DPI for $\alpha - z$ Rényi divergence

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Let ψ be a faithful normal state on a von Neumann algebra $\mathcal{M}$. We will prove the following inequality:

$$\|h_{\psi\circ\gamma}^{\frac{1}{2p}} b h_{\psi\circ\gamma}^{\frac{1}{2p}}\|_p \leq \|h_\psi^{\frac{1}{2p}} \gamma(b) h_\psi^{\frac{1}{2p}}\|_p \tag{1}$$

for all $p \in [1/2, 1]$, all $b \in \mathcal{N}^+$ and any unital positive map $\gamma : \mathcal{N} \to \mathcal{M}$ (Eq. (19) in [2]). This then implies DPI for the $\alpha - z$-Rényi divergence for $\alpha/2, \alpha - 1 \leq z \leq \alpha$.

Let γ_ψ^* be the Petz dual of γ with respect to ψ, then its predual satisfies

$$(\gamma_\psi^*)_*(h_{\psi\circ\gamma}^{1/2} b h_{\psi\circ\gamma}^{1/2}) = h_\psi^{1/2} \gamma(b) h_\psi^{1/2}$$

(this is eq. (21) in [2]). Put $h_\omega := h_{\psi\circ\gamma}^{1/2} b h_{\psi\circ\gamma}^{1/2} \in L_1(\mathcal{N})^+$. We then have, using Thm. 4.1 in [1]

$$\begin{aligned}
\|h_\psi^{\frac{1}{2p}} \gamma(b) h_\psi^{\frac{1}{2p}}\|_p^p &= \|h_\psi^{\frac{1-p}{2p}} (\gamma_\psi^*)_*(h_\omega) h_\psi^{\frac{1-p}{2p}}\|^p = \tilde{Q}_p((\gamma_\psi^*)_*(h_\omega)\|(\gamma_\psi^*)_*(h_{\psi\circ\gamma})) \\
&\geq \tilde{Q}_p(h_\omega\|h_{\psi\circ\gamma}) = \|h_{\psi\circ\gamma}^{\frac{1-p}{2p}} h_\omega h_{\psi\circ\gamma}^{\frac{1-p}{2p}}\|^p = \|h_{\psi\circ\gamma}^{\frac{1}{2p}} b h_{\psi\circ\gamma}^{\frac{1}{2p}}\|_p^p.
\end{aligned}$$

References

[1] A. Jenčová, Rényi relative entropies and noncommutative L_p-spaces II, Ann. Henri Poincaré 22, 3235–3254 (2021)

[2] Shinya Kato, On $\alpha - z$-Rényi divergence in the von Neumann algebra setting, arXiv:2311.01748.