On the properties $\alpha - z$ Rényi divergences on general von Neumann algebras

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1 Introduction

2 Preliminaries

2.1 Basic definitions

Let $\mathcal{M}$ be a von Neumann algebra and let $\mathcal{M}^+$ be the cone of positive elements in $\mathcal{M}$. We denote the predual by $\mathcal{M}_*$, its positive part by $\mathcal{M}_*^+$ and the set of normal states by $\mathfrak{S}_*(\mathcal{M})$. For $\psi \in \mathcal{M}_*^+$, we will denote by $s(\psi)$ the support projection of ψ.

For $0 < p \leq \infty$, let $L_p(\mathcal{M})$ be the Haagerup L_p-space over $\mathcal{M}$ and let $L_p(\mathcal{M})$ its positive cone, [?]. We will use the identifications $\mathcal{M} \simeq L_\infty(\mathcal{M})$, $\mathcal{M}_* \ni \psi \leftrightarrow h_\psi \in L_1(\mathcal{M})$ and the notation $\mathrm{Tr}\, h_\psi = \psi(1)$ for the trace in $L_1(\mathcal{M})$. It this way, $\mathcal{M}_*^+$ is identified with the positive cone $L_1(\mathcal{M})^+$ and $\mathfrak{S}_*(\mathcal{M})$ with subset of elements in $L_1(\mathcal{M})^+$ with unit trace. Precise definitions and further details on the spaces $L_p(\mathcal{M})$ can be found in the notes [?].

2.2 The $\alpha - z$-Rényi divergences

In [? ?], the $\alpha - z$-Rényi divergence for $\psi, \varphi \in \mathcal{M}_*^+$ was defined as follows:

Definition 2.1. Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$ and let $\alpha, z > 0$, $\alpha \neq 1$. The $\alpha - z$-Rényi divergence is defined as

$$D_{\alpha,z}(\psi\|\varphi) := \frac{1}{\alpha - 1} \log \frac{Q_{\alpha,z}(\psi\|\varphi)}{\psi(1)},$$

where

$$Q_{\alpha,z} = \begin{cases} \mathrm{Tr}\, \left(h_\varphi^{\underline{(1-\alpha)/2z}\,\frac{1-\alpha}{2z}} h_\psi^{\underline{\alpha/z}\,\frac{\alpha}{z}} h_\varphi^{\underline{(1-\alpha)/2z}\,\frac{1-\alpha}{2z}} \right)^z, & \text{if } 0 < \alpha < 1 \\ \|x\|_z^z, & \text{if } \alpha > 1 \text{ and} \\ & h_\psi^{\underline{\alpha/z}\,\frac{\alpha}{z}} = h_\varphi^{\underline{(\alpha-1)/2z}\,\frac{\alpha-1}{2z}} x h_\varphi^{\underline{(\alpha-1)/2z}\,\frac{\alpha-1}{2z}}, \text{ with } x \in s(\varphi)L_z(\\ \infty & \text{otherwise.} \end{cases}$$

In the case $\alpha > 1$, the following alternative form will be useful.

Lemma 2.2. ~~[?]~~ [? , Lemma 7] *Let $\alpha > 1$ and $\psi, \varphi \in \mathcal{M}_*^+$. Then $Q_{\alpha,z}(\psi\|\varphi) < \infty$ if and only if there is some $y \in L_{2z}(\mathcal{M})s(\varphi)$ such that*

$$h_{\psi}^{\cancel{\alpha/2z}\,\frac{\alpha}{2z}} = y h_{\varphi}^{\cancel{(\alpha-1)/2z}\,\frac{\alpha-1}{2z}}.$$

Moreover, in this case, such y is unique and we have $Q_{\alpha,z}(\psi\|\varphi) = \|y\|_{2z}^{2z}$.

The standard Rényi divergence [? ? ?] is contained in this range as $D_\alpha(\psi\|\varphi) = D_{\alpha,1}(\psi\|\varphi)$. The sandwiched Rényi divergence is obtained as $\tilde{D}_\alpha(\psi\|\varphi) = D_{\alpha,\alpha}(\psi\|\varphi)$, see [2? ? ?] for some alternative definitions and properties of $\tilde{D}_\alpha$. The definition in [2] and [?] is based on the Kosaki interpolation spaces $L_p(\mathcal{M},\varphi)$ with respect to a state [?]. These spaces and complex interpolation method will be used frequently also in the present work.

Many of the properties of $D_{\alpha,z}(\psi\|\varphi)$ were extended from the finite dimensional case in [?]. In particular, ~~the following variational expressions will be an important tool for our work.~~ a variational expression for $Q_{\alpha,z}$ in the case $0 < \alpha < 1$ was proved there, see part (i) in the theorem below. We will prove a similar variational expression also in the case when $\alpha > 1$.

Theorem 2.3 (Variational expressions)**.** *Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$.*

(i) Let $0 < \alpha < 1$ and $\max\{\alpha, 1-\alpha\} \leq z$. Then

$$Q_{\alpha,z}(\psi\|\varphi) = \inf_{a \in \mathcal{M}^{++}} \left\{ \alpha \mathrm{Tr}\left((a^{\frac{1}{2}} h_\psi^{\frac{\alpha}{z}} a^{\frac{1}{2}})^{\frac{z}{\alpha}} \right) + (1-\alpha)\mathrm{Tr}\left((a^{\frac{1}{2}} h_\varphi^{\frac{1-\alpha}{z}} a^{\frac{1}{2}})^{\frac{z}{1-\alpha}} \right) \right\}.$$

~~*Moreover, if $\lambda^{-1}\varphi \leq \psi \leq \lambda\varphi$ for some $\lambda > 0$, then the infimum is attained at a unique element $\bar{a} \in \mathcal{M}^{++}$. This element satisfies*~~

$$\cancel{h_\psi^{\frac{\alpha}{2z}} \bar{a} h_\psi^{\frac{\alpha}{2z}} = \left(h_\psi^{\frac{\alpha}{2z}} h_\varphi^{\frac{1-\alpha}{z}} h_\psi^{\frac{\alpha}{2z}} \right)^{\alpha}}$$

~~*and*~~

$$\cancel{h_\varphi^{\frac{1-\alpha}{2z}} \bar{a}^{-1} h_\varphi^{\frac{1-\alpha}{2z}} = \left(h_\varphi^{\frac{1-\alpha}{2z}} h_\psi^{\frac{\alpha}{z}} h_\varphi^{\frac{1-\alpha}{2z}} \right)^{1-\alpha}.}$$

(ii) Let ~~$1 < \alpha \leq 2z$, then~~ $1 < \alpha$, $\max\{\frac{\alpha}{2}, \alpha - 1\} \leq z$. Then

$$Q_{\alpha,z}(\psi\|\varphi) = \sup_{a \in \mathcal{M}_+} \left\{ \alpha \mathrm{Tr}\left((a^{\frac{1}{2}} h_\psi^{\frac{\alpha}{z}} a^{\frac{1}{2}})^{\frac{z}{\alpha}} \right) - (\alpha-1)\mathrm{Tr}\left((a^{\frac{1}{2}} h_\varphi^{\frac{\alpha-1}{2z}} a^{\frac{1}{2}})^{\frac{z}{\alpha-1}} \right) \right\}.$$

Proof. For part (i) see [? , Theorem 1 (vi)]~~and its proof~~. The inequality $\geq$ in part (ii) holds for all α and z and was proved in [? , Theorem 2 (vi)]. We now prove the opposite inequality.

Assume first that $Q_{\alpha,z}(\psi\|\varphi) < \infty$, so that there is some $x \in s(\varphi)L_z(\mathcal{M})^+ s(\varphi)$ such that $h_\psi^{\frac{\alpha}{z}} = h_\varphi^{\frac{\alpha-1}{2z}} x h_\varphi^{\frac{\alpha-1}{2z}}$. Plugging this into the right hand side, we obtain

$$\begin{aligned}
&\sup_{a \in \mathcal{M}_+} \left\{ \alpha \mathrm{Tr}\left((a^{\frac{1}{2}} h_\psi^{\frac{\alpha}{z}} a^{\frac{1}{2}})^{\frac{z}{\alpha}} \right) - (\alpha-1)\mathrm{Tr}\left((a^{\frac{1}{2}} h_\varphi^{\frac{\alpha-1}{2z}} a^{\frac{1}{2}})^{\frac{z}{\alpha-1}} \right) \right\} \\
&= \sup_{a \in \mathcal{M}_+} \left\{ \alpha \mathrm{Tr}\left((a^{\frac{1}{2}} h_\varphi^{\frac{\alpha-1}{2z}} x h_\varphi^{\frac{\alpha-1}{2z}} a^{\frac{1}{2}})^{\frac{z}{\alpha}} \right) - (\alpha-1)\mathrm{Tr}\left((a^{\frac{1}{2}} h_\varphi^{\frac{\alpha-1}{2z}} a^{\frac{1}{2}})^{\frac{z}{\alpha-1}} \right) \right\} \\
&= \sup_{a \in \mathcal{M}_+} \left\{ \alpha \mathrm{Tr}\left((x^{\frac{1}{2}} h_\varphi^{\frac{\alpha-1}{2z}} a h_\varphi^{\frac{\alpha-1}{2z}} x^{\frac{1}{2}})^{\frac{z}{\alpha}} \right) - (\alpha-1)\mathrm{Tr}\left((h_\varphi^{\frac{\alpha-1}{2z}} a h_\varphi^{\frac{\alpha-1}{2z}})^{\frac{z}{\alpha-1}} \right) \right\} \\
&= \sup_{w \in L_{\frac{z}{\alpha-1}}(\mathcal{M})^+} \left\{ \alpha \mathrm{Tr}\left((x^{\frac{1}{2}} w x^{\frac{1}{2}})^{\frac{z}{\alpha}} \right) - (\alpha-1)\mathrm{Tr}\left(w^{\frac{z}{\alpha-1}} \right) \right\},
\end{aligned}$$

where we used the fact that ~~$\mathrm{Tr}\,((a^*a)^p) = \mathrm{Tr}\,((aa^*)^p)$~~ $\mathrm{Tr}\,((h^*h)^p) = \mathrm{Tr}\,((hh^*)^p)$ for $p > 0$ and ~~$a \in L_{\frac{p}{2}}(\mathcal{M})$ and the fact that the set of elements of the form $h_\varphi^{\frac{\alpha-1}{2z}} a h_\varphi^{\frac{\alpha-1}{2z}}$ with $a \in \mathcal{M}^+$ is dense in the positive cone $L_{\frac{z}{\alpha-1}}(\mathcal{M})^+$~~ $h \in L_{\frac{p}{2}}(\mathcal{M})$, and Lemma A.1. Putting $w = x^{\alpha-1}$ we get

$$\sup_{w \in L_{\frac{z}{\alpha-1}}(\mathcal{M})^+} \left\{ \alpha \mathrm{Tr}\, \left((x^{\frac{1}{2}} w x^{\frac{1}{2}})^{\frac{z}{\alpha}} \right) - (\alpha - 1) \mathrm{Tr}\, \left(w^{\frac{z}{\alpha-1}} \right) \right\} \geq \mathrm{Tr}\,(x^z) = \|x\|_z^z = Q_{\alpha,z}(\psi\|\varphi).$$

This finishes the proof in the case that $Q_{\alpha,z}(\psi\|\varphi) < \infty$. Note that this holds if $\psi \leq \lambda\varphi$ for some $\lambda > 0$. Indeed, since $\frac{\alpha}{2z} \in (0,1]$ by the assumption, we then have

$$h_\psi^{\frac{\alpha}{2z}} \leq \lambda^{\frac{\alpha}{2z}} h_\varphi^{\frac{\alpha}{2z}},$$

hence by [**?** , Lemma A.58] there is some $b \in \mathcal{M}$ such that

$$h_\psi^{\frac{\alpha}{2z}} = b h_\varphi^{\frac{\alpha}{2z}} = y h_\varphi^{\frac{\alpha-1}{2z}},$$

where $y = b h_\varphi^{\frac{1}{2z}} \in L_{2z}(\mathcal{M})$. By Lemma 2.2 we get $Q_{\alpha,z}(\psi\|\varphi) = \|y\|_{2z}^{2z} < \infty$.

In the general case, ~~note that lower semicontinuity [**?**], we have~~

$$\cancel{Q_{\alpha,z}(\psi\|\varphi) \leq \liminf_{\epsilon \searrow 0} Q_{\alpha,z}(\psi\|\varphi + \epsilon\psi)}$$

~~and~~ the variational expression holds for $Q_{\alpha,z}(\psi\|\varphi + \epsilon\psi)$ for all $\epsilon > 0$~~. The proof is finished by using norm continuity of the map $L_1(\mathcal{M})^+ \ni h \mapsto h^{\frac{1}{p}} \in L_p(\mathcal{M})^+$ for $p > 1$.~~ , so that we have

$$\begin{aligned} Q_{\alpha,z}(\psi\|\varphi + \epsilon\psi) &= \sup_{a \in \mathcal{M}_+} \left\{ \alpha \mathrm{Tr}\, \left((a^{\frac{1}{2}} h_\psi^{\frac{\alpha}{z}} a^{\frac{1}{2}})^{\frac{z}{\alpha}} \right) - (\alpha - 1) \mathrm{Tr}\, \left((a^{\frac{1}{2}} h_{\varphi+\epsilon\psi}^{\frac{\alpha-1}{2z}} a^{\frac{1}{2}})^{\frac{z}{\alpha-1}} \right) \right\} \\ &\leq \sup_{a \in \mathcal{M}_+} \left\{ \alpha \mathrm{Tr}\, \left((a^{\frac{1}{2}} h_\psi^{\frac{\alpha}{z}} a^{\frac{1}{2}})^{\frac{z}{\alpha}} \right) - (\alpha - 1) \mathrm{Tr}\, \left((a^{\frac{1}{2}} h_\varphi^{\frac{\alpha-1}{2z}} a^{\frac{1}{2}})^{\frac{z}{\alpha-1}} \right) \right\}, \end{aligned}$$

where the inequality above follows by Lemma A.2. Therefore, since lower semicontinuity [**?** , Theorem 2 (iv)] gives

$$Q_{\alpha,z}(\psi\|\varphi) \leq \liminf_{\epsilon \searrow 0} Q_{\alpha,z}(\psi\|\varphi + \epsilon\psi)$$

the desired inequality follows.

~~I am not entirely sure about this proof , since the convergence of the expressions~~

$$\cancel{\mathrm{Tr}\, \left((a^{\frac{1}{2}} (h_\varphi + \epsilon h_\psi)^{\frac{\alpha-1}{2z}} a^{\frac{1}{2}})^{\frac{z}{\alpha-1}} \right) \to \mathrm{Tr}\, \left((a^{\frac{1}{2}} h_\varphi^{\frac{\alpha-1}{2z}} a^{\frac{1}{2}})^{\frac{z}{\alpha-1}} \right)}$$

□

Lemma 2.4. *Assume that $\lambda^{-1}\varphi \leq \psi \leq \lambda\varphi$ for some $\lambda > 0$. Then the infimum in the variational expression in Theorem 2.3(i) is attained at a unique element $\bar{a} \in \mathcal{M}^{++}$. This element satisfies*

$$h_\psi^{\frac{\alpha}{2z}} a h_\psi^{\frac{\alpha}{2z}} = \left(h_\psi^{\frac{\alpha}{2z}} h_\varphi^{\frac{1-\alpha}{z}} h_\psi^{\frac{\alpha}{2z}} \right)^\alpha \tag{1}$$

$$h_\varphi^{\frac{1-\alpha}{2z}} a^{-1} h_\varphi^{\frac{1-\alpha}{2z}} = \left(h_\varphi^{\frac{1-\alpha}{2z}} h_\psi^{\frac{\alpha}{z}} h_\varphi^{\frac{1-\alpha}{2z}} \right)^{1-\alpha}. \tag{2}$$

~~*also depends on $\|a\|$, which is not bounded. But probably I misunderstand something, or I do not see something trivial.*~~

Proof. We may assume that φ and hence also ψ is faithful. Following the proof of [? , Theorem 1 (vi)] , we may use the assumptions and [? , Lemma A.58] to show that there are $b, c \in \mathcal{M}$ such that

$$h_\varphi^{\frac{1-\alpha}{2z}} = b\big(h_\varphi^{\frac{1-\alpha}{2z}} h_\psi^{\frac{\alpha}{z}} h_\varphi^{\frac{1-\alpha}{2z}}\big)^{\frac{1-\alpha}{2}}, \qquad \big(h_\varphi^{\frac{1-\alpha}{2z}} h_\psi^{\frac{\alpha}{z}} h_\varphi^{\frac{1-\alpha}{2z}}\big)^{\frac{1-\alpha}{2}} = c h_\varphi^{\frac{1-\alpha}{2z}}. \tag{3}$$

Put $\bar{a} := bb^* \in \mathcal{M}^{++}$, then we have $\bar{a}^{-1} = c^*c$ and $\bar{a}$ is indeed a minimizer of

$$Q_{\alpha,z}(\psi\|\varphi) = \inf_{a\in\mathcal{M}^{++}} \Big\{ \alpha \Big\| h_\psi^{\frac{\alpha}{2z}} a h_\psi^{\frac{\alpha}{2z}} \Big\|_{\frac{z}{\alpha}}^{\frac{z}{\alpha}} + (1-\alpha) \Big\| h_\varphi^{\frac{1-\alpha}{2z}} a^{-1} h_\varphi^{\frac{1-\alpha}{2z}} \Big\|_{\frac{z}{1-\alpha}}^{\frac{z}{1-\alpha}} \Big\}. \tag{4}$$

We next observe that such minimizer is unique. Indeed, suppose that the infimum is attained at some $a_1, a_2 \in \mathcal{M}^{++}$. Let $a_0 := (a_1 + a_2)/2$. Since the map $L^p(\mathcal{M}) \ni k \mapsto \|k\|_p^p$ is convex for any $p \geq 1$ and $a_0^{-1} \leq (a_1^{-1} + a_2^{-1})/2$, we have

$$\Big\| h_\psi^{\frac{\alpha}{2z}} a_0 h_\psi^{\frac{\alpha}{2z}} \Big\|_{\frac{z}{\alpha}}^{\frac{z}{\alpha}} \leq \frac{1}{2} \Big\{ \Big\| h_\psi^{\frac{\alpha}{2z}} a_1 h_\psi^{\frac{\alpha}{2z}} \Big\|_{\frac{z}{\alpha}}^{\frac{z}{\alpha}} + \Big\| h_\psi^{\frac{\alpha}{2z}} a_2 h_\psi^{\frac{\alpha}{2z}} \Big\|_{\frac{z}{\alpha}}^{\frac{z}{\alpha}} \Big\},$$

$$\begin{aligned}
\Big\| h_\varphi^{\frac{1-\alpha}{2z}} a_0^{-1} h_\varphi^{\frac{1-\alpha}{2z}} \Big\|_{\frac{z}{1-\alpha}}^{\frac{z}{1-\alpha}} &\leq \Big\| h_\varphi^{\frac{1-\alpha}{2z}} \Big(\frac{a_1^{-1} + a_2^{-1}}{2} \Big) h_\varphi^{\frac{1-\alpha}{2z}} \Big\|_{\frac{z}{1-\alpha}}^{\frac{z}{1-\alpha}} \\
&\leq \frac{1}{2} \Big\{ \Big\| h_\varphi^{\frac{1-\alpha}{2z}} a_1^{-1} h_\varphi^{\frac{1-\alpha}{2z}} \Big\|_{\frac{z}{1-\alpha}}^{\frac{z}{1-\alpha}} + \Big\| h_\varphi^{\frac{1-\alpha}{2z}} a_2^{-1} h_\varphi^{\frac{1-\alpha}{2z}} \Big\|_{\frac{z}{1-\alpha}}^{\frac{z}{1-\alpha}} \Big\}.
\end{aligned}$$

Hence we have

$$\Big\| h_\varphi^{\frac{1-\alpha}{2z}} a_0^{-1} h_\varphi^{\frac{1-\alpha}{2z}} \Big\|_{\frac{z}{1-\alpha}} = \Big\| h_\varphi^{\frac{1-\alpha}{2z}} \Big(\frac{a_1^{-1} + a_2^{-1}}{2} \Big) h_\varphi^{\frac{1-\alpha}{2z}} \Big\|_{\frac{z}{1-\alpha}},$$

which implies that $a_0^{-1} = \frac{a_1^{-1} + a_2^{-1}}{2}$, as easily verified. From this we easily have $a_1 = a_2$.

The equality (2) is obvious from the second equality in (3) and $\bar{a}^{-1} = c^*c$. Since $Q_{\alpha,z}(\psi\|\varphi) = Q_{1-\alpha,z}(\varphi\|\psi)$, we see by uniqueness that the minimizer of the infimum expression for $Q_{1-\alpha,z}(\varphi\|\psi)$ (instead of (4)) is $\bar{a}^{-1}$ (instead of $\bar{a}$). This says that (1) is the equality corresponding to (2) when ψ, φ, α are replaced with $\varphi, \psi, 1-\alpha$, respectively.

□

3 Data processing inequality and reversibility of channels

Let $\gamma : \mathcal{N} \to \mathcal{M}$ be a normal positive unital map. Then the predual of γ defines a positive linear map $\gamma_* : L_1(\mathcal{M}) \to L_1(\mathcal{N})$ that preserves the trace, acting as

$$L_1(\mathcal{M}) \ni h_\rho \mapsto h_{\rho\circ\gamma} \in L_1(\mathcal{N}).$$

The support of γ will be denoted by $s(\gamma)$, recall that this is defined as the largest projection $p \in \mathcal{N}$ such that $\gamma(p) = 1$. For any $\rho \in \mathcal{M}_*^+$ we clearly have $s(\rho\circ\gamma) \leq s(\gamma)$, with equality if ρ is faithful. It follows that γ_* maps $L_1(\mathcal{M})$ to $s(\gamma)L_1(\mathcal{N})s(\gamma) \equiv L_1(s(\gamma)\mathcal{N}s(\gamma))$. For any $\rho \in \mathcal{M}^*_+$, $\rho \neq 0$, the map

$$s(\gamma)\mathcal{N}s(\gamma) \to s(\rho)\mathcal{M}s(\rho), \qquad a \mapsto s(\rho)\gamma(a)s(\rho)$$

is a faithful normal positive unital map, so using such restrictions we may always assume that both ρ and $\rho \circ \gamma$ are faithful.

The Petz dual of γ with respect to a faithful $\rho \in \mathcal{M}_*^+$ is a map $\gamma_\rho^* : \mathcal{M} \to \mathcal{N}$, introduced in [**?**]. It was proved that it is again normal, positive and unital, in addition, it is n-positive whenever γ is. As explained in [2] γ_ρ^* is determined by the equality

$$(\gamma_\rho^*)_*(h_{\rho\circ\gamma}^{\frac{1}{2}} b h_{\rho\circ\gamma}^{\frac{1}{2}}) = h_\rho^{\frac{1}{2}} \gamma(b) h_\rho^{\frac{1}{2}}, \tag{5}$$

for all $b \in \mathcal{N}^+$, here $(\gamma_\rho^*)_*$ is the predual map of γ_ρ^*. We also have

$$(\gamma_\rho^*)_*(h_{\rho\circ\gamma}) = (\gamma_\rho^*)_* \circ \gamma_*(h_\rho) = h_\rho$$

and $(\gamma_\rho^*)_{\rho\circ\gamma}^* = \gamma$.

3.1 Data processing inequality

In this paragraph we prove the data processing inequality (DPI) for $D_{\alpha,z}$ with respect to normal positive unital maps. In the case of the sandwiched divergences $\tilde{D}_\alpha$ with $1/2 \leq \alpha \neq 1$, DPI was proved in [2**?**], see also [**?**] for an alternative proof in the case when the maps are also completely positive.

Lemma 3.1. *Let $\gamma : \mathcal{N} \to \mathcal{M}$ be a normal positive unital map and let $\rho \in \mathcal{M}_*^+$, $b \in \mathcal{N}^+$.*

(i) If $p \in [1/2, 1)$, then

$$\|h_{\rho\circ\gamma}^{\frac{1}{2p}} b h_{\rho\circ\gamma}^{\frac{1}{2p}}\|_p \leq \|h_\rho^{\frac{1}{2p}} \gamma(b) h_\rho^{\frac{1}{2p}}\|_p.$$

(ii) If $p \in [1, \infty]$, the inequality reverses.

Proof. Let us denote $\beta := \gamma_\rho^*$ and let $\omega \in \mathcal{M}_*^+$ be such that $h_\omega := h_{\rho\circ\gamma}^{\frac{1}{2}} b h_{\rho\circ\gamma}^{\frac{1}{2}} \in L_1(\mathcal{N})^+$. Then β is a normal positive unital map and we have

$$\beta_*(h_\omega) = h_\rho^{\frac{1}{2}} \gamma(b) h_\rho^{\frac{1}{2}}, \qquad \beta_*(h_{\rho\circ\gamma}) = h_\rho.$$

Let $p \in [1/2, 1)$, then

$$\begin{aligned} \|h_\rho^{\frac{1}{2p}} \gamma(b) h_\rho^{\frac{1}{2p}}\|_p^p &= \|h_\rho^{\frac{1-p}{2p}} \beta_*(h_\omega) h_\rho^{\frac{1-p}{2p}}\|_p^p = Q_{p,p}(\beta_*(h_\omega)\|h_\rho) = Q_{p,p}(\beta_*(h_\omega)\|\beta_*(h_{\rho\circ\gamma})) \\ &\geq Q_{p,p}(h_\omega\|h_{\rho\circ\gamma}) = \|h_{\rho\circ\gamma}^{\frac{1-p}{2p}} h_\omega h_{\rho\circ\gamma}^{\frac{1-p}{2p}}\|_p^p = \|h_{\rho\circ\gamma}^{\frac{1}{2p}} b h_{\rho\circ\gamma}^{\frac{1}{2p}}\|_p^p. \end{aligned}$$

Here we have used the DPI for the sandwiched Rényi divergence $D_{\alpha,\alpha}$ for $\alpha \in [1/2, 1)$, [**?** , Theorem 4.1]. This proves (i). The case (ii) was proved in [**?**] (see Eq. (22) therein), using the relation of the sandwiched Rényi divergence to the Kosaki L_p norms. In our setting, the proof can be written as

$$\begin{aligned} \|h_\rho^{\frac{1}{2p}} \gamma(b) h_\rho^{\frac{1}{2p}}\|_p^p &= Q_{p,p}(h_\rho^{\frac{1}{2}} \gamma(b) h_\rho^{\frac{1}{2}}\|h_\rho) = Q_{p,p}(\beta_*(h_\omega)\|\beta_*(h_{\rho\circ\gamma})) \\ &\leq Q_{p,p}(h_\omega\|h_{\rho\circ\gamma}) = \|h_{\rho\circ\gamma}^{\frac{1}{2p}} b h_{\rho\circ\gamma}^{\frac{1}{2p}}\|_p^p, \end{aligned}$$

here the inequality follows from the DPI for sandwiched Rényi divergence $D_{\alpha,\alpha}$ with $\alpha > 1$, [2]. ☐

Theorem 3.2 (DPI)**.** *Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$ and let $\gamma : \mathcal{N} \to \mathcal{M}$ be a normal positive unital map. Assume either of the following conditions:*

(i) $0 < \alpha < 1$, $\max\{\alpha, 1-\alpha\} \leq z$

(ii) $\alpha > 1$, $\max\{\alpha/2, \alpha - 1\} \leq z \leq \alpha$.

Then we have

$$D_{\alpha,z}(\psi \circ \gamma \| \varphi \circ \gamma) \leq D_{\alpha,z}(\psi \| \varphi).$$

Proof. Under the conditions (i), the DPI was proved in [**?** , Theorem 1 (viii)]. Since parts of this proof will be used below, we repeat it here.

Assume the conditions in (i) and put $p := \frac{z}{\alpha}$, $r := \frac{z}{1-\alpha}$, so that $p, r \geq 1$. For any $b \in \mathcal{N}^{++}$, we have by the Choi inequality [**?**] that $\gamma(b)^{-1} \leq \gamma(b^{-1})$, so that

$$\|h_\varphi^{\frac{1}{2r}} \gamma(b)^{-1} \varphi^{\frac{1}{2r}}\|_r \leq \|h_\varphi^{\frac{1}{2r}} \gamma(b^{-1}) \varphi^{\frac{1}{2r}}\|_r.$$

Using the variational expression in Theorem 2.3 (i), we have

$$\begin{aligned} Q_{\alpha,z}(\psi\|\varphi) &\leq \alpha \|h_\psi^{\frac{1}{2p}} \gamma(b) h_\psi^{\frac{1}{2p}}\|_p^p + (1-\alpha)\|h_\varphi^{\frac{1}{2r}} \gamma(b)^{-1} h_\varphi^{\frac{1}{2r}}\|_r^r & (6)\\ &\leq \alpha \|h_\psi^{\frac{1}{2p}} \gamma(b) h_\psi^{\frac{1}{2p}}\|_p^p + (1-\alpha)\|h_\varphi^{\frac{1}{2r}} \gamma(b^{-1}) h_\varphi^{\frac{1}{2r}}\|_r^r & (7)\\ &\alpha \|h_{\psi\circ\gamma}^{\frac{1}{2p}} b h_{\psi\circ\gamma}^{\frac{1}{2p}}\|_p^p + (1-\alpha)\|h_{\varphi\circ\gamma}^{\frac{1}{2r}} b^{-1} h_{\varphi\circ\gamma}^{\frac{1}{2r}}\|_r^r, & (8) \end{aligned}$$

here we used Lemma 3.1 (ii) for the last inequality. Since this holds for all $b \in \mathcal{N}^+ +$, it follows that $Q_{\alpha,z}(\psi\|\varphi) \leq Q_\alpha(\psi \circ \gamma \| \varphi \circ \gamma)$, which proves the DPI in this case.

Assume next the condition (ii), and put $p := \frac{z}{\alpha}$, $q := \frac{z}{\alpha - 1}$, so that $p \in [1/2, 1)$ and $q \geq 1$. Using Theorem 2.3 (ii), we get for any $b \in \mathcal{N}^+$,

$$\begin{aligned} Q_{\alpha,z}(\psi\|\varphi) &\geq \alpha \|h_\psi^{\frac{1}{2p}} \gamma(b) h_\psi^{\frac{1}{2p}}\|_p^p - (\alpha - 1)\|h_\varphi^{\frac{1}{2q}} \gamma(b) h_\varphi^{\frac{1}{2q}}\|_q^q \\ &\geq \alpha \|h_{\psi\circ\gamma}^{\frac{1}{2p}} b h_{\psi\circ\gamma}^{\frac{1}{2p}}\|_p^p - (\alpha - 1)\|h_{\varphi\circ\gamma}^{\frac{1}{2r}} b h_{\varphi\circ\gamma}^{\frac{1}{2q}}\|_q^q, \end{aligned}$$

here we used both (i) and (ii) in Lemma 3.1. Again, since this holds for all $b \in \mathcal{N}^+$, we get the desired inequality.

□

3.2 Martingale convergence

3.3 Equality in DPI and reversibility of channels

In what follows, a channel is a normal 2-positive unital map $\gamma : \mathcal{N} \to \mathcal{M}$.

Definition 3.3. Let $\gamma : \mathcal{N} \to \mathcal{M}$ be a channel and let $\mathcal{S} \subset \mathcal{M}_*^+$. We say that γ is reversible (or sufficient) with respect to $\mathcal{S}$ if there exists a channel $\beta : \mathcal{M} \to \mathcal{N}$ such that

$$\rho \circ \gamma \circ \beta = \rho, \qquad \forall \rho \in \mathcal{S}.$$

The notion of sufficient channels was introduced by Petz [**?** **?**], who also obtained a number of conditions characterizing this situation. It particular, it was proved in [**?**] that sufficient channels can be characterized by equality in DPI for the relative entropy $D(\psi\|\varphi)$: if $D(\psi\|\varphi)<\infty$, then a channel γ is sufficient with respect to $\{\psi,\varphi\}$ if and only if

$$D(\psi\circ\gamma\|\varphi\circ\gamma)=D(\psi\|\varphi).$$

This result has been proved for a number of other divergence measures, including the standard Rényi divergences $D_{\alpha,1}$ with $0<\alpha<2$ ([]) and the sandwiched Rényi divergences $D_{\alpha,\alpha}$ for $\alpha>1/2$ ([2**?**]). Our aim in this section is to prove that a similar statement holds for $D_{\alpha,z}$ for values of the parameters strictly contained in the DPI bounds of Theorem 3.2.

Throughout this section, we will assume that $\psi,\varphi\in\mathcal{M}_*^+$ are such that $s(\psi)\le s(\varphi)$. As noted above, we may replace the channel γ by its restriction so that we may assume that both φ and $\varphi_0:=\varphi\circ\gamma$ are faithful.

Another important result of [**?**] shows that the Petz dual γ^*_φ is a universal recovery map, in the sense given in the proposition below.

Proposition 1. *Let $\varphi\in\mathcal{M}_*^+$ be faithful and let $\gamma:\mathcal{N}\to\mathcal{M}$ be a faithful channel. Then for any $\psi\in\mathcal{M}_*^+$, γ is reversible with respect to $\{\psi,\varphi\}$ if and only if $\psi\circ\gamma\circ\gamma^*_\varphi=\psi$.*

Consequently, there is a faithful normal conditional expectation $\mathcal{E}$ on $\mathcal{M}$ such that $\varphi\circ\mathcal{E}=\varphi$ and γ is sufficient with respect to $\{\psi,\varphi\}$ if and only if also $\psi\circ\mathcal{E}=\psi$.

Note that the range of the conditional expectation $\mathcal{E}$ in the above proposition is the set of fixed points of the channel $\gamma\circ\gamma^*_\varphi$.

3.3.1 The case $\alpha\in(0,1)$

Theorem 3.4. *Let $0<\alpha<1$ and $\alpha,1-\alpha\le z$ where at least one of the inequalities is strict. Let $\psi,\varphi\in\mathcal{M}_*^+$ be such that $s(\psi)\le s(\varphi)$. Then γ is reversible with respect to $\{\psi,\varphi\}$ if and only if*

$$D_{\alpha,z}(\psi\|\varphi)=D_{\alpha,z}(\psi\circ\gamma\|\varphi\circ\gamma).$$

Proof. Let us denote $\psi_0:=\psi\circ\gamma$, $\varphi_0:=\varphi\circ\gamma$. Using restrictions as before, we may assume that both φ and φ_0 are faithful.

We first treat the case when $\lambda^{-1}\varphi\le\psi\le\lambda\varphi$ for some $\lambda>0$, then ψ_0 and φ_0 also satisfy this condition an all the states $\psi,\varphi,\psi_0,\varphi_0$ are faithful. By Theorem 2.3 (i), there are some $\bar a\in\mathcal{M}^{++}$ and $\bar a_0\in\mathcal{N}^{++}$ such that the infimum in the variational formula for $D_{\alpha,z}(\psi\|\varphi)$ resp. $D_{\alpha,z}(\psi_0\|\varphi_0)$ is attained. Using the inequalities in (6) - (8), we obtain

$$\begin{aligned}
Q_{\alpha,z}(\psi\|\varphi)&\le\alpha\|h_\psi^{\frac{1}{2p}}\gamma(\bar a_0)h_\psi^{\frac{1}{2p}}\|_p^p+(1-\alpha)\|h_\varphi^{\frac{1}{2r}}\gamma(\bar a_0)^{-1}h_\varphi^{\frac{1}{2r}}\|_r^r\\
&\le\alpha\|h_{\psi_0}^{\frac{1}{2p}}\bar a_0h_{\psi_0}^{\frac{1}{2p}}\|_p^p+(1-\alpha)\|h_{\varphi_0}^{\frac{1}{2r}}\bar a_0^{-1}h_{\varphi_0}^{\frac{1}{2r}}\|_r^r\\
&=Q_{\alpha,z}(\psi_0\|\varphi_0),
\end{aligned}$$

where we again put $p=\frac{z}{\alpha}$, $r=\frac{z}{1-\alpha}$. Assume $D_{\alpha,z}(\psi\|\varphi)=D_{\alpha,z}(\psi_0\|\varphi_0)$, then all the above inequalities must be equalities.

This has several consequences. First, by uniqueness of $\bar a$ in Theorem 2.3 (i), we have $\gamma(\bar a_0)=\bar a$. Furthermore, by Lemma 3.1 (ii), we obtain that

$$\|h_\psi^{\frac{1}{2p}}\gamma(\bar a_0)h_\psi^{\frac{1}{2p}}\|_p^p=\|h_{\psi_0}^{\frac{1}{2p}}\bar a_0h_{\psi_0}^{\frac{1}{2p}}\|_p^p,\qquad\|h_\varphi^{\frac{1}{2r}}\gamma(\bar a_0)^{-1}h_\varphi^{\frac{1}{2r}}\|_r^r=\|h_{\varphi_0}^{\frac{1}{2r}}\bar a_0^{-1}h_{\varphi_0}^{\frac{1}{2r}}\|_r^r.$$

By the assumptions, at least one of p and r must be strictly larger than 1. Assume that $r > 1$ (the case $p > 1$ is similar, even slightly easier). Since $h_\varphi^{\frac{1}{2r}}\gamma(\bar a_0)^{-1}h_\varphi^{\frac{1}{2r}} \le h_\varphi^{\frac{1}{2r}}\gamma(\bar a_0^{-1})h_\varphi^{\frac{1}{2r}}$, Lemma 3.1 and the equality above imply that

$$\|h_\varphi^{\frac{1}{2r}}\gamma(\bar a_0)^{-1}h_\varphi^{\frac{1}{2r}}\|_r^r = \|h_\varphi^{\frac{1}{2r}}\gamma(\bar a_0^{-1})h_\varphi^{\frac{1}{2r}}\|_r^r = \|h_{\varphi_0}^{\frac{1}{2r}}\bar a_0^{-1}h_{\varphi_0}^{\frac{1}{2r}}\|_r^r. \tag{9}$$

Using [? , Lemma 5.1], this shows that we must have

$$h_\varphi^{\frac{1}{2r}}\bar a^{-1}h_\varphi^{\frac{1}{2r}} = h_\varphi^{\frac{1}{2r}}\gamma(\bar a_0)^{-1}h_\varphi^{\frac{1}{2r}} = h_\varphi^{\frac{1}{2r}}\gamma(\bar a_0^{-1})h_\varphi^{\frac{1}{2r}}.$$

Put $h_\omega := h_\varphi^{\frac12}\bar a^{-1}h_\varphi^{\frac12}$, $h_{\omega_0} := h_{\varphi_0}^{\frac12}\bar a_0^{-1}h_{\varphi_0}^{\frac12}$. Then we have

$$(\gamma_\varphi^*)_*(h_{\omega_0}) = h_\varphi^{\frac12}\gamma(\bar a_0^{-1})h_\varphi^{\frac12} = h_\varphi^{\frac12}\gamma(\bar a_0)^{-1}h_\varphi^{\frac12} = h_\varphi^{\frac12}\bar a^{-1}h_\varphi^{\frac12} = h_\omega. \tag{10}$$

Using (9), we obtain

$$Q_{r,r}((\gamma_\varphi^*)_*(h_{\omega_0})\|(\gamma_\varphi^*)_*(h_{\varphi_0})) = \|h_\varphi^{\frac{1}{2r}}\gamma(\bar a_0^{-1})h_\varphi^{\frac{1}{2r}}\|_r^r = \|h_{\varphi_0}^{\frac{1}{2r}}\bar a_0^{-1}h_{\varphi_0}^{\frac{1}{2r}}\|_r^r = Q_{r,r}(h_{\omega_0}\|h_{\varphi_0}),$$

which by the properties of the sandwiched Rényi divergence [2, Thm.] implies that γ_φ^* is sufficient with respect to $\{\omega_0,\varphi_0\}$. By Proposition 1 and the fact that the Petz dual $(\gamma_\varphi^*)_{\varphi_0}^*$ is γ itself, this is equivalent to

$$\gamma_* \circ (\gamma_\varphi^*)_*(h_{\omega_0}) = h_{\omega_0},$$

so that by (10),

$$(\gamma_\varphi^*)_* \circ \gamma_*(h_\omega) = (\gamma_\varphi^*)_* \circ \gamma_* \circ (\gamma_\varphi^*)_*(h_{\omega_0}) = (\gamma_\varphi^*)_*(h_{\omega_0}) = h_\omega.$$

Hence γ is sufficient with respect to $\{\omega,\varphi\}$. Let $\mathcal{E}$ be the faithful normal conditional expectation as in Proposition 1. Then $\mathcal{E}$ preserves both h_ω and h_φ, which by [?] implies that

$$h_\omega = \mathcal{E}_*(h_\omega) = h_\varphi^{\frac12}\mathcal{E}(\bar a^{-1})h_\varphi^{\frac12},$$

so that $\mathcal{E}(\bar a^{-1}) = \bar a^{-1}$. It follows that

$$\left(h_\varphi^{\frac{1}{2r}}h_\psi^{\frac1p}h_\varphi^{\frac{1}{2r}}\right)^{1-\alpha} = h_\varphi^{\frac{1}{2r}}\bar a^{-1}h_\varphi^{\frac{1}{2r}} \in L_r(\mathcal{E}(\mathcal{M}))$$

and consequently $|h_\psi^{\frac{1}{2p}}h_\varphi^{\frac{1}{2r}}| \in L_{2z}(\mathcal{E}(\mathcal{M}))$. Note that by the assumptions $2z > 1$, so that we may use the multiplicativity properties of the extension of $\mathcal{E}$ [?]. Let

$$h_\psi^{\frac{1}{2p}}h_\varphi^{\frac{1}{2r}} = u|h_\psi^{\frac{1}{2p}}h_\varphi^{\frac{1}{2r}}|$$

be the polar decomposition in $L_{2z}(\mathcal{M})$, then we have

$$u^*h_\psi^{\frac{1}{2p}}h_\varphi^{\frac{1}{2r}} = \mathcal{E}_{2z}(u^*h_\psi^{\frac{1}{2p}}h_\varphi^{\frac{1}{2r}}) = \mathcal{E}_{2p}(u^*h_\psi^{\frac{1}{2p}})h_\varphi^{\frac{1}{2r}},$$

which implies that

$$\mathcal{E}_p(h_\psi^{\frac1p}) = \mathcal{E}_p(h_\psi^{\frac{1}{2p}}uu^*h_\psi^{\frac{1}{2p}}) = h_\psi^{\frac{1}{2p}}uu^*h_\psi^{\frac{1}{2p}} = h_\psi^{\frac1p}$$

Consequently, $\psi\circ\mathcal{E} = \psi$ and γ is sufficient with respect to $\{\psi,\varphi\}$.

□

3.3.2 The case $\alpha > 1$

~~//////////////////////~~

A ~~Monotonicity in the parameter z~~Haagerup L_p-spaces

~~It is well known[2? ?] that the standard Rényi divergence $D_{\alpha,1}(\psi\|\varphi)$ is monotone increasing in $\alpha \in (0,1) \cup (1,\infty)$ and the sandwiched Rényi divergence $D_{\alpha,\alpha}(\psi\|\varphi)$ is monotone increasing in $\alpha \in [1/2,1) \cup (1,\infty)$. It is also known [2? ?] that~~

$$\lim_{\alpha\nearrow 1} D_{\alpha,1}(\psi\|\varphi) = \lim_{\alpha\nearrow 1} D_{\alpha,\alpha}(\psi\|\varphi) = D_1(\psi\|\varphi),$$

~~and if $D_{\alpha,1}(\psi\|\varphi) < \infty$ (resp., $D_{\alpha,\alpha}(\psi\|\varphi) < \infty$) for some $\alpha > 1$, then~~

$$\lim_{\alpha\searrow 1} D_{\alpha,1}(\psi\|\varphi) = D_1(\psi\|\varphi) \quad \Big(\text{resp., } \lim_{\alpha\searrow 1} D_{\alpha,\alpha}(\psi\|\varphi) = D_1(\psi\|\varphi)\Big).$$

~~In the rest of the paper we will discuss similar monotonicity properties and limits for $D_{\alpha,z}(\psi\|\varphi)$. We consider monotonicity in the parameter z in Sec. 4 and monotonicity in the parameter α in Sec. 5.~~ The following lemmas are well known, proofs are given for completeness.

A.1 ~~The finite von Neumann algebra case~~

~~Assume that $(\mathcal{M},\tau)$ is a semi-finite von Neumann algebra $\mathcal{M}$ with a faithful normal semi-finite trace τ. Then the Haagerup L_p-space $L_p(\mathcal{M})$ is identified with the L_p-space $L_p(\mathcal{M},\tau)$~~

Lemma A.1. *For any $0 < p < \infty$ and $\varphi \in \mathcal{M}_*^+$, $h_\varphi^{\frac{1}{2p}}\mathcal{M}^+ h_\varphi^{\frac{1}{2p}}$ is dense in $L_p(\mathcal{M})^+$ with respect to* ~~*τ [1, Example 9.11]. Hence one can define $Q_{\alpha,z}(\psi\|\varphi)$ for $\psi,\varphi \in \mathcal{M}_*^+$ by replacing, in Definition 2.1, $L_p(\mathcal{M})$ with $L_p(\mathcal{M},\tau)$ and $h_\psi \in L_1(\mathcal{M})_+$ with the Radon–Nikodym derivative $d\psi/d\tau \in L_1(\mathcal{M},\tau)^+$. Below we use the symbol h_ψ to denote $d\psi/d\tau$ as well. Note that τ on $\mathcal{M}_+$ is naturally extended to the positive part $\widetilde{\mathcal{M}}^+$ of the space $\widetilde{\mathcal{M}}$ of τ-measurable operators. We then have [1, Proposition 4.20]*~~

$$\tau(a) = \int_0^\infty \mu_s(a)\,ds, \qquad a \in \widetilde{\mathcal{M}}^+,$$

~~*where $\mu_s(a)$ is the generalized s-number of a [?].*~~ *the (quasi)-norm $\|\cdot\|_p$.*

~~*Throughout this subsection we assume that $\mathcal{M}$ is a finite von Neumann algebra with a faithful normal finite trace τ; then $\widetilde{\mathcal{M}}^+$ consists of all positive self-adjoint operators affiliated with $\mathcal{M}$.*~~

~~*For every $\psi,\varphi \in \mathcal{M}_*^+$ with $\psi \neq 0$ and for any $\alpha, z > 0$ with $\alpha \neq 1$,*~~

$$D_{\alpha,z}(\psi\|\varphi) = \lim_{\searrow 0} D_{\alpha,z}(\psi\|\varphi + \tau) \quad \textit{increasingly},$$

~~*and hence $D_{\alpha,z}(\psi\|\varphi) = \sup_{>0} D_{\alpha,z}(\psi\|\varphi + \tau)$.*~~

Proof. ~~Case $0 < \alpha < 1$.We need to prove that~~

$$Q_{\alpha,z}(\psi\|\varphi) = \lim_{\searrow 0} Q_{\alpha,z}(\psi\|\varphi + \tau) \quad \text{decreasingly.}$$

~~In the present setting we have by (??)~~

$$Q_{\alpha,z}(\psi\|\varphi) = \tau\Big(\big(h_\psi^{\alpha/2z} h_\varphi^{\frac{1-\alpha}{z}} h_\psi^{\alpha/2z}\big)^z\Big) = \int_0^\infty \mu_s\big(h_\psi^{\alpha/2z} h_\varphi^{\frac{1-\alpha}{z}} h_\psi^{\alpha/2z}\big)^z \, ds,$$

~~and similarly~~

$$Q_{\alpha,z}(\psi\|\varphi + \tau) = \int_0^\infty \mu_s\big(h_\psi^{\alpha/2z} h_{\varphi+\tau}^{\frac{1-\alpha}{z}} h_\psi^{\alpha/2z}\big)^z \, ds.$$

~~Since $h_{\varphi+\tau}^{\frac{1-\alpha}{z}} = (h_\varphi +)^{\frac{1-\alpha}{z}}$ decreases to $h_\varphi^{\frac{1-\alpha}{z}}$ in the measure topology as $\searrow 0$, it follows that $h_\psi^{\alpha/2z} h_{\varphi+\tau}^{\frac{1-\alpha}{z}} h_\psi^{\alpha/2z}$ decreases to $h_\psi^{\alpha/2z} h_\varphi^{\frac{1-\alpha}{z}} h_\psi^{\alpha/2z}$ in the measure topology. Hence by [? , Lemma 3.4] we have $\mu_s\big(h_\psi^{\alpha/2z} h_{\varphi+\tau}^{\frac{1-\alpha}{z}} h_\psi^{\alpha/2z}\big) \searrow$ as $\searrow 0$ for almost every $s > 0$. Since $s \mapsto \mu_s\big(h_\psi^{\alpha/2z} h_{\varphi+\tau}^{\frac{1-\alpha}{z}} h_\psi^{\alpha/2z}\big)$ is integrable on $(0, \infty)$, the Lebesgue convergence theorem gives (??).~~

~~Case $\alpha > 1$.We need to prove that~~

$$Q_{\alpha,z}(\psi\|\varphi) = \lim_{\searrow 0} Q_{\alpha,z}(\psi\|\varphi + \tau) \quad \text{increasingly.}$$

~~For any > 0, since $h_{\varphi+\tau} = h_\psi +$ has the bounded inverse $h_{\varphi+\tau}^{-1} = (h_\varphi +)^{-1} \in \mathcal{M}^+$, one can define $x_: = (h_\varphi +)^{-\frac{\alpha-1}{2z}} h_\psi^{\alpha/z} (h_\varphi +)^{-\frac{\alpha-1}{2z}} \in \widetilde{\mathcal{M}}^+$ so that~~

$$h_\psi^{\alpha/z} = (h_\varphi +)^{\frac{\alpha-1}{2z}} x_(h_\varphi +)^{\frac{\alpha-1}{2z}}.$$

~~In the present setting one can write by (??)~~

$$Q_{\alpha,z}(\psi\|\varphi + \tau) = \tau(x_{z)=\int_0^\infty \mu_s(x_)^z\, ds \ (\in[0,\infty]).}$$

We may assume that φ is faithful. By [? , Lemma 1.1], $\mathcal{M}h_\varphi^{\frac{1}{2p}}$ is dense in $L_{2p}(\mathcal{M})$ for any $0 < p < \infty$. Let ~~$0 << \leq'$. Since $(h_\varphi +)^{-\frac{\alpha-1}{z}} \geq (h_\varphi +')^{-\frac{\alpha-1}{z}}$, one has $\mu_s(x_) \geq \mu_s(x_{'})$ for all $s > 0$, so that~~

$$Q_{\alpha,z}(\psi\|\varphi + \tau) \geq Q_{\alpha,z}(\psi\|\varphi +' \tau).$$

~~Hence $> 0 \mapsto D_{\alpha,z}(\psi\|\varphi + \tau)$ is decreasing.~~

~~First, assume that $s(\psi) \not\leq s(\varphi)$. Then $\mu_{s_0}(h_\psi^{\alpha/2z} s(\varphi)^\perp h_\psi^{\alpha/2z}) > 0$ for some $s_0 > 0$; indeed, otherwise, $h_\psi^{\alpha/2z} s(\varphi)^\perp h_\psi^{\alpha/2z} = 0$ so that $s(\psi) \leq s(\varphi)$. Hence we have~~

$$\mu_s(x_) = \mu_s\big(h_\psi^{\alpha/2z} (h_\varphi +)^{-\frac{\alpha-1}{z}} h_\psi^{\alpha/2z}\big) \geq^{-\frac{\alpha-1}{z}} \mu_s(h_\psi^{\alpha/2z} s(\varphi)^\perp h_\psi^{\alpha/2z}) \nearrow \infty \quad \text{as} \searrow 0$$

~~for all $s \in (0, s_0]$. Therefore, it follows from (??) that $Q_{\alpha,z}(\psi\|\varphi + \tau) \nearrow \infty = Q_{\alpha,z}(\psi\|\varphi)$.~~

~~Next, assume that $s(\psi) \le s(\varphi)$. Take the spectral decomposition $h_\varphi = \int_0^\infty t\,de_t$ and define $y, x \in \widetilde{\mathcal{M}}_+$ by~~

$$y := h_\varphi^{-\frac{\alpha-1}{z}} s(\varphi) = \int_{(0,\infty)} t^{-\frac{\alpha-1}{z}}\,de_t, \qquad x := y^{1/2} h_\psi^{\alpha/z} y^{1/2}.$$

~~Since~~

$$h_\psi^{\alpha/z} = s(\varphi) h_\psi^{\alpha/z} s(\varphi) = h_\varphi^{\frac{\alpha-1}{2z}} y^{1/2} h_\psi^{\alpha/z} y^{1/2} h_\varphi^{\frac{\alpha-1}{2z}} = h_\varphi^{\frac{\alpha-1}{2z}} x h_\varphi^{\frac{\alpha-1}{2z}},$$

~~one has, similarly to **??**,~~

$$Q_{\alpha,z}(\psi\|\varphi) = \tau(x^z) = \int_0^\infty \mu_s(x)^z\,ds.$$

~~We write $(h_\varphi+)^{-\frac{\alpha-1}{z}} s(\varphi) = \int_{(0,\infty)} (t+)^{-\frac{\alpha-1}{z}}\,de_t$, and for any $\delta > 0$ choose a $t_0 > 0$~~ $y \in L_p(\mathcal{M})^+$, then $y^{\frac{1}{2}} \in L_{2p}(\mathcal{M})$, hence there is a sequence $a_n \in \mathcal{M}$ such that ~~$\tau(e_{(0,t_0)}) < \delta$. Then, since $\int_{[t_0,\infty)} (t+)^{-\frac{\alpha-1}{z}}\,de_t \to \int_{[t_0,}$ in the operator norm as $\searrow 0$, we obtain $(h_\varphi+)^{-\frac{\alpha-1}{z}} s(\varphi) \nearrow y$ in the measure topology (see [? , 1.5]), so that $h_\psi^{\alpha/2z} (h_\varphi+)^{-\frac{\alpha-1}{z}} h_\psi^{\alpha/2z} \nearrow h_\psi^{\alpha/2z} y h_\psi^{\alpha/2z}$ in the measure topology as $\searrow 0$. Hence we have by [? , Lemma 3.4]~~

$$\mu_s(x_) = \mu_s\big(h_\psi^{\alpha/2z} (h_\varphi+)^{-\frac{\alpha-1}{z}} h_\psi^{\alpha/2z}\big) \nearrow \mu_s(h_\psi^{\alpha/2z} y h_\psi^{\alpha/2z}) = \mu_s(x)$$

$\|a_n h_\varphi^{\frac{1}{2p}} - y^{\frac{1}{2}}\|_{2p} \to 0$. Then also

$$\|h_\varphi^{\frac{1}{2p}} a_n^* - y^{\frac{1}{2}}\|_p = \|(a_n h_\varphi^{\frac{1}{2p}} - y^{\frac{1}{2}})^*\|_p = \|a_n h_\varphi^{\frac{1}{2p}} - y^{\frac{1}{2}}\|_p \to 0$$

~~for all $s > 0$. Therefore, by (??) and (??) the monotone convergence theorem gives (??).~~

~~Let $(\mathcal{M}, \tau)$~~ and ~~ψ, φ be as above, and let $0 < z \le z'$. Then~~

$$\begin{cases} D_{\alpha,z}(\psi\|\varphi) \le D_{\alpha,z'}(\psi\|\varphi), & 0 < \alpha < 1, \\ D_{\alpha,z}(\psi\|\varphi) \ge D_{\alpha,z'}(\psi\|\varphi), & \alpha > 1. \end{cases}$$

$$\|h_\varphi^{\frac{1}{2p}} a_n^* a_n h_\varphi^{\frac{1}{2p}} - y\|_p = \|(h_\varphi^{\frac{1}{2p}} a_n^* - y^{\frac{1}{2}}) a_n h_\varphi^{\frac{1}{2p}} + y^{\frac{1}{2}} (a_n h_\varphi^{\frac{1}{2p}} - y^{\frac{1}{2}})\|_p$$

~~The case $0 < \alpha < 1$ was shown in [? , Theorem 1(x)] for general von Neumann algebras. For the case $\alpha > 1$, by Lemma **??** it suffices to show that , for every > 0,~~

$$\tau\Big(\Big(y_{\frac{\alpha-1}{2z}} h_\psi^{\alpha/z} y_{\frac{\alpha-1}{2z}}\Big)^z\Big) \ge \tau\Big(\Big(y_{\frac{\alpha-1}{2z'}} h_\psi^{\alpha/z'} y_{\frac{\alpha-1}{2z'}}\Big)^z\Big),$$

~~where $y_: = (h_\varphi+)^{-1} \in \mathcal{M}_+$. The above is equivalently written as~~

$$\tau\Big(\big|(h_\psi^{\alpha/2z'})^r (y^{(\alpha-1)/2z'})^r\big|^{2z}\Big) \ge \tau\Big(\big|h_\psi^{\alpha/2z'} y^{(\alpha-1)/2z'}\big|^{2zr}\Big),$$

~~where $r := z'/z \geq 1$. Hence the desired inequality follows from Kosaki's ALT inequality [? , Corollary 3] .~~

~~When $(\mathcal{M}, \tau)$ and ψ, φ are as in Lemma **??**, one can define, thanks to Lemma **??**, for any $\alpha \in (0, \infty) \setminus \{1\}$,~~

$$Q_{\alpha,\infty}(\psi\|\varphi) := \lim_{z\to\infty} Q_{\alpha,\infty}(\psi\|\varphi) = \inf_{z>0} Q_{\alpha,z}(\psi\|\varphi),$$

$$D_{\alpha,\infty}(\psi\|\varphi) := \frac{1}{\alpha-1}\log\frac{Q_{\alpha,\infty}(\psi\|\varphi)}{\psi()}$$

$$= \lim_{z\to\infty} D_{\alpha,z}(\psi\|\varphi) = \begin{cases} \sup_{z>0} D_{\alpha,z}(\psi\|\varphi), & 0<\alpha<1, \\ \inf_{z>0} D_{\alpha,z}(\psi\|\varphi), & \alpha>1. \end{cases}$$

~~If $h_\psi, h_\varphi \in \mathcal{M}^{++}$ (i. e., $\delta\tau \leq \psi, \varphi \leq \delta^{-1}\tau$ for some $\delta \in (0,1)$), then the Lie–Trotter formula gives~~

$$Q_{\alpha,\infty}(\psi\|\varphi) = \tau\big(\exp(\alpha\log h_\psi + (1-\alpha)\log h_\varphi)\big).$$

~~Let $(\mathcal{M}, \tau)$ and ψ, φ be as above. Then for any $z > 0$,~~

$$\begin{cases} D_{\alpha,z}(\psi\|\varphi) \leq D_1(\psi\|\varphi), & 0<\alpha<1, \\ D_{\alpha,z}(\psi\|\varphi) \geq D_1(\psi\|\varphi), & \alpha>1. \end{cases}$$

~~First, assume that $h_\psi, h_\varphi \in \mathcal{M}^{++}$. Set self-adjoint $H := \log h_\psi$ and $K := \log h_\varphi$ in $\mathcal{M}$ and define $F(\alpha) := \log\tau\big(e^{\alpha H+(1-\alpha)K}\big)$ for $\alpha > 0$. Then by (**??**), $F(\alpha) = \log Q_{\alpha,\infty}(\psi\|\varphi)$ for all $\alpha \in (0,\infty)\setminus\{1\}$, and we compute~~

$$F'(\alpha) = \frac{\tau\big(e^{\alpha H+(1-\alpha)K}(H-K)\big)}{\tau\big(e^{\alpha H+(1-\alpha)K}\big)},$$

$$F''(\alpha) = \frac{\big\{\tau\big(e^{\alpha H+(1-\alpha)K}(H-K)\big)\big\}^2 - \tau\big(e^{\alpha H+(1-\alpha)K}(H-K)^2\big)}{\big\{\tau\big(e^{\alpha H+(1-\alpha)K}\big)\big\}^2}.$$

Since ~~$F''(\alpha) \geq 0$ on $(0,\infty)$ thanks to the Schwarz inequality, we see that $F(\alpha)$ is convex on $(0,\infty)$ and hence~~

$$D_{\alpha,\infty}(\psi\|\varphi) = \frac{F(\alpha)-F(1)}{\alpha-1}.$$

~~is increasing in $\alpha \in (0,\infty)$, where for $\alpha = 1$ the above RHS is understood as~~

$$F'(1) = \frac{\tau(e^H(H-K))}{\tau(e^H)} = \frac{\tau\big(h_\psi(\log h_\psi - \log h_\varphi)\big)}{\tau(h_\psi)} = D_1(\psi\|\varphi).$$

~~Hence by (**??**) the assertion holds when $h_\psi, h_\varphi \in \mathcal{M}^{++}$. Below we extend it to general $\psi, \varphi \in \mathcal{M}_*^+$~~$\|\cdot\|_p$ is a (quasi)-norm, the above expression goes to 0 by the Hölder inequality.

~~Case $0 < \alpha < 1$.Let $\psi, \varphi \in \mathcal{M}_*^+$ and $z > 0$. From [? , Theorem 1(iv)] and [? , Corollary 2.8(3)] we have~~

$$
\begin{aligned}
D_{\alpha,z}(\psi\|\varphi) &= \lim_{\searrow 0} D_{\alpha,z}(\psi+\tau\|\varphi+\tau), \\
D_1(\psi\|\varphi) &= \lim_{\searrow 0} D_1(\psi+\tau\|\varphi+\tau),
\end{aligned}
$$

~~so that we may assume that $\psi, \varphi \geq \tau$ for some > 0. Take the spectral decompositions $h_\psi = \int_0^\infty t\, de_t^\psi$ and $h_\varphi = \int_0^\infty t\, de_t^\varphi$, and define $e_n := e_n^\psi \wedge e_n^\varphi$ for each $n \in$. Then $\tau(e_n^\perp) \leq \tau((e_n^\psi)^\perp) + \tau((e_n^\varphi)^\perp) \to 0$ as $n \infty$, so that $e_n \nearrow$. We set $\psi_n := \psi(e_n \cdot e_n)$ and $\varphi_n := \varphi(e_n \cdot e_n)$; then $h_{\psi_n} = e_n h_\psi e_n$ and $h_{\varphi_n} = e_n h_\varphi e_n$ are in $(e_n \mathcal{M} e_n)^{++}$. Note that~~

$$
\begin{aligned}
\|h_\psi - e_n h_\psi e_n\|_1 &\leq \|(-e_n)h_\psi\|_1 + \|e_n h_\psi(-e_n)\|_1 \\
&\leq \|(-e_n)h_\psi^{1/2}\|_2 \|h_\psi^{1/2}\|_2 + \|e_n h_\psi^{1/2}\|_2 \|h_\psi^{1/2}(-e_n)\|_2 \\
&= \psi(-e_n)^{1/2}\psi()^{1/2} + \psi(e_n)^{1/2}\psi(-e_n)^{1/2} \to 0 \quad \text{as } n \to \infty,
\end{aligned}
$$

~~and similarly $\|h_\varphi - e_n h_\varphi e_n\|_1 \to 0$. Hence by [? , Theorem 1(iv)] one has $D_{\alpha,z}(e_n\psi e_n\|e_n\varphi e_n) \to D_{\alpha,z}(\psi\|\varphi)$. On the other hand, one has $D_1(e_n\psi e_n\|e_n\varphi e_n) \to D_1(\psi\|\varphi)$ by [? , Proposition 2.10]. Since $D_{\alpha,z}(e_n\psi e_n\|e_n\varphi e_n)$ holds by regarding $e_n\psi e_n, e_n\varphi e_n$ as functionals on the reduced von Neumann algebra $e_n\mathcal{M}e_n$, we obtain the desired inequality for general $\psi, \varphi \in \mathcal{M}_*^+$.~~

~~Case $\alpha > 1$.We show the extension to general $\psi, \varphi \in \mathcal{M}_*^+$ by dividing four steps as follows, where $h_\psi = \int_0^\infty t\, e_t^\psi$ and $h_\varphi = \int_0^\infty t\, de_t^\varphi$ are the spectral decompositions.~~

~~(1)Assume that $h_\psi \in \mathcal{M}^+$ and $h_\varphi \in \mathcal{M}^{++}$. Set $\psi_n \in \mathcal{M}_*^+$ by $h_{\psi_n} = (1/n)e_{[0,1/n]}^\psi + \int_{(1/n,\infty)} t\, de_t^\psi$ $(\in \mathcal{M}^{++})$. Since $h_{\psi_n}^{\alpha/z} \searrow h_\psi^{\alpha/z}$ in the operator norm, we have by (??) and [? , Lemma 3.4]~~

$$
\begin{aligned}
Q_{\alpha,z}(\psi\|\varphi) &= \int_0^\infty \mu_s\big((h_\varphi^{-1})^{\frac{\alpha-1}{2z}} h_\psi^{\alpha/z} (h_\varphi^{-1})^{\frac{\alpha-1}{2z}}\big)^z\, ds \\
&= \lim_{n\to\infty} \int_0^\infty \mu_s\big((h_\varphi^{-1})^{\frac{\alpha-1}{2z}} h_{\psi_n}^{\alpha/z} (h_\varphi^{-1})^{\frac{\alpha-1}{2z}}\big)^z\, ds = \lim_{n\to\infty} Q_{\alpha,z}(\psi_n\|\varphi).
\end{aligned}
$$

~~From this and the lower semicontinuity of D_1 the extension holds in this case.~~

~~(2)Assume that $h_\psi \in \mathcal{M}^+$ and $h_\varphi \geq \delta$ for some $\delta > 0$. Set $\varphi_n \in \mathcal{M}_*^+$ by $h_{\varphi_n} = \int_{[\delta,n]} t\, de_t^\varphi + n e_{(n,\infty)}^\varphi$ $(\in \mathcal{M}^{++})$. Since $h_{\varphi_n}^{-\frac{\alpha-1}{z}} \searrow h_\varphi^{-\frac{\alpha-1}{z}}$ in the operator norm, we have by (??) and [? , Lemma 3.4] again~~

$$
\begin{aligned}
Q_{\alpha,z}(\psi\|\varphi) &= \int_0^\infty \mu_s\big(h_\psi^{\alpha/2z} h_\varphi^{-\frac{\alpha-1}{z}} h_\psi^{\alpha/2z}\big)^z\, ds \\
&= \lim_{n\to\infty} \int_0^\infty \mu_s\big(h_\psi^{\alpha/2z} h_{\varphi_n}^{-\frac{\alpha-1}{z}} h_\psi^{\alpha/2z}\big)^z\, ds = \lim_{n\to\infty} Q_{\alpha,z}(\psi, \varphi_n).
\end{aligned}
$$

~~From this and (1) above the extension holds in this case too.~~

~~(3)Assume that ψ is general and $\varphi \geq \delta\tau$ for some $\delta > 0$. Set $\psi_n \in \mathcal{M}_*^+$ by $h_{\psi_n} = \int_{[0,n]} t\, de_t^\psi + ne_{(n,\infty)}^\varphi$ $(\in \mathcal{M}_+)$. Since $h_{\psi_n}^{\alpha/z} \nearrow h_\psi^{\alpha/z}$ in the measure topology, one can argue as in (??) with use of the monotone convergence theorem to see from (2) that the extension holds in this case too.~~

~~(4)Finally, from (3) with Lemma ?? and [? , Corollary 2.8(3)] it follows that the desired extension hods for general $\psi, \varphi \in \mathcal{M}_*^+$.~~ □

~~In the next proposition, we summarize inequalities for $D_{\alpha,z}$ obtained so far in Lemmas ?? and ??.~~

~~Assume that $\mathcal{M}$ is a finite von Neumann algebra with a faithful normal finite trace τ.~~

Lemma A.2. *Let* ~~*$\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$. If $0 < \alpha < 1 < \alpha'$ and $0 < z \leq z' \leq \infty$, then*~~

$$\underline{D_{\alpha,z}(\psi\|\varphi) \leq D_{\alpha,z'}(\psi\|\varphi) \leq D_1(\psi\|\varphi) \leq D_{\alpha',z'}(\psi\|\varphi) \leq D_{\alpha',z}(\psi\|\varphi).}$$

~~*Let $(\mathcal{M}, \tau)$ and ψ, φ be as in Proposition ??.*~~ *$\psi, \varphi \in \mathcal{M}_*^+$ with $\psi \leq \varphi$. Then for any* ~~$z \in [1, \infty]$,~~

$$\underline{\lim_{\alpha \nearrow 1} D_{\alpha,z}(\psi\|\varphi) = D_1(\psi\|\varphi).}$$

~~*Moreover, if $D_{\alpha,\alpha}(\psi\|\varphi) < \infty$ for some $\alpha > 1$ then for any $z \in (1, \infty]$,*~~

$$\underline{\lim_{\alpha \searrow 1} D_{\alpha,z}(\psi\|\varphi) = D_1(\psi\|\varphi).}$$

~~*Let $z \geq 1$. For every $\alpha \in (0,1)$, Proposition ?? gives*~~

$$\underline{D_{\alpha,1}(\psi\|\varphi) \leq D_{\alpha,z}(\psi\|\varphi) \leq D_1(\psi\|\varphi).}$$

~~*Hence (??) follows since it holds for $D_{\alpha,1}$ [? , Proposition 5.3(3)].*~~

~~*Next, assume that $D_{\alpha,\alpha}(\psi\|\varphi) < \infty$ for some $\alpha > 1$. Let $z > 1$. For every $\alpha \in (1, z]$, Proposition ?? gives*~~

$$\underline{D_1(\psi\|\varphi) \leq D_{\alpha,z}(\psi\|\varphi) \leq D_{\alpha,\alpha}(\psi\|\varphi).}$$

~~*Hence (??) follows since it holds for $D_{\alpha,\alpha}$ [2, Proposition 3.8(ii)].*~~

~~*In this subsection, in the specialized setting of finite von Neumann algebras, we have given monotonicity of $D_{\alpha,z}$ in the parameter z in an essentially similar way to the finite-dimensional case [?]. In the next subsection we will extend it to general von Neumann algebras under certain restrictions of α, z.*~~

A.1 ~~The general von Neumann algebra case~~

~~*For every $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$, and $0 < \alpha < 1$, we have: If $0 < \alpha < 1$*~~ *$a \in \mathcal{M}$ and* ~~$\max\{\alpha, 1-\alpha\} \leq z \leq z'$, *then*~~

$$\underline{D_{\alpha,z}(\psi\|\varphi) \leq D_{\alpha,z'}(\psi\|\varphi) \leq D_1(\psi\|\varphi).}$$

$p \in [1, \infty)$,

$$\mathrm{Tr}\left((a^* h_\psi^{\frac{1}{p}} a)^p\right) \leq Tr\left((a^* h_\varphi^{\frac{1}{p}} a)^p\right)$$

~~*If* $\alpha > 1$ *and* $\max\{\alpha/2, \alpha - 1\} \le z \le z' \le \alpha$*, then*~~

$$D_1(\psi\|\varphi) \le D_{\alpha,z'}(\psi\|\varphi) \le D_{\alpha,z}(\psi\|\varphi).$$

~~*Hiai (12/8/2023)In fact, (2) is improved in Theorem 6.*~~

~~*For every* $\psi, \varphi \in \mathcal{M}_*^+$*,* $\psi \ne 0$*, and* $\alpha > 1$*, the function* $z \mapsto D_{\alpha,z}(\psi\|\varphi)$ *is monotone decreasing on* $[\alpha/2, \infty)$*. Anna (Jan. 23, 2024)*~~

B ~~Monotonicity in the parameter α~~

A.1 ~~The case $\alpha < 1$ and all $z > 0$~~

~~Let $\psi, \varphi \in \mathcal{M}_*^+$ and $z > 0$. Then we have~~

~~(1) $\alpha \mapsto \log Q_{\alpha,z}(\psi\|\varphi)$ is convex on $(0, 1)$,~~

~~(2) $\alpha \mapsto D_{\alpha,z}(\psi\|\varphi)$ is monotone increasing on $(0, 1)$.~~

~~Anna (Jan. 10, 2024), Hiai (1/16/2024)~~

A.1 ~~The case $1 < \alpha \le 2z$~~

~~Let $\psi, \varphi \in \mathcal{M}_*^+$ and $z > 1/2$. Then we have~~

~~(1) $\alpha \mapsto \log Q_{\alpha,z}(\psi\|\varphi)$ is convex on $(1, 2z]$,~~

~~(2) $\alpha \mapsto D_{\alpha,z}(\psi\|\varphi)$ is monotone increasing on $(1, 2z]$.~~

~~Anna (Jan. 23, 2024), Hiai (12/31/2023)~~

A.1 ~~Limits as $\alpha \nearrow 1$ and $\alpha \searrow 1$~~

~~Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \ne 0$. For every $z \in (0, 1]$ we have~~

$$\lim_{\alpha \nearrow 1} D_{\alpha,z}(\psi\|\varphi) = D_1(\psi\|\varphi).$$

~~Anna (Dec. 7, 2023)~~

~~Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \ne 0$, and $z > 1/2$. Assume that $D_{\alpha,z}(\psi\|\varphi) < \infty$ for some $\alpha \in (1, 2z]$. Then we have~~

$$\lim_{\alpha \searrow 1} D_{\alpha,z}(\psi\|\varphi) = D_1(\psi\|\varphi).$$

~~Anna (Jan. 23, 2024)~~

~~M. Berta, V. B. Scholz, and M. Tomamichel. Rényi divergences as weighted noncommutative vector valued *Lp*-spaces. Annales Henri Poincaré, 19:1843–1867, 2018. doi:https://doi.org/10.48550/arXiv.160~~

~~M.-D. Choi. A Schwarz inequality for positive linear maps on C^*-algebras. Illinois Journal of Mathematics, 18~~

Proof. Since $1/p \in (p, 1]$, it follows (~~4):565–574, 1974. doi:10.1215/ijm/1256051007.~~

~~T. Fack and H. Kosaki. Generalized s-numbers of~~ see [? , Lemma B.7] and [? , Lemma 3.2]) that $h_\psi^{1/p} \le h_\varphi^{1/p}$ as τ-measurable operators ~~. Pacific Journal of Mathematics, 123(2):269–300, 1986.~~

~~U. Haagerup. L_p-spaces associated with an arbitrary von Neumann algebra. In Algebres d'opérateurs et leurs applications en physique mathématique (Proc. Colloq., Marseille, 1977), volume 274, pages 175–184, 1979.~~

~~F. Hiai. Quantum f-divergences in von Neumann algebras. I. Standard f-divergences. Journal of Mathematical Physics, 59(10):102202, 2018.~~

~~F. Hiai. Quantum f-Divergences in von Neumann Algebras: Reversibility of Quantum Operations. Mathematical Physics Studies. Springer, Singapore, 2021. ISBN 9789813341999. doi:10.1007/978-981-33-4199~~

with $\mathcal{M} \rtimes_{\sigma^\phi} \mathbb{R}$ (in which $L_p(\mathcal{M})$ lives). Hence $a^* h_\psi^{1/p} a \le a^* h_\varphi^{1/p} a$ in the same sense. Therefore, by [? , Lemma 2.5 (iii), Lemma 4.8], we have the statement.

[1] ~~F. Hiai. Lectures on Selected Topics in Von Neumann Algebras. EMS Series of Lectures in Mathematics. EMS Press, Berlin, 2021.~~ □

~~A. Jenčová. Rényi relative entropies and noncommutative Lp-spaces. Annales Henri Poincaré, 19:2513–2542, 2018. doi:10.1007/s00023-018-0683-5.~~

~~A. Jenčová. Rényi relative entropies and noncommutative Lp-spaces II. Annales Henri Poincaré, 22:3235–3254, 2021. doi:10.1007/s00023-021-01074-9.~~

~~M. Junge and Q. Xu. Noncommutative Burkholder/Rosenthal inequalities. The Annals of Probability, 31(2):948–995, 2003.~~

~~S. Kato. On α-z-Rényi divergence in the von Neumann algebra setting. arXiv preprint arXiv:2311.01748, 2023.~~

~~S. Kato and Y. Ueda. A remark on non-commutative Lp-spaces. arXiv preprint arXiv:2307.01790, 2023.~~

~~H. Kosaki. Applications of the complex interpolation method to a von Neumann algebra: Noncommutative L^p-spaces. J. Funct. Anal., 56:26–78, 1984. doi:https://doi.org/10.1016/0022-1236(84)90025~~

~~H. Kosaki. An inequality of Araki–Lieb–Thirring (von Neumann algebra case). Proc. Amer. Math. Soc., 114:477–481, 1992.~~

~~M. Mosonyi and F. Hiai. Some continuity properties of quantum Rényi divergences. IEEE Transactions on Information Theory, to appear, DOI 10.1109/TIT.2023.3324758.~~

~~D. Petz. Quasi-entropies for states of a von Neumann algebra. Publications of the Research Institute for Mathematical Sciences, 21(4):787–800, 1985. doi:10.2977/prims/1195178929.~~

~~D. Petz. Sufficient subalgebras and the relative entropy of states of a von Neumann algebra. Communications in Mathematical Physics, 105(1):123–131, 1986. doi:10.1007/BF01212345.~~

~~D. Petz. Sufficiency of channels over von Neumann algebras. The Quarterly Journal of Mathematics, 39(1):97–108, 1988. doi:10.1093/qmath/39.1.97.8~~

~~M. Terp. L_p-spaces associated with von Neumann algebras. Notes, Copenhagen University, 1981.~~