# Note on monotonicity of $\alpha \mapsto D_{\alpha,z}$ for $\alpha \in (0,1)$

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We will show that we can use complex interpolation and Kosaki $L_p$-spaces also in the case $\alpha \in (0,1)$ if $z > 1/2$. Put $p = 2z$ and $q = \frac{2z-1}{2z}$ the dual parameter. Let $e := s(\varphi)$ and $f := s(\psi)$, and let $\sigma, \tau \in \mathcal{M}_*^+$ be such that $s(\sigma) = 1 - e$, $s(\tau) = 1 - f$. Put $\psi_0 := \psi + \tau$, $\varphi_0 = \varphi + \sigma$, then $\psi_0, \varphi_0$ are faithful positive normal functionals on $\mathcal{M}$ and we have $h_\varphi^\theta = eh_{\varphi_0}^\theta = h_{\varphi_0}^\theta e$ and $h_\psi^\theta = fh_{\psi_0}^\theta = h_{\psi_0}^\theta f$ for any $\theta > 0$. We will use the notations $L_L^p := L^p(\mathcal{M}; \varphi_0)_L$, $L_R^p := L^p(\mathcal{M}; \psi_0)_R$ and $L_\eta^p := C_\eta(L_L^p, L_R^p)$.

The proof is easier in the case $z \geq 1$, which we prove first.

**Proposition 1.** *Assume that $z \geq 1$. Then*

1. *$\alpha \mapsto \log Q_{\alpha,z}(\psi\|\varphi)$ is convex on $(0,1)$*
2. *$\alpha \mapsto D_{\alpha,z}(\psi\|\varphi)$ is monotone increasing on $(0,1)$.*

*Proof.* Put $\xi := h_\psi^{1/2} h_\varphi^{1/2} \in L_1(\mathcal{M})$. Let $\alpha \in (0,1)$ and put $\eta := \frac{z-\alpha}{2z-1}$, so that we have

$$0 \leq 1 - \frac{q}{2} = \frac{z-1}{2z-1} < \eta < \frac{z}{2z-1} = \frac{q}{2} \leq 1.$$

Then

$$\xi = h_\psi^{\frac{\eta}{q}} (h_\psi^{\frac{\alpha}{2z}} h_\varphi^{\frac{1-\alpha}{2z}}) h_\varphi^{\frac{1-\eta}{q}} = h_{\psi_0}^{\frac{\eta}{q}} (h_\psi^{\frac{\alpha}{2z}} h_\varphi^{\frac{1-\alpha}{2z}}) h_{\varphi_0}^{\frac{1-\eta}{q}} \in L_\eta^p$$

and $Q_{\alpha,z}(\psi\|\varphi) = \|\xi\|_{p,\psi_0,\varphi_0,\eta}^p$. The proof can be finished by reiteration theorem, similarly as in [2, Prop. 0.1].

□

Now we turn to the case $1/2 < z < 1$. Note that a similar strategy as in the above proof works only for restricted values of $\alpha$. We will need a bit more of the complex interpolation method. Let us denote $\Sigma := \Sigma(L_L^p, L_R^p) = L_L^p + L_R^p$ and let $\mathcal{F} := \mathcal{F}(L_L^p, L_R^p)$ be the set of functions $S := \{w \in \mathbb{C},\ \mathrm{Re}\,(w) \in [0,1]\} \to \Sigma$ that are

(i) bounded, continuous and analytic in the interior of $S$ (with respect to the norm in $\Sigma$),

(ii) $f(it) \in L_L^p$, $f(1+it) \in L_R^p$, $t \in \mathbb{R}$,

(iii) the maps $t \mapsto f(it) \in L_L^p$ and $t \mapsto f(1+it) \in L_R^p$ are continuous and

$$\max\{\sup_t \|f(it)\|_{p,\varphi_0,L}, \sup_t \|f(1+it)\|_{p,\psi_0,R}\} < \infty.$$

We will use the following functions, defined on the strip $S$:

$$f(w) = h_\psi^{\frac{w}{q}+\frac{1-w}{p}} h_\varphi^{\frac{1-w}{q}+\frac{w}{p}}, \qquad w \in S. \tag{1}$$

Note that $f(w)$ is an element in $L_1(\mathcal{M})$. The next lemma shows that $f$ has values in $\Sigma$.

**Lemma 1.** *We have $f \in \mathcal{F}$ and for each $\eta \in (0,1)$, we have*

$$\|f(\eta+it)\|^p_{p,\varphi_0,\psi_0,\eta} = Q_{1-\eta,z}(\psi\|\varphi).$$

*Proof.* For $\eta \in [0,1]$ we have

$$f(\eta+it) = h_\psi^{\frac{\eta}{q}} h_\psi^{i(\frac{1}{q}-\frac{1}{p})t} h_\psi^{\frac{1-\eta}{p}} h_\varphi^{\frac{\eta}{p}} h_\varphi^{i(\frac{1}{p}-\frac{1}{q})t} h_\varphi^{\frac{1-\eta}{q}} = h_{\psi_0}^{\frac{\eta}{q}} \big(h_{\psi_0}^{i(\frac{1}{q}-\frac{1}{p})t} h_\psi^{\frac{1-\eta}{p}} h_\varphi^{\frac{\eta}{p}} h_{\varphi_0}^{i(\frac{1}{p}-\frac{1}{q})t}\big) h_{\varphi_0}^{\frac{1-\eta}{q}}$$

By [3, Lemmas 10.1 and 10.2], $h_{\psi_0}^{it}\cdot h_{\varphi_0}^{-it}$ defines a strongly continuous group of isometries on $L_p(\mathcal{M})$ for every $1 \le p \le \infty$. This implies the property (iii) in the definition of $\mathcal{F}$. Also for $\eta \in (0,1)$, we see that $f(\eta+it) \in L^p_\eta$ and

$$\|f(\eta+it)\|^p_{p,\varphi_0,\psi_0,\eta} = \|h_{\psi_0}^{i(\frac{1}{q}-\frac{1}{p})t} h_\psi^{\frac{1-\eta}{p}} h_\varphi^{\frac{\eta}{p}} h_{\varphi_0}^{i(\frac{1}{p}-\frac{1}{q})t}\|_p^p = \|h_\psi^{\frac{1-\eta}{p}} h_\varphi^{\frac{\eta}{p}}\|_p^p = Q_{1-\eta,z}(\psi\|\varphi).$$

Since $L^p_\eta$ for each $\eta$ is continuously embedded in $\Sigma$, this implies that $f$ is $\Sigma$-valued. Since by Hölder $\|h_\psi^{\frac{1-\eta}{p}} h_\varphi^{\frac{\eta}{p}}\|_p \le \psi(1)\varphi(1)$ for any $\eta$, $f$ is also bounded. Note that as a function with values in $L_1(\mathcal{M})$, $f$ is bounded, continuous on $S$ and analytic in the interior. We now prove that the continuity and analyticity properties also hold in $\Sigma$ (maybe this is already obvious, but I will give an argument similar to that in [1, Sec. 9.1,29.1] just for the case). Let $\mu_0(w,t)$ and $\mu_1(w,t)$ be the Poisson kernels associated with $S$. We then have

$$f(w) = \int_{\mathbb{R}} f(it)\mu_0(w,t)dt + \int f(1+it)\mu_1(w,t)dt.$$

The integrals are in $L_1(\mathcal{M})$, but since $t \mapsto f(it) \in L^p_L$ and $t \mapsto f(1+it) \in L^p_R$ are continuous and bounded in the respective norms, we see that the integrals also exist in $\Sigma$ and since $\Sigma$ is continuously embedded in $L_1(\mathcal{M})$, the above equality holds. This shows that $f : S \to \Sigma$ is continuous. Therefore, the expressions

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi - w} d\xi$$

for a suitable circle $\Gamma$ around a point $w$ in the interior of $S$ are defined in $\Sigma$. Since $f$ is analytic in $L_1(\mathcal{M})$, this expression is equal to $f(w)$, hence $f$ is analytic in the interior of $S$.

□

**Proposition 2.** *Assume that $1/2 < z < 1$. Then*

1. *$\alpha \mapsto \log Q_{\alpha,z}(\psi\|\varphi)$ is convex on $(0,1)$*
2. *$\alpha \mapsto D_{\alpha,z}(\psi\|\varphi)$ is monotone increasing on $(0,1)$.*

*Proof.* Let $\alpha_1, \alpha_2 \in (0,1)$ and let $\alpha := (1-\theta)\alpha_1 + \theta\alpha_2$. Put $\eta_i = 1 - \alpha_i$, $i = 1,2$ so that $\eta := 1-\alpha = (1-\theta)\eta_1 + \theta\eta_2$. By the reiteration theorem, $L^p_\eta = C_\theta(L^p_{\eta_1}, L^p_{\eta_2})$. Let $f$ be the function given by (1). Then $f_1 : w \mapsto f((1-w)\eta_1 + w\eta_2) \in \mathcal{F}(L^p_{\eta 1}, L^p_{\eta_2})$ and by usual arguments, we have

$$\|f(\eta)\|_{p,\varphi_0,\psi_0,\eta} = \|f_1(\theta)\|_{C_\theta(L^p_{\eta_1},L^p_{\eta_2})} \leq (\sup_t \|f_1(it)\|_{L^p_{\eta_1}})^{1-\theta}(\sup_t \|f_1(1+it)\|_{L^p_{\eta_1}})^{\theta}.$$

Since $f_1(it) = f(\eta_1 + i(\eta_2 - \eta_1)t)$ and $f_1(1+it) = f(\eta_2 + i(\eta_2 - \eta_1)t)$, we get from Lemma 1 that

$$Q_{1-\eta,z}(\psi\|\varphi) \leq Q_{1-\eta_1,z}(\psi\|\varphi)^{1-\theta} Q_{1-\eta_2,z}(\psi,\varphi)^{\theta}.$$

The proof is finished as in [2].

□

# References


[1] A. Calderón. Intermediate spaces and interpolation, the complex method. *Studia Mathematica*, 24(2):113–190, 1964.

[2] F. Hiai, Monotonicity of $\alpha \mapsto D_{\alpha,z}$, (12/31/2023) notes.

[3] H. Kosaki. Applications of the complex interpolation method to a von Neumann algebra: Noncommutative $L_p$-spaces. *J. Funct. Anal.*, 56:26–78, 1984. doi:https://doi.org/10.1016/0022-1236(84)90025-9.