# Some useful categories

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## 1 The category FinVect

Let FinVect be the category of finite dimensional real vector spaces with linear maps. We will denote the usual tensor product by $\otimes$, then $(\mathrm{FinVect}, \otimes, I = \mathbb{R})$ is a symmetric monoidal category, with the associators, unitors and symmetries given by the obvious isomorphisms

$$\begin{aligned} \alpha_{U,V,W} &: (U \otimes V) \otimes W \simeq U \otimes (V \otimes W), \\ \lambda_V &: I \otimes V \simeq V, \qquad \rho_V : V \otimes I \simeq V, \\ \sigma_{U,V} &: U \otimes V \simeq V \otimes U. \end{aligned}$$

Let $(-)^* : V \mapsto V^*$ be the usual vector space dual, with duality denoted by $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}$. We will use the canonical identification $V^{**} = V$ and $(V_1 \otimes V_2)^* = V_1^* \otimes V_2^*$. With this duality, FinVect is compact closed. This means that for each object $V$, there are maps $\eta_V : I \to V^* \otimes V$ (the "cup") and $\epsilon_V : V \otimes V^* \to I$ (the "cap") such that the following snake identities hold:

$$(\epsilon_V \otimes V) \circ (V \otimes \eta_V) = V, \qquad (V^* \otimes \epsilon_V) \circ (\eta_V \otimes V^*) = V^*, \tag{1}$$

here we denote the identity map on the object $V$ by $V$. Indeed, $\eta_V$ can be identified with an element $\eta_V(1) \in V^* \otimes V$ and $\epsilon_V \in (V \otimes V^*)^* = V^* \otimes V$ is again an element of the same space. Choose a basis $\{e_i\}$ of $V$, let $\{e_i^*\}$ be the dual basis of $V^*$, that is, $\langle e_i^*, e_j \rangle = \delta_{i,j}$. Let us then define

$$\eta_V(1) = \epsilon_V := \sum_i e_i^* \otimes e_i.$$

It is easy to see that this definition does not depend on the choice of the basis, indeed, $\epsilon_V$ is the linear functional on $V \otimes V^*$ defined by

$$\langle \epsilon_V, x \otimes x^* \rangle = \langle x^*, x \rangle, \qquad x \in V,\ x^* \in V^*.$$

It is also easily checked that the snake identities (1) hold.

For two objects $V$ and $W$ in FinVect, let $L(V, W)$ be the space of all linear maps $V \to W$. Then $L(V, W)$ is itself an object in FinVect and we have the well-known identification $L(V, W) \simeq V^* \otimes W$. This can be given as follows: for each $f \in L(V, W)$, we have $C_f := (V^* \otimes f)(\epsilon_V) = \sum_i e_i^* \otimes f(e_i) \in V^* \otimes W$. Conversely, since $\{e_i^*\}$ is a basis of $V^*$, any element $w \in V^* \otimes W$ can be uniquely written as $w = \sum_i e_i^* \otimes w_i$ for $w_i \in W$, and since $\{e_i\}$ is a basis of $V$, the assignment $f(e_i) := w_i$ determines a unique map $f : V \to W$. The relations between $f \in L(V, W)$ and $C_f \in V^* \otimes V$ can be also determined as

$$\langle C_f, x \otimes y^* \rangle = \langle \epsilon_V, x \otimes f^*(y^*) \rangle = \langle f^*(y^*), x \rangle = \langle y^*, f(x) \rangle, \qquad x \in V,\ y^* \in W^*,$$

here $f^* : W^* \to V^*$ is the adjoint of $f$. Note that by compactness, $[V, W] = V^* \otimes W$ is the internal hom, and in the case of FinVect, the object $[V, W]$ can be identified with the space of linear maps $V \to W$.

*Example* 1. Let $V = \mathbb{R}^N$. In this case, we fix the canonical basis $\{|i\,\rangle,\ i = 1, \ldots, N\}$. We will identify $(\mathbb{R}^N)^* = \mathbb{R}^N$, with duality $\langle\, x, y\,\rangle = \sum_i x_i y_i$, in particular, we identify $I = I^*$. We then have $\epsilon_V = \sum_i |i\,\rangle \otimes |i\,\rangle$ and if $f : \mathbb{R}^N \to \mathbb{R}^M$ is given by the matrix $A$ in the two canonical bases, then $C_f = \sum_i |i\,\rangle \otimes A|i\,\rangle$ is the vectorization of $A$.

*Example* 2. Let $V = M_n^h$ be the space of $n \times n$ complex hermitian matrices. We again identify $(M_n^H)^* = M_n^h$, with duality $\langle\, A, B\,\rangle = \mathrm{Tr}\, A^T B$, where $A^T$ is the usual transpose of the matrix $A$. Let us choose the basis in $M_n^h$, given as

$$\left\{ |j\,\rangle\langle\, k| + |k\,\rangle\langle\, j|,\ j \leq k,\ i\Big( |j\,\rangle\langle\, k| - |k\,\rangle\langle\, j| \Big),\ j < k \right\}.$$

Then one can check that

$$\left\{ \frac{1}{2}\Big( |j\,\rangle\langle\, k| + |k\,\rangle\langle\, j| \Big),\ j \leq k,\ \frac{i}{2}\Big( |k\,\rangle\langle\, j| - |j\,\rangle\langle\, k| \Big),\ j < k \right\}$$

is the dual basis and we have

$$\epsilon_V = \sum_{j,k} |j\,\rangle\langle\, k| \otimes |j\,\rangle\langle\, k|.$$

For any $f : M_n^h \to M_m^h$,

$$C_f = \sum_{j,k} |j\,\rangle\langle\, k| \otimes f(|j\,\rangle\langle\, k|)$$

is the Choi matrix of $f$.