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Fuzzy observables and the universal family of fuzzy events

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Then, we show that every weak Markov kernel is functionally subordinated to μ^U; in some sense the randomization encoded in μ^U includes all the possible randomizations. Finally, we show that the universal family of fuzzy events corresponding to the universal Markov kernel includes all the possible fuzzy events of which a POVM can represent the probabilities. Moreover, the probabilities associated to a fuzzy observable F coincide with the probabilities of the universal family of fuzzy events with respect to E^F.

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Fuzzy observables and the universal family of fuzzy events

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1. Introduction

In the present section we briefly recall the concept of fuzzy observable and its connections with fuzzy sets. As a consequence of the statistical nature of the measurement process [19], quantum observables are represented by positive operator valued measures [12, 15, 19, 27, 29, 32]. Spectral measures are particular examples of POVMs and, by the spectral theorem, they are in a one-to-one correspondence with self-adjoint operators. Therefore, the analysis of the measurement process reveals that the set of self-adjoint operators is too small in order to represent quantum observables. A very relevant feature of positive operator valued measures (POVMs) is that there are couples of non-commuting POVMs which are the marginals of a joint POVM (they are jointly measurable). We know that in the particular case of self-adjoint operators joint measurability and commutativity coincide [12, 25].

As a relevant example one can consider the spectral measures of the position and momentum operators, E^Q and E^P for which a joint spectral measure does not exist since they do not commute. Nevertheless, it is possible to randomize E^Q and E^F by means of Markov kernels μ^Q and μ^P. That provides two POVMs F^Q and F^P for which a joint POVM does exist [6, 9, 15, 29]. In particular, for every Δ in the Borel σ-algebra of the reals, $\mathcal{B}(\mathbb{R})$,

$$
\begin{aligned}
F^Q(\Delta) &= \int_{\mathbb{R}} \mu^Q_\Delta(\lambda)\, dE^Q_\lambda, \qquad (1)\\
F^P(\Delta) &= \int_{\mathbb{R}} \mu^P_\Delta(\lambda)\, dE^P_\lambda.
\end{aligned}
$$

The POVMs F^Q and F^P are called the unsharp or fuzzy version of E^Q and E^P respectively [2, 17, 29] and are commutative since they are contained in the commutative von Neumann algebras generated by E^Q and E^P respectively. It is relevant that there is a third POVM, F, of which F^Q and F^P are the marginals [6, 15, 29, 32],

$$F(\Delta_q \times \mathbb{R}) = F^Q(\Delta_q)$$
$$F(\mathbb{R} \times \Delta_p) = F^P(\Delta_p).$$

That allows a representation of quantum mechanics on a phase space which should be interpreted as a stochastic phase space whose points are fuzzy points [29].

There are other examples of the power of POVMs as the mathematical representative of quantum observables [1, 2, 12, 15, 19, 27, 29, 32].

The previous example provides some insight about the relevance of commutative POVMs to quantum physics. We add that they model certain standard forms of noise in quantum measurements and provide optimal approximators as marginals in joint measurements of incompatible observables (e.g., Position and Momentum) [13].

In (1) we obtained the fuzzy position and momentum POVMs (that are commutative POVMs) as the randomization of the sharp position and momentum operators with the randomization realized through Markov kernels. That is a general proprety of commutative POVMs, i.e., every commutative POVM $F : \mathcal{B}(X) \to \mathcal{L}_s^+(\mathcal{H})$ from the σ-algebra of a topological space X to the space of linear positive self-adjoint operators on a Hilbert space $\mathcal{H}$ is the random version of a spectral measure E^F (the sharp version of F); the randomization being represented by a Markov kernel μ [8, 10, 16, 20, 21],

$$\langle \psi, F(\Delta)\psi \rangle := \int \mu_\Delta(\lambda)\, d\langle \psi, E_\lambda \psi \rangle, \quad \Delta \in \mathcal{B}(X), \quad \psi \in \mathcal{H}. \tag{2}$$

Consider for example the unsharp position observable in equation (1). It can be represented as follows [15],

$$\langle\psi, F^Q(\Delta)\psi\rangle := \int_{\mathbb{R}} \mu_\Delta(\lambda)\, d\langle\psi, E^Q_\lambda\psi\rangle, \quad \Delta\in\mathcal{B}(\mathbb{R}), \quad \psi\in L^2(\mathbb{R}), \tag{3}$$
$$\mu_\Delta(\lambda) := \int_{\mathbb{R}} \chi_\Delta(\lambda - y)\, f(y)\, dy, \quad \lambda\in\mathbb{R}$$

where, f is a positive, bounded, Borel function such that $\int_{\mathbb{R}} f(y)dy = 1$, while E^Q is the spectral measure corresponding to the position operator

$$Q : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$
$$(Q\psi)(x) := x\psi(x), \quad a.a. \quad x\in\mathbb{R}.$$

The quantity $\langle\psi, E^Q(\Delta)\psi\rangle$ can be interpreted as the probability that a perfectly accurate measurement (sharp measurement) of the position gives a result in Δ. A possible interpretation of equation (3) is that, due to measurement imprecision[1], the outcomes of the measurement of E^Q are randomized: if the sharp value of the outcome of the measurement of E^Q is λ then the apparatus produces with probability $\mu_\Delta(\lambda)$ a reading in Δ. As a result, the probability of an outcome in Δ is given by $\langle\psi, F^Q(\Delta)\psi\rangle$ so that F^Q represents an unsharp measurement of E^Q.

An interpretation of equation (2) in the framework of fuzzy sets theory [34, 35] has been suggested in Ref. [17]. A fuzzy set is a pair $A = (\mathbb{R}, \mu_A)$ where $\mu_A : \mathbb{R} \to [0, 1]$ is a membership function. The value $\mu_A(x)$ is interpreted as the membership degree of x in A. A fuzzy event is a fuzzy set such that the membership function μ_A is Borel measurable. If ν is a probability measure on $\mathbb{R}$, the probability of a fuzzy event with respect to ν is defined as

$$P(A) = \int \mu_A(x)\, d\nu(x).$$

Going back to equation (3), we have the following interpretation in terms of fuzzy sets: the Markov kernel μ provides a family of fuzzy events $\{(\mathbb{R}, \mu_\Delta)\}_{\Delta\in\mathcal{B}(\mathbb{R})}$.

[1]Which can be thought to be intrinsic to the quantum measurement process and then to be unavoidable

For every point $x \in \mathbb{R}$, the family of membership functions $\{\mu_\Delta\}_{\Delta\in\mathcal{B}(\mathbb{R})}$ defines a probability measure. That is a very peculiar situation with respect to the general definition of fuzzy events. For every $\psi \in \mathcal{H}$, the expression

$$\langle\psi, F^Q(\Delta)\psi\rangle = \int_{\mathbb{R}} \mu_\Delta(x)\, d\langle\psi, E^Q_x\psi\rangle$$

can then be interpreted as the probability of the fuzzy event $(\mathbb{R}, \mu_\Delta)$ with respect to the probability measure $\langle\psi, E^Q(\cdot)\psi\rangle$. Such a probability coincides with the probability of the event Δ (which is not a fuzzy event) with respect to the probability $\langle\psi, F^Q(\cdot)\psi\rangle$. In other words, the unsharp observable F^Q gives the probabilities of the fuzzy events $(\mathbb{R}, \mu_\Delta)$ with respect to the probability measures corresponding to E^Q.

The present paper focuses on the analysis of the fuzzification process that connects E and F. In particular, we prove that there is a universal Markov kernel μ^U that connects every commutative POVM F (and then every fuzzy observable) to its sharp version E^F (we just require the POVM to be defined on a Hausdorff, locally compact topological space whose topology is countably generated) and that every Markov kernel is functionally subordinated to the universal Markov kernel. That generalizes some previous results [4, 6] where the existence of a universal Markov kernel is proved in the case $X = \mathbb{R}$. Moreover, we provide a general procedure for the construction of both the sharp version and the universal Markov kernel μ^U which is based on a modified version of Jessen's transfering principle [22]. The transferring principle was introduced by Lebesgue [26], Riesz [30] and de la Vallée Poussin [33] in the n-dimensional case. Later Jessen generalized the transferring principle in order to define integration of functions with a countable number of variables [22].

Concerning the possible interpretations of the universal Markov kernel we have that, if we interpret a Markov kernel as a measure of the randomization due to the measurement imprecision, than the existence of the universal Markov kernel means that there is a randomization which includes all the others (see sections 2.4). On the other hand, if we interpret a Markov kernel as a family of fuzzy events, then the existence of the universal Markov kernel μ^U means that every POVM F gives the probabilities of the fuzzy events, $\{(\mathbb{R}, \mu^U_\Delta)\}_{\Delta\in\mathcal{B}(X)}$, with respect to E^F and that $\{(\mathbb{R}, \mu^U_\Delta)\}_{\Delta\in\mathcal{B}(X)}$ can be interpreted as a universal family of fuzzy events which includes (up to functional subordination) all the possible fuzzy events of which a POVM can represent the probabilities (see section 3).

Now, we briefly recall the main definitions and properties of POVMs. In what follows, $\mathcal{B}(X)$ denotes the Borel σ-algebra of a topological space X and $\mathcal{L}_s^+(\mathcal{H})$ the space of all bounded positive self-adjoint linear operators acting in a Hilbert space $\mathcal{H}$ with scalar product $\langle\cdot,\cdot\rangle$.

Definition 1. A Positive Operator Valued measure (for short, POVM) is a map $F:\mathcal{B}(X)\to\mathcal{L}_s^+(\mathcal{H})$ such that:

$$F\Big(\bigcup_{n=1}^{\infty}\Delta_n\Big)=\sum_{n=1}^{\infty}F(\Delta_n).$$

where, $\{\Delta_n\}$ is a countable family of disjoint sets in $\mathcal{B}(X)$ and the series converges in the weak operator topology. It is said to be normalized if

$$F(X)=\mathbf{1}$$

where $\mathbf{1}$ is the identity operator.

Definition 2. A POVM is said to be commutative if

$$\big[F(\Delta_1),F(\Delta_2)\big]=\mathbf{0},\quad \forall\,\Delta_1\,,\Delta_2\in\mathcal{B}(X). \tag{4}$$

Definition 3. A POVM is said to be orthogonal if $\Delta_1\cap\Delta_2=\emptyset$ implies

$$F(\Delta_1)F(\Delta_2)=\mathbf{0} \tag{5}$$

where $\mathbf{0}$ is the null operator. An orthogonal POVM is called projection valued measure (for short, PVM).

Note that if F is an orthogonal POVM, the operators $F(\Delta)$ are projection operators.

Definition 4. A Spectral measure is a real, normalized PVM.

In quantum mechanics, non-orthogonal normalized POVMs represent **generalised** or **unsharp** or **fuzzy** observables while PVMs represent **standard** or **sharp** observables.

We recall that $\langle \psi, F(\Delta)\psi \rangle$ is interpreted as the probability that a measurement of the observable represented by F gives a result in Δ.

The following theorem gives a characterization of commutative POVMs as the randomization of spectral measures with the randomization realized by means of Markov kernels.

Definition 5. Let Λ be a topological space. A Markov kernel is a map $\mu : \Lambda \times \mathcal{B}(X) \to [0,1]$ such that,

1. $\mu_\Delta(\cdot)$ is a measurable function for each $\Delta \in \mathcal{B}(X)$,
2. $\mu_{(\cdot)}(\lambda)$ is a probability measure for each $\lambda \in \Lambda$.

Definition 6. Let ν be a measure on Λ. A map $\mu : (\Lambda, \nu) \times \mathcal{B}(X) \to [0,1]$ is a weak Markov kernel with respect to the measure ν if:

1. $\mu_\Delta(\cdot)$ is a measurable function for each $\Delta \in \mathcal{B}(X)$,
2. for every $\Delta \in \mathcal{B}(X)$, $0 \leq \mu_\Delta(\lambda) \leq 1, \quad \nu - a.e.$,
3. $\mu_\emptyset(\lambda) = 0, \mu_\Lambda(\lambda) = I \quad \nu - a.e.$,
4. for any sequence $\{\Delta_i\}_{i\in\mathbb{N}}$, $\Delta_i \cap \Delta_j = \emptyset$,

$$\sum_i \mu_{(\Delta_i)}(\lambda) = \mu_{(\cup_i \Delta_i)}(\lambda), \quad \nu - a.e.$$

In the following the symbol $\mathcal{A}(F)$ denotes the von Neumann algebra generated by the POVM F, i.e., the von Neumann algebra generated by the set $\{F(\Delta)\}_{\mathcal{B}(X)}$. Analogously $\mathcal{A}(B)$ denotes the von Neumann algebra generated by the self-adjoint operator B. Hereafter, we assume that X is a Hausdorff, locally compact, second countable topological space. The connection between commutative POVMs and randomization of spectral measures has been pointed out by several authors [7, 8, 10, 14, 16, 20, 21]. One of the possible formulation of their equivalence is provided in the following theorem.

Theorem 1.1 ([8, 10]). *A POVM $F : \mathcal{B}(X) \to \mathcal{L}_s^+(\mathcal{H})$ is commutative if and only if there exists a bounded self-adjoint operator $A = \int \lambda \, dE_\lambda$ with spectrum $\sigma(A) \subset [0, 1]$, and a Markov Kernel $\mu : \sigma(A) \times \mathcal{B}(X) \to [0, 1]$ such that*

1) $F(\Delta) = \int_{\sigma(A)} \mu_\Delta(\lambda) \, dE_\lambda, \quad \Delta \in \mathcal{B}(X)$.

2) $\mathcal{A}(F) = \mathcal{A}(A)$.

3) *there are a ring $\mathcal{R}(\mathcal{S}) \subset \mathcal{B}(X)$ and a set $\Gamma \subset \sigma(A)$, $E(\Gamma) = \mathbf{1}$, such that $\mu_\Delta(\lambda)$ is continuous for every $\Delta \in \mathcal{R}(\mathcal{S})$ and $\lambda \in \Gamma$. In particular, $\mu : \Gamma \times \mathcal{B}(X) \to [0, 1]$ is a Feller Markov kernel.*

The operator A introduced in theorem 1.1 is called the **sharp version** of F and is unique up to almost everywhere bijections [7, 8]. The POVM F is said to be a smearing of the spectral measure E or equivalently a smearing of A. It can be interpreted as a fuzzy version of E [10].
Other equivalent versions of these result exist [3, 5, 18, 21]. In [21] and [5] their equivalence has been proved.

2. Universal Markov kernel

In the present section we prove the existence of a universal Markov kernel for commutative POVMs (see subsection 2.4). That generalizes the result in [4, 6] where the existence of a universal Markov kernel has been proved for real commutative POVMs.

The proof is based on a modified version of the algorithm developed by Lebesgue, Riesz, Jessen and Sz-Nagy [22, 26, 28, 31] that they used in order to prove the transferring principle. In particular it was used by Jessen in order to introduce integration in infinite dimensional spaces. Later, the algorithm was used by Sz-Nagy in order to prove a theorem previously proved by von

Neumann, i.e., that to an arbitrary family $\{A_i\}_{i\in I}$ of bounded commuting self-adjoint operators there corresponds a self-adjoint operator A and a family of measurable functions $\{f_i\}_{i\in I}$ such that $A_i = f_i(A)$, for all $i \in I$ (see section 130 in Ref. [30]). A similar algorithm has been used in Ref. [14] in order to show that every real commutative POVM is the smearing of a spectral measure (see also [7]).

In particular, Jessen [22] proposed a quite general procedure to define a bijective function from a subset of the infinite dimensional torus to a subset of the unit interval (transferring principle). Here we modify Jessen's algorithm in order to obtain a bijective function $f : [0,1]^\omega \to [0,1]$ from the infinite dimentional cube to the unit interval and make use of such a function in order to define the sharp version of a commutative POVM $F : \mathcal{B}(X) \to \mathcal{L}_s(\mathcal{H})$ where X is a second countable topological space X. That generalizes some previous results, uses explicitly and in its full generality the transferring principle and provides a more powerful and compact formulation based on the bijective function $f : [0,1]^\omega \to [0,1]$. Then, we proceed to prove the existence of the universal Markov kernel on a Hausdorff, locally compact, second countable space.

2.1. Transferring Principle

We diverge from Jessen's construction in order to obtain a one-to-one function between $[0,1]^\omega$ and $[0,1]$. In particular, we use left-closed subintervals and provide two related nets of dissections of $[0,1]^\omega$ and $[0,1]$ in the following recursive way.

Set $D_1 := \{[0,1]^\omega\}$ and $d_1 := \{[0,1]\}$. At step $n = 2$, set $D_2 := \{I_1^{(2)}, I_2^{(2)}\}$ and $d_2 := \{i_1^{(2)}, i_2^{(2)}\}$ where $I_1^{(2)} = [0, \frac{1}{2}) \times [0,1] \times [0,1]^\omega$, $I_2^{(2)} = [\frac{1}{2}, 1] \times [0,1] \times [0,1]^\omega$, $i_1^{(2)} = [0, \frac{1}{2})$, $I_2^{(2)} = [\frac{1}{2}, 1]$. At step n, $D_n = \{I_1^{(n)}, \dots, I_{k_n}^{(n)}\}$ and $d_n = \{i_1^{(n)}, \dots, i_{k_n}^{(n)}\}$, where $I_l^{(n)} = b_{l_1}^1 \times b_{l_2}^2 \times \cdots \times b_{l_n}^n \times [0,1]^\omega$ and:

1. each set $b_{l_j}^j$ is a left-closed interval with the exception of those sets with right endpoint equal to 1;
2. $b_{l_{n-1}}^{n-1} \neq [0,1]$;

3. $b^n_{l_n} = [0,1]$.

At step $n+1$ the dissections are defined as follows. The dissection D_{n+1} is obtained by subdividing each $b^j_{l_j}$ into 2 subintervals of the same size and the dissection d_{n+1} is obtained by subdividing each interval $i^{(n)}_l$ into 2^n subintervals of the same size. It follows that, for each $n \in \mathbb{N}$, $k_n = 2^{\frac{n(n-1)}{2}}$ and the intervals $I^{(n)}_l$ and $i^{(n)}_l$ have the same Lebesgue measure. Moreover there is a one-to-one correspondence between the intervals $\{I^{(n)}_1, \ldots, I^{(n)}_{k_n}\}$ of D_n and the intervals $\{i^{(n)}_1, \ldots, i^{(n)}_{k_n}\}$ of d_n such that, up to permutations of the elements, $I^{(n)}_j$ corresponds to $i^{(n)}_j$, the measure of corresponding intervals are the same and $I^{(n+1)}_j \subseteq I^{(n)}_l$ if and only if $i^{(n+1)}_j \subseteq i^{(n)}_l$. Moreover, we add the following **constraint** to the iterative procedure. Let $I^{(n)}_k = b^1_{k_1} \times b^2_{k_2} \times \cdots \times b^n_{k_n} \times [0,1]^\omega$. Let $v^{(n)}_k$ be the coordinate of the vertex of $b^1_{k_1} \times b^2_{k_2} \times \cdots \times b^n_{k_n}$ with the greatest distance from the origin. Let $a^{(n)}_k$ be the right extreme of $i^{(n)}_k$. At step $n+1$, the interval $I^{(n+1)}_l \subset I^{(n)}_k$ such that $v^{(n+1)}_l = \min_j\{v^{(n+1)}_j\}$ will correspond to the interval $i^{(n+1)}_r \subset i^{(n)}_k$ such that $a^{(n+1)}_r = \min_j\{a^{(n+1)}_j\}$.

Note that the two sequences of dissections have the following useful property. Let $\{I_n\}$ be a sequence of subintervals of $[0,1]^\omega$ such that $I_n \in D_n$ and $I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$. Then, the cardinality of their intersection is

$$\left|\bigcap_n I_n\right| \leq 1.$$

Nevertheless, for every point $x \in [0,1]^\omega$, there is exactly one sequence of nested intervals $\{I_n\}$, $I_n \in D_n$, whose intersection is $\{x\}$. Analogously, every point in $[0,1]$ is the intersection of a unique sequence of nested intervals, $\{i_n\}$, $i_n \in d_n$. Now, let $\{I_n\}$, $I_n \in D_n$, be the descending chain of intervals corresponding to the point $x \in [0,1]^\omega$. Let $\{i_n\}$, $i_n \in d_n$ be the corresponding chain of descending intervals contained in $[0,1]$. The constraint we introduced above ensures the existence[2] of a point $y_x \in [0,1]$ such that $y_x \in i_n$, for all $n \in \mathbb{N}$. At last let us

[2] That is always true for each point $x \in [0,1]^\omega$ which is not in the set S^ω where

$$S := \left\{ \frac{k}{2^n} : n \in \mathbb{N}, k \in \mathbb{Z}, 0 \leq k \leq 2^n \right\}.$$

notice that $x \in I_j^{(n)}$ if and only if $y_x \in i_j^{(n)}$. We then have the following theorem which slightly modify the transferring principle introduced by Jessen (see [22] for more details).

Theorem 2.1 (Transferring Principle)**.** *There exists a one-to-one application*

$$f : [0,1]^\omega \to [0,1],$$

with the property that corresponding sets have always the same Lebesgue measure. Moreover, $f^{-1}(i_l^{(n)}) = I_l^{(n)}$.

At variance with Jessen's construction we have used half-open intervals and added more constraints in the definition of the dissections. That ensured the bijectivity of the function $f : [0,1]^\omega \to [0,1]$ that in Jessen's version is bijectvie on a subset of $[0,1]^\omega$.

2.2. The Sharp Version

Now, we can use the transferring principle in order to define the spectral measure (sharp version) associated to a commutative POVM. Let X be a second countable space and $\mathcal{S}$ a countable basis for the topology of X (we suppose $\emptyset, X \in \mathcal{S}$). Let $\mathcal{R}(\mathcal{S})$ be the ring generated by $\mathcal{S}$ and $\mathcal{B}(X)$ the Borel σ-algebra generated by $\mathcal{R}(\mathcal{S})$.

Let $\{\Delta_n\}_{n\in\mathbb{N}}$ be an enumeration of $\mathcal{R}(\mathcal{S})$. Let F be a commutative POVM on $\mathcal{B}(X)$ and E_n the spectral resolution of $F(\Delta_n)$. Let $f_j^{-1}(x)$ be the j-th coordinate of $f^{-1}(x)$, $x \in [0,1]$. In particular $f_j^{-1}(i_l^{(n)}) = b_{l_j}^j$ where $b_{l_j}^j$ is the j-th edge of the subinterval $I_l^{(n)} = b_{l_1}^1 \times b_{l_2}^2 \times \cdots \times b_{l_j}^j \times \cdots$.

We can define a projection valued map on the set of subintervals of $[0,1]$ as follows. Note that $\mathcal{I} := \cup_{n=1}^\infty d_n$ is a semiring. Then, thanks to the commutativity of the POVM F,

The constraint ensures that it holds for points in the set S^ω as well.

$$E^F : \mathcal{I} \to \mathcal{L}_s^+(\mathcal{H})$$
$$E^F(i_l^{(n)}) := \prod_{j=1}^{n} E_j[f_j^{-1}(i_l^{(n)})] \tag{6}$$

defines a projection valued map. It is straightforward to show that $E^F(i_1^{(n)})E^F(i_2^{(n)}) = \mathbf{0}$ if $l \neq j$ and that $E^F([0,1]) = \mathbf{1}$ (concerning this last property, note that $f_j^{-1}([0,1]) = [0,1]$).

Moreover, E^F is additive on $\mathcal{I}$. It is sufficient to show the additivity in the case $i_l^{(n)} = \sqcup_r i_{r_l}^{(k)}$, with $k > n$. Indeed, the general case where $i_l^{(n)}$ is the union of sets from different dissections D_k, i.e., $i_l^{(n)} = \sqcup_k \sqcup_r i_{r_l^k}^{(k)}$, can be reduced to this case as follows. Let $a = max\{k\}$. Then, each set $i_{r_l^k}^{(k)}$ can be decomposed as the disjoint union of sets from D_a, i.e., $i_{r_l^k}^{(k)} = \sqcup_j i_{j_{r_l^k}}^{(a)}$, and $i_l^{(n)} = \sqcup_k \sqcup_r \sqcup_j i_{j_{r_l^k}}^{(a)} = \sqcup_p i_{p_l}^{(a)}$. Then, supposing $E^F(i_l^{(n)}) = \sum_{k,r,j} E^F(i_{j_{r_l^k}}^{(a)})$ and $E^F(i_{r_l^k}^{(k)}) = \sum_j E^F(i_{j_{r_l^k}}^{(a)})$, we obtain $E^F(\sqcup_k \sqcup_r i_{r_l^k}^{(k)}) = E^F(i_l^{(n)}) = \sum_{k,r,j} E^F(i_{j_{r_l^k}}^{(a)}) = \sum_{k,r} E^F(i_{r_l^k}^{(k)})$.

We use the following notation. Let

$$I_l^{(n)} = b_{l_1}^1 \times b_{l_2}^2 \times \dots b_{l_n}^n \times [0,1]^\omega \in D_n$$
$$I_l^{(n)} = \cup_{r=1}^p I_{r_l}^{(k)}, \quad p = 2^{\frac{k(k-1)-n(n-1)}{2}}$$
$$I_{r_l}^{(k)} = c_{r_{l\,1}}^1 \times \dots c_{r_{l\,k}}^k \times [0,1]^\omega \in D_k, \quad k > n.$$

Note that, $\cup_r c_{r_{l\,s}}^s = b_{l_s}^s$ for every $s \leq n$ while $\cup_r c_{r_{l\,s}}^s = [0,1]$, for every $s > n$. Let $B_r^m := \{s \,|\, c_{s_{l\,i}}^i = c_{r_{l\,i}}^i, \forall i \neq m\}$ and $|B_r^m|$ the cardinality of B_r^m. Then,

$$
\begin{aligned}
\sum_{r=1}^{p} E^F(i_{r_l}^{(k)}) &= \sum_{r=1}^{p}\prod_{j=1}^{k} E_j(f_j^{-1}(i_{r_l}^{(k)})) = \sum_{r=1}^{p}\prod_{j=1}^{k} E_j(c^j_{r_{l_j}}) \\
&= \sum_{r=1}^{p}\prod_{j=1}^{k-1} E_j(c^j_{r_{l_j}})E_k[0,1] = \sum_{r=1}^{p}\prod_{j=1}^{k-1} E_j(c^j_{r_{l_j}}) \\
&= \sum_{r=1}^{p}\frac{1}{|B_r^{k-1}|}\sum_{s\in B_r^{k-1}}\prod_{j=1}^{k-1} E_j(c^j_{s_{l_j}}) = \sum_{r=1}^{\frac{p}{2}}\prod_{j=1}^{k-2} E_j(c^j_{r_{l_j}})\sum_{i=1}^{2} E_{k-1}(c^{k-1}_{i_{l_{k-1}}}) \\
&= \sum_{r=1}^{\frac{p}{2}}\prod_{j=1}^{k-2} E_j(c^j_{r_{l_j}})E_{k-1}([0,1]) \\
&= \sum_{r=1}^{\frac{p}{2}}\frac{1}{|B_r^{k-2}|}\sum_{s\in B_r^{k-2}}\prod_{j=1}^{k-2} E_j(c^j_{s_{l_j}}) = \sum_{r=1}^{\frac{p}{8}}\prod_{j=1}^{k-3} E_j(c^j_{r_{l_j}})\sum_{i=1}^{8} E_{k-2}(c^{k-2}_{i_{l_{k-2}}}) \\
&= \sum_{r=1}^{\frac{p}{8}}\prod_{j=1}^{k-3} E_j(c^j_{r_{l_j}}) \\
&= \dots \\
&= \sum_{r=1}^{q}\prod_{j=1}^{n} E_j(c^j_{r_{l_j}}), \qquad q = \frac{p}{2^{\frac{(k-n)(k-n+1)}{2}}} \\
&= \sum_{r=1}^{q}\frac{1}{|B_r^{n}|}\sum_{s\in B_r^{n}}\prod_{j=1}^{n} E_j(c^j_{s_{l_j}}) = \sum_{r=1}^{\frac{q}{2^{(k-n)}}}\prod_{j=1}^{n-1} E_j(c^j_{r_{l_j}})\sum_{i=1}^{2^{k-n}} E_n(c^n_{i_{l_n}}) \\
&= \sum_{r=1}^{\frac{q}{2^{k-n}}}\prod_{j=1}^{n-1} E_j(c^j_{r_{l_j}})E_n(b^n_{l_n}) \\
&= \sum_{r=1}^{\frac{q}{2^{k-n}}}\prod_{j=1}^{n-1} E_j(c^j_{r_{l_j}}) \\
&= \sum_{r=1}^{\frac{q}{2^{k-n}}}\frac{1}{|B_r^{n-1}|}\sum_{s\in B_r^{n-1}}\prod_{j=1}^{n-1} E_j(c^j_{s_{l_j}}) = \sum_{r=1}^{\frac{q}{2^{k-n}}}\prod_{j=1}^{n-2} E_j(c^j_{r_{l_j}})\sum_{i=1}^{2^{k-n}} E_{n-1}(c^{n-1}_{i_{l_{n-1}}}) \\
&= \sum_{r=1}^{\frac{q}{2^{k-n}}}\prod_{j=1}^{n-2} E_j(c^j_{r_{l_j}})E_{n-1}(b^{n-1}_{l_{n-1}}) \\
&= \sum_{r=1}^{\frac{q}{2^{k-n}}}\prod_{j=1}^{n-2} E_j(c^j_{r_{l_j}})E_{n-1}(b^{n-1}_{l_{n-1}}) = \sum_{r=1}^{\frac{q}{2^{k-n}}}\frac{1}{|B_r^{n-2}|}\sum_{s\in B_r^{n-2}}\prod_{j=1}^{n-2} E_j(c^j_{s_{l_j}})E_{n-1}(b^{n-1}_{l_{n-1}}) \\
&= \sum_{r=1}^{\frac{q}{2^{k-n}}}\prod_{j=1}^{n-3} E_j(c^j_{r_{l_j}})\sum_{i=1}^{2^{k-n}} E_{n-2}(c^{n-1}_{i_{l_{n-1}}})E_{n-1}(b^{n-1}_{l_{n-1}}) \\
&= \sum_{r=1}^{\frac{q}{2^{k-n}}}\prod_{j=1}^{n-3} E_j(c^j_{r_{l_j}})E_{n-2}(b^{n-2}_{l_{n-2}})E_{n-1}(b^{n-1}_{l_{n-1}}) \\
&= \dots \\
&= E_1(b^1_{l_1})\cdots E_{n-2}(b^{n-2}_{l_{n-2}})E_{n-1}(b^{n-1}_{l_{n-1}})
\end{aligned}
$$

The map E^F can be additively extended to the ring $\mathcal{R}(\mathcal{I})$ generated by $\mathcal{I}$. Moreover E^F is σ-additive on $\mathcal{R}(\mathcal{I})$. In order to prove the σ-additivity it is sufficient to prove that for every sequence $B_n \in \mathcal{R}(\mathcal{I})$ such that $B_n \downarrow \emptyset$,

$$\lim_{n\to\infty} E^F(B_n) = \mathbf{0}.$$

Every set $B \in \mathcal{R}(\mathcal{I})$ is the finite union of sets in $\mathcal{I}$. In particular (see lemma 2, page 33, in [23]), it can be decomposed as the disjoint union of sets in $\mathcal{I}$, $B = \sqcup_k i_k^{n_k}$, $i_k^{n_k} \cap i_j^{n_j} = \emptyset$, $k \neq j$. We can say more. Let $p = \max_k\{n_k\}$. Then, each $i_k^{n_k}$ can be decomposed as $i_k^{n_k} = \sqcup_r i_{r_k}^p$, $i_{r_k}^p \cap i_{r_j}^p = \emptyset$ so that $\Delta = \sqcup_l i_l^p$ with l in a subset of $\{1, \cdots, 2^{\binom{p}{2}}\}$. Now, let $\{B_n\}_{n\in\mathbb{N}} \subset \mathcal{R}(\mathcal{I})$ be such that $B_n \downarrow \emptyset$. For each $n \in \mathbb{N}$, $B_n = \sqcup_{l\in N_n} i_l^{p_n}$, $N_n \subset \{1, \cdots, 2^{\binom{p_n}{2}}\}$, $i_l^{p_n} \cap i_j^{p_n} = \emptyset$, $l \neq j$. For every k, let $A_k^n := \left\{l \in N_n \,|\, f_1^{-1}(i_l^{p_n}) = [\frac{k-1}{2^{p_n-1}}, \frac{k}{2^{p_n-1}})\right\}$ and l_k an arbitrary element of A_k. We have $\cup_k A_k^n = N_n$ and,

$$\begin{aligned}
\lim_{n\to\infty} E^F(B_n) &= \lim_{n\to\infty} E^F(\sqcup_{l\in N_n} i_l^{p_n}) = \lim_{n\to\infty} \sum_{l\in N_n} E^F(i_l^{p_n}) \\
&= \lim_{n\to\infty} \sum_{l\in N_n} \Pi_{j=1}^{p_n} E_j(f_j^{-1}(i_l^{p_n})) \\
&= \lim_{n\to\infty} \sum_{l\in N_n} E_1(f_1^{-1}(i_l^{p_n}))\Pi_{j=2}^{p_n} E_j(f_j^{-1}(i_l^{p_n})) \\
&= \lim_{n\to\infty} \sum_k E_1(f_1^{-1}(i_{l_k}^{p_n})) \sum_{l\in A_k^n} \Pi_{j=2}^{p_n} E_j(f_j^{-1}(i_l^{p_n})) \\
&\leq \lim_{n\to\infty} \sum_k E_1(f_1^{-1}(i_{l_k}^{p_n})) \\
&= \lim_{n\to\infty} E_1[f_1^{-1}(\sqcup_k i_{l_k}^{p_n})] \leq \lim_{n\to\infty} E_1[f_1^{-1}(B_n)] \\
&= \lim_{n\to\infty} E_1[f_1^{-1}(\cap_{i=1}^n B_i)] = \lim_{n\to\infty} E_1[\cap_{i=1}^n f_1^{-1}(B_i)] \\
&= E_1[\cap_{i=1}^\infty f_1^{-1}(B_i)] = E_1[f_1^{-1}(\cap_{i=1}^\infty B_i)] = E_1(\emptyset) = \mathbf{0}.
\end{aligned}$$

where, we have used the continuity of E_1, the identity $f_1^{-1}(\cap_{i=1}^\infty B_i) = \cap_{i=1}^\infty f_1^{-1}(B_i)$ and the fact that $f_1^{-1}(i_{l_k}^{p_n}) \cap f_1^{-1}(i_{l_m}^{p_n}) = \emptyset$, $k \neq m$.

Since $\mathcal{R}(\mathcal{I})$ generates the Borel σ-algebra $\mathcal{B}([0,1])$, the map E^F can be extended uniquely to a spectral measure on $\mathcal{B}([0,1])$ (see theorem 7 in Ref. [11]).

By (6), for every $B \in \mathcal{R}(\mathcal{I})$, $E^F(B) \in \mathcal{A}^W(F)$ where $\mathcal{A}^W(F)$ denotes the von Neumann algebra generated by $\{F(\Delta)\}_{\Delta\in\mathcal{B}(X)}$. Actually we can prove the following lemma.

Lemma 2.1. $\mathcal{A}^W(E^F) \subset \mathcal{A}^W(F)$.

Proof. We have, $\mathcal{A}^W\big(\{E^F(\Delta)\}_{\Delta\in\mathcal{R}(\mathcal{I})}\big) \subset \mathcal{A}^W(F)$. Moreover, $\mathcal{I}$ is a basis for the topology of $[0,1]$ and by proposition 3.1 in the appendix, $\mathcal{A}^W\big(\{E^F(\Delta)\}_{\mathcal{R}(\mathcal{I})}\big) = \mathcal{A}^W(E^F)$. □

2.3. The kernel

In order to define the universal Markov kernel we first define a family of functions ω_l whose integral with respect to E^F gives $F(\Delta_l)$, $\Delta_l \in \mathcal{R}(\mathcal{S})$. Let $\nu_l(i_j^{(n)})) = \sup f_l^{-1}(i_j^{(n)})$ and $\chi_{i_l^{(n)}}(\lambda)$ the characteristic function of the interval $i_l^{(n)}$. The sequence of non-increasing functions

$$\omega_{\Delta_l}^{(n)}(\lambda) := \sum_{j=1}^{2^{\binom{n}{2}}} \upsilon_l(i_j^{(n)})\,\chi_{i_j^{(n)}}(\lambda) \geq 0$$

converges uniformly to a function $\omega_{\Delta_l}^U(\lambda) := \lim_{n\to\infty}\omega_{\Delta_l}^{(n)}(\lambda)$. Let $B_{k,l}^n := \{j \,|\, \nu_l(i_j^{(n)}) = \frac{k}{2^{n-l}}\}$. We have,

$$\begin{aligned}
\int_0^1 \omega_{\Delta_l}^U(\lambda)\,dE_\lambda^F &= \lim_{n\to\infty}\int_0^1 \omega_{\Delta_l}^{(n)}(\lambda)\,dE_\lambda^F = \lim_{n\to\infty}\sum_{j=1}^{2^{\binom{n}{2}}} \nu_l((i_j^{(n)})E^F(i_j^{(n)}) = \\
&= \lim_{n\to\infty}\sum_{k=1}^{2^{n-l}} \frac{k}{2^{n-l}} \sum_{j\in B_{k,l}^n}\prod_r E_r(f_r^{-1}(i_j^{(n)})) = \\
&= \lim_{n\to\infty}\sum_{k=1}^{2^{n-l}} \frac{k}{2^{n-l}}\, E_l\left(\left[\frac{k-1}{2^{n-l}},\frac{k}{2^{n-l}}\right)\right) \sum_{j\in B_{k,l}^n}\prod_{r\neq l} E_r(f_r^{-1}(i_j^{(n)})) \\
&= \lim_{n\to\infty}\sum_{k=1}^{2^{n-l}} \frac{k}{2^{n-l}}\, E_l\left(\left[\frac{k-1}{2^{n-l}},\frac{k}{2^{n-l}}\right)\right) \\
&= \int_0^1 \lambda\, E_l(d\lambda) = F(\Delta_l), \quad \Delta_l \in \mathcal{R}(\mathcal{S})
\end{aligned} \tag{8}$$

where we have used the identity $\sum_{j\in B^n_{k,l}}\prod_{r\neq l} E_r(f_r^{-1}(i_j^{(n)})) = \mathbf{1}$ which can be derived by noting that the set $B^n_{k,l}$ includes all the indexes j such that the l-th edge of $f^{-1}(i_j^n)$ is fixed to be $[\frac{k-1}{2^{n-l}}, \frac{k}{2^{n-l}})$. Then, the other edges of $\{f^{-1}(i_j^n)\}_{j\in B^n_{k,l}}$ are arbitrary. Hence,

$$\Big\{\prod_{i\neq l} f_i^{-1}(i_j^{(n)})\Big\}_{j\in B^n_{k,l}} = \Big\{\prod_{i\neq l}\Big[\frac{k_i-1}{2^{n-i}}, \frac{k_i}{2^{n-i}},\Big)\Big\}_{k_i=1,\ldots,2^{n-i}}.$$

and (supposing $l\neq n-1$)

$$\begin{aligned}\Big\{\prod_{i\neq l}\Big[\frac{k_i-1}{2^{n-i}}, \frac{k_i}{2^{n-i}}\Big)\Big\}_{k_i=1,\ldots,2^{n-i}} &= \Big\{\prod_{i\neq l,n-1}\Big[\frac{k_i-1}{2^{n-i}}, \frac{k_i}{2^{n-i}}\Big)\Big\}_{k_i=1,\ldots,2^{n-i}} \times \Big[0,\frac{1}{2}\Big)\\ &\bigcup\Big\{\prod_{i\neq l,n-1}\Big[\frac{k_i-1}{2^{n-i}}, \frac{k_i}{2^{n-i}}\Big)\Big\}_{k_i=1,\ldots,2^{n-i}} \times \Big[\frac{1}{2},1\Big]\\ &= \Big\{\prod_{i\neq l,n-1}\Big[\frac{k_i-1}{2^{n-i}}, \frac{k_i}{2^{n-i}}\Big)\Big\}_{k_i=1,\ldots,2^{n-i}} \times [0,1]\end{aligned}$$

so that

$$\sum_{j\in B^n_{k,l}}\prod_{r\neq l} E_r(f_r^{-1}(i_j^{(n)})) = \sum_{j\in B^n_{k,l}}\prod_{r\neq l,n-1} E_r(f_r^{-1}(i_j^{(n)})).$$

In the case $l = n-1$,

$$\begin{aligned}\Big\{\prod_{i\neq l}\Big[\frac{k_i-1}{2^{n-i}}, \frac{k_i}{2^{n-i}}\Big)\Big\}_{k_i=1,\ldots,2^{n-i}} &= \Big\{\prod_{i\neq l,n-2}\Big[\frac{k_i-1}{2^{n-i}}, \frac{k_i}{2^{n-i}}\Big)\Big\}_{k_i=1,\ldots,2^{n-i}} \times \Big[0,\frac{1}{4}\Big)\\ &\bigcup\Big\{\prod_{i\neq l,n-2}\Big[\frac{k_i-1}{2^{n-i}}, \frac{k_i}{2^{n-i}}\Big)\Big\}_{k_i=1,\ldots,2^{n-i}} \times \Big[\frac{1}{4},\frac{1}{2}\Big)\\ &\bigcup\Big\{\prod_{i\neq l,n-2}\Big[\frac{k_i-1}{2^{n-i}}, \frac{k_i}{2^{n-i}}\Big)\Big\}_{k_i=1,\ldots,2^{n-i}} \times \Big[\frac{1}{2},\frac{3}{4}\Big)\\ &\bigcup\Big\{\prod_{i\neq l,n-2}\Big[\frac{k_i-1}{2^{n-i}}, \frac{k_i}{2^{n-i}}\Big)\Big\}_{k_i=1,\ldots,2^{n-i}} \times \Big[\frac{3}{4},1\Big]\\ &= \Big\{\prod_{i\neq l,n-2}\Big[\frac{k_i-1}{2^{n-i}}, \frac{k_i}{2^{n-i}}\Big)\Big\}_{k_i=1,\ldots,2^{n-i}} \times [0,1]\end{aligned}$$

so that

$$\sum_{j\in B^n_{k,l}}\prod_{r\neq l}E_r(f_r^{-1}(i_j^{(n)}))=\sum_{j\in B^n_{k,l}}\prod_{r\neq l,n-2}E_r(f_r^{-1}(i_j^{(n)})).$$

By iterating the procedure, one arrives at $\sum_{j\in B^n_{k,l}}\prod_{r\neq l}E_r(f_r^{-1}(i_j^{(n)}))=\mathbf{1}$. It is worth remarking that the functions $\omega^U_{\Delta_i}$ do not depend on the POVM F and this is at the root of the proof of the existence of a universal Markov kernel.

In the following, we use the symbol $\mathcal{D}(X)$ to denote the set of POVMs from the Borel σ-algebra $\mathcal{B}(X)$ to $\mathcal{L}_s^+(\mathcal{H})$. We then have a map $\omega^U:[0,1]\times\mathcal{R}(\mathcal{S})\to[0,1]$ such that, for every $F\in\mathcal{D}(X)$, $F(\Delta)=\int\omega^U_\Delta(\lambda)\,dE^F_\lambda$, $\Delta\in\mathcal{R}(\mathcal{S})$.

Lemma 2.2. $\mathcal{A}^W(E^F)=\mathcal{A}^W(F)$.

Proof. By lemma 2.1, $\mathcal{A}^W(E^F)\subset\mathcal{A}^W(F)$. By equation (8) $\mathcal{A}^W(\{F(\Delta)\}_{\Delta\in\mathcal{R}(\mathcal{S})})\subset\mathcal{A}^W(E^F)$. By proposition 3.1 in the appendix $\mathcal{A}^W(\{F(\Delta)\}_{\Delta\in\mathcal{R}(\mathcal{S})})=\mathcal{A}^W(F)$. □

We have proved the following proposition.

Proposition 2.2. *There is a map $\omega^U:[0,1]\times\mathcal{R}(\mathcal{S})\to[0,1]$ with the following property. For every $F\in\mathcal{D}(X)$, there is a spectral measure E^F with spectrum in $[0,1]$ which generates $\mathcal{A}^W(F)$ and is such that*

$$F(\Delta)=\int\omega^U_\Delta(\lambda)\,dE^F_\lambda,\quad\Delta\in\mathcal{R}(\mathcal{S}).$$

2.4. The Universal Markov Kernel

We are now ready to prove the existence of a universal Markov kernel.

Theorem 2.3. *There are a subset $I\subset[0,1]$ and a Markov kernel $\mu^U:\mathcal{B}(X)\times I\to[0,1]$ such that, for every $F\in\mathcal{D}(X)$, $E^F(I)=\mathbf{1}$ and*

$$F(\Delta)=\int\mu^U_\Delta(\lambda)\,dE^F_\lambda$$

where E^F is the spectral measure whose existence has been proved in subsection 2.2

Proof. Let $F \in \mathcal{D}(X)$. By theorem 1.1 there are a self-adjoint operator B with spectrum in $[0,1]$ and a Markov kernel μ such that B generates $\mathcal{A}^W(F)$ and $\int \mu_\Delta(\lambda)\, dE^B_\lambda = F(\Delta)$ for every $\Delta \in \mathcal{B}(X)$. Let $\nu(\cdot) = \langle \psi_0, E^B(\cdot)\psi_0 \rangle$ where E^B is the spectral measure corresponding to B and ψ_0 is a separating vector for $\mathcal{A}^W(F)$.

By lemma 2.2 there is a spectral measure E^F which generates $\mathcal{A}^W(F)$. Let A^F be the corresponding self-adjoint operator and $\sigma(A^F) \subset [0,1]$ its spectrum. We have

$$\int_{\sigma(A^F)} \omega^U_{\Delta_i}(\lambda)\, dE^F_\lambda = F(\Delta_i), \quad i \in \mathbb{N}.$$

Since B and A^F generate $\mathcal{A}^W(F)$, there are two Borel functions $f_F : \sigma(B) \to \sigma(A^F)$ and $h_F : \sigma(A^F) \to \sigma(B)$ such that $E^F(G) = E^B(f_F^{-1}(G))$, $G \in \mathcal{B}([0,1])$ and $E^B(D) = E^F(h_F^{-1}(D))$, $D \in \mathcal{B}([0,1])$. Then, there is a set N such that $E^B(N) = \mathbf{1}$ and $(h_F \circ f_F)(\lambda) = \lambda$ for every $\lambda \in N$. In other words $f_F : N \to f_F(N)$ is injective. By Souslin's theorem ([24], page 440-442) $f_F(N)$ is a Borel set and $E^F(f_F(N)) = E^B[f_F^{-1}(f_F(N))] = \mathbf{1}$.

Then, by the change of measure principle,

$$\int_{\sigma(B)} \omega^U_{\Delta_i}(f_F(\lambda))\, dE^B_\lambda = \int_{\sigma(A^F)} \omega^U_{\Delta_i}(\lambda)\, dE^F_\lambda = F(\Delta_i) = \int_{\sigma(B)} \mu_{\Delta_i}(\lambda)\, dE^B_\lambda$$

Therefore, $\omega^U_{\Delta_i}(f_F(\lambda)) = \mu_{\Delta_i}(\lambda)$, $E^B - a.e.$. Since, E^B and ν are mutually absolutely continuous,

$$\omega^U_{\Delta_i}(f_F(\lambda)) = \mu_{\Delta_i}(\lambda), \quad \nu - a.e. \tag{9}$$

Let $M^F \subset \sigma(B)$, $\nu(M^F) = 1$ be such that

$$\omega^U_{\Delta_i}(f_F(\lambda)) = \mu_{\Delta_i}(\lambda), \quad \lambda \in M^F, i \in \mathbb{N}. \tag{10}$$

Thus, thanks to the σ-additivity of μ, $(\omega^U_{(\cdot)} \circ f_F)(\lambda))$ is σ-additive on $\mathcal{R}(\mathcal{S})$ for every $\lambda \in N^F := M^F \cap N$. As a consequence, $\omega^U_{(\cdot)}(\lambda)$ is σ-additive on $\mathcal{R}(\mathcal{S})$ for every $\lambda \in f_F(N^F)$. Note that $E^F[f_F(N^F)] = E^B[f_F^{-1}(f_F(N^F))] = \mathbf{1}$.

By repeating the reasoning for every $F \in \mathcal{D}(X)$, one proves that the set $I := \cup_{F\in\mathcal{D}(X)} f_F(N^F)$ is such that $\omega^U_{(\cdot)}(\lambda)$ is σ-additive on $\mathcal{R}(\mathcal{S})$ for every $\lambda \in I$.

In what follows we need the set I to be measurable. Thus, if I is not a Borel set we enlarge the Borel σ-algebra in order to include I. In particular, we consider the σ-algebra $\mathfrak{S}$ generated by I and $\mathcal{B}([0,1])$.

Since, $\forall F \in \mathcal{D}(X)$, $[0,1] \setminus I \subset [0,1] \setminus f_F(N^F)$ and $E^F\big([0,1] \setminus f_F(N^F)\big) = \mathbf{0}$, the set $[0,1] \setminus I$ is a subset of a E^F-null set for any $F \in \mathcal{D}(X)$. Then, each PVM E^F can be extended to $\mathfrak{S}$. The extension $\widetilde{E}^F : \mathfrak{S} \to \mathcal{L}_s(\mathcal{H})$ satisfies the following relations:

$$\widetilde{E}^F(\sigma(A^F)) = \widetilde{E}^F(I) = \mathbf{1}$$

$$\widetilde{E}^F(I \cap \Delta) = E^F(\Delta), \quad \forall \Delta \in \mathcal{B}[0,1]$$

$$A^F = \int_{[0,1]} \lambda \, d\widetilde{E}^F_\lambda.$$

The space $([0,1], \mathfrak{S})$ is a measurable space and I is a measurable subset of $\mathfrak{S}$. Moreover, for each $\Delta \in \mathcal{R}(\mathcal{S})$, the function $\omega^U_\Delta : ([0,1], \mathfrak{S}) \to ([0,1], \mathcal{B}[0,1])$ is $\mathfrak{S}$-measurable and

$$\int_{[0,1]} \omega^U_\Delta(\lambda)\, d\widetilde{E}^F_\lambda = \int_{[0,1]} \omega^U_\Delta(\lambda)\, dE^F_\lambda = F(\Delta), \quad \forall F \in \mathcal{D}(X). \tag{11}$$

Now, for every $\lambda \in I$, the measure $\omega^U_{(\cdot)}(\lambda) : \mathcal{R}(\mathcal{S}) \to [0,1]$ can be extended to the Borel σ-algebra $\mathcal{B}(X)$. Let $\mu_{(\cdot)}(\lambda) : \mathcal{B}(X) \to [0,1]$ denotes such an extension. We want to show that, for each $\Delta \in \mathcal{B}(X)$, μ_Δ is $\mathfrak{S}$-measurable and $\int \mu^U_\Delta(\lambda)\, dE_\lambda = F(\Delta)$. That can be proved by using transfinite induction. Let Δ be an open set. Then, there is an increasing sequence of open sets $\Delta_{k_i} \in \mathcal{S}$ such that $\Delta_{k_i} \uparrow \Delta$. Then, for every λ, $\omega^U_{\Delta_{k_i}}(\lambda) = \mu^U_{\Delta_{k_i}}(\lambda) \uparrow \mu^U_\Delta(\lambda)$ so that μ^U_Δ is $\mathfrak{S}$-measurable. Let Δ be a G_δ set. Then there is a decreasing sequence of open sets G_i such that $G_i \downarrow \Delta$. Moreover, for every λ, $\mu^U_{G_i}(\lambda) \downarrow \mu^U_\Delta(\lambda)$ so that μ^U_Δ is $\mathfrak{S}$-measurable.

Let G_0 be the family of open subsets of X, ω_1 the first uncountable ordinal and G_α, $\alpha < \omega_1$ the Borel hierarchy [24]. In particular, $G_1 = G_\delta$, $G_2 = G_{\delta\sigma}$, $G_3 = G_{\delta\sigma\delta}, \dots$ and $G_\alpha = (\cup_{\beta<\alpha} G_\beta)_\sigma$ for each limit ordinal α. By means of the same reasoning that we used in the case of open and G_δ sets, one can prove the $\mathfrak{S}$-measurability of μ_Δ for every Δ of the kind $G_{\delta,\sigma}, G_{\delta\sigma\delta}\dots$. Analogously, if μ^U_Δ is $\mathfrak{S}$-measurable for each $\Delta \in G_\alpha$ then, μ^U_Δ is $\mathfrak{S}$-measurable for each Δ in $G_{\alpha+1}$. Indeed, each set in $G_{\alpha+1}$ is either the countable union or the countable intersection of sets in G_α and the previous reasoning can be used. If α is a limit ordinal and μ^U_Δ is $\mathfrak{S}$-measurable for each $\Delta \in G_\beta$, $\beta < \alpha$, then, μ^U_Δ is $\mathfrak{S}$-measurable for each $\Delta \in G_\alpha = (\cup_{\beta<\alpha} G_\beta)_\sigma$. Indeed, each set in G_α is the countable union of sets in $\cup_{\beta<\alpha} G_\beta$ and the previous reasoning can be used. Therefore, by transfinite induction, $\mu^U_\Delta(\cdot) : I \to [0,1]$ is $\mathfrak{S}$-measurable for each $\Delta \in \cup_{\alpha<\omega_1} G_\alpha = \mathcal{B}(X)$.

Moreover, since $\mu^U_\Delta = \omega^U_\Delta$, $\Delta \in \mathcal{R}(\mathcal{S})$, the POVM $F'(\Delta) = \mu^U_\Delta(A^F)$, $\Delta \in \mathcal{B}(X)$, coincides with $F(\Delta)$ for every $\Delta \in \mathcal{R}(\mathcal{S})$. Since $F : \mathcal{R}(\mathcal{S}) \to \mathcal{E}(\mathcal{H})$ has a unique extension [11] to the Borel σ-algebra $\mathcal{B}(X)$, it must be $F(\Delta) = F'(\Delta) = \mu^U_\Delta(A^F)$.

□

Definition 7. The Markov kernel $\mu^U : \mathcal{B}(X) \times I \to [0,1]$ whose existence has been proved in theorem 2.3 is called the universal Markov kernel.

2.5. Functional subordination

The following definition establishes a weak functional relationship between weak Markov kernels. In the following Λ and Γ denote compact subsets of $[0,1]$ while ν denotes a probability measure.

Definition 8. A weak Markov kernel $\gamma : (\Lambda, \nu) \times \mathcal{B}(X) \to [0,1]$ is functionally subordinated to a weak Markov kernel $\mu : \big(\Gamma, \nu\big) \times \mathcal{B}((X) \to [0,1]$ if there is a measurable function $f : \Lambda \to \Gamma$ such that, $\gamma_\Delta(\lambda) = \mu_\Delta(f(\lambda))$, ν-a.e.

The following theorem has been established in references [4, 6] in the case of real POVMs. Once theorem 2.3 has been proved, the proof can be straightforwardly extended to the case of an arbitrary POVM. For completness we provide the proof below.

Theorem 2.4. *Every weak Markov kernel* $\gamma : \big(\Lambda, \nu\big) \times \mathcal{B}(X) \to [0,1]$ *is functionally subordinated to the universal Markov kernel.*

Proof. We proceed as in the proof of theorem 6 in Ref. [4]. Without loss of generality, we can assume Λ to be the support of ν. Let $L_\infty(\Lambda, \nu)$ be the space of essentially bounded measurable functions (with two functions identified if they coincide up to ν-null sets) and $\mathcal{A}_\nu$ the von Neumann algebra of multiplication operators on $\mathcal{H} = L_2(\Lambda, \nu)$ which corresponds to $L_\infty(\Lambda, \nu)$. In particular, for every function $f \in L_\infty(\Lambda, \nu)$ there is a multiplication operator

$$M_f : L_2(\Lambda, \nu) \to L_2(\Lambda, \nu)$$

$$[M_f(h)](x) = f(x)h(x).$$

The generator of $\mathcal{A}_\nu$ is $B := M_x$, $[Bh](x) = [M_x(h)](x) = xh(x)$, $x \in \Lambda$. The spectrum of B, $\sigma(B)$, coincides with the support, Λ, of ν and the spectral measure corresponding to B is $E^B(\Delta) = M_{\chi_\Delta}$. Moreover, ν is a scalar-valued spectral measure for B, i.e., ν and E^B are mutually absolutely continuous.

Now, we can define a commutative POVM,

$$F(\Delta) := \gamma_\Delta(B) = \int_{\sigma(B)} \gamma_\Delta(\lambda)\, dE^B_\lambda = M_{\gamma_\Delta}, \quad \Delta \in \mathcal{B}(X). \tag{12}$$

By lemma 2.2, there is a generator A^F of $\mathcal{A}^W(F)$ with spectral resolution E^F and a Markov kernel μ^U such that

$$\int_{\sigma(A^F)} \mu^U_\Delta(\lambda)\, dE^F_\lambda = F(\Delta), \quad \Delta \in \mathcal{B}(X).$$

Since B generates $\mathcal{A}_\nu \supset \mathcal{A}^W(F)$, there is a Borel function $f : \sigma(B) \to \sigma(A^F)$ such that $E^F(G) = E^B(f^{-1}(G))$, $G \in \mathcal{B}([0,1])$. Then,

$$\int_{\sigma(B)} \mu^U_\Delta(f(\lambda))\, dE^B_\lambda = \int_{\sigma(A^F)} \mu^U_\Delta(\lambda)\, dE^F_\lambda = F(\Delta) = \int_{\sigma(B)} \gamma_\Delta(\lambda)\, dE^B_\lambda$$

Therefore, $\mu^U_\Delta(f(\lambda)) = \gamma_\Delta(\lambda)$, $E^B - a.e.$. Since, E^B and ν are mutually absolutely continuous,

$$\mu^U_\Delta(f(\lambda)) = \gamma_\Delta(\lambda), \quad \nu - a.e.$$

□

3. The universal family of fuzzy events

We know that a commutative POVM is the randomization of a spectral measure (theorem 1.1). We have shown that the universal Markov kernel μ^U is the source of the randomness for every commutative POVM F: if we take two different commutative POVMs F_1 and F_2, they will be the random version of two spectral measures E_1 and E_2 with the randomization realized by μ^U in both the cases.

The commutative POVM F_1 can also be seen as the randomization of a spectral measure E_1' with the randomization realized by a Markov kernel μ'. We have shown that there is a function f such that $\mu' = \mu^U \circ f$. Thus, the randomization which characterizes every other Markov kernel μ' can be replicated by μ^U by a change in the variable $\mu^U \circ f$.

In the introduction we noted that a Markov kernel $\mu : \mathbb{R} \times \mathcal{B}(X) \to [0,1]$ provides a family of fuzzy events $\{(\mathbb{R}, \mu_\Delta)\}_{\Delta \in \mathcal{B}(X)}$. For every point $x \in \mathbb{R}$, the family $\{\mu_\Delta(x)\}_{\Delta \in \mathcal{B}(X)}$ defines a probability measure. Moreover, we pointed out that the expression

$$\langle \psi, F(\Delta)\psi \rangle = \int_{\mathbb{R}} \mu_\Delta(\lambda)\, d\langle \psi, E_\lambda \psi \rangle \tag{13}$$

can be interpreted as the probability of the fuzzy event $(\mathbb{R}, \mu_\Delta)$ with respect to the probability measure $\langle \psi, E(\cdot)\psi \rangle$. Such a probability coincides with the probability of the event Δ (which is not a fuzzy event) with respect to the probability $\langle \psi, F(\cdot)\psi \rangle$. On the other hand we proved that for every commutative POVM F,

$$\langle \psi, F(\Delta)\psi \rangle = \int_{\mathbb{R}} \mu^U_\Delta(\lambda)\, d\langle \psi, E^F_\lambda \psi \rangle, \quad \psi \in \mathcal{H}.$$

where E^F is the sharp version of F and μ^U is the universal Markov kernel. Therefore, the existence of the universal Markov kernel μ^U means that every POVM F gives the probabilities of the same fuzzy events $(\mathbb{R}, \mu^U_\Delta)$, $\Delta \in \mathcal{B}(X)$, with respect to the probability measures corresponding to E^F. In other words, $\langle\psi, F_1(\Delta)\psi\rangle$ and $\langle\psi, F_2(\Delta)\psi\rangle$ are the probabilities of the same fuzzy event, $(\mathbb{R}, \mu^U_\Delta)$, with respect to the probability measures $\langle\psi, E^{F_1}(\cdot)\psi\rangle$ and $\langle\psi, E^{F_2}\psi\rangle$ respectively.

If we now recall that every Markov kernel μ is functionally subordinated to the universal Markov kernel, $\mu_\Delta = \mu^U_\Delta \circ f$, we see that, in a certain sense, $\{(\mathbb{R}, \mu^U_\Delta)\}_{\Delta\in\mathcal{B}(X)}$ is a universal family of fuzzy events since it includes (up to functional subordination) all the possible fuzzy events of which a POVM can represent the probabilities. Indeed, from equation (13) we see that F gives the probabilities of the fuzzy events $\{(\mathbb{R}, \mu_\Delta)\}_{\Delta\in\mathcal{B}(X)}$ with respect to E which, by functional subordination, coincides with the probabilities of the fuzzy events $\{(\mathbb{R}, \mu^U_\Delta \circ f)\}_{\Delta\in\mathcal{B}(X)}$ with respect to E,

$$\langle\psi, F(\Delta)\psi\rangle = \int_{\mathbb{R}} \mu^U_\Delta(f(\lambda))\, d\langle\psi, E_\lambda\psi\rangle$$

Appendix

Proposition 3.1. *Let X be second countable. Let $\mathcal{S}$ be a basis for the topology of X. Let $\mathcal{R}(\mathcal{S})$ be the ring generated by $\mathcal{S}$. Let $F : \mathcal{B}(X) \to \mathcal{L}^+_s(\mathcal{H})$ be a POVM. Then, the von Neumann algebra $\mathcal{A}^W(\mathcal{R}(\mathcal{S}))$ generated by $\{F(\Delta)\}_{\Delta\in\mathcal{R}(\mathcal{S})}$ coincides with the von Neumann algebra $\mathcal{A}^W(F)$.*

Proof. Let $G \subset X$ be an open set. Then, there is an increasing sequence of sets $\Delta_k \in \mathcal{S}$ such that $\Delta_k \uparrow G$. By the continuity of F, $F(\Delta_k) \uparrow F(G)$. Since $F(\Delta_k) \in \mathcal{A}^W(\mathcal{R}(\mathcal{S}))$ and $\mathcal{A}^W(\mathcal{R}(\mathcal{S}))$ is weakly closed, $F(G) \in \mathcal{A}^W(\mathcal{R}(\mathcal{S}))$. Let $\Delta \subset X$ be a G_δ set. Then there is a decreasing sequence of open sets G_k such that $G_k \downarrow \Delta$ and, by the continuity of F and the weak closure of $\mathcal{A}^W(\mathcal{R}(\mathcal{S}))$, $F(G_k) \downarrow F(\Delta) \in \mathcal{A}^W(\mathcal{R}(\mathcal{S}))$. Let G_0 be the family of open subsets of X,

ω_1 the first uncountable ordinal and G_α, $\alpha < \omega_1$ the Borel hierarchy [24]. In particular, $G_1 = G_\delta$, $G_2 = G_{\delta\sigma}$, $G_3 = G_{\delta\sigma\delta}, \dots$ and $G_\alpha = (\cup_{\beta<\alpha} G_\beta)_\sigma$ for each limit ordinal α. Suppose $F(\Delta) \in \mathcal{A}^W(\mathcal{R}(\mathcal{S}))$ for each $\Delta \in G_\alpha$. Then, $F(\Delta) \in \mathcal{A}^W(\mathcal{R}(\mathcal{S}))$ for each $\Delta \in G_{\alpha+1}$. Indeed, each set in $G_{\alpha+1}$ is either the countable union or the countable intersection of sets in G_α and the previous reasoning can be used. If α is a limit ordinal and $F(\Delta) \in \mathcal{A}^W(\mathcal{R}(\mathcal{S}))$ for each $\Delta \in G_\beta$, $\beta < \alpha$, then $F(\Delta) \in \mathcal{A}^W(\mathcal{R}(\mathcal{S}))$ for each $\Delta \in G_\alpha = (\cup_{\beta<\alpha} G_\beta)_\sigma$. Indeed, each set in G_α is the countable union of sets in $\cup_{\beta<\alpha} G_\beta$ and the previous reasoning can be used. Therefore, by transfinite induction, $F(\Delta) \in \mathcal{A}^W(\mathcal{R}(\mathcal{S}))$ for each $\Delta \in \cup_{\alpha<\omega_1} G_\alpha = \mathcal{B}(X)$ so that $\mathcal{A}^W(\mathcal{R}(\mathcal{S})) = \mathcal{A}^W(F)$.

□

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References

References

[1] S.T. Ali, La Rivista Nuovo Cimento 8 (1985) 1-127.

[2] S.T. Ali, G.G. Emch, Fuzzy observables in quantum mechanics, J. Math. Phys. 15 (1974) 176.

[3] S.T. Ali, A geometrical property of POV-measures and systems of covariance, Lecture Notes in Mathematics 905 (1982) 207-228.

[4] R. Beneduci, Universal Randomization of Quantum Observables, Int. J. Theor. Phys., doi.org/10.1007/s10773-019-04090-y (2019).

[5] R. Beneduci, Commutative POVM-measures: from the Choquet representation to the Markov kernel and back, Russian Journal of Mathematical Physics, 25 (2018) 158-182.

[6] R. Beneduci, Universal Markov Kernels for Quantum Observables, Geometric Methods in Physics. XXXVI Workshop 2017, Trends in Mathematics, 21-29 (2018).

[7] R. Beneduci, G. Nisticó, Sharp reconstruction of unsharp quantum observables, J. Math. Phys. 44 (2003) 5461.

[8] R. Beneduci, A geometrical characterizations of commutative positive operator valued measures, J. Math. Phys. 47 (2006) 062104.

[9] R. Beneduci, Joint measurability through Naimark's dilation theorem, Rep. Math. Phys. 79 (2017) 197-213.

[10] R. Beneduci, Positive Operator Valued Measures and Feller Markov Kernels, J. Math. Anal. Appl. 442 (2016) 50-71.

[11] S.K. Berberian, Notes on Spectral theory, Van Nostrand Mathematical Studies, New York, 1966.

[12] P. Busch, M. Grabowski, P. Lahti, Operational quantum physics, Lecture Notes in Physics 31 (1995).

[13] P. Busch, P. Lahti, R. Werner, Proof of Heisenberg's Error-Disturbance Relation, Phys. Rev. Lett. 111 (2013) 160405.

[14] G. Cattaneo, G. Nisticò, From unsharp to sharp quantum observable: The general Hilbert space case, J. Math. Phys. 41 (2000) 4365.

[15] E.B. Davies, Quantum mechanics of Open Systems, Academic Press, London, 1976.

[16] A. Dvurečenskij, Smearing of Observables and Spectral Measures on Quantum Structures, Found. Phys. 43 (2013) 210-224.

[17] T. Heinonen, P. Lahti, K.Ylinen, Covariant Fuzzy Observables and Coarse-Graining, Rep. Math. Phys. 53 (2004) 425-441.

[18] A.S. Holevo, An analog of the theory of statistical decisions in non-commutative probability theory, Trans, Moscow Math. Soc. 26 (1972) 133.

[19] A.S. Holevo, Probabilistics and statistical aspects of quantum theory, North Holland, Amsterdam, 1982.

[20] A. Jenčová, S. Pulmannová, How sharp are PV measures, Rep. Math. Phys. 59 (2007) 257-266.

[21] A. Jenčová, S. Pulmannová, Characterizations of Commutative POV Measures, Found. Phys. 39 (2009) 613-624.

[22] B. Jessen, The theory of integration in a space of an infinite number of dimensions, Acta Mathematica 63 (1935) 249-323.

[23] A. N. Kolmogorov, S. V. Fomin, Introductory Real Analysis, Dover, New York, 1970.

[24] K. Kuratowski, A. Mostowski, Set Theory with an introduction to descriptive set theory, North-Holland, New York, 1976.

[25] P. Lahti, Coexistence and joint measurability in quantum mechanics, Int. J. Theor. Phys., 42 (2003) 893-906.

[26] E. Lebesgue, Sur l'intégration des fonctions discontinues, Annales scientifiques de l'É.N.S. 3^e serie, tome 27 (1910) 361-450.

[27] G. Ludwig, Foundations of quantum mechanics I, Springer-Verlag, New York, 1983.

[28] B. Sz-Nagy, Spektraldarstellung linearer transformationens des Hilbertschen Raumes, Ergebnisse, d. Math, V/5, Berlin, 1942.

[29] E. Prugovečki, Stochastic Quantum Mechanics and Quantum Spacetime, D. Reidel Publishing Company, Dordrecht, Holland, 1984.

[30] F. Riesz, B. Sz. Nagy, Functional Analysis, Dover, 1990.

[31] F. Riesz, Untersuchungen *über* systeme integrierbarer funktionen, Math. Annalen 69 (1910) 449-497.

[32] F.E. Schroeck Jr., Quantum Mechanics on Phase Space, Kluwer Academic Publishers, Dordrecht, 1996.

[33] de la Vallée-Poussin, Sur la transformation d'une intégrale multiple en une intégrale simple, Ann. Soc. Sci. Bruxelles 35 (1911) 189-190.

[34] L.A. Zadeh, Fuzzy Sets, Information and Control 8 (1965) 338-353.

[35] L.A. Zadeh, Probability Measures on Fuzzy Events, J. Math. Anal. Appl. 23 (1968) 421-427.