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STRONG SUBADDITIVITY OF SEGAL'S ENTROPY

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ABSTRACT. Let $\mathcal{M}$ be a semifinite von Neumann algebra with a normal faithful semifinite trace τ, and let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{R}$ be its subalgebras such that $\mathcal{R} \subset \mathcal{A} \cap \mathcal{B}$ and that τ restricted to any of these subalgebras is semifinite. Denote by $\mathbb{E}_{\mathcal{A}}$, $\mathbb{E}_{\mathcal{B}}$ and $\mathbb{E}_{\mathcal{R}}$ the normal conditional expectations from $\mathcal{M}$ onto $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{R}$, respectively, such that τ is invariant with respect to any of them. The quadruple $\mathcal{M}, \mathcal{A}, \mathcal{B}, \mathcal{R}$ is said to be a commuting square if

$$\mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{B}} = \mathbb{E}_{\mathcal{B}}\mathbb{E}_{\mathcal{A}} = \mathbb{E}_{\mathcal{R}}.$$

In this note, we show that the property of being a commuting square is characterised by the inequality

$$H(\rho|\mathcal{A}) + H(\rho|\mathcal{B}) \leqslant H(\rho) + H(\rho|\mathcal{R})$$

for an arbitrary normal state ρ on $\mathcal{M}$, where $H(\varphi)$ denotes the Segal entropy of the state φ.

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INTRODUCTION

In the paper, we show how the notion of Segal entropy in semifinite von Neumann algebras can be used to characterise the property of being a so-called commuting square for a given von Neumann algebra and its subalgebras. This generalises a corresponding result obtained earlier for von Neumann algebras acting on a finite dimensional Hilbert space. As a corollary, we obtain the celebrated strong subadditivity property for the tensor product of finite von Neumann algebras.

1. PRELIMINARIES AND NOTATION

Let $\mathcal{M}$ be a semifinite von Neumann algebra with a normal semifinite faithful trace τ, identity $\mathbb{1}$, and predual $\mathcal{M}_*$. The operator norm on $\mathcal{M}$ shall be denoted by $\|\cdot\|_\infty$. By $\mathcal{M}^+$ we shall denote the set of positive operators in $\mathcal{M}$, and by $\mathcal{M}_*^+$ — the set of positive functionals in $\mathcal{M}_*$. These functionals will be referred to as normal states.

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For each $\rho \in \mathcal{M}_*$, there is an operator $h \in L^1(\mathcal{M}, \tau)$ such that

$$\rho(x) = \tau(xh) = \tau(hx), \quad x \in \mathcal{M}.$$

The correspondence between $\mathcal{M}_*$ and $L^1(\mathcal{M}, \tau)$ defined above is one-to-one and isometric. Recall that the norm on $L^1(\mathcal{M}, \tau)$, denoted by $\|\cdot\|_1$, is defined as

$$\|h\|_1 = \tau(|h|), \quad h \in L^1(\mathcal{M}, \tau).$$

For a normal state ρ, the corresponding element in $L^1(\mathcal{M}, \tau)^+$ will be denoted by h_ρ and called the *density* of ρ, thus

$$\rho(x) = \tau(xh_\rho) = \tau(h_\rho x) = \tau\big(h_\rho^{\frac{1}{2}} x h_\rho^{\frac{1}{2}}\big), \quad x \in \mathcal{M}.$$

Let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$ such that $\tau|\mathcal{N}$ is semifinite. Then there exists a normal conditional expectation $\mathbb{E}\colon \mathcal{M} \to \mathcal{N}$, which is a unital completely positive map, such that

$$\tau \circ \mathbb{E} = \tau.$$

This expectation can also be defined as a map from $L^1(\mathcal{M}, \tau)$ onto $L^1(\mathcal{N}, \tau|\mathcal{N})$, denoted by the same letter, which is again a positive map of $\|\cdot\|_1$-norm one. Of course on the set $\mathcal{M} \cap L^1(\mathcal{M}, \tau)$ these two expectations coincide.

Lemma 1. *Let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$ such that $\tau|\mathcal{N}$ is semifinite. For each normal state ρ on $\mathcal{M}$ and the conditional expectation $\mathbb{E}$ from $\mathcal{M}$ onto $\mathcal{N}$, we have for the densities $h_{\rho\circ\mathbb{E}}$ and $h_{\rho|\mathcal{N}}$ the following formula*

$$h_{\rho\circ\mathbb{E}} = h_{\rho|\mathcal{N}} = \mathbb{E}h_\rho.$$

Proof. For any $x \in \mathcal{M}$ and $h \in L^1(\mathcal{M}, \tau)$, we have

$$\tau((\mathbb{E}x)h) = \tau(\mathbb{E}((\mathbb{E}x)h)) = \tau((\mathbb{E}x)(\mathbb{E}h)),$$

and

$$\tau(x\mathbb{E}h) = \tau(\mathbb{E}(x\mathbb{E}h)) = \tau((\mathbb{E}x)(\mathbb{E}h)),$$

thus

$$\tau((\mathbb{E}x)h) = \tau(x\mathbb{E}h).$$

Consequently, for any $x \in \mathcal{M}$, we have

$$(\rho \circ \mathbb{E})(x) = \rho(\mathbb{E}x) = \tau((\mathbb{E}x)h_\rho) = \tau(x\mathbb{E}h_\rho),$$

which yields

$$h_{\rho\circ\mathbb{E}} = \mathbb{E}h_\rho.$$

For any $x \in \mathcal{N}$, we have

$$(\rho|\mathcal{N})(x) = \rho(x) = \tau(xh_\rho) = \tau(\mathbb{E}(xh_\rho)) = \tau(x\mathbb{E}h_\rho),$$

which yields

$$h_{\rho|\mathcal{N}} = \mathbb{E}h_\rho. \qquad \square$$

2. Segal entropy and information

Let ρ be a normal state on $\mathcal{M}$. The *Segal entropy* $H(\rho)$ of ρ is defined as

$$H(\rho) = \tau(h_\rho \log h_\rho).$$

(In the original Segal definition [9], there is a minus sign before the trace; we choose the version as above for simplicity and in order that $H(\rho)$ be nonnegative for a normalised state and finite trace.) The notion of entropy can be, in a natural way, defined for $h \in L^1(\mathcal{M}, \tau)^+$, namely,

$$H(h) = \tau(h \log h),$$

thus the entropy of a state is the entropy of its density.

Let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$ such that $\tau|\mathcal{N}$ is semifinite. For each normal state ρ on $\mathcal{M}$ and the conditional expectation $\mathbb{E}$ from $\mathcal{M}$ onto $\mathcal{N}$, we obtain, on account of Lemma 1, the following equality

$$H(\rho|\mathcal{N}) = H(\rho \circ \mathbb{E}). \tag{1}$$

For the normal states ρ and ω on $\mathcal{M}$, the *information* $I(\omega, \rho)$, denoted also by $D(\omega||\rho)$, between these states is defined in [10] by the formula

$$I(\omega, \rho) = \tau(h_\omega \log h_\omega - h_\omega \log h_\rho),$$

under the assumption that $\mathrm{s}^{\mathcal{M}}(\omega) \leqslant \mathrm{s}^{\mathcal{M}}(\rho)$. It should be noted that this definition is a little formal, especially for a semifinite and not finite trace, since then the operators $\log h_\omega$ and $\log h_\rho$ need not be even measurable let alone the relation $h_\omega(\log h_\omega - \log h_\rho) \in L^1(M, \tau)$. The proper formula for the information reads as

$$I(\omega, \rho) = \text{“}\tau(h_\omega \log h_\omega) - \tau(h_\omega \log h_\rho)\text{”} = \omega(\log h_\omega) - \omega(\log h_\rho)$$

with an appropriate definition of $\omega(\log h_\omega)$ and $\omega(\log h_\rho)$, see [4] for a more detailed explanation. In particular, $I(\omega, \rho)$ is well-defined if, for example, ω has finite entropy. Since for a seladjoint x affiliated with $\mathcal{M}$ with the spectral decomposition

$$x = \int_{-\infty}^{+\infty} \lambda \, e(d\lambda),$$

$\omega(x)$ is defined as

$$\omega(x) = \int_{-\infty}^{+\infty} \lambda \, \omega(e(d\lambda)),$$

it is obvious that if x is affiliated with a von Neumann subalgebra $\mathcal{N}$, then

$$(\omega|\mathcal{N})(x) = \omega(x).$$

The following result was proved in [4, Lemma 18] under the assumption that x is measurable. However, this assumption is redundant. For the sake of completeness we repeat the proof here.

Lemma 2. *Let ω be a normal state on $\mathcal{M}$, let x be selfadjoint, and assume that $h_\omega x \in L^1(\mathcal{M}, \tau)$. Then*

$$\omega(x) = \tau(h_\omega x).$$

Proof. For the spectral decomposition

$$x = \int_{-\infty}^{\infty} \lambda\, e(d\lambda),$$

put

$$p_n = e([-n, n]) \uparrow \mathbb{1},$$

and let $x_{[n]}$ be the truncation

$$x_{[n]} = \int_{-n}^{n} \lambda\, e(d\lambda).$$

Let ρ be the normal functional on $\mathcal{M}_*$ having the density $h_\omega x$, i.e. for each $z \in \mathcal{M}$

$$\rho(z) = \tau(h_\omega x z).$$

Then

$$\begin{aligned}\tau(h_\omega x) = \rho(\mathbb{1}) &= \lim_{n\to\infty} \rho(p_n) = \lim_{n\to\infty} \tau(h_\omega x p_n) \\ &= \lim_{n\to\infty} \tau(h_\omega x_{[n]}) = \lim_{n\to\infty} \omega(x_{[n]}) = \omega(x).\end{aligned}$$ □

From the above lemma, we obtain the following corollary.

Corollary 3. *Let a normal state ρ have finite entropy. Then*

$$H(\rho) = \tau(h_\rho \log h_\rho) = \rho(\log h_\rho).$$

An important fact for the information, proved in [4], is the equality

$$I(\omega, \rho) = S(\rho, \omega),$$

where $S(\rho, \omega)$ is the Araki relative entropy. Due to this equivalence, we have the following basic properties of the information between states.

Theorem 4. *Let ω and ρ be normal states on a semifinite von Neumann algebra $\mathcal{M}$ such that $\|\omega\| = \omega(\mathbb{1}) = \rho(\mathbb{1}) = \|\rho\|$. Then*

(i) *$I(\omega, \rho) \geqslant 0$ and $I(\omega, \rho) = 0$ if and only if $\omega = \rho$.*
(ii) *Let $\mathcal{N}$ be another semifinite von Neumann algebra, and let $\alpha\colon \mathcal{N} \to \mathcal{M}$ be a unital normal Schwarz mapping. Then*

$$I(\omega \circ \alpha, \rho \circ \alpha) \leqslant I(\omega, \rho)$$

(*see* [5, Chapter 5]).

3. SUBADDITIVITY OF ENTROPY

Let us note the following important relation.

Proposition 5. *Let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$ such that $\tau|\mathcal{N}$ is semifinite, let $\mathbb{E}$ be the conditional expectation from $\mathcal{M}$ onto $\mathcal{N}$ such that τ is $\mathbb{E}$-invariant, and let ρ be a normal state on $\mathcal{M}$ such that the entropies of ρ and $\rho \circ \mathbb{E}$ are finite. Then*

$$I(\rho, \rho \circ \mathbb{E}) = H(\rho) - H(\rho \circ \mathbb{E}).$$

Proof. We have

$$I(\rho, \rho \circ \mathbb{E}) = \rho(\log h_\rho) - \rho(\log h_{\rho \circ \mathbb{E}}) = \rho(\log h_\rho) - \rho(\log \mathbb{E} h_\rho).$$

Let

$$\mathbb{E} h_\rho = \int_0^{+\infty} \lambda \, e(d\lambda)$$

be the spectral decomposition of $\mathbb{E} h_\rho$. Then the spectral projections $e(\Delta)$ belong to $\mathcal{N}$, consequently $\mathbb{E} e(\Delta) = e(\Delta)$. Hence

$$\begin{aligned}(\rho \circ \mathbb{E})(\log \mathbb{E} h_\rho) &= \int_0^{+\infty} \log \lambda \, (\rho \circ \mathbb{E})(e(d\lambda)) \\ &= \int_0^{+\infty} \log \lambda \, \rho(e(d\lambda)) = \rho(\log \mathbb{E} h_\rho),\end{aligned}$$

and thus

$$\begin{aligned}I(\rho, \rho \circ \mathbb{E}) =& \rho(\log h_\rho) - \rho(\log \mathbb{E} h_\rho) = \rho(\log h_\rho) - (\rho \circ \mathbb{E})(\log \mathbb{E} h_\rho) \\ =& \rho(\log h_\rho) - (\rho \circ \mathbb{E})(\log h_{\rho \circ \mathbb{E}}) = H(\rho) - H(\rho \circ \mathbb{E}). \qquad \square\end{aligned}$$

Let $\mathcal{M}$ be a semifinite von Neumann algebra with a normal faithful semifinite trace τ, and let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{R}$ be von Neumann subalgebras of $\mathcal{M}$ satisfying the quadrilateral of inclusions

$$\begin{array}{ccc} \mathcal{A} & \subset & \mathcal{M} \\ \cup & & \cup \\ \mathcal{R} & \subset & \mathcal{B} \end{array} \tag{2}$$

Assume that the trace τ restricted to each of these subalgebras is semifinite, and denote by $\mathbb{E}_\mathcal{A}$, $\mathbb{E}_\mathcal{B}$, $\mathbb{E}_\mathcal{R}$ the conditional expectations from $\mathcal{M}$ onto $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{R}$, respectively, such that

$$\tau \circ \mathbb{E}_\mathcal{A} = \tau \circ \mathbb{E}_\mathcal{B} = \tau \circ \mathbb{E}_\mathcal{R} = \tau.$$

The quadrilateral is said to be a *commuting square* if

$$\mathbb{E}_\mathcal{A} \mathbb{E}_\mathcal{B} = \mathbb{E}_\mathcal{B} \mathbb{E}_\mathcal{A} = \mathbb{E}_\mathcal{R}.$$

The following result is a generalisation of the one obtained in [1] for von Neumann algebras acting on a finite dimensional Hilbert space.

Theorem 6. *Let $\mathcal{M}$ be a semifinite von Neumann algebra with a normal faithful semifinite trace τ, and let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{R}$ be von Neumann subalgebras of $\mathcal{M}$ such that the trace τ restricted to each of these subalgebras is semifinite. Assume that these algebras satisfy the quadrilateral of inclusions* (2), *and denote by $\mathbb{E}_{\mathcal{A}}$, $\mathbb{E}_{\mathcal{B}}$, $\mathbb{E}_{\mathcal{R}}$ the conditional expectations from $\mathcal{M}$ onto $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{R}$, respectively, such that*

$$\tau \circ \mathbb{E}_{\mathcal{A}} = \tau \circ \mathbb{E}_{\mathcal{B}} = \tau \circ \mathbb{E}_{\mathcal{R}} = \tau.$$

The algebras $\mathcal{M}$, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{R}$ form a commuting square if and only if for any normal state ρ on $\mathcal{M}$ such that the entropies $H(\rho)$, $H(\rho|\mathcal{A})$, $H(\rho|\mathcal{B})$ and $H(\rho|\mathcal{R})$ are finite the following inequality holds

$$H(\rho|\mathcal{A}) + H(\rho|\mathcal{B}) \leqslant H(\rho) + H(\rho|\mathcal{R}). \tag{3}$$

Proof. Assume first that the algebras $\mathcal{M}$, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{R}$ form a commuting square, and let ρ be an arbitrary normal state on $\mathcal{M}$ such that the entropies $H(\rho)$, $H(\rho|\mathcal{A})$, $H(\rho|\mathcal{B})$ and $H(\rho|\mathcal{R})$ are finite. On account of the equality (1), Proposition 5 and Theorem 4, we get

$$\begin{aligned} H(\rho) - H(\rho|\mathcal{A}) =& H(\rho) - H(\rho \circ \mathbb{E}_{\mathcal{A}}) = I(\rho, \rho \circ \mathbb{E}_{\mathcal{A}}) \\ \geqslant& I(\rho \circ \mathbb{E}_{\mathcal{B}}, \rho \circ \mathbb{E}_{\mathcal{A}} \circ \mathbb{E}_{\mathcal{B}}) = I(\rho \circ \mathbb{E}_{\mathcal{B}}, \rho \circ \mathbb{E}_{\mathcal{R}}) \\ =& I(\rho \circ \mathbb{E}_{\mathcal{B}}, \rho \circ \mathbb{E}_{\mathcal{B}} \circ \mathbb{E}_{\mathcal{A}}) \\ =& H(\rho \circ \mathbb{E}_{\mathcal{B}}) - H(\rho \circ \mathbb{E}_{\mathcal{B}} \circ \mathbb{E}_{\mathcal{A}}) \\ =& H(\rho|\mathcal{B}) - H(\rho \circ \mathbb{E}_{\mathcal{R}}) = H(\rho|\mathcal{B}) - H(\rho|R), \end{aligned} \tag{4}$$

which shows the claim.

Now assume that the inequality (3) holds. It can be written in the form

$$H(\rho \circ \mathbb{E}_{\mathcal{A}}) + H(\rho \circ \mathbb{E}_{\mathcal{B}}) \leqslant H(\rho) + H(\rho \circ \mathbb{E}_{\mathcal{R}})$$

for $\rho \in \mathcal{M}_*^+$.

Let $h_0 \in L^1(\mathcal{M}, \tau)^+$ with finite entropy be of the form

$$h_0 = \int_m^M \lambda\, e(d\lambda) \tag{5}$$

for some $0 < m < M$, and let $\rho \in \mathcal{M}_*^+$ have density h_0. Let $\mathbb{E}$ be any of the conditional expectations $\mathbb{E}_{\mathcal{A}}$, $\mathbb{E}_{\mathcal{B}}$ or $\mathbb{E}_{\mathcal{R}}$. Since

$$m\mathbb{1} \leqslant h_0 \leqslant M\mathbb{1},$$

we have

$$m\mathbb{1} \leqslant \mathbb{E}h_0 \leqslant M\mathbb{1}$$

and thus

$$(\log m)\mathbb{1} \leqslant \log \mathbb{E}h_0 \leqslant (\log M)\mathbb{1}.$$

This yields

$$(\log m)\mathbb{E}h_0 \leqslant \mathbb{E}h_0 \log \mathbb{E}h_0 \leqslant (\log M)\mathbb{E}h_0,$$

which implies

$$\log m\tau(h_0) = \tau((\log m)\mathbb{E}h_0) \leqslant \tau(\mathbb{E}h_0 \log \mathbb{E}h_0) \\ \leqslant \tau((\log M)\mathbb{E}h_0) = \log M\tau(h_0),$$

showing that the entropy of $\mathbb{E}h_0$ is finite. The inequality (3) can be rewritten in the form

$$H(\mathbb{E}_{\mathcal{A}}h) + H(\mathbb{E}_{\mathcal{B}}h) \leqslant H(h) + H(\mathbb{E}_{\mathcal{R}}h)$$

for every $h \in L^1(\mathcal{M}, \tau)^+$ such that all the entropies in the formula above are finite. Putting $\mathbb{E}_{\mathcal{B}}h_0$ in place of h in this formula, we get

$$H(\mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{B}}h_0) + H(\mathbb{E}_{\mathcal{B}}h_0) \leqslant H(\mathbb{E}_{\mathcal{B}}h_0) + H(\mathbb{E}_{\mathcal{R}}\mathbb{E}_{\mathcal{B}}h_0),$$

i.e.

$$H(\mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{B}}h_0) \leqslant H(\mathbb{E}_{\mathcal{R}}\mathbb{E}_{\mathcal{B}}h_0). \tag{6}$$

Since obviously $\mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{R}} = \mathbb{E}_{\mathcal{R}}$, which follows from the inclusion $\mathcal{R} \subset \mathcal{A}$, we have

$$(\mathbb{E}_{\mathcal{R}}\mathbb{E}_{\mathcal{A}})^2 = \mathbb{E}_{\mathcal{R}}\mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{R}}\mathbb{E}_{\mathcal{A}} = (\mathbb{E}_{\mathcal{R}})^2\mathbb{E}_{\mathcal{A}} = \mathbb{E}_{\mathcal{R}}\mathbb{E}_{\mathcal{A}},$$

which means that $\mathbb{E}_{\mathcal{R}}\mathbb{E}_{\mathcal{A}}$ is a projection. Moreover, $\mathbb{E}_{\mathcal{R}}\mathbb{E}_{\mathcal{A}}(\mathcal{M}) = \mathcal{R}$, and τ is $\mathbb{E}_{\mathcal{R}}\mathbb{E}_{\mathcal{A}}$-invariant, thus the uniqueness of the invariant projection ($\equiv$ conditional expectation) onto $\mathcal{R}$ yields the equality

$$\mathbb{E}_{\mathcal{R}}\mathbb{E}_{\mathcal{A}} = \mathbb{E}_{\mathcal{R}}.$$

Now the inequality (6) can be rewritten in the form

$$H(\mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{B}}h_0) \leqslant H(\mathbb{E}_{\mathcal{R}}\mathbb{E}_{\mathcal{B}}h_0) = H(\mathbb{E}_{\mathcal{R}}\mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{B}}h_0). \tag{7}$$

Denote by φ the normal state with density $h' = \mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{B}}h_0$. The inequality (7) takes the form

$$H(\varphi) = H(h') \leqslant H(\mathbb{E}_{\mathcal{R}}h') = H(\varphi \circ \mathbb{E}_{\mathcal{R}}),$$

and since by virtue of Theorem 4 and Proposition 5 we have $H(\varphi) \geqslant H(\varphi \circ \mathbb{E}_{\mathcal{R}})$, the equality

$$H(h') = H(\mathbb{E}_{\mathcal{R}}h')$$

follows. From [3, Theorem 12 (alternatively Theorem 13)], we obtain the equality

$$\mathbb{E}_{\mathcal{R}}h' = h',$$

i.e.

$$\mathbb{E}_{\mathcal{R}}h_0 = \mathbb{E}_{\mathcal{R}}\mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{B}}h_0 = \mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{B}}h_0.$$

Since, by virtue of [2, Theorem 13], the elements h_0 of the form (5) with finite entropy are dense in $L^1(\mathcal{M}, \tau)^+$, and the maps $\mathbb{E}_{\mathcal{R}}$ and $\mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{B}}$ are bounded in $\|\cdot\|_1$-norm, we obtain the equality

$$\mathbb{E}_{\mathcal{R}} = \mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{B}}$$

on $L^1(\mathcal{M},\tau)^+$, consequently, on the whole of $L^1(\mathcal{M},\tau)$. Since $\mathcal{M}\cap L^1(\mathcal{M},\tau)$ is σ-weakly dense in $\mathcal{M}$ and the maps $\mathbb{E}_{\mathcal{R}}$ and $\mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{B}}$ are normal, we get the same equality also on $\mathcal{M}$. The equality

$$\mathbb{E}_{\mathcal{R}} = \mathbb{E}_{\mathcal{B}}\mathbb{E}_{\mathcal{A}}$$

is obtained in a similar way. □

An interesting question is when we have equality in the inequality (3). In this case the following equality holds (cf. the relation (4))

$$\begin{aligned} I(\rho,\rho\circ\mathbb{E}_{\mathcal{A}}) =& H(\rho) - H(\rho\circ\mathbb{E}_{\mathcal{A}}) = H(\rho) - H(\rho|\mathcal{A}) \\ =& H(\rho|\mathcal{B}) - H(\rho|R) = H(\rho|\mathcal{B}) - H(\rho\circ\mathbb{E}_{\mathcal{R}}) \\ =& H(\rho\circ\mathbb{E}_{\mathcal{B}}) - H(\rho\circ\mathbb{E}_{\mathcal{A}}\circ\mathbb{E}_{\mathcal{B}}) \\ =& I(\rho\circ\mathbb{E}_{\mathcal{B}},\rho\circ\mathbb{E}_{\mathcal{A}}\circ\mathbb{E}_{\mathcal{B}}) = I(\rho|\mathcal{B},(\rho\circ\mathbb{E}_{\mathcal{A}})|\mathcal{B}). \end{aligned}$$

According to [6, Theorem 4], this equality is equivalent to the relation

$$[D\rho : D(\rho\circ\mathbb{E}_{\mathcal{A}})]_t = [D(\rho|\mathcal{B}) : D((\rho\circ\mathbb{E}_{\mathcal{A}})|\mathcal{B})]_t \quad \text{for all } t\in\mathbb{R},$$

where $[D\varphi : D\omega]_t$ is the Connes-Radon-Nikodym derivative of the states φ and ω. Since

$$[D\varphi : D\omega]_t = h_\varphi^{it}h_\omega^{-it},$$

we obtain, taking into account Lemma 1,

Theorem 7. *Let the algebras $\mathcal{M}$, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{R}$ be as before, and let ρ be a normal state on $\mathcal{M}$ such that the entropies $H(\rho)$, $H(\rho|\mathcal{A})$, $H(\rho|\mathcal{B})$ and $H(\rho|\mathcal{R})$ are finite. Then the equality*

$$H(\rho|\mathcal{A}) + H(\rho|\mathcal{B}) = H(\rho) + H(\rho|\mathcal{R}) \tag{8}$$

holds if and only if

$$h_\rho^{it}(\mathbb{E}_{\mathcal{A}}h_\rho)^{-it} = (\mathbb{E}_{\mathcal{B}}h_\rho)^{it}(\mathbb{E}_{\mathcal{R}}h_\rho)^{-it} \quad \textit{for all } t\in\mathbb{R}.$$

The condition for equality can be further simplified if we assume that the algebra $\mathcal{M}$ is finite. Then the unitary groups $\left(h_\rho^{it}\right)$, $\left((\mathbb{E}_{\mathcal{A}}h_\rho)^{-it}\right)$, $\left((\mathbb{E}_{\mathcal{B}}h_\rho)^{it}\right)$ and $\left((\mathbb{E}_{\mathcal{R}}h_\rho)^{-it}\right)$ have generators $\log h_\rho$, $-\log\mathbb{E}_{\mathcal{A}}h_\rho$, $\log\mathbb{E}_{\mathcal{B}}h_\rho$ and $-\log\mathbb{E}_{\mathcal{R}}h_\rho$, respectively, which are *measurable* operators, in particular, their common domain is dense. Denoting for simplicity $u_t = h_\rho^{it}$ and $v_t = \left(\mathbb{E}_{\mathcal{A}}h_\rho\right)^{-it}$, we have for ξ belonging to this domain

$$\frac{u_tv_t-\mathbb{1}}{t}\xi = u_t\frac{v_t-\mathbb{1}}{t}\xi + \frac{u_t-\mathbb{1}}{t}\xi \xrightarrow[t\to 0]{} -(\log\mathbb{E}_{\mathcal{A}}h_\rho)\xi + (\log h_\rho)\xi,$$

and similarly for the other two unitary groups. This yields the equality

$$\log h_\rho - \log\mathbb{E}_{\mathcal{A}}h_\rho = \log\mathbb{E}_{\mathcal{B}}h_\rho - \log\mathbb{E}_{\mathcal{R}}h_\rho$$

or

$$\log \mathbb{E}_{\mathcal{A}} h_\rho + \log \mathbb{E}_{\mathcal{B}} h_\rho = \log h_\rho + \log \mathbb{E}_{\mathcal{R}} h_\rho. \tag{9}$$

(Remember that the addition above is performed in the algebra $\widetilde{\mathcal{M}}$ of measurable operators, i.e. it is a *strong* addition which means that $x + y$ is in fact a closure of the sum.)

On the other hand, if the equality (9) holds, then under the assumption of finite entropy we get, multiplying both sides by h_ρ and taking the trace,

$$\tau(h_\rho \log \mathbb{E}_{\mathcal{A}} h_\rho) + \tau(h_\rho \log \mathbb{E}_{\mathcal{B}} h_\rho) = \tau(h_\rho \log h_\rho) + \tau(h_\rho \log \mathbb{E}_{\mathcal{R}} h_\rho),$$

and now it is enough to observe that we have e.g.

$$\tau(h_\rho \log \mathbb{E}_{\mathcal{A}} h_\rho) = \tau(\mathbb{E}_{\mathcal{A}}(h_\rho \log \mathbb{E}_{\mathcal{A}} h_\rho)) = \tau(\mathbb{E} h_\rho \log \mathbb{E}_{\mathcal{A}} h_\rho) = H(\rho|\mathcal{A}),$$

which yields the equality (8). Thus we obtain

Theorem 8. *Let the algebras $\mathcal{M}$, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{R}$ be as before with $\mathcal{M}$ finite, and let ρ be a normal state on $\mathcal{M}$ such that the entropies $H(\rho)$, $H(\rho|\mathcal{A})$, $H(\rho|\mathcal{B})$ and $H(\rho|\mathcal{R})$ are finite. Then the equality*

$$H(\rho|\mathcal{A}) + H(\rho|\mathcal{B}) = H(\rho) + H(\rho|\mathcal{R})$$

holds if and only if

$$\log \mathbb{E}_{\mathcal{A}} h_\rho + \log \mathbb{E}_{\mathcal{B}} h_\rho = \log h_\rho + \log \mathbb{E}_{\mathcal{R}} h_\rho.$$

Remark. It should be noted that a condition of the type like (9) was obtained in [8] for strong subadditivity of entropy in $\mathbb{B}(\mathcal{H})$ with finite-dimensional $\mathcal{H}$.

In the rest of the paper, we assume that $\mathcal{M}$ is a finite von Neumann algebra with a normal finite faithful unital trace τ.

An interesting situation appears when $\mathcal{R}$ is a trivial algebra which in our situation can be expressed as independence of the algebras $\mathcal{A}$ and $\mathcal{B}$. There are many notions of independence in the setting of operator algebras, we adopt the simplest and, in many respects, the most natural one. Subalgebras $\mathcal{A}$ and $\mathcal{B}$ of a von Neumann algebra $\mathcal{M}$ are said to be *independent* if for any $a \in \mathcal{A}$ and $b \in \mathcal{B}$ we have

$$\tau(ab) = \tau(a)\tau(b).$$

Lemma 9. *The following conditions are equivalent.*

(i) *For every $x \in \mathcal{M}$, the following equality holds*

$$\mathbb{E}_{\mathcal{A}} \mathbb{E}_{\mathcal{B}} x = \mathbb{E}_{\mathcal{B}} \mathbb{E}_{\mathcal{A}} x = \tau(x)\mathbb{1}.$$

(ii) *The algebras $\mathcal{A}$ and $\mathcal{B}$ are independent.*

(iii) $\mathcal{A} \cap \mathcal{B} = \mathbb{C} \cdot \mathbb{1}$ *and* $\mathbb{E}_{\mathcal{A}} \mathbb{E}_{\mathcal{B}} = \mathbb{E}_{\mathcal{B}} \mathbb{E}_{\mathcal{A}}$.

Proof. Observe first that if $\mathbb{E}$ is a conditional expectation such that $\tau \circ \mathbb{E} = \tau$, then we have for arbitrary $x, y \in \mathcal{M}$

$$\tau((\mathbb{E}x)y) = \tau(\mathbb{E}(\mathbb{E}x)y) = \tau(\mathbb{E}x\mathbb{E}y) = \tau(\mathbb{E}(x\mathbb{E}y)) = \tau(x\mathbb{E}y).$$

(i)$\Longrightarrow$(ii) For arbitrary $a \in \mathcal{A}$ and $b \in \mathcal{B}$ we have

$$\begin{aligned}\tau(ab) =& \tau(\mathbb{E}_{\mathcal{A}}(ab)) = \tau(a\mathbb{E}_{\mathcal{A}}b)\\ &= \tau(a\mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{B}}b) = \tau(a\tau(b)\mathbb{1}) = \tau(a)\tau(b),\end{aligned}$$

thus $\mathcal{A}$ and $\mathcal{B}$ are independent.

(ii)$\Longrightarrow$(iii) For arbitrary $y \in \mathcal{M}$, we have

$$\begin{aligned}&\tau(y(\mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{B}}x - \tau(x)\mathbb{1})) = \tau(y\mathbb{E}_{\mathcal{A}}(\mathbb{E}_{\mathcal{B}}x - \tau(x)\mathbb{1}))\\ =&\tau(\mathbb{E}_{\mathcal{A}}y(\mathbb{E}_{\mathcal{B}}x - \tau(x)\mathbb{1})) = \tau(\mathbb{E}_{\mathcal{A}}y\mathbb{E}_{\mathcal{B}}(x - \tau(x)\mathbb{1}))\\ =&\tau(\mathbb{E}_{\mathcal{A}}y)\tau(\mathbb{E}_{\mathcal{B}}(x - \tau(x)\mathbb{1})) = \tau(y)\tau(x - \tau(x)\mathbb{1}) = 0\end{aligned}$$

showing that

$$\mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{B}}x - \tau(x)\mathbb{1} = 0.$$

In the same way we obtain the equality

$$\mathbb{E}_{\mathcal{B}}\mathbb{E}_{\mathcal{A}}x = \tau(x)\mathbb{1}.$$

Let p be a projection in $\mathcal{A} \cap \mathcal{B}$. Then

$$\tau(p) = \tau(p \cdot p) = \tau(p)\tau(p),$$

thus $\tau(p)$ equals 0 or 1. It follows that $p = 0$ or $p = \mathbb{1}$ which means that in the algebra $\mathcal{A} \cap \mathcal{B}$ there are only trivial projections, consequently, (iii) follows.

(iii)$\Longrightarrow$(i) It follows that

$$\mathbb{E}_{\mathcal{A}}(\mathbb{E}_{\mathcal{B}}(\mathcal{M})) = \mathbb{E}_{\mathcal{B}}(\mathbb{E}_{\mathcal{A}}(\mathcal{M})) = \mathcal{A} \cap \mathcal{B} = \mathbb{C} \cdot \mathbb{1},$$

thus for every $x \in \mathcal{M}$ we have

$$\mathbb{E}_{\mathcal{A}}(\mathbb{E}_{\mathcal{B}}x) = \alpha(x)\mathbb{1},$$

and since τ is $\mathbb{E}_{\mathcal{A}}$- and $\mathbb{E}_{\mathcal{B}}$-invariant, we get, applying τ to both sides of the equality above, $\alpha(x) = \tau(x)$. □

From the lemma above, it follows that for subalgebras $\mathcal{A}$ and $\mathcal{B}$ their independence is equivalent to the fact that the quadrilateral $\mathcal{M}$, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{R}$ forms a commuting square with $\mathcal{R} = \mathcal{A} \cap \mathcal{B} = \mathbb{C} \cdot \mathbb{1}$. In such a case, for an arbitrary normal state ρ we have

$$\mathbb{E}_{\mathcal{R}}h_\rho = \tau(h_\rho)\mathbb{1}.$$

Theorem 10. *Let subalgebras $\mathcal{A}$ and $\mathcal{B}$ of the algebra $\mathcal{M}$ be independent. Then the equality* (8) *holds for a normal state ρ if and only if*

$$\tau(h_\rho)h_\rho = (\mathbb{E}_{\mathcal{A}}h_\rho)(\mathbb{E}_{\mathcal{B}}h_\rho) = (\mathbb{E}_{\mathcal{B}}h_\rho)(\mathbb{E}_{\mathcal{A}}h_\rho).$$

Proof. On account of Theorem 7, the equality (8) holds if and only if

$$h_\rho^{it}(\mathbb{E}_{\mathcal{A}}h_\rho)^{-it} = (\mathbb{E}_{\mathcal{B}}h_\rho)^{it}(\mathbb{E}_{\mathcal{R}}h_\rho)^{-it} \quad \text{for all } t \in \mathbb{R}.$$

which in our case amounts to

$$h_\rho^{it}(\mathbb{E}_{\mathcal{A}}h_\rho)^{-it} = \tau(h_\rho)^{-it}(\mathbb{E}_{\mathcal{B}}h_\rho)^{it} \quad \text{for all } t \in \mathbb{R},$$

that is

$$\big(\tau(h_\rho)h_\rho\big)^{it} = (\mathbb{E}_{\mathcal{B}}h_\rho)^{it}(\mathbb{E}_{\mathcal{A}}h_\rho)^{it} \quad \text{for all } t \in \mathbb{R}.$$

Since on the left-hand side of the equality above we have a unitary group, it follows that the two unitary groups on the right-hand side commute, consequently,

$$\big(\tau(h_\rho)h_\rho\big)^{it} = \big((\mathbb{E}_{\mathcal{B}}h_\rho)(\mathbb{E}_{\mathcal{A}}h_\rho)\big)^{it} = \big((\mathbb{E}_{\mathcal{A}}h_\rho)(\mathbb{E}_{\mathcal{B}}h_\rho)\big)^{it} \quad \text{for all } t \in \mathbb{R},$$

which shows the claim. □

As a corollary to Theorem 6, the strong subadditivity theorem for the tensor product of finite von Neumann algebras can be obtained. Let $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$ be von Neumann algebras with normal faithful finite normalised traces τ_1, τ_2, τ_3, respectively. Define maps

$$\begin{aligned}
\pi_{12}&\colon (\mathcal{M}_1\overline{\otimes}\mathcal{M}_2\overline{\otimes}\mathcal{M}_3)_* \to (\mathcal{M}_1\overline{\otimes}\mathcal{M}_2)_*,\\
\pi_{23}&\colon (\mathcal{M}_1\overline{\otimes}\mathcal{M}_2\overline{\otimes}\mathcal{M}_3)_* \to (\mathcal{M}_2\overline{\otimes}\mathcal{M}_3)_*,\\
\pi_{2}&\colon (\mathcal{M}_1\overline{\otimes}\mathcal{M}_2\overline{\otimes}\mathcal{M}_3)_* \to (\mathcal{M}_2)_*,
\end{aligned}$$

by the formulae

$$\begin{aligned}
(\pi_{12}\rho_{123})(x_{12}) &= \rho_{123}(x_{12}\otimes \mathbb{1}), \quad x_{12}\in \mathcal{M}_1\overline{\otimes}\mathcal{M}_2,\\
(\pi_{23}\rho_{123})(x_{23}) &= \rho_{123}(\mathbb{1}\otimes x_{23}), \quad x_{23}\in \mathcal{M}_2\overline{\otimes}\mathcal{M}_3,\\
(\pi_{2}\rho_{123})(x_{2}) &= \rho_{123}(\mathbb{1}\otimes x_2\otimes \mathbb{1}), \quad x_{2}\in \mathcal{M}_2,
\end{aligned}$$

where $\rho_{123} \in (\mathcal{M}_1\overline{\otimes}\mathcal{M}_2\overline{\otimes}\mathcal{M}_3)_*$. (These maps are counterparts of partial traces.) For simplicity, denote

$$\pi_{12}\rho_{123} = \rho_{12}, \quad \pi_{23}\rho_{123} = \rho_{23}, \quad \pi_{2}\rho_{123} = \rho_{2}.$$

Let

$$\begin{aligned}
\mathbb{E}_{12}&\colon \mathcal{M}_1\overline{\otimes}\mathcal{M}_2\overline{\otimes}\mathcal{M}_3 \to \mathcal{M}_1\overline{\otimes}\mathcal{M}_2\overline{\otimes}\mathbb{1},\\
\mathbb{E}_{23}&\colon \mathcal{M}_1\overline{\otimes}\mathcal{M}_2\overline{\otimes}\mathcal{M}_3 \to \mathbb{1}\overline{\otimes}\mathcal{M}_2\overline{\otimes}\mathcal{M}_3,\\
\mathbb{E}_{2}&\colon \mathcal{M}_1\overline{\otimes}\mathcal{M}_2\overline{\otimes}\mathcal{M}_3 \to \mathbb{1}\overline{\otimes}\mathcal{M}_2\overline{\otimes}\mathbb{1},
\end{aligned}$$

be defined for $x_1 \in \mathcal{M}_1$, $x_2 \in \mathcal{M}_2$, $x_3 \in \mathcal{M}_3$ as

$$\begin{aligned}
\mathbb{E}_{12}(x_1\otimes x_2\otimes x_3) &= \tau_3(x_3)x_1\otimes x_2\otimes \mathbb{1},\\
\mathbb{E}_{23}(x_1\otimes x_2\otimes x_3) &= \tau_1(x_1)\mathbb{1}\otimes x_2\otimes x_3\\
\mathbb{E}_{2}(x_1\otimes x_2\otimes x_3) &= \tau_1(x_1)\tau_3(x_3)\mathbb{1}\otimes x_2\otimes \mathbb{1}.
\end{aligned}$$

Then $\mathbb{E}_{12}$, $\mathbb{E}_{23}$, and $\mathbb{E}_2$ are normal conditional expectations such that

$$\begin{aligned}(\tau_1 \otimes \tau_2 \otimes \tau_3) \circ \mathbb{E}_{12} &= (\tau_1 \otimes \tau_2 \otimes \tau_3) \circ \mathbb{E}_{23} \\ &= (\tau_1 \otimes \tau_2 \otimes \tau_3) \circ \mathbb{E}_2 = \tau_1 \otimes \tau_2 \otimes \tau_3.\end{aligned}$$

For arbitrary normal state ρ_{123} on $\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2 \overline{\otimes} \mathcal{M}_3$, and arbitrary $x_1 \in \mathcal{M}_1$, $x_2 \in \mathcal{M}_2$, we have

$$\begin{aligned}&\tau_1 \otimes \tau_2 \otimes \tau_3((\mathbb{E}_{12} h_{\rho_{123}}) x_1 \otimes x_2 \otimes \mathbb{1}) = \tau_1 \otimes \tau_2 \otimes \tau_3(h_{\rho_{123}}(x_1 \otimes x_2 \otimes \mathbb{1})) \\ =&\rho_{123}(x_1 \otimes x_2 \otimes \mathbb{1}) = \rho_{12}(x_1 \otimes x_2) = \tau_1 \otimes \tau_2(h_{\rho_{12}}(x_1 \otimes x_2)) \\ =&\tau_1 \otimes \tau_2 \otimes \tau_3((h_{\rho_{12}} \otimes \mathbb{1})(x_1 \otimes x_2 \otimes \mathbb{1})),\end{aligned}$$

which shows that

$$\mathbb{E}_{12} h_{\rho_{123}} = h_{\rho_{12}} \otimes \mathbb{1}.$$

Assume that the entropy $H(\rho_{12})$ is finite. Then

$$\begin{aligned}&H(\rho_{12}) = \rho_{12}(\log h_{\rho_{12}}) = \rho_{123}(\log h_{\rho_{12}} \otimes \mathbb{1}) = \rho_{123}(\log(h_{\rho_{12}} \otimes \mathbb{1}) \\ =&\rho_{123}(\log \mathbb{E}_{12} h_{\rho_{123}}) = (\rho_{123} \circ \mathbb{E}_{12})(\log \mathbb{E}_{12} h_{\rho_{123}}) \\ =&(\rho_{123} \circ \mathbb{E}_{12})(\log h_{\rho_{123} \circ \mathbb{E}_{12}}) = H(\rho_{123} \circ \mathbb{E}_{12}).\end{aligned}$$

Analogously we obtain, under the assumption of finiteness of $H(\rho_{23})$ and $H(\rho_2)$, the equalities

$$H(\rho_{23}) = H(\rho_{123} \circ \mathbb{E}_{23}),$$

and

$$H(\rho_2) = H(\rho_{123} \circ \mathbb{E}_2).$$

Now Theorem 6 with

$$\begin{aligned}\mathcal{M} = \mathcal{M}_1 \overline{\otimes} \mathcal{M}_2 \overline{\otimes} \mathcal{M}_3, \quad \mathcal{A} = \mathcal{M}_1 \overline{\otimes} \mathcal{M}_2 \overline{\otimes} \mathbb{1}, \\ \mathcal{B} = \mathbb{1} \overline{\otimes} \mathcal{M}_2 \overline{\otimes} \mathcal{M}_3, \quad \mathcal{R} = \mathbb{1} \overline{\otimes} \mathcal{M}_2 \overline{\otimes} \mathbb{1},\end{aligned}$$

gives the inequality

$$H(\rho_{12}) + H(\rho_{23}) \leqslant H(\rho_{123}) + H(\rho_2),$$

which is the strong subadditivity of entropy.

Remark. Note that the assumption of finiteness of the algebras $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$ is essential since for semifinite algebras the trace $\tau_1 \otimes \tau_2 \otimes \tau_3$ restricted to the subalgebra $\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2 \overline{\otimes} \mathbb{1}$ (or $\mathbb{1} \overline{\otimes} \mathcal{M}_2 \overline{\otimes} \mathcal{M}_3$ or $\mathbb{1} \overline{\otimes} \mathcal{M}_2 \overline{\otimes} \mathbb{1}$) need not be semifinite. The general strong subadditivity theorem for semifinite algebras is proved in [7].

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