Rényi divergences in quantum information theory

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What is a divergence?

- A "dissimilarity measure" on probability distributions:

  For probability distributions p, q

  $$D(p\|q) \equiv \text{how different } p \text{ is from } q.$$

- A contrast functional:

  $$D(p\|q) \geq 0, \qquad D(p\|q) = 0 \iff p = q.$$

- Not a metric (not necessarily symmetric)
- Other properties?

Axiomatic approach (A. Rényi, 1961)

Let $p = (p_1, \ldots, p_m)$, $q = (q_1, \ldots, q_m)$, $p_i \geq 0$, $q_i > 0$

A divergence D should satisfy the postulates:

- invariance under permutations: $D(\pi(p)\|\pi(q)) = D(p\|q)$
- continuity
- additivity: $D(p_1 \otimes p_1\|q_1 \otimes q_2) = D(p_1\|q_1) + D(p_2\|q_2)$
- generalized mean: for a continuous, strictly increasing real function g

$$D(p_1 \oplus p_2\|q_1 \oplus q_2) = g^{-1}\left(\frac{g(D(p_1\|q_1) + g(D(p_2\|q_2))}{2}\right)$$

- order: if p is not normalized

$$p_i \leq q_i, \forall i \implies D(p\|q) \geq 0, \qquad p_i \geq q_i, \forall i \implies D(p\|q) \leq 0$$

- normalization: $D(\{1\}\|\{1/2\}) = 1$

Rényi divergences

There is a unique family of divergences $\{D_\alpha\}_{\alpha>0}$, satisfying the Rényi postulates:

$$D_\alpha(p\|q) = \frac{1}{\alpha-1}\log\left(\sum_k p_k^\alpha q_k^{1-\alpha}\right), \qquad 1 \neq \alpha > 0$$

$$D_1(p\|q) = \lim_{\alpha\to 1} D_\alpha(p\|q) = \sum_k p_k \log\left(\frac{p_k}{q_k}\right)$$

- Fundamental quantities in information theory
- For $\alpha = 1$, we get the Kullback-Leibler divergence (relative entropy, I-divergence)

Application: Asymptotic hypothesis testing

Testing simple hypothesis $H_0 = p$ against simple alternative $H_1 = q$:

- A test statistic: T
- Two kinds of error probabilities:

$$\alpha(T) \text{ - rejecting true} \qquad \beta(T) \text{ - accepting false}$$

- Cannot minimize both errors simultaneously
- i.i.d. repetitions: a sequence of tests $\{T_n\}$
- We can obtain $\alpha(T_n) \to 0$, $\beta(T_n) \to 0$ exponentially
- Rate of the convergence?

Stein's lemma and error exponents

Direct domain

$\exists\ \{T_n\}$: $\beta(T_n) \approx e^{-Rn}$

and

$$\alpha(T_n) \to 0$$

optimal exponent (largest):

$$\alpha(T_n) \approx e^{-R'n}$$

Converse domain

$\forall \{T_n\}$, $\beta(T_n) \approx e^{-Rn}$

$$\Downarrow$$

$$\alpha(T_n) \to 1$$

optimal exponent (smallest):

$$\alpha(T_n) \approx 1 - e^{-R'n}$$

$$D(p\|q)$$

Tradeoff between R and R':

- Direct domain - $D_\alpha(p\|q)$, $\alpha \in (0,1)$
- Converse domain - $D_\alpha(p\|q)$, $\alpha > 1$

A basic property: DPI and sufficient statistics

Data processing inequality: For a transformation

$T : \{1, \ldots, m\} \to \{1, \ldots, k\}$, with p^T, q^T induced distributions

$$D_\alpha(p^T \| q^T) \leq D_\alpha(p \| q)$$

- Any reasonable divergence should satisfy DPI!

Kullback-Leibler-Csiszár Theorem: If $\mathrm{supp}(p) \subseteq \mathrm{supp}(q)$, $\alpha > 1$

$D_\alpha(p^T \| q^T) = D_\alpha(p \| q) \iff T$ is a sufficient statistic for $\{p, q\}$:

- conditional expectations $E_p[\cdot|T] = E_q[\cdot|T]$
- T contains all information needed to distinguish p from q.

Quantum divergences

Quantum information theory:

- quantum states instead of probability measures
- simplest case: density matrices

$$\rho \in M_n(\mathbb{C}),\ \ \rho \geq 0,\ \ \mathrm{Tr}\,[\rho] = 1$$

- general case: normal states of a von Neumann algebra
  - covers most of interesting situations
  - powerful technical tools

Quantum divergences: dissimilarity measures for quantum states

Postulates for quantum divergences?

- Postulates similar to Rényi (Müller-Lennert et al, 2013)
- In the classical case (commuting density matrices) we get the unique family of Rényi divergences $\{D_\alpha\}_{\alpha>0}$
- In general quantum case: no unique solution

Quantum DPI

Quantum channel: a linear map $\Phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$

- completely positive: $\mathrm{id}_k : M_k(\mathbb{C}) \to M_k(\mathbb{C})$ identity map

$$\Phi \otimes \mathrm{id}_k \text{ is positive for any } k \geq 1$$

- trace-preserving: $\mathrm{Tr}\,[\Phi(\rho)] = \mathrm{Tr}\,[\rho]$

Equivalently: $\Phi \otimes \mathrm{id}_k$ maps states to states, for all k.

Data processing inequality for quantum divergences:

$$D(\Phi(\rho)\|\Phi(\sigma)) \leq D(\rho\|\sigma)$$

for any quantum channel Φ and any pair of states ρ, σ.

An important quantum divergence

Quantum relative entropy (Umegaki, 1962)

$$S(\rho\|\sigma) = \mathrm{Tr}\left[\rho\left(\log(\rho) - \log(\sigma)\right)\right]$$

- satisfies postulates, DPI (Lindblad, 1975)
- fundamental in quantum information theory
- operational interpretations: quantum communication, asymptotic hypothesis testing
- related to many important quantities
- entanglement measures, uncertainty relations

Quantum Rényi divergences

Petz-type (standard) quantum Rényi divergence: (Petz, 1985,1986)

$$D_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1} \log \mathrm{Tr}\, \left[\rho^\alpha \sigma^{1-\alpha}\right], \qquad 1 \neq \alpha > 0$$

- satisfies postulates, DPI for $\alpha \in (0, 2]$
- $\lim_{\alpha\to 1} D_\alpha(\rho\|\sigma) = S(\rho\|\sigma)$
- operational interpretation for $\alpha \in (0, 1)$: (Audenaert et al., 2008, Nagaoka, 2006)

  asymptotic hypothesis testing (error exponents, direct part)

Quantum Rényi divergences

Minimal (sandwiched) quantum Rényi divergence: (Müller-Lennert et al, 2013, Wilde et al, 2014)

$$\tilde{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1} \log \mathrm{Tr} \left[\left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}} \right)^\alpha \right], \qquad 1 \neq \alpha > 0$$

- satisfies postulates, DPI for $\alpha \in [1/2, \infty)$ (Frank & Lieb, 2013)
- $\lim_{\alpha \to 1} \tilde{D}_\alpha(\rho\|\sigma) = S(\rho\|\sigma)$
- operational interpretation for $\alpha > 1$: (Mosonyi & Ogawa, 2015)

  asymptotic hypothesis testing (error exponents, converse part)

Quantum Rényi divergences

$\alpha - z$-Rényi divergence: (Jaksic et al, 2011, Audenaert & Datta, 2015)

$$D_{\alpha,z}(\rho\|\sigma) = \frac{1}{\alpha-1}\log \mathrm{Tr}\left[\left(\sigma^{\frac{1-\alpha}{2z}}\rho^{\frac{\alpha}{z}}\sigma^{\frac{1-\alpha}{2z}}\right)^{z}\right], \qquad 1 \neq \alpha > 0, z > 0$$

- satisfies postulates, DPI if: (Zhang, 2020)

  - $\alpha \in (0,1),\ \max\{\alpha, 1-\alpha\} \leq z$
  - $\alpha > 1,\ \max\{\frac{\alpha}{2}, \alpha - 1\} \leq z \leq \alpha$

- $\lim_{\alpha\to 1} D_{\alpha,z}(\rho\|\sigma) = S(\rho\|\sigma)$, $z > 1$
- Petz type: $D_{\alpha,1}(\rho\|\sigma) = D_\alpha(\rho\|\sigma)$
- Minimal: $D_{\alpha,\alpha}(\rho\|\sigma) = \tilde{D}_\alpha(\rho\|\sigma)$

Quantum Rényi divergences

Maximal Rényi divergence: (Matsumoto, 2018)

$$D_\alpha^{max}(\rho\|\sigma) = \frac{1}{\alpha - 1} \log \mathrm{Tr}\left[\sigma\left(\sigma^{-1/2}\rho\sigma^{-1/2}\right)^{\alpha}\right], \qquad 1 \neq \alpha > 0$$

- satisfies postulates, DPI if $\alpha \in (0, 2]$
- Belawkin-Staszewski relative entropy as limit

$$\lim_{\alpha\to 1} D_\alpha^{max}(\rho\|\sigma) = \mathrm{Tr}\left[\rho \log\left(\rho^{1/2}\sigma^{-1}\rho^{1/2}\right)\right]$$

Extensions to von Neumann algebras

- In some infinite dimensional situations the previous definitions do not work.
- Useful also in e.g. QFT
- Technical problems: no density matrices (operators) in general, no matrix analysis tools...
- Other tools: modular theory, non-commutative L_p-spaces, complex interpolation

Extensions to von Neumann algebras

- Relative entropy (Araki, 1976)

  - relative modular operator

- Petz-type Rényi divergences (Petz, 1985)

  - relative modular operator, operator convex functions

- Minimal Rényi divergences (Berta et al, 2018, AJ 2018, 2021)

  - weighted L_p-norms, interpolation

- $\alpha - z$-Rényi divergences (Hiai & AJ, 2024)

  - weighted L_p-norms, variational formulas

- Maximal Rényi divergences (Hiai, 2019)

  - operator means, generalized connections

Quantum Rényi divergences and L_p-spaces

Rényi divergence: $D_\alpha(\rho\|\sigma) = \frac{1}{\alpha-1}\log Q_\alpha(\rho\|\sigma)$

- Classical case: Q-weighted L_p-norm

$$Q_\alpha(P\|Q) = \int (dP/dQ)^\alpha \, dQ = \|dP/dQ\|_{\alpha,Q}^\alpha$$

- Quantum sandwiched case: σ-weighted L_p-norm

$$Q_\alpha(\rho\|\sigma) = \mathrm{Tr}\left[\left(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha\right] = \|\rho\|_{\alpha,\sigma}^\alpha,$$

- For $\alpha > 1$: complex interpolation norm (Kosaki, 1984)

  $\|\sigma^{-1/2}\rho\sigma^{-1/2}\|$ (operator norm), $\|\rho\|_1$ (trace norm)
- works in general von Neumann algebras

$\alpha - z$-Rényi divergences and L_p-spaces

Variational formula: (Kato, 2024, Hiai & AJ, 2024)

- For $\alpha > 1$, $p = \frac{z}{\alpha}$, $r = \frac{z}{1-\alpha}$:

$$Q_{\alpha,z}(\rho\|\sigma) = \inf_{a \text{ p.d.}} \left\{\alpha\|\rho^{\frac{1}{2}} a \rho^{\frac{1}{2}}\|_{p,\rho}^{p} + (1-\alpha)\|\sigma^{\frac{1}{2}} a^{-1} \sigma^{\frac{1}{2}}\|_{r,\sigma}^{r}\right\}$$

- For $\alpha > 1$, $p = \frac{z}{\alpha}$, $q = \frac{z}{\alpha-1}$:

$$Q_{\alpha,z}(\rho\|\sigma) \ = \ \sup_{a\geq 0} \left\{\alpha\|\rho^{\frac{1}{2}} a \rho^{\frac{1}{2}}\|_{p,\rho}^{p} - (\alpha-1)\|\sigma^{\frac{1}{2}} a \sigma^{\frac{1}{2}}\|_{q,\sigma}^{q}\right\}$$

- Connects to the weighted L_p-norms for all α, z, DPI holds whenever $p, q, r \geq 1$

- Extends many results to von Neumann algebras

Quantum sufficient statistics?

- Quantum statistics - quantum channels
- When is a channel Φ sufficient w. r. to a set of states $\mathcal{S}$?
- Conditional expectations do not exist in most situations

Sufficient quantum channels: (Petz, 1986)

A channel Φ is sufficient with respect to $\mathcal{S}$ if there is another channel Ψ such that

$$\Psi \circ \Phi(\rho) = \rho, \qquad \rho \in \mathcal{S}.$$

- Φ is reversible on $\mathcal{S}$, Ψ - recovery map

- sufficient statistics in classical case

Sufficient quantum channels

Characterizations of sufficient quantum channels: (Petz, 1986, 1988)

- Petz theorem: if $\operatorname{supp}\rho \leq \operatorname{supp}\sigma$ for all $\rho \in \mathcal{S}$

$$S(\Phi(\rho)\|\Phi(\sigma)) = S(\rho\|\sigma), \qquad \rho \in \mathcal{S}$$

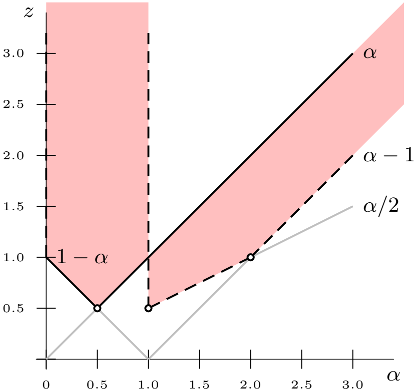
- There is a universal recovery map: Φ_σ (Petz recovery map)

$$\Phi_\sigma \circ \Phi(\rho) = \rho, \qquad \rho \in \mathcal{S}$$

- structure of the states $\rho \in \mathcal{S}$, strong conditions.
- For classical statistics: $E_q[\cdot|T] = E_p[\cdot|T]$ is the Petz recovery.

Quantum Rényi divergences and sufficient channels

Assume that α, z belong to the folllowing set:



Then Φ is sufficient w.r. to $\{\rho, \sigma\}$ if and only if (Hiai & AJ,2024)

$$D_{\alpha,z}(\Phi(\rho)\|\Phi(\sigma)) = D_{\alpha,z}(\rho\|\sigma).$$

- holds in general von Neumann algebras

Classical to quantum and in between

Classical	Classical/quantum	Quantum
discrete probability measures p, q	commuting density matrices ρ, σ	density matrices ρ, σ in $M_n(\mathbb{C})$
probability measures $P, Q \ll \mu$ on a measure space (X, Ω, μ)	$L_\infty(X, \Omega, \mu)$, densities $p, q \in L_1(X, \Omega, \mu)$	normal states ρ, σ of a von Neumann algebra
$T : X \to Y$ statistic, Markov kernel $X \times Y \to [0, 1]$	Positive trace preserving map	Quantum channel $M_n(\mathbb{C}) \to M_m(\mathbb{C})$
transformation of probability measures	A Markov map $L_\infty(X, \Omega, \mu) \to L_\infty(Y, \Sigma, \nu)$	Unital normal cp map $\mathcal{M} \to \mathcal{N}$
conditional expectation $E_p[\cdot\|T]$	unital projection with norm 1 preserving p	Petz recovery map Φ_σ