This is a note on the tensor product of finite MV-algebras. Let $E = \mathbb{Z}^k[0,u]$, $F = \mathbb{Z}^l[0,v]$. Let $e_1, \dots, e_k$ be the atoms in E and $f_1, \dots, f_l$ the atoms in F. Let

$$u = \sum_i u_i e_i, \qquad v = \sum_j v_j f_j.$$

Let also G be the finite MV-algebra with atoms $e_i \otimes f_j$ and order unit

$$w = \sum_{i,j} u_i v_j (e_i \otimes f_j).$$

We show that $G \simeq E \otimes F$ (in the category of effect algebras).

So let $\otimes : E \times F \to G$ be the bimorphism determined by $(e_i, f_j) \mapsto e_i \otimes f_j \in G$. Then we have

$$a \otimes b = \sum_i a_i b_j (e_i \otimes f_j).$$

Let H be an effect algebra and let $\beta : E \times F \to H$ be a bimorphism. Any morphism $\psi : G \to H$ such that $\psi \circ \otimes = \beta$ must satisfy

$$\psi(e_i \otimes f_j) = \beta(e_i, f_j).$$

We need to show that this prescription extends to a morphism $G \to H$, which is then necessarily unique. So let $y \in G$, then

$$y = \sum_{i,j} y_{ij}(e_i \otimes f_j), \qquad y_{ij} \leq u_i v_j, \ \forall i, j.$$

For each i, j, let

$$y_{ij} = v_j q_{ij} + r_{ij}, \qquad r_{ij} < v_j,$$

then since $v_j q_{ij} \leq y_{ij} \leq u_i v_j$, we have $q_{ij} \leq u_i$, with equality only if $r_{ij} = 0$. We then have

$$\begin{aligned} y &= \sum_j (\sum_i q_{ij} e_i \otimes v_j f_j + \sum_i e_i \otimes r_{ij} f_j) \\ &= \sum_j (a_j \otimes v_j f_j + \sum_{i, r_{ij} > 0} e_i \otimes r_{ij} f_j) \end{aligned}$$

where $a_j := \sum_i q_{ij}e_i \in E$ and $r_{ij}f_j \in F$. Put $a'_j := \sum_{i,r_{ij}>0} e_i$, then $a_j \perp a'_j$. Now we write

$$\begin{aligned}\beta(u,v) &= \sum_j \beta(u, v_j f_j) = \sum_j [\beta(a_j, v_j f_j) + \beta(u - a_j, v_j f_j)] \\ &= \sum_j \big[\beta(a_j, v_j f_j) + \beta(a'_j, v_j f_j) + \beta(u - (a_j + a'_j), v_j f_j)\big] \\ &= \sum_j [\beta(a_j, v_j f_j) + \sum_i \beta(e_i, r_{ij} f_j) + \sum_{i,r_{ij}>0} \beta(e_i, (v_j - r_{ij}) f_j) \\ &+ \beta(u - (a_j + a'_j), v_j f_j)]\end{aligned}$$

It follows that

$$\begin{aligned}\psi(y) &= \sum_j [\beta(a_j, v_j f_j) + \sum_i \beta(e_i, r_{ij} f_j)] = \sum_{i,j} [q_{ij} v_j \beta(e_i, f_j) + r_{ij} \beta(e_i, f_j)] \\ &= \sum_{i,j} y_{ij} \beta(e_i, f_j)\end{aligned}$$

is a well defined element in H.