Notes on DPI

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1 Dual maps

Let $\varphi \in \mathfrak{S}_*(\mathcal{M})$, $\Phi : L_1(\mathcal{M}) \to L_1(\mathcal{N})$. Put $\varphi_0 := \Phi(\varphi)$. We will also use the notations $H := h_\varphi$, $H_0 := h_{\varphi_0}$.

Any positive map $\Phi : L_1(\mathcal{M}) \to L_1(\mathcal{N})$ restricts to a positive contraction $L_p(\mathcal{M}, \varphi) \to L_p(\mathcal{N}, \varphi_0)$. The dual map $\Phi_\varphi : L_1(\mathcal{N}) \to L_1(\mathcal{M})$ satisfies

$$\langle \Phi(h), k_0 \rangle = \langle h, \Phi_\varphi(k_0) \rangle, \qquad h \in L_p(\mathcal{M}, \varphi), k_0 \in L_p(\mathcal{N}, \varphi_0).$$

This map restricts to a contraction $L_p(\mathcal{N}, \varphi_0) \to L_p(\mathcal{M}, \varphi)$.

Let $h \in L_p(\mathcal{M}, \varphi)$, $h = H^{1/2q} k H^{1/2q}$ for some $k \in L_p(\mathcal{M})$. Then $\Phi(h) \in L_p(\mathcal{N}, \varphi_0)$, so that there is some $\tilde{k} \in L_p(\mathcal{N})$ such that

$$\Phi(h) = H_0^{1/2q} \tilde{k} H_0^{1/2q}.$$

Clearly, the map $k \mapsto \tilde{k}$ is a linear and positive map $L_p(\mathcal{M}) \to L_p(\mathcal{N})$, which will be denoted by $\Phi_{p,\varphi}$. Note that then we have $\Phi_{\infty,\varphi} = \Phi_\varphi^*$.

Lemma 1.1. *Let $\frac{1}{p} + \frac{1}{q} = 1$, $1 \le p \le \infty$. Then*

$$(\Phi_\varphi)_{p,\varphi_0} = \Phi_{q,\varphi}^*.$$

Proof. Let $k_0 \in L_p(\mathcal{N})$. Then

$$\Phi_\varphi(H_0^{1/2q} k_0 H_0^{1/2q}) = H^{1/2q} (\Phi_\varphi)_{p,\varphi_0}(k_0) H^{1/2q}.$$

For any $k \in L_q(\mathcal{M})$, we have

$$\begin{aligned} \mathrm{Tr}\,[(\Phi_\varphi)_{p,\varphi_0}(k_0)^* k] &= \langle \Phi_\varphi(H_0^{1/2q} k_0 H_0^{1/2q}), H^{1/2p} k H^{1/2p} \rangle \\ &= \langle H_0^{1/2q} k_0 H_0^{1/2q}, \Phi(H^{1/2p} k H^{1/2p}) \rangle \\ &= \langle H_0^{1/2q} k_0 H_0^{1/2q}, H_0^{1/2p} \Phi_{q,\varphi}(k) H_0^{1/2p} \rangle = \mathrm{Tr}\, k_0^* \Phi_{q,\varphi}(k). \end{aligned}$$

□

2 The DPI bounds

We will use the notation as in [**?**, Theorem 3.11]. Let $\psi \in \mathcal{M}_*^+$, $h_\psi \in L_p(\mathcal{M}, \varphi)$. Then there is some $\omega \in \mathcal{M}_*^+$ such that

$$h_\psi = H^{1/2q} h_\omega^{1/p} H^{1/2q}, \qquad \|h_\psi\|_{p,\varphi} = \|h_\omega^{1/p}\|_p = \omega(1)^{1/p}.$$

It follows that

$$h = T_{q,\varphi}(h_\psi) = \omega(1)^{-1/q} H^{1/2p} h_\omega^{1/q} H^{1/2p}.$$

Furthermore,

$$\Phi(h_\psi) = H_0^{1/2q} \Phi_{p,\varphi}(h_\omega^{1/p}) H_0^{1/2q}, \qquad \|\Phi(h_\psi)\|_{p,\varphi_0} = \|\Phi_{p,\varphi}(h_\omega^{1/p})\|_p$$

There is some element $\omega_0 \in \mathcal{N}_*^+$ such that

$$\Phi_{p,\varphi}(h_\omega^{1/p}) = h_{\omega_0}^{1/p}, \qquad \|\Phi_{p,\varphi}(h_\omega^{1/p})\|_p = \omega_0(1)^{1/p}$$

and

$$h_0 = T_{q,\varphi_0}(\Phi(h_\psi)) = \omega_0(1)^{-1/q} H_0^{1/2p} h_{\omega_0}^{1/q} H_0^{1/2p}.$$

Let $\ell_{p,\mathcal{M}} : \mathcal{M}_*^+ \to L_p(\mathcal{M})^+$ be the map $\omega \mapsto h_\omega^{1/p}$, then $\ell_{p,\mathcal{M}}$ is a (norm) homeomorphism (?) [**?**] and the map $\omega \mapsto \omega_0$ is then given by

$$\ell_{p,\mathcal{N}}^{-1} \circ \Phi_{p,\varphi} \circ \ell_{p,\mathcal{M}}.$$

Now

$$\Phi_\varphi(h_0) = \omega_0(1)^{-1/q} H^{1/2p} \Phi_{q,\varphi}^*(h_{\omega_0}^{1/q}) H^{1/2p}$$

so that

$$\|h - \Phi_\varphi(h_0)\|_{q,\varphi} = \|\omega(1)^{-1/q} h_\omega^{1/q} - \omega_0(1)^{-1/q} \Phi_{q,\varphi}^*(h_{\omega_0}^{1/q})\|_q$$

The following is just as in the proof of [**?**, Theorem 3.11].

Lemma 2.1. *We have*

$$1 - \omega(1)^{-1/q} \|h_\omega^{1/q} - \Phi_{p,\varphi}^*(h_{\omega_0}^{1/q})\|_q \le \frac{\omega_0(1)}{\omega(1)} \le \omega(1)^{-1/q} \|(h_\omega^{1/q} + \Phi_{p,\varphi}^*(h_{\omega_0}^{1/q})\|_q - 1$$

Proof. Let us compute

$$\begin{aligned}
\frac{\omega_0(1)}{\omega(1)} &= \omega(1)^{-1/q} \mathrm{Tr}\, \frac{h_{\omega_0}^{1/p}}{\omega(1)^{1/p}} h_{\omega_0}^{1/q} = \omega(1)^{-1/q} \mathrm{Tr}\, \Phi_{p,\varphi}(\frac{h_\omega^{1/p}}{\omega(1)^{1/p}}) h_{\omega_0}^{1/q} \\
&= \omega(1)^{-1/q} \mathrm{Tr}\, \frac{h_\omega^{1/p}}{\omega(1)^{1/p}} \Phi_{p,\varphi}^*(h_{\omega_0}^{1/q}) \\
&= \omega(1)^{-1/q} \mathrm{Tr}\, \frac{h_\omega^{1/p}}{\omega(1)^{1/p}} (h_\omega^{1/q} + \Phi_{p,\varphi}^*(h_{\omega_0}^{1/q}) - h_\omega^{1/q}) \\
&= 1 - \omega(1)^{-1/q} \mathrm{Tr}\, \frac{h_\omega^{1/p}}{\omega(1)^{1/p}} (h_\omega^{1/q} - \Phi_{p,\varphi}^*(h_{\omega_0}^{1/q})) \\
&\ge 1 - \omega(1)^{-1/q} \|h_\omega^{1/q} - \Phi_{p,\varphi}^*(h_{\omega_0}^{1/q})\|_q
\end{aligned}$$

At the same time, we have

$$\begin{aligned}\frac{\omega_0(1)}{\omega(1)} &= \omega(1)^{-1/q}\mathrm{Tr}\,\frac{h_\omega^{1/p}}{\omega(1)^{1/p}}(h_\omega^{1/q}+\Phi_{p,\varphi}^*(h_{\omega_0}^{1/q})-h_\omega^{1/q})\\ &\leq \omega(1)^{-1/q}\|(h_\omega^{1/q}+\Phi_{p,\varphi}^*(h_{\omega_0}^{1/q})\|_q-1\end{aligned}$$

□

Corollary 2.2. $\omega(1)=\omega_0(1)$ *if and only if* $h_\omega^{1/q}=\Phi_{p,\varphi}^*(h_{\omega_0}^{1/q})$.

Proof. Let $\omega_0(1)=\omega(1)$, then

$$1\leq \omega(1)^{-1/q}\|(h_\omega^{1/q}+\Phi_{p,\varphi}^*(h_{\omega_0}^{1/q})\|_q-1\leq \omega(1)^{-1/q}2\|h_\omega^{1/q}\|_q-1\leq 1,$$

so that $\|\frac{h_\omega^{1/q}+\Phi_{p,\varphi}^*(h_{\omega_0}^{1/q})}{2}\|_q=\omega(1)^{1/q}$. Since both $h_\omega^{1/q}$ and $\Phi_{p,\varphi}^*(h_{\omega_0}^{1/q})$ are elements in the ball with radius $\omega(1)^{1/q}$ in $L_1(\mathcal{M})$, the result follows from strict convexity of $L_q(\mathcal{M})$. Converse is clear from Lemma 2.1. □