Notes on asymptotics of quantum hypothesis testing

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1 Preliminaries

Let $\mathcal{H}$ be a finite dimensional Hilbert space.

1.1 Pinching

Let $A \in B(\mathcal{H})$ be self-djoint, with spectral decomposition $A = \sum_i \lambda_i P_i$. We will need the pinching map $B(\mathcal{H}) \to B(\mathcal{H})$, defined as

$$\mathcal{E}_A(X) = \sum_i P_i X P_i.$$

Then A is a cp unital map. Moreover, $\mathcal{E}_A(X)$ commutes with X and we have the pinching inequality [**?**]

$$\mathcal{E}_A(X) \leq |\mathrm{spec}(A)| X, \qquad X \geq 0. \tag{1}$$

1.2 Relative entropies

Let ρ and σ be density operators. The (Umegaki) relative entropy is defined as

$$D(\rho\|\sigma) := \begin{cases} \mathrm{Tr}\,[\rho(\log\rho - \log\sigma)], & \mathrm{supp}(\rho) \leq \mathrm{supp}(\sigma) \\ \infty, & \text{otherwise.} \end{cases}$$

The standard Rényi relative entropy for $\alpha \in [0,1] \setminus \{1\}$ is defined as

$$D_\alpha(\rho\|\sigma) := \begin{cases} \dfrac{1}{\alpha - 1} \log \mathrm{Tr}\,[\rho^\alpha \sigma^{1-\alpha}], & \mathrm{supp}(\rho) \leq \mathrm{supp}(\sigma) \text{ or } \alpha \in (0,1) \\ \infty, & \text{otherwise.} \end{cases}$$

The sandwiched Rényi relative entropy for $\alpha \in [1/2, \infty] \setminus \{1\}$ is defined as

$$\hat{D}_\alpha(\rho\|\sigma) := \begin{cases} \dfrac{1}{\alpha - 1} \log \mathrm{Tr}\,[\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}}], & \mathrm{supp}(\rho) \leq \mathrm{supp}(\sigma) \text{ or } \alpha \in [1/2, 1) \\ \infty, & \text{otherwise.} \end{cases}$$

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1.3 The functions ϕ and ψ

We assume $\mathrm{supp}(\rho) \le \mathrm{supp}(\sigma)$**, so that** $D(\rho\|\sigma) < \infty$**.**

Let us define

$$\phi(s) = \log \mathrm{Tr}\,[\rho^{1-s}\sigma^s], \qquad s \in \mathbb{R}.$$

Then ϕ is a strictly convex and smooth function, with derivative

$$\phi'(s) = (\mathrm{Tr}\,[\rho^{1-s}\sigma^s])^{-1}\mathrm{Tr}\,[\rho^{1-s}\sigma^s(\log\sigma - \log\rho)],$$

[**?** , Exercise 3.5] In particular, $\phi'(0) = -D(\rho\|\sigma)$ and $\phi'(1) = D(\sigma\|\rho)$ Let us define

$$\psi(\lambda) = \min_{s\in\mathbb{R}} \lambda s + \phi(s).$$

Lemma 1. *Let* $-D(\sigma\|\rho) = -\phi'(1) \le \lambda \le -\phi'(0) = D(\rho\|\sigma)$*. Then*

1. $\psi(\lambda) = \inf_{s\in[0,1]} \lambda s + \phi(s)$
2. ψ *is monotone increasing*
3. $\psi(-D(\sigma\|\rho)) = -D(\sigma\|\rho)$, $\psi(D(\rho\|\sigma)) = 0$.

Proof. By strict convexity, the derivative $\phi'(s)$ is increasing, so that there is some $s_0 \in [0,1]$ such that $\lambda = -\phi'(s_0)$ and the function $s \mapsto \lambda s + \phi(s)$ is decreasing for $s \le s_0$ and increasing for $s_0 \le s$. It follows that the infimum is attained at $s_0 \in [0,1]$. This also implies (3).

Assume that $\lambda \in (-\phi'(1), \phi'(0))$, then $\lambda = -\phi'(s_0)$ for some $s_0 \in (0,1)$ and we have

$$\psi(\lambda) = \lambda s_0 + \phi(s_0) = \phi(s_0) - \phi'(s_0)s_0 < \phi(0) = 0 = \psi(-\phi(0)),$$

the inequality follows by strict convexity of ϕ. If $-\phi'(1) \le \lambda_1 < \lambda$, then clearly

$$\psi(\lambda_1) \le \lambda_1 s_0 + \phi(s_0) < \lambda s_0 + \phi(s_0) = \psi(\lambda).$$

This proves (2). □

Lemma 2. *Let* $\lambda \in [-D(\sigma\|\rho), D(\rho\|\sigma)]$ *and* $0 \le r \le D(\rho\|\sigma)$*. Then*

$$\lambda - \psi(\lambda) = r \iff \psi(\lambda) = b(r) := \inf_{s\in[0,1]} \frac{s}{1-s}r + \frac{1}{1-s}\phi(s).$$

Proof. Let $s_\lambda \in [0,1]$ be such that $\lambda = -\phi'(s_\lambda)$, then the assuptions imply that

$$\psi(\lambda) = -\phi'(s_\lambda)s_\lambda + \phi(s_\lambda) = -\phi'(s_\lambda) - r.$$

Solving for $\phi'(s_\lambda)$, we get $-\phi'(s_\lambda) = \frac{1}{1-s_\lambda}(r + \phi(s_\lambda))$, so that

$$\psi(\lambda) = \frac{s_\lambda}{1-s_\lambda}r + \frac{1}{1-s_\lambda}\phi(s_\lambda).$$

Let

$$g(s) = \frac{s}{1-s}r + \frac{1}{1-s}\phi(s).$$

Then $g'(s) = \frac{1}{(1-s)^2}(r + \phi(s) + \phi'(s)(1-s))$. Now note that $h(s) := \phi(s) + \phi'(s)(1-s)$ satisfies $h'(s) = \phi''(s)(1-s)$ so that h is increasing on $(0,1)$ (strict) convexity of ϕ, so that $-D(\rho\|\sigma) = h(0) \leq h(s) \leq h(1) = 0$. It follows that the derivative $g'(s)$ changes sign at a unique point $s_r \in [0,1]$, such that $r + \phi(s_r) + \phi'(s_r)(1-s_r) = 0$. Comparing this to the above computation, we see that $s_r = s_\lambda$ and

$$\psi(\lambda) = \min_{s\in[0,1]} g(s) = b(r).$$

□

We define

$$\psi^*(\lambda) = \inf_{t\in[-1,0]} t\lambda + \phi(t).$$

Again, if $\lambda > D(\rho\|\sigma) = -\phi'(0)$, then $t \mapsto t\lambda + \phi(t)$ is strictly increasing at $t = 0$, which implies that $\phi^*(\lambda) < 0$.

1.4 Inequalities

We have two basic inequalities. For $A, B \geq 0$, let $\{A \geq B\}$ be the sum of eigenprojections of $A - B$ corresponding to nonnegative eigenvalues, similarly $\{A \leq B\}$, $\{A > B\}$ etc. Then

Lemma 3 (Quantum Neyman-Pearson)**.** *We have*

$$\min_{0\leq T\leq I} \mathrm{Tr}\,[A(I-T)] + \mathrm{Tr}\,[BT] = \mathrm{Tr}\,[A\{A \leq B\}] + \mathrm{Tr}\,[B\{A > B\}].$$

Lemma 4 (Audenaert et al)**.** *We have for any $s \in [0,1]$,*

$$\mathrm{Tr}\,[A\{A \leq B\}] + \mathrm{Tr}\,[B\{A > B\}] \leq \mathrm{Tr}\,[A^{1-s}B^s].$$

These statements hold in the von Neumann algebra case as well.

1.5 Nussbaum-Szkola probability distributions

Let $\rho = \sum_i \lambda_i |x_i\rangle\langle x_i|$ and $\sigma = \sum_j \mu_j |y_j\rangle\langle y_j|$ be the spectral decompositions. The pair (P,Q) of Nussbaum-Szkola probability distributions related to (ρ,σ) is defined on $[n]\times[n]$, here $n = \dim(\mathcal{H})$ and $[n] = \{1,\ldots,n\}$. We put

$$P_{ij} = \lambda_i |\langle x_i|y_j\rangle|^2, \qquad Q_{ij} = \mu_j |\langle x_i|y_j\rangle|^2.$$

We then have $D(\rho\|\sigma) = D(P\|Q)$ and $D_\alpha(\rho\|\sigma) = D_\alpha(P\|Q)$ for all α.

For $(\rho^{\otimes n}, \sigma^{\otimes n})$ we get the iid distributions (P^n, Q^n). We also have the following result:

Lemma 5 (Nussbaum-Szkola)**.** *For any test T and $c > 0$ we have*

$$\alpha(T) + c\beta(T) \geq \frac{1}{2}(P(\{P \leq cQ\}) + cQ(\{P > cQ\})) = \frac{1}{2}\sum_{ij} \min\{P_{ij}, cQ_{ij}\}$$

Proof. Let T be a projection, then

$$\mathrm{Tr}\,[\rho T] = \sum_i \lambda_i \langle\, x_i | TT | x_i \,\rangle = \sum_{ij} \lambda_i \langle\, x_i | T | y_j \,\rangle \langle\, y_j | T | y_j \,\rangle = \sum_{ij} \lambda_i |\langle\, x_i | T | y_j \,\rangle|^2$$

and similarly for σ. It follows that

$$\alpha(T) + c\beta(T) = \sum_{ij} \lambda_i |\langle\, x_i | I - T | y_j \,\rangle|^2 + c \sum_{ij} \mu_j |\langle\, x_i | T | y_j \,\rangle|^2.$$

Now we use the inequality

$$a|u-v|^2 + b|v|^2 \geq \frac{1}{2}|u|^2 \min\{a, b\}$$

to lower bound

$$\alpha(T) + c\beta(T) \geq \frac{1}{2} \sum_{ij} |\langle\, x_i | y_j \,\rangle|^2 \min\{\lambda_i, c\mu_j\} = \frac{1}{2} \sum_{i,j} \min\{P_{ij}, cQ_{ij}\}$$

By Lemma 3, this inequality holds for all tests. The equality $\min\{P_{ij}, cQ_{ij}\} = P(\{P \leq cQ\}) + cQ(\{P > cQ\})$ can be easily seen.

□

2 QHT

Let ρ, σ be a pair of density matrices. We test the hypothesis $H_0 = \rho$ against the alternative $H_1 = \sigma$. A test is given by an operator $0 \leq T \leq I$, corresponding to accepring H_0. The two error probabilities are

$$\alpha(T) = \mathrm{Tr}\,[(I - T)\rho], \qquad \beta(T) = \mathrm{Tr}\,[T\sigma].$$

We will consider the asymptotic behaviour of the error probabilities

$$\alpha_n(T_n) = \mathrm{Tr}\,[(I - T_n)\rho_n], \qquad \beta_n(T_n) = \mathrm{Tr}\,[T_n \sigma_n]$$

in testing $H_0 = \rho_n := \rho^{\otimes n}$ against $H_1 = \sigma_n := \sigma^{\otimes n}$.

2.1 Quantum Stein's lemma

We assume $\mathrm{supp}(\rho) \leq \mathrm{supp}(\sigma)$**, so that** $D(\rho\|\sigma) < \infty$**.**

- Quantum Stein's lemma states that if the I. kind error probabilities are constrained as $\alpha_n(T_n) \leq \epsilon$, then the II. kind error probabilities go to zero exponentially, with optimal decay rate equal to the relative entropy $D(\rho\|\sigma)$.
- The strong converse says that if the decay rate of $\beta_n(T_n)$ is greater than $D(\rho\|\sigma)$, then $\alpha_n(T_n) \to 0$ exponentially.

We will need the following inequalities (3) and (4).

Let $\lambda \in \mathbb{R}$ and let $S_n := \{\rho^{\otimes n} > e^{n\lambda}\sigma^{\otimes n}\}$. Then using Lemma 4 (Audenaert) with $A = \rho^{\otimes n}$ and $B = e^{\lambda n}\sigma^{\otimes n}$, we get for any $s \in [0, 1]$

$$\alpha_n(S_n) + e^{n\lambda}\beta_n(S_n) \leq \mathrm{Tr}\, e^{n\lambda s}[(\rho^{\otimes n})^{1-s}(\sigma^{\otimes n})^s] = e^{n\lambda s}(\mathrm{Tr}\,[\rho^{1-s}\sigma^s])^n = e^{n(\lambda s + \phi(s))}. \tag{2}$$

Hence by taking the infimum over $s \in [0,1]$,

$$\alpha_n(S_n) \le e^{n\psi(\lambda)}, \qquad \beta_n(S_n) \le e^{n(-\lambda+\psi(\lambda))} \tag{3}$$

On the other hand, put $p_n = \mathrm{Tr}\,[\rho^{\otimes n} S_n]$ and $q_n = \mathrm{Tr}\,[\sigma^{\otimes n} S_n]$. Then $p_n \ge e^{n\lambda} q_n$ and therefore $p_n^t \le e^{n\lambda t} q_n^t$ for for any $t \in [-1,0]$. We get

$$\begin{aligned} 1-\alpha_n(S_n) = p_n \le e^{n\lambda t} p_n^{1-t} q_n^t &\le e^{n\lambda t}(p_n^{1-t}q_n^t + (1-p_n)^{1-t}(1-q_n)^t) \le e^{n\lambda t}\mathrm{Tr}\,[(\rho^{\otimes n})^{1-t}(\sigma^{\otimes n})^t] \\ = e^{n(\lambda t+\phi(t))} & \end{aligned}$$

for all $t \in [-1,0]$. It follows that for any test T_n, we have

$$\begin{aligned} 1-\alpha_n(T_n) = \mathrm{Tr}\,[\rho^{\otimes n} T_n] &= \mathrm{Tr}\,[(\rho^{\otimes n} - e^{\lambda n}\sigma^{\otimes n})T_n] + e^{\lambda n}\beta_n(T_n) \le \mathrm{Tr}\,[(\rho^{\otimes n} - e^{\lambda n}\sigma^{\otimes n})S_n] + e^{\lambda n}\beta_n(T_n) \\ &\le 1-\alpha_n(S_n) + e^{\lambda n}\beta_n(T_n) \le e^{n(\lambda t+\phi(t))} + e^{\lambda n}\beta_n(T_n) \end{aligned}$$

and hence

$$\beta_n(T_n) \ge e^{-n\lambda}(1-\alpha_n(T_n) - e^{n\psi^*(\lambda)}) \tag{4}$$

Let

$$\beta_n(\epsilon) := \min_{0\le T_n \le I}\{\beta_n(T_n) \mid \alpha_n(T_n) \le \epsilon\}, \qquad \epsilon > 0.$$

Lemma 6 (Quantum Stein's lemma)**.** *[**? ?**] For all $\epsilon \in (0,1)$, we have*

$$\lim_{n\to\infty} -\frac{1}{n}\log\beta_n(\epsilon) = D(\rho\|\sigma).$$

In particular, there exists a sequence T_n of tests such that $\alpha_n(T_n) \to 0$ and $\lim_n \frac{1}{n}\log\beta_n(T_n) = -D(\rho\|\sigma)$.

Proof. Let $\lambda < D(\rho\|\sigma)$, then by Lemma 1, $\psi(\lambda) < 0$, so that in this case (3), $\alpha_n(S_n) \to 0$ and

$$-\frac{1}{n}\log\beta_n(S_n) \ge \lambda - \psi(\lambda) > \lambda.$$

For $\epsilon \in (0,1)$ we have $\alpha_n(S_n) \le \epsilon$ for large enough n, so that $\beta_n(\epsilon) \le \beta_n(S_n)$. It follows that

$$\liminf_n -\frac{1}{n}\log\beta_n(\epsilon) \ge \liminf_n -\frac{1}{n}\log\beta_n(S_n) \ge \lambda.$$

Conversely, by (4) we have for any sequence of tests such that $\alpha_n(T_n) \le \epsilon$ that

$$\beta_n(T_n) \ge e^{-n\lambda}(1-\epsilon - e^{n\psi^*(\lambda)}).$$

Since $\psi^*(\lambda) < 0$ for $\lambda > D(\rho\|\sigma)$, this implies that, for such λ,

$$\limsup_n -\frac{1}{n}\beta_n(\epsilon) \le \lambda.$$

Choosing any $\delta > 0$, we obtain

$$D(\rho\|\sigma) - \delta \le \liminf_n -\frac{1}{n}\beta_n(\epsilon) \le \limsup_n -\frac{1}{n}\beta_n(\epsilon) \le D(\rho\|\sigma) + \delta.$$

Since δ was arbitrary, this implies the first statement. For the second statement, we can choose sequences $\delta_n, \epsilon_n > 0$, $\delta_n, \epsilon_n \to 0$, then we can find a sequence of tests T_n such that $\alpha_n(T_n) \le \epsilon_n$ and $|\frac{1}{n}\log\beta_n(T_n) + D(\rho\|\sigma)| < \delta_n$.

□

Lemma 7 (Strong converse). *[?] Let T_n be a sequence of tests and let $r > D(\rho\|\sigma)$. If*

$$\limsup_n \frac{1}{n}\log\beta_n(T_n) \le -r$$

then $\alpha_n(T_n) \to 1$ exponentially fast.

Proof. By (4), we have for any $\lambda \in \mathbb{R}$, $\delta > 0$ and large enough n,

$$1-\alpha_n(T_n) \le e^{n\psi^*(\lambda)} + e^{n\lambda}\beta_n(T_n) \le e^{n\psi^*(\lambda)} + e^{n(\lambda-r+\delta)}.$$

We then may choose δ and λ such that $D(\rho\|\sigma) < \lambda < r-\delta$, in which case both of the above exponents are negative.

□

2.2 Hoeffding bound

The Hoeffding bound studies the exponential decay of α_n under an exponential constraint on β_n. Specifically, we look at the value of

$$B_e(r) := \sup\{-\limsup_n \frac{1}{n}\log\alpha_n(T_n) \mid \limsup_n \frac{1}{n}\log\beta_n(T_n) \le -r\}, \qquad r > 0$$

The following result was proved in [?] (direct part) and [?].

Lemma 8 (Quantum Hoeffding bound). *For $0 < r \le D(\rho\|\sigma)$, we have*

$$B_e(r) = \sup_{0\le s\le 1} \frac{-sr-\phi(s)}{1-s} = -b(r).$$

Proof. By (2), we see that for any λ,

$$\alpha_n(S_n) \le e^{n(s\lambda+\phi(s))}, \qquad \beta_n(S_n) \le e^{n(-(1-s)\lambda+\phi(s))}$$

Let us choose λ such that $-(1-s)\lambda+\phi(s) = -r$, that is, $\lambda = \frac{r+\phi(s)}{1-s}$, then we obtain

$$\alpha_n(S_n) \le e^{-n\frac{-sr-\phi(s)}{1-s}}, \qquad \beta_n(S_n) \le e^{-nr}.$$

. This implies that $B_e(r) \ge \sup_{0\le s\le 1} \frac{-sr-\phi(s)}{1-s} = -b(r)$. To show the lower bound, we will use the pair of Nussbaum-Szkola probability distributions (P, Q) related to the pair (ρ, σ) and Lemma 5.

What we need to prove is that, for a sequence of tests T_n,

$$\limsup_n \frac{1}{n}\log\beta_n(T_n) \le -r \implies \limsup_n \frac{1}{n}\log\alpha_n(T_n) \ge b(r).$$

By Lemma 5, we have for any $b \in \mathbb{R}$,

$$\alpha_n(T_n) + e^{nb}\beta_n(T_n) \ge \frac{1}{2}[P^n(\{P^n \le e^{nb}Q^n\}) + e^{-nb}Q^n(\{P^n > e^{nb}Q^n\})] \tag{5}$$

Now note that $\{P^n \le e^{nb}Q^n\} = \{\frac{1}{n}\log\frac{Q^n}{P^n} \ge -b\}$ and

$$\log\frac{Q^n(\omega^n)}{P^n(\omega^n)} = \sum_k \log\frac{Q(\omega_k)}{P(\omega_k)}, \qquad \omega^n = (\omega_1,\dots,\omega_n) \in \Omega^n,$$

where $\Omega = [n]\times[n]$. Put $X(\omega) = \log\frac{Q(\omega)}{P(\omega)}$, then $E_P[X] = -D(P\|Q) = -D(\rho\|\sigma)$ and the cumulant generating function of X at P is

$$\log E_P[e^{sX}] = \log E_P[Q^s P^{-s}] = \phi_{P\|Q}(s) = \phi_{\rho\|\sigma}(s) = \phi(s).$$

Now the Cramér theorem of tha large deviation theory implies that

$$\lim_n \frac{1}{n}\log(P(\frac{1}{n}\sum_i X_i \geq -b)) = \psi(b)$$

for all $-b > E_P(X) = -D(\rho\|\sigma)$. Similarly, $\{P^n > e^{nb}Q^n\} = \{\frac{1}{n}\sum_i X_i(\omega_i) < -b\}$, and we have $E_Q[X] = D(Q\|P) = D(\sigma\|\rho)$ and

$$\log E_Q[e^{sX}] = \log \mathrm{Tr}\,[\sigma^{1+s}\rho^{-s}] = \phi(1+s).$$

Now note that

$$\inf_s bs + \phi(1+s) = \inf_s b(1+s) + \phi(1+s) - b = \psi(b) - b$$

so that

$$\lim_n \frac{1}{n}\log(e^{nb}Q(\frac{1}{n}\sum_i X_i < -b)) = \psi(b)$$

for $-b < E_Q[X] = D(\sigma\|\rho)$. It follows that for any $b\in(-D(\sigma\|\rho), D(\rho\|\sigma))$, we have

$$\lim_n \frac{1}{n}\log[P^n(\{P^n \leq e^{nb}Q^n\}) + e^{-nb}Q^n(\{P^n > e^{nb}Q^n\})] = \psi(b),$$

this follows from

$$\log 2 + \min\{\log x, \log y\} = \log(2\min\{x,y\}) \leq \log(x+y) \leq \log 2 + \max\{\log x, \log y\}$$

and the fact that the two limits above are the same. From (5) and the assumption on $\beta_n(T_n)$, we now get

$$\psi(b) \leq \liminf_n \frac{1}{n}\log(\alpha_n(T_n) + e^{nb}\beta_n(T_n)) \leq \max\{\limsup_n \frac{1}{n}\log\alpha_n(T_n), b - r\}.$$

Let us now assume that $0 < r \leq D(\rho\|\sigma)$ and let $\lambda\in(-D(\sigma\|\rho), D(\rho\|\sigma)]$ be such that $r = \lambda - \psi(\lambda)$. Choose a small $\epsilon > 0$ such that $b = \lambda - \epsilon > -D(\sigma\|\rho)$ (we clearly must have $\lambda > -D(\sigma\|\rho)$, since otherwise $r = 0$). Then we get

$$\psi(\lambda-\epsilon) \leq \max\{\limsup_n \frac{1}{n}\log\alpha_n(T_n), \psi(\lambda) - \epsilon\}.$$

Since clearly

$$(\lambda-\epsilon)s + \phi(s) > \lambda s + \phi(s) - \epsilon, \qquad s\in(0,1)$$

and by the assumptions both infima are attained in $(0,1)$, we see that $\phi(\lambda-\epsilon) > \phi(\lambda) - \epsilon$, so that we must have

$$\phi(\lambda-\epsilon) \leq \limsup_n \frac{1}{n}\log\alpha_n(T_n)$$

Taking the limit $\epsilon\to 0$ and noting that $\phi(\lambda) = b(r)$ finishes the proof.

$\square$

2.3 Strong converse exponents

By the strong converse of Stein's lemma (Lemma 7), we know that if $\limsup_n \frac{1}{n}\log\beta_n(T_n)\le -r$ for $r>D(\rho\|\sigma)$, then the success probabilities $1-\alpha_n(T_n)\to 0$ exponentially. We are now interested in the optimal (smallest possible) decay rate, called the strong converse exponent. We put

$$B^*(r) := \inf\{-\liminf_n \frac{1}{n}\log(1-\alpha_n(T_n))\ |\ \limsup_n \frac{1}{n}\log\beta_n(T_n)\le -r\}$$

Let $\mathcal{M}$ be a von Neumann algebra and let ρ,σ be a pair of normal states. We consider the problem of testing the hypothesis $H_0=\rho$ againts alternative $H_1=\sigma$. Any test is represented by an operator $M\in\mathcal{M}$, $0\le M\le I$, such that the value $\omega(M)$ is interpreted as the probability of accepting the hypothesis H_0 in the state ω.

The two error probabilities related to a test M are given as

$$\alpha(M)=\rho(1-M),\qquad \beta(M)=\sigma(M).$$

In the Bayes approach, we choose a prior probability $\pi\in(0,1)$ and minimize the average error probability over all tests, obtaining the optimal value

$$\begin{aligned}
b_\pi(\rho\|\sigma) &:= \inf_M \pi\alpha(M)+(1-\pi)\beta(M)\\
&= \inf_M \pi\rho(1-M)+(1-\pi)\sigma(M)\\
&= \pi-\sup_M(\pi\rho-(1-\pi)\sigma)(M)=\pi(1-(\rho-s\sigma)_+(1))\\
&= \frac{1}{2}(1-\pi\|\rho-s\sigma\|_1),\qquad s=\frac{1-\pi}{\pi}.
\end{aligned}$$

The optimal test for $\pi\in(0,1)$ is the quantum Neyman-Pearson test M_π, which is of the form

$$M_\pi=\{\rho>s\sigma\}+X,\qquad 0\le X\le\{\rho=s\sigma\},\qquad s=\frac{1-\pi}{\pi},$$

here for $\varphi,\psi\in\mathcal{M}_*^+$ we define $\{\varphi>\psi\}$ as the projection ionto the support of $(\varphi-\psi)_+$, similarly $\{\varphi<\psi\}$ is the projectio onto the support of $(\psi-\varphi)_+=(\varphi-\psi)_-$. We then have $\{\varphi\le\psi\}=1-\{\varphi>\psi\}$, $\{\varphi\ge\psi\}=1-\{\varphi<\psi\}$ and $\{\varphi=\psi\}=1-\{\varphi<\psi\}-\{\varphi>\psi\}$.

It is clear from these expressions that the function $\pi\mapsto b_\pi(\rho\|\sigma)$ is continuous. Furthermore, let us look at the error probabilities for the QNP test $\{\rho>s\sigma\}$. Put $\alpha(s):=\rho(\{\rho\le s\sigma\})$ and $\beta(s):=\sigma(\{\rho>s\sigma\})$. Then By the QNP lemma, we have for any $s,s'\in(0,\infty)$ that

$$\alpha(s)+s\beta(s)\le\alpha(s')+s\beta(s') \tag{6}$$

hence

$$s(\beta(s)-\beta(s'))\le\alpha(s')-\alpha(s)\le s'(\beta(s)-\beta(s')),$$

the second inequality is obtained by exchanging s and s' in (6). It follows that if $s'>s$, then $\beta(s)\ge\beta(s')$ and therefore also $\alpha(s)\le\alpha(s')$.

In the asymmetric QHT approach, we fix the maximum acceptable value ϵ of the error $\alpha(M)\le\epsilon$ and minimize the error $\beta(M)$:

$$d_H^\epsilon(\rho\|\sigma):=\inf\{\sigma(M),\ 0\le M\le I,\ \rho(1-M)\le\epsilon\}.$$

The following result is proved similarly as in the classical case (cf. [**?**]).

Lemma 9. *We have*

$$b_\pi(\rho\|\sigma) = \inf_{0<\epsilon<1} [\pi\epsilon + (1-\pi)d_H^\epsilon(\rho\|\sigma)], \qquad \pi \in (0,1)$$

Proof. Let $\pi, \epsilon \in (0,1)$ and let M be any test such that $\rho(1-M) \le \epsilon$, then we have

$$b_\pi(\rho\|\sigma) \le \pi\rho(1-M) + (1-\pi)\sigma(M) \le \pi\epsilon + (1-\pi)\sigma(M).$$

Taking infimum over all such tests, we obtain

$$b_\pi(\rho\|\sigma) \le \pi\epsilon + (1-\pi)d_H^\epsilon(\rho\|\sigma), \qquad \forall \epsilon, \pi \in (0,1). \tag{7}$$

For a fixed $\pi \in (0,1)$ and a Neyman-Pearson test M_π, let $\epsilon_\pi = \rho(1-M_\pi)$, then

$$b_\pi(\rho\|\sigma) = \pi\epsilon_\pi + (1-\pi)\sigma(M_\pi) \ge \pi\epsilon_\pi + (1-\pi)d_H^{\epsilon_\pi}(\rho\|\sigma),$$

this proves equality.

□

2.4 In finite dimensions...

Lemma 10. *Let $s(\rho) \le s(\sigma)$. Then for any $\epsilon \in (0,1)$ there is some $\pi_\epsilon \in (0,1)$ and a Neyman-Pearson test $M_\epsilon = M_{\pi_\epsilon}$ such that $\rho(1-M_\epsilon) = \epsilon$.*

Proof. Let

$$s_\epsilon := \sup\{s \ge 0, \mathrm{Tr}\,[\rho P_{s,-}] < \epsilon\}.$$

Under the assumption, there is some $\lambda > 0$ such that $\rho \le \lambda\sigma$, and then $P_{s,-} = I$ for all $s \ge \lambda$. It follows that we must have $c_\epsilon < \lambda$, in particular, c_ϵ is finite. Further, by [**?** , Lemma], we have for $s \in [0,\infty)$:

$$\lim_{t\to s^-} P_{t,-} = P_{s,-}, \qquad \lim_{t\to s+} P_{t,-} = P_{s,-} + P_{s,0}$$

and $P_{s,0} \neq 0$ for a finite number of values of s. If follows that

$$\mathrm{Tr}\,[\rho P_{c_\epsilon,-}] = \lim_{t\to c_\epsilon -} \mathrm{Tr}\,[\rho P_{t,-}] \le \epsilon \le \lim_{t\to c_\epsilon +} \mathrm{Tr}\,[\rho P_{t,-}] = \mathrm{Tr}\,[\rho(P_{c_\epsilon,-} + P_{c_\epsilon,0})]$$

and

$$M_\epsilon = P_{c_\epsilon,-} + \frac{\epsilon - \mathrm{Tr}\,[\rho P_{c_\epsilon,-}]}{\mathrm{Tr}\,[\rho P_{c_\epsilon,0}]} P_{c_\epsilon,0}$$

is a Neyman-Pearson test for $\pi_\epsilon := (c_\epsilon + 1)^{-1}$ such that $\mathrm{Tr}\,[\rho M_\epsilon] = \epsilon$.

□

Lemma 11. *If $s(\rho) \le s(\sigma)$ then also*

$$d_H^\epsilon(\rho\|\sigma) = \sup_{0<\pi<1} \frac{1}{1-\pi}[b_\pi(\rho\|\sigma) - \pi\epsilon], \qquad \epsilon \in (0,1).$$

Proof. We see that (7) implies

$$d_H^\epsilon(\rho\|\sigma) \ge \frac{1}{1-\pi}[b_\pi(\rho\|\sigma) - \pi\epsilon]$$

for all ϵ and π. If $s(\rho) \le s(\sigma)$, then the second equality follows by Lemma 9.

□

3 Measured Rényi relative entropy

Let $\mathcal{M}$ be a von Neumann algebra. A measurement on $\mathcal{M}$ (with values in a finite set Ω) is defined as a positive linear map $M : L_\infty(\Omega) \to \mathcal{M}$ such that $M(1) = 1$. Equivalently, a measurement is given by a set of positive operators $\{a_\omega\}_{\omega\in\Omega}$ such that $\sum_\omega a_\omega = 1$. More generally, a measurement can be defined as a unital positive map from a commutative von Neumann algebra to $\mathcal{M}$.

Let ρ, σ be normal states on $\mathcal{M}$. The measured Rényi relative entropy is defined as

$$D_\alpha^M(\rho\|\sigma) = \sup\{D_\alpha(\rho \circ M\|\sigma \circ M),\ M \text{ is a measurement}\}.$$

By [**?** , Thm. 5.2], we may restrict to finite valued measurements.

It is easily seen that D_α^M is monotone under positive unital normal maps. Indeed, let $\Phi : \mathcal{N} \to \mathcal{M}$ be such a map and let $M : L_\infty(\Omega) \to \mathcal{N}$ be a measurement on $\mathcal{N}$, then $\Phi\circ M$ is a measurement on $\mathcal{M}$ and we have

$$\begin{aligned} D_\alpha^M(\rho \circ \Phi\|\sigma \circ \Phi) &= \sup\{D_\alpha(\rho \circ \Phi \circ M\|\sigma \circ \Phi \circ M),\ M \text{ is a measurement on } \mathcal{N}\} \\ &\leq \sup\{D_\alpha(\rho \circ M\|\sigma \circ M),\ M \text{ is a measurement on } \mathcal{M}\} \\ &= D_\alpha^M(\rho\|\sigma). \end{aligned}$$

Further, by monotonicity of the sandwiched Rényi relative entropy $\tilde{D}_\alpha$ for $\alpha \in [1/2, 1)\cup(1, \infty]$, and by additivity with respect to tensor products, we have

$$\frac{1}{n} D_\alpha^M(\rho^{\otimes n}\|\sigma^{\otimes n}) \leq \frac{1}{n}\tilde{D}_\alpha(\rho^{\otimes n}\|\sigma^{\otimes n}) = \tilde{D}_\alpha(\rho\|\sigma).$$

We are interested in the equality

$$\lim_n \frac{1}{n} D_\alpha^M(\rho^{\otimes n}\|\sigma^{\otimes n}) = \tilde{D}_\alpha(\rho\|\sigma), \qquad \alpha > 1. \tag{8}$$

This equality was proved in [**?**] in the finite dimensional case, in [**?**] in the approximately finite (injective) case and in [**?**] it was extended to semifinite von Neumann algebras. We will use the Haagerup reduction to show that this equality holds for all von Neumann algebras.

3.1 Haagerup reduction

We just give an outline:

- There exists a von Neumann algebra $\hat{\mathcal{M}}$ (a crossed product) such that we may identify $\mathcal{M}$ as a subalgebra in $\hat{\mathcal{M}}$ and there is a canonical normal conditional expectation $E_{\mathcal{M}}$ of $\hat{\mathcal{M}}$ onto $\mathcal{M}$,
- There is an increasing family of subalgebras $\mathcal{M}_n \subseteq \hat{\mathcal{M}}$ such that
  - Each $\mathcal{M}_n$ is finite (equipped with a faithful normal tracial state τ_n),
  - $\bigcup_{n\geq 1} \mathcal{M}_n$ is weak*-dense in $\hat{\mathcal{M}}$
  - For each n, there exists a faithful normal conditional expectation E_n of $\hat{\mathcal{M}}$ onto $\mathcal{M}_n$ and for all $x \in \hat{\mathcal{M}}$, $E_n(x) \to x$ in the σ-strong topology.

- Any normal state ρ on $\mathcal{M}$ extends to a normal state on $\hat{\mathcal{M}}$ as $\hat{\rho} := \rho \circ E_{\mathcal{M}}$. Put $\rho_n := \hat{\rho}|_{\mathcal{M}_n}$, then ρ_n is a normal state on $\mathcal{M}_n$. Using the conditional expectation E_n, ρ_n extends to a state $\hat{\rho}_n = \rho_n \circ E_n = \hat{\rho} \circ E_n$. We have $\|\hat{\rho}_n - \hat{\rho}\|_1 \to 0$.

We will use this result to prove (8) for any von Neumann algebra $\mathcal{M}$. First, note that since $\rho = \hat{\rho}|_{\mathcal{M}}$ and $\hat{\rho} = \rho \circ E_{\mathcal{M}}$, we have by DPI that

$$\tilde{D}_\alpha(\rho\|\sigma) = \tilde{D}_\alpha(\hat{\rho}\|\hat{\sigma}), \qquad D_\alpha^M(\rho^{\otimes n}\|\sigma^{\otimes n}) = D_\alpha^M(\hat{\rho}^{\otimes n}\|\hat{\sigma}^{\otimes n}).$$

Hence it is enough to work with $\hat{\rho}$ and $\hat{\sigma}$. Notice that by th DPI for D_α^M, we have for any m and $\alpha > 1$

$$\lim_n \frac{1}{n} D_\alpha^M(\hat{\rho}^{\otimes n}\|\sigma^{\otimes n}) \geq \lim_n \frac{1}{n} D_\alpha^M(\hat{\rho}_m^{\otimes n}\|\hat{\sigma}_m^{\otimes n}) = \tilde{D}_\alpha(\hat{\rho}_m\|\hat{\sigma}_m),$$

since $\mathcal{M}_m$ is finite. We therefore have

$$\lim_n \frac{1}{n} D_\alpha^M(\hat{\rho}^{\otimes n}\|\hat{\sigma}^{\otimes n}) \geq \lim_m \tilde{D}_\alpha(\hat{\rho}_m\|\hat{\sigma}_m) = \tilde{D}_\alpha(\hat{\rho}\|\hat{\sigma}).$$

The last equality holds by DPI and lower semicontinuity of $\tilde{D}_\alpha$. Since the opposite inequality always holds, this proves (8).

3.2 Strong converse exponents