologies) from K_2 onto K_1.*

Proof. Assume that M_1, M_2 are JBW-algebras with normal state spaces K_1, K_2. Then $T \mapsto T^*|K_2$ is a 1-1 correspondence of unital order isomorphisms from M_1 onto M_2 and affine isomorphisms from K_2 onto K_1 (cf. Lemma 5.13). Note that T is a unital order isomorphism iff T is a Jordan isomorphism. In fact, unital order isomorphisms are Jordan isomorphisms (Theorem 2.80), and conversely, each Jordan isomorphism preserves squares, so is an order isomorphism. This completes the proof of the first statement of the proposition.

The proof of the second statement is similar. □

5.17. Corollary. *Let M be a JBW-algebra with normal state space K. Let F and G be projective faces with associated projections p, q respectively. Let T be a Jordan automorphism of M. Then $T^*(F) = G$ iff $Tq = p$.*

Proof. If $Tq = p$, then $T^*(F) = G$ follows from Lemma 5.14((i)$\Rightarrow$ (iii)). Conversely, suppose $T^*(F) = G$. Since T^* is an affine automorphism of K (Proposition 5.16), it preserves complements of projective faces (see the remarks following (5.6)). Thus $T^*(F') = G'$. Now by Lemma 5.14 ((iii)$\Rightarrow$ (i)), $Tq = p$. □

Traces on JB-algebras

5.18. Lemma. *Let M be a JBW-algebra, and let τ be a normal state on M. The following four properties are equivalent.*

(i) $\tau((a \circ b) \circ c) = \tau(a \circ (b \circ c))$ *for all* $a, b, c \in M$,
(ii) $\tau(b^2 \circ a) = \tau(\{bab\})$ *for all* $a, b \in M$,
(iii) $(U_p^* + U_{p'}^*)\tau = \tau$ *for all projections* $p \in M$,
(iv) $U_s^*\tau = \tau$ *for all symmetries* $s \in M$.

Proof. (i) $\Rightarrow$ (ii) Let $a, b \in M$. By the definition of $\{bab\}$ (cf. (1.13)),

$$\tau(\{bab\} = \tau(2b \circ (b \circ a) - b^2 \circ a). \tag{5.8}$$

If (i) holds, then

$$\tau(2b \circ (b \circ a)) = \tau((2b \circ b) \circ a) = \tau(2b^2 \circ a)$$

which when substituted into (5.8) gives (ii).

(ii) $\Rightarrow$ (iii). Let $a \in M$. For a projection p, taking $b = p$ and then $b = p'$ in (ii) gives

$$\tau((U_p + U_{p'})(a)) = \tau(p \circ a) + \tau(p' \circ a) = \tau(a),$$

which proves (iii).

(iii) $\Leftrightarrow$ (iv) By spectral theory, the symmetries s in M are the elements of the form $s = 2p - 1 = p - p'$, for p a projection. Now the equivalence of (iii) and (iv) follows from $U_s = 2(U_p + U_{p'}) - I$ (cf. (2.25)).

(iii) $\Rightarrow$ (i) Let $a, b, c \in M$. From the definition (1.12) of the triple product we have

$$(a \circ b) \circ c - a \circ (b \circ c) = \tfrac{1}{2}(\{bac\} - \{acb\}). \tag{5.9}$$

Thus to prove (i) it suffices to show

$$\tau(\{bac\} - \{acb\}) = 0. \tag{5.10}$$

By linearity of the triple product in each factor, the spectral theorem, and norm continuity of Jordan multiplication, it suffices to prove (5.10) for the case where $b = p$ is a projection. By (iii)

$$\tau(\{pac\} - \{acp\}) = \tau(U_p + U_{p'})(\{pac\} - \{acp\}),$$

so we will be done if we can show

$$(U_p + U_{p'})(\{pac\} - \{acp\}) = 0 \tag{5.11}$$

for all projections p in M and all elements $a, c \in M$. To prove (5.11), we begin with the identity

$$2\{bac\} \circ b = \{b(a \circ c)b\} + \{b^2ac\}. \tag{5.12}$$

(This is easily verified for any special Jordan algebra, and then holds in all Jordan algebras by Macdonald's theorem (Theorem 1.13).) Applying (5.12) with $b = p$ gives

$$2\{pac\} \circ p = \{p(a \circ c)p\} + \{pac\}. \tag{5.13}$$

Interchanging the roles of a and c and subtracting gives

$$2p \circ (\{pac\} - \{pca\}) = \{pac\} - \{pca\}. \tag{5.14}$$

Let $x = \{pac\} - \{pca\}$. Then by (5.14), $x = 2p \circ x$. Since $p \circ x = \frac{1}{2}(I + U_p - U_{p'})$ (cf. (1.47)), then

$$x = (I + U_p - U_{p'})x. \tag{5.15}$$

Applying U_p to both sides of (5.15) gives $U_p x = 2U_p x$ so $U_p x = 0$. Applying U_p' to both sides of (5.15) gives $U_{p'}x = 0$. This proves (5.11), which completes the proof of (i). □

5.19. Definition. Let M be a JB-algebra. A state τ on A is a *tracial state* if $\tau((a \circ b) \circ c) = \tau(a \circ (b \circ c))$ for all $a, b, c \in A$, or (when M is a JBW-algebra) if τ satisfies any of the equivalent conditions of Lemma 5.18. A tracial state τ is *faithful* if whenever $a \geq 0$ and $\tau(a) = 0$, then $a = 0$.

Remark. Recall that a tracial state on a von Neumann algebra $\mathcal{M}$ is a state τ satisfying $\tau(ab) = \tau(ba)$ for all $a, b \in \mathcal{M}$. If s is a symmetry in $\mathcal{M}_{\mathrm{sa}}$ and τ is a tracial state on $\mathcal{M}$, then $\tau(sas) = \tau(as^2) = \tau(a)$ so τ is a tracial state on the JBW-algebra $\mathcal{M}_{\mathrm{sa}}$. Conversely, let τ be a tracial state on $M = \mathcal{M}_{\mathrm{sa}}$ and also denote by τ its complex linear extension to a state on $\mathcal{M}$. Then for all symmetries s, t, we have $\tau(st) = \tau(t(st)t) = \tau(ts)$. Then $\tau(pq) = \tau(qp)$ for all projections $p, q \in \mathcal{M}$. By linearity and the spectral theorem, $\tau(ab) = \tau(ba)$ for all a and b in $\mathcal{M}$. Thus τ is a tracial state on $\mathcal{M}$.

5.20. Corollary. *Every tracial state τ on a JBW-algebra M satisfies*

$$a, b \geq 0 \quad \Rightarrow \quad \tau(a \circ b) \geq 0.$$

Proof. By Lemma 5.18(ii)

$$\tau(a \circ b) = \tau(\{a^{1/2} b a^{1/2}\}),$$

and $\{a^{1/2} b a^{1/2}\} \geq 0$ by Theorem 1.25. □

5.21. Lemma. *A spin factor admits a unique tracial state. This state is faithful and normal.*

Proof. Write $M = \mathbf{R}1 \oplus N$ as in Definition 3.33, where N is a real Hilbert space and $x \circ y = (x|y)1$ for $x, y \in N$. Recall that N consists of the scalar multiples of non-zero symmetries other than ± 1 (Lemma 3.34). Define $\tau : A \to \mathbf{R}1$ by

$$\tau(\lambda 1 + a) = \lambda \quad \text{for } \lambda \in \mathbf{R} \text{ and } a \in N.$$

If s is any symmetry, then U_s is a Jordan automorphism (Proposition 2.34), and so U_s takes multiples of symmetries to multiples of symmetries, and therefore maps N onto itself. Thus

$$\tau(U_s(\lambda 1 + a)) = \lambda 1 + \tau(U_s(a)) = \lambda 1 = \tau(\lambda 1 + a).$$

τ is positive since

$$\tau((\lambda 1 + a)^2) = \tau(\lambda^2 1 + 2\lambda a + a^2) = \lambda^2 + (a \,|\, a) \geq 0.$$

Thus τ is a tracial state on M. Furthermore, $\tau((\lambda 1 + a)^2) = 0$ evidently implies $\lambda = 0$ and $a = 0$, so τ is faithful.

We now prove uniqueness. The smallest possible dimension of N is 2, so if s is a symmetry in N there is at least one unit vector t in N orthogonal to s, and then $s \circ t = 0$. By the definition of the triple product

$$U_t s = 2t \circ (t \circ s) - t^2 \circ s = -s. \tag{5.16}$$

Thus for any tracial state τ_1,

$$\tau_1(s) = \tau_1(U_t s) = \tau_1(-s) = -\tau_1(s),$$

so $\tau_1(s) = 0$. This holds for all symmetries in N. Since every element of N is a multiple of a symmetry, then $\tau_1 = 0$ on N. Then $\tau = \tau_1$ follows.

Finally, by Proposition 3.38 every state on a spin factor is normal. □

5.22. Proposition. *Every JBW-factor M of type I_n $(1 \leq n < \infty)$ admits a unique tracial state. This state takes the value $1/n$ on every minimal projection.*

Proof. If M is of type I_2, then M is a spin factor by Proposition 3.37, and so there is a unique tracial state on M (Lemma 5.21). Now consider the case where M is of type I_n with $n \neq 2$. Then M is finite dimensional (Theorem 3.32). Let G be the group of affine automorphisms of the state space K of M. Since the unit ball of M^* is $\mathrm{co}(K \cup -K)$, then G extends to a group of isometries of M^*. Thus G forms an equicontinuous group of affine automorphisms of the compact convex set K. By the Kakutani fixed point theorem [108, Thm. 5.11], there is a fixed point τ of G. Since for each symmetry s the dual map U_s^* is in G, then $U_s^* \tau = \tau$, so τ is a tracial state.

Now we prove uniqueness. Let M be of type I_n with $1 \leq n < \infty$, and let τ be any tracial state on M. Let p and q be any two minimal projections. Recall that minimal projections and abelian projections coincide in a JBW-factor (Lemma 3.30), so p and q are abelian. Since p and q have the same central cover (namely 1), then p and q are exchangeable by a symmetry (Lemma 3.19). It follows that $\tau(p) = \tau(q)$. Since M is a factor of type I_n, by definition there is a set S of n abelian (therefore minimal) projections with sum 1. Since τ takes the same value on each projection in S, it must have the value $1/n$ on each. Since all minimal projections in M are exchangeable, τ takes the value $1/n$ on all minimal projections. By Lemma 3.22 any set of orthogonal non-zero projections has cardinality at most n, so every projection in M is a finite sum of minimal projections. By the spectral theorem τ is determined by its value on minimal projections. Thus τ is unique. □

5.23. Definition. Let M be a JBW-algebra. A positive linear map T from M onto the center Z of M is a *center-valued trace* if

(i) T is the identity map on Z, and
(ii) $T(U_s x) = Tx$ for all $x \in M$ and all symmetries s.

We say T is *faithful* if $x \geq 0$ and $Tx = 0$ imply $x = 0$. Recall T is *normal* if it is σ-weakly continuous (Proposition 2.64).

Note that if A is a JBW-factor, then $Z = \mathbf{R}1$, so a center valued trace is of the form $x \mapsto \tau(x)1$ for a tracial state τ.

5.24. Lemma. *If T is a center-valued trace or a tracial state on a JBW-algebra M and $p_1, \ldots, p_n$ are orthogonal projections with $p = p_1 + \cdots + p_n$, then for $x \in M$*

$$T(U_p x) = T((U_{p_1} + \cdots + U_{p_n})x).$$

Proof. We will prove the lemma for center valued traces; the proof for tracial states is essentially the same. It suffices to prove the lemma for the case $n = 2$. (The general case follows by induction.) Let p be any projection and let $s = p - p'$. Since $U_s = 2(U_p + U_{p'} - I)$ (cf. (2.25)), by the invariance of center-valued traces under the map U_s, for each $x \in M$ we have

$$Tx = T(U_s x) = T(2U_p x + 2U_{p'} x - x)$$

which implies

$$Tx = T((U_p + U_{p'})x). \tag{5.17}$$

Let p_1 and p_2 be orthogonal projections in M. Applying (5.17) with p_1 in place of p and $U_{p_1+p_2}x$ in place of x gives

$$T(U_{p_1+p_2}x) = T((U_{p_1} + U_{p_1'})(U_{p_1+p_2}x)). \tag{5.18}$$

Since $p_1 \wedge (p_1+p_2) = p_1$ and $p_1' \wedge (p_1+p_2) = p_1+p_2-p_1 = p_2$ (cf. (2.17)), then by Proposition 2.28,

$$U_{p_1}U_{p_1+p_2} = U_{p_1} \quad \text{and} \quad U_{p_1'}U_{p_1+p_2} = U_{p_2}. \tag{5.19}$$

Combining (5.18) and (5.19) gives

$$T(U_{p_1+p_2}x) = T((U_{p_1} + U_{p_2})x).$$

This completes the proof. □

5.25. Proposition. *Every JBW-algebra M of type I_n $(1 \le n < \infty)$ admits a unique center-valued trace. This trace is faithful and normal.*

Proof. Let $p_1, \ldots, p_n$ be abelian exchangeable projections with sum 1. Let $\{s_{ij}\}$ be symmetries such that s_{ij} exchanges p_i and p_j for $1 \le i, j \le n$. For each i, let $M_{p_i} = \{p_i M p_i\}$. Define $\phi_i : M_{p_i} \to M$ by

$$\phi_i(x) = \sum_j \{s_{ij} x s_{ij}\}, \tag{5.20}$$

and $\Phi : M \to M$ by

$$\Phi(x) = \frac{1}{n} \sum_i \phi_i(\{p_i x p_i\}). \tag{5.21}$$

We will show that Φ is a center valued trace. (For motivation, consider the example where M consists of the hermitian $n \times n$ matrices with entries in an abelian von Neumann algebra, so that each $x \in M$ has the form $x = \sum_{i,j} z_{ij} e_{ij}$. Let $p_i = e_{ii}$ for $1 \le i \le n$. Then $\phi_i(p_i x p_i) = \phi_i(z_{ii} e_{ii}) = z_{ii}$ and $\Phi(x) = \sum_i z_{ii}$.)

Let Z be the center of M. It is clear that Φ is a positive map. We will now show that $\Phi(M) \subset Z$, and that Φ is the identity map on Z. Fix i. We first show $\phi_i(M_{p_i}) \subset Z$ for $i = 1, \ldots, n$. Let π be an arbitrary factor representation of M, and let $M_0 = \overline{\pi(M)}$ (σ-weak closure). Since p_i is an abelian projection, $\pi(p_i)$ is an abelian projection in M_0. An abelian projection in a factor is minimal (Lemma 3.30), so for $x \in M_{p_i}$,

$$\pi(\{s_{ij} x s_{ij}\}) \in \pi(M_{p_j}) \subset \{\pi(p_j) M_0 \pi(p_j)\} = \mathbf{R}\pi(p_j).$$

Fix $x \in M_{p_i}$. Then for each j there exists $\lambda_j \in \mathbf{R}$ such that

$$\pi(\{s_{ij} x s_{ij}\}) = \lambda_j \pi(p_j).$$

Since

$$U_{\pi(s_{ii})} U_{\pi(s_{ij})}^{-1} \pi(\{s_{ij} x s_{ij}\}) = \pi(\{s_{ii} x s_{ii}\}),$$

the Jordan automorphism $U_{\pi(s_{ii})} U_{\pi(s_{ij})}^{-1}$ takes $\lambda_j \pi(p_j)$ to $\lambda_i \pi(p_i)$. We conclude that $\lambda_j = \lambda_i$ for all j. Thus

$$\pi(\phi_i(x)) = \sum_j \pi(\{s_{ij} x s_{ij}\}) = \sum_j \lambda_j \pi(p_j) = \lambda_i \sum_j \pi(p_j) = \lambda_i 1,$$

where λ_i depends on x. Therefore for all symmetries t in M, and all factor representations π of M,

$$\pi(U_t \phi_i(x)) = U_{\pi(t)} \lambda_i 1 = \lambda_i 1 = \pi(\phi_i(x)).$$

Since factor representations separate points of M (Lemma 4.14), $U_t \phi_i(x) = \phi_i(x)$. It follows that $\phi_i(x)$ is central in M (Lemma 2.35), which completes the proof that ϕ_i maps M_{p_i} into the center Z of M. Now it follows at once from the definition (5.21) of Φ that Φ maps M into Z.

Now let $z \in Z$. Then

$$\begin{aligned}\phi_i(\{p_i z p_i\}) &= \sum_j \{s_{ij}\{p_i z p_i\} s_{ij}\} \\ &= z \sum_j \{s_{ij} p_i s_{ij}\} = z \sum_j p_j = z,\end{aligned}$$

so

$$\Phi(z) = \frac{1}{n} \sum_i \phi_i(\{p_i z p_i\}) = z.$$

Thus Φ is the identity map on Z.

We next show $\Phi \circ U_s = \Phi$ for all symmetries s. We first restrict to the case where M is a type I_n factor. Let τ be the unique trace whose existence is guaranteed by Proposition 5.22. Let $x \in M$. Then we are going to show

$$\Phi(x) = \tau(x)1. \tag{5.22}$$

Note that since the center of M is $\mathbf{R}1$, by the results we have just established, $\Phi(x) = \lambda 1$ for some scalar λ. Then (5.22) will follow if we show that $\tau(\Phi(x)) = \tau(x)$. By the definitions of ϕ_i and Φ, and the definition of a tracial state,

$$\begin{aligned}\tau(\Phi(x)) &= \frac{1}{n}\sum_i\sum_j \tau(\{s_{ij}\{p_i x p_i\}s_{ij}\}) \\ &= \frac{1}{n}\sum_i\sum_j \tau(\{p_i x p_i\}) \\ &= \sum_i \tau(\{p_i x p_i\}).\end{aligned}$$

Since $\sum_i p_i = 1$, by Lemma 5.24 the last sum above is $\tau(x)$, so $\tau(\Phi(x)) = \tau(x)$, which proves (5.22). From the definition of Φ and (5.22), this also proves that when M is a factor the unique tracial state is given by

$$\tau(x)1 = \frac{1}{n}\sum_i\sum_j \{s_{ij}\{p_i x p_i\}s_{ij}\}. \tag{5.23}$$

The uniqueness of the trace also proves that the definition of Φ in this case is independent of the choice of the projections $\{p_i\}$ and the symmetries $\{s_{ij}\}$.

We now let M be an arbitrary JBW-algebra of type I_n, and will show that $\Phi \circ U_s = \Phi$ for all symmetries s. Let x be any element of M, and π any factor representation of M, and $M_0 = \overline{\pi(M)}$. By Lemma 4.8, M_0 is a type I_n factor. Let τ be the unique trace on M_0. Then

$$\begin{aligned}\pi(\Phi(x)) &= \pi\left(\frac{1}{n}\sum_i\sum_j \{s_{ij}\{p_i x p_i\}s_{ij}\}\right) \\ &= \frac{1}{n}\sum_i\sum_j \{\pi(s_{ij})\{\pi(p_i)\pi(x)\pi(p_i)\}\pi(s_{ij})\}.\end{aligned}$$

By (5.23) applied to M_0 (with $\pi(s_{ij})$ in place of s_{ij}, $\pi(x)$ in place of x, and $\pi(p_i)$ in place of p_i), the last expression is just $\tau(\pi(x))1$, so

$$\pi(\Phi(x)) = \tau(\pi(x))1.$$

Thus if s is any symmetry in M, then

$$\pi(\Phi(U_s x)) = \tau(\pi(U_s x))1 = \tau(U_{\pi(s)}\pi(x))1 = \tau(\pi(x))1 = \pi(\Phi(x)).$$

Since factor representations separate points, we conclude that $\Phi(U_s x) = \Phi(x)$ for all $x \in M$ and all symmetries s. Thus Φ is a center-valued trace.

All maps involved in the definition of Φ are σ-weakly continuous, so the same is true of Φ. Thus Φ is normal. To prove Φ is faithful, suppose $x \geq 0$

and $\Phi(x) = 0$. Then $\phi_i(\{p_i x p_i\}) = 0$ for all i, and so $\{s_{ij}\{p_i x p_i\}s_{ij}\} = 0$ for all i, j. Since $U_{s_{ij}}$ is a Jordan isomorphism, then $U_{p_i} x = 0$ for all i. By Lemma 1.26, $x \circ p_i = 0$ for all i. It follows that $x = \sum_i x \circ p_i = 0$. Thus Φ is faithful.

Finally, we prove uniqueness. Let $T : M \to Z$ be any center-valued trace. For each $x \in M$, since $\Phi(x) \in Z$, then

$$\begin{aligned}\Phi(x) = T(\Phi(x)) &= T\Big(\frac{1}{n}\sum_i\sum_j \{s_{ij}\{p_i x p_i\}s_{ij}\}\Big)\\ &= \frac{1}{n}\sum_i\sum_j T(\{s_{ij}\{p_i x p_i\}s_{ij}\})\\ &= \frac{1}{n}\sum_i\sum_j T(\{p_i x p_i\})\\ &= T\Big(\sum_i \{p_i x p_i\}\Big).\end{aligned}$$

By Lemma 5.24, $T(\sum_i \{p_i x p_i\}) = Tx$ so $\Phi(x) = Tx$, proving uniqueness. $\square$

5.26. Lemma. *If σ is a faithful normal state on a JBW-algebra M of type I_n $(1 \le n < \infty)$, then there exists a faithful normal tracial state τ on M such that $\sigma \le n\tau$.*

Proof. Let T be the unique center-valued trace on M, and define τ on M by $\tau(x) = \sigma(Tx)$. Then it follows from the properties of T and σ that τ is a faithful normal tracial state on M. It remains to prove $\sigma \le n\tau$.

Let p be an abelian projection in M. Our first step is to show

$$Tp = \tfrac{1}{n}c(p), \tag{5.24}$$

where $c(p)$ is the central cover of p, cf. Definition 2.38. Let $p_1, \dots, p_n$ be abelian exchangeable projections with sum 1. Define $q_i = c(p)p_i$ for $i = 1, \dots, n$. Then

$$q_1 + \cdots + q_n = c(p), \tag{5.25}$$

and $q_1, \dots, q_n$ are abelian projections (since $q_i \le p_i$), and are exchangeable (since if s is any symmetry exchanging p_i and p_j, then the Jordan automorphism U_s fixes $c(p)$, and so exchanges q_i and q_j). By the definition of a center-valued trace, T takes the same value on exchangeable projections. Thus (5.25) implies

$$Tq_1 = \tfrac{1}{n}c(p). \tag{5.26}$$

Since $q_1, \ldots, q_n$ are exchangeable, then $c(q_1) = c(q_2) = \cdots = c(q_n)$. Therefore $c(q_1) \geq q_i$ for $1 \leq i \leq n$, so

$$c(q_1) \geq q_1 \vee q_2 \vee \cdots \vee q_n = q_1 + q_2 + \cdots + q_n = c(p).$$

On the other hand, $q_1 \leq c(p)$ implies $c(q_1) \leq c(p)$, so $c(q_1) = c(p)$. Abelian projections with the same central cover are exchangeable, so $Tq_1 = Tp$, which when combined with (5.26) proves (5.24).

Finally, by the definition of τ,

$$\sigma(p) \leq \sigma(c(p)) = \sigma(nT(p)) = n\tau(p).$$

Thus $\sigma \leq n\tau$ on abelian projections. However, in a type I JBW-algebra, every projection dominates a non-zero abelian projection (Lemma 3.18), so by a Zorn's lemma argument is the least upper bound of an orthogonal family of abelian projections. Since both σ and T are normal, so is τ, and thus $\sigma \leq n\tau$ holds on all projections. By the spectral theorem, $\sigma \leq n\tau$ holds in general. □

The following result generalizes a well known Radon–Nikodym type theorem for von Neumann algebras.

5.27. Lemma. *Let M be a JBW-algebra and let σ, τ be two normal positive functionals on M such that $\sigma \leq \tau$. Then there exists $h \in M^+$ with $0 \leq h \leq 1$ such that*

$$\sigma(a) = \tau(h \circ a) \text{ for all } a \in M.$$

Proof. (We follow the similar argument for von Neumann algebras in [112, Prop. 1.24.4]). For each $x \in M$, let T_x be the map $T_x(y) = x \circ y$. The map $h \mapsto T_h^* \tau$ is continuous from the σ-weak topology on M to the weak topology on M_* (i.e., the weak topology for the duality of M_* and M). Since the closed unit ball M_1 of M is σ-weakly compact (Corollary 2.56), then the set $K_0 = \{T_h^* \tau \mid h \in M_1\}$ is the continuous image of a σ-weakly compact convex set, and so is a weakly compact convex subset of M_* containing 0.

We are going to show that $\sigma \in K_0$. Let b be an arbitrary element of M such that $\rho(b) \leq 1$ for all $\rho \in K_0$. We will show that $\sigma(b) \leq 1$ in order to apply the bipolar theorem. Let $b = b^+ - b^-$ be the orthogonal decomposition of b into the difference of orthogonal positive elements (Proposition 1.28). Let p be the range projection of b^+. Then by definition of the range projection, $b^+ \in M_p$. Since b^+ and b^- are orthogonal, by (2.9) $b^- \in M_{p'}$. We also have $M_p \circ M_{p'} = \{0\}$ (Lemma 1.45), so

$$p \circ b^+ = b^+, \quad p \circ b^- = 0, \quad p' \circ b^+ = 0, \quad p' \circ b^- = b^-.$$

Thus

$$(5.27) \qquad (p - p') \circ b = b^+ + b^-.$$

Since $\|p - p'\| = 1$, and by hypothesis $\sigma \leq \tau$, then $T^*_{p-p'}\tau \in K_0$, so

$$1 \geq (T^*_{p-p'}\tau)(b) = \tau(b^+ + b^-) \geq \sigma(b^+ + b^-) \geq \sigma(b^+ - b^-) = \sigma(b).$$

By the bipolar theorem (or a direct application of the Hahn–Banach theorem), we conclude that $\sigma \in K_0$.

It remains to show that we can choose $0 \leq h \leq 1$. Let $h = h^+ - h^-$ be the orthogonal decompositon of H, and let $p = r(h^+)$, and $s = p - p'$. As in the calculation above, $s \circ h = h^+ + h^-$. Then

$$\tau(h^+ - h^-) = \tau(h) = \sigma(1) \geq \sigma(s) = \tau(s \circ h) = \tau(h^+ + h^-),$$

so $\tau(h^-) = 0$. Since $(h^-)^2 \leq \|h^-\|\, h^-$, then $\tau((h^-)^2) = 0$. Now by the Cauchy–Schwarz inequality, $\tau(h^- \circ a) = 0$ for all $a \in M$. Thus $\sigma(a) = \tau(h^+ \circ a) = \tau(h^+ \circ a)$, and $\|h^+\| \leq \|h\| \leq 1$, so $0 \leq h^+ \leq 1$. □

Norm closed faces of JBW normal state spaces

The main result of this section is Theorem 5.32, which shows that every norm closed face of the normal state space K of a JBW-algebra M is projective, i.e., of the form $F_p = \{\, \sigma \in K \mid \sigma(p) = 1 \,\}$ for a projection p in K. We then establish various properties of the norm closed faces of K.

5.28. Definition. If X is a subset of the normal state space of a JBW-algebra, the intersection of the faces containing X is called the *face generated by* X, and is denoted face(X). Similarly, the intersection of the norm closed faces containing X is called the *norm closed face generated by* X. If $X = \{\sigma\}$, then we write face(σ) instead of face($\{\sigma\}$), and similarly we write face(σ, τ) instead of face($\{\sigma, \tau\}$).

5.29. Lemma. *Let M be a JBW-algebra with normal state space K and τ a normal tracial state on M. Let $h \in M^+$, with range projection satisfying $r(h) = 1$. Define ω on M by $\omega(a) = \tau(h \circ a)$. Then τ is in the norm closure of the face of K generated by ω.*

Proof. Let $\{e_\lambda\}$ be the spectral resolution for h (cf. Definition 2.21). By the definition of a spectral resolution and Theorem 2.20, for each $\lambda \in \mathbf{R}$

$$U_{e'_\lambda} h \geq \lambda e'_\lambda.$$

On the other hand, each e'_λ operator commutes with h, so by (1.62)

$$0 \le U_{e'_\lambda} h \le h.$$

Combining these gives

$$0 \le \lambda e'_\lambda \le h.$$

Now for each $\lambda \in \mathbf{R}$ define a functional τ_λ by

$$\tau_\lambda(a) = \tau(e'_\lambda \circ a).$$

By Corollary 5.20, τ_λ is a positive functional, and for $a \in M^+$,

$$\lambda\tau_\lambda(a) = \tau(\lambda e'_\lambda \circ a) \le \tau(h \circ a) = \omega(a).$$

Thus for any $\lambda > 0$, we conclude that τ_λ is in the face of M_*^+ generated by ω, and so $\|\tau_\lambda\|^{-1}\tau_\lambda$ is in the face of K generated by ω. Finally, we show that $\{\|\tau_\lambda\|^{-1}\tau_\lambda\}$ converges to τ in norm as $\lambda \searrow 0$. By the definition of a spectral resolution, as $\lambda \searrow 0$, $e'_\lambda \nearrow e'_0$. Since $e'_0 = r(h)$ (cf. (2.15)), and by hypothesis, $r(h) = 1$, then

$$e'_\lambda \nearrow 1. \tag{5.28}$$

By the Cauchy–Schwarz inequality (1.53), if $a \in M$ with $\|a\| \le 1$, then

$$\begin{aligned}
|\tau(a) - \tau_\lambda(a)| &= |\tau(a) - \tau(e'_\lambda \circ a)| \\
&= |\tau((1 - e'_\lambda) \circ a)| \\
&\le \tau(1 - e'_\lambda)^{1/2}\tau(a^2)^{1/2} \\
&\le \tau(1 - e'_\lambda)^{1/2}\tau(1)^{1/2} \\
&= \tau(1 - e'_\lambda)^{1/2}.
\end{aligned}$$

Then

$$\|\tau - \tau_\lambda\| \le \tau(1 - e'_\lambda)^{1/2},$$

and the last term goes to zero by (5.28) and normality of τ. □

5.30. Lemma. *Let M be a JBW-algebra with normal state space K. If τ is a faithful normal tracial state on M, then the face of K generated by τ is norm dense in K.*

Proof. Let F be the face of K generated by τ. For $b \in M$ define $\tau_b \in M_*$ by $\tau_b(a) = \tau(b \circ a)$. Recall that if $a, b \ge 0$, then $\tau(a \circ b) \ge 0$

(Corollary 5.20). Thus if $b \geq 0$, then $\tau_b \geq 0$. Note also that if $b \in M^+$, then $b \leq \|b\|1$, so $\tau_b \leq \tau_{\|b\|1} = \|b\|\tau$. Thus

$$0 < b \quad \Rightarrow \quad \|\tau_b\|^{-1}\tau_b \in F. \tag{5.29}$$

If $b = b^+ - b^-$ is the orthogonal decomposition of b, we will prove that the orthogonal decomposition of τ_b is given by $\tau_b = \tau_{b^+} - \tau_{b^-}$. Let p be the range projection of b^+ and $p' = 1 - p$. Then as in the proof of Lemma 5.27 (cf. (5.27)), we have $(p - p') \circ b = b^+ + b^-$. Thus

$$\|\tau_b\| \geq \tau_b(p - p') = \tau(b^+) + \tau(b^-) = \tau_{b^+}(1) + \tau_{b^-}(1) = \|\tau_{b^+}\| + \|\tau_{b^+}\|.$$

The opposite inequality is clear, so it follows that $\tau_b = \tau_{b^+} - \tau_{b^-}$ is the orthogonal decomposition of τ_b.

Now let X be the linear space of all functionals τ_b for $b \in M$. Note that by the previous paragraph

$$\sigma \in X \quad \Rightarrow \quad \sigma^+, \sigma^- \in X.$$

Furthermore, since $(\tau_b)^+ = \tau_{b^+}$, then by (5.29) we have

$$\sigma \in X \quad \Rightarrow \quad \|\sigma^+\|^{-1}\sigma^+ \in F, \tag{5.30}$$

as long as $\sigma^+ \neq 0$.

We next show X is norm dense in M_*. If $a \in M$ annihilates X, then $\tau_b(a) = \tau(b \circ a) = 0$ for all $b \in M$. In particular, $\tau(a^2) = 0$. Since τ is faithful, then $a^2 = 0$, and so $a = 0$. Thus no non-zero element of M annihilates X, so X is norm dense in M_*. Now fix $\omega \in K$ and choose a sequence σ_n in X such that $\sigma_n \to \omega$ (in norm). Then $\|\sigma_n\| \to \|\omega\| = 1$, so

$$\|\sigma_n^+\| + \|\sigma_n^-\| \to 1. \tag{5.31}$$

On the other hand, $\sigma_n(1) \to \omega(1) = 1$, so

$$\|\sigma_n^+\| - \|\sigma_n^-\| = \sigma_n^+(1) - \sigma_n^-(1) = \sigma_n(1) \to 1. \tag{5.32}$$

Combining (5.31) and (5.32) shows $\|\sigma_n^-\| \to 0$, so $\sigma_n^+ \to \omega$ in norm. Let $\omega_n = \|\sigma_n^+\|^{-1}\sigma_n^+$. Then by (5.30), ω_n is a sequence in F converging to ω, which completes the proof that F is norm dense in K. □

5.31. Lemma. *Let M be a JBW-algebra with normal state space K. Let $\sigma \in K$, and let p be the carrier projection of σ. Then the face generated by σ is norm dense in $F_p = \{\omega \in K \mid \omega(p) = 1\}$.*

Proof. We first reduce to the case where $p = 1$. Let $M_p = \{pMp\}$. Recall from Proposition 2.62 that we can identify the normal state space of

M_p with F_p. Note that since F_p is a face of K, then face(σ) is the same whether calculated as a subset of F_p or of K. Similarly by Proposition 2.62 the norm topology on F_p, viewed as the normal state space of M_p, is just the norm topology inherited as a subset of K. Finally, the carrier projection of σ viewed as a normal state on M_p will still be p. Therefore it suffices to prove the lemma with M replaced by M_p, and we may assume hereafter that $p = 1$.

We then have carrier(σ) $= 1$. We now show this implies that σ is faithful. If $0 \le a \in M$ and $\sigma(a) = 0$, then $\sigma(r(a)) = 0$ by equation (2.6). Then $\sigma(1 - r(a)) = 1$, so by the definition of the carrier projection of σ, we must have $1 \le 1 - r(a)$. Therefore $r(a) = 0$ and so $a = 0$. Thus σ is faithful as claimed. It remains to show that the face generated by σ is norm dense in K.

(i) First consider the case where M has no part of type I_2 or I_3. By Theorem 4.23, M is a JW-algebra. Let $\mathcal{M}$ be the universal von Neumann algebra for M (cf. Definition 4.43); we may view M as a JW-subalgebra of $\mathcal{M}_{\mathrm{sa}}$. M acts reversibly in $\mathcal{M}$ (Corollary 4.30), so there is a *-anti-automorphism Φ of period 2 of $\mathcal{M}$ such that M coincides with the set of self-adjoint elements of $\mathcal{M}$ fixed by Φ (Proposition 4.45). Observe that $\frac{1}{2}(I + \Phi)$ will be a positive idempotent map from $\mathcal{M}_{\mathrm{sa}}$ onto M.

Note that Φ preserves the Jordan product on $\mathcal{M}_{\mathrm{sa}}$, and so is a Jordan isomorphism. Thus it is σ-weakly continuous. For each normal state ω on M, define $\widetilde{\omega} = \frac{1}{2}\omega \circ (I + \Phi)$. Then $\widetilde{\omega}$ is an extension of ω to a normal state on $\mathcal{M}_{\mathrm{sa}}$. We also denote by $\widetilde{\omega}$ the extension of $\widetilde{\omega}$ to a complex linear normal functional on $\mathcal{M}$, which will then be a normal state on $\mathcal{M}$.

As shown above, σ is faithful on M. We now show that $\widetilde{\sigma}$ is faithful on $\mathcal{M}$. Let $a \in \mathcal{M}^+$ and suppose $\widetilde{\sigma}(a) = 0$. Then by definition $\sigma(\frac{1}{2}(a + \Phi(a)) = 0$. Since $\frac{1}{2}(a+\Phi(a))$ is in M, by faithfulness of σ on M, $a+\Phi(a) = 0$. Since Φ is a *-anti-isomorphism of $\mathcal{M}$, it preserves order, so $\Phi(a) \ge 0$. It follows that $a = \Phi(a) = 0$. Thus $\widetilde{\sigma}$ is faithful on $\mathcal{M}$.

Let $\mathcal{K}$ be the normal state space of $\mathcal{M}$. Since $\widetilde{\sigma}$ is faithful on $\mathcal{M}$, its carrier projection is the identity, so $F_{\mathrm{carrier}(\widetilde{\sigma})} = \mathcal{K}$. Thus the norm closure of the face generated by $\widetilde{\sigma}$ equals $\mathcal{K}$, cf. (A 107). As shown above, each $\omega \in K$ has an extension $\widetilde{\omega} \in \mathcal{K}$. Therefore the restriction map is an affine map from $\mathcal{K}$ onto K. The restriction map is norm continuous and sends face($\widetilde{\sigma}$) into face(σ), so it follows that face(σ) is dense in K.

(ii) Now consider the case where M is either of type I_2 or I_3. Let $\sigma \in K$ be faithful. By Lemma 5.26 there is a faithful normal tracial state τ on M such that $\sigma \le n\tau$ where $n = 2$ or $n = 3$. By the JBW Radon–Nikodym type theorem (Lemma 5.27), there is a positive element $h \in M$ such that $\sigma(a) = \tau(h \circ a)$ for all $a \in M$. We will show the range projection $r(h)$ equals 1. Suppose q is a projection with $U_q h = h$. Then by (1.47)

$$q' \circ h = \tfrac{1}{2}(I + U_{q'} - U_q)h = 0.$$

Thus $\sigma(q') = \tau(h \circ q') = 0$. By faithfulness of σ, $q' = 0$, so $q = 1$. Thus the range projection of h is 1. By Lemma 5.29, τ is in the norm closure of face(σ). Since $\sigma \le n\tau$, then σ is in face(τ), so $\overline{\text{face}(\sigma)} = \overline{\text{face}(\tau)}$. Since σ is faithful and $\tau \ge (1/n)\sigma$, then τ is also faithful. By Lemma 5.30, $\overline{\text{face}(\sigma)} = \overline{\text{face}(\tau)} = K$.

(iii) Finally, let M be an arbitrary JBW-algebra. Write $M = M_2 \oplus M_3 \oplus M_0$ where M_2 is of type I_2, M_3 is of type I_3, and M_0 has no summand of types I_2 or I_3 (cf. Lemma 3.16 and Theorem 3.23). Let σ be a normal faithful state on M. Then we can write σ as a convex combination of normal faithful states σ_2, σ_3, σ_0 on M_2, M_3, and M_0 respectively. To verify this, let $c_2, c_3, c_0 = 1 - c_2 - c_3$ be the central projections such that $M_i = c_i M$ for $i = 0, 2, 3$. Let $\lambda_i = \sigma(c_i)$ and $\sigma_i = \lambda_i^{-1} U_{c_i}^* \sigma$ for $i = 0, 2, 3$. Then $\sigma = \lambda_0\sigma_0 + \lambda_2\sigma_2 + \lambda_3\sigma_3$ is the desired convex combination.

If ω is any normal state on M, we can similarly decompose ω into a convex combination of normal states ω_2, ω_3, ω_0 on M_2, M_3, and M_0. By (i) and (ii), $\omega_i \in \overline{\text{face}(\sigma_i)}$ for $i = 0, 2, 3$. Since $\sigma = \lambda_0\sigma_0 + \lambda_2\sigma_2 + \lambda_3\sigma_3$, each of σ_2, σ_3, σ_0 is in face(σ), so it follows that $\omega \in \overline{\text{face}(\sigma)}$. $\square$

Recall that a face F of K is *norm exposed* if there is a positive bounded affine functional a on K whose zero set equals F, cf. (A 1).

5.32. Theorem. *Let M be a JBW-algebra with normal state space K, and let F be a norm closed face of K. If p is the carrier projection of F, then $F = F_p = \{\sigma \in K \mid \sigma(p) = 1\}$. Thus every norm closed face of K is norm exposed and projective.*

Proof. By the definition of the carrier projection, $F \subset F_p$. To prove the opposite inclusion, let $\sigma \in F_p$. For ω_1 and ω_2 in F, the carrier projection of $\frac{1}{2}(\omega_1+\omega_2)$ dominates the carrier projections of ω_1 and ω_2. Thus the set of carrier projections of states in F is directed upward. The least upper bound of this set of projections is carrier(F) (cf. (5.2)). Therefore there is a net $\{\omega_\alpha\}$ in F such that $\{\text{carrier}(\omega_\alpha)\}$ is an increasing net with least upper bound carrier(F). Then $\sigma(\text{carrier}(\omega_\alpha)) \to \sigma(\text{carrier}(F)) = 1$. Thus we can choose a sequence $\{\omega_i\}$ in F such that $\sigma(\text{carrier}(\omega_i)) \to 1$. Define an element ω of F by $\omega = \sum_i 2^{-i}\omega_i$, and observe that $\text{carrier}(\omega) \ge \text{carrier}(\omega_i)$ for each i. Thus

$$\sigma(\text{carrier}(\omega)) \ge \sigma(\text{carrier}(\omega_i)) \to 1.$$

Then $\sigma(\text{carrier}(\omega)) = 1$, so $\sigma \in F_{\text{carrier}(\omega)}$. By Lemma 5.31, this implies that σ is in the norm closed face generated by ω, and thus is in F. Thus we have shown $F_p \subset F$, which completes the proof that F is projective. Viewing p as an affine function on K, then $1-p$ takes the value 0 precisely on F, so F is norm exposed. $\square$

5.33. Corollary. *Let M be a JBW-algebra with normal state space K. Then $p \mapsto F_p$ is an isomorphism from the lattice of projections of M onto the lattice of norm closed faces of K (where $\bigwedge_\alpha F_\alpha = \bigcap_\alpha F_\alpha$). The inverse map takes a norm closed face to its carrier projection. Thus the norm closed faces of K form an orthomodular lattice with the orthocomplementation $F \mapsto F'$.*

Proof. This follows from the isomorphism of the lattice of projections with the lattice of projective faces (Proposition 5.10), together with the fact that the collections of projective faces and norm closed faces coincide (by Theorem 5.32 and the observation that projective faces are norm closed). The statement $\bigwedge_\alpha F_\alpha = \bigcap_\alpha F_\alpha$ follows from the fact that the intersection of norm closed faces is a norm closed face. □

5.34. Corollary. *Let K be the normal state space of a JBW-algebra M. Let p be a projection in M, and $F = F_p$. Then the following are equivalent:*

(i) *p is central,*
(ii) *F is a split face,*
(iii) *$co(F \cup F') = K$.*

Proof. (i) $\Rightarrow$ (ii) If p is central, then by definition p operator commutes with every element of M, so $U_p + U_{p'} = I$ by Proposition 1.47. Then $U_p^* + U_{p'}^* = I$. If $\sigma \in K$, let $\lambda = \sigma(p) = \|U_p^*\sigma\|$. Then $1 - \lambda = \sigma(p') = \|U_{p'}^*\sigma\|$, so

$$\sigma = \lambda(\lambda^{-1}U_p^*\sigma) + (1-\lambda)((1-\lambda)^{-1}U_{p'}^*\sigma)$$

expresses σ as a convex combination of states in F and F', cf. (5.4). Since the intersection of $\operatorname{im} U_p^*$ and $\operatorname{im} U_{p'}^*$ is $\{0\}$, K is the direct convex sum of F and F'. Thus F is a split face of K.

(ii) $\Rightarrow$ (iii) follows at once from the definition of a split face (A 5).

(iii) $\Rightarrow$ (i) If $\operatorname{co}(F \cup F') = K$, then

$$M_* = \operatorname{im} U_p^* + \operatorname{im} U_{p'}^*.$$

Since the intersection of these subspaces is zero, the sum is a direct sum, and so $U_p^* + U_{p'}^* = I$. Dualizing gives $U_p + U_{p'} = I$. By Proposition 1.47, p operator commutes with all elements of M, i.e., is central. □

5.35. Corollary. *Let K be the normal state space of a JBW-algebra M. If F is a split face of K, then $F = F_c$, where c is its carrier projection and is central. Furthermore, M will be a JBW-factor iff K has no proper split faces.*

Proof. A split face is norm closed (A 28), and so has the form F_c where c is its carrier projection (Theorem 5.32). By Corollary 5.34, c is central. Finally, M is a JBW-factor iff M has no non-trivial central projections, which then corresponds to K having no proper split faces. □

5.36. Proposition. *Let M be a JBW-algebra with normal state space K. The map that takes split faces of K to their annihilators in M gives a 1-1 correspondence of split faces and σ-weakly closed Jordan ideals. The inverse map takes a σ-weakly closed ideal to its annihilator in K.*

Proof. Let F be a split face of K. By Corollary 5.34 the carrier projection c of F is central, and by (5.4), $F = \operatorname{im} U_c^* \cap K$. Since every normal functional is a difference of positive normal functionals (Proposition 2.52), and U_c^* is a positive map, then $\operatorname{im} U_c^*$ is positively generated, i.e., $\operatorname{im} U_c^* = \operatorname{im}^+ U_c^* - \operatorname{im}^+ U_c^*$. Thus the annihilator of F is the annihilator of $\operatorname{im} U_c^*$, which in turn is the kernel of U_c. By Proposition 2.41, the latter is exactly the σ-weakly closed ideal $(1-c)M$.

Conversely, if J is a σ-weakly closed Jordan ideal, then by Proposition 2.39, $J = \operatorname{im} U_{1-c}$ for some central projection M. Then the annihilator of J in K is the annihilator of $\operatorname{im} U_{1-c}$, which is the kernel of U_{1-c}^*. Since $U_c + U_{1-c} = 1$, the kernel of U_{1-c}^* is the image of U_c^*. Thus the annihilator of J in K is $\operatorname{im} U_c^* \cap K = F_c$. By Corollary 5.34, F_c is a split face, which completes the proof. □

5.37. Corollary. *Let A be a JB-algebra with state space K. The map that takes w*-closed split faces of K to their annihilators in A gives a 1-1 correspondence of w*-closed split faces and norm closed Jordan ideals. The inverse map takes a norm closed ideal to its annihilator in K.*

Proof. Let J be a norm closed Jordan ideal of A. View A as a subset of A^{**} and denote annihilators in the duality of A^{**} and A^* by $^\circ$. Then the σ-weak closure $\bar{J}$ of J in A^{**} is a Jordan ideal, and by the bipolar theorem satisfies $\bar{J} = J^{\circ\circ}$. Since J is norm closed, we also have

$$J = J^{\circ\circ} \cap A = \bar{J} \cap A.$$

Let $F = J^\circ \cap K$. Since each element of F is σ-weakly continuous on A^{**}, then $F = (\bar{J})^\circ \cap K$. By Proposition 5.36, F is a split face of K and $F^\circ = \bar{J}$. Thus

$$F^\circ \cap A = \bar{J} \cap A = J.$$

Therefore starting with a norm closed ideal J of A and taking the annihilator in K gives a split face whose annihilator in A is J again.

Now suppose that F is a w*-closed split face of K. If $c \in A^{**}$ is the carrier projection of F, then by Theorem 5.32, $F = F_c$. By Proposition

2.62 we can identify F with the normal state space of $\operatorname{im} U_c$, and $\operatorname{im} U_c^*$ with the predual of $\operatorname{im} U_c$. By Corollary 2.60, $\operatorname{im} U_c^*$ is a base norm space with distinguished base F, so by definition of a base norm space its unit ball is $\operatorname{co}(F \cup -F)$. Since F is w*-compact so is the Cartesian product $F \times (-F) \times [0,1]$, and the map $(\sigma, \tau, \lambda) \mapsto \lambda\sigma + (1-\lambda)\tau$ is a continuous map onto $\operatorname{co}(F \cup -F)$, so it follows that the unit ball of $\operatorname{im} U_c^*$ is w*-compact. By the Krein–Šmulian theorem, $\operatorname{im} U_c^*$ is w*-closed.

Let

$$J = F^\circ \cap A = (\operatorname{im} U_c^*)^\circ \cap A$$

be the annihilator of F in A. Since F° is a Jordan ideal of A^{**} by Proposition 5.36, then J is a norm closed ideal in A. Since $\operatorname{im} U_c^*$ is w*-closed, the annihilator of J in A^* will then equal $\operatorname{im} U_c^*$. The annihilator of J in K is then $\operatorname{im} U_c^* \cap K = F_c = F$. This completes the proof that taking annihilators gives a 1-1 correspondence of norm closed ideals of A and w*-closed split faces of K. □

5.38. Corollary. *Let A be a JB-algebra with state space K, let J be a norm closed Jordan ideal of A and let $\pi : A \to A/J$ be the associated homomorphism. Then the dual map $\pi^* : (A/J)^* \to A^*$ is an affine isomorphism and a w*-homeomorphism from the state space $S(A/J)$ of A/J onto the w*-closed split face $F = J^\circ \cap K$ corresponding to J.*

Proof. Let $\sigma \in S(A/J)$. For each $a \in J$, $(\pi^*\sigma)(a) = \sigma(\pi(a)) = 0$, so $\pi^*\sigma \in J^\circ$. Clearly also $\pi^*\sigma \in K$, so $\pi^*\sigma \in F$. Thus π^* maps $S(A/J)$ into F.

Now let σ_1 and σ_2 be two distinct elements of $S(A/J)$. Then there is an $a \in A$ such that $\sigma_1(\pi(a)) \neq \sigma_2(\pi(a))$. Thus π^* is injective on $S(A/J)$.

Next let $\omega \in F$. Thus $\omega \in K$, and $\omega = 0$ on the ideal $J = \ker \pi$. Hence there exists a linear functional σ on A/J such that $\sigma(\pi(a)) = \omega(a)$ for all $a \in A$. Every positive element of A/J is a square, and so is the image of a positive element of A. Thus $\sigma \in S(A)$, and $\pi^*\sigma = \omega$. Therefore π^* maps $S(A/J)$ onto F.

Clearly the map π^* is affine and w*-continuous, so it is an affine isomorphism and a w*-homeomorphism. □

Recall that an *atom* in a JBW-algebra is a minimal projection.

5.39. Proposition. *Let M be a JBW-algebra with normal state space K. The map $p \mapsto F_p$ is a 1-1 correspondence of atoms of M and faces of K consisting of a single extreme point.*

Proof. By Corollary 5.33, there is a 1-1 correspondence of atoms of M and minimal norm closed faces of K. We need to show that the latter consist exactly of the faces $\{\sigma\}$ for σ an extreme point of K. Such faces are clearly minimal. Thus it remains to prove that for p an atom, F_p is a single point. This follows from the fact that M_p is one dimensional (Lemma 3.29), together with the observation that $\operatorname{im} U_p^*$ is the predual of M_p (Proposition 2.62), and thus is also one dimensional. □

5.40. Definition. Let p be an atom in a JBW-algebra M with normal state space K. We will denote by $\widehat{p}$ the extreme point of K satisfying $F_p = \{\widehat{p}\}$.

Note that if p is an atom, then $\widehat{p}$ is the unique normal state taking the value 1 at p. Recall that a state on a JB-algebra A is *pure* if it is an extreme point of the state space of A.

5.41. Corollary. *Let M be a JBW-factor. Then M is of type I iff M admits a pure normal state.*

Proof. Let K be the normal state space of M and let N be the (full) state space of M. By Proposition 2.52, K is a split face of N. Since an extreme point of a face of a convex set is also an extreme point of the full convex set, a normal state is extreme in K iff it is extreme in N. By Proposition 3.44, M is of type I iff it contains an atom. By Proposition 5.39, this is equivalent to K containing an extreme point, which is then also extreme in N, i.e., a pure normal state. □

Recall that if σ is a pure state on a C*-algebra $\mathcal{A}$, and $\pi_\sigma : \mathcal{A} \to \mathcal{B}(H)$ is the associated GNS-representation, then $\pi_\sigma(\mathcal{A})$ is irreducible (A 81). Therefore $\pi_\sigma(\mathcal{A})' = \mathbf{C}1$ (A 80), so by the bicommutant theorem $\overline{\pi_\sigma(\mathcal{A})} = \mathcal{B}(H)$.

For each state σ on a JB-algebra A, we have defined a homomorphism $\pi_\sigma : A \to c(\sigma)A^{**}$ (Definition 4.11). We earlier showed that $\overline{\pi_\sigma(A)}$ is a JBW-factor. Now we are able to show that this factor is of type I, as one would expect by analogy with the C*-algebra example just discussed.

5.42. Corollary. *If σ is a pure state on a JB-algebra A, then $\overline{\pi_\sigma(A)} = c(\sigma)A^{**}$ is a type I JBW-factor.*

Proof. By Proposition 5.39, the carrier projection of σ is an atom in A^{**}, and therefore also in $c(\sigma)A^{**}$. By Lemma 4.14, $c(\sigma)A^{**}$ is a JBW-factor, which is type I by Proposition 3.44. □

5.43. Corollary. *Let F and G be norm closed faces of the normal state space of a JBW-algebra M. Then these are equivalent:*

(i) *$F \subset G'$ (i.e., F and G are orthogonal in the lattice $\mathcal{F}$).*
(ii) *The carrier projections of F and G are orthogonal*

(iii) *Each state in F is orthogonal to each state in G.*

Furthermore, F' consists exactly of the normal states orthogonal to every element of F.

Proof. The equivalence of (i) and (ii) follows at once from Corollary 5.33, since projections p and q are orthogonal precisely if $p \leq q'$ (Proposition 2.18).

(ii) $\Rightarrow$ (iii) If carrier$(F) \perp$ carrier(G), then carrier$(\sigma) \perp$ carrier(τ) for all $\sigma \in F$, $\tau \in G$. Normal states are orthogonal iff their carrier projections are orthogonal (Lemma 5.4), so each $\sigma \in F$ is orthogonal to each $\tau \in G$.

(iii) $\Rightarrow$ (ii) If (iii) holds, then by Lemma 5.4, carrier$(\sigma) \perp$ carrier(τ) for all $\sigma \in F$, $\tau \in G$. Then taking least upper bounds in the lattice of projections we have

$$\bigvee\{\text{carrier}(\sigma) \mid \sigma \in F\} \quad \perp \quad \bigvee\{\text{carrier}(\tau) \mid \tau \in G\}.$$

The left side is carrier(F) and the right side is carrier(F) (cf. (5.2)), so (ii) follows.

Now we will prove the final statement of the lemma. Suppose first that $\sigma \in F'$. Since F and F' are orthogonal, then by the equivalence of (i) and (iii), σ is orthogonal to every state in F. Conversely, suppose that σ is orthogonal to every state in F. Then carrier(σ) is orthogonal to carrier(τ) for every $\tau \in F$, and thus carrier(σ) is orthogonal to carrier(F). Since (ii) implies (i), then

$$F_{\text{carrier}(\sigma)} \subset F'.$$

Since σ is in the left side, then $\sigma \in F'$. □

Recall that $c(\sigma)$ denotes the central carrier of a normal state σ on a JBW-algebra (Definition 4.10).

5.44. Proposition. *Let K be the normal state space of a JBW-algebra M. Let $\sigma \in K$. Then $F_{c(\sigma)}$ is the smallest split face containing σ.*

Proof. By Corollary 5.34, $F_{c(\sigma)}$ is a split face containing σ. Any split face is norm closed (A 28), and so has the form F_c, where c is its carrier projection and is central by Corollary 5.34. If a split face $F = F_c$ contains σ, then $\sigma(c) = 1$, so by definition $c \geq c(\sigma)$. Then $F_{c(\sigma)} \subset F_c$, and so $F_{c(\sigma)}$ is the smallest split face containing σ. □

5.45. Definition. Let K be the normal state space of a JBW-algebra M, and let $\sigma \in K$. We will denote the smallest split face containing σ by

F_σ, and we will refer to F_σ as the *split face generated by* σ. (Note that by Proposition 5.44, $F_\sigma = F_{c(\sigma)}$.) We use the same notation F_σ if σ is a state of a JB-algebra A. (In that case, by Corollary 2.61 we can identify the state space of A with the normal state space K of A^{**}.)

If σ is a normal state on a JBW-algebra M, we note that the central cover of the carrier projection of σ equals the central carrier of σ:

$$c(\sigma) = c(\text{carrier}(\sigma)), \tag{5.33}$$

which follows from the observation that σ takes the value 1 on a central projection iff that projection dominates the carrier of σ.

5.46. Lemma. *Let M be a JBW-algebra with normal state space K. If σ is an extreme point of K, then the split face F_σ generated by σ is a minimal split face among all split faces of K.*

Proof. Let p be the carrier projection of σ. Since $\{\sigma\}$ is a minimal norm closed face of K, then p is a minimal projection in M. Then $c(p)$ is a minimal projection in the center of M (Lemma 3.43). By (5.33), $c(\sigma) = c(p)$, so $c(\sigma)$ is a minimal projection in the center of M. By the correspondence of split faces and central projections, it follows that $F_\sigma = F_{c(\sigma)}$ is minimal among split faces of K. □

Recall that a state σ on a JB-algebra A is a *factor state* if $c(\sigma)$ is a minimal projection in the center of A^{**} (cf. Definition 4.13), and that this is equivalent to $c(\sigma)A^{**}$ being a JBW-factor. If σ is a state on a C*-algebra $\mathcal{A}$, and $\pi_\sigma : \mathcal{A} \to \mathcal{B}(H)$ is the associated GNS-represenation, then $\overline{\pi_\sigma(\mathcal{A})} \cong c(\sigma)\mathcal{A}^{**}$, so σ will be a factor state iff $\overline{\pi_\sigma(\mathcal{A})}$ is a von Neumann algebra factor.

5.47. Proposition. *Let A be a JB-algebra, and σ a state on A. Then σ is a factor state iff the split face F_σ generated by σ is a minimal split face of the state space of A.*

Proof. This follows at once from the definition of a factor state, the identification of the state space of A with the normal state space of A^{**} (Corollary 2.61), and the correspondence of central projections in a JBW-algebra and split faces of the normal state space (Corollary 5.34). □

The following is a dual version to the analogous result in the projection lattice (Proposition 2.31).

5.48. Lemma. *Let p be a projection in a JBW-algebra M and σ a normal state on M. Then*

$$\text{carrier}(U_p^*\sigma) = (\text{carrier}(\sigma) \vee p') \wedge p. \tag{5.34}$$

Proof. The proof is exactly the same as the proof of Proposition 2.31, with the positive element a there replaced by the normal state σ, range projections replaced by carrier projections, and compressions such as U_p replaced by their dual maps U_p^*. □

5.49. Corollary. *Let p be a projection in a JBW-algebra M and σ an extreme point of the normal state space K of M. Then $U_p^*\sigma$ is a multiple of an extreme point of K.*

Proof. Let $q = \text{carrier}(U_p^*\sigma)$, and let $u = \text{carrier}(\sigma)$. Then by the correspondence of atoms and extreme points of K (Proposition 5.39), u is an atom. By Lemma 5.48,

$$q = (\text{carrier}(\sigma) \vee p') \wedge p = (u \vee p') \wedge p = (u \vee p') - p',$$

with the last equality following from (2.17). By the covering property of the projection lattice (Lemma 3.50), the last expression is either an atom or zero. If $q = 0$, then $U_p^*\sigma$ equals zero and thus is certainly equal to a multiple of an extreme point. If q is an atom, then $\|U_p^*\sigma\|^{-1}U_p^*\sigma$ has carrier q, so is an extreme point of K. Thus again $U_p^*\sigma$ is a multiple of an extreme point, which completes the proof of the corollary. □

The Hilbert ball property

Recall that the JBW-factors of type I_2 are exactly the spin factors. (See Definition 3.33 and Proposition 3.37). We begin this section by describing the normal state spaces of such factors.

5.50. Definition. A *Hilbert ball* is the closed unit ball of a real Hilbert space of finite or infinite dimension. For convenience, we include the "zero dimensional Hilbert ball", i.e., the ball consisting of a single point.

5.51. Proposition. *If M is a spin factor, then every state on M is normal and the state space of M is affinely isomorphic to a Hilbert ball.*

Proof. Recall that since M is a spin factor, then $M = N \oplus \mathbf{R}1$ (vector space direct sum) where N is a real Hilbert space, with the norm given by

$$\|a + \lambda 1\| = \|a\|_2 + |\lambda| \tag{5.35}$$

for $a \in N$ and $\lambda \in \mathbf{R}$. (Here $\|a\|_2$ denotes the Hilbert space norm.) Note that the Hilbert and JBW norms coincide on N. Recall from Proposition 3.38 that every state on M is normal.

For each b in the closed unit ball of N, we define $\sigma_b : M \to \mathbf{R}$ by

$$\sigma_b(a + \lambda 1) = (a \mid b) + \lambda. \tag{5.36}$$

We will show that $b \mapsto \sigma_b$ is an affine isomorphism from the closed unit ball of the Hilbert space N onto the state space of M. Let $b \in N$ with $\|b\| \leq 1$. Note first that $\sigma_b(1) = 1$. Furthermore, $\|\sigma_b\| = 1$, because for each $a \in N$ we have

$$|\sigma_b(a + \lambda 1)| \leq |(a \mid b)| + |\lambda| \leq \|a\|\|b\| + |\lambda| \leq \|a\| + |\lambda| = \|a + \lambda 1\|.$$

To show σ_b is positive, let $0 \leq x \in M$ with $\|x\| \leq 1$. By spectral theory, $\|1 - x\| \leq 1$, so because $\|\sigma_b\| \leq 1$ we have

$$\sigma_b(1 - x) = 1 - \sigma_b(x) \leq 1.$$

Thus $\sigma_b(x) \geq 0$ as claimed, so σ_b is a state. (Alternatively, this follows from (A 16).)

If $\sigma_{b_1} = \sigma_{b_2}$, then for all $a \in N$ we have $(a \mid b_1 - b_2) = 0$, so $b_1 = b_2$. Thus the map $b \mapsto \sigma_b$ is 1-1. Finally, we show this map is surjective. If σ is any state, then σ restricts to a functional on N of norm at most 1. Since the JBW and Hilbert norms coincide on N, then there exists b in the closed unit ball of N such that $\sigma(a) = (a \mid b)$ for all $a \in N$. Since $\sigma(1) = 1$, then $\sigma = \sigma_b$. □

We illustrate Proposition 5.51 with a particular example. Recall that $M = (M_2(\mathbf{C}))_{\mathrm{sa}}$ is a spin factor, cf. Example 3.36. In this case, the state space will be a 3-ball. States are given by the maps $a \mapsto \mathrm{tr}(da)$ where d is a positive matrix of trace 1. According to (A 119), an affine isomorphism from the state space of M onto the Euclidean ball $\mathbf{B}^3$ is given by the map $\omega \mapsto (\beta_1, \beta_2, \beta_3)$, where

$$\omega = \begin{pmatrix} 1 + \beta_1 & \beta_2 + \imath\beta_3 \\ \beta_2 - \imath\beta_3 & 1 - \beta_1 \end{pmatrix}.$$

For later use, suppose that p is an atom in a spin factor M. Then, $s = p - p'$ is a symmetry in N, since N includes all symmetries not equal to ± 1. Thus the state σ_b defined in (5.36) for $b = s$ satisfies

$$\sigma_s(p) = \sigma_s(\tfrac{1}{2}(1 + s)) = \tfrac{1}{2} + \tfrac{1}{2}(s \mid s) = 1.$$

It follows that σ_s is the unique (normal) state with value 1 at p, or in the notation of Definition 5.40, for each atom p in a spin factor,

$$\widehat{p} = \sigma_s \quad \text{where } s = p - p' . \tag{5.37}$$

We are now going to describe the circumstances under which the face generated by a pair of extreme points in the normal state space of a JBW-algebra degenerates to a line segment.

5.52. Definition. Let σ and τ be points in a convex set K. We say σ and τ are *separated by a split face* if there is a split face F of K such that $\sigma \in F$ and $\tau \in F'$.

Note that for any split face F of a convex set K, since the convex hull of F and F' is all of K, then each extreme point must lie either in F or in F'. Thus two extreme points not separated by a split face always lie in the same split faces.

5.53. Lemma. *Let M be a JBW-algebra, and p, q distinct atoms in M. Then either $c(p) = c(q)$ or $c(p) \perp c(q)$. In the former case, $M_{p\vee q}$ is a spin factor, and in the latter case, it is isomorphic to $\mathbf{R}1 \oplus \mathbf{R}1$.*

Proof. Since $c(p)$ and $c(q)$ are minimal projections in the center of M (Lemma 3.43), then either $c(p) = c(q)$ or $c(p) \perp c(q)$.

We first consider the case $c(p) = c(q)$. Note that by minimality of $c(p)$ in the center of M, $c(p)M$ is a JBW-factor. Since

$$M_{p\vee q} = \{(p \vee q)(c(p)M)(p \vee q)\},$$

then $M_{p\vee q}$ is a JBW-factor (Proposition 3.13). Define $r = p \vee q - p$. By the covering property (Lemma 3.50), r is an atom, and $p + r = p \vee q$. Minimal projections are abelian, and in a factor have the same central cover (namely, the identity), so are exchangeable (Corollary 3.20). Thus $p \vee q = p + r$ is the sum of two abelian exchangeable projections, so $M_{p\vee q}$ is by definition a type I_2 factor, and therefore is a spin factor.

Now consider the case $c(p) \perp c(q)$. Then p and q are orthogonal, and $qc(p) \le c(q)c(p) = 0$ implies $qc(p) = 0$. Thus $p = (p + q)c(p)$ is central in $M_{p\vee q}$ (Proposition 3.13), as is q (by the same argument). Therefore by Proposition 2.41 $M_{p\vee q}$ is the direct sum of the algebras $M_p = U_p(M_{p\vee q})$ and $M_q = U_q(M_{p\vee q})$. Since p and q are minimal projections, then $M_p = \mathbf{R}p$ and $M_q = \mathbf{R}q$. Thus $M = \mathbf{R}p \oplus \mathbf{R}q$. □

5.54. Lemma. *Let M be a JBW-algebra with normal state space K. Let σ and τ be distinct extreme points of K. Then* $\operatorname{face}(\sigma,\tau)$ *coincides with the line segment $[\sigma, \tau]$ iff σ and τ are separated by a split face.*

Proof. Let $F = \mathrm{face}(\sigma, \tau)$. Let p and q be the carrier projections of σ and τ respectively.

Suppose $F = [\sigma, \tau]$. Then F is norm closed, and so equals the minimal norm closed face containing σ and τ. Then by Corollary 5.33 the carrier projection of F is $p \vee q$ and $F = F_{p \vee q}$. Since the linear span of F is two dimensional, and F can be identified with the normal state space of $M_{p \vee q}$ (Proposition 2.62), then $M_{p \vee q}$ has dimension 2. Since spin factors have dimension at least 3, then $M_{p \vee q}$ is not a spin factor. By Lemma 5.53, $c(p) \perp c(q)$. Then $c(\sigma) \perp c(\tau)$, so $F_\sigma \subset F'_\tau$ (cf. (5.33) and Corollary 5.43). Thus σ and τ are separated by the split face F_τ.

Suppose now that $F \neq [\sigma, \tau]$. Then the norm closed face generated by σ and τ has affine dimension 2 or more, so $M_{p \vee q}$ has dimension 3 or more. By Lemma 5.53, $c(p) = c(q)$. Then $c(\sigma) = c(\tau)$, so $F_\sigma = F_\tau$. Thus σ and τ generate the same split face, so are not separated by any split face. (Alternatively, this follows from the fact that in any convex set, the face generated by two extreme points σ, τ separated by a split face is the line segment $[\sigma, \tau]$, cf. (A 29).) □

5.55. Proposition. *Let M be a JBW-algebra, and σ, τ extreme points of the normal state space K. Then* $\mathrm{face}(\sigma, \tau)$ *is norm exposed and is affinely isomorphic to a Hilbert ball.*

Proof. Assume first that $\sigma \neq \tau$. Let G be the norm closed face generated by σ and τ. $p = \mathrm{carrier}(\sigma)$ and $q = \mathrm{carrier}(\tau)$. Then the carrier of G is $p \vee q$. By Proposition 2.62 the normal state space of $M_{p \vee q}$ is affinely isomorphic to $F_{p \vee q}$. By Lemma 5.53, $M_{p \vee q}$ is either a spin factor or isomorphic to $\mathbf{R}1 \oplus \mathbf{R}1$. In the first case, by Proposition 5.51 its normal state space is affinely isomorphic to a Hilbert ball. In the second case, the dual space of $M_{p \vee q}$ is two dimensional and so its state space is a line segment, which is again a Hilbert ball. Thus G is a Hilbert ball, and $\mathrm{face}(\sigma, \tau)$ is a face of G. Since the only proper faces of a Hilbert ball are extreme points, we must have $\mathrm{face}(\sigma, \tau) = G$. Thus $\mathrm{face}(\sigma, \tau)$ is a Hilbert ball, and is norm exposed since $G = F_{p \vee q}$.

Finally, if $\sigma = \tau$, then $\mathrm{face}(\sigma, \tau) = \{\sigma\}$, which is a zero dimensional Hilbert ball. If p is the carrier projection of the norm closed face $\{\sigma\}$, then $\{\sigma\} = F_p$, so $\{\sigma\}$ is norm exposed. □

5.56. Corollary. *Let A be a JB-algebra, and σ, τ extreme points of the state space K. Then* $\mathrm{face}(\sigma, \tau)$ *is norm exposed and is affinely isomorphic to a Hilbert ball. In particular, extreme points of K are norm exposed.*

Proof. By Corollary 2.61 we can identify the state space of A with the normal state space of the JBW-algebra A^{**}, so the result follows from Proposition 5.55. □

In the last part of this book, we will abstract the property of state spaces described in Corollary 5.56, and call it the *Hilbert ball property*. This property of JB state spaces will turn out to be a key tool in our characterization of these state spaces.

Recall that if p is an atom in a JBW-algebra M, then $\widehat{p}$ denotes the unique normal state with value 1 at p (Definition 5.40).

5.57. Corollary. (Symmetry of transition probabilities) Let p and q be atoms in a JBW-algebra M. Then

$$\widehat{p}(q) = \widehat{q}(p).$$

Proof. The result is clear if $p = q$, so we may assume $p \neq q$. Since $\widehat{p}$ and $\widehat{q}$ are in $F_{p \vee q}$, we can replace M by $M_{p \vee q}$, and thus arrange that $p \vee q = 1$. Then by Lemma 5.53, M is either the direct sum of two one dimensional algebras or else is a spin factor. In the first case, there are only two distinct atoms p, q and one easily sees $\widehat{p}(q) = 0 = \widehat{q}(p)$. Suppose instead that M is a spin factor, and define $s = p - p'$ and $t = q - q'$. By equation (5.37)

$$\widehat{p}(q) = \sigma_s(\tfrac{1}{2}(t+1)) = \tfrac{1}{2} + \tfrac{1}{2}(t \mid s) = \widehat{q}(p),$$

which completes the proof. □

Remark. The property described in Corollary 5.57 has an interesting physical interpretation. If σ and τ are pure states, and p, q the atoms such that $\widehat{p} = \sigma$ and $\widehat{q} = \tau$, then the probability $\widehat{p}(q)$ can be thought of as the "transition probability" for a system prepared in the pure state $\sigma = \widehat{p}$ to be found in the pure state $\tau = \widehat{q}$. Then Corollary 5.57 is a statement about symmetry of transition probabilities. We will return to this concept in the axiomatic portion of this book. We also observe that a purely geometric proof can be given for Corollary 5.57, based on the geometry of Hilbert balls. We will give such a proof in the axiomatic context later, cf. Proposition 9.14.

The σ-convex hull of extreme points

In this section we will show that every normal state on an atomic JBW-algebra can be written as a countable convex combination of an orthogonal set of extreme points, and that more generally the set of such countable convex combinations forms a split face of the normal state space of a JBW-algebra M.

Recall that $\partial_e K$ denotes the set of extreme points of a convex set K. Recall also that a JBW-algebra is atomic if every non-zero projection dominates a minimal projection, i.e., an atom, and that this is equivalent to every projection being the least upper bound of an orthogonal collection of atoms. (See the remark after Definition 3.40.)

5.58. Lemma. *If M is an atomic JBW-algebra, then its normal state space K is the norm closed convex hull of its extreme points.*

Proof. It suffices to show the cone generated by $\partial_e K$ is norm dense in the positive cone of M_*. By the Hahn–Banach theorem, this will follow if we show $\partial_e K$ determines the order on M, i.e., that for $a \in M$,

$$\sigma(a) \geq 0 \quad \text{for all} \quad \sigma \in \partial_e K \quad \Rightarrow \quad a \geq 0. \tag{5.38}$$

Let $a \in M$ satisfy the left side of (5.38). Let $a = a^+ - a^-$ be the orthogonal decomposition of a (Proposition 1.28), and let $p = r(a^-)$. Then by (2.9) $U_p a^+ = 0$, so $a^- = -U_p a$. For each $\sigma \in \partial_e K$, $U_p^* \sigma$ is a multiple of an element of $\partial_e K$ (Corollary 5.49), so

$$\sigma(a^-) = \sigma(-U_p a) = -(U_p^* \sigma)(a) \leq 0. \tag{5.39}$$

Since $a^- \geq 0$, by (5.39) we must have $\sigma(a^-) = 0$ for all $\sigma \in \partial_e K$. Since M is atomic, then 1 is the least upper bound of atoms. By Corollary 5.33, the lattices of projections and norm closed faces are isomorphic, with atoms corresponding to extreme points (Proposition 5.39), so the fact that 1 is the least upper bound of atoms implies that the norm closed face of K generated by the extreme points of K is K itself. Since the set of normal states annihilated by a^- is a norm closed face of K, we conclude that a^- is zero on K, and thus $a^- = 0$. We've then shown $a = a^+ - a^- = a^+ \geq 0$. This completes the proof of (5.38), and thus of the lemma itself. □

Recall that a JBW-algebra M has a unique predual M_*, i.e., a Banach space such that M is isometrically isomorphic to $(M_*)^*$, with M_* consisting of the normal linear functionals on M (Theorem 2.55). If $\{\sigma_i\}$ is a sequence in M_* with $\sum_i \|\sigma_i\| < \infty$, then by completeness of M_*, $\sum_i \sigma_i$ converges in norm to an element of M_*.

5.59. Lemma. *Let M be a JBW-algebra, $p_1, p_2, \ldots$ orthogonal atoms, and $\lambda_1, \lambda_2, \ldots$ scalars such that $\sum_i |\lambda_i| < \infty$. Define $\sigma = \sum_i \lambda_i \widehat{p}_i \in M_*$. Then $\|\sigma\| = \sum_i |\lambda_i|$ and the unique orthogonal decomposition $\sigma = \sigma^+ - \sigma^-$ is given by*

$$\sigma^+ = \sum \{\lambda_i \widehat{p}_i \mid \lambda_i > 0\} \quad \textit{and} \quad \sigma^- = -\sum \{\lambda_i \widehat{p}_i \mid \lambda_i < 0\}. \tag{5.40}$$

Proof. Define σ_1 and σ_2 by

$$\sigma_1 = \sum \{\lambda_i \widehat{p}_i \mid \lambda_i > 0\} \quad \text{and} \quad \sigma_2 = -\sum \{\lambda_i \widehat{p}_i \mid \lambda_i < 0\}.$$

Clearly $\sigma = \sigma_1 - \sigma_2$, and each of σ_1 and σ_2 is a positive linear functional Define

$$s = \sum \{p_i \mid \lambda_i > 0\} - \sum \{p_i \mid \lambda_i < 0\}.$$

Note that for distinct i, j, by orthogonality of p_i and p_j, $\widehat{p}_i(p_j) \leq \widehat{p}_i(p_i') = 0$, so $\widehat{p}_i(p_j) = 0$. Then $\sigma(s) = \sum_i |\lambda_i|$, and by spectral theory $\|s\| = 1$. Thus

$$\sum_i |\lambda_i| = \sigma(s) \leq \|\sigma\| \leq \sum_i |\lambda_i| \, \|\widehat{p}_i\| = \sum_i |\lambda_i|,$$

which implies that $\|\sigma\| = \sum_i |\lambda_i|$. Furthermore,

$$\|\sigma_1\| + \|\sigma_2\| = \sigma_1(1) + \sigma_2(1) = \sum_i |\lambda_i\| = \|\sigma\|,$$

which completes the proof that $\sigma = \sigma_1 - \sigma_2$ is the orthogonal decomposition of σ. □

5.60. Lemma. *Let M be an atomic JBW-algebra M with normal state space K. There exists a unique linear order isomorphism Ψ from M_* into M, such that $\|\Psi\| \leq 1$ and $\Psi(\widehat{p}) = p$ for all atoms p.*

Proof. Define Ψ on $\operatorname{lin}(\partial_e K)$ by

$$\Psi\left(\sum_{i=1}^{n} \alpha_i \widehat{p}_i\right) = \sum_{i=1}^{n} \alpha_i p_i \,,$$

where $p_1, \ldots, p_n$ are atoms and $\alpha_1, \ldots, \alpha_n$ are scalars. We will show that Ψ is well defined. By atomicity, any normal functional that annihilates all atoms will annihilate all projections, and then by the spectral theorem (Theorem 2.20) must be zero. Thus for scalars $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$ and atoms $p_1, \ldots, p_n, q_1, \ldots, q_m$,

$$\sum_{i=1}^{n} \alpha_i \widehat{p}_i = \sum_{j=1}^{m} \beta_j \widehat{q}_j \iff \left(\sum_{i=1}^{n} \alpha_i \widehat{p}_i - \sum_{j=1}^{m} \beta_j \widehat{q}_j\right)(u) = 0 \text{ for all atoms } u.$$

By symmetry of transition probabilities (Corollary 5.57), this is equivalent to

$$\widehat{u}\left(\sum_{i=1}^{n} \alpha_i p_i - \sum_{j=1}^{m} \beta_j q_j\right) = 0 \text{ for all atoms } u. \tag{5.41}$$

As u varies over the atoms of M, then $\widehat{u}$ will vary over the extreme points of K (Proposition 5.39). Thus (5.41) is equivalent to

$$\sigma\Big(\sum_{i=1}^{n}\alpha_i p_i - \sum_{j=1}^{m}\beta_j q_j\Big) = 0 \quad \text{for all } \sigma \in \partial_e K .$$

By Lemma 5.58, this is equivalent to $\sum_{i=1}^n \alpha_i p_i = \sum_{j=1}^m \beta_j q_j$. Thus we have shown

$$\sum_{i=1}^{n}\alpha_i \widehat{p}_i = \sum_{j=1}^{m}\beta_j \widehat{q}_j \iff \sum_{i=1}^{n}\alpha_i p_i = \sum_{j=1}^{m}\beta_j q_j .$$

It follows that Ψ is well defined and 1-1 on $\operatorname{lin}(\partial_e K)$.

Next we show $\|\Psi\| \leq 1$ on $\operatorname{lin}(\partial_e K)$. Let $\sigma \in \operatorname{lin}(\partial_e K)$. Since Ψ maps linear combinations of extreme points to linear combinations of atoms, by Lemma 3.53, there are orthogonal atoms $p_1, \dots, p_n$ and scalars $\lambda_1, \dots, \lambda_n$ such that

$$\Psi(\sigma) = \sum_{i=1}^{n} \lambda_i p_i .$$

Since Ψ is 1-1 on $\operatorname{lin}(\partial_e K)$, $\sigma = \sum_{i=1}^n \lambda_i \widehat{p}_i$. Now by Lemma 5.59, $\|\sigma\| = \sum_{i=1}^n |\lambda_i|$ so

$$\|\Psi(\sigma)\| = \Big\|\sum_{i=1}^{n}\lambda_i p_i\Big\| = \max_{1\leq i\leq n}\{|\lambda_i|\} \leq \sum_{i=1}^{n}|\lambda_i| = \|\sigma\| .$$

This shows that $\|\Psi\| \leq 1$ on $\operatorname{lin}(\partial_e K)$. By density of $\operatorname{lin}(\partial_e K)$ in M_* (Lemma 5.58), we can extend Ψ to a linear map of norm 1 from M_* into M.

Now we show that Ψ is bipositive. Note that by symmetry of transition probabilities,

$$\tau(\Psi(\sigma)) = \sigma(\Psi(\tau)) \tag{5.42}$$

holds for all σ and τ in $\operatorname{lin}(\partial_e K)$, and then by density and continuity holds for all σ, τ in M_*. By definition Ψ is positive on the convex hull of $\partial_e K$, and thus on K by Lemma 5.58, so Ψ is a positive map.

If $\sigma \in M_*$ and $\Psi(\sigma) \geq 0$, then by (5.42)

$$\sigma(\Psi(\tau)) = \tau(\Psi(\sigma)) \geq 0$$

for all $\tau \in \partial_e K$. Since Ψ maps $\partial_e K$ onto the set of atoms in M, then σ is positive on all atoms. By atomicity, σ is positive on all projections, and so by the spectral theorem, $\sigma \geq 0$. Thus Ψ^{-1} is a positive map.

Finally, since $\text{lin}\,(\partial_e K)$ is norm dense in M_*, and every extreme point of K has the form $\widehat{p}$ for some atom $p \in M$, Ψ is determined by the requirement that $\psi(\widehat{p}) = p$ on all atoms p in M, and so is unique. $\square$

5.61. Theorem. *Let M be an atomic JBW-algebra with normal state space K. If $\sigma \in K$, then there exists a (possibly finite) sequence of positive scalars $\{\lambda_i\}$ with sum 1 and orthogonal states σ_i in $\partial_e K$ such that $\sigma = \sum_i \lambda_i \sigma_i$.*

Proof. Let $\sigma \in K$, and let Ψ be the map in Lemma 5.60. Let $\{e_\lambda\}$ be the spectral resolution for $\Psi(\sigma)$ (cf. Theorem 2.20). Then for each $\lambda \in \mathbf{R}$, $e_\lambda = 1 - r((\Psi(\sigma) - \lambda 1)^+)$. Since $\Psi(\sigma) \geq 0$, then for $\lambda < 0$, we have $\Psi(\sigma) - \lambda 1 \geq -\lambda 1 = |\lambda| 1$, so $e_\lambda = 0$ for $\lambda < 0$. By Theorem 2.20(ii), for each α we have

$$U_{e'_\alpha} \Psi(\sigma) \geq \alpha e'_\alpha. \tag{5.43}$$

On the other hand, by Theorem 2.20(i) e_α operator commutes with $\Psi(\sigma)$, and so by (1.62) $U_{e'_\alpha} \Psi(\sigma) \leq \Psi(\sigma)$. Combining this with (5.43) gives

$$\Psi(\sigma) \geq \alpha(1 - e_\alpha).$$

We will show that for each $\alpha > 0$ the projection $(1 - e_\alpha)$ has finite dimension in the lattice of projections (cf. (A 42) and Proposition 3.51). Suppose that $\alpha > 0$ and that $p_1, \ldots, p_n$ are orthogonal atoms under $1 - e_\alpha$. Then

$$\Psi(\widehat{p}_1 + \cdots + \widehat{p}_n) = p_1 + \cdots + p_n \leq 1 - e_\alpha \leq \alpha^{-1} \Psi(\sigma).$$

Since Ψ is bipositive, then

$$\widehat{p}_1 + \cdots + \widehat{p}_n \leq \alpha^{-1} \sigma.$$

Applying both sides to the identity in M, we conclude that $n \leq \alpha^{-1}$. By atomicity, $1 - e_\alpha$ is the least upper bound of an orthogonal set of atoms, and by the result just established this set of orthogonal atoms has cardinality $\leq \alpha^{-1}$. Thus we conclude that $1 - e_\alpha$ has finite dimension for each $\alpha > 0$.

This implies that there is a strictly decreasing sequence of positive scalars $\{\gamma_i\}$ (which may be a finite sequence), such that each e'_{γ_i} is finite dimensional, and such that e_λ is constant for $\lambda > 0$ except for jumps at the γ_i. For simplicity of notation we will consider the case where the sequence $\{\lambda_i\}$ is infinite, and leave to the reader the minor changes in wording needed for the case where the sequence terminates.

Let $\gamma = \lim_i \gamma_i$. Then $\gamma > 0$ would contradict finite dimensionality of e'_γ for $\gamma > 0$, so $\lim_i \gamma_i = 0$. Let q_i be the jump in the spectral resolution e_λ at $\lambda = \gamma_i$, i.e., $q_i = e_{\gamma_i} - e_{\gamma_i - \epsilon}$ for any ϵ with $0 < \epsilon < \gamma_i - \gamma_{i+1}$. The spectral theorem (Theorem 2.20) now gives $\Psi(\sigma) = \sum_i \gamma_i q_i$. Each q_i has finite lattice dimension (since $q_i \le e_{\gamma_i} - e_{\gamma_{i+1}} \le e'_{\gamma_{i+1}}$). Replacing each q_i by a finite sum of atoms leads to an expression $\Psi(\sigma) = \sum_i \lambda_i p_i$ for orthogonal atoms p_i and a sequence of positive scalars λ_i.

Since Ψ is bipositive, for each n,

$$\sum_1^n \lambda_i p_i \le \Psi(\sigma) \quad \Rightarrow \quad \sum_{i=1}^n \lambda_i \widehat{p}_i \le \sigma.$$

Applying both sides of the inequality on the right above to the identity element 1, we conclude that $\sum_{i=1}^n \lambda_i \le 1$ for each n, and so $\sum_{i=1}^\infty \lambda_i \le 1$. Thus $\sum_{i=1}^\infty \lambda_i \widehat{p}_i$ converges in norm to an element of M_*. Since

$$\Psi\left(\sum_{i=1}^\infty \lambda_i \widehat{p}_i\right) = \Psi(\sigma),$$

and Ψ is bijective, we conclude that

$$\sigma = \sum_{i=1}^\infty \lambda_i \widehat{p}_i.$$

Finally, applying both sides of this equation to the identity element 1 gives $\sum_{i=1}^\infty \lambda_i = 1$. $\square$

5.62. Definition. The *σ-convex hull* of a bounded set F of elements in a Banach space is the set of all sums $\sum_i \lambda_i \sigma_i$ where $\lambda_1, \lambda_2, \dots$ are positive scalars with sum 1 and $\sigma_1, \sigma_2, \dots \in F$.

5.63. Corollary. *If M is a JBW-algebra with normal state space K, then the σ-convex hull of the extreme points of K is a split face of K.*

Proof. Let z be the least upper bound of the atoms in M. Recall that z is a central projection, zM is atomic, and $z'M$ contains no atoms (Lemma 3.42). Then since every atom is under z, by the correspondence of atoms and extreme points (Proposition 5.39), every extreme point of K is in the split face F_z. By Proposition 2.62, F_z is affinely isomorphic to the normal state space of zM, and thus equals the σ-convex hull of its extreme points by Theorem 5.61. Every extreme point of a face of K is also an extreme point of K, so F_z is the σ-convex hull of the extreme points of K. $\square$

5.64. Corollary. *If A is a JB-algebra with state space K, then the σ-convex hull of the extreme points of K is a split face of K.*

Proof. The state space of A can be identified with the normal state space of the JBW-algebra A^{**} by Corollary 2.61. $\square$

Trace class elements of atomic JBW-algebras

5.65. Definition. Let M be an atomic JBW-algebra. An element $a \in M$ is *trace class* if there are orthogonal atoms $p_1, p_2, \dots$ and $\lambda_1, \lambda_2, \dots \in \mathbf{R}$ such that

$$a = \sum_i \lambda_i p_i \quad \text{and} \quad \sum_i |\lambda_i| < \infty. \tag{5.44}$$

We will write M_{tr} for the set of trace class elements, and we define the *trace* on M_{tr} by

$$\operatorname{tr}\Bigl(\sum_i \lambda_i p_i\Bigr) = \sum_i \lambda_i. \tag{5.45}$$

To see that the trace is well defined on M_{tr}, let a be defined as in (5.44), and let $\sigma = \sum_i \lambda_i \widehat{p}_i \in M_*$. Let Ψ be the map defined in Lemma 5.60. Then $\Psi(\sigma) = a$, and since Ψ is 1-1, then σ is the unique element of M_* with $\Psi(\sigma) = a$. (Note for use below that this also shows $\psi(M_*) \supset M_{\mathrm{tr}}$.) Since $\sum_i \lambda_i = \sigma(1)$, then

$$\operatorname{tr}(a) = \sigma(1) \text{ where } \Psi(\sigma) = a, \tag{5.46}$$

which proves that the definition of tr is independent of the choice of the representation (5.44).

Note that so far it is not obvious that M_{tr} is a linear subspace, but we now will establish that. Soon we also will show that M_{tr} is an ideal, and that τ satisfies $\tau((a \circ b) \circ c) = \tau(a \circ (b \circ c))$ for $a, b, c \in M_{\mathrm{tr}}$.

5.66. Proposition. *Let M be an atomic JBW-algebra. Then the order isomorphism Ψ defined in Lemma 5.60 maps M_* onto M_{tr}. Furthermore,* tr *is a positive linear functional on M_{tr}, and M_{tr} is a Banach space with respect to the "trace norm" given by*

$$\|a\|_{\mathrm{tr}} = \operatorname{tr}(|a|) \tag{5.47}$$

(where $|a| = a^+ + a^-$). The map Ψ is an isometry from M_* onto M_{tr} for the trace norm on M_{tr}.

Proof. If $\sigma \in M_*$, let $\sigma = \sigma^+ - \sigma^-$ be the orthogonal decomposition of σ. By Theorem 5.61 we can write

$$\sigma^+ = \sum_i \alpha_i \widehat{p}_i \quad \text{and} \quad \sigma^- = \sum_j \beta_j \widehat{q}_j,$$

where $\alpha_1, \alpha_2, \dots$ and $\beta_1, \beta_2, \dots$ are positive numbers such that $\sum_i \alpha_i = \|\sigma^+\|$ and $\sum_j \beta_j = \|\sigma^-\|$, and $p_1, p_2, \dots$ are orthogonal atoms and $q_1, q_2, \dots$ are orthogonal atoms. Note that

$$\Psi(\sigma) = \sum_i \alpha_i p_i - \sum_j \beta_j q_j. \tag{5.48}$$

Let $p = \text{carrier}(\sigma^+)$ and $q = \text{carrier}(\sigma^-)$. Since σ^+ and σ^- are orthogonal, by Lemma 5.4 their carrier projections p and q are orthogonal. Since $\sigma^+(p) = \|\sigma^+\| = \sum_i \alpha_i$, then the carrier projection p of σ^+ must take the value 1 at each $\widehat{p}_i$, so $p \geq p_i$ for each i, and similarly $q \geq q_j$ for each j. Since p and q are orthogonal, then each p_i is orthogonal to each q_j. Therefore by (5.48), $\Psi(\sigma) \in M_{\text{tr}}$, so Ψ maps M_* into M_{tr}. In the remarks preceding this lemma we observed that $\Psi(M_*)$ contains M_{tr}, so Ψ maps M_* onto M_{tr}. Since Ψ is linear, then M_{tr} is a linear subspace of M. Since Ψ is a bijection from M_* onto M_{tr}, by (5.46) $\text{tr}(a) = (\Psi^{-1}(a))(1)$ for each $a \in M_{\text{tr}}$. Since Ψ is bipositive and linear, then tr is a positive linear functional on M_{tr}.

Shifting notation, we have shown above that every element $\sigma \in M_*$ admits a representation $\sigma = \sum_i \lambda_i \widehat{p}_i$ where $p_1, p_2, \dots$ are orthogonal atoms in M and $\sum_i |\lambda_i| < \infty$. Fix $a \in M_{\text{tr}}$, let $\sigma = \Psi^{-1}(a)$, and let $\sigma = \sum_i \lambda_i \widehat{p}_i$ be such a representation of σ. Then $a = \Psi(\sigma) = \sum_i \lambda_i p_i$, and the unique orthogonal decomposition of a in M is given by $a = a^+ - a^-$ where

$$a^+ = \sum \{\lambda_i p_i \mid \lambda_i > 0\} \quad \text{and} \quad a^- = -\sum \{\lambda_i p_i \mid \lambda_i < 0\}.$$

Therefore

$$|a| = a^+ + a^- = \sum_i |\lambda_i| p_i, \tag{5.49}$$

so

$$\text{tr}(|a|) = \text{tr}(\sum_i |\lambda_i| p_i) = \sum |\lambda_i|. \tag{5.50}$$

By Lemma 5.59,

$$\text{tr}(|a|) = \|\sigma\| = \|\Psi^{-1}(a)\|,$$

which proves that $a \mapsto \text{tr}(|a|)$ is a norm on M_{tr}, and that Ψ is an isometry from M_* onto M_{tr} for this norm. Since M_* is a Banach space, then M_{tr} is complete with respect to the trace norm.

Finally by (5.50),

$$|\text{tr}(a)| = \left|\sum_i \lambda_i\right| \leq \sum |\lambda_i| = \text{tr}(|a|). \tag{5.51}$$

Hence tr is a linear functional of norm at most 1 with respect to the trace norm on M_{tr}. Since $\mathrm{tr}(p) = 1$ for each atom $p \in M$, the norm of tr is equal to 1. □

Note that if $a \in M_{\mathrm{tr}}$ is represented in the form (5.44), then by (5.50)

$$\|a\| = \Big\|\sum_i \lambda_i p_i\Big\| = \max_i\{|\lambda_i|\} \le \sum_i |\lambda_i| = \mathrm{tr}(|a|), \tag{5.52}$$

so the order unit norm restricted to M_{tr} is dominated by the trace norm.

5.67. Lemma. *Let M be an atomic JBW-algebra. If $\{a_i\}$ is a sequence of trace class elements of M with bounded trace norm, and $\lambda_1, \lambda_2, \dots$ are real numbers such that $\sum_i |\lambda_i| < \infty$, then $\sum_i \lambda_i a_i$ converges in the trace norm, and then also in the usual (order unit) norm, to an element a in M_{tr}, and $\mathrm{tr}(a) = \sum_i \lambda_i \mathrm{tr}(a_i)$.*

Proof. Clearly the sequence of partial sums of the series $\sum_i \lambda_i a_i$ is Cauchy for the trace norm. By Proposition 5.66, M_{tr} is a Banach space for this norm, so $\sum_i \lambda_i a_i$ converges in trace norm to an element a in M_{tr}. By (5.52), $\sum_i \lambda_i a_i$ also converges to a in the (order unit) norm induced from M. By (5.51), the linear functional tr is continuous with respect to the trace norm; hence $\mathrm{tr}(a) = \sum_i \lambda_i \mathrm{tr}(a_i)$. □

Remark. Note that if $M = \mathcal{B}(H)_{\mathrm{sa}}$, then the atoms are one dimensional projections, so if an element $a \in M_{\mathrm{tr}}$ is represented in the form (5.44), then there is an orthonormal set $\{\xi_i\}$ in H such that p_i is the projection onto the complex line $[\xi_i]$ for each index i. By (5.49) $|a|\xi_i = |\lambda_i|\xi_i$ for each i. Hence by (5.50),

$$\mathrm{tr}(|a|) = \sum_i |\lambda_i| = \sum_i (\,|\lambda_i|\xi_i \mid \xi_i) = \sum_i (\,|a|\xi_i \mid \xi_i). \tag{5.53}$$

From this we conclude that the space M_{tr} of trace class elements as defined here is contained in the space $\mathcal{T}(H)$ of trace class elements in $\mathcal{B}(H)$ as defined in (A 86). In fact, we will see in the remarks after Theorem 5.70 that $M_{\mathrm{tr}} = \mathcal{T}(H)_{\mathrm{sa}}$. It follows from a similar argument based on (5.45) that our definition of tr on M_{tr} coincides with the trace inherited from $\mathcal{T}(H)_{\mathrm{sa}}$.

In Lemma 3.41, we showed that if b is a projection in a JBW-algebra, then U_b sends atoms to multiples of atoms. The following is a generalization of that result.

5.68. Lemma. *Let M be a JBW-algebra, b any element of M, and p any atom in M. Then $U_b p$ is a multiple of an atom, say q, and*

$$U_b p = \widehat{p}(b^2)q. \tag{5.54}$$

Proof. By the identity (1.15),

$$\{bpb\}^2 = \{b\{pb^2p\}b\}. \tag{5.55}$$

Since p is an atom, then $\operatorname{im} U_p = \mathbf{R}p$ by Lemma 3.29, so

$$\{pb^2p\} = \lambda p$$

for some $\lambda \in \mathbf{R}$. Evaluating both sides with $\widehat{p}$ and using $U_p^*\widehat{p} = \widehat{p}$ (cf. (1.55)) gives $\lambda = \widehat{p}(b^2)$, and so

$$\{pb^2p\} = \widehat{p}(b^2)p.$$

Substituting this into (5.55) gives

$$\{bpb\}^2 = \widehat{p}(b^2)\{bpb\}. \tag{5.56}$$

If $\widehat{p}(b^2) = 0$, then $\{bpb\}^2 = 0$, so $\{bpb\} = 0$, and thus (5.54) holds for any projection q. If $\widehat{p}(b^2) \neq 0$, we write $q = \lambda^{-1}\{bpb\}$ (where $\lambda = \widehat{p}(b^2)$ as above). Then (5.54) holds, and by (5.56)

$$q^2 = \lambda^{-2}\{bpb\}^2 = \lambda^{-1}\{bpb\} = q,$$

so q is a projection. It remains to show that q is an atom, which will follow if we can prove $\operatorname{im} U_q = \mathbf{R}q$ (cf. Lemma 3.29). Let $a \in M$. By the identity (1.14) and the fact that $\operatorname{im} U_p = \mathbf{R}p$,

$$\{qaq\} = \lambda^{-2}\{\{bpb\}a\{bpb\}\} = \lambda^{-2}\{b\{p\{bab\}p\}b\} \in \{b(\mathbf{R}p)b\} = \mathbf{R}q.$$

Thus $\operatorname{im} U_q = \mathbf{R}q$, so the projection q is an atom. □

5.69. Lemma. *Let M be an atomic JBW-algebra. If $a \in M$ is trace class, and $b \in M$, then $b \circ a$ is trace class.*

Proof. We have the polarization identity

$$2b \circ a = \{(1+b)a(1+b)\} - \{bab\} - a \tag{5.57}$$

(cf. (2.1)). Thus in order to prove that $b \circ a$ is trace class, it suffices to show $U_b a$ is trace class for all $b \in M$. Let $a = \sum_i \lambda_i p_i$ as in equation (5.44). Then by Lemma 5.68

$$U_b a = \sum_i \lambda_i U_b p_i = \sum_i \lambda_i \widehat{p}_i(b^2) q_i \tag{5.58}$$

for suitable atoms q_i. Since $\widehat{p}_i(b^2) \leq \|b^2\|$ for each i, the sequence $\{\widehat{p}_i(b^2) q_i\}$ is bounded in trace norm, so $U_b a$ is of trace class (Lemma 5.67). □

5.70. Theorem. *Let M be an atomic JBW-algebra. Then for every $\sigma \in M_*$, there is a unique trace class element $d_\sigma \in M$ such that*

$$\sigma(b) = \operatorname{tr}(d_\sigma \circ b) \quad \textit{for all } b \in M. \tag{5.59}$$

The element d_σ is $\Psi(\sigma)$, where Ψ is the map defined in Lemma 5.60.

Proof. We are going to show that

$$\sigma(b^2) = \operatorname{tr}\big(U_b(\Psi(\sigma))\big) \quad \text{for all } b \in M. \tag{5.60}$$

To see why this is of interest, note that by the polarization identity (5.57)

$$\operatorname{tr}(\Psi(\sigma) \circ b) = \operatorname{tr}\big((\tfrac{1}{2}(U_{1+b} - U_b - U_1)\Psi(\sigma)\big). \tag{5.61}$$

Applying (5.60) to (5.61) would prove the existence of d_σ as stated in the theorem, since then

$$\operatorname{tr}(\Psi(\sigma) \circ b) = \sigma\big(\tfrac{1}{2}((1+b)^2 - b^2 - 1)\big) = \sigma(b).$$

Let $\sigma \in M_*$. By Theorem 5.61, we can write $\sigma = \sum_i \lambda_i \widehat{p}_i$ where $\sum_i |\lambda_i| < \infty$ and where each p_i is an atom. Then by Lemma 5.68

$$\begin{aligned}
\operatorname{tr}(U_b(\Psi(\sigma))) &= \operatorname{tr}\left(U_b \sum_i \lambda_i p_i\right) \\
&= \operatorname{tr}\left(\sum_i \lambda_i U_b p_i\right) \\
&= \operatorname{tr}\left(\sum_i \lambda_i \widehat{p}_i(b^2) q_i\right),
\end{aligned}$$

where each q_i is an atom. Then by Lemma 5.67

$$\operatorname{tr}(U_b(\Psi(\sigma))) = \sum_i \lambda_i \widehat{p}_i(b^2) \operatorname{tr}(q_i) = \sum_i \lambda_i \widehat{p}_i(b^2) = \sigma(b^2).$$

This completes the proof of (5.60).

Finally, to prove uniqueness, suppose that

$$\operatorname{tr}(d_1 \circ b) = \operatorname{tr}(d_2 \circ b)$$

for all $b \in M$. Then taking $b = d_1 - d_2$ gives

$$\operatorname{tr}((d_1 - d_2)^2) = 0.$$

From its definition the trace on M_{tr} annihilates no positive element, so $d_1 = d_2$. □

Remark. We can now show that when $M = \mathcal{B}(H)_{\mathrm{sa}}$, then $M_{\mathrm{tr}} = \mathcal{T}(H)_{\mathrm{sa}}$. Recall that in the remarks before Lemma 5.68, we showed that $M_{\mathrm{tr}} \subset \mathcal{T}(H)_{\mathrm{sa}}$. Now let $r \in \mathcal{T}(H)_{\mathrm{sa}}$. Define $\omega_r \in \mathcal{B}(H)_*$ by $\omega_r(a) = \tau(ra)$, where τ is the usual trace on $\mathcal{B}(H)$, cf. (A 87). By Theorem 5.70, there exists $\widetilde{r} \in M_{\mathrm{tr}}$ such that $\operatorname{tr}(\widetilde{r} \circ a) = \omega_r(a) = \tau(ra)$ for all $a \in \mathcal{B}(H)$. By Lemma 5.69, $a \circ \widetilde{r} \in M_{\mathrm{tr}}$, and by the remarks following Lemma 5.67, tr and τ agree on M_{tr}. Thus $\tau(\widetilde{r} \circ a) = \tau(ra)$ for all $a \in \mathcal{B}(H)$. Since $\tau(\widetilde{r} \circ a) = \frac{1}{2}(\tau(ra) + \tau(ar)) = \tau(ra)$, it follows that $r = \widetilde{r}$, and so r is in M_{tr}. Thus $M_{\mathrm{tr}} = \mathcal{T}(H)_{\mathrm{sa}}$.

We conclude this section by noting that although we have used the term trace, we have not actually verified that tr is a tracial state. Of course, M_{tr} is not a JB-algebra, (since it is not norm complete), so our previous Definition 5.19 of a tracial state is not applicable, nor is Lemma 5.18. Nevertheless, for each symmetry $s \in M$, U_s is a Jordan automorphism (Proposition 2.34), so for $a = \sum_i \lambda_i p_i$ as in (5.44), then $U_s a = \sum_i \lambda_i U_s(p_i)$ and $\{U_s(p_i)\}$ will be a sequence of orthogonal atoms. For $a \in M_{\mathrm{tr}}$, we then have $U_s a \in M_{\mathrm{tr}}$ and $\operatorname{tr}(U_s(a)) = \operatorname{tr}(a)$. A careful examination of the proof of Lemma 5.18 shows that the argument there still goes through with M_{tr} in place of M, so that

$$\operatorname{tr}((a \circ b) \circ c) = \operatorname{tr}(a \circ (b \circ c))$$

holds for all trace class a, b, c. Since we will not need this result in the sequel, we leave the details to the interested reader.

Symmetry and ellipticity

In this section we will discuss two properties having to do with the existence of affine automorphisms of the state space, and thus with its degree of symmetry or homogeneity. We start with symmetry with respect to certain reflections.

5.71. Definition. An affine automorphism R of a convex set K is a *reflection* if $R^2 = I$. A convex set K is *symmetric with respect to a convex subset* K_0 if there is a reflection R of K whose fixed point set is exactly K_0.

Note that if R is a reflection with fixed point set K_0, then the image of a point σ under R is obtained by reflecting the line segment $[\sigma, R\sigma]$ about its midpoint $\frac{1}{2}(\sigma + R\sigma) \in K_0$. This justifies both the term "reflection" and "symmetric" above.

5.72. Proposition. *If M is a JBW-algebra with normal state space K, and F is a norm closed face with complement F', then K is symmetric with respect to $co(F \cup F')$. In particular, if p is the carrier projection of F, then $\Psi = (U_p + U_{p'})^*$ is an affine projection (idempotent map) from K onto $co(F \cup F')$, and $R = U^*_{p-p'}$ is a reflection whose fixed point set is $co(F \cup F')$.*

Proof. Recall that $F = F_p = (\operatorname{im} U_p^*) \cap K$. Let $s = p - p'$, and $R = U_s^*|K$. By Proposition 2.34, U_s is a Jordan automorphism of M satisfying $(U_s)^2 = I$, and $U_s^*(K) \subset K$ (Proposition 5.16), so R is a reflection of K. By equation (2.25), $U_s = 2U_p + 2U_{p'} - I$. Thus if $\omega \in K$, then $R\omega = \omega$ iff $(U_p + U_{p'})^*\omega = \omega$. Then

$$\omega = \lambda \|U_p^*\omega\|^{-1} U_p^*\omega + (1 - \lambda)\|U_{p'}^*\omega\|^{-1} U_{p'}^*\omega \in \operatorname{co}(F \cup F'),$$

where $\lambda = \|U_p^*\omega\|$. Since each element of $\operatorname{co}(F \cup F')$ is clearly fixed by R, the fixed point set of R equals $\operatorname{co}(F \cup F')$. $\square$

We will refer to $(U_p + U_{p'})^*$ as the *canonical affine projection* of K onto $\operatorname{co}(F \cup F')$. We will see in Part II of this book that this is the unique affine projection of K onto $\operatorname{co}(F \cup F')$, and that $\Psi = U_s^*$ is the unique reflection of K with fixed point set $\operatorname{co}(F \cup F')$. (The analogous uniqueness result in the von Neumann context is given in (A 174).)

We now turn to showing that certain orbits are ellipses. Recall that for each element a in a JB-algebra, the map T_a given by $T_a x = a \circ x$ is an order derivation (Lemma 1.56). Thus for an element a in a JBW-algebra M, since T_a is σ-weakly continuous, then $t \mapsto \exp(tT_a)^*$ restricts to give a one-parameter group of order automorphisms of M_*. Orbits of normal states then stay inside the positive cone of M_*, and if normalized the orbit will lie in K. We are going to show, for the special case where $a = p - p'$ for a projection p, that these normalized orbits trace out half of an ellipse. Since $T_{p-p'} = U_p - U_{p'}$ (cf. (1.47)), for later applications we will express our results in terms of the maps $U_p - U_{p'}$.

The following is a simple technical lemma that we will use both now and later, and so we state it separately for easy reference.

5.73. Lemma. *Let σ, τ and ρ be linearly independent elements of a vector space, let α be a real number with $0 < \alpha < 1$, and set $\widetilde{\rho} = \frac{1}{2}(\alpha(1-\alpha))^{-1/2}\rho$. For each $t \in \mathbf{R}$ the point*

$$\omega_t = \big(\alpha e^t + (1-\alpha)e^{-t}\big)^{-1}\big(\alpha e^t \sigma + (1-\alpha)e^{-t}\tau + \rho\big) \tag{5.62}$$

lies on the ellipse consisting of all points

$$\omega = \tfrac{1}{2}(\sigma+\tau) + x\,(\tfrac{1}{2}(\sigma-\tau)) + y\,\widetilde{\rho} \quad \textit{where} \quad x^2+y^2 = 1. \tag{5.63}$$

This ellipse has the diameter $[\sigma, \tau]$. When t varies from $-\infty$ to ∞, the parameterized curve ω_t traces out, in the direction from τ to σ, the half-ellipse between these points which contains the point $\alpha\sigma + (1-\alpha)\tau + \rho$ (attained for $t = 0$). The other half of the ellipse is traced out in the same fashion by the parameterized curve obtained by replacing ρ by $-\rho$ in (5.62).

Proof. It is easily verified that when t varies from $-\infty$ to ∞, the point $(x_t, y_t) \in \mathbf{R}^2$ defined by

$$x_t = \frac{\alpha e^t - (1-\alpha)e^{-t}}{\alpha e^t + (1-\alpha)e^{-t}}, \quad y_t = \frac{2(\alpha(1-\alpha))^{1/2}}{\alpha e^t + (1-\alpha)e^{-t}} \tag{5.64}$$

traces out the upper half of the circle $x^2 + y^2 = 1$ in the direction from left to right. The affine map $\Phi : \mathbf{R}^2 \mapsto M_*$ defined by

$$\Phi(x,y) = \tfrac{1}{2}(\sigma+\tau) + x\,(\tfrac{1}{2}(\sigma-\tau)) + y\,\widetilde{\rho} \tag{5.65}$$

carries this circle to an ellipse. Rewriting (5.65) in the form

$$\Phi(x,y) = \tfrac{1}{2}(x+1)\,\sigma + \tfrac{1}{2}(1-x)\,\tau + y\,\widetilde{\rho}$$

and substituting $x = x_t$, $y = y_t$, we find that $\Phi(x_t, y_t) = \omega_t$. Thus ω_t lies on the specified ellipse for all $t \in \mathbf{R}$.

Since $\Phi(1,0) = \sigma$ and $\Phi(-1,0) = \tau$, the line segment $[\sigma,\tau]$ is a diameter of this ellipse. By (5.64), $(x_t, y_t) \to (\pm 1, 0)$ when $t \to \pm\infty$, so $\omega_t \to \tau$ when $t \to -\infty$ and $\omega_t \to \sigma$ when $t \to \infty$. Thus ω_t traces out, in the direction from τ to σ, the half-ellipse between these points which contains the point $\omega_0 = \alpha\sigma + (1-\alpha)\tau + \rho$.

In the same fashion, the parameterized curve obtained by changing the sign of ρ in (5.62) traces out the half-ellipse containing the "opposite" point $\alpha\sigma + (1-\alpha)\tau - \rho$ (i.e., the image of ω_0 under reflection about the diameter $[\sigma,\tau]$). This proves the last statement of the lemma. □

5.74. Lemma. *Let M be a JBW-algebra with normal state space K, let p be a projection in M with associated face $F = F_p$ and reflection $R = U^*_{p-p'}$ (cf. Proposition 5.72), and let $\omega \in K$. Assume $\omega \notin F \cup F'$, and define $\sigma = \|U^*_p\omega\|^{-1}U^*_p\omega$, $\tau = \|U^*_{p'}\omega\|^{-1}U^*_{p'}\omega$. Consider first the orbit $\{\widetilde{\omega}_t\}_{t\in\mathbf{R}}$ of ω under the one-parameter group of order automorphisms of M_* given by $t \mapsto exp(t(U^*_p - U^*_{p'}))$. This orbit is one half of a hyperbola. Consider also the normalized orbit $\{\omega_t\}_{t\in\mathbf{R}}$ of ω, given by $\omega_t = \|\widetilde{\omega}_t\|^{-1}\,\widetilde{\omega}_t$. If $\omega \notin \mathrm{co}(F \cup F')$, then this orbit is the (unique) half-ellipse with diameter $[\sigma, \tau]$ which contains the point ω, traced out in the direction from τ to σ, and the other half of this ellipse is the normalized orbit of $R\omega$. If $\omega \in \mathrm{co}(F \cup F')$, (but still $\omega \notin F \cup F'$), then the normalized orbit of ω degenerates to the line segment from τ to σ, traced out in the same direction.*

Proof. Using $U_p^2 = U_p$, $U_{p'}^2 = U_{p'}$ and $U_pU_{p'} = U_{p'}U_p = 0$, we find

$$\widetilde{\omega}_t = \exp(t(U_p - U_{p'})^*)\omega = e^tU^*_p\omega + e^{-t}U^*_{p'}\omega + (I - U^*_p - U^*_{p'})\omega.$$

Since $\omega \notin F \cup F'$, then $\|U^*_p\omega\| = (U^*_p\omega)(1) = \omega(p) \neq 0$, and likewise $\|U^*_{p'}\omega\| = \omega(p') \neq 0$. Therefore the two points σ and τ of the lemma are well defined. Now we can rewrite the equation above in the form

$$\widetilde{\omega}_t = \alpha e^t\sigma + (1-\alpha)e^{-t}\tau + \rho \tag{5.66}$$

where $\alpha = \|U^*_p\omega\|$ and $\rho = (I - U_p - U_{p'})^*\omega$. This equation shows that the orbit $\{\widetilde{\omega}_t\}_{t\in\mathbf{R}}$ is one half of a hyperbola.

For each $t \in \mathbf{R}$, $\|\widetilde{\omega}_t\| = \widetilde{\omega}_t(1) = \alpha e^t + (1-\alpha)e^{-t}$, so the normalized orbit of ω is given by (5.62).

Assume now $\omega \notin \mathrm{co}(F \cup F')$. Then σ, τ, ρ are linearly independent; otherwise ρ would be a linear combination of the two (linearly independent) elements σ and τ, and then $\omega = \widetilde{\omega}_0$ would also be a linear combination of $\sigma \in F$ and $\tau \in F'$, so ω would be in $\mathrm{im}\,(U^*_p + U^*_{p'}) \cap K = \mathrm{co}(F \cup F')$. Thus we can apply Lemma 5.73, from which the description of the normalized orbit of ω follows. Since $U_{p-p'} = 2U_p + 2U_{p'} - 1$, the reflection $R = U^*_{p-p'}$ sends $\rho = (I - U_p - U_{p'})^*\omega$ to $-\rho$, and fixes σ and τ. Since U^*_p and $U^*_{p'}$ commute with R, the orbit of $R\omega$ equals R applied to the orbit of ω. Therefore the description of the normalized orbit of $R\omega$ also follows from Lemma 5.73.

Finally, if $\omega \in \mathrm{co}(F \cup F')$, then $\rho = 0$ and it follows from (5.62) that ω_t can be written as a convex combination $\omega_t = \lambda_t\sigma + (1-\lambda_t)\tau$ where $\lambda_t \to 0$ when $t \to -\infty$ and $\lambda_t \to 1$ when $t \to \infty$. Thus ω_t traces out the line segment from τ to σ in this case. □

Note that the fact that the normalized orbit of the point ω of Lemma 5.74 is part of an ellipse follows easily from the fact that the non-normalized orbit is half of a hyperbola. Actually, the passage from the non-normalized orbit to the normalized orbit is just the projection via the origin onto the plane through ρ, σ, τ, and since the image of the hyperbola under this projection is a conic section in the bounded set K, it must be an ellipse.

Note also that if the point ω of Lemma 5.74 lies in either F or F', then it is easily verified that ω is a fixed point under the one-parameter group $t \mapsto \exp(t(U_p^* - U_p'^*))$.

5.75. Proposition. *Let M be a JBW-algebra, and F a norm closed face of the normal state space K. Let $\Psi = U_p^* + U_{p'}^*$ be the canonical affine projection of K onto $co(F \cup F')$, and $\sigma \in F$, $\tau \in F'$. Then $\Psi^{-1}([\sigma, \tau])$ has elliptical cross-sections, i.e., the intersection of this set with every plane through the line segment $[\sigma, \tau]$ is an elliptical disk (which may degenerate to a line segment).*

Proof. Let p be the carrier projection of F. As observed in the remarks preceding Lemma 5.73, for each $\omega \in K$ the normalized elliptical orbit ω_t described in Lemma 5.74 stays inside K. Let L_0 be the intersection of $\Psi^{-1}([\sigma, \tau])$ with a plane L through $[\sigma, \tau]$. If L_0 is a line segment, then L_0 is a degenerate ellipse, so we are done. Hereafter, we assume this is not the case. Since K is closed and bounded for the base norm, and all norms on the finite dimensional subspace L are equivalent (cf., e.g., [108, Thm. 1.21]), $L_0 = K \cap L$ is closed and bounded and thus is compact. Let ω be a point on the boundary of L_0 in L. Then since $\Psi(\omega) \in [\sigma, \tau]$, there is a scalar λ with $0 < \lambda < 1$ such that

$$U_p^*\omega + U_{p'}^*\omega = \lambda\sigma + (1 - \lambda)\tau.$$

Applying U_p^* to both sides, we find that $\sigma = \|U_p^*\omega\|^{-1}U_p^*\omega$. Similarly $\tau = \|U_{p'}^*\omega\|^{-1}U_{p'}^*\omega$, so σ and τ have the same meaning as in Lemma 5.74. Let $E(\omega)$ denote the ellipse with diameter $[\sigma, \tau]$ passing through ω, together with its interior. Then each half of this ellipse is the normalized orbit of either ω or the reflected point $U_s^*\omega$ (where $s = p - p'$) by Lemma 5.74, and so this ellipse is contained in K. Thus $E(\omega)$ is contained in L_0. We will show that $E(\omega) = L_0$. If ω_1 were another element of L_0 not in $E(\omega)$, then the ellipse through σ, τ, ω_1 would not meet the ellipse through σ, τ, ω except at the points σ and τ. Therefore $E(\omega_1)$ would be contained in L_0 and would contain ω in its interior. But ω is a boundary point, so this is a contradiction. We conclude that $L_0 = E(\omega)$ as claimed. □

We will refer to the elliptical cross-section property described in Proposition 5.75 as *ellipticity* of K. We will define this in an abstract context later, and show that the combination of admitting a suitable spectral theory and ellipticity characterize normal state spaces of JBW-algebras.

Both of the properties symmetry and ellipticity can be viewed as illustrations that there are "many" affine automorphisms of the normal state space of a JBW-algebra (or of the state space of a JB-algebra). Dually, the positive cone of every JB-algebra admits "many" Jordan automorphisms. One way to make this precise is to observe that the affine automorphisms of the interior of the positive cone of a JB-algebra A are transitive. For a more complete discussion of such *homogeneity*, see the discussion in the notes to Chapter 6. As discussed there, in the context of self-dual cones, another notion of homogeneity (sometimes called "facial homogeneity"), is due to Connes [36]. The definition is that $\exp(t(P_F - P_{F^\perp})) \geq 0$ for all t and F, where his maps P_F are projections onto faces of the associated cone, and thus are closely related to our maps U_p. In the JBW-context, positivity of the analogous maps $\exp(t(U_p - U_{p'})^*)$ is what we used above to establish ellipticity, and we will see in an abstract context that positivity of such maps is equivalent to ellipticity.

Notes

The introductory results of this chapter on the lattice of projective faces can be found in [10]. The characterization of Jordan homomorphisms via their dual maps on the state space (Lemma 5.15) is from [117]. The result describing alternative characterizations of traces on JB-algebras (Lemma 5.18) is due to Pedersen and Størmer [101]. The result that type I_n JBW-algebras have a center-valued trace (Proposition 5.25) can be found in [62]; our proof is modeled on the similar proof for type I_n von Neumann algebras in [80]. For a more detailed treatment of traces on Jordan operator algebras, see [23]. The Radon–Nikodym type theorem for JBW-algebras in Lemma 5.27 generalizes a well known result for von Neumann algebras [112], and the proof is essentially the same. The result that norm closed faces of the normal state space of a JBW-algebra are projective (Theorem 5.32) is due to Iochum [68]. After quite a bit of work in the lemmas preceding Theorem 5.32, the proof eventually reduces to using the analogous result for von Neumann algebras (A 107), which is due to Effros [46] and to Prosser [105].

In the sequel, we will only use a special case of this result: namely, that extreme points of the state space of a JB-algebra are norm exposed. This is used to show that the factor representations associated with pure states are type I (Corollary 5.42), and will play an important role in our characterization of JB-algebra and C*-algebra state spaces. In [8] and in [10] the result that extreme points are norm exposed is based on the characterization of pure states on JC-algebras due to Størmer [126].

The general correspondence of ideals and split faces (Proposition 5.36 and Corollary 5.37) is quite analogous to the similar results for C*-algebras and von Neumann algebras, cf. (A 112) and (A 114). For C*-algebras, this result is due to Segal [115]. For JB-algebras, this result is in [65].

The Hilbert ball property for JB-algebra state spaces (Corollary 5.56) was first discussed in [10] as part of the characterization of JB state spaces

(Theorem 9.38), as were the related facts that the state space of a spin factor is a Hilbert ball (Proposition 5.51), and the symmetry of transition probabilities (Corollary 5.57). The representation of normal states on an atomic JBW-factor as a countable convex combination of orthogonal pure states (Theorem 5.61) also is in [10], and generalizes the analogous result for $\mathcal{B}(H)$, which can be found in [100]. The use of trace class elements to describe the normal state space of atomic JBW-algebras (Theorem 5.70) parallels the corresponding result for $\mathcal{B}(H)$ (A 87). The symmetry property (Proposition 5.72) first appears in [10]. As discussed at the end of this chapter, ellipticity as described in Proposition 5.75 is formally similar to the notion of *facial homogeneity* due to Connes [36]. The geometric interpretation of ellipticity given in Proposition 5.75 appeared in [71].

6 Dynamical Correspondences

In this chapter we will discuss the notion of a "dynamical correspondence", which is applicable in both JB and JBW contexts. This generalizes the correspondence of observables and generators of one-parameter groups of automorphisms in quantum mechanics. It is closely related to Connes' concept of orientation [36, Definition 4.11] that he used as a key property in his characterization of the natural self-dual cones associated with von Neumann algebras (i.e., the cones $P_\xi^\natural$ of Tomita–Takesaki theory). Both can be thought of as ways to specify possible Lie structures compatible with a given Jordan structure.

In the first section we introduce the general notion of a "Connes orientation", which carries over to JBW-algebras the concept Connes introduced. In the second section we introduce the notion of a "dynamical correspondence" (first defined in [12, Def. 17]). We show that a JB-algebra is the self-adjoint part of a C*-algebra iff it admits a dynamical correspondence, with a similar result for JBW-algebras, and that the dynamical correspondences are in both cases in 1-1 correspondence with "Jordan compatible" associative products (Definition 6.14). We also show that for JBW-algebras there is a 1-1 correspondence of Connes orientations and dynamical correspondences. The last section sketches a geometric interpretation of these concepts, and relates them to the concepts of orientation for C*-algebra state spaces and von Neumann algebra normal state spaces discussed in [AS].

Connes orientations

Both Connes' concept of orientation and our notion of a dynamical correspondence refer to order derivations. If A is an ordered Banach space, a bounded linear map $\delta : A \to A$ is an *order derivation* if $\exp(t\delta) \geq 0$ for all $t \in \mathbf{R}$, i.e., if δ is the generator of a one-parameter group of order automorphisms of A. A bounded linear operator δ acting on a C*-algebra $\mathcal{A}$ is said to be an *order derivation* if it leaves $\mathcal{A}_{\rm sa}$ invariant and restricts to an order derivation on $\mathcal{A}_{\rm sa}$ (A 181).

6.1. Definition. If $\mathcal{A}$ is a C*-algebra $\mathcal{A}$, and $a \in \mathcal{A}$, then we define $\delta_a : \mathcal{A} \to \mathcal{A}$ by

$$\delta_a x = \tfrac{1}{2}(ax + xa^*). \tag{6.1}$$

We recall the following results from [AS].

6.2. Theorem. *If $\mathcal{A}$ is a C^*-algebra and $a \in \mathcal{A}$, then δ_a is an order derivation on $\mathcal{A}$. If $\mathcal{A}$ is a von Neumann algebra, then every order derivation on $\mathcal{A}$ has the form δ_a for some $a \in \mathcal{A}$.*

Proof. The first statement is (A 182), and the second is (A 183). □

6.3. Proposition. *Let $\mathcal{A}$ be a C^*-algebra. If $a \in \mathcal{A}$, then the one-parameter group $\alpha_t = \exp(t\delta_a)$ is given by*

$$\alpha_t(x) = e^{ta/2} x e^{ta^*/2}. \tag{6.2}$$

In particular, if $h \in \mathcal{A}_{\mathrm{sa}}$ and $a = h$ (the self-adjoint case), then

$$\alpha_t(x) = e^{th/2} x e^{th/2}, \tag{6.3}$$

and if $a = ih$ (the skew case), then

$$\alpha_t(x) = e^{ith/2} x e^{-ith/2}. \tag{6.4}$$

Proof. Consider the left and right multiplication operators $L : x \mapsto ax$ and $R : x \mapsto xa^*$ defined on $\mathcal{A}$ for $a \in \mathcal{A}$. Since L and R commute,

$$\exp(2t\delta_a)(x) = \exp(tL + tR)(x) = \exp(tL)\exp(tR)(x) = e^{ta} x e^{ta^*}.$$

This proves (6.2), from which (6.3) and (6.4) both follow. □

The orbits of the one-parameter group $\alpha_t^* = \exp(\delta_a)^*$ can be easily visualized in the simplest non-commutative case where $\mathcal{A} = M_2(\mathbf{C})$. Here the state space is a Euclidean 3-ball, and the pure state space is the surface of the ball, i.e., a Euclidean 2-sphere. If $a \in \mathcal{A}$ is self-adjoint and has two distinct eigenvalues $\lambda_1 < \lambda_2$ corresponding to (unit) eigenvectors ξ_1, ξ_2, then the vector states $\omega_{\xi_1}, \omega_{\xi_2}$ are antipodal points on the sphere (South Pole and North Pole in Fig. 6.1). If $a = ih$ where $h \in \mathcal{A}_{\mathrm{sa}}$ (the skew case), then it can be seen from (6.4) that α_t^* is a rotation of the ball by an angle $t(\lambda_2 - \lambda_1)/2$ about the diameter $[\omega_{\xi_1}, \omega_{\xi_2}]$, cf. (A 129). Thus the one-parameter group α_t^* represents a rotational motion with rotational velocity $(\lambda_2 - \lambda_1)/2$ about this diameter, and the orbits on the sphere are the "parallel circles" (in planes orthogonal to $[\omega_{\xi_1}, \omega_{\xi_2}]$). If $a = h$ where $h \in \mathcal{A}_{\mathrm{sa}}$ (the self-adjoint case), then the orbits will take us out of the state space. But this can be remedied by a normalization, i.e., by considering the parametric curves $t \mapsto \|\alpha_t^*(\sigma)\|^{-1}\alpha_t^*(\sigma)$ instead of $t \mapsto \alpha_t^*(\sigma)$. Now it can be seen from (6.3) that the (normalized) orbits are the "longitudinal semi-circles" on the sphere (in planes through $[\omega_{\xi_1}, \omega_{\xi_2}]$), cf. Lemma 5.74.

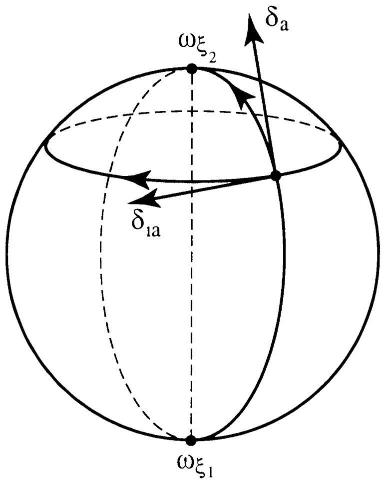

Fig. 6.1

In the example above we can easily see how the one-parameter group is determined by the geometry in the self-adjoint case, and we can also see what indeterminacy there is in the skew case. The element $a \in \mathcal{A}_{\mathrm{sa}}$ determines a real valued affine function $\widehat{a} : \omega \mapsto \omega(a)$ on the state space. This function attains its minimum λ_1 at ω_{ξ_1} and its maximum λ_2 at ω_{ξ_2}. In the self-adjoint case the orbits are the longitudinal semi-circles traced out in the direction from ω_{ξ_1} to ω_{ξ_2}. In the skew case the orbits are the parallel circles, but they can be traced out in two possible directions, "eastbound" and "westbound". The mere knowledge of the affine function $\widehat{a}$ does not tell us which direction to choose. This would require a specific orientation of the ball (right-handed or left-handed around $\overrightarrow{\omega_{\xi_1}\omega_{\xi_2}}$).

If a, b are elements of a C*-algebra, then we will often make use of the identity

$$[\delta_a, \delta_b] = \tfrac{1}{2}\delta_{[a,b]}. \tag{6.5}$$

We are going to transfer Connes' concept of orientation [36, Definition 4.11], originally defined in the context of certain self-dual cones, to the context of JBW-algebras. For motivation, note that if $\mathcal{M}$ is a von Neumann algebra and $a \in \mathcal{M}_{\mathrm{sa}}$, then

$$\delta_{ia}x = \frac{i}{2}(ax - xa),$$

so that δ_{ia} essentially gives the Lie product. Thus we can express the ordinary left multiplication by a as

$$ax = \delta_a x - \imath\delta_{ia} x.$$

Now the idea is to axiomatize the map $\delta_d \mapsto \delta_{id}$ from the set $D(\mathcal{M}_{\mathrm{sa}})$ of order derivations into itself. Note that as it stands this map is not well defined on the set of all order derivations, since if d is not self-adjoint, then δ_d does not determine d. In fact, for any central skew-adjoint d the map δ_d will be zero. Thus we will define the map instead on $D(\mathcal{M}_{\mathrm{sa}})$ modulo the *center* of the set of order derivations, which we now define.

6.4. Definition. Let A be a JB-algebra. The *center* of the Lie algebra $D(A)$ consists of all the order derivations δ in $D(A)$ that commute with all members of $D(A)$. We will denote the center of $D(A)$ by $Z(D(A))$.

We review some basic facts about order derivations on JB-algebras. If a is an element of a JB-algebra A, we let δ_a denote Jordan multiplication by a, i.e., $\delta_a x = a \circ x$ for $x \in A$. (Note that this agrees with (6.1) when A is the self-adjoint part of a C*-algebra $\mathcal{A}$.) Each such map δ_a is an order derivation on A (Lemma 1.56). The center of a JB-algebra A consists of those $z \in A$ such that $\delta_z\delta_a = \delta_a\delta_z$ for all $a \in A$ (Definition 1.51). The space $D(A)$ of order derivations of A is a Lie algebra with respect to the usual Lie bracket given by commutators: $[\delta_1, \delta_2] = \delta_1\delta_2 - \delta_2\delta_1$ (Proposition 1.59). An order derivation on A is *self-adjoint* if it has the form δ_a for some $a \in A$. An order derivation δ on A is *skew* if $\delta(1) = 0$. Every order derivation admits a unique decomposition as the sum of a self-adjoint derivation and a skew derivation (Proposition 1.60). A skew order derivation is a Jordan derivation, i.e., satisfies the Leibniz rule

$$\delta(a \circ b) = \delta(a) \circ b + a \circ \delta(b), \tag{6.6}$$

and the skew order derivations are precisely the bounded linear maps that generate one-parameter groups of Jordan automorphisms (Lemma 2.81).

6.5. Lemma. *If A is a JB-algebra, then*

$$Z(D(A)) = \{\delta_z \mid z \in Z(A)\}. \tag{6.7}$$

Proof. Assume first that $\delta \in Z(D(A))$. Then $[\delta, \delta_a] = 0$ for every $a \in A$. Let $z = \delta(1)$. Then for every $a \in A$,

$$\delta(a) = \delta\delta_a(1) = \delta_a\delta(1) = \delta_a(z) = a \circ z = z \circ a.$$

Hence $\delta = \delta_z$. Also $\delta_z\delta_a = \delta_a\delta_z$, so $z \in Z(A)$.

Assume next that $z \in Z(A)$. By the definition of $Z(A)$, δ_z commutes with every self-adjoint order derivation δ_a. Therefore we only have to show that δ_z commutes with every skew derivation δ. But such a derivation is a Jordan derivation, so it follows from the Leibniz rule (6.6) that

$$\delta\delta_z(x) = \delta(z \circ x) = (\delta z) \circ x + z \circ (\delta x) \tag{6.8}$$

for every $x \in A$. Since δ is skew, $\delta z = 0$ (Proposition 2.82), so (6.8) shows that $(\delta\delta_z)(x) = (\delta_z\delta)(x)$. Thus $\delta\delta_z = \delta_z\delta$ as desired. □

6.6. Definition. Let A be a JB-algebra and $\delta \in D(A)$. Write $\delta = \delta_1 + \delta_2$ where δ_1 is self-adjoint and δ_2 is skew. Then we define $\delta^\dagger = \delta_1 - \delta_2$.

Note that if A is a JB-algebra and $\delta \in D(A)$, then $(\delta^\dagger)^\dagger = \delta$, so $\delta \to \delta^\dagger$ is linear and is an involutive map, i.e., has period 2. Note also that δ is self-adjoint iff $\delta^\dagger = \delta$, and skew iff $\delta^\dagger = -\delta$.

6.7. Definition. Let M be a JBW-algebra. We write $\widetilde{D}(M)$ in place of $D(M)/Z(D(M))$, and denote the equivalence class of an element δ of $D(M)$ modulo $Z(D(M))$ by $\widetilde{\delta}$.

Note that the involution $(\widetilde{\delta})^\dagger = \widetilde{(\delta^\dagger)}$ is well defined on $\widetilde{D}(M)$, for if $\widetilde{\delta}_1 = \widetilde{\delta}_2$, then $\delta_1 - \delta_2 = \delta_z$ for some $z \in Z(M)$ (Lemma 6.5). Hence $\delta_1^\dagger - \delta_2^\dagger = \delta_z^\dagger = \delta_z$, so $\widetilde{(\delta_1^\dagger)} = \widetilde{(\delta_2^\dagger)}$.

6.8. Definition. A *Connes orientation* on a JBW-algebra M is a complex structure on $\widetilde{D}(M)$, which is compatible with Lie brackets and involution, i.e., a linear operator I on $\widetilde{D}(M)$ which satisfies the requirements:

(i) $I^2 = -1$ (where 1 is the identity map),
(ii) $[I\widetilde{\delta}_1, \widetilde{\delta}_2] = [\widetilde{\delta}_1, I\widetilde{\delta}_2] = I[\widetilde{\delta}_1, \widetilde{\delta}_2]$, and
(iii) $I(\widetilde{\delta}^\dagger) = -(I\widetilde{\delta})^\dagger$.

6.9. Proposition. *Let $\mathcal{M}$ be a von Neumann algebra. Then*

$$I(\widetilde{\delta}_x) = \widetilde{\delta}_{ix} \quad \textit{for all } x \in \mathcal{M}$$

defines a Connes orientation on $\mathcal{M}_{\mathrm{sa}}$.

Proof. Let $M = \mathcal{M}_{\mathrm{sa}}$. We first verify that I is well defined. For this purpose, it suffices to show that if $x \in \mathcal{M}$ and $\widetilde{\delta}_x = 0$, then $\widetilde{\delta}_{ix} = 0$. We first verify

$$\widetilde{\delta}_x = 0 \quad \Rightarrow \quad x \in Z(\mathcal{M}), \tag{6.9}$$

where $Z(\mathcal{M})$ is the center of $\mathcal{M}$. Write $x = a + \imath b$ with $a, b \in M$. If $\widetilde{\delta}_x = 0$, then $\delta_x \in Z(D(M))$, so $\delta_x = \delta_z$ for some z in the center of the JBW-algebra M (Lemma 6.5). Then z is also central in the von Neumann algebra $\mathcal{M}$ (Corollary 1.53). Note that the self-adjoint and skew parts of the order derivation δ_x are δ_a and $\delta_{\imath b}$ respectively. Equating the self-adjoint and skew parts of the order derivations δ_x and δ_z, we conclude that $\delta_a = \delta_z$ (so $a = z$), and $\delta_{\imath b} = 0$ (so b is central). Thus $x = a + \imath b$ is central in $\mathcal{M}$, so (6.9) is verified.

Now let $x \in \mathcal{M}$ with $\widetilde{\delta}_x = 0$. Then x is central in $\mathcal{M}$, so we can write $x = a + \imath b$ with a, b self-adjoint elements of the center of $\mathcal{M}$. Then $\delta_{\imath x} = \delta_{\imath a - b} = \delta_{\imath a} - \delta_b$. Thus $\delta_{\imath a} = 0$ and $\delta_b \in Z(D(M))$, so $\widetilde{\delta}_{\imath x} = 0$, which completes the proof that I is well defined. Since every order derivation on $\mathcal{M}$ has the form δ_x for some $x \in \mathcal{M}$ (Theorem 6.2), then the domain of I is all of $\widetilde{D}(M)$.

Now we verify the three properties of Definition 6.8. The property (i) follows at once from the definition of I, and (ii) is an immediate consequence of the identity (6.5). To verify (iii), note that if $a, b \in \mathcal{M}_{\rm sa}$, then δ_a is self-adjoint and $\delta_{\imath b}$ is skew, so

$$(\delta_{a+\imath b})^{\dagger} = (\delta_a + \delta_{\imath b})^{\dagger} = \delta_a - \delta_{\imath b} = \delta_{a-\imath b}.$$

Thus

$$\delta_x^{\dagger} = \delta_{x^*} \quad \text{for all } x \in \mathcal{M}.$$

Now it follows that I satisfies Definition 6.8(iii). □

We will see later (in Corollary 6.19) that a JBW-algebra M is the self-adjoint part of a von Neumann algebra iff M admits a Connes orientation, and that such orientations are in 1-1 correspondence with W*-products on $M + \imath M$.

Dynamical correspondences in JB-algebras

When a C*-algebra or a von Neumann algebra is used as an algebraic model of quantum mechanics, then it is only the self-adjoint part of the algebra that represents observables. However, the self-adjoint part of such an algebra is not closed under the given associative product, but only under the Jordan product. Therefore it has been proposed to model quantum mechanics on Jordan algebras rather than associative algebras [75], [98]. This approach is corroborated by the fact that many physically relevant properties of observables are adequately described by Jordan constructs.

However, it is an important feature of quantum mechanics that the physical variables play a dual role, as observables and as generators of transformation groups. The observables are random variables with a specified probability law in each state of the quantum system, while the generators determine one-parameter groups of transformations of observables (Heisenberg picture) or states (Schrodinger picture).

Both aspects can be adequately dealt with in a C*-algebra (or a von Neumann algebra) $\mathcal{A}$. But unlike the observables whose probability distribution is determined by the functional calculus in $\mathcal{A}_{\mathrm{sa}}$ and thus by Jordan structure, the generators cannot be expressed in terms of the Jordan product. Each self-adjoint element $a = ih$ of a C*-algebra defines a derivation δ_{ih}, and these derivations generate one-parameter groups as in (6.4) which describe the time development of the physical system. But generators are determined not by the symmetrized product $a \circ b = \frac{1}{2}(ab + ba)$ in $\mathcal{A}_{\mathrm{sa}}$, but instead by the antisymmetrized product $a \star b := \frac{i}{2}(ab - ba)$ on $\mathcal{A}_{\mathrm{sa}}$. Therefore both the Jordan product and the Lie product of a C*-algebra are needed for physics, and the decomposition

$$ab = a \circ b - i\,(a \star b) \tag{6.10}$$

separates the two aspects of a physical variable.

It is of interest to find appropriate constructs, defined in terms of the Jordan structure of a JB-algebra (or the geometry of its state space), which makes it the self-adjoint part of a C*-algebra. One such construct is the concept of a *dynamical correspondence* (defined below), which axiomatizes the transition $h \mapsto \delta_{ih}$ from the self-adjoint part of a C*-algebra to the set of skew order derivations on the algebra.

6.10. Definition. A *dynamical correspondence* on a JB-algebra A is a linear map $\psi : a \mapsto \psi_a$ from A into the set of skew order derivations on A which satisfies the requirements

(i) $[\psi_a, \psi_b] = -[\delta_a, \delta_b]$ for $a, b \in A$,
(ii) $\psi_a a = 0$ for all $a \in A$.

A dynamical correspondence on a JB-algebra A will be called *complete* if it maps A onto the set of all skew order derivations on A.

6.11. Proposition. *If A is the self-adjoint part of a C*-algebra or a von Neumann algebra, then the map $a \mapsto \psi_a = \delta_{ia}$ is a dynamical correspondence. In the von Neumann algebra case, this dynamical correspondence is complete.*

Proof. In both cases, property (i) above follows from the identity (6.5), and (ii) is immediate, so ψ is a dynamical correspondence. A skew order derivation on a von Neumann algebra has the form δ_{ia} for some self-adjoint element a (A 184). Thus in a von Neumann algebra the map $a \mapsto \psi_a = \delta_{ia}$ is a complete dynamical correspondence. □

The skew order derivations on a JB-algebra A determine one-parameter groups of Jordan automorphisms of A (Lemma 2.81). Therefore the dual maps determine affine automorphisms of the state space of A (and also of the normal state space in the JBW case), cf. Proposition 5.16. Thus a dynamical correspondence gives the elements of A a double identity, which reflects the dual role of physical variables as observables and as generators of a one-parameter group of motions. Hence the name "dynamical correspondence".

Since the Jordan product is commutative, there is no useful concept of "commutator" for elements in a JB-algebra, but the commutators of the associated Jordan multipliers can be used as a substitute in view of the identity $[\delta_a, \delta_b] = \frac{1}{2}\delta_{[a,b]}$ for elements a, b in a C*-algebra. Thus the condition (i) above is a kind of quantization requirement, relating commutators of elements to the commutators of the associated generators. Note also that the equation $\psi_a(a) = 0$ is equivalent to $\exp(t\psi_a)(a) = a$ for all $t \in \mathbf{R}$. Thus condition (ii) says that the time evolution associated with an observable fixes that observable

We will also state the definition of a dynamical correspondence in another form. In this connection we shall need the following lemma.

6.12. Lemma. *Let A be a JB-algebra and let $a \mapsto \psi_a$ be a map from A into the set of all skew order derivations on A. Then for all pairs $a, b \in A$,*

$$[\psi_a, \delta_b] = \delta_{\psi_a b}. \tag{6.11}$$

Proof. Since ψ_a is skew, it is a Jordan derivation. Hence for all $c \in A$,

$$\psi_a(b \circ c) = (\psi_a b) \circ c + b \circ (\psi_a c),$$

which can be rewritten

$$\psi_a \delta_b c - \delta_b \psi_a c = \delta_{\psi_a b} c.$$

This gives (6.11). □

Note that linearity of ψ is not listed among the requirements in the proposition below, since it follows from the other requirements.

6.13. Proposition. *Let A be a JB-algebra and let $\psi : a \mapsto \psi_a$ be a map from A into the set of skew order derivations of A. Then ψ is a dynamical correspondence iff the following requirements are satisfied for $a, b \in A$:*

(i) $[\psi_a, \psi_b] = -[\delta_a, \delta_b]$, *and*
(ii) $[\psi_a, \delta_b] = [\delta_a, \psi_b]$.

Proof. Assume first that ψ is a dynamical correspondence. Condition (i) above is satisfied as it is the same as condition (i) of the definition of a dynamical correspondence (Definition 6.10).

By condition (ii) of Definition 6.10 and Lemma 6.12, for all $a \in A$

$$[\psi_a, \delta_a] = \delta_{\psi_a a} = \delta_0 = 0,$$

where the last equality follows from linearity of the map $a \mapsto \psi_a$. Therefore, again by linearity of ψ, for all $a, b \in A$,

$$0 = [\psi_{a+b}, \delta_{a+b}] = [\psi_a, \delta_b] + [\psi_b, \delta_a],$$

which gives

$$[\psi_a, \delta_b] = -[\psi_b, \delta_a] = [\delta_a, \psi_b].$$

This proves condition (ii) above.

Assume next that ψ satisfies conditions (i) and (ii) of the proposition. Condition (i) of Definition 6.10 is trivially satisfied. It follows from condition (ii) above that for all $a \in A$,

$$[\psi_a, \delta_a] = [\delta_a, \psi_a] = -[\psi_a, \delta_a],$$

so $[\psi_a, \delta_a] = 0$ for all $a \in A$. By Lemma 6.12, $\delta_{\psi_a a} = [\psi_a, \delta_a]$, so $\delta_{\psi_a a} = 0$. Hence $\psi_a a = \delta_{\psi_a a} 1 = 0$, so condition (ii) of Definition 6.10 is also satisfied.

It remains to show that ψ is linear. By Lemma 6.12 and condition (ii) above, for all $a, b \in A$

$$\delta_{\psi_a b}(1) = [\psi_a, \delta_b](1) = [\delta_a, \psi_b](1) = -[\psi_b, \delta_a](1) = -\delta_{\psi_b a}(1).$$

Hence $\psi_a b = -\psi_b a$. Since $a \mapsto \psi_b a$ is linear, it follows that $a \mapsto \psi_a$ is a linear map from A into $D(A)$. □

In the last paragraph of the proof above we have shown that if ψ is a dynamical correspondence on A, then

$$\psi_a b = -\psi_b a \quad \text{for all } a, b \in A. \tag{6.12}$$

We shall make more use of this equation later.

6.14. Definition. Let A be a JB-algebra. A bilinear associative product $\star$ on the complexified space $A + iA$ is a *Jordan compatible product* if

(i) $\frac{1}{2}(a \star b + b \star a) = a \circ b$ for all $a, b \in A$,

(ii) $(a + \imath b)^* = a - \imath b$ is an involution for $(A + \imath A, \star)$, i.e., $(x \star y)^* = y^* \star x^*$ for all $x, y \in A + \imath A$.

Such a product will also be called a *C^*-product* on $A + \imath A$. If A is a JBW-algebra, then we call such a product a *W^*-product* or a *von Neumann product.*

Remark. If A is a JB-algebra and $\star$ is a Jordan compatible product on $A + \imath A$, then we can define a norm on $A + \imath A$ by $\|x\| = \|x^* x\|^{1/2}$. With this norm, $A + \imath A$ will become a C*-algebra (A 59), justifying our use of the term "C*-product". In principle this norm could depend on the associative product as well as on the norm on A, but in fact it is uniquely determined by the JB-algebra A (A 160). (This follows from Kadison's theorem that a Jordan isomorphism between C*-algebras is an isometry (A 159).) If A is a JBW-algebra and $A + \imath A$ is equipped with a Jordan compatible product, then $A + \imath A$ will be a C*-algebra whose self-adjoint part A is monotone complete with a separating set of normal states, so will be a von Neumann algebra (A 95), justifying the term W*-product (or von Neumann product).

We are now ready to state and prove our main theorem, which relates dynamical correspondences to associative products.

6.15. Theorem. *A JB-algebra A is (Jordan isomorphic to) the self-adjoint part of a C^*-algebra iff there exists a dynamical correspondence on A. In this case there is a 1-1 correspondence of C^*-products on $A + \imath A$ and dynamical correspondences on A. The dynamical correspondence on A associated with a C^*-product $(a, b) \mapsto ab$ is*

$$\psi_a b = \frac{\imath}{2}(ab - ba), \tag{6.13}$$

and the C^*-product on $A + \imath A$ associated with a dynamical correspondence ψ on A is the complex bilinear extension of the product defined on A by

$$ab = a \circ b - \imath\, \psi_a b. \tag{6.14}$$

Proof. Assume first that A admits a dynamical correspondence ψ. By equation (6.12) we can define an anti-symmetric bilinear product $(a, b) \mapsto a \times b$ on A by writing

$$a \times b = \psi_a b \tag{6.15}$$

Next define a bilinear map $(a, b) \mapsto ab$ from the Cartesian product $A \times A$ into $A + \imath A$ (considered as a real linear space) by writing

$$ab = a \circ b - \imath(a \times b). \tag{6.16}$$

This map can be uniquely extended to a bilinear product on $A + \imath A$ (considered as a complex linear space). We will show that this product is associative. By linearity, it suffices to prove the associative law

$$a(cb) = (ac)b \tag{6.17}$$

for $a, b, c \in A$.

Writing out (6.17) by means of (6.16), we get

$$\begin{aligned} & a \circ (c \circ b) - \imath(a \times (c \circ b)) - \imath(a \circ (c \times b)) - a \times (c \times b) \\ &= (a \circ c) \circ b - \imath((a \circ c) \times b) - \imath((a \times c) \circ b) - (a \times c) \times b. \end{aligned}$$

Separating real and imaginary terms (and using the anti-symmetry of the $\times$-product), we get two equations. The first one can be written as

$$a \times (b \times c) - b \times (a \times c) = -a \circ (b \circ c) + b \circ (a \circ c), \tag{6.18}$$

and the second one as

$$a \times (b \circ c) - b \circ (a \times c) = a \circ (b \times c) - b \times (a \circ c). \tag{6.19}$$

The left-hand side of (6.18) is just $[\psi_a, \psi_b](c)$ and the right-hand side of (6.18) is $-[\delta_a, \delta_b](c)$. Similarly the left-hand side of (6.19) is $[\psi_a, \delta_b](c)$ and the right-hand side of (6.19) is $[\delta_a, \psi_b](c)$. Thus these two equations follow directly from the characterization of a dynamical correspondence in Proposition 6.13.

We must also show that the bilinear product on $A + \imath A$ is compatible with the natural involution $(a + \imath b)^* = a - \imath b$ on $A + \imath A$, i.e., that $(xy)^* = y^* x^*$ for $x, y \in A + \imath A$. By linearity it suffices to show that $(ab)^* = ba$ for $a, b \in A$. But this follows directly from the anti-symmetry of the $\times$-product, as

$$(ab)^* = \big(a \circ b - \imath(a \times b)\big)^* = a \circ b + \imath(a \times b) = b \circ a - \imath(b \times a) = ba.$$

We have now shown that $A + \imath A$ is an associative *-algebra.

By the definition of the involution, the self-adjoint part of $A + \imath A$ is A. Thus by (6.16) and the anti-symmetry of the $\times$-product,

$$\tfrac{1}{2}(ab + ba) = a \circ b$$

for all $a, b \in A$. Therefore the associative product in $A + \imath A$ induces the given Jordan product on A, and thus is a C*-product.

Assume next that A is Jordan isomorphic to the self-adjoint part of a C*-algebra $\mathcal{A}$. Extending the isomorphism from A onto $\mathcal{A}_{\mathrm{sa}}$ to a complex linear map from $A + iA$ onto $\mathcal{A}$ and pulling back the associative product in $\mathcal{A}$, we get a C*-product $(a,b) \mapsto ab$ in $A + \imath A$. Defining a map ψ from A into $D(A)$ by equation (6.13), we get a dynamical correspondence on A (cf. the remark following Definition 6.14, and Proposition 6.11). Then we define a new product $\star$ on $A + \imath A$ to be the complex bilinear extension of (6.14). By the definition of ψ, we have $a \star b = a \circ b - \imath \psi_a(b) = ab$ for $a, b \in A$, so the new product is the same as the original.

It remains to prove that if we start out from a dynamical correspondence ψ on A and construct first the C*-product in $A+iA$ via (6.14) as in the first part of the proof, and then from this product the dynamical correspondence given by (6.13) as above, then we will end with the dynamical correspondence with which we began. Let $a, b \in A$. By (6.15), (6.16) and the anti-symmetry of the $\times$-product, we find that

$$\tfrac{1}{2}i(ab - ba) = \tfrac{1}{2}(a \times b - b \times a) = a \times b = \psi_a(b),$$

which completes the proof. □

Note that it follows from equation (6.13) of Theorem 6.15 that a dynamical correspondence ψ on a JB-algebra A can be recovered from the associated C*-product on $A + \imath A$ through the equation

$$\psi_a = \delta_{\imath a} \quad \text{for all } a \in A. \tag{6.20}$$

6.16. Corollary. *A JBW-algebra M is (Jordan isomorphic to) the self-adjoint part of a von Neumann algebra iff there exists a dynamical correspondence on M. In this case the construction in Theorem 6.15 provides a 1-1 correspondence of W*-products on $M + \imath M$ and dynamical correspondences on M.*

Proof. This follows from Theorem 6.15, together with the observation in the remarks preceding Theorem 6.15 showing that if M is a JBW-algebra, then a Jordan compatible product on $M + \imath M$ makes $M + \imath M$ into a von Neumann algebra. □

We will now explain the relationship between Connes orientations and dynamical correspondences, and we begin with the following:

6.17. Lemma. *Let I be a Connes orientation on a JBW-algebra M. If $a \in M$, then there exists a unique skew order derivation δ in $I(\widetilde{\delta}_a)$.*

Proof. Choose $\delta_1 \in I(\widetilde{\delta}_a)$. By Definition 6.8 (iii), since $(\delta_a)^\dagger = \delta_a$,

$$\widetilde{\delta}_1^\dagger = (I(\widetilde{\delta}_a))^\dagger = -I(\widetilde{\delta}_a^\dagger) = -I(\widetilde{\delta}_a) = -\widetilde{\delta}_1.$$

Hence $\widetilde{\delta_1^\dagger} + \widetilde{\delta_1} = \widetilde{0}$, so $\delta_1^\dagger + \delta_1 = \delta_z$ for some $z \in Z(M)$ (Lemma 6.5). Now define $\delta = \delta_1 - \frac{1}{2}\delta_z = \frac{1}{2}(\delta_1 - \delta_1^\dagger)$. Then $\delta^\dagger = -\delta$, so δ is skew. Also

$$\delta = \delta_1 - \tfrac{1}{2}\delta_z \in \widetilde{\delta_1} = I(\widetilde{\delta_a}).$$

If δ' is an arbitrary skew order derivation such that $\delta' \in I(\widetilde{\delta_a})$, then $\delta - \delta'$ is both central and skew. By Lemma 6.5, a central order derivation is self-adjoint. Thus $\delta - \delta'$ is both skew and self-adjoint, so $\delta - \delta' = 0$. Hence δ is the unique skew order derivation in $I(\widetilde{\delta_a})$. □

The concept of a Connes orientation is defined for JBW-algebras, while the concept of a dynamical correspondence is defined for all JB-algebras. However, in the context of JBW-algebras, they are equivalent, as we now show. In this connection we shall need the Kleinecke–Shirokov theorem, which says that if P and Q are bounded operators on a Banach space and their commutator $C = [P, Q]$ commutes with P (or Q), then C is a quasi-nilpotent, i.e., $\|C^n\|^{1/n} \to 0$ when $n \to \infty$ (cf. [63, p. 128]). (In this reference the theorem is stated for Hilbert space operators, but the proof works equally well for operators on a general Banach space.)

6.18. Theorem. *If M is a JBW-algebra, then there is a 1-1 correspondence between Connes orientations on M and dynamical correspondences on M, and any dynamical correspondence on M is complete. If I is a Connes orientation on M, then the associated dynamical correspondence is the map $a \mapsto \psi_a \in D(M)$, where ψ_a is the unique skew adjoint order derivation such that*

$$\psi_a \in I(\widetilde{\delta_a}) \quad \textit{for all } a \in M. \tag{6.21}$$

For each dynamical correspondence ψ, the associated Connes orientation I is given by

$$I(\widetilde{\delta_x}) = \widetilde{\delta_{ix}} \quad \textit{for all } x \in M + iM, \tag{6.22}$$

where δ_x and δ_{ix} are defined with respect to the W^*-product corresponding to ψ (as in (6.14)).

Proof. Let I be a Connes orientation on M. Let $a \in M$. Denote the unique skew order derivation in $I(\widetilde{\delta_a})$ by ψ_a for each $a \in M$. Clearly $\psi : a \mapsto \psi_a$ is a linear map from M into $D(M)$. Let $a, b \in M$. By Definition 6.8 (i),(ii),

$$[I\widetilde{\delta_a}, I\widetilde{\delta_b}] = I[\widetilde{\delta_a}, I\widetilde{\delta_b}] = I^2[\widetilde{\delta_a}, \widetilde{\delta_b}] = -[\widetilde{\delta_a}, \widetilde{\delta_b}].$$

Thus for some $z \in Z(M)$,

$$[\psi_a, \psi_b] = -[\delta_a, \delta_b] + \delta_z.$$

Since ψ_a and ψ_b are skew, then $[\psi_a, \psi_b](1) = 0$. Also $[\delta_a, \delta_b](1) = a \circ b - b \circ a = 0$. Hence $z = \delta_z(1) = 0$. Thus $[\psi_a, \psi_b] = -[\delta_a, \delta_b]$, so ψ satisfies condition (i) of Definition 6.10.

To show that ψ also satisfies condition (ii) of Definition 6.10, we first observe that for all $a \in M$, by Definition 6.8 (ii),

$$[\widetilde{\psi}_a, \widetilde{\delta}_a] = [I\widetilde{\delta}_a, \widetilde{\delta}_a] = I[\widetilde{\delta}_a, \widetilde{\delta}_a] = 0.$$

Therefore there exists $z \in Z(M)$ such that $[\psi_a, \delta_a] = \delta_z$. Thus δ_z is a commutator of two bounded linear operators on M, and it commutes with each of them, so it follows from the Kleinecke–Shirokov Theorem that δ_z is quasi-nilpotent, i.e.,

$$\lim_{n \to \infty} \|\delta_z^n\|^{1/n} = 0.$$

But $\delta_z^n(1) = z^n$ for all n. Since $\|a^2\| = \|a\|^2$ for all $a \in M$, then $\|z^{2^n}\| = \|z\|^{2^n}$, so

$$\|\delta_z^{2^n}\|^{2^{-n}} \geq \|\delta_z^{2^n} 1\|^{2^{-n}} = \|z^{2^n}\|^{2^{-n}} = \|z\|,$$

hence $z = 0$. Thus $[\psi_a, \delta_a] = 0$. Now it follows from Lemma 6.12 that $\delta_{\psi_a a} = 0$, and hence also $\psi_a a = 0$, so we have verified condition (ii) of Definition 6.10. Thus ψ is a dynamical correspondence.

Now let ψ be any dynamical correspondence on M, and let $(a, b) \mapsto ab$ be the associated W*-product on $M + \imath M$ (Corollary 6.16). By Proposition 6.9, there is a Connes orientation I on M given by (6.22). With this we have from each given Connes orientation constructed a dynamical correspondence, and from each given dynamical correspondence constructed a Connes orientation.

To show that these two constructions are inverses, start with a Connes orientation I on M. Let ψ be the dynamical correspondence constructed from I, so that $I(\widetilde{\delta}_a) = \widetilde{\psi}_a$ holds for all $a \in M$. Note also that this equation implies $I(\widetilde{\psi}_a) = I^2(\widetilde{\delta}_a) = -\widetilde{\delta}_a$ for all $a \in M$. From ψ, construct the corresponding W*-product $(a, b) \mapsto ab$ on $M + \imath M$ as in Theorem 6.15, so that $\psi_a = \delta_{ia}$ for all $a \in M$ (cf. (6.20)). Then for $a, b \in M$,

$$I(\widetilde{\delta}_{a+ib}) = I(\widetilde{\delta}_a) + I(\widetilde{\delta}_{ib}) = \widetilde{\psi}_a + I(\widetilde{\psi}_b) = \widetilde{\delta}_{ia} - \widetilde{\delta}_b = \widetilde{\delta}_{ia-b} = \widetilde{\delta}_{i(a+ib)}.$$

Thus we have $I(\widetilde{\delta}_x) = \widetilde{\delta}_{ix}$ for all $x \in M + \imath M$. By (6.22) this shows that I is the same as the Connes orientation constructed from ψ.

Now let ψ be any dynamical correspondence on M, construct the corresponding W*-product $(a, b) \mapsto ab$ on $M + iM$ as in Theorem 6.15, so that $\psi_a = \delta_{ia}$ for all $a \in M$. Let I be the Connes orientation defined by (6.22). Then for $a \in M$, we have $\widetilde{\psi}_a = \widetilde{\delta}_{ia} = I(\widetilde{\delta}_a)$, so ψ satisfies (6.21). Thus ψ coincides with the dynamical orientation constructed from the Connes orientation I.

Finally, we prove that every dynamical correspondence ψ on M is complete. For the W*-product on $M + iM$ associated with ψ, we have $\psi_a = \delta_{ia}$ for all $a \in M + iM$, i.e., ψ is the dynamical correspondence associated with this product. Then ψ is complete (cf. Proposition 6.11). □

6.19. Corollary. *Let M be a JBW-algebra. Then M is Jordan isomorphic to the self-adjoint part of a von Neumann algebra iff M admits a Connes orientation. There is a 1-1 correspondence of Connes orientations on M and W*-products on $M + iM$. The Connes orientation corresponding to the W*-product $(a, b) \mapsto ab$ is given by*

$$I(\widetilde{\delta}_x) = \widetilde{\delta}_{ix} \quad \textit{for all } x \in M + iM.$$

Proof. This follows from Theorem 6.18 and Corollary 6.16. □

Comparing notions of orientation

In [AS], two notions of orientation were defined: one for C*-algebra state spaces (A 146), and the other for von Neumann algebras and their normal state spaces, cf. (A 200) and (A 201). Here we will sketch how these are related to our current notion of a dynamical correspondence (and thus also to Connes orientations.) The relationships are easiest to see for the special example of the C*-algebra $\mathcal{M}$ of 2×2 matrices over $\mathbf{C}$. In this case, the state space is affinely isomorphic to a Euclidean 3-ball (A 119). An orientation in the C* sense consists of an equivalence class of parameterizations, i.e., affine isomorphisms from the standard Euclidean 3-ball onto the state space, with two parameterizations being equivalent if they differ by composition with a rotation. An orientation in the von Neumann sense corresponds to an equivalence class of Cartesian frames for the state space, with two frames being equivalent if one can be rotated into the other. Note in each case there are two equivalence classes; these correspond to the standard multiplication on $\mathcal{M}$, and to the opposite multiplication $(a, b) \mapsto ba$.

Now let s be a symmetry in $\mathcal{M}$. Then s can be viewed as an affine function on the state space, and will take its maximum and minimum values at antipodal points on the boundary of the 3-ball. Let ψ be the dynamical correspondence associated with the standard multiplication on $\mathcal{M}$. The dual of the map $\psi_s = \delta_{is}$ generates a rotation about the axis determined by these two antipodal points. (See Figure 6.1 and the accompanying remarks.) As s varies, the axis of rotation varies, and each map ψ_s determines an orientation of the 3-ball. However, as s varies continuously, these rotations will vary continuously, and thus give the same orientation. If we replace the multiplication on $\mathcal{M}$ by the opposite one, the effect is to replace ψ_s by $-\psi_s$, and so the associated rotation moves in the opposite direction. Thus in all cases, one can think of the orientation as determining a direction of rotation about diameters of this 3-ball, with consistency conditions (e.g., continuity) as the axis varies. The rotations ψ_s are determined by (and determine) the dynamical correspondence $a \mapsto \psi_a$.

Now we sketch how this extends to arbitrary C* and von Neumann algebras. In the case of C*-algebra orientations, each pair of equivalent pure states determine a face of the state space which is a 3-ball, and a requirement of continuity is imposed, so that the orientation of these 3-balls is required to vary continuously with the ball (A 146). In the von Neumann algebra case, there may be no pure normal states, and thus no facial 3-balls. Instead one works with "blown up 3-balls" (A 194), which also will appear later in our characterization of von Neumann algebra normal state spaces. One can describe an orientation of such blown up 3-balls, cf. (A 206), in terms of pairs (s, ψ_s), where s is a partial symmetry, and $\psi_s = \delta_{is}$ is the generator of a "generalized rotation". Thus in all contexts, the notion of orientation can be expressed in terms of maps $s \mapsto \psi_s$ that are related to (generalized) rotations of (blown up) 3-balls, and then one needs to impose consistency conditions, as specified in the various definitions of orientation. Piecing these orientations together consistently is equivalent to choosing the maps ψ_s so that they can be extended to a dynamical correspondence $a \mapsto \psi_a$.

Notes

The description of the order derivations on a von Neumann algebra (Theorem 6.2) is in [36], where Connes introduced the notion of an order derivation. The proof of Theorem 6.2 quickly reduces to the result that derivations on a von Neumann algebra are inner, due to Kadison [79] and Sakai [111].

Part of the history of normed Jordan algebras is tied to homogeneous self-dual cones. A cone A^+ in a vector space A is *self-dual* if there exists an inner product on A such that $a \geq 0$ iff $(a \,|\, b) \geq 0$ for all $b \geq 0$. It is *homogeneous* if the order automorphisms are transitive on the interior of the positive cone. Note that for any JB-algebra A, if $a \in A$ is positive and invertible, then $U_{a^{1/2}}$ is an automorphism of the interior of A^+ taking 1 to a, so the group of affine automorphisms of the interior of A^+ is transitive. Koecher [86] and Vinberg [134], by rather different methods, showed that the finite dimensional homogeneous self-dual cones are exactly the positive cones of formally real Jordan algebras. As we remarked in the notes to Chapter 1, these are exactly the finite dimensional JB-algebras. The book of Faraut and Korányi [52] discusses in detail finite dimensional homogeneous self-dual cones and their connections with Jordan algebras and with symmetric tube domains.

The results in this chapter are related to the general problem of characterizing von Neumann algebras as ordered linear spaces. Sakai [109] characterized the L^2-completion (with respect to a trace) of a finite von Neumann algebra by means of self-duality of its positive cone and existence of an "absolute value", which models the left absolute value $a \mapsto (a^*a)^{1/2}$. Connes, in his order-theoretic characterization of σ-finite von Neumann algebras [36], showed that these algebras are in 1-1 correspondence with cones that are self-dual, "homogeneous", and have an "orientation". (The latter two are new concepts defined in Connes' paper.) Later on, Bellissard and Iochum showed that the σ-finite JBW-algebras are in 1-1 correspondence with the cones that have the first two of Connes' three properties, i.e., are self-dual and homogeneous [27]. Such cones were studied in greater detail by Iochum [68], who introduced the name "facial homogeneity" for Connes' concept. As remarked at the end of Chapter 5, facial homogeneity is closely related to ellipticity. Homogeneity (in the sense of transitivity of the automorphism group on the interior of the positive cone) is equivalent to facial homogeneity in finite dimensions as shown in the paper [26] by Bellissard, Iochum, and Lima.

Once one has Jordan structure, something must be added to get the C*-structure, since the Jordan product does not determine the associative multiplication. Connes' definition of orientation is one way to proceed; dynamical correspondences are another approach, closely related to Connes', and in Chapter 11 we will discuss yet a third (more geometric) approach. The results in this chapter are taken from [12] and [13]; the latter also discusses the relationship between the concepts of orientation in Connes' paper [36] and that of the authors in [10]. The notion of a "Connes orientation" (Definition 6.8) carries Connes' definition of an orientation from the natural cone of a JBW-algebra to the algebra itself. For JBW-algebras, the concept of a dynamical correspondence is closely related to that of a "Connes orientation", as illustrated by Theorem 6.18. However, the notion of a dynamical correspondence also makes sense in the broader context of a general JB-algebra.

Grgin and Petersen [58] axiomatize an observable-generator duality in terms of what they call a Hamilton algebra, which is a linear space H equipped with two bilinear operations: a product τ and a Lie product α, satisfying certain conditions. Emch [49] discussed "Jordan-Lie" algebras, in which both a Jordan product and a Lie product are given, satisfying certain conditions relating the two products. There is also a discussion of such algebras in the book of Landsman [91]. In each case the focus is on deformations of the algebraic structure that approach a kind of classical limit as a parameter approaches zero.

PART II

Convexity and Spectral Theory

7 General Compressions

We will now approach the theory of state spaces from a new angle, starting with geometric axioms which are at first very general, but which will later be strengthened to characterize the state spaces of Jordan and C*-algebras. The first part of this program is to establish a satisfactory spectral theory and functional calculus. Here the guiding idea is to replace projections p by "projective units" $p = P1$ determined by "general compressions" P defined by properties similar to those characterizing the compressions U_p in (A 116) and Theorem 2.83.

The first section of this chapter presents basic results on projections in cones. The second section defines general compressions. The third section contains results on projective units and projective faces. The fourth and last section gives a geometric characterization of projective faces together with some concrete examples in low dimension.

Projections in cones

We will first work in a very general setting, assuming only that we have a pair X, Y of two positively generated ordered vector spaces in separating ordered duality under a bilinear form $\langle \cdot\,, \cdot \rangle$ (A 3). (An ordered vector space X is said to be positively generated if $X = X^+ - X^+$.) Unless otherwise stated we will use words like "continuous", "closed", etc. with reference to the weak topologies defined by the given duality.

7.1. Definition. If F is a subset of a convex set $C \subset X$, then the intersection of all closed supporting hyperplanes of C which contain F will be called the *tangent space* of C at F, and it will be denoted by $\mathrm{Tan}_C F$, or simply by $\mathrm{Tan}\, F$ when there is no need to specify C. If there is no closed supporting hyperplane of C which contains F, then $\mathrm{Tan}_C F = X$ by convention.

Clearly a tangent space of C is a supporting (affine) subspace, but a supporting subspace is not a tangent space in general. In Fig. 7.1 we have shown one supporting subspace which is a tangent space and one which is not.

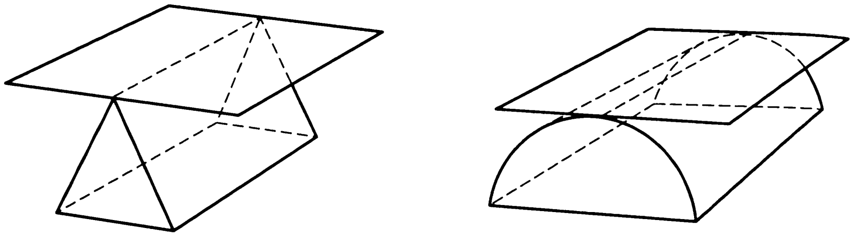

Fig. 7.1

By (A 1) a face F of a convex set $C \subset X$ is *semi-exposed* if it is the intersection of C and a collection of closed supporting hyperplanes containing F. Thus F is semi-exposed iff

$$F = C \cap \mathrm{Tan}_C F\,. \tag{7.1}$$

We will use the symbol B° to denote the *annihilator* of a subset B of X in the space Y. We will now also use the symbol $B^\bullet$ to denote the *positive annihilator* of B. Thus

$$B^\bullet = \{y \in Y^+ \mid \langle x, y\rangle = 0 \ \text{ for all } \ x \in B\}. \tag{7.2}$$

Similarly we use the same notation with $B \subset Y$ and X, Y interchanged.

For each $x \in B \subset X^+$, the set $x^{-1}(0) = \{y \in Y \mid \langle x, y\rangle = 0\}$ is a closed supporting subspace of Y^+, so the positive annihilator

$$B^\bullet = Y^+ \cap \bigcap_{x \in B} x^{-1}(0)$$

is a semi-exposed face of Y^+.

We also make the following observation, which we state as a proposition for later reference.

7.2. Proposition. *If X, Y are ordered vector spaces in separating ordered duality and $B \subset X^+$, then $B^{\bullet\bullet}$ is the semi-exposed face generated by B in the cone X^+ and $B^{\bullet\circ}$ is the tangent space to X^+ at B.*

Proof The closed supporting hyperplanes of X^+ are precisely the sets of the form $y^{-1}(0)$ for $y \in Y^+$. Thus if $B \subset X^+$, then

$$B^{\bullet\bullet} = X^+ \cap \Big(\bigcap_{y \in B^\bullet} y^{-1}(0)\Big) \quad \text{and} \quad B^{\bullet\circ} = \bigcap_{y \in B^\bullet} y^{-1}(0).$$

The intersections above are extended over all supporting hyperplanes of X^+ which contain B, so the proposition follows from the definition of a semi-exposed face and of a tangent space. $\square$

If $P : X \to X$ is a *positive projection*, i.e., if P is a linear map such that $P(X^+) \subset X^+$ and $P^2 = P$, then we will use the notation $\ker^+ P = X^+ \cap \ker P$ and $\mathrm{im}^+ P = X^+ \cap \ \mathrm{im}\, P$. Now it follows from the simple characterization (A 2) of faces in cones, and from the assumption that X is positively generated, that if $P : X \to X$ is a positive projection, then $\ker^+ P$ is a face of X^+ and $\mathrm{im}\, P$ is a positively generated linear subspace of X. We will now study the tangent spaces of the cone X^+ at $\ker^+ P$ and $\mathrm{im}^+ P$, and for simplicity we will omit the subscript X^+ and just write $\mathrm{Tan}(\ker^+ P)$ and $\mathrm{Tan}(\mathrm{im}^+ P)$.

If P is a continuous positive projection on X, then we denote by P^* the (continuous) *dual projection* on Y, defined by

$$\langle Px, y\rangle = \langle x, P^* y\rangle \quad \text{when } x \in X \text{ and } y \in Y. \tag{7.3}$$

Clearly X and Y may be interchanged in all the statements above.

7.3. Proposition. *Let X, Y be positively generated ordered vector spaces in separating ordered duality. If P is a continuous positive projection on X, then*

$$\mathrm{Tan}(\ker^+ P) = (\ker^+ P)^{\bullet\circ} \subset \ker P. \tag{7.4}$$

Proof. By Proposition 7.2,

$$\mathrm{Tan}(\ker^+ P) = (\ker^+ P)^{\bullet\circ} \subset (\ker P)^{\bullet\circ}.$$

Since $(\ker P)^\circ = \mathrm{im} P^*$ and $\mathrm{im}\, P^*$ is positively generated,

$$(\ker P)^\circ = \mathrm{im}^+ P^* - \mathrm{im}^+ P^* = (\ker P)^\bullet - (\ker P)^\bullet.$$

Hence $(\ker P)^{\bullet\circ} = (\ker P)^{\circ\circ} = \ker P$, which completes the proof. □

7.4. Corollary. *Let X, Y be positively generated ordered vector spaces in separating ordered duality. If P is a continuous positive projection on X, then $\ker^+ P$ is a semi-exposed face of X^+.*

Proof. Clearly (7.4) implies $X^+ \cap \mathrm{Tan}(\ker^+ P) \subset \ker^+ P$. The opposite inclusion is trivial, so (7.1) holds with $\ker^+ P$ in place of F. Thus $\ker^+ P$ is a semi-exposed face of X^+. □

The inclusion opposite to the one in (7.4) is not valid in general, but projections for which this is the case form an important class containing the compressions U_p in the case of C*-algebras and JB-algebras (as we will show in Theorem 7.12).

7.5. Definition. Let X, Y be positively generated ordered vector spaces in separating ordered duality. A continuous positive projection P on X will be called *smooth* if $\ker P$ is equal to the tangent space of X^+ at $\ker^+ P$, i.e., if

$$\mathrm{Tan}(\ker^+ P) = \ker P. \tag{7.5}$$

Note that the non-trivial part of the equality (7.5) is the inclusion $\ker P \subset \mathrm{Tan}(\ker^+ P)$, which by Proposition 7.2 is equivalent to

$$\ker P \subset (\ker^+ P)^{\bullet\circ}. \tag{7.6}$$

We will now show that a smooth projection P is determined by $\ker^+ P$ and $\mathrm{im}^+ P$. (This is not the case for general positive projections. A simple example of a non-smooth projection is the following. Let $X = Y = \mathbf{R}^3$ with the standard ordering, and let $P : X \to X$ be the orthogonal projection onto the (horizontal) line consisting of all points (x_1, x_2, x_3) such that $x_1 = x_2$ and $x_3 = 0$. Then $\ker P$ is the (vertical) plane consisting of all points such that $x_1 = -x_2$, but the tangent space to $\ker^+ P$ is only the (vertical) line consisting of all points such that $x_1 = x_2 = 0$.)

7.6. Proposition. *Let X, Y be positively generated ordered vector spaces in separating ordered duality. If P is a smooth projection on X and R is another continuous positive projection on X such that* $\mathrm{im}^+ R = \mathrm{im}^+ P$ *and* $\ker^+ R = \ker^+ P$, *then* $R = P$.

Proof. By (7.5) and (7.4) (the latter with R in place of P),

$$\ker P = \mathrm{Tan}(\ker^+ P) = \mathrm{Tan}(\ker^+ R) \subset \ker R.$$

Since $\mathrm{im}^+ P = \mathrm{im}^+ R$, and the sets $\mathrm{im}\, P$ and $\mathrm{im}\, R$ are both positively generated, $\mathrm{im}\, P = \mathrm{im}\, R$. From the two relations $\ker P \subset \ker R$ and $\mathrm{im}\, P = \mathrm{im}\, R$, we conclude that $P = R$. □

Smoothness of P will also imply that $\ker^+ P$ and $\mathrm{im}^+ P$ dualize properly under positive annihilators. The precise meaning of this statement is explained in our next proposition.

7.7. Proposition. *Let X, Y be positively generated ordered vector spaces in separating ordered duality. Let P be a continuous positive projection on X and consider the equations*

$$\mathrm{im}^+ P)^\bullet = \ker^+ P^*, \tag{7.7}$$

$$(\ker^+ P)^\bullet \supset \mathrm{im}^+ P^*, \tag{7.8}$$

$$(\ker^+ P)^\bullet = \mathrm{im}^+ P^*. \tag{7.9}$$

Of these, (7.7) and (7.8) hold generally, and (7.9) holds iff P is smooth.

Proof. Since $\operatorname{im} P$ is positively generated, $(\operatorname{im}^+ P)^\circ = (\operatorname{im} P)^\circ = \ker P^*$, from which (7.7) follows.

Generally

$$\operatorname{im} P^* = (\ker P)^\circ \subset (\ker^+ P)^\circ,$$

so $\operatorname{im}^+ P^* \subset (\ker^+ P)^\bullet$, which proves (7.8).

To prove the last statement of the proposition, we assume that P is smooth. By (7.5), $(\ker^+ P)^{\bullet\circ} = \ker P$. The set $(\ker^+ P)^\bullet$ is trivially contained in its bi-annihilator, so

$$(\ker^+ P)^\bullet \subset (\ker P)^\circ = \operatorname{im} P^*.$$

Hence $(\ker^+ P)^\bullet \subset \operatorname{im}^+ P^*$. The opposite relation is (7.8), so the equation (7.9) is satisfied.

Finally, (7.9) implies (7.5), so the equation (7.9) is only satisfied if P is smooth. □

We saw in Corollary 7.4 that $\ker^+ P$ was a semi-exposed face of X^+ for every continuous positive projection P on X. The corresponding statement for $\operatorname{im}^+ P$ is false. In fact, $\operatorname{im}^+ P$ need not be a face of X^+, and if it is a face, it need not be semi-exposed. Examples in which $\operatorname{im}^+ P$ is not a face of X^+ are easily found already in $\mathbf{R}^2$, where it suffices to consider the orthogonal projection P onto a line through the origin and some interior point of the positive cone X^+. An example in which $\operatorname{im}^+ P$ is a non-semi-exposed face of the cone X^+, can be obtained by ordering $\mathbf{R}^3$ by a positive cone X^+ generated by a convex base with a non-exposed extreme point. A standard example of a convex set with non-exposed extreme points is a square with two circular half-disks attached at opposite edges. (Such a set may be thought of as a sports stadium with a soccer field inside the running tracks. Then the non-exposed extreme points will be located at the four corner flags of the soccer field). Now the orthogonal projection P onto the line through the origin and such a non-exposed extreme point of the base will give the desired example.

It turns out that smoothness of P^* is a necessary and sufficient condition for $\operatorname{im}^+ P$ to be a semi-exposed face of X^+, as we will now show.

7.8. Corollary. *Let X, Y be positively generated ordered vector spaces in separating ordered duality, and let P be a continuous positive projection on X. The dual projection P^* is smooth iff the cone $\operatorname{im}^+ P$ is a semi-exposed face of X^+.*

Proof. By Proposition 7.2, $\operatorname{im}^+ P$ is a semi-exposed face of X^+ iff

$$\operatorname{im}^+ P = (\operatorname{im}^+ P)^{\bullet\bullet},$$

and by (7.7) this is equivalent to

$$\mathrm{im}^+ P = (\ker^+ P^*)^\bullet.$$

But this equality is (7.9) with P^* in place of P, which is satisfied iff P^* is smooth. □

Recall from (A 4) that two continuous positive projections P, Q on X (or on Y) are said to be *complementary* (and Q is said to be a *complement* of P and vice versa) if

$$\ker^+ Q = \mathrm{im}^+ P, \qquad \ker^+ P = \mathrm{im}^+ Q. \tag{7.10}$$

Note that since X is positively generated, (7.10) implies that $PQ = QP = 0$. Recall also that the two projections P, Q are said to be *strongly complementary* if (7.10) holds with the plus signs removed, which is equivalent to

$$PQ = QP = 0, \qquad P + Q = I. \tag{7.11}$$

Thus P admits a strong complement iff $I - P \geq 0$.

A continuous positive projection P on X (or on Y) is to be *complemented* if there exists a continuous positive projection Q on X such that P, Q are complementary, in which case Q is said to be *a complement* of P. Also P is said to be *bicomplemented* if there exists a continuous positive projection Q on X such that P, Q are complementary and P^*, Q^* are also complementary.

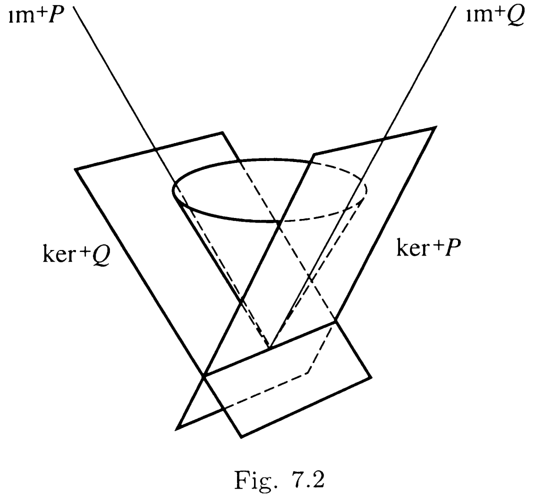

Fig. 7.2

A pair of projections which are complementary, but not strongly complementary, will satisfy the first, but not the second, of the two equalities in (7.11). An example of such a pair is shown in Fig. 7.2 where $X = \mathbf{R}^3$ and X^+ is a circular cone.

7.9. Proposition. *Let X, Y be positively generated ordered vector spaces in separating ordered duality, and let P be a continuous positive projection on X (or on Y). If P is complemented, then the dual projection P^* is smooth, and if P has a smooth complement Q, then Q is the unique complement of P.*

Proof. If P has a complement Q, then the cone $\mathrm{im}^+P = \ker^+Q$ is a semi-exposed face of X^+ (Corollary 7.4), so the dual projection P^* must be smooth (Corollary 7.8).

If P has a smooth complement Q, then an arbitrary complement R of P must satisfy $\ker^+R = \mathrm{im}^+P = \ker^+Q$ and $\mathrm{im}^+R = \ker^+P = \mathrm{im}^+Q$. By Proposition 7.6 this implies $R = Q$. Thus, Q is the unique complement of P. □

Complementary projections are not unique in general. One can obtain an example of a positive projection with infinitely many complements by replacing the circular base of the cone in Fig. 7.2 by the convex set in Fig 7.3 which has a vertex at $\mathrm{im}^+P = \ker^+Q$ and also at $\mathrm{im}^+Q = \ker^+P$. Then $\ker^+Q$ will be a sharp edge of the cone X^+, and by tilting the plane $\ker^+Q$ about this edge we obtain infinitely many positive projections which are all complements of P. Similarly with P and Q interchanged. (We leave it as an exercise to picture the dual cone $\{y \in X^* \mid \langle x, y \rangle \geq 0\}$ with the subcones $\ker^+P^*$, im^+P^*, $\ker^+Q^*$, im^+Q^*, and to show that in this example P, Q are complementary but not bicomplementary.)

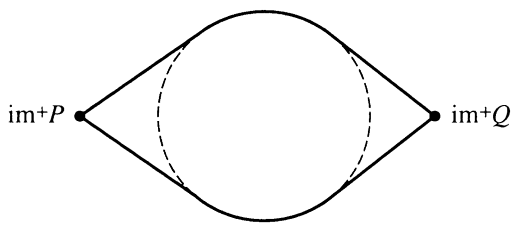

Fig. 7.3

The theorem below is a geometric characterization of bicomplementarity in terms of smoothness.

7.10. Theorem. *Let X, Y be positively generated ordered vector spaces in separating ordered duality. If P and Q are two continuous positive projections on X, then the following are equivalent:*

(i) *P, Q are complementary smooth projections,*
(ii) *P^*, Q^* are complementary smooth projections,*
(iii) *P, Q are bicomplementary projections.*

Proof. It suffices to prove (i)$\Leftrightarrow$(iii) since P^*, Q^* are bicomplementary iff P, Q are bicomplementary.

(i)$\Rightarrow$(iii) Assuming (i) and using Proposition 7.7, we get

$$\ker^+ P^* = (\operatorname{im}^+ P)^\bullet = (\ker^+ Q)^\bullet = \operatorname{im}^+ Q^*.$$

Interchanging P and Q, we also get $\ker^+ Q^* = \operatorname{im}^+ P^*$. Hence we have (iii).

(iii)$\Rightarrow$(i) Assuming (iii) we have $\operatorname{im}^+ P = \ker^+ Q$. By Corollary 7.4 $\ker^+ Q^*$ is semi-exposed, therefore $\operatorname{im}^+ P^*$ is semi-exposed. Now it follows from Corollary 7.8 (with the roles of P and P^* interchanged) that P is smooth. Similarly, Q is smooth. Hence we have (i). $\square$

7.11. Corollary. *Let X, Y be positively generated ordered vector spaces in separating ordered duality. If P is a bicomplemented continuous positive projection on X, then P has a unique complement.*

Proof. By Theorem 7.10, P has a smooth complement Q, and by Proposition 7.9, Q is the unique complement of P. $\square$

Our next result, concerning C*-algebras and JB-algebras, can be found in (A 116) and Theorem 2.83. But it is actually only the easy argument in the first part of the proofs of these results that is needed, so for the reader's convenience we will give this argument.

7.12. Theorem. *Let X be the self-adjoint part of a C*-algebra (or von Neumann algebra) and let Y be the self-adjoint part of its dual (respectively its predual). Then the compression U_p determined by a projection $p \in X$ is a smooth projection on X, the dual map U_p^* is a smooth projection on Y, and the two maps U_p and U_q where $q = p'(= 1 - p)$ are bicomplementary projections on X. The same is true when X is a JB-algebra (or JBW-algebra) and Y is its dual (respectively its predual). Thus in all these cases U_p and U_q are bicomplementary projections.*

Proof. Let p be a projection in the C*-algebra X. We will first show that U_p is a smooth projection on X by proving (7.6). Thus we must show that if $a \in \ker U_p$, then $\omega(a) = 0$ for each $\omega \in (\ker^+ U_p)^\bullet$.

Let $a \in \ker U_p$, i.e., $pap = 0$, and $\omega \in (\ker^+ U_p)^\bullet$, i.e., $\omega \geq 0$ and $\omega = 0$ on $\ker^+ U_p$. Clearly the projection $q = p'$ is in $\ker^+ U_p$, so $\omega(q) = 0$. By (A 74) ((iii)$\Rightarrow$(vi)), $\omega = p \cdot \omega \cdot p$. Thus $\omega(a) = \omega(pap) = 0$ as desired.

Next we observe that by (A 73) ((iii)$\Leftrightarrow$(iv)), $\ker^+ U_q = \operatorname{im}^+ U_p$, and likewise with p and q interchanged. Thus U_p and U_q are complementary smooth projections.

Thus the statement (i) of Theorem 7.10 holds when $P = U_p$ and $Q = U_q$. Then the statement (ii) also holds, so U_p^* and U_q^* are complementary smooth projections. This completes the proof in the C*-algebra case. (The same proof works in the von Neumann algebra case, now with ω a normal linear functional).

In the JB-algebra (and JBW-algebra) cases we can use the same argument with reference to Proposition 1.41 instead of (A 74), and Proposition 1.38 instead of (A 73). □

We will now establish a direct sum decomposition which generalizes the Pierce decomposition of Jordan algebras [67, §2.6], and we begin by a lemma which will also be needed later. Here we shall need the concept of a *direct sum* of two wedges in a linear space. Recall that a *wedge* W is a (not necessarily proper) cone, so it is defined by the relations $W + W \subset W$ and $\lambda W \subset W$ for all $\lambda \geq 0$ (cf. [AS, p. 2]). We will say a wedge W is the direct sum of two subwedges W_1 and W_2 iff each element of W can be uniquely decomposed as a sum of an element of W_1 and an element of W_2. We will use the notation $\oplus_w$ for direct sums of wedges (reserving $\oplus$ for direct sums of subspaces).

7.13. Lemma. *Let X, Y be positively generated ordered vector spaces in separating ordered duality. If F and G are subcones of X^+ such that*

$$X^+ \subset F \oplus_w \operatorname{Tan} G, \quad X^+ \subset G \oplus_w \operatorname{Tan} F, \tag{7.12}$$

and $N := \operatorname{Tan} F \cap \operatorname{Tan} G$, then

$$\begin{gathered} \operatorname{Tan} F = \operatorname{lin} F \oplus N, \quad \operatorname{Tan} G = \operatorname{lin} G \oplus N, \\ X = \operatorname{lin} F \oplus \operatorname{lin} G \oplus N. \end{gathered} \tag{7.13}$$

Proof. Clearly

$$\operatorname{lin} F + N \subset \operatorname{Tan} F + N \subset \operatorname{Tan} F.$$

To prove the converse relation, we consider an element $x \in \operatorname{Tan} F$. Since X is positively generated, $x = x_1 - x_2$ where $x_1, x_2 \in X^+$. By (7.12) we also have decompositions $x_i = a_i + b_i$ where $a_i \in F$ and $b_i \in \operatorname{Tan} G$ for $i = 1, 2$. Now $a_1 - a_2 \in \operatorname{lin} F$ and $b_1 - b_2 \in \operatorname{Tan} G$. But we also have

$$b_1 - b_2 = x - (a_1 - a_2) \in \operatorname{Tan} F,$$

so $b_1 - b_2 \in N$. Now

$$x = (a_1 - a_2) + (b_1 - b_2) \in \operatorname{lin} F + N.$$

Thus $\operatorname{Tan} F = \operatorname{lin} F + N$. Similarly $\operatorname{Tan} G = \operatorname{lin} G + N$.

Observe now that $\operatorname{lin} F \cap \operatorname{Tan}(G) = \{0\}$. In fact, if $x \in \operatorname{lin} F \cap \operatorname{Tan}(G)$, then $x = x_1 - x_2$ where $x_1, x_2 \in F$, and the equation $x_2 + x = x_1 + 0$ provides two decompositions of the element x_1 of X^+ as a sum of an element of F and an element of $\operatorname{Tan}(G)$, so $x_2 = x_1$ and $x = 0$. Similarly $\operatorname{lin} G \cap \operatorname{Tan}(F) = \{0\}$.

Now also $\operatorname{lin} F \cap N = \{0\}$ and $\operatorname{lin} G \cap N = \{0\}$, so $\operatorname{Tan} F = \operatorname{lin} F \oplus N$ and $\operatorname{Tan} G = \operatorname{lin} G \oplus N$. Since X is positively generated, it follows from (7.12) that $X = \operatorname{lin} F + \operatorname{Tan}(G)$. In fact, $X = \operatorname{lin} F \oplus \operatorname{Tan} G$, and then also $X = \operatorname{lin} F \oplus \operatorname{lin} G \oplus N$. □

7.14. Theorem. *Let X, Y be positively generated ordered vector spaces in separating ordered duality, let P, Q be a pair of bicomplementary continuous positive projections on X, and set $F = \operatorname{im}^+ P$, $G = \operatorname{im}^+ Q$ and $N = \operatorname{Tan} F \cap \operatorname{Tan} G$. Then (7.12) holds, and P and Q are the projections onto $\operatorname{lin} F$ and $\operatorname{lin} G$ determined by the direct sum*

$$X = \operatorname{lin} F \oplus \operatorname{lin} G \oplus N. \tag{7.14}$$

Thus $P + Q$ is a projection with kernel N onto the subspace $\operatorname{lin} F \oplus \operatorname{lin} G$.

Proof. By Theorem 7.10, P is a smooth projection. Hence

$$\ker P = \operatorname{Tan}(\ker^+ P) = \operatorname{Tan}(\operatorname{im}^+ Q). \tag{7.15}$$

Let x be an element of X^+. Then we can decompose x as the sum of the element Px in the cone $\operatorname{im}^+ P$ and the element $x - Px$ in the subspace $\ker P = \operatorname{Tan}(\operatorname{im}^+ Q)$. If $x = x_1 + x_2$ is an arbitrary decomposition with $x_1 \in \operatorname{im}^+ P$ and $x_2 \in \operatorname{Tan}(\operatorname{im}^+ Q)$, then $x_2 \in \ker P$, so $Px = x_1$. Thus the decomposition is unique. With this we have shown the first inclusion in (7.12). The second inclusion follows by interchanging P and Q.

Since $\operatorname{im} P$ is positively generated, $\operatorname{im} P = F - F = \operatorname{lin} F$, and by (7.15) and (7.13) $\ker P = \operatorname{Tan} G = \operatorname{lin} G \oplus N$. Thus P is the projection onto $\operatorname{lin} F$ determined by the direct sum in (7.14). Similarly Q is the projection onto $\operatorname{lin} G$ determined by this direct sum. This completes the proof. □

We will now generalize the uniqueness theorem for the conditional expectations $E = U_p + U_{p'}$ in a von Neumann algebra (A 117) to the geometric context of bicomplementary projections in cones, which by Theorem 7.12 includes compressions in general C*-algebras, von Neumann algebras, JB-algebras and JBW-algebras.

7.15. Theorem. *If X, Y are positively generated ordered vector spaces in separating ordered duality and P, Q are bicomplementary continuous positive projections on X, then the map $E = P + Q$ is the unique continuous positive projection onto the subspace* $\operatorname{im}(P+Q) = \operatorname{im} P \oplus \operatorname{im} Q$.

Proof. Note first that since $PQ = QP = 0$, E is a positive projection onto $\operatorname{im}(P+Q) = \operatorname{im} P \oplus \operatorname{im} Q$ and $\ker E = \ker P \cap \ker Q$.

To prove the uniqueness, we consider an arbitrary continuous positive projection T onto $\operatorname{im}(P+Q)$. Thus $\operatorname{im} E = \operatorname{im} T$, so $ET = T$. We will prove that we also have $\ker E \subset \ker T$, which will imply $\ker E = \ker T$ and complete the proof.

Observe that it suffices to prove

$$\ker P \subset \ker PT \quad \text{and} \quad \ker Q \subset \ker QT, \tag{7.16}$$

since this will imply

$$\begin{aligned}\ker E = \ker P \cap \ker Q &\subset \ker PT \cap \ker QT \\ &= \ker (PT + QT) = \ker ET = \ker T.\end{aligned}$$

Assume $x \in \ker P$. To show that $x \in \ker PT$, it suffices to show that $\langle PTx, y\rangle = 0$ for all $y \in Y^+$ (since Y is positively generated).

Let $y \in Y^+$ be arbitrary. Note that $T^*P^*y \in (\ker^+ P)^\bullet$. In fact, for each $a \in \ker^+ P = \operatorname{im}^+ Q \subset \operatorname{im} T$,

$$\langle a, T^*P^*y\rangle = \langle Ta, P^*y\rangle = \langle a, P^*y\rangle = \langle Pa, y\rangle = 0.$$

But since P is smooth, $x \in \ker P$ implies that $x \in \operatorname{Tan}(\ker^+ P) = (\ker^+ P)^{\bullet\circ}$, so we get

$$\langle PTx, y\rangle = \langle x, T^*P^*y\rangle = 0$$

as desired. Thus $\ker P \subset \ker PT$. Similarly $\ker Q \subset \ker QT$. With this we have proved (7.16) and have completed the proof of the theorem. □

We will briefly discuss the geometric idea behind the above proof. The key to the argument is smoothness. Since P and Q are smooth projections, the cones $\ker^+ P$ and $\ker^+ Q$ determine the subspaces $\ker P = \operatorname{Tan}(\ker^+ P)$ and $\ker Q = \operatorname{Tan}(\ker^+ Q)$, and then also their intersection, which is shown to be the kernel not only of the projection $E = P + Q$ but of any positive projection onto the subspace $M = \operatorname{im}(P+Q)$.

The fact that the uniqueness of E depends on the smoothness of P and Q, can be easily seen in the example given in Fig. 7.2. Since E is the projection onto M with kernel equal to the intersection of the tangent planes to the cone X^* at the two opposite rays $\ker^+ P$ and $\ker^+ Q$, E must be the orthogonal projection onto M, and this is the only projection onto M which maps X^+ into itself.

The situation is different in the example obtained by replacing the circular cone in Fig. 7.2 by a cone with a base as in Fig. 7.3, having two antipodal vertices. Now we can find infinitely many pairs of (non-smooth) positive projections P, Q with $\ker^+ P$ and $\ker^+ Q$ equal to the rays through these vertices (passing from one such pair to another by tilting the planes $\ker P$ and $\ker Q$ about the rays $\ker^+ P$ and $\ker^+ Q$). Two distinct pairs will determine distinct intersections $\ker P \cap \ker Q$, and then distinct positive projections of X onto M. Thus there is no uniqueness in this case.

Definition of general compressions

We will now specialize the discussion to a pair of an order unit space A and a base norm space V in separating order and norm duality (A 21). Note that order unit spaces and base norm spaces are positively generated, cf. (A 13) and (A 26), and so the previous results in this chapter are applicable. As in the previous section, we will study positive projections that are continuous in the weak topologies defined on A or V by the given duality. But we will no longer omit the words "weak" or "weakly" when referring to these topologies since we now also have to deal with the norm topologies.

We will say that a positive projection on A or V is *normalized* if it is either zero or has norm 1, or which is equivalent, if it has norm ≤ 1. Note that if P is a normalized positive projection on A, then the dual projection P^* is a normalized positive projection on V, and vice versa. Recall from (A 22) that if P is a weakly continuous positive projection on A, then

$$\|P^*\rho\| = \langle P1, \rho \rangle \quad \text{for all } \rho \in V^+. \tag{7.17}$$

We also make two observations, which we state as lemmas for later reference.

7.16. Lemma. *Let A, V be an order unit space and a base norm space in separating order and norm duality. A positive projection P on A is normalized iff $P1 \leq 1$.*

Proof. To prove the "if" part of the lemma, we assume $P1 \leq 1$ and consider $a \in A$ with $\|a\| \leq 1$, i.e., $-1 \leq a \leq 1$, cf. (A 13). Since P is positive, $-P1 \leq Pa \leq P1$; hence $-1 \leq Pa \leq 1$, so $\|Pa\| \leq 1$ as desired.

The "only if" part is trivial, for if $\|P\| \leq 1$, then $\|P1\| \leq 1$, which implies $P1 \leq 1$. □

7.17. Lemma. *Let A, V be an order unit space and a base norm space in separating order and norm duality, and let K be the distinguished* *base of V. If P and Q are weakly continuous positive projections on A, then the following are equivalent:*

(i) $P1 + Q1 = 1$,
(ii) $\|P^*\rho + Q^*\rho\| = \|\rho\|$ *for all* $\rho \in V^+$,
(iii) $(P^* + Q^*)(K) \subset K$.

Proof. Use the fact that $\|\rho\| = \langle 1, \rho \rangle$ for $\rho \in V^+$. □

7.18. Lemma. *Let A, V be an order unit space and a base norm space in separating order and norm duality, and let P, Q be two complementary weakly continuous positive projections on A. If one of the projections P, Q is normalized, then the other one is also normalized and P, Q will satisfy the requirements (i), (ii), (iii) of Lemma 7.17.*

Proof. Assume P is normalized. By Lemma 7.16, $1 - P1 \geq 0$; hence $1 - P1 \in \ker^+ P = \operatorname{im}^+ Q$, so

$$Q(1 - P1) = 1 - P1.$$

But $P1 \in \operatorname{im}^+ P = \ker^+ Q$, so $QP1 = 0$. Therefore the equation above reduces to $Q1 = 1 - P1$, which gives the desired statement (i), and then also (ii) and (iii).

By statement (i), $Q1 \leq 1$, so by Lemma 7.16, Q is normalized. □

We will now give the general definition of a concept which we have already met in the context of JB-algebras (equation (1.57)).

7.19. Definition. Let A, V be an order unit space and a base norm space in separating order and norm duality, and let P be a normalized weakly continuous positive projection on A. We will say that the dual projection P^* is *neutral* if the following implication holds for $\rho \in V^+$:

$$\|P^*\rho\| = \|\rho\| \quad \Rightarrow \quad P^*\rho = \rho. \tag{7.18}$$

Remark. The term "neutral" relates to physical filters which may let through one part of an incident beam and stop another part, for example, optical filters which are transparent for certain colors but not for others. The equation (7.18) represents the situation in which a beam can pass through with intensity undiminished only when the beam is unchanged, or otherwise stated, when the filter is completely neutral to the beam.

7.20. Proposition. *Let A, V be an order unit space and a base norm space in separating order and norm duality, and let P be a normalized weakly continuous positive projection on A. If P^* is a neutral projection,* *then P is smooth. If P is a smooth projection with smooth complement, then P^* is neutral.*

Proof. Assume first that P^* is a neutral projection on V. We will show that P is a smooth projection on A by verifying (7.9).

Let $\rho \in (\ker^+ P)^\bullet$. Since $1 - P1 \in \ker^+ P$, then $\langle 1 - P1, \rho \rangle = 0$. This is equivalent to $\langle 1, \rho \rangle = \langle 1, P^*\rho \rangle$, and in turn to $\|\rho\| = \|P^*\rho\|$. Since P is neutral, we must have $\rho = P^*\rho$. Hence $(\ker^+ P)^\bullet \subset \operatorname{im}^+ P^*$. The opposite inclusion is generally satisfied (cf. (7.8)), so (7.9) is satisfied.

Assume next that P is a smooth projection with a smooth complementary projection Q. We will show that P^* is neutral by verifying (7.18). Let $\rho \in V^+$ with $\|P^*\rho\| = \|\rho\|$, or which is equivalent, $\langle P1, \rho \rangle = \langle 1, \rho \rangle$. Since P is normalized, $P1 + Q1 = 1$ (Lemma 7.18). Hence $\langle Q1, \rho \rangle = 0$, or which is equivalent, $\|Q^*\rho\| = 0$. Thus $Q^*\rho = 0$, i.e., $\rho \in \ker^+ Q^*$. By Theorem 7.10, P^* and Q^* are complementary, so $\rho \in \operatorname{im}^+ P^*$, i.e., $P^*\rho = \rho$ as desired. □

7.21. Proposition. *Let A, V be an order unit space and a base norm space in separating order and norm duality, and let P and Q be two normalized weakly continuous positive projections on A. Then P and Q are bicomplementary iff they are complementary and P^*, Q^* are both neutral.*

Proof. Application of Theorem 7.10 and Proposition 7.20. □

7.22. Definition. If A, V is a pair of an order unit space and a base norm space V in separating order and norm duality, then a bicomplemented normalized weakly-continuous positive projection P on A will be called a *compression*.

Note that by Lemma 7.18, the complement Q of a compression P is normalized, so it is also a compression. Note also that by our previous discussion, a normalized weakly continuous positive projection P is a compression iff there is another weakly continuous positive projection Q such that any one of the three conditions in Theorem 7.10 are satisfied, or there is a normalized weakly continuous positive projection Q such that the condition in Proposition 7.21 is satisfied.

7.23. Proposition. *Let A be the self-adjoint part of a von Neumann algebra and let V be the self-adjoint part of its predual, or more generally, let A be a JBW-algebra and let V be its predual. Then A, V is a pair of an order unit space and a base norm space in separating order and norm duality, and the (abstract) compressions defined by Definition 7.22 are precisely the (concrete) compressions U_p associated with projections $p \in A$.*

Proof. In the von Neumann algebra case, the duality statement for A and V follows from (A 69), (A 94) and (A 96), and the characterization of the abstract compressions follows from (A 116). In the JBW-algebra case, the duality statement for A and V follows from Theorem 1.11 and Corollary 2.60, and the characterization of the abstract compressions follows from Theorem 2.83. □

The following description of compressions will be used later in some physical remarks.

7.24. Proposition. *Let A, V be a pair of an order unit and base norm space in separating order and norm duality. Let $\Phi : V \to V$ and $\Phi' : V \to V$ be weakly continuous positive linear projections with dual maps $P = \Phi^* : A \to A$ and $P' = \Phi'^* : A \to A$. Then P, P' are complementary compressions iff*

(i) $\|\Phi\omega\| + \|\Phi'\omega\| = \|\omega\|$ *for all* $\omega \in V^+$,
(ii) Φ *and* Φ' *are neutral,*
(iii) $\ker^+ P \subset \operatorname{im}^+ P'$ *and* $\ker^+ P' \subset \operatorname{im}^+ P$.

Proof. If P and P' are complementary compressions, then (i) follows from Lemma 7.18, (ii) follows from Proposition 7.21, and (iii) follows from the definition of complementary compressions.

Conversely, assume (i), (ii), (iii) hold. By (i) we have $\|\Phi\omega\| \leq 1$ for $\omega \in K$, where K is the distinguished base of V. Since the unit ball of V is $\operatorname{co}(K \cup -K)$, it follows that $\|\Phi\| \leq 1$, and similarly $\|\Phi'\| \leq 1$. Since A, V are in separating order and norm duality, then $\|P\| = \|\Phi^*\| \leq 1$, and similarly $\|P'\| \leq 1$, so P and P' are normalized. We next show that for $\omega \in V^+$,

$$\Phi\omega = \omega \iff \Phi'\omega = 0. \tag{7.19}$$

If $\Phi\omega = \omega$, then by (i), $\|\Phi'\omega\| = 0$, so $\Phi'\omega = 0$. If $\Phi'\omega = 0$, then $\|\Phi\omega\| = \|\omega\|$, so by (ii), $\Phi\omega = \omega$, which completes the proof of (7.19). Similarly the same conclusion with Φ and Φ' interchanged holds. Thus Φ and Φ' are complementary projections.

Finally we show that P and P' are complementary, i.e., that equality holds in (iii). Since by assumption Φ is a projection, then for $\omega \in V^+$, $\Phi(\Phi\omega) = \Phi\omega$, so $\Phi'(\Phi\omega) = 0$ (by (7.19), applied with $\Phi\omega$ in place of ω). Thus $\Phi'\Phi = 0$, and similarly $\Phi\Phi' = 0$. Dualizing gives $PP' = P'P = 0$. Therefore if $P'a = a$, applying P to both sides of this equality gives $Pa = 0$, proving that $\operatorname{im}^+ P' \subset \ker^+ P$. Similarly, $\operatorname{im}^+ P \subset \ker^+ P'$. The reverse inclusions follow from condition (iii) of this proposition, which completes the proof that P, P' are complementary compressions. □

If A is an order unit space in separating order and norm duality with a base norm space V with distinguished base

$$K = \{\, \rho \in V^+ \mid \|\rho\| = \langle 1, \rho \rangle = 1 \,\}, \tag{7.20}$$

then the map $a \mapsto \widehat{a}$ (where $\widehat{a}(\rho) = \langle a, \rho \rangle$ for $\rho \in V$) is an order and norm preserving isomorphism from A onto a point-separating subspace of the space $V^* = A_b(K)$ of all bounded affine functions on K, equipped with the pointwise ordering and the uniform norm (A 11). For simplicity we will identify elements a of A with their representing affine functions $\widehat{a}$ on K. Thus if $a, b \in K$ and $F \subset K$, then we will write $a \geq b$ on F instead of $\langle a, \rho \rangle \geq \langle b, \rho \rangle$ for all $\rho \in K$, etc. As before we wili use the notation A_1^+ for the positive part of the norm closed unit ball of A.

7.25. Lemma. *Let A, V be an order unit space and a base norm space in separating order and norm duality. If P is a compression on A, then $P1$ is the greatest element of A_1^+ which vanishes on $\ker^+ P^*$, and*

$$\operatorname{im}^+ P^* = \{ \rho \in V^+ \mid \langle P1, \rho \rangle = \langle 1, \rho \rangle \}. \tag{7.21}$$

Proof. Let P be a compression on A. If $\rho \in \ker^+ P^*$, then $\langle P1, \rho \rangle = \langle 1, P^* \rho \rangle = 0$, so $P1$ vanishes on $\ker^+ P^*$.

Consider now an arbitrary element a of A_1^+ which vanishes on $\ker^+ P^*$. Thus $0 \leq a \leq 1$ and $a \in (\ker^+ P^*)^\bullet$. By equation (7.9) (with P^* in place of P), $a \in \operatorname{im}^+ P$. Hence $a = Pa \leq P1$. Thus $P1$ is the greatest element of A_1^+ which vanishes on $\ker^+ P^*$.

The equation $\langle P1, \rho \rangle = \langle 1, \rho \rangle$ is equivalent to $\|P^* \rho\| = \|\rho\|$, and since P is neutral, this gives (7.21). □

7.26. Proposition. *Let A, V be an order unit space and a base norm space in separating order and norm duality, and let P be a compression on A. Then the two cones $\operatorname{im}^+ P$, $\ker^+ P$ are semi-exposed faces of A^+, the two cones $\operatorname{im}^+ P^*$, $\ker^+ P^*$ are semi-exposed faces of V^+, and each one of these four cones determines the compression P.*

Proof. The cones in question are semi-exposed by Corollary 7.4 and the existence of a complementary positive projection.

By Lemma 7.25, $\ker^+ P^*$ determines $P1$, and through (7.21) also $\operatorname{im}^+ P^*$. By Proposition 7.6, $\ker^+ P^*$ and $\operatorname{im}^+ P^*$ together determine P^*, and hence also P. With this, we have shown that $\ker^+ P^*$ determines P.

When $\operatorname{im}^+ P^*$ is given, we also know the cone $\ker^+ Q^* = \operatorname{im}^+ P^*$ where Q is the complement of P. Thus it follows from what we have just proved that $\operatorname{im}^+ P^*$ determines Q, and then also its complement P (Corollary 7.11).

It follows from Proposition 7.7 that $\operatorname{im}^+ P$ determines $\ker^+ P^*$ and $\ker^+ P$ determines $\operatorname{im}^+ P^*$. Thus it follows from what we have shown above that each of the two cones $\operatorname{im}^+ P$ and $\ker^+ P$ determines P. □

Remark. Proposition 7.26 fails if complementary compressions are replaced by bicomplementary weakly continuous positive projections. Thus the requirement that the projections be normalized is crucial. To see this, observe that if V^+ is a circular cone, for each pair of distinct faces F, G of V^+ there are bicomplementary projections P, Q with $\mathrm{im}^+ P^* = F$ and $\mathrm{im}^+ Q^* = G$.

Projective units and projective faces

We will continue the study of compressions defined in the context of a separating order and norm duality of an order unit space A and a base norm space V with distinguished base K. In the motivating example of a von Neumann algebra and its predual, each compression is of the form $P = U_p$ where p is a projection in the algebra, uniquely determined by the equation $p = P1$, cf. Proposition 7.23. Starting out from this equation, we will define a class of elements $p \in A$ which will play the same role as the projections p in the algebra, and a class of faces $F \subset K$ which will play the same role as their associated norm closed faces $F_p = K \cap \mathrm{im}\, P^*$ of the normal state space (cf. (A 110)).

7.27. Definition. Let A, V be an order unit space and a base norm space in separating order and norm duality. If P is a compression on A, then the element $p = P1$ of A_1^+ is called its associated *projective unit*, and the face $F = K \cap \mathrm{im}\, P^*$ is called its associated *projective face*.

In the case of von Neumann algebras and JBW-algebras, every norm closed face of the normal state space is projective, cf. (A 109) and Theorem 5.32. This is certainly not true in general. In fact, most convex sets K encountered in elementary geometry have no projective faces at all (other than K and $\emptyset$). For simplexes and smooth strictly convex sets, all faces are projective. (They are split faces in the former case, and just the extreme points in the latter.) Other examples of 3-dimensional convex sets with projective faces, are not so easy to come by. But we will soon give a geometric characterization of projective faces (Theorem 7.52), which we will use to give more examples. (Examples are pictured in Fig. 7.6 and Fig. 7.7, where F and G are complementary projective faces, and in Fig. 8.1, where all faces are projective.) We will first prove some elementary properties of projective units and projective faces.

7.28. Proposition. *Let A, V be an order unit space and a base norm space in separating order and norm duality, and let $a \in A$. Each projective unit p is associated with a unique compression P. Also each projective face F is associated with a unique compression P, and then $Pa \geq 0$ iff $a \geq 0$ on F.*

Proof. Assume first $p = P1$ where P is a compression on A. By (7.21) $\mathrm{im}^+ P^*$ is determined by p, and thus by Proposition 7.26 the compression P is itself determined by p.

Assume next $F = K \cap \operatorname{im} P^*$ where P is a compression on A. Then $\operatorname{im}^+ P^*$ is the cone generated by F (i.e., $\operatorname{im}^+ P^*$ consists of all $\lambda\omega$ where $\lambda \in \mathbf{R}^+$ and $\omega \in F$). Thus $\operatorname{im}^+ P^*$ is determined by F, and then P is determined by F.

If $a \geq 0$ on F, then $a \geq 0$ on the cone $\operatorname{im}^+ P^*$ generated by F, hence $\langle Pa, \omega \rangle = \langle a, P^*\omega \rangle \geq 0$ for all $\omega \in K$. Thus $a \geq 0$ on F implies $Pa \geq 0$. Conversely if $Pa \geq 0$, then $\langle a, \omega \rangle = \langle a, P^*\omega \rangle \geq 0$ for all $\omega \in F \subset \operatorname{im}^+ P^*$. Thus $Pa \geq 0$ implies $a \geq 0$ on F. □

If P is a compression with associated projective unit p and associated projective face F, then it follows from the proposition above that each one of P, p, F determines the others, and we will say that they are mutually *associated* with each other. We will also denote the (unique) compression which is complementary to P by P', and we will denote the projective unit and the projective face associated with P' by p' and F' respectively Also we will say that P', p', F' are the *complements* of P, p, F respectively. Note that by equation (i) of Lemma 7.17, $p' = 1 - p$, so this notation is compatible with our previous use of the prime to denote the (orthogonal) complement of projections. Note also that by (7.21), the projective faces F and F' are determined by p through the explicit formula

$$F = \{\omega \in K \mid \langle p, \omega \rangle = 1\}, \quad F' = \{\omega \in K \mid \langle p, \omega \rangle = 0\}, \tag{7.22}$$

and conversely, by Lemma 7.25, that the projective unit p is determined by F or F' through the formula

$$p = \bigvee \{a \in A_1^+ \mid a = 0 \text{ on } F'\} = \bigwedge \{a \in A_1^+ \mid a = 1 \text{ on } F\}, \tag{7.23}$$

where the second equality follows from the first by replacing p by p', F by F', and a by $1 - a$. Thus p is the least element of A_1^+ which takes the value 1 on F, and p is the greatest element of A_1^+ which takes the value 0 on F'. Hence p is the unique element of A_1^+ which takes the value 1 on F and also takes the value 0 on F'.

By Definition 7.27, we have the following formula for the projective face associated with a compression:

$$F = \{\omega \in K \mid P^*\omega = \omega\}, \quad F' = \{\omega \in K \mid P^*\omega = 0\}. \tag{7.24}$$

From this it follows that for each $a \in A$ we have $\langle Pa, \omega \rangle = \langle a, \omega \rangle$ when $\omega \in F$ and $\langle Pa, \omega \rangle = 0$ when $\omega \in F'$. Thus

$$Pa = a \text{ on } F, \quad Pa = 0 \text{ on } F'. \tag{7.25}$$

Recall from (A 2) that if $a \in A^+$, then the face generated by a in A^+, denoted by $\operatorname{face}(a)$, consists of all $b \in A^+$ such that $b \leq \lambda a$ for some

$\lambda \in \mathbf{R}^+$, and that the semi-exposed face generated by a in A^+ is equal to $\{a\}^{\bullet\bullet}$ (Proposition 7.2).

7.29. Lemma. *Let A, V be an order unit space and a base norm space in separating order and norm duality and consider a compression P with associated projective unit p. If $a \in A^+$, then the following are equivalent:*

(i) $a \in \operatorname{im}^+ P$,
(ii) $a \leq \|a\|$,
(iii) $a \in \operatorname{face}(p)$,
(iv) $a \in \{p\}^{\bullet\bullet}$.

Proof. (i)$\Rightarrow$(ii) If $a \in \operatorname{im}^+ P$, then $a \leq \|a\|\, 1$ and $a = Pa \leq \|a\| P1 = \|a\|\, p$. (ii)$\Rightarrow$(iii) and (iii)$\Rightarrow$(iv) are trivial. (iv)$\Rightarrow$(i) Assume (iv). By Proposition 7.26, $\operatorname{im}^+ P$ is a semi-exposed face of A^+. Hence $a \in \{p\}^{\bullet\bullet} \subset (\operatorname{im}^+ P)^{\bullet\bullet} = \operatorname{im}^+ P$. □

By (A 75) and Proposition 1.40, an element of a von Neumann algebra or a JBW-algebra is an extreme point of the closed unit ball iff it is a projection. Our next proposition generalizes the "if"-part of this result.

7.30. Proposition. *Let A, V be an order unit space and a base norm space in separating order and norm duality. If p is a projective unit, then p is an extreme point of A_1^+.*

Proof. Let p be the projective unit associated with a compression P. Clearly $p = P1 \in A_1^+$. Assume

$$p = \lambda a + (1 - \lambda) b \tag{7.26}$$

with $a, b \in A_1^+$ and $0 < \lambda < 1$. Then $a \leq \lambda^{-1} p$ and $b \leq (1-\lambda)^{-1} p$. Hence $a, b \in \operatorname{face}(p) = \operatorname{im}^+ P$, so $a = Pa \leq P1 = p$ and $b = Pb \leq P1 = p$. By (7.26), $a = b = p$, so p is an extreme point of A_1^+. □

In the proposition below (and in the sequel) we will use the standard square bracket notation for closed order intervals, so $[a, b]$ will denote the set of all $x \in A$ such that $a \leq x \leq b$.

7.31. Proposition. *Let A, V be an order unit space and a base norm space in separating order and norm duality. If P is a compression on A with associated projective unit p and associated projective face F, then*

(i) $[-p, p] = A_1 \cap \operatorname{im} P$,
(ii) $\operatorname{co}(F \cup -F) = V_1 \cap \operatorname{im} P^*$.

Moreover, $P(A) = \operatorname{im} P$ is an order unit space with distinguished order unit p and $P^*(V) = \operatorname{im} P^*$ is a base norm space with distinguished base

F. These two spaces are in separating order and norm duality under the ordering, norm and bilinear form $\langle\cdot,\cdot\rangle$ relativized from A and V.

Proof. (i) If $a \in [-p, p]$, then it follows from Lemma 7.29 ((iii)$\Rightarrow$(i)) that $a + p \in \operatorname{im} P$. Hence also $a \in \operatorname{im} P$, and since $\|a\| \leq \|p\| \leq 1$, then $a \in A_1^+ \cap \operatorname{im} P$. Conversely, if $a \in A_1^+ \cap \operatorname{im} P$, then $-1 \leq a \leq 1$, so $-p \leq Pa \leq p$, and also $Pa = a$.

(ii) The set at the left-hand side of (ii) is trivially contained in the set at the right. To prove the reverse containment, we consider an element ω in $V_1 \cap \operatorname{im} P^*$. Without loss of generality, we may assume $\|\omega\| = 1$. By the definition of a base norm space (cf. Definition 2.45), $V_1 = \operatorname{conv}(K \cup -K)$. Thus there exists $\sigma \in K$, $\tau \in -K$ and $\lambda \in [0,1]$ such that $\omega = \lambda\sigma + (1 - \lambda)\tau$. Then

$$\omega = P^*\omega = \lambda P^*\sigma + (1-\lambda)P^*\tau.$$

Hence

$$1 = \|\omega\| \leq \lambda\|P^*\sigma\| + (1-\lambda)\|P^*\tau\|,$$

which implies $\|P^*\sigma\| = \|P^*\tau\| = 1$. Thus $P^*\sigma \in \operatorname{im}^+ P^* \cap K = F$, and similarly $-P^*\tau \in F$, so $\omega \in \operatorname{co}(F \cup -F)$.

The rest of the proposition follows from (i) and (ii) and the fact that P is a positive projection with $\|P\| \leq 1$. $\square$

7.32. Proposition. *Let A, V be an order unit space and a base norm space in separating order and norm duality. Let P, Q be compressions with associated projective units p, q and associated projective faces F, G, respectively. Then the following statements are equivalent:*

(i) $\operatorname{im} P \subset \operatorname{im} Q$,
(ii) $QP = P$,
(iii) $p \leq q$,
(iv) $F \subset G$,
(v) $PQ = P$,
(vi) $\ker Q \subset \ker P$,
(vii) $\operatorname{im} Q' \subset \operatorname{im} P'$.

Proof. (i)$\Rightarrow$(ii) Trivial. (ii)$\Rightarrow$(iii) If $QP = P$, then $p = P1 = QP1 \leq Q1 = q$. (iii)$\Rightarrow$(iv) Clear from (7.22). (iv)$\Rightarrow$(v) If $F \subset G$, then $\operatorname{im}^+ P^* \subset \operatorname{im}^+ Q^*$ and then also $\operatorname{im} P^* \subset \operatorname{im} Q^*$, so $Q^*P^* = P^*$. Dualizing, we get $PQ = P$. (v)$\Rightarrow$(vi) Trivial. (vi)$\Rightarrow$(vii) If $\ker Q \subset \ker P$, then trivially $\ker^+ Q \subset \ker^+ P$, so $\operatorname{im}^+ Q' \subset \operatorname{im}^+ P'$, and then also $\operatorname{im} Q' \subset \operatorname{im} P'$. (vii)$\Rightarrow$(i) Use the implication (i)$\Rightarrow$(vii) with Q' in place of P and P' in place of Q. $\square$

7.33. Definition. Let A, V be an order unit space and a base norm space in separating order and norm duality and let P, Q be compressions with associated projective units p, q and associated projective faces F, G respectively. If the condition $\operatorname{im} P \subset \operatorname{im} Q$ (together with the equivalent conditions of Proposition 7.32) are satisfied, then we will write $P \preceq Q$ and $F \preceq G$.

Note that if $P \preceq Q$ and $Q \preceq P$, then $\operatorname{im} P = \operatorname{im} Q$ and $\ker P = \ker Q$, so $P = Q$. Thus the relation $\preceq$ is a partial ordering. By the equivalence of (i) and (vii) in Proposition 7.32, $P \preceq Q$ iff $Q' \preceq P'$, so $P \mapsto P'$ is an order reversing map for this relation, and similarly for $p \mapsto p'$ and $F \mapsto F'$.

7.34. Proposition. *Let A, V be an order unit space and a base norm space in separating order and norm duality, and let P, Q be compressions on A with associated projective units p, q and associated projective faces F, G, respectively. Then the following are equivalent:*

(i) $QP = 0$,
(ii) $P \preceq Q'$,
(iii) $p \leq q'$,
(iv) $F \subset G'$,
(v) $PQ = 0$.

Proof. (i)$\Rightarrow$(ii) If $QP = 0$, then $\operatorname{im} P \subset \ker Q$. Hence $\operatorname{im}^+ P \subset \ker^+ Q = \operatorname{im}^+ Q'$, so $P \preceq Q'$. (ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv) Follows directly from Proposition 7.32. (iv)$\Rightarrow$(v) If $F \subset G'$, then it follows from the implication (iv)$\Rightarrow$(v) of Proposition 7.32 that $PQ' = P$. Multiplying from the right by Q gives $0 = PQ$. (v)$\Rightarrow$(i) Use the implication (i)$\Rightarrow$(v) with P and Q interchanged. □

7.35. Definition. Let A, V be an order unit space and a base norm space in separating order and norm duality, and let P, Q be compressions with associated projective faces F, G and projective units p, q. If the condition $PQ = 0$ (or any of the equivalent conditions in Proposition 7.34) is satisfied, then we will say that P *is orthogonal to* Q and we will write $P \perp Q$, and we will also say that p *is orthogonal to* q and write $p \perp q$, and that F *is orthogonal to* G and write $F \perp G$.

The notion of orthogonality of projective faces F, G depends on the duality of A and V (since G' is defined in terms of this duality). We will now show that in the important special case where $A = V^*$, this notion corresponds to the geometric notion of antipodality defined in (A 23). Recall that two subsets F, G of a convex set are *antipodal* if there is a function $a \in A_b(K)$ with $0 \leq a \leq 1$ such that

$$a = 1 \text{ on } F, \quad a = 0 \text{ on } G. \tag{7.27}$$

7.36. Proposition. *Let A, V be an order unit space and a base norm space in separating order and norm duality, and assume $A = V^*$. Let F and G be projective faces of K. Then F, G are orthogonal iff F, G are antipodal. In particular, two extreme points $\rho, \sigma \in K$ are orthogonal iff $\|\rho - \sigma\| = 2$.*

Proof. Assume first that $F \perp G$, and let p, q be the projective units associated with F, G. Then $p \leq q'$, so $p + q \leq 1$. By (7.22), p takes the value 1 on F. Similarly q takes the value 1 on G. Hence (7.27) is satisfied when $a = p$, so F, G are antipodal.

Assume next that F, G are antipodal and let p, q be as above. Choose $a \in A_1^+$ such that (7.27) holds. Since $a = 1$ on F, then $a \geq p$ (cf. (7.23)). Similarly since $1 - a = 1$ on G, then $1 - a \geq q$, so $a \leq q'$. Hence $p \leq a \leq q'$, so $F \perp G$.

The last statement follows from (A 25). □

7.37. Lemma. *Let A, V be an order unit space and a base norm space in separating order and norm duality. If P is a compression and $a \in A^+$, then the following are equivalent:*

(i) $a = Pa + P'a$,
(ii) $Pa \leq a$.

Proof. (i)$\Rightarrow$(ii) Trivial.

(ii)$\Rightarrow$(i) Assume $Pa \leq a$. Since $a - Pa \in \ker^+ P = \operatorname{im}^+ P'$ and the two complementary projections P and P' satisfy the equation $P'P = 0$, then

$$a - Pa = P'(a - Pa) = P'a.$$

Thus $a = Pa + P'a$. □

7.38. Lemma. *Let A, V be an order unit space and a base norm space in separating order and norm duality. If P is a compression and q is a projective unit such that $Pq \leq q$, then also $Pq' \leq q'$.*

Proof. The compression P and its complementary compression P' satisfy the equation $P1 + P'1 = 1$ (Lemma 7.17). Therefore by Lemma 7.37,

$$q' = 1 - q = (P1 + P'1) - (Pq + P'q) = Pq' + P'q',$$

which gives $Pq' \leq q'$. □

7.39. Proposition. *Let A, V be an order unit space and a base norm space in separating order and norm duality. If P and Q are compressions with associated projective units p and q, then the following are equivalent:*

(i) $PQ = QP$,
(ii) $Pq \leq q$,
(iii) $Qp \leq p$.

Proof. Since (i) is symmetric in P and Q, it suffices to prove the equivalence of (i) and (ii).

(i)$\Rightarrow$(ii) If $PQ = QP$, then

$$Pq = PQ1 = QP1 = Qp \leq Q1 = q.$$

(ii)$\Rightarrow$(i) Assume now that $Pq \leq q$. Let $a \in A^+$ be arbitrary. Since $0 \leq a \leq \|a\|\, 1$, then $0 \leq Qa \leq \|a\|\, q$. Hence

$$0 \leq PQa \leq \|a\| Pq \leq \|a\|\, q,$$

from which it follows that

$$Q'PQa = 0 \quad \text{for all } a \in A^+. \tag{7.28}$$

Therefore $PQa \in \ker^+ Q' = \mathrm{im}^+ Q$, so $PQa = QPQa$ for all $a \in A^+$, and then for all $a \in A$. Thus

$$PQ = QPQ. \tag{7.29}$$

By Lemma 7.38, we also have $Pq' \leq q'$, so we can argue as above with q' in place of q and Q' in place of Q. Therefore (7.28) holds with Q and Q' interchanged, i.e., $QPQ'a = 0$ for all $a \in A^+$, and then for all $a \in A$. Thus $QPQ' = 0$. Dualizing, we get

$$Q'^* P^* Q^* = 0.$$

Thus for arbitrary $\rho \in V^+$, we have $P^*Q^*\rho \in \ker^+ Q'^* = \mathrm{im}^+ Q^*$. Therefore $P^*Q^*\rho = Q^*P^*Q^*\rho$ for all $\rho \in V^+$, and then for all $\rho \in V$. Thus

$$P^*Q^* = Q^*P^*Q^*.$$

Dualizing back to A, we get

$$QP = QPQ. \tag{7.30}$$

Combining (7.29) and (7.30) gives $PQ = QP$. $\square$

7.40. Definition. Let A, V be an order unit space and a base norm space in separating order and norm duality and let P, Q be compressions with associated projective units p, q and associated projective faces F, G. If the equation $PQ = QP$ (or any of the equivalent conditions in Proposition 7.39) is satisfied, then we will say that each of the entities P, p, F is *compatible* with each of the entities Q, q, G. (We will also say that these entities are mutually *compatible*.)

Note that it follows from Lemma 7.38 that if P (or either one of the associated entities p or F) is compatible with Q (or either one of q or G), then P is also compatible with the complementary compression Q' (or either one of the associated entities q' or G').

By definition, a compression P is compatible with a projective unit q iff $Pq \leq q$, and this is equivalent to $q = Pq + Pq'$ by Lemma 7.37. Thus we can extend the definition to a pair of a compression and an arbitrary element of A as follows.

7.41. Definition. Let A, V be an order unit space and a base norm space in separating order and norm duality, and let P be a compression with associated projective unit p and associated projective face F. We will say that P (or either one of p or F) is *compatible* with an element $a \in A$ if $a = Pa + P'a$. We will also say that a is *compatible* with P, and with p and F.

By an easy calculation (used, e.g., in the proof of [AS, Lemma 3.39]), a projection p in a von Neumann algebra commutes with an arbitrary element a of the algebra iff $a = U_p a + U_{p'} a$ (where $p' = 1 - p$). Thus the definition above generalizes not only the notion of commutation for pairs of projections in a von Neumann algebra, but also the notion of commutation for pairs consisting of a projection and an arbitrary (self-adjoint) element of the algebra.

Remark. The notion of compatibility comes from quantum physics where it relates to simultaneous measurability of observables, as we will now explain in terms of the von Neumann algebra model of quantum mechanics (described in [AS, p. 121]).

The (bounded) observables are represented by the self-adjoint elements of a von Neumann algebra $\mathcal{M}$ and the (mixed) states are represented by the elements in the normal state space K of $\mathcal{M}$. The propositions, i.e., the observables that take on only the two values 0 or 1, are represented by the elements in the set $\mathcal{P}$ of projections. If we measure an observable $a \in \mathcal{M}_{\mathrm{sa}}$ directly on a quantum system, say a beam of particles, in a state $\omega \in K$, then the expected (mean) value for a is $\langle a, \omega \rangle$. But if we first send the beam through a measuring device for a proposition $p \in \mathcal{P}$, then it will be split in two parts, the one consisting of the particles appearing with the value 1 for p, the other consisting of the particles appearing with the value 0 for p. Of these, the former will be in the transformed state $U_p^*\omega$ and the latter will be in the transformed state $U_{p'}^*\omega$, where U_p^* and $U_{p'}^*$ are the compressions associated with p and $p' = 1 - p$ respectively. In this case the expected value of a (measured on the combined output from the measuring device for p) will be $\langle a, U_p^*\omega\rangle + \langle a, U_{p'}^*\omega\rangle$. This value is equal to the expected value $\langle a, \omega\rangle$ for the unperturbed beam iff

$$\langle a, \omega\rangle = \langle U_p a + U_{p'} a, \omega\rangle. \tag{7.31}$$

Therefore the expected value of a on any beam is unaffected by the measurement of p iff $a = U_p a + U_{p'} a$. Thus a and p are simultaneously measurable iff a and U_p are compatible in the sense of Definition 7.41 (applied in the von Neumann algebra context of Proposition 7.23).

We will also present two other properties of quantum mechanical measurements, which are both easily obtained in the algebraic model, and will be needed in a later remark on physics.

Assume $p \in \mathcal{P}$ and $\omega \in K$. If $\|U_p^*\omega\| = 1$, then $\langle p, \omega\rangle = \langle U_p 1, \omega\rangle = 1$. Hence by the implication (iv) $\Rightarrow$ (vi) in (A 74),

$$\|U_p^*\omega\| = 1 \quad \Rightarrow \quad U_p^*\omega = \omega, \tag{7.32}$$

which is the defining implication for neutral projections (i.e., (7.18) with U_p^* in the place of P). As explained above, U_p^* describes the transformation of states by a measuring device for the proposition p, say a filter for a beam of particles. Therefore such quantum mechanical filters have the neutrality property described in the Remark after Definition 7.19.

Assume next that $a \in A^+$ and $p \in \mathcal{P}$. Now it follows from the implication (iii) $\Rightarrow$ (iv) in (A 73) that

$$U_{p'} a = 0 \quad \Rightarrow \quad U_p a = a. \tag{7.33}$$

Thus if $U_{p'} a = 0$, then $a = U_p a + U_{p'} a$, so a and p are compatible. Therefore the positive observable a and the proposition p are simultaneously measurable in this case.

What does this compatibility property mean in physical terms? Clearly $U_{p'} a = 0$ is equivalent to $\langle a, U_{p'}^*\omega\rangle = 0$ for all $\omega \in K$, and $U_p a = a$ is equivalent to $\langle a, U_p^*\omega\rangle = \langle a, \omega\rangle$ for all $\omega \in K$. Thus the implication (7.33) says that if the positive observable a and the proposition p are such that particles which appear with the value 0 in a measurement of p never contribute anything to a subsequent measurement of a, then the expectation of a in such a measurement will have the same value $\langle a, U_p^*\omega\rangle = \langle a, \omega\rangle$ as if it were measured directly on the unperturbed beam.

It is clear from Definition 7.41 that if the compression P is compatible with the element a, then P' (as well as p' or F') is also compatible with

a. Clearly, the set of elements compatible with a compression P is a linear subspace of A which contains 1. From this it follows that P is compatible with an element a of A iff P is compatible with $a + \lambda 1$ for one (and then all) $\lambda \in \mathbf{R}$. We can choose λ so that $a + \lambda 1 \geq 0$. Thus the problem of showing compatibility of a compression P and an element a in A can be reduced to showing compatibility of P and an element $b = a + \lambda 1$ in A^+, and by Lemma 7.37, this can be done by showing $Pb \leq b$.

We will now prove some basic properties of the compatibility relation, and we begin by an observation which we state as a lemma for later reference.

7.42. Lemma. *Let A, V be an order unit space and a base norm space in separating order and norm duality and let P, Q be a pair of compressions. If $P \preceq Q$, then P and Q are compatible. If P, Q are compatible, then P', Q are compatible and P', Q' are also compatible. If $P \perp Q$, then P and Q are compatible.*

Proof. If $P \preceq Q$, then $PQ = P$ and also $QP = P$ (Definition 7.33), so P and Q are compatible. If P, Q are compatible, then $q = Pq + P'q$, so P', Q are compatible, and then P', Q' are also compatible. If $P \perp Q$, then $P \preceq Q'$ implies that P is compatible with Q', and so P and Q are compatible. □

7.43. Lemma. *Let A, V be an order unit space and a base norm space in separating order and norm duality, and let P be a compression with associated projective face F. If a is an element of A^+ such that $a = 0$ on F', then $Pa = a$, $P'a = 0$ and a is compatible with P.*

Proof. Assume $a = 0$ on F', i.e., $\langle a, \omega \rangle = 0$ for all $\omega \in F'$. Since $F' \subset \operatorname{im}^+ P'^* = \ker^+ P^*$, we have $a \in (\ker^+ P^*)^\bullet = \operatorname{im}^+ P$ (Proposition 7.7 with P^* in place of P). Thus $Pa = a$. Applying P' gives $0 = P'a$, and then $a = Pa + P'a$. Thus a is compatible with P. □

7.44. Proposition. *Let A, V be an order unit space and a base norm space in separating order and norm duality, let P be a compression with associated projective face F, and let a be an arbitrary element of A. Then Pa is the unique element compatible with P which is equal to a on F and vanishes on F', and $(P + P')a$ is the unique element compatible with P which is equal to a on* $\operatorname{co}(F \cup F')$.

Proof. Clearly Pa is compatible with P. If $\omega \in F$, then $P^*\omega = \omega$ (7.24), so $\langle Pa, \omega \rangle = \langle a, P^*\omega \rangle = \langle a, \omega \rangle$. Thus $Pa = a$ on F. If $\omega \in F'$, then $P^*\omega = 0$, so $\langle Pa, \omega \rangle = \langle a, P^*\omega \rangle = 0$. Thus $Pa = 0$ on F'.

Now consider an element b compatible with P such that $b = a$ on F and $b = 0$ on F'. By the definition of a projective face (Definition 7.27),

$\mathrm{im}^+ P^*$ is the cone generated by F. Similarly $\mathrm{im}^+ P'^*$ is the cone generated by F'. Therefore we also have $b = a$ on $\mathrm{im}^+ P^*$ and $b = 0$ on $\mathrm{im}^+ P'^*$. By the compatibility of b and P, we have for each $\omega \in V^+$,

$$\langle b, \omega\rangle = \langle Pb, \omega\rangle + \langle P'b, \omega\rangle = \langle b, P^*\omega\rangle + \langle b, P'^*\omega\rangle = \langle a, P^*\omega\rangle = \langle Pa, \omega\rangle.$$

Thus $b = Pa$, which proves the uniqueness.

Clearly also $P'a$ is compatible with P, and by the above $P'a = a$ on F' and $Pa = 0$ on F. Therefore $(P + P')a$ is compatible with P and $(P+P')a = a$ on $\mathrm{co}(F \cup F')$. Now consider an element c compatible with P such that $c = a$ on $\mathrm{co}(F \cup F')$. Then $c = a$ on $\mathrm{im}^+ P^*$ and $c = a$ on $\mathrm{im}^+ P'^*$, so we can argue as above getting $(P + P')a = c$, which proves uniqueness also in this case. □

Note that by Proposition 7.44, an element $a \in A$ which is compatible with a compression P with associated projective face F is completely determined by its values on F and F'.

Our next proposition gives an explicit expression for the compression associated with a projective face.

7.45. Proposition. *Let A, V be an order unit space and a base norm space in separating order and norm duality. If F is a projective face with associated compression P, then for each $a \in A^+$,*

$$\begin{aligned} Pa &= \bigvee \{b \in A^+ \mid b = 0 \text{ on } F',\ b \le a \text{ on } F\} \\ &= \bigwedge \{b \in A^+ \mid b = 0 \text{ on } F',\ b \ge a \text{ on } F\}. \end{aligned} \tag{7.34}$$

Proof. Assume $a \in A^+$. By (7.25) $Pa = 0$ on F', so we can choose $b = Pa$ in (7.34). We must prove that this is the best choice, i.e., we must show that if $b \in A^+$ with $b = 0$ on F' and $b \le a$ on F (or $b \ge a$ on F), then $b \le Pa$ (respectively $b \ge Pa$).

Consider an element b of A^+ such that $b = 0$ on F' and $b \le a$ on F. By Lemma 7.43, b is compatible with P. Hence for each $\omega \in V^+$,

$$\langle b, \omega\rangle = \langle Pb, \omega\rangle + \langle P'b, \omega\rangle = \langle b, P^*\omega\rangle + \langle b, P'^*\omega\rangle \le \langle a, P^*\omega\rangle = \langle Pa, \omega\rangle.$$

Thus $b \le Pa$. (Similarly we prove that if $b \in A^+$, $b = 0$ on F' and $b \ge a$ on F, then $b \ge Pa$.) We are done. □

We will now specialize Theorem 7.15 to the present context, and simultaneously we will prove uniqueness under alternative conditions.

7.46. Theorem. *Let A, V be an order unit space and a base norm space in separating order and norm duality, let K be the distinguished base of V, and let P be a compression with associated projective face F. The map $E = P + P'$ is the unique weakly-continuous projection T of A onto the subspace* $\operatorname{im}(P + P')$ *of all elements compatible with P, which satisfies any one of the following three conditions:*

(i) *T is positive,*
(ii) $\|T\| \leq 1$,
(iii) *T commutes with P and P'.*

Dually, E^* is the unique weakly continuous projection of V which maps K onto $\operatorname{co}(F \cup F')$ *and leaves this set pointwise fixed.*

Proof. Clearly the projection $T = E$ satisfies (i), (ii),(iii), and we will prove uniqueness under each of these conditions.

(i) If T is a weakly-continuous positive projection of A onto $\operatorname{im}(P + P')$, then $T = E$ by Theorem 7.15 (with $X = A$, $Y = V$ and $Q = P'$).

(ii) Assume $\|T\| \leq 1$. Since T acts as the identity operator on the subspace $\operatorname{im}(P + P')$, then $T1 = 1$. We will show that T is also positive. Let $a \in A^+$ and assume without loss of generality that $\|a\| \leq 2$. Now it follows from the defining property of the order unit norm (A 13) that $-1 \leq a - 1 \leq 1$, and then in turn that $\|a - 1\| \leq 1$. Since $\|T\| \leq 1$ and $T1 = 1$, then $\|Ta - 1\| \leq 1$, or which is equivalent, $-1 \leq Ta - 1 \leq 1$. Hence $Ta \geq 0$. Thus T is positive, and the desired uniqueness follows from (i).

(iii) Assume that T is a projection of A onto $\operatorname{im}(P + P')$ which commutes with P and P', and then also with E. Since $\operatorname{im} T = \operatorname{im} E$, then $ET = T$ and $TE = E$, and since T commutes with E, we have $T = ET = TE = E$.

It remains to prove the dual result on E^*. Let $\omega \in K$. By Lemma 7.17, $E^*(K) \subset K$. By the definition of F and F', we have $P^*\omega = \lambda\rho$ and $P'^*\omega = \mu\sigma$ where $\rho \in F$ and $\sigma \in F'$ with $\lambda, \mu \geq 0$. Thus $E^*\omega = \lambda\rho + \mu\sigma$. Since $\langle 1, E^*\omega\rangle = \langle (P + P')1, \omega\rangle = \langle 1, \omega\rangle = 1$, then $\lambda + \mu = 1$. Hence

$$E^*\omega = \lambda\rho + (1 - \lambda)\sigma \in \operatorname{co}(F \cup F').$$

Since E^* leaves F and F' pointwise fixed, it also leaves $\operatorname{co}(F \cup F')$ pointwise fixed, so $E^*(K) = \operatorname{co}(F \cup F')$.

The uniqueness of E^* follows from Theorem 7.15 (this time with $X = V$, $Y = A$ and with P^* in place of P and P'^* in place of Q.) □

7.47. Definition. Let A, V be an order unit space and a base norm space in separating order and norm duality. A compression P is said to be *central* if it is compatible with all $a \in A$. An element $a \in A$ is *central* if it is compatible with all compressions P on A.

7.48. Proposition. *Let A, V be an order unit space and a base norm space in separating order and norm duality. If $P : A \to A$ is a weakly continuous positive projection, then the following are equivalent:*

(i) *$Pa \le a$ for all $a \in A^+$,*
(ii) *P is strongly complemented,*
(iii) *P is a compression such that $P' = I - P$,*
(iv) *P is a central compression.*

Proof. (i)$\Rightarrow$(ii) Trivial, since statement (i) means that $I - P$ is positive.

(ii)$\Rightarrow$(iii) Assume P strongly complemented and let $Q = I - P \ge 0$. Then $P1 = 1 - Q1 \le 1$, so P is normalized (Lemma 7.16). We also have $Q^* = I - P^* \ge 0$, so P^*, Q^* are also strongly complementary projections of norm ≤ 1, and thus (iii) is satisfied.

(iii)$\Rightarrow$(iv) Trivial.

(iv)$\Rightarrow$(i) Clear from Lemma 7.37. □

Recall that a convex set K is the *direct convex sum* of faces F and G if each element of K can be written uniquely as a convex combination of an element from F and one from G, in which case we write $K = F \oplus_c G$. In this case, G is uniquely determined by F, and we write $G = F'$. (We will see below that this is consistent with our previous meaning for F'.) If such a face G exists, then F is a split face of K, cf. (A 5).

7.49. Proposition. *Let A, V be an order unit space and a base norm space in separating order and norm duality. If P is a central compression, then the associated projective face F is a split face of the distinguished base K, and its complementary split face is equal to the complementary projective face F'. If $A = V^*$, then the converse statement also holds, i.e., each split face F of K is associated with a central compression P.*

Proof. The first statement of the proposition follows easily from the equation $P' = I - P$.

To prove the second statement, we consider an arbitrary split face F of K, and we denote the complementary split face by G. By the definition of a split face, each $\omega \in K$ has a unique decomposition

$$\omega = \lambda\sigma + (1 - \lambda)\tau \tag{7.35}$$

where $\sigma \in F$, $\tau \in G$ and $0 \le \lambda \le 1$. It is easily verified that $\Phi : \omega \mapsto \lambda\sigma$ is an affine map from K onto $\mathrm{co}(F \cup \{0\})$. If a is an arbitrary element of the space $A = V^*$, then the composed map $a \circ \Phi$ is a bounded affine function on K. By elementary linear algebra, each bounded affine function on K can be uniquely extended (with preservation of norm) to a linear functional $\widetilde{a} \in A$, cf. (A 11). Now $P : a \mapsto \widetilde{a}$ is a positive linear map from the space A into itself such that $\|P\| \leq 1$ and

$$\langle Pa, \omega\rangle = \langle a, \Phi(\omega)\rangle \quad \text{for all } a \in A,\ \omega \in K.$$

Observe also that P is weakly continuous (as the weak topology on A is given by the seminorms $a \mapsto |\langle a, \omega\rangle|$ where $\omega \in K$). By the equation above, $P^* = \Phi$ on K, so P^* is a positive projection with $\operatorname{im}^+ P^*$ equal to the cone generated by F.

It follows from (7.35) that for each $\omega \in K$,

$$\omega - P^*\omega = \omega - \Phi(\omega) = (1-\lambda)\tau \geq 0,$$

so $P^*\omega \leq \omega$. Hence $\langle Pa, \omega\rangle = \langle a, P^*\omega\rangle \leq \langle a, \omega\rangle$ for each $a \in A^+$. Now it follows from Proposition 7.48 that P is a central compression. By the above, $K \cap \operatorname{im}^+ P^* = F$, so F is the projective face associated with P. □

Remark. We discussed the von Neumann algebra approach to quantum mechanical measurement theory in the remark after Definition 7.41. But the basic concepts of this theory can be defined and studied also in the general order-theoretic context of this chapter. Then the (mixed) states of a quantum system are represented by the elements in the distinguished base K of a base norm space V, the (bounded) observables are represented by the elements in the order unit space $A = V^*$, and the expected value of the observable $a \in A$ when the system is in the state $\omega \in K$ is $\langle a, \omega\rangle$. The propositions are represented by the elements in the set $\mathcal{P}$ of projective units, or alternatively by the associated elements in the set $\mathcal{F}$ of projective faces or in the set $\mathcal{C}$ of compressions.

We will now discuss the physical interpretation and justification of this general order-theoretic approach. The representation of states by elements in the convex set $K \subset V$ and of observables by elements in the order unit space $A = V^*$ is a natural generalization of the standard algebraic model. But the representation of propositions is less obvious. Why should they be represented by projective units associated with compressions defined in terms of bicomplementarity?

A quantum mechanical proposition is operationally defined by a measuring device, which we may think of as a filter for a beam of particles. We have previously explained how such a device splits the beam into two partial beams, one consisting of the particles appearing with the value 1, the other consisting of the particles appearing with the value 0. If the incident beam is in the state ω, then the beam of particles exiting with the value 1 will be in a transformed state ω_1 and the beam of particles exiting with the value 0 will be in a transformed state ω_0. In the algebraic model, where the proposition is represented by a projection p in a von Neumann algebra, the transformation of states is given by the maps U_p^* and $U_{p'}^*$. More precisely, ω_1 and ω_2 are (up to a normalization factor) equal to $U_p^*\omega$ and

$U_{p'}^*\omega$ respectively (or which is equivalent, ω_1 and ω_2 are in the faces F and F' associated with p and p' respectively). In the general order-theoretic context we denote the corresponding maps on the base norm space V by Φ and Φ', and we will argue that Φ and Φ' should be of the form $\Phi = P^*$ and $\Phi' = P'^*$ for a pair of complementary compressions P, P' on the order unit space A.

The physical interpretation of convex combinations in K is "mixing of states". From this we conclude that Φ must be affine on K, hence linear on V. Beams exiting a quantum mechanical filter will be unchanged if they pass through that filter again. Therefore $\Phi^2 = \Phi$. Similarly $\Phi'^2 = \Phi'$. Thus Φ and Φ' must be positive projections on V.

If ω is a beam of particles, $\|\omega\|$ is the intensity of the beam (proportional to the number of particles in the beam). The number $\|\Phi\omega\|$ is the intensity of the partial beam of particles passing through the filter Φ. Therefore $\|\Phi\omega\| \leq \|\omega\|$, and so $\|\Phi\| \leq 1$, and similarly for Φ'. Thus Φ and Φ' are normalized. Furthermore, since every particle when measured will be found to have value 0 or 1, we must have

$$\|\Phi\omega\| + \|\Phi'\omega\| = \|\omega\|. \tag{7.36}$$

Now consider the dual maps $P = \Phi^*$ and $P' = \Phi'^*$ on $A = V^*$ and note that the duals of these maps in the given (weak) duality of A and V are $\Phi = P^*$ and $\Phi' = P'^*$. What remains to be shown is bicomplementarity, that is, that Φ and Φ' are complementary projections on V, and that P and P' are complementary projections on A.

For this purpose, we will make use of the description of compressions in Proposition 7.24. We have already verified condition (i) of that proposition (cf. (7.36)). The two remaining conditions (neutrality of Φ and Φ', and the inclusions $\ker^+ P' \subset \operatorname{im}^+ P$ and $\ker^+ P \subset \operatorname{im}^+ P'$) are less straightforward, but we have seen that they are valid in the standard algebraic model (cf. (7.32) and (7.33)), and have natural physical interpretations. By Proposition 7.24, these properties are equivalent to P, P' being complementary compressions. Thus in our order-theoretic generalization of the standard model for quantum mechanics, we will work with compressions as a fundamental notion, and think of them as an axiomization of the measurement process.

Geometry of projective faces

We will continue to work with an order unit space A and a base norm space V in separating order and norm duality. But we will now also impose the extra condition that $A = V^*$, which is satisfied in the important special cases of von Neumann algebras and JBW-algebras, and of course also in all finite dimensional cases. Under this condition, $a \mapsto \widehat{a}$ is an order and norm preserving isomorphism of A onto the space $A_b(K)$ of all bounded affine functions on K (A 11), and every bounded linear operator Φ on V is of the form T^* where T is the (weakly continuous) dual operator (defined on $A = V^*$ by $\langle a, \Phi\omega\rangle = \langle Ta, \omega\rangle$ for $a \in A$ and $\omega \in V$).

We know that compressions occur in complementary pairs P, P' and that P' is uniquely determined by P (Corollary 7.11). We will now state the corresponding result for projective faces in an explicit geometric form. For this purpose we associate with each face F of the distinguished base K of V the function k_F defined on K by

$$k_F = \inf\{a \in A_b(K) \mid a = 1 \text{ on } F,\ 0 \le a \le 1\}. \tag{7.37}$$

Note that k_F is a concave but not necessarily affine function for a general face F of K.

Remark. The upper semi-continuous upper envelope of the characteristic function χ_F of a closed face F in a compact convex set K is equal to $\inf\{a \in A(K) \mid a \ge 1 \text{ on } F,\ 0 \le a\}$ (see, e.g., [6, p. 4] or [20, p. 22]). This expression resembles the definition of k_F, but it differs from it in two respects. First, the space $A_b(K)$ of bounded affine functions on K is replaced by the space $A(K)$ of affine functions continuous in the given compact topology of K. But this difference is not essential from a geometric point of view. For example, in finite dimensions $A_b(K) = A(K)$. Second, the relation $0 \le a \le 1$ is replaced by the weaker condition $0 \le a$, and this difference is crucial. In fact, under the additional assumption $a \le 1$, the function $1 - a$ is positive and takes the value zero on F, so $1 - a$, and then also a, is compatible with F (Lemma 7.43). Thus this condition severely restricts the set of elements a permitted in (7.37).

7.50. Proposition. *Let A, V be an order unit space and a base norm space in separating order and norm duality, and let K be the distinguished base of V. Assume $A = V^*$ and let F be a projective face with associated projective unit p. Then $p = k_F$, and*

$$F' = \{\omega \in K \mid k_F(\omega) = 0\}. \tag{7.38}$$

Proof. By Lemma 7.25, p' $(= 1 - p)$ is the greatest element of A_1^+ which vanishes on the set

$$K \cap \ker^+ P'^* = K \cap \operatorname{im}^+ P^* = F.$$

Thus p is the least element of A_1^+ which is 1 on F, so $p = k_F$. Now (7.38) follows from (7.22). □

We will give a geometric characterization of pairs of complementary projective faces $F, G \in K$. Note that when we have found conditions characterizing such a pair, then we can also decide if a single face F is projective by checking if the pair F, F' where F' is defined by (7.38) satisfies these conditions.

We have already seen that two faces $F, G \subset K$ are complementary projective faces corresponding to central compressions iff they are complementary split faces, i.e., iff $K = F \oplus_c G$ (Proposition 7.49), or just $K \subset F \oplus_c G$ since the opposite relation is trivial. We will now characterize general pairs of complementary projective faces by two conditions. The first of these is a modified version of the relation $K \subset F \oplus_c G$ which consists of two relations, one where F is replaced by $\mathrm{Tan}_K F$ and one where G is replaced by $\mathrm{Tan}_K G$. The second condition is the equality $\mathrm{Tan}_K F \cap \mathrm{Tan}_K G = \emptyset$. (The latter is trivial in the case of split faces, where $\mathrm{Tan}_K F$ and $\mathrm{Tan}_K G$ are the affine spans of F and G, which are disjoint since F and G are affinely independent.)

We begin by introducing some notation which will be used in the announced characterization and its proof. We will denote by H the affine hyperplane which carries the convex set K, i.e., $H = \{\omega \in V \mid \langle 1, \omega \rangle = 1\}$. For each face F of K we will denote by $\widetilde{F}$ the cone generated by F, i.e., $\widetilde{F} = \{\lambda\omega \mid \omega \in F,\ \lambda \geq 0\}$. If F is a face of K, then the tangent space of K at F is an affine subspace of H, while the tangent space of $\widetilde{F}$ is a linear subspace of V. To simplify the notation, we will omit the subscripts in these two cases, writing

$$\mathrm{Tan}\, F := \mathrm{Tan}_K F, \quad \mathrm{Tan}\, \widetilde{F} := \mathrm{Tan}_{V^+} \widetilde{F}. \tag{7.39}$$

7.51. Lemma. *Let A, V be an order unit space and a base norm space in separating order and norm duality, assume $A = V^*$, and let F be a non-empty subset of the distinguished base K of V. Then*

$$\mathrm{Tan}\, F = H \cap \mathrm{Tan}\, \widetilde{F}, \tag{7.40}$$

$$\mathrm{Tan}\, \widetilde{F} = \mathrm{lin}\,(\mathrm{Tan}\, F). \tag{7.41}$$

Proof. We first prove

$$\mathrm{Tan}\, F = F^{\bullet\circ} \cap H. \tag{7.42}$$

Since $F^{\bullet\circ} = \widetilde{F}^{\bullet\circ}$ and $\widetilde{F}^{\bullet\circ} = \mathrm{Tan}\, \widetilde{F}$ (Proposition 7.2), the equality (7.40) will follow from (7.42). By definition, $F^{\bullet\circ}$ is an intersection of closed supporting hyperplanes for V^+ (therefore also for K) containing F. All such hyperplanes contain $\mathrm{Tan}\, F$, so $\mathrm{Tan}\, F \subset F^{\bullet\circ}$. Since H is a closed supporting hyperplane for K, then $\mathrm{Tan}\, F \subset F^{\bullet\circ} \cap H$.

Now suppose $\sigma \in F^{\bullet\circ} \cap H$. Let N be any closed supporting hyperplane for K containing F. Choose $a \in A$, $\lambda \in \mathbf{R}$, such that $a \geq \lambda$ on K, and such that $N = \{\sigma \in V \mid \langle a, \sigma \rangle = \lambda\}$. Then $a - \lambda 1 \geq 0$, and $N \supset F$ implies $a - \lambda 1 = 0$ on F. Then $a - \lambda 1 \in F^{\bullet}$, so $\langle a - \lambda 1, \sigma \rangle = 0$. Thus $\langle a, \sigma \rangle = \lambda$, so $\sigma \in N$. Hence $F^{\bullet\circ} \cap H \subset N$, so $F^{\bullet\circ} \cap H$ is contained in the intersection of all such N, i.e., in $\operatorname{Tan} F$. This completes the proof of (7.42), and then also of (7.40).

Now we prove (7.41). By Proposition 7.2, $\operatorname{Tan} \widetilde{F} = \widetilde{F}^{\bullet\circ} = F^{\bullet\circ} \supset \operatorname{Tan} F$, so $\operatorname{Tan} \widetilde{F} \supset \operatorname{lin}(\operatorname{Tan} F)$. Suppose $\sigma \in \operatorname{Tan} \widetilde{F}$. We consider two cases. If $\langle 1, \sigma \rangle = 0$, then for any $\tau \in F$, $\langle 1, \sigma + \tau \rangle = 1$, so $\sigma + \tau \in \operatorname{Tan} \widetilde{F} \cap H = \operatorname{Tan} F$. Therefore $\sigma \in \operatorname{Tan} F - \tau \subset \operatorname{lin}(\operatorname{Tan} F)$. On the other hand, if $\langle 1, \sigma \rangle \neq 0$, then $\langle 1, \sigma \rangle^{-1} \sigma \in \operatorname{Tan} \widetilde{F} \cap H = \operatorname{Tan} F$, so $\sigma \in \operatorname{lin}(\operatorname{Tan} F)$. Thus $\operatorname{Tan} \widetilde{F} \subset \operatorname{lin}(\operatorname{Tan} F)$, which completes the proof of (7.41). □

7.52. Theorem. *Let A, V be an order unit space and a base norm space in separating order and norm duality, assume $A = V^*$, and let F, G be faces of the distinguished base K of V. Then F, G is a pair of complementary projective faces iff the following two conditions are satisfied:*

(i) $K \subset F \oplus_c \operatorname{Tan} G, \quad K \subset \operatorname{Tan} F \oplus_c G,$

(ii) $\operatorname{Tan} F \cap \operatorname{Tan} G = \emptyset$.

Proof. Assume first that F, G are complementary projective faces associated with the compressions P, Q. Now we apply Theorem 7.14 (noting that $\widetilde{F}$ and $\widetilde{G}$ above are denoted by F and G respectively in that theorem). Then

$$V^+ \subset \widetilde{F} \oplus_w \operatorname{Tan} \widetilde{G}, \quad V^+ \subset \widetilde{G} \oplus_w \operatorname{Tan} \widetilde{F}. \tag{7.43}$$

We will now show (i) holds. Note that $\operatorname{im}^+ P^* = \widetilde{F}$ and $\operatorname{im}^+ Q^* = \widetilde{G}$, and by smoothness of P^* we have

$$\ker P^* = \operatorname{Tan}(\ker^+ P^*) = \operatorname{Tan}(\operatorname{im}^+ Q^*) = \operatorname{Tan} \widetilde{G}.$$

Similarly $\ker Q^* = \operatorname{Tan} \widetilde{F}$. Let $\sigma \in K$; then

$$\sigma = P^*\sigma + (\sigma - P^*\sigma) \in \operatorname{im}^+ P^* + \ker P^* = \widetilde{F} \oplus_w \operatorname{Tan} \widetilde{G}.$$

Let $\lambda = \|P^*\sigma\|$. Since $\|P^*\| \leq 1$, then $0 \leq \lambda \leq 1$. Let H be the hyperplane which carries K. If $0 < \lambda < 1$, then

$$\sigma = \lambda(\lambda^{-1}P^*\sigma) + (1-\lambda)(1-\lambda)^{-1}(\sigma - P^*\sigma) \in F \oplus_c (H \cap \operatorname{Tan} \widetilde{G}) = F \oplus_c \operatorname{Tan} G.$$

If $\lambda = 0$, then $P^*\sigma = 0$, so $\sigma \in H \cap \operatorname{Tan} \widetilde{G} = \operatorname{Tan} G$. If $\lambda = 1$, then $\|P\sigma\| = 1$, so by neutrality $P^*\sigma = \sigma$, and thus $\sigma \in F$. This completes the proof that (i) holds.

Set for brevity $N := \operatorname{Tan} \widetilde{F} \cap \operatorname{Tan} \widetilde{G}$. Thus

$$(P^* + Q^*)\,\omega = 0 \quad \text{for } \omega \in N. \tag{7.44}$$

By Lemma 7.17, for each $\omega \in K$,

$$\langle 1, (P^* + Q^*)\,\omega\rangle = \langle P1 + Q1, \omega\rangle = 1.$$

Since H is the affine span of K, then $\langle 1, (P^* + Q^*)\,\omega\rangle = 1$ for all $\omega \in H$. Now it follows from (7.44) that $N \cap H = \emptyset$, so

$$\operatorname{Tan} F \cap \operatorname{Tan} G = \operatorname{Tan} \widetilde{F} \cap \operatorname{Tan} \widetilde{G} \cap H = N \cap H = \emptyset.$$

Thus we also have (ii).

Assume next that (i) and (ii) are satisfied. By elementary linear algebra, (i) implies that $\operatorname{lin} F \cap \operatorname{lin}(\operatorname{Tan} G) = \{0\}$ (cf. [6, Prop. II.6.1]). Thus $\operatorname{lin} F \cap \operatorname{Tan} \widetilde{G} = \{0\}$, and similarly $\operatorname{lin} G \cap \operatorname{Tan} \widetilde{F} = \{0\}$. By (i) we also have $V^+ \subset \widetilde{F} + \operatorname{Tan} \widetilde{G}$ and $V^+ \subset \widetilde{G} + \operatorname{Tan} \widetilde{F}$, so (7.43) follows. Thus we can apply Lemma 7.13, by which we have the equations

$$\operatorname{Tan} \widetilde{F} = \operatorname{lin} \widetilde{F} \oplus N, \quad \operatorname{Tan} \widetilde{G} = \operatorname{lin} \widetilde{G} \oplus N, \tag{7.45}$$

$$V = \operatorname{lin} \widetilde{F} \oplus \operatorname{lin} \widetilde{G} \oplus N. \tag{7.46}$$

Let Φ, Ψ be the projections onto $\operatorname{lin} \widetilde{F}$ and $\operatorname{lin} \widetilde{G}$ determined by (7.46), i.e., $\operatorname{im} \Phi = \operatorname{lin} \widetilde{F}$, $\ker \Phi = \operatorname{lin} \widetilde{G} \oplus N = \operatorname{Tan} \widetilde{G}$, and $\operatorname{im} \Psi = \operatorname{lin} \widetilde{G}$, $\ker \Psi = \operatorname{lin} \widetilde{F} \oplus N = \operatorname{Tan} \widetilde{F}$.

By (ii) we have

$$H \cap N = H \cap \operatorname{Tan} \widetilde{F} \cap \operatorname{Tan} \widetilde{G} = \operatorname{Tan} F \cap \operatorname{Tan} G = \emptyset. \tag{7.47}$$

This implies

$$\langle 1, \tau\rangle = 0 \quad \text{for all } \tau \in N, \tag{7.48}$$

for if $\tau \in N$ and $\langle 1, \tau\rangle = \alpha \neq 0$, then $\alpha^{-1}\tau \in H \cap N$ contrary to (7.47).

Consider now an arbitrary $\omega \in K$. By (i),

$$\omega = \lambda\rho_1 + (1 - \lambda)\omega_1, \tag{7.49}$$

where $\rho_1 \in F$, $\omega_1 \in \operatorname{Tan} G$, $0 \leq \lambda \leq 1$. Here $(1-\lambda)\omega_1 \in \operatorname{Tan} \widetilde{G} = \operatorname{lin} \widetilde{G} \oplus N$, so $(1 - \lambda)\omega_1 = \sigma + \tau$ where $\sigma \in \operatorname{lin} \widetilde{G}$ and $\tau \in N$. Setting $\rho = \lambda\rho_1$, we can rewrite (7.49) as

$$\omega = \rho + \sigma + \tau,$$

where $\rho \in \operatorname{lin} \widetilde{F}$, $\sigma \in \operatorname{lin} \widetilde{G}$, $\tau \in N$. Thus the first term in the decomposition of ω determined by the direct sum (7.46) is of the form $\rho = \lambda\rho_1$ with

$\rho_1 \in F$ and $0 \leq \lambda \leq 1$. By the same argument, the second term is of the form $\sigma = \mu\sigma_1$ with $\sigma_1 \in G$ and $0 \leq \mu \leq 1$. Thus

$$\omega = \lambda\rho_1 + \mu\sigma_1 + \tau.$$

By (7.48) $\langle 1, \omega \rangle = \lambda + \mu$, so $\mu = 1 - \lambda$. Therefore this decomposition of ω takes the form

$$\omega = \lambda\rho_1 + (1-\lambda)\sigma_1 + \tau, \tag{7.50}$$

where $\rho_1 \in F,\ \sigma_1 \in G,\ \tau \in N,\ 0 \leq \lambda \leq 1$.

From this it follows that Φ is a normalized positive projection and that $\mathrm{im}^+\Phi = \widetilde{F}$, so $F = K \cap \mathrm{im}^+\Phi$. Similarly Ψ is a normalized positive projection with $\mathrm{im}^+\Psi = \widetilde{G}$, so $G = K \cap \mathrm{im}^+\Psi$.

We will now show that Φ, Ψ are complementary projections of V, i.e., that $\ker^+\Psi = \mathrm{im}^+\Phi$ and $\ker^+\Phi = \mathrm{im}^+\Psi$, or which is the same, that $\ker^+\Psi = \widetilde{F}$ and $\ker^+\Phi = \widetilde{G}$.

Clearly $\widetilde{F} \subset \ker^+\Psi$. To prove the converse relation, we will show $K \cap \ker^+\Psi \subset F$. Assume for contradiction that there exists $\omega \in K \setminus F$ such that $\Psi\omega = 0$. Consider now the decomposition of ω as in (7.49). Since $\omega \notin F$, we must have $\lambda < 1$. Consider next the decomposition of ω as in (7.50). Since $\Psi\omega = 0$, we must have $(1-\lambda)\sigma_1 = \Psi\omega = 0$. Thus $\omega = \lambda\rho_1 + \tau$. Now it follows from (7.48) that $\langle 1, \omega \rangle = \lambda$, which contradicts the inequality $\lambda < 1$.

With this we have shown $\ker^+\Psi = \mathrm{im}^+\Phi$. By the same argument $\ker^+\Phi = \mathrm{im}^+\Psi$, so Φ, Ψ are complementary projections. Observe now that

$$\ker\Phi = \mathrm{Tan}\,\widetilde{G} = \mathrm{Tan}\,(\mathrm{im}^+\Psi) = \mathrm{Tan}\,(\ker^+\Phi),$$

so Φ is a smooth projection. By the same argument, Ψ is a smooth projection.

Since $A = V^*$, we have $\Phi = P^*$ and $\Psi = Q^*$ where P, Q are weakly continuous positive projections on A. By the above, the dual projections $P^* = \Phi$ and $Q^* = \Psi$ are complementary smooth projections on V, so it follows from Theorem 7.10 that P, Q are bicomplementary projections on A. Since Φ and Ψ are normalized, so are P and Q. Thus P, Q is a pair of complementary compressions. We have already shown that $K \cap \mathrm{im}^+P^* = F$ and $K \cap \mathrm{im}^+Q^* = G$, so we are done. □

In order to specify the pair A, V of the two ordered normed spaces A and V in Theorem 7.52, it suffices to specify the distinguished base K, since this convex set determines not only V, but also $A = V^* \cong A_b(K)$. We will now use this fact to give some finite dimensional examples, starting out from a bounded convex set K, which we think of as embedded as the base of a cone V^+ in a finite dimensional linear space V, and then defining $A = A_b(K)$.

In our first example K is a simplex. Then each face is split, and therefore projective (Proposition 7.49). In this case the conditions (i), (ii) of Theorem 7.52 are trivially satisfied as remarked prior to the theorem.

In our next example K is a Euclidean ball. Then all points on the surface of K are extreme points, and it is easily seen that the conditions (i), (ii) of Theorem 7.52 are satisfied by a pair of faces $F = \{\rho\}$, $G = \{\sigma\}$ where ρ, σ are antipodal points on the surface of K. Thus each point on the surface of K is a projective face whose complementary projective face is the antipodal point on the surface of K.

Fig. 7.4 shows the base norm space V and the order unit space A in the particular case where K is a two-dimensional disk. Here the norm and the ordering are given by the unit balls V_1, A_1 and their positive parts V_1^+, A_1^+. We see that the proper projective faces of K are the points on the circular boundary of K, and the associated projective units in A are the points on the circle where the boundary of the positive cone meets the boundary of the unit ball. (The associated compressions were already indicated in Fig. 7.2)

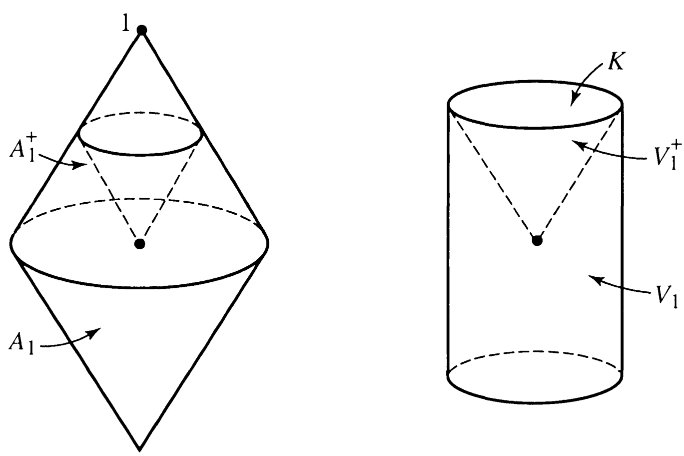

Fig. 7.4

Readers may find it instructive to compare the example above with the similar example where the circular base is replaced by the convex set in Fig. 7.3. These two convex sets are both shown in Fig. 7.5 below. While conditions (i), (ii) are satisfied in Fig. 7.5 (a), the first of those will not hold in Fig. 7.5 (b) as is easily seen. Observe also that Fig. 7.5 illustrates the importance of smoothness for the uniqueness of the projection E^* (already commented on after Theorem 7.15). In Fig. 7.5 (a) the direction of the "projection vector" (from ω to $E^*\omega$) is uniquely determined by the parallel tangents at F and G. But in Fig. 7.5 (b) it can be tilted up to 45° to either side with K still being projected onto $\mathrm{co}(F \cup G)$.

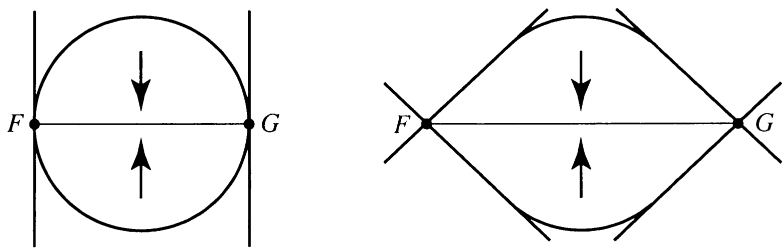

Fig. 7.5

We will also give two slightly less trivial examples. The convex set K in Fig. 7.6 is a section of a circular cylinder, determined by two planes intersecting the cylinder in two ellipses which at a point ρ on the surface of the cylinder have a common tangent orthogonal to the axis. In this example, the singleton $F = \{\rho\}$ and the opposite line segment G on the cylindrical surface will satisfy the conditions (i), (ii) of Theorem 7.52, so they are complementary projective faces.

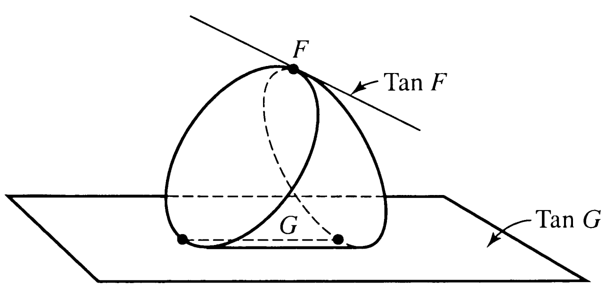

Fig. 7.6

The convex set in Fig. 7.7 is the convex hull of a line segment F and a circular disk which is orthogonal to the segment such that the midpoint of the segment lies on the periphery of the disk. In this example the line segment F and the singleton $G = \{\sigma\}$, where σ is the opposite point on the disk, will satisfy the conditions (i), (ii) of Theorem 7.52, so they are complementary projective faces.

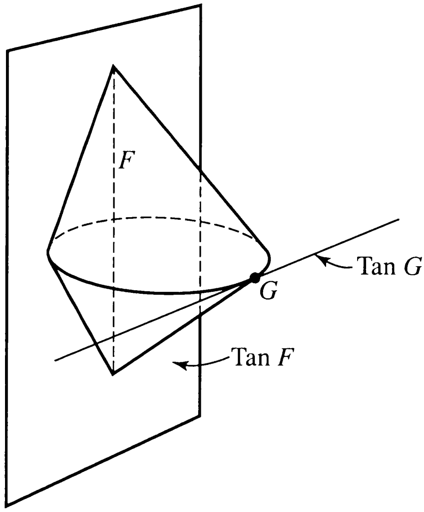

Fig. 7.7

Note, by the way, that the two examples above are closely related, as the convex sets K in Fig. 7.6 and Fig. 7.7 are in a natural way the polars of each other. (We leave the details to the interested reader.)

Notes

The concept of a general compression on an order unit space A in separating duality with a base norm space V was first introduced in [7, p. 8] under the name "P-projection". The characterization of concrete compressions (i.e., the maps $a \mapsto pap$) in order-theoretic terms as abstract compressions defined by bicomplementarity (Proposition 7.23), was first given in [7] for the von Neumann algebra case and in [9] for the JBW-algebra case. The geometric characterization of projective faces in Theorem 7.52 is new.

The correspondence of split faces and strongly complementary compressions in Propositions 7.48 and 7.49 has been known for quite some time. It was used by Wils [136] in his study of the *ideal center* and *central decomposition* in the general context of ordered linear spaces, which generalize the same concepts for C*-algebras (cf. Sakai's book [112]). The study of split faces is actually a central theme in much of the early theory of compact convex sets and the geometry of C*-algebra state spaces, which can be found in the book [6]. Related material published shortly after that book appeared include Vershik's paper [132], which among other things discusses a central decomposition akin to that of Wils (although with a different approach), and the papers by Andersen [15, 16] and Vesterstrøm [133] on split faces and linear extension operators for affine functions.

8 Spectral Theory

In this chapter we will study a pair A, V of an order unit space and a base norm space in separating order and norm duality under the additional assumption that each exposed face of the distinguished base K of V is projective. If A, V is such a pair, then we will say they satisfy *the standing hypothesis*. This hypothesis is satisfied when A is the self-adjoint part of a von Neumann algebra and V is the self-adjoint part of its predual, and also when A is a JBW-algebra and V is its predual. (In the former case, each norm closed face, and in particular each norm exposed face, is projective by virtue of (A 107). In the latter case, the same is true by Theorem 5.32.)

In the first two sections we will show that if the standing hypothesis is satisfied, then the compressions have lattice properties similar to those of the projections in von Neumann algebras and JBW-algebras. In the third section we will specialize to spaces in "spectral duality", for which we will establish a well-behaved functional calculus. In the fourth and last section we will study some concrete examples other than the motivating examples of von Neumann algebras and JBW-algebras.

Note that the two last sections of this chapter are not needed for the final results on geometric characterization of JB-algebra and C*-algebra state spaces, so they may be skipped by readers who are mainly interested in these results. Theorem 8.64 is the one result in these sections that will be used later (in the characterization of normal state spaces of JBW-algebras and von Neumann algebras.)

The lattice of compressions

If A, V is a pair of an order unit space and a base norm space in separating order and norm duality, then we will denote by $\mathcal{C}, \mathcal{F}, \mathcal{P}$ the sets of compressions, projective faces, and projective units, respectively. Recall from Chapter 7 that each of these sets has a natural ordering and complementation, and that there are natural complementation-preserving order isomorphisms mapping an element in each of these sets onto an associated element in each of the other two sets (cf. Proposition 7.32 and the explicit formulas (7.22), (7.23), (7.24) and (7.34)).

8.1. Proposition. *If the standing hypothesis of this chapter is satisfied, then $\mathcal{C}, \mathcal{P}, \mathcal{F}$ are (isomorphic) lattices, and the lattice operations are given by the following equations for each pair $F, G \in \mathcal{F}$:*

$$F \wedge G = F \cap G, \quad F \vee G = (F' \cap G')'. \tag{8.1}$$

Proof. Let $F, G \in \mathcal{F}$ and let p, q be the associated projective units. Consider the element $a = \frac{1}{2}(p' + q') \in A^+$ and an arbitrary $\omega \in K$. Then $\langle a, \omega \rangle = 0$ iff $\langle p', \omega \rangle = \langle q', \omega \rangle = 0$, which is equivalent to $\omega \in F \cap G$. Thus $F \cap G$ is an exposed face, hence is projective by the standing hypothesis of this chapter. Clearly $F \cap G$ is the greatest lower bound of F and G in $\mathcal{F}$. Since complementation reverses order, $\mathcal{F}$ is a lattice with lattice operations as in (8.1). Then $\mathcal{C}$ and $\mathcal{P}$ are lattices isomorphic to $\mathcal{F}$. □

8.2. Lemma. *Assume the standing hypothesis of this chapter. If p, q are projective units and a is an element of A^+ such that $a \le p$ and $a \le q$, then also $a \le p \wedge q$ (greatest lower bound in $\mathcal{P}$).*

Proof. Let P, Q be the compressions associated with p, q and let F, G be the projective faces associated with p, q. Since $a \le p$, then $a = 0$ on F', and since $a \le q$, then $a = 0$ on G'. Thus $a = 0$ on $F' \cup G'$. The set $H = \{\omega \mid \langle a, \omega \rangle = 0\}$ is an exposed, hence projective, face. Since $F' \cup G' \subset H$, then also $F' \vee G' \subset H$. Thus $a = 0$ on $F' \vee G' = (F \wedge G)'$. By Lemma 7.43,

$$a = (P \wedge Q)\, a \le (P \wedge Q)\, 1 = p \wedge q.$$

We are done. □

8.3. Theorem. *Assume the standing hypothesis of this chapter. If P, Q are compatible compressions, then $PQ = QP = P \wedge Q$.*

Proof. By Definition 7.40, $PQ = QP$ and also $Pq \le q$ and $Qp \le p$, where p, q are the projective units associated with P, Q. Define the element $r \in A^+$ by

$$r = Pq = PQ1 = QP1 = Qp. \tag{8.2}$$

Observe that $r = Pq \le P1 = p$, and also $r = Qq \le Q1 = q$, so $r \le p$ and $r \le q$.

Let s be an arbitrary element of A^+ such that $s \le p$ and $s \le q$. Then $s \in \mathrm{im}^+ P$ and $s \in \mathrm{im}^+ Q$ (Lemma 7.29). Hence $s = Ps \le Pq = r$. Thus r is the greatest lower bound of p and q in A^+.

In particular, since $p \wedge q$ is a lower bound for p and q, we have $r \ge p \wedge q$. By Lemma 8.2, $r \le p \wedge q$, so $r = p \wedge q$, or which is equivalent, $r = (P \wedge Q)\, 1$.

For each $a \in A_1^+$,

$$PQa \le PQ1 = r = (P \wedge Q)\, 1.$$

Hence $PQa \in \mathrm{im}^+(P \wedge Q)$ (again by Lemma 7.29). Therefore

$$PQa = (P \wedge Q)PQa = (P \wedge Q)Qa = (P \wedge Q)a.$$

Thus $PQ = P \wedge Q$, so we are done. □

8.4. Lemma. *Assume the standing hypothesis of this chapter. If P, Q are mutually compatible compressions, both compatible with an element $a \in A$, then $P \wedge Q$ and $P \vee Q$ are also compatible with this element.*

Proof. Clearly we can assume $a \in A^+$. Since a is compatible with P and Q, then $Pa \leq a$ and $Qa \leq a$ (Definition 7.41). Hence

$$(P \wedge Q)\, a = PQa \leq Pa \leq a.$$

Thus a is compatible with $P \wedge Q$ (cf. Lemma 7.37). Since P, Q are compatible, then so are P', Q' (Lemma 7.42). Also a is compatible with P' and Q', hence with $P' \wedge Q'$, and in turn with $P \vee Q = (P' \wedge Q')'$. □

Recall that compressions P and Q are orthogonal if $P \preceq Q'$ (Definition 7.35).

8.5. Proposition. *Assume the standing hypothesis of this chapter. If P, Q are orthogonal compressions, both compatible with an element $a \in A$, then $P \vee Q$ is compatible with a, and*

$$(P \vee Q)\, a = Pa + Qa. \tag{8.3}$$

Proof. Since P and Q are compatible with a, P' and Q' are also compatible with a. Since P, Q are orthogonal, P, Q are compatible, as are P' and Q' (Lemma 7 42). By Lemma 8.4, $P \vee Q = (P' \wedge Q')'$ is compatible with a, so it follows from Theorem 8.3 that

$$a = (P \vee Q)\, a + (P' \wedge Q')\, a = (P \vee Q)\, a + P'Q'a. \tag{8.4}$$

Since P and Q are compatible with a, $P'a = a - Pa$ and $Q'a = a - Qa$. Since $P \perp Q$, then $Q \preceq P'$, so $P'Q = Q$. Hence

$$P'Q'a = P'(a - Qa) = P'a - Qa = a - Pa - Qa.$$

Substituting this value for $P'Q'a$ in (8.4), gives (8.3). □

We will occasionally write $P_1 \dot{+} \cdots \dot{+} P_n$ in place of $P_1 \vee \cdots \vee P_n$ if $P_1, \ldots, P_n$ are mutually orthogonal compressions. Thus by definition

$$(8.5)\quad P = P_1 \dot{+} \cdots \dot{+} P_n \iff P = P_1 \vee \cdots \vee P_n \text{ and } P_i \perp P_j \text{ for } i \neq j.$$

8.6. Lemma. *Assume the standing hypothesis of this chapter. If P, Q, R are mutually orthogonal compressions, then $P \perp (Q \dot{+} R)$.*

Proof. Let p, q, r be the projective units associated with P, Q, R. Since $p \perp q$ and $p \perp r$, then $p \leq q'$ and $p \leq r'$. Thus $p \leq q' \wedge r' = (q \vee r)'$, so $p \perp (q \vee r)$. Therefore $P \perp (Q \vee R)$. Since Q, R are orthogonal, we can also write $P \perp (Q \dot{+} R)$. □

8.7. Proposition. *Assume the standing hypothesis of this chapter. If $P_1, \ldots, P_n$ are mutually orthogonal compressions, all compatible with $a \in A$, then $P_1 \dot{+} \cdots \dot{+} P_n$ is compatible with a, and*

$$(8.6)\qquad (P_1 \dot{+} \cdots \dot{+} P_n)\, a = P_1 a + \cdots + P_n a.$$

Proof. By Lemma 8.6, repeated use of Proposition 8.5 gives (8.6). □

8.8. Proposition. *Assume the standing hypothesis of this chapter. If $p_1, p_2, \ldots, p_n$ are projective units, then the following are equivalent:*

(i) $\sum\limits_{i=1}^{n} p_i \leq 1$,

(ii) $p_i \perp p_j$ *for* $i \neq j$,

(iii) $\bigvee\limits_{i=1}^{n} p_i = \sum\limits_{i=1}^{n} p_j$.

Proof. (i)⇒(ii) If $p_i + p_j \leq 1$, then $p_i \leq 1 - p_j = p_j'$, so $p_i \perp p_j$. (ii)⇒(iii) If $p_i \perp p_j$ for $i \neq j$, then by Proposition 8.7,

$$\bigvee_{i=1}^{n} p_i = \left(\bigvee_{i=1}^{n} P_i\right) 1 = \sum_{i=1}^{n} P_i 1 = \sum_{i=1}^{n} p_i\,.$$

(iii)⇒(i) If (iii) holds, then trivially $\sum_{i=1}^{n} p_i = \bigvee_{i=1}^{n} p_i \leq 1$. □

Observe that it follows from Proposition 8.8 that $\sum_{i=1}^{n} p_i = p$ for a projection p iff $\bigvee_{i=1}^{n} p_i = p$ and $p_i \perp p_j$ for $i \neq j$. So if $P_1, \ldots, P_n$ and P are the compressions associated with $p_1, \ldots, p_n$ and p, then

$$(8.7)\qquad p_1 + \cdots + p_n = p \iff P_1 \dot{+} \cdots \dot{+} P_n = P\,.$$

8.9. Lemma. *Assume the standing hypothesis of this chapter. If P, Q are compressions such that $Q \preceq P$ and p, q are the associated projective* *units, then* $p - q$ *is the projective unit associated with* $P \wedge Q' = PQ'$. *Moreover,* $R = P \wedge Q'$ *is the unique compression such that* $P = Q \dotplus R$, *and if* $a \in A$ *is compatible with* P *and* Q, *then* $Ra = Pa - Qa$.

Proof. Since $Q \preceq P$, then P, Q are compatible, so P, Q' are also compatible (Lemma 7.42). By Theorem 8.3, $P \wedge Q' = PQ'$. Hence $(P \wedge Q')\,1 = P(1 - Q1) = p - q$, so $p - q = p \wedge q'$.

By (8.7), for a compression R the equation $P = Q \dotplus R$ is equivalent to $p = q + r$, where r is the projective unit associated with R. Therefore the unique solution to this equation is the compression $R = P \wedge Q'$ associated with the projective unit $p - q = p \wedge q'$.

The last statement of the lemma follows from Lemma 8.4 and Proposition 8.7. □

8.10. Theorem. *Assume the standing hypothesis of this chapter. The lattice of compressions (as well as the isomorphic lattices of projective units and projective faces) is orthomodular; that is, for each pair* $P, Q \in \mathcal{C}$,

(i) $P'' = P$,
(ii) $Q \preceq P \Rightarrow P' \preceq Q'$,
(iii) $P \wedge P' = 0$ *and* $P \vee P' = I$,
(iv) $Q \preceq P \Rightarrow P = (P \wedge Q') \vee Q$.

Proof. (i), (ii), and $P \vee P' = I$ follow directly from the corresponding statements for projective units and Proposition 8.8 (iii). Now $P \wedge P' = (P' \vee P)' = I' = 0$ follows, so (iii) holds. Finally, (iv) follows from Lemma 8.9. □

We state the following simple observation as a lemma for later reference.

8.11. Lemma. *Assume the standing hypothesis of this chapter. If* P_1, P_2, Q_1, Q_2 *are compressions such that* $P_1 \preceq Q_1$, $P_2 \preceq Q_2$ *and* $Q_1 \perp Q_2$, *then also* $P_1 \perp P_2$.

Proof. Since $P_1 \preceq Q_1 \preceq Q_2' \preceq P_2'$, then $P_1 \perp P_2$. □

8.12. Lemma. *Assume the standing hypothesis of this chapter. Two compressions* P, Q *are compatible iff*

$$P = P \wedge Q \dotplus P \wedge Q'. \tag{8.8}$$

Proof. Let p, q be the projective units associated with P, Q, and observe that (by (8.7)) the equation (8.8) is equivalent to the equation

$$p = p \wedge q + p \wedge q'. \tag{8.9}$$

If P, Q, and then also P, Q', are compatible, then $P \wedge Q = PQ$ and $P \wedge Q' = PQ'$ (Theorem 8.3). Thus in this case,

$$p \wedge q + p \wedge q' = (P \wedge Q)\,1 + (P \wedge Q')\,1 = P(Q + Q')\,1 = P1 = p,$$

so (8.9) is satisfied.

Conversely, assume (8.9). Since $p \wedge q \leq q$, then $p \wedge q \in \operatorname{im}^+ Q$ (Lemma 7.29), so $Q(p \wedge q) = p \wedge q$. Since $p \wedge q' \leq q'$, then $p \wedge q' \in \operatorname{im}^+ Q' = \ker^+ Q$, so $Q(p \wedge q') = 0$. Now

$$Qp = Q(p \wedge q) + Q(p \wedge q') = p \wedge q \leq p,$$

so P, Q are compatible. □

Note that generally $(P \wedge Q) \perp (P \wedge Q')$ (Lemma 8.11), so we can replace $\dot{+}$ by $\vee$ in (8.8). Also we can replace $=$ by $\preceq$ since the opposite relation is trivial. Therefore the compatibility criterion (8.8) can be rewritten as

$$P \preceq (P \wedge Q) \vee (P \wedge Q'). \tag{8.10}$$

8.13. Theorem. *Assume the standing hypothesis of this chapter. Two compressions P, Q are compatible iff there exist three compressions R, S, T such that $S \perp T$ and*

$$P = R \dot{+} S, \quad Q = R \dot{+} T. \tag{8.11}$$

If such a decomposition exists, then it is unique; in fact $R = P \wedge Q$, $S = P \wedge Q'$, $T = Q \wedge P'$.

Proof. Assume first that P, Q are compatible and set $R = P \wedge Q$, $S = P \wedge Q'$, $T = Q \wedge P'$. Clearly $S \perp T$ (Lemma 8.11), and (8.11) follows from Lemma 8.12.

Assume next that R, S, T are three compressions such that $S \perp T$ and (8.11) is satisfied. Note first that since $R \preceq P$ and $R \preceq Q$, then $R \preceq P \wedge Q$.

Note also that since $R \perp S$, then $R \preceq S'$, and since $T \perp S$, then $T \preceq S'$. Hence $Q = R \vee T \preceq S'$, so $S \preceq Q'$. We also have $S \preceq P$, so $S \preceq P \wedge Q'$.

Combining the two inequalities just proven, we find that

$$P = R \vee S \preceq (P \wedge Q) \vee (P \wedge Q').$$

Hence P, Q are compatible by the criterion (8.10).

Next we show that $R = P \wedge Q$. Write $r = R1$, $s = S1$, $t = T1$. Then $p = r + s$. Since $S \perp R$ and $S \perp T$, then $S \perp Q$ (Lemma 8.6). Therefore $Qs = QS1 = 0$. Since $r \leq q$, then $Qr = r$ (Lemma 7.29), so

$$r = Q(r + s) = Qp = QP1 = (Q \wedge P)1 = q \wedge p,$$

where the penultimate equality follows from Theorem 8.3. Finally,

$$s = p - r = p - (p \wedge q) = p \wedge q'$$

(by (8.9)), and similarly $t = q \wedge p'$. This gives uniqueness and completes the proof. □

8.14. Corollary. *Assume the standing hypothesis of this chapter. If P, Q are compatible compressions, then*

$$P \vee Q = P \wedge Q \dot{+} P \wedge Q' \dot{+} Q \wedge P' = P \dot{+} Q \wedge P'. \tag{8.12}$$

Proof. By (8.11), $P \vee Q$ is the least upper bound of the compressions $(P \wedge Q)$, $(P \wedge Q')$, $(Q \wedge P')$. These compressions are mutually orthogonal (Lemma 8.11), so we have the first equality in (8.12). By Lemma 8.12, we also have the second equality. □

8.15. Corollary. *Assume the standing hypothesis of this chapter. If p, q are compatible projective units, then*

$$p + q = p \vee q + p \wedge q. \tag{8.13}$$

Proof. Using (8.9) (and (8.9) with p and q exchanged), we have

$$p + q = p \wedge q + p \wedge q' + q \wedge p + q \wedge p'.$$

Now (8.13) follows from (8.12). □

Recall that a *Boolean algebra* is a distributive orthocomplemented lattice, i.e., a lattice with a least element 0, a greatest element I and a complementation $P \mapsto P'$ which satisfies the conditions (i), (ii), (iii) of Theorem 8.10 and the distributive law

$$P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R). \tag{8.14}$$

Note that for an orthocomplemented lattice (8.14) is equivalent to the "dual" distributive law, obtained by interchanging $\vee$ and $\wedge$. Note also that condition (iv) of Theorem 8.10 (the "orthomodular law") follows from (8.14) by replacing R by Q'. Thus Boolean algebras are a special class of orthomodular lattices.

Generally we will say that a set of elements in an orthocomplemented lattice is a *complemented sublattice* if it contains 0 and I and is closed under the complementation $P \mapsto P'$ as well as the lattice operations $P, Q \mapsto P \wedge Q$ and $P, Q \mapsto P \vee Q$.

8.16. Theorem. *Assume the standing hypothesis of this chapter. A set $\mathcal{L}$ of compressions is a Boolean algebra (under the operations induced from the complemented lattice $\mathcal{C}$ of all compressions) iff it is a complemented sublattice of $\mathcal{C}$ and each pair $P, Q \in \mathcal{L}$ is compatible.*

Proof. Consider first a complemented sublattice $\mathcal{L} \subset \mathcal{C}$ which is a Boolean algebra. If $P, Q \in \mathcal{L}$, then by the distributive law,

$$P = P \wedge (Q \vee Q') = (P \wedge Q) \vee (P \wedge Q'),$$

so the compatibility criterion (8.8) is satisfied.

Assume next that $\mathcal{L}$ is a a complemented sublattice of $\mathcal{C}$ for which each pair $P, Q \in \mathcal{L}$ is compatible. To prove the distributive law (8.14), we consider an arbitrary triple P, Q, R of compressions in $\mathcal{L}$ with associated projective units p, q, r. By Corollary 8.14 (together with Theorem 8.3 and Proposition 8.7),

$$P \wedge (Q \vee R)\, 1 = P(Q \dotplus RQ')\, 1 = PQ1 \dotplus PRQ'1.$$

Hence

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r \wedge q') \leq (p \wedge q) \vee (p \wedge r).$$

In fact, the equality sign must hold in this relation, since $p \wedge q \leq p \wedge (q \vee r)$ and also $p \wedge r \leq p \wedge (q \vee r)$. This gives (8.14) and completes the proof. □

Recall from Proposition 7.31 that if A, V is a pair of an order unit space and a base norm space in separating order and norm duality and if P_0 is a compression with associated projective unit p_0 and associated projective face F_0, then $P_0(A) = \operatorname{im} P_0$ is an order unit space with distinguished order unit p_0, $P_0^*(V) = \operatorname{im} P_0^*$ is a base norm space with distinguished base F_0, and $P_0(A)$, $P_0^*(V)$ are in separating order and norm duality .

8.17. Proposition. *Assume the standing hypothesis of this chapter and let P_0 be a compression with associated projective unit p_0 and associated projective face F_0. Then the pair $P_0(A)$, $P_0^*(V)$ will also satisfy the standing hypothesis. For this pair the compressions are the compressions P of A such that $P \preceq P_0$ (restricted to $P_0(A)$), the projective units are the projective units p of A such that $p \leq p_0$, and the projective faces are the projective faces F of K such that $F \subset F_0$. For this pair, the complementary projection of $P \preceq P_0$ is P' (restricted to $P_0(A)$).*

Proof. Let F be an exposed face of the distinguished base F_0 of $P_0^*(V)$. This means that there exists an element of $P_0(A)^+$ which vanishes precisely on F. Otherwise stated, there exists an element a of $P_0(A)^+$ such that $a = 0$ on F and $a > 0$ on $F_0 \setminus F$.

Now define $b = a + P_0'1$. We know that $P_0'1 = 0$ on F_0, and also that $P_0'1 = p_0' > 0$ on $K \setminus F_0$ (cf. (7.25) and (7.22)). Hence $b = a = 0$ on F, $b = a > 0$ on $F_0 \setminus F$, and also $b > 0$ on $K \setminus F_0$. Thus F is an exposed, hence projective, face of K.

Let P be the compression and p be the projective unit associated with F. Since $F \subset F_0$, then $P \preceq P_0$, so P and P_0, and P' and P_0, are compatible, and therefore commute (Lemma 7.42). It follows that P and P' leave $P_0(A)$ invariant. The restrictions $P|P_0(A)$ and $P'|P_0(A)$ are seen to be complementary compressions (with duals $P^*|P_0^*(V)$ and $P'^*|P_0^*(V)$) whose associated projective units are p and $p' \wedge p_0$, and whose associated projective faces are F and $F_0 \wedge F'$. This shows that the pair $P_0(A)$, $P_0^*(V)$ satisfies the standing hypothesis, and also that the compressions, projective units and projective faces for this pair of spaces are as stated in the proposition. □

In Proposition 8.17, we observe that if $A = V^*$, then it is easy to check that the dual of $\operatorname{im} P_0^*$ is $\operatorname{im} P_0$.

8.18. Corollary. *Assume the standing hypothesis of this chapter and let P_0 be a central compression. Then a compression $P \preceq P_0$ is central for $P_0(A)$ iff it is central for A.*

Proof. By Proposition 7.48, P is central for A (or $P_0(A)$) iff $Pa \le a$ for all $a \in A^+$ (respectively, all $a \in P_0(A)^+$).

If P is central for A, then trivially $Pa \le a$ for all $a \in P_0(A)^+ \subset A^+$, so P is central for $P_0(A)$. Conversely, assume P is central for $P_0(A)$. Since $P \preceq P_0$, then $P = PP_0$, so for each $a \in A^+$,

$$Pa = PP_0a \le P_0a \le a.$$

Thus P is central for A. □

Remark. We will continue the discussion of the order theoretic approach to quantum mechanical measurement theory initiated in the Remark after Proposition 7.49 in the preceding chapter, but now under the standing hypothesis of the present chapter.

This hypothesis says that each exposed face, i.e., each set of the form $F = \{\omega \in K \mid \langle a, \omega \rangle = 0\}$ with $a \in A^+$, is a projective face. Interpreted physically, this means that for each positive observable $a \in A^+$ the question: "does a have the value 0 ?" is a quantum mechanical proposition, i.e., a question which can be answered by a quantum mechanical measuring device, say a filter for a beam of particles. This is true in standard quantum mechanics, and it is natural to assume it as an axiom in the general order-theoretic approach.

As explained before, the expected value of an observable $a \in A$ measured on a system in a state $\omega \in K$, is $\langle a, \omega \rangle$. For a proposition $p \in \mathcal{P}$, the expectation $\langle p, \omega \rangle$ is the same as the probability of the value 1. Therefore the face F associated with p is the set of all states for which the particle will certainly (i.e., with probability 1) appear with the value 1.

Now it follows from Theorem 8.10 that when the standing hypothesis is satisfied, then the propositions form an orthomodular lattice with the same physical interpretation as in standard quantum mechanics. Thus the proposition lattice has the properties of "quantum logic", which are often used as the basic assumption in axiomatic quantum theory. (See, for example, Piron's paper [87].)

The lattice of compressions when $A = V^*$

In this section we will continue to study a pair of spaces A, V satisfying the standing hypothesis of this chapter, but we will now also assume that $A = V^*$. Clearly, this condition is satisfied in the motivating examples of von Neumann algebras and JBW-algebras in duality with their preduals.

We know that if $A = V^*$, then we can identify A with the space $A_b(K)$ of all bounded affine functions on K (by the map $a \mapsto \widehat{a}$ of (A 11)). Then the ordering in A will be the same as the pointwise ordering in $A_b(K)$, and the weak topology in A (defined by the duality with V) will be the same as the topology of pointwise convergence in $A_b(K)$. Since $A = V^*$, the ordered vector space A is monotone complete. Thus each increasing net $\{a_\alpha\}$ bounded above has a least upper bound $a \in A$, which is also the weak (= pointwise) limit of $\{a_\alpha\}$, and in this case we will write $a_\alpha \nearrow a$. Similarly each decreasing net $\{a_\alpha\}$ bounded below has a greatest lower bound $a \in A$, which is also the weak (= pointwise) limit of $\{a_\alpha\}$, and in this case we will write $a_\alpha \searrow a$. Note that if $a_\alpha \nearrow a$ (or $a_\alpha \searrow a$) and P is a compression, then (by weak continuity) $Pa_\alpha \nearrow Pa$ (respectively $Pa_\alpha \searrow Pa$).

8.19. Lemma. *Assume the standing hypothesis of this chapter and $A = V^*$. If $\{p_\alpha\}$ is a decreasing net of projective units, and $p_\alpha \searrow a \in A$, then a is a projective unit, so it is also the greatest lower bound of $\{p_\alpha\}$ in $\mathcal{P}$. Similarly, if $\{p_\alpha\}$ is an increasing net of projective units, and $p_\alpha \nearrow a \in A$, then a is a projective unit, so it is also the least upper bound of $\{p_\alpha\}$ in $\mathcal{P}$.*

Proof. Assume $p_\alpha \searrow a$. Let F_α be the projective face associated with p_α for each α, and set $G = \{\omega \in K \mid \langle a, \omega \rangle = 0\}$. Now G is an exposed, hence projective, face. Let P be the compression, and p the projective unit, associated with G'. For each α,

$$G \supset \{\omega \in K \mid \langle p_\alpha, \omega \rangle = 0\} = F'_\alpha .$$

Hence $G' \subset F_\alpha$, so $p \le p_\alpha$, and then $p \le a$. But by (7.23), p is the greatest element of A_1^+ which vanishes on G, so $p = a$. Thus a is a projective unit. The remaining statement can be proven in similar fashion, or can be derived from the first by considering the net $\{1 - p_\alpha\}$. $\square$

8.20. Proposition. *If the standing hypothesis of this chapter is satisfied and $A = V^*$, then $\mathcal{C}, \mathcal{P}, \mathcal{F}$ are complete lattices, and the lattice operations are given by the following equations for each family $\{F_\alpha\} \subset \mathcal{F}$:*

$$\bigwedge_\alpha F_\alpha = \bigcap_\alpha F_\alpha, \quad \bigvee_\alpha F_\alpha = (\bigcap_\alpha F'_\alpha)'. \tag{8.15}$$

Moreover, if $\{P_\alpha\}$ is an increasing net of compressions compatible with an element $a \in A$, then the compression $P = \bigvee_\alpha P_\alpha$ is also compatible with a and $P_\alpha a \nearrow Pa$. Similarly, if $\{P_\alpha\}$ is a decreasing net of compressions compatible with $a \in A$, then $P = \bigwedge_\alpha P_\alpha$ is compatible with a and $P_\alpha a \searrow Pa$.

Proof. By Proposition 8.1, $\mathcal{C}, \mathcal{P}, \mathcal{F}$ are isomorphic lattices, so it suffices to show that $\mathcal{F}$ is a complete lattice with lattice operations as in (8.15). In fact, by (8.1) it suffices to show that the first equality in (8.15) is satisfied for each decreasing net $\{F_\alpha\}$ in $\mathcal{F}$.

Let $\{F_\alpha\}$ be a decreasing net in $\mathcal{F}$ and let p_α be the projective unit associated with F_α for each α. By Lemma 8.19, $p_\alpha \searrow p$ where p is the greatest lower bound of $\{p_\alpha\}$ in $\mathcal{P}$. Therefore the projective face F associated with p is the greatest lower bound of $\{F_\alpha\}$ in $\mathcal{F}$. For each α, F_α is the set of points in K where p_α takes the value 1. Also F is the set of points in K where p takes the value 1. If $\omega \in \bigcap_\alpha F_\alpha$, then $\langle p_\alpha, \omega\rangle = 1$ for each α, hence also $\langle p, \omega\rangle = 1$, so $\omega \in F$. Conversely, if $\omega \in F$, then $1 = \langle p, \omega\rangle \le \langle p_\alpha, \omega\rangle$ for each α, so $\omega \in \bigcap_\alpha F_\alpha$. Thus $F = \bigcap_\alpha F_\alpha$, and we are done.

It suffices to prove the last statement of the proposition for an increasing net $\{P_\alpha\}$. Without loss of generality we assume $0 \le a \le 1$. We have $P_\alpha a \le a$ for each α. If $\alpha < \beta$, then $P_\alpha \preceq P_\beta$ so $P_\alpha a = P_\beta P_\alpha a \le P_\beta a$. Thus $\{P_\alpha a\}$ is an increasing net bounded above by a. Since $P_\alpha a \le a$, applying P to both sides gives $P_\alpha a \le Pa$ for each α.

Set $P = \bigvee_\alpha P_\alpha$ and also $p = P1$ and $p_\alpha = P_\alpha 1$ for each α. By Lemma 8.19, $p_\alpha \nearrow p$, so $p - p_\alpha \searrow 0$. For each α, let $R_\alpha = P \wedge P'_\alpha$. By Lemma 8.9,

$$(P - P_\alpha)1 - (P - P_\alpha)a = P(1-a) - P_\alpha(1-a) = R_\alpha(1-a) \ge 0.$$

Hence

$$0 \le (P - P_\alpha)\, a \le (P - P_\alpha)\, 1 = p - p_\alpha \searrow 0.$$

Thus $P_\alpha a \nearrow Pa$.

By compatibility, $P_\alpha a \leq a$ for each α. Thus $Pa \leq a$, so P is also compatible with a. □

Recall from (A 1) that a subset F of K is a *semi-exposed face* if there is a family $\{a_\alpha\}$ in A^+ such that

$$F = \{\omega \in K \mid \langle a_\alpha, \omega \rangle = 0 \text{ for all } \alpha\}. \tag{8.16}$$

By our standing hypothesis, each exposed face of K is projective. With the additional assumption $A = V^*$, each semi-exposed face is also projective, (and therefore is exposed, cf. (7.22)), as we will now show.

8.21. Corollary. *If the standing hypothesis of this chapter is satisfied and $A = V^*$, then each semi-exposed face of K is projective.*

Proof. Let F be a semi-exposed face of K, given as in (8.16). For each index α, we denote by F_α the set of points in K where a_α takes the value zero. Then each F_α is an exposed, hence projective, face. By (8.16), $F = \bigcap_\alpha F_\alpha$. Now it follows from Proposition 8.20 that F is a projective face. □

8.22. Definition. Assume the standing hypothesis of this chapter and $A = V^*$. If $a \in A$, then the set of all compressions compatible with a and with all compressions compatible with a, will be called the *$\mathcal{C}$-bicommutant* of a. We will also refer to the corresponding sets of projective faces and projective units as the *$\mathcal{F}$-bicommutant* and *$\mathcal{P}$-bicommutant* respectively.

The use of the term "bicommutant" is of course motivated by its use in operator algebras. It is also partly justified by the fact that in our present context, compatibility of compressions is the same as commutation.

8.23. Proposition. *Assume the standing hypothesis of this chapter and $A = V^*$. The $\mathcal{C}$-bicommutant of an element $a \in A$ is a complete Boolean algebra. More specifically, it is closed under arbitrary (finite or infinite) lattice operations in $\mathcal{C}$ and is a Boolean algebra under these operations.*

Proof. Let $\mathcal{B}$ be the $\mathcal{C}$-bicommutant of a. Consider two compressions $P_1, P_2 \in \mathcal{B}$. Each of them is compatible with a and with all compressions compatible with a. Hence they are compatible with each other. Then by Lemma 8.4, $P_1 \wedge P_2$ and $P_1 \vee P_2$ are also compatible with a. Moreover, if Q is a compression compatible with a, then $P_1 \wedge P_2$ and $P_1 \vee P_2$ are also compatible with Q (again by Lemma 8.4, this time with the projective unit $q = Q1$ in place of a). Thus $\mathcal{B}$ is closed under finite lattice operations induced from $\mathcal{C}$. It follows by an easy application of Proposition 8.20 that $\mathcal{B}$ is also closed under infinite lattice operations.

Clearly $\mathcal{B}$ contains 0 and 1, and $P \in \mathcal{B}$ implies $P' \in \mathcal{B}$. Thus $\mathcal{B}$ is a complemented sublattice of $\mathcal{C}$. Since each pair of compressions in $\mathcal{B}$ is compatible, $\mathcal{B}$ is a Boolean algebra (Theorem 8.16). □

8.24. Proposition. *Assume the standing hypothesis of this chapter and $A = V^*$. For each $a \in A^+$ there is a least projective unit p such that $a \in \mathrm{face}(p)$, and p is the unique element of $\mathcal{P}$ such that*

$$\{\omega \in K \mid \langle a, \omega\rangle = 0\} = \{\omega \in K \mid \langle p, \omega\rangle = 0\}. \tag{8.17}$$

Proof. Let G be the set of points in K where a takes the value zero. Then G is an exposed, hence projective, face. Let p be the projective unit associated with G'. Thus G' is the set of points where p takes the value 1, and G is the set of points where p takes the value 0. Now (8.17) is satisfied, and this equation determines p uniquely (by the 1-1 correspondence of projective faces and projective units).

Leaving aside the trivial case $a = 0$, we define $a_1 = \|a\|^{-1}a \in A_1^+$. By (7.23), p is the greatest element of A_1^+ which vanishes on G, so $a_1 \le p$. Now $a \le \|a\|\, p$, so $a \in \mathrm{face}(p)$.

Let q be a projective unit such that $a \in \mathrm{face}(q)$. Then $a \le \lambda q$ for some $\lambda \ge 0$, so the projective face F of points where q takes the value zero is contained in the projective face G of points where a takes the value zero. Hence $G' \subset F'$, so $q' \le p'$, and therefore $p \le q$. Thus p is the least projective unit such that $a \in \mathrm{face}(q)$. □

8.25. Definition. Assume the standing hypothesis of this chapter and $A = V^*$. For each $a \in A^+$ we denote by $r(a)$ the least projective unit p such that $a \in \mathrm{face}(p)$.

Note that by (8.17), $r(a)$ is characterized among projective units by the following equivalence where $\omega \in K$:

$$\langle a, \omega\rangle = 0 \iff \langle r(a), \omega\rangle = 0. \tag{8.18}$$

Clearly $r(a)$ is the customary range projection in the von Neumann algebra case, cf. (A 99).

8.26. Proposition. *Assume the standing hypothesis of this chapter and $A = V^*$. If $a \in A^+$, then $r(a)$ is characterized by each of the following statements.*

(i) *The compression associated with $r(a)$ is the least compression P such that $Pa = a$.*
(ii) *The complement of the compression associated with $r(a)$ is the greatest compression P such that $Pa = 0$.*
(iii) *$r(a)$ is the least projective unit p such that $a \le \|a\|\, p$.*

(iv) $r(a)$ *is the greatest projective unit contained in the semi-exposed face of A^+ generated by a.*

Proof. (i) and (iii) follow from Lemma 7.29, and (ii) is equivalent to (i) since $P \mapsto P'$ reverses order and $\ker^+ P' = \operatorname{im}^+ P$. For the proof of (iv), we recall that the semi-exposed face generated by a is equal to $\{a\}^{\bullet\bullet}$ (Proposition 7.2). Thus a projective unit p is in this face iff it is annihilated by each $\omega \in K$ that annihilates a, i.e., iff

$$\{\omega \in K \mid \langle a, \omega \rangle = 0\} \subset \{\omega \in K \mid \langle p, \omega \rangle = 0\}.$$

By (8.17) and (7.22), the left side is the projective face associated with $r(a)'$, and the right with p', so this is equivalent to $r(a)' \leq p'$. Thus $p \leq r(a)$, so $r(a)$ is the greatest projective unit in the semi-exposed face generated by a. □

By statement (iii) of the proposition above, we generally have $a \leq \|a\|\, r(a)$. Note in particular that

$$a \leq r(a) \quad \text{when} \quad 0 \leq a \leq 1. \tag{8.19}$$

Recall from (A 24) that two elements $\rho, \sigma \in V^+$ are said to be *orthogonal*, in symbols $\rho \perp \sigma$, if

$$\|\rho - \sigma\| = \|\rho\| + \|\sigma\|. \tag{8.20}$$

In particular, two elements $\rho, \sigma \in K$ are orthogonal if $\|\rho - \sigma\| = 2$, that is, if their distance is maximal in K.

8.27. Theorem. *Assume the standing hypothesis of this chapter and $A = V^*$. Every $\omega \in V$ admits a unique decomposition $\omega = \rho - \sigma$ such that $\rho, \sigma \geq 0$ and $\rho \perp \sigma$, and this decomposition is given by $\rho = P^*\omega$ and $\sigma = -P'^*\omega$ for a compression P of A.*

Proof. By (A 26), there is a decomposition $\omega = \rho - \sigma$ such that $\rho, \sigma \geq 0$ and $\rho \perp \sigma$. We will show that this decomposition is unique and of the given form, by constructing a compression P, depending only on ω, such that $\rho = P^*\omega$ and $\sigma = -P'^*\omega$.

Assume (without loss of generality) that $\|\omega\| = 1$. Since $A = V^*$, the closed unit ball $A_1 = [-1, 1]$ is w*-compact. Therefore the linear functional $a \mapsto \langle a, \omega \rangle$ attains its norm at a point $x \in A_1$. Thus $-1 \leq x \leq 1$ and $\langle x, \omega \rangle = 1$. Define $a = \frac{1}{2}(1 + x)$ and $b = \frac{1}{2}(1 - x)$. Then $0 \leq a \leq 1$ and $0 \leq b \leq 1$, and $x = a - b$. Now

$$\begin{aligned} 1 = \langle x, \omega \rangle &= \langle a, \rho \rangle - \langle b, \rho \rangle - \langle a, \sigma \rangle + \langle b, \sigma \rangle \\ &\leq \langle a, \rho \rangle + \langle b, \sigma \rangle \leq \|\rho\| + \|\sigma\| = \|\omega\|. \end{aligned}$$

Since $\|\omega\| = 1$, the equality signs must hold. Therefore we conclude that

$$\langle a, \rho \rangle = \|\rho\| \quad \text{and} \quad \langle a, \sigma \rangle = 0. \tag{8.21}$$

(Similarly $\langle b, \sigma \rangle = \|\sigma\|$ and $\langle b, \rho \rangle = 0$, but we shall not need these equalities.)

Let P be the compression and F the projective face associated with $r(a)$. Leaving aside the trivial cases $\rho = 0$ and $\sigma = 0$, we define $\rho_1 = \|\rho\|^{-1}\rho$ and $\sigma_1 = \|\sigma\|^{-1}\sigma$. We have $0 \le a \le r(a) \le 1$ (cf. (8.19)), so by (8.21)

$$1 = \langle a, \rho_1 \rangle \le \langle r(a), \rho_1 \rangle \le 1.$$

Thus $\langle r(a), \rho_1 \rangle = 1$. Hence $\rho_1 \in F$, so $\rho \in \mathrm{im}^+ P^* = \ker^+ P'^*$. Therefore $P^*\rho = \rho$ and $P'^*\rho = 0$.

By (8.21) we have $\langle a, \sigma_1 \rangle = 0$, so it follows from (8.18) that we also have $\langle r(a), \sigma_1 \rangle = 0$. Hence $\sigma_1 \in F'$, so $\sigma \in \mathrm{im}^+ P'^* = \ker^+ P^*$. Therefore $P'^*\sigma = \sigma$ and $P^*\sigma = 0$.

Combining the equalities above, we find that $P^*\omega = \rho$ and $\mathcal{P}'^*\omega = -\sigma$. The proof is complete. □

8.28. Definition. Assume the standing hypothesis of this chapter and $A = V^*$. An element $\omega \in V$ is called *central* if $(P + P')^*\omega = \omega$ for all compressions P.

This definition is motivated by C*-algebras and von Neumann algebras. It is easily seen that a bounded linear functional ω on a C*-algebra $\mathcal{A}$ satisfies the equation $\omega(ab) = \omega(ba)$ for all pairs $a, b \in \mathcal{A}$ iff its normal extension to the enveloping von Neumann algebra $\mathcal{A}^{**}$ satisfies the same requirement, and this is the case iff

$$\langle pap + p'ap', \omega \rangle = \langle a, \omega \rangle$$

for all projections $p \in \mathcal{A}^{**}$ and all $a \in \mathcal{A}^{**}$, i.e., iff ω is central as defined above (cf. [7, Lemma 12.8 and Thm. 12.9]).

8.29. Lemma. *Assume the standing hypothesis of this chapter and $A = V^*$. If ω is a central element of V, and $\omega = \rho - \sigma$ is the orthogonal decomposition described in Theorem 8.27, then ρ and σ are also central.*

Proof. For every compression P we have $\omega = (P + P')^*\omega$, so

$$\omega = (P + P')^*\rho - (P + P')^*\sigma, \tag{8.22}$$

and since $(P + P')^*$ preserves norms in V^+ (Lemma 7.17), then

$$\|\omega\| = \|\rho\| + \|\sigma\| = \|(P + P')^*\rho\| + \|(P + P')^*\sigma\|. \tag{8.23}$$

By the uniqueness of orthogonal decompositions, it follows from (8.22) and (8.23) that

$$\rho = (P + P')^*\rho \quad \text{and} \quad \sigma = (P + P')^*\sigma.$$

Hence ρ and σ are central. □

8.30. Proposition. *If the standing hypothesis of this chapter is satisfied and $A = V^*$, then the set of central elements in V is a vector lattice*

Proof. The set of central elements in V is a linear subspace of V, and is positively generated by Lemma 8.29. To prove that it is a vector lattice, it suffices to show that the element ρ in the orthogonal decomposition $\omega = \rho - \sigma$ is the least upper bound of ω and 0 within the space of central elements. Clearly $\rho \geq \omega$ and $\rho \geq 0$. Suppose τ is any central element such that $\tau \geq \omega$ and $\tau \geq 0$. Let P be a compression such that $\rho = P\omega$ and $\sigma = -P'\omega$ (Theorem 8.27). Then

$$\tau = (P + P')^*\tau \geq P^*\tau \geq P^*\omega = \rho,$$

as desired. □

By Proposition 7.30, each projective unit is an extreme point of A_1^+. In the special case of a von Neumann algebra or a JBW-algebra the converse is also true, cf. (A 75) and Proposition 1.40, but we cannot prove this in our present generality. However, we have the following partial converse.

8.31. Proposition. *If the standing hypothesis of this chapter is satisfied and $A = V^*$, then the projective units are w^*-dense in the set of extreme points of A_1^+.*

Proof. By Milman's theorem (A 33), it suffices to prove that $\overline{\text{co}}\,\mathcal{P} = A_1^+$ (w*-closure). Since the map $a \mapsto 2a - 1$ is an affine homeomorphism from $A_1^+ = [0, 1]$ onto A_1, we may as well show that $\overline{\text{co}}\,\mathcal{S} = A_1$ where $\mathcal{S} = \{2p - 1 \mid p \in \mathcal{P}\}$. This will follow (by Hahn–Banach separation as in (A 30)) if we can prove that for each $\omega \in V$,

$$\|\omega\| = \sup_{s \in \mathcal{S}} |\langle s, \omega \rangle|. \tag{8.24}$$

By Theorem 8.27, for given $\omega \in V$ we can choose an orthogonal decomposition $\omega = \rho - \sigma$ where $\rho, \sigma \geq 0$, $\rho \perp \sigma$ and $\rho = P^*\omega$, $\sigma = -P'^*\omega$ for some $P \in \mathcal{C}$. Let p be the projective unit associated with P and set $s = 2p - 1$. Then $s \in \mathcal{S}$ and

$$\|\omega\| = \|\rho\| + \|\sigma\| = \langle 1, \rho \rangle + \langle 1, \sigma \rangle = \langle P1 - P'1, \omega \rangle = \langle s, \omega \rangle,$$

which proves (8.24). □

We are now going to establish an explicit formula expressing the range projection of Pa where P is a compression and $a \in A^+$, in terms of the projective unit associated with P and the range projection of a itself. This formula, proved for JBW-algebras in Proposition 2.31, is a useful technical tool. But it is also of some intrinsic interest in that it links compressions of A with the maps $\phi_p : \mathcal{P} \to \mathcal{P}$ defined for each $p \in \mathcal{P}$ by $\phi_p(q) = (q \vee p') \wedge p$; these are the Sasaki projections of orthomodular lattice theory.

8.32. Theorem. *Assume the standing hypothesis of this chapter and $A = V^*$. If $a \in A^+$ and P is a compression with associated projective unit p, then*

$$r(Pa) = (r(a) \vee p') \wedge p. \tag{8.25}$$

Proof. Let Q be the compression associated with $r(a)$ and let R be the compression associated with $r(Pa)$. Thus $Q \vee P'$ is the compression associated with $r(a) \vee p'$. Observe that since

$$a \in \text{face}\,(r(a)) = \text{im}^+ Q \subset \text{im}\,(Q \vee P'),$$

then $(Q \vee P')\,a = a$. Observe also that since $P' \preceq Q \vee P'$, then P', and hence also P, is compatible with $Q \vee P'$. Hence

$$(Q \vee P')Pa = P(Q \vee P')\,a = Pa.$$

Thus $Q \vee P'$ fixes the element Pa. Therefore $R \preceq Q \vee P'$ (Proposition 8.26 (i)), so $r(Pa) \leq r(a) \vee p'$. Clearly P fixes Pa, so we also have $r(Pa) \leq p$. Thus

$$r(Pa) \leq (r(a) \vee p') \wedge p. \tag{8.26}$$

Since $r(Pa) \leq p$, then $R \preceq P$. Now R, and then also R', is compatible with P. Hence $R' \wedge P = R'P$, and since $R'Pa = 0$, then $(R' \wedge P)\,a = 0$. Now

$$a \in \ker^+(R' \wedge P) = \text{im}^+(R \vee P').$$

Thus $R \vee P'$ fixes a. Therefore (by Proposition 8.26 (i)) $r(a) \leq r(Pa) \vee p'$. Hence also

$$r(a) \vee p' \leq r(Pa) \vee p'. \tag{8.27}$$

We have seen above that $R \preceq P$, or equivalently $R \perp P'$, so $r(Pa) \perp p'$. Thus we can replace $\vee$ by $+$ on the right side of (8.27). Then after rearranging,

$$r(a) \vee p' - p' \leq r(Pa).$$

By Lemma 8.9, the left side of this inequality is equal to $(r(a) \vee p') \wedge p$. Thus

$$(r(a) \vee p') \wedge p \leq r(Pa). \tag{8.28}$$

The two inequalities (8.26) and (8.28) give (8.25). □

For given non-zero $\omega \in V^+$ we will denote by $\operatorname{ProjFace}(\omega)$ the projective face generated by $\omega_1 := \|\omega\|^{-1}\omega$ in K. Thus $\operatorname{ProjFace}(\omega)$ is the intersection of all $F \in \mathcal{F}$ which contain ω_1 (Proposition 8.20). We define $\operatorname{ProjFace}(0)$ to be the empty set.

8.33. Corollary. *Assume the standing hypothesis of this chapter and $A = V^*$. For each $\omega \in V^+$ and each $p \in \mathcal{P}$ with associated projective face F,*

$$\operatorname{ProjFace}(P^*\omega) = (\operatorname{ProjFace}(\omega) \vee F') \wedge F. \tag{8.29}$$

Proof. Dualize the proof of Theorem 8.32 by replacing P by P^*, a by ω, $r(a)$ by $\operatorname{ProjFace}(\omega)$ and $r(Pa)$ by $\operatorname{ProjFace}(P^*\omega)$. □

8.34. Proposition. *Assume the standing hypothesis of this chapter and $A = V^*$. For each $\omega \in K$ there is a least central projection $c(\omega)$ such that $\langle c(\omega), \omega \rangle = 1$; the associated compression is the least central compression P such that $P^*\omega = \omega$, and the associated projective face is the smallest split face which contains ω.*

Proof. Recall first that for $\omega \in K$ and $P \in \mathcal{P}$ one has $\langle P1, \omega \rangle = 1$ iff $P^*\omega = \omega$, cf. (7.21) and (7.24).

If c_1, c_2 are central projective units such that $\langle c_1, \omega \rangle = \langle c_1, \omega \rangle = 1$, then the associated compressions P_1, P_2 are also central, so $P_1 \wedge P_2 = P_1 P_2$. Hence

$$(P_1 \wedge P_2)^*\omega = (P_1P_2)^*\omega = P_2^* P_1^*\omega = \omega,$$

so $\langle c_1 \wedge c_2, \omega \rangle = 1$. Therefore the set of central projective units c such that $\langle c, \omega \rangle = 1$ is directed downward. By Lemma 8.19, this directed set has a greatest lower bound $c(\omega) \in \mathcal{P}$ which is also its weak limit. Therefore, by weak continuity, $(P + P')\,c(\omega) = c(\omega)$ for all $P \in \mathcal{C}$. Thus $c(\omega)$ is compatible with all $P \in \mathcal{C}$, i.e., $c(\omega)$ is central. By weak convergence $\langle c(\omega), \omega \rangle = 1$. Clearly $c(\omega)$ is the least central projective unit with this property.

By the introductory remark of this proof, the compression P associated with $c(\omega)$ is the least central compression such that $P^*\omega = \omega$. By the explicit characterization of the projective face F associated with $c(\omega)$ (7.24), and by Proposition 7.49, F is the smallest split face containing ω. □

8.35. Definition. Assume the standing hypothesis of this chapter and $A = V^*$. For each $\omega \in K$ the least central projective unit which takes the value 1 at ω will be called the *central support* (or *central carrier*) of ω, and it will be denoted by $c(\omega)$ (as in the proposition above).

We now turn to the study of minimal projective units.

8.36. Proposition. *Assume the standing hypothesis of this chapter and $A = V^*$. If p is a minimal (non-zero) projective unit, then the associated compression P has one dimensional range, i.e.,* $\operatorname{im} P = \mathbf{R}p$, *and the associated projective face F is a singleton, i.e., $F = \{\widehat{p}\}$ where $\widehat{p}$ is the only point of K where p takes the value 1. Moreover, the map $p \mapsto \widehat{p}$ is a 1-1 map from the set of minimal elements of $\mathcal{P}$ onto the set of exposed points of K.*

Proof. Let p be a minimal projective unit with associated compression P and associated projective face F. Relativizing to the pair of spaces $P(A), P^*(V)$ as in Proposition 8.17 and making use of Proposition 8.31, we conclude that the positive unit ball $[0,p]$ of $P(A) = \operatorname{im} P$ is the w*-closure of the projective units in $P(A)$, which are precisely those projective units q in A such that $q \le p$. By minimality of p, there are no such projective units other than $q = 0$ and $q = 1$. Thus $[0,p]$ contains only scalar multiples of p, so $\operatorname{im} P = \mathbf{R}p$. Since $P(A)$ and $P^*(V)$ are in separating duality, then $P^*(V)$ is also one dimensional, so F is a singleton.

The remaining statements of the proposition follow easily from the explicit formulas for F in terms of p and P, cf. (7.22) and (7.24), and from our standing hypothesis that all exposed faces are projective. □

We will now define orthogonality in A^+ by a condition similar to the characterization in Theorem 8.27 of orthogonality in V^+. (Like the characterization in Theorem 8.27, this condition will not be useful unless we assume our standing hypothesis, or some other hypothesis which ensures the existence of "enough" compressions.)

8.37. Lemma. *Assume the standing hypothesis of this chapter and $A = V^*$, and let $b, c \in A^+$. If P is a compression with associated projective face F, then the following are equivalent:*

(i) $Pb = b$ *and* $P'c = c$,
(ii) $P'b = 0$ *and* $Pc = 0$,
(iii) $b = 0$ *on* F' *and* $c = 0$ *on* F.

Proof. (i)$\Leftrightarrow$(ii) This follows from the fact that P and P' are complementary.

(ii)$\Rightarrow$(iii) Assume (ii). If $\omega \in F'$, then $\langle b, \omega \rangle = \langle P'b, \omega \rangle = 0$ by (7.25). Thus $b = 0$ on F'. Similarly $c = 0$ on F.

(iii)$\Rightarrow$(i) Assume (iii). By Lemma 7.43, $Pb = b$ and $P'c = c$. □

8.38. Definition. Assume the standing hypothesis of this chapter and $A = V^*$. We will say that two elements $a, b \in A^+$ are *orthogonal*, and we will write $a \perp b$, if there exists a compression P such that any one of the three equivalent conditions in Lemma 8.37 is satisfied. We will also say that an equality $a = b - c$ is an *orthogonal decomposition* of an element $a \in A$ if $b, c \in A^+$ and $b \perp c$.

Note that $Pa \perp P'a$ for all $a \in A^+$ and $P \in \mathcal{C}$. Note also that if $a = b - c$ is an orthogonal decomposition where $Pb = b$ and $P'c = c$ for a compression P, then $b = Pa$ and $c = -P'a$, from which it follows that $a = Pa + P'a$, so P is compatible with a. These facts will be used frequently and without reference in the sequel.

8.39. Proposition. *Assume the standing hypothesis of this chapter and $A = V^*$, and let $a, b \in A^+$. Then $a \perp b$ iff $r(a) \perp r(b)$, and if $a \perp b$, then*

$$r(a+b) = r(a) + r(b). \tag{8.30}$$

Proof. Let $a \perp b$, say $Pa = a$ and $P'b = b$ for a compression P. Let R_a and R_b be the compressions associated with $r(a)$ and $r(b)$ respectively. Since $Pa = a$, we have $R_a \preceq P$, and similarly $R_b \preceq P'$. Then $R_a \perp R_b$ (Lemma 8.11), so $r(a) \perp r(b)$. Conversely, if $r(a) \perp r(b)$, then $R_b R_a = 0$ (Proposition 7.34), so $R_b a = R_b R_a a = 0$. Then $R_b' a = a$ and $R_b b = b$ imply $a \perp b$.

Now assume $a \perp b$. We may assume without loss of generality that $a, b \leq 1$. Then $a \leq r(a)$ and $b \leq r(b)$ (cf. (8.19)). By Proposition 8.8,

$$a + b \leq r(a) + r(b) = r(a) \vee r(b). \tag{8.31}$$

Thus the projective unit $r(a) \vee r(b)$ majorizes $a + b$. Therefore

$$r(a+b) \leq r(a) \vee r(b).$$

Conversely, $r(a) \leq r(a+b)$ and $r(b) \leq r(a+b)$, so

$$r(a) \vee r(b) \leq r(a+b).$$

By (8.31) these two inequalities give the desired equality (8.30). $\square$

8.40. Lemma. *Assume the standing hypothesis of this chapter and $A = V^*$. If $a \in A^+$ and Q is a compression compatible with a, then*

$$r(Qa) = Q(r(a)). \tag{8.32}$$

Proof. Assume without loss that $a \leq 1$. We have $Qa \perp Q'a$, so it follows from Proposition 8.39 and compatibility that

$$r(Qa) + r(Q'a) = r(Qa + Q'a) = r(a). \tag{8.33}$$

If q is the projective unit associated with Q, then $r(Qa) \leq r(Q1) = q$, so $r(Qa) \in \text{face}(q) = \text{im}^+Q$. Similarly $r(Q'a) \in \text{im}^+Q' = \ker^+Q$. Now applying Q to (8.33) gives the desired equality (8.32). □

8.41. Proposition. *Assume the standing hypothesis of this chapter and $A = V^*$. If $a \in A^+$, then $r(a)$ is in the $\mathcal{P}$-bicommutant of a.*

Proof. If P is the compression corresponding to $r(a)$, then $Pa = a$ (Proposition 8.26), and so $P'a = 0$, which gives $(P + P')a = a$. Thus $r(a)$ is compatible with a. If Q is a compression compatible with a, then $Qa \leq a$ (Lemma 7.37). Hence $r(Qa) \leq r(a)$, and by Lemma 8.40 also $Q(r(a)) \leq r(a)$. Thus $r(a)$ is compatible with Q. □

Spaces in spectral duality

We will now establish a general spectral theory in the context of an order unit space A in separating order and norm duality with a base norm space V. This theory, first presented in [7] and [9], generalizes the spectral theory of von Neumann algebras and JBW-algebras. As in these cases, it may be regarded as a non-commutative measure theory which reduces to ordinary measure theory when $A = L^\infty(\mu)$ and $V = L^1(\mu)$ for a probability measure μ (the commutative case). In the general case, the elements of A will play the role of the bounded measurable functions, and the orthomodular lattices $\mathcal{F}, \mathcal{P}, \mathcal{C}$ will play the role of the Boolean algebras of, respectively, measurable sets, characteristic functions of measurable sets and multiplication by characteristic functions of measurable sets.

8.42. Definition. Assume that A, V are a pair of an order unit and base norm space in separating order and norm duality, with $A = V^*$. If in addition each $a \in A$ admits a least compression P such that $Pa \geq a$ and $Pa \geq 0$, then we will say that A and V are in *spectral duality.*

Generally, spectral duality will imply our standing hypothesis (that all exposed faces are projective), as we will show in Theorem 8.52. If A and V are reflexive Banach spaces, and in particular if A and V are finite dimensional, then the converse also holds, as we will show in Theorem 8.72. But in the general case, our standing hypothesis will not imply spectral duality. (A counterexample is sketched in [9, p. 508].)

The new condition of spectral duality will ensure that there are enough elements in $\mathcal{C}$ (or in $\mathcal{P}$) to approximate each $a \in A$ by elements $\sum_{i=1}^{n} \lambda_i p_i$ where $p_1, \ldots, p_n \in \mathcal{P}$, and will provide a unique spectral resolution of a by the same construction as in the motivating examples. Note in this connection that in the commutative case, where the compressions are multiplication operators $P : f \mapsto \chi_E \cdot f$, the requirements for P in the definition above characterize multiplication by χ_E, where $E = a^{-1}((0, \infty))$. Later on we will show that the self-adjoint part of a von Neumann algebra is in spectral duality with its predual, and likewise for a JBW-algebra (Proposition 8.76).

8.43. Lemma. *Let A, V be an order unit space and a base norm space in separating order and norm duality, let $a \in A$ and let P be a compression. Then $Pa \geq a$ iff P is compatible with a and $P'a \leq 0$.*

Proof. If P is compatible with a and $P'a \leq 0$, then trivially $Pa = a - P'a \geq a$. Conversely, if $Pa \geq a$, then

$$Pa - a \in \ker^+ P = \operatorname{im}^+ P'.$$

Hence $P'(Pa - a) = Pa - a$, so $-P'a = Pa - a$. From this we conclude that $P'a \leq 0$, and also that $a = Pa + P'a$, so P is compatible with a. □

We state the following observation as a lemma for later reference.

8.44. Lemma. *Let A, V be an order unit space and a base norm space in separating order and norm duality, let $a \in A$ and let P be a compression with associated projective face F. Then the following are equivalent:*

(i) *$Pa \geq a$ and $Pa \geq 0$,*
(ii) *P is compatible with a and $Pa \geq 0$ and $P'a \leq 0$,*
(iii) *F is compatible with a and $a \geq 0$ on F and $a \leq 0$ on F'.*

Thus A and V are in spectral duality iff there exists a least compression P compatible with a such that either one of (i) or (ii) holds, or there exists a least projective face F such that (iii) holds.

Proof. (i)$\Leftrightarrow$(ii) Clear from Lemma 8.43. (ii)$\Leftrightarrow$(iii) Clear from Proposition 7.28. □

We are now going to establish that spectral duality implies the standing hypothesis of this chapter. To accomplish this, we will need to re-establish several key results with the hypothesis of spectral duality instead of the standing hypothesis.

For this purpose, we now introduce a useful technical tool. For an element $b \geq 0$ in an order unit space A we write $[0, b]$ for the order interval $\{a \in A \mid 0 \leq a \leq b\}$. Recall that face$(b)$ denotes the face generated by b in A^+, i.e., the set of all $a \in A$ such that $0 \leq a \leq \lambda b$ for some $\lambda \in \mathbf{R}^+$.

8.45. Definition. Let A, V be in spectral duality. An element b of A_1^+ has the *facial property* if

$$[0, b] = A_1^+ \cap \text{face}(b). \tag{8.34}$$

The equivalence of (ii) and (iii) in Lemma 7.29 implies that every projective unit has the facial property. We now establish the converse, which is based on an argument of Riedel [106, 1.1(f)$\Rightarrow$1.1(c)].

8.46. Lemma. *Assume A, V are in spectral duality. If $b \in A_1^+$ has the facial property, then b is a projective unit.*

Proof. Let $b \in A_1^+$ have the facial property and define $a = 2b - 1$. Clearly $-1 \leq a \leq 1$. By spectral duality, there exists a compression P such that $Pa \geq a$ and $Pa \geq 0$. By Lemma 8.43, P is compatible with a, and then also with b.

Let $p = P1$ and $p' = P'1$. Since $P(2b - 1) \geq 0$, then $2Pb \geq p$. By compatibility of P and b, $2b \geq 2Pb$, so $2b \geq p$. Thus by the facial property of b, we conclude that $b \geq p$.

Since $Pa \geq a$, then

$$2b - 1 \leq P(2b - 1) \leq P1 = p.$$

Therefore $2b - p \leq 1$. Since $2b - p = b + (b - p) \geq 0$ and $2b - p \leq 2b$, then $2b - p$ is in $\text{face}(b) \cap [0, 1]$, so again by the facial property $2b - p \leq b$. This implies $b \leq p$, so $b = p$, and thus b is a projective unit. □

8.47. Lemma. *Assume A, V are in spectral duality. If P, Q are compatible compressions, then $PQ = QP = P \wedge Q$.*

Proof. Define $r = PQ1$. The first two paragraphs of the proof of Theorem 8.3 apply without change to show that r is the greatest lower bound of p and q in A^+.

Next we show that r has the facial property. To this end we consider $a \in A_1^+ \cap \text{face}(r)$. Since $r \leq p$, then $a \in \text{face}(p) = \text{im}^+ P$, so $a = Pa$. Since $r \leq q$, also $a \in \text{face}(q) = \text{im}^+ Q$, so $a = Qa \leq Q1 = q$. From this it follows that $a = Pa \leq Pq = r$. Thus, r has the facial property, and by Lemma 8.46, r is a projective unit. Then by the first paragraph, r is the greatest lower bound of p and q in $\mathcal{P}$. Thus $r = p \wedge q$, or which is equivalent, $r = (P \wedge Q)\,1$.

Now the final paragraph of the proof of Theorem 8.3 applies without change to show that $PQ = QP = P \wedge Q$. □

Note that under the assumption of spectral duality, we do not yet know that the order isomorphic sets $\mathcal{C}$, $\mathcal{P}$, $\mathcal{F}$ are lattices. However, if P and

Q are compatible compressions, by Lemma 8.47 the greatest lower bound $P \wedge Q$ exists. Furthermore, P' and Q' will be compatible, so $P' \wedge Q'$ exists, and then so does $P \vee Q = (P' \wedge Q')'$. Since orthogonal compressions are compatible (Lemma 7.42), the least upper bound of two orthogonal compressions exists.

The following two results are the same as Lemma 8.4 and Proposition 8.5 respectively, except for the hypothesis being spectral duality instead of the standing hypothesis of this chapter. The proofs are the same as those of the corresponding earlier results, except that the application of Theorem 8.3 is replaced by Lemma 8.47.

8.48. Lemma. *Assume A, V are in spectral duality. If P, Q are mutually compatible compressions, both compatible with an element $a \in A$, then $P \wedge Q$ and $P \vee Q$ are also compatible with this element.*

8.49. Proposition. *Assume A, V are in spectral duality. If P, Q are orthogonal compressions, both compatible with an element $a \in A$, then $P \vee Q$ is compatible with a, and*

$$(P \vee Q)\, a = Pa + Qa.$$

Recall that if $A = V^*$, then A is monotone complete. (See the discussion preceding Lemma 8.19). In particular, if A, V are in spectral duality, then A is monotone complete.

8.50. Lemma. *Assume A, V are in spectral duality. If $\{p_n\}$ is a decreasing (increasing) sequence of projective units, then the element $\inf_n p_n$ ($\sup_n p_n$) of A is a projective unit, hence it is the greatest lower bound (least upper bound) of $\{p_n\}$ among the projective units.*

Proof. Assume that $\{p_n\}$ is a decreasing sequence of projective units and set $b = \inf_n p_n$. We will show that b has the facial property, so we consider an arbitrary $a \in A_1^+ \cap \operatorname{face}(b)$. For every $n = 1, 2, \ldots$, then $a \in \operatorname{face}(b) \subset \operatorname{face}(p_n)$, so $a \in \operatorname{im}^+ P_n$, where P_n is the compression associated with p_n. Hence $a = P_n a \le P_n 1 = p_n$ for all n; therefore $a \le \inf_n p_n = b$, so b has the facial property. By Lemma 8.46, b is a projective unit. The case of an increasing sequence follows by taking complements. □

8.51. Lemma. *Assume A, V are in spectral duality, let $a \in A^+$, and let λ_1, λ_2 be real numbers such that $0 < \lambda_1 < \lambda_2$. If Q_2 is a compression compatible with a, with associated projective unit q_2, such that*

$$Q_i a \le \lambda_i q_i, \quad Q_i' a \ge \lambda_i q_i' \tag{8.35}$$

for $i = 2$, then there exists a compression Q_1 compatible with a, with associated projective unit q_1, such that $Q_1 \preceq Q_2$, and such that (8.35) holds also for $i = 1$.

Proof. Set $b = Q_2 a$. By Lemma 8.44 (applied to the element $b - \lambda_1 1 \in A$), there exists a compression P compatible with b, having an associated projective unit p, such that $Pb \geq \lambda_1 p$ and $P'b \leq \lambda_1 p'$. We consider now the complementary compression $Q = P'$ with associated projective unit $q = p'$, and we observe that Q is also compatible with b, and that

$$Qb \leq \lambda_1 q, \quad Q'b \geq \lambda_1 q'. \tag{8.36}$$

Using compatibility and (8.36), we observe that

$$\lambda_1 q' \leq Q'b \leq b. \tag{8.37}$$

We have $Q_2'b = Q_2'Q_2a = 0$, and so by (8.37) $Q_2'q' = 0$. Then $Q_2q' = q'$, so $Q' \preceq Q_2$. From this it follows that Q', and then also Q, is compatible with Q_2. Since Q' is compatible with b, then

$$Q'a = Q'Q_2a = Q'b \leq b \leq a. \tag{8.38}$$

Hence Q', and then also Q, is compatible with a.

Define now $Q_1 = QQ_2 = Q \wedge Q_2$, and let q_1 be the projective unit associated with Q_1. Since Q and Q_2 are mutually compatible and both are compatible with a, then $Q_1 = Q \wedge Q_2$ is also compatible with a (Lemma 8.48).

It remains to prove that we can replace Q by Q_1, b by a, and q by q_1 in (8.36). Applying Q_1 to both sides of the first inequality in (8.36) gives $Q_1 b \leq \lambda_1 q_1$. Since $Q_1 b = Q_1 Q_2 a = Q_1 a$, this gives

$$Q_1 a \leq \lambda_1 q, \tag{8.39}$$

which is the first of the desired inequalities.

We have seen that $Q' \preceq Q_2$, so $Q' \perp Q_2'$, and by definition $Q_1' = (Q \wedge Q_2)' = Q' \vee Q_2'$. Hence by Proposition 8.49,

$$Q_1'a = Q'a + Q_2'a. \tag{8.40}$$

By the same argument, $Q_1'1 = Q'1 + Q_2'1$, or otherwise stated, $q_1' = q' + q_2'$. Note also that by (8.38) $Q'a = Q'b$. Combining these two facts with (8.40), (8.36), and (8.35), we get

$$Q_1'a = Q'b + Q_2'a \geq \lambda_1 q' + \lambda_2 q_2' \geq \lambda_1(q' + q_2') = \lambda_1 q_1', \tag{8.41}$$

which is the second of the two desired inequalities. With (8.39) and (8.41) we have shown that (8.35) holds also for $i = 1$. □

8.52. Theorem. *If A, V are in spectral duality, then every exposed face of K is projective.*

Proof. Let F be an exposed face of K and choose $a \in A^+$ such that

$$F = \{\omega \in K \mid \langle a, \omega \rangle = 0\}. \tag{8.42}$$

Consider a sequence $\{\lambda_n\}$ of real numbers such that $\lambda_1 > \lambda_2 > \cdots > 0$ and $\lim_n \lambda_n = 0$. By repeated use of Lemma 8.51, we construct a decreasing sequence $\{Q_n\}$ of compressions compatible with a such that for each $n = 1, 2, \ldots$ the compression Q_n and its associated projective unit q_n satisfy the inequalities

$$Q_n a \leq \lambda_n q_n, \quad Q'_n a \geq \lambda_n q'_n. \tag{8.43}$$

Consider the compression $Q = \bigwedge_n Q_n$ with associated projective unit q. Let G_n be the projective face associated with Q_n, and G the projective face associated with Q. By (8.43), for each n we have $a \leq \lambda_n$ on G_n, and therefore $a \leq \lambda_n$ on $G = \bigwedge_m G_m \subset G_n$. Then $a \leq \lim_n \lambda_n = 0$ on G. Since $a \geq 0$, then $a = 0$ on G. By (8.42), $G \subset F$.

Conversely if $\omega \in F$, then by (8.43) and the compatibility of Q'_n and a,

$$\lambda_n \langle q'_n, \omega \rangle \leq \langle Q'_n a, \omega \rangle \leq \langle a, \omega \rangle = 0,$$

so $\langle q'_n, \omega \rangle = 0$ for $n = 1, 2, \ldots$. By Lemma 8.50, $q_n \searrow q$, so $q'_n \nearrow q'$. Hence $\langle q', \omega \rangle = 0$, which shows that $\omega \in G$. Thus $G = F$, which proves that F is projective. □

Spectral duality can be defined in various other equivalent ways (among those the ones in [7] and [9]), as we proceed to show.

In the next lemma (and in the sequel) we will write "$a > 0$ on F" when an element $a \in A$ is strictly greater than zero everywhere on a set $F \subset K$, i.e., if $\langle a, \omega \rangle > 0$ for all $\omega \in F$.

8.53. Lemma. *Assume the standing hypothesis of this chapter and $A = V^*$. Let $a \in A$, let P be a compression with associated projective face F, and assume a is compatible with F and $a \geq 0$ on F and $a \leq 0$ on F'. Then $G = \{\omega \in K \mid \langle Pa, \omega \rangle = 0\}$ is a projective face whose complement $\widetilde{F} = G'$ is compatible with a and satisfies the conditions $\widetilde{F} \subset F$, $a > 0$ on $\widetilde{F}$ and $a \leq 0$ on $\widetilde{F}'$. Moreover, $\widetilde{P}a = Pa$ where $\widetilde{P}$ is the compression associated with $\widetilde{F}$.*

Proof. Observe first that $Pa = 0$ on F' and $P'a = 0$ on F by (7.25). Furthermore, $Pa \geq 0$ and $P'a \leq 0$ follow from Proposition 7.28. Clearly

G is an exposed, hence projective, face of K. Since $Pa = 0$ on F', we have $F' \subset G$, and then $G' \subset F$.

If $\omega \in G$, then

$$\langle a, \omega \rangle = \langle Pa, \omega \rangle + \langle P'a, \omega \rangle = \langle P'a, \omega \rangle \leq 0.$$

If $\omega \in G' \subset F$, then

$$\langle a, \omega \rangle = \langle Pa, \omega \rangle + \langle P'a, \omega \rangle = \langle Pa, \omega \rangle \geq 0.$$

Here the inequality must be valid, otherwise $\omega \in G$. Thus $a \leq 0$ on G and $a > 0$ on G'.

Let Q be the compression associated with G. We have $Pa = 0$ on G, so $QPa = 0$ and $Q'Pa = Pa$ (Lemma 7.43). Since $G' \subset F$, we have $P'a = 0$ on G'. Therefore also $Q'P'a = 0$ and $QP'a = P'a$. Hence

$$Qa + Q'a = Q(Pa + P'a) + Q'(Pa + P'a) = P'a + Pa = a.$$

Thus Q, and then also G and G', are compatible with a. Now $\widetilde{F} = G'$ has all the properties announced in the lemma.

Finally, the compression $\widetilde{P} = Q'$ associated with $\widetilde{F}$ satisfies $\widetilde{P} \preceq P$, so the equality $Q'Pa = Pa$ gives $\widetilde{P}a = \widetilde{P}Pa = Pa$. $\square$

8.54. Lemma. *Assume that A, V are a pair of an order unit and base norm space in separating order and norm duality, with $A = V^*$. Then the following are equivalent:*

(i) *A and V are in spectral duality.*
(ii) *For each $a \in A$ there exists a unique projective face F compatible with a such that $a > 0$ on F and $a \leq 0$ on F'.*

If these conditions are satisfied, then the face F in (ii) is also the least projective face compatible with a such that $a \geq 0$ on F and $a \leq 0$ on F'.

Proof. (i)$\Rightarrow$(ii) Assume A and V are in spectral duality, and let $a \in A$. By Theorem 8.52, every norm exposed face is projective. By Lemma 8.44 there exists a least projective face F compatible with a such that $a \geq 0$ on F and $a \leq 0$ on F'. By Lemma 8.53, there exists a projective face $\widetilde{F} \subset F$ with the same properties as F, but now with the strict inequality $a > 0$ on $\widetilde{F}$. By minimality, $F = \widetilde{F}$. Thus $a > 0$ on F.

To prove uniqueness, we consider an arbitrary projective face F_1 compatible with a such that $a > 0$ on F_1 and $a \leq 0$ on F_1'. By minimality, $F \subset F_1$. Thus we can apply Theorem 8.10 (iv) (the orthomodular law), by which

$$F_1 = F \vee (F_1 \wedge F') = F \vee (F_1 \cap F').$$

Here we must have $F_1 \cap F' = \emptyset$. For if $\omega \in F_1 \cap F'$, then $\langle a, \omega \rangle > 0$ since $\omega \in F_1$ and $\langle a, \omega \rangle \leq 0$ since $\omega \in F'$, a contradiction. Thus $F_1 = F$ as desired.

(ii)$\Rightarrow$(i) We first show that exposed faces of K are projective. Fix $a \in A^+$, and let $G = \{\omega \mid \langle a, \omega \rangle = 0\}$. Let F be as in (ii); we will show $G = F'$.

Since $a \geq 0$ on K and $a \leq 0$ on F', then $a = 0$ on F', so $F' \subset G$. Let $\omega \in G$, and let P be the compression associated with F. Then P and a are compatible, so

$$0 = \langle a, \omega \rangle = \langle Pa + P'a, \omega \rangle = \langle a, P^*\omega \rangle + \langle a, P'^*\omega \rangle. \tag{8.44}$$

Note that $P^*\omega$ and $P'^*\omega$ are multiples of elements of F and F' respectively. Since $a = 0$ on F', then $\langle a, P'^*\omega \rangle = 0$. Thus by (8.44), $0 = \langle a, P^*\omega \rangle$. Since $a > 0$ on F, we must have $P^*\omega = 0$, and so $\omega \in F'$. Thus $G = F'$, and G is projective.

Now let $a \in A$ be arbitrary, and let F be the projective face with the properties in (ii). Let F_1 be an arbitrary projective face compatible with a such that $a \geq 0$ on F_1 and $a \leq 0$ on F_1. Consider once more the projective face $\widetilde{F} \subset F_1$ which satisfies the same requirements as F_1, but now with the strict inequality $a > 0$ on $\widetilde{F}$ (cf. Lemma 8.53). Since F is unique under these requirements, we must have $F = \widetilde{F} \subset F_1$. Thus F is the least projective face compatible with a such that $a \geq 0$ on F and $a \leq 0$ on F'. By Lemma 8.44, the proof is complete. □

8.55. Theorem. *Assume the standing hypothesis of this chapter and $A = V^*$. A and V are in spectral duality iff each $a \in A$ has a unique orthogonal decomposition.*

Proof. Assume A and V are in spectral duality. Let $a \in A$ and let F be the unique projective face compatible with a such that $a > 0$ on F and $a \leq 0$ on F' (Lemma 8.54). Then the compression P associated with F is compatible with a, and satisfies the inequalities $Pa \geq 0$ and $P'a \leq 0$ (Lemma 8.44). Thus we have the orthogonal decomposition $a = b - c$ where $b = Pa$ and $c = -P'a$.

To prove uniqueness, we consider an arbitrary orthogonal decomposition $a = b_1 - c_1$ with $b_1 = P_1 a$ and $c_1 = -P_1' a$ for a compression P_1 compatible with a. Let F_1 be the projective face associated with P_1. Since $P_1 a = b_1 \geq 0$ and $P_1' a = -c_1 \leq 0$, we have $a \geq 0$ on F_1 and $a \leq 0$ on F_1 (Lemma 8.44). By Lemma 8.53, there is a projective face $\widetilde{F} \subset F_1'$ compatible with a such that $a > 0$ on $\widetilde{F}$ and $a \leq 0$ on $\widetilde{F}'$. Moreover, $\widetilde{P}a = P_1 a$ where $\widetilde{P}$ is the compression associated with $\widetilde{F}$. But F was the unique projective face compatible with a such that $a > 0$ on F and $a \leq 0$ on F', so $F = \widetilde{F}$ and $\widetilde{P} = P$. Thus

$$b_1 = P_1 a = \widetilde{P}a = Pa = b, \quad c_1 = b_1 - a = b - a = c,$$

so $a = b - c$ is the unique orthogonal decomposition of a.

Assume next that a is an element of A with a unique orthogonal decomposition, say $b = Pa$ and $c = -P'a$ for a compression P compatible with a, and let F be the projective face associated with P. Again we consider the projective face $\widetilde{F}$ of Lemma 8.53. Thus a is compatible with $\widetilde{F}$, and $a > 0$ on $\widetilde{F}$ and $a \leq 0$ on $\widetilde{F}'$. Recall also that $\widetilde{F} = G'$ where $G = \{\omega \in K \mid \langle Pa, \omega \rangle = 0\}$.

We will prove that A and V are in spectral duality by showing that $\widetilde{F}$ is contained in an arbitrary projective face F_1 compatible with a such that $a \geq 0$ on F_1 and $a \leq 0$ on F_1 (Lemma 8.44). If P_1 is the compression associated with F_1, then we have $P_1 a \geq 0$ and $P_1' a \leq 0$ (Lemma 8.44), so $a = P_1 a - (-P_1' a)$ is an orthogonal decomposition. By uniqueness, $P_1 a = b$ and $-P_1' a = c$. We have $b = 0$ on F_1' (7.25), so $F_1' \subset G$. Hence $\widetilde{F} = G' \subset F_1$, which completes the proof. □

8.56. Definition. Assume A and V are in spectral duality. For each $a \in A$ we will write $a^+ = b$ and $a^- = c$ where $a = b - c$ is the unique orthogonal decomposition of a.

8.57. Proposition. *Assume A and V are in spectral duality, and let $a \in A$. Then the projective face F associated with the projective unit $r(a^+)$ is the unique projective face F compatible with a such that $a > 0$ on F and $a \leq 0$ on F', and it is also the least projective face F compatible with a such that $a \geq 0$ on F and $a \leq 0$ on F'. The compression P associated with $r(a^+)$ is the least compression such that $Pa \geq a$ and $Pa \geq 0$.*

Proof. Let H be the projective face associated with $r(a^+)$. By (8.18) (and (7.22)), $H' = \{\omega \in K \mid \langle a^+, \omega \rangle = 0\}$. Therefore we will show that $H = F$ where F is the unique projective face compatible with a such that $a > 0$ on F and $a \leq 0$ on F'. Let P be the compression associated with F. Then $Pa \geq 0$, $P'a \leq 0$ and $a = Pa - (-P'a)$ (Lemma 8.44). Thus $a^+ = Pa$ and $a^- = -P'a$ (by uniqueness of orthogonal decompositions). Since $a^+ = Pa$, the set G of Lemma 8.53 is equal to H', so $H = G' = \widetilde{F}$ where $\widetilde{F}$ has the same properties as those that determine F. Thus $H = F$. With this we have proved the first characterization of $r(a^+)$ in terms of its associated projective face F.

The second characterization of F follows from the first by Lemma 8.54. Then the characterization of $r(a^+)$ in terms of its associated compression P follows by the equivalence of (iii) and (i) in Lemma 8.44. □

In a von Neumann algebra the spectral projections e_λ of a self-adjoint element a are explicitly given as the complements of the range projections $r((a - \lambda 1)^+)$ (and similarly for a JBW-algebra, cf. (A 99) and Theorem 2.20. This will now be generalized to the context of an order unit space

$A = V^*$ in spectral duality with a base norm space V. For simplicity of notation we will write

$$r_\lambda(a) := r((a - \lambda 1)^+). \tag{8.45}$$

Note that by Proposition 8.57, $r_\lambda(a)$ is compatible with $a - \lambda 1$ and then also with a, and the projective face F_λ associated with $r_\lambda(a)$ is the unique projective face compatible with a such that

$$a > \lambda \text{ on } F_\lambda, \quad a \leq \lambda \text{ on } F'_\lambda. \tag{8.46}$$

It also follows from Proposition 8.57 that the compression R_λ associated with $r_\lambda(a)$ is the least compression compatible with a such that

$$R_\lambda a \geq \lambda r_\lambda(a), \quad R'_\lambda a \leq \lambda r'_\lambda(a). \tag{8.47}$$

Clearly $r_\lambda(a) = r(a)$ if $a \geq 0$ and $\lambda = 0$. In this case $r_\lambda(a)$ is compatible, not only with a, but also with all compressions compatible with a (Proposition 8.41). It turns out that the same is true for arbitrary $a \in A$ and $\lambda \in \mathbf{R}$. This is an important technical result, which we now proceed to prove (Proposition 8.62).

8.58. Lemma. *Assume the standing hypothesis of this chapter and $A = V^*$. Let F, G be a pair of projective faces both compatible with $a \in A$ and let $\lambda \in \mathbf{R}$. If $F \perp G$ and $a > \lambda$ on $F \cup G$, then $a > \lambda$ on $F \vee G$.*

Proof. Let p, q be the projective units and P, Q be the compressions associated with F, G. Let $\omega \in F \vee G$. By Proposition 8.8 (and (7.22)),

$$\langle p, \omega \rangle + \langle q, \omega \rangle = \langle p \vee q, \omega \rangle = 1. \tag{8.48}$$

Set $\mu = \langle p, \omega \rangle$, and note that by (8.48), $\langle q, \omega \rangle = 1 - \mu$. Note also that $\langle p, \omega \rangle = 1$ implies $\omega \in F$ and that $\langle q, \omega \rangle = 1$ implies $\omega \in G$. In these two cases there is nothing to prove, so we will assume $0 < \mu < 1$.

Since $\langle 1, P^*\omega \rangle = \langle p, \omega \rangle = \mu$ and $\langle 1, Q^*\omega \rangle = \langle q, \omega \rangle = 1 - \mu$, the two positive linear functionals $\rho = \mu^{-1} P^*\omega$ and $\sigma = (1 - \mu)^{-1} Q^*\omega$ are states. Moreover, $\rho \in F$ and $\sigma \in G$ (7.24). Since $\omega \in F \vee G$, then $(P \vee Q)^*\omega = \omega$. Hence by Proposition 8.5,

$$\begin{aligned} \langle a, \omega \rangle &= \langle (P \vee Q)a, \omega \rangle = \langle Pa, \omega \rangle + \langle Qa, \omega \rangle \\ &= \langle a, P^*\omega \rangle + \langle a, Q^*\omega \rangle = \mu \langle a, \rho \rangle + (1 - \mu) \langle a, \sigma \rangle > \lambda . \end{aligned}$$

We are done. $\square$

8.59. Lemma. *Assume A and V are in spectral duality. If $a \in A^+$ and $\lambda > 0$, then $r_\lambda(a) \leq r(a)$.*

Proof. We know that $r(a)$ is compatible with all compressions compatible with a (Proposition 8.41). Therefore $r(a)$ is compatible with the compression associated with $r_\lambda(a)$, or which is equivalent, the projective face F associated with $r(a)$ is compatible with the projective face $F_\lambda(a)$ associated with $r_\lambda(a)$. Hence F' is compatible with F_λ, so by Lemma 8.12,

$$F' = (F' \wedge F_\lambda) \vee (F' \wedge F'_\lambda).$$

We will show that $F' \wedge F_\lambda = \emptyset$. If $\omega \in F' \wedge F_\lambda = F' \cap F_\lambda$, then $\langle a, \omega \rangle = 0$ since $\omega \in F$, and $\langle a, \omega \rangle > \lambda > 0$ since $\omega \in F_\lambda$ (cf. (8.46)), a contradiction. Thus $F' = F' \wedge F'_\lambda \subset F'_\lambda$, so $F_\lambda \subset F$. □

8.60. Lemma. *Assume A and V are in spectral duality. Let $a, b \in A^+$ and let $\lambda > 0$. If $a \perp b$, then $r_\lambda(a) \perp r_\lambda(b)$ and*

$$r_\lambda(a + b) = r_\lambda(a) + r_\lambda(b). \tag{8.49}$$

Proof. Since $a \perp b$, then also $r(a) \perp r(b)$ (Proposition 8.39). By Lemma 8.59, $r_\lambda(a) \leq r(a)$ and $r_\lambda(b) \leq r(b)$, so we have $r_\lambda(a) \perp r_\lambda(b)$. Note that by Proposition 8.8, this implies

$$r_\lambda(a) + r_\lambda(b) = r_\lambda(a) \vee r_\lambda(b). \tag{8.50}$$

Let F and G be the projective faces associated with $r(a)$ and $r(b)$ respectively. Also let F_λ and G_λ be the projective faces associated with $r_\lambda(a)$ and $r_\lambda(b)$ respectively. Then F_λ is compatible with a and G_λ is compatible with b (Proposition 8.57). Since $r(a)$ and $r(b)$ are orthogonal and $r_\lambda(a) \leq r(a)$, then $r_\lambda(a) \perp r(b)$. If R_λ is the compression associated with $r_\lambda(a)$, then $0 \leq R_\lambda b \leq \|b\| R_\lambda r(b) = 0$, so R_λ is compatible with b (Lemma 7.37). Thus F_λ is compatible with b, and similarly G_λ is compatible with a. Hence F_λ and G_λ are compatible with both a and b, and therefore also with $a + b$. Since $F_\lambda \perp G_\lambda$, then $F_\lambda \vee G_\lambda$ is also compatible with $a + b$ (Proposition 8.5).

By (8.46) $a > \lambda$ on F_λ and $b > \lambda$ on G_λ, so $a + b > \lambda$ on $F_\lambda \cup G_\lambda$. By Lemma 8.58, also $a + b > \lambda$ on $F_\lambda \vee G_\lambda$.

We next prove

$$a \leq \lambda r(a) \text{ on } F'_\lambda \quad \text{and} \quad b \leq \lambda r(b) \text{ on } G'_\lambda. \tag{8.51}$$

To prove the first of these inequalities, let R_λ be the compression associated with $r_\lambda(a)$, and R_0 the compression associated with $r(a)$. Since $r_\lambda(a) \leq$

$r(a)$ (Lemma 8.59), then R_λ and R_0 are compatible, and then also R'_λ and R_0. Then by (8.47)

$$R'_\lambda a = R'_\lambda R_0 a = R_0 R'_\lambda a \leq \lambda R_0 r'_\lambda(a) = \lambda R_0 R'_\lambda 1 = \lambda R'_\lambda R_0 1 = \lambda R'_\lambda r(a).$$

Thus $R'_\lambda(\lambda r(a) - a) \geq 0$, which gives the first inequality in (8.51); the second follows in a similar manner. Since $r(a) \perp r(b)$, then $r(a)+r(b) \leq 1$. Then

$$a + b \leq \lambda(r(a) + r(b)) \leq \lambda \quad \text{on} \quad F'_\lambda \cap G'_\lambda = (F_\lambda \vee G_\lambda)'.$$

With this we have shown that (8.46) is satisfied with $a + b$ in place of a and $F_\lambda \vee G_\lambda$ in place of F_λ. Thus $F_\lambda \vee G_\lambda$ is the projective face associated with $r_\lambda(a + b)$. Hence

$$r_\lambda(a + b) = r_\lambda(a) \vee r_\lambda(b).$$

By (8.50) this gives (8.49). □

8.61. Lemma. *Assume A and V are in spectral duality. If $a \in A^+$ and Q is a compression compatible with a, then for each $\lambda > 0$,*

$$r_\lambda(Qa) = Q(r_\lambda(a)). \tag{8.52}$$

Proof. We have $Qa \perp Q'a$, so it follows from Lemma 8.60 (and compatibility) that

$$r_\lambda(Qa) + r_\lambda(Q'a) = r_\lambda(Qa + Q'a) = r_\lambda(a). \tag{8.53}$$

By Lemma 8.59, $r_\lambda(Qa) \leq r(Qa) \leq q$. Hence $r_\lambda(Qa) \in \text{face}\,(q) = \text{im}^+ Q$. Similarly $r_\lambda(Q'a) \in \text{im}^+ Q' = \ker^+ Q$. Now applying Q to (8.53) gives (8.52). □

8.62. Proposition. *Assume A and V are in spectral duality. If $a \in A$, then each $r_\lambda(a)$ is in the $\mathcal{P}$-bicommutant of a.*

Proof. Let Q be a compression compatible with a. Assume first $a \geq 0$. If $\lambda < 0$, then $r_\lambda(a) = 1$, so $r_\lambda(a)$ is compatible with Q. If $\lambda = 0$, then $r_\lambda(a) = r(a)$, so it follows from Proposition 8.41 that Q and $r_\lambda(a)$ are compatible in this case also. Assume now $\lambda > 0$. By Lemmas 8.61 and 8.60,

$$Q(r_\lambda(a)) + Q'(r_\lambda(a)) = r_\lambda(Qa) + r_\lambda(Q'a) = r_\lambda(Qa + Q'a) = r_\lambda(a).$$

Thus Q and $r_\lambda(a)$ are compatible in this case.

In the general case we define $b = a + \|a\| 1$ and $\mu = \lambda + \|a\|$. Then $b \in A^+$ and Q is compatible with b. By the above, Q is also compatible with $r_\mu(b)$. Let F_λ be the projective face associated with $r_\lambda(a)$ and let G_μ be the projective face associated with $r_\mu(b)$. Now G_μ is compatible with b, and then also with a. For $\omega \in K$, $\langle b, \omega\rangle > \mu$ iff $\langle a, \omega\rangle > \lambda$, so it follows from the criterion (8.46) that $G_\mu = F_\lambda$. Thus $r_\mu(b) = r_\lambda(a)$, so Q and $r_\lambda(a)$ are compatible. □

8.63. Corollary. *Assume A and V are in spectral duality. Let $a \in A$ and $\lambda \in \mathbf{R}$, and let F_λ be the projective face associated with $r_\lambda(a)$. If F is a projective face compatible with a such that $a \geq \mu$ on F where $\mu > \lambda$, then $F \subset F_\lambda$.*

Proof. By Proposition 8.62, F is compatible with F_λ. Hence (by Lemma 8.12) $F = (F \wedge F_\lambda) \vee (F \wedge F'_\lambda)$. But $F \wedge F'_\lambda = F \cap F'_\lambda = \emptyset$ since $a \geq \mu$ on F and $a \leq \lambda$ on F'_λ. Thus $F = (F \wedge F_\lambda) \subset F_\lambda$. □

We are now ready to prove our spectral theorem which generalizes the corresponding theorem for von Neumann algebras (A 99) and JBW-algebras (Theorem 2.20). As in these cases, we define the norm $\|\gamma\|$ of a finite increasing sequence $\gamma = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ of real numbers by

(8.54)
$$\|\gamma\| = \max_{1 \leq i \leq n} (\lambda_i - \lambda_{i-1}).$$

8.64. Theorem. *Assume A and V are in spectral duality, and let $a \in A$. Then there is a unique family $\{e_\lambda\}_{\lambda \in \mathbf{R}}$ of projective units with associated compressions U_λ such that*

(i) *e_λ is compatible with a for each $\lambda \in \mathbf{R}$,*
(ii) *$U_\lambda a \leq \lambda e_\lambda$ and $U'_\lambda a \geq \lambda e'_\lambda$ for each $\lambda \in \mathbf{R}$,*
(iii) *$e_\lambda = 0$ for $\lambda < -\|a\|$, and $e_\lambda = 1$ for $\lambda > \|a\|$,*
(iv) *$e_\lambda \leq e_\mu$ for $\lambda < \mu$,*
(v) *$\bigwedge_{\mu > \lambda} e_\mu = e_\lambda$ for each $\lambda \in \mathbf{R}$.*

The family $\{e_\lambda\}$ is given by $e_\lambda = 1 - r((a - \lambda 1)^+)$, each e_λ is in the $\mathcal{P}$-bicommutant of a, and the Riemann sums

(8.55)
$$s_\gamma = \sum_{i=1}^{n} \lambda_i (e_{\lambda_i} - e_{\lambda_{i-1}})$$

converge in norm to a when $\|\gamma\| \to 0$.

Proof. (i)–(v) Define $e_\lambda = 1 - r((a - \lambda 1)^+)$ and let U_λ be the compression associated with e_λ. By Proposition 8.62 each e_λ is in the $\mathcal{P}$-bicommutant of a. With the notation from (8.45) and (8.47) we have $e_\lambda = r_\lambda(a)'$ and $U_\lambda = R'_\lambda$, which gives (i) and (ii).

If $\lambda < -\|a\|$, then $a - \lambda 1 > 0$ on K, so $r_\lambda(a) = r(a - \lambda 1) = 1$ and $e_\lambda = r_\lambda(a)' = 0$. Similarly, $\lambda > \|a\|$ implies $e_\lambda = 1$. This proves (iii).

Let F_λ be the projective face associated with $r_\lambda(a)$ for each $\lambda \in \mathbf{R}$. Then assume $\lambda < \mu$. By Corollary 8.63, $F_\mu \subset F_\lambda$. Hence $r_\mu \leq r_\lambda$ which gives $e_\lambda \leq e_\mu$ and proves (iv).

We will prove (v) by establishing the corresponding equality for the projective faces F'_λ associated with the projective units $e_\lambda = r_\lambda(a)'$, i.e., $\bigwedge_{\mu > \lambda} F'_\mu = F'_\lambda$. Choose an arbitrary $\lambda \in \mathbf{R}$ and consider the decreasing net $\{F'_\mu\}_{\mu > \lambda}$. By Proposition 8.20, the projective face

$$G = \bigwedge_{\mu > \lambda} F'_\mu = \bigcap_{\mu > \lambda} F'_\mu$$

is compatible with a. Since $F'_\mu \supset F'_\lambda$ for each $\mu > \lambda$, then $G \supset F'_\lambda$. We must show that $G = F'_\lambda$.

Since $G' \subset F_\lambda$, then $a > \lambda$ on G' by (8.46). Assume now $\omega \in G$. For each $\mu > \lambda$ we have $\omega \in F'_\mu$, so $\langle a, \omega \rangle \leq \mu$. Hence $\langle a, \omega \rangle \leq \lambda$. Thus $a \leq \lambda$ on G. With this we have shown that the criterion (8.46) is satisfied with G' in place of F_λ. Thus $G' = F_\lambda$, so $G = F'_\lambda$ as desired.

To prove uniqueness, we consider an arbitrary family $\{e_\lambda\}_{\lambda \in \mathbf{R}}$ of projective units with associated compressions U_λ such that (i)–(v) are satisfied. We will show that for a given $\lambda \in \mathbf{R}$ the projective face H_λ associated with e_λ is equal to the projective face F'_λ associated with $r_\lambda(a)'$.

We know from (8.47) that the compression R_λ associated with $r_\lambda(a)$ is the least compression compatible with a such that

$$R_\lambda a \geq \lambda R_\lambda 1, \quad R'_\lambda a \leq R'_\lambda 1.$$

By (ii), this condition is satisfied with U'_λ in place of R_λ. Therefore $R_\lambda \preceq U'_\lambda$. Hence $H_\lambda \subset F'_\lambda$.

By (i) each H_μ is compatible with a, and by (ii) $H_\mu \supset H_\nu$ when $\mu > \nu$. Then by (v),

$$H_\lambda = \bigwedge_{\mu > \lambda} H_\mu = \bigcap_{\mu > \lambda} H_\mu. \tag{8.56}$$

For each μ, we have $U'_\mu a = a$ on H'_μ, cf. (7.25). By (ii), $U'_\mu a \geq \mu e'_\mu$, so $a \geq \mu$ on H'_μ. If $\mu > \lambda$, we can apply Corollary 8.63 with the projective face H'_μ in place of F. This gives $H'_\mu \subset F_\lambda$ for each $\mu > \lambda$, so $F'_\lambda \subset H_\mu$. Thus by (8.56), $F'_\lambda \subset H_\lambda$, so $H_\lambda = F'_\lambda$ as desired.

To prove convergence of the Riemann sums, we choose for given $\varepsilon > 0$ a finite increasing sequence $\{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ of real numbers such that $\lambda_0 < -\|a\|$, $\lambda_n > \|a\|$ and $\lambda_i - \lambda_{i-1} < \varepsilon$ for $i = 1, \ldots, n$.

By (iv), $e_\lambda \leq e_\mu$ when $\lambda < \mu$, so in this case $e_\lambda \perp e'_\mu$ and $e_\mu - e_\lambda = e_\mu \wedge e'_\lambda$ (Lemma 8.9). If $\lambda < \mu$, we also have $(U_\mu \wedge U'_\lambda)\, a = U_\mu a - U_\lambda a$

(Lemma 8.9). By (ii), $U_\mu a \le \mu\, e_\mu$ and $U'_\lambda a \ge \lambda\, e'_\lambda$, so applying U'_λ to the first inequality and U_μ to the second gives

$$(U'_\lambda \wedge U_\mu)a \le \mu U'_\lambda e_\mu = \mu\, e'_\lambda \wedge e_\mu \quad \text{and} \quad (U_\mu \wedge U'_\lambda)a \ge \lambda U_\mu e'_\lambda = \lambda\, e_\mu \wedge e'_\lambda.$$

Therefore

$$\lambda\,(e_\mu - e_\lambda) \le (U_\mu \wedge U'_\lambda)\, a \le \mu\,(e_\mu - e_\lambda).$$

Now define $p_i = e_{\lambda_i} - e_{\lambda_{i-1}}$ for $i = 1, \dots, n$. Then the compression $P_i = U_{\lambda_i} \wedge U'_{\lambda_{i-1}}$ associated with p_i satisfies

$$\lambda_{i-1}\,(e_{\lambda_i} - e_{\lambda_{i-1}}) \le P_i a \le \lambda_i\,(e_{\lambda_i} - e_{\lambda_{i-1}}).$$

Note that $\sum_i p_i = 1$ by (iii). Since $\sum_{i=1}^n P_i a = a$ (Proposition 8.7), this gives

$$\sum_{i=1}^{n} \lambda_{i-1}\,(e_{\lambda_i} - e_{\lambda_{i-1}}) \le a \le \sum_{i=1}^{n} \lambda_i\,(e_{\lambda_i} - e_{\lambda_{i-1}}).$$

Thus $\|s_\gamma - a\| < \varepsilon$ and the proof is complete. □

8.65. Corollary. *If A and V are in spectral duality, then each $a \in A$ can be approximated in norm by linear combinations $\sum_{i=1}^n \lambda_i p_i$ of mutually orthogonal projections p_i in the $\mathcal{P}$-bicommutant of a.*

Proof. Clear from Theorem 8.64. □

Note that if A and V are in spectral duality, then the first of the two requirements for the $\mathcal{C}$-bicommutant (compatibility with a) is redundant in Definition 8.22. In fact, if a compression is compatible with all compressions compatible with a, then it is compatible with the projective units e_λ of a, and by an easy application of the spectral theorem (Theorem 8.64) also with a itself. (Similarly for the $\mathcal{P}$-bicommutant and the $\mathcal{F}$-bicommutant.)

8.66. Definition. If A and V are in spectral duality and $a \in A$, then the family $\{e_\lambda\}_{\lambda\in\mathbf{R}}$ of Theorem 8.64 will be called the *spectral resolution* of a, and the projective units e_λ in this family will be called the *spectral units* of a.

If we define Riemann–Stieltjes integrals with respect to $\{e_\lambda\}_{\lambda\in\mathbf{R}}$ as the norm limit of approximating Riemann sums in the usual way, then we can restate the last result in Theorem 8.64 as

$$a = \int \lambda\, de_\lambda. \tag{8.57}$$

Similarly the integral with respect to $\{e_\lambda\}_{\lambda\in\mathbf{R}}$ of an arbitrary continuous function f defined on the interval $[-\|a\|, \|a\|]$ is given by

$$\int f(\lambda)\,de_\lambda = \lim_{\|\gamma\|\to 0} \sum_{i=1}^{n} f(\lambda_i)(e_{\lambda_i} - e_{\lambda_{i-1}}), \tag{8.58}$$

where the (norm) limit exists by the same argument as in the last part of the proof of Theorem 8.64. More generally we can define the integral with respect to $\{e_\lambda\}_{\lambda\in\mathbf{R}}$ of a bounded Borel function f defined on this interval as the *weak integral* in $A = V^*$, determined by the equation

$$\left\langle \int f(\lambda)\,de_\lambda, \phi \right\rangle = \int f(\lambda)\,d\langle e_\lambda, \phi\rangle \quad \text{for all } \phi \in V^+, \tag{8.59}$$

where $d\langle e_\lambda, \phi\rangle$ denotes the measure associated with the increasing, right continuous function $\lambda \mapsto \langle e_\lambda, \phi\rangle$. Using this notion of integral, we define the *spectral functional calculus* for an element $a \in A$. This is the map which assigns to each bounded Borel function f on $[-\|a\|, \|a\|]$ the element $f(a) \in A$ given by

$$f(a) = \int f(\lambda)\,de_\lambda, \tag{8.60}$$

where $\{e_\lambda\}_{\lambda\in\mathbf{R}}$ is the spectral resolution of a.

8.67. Proposition. *If A and V are in spectral duality and $a \in A$, then the spectral functional calculus for a satisfies the following requirements (where f, g and all f_n are bounded Borel functions on $[-\|a\|, \|a\|]$ and α, β are real numbers):*

(i) $\|f(a)\| \le \|f\|_\infty$.
(ii) $(\alpha f + \beta g) = \alpha f(a) + \beta g(a)$.
(iii) $f \le g \Rightarrow f(a) \le g(a)$.
(iv) *If $\{f_n\}$ is a bounded sequence and $f_n \to f$ pointwise, then $f_n(a) \to f(a)$ in the w^*-topology of $A = V^*$.*

Proof. (i) We know from (A 11) that

$$\|f(a)\| = \sup_{\omega\in K} |\langle f(a), \omega\rangle| \le \sup_{\omega\in K} \int |f(\lambda)|\,d\langle e_\lambda, \omega\rangle.$$

For each state ω, the function $\lambda \mapsto \langle e_\lambda, \omega\rangle$ is increasing and right continuous with values ranging from 0 to 1, so $d\langle e_\lambda, \omega\rangle$ denotes a probability measure in the integral above. Therefore $\|f(a)\| \le \|f\|_\infty$.

(ii), (iii) Trivial.

(iv) Easy application of the monotone convergence theorem. □

Remark. The functional calculus described in Proposition 8.67 has a natural physical interpretation. If a is an observable and f is a bounded Borel function, then $f(a)$ is the observable determined by measuring a and then applying the function f to the result. Now define $\operatorname{pos}(\lambda) = \max(\lambda, 0)$ and $\operatorname{neg}(\lambda) = -\min(\lambda, 0)$, and let $a^+ = \operatorname{pos}(a)$ and $a^- = \operatorname{neg}(a)$. Then $a = a^+ - a^-$, and a^+ and a^- are orthogonal since $a^+ \leq \|a\| p$ and $a^- \leq \|a\|(1-p)$, where $p = \chi_{(0,\infty)}(a)$. Thus the existence of orthogonal decompositions, which will be a key assumption in our characterization of state spaces of C*-algebras (Theorem 11.59), is well justified on physical grounds. Uniqueness of such decompositions is perhaps less evident, but we will see that uniqueness is automatic in the context in which we will be using it. By virtue of Theorem 8.55, this also provides another physical justification for the assumption of spectral duality.

We will now give a characterization of spectral duality which involves the concept of the $\mathcal{F}$-bicommutant.

8.68. Lemma. *Assume the standing hypothesis of this chapter and $A = V^*$. For each $a \in A$ there is a greatest compression Q in the $\mathcal{C}$-bicommutant of a such that $Qa \leq 0$; equivalently there is a largest projective face G in the $\mathcal{F}$-bicommutant of a such that $a \leq 0$ on G.*

Proof. Let $\mathcal{Q}$ be the set of compressions Q in the $\mathcal{C}$-bicommutant of a such that $Qa \leq 0$. Recall that the $\mathcal{C}$-bicommutant of a is closed under the lattice operations of $\mathcal{C}$, and is a Boolean algebra (Proposition 8.23). Thus if $Q_1, Q_2 \in \mathcal{Q}$, then $Q_1 \vee Q_2$ is in the $\mathcal{C}$-bicommutant of a, and by the distributive law for Boolean algebras, $(Q_1 \vee Q_2) = (Q_1 \dot{+} Q_1' \wedge Q_2)$. By Proposition 8.5

$$(Q_1 \vee Q_2)\, a = (Q_1 \dot{+} Q_1' \wedge Q_2)\, a = Q_1 a + Q_1' Q_2 a \leq 0.$$

Hence $Q_1 \vee Q_2 \in \mathcal{Q}$. Thus $\mathcal{Q}$ is directed upwards, so we can organize $\mathcal{Q}$ to an increasing net $\{Q_\alpha\}$ such that $Q_\alpha \nearrow Q := \bigvee_\alpha Q_\alpha$. By Proposition 8.20, $Q_\alpha a \nearrow Qa$, so $Qa \leq 0$. By Proposition 8.23, Q is in the $\mathcal{Q}$-bicommutant of a. Thus Q is in $\mathcal{Q}$, and in fact is the greatest element of $\mathcal{Q}$. This proves the result for compressions. The result for projective faces follows by Proposition 7.28. □

8.69. Theorem. *Assume the standing hypothesis of this chapter and $A = V^*$. Then the following are equivalent:*

(i) *A and V are in spectral duality.*
(ii) *For each $a \in A$ there is a projective face G in the $\mathcal{F}$-bicommutant of a such that $a \leq 0$ on G and $a > 0$ on G'.*
(iii) *For each $a \in A$ the largest projective face G in the $\mathcal{F}$-bicommutant of a such that $a \leq 0$ on G satisfies the inequality $a > 0$ on G'.*

Proof. (i)$\Rightarrow$(ii) If A and V are in spectral duality and $a \in A$, then the projective face G associated with the spectral unit $e_0 = r(a^+)'$ has the properties in (ii) by virtue of Proposition 8.62 and equation (8.46).

(ii)$\Rightarrow$(iii) Let G_0 be a projective face in the $\mathcal{F}$-bicommutant of a such that $a \leq 0$ on G_0 and $a > 0$ on G_0'. If G is the largest projective face in the $\mathcal{F}$-bicommutant of A such that $a \leq 0$ on G (cf. Lemma 8.68), then $G_0 \subset G$. Hence $G' \subset G_0'$, so we also have $a > 0$ on G'.

(iii)$\Rightarrow$(ii) Trivial.

(ii)$\Rightarrow$(i) Suppose F is a projective face compatible with a such that $a > 0$ on F and $a \leq 0$ on F'. Then $F' \cap G' = \emptyset$ and $G \cap F = \emptyset$. Since G is in the $\mathcal{F}$-bicommutant of a, then G and F are compatible. By (8.8)

$$G = (G \wedge F) \vee (G \wedge F') = G \wedge F',$$

so $G \subset F'$. We also have

$$F' = (F' \wedge G) \vee (F' \wedge G') = F' \wedge G,$$

so $F' \subset G$. Therefore $F' = G$, and so $F = G'$, which proves condition (ii) of Lemma 8.54. Thus A and V are in spectral duality. □

Note that statement (iii) in Theorem 8.69 differs from all the preceding characterizations of spectral duality in that it imposes conditions on a specific face G, whereas the other ones just demand the existence of a projective face or compression with certain properties without telling how it can be found. Therefore it is easier to apply statement (iii) when proving spectral duality in special cases, as we will do in Theorem 8.72. In this theorem V is supposed to have a weakly compact base, and we shall need the following elementary result on compact convex sets.

8.70. Lemma. *Let K be a compact convex set (in some locally convex space), and let $A(K)$ be the order unit space of all continuous affine functions on K. If ϕ is a state on $A(K)$, (i.e., $\phi \in A(K)^*$, $\phi \geq 0$ and $\phi(1) = 1$), then there is a point $\omega \in K$ such that $\phi(a) = a(\omega)$ for all $a \in A(K)$.*

Proof. Recall that a linear functional ψ on the order unit space $A(K)$ is positive iff $\psi(1) = \|\psi\|$, cf. (A 16). Therefore the linear functional ϕ on $A(K)$ can be extended with preservation of the properties $\phi \geq 0$ and $\phi(1) = 1$ to a linear functional on $C(K)$ by the Hahn–Banach theorem. This functional on $C(K)$ can be represented by a probability measure μ on K (Riesz representation theorem), and this measure has a barycenter

$\omega \in K$ (cf. [6, Prop. I.2.1]) such that

$$\phi(a) = \int a \, d\mu = a(\omega)$$

for all $a \in A(K)$. $\square$

A subset of a Banach space is *weakly compact* if it is compact in the weak topology determined by the dual space. Thus the distinguished base K of a base norm space V is weakly compact iff it is compact in the weak topology determined by the order unit space V^*. By elementary linear algebra, V^* is isomorphic to the order unit space $A_b(K)$ of all bounded affine functions on K, cf. (A 11). Hence K is weakly compact iff it is compact in the weak topology determined by $A_b(K)$, i.e., in the weakest topology for which all bounded affine functions are continuous. Thus if K is weakly compact, then $A(K) = A_b(K)$.

Recall also that a Banach space is *reflexive* if it is canonically isomorphic to its bidual. Thus a base norm space V with distinguished base K is reflexive iff it is isomorphic to the space $V^{**} \cong A_b(K)^*$ under the map $\omega \mapsto \widehat{\omega}$ (where $\widehat{\omega}(a) = a(\omega)$ for all $a \in A_b(K)$), or equivalently, iff the map $\omega \mapsto \widehat{\omega}$ from V into V^{**} is surjective.

8.71. Lemma. *A base norm space V is reflexive iff its distinguished base K is weakly compact.*

Proof. Assume K is weakly compact. By Lemma 8.70, each state ϕ on $A(K) = A_b(K)$ is of the form $\phi = \widehat{\omega}$ for some $\omega \in K$. The dual of the order unit space $A_b(K)$ is a base norm space (A 19), so the state space of $A_b(K)$ spans $A_b(K)^* = V^{**}$. Therefore the map $\omega \mapsto \widehat{\omega}$ maps V onto V^{**}, so V is reflexive.

Conversely, if V is reflexive, then the closed unit ball V_1 of V is weakly compact (Banach–Alaoglu), so the weakly closed subset K of V_1 is also weakly compact. $\square$

We will now specialize the discussion to dual pairs (A, V) for which A and V are reflexive spaces with $A = V^*$. Clearly this is true in all finite dimensional cases.

8.72. Theorem. *If the dual pair (A, V) satisfies the standing hypothesis of this chapter with $A = V^*$, and A and V are reflexive, then A and V are in spectral duality.*

Proof. For given $a \in A$ we will show that the largest projective face G in the $\mathcal{F}$-bicommutant of a such that $a \leq 0$ on G satisfies the inequality

$a > 0$ on G'. Assume for contradiction that $\langle a, \omega \rangle \le 0$ for some $\omega \in G'$ and define

$$\beta = \inf_{\omega \in G'} \langle a, \omega \rangle \le 0. \tag{8.61}$$

Let $\widetilde{a} = a - \beta 1$. Then $\widetilde{a} \ge 0$ on G'. Let Q be the compression associated with G and observe that $Q'\widetilde{a} \ge 0$ (Proposition 7.28). Then define

$$H = \{\omega \in K \mid \langle Q'\widetilde{a}, \omega \rangle = 0\}. \tag{8.62}$$

Since V is reflexive, the distinguished base K, and then also the set $G' \subset K$, is weakly compact. Hence there is an $\omega_0 \in G'$ such that $\langle a, \omega_0 \rangle = \beta$. Therefore

$$\langle Q'\widetilde{a}, \omega_0 \rangle = \langle \widetilde{a}, Q'^* \omega_0 \rangle = \langle \widetilde{a}, \omega_0 \rangle = \langle a, \omega_0 \rangle - \beta = 0.$$

Thus $\omega_0 \in H$, so $H \cap G' \ne \emptyset$.

We will show that $H \cap G'$ is in the $\mathcal{F}$-bicommutant of a. Observe first that H' is the projective face associated with $r(Q'\widetilde{a})$ (8.18). It follows from Proposition 8.41 that H' is in the $\mathcal{F}$-bicommutant of $Q'\widetilde{a}$. Therefore the compression R associated with H is in the $\mathcal{C}$-bicommutant of $Q'\widetilde{a}$.

Clearly Q is compatible with $Q'\widetilde{a}$, so R is compatible with Q and then also with Q'. Since R and Q' are mutually compatible and both are compatible with $Q'\widetilde{a}$, then $R \wedge Q'$ is compatible with $Q'\widetilde{a}$ (Lemma 8.4). Since $Q'\widetilde{a} = Q'a - \beta Q'1$, and $R \wedge Q' \preceq Q'$ is compatible with $Q'1$, then $R \wedge Q'$ is compatible with $Q'a$.

Since R and Q' are compatible, $R \wedge Q' = RQ'$, so $(R \wedge Q')Qa = 0$. Observe now that $Qa \le 0$ since $a \le 0$ on G (Proposition 7.28), and note that a compression which annihilates a positive or negative element of A is compatible with this element (Lemma 7.37). Therefore $R \wedge Q'$ is compatible with Qa. Thus $R \wedge Q'$ is compatible with both $Q'a$ and Qa, so is compatible with $a = Qa + Q'a$.

Let P be a compression compatible with a. Then P and also P' are compatible with the compression Q' in the $\mathcal{C}$-bicommutant of a, so

$$P(Q'a) + P'(Q'a) = Q'(Pa + P'a) = Q'a.$$

Thus P is compatible with $Q'a$, and then is also compatible with $Q'\widetilde{a} = Q'a - \beta Q'1$. Therefore P is compatible with the compression R in the $\mathcal{C}$-bicommutant of $Q'\widetilde{a}$. Hence

$$RQ'P = RPQ' = PRQ'.$$

Thus the compression $R \wedge Q' = RQ'$ is compatible with P. With this we have shown that $R \wedge Q'$ is in the $\mathcal{C}$-bicommutant of a. Now $H \wedge G'$ is in the $\mathcal{F}$-bicommutant of a, as claimed.

Consider the projective face $\widetilde{G} := G \vee (H \wedge G')$ and note that $\widetilde{G}$ properly contains G since $H \wedge G' = H \cap G' \neq \emptyset$. The projective faces G and $H \wedge G'$ are both in the $\mathcal{F}$-bicommutant of a, and therefore $\widetilde{G}$ is also in the $\mathcal{F}$-bicommutant of a (Proposition 8.23).

We have $a \leq 0$ on G, so $Qa \leq 0$ (Proposition 7.28). Since $a \leq \widetilde{a}$, then $Q'a \leq Q'\widetilde{a}$. Since $Q'\widetilde{a} = 0$ on H, then $Q'a \leq 0$ on H, and so we also have $RQ'a \leq 0$. Now by Proposition 8.5 the compression $Q \dot{+} RQ'$ associated with $\widetilde{G}$ satisfies the inequality

$$(Q \dot{+} RQ')\, a = Qa + RQ'a \leq 0.$$

Hence $a \leq 0$ on $\widetilde{G}$. This contradicts the maximality of G. By Theorem 8.69, the proof is complete. □

Remark. Theorem 8.72 is not true without the condition that V be reflexive, by which K is weakly compact. A counterexample is given in [7, Prop. 6.11]. We will just sketch the idea leaving the details to the interested reader. In the example, K is a smooth and strictly convex set, and the space $A = V^*$ has elements not attaining their maximum and minimum on K. (Actually, K is affinely isomorphic to the set of sequences that are eventually zero in the closed unit ball of the Banach space l_2 of square summable sequences.) The standing hypothesis of this chapter is trivially satisfied, as the only faces are the extreme points and each of them is a projective face (whose complement is the unique antipodal point). But no such face is compatible with an element $a \in A$ that does not attain its maximum and minimum, so the only projective faces compatible with this element are K and $\emptyset$. Thus by Lemma 8.44 (iii), A and V are not in spectral duality.

In this example K is not norm complete. But spectral duality can not be rescued just by demanding norm completeness. In fact, one can construct a similar example where K is a norm complete (but not weakly compact) smooth and strictly convex set, as explained in [9, §2].

Spectral convex sets

In our definition of spectral duality for a pair of spaces A, V we assumed $A = V^*$. Then the pair A, V is completely determined by the distinguished base K of V, as we will show in Proposition 8.73 below. In this section we will shift the focus from the pair of ordered normed spaces A, V to the geometry of K, and we will give various examples of convex sets which in this way determine spaces in spectral duality.

The proof of Proposition 8.73 involves elementary properties of order unit and base norm spaces, presented in [AS, Chpt. 1]. For the readers convenience we will briefly review what is needed.

If K is a convex set for which the bounded affine functions separate points, then the set $A_b(K)$ of all such functions is a norm complete order unit space (whose distinguished order unit is the constant unit function), the dual space $A_b(K)^*$ is a norm complete base norm space (whose distinguished base is the "state space" consisting of all linear functionals ϕ such that $\phi \geq 0$, or equivalently $\|\phi\| = \phi(1)$), and these two spaces are in separating order and norm duality. Observe that $A_b(K)$ is also in separating duality with the subspace of $A_b(K)^*$ which consists of all *discrete* linear functionals, i.e., functionals of the form $\sum_{i=1}^n \lambda_i \widehat{\omega}_i$ with $\omega_1, \ldots, \omega_n \in K$. (Here, and in the sequel, $\widehat{\omega}$ denotes the evaluation functional at a point ω, i.e., $\widehat{\omega}(a) = a(\omega)$ for all $a \in A_b(K)$.) Recall also that an *isomorphism* of order unit spaces, or base norm spaces, is a bijective map that preserves all relevant structure (i.e., linear structure, ordering and norm).

8.73. Proposition. *Let K be the distinguished base of a base norm space V, and set $A = V^*$ (so A is a norm complete order unit space in separating order and norm duality with V). Then the dual pair A, V is canonically isomorphic to the pair consisting of the space $A_b(K)$ and the space of all discrete linear functionals on $A_b(K)$.*

Proof. Each element ω in the base norm space V can be written as

$$\omega = \alpha\sigma - \beta\tau \quad \text{where } \sigma, \tau \in K \text{ and } \alpha, \beta \in \mathbf{R}^+. \tag{8.63}$$

This representation is of course not unique, but by elementary linear algebra each $a_0 \in A_b(K)$ has a well defined extension to a linear functional $a \in V^*$ given by the equation

$$\langle a, \omega \rangle = \alpha\, a_0(\sigma) - \beta\, a_0(\tau),$$

cf. (A 11). Since $V^* = A$ and the spaces A and V are in separating order and norm duality, the extension map $\Phi : a_0 \mapsto a$ is an isomorphism of the order unit space $A_b(K)$ onto the order unit space A.

By linearity, the dual map $\Phi^* : V \to A_b(K)^*$ is given by $\Phi^*\omega = \alpha\widehat{\sigma} - \beta\widehat{\tau}$ when ω is represented as in (8.63). Thus $\Phi^*\omega$ is a discrete linear functional on $A_b(K)$. Clearly each discrete linear functional $\phi = \sum_{i=1}^n \lambda_i \widehat{\omega}_i$ with $\omega_i \in K$ and $\lambda_i \in \mathbf{R}$ for $i = 1, \ldots, n$ can be expressed as a difference $\phi = \alpha\widehat{\sigma} - \beta\widehat{\tau}$ where $\sigma, \tau \in K$ and $\alpha, \beta \in \mathbf{R}^+$. Therefore Φ^* is an isomorphism of the base norm space V onto the base norm space of all discrete linear functionals on $A_b(K)$. □

8.74. Definition. A convex set K will be called *spectral* if it is affinely isomorphic to the distinguished base of a complete base norm space V in spectral duality with the order unit space $A = V^*$.

By Proposition 8.73, this definition is intrinsic in that the spaces A and V are determined by the convex set K. Note however, that A and V will be given in all of our concrete examples, so the identification of these spaces in Proposition 8.73 will not be needed for this purpose.

The requirement that K be affinely isomorphic to the distinguished base of a base norm space is equivalent to the assumption that the bounded affine functions on K separate points of K, together with a weak kind of compactness requirement, namely, that $\mathrm{co}(K \cup -K)$ be radially compact when K sits on a hyperplane missing the origin, cf. [6]. In particular, any compact convex subset of a locally convex space will satisfy this requirement.

We begin with the motivating examples already mentioned in connection with the definition of spectral duality.

8.75. Proposition. *If μ is a measure on a σ-field of subsets of a set X, and K is the set of all $f \in L^1(\mu)$ such that $f \geq 0$ and $\int f\, d\mu = 1$, then K is the distinguished base of the base norm space $L^1(\mu)$ which is in spectral duality with the order unit space $L^\infty(\mu)$, so K is a spectral convex set.*

Proof. If $f \in L^1(\mu)$ and $f = f^+ - f^-$ is the usual decomposition of f into positive and negative parts, then $\|f\| = \|f^+\| + \|f^-\|$, so $L^1(\mu)$ is a base norm space. Each measurable set $E \subset X$ determines a face F_E of K which consists of all $f \in K$ such that $f = 0$ on $X \setminus E$. This face is easily seen to be a projective face associated with the projective unit $p = \chi_E$ and with the compression $P : g \mapsto \chi_E \cdot g$ for $g \in L^\infty(\mu)$. Clearly also, each $f \in L^1(\mu)$ admits a least compression P such that $Pf \geq f$ and $Pf \geq 0$, namely the compression associated with the face F_E where $E = f^{-1}((0, \infty))$. Thus P satisfies the requirements in the definition of spectral duality. □

8.76. Proposition. *If M is a JBW-algebra with normal state space K, then K is the distinguished base of the predual M_* which is a base norm space in spectral duality with the order unit space M, so K is a spectral convex set. The same is true when we specialize to the self-adjoint part of a von Neumann algebra.*

Proof. M_* is a base norm space with distinguished base K by Corollary 2.60. M is in spectral duality with M_* by Proposition 2.19. (Note that in the von Neumann algebra case no knowledge of Jordan algebras is needed for the proof of Proposition 2.19.) □

From now on we will concentrate on compact convex sets K. These may initially be given as subsets of an arbitrary locally convex (Hausdorff) space. But this ambient vector space will play no role in our discussion since we will always pass to the *regular imbedding* of K as the distinguished base of a complete base norm space V, as explained in [6, Thm. II.2.4].

We will briefly describe this imbedding. As usual we denote by $A(K)$ the order unit space consisting of all affine functions on K that are continuous in the given compact topology. Then the canonical map $\omega \mapsto \widehat{\omega}$ is an affine isomorphism from K into the dual space $V = A(K)^*$. In fact, this map is a homeomorphism from the topology on K to the w*-topology on $A(K)^*$, since the latter is pulled back to a Hausdorff topology which must be identical with the given compact topology on K. The space $V = A(K)^*$ is a complete base norm space for the natural ordering and norm (A 19), and the map $\omega \mapsto \widehat{\omega}$ carries K onto the distinguished base of V, which consists of all states on $A(K)$ (Lemma 8.70). Now the homeomorphic affine isomorphism of K onto the distinguished base of the complete base norm space $V = A(K)^*$ is the regular imbedding referred to above.

When a compact convex set K is regularly imbedded in $V = A(K)^*$, then V is in separating order and norm duality with both $A(K)$ and $A_b(K)$, and we will now specify some terminology to distinguish between concepts defined by these two dualities. The duality with $A(K)$ determines the w*-topology on $V = A(K)^*$. On K this topology coincides with the given compact topology, which we will henceforth refer to as the *w*-topology on K*. In particular, we will say that a face F of K is *w*-closed* if it is closed in this topology, and we will say that F is *w*-exposed* if it is exposed by a w*-continuous function $a \in A(K)^+$ (i.e., if $F = \{\omega \in K \mid \langle a, \omega\rangle = 0\}$). The duality with $A_b(K) \cong V^*$ determines a (stronger) weak topology on V, which determines the same continuous linear functionals and closed convex sets as the norm topology. We will say that a face F of K is *norm closed* if it is closed in this topology, and we will say that F is *norm exposed* if it is exposed by a norm continuous function $a \in A_b(K)^+$. Note that this is all standard terminology when K is the state space of a C*-algebra $\mathcal{A}$ with enveloping von Neumann algebra $\mathcal{A}^{**}$ (then $A(K) \cong \mathcal{A}_{\mathrm{sa}}$ and $A_b(K) \cong (\mathcal{A}^{**})_{\mathrm{sa}}$, cf. (A 70) and (A 102)), or when K is the state space of a JB-algebra A with enveloping JBW- algebra A^{**} (then $A(K) \cong A$ and $A_b(K) \cong A^{**}$ (Lemma 5.12 and Corollary 2.50)).

It is not hard to show that each finite dimensional simplex is spectral. But it is also true that a general Choquet simplex is spectral, as we now proceed to show. Recall first that a convex compact set K (regularly embedded in $V = A(K)^*$) is a *Choquet simplex* if V is a vector lattice with the ordering defined by the positive cone $V^+ = \mathrm{cone}(K)$, cf. (A 8) or [6, Ch. II, §3]. In fact, in this definition, rather than assuming that K is regularly imbedded, we may just assume that K is a compact convex subset of a locally convex space V, such that the linear span of K equals V and the affine span of K does not contain the origin, cf. [85].

8.77. Lemma. *If K is a Choquet simplex, then the space $A_b(K)$ is a vector lattice.*

Proof. The proof makes use of the Riesz decomposition property, i.e., if τ_1, τ_2, $\sigma \geq 0$, then

$$\sigma \leq \tau_1 + \tau_2 \;\Rightarrow\; \exists\, \sigma_1, \sigma_2,\; 0 \leq \sigma_i \leq \tau_i \text{ for } i = 1, 2, \text{ with } \sigma = \sigma_1 + \sigma_2,$$

and can be found, e.g., in [84, Prop. 23.9].) □

Note that the space $A(K)$ of a Choquet simplex is not a vector lattice in general. Nevertheless it enjoys the Riesz decomposition property. In fact, a compact convex set K is a Choquet simplex iff $A(K)$ enjoys the Riesz decomposition property (or the equivalent Riesz interpolation property, cf., e.g., [6, Cor. II.3.11]).

The following is the well known theorem of Stone–Kakutani–Krein–Yosida.

8.78. Theorem. *If a complete order unit space A is a vector lattice, then $A \cong C_{\mathbf{R}}(X)$ for some compact Hausdorff space X.*

Proof. We sketch the proof. Let X be the set of lattice homomorphisms from A into $\mathbf{R}$ that take the value 1 at the distinguished order unit. Then X is compact and Hausdorff (when equipped with the w*-topology), and coincides with the set of pure states on A. The evaluation map is then an isomorphism from A into $C_{\mathbf{R}}(X)$, which is surjective by the lattice version of the Stone–Weierstrass theorem. The details can be found in [6, Cor. II.1.11] or [84, Thm. 24.6]. □

8.79. Corollary. *If K is a Choquet simplex, then $A_b(K) \cong C_{\mathbf{R}}(X)$ for a compact Hausdorff space X, and K is affinely isomorphic to the normal state space of the JBW-algebra $C_{\mathbf{R}}(X)$, and to the normal state space of the abelian von Neumann algebra $C_{\mathbf{C}}(X)$.*

Proof. Assume K is regularly imbedded in $V = A(K)^*$. The first statement follows from Lemma 8.77 and Theorem 8.78. Thus $A_b(K) \cong C_{\mathbf{R}}(X)$ can be equipped with a product making it a JBW-algebra with predual V. Therefore by the uniqueness of the predual (Theorem 2.55), K is affinely isomorphic to the normal state space of $C_{\mathbf{R}}(X)$. Alternatively, $C_{\mathbf{C}}(X)$ will be a C*-algebra with a separating set of normal states, so will be an abelian von Neumann algebra (A 95), and the self-adjoint part of the unique predual will be V (Proposition 2.59), so K will be affinely isomorphic to the normal state space of $C_{\mathbf{C}}(X)$. □

In the proof above, (A 95) refers back to a theorem abstractly characterizing von Neumann algebras among C*-algebras (due to Kadison and to Sakai). The proof is non-trivial, and only a reference to the proof of the hard part is given in [AS, Thm. 2.93]. But what is needed here is only the commutative case, for which much less difficult arguments suffice (see [45, Cor. 5.2]).

8.80. Proposition. *Every Choquet simplex K is a spectral convex set for which every norm closed face is a split face, and thus is norm exposed*

Proof. By Corollary 8.79, K is affinely isomorphic to the normal state space of the JBW-algebra $C_{\mathbf{R}}(X)$ and to the normal state space of the abelian von Neumann algebra $C_{C}(X)$, so is spectral by Proposition 8.76. By Theorem 5.32 or (A 109), every norm closed face F of K is a projective face associated with a projection p, and is then exposed by $p' = 1-p$. Every projection p in $C_{\mathbf{R}}(X)$ or $C_{C}(X)$ is central, so the associated compression F will be a split face, cf. Corollary 5.34 or (A 112). □

Remark. If K is a compact spectral convex set regularly imbedded in $V = A(K)^*$, then every $a \in A \cong A_b(K)$ has a unique orthogonal decomposition $a = a^+ - a^-$ (Theorem 8.55). But it is not generally true that $a^+, a^- \in A(K)$ for all $a \in A(K)$. This is the case iff all spectral units e_λ (Definition 8.66) of continuous affine functions are upper semicontinuous [9, Prop. 2.4]. A compact spectral convex set K with this property was called *strongly spectral* in [7]. There it was shown that K is strongly spectral iff $A(K)$ is closed under functional calculus by continuous functions [7, Thm. 10.6]. The state space of every C*-algebra is a strongly spectral convex set [7, Thm. 11.6], and the state space of every JB-algebra is also a strongly spectral convex set [9, Cor. 3.2]. But it is not true that every Choquet simplex is strongly spectral. In fact, a Choquet simplex K is a strongly spectral convex set iff the extreme boundary $\partial_e K$ is closed, i.e., iff K is a Bauer simplex [7, Prop. 10.9].

We will now define a general concept of "trace" in the context of spectral convex sets, and then show that such traces form a Choquet simplex.

8.81. Definition. Let K be a spectral convex set and assume K is imbedded as the distinguished base of a base norm space V in duality with $A = V^*$. Then we will say that $\omega \in K$ is a *tracial state* if it is central in V, i.e., if $(P + P')^*\omega = \omega$ for all compressions P on A.

Clearly this notion generalizes the notion of a tracial state for C*-algebras (cf. the remarks after Definition 8.28).

Remark. By definition each tracial state is "central" in the sense of Definition 8.28. In the special case of the C*-algebra $M_2(\mathbf{C})$ it is also central in the usual geometric sense, since the state space of this algebra is a 3-dimensional Euclidean ball and the center of this ball is seen to be the unique tracial state (A 119). Actually, this statement can be generalized, as we will now explain. If we define a *generalized diameter* of a compact spectral convex set K to be a subset of the form $\mathrm{co}(F \cup F')$ where F is a projective face, then it follows easily from Definition 8.81 that the set of all tracial states in K is the intersection of all generalized diameters. Note that such generalized diameters of the normal state space of a von Neumann algebra were studied (but not named) in (A 174), and also that they are closely related to the *generalized axes* (A 185), which were used in the definition of *generalized rotations* (A 186) and *3-frames* (A 190).

8.82. Proposition. *If K is a compact convex set for which every norm exposed face is projective, then the set of tracial states in K is either empty or a Choquet simplex.*

Proof. Let K be imbedded as the distinguished base of a base norm space as in Definition 8.81, and assume that there exists at least one tracial state in K. Then the set of all central elements is a vector lattice (Proposition 8.30). The set of central elements is the linear span of the set of tracial states (Lemma 8.29). Hence K is a Choquet simplex. □

The proposition above gives the following alternative definition of a Choquet simplex which is perhaps more geometrically appealing than the original lattice definition, as it immediately shows that ordinary finite dimensional simplexes are Choquet simplexes.

8.83. Proposition. *A compact convex set K is a Choquet simplex iff every norm closed face is split.*

Proof. If K is a Choquet simplex, then every norm closed face is split (Proposition 8.80). On the other hand, if every norm closed face of K is split, since every split face is projective (Proposition 7.49), then every norm closed face of K is projective. For every compression P, the associated norm closed face is then split, so all compressions P are strongly complemented (Proposition 7.49), i.e., $P + P' = I$. Thus in this case every $\rho \in K$ is a tracial state. By Proposition 8.82, K is a Choquet simplex. □

Remark. It follows from Proposition 8.82 that the set of tracial states of a C*-algebra (or more generally, a JB-algebra) is a Choquet simplex. There is also a remarkable converse result, proved independently by Blackadar [30] and Goodearl [57], by which every metrizable Choquet simplex is the trace space of a C*-algebra A. (See the notes at the end of this chapter for more references on this topic.)

We will now specialize to weakly compact convex sets. Recall from the discussion in the previous section that such a set is compact in the weak topology determined by the space $A_b(K)$ of all bounded affine functions on K, which is equal to the space $A(K)$ of all continuous affine functions on K. In this case there is no difference between w*-closed and norm closed faces of K, or between w*-exposed and norm exposed faces of K. Therefore we will use the terms "closed face" and "exposed face" without further specification. Similarly we will use the term "tangent space" and the notation $\operatorname{Tan}(F)$ for a face F of K without further specification (cf. Definition 7.1).

We say that a weakly compact (and in particular a finite dimensional) convex set K is *strictly convex* if the only proper faces (i.e., non-empty faces $\neq K$) are the extreme points, and we say that K is *smooth* at an extreme point $\omega \in K$ if $\operatorname{Tan}(\omega)$ is a hyperplane (or otherwise stated, if ω is contained in a unique closed supporting hyperplane). We will also say a strictly convex weakly compact convex set K is *smooth* if every extreme point of K is smooth.

8.84. Proposition. *If K is a strictly convex and smooth weakly compact convex set, then K is spectral.*

Proof. As usual we assume K is regularly imbedded as the distinguished base of the base norm space $V = A(K)^*$ which is in separating duality with the order unit space $A = V^* \cong A_b(K)$. Then V is a reflexive space (Lemma 8.71), so we also have $V \cong V^{**} \cong A^*$. Thus we can apply Theorem 8.72, by which it suffices to show that the dual pair (A, V) satisfies the standing hypothesis of this chapter, i.e., that each exposed face of K is projective.

By strict convexity, each face of K is of the form $F = \{\omega\}$, where ω is an extreme point, and by smoothness its tangent space $H = \operatorname{Tan}(\omega)$ is a hyperplane, i.e.,

$$H = \{\rho \in \operatorname{aff}(K) \mid \langle h, \rho \rangle = 0\}$$

where $h \in A^+$, $\operatorname{aff}(K)$ is the affine span of K, and $H \cap K = \{\omega\}$. By weak compactness, h attains its maximum value at a non-empty face G of K. (We may assume this maximum value is 1.) By strict convexity, $G = \{\omega'\}$ where ω' is an extreme point of K, in fact, by smoothness ω' is the unique antipodal point of ω.

Let ρ be an arbitrary point of K. By smoothness of K, the tangent space of K at ω' is the set of points in the affine span of K where h takes on the value 1. Then there is a unique scalar λ such that $\langle h, \lambda\omega + (1-\lambda)\rho \rangle = 1$, so K is contained in the direct convex sum of $F = \{\omega\}$ and the tangent space of K at $G = \{\omega'\}$. A similar result holds with the roles of ω and ω' interchanged. Thus the conditions (i) and (ii) of Theorem 7.52 are satisfied, so F and G are complementary projective faces of K. Thus each face of K is projective, and we are done. □

8.85. Corollary. *The closed unit ball of a Hilbert space, and in particular any finite dimensional Euclidean ball, is a spectral convex set. Also the closed unit ball of $L^p(\mu)$ is a spectral convex set for each measure μ and each exponent p such that $1 < p < \infty$.*

Proof. The closed unit ball of $L^p(\mu)$ is weakly compact, and is smooth and strictly convex (cf. [87, p. 351]). Now the result is clear from Proposition 8.84. □

If V_1 and V_2 are base norm spaces with distinguished bases K_1 and K_2 respectively, then it is easily seen that their direct sum $V = V_1 \oplus V_2$ is a base norm space with distinguished base $K = K_1 \oplus_c K_2$ (direct convex sum), and that the direct sum $A = A_1 \oplus A_2$ of the order unit spaces $A = V_1^*$ and $A_2 = V_2^*$ is an order unit space isomorphic to V^*, in separating order and norm duality with V under the bilinear form

$$\langle a_1 + a_2, \omega_1 + \omega_2 \rangle = \langle a_1, \omega_1 \rangle + \langle a_2, \omega_2 \rangle. \tag{8.64}$$

Observe that the faces of K are the sets $F_1 \oplus_c F_2$ where F_1 and F_2 are faces of K_1 and K_2 (possibly improper, i.e., $F_i = \emptyset$ or $F_i = K_i$ for $i = 1, 2$). In fact, each face of K is of the form

$$F = F_1 \oplus_c F_2 \quad \text{where} \quad F_i = F \cap K_i \ \text{ for } i = 1, 2. \tag{8.65}$$

Assume for $i = 1, 2$ that F_i is a projective face of K_i, with associated projective unit p_i and associated compression P_i, and let F_i', p_i', P_i' be the complementary projective face, projective unit, and compression, respectively. Then the direct sum $P_1 \oplus P_2$ is seen to be a compression of $A_1 \oplus A_2$ with complementary compression $P_1' \oplus P_2'$. Its associated projective unit is $p_1 \oplus p_2$ with complementary projective unit $p_1' \oplus p_2'$, and its associated projective face is $F_1 \oplus_c F_2$ with complementary projective unit $F_1' \oplus_c F_2'$. Thus if F_i is a projective face of K_i for $i = 1, 2$, then $F_1 \oplus_c F_2$ is a projective face of $K = K_1 \oplus_c K_2$, and its complementary projective face is $F_1' \oplus_c F_2'$. From this we draw the following conclusion.

8.86. Proposition. *The direct convex sum $K = K_1 \oplus_c K_2$ of two spectral convex sets K_1 and K_2 is a spectral convex set. A subset of K is a projective face iff it is of the form $F_1 \oplus_c F_2$ where F_i is a projective face of K_i (possibly improper) for $i = 1, 2$, and then its complementary projective face is given by*

$$(F_1 \oplus_c F_2)' = F_1' \oplus_c F_2'. \tag{8.66}$$

Proof. Clear from the argument above. □

Proposition 8.86 provides many new spectral convex sets. For example, in dimension 3, in addition to Euclidean balls and tetrahedrons (3-simplexes), there are also circular cones (direct convex sums of a circular disk and a single point). Such direct convex sums are perhaps not so interesting since their facial structure is completely described by the facial structure of their direct summands by virtue of Proposition 8.86. But there exist other examples of spectral convex sets already in finite dimensions. Some examples are given in our next theorem.

8.87. Theorem. *If $\widetilde{K}$ is the set of all points*

$$(x;\, y_1, \dots, y_m;\, z_1, \dots, z_n;\, t\,) \in \mathbf{R}^{m+n+2} \tag{8.67}$$

where $m, n = 0, 1, 2, \dots$ (if $m = 0$ or $n = 0$, then there are no y-terms or z-terms) which satisfy the inequality

$$t^4 \leq \left(x^2 - \sum_{i=1}^{m} y_i^2\right)\left((1-x)^2 - \sum_{j=1}^{n} z_j^2\right), \tag{8.68}$$

together with the inequalities

$$0 \leq x \leq 1, \quad \sum_{i=1}^{m} y_i^2 \leq x^2, \quad \sum_{j=1}^{n} z_j^2 \leq (1-x)^2, \tag{8.69}$$

then $\widetilde{K}$ is a non-decomposable spectral convex set.

Proof. $\widetilde{K}$ is constructed from the direct convex sum $K = B \oplus_c C$ of a Euclidean m-ball B and a Euclidean n-ball C by "blowing up" each line segment between a point in B and a point in C to an elliptical disk in a plane orthogonal to the affine span of B and C. This construction involves some rather lengthy convexity arguments which are not needed in the sequel. Therefore we have relegated the details of the proof to the appendix of this chapter. □

If $m = n = 0$, then (8.68) reduces to

$$t^2 \leq x(1-x), \tag{8.70}$$

which defines a circular disk in $\mathbf{R}^2$ (obtained by "blowing up" the line segment between the points $(0, 0)$ and $(0, 1)$).

If $m = 1$ and $n = 0$, then (8.68) reduces to

$$t^4 \leq (x^2 - y^2)(1-x)^2. \tag{8.71}$$

This inequality determines a convex set in $\mathbf{R}^3$ which is shown in Fig. 8.1.

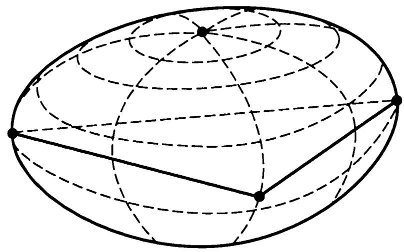

Fig. 8.1

Note that the convex set defined by (8.71) is constructed from the direct convex sum of the origin $(0,0,0)$ and the line segment between $(1,1,0)$ and $(1,-1,0)$) by blowing up each line segment $[(0,0,0),(0,y_0,0)]$ with $|y_0| \le 1$ to the elliptical disk consisting of all points (x,y,t) where $y = xy_0$ and t satisfies the inequality

$$t^2 \le \delta^2 x(1-x),$$

where $\delta = \sqrt{1-y_0^2}$. The longer axis of the ellipse is the line segment $[(0,0,0),(0,y_0,0)]$ in the x,y-plane and the shorter axis is a vertical line segment of length δ, so the ellipse degenerates to a line segment iff $y_0 = \pm 1$.

In the present case the convex set $\widetilde{K}$ looks like a "triangular pillow". The faces are the three edges and the three vertices of the triangle together with all boundary points on the curved surface off the triangle. The affine tangent spaces at the edges are the vertical planes which they determine; the affine tangent spaces at the vertices are the vertical lines which they determine; and the affine tangent spaces at the (smooth) points on the curved surface are the ordinary tangent planes at these points. An edge of the triangle and its opposite vertex are seen to satisfy the requirements for the faces F and G of Theorem 7.52, so they are complementary projective faces. Similarly, a boundary point on the curved surface and its antipodal boundary point (with parallel tangent plane) are complementary projective faces. Thus all faces are projective, and by Theorem 8.72 that is all that is needed to show this convex set is spectral.

We will also describe in some detail the two four dimensional spectral convex sets obtained from Theorem 8.87.

If $m = 1$ and $n = 1$, then (8.67) reduces to

$$t^4 \le (x^2 - y^2)\,((1-x)^2 - z^2). \tag{8.72}$$

This inequality determines a convex set in $\mathbf{R}^4$ which is constructed in the same way as above from the direct convex sum of the two line segments

$\{x, y, z, t) \mid x = y = t = 0, |z| \leq 1\}$ and $\{x, y, z, t) \mid x = 1, |y| \leq 1, z = t = 0\}$, so a tetrahedron in $\mathbf{R}^3$ will play the same role as the triangle in the example above.

If $m = 2$ and $n = 0$, then (8.67) reduces to

$$t^4 \leq (x^2 - (y_1^2 + y_2^2))\,(1 - x)^2. \tag{8.73}$$

This inequality determines a convex set in $\mathbf{R}^4$ which is constructed in the same way from the direct convex sum of the circular disk $\{(x, y_1, y_2, t) \mid x = 1, y_1^2 + y_2^2 \leq 1, t = 0\}$ and the single point $(0, 0, 0, 0)$, so a circular cone in $\mathbf{R}^3$ will play the same role as the triangle or the tetrahedron in the examples above. (The spectral convex set obtained for $m = 0$ and $n = 2$ is isomorphic to the one in this example under the map $x \mapsto 1 - x$, $y_\imath \mapsto z_i$, $z_i \mapsto y_i$ with $\imath = 1, 2$).

It is possible to construct many other examples of non-decomposable spectral convex sets by generalizing the construction in the proof of Theorem 8.87 in various ways, but a systematic study of such examples is beyond the scope of this book.

Remark. We will now continue the discussion of the order theoretic approach to quantum mechanics under the assumption that A and V are in spectral duality, and we begin by explaining how this assumption is motivated by physics.

Assume $A = V^*$, and recall that by Lemma 8.54, A and V are in spectral duality iff for all $a \in A$ there exists a unique projective face F compatible with a such that

$$a > 0 \text{ on } F \quad \text{and} \quad a \leq 0 \text{ on } F'. \tag{8.74}$$

Note that since the projective unit p associated with F is compatible with a, the proposition $p \in \mathcal{P}$ and the general observable $a \in A$ are simultaneously measurable.

We know that a measuring device for the proposition p will split a beam of particles in a state ω into two partial beams, one consisting of particles that have the value 1 on p and are in a transformed state $\omega_1 \in F$, and one consisting of particles that have the value 0 and are in a transformed state $\omega_0 \in F'$. By (8.74) the value of the observable a (which is unaffected by measurement of p) is strictly greater than zero on the first partial beam and less than or equal to zero on the second. Therefore the proposition p is the question that asks if the observable a is strictly positive. Thus, the spectral axiom means that for each observable $a \in A$, the question: "does a have a value > 0?" is a quantum mechanical proposition, i.e., a question that can be answered by a quantum mechanical measuring device.

By Theorem 8.52, the spectral axiom implies the standing hypothesis of this chapter. Thus if A and V are in spectral duality, then the propositions form a complete orthomodular lattice as in standard quantum mechanics (Theorem 8.10 and Proposition 8.20). But the standing hypothesis is not enough to give all aspects of quantum mechanical measurement theory. An important ingredient that is missing is a mathematical procedure to determine the entire probability distribution (not only the expectation) of an observable $a \in A$. Thus what we would like to have is a method to compute the probability that the value of an observable is in a given Borel set $E \subset \mathbf{R}$ when the observable is measured on a system in a given state $\omega \in K$. We will show that this can be achieved if we assume that A and V are in spectral duality.

Spectral duality of A and V provides a spectral resolution $\{e_\lambda\}_{\lambda \in \mathbf{R}}$ of each $a \in A$ (Theorem 8.64 and Definition 8.66). This spectral resolution provides a functional calculus for a, i.e., a map $f \mapsto f(a)$ from the space $B(\mathbf{R})$ of bounded Borel functions into A satisfying the conditions (i)–(iv) of Proposition 8.67. As explained in the remark after Proposition 8.67, the element $f(a)$ of A represents the observable operationally defined by a measurement of the given observable a followed by evaluation of the function f on the recorded result. Thus we can for each Borel set $B \subset \mathbf{R}$ compute the observable $\chi_E(a) \in A$, which has the value 1 when the value of a is in E and the value 0 when the value of a is in $\mathbf{R} \setminus E$. This observable is a proposition, more specifically, it is the question that asks if the value of a is in E. The expected value $\langle \chi_E(a), \omega \rangle$ is the same as the probability that the proposition $\chi_E(a)$ has the value 1, and therefore that the observable a has a value in E, when the system is in the state ω. In this way, the functional calculus gives the desired probability distribution of the observable in every state of the quantum system.

Note, however, that although all aspects of quantum mechanical measurement theory which depend on spectral theory and functional calculus can be adequately described in the context of spectral duality, there are also features which require additional assumptions. This is the case, for example, for the pure state properties (among those the symmetry of transition probabilities) which will be introduced and studied in the next chapter.

Appendix. Proof of Theorem 8.87

In this appendix S will denote a finite dimensional real linear space, i.e., $S \cong \mathbf{R}^k$ for a natural number k. If N is an affine subspace of S, then the *vertical subspace* over N is the affine subspace $N \times \mathbf{R}$ of $S \times \mathbf{R} \cong \mathbf{R}^{k+1}$ which consists of all pairs (w, λ) with $w \in N$ and $\lambda \in \mathbf{R}$. An affine subspace M of $S \times \mathbf{R}$ is said to be *vertical* if it is the vertical subspace over the subspace $N = M \cap S$ of S. Note that in order to prove that an affine subspace M of $S \times \mathbf{R}$ is vertical, it suffices to show that M contains a single vertical line $\{w\} \times \mathbf{R}$. (This observation is based on the simple fact that if an affine subspace contains three of the vertices of a parallelogram, then it also contains the fourth vertex.)

8.88. Lemma. *Let K be a compact convex set in a finite dimensional space S and let $\widetilde{K}$ be a compact convex set in $S \times \mathbf{R}$ such that $\widetilde{K} \cap S = K$. Assume that F_1 and F_2 are complementary projective faces of K and that the following conditions are satisfied:*

(i) *$(w, t) \in \widetilde{K}$ implies $w \in K$.*
(ii) *F_1 and F_2 are faces of $\widetilde{K}$ as well as of K.*
(iii) *The affine tangent spaces of $\widetilde{K}$ at F_1 and F_2 are the vertical subspaces over the affine tangent spaces of K at F_1 and F_2 respectively.*

Then F_1 and F_2 are complementary projective faces of $\widetilde{K}$.

Proof. Let N_1 and N_2 be the affine tangent spaces of F_1 and F_2 respectively. By Theorem 7.52,

$$K \subset F_1 \oplus_c N_2, \quad K \subset F_2 \oplus_c N_1, \quad N_1 \cap N_2 = \emptyset. \tag{8.75}$$

We must show that also

$$\begin{gathered} \widetilde{K} \subset F_1 \oplus_c (N_2 \times \mathbf{R}), \quad \widetilde{K} \subset F_2 \oplus_c (N_1 \times \mathbf{R}), \\ (N_1 \times \mathbf{R}) \cap (N_2 \times \mathbf{R}) = \emptyset. \end{gathered} \tag{8.76}$$

To prove the first inclusion in (8.76), we consider an arbitrary point $(w, t) \in \widetilde{K}$ and we will show that it can be uniquely expressed as a convex combination of a point in F_1 and a point in $N_2 \times \mathbf{R}$. By condition (i), $w \in K$, and by (8.75) the point w can be uniquely expressed as a convex combination

$$w = \lambda w_1 + (1 - \lambda) w_2 \tag{8.77}$$

where $w_1 \in F_1$, $w_2 \in N_2$, and $0 \leq \lambda \leq 1$. Here we can assume $\lambda < 1$; otherwise the point $(w, 0)$ is in the face F_1 of $\widetilde{K}$ and there is nothing to prove. Assuming $\lambda < 1$, we define $t_2 = (1 - \lambda)^{-1} t$. Then

$$(w, t) = \lambda(w_1, 0) + (1 - \lambda)(w_2, t_2),$$

and by the uniqueness of the convex combination (8.77), this expression of (w, t) as a convex combination of a point in F_1 and a point in $N_2 \times \mathbf{R}$ is also unique.

The second inclusion in (8.76) is proved in the same way, and the last equality in (8.76) follows trivially from the last equality in (8.75). □

8.89. Lemma. *Let K be a compact convex set in a finite dimensional space S and let $\widetilde{K}$ be a compact convex set in $S \times \mathbf{R}$ such that $\widetilde{K} \cap S = K$.*

Assume that F_1 and F_2 are complementary projective faces of K and that the following conditions are satisfied:

(i) $(w,t) \in \widetilde{K}$ *implies* $w \in K$.
(ii) $(w,t) \in \widetilde{K}$ *and* $w \in F_1$ *or* $w \in F_2$ *implies* $t = 0$.
(iii) *There is a line L through two points $w_1 \in F_1$ and $w_2 \in F_2$ such that the plane set $D = \widetilde{K} \cap (L \times \mathbf{R})$ has vertical tangents at F_1 and F_2.*

Then F_1 and F_2 are complementary projective faces of $\widetilde{K}$.

Proof. We will first prove statement (ii) of Lemma 8.88. Assume $w \in F_1$ and

$$(w,0) = \lambda(u_1,t_1) + (1-\lambda)(u_2,t_2),$$

where $(u_i,t_i) \in \widetilde{K}$ for $i = 1,2$ and $0 < \lambda < 1$. By condition (i) above, $u_1, u_2 \in K$. Since $w = \lambda u_1 + (1-\lambda)u_2 \in F_1$, then $u_1, u_2 \in F_1$. By condition (ii) above, $t_i = 0$ so that $(u_i,t_i) \in F_i \times \{0\}$ for $i = 1,2$. Thus F_1 is a face of $\widetilde{K}$. Similarly F_2 is a face of $\widetilde{K}$.

We will now prove statement (iii) of Lemma 8.88. Let M be any supporting hyperplane of $\widetilde{K}$ at F_1. Either the hyperplane M contains the plane $L \times \mathbf{R}$ or it intersects it in an affine subspace which will be a supporting affine subspace of the convex set D in $L \times \mathbf{R}$, so it must contain the vertical tangent $\{w_1\} \times \mathbf{R}$ of D at w_1. In either case, by the observation preceding Lemma 8.88, M will be a vertical subspace of $S \times \mathbf{R}$.

Observe that the vertical hyperplane M in $S \times \mathbf{R}$ supports $\widetilde{K}$ at F_1 iff it is of the form $M = H \times \mathbf{R}$ where H is a hyperplane in S that supports K at F_1. (If ϕ is a defining function for H, then $(w,t) \mapsto \phi(w)$ is a defining function for $H \times \mathbf{R}$.) The intersection of all such hyperplanes $H \times \mathbf{R}$ is equal to $N_1 \times \mathbf{R}$ (since N_1 is the intersection of all hyperplanes H in S that supports K at F_1). Therefore the tangent space for $\widetilde{K}$ at F_1 is equal to $N_1 \times \mathbf{R}$. By the same argument, the tangent space for $\widetilde{K}$ at F_2 is equal to $N_2 \times \mathbf{R}$. This completes the proof. □

Consider a direct convex sum $K = B \oplus_c C$ of two Euclidean balls, say that B and C are the closed unit balls of $\mathbf{R}^m$ and $\mathbf{R}^n$ respectively, and admit also the case where m or n is zero so that B or C is an "improper ball" consisting of a single point. Let $X = \mathbf{R}$, $Y = \mathbf{R}^m$, $Z = \mathbf{R}^n$ and consider the Euclidean space $S = X \times Y \times Z \cong \mathbf{R}^{m+n+1}$. Assume (without loss) that K is imbedded in S by the affine isomorphism which identifies the point $b \in B$ with the point $(1,b,0) \in S$ and the point $c \in C$ with the point $(0,0,c) \in S$. Then the point $\lambda\, b \oplus (1-\lambda)\, c \in K$ with $b \in B$, $c \in C$ and $0 \le \lambda \le 1$ will be identified with the point $(x,y,z) \in S$ where $x = \lambda \in X$, $y = \lambda\, b \in Y$, $z = (1-\lambda)\, c$.

Thus we can represent K as the set of all points

$$w = (x, y, z) \in S, \tag{8.78}$$

where $S = X \times Y \times Z$ with $X = \mathbf{R}$, $Y = \mathbf{R}^m$, $Z = \mathbf{R}^n$ and

$$0 \le x \le 1, \quad ||y|| \le x, \quad ||z|| \le 1 - x. \tag{8.79}$$

8.90. Lemma. *Let K be as above. Then all proper faces of K are projective, and they are of the following types where u and v are boundary points of B and C respectively:*

(i) *B or C (balls),*
(ii) *$[u, v] = \{u\} \oplus_c \{v\}$ (line segments),*
(iii) *$B \oplus_c \{v\}$ or $\{u\} \oplus_c C$ (caps of cones),*
(iv) *$\{u\}$ or $\{v\}$ (singletons).*

B and C are complementary split faces. If u' is the antipodal point of u on the boundary of the ball B and v' is the antipodal point of v on the boundary of the ball C, then the line segments $[u, v]$ and $[u', v']$ are complementary projective faces, the circular cone $B \oplus_c \{v\}$ and the singleton $\{v'\}$ are complementary projective faces, similarly the circular cone $\{u\} \oplus_c C$ and the singleton $\{u'\}$ are complementary projective faces.

Proof. Clear from Proposition 8.86. □

Note that in Lemma 8.90, if B is a single point, then only (i) and the first part of (iii) need be considered, and similarly if C is a single point, then only (i) and the second part of (iii) apply.

The theorem below is Theorem 8.87 restated in the terminology of this appendix.

8.91. Theorem. *If K is the direct convex sum of two Euclidean balls B and C represented as a subset of the Euclidean space $S = X \times Y \times Z$ where $X = \mathbf{R}$, $Y = \mathbf{R}^m$, $Z = \mathbf{R}^n$ as in (8.78) and (8.79), then the set $\widetilde{K} \subset S \times \mathbf{R}$ consisting of all points (w, t) with $w = (x, y, z) \in K$, $t \in \mathbf{R}$ and*

$$t^4 \le (x^2 - \|y\|^2)\big((1 - x)^2 - \|z\|^2\big) \tag{8.80}$$

is a non-decomposable spectral convex set.

Proof. Let f be the positive function defined on K by the equation $f(w) = t$ where $w = (x, y, z)$ and

$$t^4 = (x^2 - \|y\|^2)\big((1 - x)^2 - \|z\|^2\big). \tag{8.81}$$

Now $\widetilde{K}$ consists of all points $(w,t) \in S \times \mathbf{R}$ such that $w \in K$, $t \in \mathbf{R}$ and

$$|t| \leq f(w). \tag{8.82}$$

Clearly f is a continuous function, so $\widetilde{K}$ is compact. We will show that f is a concave function, which implies that $\widetilde{K}$ is convex.

Consider the two functions

$$g(x,y) = \sqrt{x^2 - \|y\|^2}, \quad h(x,z) = \sqrt{(1-x)^2 - \|z\|^2}, \tag{8.83}$$

where g is defined on the set of all points $(x,y) \in X \times Y$ such that $0 \leq x \leq 1$ and $\|y\| \leq x$, and where h is defined on the set of all points $(x,z) \in X \times Z$ such that $0 \leq x \leq 1$ and $\|z\| \leq 1-x$. Observe that for $u \geq 0$ the equation $u = g(x,y)$ is equivalent to $u^2 + \|y\|^2 = x^2$, which defines the boundary of a convex cone in $X \times Y \times \mathbf{R}$ (with top-point $(0,0,0)$ and with base $\{(1,y,u) \mid u^2 + \|y\|^2 \leq 1\}$ that is a Euclidean ball). The subgraph of g (i.e., the set of all points (x,y,t) such that $0 \leq t \leq g(x,y)$) is the "upper half" of this cone (i.e., the intersection with the upper half-space consisting of all points $(x,y,t) \in X \times Y \times \mathbf{R}$ such that $t \geq 0$). Thus the subgraph of g is convex. Therefore g is concave. Similarly, the subgraph of h is the upper half of a convex cone in $X \times Y \times \mathbf{R}$) (with top-point $(1,0,0)$), from which it follows that h is concave.

Note that $f(x,y,z) = \sqrt{g(x,y)h(y,z)}$. From this and the concavity of g and h it follows that f is concave, as we will now show. Let $w_1 = (x_1,y_1,z_1)$ and $w_2 = (x_2,y_2,z_2)$ be two distinct points of K, let $0 < \lambda < 1$, and define $w_0 = (x_0,y_0,z_0)$ by

$$w_0 = \lambda w_1 + (1-\lambda)w_2. \tag{8.84}$$

For simplicity of notation, set $f_i = f(w_i)$, $g_i = g(x_i,y_i)$ and $h_i = h(y_i,z_i)$ for $\imath = 0,1,2$. By the Cauchy–Schwarz inequality and the concavity of g and h, we have

$$\begin{aligned} \lambda f_1 + (1-\lambda)f_2 &= \sqrt{\lambda g_1}\,\sqrt{\lambda h_1} + \sqrt{(1-\lambda)g_2}\,\sqrt{(1-\lambda)h_2} \\ &\leq \sqrt{\lambda g_1 + (1-\lambda)g_2}\,\sqrt{\lambda h_1 + (1-\lambda)h_2} \\ &\leq \sqrt{g_0}\,\sqrt{h_0} = f_0. \end{aligned} \tag{8.85}$$

Thus f is concave on K, so $\widetilde{K}$ is convex.

We will now show that f is strictly concave in the interior of the set $K \subset S$. By the definition of K and f (i.e., by (8.79) and (8.81)), the boundary of K consists of all points $w \in K$ such that $f(w) = 0$. Thus $f(w) > 0$ for w in the interior of K.

Assume for contradiction that w_0 is an interior point of K for which f is not strictly convex. Then w_0 can be written as a convex combination

(8.84) with $0 < \lambda < 1$ and $w_1 \neq w_2$ for which the equality signs hold in (8.85). Moreover, we can and will choose this convex combination such that w_1 and w_2 are interior points of K. Then for $\imath = 0, 1, 2$ the variables f_i, g_i, h_i are all non-zero.

Since the last $\leq$ relation in (8.85) holds with equality,

$$\lambda g_1 + (1-\lambda) g_2 = g_0, \quad \lambda h_1 + (1-\lambda) h_2 = h_0. \tag{8.86}$$

From the first of these equalities, it follows that the line segment between the points $((x_1, y_1), g_1)$ and $((x_2, y_2), g_2)$ in $(X \times Y) \times \mathbf{R}$ lies on the boundary of the (conical) subgraph of g, hence on a ray from the point $(0,0,0)$. By the second equality in (8.86), the line segment between the points $((x_1, z_1), h_1)$ and $((x_2, z_2), h_2)$ in $(X \times Z) \times \mathbf{R}$ lies on the boundary of the (conical) subgraph of h, hence on a ray from the point $(1,0,0)$. Thus there exist two scalar factors α, β such that

$$\begin{gathered} x_2 = \alpha x_1, \quad y_2 = \alpha y_1, \quad g_2 = \alpha g_1, \\ 1 - x_2 = \beta(1 - x_1), \quad z_2 = \beta z_1, \quad h_2 = \beta h_1. \end{gathered} \tag{8.87}$$

Since the first $\leq$ relation in (8.85) holds with equality and this relation follows from Cauchy–Schwarz, the two $\mathbf{R}^2$-vectors $(\sqrt{\lambda g_1}, \sqrt{(1-\lambda) g_2})$ and $(\sqrt{\lambda h_1}, \sqrt{(1-\lambda) h_2})$ are proportional. Thus $h_1/g_1 = h_2/g_2$, so $g_2/g_1 = h_2/h_1$. Therefore there exists a scalar factor κ such that $g_2 = \kappa g_1$, $h_2 = \kappa h_1$. Combining with (8.87), we find that $\alpha = \beta = \kappa$ and that

$$x_2 = \kappa x_1, \quad 1 - x_2 = \kappa(1 - x_1).$$

Adding these two equations, we conclude that $\alpha = \beta = \kappa = 1$ so that the two points $w_1 = (x_1, y_1, z_1)$ and $w_2 = (x_2, y_2, z_2)$ coincide, contrary to assumption.

The interior of the set $\widetilde{K}$ consists of all points $(w, t) \in K \times \mathbf{R}$ for which the inequality (8.80) is strict, that is, the ones for which $|t| < f(w)$. Therefore the boundary of $\widetilde{K}$ consists of all $(w, t) \in K \times \mathbf{R}$ for which $f(w) = |t|$, and the boundary points with w in the interior of K are those for which $|t| = f(w) > 0$. Since f is strictly concave in the interior of K, all such boundary points are seen to be extreme points of $\widetilde{K}$.

We will now determine the faces of $\widetilde{K}$. By the above, they include all singletons $\{(w, t)\}$ on the boundary of $\widetilde{K}$ for which w is an interior point of K. We will show that the only other faces of $\widetilde{K}$ are the subsets that are proper faces of K. Let F be a proper face of K (i.e., $F \neq K$). To prove F is a face of $\widetilde{K}$, let $(w, 0)$ be an arbitrary point of F and assume

$$(w, 0) = \lambda(w_1, t_1) + (1-\lambda)(w_2, t_2)$$

where $0 < \lambda < 1$ and $(w_i, t_i) \in \widetilde{K}$ for $i = 1, 2$. Then $w = \lambda w_1 + (1-\lambda)w_2$, so w_1 and w_2 are in F. Since F is a proper face, it contains no interior point of K. Hence $|t_i| \le f(w_i) = 0$ for $i = 1, 2$. Thus $|t_1| = |t_2| = 0$, so that (w_1, t_1) and (w_2, t_2) are in F. This shows F is a face of $\widetilde{K}$. Clearly every face of $\widetilde{K}$ which is contained in K must be a face of K, so it only remains to show that a face of $\widetilde{K}$ that is not contained in K is a singleton. But this is clear, for otherwise such a face would contain an open line segment disjoint from K, and for a point (w, t) of this segment we would have $|t| > 0$, so that w would be an interior point of K and then (w, t) would be an extreme point of $\widetilde{K}$, contrary to the fact that all these points are located on an open line segment in $\widetilde{K}$.

With this we have shown that the faces of $\widetilde{K}$ are those extreme points of $\widetilde{K}$ that are not in K together with the faces of K. We will show $\widetilde{K}$ is spectral by proving that all these faces are projective (Theorem 8.72).

The non-trivial faces of K are the ones described in Lemma 8.90. We will show they are all projective in $\widetilde{K}$ by verifying the assumptions in Lemma 8.89. Conditions (i) and (ii) are easily verified, so we only have to construct lines L with the property demanded in condition (iii) for pairs F_1, F_2 of projective faces of K.

First assume $F_1 = B$ and $F_2 = C$. Let L be the line through the centers $w_1 = (1, 0, 0)$ and $w_2 = (0, 0, 0)$ of the two balls B and C. Thus L consists of the points $w_\lambda = (\lambda, 0, 0)$ with $\lambda \in \mathbf{R}$, and $L \times \mathbf{R}$ consists of the points (w_λ, t) with $(\lambda, t) \in \mathbf{R}^2$. By (8.80), the convex set $D = \widetilde{K} \cap (L \times \mathbf{R})$ consists of all points (w_λ, t) for which $t^4 \le \lambda^2(1-\lambda)^2$. Thus D is the circular disk defined by the inequality

$$t^2 \le \lambda(1-\lambda). \tag{8.88}$$

Clearly, condition (iii) of Lemma 8.89 is satisfied in this case.

Next assume that $F_1 = [u, v]$ and $F_2 = [u', v']$ where u and v are boundary points of B and C respectively, and where u' and v' are the antipodal boundary points. Now let L be the line through the mid-points $w_1 = \frac{1}{2}(u+v)$ and $w_2 = \frac{1}{2}(u'+v')$ of the line segments $[u, v]$ and $[u', v']$. Therefore $u = (1, b, 0)$ and $u' = (1, -b, 0)$ where $b \in B$ and $\|b\| = 1$. Similarly $v = (0, 0, c)$ and $v' = (0, 0, -c)$ where $c \in C$ and $\|c\| = 1$. Hence $w_1 = (\frac{1}{2}, \frac{1}{2}b, \frac{1}{2}c)$ and $w_2 = (\frac{1}{2}, -\frac{1}{2}b, -\frac{1}{2}c)$. L consists of all points

$$w_\lambda = \lambda w_1 + (1-\lambda)w_2 \quad \text{with } \lambda \in \mathbf{R}, \tag{8.89}$$

and with the above values for w_1 and w_2,

$$w_\lambda = (\,\tfrac{1}{2}, (\lambda - \tfrac{1}{2})b, (\lambda - \tfrac{1}{2})c\,).$$

Substituting $x = \frac{1}{2}$, $y = (\lambda - \frac{1}{2})b$, $z = (\lambda - \frac{1}{2})c$ into (8.80), we get the same inequality (8.88) as in the preceding case. Thus D is a circular disk and condition (iii) of Lemma 8.89 is satisfied also in this case.

Finally assume $F_1 = B \oplus_c \{v\}$ and $F_2 = \{v'\}$ where v is a boundary point of C and v' is the antipodal boundary point. (The case where $F_1 = \{u\} \oplus_c C$ and $F_2 = \{u'\}$ is similar.) Let u be the center of B and let L be the line through the points $w_1 = \frac{1}{2}(u + v)$ and $w_2 = v'$. Now $u = (1, 0, 0)$, $v = (0, 0, c)$ and $v' = (0, 0, -c)$ where $c \in C$ and $\|c\| = 1$. Hence $w_1 = (\frac{1}{2}, 0, \frac{1}{2}c)$ and $w_2 = (0, 0, -c)$. The points of L are expressed in terms of w_1 and w_2 as in (8.89). Therefore we find that

$$w_\lambda = (\,\tfrac{1}{2}\,\lambda,\ 0,\ \tfrac{3}{2}\,\lambda - 1\,).$$

Substituting $x = \frac{1}{2}\,\lambda$, $y = 0$, $z = \frac{3}{2}\,\lambda - 1$ into (8.80), we get the inequality

$$t^4 \le \tfrac{1}{2}\,\lambda^3(1 - \lambda). \tag{8.90}$$

This inequality defines an oval disk (of degree 4) with vertical tangents for $\lambda = 0, 1$. Thus D has vertical tangents at w_1 and w_2, so condition (iii) of Lemma 8.89 is satisfied also in this case.

Now we consider extreme points of $\widetilde{K}$. Clearly f is continuously differentiable in the interior of K, from which it follows that $\widetilde{K}$ has a unique tangent hyperplane at each boundary point (w, t) with w in the interior of K, and this hyperplane is equal to $\mathrm{Tan}\{(w, t)\}$ (as defined in Definition 7.1). Clearly (w, t) has a unique antipodal point (w', t') (also with w' in the interior of K) defined by the property that the tangent space of $\widetilde{K}$ at (w', t') is parallel to the tangent space at (w, t). (By the results of the previous paragraph, this antipodal point is not in K.) Now it follows by an elementary argument in plane geometry that each point in $\widetilde{K}$ can be expressed in a unique way as a convex combination of the point (w, t) and a point in $\mathrm{Tan}\{(w', t')\}$ (cf. the proof of Proposition 8.84). Thus the conditions (i) and (ii) of Theorem 7.52 are satisfied by the pair of faces $F = \{(w, t)\}$ and $G = \{(w', t')\}$. Therefore all extreme points of $\widetilde{K}$ that are not in K are projective faces of $\widetilde{K}$. The proof is complete. □

It can be shown that the set $D = \widetilde{K} \cap (L \times \mathbf{R})$ defined by the line L between arbitrary interior points w_1 and w_2 of B and C is an elliptical disk (which degenerates to a line segment if w_1 or w_2 are boundary points.) Thus $\widetilde{K}$ is obtained by "blowing up" the line segments between points in B and C to elliptical disks in an extra dimension.

If $m = n = 0$, then $\widetilde{K}$ is a circular disk, so it is the state space of the spin factor $M_2(\mathbf{R})_{\mathrm{sa}}$. In all other cases $\widetilde{K}$ is different from the state space of a JB-algebra. (The inequality (8.90) violates the elliptical cross-section property of Proposition 5.75.)

The convex sets $\widetilde{K}$ in Theorem 8.91 are mutually non-isomorphic except for the trivial isomorphism between sets with transposed pairs of parameters m, n and n, m (under the map $x \mapsto 1 - x$, $y \mapsto z$, $z \mapsto y$).

Notes

The results in this chapter on the lattice of compressions, and on spectral duality, were established either in [7] or [9], with a few exceptions noted below. Loosely speaking, spectral duality requires that there be "enough" projective faces or compressions. The concept of spectral duality in [7] was defined in terms of projective faces by statement (ii) of our present Theorem 8.69, which involves projective faces in the bicommutant. The alternate characterization of spectral duality in statement (ii) of Lemma 8.54 was given in [7, Thm. 7.5], and was used to prove the existence of unique orthogonal decompositions (Theorem 8.55) in [9, Thm. 2.2]. Our present definition of spectral duality in terms of compressions (Definition 8.42) does not involve the bicommutant, but we have to work harder to prove the key result that norm exposed faces are projective (Theorem 8.52). An important first step in the argument that leads up to this theorem is Lemma 8.46, which was originally proved by Riedel [106] in a slightly different context (with his "fundamental units" in place of our projective units).

Note that in [7] there is also a concept of "weak spectral duality", which is shown to imply existence, but not, *a priori*, uniqueness, of spectral resolutions. It is still an open question if weak spectral duality implies spectral duality (or, which can be seen to be equivalent, if the minimality requirement is redundant in Definition 8.42).

The proof given here that Choquet simplexes are spectral (Proposition 8.80) reduces to the fact that normal state spaces of a JBW-algebra or a von Neumann algebra are spectral. A more direct lattice-theoretic proof is given in [7]. The "exotic" examples in Theorem 8.87 are presented here for the first time.

What was known about the relationship of split faces to Choquet simplexes evolved over several years. It was shown by Alfsen [5] that every w*-closed face of a Choquet simplex is a split face (where for a regularly imbedded compact convex set we refer to the given compact topology as the w*-topology). Then Asimow and Ellis [19] showed that every norm closed face of a Choquet simplex is split. The fact that a compact convex set is a Choquet simplex iff every norm closed face is split (Proposition 8.83) is proved in [9], and the result that a compact convex set is a Choquet simplex iff every w*-closed face is split was shown by Ellis [51].

The fact that the traces of a C*-algebra form a Choquet simplex is due to Thoma [127]. The converse result by which every metrizable Choquet simplex is the trace simplex of a C*-algebra, was found by Blackadar [30] and Goodearl [57], who independently proved that for every metrizable Choquet simplex K there is a simple, unital C*-algebra whose trace simplex is affinely homeomorphic to K. This result can also be obtained from the characterization of dimension groups among all ordered abelian groups, which was found slightly later by Effros, Handelman and Shen [47]. Note that the trace simplex has recently found application as an important invariant for C*-algebras, and that a series of interesting new results on the interplay between the theory of infinite dimensional simplexes and C*-algebras have been found in connection with the Elliott program for a K-theoretic classification of various kinds of C*-algebras. (See [107], and in particular the references to Rørdam, Thomsen and Villadsen in that book, and the book [92].)

Many authors have studied spectral theory from an order-theoretic point of view, but we will just mention a few papers that relate fairly closely to the approach in this book. Some of those are the papers by Abbati and Manià [1], and by Bonnet [31], which are based on axiom systems similar to (but not identical with) that in the paper by Riedel [106] mentioned above. Another relevant reference is the paper by C. M. Edwards [41], which extends spectral duality from order unit spaces to the wider class of GM-spaces (not necessarily with unit).

There is also a rich literature on quantum mechanics modeled on ordered vector spaces, and again we will only give a few references that are directly related to the material in this book. The "operational approach" to axiomatic quantum mechanics (consistent with the point of view discussed in the remark after Proposition 7.49) goes back at least as far as Davies and Lewis [39], who proposed analyzing the space of observables and the convex set of states, expressed as an order unit and a base norm space in duality. In this context, it is natural to axiomatize filters, and numerous authors have taken that approach. We will just point out the papers of Mielnik [97], Gunson [59], Araki [17], and Guz [60]. The attempt to axiomatize quantum mechanics by assuming a linear space of observables equipped with a suitable functional calculus was the basis of Segal's paper [114], and is also present in von Neumann's paper [98]. In both cases one is led to define a candidate for a Jordan product given by squares: $a \circ b = \frac{1}{4}((a+b)^2 - (a-b)^2)$. However, distributivity of such a product is not automatic, so additional axioms must be assumed to guarantee it. This is exactly what is needed to make the transition from spectral theory to Jordan algebras, and is the topic of the next chapter.

PART III

State Space Characterizations

9 Characterization of Jordan Algebra State Spaces

In this chapter we will give a characterization of the state spaces of JB-algebras and of normal state spaces of JBW-algebras. As a preliminary step, we will characterize the normal state spaces of JBW-factors of type I. (This includes as a special case a characterization of the normal state space of $\mathcal{B}(H)$.)

Our standing hypothesis for this chapter will be that A, V is a pair of an order unit space and a complete base norm space in separating order and norm duality with $A = V^*$, with the additional assumption that each exposed face of the distinguished base K of V is projective. (This is the same as the standing hypothesis of the last chapter, except for the additional requirements that V be complete and that $A = V^*$.) By Proposition 8.76, the normal state space of a JBW-algebra A is spectral, i.e., A is in spectral duality with its predual $V = A_*$. Thus by Theorem 8.52, such a pair A, V satisfies the standing hypothesis of this chapter. If instead B is a JB-algebra with state space K, then K can be identified with the normal state space of the JBW-algebra B^{**} (Corollary 2.61), so again K is spectral and thus satisfies the standing hypothesis for the duality of $V = B^*$ and $A = B^{**}$.

We will add to the standing hypothesis two equivalent sets of axioms. One set (the "pure state properties") is of particular physical interest, while the other set is geometric in nature. We will use these axioms to characterize the normal state spaces of JBW-factors of type I, and then use this result to achieve the characterization of state spaces of JB-algebras.

The axioms just discussed involve the extreme points of the state space, and so are not relevant in the more general situation of normal state spaces of arbitrary JBW-algebras, where there may be no extreme points. The ellipticity property (cf. Proposition 5.75 and the remarks following) will be used to characterize these normal state spaces, and we will also give a characterization in which ellipticity is replaced by a more physically relevant assumption.

The pure state properties

9.1. Definition. If V is a base norm space with distinguished base K and $T : V \to V$ a positive map, we say T *preserves extreme rays* if T maps each extreme point of K to a multiple of an extreme point.

Recall that under the standing hypothesis of this chapter, the sets $\mathcal{C}$, $\mathcal{P}$, $\mathcal{F}$ of compressions, projective units, and projective faces form isomorphic lattices. Each is a complete orthomodular lattice (Theorem 8.10 and Proposition 8.20).

We now briefly review some lattice terminology. The least element of a complete lattice is called zero. A projective unit that is minimal among the non-zero projective units is called an *atom*. The same term is used for any element of a complete lattice that is minimal among the non-zero elements of the lattice, cf. (A 41). A complete lattice is *atomic* if each non-zero element is the least upper bound of atoms. In this section, we will say an element in a complete lattice is *finite* if it is the least upper bound of a finite set of atoms. Note that this lattice-theoretic concept may differ from the notion of finiteness for projections in von Neumann algebras (A 166), as defined in terms of Murray–von Neumann equivalence, if the projection lattice is not atomic.

Recall that under the standing hypothesis, for each atom u in A there is a unique extreme point $\widehat{u}$ of K such that $\widehat{u}(u) = 1$ (Proposition 8.36).

9.2. Definition. Assume the standing hypothesis of this chapter. Then K has the *pure state properties* if

(i) every extreme point of K is norm exposed,
(ii) for every compression P, the map P^* preserves extreme rays, and
(iii) $\langle u, \widehat{v}\rangle = \langle v, \widehat{u}\rangle$ for all atoms u, v in A.

9.3. Proposition. *The normal state space of a JBW-algebra and the state space of a JB-algebra have the pure state properties.*

Proof. Since the state space of a JB-algebra can be identified with the normal state space of its bidual (Corollary 2.61), it suffices to establish the pure state properties for normal state spaces of JBW-algebras. By Theorem 5.32 every norm closed face is projective, so (i) holds. By Corollary 5.49, each compression on a JBW-algebra preserves extreme rays of the normal state space. Finally, (iii) follows from Corollary 5.57. □

Remark. If we interpret K as the set of states of a physical system, we can view each $\sigma \in V^+$ as representing a beam of particles with intensity $\|\sigma\|$. We further interpret compressions P as representing physical filters. (See the remark after Proposition 7.49.) If each norm exposed face is projective, then (i) says that for each pure state there is a filter that prepares that state. The property (ii) says that filters transform pure states to pure states. Finally, (iii) is a statement of "symmetry of transition probabilities". See the remarks after Corollary 5.57, and [AS, remarks following Lemma 4.10 and Cor. 6.43], for further discussion of this interpretation.

In a general finite dimensional compact convex set, there may be faces that are not exposed. However, we have the following result, due to Minkowksi, cf. [123, Thm. 3.4.11].

9.4. Theorem. *If F is a proper face of a non-empty convex set K in $\mathbf{R}^n$, then there is a proper exposed face G of K containing F.*

Proof. By working in the affine span of K, we may assume without loss of generality that the affine span of K is $\mathbf{R}^n$. The interior of K is non-empty (e.g., it contains the arithmetic mean of any finite number of points in K whose affine span equals the affine span of K), and is convex and dense in K. Since F is a proper face, then F does not meet the interior of K. By Hahn–Banach separation, there is an affine function a with $a > 0$ on the interior of K and $a \leq 0$ on F. Since the interior of K is dense in K, then $a = 0$ on F. Then $G = a^{-1}(0) \cap K$ is the desired proper exposed face. □

In the current context, we can say more. The following result shows that the first pure state property is automatic in finite dimensions. (This result will not be needed in the sequel.)

9.5. Proposition. *Let K be a finite dimensional compact convex set in which every exposed face is projective. Then every face F of K is exposed, and therefore projective.*

Proof. Let G be the intersection of the exposed faces of K that contain F. By finite dimensionality, G is itself an exposed face of K. (An intersection of any family of hyperplanes will equal the intersection of some finite subfamily, and if $a_1, \ldots, a_m$ are positive affine functions on K such that $\bigcap_i a_i^{-1}(0) = F$, then F is exposed by $a_1 + \cdots + a_m$.) Suppose $F \neq G$. By Minkowksi's theorem (Theorem 9.4) applied to $F \subset G$, there is an affine function a on G, not the zero function, such that $a = 0$ on F, and $a \geq 0$ on G. Let P be the compression associated with G, so that $G = \operatorname{im} P^* \cap K$. Note that G is a base for the cone $\operatorname{im}^+ P^*$, and so we can extend a to a linear function on $\operatorname{lin} G = \operatorname{im}^+ P^*$. Let $b = a \circ P^*$. Then $b = 0$ on F, and $b \geq 0$ on K. Thus $H = \{\sigma \in K \mid \langle b, \sigma \rangle = 0\}$ is an exposed face of K containing F and properly contained in G, contrary to the definition of G. We conclude that $F = G$. □

The second of the pure state properties is closely related to the covering property for lattices, cf. (A 43), which we now review for the reader's convenience.

9.6. Definition. Let p and q be elements in a complete lattice L. We say q *covers* p if $p < q$ and there is no element strictly between p

and q. We say L has the *covering property* if for all $p \in L$ and all atoms $u \in L$,

$$p \vee u = p \quad \text{or} \quad p \vee u \text{ covers } p. \tag{9.1}$$

We say L has the *finite covering property* if (9.1) holds for all finite p in L and all atoms $u \in L$.

Recall that by Lemma 8.9, $(p \vee u) \wedge p' = (p \vee u) - p$ for each $p, u \in \mathcal{P}$. Thus $\mathcal{P}$ has the covering property iff for each atom $u \in \mathcal{P}$ and each $p \in \mathcal{P}$,

$$(p \vee u) \wedge p' \text{ is an atom or zero.} \tag{9.2}$$

We will use the following result to relate the covering property to the second of the pure state properties.

9.7. Proposition. *Assume the standing hypothesis of this chapter, and let P be a compression with associated projective unit p. Assume also that each extreme point of K is norm exposed. The following are equivalent:*

(i) *P^* preserves extreme rays.*
(ii) *For each atom $u \in \mathcal{P}$, $(u \vee p') \wedge p$ is either an atom or zero.*
(iii) *P maps atoms to multiples of atoms.*

Proof. Let F be the projective face associated with p. Let u be an atom, and σ the associated extreme point of K. By Theorem 8.32,

$$r(Pu) = (p' \vee u) \wedge p, \tag{9.3}$$

and by Corollary 8.33,

$$\operatorname{ProjFace}(P^*\sigma) = (F' \vee \{\sigma\}) \wedge F, \tag{9.4}$$

where $\operatorname{ProjFace}(P^*\sigma)$ denotes the least projective face containing $\|P^*\sigma\|^{-1}P^*\sigma$, or the empty set if $P^*\sigma = 0$.

(i) $\Rightarrow$ (ii) By (i), $P^*\sigma$ is a multiple of an extreme point, and thus of an exposed point of K. Therefore $\operatorname{ProjFace}(P^*\sigma)$ is an atom in the lattice $\mathcal{F}$, or else the empty set. Now (ii) follows from (9.4) and the isomorphism of the lattices $\mathcal{P}$ and $\mathcal{F}$.

(ii) $\Rightarrow$ (iii) By (ii) and (9.3), $r(Pu)$ is zero or an atom. If $Pu = 0$, then Pu is certainly a multiple of an atom. Otherwise $q = r(Pu)$ is an atom. Let Q be the associated compression. By Proposition 8.36, $\operatorname{im} Q = \mathbf{R}q$. By Lemma 7.29, $Pu \in \operatorname{face}(q) = \operatorname{im}^+ Q \subset \mathbf{R}q$, which proves (iii).

(iii) $\Rightarrow$ (i) By (iii), Pu is a multiple of an atom, so $r(Pu)$ is an atom or zero. Thus by (9.3), $(p' \vee u) \wedge p$ is an atom or zero. By the isomorphism of $\mathcal{P}$ and $\mathcal{F}$, $(F' \vee \{\sigma\}) \wedge F$ is an atom in $\mathcal{F}$ or equals the empty set. By (9.4) the same is true of ProjFace$(P^*\sigma)$. Since minimal projective faces are sets consisting of a single extreme point (Proposition 8.36), then $P^*\sigma$ is a multiple of an extreme point. □

9.8. Corollary. *Assume the standing hypothesis of this chapter, and assume that every extreme point of K is norm exposed. Then the second of the pure state properties is equivalent to the covering property for the lattices $\mathcal{C}$, $\mathcal{P}$, $\mathcal{F}$.*

Proof. Let u be an atom in $\mathcal{P}$ and let $p \in \mathcal{P}$. By (9.2), the covering property for $\mathcal{P}$ is equivalent to (ii) in Proposition 9.7 holding for all $p \in \mathcal{P}$ and all atoms u. The corollary now follows from the equivalence of (i) and (ii) in Proposition 9.7. □

The Hilbert ball property

Recall that a *Hilbert ball* is a convex set affinely isomorphic to the closed unit ball of a finite or infinite dimensional real Hilbert space.

9.9. Definition. A convex set K has the *Hilbert ball property* if the face generated by each pair of extreme points of K is a norm exposed Hilbert ball.

Note that in Definition 9.9 we allow the pair of extreme points to coincide, so the Hilbert ball property implies that each extreme point of K is norm exposed. Note also that by the standing hypothesis of this chapter, the Hilbert ball property implies that the face generated by each pair of extreme points of K is projective.

9.10. Proposition. *State spaces of JB-algebras and C*-algebras, and normal state spaces of JBW and von Neumann algebras, have the Hilbert ball property.*

Proof. By Proposition 5.55 and Corollary 5.56, normal state spaces of JBW-algebras and state spaces of JB-algebras have the Hilbert ball property. This implies as a special case the stated results for C*-algebra state spaces and von Neumann algebra normal state spaces, but the fact that the face generated by two distinct pure states of a C*-algebra is a Hilbert ball (of dimension 1 or 3) is also stated directly in (A 143). Since norm closed faces of the normal state space of a von Neumann algebra are norm exposed (A 108), and the state space of a C*-algebra can be identified with the normal state space of the enveloping von Neumann algebra (A 101), it follows that the face generated by a pair of pure states in a C*-algebra state space is norm exposed. This completes the proof that C*-algebra state spaces have the Hilbert ball property. □

We will now derive some consequences of the Hilbert ball property. As remarked above, the first of the pure state properties follows from the definition of the Hilbert ball property. Recall that the second pure state property is equivalent to the covering property (Corollary 9.8). We now show that the Hilbert ball property implies the finite covering property.

9.11. Proposition. *Assume the standing hypothesis of this chapter. If K has the Hilbert ball property, then the lattices $\mathcal{C}$, $\mathcal{P}$, $\mathcal{F}$ have the finite covering property.*

Proof. Assume K has the Hilbert ball property. We first establish the implication for atoms u, v and projective units q,

$$q < u \vee v \quad \Rightarrow \quad q \text{ is an atom or zero.} \tag{9.5}$$

If $u = v$, then $u \vee v = u$ is an atom, so by definition q must be an atom or zero. If $u \neq v$, the projective face corresponding to $u \vee v$ is generated by the pair of distinct extreme points associated with u and v, so by assumption is a Hilbert ball. The projection q corresponds to a projective face properly contained in this Hilbert ball. The only proper faces of a Hilbert ball are the extreme points, so q must be an atom or zero.

Now let u be any atom in $\mathcal{P}$ and p any finite projective unit. By (9.1) we must show $u \vee p$ covers or equals p, or equivalently,

$$(u \vee p) - p \text{ is an atom or zero.} \tag{9.6}$$

We will prove (9.6) by induction on the lattice dimension of p. (Recall that the lattice dimension of p is the smallest number of atoms whose l.u.b. is p, cf. (A 42).) If $\dim(p) = 1$, then p is an atom, so (9.6) follows from (9.5) (with p in place of v and $(u \vee p) - p$ in place of q).

Let $\dim(p) = n \geq 2$, and assume (9.6) holds for all smaller dimensions of p. Choose atoms $u_1, \ldots, u_n$ such that $p = u_1 \vee \cdots \vee u_n$. By definition of dimension, the element $q = u_1 \vee \cdots \vee u_{n-1}$ satisfies $\dim(q) < n$ and $q < p$ (since $\dim(p) = n$). The map $x \mapsto x - q$ is order preserving with order preserving inverse, and thus induces a lattice isomorphism from the lattice interval $[q, 1]$ onto the interval $[0, q']$. Since

$$p \vee u = (q \vee u) \vee (q \vee u_n),$$

then

$$(p \vee u) - q = \big((q \vee u) - q\big) \vee \big((q \vee u_n) - q\big).$$

By our induction hypothesis, since $\dim(q) < n$, both terms on the right are atoms or zero, so we conclude that

$$\dim((p \vee u) - q) \leq 2.$$

Note that $p > q$ implies

$$(p \vee u) - p < (p \vee u) - q.$$

By (9.5), the only elements under an element of dimension 2 are atoms or zero, so this finishes the proof that $(p \vee u) - p$ is an atom or zero. Thus we have shown that the finite covering property follows from the Hilbert ball property. □

Remark. In the proof above, the only property of the Hilbert ball that is used is that it has no proper faces other than singletons. Thus the finite covering property would be implied by the weaker assumption that the face generated by each pair of extreme points is norm exposed and strictly convex.

If an additional geometric assumption is added to the Hilbert ball property, we will see in Proposition 9.13 below that the second pure state property follows. Recall from Definition 5.71 that a convex set K is *symmetric with respect to a convex subset* K_0 if there exists a *reflection* of K whose set of fixed points is precisely K_0.

9.12. Lemma. *Assume the standing hypothesis of this chapter. If F is a projective face of K and P is the corresponding compression on A, then the following are equivalent:*

(i) *K is symmetric with respect to* $\mathrm{co}(F \cup F')$,
(ii) $2P + 2P' - I \geq 0$.

If these equivalent conditions hold, then $(2P + 2P' - I)^*$ is the unique reflection of K with $\mathrm{co}(F \cup F')$ as its set of fixed points.

Proof. Let $T = 2P+2P'-I$. If (ii) holds, then T is a positive map such that $T^2 = I$, and $T1 = 1$. It follows that T^* is an affine automorphism of K of period 2. For $\sigma \in K$ we have

$$T^*\sigma = \sigma \iff (P + P')^*\sigma = \sigma \iff \sigma \in \mathrm{co}(F \cup F').$$

(The last equivalence follows from Theorem 7.46). Thus T^* is a reflection of K with fixed point set $\mathrm{co}(F \cup F')$, so by definition (i) holds.

Conversely, suppose (i) holds, and let R be a reflection of K with fixed point set $\mathrm{co}(F \cup F')$. Then $\frac{1}{2}(R + I)$ is an affine projection of K onto $\mathrm{co}(F \cup F')$. By Theorem 7.46, there is just one affine projection of K onto $\mathrm{co}(F \cup F')$, namely $(P + P')^*$, so we must have $\frac{1}{2}(R+I) = (P+P')^*$. Thus $R = (2P+2P'-I)^*$, and (ii) follows. This also establishes uniqueness of the reflection that has $\mathrm{co}(F \cup F')$ as its set of fixed points. □

Remark. By Proposition 5.72, the normal state space K of a JBW-algebra (and the state space of a JB-algebra) is symmetric with respect to $\mathrm{co}(F \cup F')$ for every norm closed face F. The same holds for von Neumann algebra normal state spaces, since the self-adjoint part of a von Neumann algebra is also a JBW-algebra. In the von Neumann context, let $p = P1$ be the projection associated with the face F, let s be the symmetry $2p - 1$, and let U_s be the map $a \mapsto sas$. Then $U_s = 2U_p + 2U_{p'} - I$, so from Lemma 9.12 it follows that U_s^* is the reflection whose fixed point set is $\mathrm{co}(F \cup F')$.

The following result gives one circumstance in which the Hilbert ball property implies the second of the pure state properties, but will not be needed in the sequel.

9.13. Proposition. *Assume the standing hypothesis of this chapter. If K has the Hilbert ball property and is symmetric with respect to $\mathrm{co}(F \cup F')$ for every projective face F, then P^* preserves extreme rays for every compression P.*

Proof. Let σ be an extreme point of K, and let P be a compression. We will show that $P^*\sigma$ is a multiple of an extreme point of K. If $P^*\sigma = \sigma$ or $P^*\sigma = 0$ this is trivial, and so we assume neither occurs. By bicomplementarity of P and P', it follows that neither $P'^*\sigma = \sigma$ nor $P'^*\sigma = 0$ occurs. Let $\lambda = \|P^*\sigma\|$. Note that

$$\|P'^*\sigma\| = \langle 1, P'^*\sigma\rangle = \langle 1 - P1, \sigma\rangle = 1 - \|P^*\sigma\| = 1 - \lambda.$$

By neutrality of P^* (Proposition 7.21) and our assumptions, $0 < \lambda < 1$. Define

$$\sigma_1 = \lambda^{-1}P^*\sigma \quad \text{and} \quad \sigma_2 = (1-\lambda)^{-1}P'^*\sigma.$$

Note that σ_1 and σ_2 are positive and of norm 1, hence are in K. Now define $\tau = (2P + 2P' - I)^*\sigma$. By Lemma 9.12, $(2P + 2P' - I)^*$ is an affine automorphism of K of period 2, and so τ is an extreme point of K. Now we have

$$\tfrac{1}{2}(\sigma + \tau) = (P + P')^*\sigma = \lambda\sigma_1 + (1-\lambda)\sigma_2.$$

It follows that σ_1 and σ_2 are in the face of K generated by σ and τ. Furthermore, σ_1 and σ_2 are in the disjoint faces F and F', where F is the projective face associated with P. Therefore the line segment $[\sigma_1, \sigma_2]$ cannot be extended beyond σ_1 or σ_2 and remain in K. (Otherwise, one of σ_1, σ_2 would be in the face generated by the other, contradicting $F \cap F' = \emptyset$.) Hence both are on the boundary of the Hilbert ball $\mathrm{face}(\sigma, \tau)$. For a

Hilbert ball, boundary points are extreme points. Thus σ_1 is an extreme point of this face, and then also of K. Hence $P^*\sigma = \lambda\sigma_1$ is a multiple of an extreme point. $\square$

Now we show that the third pure state property (symmetry of transition probabilities) follows from the Hilbert ball property.

9.14. Proposition. *Assume the standing hypothesis of this chapter. If K has the Hilbert ball property, then for every pair of atoms $u, v \in \mathcal{P}$,*

$$\langle u, \widehat{v}\rangle = \langle v, \widehat{u}\rangle. \tag{9.7}$$

Proof. By hypothesis $F = \operatorname{face}(\widehat{u}, \widehat{v})$ is a norm exposed face affinely isomorphic to a Hilbert ball. Being norm exposed, the face F is also projective, therefore of the form $F = K \cap \operatorname{im} Q^*$ for a compression Q. Since $\widehat{u}, \widehat{v} \in F$, we can pass to the corresponding projective units and obtain $u, v \leq Q1$. Now by Proposition 8.17, u and v will be projective units for the relativized duality of $(\operatorname{im} Q, Q1)$ and $(\operatorname{im} Q^*, F)$. Thus the restriction of u and v to F will be affine functions with values in $[0, 1]$; the former with the value 1 at $\widehat{u}$ and 0 at its antipodal point, and the latter with the value 1 at $\widehat{v}$ and 0 at its antipodal point. Denoting by ω the center of the Hilbert ball F, and by θ the angle between the vectors $\widehat{u} - \omega$ and $\widehat{v} - \omega$, one easily computes that $\langle \widehat{u}, v\rangle = \langle \widehat{v}, u\rangle = \frac{1}{2} + \frac{1}{2}\cos\theta$. $\square$

Atomicity

Most of our effort in this chapter will be spent to characterize the normal state spaces of atomic JBW-algebras (i.e., those for which the lattice of projections is atomic). These are precisely the JBW-algebras that are the direct sum of type I factors (Proposition 3.45), so this also will lead to a characterization of normal state spaces of type I factors. In this section, we give a geometric property that is closely related to atomicity.

Recall that the *σ-convex hull* of a set X in a Banach space consists of all (necessarily norm convergent) countable sums $\sum_i \lambda_i x_i$ where $\lambda_1, \lambda_2, \ldots$ are non-negative real numbers such that $\sum_i \lambda_i = 1$, and $x_1, x_2, \ldots$ are points in X. By Theorem 5.61, the normal state space of an atomic JBW-algebra is the σ-convex hull of its extreme points. By virtue of the next result, this property for JBW-algebras is equivalent to atomicity, and will play a role in our state space characterizations.

9.15. Proposition. *Assume the standing hypothesis of this chapter. If every extreme point of K is norm exposed and the σ-convex hull of the extreme points of K equals K, then the lattices $\mathcal{C}$, $\mathcal{P}$, and $\mathcal{F}$ are atomic.*

Proof. Let G be a non-empty projective face in $\mathcal{F}$. We will show that G contains an extreme point. Let $\omega \in G$. By assumption, we can find a sequence of extreme points $\omega_1, \omega_2, \dots$ and non-negative scalars $\lambda_1, \lambda_2, \dots$ with sum 1 such that $\omega = \sum_i \lambda_i \omega_i$. We may assume without loss of generality that λ_1 is not zero. Then

$$\omega = \lambda_1 \omega_1 + (1-\lambda_1)\big((1-\lambda_1)^{-1} \sum_{i=2}^{\infty} \lambda_i \omega_i\big). \tag{9.8}$$

Since G is a face of K, this implies that $\omega_1 \in G$.

By assumption, the extreme point ω_1 is exposed, so $\{\omega_1\}$ is an atom in $\mathcal{F}$ dominated by G. It follows that in the isomorphic lattice $\mathcal{P}$, every non-zero projective unit dominates an atom. Let p be any projective unit, and let q be the least upper bound of the atoms dominated by p. If $q \neq p$, then $p - q$ must dominate an atom, and this atom is then dominated by both q and $1-q$, contrary to $q \wedge q' = 0$. We conclude that $p = q$, so every projective unit is a least upper bound of atoms. Thus $\mathcal{P}$ and the isomorphic lattices $\mathcal{F}$ and $\mathcal{C}$ are atomic. □

Recall that by Lemma 3.42 the least upper bound of the atoms in a JBW-algebra M is a central projection z such that zM is atomic and $(1-z)M$ contains no atoms. We will now show that given the standing hypothesis of this chapter, the covering property implies a similar splitting into atomic and non-atomic parts.

9.16. Proposition. *Assume the standing hypothesis of this chapter. Assume also that each extreme point of K is norm exposed, and that the lattices $\mathcal{C}$, $\mathcal{P}$, $\mathcal{F}$ have the covering property. Let z be the least upper bound of the atoms in $\mathcal{P}$. Then z is central, every non-zero projection under z dominates an atom, and $z' = (1-z)$ dominates no atom.*

Proof. We first establish the following implication involving the projective unit $r(a)$ associated with an element $a \in A^+$, and arbitrary compressions P and Q:

$$Pa \in \operatorname{im} Q \quad \Rightarrow \quad P(r(a)) \in \operatorname{im} Q. \tag{9.9}$$

Indeed, if $Pa \in \operatorname{im} Q$, then $Q'Pa = 0$, so a is zero on $P^*Q'^*\sigma$ for all $\sigma \in K$. By (8.18), a and $r(a)$ annihilate the same elements of K, so $r(a)$ is zero on $P^*Q'^*\sigma$ for all $\sigma \in K$. Thus $Q'Pr(a) = 0$, which implies $Pr(a) \in \operatorname{im} Q$.

Now let z be the least upper bound of the atoms in A, which exists since the lattice of projective units is complete (Proposition 8.20). We will prove that z is central. Let Q be the compression corresponding to z; note that $\operatorname{im} Q$ contains all atoms. Let P be any compression, and $u_1, u_2, \dots, u_n$ atoms. By the covering property (9.2) and Proposition 9.7

((ii) $\Rightarrow$ (iii)), $Pu_1, \dots Pu_n$ are multiples of atoms. Therefore

$$P(u_1 + \cdots + u_n) \in \operatorname{im} Q.$$

By (9.9)

$$P(r(u_1 + \cdots + u_n)) \in \operatorname{im} Q.$$

Since $u_i = r(u_i) \leq r(u_1 + \cdots + u_n)$ for $i = 1, \dots, n$, then $u_1 \vee \cdots \vee u_n \leq r(u_1 + \cdots + u_n)$. Therefore

$$P(u_1 \vee \cdots \vee u_n) \leq P\big(r(u_1 + \cdots + u_n)\big) \in \operatorname{im} Q.$$

Since $\operatorname{im}^+ Q$ is a face of A^+, then $P(u_1 \vee \cdots \vee u_n) \in \operatorname{im} Q$. By Lemma 7.29, $P(u_1 \vee \cdots \vee u_n) \leq z$. The family of all l.u.b.'s of finite sets of atoms is directed upwards with l.u.b. equal to z. By Lemma 8.19, z is also the weak limit of this directed set. By weak continuity of compressions, we conclude that $Pz \leq z$. By Proposition 7.39, P is compatible with z. Since P was an arbitrary compression, then z is central.

Let p be any non-zero projective unit such that $p \leq z$, and let P be the corresponding compression. We will show p dominates an atom. We are going to show that there exists an atom u such that $Pu \neq 0$. Suppose instead that $Pu = 0$ for each atom u. Then $P'u = u$, so $u \leq p'$. Since this holds for all atoms u, and the least upper bound of the set of atoms is z, it follows that $z \leq p'$ and so $p \leq z'$, which contradicts $p \leq z$.

Thus there exists an atom u such that $Pu \neq 0$. By the covering property and Proposition 9.7, $Pu = \lambda v$ for some atom v and some $\lambda \neq 0$. Then $v \in \operatorname{im} P$, so by Lemma 7.29, $v \leq p$, and we are done.

Finally, if u is any atom and $u \leq z'$, then $u \leq z \wedge z' = 0$, a contradiction, so z' dominates no atoms. $\square$

The type I factor case

In this section we will give two characterizations of the normal state spaces of JBW-factors of type I, and more generally of atomic JBW-algebras. One characterization (Theorem 9.33) will rely on the pure state properties, while the other (Theorem 9.34) will rely on the Hilbert ball property and the property that K is the σ-convex hull of its extreme points.

Our approach is to work with the following set of properties, which will be shown to be implied by either of the aforementioned sets of axioms, and to imply both of the said characterizations. In this section we will assume that the following conditions are satisfied.

(i) A, V satisfy the standing hypothesis of this chapter.
(ii) The lattices $\mathcal{C}$, $\mathcal{P}$, $\mathcal{F}$ are atomic.

(iii) The lattices $\mathcal{C}$, $\mathcal{P}$, $\mathcal{F}$ have the finite covering property.
(iv) Every extreme point of K is norm exposed.
(v) $\langle u, \widehat{v}\rangle = \langle v, \widehat{u}\rangle$ for each pair of atoms $u, v \in A$.

We will refer to (i) – (v) as the *working hypotheses of this section.*

9.17. Proposition. *Assume that the pair A, V satisfy the working hypotheses of this section, and let P be a compression. Then the pair* $\operatorname{im} P$, $\operatorname{im} P^*$ *also satisfy the working hypotheses of this section.*

Proof. The properties (i) and (iv) follow from Proposition 8.17, and the remarks following that proposition. By the same result, the lattice of compressions for the pair $\operatorname{im} P$, $\operatorname{im} P^*$ can be identified with the compressions dominated by P, and (ii) and (iii) follow. In order to establish (v) for this duality, we first show that for atoms u and compressions P we have the equivalence

$$u \in \operatorname{im} P \iff \widehat{u} \in \operatorname{im} P^*. \tag{9.10}$$

To see this, note that $u \in \operatorname{im} P$ is equivalent to $u \le P1$ (Lemma 7.29). By the order preserving correspondence of projective units and projective faces (Proposition 7.32), $u \le P1$ is equivalent to $\{\widehat{u}\} \subset (\operatorname{im} P^*) \cap K$, which proves (9.10).

Now let u and v be atoms under $P1$. Then $\widehat{u}$ is in $\operatorname{im} P^*$, so is the unique element of $K \cap \operatorname{im} P^*$ with the value 1 on the atom u. Thus (v) follows for the duality of $\operatorname{im} P, \operatorname{im} P^*$, which completes the proof that the working hypotheses of this section hold for the pair $\operatorname{im} P$, $\operatorname{im} P^*$. □

We observe that we have proven slightly more than was stated in the proposition. If A, V satisfy the standing hypothesis of this chapter, and if P is any compression, then the pair $\operatorname{im} P$, $\operatorname{im} P^*$ also satisfies the standing hypothesis of this chapter. Furthermore, if any one of the properties (ii) – (v) holds for A, V, then it holds for the pair $\operatorname{im} P$, $\operatorname{im} P^*$.

The finite covering property assures that there is a well-behaved notion of dimension for elements of the lattices $\mathcal{C}$, $\mathcal{P}$, $\mathcal{F}$. In an atomic lattice L, the *dimension* of a finite element p is the least number of atoms whose least upper bound is p (A 42). We will make frequent use of the following result.

9.18. Proposition. *Assume the standing hypothesis of this chapter, and that $\mathcal{P}$ is atomic and has the finite covering property. Let p be a finite element of $\mathcal{P}$. Then p can be expressed as a sum of atoms. In fact, $p = p_1 + \cdots + p_k$ for each maximal set of orthogonal atoms $p_1, \ldots, p_k$ under p and the cardinality of any set of atoms with sum p is* $\dim(p)$*. Furthermore, every element $q \le p$ is finite, and if $q \le p$ with $q \ne p$, then* $\dim(q) < \dim(p)$.

Proof. The lattice $\mathcal{P}$ is a complete atomic orthomodular lattice with the finite covering property, so all but the last strict inequality follows from (A 44). So suppose now that $q \leq p$ with $q \neq p$. Then each of q and $p-q$ can be written as finite sums of $\dim(q)$ and $\dim(p-q)$ atoms respectively, and so p can be expressed as the sum of $\dim(q)+\dim(p-q)$ atoms. It follows that $\dim(p)=\dim(q)+\dim(p-q)>\dim(q)$. □

We say a compression P is *cofinite* if its complement P' is finite (and we define *cofinite* similarly in the lattices $\mathcal{P}$ and $\mathcal{F}$).

9.19. Lemma. *Assume A, V satisfy the working hypotheses of this section. Then each finite or cofinite compression P maps atoms to multiples of atoms and P^* maps extreme rays to extreme rays.*

Proof. Let P be a cofinite compression and $p = P1$. By the working hypotheses, $\mathcal{P}$ has the finite covering property. Thus for each atom $u \in \mathcal{P}$, $p' \vee u$ covers or equals p', and so by Lemma 8.9, $(p' \vee u) \wedge p = (p' \vee u) - p'$ is an atom or zero. By Proposition 9.7, P^* maps extreme points to multiples of extreme points and P maps atoms to multiples of atoms.

Now suppose that P is finite. Let $q = p \vee u$, where u is an atom. By definition q is finite. By Proposition 9.17, the working hypotheses also hold for the duality of $\operatorname{im} Q$ and $\operatorname{im} Q^*$. Since Q is finite, then so are all compressions below Q (Proposition 9.18.) In particular, the complementary compression for P viewed as a compression on $\operatorname{im} Q$ will be finite, so P will be cofinite. It follows from the first part of this proof that Pu will be a multiple of an atom. Thus P maps atoms to multiples of atoms, and so P^* maps extreme rays to extreme rays (Proposition 9.7). □

We remind the reader of our convention that the weak topology on A refers to that given by the duality of A and V. (In the context of JBW-algebras or von Neumann algebras, this would be the σ-weak topology.) Since we are always working in the context where $A = V^*$, this coincides with the w*-topology. In particular, the unit ball of A is weakly compact.

9.20. Lemma. *Assume the working hypotheses of this section. If $\sigma \in V$ and $\langle u, \sigma\rangle \geq 0$ for all atoms u, then $\sigma \geq 0$.*

Proof. If p is any projective unit, then atomicity of the lattice $\mathcal{P}$ implies that each maximal orthogonal collection of atoms under p has l.u.b. equal to p. The finite sums from this collection form an increasing net of projective units with l.u.b. p. Thus p is the weak limit of finite sums of atoms (Lemma 8.19), and so $\langle p, \sigma\rangle \geq 0$. By Proposition 8.31, the projective units are weakly dense in the set of extreme points of the order interval $[0,1]$. Since $[0,1]$ is convex and weakly compact, by the Krein–Milman theorem, $\sigma \geq 0$ on $[0,1]$, and so $\sigma \geq 0$ on A^+. □

9.21. Definition. Assume the working hypotheses of this section. Then A_{f} denotes the linear span of the finite projective units in A.

Recall that if C is a convex set, $\partial_e C$ denotes the set of its extreme points.

9.22. Lemma. *Assume the working hypotheses of this section. Then A_{f} and V are in separating order and norm duality.*

Proof. A_{f} and V are in separating ordered duality (cf. (A 3)) by Lemma 9.20. To complete the proof we must show that the norm of an element of V is equal to its supremum on the unit ball of A_{f}, cf. (A 21). Let $\sigma \in V$ and assume σ restricted to A_{f} has norm 1. Since the projective units are weakly dense in $\partial_e[0,1]$, and $a \mapsto 2a-1$ is an affine homeomorphism of $[0,1]$ onto $[-1,1]$, then $\{2p-1 \mid p \in \mathcal{P}\}$ is weakly dense in $\partial_e[-1,1]$. As in the proof of Lemma 9.20, for each $p \in \mathcal{P}$ we can choose a net p_α of finite projective units converging weakly to p and a net q_α of finite projective units converging to $1-p$. Then $1 \geq \langle p_\alpha - q_\alpha, \sigma\rangle$ for each α, and $\langle p_\alpha - q_\alpha, \sigma\rangle \to \langle 2p-1, \sigma\rangle$. Hence $\langle 2p-1,\sigma\rangle \leq 1$. By weak compactness of $[-1,1]$ and the Krein–Milman theorem, we conclude that $\|\sigma\| \leq 1$. □

9.23. Lemma. *Assume the working hypotheses of this section. Then there exists a unique positive linear map $\phi : A_{\mathrm{f}} \to V$ such that $\phi(u) = \widehat{u}$ for all atoms u. The equation $(a|b) = \langle a, \phi(b)\rangle$ defines a symmetric bilinear form on A_{f} such that*

$$(u|v) = \langle u, \widehat{v}\rangle \tag{9.11}$$

for all pairs of atoms u, v. Furthermore, each finite or cofinite compression P maps A_{f} into itself and satisfies the symmetry condition

$$(Pa \,|\, b) = (a \,|\, Pb) \tag{9.12}$$

for all $a, b \in A_{\mathrm{f}}$.

Proof. First let $u_1, \dots, u_n$ be atoms and $\lambda_1, \dots, \lambda_n$ scalars such that $\sum_{i=1}^n \lambda_i u_i \geq 0$. By (v) of the working hypotheses,

$$\left\langle v, \sum_{i=1}^n \lambda_i \widehat{u}_i \right\rangle = \left\langle \sum_{i=1}^n \lambda_i u_i, \widehat{v} \right\rangle \geq 0$$

for each atom v. Hence $\sum_{i=1}^n \lambda_i \widehat{u}_i \geq 0$ by Lemma 9.20. In particular, if $\sum_{i=1}^n \lambda_i u_i = 0$, then $\sum_{i=1}^n \lambda_i \widehat{u}_i = 0$. Thus there is a well defined positive linear map $\phi : A_{\mathrm{f}} \to V$ given by

$$\phi\left(\sum_{i=1}^{n} \lambda_i u_i\right) = \sum_{i=1}^{n} \lambda_i \widehat{u}_i. \tag{9.13}$$

Bilinearity of the form $(a|b) = \langle a, \phi(b)\rangle$ is trivial. So is (9.11), and symmetry of the bilinear form follows from (9.13) and (v) of the working hypotheses.

Now assume that P is a finite or cofinite compression. By Lemma 9.19, P maps atoms to multiples of atoms, hence P maps A_{f} into A_{f}. Let u and v be atoms. In order to prove that P is symmetric with respect to the bilinear form on A_{f}, we start by showing that

$$((I - P)u \,|\, Pv) = 0. \tag{9.14}$$

Write $Pv = \lambda w$, where w is an atom. We may assume $\lambda \neq 0$, since otherwise (9.14) is clear. Since $w \in \operatorname{im} P$, then by (9.10), $\widehat{w} \in \operatorname{im} P^*$. This together with the definition of the bilinear form on A_{f} gives

$$\big((I-P)u \,|\, Pv\big) = \lambda\big((I-P)u \,|\, w\big) = \lambda\langle (I-P)u, \widehat{w}\rangle = \lambda\langle u, (I-P)^*\widehat{w}\rangle = 0,$$

which proves (9.14). From (9.14) we conclude that $(u \,|\, Pv) = (Pu \,|\, Pv)$ for all atoms u, v. Exchanging the roles of u and v, and using $(Pu \,|\, Pv) = (Pv \,|\, Pu)$, we get

$$(Pu \,|\, v) = (Pu \,|\, Pv) = (u \,|\, Pv)$$

for all atoms u, v. Now (9.12) follows by linearity, since by definition A_{f} is the linear span of atoms. □

9.24. Proposition. *Assume the working hypotheses of this section. Let P be a compression, and let ϕ be the map defined in Lemma 9.23. If P is finite or cofinite , then*

$$\phi(Pa) = P^*(\phi(a)) \quad \textit{for all } a \in A_{\mathrm{f}}. \tag{9.15}$$

If P is finite and Q is any compression such that $Q \preceq P$, then

$$(Q + Q')^*\phi(P1) = \phi(P1). \tag{9.16}$$

Proof. Assume that P is finite or cofinite. By (9.12) and the definition of the bilinear form $(\cdot \,|\, \cdot)$, for all $a, b \in A_{\mathrm{f}}$,

$$\langle b, \phi(Pa)\rangle = (b \,|\, Pa) = (Pb \,|\, a) = \langle Pb, \phi(a)\rangle = \langle b, P^*\phi(a)\rangle.$$

Since A_{f} and V are in separating duality (Lemma 9.22), then $\phi(Pa) = P^*\phi(a)$ follows.

Assume now that P is finite and Q is any compression such that $Q \preceq P$. By Lemma 7.42, P and Q are compatible. Since P is finite and $Q \preceq P$, then Q is finite (Proposition 9.18). Applying (9.15) and compatibility of Q with $P1$, we get

$$(Q+Q')^*\phi(P1) = \phi\big((Q+Q')(P1)\big) = \phi(P1),$$

which proves (9.16). □

Note that (9.16) says that $\phi(P1)$ is a tracial state of $\operatorname{im} P$ for the duality of $\operatorname{im} P$ and $\operatorname{im} P^*$, cf. Definition 8.81.

Recall that $\sigma, \tau \in K$ are defined to be orthogonal if $\|\sigma - \tau\| = 2$ (cf. (8.20)).

9.25. Lemma. *Assume the working hypotheses of this section. Let u, v be atoms. Then u, v are orthogonal iff $\widehat{u}$, $\widehat{v}$ are orthogonal.*

Proof. If $u \perp v$, then $\langle v, \widehat{u}\rangle \le \langle 1 - u, \widehat{u}\rangle = 0$, so $\langle v, \widehat{u}\rangle = 0$, and similarly $\langle u, \widehat{v}\rangle = 0$. Then $\langle u - v, \widehat{u} - \widehat{v}\rangle = 2$. Since $\|u - v\| \le 1$, then $\|\widehat{u} - \widehat{v}\| = 2$, so $\widehat{u}$, $\widehat{v}$ are orthogonal.

Conversely, suppose that $\widehat{u} \perp \widehat{v}$. Define $\omega = \widehat{u} - \widehat{v}$; then this must coincide with the unique orthogonal decomposition of ω (cf. Theorem 8.27). Thus by the same result, there exists a compression P such that $P^*\widehat{u} = \widehat{u}$ and $\mathcal{P}'^*\widehat{v} = \widehat{v}$. Then by (9.10), $Pu = u$ and $P'v = v$, so u and v are orthogonal. □

9.26. Lemma. *Assume the working hypotheses of this section. Let P be a finite compression. Then every $\sigma \in \operatorname{im} P^*$ can be written as a finite linear combination $\sigma = \sum_{i=1}^n \lambda_i \sigma_i$ of orthogonal points $\sigma_1, \ldots, \sigma_n \in (\partial_e K) \cap \operatorname{im} P^*$.*

Proof. By working in the duality of $\operatorname{im} P$ and $\operatorname{im} P^*$ (cf. Proposition 8.17), we may assume that $P1$ is equal to the order unit 1, (i.e., that P is the identity map), so that 1 is finite. We will use induction on the lattice-theoretic dimension of P. If $\dim(P) = 1$, then P is a minimal non-zero compression, so by Proposition 8.36 the corresponding projective face consists of a single extreme point, and thus the proposition holds in this case.

Now assume that the result is true for $\dim(P) < n$. Let $\dim(P) = n$, and let $\sigma \in \operatorname{im} P^*$ be arbitrary. By assumption 1 is finite, and thus is a finite sum of orthogonal atoms (Proposition 9.18). Define $\tau = \phi(1)$. By Lemma 9.25, τ is a finite sum of orthogonal extreme points of K. By Proposition 9.24, for all compressions Q we also have

$$(Q+Q')^*\tau = \tau. \tag{9.17}$$

If σ is a multiple of τ, then we are done. If not, then there is a scalar $\lambda \in \mathbf{R}$ such that neither $\sigma \leq \lambda\tau$ nor $\sigma \geq \lambda\tau$. Thus for that scalar λ, if we define $\omega = \sigma - \lambda\tau$, then ω is neither positive nor negative. By Theorem 8.27, σ can be written as a difference of positive elements $\omega = \omega^+ - \omega^-$ in such a way that there is a compression Q such that $\omega^+ = Q^*\omega$, $\omega^- = -Q'^*\omega$. Then $(Q+Q')^*\omega = \omega$. Thus by equation (9.17)

$$(Q+Q')^*\sigma = (Q+Q')^*(\omega + \lambda\tau) = (\omega + \lambda\tau) = \sigma.$$

Now let $\sigma_1 = Q^*\sigma$ and $\sigma_2 = Q'^*\sigma$; note that $\sigma = \sigma_1 + \sigma_2$. Since $\omega \not\geq 0$ and $Q^*\omega = \omega^+ \geq 0$, then we cannot have $Q = P$ (since P is the identity map), and similarly $Q' \neq P$. By Proposition 9.18, the lattice dimensions of Q and of Q' are strictly smaller than that of P. By our induction hypothesis, σ_1 is a finite linear combination of orthogonal extreme points of K in $\operatorname{im} Q^*$, which we can write in the form $\widehat{u}_1, \ldots, \widehat{u}_k$ for atoms $u_1, \ldots, u_k$ in $\operatorname{im} Q$ (cf. (9.10)). Similarly we can write σ_2 as a finite linear combination of orthogonal elements $\widehat{v}_1, \ldots, \widehat{v}_m$ in $\operatorname{im} Q'^* \cap \partial_e K$, where $v_1, \ldots, v_m$ are atoms in $\operatorname{im} Q'$. Then each u_i is dominated by $Q1$ and each v_j is dominated by $Q'1$, so $u_i \perp v_j$ for $1 \leq i \leq k$ and $1 \leq j \leq m$. By Lemma 9.25, the corresponding extreme points of K are orthogonal. Substituting these decompositions into $\sigma = \sigma_1 + \sigma_2$ gives the desired decomposition of σ. □

9.27. Corollary. *Assume the working hypotheses of this section. Then every $a \in A_f$ can be written as a finite linear combination $a = \sum_{i=1}^n \lambda_i u_i$ of orthogonal atoms $u_1, \ldots, u_n$.*

Proof. We first show that the map $\phi : A_f \to V$ is one-to-one. Let $a = \sum_{i=1}^n \lambda_i u_i$ be any finite linear combination of atoms $u_1, \ldots, u_n$. If $\phi(a) = 0$, then for all atoms v in A,

$$\langle v, \phi(a)\rangle = \left\langle v, \sum_{i=1}^n \lambda_i \widehat{u}_i \right\rangle = \left\langle \sum_{i=1}^n \lambda_i u_i, \widehat{v} \right\rangle = \langle a, \widehat{v}\rangle.$$

Let P be the finite compression corresponding to $u_1 \vee \cdots \vee u_n$. Then a annihilates every $\widehat{v} \in \partial_e K$ in $\operatorname{im} P^*$, so by Lemma 9.26, a annihilates all of $\operatorname{im} P^*$. Since $a \in \operatorname{im} P$, the separating duality of $\operatorname{im} P$ and $\operatorname{im} P^*$ implies that $a = 0$.

Finally, applying Lemma 9.26 to $\phi(a)$ shows that we can write $\phi(a) = \sum_{i=1}^m \beta_i \sigma_i$ for orthogonal points $\sigma_1, \ldots, \sigma_m \in \partial_e K$. If we choose atoms $v_1, \ldots, v_m$ such that $\widehat{v}_i = \sigma_i$ for each i, then $\phi(a) = \phi(\sum_{i=1}^m \beta_i v_i)$ and so $a = \sum_{i=1}^m \beta_i v_i$ is our desired representation. □

9.28. Definition. Assume the working hypotheses of this section. For each atom $u = Pe$ (with P a compression), and each $b \in A$ we define

$$u * b = \tfrac{1}{2}(I + P - P')b. \tag{9.18}$$

Note that by (1.47), $u * b = u \circ b$ when A is a JBW-algebra.

9.29. Lemma. *Assume the working hypotheses of this section. Let u, v be atoms in A. Then:*

(i) $u * v = v * u$.
(ii) *If $u \perp v$ and $b \in A$, then $u * (v * b) = v * (u * b)$.*
(iii) *If $u \perp v$, then $u * v = 0$.*
(iv) $u * u = u$.

Proof. (i) Let P, Q be the compressions such that $P1 = u$ and $Q1 = u \vee v$. On $\operatorname{im} Q$ the complement of P is the restriction of P' (Proposition 8.17), and thus $u * v$ is the same whether calculated in A or in $\operatorname{im} Q$. Hence without loss of generality we may assume that $u \vee v = 1$. If $u = v$ then (i) is obvious, so we may also assume that $u \neq v$. Then $\dim(1) = \dim(u \vee v) = 2$. Now $\dim(1 - u) = 1$, so $1 - u$ is also an atom.

Since u is an atom, then the image of P is one dimensional (Proposition 8.36), and thus $\operatorname{im} P$ consists of all multiples of u and $\operatorname{im} P^*$ consists of all multiples of $\widehat{u}$ (cf. (9.10)). Therefore for each atom w,

$$Pw = \lambda u$$

for some scalar λ. Applying $\widehat{u}$ to both sides and using $P^* \widehat{u} = \widehat{u}$ gives $\langle w, \widehat{u} \rangle = \lambda$, and so

$$Pw = \langle w, \widehat{u} \rangle u = (u|w)u. \tag{9.19}$$

Since $u' = 1 - u$ is also an atom, we have

$$P'w = \langle w, \widehat{u'} \rangle u' = (u'|w)u'.$$

By (9.18) we can compute $u * v$ as follows.

$$\begin{aligned} u * v &= \tfrac{1}{2}(v + (u|v)u - (u'|v)u') \\ &= \tfrac{1}{2}\big(v + (u|v)u - (u'|v)(1 - u)\big) \\ &= \tfrac{1}{2}\big(v + (u|v)u + (u'|v)u - (u'|v)1\big) \\ &= \tfrac{1}{2}\big(v + (1|v)u - (u'|v)1\big). \end{aligned}$$

Since $(1|v) = \langle 1, \widehat{v} \rangle = 1$, then

$$u * v = \tfrac{1}{2}(v + u - (1 - (u|v))1).$$

Since this last expression is symmetric in u and v, (i) follows.

(ii), (iii) Now assume $u \perp v$, and let P, Q be the compressions corresponding to u, v. Then P, Q, P', Q' commute (Lemma 7.42), so (ii) follows. Also, $v \leq u'$ implies that $Pv = 0$, and then $P'v = v$, so (iii) follows from the definition (9.18).

(iv) follows at once from (9.18). □

In the proof of several results below we shall also need the following elementary result on order unit spaces: If A_0 is a linear subspace of an order unit space A and $\circ$ is commutative bilinear product on A_0 with values in A such that

$$-1 \leq a \leq 1 \quad \Rightarrow \quad 0 \leq a \circ a \leq 1 \tag{9.20}$$

for $a \in A_0$, then

$$\|a \circ b\| \leq \|a\| \, \|b\| \tag{9.21}$$

for all pairs $a, b \in A_0$ (A 50). This can be readily seen from (9.20) and the identity

$$a \circ b = \tfrac{1}{4}(a+b)^2 - \tfrac{1}{4}(a-b)^2.$$

9.30. Proposition. *Assume the working hypotheses of this section. Let A_0 be the norm closure of $A_{\mathrm{f}} + \mathbf{R}1$. Then there is a unique product $a \circ b$ on A_0 such that A_0 is a JB-algebra with identity 1 and such that $u \circ v = u * v$.*

Proof. We first define the product on A_{f} by

$$a \circ b = \sum_{i,j} \alpha_i \beta_j u_i * v_j \tag{9.22}$$

where $a = \sum_i \alpha_i u_i$ and $b = \sum_j \beta_j v_j$ with $u_1, \ldots, u_n$ and $v_1, \ldots, v_m$ atoms. From Definition 9.28 it follows that the right side of (9.22) is independent of the representation of b as a linear combination of atoms. From Lemma 9.29 (i) it follows that it is also independent of the representation of a, and so the product is well defined on A_{f}. There is clearly a unique extension of this product to a commutative bilinear product on $A_{\mathrm{f}} + \mathbf{R}1$ such that 1 acts as the identity. (If it happens that 1 is in A_{f}, then from the definition (9.18) it is easy to check that 1 already acts as the identity on A_{f}.)

Next we show that this product on $A_{\mathrm{f}} + \mathbf{R}1$ satisfies

$$-1 \leq a \leq 1 \quad \Rightarrow \quad 0 \leq a^2 \leq 1. \tag{9.23}$$

By Corollary 9.27, each element b in A_{f} is a finite linear combination of orthogonal projective units, say $v_1, \dots, v_n$. Then the same is true of each element $b + \lambda 1$ with $\lambda \in \mathbf{R}$, since $b + \lambda 1$ is a linear combination of the orthogonal projective units $v_1, \dots, v_n, q$ where $q = 1 - u_1 - \cdots - u_n$. It is straightforward to check that $q^2 = q$, and that $q * u_j = 0$ for $j = 1, \dots, n$. Thus for $a \in A_{\mathrm{f}} + \mathbf{R}1$ we can find real scalars $\lambda_1, \dots, \lambda_n$ and λ such that

$$a = \sum_i \lambda_i u_i + \lambda q \quad \text{and} \quad a^2 = \sum_i \lambda_i^2 u_i + \lambda^2 q.$$

Now by applying the compressions associated with $u_1, \dots, u_n, q$ to an element $a \in A_{\mathrm{f}} + \mathbf{R}1$ such that $-1 \leq a \leq 1$, we conclude that $\lambda_1, \dots, \lambda_n$ and λ are between -1 and 1, and thus that (9.23) holds. A similar calculation shows that all groupings $a * (a * \cdots * (a * a) * \cdots)$ of n factors of a give the same product, namely

$$a^n = \sum_i \lambda_i^n u_i + \lambda^n q,$$

and thus the product is power associative.

Now by the remarks preceding this proposition, (9.23) implies that (9.21) holds in $A_{\mathrm{f}} + \mathbf{R}1$. Hence there is a unique extension of the product $\circ$ to the norm closure A_0 of $A_{\mathrm{f}} + \mathbf{R}1$. By continuity, the resulting product on A_0 is commutative, power associative, squares are positive, the order unit 1 acts as an identity, and (9.21) holds. Then by definition A_0 is a commutative order unit algebra (cf. Definition 1.9), and thus A_0 is a JB-algebra by Theorem 2.49. □

9.31. Lemma. *Assume the working hypotheses of this section. For each $a \in A_0$ and each $\sigma \in V$ there is a unique element $a \circ \sigma \in V$ such that*

$$\langle b, a \circ \sigma \rangle = \langle a \circ b, \sigma \rangle \quad \textit{for all } b \in A_0 . \tag{9.24}$$

Proof. Suppose first that a is an atom, and let P be the compression such that $a = P1$. Then by definition of the Jordan product on A_0, for $b \in A_0$,

$$a \circ b = \tfrac{1}{2}(I + P - P')b.$$

Then

$$\langle a \circ b, \sigma \rangle = \langle \tfrac{1}{2}(I + P - P')b, \sigma \rangle = \langle b, \tfrac{1}{2}(I + P - P')^* \sigma \rangle,$$

and so (9.24) holds with $a \circ \sigma = \frac{1}{2}(I + P - P')^* \sigma$. We can take $1 \circ \sigma$ to be σ, and so by linearity an element $a \circ \sigma$ satisfying (9.24) exists for all a in $A_{\mathrm{f}} + \mathbf{R}1$. For an arbitrary element a of A_0, choose a sequence $\{a_n\}$ in $A_{\mathrm{f}} + \mathbf{R}1$ converging in norm to a. By the norm property (9.21), $\{a_n \circ \sigma\}$ is a Cauchy sequence in V. By completeness of V, this sequence converges in norm to an element of V; we define that element to be $a \circ \sigma$. It is then evident that (9.24) holds. Since A_{f} and V are in separating duality, the element $a \circ \sigma$ satisfying (9.24) is unique. □

9.32. Lemma. *Assume the working hypotheses of this section. Then A can be equipped with a product that makes it an atomic JBW-algebra with predual V (with the given order and norm on A and V).*

Proof. By Lemma 9.22, A_{f} and V are in separating norm duality, as are A and V by hypothesis. Thus A_0 and V are also in separating norm duality, so we can imbed V isometrically into A_0^*. We therefore identify V with a subspace of A_0^*. By Corollary 2.50, A_0^{**} is a JBW-algebra, with a separately w*-continuous product that extends the product on A_0. By w*-density of A_0 in A_0^{**}, together with (9.24), we conclude that for each $\sigma \in V$,

$$\langle b, a \circ \sigma \rangle = \langle a \circ b, \sigma \rangle \quad \text{for all } a \in A_0 \text{ and all } b \in A_0^{**}. \tag{9.25}$$

Now let J be the annihilator of V in A_0^{**}. By (9.25), $a \circ J \subset J$ for all $a \in A_0$. Since J is w*-closed ($=$ σ-weakly closed), it follows that $a \circ J \subset J$ for all $a \in A_0^{**}$, i.e., J is a σ-weakly closed ideal of A_0^{**}. By Proposition 2.39 there is a central projection $c \in A_0^{**}$ such that $J = \operatorname{im} U_c = cA_0^{**}$. Since V is complete, it is a norm closed subspace of A_0^*, so by the bipolar theorem, V is the annihilator of $J = \operatorname{im} U_c$ in A_0^*. The annihilator of $\operatorname{im} U_c$ is $\ker U_c^*$, which coincides with $\operatorname{im} U_{1-c}^*$. By Propositions 2.9 and 2.62, $U_{1-c}^*(A_0^*)$ is the predual of the JBW-algebra $\operatorname{im} U_{1-c}$, and thus we have the isometric isomorphism

$$A \cong V^* \cong (U_{1-c}^*(A_0^*))^* \cong \operatorname{im} U_{1-c}.$$

Hence A can be equipped with a product making it a JBW-algebra with predual V. The order on an order unit space is determined by the norm and the order unit (since for an element a with norm 1, $a \geq 0$ iff $\|1 - a\| \leq 1$), so the order on A as a JBW-algebra coincides with that inherited as the dual of the base norm space V, and thus matches the given order on A. By uniqueness of the predual of a JBW-algebra (Theorem 2.55), K will be affinely isomorphic to the normal state space of A. By assumption the lattice F of norm exposed faces of K is atomic, and therefore so is the projection lattice of A, i.e., A is an atomic JBW-algebra. □

Remark. For the interested reader, we sketch the structure of the proof above in a familiar context, without providing details. In the case where $A = \mathcal{B}(H)_{\mathrm{sa}}$, then A_{f} consists of the finite rank operators, and A_0 will be $K + \mathbf{R}1$ where K is the space of (self-adjoint) compact operators Then $A_0^* = V \oplus \mathbf{R}\omega_0$, where ω_0 is the unique state on $K + \mathbf{R}1$ that annihilates K, and $A_0^{**} \cong \mathcal{B}(H)_{\mathrm{sa}} + \mathbf{R}c_0$, where c_0 is a central projection and an atom. Thus essentially we have defined the Jordan product on the finite rank operators and then recovered the Jordan product on $\mathcal{B}(H)_{\mathrm{sa}}$ as the bidual.

9.33. Theorem. *Let K be the base of a complete base norm space. Then K is affinely isomorphic to the normal state space of an atomic JBW-algebra iff all of the following hold.*

(i) *Every norm exposed face of K is projective.*
(ii) *Every non-empty split face of K contains an extreme point.*
(iii) *K has the pure state properties (cf. Definition 9.2).*

Furthermore, K will be affinely isomorphic to the normal state space of a JBW-factor of type I iff K satisfies the three conditions above and in addition K has no proper split face (i.e., one other than K or $\emptyset$).

Proof. Assume first that K is affinely isomorphic to the normal state space of an atomic JBW-factor M. By Theorem 5.32 every norm closed face of K is projective, so (i) holds.

By definition of an atomic JBW-algebra, the lattice of projections of M is atomic. By Proposition 5.39, atoms in the projection lattice correspond to projective faces consisting of a single extreme point, so every projective face of K contains an extreme point. By Proposition 7.49 every split face of K is a projective face. (Alternatively, every split face is norm closed (A 28), and therefore projective (Theorem 5.32)). Thus (ii) holds. By Proposition 9.3, the normal state space of M has the pure state properties, so (iii) holds.

Now suppose that K is the distinguished base of a complete base norm space V, and satisfies (i), (ii), and (iii). Let $A = V^*$. We are going to show that the working hypotheses of this section hold. The pair A, V are in separating order and norm duality (A 27). Then (i) implies that the standing hypothesis for this chapter holds. The properties (iv) and (v) of the working hypotheses of this section are part of the pure state properties. The covering property follows from the second of the pure state properties (Corollary 9.8).

It remains to show that the lattices $\mathcal{C}$, $\mathcal{P}$, $\mathcal{F}$ are atomic. Let z be the least upper bound of the atoms in $\mathcal{P}$. By the covering property and Proposition 9.16, z is central, every projection under z dominates an atom, and $1 - z$ dominates no atom. We will show $z = 1$, which will then imply that the lattices $\mathcal{P}$, $\mathcal{C}$, $\mathcal{F}$ are atomic.

The projective face F corresponding to z is a split face by Proposition 7.49. Let F' be the complementary split face. If F' is not empty, then by (ii) it contains an extreme point σ, which is norm exposed by the first of the pure state properties. Then $z' = 1 - z$ must dominate the atom corresponding to σ (cf. Proposition 8.36), which contradicts the fact that z' dominates no atom. Thus F' must be the empty set, and F must be all of K, so $z = 1$, which completes the proof of atomicity.

Therefore the working hypotheses of this section are satisfied. By Lemma 9.32, $A = V^*$ can be equipped with a product making it a JBW-algebra. By the uniqueness of the predual (Theorem 2.55), K is affinely isomorphic to the normal state space of A.

Finally, suppose that K satisfies (i), (ii), (iii), and has no proper split faces. Then as established in the first part of this proof, K is the normal state space of a JBW-algebra M. Since K has no proper split faces, then M is a JBW-factor (Corollary 5.35). Since K contains extreme points, then M is a JBW-factor of type I (Corollary 5.41). Conversely, if K is the normal state space of a JBW-factor M of type I, then M is atomic, so (i), (ii), (iii) hold, and since M is a factor, then K has no proper split face (Corollary 5.35). This completes the proof of the last statement of the theorem. □

9.34. Theorem. *Let K be the base of a complete base norm space. Then K is affinely isomorphic to the normal state space of an atomic JBW-algebra iff all of the following hold.*

(i) *Every norm exposed face of K is projective.*
(ii) *The σ-convex hull of the extreme points of K equals K.*
(iii) *K has the Hilbert ball property.*

Furthermore, K will be affinely isomorphic to the normal state space of a JBW-factor of type I iff K satisfies the three conditions above and in addition K has no proper split face.

Proof. Assume first that K is affinely isomorphic to the normal state space of an atomic JBW-factor M. By Theorem 5.32 every norm closed face of K is projective, so (i) holds. By Theorem 5.61, (ii) holds. By Proposition 9.10, (iii) holds.

Now suppose that K is the distinguished base of a complete base norm space V, and satisfies (i), (ii), and (iii). Let $A = V^*$. We are going to show that the working hypotheses of this section hold. The pair A, V are in separating order and norm duality (A 27). Every extreme point of K is norm exposed by the Hilbert ball property. Property (ii) implies that the lattices $\mathcal{C}$, $\mathcal{P}$, $\mathcal{F}$ are atomic (Proposition 9.15). The Hilbert ball property implies the finite covering property (Proposition 9.11). The symmetry of transition probabilities, i.e., (v) of the working hypotheses of this section, also follows from the Hilbert ball property (Proposition 9.14).

Thus the working hypotheses of this section are satisfied. The remainder of the proof of the current theorem is the same as the last two paragraphs of the proof of Theorem 9.33. □

Remark. Instead of assuming in Theorems 9.33 and 9.34 that every norm exposed face of K is projective, we could instead assume the stronger property that K is a spectral convex set, cf. Definition 8.74 and Theorem 8.52.

Recall that if σ, τ are extreme points of the normal state space of a JBW-algebra, the face they generate is a norm exposed Hilbert ball (Proposition 5.55). We will see that the affine dimension of this ball distinguishes the normal state spaces of type I JBW-factors. Recall from Theorem 3.39 that type I JBW-factors are either $\mathbf{R}$, spin factors, $H_3(\mathbf{O})$, or the bounded self-adjoint operators on a real, complex, or quaternionic Hilbert space.

9.35. Definition. A type I JBW-factor is said to be *real* (resp. *complex* resp. *quaternionic*) if it is isomorphic to $\mathcal{B}(H)_{\mathrm{sa}}$ for some real (resp. complex resp. quaternionic) Hilbert space H.

9.36. Proposition. *Let σ, τ be distinct extreme points of the normal state space K of a JBW-factor M of type I.*

(i) *If M is real, then $\dim \operatorname{face}(\sigma,\tau) = 2$.*
(ii) *If M is complex, then $\dim \operatorname{face}(\sigma,\tau) = 3$.*
(iii) *If M is quaternionic, then $\dim \operatorname{face}(\sigma,\tau) = 5$.*
(iv) *If M is a spin factor, then $\dim \operatorname{face}(\sigma,\tau) = \dim M - 1$.*
(v) *If $M = H_3(\mathbf{O})$, then $\dim \operatorname{face}(\sigma,\tau) = 9$.*

Proof. Let p and r be the carrier projections of σ and τ. Then by the isomorphism of the lattice of norm closed faces and the lattice of projections (Corollary 5.33), p and r are atoms (i.e., minimal projections), and the carrier of $\operatorname{face}(\sigma,\tau)$ is $p \vee r$. Let $q = p \vee r - p$. Then p and q are orthogonal atoms (Lemma 3.50), with $p \vee q = p \vee r$. Thus $\operatorname{face}(\sigma,\tau)$ can be identified with the normal state space of M_{p+q}. The affine dimension of $\operatorname{face}(\sigma,\tau)$ is then one less than the linear dimension of M_{p+q}.

By the classification of type I JBW-factors, M is isomorphic to a spin factor, to $H_3(\mathbf{O})$, or to $\mathcal{B}(H)_{\mathrm{sa}}$ for a real, complex, or quaternionic Hilbert space. If M is a spin factor, then M is of type I_2, so any pair of orthogonal projections must add to 1 (Lemma 3.22). Thus in this case $M_{p+q} = M$, and hence (iv) holds. So it remains to consider the case where M is $H_3(\mathbf{O})$, or $\mathcal{B}(H)_{\mathrm{sa}}$ for a real, complex, or quaternionic Hilbert space. In each case, there are orthogonal minimal projections $\widetilde{p}$ and $\widetilde{q}$ such that $M_{\widetilde{p}+\widetilde{q}}$ is isomorphic to $M_2(\mathbf{R})_{\mathrm{sa}}$, $M_2(\mathbf{C})_{\mathrm{sa}}$, $M_2(\mathbf{H})_{\mathrm{sa}}$, or $M_2(\mathbf{O})_{\mathrm{sa}}$. The results stated in the proposition will follow if we show that $p+q$ is equivalent to $\widetilde{p}+\widetilde{q}$, so that M_{p+q} is isomorphic to $M_{\widetilde{p}+\widetilde{q}}$.

Since p and $\widetilde{p}$ are minimal projections in a JBW-factor, they are exchangeable by a symmetry s (Lemma 3.19). Then $U_s q$ is orthogonal to $U_s p = \widetilde{p}$, as is $\widetilde{q}$, so both are minimal projections in $M_{1-\widetilde{p}}$, which is a JBW-factor (Proposition 3.13). Thus there is a $(1-\widetilde{p})$-symmetry t_0 that exchanges $U_s q$ and $\widetilde{q}$. Then $t = \widetilde{p} + t_0$ is a symmetry that also exchanges $U_s q$ and $\widetilde{q}$ (since Jordan multiplication by $\widetilde{p}$ on $M_{1-\widetilde{p}}$ is the zero operator), while U_t fixes $\widetilde{p}$. Thus $U_t U_s$ takes p to $\widetilde{p}$ and q to $\widetilde{q}$, so takes $p+q$ to $\widetilde{p}+\widetilde{q}$. This completes the proof. □

Characterization of state spaces of JB-algebras

We will continue to assume the standing hypothesis of this chapter, namely, that A is an order unit space and V a complete base norm space with distinguished base K, such that every norm exposed face of K is projective. Recall from Proposition 8.34 and Definition 8.35 that for each $\sigma \in K$ there is a smallest central projective unit $c(\sigma)$ (called the *central carrier* of σ) such that $\langle c(\sigma), \sigma\rangle = 1$. The corresponding face $F_{c(\sigma)}$ is the smallest split face containing σ. We will write F_σ in place of $F_{c(\sigma)}$.

9.37. Lemma. *Assume the standing hypothesis of this chapter. If σ is an extreme point of K, then F_σ contains no proper split face.*

Proof. Suppose that G is a non-empty split face of K properly contained in F_σ. Since K is the direct convex sum of G and the complementary split face G', and σ is an extreme point of K, then σ must be in either G or G'. By minimality of F_σ, σ cannot be contained in G, so $\sigma \in G'$. Then by (A 7), $G' \cap F_\sigma$ is a split face of K containing σ and properly contained in F_σ (since it contains no point of G). This violates the fact that F_σ is the smallest split face containing σ. Thus we conclude that no such split face G of K exists.

Finally, a split face of F_σ would also be a split face of K, as can be verified directly from the definition of a split face (A 5), or from Corollary 8.18. Thus F_σ contains no proper split face. □

Recall that positive elements a_1, a_2 of A are *orthogonal* if there exists a projective face F of K such that $a_1 = 0$ on F and $a_2 = 0$ on F' (Definition 8.38). Recall that under the standing hypotheses of this chapter, every element $a \in A$ admits a unique decomposition as a difference of orthogonal positive elements iff A, V are in spectral duality (Theorem 8.55).

In the rest of this section K will be a compact convex set. Without loss of generality, we may assume that K is regularly imbedded as the base of the base norm space $V = A(K)^*$, where $A(K)$ denotes the continuous affine functions on K. (See the description of this imbedding following

Proposition 8.76.) We let $A = A_b(K) = V^*$ be the space of bounded affine functions on K (A 11). Then A and V are in separating order and norm duality (A 27). Note that $A(K) \subset A_b(K)$, and so notions such as orthogonality of positive elements of $A(K)$ are to be interpreted with respect to the duality of $A = A_b(K)$ and V.

One of the key properties that we will use to characterize JB-state spaces is that each $a \in A(K)$ admits a decomposition as a difference of positive orthogonal elements of $A(K)$. (For a discussion of a physical interpretation of this property, see the remarks after Proposition 8.67.) Note that we will not require uniqueness of this decomposition, but we will require that the orthogonal elements be in $A(K)$ and not just in $A = A_b(K)$. Combining this property with the properties previously discussed, we can now characterize the state spaces of JB-algebras.

9.38. Theorem. *A compact convex set K is affinely homeomorphic to the state space of a JB-algebra (with the w^*-topology) iff K satisfies the conditions:*

(i) *Every $a \in A(K)$ admits a decomposition $a = b - c$ with $b, c \in A(K)^+$ and $b \perp c$.*
(ii) *Every norm exposed face of K is projective.*
(iii) *K has the pure state properties.*

Here the condition (iii) can be replaced by the alternative condition

(iii′) *The σ-convex hull of the extreme points of K is a split face of K and K has the Hilbert ball property.*

Proof. Let B be a JB-algebra with state space K, and let $V = B^*$. We will show K satisfies the properties above. Recall that we can identify K with the normal state space of the JBW-algebra B^{**} (Corollary 2.61), and so we work in the duality of V and $A = V^* = B^{**}$.

(i) By Proposition 1.28, each element of A admits a unique decomposition as the difference of orthogonal positive elements. Those elements will also be orthogonal as elements of the JBW-algebra A^{**}. By Proposition 2.16, positive elements a, b of the JBW-algebra B^{**} are orthogonal iff their range projections are orthogonal. Viewing a, b as elements of $A = A_b(K)$, this is the same as the notion of orthogonality with respect to the duality of A and V, cf. Definition 8.38 and Proposition 8.39. Thus (i) holds.

(ii) Norm closed faces of the normal state space of a JBW-algebra are projective (Theorem 5.32), so (ii) follows. This also follows more simply from the fact that positive elements of the JBW-algebra B^{**} and their range projections have the same annihilators in the normal state space, cf. (2.6).

(iii) follows from Proposition 9.3.

(iii′) follows from Corollary 5.63 and Corollary 5.56.

Conversely, assume that (i), (ii), and either (iii) or (iii ′) hold. For each extreme point σ of K, let $P_{c(\sigma)}$ denote the compression associated with $c(\sigma)$, and let $A_\sigma = \operatorname{im} P_{c(\sigma)}$ and $V_\sigma = \operatorname{im} P^*_{c(\sigma)}$. By Proposition 8.17 and Proposition 8.34, we can identify A_σ with the space of all bounded affine functions on F_σ (the minimal split face generated by σ).

Fix an extreme point σ of K. By Lemma 9.37, F_σ contains no proper split faces. We are going to show that F_σ satisfies the conditions (i), (ii), and (iii) of Theorem 9.33 or of Theorem 9.34, and thus that A_σ can be equipped with a product making it a type I JBW-factor.

By (ii) of the current theorem, A, V satisfy the standing hypothesis of this chapter. Thus by Proposition 8.17, we can identify A_σ with the dual of the complete base norm space V_σ (with distinguished base F_σ), and every norm exposed face of F_σ is projective for the duality of A_σ and V_σ. The projective faces for this restricted duality will be precisely the projective faces of K that are contained in F_σ.

Assume that the pure state properties hold for K. Then by the remarks following Proposition 9.17, the first and third of the pure state properties will also hold for F_σ. By the second pure state property and Corollary 9.8, the covering property holds for the lattice of projective faces of K, and therefore also for the lattice of projective faces of F_σ. Therefore again by Corollary 9.8, the second of the pure state properties holds for F_σ. Since F_σ contains the extreme point σ, and has no proper split faces, then we have shown that F_σ satisfies the hypotheses of Theorem 9.33, and so we conclude that A_σ admits a product making it a JBW-factor of type I, with the given order and norm.

Now assume alternatively that (iii ′) holds. If ω, τ are extreme points of F_σ, then the face G generated by ω, τ in K is the same as in F_σ, so the fact that G is norm exposed and a Hilbert ball in K implies the same in F_σ. Thus the Hilbert ball property holds in F_σ. Now let G be the split face of K that is the σ-convex hull of the extreme points of K. Then $G \cap F_\sigma$ is a split face of K that contains σ, so by minimality of F_σ, the split face G must contain F_σ. Thus if ω is any element of F_σ, then ω is a σ-convex combination of extreme points of K. Each of those extreme points of K must in fact be in the face F_σ (cf. (9.8)), so must be extreme points of F_σ. Thus F_σ is the σ-convex hull of its extreme points. Hence (iii ′) holds for F_σ. Now by Theorem 9.34, A_σ can be equipped with a product making it a JBW-factor of type I, with F_σ as its normal state space.

Thus we've shown that (i), (ii), and (iii) or (iii ′) imply that each F_σ is the normal state space of a type I JBW-factor. For each extreme point σ of K, define π_σ to be the restriction map which sends $a \in A(K)$ to $a|F_\sigma \in A_b(F_\sigma)$. We have identified A_σ with $A_b(F_\sigma)$, so π_σ maps A into A_σ. Define $B = \bigoplus_{\sigma \in \partial_e K} A_\sigma$ (cf. Definition 2.42) where $\partial_e K$ denotes the set of extreme points of K. Then B is a JB-algebra with pointwise operations and the supremum norm. By the Krein–Milman theorem, the map π that takes a to $\bigoplus_{\sigma \in \partial_e K} \pi_\sigma(a)$ is an isometric order isomorphism of $A(K)$ into B.

Next, we are going to show that $\pi(A(K))$ is a JB-subalgebra of B. We first will show that $\pi(A(K))$ is closed under the map $x \mapsto x^+$ where $x = x^+ - x^-$ is the unique orthogonal decomposition in the JB-algebra B (cf. Proposition 1.28). For that purpose, let $a \in A(K)$ and let $a = b - c$ be a decomposition of the type described in (i). By the definition of orthogonality, there is a projective face G such that $b = 0$ on G and $c = 0$ on G'. For each $\sigma \in \partial_e K$, by (A 6) the faces $F_\sigma \cap G$ and $F_\sigma \cap G'$ are complementary split faces of F_σ, and thus are complementary projective faces of F_σ by Proposition 7.49. Thus b and c restricted to F_σ are orthogonal elements of the order unit space A_σ. As discussed at the start of this proof, b and c restricted to F_σ will also be orthogonal as elements of A_σ viewed as a JBW-algebra. Thus by the uniqueness of the orthogonal decomposition $\pi(a) = \pi(a)^+ - \pi(a)^-$ in the JB-algebra A_σ, we have $\pi_\sigma(b) = \pi_\sigma(a)^+$ for all $\sigma \in \partial_e K$. Hence

$$\pi(b) = \bigoplus_\sigma \pi_\sigma(b) = \bigoplus_\sigma (\pi_\sigma(a)^+) = \pi(a)^+,$$

and in particular $\pi(A)$ is closed in B under the map $x \mapsto x^+$. Now let X be the spectrum of $\pi(a)$ (cf. Definition 1.18). We will show $f(\pi(a)) \in \pi(A)$ for each $f \in C_{\mathbf{R}}(X)$. Let C_0 denote the set of all $f \in C_{\mathbf{R}}(X)$ such that $f(\pi(a)) \in \pi(A)$, where f acts on $\pi(a)$ by the continuous functional calculus (Definition 1.20). Since $A(K)$ is complete, and π is an isometry, then $\pi(A(K))$ is complete. Therefore C_0 is a linear subspace of $C_{\mathbf{R}}(X)$ closed under the map $f \mapsto f^+$, and so is a vector sublattice of $C_{\mathbf{R}}(X)$. Since $f \mapsto f(\pi(a))$ is an isometry on $C_{\mathbf{R}}(X)$ (Corollary 1.19), then C_0 is a norm closed vector sublattice of $C_{\mathbf{R}}(X)$ containing the constants and the identity function. Hence by the lattice version of the Stone–Weierstrass theorem (A 36), $C_0 = C_{\mathbf{R}}(X)$. In particular, C_0 contains the squaring map, so $\pi(A(K))$ is a JB-subalgebra of B. Thus π is a unital order isomorphism from $A(K)$ onto a JB-algebra, and so its dual gives an affine homeomorphism of the state space of a JB-algebra onto K. □

Remark. Instead of assuming (i) and (ii) in Theorem 9.38, we could assume the stronger property that K is a strongly spectral convex set. (See the remark after Proposition 8.80.)

Recall that a JC-algebra is a JB-algebra which is isomorphic to a norm closed Jordan subalgebra of $\mathcal{B}(H)_{\mathrm{sa}}$ (Definition 1.7).

9.39. Corollary. *A compact convex set K is affinely homeomorphic to the w^*-compact state space of a JC-algebra iff it satisfies, not only the conditions of Theorem 9.38, but also the additional requirement that every minimal split face of dimension 26 is a Hilbert ball.*

Proof. By Lemma 4.17 and Corollary 4.20, a JB-algebra A is a JC-algebra iff there is no closed Jordan ideal J in A such that A/J is isomorphic to the exceptional Jordan factor $H_3(\mathbf{O})$. Recall from Corollary 5.38 that for every closed ideal J of A there is a natural affine isomorphism of the state space of A/J onto the split face $F = J^\circ \cap K$ corresponding to J. Thus it remains to show that $H_3(\mathbf{O})$ is the only JB-algebra whose state space is a 26-dimensional convex set which has no proper split face and is not a Hilbert ball.

Counting (real) parameters in self-adjoint 3×3-matrices over the octonions, we find that $H_3(\mathbf{O})$ is a 27-dimensional vector space (over $\mathbf{R}$). Therefore the state space of $H_3(\mathbf{O})$ is a 26-dimensional convex set. Since $H_3(\mathbf{O})$ is a factor, it contains no non-trivial central projection. Therefore its state space contains no proper split face. The state space of $H_3(\mathbf{O})$ is not a Hilbert ball since it has proper faces that are not singletons, e.g., the projective faces associated with projections p such that $pH_3(\mathbf{O})p$ is isomorphic to $H_2(\mathbf{O})$.

If B is any JB-algebra whose state space is a 26-dimensional convex set which has no proper split face and is not a Hilbert ball, then B is a JBW-factor (Corollary 5.35), and is of type I (since it is finite dimensional). We conclude from Theorem 3.39 that the only type I factors with 26-dimensional state space are the 27-dimensional spin factor and $H_3(\mathbf{O})$. The state space of a spin factor is a Hilbert ball (Proposition 5.51), so $B = H_3(\mathbf{O})$. We are done. $\square$

The state spaces of JC-algebras can also be characterized among JB-state spaces by a condition on the dimensions of the Hilbert balls generated by pairs of pure states: cf. [10, Prop. 7.6].

Characterization of normal state spaces of JBW algebras

In this section we will always be working with a spectral convex set K which is then the distinguished base of a base norm space V, in spectral duality with $A = V^*$ (cf. Definition 8.74). Note then that $A = A_b(K)$ (Proposition 8.73). By Theorem 8.52, every norm exposed face of K will be projective, so the standing hypothesis of the previous chapter is satisfied.

9.40. Lemma. *Let K be a spectral convex set, and $A = A_b(K)$. Let $\{e_\lambda\}$ be a spectral resolution for $a \in A$, and let $E = \{e_{\lambda_2} - e_{\lambda_1} \mid \lambda_1 < \lambda_2\}$. Then:*

(i) *Each pair e, f in E is compatible.*
(ii) *E is closed under the map $(e, f) \mapsto e \wedge f$.*
(iii) *Each element $b \in \operatorname{lin} E$ can be expressed as a linear combination of orthogonal elements of E.*

Proof. Let E_0 be a finite subset of E, say

$$E_0 = \{e_{\beta_i} - e_{\alpha_i} \mid 1 \le i \le n\},$$

where $\alpha_i < \beta_i$ for each i. Let $\lambda_1, \ldots, \lambda_{2n}$ be the set $\{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n\}$ arranged in increasing order. Let $F = \{e_{\lambda_{i+1}} - e_{\lambda_i} \mid 1 \le i \le 2n-1\}$. The projective units in F are orthogonal, and each member of E_0 is a sum of elements of F. The items (i), (ii), (iii) follow. □

9.41. Lemma. *Let K be a spectral convex set, and $A = A_b(K)$. Let $\{e_\lambda\}$ be a spectral resolution for $a \in A$, let $E = \{e_{\lambda_2} - e_{\lambda_1} \mid \lambda_1 < \lambda_2\}$, and let M be the norm closure of the linear span of E. Then there is a compact Hausdorff space X and an isometric order isomorphism from $C_{\mathbf{R}}(X)$ onto M such that the induced product $\star$ on M satisfies $e \star f = e \wedge f$ for all $e, f \in \mathcal{P}$.*

Proof. Let $M_0 = \operatorname{lin} E$. Define $\star$ on M_0 by

$$a \star b = \sum_{i,j} \alpha_i \beta_j e_i \wedge f_j$$

where $a = \sum_i \alpha_i e_i$ and $b = \sum_j \beta_j f_j$ with $e_1, \ldots, e_n, f_1, \ldots, f_m \in E$. We will show this product is well defined. By Lemma 9.40 all elements of E are compatible. Thus if P_{e_i} is the compression associated with e_i for $i = 1, \ldots, n$, then by Theorem 8.3,

$$e_i \wedge f_j = (P_{e_i} \wedge P_{f_j})1 = P_{e_i} P_{f_j} 1 = P_{e_i} f_j$$

for $1 \le i, j \le n$. Hence

$$\sum_{i,j} \alpha_i \beta_j e_i \wedge f_j = \Big(\sum_i \alpha_i P_{e_i}\Big)\Big(\sum_j \beta_j f_j\Big) = \Big(\sum_i \alpha_i P_{e_i}\Big)(b).$$

Thus the definition of $a \star b$ is independent of the representation of b, and similarly of the representation of a. Since the operation $\wedge$ is commutative and associative, the same is true of $\star$ on M_0. Clearly $\star$ is bilinear.

Next we show that for $a \in M_0$,

$$-1 \le a \le 1 \quad \Rightarrow \quad 0 \le a \star a \le 1. \tag{9.26}$$

By Lemma 9.40, we can write a as a linear combination of orthogonal elements $e_1, \ldots, e_n$ of E, say $a = \sum_i \lambda_i e_i$. Since $-1 \le a \le 1$, then $|\lambda_i| \le 1$ for all i (as can be seen by applying suitable compressions to both sides of the equation $a = \sum_i \lambda_i e_i$). Then $a \star a = \sum_i \lambda_i^2 e_i$, so $0 \le a \star a \le 1$, which proves (9.26).

Thus (9.20) holds for $a \in M_0$, and $\star$ is commutative, so (9.21) also holds, i.e., for $a, b \in M_0$,

$$\|a \star b\| \le \|a\|\,\|b\|. \tag{9.27}$$

(See also the argument after (9.21).)

Now we can extend $\star$ to M (the norm closure of M_0) by continuity. The resulting product on M still satisfies (9.27) and is commutative and associative. By norm continuity of the map $\star$ and the fact that A^+ is norm closed, it follows that $a \star a \ge 0$ for all $a \in M$. It is clear from the definition of the product that the order unit acts as an identity for M. Thus M is a commutative, associative order unit algebra, and so $M \cong C_{\mathbf{R}}(X)$ follows from (A 48). □

9.42. Definition. Let K be a spectral convex set, and $A = A_b(K)$. Let $e \in \mathcal{P}$ and let P_e be the corresponding compression. Then we define $T_e : A \to A$ by

$$T_e = \tfrac{1}{2}(I + P_e - P_e').$$

Note that if e and f are compatible projective units, then $(P_e + P_e')f = f$, so

$$T_e f = \tfrac{1}{2}(P_e + P_e' + P_e - P_e')f = P_e f.$$

Thus for compatible e, f we have

$$T_e f = e \wedge f. \tag{9.28}$$

Recall from (1.47) that if A is a JB-algebra, then T_e is the Jordan multiplication operator $a \mapsto e \circ a$. Thus in this case $T_e f = T_f e = e \circ f$ for every pair of projections e, f. For a general spectral convex set K, the equation $T_e f = T_f e$ may fail for a pair of projective units e, f. But we will now show that if it holds, then the map $(e, f) \mapsto T_e f = T_f e$ from $\mathcal{P} \times \mathcal{P}$ into A extends to a bilinear map from $A \times A$ into A which organizes A to a JBW-algebra with normal state space affinely isomorphic to K. This result provides our first characterization of normal state spaces of JBW-algebras.

Note that this construction of a bilinear product in A is similar to that in Proposition 9.30 with atoms replaced by general projective units. However, in the proof of Proposition 9.30 we made use of the fact that every linear combination of atoms can be written as a linear combination of orthogonal atoms (Corollary 9.27), which is not true if atoms are replaced by arbitrary projective units. Thus we take a different tack, using the spectral theorem (Theorem 8.64) and Lemma 9.41.

9.43. Theorem. *A spectral convex set K is the normal state space of a JBW-algebra iff*

$$T_e f = T_f e \tag{9.29}$$

for all pairs of projective units $e, f \in A = A_b(K)$. More specifically, if this condition is satisfied, then there is a unique bilinear product $\circ$ on A which satisfies the norm condition (9.21) for $a, b \in A$ and the equation $e \circ e = e$ for $e \in \mathcal{P}$, and this product organizes A to the unique JBW-algebra whose normal state space is affinely isomorphic to K.

Proof. Clearly (9.29) is a necessary condition that K be the normal state space JBW-algebra since T_e is the Jordan multiplication operator $a \mapsto e \circ b$ in this case.

To prove sufficiency, we assume (9.29). Define $\Phi(e, f) = T_e f = T_f e$ for each pair $e, f \in \mathcal{P}$. Let $a = \sum_i \alpha_i e_i$ for $e_i \in \mathcal{P}$, $\alpha_i \in \mathbf{R}$ where $1 \leq i \leq n$, and $b = \sum_i \beta_j f_j$ for $f_j \in \mathcal{P}$, $\beta_j \in \mathbf{R}$ where $1 \leq j \leq m$. Then

$$\Big(\sum_i \alpha_i T_{e_i}\Big)(b) = \sum_{i,j} \alpha_i \beta_j \Phi(e_i, f_j) = \Big(\sum_j \beta_j T_{f_j}\Big)(a).$$

Thus there is a commutative bilinear product $\circ$ on the space $A_0 := \operatorname{lin} \mathcal{P}$ with values in A given by the following definition, which is independent of the representation of a and b:

$$a \circ b = \Big(\sum_i \alpha_i T_{e_i}\Big)(b) = \Big(\sum_j \beta_j T_{f_j}\Big)(a). \tag{9.30}$$

Clearly, this product is norm continuous in each variable separately.

We next show that the implication (9.20) holds in A_0. Fix $a \in A_0$. Define E, M, and the product $\star$ as in Lemma 9.41. By Lemma 9.40, each pair $e, f \in E$ is compatible, so by (9.28)

$$e \circ f = T_e f = e \wedge f = e \star f.$$

Thus $\circ$ and the product $\star$ agree on $\operatorname{lin} E$. Let $\{a_n\}$ be a sequence in $\operatorname{lin} E$ converging to a. (Such a sequence exists by the spectral theorem (Theorem 8.64)). By separate norm continuity of $\circ$ on A_0 and norm continuity of $\star$ on $M \cong C_{\mathbf{R}}(X)$,

$$a \circ a = \lim_n \lim_m a_n \circ a_m = \lim_n \lim_m a_n \star a_m = a \star a. \tag{9.31}$$

Now in $M \cong \mathbf{C}_{\mathbf{R}}(X)$ the implication (9.20) holds, so it follows from (9.31) that it also holds in A_0. Thus

$$-1 \leq a \leq 1 \quad \Rightarrow \quad 0 \leq a \circ a \leq 1 \tag{9.32}$$

for $a \in A_0$. Now (9.21) also holds, i.e.,

$$\|a \circ b\| \leq \|a\| \, \|b\| \tag{9.33}$$

for all pairs $a, b \in A_0$.

By (9.33) and the spectral theorem (Theorem 8.64), there is a unique continuous extension of $\circ$ to a product on A. By continuity, the extended product is still bilinear and commutative, and satisfies (9.32) and (9.33). To verify that A is a JB-algebra for this product, by Theorem 1.11 we need only show that A satisfies the Jordan identity

$$a^2 \circ (b \circ a) = (a^2 \circ b) \circ a. \tag{9.34}$$

So suppose that $e_1, \ldots, e_n$ are orthogonal projective units and $\lambda_1, \ldots, \lambda_n$ scalars such that $a = \sum_i \lambda_i e_i$. Then for each i, j, the compressions $P_{e_i}, P'_{e_i}, P_{e_j}, P'_{e_j}$ associated with e_i, e_j commute (Lemma 7.42), so T_{e_i} and T_{e_j} commute. From the definition (9.30) together with commutativity, we have $(e_i \circ b) \circ e_j = e_i \circ (b \circ e_j)$, so

$$a^2 \circ (b \circ a) = \sum_{i,j} \lambda_i^2 \lambda_j e_i \circ (b \circ e_j) = \sum_{i,j} \lambda_i^2 \lambda_j (e_i \circ b) \circ e_j = (a^2 \circ b) \circ a.$$

We have then shown that the Jordan identity (9.34) holds for $a \in A_0$ with finite spectral decomposition and all $b \in A_0$. By continuity and the spectral theorem (Theorem 8.64), the Jordan identity holds for all $a, b \in A$. By Theorem 1.11, the order from the JB-algebra structure (given by the cone of squares) coincides with the given order from the order unit space A.

The spectral convex set K is the distinguished base of a complete base norm space V (Definition 8.74) whose dual space is the order unit space $A = A_b(K)$ (Proposition 8.73). Thus A is a dual space, hence a JBW-algebra. The JBW-algebra A has a unique predual space A_*, which consists of all normal linear functionals on A (Theorem 2.55). Therefore the Banach space V is isometrically isomorphic to the space of all normal linear functionals on A. Since V as well as A_* are in separating order duality with A, these two base norm spaces are also order isomorphic. Therefore the distinguished bases of these two base norm spaces are affinely isomorphic. Thus K is affinely isomorphic to the normal state space of A.

The uniqueness of the product $\circ$ follows from the fact that two bilinear products on A which satisfy the norm condition (9.21) and the equation $e \circ e = e$ for $e \in \mathcal{P}$, must agree on $\mathcal{P}$ and by linearity on A_0, and then by norm continuity and the spectral theorem also on A.

If $\widetilde{A}$ is any JBW-algebra whose normal state space is affinely isomorphic to K, then $\widetilde{A}$ is isomorphic to A by Proposition 5.16. $\square$

The corollary below gives another characterization of normal state spaces of JBW-algebras. It is of interest mainly in showing that the functional calculus for affine functions on a spectral convex set provides a natural candidate for a Jordan product, which will give a JBW-algebra exactly under the simple but unappealing requirement that this product be bilinear.

Let $a \in A$ and let E, M, and $\star$ be defined as in Lemma 9.41. We observe for use below that the proof above showed that the products $\star$ and $\circ$ agree on $\operatorname{lin} E$, and thus by norm continuity of each product, agree on M.

9.44. Corollary. *Let K be a spectral convex set, and $A = A_b(K)$. Define a product $\circ$ on A by*

$$a \circ b = \tfrac{1}{4}(a+b)^2 - \tfrac{1}{4}(a-b)^2, \tag{9.35}$$

where the squares are given by the functional calculus on A defined by equation (8.60). A with this product and the order unit norm is a JBW-algebra iff this product is bilinear.

Proof. Suppose that this product is bilinear, and let $a \in A$. Then squares for this product by definition coincide with those given by the functional calculus, so by Proposition 8.67,

$$-1 \le a \le 1 \quad \Rightarrow \quad 0 \le a^2 \le 1. \tag{9.36}$$

Furthermore, with the notation of Lemma 9.41, $a \star a = a \circ a$ (see the remarks preceding this corollary), so in particular $a \circ a$ is again in M. Then we have $(a \circ a) \circ a = (a \star a) \star a$, and similarly for any product of factors of a grouped in any fashion. Since $M \cong C_{\mathbf{R}}(X)$ is power associative, the same follows for the product $\circ$. Thus A is a complete order unit space with a commutative, power associative product for which 1 is an identity, and for which (9.36) holds. By (A 51), A is a commutative order unit algebra with positive cone A^2 (A 47), and thus by Theorem 2.49, A is a JB-algebra. Since A is monotone complete and admits a separating set of normal states, then A is a JBW-algebra. □

The corollary below shows that the condition $T_e f = T_f e$ in Theorem 9.43 can be restated in terms of the general version of the "closeness operator"

$$c(e, f) = efe + e'f'e' \tag{9.37}$$

associated with a pair of projections e, f in a von Neumann algebra. This operator, which was first defined in $\mathcal{B}(H)$ by Chandler Davis [40], is a useful tool in the theory of algebras generated by a pair of two non-commuting projections, sometimes referred to as "non-commutative trigonometry". (If e and f are projections onto two lines in $\mathbf{C}^2$ with an angle α, then $c(e,f) = (\cos^2\alpha)I$. The element $c(e,f)$ has an interesting physical intepretation, discussed in the remark after Corollary 9.46. For more information on this topic, see the discussion in [AS, Chpt. 6] and also the references in the notes to that chapter.)

The compression P_e associated with a projection e in a von Neumann algebra is the map $a \mapsto eae$, and the compression P'_e associated with the complementary projection $e' = 1 - e$ is the map $a \mapsto e'ae'$. Therefore we define $c(e,f)$ for a pair e, f of two projective units in an order unit space A in spectral duality with a base norm space V by

$$c(e,f) = P_e f + P'_e f'. \tag{9.38}$$

9.45. Corollary. *A spectral convex set K is the normal state space of a JBW-algebra iff*

$$c(e,f) = c(f,e) \tag{9.39}$$

for all pairs of projective units $e, f \in A := A_b(K)$.

Proof. Let e, f be a pair of projective units in A. Using the equality

$$P'_e f' = P'_e(1-f) = 1 - e - P'_e f,$$

we find that

$$T_e f = \tfrac{1}{2}(f + P_e f - P'_e f) = \tfrac{1}{2}(e + f - 1 + P_e f + P'_e f').$$

Subtracting the corresponding equation with e and f interchanged, we find that

$$T_e f - T_f e = \tfrac{1}{2}(P_e f + P'_e f' - P_f e - P'_f e').$$

Thus the equation $T_e f = T_f e$ is equivalent to the equation

$$P_e f + P'_e f' = P_f e + P'_f e', \tag{9.40}$$

which is the same as $c(e,f) = c(f,e)$. Now the corollary follows from Theorem 9.43. □

The above characterization of the normal state space of a JBW-algebra was first proved in [9, Cor. 3.7] under the condition (9.40), written as the commutator relation $[P_e, P_f]\,1 = [P'_f, P'_e]\,1$, and with the Jordan product defined as in (9.35).

The symmetry condition $c(e,f) = c(f,e)$ for projective units in a spectral convex set is related to symmetry of transition probabilities (the third of the pure state properties). In fact, if e and f are minimal projections in a von Neumann algebra or JBW-algebra M, computing in the subalgebra $M_{e\vee f}$ gives

$$c(e,f) = \langle e, \widehat{f}\rangle\,(e \vee f) = \langle f, \widehat{e}\rangle\,(e \vee f).$$

Of course, by Corollary 9.45 and Corollary 5.57, the symmetry condition $c(e,f) = c(f,e)$ for projective units in a spectral convex set K implies that $A_b(K)$ can be equipped with a product making it a JBW-algebra with normal state space K, and so in particular that K has the property of symmetry of transition probabilities.

9.46. Corollary. *A compact spectral convex set K is the state space of a JB-algebra iff it satisfies the conditions*

(i) $c(e,f) = c(f,e)$ *for all pairs* $e, f \in \mathcal{P}$,
(ii) $a^2 \in A(K)$ *for all* $a \in A(K)$.

Proof. Clearly these are necessary conditions that K be the state space of a JB-algebra (which must be isomorphic to $A(K)$ equipped with the product $\circ$ of (9.35)). They are also sufficient, for if (i) is satisfied, then $A_b(K)$ is a JBW-algebra with the product $\circ$ of (9.35); and if (ii) is also satisfied, then $A(K)$ is closed under this product which will organize $A(K)$ to a JB-algebra with state space affinely isomorphic and homeomorphic to K. □

Remark. The "closeness operator" $c(e,f)$ associated with projections in a von Neumann algebra has a physical interpretation discussed in the remark following [AS, Cor. 6.43]. We will show that the element $c(e,f)$ of A^+ defined by (9.38) for a pair of projective units in the space $A = A_b(K)$ of a spectral convex set K can be interpreted in a similar way. Then we assume that e and f represent propositions (or questions) and that the compressions P_e and P_f represent measuring devices for these propositions, as explained in the remark after Proposition 7.49. Thus each of these measuring devices records if particles in a beam appear with the value 1 or the value 0 (the two possible values for the proposition). What happens if the beam from one apparatus is sent through the other one?

Clearly, there are four possible outcomes with probabilities $p(1,1)$, $p(1,0)$, $p(0,1)$, $p(0,0)$ respectively. (Here the first variable of the function $p(\cdot,\cdot)$ refers to the first apparatus and the second variable refers to the second apparatus). We will now determine these probabilities.

If a beam of particles in a state $\omega \in K$ is sent into the apparatus P_e, then the particles that are recorded with the value 1 will leave the apparatus in a new state represented by $P_e^*\omega \in V^+$. More specifically, they will be in the state $\|P_e^*\omega\|^{-1}P_e^*\omega \in K$. Here $\|P_e^*\omega\|$ is the relative intensity of the partial beam consisting of all particles that emerge with the value 1. Otherwise stated, $\|P^*\omega\|$ is the probability that a particle in the state ω is recorded with the value 1 by the apparatus P_e.

Assume now that the beam from the apparatus P_e is sent into the apparatus P_f. Then the states are transformed once more in the same fashion as in the first apparatus. Therefore the particles that are recorded with the value 1 by both apparata will emerge in a state represented by $P_f^* P_e^* \omega$, and the partial beam of such particles will have the relative intensity $\|P_f^* P_e^* \omega\|$. Thus the probability that a particle is recorded with the value 1 by both apparata is

$$p(1,1) = \|P_f^* P_e^* \omega\| = \langle 1, P_f^* P_e^* \omega\rangle = \langle P_e P_f 1, \omega\rangle = \langle P_e f, \omega\rangle.$$

In the same way we find that

$$p(1,0) = \langle P_e f', \omega\rangle, \quad p(0,1) = \langle P_e' f, \omega\rangle, \quad p(0,0) = \langle P_e' f', \omega\rangle.$$

Note that in general these probabilities will depend on the order in which the measurements are performed. However, it follows from the above that

$$p(1,1) + p(0,0) = \langle P_e f + P_e' f', \omega\rangle = \langle c(e,f), \omega\rangle.$$

Thus $\langle c(e,f), \omega\rangle$ is the probability that two consecutive measurements on particles in the state ω by the apparata P_e and P_f in that order coincide (both with the value 1 or both with the value 0). Therefore the equation $c(e,f) = c(f,e)$ says that for all ω this probability is the same when the order of the measurements is reversed. (Thus we might refer to this equation as "the symmetry of coincidence probabilities".)

We now give an alternate characterization of normal state spaces that is more geometric in nature, replacing the condition (9.29) by ellipticity. We have previously discussed this concept in the context of normal state spaces of JBW-algebras (cf. Proposition 5.75). We now define the corresponding concept for spectral convex sets and show that it characterizes normal state spaces of JBW-algebras.

9.47. Definition. Let K be a spectral convex set. Let F and F' be complementary projective faces of K, with corresponding compressions P and P'. Let $\Psi = (P + P')^*$ be the canonical projection of K onto $\text{co}(F \cup F')$, let $\sigma \in F$, $\tau \in F'$, and let $[\sigma, \tau]$ be the line segment with endpoints σ, τ. Then we say $\Psi^{-1}([\sigma,\tau])$ has *elliptical cross-sections* if the intersection of this set with every plane through the line segment $[\sigma,\tau]$ is an ellipse together with its interior. We say K is *elliptic* if such cross-sections are elliptical for all F, F', $\sigma \in F$, $\tau \in F'$.

By Proposition 5.75, the normal state spaces of JBW-algebras are elliptic. Now let K be a spectral convex set, represented as the distinguished base of a complete base norm space V. Let F and F' be complementary projective faces of K, with corresponding compressions P and P', and fix $\sigma \in F$, $\tau \in F'$, and $\omega \in K \setminus \mathrm{co}(F \cup F')$. Consider the orbit ω_t of ω under the action of the one-parameter group $t \mapsto \exp\bigl(t(P - P')\bigr)^*$, and let $\omega_t' = \|\omega_t\|^{-1}\omega_t$ denote the normalized orbit. Then the proof of Lemma 5.74 (with the concrete compression U_p replaced by the abstract compression P) shows that the normalized orbit of any such ω is the unique half ellipse that passes through ω and has σ and τ as endpoints. If the cross-sections described in Definition 9.47 are ellipses, then each such normalized orbit will stay inside K, or equivalently, $t \mapsto \exp\bigl(t(P - P')\bigr)^*$ will be a one-parameter group of order automorphisms of V. Thus we have for $t \in \mathbf{R}$, $P \in \mathcal{C}$,

$$K \text{ is elliptic} \quad \Rightarrow \quad \exp\bigl(t(P - P')\bigr)^* \geq 0. \tag{9.41}$$

This is equivalent to requiring $\exp\bigl(t(P - P')\bigr) \geq 0$ for all $t \in \mathbf{R}$. This says that $P - P'$ is an order derivation of the order unit space $A = A_b(K)$, cf. (A 66).

Note that in the case of a JBW-algebra, $\frac{1}{2}(I + P - P')b = (P1) \circ b$, so the fact that $P - P'$ is an order derivation follows from the fact that Jordan multiplication by any element of a JB-algebra is an order derivation (Lemma 1.56.) For use below, we recall that if A is any complete order unit space and $\delta : A \to A$ is a bounded linear map, then by (A 67), δ will be an order derivation iff for $a \in A^+, \sigma \in (A^*)^+$,

$$\langle a, \sigma \rangle = 0 \quad \Rightarrow \quad \langle \delta a, \sigma \rangle = 0. \tag{9.42}$$

We will also need the fact that the set of order derivations is closed under commutators (A 68).

9.48. Theorem. *A convex set K is affinely isomorphic to the normal state space of a JBW-algebra iff K is spectral and elliptic.*

Proof. We have already shown that the normal state space of a JBW-algebra is spectral and elliptic (Proposition 5.75 and Proposition 8.76). Conversely, assume that K is spectral and elliptic, and let $A = A_b(K)$. We are going to prove that K satisfies the condition (9.29), which by Theorem 9.43 will complete the proof.

Let P and Q be arbitrary compressions. We will show that

$$[P - P', Q - Q']1 = 0, \tag{9.43}$$

which is evidently equivalent to $[T_e, T_f]1 = 0$, and then to (9.29), where e and f are the projective units associated with P and Q.

By (9.41) the operators $D_1 := P - P'$ and $D_2 := Q - Q'$ are order derivations. Consider also the positive operators $E_1 := P + P'$ and $E_2 := Q + Q'$. Note that

$$D_1^2 = E_1, \quad D_2^2 = E_2. \tag{9.44}$$

For an arbitrary $a \in A^+$ and $\sigma \in K$ we have $\langle Qa, Q'^*\sigma\rangle = 0$, so we can use (9.42) with Qa in the place of a and $Q'\sigma$ in the place of σ to conclude that $\langle D_1 Qa, Q'^*\sigma\rangle = 0$. Hence $Q'D_1Q = 0$. Similarly $QD_1Q' = 0$. An immediate consequence is

$$(Q - Q')(P - P')(Q - Q') = (Q + Q')(P - P')(Q + Q')$$

(both sides being equal to $QD_1Q + Q'D_1Q'$). Clearly the same equality holds with P in the place of Q and D_2 in the place of D_1. Thus

$$D_2D_1D_2 = E_2D_1E_2, \quad D_1D_2D_1 = E_1D_2E_1. \tag{9.45}$$

We next establish

$$[D_1, D_2]^2 1 = 0. \tag{9.46}$$

To verify this, we expand the left side using (9.44):

$$\begin{aligned}[D_1, D_2]^2 1 &= D_1D_2D_1D_2 1 + D_2D_1D_2D_1 1 - D_1D_2^2D_1 1 - D_2D_1^2D_2 1\\ &= D_1D_2D_1D_2 1 + D_2D_1D_2D_1 1 - D_1E_2D_1 1 - D_2E_1D_2 1.\end{aligned}$$

Clearly $E_1 1 = 1$ and $E_2 1 = 1$, so in the last two terms above we can replace 1 by $E_2 1$ and $E_1 1$ (in this order), getting

$$[D_1, D_2]^2 1 = D_1D_2D_1D_2 1 + D_2D_1D_2D_1 1 - D_1E_2D_1E_2 1 - D_2E_1D_2E_1 1.$$

By (9.45) this gives (9.46).

From (9.46) we have

$$\exp\big(t\,[D_1, D_2]\big)1 = 1 + t\,[D_1, D_2]1. \tag{9.47}$$

By the fact that the set of order derivations is closed under commutators, $[D_1, D_2]$ is an order derivation. Therefore the left side of (9.47) is positive for all t. From this it follows that $[D_1, D_2]1 = 0$, which proves (9.43) and completes the proof. □

Notes

The geometric characterization of the normal state spaces of type I JBW-factors (in Theorem 9.33) was first given in [10], and it was then followed by the characterization of the state spaces of JB-algebras (Theorem 9.38) in the same paper. (Note that the condition in Theorem 9.38 that requires the σ-convex hull of the extreme points to be a split face may be redundant, as there are no known examples to the contrary.) In a paper proposing axioms for quantum mechanics, Gunson [59] showed under various assumptions that the linear span of the atoms formed a Jordan algebra, and the proof of Proposition 9.30 is related to his. The characterization of normal state spaces of general JBW-algebras (Theorem 9.48) is due to Iochum and Shultz [71]. The Hilbert ball property and the pure state properties were introduced in [10], and the ellipticity property in [71]. Note that the equivalent condition on the right side of (9.41) formally is very similar to the notion of (facial) homogeneity in self-dual cones due to Connes [32]. (See the discussion following Proposition 5.75.) The characterization of normal state spaces of JBW-algebras and state spaces of JB-algebras by means of the condition (9.39) was first given (in a slightly different form) in [9], where there is also a brief discussion of the physical interpretation. For a thorough discussion of an approach to axiomatic quantum mechanics via JBW-algebras, based on Corollary 9.45, see the papers of Kummer [88, 89] and Guz [60].

In the paper [21], the authors Ayupov, Iochum and Yadgorov characterize the state spaces of finite dimensional JB-algebras by three properties. The first one, called "projectivity", is the same as the "standing hypothesis" of Chapter 8 (exposed faces being projective), and the two others are "symmetry" and "modularity". In [22] the same authors show that a self-dual cone in a real Hilbert space is facially homogeneous iff it is symmetric and modular.

Transition probabilities have long played a central role in quantum mechanics. For example, in [135] Wigner shows that a bijection of the set of pure normal states of $\mathcal{B}(H)$ onto itself that preserves transition probabilities is implemented by a unitary or a conjugate linear isometry, and thus induces a *-isomorphism or *-anti-isomorphism of $\mathcal{B}(H)$.

Araki has given a careful justification of JB-algebras as a model for quantum mechanics in finite dimensions [17]. In this paper he discusses the role of symmetry of transition probabilities (cf. (9.7)), which is one of his axioms, and he also assumes that filters send pure states to multiples of pure states. In the finite dimensional context of Araki's paper, his axioms force the filters to be compressions.

The assumption that filters should send pure states to multiples of pure states is also in the paper of Pool [104], and "pure operations" (those that send pure states to pure states) were discussed in Haag and Kastler's influential paper [61]. The lattice-theoretic covering property (Definition 9.6), which is equivalent to compressions sending pure states to multiples of pure states (Corollary 9.8), plays a key role in the "quantum logic" approach to axioms for quantum mechanics, e.g., in the work of Piron [102].

Other relevant references for axiomatic foundations of quantum mechanics related to our presentation here can be found in the books of Busch, Grabowski, and Lahti [35], Emch [49], Landsman [91], Ludwig [94], and Upmeier [130].

Finally, we mention briefly JB*-triples. These generalize JB-algebras, and axiomatize the triple product $\{a, b, c\} \mapsto \frac{1}{2}(abc + cba)$. Recall that JB-algebras are in 1-1 correspondence with symmetric tube domains. By a theorem of Kaup [83], JB*-triples are in 1-1 correspondence with bounded symmetric domains. Thus they are of considerable geometric interest.

There is a Gelfand–Naimark type theorem for JB*-triples, due to Friedman and Russo [54]. Friedman and Russo also showed that most of the geometric properties of JB-algebra state spaces in Theorem 9.38 generalize to preduals of JBW*-triples [53], and they gave a characterization of the preduals of atomic JBW*-triples [55], which generalizes Theorem 9.33. These results are quite remarkable, since they generalize order-theoretic concepts and results to a context where there is no order, requiring totally different proofs.

10 Characterization of Normal State Spaces of von Neumann Algebras

We begin this chapter by characterizing the normal state space of the von Neumann algebra $\mathcal{B}(H)$. Our starting point is Theorem 9.34 which characterizes normal state spaces of JBW-factors of type I by geometric axioms, among those the Hilbert ball property by which the face generated by each pair of extreme points is a Hilbert ball. In the case of $\mathcal{B}(H)$ these balls will be 3-dimensional (A 120), and this "3-ball property" is the single additional property we need to characterize the normal state space of $\mathcal{B}(H)$ (Theorem 10.2).

Since general von Neumann algebras may admit no pure normal states, assumptions involving pure states are not useful to characterize their normal state spaces. Our starting point for general von Neumann algebras is Theorem 9.48 which characterizes normal state spaces of general JBW-algebras, and we show that a "generalized 3-ball property" that involves pairs of norm closed faces rather than pairs of extreme points (Definition 10.20) is the single additional property needed to characterize the normal state spaces of general von Neumann algebras (Theorem 10.25).

The normal state space of $\mathcal{B}(H)$

10.1. Definition. A *3-ball* is a convex set affinely isomorphic to the *standard 3-ball*

$$\mathbf{B}^3 = \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbf{R}^3 \mid \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1\}.$$

Note that a 3-ball is a special case of a Hilbert ball.

10.2. Theorem. *Let K be the base of a complete base norm space. Then K is affinely isomorphic to the normal state space of $\mathcal{B}(H)$ for a complex Hilbert space H iff all of the following hold.*

(i) *Every norm exposed face of K is projective.*
(ii) *The σ-convex hull of the extreme points of K equals K.*
(iii) *The face generated by every pair of extreme points of K is a 3-ball and is norm exposed.*

Proof. Assume first that K is affinely isomorphic to the normal state space of $\mathcal{B}(H)$. Since $\mathcal{B}(H)_{\mathrm{sa}}$ is an atomic JBW-algebra, then properties (i) and (ii) follow from Theorem 9.34, as does the Hilbert ball property. If

σ, τ are extreme points of K, then the face they generate is norm exposed by the Hilbert ball property. The fact that the face generated by σ, τ is a 3-ball is (A 120).

Now assume K satisfies (i), (ii), (iii). By Theorem 9.34, K is affinely isomorphic to the normal state space of an atomic JBW-algebra M. Suppose that K contains a proper split face F. Since M is atomic, then F and F' contain minimal projective faces, which are then singleton extreme points (Propositions 8.36 or 5.39). Let $\sigma \in F$ and $\tau \in F'$ be such extreme points. Then the face of K generated by σ and τ is the line segment joining these two points (Lemma 5.54 or (A 29)), which contradicts (iii). Thus K contains no proper split face, and so by Theorem 9.34, K is affinely isomorphic to the normal state space of a type I JBW-factor M. By (iii), M is isomorphic to $\mathcal{B}(H)_{\mathrm{sa}}$ for a complex Hilbert space H (Proposition 9.36). □

3-frames, Cartesian triples, and blown up 3-balls

Let (σ, τ) be a pair of antipodal points in the standard Euclidean 3-ball $\mathbf{B}^3$. Then there is a unique reflection (i.e., period 2 automorphism) of $\mathbf{B}^3$ whose set of fixed points is the axis determined by (σ, τ) (i.e., the line segment with endpoints σ, τ). Two such axes are orthogonal iff the reflection associated with each axis reverses the other axis, i.e., exchanges the corresponding pair of antipodal points of that axis. If we have three orthogonal axes, then the product of the associated reflections will be the identity map. This is characteristic of dimension 3: for higher dimensional Hilbert balls the product of any such triple of reflections will invert any fourth axis orthogonal to the first three, and thus cannot be the identity map.

These observations motivate our generalization of 3-balls. We are going to define *generalized axes*, *3-frames*, and *blown up 3-balls.* (See the definitions (A 185), (A 190), and (A 194) for a discussion of these concepts for normal state spaces of von Neumann algebras.)

Our context in the rest of this section will be a JBW-algebra M and its normal state space K. Recall that there is a 1-1 correspondence of projections in M and norm closed faces in K given by $p \mapsto F_p = \{\sigma \in K \mid \sigma(p) = 1\}$ (Corollary 5.33). The faces F_p and $F_p' = F_{p'}$ (where $p' = 1 - p$) are said to be *complementary*.

10.3. Definition. If K is the normal state space of a JBW-algebra, an ordered pair (F, F') of complementary norm closed faces is called a *generalized axis* of K.

Observe that if p is a projection and $p' = 1 - p$, then $s = p - p'$ is a symmetry (and every symmetry arises in this way). If $F = F_p$, we will call (F, F') the *generalized axis associated with the symmetry* $s = p - p'$.

Recall that a reflection of a convex set is an affine automorphism of period 2, and that the map U_s^* (dual to $U_s : x \mapsto \{sxs\}$) acts as a reflection of K with fixed point set $\mathrm{co}(F \cup F')$ (Proposition 5.72). We now show that this reflection is unique, and will then refer to it as *the reflection about the generalized axis* (F, F'). This uniqueness result in the von Neumann algebra context is (A 174).

10.4. Proposition. *If p is a projection in a JBW-algebra M with associated projective face F, and $s = p - p'$ where $p' = 1 - p$, then U_s^* is the unique reflection of the normal state space K with fixed point set $\mathrm{co}(F \cup F')$.*

Proof. Let R be any reflection of K with fixed point set $\mathrm{co}(F \cup F')$. Then $\frac{1}{2}(I+R)$ is an affine projection of K onto $\mathrm{co}(F \cup F')$. Since U_p, $U_{p'}$ are complementary compressions (Theorem 2.83) with associated projective faces F, F' (cf. (5.4)), then $(U_p + U_{p'})^*$ is the unique affine projection of K onto $\mathrm{co}(F \cup F')$ (Theorem 7.46). Thus $\frac{1}{2}(I + R) = (U_p + U_{p'})^*$, so $R = (2(U_p + U_{p'}) - I)^*$. Then $R = U_s^*$ follows from (2.25). □

If F is a projective face of K with complementary face F', and an affine automorphism ϕ of K exchanges F and F', then we will say that *ϕ reverses the generalized axis* (F, F'). We can use this to define a notion of orthogonality for generalized axes.

10.5. Definition. If M is a JBW-algebra with normal state space K, a pair of generalized axes is *orthogonal* if the reflection about the first generalized axis reverses the second generalized axis.

By the following result, the reflection about the second generalized axis will also reverse the first, so orthogonality is a symmetric relation as one would hope.

10.6. Lemma. *Let p and q be projections of a JBW-algebra M with complementary projections p' and q'. Let F and G be the norm closed faces associated with p and q, and let R_F and R_G be the reflections of K about the generalized axes (F, F') and (G, G'). The following are equivalent.*

(i) *The reflection R_F reverses the generalized axis (G, G').*
(ii) *The reflection R_G reverses the generalized axis (F, F').*
(iii) $(p - p') \circ (q - q') = 0$.

Proof. Let $s = p - p'$ and $t = q - q'$. Then s and t are symmetries and $R_F = U_s^*$ and $R_G = U_t^*$ (Proposition 10.4). By Corollary 5.17, (i) is equivalent to (i′) U_s exchanges q and q'

while (ii) is equivalent to

(ii′) U_t exchanges p and p'.

Each of (i') and (ii') is equivalent to (iii) by Lemma 3.6. □

We say that symmetries s, t are *Jordan orthogonal* if $s \circ t = 0$. Thus orthogonality of generalized axes is equivalent to Jordan orthogonality of the associated symmetries.

The following definition abstracts the key properties of the set of three orthogonal axes of $\mathbf{B}^3$.

10.7. Definition. Let K be the normal state space of a JBW-algebra M. A *3-frame* is an ordered triple of mutually orthogonal generalized axes of K such that the product of the associated reflections of K is the identity map.

Recall that if F is the projective face associated with a projection p, then F can be identified with the normal state space of the JBW-subalgebra $\operatorname{im} U_p$, cf. Proposition 2.62. Therefore we can apply the notions of generalized axes and 3-frames to F as well as to K.

Recall that a Cartesian triple of symmetries is an ordered triple (r, s, t) of Jordan orthogonal symmetries such that $U_r U_s U_t$ is the identity map (Definition 4.50). We now show that Cartesian triples are the algebraic objects corresponding to 3-frames.

10.8. Definition. If $\alpha = (r, s, t)$ is a Cartesian triple, then α_* denotes the ordered triple of generalized axes associated with r, s, t.

10.9. Lemma. *Let M be a JBW-algebra, The map $\alpha \mapsto \alpha_*$ is a 1-1 correspondence of Cartesian triples of symmetries and 3-frames of K.*

Proof. Let $\alpha = (r, s, t)$. By Lemma 10.6, the orthogonality of the generalized axes of α_* is equivalent to Jordan orthogonality of r, s, t. The reflections about the three generalized axes are U_r^*, U_s^*, U_t^* respectively. Then the product of the three reflections is the identity iff $U_r^* U_s^* U_t^* = Id$, which in turn is equivalent to $U_t U_s U_r = Id$. By the remarks after Definition 4.50, if (r, s, t) is a Cartesian triple of symmetries, then the maps U_r, U_s, U_t commute, so this is equivalent to (r, s, t) being a Cartesian triple. □

From the proof above and Proposition 10.4, it follows that the reflections associated with the generalized axes in a 3-frame commute (and therefore their product in any order is the identity).

Remark. By Lemma 10.9, the existence of 3-frames in K is equivalent to the existence of Cartesian triples in M. By Definition 10.5 and

Lemma 10.6, a necessary condition for this is the existence of a projection $p \in M$ such that p and p' can be exchanged by a symmetry (as in (ii$'$) of the proof of Lemma 10.6). (In the case of von Neumann algebras, the above condition is sufficient as well as necessary for the existence of 3-frames in K, cf. (A 193) where this condition is referred to as "halvability of the identity element".)

Note that the above exchangeability condition is satisfied in the Jordan matrix algebra $M_n(\mathbf{C})_{\mathrm{sa}}$ iff n is an even number. Therefore the state space K of $M_n(\mathbf{C})_{\mathrm{sa}}$ has no 3-frame when n is odd. But K has closed faces with 3-frames for all $n > 1$, namely all faces that are isomorphic to state spaces of subalgebras of the form $\operatorname{im} U_p \cong M_k(\mathbf{C})_{\mathrm{sa}}$ where $k = \operatorname{rank}(p)$ is even.

We are now ready to define our generalization of 3-balls.

10.10. Definition. Let M be a JBW-algebra with normal state space K. A face F of K (possibly $F = K$) is a *blown up 3-ball (of K)* if it is a norm closed face admitting a 3-frame.

We will see later that this implies that the facial structure and automorphism group of F are quite similar to those of the standard 3-ball $\mathbf{B}^3$.

10.11. Example. In the special case of $M = M_2(\mathbf{C})_{\mathrm{sa}}$, we can identify the state space of M with the positive matrices of trace 1, and thus with $\mathbf{B}^3$ as follows (see (A 119)):

$$\frac{1}{2}\begin{pmatrix} 1+\beta_1 & \beta_2 + \imath\beta_3 \\ \beta_2 - \imath\beta_3 & 1-\beta_1 \end{pmatrix} \mapsto (\beta_1, \beta_2, \beta_3). \tag{10.1}$$

Let α_{0*} be the ordered triple of pairs of singleton faces of $\mathbf{B}^3$ located along each of the three coordinate axes. Each of these pairs is a "generalized axis" in our current terminology. We will refer to α_{0*} as the *standard 3-frame of* $\mathbf{B}^3$. The reflections corresponding to each generalized axis will be exactly the reflections of $K \cong \mathbf{B}^3$ in each of the coordinate axes. The associated 3-frame is

$$\alpha_0 = \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \imath \\ -\imath & 0 \end{pmatrix} \right), \tag{10.2}$$

which are the Pauli spin matrices.

If a JBW-algebra M possesses a Cartesian triple of symmetries, then M is isomorphic to the self-adjoint part of a von Neumann algebra $\mathcal{M}$ (Theorem 4.56). Furthermore, this von Neumann algebra $\mathcal{M}$ will contain a set of 2×2 matrix units, so that $\mathcal{M} \cong M_2(\mathbf{C}) \otimes \mathcal{B}$ for a von Neumann algebra $\mathcal{B}$. Using the correspondence of 3-frames and Cartesian triples, we can characterize normal state spaces of such von Neumann algebras.

10.12. Proposition. *If the normal state space K of a JBW-algebra M contains a 3-frame, with associated Cartesian triple of symmetries (r,s,t), then $\mathcal{M} = M + iM$ can be equipped with a unique von Neumann algebra product compatible with the given Jordan product and satisfying $irst = 1$. For this product, the complex linear span of $1, r, s, t$ is a *-subalgebra *-isomorphic to $M_2(\mathbf{C})$, and thus $\mathcal{M} \cong M_2(C) \otimes \mathcal{B}$ for a von Neumann algebra $\mathcal{B}$.*

Proof. The proposition follows from the correspondence of 3-frames and Cartesian triples (Lemma 10.9) and Theorem 4.56. □

The following result generalizes the fact that the state space of $M_2(\mathbf{C})$ is a 3-ball, cf. (A 119).

10.13. Corollary. *The normal state space K of a JBW-algebra M is a blown up 3-ball iff M is isomorphic to the self-adjoint part of a von Neumann algebra $\mathcal{M}$ of the form $\mathcal{M} \cong M_2(\mathbf{C}) \otimes \mathcal{B}$ for a von Neumann algebra $\mathcal{B}$.*

Proof. This follows at once from Proposition 10.12 and the definition of a blown up 3-ball. □

We will now present some elementary concepts and facts that will be used in the proof of our next result (Proposition 10.15). This result is not needed in the sequel, but provides a further illustration of the appropriateness of the term "blown up 3-ball".

10.14. Definition. If K is the normal state space of a JBW-algebra, then $\mathrm{Aut}_0(K)$ is the (norm-continuous) path component of the identity in the group of affine automorphisms of K. (Here we extend affine automorphisms of K to $V = \operatorname{lin} K$ and give them the norm topology as bounded linear maps on V.)

Note that affine automorphisms of the standard 3-ball $\mathbf{B}^3$ extend uniquely to orthogonal transformations of $\mathbf{R}^3$. The latter have determinant ± 1. Those with determinant $+1$ are precisely the rotations, while those with determinant -1 are the reversals (i.e., compositions of a rotation and a reflection in the origin.) Rotations are evidently connected by a path to the identity, while reversals are not (by continuity of the determinant map.) Thus $\mathrm{Aut}_0(\mathbf{B}^3)$ can be identified with the rotation group SO(3). By (A 131), the rotations of $\mathbf{B}^3$ are precisely the duals of the maps $x \mapsto uxu^*$ for unitaries u in $M_2(\mathbf{C})$, while the reversals arise as the duals of the maps $x \mapsto vx^*v^{-1}$, where v is conjugate unitary. Thus rotations of $\mathbf{B}^3$ are the duals of $*$-automorphisms of $M_2(\mathbf{C})$, while reversals correspond to $*$-anti-automorphisms (A 124).

Recall that if K is the normal state space of a JBW-algebra, then the projective faces and norm closed faces of K coincide, and the lattice of norm closed faces of K is an orthomodular lattice (Corollary 5.33). An *ortho-isomorphism* from one orthomodular lattice into another is an injective map that preserves complements and (finite) least upper bounds and greatest lower bounds.

Let M_1 and M_2 be JBW-algebras with normal state spaces K_1, K_2. Recall that the map $T \mapsto T^*$ is a 1-1 correspondence of positive unital σ-weakly continuous maps $T : M_1 \to M_2$ and affine maps from K_2 into K_1 (Lemma 5.13). If T is also a Jordan homomorphism, and p is a projection in M_1 with associated projective face F_p, then by Lemma 5.14,

$$(T^*)^{-1}(F_p) = F_{Tp}, \tag{10.3}$$

and if T is a Jordan isomorphism from M_1 onto M_2, and q is a projection in M_2, then

$$T^*(F_q) = F_{T^{-1}(q)}. \tag{10.4}$$

We will make use of these results several times in the next proof, in a context where T will be a unital $*$-homomorphism between von Neumann algebras.

Recall that for each closed face F of the normal state space of a JBW-algebra, there is a reflection R_F with fixed point set $\mathrm{co}(F \cup F')$, cf. Proposition 10.4.

10.15. Proposition. *Let K be the normal state space of a JBW-algebra M. If K is a blown up 3-ball, then there is an affine map ϕ from K onto $\mathbf{B}^3$ such that*

(i) *ϕ^{-1} is an ortho-isomorphism from the lattice of faces of $\mathbf{B}^3$ into the lattice of norm closed faces of K,*
(ii) *there is an isomorphism $\sigma \mapsto \widehat{\sigma}$ of the rotation group $Aut_0(\mathbf{B}^3)$ into the group of affine automorphisms of K which takes R_F to $R_{\phi^{-1}(F)}$ for each face F of $\mathbf{B}^3$, and which satisfies $\sigma \circ \phi = \phi \circ \widehat{\sigma}$.*

Proof. Since K is a blown up 3-ball, there is a von Neumann algebra $\mathcal{B}$ such that M is isomorphic to the self-adjoint part of $\mathcal{M} = M_2(\mathbf{C}) \otimes \mathcal{B}$ (Corollary 10.13). Identify M and $\mathcal{M}_{\mathrm{sa}}$, and identify K with the normal state space of $\mathcal{M}$. Define $T : M_2(\mathbf{C}) \to \mathcal{M}$ by $Tx = x \otimes 1$. We identify the state space of $M_2(\mathbf{C})$ with $\mathbf{B}^3$, and let $\phi : K \to \mathbf{B}^3$ be T^* restricted to K.

To see that ϕ is surjective, it suffices to show $\phi(K)$ contains every pure state on $M_2(\mathbf{C})$. Recall that there is a 1-1 correspondence of pure states on $M_2(\mathbf{C})$ and minimal projections in $M_2(\mathbf{C})$ (e.g., Proposition 5.39). Let p be a minimal projection in $M_2(\mathbf{C})$ and ω the unique state with $\omega(p) = 1$. If τ is any normal state on M with $\tau(p \otimes 1) = 1$, then $\phi(\tau)(p) = \tau(Tp) = \tau(p \otimes 1) = 1$, so $\phi(\tau) = \omega$. Thus ϕ is surjective.

Since ϕ is affine and norm continuous, then ϕ^{-1} maps faces of $\mathbf{B}^3$ to norm closed faces of K. For faces F and G of $\mathbf{B}^3$,

$$(10.5)\quad \phi^{-1}(F \wedge G) = \phi^{-1}(F \cap G) = \phi^{-1}(F) \cap \phi^{-1}(G) = \phi^{-1}(F) \wedge \phi^{-1}(G).$$

Thus ϕ^{-1} preserves greatest lower bounds. Since $x \mapsto x \otimes 1$ is a unital *-homomorphism from $M_2(\mathbf{C})$ into $M_2(\mathbf{C}) \otimes \mathcal{B}$, then ϕ^{-1} preserves complements (Lemma 5.15). Since $(F \wedge G)' = F' \vee G'$ for projective faces F, G, then ϕ^{-1} also preserves least upper bounds, and thus is an ortho-isomorphism from the lattice of faces of $\mathbf{B}^3$ into the lattice of norm closed faces of K. This proves (i).

For each $\sigma \in \mathrm{Aut}_0(\mathbf{B}^3)$ there is a unique *-automorphism S of $M_2(\mathbf{C})$ such that $\sigma = S^*$. (See the discussion preceding this proposition.) Let Id be the identity automorphism of $\mathcal{B}$. Then $S \otimes Id$ is a *-automorphism of $\mathcal{M}$, so $(S \otimes Id)^*$ is in the space $\mathrm{Aut}(K)$ of affine automorphisms of K. Observe that $\sigma \mapsto S$ is an anti-isomorphism from the group $\mathrm{Aut}_0(\mathbf{B}^3)$ onto the group of all *-automorphisms of $M_2(\mathbf{C})$.

Define $\pi : \mathrm{Aut}_0(\mathbf{B}^3) \to \mathrm{Aut(K)}$ by

$$\pi(\sigma) = (S \otimes Id)^*,$$

where $\sigma = S^*$. Then π is a group isomorphism into Aut(K).

Let F be an arbitrary face of $\mathbf{B}^3$, with associated projection p. Now we show that $\pi(R_F) = R_{\phi^{-1}(F)}$. Then R_F is the reflection U_s^*, where $s = p - p'$ (Proposition 10.4). Therefore

$$\pi(R_F) = \pi(U_s^*) = (U_s \otimes Id)^* = U_{s\otimes 1}^*.$$

Note that $s \otimes 1 = p \otimes 1 - p' \otimes 1$. Thus $\pi(R_F)$ is the reflection R_G where G is the face of K associated with the projection $p \otimes 1 = Tp$. By (10.3), $G = F_{Tp} = (T^*)^{-1}(F_p) = \phi^{-1}(F)$, so $\pi(R_F) = R_{\phi^{-1}(F)}$.

Next we show $\sigma \circ \phi = \phi \circ \pi(\sigma)$ for each $\sigma \in \mathrm{Aut}_0(\mathbf{B}^3)$. Write $\sigma = S^*$, with S a *-automorphism of $M_2(\mathbf{C})$. For $x \in M_2(\mathbf{C})$, we have

$$T(S(x)) = S(x) \otimes 1 = (S \otimes Id)(x \otimes 1) = (S \otimes Id)(Tx),$$

so $T \circ S = (S \otimes Id) \circ T$. Dualizing gives

$$\sigma \circ \phi = \phi \circ (S \otimes Id)^* = \phi \circ \pi(\sigma).$$

Setting $\widehat{\sigma} = \pi(\sigma)$ proves (ii) and completes the proof of the proposition. □

Remark. By Proposition 10.15 a blown up 3-ball can be mapped onto the standard 3-ball $\mathbf{B}^3$ by an affine map which "pulls back" the face lattice of $\mathbf{B}^3$ into the face lattice of K and "lifts" the rotation group of $\mathbf{B}^3$ into the group of all affine automorphisms of K. This shows that a blown up 3-ball possesses much of the affine structure of the standard 3-ball, and is another reason for our choice of that name.

Note that it is not the case that each affine surjection $\phi : K \to \mathbf{B}^3$ which admits a group isomorphism from $\mathrm{Aut}_0(\mathbf{B}^3)$ into $\mathrm{Aut}(K)$ with these properties will admit a group isomorphism from $\mathrm{Aut}(\mathbf{B}^3)$ with the same properties. We will show this by an example where ϕ is as in the proof above and $\mathcal{B}$ is chosen to be a von Neumann factor not *-anti-isomorphic to itself, (e.g., [37]). Then it is straightforward to check that the von Neumann algebra $M_2(\mathbf{C}) \otimes \mathcal{B}$ is also a von Neumann factor. As in the proof above, define $T : M_2(\mathbf{C}) \to \mathcal{M}$ by $Tx = x \otimes 1$, and recall that ϕ is the dual map from K onto $\mathbf{B}^3$, and has the properties described in (ii). Now let ψ be the transpose map on $M_2(\mathbf{C})$. Then ψ^* is an affine automorphism of $\mathbf{B}^3$, but since ψ is a *-anti-automorphism of $\mathbf{B}^3$, then $\sigma = \psi^*$ is in $\mathrm{Aut}(\mathbf{B}^3)$ but is not in $\mathrm{Aut}_0(\mathbf{B}^3)$. Now suppose that there is a lift $\widehat{\sigma} \in \mathrm{Aut(K)}$, i.e., $\phi \circ \widehat{\sigma} = \sigma \circ \phi$. Let $\Psi : \mathcal{M} \to \mathcal{M}$ be the Jordan isomorphism such that $\Psi^* = \widehat{\sigma}$ (Proposition 5.16). By Kadison's theorem, cf. (A 158), since $\mathcal{M}$ is a factor, Ψ is either a *-isomorphism or an *-anti-isomorphism.

Since $\phi \circ \widehat{\sigma} = \sigma \circ \phi$, then $T^* \circ \Psi^* = \psi^* \circ T^*$. Therefore $\Psi \circ T = T \circ \psi$, so for all $x \in M_2(\mathbf{C})$ we have $\Psi(x \otimes 1) = \psi(x) \otimes 1$. Hence Ψ restricts to a *-anti-isomorphism on the subalgebra $M_2(\mathbf{C}) \otimes \mathbf{C}1$. Since Ψ is either a *-isomorphism or a *-anti-isomorphism on $\mathcal{M}$, the latter must hold. Since Ψ leaves invariant the *-subalgebra $M_2(\mathbf{C}) \otimes \mathbf{C}1$, it must leave invariant the relative commutant of this subalgebra, namely $1 \otimes \mathcal{B}$. Then Ψ induces a *-anti-isomorphism of $\mathcal{B}$, a contradiction.

Thus in Proposition 10.15, $\mathrm{Aut}_0(\mathbf{B}^3)$ cannot be replaced by $\mathrm{Aut}(\mathbf{B}^3)$. It follows that in the definition of global 3-balls in [71], $\mathrm{Aut}(\mathbf{B}^3)$ should be replaced by $\mathrm{Aut}_0(\mathbf{B}^3)$. With this change in definition, the results and proofs in [71] hold without other change.

The generalized 3-ball property

In this section we will define the key property needed to insure that a JBW-algebra normal state space is in fact a von Neumann algebra normal state space. We start by establishing some useful properties concerning equivalence of orthogonal projections. These properties generalize known properties of projections in von Neumann algebras, cf. (A 175) for Lemma 10.16, and (A 179) and (A 195) for Proposition 10.17.

10.16. Lemma. *If p and q are orthogonal, equivalent projections in a JBW-algebra M, then p and q are exchanged by a symmetry.*

Proof. Suppose first that M has no I_2 or I_3 direct summand. By Theorem 4.23, we may assume M is represented as a concrete JW-subalgebra of $\mathcal{B}(H)_{\mathrm{sa}}$, and is reversible (Corollary 4.30). By assumption there are symmetries $s_1, \dots, s_n$ with $s_1 s_2 \cdots s_n p s_n \cdots s_1 = q$. Let $v = s_1 s_2 \dots s_n p$. Then $vv^* = q$ and $v^*v = p$, so v is a partial isometry with initial projection p and final projection q. Let $s = v + v^*$. Then s is a $(p+q)$-symmetry in $\mathcal{B}(H)_{\mathrm{sa}}$ that exchanges p and q. By reversibility of M, $s = s_1 s_2 \cdots s_n p + p s_n \cdots s_2 s_1 \in M$. By Lemma 3.5, p and q are exchangeable by a symmetry.

It remains to establish the lemma's conclusion when p and q are orthogonal, equivalent projections in a JBW-algebra M of type I_2 or I_3. We claim that in this case p and q are abelian projections, and so are exchangeable (Corollary 3.20). Since factor representations separate the points of M (Lemma 4.14), it suffices to show that $\pi(p)$ and $\pi(q)$ are abelian projections for every factor representation π. Let π be a factor representation of M, and define $N = \overline{\pi(M)}$ (σ-weak closure). Here N will be a type I_2 or I_3 JBW-factor (Lemma 4.8). Then $\pi(p)$ and $\pi(q)$ are exchangeable orthogonal projections in N. If they were not zero and not minimal projections, then each would dominate at least two non-zero orthogonal projections, so N would contain at least four orthogonal non-zero projections, contrary to the properties of factors of type I_2 or I_3 (Lemma 3.22). Thus $\pi(p)$ and $\pi(q)$ must be either minimal projections or zero, and so in either case are abelian. This completes the proof that p and q are abelian, and thus are exchangeable by a symmetry. □

A key step in the next proof is to show that if p and q are projections with $\|p - q\| < 1$, then p and q can be exchanged by a symmetry. (See (A 178) for the corresponding von Neumann algebra result.) We will see that this follows from the fact that for any projections p and q, the projections $p - p \wedge q'$ and $q - q \wedge p'$ can always be exchanged by a symmetry. For von Neumann algebras, a proof of the latter fact can be found in (A 177). For JBW-algebras, note that

$$p - (p \wedge q') = p \wedge (p' \vee q) = r(U_p q),$$
$$q - (q \wedge p') = q \wedge (q' \vee p) = r(U_q p),$$

where the last equality in each equation follows from Proposition 2.31. There is a symmetry s that exchanges $U_p q$ and $U_q p$ (Lemma 3.46). Since U_s is a Jordan automorphism, it will also exchange the range projections of $U_p q$ and $U_q p$, and therefore will exchange $p - p \wedge q'$ and $q - q \wedge p'$.

10.17. Proposition. *If p and q are orthogonal projections in a JBW-algebra M, then p and q are equivalent iff there is a (norm continuous) path of projections from p to q.*

Proof. Suppose first that p and q are equivalent, orthogonal projections. By Lemma 10.16, p and q are exchanged by a symmetry. If $e = p+q$, by Lemma 3.5 we can choose an e-symmetry s that exchanges p and q, i.e., $U_s(p) = q$. Then $U_s(p-q) = q-p$, so by the equivalence of (iii) and (iv) in Lemma 3.6 (applied to M_e), we have $s \circ (p-q) = 0$. Now let $s_t = (\cos \pi t)(p-q) + (\sin \pi t)s$ for $0 \le t \le 1$. Then for each t, the element s_t is an e-symmetry, and $s_0 = p-q$, while $s_1 = q-p$. Thus there is a path of e-symmetries from $p-q$ to $q-p$. Then $p_t = \frac{1}{2}(s_t + e)$ will be a path of projections from p to q.

Conversely, suppose that p and q are projections such that there exists a path of projections from p to q. Then we can find a finite sequence of projections $p_0 = p, p_1, \dots, p_n = q$ such that $\|p_i - p_{i+1}\| < 1$ for $0 \le i \le n-1$. We will be done if we show that there is a symmetry s_i such that $U_{s_i} p_i = p_{i+1}$ for $0 \le i \le n-1$. For simplicity of notation we may assume without loss of generality that $n = 1$, so that $\|p - q\| < 1$.

We are going to show that $p \wedge q' = q \wedge p' = 0$. Since $p \wedge q' \le q'$, then $p \wedge q'$ is orthogonal to q, so $(p \wedge q') \circ q = 0$. Thus by the JB-algebra norm requirement (1.6),

$$\|p \wedge q'\| = \|(p \wedge q') \circ (p-q)\| \le \|(p \wedge q')\| \, \|p-q\| \le \|p-q\| < 1.$$

Since the projection $p \wedge q'$ must have norm 1 or 0, then $p \wedge q' = 0$. By a similar argument, $q \wedge p' = 0$. As observed before the statement of this proposition, the projections $p - p\wedge q'$ and $q - q \wedge p'$ can always be exchanged, so we conclude that p and q can be exchanged by a symmetry whenever $\|p-q\| < 1$. This completes the proof of the proposition. □

10.18. Corollary. *If F and G are orthogonal norm closed faces of the normal state space of a JBW-algebra M, with associated projections p and q, the following are equivalent.*

(i) *p and q are equivalent.*
(ii) *p and q are exchangeable by a symmetry.*
(iii) *There is a (norm continuous) path of projections from p to q.*
(iv) *There is a norm closed face H with associated reflection R_H such that R_H exchanges F and G.*

Proof. The equivalence of (i) and (ii) is Lemma 10.16. The equivalence of (i) and (iii) is Proposition 10.17. A symmetry s exchanges p and q iff U_s^* exchanges F and G (Corollary 5.17). Since the set of maps U_s^* for s a symmetry is the same as the set of reflections R_H for H a projective face (Proposition 10.4), this establishes the equivalence of (ii) and (iv). □

In a von Neumann algebra, projections can be exchanged by a finite product of symmetries iff the projections are unitarily equivalent (A 179), so the statements above correspond to unitary equivalence of F and G. For von Neumann algebras, the statement (iii) above is equivalent to there being a (norm continuous) path from F to G in the Hausdorff metric (A 196).

10.19. Definition. Let M be a JBW-algebra with normal state space K. Projective faces F, G of K are *equivalent* if their associated projections are equivalent.

Note that if F and G are orthogonal, then F and G will be equivalent iff any of the conditions (i), (ii), (iii), or (iv) of Corollary 10.18 hold.

If K is the normal state space of a JBW-algebra, we topologize the norm closed faces of K by using the 1-1 correspondence of norm closed faces and projections and carrying over the norm topology from the set of projections. Then by Corollary 10.18 orthogonal norm closed faces are equivalent iff there is a norm continuous path from one face to the other.

We are now ready for the key definition. By (A 120), the face generated by any two extreme points of the normal state space of $\mathcal{B}(H)$ is a 3-ball, and by Theorem 10.2 and Theorem 9.34, this property distinguishes the normal state space of $\mathcal{B}(H)$ among the normal state spaces of JBW-factors of type I. More generally, if two pure states on a C*-algebra are unitarily equivalent, then the face they generate in the state space will be a 3-ball (A 143). The following definition can be thought of as generalizing this property to the context of general norm closed faces, and is useful in situations where there may be no extreme points. Recall that norm closed faces F, G are *orthogonal* if they are orthogonal in the orthomodular lattice $\mathcal{F}$, i.e., if $F \subset G'$, or equivalently, if the associated projections are orthogonal.

10.20. Definition. Let K be the normal state space of a JBW-algebra M. Then K has the *generalized 3-ball property* if every norm closed face $E = F \vee G$ generated by a pair F, G of orthogonal, equivalent projective faces is a blown up 3-ball.

10.21. Lemma. *Let M be a JBW-algebra with normal state space K with the generalized 3-ball property, let p and q be orthogonal projections exchanged by a symmetry, and let $e = p+q$, and $r = p-q$. Then there is an e-symmetry s that exchanges p and q, and for each such partial symmetry s there is a partial symmetry t such that (r, s, t) is a Cartesian triple of e-symmetries.*

Proof. Let F, G be the projective faces associated with p, q respectively. By Corollary 5.17, since p and q are exchanged by a symmetry w, then U_w^* exchanges G and F, so F and G are equivalent. We can identify $F \vee G$ with the normal state space of M_e (Proposition 2.62). By the generalized 3-ball property, $F \vee G$ is a blown up 3-ball, and so by Corollary 10.13, M_e is Jordan isomorphic to the self-adjoint part of a von Neumann algebra $\mathcal{M}$. We identify M_e with $\mathcal{M}_{\mathrm{sa}}$.

Let $r = p - q$. Since p and q are exchanged by a symmetry, by Lemma 3.5 there is an e-symmetry s exchanging p and q, so by Lemma 3.6, $r \circ s = 0$. Then r, s anticommute in $\mathcal{M}$. Define $t = irs \in \mathcal{M}_{\mathrm{sa}} = M_e$. It is straightforward to verify that (r, s, t) is a Cartesian triple of e-symmetries, cf. (A 192). □

10.22. Lemma. *If M is a JBW-algebra whose normal state space K has the generalized 3-ball property, and M_1 is any direct summand of M, then the normal state space K_1 of M_1 also has the generalized 3-ball property.*

Proof. Let F and G be orthogonal equivalent norm closed faces of K_1. If p and q are the projections in M_1 corresponding to F and G, then p and q are orthogonal and equivalent in M_1, and then also in M (e.g., by Lemma 3.5). Thus F and G are equivalent in K, so $F \vee G$ is a blown up 3-ball. (Note that the latter property is the same whether $F \vee G$ is viewed as a projective face of K or of K_1, cf. Definition 10.10). □

10.23. Proposition. *If $\mathcal{M}$ is a von Neumann algebra with normal state space K, then K has the generalized 3-ball property.*

Proof. Let F and G be orthogonal equivalent projective faces of K with associated projections p and q. Let $e = p + q$. There is a symmetry that exchanges p and q (Lemma 10.16), and then by Lemma 3.5 there is an e-symmetry s that exchanges p and q. By Lemma 3.6 the e-symmetries $r = p - q$ and s satisfy $r \circ s = 0$. By direct calculation or (A 192), if we define $t = irs$, then (r, s, t) is a Cartesian triple of e-symmetries. It follows that the face $E = F \vee G$ associated with e is a blown up 3-ball (Lemma 10.9). Thus K has the generalized 3-ball property. □

10.24. Lemma. *If a JBW-algebra M has the generalized 3-ball property, then M is a JW-algebra, and is reversible in every representation as a concrete JW-algebra.*

Proof. We first show that M is a JW-algebra. Write $M = M_0 \oplus M_3$ where M_3 is the type I_3 summand of M. Then M_0 is a JW-algebra (Theorem 4.23). Let p_1, p_2, p_3 be equivalent abelian projections in M_3 with sum the identity of M_3. By Corollary 3.20, p_1 and p_2 can be exchanged by a single symmetry. By Lemma 10.21 there exists a Cartesian triple (r, s, t) of $(p_1 + p_2)$-symmetries with $r = p_1 - p_2$.

Now let π be a factor representation of M_3, and define $N = \overline{\pi(M_3)}$ (σ-weak closure). Since p_1, p_2, p_3 are exchangeable and have sum the identity (of M_3), then their images are exchangeable with sum 1. In particular, none of the images can be zero, and so $(\pi(r), \pi(s), \pi(t))$ is a non-zero Cartesian triple of partial symmetries in N. By Lemma 4.52, $H_3(\mathbf{O})$ contains no such triple, so N cannot be the exceptional JBW-factor. By Corollary 4.20, M_3 is a JW-algebra, and thus so is M.

Now assume that M is represented as a concrete JW-algebra, and let M_2 be the I_2 direct summand of M. By Corollary 4.30, M is reversible iff M_2 is reversible. By definition of type I_2, the identity of M_2 is the sum of equivalent abelian projections p and q, which can be exchanged by a symmetry. By the generalized 3-ball property and Lemma 10.21, M contains a Cartesian triple of symmetries. Thus by Lemma 4.53, M_2 is reversible, and hence M is reversible. □

10.25. Theorem. *A convex set K is affinely isomorphic to the normal state space of a von Neumann algebra iff K is spectral, elliptic, and has the generalized 3-ball property.*

Proof. The necessity of the first two of these conditions follows from Theorem 9.48, and the third follows from Proposition 10.23. Conversely, assume that K is spectral, elliptic, and has the generalized 3-ball property. From Theorem 9.48, K is affinely isomorphic to the normal state space of a JBW-algebra M. Identify K with the normal state space of M. We will show that M is isomorphic to the self-adjoint part of a von Neumann algebra.

By Lemma 10.24, we may assume that M is represented as a reversible concrete JW-subalgebra of $\mathcal{B}(H)_{\mathrm{sa}}$. Let $R_0(M)$ denote the real *-algebra generated by M; by reversibility, $R_0(M)_{\mathrm{sa}} = M$. Recall that by Lemma 10.22 the normal state space of each direct summand of M also has the generalized 3-ball property.

If M is of type I_1, then $M \cong C_{\mathbf{R}}(X)$ for a compact Hausdorff space X, so $M \cong \mathcal{M}_{\mathrm{sa}}$ where $\mathcal{M} = C_{\mathbf{C}}(X)$. Since M is monotone complete and has a separating set of normal states, then $\mathcal{M}$ is a von Neumann algebra (A 95).

If M is of type I_k for $2 \leq k \leq \infty$, or has no type I part, then by Propositions 3.24 and 3.17, the identity is the sum of a finite set $p_1, p_2, \ldots, p_n$ of n exchangeable projections, with $n \geq 2$. For $i = 1, 2, \ldots, n$ let s_i be a symmetry that exchanges p_1 and p_i. Define

$$e_{ij} = s_i p_1 s_j \quad \text{for } 1 \leq i, j \leq n.$$

By Lemma 4.28 the family $\{e_{ij}\}$ is a system of matrix units for $R_0(M)$, and $e_{ij} + e_{ji} = s_i p_1 s_j + s_j p_1 s_i \in M$ (by reversibility, or by (1.12)). Now $s = e_{12} + e_{21}$ exchanges e_{11} and e_{22}, and is a $(e_{11} + e_{22})$-symmetry. By the generalized 3-ball property and Lemma 10.21, there is a partial symmetry t such that $(e_{11}-e_{22}, e_{12}+e_{21}, t)$ is a Cartesian triple of partial symmetries. Thus we can apply Lemma 4.57 to find $\jmath$ central in $R_0(M)$ such that $\jmath^* = -\jmath$ and $\jmath^2 = -1$. Now by Lemma 4.55, M is isomorphic to the self-adjoint part of a von Neumann algebra.

Finally we consider the case where M is an arbitrary JBW-algebra. Recall that M can be written as the direct sum of JBW-algebras of type I and a JBW-subalgebra with no summand of type I (Lemma 3.16), and the former can in turn be written as the direct sum of JBW-algebras of type I_k for $1 \le k \le \infty$ (Theorem 3.23). We have shown that each of these summands is isomorphic to the self-adjoint part of a von Neumann algebra, so it follows that the same is true of M. This completes the proof of the theorem. □

Remark. If K is affinely isomorphic to the normal state space of a von Neumann algebra $\mathcal{M}$, then K determines the Jordan product on $\mathcal{M}$. In fact, the evaluation map is a bijection from $\mathcal{M}_{\mathrm{sa}}$ onto the bounded affine functions on K (A 97), and the functional calculus for such bounded affine functions was defined in Proposition 8.67. Then the Jordan product on $\mathcal{M}_{\mathrm{sa}}$ can be recovered from $\frac{1}{2}(ab+ba) = \frac{1}{4}((a+b)^2-(a-b)^2)$. However, the associative product on $\mathcal{M}$ is not determined by the affine structure of K. For example, the transpose map on $M_2(\mathbf{C})$ is a *-anti-automorphism, although its dual map is an affine automorphism of the state space. Thus some additional structure on K is needed to determine the associative product on $\mathcal{M}$. One such structure is that of an *orientation* of K (A 201). We recall:

10.26. Theorem. *If $\mathcal{M}$ is a von Neumann algebra, then there is 1-1 correspondence between Jordan compatible associative products in $\mathcal{M}$ and global orientations of the normal state space K of $\mathcal{M}$.*

Proof. [AS, Thm. 7.103]. □

For a discussion of the geometric interpretation of orientations on the normal state space of a von Neumann algebra, see the remarks at the end of Chapter 6.

State spaces of C* and von Neumann algebras can also be characterized among state spaces of Jordan algebras (JB and JBW) by the existence of a dynamical correspondence (cf. Definition 6.10), which also determines the associative product. This is explained in the proposition below, which is essentially a reformulation of information from Theorem 6.15 and Corollary 6.16, restated in terms of state spaces instead of algebras.

10.27. Proposition. *The state space K of a JB-algebra A is affinely homeomorphic to the state space of a C*-algebra $\mathcal{A}$ iff there is a dynamical* *correspondence on A. The normal state space K of a JBW-algebra M is affinely isomorphic to the state space of a von Neumann algebra $\mathcal{M}$ iff there is a dynamical correspondence on M. In these cases, the dynamical correspondence determines a unique C^*-product on $A + iA$ (respectively, W^*-product on $M + iM$).*

Proof. K is affinely homeomorphic to the state space of a C*-algebra $\mathcal{A}$ iff A is isomorphic (as a JB-algebra) to the self-adjoint part of $\mathcal{A}$ (Proposition 5.16). Furthermore, this occurs iff A admits a dynamical correspondence, and dynamical correspondences are in 1-1 correspondence with C*-products on $A + iA$ (Theorem 6.15).

The JBW-algebra result is proven in the same way by means of Corollary 6.16. □

Recall from Theorem 6.18 that there is a 1-1 correspondence between dynamical correspondences and Connes orientations (Definition 6.8) on a JBW-algebra M. Thus we also have the following characterization.

10.28. Proposition. *The normal state space K of a JBW-algebra M is affinely isomorphic to the normal state space of a JBW-algebra $\mathcal{M}$ iff there is a Connes orientation on M. In this case such a Connes orientation determines a unique W^* product on $M + iM$.*

Proof. This follows from Proposition 10.27 and Theorem 6.18. □

Note that a Connes orientation holds more information than the generalized 3-ball property, since it provides a uniquely determined von Neumann algebra whereas the W*-product constructed from the generalized 3-ball property is not unique. (It depends on the choice of symmetries used to construct matrix units in the proof of Theorem 10.25.)

Since a Connes orientation determines the associative product when the Jordan product is known, then in this respect it serves the same purpose as the concept of orientation defined for a von Neumann algebra (or its normal state space) in (A 200) and (A 201). Otherwise the two concepts of orientation have little in common. The concept of a Connes orientation is algebraic and global in nature, while the concept of orientation from [AS] is geometric and locally defined in terms of facial blown up 3-balls (A 194). The relationship between these concepts of orientation was discussed in greater detail in the last section of Chapter 6.

A dynamical correspondence provides a physically meaningful way to single out C*-algebra state spaces among those of JB-algebras. In the next chapter we will show how one can characterize state spaces of general C*-algebras by a set of geometric axioms involving a concept of "orientability" defined in terms of facial 3-balls as in (A 146), but for convex sets more general than state spaces of C*-algebras.

Notes

The geometric characterization of the normal state space of $\mathcal{B}(H)$ (Theorem 10.2) follows easily from the characterization of normal state spaces of type I JBW-factors given in Theorem 9.34, which first appeared in [10]. The characterization of von Neumann algebra state spaces (Theorem 10.25) is due to Iochum and Shultz [62]. It was actually preceded by the characterization of state spaces of C*-algebras (Theorems 11.58 and 11.59 in the next chapter), which appeared in [11]. Theorem 10.26 was announced in [13], and the proof first appeared in [AS].

In the C*-algebra theorem in Chapter 11, we will see that the Hilbert ball property used in the characterization of JB-algebra state spaces is replaced by the 3-ball property. However, this property is of no use in characterizing normal state spaces of von Neumann algebras, since there may be no extreme points for the normal state space, and thus no facial 3-balls. Therefore, the 3-ball property has been replaced by the concepts of a "blown up 3-ball" and of the "generalized 3-ball property" (Definitions 10.10 and 10.20), which were first given in [71]. There the blown up 3-balls were called "global 3-balls", and they were defined, not by the existence of a 3-frame as in this book, but by the existence of an affine surjection onto a bona fide 3-ball which pulls back the lattice of faces and the group of affine automorphisms in an appropriate way, as in Proposition 10.15.

11 Characterization of C*-algebra State Spaces

In this chapter we will reach our final goal: a characterization of the compact convex sets that are state spaces of C*-algebras. (We will assume our C*-algebras have an identity, but our characterization can easily be adapted for non-unital algebras). We will start with our previous characterization of state spaces of JB-algebras (Theorem 9.38). Then we will add two additional properties that characterize C*-state spaces among state spaces of JB-algebras.

A key axiom for state spaces of JB-algebras is the Hilbert ball property, by which the face generated by any two pure states is a norm exposed Hilbert ball (Definition 9.9). These balls may be of any finite or infinite dimension (as can be seen from Proposition 5.51). But in the case of C*-algebras, they have dimension 0, 1 or 3, with the last occurring iff the two states are distinct and unitarily equivalent, or which is the same, iff they generate the same split face, cf. (A 142) and (A 143). This is the *3-ball property* (Definition 11.17), which will be a key axiom for our characterization of state spaces of C*-algebras.

We know (from Corollary 5.42) that each pure state ρ on a JB-algebra A determines a unital Jordan homomorphism $\pi_\rho : A \to M$ where M is a JBW-factor of type I and $\pi_\rho(A)$ is σ-weakly dense in M. Briefly stated: π_ρ is a *dense type I factor representation.* The dense type I factor representations separate the points of A (Lemma 4.14), and each of them is Jordan equivalent (Definition 11.4) to π_ρ for a pure state ρ on A (Lemma 11.16).

We show in this chapter that the 3-ball property for the state space K of a JB-algebra A is equivalent to A being *of complex type*, i.e., that for each dense type I factor representation $\pi : A \to M$ we have $M \cong \mathcal{B}(H)_{\mathrm{sa}}$ with H a complex Hilbert space (Theorem 11.19). Thus if the state space K of A has the 3-ball property, then we can associate to each pure state $\rho \in K$ a concrete representation of A on a complex Hilbert space. This concrete representation is irreducible (Lemma 11.3), but it is not generally unique (up to unitary equivalence).

By associating an irreducible concrete representation to each pure state as described above, and by forming the direct sum of all these representations, we can obtain a faithful representation $\pi : A \to \mathcal{B}(H)_{\mathrm{sa}}$ on a complex Hilbert space. But this does not solve our characterization problem, since the JB-algebra $\pi(A)$ is not in general equal to the self-adjoint part of the C*-algebra it generates in $\mathcal{B}(H)$. However, after imposing a second key axiom on K, we can choose concrete representations associated with the pure states in such a way that this is in fact the case.

This second axiom involves the concept of a "global orientation", which is a direct generalization of the same notion for the state space of a C*-algebra (A 146). Thus a *global orientation* of the state space K of a JB-algebra is a continuous choice of orientation of all those faces of K that are 3-balls, and K is said to be *orientable* if it admits such a global orientation (Definition 11.50). Together, the 3-ball property and orientability will characterize state spaces of C*-algebras among all state spaces of JB-algebras.

Our proof of this result involves the non-trivial fact that a JB-algebra of complex type is a universally reversible JC-algebra (Theorem 11.26). A universally reversible JC-algebra is imbedded in its universal C*-algebra in a particularly nice fashion (Corollary 4.42), and we use this fact to show that in our case each pure state is associated with no more than two inequivalent concrete representations. More specifically, if the state space K of a JB-algebra A has the 3-ball property, then each pure state ρ on A is the restriction to A of just two pure states σ and τ on the universal C*-algebra $\mathcal{U}$ of A (with $\sigma = \tau$ iff ρ is "abelian"). Then the GNS-representations of $\mathcal{U}$ associated with σ and τ restrict to a pair of conjugate irreducible representations of A, and those are (up to unitary equivalence) the only irreducible concrete representations associated with the pure state ρ on A (Propositions 11.32 and 11.36).

Now the problem is to select for each pure state ρ on A either the one or the other of the two pure states σ and τ on $\mathcal{U}$ in such a way that the direct sum of the corresponding irreducible concrete representations maps A onto the self-adjoint part of a C*-algebra. We show that if K is orientable, then this is possible; in fact, each global orientation of K will determine such a selection, so in this case A is isomorphic to the self-adjoint part of a C*-algebra. Here the orientability condition is necessary since the state space of every C*-algebra is orientable (A 148), and is irredundant since there are examples of JB-algebras of complex type with non-orientable state space (Proposition 11.51). Thus a JB-algebra is isomorphic to the self-adjoint part of a C*-algebra iff its state space has the 3-ball property and is orientable (Theorem 11.58).

Combining this result with the characterization of state spaces of JB-algebras (Theorem 9.38), we obtain a complete geometric characterization of the compact convex sets that are state spaces of C*-algebras (Theorem 11.59). By a short argument this also gives the 1-1 correspondence of Jordan compatible C*-products and global orientations (Corollary 11.60), which was proven without use of Jordan algebra theory in (A 156).

States and representations of JB-algebras

Recall that a factor representation of a JB-algebra A is a unital homomorphism from A into a JBW-algebra such that the σ-weak closure of $\pi(A)$

is a JBW-factor (Definition 4.7). A factor representation is *faithful* if it is 1-1. Since there are JB-algebras that admit no non-zero homomorphisms into $\mathcal{B}(H)$ (e.g., the exceptional algebra $H_3(\mathbf{O})$), we need a substitute for the notion of irreducibility of a representation. For a C*-algebra $\mathcal{A}$, a representation $\pi : \mathcal{A} \to \mathcal{B}(H)$ is irreducible iff the image is σ-weakly dense, cf. (A 80) and (A 88). That motivates singling out the following class of representations.

11.1. Definition. Let A be a JB-algebra and M a JBW-algebra. A factor representation $\pi : A \to M$ is *dense* if the σ-weak closure of $\pi(A)$ equals M.

Recall from Proposition 2.68 that the σ-weak and σ-strong closures of convex sets in a JBW-algebra coincide, so we could use either topology in the definition above. In general, for subspaces X of a JBW-algebra, $\overline{X}$ will denote the σ-weak closure.

11.2. Definition. A *concrete representation* of a JB-algebra A is a Jordan homomorphism π from A into $\mathcal{B}(H)_{\text{sa}}$ for a complex Hilbert space H. Such a representation is *irreducible* if there are no non-trivial closed subspaces invariant under $\pi(A)$.

11.3. Lemma. *Let A be a JB-algebra. Then every dense concrete representation $\pi : A \to \mathcal{B}(H)_{\text{sa}}$ is irreducible.*

Proof. If p is a projection in $\mathcal{B}(H)_{\text{sa}}$ such that the closed subspace pH is invariant under $\pi(A)$, then $p\pi(a)p = \pi(a)p$ for all a in A. Then taking adjoints gives $\pi(a)p = p\pi(a)$, and by density of $\pi(A)$ in $\mathcal{B}(H)_{\text{sa}}$, p must be central in $\mathcal{B}(H)$, and thus equal to 0 or 1. Thus $\pi(A)$ has no non-trivial closed invariant subspaces. □

Note that the converse to Lemma 11.3 is not true. For example, the JB-algebra of real $n \times n$ symmetric matrices acts irreducibly on $\mathbf{C}^n$ (since the complex algebra generated includes all matrix units and thus all of $M_n(\mathbf{C})$), and yet is not dense in $M_n(\mathbf{C})_{\text{sa}}$.

Recall that representations π_1, π_2 of a C*-algebra $\mathcal{A}$ are said to be *quasi-equivalent* if there is a *-isomorphism from $\overline{\pi_1(\mathcal{A})}$ onto $\overline{\pi_2(\mathcal{A})}$ carrying π_1 to π_2 (A 136). Now we define the analogous Jordan notion.

11.4. Definition. Let A be a JB-algebra, M_1 and M_2 JBW-algebras, and let $\pi_1 : A \to M_1$ and $\pi_2 : A \to M_2$ be homomorphisms. Then π_1 and π_2 are *Jordan equivalent* if there exists a Jordan isomorphism Φ from $\overline{\pi_1(A)}$ onto $\overline{\pi_2(A)}$ such that $\Phi(\pi_1(a)) = \pi_2(a)$ for all $a \in A$.

Recall that if A is a JB-algebra and M a JBW-algebra, then each homomorphism $\pi : A \to M$ admits a unique extension to a normal homomorphism $\widetilde{\pi} : A^{**} \to M$ (Theorem 2.65). Then $\widetilde{\pi}$ is σ-weakly continuous (Proposition 2.64), so its kernel is a σ-weakly closed ideal of A^{**}. Thus by Proposition 2.39, the kernel has the form cA^{**} for some central projection c in A^{**}.

11.5. Definition. Let A be a JB-algebra, M a JBW-algebra, and $\pi : A \to M$ a homomorphism. The *central cover of* π is the central projection $c(\pi)$ in A^{**} such that $\ker \widetilde{\pi} = (1 - c(\pi))A^{**}$.

Note that $1 - c(\pi)$ is the largest central projection in A^{**} killed by $\widetilde{\pi}$, and thus $c(\pi)$ is the smallest central projection c such that $\widetilde{\pi}(c) = 1$.

Most of the remaining results in this section are close analogs of similar results for C*-algebras, cf. [AS, Chpt. 5].

11.6. Lemma. *Let A be a JB-algebra, M a JBW-algebra, $\pi : A \to M$ a homomorphism, and $\widetilde{\pi} : A^{**} \to M$ the unique normal extension. Then $\widetilde{\pi}$ restricted to $c(\pi)A^{**}$ is a *-isomorphism onto $\overline{\pi(A)} \subset M$.*

Proof. By Proposition 2.67, $\widetilde{\pi}(A^{**}) = \overline{\pi(A)}$. Since $(\ker \widetilde{\pi}) \cap c(\pi)A^{**} = (1 - c(\pi))A^{**} \cap c(\pi)A^{**} = \{0\}$, then $\widetilde{\pi}$ is faithful on $c(\pi)A^{**}$. For each element $a \in A^{**}$ we have $\widetilde{\pi}(a) = \widetilde{\pi}(c(\pi)a + (1 - c(\pi))a) = \widetilde{\pi}(c(\pi)a)$, so $\widetilde{\pi}$ maps $c(\pi)A^{**}$ onto $\widetilde{\pi}(A^{**}) = \overline{\pi(A)}$. Thus $\widetilde{\pi}$ restricted to $c(\pi)A^{**}$ is a *-isomorphism onto $\overline{\pi(A)}$. □

11.7. Proposition. *Let A be a JB-algebra and π_1 and π_2 homomorphisms of A into JBW-algebras. Then π_1 and π_2 are Jordan equivalent iff $c(\pi_1) = c(\pi_2)$.*

Proof. Let $M_i = \overline{\pi_i(A)}$ for $i = 1, 2$. Let Φ be a Jordan isomorphism of M_1 onto M_2 such that

$$\Phi(\pi_1(a)) = \pi_2(a) \tag{11.1}$$

for all a in A. Every Jordan isomorphism between JBW-algebras is normal, and therefore σ-weakly continuous (Proposition 2.64). By σ-weak continuity of $\widetilde{\pi}_1$ and $\widetilde{\pi}_2$, and σ-weak density of A in A^{**}, the equality (11.1) extends to $a \in A^{**}$ if we replace π_i by $\widetilde{\pi}_i$ for $i = 1, 2$. It follows that $\ker \widetilde{\pi}_1 = \ker \widetilde{\pi}_2$, and so by definition $c(\pi_1) = c(\pi_2)$.

Conversely, if $c(\pi_1) = c(\pi_2)$, then $\ker \widetilde{\pi}_1 = \ker \widetilde{\pi}_2$, and so we can define $\Phi : \widetilde{\pi}_1(A^{**}) \to \widetilde{\pi}_2(A^{**})$ by

$$\Phi(\widetilde{\pi}_1(x)) = \widetilde{\pi}_2(x).$$

The agreement of the kernels is exactly what is needed to make Φ well defined and a *-isomorphism. By Proposition 2.67, $\widetilde{\pi}_i(A^{**}) = \overline{\pi_i(A)} = M_i$ for $i = 1, 2$. Thus Φ is a Jordan isomorphism from M_1 onto M_2 carrying π_1 to π_2. □

Proposition 11.7 can be thought of as generalizing the fact that representations of a C*-algebra are quasi-equivalent iff their central covers coincide (A 137).

11.8. Lemma. *Let A, B be JB-algebras and $\pi : A \to B$ a Jordan homomorphism from A onto B. If J is the kernel of π and F is the annihilator of J in the state space of A, then π^* is an affine homeomorphism from the state space of B onto the split face F.*

Proof. This follows from Corollary 5.38. □

Recall that if A is a JB-algebra, each state on A has a unique extension to a normal state on A^{**}, so the restriction map is an affine isomorphism from the normal state space of A^{**} onto the state space of A (Corollary 2.61). As usual we identify the state space of A and the normal state space of A^{**}. Thus if p is a projection in A^{**}, F_p denotes the set of normal states on A^{**} (or equivalently, states on A) with value 1 on p.

11.9. Lemma. *Let A be a JB-algebra with state space K and let $\pi : A \to M$ be a dense factor representation into a JBW-algebra M. Then π^* is an affine isomorphism from the normal state space of M onto $F_{c(\pi)} \subset K$.*

Proof. Let $\widetilde{\pi} : A^{**} \to M$ be the normal extension of π. For each normal state σ on M, $\sigma \circ \widetilde{\pi}$ is the unique normal extension of $\sigma \circ \pi = \pi^* \sigma$. Thus to prove the lemma we need to show that $\sigma \mapsto \sigma \circ \widetilde{\pi}$ is an affine isomorphism from the normal state space of M onto $F_{c(\pi)}$. Since the annihilator of $\ker \widetilde{\pi}$ in K is

$$((1 - c(\pi))A^{**})^\circ \cap K = (\operatorname{im} U_{1-c(\pi)})^\circ \cap K = (\ker U^*_{1-c(\pi)}) \cap K = F_{c(\pi)}$$

(cf. equation (5.4)), the lemma follows by an argument similar to that in the proof of Corollary 5.38. □

11.10. Proposition. *Let A be a JB-algebra, and let $\pi_1 : A \to M_1$ and $\pi_2 : A \to M_2$ be dense factor representations. Let K_1 and K_2 be the normal state spaces of M_1 and M_2 respectively. Then π_1 and π_2 are Jordan equivalent iff $\pi_1^*(K_1) = \pi_2^*(K_2)$.*

Proof. We have $\pi_1^*(K_1) = \pi_2^*(K_2)$ iff $F_{c(\pi_1)} = F_{c(\pi_2)}$ (Lemma 11.9), i.e., iff $c(\pi_1) = c(\pi_2)$. This in turn holds iff π_1 and π_2 are Jordan equivalent (Proposition 11.7). □

For our purposes, we will be most interested in concrete dense representations. In this case, Jordan equivalence is closely related to unitary equivalence as we shall now see.

11.11. Definition. Let A be a JB-algebra. Then concrete representations $\pi_1 : A \to B(H_1)$ and $\pi_2 : A \to B(H_2)$ are *unitarily equivalent* (respectively, *conjugate*) if there is a complex linear (respectively, conjugate linear) isometry $u : H_1 \to H_2$ such that

$$\pi_2(a) = u\pi_1(a)u^{-1}$$

for all $a \in A$.

11.12. Proposition. *Let $\pi_1 : A \to B(H_1)_{\mathrm{sa}}$ and $\pi_2 : A \to B(H_2)_{\mathrm{sa}}$ be concrete representations of a JB-algebra A. If π_1 and π_2 are unitarily equivalent or conjugate, then they are Jordan equivalent. If π_1 and π_2 are Jordan equivalent and dense, then they are either unitarily equivalent or conjugate, with both possibilities occurring iff the representations are one dimensional.*

Proof. Suppose that π_1 and π_2 are either unitarily equivalent or conjugate. Let u be as in Definition 11.11. Define $\Phi : B(H_1) \to B(H_2)$ by $\Phi(x) = uxu^{-1}$ if u is complex linear, and by $\Phi(x) = ux^*u^{-1}$ if u is conjugate linear. In the former case, Φ is a *-isomorphism, and in the latter case it is a *-anti-isomorphism (A 124). Thus in either case Φ is a Jordan isomorphism of $B(H_1)_{\mathrm{sa}}$ onto $B(H_2)_{\mathrm{sa}}$. A Jordan isomorphism is normal, and thus Φ and Φ^{-1} are σ-weakly continuous (Proposition 2 64). Hence Φ maps the σ-weak closure of $\pi_1(A)$ onto the σ-weak closure of $\pi_2(A)$. It follows that concrete representations that are either unitarily equivalent or conjugate are also Jordan equivalent.

Let $\pi_1 : A \to B(H_1)_{\mathrm{sa}}$ and $\pi_2 : A \to B(H_2)_{\mathrm{sa}}$ be dense and Jordan equivalent. Let Φ be a Jordan isomorphism from $B(H_1)_{\mathrm{sa}}$ onto $B(H_2)_{\mathrm{sa}}$ carrying π_1 to π_2. Extend Φ to a complex linear Jordan isomorphism from $B(H_1)$ onto $B(H_2)$. By Kadison's decomposition theorem for Jordan isomorphisms of von Neumann algebras (A 158), Φ is either a *-isomorphism or a *-anti-isomorphism. Thus by (A 126) there is an isometry $v : H_1 \to H_2$ which is either complex linear or conjugate linear such that

$$\Phi(a) = vav^{-1} \quad \text{for } a \in B(H_1)_{\mathrm{sa}}.$$

Therefore

$$\pi_2(a) = \Phi(\pi_1(a)) = v\pi_1(a)v^{-1} \quad \text{for all } a \in A,$$

and so π_1 and π_2 are either unitarily equivalent or conjugate.

Finally, if π_1 and π_2 are unitarily equivalent and conjugate, then there is a *-isomorphism Φ and a *-anti-isomorphism Ψ from $B(H_1)$ onto $B(H_2)$ such that

$$\Phi(\pi_1(a)) = \Psi(\pi_1(a)) = \pi_2(a) \quad \text{for all } a \in A$$

(A 124). Then $\Psi^{-1} \circ \Phi$ fixes $\pi_1(A)$. By density of $\pi_1(A)$ in $B(H_1)_{\mathrm{sa}}$ and complex linearity, $\Psi^{-1} \circ \Phi$ is the identity on $\mathcal{B}(H_1)$. Since $\Psi^{-1} \circ \Phi$ is a *-anti-isomorphism, this implies that $\mathcal{B}(H_1)$ is abelian. Thus H_1 and H_2 must be one dimensional.

Conversely, suppose π_1 and π_2 are Jordan equivalent, and that H_1 and H_2 are one dimensional. Let $\Phi : B(H_1)_{\mathrm{sa}} \to B(H_2)_{\mathrm{sa}}$ be a Jordan isomorphism carrying π_1 to π_2. Then it is straightforward to check that any complex linear or conjugate linear isometry from H_1 onto H_2 implements Φ, so π_1 and π_2 are both unitarily equivalent and conjugate. □

Let σ be a state on a JB-algebra A. Recall that the central carrier $c(\sigma)$ is the least central projection with value 1 on σ (Definition 4.10), and that $\pi_\sigma : A \to c(\sigma)A^{**}$ is defined by $\pi_\sigma(a) = c(\sigma)a$ (Definition 4.11).

11.13. Lemma. *Let σ be a state on a JB-algebra A. Then $c(\pi_\sigma)$ equals $c(\sigma)$.*

Proof. The unique normal extension of π_σ to A^{**} (cf. Theorem 2.65) is given by $\widetilde{\pi}_\sigma(a) = c(\sigma)a$ for $a \in A^{**}$. The kernel of this extension is $(1 - c(\sigma))A^{**}$, which must by definition equal $(1 - c(\pi_\sigma))A^{**}$, and so $c(\pi_\sigma) = c(\sigma)$ follows. □

11.14. Corollary. *If A is a JB-algebra with state space K, and σ and τ are states on A, then the following are equivalent.*

(i) *π_σ and π_τ are Jordan equivalent.*
(ii) *$c(\sigma) = c(\tau)$.*
(iii) *The split faces of K generated by σ and τ coincide.*

Proof. By Proposition 11.7, π_σ and π_τ are Jordan equivalent iff $c(\pi_\sigma) = c(\pi_\tau)$. By Lemma 11.13, this is equivalent to (ii). By Proposition 5.44, the minimal split faces generated by σ and τ are $F_{c(\sigma)}$ and $F_{c(\tau)}$ respectively. By the correspondence of projections and norm closed faces (Corollary 5.33), $F_{c(\sigma)} = F_{c(\tau)}$ iff $c(\sigma) = c(\tau)$, which implies the equivalence of (ii) and (iii). □

Remark. If σ is a state on a C*-algebra $\mathcal{A}$, temporarily denote the associated GNS representation (A 63) by $\psi_\sigma : \mathcal{A} \to \mathcal{B}(H)$. Identify $(\mathcal{A}^{**})_{\mathrm{sa}}$ with $(\mathcal{A}_{\mathrm{sa}})^{**}$ (cf. Lemma 2.76). Then the central cover of the Jordan representation $\psi_\sigma | \mathcal{A}_{\mathrm{sa}}$ is $c(\sigma)$ (cf. (A 135)). Define $\pi_\sigma : \mathcal{A}_{\mathrm{sa}} \to (\mathcal{A}_{\mathrm{sa}})^{**}$ by $\pi_\sigma(a) = c(\sigma)a$. The central cover of π_σ is $c(\sigma)$ (Lemma 11.13), so by Proposition 11.7, $\psi_\sigma | \mathcal{A}_{\mathrm{sa}}$ is Jordan equivalent to π_σ.

11.15. Definition. Let A be a JB-algebra, and M a JBW-algebra. A homomorphism $\pi : A \to M$ is a *type I factor representation* if the σ-weak closure of $\pi(A)$ is a type I JBW-factor.

Recall that if σ is a pure state, then π_σ is a dense type I factor representation, i.e., $\overline{\pi_\sigma(A)}$ is a type I JBW-factor (Corollary 5.42). In fact, up to Jordan equivalence all dense type I factor representations arise in this way, as we now show. Recall that F_σ denotes the smallest split face containing σ.

11.16. Lemma. *Let A be a JB-algebra with state space K, and $\pi : A \to M$ a dense type I factor representation into a JBW-algebra M. Then $F_{c(\pi)}$ contains at least one pure state, and for every such pure state σ in $F_{c(\pi)}$, π is Jordan equivalent to π_σ, and $F_{c(\pi)} = F_\sigma$.*

Proof. By Lemma 11.9, π^* is an affine isomorphism from the normal state space of M onto $F_{c(\pi)}$. Since M is of type I, by Corollary 5.41 the normal state space of M contains an extreme point, and therefore so does $F_{c(\pi)}$.

Let σ be an extreme point of $F_{c(\pi)}$. Since M is a factor, it contains no non-trivial central projections, and thus its normal state space contains no proper split face (Corollary 5.35). Hence the split face $F_\sigma = F_{c(\sigma)}$ generated by σ must equal $F_{c(\pi)}$, so $c(\sigma) = c(\pi)$. Since $c(\sigma) = c(\pi_\sigma)$ (Lemma 11.13), then $c(\pi) = c(\pi_\sigma)$. Therefore π is Jordan equivalent to π_σ (Proposition 11.7). $\square$

Reversibility of JB-algebras of complex type

We now define one of the key properties that distinguishes C*-algebra state spaces. Recall that a *3-ball* is a convex set affinely isomorphic to the closed unit ball $\mathbf{B}^3$ of $\mathbf{R}^3$ (Definition 10.1). In the case of C*-algebras, the faces generated by distinct pure states are either line segments (iff the states are unitarily inequivalent), or 3-balls (A 143). We now abstract this property.

Recall that for the state space of a JB-algebra, the face generated by any pair of pure states is a Hilbert ball (Corollary 5.56), but that this Hilbert ball might be of any dimension (cf. Proposition 5.51). If σ and τ are distinct pure states, the split faces F_σ and F_τ generated by σ and τ will be minimal split faces (Lemma 5.46). These must either coincide or be disjoint, since the intersection of split faces is also a split face (A 7). The face generated by σ and τ will degenerate to a line segment iff F_σ and F_τ are disjoint (Lemma 5.54), or equivalently, iff π_σ and π_τ are not Jordan equivalent (Corollary 11.14).

11.17. Definition. If K is a convex set, then K has the *3-ball property* if the face generated by every pair of extreme points of K is norm exposed, and is either a point, a line segment or a 3-ball.

Note that in Definition 11.17 we allow the two extreme points to coincide, in which case the generated face will be a single norm exposed extreme point. Note also that the convex sets with the 3-ball property are a subclass of the convex sets with the Hilbert ball property (for which every extreme point is norm exposed, cf. Definition 9.9). Of course, the 3-ball property will be useful only when there are many extreme points, e.g., when K is compact, or K is the normal state space of an atomic JBW-algebra (cf. Lemma 5.58).

11.18. Definition. A JB-algebra is of *complex type* if for every dense type I factor representation $\pi : A \to M$ we have $M \cong \mathcal{B}(H)_{\mathrm{sa}}$ for a complex Hilbert space H.

11.19. Theorem. *A JB-algebra A is of complex type iff its state space has the 3-ball property.*

Proof. Suppose first that A is of complex type, and let σ and τ be pure states. If σ, τ are separated by a split face, then face(σ,τ) is the line segment $[\sigma,\tau]$, cf. Lemma 5.54 or (A 29).

Now suppose σ and τ are not separated by any split face, and let $F = F_\sigma = F_\tau$. Let $\pi_\sigma : A \to c(\sigma)A^{**}$ be multiplication by $c(\sigma)$. Then π_σ is a type I factor representation (Corollary 5.42). Since A is of complex type, then $c(\sigma)A^{**} \cong \mathcal{B}(H)_{\mathrm{sa}}$. The normal state space of $c(\sigma)A^{**}$ is $F = F_{c(\sigma)} = F_\sigma$ (Proposition 2.62). Hence we conclude that F is affinely isomorphic to the normal state space of $\mathcal{B}(H)_{\mathrm{sa}}$. The latter has the 3-ball property (A 120), and thus so does F. The face of K generated by τ and σ is also a face of F, and so we conclude that it is a 3-ball. Thus we have shown that K has the 3-ball property.

Conversely, assume K has the 3-ball property, and let M be a JBW-algebra, and $\pi : A \to M$ a dense type I factor representation. Then $M = \overline{\pi(A)}$ is a type I JBW-factor. By Lemma 11.9, π^* is an affine isomorphism from the normal state space of M onto $F_{c(\pi)}$. The normal state space of M is a split face of K, and thus inherits the 3-ball property from K.

Let σ and τ be distinct extreme points of the normal state space of M. Since M is a factor, its normal state space has no proper split faces (Corollary 5.35). Then σ and τ cannot be separated by a split face, so face(σ,τ) is not a line segment (Lemma 5.54). By the 3-ball property, face(σ,τ) is a 3-ball. Thus M is a type I JBW-factor for which the face of the normal state space generated by each pair of distinct pure normal states is a 3-ball. Therefore M is isomorphic to $\mathcal{B}(H)_{\mathrm{sa}}$ for some complex Hilbert space H (Proposition 9.36). Hence A is a JB-algebra of complex type. $\square$

11.20. Corollary. *The self-adjoint part of a C*-algebra is a JB-algebra of complex type.*

Proof. This follows from Theorem 11.19 and the fact that the state space of a C*-algebra has the 3-ball property, cf. (A 143) and (A 108). □

The following simple observation will be used frequently.

11.21. Lemma. *Every JBW-algebra M of type I_2 contains a pair of Jordan orthogonal symmetries.*

Proof. By the definition of type I_2 (Definition 3.21), there are exchangeable projections p_1 and p_2 in M with sum 1 (Corollary 3.20). Let s be a symmetry that exchanges p_1 and p_2. Then by Lemma 3.6, s and the symmetry $r = p_1 - p_2$ are Jordan orthogonal. □

Our next lemma involves Jordan polynomials in several variables. Such a polynomial is a sum of products, just like a standard polynomial. But by non-associativity, parentheses are needed to indicate how these products are to be multiplied out. Therefore each standard polynomial corresponds to a whole class of Jordan polynomials (obtained by inserting parentheses). Clearly all those take the same values as the given standard polynomial on every associative subalgebra, and then in particular on the center.

11.22. Lemma. *Let S be any spin factor and let s_1 and s_2 be any two Jordan orthogonal symmetries in S. For each natural number k we can find a Jordan polynomial P_k in $k+2$ variables, with P_k independent of the choices s_1, s_2, such that k elements $a_1, \ldots, a_k$ of S are linearly dependent iff*

$$P_k(s_1, s_2, a_1, \ldots, a_k) = 0. \tag{11.2}$$

Proof. Recall that $S = \mathbf{R}1 \oplus N$ (vector space direct sum), where N is a real Hilbert space and the Jordan product in S is defined by

$$(\lambda 1 + x) \circ (\mu 1 + y) = \big(\lambda\mu + (x|y)\big)1 + (\lambda y + \mu x) \tag{11.3}$$

for $x, y \in N$, and that N consists of all scalar multiples of symmetries in S other than ± 1 (Lemma 3.34).

We extend the inner product from N to all of S by setting

$$(\lambda 1 + x \mid \mu 1 + y) = \lambda\mu + (x|y) \tag{11.4}$$

for $x, y \in N$, so that S becomes the Hilbert space direct sum of $\mathbf{R}1$ and N. By the Gram criterion for inner product spaces, k elements $a_1, \ldots, a_k$

of S are linearly dependent iff $\det((a_i|a_j)) = 0$. We will now use this criterion to construct P_k.

Observe first that if $a_i = \lambda_i 1 + x_i$ with $x_i \in N$ and $\lambda_i \in \mathbf{R}$ for $i = 1, \ldots, k$, then by (11.4)

$$(a_i|a_j)1 = (\lambda_i 1) \circ (\lambda_j 1) + x_i \circ x_j \quad \text{for } i, j = 1, \ldots, k. \tag{11.5}$$

Furthermore, for any $a = \lambda 1 + x$ with $x \in N$ and $\lambda \in \mathbf{R}$, multiplying out by means of (11.3) gives

$$(((a \circ s_1) \circ s_1) \circ s_2) \circ s_2 = \lambda 1, \tag{11.6}$$

and then also

$$x = a - (((a \circ s_1) \circ s_1) \circ s_2) \circ s_2. \tag{11.7}$$

Now it follows from (11.5), (11.6) and (11.7) that there is a Jordan polynomial Q such that

$$(a_i|a_j)1 = Q(s_1, s_2, a_i, a_j) \quad \text{for } i, j = 1, \ldots, k. \tag{11.8}$$

Choose a Jordan polynomial f in k^2 variables which takes the same values as the determinant of order k on the center $\mathbf{R}1$ of S, i.e.,

$$f(\alpha_{11}1, \alpha_{12}1, \ldots, \alpha_{kk}1) = \det(\alpha_{ij})1$$

for all scalars α_{ij} with $1 \leq i, j \leq k$. (See the remark in the paragraph preceding this lemma). Substitute $Q(s_1, s_2, a_i, a_j)$ for the $(ij)^{th}$ variable of f for $1 \leq i, j \leq k$, and then denote the resulting Jordan polynomial by P_k. By (11.8),

$$P_k(s_1, s_2, a_1, \ldots, a_k) = \det((a_i|a_j))1.$$

Thus $a_1, \ldots, a_k$ are linearly dependent iff $P_k(s_1, s_2, a_1, \ldots, a_k) = 0$. □

Recall that the JB norm and the Hilbert space norm on a spin factor are equivalent (cf. equation (3.10)), so closed subspaces for the two norms coincide. Observe for use in the next proof that if S is a spin factor and B is a norm closed subspace of dimension 3 or more, containing the identity 1, then B is also a spin factor for the inherited product and norm. In fact, if $S = \mathbf{R}1 \oplus N$ where N is a real Hilbert space, then B is seen to be the spin factor $\mathbf{R}1 \oplus (B \cap N)$.

11.23. Lemma. *Let A be a JB-algebra of complex type, and c the central projection such that cA^{**} is the type I_2 summand of A^{**}. Then every factor representation of cA^{**} is onto a spin factor of dimension at most 4.*

Proof. Let π be a factor representation of cA^{**}, and let $S = \overline{\pi(cA^{**})}$ Note that S is of type I_2 (Lemma 4.8), and thus is a spin factor (Proposition 3.37). We must show S has dimension 4 or less.

We will first show $\pi(cA)$ has dimension 4 or less. Suppose $\pi(cA)$ has dimension 4 or more. The homomorphic image $\pi(cA)$ of the JB-algebra cA is a norm closed unital subalgebra of the spin factor S (Proposition 1.35), so it is itself a spin factor. (See the observation preceding this lemma.) Therefore $a \mapsto \pi(ca)$ is a dense type I factor representation mapping A onto the spin factor $\pi(cA)$. Since A is of complex type, $\pi(cA) \cong \mathcal{B}(H)_{\mathrm{sa}}$ for a complex Hilbert space H. But $\mathcal{B}(H)_{\mathrm{sa}}$ is a spin factor iff H is two dimensional (Theorem 3.39). Hence $\pi(cA)$ is four dimensional.

We have now shown that $\pi(cA)$ is of dimension 4 or less. It remains to prove the same result for $\pi(cA^{**})$ in place of $\pi(cA)$. Since cA^{**} is of type I_2, it contains a pair r, s of Jordan orthogonal c-symmetries (Lemma 11.21). Then since π is unital, $\pi(r)$ and $\pi(s)$ are seen to be Jordan orthogonal symmetries in S. Let $a_1, \ldots, a_5 \in A$. Since the dimension of $\pi(cA)$ is 4 or less, then $\pi(a_1), \ldots, \pi(a_5)$ will be linearly dependent in S, so by Lemma 11.22, with P_5 as described in Lemma 11.22,

$$\pi(P_5(r, s, a_1, \ldots, a_5)) = P_5(\pi(r), \pi(s), \pi(a_1), \ldots, \pi(a_5)) = 0. \tag{11.9}$$

Since π is an arbitrary factor representation of cA^{**} and the factor representations of a JB-algebra separate points (Lemma 4.14), then

$$P_5(r, s, a_1, \ldots, a_5) = 0. \tag{11.10}$$

By the Kaplansky density theorem for JB-algebras (Proposition 2.69), the unit ball of A is σ-strongly dense in the unit ball of A^{**}, so by σ-strong continuity of Jordan multiplication on bounded sets (Proposition 2.4), the equality (11.10) will hold not only for arbitrary $a_1, \ldots, a_5$ in cA, but also for all $a_1, \ldots, a_5$ in cA^{**}. Therefore (11.9) will hold for every factor representation π of cA^{**} and all $a_1, \cdots, a_5$ in cA^{**}. By Lemma 11.22, $\pi(a_1), \ldots, \pi(a_5)$ are linearly dependent, so the dimension of $\pi(cA^{**})$ is at most 4. We are done. □

Recall that an element x of a JBW-algebra M is central iff $U_s x = x$ for all symmetries s in M (Lemma 2.35). We need the following slightly stronger result for JBW-algebras of type I_2.

11.24. Lemma. *Let M be a JBW-algebra of type I_2, and let r, s be a pair of Jordan orthogonal symmetries in M. If $x \in M$ satisfies $U_r x = U_s x = x$, then x is central in M.*

Proof. Let π be a factor representation of M, and $S = \overline{\pi(M)}$. We will show $\pi(x)$ is central in S.

By Lemma 4.8, S is of type I_2, and thus is a spin factor. Recall that $S = \mathbf{R}1 \oplus N$ where N consists of all scalar multiples of symmetries in S other than ± 1 (Lemma 3.34). Since π is unital, $\pi(r)$ and $\pi(s)$ are Jordan orthogonal symmetries. Now $\pi(r) \circ \pi(s) = 0$ implies that neither $\pi(r)$ nor $\pi(s)$ can equal ± 1, and so in particular $\pi(r) \in N$. Thus we can write

$$\pi(x) = \lambda 1 + \alpha \pi(r) + w, \tag{11.11}$$

where $\alpha, \lambda \in \mathbf{R}$, and $w \in N$ is orthogonal (and then Jordan orthogonal) to $\pi(r)$. By Lemma 3.6, $U_{\pi(r)} w = -w$. By hypothesis, $x = U_r x = \{rxr\}$, and since π preserves Jordan products, then $\pi(x) = U_{\pi(r)} \pi(x)$. Substituting (11.11) into this last equality gives

$$\lambda 1 + \alpha \pi(r) + w = \lambda 1 + \alpha \pi(r) - w.$$

Thus $w = 0$, so $\pi(x) \in \mathbf{R}1 + \mathbf{R}\,\pi(r)$. Similarly $\pi(x) \in \mathbf{R}1 + \mathbf{R}\,\pi(s)$, so $\pi(x) \in \mathbf{R}1$. Thus $\pi(x)$ is central for all factor representations of M, and therefore x is central in M. $\square$

We will now prove a technical result that will play a key role in showing that JB-algebras of complex type are universally reversible, cf. Definition 4.32. We begin by listing a few elementary general facts.

By common usage, $\circ$ is often omitted when multiplying by a central element in a Jordan algebra. (See the remarks following Proposition 1.52). Let q be a central projection in a JBW-algebra M. If x is a general element of M, then

$$qx = q \circ x = \{qxq\} = U_q x. \tag{11.12}$$

(cf. (1.65)). Furthermore, multiplication by q is a unital Jordan homomorphism from M onto the JBW-algebra M_q (cf. (1.66)).

Note also that if r and s are orthogonal symmetries in a JB-algebra, then U_r and U_s will commute. In fact, $U_r s = -s$ (Lemma 3.6), so by the identity (1.16)

$$U_s = U_{-s} = U_{\{rsr\}} = U_r U_s U_r, \tag{11.13}$$

and then multiplying both sides by U_r gives $U_r U_s = U_s U_r$.

Recall that every JBW-algebra of type I_2 contains a pair of Jordan orthogonal symmetries (Lemma 11.21).

11.25. Lemma. *Let M be a JBW-algebra of type I_2, and let r, s be a pair of Jordan orthogonal symmetries in M. If M admits a homomorphism onto the four dimensional spin factor, then M contains a central projection q and an element t such that qr, qs, t are Jordan orthogonal q-symmetries.*

Proof. Let $\pi : M \to S$ be a representation onto a four dimensional spin factor S. Recall that $S = \mathbf{R}1 \oplus N$, where N is the span of the symmetries other than ± 1, and N is a three dimensional Hilbert space such that $a \circ b = (a|b)1$ for $a, b \in N$.

We begin by constructing an element $y \in M$, Jordan orthogonal to r and s. Note that $\pi(r)$ and $\pi(s)$ are not equal to ± 1 (since $\pi(r)$ and $\pi(s)$ are Jordan orthogonal symmetries), and so $\pi(r)$ and $\pi(s)$ are in N. Let $v \in N$ be a non-zero vector orthogonal to $\pi(r)$ and $\pi(s)$, and choose $x \in M$ such that $\pi(x) = v$. By the definition of the Jordan product on S, $\pi(x) = v$ is Jordan orthogonal to $\pi(r)$ and $\pi(s)$. Note then by Lemma 3.6, $U_{\pi(r)}\pi(x) = -\pi(x)$ and $U_{\pi(s)}\pi(x) = -\pi(x)$. Thus

$$(1 - U_{\pi(r)})(1 - U_{\pi(s)})\pi(x) = 4\pi(x) \neq 0. \tag{11.14}$$

Define

$$y = (1 - U_r)(1 - U_s)x. \tag{11.15}$$

By (11.14), y is not zero. Since $U_r^2 = Id$, then $U_r y = -y$.

Now (11.15), together with the fact that U_r and U_s commute (see the remarks preceding this lemma), implies that $U_s y = -y$. Then $U_r y = -y$ and $U_s y = -y$ imply that y is Jordan orthogonal to r and s (Lemma 3.6). Thus we have succeeded in finding an element of M Jordan orthogonal to r and s.

Next we construct a non-zero central projection q, and a q-symmetry $t \in M$ Jordan orthogonal to r and s. Define

$$t = r(y^+) - r(y^-) \quad \text{and} \quad q = r(y^+) + r(y^-),$$

where $r(a)$ denotes the range projection of a positive element a (cf. Definition 2.14), and $y = y^+ - y^-$ is the unique orthogonal decomposition of y (Proposition 1.28). By Proposition 2.16, $r(y^+)$ and $r(y^-)$ are orthogonal, and since $y \neq 0$, they are not both zero. Hence q is a non-zero projection, and t is a q-symmetry. Since U_r and U_s are Jordan automorphisms of M (Proposition 2.34), they preserve orthogonal decompositions. The unique orthogonal decomposition of $-y$ is $y^- - y^+$, and as shown above U_r exchanges y with $-y$, so it follows that U_r exchanges y^+ and y^-, and then also exchanges their range projections. Hence $U_r t = -t$, and similarly $U_s t = -t$. Thus t is a q-symmetry, Jordan orthogonal to r and s.

We finally show that q is central. Since U_r and U_s exchange $r(y^+)$ and $r(y^-)$, then by the definition of q,

$$U_r q = q \quad \text{and} \quad U_s q = q. \tag{11.16}$$

Thus q is central (Lemma 11.24), and so $a \mapsto qa$ is a unital Jordan homomorphism from M onto the JBW-algebra M_q. A unital Jordan homomorphism preserves Jordan orthogonality, so qr, qs, $qt = t$ are Jordan orthogonal. A unital homomorphism also takes symmetries to symmetries, so qr, qs, $qt = t$ are the desired Jordan orthogonal q-symmetries. □

The following theorem is a key property of JB-algebras of complex type that will play a crucial role in what follows. For use in the proof below, we review the notion of a Cartesian triple (Definition 4.50). If e is a projection in a JBW-algebra M, a *Cartesian triple of e-symmetries* is a triple (r, s, t) such that (i) $r \circ s = s \circ t = t \circ r = 0$, (ii) $r^2 = s^2 = t^2 = e$, and (iii) $U_r U_s U_t$ is the identity on M_e. Any JBW-algebra admitting a Cartesian triple of symmetries is a JW-algebra, and is reversible in any faithful representation as a concrete JW-algebra (Lemma 4.53).

The condition $U_r U_s U_t = I$ in the definition of a Cartesian triple of symmetries is not redundant. (A counterexample is given in the remark after [AS, Thm. 7.29]). But note for use in the next proof that it is redundant in case M is a four dimensional spin factor. Indeed, if $M = \mathbf{R}1 \oplus N$ where N is a three dimensional Hilbert space, then by using the general fact that $U_r s = -s$ for any pair r, s of Jordan orthogonal symmetries (Lemma 3.6), and the fact that a given Jordan orthogonal triple r, s, t of symmetries is a basis for N, one readily computes that $U_r U_s U_t x = x$ for $x \in N$, so that $U_r U_s U_t a = a$ for every $a = \lambda 1 + x \in N$.

11.26. Theorem. *Every JB-algebra A of complex type is a universally reversible JC-algebra.*

Proof. We first show that A is a JC-algebra. For each pure state σ of the state space of A, the map $\pi_\sigma : A \to c(\sigma)A^{**}$ is a dense type I factor representation (Corollary 5.42), and such representations separate elements of A by Lemma 4.14. Since A is of complex type, then by definition $c(\sigma)A^{**}$ must be isomorphic to $\mathcal{B}(H)_{\mathrm{sa}}$ for some complex Hilbert space H. The direct sum of such representations then provides a faithful representation of A as a JC-algebra.

The bidual of a JC-algebra is a JW-algebra, so A^{**} is a JW-algebra (Proposition 2.77). We will show that A^{**} is reversible in any faithful representation as a concrete JW-algebra on a Hilbert space H. By Lemma 4.33, this implies that A is universally reversible, which will complete the proof. Assume $A^{**} \subset \mathcal{B}(H)_{\mathrm{sa}}$. Let c be the central projection such that cA^{**} is the I_2 summand of A^{**}. By Corollary 4.30 it suffices to prove that

cA^{**} is reversible. We are going to show cA^{**} contains a Cartesian triple of symmetries, and thus is reversible (Lemma 4.53), which will complete the proof.

Let $\{(r_\alpha, s_\alpha, t_\alpha)\}$ be a maximal collection of Jordan orthogonal triples of partial symmetries in cA^{**} such that the projections $c_\alpha = r_\alpha^2 = s_\alpha^2 = t_\alpha^2$ are orthogonal and central. Let $r_0 = \sum_\alpha r_\alpha$, $s_0 = \sum_\alpha s_\alpha$, $t_0 = \sum_\alpha t_\alpha$, $c_0 = \sum_\alpha c_\alpha$. (For each of these sums, the terms live on orthogonal subspaces of H, so the sums converge strongly. Therefore the sums also converge σ-weakly in $\mathcal{B}(H)_{\mathrm{sa}}$, and then in M as well.) Then r_0, s_0, t_0 are Jordan orthogonal c_0-symmetries. We will show that $c_0 = c$, and then show that r_0, s_0, t_0 form a Cartesian triple.

Let $w = c - c_0$ and suppose that w is not zero. Since wA^{**} is a direct summand of cA^{**}, it is also of type I_2. We now show that wA^{**} admits a homomorphism onto the four dimensional spin factor. Suppose not, and let π be any factor representation of wA^{**}. Since wA^{**} is of type I_2, then $S = \overline{\pi(wA^{**})}$ (σ-weak closure) is an I_2 factor (Lemma 4.8), i.e., a spin factor. Then $a \mapsto \pi(wa)$ is a spin factor representation of cA^{**}, so by Lemma 11.23 the dimension of S is 4 or less. We are assuming the dimension of S is not 4, so the dimension is 3 or less. Since A is of complex type, it has no spin factor representations onto the three dimensional spin factor. Hence $\pi(wA)$ is of dimension 2 or less. Then $\pi(wA)$ is spanned by the identity and at most one other element, so must be associative. Thus $\pi(wA)$ is associative for all factor representations π of wA^{**}. Since factor representations of wA^{**} separate points, we conclude that wA is associative. Then wA^{**} must also be associative (by σ-weak density of wA in wA^{**} and separate σ-weak continuity of Jordan multiplication). This contradicts wA^{**} being of type I_2. Thus wA^{**} admits at least one representation π onto a four dimensional spin factor.

Let r, s be a pair of Jordan orthogonal w-symmetries in wA^{**} (cf. Lemma 11.21). Since wA^{**} admits a homomorphism onto the four dimensional spin factor, wA^{**} contains a non-zero central projection q and an element t such that qr, qs, t are Jordan orthogonal q-symmetries (Lemma 11.25). This contradicts the maximality of the collection $\{(r_\alpha, s_\alpha, t_\alpha)\}$, and thus proves that $w = 0$. Hence $c_0 = c$.

We finally show that r_0, s_0, t_0 form a Cartesian triple. All that remains is to show that $U_{r_0}U_{s_0}U_{t_0}$ is the identity on cA^{**}. We prove this by considering factor representations of cA^{**}. By Lemma 11.23, an arbitrary factor representation π maps cA^{**} onto a spin factor S of dimension 4 or less. Thus $S = \mathbf{R}1 \oplus N$ where N is a real Hilbert space of dimension 3 or less. In fact, N must be of dimension 3, since $\pi(r_0)$, $\pi(s_0)$, $\pi(t_0)$ are Jordan orthogonal symmetries in S, hence orthogonal vectors in N. Therefore S is four dimensional. Now it follows from the observation in the paragraph before the theorem that $(\pi(r_0), \pi(s_0), \pi(t_0))$ will be a Cartesian triple in S. Therefore for all $x \in cA^{**}$,

$$\pi(U_{r_0}U_{s_0}U_{t_0}x) = U_{\pi(r_0)}U_{\pi(s_0)}U_{\pi(t_0)}\pi(x) = \pi(x).$$

Hence $U_{r_0}U_{s_0}U_{t_0} = Id$ on cA^{**}.

This completes the proof that cA^{**} admits a Cartesian triple. We are done. □

We have worked hard to prove reversibility of JB-algebras of complex type. The following corollary is the key consequence of reversibility that we will use in this chapter. Recall (Definition 4.37) that with each JB-algebra there is associated its universal C*-algebra $C_u^*(A)$, and a canonical homomorphism from A into $C_u^*(A)$ such that the image of A generates $C_u^*(A)$. If A is a JC-algebra, this homomorphism is 1-1, and we identify A with its image under this imbedding. There is also a canonical *-anti-automorphism Φ of $C_u^*(A)$ of period 2, which fixes A (Proposition 4.40).

11.27. Corollary. *Let A be a JB-algebra of complex type, and $C_u^*(A)$ the universal C*-algebra with canonical *-anti-automorphism Φ. Then A coincides with the set of self-adjoint fixed points of Φ.*

Proof. A is universally reversible by Theorem 11.26, and thus $A = C_u^*(A)_{\mathrm{sa}}^{\Phi}$ by Corollary 4.42. □

We will now develop some properties of representations of JB-algebras of complex type. Recall that a dense concrete representation of a JB-algebra is irreducible (Lemma 11.3). We have already observed that the converse is not true in general, but now we show that the converse holds for JB-algebras of complex type.

11.28. Proposition. *Let A be a JB-algebra of complex type. If $\pi : A \to \mathcal{B}(H)_{\mathrm{sa}}$ is an irreducible concrete representation, then π is dense, i.e., the σ-weak closure of $\pi(A)$ equals $\mathcal{B}(H)_{\mathrm{sa}}$.*

Proof. Let $M = \overline{\pi(A)}$ (σ-weak closure). We will show first that M is a JW-factor, then that it is reversible, and finally that it is of type I. Since A is of complex type, then it will follow that $M \cong B(K)_{\mathrm{sa}}$ for some Hilbert space K. Since M is irreducible, Proposition 4.59 will imply that $M = \mathcal{B}(H)_{\mathrm{sa}}$.

If there were a non-trivial central projection c in M, then cH would be a closed invariant subspace for M, contradicting the irreducibility of $\pi(A)$. Thus M is a JW-factor.

We next show that M is reversible in every representation as a concrete JW-algebra. Recall that a concrete JW-algebra is reversible iff its I_2 summand is reversible (Corollary 4.30). Since M is a factor, we need only deal with the case where M is of type I_2, i.e., a spin factor. In this case M is a type I factor, and A is of complex type, so we must have $M \cong B(K)_{\mathrm{sa}}$ for some Hilbert space K. The self-adjoint part of a C*-algebra is reversible in every concrete representation (Proposition 4.34), so M is reversible in every concrete representation. This completes the proof that the I_2 summand of M is reversible in every concrete representation, and thus that M is reversible in every faithful representation as a concrete JW-algebra. We note for use below that in particular M is reversible in its universal von Neumann algebra $W_u^*(M)$ (Definition 4.43).

Finally we show that M is a type I factor by showing that M contains a minimal projection. We start by showing that $W_u^*(M)$ contains a minimal projection. Let $\mathcal{M}$ be the von Neumann algebra generated by M in $\mathcal{B}(H)$. Then $\mathcal{M}$ is irreducible, and so its commutant is $\mathbf{C}1$ (A 80). Thus, by the bicommutant theorem (A 88), $\mathcal{M} = \mathcal{B}(H)$. By the defining property of $W_u^*(M)$, the inclusion of M in $\mathcal{B}(H)_{\mathrm{sa}}$ can be lifted to a normal *-homomorphism π from $W_u^*(M)$ onto $\mathcal{B}(H)$. Then the kernel of π is a σ-weakly closed ideal of $W_u^*(M)$, and so there is a central projection c in $W_u^*(M)$ such that the kernel of π is $(1-c)W_u^*(M)$ (A 105). Thus π is a *-isomorphism from $cW_u^*(M)$ onto $\mathcal{B}(H)$. It follows that $cW_u^*(M)$ contains a minimal projection p. Then p also is minimal in $W_u^*(M)$.

Now we show that M contains a minimal projection. Let Φ be the canonical *-anti-isomorphism of $W_u^*(M)$. Since M is reversible in $W_u^*(M)$, by Proposition 4.45 the set of self-adjoint fixed points of Φ is M. Let $q = p \vee \Phi(p)$ (the least upper bound in the projection lattice of $W_u^*(M)$). Since Φ is a *-anti-isomorphism, it is also a Jordan isomorphism and an order isomorphism, and thus is a lattice isomorphism on the projection lattice. Since Φ is of period 2, $\Phi(q) = \Phi(p) \vee p = q$. Thus q is in M. Since p and $\Phi(p)$ are minimal projections ("atoms"), in the lattice of projections of $W_u^*(M)$, the lattice-theoretic dimension of $q = p \vee \Phi(p)$ is at most 2, cf. (A 42) (applied to the atomic part of $W_u^*(M)$, cf. Lemma 3.42). If q is an atom in M we are done. If not, let v be any non-zero projection in M with $v < q$. By Proposition 3.51, v is a finite projection in $W_u^*(M)$ with $\dim(v) < 2$. Thus v is an atom in $W_u^*(M)$, and therefore also is a minimal projection in M. This completes the proof that M is a type I factor. As remarked at the start of this proof, this shows that $M = \mathcal{B}(H)_{\mathrm{sa}}$ and thus that π is dense. □

Next we study the relationship of pure states and irreducible representations for JB-algebras. Recall that if K is any convex set, then $\partial_e K$ denotes the set of extreme points of K. Recall also that if σ is a pure state of a C*-algebra $\mathcal{A}$ and π_σ the associated GNS representation (A 63), then there is a unit vector ξ such that $\sigma(a) = (\pi_\sigma(a)\xi \mid \xi)$ for all $a \in \mathcal{A}$. Such a vector ξ is said to be a *representing vector* for σ.

11.29. Definition. Let A be a JB-algebra of complex type, and σ a pure state on A. An irreducible concrete representation $\pi : A \to \mathcal{B}(H)_{\text{sa}}$ is *associated with* σ if there is a unit vector ξ in H such that

$$\sigma(a) = (\pi(a)\xi \mid \xi) \quad \text{for all } a \in A. \tag{11.17}$$

Then we also say that σ is *associated with* π. By Proposition 11.28, $\pi(A)$ is σ-weakly dense in $\mathcal{B}(H)_{\text{sa}}$, and so the vector ξ is uniquely determined up to multiplication by a scalar of modulus 1. We will say the vector ξ *represents* σ *with respect to* π.

Recall that we denote the vector state on $\mathcal{B}(H)$ associated with a vector ξ by ω_ξ.

11.30. Lemma. *Let A be a JB-algebra of complex type, $\pi : A \to \mathcal{B}(H)_{\text{sa}}$ an irreducible concrete representation. The pure states associated with π are precisely the states of the form $\omega_\xi \circ \pi$ for unit vectors $\xi \in H$. Furthermore, π is associated with a pure state σ iff $c(\pi) = c(\sigma)$, or equivalently, iff σ is an extreme point of $F_{c(\pi)}$. In that case, π^* is an affine isomorphism from the normal state space of $\mathcal{B}(H)_{\text{sa}}$ onto the split face F_σ generated by σ.*

Proof. By definition, π is associated with σ iff there is a vector state ω_ξ such that $\sigma = \pi^*\omega_\xi (= \omega_\xi \circ \pi)$. By Proposition 11.28, π is a dense type I factor representation, and so π^* is an affine isomorphism of the normal state space of $\mathcal{B}(H)_{\text{sa}}$ onto $F_{c(\pi)}$ (Lemma 11.9). Since the pure normal states on $\mathcal{B}(H)$ are the vector states (A 118), the image of the set of vector states under π^* is precisely the set of extreme points of $F_{c(\pi)}$; these are then the pure states with which π is associated. In particular, there is at least one pure state associated with π.

Since $\mathcal{B}(H)_{\text{sa}}$ is a factor, its normal state space has no non-trivial split faces (Corollary 5.35). Thus the split face generated by any state in $F_{c(\pi)}$ is $F_{c(\pi)}$. Since the split face generated by σ is $F_{c(\sigma)}$ (Proposition 5.44), then a pure state σ is in $F_{c(\pi)}$ (and therefore is associated with π) iff $c(\sigma) = c(\pi)$. □

The state space of the universal C*-algebra of a JB-algebra

In this section, A will be a JB-algebra of complex type, with state space K. We will let $\mathcal{U}$ denote the universal C*-algebra for A (cf. Definition 4.37), and $\mathcal{K}$ will denote the state space of $\mathcal{U}$. Φ will be the canonical *-anti-automorphism of $\mathcal{U}$ (Proposition 4.40), and $r : \mathcal{K} \to K$ will be the restriction map. Recall that Φ has period 2, and that the set of self-adjoint fixed points of Φ is A (Corollary 11.27).

11.31. Proposition. *Let A be a JB-algebra of complex type, and let $\mathcal{U}$, K, $\mathcal{K}$, r be as defined above. Then each pure state of $\mathcal{U}$ restricts to a pure state of A, and for each pure state σ in $\mathcal{K}$, r is a bijection from the split face generated by σ onto the split face generated by $r(\sigma)$.*

Proof. Let σ be a pure state on $\mathcal{U}$, and let $\pi : \mathcal{U} \to \mathcal{B}(H)$ be the GNS-representation of $\mathcal{U}$ associated with σ. Then π is an irreducible representation of $\mathcal{U}$ (A 81). By the definition of $\mathcal{U}$, the C*-algebra generated by A is $\mathcal{U}$, so $\pi(A)$ is irreducible. Since A is of complex type, irreducibility of $\pi(A)$ implies that $\pi : A \to \mathcal{B}(H)_{\mathrm{sa}}$ is dense (Proposition 11.28). Thus π^* is a bijection of the normal state space N of $\mathcal{B}(H)$ onto a split face of $\mathcal{K}$ (Lemma 11.9), and $(\pi|A)^*$ is a bijection from N onto a split face of K. Since $(\pi|A)^* = r \circ \pi^*$, then r is an affine isomorphism from $\pi^*(N)$ onto $(\pi|A)^*(N)$.

By the definition of the GNS-representation, $\sigma = \pi^*\omega_\xi$ for a representing vector ξ. Hence σ is in $\pi^*(N)$ and $r(\sigma)$ is in $(\pi|A)^*(N)$. An affine isomorphism of a split face of $\mathcal{K}$ onto a split face of K will take minimal split faces to minimal split faces and extreme points to extreme points, and so the proposition follows. □

We now show that for a JB-algebra of complex type, a pure state is associated with two irreducible representations which are not unitarily equivalent (unless the representations are one dimensional), but rather are conjugate (cf. Definition 11.11).

11.32. Proposition. *Let A be a JB-algebra of complex type, and let r, Φ and $\mathcal{U}$ be as defined at the beginning of this section. If σ is a pure state of $\mathcal{U}$ and π_σ is the associated GNS representation of $\mathcal{U}$, then the representations $\pi_\sigma|A$ and $\pi_{\Phi^*\sigma}|A$ are conjugate irreducible representations of A associated with $r(\sigma) = r(\Phi^*\sigma)$. Every irreducible representation of A associated with $r(\sigma)$ is unitarily equivalent to $\pi_\sigma|A$ or to $\pi_{\Phi^*\sigma}|A$, with both possibilities occurring iff they are one dimensional.*

Proof. Since Φ is a *-anti-isomorphism, it is a unital order automorphism, so Φ^* is an affine automorphism of $\mathcal{K}$ (A 100). Since σ is a pure state on $\mathcal{U}$, then $\Phi^*\sigma$ is also a pure state on $\mathcal{U}$. Thus the GNS representations π_σ and $\pi_{\Phi^*\sigma}$ are irreducible representations of $\mathcal{U}$ (A 81). Since A generates $\mathcal{U}$, then π_σ and $\pi_{\Phi^*\sigma}$ are also irreducible when restricted to A.

It remains to show that π_σ and $\pi_{\Phi^*\sigma}$ restricted to A are conjugate. We will do this by constructing an auxiliary representation ψ of $\mathcal{U}$ conjugate to π_σ, and then showing ψ is unitarily equivalent to $\pi_{\Phi^*\sigma}$ as a representation of $\mathcal{U}$ (and then also when restricted to $\mathcal{A}$).

Let ξ be a cyclic vector for π_σ (A 61) such that

$$\sigma(a) = (\pi_\sigma(a)\xi \mid \xi) \tag{11.18}$$

for all $a \in \mathcal{U}$. Recall that a conjugation of H is a conjugate linear isometry $\jmath$ such that $\jmath^2 = 1$ (A 121). Extend $\{\xi\}$ to an orthonormal basis of H, and let $\jmath$ be the associated conjugation of H fixing that basis, cf. (A 123). Let $a \mapsto a^t$ be the associated transpose map (A 125), i.e.,

$$a^t = \jmath a^* \jmath \quad \text{for all } a \in \mathcal{B}(H). \tag{11.19}$$

Define $\psi : \mathcal{U} \to \mathcal{B}(H)$ by $\psi(a) = \pi_\sigma(\Phi(a))^t$. Since Φ and the transpose map are *-anti-isomorphisms, then ψ is a *-homomorphism.

Note that for $a \in A$, since Φ fixes each element of A, we have

$$\psi(a) = \pi_\sigma(\Phi(a))^t = \pi_\sigma(a)^t = \jmath\pi_\sigma(a)\jmath.$$

Thus by the definition of conjugacy, ψ restricted to A is conjugate to π_σ restricted to A.

It remains to show that ψ and $\pi_{\Phi^*\sigma}$ are unitarily equivalent. For $a \in \mathcal{U}_{\mathrm{sa}}$, by the definition (11.19), and the identity $(\jmath\eta_1 \mid \jmath\eta_2) = \overline{(\eta_1 \mid \eta_2)}$ for $\eta_1, \eta_2 \in H$ (A 122), we get

$$\begin{aligned}(\psi(a)\xi \mid \xi) &= (\pi_\sigma(\Phi(a))^t\xi \mid \xi) = (\jmath\pi_\sigma(\Phi(a))\jmath\xi \mid \xi) \\ &= \overline{(\pi_\sigma(\Phi(a))\jmath\xi \mid \jmath\xi)} = (\pi_\sigma(\Phi(a))\xi \mid \xi) = (\Phi^*\sigma)(a).\end{aligned} \tag{11.20}$$

If we can show that ξ is a cyclic vector for ψ, then the unitary equivalence of ψ and $\pi_{\Phi^*\sigma}$ will follow from (11.20) and (A 77). Since $\pi_\sigma(\mathcal{U})$ is irreducible, so is $\psi(\mathcal{U}) = \jmath\pi_\sigma(\Phi(\mathcal{U}))^*\jmath = \jmath\pi_\sigma(\mathcal{U})\jmath$. (If H_0 were a proper closed subspace invariant for $\psi(\mathcal{U})$, then $\jmath H_0$ would be a proper closed invariant subspace for $\pi_\sigma(\mathcal{U}) = \jmath\psi(\mathcal{U})\jmath$.) It follows that ξ is a cyclic vector for ψ. Thus ψ is unitarily equivalent to $\pi_{\Phi^*\sigma}$ as a representation of $\mathcal{U}$. Since $\psi|A$ is conjugate to π_σ, this completes the proof that $\pi_\sigma|A$ and $\pi_{\Phi^*\sigma}|A$ are conjugate.

By (11.18) $\pi_\sigma|A$ is associated with $r(\sigma)$. Since Φ fixes A, then $r(\Phi^*\sigma) = r(\sigma)$.

Next, let π be any irreducible representation of A associated with $r(\sigma)$. Then by Lemma 11.30, π and $\pi_\sigma|A$ have the same central covers, and so are Jordan equivalent (Proposition 11.7). Since π and π_σ are dense (Proposition 11.28), then π and $\pi_\sigma|A$ are unitarily equivalent or conjugate (Proposition 11.12). In the latter case, $\pi_{\Phi^*\sigma}|A$ and π are both conjugate to $\pi_\sigma|A$, and thus are unitarily equivalent to each other.

Recall that conjugate dense representations are Jordan equivalent, and are also unitarily equivalent iff they are one dimensional (Proposition 11.12). Density of π_σ and $\pi_{\Phi^*\sigma}$ follows from Proposition 11.28. Hence the last statement of the proposition holds. □

11.33. Corollary. *If A is a JB-algebra of complex type, then every pure state on A is associated with at least one irreducible representation.*

Proof. Let $\mathcal{U}$ be the universal enveloping C*-algebra for A, and let σ be a pure state on A. The set of states on $\mathcal{U}$ that extend σ form a w*-closed face of the state space of $\mathcal{U}$. Thus by the Krein–Milman theorem, there is a pure state extension $\widetilde{\sigma}$. Let π be the GNS representation of $\mathcal{U}$ associated with $\widetilde{\sigma}$. By Proposition 11.32, $\pi|A$ is an irreducible representation of A associated with σ. □

11.34. Definition. Let A be a JB-algebra. A state σ is *abelian* if $\pi_\sigma(A)$ is associative.

For motivation, let $\mathcal{A}$ be a C*-algebra, and let π_σ be the GNS representation associated with a state σ. Then $\pi_\sigma(\mathcal{A})$ is abelian iff $\pi_\sigma(\mathcal{A})_{\mathrm{sa}}$ is an associative JB-algebra (Corollary 1.50). As remarked after Corollary 11.14, $\pi_\sigma|\mathcal{A}_{\mathrm{sa}}$ is Jordan equivalent to the Jordan homomorphism described in Definition 4.11, so $\pi_\sigma(\mathcal{A}_{\mathrm{sa}}) \cong c(\sigma)\mathcal{A}_{\mathrm{sa}}$. Thus σ is an abelian state on $\mathcal{A}_{\mathrm{sa}}$ iff $\pi_\sigma(\mathcal{A})$ is abelian. (See also (A 155).)

11.35. Lemma. *Let σ be a pure state on a JB-algebra A. Then the following are equivalent:*

(i) *σ is abelian,*
(ii) *$\pi_\sigma(A)$ is one dimensional,*
(iii) *$\{\sigma\}$ is a split face of the state space of A.*

Proof. (i) $\Leftrightarrow$ (ii) If σ is abelian, then $\pi_\sigma(A) = c(\sigma)A \subset c(\sigma)A^{**}$ is associative, and by density so is $c(\sigma)A^{**}$. Then each projection in $c(\sigma)A^{**}$ operator commutes with all elements of $c(\sigma)A^{**}$ (Proposition 1.47), i.e., is central. Since σ is a pure state, then σ is a factor state, i.e., $c(\sigma)A^{**}$ is a factor (Lemma 4.14). Therefore, the only central projections are 0 and $c(\sigma)$. Since we have shown every projection is central, then $c(\sigma)A^{**}$ has no non-trivial projections. By the spectral theorem (Theorem 2.20), $c(\sigma)A^{**}$ must consist of multiples of the identity $c(\sigma)$, which proves (ii). Conversely, if (ii) holds, then $\pi_\sigma(A)$ is associative, so σ is abelian.

(ii) $\Rightarrow$ (iii) If $\pi_\sigma(A)$ is one dimensional, since $\pi_\sigma(A) = c(\sigma)A$ is dense in $c(\sigma)A^{**}$, then $c(\sigma)A^{**}$ is one dimensional. Hence the normal state space of $c(\sigma)A^{**}$ is a single point. By Proposition 2.62, the normal state space of $c(\sigma)A^{**}$ can be identified with $F_{c(\sigma)}$, which equals F_σ (Proposition 5.44). It follows that F_σ is a single point, and thus must equal $\{\sigma\}$.

(iii) $\Rightarrow$ (ii) If $\{\sigma\}$ is a split face, then $F_{c(\sigma)} = F_\sigma = \{\sigma\}$. Hence the normal state space of $c(\sigma)A^{**}$ is a single point, so $c(\sigma)A^{**}$ is one dimensional. Now (ii) follows. □

11.36. Proposition. *Let A be a JB-algebra of complex type, let $\mathcal{U}$, Φ and r be as defined at the beginning of this section, and let σ be a pure state on $\mathcal{U}$. Then*

(i) $\Phi^*\sigma = \sigma$ *iff* σ *is abelian.*

(ii) *If* σ *is not abelian, then the split faces generated by* σ *and* $\Phi^*\sigma$ *are disjoint.*

(iii) *The only pure states of* $\mathcal{U}$ *that restrict to* $r(\sigma)$ *on* A *are* σ *and* $\Phi^*\sigma$*, and the inverse image of* $r(\sigma)$ *consists of the line segment with endpoints* σ, $\Phi^*\sigma$.

Proof. (i) Assume $\Phi^*\sigma = \sigma$. The GNS representations π_σ and $\pi_{\Phi^*\sigma}$ of $\mathcal{U}$ when restricted to A are conjugate irreducible representations (Proposition 11.32). Since $\Phi^*\sigma = \sigma$, these representations coincide, and by Proposition 11.32, $\pi_\sigma(A)$ must be one dimensional. Since A generates $\mathcal{U}$, then $\pi_\sigma(\mathcal{U})$ is one dimensional, so σ is an abelian state on $\mathcal{U}$.

Conversely, suppose that σ is abelian, so that $\{\sigma\}$ is a split face of $\mathcal{K}$ (Lemma 11.35). Let $(\pi_\sigma, H_\sigma, \xi_\sigma)$ be the associated GNS-representation of $\mathcal{U}$ on the Hilbert space H_σ, with the cyclic vector ξ_σ. Then the normal state space of $B(H_\sigma)$ is affinely isomorphic to $F_{c(\sigma)} = \{\sigma\}$ (A 138), and so H_σ is one dimensional. Thus for each $a \in \mathcal{U}$, the vector $\pi_\sigma(a)\xi_\sigma$ is a multiple of ξ_σ. Since $\sigma(a) = (\pi_\sigma(a)\xi_\sigma \mid \xi_\sigma)$, one has

$$\pi_\sigma(a)\xi_\sigma = \sigma(a)\xi_\sigma.$$

This holds for all $a \in \mathcal{U}$, but we will make use of it for $a \in A$.

Similarly, since Φ^* is an affine automorphism of $\mathcal{K}$, then $\{\Phi^*\sigma\} = \Phi^*(\{\sigma\})$ is a split face. For simplicity of notation, set $\Phi^*\sigma = \tau$. If $(\pi_\tau, H_\tau, \xi_\tau)$ is the GNS-representation of $\mathcal{U}$ associated with τ, then for $a \in A$,

$$\pi_\tau(a)\xi_\tau = \tau(a)\xi_\tau = \sigma(\Phi(a))\xi_\tau = \sigma(a)\xi_\tau,$$

where the last equality uses the fact that $\Phi(a) = a$ for $a \in A$. It is easily verified that $\lambda\xi_\sigma \mapsto \lambda\xi_\tau$ is a unitary intertwining these representations of A, and so π_σ and π_τ are unitarily equivalent as representations of A. Since A generates $\mathcal{U}$, then they are also unitarily equivalent as representations of $\mathcal{U}$. Therefore σ and τ generate the same (singleton) split face (A 142), and so $\tau = \sigma$.

(ii) Suppose that the split faces generated by σ and τ are not disjoint. Since σ and τ are pure, they generate minimal split faces (Lemma 5.46). The intersection of two split faces is a split face (A 7), and so minimal split faces must either coincide or be disjoint. Thus the split faces generated by σ and τ coincide. Recall that $r(\sigma) = r(\tau)$ (Proposition 11.32), and that r is bijective on the split face generated by σ (Proposition 11.31). Thus we conclude that $\tau = \sigma$. By (i) this implies that σ is abelian, and now (ii) follows.

(iii) Suppose ω is a pure state of $\mathcal{U}$ such that $r(\omega) = r(\sigma)$. Let π_ω be the associated GNS representation of $\mathcal{U}$. Then $\pi_\omega|A$ is an irreducible representation associated with $r(\omega) = r(\sigma)$ (Proposition 11.32), and so is unitarily equivalent to either $\pi_\sigma|A$ or to $\pi_\tau|A$. Since A generates $\mathcal{U}$, the unitary that carries $\pi_\omega|A$ to either $\pi_\sigma|A$ or to $\pi_\tau|A$ will also carry π_ω to either π_σ or π_τ. Therefore the split face generated by ω coincides with the split face generated by one of σ or τ (A 142). Since r is bijective on each of these split faces (Proposition 11.31), and $r(\omega) = r(\sigma) = r(\tau)$, we must have $\omega = \sigma$ or $\omega = \tau$.

Finally, let $F = r^{-1}(r(\sigma))$. Since $r(\sigma)$ is a pure state of A, and $r : \mathcal{K} \to K$ is surjective (by a Hahn–Banach argument), then F is a non-empty w*-closed face of $\mathcal{K}$. By the Krein–Milman theorem F is the w*-convex hull of its extreme points. The latter are pure states of $\mathcal{K}$ that restrict to σ, and so by the last paragraph, the only possibilities are σ and τ. It follows that F is the line segment with these points as endpoints. □

If $\mathcal{K}$ is any convex set, we let $\partial_{e,0}\mathcal{K}$ denote the extreme points of $\mathcal{K}$ that generate singleton split faces. If $\mathcal{K}$ is the state space of $\mathcal{U}$ as above, then $\partial_{e,0}\mathcal{K}$ will be the set of abelian pure states of $\mathcal{U}$.

11.37. Corollary. *Let A be a JB-algebra of complex type, and let r, K and $\mathcal{K}$ be as defined at the beginning of this section. The restriction map $r : \mathcal{K} \to K$ maps $\partial_{e,0}\mathcal{K}$ bijectively onto $\partial_{e,0}K$ and maps $\partial_e\mathcal{K} \setminus \partial_{e,0}\mathcal{K}$ two-to-one onto $\partial_e K \setminus \partial_{e,0}K$.*

Proof. By Proposition 11.31, $r(\partial_e\mathcal{K}) \subset \partial_e K$, and for $\sigma \in \partial_e\mathcal{K}$, $\{\sigma\}$ is a split face iff $\{r(\sigma)\}$ is a split face. By the Krein–Milman theorem, each $\sigma \in \partial_e K$ has an extension $\widetilde{\sigma} \in \partial_e\mathcal{K}$. Thus r maps $\partial_{e,0}\mathcal{K}$ onto $\partial_{e,0}K$ and $\partial_e\mathcal{K} \setminus \partial_{e,0}\mathcal{K}$ onto $\partial_e K \setminus \partial_{e,0}K$. The last statement of the corollary follows from Proposition 11.36 (iii). □

Remark. The abelian states on $\mathcal{U}$ form a w*-closed split face of $\mathcal{K}$ (A 154). Thus $\partial_{e,0}\mathcal{K}$ is the set of extreme points of this split face, and by the Krein–Milman theorem, the w*-closed convex hull of $\partial_{e,0}\mathcal{K}$ is the set of all abelian states.

11.38. Lemma. *If A is a JB-algebra of complex type, and r, K, $\mathcal{K}$ and Φ are defined as at the beginning of this section, then the fixed point set of Φ^* in $\mathcal{K}$ is mapped bijectively onto K by r.*

Proof. Let $\sigma \in \mathcal{K}$. Since Φ fixes A, then $r(\Phi^*\sigma) = r(\sigma)$. Let $\mathcal{K}_1$ denote the set of fixed points of Φ^*. Note that $\frac{1}{2}(\sigma + \Phi^*\sigma)$ is in $\mathcal{K}_1$, and has image $r(\sigma) = r(\Phi^*\sigma)$. Thus r maps the fixed point set of Φ^* onto $r(\mathcal{K}) = K$. To show that r is injective, let σ and ω be states in $\mathcal{K}_1$ such that $r(\sigma) = r(\omega)$, and let $x \in \mathcal{U}_{sa}$. Note that $\frac{1}{2}(x + \Phi x)$ is fixed by Φ, and thus is in A (Corollary 11.27). Then $r(\sigma) = r(\omega)$ implies that σ and

ω agree on $\frac{1}{2}(x + \Phi x)$. By assumption, σ and ω are fixed by Φ^*, so

$$\sigma(x) = \tfrac{1}{2}(\sigma + \Phi^*\sigma)(x) = \sigma\big(\tfrac{1}{2}(x + \Phi x)\big) = \omega\big(\tfrac{1}{2}(x + \Phi x)\big) = \omega(x).$$

Thus $\sigma = \omega$, which completes the proof that r is injective on $\mathcal{K}_1$. □

11.39. Lemma. *Let A be a JB-algebra of complex type, let r and $\mathcal{K}$ be as defined at the beginning of this section, and let F be a w*-closed split face of $\mathcal{K}$. If r is one-to-one on $\partial_e F$, then r is one-to-one on F.*

Proof. Let $G = \Phi^*(F)$. Since Φ^* is of period 2, then $\Phi^*(G) = F$, so that Φ^* exchanges F and G. Since Φ^* is an affine isomorphism, it preserves complementation of split faces, so Φ^* also exchanges F' and G'.

We first show that $F \cap G$ is pointwise fixed by Φ^*. Let σ be a pure state in $F \cap G$. Then $\sigma \in G$ implies that $\Phi^*\sigma$ is a pure state in F. Thus σ and $\Phi^*\sigma$ are pure states in F that restrict to the same state on A, so by our hypothesis $\Phi^*\sigma = \sigma$. Hence Φ^* fixes each pure state of the w*-closed split face $F \cap G$. By w*-continuity of Φ^* and the Krein–Milman theorem, Φ^* fixes every state in $F \cap G$.

Now let σ and ω be states in F such that $r(\sigma) = r(\omega)$. Then since $r(\sigma) = r(\Phi^*\sigma)$ and $r(\omega) = r(\Phi^*\omega)$, we have

$$r\big(\tfrac{1}{2}(\sigma + \Phi^*\sigma)\big) = r\big(\tfrac{1}{2}(\omega + \Phi^*\omega)\big).$$

By Lemma 11.38,

$$\tfrac{1}{2}(\sigma + \Phi^*\sigma) = \tfrac{1}{2}(\omega + \Phi^*\omega). \tag{11.21}$$

Since F is a face, by the definition of a split face of $\mathcal{K}$, F is the direct convex sum of $F \cap G$ and $F \cap G'$. Write

$$\sigma = s\sigma_1 + (1 - s)\sigma_2 \quad \text{and} \quad \omega = t\omega_1 + (1 - t)\omega_2 \tag{11.22}$$

where $\sigma_1, \omega_1 \in F \cap G$, and $\sigma_2, \omega_2 \in F \cap G'$, and s, t are scalars with $0 \le s, t \le 1$. Substituting (11.22) into (11.21) gives

$$\begin{aligned}&\tfrac{1}{2}\big(s\sigma_1 + (1 - s)\sigma_2 + s\Phi^*\sigma_1 + (1 - s)\Phi^*\sigma_2\big)\\ &\quad = \tfrac{1}{2}\big(t\omega_1 + (1 - t)\omega_2 + t\Phi^*\omega_1 + (1 - t)\Phi^*\omega_2\big).\end{aligned}$$

Since Φ^* exchanges F with G and F' with G', the four terms in each sum above are in $F \cap G$, $F \cap G'$, $G \cap F$, $G \cap F'$ respectively. Since Φ^* fixes $F \cap G$ pointwise, then

$$\tfrac{1}{2}(2s\sigma_1 + (1 - s)\sigma_2 + (1 - s)\Phi^*\sigma_2) = \tfrac{1}{2}(2t\omega_1 + (1 - t)\omega_2 + (1 - t)\Phi^*\omega_2).$$

By affine independence of $F \cap G$, $F \cap G'$, $G \cap F'$, cf. (A 6) or [6, Prop. II.6.6], we conclude that for $0 < s < 1$, then $s = t$, $\sigma_1 = \omega_1$, $\sigma_2 = \omega_2$, and so $\sigma = \omega$. We leave it to the reader to show that $\sigma = \omega$ also follows for $s = 0$ or $s = 1$. Thus r is one-to-one on F. □

Let A be a JB-algebra, $A + iA$ its complexification. Recall from Definition 6.14 that an associative product $\star$ on $A + iA$ is *Jordan compatible* if the associated Jordan product coincides with the original one on A (i.e., $a \star b + b \star a = ab + ba$ for $a, b \in A$), and if the map $a + ib \mapsto a - ib$ for $a, b \in A$ is an involution for the product $\star$. We refer to such a product as a *C*-product* on $A + iA$.

11.40. Lemma. *If A is a JB-algebra and $A+iA$ admits a C^*-product $\star$, then $\|x\| = \|x^*x\|^{1/2}$ is a norm on $A + iA$, and makes $A + iA$ into a C^*-algebra.*

Proof. A JB-algebra is also an order unit algebra (Lemma 1.10), so the lemma follows from (A 59). □

Thus if A is a JB-algebra and $A + iA$ admits a C*-product, then A is isomorphic to the self-adjoint part of a C*-algebra. Conversely, every isomorphism of A with the self-adjoint part of a C*-algebra induces a C*-product on $A + iA$, as we now will see.

In what follows, if A is a JB-algebra and π is a Jordan isomorphism onto the self-adjoint part of a C*-algebra, then we will also denote by π the complex linear extension of π to $A + iA$.

11.41. Proposition. *Let $\pi : A \to \mathcal{A}_{\mathrm{sa}}$ be a Jordan isomorphism from a JB-algebra A onto the self-adjoint part of a C^*-algebra $\mathcal{A}$. Then there is a unique C^*-product on $A + iA$ making $\pi : A + iA \to \mathcal{A}$ into a *-isomorphism. Two such *-isomorphisms $\pi_1 : A + iA \to \mathcal{A}_1$ and $\pi_2 : A + iA \to \mathcal{A}_2$ induce the same C^*-product on $A + iA$ iff $\pi_2 \circ \pi_1^{-1}$ is a *-isomorphism from $\mathcal{A}_1$ onto $\mathcal{A}_2$.*

Proof. The product $(x, y) \mapsto \pi^{-1}(\pi(x)\pi(y))$ on $A + iA$ is easily seen to be a C*-product making $\pi : A + iA \to \mathcal{A}$ into a *-isomorphism. In fact, it is the only such product, for if $\star$ is any product on $A + iA$ such that

$$\pi(x \star y) = \pi(x)\pi(y)$$

for $x, y \in A + iA$, then applying π^{-1} to both sides gives $x \star y = \pi^{-1}(\pi(x), \pi(y))$.

For the second statement of the proposition, define $\Phi = \pi_2 \circ \pi_1^{-1}$ and let Ψ be the identity map from $A + iA$ equipped with the C*-product determined by π_1 onto $A + iA$ equipped with the C*-product determined by π_2. If these two products coincide, then it follows from the equality $\Phi = \pi_2 \circ \Psi \circ \pi_1^{-1}$ that Φ is a *-isomorphism. Conversely, if Φ is a *-isomorphism, then it follows from the equality $\Psi = \pi_2^{-1} \circ \Phi \circ \pi_1$ that the identity map Ψ is a *-isomorphism, so that the two products coincide. □

Thus for a JB-algebra A, specifying a C*-product on $A + iA$ is equivalent to giving a Jordan isomorphism of A onto the self-adjoint part of a C*-algebra. We next give a necessary and sufficient condition for a JB-algebra of complex type to admit such a C*-product, in terms of the state space $\mathcal{K}$ of the universal C*-algebra.

11.42. Proposition. *Let A be a JB-algebra of complex type, and let $\mathcal{U}$, $\mathcal{K}$, K and r be as defined at the beginning of this section. Then A is isomorphic to the self-adjoint part of a C*-algebra iff there exists a w*-closed split face F of $\mathcal{K}$ such that r maps F bijectively onto K. Furthermore, there is a 1-1 correspondence of such split faces F and C*-products on $A + iA$.*

Proof. Let $\mathcal{F}_0$ be the set of w*-closed split faces F of $\mathcal{K}$ that are mapped bijectively by r onto K. We will establish the 1-1 correspondence of C*-products and split faces F in $\mathcal{F}_0$.

Fix a C*-product on $A + iA$. This product organizes $A + iA$ to a C*-algebra, which we will denote by $\mathcal{A}$ for short. Let ι be the imbedding of the JB-algebra A into its universal C*-algebra $\mathcal{U}$, and let ψ be the unique lift of the inclusion $\pi : A \to A + iA$ to a *-homomorphism from $\mathcal{U}$ onto $\mathcal{A}$. Thus $\psi \circ \iota = \pi$. Let $\mathcal{J}$ be the kernel of ψ. Then ψ^* is an affine isomorphism from the state space $S(\mathcal{A})$ of $\mathcal{A}$ onto the annihilator F of $\mathcal{J}$ in $\mathcal{K}$, which is a w*-closed split face of $\mathcal{K}$ (A 133). On the other hand, π^* is an affine isomorphism from $S(\mathcal{A})$ onto K (Proposition 5.16). Dualizing $\psi \circ \iota = \pi$ gives $\iota^* \circ \psi^* = \pi^*$. But $\iota^* = r$, so $r \circ \psi^* = \pi^*$. Since ψ^* is an affine isomorphism from $S(\mathcal{A})$ onto F and π^* is an affine isomorphism from $S(\mathcal{A})$ onto K, we conclude from this that r is an affine isomorphism from F onto K. Thus to each C*-product we have associated a split face in $\mathcal{F}_0$.

Suppose that two C*-products lead to the same split face $F \in \mathcal{F}_0$, and let $\pi_j : A \to A + iA$ for $j = 1, 2$ be the inclusions of A into the C*-algebra $A + iA$ equipped with each product. Then the kernels of the associated *-homomorphisms $\psi_1 : \mathcal{U} \to A + iA$ and $\psi_2 : \mathcal{U} \to A + iA$ must coincide, since both are the annihilators of F in $\mathcal{U}$ (cf. (A 114)). Hence there is a *-isomorphism Φ from $A + iA$ with the first product, to $A + iA$ with the second product, such that $\Phi \circ \psi_1 = \psi_2$. By definition of ψ_1 and ψ_2, we have $\pi_1 = \psi_1 \circ \iota$ and $\pi_2 = \psi_2 \circ \iota$. Thus $\Phi \circ \pi_1 = \pi_2$, so $\Phi = \pi_2 \circ \pi_1^{-1}$. By Proposition 11.41, the two C*-products must coincide. Thus we have shown that the map from C*-products into $\mathcal{F}_0$ is injective.

To prove surjectivity, let $F \in \mathcal{F}_0$ and let $\mathcal{J}$ be the annihilator of F in $\mathcal{U}$. Recall that $\mathcal{J}$ is a closed ideal in $\mathcal{U}$ and that F is the annihilator of $\mathcal{J}$ in $\mathcal{K}$ (A 114). Let $\widetilde{\psi} : \mathcal{U} \to \mathcal{U}/\mathcal{J}$ be the quotient map, and define $\widetilde{\pi} = \widetilde{\psi} \circ \iota : A \to \mathcal{U}/\mathcal{J}$. We will show that $\widetilde{\pi}$ maps A bijectively onto $(\mathcal{U}/\mathcal{J})_{\mathrm{sa}}$. Recall that $\widetilde{\psi}^*$ is an affine isomorphism from the state space of $\mathcal{U}/\mathcal{J}$ onto F (A 133). Since $F \in \mathcal{F}_0$, then r is an affine isomorphism from F onto K, so $\widetilde{\pi}^* = r \circ \widetilde{\psi}^*$ is an affine isomorphism from the state space of $\mathcal{U}/\mathcal{J}$ onto K. It follows that $\widetilde{\pi}$ is a Jordan isomorphism from A onto $(\mathcal{U}/\mathcal{J})_{\mathrm{sa}}$ (Proposition 5.16).

Give $A + \imath A$ the C*-product induced by $\widetilde{\pi}$ (cf. Proposition 11.41). Then $\widetilde{\pi} : A \to \mathcal{U}/\mathcal{J}$ extends to a *-isomorphism $\widetilde{\pi} : A + \imath A \to \mathcal{U}/\mathcal{J}$ Define $\phi = (\widetilde{\pi})^{-1} : \mathcal{U}/\mathcal{J} \to A + \imath A$, and $\psi = \phi \circ \widetilde{\psi} : \mathcal{U} \to A + \imath A$. Let $\pi : A \to A + \imath A$ be the inclusion map.

$$\begin{tikzcd}
 & \mathcal{U} \arrow[d, "\widetilde{\psi}"] \\
A \arrow[ur, "\iota"] \arrow[r, "\widetilde{\pi}"] \arrow[dr, "\pi"'] & \mathcal{U}/\mathcal{J} \arrow[d, "\phi"] \\
 & A + \imath A
\end{tikzcd}$$

Then ψ is a *-homomorphism, and for all $a \in A$,

$$(\psi \circ \iota)(a) = \phi(\widetilde{\psi}(\iota(a))) = \phi(\widetilde{\pi}(a)) = a = \pi(a),$$

so $\psi \circ \iota = \pi$. Since ϕ is a *-isomorphism, the kernels of $\psi = \phi \circ \widetilde{\psi}$ and $\widetilde{\psi}$ coincide, and thus both equal F. Hence the split face associated with the given product on $A + \imath A$ is F, which completes the proof of a 1-1 correspondence of C*-products and split faces in $\mathcal{F}_0$. □

Lemma 11.39 and Proposition 11.42 will play a key role in the proof of our characterization of C*-algebra state spaces.

Orientability

The 3-ball property is not sufficient to guarantee that a JB-algebra state space is a C*-algebra state space, since there are JB-algebras of complex type that are not the self-adjoint part of C*-algebras, as we will see in Proposition 11.51. An extra condition is needed ("orientability"), which we will now discuss.

The notion of orientation of 3-balls and of C*-algebra state spaces was introduced in [AS, Chpt. 5], where it was shown that orientations of the state space of a C*-algebra are in 1-1 correspondence with C*-products

(A 156). C*-algebra state spaces are always orientable (A 148), and we will see that this property (extended to JB-algebra state spaces) is exactly what singles out C*-algebra state spaces among all JB-algebra state spaces satisfying the 3-ball property. We briefly review the relevant concepts, and extend them to our current context.

Recall that $\mathbf{B}^3$ denotes the closed unit ball of $\mathbf{R}^3$, and that a *3-ball* is a convex set affinely isomorphic to $\mathbf{B}^3$.

11.43. Definition. Let K be a convex set. A *facial 3-ball* is a 3-ball that is a face of K.

Our primary interest will be in the facial 3-balls of the state space of a JB-algebra of complex type. However, we will first review some elementary notions concerning general 3-balls, cf. (A 128).

11.44. Definition. If B is a 3-ball, a *parameterization* of B is an affine isomorphism from $\mathbf{B}^3$ onto B.

Each parameterization ϕ of a 3-ball B is determined by an orthogonal frame (i.e, an ordered triple of orthogonal axes), given by the image under ϕ of the usual x, y, z coordinate axes of $\mathbf{B}^3$. If ϕ_1 and ϕ_2 are parameterizations of a 3-ball B, then $\phi_2^{-1} \circ \phi_1$ is an affine isomorphism of $\mathbf{B}^3$, which extends uniquely to an orthogonal transformation of $\mathbf{R}^3$.

11.45. Definition. If B is a 3-ball, an *orientation of B* is an equivalence class of parameterizations of B, where two such parameterizations ϕ_1 and ϕ_2 are considered equivalent if $\det(\phi_2^{-1} \circ \phi_1) = 1$. If $\phi : \mathbf{B}^3 \to B$ is a parameterization, we denote by $[\phi]$ the associated orientation of B.

An orientation of B corresponds to an equivalence class of orthogonal frames, with two frames being equivalent if there is a rotation that moves one into the other. Note that each 3-ball admits exactly two orientations, which we will refer to as "opposite" to each other.

11.46. Definition. If B_1 and B_2 are 3-balls equipped with orientations $[\phi_1]$ and $[\phi_2]$ respectively, and $\psi : B_1 \to B_2$ is an affine isomorphism, we say ψ *preserves orientation* if $[\psi \circ \phi_1] = [\phi_2]$, and *reverses orientation* if the orientation $[\psi \circ \phi_1]$ is the opposite of $[\phi_2]$.

We now outline how we will proceed. In the special case of the JB-algebra $A = M_2(\mathbf{C})_{\mathrm{sa}}$, the state space is a 3-ball. The affine structure of the state space determines the Jordan product on A uniquely. There are two possible C*-products on $A + iA$ compatible with this Jordan product, namely, the usual multiplication on $M_2(\mathbf{C})$ and the opposite one. This choice of products corresponds to a choice of one of the two possible orientations on the state space (A 132).

More generally, if K is the state space of a JB-algebra A of complex type, then a choice of orientation for each facial 3-ball may be thought of as determining a possible associative multiplication locally (i.e., on the affine functions on that 3-ball). It isn't necessarily the case that these choices of local multiplications can be extended to give an associative multiplication for A. We will see that this is possible iff the orientations of the facial 3-balls can be chosen in a continuous fashion.

To make sense of a continuous choice of orientation, we need to define topologies on the set of facial 3-balls and on the set of oriented facial 3-balls. We will then have a $\mathbf{Z_2}$ bundle with the two possible orientations (or oriented balls) sitting over each facial 3-ball. A continuous choice of orientation then will be a continuous cross-section of this bundle.

The precise definition of this concept will be given for convex sets with the 3-ball property. Since each face of a convex set with the 3-ball property also has the 3-ball property, this definition will apply not only to state spaces of JB-algebras of complex type, but also to all faces of such state spaces.

11.47. Definition. Let K be a compact convex set with the 3-ball property in a locally convex linear space. Param(K) denotes the set of all affine isomorphisms from the standard 3-ball $\mathbf{B}^3$ onto facial 3-balls of K. We equip Param(K) with the topology of pointwise convergence of maps from $\mathbf{B}^3$ into K.

Note that O(3) and SO(3) acting on $\mathbf{B}^3$ induce continuous actions on Param(K) by composition. Two maps ϕ_1, ϕ_2 in Param(K) differ by an element of $O(3)$ iff their range is the same facial 3-ball B, and differ by an element of SO(3) iff they induce the same orientation on B. Thus we are led to the following definitions.

11.48. Definition. Let K be a compact convex set with the 3-ball property in a locally convex linear space. We call $\mathcal{OB}_K =$ Param$(K)/SO(3)$ the *space of oriented facial 3-balls of* K. We equip it with the quotient topology. If $\phi \in$ Param(K), then we denote its equivalence class by $[\phi]$. (Note that $[\phi]$ is an orientation of the 3-ball $\phi(\mathbf{B}^3)$.) We call $\mathcal{B}_K =$ Param$(K)/SO(3)$ the *space of facial 3-balls*, equipped with the quotient topology from the map $\phi \mapsto \phi(\mathbf{B}^3)$. When there is no danger of confusion we will write $\mathcal{OB}$ for $\mathcal{OB}_K$ and $\mathcal{B}$ for $\mathcal{B}_K$.

Note that the definition above generalizes the corresponding one for C*-algebra state spaces (A 144). But the proposition below, which involves the canonical map from $\mathcal{OB}$ to $\mathcal{B}$ that takes an oriented facial ball to its underlying (non-oriented) facial ball, is only a partial generalization of the corresponding result for C*-algebra state spaces (A 145) and (A 148), since in the C*-algebra case, this canonical map defines a trivial bundle, and not merely a locally trivial one.

11.49. Proposition. *Let K be a compact convex set with the 3-ball property in a locally convex linear space V. The spaces $\mathcal{OB}_K$ and $\mathcal{B}_K$ are Hausdorff, the canonical map from $\mathcal{OB}_K$ onto $\mathcal{B}_K$ is continuous and open, and $\mathcal{OB}_K \to \mathcal{B}_K$ is a locally trivial principal $\mathbf{Z_2}$ bundle.*

Proof. By definition of the quotient topologies, the quotient maps from $\mathrm{Param}(K)$ onto $\mathrm{Param}(K)/SO(3)$ and onto $\mathrm{Param}(K)/O(3)$ are continuous. Since these are quotients of a Hausdorff space by the action of compact groups, both are Hausdorff [32, Props. III.4.1.2 and III.4.2.3]). Since the relevant equivalence relations on $\mathrm{Param}(K)$ are given by continuous group actions, the quotient maps are open. Since the quotient maps $\mathrm{Param}(K) \to \mathcal{OB}$ and $\mathrm{Param}(K) \to \mathcal{B}$ are continuous and open, it is straightforward to check that the canonical map $\mathcal{OB} \to \mathcal{B}$ is continuous and open.

Now let ϕ_0 be any element of O(3) that has determinant -1 and whose square is the identity. Then the continuous action of the group $\{\phi_0, 1\}$ on $\mathrm{Param}(K)$ induces a continuous $\mathbf{Z_2}$ action on $\mathcal{OB} = \mathrm{Param}(K)/SO(3)$ that exchanges the two elements in each fiber. To finish the proof we need to show that for each 3-ball B in $\mathcal{B}$ there is an open neighborhood W containing B such that the bundle $\mathcal{OB} \to \mathcal{B}$ restricted to W is isomorphic to the trivial bundle $\mathbf{Z_2} \times W \to W$. This is equivalent to showing that for each $\phi \in \mathrm{Param}(K)$ there is an open neighborhood S of $[\phi]$ that meets each fiber of the bundle $\mathcal{OB} \to \mathcal{B}$ just once.

Let $A(\mathbf{B}^3, K)$ denote the set of affine maps from $\mathbf{B}^3$ into K, with the topology of pointwise convergence. We may assume without loss of generality that 0 is not in the affine span of K. (Otherwise translate K so that this is true). We claim that the set of injective maps is open. To see this, let $\phi_0 \in A(\mathbf{B}^3, K)$ be injective. Let $\{\sigma_i,\ i = 1, 2, 3, 4\}$ be affinely independent elements of $\mathbf{B}^3$. Then $\{\phi_0(\sigma_i),\ i = 1, 2, 3, 4\}$ are affinely independent, and since 0 is not in the affine span of K, then these vectors in V are also linearly independent. Thus we can pick continuous linear functionals a_1, a_2, a_3, a_4 such that $a_i(\phi_0(\sigma_j)) = \delta_{ij}$ (the Kronecker delta). Then the determinant of the matrix $(a_i(\phi_0(\sigma_j)))$ will be 1. If ϕ is near enough to ϕ_0 then the determinant of the matrix $(a_i(\phi(\sigma_j)))$ will be nonzero, and so $\{\phi(\sigma_i),\ i = 1, 2, 3, 4\}$ must be linearly independent. It follows that the affine span of $\{\phi(\sigma_i),\ i = 1, 2, 3, 4\}$ must be three dimensional, and so ϕ must be injective. Note further that for each $\epsilon > 0$ the set of ϕ such that $|\delta_{ij} - (a_i(\phi(\sigma_j)))| < \epsilon$ is a convex open neighborhood of ϕ_0 consisting of injective maps for ϵ sufficiently small.

Now choose a convex open neighborhood N_0 of ϕ_0 in $A(\mathbf{B}^3, K)$ consisting of injective maps, and let N be the intersection of this neighborhood with $\mathrm{Param}(K)$. Let $S = \{[\phi] \mid \phi \in N\}$. Then S is an open subset of $\mathcal{OB}_K$, which we claim meets each fiber of our bundle at most once. Suppose that ϕ_1 and ϕ_2 are in N and $[\phi_1]$ and $[\phi_2]$ are in the same fiber. Then ϕ_1 and ϕ_2 map $\mathbf{B}^3$ onto the same facial 3-ball B. For each $t \in [0, 1]$ define

$\gamma_t : \mathbf{B}^3 \to K$ by

$$\gamma_t = t\phi_1 + (1-t)\phi_2.$$

Then each γ_t is in the convex neighborhood N_0, and so must be an injective map of $\mathbf{B}^3$ into B. For each $t \in [0,1]$, $\phi_2^{-1} \circ \gamma_t$ is an injective affine map from $\mathbf{B}^3$ into $\mathbf{B}^3$, taking 0 to 0, which then admits a unique extension to an invertible linear map from $\mathbf{R}^3$ into $\mathbf{R}^3$. Thus $t \mapsto \det(\phi_2^{-1} \circ \gamma_t)$ is a continuous map from $[0,1]$ into $\mathbf{R} \setminus \{0\}$. Since its value at $t=0$ is 1, its value at $t=1$ cannot be -1, and so ϕ_1 and ϕ_2 must determine the same orientation. We have shown the bundle $\mathcal{OB} \to \mathcal{B}$ over $W = \{\phi(\mathbf{B}^3) \mid \phi \in N\}$ is isomorphic to the trivial $\mathbf{Z_2}$ bundle $\mathbf{Z_2} \times W \to W$, and thus is a locally trivial $\mathbf{Z_2}$ principal bundle. □

Let $\pi : \mathcal{OB} \to \mathcal{B}$ be the bundle map that takes each oriented 3-ball to the underlying facial 3-ball. To specify an orientation for each facial 3-ball, we can give a map $\Theta : \mathcal{B} \to \mathcal{OB}$ such that $\pi(\Theta(F)) = F$ for all $F \in \mathcal{B}$. Such a map is determined by its range, which will be a subset of $\mathcal{OB}$ that meets each fiber exactly once, i.e., includes exactly one of the two oriented 3-balls sitting over each facial 3-ball. We refer to both the map Θ and its range as *cross-sections.* If Θ is a continuous cross-section, then its range is a closed cross-section, since $\mathcal{OB}$ is Hausdorff and $\Theta(\mathcal{B}) = \{[\phi] \mid \Theta(\pi([\phi])) = [\phi]\}$. Conversely, suppose $\mathcal{X} \subset \mathcal{OB}$ is a closed cross-section. Then the complementary set $\mathcal{Y}$ of oriented 3-balls is also closed (since it is the image of $\mathcal{X}$ under the Z_2 action of the bundle). We refer to $\mathcal{Y}$ as the *opposite cross-section* to $\mathcal{X}$. Then $\mathcal{X}$ and its opposite cross-section $\mathcal{Y}$ are both open and closed. If $\Theta : \mathcal{B} \to \mathcal{OB}$ is the cross-section such that $\Theta(\mathcal{B}) = \mathcal{X}$, then Θ will be continuous, and there is a 1-1 correspondence between closed cross-sections and continuous cross-sections. Note that a closed cross-section $\mathcal{X}$ provides a trivialization of the bundle: the bundle is isomorphic to the trivial bundle $\mathcal{X} \times Z_2 \to \mathcal{X}$. Clearly each trivialization of the bundle gives a closed cross-section.

11.50. Definition. Let K be a convex set with the 3-ball property in a locally convex space. Then K is *orientable* if the bundle $\mathcal{OB}_K \to \mathcal{B}_K$ is trivial. A continuous cross-section of this bundle is called a *global orientation*, or simply an *orientation*, of K.

Note that the above definition is a direct generalization of the corresponding definition for C*-algebra state spaces (A 146).

Characterization of C*-algebra state spaces among JB-algebra state spaces

We now pause to give an example showing that orientability is not automatic for state spaces of JB-algebras of complex type.

11.51. Proposition. *Let* $\mathbf{T}$ *be the unit circle and let* A *consist of all continuous functions* f *from* $\mathbf{T}$ *into* $M_2(\mathbf{C})_{\mathrm{sa}}$ *such that* $f(-\lambda) = f(\lambda)^t$ *for all* $\lambda \in \mathbf{T}$*, where* $f(\lambda)^t$ *denotes the transpose of* $f(\lambda)$*. Then* A *is a JB-algebra of complex type whose state space is not orientable.*

Proof. A is a norm closed Jordan subalgebra of the self-adjoint part of the C*-algebra $\mathcal{C}$ of all continuous functions from $\mathbf{T}$ into $M_2(\mathbf{C})$, and thus is a JB-algebra. Let σ be a pure state of A. We will show that the split face of the state space K of A generated by σ is a 3-ball.

The set of states of $\mathcal{C}$ that restrict to σ form a w*-closed face of the state space of C, and so by the Krein–Milman theorem there is a pure state $\widetilde{\sigma}$ of $\mathcal{C}$ that restricts to σ. Let π be the GNS representation and ξ the cyclic vector associated with $\widetilde{\sigma}$. Since $\widetilde{\sigma}$ is pure, then π is an irreducible representation of $\mathcal{C}$ (A 81).

We will now verify that π is unitarily equivalent to evaluation at some point in $\mathbf{T}$. Let $\mathcal{C}_0$ be the subalgebra of $\mathcal{C}$ consisting of all $f \in \mathcal{C}$ whose values are scalars (i.e., are in $\mathbf{C}1$ where 1 is the identity matrix) at each point in $\mathbf{T}$. Then $\pi(\mathcal{C}_0)$ is central in $\pi(\mathcal{C})$. Since π is irreducible, $\pi(\mathcal{C})$ is σ-weakly dense in $\mathcal{B}(H)$ (A 141), so $\pi(\mathcal{C}_0)$ is also central in $\mathcal{B}(H)$. Thus $\pi(\mathcal{C}_0)$ must consist of scalar multiples of the identity. Therefore π is a multiplicative linear functional on $\mathcal{C}_0$. By a well known theorem, every multiplicative linear functional on the commutative C*-algebra of all continuous complex valued functions on a compact set is evaluation at a point. Therefore there exists $\lambda_0 \in \mathbf{T}$ such that $\pi(f) = f(\lambda_0)$ for all $f \in \mathcal{C}_0$. Write M_2 for $M_2(\mathbf{C})$. For $1 \le i,j \le 2$ let $m_{ij} \in \mathcal{C}$ be the constant function whose value at each point of $\mathbf{T}$ is the standard matrix unit e_{ij} of M_2. Then every element of $\mathcal{C}$ admits a unique representation in the form $\sum_{ij} f_{ij} m_{ij}$ where each $f_{ij} \in \mathcal{C}_0$. Thus

$$\pi\Big(\sum_{ij} f_{ij} m_{ij}\Big) = \sum_{ij} f_{ij}(\lambda_0)\pi(m_{ij}).$$

Then $\pi(m_{ij})$ are 2×2 matrix units and generate the image $\pi(\mathcal{C})$ of $\mathcal{C}$, so $\pi(\mathcal{C})$ must be isomorphic to M_2. Identify $\pi(\mathcal{C})$ with M_2. Choose a unitary $v \in M_2$ such that $v\pi(m_{ij})v^* = e_{ij}$ for all i,j. Then $v\pi v^*$ is exactly evaluation at λ_0. Thus π is unitarily equivalent to evaluation at some point in $\mathbf{T}$, and we assume hereafter that π is given by evaluation at such a point λ_0.

We claim that $\pi(A) = M_2(\mathbf{C})_{\mathrm{sa}}$. Choose a continuous function $f : \mathbf{T} \to \mathbf{R}$ so that $f(\lambda_0) = 1$ and $f(-\lambda_0) = 0$. Given $m \in (M_2)_{\mathrm{sa}}$, the map $a : \lambda \mapsto f(\lambda)m + f(-\lambda)m^t$ is in A, and satisfies $\pi(a) = m$. Thus $\pi(A) = M_2(\mathbf{C})_{\mathrm{sa}}$.

Therefore $(\pi|A)^*$ is an affine isomorphism of the state space of M_2 (i.e., the 3-ball $\mathbf{B}^3$) onto a split face F of K (Lemma 11.8). Since $\sigma(a) = \widetilde{\sigma}(a) = (\pi(a)\xi \mid \xi)$ for $a \in A$, this split face contains σ. Thus the split face generated by each pure state σ is a 3-ball. If τ is another pure state of A, then either $\tau \in F$, in which case $\operatorname{face}(\tau, \sigma) = F$ is a 3-ball, or else τ and σ are separated by the split face F, in which case $\operatorname{face}(\tau, \sigma)$ is the line segment $[\sigma, \tau]$, cf. Lemma 5.54 or (A 29). It follows that the state space K of A has the 3-ball property, and so A is of complex type (Theorem 11.19).

Next we are going to construct a path between two distinct oriented facial 3-balls of A in the same fiber of $\mathcal{OB} \to \mathcal{B}$. We will see that this implies that K is not orientable. We identify the state space $S(M_2)$ with $\mathbf{B}^3$, and thus view an affine isomorphism from the state space of M_2 onto a 3-ball of K as an element of $\operatorname{Param}(K)$. For each $s \in [0, 1]$, define a homomorphism $\pi_s : A \to M_2$ by $\pi_s(f) = f(e^{i\pi s})$. Then π_s^* will be an affine isomorphism from $S(M_2) = \mathbf{B}^3$ onto a split face of K, which will also be a facial 3-ball of K. Thus π_s^* is a parameterization of a facial 3-ball of K; as usual we denote the associated oriented 3-ball by $[\pi_s^*]$.

Thus $s \mapsto \pi_s^*$ is a continuous map from $[0, 1]$ into $\operatorname{Param}(K)$, and so $s \mapsto [\pi_s^*]$ is a continuous path in $\mathcal{OB}$. If $\psi : M_2 \to M_2$ denotes the transpose map, then by the definition of A, $\pi_1(f) = \psi(\pi_0(f))$ for all $f \in A$. Thus $\pi_0^* \circ \psi^* = \pi_1^*$. It follows that the ranges of the orientations π_0^* and π_1^* are the same facial 3-ball. Furthermore, since the dual of the transpose map, extended from $S(M_2) = \mathbf{B}^3$ to $\mathbf{R}^3$, has determinant -1 (A 130), then the orientations given by π_0^* and π_1^* are different, so $[\pi_0^*]$ and $[\pi_1^*]$ are the two distinct elements of a single fiber of $\mathcal{OB} \to \mathcal{B}$. However, if K were orientable, then the bundle of oriented 3-balls would be trivial, and so $[\pi_0^*]$ and $[\pi_1^*]$ would lie in different connected components of $\mathcal{OB}$. This contradicts the fact that we have just found a path in $\mathcal{OB}$ from $[\pi_0^*]$ to $[\pi_1^*]$, so K is not orientable. □

Remark. In the example above, K is actually affinely isomorphic to a C*-algebra state space, but by Proposition 11.51 the affine isomorphism cannot be chosen to be a homeomorphism, since the state space of a C*-algebra is orientable (A 148). We sketch how to construct such an affine isomorphism. Let $\mathcal{D}$ be the C*-algebra of continuous functions f from $\mathbf{T}$ into $M_2(\mathbf{C})$ such that $f(-\lambda) = f(\lambda)$ for all $\lambda \in \mathbf{T}$. Let $\mathcal{J}$ be an ideal of $\mathcal{D}$ consisting of functions vanishing at 1, and similarly J the Jordan ideal of A consisting of functions vanishing at 1. For each $f \in \mathcal{J}_{\mathrm{sa}}$ or $f \in J$, we associate the map $s \mapsto f(e^{i\pi s})$ from $[0, 1]$ into $M_2(\mathbf{C})_{\mathrm{sa}}$. In this fashion, $\mathcal{J}_{\mathrm{sa}}$ and J both can be identified with the Jordan algebra of all continuous functions from $[0, 1]$ into $M_2(\mathbf{C})_{\mathrm{sa}}$ vanishing at 0 and 1. Let $F_{\mathcal{D}}$ and F_A be the annihilators of $\mathcal{J}_{\mathrm{sa}}$ and J in the respective state spaces of $\mathcal{D}$ and A. Since $\mathcal{J}$ and J are the kernels of homomorphisms onto $M_2(\mathbf{C})$ and $\mathcal{M}_2(\mathbf{C})_{\mathrm{sa}}$ respectively, it follows that $F_{\mathcal{D}}$ and F_A are affinely isomorphic to 3-balls. Their complementary split faces $F'_{\mathcal{D}}$ and F'_A are then affinely isomorphic to the state space of $\mathcal{J}$ and J respectively, i.e., the set of positive linear functionals of norm 1 on $\mathcal{J}$ or J respectively.

In particular, since $\mathcal{J}_{sa} \cong J$, then $F'_{\mathcal{D}}$ and F'_A are affinely isomorphic. This isomorphism, combined with the affine isomorphism of $F_{\mathcal{D}}$ and F_A, extends uniquely to an affine isomorphism of the state spaces of $\mathcal{D}$ and A.

We do not know if the state space of an arbitrary JB-algebra A of complex type admits a (possibly discontinuous) affine isomorphism onto the state space of a C*-algebra; nor do we know if A^{**} is isomorphic to the self-adjoint part of a von Neumann algebra for each JB-algebra A of complex type.

If K is the state space of a JB-algebra of complex type and if F is a split face of K, then the subspace $\mathrm{Param}(F)$ denotes the subset of $\mathrm{Param}(K)$ consisting of maps with range in F, with the relative topology from $\mathrm{Param}(K)$. $\mathcal{B}_F$ denotes the set of facial 3-balls contained in F with the quotient topology from $\mathrm{Param}(F)$. We define $\mathcal{OB}_F$ in an analogous way. Since $\mathrm{Param}(F)$ is saturated with respect to the action of $O(3)$ and $SO(3)$, and the actions of $O(3)$ and $SO(3)$ induce open quotient maps, $\mathcal{B}_F$ has the topology inherited from $\mathcal{B}_K$, and $\mathcal{OB}_F$ has the topology inherited from $\mathcal{OB}_K$ ([32, I.5.2, Prop. 4]).

If N is the normal state space of $\mathcal{B}(H)$, we can view N as a split face of the state space of $\mathcal{B}(H)$ (A 113). Then $\mathcal{B}_N$ is path connected (A 152), and more generally we have the following result for state spaces of JB-algebras of complex type.

11.52. Lemma. *If A is a JB-algebra of complex type, and F the split face generated by a pure state, then the space $\mathcal{B}_F$ of facial 3-balls in F is path connected.*

Proof. Let σ be a pure state on A, and let π be an irreducible concrete representation of A on a Hilbert space H associated with σ (cf. Corollary 11.33). Then π^* is an affine isomorphism from the normal state space N of $\mathcal{B}(H)$ onto the split face F generated by σ (Lemma 11.30). Furthermore, π^* is continuous for the weak topology on N given by the duality of $\mathcal{B}(H)$ and $\mathcal{B}(H)_*$ and the w*-topology on F. It follows that π^* induces a continuous surjective map from $\mathrm{Param}(N)$ to $\mathrm{Param}(F)$, and then also between the quotient spaces $\mathcal{B}_N$ and $\mathcal{B}_F$. Since $\mathcal{B}_N$ is path connected (A 152), then $\mathcal{B}_F$ is the continuous image of a path connected set, and therefore is itself path connected. $\square$

11.53. Definition. Let K_1 and K_2 be arbitrary convex sets, with each facial 3-ball in K_1 and K_2 given an orientation. Let $\phi : K_1 \to K_2$ be an affine isomorphism of K_1 onto a face of K_2. Then ϕ *preserves orientation* if ϕ carries the given orientation of each facial 3-ball F of K_1 onto the given orientation of $\phi(F)$, and *reverses orientation* if ϕ carries the given orientation of each facial 3-ball F of K_1 onto the opposite of the given orientation of $\phi(F)$.

Let A be a JB-algebra of complex type. We continue the notation of the previous section: $\mathcal{U}$ is the universal C*-algebra for A, Φ is the canonical *-anti-automorphism of $\mathcal{U}$, $\mathcal{K}$ is the state space of $\mathcal{U}$, K is the state space of A, and $r : \mathcal{K} \to K$ is the restriction map.

11.54. Lemma. *Let A be a JB-algebra of complex type, and let $\mathcal{U}$, Φ, $\mathcal{K}$ and r be as defined above. Then the map $\phi \mapsto r \circ \phi$ is a continuous map from Param($\mathcal{K}$) onto Param(K), and induces continuous maps $[\phi] \mapsto [r \circ \phi]$ from $\mathcal{OB}_{\mathcal{K}}$ to $\mathcal{OB}_K$, and $B \mapsto r(B)$ from $\mathcal{B}_{\mathcal{K}}$ onto $\mathcal{B}_K$*

Proof. Recall that r maps $\partial_e \mathcal{K}$ onto $\partial_e K$ (Corollary 11.37), and for each pure state σ on $\mathcal{U}$, r maps the minimal split face F_σ bijectively onto the minimal split face $F_{r(\sigma)}$ (Proposition 11.31). Therefore r maps facial 3-balls of $\mathcal{K}$ bijectively onto facial 3-balls of K, and every facial 3-ball of K is the image of a facial 3-ball of $\mathcal{K}$. Thus the map $\phi \mapsto r \circ \phi$ is a continuous map from Param($\mathcal{K}$) onto Param(K). This map commutes with the action of $O(3)$ and $SO(3)$, and hence induces a continuous map $[\phi] \mapsto [r \circ \phi]$ from $\mathcal{OB}_{\mathcal{K}}$ to $\mathcal{OB}_K$, and a continuous map $B \mapsto r(B)$ from $\mathcal{B}_{\mathcal{K}}$ to $\mathcal{B}_K$. □

We now make use of the fact that the state space of any C*-algebra (in particular, that of $\mathcal{U}$) has a global orientation induced by the algebra (A 148), i.e., an orientation assigned to each facial 3-ball in such a way as to give a continuous cross-section to the bundle $\mathcal{OB}_{\mathcal{K}} \to \mathcal{B}_K$. This orientation is defined in (A 147), and a key property of this orientation, which will be used below, is that the dual Φ^* of a *-anti-isomorphism Φ of $\mathcal{U}$ reverses the orientation of $\mathcal{K}$ (A 157).

11.55. Lemma. *Let A be a JB-algebra of complex type whose state space K is orientable, let $\mathcal{U}$, Φ, $\mathcal{K}$ and r be as defined before Lemma 11.54, and let a global orientation of K be given. Then for each pure state σ on A, there is a unique pure state $\tilde{\sigma}$ of $\mathcal{U}$ such that the restriction map r is an orientation preserving affine isomorphism from the split face $F_{\tilde{\sigma}}$ onto the split face F_σ, and orientation reversing from $F_{\Phi^*\tilde{\sigma}}$ onto F_σ.*

Proof. Let ω be any extension of σ to a pure state on $\mathcal{U}$. Then r is an affine isomorphism from F_ω onto F_σ (Proposition 11.31). Equip each facial 3-ball in $\mathcal{K}$ with the orientation induced by $\mathcal{U}$ (A 148). Let $[\phi_0]$ and $[\phi_1]$ be two of the resulting oriented 3-balls in F_ω. By Lemma 11.52 there is a path of facial 3-balls from $\phi_0(\mathbf{B}^3)$ to $\phi_1(\mathbf{B}^3)$. Composing with the orientation induced by $\mathcal{U}$ gives a path of oriented 3-balls $[\phi_t]$ from $[\phi_0]$ to $[\phi_1]$.

By continuity of the map $[\phi] \mapsto [r \circ \phi]$ (Lemma 11.54), $[r \circ \phi_t]$ is a continuous path in $\mathcal{OB}_K$. Then the path must lie either in the closed cross-section given by the specified orientation of K, or in its opposite, since these cross-sections are both open and closed. Thus $[r \circ \phi_0]$ and $[r \circ \phi_1]$ either both match the given orientation or both match its opposite. Hence r either preserves orientation or reverses orientation for all facial 3-balls in F_ω.

Finally, since Φ is a *-anti-isomorphism, then Φ^* reverses the orientation on $\mathcal{K}$ given by $\mathcal{U}$. Since $r \circ \Phi^* = r$, then r preserves orientation on one of F_ω, $F_{\Phi^*\omega}$, and reverses orientation on the other. Since ω and $\Phi^*\omega$ are the unique pure states of $\mathcal{U}$ that restrict to σ (Proposition 11.36), the proof of the lemma is complete. □

Note that if σ is an abelian pure state in Lemma 11.55, then σ has a unique pure state extension $\widetilde{\sigma}$, and $\Phi^*\widetilde{\sigma} = \widetilde{\sigma}$ (Proposition 11.36). In that case, the split face $F_{\widetilde{\sigma}}$ is the single point $\{\widetilde{\sigma}\}$ (Lemma 11.35), so there are no facial 3-balls in $F_{\widetilde{\sigma}}$. Thus the statements about orientation in Lemma 11.55 still hold for $\widetilde{\sigma}$, but vacuously.

Note also that the above proof relies upon the non-trivial fact that a JB-algebra A of complex type is universally reversible (Theorem 11.26). This fact was used to show that A is imbedded as the set of self-adjoint fixed points of the canonical *-anti-automorphism Φ of $\mathcal{U}$ (Corollary 11.27), which was then used to prove that a pure state σ of K is the restriction of no more than two pure states of $\mathcal{K}$ (in Proposition 11.36, referred to in the proof above).

We are going to show that every JB-algebra of complex type whose state space is orientable is isomorphic to the self-adjoint part of a C*-algebra. For the reader's convenience we will first reproduce from [AS] some constructs and results on C*-algebras that will be needed.

Let $\mathcal{A}$ be a C*-algebra, $a \in \mathcal{A}$, and let σ be any state on $\mathcal{A}$ such that $\sigma(a^*a) \neq 0$. Then we define a state σ_a by

$$\sigma_a(b) = \sigma(a^*a)^{-1}\sigma(a^*ba), \tag{11.23}$$

cf. (A 149). Note that if $(\pi_\sigma, H_\sigma, \xi_\sigma)$ is the GNS-representation of $\mathcal{A}$ associated with σ, then

$$\sigma(a^*a) = (\pi_\sigma(a^*a)\xi_\sigma \mid \xi_\sigma) = \|\pi_\sigma(a)\xi_\sigma\|^2, \tag{11.24}$$

and then, for $b \in \mathcal{A}$,

$$\sigma_a(b) = (\pi_\sigma(b)\eta \mid \eta) \quad \text{with} \quad \eta = \|\pi_\sigma(a)\xi_\sigma\|^{-1}\pi_\sigma(a)\xi_\sigma. \tag{11.25}$$

This shows that a representing vector for σ_a is obtained by applying the representing operator $\pi_\sigma(a)$ for a to the representing vector ξ_σ for σ, and then normalizing to get a unit vector. In this way, the action $\sigma \mapsto \sigma_a$ of a on states corresponds to the action of $\pi_\sigma(a)$ on their representing vectors.

11.56. Lemma. *Let $\mathcal{A}$ be a C*-algebra, a an element of $\mathcal{A}$, and σ a pure state on $\mathcal{A}$ such that $\sigma(a^*a) \neq 0$. Then σ_a is also a pure state. The two pure states σ_a and σ are unitarily equivalent, and are distinct iff*

$$|\sigma(a)|^2 < \sigma(a^*a), \tag{11.26}$$

in which case the face generated by σ and σ_a is a facial 3-ball $B(\sigma, \sigma_a)$.

Proof. By (11.25), we have $\sigma_a = \omega_\eta \circ \pi_\sigma$. Since vector states are pure normal states on $\mathcal{B}(H)$ (A 118), and π_σ^* is an affine isomorphism from the normal state space of $\mathcal{B}(H)$ onto the split face F_σ generated by σ (A 138), then $\sigma_a = \omega_\eta \circ \pi_\sigma$ is a pure state in F_σ. Since the split face F_σ is minimal among all split faces (not only those containing σ) (A 140), the split face generated by σ_a must be all of F_σ. By (A 142), σ_a and σ are unitarily equivalent.

Since $\pi = \pi_\sigma$ is irreducible, $\pi_\sigma(\mathcal{A})$ is σ-weakly dense in $\mathcal{B}(H)$ (A 141). Therefore σ_a and σ will be distinct iff the representing vectors η and ξ_σ are linearly independent. This will be the case iff strict inequality holds in the Schwarz inequality, i.e., by (11.25) and (11.24), iff

$$1 > |(\eta|\xi_\sigma)| = \|\pi_\sigma(a)\xi_\sigma\|^{-1}|(\pi_\sigma(a)\xi_\sigma \mid \xi_\sigma)| = \sigma(a^*a)^{-1/2}|\sigma(a)|.$$

Thus σ_a and σ are distinct iff (11.26) is satisfied.

We know that the face generated by a pair of distinct pure states of a C*-algebra is a 3-ball (A 143). Therefore the face generated by σ and σ_a is a 3-ball when the equality (11.26) is satisfied. □

By the lemma above, each $a \in \mathcal{A}$ determines a map $\sigma \mapsto B(\sigma, \sigma_a)$ from pure states to facial 3-balls whose domain is the w*-open set of all pure states for which (11.26) is satisfied.

Observe for use in the theorem below that if σ is any non-abelian pure state, then there exists $a \in \mathcal{A}$ such that σ is in the domain of the map $\sigma \mapsto B(\sigma, \sigma_a)$. In fact, in this case π_σ is not one dimensional, so there exists $a \in \mathcal{A}$ such that $\pi_\sigma(a)\xi_\sigma$ and ξ_σ are linearly independent, and then strict inequality will hold in the Schwarz inequality, so by (11.24)

$$|\sigma(a)|^2 = |(\pi_\sigma(a)\xi_\sigma \mid \xi_\sigma)|^2 < \|\pi_\sigma(a)\xi_\sigma\|^2 = \sigma(a^*a).$$

11.57. Lemma. *If $\mathcal{A}$ is a C*-algebra and $a \in \mathcal{A}$, the map $\sigma \mapsto B(\sigma, \sigma_a)$ is continuous from the w*-topology of its domain (defined by (11.26)) to the topology of the space $\mathcal{B}$ of facial 3-balls.*

Proof. Cf. (A 151). □

Now we can prove the main result of this section.

11.58. Theorem. *Let A be a JB-algebra with state space K. Then A is isomorphic to the self-adjoint part of a C*-algebra iff K has the 3-ball property and is orientable.*

Proof. The necessity of the latter two conditions follows from (A 108), (A 143) and (A 148). Suppose conversely that K has the 3-ball property and is orientable. Then A is of complex type (Theorem 11.19). Fix a global orientation of K. Let $\mathcal{U}$ be the universal enveloping C*-algebra of A, and $\mathcal{K}$ the state space of $\mathcal{U}$ with the global orientation induced by the C*-algebra $\mathcal{U}$. We are going to construct a w*-closed split face F of $\mathcal{K}$, such that the restriction map r is a bijection from F onto K. By Proposition 11.42, that will complete the proof.

For each $\sigma \in \partial_e K$, let $\phi(\sigma)$ be the unique pure state in $\mathcal{K}$ such that the restriction map r is an orientation preserving affine isomorphism from $F_{\phi(\sigma)}$ onto F_σ (cf. Lemma 11.55). Let F be the w*-closed convex hull of $\{\phi(\sigma) \mid \sigma \in \partial_e K\}$. We are going to show that F is the w*-closed convex hull of the collection $\{F_{\phi(\sigma)} \mid \sigma \in \partial_e K\}$ of split faces, and thus is itself a split face (A 115).

The w*-closed convex hull of this collection evidently contains F, so to prove equality it suffices to show that $F_{\phi(\sigma)} \subset F$ for each $\sigma \in \partial_e K$. Fix $\sigma \in \partial_e K$. If ω is a pure state in $F_{\phi(\sigma)}$, then $r(\omega)$ is a pure state on A (Proposition 11.31). Since pure states generate minimal split faces, $F_\omega = F_{\phi(\sigma)}$. By definition ϕ maps $F_{\phi(\sigma)}$ onto F_σ, so in particular $r(\omega) \in F_\sigma$. It follows that $F_{r(\omega)} = F_\sigma$. Thus r is an orientation preserving affine isomorphism from F_ω onto $F_{r(\omega)}$, so by the definition of ϕ, $\phi(r(\omega)) = \omega$, which implies that $\omega \in F$. Thus $\partial_e F_{\phi(\sigma)} \subset F$. The split face $F_{\phi(\sigma)}$ is the σ-convex hull of its extreme points (A 139). Therefore $F_{\phi(\sigma)} \subset F$, which completes the proof that F is a split face of $\mathcal{K}$.

Now we will show that $\partial_e F = \{\phi(\sigma) \mid \sigma \in \partial_e K\}$. Since $r(\phi(\sigma)) = \sigma$, this will show that the restriction map r is a bijection from $\partial_e F$ onto $\partial_e K$, and thus that r is a bijection from F onto K (Lemma 11.39). By Proposition 11.42, this will imply that A is the self-adjoint part of a C*-algebra and complete the proof.

Suppose for contradiction that $\omega \in \partial_e F$ and $\omega \notin \phi(\partial_e K)$. If ω is abelian, then $r(\omega)$ is abelian and $\omega = \phi(r(\omega))$ (Corollary 11.37), contrary to our supposition. Thus we may assume ω is not abelian. By Proposition 11.36, the pure state $r(\omega)$ of K has exactly two pure state extensions, namely ω and $\Phi^*\omega$ where Φ is the canonical *-anti-automorphism of $\mathcal{U}$. The pure state extension $\phi(r(\omega))$ cannot be equal to ω since $\omega \notin \phi(\partial_e K)$. Therefore $\phi(r(\omega)) = \Phi^*\omega$. Recall from Lemma 11.55 that the restriction map r is orientation preserving on one of the two split faces F_ω and $F_{\Phi^*\omega}$ and orientation reversing on the other. Since $\Phi^*\omega = \phi(r(\omega))$, then r is orientation preserving on $F_{\Phi^*\omega}$. Hence r is orientation reversing on F_ω.

By Milman's theorem (A 33), $\partial_e F$ is contained in the w*-closure of $\phi(\partial_e K)$, so there is a net $\{\rho_\gamma\}$ in $\phi(\partial_e K)$ such that $\rho_\gamma \to \omega$ (in the w*-topology). As observed above, there exists $a \in \mathcal{U}$ such that ω is in the domain of the map $\rho \mapsto B(\rho, \rho_a)$. Since this domain is a w*-open subset of $\partial_e \mathcal{K}$, it contains ρ_γ for all γ large enough. By Lemma 11.57, $B(\rho_\gamma, (\rho_\gamma)_a) \mapsto B(\omega, \omega_a)$) in the topology of $\mathcal{B}_\mathcal{K}$.

Write B_γ in place of $B(\widetilde{\sigma}_\gamma, (\widetilde{\sigma}_\gamma)_a)$ and B_0 in place of $B(\omega, \omega_a)$. Then $B_\gamma \to B_0$ implies that $r(B_\gamma) \to r(B_0)$ (Lemma 11.54). Therefore, by continuity of the given orientation on K, the orientation of $r(B_\gamma)$ converges to the orientation of $r(B_0)$. On the other hand, the map induced by r on the space of oriented 3-balls is continuous (Lemma 11.55), and r preserves orientation on B_γ and reverses it on B_0 (as shown above), so the orientation of $r(B_\gamma)$ must converge to the opposite of the orientation on $r(B_0)$. This contradiction completes the proof that $\partial_e F$ equals $\{\widetilde{\sigma} \mid \sigma \in \partial_e K\}$. We are done. □

We record for use below some additional facts established in the proof above. Let F be the w*-closed convex hull of $\{\phi(\sigma) \mid \sigma \in \partial_e K\}$, where $\phi(\sigma)$ is the state $\widetilde{\sigma}$ defined in Lemma 11.55. Then F is a split face, every pure state in F has the form $\phi(\sigma)$ for some $\sigma \in \partial_e K$, and the restriction map is an orientation preserving affine isomorphism from the split face F onto K.

Characterization of C*-algebra state spaces among convex sets

Our main result now follows by combining previous results.

11.59. Theorem. *A compact convex set K is affinely homeomorphic to the state space of a C*-algebra (with the w*-topology) iff K satisfies the following properties:*

(i) *Every norm exposed face of K is projective.*
(ii) *Every $a \in A(K)$ admits a decomposition $a = b - c$ with $b, c \in A(K)^+$ and $b \perp c$.*
(iii) *The σ-convex hull of the extreme points of K is a split face of K, and K has the 3-ball property.*
(iv) *K is orientable.*

Proof. This follows by combining our characterization of the state spaces of JB-algebras (Theorem 9.38), and our characterization of those JB-algebra state spaces that are state spaces of C*-algebras (Theorem 11.58). □

Remark. In (iii) above, we could replace the assumption that the σ-convex hull of the extreme points of K is a split face of K by the assumption that for each compression P, the dual map P^* sends pure states to multiples of pure states. The latter is one of the three pure state properties, cf. Definition 9.2. The other two pure state properties follow from the 3-ball property, cf. Proposition 9.14, and now the proof of Theorem 11.59 still is valid, if we use the part of Theorem 9.38 that involves the pure state properties.

According to (A 156), there is a 1-1 correspondence of Jordan compatible C*-products on a C*-algebra and orientations of its state space. The following result essentially says the same thing, adapted to our present context, but the results in this chapter allow us to give a much shorter proof than for (A 156) in [AS].

11.60. Corollary. *If A is a JB-algebra whose state space K has the 3-ball property and is orientable, then C*-products on $A \oplus iA$ are in 1-1 correspondence with global orientations of K.*

Proof. There is a 1-1 correspondence between C*-products on $A \oplus iA$ and those w*-closed split faces F of the state space $\mathcal{K}$ of the universal C*-algebra of A for which r maps F bijectively onto K (Proposition 11.42). Thus what remains is to establish a 1-1 correspondence of such split faces of $\mathcal{K}$ and global orientations of K.

Suppose first that we are given a global orientation Θ of K. As in the proof of Theorem 11.58 we equip $\mathcal{K}$ with the global orientation induced by the C*-algebra $\mathcal{U}$, and assign to each pure state $\sigma \in K$ the unique pure state $\phi(\sigma) \in \mathcal{K}$ for which the restriction map r is orientation preserving from the orientation of $F_{\phi(\sigma)}$ to the orientation Θ on K. Let F_Θ be the w*-closed convex hull of $\{\phi(\sigma) \mid \sigma \in \partial_e K\}$. Then it follows from the remarks following Theorem 11.58, that F_Θ is a split face, and that r is an orientation preserving affine isomorphism from F_Θ onto K. Hence the orientation Θ of K can be recovered from the orientation of $\mathcal{K}$ restricted to F_Θ, so the map $\Theta \mapsto F_\Theta$ is 1-1 from the set of global orientations of K into the set of w*-closed split faces F of $\mathcal{K}$ for which r maps F bijectively onto K.

To prove surjectivity, we suppose that we are given a w*-closed split face F of $\mathcal{K}$ which is mapped bijectively by r onto K. The global orientation on $\mathcal{K}$ restricts to give a global orientation of F. Since $r : F \to K$ is an affine homeomorphism, there is a unique global orientation Θ of K such that $r : F \to K$ is orientation preserving. For each $\omega \in \partial_e F$, the restriction map r is an orientation preserving affine isomorphism on the generated split face $F_\omega \subset F$. Thus ω has the defining property of the pure state $\phi(r(\omega))$, so $\omega = \phi(r(\omega))$. Hence $\partial_e F \subset \{\phi(\sigma) \mid \sigma \in \partial_e K\}$. On the other hand, if $\sigma \in \partial_e K$, and $\omega \in \partial_e F$ is the unique pure state such that $r(\omega) = \sigma$, then $\omega = \phi(r(\omega) = \phi(\sigma)$, so $\{\phi(\sigma) \mid \sigma \in \partial_e K\} \subset \partial_e F$, and thus equality holds. Since F_Θ is by definition the w*-closed convex hull of $\{\phi(\sigma) \mid \sigma \in \partial_e K\}$, we conclude by the Krein–Milman theorem that $F = F_\Theta$, as desired. Thus the map $\Theta \mapsto F_\Theta$ is a 1-1 correspondence from global orientations of K to w*-closed split faces F mapped bijectively onto K by r. This completes the proof of the 1-1 correspondence of C*-products on A and global orientations of K. □

The above characterization of C*-algebra state spaces was obtained by adding new assumptions to those characterizing JB-algebra state spaces. We did the same thing in Proposition 10.27, where we added the assumption about the existence of a dynamical correspondence. However, the two sets of extra assumptions are quite different in character. The one in Theorem 11.59 is "geometric and local", whereas the one in Proposition 10.27 is "algebraic and global". The former proceeds from local data of geometric nature (a continuous choice of orientations of facial 3-balls) from which a C*-product is constructed, while the latter postulates the existence of a globally defined map $a \mapsto \psi_a$ from elements to skew order derivations, from which the Lie part of the C*-product is obtained by an explicit formula (equation (6.14)). Thus, the starting point is closer to the final goal in the latter than in the former of the two characterizations. But the characterization in terms of dynamical correspondences is nevertheless of considerable interest, both because it explains the relationship with Connes' approach to the characterization of σ-finite von Neumann algebras, and because of its physical interpretation, which gives the elements of A the "double identity" of observables and generators of one-parameter groups (explained in Chapter 6).

Note, however, that the conditions of Theorem 11.59 also relate to physics. We know that a Euclidean 3-ball is (affinely isomorphic to) the state space of $M_2(\mathbf{C})$, which models a two-level quantum system (A 119). Thus the key role that is played by 3-balls in Theorem 11.59 reflects the fact that properties that distinguish quantum theory from classical physics are present already in the simplest non-classical quantum systems, the two-level systems, and that multilevel systems do not introduce any new features in this respect.

The state space of a C*-algebra has at least two different global orientations. One is the orientation induced by the given C*-product (A 147), and the other is the opposite orientation, obtained by reversing the orientation of each facial 3-ball. But there may be many other global orientations. For example, if we have a general finite dimensional C*-algebra $\mathcal{A} = \sum_j^n M_j(\mathbf{C})$, then each summand $\mathcal{M}_j(\mathbf{C})$ has just two possible associative Jordan compatible multiplications, each associated with one of the two orientations of the finite dimensional split face corresponding to that summand. In each such split face, there are just two ways to choose a continuous orientation, and so the orientation for each such split face is determined by the orientation of a single facial 3-ball. There are n connected components of the space of facial 3-balls, each consisting of those 3-balls contained in a single minimal split face, and there are 2^n possible global orientations.

Finally we observe that if the global orientation of a C*-algebra state space is changed to the opposite orientation, then the associated dynamical correspondence $a \mapsto \psi_a$ (determined by the associated Lie product) will be changed to the opposite dynamical correspondence $a \mapsto -\psi_a$, and then the generated one-parameter group $\exp(t\psi_a)$ will be changed to the one-parameter group $\exp(-t\psi_a)$, which has opposite sign for the time parameter t. In this sense, changing to the opposite orientation can be thought of as time reversal.

Notes

The results in this chapter first appeared in [11]. These results can be thought of as saying that the state space together with an orientation is a dual object for C*-algebras. Instead of the state space, some papers focus on the set of pure states as a dual object, e.g., [118], and the papers of Brown [34] and Landsman [90].

Appendix

Below are the results from *State spaces of operator algebras: basic theory, orientations and C*-products*, referenced as [AS] below, that have been referred to in the current book. Many of these are well known results; see [AS] for attribution.

A 1. A face F of a convex set $K \subset X$ is said to be *semi-exposed* if there exists a collection $\mathcal{H}$ of closed supporting hyperplanes of K such that $F = K \cap \bigcap_{H \in \mathcal{H}} H$, and it is said to be *exposed* if $\mathcal{H}$ can be chosen to consist of a single hyperplane. Thus, F is exposed iff there exists a $y \in Y$ and an $\alpha \in \mathbf{R}$ such that $\langle x, y \rangle = \alpha$ for all $x \in F$ and $\langle x, y \rangle > \alpha$ for all $x \in K \setminus F$. A point $x \in K$ is a *semi-exposed point* if $\{x\}$ is a semi-exposed face of K, and it is an *exposed point* if $\{x\}$ is an exposed face of K. The intersection of all closed supporting hyperplanes containing a given set $B \subset K$ will meet K in the smallest semi-exposed face containing B, called *the semi-exposed face generated by* B. [AS, Def. 1.1]

A 2. A non-empty subset of a linear space X is a *cone* if $C + C \subset C$ and $\lambda C \subset C$ for all scalars $\lambda \geq 0$. A cone C is *proper* if $C \cap (-C) = \{0\}$. Let C be a proper cone C of a linear space $\subset X$, and order X by $x \leq y$ if $y - x \in C$. If F is a non-empty subset of C other than $\{0\}$, then F is a face of C iff F is a subcone of C (i.e., $F + F \subset F$ and $\lambda F \subset F$ for all $\lambda \geq 0$) for which the following implication holds:

$$0 \leq y \leq x \in F \quad \Rightarrow \quad y \in F.$$

It follows that if $x \in C$, then

$$\text{face}_C(x) = \{y \in C \mid y \leq \lambda x \quad \text{for some} \quad \lambda \in \mathbf{R}^+\}.$$

[AS, p. 3]

A 3. Two ordered vector spaces X and Y are in *separating ordered duality* if they are in separating duality and the following statements hold for $x \in X$ and $y \in Y$:

$$x \geq 0 \quad \Leftrightarrow \quad \langle x, y \rangle \geq 0 \text{ all } y \geq 0,$$
$$y \geq 0 \quad \Leftrightarrow \quad \langle x, y \rangle \geq 0 \text{ all } x \geq 0.$$

[AS, Def. 1.2]

A 4. We will say that two positive projections P, Q on an ordered vector space X are *complementary* (and also that Q is a complement of P and vice versa) if

$$\ker^+ Q = \operatorname{im}^+ P, \quad \ker^+ P = \operatorname{im}^+ Q.$$

We will say that P, Q are *complementary in the strong sense* if

$$\ker Q = \operatorname{im} P, \quad \ker P = \operatorname{im} Q.$$

[AS, Def. 1.3]

A 5. We say that a convex set $K \subset X$ is *the free convex sum* of two convex subsets F and G, and we write $K = F \oplus_c G$, if $K = \operatorname{co}(F \cup G)$ and F, G are affinely independent. Observe that if $K = F \oplus_c G$, then the two sets F and G must be faces of K. We say that a face F of K is a *split face* if there exists another face G such that $K = F \oplus_c G$. In this case G is unique; we call it the *complementary split face* of F, and we will use the notation $F' = G$. More specifically, F' consists of all points $x \in K$ whose generated face in K is disjoint from F, in symbols

$$F' = \{x \in K \mid \operatorname{face}_K(x) \cap F = \emptyset\}.$$

[AS, Def. 1.4]

A 6. If F and G are split faces of a convex set K, then every $x \in K$ can be decomposed as $x = \sum_{i,j=1}^{2} \alpha_{ij} x_{ij}$ where $\alpha_{ij} \geq 0$ for $i, j = 1, 2$ and $x_{11} \in F \cap G$, $x_{12} \in F \cap G'$, $x_{21} \in F' \cap G$, $x_{22} \in F' \cap G'$. This decomposition is unique in that every α_{ij} is uniquely determined and every x_{ij} with non-vanishing coefficient α_{ij} is uniquely determined. [AS, Lemma 1.5]

A 7. If F and G are split faces of a convex set K, then $F \cap G$ and $\operatorname{co}(F \cup G)$ are also split faces and

$$(F \cap G)' = \operatorname{co}(F' \cup G').$$

[AS, Prop. 1.6]

A 8. Let K be a compact convex set regularly embedded in a locally convex vector space X. K is said to be a *Choquet simplex* if X is a lattice with the ordering defined by the cone X^+ generated by K. [AS, Def. 1.8]

A 9. An ordered normed vector space V with a generating cone V^+ is said to be a *base norm space* if V^+ has a base K located on a hyperplane

H ($0 \notin H$) such that the closed unit ball of V is $\mathrm{co}(K \cup -K)$. The convex set K is called the *distinguished base* of V. [AS, Def. 1.10]

A 10. If V is an ordered vector space with a generating cone V^+, and if V^+ has a base K located on a hyperplane H ($0 \notin H$) such that $B = \mathrm{co}(K \cup -K)$ is *radially compact*, then the *Minkowski functional*

$$\|\rho\| = \inf\{\alpha > 0 \mid \rho \in \alpha B\}$$

is a norm on V making V a base norm space with distinguished base K. [AS, p. 9]

A 11. If V is a base norm space with distinguished base K, then the restriction map $f \mapsto f|K$ is an order and norm preserving isomorphism of V^* onto the space $A_b(K)$ of all real valued bounded affine functions on K equipped with pointwise ordering and supremum norm. [AS, Prop. 1.11]

A 12. A positive element e of an ordered vector space A is said to be an *order unit* if for all $a \in A$ there exists $\lambda \geq 0$ such that

$$-\lambda e \leq a \leq \lambda e.$$

The order unit is called *Archimedean* if for all $a \in A$ there exists $\lambda \geq 0$ such that

$$na \leq e \ \text{ for } n = 1, 2, \ldots \quad \Rightarrow \quad a \leq 0.$$

[AS, Def. 1.12]

A 13. An ordered linear space A with an Archimedean order unit e admits a norm

$$\|a\| = \inf\{\lambda > 0 \mid -\lambda e \leq a \leq \lambda e\} \tag{1}$$

which satisfies the following relation for all $a \in A$:

$$-\|a\|e \leq a \leq \|a\|e.$$

[AS, p. 10]

A 14. An ordered normed linear space A is said to be an *order unit space* if the norm can be obtained as in equation (1) from an Archimedean order unit, which is called *the distinguished order unit* and will be denoted by 1. [AS, Def. 1.13]

A 15. If A is an order unit space, then every positive linear map $T: A \to A$ is norm continuous with norm $\|T\| = \|T1\|$. [AS, Lemma 1.15]

A 16. A linear functional ρ on an order unit space A is positive iff it is bounded with $\|\rho\| = \rho(1)$. [AS, Lemma 1.16]

A 17. A linear functional ρ on an order unit space A is called a *state* if it is positive and $\rho(1) = 1$. The set K of all states on A is called the *state space* of A. An extreme point of the state space is called a *pure state.* [AS, Def. 1.17]

A 18. If a is an element of an order unit space A with state space K, then

$$a \in A^+ \quad \Longleftrightarrow \quad \rho(a) \geq 0 \text{ for all } \rho \in K,$$

and

$$\|a\| = \sup\{|\rho(a)| \,|\, \rho \in K\}.$$

[AS, Lemma 1.18]

A 19. The dual of an order unit space A is a base norm space and the dual of a base norm space V is an order unit space. More specifically, the distinguished base of A^* is the state space of A, and the distinguished order unit of V^* is the unit functional on V. [AS, Thm. 1.19]

A 20. Let A be an order unit space and define for each $a \in A$ the function $\widehat{a}$ on the state space K by writing $\widehat{a}(\rho) = \rho(a)$ for all $\rho \in K$. Then $a \mapsto \widehat{a}$ is an order and norm preserving isomorphism of A onto a dense subspace of the space $A(K)$ of all real valued w*-continuous affine functions on the w*-compact convex set K. If A is complete in the order unit norm, then the range of the map $a \mapsto \widehat{a}$ will be all of $A(K)$. [AS, Thm. 1.20]

A 21. An order unit space A and a base norm space V are said to be in separating *order and norm duality* if they are in separating ordered duality and if for $a \in A$ and $\rho \in V$,

$$\|a\| = \sup_{\|\sigma\| \leq 1} |\langle a, \sigma \rangle| \quad \text{and} \quad \|\rho\| = \sup_{\|b\| \leq 1} |\langle b, \rho \rangle|.$$

[AS, Def. 1.21]

A 22. If an order unit space A and a base norm space V are in separating order and norm duality, then $\langle 1, \rho \rangle = \|\rho\|$ for all $\rho \in V^+$, the norm of V is additive on V^+, and the distinguished base K is located on the hyperplane $H = \{\rho \in V \mid \langle 1, \rho \rangle = 1\}$. Moreover, if $T : A \to A$ is a continuous positive linear map, then the equality $\langle T1, \rho \rangle = \|T^* \rho\|$ holds for an arbitrary $\rho \in V^+$. [AS, Lemma 1.22]

A 23. Two subsets F and G of a convex set K in a vector space X are said to be *antipodal* if they are located on parallel supporting hyperplanes on opposite sides of K, i.e., if there exists a linear functional h on X with values in $[0, 1]$ such that $h(\rho) = 1$ for $\rho \in F$ and $h(\rho) = 0$ for $\rho \in G$. If we have two antipodal singletons $F = \{\rho\}$, $G = \{\sigma\}$, we will also say that the two points ρ and σ are *antipodal.* [AS, Def. 1.23]

A 24. Two points ρ and σ in the positive cone V^+ of a base norm space V are said to be *orthogonal* (written $\rho \perp \sigma$) if

$$\|\rho - \sigma\| = \|\rho\| + \|\sigma\|.$$

In particular, two points ρ and σ in the distinguished base K are orthogonal if $\|\rho - \sigma\| = 2$, i.e., if the distance between ρ and σ is maximal. [AS, Def. 1.24]

A 25. Let V be a base norm space, and let $\rho, \sigma \in V^+$ with $\rho \neq 0$, $\sigma \neq 0$. Then $\rho \perp \sigma$ iff $\|\rho\|^{-1}\rho$ and $\|\sigma\|^{-1}\sigma$ are antipodal points of the distinguished base K. [AS, Prop. 1.25]

A 26. Each point $\omega \neq 0$ in a base norm space V can be decomposed as a difference of two orthogonal positive components, i.e., there exists $\rho, \sigma \in V^+$ such that

$$\omega = \rho - \sigma \quad \text{and} \quad \|\omega\| = \|\rho\| + \|\sigma\|.$$

[AS, Prop. 1.26]

A 27. If V is a base norm space with dual space $A = V^*$, then A and V are in separating order and norm duality. [AS, Cor. 1.27]

A 28. Each split face F of the distinguished base K of a base norm space is norm closed. [AS, Prop. 1.29]

A 29. Let K be the base of a base norm space V, and let σ and τ be extreme points of K. If there is a split face F such that $\sigma \in F$ and

$\tau \notin F$, then the face of K generated by σ and τ is the line segment $[\sigma, \tau]$. [AS, Prop. 1.30]

A 30. (Hahn–Banach separation) Let X be a locally convex space and let $B \subset X$ be closed, convex and balanced (i.e., $x \in B$ and $|\lambda| \leq 1$ implies $\lambda x \in B$). If $x_0 \in X$ and $x_0 \notin B$, then there exists an $\omega \in X^*$ such that $|\omega(x)| \leq 1$ for all $x \in B$ and $\omega(x_0) > 1$. [AS, Prop. 1.31]

A 31. (Bipolar theorem) Let X and Y be vector spaces over $\mathbf{R}$ or $\mathbf{C}$ in separating duality under a bilinear form $\langle\ ,\ \rangle$. If M is a weakly closed subspace of X with the annihilator M° in Y and if $M^{\circ\circ}$ is the annihilator of M° in X, then $M^{\circ\circ} = M$. [AS, Thm. 1.35]

A 32. If Y is a norm closed subspace of a Banach space X and $\phi : X \to X/Y$ is the quotient map, then the dual map ϕ^* is an isometric isomorphism from $(X/Y)^*$ onto the annihilator of Y in X^*. [AS, Prop. 1.36]

A 33. (Milman's theorem) If E is a closed subset of a compact convex set K in a locally convex vector space, then the only extreme points in $\overline{\mathrm{co}}(E)$ are points in E. Thus $K = \overline{\mathrm{co}}(E)$ only if $\partial_e K \subset E$. [AS, Thm. 1.37]

A 34. (Krein–Šmulian theorem) A convex set in the dual X^* of a Banach space X is w^*-closed iff its intersection with each positive multiple of the closed unit ball in X^* is w^*-closed. [AS, Thm. 1.38]

A 35. In this book we will use the word *algebra* to denote a real or complex vector space with a bilinear product (not necessarily associative or commutative). An algebra A will be called *normed* if it is equipped with a norm such that for $a, b \in A$,

$$\|ab\| \leq \|a\|\|b\|. \tag{2}$$

As usual, an associative normed algebra which is (norm) complete, is called a *Banach algebra.* [AS, Def. 1.42]

A 36. (Lattice version of the Stone–Weierstrass theorem) Let X be a compact Hausdorff space and let A be a closed linear subspace of $C_{\mathbf{R}}(X)$ which is closed under the lattice operations and contains the unit function (taking the value 1 at all points of X). Then $A = C_{\mathbf{R}}(X)$ iff A separates the points of X. [AS, Thm. 1.43]

A 37. An ordered vector space X is said to be *monotone complete (monotone σ-complete)* if every increasing net (sequence) in X which is bounded above has a least upper bound in X. (Then also every decreasing net (sequence) in X which is bounded below has a greatest lower bound in X.) [AS, p. 3]

A 38. If X is a compact Hausdorff space and $C_{\mathbf{R}}(X)$ is monotone complete, then for each $a \in C_{\mathbf{R}}(X)$, $E = \overline{\{s \in X \mid a(s) > 0\}}$ is simultaneously closed and open in X. Furthermore,

(i) $a \geq 0$ on E, $a \leq 0$ on $X \backslash E$, and E is the smallest closed subset of X such that $a \leq 0$ on $X \setminus E$,
(ii) E is the smallest closed subset of X for which $\chi_E a^+ = a^+$ (pointwise product),
(iii) χ_E is the supremum in $C_{\mathbf{R}}(X)$ of an increasing sequence in face(a^+).

If $a \geq 0$, then we write $r(a)$ for χ_E defined as above. [AS, Lemma 1.55]

A 39. Let X be a compact Hausdorff space such that $C_{\mathbf{R}}(X)$ is monotone complete and let $a \in C_{\mathbf{R}}(X)$. Then there is a unique family $\{e_\lambda\}_{\lambda \in \mathbf{R}}$ of projections in $C_{\mathbf{R}}(X)$ such that

(i) $e_\lambda a \leq \lambda e_\lambda$ and $e'_\lambda a \geq \lambda e'_\lambda$ for all $\lambda \in \mathbf{R}$,
(ii) $e_\lambda = 0$ for $\lambda < -\|a\|$ and $e_\lambda = 1$ for $\lambda > \|a\|$,
(iii) $e_\lambda \leq e_\mu$ for $\lambda < \mu$,
(iv) $\bigwedge_{\mu > \lambda} e_\mu = e_\lambda$ for all $\lambda \in \mathbf{R}$.

The family $\{e_\lambda\}$ is given by $e_\lambda = 1 - r\big((a - \lambda 1)^+\big)$. For each increasing finite sequence $\gamma = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ with $\lambda_0 < -\|a\|$ and $\lambda_n > \|a\|$, define $\|\gamma\| = \max(\lambda_i - \lambda_{i-1})$ and $s_\gamma = \sum_{i=1}^n \lambda_i (e_{\lambda_i} - e_{\lambda_{i-1}})$. Then

$$\lim_{\|\gamma\| \to 0} \|s_\gamma - a\| = 0.$$

[AS, Thm. 1.58]

A 40. A lattice L with least element 0 and greatest element 1 is *orthomodular* if there is a map $p \mapsto p'$, called the *orthocomplementation*, that satisfies

(i) $p'' = p$,
(ii) $p \leq q$ implies $p' \geq q'$,
(iii) $p \vee p' = 1$ and $p \wedge p' = 0$,
(iv) If $p \leq q$, then $q = p \vee (q \wedge p')$.

If $p \leq q'$ in an orthomodular lattice L, then we say p is *orthogonal* to q and write $p \perp q$. If $p \perp q$, then we sometimes write $p + q$ in place of $p \vee q$. [AS, Def. 1.60]

A 41. Let L be a complete lattice. A non-zero element p of L is an *atom* if each element $q \leq p$ either equals p or is the zero element. L is *atomic* if every non-zero element is the least upper bound of atoms, and an element p of L is *finite* if p is the least upper bound of a finite set of atoms. [AS, Def. 1.61]

A 42. Let p be a finite element in a complete atomic lattice. The minimum number of atoms whose least upper bound is p is called the *dimension* of p and is denoted $\dim(p)$. [AS, Def. 1.62]

A 43. Let p and q be elements in a complete lattice L. We say q *covers* p if $p < q$ and there is no element strictly between p and q. We say L has the *covering property* if for all $p \in L$ and all atoms $u \in L$,

$$p \vee u = p \quad \text{or} \quad p \vee u \text{ covers } p. \tag{3}$$

We say L has the *finite covering property* if (3) holds for all finite p in L and all atoms $u \in L$. [AS, Def. 1.63]

A 44. Let p be a finite element in a complete atomic orthomodular lattice L with the finite covering property. Then p can be expressed as a sum of atoms. In fact, $p = p_1 + \cdots + p_k$ for each maximal set of orthogonal atoms $p_1, \ldots, p_k$ under p, and the cardinality of any set of atoms with sum p is $\dim(p)$. Furthermore, every element $q \leq p$ is finite with $\dim(q) \leq \dim(p)$. [AS, Prop. 1.66]

A 45. An algebra is said to be *power associative* if parentheses can be inserted freely in products with identical factors. Thus the n-th power a^n of an element a is well defined in this case. We will write A^n for the set of all n-th powers of elements in a power associative algebra A. [AS, Def. 1.70]

A 46. An order unit space A which is also a power associative complete normed algebra (for the order unit norm) and satisfies the following requirements:

(i) the distinguished order unit 1 is a multiplicative identity,
(ii) $a^2 \in A^+$ for each $a \in A$,

will be called an *order unit algebra*.

[AS, Def. 1.72]

A 47. If A is an order unit algebra, then $A^+ = A^2$. [AS, Lemma 1.73]

A 48. If A is an order unit algebra, then the following are equivalent:

(i) A is associative and commutative,
(ii) $ab \in A^+$ for each pair $a, b \in A^+$,

(iii) $A \cong C_{\mathbf{R}}(X)$ for a compact Hausdorff space X.

[AS, Thm. 1.74]

A 49. The *Jordan product* in an associative (real or complex) algebra A is given by

$$a \circ b = \tfrac{1}{2}(ab + ba) \qquad \text{for } a, b \in A\ .$$

[AS, Def. 1.76]

A 50. Let A_0 be a linear subspace of an order unit space A, and suppose that $a, b \mapsto ab$ is a bilinear map from $A_0 \times A_0$ into A such that for $a, b \in A_0$ (and with the standard notation $a^2 = aa$),

(i) $ab = ba$,
(ii) $-1 \leq a \leq 1 \;\Rightarrow\; 0 \leq a^2 \leq 1$.

Then for all $a, b \in A_0$,

$$\|ab\| \leq \|a\|\ \|b\|.$$

[AS, Lemma 1.79]

A 51. Suppose A is a complete order unit space which is a power associative commutative algebra where the distinguished order unit 1 acts as an identity. Then A is an order unit algebra iff the following implication holds for $a \in A$:

$$-1 \leq a \leq -1 \quad \Rightarrow \quad 0 \leq a^2 \leq 1,$$

[AS, Lemma 1.80]

A 52. Suppose A is a real Banach space which is equipped with a power associative and commutative bilinear product with identity element 1. Then A is an order unit algebra with positive cone consisting of all squares, distinguished order unit 1 and the given norm, iff for $a, b \in A$,

(i) $\|ab\| \leq \|a\|\|b\|$,
(ii) $\|a^2\| = \|a\|^2$,
(iii) $\|a^2\| \leq \|a^2 + b^2\|$.

Moreover, the ordering of A is uniquely determined by the norm and the identity element 1. [AS, Thm. 1.81]

A 53. Let A be a commutative order unit algebra that is the dual of a base norm space V such that multiplication in A is separately w*-continuous. Then for each $a \in A$ and each $\varepsilon > 0$ there are orthogonal projections $p_1, \dots, p_n$ in the w*-closed subalgebra $W(a, 1)$ generated by a and 1 and scalars $\lambda_1, \dots, \lambda_n$ such that

$$\|a - \sum_{i=1}^{n} \lambda_i p_i\| < \varepsilon.$$

[AS, Thm. 1.84]

A 54. For each pair of elements x, y in a unital *-algebra,

$$\operatorname{sp}(yx) \setminus \{0\} = \operatorname{sp}(xy) \setminus \{0\} .$$

[AS, Lemma 1.90]

A 55. If a is a continuous complex valued function on a compact Hausdorff space X, then the spectrum of a relative to the Banach algebra $C_{\mathbf{C}}(X)$ is equal to the range of a, i.e., $\operatorname{sp}(a) = \{a(s) \mid s \in X\}$. Moreover, the spectrum of a relative to $C_{\mathbf{C}}(X)$ is the same as the spectrum of a relative to the norm closed *-subalgebra $C(a, 1)$ generated by a and the unit function 1. The same statements hold with $C_{\mathbf{R}}(X)$ in the place of $C_{\mathbf{C}}(X)$. [AS, Lemma 1.91]

A 56. If $\mathcal{A}$ is a unital complex *-algebra and the real vector space $A = \mathcal{A}_{\mathrm{sa}}$ is an order unit algebra with distinguished order unit equal to the multiplicative identity and with the Jordan product induced from $\mathcal{A}$, then $A^+ = A^2 = \{x^*x \mid x \in \mathcal{A}\}$. [AS, Lemma 1.92]

A 57. A *C*-algebra* is a complex Banach *-algebra $\mathcal{A}$ such that $\|x^*x\| = \|x\|^2$ for all $x \in \mathcal{A}$. Unless otherwise stated, we will assume that C*-algebras mentioned have an identity. [AS, Def. 1.93]

A 58. If $\mathcal{A}$ is a unital C*-algebra, then its self-adjoint part $A = \mathcal{A}_{\mathrm{sa}}$ is an order unit algebra for the Jordan product and the norm induced from $\mathcal{A}$, and with distinguished order unit equal to the multiplicative identity and positive cone $A^+ = A^2 = \{x^*x \mid x \in \mathcal{A}\}$. [AS, Thm. 1.95]

A 59. If $\mathcal{A}$ is a unital complex *-algebra and $\mathcal{A}_{\mathrm{sa}}$ is a complete order unit algebra for the Jordan product induced from $\mathcal{A}$ with distinguished order unit equal to the multiplicative identity, then the positive cone consists of all elements x^*x where $x \in \mathcal{A}$, and $\mathcal{A}$ is a C*-algebra for a unique norm which restricts to the order unit norm on $\mathcal{A}_{\mathrm{sa}}$. [AS, Thm. 1.96], where "order unit space" should read "order unit algebra".

A 60. An element of a C*-algebra $\mathcal{A}$ is said to be *positive* if it is of the form x^*x for some $x \in \mathcal{A}$, and we will denote the set of all positive elements in $\mathcal{A}$ by $\mathcal{A}^+$. A linear functional ρ on a C*-algebra $\mathcal{A}$ with identity element 1 is called a *state* if it is positive on positive elements and $\rho(1) = 1$. The set of all states on $\mathcal{A}$ is called the *state space* of $\mathcal{A}$, and it will be denoted by $S(\mathcal{A})$, or just by K when there is no need to specify $\mathcal{A}$, and the extreme points of K are called *pure states.* [AS, Def. 1.97]

A 61. A *representation* of a *-algebra $\mathcal{A}$ is a *-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(H)$ where H is a real or complex Hilbert space. If we want to specify H, then we say that π is a *representation of* $\mathcal{A}$ *on* H. If π is injective, then we say that π is *faithful.* If $\xi \in H$ and the linear subspace $\pi(\mathcal{A})\xi = \{\pi(x)\xi \mid x \in \mathcal{A}\}$ is dense in H, then we say that ξ is a *cyclic vector* (or a *generating vector*) for π. A representation with a cyclic vector is called a *cyclic representation.* [AS, Def. 1.98]

A 62. Suppose $\mathcal{A}$ is a (real or complex) unital *-algebra whose self-adjoint part $A = \mathcal{A}_{\mathrm{sa}}$ is a complete order unit space with distinguished order unit equal to the multiplicative identity and with the positive cone $A^+ = \{x^*x \mid x \in \mathcal{A}\}$. For each state ρ on $\mathcal{A}$ there is a Hilbert space H_ρ, a cyclic unit vector $\xi_\rho \in H_\rho$ and a representation π_ρ of $\mathcal{A}$ on H_ρ such that $\rho(x) = (\pi_\rho(x)\xi_\rho \mid \xi_\rho)$ for all $x \in \mathcal{A}$. Moreover, $\|\pi_\rho(x)\| \le \|x^*x\|^{1/2}$ for all $x \in \mathcal{A}$. [AS, Prop. 1.100]

A 63. If $\mathcal{A}$ is a C*-algebra and ρ is a state on $\mathcal{A}$, then the representation described above is the *GNS-representation* associated with ρ. We will denote it by $(\pi_\rho, H_\rho, \xi_\rho)$, or just by π_ρ when there is no need to specify H_ρ or ξ_ρ. Thus

$$\rho(x) = (\pi_\rho(x)\xi_\rho|\xi_\rho) \quad \text{for all } x \in \mathcal{A}.$$

[AS, Def. 1.101]

A 64. (Gelfand–Naimark Theorem) Each unital C*-algebra $\mathcal{A}$ is isometrically *-isomorphic to a C*-subalgebra of $\mathcal{B}(H)$ for a complex Hilbert space H. One such isomorphism is the direct sum of all GNS-representations associated with states on $\mathcal{A}$. [AS, Thm. 1.102]

A 65. Suppose $\mathcal{A}$ is a real *-algebra with identity 1 whose self-adjoint part $\mathcal{A}_{\mathrm{sa}}$ is a complete order unit space with the distinguished order unit 1 and the positive cone $A^+ = \{x^*x \mid x \in \mathcal{A}\}$. Then there is a representation π of $\mathcal{A}$ on a (complex) Hilbert space H which is an isometry on $\mathcal{A}_{\mathrm{sa}}$. [AS, Prop. 1.103]

A 66. Let A be an ordered Banach space (i.e., a real Banach space ordered by a positive cone A^+). A bounded linear operator δ on A is an *order derivation* if $e^{t\delta}$ is a positive operator on A, i.e., if $e^{t\delta}(A^+) \subset A^+$

for all $t \in \mathbf{R}$, or equivalently, if $\{e^{t\delta}\}$ is a one-parameter group of order automorphisms. We will write $D(A)$ for the set of all order derivations on A. [AS, Def. 1.104]

A 67. Let A be a complete order unit space, and δ a bounded linear map from A into A. The following are equivalent:

(i) δ is an order derivation.
(ii) If $x \in A^+$, $0 \leq \sigma \in A^*$ and $\sigma(x) = 0$, then $\sigma(\delta x) = 0$.

[AS, Prop. 1.108]

A 68. The set $D(A)$ of order derivations of a complete order unit space A is a real linear space closed under Lie brackets $[\delta_1, \delta_2] = \delta_1\delta_2 - \delta_2\delta_1$. [AS, Prop. 1.114]

A 69. Let $\mathcal{A}$ be a (unital) C*-algebra with state space K. Its self-adjoint part $\mathcal{A}_{\rm sa}$ is a complete order unit space under the norm induced from $\mathcal{A}$, the ordering determined by the positive cone $\mathcal{A}^+ = A^2 = \{x^*x \mid x \in A\}$, and with the distinguished order unit equal to the multiplicative identity 1. [AS, Prop. 2.2]

A 70. Let $\mathcal{A}$ be a C*-algebra with state space K, and define for each $a \in \mathcal{A}_{\rm sa}$ the function $\widehat{a}$ on K by $\hat{a}(\rho) = \rho(a)$ for $\rho \in K$. Then the map $a \mapsto \widehat{a}$ is an order and norm preserving linear isomorphism of $\mathcal{A}_{\rm sa}$ onto the space $A(K)$ of all continuous affine functions on K. [AS, Prop. 2.3]

A 71. A linear functional ρ on a C*-algebra $\mathcal{A}$ is positive iff it is bounded with $\|\rho\| = \rho(1)$. [AS, Prop. 2.11]

A 72. Let $\mathcal{A}$ and $\mathcal{B}$ be two possibly non-unital C*-algebras. If Φ is a *-homomorphism from $\mathcal{A}$ into $\mathcal{B}$, then $\Phi(\mathcal{A})$ is a C*-subalgebra of $\mathcal{B}$. [AS, Prop. 2.17]

A 73. Let p be a projection in a C*-algebra $\mathcal{A}$ and set $q = p'$. If $a \in \mathcal{A}$ is self-adjoint, then the following two statements are equivalent:

(i) $aq = 0$ (by taking adjoints also $qa = 0$),
(ii) $ap = a$ (by taking adjoints also $pa = a$),

and if $a \geq 0$, then they are also equivalent to each of the following four statements:

(iii) $qaq = 0$,
(iv) $pap = a$,
(v) $a \leq \|a\|p$,
(vi) $a \in \text{face}(p)$.

[AS, Lemma 2.20]

A 74. Let p be a projection in a C^*-algebra $\mathcal{A}$ and set $q = p'$. If $\omega \in \mathcal{A}^*$ is self-adjoint, then the following four statements are equivalent:

(i) $q \cdot \omega = 0$ (by taking adjoints also $\omega \cdot q = 0$),
(ii) $p \cdot \omega = \omega$ (by taking adjoints also $\omega \cdot p = \omega$),

and if $\omega \geq 0$, then they are also equivalent to each of the following four statements,

(iii) $\omega(q) = 0$,
(iv) $\omega(p) = \|\omega\|$,
(v) $q \cdot \omega \cdot q = 0$,
(vi) $p \cdot \omega \cdot p = \omega$.

[AS, Lemma 2.22]

A 75. Let $\mathcal{A}$ be a C*-algebra. Then the extreme points of $\mathcal{A}_1^+$ are precisely the projections in $\mathcal{A}$. [AS, Prop. 2.23]

A 76. If $\{a_\gamma\}$ is an increasing net bounded above in $\mathcal{B}(H)_{\mathrm{sa}}$, then $\{a_\gamma\}$ has a least upper bound $a \in \mathcal{B}(H)_{\mathrm{sa}}$, and a is also a strong (and weak) limit of $\{a_\gamma\}$. Similarly for a decreasing net and its greatest lower bound. [AS, Prop. 2.29]

A 77. Let π_1 be a representation of a C*-algebra $\mathcal{A}$ on a Hilbert space H_1 with cyclic vector ξ. Another representation π_2 of $\mathcal{A}$ on a Hilbert space H_2 is unitarily equivalent to π_1 iff it has a cyclic vector ξ_2 such that

$$(\pi_1(x)\xi_1|\xi_1) = (\pi_2(x)\xi_2|\xi_2) \quad \text{for all } x \in \mathcal{A},$$

and then the equivalence can be achieved by a unitary $u \in \mathcal{B}(H_1, H_2)$ such that $u\xi_1 = \xi_2$. [AS, Prop. 2.31]

A 78. A representation π of a C*-algebra $\mathcal{A}$ on a Hilbert space H is said to be *irreducible* if $\{0\}$ and H are the only closed subspaces invariant under $\pi(\mathcal{A})$. [AS, Def. 2.34]

A 79. For each subset $\mathcal{S}$ of $\mathcal{B}(H)$ for a Hilbert space H, the set of operators in $\mathcal{B}(H)$ that commute with all operators in $\mathcal{S}$ is called the *commutant* of $\mathcal{S}$ and is denoted by $\mathcal{S}'$. [AS, Def. 2.35]

A 80. A representation π of a C*-algebra $\mathcal{A}$ on a Hilbert space H is irreducible iff it has trivial commutant, i.e., iff $\pi(\mathcal{A})' = \mathbf{C}1$. [AS, Lemma 2.39]

A 81. The GNS-representation associated with a state ρ on a C*-algebra is irreducible iff ρ is a pure state. [AS, Cor. 2.41]

A 82. The irreducible representations of a C*-algebra $\mathcal{A}$ separate the points of $\mathcal{A}$. [AS, Cor. 2.43]

A 83. If $\mathcal{A}$ is a C*-algebra with state space K, then $\bigoplus_{\rho\in K}\pi_\rho$ (the direct sum of all GNS-representations) is called the *universal representation* of $\mathcal{A}$. [AS, Def. 2.45]

A 84. If $\mathcal{A}$ is a *-subalgebra of $\mathcal{B}(H)$, then the *weak*, respectively *strong*, (operator) topology of $\mathcal{A}$ is the locally convex topology determined by the semi-norms $a \mapsto |(a\xi|\eta)|$, respectively $a \mapsto \|a\xi\|$, where $\xi,\eta \in H$; the σ*-weak* topology is determined by the semi-norms

$$a \mapsto \left|\sum_{i=1}^{\infty}(a\xi_i|\eta_i)\right|$$

where $\{\xi_i\}$ and $\{\eta_i\}$ are two sequences in H such that $\sum_{i=1}^{\infty}\|\xi_i\|^2 < \infty$ and $\sum_{i=1}^{\infty}\|\eta_i\|^2 < \infty$, and the σ*-strong* topology is determined by the semi-norms

$$a \mapsto \left(\sum_{i=1}^{\infty}\|a\xi_i\|^2\right)^{1/2}$$

where $\{\xi_i\}$ is a sequence in H such that $\sum_{i=1}^{\infty}\|\xi_i\|^2 < \infty$. Note that $a_\alpha \to a$ σ-strongly iff $(a_\alpha - a)^*(a_\alpha - a) \to 0$ σ-weakly. [AS, Def. 2.27 and Def. 2.64]

A 85. For each positive operator $a \in \mathcal{B}(H)$ we define the *trace* of a as the sum $\operatorname{tr}(a) = \sum_\gamma (a\xi_\gamma|\xi_\gamma)$ (with values in $[0,\infty]$), where $\{\xi_\gamma\}$ is any orthonormal basis in H. For arbitrary $a \in \mathcal{B}(H)$ we define $\|a\|_1 = \operatorname{tr}(|a|)$. [AS, Def. 2.51]

A 86. An operator $a \in \mathcal{B}(H)$ is of *trace class* if $\|a\|_1 < \infty$. The set of trace class operators is denoted by $\mathcal{T}(H)$ (or by $L^1(H)$). By Lemma 2.53, $\mathcal{T}(H)$ is a two-sided ideal in $\mathcal{B}(H)$ and $\|a\|_1$ is a norm on this ideal; we call it the *trace norm*. [AS, Def. 2.54]

A 87. If r is a trace class operator, we define a linear functional ω_r on $\mathcal{B}(H)$ by $\omega_r(a) = \operatorname{tr}(ar)$. The map $r \mapsto \omega_r$ is an isometric order isomorphism of the ordered Banach space $\mathcal{T}(H)$ of trace class operators onto the subspace of $\mathcal{B}(H)^*$ which consists of all σ-weakly continuous linear functionals. [AS, Thm. 2.68]

A 88. If $\mathcal{A}$ is a unital *-subalgebra of $\mathcal{B}(H)$, then the bicommutant $\mathcal{A}''$ is the closure of $\mathcal{A}$ in any of the weak, strong, σ-weak, or σ-strong topologies. [AS, Cor. 2.78]

A 89. A unital *-subalgebra $\mathcal{M}$ of $\mathcal{B}(H)$ is called a *concrete von Neumann algebra* if it is weakly closed. A C*-algebra which admits a faithful representation as a concrete von Neumann algebra is called an *abstract von Neumann algebra.* [AS, Def. 2.80]

A 90. A positive linear functional (or state) ϕ on a C*-algebra $\mathcal{A}$ is said to be *normal* if $\phi(a) = \lim_\gamma \phi(a_\gamma)$ for each increasing net $\{a_\gamma\}$ with least upper bound a in $\mathcal{A}$. More generally, a linear functional on $\mathcal{A}$ is said to be *normal* if it is a linear combination of normal positive linear functionals on $\mathcal{A}$. [AS, Def. 2.82]

A 91. Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space H. The σ-weakly continuous linear functionals on $\mathcal{M}$ form a norm closed subspace of $\mathcal{M}^*$. [AS, Lemma 2.85]

A 92. If $\mathcal{M}$ is a von Neumann algebra acting on a Hilbert space, then the σ-weakly (and σ-strongly) continuous linear functionals on $\mathcal{M}$ are precisely the normal linear functionals on $\mathcal{M}$. [AS, Cor. 2.87]

A 93. The σ-weak topology on a von Neumann algebra $\mathcal{M}$ is determined by the seminorms $x \mapsto |\omega(x)|$ where ω is a normal state on $\mathcal{M}$, and the σ-strong topology on $\mathcal{M}$ is determined by the seminorms $x \mapsto \omega(x^*x)^{1/2}$ where ω is a normal state on $\mathcal{M}$. [AS, Cor. 2.90]

A 94. If $\mathcal{M}$ is a von Neumann algebra, then the map $\Psi : \mathcal{M} \mapsto (\mathcal{M}_*)^*$ defined by $(\Psi a)(\phi) = \phi(a)$ for $a \in \mathcal{M}$ and $\phi \in \mathcal{M}_*$ is a surjective isometric isomorphism and a homeomorphism from the σ-weak topology on $\mathcal{M}$ to the w*-topology on $(\mathcal{M}_*)^*$. Moreover, $\Psi a \geq 0$ iff $a \geq 0$. [AS, Thm. 2.92]

A 95. A C*-algebra $\mathcal{A}$ is a von Neumann algebra iff it satisfies either one of the following two conditions:

(i) $\mathcal{A}_{\mathrm{sa}}$ is monotone complete and the normal states separate the points of $\mathcal{A}$.
(ii) There is a Banach space $\mathcal{B}$ such that $\mathcal{A}$ is (isometrically isomorphic to) the dual of $\mathcal{B}$.

Moreover, if condition (ii) is satisfied, then $\mathcal{B}$ is unique, in fact it is (up to an isometry) the Banach space of normal linear functionals on $\mathcal{A}$. [AS, Thm. 2.93]

A 96. The self-adjoint part $(\mathcal{M}_*)_{\mathrm{sa}}$ of the predual of a von Neumann algebra $\mathcal{M}$ is a base norm space whose distinguished base is the normal state space K of $\mathcal{M}$. [AS, Cor. 2.96]

A 97. Let $\mathcal{M}$ be a von Neumann algebra with normal state space K, and define for each $a \in \mathcal{M}_{\mathrm{sa}}$ the function $\widehat{a}$ on K by $\hat{a}(\rho) = \rho(a)$

for $\rho \in K$. Then the map $a \mapsto \widehat{a}$ is an order and norm preserving linear isomorphism of $\mathcal{M}_{\rm sa}$ onto the space $A_b(K)$ of all bounded affine functions on K. [AS, Cor. 2.97]

A 98. The set of projections in a von Neumann algebra is an orthomodular lattice (cf. (A 40)) with $p \wedge q = pq$ and $p \vee q = p + q - pq$ for a pair of commuting projections p, q. [AS, Thm. 2.104]

A 99. To each self-adjoint element a in a von Neumann algebra $\mathcal{M}$ there exists a unique resolution of the identity $\{e_\lambda\}_{\lambda \in \mathbf{R}}$ such that

(i) $e_\lambda a \leq \lambda e_\lambda$ and $(1 - e_\lambda)a \geq \lambda(1 - e_\lambda)$ for all $\lambda \in \mathbf{R}$,
(ii) $e_\lambda = 0$ for $\lambda < -\|a\|$ and $e_\lambda = 1$ for $\lambda > \|a\|$,
(iii) e_λ commutes with a for all $\lambda \in \mathbf{R}$.

The projections e_λ are given by $e_\lambda = 1 - r\big((a - \lambda 1)^+\big)$, and

(iv) $a = \int \lambda \, de_\lambda$.

Here for $x \in \mathcal{M}^+$, $r(x)$ denotes the least projection p such that $pxp = x$, and is called the *range projection* of x. [AS, Def. 2.107 and Thm. 2.110]

A 100. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be C*-algebras. Then $T \mapsto T^*$ is a 1-1 correspondence of unital order isomorphisms T from $\mathcal{A}_1$ onto $\mathcal{A}_2$, and affine homeomorphisms from the state space of $\mathcal{A}_2$ onto the state space of A_1. Similarly, if $\mathcal{M}_1$ and $\mathcal{M}_2$ are von Neumann algebras, then there is a 1-1 correspondence of unital order isomorphisms from $\mathcal{M}_1$ onto M_2 and affine isomorphisms from the normal state space of $\mathcal{M}_2$ onto the normal state space of M_1. [AS, Cor. 2.122]

A 101. If $\mathcal{A}$ is a C*-algebra with the universal representation π (cf. (A 83)), then the von Neumann algebra $\widetilde{\mathcal{A}} = \overline{\pi(\mathcal{A})}$ (weak closure) is called the *enveloping von Neumann algebra* of $\mathcal{A}$. $\widetilde{\mathcal{A}}$ is isomorphic (as a Banach space) to the bidual $\mathcal{A}^{**}$ of $\mathcal{A}$, and the normal state space of $\widetilde{\mathcal{A}}$ is affinely isomorphic to the state space of $\mathcal{A}$. By common usage, we will denote the enveloping von Neumann algebra by $\mathcal{A}^{**}$. Thus we identify $\widetilde{\mathcal{A}}$ with $\mathcal{A}^{**}$ equipped with the induced involution and product. [AS, Def. 2.123, Cor. 2.126 and Cor. 2.127]

A 102. Let $\mathcal{A}$ be a C*-algebra with state space K and enveloping von Neumann algebra $\mathcal{A}^{**}$ (cf. (A 101)). Then $(\mathcal{A}^{**})_{\rm sa}$ is isomorphic (as an order unit space) to the space $A_b(K)$ of all bounded affine functions on K under the map $a \mapsto \widehat{a}$ (with $\hat{a}(\rho) = \rho(a)$ for $\rho \in K$) which is the unique σ-weakly continuous extension of the corresponding isomorphism of $\mathcal{A}_{\rm sa}$ onto $A(K)$ (cf. (A 20)). [AS, Prop. 2.128]

A 103. A unital *-homomorphism $\phi : \mathcal{A} \to \mathcal{M}$ from a C*-algebra $\mathcal{A}$ into a von Neumann algebra $\mathcal{M}$ has a unique extension to a normal *-homomorphism $\widetilde{\phi} : \mathcal{A}^{**} \to \mathcal{M}$. [AS, Thm. 2.129]

A 104. If σ is a normal state on a von Neumann algebra $\mathcal{M}$, then there is a smallest central projection $c \in \mathcal{M}$ such that $\sigma(c) = 1$. This projection c is called the *central carrier* of σ, and is denoted $c(\sigma)$. [AS, Prop. 2.130 and Def. 2.131]

A 105. Let J be a σ-weakly closed two-sided ideal in a von Neumann algebra $\mathcal{M}$. Then there is a unique central projection c such that $J = c\mathcal{M}$; in fact, c is the unique two-sided identity of J. [AS, Cor. 3.17]

A 106. If $\mathcal{M}$ is a von Neumann algebra and F is a subset of $\mathcal{M}_*^+$, then the least projection $p \in \mathcal{M}$ such that $\omega(p) = \|\omega\|$ (or equivalently $\omega(p') = 0$) for all $\omega \in F$ is called the *carrier projection* of F and is denoted by $\text{carrier}(F)$. [AS, Def. 3.20]

A 107. Let F be a face of the normal state space K of a von Neumann algebra $\mathcal{M}$ and set $p = \text{carrier}(F)$. Then the norm closure $\overline{F}$ of F consists of all $\sigma \in K$ such that $\sigma(p) = 1$. [AS, Prop. 3.30]

A 108. A norm closed face F of the normal state space of a von Neumann algebra is norm exposed. [AS, Prop. 3.34]

A 109. Let $\mathcal{M}$ be a von Neumann algebra with normal state space K, and denote by $\mathcal{F}$ the set of all norm closed faces of K, by $\mathcal{P}$ the set of all projections in $\mathcal{M}$, and by $\mathcal{J}$ the set of all σ-weakly closed left ideals in $\mathcal{M}$, each equipped with the natural ordering. Then there are an order preserving bijection $\Phi : p \mapsto F$ from $\mathcal{P}$ to $\mathcal{F}$ and an order reversing bijection $\Psi : p \mapsto J$ from $\mathcal{P}$ to $\mathcal{J}$, and hence also an order reversing bijection $\Theta = \Psi \circ \Phi^{-1}$ from $\mathcal{F}$ to $\mathcal{J}$. The maps Φ, Ψ, Θ and their inverses are explicitly given by the equations

(i) $F = \{\sigma \in K \mid \sigma(p) = 1\}$, $p = \text{carrier}(F)$,

(ii) $J = \{a \in \mathcal{M} \mid ap = 0\}$, $p = r(J)'$,

(iii) $$J = \{a \in \mathcal{M} \mid \sigma(a^*a) = 0 \ \textit{all}\ \sigma \in F\},$$ $$F = \{\sigma \in K \mid \sigma(a^*a) = 0 \ \textit{all}\ a \in J\}.$$

Here in (ii) $r(J)$ denotes the right identity of J. [AS, Thm. 3.35]

A 110. If p is a projection in a von Neumann algebra $\mathcal{M}$ with normal state space K, the associated face $F_p = \{\sigma \in K \mid \sigma(p) = 1\}$ satisfies

$$F_p = \{\sigma \in K \mid p \cdot \sigma \cdot p = \sigma\}.$$

[AS, equation (3.14)]

A 111. If p is a projection in a von Neumann algebra $\mathcal{M}$, then the following are equivalent:

(i) p is central,
(ii) $a = pap + p'ap'$ for all $p \in \mathcal{M}$,
(iii) $\omega = p \cdot \omega \cdot p + p' \cdot \omega \cdot p'$ for all $\omega \in \mathcal{M}_*$.

[AS, Lemma 3.39]

A 112. Let p be a projection in a von Neumann algebra $\mathcal{M}$, let J be the associated σ-weakly closed left ideal in $\mathcal{M}$, and let F be the associated norm closed face of the normal state space K of $\mathcal{M}$. Then the following are equivalent:

(i) p is a central projection,
(ii) J is a two-sided ideal,
(iii) F is a split face.

If these conditions are satisfied, then the complementary split face of F is the norm closed face F' associated with p'. [AS, Prop. 3.40]

A 113. The normal state space of a von Neumann algebra $\mathcal{M}$ is a split face of the state space K of $\mathcal{M}$. [AS, Cor. 3.42]

A 114. Let $\mathcal{A}$ be a C*-algebra with state space K. Then the canonical 1-1 correspondence between closed two-sided ideals J in $\mathcal{A}$ and w*-closed split faces F in K maps the ideal J to its annihilator $J^\circ \cap K$ in K and the face F to its annihilator F° in $\mathcal{A}$. [AS, Cor. 3.63]

A 115. If $\{F_\alpha\}$ is a collection of split faces in the state space K of a C*-algebra $\mathcal{A}$, then the w*-closed convex hull $\overline{\text{co}}(\bigcup_\alpha F_\alpha)$ is a split face. [AS, Prop. 3.77]

A 116. Let $\mathcal{M}$ be a von Neumann algebra and let P be a positive, σ-weakly continuous, normalized projection on $\mathcal{M}_{\text{sa}}$. There exists a projection $p \in \mathcal{M}$ such that $P = U_p$ iff P is bicomplemented; in this case p is unique: $p = P1$, and the complement Q of P is also unique: $Q = U_q$ (where $q = 1 - p$). [AS, Thm. 3.81]

A 117. The conditional expectation $E = U_p + U_q$ associated with a projection p with complement $q = p'$ in a von Neumann algebra $\mathcal{M}$ is the unique normal positive projection of $\mathcal{M}$ onto the relative commutant $\{p\}^c = p\mathcal{M}p + q\mathcal{M}q$. [AS, Thm. 3.85]

A 118. The extreme points of the normal state space K of $\mathcal{B}(H)$ are the vector states ω_η (with η a unit vector in H), and each $\sigma \in K$ is an infinite convex combination of vector states, i.e., $\sigma = \sum_{i=1}^\infty \lambda_i \omega_{\eta_i}$ (norm convergent sum in $\mathcal{B}(H)_*$) where $\sum_{i=1}^\infty \lambda_i = 1$ and where $\lambda_i \geq 0$ and $\|\eta_i\| = 1$ for $i = 1, 2, \ldots$. [AS, Prop. 4.1]

A 119. For each Hilbert space H the normal state space K of $\mathcal{B}(H)$ can be inscribed in a Hilbert ball; in fact, K can be inscribed in a ball defined by the norm of the space $\mathcal{HS}(H)$ of Hilbert–Schmidt operators by the map $\omega_r \mapsto r$. If $\dim H = n < \infty$, then K is a convex set of dimension $n^2 - 1$ which can be inscribed in a Euclidean ball of the same dimension. If $n = 2$, then K is affinely isomorphic to the full Euclidean ball B^3; in fact, the positive trace-one matrix representing a state $\omega \in K$ relative to an orthonormal basis $\{\xi_1, \xi_2\}$ can be written in the form

$$\frac{1}{2}\begin{bmatrix} 1+\beta_1 & \beta_2 + i\beta_3 \\ \beta_2 - i\beta_3 & 1-\beta_1 \end{bmatrix}$$

where $\omega \mapsto (\beta_1, \beta_2, \beta_3)$ is an affine isomorphism of K onto B^3. [AS, Thm. 4.4]

A 120. The face generated by two distinct extreme points of the normal state space of $\mathcal{B}(H)$ is a Euclidean 3-ball. [AS, Cor. 4.8]

A 121. A map v from a Hilbert space H_1 into a Hilbert space H_2 is said to be *conjugate linear* if it is additive and $v(\lambda\xi) = \bar{\lambda} v\xi$ for all $\lambda \in \mathbf{C}$ and $\xi \in H_1$. A conjugate linear isometry from H_1 into H_2 is called *a conjugate unitary.* If j is a conjugate unitary map from the Hilbert space H to itself and $j^2 = 1$, then j is said to be a *conjugation* of H. [AS, Def. 4.21]

A 122. If v is a conjugate linear isometry from a Hilbert space H_1 into a Hilbert space H_2, then $(v\xi|v\eta) = \overline{(\xi|\eta)}$ for all pairs $\xi, \eta \in H_1$. [AS, Lemma 4.22]

A 123. Every orthonormal basis $\{\xi_i\}$ in a Hilbert space H determines a conjugation, namely

$$j \sum_i \alpha_i \xi_i = \sum_i \bar{\alpha}_i \xi_i .$$

We will refer to this as the *conjugation associated with the orthonormal basis* $\{\xi_i\}$. [AS, equation (4.13)]

A 124. Let H_1 and H_2 be Hilbert spaces and let K_1 and K_2 be the normal state spaces of $\mathcal{B}(H_1)$ and $\mathcal{B}(H_2)$ respectively. If u is a unitary from H_1 to H_2, then the map $\Phi : a \mapsto \mathrm{Ad}_u(a)$ is a *-isomorphism from $\mathcal{B}(H_1)$ onto $\mathcal{B}(H_2)$ and Φ^* is an affine isomorphism from K_2 onto K_1. If v is a conjugate unitary from H_1 onto H_2, then the map $\Psi : a \mapsto \mathrm{Ad}_v(a^*)$ is a *-anti-isomorphism from $\mathcal{B}(H_1)$ onto $\mathcal{B}(H_2)$ and Ψ^* is also an affine isomorphism from K_2 onto K_1. [AS, Prop. 4.24]

A 125. Let H be a complex Hilbert space, and $\{\xi_\alpha\}$ an orthonormal basis. Define the map $a \mapsto a^t$ on $\mathcal{B}(H)$ by $a^t = ja^*j$ where j is the conjugation associated with the orthonormal basis. The map $a \mapsto a^t$ is called the *transpose map* with respect to the orthonormal basis $\{\xi_\alpha\}$. [AS, Def. 4.25]

A 126. Let H_1 and H_2 be Hilbert spaces. A map Φ from $\mathcal{B}(H_1)$ onto $\mathcal{B}(H_2)$ is a *-isomorphism iff it is implemented by a unitary and is a *-anti-isomorphism iff it is implemented by a conjugate unitary, cf. (A 124). [AS, Thm. 4.27]

A 127. The convex set $\mathbf{B}^3$ is the closed unit ball of $\mathbf{R}^3$. A convex set (in a linear space) is called a *3-ball* if there is an affine isomorphism of B^3 onto the set. A *parameterization* of a 3-ball F is an affine isomorphism ϕ from $\mathbf{B}^3$ onto F. [AS, p. 193]

A 128. The relation $\phi_1 \sim \phi_2 \mod SO(3)$ divides the set of all parametrizations of a 3-ball B into two equivalence classes, each of which is called an *orientation* of B. We refer to each of the two orientations of a 3-ball as the *opposite* of the other. We denote the orientation associated with a parametrization ϕ by $[\phi]$. An affine isomorphism $\psi : B_1 \to B_2$ between two 3-balls oriented by parametrizations ϕ_1 and ϕ_2, is said to be *orientation preserving* if $[\phi_2] = [\psi \circ \phi_1]$, and it is said to be *orientation reversing* if $[\psi \circ \phi_1]$ is the opposite of $[\phi_2]$. [AS, Def. 4.28]

A 129. Let a be a non-scalar self-adjoint operator on a two dimensional Hilbert space H, consider the w*-continuous affine function $\widehat{a}$ on the state space K of $\mathcal{B}(H)$ (defined by $\widehat{a}(\omega) = \omega(a)$), and assume that $\widehat{a}$ attains its maximum μ at the point $\sigma \in K$ and its minimum ν at the point $\tau \in K$. Then the unitary $u = e^{ia}$ determines a rotation Ad_u^* with the angle $\mu - \nu$ about the axis $\overrightarrow{\sigma\tau}$. [AS, Thm. 4.30]

A 130. If $\{\xi_1, \xi_2\}$ is an orthonormal basis in a two dimensional Hilbert space H, then the dual of the transpose map with respect to this basis is a reflection about the plane orthogonal to the axis $\overrightarrow{\omega_{\eta_1}\omega_{\eta_2}}$ where

$$\eta_1 = \tfrac{1}{\sqrt{2}}(\xi_1 + i\xi_2) \quad \text{and} \quad \eta_2 = \tfrac{1}{\sqrt{2}}(\xi_1 - i\xi_2).$$

Thus the dual of the transpose map reverses orientation. [AS, Lemma 4.33]

A 131. Let K be the state space of $\mathcal{B}(H)$ for a two dimensional Hilbert space H, so that K is a 3-ball. Then the rotations of K are precisely the maps Φ^* where Φ is implemented by a unitary operator u, i.e., $\Phi : a \mapsto uau^* = \mathrm{Ad}_u(a)$, and the reversals of K (i.e., the orientation reversing maps, which are the compositions of a reflection with respect to any diametral plane, and a rotation), are precisely the maps Φ^* where Φ

is implemented by a conjugate unitary operator v, i.e., $\Phi : a \mapsto va^*v^{-1} = \mathrm{Ad}_v(a^*)$. [AS, Thm. 4.34]

A 132. Let H be a two dimensional Hilbert space and let K be the state space of $\mathcal{B}(H)$. If Φ is a unital order automorphism of $\mathcal{B}(H)$, then there are two possibilities:

(i) Φ^* is an orientation preserving affine automorphism of K; in this case Φ is a *-isomorphism.

(ii) Φ^* is an orientation reversing affine automorphism of K; in this case Φ is a *-anti-isomorphism.

[AS, Thm. 4.35]

A 133. Let $\mathcal{A}$, $\mathcal{B}$ be C*-algebras and π a *-homomorphism from $\mathcal{A}$ onto $\mathcal{B}$. If J is the kernel of π and $F = J^\circ \cap K$ is the annihilator of J in the state space K of $\mathcal{A}$, then π^* is an affine homeomorphism from the state space of $\mathcal{B}$ onto the split face F. In particular, if $\sigma \in K$ and π_σ is the associated GNS representation, then

$$(\ker \pi_\sigma)^\circ \cap K = \overline{F}_\sigma,$$

where F_σ is the split face generated by σ, and $\overline{F}_\sigma$ is its w*-closure. [AS, Prop. 5.3]

A 134. Let $\mathcal{A}$ be a C*-algebra and $\pi : A \to \mathcal{B}(H)$ a representation. The *central cover of* π is the central projection $c(\pi)$ in $\mathcal{A}^{**}$ such that $\ker \widetilde{\pi} = (1 - c(\pi))\mathcal{A}^{**}$, where $\widetilde{\pi} : \mathcal{A}^{**} \to \mathcal{B}(H)$ is the normal extension of π, cf. (A 103). [AS, Def. 5.6]

A 135. If σ is any state on a C*-algebra, and π_σ the associated GNS representation, then the central cover $c(\pi_\sigma)$ and the central carrier $c(\sigma)$ are equal. [AS, equation (5.6)]

A 136. Two representations π_1 and π_2 of a C*-algebra $\mathcal{A}$ on Hilbert spaces H_1 and H_2 are *quasi-equivalent* if there is a *-isomorphism Φ from $\overline{\pi_1(\mathcal{A})}$ onto $\overline{\pi_2(\mathcal{A})}$ (weak closures) such that $\pi_2(a) = \Phi(\pi_1(a))$ for all $a \in \mathcal{A}$. [AS, Def. 5.8]

A 137. Let π_1 and π_2 be representations of a C*-algebra $\mathcal{A}$. Then the following are equivalent:

(i) π_1 and π_2 are quasi-equivalent.

(ii) The normal extensions $\widetilde{\pi}_1$ and $\widetilde{\pi}_2$ to $\mathcal{A}^{**}$ have the same kernel.

(iii) $c(\pi_1) = c(\pi_2)$.

[AS, Prop. 5.10]

A 138. If $\mathcal{A}$ is a C*-algebra and $\pi : \mathcal{A} \to \mathcal{B}(H)$ is an irreducible representation, then π^* is an affine isomorphism of the normal state space of $\mathcal{B}(H)$ onto $F_{c(\pi)}$. In particular, if σ is a pure state on $\mathcal{A}$, then π_σ^* is an affine isomorphism of the normal state space of $\mathcal{B}(H_\sigma)$ onto F_σ, where F_σ is the split face generated by σ. [AS, Cor. 5.16]

A 139. If σ is a pure state on a C*-algebra $\mathcal{A}$, then the split face F_σ is the σ-convex hull of its extreme points. [AS, Cor. 5.17]

A 140. Let $\mathcal{A}$ be a C*-algebra with state space K. The split face F_σ generated by each pure state σ properly contains no non-empty split face of K. [AS, Cor. 5.18]

A 141. If π is a representation of a C*-algebra $\mathcal{A}$ on a Hilbert space H, then the following are equivalent:

(i) π is irreducible.
(ii) $\pi(\mathcal{A})' = \mathbf{C}1$.
(iii) $\pi(\mathcal{A})'' = \mathcal{B}(H)$.
(iv) $\pi(\mathcal{A})$ is weakly (or σ-weakly) dense in $\mathcal{B}(H)$.
(v) The normal extension $\widetilde{\pi}$ of π maps $\mathcal{A}^{**}$ onto $\mathcal{B}(H)$.

[AS, Prop. 5.15]

A 142. Let σ and τ be pure states on a C*-algebra $\mathcal{A}$ with state space K. The following are equivalent:

(i) σ is unitarily equivalent to τ.
(ii) π_σ is unitarily equivalent to π_τ.
(iii) π_σ is quasi-equivalent to π_τ.
(iv) σ and τ generate the same split face of K.
(v) $c(\sigma) = c(\tau)$.
(vi) There is a unit vector η in H_σ such that $\tau = \omega_\eta \circ \pi_\sigma$.

[AS, Thm. 5.19]

A 143. Let σ, τ be distinct pure states on a C*-algebra $\mathcal{A}$. If the GNS-representations π_σ and π_τ are unitarily equivalent, then the face generated by σ and τ is a 3-ball. If these representations are not unitarily equivalent, then the face they generate is the line segment $[\sigma, \tau]$. [AS, Thm. 5.36]

A 144. Let K be the state space of a C*-algebra. Param(K) denotes the set of all parametrizations of facial 3-balls of K, cf. (A 127). We equip Param(K) with the topology of pointwise convergence of maps from $\mathbf{B}^3$ into the space K with the w*-topology. We call $\mathcal{OB}_K = \mathrm{Param}(K)/SO(3)$ the *space of oriented facial 3-balls of* K. We equip it with the quotient topology. We let $\mathcal{B}_K$ denote the set of facial 3-balls, equipped with the quotient topology from the map of Param(K) onto $\mathcal{B}_K$ given by $\phi \mapsto \phi(\mathbf{B}^3)$. Thus $\mathcal{B}_K$ is homeomorphic to Param$(K)/O(3)$ and we will often identify these two spaces. [AS, Def. 5.38 and Def. 5.39]

A 145. Let K be the state space of a C*-algebra. The spaces $\mathcal{OB}_K$ and $\mathcal{B}_K$ are Hausdorff, the canonical map from $\mathcal{OB}_K$ onto $\mathcal{B}_K$ is continuous and open, and $\mathcal{OB}_K \to \mathcal{B}_K$ is a $\mathbf{Z_2}$ bundle. [AS, Prop. 5.40]

A 146. Let K be the state space of a C*-algebra. A continuous cross-section of the bundle $\mathcal{OB}_K \to \mathcal{B}_K$ (of oriented facial 3-balls over facial 3-balls) is called a *global orientation*, or simply an *orientation*, of K. [AS, Def. 5.41]

A 147. Let $\mathcal{A}$ be a C*-algebra with state space K, and F a facial 3-ball of K with carrier projection p. Let π be any *-isomorphism from $p\mathcal{A}^{**}p$ onto M_2. The *orientation on* F *induced by* $\mathcal{A}$ is the (equivalence class) of the map π^* from $\mathbf{B^3}$ onto F, i.e., the orientation determined by the parameterization π^*. [AS, Def. 5.44]

A 148. The state space K of a C*-algebra $\mathcal{A}$ is orientable. Specifically, the orientation of each facial 3-ball induced by $\mathcal{A}$ gives a global orientation of K. [AS, Thm. 5.54]

A 149. Let $\mathcal{A}$ be a C*-algebra, $a \in \mathcal{A}$, and let σ be any state on $\mathcal{A}$ such that $\sigma(a^*a) \neq 0$. Then we define a state σ_a by

$$\sigma_a(b) = \sigma(a^*a)^{-1}\sigma(a^*ba).$$

[AS, Def. 5.57]

A 150. Let $\mathcal{A}$ be a C*-algebra, a an element of $\mathcal{A}$, and σ any pure state on $\mathcal{A}$ such that $\sigma(a^*a) \neq 0$. Then σ_a is a pure state, σ_a and σ generate the same split face, and their associated GNS-representations are unitarily equivalent. [AS, Lemma 5.58]

A 151. Let $\mathcal{A}$ be a C*-algebra with state space K and let $a \in \mathcal{A}$. Then the map $\sigma \mapsto B(\sigma, \sigma_a)$ is continuous from its domain into the space of facial 3-balls $\mathcal{B}_K$. [AS, Lemma 5.59]

A 152. If H is a complex Hilbert space, and N the normal state space of $\mathcal{B}(H)$, then the space $\mathcal{B}_N$ of facial 3-balls is path connected. [AS, Lemma 5.60]

A 153. The *commutator ideal* of a C*-algebra $\mathcal{A}$ is the (norm closed) ideal generated by all commutators of elements in $\mathcal{A}$. It is denoted $[\mathcal{A}, \mathcal{A}]$. [AS, Def. 5.62]

A 154. The annihilator $F_0 = [\mathcal{A}, \mathcal{A}]^\circ \cap K$ of the commutator ideal of a C*-algebra $\mathcal{A}$ in the state space K is a w*-closed split face such that

(i) F_0 consists of all $\sigma \in K$ with $\pi_\sigma(\mathcal{A})$ abelian,
(ii) $\partial_e F_0$ consists of all $\sigma \in \partial_e K$ with $\pi_\sigma(\mathcal{A})$ one dimensional, or equivalently, with $F_\sigma = \{\sigma\}$.

[AS, Prop. 5.63]

A 155. Let $\mathcal{A}$ be a C*-algebra with state space K. We will denote by F_0 the split face that is the annihilator of the commutator ideal, i.e., $F_0 = K \cap [\mathcal{A}, \mathcal{A}]^\circ$. We will say that a state σ is *abelian* if $\pi_\sigma(\mathcal{A})$ is abelian. Thus F_0 consists of all abelian states. [AS, Def. 5.64]

A 156. If $\mathcal{A}$ is a C*-algebra with state space K, then there is a 1-1 correspondence between

(i) Jordan compatible associative products on $\mathcal{A}$,
(ii) global orientations of K,
(iii) ordered pairs (F_1, F_2) of w*-closed split faces with $\mathrm{co}(F_1 \cup F_2) = K$ and $F_1 \cap F_2 = F_0$ (where F_0 is defined as in (A 155),
(iv) direct sum decompositions $[\mathcal{A}, \mathcal{A}] = J_1 \oplus J_2$.

[AS, Thm. 5.73]

A 157. If $\Phi : \mathcal{A} \to \mathcal{B}$ is a Jordan isomorphism between C*-algebras, then Φ is a *-isomorphism iff Φ^* preserves orientation, and is a *-anti-isomorphism iff Φ^* reverses orientation. [AS, Thm. 5.71]

A 158. Let Φ be a Jordan isomorphism of a von Neumann algebra $\mathcal{M}$ onto a von Neumann algebra $\mathcal{N}$. Then there exists a central projection c such that Φ restricted to $c\mathcal{M}$ is a *-isomorphism onto $\Phi(c)\mathcal{N}$, and Φ restricted to $(1-c)\mathcal{M}$ is a *-anti-isomorphism onto $(1-\Phi(c))\mathcal{N}$. [AS, Cor. 5.76]

A 159. Every Jordan isomorphism Φ of a C*-algebra $\mathcal{A}$ onto a C*-algebra $\mathcal{B}$ is an isometry. [AS, Cor. 5.77]

A 160. If $\star$ is a Jordan compatible associative product on a C*-algebra $\mathcal{A}$, then the norm associated with $\star$ coincides with the original norm, so that $\mathcal{A}$ is a C*-algebra when equipped with the product $\star$ and the original involution and norm. [AS, Cor. 5.78]

A 161. Two projections e and f in a von Neumann algebra $\mathcal{M}$ are said to be *equivalent* (in the sense of Murray and von Neumann) if there exists a partial isometry $v \in \mathcal{M}$ with initial projection e and final projection f, i.e., if there exists $v \in \mathcal{M}$ such that $v^*v = e$ and $vv^* = f$. [AS, Def. 6.2]

A 162. Let e and f be projections in a von Neumann algebra $\mathcal{M}$. We write $e \preceq f$ if e is equivalent to a subprojection of f, and we write $e \prec f$ if e is equivalent to a subprojection of f but not to f itself. [AS, Def. 6.6]

A 163. Let e and f be two projections in a von Neumann algebra $\mathcal{M}$ with decompositions $e = \sum_\alpha e_\alpha$ and $f = \sum_\alpha f_\alpha$ where e_α and f_α are projections in $\mathcal{M}$. If $e_\alpha \sim f_\alpha$ for all α, then $e \sim f$. In fact, if $e_\alpha = v_\alpha^* v_\alpha$ and $f_\alpha = v_\alpha v_\alpha^*$ for all α and $v = \sum_\alpha v_\alpha$, then $e = v^*v$ and $f = vv^*$. Also, $ve_\alpha v^* = f_\alpha$ for all α. [AS, Lemma 6.7]

A 164. If e and f are projections in a von Neumann algebra $\mathcal{M}$, then there exists a central projection $c \in \mathcal{M}$ (with complementary projection $c' = 1 - c$) such that

$$ce \preceq cf \quad \text{and} \quad c'f \preceq c'e.$$

[AS, Thm. 6.11]

A 165. A projection e in a von Neumann algebra $\mathcal{M}$ is said to be an *abelian projection* if $e\mathcal{M}e$ is an abelian subalgebra of $\mathcal{M}$. [AS, Def. 6.18]

A 166. A projection e in a von Neumann algebra $\mathcal{M}$ is said to be *infinite* (relative to $\mathcal{M}$) if it has a non-zero subprojection $e_0 \neq e$ such that $e_0 \sim e$. Otherwise, e is said to be *finite* (relative to $\mathcal{M}$). If e is infinite and ce is also infinite for each central projection c such that $ce \neq 0$, then e is said to be *properly infinite*. The von Neumann algebra $\mathcal{M}$ is said to be *finite* or *properly infinite* if the identity element 1 is finite, respectively, properly infinite. [AS, Def. 6.12]

A 167. Let $\mathcal{M}$ be a von Neumann algebra. $\mathcal{M}$ is said to be of *type I* if it has an abelian projection e such that $c(e) = 1$, and is said to be of *type* I_n (where n is a cardinal number) if 1 is the sum of n equivalent abelian projections. $\mathcal{M}$ is said to be of *type II* if it has no non-zero abelian projection but has a finite projection e such that $c(e) = 1$, (and then is said to be of *type* II_1 if 1 is a finite projection, and of *type* II_∞ if 1 is a properly infinite projection). $\mathcal{M}$ is said to be of *type III* if it has no non-zero finite projections. [AS, Def. 6.22]

A 168. If $\mathcal{M}$ is a von Neumann algebra, then there is a cardinal number n_0 and unique orthogonal central projections p_n (for each $n \leq n_0$), c_1, c_∞, and q, summing to 1, and such that $p_n\mathcal{M}$ is of type I_n (or zero) for all n, $c_1\mathcal{M}$ is of type II_1 (or zero), $c_\infty\mathcal{M}$ is of type II_∞ (or zero), and $q\mathcal{M}$ is of type III (or zero). [AS, Thm. 6.23]

A 169. Let $\{e_{ij}\}$ be a self-adjoint system of $n \times n$ matrix units in a unital *-algebra $\mathcal{M}$ and let $\mathcal{T}$ be the subalgebra of $\mathcal{M}$ consisting of those elements commuting with all e_{ij}. For each $t \in \mathcal{M}$ and each pair i, j, the sum $t_{ij} = \sum_k e_{ki} t e_{jk}$ is in $\mathcal{T}$, and the map $\phi : t \mapsto [t_{ij}]$ is a *-isomorphism of $\mathcal{M}$ onto $\mathcal{M}_n(\mathcal{T})$. Moreover, $e_{ii}\mathcal{M}e_{ii}$ is *-isomorphic to $\mathcal{T}$ for each i, so $\mathcal{M}$ is also *-isomorphic to $M_n(e_{ii}\mathcal{M}e_{ii})$ for each i. [AS, Lemma 6.26]

A 170. If $\mathcal{M}$ is a von Neumann algebra of type I_n $(n < \infty)$ with center $\mathcal{Z}$ and e is an abelian projection with central cover 1, then $\mathcal{M}$ is *-isomorphic to both $M_n(\mathcal{Z})$ and $M_n(e\mathcal{M}e)$. [AS, Thm. 6.27]

A 171. We say two projections p and q in a C*-algebra are *unitarily equivalent*, and we write $p \sim_{\mathrm{u}} q$, if there is a unitary v in the algebra such that $vpv^* = q$. [AS, Def. 6.28]

A 172. An element s in a von Neumann algebra $\mathcal{M}$ is a *symmetry* if $s^* = s$ and $s^2 = 1$, and is a *partial symmetry* (or *e-symmetry*) if $s^* = s$ and $s^2 = e$ for a projection $e \in \mathcal{M}$. We say two projections p and q in $\mathcal{M}$ are *exchanged by a symmetry*, and we write $p \sim_{\mathrm{s}} q$, if there is a symmetry $s \in \mathcal{M}$ such that $sps = q$. [AS, pp. 251–2]

A 173. Each e-symmetry s in a von Neumann algebra $\mathcal{M}$ can be uniquely decomposed as a difference $s = p - q$ of two orthogonal projections, namely $p = \frac{1}{2}(e + s)$ and $q = \frac{1}{2}(e - s)$. This is called the *canonical decomposition* of s. Conversely, each pair p, q of two orthogonal projections determines an e-symmetry $s = p - q$ with $e = p + q$. Moreover, if s is an e-symmetry, then $s = ese = es = se$, and s is a symmetry in the von Neumann subalgebra $e\mathcal{M}e$. [AS, Lemma 6.33]

A 174. Let s be a symmetry with canonical decomposition $s = p - q$ (where $q = p'$) in a von Neumann algebra $\mathcal{M}$ with normal state space K, and let F and G be the norm closed faces in K associated with p and q respectively. Then the set of fixed points of U_s is equal to the relative commutant $\{s\}^c$ of s in $\mathcal{M}$ and also to the range space $\operatorname{im} E = p\mathcal{M}p + q\mathcal{M}q$ of the conditional expectation $E = U_p + U_q$, and U_s is the unique normal order preserving linear map of period 2 whose set of fixed points is equal to $\operatorname{im} E$. Furthermore, F and G are antipodal and affinely independent faces of K, the restriction of the map E^* to K is the unique affine projection (idempotent map) of K onto $\operatorname{co}(F \cup G)$ and the reflection U_s^* determined by s is the unique affine automorphism of K of period 2 whose set of fixed points is equal to $\operatorname{co}(F \cup G)$. [AS, Thm. 6.36]

A 175. Let e and f be two orthogonal projections in a von Neumann algebra $\mathcal{M}$. Then e and f are unitarily equivalent iff e and f are exchanged by a symmetry iff e and f are Murray–von Neumann equivalent. [AS, Prop. 6.38]

A 176. To each pair of projections p, q in a von Neumann algebra we assign the element $c(p,q) = pqp + p'q'p'$ (which is a "generalized squared cosine", or "closeness operator" in the terminology of Chandler Davis). [AS, Def. 6.40 and Lemma 6.41]

A 177. If p, q is a pair of projections in a von Neumann algebra $\mathcal{M}$, then $p - p \wedge q'$ and $q - q \wedge p'$ can be exchanged by a partial symmetry. [AS, Prop. 6.49]

A 178. Let p and q be distinct projections in a von Neumann algebra $\mathcal{M}$. If $\|p - q\| < 1$, then p and q can be exchanged by a unitary $u \in C^*(p, q, 1)$ which is the product of two symmetries in $\mathcal{M}$ and satisfies the inequality

$$\|1 - u\| < \sqrt{2}\|p - q\|.$$

[AS, Thm. 6.54]

A 179. Two unitarily equivalent projections p, q in a von Neumann algebra $\mathcal{M}$ can be exchanged by a finite product of symmetries. [AS, Prop. 6.56]

A 180. If p and q are projections in a von Neumann algebra $\mathcal{M}$, then there is a central symmetry $c \in \mathcal{M}$ such that

$$cp \preceq_{\mathrm{s}} cq \quad \text{and} \quad c'q \preceq_{\mathrm{s}} c'p,$$

i.e., cp can be exchanged with a subprojection of cq by a symmetry, and likewise for $c'q$ and $c'p$. [AS, Thm. 6.60]

A 181. A bounded linear operator δ acting on a C*-algebra $\mathcal{A}$ is said to be an *order derivation* if it leaves $\mathcal{A}_{\mathrm{sa}}$ invariant and restricts to an order derivation on $\mathcal{A}_{\mathrm{sa}}$, or which is equivalent, if it is *-preserving (i.e., $\delta(x^*) = \delta(x)^*$ for all $x \in \mathcal{A}$) and $\exp(t\delta)(x) \in \mathcal{A}^+$ for all $x \in \mathcal{A}^+$ and all $t \in \mathbf{R}$. [AS, Def. 6.61]

A 182. If a is an element of a C*-algebra $\mathcal{A}$, then the operator $\delta_a : \mathcal{A} \to \mathcal{A}$ defined by $\delta_a x = \frac{1}{2}(ax + xa^*)$ is an order derivation, and the associated one-parameter group is given by

$$\alpha_t(x) = e^{ta/2} x e^{ta^*/2} \quad \text{for all } x \in \mathcal{A}.$$

[AS, Prop. 6.65]

A 183. A bounded linear operator δ on a von Neumann algebra $\mathcal{M}$ is an order derivation iff $\delta = \delta_m$ for some $m \in \mathcal{M}$; in particular δ is an order derivation such that $\delta(1) = 0$ iff $\delta = \delta_{ia}$ for some $a \in \mathcal{M}_{\text{sa}}$. [AS, Thm. 6.68]

A 184. If δ is a linear operator on a von Neumann algebra $\mathcal{M}$, then the following are equivalent:

(i) δ is an order derivation with $\delta(1) = 0$,
(ii) $\delta = \delta_{ia}$ for some $a \in \mathcal{M}_{\text{sa}}$,
(iii) δ is a *-derivation (i.e., $\delta(x^*) = \delta(x)^*$, and $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in \mathcal{M}$),
(iv) δ is a *-preserving Jordan derivation (i.e., $\delta(x^*) = \delta(x)^*$, and $\delta(x \circ y) = \delta(x) \circ y + x \circ \delta(y)$ for all $x, y \in \mathcal{M}$).

[AS, Cor. 6.69]

A 185. If p and q are complementary projections in a von Neumann algebra $\mathcal{M}$ (i.e., $p + q = 1$), then the associated norm closed faces F_p and F_q of the normal state space K (cf. (A 110)) are said to be *complementary*. We will say that an ordered pair (F_p, F_q) of complementary faces is a *generalized axis* of K, and we will call (F_q, F_p) the *opposite generalized axis* of (F_p, F_q). [AS, Def. 6.77]

A 186. Let (F, G) be a generalized axis of the normal state space K of a von Neumann algebra $\mathcal{M}$. A one-parameter group of affine automorphisms of K will be called a *generalized rotation* of K about (F, G) if the set of fixed points is $\mathrm{co}(F \cup G)$ and the orbit of each $\omega \notin \mathrm{co}(F \cup G)$ is (affinely isomorphic to) a circle with center in $\mathrm{co}(F \cup G)$. [AS, Def. 6.78]

A 187. A partial symmetry s in a von Neumann algebra $\mathcal{M}$ will be called *balanced* if it has a canonical decomposition $s = p - q$ where $p \sim q$. [AS, Def. 7.4]

A 188. Let r and s be e-symmetries for a given projection e in a von Neumann algebra and let $r = p - q$ be the canonical decomposition of r (cf. (A 173)). Then the following are equivalent:

(i) $r \circ s = 0$,
(ii) $U_s r = -r$,
(iii) $U_s p = q$.

[AS, Prop. 7.5]

A 189. An e-symmetry r in a von Neumann algebra $\mathcal{M}$ is balanced (cf. (A 187)) iff there is another e-symmetry $s \in \mathcal{M}$ which is Jordan orthogonal to r. [AS, Cor. 7.6]

A 190. A *3-frame* in the normal state space of a von Neumann algebra is an ordered triple of generalized axes whose associated reflections satisfy the following two requirements:

(i) The reflection about each of the three generalized axes reverses the direction of the other two generalized axes.
(ii) The product of all three reflections in any order is the identity map.

[AS, Def. 7.27]

A 191. A *Cartesian triple of symmetries* (r, s, t) in a von Neumann algebra $\mathcal{M}$ satisfies the two requirements:

(i) $r \circ s = s \circ t = t \circ r = 0$,
(ii) $U_r U_s U_t = I$

(where I is the identity operator and $U_s x := sxs$ for all $x \in \mathcal{M}$ and likewise for r and t). The Cartesian triples in $\mathcal{M}$ are in 1-1 correspondence with the 3-frames in the normal state space of $\mathcal{M}$ (cf. (A 190)) under the map which assigns to s the generalized axis (F_p, F_q) determined by the canonical decomposition $s = p - q$ (cf. (A 173)) and likewise for r and t. If e is a projection in M, then a *Cartesian triple of e-symmetries* is a Cartesian triple of symmetries in the von Neumann algebra $\mathcal{M}_e$, and a *Cartesian triple* is a Cartesian triple of e-symmetries for any projection e in $\mathcal{M}$. [AS, Prop. 7.28 and Def. 7.29]

A 192. If (r, s) is a Jordan orthogonal pair of symmetries in a von Neumann algebra $\mathcal{M}$ (i.e., if $r \circ s = 0$), then this pair can be extended to a Cartesian triple of symmetries, and the possible extensions are those triples (r, s, t) where $t = izrs$ for a central symmetry $z \in \mathcal{M}$. [AS, Prop. 7.31]

A 193. There exists a Cartesian triple of symmetries in a von Neumann algebra $\mathcal{M}$ iff the identity element 1 is halvable, i.e., is the sum of two equivalent projections. [AS, Cor. 7.33]

A 194. Let K be the normal state space of a von Neumann algebra. If K has a 3-frame, then K will be called a *blown-up 3-ball.* If a norm closed face F of K has a 3-frame, then F will be called a *facial blown-up 3-ball.* [AS, Def. 7.37]

A 195. Two projections p and q in a von Neumann algebra $\mathcal{M}$ are homotopic iff they are unitarily equivalent. [AS, Prop. 7.49]

A 196. If p and q are projections in a von Neumann algebra $\mathcal{M}$ with normal state space K, then

$$\tfrac{1}{4}\|p-q\|^2 \leq d(F_p, F_q) \leq 4\|p-q\| ,$$

so the map $e \mapsto F_e$ is a homeomorphism from the set $\mathcal{P}$ of projections in $\mathcal{M}$ to the set $\mathcal{F}$ of norm closed faces in K (with the Hausdorff metric). [AS, Lemma 7.54]

A 197. Let $\mathcal{M}$ be a von Neumann algebra with normal state space K and assume that $\mathcal{M}$ has halvable identity, or which is equivalent, that K is a blown-up 3-ball. An *orientation* of $\mathcal{M}$ is a homotopy class of Cartesian triples of symmetries in $\mathcal{M}$, or which is the same, a unitary equivalence class of such triples. Similarly, an *orientation* of K is a homotopy class of 3-frames in K, or which is the same, a unitary equivalence class of such frames. [AS, Def. 7.67]

A 198. We will denote the set of all halvable projections in a von Neumann algebra $\mathcal{M}$ by $\mathcal{X}$, and we will assume that this set is equipped with the norm topology. We also will assume that the set $\mathcal{T}$ of all Cartesian triples in $\mathcal{M}$ is equipped with the topology inherited from $\mathcal{M} \times \mathcal{M} \times \mathcal{M}$, where $\mathcal{M}$ has the norm topology. We will denote the set of all local orientations of $\mathcal{M}$, i.e., the set of all equivalence classes $[\alpha]$ of Cartesian triples in $\mathcal{M}$, by $\mathcal{O}$, and we will equip this set with the quotient topology induced from $\mathcal{T}$. Clearly, the set $\mathcal{O}$ of all local orientations of $\mathcal{M}$ is a bundle over $\mathcal{X}$ with the bundle map π which assigns the projection e to each orientation of a local subalgebra $e\mathcal{M}e$. Thus $\pi([\alpha]) = e_\alpha$ for each $[\alpha] \in \mathcal{O}$. [AS, Def. 7.70]

A 199. Let e_1 and e_2 be halvable projections with $e_1 \leq e_2$ in a von Neumann algebra $\mathcal{M}$, and let $[\alpha_1]$ and $[\alpha_2]$ be orientations of $e_1\mathcal{M}e_1$ and $e_2\mathcal{M}e_2$ respectively. We will say that $[\alpha_1]$ is *a restriction* of $[\alpha_2]$ and that $[\alpha_2]$ is *an extension* of $[\alpha_1]$, and we will write $[\alpha_1] \ll [\alpha_2]$, if the representing Cartesian triples $\alpha_1 = (r_1, s_1, t_1)$ and $\alpha_2 = (r_2, s_2, t_2)$ can be chosen such that

(i) $r_1 \ll r_2$,
(ii) $\psi_{\alpha_1} = \psi_{\alpha_2}$ on $e_1\mathcal{M}e_1$. [AS, Def. 7.81]

A 200. A cross-section $\theta : \mathcal{X} \to \mathcal{O}$ of the bundle $\mathcal{O}$ of local orientations of a von Neumann algebra $\mathcal{M}$ is called *consistent* if $\theta(e_1) \ll \theta(e_2)$ for all pairs $e_1, e_2 \in \mathcal{X}$ with $e_1 \leq e_2$, and a consistent continuous cross-section of $\mathcal{O}$ is called a *global orientation* of $\mathcal{M}$. [AS, Def. 7.85]

A 201. A cross-section $\theta : \mathcal{B} \to \mathcal{OB}$ of the bundle $\mathcal{OB}$ of oriented facial blown-up 3-balls of the normal state space K of a von Neumann algebra $\mathcal{M}$ is called *consistent* if $\theta(B_1) \ll \theta(B_2)$ for all pairs $B_1, B_2 \in \mathcal{B}$ with $B_1 \subset B_2$, and a consistent continuous cross-section of $\mathcal{OB}$ is called a *global orientation* of K. [AS, Def. 7.93]

A 202. A one-parameter group $\{U_\tau\}_{\tau\in\mathbf{R}}$ of linear operators on a real or complex vector space X will be called *rotational* if the orbits are either fixed points or (affinely isomorphic to) circles traced out with common minimal period 2π. [AS, Def. 1.109]

A 203. An order derivation δ of an ordered Banach space A will be called a *rotational derivation* if $\delta^3 + \delta = 0$, or, which is equivalent, if the associated one-parameter group $\{e^{\tau\delta}\}_{\tau\in\mathbf{R}}$ is rotational. [AS, Def. 1.112]

A 204. If δ is a *-derivation of a von Neumann algebra $\mathcal{M}$, then the following are equivalent:

(i) δ is a rotational derivation,
(ii) $\mathrm{sp}(\delta) \subset \{-\imath, 0, \imath\}$ and the spectral points are eigenvalues whose eigenspaces span $\mathcal{M}$,
(iii) $\delta = \delta_{is}$ for a symmetry $s \in \mathcal{M}$. [AS, Thm. 6.76]

A 205. If θ is a global orientation of a von Neumann algebra $\mathcal{M}$, then we can assign to each balanced e-symmetry r (where e is halvable) a unique rotational derivation $\psi(r)$ of the local subalgebra $e\mathcal{M}e$ such that $\psi(r) = \psi_\alpha | e\mathcal{M}e$ for each Cartesian e-triple α such that the first entry of α is r and $[\alpha] = \theta(e)$. [AS, Lemma 7.95]

A 206. If θ is a global orientation of a von Neumann algebra $\mathcal{M}$, then the map ψ from balanced e-symmetries in $\mathcal{M}$ to rotational derivations of $e\mathcal{M}e$ defined above, satisfies the following conditions:

(i) For each r, $\ker\psi(r)$ is equal to the relative commutant $\{r\}^c \cap e\mathcal{M}e$.
(ii) If a unitary $v \in \mathcal{M}$ carries r_1 to r_2, then it carries $\psi(r_1)$ to $\psi(r_2)$, i.e., $\mathrm{Ad}_v\psi(r_1)\mathrm{Ad}_v^{-1} = \psi(r_2)$ on $e_2\mathcal{M}e_2$.
(iii) If $r_1 \ll r_2$, then $\psi(r_1) = \psi(r_2)$ on $e_1\mathcal{M}e_1$ (where $e_1 = r_1^2$).

Conversely, each map ψ which assigns rotational derivations on $e\mathcal{M}e$ to balanced e-symmetries in such a way that the conditions above are satisfied, arises in this way from a unique global orientation θ. [AS, Thm. 7.96]

A 207. If $\mathcal{M}$ is a von Neumann algebra, then there is 1-1 correspondence between

(i) central symmetries in $\mathcal{M}$ which are 1 on the abelian part of $\mathcal{M}$,
(ii) Jordan compatible associative products in $\mathcal{M}$,

(iii) global orientations of $\mathcal{M}$,
(iv) global orientations of the normal state space K of $\mathcal{M}$.

The orientation associated with the given product on $\mathcal{M}$ assigns to each halvable projection e the equivalence class of the Cartesian triple (r, s, t) determined by $e = \imath rst$, and this orientation is called the *natural global orientation* associated with $\mathcal{M}$. [AS, Thm. 7.103]

A 208. Let p_1 be the largest central abelian projection in a von Neumann algebra $\mathcal{M}$. We will say a Cartesian e-triple in $\mathcal{M}$ is *full* iff $c(e) = 1 - p_1$. [AS, p. 338]

A 209. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be von Neumann algebras with natural global orientations θ_1 and θ_2 respectively. Let $\Phi : \mathcal{M}_1 \to \mathcal{M}_2$ be a Jordan isomorphism, and let $\alpha = (r, s, t)$ be a full Cartesian e-triple such that $\imath rst = e$. Then the following are equivalent:

(i) Φ is a *-isomorphism.
(ii) Φ is orientation preserving.
(iii) $\Phi(rst) = \Phi(r)\Phi(s)\Phi(t)$.
(iv) Φ carries $\theta_1(e)$ to $\theta_2(\Phi(e))$.

Similarly, the following are equivalent:

(v) Φ is a *-anti-isomorphism.
(vi) Φ reverses orientation.
(vii) $\Phi(rst) = \Phi(t)\Phi(s)\Phi(r)$ $(= -\Phi(r)\Phi(s)\Phi(t))$.
(viii) Φ carries $\theta_1(e)$ to the opposite of $\theta_2(\Phi(e))$.

[AS, Cor. 7.108]

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Index

Erik M. Alfsen and Frederic W. Shultz

Geometry of State Spaces of Operator Algebras

This monograph presents a complete and self-contained solution to the long-standing problem of giving a geometric description of state spaces of C*-algebras and von Neumann algebras, and of their Jordan algebraic analogs (JB-algebras and JBW-algebras). The material, which previously has appeared only in research papers and required substantial prerequisites for a reader's understanding, is made accessible here to a broad mathematical audience.

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