Comparison of quantum channels with superchannels, by Gilad Gour

Referee report

This paper can be divided into two parts. The first part discusses the superchannels, that is, transformations on the space of maps that are completely positive and transform channels into channels. The structure of superchannels is described and several subclasses are investigated, such as the random unitary superchannels and more general doubly stochastic superchannels. These results are then applied in the main part, where entropies of quantum channels are defined axiomatically. Two particular entropy functions are studied, extending the min-entropy and conditional min entropy for states. The extended conditional min-entropy is used in a characterization of a majorization preorder for bipartite quantum channels, defined by the property that one channel is related to the other by a superchannel acting on one subsystem.

These results generalize previously obtained results for bipartite quantum states [11] and are expected to have similar applications, in particular in resource theories for quantum channels. It is also proved that the relevant values can be formulated as an SDP problem and therefore can be computed efficiently. I find the results very interesting and important and strongly recommend publication in IEEE Transactions on Information Theory, after dealing with some small issues described below.

General remarks

- Mistakes: there is a number of them. These are not serious, mostly just typos, but can be confusing and make the reading more difficult, in particular when one needs to keep track of the various input and output spaces. Some of these are listed below, but the author should carefully re-read and check the paper.

- Proofs: it often happens that in a proof of equivalence of two statements, only one direction is proved. It would be better to say at least that the converse is easy. Or, better, add something like "The converse follows from the form of the Choi matrix", etc.

- Notations: it happens at some places that the same symbols denote different things, even within the same proof on the same page, see the detailed comments below.

- A question: is there an operational interpretation to the conditions in Theorem 7 or Corollary 1?

- This last point is more a suggestion to the author. When describing the spaces of maps and supermaps, it might be useful to stress that the Choi matrix provides an isomorphism between the spaces $\mathcal{L}^A$ and $B(\mathcal{H}^A)$, in fact a unitary with respect to the given inner products. The Choi isomorphism, say J, and its inverse, can be interpreted as bending of some of the input or output wires. If Θ is a supermap, then the map $\Delta_\Theta : B(\mathcal{H}^A) \to B(\mathcal{H}^B)$ is the lifting $\Delta_\Theta = J\Theta J^*$ of Θ. This map is in $\mathcal{L}^{AB}$ and it is very natural to consider its Choi matrix J_Θ^{AB} in $B(\mathcal{H}^{AB})$. Some of the results proved in the preliminary section II.B follow just from the usual properties of the Choi isomorphism. J_Θ^{AB} is also the Choi matrix of other maps, like Λ_Θ and Γ_Θ, specified by the choice of input and output spaces, all such maps are related by some bending wires isomorphisms. It might be even useful to adopt some diagrammatic language such as the string diagrams used e.g. in arXiv:1701.04732.

More specific remarks

1. title: the title seems to suggest that quantum channels are compared with superchannels. Better replace "with" by "by" or "via".
2. p. 3., line 3 in Sec. II.A: "quipped" ... equipped;
3. p. 3, line 4 in Sec. II.A: strange word order in the sentence beginning with: "In this paper, quite often..."
4. p. 4: it seems that the adjoint is denoted by *, but a † appears in Eq. (5);
5. p. 8, line 10: "purification"... purification of;
6. p. 9, last line of the 3. block of displayed equations: $\Delta_\Theta^{\tilde{A}\to B_0}$ should be $\Delta_\Theta^{\tilde{A}_0\to B_0}$;
7. p. 11: it is better to add some comments to the claim in lines 7-8;
8. p. 13, line 11 from below: "the marginals"..."satisfies" ... satisfy;
9. p 14. Thm. 2: better skip the condition (45), since this is the same as the conditions (19), characterizing superchannels (as it is stated in Remark 2). Here the condition is superfluous and confusing;
10. p. 15, line 4 (the proof of Thm. 2): "(recall that $d_{A_1} = d_{B_1}$)"... here the Eq. (44) is assumed, so this equality may not hold. It is also not needed: it seems that it would be enough to replace I^{B_1} by u^{B_1} and I^{A_1} by u^{A_1} in the displayed equations below;
11. p. 15, line 11 (the proof of Thm. 2): It is claimed that equivalence between the equality $J_\Theta^{A_0B_1} = I^{A_0B_1}$ and condition (48) is proved. To be precise, this equivalence is not proved, since the derivation in Eq. (51) uses also the condition (47);
12. p. 15, last equality of (51): one extra $\otimes$;
13. p. 16, first displayed eq.: this form suggests that σ^{C_1} is fixed for all ρ, which is obviously not the case;
14. p. 17, first line below Eq. (55) (proof of Thm. 3): "We show that..." add also ..."and for all systems C...", this is also used below;
15. p. 17, line 2 line below Eq. (55) (and one more place): M^{A_0C} turns into $M^{R_0B_0}$

16. p.22, line 8: better stress that Λ_{Θ}^{AB} is a channel if Θ is a superchannel (e.g. write something like "Note that...")

17. p. 24, lines 8 and 20: "the conditions follows"... follow;

18. p. 24, line 12: $2^{-H_{min}^{ext}(BB'|AA')_{\Omega}} \geq$ ... here $\otimes\Gamma$ is missing. It also seems to me that the inequality should be opposite, since *min* on the RHS is over a smaller set;

19. p. 24, line 10 from below (displayed equation): similarly as above, the inequality should be opposite, here *max* is taken over a smaller set. (So the proof of equality is saved);

20. p. 26, line 1: "over all bipartite pure state"... states

21. p. 28, line 2 from below: "contraction"... this expression for a divergence seems strange to me (I would rather say that Φ is a contraction w.r. to D);

22. p. 31, line 9 from below: $\mathcal{E}_{xy}^{B_0\to B_1}$ better find some different notation, this was used for some basis elements before (Eq. (6));

23. p. 32, line 14: $\tilde{\Lambda}^{XYZ}$ ... Z instead of B;

24. p. 32, last line: $\tilde{\Lambda}^{XYB}$ here denotes something different than just half a page before, in the same proof, different notation is needed;

25. p. 33, last displayed equation: one of the Λ_1^R's should perhaps be Λ_2^R?

26. p. 35, line 7: here F_j denotes something else than just 2 lines above. Also, it would be enough to take $\{E_j\}$ to be a basis of $B_h(\mathcal{H}^{AB})$.