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Base norms and discrimination of generalized quantum channels

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We introduce and study norms in the space of hermitian matrices, obtained from base norms in positively generated subspaces. These norms are closely related to discrimination of so-called generalized quantum channels, including quantum states, channels, and networks. We further introduce generalized quantum decision problems and show that the maximal average payoffs of decision procedures are again given by these norms. We also study optimality of decision procedures, in particular, we obtain a necessary and sufficient condition under which an optimal 1-tester for discrimination of quantum channels exists, such that the input state is maximally entangled. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4863715]

I. INTRODUCTION AND PRELIMINARIES

It is well known that in the problem of discrimination of quantum states, the best possible distinguishability of two states ρ_0 and ρ_1 is given by the trace norm $\|\rho_0 - \rho_1\|_1$.[10,11] The set of states forms a base of the convex cone of positive operators and the restriction of the trace norm to hermitian operators is the corresponding base norm. Similarly, it was shown in Ref. 20 that more general distinguishability measures, obtained by specification of the allowed measurements, e.g., for bipartite states, are obtained from base norms associated with more general positive cones. This correspondence is related to duality of the base norm and the order unit norm, with respect to a given positive cone.

In a similar problem for quantum channels, and recently also quantum networks, the diamond norm $\|\cdot\|_\diamond$ for channels,[14] resp. the strategy N-norm $\|\cdot\|_{N\diamond}$[9,3] for networks is obtained. Via the Choi isomorphism, quantum networks are represented by certain positive operators on the tensor product of the input and output spaces, so-called N-combs,[2,4] see also Ref. 8. The set of N-combs is the intersection of the multipartite state space by a positively generated subspace of the real vector space of hermitian operators. Since this subspace inherits the order structure and the set of N-combs forms a base of its positive cone, it is natural to expect that the distinguishability norm $\|\cdot\|_{N\diamond}$ is in fact the corresponding base norm.

Motivated by this question, we study positively generated subspaces of the space of hermitian operators $B_h(\mathcal{H})$ acting on a finite dimensional Hilbert space $\mathcal{H}$. For a given base B of the positive cone, we define a distinguishability measure in terms of tests that are defined as affine maps $B \to [0, 1]$ and show that this measure is given by the base norm. This, in fact, is easy to see for any finite dimensional ordered vector space. We then study a natural extension of this norm to $B_h(\mathcal{H})$ and its dual norm. An example of such a base is the set of Choi matrices of so-called generalized channels. The set of N-combs is a special case. For N-combs, the obtained norm coincides with $\|\cdot\|_{N\diamond}$ and we recover some of the results of Ref. 9 concerning the dual norm. Moreover, we find a suitable expression for this norm, closely related to the definition of $\|\cdot\|_\diamond$.

In Sec. IV, we introduce generalized quantum decision problems with respect to a base B. We show that the maximal average payoff (or minimal average loss) of a generalized decision procedure is again given by a base norm. We find optimality conditions for generalized decision procedures, in particular, for quantum measurements and testers. In the case of multiple hypothesis testing for states, we get the results obtained previously in Refs. 17, 7, and 12. In the case of discrimination of quantum channels, we find a necessary and sufficient condition for existence of an optimal tester such that the input state is maximally entangled.

The rest of the present section contains some basic definitions and preliminary results on discrimination of quantum devices, as well as convex cones, bases, and base norms.

A. Discrimination of quantum states, channels, and networks

Let $\mathcal{H}$ be a finite dimensional Hilbert space and let $B(\mathcal{H})$ be the set of bounded operators on $\mathcal{H}$. We denote by $B_h(\mathcal{H})$ the set of self-adjoint operators, $B(\mathcal{H})^+$ the cone of positive operators and $\mathfrak{S}(\mathcal{H}) := \{\rho \geq 0, \operatorname{Tr}\rho = 1\}$ the set of states in $B(\mathcal{H})$. We will also use the notation $B(\mathcal{H})^{++}$ for the set of strictly positive elements in $B(\mathcal{H})$. Let $\mathcal{K}$ be another finite dimensional Hilbert space. It is well known that $B(\mathcal{K}\otimes\mathcal{H})$ corresponds to the set of all linear maps $B(\mathcal{H}) \to B(\mathcal{K})$, via the Choi representation:[1]

$$X_\Phi = (\Phi \otimes id_{\mathcal{H}})(\Psi), \qquad \Phi_X(a) = \operatorname{Tr}_{\mathcal{H}}[(I_{\mathcal{K}} \otimes a^{\mathsf{T}})X], \tag{1}$$

here $\Psi = |\psi\rangle\langle\psi|$ and $|\psi\rangle = \sum_i |i\rangle \otimes |i\rangle$ for an orthonormal basis (ONB) $\{|i\rangle, i = 1, \ldots, \dim(\mathcal{H})\}$ in $\mathcal{H}$, a^{T} denotes transpose of a with respect to this basis. In this correspondence, $B(\mathcal{K}\otimes\mathcal{H})^+$ is identified with the set of completely positive maps and $B_h(\mathcal{K}\otimes\mathcal{H})$ with hermitian maps, that is, maps satisfying $\Phi(a^*) = \Phi(a)^*$.

Consider the problem of quantum state discrimination: suppose the quantum system represented by $\mathcal{H}$ is known to be in one of two given states ρ_0 or ρ_1 and the task is to decide which of them is the true state. This is done by using a test, that is a binary positive operator valued measure (POVM). This is given by an operator $0 \leq M \leq I$, with the interpretation that $\operatorname{Tr} M\rho$ is the probability of deciding for ρ_0 if the true value of the state is ρ. Equivalently, a test can be defined as an affine map $\mathfrak{S}(\mathcal{H}) \to [0, 1]$.

Given an *a priori* probability $0 \leq \lambda \leq 1$ that the true state is ρ_0, we need to minimize the average probability of error over all tests, that is to find the value of

$$\Pi_\lambda(\rho_0, \rho_1) := \min_{0\leq M\leq I} \lambda \operatorname{Tr}(I - M)\rho_0 + (1-\lambda)\operatorname{Tr} M\rho_1,$$

this is the minimum Bayes error probability. Then[10,11]

$$\Pi_\lambda(\rho_0, \rho_1) = \frac{1}{2} - \frac{1}{2}\|\lambda\rho_0 - (1-\lambda)\rho_1\|_1,$$

where $\|a\|_1 := \operatorname{Tr}|a|$, $a \in B(\mathcal{H})$ is the trace norm.

Let now $\mathcal{H}$ and $\mathcal{K}$ be two finite dimensional Hilbert spaces and consider the problem of discrimination of channels. Here we have to decide between two channels Φ_0 and Φ_1 and this time the tests are given by binary quantum 1-testers,[3] or PPOVMs,[21] which are positive operators $T \in B(\mathcal{K}\otimes\mathcal{H})^+$, such that $T \leq I_{\mathcal{K}} \otimes \sigma$ for some $\sigma \in \mathfrak{S}(\mathcal{H})$. These correspond to triples $(\mathcal{H}_A, \rho, M)$, where $\mathcal{H}_A$ is an ancillary Hilbert space, $\rho \in \mathfrak{S}(\mathcal{H}\otimes\mathcal{H}_A)$ and $0 \leq M \leq I$, $M \in B(\mathcal{K}\otimes\mathcal{H}_A)$. The probability of choosing Φ_0 if the true value is Φ for a tester T is given by

$$p(T, \Phi) := \operatorname{Tr} T X_\Phi = \operatorname{Tr} M(\Phi \otimes id_A)(\rho).$$

The minimum Bayes error probability is now

$$\Pi^1_\lambda(\Phi_0, \Phi_1) := \min_T \lambda(1 - p(T, \Phi_0)) + (1-\lambda)p(T, \Phi) = \frac{1}{2} - \frac{1}{2}\|\lambda\Phi_0 - (1-\lambda)\Phi_1\|_\diamond,$$

where the diamond norm $\|\Phi\|_\diamond$ for a hermitian map Φ is defined as[14,19]

$$\begin{aligned}\|\Phi\|_\diamond &= \sup_{\dim(\mathcal{L}')<\infty} \sup_{\rho\in\mathfrak{S}(\mathcal{H}\otimes\mathcal{L}')} \|\Phi \otimes id_{\mathcal{L}'}(\rho)\|_1 \\ &= \sup_{\rho\in\mathfrak{S}(\mathcal{H}\otimes\mathcal{L})} \|\Phi \otimes id_{\mathcal{L}}(\rho)\|_1, \qquad \dim(\mathcal{L}) = \dim(\mathcal{H}).\end{aligned}$$

By duality, this norm is related to the *cb*-norm for completely bounded linear maps, see Ref. 16.

Let now $\{\mathcal{H}_0, \mathcal{H}_1, \ldots, \mathcal{H}_{2N-1}\}$ be finite dimensional Hilbert spaces. Consider a sequence of channels $\Phi_i : B(\mathcal{H}_{2i-2} \otimes \mathcal{H}_A) \to B(\mathcal{H}_{2i-1} \otimes \mathcal{H}_A)$, $i = 1, \ldots, N$, connected by the ancilla $\mathcal{H}_A$ as

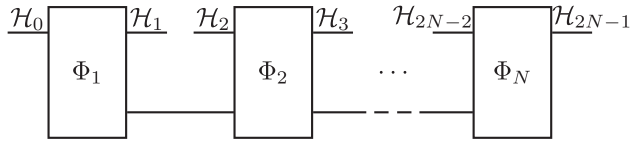

FIG. 1. A deterministic quantum *N*-comb.

indicated in Fig. 1 (the first and last ancilla are traced out). This defines a channel $\Phi : B(\mathcal{H}_0 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_{2N-2}) \to B(\mathcal{H}_1 \otimes \mathcal{H}_3 \otimes \cdots \otimes \mathcal{H}_{2N-1})$, such channels describe quantum networks. The channels $\Phi_1, \ldots, \Phi_N$ are not unique, in fact, these can always be supposed to be isometries. A (deterministic) quantum *N*-comb is defined as the Choi matrix X_Φ of such a channel, see Ref. 4 for more about quantum networks and *N*-combs. The same definition, called a (non-measuring) quantum *N*-round strategy, was also introduced in Ref. 8. A (non-measuring) quantum *N*-round co-strategy can be defined as an $(N+1)$-strategy for the sequence of spaces $\{\mathbb{C}, \mathcal{H}_0, \ldots, \mathcal{H}_{2N-1}, \mathbb{C}\}$.

The tests for discrimination of two networks Φ^0 and Φ^1 are given by quantum *N*-testers, which are obtained by an $(N+1)$-comb such that the first channel has 1-dimensional input space (hence is a state) and a (binary) POVM is applied to the ancilla,[4,3] see Figs. 2 and 3. This can be represented by a pair (T_0, T_1) of positive operators, such that $T_0 + T_1$ is an $(N+1)$-round co-strategy.[4,8,9]

The minimal Bayes error probability now has the form

$$\Pi_\lambda^N(\Phi^0, \Phi^1) = \frac{1}{2} - \frac{1}{2}\|\lambda\Phi^0 - (1-\lambda)\Phi^1\|_{N\diamond},$$

where the norm $\|\cdot\|_{N\diamond}$ was introduced in Ref. 3 as

$$\|\Phi\|_{N\diamond} = \sup_T \|(T_0 + T_1)^{1/2} X_\Phi (T_0 + T_1)^{1/2}]\|_1, \tag{2}$$

for any hermitian Φ. Another expression for this norm was found in Ref. 9:

$$\|\Phi\|_{N\diamond} = \sup_T \operatorname{Tr} X_\Phi (T_0 - T_1). \tag{3}$$

In both cases, the supremum is taken over all *N*-testers. The dual norm was also obtained in Ref. 9 as

$$\|\Phi\|_{N\diamond}^* = \sup_S \operatorname{Tr} X_\Phi (S_1 - S_0),$$

where the supremum is taken over the set of pairs of positive operators such that $S_0 + S_1$ is an *N*-round strategy (*N*-comb).

B. Convex cones, bases, and base norms

Let $\mathcal{V}$ be a finite dimensional real vector space and let $\mathcal{V}^*$ be the dual space, with duality $\langle\cdot, \cdot\rangle$. A subset $Q \subset \mathcal{V}$ is a convex cone if $\lambda q_1 + \mu q_2 \in Q$ whenever $q_1, q_2 \in Q$ and $\lambda, \mu \geq 0$. The cone is pointed if $Q \cap -Q = \{0\}$ and generating if $\mathcal{V} = Q - Q$. Closed pointed convex cones are in one-to-one correspondence with partial orders in $\mathcal{V}$, by $x \leq_Q y \Leftrightarrow y - x \in Q$.

The dual cone of Q is defined as

$$Q^* = \{f \in \mathcal{V}^*, \langle f, q\rangle \geq 0, q \in Q\}.$$

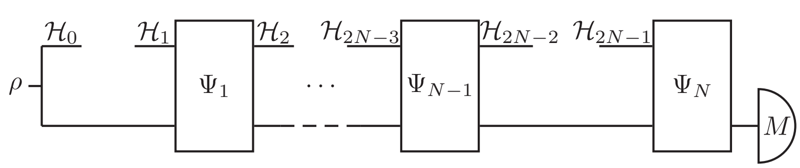

FIG. 2. A quantum *N*-tester.

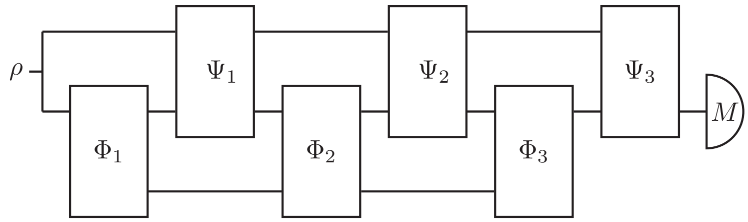

FIG. 3. A 3-tester Ψ applied to a 3-comb Φ.

This is a closed convex cone and $Q^{**} = Q$ if Q is closed. Moreover, a closed convex cone Q is pointed if and only if Q^* is generating . A closed pointed generating convex cone is called a proper cone.

A base of the proper cone Q is a compact convex subset $B \subset Q$, such that each nonzero element $q \in Q$ has a unique representation in the form $q = \lambda b$ with $b \in B$ and $\lambda > 0$. It is clear that any base generates the cone Q, in the sense that $Q = \bigcup_{\lambda \geq 0} \lambda B$. Then any element $v \in \mathcal{V}$ can be written as $v = \lambda b_1 - \mu b_2$, $\lambda, \mu \geq 0$, $b_1, b_2 \in B$.

For any base B, the map $Q \ni q = \lambda b \mapsto \lambda$ extends uniquely to a linear functional $e_B \in Q^*$ and we have $B = \{q \in Q, \langle e_B, q\rangle = 1\}$.

Lemma 1. Let $f \in Q^*$. Then $f \in int(Q^*)$ if and only if

$$B_f := \{q \in Q, \langle f, q\rangle = 1\}$$

is a base of Q.

Proof. It is quite clear that B_f is a base of Q if and only if $\langle f, q\rangle > 0$ for any nonzero $q \in Q$. By Theorem 11.6 of Ref. 18, this is equivalent with $f \in int(Q^*)$. □

Let $\leq$ denote the order in $\mathcal{V}$ given by Q. An element $e \in \mathcal{V}$ is an order unit in $\mathcal{V}$ if for any $v \in \mathcal{V}$, there is some $r > 0$ such that $re \geq v$. It is easy to see that e is an order unit if and only if $e \in int(Q)$. Consequently,

Corollary 1. Any base B of Q defines an order unit e_B in $\mathcal{V}^*$ and, conversely, any order unit e in $\mathcal{V}^*$ defines a base B_e of Q. We have $e_{B_e} = e$ and $B_{e_B} = B$.

Let B be a base of Q. The corresponding base norm in $\mathcal{V}$ is defined by

$$\|v\|_B = \inf\{\lambda + \mu,\ v = \lambda b_1 - \mu b_2, \lambda, \mu \geq 0,\ b_1, b_2 \in B\}.$$

It is clear that $\|q\|_B = \langle e_B, q\rangle$ for all $q \in Q$. Let $\mathcal{V}_1$ be the unit ball of $\|\cdot\|_B$ in $\mathcal{V}$, then

$$\mathcal{V}_1 = \{\lambda b_1 - \mu b_2,\ b_1, b_2 \in B, \lambda, \mu \geq 0, \lambda + \mu = 1\} = co(B \cup -B),$$

where $co(A)$ denotes the convex hull of $A \subset \mathcal{V}$. Let $\|\cdot\|_B^*$ be the dual norm in $\mathcal{V}^*$, then the unit ball $\mathcal{V}_1^*$ for $\|\cdot\|_B^*$ is given by

$$\mathcal{V}_1^* = \mathcal{V}_1^\circ = (co(B \cup -B))^\circ = (B \cup -B)^\circ = B^\circ \cap (-B)^\circ,$$

where $A^\circ := \{f \in \mathcal{V}^*,\ \langle f, a\rangle \leq 1, \forall a \in A\}$ is the polar of $A \subset \mathcal{V}$, see Ref. 18. We have

$$\mathcal{V}_1^* = \{f \in \mathcal{V}^*, -1 \leq \langle f, b\rangle \leq 1, \forall b \in B\} = \{f \in \mathcal{V}^*, -e_B \leq_{Q^*} f \leq_{Q^*} e_B\},$$

where e_B is the order unit. Hence the dual norm is given by

$$\|f\|_B^* = \inf\{\lambda > 0, -\lambda e_B \leq_{Q^*} f \leq_{Q^*} \lambda e_B\} =: \|f\|_{e_B}.$$

In general, if e is an order unit, then $\|\cdot\|_e$ defines a norm called the order unit norm in $\mathcal{V}^*$.

Since $\|\cdot\|_B$ is the dual norm for $\|\cdot\|_{e_B}$, we get for $v \in \mathcal{V}$,

$$\|v\|_B = \|v\|_{e_B}^* = \sup_{-e_B \leq_{Q^*} f \leq_{Q^*} e_B} \langle f, v\rangle = 2 \sup_{f \in Q^*, f \leq_{Q^*} e_B} \langle f, v\rangle - \langle e_B, v\rangle, \tag{4}$$

where the last equality follows by replacing f by $\frac{1}{2}(f + e_B)$.

Example 1. Let $\mathcal{V} = B_h(\mathcal{H})$ and $Q = B(\mathcal{H})^+$. We identify $\mathcal{V}^*$ with $\mathcal{V}$, with duality $\langle a, b\rangle = \mathrm{Tr}\, ab$, then Q is a self-dual proper cone and $B = \mathfrak{S}(\mathcal{H})$ is a base of Q, with $e_B = I$. The order unit norm $\|\cdot\|_I$ is the operator norm $\|\cdot\|$ in $B(\mathcal{H})$ and its dual $\|\cdot\|_B$ is the trace norm $\|\cdot\|_1$.

We will finish this section by showing that the base norm is naturally related to a distinguishability measure for elements of the base. By analogy with the set of quantum states, let us define a test on a base B as an affine map $\mathbf{t} : B \to [0, 1]$. It is easy to see that there is a one-to-one correspondence between tests on B and elements $f \leq_{Q^*} e_B$ in Q^*. Let b_0, b_1 be two elements of B and let us interpret the value $\mathbf{t}(b) = \langle f, b\rangle$ as the probability of choosing b_0 if the "true value" is b. Then $\langle f, b_1\rangle$ and $1 - \langle f, b_0\rangle$ are probabilities of making an error. Let $\lambda \geq 0$, then we define the minimal average error probability as

$$\Pi^B_\lambda(b_0, b_1) := \min_{0\leq_{Q^*} f \leq_{Q^*} e_B} \lambda(1 - \langle f, b_0\rangle) + (1 - \lambda)\langle f, b_1\rangle.$$

We obtain by (4) that

$$\begin{aligned}\Pi^B_\lambda(b_0, b_1) &= \lambda - \max_{0\leq_{Q^*} f \leq_{Q^*} e_B} \langle f, \lambda b_0 - (1 - \lambda)b_1\rangle \\ &= \frac{1}{2}(1 - \|\lambda b_0 - (1 - \lambda)b_1\|_B).\end{aligned}$$

II. BASE NORMS ON SUBSPACES OF $B_h(\mathcal{H})$

We now put $\mathcal{V} = B_h(\mathcal{H})$, with the self-dual proper cone $B(\mathcal{H})^+$ as in Example 1. We will describe all possible bases of this cone.

It is clear that $int(B(\mathcal{H})^+) = B(\mathcal{H})^{++}$, hence the strictly positive elements are the order units in $B_h(\mathcal{H})$. By Corollary 1, there is a one-to-one correspondence between strictly positive elements and bases of $B(\mathcal{H})^+$, given by

$$B(\mathcal{H})^{++} \ni b \leftrightarrow S_b := \{a \in B(\mathcal{H})^+, \mathrm{Tr}\, ab = 1\} = B(\mathcal{H})^+ \cap \mathcal{T}_b, \tag{5}$$

where $\mathcal{T}_b = \{x \in B_h(\mathcal{H}), \mathrm{Tr}\, xb = 1\}$. By (4) and Example 1, the corresponding base norm is

$$\|x\|_{S_b} = \sup_{-b\leq a\leq b} \mathrm{Tr}\, ax = \sup_{-I\leq a\leq I} \mathrm{Tr}\, ab^{1/2}xb^{1/2} = \|b^{1/2}xb^{1/2}\|_1 \tag{6}$$

and the dual order unit norm is

$$\|x\|_b = \inf\{\lambda > 0, -\lambda b \leq x \leq \lambda b\} = \|b^{-1/2}xb^{-1/2}\|. \tag{7}$$

If $b \in B(\mathcal{H})^+$, we define

$$\|b^{-1/2}xb^{-1/2}\| := \lim_{\varepsilon\to 0^+} \|(b + \varepsilon)^{-1/2}x(b + \varepsilon)^{-1/2}\|.$$

Note that the expression on the RHS is bounded for all $\varepsilon > 0$ if and only if $\mathrm{supp}\,(x) \leq \mathrm{supp}\,(b)$ and in this case the norm on the LHS is defined by restriction to the support of b. Otherwise, the limit is infinite. Moreover, for $a, b \in B(\mathcal{H})^+$, we define

$$D_{max}(a\|b) := \log\inf\{\lambda > 0, a \leq \lambda b\} = \inf\{\gamma > 0, a \leq 2^\gamma b\}.$$

For a pair of states ρ and σ, $D_{max}(\rho\|\sigma)$ is the max-relative entropy of ρ and σ, introduced in Ref. 6. (Note that D_{max} was denoted by D_∞ in Ref. 17.) If $b \in B(\mathcal{H})^{++}$, then

$$D_{max}(a\|b) = \log(\|a\|_b).$$

In general, if $\mathrm{supp}\,(a) \leq \mathrm{supp}\,(b)$, then we may restrict to the support of b and with this restriction $D_{max}(a\|b) = \log\,(\|a\|_b)$, otherwise $D_{max}(a\|b) = \infty$.

A. Sections of a base of $B(\mathcal{H})^+$

Let $J \subset B_h(\mathcal{H})$ be a subspace and let $Q = J \cap B(\mathcal{H})^+$ be the convex cone of positive elements in J. It is obvious that Q is closed and pointed. We will suppose that J is positively generated, then J

$= Q - Q$ and Q is a proper cone in J. Let $b \in Q$ be such that $\operatorname{supp} a \leq \operatorname{supp} b =: p$ for all $a \in Q$, then $J \subseteq B_h(p\mathcal{H})$ and by restricting to $B_h(p\mathcal{H})$, we may suppose that b is strictly positive. Conversely, if J contains a strictly positive element, then J is positively generated.

Let $J^\perp = \{y \in B_h(\mathcal{H}), \operatorname{Tr} xy = 0,\ \ x \in J\}$, let $B_h(\mathcal{H})|_{J^\perp}$ be the quotient space and let $\pi : B_h(\mathcal{H}) \to B_h(\mathcal{H})|_{J^\perp}$ be the quotient map $a \mapsto a + J^\perp$. We may identify the dual space J^* with $B_h(\mathcal{H})|_{J^\perp}$, with duality

$$\langle x, \pi(a)\rangle = \operatorname{Tr} xa, \qquad x \in J,\ a \in B_h(\mathcal{H}).$$

It was shown in Theorem 2 of Ref. 13 that the dual cone of Q is $Q^* = \pi(B(\mathcal{H})^+)$, moreover, since π is a linear map, we have $int(Q^*) = int(\pi(B(\mathcal{H})^+)) = \pi(B(\mathcal{H})^{++})$ by Theorem 6.6 of Ref. 18. In other words, any element $f \in Q^*$ has the form

$$f(x) = \operatorname{Tr} ax, \qquad x \in J,$$

for some (in general non-unique) element $a \in B(\mathcal{H})^+$ and f is an order unit in J^* if and only if a may be chosen strictly positive. Now we can use Corollary 1 to describe all bases of Q.

Lemma 2. A subset $B \subset Q$ is a base of Q if and only if $B = J \cap S_{\tilde{b}}$, where $\tilde{b} \in B(\mathcal{H}^{++})$. In this case, $\pi(\tilde{b}) = e_B$.

Proof. Let B be a base of Q. Since $e_B \in int(Q^*)$, there is some $\tilde{b} \in B(\mathcal{H})^{++}$ such that $e_B = \pi(\tilde{b})$ and

$$B = \{q \in Q, \operatorname{Tr} q\tilde{b} = \langle e_B, q\rangle = 1\rangle\} = Q \cap \mathcal{T}_{\tilde{b}} = J \cap S_{\tilde{b}}$$

(see (5)). Conversely, it is quite clear that $B = J \cap S_{\tilde{b}}$ is a base of Q and $e_B = \pi(\tilde{b})$. □

A set of the form $B = L \cap S_{\tilde{b}}$ where $\tilde{b} \in B(\mathcal{H})^{++}$ and $L \subseteq B_h(\mathcal{H})$ is a subspace will be called a section of a base of $B(\mathcal{H})^+$, or simply a section. Let span(B) be the real linear span of B, then

$$B \subseteq \operatorname{span}(B) \cap S_{\tilde{b}} \subseteq L \cap S_{\tilde{b}} = B,$$

so that $B = \operatorname{span}(B) \cap S_{\tilde{b}}$ and B is a base of $\operatorname{span}(B) \cap B(\mathcal{H})^+$. If moreover B contains a positive definite element, we say that B is a faithful section. In this case, we have $B \cap B(\mathcal{H})^{++} = ri(B)$, where $ri(B)$ denotes the relative interior of B, Section 6 of Ref. 18. Indeed, since $B = L_{\tilde{b}} \cap B(\mathcal{H})^+$, where $L_{\tilde{b}} =: L \cap \mathcal{T}_{\tilde{b}}$ is an affine subspace containing an interior point of $B(\mathcal{H})^+$, we have by Corollary 6.5.1 of Ref. 18 that

$$ri(B) = ri(L_{\tilde{b}} \cap B(\mathcal{H})^+) = L_{\tilde{b}} \cap B(\mathcal{H})^{++} = B \cap B(\mathcal{H})^{++}.$$

For example, note that if $B = \{b\}$ for some $b \in B(\mathcal{H})^+$, then B is a section and B is faithful if and only if b is strictly positive. If a section B is not faithful, then there is some element $b \in B$ such that $p = \operatorname{supp}(b)$ and $B \subset B(p\mathcal{H})$. Then B is a faithful section of a base of $B(p\mathcal{H})^+$, in this case, $ri(B) = B \cap ri(B(p\mathcal{H})^+)$. From now on, we will suppose that B is a faithful section of a base of $B(\mathcal{H})^+$.

Note that in Lemma 2, the correspondence between the base B and the element $\tilde{b}$ such that $B = \operatorname{span}(B) \cap S_{\tilde{b}}$ is not one-to-one, since the order unit $e_B = \pi(\tilde{b})$ may contain more different strictly positive elements. We will now look at the set of all such elements. Let

$$\tilde{B} := \{\tilde{b} \in B(\mathcal{H})^+, \operatorname{Tr} b\tilde{b} = 1, \forall b \in B\}.$$

Then

$$\tilde{B} = \pi^{-1}(e_B) \cap B(\mathcal{H})^+ = (\tilde{b} + B^\perp) \cap B(\mathcal{H})^+, \tag{8}$$

where $\tilde{b}$ is any element in $\tilde{B}$. Note that $\tilde{B}$ always contains a strictly positive element. Since by (8) $\tilde{B}$ is an intersection of $B(\mathcal{H})^+$ by an affine subspace, we have

$$\{\tilde{b} \in B(\mathcal{H})^{++},\ B = \operatorname{span}(B) \cap S_{\tilde{b}}\} = \tilde{B} \cap B(\mathcal{H})^{++} = ri(\tilde{B}).$$

Lemma 3.

(i) $\tilde{B}$ *is a faithful section of a base of* $B(\mathcal{H})^+$.
(ii) $\tilde{\tilde{B}} = B$.
(iii) $B = \bigcap_{\tilde{b}' \in ri(\tilde{B})} S_{\tilde{b}'}$.

Proof. (i) Let $b \in ri(B)$. Since $\tilde{B}$ is convex, any element $y \in \mathrm{span}(\tilde{B})$ has the form $y = \lambda \tilde{b}_1 - \mu \tilde{b}_2$, with $\tilde{b}_1, \tilde{b}_2 \in \tilde{B}$ and $\lambda, \mu \geq 0$. Hence by (8), $y = (\lambda - \mu)\tilde{b} + z$ for some $z \in B^\perp$. If y is also in S_b, we must have $1 = \mathrm{Tr}\, yb = \lambda - \mu$, so that $y \in (\tilde{b} + B^\perp) \cap B(\mathcal{H})^+ = \tilde{B}$. It follows that $\tilde{B} = \mathrm{span}(\tilde{B}) \cap S_b$.

(ii) It is clear that $B \subseteq \tilde{\tilde{B}}$ and $\tilde{\tilde{B}} = (b + \tilde{B}^\perp) \cap B(\mathcal{H})^+$. Let $\tilde{b} \in ri(\tilde{B})$, then $\tilde{B} = (\tilde{b} + B^\perp) \cap B(\mathcal{H})^+$. Since $\tilde{b} \in B(\mathcal{H})^{++}$, for each $z \in B^\perp$ there is some $t > 0$ such that $\tilde{b} + tz \in \tilde{B}$ and this implies that $\tilde{B}^\perp \subseteq (B^\perp)^\perp = \mathrm{span}(B)$, hence also $\tilde{\tilde{B}} \subseteq \mathrm{span}(B)$. It follows that $\mathrm{span}(B) = \mathrm{span}(\tilde{\tilde{B}})$, so that B and $\tilde{\tilde{B}}$ are two bases of the same cone. This implies (ii).

(iii) Obviously $B \subseteq \bigcap_{\tilde{b}' \in ri(\tilde{B})} S_{\tilde{b}'}$. If $a \in \bigcap_{\tilde{b}' \in ri(\tilde{B})} S_{\tilde{b}'}$, then a is a positive element such that $\mathrm{Tr}\, a\tilde{b}' = 1$ for all $\tilde{b}' \in cl(ri(\tilde{B})) = \tilde{B}$, hence $a \in \tilde{\tilde{B}} = B$. □

We call $\tilde{B}$ the dual section of B. The section B defines a base norm $\|\cdot\|_B$ in span(B). Next we show that this norm can be naturally extended to all $B_h(\mathcal{H})$. For this, let us define

$$\mathcal{O}_B := \{x \in B_h(\mathcal{H}), x = x_1 - x_2,\ x_1, x_2 \in B(\mathcal{H})^+, x_1 + x_2 \in B\}. \tag{9}$$

For $b \in B(\mathcal{H})^+$, we define $\mathcal{O}_b := \mathcal{O}_{\{b\}}$.

Lemma 4. We have

(i) $\mathcal{O}_B = \{x \in B_h(\mathcal{H}), \exists b' \in B,\ -b' \leq x \leq b'\} = \bigcup_{b' \in B} \mathcal{O}_{b'}$.
(ii) *The unit ball of the base norm* $\|\cdot\|_B$ *is* $\mathcal{O}_B \cap \mathrm{span}(B)$.

Proof. (i) Let $x = x_1 - x_2$ with $x_1 + x_2 = b' \in B$, then $-b' = -(x_1 + x_2) \leq x \leq x_1 + x_2 = b'$. Conversely, let $-b' \leq x \leq b'$ for some $b' \in B$. Put $x_\pm = 1/2(b' \pm x)$, then $x_\pm \in B(\mathcal{H})^+$, $x_+ - x_- = x$, and $x_+ + x_- = b' \in B$.

(ii) By definition, the unit ball of $\|\cdot\|_B$ is the set of elements of the form $x = \lambda b_1 - (1-\lambda)b_2$, $b_1, b_2 \in B$, $0 \leq \lambda \leq 1$. Then clearly $x \in \mathcal{O}_B$, by putting $x_1 = \lambda b_1$ and $x_2 = (1-\lambda)b_2$. Conversely, let $x \in \mathrm{span}(B)$ be such that $-b' \leq x \leq b'$ for some $b' \in B$, then $x_\pm = 1/2(b' \pm x)$ are positive elements in span(B) and we have $x_\pm = \lambda_\pm b_\pm$, for $\lambda_\pm \geq 0$, $b_\pm \in B$. By applying the order unit e_B to the equality $b' = x_+ + x_-$, we see that we must have $\lambda_+ + \lambda_- = 1$, so that $\|x\|_B \leq 1$. □

Theorem 1. *Let B be a faithful section and let $\tilde{B}$ be the dual section. Then $\mathcal{O}_B$ is the unit ball of a norm in $B_h(\mathcal{H})$. The unit ball of the dual norm is $\mathcal{O}_{\tilde{B}}$.*

We will denote this norm by $\|\cdot\|_B$, note that Lemma 4 (ii) justifies this notation.

Proof. It is clear that $\mathcal{O}_B$ is convex and symmetric, that is, $-\mathcal{O}_B \subseteq \mathcal{O}_B$. Since B is compact, $\mathcal{O}_B$ is closed. If $x \in \mathcal{O}_B$, then $x = x_1 - x_2$ with $x_1, x_2 \geq 0$, $x_1 + x_2 \in B$ and by (6),

$$\|x\|_{S_{\tilde{b}}} \leq \|x_1\|_{S_{\tilde{b}}} + \|x_2\|_{S_{\tilde{b}}} = \mathrm{Tr}\,(x_1 + x_2)\tilde{b} = 1,$$

for any $\tilde{b} \in ri(\tilde{B})$, hence $\mathcal{O}_B$ is bounded. Moreover, since $b \in ri(B)$ is an order unit, for every $x \in B_h(\mathcal{H})$ there is some $t > 0$ such that $-tb \leq x \leq tb$, so that $x \in t\mathcal{O}_B$ (see Lemma 4 (i)). This means that $\mathcal{O}_B$ is absorbing . These facts imply that $\mathcal{O}_B$ is the unit ball of a norm.

To show duality of the norms $\|\cdot\|_B$ and $\|\cdot\|_{\tilde{B}}$, let $\mathcal{H}_2 = \mathcal{H} \oplus \mathcal{H}$ and let $\Phi : B_h(\mathcal{H}_2) \to B_h(\mathcal{H})$ be the map defined by $\Phi(a \oplus b) = a + b$. Let $J_2 = \Phi^{-1}(\mathrm{span}(B))$, then J_2 is a subspace in $B_h(\mathcal{H}_2)$ and

$$J_2^\perp = \Phi^*(B^\perp) = \{x \oplus x,\ x \in B^\perp\},$$

see Ref. 13. Let $\pi_2 : B(\mathcal{H}_2) \to J_2^* = B(\mathcal{H}_2)|_{J_2^\perp}$ be the quotient map.

Let $\tilde{b} \in ri(\tilde{B})$ and put $B_2 = J_2 \cap S_{\tilde{b}\oplus\tilde{b}}$. Then B_2 is a base of $Q_2 = J_2 \cap B(\mathcal{H}_2)^+$ and it is clear that for $w_1, w_2 \in B(\mathcal{H})^+$, $w_1 \oplus w_2 \in B_2$ if and only if $w_1 + w_2 \in B$. Let now $a \in B_h(\mathcal{H})$, then $a \in \mathcal{O}_B^\circ$ if and only if $\mathrm{Tr}\,(a \oplus -a)w \leq 1$ for all $w \in B_2$. Equivalently,

$$\pi_2(a \oplus -a) \leq_{Q_2^*} e_{B_2} = \pi_2(\tilde{b} \oplus \tilde{b}),$$

that is, there is some $v \in J_2^\perp$ such that $a \oplus -a \leq \tilde{b} \oplus \tilde{b} + v$. Since $v = x \oplus x$, $x \in B^\perp$, we obtain $\pm a \leq \tilde{b} + x$. Note that we must have $\tilde{b} + x \geq 0$: if c is any element in $B(\mathcal{H})^+$, then we have $\pm\,\mathrm{Tr}\,ca \leq \mathrm{Tr}\,c(b + x)$, so that $\mathrm{Tr}\,c(b + x)$ cannot be negative. Hence $\pm a \leq \tilde{b} + x \in \tilde{B}$, so that $a \in \mathcal{O}_{\tilde{B}}$, by Lemma 4 (i). This shows that $\mathcal{O}_B^\circ \subseteq \mathcal{O}_{\tilde{B}}$. Conversely, it is easy to see that if $-b \leq x \leq b$ and $-\tilde{b} \leq y \leq \tilde{b}$ for $b \in B$, $\tilde{b} \in \tilde{B}$, then $\mathrm{Tr}\,xy \leq \mathrm{Tr}\,b\tilde{b} = 1$, this implies the opposite inclusion. □

Corollary 2. Let $x \in B_h(\mathcal{H})$. Then

(i) $\mathcal{O}_B = \bigcap_{\tilde{b}\in ri(\tilde{B})} \mathcal{O}_{S_{\tilde{b}}}$,
(ii) $\|x\|_B = \sup_{\tilde{b}\in ri(\tilde{B})} \|x\|_{S_{\tilde{b}}} = \sup_{\tilde{b}\in\tilde{B}} \|\tilde{b}^{1/2} x \tilde{b}^{1/2}\|_1$,
(iii) $\|x\|_B = \in f_{b\,\in\, ri(B)} \|x\|_b = \in f_{b\,\in\, B} \|b^{-1/2} x b^{-1/2}\|$.

Proof. (i) It is easy to see from Lemma 4 that

$$\mathcal{O}_B = \bigcup_{b\in B} \mathcal{O}_b = cl(\bigcup_{b\in ri(B)} \mathcal{O}_b). \tag{10}$$

Indeed, let $x \in B_h(\mathcal{H})$ be such that $-b \leq x \leq b$ for some $b \in B$ and let $b' \in ri(B)$, then $b_\epsilon := \epsilon b' + (1 - \epsilon)b \in ri(B)$ for all $0 < \epsilon < 1$. Let $x' \in \mathcal{O}_{b'}$ be any element, then $x_\epsilon := \epsilon x' + (1 - \epsilon)x \in \mathcal{O}_{b_\epsilon}$ and $x = \lim_{\epsilon\to 0^+} x_\epsilon \in cl(\bigcup_{b\in ri(B)} \mathcal{O}_b)$.

Since $A^\circ = (cl(conv(A)))^\circ$ for any subset $A \in B_h(\mathcal{H})$ containing 0, we obtain by Theorem 1 that

$$\mathcal{O}_B = \mathcal{O}_{\tilde{B}}^\circ = (\bigcup_{\tilde{b}\in ri(\tilde{B})} \mathcal{O}_{\tilde{b}})^\circ = \bigcap_{\tilde{b}\in ri(\tilde{B})} \mathcal{O}_{\tilde{b}}^\circ = \bigcap_{\tilde{b}\in ri(\tilde{B})} \mathcal{O}_{S_{\tilde{b}}}.$$

(ii) Since $\mathcal{O}_B$ is the unit ball of $\|\cdot\|_B$, we get from (i)

$$\begin{aligned}\|x\|_B &= \inf\{\lambda > 0, x \in \lambda\mathcal{O}_B\} = \inf\{\lambda > 0, x \in \lambda\mathcal{O}_{S_{\tilde{b}}}, \forall\tilde{b} \in ri(\tilde{B})\}\\ &= \inf\{\lambda > 0, \lambda \geq \|x\|_{S_{\tilde{b}}}, \forall\tilde{b} \in ri(\tilde{B})\} = \sup_{\tilde{b}\in ri(\tilde{B})} \|x\|_{S_{\tilde{b}}} = \sup_{\tilde{b}\in\tilde{B}} \|\tilde{b}^{1/2} x \tilde{b}^{1/2}\|_1,\end{aligned}$$

the last equality follows from (6) and continuity of the norm $\|\cdot\|_1$.

(iii) On the other hand, we get from Lemma 4 and (10)

$$\begin{aligned}\|x\|_B &= \inf\{\lambda > 0, x \in \lambda\mathcal{O}_B\} = \inf\{\lambda > 0, x \in \lambda \cup_{b\in ri(B)} \mathcal{O}_b\}\\ &= \inf_{b\in ri(B)} \inf\{\lambda > 0, x \in \lambda\mathcal{O}_b\} = \inf_{b\in ri(B)} \|x\|_b = \inf_{b\in B} \|b^{-1/2} x b^{-1/2}\|,\end{aligned}$$

where the last equality follows by (7). □

Corollary 3. For $a \in B(\mathcal{H})^+$, we have

$$\|a\|_B = \sup_{\tilde{b}\in\tilde{B}} \mathrm{Tr}\,a\tilde{b} = \inf_{b\in B} 2^{D_{max}(a\|b)}.$$

Proof. We have

$$\|a\|_B = \sup_{x\in\mathcal{O}_{\tilde{B}}} \mathrm{Tr}\,ax.$$

Let $x \in \mathcal{O}_{\tilde{B}}$, then $x = x_1 - x_2$, $x_1, x_2 \in B(\mathcal{H})^+$ and $x_1 + x_2 =: \tilde{b}_x \in \tilde{B}$, so that

$$\mathrm{Tr}\,ax \leq \mathrm{Tr}\,ax_1 \leq \mathrm{Tr}\,a\tilde{b}_x \leq \sup_{\tilde{b}\in\tilde{B}} \mathrm{Tr}\,a\tilde{b} \leq \sup_{y\in\mathcal{O}_{\tilde{B}}} \mathrm{Tr}\,ay = \|a\|_B.$$

Hence $\|a\|_B = \sup_{\tilde{b}\in\tilde{B}} \mathrm{Tr}\, a\tilde{b}$. The second equality follows directly from Corollary 2 (iii) and the definition of D_{max}. □

We can also characterize the maximizer resp. minimizer in Corollary 3.

Corollary 4. Let $a \in B(\mathcal{H})^+$.

(i) *Let $\tilde{b}_0 \in \tilde{B}$, then $\|a\|_B = \mathrm{Tr}\, a\tilde{b}_0$ if and only if there exists some $q \in \mathrm{span}(B)$, such that $a \leq q$ and $(q-a)\tilde{b}_0 = 0$. In this case, $q = \|a\|_B b_0$, $b_0 \in B$, and $\|a\|_B = 2^{D_{max}(a\|b_0)}$.*

(ii) *Let $b_0 \in B$, then $\|a\|_B = 2^{D_{max}(a\|b_0)}$ if and only if there exists some $t > 0$ and $\tilde{b}_0 \in \tilde{B}$, such that $a \leq tb_0$ and $(tb_0 - a)\tilde{b}_0 = 0$. In this case, $t = \|a\|_B = \mathrm{Tr}\, a\tilde{b}_0$.*

Proof. (i) Let $\tilde{b}_0 \in \tilde{B}$ be such that $\|a\|_B = \mathrm{Tr}\, a\tilde{b}_0$. Let $b_0 \in B$ be such that $\|a\|_B = 2^{D_{max}(a\|b_0)}$, in particular, $a \leq \|a\|_B b_0$. Put $q = \|a\|_B b_0$, then $q - a \geq 0$ and $\mathrm{Tr}\,(q-a)\tilde{b}_0 = 0$. Since also $\tilde{b}_0 \geq 0$, it follows that $(q-a)\tilde{b}_0 = 0$.

Conversely, suppose $q \in \mathrm{span}(B)$ satisfies $a \leq q$ and $(q-a)\tilde{b}_0 = 0$. Then $q = sb_0$ for some $b_0 \in B$, $s \geq 0$. Since $a \leq sb_0$, we have

$$\|a\|_B \leq s = \mathrm{Tr}\, a\tilde{b}_0 \leq \|a\|_B,$$

so that $\mathrm{Tr}\, a\tilde{b}_0 = \|a\|_B = s = 2^{D_{max}(a\|b_0)}$.

(ii) is proved similarly. □

III. GENERALIZED CHANNELS

Let B be a section of a base of $B(\mathcal{H})^+$. A generalized channel with respect to B (or a B-channel) is a completely positive map $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ such that $\Phi(B) \subseteq \mathfrak{S}(\mathcal{K})$. Let X_Φ be the Choi matrix of Φ, then Φ is a generalized channel with respect to B if and only if $X_\Phi \geq 0$ and

$$1 = \mathrm{Tr}\,\Phi(b) = \mathrm{Tr}\,\mathrm{Tr}_{\mathcal{H}}[(I\otimes b^{\mathsf{T}})X_\Phi] = \mathrm{Tr}\,(I\otimes b^{\mathsf{T}})X_\Phi = \mathrm{Tr}\, b^{\mathsf{T}}\mathrm{Tr}_{\mathcal{K}} X_\Phi,$$

for all $b \in B$. Let $\mathcal{C}_B(\mathcal{H},\mathcal{K})$ denote the set of Choi matrices of all generalized channels with respect to B, then

$$\mathcal{C}_B(\mathcal{H},\mathcal{K}) = \{X \in B(\mathcal{K}\otimes\mathcal{H})^+, \mathrm{Tr}_{\mathcal{K}} X \in \tilde{B}^{\mathsf{T}}\}.$$

Let us remark that if B is a section, then $B^{\mathsf{T}} := \{b^{\mathsf{T}}, b \in B\}$ is a section as well, here b^{T} denotes the transpose of b. Moreover, $\widetilde{B^{\mathsf{T}}} = \tilde{B}^{\mathsf{T}}$. Note also that we have

$$\mathcal{C}_B(\mathcal{H},\mathbb{C}) = \tilde{B}^{\mathsf{T}}, \tag{11}$$

so that, in particular, $\mathcal{C}_B(\mathcal{H},\mathbb{C})$ is a section.

Proposition 1. Let B be a faithful section of a base of $B(\mathcal{H})^+$. Then $\mathcal{C}_B(\mathcal{H},\mathcal{K})$ is a faithful section of a base of $B(\mathcal{K}\otimes\mathcal{H})^+$ and $\widetilde{\mathcal{C}_B(\mathcal{H},\mathcal{K})} = \{I_{\mathcal{K}}\otimes b^{\mathsf{T}},\ b \in B\}$.

Proof. It is easy to see that $I_{\mathcal{K}}\otimes B^{\mathsf{T}} = \{I_{\mathcal{K}}\otimes b^{\mathsf{T}}, b\in B\}$ is a faithful section of a base of $B(\mathcal{K}\otimes\mathcal{H})^+$ and

$$\mathcal{C}_B(\mathcal{H},\mathcal{K}) = \{X \in B(\mathcal{K}\otimes\mathcal{H})^+, \mathrm{Tr}\, X(I\otimes b^{\mathsf{T}}) = 1, \forall b\in B\} = \widetilde{I_{\mathcal{K}}\otimes B^{\mathsf{T}}}.$$

The proof now follows by Lemma 3 (i) and (ii). □

Let now $X \in B_h(\mathcal{K}\otimes\mathcal{H})$ and let $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ be the corresponding Hermitian map. By Corollary 2 and Proposition 1,

$$\|X\|_{\mathcal{C}_B(\mathcal{H},\mathcal{K})} = \sup_{b\in B} \|(I\otimes(b^{\mathsf{T}})^{1/2})X(I\otimes(b^{\mathsf{T}})^{1/2})\|_1$$

and we have

$$(I\otimes(b^{\mathsf{T}})^{1/2})X(I\otimes(b^{\mathsf{T}})^{1/2}) = (\Phi\otimes id_{\mathcal{H}})(\sigma_b),$$

where $\sigma_b = |\psi_b\rangle\langle\psi_b|$, with

$$|\psi_b\rangle = \sum_i |i\rangle \otimes (b^{\mathsf{T}})^{1/2}|i\rangle = \sum_i b^{1/2}|i\rangle \otimes |i\rangle \in \mathcal{H}\otimes\mathcal{H}.$$

Hence $\sigma_b \in B(\mathcal{H}\otimes\mathcal{H})^+$ and $\mathrm{Tr}\,_1\sigma_b = b^{\mathsf{T}} \in B^{\mathsf{T}}$, so that $\sigma_b \in \mathcal{C}_{\tilde{B}}(\mathcal{H},\mathcal{H})$. Conversely, if $\sigma = |\varphi\rangle\langle\varphi| \in \mathcal{C}_{\tilde{B}}(\mathcal{H},\mathcal{L})$ for some Hilbert space $\mathcal{L}$, then there is some linear map $R:\mathcal{H}\to\mathcal{L}$ satisfying $R^*R = b \in B$ and such that $|\varphi\rangle = \sum_i R|i\rangle\otimes|i\rangle$. Let $U:\mathcal{H}\to\mathcal{L}$ be an isometry such that $R = Ub^{1/2}$, then

$$|\varphi\rangle = \sum_i R|i\rangle\otimes|i\rangle = \sum_i Ub^{1/2}|i\rangle\otimes|i\rangle = (U\otimes I)|\psi_b\rangle.$$

Theorem 2. *Let $X \in B_h(\mathcal{K}\otimes\mathcal{H})$ and let Φ be the corresponding Hermitian map $B(\mathcal{H})\to B(\mathcal{K})$. Let $\mathcal{L}$ be any Hilbert space with* $\dim(\mathcal{L}) = \dim(\mathcal{H})$. *Then*

$$\|X\|_{\mathcal{C}_B(\mathcal{H},\mathcal{K})} = \sup_{\dim(\mathcal{L}')<\infty}\ \sup_{\sigma\in\mathcal{C}_{\tilde{B}}(\mathcal{H},\mathcal{L}')} \|(\Phi\otimes id_{\mathcal{L}'})(\sigma)\|_1$$
$$= \sup_{\sigma\in\mathcal{C}_{\tilde{B}}(\mathcal{H},\mathcal{L})} \|(\Phi\otimes id_{\mathcal{L}})(\sigma)\|_1,$$

and the dual norm is $\|X\|^*_{\mathcal{C}_B(\mathcal{H},\mathcal{K})} = \|X\|_{I_{\mathcal{K}}\otimes B^{\mathsf{T}}}$. Moreover, if $X\geq 0$ then

$$\|X\|_{\mathcal{C}_B(\mathcal{H},\mathcal{K})} = \sup_{b\in B}\mathrm{Tr}\,\Phi(b) = \inf_{Y\in\mathcal{C}_B(\mathcal{H},\mathcal{K})} 2^{D_{max}(X\|Y)}$$

and

$$\|X\|_{I\otimes B^{\mathsf{T}}} = \inf_{b\in B} 2^{D_{max}(X\|I\otimes b^{\mathsf{T}})} = \sup_{Y\in\mathcal{C}_B(\mathcal{H},\mathcal{K})}\mathrm{Tr}\,XY = \sup_S\langle\psi|X_{S^*\circ\Phi}|\psi\rangle,$$

where the last supremum is taken over the set of all B-channels $B(\mathcal{H})\to B(\mathcal{K})$.

Proof. From what was said above, it is easy to see that

$$\|X\|_{\mathcal{C}_B(\mathcal{H},\mathcal{K})} = \sup_{|\varphi\rangle\langle\varphi|\in\mathcal{C}_{\tilde{B}}(\mathcal{H},\mathcal{L})}\|(\Phi\otimes id_{\mathcal{L}})(|\varphi\rangle\langle\varphi|)\|_1,$$

with $\dim(\mathcal{L}) = \dim(\mathcal{H})$. We will show that

$$\sup_{|\varphi\rangle\langle\varphi|\in\mathcal{C}_{\tilde{B}}(\mathcal{H},\mathcal{L}')}\|(\Phi\otimes id_{\mathcal{L}'})(|\varphi\rangle\langle\varphi|)\|_1 \leq \sup_{|\varphi\rangle\langle\varphi|\in\mathcal{C}_{\tilde{B}}(\mathcal{H},\mathcal{L})}\|(\Phi\otimes id_{\mathcal{L}})(|\varphi\rangle\langle\varphi|)\|_1,$$

whenever $\dim(\mathcal{L}')\geq\dim(\mathcal{L})$. The proof is almost the same as the proof of Theorem 5 of Ref. 19, we include it here for completeness.

So let $\dim(\mathcal{L}')\geq\dim(\mathcal{L}) = \dim(\mathcal{H})$, then there is some $\varphi_0\in\mathcal{H}\otimes\mathcal{L}'$, with $|\varphi_0\rangle\langle\varphi_0|\in\mathcal{C}_{\tilde{B}}(\mathcal{H},\mathcal{L}')$ such that

$$\sup_{|\varphi\rangle\langle\varphi|\in\mathcal{C}_{\tilde{B}}(\mathcal{H},\mathcal{L}')}\|\Phi\otimes id_{\mathcal{L}'}(|\varphi\rangle\langle\varphi|)\|_1 = \|\Phi\otimes id_{\mathcal{L}'}(|\varphi_0\rangle\langle\varphi_0|)\|_1.$$

Let $|\varphi_0\rangle = \sum_{i=1}^m s_i|\varphi_i\rangle\otimes|\xi_i\rangle$ be the Schmidt decomposition of φ_0, with $\{|\varphi_i\rangle\}$ and $\{|\xi_i\rangle\}$ orthonormal sets in $\mathcal{H}$ resp. $\mathcal{L}'$ and $m = \dim(\mathcal{H})$. Then $|\varphi_0\rangle\langle\varphi_0| = \sum_{i,j}|\varphi_i\rangle\langle\varphi_j|\otimes|\xi_i\rangle\langle\xi_j|$ and

$$(\mathrm{Tr}\,_{\mathcal{L}'}|\varphi_0\rangle\langle\varphi_0|)^{\mathsf{T}} = (\sum_i s_i|\varphi_i\rangle\langle\varphi_i|)^{\mathsf{T}} \in \tilde{\tilde{B}} = B.$$

Let $\{|e_i\rangle,\ i = 1,\ \ldots,\ m\}$ be an ONB in $\mathcal{L}$. Define the linear map $U:\mathcal{L}'\to\mathcal{L}$ by $U = \sum_{i=1}^m|e_i\rangle\langle\xi_i|$, then $U^*U = \sum_i|\xi_i\rangle\langle\xi_i|$ is the projection in $\mathcal{L}'$ onto the subspace spanned by the vectors $|\xi_i\rangle$, $i = 1,\ldots,m$, and $(I\otimes U^*U)|\varphi_0\rangle = |\varphi_0\rangle$. Put $\varphi_U := (I\otimes U)|\varphi_0\rangle = \sum_i|\varphi_i\rangle\otimes|e_i\rangle$, then it is easy to see that $|\varphi_U\rangle\langle\varphi_U| \in \mathcal{C}_{\tilde{B}}(\mathcal{H}, \mathcal{L})$. Now we have

$$\begin{aligned}\sup_{|\varphi\rangle\langle\varphi| \in \mathcal{C}_{\tilde{B}}(\mathcal{H},\mathcal{L})} \|\Phi \otimes id_{\mathcal{L}}(|\varphi\rangle\langle\varphi|)\|_1 &\geq \|\Phi \otimes id_{\mathcal{L}}(|\varphi_U\rangle\langle\varphi_U|)\|_1 \\ &\geq \|(I \otimes U^*)(\Phi \otimes id_{\mathcal{L}})(|\varphi_U\rangle\langle\varphi_U|)(I \otimes U)\|_1 \\ &= \|\Phi \otimes id_{\mathcal{L}'}((I \otimes U^*)|\varphi_U\rangle\langle\varphi_U|(I \otimes U))\|_1 \\ &= \|\Phi \otimes id_{\mathcal{L}'}(|\varphi_0\rangle\langle\varphi_0|)\|_1 \\ &= \sup_{|\varphi\rangle\langle\varphi| \in \mathcal{C}_{\tilde{B}}(\mathcal{H},\mathcal{L}')} \|\Phi \otimes id_{\mathcal{L}'}(|\varphi\rangle\langle\varphi|)\|_1.\end{aligned}$$

Next, let Y be any element in $\mathcal{C}_{\tilde{B}}(\mathcal{H}, \mathcal{L}')$, then the corresponding map $\xi : B(\mathcal{H}) \to B(\mathcal{L}')$ has the form

$$\xi(a) = \sum_{i=1}^{N} V_i a V_i^*, \qquad a \in B(\mathcal{H}),$$

where $V_i : \mathcal{H} \to \mathcal{L}'$ are linear maps such that $\sum_i V_i^* V_i \in B$. Let $\mathcal{L}'_0$ be a Hilbert space with $\dim(\mathcal{L}'_0) = N$ and let $\{|f_j\rangle, j = 1, \ldots, N\}$ be an ONB in $\mathcal{L}'_0$. Define $V = \sum_{j=1}^{N} V_j \otimes |f_j\rangle$, then V is a linear map $\mathcal{H} \to \mathcal{L}' \otimes \mathcal{L}'_0$ with $V^*V = \sum_i V_i^* V_i \in B$. Let $\mathcal{V}(a) = VaV^*$ and let Z be the Choi matrix of $\mathcal{V}$, then Z is a rank one element in $\mathcal{C}_{\tilde{B}}(\mathcal{H}, \mathcal{L}' \otimes \mathcal{L}'_0)$. Moreover, $\xi(a) = \mathrm{Tr}_{\mathcal{L}'_0} VaV^*$ and $Y = \mathrm{Tr}_{\mathcal{L}'_0} Z$. It follows that

$$\begin{aligned}\|(\Phi \otimes id_{\mathcal{L}'})(Y)\|_1 &= \|(\Phi \otimes id_{\mathcal{L}'})(\mathrm{Tr}_{\mathcal{L}'_0} Z)\|_1 = \|\mathrm{Tr}_{\mathcal{L}'_0}(\Phi \otimes id_{\mathcal{L}' \otimes \mathcal{L}'_0})(Z)\|_1 \\ &\leq \|(\Phi \otimes id_{\mathcal{L}' \otimes \mathcal{L}'_0})(Z)\|_1 \leq \|X\|_{\mathcal{C}_B(\mathcal{H},\mathcal{K})}.\end{aligned}$$

We now have

$$\begin{aligned}\|X\|_{\mathcal{C}_B(\mathcal{H},\mathcal{K})} &= \sup_{|\varphi\rangle\langle\varphi| \in \mathcal{C}_{\tilde{B}}(\mathcal{H},\mathcal{L})} \|(\Phi \otimes id_{\mathcal{L}})(|\varphi\rangle\langle\varphi|)\|_1 \leq \sup_{\sigma \in \mathcal{C}_{\tilde{B}}(\mathcal{H},\mathcal{L})} \|(\Phi \otimes id_{\mathcal{L}})(\sigma)\|_1 \\ &\leq \sup_{\dim(\mathcal{L}')<\infty} \sup_{\sigma \in \mathcal{C}_{\tilde{B}}(\mathcal{H},\mathcal{L}')} \|(\Phi \otimes id_{\mathcal{L}'})(\sigma)\|_1 \leq \|X\|_{\mathcal{C}_B(\mathcal{H},\mathcal{K})}.\end{aligned}$$

The expression for the dual norm follows by Proposition 1. Suppose now that $X \geq 0$, then by Corollary 3

$$\|X\|_{\mathcal{C}_B(\mathcal{H},\mathcal{K})} = \sup_{b \in B} \mathrm{Tr}\, X(I \otimes b^{\mathsf{T}}) = \inf_{Y \in \mathcal{C}_B(\mathcal{H},\mathcal{K})} 2^{D_{max}(X\|Y)},$$

$$\|X\|_{I \otimes B^{\mathsf{T}}} = \sup_{Y \in \mathcal{C}_B(\mathcal{H},\mathcal{K})} \mathrm{Tr}\, XY = \inf_{b \in B} 2^{D_{max}(X\|I \otimes b^{\mathsf{T}})}.$$

By (1), $\mathrm{Tr}\, X(I \otimes b^{\mathsf{T}}) = \mathrm{Tr}\,\mathrm{Tr}_{\mathcal{H}} X(I \otimes b^{\mathsf{T}}) = \mathrm{Tr}\,\Phi(b)$. Moreover, let $Y \in \mathcal{C}_B(\mathcal{H}, \mathcal{K})$ and let S be the corresponding B-channel, then

$$\mathrm{Tr}\, XY = \mathrm{Tr}\, X(S \otimes id)(\Psi) = \mathrm{Tr}\,(S^* \otimes id)(X)\Psi = \langle\psi, X_{S^* \circ \Phi}\psi\rangle.$$

□

A. Channels

Let $B = \mathfrak{S}(\mathcal{H})$, then generalized channels are the usual channels. In this case, we denote $\mathcal{C}_B(\mathcal{H}, \mathcal{K})$ by $\mathcal{C}(\mathcal{H}, \mathcal{K})$. Note that $\tilde{B} = \{I\}$ and $\mathcal{C}_{\tilde{B}}(\mathcal{H}, \mathcal{K}) = \mathfrak{S}(\mathcal{K} \otimes \mathcal{H})$.

By Proposition 1, $\mathcal{C}(\mathcal{H}, \mathcal{K})$ is a faithful section of a base of $B(\mathcal{K} \otimes \mathcal{H})^+$ and

$$\widetilde{\mathcal{C}(\mathcal{H}, \mathcal{K})} = \{I_{\mathcal{K}} \otimes \rho, \rho \in \mathfrak{S}(\mathcal{H})\}.$$

Furthermore, let $X \in B_h(\mathcal{K} \otimes \mathcal{H})$ and let $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ be the corresponding Hermitian map. Then by Theorem 2,

$$\|X\|_{\mathcal{C}(\mathcal{H},\mathcal{K})} = \sup_{\sigma \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{L})} \|(\Phi \otimes id_{\mathcal{L}})(\sigma)\|_1 = \|\Phi\|_{\diamond},$$

with $\dim(\mathcal{L}) = \dim(\mathcal{H})$. For the dual norm, we have

$$\|X\|_{I\otimes\mathfrak{S}(\mathcal{H})} = \inf_{\rho\in\mathfrak{S}(\mathcal{H})} \inf\{\lambda > 0,\ -\lambda(I\otimes\rho) \le X \le \lambda(I\otimes\rho)\}.$$

If $\sigma \in B(\mathcal{K}\otimes\mathcal{H})^+$, we obtain

$$\|\sigma\|_{I\otimes\mathfrak{S}(\mathcal{H})} = \inf_{\rho\in\mathfrak{S}(\mathcal{H})} 2^{D_{max}(\sigma\|I\otimes\rho)} = 2^{-H_{min}(\mathcal{K}|\mathcal{H})_\sigma},$$

where $H_{min}(\mathcal{K}|\mathcal{H})_\sigma$ is the conditional min-entropy, see Ref. 17.

B. Quantum supermaps

Let $\mathcal{H}_0, \mathcal{H}_1, \dots$ be a sequence of finite dimensional Hilbert spaces. For each $n \ge 1$, we define the sets $\mathcal{C}(\mathcal{H}_0, \dots, \mathcal{H}_n)$ as follows: $\mathcal{C}(\mathcal{H}_0, \mathcal{H}_1)$ is, as before, the set of Choi matrices of channels $B(\mathcal{H}_0) \to B(\mathcal{H}_1)$. For $n > 1$, we define $\mathcal{C}(\mathcal{H}_0, \dots, \mathcal{H}_n)$ as the set of Choi matrices of cp maps $B(\mathcal{H}_{n-1}\otimes\cdots\otimes\mathcal{H}_0) \to B(\mathcal{H}_n)$ that map $\mathcal{C}(\mathcal{H}_0, \dots, \mathcal{H}_{n-1})$ into $\mathfrak{S}(\mathcal{H}_n)$. Such maps were called quantum supermaps in Ref. 13. (Note that this definition is slightly different from the notion of supermap introduced in Ref. 5, which is a cp map that maps Choi matrices of channels to Choi matrices of channels.) and it was proved that for $n = 2N - 1$ we get precisely the set of deterministic quantum N-combs for the sequence $\{\mathcal{H}_0, \dots, \mathcal{H}_{2N-1}\}$, Theorem 7 of Ref. 13. If $n = 2N$, we get the set of $N+1$-combs for $\{\mathbb{C}, \mathcal{H}_0, \dots, \mathcal{H}_{2N}\}$.

Let us fix the sequence $\mathcal{H}_0, \mathcal{H}_1, \dots$ and for this, put $\mathcal{C}_n = \mathcal{C}(\mathcal{H}_0, \dots, \mathcal{H}_n)$. By using repeatedly Proposition 1, we see that $\mathcal{C}_n$ is a faithful section of a base of $B(\mathcal{H}_n\otimes\cdots\otimes\mathcal{H}_0)^+$ and

$$\mathcal{C}_{n+1} = \mathcal{C}_{\mathcal{C}_n}(\mathcal{H}_n\otimes\cdots\otimes\mathcal{H}_0, \mathcal{H}_{n+1}).$$

Moreover, by Proposition 1,

$$\widetilde{\mathcal{C}_n} = I_{\mathcal{H}_n}\otimes\mathcal{C}_{n-1} = \mathcal{C}(\mathcal{H}_0, \dots, \mathcal{H}_n, \mathbb{C})$$

(note that $\mathcal{C}_{n-1}^{\mathsf{T}} = \mathcal{C}_{n-1}$, the last equality above follows from (11)). For $n = 2N - 1$, this corresponds to the set of N-round nonmeasuring co-strategies of Refs. 8 and 9. Note also that for any finite dimensional Hilbert space $\mathcal{L}'$,

$$\begin{aligned}
\mathcal{C}_{\tilde{\mathcal{C}}_n}(\mathcal{H}_n\otimes\cdots\otimes\mathcal{H}_0, \mathcal{L}') &= \{Y \ge 0, \mathrm{Tr}_{\mathcal{L}'} Y \in \mathcal{C}_n = \mathcal{C}_{\mathcal{C}_{n-1}}(\mathcal{H}_{n-1}\otimes\cdots\otimes\mathcal{H}_0, \mathcal{H}_n)\}\\
&= \{Y \ge 0, \mathrm{Tr}_{\mathcal{H}_n}(\mathrm{Tr}_{\mathcal{L}'} Y) \in \widetilde{\mathcal{C}_{n-1}}\}\\
&= \mathcal{C}(\mathcal{H}_0, \dots, \mathcal{H}_n\otimes\mathcal{L}').
\end{aligned}$$

Now we obtain the following expressions for the corresponding norm and its dual.

Theorem 3. *Let $n \ge 2$. Let $X \in B_h(\mathcal{H}_n\otimes\cdots\otimes\mathcal{H}_0)$ and let $\Phi : B(\mathcal{H}_{n-1}\otimes\cdots\otimes\mathcal{H}_0) \to B(\mathcal{H}_n)$ be the corresponding map. We have*

$$\begin{aligned}
\|X\|_{\mathcal{C}(\mathcal{H}_0,\dots,\mathcal{H}_n)} &= \sup_{Y_1, Y_2 \ge 0, Y_1+Y_2 \in \mathcal{C}(\mathcal{H}_0,\dots,\mathcal{H}_n,\mathbb{C})} \mathrm{Tr}\, X(Y_1 - Y_2)\\
&= \sup_{Y\in\mathcal{C}(\mathcal{H}_0,\dots,\mathcal{H}_n,\mathbb{C})} \|Y^{1/2}XY^{1/2}\|_1\\
&= \inf_{Y\in\mathcal{C}(\mathcal{H}_0,\dots,\mathcal{H}_n)} \inf\{\lambda > 0, -\lambda Y \le X \le \lambda Y\}\\
&= \sup_{\dim(\mathcal{L}')<\infty}\ \sup_{Y\in\mathcal{C}(\mathcal{H}_0,\dots,\mathcal{H}_{n-2},\mathcal{H}_{n-1}\otimes\mathcal{L}')} \|(\Phi\otimes id_{\mathcal{L}'})(Y)\|_1\\
&= \sup_{Y\in\mathcal{C}(\mathcal{H}_0,\dots,\mathcal{H}_{n-2},\mathcal{H}_{n-1}\otimes\mathcal{L})} \|(\Phi\otimes id_{\mathcal{L}})(Y)\|_1,
\end{aligned}$$

where $\dim(\mathcal{L}) = \dim(\mathcal{H}_{n-1}\otimes\cdots\otimes\mathcal{H}_0)$. Moreover, the dual norm is

$$\|X\|_{I_{\mathcal{H}_n}\otimes\mathcal{C}(\mathcal{H}_0,\dots,\mathcal{H}_{n-1})} = \|X\|_{\mathcal{C}(\mathcal{H}_0,\dots,\mathcal{H}_n,\mathbb{C})}.$$

Proof. Duality of the norms is obtained from Theorem 2, this also implies the first equality. Next two equalities follow by Corollary 2. The rest follows by Theorem 2. □

For $n = 2N - 1$, first two expressions are exactly the $N\diamond$-norm as obtained in Refs. 9 and 3. Duality of the norms corresponding to strategies and co-strategies was also obtained in Ref. 9.

IV. A GENERAL QUANTUM DECISION THEORY

As before, let B be a faithful section of a base of $B(\mathcal{H})^+$. As we have seen, elements of B may represent certain quantum devices and it is therefore reasonable to consider the following definitions.

Let $\{b_\theta, \theta \in \Theta\} \subset B$ be a parametrized family, for simplicity, we will suppose that the set of parameters Θ is finite. If B is the set of states, the pair $\mathcal{E} = (\mathcal{H}, \{b_\theta, \theta \in \Theta\})$ is called an experiment and is interpreted as an *a priori* information on the true state of the system. Accordingly, for a section B, we define a generalized experiment as a triple $\mathcal{E} = (\mathcal{H}, B, \{b_\theta, \theta \in \Theta\})$.

Another ingredient of decision theory is a (finite) set D, the set of possible decisions. A decision procedure $\mathbf{m}$ is a procedure by which we pick some decision $d \in D$, with probability based on the "true value" of b. That is, $\mathbf{m}$ is a map $B \to \mathcal{P}(D)$, where $\mathcal{P}(D)$ is the set of probability measures on D, such a map will be called a measurement on B, with values in D. The payoff obtained if $d \in D$ is chosen while the true value is $\theta \in \Theta$ is given by the payoff function $w : \Theta \times D \to [0, 1]$, the pair (D, w) is called a (classical) decision problem. Let λ be an *a priori* probability distribution on Θ. The task is to maximize the average payoff, that is the value of

$$\mathcal{L}_{\mathcal{E},\lambda,w}(\mathbf{m}) := \sum_{\theta,d} \lambda_\theta w(\theta, d)\mathbf{m}(b_\theta)_d \tag{12}$$

over all measurements $\mathbf{m} : B \to \mathcal{P}(D)$.

It is quite clear that any measurement $\mathbf{m}$ on B is given by a collection $\{\mathbf{m}_d, d \in D\}$ of elements in Q^* such that $\mathbf{m}(b)_d = \langle \mathbf{m}_d, b\rangle$ and that we must have $\sum_d \mathbf{m}_d = e_B$. Similarly as it was shown in Ref. 13, any measurement is given by a collection $\{M_d, d \in D\} \subset B(\mathcal{H})^+$ such that $\mathbf{m}_d = \pi(M_d)$ and $\pi(\sum_d M_d) = e_B$, that is

$$\sum_d M_d \in \pi^{-1}(e_B) \cap B(\mathcal{H})^+ = \tilde{B}.$$

Any such collection of positive operators will be called a generalized POVM (with respect to B), or a B-POVM. It is also clear that any B-POVM defines a measurement on B (but it may happen that different generalized POVMs define the same measurement, see Ref. 13). If $B = \mathfrak{S}(\mathcal{H})$, we obtain a (usual) POVM $M = \{M_d, d \in D\} \subset B(\mathcal{H})^+$, $\sum_d M_d = I$.

Let us denote by $\mathcal{M}_B(\mathcal{H}, D)$ the set of all generalized POVMs with respect to B with values in D and let $\{M_d, d \in D\} \in \mathcal{M}_B(\mathcal{H}, D)$. Let us denote

$$M = \sum_{d\in D} |d\rangle\langle d| \otimes M_d^{\mathsf{T}} \in B(\mathcal{H}_D \otimes \mathcal{H})^+, \tag{13}$$

where $\mathcal{H}_D$ is a Hilbert space with $\dim(\mathcal{H}_D) = |D|$ and $\{|d\rangle, d \in D\}$ an ONB in $\mathcal{H}_D$. Then it is clear that M is a block-diagonal element in $\mathcal{C}_B(\mathcal{H}, \mathcal{H}_D)$. Conversely, it is clear that if $X = \sum_d |d\rangle\langle d| \otimes X_d \in \mathcal{C}_B(\mathcal{H}, \mathcal{H}_D)$, then $\{X_d^{\mathsf{T}}, d \in D\} \in \mathcal{M}_B(\mathcal{H}, D)$. In this way, we identify $\mathcal{M}_B(\mathcal{H}, D)$ with the subset of block-diagonal elements in $\mathcal{C}_B(\mathcal{H}, \mathcal{H}_D)$.

Let now (D, w) be a decision problem and let $\mathbf{m}$ be a decision procedure with corresponding B-POVM M. Then the average payoff is computed as

$$\mathcal{L}_{\mathcal{E},\lambda,w}(\mathbf{m}) = \mathcal{L}_{\mathcal{E},\lambda,w}(M) := \sum_{\theta,d} \lambda_\theta w(\theta, d)\mathrm{Tr}\, M_d b_\theta = \mathrm{Tr}\, \xi_{\mathcal{E},\lambda,w} M^{\mathsf{T}},$$

where

$$\xi_{\mathcal{E},\lambda,w} = \sum_\theta \sum_d \lambda_\theta w(\theta, d)|d\rangle\langle d| \otimes b_\theta = \sum_d |d\rangle\langle d| \otimes \bar{b}_d \in B(\mathcal{H}_D \otimes \mathcal{H})^+,$$

where $\bar{b}_d := \sum_\theta \lambda_\theta w(\theta, d) b_\theta$.

More generally, let $\mathcal{D}$ be a Hilbert space, $\dim(\mathcal{D}) = k$ and let W be a function $W : \theta \mapsto W_\theta \in B(\mathcal{D})^+$, with $W_\theta \le I$. We call the pair $(\mathcal{D}, W)$ a quantum decision problem.[15] Mathematically, this is a natural extension of classical decision problems, but at present its operational relevance is not clear.

A decision procedure is now a B-channel $\Phi : B(\mathcal{H}) \to B(\mathcal{D})$ and the average payoff of Φ is given by

$$\mathcal{L}_{\mathcal{E},\lambda,W}(\Phi) = \sum_\theta \lambda_\theta \mathrm{Tr}\, \Phi(b_\theta) W_\theta.$$

If $X \in \mathcal{C}_B(\mathcal{H}, \mathcal{D})$ is the Choi matrix of Φ, then the average payoff has the form

$$\begin{aligned}\mathcal{L}_{\mathcal{E},\lambda,W}(\Phi) = \mathcal{L}_{\mathcal{E},\lambda,W}(X) &:= \sum_\theta \lambda_\theta \mathrm{Tr}\,(W_\theta \mathrm{Tr}_{\mathcal{H}}[(I_{\mathcal{D}} \otimes b_\theta^{\mathsf{T}})X]) \\ &= \sum_\theta \mathrm{Tr}\,(\lambda_\theta W_\theta \otimes b_\theta^{\mathsf{T}})X = \mathrm{Tr}\,\xi_{\mathcal{E},\lambda,W} X^{\mathsf{T}}, \end{aligned} \tag{14}$$

where

$$\xi_{\mathcal{E},\lambda,W} = \sum_\theta \lambda_\theta W_\theta^{\mathsf{T}} \otimes b_\theta \in B(\mathcal{D} \otimes \mathcal{H})^+.$$

It is easy to see that the set of quantum decision problems contains also classical ones: Let (D, w) be a classical decision problem and let $\mathcal{H}_D$ be as before. Let $W_\theta := \sum_{d\in D} w(\theta, d)|d\rangle\langle d|$, then $(\mathcal{H}_D, W)$ is a quantum decision problem and $\xi_{\mathcal{E},\lambda,W} = \xi_{\mathcal{E},\lambda,w}$. Let $X \in \mathcal{C}_B(\mathcal{H}, \mathcal{H}_D)$ and $X = \sum_{c,\, d\in D} |c\rangle\langle d| \otimes X_{cd}$ $X_{cd} \in B(\mathcal{H})$. Since $\xi_{\mathcal{E},\lambda,w}$ is block-diagonal, we have

$$\mathcal{L}_{\mathcal{E},\lambda,W}(X) = \mathcal{L}_{\mathcal{E},\lambda,w}(M),$$

where $M = \sum_d |d\rangle\langle d| \otimes X_{dd}$ is a B-POVM. In other words, for a classical decision problem one cannot get better results by considering quantum decision procedures. Conversely, let $(\mathcal{D}, W)$ be a quantum decision problem such that all the operators W_θ commute. Then there is a basis of $\mathcal{D}$ with respect to which all the operators W_θ are given by diagonal matrices, and the problem is equivalent to a classical problem, in the sense that we obtain the same average payoffs. Hence we can view the set of classical decision problems as the subset of quantum decision problems such that the payoff function W has commutative range.

Theorem 4. *Let $\mathcal{E} = (\mathcal{H}, B, \{b_\theta, \theta \in \Theta\})$ be a generalized experiment and let $(\mathcal{D}, W)$ be a quantum decision problem. Then the maximal average payoff is given by*

$$\mathcal{L}_{\mathcal{E},\lambda,W} := \max_{X\in\mathcal{C}_B(\mathcal{H},\mathcal{D})} \mathcal{L}_{\mathcal{E},\lambda,W}(X) = \|\xi_{\mathcal{E},\lambda,W}\|_{I_{\mathcal{D}}\otimes B}.$$

If $(\mathcal{D}, W)$ is classical, then

$$\mathcal{L}_{\mathcal{E},\lambda,W} = \inf_{b\in B} \sup_{d\in D} 2^{D_{\max}(\bar{b}_d \| b)}.$$

Proof. By (14), the maximal average payoff is given by

$$\mathcal{L}_{\mathcal{E},\lambda,W} = \max_{X\in\mathcal{C}_B(\mathcal{H},\mathcal{D})} \mathrm{Tr}\,\xi_{\mathcal{E},\lambda,W} X^{\mathsf{T}} = \|\xi_{\mathcal{E},\lambda,W}\|_{I_{\mathcal{D}}\otimes B},$$

the last equality follows by Corollary 3 and Proposition 1. If $(\mathcal{D}, W)$ is classical, then we may suppose that the matrices W_θ are diagonal. Then $\xi_{\mathcal{E},\lambda,W} = \sum_d |d\rangle\langle d| \otimes \bar{b}_d$ is block-diagonal. By Corollary 3, and definition of D_{max},

$$\begin{aligned}\|\xi_{\mathcal{E},\lambda,W}\|_{I_{\mathcal{D}}\otimes B} &= \inf_{b\in B} 2^{D_{max}(\xi_{\mathcal{E},\lambda,W}\|I_{\mathcal{D}}\otimes b)} = \inf_{b\in B} \inf\{\gamma > 0, \bar{b}_d \le 2^\gamma b, \forall d \in D\} \\ &= \inf_{b\in B} \sup_{d\in D} 2^{D_{max}(\bar{b}_d\|b)}.\end{aligned}$$

□

We can also use Corollary 4 to characterize decision procedures that maximize average payoff, we will call such procedures optimal with respect to $(\mathcal{E}, \lambda, W)$.

Corollary 5. Let $(\mathcal{D}, W)$ be a decision problem and let $X \in \mathcal{C}_B(\mathcal{H}, \mathcal{D})$. Then X is optimal with respect to $(\mathcal{E}, \lambda, W)$ if and only if there is some element $q \in \mathrm{span}(B)$ such that $\xi_{\mathcal{E},\lambda,W} \leq I_{\mathcal{D}} \otimes q$ and

$$((I \otimes q) - \xi_{\mathcal{E},\lambda,W})X^{\mathsf{T}} = 0. \tag{15}$$

If $(\mathcal{D}, W)$ is classical, then a B-POVM $(M_1, \ldots, M_{\dim(\mathcal{D})})$ is optimal if and only if there is some $q \in \mathrm{span}(B)$ such that $\bar{b}_d \leq q$ for all d and

$$q \sum_d M_d = \sum_d \bar{b}_d M_d. \tag{16}$$

Proof. The first part follows directly by Theorem 4 and Corollary 4. If $(\mathcal{D}, W)$ is classical, then $\xi_{\mathcal{E},\lambda,W}$ is block-diagonal, so that $\xi_{\mathcal{E},\lambda,W} \leq I \otimes q$ if and only if each block is majorized by q, that is, $\bar{b}_d \leq q$. Moreover, (16) implies that

$$\sum_d \mathrm{Tr}\,(q - \bar{b}_d)M_d = 0.$$

Since this is a sum of nonnegative elements, it is zero if and only if each summand is equal to zero. Again by positivity, this is equivalent to (15). □

In particular, in the case $B = \mathfrak{S}(\mathcal{H})$, we obtain the following optimality condition for POVMs.

Corollary 6. Let $\mathcal{E} = \{\sigma_\theta, \theta \in \Theta\}$ be an experiment and let (D, w) be a classical decision problem. Then a POVM $\{M_d, d \in D\}$ is optimal with respect to $(\mathcal{E}, \lambda, W)$ if and only if $q := \sum_d \bar{\sigma}_d M_d$ is hermitian and such that $\bar{\sigma}_\theta \leq q$ for all d, here $\bar{\sigma}_\theta := \sum_\theta \lambda_\theta \sigma_\theta w(\theta, d)$.

Remark 1. Sometimes the function W is interpreted as loss rather than payoff, then $\mathcal{L}_{\mathcal{E},\lambda,W}(\Phi)$ is the average loss of the procedure Φ which has to be minimized. Let $W'_\theta = I_{\mathcal{D}} - W_\theta$, then $\theta \mapsto W_\theta$ is again a payoff (or loss) function and we have

$$\begin{aligned}\min_\Phi \mathcal{L}_{\mathcal{E},\lambda,W} &= \min_\Phi \sum_\theta \lambda_\theta \mathrm{Tr}\,\Phi(b_\theta)W_\theta = \min_\Phi \sum_\theta \lambda_\theta \mathrm{Tr}\,\Phi(b_\theta)(I - W'_\theta)\\ &= 1 - \max_\Phi \mathcal{L}_{\mathcal{E},\lambda,W'}(\Phi) = 1 - \|\xi_{\mathcal{E},\lambda,W'}\|_{I_{\mathcal{D}}\otimes B}.\end{aligned}$$

Moreover, an optimal procedure Φ that minimizes the loss is a maximizer for $\mathcal{L}_{\mathcal{E},\lambda,W'}$, hence satisfies the conditions of Corollary 5, with W replaced by W'. Note that then the condition from Corollary 6 is the same as obtained in Ref. 12.

Let $\{M_d, d \in D\}$ be a B-POVM with $\sum_d M_d = c \in \tilde{B}$. Then since $0 \leq M_d \leq c$ for all d, we have

$$M_d = c^{1/2}\Lambda_d c^{1/2}, \qquad d \in D,$$

where $\Lambda_d := c^{-1/2}M_d c^{-1/2}$ defines a (usual) POVM on the support $\mathrm{supp}\,c$ of c. It follows that $\mathrm{Tr}\,xM_d = \mathrm{Tr}\,c^{1/2}xc^{1/2}\Lambda_d$, that is, we can decompose the measurement defined by $\{M_d\}$ into a cp map χ_c: $x \mapsto c^{1/2}xc^{1/2}$ followed by the usual measurement given by $\{\Lambda_d\}$. Note that $\chi_c \in \mathcal{C}_B(\mathcal{H}, \mathrm{supp}\,c)$ so that χ_c maps a generalized experiment $\mathcal{E} = (\mathcal{H}, B, \{b_\theta, \theta \in \Theta\})$ onto an ordinary experiment $\mathcal{E}_c := \{\mathrm{supp}\,c, \mathfrak{S}(\mathrm{supp}\,c), \{\chi_c(b_\theta), \theta \in \Theta\}\}$. We write this decomposition as $M = \Lambda \circ \chi_c$. Such a decomposition was also used in Ref. 3 in the case of testers and in Ref. 13 for generalized POVMs. Using this decomposition, we obtain the following optimality condition for B-POVMs.

Corollary 7. Let (D, w) be a classical decision problem and let $M \in \mathcal{M}_B(\mathcal{H}, D)$ with decomposition $M = \Lambda \circ \chi_c$. Suppose c is invertible and let $\mathcal{E}_c := (\mathcal{H}, \{\sigma_\theta := \chi_c(b_\theta), \theta \in \Theta\})$. Then M is optimal for $(\mathcal{E}, \lambda, w)$ if and only if Λ is optimal for $(\mathcal{E}_c, \lambda, w)$ and

$$\sum_d \bar{\sigma}_d \Lambda_d \in \mathrm{span}(\chi_c(B)),$$

where $\bar{\sigma}_d = \sum_\theta \lambda_\theta w(\theta, d)\sigma_\theta$.

Proof. Directly by Corollaries 5 and 6. □

Example 2 (Multiple hypothesis testing). Suppose a family $\{b_1, \dots, b_k\}$ of elements in B is given and the task is to decide which is the true one, moreover, given some $\lambda \in \mathcal{P}(\{1, \dots k\})$, we want to minimize the average probability of making an error. In this case, put $\mathcal{E} = (\mathcal{H}, B, \{b_1, \dots, b_k\})$, $\Theta = D = \{1, \dots, k\}$ and the loss function is $w(i, j) = 1 - \delta_{ij}$, where δ is the Kronecker symbol. A decision procedure is a B-POVM $\{M_1, \dots, M_k\}$, where M_i corresponds to the choice b_i. Then the average loss is the average error probability

$$\mathcal{L}_{\mathcal{E},\lambda,w}(M) = \sum_{i,j} \lambda_i (1 - \delta_{ij}) \mathrm{Tr}\, b_i M_j = \sum_{i \neq j} \lambda_i \mathrm{Tr}\, b_i M_j.$$

We can use Remark 1 to compute the minimal average error probability $\Pi^B_\lambda(b_1, \dots, b_k) := \min_M \mathcal{L}_{\mathcal{E},\lambda,w}(M)$. We obtain $\xi_{\mathcal{E},\lambda,w'} = \sum_i |i\rangle\langle i| \otimes \lambda_i b_i$, so that the minimal average error probability is

$$\Pi^B_\lambda(b_1, \dots, b_k) = 1 - \|\xi_{\lambda,w'}\|_{I\otimes B} = 1 - \inf_{b\in B} \sup_{1\le i\le k} 2^{D_{max}(\lambda_i b_i \| b)}.$$

For $B = \mathfrak{S}(\mathcal{H})$, the last equality was obtained in Ref. 7, see also Ref. 17.

Let us now look at an optimal decision procedure. Let $\{M_i\}$ be a B-POVM with decomposition $M = \Lambda \circ \chi_c$ and let us suppose that $c = \sum_i M_i$ is strictly positive. Let $\sigma_i = \chi_c(b_i)$ and $\mathcal{E}_c = (\mathcal{H}, \mathfrak{S}(\mathcal{H}), \{\sigma_1, \dots, \sigma_k\})$. Suppose that $\{\Lambda_i\}$ is optimal for $(\mathcal{E}_c, \lambda, w)$, this is equivalent to the fact that $\sum_i \lambda_i \sigma_i \Lambda_i =: p$ is a hermitian element that majorizes $\lambda_i \sigma_i$ for all i. By Remark 1 and Corollary 7, $\{M_i\}$ is then optimal for $(\mathcal{E}, \lambda, w)$ if and only if $p \in \mathrm{span}(\chi_c(B))$, note that $\sigma_i \in \chi_c(B)$ for all i.

Example 3 (Hypothesis testing). Let $k = 2$ in the previous example, then we obtain the hypothesis testing or discrimination problem, considered at the end of Sec. I B. Here we have

$$\| |0\rangle\langle 0| \otimes s b_0 + |1\rangle\langle 1| \otimes t b_1 \|_{I_2 \otimes B} = \frac{1}{2}(\|s b_0 - t b_1\|_B + s + t),$$

for $s, t > 0$, so that indeed, $1 - \|\xi_{\mathcal{E},\lambda,w'}\|_{I_2\otimes B} = \frac{1}{2}(1 - \|\lambda b_0 - (1-\lambda) b_1\|_B)$ is the minimal Bayes error probability. Let $\{M_0, M_1\}$ be a B-POVM such that $c = M_0 + M_1$ is strictly positive and let $\sigma_i = \chi_c(b_i)$. Suppose $\lambda = 1/2$ and let $\Lambda_i = c^{-1/2} M_i c^{-1/2}$ be a POVM which is optimal for $(\mathcal{E}_c, \lambda, w)$, then Λ_0 is the projection onto the support of $(\sigma_0 - \sigma_1)_+$ and $\sum_i \lambda_i \sigma_i \Lambda_i = \frac{1}{2}((\sigma_0 - \sigma_1)_+ + \sigma_1)$. From the previous example, it is clear that $\{M_0, M_1\}$ is then an optimal test for $(\mathcal{E}, \lambda, w)$ if and only if any of (and therefore all of) $(\sigma_0 - \sigma_1)_+, (\sigma_0 - \sigma_1)_-, |\sigma_0 - \sigma_1|$ is an element in $\mathrm{span}(\chi_c(B))$.

In particular, let $B = \mathcal{C}(\mathcal{H}, \mathcal{K})$. In this case, the B-POVMs are exactly the quantum 1-testers of Refs. 3 and 21, see also Ref. 13. More precisely, the B-POVMs $M = \{M_d, d \in D\} \subset B(\mathcal{K} \otimes \mathcal{H})^+$ satisfy $\sum_d M_d = I \otimes \sigma$ for some $\sigma \in \mathfrak{S}(\mathcal{H})$. Let $M = \Lambda \circ \chi_{I\otimes\sigma}$ be the decomposition of M, then for X_Φ,

$$\mathrm{Tr}\, M_d X_\Phi = \mathrm{Tr}\, \Lambda_d \chi_{I\otimes\sigma}(X_\Phi) = \mathrm{Tr}\, \Lambda_d (\Phi \otimes id_A)(\rho),$$

where $\rho = \chi_{I\otimes\sigma}(\Psi)$ is a pure state in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{H}_A)$ and $\mathcal{H}_A = \mathrm{supp}\,(\sigma)$. This means that the tester M is implemented by the triple $(\mathcal{H}_A, \rho, \Lambda)$. If $\sigma = \dim(\mathcal{H})^{-1} I$, then $\rho = \dim(\mathcal{H})^{-1}\Psi$ is the maximally entangled state in $\mathcal{H} \otimes \mathcal{H}$. By the results of Example 3, we have the following.

Corollary 8. Let $b_i = X_{\Phi_i}$ be Choi matrices of the channels $\Phi_0, \Phi_1 : B(\mathcal{H}) \to B(\mathcal{K})$. Consider the problem of testing the hypothesis Φ_0 against Φ_1, with a priori probability $\lambda \in [0, 1]$. Then there exists an optimal 1-tester implemented by a triple $(\mathcal{H}, \Lambda, \rho)$ with maximally entangled input state ρ if and only if $\mathrm{Tr}_{\mathcal{K}} |\lambda X_{\Phi_0} - (1-\lambda) X_{\Phi_1}|$ is a multiple of $I_{\mathcal{H}}$.

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[1] M. D. Choi, "Completely positive maps on complex matrices," Lin. Alg. Appl. **10**, 285–290 (1975).
[2] G. Chiribella, G. M. D'Ariano, and P. Perinotti, "Quantum circuit architecture," Phys. Rev. Lett. **101**, 060401 (2008).
[3] G. Chiribella, G. M. D'Ariano, and P. Perinotti, "Memory effects in quantum channel discrimination," Phys. Rev. Lett. **101**, 180501 (2008).
[4] G. Chiribella, G. M. D'Ariano, and P. Perinotti, "Theoretical framework for quantum networks," Phys. Rev. A **80**, 022339 (2009).
[5] G. Chiribella, G. M. D'Ariano, and P. Perinotti, "Transforming quantum operations: Quantum supermaps," Europhys. Lett. **83**, 30004 (2008).
[6] N. Datta, "Min- and max- relative entropies and a new entanglement monotone," IEEE Trans. Inf. Theory **55**, 2816–2826 (2009).
[7] N. Datta and M. Mosonyi, "Generalized relative entropies and the capacity of classical-quantum channels," J. Math. Phys. **50**, 072104 (2009).
[8] G. Gutoski and J. Watrous, "Toward a general theory of quantum games," in *Proceedings of the 39th ACM Symposium on Theory of Computing (STOC 2007)* (ACM, 2007), pp. 565–574; e-print arXiv:quant-ph/0611234v2.
[9] G. Gutoski, "On a measure of distance for quantum strategies," J. Math. Phys. **53**, 032202 (2012).
[10] C. E. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976).
[11] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North Holland, Amsterdam, 1982).
[12] A. Holevo, *Statistical Structure of Quantum Theory* (Springer-Verlag, Berlin, 2001).
[13] A. Jenčová, "Generalized channels: channels for convex subsets of the state space," J. Math. Phys. **53**, 012201 (2012); e-print arxiv:1105.1899.
[14] A. Kitaev, "Quantum computations: algorithms and error correction," Russian Math. Surv. **52**, 1191–1249 (1997).
[15] K. Matsumoto, "A quantum version of randomization criterion," preprint arxiv:1012.2650v6 (2012).
[16] V. Paulsen, *Completely Bounded Maps and Operator Algebras* (Cambridge University Press, 2003).
[17] R. Koenig, R. Renner, and C. Schaffner, "The operational meaning of min- and max-entropy," IEEE Trans. Inf. Theory **55**, 4337 (2009).
[18] T. Rockafellar, *Convex Analysis* (Princeton University Press, Princeton, 1970).
[19] J. Watrous, "Notes on super-operator norms induced by Schatten norms," Quantum Inf. Comput. **5**, 58–68 (2005); e-print arxiv:quant-ph/0411077v1.
[20] D. Reeb, M. J. Kastoryano, and M. M. Wolf, "Hilbert's projective metric in quantum information theory," J. Math. Phys. **52**, 082201 (2011).
[21] M. Ziman, "Process POVM: A mathematical framework for the description of process tomography experiments," Phys. Rev. A **77**, 062112 (2008).