Fumio Hiai: Quantum f-divergences in von Neumann Algebras. Reversibility of Quantum Operations

Referee report.

Quantum information theory experienced a rapid development mainly in the last two decades. A significant role in the theory is played by quantum divergences. These quantities have numerous applications in various tasks, such as state discrimination or channel coding, and also provide an important theoretical tool. The literature describing these developments, including various types of quantum divergences, is quite large. However, most of these works are concentrated in the finite dimensional or matrix algebra setting, whereas it is understood that the framework for description of most general quantum systems is provided by von Neumann algebras and their normal states. In this setting, the quantum divergences are less covered in the literature, especially in the more recent works.

The present monograph fills this gap by providing a comprehensive and detailed study of several versions of quantum f-divergences in von Neumann algebras. Apart from the standard quantum f-divergence based on the relative modular operator (similar to the definition of Araki relative entropy or Petz quasi-entropy), two more versions are studied: the maximal and the minimal, or measured, quantum f-divergence. A separate chapter is devoted to extensions of two important quantum versions of the Rényi divergences (standard and sandwiched) from finite dimensions to von Neumann algebras. The last three chapters contain a variety of characterizations of the situations when the data processing inequality becomes an equality (i.e. when the divergence is preserved by a quantum channel), which in the strongest cases is equivalent to reversibility of the channel with respect to the given pair of states. Some parts of the monograph (Chaps. 2 and 4) survey very recently published results by the author, some parts (Chaps. 3 and 6) review, extend and complete previously known results, Chapters 5, 7 and 8 discuss results that are new and appear here for the first time. The monograph also contains four Appendices where the necessary technical tools are introduced and developed.

The material of the monograph is technically rather involved. One of the most important tools is the Tomita-Takesaki modular theory, which is unavoidable in the general von Neumann algebra setting, as well as the theory of noncommutative L_p-spaces. A widely used technique is also the application of integral representations of operator convex/monotone functions. The Appendices provide concise but well written accounts on these techniques and serve as a good reference to the various results applied throughout the text. Nevertheless, some background especially in operator algebra theory is necessary, so the monograph is less accessible to a wide readership in quantum information theory.

I am convinced this book will become an invaluable resource for anyone working on quantum information theory in the von Neumann algebra setting. Quantum f-divergences and their properties are studied in detail and alternative definitions, integral formulas and variational expressions are provided, which is useful for various applications and theoretical developments. The content is of interest also for readers from operator algebra theory.

In conclusion, the proposed monograph gives a detailed account on fundamental concepts of quantum information theory, in a setting not sufficiently covered by the existing literature. It is my pleasure to highly recommend it for publication.

Although the text is written rather carefully, I spotted a few mistakes and typos, listed among the minor comments below.

Specific comments:

- Sec. 3.3: the index α is missing in $\tilde{D}_\alpha$ and $\tilde{D}_\alpha$ in some places in Sec. 3.3;
- Lemma 4.19: φ_t should be σ_t;
- Lemma 5.6: it seems that s should be replaced by t (and vice versa) in some of the expressions $s \in (0, \infty)$ in displayed equations;
- page 78: it seems that S_{h_s} should be S_{g_s} and S_{φ_s} should be S_{h_s}
- Chapter 7: some introduction is needed at the beginning of this chapter, explaining what is going to happen;
- Lemma 7.1: some more explicit comments on the relation to the results in [95] and [88] are needed, also for other results in this chapter;
- p. 93, first line of Sec. 7.2: a parenthesis (is missing;
- p. 95, the statement of Lemma 7.4, last line: the full stop between "unitary" and "Them" should be a comma;
- p. 97, last line: $S_f(\alpha \| \varphi)$ should be $S_f(\psi \| \varphi)$;
- p. 99, line 6: $L^1(M)_+$ should be $L^1(M)$;
- p. 99, before Definition 8.2, maybe recall the definition of an operator connection here, or refer to Appendix D;
- Definition 8.2: we have to assume again that $(\psi, \varphi) \in (M_*^+ \times M_*^+)_\leq$;
- page 99-100: it would be good to give the relation to $\hat{S}_f$ (Prop. D.11) somewhere before Thm. 8.4
- page 103, Eq. (8.7): s is missing on the left hand side;
- I found some strange English phrases, like "definition is well defined" p. 36, "question holds true" p. 85, "problem holds true" p. 98, p. 107