On the category of affine subspaces

Anna Jenčová

April 15, 2024

1 Definitions and properties

1.1 Base sections

Let V be a finite dimensional vector space. A subset $C \subseteq V$ is called a **proper cone** if

- C is a convex cone: for any $a, b \in C$ and $\lambda, \mu \geq 0$ we have $\lambda a + \mu b \in C$;
- C is closed;
- C is generating in V: $V = C - C = \{a - b,\ a, b \in C\}$;
- C is pointed: $C \cap -C = \{0\}$.

For a convex cone C, the **dual cone** C^* is the set of positive linear functionals

$$C^* = \{\varphi \in V^*, \langle \varphi, a \rangle \geq 0, \forall a \in C\}.$$

If C is a proper cone, then C^* is a proper cone and we have $C^{**} = C$. From now on C will be a proper cone.

Let $J \subseteq V$ be a subspace. Then the dual space $J^* \equiv V^*|_{J^\perp}$. Let $g_J : V^* \to J^*$, $q_J(\psi) = \psi + J^\perp$ be the quotient map. Then $J \cap C$ is a closed pointed convex cone. If $J \cap int(C) \neq \emptyset$, $J \cap C$ is also generating in J, and we have

$$(J \cap C)^* = q_J(C^*) = \{\varphi + J^\perp,\ \varphi \in C^*\}.$$

In other words, any positive linear functional on $(J, J \cap C)$ extends to a positive linear functional on (V, C). Moreover, we have

$$int((J \cap C)^*) = q_J(int(C^*)).$$

A subset $K \subseteq C$ is a **base** of C if

- K is convex;
- for any $a \in C$, there are unique $x \in K$ and $\lambda \geq 0$ such that $a = \lambda x$.

Any base of C is determined by a (unique) element $u \in int(C^*)$:

$$K = \{x \in C,\ \langle\, u, x\,\rangle = 1\}.$$

Given such an element $u \in int(C^*)$, the corresponding base will be denoted by K_u.

A subset $B \subseteq C$ is called a **base section** if

- B is a base of the cone $span(B) \cap C$;
- $B \cap int(C) \neq \emptyset$.

For relative interiors, we have $ri(B) = B \cap int(C)$.

Lemma 1. *A subset $B \subseteq C$ is a base section if and only if $B = span(B) \cap K$ for some base K of the cone C.*

Proof. Let B be a base section and let $J = span(B)$, then B is a base of the cone $J \cap C$, hence there is some (unique) $[\varphi] \in int((J \cap C)^*) = q_J(int(C^*))$, that is, $[\varphi] = \varphi + J^\perp$ for some $\varphi \in int(C^*)$, such that

$$B = \{x \in J \cap C,\ \langle\, \varphi, x\,\rangle = 1\} = J \cap C \cap \varphi^{-1}(1) = J \cap K_\varphi.$$

The converse is obvious. □

Lemma 2. *Let K be a base of C and let $B \subseteq K$ be such that $B \cap int(C) \neq \emptyset$. Then*

(i) The set $\tilde{B} := \{\varphi \in C^*,\ \langle\, \varphi, x\,\rangle = 1,\ \forall x \in B\}$ is a base section in (V^*, C^*).

(ii) Let $\tilde{J} = span(\tilde{B})$, then $\dim(\tilde{J}) = 1 + \dim(V) - \dim(J)$.

(iii) B is a base section if and only if $\tilde{\tilde{B}} = B$.

(iv) $\tilde{\tilde{B}}$ is the smallest base section containing B.

Proof. Let $\tilde{b} \in int(C^*)$ be such that $K = K_{\tilde{b}}$, then clearly $\tilde{b} \in \tilde{B}$, so that $\tilde{B} \cap int(C^*) \neq \emptyset$. Let $b \in B \cap int(C)$ be any element and let K_b be the corresponding base of C^*. Then it is clear that $\tilde{B} \subseteq K_b$, so that $\tilde{B} \subseteq \tilde{J} \cap K_b$, where $\tilde{J} = span(\tilde{B})$. Since $\tilde{B}$ is obviously convex, any element $y \in \tilde{J}$ has the form $s\tilde{b}_1 - t\tilde{b}_2$ for some $\tilde{b}_1, \tilde{b}_2 \in \tilde{B}$ and $s, t \geq 0$. If $y \in \tilde{J} \cap K_b$, then $y \in C^*$ and $\langle\, y, b\,\rangle = 1$, so that $s - t = 1$ and hence for any $b' \in B$, we must have

$$\langle\, y, b'\,\rangle = \langle\, s\tilde{b}_1 - t\tilde{b}_2, b'\,\rangle = s - t = 1.$$

Hence $y \in \tilde{B}$. This shows that $\tilde{B} = \tilde{J} \cap K_b$, so that $\tilde{B}$ is a base section in (V^*, C^*).

To show (ii), let $\tilde{b} \in ri(\tilde{B})$ be any fixed element. Note that as in the firts part of the proof, $y \in \tilde{J}$ has the form

$$y = s\tilde{b}_1 - t\tilde{b}_2 = (s-t)\tilde{b} + s\tilde{b}_1 - t\tilde{b}_2 - (s-t)\tilde{b} = (s-t)\tilde{b} + x$$

where $x \in J^\perp$. Conversely, let $y = \alpha\tilde{b} + x$ for some $x \in J^\perp$, $\alpha \in \mathbb{R}$. Since $\tilde{b} \in ri(\tilde{B}) = \tilde{B} \cap int(C^*)$, there is some $\lambda > 0$ such that $\lambda\tilde{b} - y \in C^*$, which means that $y = \lambda\tilde{b} - (\lambda\tilde{b} - y) \in \tilde{J}$. It follows that $\tilde{J} = span(\tilde{b}) \wedge J^\perp$, from this (ii) follows.

For (iii), assume that B is a base section. It is clear that $B \subset \tilde{\tilde{B}}$ and $\tilde{\tilde{B}}$ is a base setcion as well. Hence $J \subseteq \tilde{\tilde{J}}$, and we have by (ii)

$$\dim(\tilde{\tilde{J}}) = 1 + \dim(V^*) - \dim(\tilde{J}) = 1 + \dim(V^*) - (1 + \dim(V) - \dim(J)) = \dim(J),$$

so that $J = \tilde{\tilde{J}}$. Since B and $\tilde{\tilde{B}}$ are bases of the same cone $J \cap C$, we must have $\tilde{\tilde{B}} = B$. The converse statement is clear from (i).

Finally, it is clear that $\tilde{\tilde{B}}$ is a base section containing B. Let B' be a base section such that $B \subseteq B'$, then clearly $\tilde{B}' \subseteq \tilde{B}$ and $\tilde{\tilde{B}} \subseteq \tilde{\tilde{B}}' = B'$, by (iii).

□

For an affine subspace $A \subseteq V$, we put

$$\tilde{A} = \{y \in V^*,\ \langle\, y, x \,\rangle = 1,\ \forall x \in A\}.$$

Lemma 3. *Let $A = Aff(B)$, $\tilde{b} \in ri(B)$. Then*

(i) $B = A \cap C$.

(ii) $\tilde{A} = Aff(\tilde{B}) = b + J^{\perp}$.

Proof. (i) It is clear that $B \subseteq A \cap C$. Conversely, let $x \in A \cap C$, then for every $y \in \tilde{B}$ we have $\langle\, y, x \,\rangle = 1$, so that $x \in \tilde{\tilde{B}} = B$.

(ii) It is straightforward to verify that $Aff(\tilde{B}) \subseteq \tilde{A} \subseteq \tilde{b} + J^{\perp}$. We now prove $\tilde{b} + J^{\perp} \subseteq Aff(\tilde{B})$. So let $z \in J^{\perp}$. Since $\tilde{B} \in int(C^*)$, there is some $s > 0$ such that $\tilde{c} := \tilde{b} + sz \in C^*$, so that clearly $\tilde{c} \in \tilde{B}$. We thena have $\tilde{b} + z = (1 - s^{-1})\tilde{b} + s^{-1}\tilde{c} \in Aff(\tilde{B})$.

□

1.2 Affine subspaces

A subset $A \subseteq V$ of a finite dimensional vector space V is an affine subspace if $\sum_i \alpha_i a_i \in A$ whenever all $a_i \in A$ and $\sum_i \alpha_i = 1$.

Let $A \subseteq V$ be an affine subspace:

- Let $a_0 \in A$ be an arbitrary element. Then A has the form $A = a_0 + L$, where

  $$L = Lin(A) := \{a - b,\ a, b \in A\} = \{a - a_0,\ a \in A\}$$

  is a vector subspace. The dimension of A is $\dim(A) := \dim(Lin(A))$.

- If $0 \in A$, then $Lin(A) = span(A) = A$ and A is a vector subspace. If $0 \notin A$, then $A \cap Lin(A) = \emptyset$ and $\dim(span(A)) = \dim(A) + 1$.

- Put $\tilde{A} := \{a^* \in V^*,\ \langle\, a^*, a \,\rangle = 1,\ \forall a \in A\}$. Then $\tilde{A} \subseteq V^*$ is an affine subspace and we have

  $$Lin(\tilde{A}) = span(A)^{\perp}.$$

  Consequently, if $0 \notin A$, $\dim(\tilde{A}) = \dim(V^*) - \dim(span(A)) = \dim(V) - \dim(A) - 1$.

- $\tilde{\tilde{A}} = A$.

1.3 The category BS

Let BS be the category whose objects are triples $\mathbf{V} = (V, C, B)$, where V is a finite dimensional real vector space, $C \subseteq V$ a proper cone and B a base section in (V, C). Morphisms $\mathbf{V} \to \mathbf{W}$ are positive linear maps preserving the base section.

Then BS is a symmetric monoidal category, inheriting the monoidal structure from FinVect. Let us define

$$\mathbf{V}_1 \otimes \mathbf{V}_2 = (V_1 \otimes V_2, C_1 \otimes C_2, B_1 \otimes B_2),$$

where $V_1 \otimes V_2$ the tensor product in FinVect,

$$C_1 \otimes C_2 = \left\{ \sum_i x_i \otimes y_i,\ x_i \in V_1,\ y_i \in V_2 \right\}$$

is the (minimal) tensor product of cones and

$$B_1 \otimes B_2 := \{b_1 \otimes b_2,\ b_1 \in B_1,\ b_2 \in B_2\}^{\sim}.$$

The monoidal unit $I = \mathbf{I} = (\mathbb{R}, \mathbb{R}^+, \{1\})$. All the isomorphisms are those obtained from FinVect. We have already shown that this is a symmetric monoidal structure.

We define the dual of $\mathbf{V}$ as $\mathbf{V}^* = (V^*, C^*, \tilde{B})$. For a morphism $f : \mathbf{V}_1 \to \mathbf{V}_2$, $f^* : \mathbf{V}_2^* \to \mathbf{V}_1^*$ is defined as the adjoint map in FinVect, it is easily checked that f^* is indeed a morphism in BS and $(-)^*$ is a functor $\mathrm{BS}^{op} \to \mathrm{BS}$. We moreover have $\mathbf{V}^{**} = \mathbf{V}$ and $(-)^*$ is full and faithful.

We next want to show that there is a natural isomorphism

$$\mathrm{BS}(X \otimes Y, Z^*) \simeq \mathrm{BS}(X, (Y \otimes Z)^*).$$

Note that we have natural iso

$$\mathrm{FinVect}(V_X \otimes V_Y, V_Z^*) \simeq \mathrm{FinVect}(V_X, (V_Y \otimes V_Z)^*)$$

Since the morphisms in BS are special morphisms in FinVect, it is enough to show that the above iso maps BS-morphisms onto respective BS-morphisms. We see that the relation between $f \in \mathrm{FinVect}(V_X \otimes V_Y, V_Z^*)$ and $\hat{f} \in \mathrm{FinVect}(V_X, (V_Y \otimes V_Z)^*)$ is given as

$$\langle\, f(x \otimes y), z\, \rangle = \langle\, \hat{f}(x), y \otimes z\, \rangle, \qquad x \in V_X, y \in V_Y, z \in V_Z.$$

1.4 The category Af

The objects of this category are pairs $X = (V, A)$, where V is in FinVect and $A \subseteq V$ is an affine subspace. Unless $V = \{0\}$, we always assume that $A \neq \emptyset$ and $0 \notin A$.

We sometimes use the notation $X = (V_X, A_X)$ is necessary. Morphisms $X \to Y$ are linear maps $f : V_X \to V_Y$ such that $f(A_X) \subseteq f(A_Y)$.

1.4.1 The monoidal structure

We define the monoidal structure as follows. We put

$$X \otimes Y := (V_X \otimes V_Y, \{x \otimes y,\ x \in A_X, y \in A_Y\}^{\approx})$$

The unit is given as $I := (\mathbb{R}, \mathbb{R}^+, \{1\})$. One can check that $(\mathrm{Af}, \otimes, I)$ is a symmetric monoidal category, with all the structures inherited from FinVect. It only remains to check that α, λ, ρ and σ from FinVect are morphisms in Af. We will do it some other time, maybe.

Lemma 4. *Let $x_0 \in A_X$, $y_0 \in A_Y$. Put $L_X = Lin(A_X)$.*

(1) $\boldsymbol{\emptyset} \otimes X = \boldsymbol{\emptyset}$, $\mathbf{0} \otimes X = \mathbf{0}$.

(2) $A_{X\otimes Y} = Aff(\{x \otimes y, x \in A_X, y \in A_Y\})$.

(3) $L_{X\otimes Y} = (x_0 \otimes L_Y) \vee (L_X \otimes y_0) \vee (L_X \otimes L_Y)$.

(4) $\dim(A_{X\otimes Y}) = (\dim(A_X) + 1)(\dim(A_Y + 1) - 1$.

Proof. (1) is quite obvious from the definition.

Let $0 \notin C \subseteq V$ be any subset of a finite dimensional vector space V. Then clearly $\tilde{C} = Aff(C)^{\sim}$ and $\tilde{\tilde{C}} = Aff(C)^{\approx} = Aff(C)$, this proves (2).

Since $A_X = x_0 + L_X$ and $A_Y = y_0 + L_Y$, we see from (2) that $A_{X\otimes Y}$ is an affine span of elements of the form

$$(x_0 + u) \otimes (y_0 + v) = x_0 \otimes y_0 + u \otimes y_0 + x_0 \otimes v + u \otimes v, \qquad u \in L_X,\ v \in L_Y.$$

Clearly, any such element is in $x_0 \otimes y_0 + L_{X\otimes Y}$ as defined in (3). Moreover, since $x_0 \notin L_X$ and $y_0 \notin L_Y$, the subspaces $x_0 \otimes L_Y$, $L_X \otimes y_0$ and $L_X \otimes L_Y$ are mutually linearly independent. For any $u \in L_X$,

$$x_0 \otimes y_0 + u \otimes y_0 = (x_0 + u) \otimes y_0 \in A_{X\otimes Y},$$

similarly $x_0 \otimes y_0 + x_0 \otimes v \in A_{X\otimes Y}$ for any $v \in L_Y$. Moreover,

$$x_0 \otimes y_0 + u \otimes v = \frac{1}{2}\left((x_0 + u) \otimes (y_0 + v) + (x_0 - u) \otimes (y_0 - v)\right) \in A_{X\otimes Y}.$$

Since any element $w \in L_X \otimes L_Y$ has the form $w = \sum_i u_i \otimes v_i$ with $u_i \in L_X$, $v_i \in L_Y$, we see that

$$A_{X\otimes Y} = x_0 \otimes y_0 + L_{X\otimes Y},$$

this proves (3). (4) is quite obvious from (3). □

1.4.2 Duality

We define the dual object

$$X^* = (V_X^*, \tilde{A}_X).$$

Note that

$$\boldsymbol{\emptyset}^* = \mathbf{0}, \qquad \mathbf{0}^* = \boldsymbol{\emptyset}.$$

We also have by Lemma 3

$$L_{X^*} = span(A_X)^{\perp} = (\mathbb{R}x_0 \vee L_X)^{\perp} = \{x_0\}^{\perp} \wedge L_X^{\perp}.$$

This means that

$$\dim(A_{X^*}) = \dim(V_X) - \dim(A_X) - 1.$$

Further, for $f : X \to Y$ we define $f^* : Y^* \to X^*$ as the usual adjoint of the map $f : V_X \to V_Y$. Let us check that $f^*(A_{Y^*}) \subseteq A_{X^*}$. So let $y^* \in A_{Y^*} = \tilde{A}_Y$, we have to check that $\langle f^*(y^*), x \rangle = 1$ for all $x \in A_X$. Indeed,

$$\langle f^*(y^*), x \rangle = \langle y^*, f(x) \rangle = 1.$$

It follows that $(-)^* : \mathrm{Af}^{op} \to \mathrm{Af}$ is a functor, which is full and faithful since the dual $(-)^*$ on FinVect is such.

We want to show that $(\mathrm{Af}, \otimes, I)$ is *-autonomous. For this we need to show that there is a natural isomorphism

$$\mathrm{Af}(X \otimes Y, Z^*) \simeq \mathrm{Af}(X, (Y \otimes Z)^*).$$

Note that we have natural iso

$$\mathrm{FinVect}(V_X \otimes V_Y, V_Z^*) \simeq \mathrm{FinVect}(V_X, (V_Y \otimes V_Z)^*)$$

Since the morphisms in Af are special morphisms in FinVect, it is enough to show that the above iso maps Af-morphisms onto respective Af-morphisms. We see that the relation between $f \in \mathrm{FinVect}(V_X \otimes V_Y, V_Z^*)$ and $\hat{f} \in \mathrm{FinVect}(V_X, (V_Y \otimes V_Z)^*)$ is given as

$$\langle\, f(x \otimes y), z\,\rangle = \langle\, \hat{f}(x), y \otimes z\,\rangle, \qquad x \in V_X, y \in V_Y, z \in V_Z.$$

Assume that $f : X \otimes Y \to Z^*$, we need to show that $\hat{f} : X \to (Y \otimes Z)^*$. Let $x \in A_X$ and let $y \in A_Y$, $z \in A_Z$, then

$$\langle\, \hat{f}(x), y \otimes z\,\rangle = \langle\, f(x \otimes y), z\,\rangle = 1,$$

since f maps $x \otimes y \in A_{X \otimes Y}$ into A_Z. It follows that

$$\hat{f}(x) \in (A_Y \otimes A_Z)^{\sim} = \tilde{A}_{Y \otimes Z}.$$

The converse is similar: we see that f maps all elements $x \otimes y$ into $\tilde{A}_Z$, hence also the affine subspace generated by $x \otimes y$ is mapped to $\tilde{A}_Z$. But this affine subspace is exactly $A_{X \otimes Y}$.

1.4.3 First order objects (states)

We say that an object X in Af is a first order object if $\dim(A_X) = \dim(V_X) - 1$. We will use ordinary capital letters (not bold) to indicate first order objects.

If X is first order, A_X is determined by a single element $u_X \in V_{X^*}$ and we have $X = (V_X, \{u_X\}^{\sim})$, $X^* = (V_X^*, \{u_X\})$. Note that if X and Y are first order objects, then $X \otimes Y$ is a first order object as well, and we have

$$u_{X \otimes Y} = u_X \otimes u_Y.$$

Indeed, this follows easily from Lemma 4. Note also that the tensor unit I is first order, but $\boldsymbol{\emptyset}$ and $\mathbf{0}$ are not.

Lemma 5. *We $(X \otimes Y)^* \simeq X^* \otimes Y^*$, if and only if some of the following holds*

1. *one of the objects is $\boldsymbol{\emptyset}$, $\mathbf{0}$ or I,*
2. *both X, Y are firts order objects,*
3. *both X^*, Y^* are first order objects.*

Proof. Using the expression for $\dim(A_{X \otimes Y})$ and $\dim(\tilde{A}_X)$, we get

$$
\begin{aligned}
\dim(A_{(X\otimes Y)^*}) &= \dim(V_{X\otimes Y}) - \dim(A_{X\otimes Y}) - 1\\
&= \dim(V_{X\otimes Y}) - (\dim(A_X)+1)(\dim(A_Y)+1)\\
\dim(A_{X^*\otimes Y^*}) &= (\dim(A_{X^*})+1)(\dim(A_{Y^*})+1) - 1\\
&= (\dim(V_X) - \dim(A_X))(\dim(V_Y) - \dim(A_Y)) - 1
\end{aligned}
$$

From this one can check that

$$\dim(A_{(X\otimes Y)^*}) - \dim(A_{X^*\otimes Y^*}) = \dim(A_X)\dim(\tilde{A}_Y) + \dim(\tilde{A}_X)\dim(A_Y).$$

It is clear that this is equal to 0 if and only if some of the conditions holds. Since we always have $A_{X^*\otimes Y^*} \subseteq \tilde{A}_{X\otimes Y}$, the statement follows.

□

The above result also shows that the dual monoidal structure

$$X \odot Y := (X^* \otimes Y^*)^*$$

coincides with $\otimes$ if and only if X and Y, or their duals, are first order.

Let us also note that the unique dualizable object in this category is I, so Af is very noncompact.

Lemma 6. *An object X in* Af *is dualizable if and only if $X \simeq I$.*

Proof. Assume that X is dualizable, then there must be some $\eta : I \to X^*\otimes X$ and $\epsilon : X\otimes X^* \to I$ such that

$$(\epsilon \otimes X)\circ(X\otimes \eta) = id_X.$$

This means that $\epsilon \in \tilde{A}_{X\otimes X^*}$ and $\eta(1) \in A_{X^*\otimes X}$, so that $\eta(1) = \sum \alpha_i x_i^* \otimes x_i$ for some $x_i^* \in \tilde{A}_X$, $x_i \in A_X$ are such that we have for any $x \in V_X$

$$\sum_i \alpha_i \langle\, \epsilon, x\otimes x_i^* \,\rangle x_i = x$$

This implies that $\{x_i\}$ must be a basis of V_X, which implies that A_X must have codimension 1, that is, X is first order. But then $\tilde{A}_X = \{u_X\}$, so that $\eta(1) = u_X \otimes \alpha_i x_i = u_X \otimes \bar{x}$, so we get

$$\sum_i \alpha_i \langle\, \epsilon, x\otimes x_i^* \,\rangle x_i = \langle\, \epsilon, x\otimes u_X \,\rangle \bar{x},$$

so any element in V_X is a multiple of $\bar{x}$, so $\dim(V_X) = 1$ and $X \simeq I$. □

1.4.4 The subspaces

For X in Af, we put $L_X = Lin(A_X)$, $S_X = span(A_X)$. Let $x_0 \in A_X$, $x_0^* \in \tilde{A}_X$, $y_0 \in A_Y$, $y_0^* \in \tilde{A}_Y$.

We have

$$L_{X^*} = S_X^{\perp}, \quad S_{X^*} = L_X^{\perp}$$

and

$$
\begin{aligned}
A_{X\otimes Y} &= \{w \in S_X\otimes S_Y,\ \langle\, w, x_0^*\otimes y_0^* \,\rangle = 1\} = S_X\otimes S_Y \cap \{x_0^*\otimes y_0^*\}^{\sim}\\
&= (x_0\otimes y_0) + (x_0\otimes L_Y)\times(L_X\otimes y_0)\times(L_X\otimes L_Y).
\end{aligned}
$$

From this we see that

$$S_{X\otimes Y} = S_X \otimes S_Y$$

and

$$\tilde{A}_{X\otimes Y} = x_0^* \otimes y_0^* + (S_X \otimes S_Y)^\perp = x_0^* \otimes y_0^* + (L_{X^*}^\perp \otimes L_{Y^*}^\perp)^\perp.$$

Note that then

$$A_{X\odot Y} = \tilde{A}_{X^*\otimes Y^*} = x_0 \otimes y_0 + (L_X^\perp \otimes L_Y^\perp)^\perp$$

1.4.5 Internal homs

Since Af is *-autonomous, the internal hom is

$$[X, Y] = (X \otimes Y^*)^* = X^* \odot Y.$$

As we have seen above,

$$A_{[X,Y]} = x_0^* \otimes y_0 + (S_X \otimes L_Y^\perp)^\perp$$

Note that if X and Y are first order, then

$$A_{[X,Y]} = u_X \otimes y_0 + V_X^* \otimes L_Y, \qquad A_{[X,Y]} = x_0^* \otimes y_0 + (S_X \otimes u_Y)^\perp$$

for all X, Y, and

$$\begin{aligned} A_{[X,Y]} &= u_X \otimes y_0 + (V_X \otimes u_Y)^\perp = \{w \in V_X^* \otimes V_Y,\ \varphi_{u_Y}(w) = \hat{w}^*(u_Y) = u_X\}, \\ \tilde{A}_{[X,Y]} &= A_{X\otimes Y^*} = A_X \otimes u_Y. \end{aligned}$$

Note that through the usual identification in FinVect, any $w \in U^* \otimes V$ is identified with a linear map $\hat{w} : U \to V$, given by

$$\langle\, w, u \otimes v^* \,\rangle = \langle\, v^*, \hat{w}(u) \,\rangle = \langle\, \hat{w}^*(v^*), u \,\rangle,$$

where $\hat{w}^*$ is the adjoint of $\hat{w}$. For $u^* \in U^*$, we define the map $\varphi_u^V : U \otimes V \to V$ by

$$\varphi_{u^*}^V(u \otimes v) = \langle\, u^*, u \,\rangle v, \qquad u \in U, v \in V.$$

We omit the index V if not necessary. Let $A \subseteq V$ be an affine subspace, $0 \notin A$, $A \neq \emptyset$. Let $u^* \in U^*$ and put

$$B = \{w \in U \otimes V, \varphi_u(w) \in A\}$$

Then B is an affine subspace, $0 \notin B \neq \emptyset$, indeed, for $a_0 \in A$ and $u \in \{u^*\}^\sim$

$$B = u \otimes a_0 + (\{u^*\}^\perp \otimes V) \vee (u \otimes Lin(A))$$

Lemma 7. *1.* $A_X = x_0 + L_X$*,*

2. $\tilde{A}_X = x_0^* + S_X^\perp$*,*

3. $A_{X\otimes Y} = x_0 \otimes y_0 + ...$

—

2 Once more from the top

We present some important categories.

2.1 The category FinVect

Let FinVect be the category of finite dimensional real vector spaces with linear maps. Then $(\mathrm{FinVect}, \otimes, \mathbb{R})$ is a symmetric monoidal category, with the usual tensor product of vector spaces. With the usual duality $(-)^* : V \mapsto V^*$ of vector spaces, FinVect is compact closed. Put

$$e_U : U \otimes U^* \to \mathbb{R}, \qquad e_u(u \otimes u^*) = \langle\, u^*, u \,\rangle,$$

then e_U is the cap for the duality of U and U^*. The corresponding element $\eta_U \in (U \otimes U^*) = U^* \otimes U$ is the cup, given by

$$\eta_U = \sum_i e_i^* \otimes e_i$$

where $\{e_i\}$ is a basis of U and $\{e_i^*\}$ the dual basis of U^*, determined by $\langle\, e_i^*, e_j \,\rangle = \delta_{ij}$. It is easily verified that η_U does not depend on the choice of the basis $\{e_i\}$.

By compactness the internal hom is $[U, V] = U^* \otimes V$ and the evaluation map $U \otimes [U, V] \to V$ is given by

$$eval_{U,V} = e_U \otimes V : U \otimes U^* \otimes V \to V.$$

For any $w \in U^* \otimes V$, we obtain a linear map $\hat{w} : U \to V$ by

$$\hat{w}(u) = (e_U \otimes V)(u \otimes w),$$

(we write V for the identity map id_V). Conversely, for any $f : U \to V$ we define $\tilde{f} \in U^* \otimes V$ as

$$\tilde{f} = (f^* \otimes V)(\eta_V).$$

Note that this gives the usual identification

$$\langle\, \hat{w}(u), v^* \,\rangle = \langle\, w, u \otimes v^* \,\rangle, \qquad u \in U,\ v^* \in V^*$$

between maps $U \to V$ and elements of $U^* \otimes V$. Put $\circ_{U,V,W} := U^* \otimes e_V \otimes W$, then $\circ_{U,V,W}$ is a linear map

$$[U, V] \otimes [V, W] \to [U, W]$$

which corresponds to composition of maps: for $f : U \to V$ and $g : V \to W$, we get

$$\circ_{U,V,W} : \tilde{f} \otimes \tilde{g} \mapsto (g \circ f)^{\sim}.$$

Similarly, e_V (tensored with identity maps and composed with symmetries as necessary) defines a partial composition map

$$[U, V \otimes X] \otimes [V \otimes Y, W] \to [U \otimes Y, X \otimes W].$$

This can be depicted graphically in a nice way.

2.2 Affine subspaces

A subset $A \subseteq V$ of a finite dimensional vector space V is an affine subspace if $\sum_i \alpha_i a_i \in A$ whenever all $a_i \in A$ and $\sum_i \alpha_i = 1$. We say that A is proper if $0 \neq A$ and $A \neq \emptyset$. We will always mean that an affine subspace is proper (if not explicitly stated otherwise).

An affine subspace can be determined in two ways:

(i) Let $L \subseteq V$ be a linear subspace and $a_0 \neq L$. Then

$$A = a_0 + L$$

is an affine subspace. Note that $a_0 \in A$ and $A \cap L = \emptyset$. Conversely, any affine subspace A can be given in this way, with a_0 an arbitrary element in A and

$$L = Lin(A) := \{a_1 - a_2,\ a_1, a_2 \in A\} = \{a - a_0,\ a \in A\}.$$

(ii) Let $S \subseteq V$ be a linear subspace and $a_0^* \in V^* \setminus S^\perp$. Then

$$A = \{a \in S, \langle\, a_0^*, a \,\rangle = 1\}$$

is an affine subspace. Conversely, any affine subspace A is given in this way, with $S = span(A)$ and a_0^* an arbitrary element in

$$\tilde{A} = \{a^* \in V^*,\ \langle\, a^*, a \,\rangle = 1,\ \forall a \in A\}.$$

For an affine subspace A, $\tilde{A}$ is an affine subspace as well and we have $\tilde{\tilde{A}} = A$. More generally, if $\emptyset \neq C \subseteq A$ is any subset of an affine subspace A, then $\tilde{C}$ is an affine subspace and $\tilde{\tilde{C}}$ is the smallest affine subspace containing C, that is,

$$\tilde{\tilde{C}} = \{\sum_i \alpha_i c_i,\ c_i \in C,\ \sum_i \alpha_i = 1\}.$$

In this case, we may write $\tilde{\tilde{C}}$ as

$$\tilde{\tilde{C}} = c_0 + Lin(C) = c_0 + span(\{c_1 - c_2,\ c_1, c_2 \in C\})$$

with an arbitrary element $c_0 \in C$, or as

$$\tilde{\tilde{C}} = \{c \in span(C),\ \langle\, a_0^*, c \,\rangle = 1\}$$

for an arbitrary element $a_0^* \in \tilde{A}$. We clearly have

$$Lin(\tilde{C}) = C^\perp = span(C)^\perp, \qquad Lin(C) = Lin(\tilde{\tilde{C}}) = \tilde{C}^\perp = span(\tilde{C})^\perp$$

and by duality also

$$span(C) = C^{\perp\perp} = Lin(\tilde{C})^\perp, \qquad span(\tilde{C}) = Lin(C)^\perp.$$

2.3 The category Af

The objects of Af are of the form $X = (V_X, A_X, a_X, \tilde{a}_X)$, where V_X is in FinVect, $A_X \subseteq V_X$ an affine subspace, $a_X \in A_X$ and $\tilde{a}_X \in \tilde{A}_X$ are some elements. Morphisms $X \to Y$ are linear maps $f : V_X \to V_Y$ such that $f(A_X) \subseteq A_Y$. Note that by definition A_X is proper for any object X. We may also add two special objects: the initial object $\emptyset := (\{0\}, \emptyset, -, 0)$ and the terminal object $0 := (\{0\}, \{0\}, 0, -)$, here the affine subspaces are obviously not proper. The products and coproducts with these element do not work, however!

For any object X, we also put

$$L_X := Lin(A_X) \qquad S_X := span(A_X), \qquad d_X := \dim(L_X), \qquad D_X := \dim(V_X).$$

Note that X is uniquely determined also when A_X is replaced by L_X or S_X.

2.3.1 Limits and colimits

Limits and colimits should be obtained from those in FinVect, we have to spectify the other structures and check whether the corresponding arrows are in Af.

Let X, Y be two objects in Af. We define their product as

$$X \times Y := (V_X \times V_Y, A_X \times A_Y, (a_X, a_Y), \frac{1}{2}(\tilde{a}_X, \tilde{a}_Y)),$$

where

$$A_X \times A_Y := \{(x, y) \in V_X \times V_Y,\ x \in A_X, y \in A_Y\}$$

is the direct product of A_X and A_Y. It is easily verified that this is indeed an affine subspace and the usual projections $\pi_X : V_X \times V_Y \to V_X$ and $\pi_Y : V_X \times V_Y \to V_Y$ are in Af. Moreover, for $f : Z \to X$ and $g : Z \to Y$, the map $f \times g(z) = (f(z), g(z))$ is also clearly a morphism $Z \to X \times Y$ in Af. We have

$$L_{X\times Y} = L_X \times L_Y, \qquad S_{X\times Y} = (S_X \times S_Y) \wedge \{(\tilde{a}_X, -\tilde{a}_Y)\}^{\perp}.$$

The coproduct is defined as

$$X \oplus Y = (V_X \times V_Y, A_X \oplus A_Y, \frac{1}{2}(a_X, a_Y), (\tilde{a}_X, \tilde{a}_Y)),$$

where

$$A_X \oplus A_Y := \{(tx, (1-t)y),\ x \in A_X, y \in A_Y,\ t \in \mathbb{R}\}$$

is the direct sum. To check that this is an affine subspace, let $x_i \in A_X$, $y_i \in A_Y$, $s_i \in \mathbb{R}$ and let $\sum_i \alpha_i = 1$, then

$$\sum_i \alpha_i(s_i x_i, (1-s_i)y_i) = (\sum_i s_i\alpha_i x_i, \sum_i (1-s_i)\alpha_i y_i) = (sx, (1-s)y) \in A_X \oplus A_Y,$$

where $s = \sum_i s_i\alpha_i$, $x = s^{-1}\sum_i s_i\alpha_i x_i$ if $s \neq 0$ and is arbitrary in A_X otherwise, similarly $y = (1-s)^{-1}\sum_i (1-s_i)\alpha_i y_i$ if $s \neq 1$ and is arbitrary otherwise. The usual embeddings $p_X : V_X \to V_X \times V_Y$ and $p_Y : V_Y \to V_X \times V_Y$ are easily seen to be morphsims in Af.

Let $f : X \to Z$, $g : Y \to Z$ be any morphisms in Af and consider the map $V_X \times V_Y \to Z$ given as $f \oplus g(u, v) = f(u) + g(v)$. We need to show that it preserves the affine subspaces. So let $x \in A_X$, $y \in A_Y$, then since $f(x), g(y) \in A_Z$, we have for any $s \in \mathbb{R}$,

$$f \oplus g(sx, (1-s)y) = sf(x) + (1-s)g(y) \in A_Z.$$

We also have

$$L_{X\oplus Y} = (L_X \times L_Y) \vee \mathbb{R}\{(a_X, -a_Y)\}, \qquad S_{X\oplus Y} = S_X \times S_Y.$$

Let us turn to equalizers. So let $f, g : X \to Y$ and let

$$V_E = \{v \in V_X,\ f(v) = g(v)\}.$$

Let $h : Z \to X$ equalize f, g, then $h(V_Z) \subseteq V_E$ and $h(A_Z) \subseteq A_X \cap V_E$, so that $A_X \cap V_E$ must be nonempty. In this case,

$$E = (V_E, A_E := V_E \cap A_X, a_E, \tilde{a}_E := \tilde{a}_X)$$

with the inclusion map $V_E \hookrightarrow V_X$ is an equalizer of f, g for any choice of $a_E \in A_E$ (note that choosing another a_E gives us an isomorphic object in Af). If the intersection $V_E \cap A_X$ is empty, then the only equalizing arrow for f and g is $\emptyset \to X$, which is then the equalizer.

For the coequalizer, let V_Q be the quotient space $V_Q := V_Y|_{Im(f-g)}$ and let $q : V_Y \to V_Q$ be the quotient map. If some $h : Y \to Z$ coequalizes f and g, then h maps $Im(f-g)$ to 0, so that $Im(f-g) \cap A_Y = \emptyset$, unless Z is the terminal object. It is easily checked that if $Im(f-g) \cap A_Y = \emptyset$, then

$$Q = (V_Q, A_Q := q(A_Y), a_Q := q(a_Y), \tilde{a}_Q)$$

together with the quotient map q is the coequalizer of f and g for any choice of $\tilde{a}_Q \in \tilde{A}_Q$. If the intersection is nonempty, then the unique coeqalizing arrow is $Y \to 0$, which is then the coequalizer.

Let us mention pullbacks and pushouts. Since pullbacks can be obtained from products and equalizers, we see that we have a similar situation: if a pullback is "well defined", then it coincides with the pullback in FinVect, otherwise it is trivial. More precisely, if $f : X \to Z$ and $g : Y \to Z$, then we put

$$V_P := \{(x, y) \in V_X \times V_Y, f(x) = g(y)\}.$$

If $V_P \cap A_X \times A_Y \neq \emptyset$, that is, there are some $x \in A_X$ and $y \in A_Y$ such that $f(x) = g(y)$, then

$$(V_P, A_P := (A_X \times A_Y) \cap V_P, a_P, \frac{1}{2}(\tilde{a}_X, \tilde{a}_Y))$$

with the two projections is a pullback of f and g for any choice of $a_P \in A_P$, otherwise the pullback is just the initial object $\emptyset$.

Similarly, let $f : Z \to X$, $g : Z \to Y$, then let V_Q be the quotient of $V_X \times V_Y$ by the subspace

$$\{(f(z), -g(z)),\ x \in V_Z\}.$$

If this subspace does not contain any element of $A_X \oplus A_Y$, that is, there is no $z \in V_Z$ such that for some $t \in \mathbb{R}$,

$$f(tz) \in A_X, \qquad g((t-1)z) \in A_Y,$$

then

$$Q = (V_Q, A_Q := q(A_X \oplus A_Y), \frac{1}{2}q(a_X, a_Y), \tilde{a}_Q)$$

with maps $x \mapsto q(x, 0)$ and $y \mapsto q(0, y)$ is the pushout of f and g. Otherwise the pushout is just 0.

2.3.2 Tensor products

Let X, Y be objects in Af. Let us define

$$A_{X\otimes Y} := \{x\otimes y, x\in A_X, y\in A_Y\}^{\approx}.$$

In other words, $A_{X\otimes Y}$ is the affine subspace in $V_X\otimes V_Y$ containing $A_X\otimes A_Y$. We have

$$\begin{aligned} L_{X\otimes Y} &= Lin(A_X\otimes A_Y) = span(\{x\otimes y - a_X\otimes a_Y,\ x\in A_X,\ y\in A_Y\}) \\ &= (a_X\otimes L_Y) + (L_X\otimes a_Y) + (L_X\otimes L_Y) \end{aligned} \tag{1}$$

(here $+$ denotes the direct sum of subspaces). We also have

$$S_{X\otimes Y} = S_X\otimes S_Y.$$

Proof. Let $x\in A_X$, $y\in A_Y$, then

$$x\otimes y - a_X\otimes a_Y = a_X\otimes(y-a_Y) + (x-a_X)\otimes a_Y + (x-a_X)\otimes(y-a_Y),$$

so that $L_{X\otimes Y} = Lin(A_X\otimes A_Y)$ is contained in the subspace on the RHS of (1). Let d be the dimension of this subspace, then clearly

$$d_{X\otimes Y}\le d\le d_X + d_Y + d_X d_Y.$$

On the other hand, any element of S_X has the form tx for some $t\in\mathbb{R}$ and $x\in A_X$, so that it is easily seen that $S_X\otimes S_Y = S_{X\otimes Y}$. Hence

$$\begin{aligned} d_{X\otimes Y} &= \dim(L_{X\otimes Y}) = \dim(S_{X\otimes X}) - 1 = \dim(S_X)\dim(S_Y) - 1 = (d_X+1)(d_Y+1) - 1 \\ &= d_X + d_Y + d_X d_Y. \end{aligned}$$

This completes the proof.

$\square$

For X, Y in Af, put

$$X\otimes Y := (V_X\otimes V_Y, A_{X\otimes Y}, a_X\otimes a_Y, \tilde{a}_X\otimes\tilde{a}_Y).$$

Also let $I := (\mathbb{R}, \{1\}, \{1\}, \{1\})$. Then $(\mathrm{Af}, \otimes, I)$ is a symmetric monoidal category. We only have to check that the associators, unitors and symmetries from FinVect are morphisms in Af. We leave this for some other day.

2.3.3 Duality

We define $X^* := (V_X^*, \tilde{A}_X, \tilde{a}_X, a_X)$. Note that we have

$$L_{X^*} = S_X^{\perp}, \qquad S_{X^*} = L_X^{\perp}.$$

It follows that

$$d_{X^*} = D_X - d_X - 1.$$

It is easily seen that $(-)^*$ defines a full and faithful functor $\mathrm{Af}^{op}\to\mathrm{Af}$, moreover, $X^{**} = X$ (if we us the canonical identification of any V in FinVect with its second dual).

Theorem 1. $(\mathrm{Af}, \otimes, I)$ *is a *-autonomous category, with duality* $(-)^*$.

Proof. ... ∎

Let us define the dual tensor product by $\odot$, that is

$$X \odot Y = (X^* \otimes Y^*)^*.$$

We then have

$$\begin{aligned} L_{X\odot Y} &= S^{\perp}_{X^*\otimes Y^*} = (S_{X^*} \otimes S_{Y^*})^{\perp} = (L^{\perp}_X \otimes L^{\perp}_Y)^{\perp} \\ S_{X\odot Y} &= L^{\perp}_{X^*\otimes Y^*} = (\tilde{a}_X \otimes S^{\perp}_Y)^{\perp} \wedge (S^{\perp}_X \otimes \tilde{a}_Y)^{\perp} \wedge (S^{\perp}_X \otimes S^{\perp}_Y)^{\perp} \end{aligned}$$

In particular,

$$d_{X\odot Y} = D_X d_Y + d_X D_Y - d_X d_Y.$$

Lemma 8. *Let X, Y be nontrivial. Then $X \otimes Y = X \odot Y$ if and only if $D_X = d_X + 1$ and $D_Y = d_Y + 1$.*

Proof. It is easy to see that (when identifying $X = X^{**}$), we have $A_X \otimes A_Y \subseteq \tilde{A}_{X^*\otimes Y^*}$, hence $A_{X\otimes Y} \subseteq A_{X\odot Y}$. We see from the above computatons that

$$d_{X\odot Y} - d_{X\otimes Y} = d_X(D_Y - 1) + (D_X - 1)d_Y - 2d_X d_Y \geq 0,$$

with equality if and only if the conditions of the lemma hold. ∎

The internal hom has the form

$$[X, Y] = (X \otimes Y^*)^* = X^* \odot Y.$$

We then have

$$L_{[X,Y]} = (S_X \otimes L^{\perp}_Y)^{\perp}, \qquad S_{[X,Y]} = (\tilde{a}_X \otimes S^{\perp}_Y)^{\perp} \wedge (L_X \otimes \tilde{a}_Y)^{\perp} \wedge (L_X \otimes S^{\perp}_Y)^{\perp}$$

and

$$d_{[X,Y]} = D_X D_Y - (d_X + 1)(D_Y - d_Y).$$

We say $X \hookrightarrow Y$ (X is embedded in Y) if $V_X = V_Y$ and $A_X \subseteq A_Y$, $a_X = a_Y$, $\tilde{a}_X = \tilde{a}_Y$.

2.3.4 The category AfH

It is easily seen that the following are equivalent:

1. $D_X = d_X + 1$;
2. $S_X = V_X$;
3. $L_X = \{\tilde{a}_X\}^{\perp}$;
4. $S_{X^*} = \mathbb{R}\tilde{a}_X$;

5. $L_{X^*} = \{0\}$.

We say that an object X is first order if any of these conditions is fulfilled. A channel is an object $[X, Y]$ where X and Y are first order.

Lemma 9. *An object Z is embedded in a channel $[X, Y]$ if and only if $V_Z = V_X^* \otimes V_Y$, $a_Z = \tilde{a}_X \otimes a_Y$, $\tilde{a}_Z = a_X \otimes \tilde{a}_Y$ and*

...

Pullbacks are intersections, pushouts the affine mixture.

Channels into (from) products and coproducts

We define AfH as the full subcategory of Af containing all first order objects and closed under (finite products,) duals and tensor products .