A note on monotonicity of $z \mapsto D_{\alpha,z}$ for $1 < \alpha \leq 2z$ (Corrected)

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Assume that $1 < \alpha \leq 2z \leq 2z'$. We will prove that $Q_{\alpha,z}(\psi\|\varphi) \geq Q_{\alpha,z'}(\psi\|\varphi)$. We may clearly assume that $Q_{\alpha,z}(\psi\|\varphi) < \infty$, in which case there is some $y \in L^{2z}(\mathcal{M})$ such that

$$h_\psi^{\frac{\alpha}{2z}} = y h_\varphi^{\frac{\alpha-1}{2z}}, \qquad Q_{\alpha,z}(\psi\|\varphi) = \|y\|_{2z}^{2z}.$$

In particular, $e := s(\psi) \leq s(\varphi)$, so that we may assume that φ is faithful. Let $\sigma \in \mathcal{M}_*^+$ be such that $s(\sigma) = 1 - e$ and put $\psi_0 := \psi + \sigma$, so that ψ_0 is faithful as well. We will use the notation $L_L^p := L^p(\mathcal{M};\varphi)_L$, $1 \leq p \leq \infty$.

Consider the function

$$f(w) = h_{\psi_0}^{\frac{\alpha}{2z}w} e h_\varphi^{1-\frac{\alpha}{2z}w}, \qquad w \in S,$$

where $S := \{w \in \mathbb{C},\ 0 \leq \operatorname{Re} w \leq 1\}$. Then f is a bounded continuous function $S \to L^1(\mathcal{M})$, analytic in the interior. Further,

$$f(it) = h_{\psi_0}^{\frac{\alpha}{2z}it} e h_\varphi^{-\frac{\alpha}{2z}it} h_\varphi \in L_L^\infty, \qquad t \in \mathbb{R},$$

and $\|f(it)\|_{L_L^\infty} = 1$ for all t. We also have

$$f(1+it) = h_{\psi_0}^{\frac{\alpha}{2z}it} h_\psi^{\frac{\alpha}{2z}} h_\varphi^{1-\frac{\alpha}{2z}} h_\varphi^{-\frac{\alpha}{2z}it} = (h_{\psi_0}^{\frac{\alpha}{2z}it} y h_\varphi^{-\frac{\alpha}{2z}it}) h_\varphi^{\frac{2z-1}{2z}} \in L_L^{2z}, \qquad t \in \mathbb{R}.$$

By [1, Lemmas 10.1 and 10.2],

$$\|f(1+it)\|_{L_L^{2z}} = \|h_{\psi_0}^{\frac{\alpha}{2z}it} y h_\varphi^{-\frac{\alpha}{2z}it}\|_{2z} = \|y\|_{2z}$$

and the functions $t \mapsto f(it)$ and $t \mapsto f(1+it)$ are continuous in L_L^{2z}. Therefore $f \in \mathcal{F}'(L_L^\infty, L_L^{2z})$, that is, f is a function $S \to L_L^{2z}$, bounded and continuous on S and analytic in the interior of S, such that the boundary values define bounded functions to L_L^∞ resp. L_L^{2z} (see Definition 1.4 and Remark 3.4 in [1]), so that for any $\theta \in (0,1)$, $f(\theta) \in C_\theta(L_L^\infty, L_L^{2z})$ and

$$\|f(\theta)\|_{C_\theta} \leq \|y\|_{2z}^\theta.$$

By the reiteration theorem, $C_\theta = L_L^{2z/\theta}$. Putting $\theta = z/z'$, we get

$$f(z/z') = h_\psi^{\frac{\alpha}{2z'}} h_\varphi^{1-\frac{\alpha}{2z'}} = y' h_\varphi^{\frac{2z'-1}{2z'}}$$

for some $y' \in L^{2z'}(\mathcal{M})$, and $\|y'\|_{2z'} \leq \|y\|_{2z}^{z/z'}$, this proves the result.

Remark 1. Note that this allows us also to prove monotonicity in α. Indeed, let $1 < \alpha' < \alpha$. For any $t \in \mathbb{R}$,

$$f(\frac{1}{\alpha} + it) = h_{\psi_0}^{it} h_\psi^{\frac{1}{2z}} h_\varphi^{-it} h_\varphi^{\frac{2z-1}{2z}},$$

so that $\|f(\frac{1}{\alpha} + it)\|_{L_L^{2z}} \leq \psi(1)^{\frac{1}{2z}}$. Further, since $\frac{\alpha'}{\alpha} < 1$, we get $f(\frac{\alpha'}{\alpha}) \in L_L^{2z}$, so that there is some $y' \in L^{2z}(\mathcal{M})$ such that

$$f(\frac{\alpha'}{\alpha}) = h_\psi^{\frac{\alpha'}{2z}} h_\varphi^{1-\frac{\alpha'}{2z}} = y' h_\varphi^{\frac{2z-1}{2z}}$$

so that $h_\psi^{\frac{\alpha'}{2z}} = y' h_\varphi^{\frac{\alpha'-1}{2z}}$ and $Q_{\alpha',z}(\psi\|\varphi) = \|y'\|_{2z}^{2z} = \|f(\frac{\alpha'}{\alpha})\|_{L_L^{2z}}^{2z}$. Now let λ be such that $(1-\lambda)+\lambda\alpha = \alpha'$, so that $\lambda = \frac{\alpha'-1}{\alpha-1}$, then by the Hadamard three lines theorem, we get

$$Q_{\alpha',z}(\psi\|\varphi) = \|f(\frac{\alpha'}{\alpha})\|_{L_L^{2z}}^{2z} \leq \left(\max_t \|f(\frac{1}{\alpha} + it)\|_{L_L^{2z}}^{1-\lambda} \max_t \|f(1+it)\|_{L_L^{2z}}^{\lambda}\right)^{2z} \leq \psi(1)^{1-\lambda} Q_{\alpha,z}(\psi\|\varphi)^\lambda,$$

this proves that $D_{\alpha',z}(\psi\|\varphi) \leq D_{\alpha,z}(\psi\|\varphi)$.

Remark 2. We now try to prove the limit $\lim_{\alpha\searrow 1} D_{\alpha,z}(\psi\|\varphi)$. The function f is analytic in a neighborhood of $\frac{1}{\alpha}$.

References

[1] H. Kosaki. Applications of the complex interpolation method to a von Neumann algebra: Non-commutative L_p-spaces. *J. Funct. Anal.*, 56:26–78, 1984. doi:https://doi.org/10.1016/0022-1236(84)90025-9.