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CLONING AND BROADCASTING IN OPERATOR ALGEBRAS

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Abstract

We investigate cloning and broadcasting in the general operator algebra framework in arbitrary dimension, generalizing thus results obtained so far in finite dimension, and for the full algebra of operators on a Hilbert space. Moreover, we also weaken the assumption of complete positivity of the broadcasting (cloning) operation. It turns out that some results can be expressed in terms of modular theory for W^*-algebras, in particular, the Connes cocycles are used for a characterization of broadcastability.

1. Introduction

Since its first appearance in [4, 15], the celebrated no-cloning theorem for quantum states has been generalized in several directions including also the no-broadcasting variant. Among numerous works in this area, let us mention [1, 7] in which the problem is analyzed in the Hilbert space setup, and [2, 3] where it is considered in generic probabilistic models. A common feature of these approaches consists of restricting attention to the finite-dimensional models; moreover, in the Hilbert space setup the map defining cloning or broadcasting is assumed to be completely positive. In this paper, we investigate cloning and broadcasting in the general operator algebra framework, i.e. instead of the full algebra of all bounded operators on a finite-dimensional Hilbert space we consider an arbitrary von Neumann algebra on a Hilbert space of arbitrary dimension. We also weaken the assumption of complete positivity of the broadcasting (cloning) operation to a weaker form of the so-called $1\frac{1}{2}$-positivity, i.e. the operation being a Schwarz (or strongly positive) map. This point deserves probably closer attention. Namely, for completely positive maps on the full algebra of all bounded operators on a Hilbert space, we have at our disposal the Krauss representation. However, this is not the case when we are dealing with Schwarz maps on an arbitrary von Neumann algebra. It turns out that in this more general setup a proper tool is ergodic theory in von Neumann algebras, in particular its part concerning the analysis of the fixed-point space of a semigroup of Schwarz maps acting on such an algebra.

It is probably worth mentioning that, although the origin of broadcasting and cloning lies in quantum physics, in particular quantum information theory, these operations have nevertheless a purely mathematical character, being simply some natural maps between states of von Neumann algebras. Thus, their investigation seems to be of independent mathematical interest, still with the hope for possible physical applications.

The main results of this work are as follows. For an arbitrary subset Γ of the normal states of a von Neumann algebra, it is shown that Γ is broadcastable if and only if there is a countable family

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of normal states having mutually orthogonal supports, such that each $\rho \in \Gamma$ is a convex combination of these states. If the von Neumann algebra in question is the full algebra of all bounded operators, it is proved that the broadcastability of Γ is equivalent to the commutativity of the density matrices of the states from it. Next, the problem of broadcastability is investigated in terms of modular theory for W^*-algebras. In particular, it is proved that Γ is broadcastable if and only if the W^*-algebra generated by the Connes cocycles of the states from Γ is abelian and atomic. Finally, it is shown that Γ is cloneable if and only if the states from it have mutually orthogonal supports.

2. Preliminaries and notation

Let $\mathscr{A}$ be an arbitrary von Neumann algebra with identity $\mathbb{1}$ acting on a Hilbert space $\mathcal{H}$. The predual $\mathscr{A}_*$ of $\mathscr{A}$ is a Banach space of all *normal*, i.e. continuous in the σ-weak topology, linear functionals on $\mathscr{A}$.

A *state* on $\mathscr{A}$ is a bounded positive linear functional $\rho\colon \mathscr{A} \to \mathbb{C}$ of norm 1. For a normal state ρ its *support*, denoted by $s(\rho)$, is defined as the smallest projection in $\mathscr{A}$ such that $\rho(s(\rho)) = \rho(\mathbb{1})$. In particular, we have

$$\rho(s(\rho)a) = \rho(a\, s(\rho)) = \rho(a), \quad a \in \mathscr{A},$$

and if $\rho(s(\rho)a\, s(\rho)) = 0$ for $s(\rho)a\, s(\rho) \geqslant 0$, then $s(\rho)a\, s(\rho) = 0$.

A normal state ρ is said to be *faithful* if for each positive element $a \in \mathscr{A}$, from the equality $\rho(a) = 0$ it follows that $a = 0$. It is easily seen that the faithfulness of ρ is equivalent to the relation $s(\rho) = \mathbb{1}$. It is well known that the existence of a faithful normal state is equivalent to the so-called *σ-finiteness* (or *countable decomposability*) of the algebra $\mathscr{A}$; however, we shall assume this property of the underlying algebra only when performing the analysis in terms of modular theory and the Connes cocycles.

Let $\{\rho_\theta : \theta \in \Theta\}$ be a family of normal states on a von Neumann algebra $\mathscr{A}$. Define the support of this family by

$$e = \bigvee_{\theta \in \Theta} s(\rho_\theta).$$

The family $\{\rho_\theta : \theta \in \Theta\}$ is said to be *faithful* if for each positive element $a \in \mathscr{A}$ from the equality $\rho_\theta(a) = 0$ for all $\theta \in \Theta$ it follows that $a = 0$. Similarly, to the case of one state it is seen that the faithfulness of this family is equivalent to the relation $e = \mathbb{1}$; moreover, if $\rho_\theta(eae) = 0$ for all $\theta \in \Theta$ and $eae \geqslant 0$, then $eae = 0$.

By a W^*-algebra of operators acting on a Hilbert space, we shall mean a C^*-subalgebra of $\mathbb{B}(\mathcal{H})$ with identity, closed in the weak-operator topology. A typical example (and in fact the only one utilized in the paper) is the algebra $p\mathscr{A}p$, where p is a projection in $\mathscr{A}$. For arbitrary $\mathscr{R} \subset \mathbb{B}(\mathcal{H})$, we denote by $W^*(\mathscr{R})$ the smallest W^*-algebra of operators on $\mathcal{H}$ containing $\mathscr{R}$.

A projection in a W^*-algebra is said to be *minimal* if it majorizes no other non-zero projection in this algebra. A W^*-algebra is said to be *atomic* if the supremum of all its minimal projections equals the identity of this algebra.

Let $\mathscr{A}$ and $\mathscr{B}$ be W^*-algebras. A linear map $T\colon \mathscr{A} \to \mathscr{B}$ is said to be *normal* if it is continuous in the σ-weak topologies on $\mathscr{A}$ and $\mathscr{B}$, respectively. It is called *unital* if it maps the unit of $\mathscr{A}$ to the unit of $\mathscr{B}$. The map T is said to be *Schwarz* (or *$1\frac{1}{2}$-positive* or *strongly positive*) if for each $a \in \mathscr{A}$ the following Schwarz inequality holds:

$$T(a)^*T(a) \leqslant \|T\|T(a^*a),$$

which for a unital map amounts simply to

$$T(a)^*T(a) \leqslant T(a^*a).$$

In particular, a completely or two-positive map is Schwarz. For a normal unital Schwarz map T, we define its *multiplicative domain* as

$$\mathscr{N} = \{a \in \mathscr{A} : T(a^*a) = T(a)^*T(a),\ T(aa^*) = T(a)T(a)^*\}.$$

It is known that $\mathscr{N}$ is a W^*-subalgebra of $\mathscr{A}$, and $T \mid \mathscr{N}$ is an *-homomorphism.

Let $\mathscr{A}$ be a von Neumann algebra and consider the tensor product $\mathscr{A}\bar{\otimes}\mathscr{A}$. We have obvious counterparts $\Pi_{1,2} \colon (\mathscr{A}\bar{\otimes}\mathscr{A})_* \to \mathscr{A}_*$ of the customary notion of partial trace, employed in the case $\mathscr{A} = \mathbb{B}(\mathcal{H})$, defined as

$$(\Pi_1\tilde{\rho})(a) = \tilde{\rho}(a \otimes \mathbb{1}), \quad (\Pi_2\tilde{\rho})(a) = \tilde{\rho}(\mathbb{1} \otimes a), \quad \tilde{\rho} \in (\mathscr{A}\bar{\otimes}\mathscr{A})_*,\ a \in \mathscr{A}.$$

The main objects of our interest are the following two operations of *broadcasting* and *cloning* of states.

A linear map $K_* \colon \mathscr{A}_* \to (\mathscr{A}\bar{\otimes}\mathscr{A})_*$ sending states to states, and such that its dual $K : \mathscr{A}\bar{\otimes}\mathscr{A} \to \mathscr{A}$ is a unital Schwarz map will be called a *channel*. (This terminology is almost standard, because by a 'channel' usually means a completely positive unital map between two von Neumann algebras.) A state $\rho \in \mathscr{A}_*$ is *broadcast* by channel K_* if $(\Pi_i \circ K_*)(\rho) = \rho,\ i = 1, 2$; in other words, ρ is broadcast by K_* if for each $a \in \mathscr{A}$

$$\rho(K(a \otimes \mathbb{1})) = \rho(K(\mathbb{1} \otimes a)) = \rho(a).$$

A family of states is said to be *broadcastable* if there is a channel K_* that broadcasts each member of this family.

A state $\rho \in \mathscr{A}_*$ is *cloned* by channel K_* if $K_*\rho = \rho \otimes \rho$. A family of states is said to be *cloneable* if there is a channel K_* that clones each member of this family.

3. Broadcasting

In this section, we aim at a description of an arbitrary set of broadcastable states, thus let $\Gamma \subset \mathscr{A}_*$ be such. Then there is a channel K_* which broadcasts the states in Γ. Define maps $L, R \colon \mathscr{A} \to \mathscr{A}$ as

$$L(a) = K(a \otimes \mathbb{1}), \quad R(a) = K(\mathbb{1} \otimes a), \quad a \in \mathscr{A}.$$

Then L and R are unital normal Schwarz maps on $\mathscr{A}$. Observe that for a state ρ broadcast by K_* we have, for each $a \in \mathscr{A}$,

$$(\rho \circ L)(a) = \rho(K(a \otimes \mathbb{1})) = \rho(a), \quad (\rho \circ R)(a) = \rho(K(\mathbb{1} \otimes a)) = \rho(a),$$

i.e.

$$\rho \circ L = \rho \circ R = \rho, \tag{1}$$

and by the very definition of broadcastability if (1) holds, then ρ is broadcast by K_*. Consequently,

$$\Gamma \subset \{\rho\text{—normal state} : \rho \circ L = \rho \circ R = \rho\}.$$

Set

$$p = \bigvee_{\rho \in \Gamma} s(\rho). \tag{2}$$

Then Γ is a faithful family of normal states on the W^*-algebra $p\mathscr{A}p$.

The idea of our further considerations, up to Proposition 3.6, lies in the following. We want to have at our disposal a faithful family of normal states, consequently, we 'transfer' the setup to the algebra $p\mathscr{A}p$, use the ergodic theorem for W^*-algebras there, and then come back to algebra $\mathscr{A}$. This is most clearly seen in Proposition 3.4 which, under the assumption that Γ is a faithful family of states on $\mathscr{A}$, i.e. that $p = \mathbb{1}$, follows at once from the ergodic theorem. Thus, the reader who does not wish to follow all this simple, but sometimes tedious procedure may assume that $p = \mathbb{1}$ without losing the essence of further considerations.

Define maps $K^{(p)} : \mathscr{A}\bar{\otimes}\mathscr{A} \to p\mathscr{A}p$ and $L^{(p)}, R^{(p)} : \mathscr{A} \to p\mathscr{A}p$ by

$$K^{(p)}(\tilde{a}) = pK(\tilde{a})p, \quad \tilde{a} \in \mathscr{A}\bar{\otimes}\mathscr{A},$$
$$L^{(p)}(a) = pL(a)p, \quad R^{(p)}(a) = pR(a)p, \quad a \in \mathscr{A}.$$

Then $K^{(p)},\ L^{(p)},\ R^{(p)}$ are normal Schwarz maps of norm 1 such that for each $a \in \mathscr{A}$

$$K^{(p)}(a \otimes \mathbb{1}) = L^{(p)}(a), \quad K^{(p)}(\mathbb{1} \otimes a) = R^{(p)}(a)$$

and

$$K^{(p)}(\mathbb{1} \otimes \mathbb{1}) = L^{(p)}(\mathbb{1}) = R^{(p)}(\mathbb{1}) = p.$$

Moreover, we have

$$p - pL(p)p = p(\mathbb{1} - L(p))p \geqslant 0,$$

and for each $\rho \in \Gamma$ the L-invariance of ρ yields

$$\rho(p - pL(p)p) = 0,$$

which means that

$$pL(p)p = p, \tag{3}$$

since Γ is faithful on $p\mathscr{A}p$. The same relation holds also for R, thus

$$L^{(p)}(p) = R^{(p)}(p) = p.$$

Consequently,

$$K^{(p)}(\mathbb{1} \otimes \mathbb{1}) = L^{(p)}(\mathbb{1}) = L^{(p)}(p) = R^{(p)}(\mathbb{1}) = R^{(p)}(p) = p.$$

It is almost immediate that the states in Γ are also $L^{(p)}$- and $R^{(p)}$-invariant, namely we have the following lemma.

LEMMA 3.1 *The following inclusion holds true*

$$\Gamma \subset \{\rho\text{—}\textit{normal state} : \rho \circ L^{(p)} = \rho \circ R^{(p)} = \rho\}.$$

Proof. Assume that $\rho \in \Gamma$. Then, since $s(\rho) \leqslant p$, we have for each $a \in \mathscr{A}$

$$\rho(a) = \rho(L(a)) = \rho(pL(a)p) = \rho(L^{(p)}(a)),$$

and the same holds for $R^{(p)}$. □

For a map T on $\mathscr{A}$, denote by $\mathcal{F}(T)$ its fixed-point space, i.e.

$$\mathcal{F}(T) = \{a \in \mathscr{A} : T(a) = a\}.$$

Let $\mathscr{N}$ be the multiplicative domain of $K^{(p)}$. We have the following lemma.

LEMMA 3.2 *The following relations hold*:

(i) *for each* $a \in \mathcal{F}(L^{(p)})$, *we have* $a \otimes \mathbb{1} \in \mathscr{N}$,
(ii) *for each* $a \in \mathcal{F}(R^{(p)})$, *we have* $\mathbb{1} \otimes a \in \mathscr{N}$.

Proof. It is enough to prove (i) since the proof of (ii) is analogous. Let $a \in \mathcal{F}(L^{(p)})$. Then $L^{(p)}(a) = a$ and $a = pa = ap$. The Schwarz inequality for the map $L^{(p)}$ yields

$$a^*a = L^{(p)}(a)^*L^{(p)}(a) \leqslant L^{(p)}(a^*a),$$

hence

$$pa^*ap = a^*a \leqslant L^{(p)}(a^*a) = pL^{(p)}(a^*a)p,$$

or in other words

$$p(L^{(p)}(a^*a) - a^*a)p \geqslant 0.$$

For an arbitrary $\rho \in \Gamma$, we have on account of the $L^{(p)}$-invariance of ρ

$$\begin{aligned}\rho(p(L^{(p)}(a^*a) - a^*a)p) &= \rho(L^{(p)}(a^*a) - a^*a)\\ &= \rho(L^{(p)}(a^*a)) - \rho(a^*a) = 0,\end{aligned}$$

and since Γ is faithful on the algebra $p\mathscr{A}p$ we obtain

$$p(L^{(p)}(a^*a) - a^*a)p = 0,$$

which amounts to the equality

$$L^{(p)}(a^*a) = a^*a.$$

Taking into account the definition of $L^{(p)}$, we get

$$\begin{aligned}K^{(p)}(a^* \otimes \mathbb{1})K^{(p)}(a \otimes \mathbb{1}) &= L^{(p)}(a^*)L^{(p)}(a) = a^*a = L^{(p)}(a^*a)\\ &= pK(a^*a \otimes \mathbb{1})p = K^{(p)}(a^*a \otimes \mathbb{1}) = K^{(p)}((a^* \otimes \mathbb{1})(a \otimes \mathbb{1})).\end{aligned}$$

By the same token, we find that

$$K^{(p)}(a \otimes \mathbb{1})K^{(p)}(a^* \otimes \mathbb{1}) = K^{(p)}((a \otimes \mathbb{1})(a^* \otimes \mathbb{1})),$$

showing that $a \otimes \mathbb{1}$ belongs to $\mathscr{N}$. □

To simplify the notation, let us agree to the following convention. For a Schwarz map $T : \mathscr{A} \to p\mathscr{A}p$ such that

$$T(\mathbb{1}) = T(p) = p$$

denote by T_p the restriction $T|p\mathscr{A}p$, so that $T_p : p\mathscr{A}p \to p\mathscr{A}p$. Now the unital Schwarz maps from $p\mathscr{A}p$ to $p\mathscr{A}p$ will also be denoted with the use of index p, thus T_p will stand for a Schwarz map on the algebra $p\mathscr{A}p$ such that $T_p(p) = p$. To justify this abuse of notation, let us define for such a map T_p the map $T : \mathscr{A} \to p\mathscr{A}p$ as follows:

$$T(a) = T_p(pap), \quad a \in \mathscr{A}. \tag{4}$$

It is clear that we have $T|p\mathscr{A}p = T_p$, so for the consistency of our notation we only need the relation

$$T(a) = T(pap), \quad a \in \mathscr{A},$$

which is a consequence of the following well-known fact whose proof can be found, e.g. in [**8**, Lemma 2].

LEMMA 3.3 *Let $T : \mathscr{A} \to \mathscr{A}$ be positive, and let e be a projection in $\mathscr{A}$ such that*

$$T(\mathbb{1}) = T(e) = e.$$

Then for each $a \in \mathscr{A}$

$$T(a) = T(ea) = T(ae) = eT(a) = T(a)e.$$

In the sequel while dealing with two maps denoted by the same capital letter, one with and the other without index p, we shall always assume that they are connected by relation (4). If T, T_p, V and V_p are maps as above, then it is easily seen that

$$(TV)_p = T_pV_p,$$

in particular, for each positive integer m we have

$$(T^m)_p = (T_p)^m.$$

The same convention will be adopted also for states with supports contained in p, i.e. if φ is a state on $\mathscr{A}$ such that $s(\varphi) \leqslant p$, then φ_p will denote its restriction to $p\mathscr{A}p$, and for an arbitrary state φ_p on $p\mathscr{A}p$ the state φ on $\mathscr{A}$ will be defined as

$$\varphi(a) = \varphi_p(pap), \quad a \in \mathscr{A}.$$

Now we fix attention on the algebra $p\mathscr{A}p$. In accordance with our convention, we define maps $L_p^{(p)}, R_p^{(p)} \colon p\mathscr{A}p \to p\mathscr{A}p$ as

$$L_p^{(p)} = L^{(p)}|p\mathscr{A}p, \quad R_p^{(p)} = R^{(p)}|p\mathscr{A}p.$$

Clearly, $L_p^{(p)}$ and $R_p^{(p)}$ are normal unital Schwarz maps such that for $\rho \in \Gamma$ the ρ_p are $L_p^{(p)}$- and $R_p^{(p)}$-invariant. It is obvious that $\mathcal{F}(L_p^{(p)}) = \mathcal{F}(L^{(p)})$ and $\mathcal{F}(R_p^{(p)}) = \mathcal{F}(R^{(p)})$.

Let $\mathfrak{S}_p$ be the semigroup of normal Schwarz maps on $p\mathscr{A}p$ generated by $L_p^{(p)}$ and $R_p^{(p)}$. Then Γ_p defined as $\Gamma_p = \{\rho_p : \rho \in \Gamma\}$ is a faithful family of $\mathfrak{S}_p$-invariant normal states on $p\mathscr{A}p$. Denote by $\mathcal{F}(\mathfrak{S}_p)$ the fixed-point space of $\mathfrak{S}_p$, i.e.

$$\mathcal{F}(\mathfrak{S}_p) = \{x \in p\mathscr{A}p : S_p(x) = x \text{ for each } S_p \in \mathfrak{S}_p\}.$$

From the ergodic theorem for W^*-algebras (cf. [6]), we infer that $\mathcal{F}(\mathfrak{S}_p)$ is a W^*-algebra, and there exists a normal faithful conditional expectation $\mathbb{E}_p \colon p\mathscr{A}p \to \mathcal{F}(\mathfrak{S}_p)$ such that

$$\mathbb{E}_p S_p = S_p \mathbb{E}_p = \mathbb{E}_p \quad \text{for each } S_p \in \mathfrak{S}_p \tag{5}$$

and

$$\rho_p \circ \mathbb{E}_p = \rho_p \quad \text{for each } \rho \in \Gamma.$$

Moreover, if φ_p is an arbitrary $\mathbb{E}_p$-invariant normal state on $p\mathscr{A}p$, then from relation (5) we see that φ_p is $\mathfrak{S}_p$-invariant. Conversely, if φ_p is an arbitrary $\mathfrak{S}_p$-invariant normal state on $p\mathscr{A}p$, then another consequence of the ergodic theorem is that φ_p is also $\mathbb{E}_p$-invariant (this follows from the fact that for each $x \in p\mathscr{A}p$, $\mathbb{E}_p x$ lies in the σ-weak closure of the convex hull of $\{S_p x : S_p \in \mathfrak{S}_p\}$). Consequently, we have the following equivalence for a normal state φ_p on $p\mathscr{A}p$:

$$\varphi_p \text{ is } \mathfrak{S}_p\text{-invariant if and only if it is } \mathbb{E}_p\text{-invariant.} \tag{6}$$

Now we want to transfer these results from the algebra $p\mathscr{A}p$ back to the algebra $\mathscr{A}$. Each element S_p of $\mathfrak{S}_p$ has the form

$$S_p = (L_p^{(p)})^{r_1}(R_p^{(p)})^{r_2} \cdots (L_p^{(p)})^{r_{m-1}}(R_p^{(p)})^{r_m},$$

where the integers $r_1, \dots, r_m$ satisfy $r_1, r_m \geqslant 0$ and $r_2, \dots, r_{m-1} > 0$, $m = 1, 2, \dots$. Consequently,

$$\begin{aligned} S_p &= ((L^{(p)})^{r_1})_p((R^{(p)})^{r_2})_p \cdots ((L^{(p)})^{r_{m-1}})_p((R^{(p)})^{r_m})_p \\ &= ((L^{(p)})^{r_1}(R^{(p)})^{r_2} \cdots (L^{(p)})^{r_{m-1}}(R^{(p)})^{r_m})_p, \end{aligned}$$

showing that S defined in accordance with our convention as

$$S(a) = S_p(pap), \quad a \in \mathscr{A},$$

is an element of the semigroup $\mathfrak{S}$ generated by the maps $L^{(p)}$ and $R^{(p)}$. Thus, we have

$$\mathfrak{S}_p = \{S_p : S \in \mathfrak{S}\}.$$

It is easily seen that $\mathcal{F}(\mathfrak{S}) = \mathcal{F}(\mathfrak{S}_p)$, where $\mathcal{F}(\mathfrak{S})$ denotes the fixed-points of $\mathfrak{S}$. In accordance with our convention, we define a map $\mathbb{E}\colon \mathscr{A} \to \mathcal{F}(\mathfrak{S})$ by the formula

$$\mathbb{E}a = \mathbb{E}_p(pap), \quad a \in \mathscr{A}. \tag{7}$$

Then $\mathbb{E}$ is a normal conditional expectation onto the W^*-algebra $\mathcal{F}(\mathfrak{S})$ such that $\mathbb{E}(\mathbb{1}) = p$.

In the following proposition, we obtain relations between broadcastable states and the states invariant with respect to $\mathfrak{S}$ or $\mathbb{E}$.

PROPOSITION 3.4 *Let Γ be a set of normal states on $\mathscr{A}$ broadcast by a channel K_*, and let φ be a normal state on $\mathscr{A}$. Consider the following conditions*:

(i) *φ belongs to Γ*,
(ii) *φ is $\mathfrak{S}$-invariant*,
(iii) *$\varphi = \varphi \circ \mathbb{E}$*.

Then (i) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (iii).

Proof. (i) $\Longrightarrow$ (ii). It follows from Lemma 3.1.

(ii) $\Longrightarrow$ (iii). We have

$$\varphi(\mathbb{1}) = \varphi(L^{(p)}(\mathbb{1})) = \varphi(p),$$

which means that $s(\varphi) \leqslant p$. Consider the state φ_p. We have for each $a \in \mathscr{A}$

$$\varphi_p(L_p^{(p)}(pap)) = \varphi(L^{(p)}(pap)) = \varphi(pap) = \varphi_p(pap),$$

showing that φ_p is $L_p^{(p)}$-invariant. In the same way, it is shown that φ_p is $R_p^{(p)}$-invariant, thus φ_p is $\mathfrak{S}_p$-invariant. Since the $\mathfrak{S}_p$-invariance of φ_p is by (6) equivalent to its $\mathbb{E}_p$-invariance, we have for each $a \in \mathscr{A}$

$$\varphi(a) = \varphi(pap) = \varphi_p(pap) = \varphi_p(\mathbb{E}_p(pap)) = \varphi(\mathbb{E}_p(pap)) = \varphi(\mathbb{E}a).$$

(iii) $\Longrightarrow$ (ii). Observe first that

$$\varphi(\mathbb{1}) = \varphi(\mathbb{E}(\mathbb{1})) = \varphi(p),$$

which means that $s(\varphi) \leqslant p$. Further, we have

$$\varphi_p(pap) = \varphi(a) = \varphi(\mathbb{E}a) = \varphi_p(\mathbb{E}_p(pap)),$$

showing that φ_p is $\mathbb{E}_p$-invariant. From the relation (6), it follows that φ_p is $\mathfrak{S}_p$-invariant which yields the $\mathfrak{S}$-invariance of φ. □

It turns out that the map $K^{(p)}$ has a special form on the tensor product W^*-algebra $\mathcal{F}(\mathfrak{S})\bar{\otimes}\mathcal{F}(\mathfrak{S})$ (this is the weak closure of the algebra of *operators* $\{\sum_{i=1}^m a_i \otimes b_i : a_i, b_i \in \mathcal{F}(\mathfrak{S})\}$ acting on $\mathcal{H} \otimes \mathcal{H}$).

PROPOSITION 3.5 *For each $a, b \in \mathcal{F}(\mathfrak{S})$, we have*

$$K^{(p)}(a \otimes b) = ab; \tag{8}$$

moreover, the W^*-algebra $\mathcal{F}(\mathfrak{S})$ is abelian.

Proof. Considering the semigroups $\{(L_p^{(p)})^n : n = 0, 1, \ldots\}$ and $\{(R_p^{(p)})^n : n = 0, 1, \ldots\}$ generated by $L_p^{(p)}$ and $R_p^{(p)}$, we immediately note that the fixed-point spaces of these semigroups are equal to $\mathcal{F}(L_p^{(p)})$ and $\mathcal{F}(R_p^{(p)})$, respectively, and the above-mentioned ergodic theorem shows that $\mathcal{F}(L_p^{(p)})$ and $\mathcal{F}(R_p^{(p)})$ are W^*-algebras. Moreover,

$$\mathcal{F}(\mathfrak{S}) = \mathcal{F}(\mathfrak{S}_p) = \mathcal{F}(L_p^{(p)}) \cap \mathcal{F}(R_p^{(p)}).$$

Let $a, b \in \mathcal{F}(\mathfrak{S})$. Then by virtue of Lemma 3.2, we have $a \otimes \mathbb{1}$, $\mathbb{1} \otimes b \in \mathscr{N}$, and thus from the homomorphic nature of $K^{(p)}$ on $\mathscr{N}$ we get

$$\begin{aligned} K^{(p)}(a \otimes b) &= K^{(p)}((a \otimes \mathbb{1})(\mathbb{1} \otimes b)) = K^{(p)}(a \otimes \mathbb{1})K^{(p)}(\mathbb{1} \otimes b) \\ &= L^{(p)}(a)R^{(p)}(b) = ab, \end{aligned}$$

and by the same token

$$K^{(p)}(a \otimes b) = K^{(p)}((\mathbb{1} \otimes b)(a \otimes \mathbb{1})) = ba$$

showing the claim. □

Now we take a closer look at a map satisfying equation (8).

PROPOSITION 3.6 *Let $\mathscr{M}$ be an abelian W^*-algebra. The following conditions are equivalent:*

(i) *there exists a bounded normal Schwarz map $\tilde{K} : \mathscr{M} \bar{\otimes} \mathscr{M} \to \mathscr{M}$ satisfying the relation*

$$\tilde{K}(a \otimes b) = ab, \quad a, b \in \mathscr{M},$$

(ii) *$\mathscr{M}$ is atomic, i.e. it is generated by minimal projections.*

Proof. (i) $\Longrightarrow$ (ii). We shall proceed by contradiction. Assume first that $\mathscr{M}$ has no minimal projections. Fix an arbitrary normal state φ. Let n be an arbitrary positive integer. From the assumption, it follows that there exists a decomposition

$$\sum_{k=1}^{\infty} e_k^{(n)} = \mathbb{1}$$

of the identity of $\mathscr{M}$ into orthogonal projections such that

$$\varphi(e_k^{(n)}) < \frac{1}{n}, \quad k = 1, 2, \ldots.$$

Put

$$\tilde{e}_n = \sum_{k=1}^{\infty} e_k^{(n)} \otimes e_k^{(n)}.$$

We have

$$\varphi \otimes \varphi(\tilde{e}_n) = \sum_{k}^{\infty} \varphi(e_k^{(n)})^2 < \frac{1}{n} \sum_{k=1}^{\infty} \varphi(e_k^{(n)}) = \frac{1}{n}. \tag{9}$$

Now, there exists a subnet $\{n'\}$ such that

$$\tilde{e}_{n'} \xrightarrow[\sigma\text{-weakly}]{} \tilde{a} \geqslant 0,$$

and relation (9) yields

$$\varphi \otimes \varphi(\tilde{a}) = 0. \tag{10}$$

On the other hand, we have

$$\tilde{K}(\tilde{e}_n) = \sum_{k=1}^{\infty} \tilde{K}(e_k^{(n)} \otimes e_k^{(n)}) = \sum_{k=1}^{\infty} e_k^{(n)} = \mathbb{1},$$

so

$$\tilde{K}(\tilde{a}) = \mathbb{1}.$$

Denoting by $s(\tilde{a})$ the support of $\tilde{a}$, we obtain

$$\tilde{K}(s(\tilde{a})) = \mathbb{1},$$

i.e.

$$\tilde{K}(s(\tilde{a})^{\perp}) = 0.$$

Relation (10) yields

$$\varphi \otimes \varphi(s(\tilde{a})) = 0,$$

thus on account of [**13**, Corollary V.5.12] we obtain

$$s(\tilde{a})^{\perp} \geqslant s(\varphi \otimes \varphi) = s(\varphi) \otimes s(\varphi).$$

Consequently,

$$s(\varphi) = \tilde{K}(s(\varphi) \otimes s(\varphi)) \leqslant \tilde{K}(s(\tilde{a})^{\perp}) = 0,$$

which is clearly impossible.

Let now e be the supremum of all minimal projections in $\mathscr{M}$. According to our assumption, we have $e^{\perp} = \mathbb{1} - e \neq 0$. Consider the W^*-algebra $e^{\perp}\mathscr{M}e^{\perp}$. This algebra has no minimal projections; moreover, for arbitrary $a, b \in \mathscr{M}$ we have

$$\tilde{K}(e^{\perp}ae^{\perp} \otimes e^{\perp}be^{\perp}) = e^{\perp}ae^{\perp}be^{\perp},$$

showing that $\tilde{K}$ maps $e^{\perp}\mathscr{M}e^{\perp}\bar{\otimes}e^{\perp}\mathscr{M}e^{\perp}$ into $e^{\perp}\mathscr{M}e^{\perp}$. Since $\tilde{K} \mid e^{\perp}\mathscr{M}e^{\perp}\bar{\otimes}e^{\perp}\mathscr{M}e^{\perp}$ is obviously normal, the first part of the proof yields a contradiction.

(ii) $\Longrightarrow$ (i). Since $\mathscr{M}$ is an abelian W^*-algebra generated by minimal projections, it has the form

$$\mathscr{M} = \left\{ \sum_i \alpha_i e_i : \alpha_i \in \mathbb{C}, \sup_i |\alpha_i| < \infty \right\},$$

where the e_i are pairwise orthogonal minimal projections. Consequently, each $x \in \mathscr{M}$ is represented as

$$x = \sum_i \varphi_i(x) e_i,$$

where $\varphi_i(x) \in \mathbb{C}$ are coefficients dependent on x. Thus,

$$x e_i = \varphi_i(x) e_i,$$

which shows that the φ_i are normal states on $\mathscr{M}$ with supports e_i. Define a map $\tilde{K}_*$ on $\mathscr{M}_*$ by

$$\tilde{K}_* \varphi = \sum_i \varphi(e_i) \varphi_i \otimes \varphi_i, \quad \varphi \in \mathscr{M}_*.$$

The series on the right-hand side is clearly norm-convergent for $\varphi \in \mathscr{M}_*^+$, so it is norm-convergent for all $\varphi \in \mathscr{M}_*$. Moreover, for $\varphi \in \mathscr{M}_*^+$ we have

$$\|\tilde{K}_* \varphi\| \leqslant \sum_i \varphi(e_i) \leqslant \varphi(\mathbb{1}) = \|\varphi\|,$$

so $\tilde{K}_*$ is bounded and maps $\mathscr{M}_*$ into $(\mathscr{M} \bar{\otimes} \mathscr{M})_*$. Its adjoint $\tilde{K} : \mathscr{M} \bar{\otimes} \mathscr{M} \to \mathscr{M}$ is a bounded normal Schwarz map having the form

$$\tilde{K}(\tilde{x}) = \sum_i \varphi_i \otimes \varphi_i(\tilde{x}) e_i, \quad \tilde{x} \in \mathscr{M} \bar{\otimes} \mathscr{M},$$

so in particular,

$$\tilde{K}(a \otimes b) = \sum_i \varphi_i(a) \varphi_i(b) e_i, \quad a, b \in \mathscr{M}.$$

On the other hand, we have

$$a = \sum_i \varphi_i(a) e_i, \quad b = \sum_i \varphi_i(b) e_i,$$

hence

$$ab = \sum_i \varphi_i(a) \varphi_i(b) e_i,$$

showing that

$$\tilde{K}(a \otimes b) = ab,$$

which ends the proof. □

REMARK 3.7 As is easily seen, $\tilde{K}$ in the above proposition is actually a normal *-homomorphism from $\mathscr{M}\bar{\otimes}\mathscr{M}$ onto $\mathscr{M}$.

From Propositions 3.5 and 3.6, we obtain an important corollary.

PROPOSITION 3.8 *The algebra $\mathcal{F}(\mathfrak{S})$ is atomic and abelian.*

Proof. Indeed, if K_* is a channel, then $K^{(p)}$ maps $\mathcal{F}(\mathfrak{S})\bar{\otimes}\mathcal{F}(\mathfrak{S})$ into $\mathcal{F}(\mathfrak{S})$ and satisfies equation (8), which yields the claim. □

The next result is well known in the case $p = \mathbf{1}$ (cf. [**14**, Lemma1]). Its proof for arbitrary p is similar, so we omit it.

LEMMA 3.9 *Let Γ be a set of normal states on $\mathscr{A}$ broadcast by a channel K_*. For each $\rho \in \Gamma$, we have $s(\rho) \in \mathcal{F}(\mathfrak{S})$.*

We have the following characterization of an arbitrary set of broadcastable states.

THEOREM 3.10 *Let Γ be an arbitrary family of normal states on $\mathscr{A}$. The following conditions are equivalent*:

(i) *Γ is broadcastable*,
(ii) *there exists a family $\{\omega_i\}$ of normal states with pairwise orthogonal supports such that each $\rho \in \Gamma$ is a (possibly infinite) convex combination of ω_i.*

Moreover, the states ω_i are cloneable by some channel which also broadcasts the states in Γ.

Proof. (i)$\Longrightarrow$(ii). Let K_* be a channel broadcasting the states in Γ. We apply to K_* all previous considerations. Since $\mathcal{F}(\mathfrak{S})$ is an abelian W^*-algebra generated by minimal projections, it has the form

$$\mathcal{F}(\mathfrak{S}) = \left\{ \sum_{i\in I} \alpha_i e_i : \alpha_i \in \mathbb{C}, \sup_{i\in I} |\alpha_i| < \infty \right\},$$

where the e_i are pairwise orthogonal minimal projections such that

$$\sum_{i\in I} e_i = p.$$

Let $\mathbb{E}$ be defined by formula (7). For each $i \in I$, pick a state $\omega_i' \in \mathscr{A}_*$ such that $\omega_i'(e_i) \neq 0$ and put

$$\omega_i(a) = \frac{\omega_i'(e_i(\mathbb{E}a)e_i)}{\omega_i'(e_i)}, \quad a \in \mathscr{A}.$$

Then the ω_i are $\mathbb{E}$-invariant, and $s(\omega_i) \leqslant e_i$. Consequently, we have

$$\omega_i\left(\sum_j \alpha_j e_j\right) = \alpha_i.$$

Take an arbitrary $\rho \in \Gamma$. Then on account of Lemma 3.9 its support belongs to $\mathcal{F}(\mathfrak{S})$, thus it has the form

$$s(\rho) = \sum_{i\in J} e_i,$$

for some $J \subset I$. It follows that

$$\rho(a) = \sum_{j\in J} \rho(e_j a),$$

and thus

$$\begin{aligned}\rho\left(\sum_{i\in I}\alpha_i e_i\right) &= \sum_{j\in J}\rho\left(e_j\sum_{i\in I}\alpha_i e_i\right) = \sum_{j\in J}\alpha_j\rho(e_j)\\ &= \sum_{j\in J}\rho(e_j)\omega_j\left(\sum_{i\in I}\alpha_i e_i\right) = \left(\sum_{j\in J}\lambda_j\omega_j\right)\left(\sum_{i\in I}\alpha_i e_i\right),\end{aligned}$$

where $\lambda_j = \rho(e_j)$, so

$$\sum_{j\in J}\lambda_j = 1.$$

Consequently, we have

$$\rho(a) = \sum_{j\in J}\lambda_j\omega_j(a) = \sum_{i\in I}\lambda_i\omega_i(a)$$

for $a \in \mathcal{F}(\mathfrak{S})$. Since $\rho = \rho\circ\mathbb{E}$, and

$$\left(\sum_{i\in I}\lambda_i\omega_i\right)\circ\mathbb{E} = \sum_{i\in I}\lambda_i\omega_i,$$

we obtain

$$\rho = \rho\circ\mathbb{E} = \left(\sum_{i\in I}\lambda_i\omega_i\right)\circ\mathbb{E} = \sum_{i\in I}\lambda_i\omega_i,$$

showing the claim.

(ii) $\Longrightarrow$ (i). Assume that for each $\rho \in \Gamma$, we have

$$\rho = \sum_i \lambda_i\omega_i, \tag{11}$$

where $\lambda_i \geqslant 0$,

$$\sum_i \lambda_i = 1,$$

and that the states ω_i have pairwise orthogonal supports e_i. Put

$$e = \sum_i e_i.$$

Take an arbitrary normal state $\tilde{\omega}$ on $\mathscr{A}\bar{\otimes}\mathscr{A}$ and define a channel $\tilde{K}_*\colon \mathscr{A}_* \to (\mathscr{A}\bar{\otimes}\mathscr{A})_*$ similarly as in Proposition 3.6, i.e.

$$\tilde{K}_*\varphi = \sum_i \varphi(e_i)\omega_i \otimes \omega_i + \varphi(e^\perp)\tilde{\omega}, \quad \varphi \in \mathscr{A}_*.$$

Its adjoint is a normal unital completely positive map from $\mathscr{A}\bar{\otimes}\mathscr{A}$ into $\mathscr{A}$ having the form

$$\tilde{K}(\tilde{a}) = \sum_i \omega_i \otimes \omega_i(\tilde{a})e_i + \tilde{\omega}(\tilde{a})e^\perp, \quad \tilde{a} \in \mathscr{A}\bar{\otimes}\mathscr{A}.$$

For ρ given by (11), we have, taking into account that $\rho(e^\perp) = 0$,

$$\begin{aligned}\rho(\tilde{K}(a \otimes \mathbb{1})) &= \sum_j \lambda_j \left(\sum_i \omega_i(a)\omega_j(e_i)\right) = \sum_j \lambda_j \omega_j(a)\omega_j(e_j) \\ &= \sum_j \lambda_j \omega_j(a) = \rho(a),\end{aligned}$$

and in the same way we get

$$\rho(\tilde{K}(\mathbb{1} \otimes a)) = \rho(a),$$

which shows that each $\rho \in \Gamma$ is broadcastable by $\tilde{K}_*$.

It is immediately seen that on account of the orthogonality of the supports of the ω_i we have

$$\tilde{K}_*\omega_i = \omega_i \otimes \omega_i,$$

hence the ω_i are cloned by $\tilde{K}_*$. □

REMARK 3.11 It is known that states are jointly distinguishable with certainty by a measurement if and only if they have orthogonal supports, so condition (ii) in Theorem 3.10 can be reformulated as Γ being a subset of the set of all (infinite) convex combinations of jointly distinguishable states. This is the form in which it was formulated, in an operational framework in finite dimension, in [**3**, Theorem 3]. (Of course, the convex combinations there were finite.)

REMARK 3.12 An interesting observation which can be seen from the proof of Theorem 3.10 is that the stronger assumption of complete positivity of a (dual) channel gives in fact the same set of broadcastable states. Indeed, if Γ is broadcastable by a Schwarz channel, then the channel K_* as defined in the proof of the implication (ii)$\Longrightarrow$(i) in the theorem has completely positive dual and broadcasts the states from Γ.

Let K_* be a channel. Denote by $\mathcal{B}(K_*)$ the set of all normal states on $\mathscr{A}$ broadcast by K_*. Then putting $\Gamma = \mathcal{B}(K_*)$ in Theorem 3.10, we obtain the following description of $\mathcal{B}(K_*)$.

COROLLARY 3.13 *There exists a family* $\{\omega_i : i \in I\}$ *of normal states with pairwise orthogonal supports such that*

$$\mathcal{B}(K_*) = \left\{\sum_{i\in I} \lambda_i\omega_i : \lambda_i \geqslant 0, \sum_{i\in I} \lambda_i = 1\right\}.$$

Consider now the case of the full algebra, i.e. $\mathscr{A} = \mathbb{B}(\mathcal{H})$. A criterion of broadcastability in this case is given by the following theorem, which together with Theorem 3.10 are infinite-dimensional counterparts of [3, Theorem 3] (cf. also [2, Theorem 2; 7]).

THEOREM 3.14 *Let Γ be an arbitrary family of normal states on $\mathbb{B}(\mathcal{H})$. The following conditions are equivalent*:

(i) Γ *is broadcastable*,
(ii) *the density matrices D_ρ of the states $\rho \in \Gamma$ commute.*

Proof. (i) $\Longrightarrow$ (ii). According to Theorem 3.10, there is a family $\{\omega_i\}$ of normal states with pairwise orthogonal supports e_i such that each $\rho \in \Gamma$ has the form

$$\rho = \sum_i \lambda_i^\rho \omega_i,$$

so for the density matrices we obtain

$$D_\rho = \sum_i \lambda_i^\rho D_{\omega_i}.$$

For the density matrices D_{ω_i} of ω_i, we have $D_{\omega_i} \leqslant e_i$, thus $D_{\omega_i} D_{\omega_j} = 0$ for $i \neq j$, hence the D_{ω_i} commute which yields that the D_ρ commute as well.

(ii) $\Longrightarrow$ (i). Assume that density matrices D_ρ commute, and let

$$D_\rho = \sum_i \alpha_i^\rho p_i^\rho$$

be the spectral decomposition of D_ρ, where we assume that $\alpha_i^\rho > 0$. Consider the abelian W^*-algebra $\mathscr{B}$ generated by all D_ρ, i.e.

$$\mathscr{B} = W^*(\{p_i^\rho : \rho \in \Gamma,\ i = 1, 2, \ldots\}).$$

Since all the p_i^ρ are finite-dimensional and commute, it follows that $\mathscr{B}$ contains minimal projections; moreover, these projections are pairwise orthogonal, and each p_i^ρ is a finite sum of some. Denoting by $\{e_j\}$ the set of all these minimal projections we obtain a representation

$$D_\rho = \sum_j \beta_j^\rho e_j = \sum_j \lambda_j^\rho \left(\frac{1}{\operatorname{tr} e_j} e_j \right), \tag{12}$$

where

$$\lambda_j^\rho = \beta_j^\rho \operatorname{tr} e_j, \quad \lambda_j^\rho \geqslant 0, \quad \sum_j \lambda_j^\rho = 1.$$

Let ω_j be the states with density matrices $(1/\operatorname{tr} e_j)e_j$, respectively. Then $s(\omega_j) = e_j$, and from representation (12) we get

$$\rho = \sum_j \lambda_j^\rho \omega_j. \tag{13}$$

Define a map $\tilde{K}_*\colon \mathbb{B}(\mathcal{H})_* \to (\mathbb{B}(\mathcal{H})\bar{\otimes}\mathbb{B}(\mathcal{H}))_*$ as in the proof of Theorem 3.10, i.e.

$$\tilde{K}_*\varphi = \sum_i \varphi(e_i)\omega_i \otimes \omega_i + \varphi(e^\perp)\tilde{\omega}, \quad \varphi \in \mathbb{B}(\mathcal{H})_*,$$

where

$$e = \sum_i e_i,$$

and $\tilde{\omega}$ is an arbitrary fixed normal state on $\mathbb{B}(\mathcal{H})\bar{\otimes}\mathbb{B}(\mathcal{H})$. Now exactly as in the proof of that theorem we find that $\tilde{K}_*$ broadcasts the states from Γ. □

REMARK 3.15 In the case of $\mathbb{B}(\mathcal{H})$ with finite-dimensional $\mathcal{H}$, it was shown in [7] that if states are broadcastable, then their density matrices commute. Our theorem gives this result as well as the reverse implication in arbitrary dimension.

It turns out that also in the general von Neumann algebra setup there is a counterpart of Theorem 3.14. However, to formulate it we have to refer to modular theory in W^*-algebras, in particular to the notion of the Connes cocycles. For an exhaustive account of this theory, as well as the relevant definitions, the reader is advised to consult [11, 12]. Those who do not wish to plunge into it may skip the remainder of this section without losing consistency.

Let Γ be an arbitrary family of normal states on $\mathscr{A}$. Put

$$p = \bigvee_{\rho\in\Gamma} s(\rho),$$

and consider the W^*-algebra $p\mathscr{A}p$. In order to define the Connes cocycles, we must have a faithful state on this algebra, but it may happen that Γ does not have one. Take into account, the following sets $\Gamma_1 = \operatorname{conv}\Gamma$, $\Gamma_2 = \{\sum_{i=1}^\infty \lambda_i\rho_i : \lambda_i \geqslant 0,\ \sum_{i=1}^\infty \lambda_i = 1,\ \rho_i \in \Gamma\}$, $\tilde{\Gamma} = \overline{\operatorname{conv}\Gamma}$. We have $\Gamma \subset \Gamma_1 \subset \Gamma_2 \subset \tilde{\Gamma}$, and it is easily seen that Γ is broadcastable if and only if each of the sets Γ_1, Γ_2, $\tilde{\Gamma}$ is broadcastable (moreover, by the same channel). Thus, from the point of view of broadcastability it does not matter which of the above sets will be considered, and the least demanding assumption is the existence of a faithful on the algebra $p\mathscr{A}p$ state in $\tilde{\Gamma}$. But this means that $p\mathscr{A}p$ is σ-finite, which yields that such a state can be found already in Γ_2 (this is essentially [5, Lemma 2]). Consequently, the only assumption we need is that $p\mathscr{A}p$ is σ-finite.

THEOREM 3.16 *Let Γ be an arbitrary family of normal states on $\mathscr{A}$, let $\tilde{\Gamma}$ and p be as above, and assume that the algebra $p\mathscr{A}p$ is σ-finite. The following conditions are equivalent*:

(i) Γ *is broadcastable*,

(ii) *for an arbitrary faithful on the algebra $p\mathscr{A}p$ state $\omega \in \tilde{\Gamma}$, the W^*-algebra $\tilde{\mathscr{T}} = W^*(\{[D\rho : D\omega]_t, t \in \mathbb{R}, \rho \in \tilde{\Gamma}\})$, is abelian and atomic*,

(iii) *for an arbitrary faithful on the algebra $p\mathscr{A}p$ state $\omega \in \tilde{\Gamma}$, the W^*-algebra $\mathscr{T} = W^*(\{[D\rho : D\omega]_t, t \in \mathbb{R}, \rho \in \Gamma\})$, is abelian and atomic.*

Proof. (i) $\Longrightarrow$ (ii). Let K_* be a channel broadcasting the states in Γ, and thus in $\tilde{\Gamma}$. We employ the notation introduced before, thus we are dealing with the W^*-algebra $p\mathscr{A}p$, the faithful family of normal states $\tilde{\Gamma}_p = \{\rho_p : \rho \in \tilde{\Gamma}\}$ on $p\mathscr{A}p$, normal unital Schwarz maps $L_p^{(p)}, R_p^{(p)} : p\mathscr{A}p \to p\mathscr{A}p$ such that the states ρ_p are $L_p^{(p)}$- and $R_p^{(p)}$-invariant, the semigroup $\mathfrak{S}_p$ of normal Schwarz maps on $p\mathscr{A}p$ generated by $L_p^{(p)}$ and $R_p^{(p)}$ and the faithful normal conditional expectation $\mathbb{E}_p : p\mathscr{A}p \to \mathcal{F}(\mathfrak{S}_p)$ such that all states from $\tilde{\Gamma}_p$ are invariant with respect to $\mathbb{E}_p$.

According to [5, Theorem 1], algebra $\tilde{\mathscr{T}}$ is minimal sufficient for the set of states $\tilde{\Gamma}_p$. For the algebra $\mathcal{F}(\mathfrak{S}_p)$, we have the conditional expectation $\mathbb{E}_p : p\mathscr{A}p \to \mathcal{F}(\mathfrak{S}_p)$ such that all states in $\tilde{\Gamma}_p$ are $\mathbb{E}_p$-invariant, which means that $\mathcal{F}(\mathfrak{S}_p)$ is also sufficient for $\tilde{\Gamma}_p$. Consequently, $\tilde{\mathscr{T}} \subset \mathcal{F}(\mathfrak{S}_p)$, showing that $\tilde{\mathscr{T}}$ is abelian and atomic since, by virtue of Proposition 3.8, $\mathcal{F}(\mathfrak{S}_p)$ is such.

(ii) $\Longrightarrow$ (iii). Obvious.

(iii) $\Longrightarrow$ (i). Again on account of [5, Theorem 1], algebra $\mathscr{T}$ is minimal sufficient for the set of states Γ_p, so on account of [9, Remark 1] there exists a normal faithful conditional expectation $\mathbb{F}_p$: $p\mathscr{A}p \to \mathscr{T}$ such that all states in Γ_p are $\mathbb{F}_p$-invariant. Since $\mathscr{T}$ is abelian and atomic, Proposition 3.5 yields the existence of a normal Schwarz map $\tilde{K} : \mathscr{T}\bar{\otimes}\mathscr{T} \to \mathscr{T}$ such that

$$\tilde{K}(x \otimes y) = xy \quad \text{for all } x, y \in \mathscr{T}.$$

Let

$$\mathbb{F}(a) = \mathbb{F}_p(pap), \quad a \in \mathscr{A}.$$

Then $\mathbb{F}$ is a normal completely positive map. Define $K : \mathscr{A}\bar{\otimes}\mathscr{A} \to \mathscr{A}$ by the formula

$$K = \tilde{K}(\mathbb{F} \otimes \mathbb{F}).$$

Then K is a normal Schwarz map, and for each $\rho \in \Gamma$ and each $a \in \mathscr{A}$ we have

$$\begin{aligned}\rho(K(a \otimes \mathbb{1})) &= \rho(\tilde{K}(\mathbb{F}_p(pap) \otimes p)) = \rho(\mathbb{F}_p(pap)p) = \rho(\mathbb{F}_p(pap)) \\ &= \rho(pap) = \rho(a),\end{aligned}$$

and by the same token

$$\rho(K(\mathbb{1} \otimes a)) = \rho(a),$$

showing that the states in Γ are broadcast by K_*. □

Remark 3.17 In the case of the full algebra, Theorems 3.14 and 3.16 show that there is a connection (in fact, equivalence) between the commutativity of the density matrices and the commutativity of the Connes cocycles. For a more elaborate discussion of these questions, the reader is referred to [10] where such relations are investigated in connection with a general notion of commutativity of states. In particular, from [10, Theorem 5] it follows that the abelianess of the algebra $\tilde{\mathscr{T}}$ is equivalent to the abelianess of each of the algebras $W^*(\{[D\rho : D\omega]_t : t \in \mathbb{R}\})$ for $\rho \in \tilde{\Gamma}$.

4. Cloning

Let $\mathscr{A}$ be an arbitrary von Neumann algebra, and let $K_* : \mathscr{A}_* \to (\mathscr{A}\bar{\otimes}\mathscr{A})_*$ be a channel. Denote by $\mathcal{C}(K_*)$ the set of all states cloned by K_*.

THEOREM 4.1 *The states in $\mathcal{C}(K_*)$ have pairwise orthogonal supports, and are extreme points of $\mathcal{B}(K_*)$.*

Proof. Since $\mathcal{C}(K_*) \subset \mathcal{B}(K_*)$, we may use the analysis of the preceding section. In particular, we adopt the setup and notation of Theorem 3.10. For each $\rho \in \mathcal{C}(K_*)$, we have $K_*\rho = \rho \otimes \rho$, so taking into account Proposition 3.5 we obtain the equality

$$\begin{aligned}\rho(a)\rho(b) &= \rho \otimes \rho(a \otimes b) = (K_*\rho)(a \otimes b) = \rho(K(a \otimes b)) \\ &= \rho(pK(a \otimes b)p) = \rho(K^{(p)}(a \otimes b)) = \rho(ab)\end{aligned} \tag{14}$$

for all $a, b \in \mathcal{F}(\mathfrak{S})$. The equality above yields that for each projection $e \in \mathcal{F}(\mathfrak{S})$ and any $\rho \in \mathcal{C}(K_*)$ we have

$$\rho(e) = 0 \quad \text{or} \quad 1.$$

By virtue of Lemma 3.9, $s(\rho) \in \mathcal{F}(\mathfrak{S})$ for each $\rho \in \mathcal{C}(K_*)$; moreover, $s(\rho)$ is a minimal projection in $\mathcal{F}(\mathfrak{S})$. Indeed, for any projection $e \in \mathcal{F}(\mathfrak{S})$ such that $e \leqslant s(\rho)$ and $e \neq s(\rho)$ we cannot have $\rho(e) = 1$, thus $\rho(e) = 0$, and the faithfulness of ρ on the algebra $s(\rho)\mathscr{A}\, s(\rho)$ yields $e = 0$.

Clearly, the e_i are the only minimal projections in $\mathcal{F}(\mathfrak{S})$, thus for each $\rho \in \mathcal{C}(K_*)$ we have $s(\rho) = e_j$ for some e_j. Now if ρ and φ are distinct states in $\mathcal{C}(K_*)$, then their supports being minimal must be either orthogonal or equal. But if $s(\rho) = s(\varphi) = e_j$, then we would have

$$\rho\left(\sum_i \alpha_i e_i\right) = \alpha_j$$

and

$$\varphi\left(\sum_i \alpha_i e_i\right) = \alpha_j,$$

showing that

$$\rho \mid \mathcal{F}(\mathfrak{S}) = \varphi \mid \mathcal{F}(\mathfrak{S}). \tag{15}$$

Let $\mathbb{E}$ be the projection onto $\mathcal{F}(\mathfrak{S})$ defined by formula (7). We have $\rho = \rho \circ \mathbb{E}$ and $\varphi = \varphi \circ \mathbb{E}$, thus equality (15) yields $\rho = \varphi$ contrary to the assumption that ρ and φ are distinct. Consequently, ρ and φ have orthogonal supports.

Equality (14) shows that for each $\rho \in \mathcal{C}(K_*)$, $\rho \mid \mathcal{F}(\mathfrak{S})$ is a character of the abelian algebra $\mathcal{F}(\mathfrak{S})$, thus it is a pure state on this algebra, which means that it is an extreme point of the set of all states of this algebra. Now if we have

$$\rho = \lambda\varphi_1 + (1 - \lambda)\varphi_2$$

for some $0 < \lambda < 1$ and $\varphi_1, \varphi_2 \in \mathcal{B}(K_*)$, then

$$\rho \mid \mathcal{F}(\mathfrak{S}) = \lambda\varphi_1 \mid \mathcal{F}(\mathfrak{S}) + (1 - \lambda)\varphi_2 \mid \mathcal{F}(\mathfrak{S}),$$

yielding the relation

$$\rho \mid \mathcal{F}(\mathfrak{S}) = \varphi_1 \mid \mathcal{F}(\mathfrak{S}) = \varphi_2 \mid \mathcal{F}(\mathfrak{S}).$$

From Proposition 3.4, it follows that each state in $\mathcal{B}(K_*)$ is $\mathbb{E}$-invariant, thus we get for each $a \in \mathscr{A}$

$$\rho(a) = \rho(\mathbb{E}a) = \varphi_{1,2}(\mathbb{E}a) = \varphi_{1,2}(a),$$

showing that ρ is an extreme point of $\mathcal{B}(K_*)$. □

REMARK 4.2 From the description of $\mathcal{B}(K_*)$ obtained in Corollary 3.13, it follows that the states ω_i defined there are the extreme points of $\mathcal{B}(K_*)$, hence $\mathcal{C}(K_*) \subset \{\omega_i\}$. However, it is not clear whether we can have equality instead of inclusion above. The next theorem shows that the ω_i are indeed cloneable, though perhaps not necessarily by the channel K_*.

THEOREM 4.3 *Let Γ be an arbitrary subset of normal states on a von Neumann algebra $\mathscr{A}$. The following conditions are equivalent*:

(i) Γ *is cloneable*,
(ii) *the states in Γ have pairwise orthogonal supports.*

Proof. The implication (i) $\Longrightarrow$ (ii) follows from Theorem 4.1.

To prove (ii) $\Longrightarrow$ (i), assume that the states from Γ have pairwise orthogonal supports $\{e_i\}$, that is, $\Gamma = \{\rho_i\}$ and $s(\rho_i) = e_i$. Define a channel $K_* \colon \mathscr{A}_* \to (\mathscr{A}\bar{\otimes}\mathscr{A})_*$ as in the proof of Theorem 3.10, i.e.

$$K_*\varphi = \sum_i \varphi(e_i)\, \rho_i \otimes \rho_i + \varphi(e^{\perp})\tilde{\omega}, \quad \varphi \in \mathscr{A}_*,$$

where

$$e = \sum_i e_i,$$

and $\tilde{\omega}$ is a fixed normal state on $\mathscr{A}\bar{\otimes}\mathscr{A}$. As shown before, it is immediate that K_* clones the ω_i. □

REMARK 4.4 Theorems 4.1 and 4.3 together generalize the known fact that in finite dimension the extreme points of the set of *all* states broadcast by a channel are cloneable (cf. the proof of [3, Theorem 3]). If Γ is an arbitrary (convex) set of broadcastable states, then its extreme points need not be cloneable. To see this, take two broadcastable states which do not have orthogonal supports and let Γ be the segment between these states. Then all points of Γ are broadcastable, but its extreme points are not jointly cloneable.

Finally, let us say a few words about the uniqueness of the cloning operation.

PROPOSITION 4.5 *Let $\Gamma = \{\rho_i\}$ be an arbitrary faithful family of normal states on a von Neumann algebra $\mathscr{A}$ such that the states in Γ have pairwise orthogonal supports. Put $e_i = s(\rho_i)$, and define the cloning channel $\hat{K}_*$ as in Theorem* 4.3, *i.e.*

$$\hat{K}_*\varphi = \sum_i \varphi(e_i)\rho_i \otimes \rho_i, \quad \varphi \in \mathscr{A}_*.$$

Then for each channel K_* that clones Γ we have

$$\hat{K}_* = (\mathbb{E} \otimes \mathbb{E})_* K_*,$$

where $\mathbb{E}$ is the conditional expectation from $\mathscr{A}$ onto $\mathcal{F}(\mathfrak{S})$ defined by means of the dual K of K_* as in Section 3.

Proof. Let K_* be a channel cloning Γ. Theorem 4.1 asserts that the ρ_i are extreme points of $\mathcal{B}(K_*)$. From the description of $\mathcal{B}(K_*)$ obtained in Corollary 3.13, we know that the states ω_i as defined there form the set of all extreme points of $\mathcal{B}(K_*)$, thus we have $\{\rho_i\} \subset \{\omega_i\}$. Since

$$\mathbb{1} = \sum_i e_i \leqslant \sum_i s(\omega_i) \leqslant \mathbb{1},$$

we infer that $\{\rho_i\} = \{\omega_i\}$, and we may clearly assume that $\rho_i = \omega_i$. Consequently,

$$\hat{K}_*\varphi = \sum_i \varphi(e_i)\omega_i \otimes \omega_i, \quad \varphi \in \mathscr{A}_*.$$

In particular, for $a, b \in \mathscr{A}$ we have, denoting by $\hat{K}$ the dual of $\hat{K}_*$,

$$\hat{K}(a \otimes b) = \sum_i \omega_i(a)\omega_i(b)e_i. \tag{16}$$

For arbitrary $a \in \mathscr{A}$, we have

$$\mathbb{E}a = \sum_i \alpha_i e_i,$$

for some coefficients $\alpha_i \in \mathbb{C}$ depending on a, since $\mathbb{E}$ maps $\mathscr{A}$ onto $\mathcal{F}(\mathfrak{S})$. Consequently,

$$\omega_j(a) = \omega_j(\mathbb{E}a) = \sum_i \alpha_i \omega_j(e_i) = \alpha_j,$$

thus

$$\mathbb{E}a = \sum_i \omega_i(a)e_i,$$

and we obtain the formula

$$\mathbb{E}a\mathbb{E}b = \sum_i \omega_i(a)\omega_i(b)e_i, \tag{17}$$

for all $a, b \in \mathscr{A}$.

By virtue of (8), for arbitrary cloning channel K_* and any $a, b \in \mathscr{A}$ the following equality holds:

$$K(\mathbb{E} \otimes \mathbb{E}(a \otimes b)) = K(\mathbb{E}a \otimes \mathbb{E}b) = \mathbb{E}a\mathbb{E}b. \tag{18}$$

Formulas (16)–(18) yield

$$\hat{K}(a \otimes b) = K(\mathbb{E} \otimes \mathbb{E}(a \otimes b)),$$

for any $a, b \in \mathscr{A}$, which means that

$$\hat{K} = K(\mathbb{E} \otimes \mathbb{E}),$$

and thus

$$\hat{K}_* = (\mathbb{E} \otimes \mathbb{E})_* K_*,$$

finishing the proof. □

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References

1. H. Barnum, C. M. Caves, C. A. Fuchs, R. Jozsa and B. Schumacher, Noncommuting mixed states cannot be broadcast, *Phys. Rev. Lett.* **76** (1996), 2818–2821.
2. H. Barnum, J. Barrett, M. Leifer and A. Wilce, *Cloning and broadcasting in generic probability models*, preprint, 2006, arXiv: quant-ph/0611295.
3. H. Barnum, J. Barrett, M. Leifer and A. Wilce, Generalized no-broadcasting theorem, *Phys. Rev. Lett.* **99** (2007), 240501.
4. D. Dieks, Communication by EPR devices, *Phys. Lett. A* **92** (1982), 271–272.
5. A. Jenčová and D. Petz, Sufficiency in quantum statistical inference, *Comm. Math. Phys.* **263** (2006), 259–276.
6. B. Kümmerer and R. Nagel, Mean ergodic semigroups on W^*-algebras, *Acta Sci. Math. (Szeged)* **41** (1979), 151–159.
7. G. Lindblad, A general no-cloning theorem, *Lett. Math. Phys.* **47** (1999), 189–196.
8. A. Łuczak, Ergodic projection for quantum dynamical semigroups, *Int. J. Theor. Phys.* **34** (1995), 1533–1540.
9. A. Łuczak, Quantum sufficiency in the operator algebra framework, *Int. J. Theor. Phys.* **53** (2014), 3423–3433.
10. A. Łuczak, *On the commutativity of states in von Neumann algebras*, 2014, arXiv: 1409.7857.
11. Ş. Strătilă, *Modular Theory in Operator Algebras*, Editura Academiei and Abacus Press, Bucureşti, Kent, 1981.
12. Ş. Strătilă and L. Zsidó, *Lectures on von Neumann Algebras*, Editura Academiei, Bucureşti and Abacus Press, Tunbridge Wells, Kent, 1979.
13. M. Takesaki, *Theory of Operator Algebras I*, Springer, New York, 1979.
14. K. E. Thomsen, Invariant states for operator semigroups, *Studia Math.* **81** (1985), 285–291.
15. W. K. Wooters and W. H. Zurek, A single quantum state cannot be cloned, *Nature* **299** (1982), 802.