Rényi relative entropies and noncommutative L_p-spaces II

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Let $\mathcal{M}$ be a von Neumann algebra and $L_p(\mathcal{M})$, the Haagerup L_p-spaces, $1 \leq p \leq \infty$. We will (mostly) work in the standard representation

$$(\lambda(\mathcal{M}), L_2(\mathcal{M}), J, L_2(\mathcal{M})^+). \tag{1}$$

1 Interpolation norms in $L_2(\mathcal{M})$

Let $\varphi \in \mathcal{M}_*^+$ be faithful. Let us consider the continuous positive embedding $\mathcal{M} \to L_2(\mathcal{M})$ by

$$x \mapsto h_\varphi^{1/2} x, \qquad x \in \mathcal{M}.$$

The image of $\mathcal{M}$ is dense in $L_2(\mathcal{M})$. Using this embedding, we define the interpolation spaces

$$L_p^2(\mathcal{M}, \varphi) := C_{2/p}(\mathcal{M}, L_2(\mathcal{M})), \qquad 2 \leq p < \infty.$$

Let us denote the norm in this space by $\|\cdot\|_{p,\varphi}^{BST}$, the reason for this notation will become clear later. The folowing polar decomposition in $L_p^2(\mathcal{M}, \varphi)$ is easily proved using the results of Kosaki [6] and the fact that the map

$$i_2^R : L_2(\mathcal{M}) \ni \xi \mapsto h_\varphi^{1/2} \xi \in L_1(\mathcal{M})$$

provides an isometric isomorphism of $L_p^2(\mathcal{M}, \varphi)$ onto the space $L_p(\mathcal{M}, \varphi)^R$, defined therein (ref).

Theorem 1. *Let $\xi \in L_2(\mathcal{M})$. Then $\xi \in L_p^2(\mathcal{M}, \varphi)$ if and only if there is some $\mu \in \mathcal{M}_*^+$ and a partial isometry $u \in \mathcal{M}$ with $uu^* = s(\mu)$ such that*

$$\xi = h_\varphi^{1/2-1/p} h_\mu^{1/p} u.$$

Moreover, in this case, μ and u are unique and $\|\xi\|_{p,\varphi}^{BST} = \mu(1)^{1/p}$.

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Let us now define for all $\xi \in L_2(\mathcal{M})$:

$$\|\xi\|_{p,\varphi}^{BST} := \begin{cases} \|\xi\|_{p,\varphi}^{BST}, & \text{if } \xi \in L_p^2(\mathcal{M},\varphi) \\ \infty, & \text{otherwise.} \end{cases}$$

As remarked in [4], this norm coincides with the norm defined in [**?**] by the variational formula

$$\|\xi\|_{p,\varphi}^{BST} = \sup_{\zeta \in L_2(\mathcal{M}), \|\zeta\|_2=1} \|\Delta(\zeta/\varphi)^{1/2-1/p}\xi\|_2 = \sup_{\omega \in \mathfrak{S}_*(\mathcal{M})} \|\Delta_{\omega,\varphi}^{1/2-1/p}\xi^*\|_2.$$

The norm can be extended to non-faithful functionals φ by restriction to the support. More precisely, let $e = s(\varphi)$ and let

$$\varphi_0 := \varphi + \sigma \tag{2}$$

where $\sigma \in \mathcal{M}_*^+$ is any functional such that $s(\sigma) = 1 - e$. We then define

$$\|\xi\|_{p,\varphi}^{BST} := \begin{cases} \|\xi\|_{p,\varphi_0}^{BST}, & \text{if } e\xi = \xi \\ \infty, & \text{otherwise,} \end{cases}$$

this again agrees with the definition in [**?**]. We also have a unique polar decomposition in this case.

Proposition 2. *Let $\varphi \in \mathcal{M}_*^+$, $s(\varphi) = e$ and $\xi \in L_2(\mathcal{M})$. Then $\|\xi\|_{p,\varphi}^{BST} < \infty$ if and only if*

$$\xi = h_\varphi^{1/2-1/p} h_\mu^{1/p} u$$

for some $\mu \in \mathcal{M}_*^+$ with $s(\mu) \leq e$ and a partial isometry $u \in \mathcal{M}$ such that $u^*u = s(\omega_{\xi^*})$ and $uu^* = s(\mu)$. Moreover, such μ and u are unique and we have $\|\xi\|_{p,\varphi}^{BST} = \mu(1)^{1/p}$.

Proof. Assume $\|\xi\|_{p,\varphi}^{BST} < \infty$, then we must have $e\xi = \xi$ and $\|\xi\|_{p,\varphi_0}^{BST} = \|\xi\|_{p,\varphi}^{BST} < \infty$. Since φ_0 is faithful, we have a polar decomposition as in Theorem 1, with $\mu \in \mathcal{M}_*^+$ and a partial isometry $u \in \mathcal{M}$ such that $u^*u = s(\omega_{\xi^*})$, $uu^* = s(\mu)$ and $\|\xi\|_{p,\varphi_0}^{BST} = \mu(1)^{1/2}$. From $\xi = e\xi$, we obtain

$$h_{\varphi_0}^{1/2-1/p} h_\mu^{1/p} u = e h_{\varphi_0}^{1/2-1/p} h_\mu^{1/p} u = h_\varphi^{1/2-1/p} h_\mu^{1/p} u$$

which implies that $h_\sigma^{1/2-1/p} h_\mu^{1/p} = 0$. Notice that the function

$$f : \{z \in \mathbb{C}, 0 \leq Re(z) \leq 1/p\} \ni z \mapsto h_\sigma^{1/2-z} h_\mu^z \in L_2(\mathcal{M})$$

is bounded and continuous and analytic on the interior of the strip, and we have $f(1/p + it) = 0$ for all $t \in \mathbb{R}$. Hadamard three lines theorem now implies that $f(z) = 0$ for all z, in particular, for $z = 0$ we obtain $h_\sigma^{1/2} s(\mu) = 0$. It follows that $s(\mu) \leq e$.

Conversely, assume that ξ has the polar decomposition as required. Then since $s(\mu) \le e$, we have

$$\xi = h_\varphi^{1/2-1/p} h_\mu^{1/2} u = h_{\varphi_0}^{1/2-1/p} h_\mu^{1/p} u$$

and Theorem 1 implies that $\|\xi\|_{p,\varphi_0}^{BST} < \infty$. Since the decomposition also implies that $e\xi = \xi$, the statement follows. Uniqueness follows by the uniqueness in Theorem 1.

□

Let us now turn to the case $1 < p \le 2$. Let us again suppose first that $\varphi \in \mathcal{M}_*^+$ is faithful and use complex interpolation, but this time with the continuous embedding $i_2^R : L_2(\mathcal{M}) \to L_1(\mathcal{M})$, defined above. By [6,], we have

$$C_{2/p-1}(L_2(\mathcal{M}), L_1(\mathcal{M})) = L_p(\mathcal{M}, \varphi)^R,$$

notice that $i_2^R(L_2(\mathcal{M})) \subseteq L_p(\mathcal{M}, \varphi)^R$. We will denote by $\|\cdot\|_{p,\varphi}^{BST}$ the norm in $L_2(\mathcal{M})$ induced by this embedding, that is

$$\|\xi\|_{p,\varphi}^{BST} := \|i_2^R(\xi)\|_{p,\varphi}^R.$$

If $s(\varphi) = e$, we put

$$\|\xi\|_{p,\varphi}^{BST} := \|e\xi\|_{p,\varphi_0}^{BST},$$

where φ_0 is given by (2). The following result gives a unique polar decomposition with respect to this norm.

Proposition 3. *Let $1 < p < 2$ and let $\varphi \in \mathcal{M}_*^+$, $\xi \in L_2(\mathcal{M})$. Then $\|\xi\|_{p,\varphi}^{BST} = \mu(1)^{1/p}$, where $\mu \in \mathcal{M}_*^+$ is obtained from the (unique) polar decomposition*

$$h_\varphi^{1/p-1/2}\xi = h_\mu^{1/p} u$$

in $L_p(\mathcal{M})$. Moreover, we have $s(\mu) \le e$.

Proof. Since $\xi \in L_2(\mathcal{M})$, we have $h_\varphi^{1/p-1/2}\xi \in L_p(\mathcal{M})$, so that $h_\varphi^{1/p-1/2}\xi = h_\mu^{1/p}u$ for some $\mu \in \mathcal{M}_*^+$ and a partial isometry $u \in \mathcal{M}$, clearly, $s(\mu) \le e$. Moreover,

$$i_2^R(e\xi) = h_{\varphi_0}^{1/2} e\xi = h_{\varphi_0}^{1/q} h_{\varphi_0}^{1/p-1/2} e\xi = h_{\varphi_0}^{1/q} h_\varphi^{1/p-1/2}\xi$$

and by [6, Theorem],

$$\|h_{\varphi_0}^{1/q} h_\varphi^{1/p-1/2}\xi\|_{p,\varphi_0}^R = \|h_\varphi^{1/p-1/2}\xi\|_p = \mu(1)^{1/p}.$$

□

We again show that this norm coincides with the one given in [**?**]

Proposition 4. *Let $1 < p < 2$. The norm $\|\cdot\|_{p,\varphi}^{BST}$ satisfies the variational formula*

$$\|\xi\|_{p,\varphi}^{BST} = \inf_{\zeta \in L_2(\mathcal{M}), \|\zeta\|_2 = 1, s(\omega'_\zeta) \ge s(\omega'_\xi)} \|\Delta(\zeta/\varphi)^{1/2-1/p}\xi\|_2 = \inf_{\omega \in \mathfrak{S}_*(\mathcal{M}), s(\omega) \ge s(\omega_{\xi^*})} \|\Delta_{\omega,\varphi}^{1/2-1/p}\xi^*\|_2.$$

Proof. Let μ and u be as in Proposition 3. Assume that $\omega \in \mathfrak{S}_*(\mathcal{M})$ is such that $s(\omega_{\xi^*}) \leq s(\omega)$ and $\xi^* \in \mathcal{D}(\Delta_{\omega,\varphi}^{1/2-1/p})$. By [4, Appendix A.1], we have $\xi \in \mathcal{D}(\Delta_{\varphi,\omega}^{1/p-1/2})$ and

$$\|\Delta_{\omega,\varphi}^{1/2-1/p}\xi^*\|_2 = \|J\Delta_{\varphi,\omega}^{1/p-1/2}J\xi^*\|_2 = \|\Delta_{\varphi,\omega}^{1/p-1/2}\xi\|_2.$$

Let $k = \Delta_{\varphi,\omega}^{1/p-1/2}\xi$, then

$$h_\mu^{1/p}u = h_\varphi^{1/p-1/2}\xi = h_\varphi^{1/p-1/2}\xi s(\omega) = kh_\omega^{1/p-1/2}.$$

By Hölder's inequality, we obtain

$$\|\xi\|_{p,\varphi}^{BST} = \mu(1)^{1/p} = \|h_\mu^{1/p}u\|_p \leq \|k\|_2\|h_\omega^{1/p-1/2}\|_{2p/(2-p)} = \|k\|_2 = \|\Delta_{\omega,\varphi}^{1/2-1/p}\xi^*\|_2. \tag{3}$$

On the other hand, assume first that φ is faithful and put $\omega(a) = \mu(1)^{-1}\mu(uau^*)$. Then $\omega \in \mathfrak{S}_*(\mathcal{M})$, but note that in general we have $s(\omega) = u^*u \leq s(\omega_{\xi^*})$. Let $\omega_0 \in \mathfrak{S}_*(\mathcal{M})$ be any state with $s(\omega_0) = s(\omega_{\xi^*}) - s(\omega)$ and put

$$\omega_\epsilon := \epsilon\omega + (1-\epsilon)\omega_0, \qquad \epsilon \in (0,1).$$

Then we have $s(\omega_\epsilon) = s(\omega_{\xi^*})$. Moreover, $\xi \in \mathcal{D}(\Delta_{\varphi,\omega_\epsilon}^{1/p-1/2})$ with

$$\Delta_{\varphi,\omega_\epsilon}^{1/p-1/2}\xi = \epsilon^{1/2-1/p}\mu(1)^{1/p-1/2}h_\mu^{1/2}u$$

so that

$$\|\Delta_{\varphi,\omega_\epsilon}^{1/p-1/2}\xi\|_2 = \epsilon^{1/2-1/p}\mu(1)^{1/p}.$$

Letting $\epsilon \to 1$, we obtain the result.

□

Note that the variational definitions with spatial derivative can be applied to any representing Hilbert space $\mathcal{H}$ and any *-representation $\pi : \mathcal{M} \to B(\mathcal{H})$, as was originally done in [**?**].

Let $\varphi \in \mathcal{M}_*^+$ and let $\pi : \mathcal{M} \to B(\mathcal{H})$ be any $*$-representation. For $\xi \in \mathcal{H}$, let ω_ξ be the functional given by ξ, that is $\omega_\xi(a) = (\xi, \pi(a)\xi)$. We also denote by ω'_ξ the corresponding functional on the commutant: $\omega'_\xi(a') = (\xi, a'\xi)$, $a' \in \pi(\mathcal{M})'$. Let $\Delta(\xi/\varphi)$ denote the spatial derivative as defined in [**?**, Sec. 2.2] (see also [**?**, Appendix A.2]). The φ-weighted p-norm of $\xi \in \mathcal{H}$ is defined as:

1. for $2 \leq p < \infty$,
$$\|\xi\|_{p,\varphi}^{BST} := \sup_{\zeta \in \mathcal{H}, \|\zeta\|=1} \|\Delta(\zeta/\varphi)^{1/2-1/p}\xi\|$$
if $s(\omega_\xi) \leq s(\varphi)$ and $+\infty$ otherwise. Note that the supremum can be infinite also if the condition on the supports holds.

2. for $1 < p < 2$, we define
$$\|\xi\|_{p,\varphi}^{BST} := \inf_{\zeta \in \mathcal{H}, \|\zeta\|=1, s(\omega'_\zeta) \geq s(\omega'_\xi)} \|\Delta(\zeta/\varphi)^{1/2-1/p}\xi\|.$$

According to [**?**], this quantity depends only on the functionals φ and ω_ξ and not on the representation π or the representing vector ξ. For a faithful φ and a standard form for $\mathcal{M}$, the BST-norm is the Araki-Masuda L_p-norm (with respect to the commutant $\mathcal{M}'$).

We start by writing the BST-norm in the standard representation on $L_2(\mathcal{M})$. By [**?**, Appendix A.2] (notice a small mistake there)

$$\Delta(\eta/\varphi) = F^*_{\eta,h_\varphi^{1/2}}\bar{F}_{\eta,h_\varphi^{1/2}}.$$

Let $\omega = \omega_\eta$ and let $\eta = h_\omega^{1/2}u$ for a partial isometry $u \in \mathcal{M}$. Then using [**?**, (C12)], we have

$$\bar{F}_{\eta,h_\varphi^{1/2}} = \Delta^{1/2}_{\eta,h_\varphi^{1/2}}J_{\eta,h_\varphi^{1/2}} = \Delta^{1/2}_{\eta,h_\varphi^{1/2}}\rho(u)J$$

where $\rho(u) \in B(L_2(\mathcal{M}))$ is the right multiplication operator: $\rho(u)\xi = \xi u$, $\xi \in L_2(\mathcal{M})$.

where $\omega = \omega_{\eta^*}$. It follows that for all $\xi \in L_2(\mathcal{M})$, we have

$$\|\xi\|^{BST}_{p,\varphi} = \begin{cases} \sup_{\omega\in\mathfrak{S}_*(\mathcal{M})}\|\Delta^{1/2-1/p}_{\omega,\varphi}\xi^*\|_2 & \text{if } s(\omega_\xi)\le s(\varphi) \\ +\infty & \text{otherwise} \end{cases}, \qquad 2 \le p < \infty, \tag{4}$$

$$\|\xi\|^{BST}_{p,\varphi} = \inf_{\omega\in\mathfrak{S}_*(\mathcal{M}), s(\omega)\ge s(\omega_{\xi^*})}\|\Delta^{1/2-1/p}_{\omega,\varphi}\xi^*\|, \qquad 1 < p < 2. \tag{5}$$

The next lemma shows that we may always assume that the state φ is faithful (or reduce the norms to this case). The proof follows easily from the expressions (4), (5) and the results in [**?**, Appendix A.1].

Lemma 5. *Let $e := s(\varphi)$ and let $\sigma \in \mathcal{M}_*^+$ be such that $s(\sigma) = 1-e$. Put $\varphi_0 = \varphi+\sigma$, then φ_0 is faithful and*

(i) for $2 \le p < \infty$, we have

$$\|\xi\|^{BST}_{p,\varphi} = \|\xi\|^{BST}_{p,\varphi_0}, \qquad \forall \xi \in L_2(\mathcal{M}),\ s(\omega_\xi) \le e;$$

(ii) for $1 < p < 2$, we have

$$\|\xi\|^{BST}_{p,\varphi} = \|e\xi\|^{BST}_{p,\varphi_0}, \qquad \forall \xi \in L_2(\mathcal{M}).$$

1.1 Polar decomposition and duality

As noted in [**?**, Lemma 3.2] for $p \ge 2$ and faithful φ, the relation to the Araki-Masuda L_p-norm gives a form of polar decomposition for elements in $L_2(\mathcal{M})$ with finite BST-norm. The next two lemmas complete this result for all $\varphi \in \mathcal{M}_*^+$ and $1 < p < \infty$.

The following duality relation was mentioned also in [**?**].

Lemma 6. *Let $\xi, \eta \in L_2(\mathcal{M})$, $\varphi \in \mathcal{M}_*^+$, $1 < p < \infty$, $1/p + 1/q = 1$. Then*

(i) $|(\xi,\eta)| \le \|\xi\|^{BST}_{p,\varphi}\|\eta\|^{BST}_{q,\varphi}$;

(ii) *if $s(\omega_\xi) \leq s(\varphi)$ or $1 < p \leq 2$, then*

$$\|\xi\|^{BST}_{p,\varphi} = \sup\{|(\xi,\eta)|,\ \|\eta\|^{BST}_{q,\varphi} \leq 1\};$$

(iii) *Let $1 < p < 2$ and let $h_\varphi^{1/p-1/2}\xi = h_\mu^{1/p}u$. Put $\tilde{\xi} := \mu(1)^{-1/q}h_\varphi^{1/2-1/q}h_\mu^{1/q}u$, then $\|\tilde{\xi}\|^{BST}_{q,\varphi} = 1$ and*

$$\|\xi\|^{BST}_{p,\varphi} = (\xi, \tilde{\xi}).$$

Moreover, $\tilde{\xi}$ is the unique element in $s(\varphi)L_2(\mathcal{M})$ with these properties.

Proof. If φ is faithful, (i) and (ii) follow from duality of Araki-Masuda L_p-spaces, [**?**, Theorem 1]. The equality in (iii) is easy to see and uniqueness follows by uniform convexity of the L_p-spaces [**?**, Theorem].

Let now $s(\varphi) = e$ and let φ_0 be as in Lemma 5. For (i), let, say, $p \geq 2$ and assume that $\|\xi\|^{BST}_{p,\varphi} < \infty$. Then $\xi = e\xi$ and

$$|(\xi,\eta)| = |(\xi, e\eta)| \leq \|\xi\|^{BST}_{p,\varphi_0}\|e\eta\|^{BST}_{q,\varphi_0} = \|\xi\|^{BST}_{p,\varphi}\|\eta\|^{BST}_{q,\varphi}.$$

For (ii), let $1 < p < 2$ and let $\|\eta\|^{BST}_{q,\varphi_0} \leq 1$. Then by Lemma **??**, $\eta = h_{\varphi_0}^{1/2-1/q}k$ for some $k \in L_q(\mathcal{M})$ with $\|k\|_q \leq 1$. It follows that

$$e\eta = h_\varphi^{1/2-1/q}ek,$$

so that $\|e\eta\|^{BST}_{q,\varphi} = \|ek\|_q \leq \|k\|_q \leq 1$. We obtain

$$\begin{aligned}\|\xi\|^{BST}_{p,\varphi} &= \|e\xi\|^{BST}_{p,\varphi_0} = \sup\{|(e\xi,\eta)|,\ \|\eta\|^{BST}_{q,\varphi_0} \leq 1\}\\ &= \sup\{|(\xi,e\eta)|,\ \|\eta\|^{BST}_{q,\varphi_0} \leq 1\} \leq \sup\{|(\xi,\eta)|,\ \|\eta\|^{BST}_{q,\varphi} \leq 1\} \leq \|\xi\|^{BST}_{p,\varphi}.\end{aligned}$$

If $2 \leq p < \infty$ and $s(\omega_\xi) \leq e$, the statement (ii) is obtained similarly from Lemma 5 and

$$\|\eta\|^{BST}_{q,\varphi} = \|h_\varphi^{1/q-1/2}\eta\|_q = \|eh_{\varphi_0}^{1/q-1/2}\eta\|_q \leq \|h_{\varphi_0}^{1/q-1/2}\eta\|_q = \|\eta\|^{BST}_{q,\varphi_0}.$$

The only thing left to prove is the uniqueness in (iii). So let $\hat{\xi} \in eL_2(\mathcal{M})$ be such that $\|\hat{\xi}\|^{BST}_{q,\varphi} = 1$ and $(\xi,\hat{\xi}) = \|\xi\|^{BST}_{p,\varphi}$. Then also $\|\hat{\xi}\|^{BST}_{q,\varphi_0} = \|\hat{\xi}\|^{BST}_{q,\varphi} = 1$ and

$$(e\xi,\hat{\xi}) = (\xi,\hat{\xi}) = \|\xi\|^{BST}_{p,\varphi} = \|e\xi\|^{BST}_{p,\varphi_0}.$$

By uniqueness in the faithful case, we obtain $\hat{\xi} = \tilde{\xi}$.

□

1.2 Interpolation

Using the polar decompositions, the BST norms can be written in terms of interpolation L_p-spaces studied in [6]. Let $1 < p < \infty$ and let φ_0 be the faithful positive functional as in Lemma 5. Let us consider the the interpolation spaces and the corresponding norms (see [**?**, Appendix C] for the notations)

$$L_p^R(\mathcal{M},\varphi_0) := C_{1/p}(i_\infty^R(\mathcal{M}), L_1(\mathcal{M})), \qquad \|\cdot\|^R_{p,\varphi_0} := \|\cdot\|_{1/p},$$

where $i^R_\infty : \mathcal{M} \to L_1(\mathcal{M})$ is the embedding

$$i^R_\infty : x \mapsto h_{\varphi_0} x.$$

By [6, Theorem 9.1], for $1 < p < \infty$,

$$i^R_p : L_p(\mathcal{M}) \ni h \mapsto h^{1/q}_{\varphi_0} h$$

is an isometric isomorphism of $L_p(\mathcal{M})$ onto $L^R_p(\mathcal{M}, \varphi_0)$.

Proposition 7. *Let $\xi \in L_2(\mathcal{M})$. Then*

(i) for $2 \le p < \infty$, $\|\xi\|^{BST}_{p,\varphi} < \infty$ if and only if $\xi = e\xi$ and $i^R_2(\xi) \in L^R_p(\mathcal{M}, \varphi_0)$, in this case

$$\|\xi\|^{BST}_{p,\varphi} = \|i^R_2(\xi)\|^R_{p,\varphi_0};$$

(ii) for $1 < p < 2$, $\|\xi\|^{BST}_{p,\varphi} = \|i^R_2(e\xi)\|^R_{p,\varphi_0}$.

Proof. Using Lemma 5, we may assume that $\varphi = \varphi_0$ is faithful. The statement (i) follows immediately from Lemma **??**, (ii) is obtained from Lemma **??** and the fact that

$$i^R_2(\xi) = h^{1/2}_\varphi \xi = h^{1/q}_\varphi (h^{1/p-1/2}_\varphi \xi).$$

□

Let $1 < p < \infty$ and let $S \subset \mathbb{C}$ be the strip $S = \{z \in \mathbb{C}, 0 \le Re(z) \le 1\}$. Let $\xi \in L_2(\mathcal{M})$ with $\|\xi\|^{BST}_{p,\varphi} < \infty$ and let μ and u be as in the polar decomposition in Lemma **??** or **??**. Put

$$f^R_{p,\varphi;\xi}(z) := \begin{cases} \mu(1)^{1/p-z} h^{1-z}_\varphi h^z_\mu u & \text{if } \mu(1) > 0 \\ 0 & \text{otherwise,} \end{cases} \qquad z \in S. \tag{6}$$

Lemma 8. *We have $f^R_{p,\varphi;\xi} \in \mathcal{F}(i^R_\infty(\mathcal{M}), L_1(\mathcal{M}))$ and*

$$\|\xi\|^{BST}_{p,\varphi} = \left|\left|\left| f^R_{p,\varphi;\xi} \right|\right|\right|_{\mathcal{F}}.$$

Proof. If φ is faithful, the statement follows from Proposition 7 and [6, proof of Thm. 9.1]. In the general case, we use Lemma 5 and the fact that under the above assumptions, $f^R_{p,\varphi;\xi} = f^R_{p,\varphi_0;e\xi}$.

□

One can prove similar statements for such functions as in [**?**, Section 2.], by very much the same methods.

2 Rényi relative entropies

We now recall the definition of the divergences in [**?**].

Definition 1. [**?**] Let $\psi, \varphi \in \mathcal{M}_*^+$ and $\alpha \in [1/2, 1) \cup (1, \infty)$. Let ξ_ψ be any vector representative of ψ for a $*$-representation $\pi : \mathcal{M} \to B(\mathcal{H})$. Then

$$D_\alpha^{BST}(\psi \| \varphi) = \begin{cases} \frac{2\alpha}{\alpha-1} \log \|\xi_\psi\|_{2\alpha,\varphi}^{BST} & \text{if } \|\xi_\psi\|_{2\alpha,\varphi}^{BST} > 0 \\ \infty & \text{otherwise.} \end{cases} \tag{7}$$

It was proved in [**?**] that for $\alpha > 1$, this quantity coincides with $\tilde{D}_\alpha$. In the sequel, we will use the notation $\tilde{D}_\alpha := D_\alpha^{BST}$ also for $\alpha \in [1/2, 1)$. The following expression follows easily from Lemma **??**, using the vetcor representative $h_\psi^{1/2} \in L_2(\mathcal{M})$ for ψ.

Theorem 9. *Let $\psi \in \mathcal{M}_*^+$, $\alpha \in [1/2, 1)$. Then*

$$\tilde{D}_\alpha(\psi \| \varphi) = \frac{1}{\alpha - 1} \log \|h_\varphi^{\frac{1-\alpha}{2\alpha}} h_\psi^{1/2}\|_{2\alpha}^{2\alpha}.$$

2.1 Properties of the function $\alpha \mapsto \tilde{D}_\alpha$

We will consider $\alpha \in [1/2, 1)$, the case of $\alpha > 1$ was treated in [**?**].

Proposition 10. *Let $\psi, \varphi \in \mathfrak{S}_*(\mathcal{M})$, $e := s(\varphi)$ and let $\alpha \in [1/2, 1)$. Then*

(i) $\tilde{D}_\alpha(\psi, \varphi) \geq 0$, *with equality if and only if* $\varphi = \psi$.

(ii) $\tilde{D}_\alpha(\psi \| \varphi)$ *is finite whenever* $eh_\psi^{1/2} \neq 0$.

(iii) If $eh_\psi^{1/2} \neq 0$ *and* $\psi \neq \varphi$, *the function* $\alpha \mapsto \tilde{D}_\alpha(\psi \| \varphi)$ *is continuous and strictly increasing.*

Proof. The inequality in (i) follows easily from Theorem 9 an Hölder inequality:

$$\|h_\varphi^{\frac{1-\alpha}{2\alpha}} h_\psi^{1/2}\|_{2\alpha}^{2\alpha} \leq \|h_\varphi^{\frac{1-\alpha}{2\alpha}}\|_{2\alpha/(1-\alpha)} \|h_\psi^{1/2}\|_2 = 1$$

For the equality, assume first that $\alpha = 1/2$, then $\tilde{D}_\alpha(\psi, \varphi) = 0$ iff $\|h_\varphi^{1/2} h_\psi^{1/2}\|_1 = 1$. Let $v|h_\varphi^{1/2} h_\psi^{1/2}|$ be the polar decomposition of $h_\varphi^{1/2} h_\psi^{1/2}$, then we obtain

$$1 = \|h_\varphi^{1/2} h_\psi^{1/2}\|_1 = \operatorname{Tr} v^* h_\varphi^{1/2} h_\psi^{1/2} = (h_\psi^{1/2}, h_\varphi^{1/2} v) \leq \|h_\psi^{1/2}\|_2 \|h_\varphi^{1/2} v\|_2 \leq 1,$$

which implies that $h_\psi^{1/2} = h_\varphi^{1/2} v$. From this, we see that $eh_\psi^{1/2} = h_\psi^{1/2}$, so that $s(\psi) \leq e$. Moreover,

$$1 = \|h_\varphi^{1/2} h_\psi^{1/2}\|_1 = \|h_\varphi^{1/2} s(\psi) h_\psi^{1/2}\| \leq \|h_\varphi^{1/2} s(\psi)\|_2 \leq 1,$$

which implies that $s(\psi) = e$. By uniqueness of the polar decomposition, we obtain $h_\psi = h_\varphi$.

Let now $\alpha \in (1/2, 1)$ and let $p = 2\alpha$. Let $h_\varphi^{1/p-1/2} h_\psi^{1/2} = h_\mu^{1/p} u$, then equality implies that $\mu(1) = 1$. Let

$$f(z) = \mathrm{Tr}\, u^* h_\mu^{z/2} h_\varphi^{(1-z)/2} h_\psi^{1/2}, \qquad z \in S,$$

then $f : S \to \mathbb{C}$ is continuous, $|f(z)| \leq 1$ on S and analytic in the interior of S, moreover, $f(1/q) = 1$, so that we must have $f(z) = 1$ for all z. In particular, for $z = 0$, we obtain

$$1 = \mathrm{Tr}\, u^* h_\varphi^{1/2} h_\psi^{1/2} \leq \|h_\varphi^{1/2} h_\psi^{1/2}\|_1 \leq 1.$$

The equality $\varphi = \psi$ is now obtained as before.

The statement (ii) is clear from Theorem 9, continuity and monotonicity in (iii) was already proved in [**?**]. We give a similar proof in our setting, since it is used in the proof of the fact the monotonicity is strict. So let $1/2 < \alpha < \alpha' < 1$ and let $p = 2\alpha$, $p' = 2\alpha'$, so that $1 < p < p' < 2$. Let $\eta \in [0, 1]$ be such that $1/p' = \eta/p + (1-\eta)/2$. Consider the constant function

$$f(z) \equiv h_\varphi^{1/2} h_\psi^{1/2} = h_{\varphi_0}^{1/2} e h_\psi^{1/2} \in L_1(\mathcal{M}), \quad z \in S.$$

Then $f \in \mathcal{F}(L_2^R(\mathcal{M}, \varphi_0)), L_p^R(\mathcal{M}, \varphi_0))$, see Section **??**. By Proposition 7 and Hadamard three lines, we have

$$\|h_\psi^{1/2}\|_{p',\varphi}^{BST} = \|h_\varphi^{1/2} h_\psi^{1/2}\|_{p',\varphi_0}^R \leq (\|h_\varphi^{1/2} h_\psi^{1/2}\|_{p,\varphi_0}^R)^\eta = (\|h_\psi^{1/2}\|_{p,\varphi}^{BST})^\eta, \tag{8}$$

this implies that the function is nondecreasing. Now assume that $D_\alpha(\psi\|\varphi) = D_{\alpha'}(\psi\|\varphi)$ It can be proved similarly as in [**?**, Lemma 2.10] that equality in (8) is attained if and only if

$$f(z) = f_{p',\varphi;h_\psi^{1/2}}(z/p + (1-z)/2) M^{z-\eta}, \qquad \forall z \in S$$

for some constant $M > 0$ and we can see from the proof of that lemma that $M = 1$. In particular, by putting $z = 0$ and $z = 1$, we obtain

$$h_\varphi^{1/2} h_\psi^{1/2} = \mu(1)^{1/p} h_\varphi u = \mu(1)^{1/p-1} h_\mu u = \mu(1)^{1/p} h_{\tilde{\mu}} u,$$

where $\tilde{\mu} = \mu(1)^{-1}\mu \in \mathfrak{S}_*(\mathcal{M})$. Since $uu^* = s(\tilde{\mu})$ and both φ and $\tilde{\mu}$ are states, we must have $uu^* = e$ and $\varphi = \tilde{\mu}$, moreover, $h_\varphi^{1/2} u$ is a vector representative of φ. The above equality also implies that

$$e h_\psi^{1/2} = c h_\varphi^{1/2} u, \qquad c := \mu(1)^{1/p}.$$

It follows that for any $1 < p'' < \infty$,

$$\|h_\psi^{1/2}\|_{p'',\varphi}^{BST} = \|h_\varphi^{1/2} h_\psi^{1/2}\|_{p'',\varphi_0}^R = c\|h_\varphi^{1/p''} u\|_{p''} = c,$$

so that $D_{\alpha''}(\psi\|\varphi) = \frac{2\alpha''}{\alpha''-1} \log c$. But, by assumption, $D_{\alpha'}(\psi\|\varphi) = D_\alpha(\psi\|\varphi)$, so that we must have $c = 1$ and $\psi = \varphi$.

□

Remark 11. For $\alpha = 1/2$, it was observed in [**?**] that $\tilde{D}_{1/2}(\psi\|\varphi) = \log F(\psi\|\varphi)$, where $F(\psi\|\varphi)$ is the fidelity. The statement (i) also follows from properties of F.

2.2 Relation to standard Rényi relative entropy

Recall that the standard Rényi relative entropy for $\alpha \in (0,1)$ is defined as

$$D_\alpha(\psi\|\varphi) = \frac{1}{\alpha-1}\log(\operatorname{Tr} h_\psi^\alpha h_\varphi^{1-\alpha}) = \frac{1}{\alpha-1}\log \|h_\varphi^{\frac{1-\alpha}{2}} h_\psi^{\frac{\alpha}{2}}\|_2^2.$$

Let $p = 2\alpha$, $\alpha \in [1/2, 1)$ and let $1/p + 1/q = 1$. Let $\varphi, \psi \in \mathcal{M}_*^+$.

Proposition 12. *Let $\varphi, \psi \in \mathcal{M}_*^+$, $\alpha \in [1/2, 1)$. Then we have*

$$\|h_\varphi^{\frac{1-\alpha}{2}} h_\psi^{\frac{\alpha}{2}}\|_2^2 \le \|h_\varphi^{\frac{1-\alpha}{2\alpha}} h_\psi^{1/2}\|_{2\alpha}^{2\alpha} \le \psi(1)^{1-\alpha}\|h_\varphi^{\frac{1-\alpha}{2\alpha}} h_\psi^{1-\frac{1}{2\alpha}}\|_2^{2\alpha}$$

Proof. By Hölder,

$$\|h_\varphi^{\frac{1-\alpha}{2\alpha}} h_\psi^{1/2}\|_{2\alpha} = \|h_\varphi^{\frac{1-\alpha}{2\alpha}} h_\psi^{1-\frac{1}{2\alpha}} h_\psi^{\frac{1-\alpha}{2\alpha}}\|_{2\alpha} \le \psi(1)^{\frac{1-\alpha}{2\alpha}}\|h_\varphi^{\frac{1-\alpha}{2\alpha}} h_\psi^{1-\frac{1}{2\alpha}}\|_2,$$

this implies the second inequality. For the first one, let us define a function

$$f(z) = h_\varphi^{1-\alpha z} h_\psi^{\alpha z} = h_{\varphi_0}^{1-\alpha z} e h_\psi^{\alpha z} \in L_1(\mathcal{M}), \qquad z \in S.$$

Then $f \in \mathcal{F}(i_\infty^R(\mathcal{M}), L_1(\mathcal{M}))$, so that we can use the properties of the interpolation spaces $L_p^R(\mathcal{M}, \varphi_0)$. Note that $\|f(1/2)\|_{2,\varphi}^R = \|h_\varphi^{\frac{1-\alpha}{2}} h_\psi^{\frac{\alpha}{2}}\|_2$. Since $1/2 = \alpha\frac{1}{2\alpha} + (1-\alpha)0$, we obtain by Hadamard three lines that

$$\|f(1/2)\|_{2,\varphi}^R \le (\sup_{t\in\mathbb{R}} \|f(it)\|_{\infty,\varphi}^R)^{1-\alpha} (\sup_{t\in\mathbb{R}} \|f(\frac{1}{2\alpha} + it)\|_{2\alpha,\varphi}^R)^\alpha$$

Let $u_t = h_\varphi^{-i\alpha t} h_\psi^{i\alpha t}$, then $u_t \in \mathcal{M}$ is a partial isometry, so that

$$\|f(it)\|_{\infty,\varphi}^R = \|h_\varphi u_t\|_{\infty,\varphi}^R = \|u_t\| = 1.$$

Let ψ_0 be a faithful state obtained from ψ similarly as φ_0 from φ. Then for $t \in \mathbb{R}$,

$$\|f(\frac{1}{2\alpha} + it)\|_{2\alpha,\varphi}^R = \|h_{\varphi_0}^{-i\alpha t} h_\varphi^{\frac{1-\alpha}{2\alpha}} h_\psi^{1/2} h_{\psi_0}^{i\alpha t}\|_{2\alpha} = \|h_\varphi^{\frac{1-\alpha}{2\alpha}} h_\psi^{1/2}\|_{2\alpha},$$

the last equality holds by [6, Lemma 10.1].

□

The next statement is an extension of [**?**, Coro] to all values of α. Note that this result for states of a finite dimensional algebra was proved in [**?**]. The proof follows easily from Proposition 12.

Theorem 13. *Let $\psi, \varphi \in \mathfrak{S}_*(\mathcal{M})$ and let $\alpha \in (1/2, 1)$. Then*

$$D_{2-1/\alpha}(\psi\|\varphi) \le \tilde{D}_\alpha(\psi\|\varphi) \le D_\alpha(\psi\|\varphi).$$

The next result is immediate from the properties of D_α.

Corollary 14. $\lim_{\alpha\nearrow 1} \tilde{D}_\alpha(\psi\|\varphi) = D_1(\psi\|\varphi)$.

2.3 Order relations and joint lower semicontinuity

Proposition 15. *Let $\psi, \psi_0, \varphi, \varphi_0 \in \mathcal{M}_*^+$ and $\psi_0 \le \psi$, $\varphi_0 \le \varphi$. Then for $\alpha \in (1/2, 1)$, we have $\tilde{D}_\alpha(\psi_0\|\varphi)$???*

Let $\psi, \psi' \in \mathcal{M}_*^+$, $\psi' \le \psi$. By the Radon-Nikodym theorem [?], $h_{\psi'}^{1/2} = h_\psi^{1/2} a$ for some $a \in \mathcal{M}$ with $\|a\| \le 1$. Hence for any $\alpha \in [1/2, 1)$, we have

$$\|h_{\psi'}^{1/2}\|_{2\alpha,\varphi}^{BST} = \|h_\varphi^{\frac{1-\alpha}{2\alpha}} h_{\psi'}^{1/2}\|_{2\alpha} \le \|h_\varphi^{\frac{1-\alpha}{2\alpha}} h_\psi^{1/2}\|_{2\alpha} = \|h_\psi^{1/2}\|_{2\alpha,\varphi}^{BST}.$$

It follows that $\tilde{D}_\alpha(\psi'\|\varphi) \ge \tilde{D}_\alpha(\psi\|\varphi)$. (HMMM)

Let now $\varphi, \varphi' \in \mathcal{M}_*^+$, $\varphi' \le \varphi$. Let us first assume that both are faithful. By the (commutant) Radon-Nikodym theorem []??, there is some $a \in \mathcal{M}$ such that $\|a\| \le 1$ and $h_{\varphi'}^{1/2} = a h_\varphi^{1/2}$. Let now $f = f_{2\alpha,\varphi,h_\psi^{1/2}}$, then $z \mapsto af(z)$ is a bounded continuous function, analytic in the interior of S and $af(1/2\alpha) = a h_\varphi^{1/2} h_\psi^{1/2} = h_{\varphi'}^{1/2} h_\psi^{1/2}$. For any $t \in \mathbb{R}$,

$$af(1/2 + it) = \mu(1)^{1/p-1/2-it} h_{\varphi'}^{1/2} h_\varphi^{-it} h_\mu^{it} h_\mu^{1/2} u \in L_2^R(\mathcal{M}, \varphi')$$

and $\|af(1/2+it)\|_{2,\varphi'}^R \le \mu(1)^{1/2\alpha}$. Moreover, for all $t \in \mathbb{R}$,

$$\|af(1+it)\|_1 = \mu(1)^{1/p-1-it}\|a h_\varphi^{-it} h_\mu^{it} h_\mu u\| \le \mu(1)^{1/2\alpha}.$$

By reiteration theorem and the definition of the interpolation norm, we have

$$\begin{aligned} \|h_\psi^{1/2}\|_{2\alpha,\varphi'}^{BST} &= \|h_{\varphi'}^{1/2} h_\psi^{1/2}\|_{2\alpha,\varphi'}^R \le \max\{\sup_t \|af(1/2+it)\|_{2,\varphi'}^R, \sup_t \|af(1+it)\|_1\} \\ &\le \mu(1)^{1/2\alpha} = \|h_\psi^{1/2}\|_{2\alpha,\varphi}^{BST} \end{aligned}$$

(hmmm)

3 Monotonicity, equality and sufficiency

Let $\Phi : L_1(\mathcal{M}) \to L_1(\mathcal{N})$ be a quantum channel (that is, a completely positive trace preserving map). Then the dual map $\Phi^* : \mathcal{N} \to \mathcal{M}$ is a completely positive unital normal map. Using Stinespring representation, there exists a Hilbert space $\mathcal{K}$, a normal *-representation $\pi : \mathcal{N} \to B(\mathcal{K})$ and an isometry $T : L_2(\mathcal{M}) \to \mathcal{K}$ such that

$$\Phi^*(a) = T^*\pi(a)T, \qquad a \in \mathcal{N}.$$

Let $k \in L_2(\mathcal{M})$ be a representing vector for $\psi \in \mathcal{M}_*^+$, then $Tk \in \mathcal{K}$ is a representing vector for $\Phi(\psi)$, hence we have

$$D_\alpha^{BST}(\Phi(\psi), \Psi(\varphi)) = \frac{2\alpha}{\alpha - 1} \log \|Tk\|_{2\alpha,\Phi(\varphi)}^{BST}.$$

The following data processing inequality (DPI) for D_α^{BST} was proved in [2]:

$$D_\alpha^{BST}(\psi\|\varphi) \ge D_\alpha^{BST}(\Phi(\psi)\|\Phi(\varphi)), \qquad \alpha \in [1/2, 1) \cup (1, \infty].$$

This is equivalent to

$$\|Tk\|_{p,\Phi(\varphi)}^{BST} \leq \|k\|_{p,\varphi}^{BST},\ 2 < p \leq \infty; \qquad \|Tk\|_{p,\Phi(\varphi)}^{BST} \geq \|k\|_{p,\varphi}^{BST},\ 1 \leq p < 2 \tag{9}$$

for any Stinespring dilation $(\mathcal{K}, \pi, T)$. We next show that equality in DPI implies that the channel Φ is sufficient with respect to $\{\psi, \varphi\}$.

Theorem 16. *Assume that $s(\psi) \leq s(\varphi)$ and let $\alpha \in (1/2, 1)$. Then $D_\alpha^{BST}(\psi\|\varphi) = D_\alpha^{BST}(\Phi(\psi)\|\Phi(\varphi))$ if and only if Φ is sufficient for $\{\psi, \varphi\}$.*

Proof. Because of the assumption on the supports, we may suppose that both φ and $\Phi(\varphi)$ are faithful. Assume that the equality holds, so that $\|h_\psi^{1/2}\|_{p,\varphi}^{BST} = \|Th_\psi^{1/2}\|_{p,\Phi(\varphi)}^{BST}$, here $p = 2\alpha \in (1, 2)$. Let $h_\psi^{1/2} = u\rho^{1/p}$ be the polar decomposition in $L_p^{AM}(\mathcal{M}, \varphi)$, then

$$\|h_\psi^{1/2}\|_{p,\varphi}^{BST} = \|h_\psi^{1/2}\|_{p,\varphi}^{AM} = (\|k\|_{q,\varphi}^{AM})^{-1}(k, h_\psi^{1/2})_{L_2(\mathcal{M})},$$

where $1/p + 1/q = 1$ and $k \in L_q^{AM}(\mathcal{M}, \varphi)$ has polar decomposition $k = u\rho^{1/q}$. By Lemma **??**, $h_\psi^{1/2}h_\varphi^{1/p-1/2} = uh_\rho^{1/p}$ and we have $k = uh_\rho^{1/q}h_\varphi^{1/2-1/q}$. Since T is an isometry, we get using the norm duality in [2, Sec. 3.2]

$$\begin{aligned}(k, h_\psi^{1/2})_{L_2(\mathcal{M})} = (h_\psi^{1/2}, k^*)_{L_2(\mathcal{M})} &= (Th_\psi^{1/2}, Tk^*)_{\mathcal{K}} \\ &\leq \|Th_\psi^{1/2}\|_{p,\Phi(\varphi)}^{BST}\|Tk^*\|_{q,\Phi(\varphi)}^{BST}\end{aligned}$$

By the assumption and Proposition **??**,

$$\|Th_\psi^{1/2}\|_{p,\Phi(\varphi)}^{BST} = \|h_\psi^{1/2}\|_{p,\varphi}^{BST} \leq (\|k^*\|_{q,\varphi}^{BST})^{-1}\|Tk^*\|_{q,\Phi(\varphi)}^{BST}\|Th_\psi^{1/2}\|_{p,\Phi(\varphi)}^{BST},$$

which implies that $\|Tk^*\|_{q,\Phi(\varphi)}^{BST} \geq \|k^*\|_{q,\varphi}^{BST}$. By (9) for $q > 2$, we get the equality $\|Tk^*\|_{q,\Phi(\varphi)}^{BST} = \|k^*\|_{q,\varphi}^{BST}$ which by Theorem **??** yields

$$\tilde{D}_\beta(\omega\|\varphi) = D_\beta^{BST}(\omega\|\varphi) = D_\beta^{BST}(\Phi(\omega)\|\Phi(\varphi)) = \tilde{D}_\beta(\Phi(\omega)\|\Phi(\varphi)),$$

where $\beta := q/2$ and $h_\omega = \|k\|_2^{-2}k^*k$ is the state given by the (normalized) vector k^*. By [4, Thm. 7], this equality implies that Φ is sufficient with respect to $\{\omega, \varphi\}$. Since $h_\omega = \|k\|_2^{-2}h_\varphi^{1/2\alpha}h_\rho^{1/\beta}h_\varphi^{1/2\alpha}$, [4, Lemma 8] implies that Φ is sufficient with respect to $\{\rho(1)^{-1}\rho, \varphi\}$.

Let $E : \mathcal{M} \to \mathcal{M}$ be a faithful normal conditional expectation as in [4, Lemma 7], so that $\varphi \circ E = \varphi$ and Φ is sufficient for $\{\psi, \varphi\}$ if and only if $\psi \circ E = \psi$. Let E_p be the extension of E to $L_p(\mathcal{M})$ ([5], [4, Appendix A.3]). We have by [4, Eq. (A.5)],

$$u^*h_\psi^{1/2}h_\varphi^{1/p-1/2} = h_\rho^{1/p} = E_p(h_\rho^{1/p}) = E_2(u^*h_\psi^{1/2})h_\varphi^{1/p-1/2}.$$

Since φ is faithful, we have $uu^* = s(\psi)$ by the properties of polar decomposition, and the above equalities imply that $u^*h_\psi^{1/2} = E_2(u^*h_\psi^{1/2})$, hence

$$h_{\psi\circ E} = E_1(h_\psi) = h_\psi^{1/2}uu^*h_\psi^{1/2} = h_\psi$$

so that Φ is sufficient for $\{\psi, \varphi\}$. The converse is obvious from DPI.

□

Appendix: The spatial derivative

We recall the definition of the spatial derivative $\Delta(\eta/\varphi)$ of [2], using the standard representation $(l(\mathcal{M}), L_2(\mathcal{M}), L_2(\mathcal{M})^+, \cdot^*)$. Let $\mathcal{H}_\varphi := [\mathcal{M} h_\varphi^{1/2}] = L_2(\mathcal{M})s(\varphi)$ and let $k \in L_2(\mathcal{M})$ be such that the corresponding functional is majorized by φ:

$$\omega_k(a^*a) = \|ak\|^2 \leq C_k \varphi(a^*a), \qquad \forall a \in \mathcal{M},$$

for some positive constant C_k. Then

$$R^\varphi(k) : a h_\varphi^{1/2} \mapsto ak, \qquad a \in \mathcal{M}$$

extends to a bounded linear operator $\mathcal{H}_\varphi \to L_2(\mathcal{M})$. Obviously, $R^\varphi(k)$ extends to a bounded linear operator on $L_2(\mathcal{M})$ by putting it equal to 0 on $L_2(\mathcal{M})(1 - s(\varphi))$. Moreover, this operator commutes with the left action of $\mathcal{M}$, so that it belongs to $l(\mathcal{M})' = r(\mathcal{M})$, where r is the right action $r(a) : h \mapsto ha$, $h \in L_2(\mathcal{M})$. In fact, ω_k is majorized by φ if and only if $k \in h_\varphi^{1/2}\mathcal{M}$, so that there is some $y_k \in \mathcal{M}$ such that $k = h_\varphi^{1/2} y_k$, $s(\varphi)y_k = y_k$ and we have $R^\varphi(k) = r(y_k)$.

Let now $h \in L_2(\mathcal{M})$, $\omega := \omega_h$. The spatial derivative $\Delta(h/\varphi)$ is a positive self-adjoint operator associated with the quadratic form $k \mapsto (h, R^\varphi(k)R^\varphi(k)^*h)$ as

$$\begin{aligned}(k, \Delta(h/\varphi)k) &= (\Delta(h/\varphi)^{1/2}k, \Delta(h/\varphi)^{1/2}k) = (h, R^\varphi(k)R^\varphi(k)^*h) \\ &= (R^\varphi(k)^*h, R^\varphi(k)^*h) = (hy_k^*s(\varphi), hy_k^*s(\varphi)) = (F_{h,h_\varphi^{1/2}}k, F_{h,h_\varphi^{1/2}}k),\end{aligned}$$

(see [4, Appendix A], for the definition of $F_{\eta,\xi}$). Since $h_\varphi^{1/2}\mathcal{M} + (1 - s(\varphi))L_2(\mathcal{M})$ is a core for both $\Delta(h/\varphi)$ and $F_{h,h_\varphi^{1/2}}$, it follows that

$$\Delta(h/\varphi) = F^*_{h,h_\varphi^{1/2}} F_{h,h_\varphi^{1/2}} = J\Delta_{\omega,\varphi}J.$$

This implies that for any $k \in L_2(\mathcal{M})$ and $\gamma \in \mathbb{C}$, we have

$$\|\Delta(h/\varphi)^\gamma k\|_2 = \|\Delta^\gamma_{\omega,\varphi} Jk\|_2 = \|\Delta^\gamma_{\omega,\varphi} k^*\|_2.$$

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