Sufficiency of quantum channels by Rényi divergences

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  for any quantum channel Φ;
- operational significance: relation to performance of some procedures in information - theoretic tasks

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- relative entropy:

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- more general f-divergences;
- distinguishability measures from hypothesis testing:

$$P_t(\rho\|\sigma) = \|\rho - t\sigma\|_1, \qquad t > 0;$$

- distinguishability measures for n copies:

$$P_{t,n}(\rho\|\sigma) = \|\rho^{\otimes n} - t\sigma^{\otimes n}\|_1, \qquad t > 0,\ n \in \mathbb{N}.$$

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Definition

We say that Φ is sufficient with respect to $\mathcal{S}$ if there is a channel Ψ (recovery map) such that

$$\Psi \circ \Phi(\rho) = \rho \qquad \forall \rho \in \mathcal{S}.$$

D. Petz, *Commun. Math. Phys.*, 1986

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Theorem

Let $\sigma \in \mathcal{S}$ be faithful. Φ is sufficient with respect to $\mathcal{S}$ if and only if

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Theorem

The same holds for a large class of f-divergences, e.g. the standard Rényi divergences.

D. Petz, M. Mosonyi, F. Hiai

Classical Rényi divergences

For p, q probability measures over a finite set X, $0 < \alpha \neq 1$:

$$D_\alpha(p\|q) := \frac{1}{\alpha - 1} \log \sum_x p(x)^\alpha q(x)^{1-\alpha}$$

A. Rényi, *Proc. Symp. on Math., Stat. and Probability*, 1961.

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- relative entropy as a limit $\alpha \to 1$

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Sandwiched:

$$\tilde{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1} \log \mathrm{Tr} \left[\left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}} \right)^\alpha \right]$$

M. Müller-Lennert et al., *J. Math. Phys.*, 2013
M. M. Wilde et al., *Commun. Math. Phys.*, 2014

Quantum Rényi divergences: properties

Standard version D_α,

- strict positivity, monotonicity: $\alpha \in (0, 2]$;

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Both versions: relative entropy as a limit for $\alpha \to 1$.

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The standard Rényi divergence D_α determines sufficiency for all $\alpha \in (0, 2)$.

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This talk

The same holds for the sandwiched Rényi divergence $\tilde{D}_\alpha$ with $\alpha \in (1/2, 1)$ and $\alpha > 1$.

AJ, Ann. H. Poincaré, 2018
AJ, arXiv:1707.00047

Sandwiched Rényi divergences and weighted L_p-spaces

$\mathcal{M}$ a von Neumann algebra, $L_p(\mathcal{M})$ - Haagerup L_p-space,

σ a faithful state, identified with an element in $L_1(\mathcal{M})$.

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Kosaki L_p-spaces: complex interpolation

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- continuous embedding

$$\mathcal{M} \to L_1(\mathcal{M}), \quad x \mapsto \sigma^{1/2} x \sigma^{1/2}$$

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- for $1/p + 1/q = 1$, the map

$$i_p : L_p(\mathcal{M}) \to L_1(\mathcal{M}), \qquad k \mapsto \sigma^{1/2q} k \sigma^{1/2q}$$

is an isometric isomorphism of $L_p(\mathcal{M})$ onto $L_p(\mathcal{M}, \sigma)$.

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Extension to non-faithful σ: by restriction to support $s(\sigma) = e$

$$L_p(\mathcal{M}, \sigma) = \{h \in L_1(\mathcal{M}),\ h = ehe \in L_p(e\mathcal{M}e, \sigma|_{e\mathcal{M}e})\}.$$

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For normal states ρ, σ and $1 < \alpha < \infty$:

$$\tilde{D}_\alpha(\rho\|\sigma) = \begin{cases} \frac{\alpha}{\alpha-1}\log(\|\rho\|_{\alpha,\sigma}) & \text{if } \rho \in L_\alpha(\mathcal{M}, \sigma) \\ \infty & \text{otherwise.} \end{cases}$$

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Extends the sandwiched Rényi divergence to von Neumann algebras.

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Corollary

S is monotone under positive trace preserving maps.

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Φ_σ - Petz dual, universal recovery map.

Universal recovery map

Theorem

Φ is sufficient with respect to $\mathcal{S}$ if and only if all $\rho \in \mathcal{S}$ are invariant states for the channel $\Phi_\sigma \circ \Phi$.

D. Petz, *Quart. J. Math. Oxford*, 1988

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Mean ergodic theorem: there is a faithful normal conditional expectation E such that

$$\Phi_\sigma \circ \Phi(\rho) = \rho \iff \rho \circ E = \rho.$$

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Let $\rho \in \mathcal{S} \subset L_\alpha(\mathcal{M}, \sigma)$, then

$$\rho = t\sigma^{1/2\beta}\tau^{1/\alpha}\sigma^{1/2\beta}$$

for a normal state τ, $t > 0$, $\alpha^{-1} + \beta^{-1} = 1$.

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$$\rho_{\alpha'} := t_{\alpha'}\sigma^{1/2\beta'}\tau^{1/\alpha'}\sigma^{1/2\beta'} \in L_{\alpha'}(\mathcal{M}, \sigma), \qquad \alpha' > 1$$

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By properties of conditional expectations:

If $\Phi_\sigma \circ \Phi(\rho_{\alpha'}) = \rho_{\alpha'}$ for some $\alpha' > 1$, then for all.

The case $\alpha \in (1/2, 1)$

For $\alpha \in (1/2, 1)$, we have $\sigma^{\frac{1-\alpha}{2\alpha}}\rho^{1/2} \in L_{2\alpha}(\mathcal{M})$. Put

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M. Berta, V. B. Scholz, and M. Tomamichel, Ann. H. Poincaré, 2018

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M. Berta, V. B. Scholz, and M. Tomamichel, Ann. H. Poincaré, 2018

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Sufficiency of channels by $\tilde{D}_\alpha$, $\alpha \in (1/2, 1)$

Let $\alpha \in (1/2, 1)$, then by polar decomposition:

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- Since $\alpha^* > 1$, this implies that $\Phi_\sigma \circ \Phi(\omega) = \omega$;
- the same is true for τ;
- using properties of conditional expectations, we get that also $\Phi_\sigma \circ \Phi(\rho) = \rho$.