Incompatibility in general probabilistic theories, generalized spectrahedra, and tensor norms

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The aim of this work is to study measurement incompatibility in general probabilistic theories (GPTs). The GPTs provide a framework for the study of physical theories permitting probabilistic mixtures. Important examples of GPTs are classical probability theory, quantum mechanics and the GPT of quantum channels. Measurement incompatibility is necessary for many tasks in quantum information theory, because compatible measurements cannot exhibit non-locality in terms of violation of a Bell inequality [1, 2] or steering [3]. In this sense, incompatibility is a resource for quantum processing tasks similar to entanglement [4]. It is thus natural to ask how much of this resource is available in a given situation. Here, we quantify this resource using robustness to white noise [5]. This leads to the *compatibility regions* studied in [6, 7, 8, 9].

Any GPT is built on basic operational notions of *states* (or preparation procedures) and *effects* (or dichotomic measurements) of the theory, which are identified with certain positive elements in a pair of dual ordered vector spaces, (V, V^+) and (A, A^+), respectively. The space of effects, (A, A^+), contains a distinguished order unit $\mathbb{1}$. Elements $f \in A$ for which $0 \leqslant f \leqslant \mathbb{1}$ correspond to effects, while measurements are k-tuples of effects summing up to $\mathbb{1}$.

The first part of the paper is dedicated to finding equivalent characterizations of measurement compatibility. The first main result is a set of characterizations which comes in terms of the positivity of maps associated to the measurements, extending the results of [10]. Here, the minimal and maximal tensor product of convex cones play an important role.

Next, we lift the results of [8, 9] to GPTs. To this end, we generalize the free spectrahedra studied e.g. in [11, 12, 13]. A *generalized spectrahedron* $\mathcal{D}_a(g; L, C)$ is determined by a tuple $a := (a_1, \ldots, a_g) \in M^g$, where (M, M^+) is an ordered vector space. It is the set of g-tuples in some other ordered vector space (L, L^+) which fulfil a linear inequality specified by the a_i, which asks for membership in a given tensor cone C. Let $\mathbf{k} = (k_1, \ldots, k_g) \in \mathbb{N}^g$ be a g-tuple determining the number of outcomes of each measurement we are interested in. We define a certain universal ordered vector space $(E_{\mathbf{k}}, E_{\mathbf{k}}^+)$ and associate to it a generalized spectrahedron $\mathcal{D}_{\mathrm{GPT}\diamondsuit}(\mathbf{k}; V, V^+)$, called the *GPT jewel*. We also define the generalized spectrahedron $\mathcal{D}_f(\mathbf{k}; V, V^+)$ determined by the measurements f under study. Our second main result is the following equivalent condition for compatibility of a g-tuple of measurements f, which is of geometrical nature: $\mathcal{D}_{\mathrm{GPT}\diamondsuit}(\mathbf{k}; V, V^+) \subseteq \mathcal{D}_f(\mathbf{k}; V, V^+)$.

A third characterization is obtained in the case of dichotomic measurements. We use the symmetry of the cone $(E_{\mathbf{k}}^+)^*$ in this case to relate compatibility of effects to *reasonable crossnorms* on the tensor product $\ell_\infty^g \otimes A$ (where we always endow A with its order unit norm). Namely, for any $f = (f_1, \ldots, f_g) \in A^g$ we study the associated tensor $\bar{\varphi}^{(f)} \in \ell_\infty^g \otimes A$. Then g-tuples of effects are determined by the condition $\|\bar{\varphi}^{(f)}\|_\varepsilon \leqslant 1$, where $\|\cdot\|_\varepsilon$ is the injective crossnorm in $\ell_\infty^g \otimes A$. As a third main result, we find a reasonable crossnorm $\|\cdot\|_\rho$ in $\ell_\infty^g \otimes A$, such that compatibility of a g-tuple f of dichotomic measurements is equivalent to $\|\bar{\varphi}^{(f)}\|_\rho \leqslant 1$.

The second part of the paper is concerned with the amount of incompatibility available in a given GPT. We are thus interested in the minimal amount of white noise $\gamma(f) \subseteq [0, 1]$ such that the noisy versions of f are compatible. The minimal noise such that this is true for any collection of g measurements with outcomes $\mathbf{k}$ is called the *compatibility degree* for the GPT and denoted by $\gamma(\mathbf{k}; V, V^+) := \min_f \gamma(f)$. Allowing for asymmetric noise leads to the aforementioned *compatibility regions*. In particular, we show that

$$\gamma(g; V, V^+) = 1/\max_{\|\varphi\|_\varepsilon \leqslant 1} \|\varphi\|_\rho \geqslant 1/\rho(\ell_\infty^g, A) \geqslant 1/\min\{g, \dim(V)\}, \tag{1}$$

where the quantity $\rho(X, Y)$ for a pair of Banach spaces X, Y was introduced in [14] as the maximal ratio of the projective over the injective tensor norm.

Finally, we consider a special class of GPTs for which our results have a simpler form: the *centrally symmetric* GPTs. For these, $A = \mathbb{R}\mathbb{1} \oplus \bar{A}$ and a similar decomposition exits for V. An important example is the Bloch ball (corresponding to quantum mechanics for 2-level systems), where $\bar{A} = \ell_2^3$. We show that for this class of GPTs, we can replace the space A in Eq. (1) by $\bar{A}$ and the norm $\|\cdot\|_\rho$ by the projective norm in $\ell_\infty^g \otimes_\pi \bar{A}$. The compatibility degree is thus $\gamma(g; V, V^+) = 1/\rho(\ell_\infty^g, \bar{A}) \geqslant 1/\min\{g, \dim(\bar{A})\}$. We also put forward a connection to 1-summing norms: $\lim_{g\to\infty} \gamma(g; V, V^+) = 1/\pi_1(\bar{V})$, where π_1 is the 1-summing norm of the Banach space $(\bar{V}, \|\cdot\|_{\bar{V}})$. This relation allows us to compute the compatibility degree for many GPTs of interest. In particular, we prove new lower bounds on the compatibility degree of qubit effects, namely $0.58 \approx 1/\sqrt{3} \geqslant \gamma(g; \mathrm{QM}_2) \geqslant 1/2$ for $g \geqslant 4$. This is the first bound on the compatibility degree of more than 3 qubit effects, whereas the case for $g \leqslant 3$ was already known (see [8] and references therein).

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