Rényi relative entropies and noncommutative L_p-spaces

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Classical relative entropy

X a finite set, p, q probability distributions on X

- Relative entropy (divergence): a measure of "dissimilarity" of p and q;
- Axiomatic approach (Rényi, 1961): postulates for relative entropy
- a unique family of relative entropies, satisfying the postulates: Rényi relative entropies
- operational significance important quantities in information theory

Rényi relative entropies

Rényi relative α-entropy, $1 \neq \alpha > 0$

$$D_\alpha(p\|q) := \frac{1}{\alpha - 1} \log \sum_x p(x)^\alpha q(x)^{1-\alpha}$$

Kullback-Leibler divergence limit value $\alpha \to 1$:

$$D_1(p\|q) := \sum_x p(x) \log(p(x)/q(x))$$

Quantum relative entropies

The basic setting of quantum information theory:

- matrix algebras: $\mathcal{M} = \mathcal{M}_n(\mathbb{C})$
- density matrices: $0 \leq \rho \in \mathcal{M}$, $\mathrm{Tr}\,\rho = 1$
- quantum channels: completely positive trace preserving maps

Relative entropies: extension of classical

- Rényi postulates: not a unique extension
- other useful properties
- operational significance

Two extensions of Rényi relative α-entropies, $\alpha \neq 1$

ρ, σ density matrices, $0 < \alpha \neq 1$

Standard:

$$D_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1} \log \left(\mathrm{Tr}\, \rho^\alpha \sigma^{1-\alpha}\right)$$

D. Petz, *Rep. Math. Phys., 1984*

Sandwiched:

$$\tilde{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1} \log \mathrm{Tr} \left[\left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha\right]$$

M. Müller-Lennert et al., *J. Math. Phys.*, 2013

M. M. Wilde et al., *Commun. Math. Phys.*, 2014

Useful properties of D_α and $\tilde{D}_\alpha$

- strict positivity:

$$D(\rho\|\sigma) \geq 0 \text{ and } D(\rho\|\sigma) = 0 \text{ iff } \rho = \sigma$$

- data processing inequality (monotonicity):

$$D(\rho\|\sigma) \geq D(\Phi(\rho)\|\Phi(\sigma))$$

  for any quantum channel Φ

Holds for restricted values of α:

standard: $\alpha \in (0, 2]$, sandwiched: $\alpha \in [1/2, \infty)$

Useful properties of D_α and $\tilde{D}_\alpha$

Umegaki relative entropy as limit value:

$$\begin{aligned}\lim_{\alpha\to 1} D_\alpha(\rho\|\sigma) &= \lim_{\alpha\to 1} \tilde{D}_\alpha(\rho\|\sigma) \\ &= D_1(\rho\|\sigma) := \mathrm{Tr}\,\rho(\log(\rho) - \log(\sigma))\end{aligned}$$

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Operational significance:

standard: $\alpha \in (0,1)$, sandwiched: $\alpha > 1$

important quantities in quantum information theory

Extension to von Neumann algebras

A more general setting:

- von Neumann algebras: $\mathcal{M}$
- normal states: $\rho \in \mathfrak{S}_*(\mathcal{M})$
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Goals (outline):

- extend the Rényi relative entropies to this setting
- prove some properties
- characterization of sufficient channels

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Tools:

- noncommutative L_p-spaces
- interpolation
- conditional expectations

Haagerup L_p-spaces

For $1 \leq p \leq \infty$, $L_p(\mathcal{M})$ - Haagerup L_p-space:

- Banach space of (unbounded) operators;

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- if $\mathcal{M} = B(\mathcal{H})$: $L_p(\mathcal{M}) \simeq S_p(\mathcal{H})$ Schatten class:

$$S_p(\mathcal{H}) = \{T \in B(\mathcal{H}),\ \mathrm{Tr}\,|T|^p < \infty\},\ \|T\|_p = (\mathrm{Tr}\,|T|^p)^{1/p}$$

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- $L_\infty(\mathcal{M}) \simeq \mathcal{M}$, $\|x\|_\infty = \|x\|$, $x \in \mathcal{M}$;
- $L_1(\mathcal{M}) \simeq \mathcal{M}_*$:

$$\mathcal{M}_* \ni \rho \mapsto h_\rho \in L_1(\mathcal{M}), \quad h_\rho \in L_1(\mathcal{M})^+ \text{ iff } \rho \geq 0$$

this defines a trace in $L_1(\mathcal{M})$: $\mathrm{Tr}\, h_\rho := \rho(1)$.

Haagerup L_p-spaces

For $1 \leq p \leq \infty$, $L_p(\mathcal{M})$ - Haagerup L_p-space:

- unique polar decomposition: $S \in L_p(\mathcal{M})$, $1 \leq p < \infty$:

$$S = uh_\psi^{1/p}, \qquad \psi \in \mathcal{M}_*^+, \; u \in \mathcal{M} \text{ partial isometry}$$

and then $\|S\|_p = (\mathrm{Tr}\,|S|^p)^{1/p} = \psi(1)^{1/p}$.

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  and then $\|S\|_p = (\mathrm{Tr}\,|S|^p)^{1/p} = \psi(1)^{1/p}$.
- Hölder inequality: if $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, then

  $$ST \in L_r(\mathcal{M}), \quad \|ST\|_r \leq \|S\|_p \|T\|_q$$

  for $S \in L_p(\mathcal{M})$ and $T \in L_q(\mathcal{M})$.

Haagerup L_p-spaces

For $1 \leq p \leq \infty$, $L_p(\mathcal{M})$ - Haagerup L_p-space:

- for $\frac{1}{p} + \frac{1}{q} = 1$, we have the duality

$$\langle\, S, T \,\rangle = \mathrm{Tr}\, ST, \quad S \in L_p(\mathcal{M}),\ T \in L_q(\mathcal{M}),$$

for $1 \leq p < \infty$, $L_p(\mathcal{M})^* \simeq L_q(\mathcal{M})$.

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- $L_2(\mathcal{M})$ is a Hilbert space, with inner product

$$\langle\, \xi, \eta\,\rangle := \mathrm{Tr}\, \eta^* \xi, \qquad \xi, \eta \in L_2(\mathcal{M})$$

A standard form for $\mathcal{M}$

- a representation $\lambda : \mathcal{M} \to B(L_2(\mathcal{M}))$:

$$\lambda(x)\xi = x\xi, \qquad x \in \mathcal{M},\ \xi \in L_2(\mathcal{M});$$

- a conjugation $J : L_2(\mathcal{M}) \to L_2(\mathcal{M})$:

$$J\xi = \xi^*, \qquad \xi \in L_2(\mathcal{M});$$

- positive cone $L_2(\mathcal{M})^+ =$ positive operators in $L_2(\mathcal{M})$.

Any $\rho \in \mathcal{M}_*^+$ has a unique vector representative $\xi \in L_2(\mathcal{M})^+$:

$$\rho(x) = \langle\, x\xi, \xi\,\rangle,\ x \in \mathcal{M}; \qquad \xi = h_\rho^{1/2}.$$

An extension of the standard Rényi relative entropies

Let $\rho, \sigma \in \mathfrak{S}_*(\mathcal{M})$. We use relative modular operator $\Delta_{\rho,\sigma}$:

$$\Delta_{\rho,\sigma}^t \xi = h_\rho^t \xi h_\sigma^{-t}, \qquad t \in \mathbb{R}$$

(unbounded) operator on $L_2(\mathcal{M})$, and we put for $1 \neq \alpha > 0$

$$D_\alpha(\rho \| \sigma) := \frac{1}{\alpha - 1} \log \| \Delta_{\rho,\sigma}^{\alpha/2} h_\rho^{1/2} \|_2^2 \quad \text{if} \quad h_\rho^{1/2} \in \mathcal{D}(\Delta_{\rho,\sigma}^{\alpha/2})$$

and is ∞ otherwise.

D. Petz, 1985

Properties of the standard Rényi relative entropies

For $\alpha \in (0, 2]$:

- strict positivity;
- data processing inequality: for all quantum channels Φ
$$D_\alpha(\rho\|\sigma) \geq D_\alpha(\Phi(\rho)\|\Phi(\sigma));$$
- limit value: $\lim_{\alpha\to 1} D_\alpha(\rho\|\sigma) = D_1(\rho\|\sigma)$,

Araki relative entropy

$$D_1(\rho\|\sigma) = \begin{cases} \langle h_\rho^{1/2}, \log(\Delta_{\rho,\sigma}) h_\rho^{1/2} \rangle, & \text{if } s(\rho) \leq s(\sigma) \\ \infty & \text{otherwise} \end{cases}$$

Araki, 1976

Kosaki L_p-spaces with respect to a faithful normal state

Let σ be a faithful normal state. We use complex interpolation:

- continuous embedding

$$\mathcal{M} \to L_1(\mathcal{M}), \quad x \mapsto h_\sigma^{1/2} x h_\sigma^{1/2}$$

- interpolation spaces

$$L_p(\mathcal{M}, \sigma) := C_{1/p}(\mathcal{M}, L_1(\mathcal{M})), \qquad 1 \leq p \leq \infty$$

  with norm $\|\cdot\|_{p,\sigma}$.

Kosaki L_p-spaces with respect to a faithful normal state

For $1/p + 1/q = 1$, the map

$$i_p : L_p(\mathcal{M}) \to L_1(\mathcal{M}), \qquad S \mapsto h_\sigma^{1/2q} S h_\sigma^{1/2q}$$

is an isometric isomorphism of $L_p(\mathcal{M})$ onto $L_p(\mathcal{M}, \sigma)$.

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For $p = \infty$:

$$L_\infty(\mathcal{M}, \sigma) = \{h_\sigma^{1/2} x h_\sigma^{1/2},\ x \in \mathcal{M}\}, \quad \|h_\sigma^{1/2} x h_\sigma^{1/2}\|_{\infty,\sigma} = \|x\|.$$

Positive elements: if $\psi \in \mathcal{M}_*^+$, $h_\psi \in L_\infty(\mathcal{M}, \sigma)$ if and only if

$$\|h_\psi\|_{\infty,\sigma} = \inf\{\lambda > 0, \psi \le \lambda\sigma\} < \infty.$$

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is an isometric isomorphism of $L_p(\mathcal{M})$ onto $L_p(\mathcal{M}, \sigma)$.

- for $1 \leq p \leq p' \leq \infty$:

$$L_\infty(\mathcal{M}, \sigma) \subseteq L_{p'}(\mathcal{M}, \sigma) \subseteq L_p(\mathcal{M}, \sigma) \subseteq L_1(\mathcal{M})$$

  and for $h \in L_{p'}(\mathcal{M}, \infty)$,

$$\|h\|_1 \leq \|h\|_{p,\sigma} \leq \|h\|_{p',\sigma}.$$

Interpolation techniques

Complex interpolation: functions on a strip

$$\mathbb{S} = \{z \in \mathbb{C}, 0 \leq Re(z) \leq 1\}$$

For $1 \leq p \leq p' \leq \infty$, let

$\mathcal{F}_{p,p'} = \{f : \mathbb{S} \to L_p(\mathcal{M}, \sigma)$, such that:

(a) f is bounded, continuous on $\mathbb{S}$, analytic in $int(\mathbb{S})$;

(b) $f(it) \in L_{p'}(\mathcal{M}, \sigma)$, $\forall t \in \mathbb{R}$;

(c) $t \mapsto f(it)$ is continuous and bounded $\mathbb{R} \to L_{p'}(\mathcal{M}, \sigma)\}$

A norm in $\mathcal{F}_{p,p'}$:

$$|||f||| := \max\{\sup_{t\in\mathbb{R}} \|f(it)\|_{p',\sigma}, \sup_{t\in\mathbb{R}} \|f(1+it)\|_{p,\sigma}\}$$

Interpolation techniques

A special function for $h = h_\sigma^{1/2q} u h_\mu^{1/p} h_\sigma^{1/2q} \in L_p(\mathcal{M}, \sigma)$:

$$f_{h,p}(z) = \mu(1)^{1/p - z} h_\sigma^{(1-z)/2} u h_\mu^{z} h_\sigma^{(1-z)/2}, \qquad z \in \mathbb{S}$$

We have

- $f_{h,p} \in \mathcal{F}_{1,\infty}$;
- $f_{h,p}(1/p) = h$;
- $\|h\|_{p,\sigma} = \|\!|f_{h,p}|\!\|$.

Interpolation techniques

Let $f \in \mathcal{F}_{p,p'}$, for $\eta \in (0,1)$, put $1/p_\eta := \eta/p + (1-\eta)/p'$.

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$$\|f(\eta)\|_{p_\eta,\sigma} \leq |||f|||$$

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$$\|f(\eta)\|_{p_\eta,\sigma} \leq \|\|f\|\|$$

with equality if and only if

$$f(z) = f_{h,p_\eta}(z/p + (1-z)/p'), \qquad z \in \mathbb{S}$$

and then equality holds for all $\eta \in (0,1)$.

Interpolation techniques

Hadamard 3 lines theorem

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Then

$$\|f(\eta)\|_{p_\eta,\sigma} \leq \left(\sup_{t\in\mathbb{R}} \|f(it)\|_{p',\sigma}\right)^{1-\eta} \left(\sup_{t\in\mathbb{R}} \|f(1+it)\|_{p,\sigma}\right)^{\eta}$$

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with equality if and only if for some $M > 0$,

$$f(z) = f_{h,p_\eta}(z/p + (1-z)/p')M^{\eta-z}, \qquad z \in \mathbb{S}$$

and then equality holds for all $\eta \in (0,1)$.

Interpolation techniques

Riesz-Thorin interpolation theorem

Let $\Phi : L_1(\mathcal{M}) \to L_1(\mathcal{N})$ be a linear map such that

- Φ is bounded with norm $\|\Phi\|_1$;
- Φ restricts to a bounded linear map

$$L_\infty(\mathcal{M}, \sigma) \to L_\infty(\mathcal{N}, \Phi(\sigma))$$

with norm $\|\Phi\|_\infty$.

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  with norm $\|\Phi\|_\infty$.

Then Φ restricts to a bounded linear map

$$L_p(\mathcal{M}, \sigma) \to L_p(\mathcal{N}, \Phi(\sigma))$$

with norm

$$\|\Phi\|_p \leq \|\Phi\|_\infty^{1/q} \|\Phi\|_1^{1/p}.$$

An extension of the sandwiched Rényi relative entropies

For $\rho, \sigma \in \mathfrak{S}_*(\mathcal{M})$ and $1 < \alpha < \infty$:

$$\tilde{D}_\alpha(\rho\|\sigma) = \begin{cases} \frac{1}{\alpha-1}\log(\|h_\rho\|_{\alpha,\sigma}^\alpha) & \text{if } h_\rho \in L_\alpha(\mathcal{M},\sigma) \\ \infty & \text{otherwise.} \end{cases}$$

AJ, 2018

Extension to non-faithful σ: by restriction to support $s(\sigma) =: e$

$$L_p(\mathcal{M},\sigma) = \{h \in L_1(\mathcal{M}),\ h = ehe \in L_p(e\mathcal{M}e, \sigma|_{e\mathcal{M}e})\}.$$

Properties of the sandwiched Rényi relative entropies

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$$\tilde{D}_\alpha(\rho\|\sigma) \geq \tilde{D}_\alpha(\Phi(\rho)\|\Phi(\sigma));$$

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- relation to the standard version:

For normal states ρ, σ, $\alpha > 1$:

$$D_{2-1/\alpha}(\rho\|\sigma) \leq \tilde{D}_\alpha(\rho\|\sigma) \leq D_\alpha(\rho\|\sigma).$$

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- Limit value: $\lim_{\alpha\to 1} \tilde{D}_\alpha(\rho\|\sigma) = D_1(\rho\|\sigma)$

Properties of the sandwiched Rényi relative entropies

Theorem

The Araki relative entropy D_1 satisfies DPI with respect to all positive quantum channels.

A. Müller-Hermes, D. Reeb, 2017; AJ, 2018

The Araki-Masuda divergences

For $\alpha \in [1/2, 1)$, we need another definition:

- uses weighted L_p-norms on a representing Hilbert space;
- based on the Araki-Masuda L_p-spaces;
- sandwiched Rényi entropies for all $1 \neq \alpha \in [1/2, \infty)$;

Berta, Scholz & Tomamichel, AHP, 2018

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We will use the standard form $(\lambda, L_2(\mathcal{M}), J, L_2(\mathcal{M})^+)$.

Berta, Scholz & Tomamichel, AHP, 2018

The BST-norms: variational definition

For $\xi \in L_2(\mathcal{M})$:

- for $2 \leq p \leq \infty$,

$$\|\xi\|_{p,\sigma}^{BST} := \sup_{\omega \in \mathfrak{S}_*(\mathcal{M})} \|\Delta_{\omega,\sigma}^{1/2-1/p} \xi^*\|_2$$

  if $s(\omega_\xi) \leq s(\sigma)$ and is infinite otherwise

- for $1 \leq p < 2$,

$$\|\xi\|_{p,\sigma}^{BST} := \inf_{\omega \in \mathfrak{S}_*(\mathcal{M}), s(\omega) \geq s(\omega_\xi)} \|\Delta_{\omega,\sigma}^{1/2-1/p} \xi^*\|_2$$

here $\omega_\xi(a) = \langle a\xi, \xi \rangle$.

Berta, Scholz & Tomamichel, AHP, 2018

The BST-norms: variational definition

The original definition

- works for any representation $\pi : \mathcal{M} \to B(\mathcal{H})$;
- uses spatial derivatives;
- the obtained norms for $\xi \in \mathcal{H}$ depend only on ω_ξ;

Berta, Scholz & Tomamichel, AHP, 2018

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- the obtained norms for $\xi \in \mathcal{H}$ depend only on ω_ξ;

$\| \cdot \|_{p,\sigma}^{BST}$ and $\| \cdot \|_{q,\sigma}^{BST}$ are dual, $1/p + 1/q = 1$ and

$$|\langle \xi, \eta \rangle| \leq \|\xi\|_{p,\sigma}^{BST} \|\eta\|_{q,\sigma}^{BST}, \qquad \xi, \eta \in \mathcal{H}$$

Berta, Scholz & Tomamichel, AHP, 2018

The Araki-Masuda divergences

For normal states ρ, σ and $\alpha \in [1/2, 1) \cup (1, \infty)$:

$$\boxed{\tilde{D}_\alpha^{AM}(\rho\|\sigma) = \frac{1}{\alpha - 1} \log\left((\|h_\rho^{1/2}\|_{2\alpha,\sigma}^{BST})^{2\alpha} \right)}$$

Berta, Scholz & Tomamichel, AHP, 2018

The BST norms: interpolation

We can define the BST norms on $L_2(\mathcal{M})$ by interpolation:

- For $2 \leq p \leq \infty$: a continuous embedding

$$\mathcal{M} \ni x \mapsto h_\sigma^{1/2} x \in L_2(\mathcal{M})$$

  then $\| \cdot \|_{p,\sigma}^{BST}$ is the norm in $C_{2/p}(\mathcal{M}, L_2(\mathcal{M}))$.

- For $1 \leq p \leq 2$: a continuous embedding

$$L_2(\mathcal{M}) \ni \xi \mapsto h_\sigma^{1/2} \xi \in L_1(\mathcal{M})$$

  then $\|\xi\|_{p,\sigma}^{BST}$ is the norm of $h_\sigma^{1/2}\xi$ in $C_{2/p-1}(L_2(\mathcal{M}), L_1(\mathcal{M}))$.

The Araki-Masuda divergences, $\alpha > 1$

For $2 \leq p \leq \infty$, $h_\rho^{1/2} \in C_{2/p}(\mathcal{M}, L_2(\mathcal{M}))$ if and only if

$$h_\rho^{1/2} = h_\sigma^{1/2-1/p} h_\mu^{1/p} u, \qquad \mu \in \mathcal{M}_*^+,\ u \in \mathcal{M} \text{ partial isometry}$$

and then $\|h_\rho^{1/2}\|_{p,\sigma}^{BST} = \mu(1)^{1/p}$.

For $\alpha > 1$, $\rho, \sigma \in \mathfrak{S}_*(\mathcal{M})$,

$$D_\alpha^{AM}(\rho\|\sigma) = \tilde{D}_\alpha(\rho\|\sigma).$$

The Araki-Masuda divergences, $\alpha \in [1/2, 1)$

For $2 \leq p \leq \infty$, $h \in L_1(\mathcal{M})$, then $h \in C_{2/p-1}(L_2(\mathcal{M}), L_1(\mathcal{M}))$ if and only if

$$h = h_\sigma^{1/q} h_\mu^{1/p} u, \qquad \mu \in \mathcal{M}_*^+, \ u \in \mathcal{M} \text{ partial isometry}$$

and its norm is $\mu(1)^{1/p}$. Putting

$$h = h_\sigma^{1/2} h_\rho^{1/2} = h_\sigma^{1/q} h_\sigma^{1/2-1/q} h_\rho^{1/2},$$

we obtain $\|h_\rho^{1/2}\|_{p,\sigma}^{BST} = \|h_\sigma^{1/2-1/q} h_\rho^{1/2}\|_p$, so that:

For $\alpha \in [1/2, 1)$, $\rho, \sigma \in \mathfrak{S}_*(\mathcal{M})$,

$$D_\alpha^{AM}(\rho\|\sigma) = \frac{1}{\alpha - 1} \log \|h_\sigma^{(1-\alpha)/2\alpha} h_\rho^{1/2}\|_{2\alpha}^{2\alpha} =: \tilde{D}_\alpha(\rho\|\sigma).$$

Properties of $\tilde{D}_\alpha$, $\alpha \in [1/2, 1)$

- extension of the sandwiched Rényi relative entropy;
- strictly positive;
- relation to the standard version:

For normal states ρ, σ, $\alpha \in (1/2, 1)$:

$$D_{2-1/\alpha}(\rho\|\sigma) \leq \tilde{D}_\alpha(\rho\|\sigma) \leq D_\alpha(\rho\|\sigma).$$

- Limit value: $\lim_{\alpha\to 1} \tilde{D}_\alpha(\rho\|\sigma) = D_1(\rho\|\sigma)$

Data processing inequality for $\tilde{D}_\alpha$, $\alpha \in [1/2, 1)$

Let $\Phi : L_1(\mathcal{M}) \to L_1(\mathcal{N})$ be a quantum channel. Here we have to assume that Φ is completely positive.

Berta, Scholz & Tomamichel, AHP, 2018

Data processing inequality for $\tilde{D}_\alpha$, $\alpha \in [1/2, 1)$

Let $\Phi : L_1(\mathcal{M}) \to L_1(\mathcal{N})$ be a quantum channel. Here we have to assume that Φ is completely positive.

- Stinespring representation: $\pi : \mathcal{N} \to B(\mathcal{H})$, isometry $V : L_2(\mathcal{M}) \to \mathcal{H}$ such that

$$\Phi^*(x) = V^*\pi(x)V, \qquad x \in \mathcal{N};$$

Berta, Scholz & Tomamichel, AHP, 2018

Data processing inequality for $\tilde{D}_\alpha$, $\alpha \in [1/2, 1)$

Let $\Phi : L_1(\mathcal{M}) \to L_1(\mathcal{N})$ be a quantum channel. Here we have to assume that Φ is completely positive.

- Stinespring representation: $\pi : \mathcal{N} \to B(\mathcal{H})$, isometry $V : L_2(\mathcal{M}) \to \mathcal{H}$ such that

$$\Phi^*(x) = V^* \pi(x) V, \qquad x \in \mathcal{N};$$

- $Vh_\rho^{1/2}$ is a vector representation of $\Phi(\rho)$ with respect to π:

$$\langle\, Vh_\rho^{1/2}, \pi(x) Vh_\rho^{1/2} \,\rangle = \rho(\Phi^*(x)) = \Phi(\rho)(x), \qquad x \in \mathbb{N};$$

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- this means that

$$\tilde{D}_\alpha(\Phi(\rho)\|\Phi(\sigma)) = \frac{1}{\alpha - 1} \log((\| Vh_\rho^{1/2} \|_{2\alpha,\Phi(\sigma)}^{BST})^{2\alpha})$$

Berta, Scholz & Tomamichel, AHP, 2018

Data processing inequality for $\tilde{D}_\alpha$, $\alpha \in [1/2, 1)$

- for $2 \leq p < \infty$: by DPI for $\tilde{D}_\alpha$, $\alpha > 1$

$$\|Vh_\rho^{1/2}\|_{p,\Phi(\sigma)}^{BST} \leq \|h_\rho\|_{p,\sigma}^{BST}$$

Berta, Scholz & Tomamichel, AHP, 2018

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$$\| V h_\rho^{1/2} \|_{p,\Phi(\sigma)}^{BST} \leq \| h_\rho \|_{p,\sigma}^{BST}$$

- for $1 < p \leq 2$: Let $h_\sigma^{1/p-1/2} h_\rho^{1/2} = h_\mu^{1/p} u$, and put

$$\xi := \mu(1)^{-1/q} h_\sigma^{1/p-1/2} h_\mu^{1/q} u.$$

Then $\|\xi\|_{q,\sigma}^{BST} = 1$ and

$$\begin{aligned} \| h_\rho^{1/2} \|_{p,\sigma}^{BST} &= \langle\, h_\rho^{1/2}, \xi \,\rangle = \langle\, V h_\rho^{1/2}, V\xi \,\rangle \\ &\leq \| V h_\rho^{1/2} \|_{p,\Phi(\sigma)}^{BST} \| V\xi \|_{q,\Phi(\sigma)}^{BST} \\ &\leq \| V h_\rho^{1/2} \|_{p,\Phi(\sigma)}^{BST} \end{aligned}$$

Berta, Scholz & Tomamichel, AHP, 2018

Sufficient (reversible) channels

Let

- $\Phi : L_1(\mathcal{M}) \to L_1(\mathcal{N})$ be a channel (completely positive)
- ρ, σ normal states, with $s(\rho) \leq s(\sigma)$.

Definition

Φ is sufficient with respect to $\{\rho, \sigma\}$ if there exists a recovery map: a channel $\Psi : L_1(\mathcal{N}) \to L_1(\mathcal{M})$, such that

$$\Psi \circ \Phi(h_\rho) = h_\rho, \qquad \Psi \circ \Phi(h_\sigma) = h_\sigma.$$

D. Petz., *Commun. Math. Phys.*, 1986

Characterizations of sufficient channels

Universal (Petz) recovery map: There is a channel $\Phi_\sigma : L_1(\mathcal{N}) \to L_1(\mathcal{M})$, such that $\Phi_\sigma \circ \Phi(h_\sigma) = h_\sigma$ and

Φ is sufficient with respect to $\{\rho, \sigma\}$ if and only if

$$\Phi_\sigma \circ \Phi(h_\rho) = h_\rho$$

D. Petz, *Quart. J. Math. Oxford*, 1988

Characterizations of sufficient channels

There exists a conditional expectation $E : \mathcal{M} \to \mathcal{M}$ onto the fixed point subalgebra of $\Phi^* \circ \Phi_\sigma^*$ and we have

$E_*(h_\sigma) = h_\sigma$ and Φ is sufficient with respect to $\{\rho, \sigma\}$ if and only if

$$E_*(h_\rho) = h_\rho.$$

Characterization of sufficient channels by divergences

A divergence D characterizes sufficiency if

$$D(\Phi(\rho)\|\Phi(\sigma)) = D(\rho\|\sigma) < \infty$$

implies that Φ is sufficient with respect to $\{\rho, \sigma\}$.

The following divergences characterize sufficiency:

- D_1 (Araki relative entropy)
- D_α for $\alpha \in (0, 1)$ (standard Rényi relative entropies)

D. Petz, *Commun. Math. Phys.*, 1986
AJ, D. Petz, *IDAQP*, 2006

Characterizations of sufficient channels by $\tilde{D}_\alpha$

The sandwiched Rényi relative entropies $\tilde{D}_\alpha$ characterize sufficiency, for $\alpha \in (1/2, 1) \cup (1, \infty)$.

AJ, AHP, 2018; arXiv:1707.00047

For $\alpha > 1$:

If $h_\rho \in L_\alpha(\mathcal{M}, \sigma)$ and

$$\|\Phi(h_\rho)\|_{\alpha,\Phi(\sigma)} = \|h_\rho\|_{\alpha,\sigma}$$

then $\Phi_\sigma \circ \Phi(h_\rho) = h_\rho$.

Characterizations of sufficient channels by $\tilde{D}_\alpha$

An easy proof for $\alpha = 2$:

- $L_2(\mathcal{M}, \sigma)$ is a Hilbert space
- Φ_σ is the adjoint of the contraction

$$\Phi : L_2(\mathcal{M}, \sigma) \to L_2(\mathcal{N}, \Phi(\sigma))$$

- $\|\Phi(h_\rho)\|_{2,\Phi(\sigma)} = \|h_\rho\|_{2,\sigma}$

By well known properties of contractions on Hilbert spaces:

$$\Phi_\sigma \circ \Phi(h_\rho) = h_\rho.$$

Characterizations of sufficient channels by $\tilde{D}_\alpha$

For $1 < \alpha \neq 2$, let $h_\rho = h_\sigma^{1/2\beta} h_\mu^{1/\alpha} h_\sigma^{1/2\beta}$ and let

$$f = f_{h_\rho,\alpha}: \quad f(z) = \mu(1)^{1/\alpha - z} h_\sigma^{(1-z)/2} h_\mu^{z} h_\sigma^{(1-z)/2}, \quad z \in \mathbb{S}$$

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Then

$$\|\Phi(h_\rho)\|_{\alpha,\Phi(\sigma)} \leq |||\Phi(f)||| \leq |||f||| = \|h_\rho\|_{\alpha,\sigma} = \|\Phi(h_\rho)\|_{\alpha,\Phi(\sigma)}$$

and hence

$$\|\Phi(f(\eta))\|_{1/\eta,\Phi(\sigma)} = \|f(\eta)\|_{1/\eta,\sigma}, \qquad \forall \eta \in (0,1)$$

Since $f(1/2) \in L_2(\mathcal{M},\sigma)$, it follows that

$$\Phi_\sigma \circ \Phi(f(1/2)) = f(1/2)$$

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By the properties (of extensions to $L_p(\mathcal{M})$) of conditional expectations, this implies that also

$$E_*(h_\rho) = E_*(h_\sigma^{1/2\beta} h_\mu^{1/\alpha} h_\sigma^{1/2\beta}) = h_\sigma^{1/2\beta} h_\mu^{1/\alpha} h_\sigma^{1/2\beta} = h_\rho$$

so that Φ is sufficient with respect to $\{\rho, \sigma\}$.

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For $\alpha \in (1/2, 1)$, we use duality of the BST norms: Let $p = 2\alpha$.

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- there is (a unique) $\xi \in L_2(\mathcal{M})$ with $\|\xi\|_{q,\sigma}^{BST} = 1$ and

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- This also implies $\|V\xi\|_{q,\Phi(\sigma)}^{BST} = 1 = \|\xi\|_{q,\sigma}^{BST}$

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\end{aligned}
$$

- This also implies $\|V\xi\|_{q,\Phi(\sigma)}^{BST} = 1 = \|\xi\|_{q,\sigma}^{BST}$
- We can invoke the previous result for $\alpha^* = q/2 > 1$ and the state $\omega := \|\xi\|_2^{-1}\omega_\xi$:

$$
D_{\alpha^*}(\Phi(\omega)\|\Phi(\sigma)) = D_{\alpha^*}(\omega\|\sigma) < \infty
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- so $E_*(h_\mu) = h_\mu$ and this implies

$$E_*(h_\rho) = h_\rho$$

so that Φ is sufficient with respect to $\{\rho, \sigma\}$.

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Thank you for your attention.