Remarks: post-processing eqivalence of channels

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We assume finite dimensions, though the results hold in infinite dimensions.

1 Post-processing eqiuvalent channels

For channels $\mathcal{T} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$, $\mathcal{S} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K}')$ we write $\mathcal{T} \preceq_{post} \mathcal{S}$ if $\mathcal{T} = \mathcal{P} \circ \mathcal{S}$ for some channel $\mathcal{P} : \mathcal{L}(\mathcal{K}') \to \mathcal{L}(\mathcal{K})$. If also $\mathcal{S} \preceq_{post} \mathcal{T}$, then we say that $\mathcal{S}$ and $\mathcal{T}$ are post-processing equivalent, $\mathcal{T} \sim_{post} \mathcal{S}$. The equivalence class of $\mathcal{T}$ will be denoted by $[\mathcal{T}]$. We will also say that two channels are *isomorphic*, $\mathcal{S} \simeq \mathcal{T}$, if there is a *-isomorphism $\mathcal{P}$ such that $\mathcal{P} \circ \mathcal{S} = \mathcal{T}$.

1.1 Post-processing minimal channels

Here we need to specify the target subalgebra of the channel: $\mathcal{T} : \mathcal{L}(\mathcal{H}) \to \mathcal{M} \subseteq B(\mathcal{K})$. The following definition and theorem are from [3] (there the channels are called "minimal sufficient").

Definition 1. We say that a channel $\mathcal{T} : B(\mathcal{H}) \to \mathcal{M}$ is *post-processing minimal* if $\mathcal{P} \circ \mathcal{T} = \mathcal{T}$ for a channel $\mathcal{P} : \mathcal{M} \to \mathcal{M}$ implies that $\mathcal{P} = id_{\mathcal{M}}$.

Theorem 1. *Any post-processing equivalence class of channels contains a post-processing minimal channel. This channel is unique up to isomorphisms.*

Proof. Let $\mathcal{T}$ be a channel, then $\mathcal{T}$ is equivalent to a faithful channel, so that we may assume that $\mathcal{T}$ is faithful (that is, $\mathcal{T}(\rho)$ is of full rank if ρ is). We denote

$$\mathcal{S} := \{\mathcal{P}^*,\ \mathcal{P} \circ \mathcal{T} = \mathcal{T}\},$$

where $\mathcal{P}^*$ is the adjoint unital cp map (Heisenberg picture). Note that $\mathcal{S}$ is a convex and closed semigroup of ucp maps with a common faithful invariant state. By mean ergodic theorem [2], there is a conditional expectation $E \in \mathcal{S}$ such that

$$E \circ \mathcal{P}^* = \mathcal{P}^* \circ E = E, \qquad \forall \mathcal{P}^* \in \mathcal{S}. \tag{1}$$

The range of E is the subalgebra $\mathcal{F} \subseteq B(\mathcal{K})$ of fixed points of $\mathcal{S}$.

Let $F : \mathcal{L}(\mathcal{K}) \to \mathcal{F}$ be the trace preserving conditional expectation (mapping each density matrix onto the density matrix of its restriction to $\mathcal{F}$). Put

$$\mathcal{T}_{min} := F \circ \mathcal{T} : \mathcal{L}(\mathcal{H}) \to \mathcal{F}.$$

Then clearly by definition $\mathcal{T}_{min} \preceq_{post} \mathcal{T}$. On the other hand, we have $E^* \circ F = E^*$, this follows from

$$\mathrm{Tr}\,[E^* \circ F(\sigma)a] = \mathrm{Tr}\,[F(\sigma)E(a)] = \mathrm{Tr}\,[\mathrm{Tr}\,[\sigma E(a)] = \mathrm{Tr}\,[E^*(\sigma)a]$$

for all $a, \sigma \in \mathcal{L}(\mathcal{K})$. Hence $E^* \circ \mathcal{T}_{min} = E^* \circ F \circ \mathcal{T} = E^* \circ \mathcal{T} = \mathcal{T}$ (note that $E \in \mathcal{S}$), so that also $\mathcal{T} \preceq_{post} \mathcal{T}_{min}$ and $\mathcal{T}_{min} \in [\mathcal{T}]$.

Let now $\mathcal{P} : \mathcal{F} \to \mathcal{F}$ be a channel such that $\mathcal{P} \circ \mathcal{T}_{min} = \mathcal{T}_{min}$, then $F^* \circ \mathcal{P}^* \circ E \in \mathcal{S}$, so that by (1), $F^* \circ \mathcal{P}^* \circ E = E$, this means that $\mathcal{P} = id_{\mathcal{F}}$.

Let $\mathcal{S} \in [\mathcal{T}]$ be another post-processing minimal channel. Then there are channels $\mathcal{P}, \mathcal{R}$ such that $\mathcal{P} \circ \mathcal{S} = \mathcal{T}_{min}$ and $\mathcal{R} \circ \mathcal{T}_{min} = \mathcal{S}$. By the definition of post-processing minimal channels, we see that $\mathcal{P} \circ \mathcal{R}$ and $\mathcal{R} \circ \mathcal{P}$ are identity channels on the respective subalgebras. Hence $\mathcal{P}$ and $\mathcal{R}$ must be isomorphisms.

□

Proposition 1. *Let $\mathcal{T} : \mathcal{L}(\mathcal{H}) \to \mathcal{M}$ be a faithful channel. Then $\mathcal{T}$ is post-processing minimal if and only if the subalgebra generated by the subset*

$$\{\mathcal{T}(\rho)^{it}\mathcal{T}(\sigma)^{-it},\ \rho \in \mathfrak{S}(\mathcal{H}),\ t \in \mathbb{R}\}$$

is the whole algebra $\mathcal{M}$, here $\sigma \in \mathfrak{S}(\mathcal{H})$ is an arbitrary (fixed) faithful state.

Proof. This follows from some results in [6, 1], that show that the subalgebra $\mathcal{F}$ in the above theorem is generated by the given subset.

□

We now describe the general form of conditional expectations on $\mathcal{L}(\mathcal{K})$. Let $\mathcal{F} \subseteq \mathcal{L}(\mathcal{K})$ be a subalgebra and let $\mathcal{F}'$ be its commutant. Then there are Hilbert spaces $\mathcal{K}_n^L, \mathcal{K}_n^R$ and a unitary $U : \mathcal{K} \to \oplus_n \mathcal{K}_n^L \otimes \mathcal{K}_n^R$ such that we have

$$\mathcal{F} = U^* \left(\bigoplus_n \mathcal{L}(\mathcal{K}_n^L) \otimes I_n^R \right) U, \qquad \mathcal{F}' = U^* \left(\bigoplus_n I_n^L \otimes \mathcal{L}(\mathcal{K}_n^R) \right) U.$$

Let P_n be the projection of $\oplus_n \mathcal{K}_n^L \otimes \mathcal{K}_n^R$ onto the n-th component.

Let $E : \mathcal{L}(\mathcal{K}) \to \mathcal{L}(\mathcal{K})$ be a conditional expectation whith range $\mathcal{F}$. Then there are states $\sigma_n^R \in \mathfrak{S}(\mathcal{K}_n^R)$ such that

$$E(a) = U^* \sum_n (id_n^L \otimes \tau_{\sigma_n^R})(P_n U a U^* P_n) U \tag{2}$$

where $\tau_{\sigma_n^R}$ is the unital cp map on $\mathcal{L}(\mathcal{K}_n^R)$ given by $b \mapsto \mathrm{Tr}\,[b\sigma_n^R] I_n^R$.

Let $\mathcal{T} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$ be a channel and let E be a conditional expectation as in (1), so that $E^* \circ \mathcal{T} = \mathcal{T}$. By (2), we have for $\rho \in \mathcal{L}(\mathcal{H})$

$$\mathcal{T}(\rho) = U^* \left(\bigoplus_n \mathrm{Tr_R[P_n U \mathcal{T}(\rho) U^* P_n] \otimes \sigma_n^R} \right) U = U^* \left(\bigoplus_n \Phi_n^L(\rho) \otimes \sigma_n^R \right) U, \tag{3}$$

where $\Phi_n^L := \mathrm{Tr_R[P_n U \mathcal{T}(\cdot) U^* P_n]}$ is a cp map $\mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K}_n^L)$.

Proposition 2. *The channel $\bigoplus_n \Phi_n^L : \mathcal{L}(\mathcal{H}) \to \oplus_n \mathcal{L}(\mathcal{K}_n^L)$ is a post-processing minimal channel in $[\mathcal{T}]$. Any channel $\mathcal{S} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K}')$ is in $[\mathcal{T}]$ if and only if it has the form*

$$\mathcal{S}(\rho) = V^* \left(\bigoplus_n \Phi_n^L(\rho) \otimes \xi_n^R \right) V, \qquad \rho \in \mathcal{L}(\mathcal{H})$$

for some states $\xi_n^R \in \mathfrak{S}(\tilde{\mathcal{K}}_n'^R)$ and a unitary operator $V : \mathcal{K}' \to \oplus_n \mathcal{K}_n^L \otimes \mathcal{K}_n'^R$.

Proof. It is quite clear that $\bigoplus_n \Phi_n^L \in [\mathcal{T}]$ and that any channel of the given form is in $[\mathcal{T}]$. It is also clear that $\bigoplus_n \Phi_n^L$ is isomorphic to $\mathcal{T}_{min}$ so that it must be post-processing minimal. Assume that $\mathcal{S} \in [\mathcal{T}]$, then there is a subalgebra $\mathcal{F}_\mathcal{S}$, conditional expectaton $E_\mathcal{S}$ with range $\mathcal{F}_\mathcal{S}$ and a post-processing minimal channel $\mathcal{S}_{min} = E_\mathcal{S}^* \circ \mathcal{S} : \mathcal{L}(\mathcal{H}) \to \mathcal{F}_\mathcal{S}$. Since $\mathcal{S}_{min} \in [\mathcal{S}] = [\mathcal{T}]$, we have by Theorem 1 that there is an isomorphism $\mathcal{P} : \mathcal{F} \to \mathcal{F}_\mathcal{S}$ such that $\mathcal{P} \circ \mathcal{T}_{min} = \mathcal{S}_{min}$. The subalgebra $\mathcal{F}_\mathcal{S}$ is of the form

$$\mathcal{F}_\mathcal{S} = W^* \left(\bigoplus_n \mathcal{L}(\mathcal{K}_n'^L) \otimes I_{\mathcal{K}_n'^R} \right) W$$

and

$$\mathcal{P} : U^* \left(\bigoplus_n a_n \otimes I_{\mathcal{K}_n^R} \right) U \mapsto W^* \left(\bigoplus_n V_n^* a_n V_n \otimes I_{\mathcal{K}_n'^R} \right) W$$

for some unitary operators $V_n : \mathcal{K}_n'^L \to \mathcal{K}_n^L$. It follows that

$$\mathcal{S}_{min} = \mathcal{P} \circ \mathcal{T}_{min} = W^* \left(\bigoplus_n V_n^* \Phi_n^L(\cdot) V_n \otimes I_{\mathcal{K}_n'^R} \right) W,$$

this implies the result.

□

Note that $\bigoplus_n \Phi_n^L$ can be also seen as an instrument: we can embed all $\mathcal{H}_n^L$ into a common target space.

1.2 Instruments

An instrument $\mathcal{I} \in \mathrm{Ins}(X, \mathcal{H}, \mathcal{K})$ can be seen as a channel $\mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K} \otimes \mathbb{C}^{|X|})$ defined as

$$\mathcal{I}(\rho) = \sum_x \mathcal{I}_x(\rho) \otimes |x\rangle\langle x|.$$

Clearly, the post-processing preorder and equivalence coincides with the one defined in [5]. For $x \in X$, let $Q_x = I_{\mathcal{K}} \otimes |x\rangle\langle x|$ and let $D = \sum_x Q_x(\cdot)Q_x$, then D is a trace preserving conditional expectation and $D \circ \mathcal{I} = \mathcal{I}$.

Let $\bigoplus_n \Phi_n^L$ be the minimal channel in the equivalence class $[\mathcal{I}]$, then $\mathcal{I}$ has the form

$$\mathcal{I}(\rho) = U^* \left(\bigoplus_n \Phi_n^L(\rho) \otimes \sigma_n^R \right) U.$$

Let $\mathcal{F}$ and E be connected to the channel $\mathcal{I}$ as in the previous paragraph, then $D \circ \mathcal{I} = \mathcal{I}$ implies that $D \circ E = E \circ D = E$ (note that $D^* = D$). It follows that $\mathcal{F}$ is contained in the range of D, in particular, any element in $\mathcal{F}$ commutes with all the projections Q_x, so that $Q_x \in \mathcal{F}'$. This means that there are some projections $q_{n,x}^R \in \mathcal{L}(\mathcal{K}_n^R)$ such that

$$Q_x = U^* \left(\bigoplus_n I_n^L \otimes q_{n,x}^R \right) U$$

Moreover, $E \circ D = D \circ E = E$ implies that $q_{x,n}^R$ commutes with σ_n^R for all x and n. It follows that

$$\mathcal{I}_x(\rho) = Q_x \mathcal{I}(\rho) Q_x = U^* \left(\bigoplus_n \Phi_n^L(\rho) \otimes \sigma_{n,x}^R \right) U \tag{4}$$

where $\sigma_{n,x}^R := \sigma_n^R q_{n,x}^R$, so that $\sum_x \mathrm{Tr}\,[\sigma_{n,x}^R] = 1$. The induced POVM has the form

$$A^{\mathcal{I}}(x) = \sum_n \mathrm{Tr}\,[\sigma_{n,x}^R] \Phi_n^{L*}(I) \tag{5}$$

It follows that the induced POVMs of all instruments in $[\mathcal{I}]$ are post-processings of the POVM $\{\Phi_n^{L*}(I)\}$.

1.2.1 Some special cases

Identity

It is quite clear that the identity channel $id_{\mathcal{H}}$ is post-processing minimal. By Prop. 1, any channel $\mathcal{T} \sim_{post} id$ has the form

$$\mathcal{T}(\rho) = V^*(\rho \otimes \sigma^R)V, \qquad \rho \in \mathcal{L}(\mathcal{H}) \tag{6}$$

for some state $\sigma^R \in \mathfrak{S}(\mathcal{K}^R)$ and a unitary operator $V : \mathcal{K} \to \mathcal{H} \otimes \mathcal{K}^R$. Let $\sigma^R = \sum_i \lambda_i |\varphi_i\rangle\langle\varphi_i|$ be the spectral decomposition of σ^R, then we may write (6)as

$$\mathcal{T}(\rho) = \sum_i \lambda_i V_i^* \rho V_i, \tag{7}$$

where $V_i := (I \otimes |\varphi_i\rangle\langle\varphi_i|)V$ are isometries $V_i : \mathcal{K} \to \mathcal{H}$ such that $V_i^* V_j = 0$ for $i \neq j$.

If $\mathcal{I} \in [id_{\mathcal{H}}]$ is an instrument, then there are projections q_x, commuting with σ^R such that

$$\mathcal{I}_x(\rho) = V^*(\rho \otimes \sigma^R q_x)V \tag{8}$$

Since any q_x is a sum of some of the eigenvectors of σ^R and $\sum_x q_x = I_{\mathcal{K}^R}$, we see that there is a decomposition of the set $\{1, \dots, \dim(\mathcal{K}^R)\}$ into subsets Ω_x such that $q_x = \sum_{i \in \Omega_x} |\varphi_i\rangle\langle\varphi_i|$, we have

$$\mathcal{I}_x(\rho) = \sum_{i \in \Omega_x} \lambda_i V_i^* \rho V_i, \tag{9}$$

this corresponds to the findings in [5].

Indecomposable instruments

Assume that $\mathcal{I}$ is an indecomposable instrument, that is, if $\mathcal{S}$ is an operation such that $\mathcal{S} \leq \mathcal{I}_x$ then $\mathcal{S}$ must be a multiple of $\mathcal{I}_x$, for all $x \in X$. Let $\bigoplus_n \Phi_n^L$ be the corresponding minimal channel, so that $\mathcal{I}_x$ has the form (4). Since $U^*(\Phi_n^L(\cdot) \otimes \sigma_{n,x}^R)U$ is an operation majorized by $\mathcal{I}_x$ for all x and n, there are some nonnegative numbers $\alpha_{n,x}$ such that

$$U(\Phi_n^L(\cdot) \otimes \sigma_{n,x}^R)U^* = \alpha_{n,x}\mathcal{I}_x, \qquad \forall n, x.$$

For fixed x, let $N_x = \{n, \sigma_{n,x}^R \neq 0\}$. If there are $m, n \in N_x$, $n \neq m$, then it follows that Φ_n^L must be a multiple of Φ_m^L, which is not possible by post-processing minimality of $\bigoplus_n \Phi_n^L$. Therefore each N_x is a singleton (if we assume that all $\mathcal{I}_x$ are nonzero), so that there is a (surjective) map $f : x \mapsto n$ such that

$$\mathcal{I}_x = U^*(\Phi_{f(x)}^L(\cdot) \otimes \sigma_x^R)U,$$

where $\sigma_x^R = \sigma_{f(x),x}^R$ and we must have $\sigma_n^R = \sum_x \sigma_{n,x}^R = \sum_{x,f(x)=n} \sigma_x^R$. Clearly, every σ_x^R must be rank 1, so that $\sigma_x^R = \lambda_x |\varphi_x\rangle\langle\varphi_x|$, where λ_x is some eigenvalue of $\sigma_{f(x)}^R$ and $|\varphi_x\rangle$ the corresponding eigenvector. We also see that each Φ_n^L must be indecomposable, so there are some operators $K_n : \mathcal{K}_n^L \to \mathcal{H}$ such that $\Phi_n^L = K_n^*(\cdot)K_n$. We obtain

$$\mathcal{I}_x = U^* \left(K_{f(x)}^*(\cdot)K_{f(x)} \otimes \lambda_x |\varphi_x\rangle\langle\varphi_x| \right) U$$

and $\sum_{x,f(x)=n} \lambda_x = 1$ for each n.

POVMs

A POVM A_x, $x \in X$ can be seen as a channel $\mathcal{A} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathbb{C}^{|X|})$, given as

$$\mathcal{A}(\rho) = \sum_x \mathrm{Tr}\,[A_x \rho]|x\rangle\langle x|.$$

Let F be the trace preserving conditional expectation $F = \sum_x |x\rangle\langle x|(\cdot)|x\rangle\langle x|$ onto the abelian subalgebra of diagonal elements. Let $\mathcal{F}_\mathcal{A}$ be the subalgebra and $E_\mathcal{A}$ be the conditional expectation related to the post-processing minimal channel $\mathcal{A}_{min}$. Then since $F \circ E_\mathcal{A} = E_\mathcal{A} \circ F = E_\mathcal{A}$, we see that the $\mathcal{F}$ is a subalgebra of diagonal elements, hence $\dim(\mathcal{K}_n^L) = 1$. It follows that $\Phi_n^L = \mathrm{Tr}\,[M_n(\cdot)]$ for some POVM $(M_n)_n$ on $\mathcal{L}(\mathcal{H})$ and we have

$$\mathcal{A}(\rho) = U^* \left(\bigoplus_n \mathrm{Tr}\,[M_n(\cdot)]\sigma_n^R \right) U = \bigoplus_n \mathrm{Tr}\,[M_n(\cdot)]\xi_n^R$$

where $\xi_n^R = U^* \sigma_n^R U$ and as in the case of instruments

$$\mathcal{A}_x(\rho) = \bigoplus_n \mathrm{Tr}\,[M_n(\cdot)]\xi_{n,x}^R,$$

where $\xi_{n,x}^R$ are mutually orthogonal. Since $\mathcal{A}_x(\rho)$ is one dimensional, we again see that

$$\mathcal{A}_x(\rho) = \lambda_x \mathrm{Tr}\,[M_{f(x)}(\cdot)]|\psi_x\rangle\langle\psi_x|$$

for orthonormal basis vectors $|\psi_x\rangle$, some surjective function $g : x \mapsto n$ and $\lambda_x \geq 0$ such that $\sum_{x,f(x)=n} \lambda_x = 1$ for each n. It should be clear that $(M_n)_n$ is a minimal sufficient POVM which is post-processing equivalent to $(A_x)_x$.

1.3 Post-processing preorder

Using results of [4, Lemma 6], we can describe the post-processing preorder in terms of the minimal channels. So let $\mathcal{T} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$ and $\mathcal{S} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K}')$ be two channels. Let $\mathcal{F}_\mathcal{T} \subseteq \mathcal{L}(\mathcal{K})$ and $\mathcal{F}_\mathcal{S} \subseteq \mathcal{L}(\mathcal{K}')$ be the minimal subalgebras, of the form

$$\mathcal{F}_\mathcal{T} = U^* \left(\bigoplus_n \mathcal{L}(\mathcal{K}_n^L) \otimes I_{\mathcal{K}_n^R} \right) U, \qquad \mathcal{F}_\mathcal{S} = V^* \left(\bigoplus_m \mathcal{L}(\mathcal{K}_m'^L) \otimes I_{\mathcal{K}_n'^R} \right) V$$

Let $\bigoplus_n \Phi_n^L \in [\mathcal{T}]$ and $\bigoplus_m \Psi_m^L \in [\mathcal{S}]$ be the post-processing minimal channels.

Theorem 2. *We have $\mathcal{T} \preceq_{post} \mathcal{S}$ if and only if there are some Hilbert spaces $\tilde{\mathcal{K}}_n^R$ and $\tilde{\mathcal{K}}_m'^R$, a unitary operator $V : \oplus_n \mathcal{K}_n^L \otimes \tilde{\mathcal{K}}_n^R \to \oplus_m \mathcal{K}_m'^L \otimes \tilde{\mathcal{K}}_m'^R$ and states $\xi_m^R \in \mathcal{L}(\tilde{\mathcal{K}}_m'^R)$ such that*

$$\sum_m \mathrm{Tr}_R[V_{mn}^*(\Psi_m^L(\cdot) \otimes \xi_m^R)V_{mn}] = \Phi_n^L(\cdot),$$

where $V_{mn} = Q_m V P_n$, P_n is the projection of $\oplus_n(\mathcal{K}_n^L \otimes \tilde{\mathcal{K}}_n^R)$ onto its n-th component and similarly Q_m for $\oplus_m(\mathcal{K}_m'^L \otimes \tilde{\mathcal{K}}_m'^R)$.

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