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A STRONG SUBADDITIVITY-LIKE INEQUALITY FOR QUANTUM ENTROPY IN SEMIFINITE VON NEUMANN ALGEBRAS

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ABSTRACT. Let $\mathcal{M}$ be a semifinite von Neumann algebra with a normal faithful semifinite trace τ, and let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{R}$ be its subalgebras such that $\mathcal{R} \subset \mathcal{A} \cap \mathcal{B}$ and that τ restricted to any of these subalgebras is semifinite. Denote by $\mathbb{E}_{\mathcal{A}}$, $\mathbb{E}_{\mathcal{B}}$ and $\mathbb{E}_{\mathcal{R}}$ the normal conditional expectations from $\mathcal{M}$ onto $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{R}$, respectively, such that τ is invariant with respect to any of them. The quadruple $\mathcal{M}$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{R}$ is said to be a commuting square if

$$\mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{B}} = \mathbb{E}_{\mathcal{B}}\mathbb{E}_{\mathcal{A}} = \mathbb{E}_{\mathcal{R}}.$$

In this note, we show that the property of being a commuting square is characterised by a sort of the SSA-like (strong subadditivity of entropy) inequality

$$H(\rho|\mathcal{A}) + H(\rho|\mathcal{B}) \leqslant H(\rho) + H(\rho|\mathcal{R})$$

for an arbitrary normal state ρ on $\mathcal{M}$, where $H(\varphi)$ denotes the Segal entropy of the state φ. The situation when we have equality in the inequality above is also investigated, and various equivalent conditions are obtained, in an especially appealing form for finite von Neumann algebras. For such algebras, one more condition is obtained under the assumption of independence of the algebras $\mathcal{A}$ and $\mathcal{B}$.

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INTRODUCTION

In the paper, we show how the notion of Segal entropy in semifinite von Neumann algebras can be used to characterise the property of being a so-called commuting square for a given von Neumann algebra and its subalgebras. This generalises a corresponding result obtained earlier for von Neumann algebras acting on a finite dimensional Hilbert space. The property mentioned above appears in a form of an inequality between entropies similar to the strong subadditivity of entropy inequality, and the situation when this inequality is saturated is also investigated in some detail. The celebrated strong

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subadditivity property for the tensor product of finite von Neumann algebras is shown to follow from these considerations.

1. Preliminaries and notation

Let $\mathcal{M}$ be a semifinite von Neumann algebra with a normal semifinite faithful trace τ, identity $\mathbb{1}$, and predual $\mathcal{M}_*$. The operator norm on $\mathcal{M}$ shall be denoted by $\|\cdot\|_\infty$. By $\mathcal{M}^+$ we shall denote the set of positive operators in $\mathcal{M}$, and by $\mathcal{M}_*^+$ — the set of positive functionals in $\mathcal{M}_*$. These functionals will be referred to as *normal states*. Note that we do not demand that a state be normalised.

For each $\rho \in \mathcal{M}_*$, there is an operator $h \in L^1(\mathcal{M}, \tau)$ such that

$$\rho(x) = \tau(xh) = \tau(hx), \quad x \in \mathcal{M}.$$

The correspondence between $\mathcal{M}_*$ and $L^1(\mathcal{M}, \tau)$ defined above is one-to-one and isometric. Recall that the norm on $L^1(\mathcal{M}, \tau)$, denoted by $\|\cdot\|_1$, is defined as

$$\|h\|_1 = \tau(|h|), \quad h \in L^1(\mathcal{M}, \tau).$$

For a normal state ρ, the corresponding element in $L^1(\mathcal{M}, \tau)^+$ will be denoted by h_ρ and called the *density* of ρ, thus

$$\rho(x) = \tau(xh_\rho) = \tau(h_\rho x) = \tau(h_\rho^{\frac{1}{2}} x h_\rho^{\frac{1}{2}}), \quad x \in \mathcal{M}.$$

Let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$ such that $\tau|\mathcal{N}$ is semifinite. Then there exists a normal conditional expectation $\mathbb{E}\colon \mathcal{M} \to \mathcal{N}$, which is a unital completely positive map, such that

$$\tau \circ \mathbb{E} = \tau.$$

This expectation can also be defined as a map from $L^1(\mathcal{M}, \tau)$ onto $L^1(\mathcal{N}, \tau|\mathcal{N})$, denoted by the same letter, which is again a positive map of $\|\cdot\|_1$-norm one. Of course on the set $\mathcal{M} \cap L^1(\mathcal{M}, \tau)$ these two expectations coincide. Note the following module property of the conditional expectation

$$\mathbb{E}(xh) = x\mathbb{E}h,$$

whenever $x \in \mathcal{N}$, $h \in L^1(\mathcal{M}, \tau)$, and

$$\mathbb{E}(xh) = (\mathbb{E}x)h,$$

whenever $x \in \mathcal{M}$, $h \in L^1(\mathcal{N}, \tau|\mathcal{N})$.

Lemma 1. *Let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$ such that $\tau|\mathcal{N}$ is semifinite. For each normal state ρ on $\mathcal{M}$ and the conditional expectation $\mathbb{E}$ from $\mathcal{M}$ onto $\mathcal{N}$, we have for the densities $h_{\rho\circ\mathbb{E}}$ and $h_{\rho|\mathcal{N}}$ the following formula*

$$h_{\rho\circ\mathbb{E}} = h_{\rho|\mathcal{N}} = \mathbb{E}h_\rho.$$

Proof. For any $x \in \mathcal{M}$ and $h \in L^1(\mathcal{M}, \tau)$, we have

$$\tau((\mathbb{E}x)h) = \tau(\mathbb{E}((\mathbb{E}x)h)) = \tau((\mathbb{E}x)(\mathbb{E}h)),$$

and

$$\tau(x\mathbb{E}h) = \tau(\mathbb{E}(x\mathbb{E}h)) = \tau((\mathbb{E}x)(\mathbb{E}h)),$$

thus

$$\tau((\mathbb{E}x)h) = \tau(x\mathbb{E}h).$$

Consequently, for any $x \in \mathcal{M}$, we have

$$(\rho \circ \mathbb{E})(x) = \rho(\mathbb{E}x) = \tau((\mathbb{E}x)h_\rho) = \tau(x\mathbb{E}h_\rho),$$

which yields

$$h_{\rho\circ\mathbb{E}} = \mathbb{E}h_\rho.$$

For any $x \in \mathcal{N}$, we have

$$(\rho|\mathcal{N})(x) = \rho(x) = \tau(xh_\rho) = \tau(\mathbb{E}(xh_\rho)) = \tau(x\mathbb{E}h_\rho),$$

showing that

$$h_{\rho|\mathcal{N}} = \mathbb{E}h_\rho. \qquad \square$$

2. Segal entropy and information

Let ρ be a normal state on $\mathcal{M}$. The *Segal entropy* $H(\rho)$ of ρ is defined as

$$H(\rho) = \tau(h_\rho \log h_\rho).$$

(In the original Segal definition [17], there is a minus sign before the trace; we choose the version as above for simplicity and in order that $H(\rho)$ be nonnegative for a normalised state and finite unital trace.) For the full algebra $\mathbb{B}(\mathcal{H})$ and the canonical trace tr, the original Segal entropy (i.e. that with a minus sign) coincides with the von Neumann entropy. The notion of entropy can be, in a natural way, defined for $h \in L^1(\mathcal{M}, \tau)^+$, namely,

$$H(h) = \tau(h \log h),$$

thus the entropy of a state is the entropy of its density.

Let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$ such that $\tau|\mathcal{N}$ is semifinite. For each normal state ρ on $\mathcal{M}$ and the conditional expectation $\mathbb{E}$ from $\mathcal{M}$ onto $\mathcal{N}$, we obtain, on account of Lemma 1, the following equality

$$H(\rho|\mathcal{N}) = H(\rho \circ \mathbb{E}). \tag{1}$$

For the normal states ρ and ω on $\mathcal{M}$, the *information* $I(\omega, \rho)$, denoted also by $D(\omega||\rho)$ and called the *quantum relative entropy*, between these states is defined in [20] by the formula

$$I(\omega, \rho) = \tau(h_\omega \log h_\omega - h_\omega \log h_\rho),$$

under the assumption that $\mathrm{s}^{\mathcal{M}}(\omega) \leqslant \mathrm{s}^{\mathcal{M}}(\rho)$. It should be noted that this definition is a little formal, especially for a semifinite and not finite trace, since then the operators $\log h_\omega$ and $\log h_\rho$ need not be even measurable, let alone the relation $h_\omega(\log h_\omega - \log h_\rho) \in L^1(M, \tau)$. The proper formula for the information reads as

$$I(\omega, \rho) = \text{“}\tau(h_\omega \log h_\omega) - \tau(h_\omega \log h_\rho)\text{''} = \omega(\log h_\omega) - \omega(\log h_\rho)$$

with an appropriate definition of $\omega(\log h_\omega)$ and $\omega(\log h_\rho)$, see [11] for a more detailed explanation. In particular, $I(\omega, \rho)$ is well-defined if, for example, ω has finite entropy. Since for a seladjoint x affiliated with $\mathcal{M}$ with the spectral decomposition

$$x = \int_{-\infty}^{+\infty} \lambda\, e(d\lambda),$$

$\omega(x)$ is defined as

$$\omega(x) = \int_{-\infty}^{+\infty} \lambda\, \omega(e(d\lambda)),$$

it is obvious that if x is affiliated with a von Neumann subalgebra $\mathcal{N}$, then

$$(\omega|\mathcal{N})(x) = \omega(x).$$

The following result was proved in [11, Lemma 18] under the assumption that x is measurable. However, this assumption is redundant. For the sake of completeness we repeat the proof here.

Lemma 2. *Let ω be a normal state on $\mathcal{M}$, let x be selfadjoint affiliated with $\mathcal{M}$, and assume that $h_\omega x \in L^1(\mathcal{M}, \tau)$. Then*

$$\omega(x) = \tau(h_\omega x).$$

Proof. For the spectral decomposition

$$x = \int_{-\infty}^{\infty} \lambda\, e(d\lambda),$$

put

$$p_n = e([-n, n]) \uparrow \mathbb{1},$$

and let $x_{[n]}$ be the truncation

$$x_{[n]} = \int_{-n}^{n} \lambda\, e(d\lambda).$$

Let ρ be the normal functional on $\mathcal{M}_*$ having the density $h_\omega x$, i.e. for each $z \in \mathcal{M}$

$$\rho(z) = \tau(h_\omega x z).$$

Then

$$\begin{aligned}\tau(h_\omega x) &= \rho(\mathbb{1}) = \lim_{n\to\infty} \rho(p_n) = \lim_{n\to\infty} \tau(h_\omega x p_n) \\ &= \lim_{n\to\infty} \tau(h_\omega x_{[n]}) = \lim_{n\to\infty} \omega(x_{[n]}) = \omega(x). \qquad \square\end{aligned}$$

From the above lemma, we obtain the following corollary.

Corollary 3. *Let a normal state ρ have finite entropy. Then*
$$H(\rho) = \tau(h_\rho \log h_\rho) = \rho(\log h_\rho).$$

An important fact for the information, proved in [11], is the equality
$$I(\omega,\rho) = S(\rho,\omega),$$
where $S(\rho,\omega)$ is the Araki relative entropy (cf. [1, 2]). Due to this equivalence, we have the following basic properties of the information between states.

Theorem 4. *Let ω and ρ be normal states on a semifinite von Neumann algebra $\mathcal{M}$ such that $\|\omega\| = \omega(\mathbb{1}) = \rho(\mathbb{1}) = \|\rho\|$. Then*

(i) $I(\omega,\rho) \geqslant 0$ *and* $I(\omega,\rho) = 0$ *if and only if* $\omega = \rho$.

(ii) *Let $\mathcal{N}$ be another semifinite von Neumann algebra, and let $\alpha\colon \mathcal{N} \to \mathcal{M}$ be a unital normal Schwarz mapping. Then*
$$I(\omega\circ\alpha,\rho\circ\alpha) \leqslant I(\omega,\rho).$$

(see [12, Chapter 5].) The last property called the *monotonicity of the relative entropy* or the *data processing inequality* is of great importance in all considerations concerning entropy. Some earlier approaches to its proof include [20, Theorem 4] for α being a conditional expectation onto $\mathcal{N}$ and h_ω, h_ρ affiliated with $\mathcal{N}'$, [8, Theorem 1] for $\mathcal{M} = \mathbb{B}(\mathcal{H})$ and α being again a conditional expectation, [1, $$ 6,7,8] for α being a conditional expectation onto some specific subalgebras but with a general notion of relative entropy valid for arbitrary von Neumann algebras. The final solution in full generality was obtained in [19, Proposition 18]. It should be noted that quite recently in the paper [5] it was shown that the assumption of α being a Schwarz map could be weakened to mere positivity.

3. Characterisation of commuting squares

Let us note the following important relation.

Proposition 5. *Let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$ such that $\tau|\mathcal{N}$ is semifinite, let $\mathbb{E}$ be the conditional expectation from $\mathcal{M}$ onto $\mathcal{N}$ such that τ is $\mathbb{E}$-invariant, and let ρ be a normal state on $\mathcal{M}$ such that the entropies of ρ and $\rho\circ\mathbb{E}$ are finite. Then*
$$I(\rho,\rho\circ\mathbb{E}) = H(\rho) - H(\rho\circ\mathbb{E}).$$

Proof. We have
$$I(\rho,\rho\circ\mathbb{E}) = \rho(\log h_\rho) - \rho(\log h_{\rho\circ\mathbb{E}}) = \rho(\log h_\rho) - \rho(\log \mathbb{E}h_\rho).$$
Let
$$\mathbb{E}h_\rho = \int_0^{+\infty} \lambda\, e(d\lambda)$$

be the spectral decomposition of $\mathbb{E}h_\rho$. Then the spectral projections $e(\Delta)$ belong to $\mathcal{N}$, consequently $\mathbb{E}e(\Delta) = e(\Delta)$. Hence

$$\begin{aligned}(\rho \circ \mathbb{E})(\log \mathbb{E}h_\rho) &= \int_0^{+\infty} \log \lambda \, (\rho \circ \mathbb{E})(e(d\lambda)) \\ &= \int_0^{+\infty} \log \lambda \, \rho(e(d\lambda)) = \rho(\log \mathbb{E}h_\rho),\end{aligned}$$

and thus

$$\begin{aligned}I(\rho, \rho \circ \mathbb{E}) =& \rho(\log h_\rho) - \rho(\log \mathbb{E}h_\rho) = \rho(\log h_\rho) - (\rho \circ \mathbb{E})(\log \mathbb{E}h_\rho) \\ =& \rho(\log h_\rho) - (\rho \circ \mathbb{E})(\log h_{\rho\circ\mathbb{E}}) = H(\rho) - H(\rho \circ \mathbb{E}). \qquad \square\end{aligned}$$

Let $\mathcal{M}$ be a semifinite von Neumann algebra with a normal faithful semifinite trace τ, and let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{R}$ be von Neumann subalgebras of $\mathcal{M}$ satisfying the quadrilateral of inclusions

$$\begin{array}{ccc} \mathcal{A} & \subset & \mathcal{M} \\ \cup & & \cup \\ \mathcal{R} & \subset & \mathcal{B} \end{array} \tag{2}$$

Assume that the trace τ restricted to each of these subalgebras is semifinite, and denote by $\mathbb{E}_\mathcal{A}$, $\mathbb{E}_\mathcal{B}$, $\mathbb{E}_\mathcal{R}$ the conditional expectations from $\mathcal{M}$ onto $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{R}$, respectively, such that

$$\tau \circ \mathbb{E}_\mathcal{A} = \tau \circ \mathbb{E}_\mathcal{B} = \tau \circ \mathbb{E}_\mathcal{R} = \tau.$$

The quadrilateral is said to be a *commuting square* if

$$\mathbb{E}_\mathcal{A}\mathbb{E}_\mathcal{B} = \mathbb{E}_\mathcal{B}\mathbb{E}_\mathcal{A} = \mathbb{E}_\mathcal{R}.$$

The following result is a generalisation of the one obtained in [3] for von Neumann algebras acting on a finite dimensional Hilbert space.

Theorem 6. *Let $\mathcal{M}$ be a semifinite von Neumann algebra with a normal faithful semifinite trace τ, and let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{R}$ be von Neumann subalgebras of $\mathcal{M}$ such that the trace τ restricted to each of these subalgebras is semifinite. Assume that these algebras satisfy the quadrilateral of inclusions* (2), *and denote by $\mathbb{E}_\mathcal{A}$, $\mathbb{E}_\mathcal{B}$, $\mathbb{E}_\mathcal{R}$ the conditional expectations from $\mathcal{M}$ onto $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{R}$, respectively, such that*

$$\tau \circ \mathbb{E}_\mathcal{A} = \tau \circ \mathbb{E}_\mathcal{B} = \tau \circ \mathbb{E}_\mathcal{R} = \tau.$$

The algebras $\mathcal{M}$, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{R}$ form a commuting square if and only if for all normal states ρ on $\mathcal{M}$ such that the entropies $H(\rho)$, $H(\rho|\mathcal{A})$, $H(\rho|\mathcal{B})$ and $H(\rho|\mathcal{R})$ are finite the following inequality holds

$$H(\rho|\mathcal{A}) + H(\rho|\mathcal{B}) \leqslant H(\rho) + H(\rho|\mathcal{R}). \tag{3}$$

Proof. Assume first that the algebras $\mathcal{M}$, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{R}$ form a commuting square, and let ρ be an arbitrary normal state on $\mathcal{M}$ such that the entropies $H(\rho)$, $H(\rho|\mathcal{A})$, $H(\rho|\mathcal{B})$ and $H(\rho|\mathcal{R})$ are finite. On account of the equality (1), Proposition 5 and Theorem 4, we get

$$\begin{aligned} H(\rho) - H(\rho|\mathcal{A}) =& H(\rho) - H(\rho \circ \mathbb{E}_{\mathcal{A}}) = I(\rho, \rho \circ \mathbb{E}_{\mathcal{A}}) \\ &\geqslant I(\rho \circ \mathbb{E}_{\mathcal{B}}, \rho \circ \mathbb{E}_{\mathcal{A}} \circ \mathbb{E}_{\mathcal{B}}) = I(\rho \circ \mathbb{E}_{\mathcal{B}}, \rho \circ \mathbb{E}_{\mathcal{R}}) \\ &= I(\rho \circ \mathbb{E}_{\mathcal{B}}, \rho \circ \mathbb{E}_{\mathcal{B}} \circ \mathbb{E}_{\mathcal{A}}) \\ &= H(\rho \circ \mathbb{E}_{\mathcal{B}}) - H(\rho \circ \mathbb{E}_{\mathcal{B}} \circ \mathbb{E}_{\mathcal{A}}) \\ &= H(\rho|\mathcal{B}) - H(\rho \circ \mathbb{E}_{\mathcal{R}}) = H(\rho|\mathcal{B}) - H(\rho|R), \end{aligned} \tag{4}$$

which shows the claim.

Now assume that the inequality (3) holds. It can be written in the form

$$H(\rho \circ \mathbb{E}_{\mathcal{A}}) + H(\rho \circ \mathbb{E}_{\mathcal{B}}) \leqslant H(\rho) + H(\rho \circ \mathbb{E}_{\mathcal{R}})$$

for $\rho \in \mathcal{M}_*^+$.

Let $h_0 \in L^1(\mathcal{M}, \tau)^+$ with finite entropy be of the form

$$h_0 = \int_m^M \lambda\, e(d\lambda) \tag{5}$$

for some $0 < m < M$. By virtue of [9, Theorem 13], such elements are dense in $L^1(\mathcal{M}, \tau)^+$. Let $\rho \in \mathcal{M}_*^+$ have density h_0. Let $\mathbb{E}$ be any of the conditional expectations $\mathbb{E}_{\mathcal{A}}$, $\mathbb{E}_{\mathcal{B}}$ or $\mathbb{E}_{\mathcal{R}}$. Since

$$m\mathbb{1} \leqslant h_0 \leqslant M\mathbb{1},$$

we have

$$m\mathbb{1} \leqslant \mathbb{E}h_0 \leqslant M\mathbb{1}$$

and thus

$$(\log m)\mathbb{1} \leqslant \log \mathbb{E}h_0 \leqslant (\log M)\mathbb{1}.$$

This yields

$$(\log m)\mathbb{E}h_0 \leqslant \mathbb{E}h_0 \log \mathbb{E}h_0 \leqslant (\log M)\mathbb{E}h_0,$$

which implies

$$\begin{aligned} \log m\tau(h_0) = \tau((\log m)\mathbb{E}h_0) &\leqslant \tau(\mathbb{E}h_0 \log \mathbb{E}h_0) \\ &\leqslant \tau((\log M)\mathbb{E}h_0) = \log M\tau(h_0), \end{aligned}$$

showing that the entropy of $\mathbb{E}h_0$ is finite. The inequality (3) can be rewritten in the form

$$H(\mathbb{E}_{\mathcal{A}}h) + H(\mathbb{E}_{\mathcal{B}}h) \leqslant H(h) + H(\mathbb{E}_{\mathcal{R}}h)$$

for every $h \in L^1(\mathcal{M}, \tau)^+$ such that all the entropies in the formula above are finite. Putting $\mathbb{E}_{\mathcal{B}}h_0$ in place of h in this formula, we get

$$H(\mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{B}}h_0) + H(\mathbb{E}_{\mathcal{B}}h_0) \leqslant H(\mathbb{E}_{\mathcal{B}}h_0) + H(\mathbb{E}_{\mathcal{R}}\mathbb{E}_{\mathcal{B}}h_0),$$

i.e.

$$H(\mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{B}}h_0) \leqslant H(\mathbb{E}_{\mathcal{R}}\mathbb{E}_{\mathcal{B}}h_0). \tag{6}$$

Since obviously $\mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{R}} = \mathbb{E}_{\mathcal{R}}$, which follows from the inclusion $\mathcal{R} \subset \mathcal{A}$, we have

$$(\mathbb{E}_{\mathcal{R}}\mathbb{E}_{\mathcal{A}})^2 = \mathbb{E}_{\mathcal{R}}\mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{R}}\mathbb{E}_{\mathcal{A}} = (\mathbb{E}_{\mathcal{R}})^2\mathbb{E}_{\mathcal{A}} = \mathbb{E}_{\mathcal{R}}\mathbb{E}_{\mathcal{A}},$$

which means that $\mathbb{E}_{\mathcal{R}}\mathbb{E}_{\mathcal{A}}$ is a projection. Moreover, $\mathbb{E}_{\mathcal{R}}\mathbb{E}_{\mathcal{A}}(\mathcal{M}) = \mathcal{R}$, and τ is $\mathbb{E}_{\mathcal{R}}\mathbb{E}_{\mathcal{A}}$-invariant, thus the uniqueness of the invariant projection ($\equiv$ conditional expectation) onto $\mathcal{R}$ yields the equality

$$\mathbb{E}_{\mathcal{R}}\mathbb{E}_{\mathcal{A}} = \mathbb{E}_{\mathcal{R}}.$$

Now the inequality (6) can be rewritten in the form

$$H(\mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{B}}h_0) \leqslant H(\mathbb{E}_{\mathcal{R}}\mathbb{E}_{\mathcal{B}}h_0) = H(\mathbb{E}_{\mathcal{R}}\mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{B}}h_0). \tag{7}$$

Denote by φ the normal state with density $h' = \mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{B}}h_0$. The inequality (7) takes the form

$$H(\varphi) = H(h') \leqslant H(\mathbb{E}_{\mathcal{R}}h') = H(\varphi \circ \mathbb{E}_{\mathcal{R}}),$$

and since by virtue of Theorem 4 and Proposition 5 we have $H(\varphi) \geqslant H(\varphi \circ \mathbb{E}_{\mathcal{R}})$, the equality

$$H(h') = H(\mathbb{E}_{\mathcal{R}}h')$$

follows. From [10, Theorem 12 (alternatively Theorem 13)], we obtain the equality

$$\mathbb{E}_{\mathcal{R}}h' = h',$$

i.e.

$$\mathbb{E}_{\mathcal{R}}h_0 = \mathbb{E}_{\mathcal{R}}\mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{B}}h_0 = \mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{B}}h_0.$$

Since the elements h_0 with finite entropy are dense in $L^1(\mathcal{M},\tau)^+$, and the maps $\mathbb{E}_{\mathcal{R}}$ and $\mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{B}}$ are bounded in $\|\cdot\|_1$-norm, we obtain the equality

$$\mathbb{E}_{\mathcal{R}} = \mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{B}}$$

on $L^1(\mathcal{M},\tau)^+$, consequently, on the whole of $L^1(\mathcal{M},\tau)$. Since $\mathcal{M} \cap L^1(\mathcal{M},\tau)$ is σ-weakly dense in $\mathcal{M}$ and the maps $\mathbb{E}_{\mathcal{R}}$ and $\mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{B}}$ are normal, we get the same equality also on $\mathcal{M}$. The equality

$$\mathbb{E}_{\mathcal{R}} = \mathbb{E}_{\mathcal{B}}\mathbb{E}_{\mathcal{A}}$$

is obtained in a similar way. □

Remark. The relation (3) bears some resemblance to the celebrated *strong subadditivity inequality of entropy* in which the algebras $\mathcal{M}$, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{R}$ have a specific form:

$$\mathcal{M} = \mathbb{B}(\mathcal{H}_1)\overline{\otimes}\mathbb{B}(\mathcal{H}_2)\overline{\otimes}\mathbb{B}(\mathcal{H}_3), \qquad \mathcal{A} = \mathbb{B}(\mathcal{H}_1)\overline{\otimes}\mathbb{B}(\mathcal{H}_2)\overline{\otimes}\mathbb{1},$$
$$\mathcal{B} = \mathbb{1}\overline{\otimes}\mathbb{B}(\mathcal{H}_2)\overline{\otimes}\mathbb{B}(\mathcal{H}_3), \qquad \mathcal{R} = \mathbb{1}\overline{\otimes}\mathbb{B}(\mathcal{H}_2)\overline{\otimes}\mathbb{1},$$

and H is the von Neumann entropy (hence the reversed direction in the original inequality). In this form, the inequality was conjectured in [6] and proved in [7]. We shall see this tensor product setup with arbitrary finite von Neumann algebras in the last section.

4. The equality case

In this section, we investigate the interesting question, namely, when we have equality in the inequality (3). In this case the following equality holds (cf. the relation (4))

$$\begin{aligned} I(\rho, \rho \circ \mathbb{E}_{\mathcal{A}}) =& H(\rho) - H(\rho \circ \mathbb{E}_{\mathcal{A}}) = H(\rho) - H(\rho|\mathcal{A}) \\ =& H(\rho|\mathcal{B}) - H(\rho|R) = H(\rho|\mathcal{B}) - H(\rho \circ \mathbb{E}_{\mathcal{R}}) \\ =& H(\rho \circ \mathbb{E}_{\mathcal{B}}) - H(\rho \circ \mathbb{E}_{\mathcal{A}} \circ \mathbb{E}_{\mathcal{B}}) \\ =& I(\rho \circ \mathbb{E}_{\mathcal{B}}, \rho \circ \mathbb{E}_{\mathcal{A}} \circ \mathbb{E}_{\mathcal{B}}) = I(\rho|\mathcal{B}, (\rho \circ \mathbb{E}_{\mathcal{A}})|\mathcal{B}). \end{aligned}$$

According to [13, Theorem 4], this equality is equivalent to the relation

$$[D\rho : D(\rho \circ \mathbb{E}_{\mathcal{A}})]_t = [D(\rho|\mathcal{B}) : D((\rho \circ \mathbb{E}_{\mathcal{A}})|\mathcal{B})]_t \quad \text{for all } t \in \mathbb{R},$$

where $[D\varphi : D\omega]_t$ is the Connes-Radon-Nikodym derivative of the states φ and ω (cf. [18]). Since

$$[D\varphi : D\omega]_t = h_\varphi^{it} h_\omega^{-it}$$

(see e.g. [4, Example 12.15] or [18, Corollary 4.8] and its consequences) we obtain, taking into account Lemma 1,

Theorem 7. *Let $\mathcal{M}$ be a semifinite von Neumann algebra with a normal faithful semifinite trace τ, and let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{R}$ be von Neumann subalgebras of $\mathcal{M}$ such that the trace τ restricted to each of these subalgebras is semifinite. Let ρ be a normal state on $\mathcal{M}$ such that the entropies $H(\rho)$, $H(\rho|\mathcal{A})$, $H(\rho|\mathcal{B})$ and $H(\rho|\mathcal{R})$ are finite. Then the equality*

$$H(\rho|\mathcal{A}) + H(\rho|\mathcal{B}) = H(\rho) + H(\rho|\mathcal{R}) \tag{8}$$

holds if and only if

$$h_\rho^{it}(\mathbb{E}_{\mathcal{A}} h_\rho)^{-it} = (\mathbb{E}_{\mathcal{B}} h_\rho)^{it}(\mathbb{E}_{\mathcal{R}} h_\rho)^{-it} \quad \textit{for all } t \in \mathbb{R}.$$

The condition for equality can be further simplified if we assume that the algebra $\mathcal{M}$ is finite. Then the unitary groups $\left(h_\rho^{it}\right)$, $\left((\mathbb{E}_{\mathcal{A}} h_\rho)^{-it}\right)$, $\left((\mathbb{E}_{\mathcal{B}} h_\rho)^{it}\right)$ and $\left((\mathbb{E}_{\mathcal{R}} h_\rho)^{-it}\right)$ have generators $\log h_\rho$, $-\log(\mathbb{E}_{\mathcal{A}} h_\rho)$, $\log(\mathbb{E}_{\mathcal{B}} h_\rho)$ and $-\log(\mathbb{E}_{\mathcal{R}} h_\rho)$, respectively, which are *measurable* operators, in particular, their common domain is dense. Denoting for simplicity $u_t = h_\rho^{it}$ and $v_t = \left(\mathbb{E}_{\mathcal{A}} h_\rho\right)^{-it}$, we have for ξ belonging to this domain

$$\frac{u_t v_t - \mathbb{1}}{t}\xi = u_t \frac{v_t - \mathbb{1}}{t}\xi + \frac{u_t - \mathbb{1}}{t}\xi \xrightarrow[t \to 0]{} -(\log(\mathbb{E}_{\mathcal{A}} h_\rho))\xi + (\log h_\rho)\xi,$$

and similarly for the other two unitary groups. This yields the equality

$$\log h_\rho - \log(\mathbb{E}_{\mathcal{A}} h_\rho) = \log(\mathbb{E}_{\mathcal{B}} h_\rho) - \log(\mathbb{E}_{\mathcal{R}} h_\rho)$$

or

$$\log(\mathbb{E}_{\mathcal{A}}h_\rho) + \log(\mathbb{E}_{\mathcal{B}}h_\rho) = \log h_\rho + \log(\mathbb{E}_{\mathcal{R}}h_\rho). \tag{9}$$

(Remember that the addition above is performed in the algebra $\widetilde{\mathcal{M}}$ of measurable operators, i.e. it is a *strong* addition which means that $x + y$ is in fact a closure of the sum.)

On the other hand, if the equality (9) holds, then under the assumption of finite entropy we get, multiplying both sides by h_ρ and taking the trace,

$$\tau(h_\rho \log(\mathbb{E}_{\mathcal{A}}h_\rho)) + \tau(h_\rho(\log \mathbb{E}_{\mathcal{B}}h_\rho)) = \tau(h_\rho \log h_\rho) + \tau(h_\rho \log(\mathbb{E}_{\mathcal{R}}h_\rho)),$$

and now it is enough to observe that we have e.g.

$$\begin{aligned}\tau(h_\rho \log(\mathbb{E}_{\mathcal{A}}h_\rho)) &= \tau(\mathbb{E}_{\mathcal{A}}(h_\rho \log(\mathbb{E}_{\mathcal{A}}h_\rho))) \\ &= \tau(\mathbb{E}_{\mathcal{A}}h_\rho \log(\mathbb{E}_{\mathcal{A}}h_\rho)) = H(\rho|\mathcal{A}),\end{aligned}$$

which yields the equality (8). Thus we obtain

Theorem 8. *Let $\mathcal{M}$ be a finite von Neumann algebra with a normal faithful unital trace τ, and let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{R}$ be von Neumann subalgebras of $\mathcal{M}$. Let ρ be a normal state on $\mathcal{M}$ such that the entropies $H(\rho)$, $H(\rho|\mathcal{A})$, $H(\rho|\mathcal{B})$ and $H(\rho|\mathcal{R})$ are finite. Then the equality*

$$H(\rho|\mathcal{A}) + H(\rho|\mathcal{B}) = H(\rho) + H(\rho|\mathcal{R})$$

holds if and only if

$$\log(\mathbb{E}_{\mathcal{A}}h_\rho) + \log(\mathbb{E}_{\mathcal{B}}h_\rho) = \log h_\rho + \log(\mathbb{E}_{\mathcal{R}}h_\rho).$$

Remark. It should be noted that a condition of the type like (9) was obtained in [16] for strong subadditivity of entropy in $\mathbb{B}(\mathcal{H})$ with finite-dimensional $\mathcal{H}$.

In the rest of the paper, we assume that $\mathcal{M}$ is a finite von Neumann algebra with a normal finite faithful unital trace τ.

An interesting situation appears when $\mathcal{R}$ is a trivial algebra which in our situation can be expressed as independence of the algebras $\mathcal{A}$ and $\mathcal{B}$. There are many notions of independence in the setting of operator algebras, we adopt the simplest and, in many respects, the most natural one. Subalgebras $\mathcal{A}$ and $\mathcal{B}$ of a von Neumann algebra $\mathcal{M}$ are said to be *independent* if for any $a \in \mathcal{A}$ and $b \in \mathcal{B}$ we have

$$\tau(ab) = \tau(a)\tau(b).$$

Lemma 9. *The following conditions are equivalent.*

(i) *For every $x \in \mathcal{M}$, the following equality holds*

$$\mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{B}}x = \mathbb{E}_{\mathcal{B}}\mathbb{E}_{\mathcal{A}}x = \tau(x)\mathbb{1}.$$

(ii) *The algebras $\mathcal{A}$ and $\mathcal{B}$ are independent.*

(iii) *$\mathcal{A} \cap \mathcal{B} = \mathbb{C} \cdot \mathbb{1}$ and $\mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{B}} = \mathbb{E}_{\mathcal{B}}\mathbb{E}_{\mathcal{A}}$.*

Proof. Observe first that if $\mathbb{E}$ is a conditional expectation such that $\tau \circ \mathbb{E} = \tau$, then we have for arbitrary $x, y \in \mathcal{M}$

$$\tau((\mathbb{E}x)y) = \tau(\mathbb{E}(\mathbb{E}x)y) = \tau(\mathbb{E}x\mathbb{E}y) = \tau(\mathbb{E}(x\mathbb{E}y)) = \tau(x\mathbb{E}y).$$

(i)$\Longrightarrow$(ii) For arbitrary $a \in \mathcal{A}$ and $b \in \mathcal{B}$ we have

$$\begin{aligned}\tau(ab) =&\tau(\mathbb{E}_{\mathcal{A}}(ab)) = \tau(a\mathbb{E}_{\mathcal{A}}b)\\ &=\tau(a\mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{B}}b) = \tau(a\tau(b)\mathbb{1}) = \tau(a)\tau(b),\end{aligned}$$

thus $\mathcal{A}$ and $\mathcal{B}$ are independent.

(ii)$\Longrightarrow$(iii) For arbitrary $y \in \mathcal{M}$, we have

$$\begin{aligned}&\tau(y(\mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{B}}x - \tau(x)\mathbb{1})) = \tau(y\mathbb{E}_{\mathcal{A}}(\mathbb{E}_{\mathcal{B}}x - \tau(x)\mathbb{1}))\\ =&\tau(\mathbb{E}_{\mathcal{A}}y(\mathbb{E}_{\mathcal{B}}x - \tau(x)\mathbb{1})) = \tau(\mathbb{E}_{\mathcal{A}}y\mathbb{E}_{\mathcal{B}}(x - \tau(x)\mathbb{1}))\\ =&\tau(\mathbb{E}_{\mathcal{A}}y)\tau(\mathbb{E}_{\mathcal{B}}(x - \tau(x)\mathbb{1})) = \tau(y)\tau(x - \tau(x)\mathbb{1}) = 0\end{aligned}$$

showing that

$$\mathbb{E}_{\mathcal{A}}\mathbb{E}_{\mathcal{B}}x - \tau(x)\mathbb{1} = 0.$$

In the same way we obtain the equality

$$\mathbb{E}_{\mathcal{B}}\mathbb{E}_{\mathcal{A}}x = \tau(x)\mathbb{1}.$$

Let p be a projection in $\mathcal{A} \cap \mathcal{B}$. Then

$$\tau(p) = \tau(p^2) = \tau(p)\tau(p),$$

thus $\tau(p)$ equals 0 or 1. It follows that $p = 0$ or $p = \mathbb{1}$ which means that in the algebra $\mathcal{A} \cap \mathcal{B}$ there are only trivial projections, consequently, (iii) follows.

(iii)$\Longrightarrow$(i) It follows that

$$\mathbb{E}_{\mathcal{A}}(\mathbb{E}_{\mathcal{B}}(\mathcal{M})) = \mathbb{E}_{\mathcal{B}}(\mathbb{E}_{\mathcal{A}}(\mathcal{M})) = \mathcal{A} \cap \mathcal{B} = \mathbb{C} \cdot \mathbb{1},$$

thus for every $x \in \mathcal{M}$ we have

$$\mathbb{E}_{\mathcal{A}}(\mathbb{E}_{\mathcal{B}}x) = \alpha(x)\mathbb{1},$$

and since τ is $\mathbb{E}_{\mathcal{A}}$- and $\mathbb{E}_{\mathcal{B}}$-invariant, we get, applying τ to both sides of the equality above, $\alpha(x) = \tau(x)$. □

From the lemma above, it follows that for subalgebras $\mathcal{A}$ and $\mathcal{B}$ their independence is equivalent to the fact that the quadrilateral $\mathcal{M}$, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{R}$ forms a commuting square with $\mathcal{R} = \mathcal{A} \cap \mathcal{B} = \mathbb{C} \cdot \mathbb{1}$. In such a case, for an arbitrary normal state ρ we have

$$\mathbb{E}_{\mathcal{R}}h_{\rho} = \tau(h_{\rho})\mathbb{1}.$$

Moreover, for the density $h_{\rho|\mathcal{R}}$ we have

$$h_{\rho|\mathcal{R}} = \rho(\mathbb{1})\mathbb{1},$$

hence

$$H(\rho|\mathcal{R}) = \rho(\mathbb{1}) \log \rho(\mathbb{1}),$$

and in this case the equality (8) takes the form

$$H(\rho|\mathcal{A}) + H(\rho|\mathcal{B}) = H(\rho) + \rho(\mathbb{1})\log\rho(\mathbb{1}) \tag{10}$$

Theorem 10. *Let subalgebras $\mathcal{A}$ and $\mathcal{B}$ of the algebra $\mathcal{M}$ be independent. Then the equality* (10) *holds for a normal state ρ if and only if*

$$\tau(h_\rho)h_\rho = (\mathbb{E}_\mathcal{A}h_\rho)(\mathbb{E}_\mathcal{B}h_\rho) = (\mathbb{E}_\mathcal{B}h_\rho)(\mathbb{E}_\mathcal{A}h_\rho).$$

Proof. On account of Theorem 7, the equality (8) holds if and only if

$$h_\rho^{it}(\mathbb{E}_\mathcal{A}h_\rho)^{-it} = (\mathbb{E}_\mathcal{B}h_\rho)^{it}(\mathbb{E}_\mathcal{R}h_\rho)^{-it} \quad \text{for all } t \in \mathbb{R}.$$

which in our case amounts to

$$h_\rho^{it}(\mathbb{E}_\mathcal{A}h_\rho)^{-it} = \tau(h_\rho)^{-it}(\mathbb{E}_\mathcal{B}h_\rho)^{it} \quad \text{for all } t \in \mathbb{R},$$

that is

$$\left(\tau(h_\rho)h_\rho\right)^{it} = (\mathbb{E}_\mathcal{B}h_\rho)^{it}(\mathbb{E}_\mathcal{A}h_\rho)^{it} \quad \text{for all } t \in \mathbb{R}.$$

Since on the left-hand side of the equality above we have a unitary group, it follows that the two unitary groups on the right-hand side commute. Indeed, let (u_t), (v_t) and (w_t) be unitary groups such that

$$u_t = v_t w_t, \quad t \in \mathbb{R}.$$

Then

$$u_{t+s} = u_t u_s = v_t w_t v_s w_s.$$

On the other hand,

$$u_{t+s} = v_{t+s}w_{t+s} = v_t v_s w_t w_s,$$

which together with the equality above yields

$$v_t v_s w_t w_s = v_t w_t v_s w_s,$$

thus, in particular.

$$v_s w_t = w_t v_s.$$

Now for the unitary groups

$$(\mathbb{E}_\mathcal{B}h_\rho)^{it} = e^{it\log(\mathbb{E}_\mathcal{B}h_\rho)}, \quad (\mathbb{E}_\mathcal{A}h_\rho)^{it} = e^{it\log(\mathbb{E}_\mathcal{A}h_\rho)}$$

their generators are $\log(\mathbb{E}_\mathcal{B}h_\rho)$ and $\log(\mathbb{E}_\mathcal{A}h_\rho)$, respectively, and commuting of the groups means commuting of the spectral measures of their generators. Thus there exists a spectral measure m and Borel functions f, g such that

$$\log(\mathbb{E}_\mathcal{B}h_\rho) = \int_{-\infty}^{\infty} f(\lambda)\, m(d\lambda), \quad \log(\mathbb{E}_\mathcal{A}h_\rho) = \int_{-\infty}^{\infty} g(\lambda)\, m(d\lambda),$$

so

$$\mathbb{E}_\mathcal{B}h_\rho = \int_{-\infty}^{\infty} e^{f(\lambda)}\, m(d\lambda), \quad \mathbb{E}_\mathcal{A}h_\rho = \int_{-\infty}^{\infty} e^{g(\lambda)}\, m(d\lambda).$$

Consequently,

$$\begin{aligned}(\mathbb{E}_{\mathcal{B}}h_\rho)(\mathbb{E}_{\mathcal{A}}h_\rho) &= \int_{-\infty}^{\infty} e^{f(\lambda)}e^{g(\lambda)}\, m(d\lambda) \\ &= \int_{-\infty}^{\infty} e^{g(\lambda)}e^{f(\lambda)}\, m(d\lambda) = (\mathbb{E}_{\mathcal{A}}h_\rho)(\mathbb{E}_{\mathcal{B}}h_\rho).\end{aligned}$$

(Again remember that the multiplication above is performed in the algebra $\widetilde{\mathcal{M}}$ of measurable operators, i.e. it is a *strong* multiplication which means that $(\mathbb{E}_{\mathcal{B}}h_\rho)(\mathbb{E}_{\mathcal{A}}h_\rho)$ is in fact a closure of the product.) Hence

$$\big(\tau(h_\rho)h_\rho\big)^{it} = \big((\mathbb{E}_{\mathcal{B}}h_\rho)(\mathbb{E}_{\mathcal{A}}h_\rho)\big)^{it} = \big((\mathbb{E}_{\mathcal{A}}h_\rho)(\mathbb{E}_{\mathcal{B}}h_\rho)\big)^{it} \quad \text{for all } t \in \mathbb{R},$$

which shows the claim. □

5. An application

As an application of Theorem 6, the strong subadditivity theorem for the tensor product of finite von Neumann algebras can be obtained. Let $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$ be von Neumann algebras with normal faithful finite unital traces τ_1, τ_2, τ_3, respectively. Define maps

$$\begin{aligned}\pi_{12}&\colon (\mathcal{M}_1\overline{\otimes}\mathcal{M}_2\overline{\otimes}\mathcal{M}_3)_* \to (\mathcal{M}_1\overline{\otimes}\mathcal{M}_2)_*,\\ \pi_{23}&\colon (\mathcal{M}_1\overline{\otimes}\mathcal{M}_2\overline{\otimes}\mathcal{M}_3)_* \to (\mathcal{M}_2\overline{\otimes}\mathcal{M}_3)_*,\\ \pi_{2}&\colon (\mathcal{M}_1\overline{\otimes}\mathcal{M}_2\overline{\otimes}\mathcal{M}_3)_* \to (\mathcal{M}_2)_*,\end{aligned}$$

by the formulae

$$\begin{aligned}(\pi_{12}\rho_{123})(x_{12}) &= \rho_{123}(x_{12}\otimes \mathbb{1}), \quad x_{12}\in \mathcal{M}_1\overline{\otimes}\mathcal{M}_2,\\ (\pi_{23}\rho_{123})(x_{23}) &= \rho_{123}(\mathbb{1}\otimes x_{23}), \quad x_{23}\in \mathcal{M}_2\overline{\otimes}\mathcal{M}_3,\\ (\pi_{2}\rho_{123})(x_{2}) &= \rho_{123}(\mathbb{1}\otimes x_2\otimes \mathbb{1}), \quad x_{2}\in \mathcal{M}_2,\end{aligned}$$

where $\rho_{123} \in (\mathcal{M}_1\overline{\otimes}\mathcal{M}_2\overline{\otimes}\mathcal{M}_3)_*$. (These maps are counterparts of partial traces.) For simplicity, denote

$$\pi_{12}\rho_{123} = \rho_{12}, \quad \pi_{23}\rho_{123} = \rho_{23}, \quad \pi_{2}\rho_{123} = \rho_{2}.$$

Let

$$\begin{aligned}\mathbb{E}_{12}&\colon \mathcal{M}_1\overline{\otimes}\mathcal{M}_2\overline{\otimes}\mathcal{M}_3 \to \mathcal{M}_1\overline{\otimes}\mathcal{M}_2\overline{\otimes}\mathbb{1},\\ \mathbb{E}_{23}&\colon \mathcal{M}_1\overline{\otimes}\mathcal{M}_2\overline{\otimes}\mathcal{M}_3 \to \mathbb{1}\overline{\otimes}\mathcal{M}_2\overline{\otimes}\mathcal{M}_3,\\ \mathbb{E}_{2}&\colon \mathcal{M}_1\overline{\otimes}\mathcal{M}_2\overline{\otimes}\mathcal{M}_3 \to \mathbb{1}\overline{\otimes}\mathcal{M}_2\overline{\otimes}\mathbb{1},\end{aligned}$$

be defined for $x_1 \in \mathcal{M}_1$, $x_2 \in \mathcal{M}_2$, $x_3 \in \mathcal{M}_3$ as

$$\begin{aligned}\mathbb{E}_{12}(x_1\otimes x_2\otimes x_3) &= \tau_3(x_3)x_1\otimes x_2\otimes \mathbb{1},\\ \mathbb{E}_{23}(x_1\otimes x_2\otimes x_3) &= \tau_1(x_1)\mathbb{1}\otimes x_2\otimes x_3\\ \mathbb{E}_{2}(x_1\otimes x_2\otimes x_3) &= \tau_1(x_1)\tau_3(x_3)\mathbb{1}\otimes x_2\otimes \mathbb{1}.\end{aligned}$$

Then $\mathbb{E}_{12}$, $\mathbb{E}_{23}$, and $\mathbb{E}_2$ are normal conditional expectations such that

$$\begin{aligned}(\tau_1 \otimes \tau_2 \otimes \tau_3) \circ \mathbb{E}_{12} &= (\tau_1 \otimes \tau_2 \otimes \tau_3) \circ \mathbb{E}_{23} \\ &= (\tau_1 \otimes \tau_2 \otimes \tau_3) \circ \mathbb{E}_2 = \tau_1 \otimes \tau_2 \otimes \tau_3.\end{aligned}$$

For arbitrary normal state ρ_{123} on $\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2 \overline{\otimes} \mathcal{M}_3$, and arbitrary $x_1 \in \mathcal{M}_1$, $x_2 \in \mathcal{M}_2$, we have

$$\begin{aligned}&\tau_1 \otimes \tau_2 \otimes \tau_3((\mathbb{E}_{12} h_{\rho_{123}}) x_1 \otimes x_2 \otimes \mathbb{1}) = \tau_1 \otimes \tau_2 \otimes \tau_3(h_{\rho_{123}}(x_1 \otimes x_2 \otimes \mathbb{1})) \\ =&\rho_{123}(x_1 \otimes x_2 \otimes \mathbb{1}) = \rho_{12}(x_1 \otimes x_2) = \tau_1 \otimes \tau_2(h_{\rho_{12}}(x_1 \otimes x_2)) \\ =&\tau_1 \otimes \tau_2 \otimes \tau_3((h_{\rho_{12}} \otimes \mathbb{1})(x_1 \otimes x_2 \otimes \mathbb{1})),\end{aligned}$$

which shows that

$$\mathbb{E}_{12} h_{\rho_{123}} = h_{\rho_{12}} \otimes \mathbb{1}.$$

Assume that the entropy $H(\rho_{12})$ is finite. Then

$$\begin{aligned}&H(\rho_{12}) = \rho_{12}(\log h_{\rho_{12}}) = \rho_{123}(\log h_{\rho_{12}} \otimes \mathbb{1}) = \rho_{123}(\log(h_{\rho_{12}} \otimes \mathbb{1}) \\ =&\rho_{123}(\log(\mathbb{E}_{12} h_{\rho_{123}})) = (\rho_{123} \circ \mathbb{E}_{12})(\log(\mathbb{E}_{12} h_{\rho_{123}})) \\ =&(\rho_{123} \circ \mathbb{E}_{12})(\log h_{\rho_{123} \circ \mathbb{E}_{12}}) = H(\rho_{123} \circ \mathbb{E}_{12}).\end{aligned}$$

Analogously we obtain, under the assumption of finiteness of $H(\rho_{23})$ and $H(\rho_2)$, the equalities

$$H(\rho_{23}) = H(\rho_{123} \circ \mathbb{E}_{23}),$$

and

$$H(\rho_2) = H(\rho_{123} \circ \mathbb{E}_2).$$

Now Theorem 6 with

$$\begin{aligned}\mathcal{M} &= \mathcal{M}_1 \overline{\otimes} \mathcal{M}_2 \overline{\otimes} \mathcal{M}_3, & \mathcal{A} &= \mathcal{M}_1 \overline{\otimes} \mathcal{M}_2 \overline{\otimes} \mathbb{1}, \\ \mathcal{B} &= \mathbb{1} \overline{\otimes} \mathcal{M}_2 \overline{\otimes} \mathcal{M}_3, & \mathcal{R} &= \mathbb{1} \overline{\otimes} \mathcal{M}_2 \overline{\otimes} \mathbb{1},\end{aligned}$$

gives the inequality

$$H(\rho_{12}) + H(\rho_{23}) \leqslant H(\rho_{123}) + H(\rho_2),$$

which is the strong subadditivity of entropy.

Remark. Note that the assumption of finiteness of the algebras $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$ is essential here since for semifinite algebras the trace $\tau_1 \otimes \tau_2 \otimes \tau_3$ restricted to the subalgebra $\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2 \overline{\otimes} \mathbb{1}$ (or $\mathbb{1} \overline{\otimes} \mathcal{M}_2 \overline{\otimes} \mathcal{M}_3$ or $\mathbb{1} \overline{\otimes} \mathcal{M}_2 \overline{\otimes} \mathbb{1}$) need not be semifinite. The general strong subadditivity theorem for semifinite algebras is proved in [15].

It should be noted that the above result follows also from [14, Theorem 12] which proves an inequality analogous to (3) for the relative entropy $S(\varphi, \omega)$ in the C*-algebra setting, upon taking there $\mathcal{A} = \mathcal{M}_1 \overline{\otimes} \mathcal{M}_2 \overline{\otimes} \mathcal{M}_3$, $\varphi = \tau_1 \otimes \tau_2 \otimes \tau_3$, $\omega = \rho_{123}$, and observing that for the relative entropy we have $S(\varphi, \omega) = H(\rho_{123})$; (warning: the notation in [14] differs from ours). The relation between the relative entropy $S(\rho, \varphi)$, which in the semifinite case is defined as

$$S(\rho, \varphi) = \tau(h_\rho \log h_\rho - h_\rho \log h_\varphi),$$

and Segal's entropy $H(\rho)$, given by the formula

$$H(\rho) = S(\rho, \tau),$$

holds only for finite algebras with a normal faithful unital trace τ.

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