EMBEDDING IN ALGEBRAS OF TYPE I

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Introduction. The theory of AW^*-algebras was instituted by Kaplansky [6] as an attempt to isolate the purely algebraic properties of rings of operators. An AW^*-algebra is a C^*-algebra satisfying one additional axiom which automatically holds in a ring of operators. This axiom guarantees the existence of sufficiently many projections in the algebra. It was shown that many of the important algebraic properties of rings of operators, for instance generalized comparability of projections and the existence of a dimension function for a finite algebra, also hold for AW^*-algebras.

The object of this paper is to show that in a large number of cases (and perhaps always) the technique of the theory of rings of operators can be applied to the study of AW^*-algebras, indeed that these algebras can be considered rings of operators in a certain general sense. To carry this out we would like to find a Hilbert space, or something like a Hilbert space, on which our AW^*-algebra can act like a ring of operators. It does not seem that a Hilbert space is the right object to look for in general. In fact, it is shown in [1, Chapter XI, Corollary 1, Theorem 12] that there exists a commutative AW^*-algebra not isomorphic to a ring of operators on any Hilbert space. A substitute for a Hilbert space, however, does exist. The substitute, an AW^*-module, arose naturally in Kaplansky's study of AW^*-algebras of type I [7]. An AW^*-module is like a Hilbert space except that the field of complex numbers is replaced by an arbitrary commutative AW^*-algebra. It was shown that any AW^*-algebra $\mathfrak{B}$ of type I is isomorphic to the algebra of all bounded operators on some AW^*-module over the center $\mathfrak{z}$ of $\mathfrak{B}$. This result justifies an attempt to determine when an AW^*-albegra has a suitable representation as an algebra of bounded operators on an AW^*-module.

This paper is divided into two parts. In the first we develop the theory of AW^*-modules, the most important point here being that a topology can be found on a commutative AW^*-algebra to replace the ordinary topology on the complex numbers. The second part of the paper is devoted to the study of representations of an AW^*-algebra as an algebra of bounded operators on an AW^*-module.

For the basic theory of AW^*-algebras and AW^*-modules we refer the reader to the series of papers [6]–[8] of Kaplansky.

I. AW^*-Modules

1. **Topologies.** Let $\mathfrak{z}$ be a commutative AW^*-algebra, $\mathfrak{M}$ the maximal ideal space of $\mathfrak{z}$. In its induced weak topology $\mathfrak{M}$ is an extremally disconnected compact Hausdorff space, and $\mathfrak{z}$ is isomorphic to $C(\mathfrak{M})$, the algebra of continuous

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complex-valued functions on $\mathfrak{M}$. If we denote by $\mathfrak{Z}_r$ the set of self-adjoint elements of $\mathfrak{Z}$, then $\mathfrak{Z}_r$ is isomorphic to $C_r(\mathfrak{M})$, the set of continuous real-valued functions on $\mathfrak{M}$. In its natural partial ordering $C_r(\mathfrak{M})$is a conditionally complete lattice, i.e. each bounded subset of $C_r(\mathfrak{M})$ has a least upper bound and greatest lower bound. We can extend $C_r(\mathfrak{M})$ to a complete lattice as follows. Give the extended real line $[-\infty, +\infty]$ the topology of a closed interval, and let $C_r^*(\mathfrak{M})$ denote the set of continuous functions from $\mathfrak{M}$ to $[-\infty, +\infty]$. Then $C_r^*(\mathfrak{M})$ is a complete lattice (being lattice isomorphic to the unit sphere of $C_r(\mathfrak{M})$), and contains $C_r(\mathfrak{M})$ as a sublattice. The mapping from $\mathfrak{Z}$ to $\mathfrak{Z}_r \times \mathfrak{Z}_r$ defined by

$$a \rightarrow (\tfrac{1}{2}(a + a^*), -\tfrac{1}{2}i(a - a^*))$$

gives rise to an embedding of $\mathfrak{Z}$ into $C_r^*(\mathfrak{M}) \times C_r^*(\mathfrak{M})$.

In $C_r^*(\mathfrak{M})$, as in any complete lattice, the concept of order convergence may be introduced [1, Chapter IV, §8]. A net $\{a_\lambda\}$ of $C_r^*(\mathfrak{M})$ *order-converges* to a (written $a_\lambda \rightarrow a(O)$) if

$$a = \limsup a_\lambda = \liminf a_\lambda .$$

Order convergence in $C_r^*(\mathfrak{M})$ induces in a natural way a concept of convergence in $C_r^*(\mathfrak{M}) \times C_r^*(\mathfrak{M})$, and so also in $\mathfrak{Z}$. We shall also refer to this type of convergence in $\mathfrak{Z}$ as order convergence.

The following lemma gives a more algebraic, and less lattice theoretic, criterion for order convergence in $\mathfrak{Z}$.

LEMMA 1.1. *A net $\{a_\lambda\}$ of $\mathfrak{Z}$ order-converges to a if and only if given a non-zero projection $e \in \mathfrak{Z}$ and an $\epsilon > 0$ there exist a non-zero projection $f \leq e$ and a λ_0 such that $\lambda \geq \lambda_0$ implies $\| f(a_\lambda - a) \| < \epsilon$.*

Proof. We may clearly assume throughout that the a_λ are self-adjoint. Assume $a_\lambda \rightarrow a(O)$ and let a non-zero projection $e \in \mathfrak{Z}$ and $\epsilon > 0$ be given. Keeping in mind the identification of $\mathfrak{Z}_r$ with $C_r(\mathfrak{M})$, let $\mathcal{S}$ be the subset of $\mathfrak{M}$ of which e is the characteristic function. Define

$$b_\lambda = \inf [a_\mu : \mu \geq \lambda].$$

Then since $\liminf a_\lambda = a$, $\{b_\lambda\}$ is an ascending net with supremum a. Now it cannot be the case that $b_\lambda \leq a - \epsilon$ on $\mathcal{S}$ for all λ, because this would imply $\sup b_\lambda \leq a - \epsilon e$. Therefore there is some point $t_0 \in \mathcal{S}$ and some λ_1 such that $b_{\lambda_1}(t_0) > a(t_0) - \epsilon$. From the continuity of b_{λ_1} and a, we can find a non-void closed and open set $\mathcal{S}' \subset \mathcal{S}$ such that $b_{\lambda_1}(t) > a(t) - \epsilon$ for all $t \in \mathcal{S}'$. Thus $\lambda \geq \lambda_1$ implies $a_\lambda(t) > a(t) - \epsilon$ for all $t \in \mathcal{S}'$. Similarly we can find a non-void closed and open set $\mathcal{S}'' \subset \mathcal{S}'$ and a λ_2 such that $\lambda \geq \lambda_2$ implies $a_\lambda < a(t) + \epsilon$ for all $t \in \mathcal{S}''$. Therefore if f is the characteristic function of $\mathcal{S}''$ and we choose $\lambda_0 \geq \lambda_1, \lambda_2$, then $\lambda \geq \lambda_0$ implies $\| f(a_\lambda - a) \| < \epsilon$.

Conversely, assume a_λ, a satisfy the stated condition and let $b = \liminf a_\lambda$.

We shall show $b \geq a$. Assuming otherwise, there exist a non-zero projection $e \in \mathfrak{Z}$ and an $\epsilon > 0$ such that $e(a - b) \geq \epsilon e$. By assumption we can find a non-zero projection $f \leq e$ and a λ_0 such that $\| f(a_\lambda - a) \| \leq \frac{1}{2}\epsilon$ for all $\lambda \geq \lambda_0$. Thus $\lambda \geq \lambda_0$ implies $f(a_\lambda - b) \geq \frac{1}{2}\epsilon f$, or $fa_\lambda \geq (b + \frac{1}{2}\epsilon)f$. Therefore $fb \geq (b + \frac{1}{2}\epsilon)f$, which is a contradiction. Thus we do have $b \geq a$, i.e. $\liminf a_\lambda \geq a$. Similarly $\limsup a_\lambda \leq a$, so $a_\lambda \to a(O)$.

Order convergence is introduced in a commutative AW^*-algebra as a replacement for the strong topology on a ring of operators. An important property of the strong topology on a ring of operators is that the unit sphere is complete. We shall show that this carries over to order convergence.

LEMMA 1.2. *The unit sphere of $\mathfrak{Z}$ is complete relative to order convergence.*

Proof. Let $\{a_\lambda\}$ be a Cauchy net relative to order convergence and contained in the unit sphere of $\mathfrak{Z}$. We may assume the a_λ are self-adjoint. Then $\limsup_{\lambda,\mu}(a_\lambda - a_\mu) = 0$. Therefore given a non-zero projection $e \in \mathfrak{Z}$ and $\epsilon > 0$ we can find a non-zero projection $f \leq e$ and a λ_0 such that $f \sup[a_\lambda - a_\mu : \lambda, \mu \geq \lambda_0] \leq \epsilon f$. Thus $\lambda, \mu \geq \lambda_0$ implies $fa_\lambda \leq f(a_\mu + \epsilon)$, which shows

$$f \limsup a_\lambda \leq f(\liminf a_\lambda + \epsilon). \tag{1}$$

Therefore every non-zero projection $e \in \mathfrak{Z}$ contains a non-zero projection f for which (1) holds. This shows $\limsup a_\lambda \leq \liminf a_\lambda + \epsilon$. Since $\epsilon > 0$ was arbitrary we have $\limsup a_\lambda \leq \liminf a_\lambda$, so $\limsup a_\lambda = \liminf a_\lambda$ and a_λ is order-convergent.

Let $\mathcal{H}$ be an AW^*-module over $\mathfrak{Z}$. Using order convergence in $\mathfrak{Z}$ we can introduce two types of convergence in $\mathcal{H}$, analogous to strong and weak convergence in Hilbert space. A net $\{x_\lambda\}$ in $\mathcal{H}$ converges *strongly* to x if $| x_\lambda - x | \to 0$ (O) in $\mathfrak{Z}$ (where, we recall, $| x | = (x, x)^{\frac{1}{2}}$), and x_λ converges *weakly* to x if $(x_\lambda - x, y) \to 0$ (O) in $\mathfrak{Z}$ for all $y \in \mathcal{H}$. If $\mathcal{B}$ is the algebra of bounded operators on $\mathcal{H}$ we can introduce analogs of the usual strong and weak topologies on the algebra of bounded operators on Hilbert space. We say that $A_\lambda \to A$ *strongly* if $A_\lambda x \to Ax$ strongly in $\mathcal{H}$ for all $x \in \mathcal{H}$, and $A_\lambda \to A$ *weakly* if $(A_\lambda x, y) \to (Ax, y)$ (O) in $\mathfrak{Z}$ for all $x, y \in \mathcal{H}$.

The following result is a simple consequence of [8, Theorem 5] and our lemma 1.2.

LEMMA 1.3. *The unit sphere of $\mathcal{H}$ is complete relative to strong or weak convergence. The unit sphere of $\mathcal{B}$ is complete relative to strong or weak convergence.*

If $\{A_\lambda\}$ is a bounded ascending net of bounded self-adjoint operators on Hilbert space, we know that $\{A_\lambda\}$ has a supremum and even converges strongly to this supremum. In particular, if E_α are orthogonal projections with supremum E we have $E = \Sigma E_\alpha$, in the sense that E is the strong limit of the finite sums. The same result holds for AW^*-modules as well, and we set it down as a lemma without proof.

LEMMA 1.4. *Let $\{A_\lambda\}$ be a bounded ascending net of self-adjoint elements of $\mathcal{B}$. Then $\{A_\lambda\}$ has a supremum A among the self-adjoint elements of $\mathcal{B}$, and $A_\lambda \to A$ strongly. If E_α are orthogonal projections in $\mathcal{B}$ with supremum E then $E = \Sigma E_\alpha$, in the sense that E is the strong limit of the finite sums.*

The notions of strong and weak convergence in $\mathcal{H}$ and $\mathcal{B}$ were obtained from order convergence in $\mathfrak{z}$ by means of mappings from $\mathcal{H}$ and $\mathcal{B}$ respectively into $\mathfrak{z}$. For instance if we define, for $x \in \mathcal{H}$, the map $S_x : \mathcal{B} \to \mathfrak{z}$ by $S_x(A) = | Ax |$, then we have $A_\lambda \to A$ strongly if and only if $S_x(A_\lambda - A) \to 0$ (O) in $\mathfrak{z}$ for all the S_x. The mappings that one used for these purposes are all of a similar type, analogous to ordinary (numerical) semi-norms. We make a formal definition.

Let $\mathfrak{M}$ be a (complex) vector space, $\mathfrak{z}$ a commutative AW^*-algebra. A *$\mathfrak{z}$-valued semi-norm* on $\mathfrak{M}$ is a map $S: \mathfrak{M} \to \mathfrak{z}$ satisfying
(1) $S(x) \geq 0$ for all $x \in \mathfrak{M}$,
(2) $S(x + y) \leq S(x) + S(y)$ for all $x, y \in \mathfrak{M}$,
(3) $S(ax) = | a | S(x)$ for all $x \in \mathfrak{M}$ and any complex number a.

We next prove a result concerning semi-norms extending the result of [5]. The idea of the proof can be found there.

LEMMA 1.5. *Let $\mathfrak{A}$ be a C^*-algebra with unit, $\mathfrak{z}$ a commutative AW^*-algebra, and H a continuous real-valued function defined on the real line such that $H(+\infty)$ and $H(-\infty)$ both exist and are equal. Assume $\mathcal{S}$ is a set of $\mathfrak{z}$-valued semi-norms on $\mathfrak{A}$ with the following property: for any $S \in \mathcal{S}$ and $C \in \mathfrak{A}$, there exist a $k = k(S, C)$ and an $S' = S'(S, C) \in \mathcal{S}$ such that $S(ABC) \leq k \| A \| S'(B)$ for all $A, B \in \mathfrak{A}$.*

Let $S \in \mathcal{S}$, a self-adjoint element $A \in \mathfrak{A}$, and $\epsilon > 0$ be given. Then we can find $S_1', \cdots, S_m' \in \mathcal{S}$ and $M = M(S, A, \epsilon)$ such that for any self-adjoint element $B \in \mathfrak{A}$ we have

$$S[H(A) - H(B)] \leq M \sup[S_i'(A - B): i = 1, \cdots, m] + \epsilon.$$

Proof. We first prove the analogous result for unitary elements of $\mathfrak{A}$. Let $P(z) = \sum_{n=-N}^{N} a_n z^n$ be a polynomial in z and $\bar{z}$ on the unit circle. I claim that given $S \in \mathcal{S}$ and a unitary $U \in \mathfrak{A}$ we can find $S_1, \cdots, S_m \in \mathcal{S}$ and an M' such that for every unitary $V \in \mathfrak{A}$,

$$S[P(U) - P(V)] \leq M' \sup[S_i(U - V): i = 1, \cdots, m]. \tag{2}$$

A simple induction shows that for $n > 1$,

$$U^n - V^n = \sum_{r=1}^{n} V^{n-r}(U - V)U^{r-1},$$

$$U^{-n} - V^{-n} = - \sum_{r=1}^{n} V^{r-n-1}(U - V)U^{-r},$$

so

$$P(U) - P(V) = \sum_{n=-N}^{N} a_n(U^n - V^n)$$

$$= \sum_{n=1}^{N} a_n \sum_{r=1}^{n} V^{n-r}(U - V)U^{r-1} - \sum_{n=1}^{N} a_{-n} \sum_{r=1}^{n} V^{r-n-1}(U - V)U^{-r}.$$

Therefore if we set $M' = \sum_{n=-N}^{N} | a_n | \sum_{r=-N}^{N} k(S, U^r)$ and let $S_1, \cdots, S_m$ be the various $S'(S, U^r)$ $(-N \leq r \leq N)$ we have (2).

Now let G be any continuous complex-value function on the unit circle. For any $A \varepsilon \mathfrak{A}$, $S(A) \leq k(S, 1) \| A \| S'(S, 1)(1)$. Let $R = 2k(S, 1) \| S'(S, 1)(1) \|$, and find a polynomial P in z and $\bar{z}$ on the unit circle such that $| P(z) - G(z) | < \epsilon R^{-1}$. Then using (2) we derive

$$(3) \qquad S[G(U) - G(V)] \leq M' \sup S_i(U - V) + \epsilon.$$

Let S, A, ϵ be given as in the statement of the lemma. Let $F(t) = (t - i)(t + i)^{-1}$ for real t, and define

$$G(z) = H[-i(z + 1)(z - 1)^{-1}] \quad \text{for} \quad | z | = 1, \qquad z \neq 1,$$
$$G(1) = H(\pm \infty).$$

Then G is continuous on the unit circle, and for any self-adjoint $B \varepsilon \mathfrak{A}$, $F(B)$ is unitary and $H(B) = G[F(B)]$. Letting $U = F(A)$ and $V = F(B)$, find M' and S_i so that (3) holds. Then

$$(4) \qquad S[H(A) - H(B)] \leq M' \sup S_i[F(A) - F(B)] + \epsilon.$$

Now $F(A) - F(B) = 2i(B + i)^{-1}(A - B)(A + i)^{-1}$, so

$$(5) \qquad S_i[F(A) - F(B)] \leq 2k(S_i, (A + i)^{-1})S'(S_i, (A + i)^{-1})(A - B).$$

Therefore if we set $M = 2M' \max k(S_i(A + i)^{-1})$ and $S_i' = S'(S_i, (A + i)^{-1})$, (4) and (5) give the desired result.

A simple consequence of Lemma 1.5 is

LEMMA 1.6. *Let $\mathfrak{B}$ be the algebra of bounded operators on the AW^*-module $\mathfrak{H}$ over $\mathfrak{Z}$. Let H be as in Lemma 1.5. Then the map $A \to H(A)$ is strongly continuous on the self-adjoint elements of $\mathfrak{B}$.*

2. ***AW^**-completion.** In this section we show that just as a pre-Hilbert space can be completed to a Hilbert space, so can an "inner product space" over a commutative AW^*-algebra be completed to an AW^*-module. A similar construction can be found in [9].

Let $\mathfrak{Z}$ be a commutative AW^*-algebra. We call $\mathfrak{M}$ an *inner product space over* $\mathfrak{Z}$ if $\mathfrak{M}$ is a module over $\mathfrak{Z}$ (in the ordinary algebraic sense) possessing an inner product $(,): \mathfrak{M} \times \mathfrak{M} \to \mathfrak{Z}$ satisfying

(1) $(x, y) = (y, x)^*$,

(2) $(x, x) \geq 0$ and is 0 only for $x = 0$,

(3) $(ax + by, z) = a(x, z) + b(y, z)$.

The following lemma shows that if $\mathfrak{M}$ is already somewhat complete, then completing it in norm makes it an AW^*-module.

LEMMA 2.1. *Let $\hat{\mathfrak{M}}$ be an inner product space over the commutative AW^*-algebra $\mathfrak{Z}$, and assume $\hat{\mathfrak{M}}$ has the following property: if $\{x_\alpha\}$ is a founded subset of $\hat{\mathfrak{M}}$ and e_α are orthogonal projections in $\mathfrak{Z}$ with supremum 1, then there exists an $x \varepsilon \hat{\mathfrak{M}}$ such that $e_\alpha x = e_\alpha x_\alpha$ for all α.*

Let $\overline{\mathfrak{M}}$ denote the completion in norm of $\hat{\mathfrak{M}}$. Then $\overline{\mathfrak{M}}$ is an AW^*-module over $\mathfrak{Z}$.

Proof. Referring to the definition of AW^*-module [8; 482], we see that we need verify only (b), part (a) being automatic in any inner product space. Accordingly, let $\{x_\alpha\}$ be a bounded subset of $\overline{\mathfrak{M}}$ and e_α orthogonal projections in $\mathfrak{Z}$ with supremum 1. For each α we can find a sequence $\{x_{\alpha n}\} \subset \hat{\mathfrak{M}}$ such that $\| x_\alpha - x_{\alpha n} \| < 1/n$. We have $\| x_{\alpha n} \| < \sup \| x_\alpha \| + 1/n$, which is bounded in α and n. Hence by hypothesis we can find, for each n, a $y_n \in \hat{\mathfrak{M}}$ such that $e_\alpha y_n = e_\alpha x_{\alpha n}$ for all α. We have

$$\| e_\alpha(y_n - y_m) \| = \| e_\alpha(x_{\alpha n} - x_{\alpha m}) \| \leq \| x_{\alpha n} - x_\alpha \| + \| x_{\alpha m} - x_\alpha \| < n^{-1} + m^{-1}.$$

Since this holds for all α and $\sup e_\alpha = 1$, we have $\| y_n - y_m \| < n^{-1} + m^{-1}$. Therefore there exists an $x \in \overline{\mathfrak{M}}$ with $\| y_n - x \| \to 0$. Then $e_\alpha x = \lim e_\alpha y_n = \lim e_\alpha x_{\alpha n} = e_\alpha x_\alpha$ for all α.

Given an inner product space $\mathfrak{M}$ over $\mathfrak{Z}$, we want to complete $\mathfrak{M}$ to an AW^*-module. Lemma 2.1 tells us that this is easy if $\mathfrak{M}$ contains all sums $\Sigma e_\alpha x_\alpha$, so the natural thing to do is add these sums to $\mathfrak{M}$. Let $\hat{\mathfrak{M}}$ be the set of all $\{e_\alpha , x_\alpha)\}$, where $\{x_\alpha\}$ is a bounded subset of $\mathfrak{M}$ and e_α are orthogonal projections of $\mathfrak{Z}$ with supremum 1. The two expressions $\{(e_\alpha , x_\alpha)\}$ and $\{(f_\beta , y_\beta)\}$ are to be identified if $e_\alpha f_\beta x_\alpha = e_\alpha f_\beta y_\beta$ for all α, β. We make $\hat{\mathfrak{M}}$ an inner product space over $\mathfrak{Z}$ by defining

$$\{(e_\alpha , x_\alpha)\} + \{(f_\beta , y_\beta)\} = \{(e_\alpha f_\beta , x_\alpha + y_\beta)\},$$

$$a\{(e_\alpha , x_\alpha)\} = \{(e_\alpha , a x_\alpha)\},$$

$$(\{(e_\alpha , x_\alpha)\}, \{(f_\beta , y_\beta)\}) = \Sigma_{\alpha,\beta} e_\alpha f_\beta (x_\alpha , y_\beta).$$

One verifies without difficulty that these definitions are independent of the particular choice of representatives and that $\hat{\mathfrak{M}}$ is indeed an inner product space over $\mathfrak{Z}$. Note that the map $x \to (1, x)$ is an isomorphism of $\mathfrak{M}$ into $\hat{\mathfrak{M}}$, so we may consider $\mathfrak{M}$ as embedded in $\hat{\mathfrak{M}}$.

We shall show that $\hat{\mathfrak{M}}$ satisfies the hypothesis of Lemma 2.1. Let $\{X_\alpha\}$ be a bounded subset of $\hat{\mathfrak{M}}$, and let f_α be orthogonal projections of $\mathfrak{Z}$ with supremum 1. We may write $X_\alpha = \{(e_{\alpha\beta} , x_{\alpha\beta})\}$ where, for each α, $e_{\alpha\beta}$ are orthogonal with supremum 1 and $\{x_{\alpha\beta}\}$ is a bounded subset of $\mathfrak{M}$. Since $\{X_\alpha\}$ is bounded there exists an M such that

$$\Sigma_\beta e_{\alpha\beta}(x_{\alpha\beta} , x_{\alpha\beta}) \leq M^2$$

for all α, so $\| e_{\alpha\beta} x_{\alpha\beta} \| \leq M$ for all α, β. If we define $X = \{(f_\alpha e_{\alpha\beta} , x_{\alpha\beta})\}$ we see that $f_\alpha X = f_\alpha X_\alpha$ for all α.

Thus $\hat{\mathfrak{M}}$ satisfies the hypothesis of Lemma 2.1. If $\overline{\mathfrak{M}}$ is the norm completion of $\hat{\mathfrak{M}}$ then $\overline{\mathfrak{M}}$ is, by the lemma, an AW^*-module over $\mathfrak{Z}$. We shall call $\overline{\mathfrak{M}}$ the *AW^*-completion* of $\mathfrak{M}$. That $\overline{\mathfrak{M}}$ has a right to be called the AW^*-completion of $\mathfrak{M}$ should be indicated by the following lemma.

Notation: if $\mathcal{S}$ is a subset of an AW^*-module, $V(\mathcal{S})$ will represent the AW^*-submodule generated by $\mathcal{S}$.

LEMMA 2.2. *Let $\mathfrak{H}$ be an AW^*-module over $\mathfrak{Z}$, $\mathfrak{M}$ a submodule of $\mathfrak{H}$, and $\overline{\mathfrak{M}}$ the AW^*-completion of $\mathfrak{M}$. Then $\overline{\mathfrak{M}}$ and $V(\mathfrak{M})$ are (module) isomorphic by an isomorphism leaving $\mathfrak{M}$ elementwise fixed.*

Proof. Define $\phi: \hat{\mathfrak{M}} \to \mathfrak{H}$ by $\phi[\{(e_\alpha, x_\alpha)\}] = \Sigma e_\alpha x_\alpha$. Then ϕ is easily seen to be independent of the representative chosen, an isomorphism into, and elementwise fixed on $\mathfrak{M}$. Since $\mathfrak{H}$ is norm complete, we can extend ϕ to an isomorphism from $\overline{\mathfrak{M}}$ to $\mathfrak{H}$. We shall show that $\phi(\overline{\mathfrak{M}}) = \vee \mathfrak{M}$. Since $\phi(\overline{\mathfrak{M}})$ is the isomorphic image of an AW^*-module, it is an AW^*-submodule of $\mathfrak{H}$. Since also $\phi(\overline{\mathfrak{M}}) \supset \mathfrak{M}$, we have $\phi(\overline{\mathfrak{M}}) \supset \vee \mathfrak{M}$. It remains to prove the reverse inclusion. Now if $\{x_\alpha\}$ is a bounded subset of $\mathfrak{M}$ and e_α are orthogonal projections of $\mathfrak{Z}$ with supremum 1, then $\Sigma_\alpha x_\alpha \in \vee \mathfrak{M}$. Hence $\phi(\hat{\mathfrak{M}}) \subset \vee \mathfrak{M}$. Since $\vee \mathfrak{M}$ is norm complete and $\phi(\overline{\mathfrak{M}})$ is the norm closure of $\phi(\hat{\mathfrak{M}})$ we have $\phi(\overline{\mathfrak{M}}) \subset \vee \mathfrak{M}$, which completes the proof.

As a corollary of Lemma 2.2 we obtain

LEMMA 2.3. *Let $\mathfrak{M}$ be a submodule of the AW^*-module $\mathfrak{H}$ over $\mathfrak{Z}$. Then for any $x \in \vee \mathfrak{M}$ and any $\epsilon > 0$ we can find a bounded subset $\{x_\alpha\}$ of $\mathfrak{M}$ and orthogonal projections e_α of $\mathfrak{Z}$ with supremum 1 such that $\| x - \Sigma e_\alpha x_\alpha \| < \epsilon$.*

Finally we state one more result, the proof of which is left to the reader.

LEMMA 2.4. *Let $\mathfrak{M}$ and $\mathfrak{M}_1$ be submodules of $\mathfrak{H}$ and $\mathfrak{H}_1$, respectively, AW^*-modules over $\mathfrak{Z}$. Let A be a continuous module homomorphism from $\mathfrak{M}$ into $\mathfrak{M}_1$. Then A can be extended, in a unique way, to a continuous module homomorphism from $\vee \mathfrak{M}$ into $\vee \mathfrak{M}_1$.*

II. Embedding of AW^*-Algebras

3. **States and representations.** Let $\mathfrak{A}$ be an AW^*-algebra and $\mathfrak{Z}$ an AW^*-subalgebra of the center of $\mathfrak{A}$. Then $\mathfrak{A}$ may be considered a module over $\mathfrak{Z}$. By a *$\mathfrak{Z}$-valued state* on $\mathfrak{A}$ we shall mean a positive module homomorphism $r: \mathfrak{A} \to \mathfrak{Z}$.

For any state r we have

(1) $\| r(A) \| \leq k \| A \|$ for all $A \in \mathfrak{A}$, k depending only on r,
(2) $| r(A^*B) |^2 \leq r(A^*A)r(B^*B)$.
(3) $r(B^*A^*AB) \leq \| A^*A \| r(B^*B)$.

Just as a numerical state on a C^*-algebra gives rise to a representation of this algebra as an algebra of operators on a Hilbert space, that is as a subalgebra of a factor of type I, so does a $\mathfrak{Z}$-valued state on $\mathfrak{A}$ give rise to a representation of $\mathfrak{A}$ as a subalgebra of an AW^*-algebra of type I. Since the construction of this representation is so similar to that in the numerical case, we shall only outline it here.

Let r be a $\mathfrak{Z}$-valued state on $\mathfrak{A}$. The set

$$\mathfrak{J}_r = [A \in \mathfrak{A} : r(A^*A) = 0]$$

is, by (3), a left ideal in $\mathfrak{A}$, so we may form the module $\mathfrak{A}/\mathfrak{I}_r$. We make this into an inner product space $\mathfrak{H}_r^0$ over $\mathfrak{Z}$ by defining $(A + \mathfrak{I}_r, B + \mathfrak{I}_r) = r(B^*A)$. By the method of §2 we form the AW^*-completion $\mathfrak{H}_r$ of $\mathfrak{H}_r^0$. If $\mathfrak{B}_r$ is the algebra of all bounded operators on $\mathfrak{H}_r$, then $\mathfrak{B}_r$ is an algebra of type I. Note that generally the center of $\mathfrak{B}_r$ is not (isomorphic to) $\mathfrak{Z}$, but rather a direct summand of $\mathfrak{Z}$. The center of $\mathfrak{B}_r$ is $\mathfrak{Z}$ if and only if $\mathfrak{H}_r$ is a faithful module over $\mathfrak{Z}$ [8, Theorem 7], and this in turn is the case if and only if r sends no non-zero projection of $\mathfrak{Z}$ into 0.

Let $A \in \mathfrak{A}$. For $B + \mathfrak{I}_r \in \mathfrak{H}_r^0$ define $\phi_r(A)(B + \mathfrak{I}_r) = AB + \mathfrak{I}_r$. Then (3) shows that $\phi_r(A)$ is a well-defined continuous module homomorphism on $\mathfrak{H}_r^0$, so by Lemma 2.4 it can be extended to a bounded operator, which we again call $\phi_r(A)$, on $\mathfrak{H}_r$. One verifies that ϕ_r is a *-homomorphism from $\mathfrak{A}$ into $\mathfrak{B}_r$, and is a module homomorphism relative to $\mathfrak{Z}$.

Thus a state on $\mathfrak{A}$ gives rise to a representation of $\mathfrak{A}$ as a subalgebra of an algebra of type I. We shall be particularly interested in representations ϕ of $\mathfrak{A}$ such that $\phi(\mathfrak{A})$ is actually an AW^*-subalgebra of the type I algebra. States which give rise to such a representation will have to satisfy some sort of continuity condition. We call the state r on $\mathfrak{A}$ *continuous* if it has the following property: if E_α are orthogonal projections in $\mathfrak{A}$ with supremum E, then for any $A \in \mathfrak{A}$ we have $r(A^*EA) = \Sigma_\alpha r(A^*E_\alpha A)$. We shall see that a continuous state gives rise to an AW^*-homomorphism, i.e., a *-homomorphism preserving suprema of sets of orthogonal projections. First we prove a useful lemma.

LEMMA 3.1. *Let $\mathfrak{B}$ be the algebra of all bounded operators on the AW^*-module $\mathfrak{H}$ over $\mathfrak{Z}$, and $\mathfrak{M}$ a submodule of $\mathfrak{H}$ such that $\vee\, \mathfrak{M} = \mathfrak{H}$. Let E_α, E be projections in $\mathfrak{B}$, the E_α being orthogonal. Assume that for all $x \in \mathfrak{M}$ we have $(Ex, x) = \Sigma(E_\alpha x, x)$. Then $E = \sup E_\alpha$.*

Proof. Let $F = \sup E_\alpha$ and assume $F \neq E$. Then $(Fy, y) \neq (Ey, y)$ for some $y \in \mathfrak{H}$ with $\| y \| = 1$. Since $((E - F)y, y) \neq 0$ we can find a non-zero projection $e \in \mathfrak{Z}$ and an $\epsilon > 0$ such that $e \,|\, ((E - F)y, y) \,| \geq \epsilon e$. By Lemma 2.3 we can find a non-zero projection $f \leq e$ and an element $x \in \mathfrak{M}$ such that $\| f(y - x) \| < \frac{1}{2}\epsilon$. Applying Lemma 1.4 we have $Ex = Fx$, so $f(E - F)y = f(E - F)(y - x)$. Therefore $\| f((E - F)y, y) \| = \| f((E - F)(y - x), y) \| \leq 2 \| f(y - x) \| < \epsilon$, which contradicts $f \,|\, ((E - f)y, y) \,| \geq \epsilon f$.

LEMMA 3.2. *If r is a continuous $\mathfrak{Z}$-valued state on $\mathfrak{A}$ then ϕ_r is an AW^*-homomorphism.*

Proof. Let $\{E_\alpha\}$ be a set of orthogonal projections in with supremum E. We must show that $\{\phi_r(E_\alpha)\}$ has supremum $\phi_r(E)$. Now $\vee\, \mathfrak{H}_r^0 = \mathfrak{H}_r$, so by Lemma 3.1 it suffices to show that $(\phi_r(E)(A + \mathfrak{I}_r), (A + \mathfrak{I}_r)) = \Sigma_\alpha(\phi_r(E_\alpha)(A + \mathfrak{I}_r), (A + \mathfrak{I}_r))$ for all $A \in \mathfrak{A}$. Applying the definitions this amounts to showing $r(A^*EA) = \Sigma_\alpha r(A^*E_\alpha A)$, and this follows from the continuity of r.

We call a collection of states *complete* if $r(A^*A) = 0$ for all r in the collection implies $A = 0$.

THEOREM 3.1. *Let $\mathfrak{A}$ be an AW^*-algebra and $\mathfrak{Z}$ an AW^*-subalgebra of the center of $\mathfrak{A}$. Then a necessary and sufficient condition that $\mathfrak{A}$ be AW^*-embeddable in an AW^*-algebra of type I with center $\mathfrak{Z}$ is that $\mathfrak{A}$ possess a complete set of continuous $\mathfrak{Z}$-valued states.*

Proof. Assume $\mathfrak{A}$ is an AW^*-subalgebra of $\mathfrak{B}$, an algebra of type I with center $\mathfrak{Z}$. By [8, Theorem 8], $\mathfrak{B}$ may be represented as the algebra of all bounded operators on some faithful AW^*-module $\mathfrak{H}$ over $\mathfrak{Z}$. For any $x \in \mathfrak{H}$ define the state r_x on $\mathfrak{A}$ by $r_x(A) = (Ax, x)$. From Lemma 1.4 and the fact that $\mathfrak{A}$ is an AW^*-subalgebra of $\mathfrak{B}$ we may conclude that each r_x is continuous. That $[r_x : x \in \mathfrak{H}]$ is a complete set of states on $\mathfrak{A}$ is clear.

Now assume $\{r\}$ is a complete set of continuous $\mathfrak{Z}$-valued states on $\mathfrak{A}$. For each $r \in \{r\}$ we have the representation ϕ_r of $\mathfrak{A}$ as operators on the AW^*-module $\mathfrak{H}_r$ over $\mathfrak{Z}$. Let $\mathfrak{H} = \Sigma \bigoplus \mathfrak{H}_r$. (See [8], pp. 846–847 for the definition of the direct sum of AW^*-modules.) If $\mathfrak{B}$ is the algebra of bounded operators on $\mathfrak{H}$ we define $\phi: \mathfrak{A} \to \mathfrak{B}$ by $\phi(A) = \{\phi_r(A)\}$, that is for $\{X_r\} \in \mathfrak{H}$ define $\phi(A)\{X_r\} = \{\phi_r(A)X_r\}$. Then ϕ is a *-homomorphism. We shall show it is actually an AW^*-isomorphism.

Let E_α be orthogonal projections in $\mathfrak{A}$ with supremum E. We shall show $\phi(E) = \sup \phi(E_\alpha)$ in $\mathfrak{B}$. The set

$$\mathfrak{M} = [\{X_r\} \in \mathfrak{H}\text{: all but finitely many } X_r \text{ vanish}]$$

is a submodule of $\mathfrak{H}$ such that $\vee\, \mathfrak{M} = \mathfrak{H}$. Therefore by Lemma 3.1 it suffices to show that $(\phi(E)\{X_r\}, \{X_r\}) = \Sigma_\alpha(\phi(E_\alpha)\{X_r\}, \{X_r\})$ for $\{X_r\} \in \mathfrak{M}$. Now by Lemmas 3.2 and 1.4 we have $\phi_r(E)X_r = \Sigma_\alpha \phi_r(E_\alpha)X_r$ for all r. Then $\phi(E)\{X_r\} = \{\phi_r(E)X_r\} = \{\Sigma_\alpha\phi_r(E_\alpha)X_r\} = \Sigma_\alpha\{\phi_r(E_\alpha)X_r\} = \Sigma_\alpha\phi_r(E_\alpha)\{X_r\}$. Therefore certainly $(\phi(E)\{X_r\}, \{X_r\}) = \Sigma_\alpha(\phi(E_\alpha)\{X_r\}, \{X_r\})$, so $\phi(E) = \sup \phi(E_\alpha)$.

Thus ϕ is an AW^*-homomorphism. That ϕ is an isomorphism and that $\mathfrak{B}$ has center $\mathfrak{Z}$ are both easy consequences of the fact that $\{r\}$ is complete.

If $\mathfrak{A}$ is an algebra with center $\mathfrak{Z}$ possessing a complete set of continuous $\mathfrak{Z}$-valued states, then by the theorem $\mathfrak{A}$ is AW^*-embeddable in an algebra of type I with center $\mathfrak{Z}$. Algebras satisfying this property are certainly not rare. Besides algebras of type I, algebras of type II_1 with central trace have this property (Goldman [4]). It seems reasonable that actually all AW^*-algebras do. A proof of this, however, seems to be out of reach at present.

4. **Double commutors.** A ring of operators is a weakly closed self-adjoint algebra of operators on Hilbert space, or equivalently a self-adjoint subalgebra of a factor of type I which is its own double commutor in the larger algebra. It is natural to ask when an AW^*-algebra can be embedded as a double commutor in some AW^*-algebra of type I. By Lemma 4 of [7] it is certainly necessary that the algebra first be AW^*-embeddable in a type I algebra, and this case was treated in the previous section. Therefore the following question arises: Is an AW^*-subalgebra of an algebra of type I automatically its own double

commutor in the larger algebra? In this section we shall show that this question has an affirmative answer in certain cases, perhaps the most important being when the smaller algebra is finite. The work here is largely an extension of work done by Dixmier [2] and Feldman [3] for the case of Hilbert space.

To begin the study we need some sort of double commutor theorem. We first state without proof an analog of the well-known result that the strong closure of a convex subset of Hilbert space is equal to its weak closure, and then prove our version of the double commutor theorem.

LEMMA 4.1. *Let $\mathcal{K}$ be a convex subset of the AW^*-module $\mathcal{H}$ over $\mathfrak{Z}$. Assume that for any $x \in \mathcal{H}$, non-zero projection $e \in Z$, and $\epsilon > 0$, we can find a $y \in \mathcal{K}$ and a non-zero projection $f \leq e$ such that $\| f(y, x) \| < \epsilon$.*

Then given a non-zero projection $e \in \mathfrak{Z}$ and $\epsilon > 0$ we can find a non-zero projection $f \leq e$ and a $y \in \mathcal{K}$ such that $\| fy \| < \epsilon$.

LEMMA 4.2. (Double Commutor Theorem.) *Let $\mathcal{B}$ be the algebra of all bounded operators on the AW^*-module $\mathcal{H}$ over $\mathfrak{Z}$, and let $\mathcal{A}$ be a self-adjoint subalgebra of $\mathcal{B}$ containing $\mathfrak{Z}$.*

(1) *Let $A \in \mathcal{A}''$. Then given a non-zero projection $e \in \mathfrak{Z}$, $x_1, \cdots, x_n \in \mathcal{H}$, and $\epsilon > 0$, we can find a non-zero projection $f \leq e$ and $B \in \mathcal{A}$ such that $\| f(B - A)x_i \| < \epsilon$ for $i = 1, \cdots, n$. If A is self-adjoint, then B can be chosen to be self-adjoint.*

(2) *Assume in addition that $\mathcal{A}$ is an AW^*-subalgebra of $\mathcal{B}$, and let $A \in \mathcal{A}''$, $A = A^*$. Then we can find a bounded net $\{A_\lambda\}$ of self-adjoint elements of $\mathcal{A}$ such that $A_\lambda \to A$ strongly.*

Proof. (1) Let $A \in \mathcal{A}''$, and assume e, x_i, and ϵ are given as in the statement. Form the AW^*-module $\tilde{\mathcal{H}} = \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ (n times), and let $\tilde{\mathfrak{M}} = [\{Bx_i\}: B \in \mathcal{A}]$. Then $\tilde{\mathfrak{M}}$ is a submodule of $\tilde{\mathcal{H}}$. Let $\tilde{P}$ be the projection on $\vee \tilde{\mathfrak{M}}$. We may write $\tilde{P}$ in matrix form, $\tilde{P} = (P_{ij})$. For any $B \in \mathcal{A}$, the operator $\tilde{B} = (\delta_{ij}B)$ on $\tilde{\mathcal{H}}$ leaves invariant $\tilde{\mathfrak{M}}$, and so also $\vee \tilde{\mathfrak{M}}$. Hence $\tilde{P}$ commutes with $\tilde{B}$, so P_{ij} commutes with B for all i, j. Since B was an arbitrary element of $\mathcal{A}$ we have $P_{ij} \in \mathcal{A}'$. Therefore A commutes with P_{ij}, so $\tilde{P}$ commutes with $\tilde{A} = (\delta_{ij}A)$. This implies $\tilde{A}(\tilde{\mathfrak{M}}) \subset \vee \tilde{\mathfrak{M}}$. In particular $\{Ax_i\} \in \vee \tilde{\mathfrak{M}}$. By Lemma 2.3 we can find e_α, orthogonal projections in $\mathfrak{Z}$ with supremum 1, and elements B_α of $\mathcal{A}$ with $\{B_\alpha x_i\}$ bounded, such that $\Sigma_{i,\alpha} e_\alpha \mid (A - B_\alpha)x_i \mid^2 \leq \epsilon^2$. Since $\sup e_\alpha = 1$, we can find some α so that $f = ee_\alpha \neq 0$. Then for this f we have $\Sigma_i f \mid (A - B_\alpha)x_i \mid^2 \leq \epsilon^2$, and so $\| f(A - B_\alpha)x_i \| \leq \epsilon$ ($i = 1, \cdots, n$).

We must still show that B can be taken to be self-adjoint if A is self-adjoint. With x_i and $\tilde{\mathcal{H}}$ as above, let

$$\tilde{\mathcal{K}} = [\{(A - B)x_i\}: B \in \mathcal{A}, B \text{ self-adjoint}].$$

Then $\tilde{\mathcal{K}}$ is a convex subset of $\tilde{\mathcal{H}}$ and, by what we have already shown, satisfies the hypothesis of Lemma 4.1. The conclusion of this lemma gives the desired result.

(2) The directed set $\{\lambda\}$ will consist of all $(x_1, \cdots, x_n; p)$ where $x_i \in \mathcal{H}$ and

p is a positive integer. If $\mu = (y_1, \cdots, y_m; q)$ then $\lambda \geq \mu$ means $\{x_i\} \supset \{y_i\}$ and $p \geq q$. We obtain the net $\{A_\lambda\}$ as follows. Let $H(t)$ be a continuous function on the real line such that

$$H(t) = t \quad \text{for} \quad |t| \leq \|A\|,$$

$$0 \leq |H(t)| \leq \|A\| \quad \text{for all} \quad t,$$

$$H(\pm\infty) = 0.$$

The semi-norms $S_x : B \to |Bx|$ for $x \in \mathcal{H}$ satisfy the condition of Lemma 1.5. Hence we can find $x_1', \cdots, x_m' \in \mathcal{H}$ and an $M > 0$ such that $|[H(A) - H(B)]x_i| \leq M \sup |(A - B)x_j'| + \frac{1}{2}p$ $(i = 1, \cdots, n)$ for every self-adjoint $B \in \mathcal{B}$. Now given a non-zero projection $e \in \mathfrak{Z}$ we can find, by part (1) of the lemma, a non-zero projection $f \leq e$ and a self-adjoint $B \in \mathcal{B}$ such that $\|f(A - B)x_j'\| < \frac{1}{2}pM$ $(j = 1, \cdots, m)$. Let $C = H(B)$. Then since $H(A) = A$ we have $\|f(A - C)x_i\| < M(\frac{1}{2}pM) + \frac{1}{2}p = 1/p$ $(i = 1, \cdots, n)$. Furthermore $\|C\| \leq \|A\|$.

By Zorn's lemma we can find e_α, orthogonal projections in $\mathfrak{Z}$ with supremum 1, and for each α an element $C_\alpha \in \mathcal{A}$ with $\|C_\alpha\| \leq \|A\|$ and $\|e_\alpha(A - C_\alpha)x_i\| < 1/p$ $(i = 1, \cdots, n)$. Now by Lemma 2.5 of [7] we can find an element $A_\lambda \in \mathcal{A}$ with $e_\alpha A_\lambda = e_\alpha C_\alpha$ for all α. Then $\|A_\lambda\| \leq \|A\|$ and $\|(A - A_\lambda)x_i\| < 1/p$ $(i = 1, \cdots, n)$. Thus $\{A_\lambda\}$ is a bounded net of self-adjoint elements of $\mathcal{A}$ with $A_\lambda \to A$ strongly. (Actually, of course, we have a good deal more.)

The second basic result we shall prove is an extension of [2], Corollary 4 of Theorem 3. The result of [2] is, essentially, that if a ring of operators is represented as an AW^*-subalgebra of the algebra of operators on (some other) Hilbert space, then it is also a ring of operators in that representation. The proof made strong use of the weak compactness of the unit sphere of a ring of operators. We are able to avoid this argument by using completeness instead of compactness.

LEMMA 4.3. *Let $\mathcal{B}$ be the algebra of all bounded operators on the AW^*-module $\mathcal{H}$ over $\mathfrak{Z}$, $\mathcal{A}$ an AW^*-subalgebra of $\mathcal{B}$ containing $\mathfrak{Z}$. Let r be a $\mathfrak{Z}$-valued state on $\mathcal{A}$ which is completely additive on projections. Then r is continuous on bounded parts of $\mathcal{A}$ relative to strong convergence on $\mathcal{A}$ and order convergence on $\mathfrak{Z}$.*

This is an extension of Theorem 3 of [2], and the proof is entirely analogous. Note that in the statement of Theorem 3 Dixmier assumed that the state satisfied a stronger condition than complete additivity on projections, but this is all that is really needed. We now prove the analog of Corollary 4.

LEMMA 4.4. *Let $\mathcal{B}$ and $\mathcal{B}_1$ be algebras of type I with centers $\mathfrak{Z}$ and $\mathfrak{Z}_1$ respectively. Let $\mathcal{A}$ be a *-subalgebra of $\mathcal{B}$ such that $\mathcal{A} = \mathcal{A}''$, and assume $\phi: \mathcal{A} \to B_1$ is an AW^*-homomorphism with the property that $\phi \mid \mathfrak{Z}$ is an isomorphism onto $\mathfrak{Z}_1$. Then $\phi(\mathcal{A}) = \phi(\mathcal{A})''$ in $\mathcal{B}_1$.*

Proof. Since ϕ is an AW^*-homomorphism, its kernel is of the form $(1 - E)\mathcal{A}$ where E is a central projection of $\mathcal{A}$. Thus we may assume to begin with that ϕ is an isomorphism. For convenience we shall identify the algebras $\mathfrak{Z}$ and $\mathfrak{Z}_1$

by means of the isomorphism $\phi \mid \mathfrak{Z}$. Then ϕ is an AW^*-isomorphism which is also a module homomorphism relative to $\mathfrak{Z}$.

Represent $\mathfrak{B}$ and $\mathfrak{B}_1$ as the algebras of bounded operators on $\mathfrak{H}$ and $\mathfrak{H}_1$ respectively, faithful AW^*-modules over $\mathfrak{Z}$. For any $x, y \in \mathfrak{H}_1$, we have

$$\begin{aligned} 4(\phi(A)x, y) &= (\phi(A)(x + y), (x + y)) - (\phi(A)(x - y), (x - y)) \\ &\qquad + i(\phi(A)(x + iy), (x + iy)) - i(\phi(A)(x - iy), (x - iy)) \\ &= \psi_1(A) - \psi_2(A) + i\psi_3(A) - i\psi_4(A), \end{aligned}$$

say. Each ψ_i is a $\mathfrak{Z}$-valued state on $\mathfrak{A}$ which is completely additive on projections. Therefore by Lemma 4.3 each ψ_i is continuous on bounded parts of $\mathfrak{A}$, relative to strong convergence on $\mathfrak{A}$ and order convergence on $\mathfrak{Z}$. It follows that ϕ is continuous on bounded parts of $\mathfrak{A}$, relative to strong convergence on $\mathfrak{A}$ and weak convergence on $\mathfrak{B}_1$. Since ϕ is an isomorphism, we can apply the same argument to ϕ^{-1}: $\phi(\mathfrak{A}) \to \mathfrak{B}$, and deduce that ϕ^{-1} is continuous on bounded parts of $\phi(\mathfrak{A})$ relative to strong convergence in $\phi(\mathfrak{A})$ and weak convergence in $\mathfrak{B}$.

Let B be a self-adjoint element of $\phi(\mathfrak{A})''$. By Lemma 4.2 (2), B is the strong limit of a bounded net $\{B_\lambda\}$ of self-adjoint elements of $\phi(\mathfrak{A})$. Then $B_\lambda - B_\mu \to 0$ strongly as $\lambda, \mu \uparrow$, so also $| B_\lambda - B_\mu | \to 0$ strongly. By the established continuity of ϕ^{-1}, $\phi^{-1}(| B_\lambda - B_\mu |) \to 0$ weakly, that is $| \phi^{-1}(B_\lambda) - \phi^{-1}(B_\mu) | \to 0$ weakly. It follows from the boundedness of $\{\phi^{-1}(B_\lambda)\}$ that in fact $| \phi^{-1}(B_\lambda) - \phi^{-1}(B_\mu) | \to 0$ strongly. Thus $\{\phi^{-1}(B_\lambda)\}$ is a bounded strong Cauchy net in $\mathfrak{B}$. Therefore by Lemma 1.3 there exists an $A \in \mathfrak{B}$ such that $\phi^{-1}(B_\lambda) \to A$ strongly. Since each $\phi^{-1}(B_\lambda)$ is self-adjoint so is A, and since $\phi^{-1}(B_\lambda) \in \mathfrak{A}$ for all λ we have $A \in \mathfrak{A}'' = \mathfrak{A}$. Now using the continuity of ϕ we have $B = \phi(A) \in \phi(\mathfrak{A})$. Thus $\phi(\mathfrak{A})$ contains every self-adjoint element of $\phi(\mathfrak{A})''$, so $\phi(\mathfrak{A}) = \phi(\mathfrak{A})''$.

Remark. In the statement of Lemma 4.4 we required that $\phi(\mathfrak{Z}) = \mathfrak{Z}_1$. We shall see later (Theorem 4.2) that $\phi(\mathfrak{A}) \supset \mathfrak{Z}_1$ is all that is really required.

We have seen that if $\mathfrak{A}$ is an AW^*-subalgebra of the algebra of type I $\mathfrak{B}$ with center $\mathfrak{Z}$, and $\mathfrak{A}$ has some representation as a double commutor in an algebra of type I with center $\mathfrak{Z}$, then $\mathfrak{A} = \mathfrak{A}''$ in $\mathfrak{B}$ as long as $\mathfrak{A} \supset \mathfrak{Z}$. We should like to find conditions on the actual embedding of $\mathfrak{A}$ in $\mathfrak{B}$ which insures that $\mathfrak{A} = \mathfrak{A}''$. The crucial result will be Lemma 4.6, from which we can derive, rather easily, the main results of the section. The analogous results for Hilbert space are due to Feldman [3].

LEMMA 4.5. *Let $\mathfrak{A}$ be an AW^*-subalgebra of $\mathfrak{B}$, and let $\{A_\lambda\}$ be a bounded ascending net of positive elements of $\mathfrak{A}$. Assume that $\{A_\lambda\}$ has a supremum among the self-adjoint elements of $\mathfrak{B}$, and that this supremum is a projection E. Then $E \in \mathfrak{A}$.*

Proof. Let E_λ be the right projection of A_λ. Then $E_\lambda \in \mathfrak{A}$. We shall show $E = \sup E_\lambda$. Since $A_\lambda E = A_\lambda$ for all λ we have $E \geq E_\lambda$ for all λ, so $E > \sup E_\lambda$.

But since $A_\lambda \leq 1$ we have $A_\lambda = E_\lambda A_\lambda E_\lambda \leq E_\lambda \leq \sup E_\lambda$, so $E \leq \sup E_\lambda$. Therefore $E = \sup E_\lambda$.

LEMMA 4.6. *Let $\mathfrak{B}$ be the algebra of all bounded operators on the AW^*-module $\mathfrak{H}$ over $\mathfrak{Z}$, $\mathfrak{A}$ an AW^*-subalgebra of $\mathfrak{B}$ containing $\mathfrak{Z}$. Assume there exists a vector $x \in \mathfrak{H}$ which is separating for $\mathfrak{A}''$ and has the further property that for any two projections $E, F \in \mathfrak{A}$, $(E \cup Fx, x) \leq (Ex, x) + (Fx, x)$. Then we have $\mathfrak{A} = \mathfrak{A}''$.*

Proof. Let E be a projection of $\mathfrak{A}''$. By Lemma 4.2 (2) we can find a bounded net $\{A_\lambda\}$ of self-adjoint elements of $\mathfrak{A}$ which converges strongly to E. Then $|(A_\lambda - E)x| \to 0$ (O), so given $\epsilon > 0$ we can find orthogonal projections $e_\alpha \in \mathfrak{Z}$ with supremum 1 and for each α an $A_\alpha \in \{A_\lambda\}$ such that $\| e_\alpha(A_\alpha - E)x \| < \epsilon$. Since the A_α are bounded we can find an $A \in \mathfrak{A}$ such that $e_\alpha A = e_\alpha A_\alpha$ for all α. Then we have $\| (A - E)x \| < \epsilon$. This shows that we can find a sequence $\{A_n\}$ of self-adjoint elements of $\mathfrak{A}$ such that $\| (A_n - E)x \| \to 0$. By choosing a subsequence if necessary we may assume $\| (A_n - A_{n+1})x \| \leq 4^{-n}$.

Let F_n be the right projection of $(| A_n - A_{n+1} | - 2^{-n}) \cup 0$. Then $F_n \leq 2^n | A_n - A_{n+1} |$, so $(F_n x, x) \leq 2^n \| (A_n - A_{n+1})x \| \| x \| \leq 2^{-n} \| x \|$. Therefore if $E_m = \sup[F_n : n \geq m]$ we have $(E_m x, x) \leq \sum_{n=m}^{\infty} (F_n x, x) \leq 2^{-m+1} \| x \|$. Let $E = \inf E_m$. Then $(Ex, x) = 0$. Since $E \in \mathfrak{A}$ we have $E = 0$. Thus $\{E_m\}$ is a decreasing sequence of projections of $\mathfrak{A}$ with infimum 0.

Now $\| (1 - F_n)(A_n - A_{n+1}) \| \leq 2^{-n}$, so $n \geq m$ implies $\| (1 - E_m)(A_n - A_{n+1}) \| \leq 2^{-n}$. Thus for each m, $\{(1 - E_m)A_n\}$ is Cauchy in norm, and so converges to some element $B_m \in \mathfrak{A}$. We have $(1 - E_m)A_n x \to B_m x$. But $A_n x \to Ex$, so $(1 - E_m)A_n x \to (1 - E_m)Ex$. Therefore $B_m x = (1 - E_m)Ex$. Since x is separating for $\mathfrak{A}''$ we have $(1 - E_m)E = B_m$, an element of $\mathfrak{A}$. Therefore $E(1 - E_m)E \in \mathfrak{A}$. But $E(1 - E_m)E \uparrow E$ in $\mathfrak{B}$. It follows from Lemma 4.5, therefore, that $E \in \mathfrak{A}$.

Thus $\mathfrak{A}$ contains every projection of $\mathfrak{A}''$, so $\mathfrak{A} = \mathfrak{A}''$.

We can now reap the consequences of Lemma 4.6.

LEMMA 4.7. *Let r be a continuous $\mathfrak{Z}$-valued state on the AW^*-algebra $\mathfrak{A}$ with the following two properties:*

(1) *given $B \in \mathfrak{A}$ there exists a real number $k > 0$ such that $\| r(B^*A^*AB) \| \leq k \| r(A^*A) \|$ for all $A \in \mathfrak{A}$.*

(2) *for any two projections $E, F \in \mathfrak{A}$, $r(E \cup F) \leq r(E) + r(F)$.*

Then if ϕ_r is the representation of $\mathfrak{A}$ induced by r, we have $\phi_r(\mathfrak{A}) = \phi_r(\mathfrak{A})''$ in $\mathfrak{B}_r$.

Proof. By Lemma 3.2 $\phi_r(\mathfrak{A})$ is an AW^*-subalgebra of $\mathfrak{B}_r$. Therefore, by Lemma 4.6 it suffices to show that $\phi_r(\mathfrak{A})''$ has a separating vector $X \in \mathfrak{H}_r$ with the further property that $(\phi_r(E \cup F)X, X) \leq (\phi_r(E)X, X) + (\phi_r(F)X, X)$. We shall show that $X = 1 + \mathfrak{I}_r$ satisfies the requirements. For $A \in \mathfrak{A}$ define $\psi_r(A)(B + \mathfrak{I}_r) = BA + \mathfrak{I}_r$. Then by (1), $\psi_r(A)$ is a well-defined continuous module homomorphism on $\mathfrak{H}_r^0$, so by Lemma 2.4 it can be extended to a bounded operator on $\mathfrak{H}_r$. Clearly $\psi_r(A) \in \phi(\mathfrak{A})'$. Also, $\psi_r(A)X = A + \mathfrak{I}_r$, which ranges over $\mathfrak{H}_r^0$ as A runs over $\mathfrak{A}$. Hence $\phi_r(\mathfrak{A})'X \supset \mathfrak{H}_r^0$, so X is a cyclic vector for

$\phi_r(\mathfrak{A})'$ and so a separating vector for $\phi_r(\mathfrak{A})''$. As for the second requirement to be satisfied by X, we have, using (2), $(\phi_r(E \cup F)X, X) = r(E \cup F) \leq r(E) + r(F) = (\phi_r(E)X, X) + (\phi_r(F)X, X)$.

LEMMA 4.8. *Let $\mathfrak{B}$ be an algebra of type I with center $\mathfrak{Z}$, and $\mathfrak{A}$ a commutative AW^*-subalgebra of $\mathfrak{B}$ containing $\mathfrak{Z}$. Then $\mathfrak{A} = \mathfrak{A}''$ in $\mathfrak{B}$.*

Proof. Represent $\mathfrak{B}$ as the algebra of bounded operators on the AW^*-module $\mathfrak{H}$ over $\mathfrak{Z}$, and let $x \in \mathfrak{H}$. The state r on $\mathfrak{A}$ defined by $r(A) = (Ax, x)$ satisfies $r(A^*B^*BA) \leq \| B^*B \| \, r(A^*A)$, so by the commutativity of $\mathfrak{A}$, $r(B^*A^*AB) \leq \| B^*B \| \, r(A^*A)$. Thus r satisfies condition (1) of Lemma 4.7. Now if E and F are projections of $\mathfrak{A}$, we have $E \cup F = E + F - EF \leq E + F$, so $r(E \cup F) \leq r(E) + r(F)$, which is condition (2) of Lemma 4.7. An application of that lemma therefore gives $\phi_r(\mathfrak{A}) = \phi_r(\mathfrak{A})''$ in $\mathfrak{B}_r$. The kernel of ϕ_r is a direct summand of $\mathfrak{A}$, that is ϕ_r represents faithfully a direct summand of $\mathfrak{A}$.

Zorn's Lemma shows the following. We can find E_α, orthogonal projections of $\mathfrak{A}$ with supremum 1 and for each α an AW^*-module $\mathfrak{H}_\alpha$ over $\mathfrak{Z}$ such that if $\mathfrak{B}_\alpha$ is the algebra of bounded operators on $\mathfrak{H}_\alpha$ there exists a representation $\phi_\alpha : \mathfrak{A} \rightarrow \mathfrak{B}_\alpha$ satisfying

(1) ϕ_α is a module homomorphism relative to $\mathfrak{Z}$,

(2) $\phi_\alpha(\mathfrak{A}) = \phi_\alpha(\mathfrak{A})''$ in $\mathfrak{B}_\alpha$,

(3) the kernel of ϕ_α is $(1 - E_\alpha)\mathfrak{A}$.

Let $\mathfrak{H}_1 = \Sigma \oplus \mathfrak{H}_\alpha$ and let $\mathfrak{B}_1$ be the algebra of bounded operators on $\mathfrak{H}_1$. Define ϕ: $\mathfrak{A} \rightarrow \mathfrak{B}_1$ by $\phi(A) = \{\phi_\alpha(A)\}$. Then $\mathfrak{B}_1$ is an algebra of type I with center $\mathfrak{Z}$, ϕ is an isomorphism, and $\phi(\mathfrak{A}) = \phi(\mathfrak{A})''$ in B_1. Thus $\mathfrak{A}$ has some representation as a double commutor in an algebra of type I with center $\mathfrak{Z}$, and it follows from Lemma 4.4 that $\mathfrak{A} = \mathfrak{A}''$ in $\mathfrak{B}$.

THEOREM 4.1. *Let $\mathfrak{B}$ be an AW^*-algebra of type I with center $\mathfrak{Z}$, and $\mathfrak{A}$ an AW^*-subalgebra of $\mathfrak{B}$ which is of type I and contains $\mathfrak{Z}$. Then $\mathfrak{A} = \mathfrak{A}''$ in $\mathfrak{B}$.*

Proof. Let $\mathfrak{A}$ have center $\mathfrak{Z}^* \supset \mathfrak{Z}$, and let E_α be mutually orthogonal Abelian projections of $\mathfrak{A}$ with supremum 1. We have $E_\alpha \mathfrak{A} E_\alpha = E_\alpha \mathfrak{Z}^*$ and, if $\mathfrak{Z}_0^*$ is the center of $\mathfrak{A}''$, $E_\alpha \mathfrak{A}'' E_\alpha = E_\alpha \mathfrak{Z}_0^*$. Now $E_\alpha \mathfrak{A} E_\alpha$ is a commutative AW^*-subalgebra of $E_\alpha \mathfrak{B} E_\alpha$ containing the center $E_\alpha \mathfrak{Z}$ of $E_\alpha \mathfrak{B} E_\alpha$. Therefore by Lemma 4.8, $E_\alpha \mathfrak{A} E_\alpha = (E_\alpha \mathfrak{A} E_\alpha)''$ in $E_\alpha \mathfrak{B} E_\alpha$.

If $\mathfrak{B}$ is represented as the algebra of bounded operators on the AW^*-module $\mathfrak{H}$ over $\mathfrak{Z}$, then $E_\alpha \mathfrak{B} E_\alpha$ is the algebra of bounded operators on $E_\alpha \mathfrak{H}$. Now by Lemma 4.2 (2), every element of $\mathfrak{A}''$ is the strong limit of a net of elements of $\mathfrak{A}$, so that every element of $E_\alpha \mathfrak{A}'' E_\alpha$ is the strong limit (relative to $E_\alpha \mathfrak{H}$) of a net of elements of $E_\alpha \mathfrak{A} E_\alpha$. Since $E_\alpha \mathfrak{A} E_\alpha = (E_\alpha \mathfrak{A} E_\alpha)''$, we may conclude from this that $E_\alpha \mathfrak{A} E_\alpha = E_\alpha \mathfrak{A}'' E_\alpha$. Therefore $E_\alpha \mathfrak{Z}^* = E_\alpha \mathfrak{Z}_0^*$. Since $\sup E_\alpha = 1$ it follows easily from Lemma 2.2 of [6] that $\mathfrak{Z}^* = \mathfrak{Z}_0^*$.

Thus $\mathfrak{A}$ and $\mathfrak{A}''$ have the same center. Therefore by Lemma 4.4 $\mathfrak{A}$ is its own double commutor in $\mathfrak{A}''$. It follows immediately from this that $\mathfrak{A} = \mathfrak{A}''$.

We can now prove the promised extension of Lemma 4.4.

THEOREM 4.2. *Let $\mathfrak{B}$ and $\mathfrak{B}_1$ be AW^*-algebras of type I with centers $\mathfrak{Z}$ and $\mathfrak{Z}_1$ respectively. Let $\mathfrak{A}$ be a *-subalgebra of $\mathfrak{B}$ such that $\mathfrak{A} = \mathfrak{A}''$ in $\mathfrak{B}$, and assume $\phi: \mathfrak{A} \to \mathfrak{B}_1$ is an AW^*-homomorphism such that $\phi(\mathfrak{A}) \supset \mathfrak{Z}_1$. Then $\phi(\mathfrak{A}) = \phi(\mathfrak{A})''$ in $\mathfrak{B}_1$.*

Proof. As in the proof of Lemma 4.4 we may assume ϕ is an isomorphism. Now $\phi(\mathfrak{Z})$ contains $\mathfrak{Z}_1$ and is an AW^*-subalgebra of $\mathfrak{B}_1$, an algebra of type I with center $\mathfrak{Z}_1$. Therefore by Theorem 3.1 $\mathfrak{Z}$ possesses a complete set $\{r\}$ of continuous $\mathfrak{Z}_1$-valued states. Moreover $\mathfrak{B}$ possesses a complete set $\{s\}$ of continuous $\mathfrak{Z}$-valued states. Then $\{r \cdot s\}$ is a complete set of continuous $\mathfrak{Z}_1$-valued states on $\mathfrak{B}$. Thus $\mathfrak{B}$ is AW^*-embeddable in an algebra $\mathfrak{B}_2$ of type I with center $\mathfrak{Z}_1$, the embedding agreeing with ϕ on $\mathfrak{Z}$. In other words we can find an AW^*-homomorphism $\psi: \mathfrak{B} \to \mathfrak{B}_2$ such that $\psi \mid \mathfrak{Z} = \phi \mid \mathfrak{Z}$. Since $\mathfrak{A} = \mathfrak{A}''$ in $\mathfrak{B}$, $\psi(\mathfrak{A}) = \psi(\mathfrak{A})''$ in $\psi(\mathfrak{B})$. By Theorem 4.1, $\psi(\mathfrak{B}) = \psi(\mathfrak{B})''$ in $\mathfrak{B}_2$. Therefore $\psi(\mathfrak{A}) = \psi(\mathfrak{A})''$ in $\mathfrak{B}_2$. But now if we map $\psi(\mathfrak{A})$ into $\mathfrak{B}_1$ by means of the AW^*-homomorphism $\phi\psi^{-1}$, we can apply Lemma 2.6 to deduce $\phi(\mathfrak{A}) = \phi(\mathfrak{A})''$ in $\mathfrak{B}_1$.

We conclude with three theorems concerning finite AW^*-algebras. The first of these is a strengthening of a result of Yen [9].

THEOREM 4.3. *Let $\mathfrak{A}$ be a finite AW^*-algebra with center $\mathfrak{Z}$ possessing a central trace. Then $\mathfrak{A}$ can be embedded in an algebra of type I with center $\mathfrak{Z}$ so that $\mathfrak{A} = \mathfrak{A}''$.*

Proof. This is almost an immediate consequence of Lemma 4.7. The state Tr (the trace on $\mathfrak{A}$) is a complete set of continuous $\mathfrak{Z}$-valued states [4]. Moreover $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for all $A, B \in \mathfrak{A}$ so Tr satisfies condition (1) of Lemma 4.7. If E and F are any projections of $\mathfrak{A}$ we know that the projections $E \cup F - E$ and $F - E \cap F$ are equivalent. Since Tr takes the same value on equivalent projections we have $\mathrm{Tr}(E \cup F) = \mathrm{Tr}(E) + \mathrm{Tr}(F) - \mathrm{Tr}(E \cap F) \leq \mathrm{Tr}(E) + \mathrm{Tr}(F)$. Thus condition (2) of Lemma 4.7 is satisfied and we may conclude from that lemma that the representation of $\mathfrak{A}$ induced by Tr has the desired properties.

THEOREM 4.4. *Let $\mathfrak{B}$ be an AW^*-algebra of type I with center $\mathfrak{Z}$, and $\mathfrak{A}$ an AW^*-subalgebra of $\mathfrak{B}$ which is finite and contains $\mathfrak{Z}$. Then $\mathfrak{A} = \mathfrak{A}''$ in $\mathfrak{B}$.*

Proof. By Theorem 3.1 $\mathfrak{A}$ possesses a complete set of continuous $\mathfrak{Z}$-valued states, and by a theorem of Goldman [4] this is enough to insure that $\mathfrak{A}$ has a central trace. If $\mathfrak{Z}_0$ is the center of A we conclude from Theorem 4.3 that $\mathfrak{A}$ is representable as a commutor in an algebra of type I with center $\mathfrak{Z}_0$, and the result follows from Theorem 4.2.

THEOREM 4.5. *Let $\mathfrak{A}$ be a finite AW^*-algebra with center $\mathfrak{Z}_0$, and let $\mathfrak{Z}$ be a commutative AW^*-subalgebra of $\mathfrak{A}$ containing $\mathfrak{Z}_0$. Then $\mathfrak{Z}$ can be embedded in an algebra of type I with center $\mathfrak{Z}_0$ so that $\mathfrak{Z} = \mathfrak{Z}''$ in this embedding.*

Proof. We can find a map $\mathrm{Tr}: \mathfrak{Z} \to \mathfrak{Z}_0$ which is itself a complete set of continuous $\mathfrak{Z}_0$-valued states on $\mathfrak{Z}$. This is done as follows. If an element $A \in \mathfrak{Z}$ is approximated by $\sum_{i=1}^{n} \lambda_i E_i$, where E_i are orthogonal projections in $\mathfrak{Z}$, $\mathrm{Tr}(A)$

is approximated by $\sum_{i=1}^{n} \lambda_i \; D(E_i)$, where D is the dimension function on the projections of $\mathfrak{A}$. Since $\mathfrak{Z}$ is commutative, there is no difficulty in pushing this through in the obvious way. Note that the continuity of Tr follows from the complete additivity of D on projections. Having verified the properties of Tr, the result follows from Theorems 3.1 and 4.1.

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