Various definitions

Anna Jenčová

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1 Symmetric monoidal categories (SMC)

Monoidal category: A category C equipped with

- A functor $\otimes : C \times C \to C$;
- unit object $I \in C$;
- associator: natural iso $(a \otimes b) \otimes c \xrightarrow{\alpha_{a,b,c}} a \otimes (b \otimes c)$;
- left unitor: natural iso $I \otimes a \xrightarrow{\lambda_a} a$;
- right unitor: natural iso $a \otimes I \xrightarrow{\rho_a} a$
- **symmetric** if there is a symmetry: natural iso $a \otimes b \xrightarrow{\sigma_{a,b}} b \otimes a$ such that $\sigma_{b,a} = \sigma_{a,b}^{-1}$,

satisfying triangle, pentagon (+ symmetry) diagrams.

We will always assume that C is a SMC.

1.1 Closed SMC

A SMC C is **closed** if:

for every $b \in C$, the endofunctor $- \otimes b$ has a right adjoint $[b, -]$ (internal hom).

What does this mean?

(1) For all $a, c \in C$, $C(a \otimes b, c) \simeq C(a, [b, c])$, naturally in a, c.

(2) unit $\eta_a^b : a \to [b, a \otimes b]$, counit: $\epsilon_a^b : [b, a] \otimes b \to a$, natural transformations, triangle identities

‘ Relation of the two:

- Let i be the iso of (1):

$$\eta_a^b \in C(a, [b, a \otimes b]) \simeq C(a \otimes b, a \otimes b), \qquad \eta_a^b = i(id_{a\otimes b})$$
$$\epsilon_a^b \in C([b, a] \otimes b, a) \simeq C([b, a], [b, a]), \qquad \epsilon_a^b = i^{-1}(id_{[b,a]}).$$

- Conversely, from η^b, ϵ^b of (2), we define i as

$$g \in C(a \otimes b, c), \qquad i(g) = [b, g] \circ \eta^b_a$$

with inverse

$$h \in C(a, [b, c]), \quad i^{-1}(h) = \epsilon^b_c \circ (h \otimes b)$$

Informally, we may interpret η^b_a as 'labeling of b by a' and ϵ^b_a as 'evaluation of $[b, a]$'.

1.2 Compact SMC

A SMC is **compact** if each object $a \in C$ has a dual $a^* \in C$ such that there are maps $\cup_a : I \to a^* \otimes a$ and $\cap_a : a \otimes a^* \to I$ satisfying the snake identities

$$(\cap_a \otimes id_a) \circ (id_a \otimes \cup_a) = id_a, \qquad (id_{a^*} \otimes \cap_a) \circ (\cup_a \otimes id_{a^*}) = id_{a^*}.$$

The following are easily seen by the snake identities (pictures):

(1) a^* is determined up to iso;

(2) $a^{**} \simeq a$, indeed, we may define $\cup_{a^*} : I \to a \otimes a^*$ and $\cap_{a^*} : a^* \otimes a$ as

$$\cup_{a^*} = \sigma_{a^*,a} \circ \cup_a, \qquad \cap_{a^*} = \cap_a \circ \sigma_{a^*,a},$$

so that a is dual to a^*, and use (1);

(3) if we fix a^* and $\cup_a$ ($\cap_a$), then $\cap_a$ ($\cup_a$) is uniquely determined;

(4) any assignment $a \mapsto a^*$ defines a functor $C \to C^{op}$ (if $f : a \to b$, we can use $\cup_a$ and $\cap_b$ to "bend the wires" to obtain a map $b^* \to a^*$, this is obviously functorial);

(5) $(a \otimes b)^* \simeq a^* \otimes b^*$, we can clearly put (using symmetry)

$$\cup_{a \otimes b} = \cup_a \otimes \cup_b, \qquad \cap_{a \otimes b} = \cap_a \otimes \cap_b$$

(5) C is closed, with $[b, c] = b^* \otimes c$: the iso $i : C(a \otimes b, c) \simeq C(a, b^* \otimes c)$ can be obtain (roughly) as

$$i(g) = g \circ \cup_b, \qquad i^{-1}(h) = \cap_b \circ h,$$

the maps are inverses by the snake identities, naturality is obvious, since i does nothing on a or c. The unit and counit of the adjunction are given as

$$\eta^b_a = a \otimes \cup_b : a \to b^* \otimes a \otimes b, \qquad \epsilon^b_a = \cap_b \otimes a : b^* \otimes a \otimes b \to a.$$

(6) Can we state a theorem like: C is compact if and only if for each $b \in C$ there is some $b^* \in C$ such that $b^* \otimes -$ is the right adjoint of $- \otimes b$ and ...? What should be the additional conditions?

2 Kleisli categories and monoidal monads

A **monad** on C is a triple (P, η, μ), where:

- $P : C \to C$ is an endofunctor;
- $\eta : Id_C \to P$, $\mu : P^2 \to P$ are natural transformations satisfying some triangles and squares.

2.1 Kleisli categories

The **Kleisli category** C_P has the same objects as C, with morphisms:

$$C_p(a, b) = C(a, P(b)),$$

the identity $id_a = \eta_a$ and for $f \in C_p(a, b)$, $g \in C_p(b, c)$, the composition is defined as

$$g \circ f := \mu_c \circ P(g) \circ f.$$

We have the following adjunction:

- the **left adjoint functor** $F_P : C \to C_P$ is defined as $a \mapsto a$ and for $f : a \to b$, we put $F_P(f) \in C_P(a, b) = C(a, P(b))$ as $\eta_b \circ f$;
- the **right adjoint functor** $G_P : C_P \to C$ is given as $a \mapsto P(a)$ and for $f \in C_P(a, b) = C(a, P(b))$ we put $G_P(f) \in C(P(a), P(b))$ as $G_P(f) = \mu_b \circ P(f)$.

This is indeed an adjunction, where the unit is given by η and the counit is determined as $\epsilon_a = id_{P(a)} \in C_P(P(a), a)$.

2.2 Monoidal monads

A monad is **monoidal** [**?**] if there are maps

$$\kappa_{a,b} : Pa \otimes Pb \to P(a \otimes b), \qquad a, b \in C,$$

natural in a, b and such that

- (P, η, κ) is a **monoidal functor**, that is, some diagrams involving P, α, λ, ρ, κ and η commute;
- additional diagrams containing μ commutes;
- **symmetric**: additionally a diagram with σ commutes.

A monoidal functor is **strict** if κ is iso.

Proposition 1. *[**?** , Prop. 1.2.2] There is a bijective correspondence between:*

(i) families of morphisms $\{\kappa_{a,b}\}$ such that (P, η, μ, κ) is a (symmetric) monoidal monad;

(ii) (symmetric) monoidal structures on C_P such that the left adjoint functor $F_P : C \to C_P$ is strict monoidal.

If (P, η, μ, κ) is a symmetric monoidal monad, we define the monoidal structure on C_P as follows. The functor

$$\otimes_P : C_P \times C_P \to C_P$$

is given as as $a \otimes_P b = a \otimes b$ on objects, and for $f \in C_P(a, c) = C(a, P(c))$ and $g \in C_P(b, d) = C(b, p(d))$, we define $f \otimes_P g \in C_P(a \otimes_P b, C \otimes_P d) = C(a \otimes b, P(c \otimes d))$ as

$$f \otimes_P g := \kappa_{b,d} \circ (f \otimes g).$$

The associator and unitors and symmetry in C_P can be defined from those in C by composition with η.