EFFECT ALGEBRAS WITH COMPRESSIONS

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The notion of a CB-effect algebra as an effect algebra equipped with a compression base was recently introduced by S. Gudder as an analogue of the notion of a unital group with a compression base (CB-group) introduced by D. Foulis. The present paper extends the investigation of CB-effect algebras with the projection cover property, the Rickart projection property, and introduces the so-called b-general comparability, which is an effect algebra version of general comparability in CB-groups. Commutativity properties, blocks and C-blocks are studied, and it is shown that a CB-effect algebra with b-general comparability can be covered by its C-blocks, which are maximal sets of commuting elements, and can be organized into MV-algebras. Connections with sequential effect algebras (SEAs) are studied.

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1. Introduction

In [8], D. Foulis characterized compressions on the set of self-adjoint elements of a unital C*-algebra, and the resulting characterization inspired a study of a new class of partially ordered abelian groups, so-called compressible groups introduced in [7], and further studied in [10, 29, 30, 9]. In the further development it turned out that the concept of compressible groups was too restrictive, and many important cases were omitted. Therefore the notion of compression bases was introduced in [11], and the study of unital groups (i.e. partially ordered abelian groups with strong unit) with compression bases continued in a series of papers [12, 15, 13], etc.

Effect algebras were introduced as an algebraic abstraction of the set of the Hilbert space effects, that is, self-adjoint operators on a Hilbert space lying between the zero and identity operator [14, 17, 25]. The effects correspond to yes-no quantum measurements that can be unsharp. They play an important role in the theory of quantum measurements [2, 3, 26]. An important class of effect algebras are unit intervals of unital groups [1]. Another important class are sequential effect algebras

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[22], which are effect algebras with an additional operation of product inspired by the operation $(A, B) \to A^{1/2}BA^{1/2}$ on the Hilbert space effects.

The notion of compressions was generalized to effect algebras in [20], and effect algebras with compression bases were introduced in [21]. In the latter paper, it was shown that sequential effect algebras possess a natural compression base. The focus of a compression is called a projection, and it was shown that projections are principal, hence sharp elements of the effect algebra. Further, a projection cover property was studied there.

In the present paper we continue investigation of effect algebras with compression bases. We introduce a version of general comparability, so-called b-general comparability, in the context of effect algebras. The latter property is satisfied, e.g. in the set $\mathcal{E}(H)$ of Hilbert space effects, in σ-MV-algebras, or more generally, in unit intervals of archimedean RC-groups. We continue the study of commutants and introduce the notion of C-blocks, in analogy with [15]. We show that in an effect algebra with a compression base which has the b-general comparability property, every C-block is an MV-algebra. Moreover, the effect algebra can be covered by C-blocks, which are maximal sets of mutually commuting elements. We also show that, due to the existence of projection covers in monotone σ-complete sequential effect algebras, the set of projections in them forms an orthomodular σ-lattice, and that a commutative SEA with the b-general comparability is an MV-algebra.

2. Effect algebras

This section summarizes the basic definitions and notations concerning effect algebras. An *effect algebra* is a system $(E; \oplus, 0, 1)$ where E is a nonempty set, 0, 1 are constants and $\oplus$ is a partial binary operation on E that satisfies the following conditions:

(E1) If $a \oplus b$ exists then $b \oplus a$ exists and $b \oplus a = a \oplus b$.

(E2) If $a \oplus b$ and $(a \oplus b) \oplus c$ exist, the $b \oplus c$ and $a \oplus (b \oplus c)$ exist and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

(E3) For every $a \in E$ there exists a unique $a' \in E$ such that $a \oplus a' = 1$.

(E4) $a \oplus 1$ exists iff $a = 0$.

We write $a \perp b$ and say that a and b are *orthogonal* if $a \oplus b$ exists. In the sequel, whenever we write $a \oplus b$ we tacitly assume that $a \perp b$. We define $a \leq b$ if there exists a $c \in E$ such that $a \oplus c = b$. If such an element $c \in E$ exists, then it is unique and we write $c = b \ominus a$. In particular, $a' = 1 \ominus a$, and we call a' the *orthosupplement* of a. It can be shown that $a \perp b$ iff $a \leq b'$. Moreover, $(E; \leq, ')$ is a partially ordered set with $0 \leq a \leq 1$ for all $a \in E$, $a'' = a$ and $a \leq b$ iff $b' \leq a'$. An element $a \in E$ is called *sharp* if $a \wedge a' = 0$ and we denote the set of all sharp elements in E by E_S. An element $a \in E$ is *principal* if $b, c \leq a$ with $b \perp c$ imply that $b \oplus c \leq a$. It is easy to see that a principal element is sharp. A subset F of an effect algebra E is a *sub-effect algebra* of E if $0, 1 \in F$, $a' \in F$ whenever $a \in F$ and $a \oplus b \in F$ whenever $a, b \in F$ with $a \perp b$.

From the point of view of quantum theory, the most important example of an effect algebra comes from the set $\mathcal{E}(H)$ of all self-adjoint operators A on a Hilbert space H satisfying $0 \leq A \leq I$, where the partial ordering comes from the partial ordering of self-adjoint operators [2, 3, 26]. For $A, B \in \mathcal{E}(H)$ we define $A \perp B$ if $A + B \in \mathcal{E}(H)$, in which case $A \oplus B = A + B$. Then $(\mathcal{E}(H); \oplus, I, 0)$ is an effect algebra, which is called the *Hilbert space effect algebra*. The elements $A \in \mathcal{E}(H)$ are called *quantum effects*. They correspond to yes-no quantum measurements that may be unsharp. The set $\mathcal{P}(H)$ of projection operators on H forms an orthomodular lattice which is a sub-effect algebra of $\mathcal{E}(H)$. It can be shown that the elements of $\mathcal{P}(H)$ correspond to sharp elements of $\mathcal{E}(H)$, that is, $\mathcal{P}(H) = \mathcal{E}(H)_S$.

An effect algebra E is *monotone σ-complete* if every ascending sequence $(a_i)_{i\in\mathbb{N}}$ has a supremum $a = \bigvee_{i\in\mathbb{N}} a_i$ in E, equivalently, if any descending sequence $(b_i)_{i\in\mathbb{N}}$ has an infimum $b = \bigwedge_{i\in\mathbb{N}} b_i$.

If E and F are effect algebras, a mapping $\phi : E \to F$ is *additive* if $a \perp b$ implies $\phi(a) \perp \phi(b)$ and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$. If $\phi : E \to F$ is additive and $\phi(1) = 1$, then ϕ is a *morphism*. If $\phi : E \to F$ is a morphism and $\phi(a) \perp \phi(b)$ implies $a \perp b$, then ϕ is called a *monomorphism*. A morphism ϕ is an *isomorphism* if ϕ is bijective and ϕ^{-1} is also a morphism. It is easy to see that a morphism ϕ is an isomorphism if and only if ϕ is a surjective monomorphism.

For $a \in E$, $a \neq 0$, define the interval $F = [0, a] = \{b \in E : 0 \leq b \leq a\}$. For $b, c \in E$ define $b \oplus_F c = b \oplus c$ if $b \perp c$ and $b \oplus c \in F$. Then $(F; \oplus_F, 0, a)$ becomes an effect algebra. If a is a principal element, then $\oplus_F$ coincides with $\oplus$. Suppose that p and p' are principal elements of E. Let

$$F = [0, p] \oplus [0, p'] = \{a \oplus b : a \leq p,\ b \leq p'\}.$$

With the restriction $\oplus/F$ of $\oplus$ to F then $(F; \oplus/F, 0, 1)$ is a sub-effect algebra of E.

An additive map $J : E \to E$ is a *retraction* if $a \leq J(1)$ implies that $J(a) = a$ [8, 20]. The converse that $J(a) = a$ implies that $a \leq J(1)$ automatically holds for any additive map J. We call $J(1)$ the *focus* of the retraction J. A retraction is *direct* if $J(a) \leq a$ for all $a \in E$.

We denote the kernel of J by $\ker(J) = \{a \in E : J(a) = 0\}$ and the image of J by $J(E)$. For retractions J and I we say that I is the *supplement* of J if $\ker(J) = I(E)$ and $\ker(I) = J(E)$.

Basic properties of retractions are collected below (see also [20]).

LEMMA 2.1. *Let J be a retraction on an effect algebra E with focus p. The following statements hold.*

(i) $J \circ J = J$.
(ii) $a \leq p' \implies J(a) = 0$.
(iii) *p is principal, hence sharp.*
(iv) $p \leq a \implies J(a) = p$.
(v) *$F := \{a \in E : J(a) = a\} = [0, p]$, and F becomes an effect algebra with unit p. If I is a retraction on F, then $I \circ J$ is a retraction on E with focus $I(p)$.*

Proof: (i) $a \leq 1 \implies J(a) \leq J(1) = p \implies J(J(a)) = J(a)$.

(ii) $a \leq p' \implies a \perp p$, hence $J(a) \perp J(p)(= p)$, so that $J(a) \oplus p = J(a \oplus p) \leq J(1) = p$, whence $J(a) = 0$.

(iii) Assume $a, b \leq p$, $a \perp b$. Then $a = J(a), b = J(b)$, $a \oplus b = J(a) \oplus J(b) = J(a \oplus b) \leq J(1) = p$.

(iv) $p \leq a \implies p = J(p) \leq J(a) \leq J(1) = p$.

(v) If $a \leq p$, then $J(a) = a$ by definition. Assume that $a = J(b) \in F$, then $J(a) = J(J(b)) = J(b) = a$, whence $a \leq p$.

Since p is principal, $F = [0, p]$ is an effect algebra with unit p. Let I be a retraction on F. Then $I \circ J$ is additive, and $I \circ J(1) = I(p)$. Since $I : F \to F$, we have $I(p) \leq p$. If $a \leq I(p)$, then $I \circ J(a) = I(J(a)) = I(a) = a$. □

By Lemma 2.1 (ii), $a \leq p'$ implies $J(a) = 0$. A retraction J is a *compression* if

$$J(a) = 0 \iff a \leq p'.$$

The following lemma is easy to prove.

LEMMA 2.2. *Let J be a retraction on an effect algebra E with focus p. The following statements are equivalent.*

(i) *J is a compression.*

(ii) $J(a) = p \implies p \leq a$.

(iii) $\ker(J) = [0, p']$.

Moreover, if a retraction J has a supplement I, then both J and I are compressions and $I(1) = J(1)'$.

DEFINITION 2.1 ([11]). A sub-effect algebra S of an effect algebra E is called *normal* if for all $e, f, d \in E$ with $e \oplus f \oplus d \in E$, we have $e \oplus d$, $f \oplus d \in S \implies d \in S$.

Recall that two elements $a, b \in E$ are *coexistent* if there are $a_1, b_1, c \in E$ such that $a_1 \oplus b_1 \oplus c$ exist in E and $a = a_1 \oplus c$, $b = b_1 \oplus c$[1]. We shall write $a \leftrightarrow b$ if $a, b \in E$ are coexistent. If $F \subseteq E$, and $a, b \in E$, we say that a, b are *coexistent in* F if there are $a_1, b_1, c \in F$ with $a_1 \oplus b_1 \oplus c \in E$ such that $a = a_1 \oplus c$, $b = b_1 \oplus c$.

LEMMA 2.3 ([21]). *Let F be a normal sub-effect algebra of E and let $a, b \in F$. If a, b are coexistent in E, then a, b are coexistent in F.*

EXAMPLE 2.1. A simple example of a *sub-effect algebra that is not normal* is obtained as follows. Let $X := \{a, b, c, d\}$ and organize the set 2^X of all subsets of X into a Boolean algebra in the usual way. Then $E = (2^X, \oplus, \emptyset, X)$ is an effect algebra with $A \perp B$ iff $A \cap B = \emptyset$, in which case $A \oplus B = A \cup B$. Clearly, every two elements of E are coexistent in E. The subset F of 2^X consisting of $\emptyset, \{a, b\}, \{c, d\}, \{a, c\}, \{b, d\}, X$ is a sub-effect algebra of E. But, for example, the elements $\{a, b\}$ and $\{a, c\}$ are not coexistent in F, so that in view of Lemma 2.3, F is not a normal sub-effect algebra of E.

[1]Some authors refer to this property as 'Mackey compatibility'

DEFINITION 2.2 ([11, 21]). A *compression base* on an effect algebra E is a family $(J_p)_{p\in P}$ of compressions on E, indexed by a normal sub-effect algebra P of E, such that

(C1) each $p \in P$ is the focus of J_p,

(C2) if $p, q, r \in P$ and $p \oplus q \oplus r \in E$, then $J_{p\oplus q} \circ J_{q\oplus r} = J_q$.

An effect algebra with a compression base will be called a *CB-effect algebra.*

A compression base is *proper* if every direct compression belongs to $(J_p)_{p\in P}$; it is *direct* if it is the family of all direct compressions on E; it is *total* if every retraction on E is a compression and belongs to the family $(J_p)_{p\in P}$.

We make the following standing assumption:

Henceforth, E is a CB-effect algebra with compression base $(J_p)_{p\in P}$.

Proof of the following lemma is straightforward.

LEMMA 2.4. *Let $a, b \in E$, $p \in P$.*

(i) $J_p(a \oplus b) = J_p(a) \oplus J_p(b)$.

(ii) $J_0(a) = 0$, $J_1(a) = a$.

(iii) $J_p(a) \leq p$.

(iv) $a \leq b \implies J_p(a) \leq J_p(b)$.

(v) $a \leq p \Leftrightarrow J_p(a) = a$, $p \leq a \Leftrightarrow J_p(a) = p$.

(vi) $J_p(a) = 0 \Leftrightarrow p \perp a$.

(vii) $a \leq b \implies J_p(b \ominus a) = J_p(b) \ominus J_p(a)$.

THEOREM 2.1. *The set P as a sub-effect algebra of E, is an orthomodular poset (OMP) and if $p \in P$, then $J_{p'}$ is a supplement of J_p.*

Proof: To prove the first statement, it suffices to prove that $p, q \in P, p \perp q$ imply $p \oplus q = p \vee q$. It is clear that $p, q \leq p \oplus q$. Suppose that $p, q \leq a, a \in E$. Then there exists $b \in E$ such that $p \oplus b = a$, and then $q = J_q(a) = J_q(p \oplus b) = J_q(p) \oplus J_q(b) = J_q(b)$ (by Lemma 2.4 (vi), and by Lemma 2.4 (v)), $q \leq b$. Hence $p \oplus q \leq p \oplus b = a$. Thus $p \oplus q = p \vee q$. That is, $p \oplus q \in P$ is the supremum of p and q in E, and hence also in P. By Lemma 2.4 (v) and (vi), $J_p(a) = 0$ iff $J_{p'}(a) = a$. Hence $J_{p'}$ is a supplement of J_p. □

THEOREM 2.2 ([21, Theorem 3.6]). *If $p, q \in P$, then the following statements are equivalent.*

(i) $q \leq p$.

(ii) $J_p \circ J_q = J_q$.

(iii) $J_p(q) = q$.

(iv) $J_q \circ J_p = J_q$.

(v) $J_q(p) = q$.

THEOREM 2.3 ([21, Theorem 3.7]). *If $p, q \in P$, then the following statements are equivalent.*

(i) $J_p(q) = 0$.
(ii) $p \perp q$.
(iii) $J_q(p) = 0$.
(iv) $p \perp q$ *and* $J_{(p \oplus q)'} = J_{p'} \circ J_{q'} = J_{q'} \circ J_{p'}$.

3. Examples

EXAMPLE 3.1. *The unit interval* $\mathcal{E}(\mathcal{A})$ *in the self-adjoint part of a unital C*-algebra* $\mathcal{A}$ *with Naimark compressions* $J_p(e) := pep$*, where* p *satisfies* $p = p^2 = p^*$. It has been proved in [8] that $(J_p)_p$ is a total compression base for $\mathcal{E}(\mathcal{A})$. As special cases, we have the standard Hilbert space CB-effect algebra $\mathcal{E}(H)$ as well as the unit interval in the commutative C*-algebra of continuous functions on a compact Hausdorff space.

EXAMPLE 3.2. *MV-algebras and MV-effect algebras.* MV-algebras were introduced by Chang [4] as the algebraic bases for many-valued logic. We recall that an *MV-algebra* is an algebra $(M; \dot{+}, ', 0)$, where $\dot{+}$ is a commutative and associative binary operation on M having 0 as a neutral element, $'$ is an involutive unary operation such that $a \dot{+} 0' = 0'$ for all $a \in M$, and in addition, the Łukasiewicz identity $a \dot{+} (a \dot{+} b')' = b \dot{+} (b \dot{+} a')'$ is satisfied for all $a, b \in M$. An MV-algebra is partially ordered by the relation $a \leq b$ iff $a' \dot{+} b = 1$, where we define $1 := 0'$. In this ordering we have $0 \leq a \leq 1$ for all $a \in M$, and M becomes a distributive lattice.

Given an MV-algebra, if we restrict the total operation $\dot{+}$ to elements (a, b) for which $a \leq b'$ (equivalently, $b \leq a'$), and for those elements (a, b) we put $a \oplus b := a \dot{+} b$, then $(M; \oplus, 0, 1)$ becomes an effect-algebra with orthosupplement a'. On the other hand, an effect algebra $(E; \oplus, 0, 1)$ can be endowed with a total binary operation $\dot{+}$ extending $\oplus$ such that $(E; \dot{+}, ', 0)$ is an MV-algebra if and only if E is lattice ordered and for every $a, b \in E$, $(a \vee b) \ominus b = a \ominus (a \wedge b)$ (equivalently, if $a \leftrightarrow b$) for all $a, b \in E$ [5, 25, 6]. Such an effect algebra is called an *MV-effect algebra.* Notice that the partial order in an MV-algebra coincides with the partial order in the corresponding MV-effect algebra. In what follows, we use MV-algebras and MV-effect algebras as equivalent notions.

A standard example of an MV-algebra is the unit interval $[0, 1]$ of the real line $\mathbb{R}$ with the $\dot{+}$ operation defined as $a \dot{+} b = \min(a + b, 1)$ and $a' = 1 - a$ (here $+$ and $-$ denote the usual addition and difference of real numbers). More generally, a set $[0, 1]^X$ of functions $f : X \to [0, 1]$, where X is any nonempty set, with the operations $(f \dot{+} g)(x) = \min(f(x) + g(x), 1)$ and $f'(x) = 1(x) - f(x)$, where $1(x) = 1$ $\forall x \in X$, is an MV-algebra, which is called the MV-algebra of fuzzy sets. Every Boolean algebra $(B; \vee, \wedge, ', 0, 1)$ is an MV-algebra with the $\dot{+}$ operation defined by $a \dot{+} b = a \vee b$ and the remaining operations defined as in the Boolean algebra, is an MV-algebra in which $a \dot{+} a = a$, $a \in B$. Conversely, every MV-algebra in which the latter property is satisfied for all its elements, is a Boolean algebra.

It was proved in [27], that MV-algebras are categorically equivalent with unital lattice ordered groups (ℓ-groups), where the functor Γ assigns to every ℓ-group $(G; u)$ with unit u the MV-algebra $\Gamma(G; u)$ consisting of the unit interval $[0, u]$ of G endowed with the operations $g' = u - g$ and $g \dotplus h = (g + h) \wedge u$. An element $a \in M$ is *idempotent* if $a \dotplus a = a$. An element in $a \in M$ is *sharp* if $a \wedge a' = 0$. The idempotents and sharp elements in M coincide and form a Boolean subalgebra $B(M)(= M_S)$ of M.

THEOREM 3.1. *Let M be an MV-effect algebra.*

(i) *For every retraction J with focus p on M and every $a \in M$, $J(a) = a \wedge p$.*
(ii) *Every retraction is a direct compression, and it is uniquely defined by its focus.*
(iii) *If $(J_p)_{p\in P}$ is a compression base of M, then P is a Boolean subalgebra of $B(M)$.*
(iv) *If the compression base $(J_p)_{p\in P}$ is direct (i.e. consists of all direct compressions on M), then $(J_p)_{p\in M}$ contains all retractions on M, and $P = B(M)$.*

Proof: (i) Let p be a focus of a retraction J. Then $p \in B(M)$. It is well known that for every $a \in M$ and $p \in B(M)$, we have $a = a \wedge p \oplus a \wedge p'$. Therefore $J(a) = J(a \wedge p) \oplus J(a \wedge p') = a \wedge p$.

(ii) Let J be a retraction with focus p. If $J(a) = 0$, then $a = a \wedge p' \leq p'$, hence J is a compression. If I and J are retractions such that $J(1) = p = I(1)$, then $J(a) = I(a) = a \wedge p$ for all $a \in M$. Let $(J_p)_{p\in P}$ be a compression base for M. Then $P \subseteq B(M)$, P is a normal sub-effect algebra of M, which is an orthomodular poset. Now for every $p, q \in P$ we have $p \leftrightarrow q$ in M, hence in P, which entails that P is a Boolean algebra.

(iv) By (ii), every retraction I is a direct compression, therefore $I \in (J_p)_{p\in P}$. For every $p \in B(M)$, the mapping $J_p : M \to M$, $J_p(a) = a \wedge p$ is a retraction (hence a compression on M, [30, Proposition 3.1], and hence $P = B(M)$. □

Moreover, it was proved in [30, Proposition 3.1] that every compression on M uniquely extends to a compression on the corresponding unital ℓ-group $(G; u)$ such that $M = \Gamma(G; u)$.

EXAMPLE 3.3. *Intervals in CB-groups.* A *unital group* is a directed abelian group G with a distinguished element $u \in G^+$, called the *unit*, such that the set $E(G) := \{e \in G : 0 \leq e \leq u\}$, called the *unit interval*, generates G^+ in the sense that every element in G^+ is a finite sum of (not necessarily distinct) elements of $E(G)$. Since G is directed, we have $G = G^+ - G^+$, hence $E(G)$ generates G as a group.

The unit interval E in a unital group G with unit u forms an effect algebra with unit u under the restriction of $+$ to E [1].

DEFINITION 3.1 ([8]). Let G be a unital group with unit u and unit interval E. A mapping $J : G \to G$ is called a *retraction with focus p on G* if J is an order-preserving group endomorphism, $p = J(u) \in E$, and for all $e \in E$, $e \leq p \implies$

$J(e) = e$. A retraction $J : G \to G$ is said to be *direct* if $g \in G^+ \implies J(g) \leq g$. A retraction J on G is called a *compression* if $J^{-1}(0) \cap E = \{e \in E : e + J(u) \in E\}$ [8]. Two retractions J and J' on G are called *quasicomplements* of each other if, for all $g \in G^+$, $J(g) = g \Leftrightarrow J'(g) = 0$ and $J'(g) = g \Leftrightarrow J(g) = 0$.

If J is a retraction on G, then J is an idempotent, i.e. $J \circ J = J$ and its focus is a principal, hence sharp element of E [7, Lemma 2.3]. If J and J' are quasicomplements, they are necessarily compressions [7, Lemma 3.2 (iii)].

DEFINITION 3.2. By a *compression base* for the unital group G with unit interval E [12], we mean a family $(J_p)_{p \in P}$ of compressions on G, indexed by a normal sub-effect algebra P of E, such that (i) each $p \in P$ is the focus of J_p and (ii) if $p, q, r \in P$ and $p + q + r \in E$, then $J_{p+r} \circ J_{q+r} = J_r$. A compression base $(J_p)_{p \in P}$ for G is *proper* if every direct compression on G belongs to the family $(J_p)_{p \in P}$; it is *direct* if it is the family of all direct compressions on G; and it is *total* if every retraction on G is a compression and belongs to the family $(J_p)_{p \in P}$.
A unital group G with a compression base is called a *CB-group*.

If G is a unital group with unit u and $(J_p)_{p \in P}$ is a compression base for G, then for each $p \in P$, we have $u - p \in P$, and J_{u-p} is the unique compression in the compression base that is a quasicomplement of J_p.

DEFINITION 3.3 ([7, Definition 3.3]). A *compressible group* is a unital group for which every retraction is determined by its focus and every retraction has a quasicomplementary retraction (and hence is a compression).

Proof of the following theorem is straightforward.

THEOREM 3.2. *Let (G, u) be a unital group, and let $E = \{a \in G : 0 \leq a \leq u\}$ be the unit interval of G. Let $(J_p)_{p \in P}$ be a compression base for (G, u). Then $(\tilde{J}_p)_{p \in P}$, where $\tilde{J}_p = J_p / E$ is the restriction of J_p to E, is a compression base for E.*

We note that both Examples 3.1 and 3.2 are special cases of unit intervals in compressible groups.

EXAMPLE 3.4. *Any effect algebra E is organized into a proper CB-effect algebra by taking all direct compressions on E, indexed by their own foci, as the compression base.* To prove the latter statement, let us first recall that an element $p \in E$ is *central* if p and p' are principal, and every $a \in E$ admits a decomposition $a = b \oplus c$ with $b \leq p$, $c \leq p'$ ([16, 6]). It can be shown that the latter decomposition is unique, and $b = a \wedge p, c = a \wedge p'$. The set C of all central elements, called the *center* of E, is a sub-effect algebra of E, and if $a, b \in C$, then $a \vee b$ and $a \wedge b$ exist in E and belong to C. Moreover, C forms a Boolean algebra [16, Theorem 5.4].
(a) C is a normal sub-effect algebra of E. Indeed, assume that $e, f, d \in E$ with $e \oplus f \oplus d \in E$, and $p = e \oplus d \in C$, $q = f \oplus d \in C$. Clearly, $e \leq p$, $f \leq p'$, hence $e \wedge f = 0$. Let $c = p \wedge q \in C$. Then $d \leq p, q$ implies $d \leq c$. We have

$p = c \oplus (p \ominus c) = d \oplus (c \ominus d) \oplus (p \ominus c) = d \oplus e$, $q = c \oplus (q \ominus c) = d \oplus (c \ominus d) \oplus (q \ominus c) = d \oplus f$. It follows that $c \ominus d \le e, f$, whence $c \ominus d = 0$, so that $d = c \in C$.

(b) The focus of every direct compression belongs to C. Indeed, let $J : E \to E$ be a direct compression and let $p = J(1)$ be the focus of J. Define $J' : E \to E$ by $J'(a) = a \ominus J(a)$. Then $J'(1) = p'$ and J' is a direct compression supplementary to J. It follows that p and p' are principal, and for all $a \in E$, $a = J(a) \oplus J'(a)$, where $J(a) \le p$, $J'(a) \le p'$. Hence p is central.

(c) For every $p \in C$, the mapping $J_p : E \to E$, $J_p(a) = a \wedge p$ is a direct compression. Indeed, $a \le p \Leftrightarrow J_p(a) = a$, $a \le p' \Leftrightarrow J_p(a) = 0$. It remains to prove that J_p is additive. Let $a \perp b$, then

$$\begin{aligned} a \oplus b &= (a \oplus b) \wedge p \oplus (a \oplus b) \wedge p' \\ &= (a \wedge p \oplus a \wedge p') \oplus (b \wedge p \oplus b \wedge p') \\ &= (a \wedge p \oplus b \wedge p) \oplus (a \wedge p' \oplus b \wedge p'), \end{aligned}$$

where $a \wedge p \oplus b \wedge p \le p$, $a \wedge p' \oplus b \wedge p' \le p'$, and uniqueness of such a decomposition implies that $J_p(a \oplus b) = (a \oplus b) \wedge p = a \wedge p \oplus b \wedge p = J_p(a) \oplus J_p(b)$. Finally, for $p, q, r \in C$ with $p \oplus q \oplus r \in E$ and $a \in E$ we have $J_{p \oplus r}(J_{q \oplus r}(a)) = (p \oplus r) \wedge (q \oplus r) \wedge a = r \wedge a = J_r(a)$.

Note that Example 3.2 is a special case of Example 3.4.

4. Compatibility and commutants

We maintain our standing assumption that E is a CB-effect algebra with compression base $(J_p)_{p \in P}$.

In agreement with [7] and [21], we will say that elements a and p are *compatible*, or that a and p *commute* if

$$a = J_p(a) \oplus J_{p'}(a). \tag{1}$$

Define the *commutant* of p by $C(p) = \{a \in E : a = J_p(a) \oplus J_{p'}(a)\}$.

LEMMA 4.1. *If $p \in P$, $a \in E$, then the following statements are equivalent.*

(i) $J_p(a) \le a$,
(ii) $a \in C(p)$,
(iii) $a \in [0, p] \oplus [0, p']$,
(iv) $a \leftrightarrow p$,
(v) $J_p(a) = p \wedge a$.[2]

Proof: Equivalence of (i), (ii) and (iii) was proved in [21, Lemma 4.1]. (iii)$\Leftrightarrow$(iv). Assume (iii), then $a = a_1 \oplus a_2$ with $a_1 \le p$, $a_2 \le p'$. Let $d \in E$ be such that $a_1 \oplus d = p$. Then $a_1 \oplus d \oplus a_2$ exists in E, whence $a \leftrightarrow p$. Conversely, if $a \leftrightarrow p$, then $a = a_1 \oplus c$, $p = p_1 \oplus c$, and $a_1 \oplus p_1 \oplus c \in E$. It follows that $a_1 \le (p_1 \oplus c)' = p'$,

[2] If $a \in P$, then $J_p(a)$ is also the infimum $a \wedge_P p$ of a and p in P. Indeed, by Lemma 2.3, the coexistence of a and p in E implies the coexistence of a and p in P, and hence $a = r \oplus s$, where $r = a \wedge_P p$, $s = a \wedge_P p'$. Then $J_p(a) = J_p(r) \oplus J_p(s) = J_P(r) = r$.

and hence (iii) holds. (ii)$\Rightarrow$(v). If $a \in C(p)$, then $a = J_p(a) \oplus J_{p'}(a)$. We have $J_p(a) \leq a, p$. Assume that for $d \in E$, $d \leq a, p$. Then $d = J_p(d) \leq J_p(a)$, and hence $J_p(a) = p \wedge a$. (v)$\Rightarrow$(i) is evident. $\square$

THEOREM 4.1 ([21, Theorem 4.2]). *For $p, q \in P$ the following statements are equivalent.*

(i) $J_p \circ J_q = J_q \circ J_p$,
(ii) $J_p(q) = J_q(p)$,
(iii) $J_p(q) \leq q$,
(iv) $p \leftrightarrow q$,
(v) *There exists an $r \in P$ such that $J_p \circ J_q = J_r$,*
(vi) $J_p(q) \in P$,
(vii) $p \in C(q)$.

THEOREM 4.2 ([21, Theorem 4.4]). *Let $p \in P$, define $H = J_p(E)$, $P_H = \{q \in P : q \leq p\}$ and for every $q \in P_H$, let J_q^H be the restriction of J_q to H. Then the following statements hold.*

(i) *H is an effect algebra with unit p and*

$$H = \{a \in E : J_p(a) = a\} = [0, p].$$

(ii) *If $q \in P_H$, then J_q^H is a compression on H.*
(iii) *$(J_P^H)_{q \in P_H}$ is a compression base for H.*

THEOREM 4.3 ([21, Theorem 4.5]). *Let $p \in P$ and let $C = C(p)$. For each $q \in C \cap P$, let J_q^C be the restriction of J_q to C.*

(i) *$C = [0, p] \oplus [0, p']$ is a sub-effect algebra of E.*
(ii) *If $q \in C \cap P$, then J_q^C is a compression on C.*
(iii) *$(J_q^C)_{q \in C \cap P}$ is a compression base for C.*

LEMMA 4.2. *Suppose that $p, q \in P$ with $p \perp q$, $a \in E$, and at least one of $a \in C(p)$ or $a \in C(q)$ holds. Then $J_{p \oplus q}(a) = J_p(a) \oplus J_q(a)$.*

Proof: Suppose, for definiteness, that $a \in C(p)$ and let $r := (p \oplus q)'$, so that $p \oplus q \oplus r = 1$. Then $J_{p \oplus q}(a) = J_{p \oplus q}(J_p(a) \oplus J_{p'}(a)) = J_{p \oplus q}(J_p(a)) \oplus J_{p \oplus q}(J_{q \oplus r}(a)) = J_p(a) \oplus J_q(a)$. $\square$

By induction, we obtain the following.

COROLLARY 4.1. *If $p_1, p_2, \ldots, p_n \in P$, $p_1 \oplus p_2 \oplus \cdots \oplus p_n$ exists and $a \in \bigcap_{i=1}^n C(p_i)$, $i = 1, 2, \ldots, p_n$, then*

$$J_{p_1 \oplus p_2 \oplus \cdots \oplus p_n}(a) = J_{p_1}(a) \oplus \cdots \oplus J_{p_n}(a).$$

If in addition $p_1 \oplus p_2 \oplus \cdots \oplus p_n = 1$, then $a \in \bigcap_{i=1}^n C(p_i)$ if and only if

$$a = J_{p_1}(a) \oplus \cdots \oplus J_{p_n}(a).$$

For $a \in E$, define the *projection commutant of* a to be

$$C_P(a) = \{p \in P : a \in C(p)\}.$$

THEOREM 4.4. *For every* $a \in E$, $C_P(a)$ *is a sub-orthomodular poset of* P.

Proof: Clearly, $0, 1 \in C_P(a)$. Suppose that $p \in C_P(a)$, so that $a = J_p(a) \oplus J_{p'}(a)$. Then $a = J_{p'}(a) \oplus J_{p''}(a)$, so that $p' \in C_P(a)$. Suppose that $p, q \in C_P(a)$ with $p \perp q$. By Lemma 4.2 we have $J_{p\oplus q}(a) = J_p(a) \oplus J_q(a)$. Therefore, using Lemma 4.2 and Theorem 2.3 (iv),

$$J_{p\oplus q}(a) \oplus J_{(p\oplus q)'}(a) = J_p(a) \oplus J_q(a) \oplus J_{p'}(J_{q'}(a)).$$

From

$$J_{p'}(a) = J'_p(J_q(a) \oplus J_{q'}(a)) = J_q(a) \oplus J_{p'}(J_{q'}(a))$$

we have that

$$J_{p'}(J_{q'}(a)) = J_{p'}(a) \ominus J_q(a).$$

Hence,

$$J_{p\oplus q}(a) \oplus J_{(p\oplus q)'}(a) = J_p(a) \oplus J_{p'}(a) = a.$$

Thus, $p \oplus q \in C_P(a)$, which concludes the proof. □

For $M \subseteq P$, we write $C(M) := \bigcap_{p\in M} C(p)$.

THEOREM 4.5. *Let* $p, q \in P$, $p \leftrightarrow q$. *Then* $C(p, q) \subseteq C(p \wedge q) \cap C(p \vee q)$.

Proof: Assume $p \leftrightarrow q$, $p, q \in P$. Then there are $p_1, q_1, r \in P$ with $p_1 \oplus q_1 \oplus r = p_1 \vee q_1 \vee r \in P$ and $p = r \oplus p_1$, $q = r \oplus q_1$. Moreover, $r = p \wedge q$, $p_1 = p \wedge q'$, $q_1 = p \wedge q'$, and $p \vee q = p_1 \vee q_1 \vee r$ ([28]). We also know that $J_p(q) = J_q(p) = p \wedge q = r$. Let $p_1 \oplus q_1 \oplus r \oplus d = 1$. Now let $e \in C(p, q)$. Then we have

$$\begin{aligned} e &= J_{p_1\oplus r}(e) \oplus J_{q_1\oplus d}(e) \\ &= J_{p_1\oplus r}(J_{q_1\oplus r}(e) \oplus J_{p_1\oplus d}(e)) \oplus J_{q_1\oplus d}(J_{q_1\oplus r}(e) \oplus J_{p_1\oplus d}(e)) \\ &= J_r(e) \oplus J_{p_1}(e) \oplus J_{q_1}(e) \oplus J_d(e), \end{aligned}$$

and by Corollary 4.1, then $e \in C(p \wedge q)$. Using duality and the fact that $C(p, q) = C(p', q')$, we obtain that $e \in C(p \vee q)$. □

COROLLARY 4.2. (i) *Let* $p_1, p_2, \ldots, p_n$ *be pairwise compatible elements of* P. *Then* $p_1 \wedge p_2 \wedge \cdots \wedge p_n$ *and* $p_1 \vee p_2 \vee \cdots \vee p_n$ *exist in* E *and belong to* P, *and* $\bigcap_{i=1}^n C(p_i) \subseteq C(p_1 \wedge \cdots \wedge p_n) \cap C(p_1 \vee \cdots \vee p_n)$.

(ii) *If* $p, q, r \in P$ *are pairwise compatible, then* $p \leftrightarrow r \wedge q$ *and* $p \leftrightarrow r \vee q$. *That is,* P *is a regular orthomodular poset.*

COROLLARY 4.3. *For every* $a \in E$, $C_P(a)$ *is a normal sub-effect algebra of* P.

Proof: Let $e, f, d \in P$ with $e \oplus f \oplus d \in P$, and $e \oplus d, f \oplus d \in C_P(a)$. Then $e \oplus d \leftrightarrow f \oplus d$ (in P), and $d = (e \oplus d) \wedge (f \oplus d)$. Then, by Theorem 4.5, $a \in C(e \oplus d, f \oplus d)$ implies $a \in C(d)$, hence $d \in C_P(a)$. □

THEOREM 4.6. *Let E be monotone σ-complete CB-effect algebra, and let $M \subseteq P$. Assume that $(e_i)_{i\in\mathbb{N}} \subseteq C(M)$ is an ascending sequence of elements. Then $\bigvee_E\{e_i : i \in \mathbb{N}\} \in C(M)$. That is, $C(M)$ is monotone σ-complete.*

Proof: It suffices to prove the statement for $M = \{p\}$. So let (e_i) be an ascending sequence in $C(p)$. Put $h_i = J_p(e_i)$, $k_i = J_{p'}(e_i)$, $i \in \mathbb{N}$. As $e_i \in C(p)$, we have $e_i = h_i \oplus k_i$ $\forall i \in \mathbb{N}$. Since $(h_i)_{i\in\mathbb{N}}$, $(k_i)_{i\in\mathbb{N}}$ are ascending sequences, and E is monotone σ-complete, $h = \bigvee_E\{h_i : i \in \mathbb{N}\}$, $k = \bigvee_E\{k_i : i \in \mathbb{N}\}$ exist in E. As $h_i \leq p$ $\forall i \in \mathbb{N}$, we have $h \leq p$, and hence $J_p(h) = h$ and $J_{p'}(h) = 0$. Similarly, $J_{p'}(k) = k$, $J_p(k) = 0$. But then $J_p(h \oplus k) \oplus J_{p'}(h \oplus k) = h \oplus k$, hence $h \oplus k \in C(p)$. Clearly, $h_i \oplus k_i \leq h \oplus k$ $\forall i \in \mathbb{N}$. Let $c \in E$, $h_i \oplus k_i \leq c$ $\forall i \in \mathbb{N}$. Fix $i \in \mathbb{N}$. Let $j \in \mathbb{N}$, and let $m = \max(i, j)$. Then $h_i \oplus k_j \leq h_m \oplus k_m \leq c$. This implies that $k_j \leq c \ominus h_i$, hence $k \leq c \ominus h_i$, and hence $h_i \leq c \ominus k$. Since $i \in \mathbb{N}$ is arbitrary, we obtain that $h \leq c \ominus k$, that is, $h \oplus k \leq c$. It follows that $h \oplus k = \bigvee_E\{h_i \oplus k_i : i \in \mathbb{N}\} = \bigvee_E\{e_i : i \in \mathbb{N}\} \in C(p)$. □

5. Rickart mapping and projection cover property

We maintain our convention that E is a CB-effect algebra with compression base $(J_p)_{p\in P}$.

The following definitions and results are inspired by [7, 9, 19, 21].

DEFINITION 5.1. We say that $e \in E$ has a *projection cover* $c \in P$ if for all $p \in P$, $e \leq p \iff c \leq p$. We say that E has the *projection cover property* (PC) if every $e \in E$ has a projection cover in P.

Let $\gamma(e)$ denote the projection cover of the element $e \in E$. Clearly, $\gamma(e)$ is uniquely defined and

$$\gamma(e) = \bigwedge\{p \in P : e \leq p\}.$$

DEFINITION 5.2. Let E be an effect algebra with the compression base $(J_p)_{p\in P}$. We say that E has the *Rickart projection property* (RP) if there is a mapping $^* : E \to P$ such that

$$\forall p \in P, \ J_p(e) = 0 \iff p \leq e^*.$$

PROPOSITION 5.1. *E has the Rickart projection property if and only if it has the projection cover property, and $\gamma(e) = (e^*)^*$.*

Proof: Let E have RP. For every $e \in E$ and $p \in P$ we have $J_p(e) = 0 \iff e \leq p'$. This yields $p \leq e^* \iff p \leq e'$, whence $e^* = \bigvee\{p \in P : p \leq e'\}$. If $e \in P$, then $e^* = e'$. Clearly, $e, f \in E$, $e \leq f$ implies $f^* \leq e^*$. Therefore $e^{**} = e^{*\prime} = \bigwedge\{p \in P : e \leq p\}$. Consequently, $e^{**} = \gamma(e)$.

Now assume that E has PC. Then $p \in P$, $e \leq p \iff \gamma(e) \leq p$ implies that $J_{p'}(e) = 0 \iff p' \leq \gamma(e)'$. We see that $e^* = \gamma(e)'$ is a Rickart mapping. □

The most important property of CB-effect algebras with PC is the following theorem.

THEOREM 5.1. *If E has PC, then the set P of projections of E is an orthomodular lattice. Moreover, for every $p, q \in P$, $p \wedge q = \gamma(J_p[\gamma(J_p(q'))'])$.*

Proof: We follow [10]. Put $t := \gamma(J_p(q'))'$ and $s := \gamma(J_p(t))$. Then $\gamma(J_p(q')) \leq p$, so $J_p(t) = J_p(1 \ominus \gamma(J_p(q'))) = p \ominus \gamma(J_p(q')) \in P$. Hence $s = \gamma(J_p(t)) = J_p(t) = p \ominus \gamma(J_p(q'))$. We have $J_p(q') \leq \gamma(J_p(q')) = t'$, whence $J_t(J_p(q')) = 0$. As $\gamma(J_p(q')) \leq p$, we have $\gamma(J_p(q')) \in C(p)$, and it follows $t = \gamma(J_p(q'))' \in C(p)$. Therefore $J_p \circ J_t = J_t \circ J_p$ is a compression with focus $J_p(J_t(1)) = J_p(t) = s$. Consequently, $J_s(q') = J_t(J_p(q')) = 0$, so $s \leq q$. We also have $s = \gamma(J_p(t)) \leq p$. Let $r \in P$, $r \leq p, q$. Then $r = \gamma(J_p(r)) \leq q$, which entails $q' \leq \gamma(J_p(r))' = r'$, and then $J_p(q') \leq \gamma(J_p(q')) \leq r'$. Hence $r \leq \gamma(J_p(q'))'$, which implies that $r = \gamma(J_p(r)) \leq \gamma(J_p(\gamma(J_p(q'))')) = s$.

If $e \in E$, $e \leq p, q$, then $e \leq \gamma(e) \leq p, q$, whence $e \leq \gamma(e) \leq s$. Hence s is the g.l.b. of p, q in both P and E. □

COROLLARY 5.1. *For $p, q \in P$, $\gamma(J_p(q)) = p \wedge (p' \vee q)$.*

Proof: We have $p' \vee q = (p \wedge q')' = \gamma(J_p[\gamma(J_p(q))'])'$, hence $p \wedge (p' \vee q) = J_p(p' \vee q) = J_p(\gamma(J_p[\gamma(J_p(q))'])') = J_p(1 \ominus \gamma(J_p(1 \ominus \gamma(J_p(q))))) = p \ominus J_p(\gamma(p \ominus J_p(\gamma(J_p(q))))) = p \ominus \gamma(p \ominus \gamma(J_p(q))) = p \ominus (p \ominus \gamma(J_p(q))) = \gamma(J_p(q))$. □

COROLLARY 5.2. *Every element in E can be written as a sum of two elements, one of which is a projection, and the other one does not majorize any nonzero projection.*

Proof: Put $\delta(e) = \gamma(e')'$. Clearly, $\delta(e)$ is the greatest projection lying under e. So we may write $e = \delta(e) \oplus c$ for some $c \in E$. If there is a nonzero projection $q \leq c$, then $\delta(e) \oplus q \leq e$, and $\delta(e) \oplus q$ is a projection, contradicting maximality of $\delta(e)$. □

We note that the element c in the preceding proof is called blunt [13], or meager [24].

COROLLARY 5.3. *If E is a proper CB-effect algebra, then E is a Boolean algebra if and only if $P = E$.*

Proof: If $P = E$, then E has the projection cover property, every element in E being its own projection cover. By Theorem 5.1, E is an orthomodular lattice, and for all $p, q \in P$, $p \wedge q = J_p(1 \ominus J_p(1 \ominus q)) = p \ominus (p \ominus J_p(q)) = J_p(q) = p \wedge (q \vee p')$. Consequently, $P = E$ is a Boolean algebra, and $J_p(q) = p \wedge q$. Conversely, let E be a proper CB-effect algebra such that E is a Boolean algebra. For every $p \in E$, $J_p(q) = p \wedge q$ is a direct compression, and as E is proper, we have $p \in P$. It follows that $P = E$. □

Let A be a partially ordered set, and $B \subseteq A$. We say that B is *sup/inf-closed in* A if for any subset $\emptyset \neq M \subseteq B$, if the supremum $s := \bigvee_A M$ exists in A then $s \in B$, and if the infimum $t := \bigwedge_A M$ exists in A, then $t \in B$.

THEOREM 5.2. *Suppose that E has the projection cover property. Then we have the following.*

(i) *Let $e \in E$. Then for every $p \in P$, $e \in C(p) \implies \gamma(e) \in C(p)$.*
(ii) *P is sup/inf-closed in E.*
(iii) *If E is monotone σ-complete, then P is a σ-complete OML.*
(iv) *Let $M \subseteq P$, and assume that $s = \bigwedge\{p : p \in M\}$, $s \in P$ (respectively, $t = \bigvee\{p : p \in M\} \in P$). Then $q \in P$, $q \in C(M)$ implies $q \in C(s)$ (respectively, $q \in C(M)$, $q \in P$ implies $q \in C(s)$).*

Proof: (i) Let $e \in E$, $p \in P$ and $e \in C(p)$. Then $e = J_p(e) \oplus J_{p'}(e)$, where $J_p(e) \leq p$, $J_{p'}(e) \leq p'$. Therefore $J_p(e) \leq \gamma(J_p(e)) \leq p$, $J_{p'}(e) \leq \gamma(J_{p'}(e)) \leq p'$. Then $\gamma(J_p(e)) \oplus \gamma(J_{p'}(e)) \in P$, and $\gamma(e) \leq \gamma(J_p(e)) \oplus \gamma(J_{p'}(e))$. Also, $J_p(e), J_{p'}(e) \leq e$, and so $\gamma(J_p(e)), \gamma(J_{p'}(e)) \leq \gamma(e)$. Since $\gamma(e)$ is principal, we get $\gamma(e) = \gamma(J_p(e)) \oplus \gamma(J_{p'}(e))$. As $\gamma(J_p(e)) \leq p$, $\gamma(J_{p'}(e)) \leq p'$, we have $\gamma(J_p(e)), \gamma(J_{p'}(e)) \in C(p)$, and therefore $\gamma(e) \in C(p)$.

(ii) Let $\emptyset \neq M \subseteq P$ and suppose that $t = \bigwedge_E\{p : p \in M\}$. As $t \leq p$ for all $p \in M$, we have $\gamma(t) \leq p$ for all $p \in M$, whence $\gamma(t) \leq t$. On the other hand, $t \leq \gamma(t)$, which gives $t = \gamma(t) \in P$. By duality we obtain that P is closed under the computation of suprema in E.

(iii) Follows from (ii) and the fact that P is a lattice.

(iv) Let $s = \bigwedge\{p : p \in M\} \in P$. If $q \in C(M)$, then $q \leftrightarrow p$ for all $p \in M$. Since P is an OML, by [28, Proposition 1.3.10], this entails that $q \leftrightarrow s$, that is, $q \in C(s)$. The rest of the proof follows by duality. $\square$

6. General comparability

We maintain our convention that E is a CB-effect algebra with compression base $(J_p)_{p\in P}$.

DEFINITION 6.1. (i) We will say that an element a in E has the b-*property* (or is a b-*element*) if there is a Boolean subalgebra $B(a)$ of P such that for all $p \in P$ we have $a \in C(p) \Leftrightarrow B(a) \subseteq C(p)$. (ii) We will say that E has the b-*property* if every $a \in E$ is a b-element.

Notice that for a Hilbert space effect $A \in \mathcal{E}(H)$, the role of $B(A)$ is played by the range of the spectral measure of A.

PROPOSITION 6.1. (i) *If an element $a \in E$ is a* b-*element, then there is a block B of P such that $a \in C(B)$.*

(ii) *Every projection $q \in P$ is a* b-*element with $B(q) = \{0, q, q', 1\}$.*

Proof: (i) The Boolean subalgebra $B(a)$ is contained in a block B of P. Therefore for every $p \in B$, $B(a) \subseteq C(p)$, which by the definition of $B(a)$ implies

$a \in C(p)$, hence $a \in \bigcap_{p \in B} C(p) = C(B)$.

(ii) Clearly, $q \in C(p)$ iff $\{0, q, q', 1\} \subseteq C(p)$. □

Let $A, B \subseteq E$, we write $A \leftrightarrow B$ iff $a \leftrightarrow b$ for all $a \in A, b \in B$. The next definition extends the notion of compatibility.

DEFINITION 6.2. Let E be a CB-effect algebra with the b-property. For all $e, f \in E$, define

$$eCf \quad \Leftrightarrow \quad B(e) \leftrightarrow B(f) \tag{2}$$

and we say that e and f are *compatible*, or equivalently, that they *commute* if condition (2) is satisfied.

LEMMA 6.1. *If $p \in P$, and $a \in E$ is a* b*-element, then the following statements are equivalent:*

(i) $a \leftrightarrow p$,
(ii) $a \in C(p)$,
(iii) aCp.

Proof: Equivalence of (i) and (ii) follows by Lemma 4.1. Since a is a b-element and by (i), $a \in C(p) \Leftrightarrow B(a) \subseteq C(p) \Leftrightarrow B(a) \leftrightarrow p \Leftrightarrow B(a) \leftrightarrow \{0, p, p', 1\} = B(p) \Leftrightarrow aCp$. □

DEFINITION 6.3. Let $e, f \in E$.

(i) $CPC(e) = C(\{p \in P : e \in C(p)\}) = C(C_P(e))$, $CPC(e, f) = CPC(e) \cap CPC(f) = C\{p \in P : e, f \in C(p)\})$.
(ii) $P(e, f) = \{p \in P \cap CPC(e, f) : e, f \in C(p)$ and $J_p(e) \leq J_p(f)$, $J_{p'}(f) \leq J_{p'}(e)\}$.
(iii) E has the *b-general comparability property* (bGC), or is a b-*comparability effect algebra*, if
   (a) E has the b-property,
   (b) for all $e, f \in E$, $eCf \Rightarrow P(e, f) \neq \emptyset$.
(iv) E is an *RC-effect algebra* if it has both the Rickart property and the b-general comparability property.

To prove the next theorem, we need a lemma.

LEMMA 6.2. *Suppose that $p \in E_S$ and there exists $q \in P$ such that $p \in C(q)$, $J_q(p) \leq J_q(p')$, and $J_{q'}(p') \leq J_{q'}(p)$. Then $p \in P$.*

Proof: As $p \in C(q)$, we have $p' \in C(q)$ and $p, p' \in C(q')$. By Lemma 4.1, $J_q(p) \leq p$ and $J_q(p') \leq p'$, hence $J_q(p) \leq p, p'$, and since p is sharp, it follows that $J_q(p) = 0$. Likewise, $J_{q'}(p') = 0$, and it follows that $J_q(p') = p'$. Therefore, $p' = J_q(1 \ominus p) = q \ominus J_q(p) = q \ominus 0 = q$, so $p = q' \in P$. □

THEOREM 6.1. *Let E be a* b*-comparability effect algebra.*

(i) *For any $p \in E$, p is sharp if and only if $p \in P$.*
(ii) *E is a proper CB-effect algebra.*

Proof: If $p \in P$, then p is principal, hence sharp, so $P \subseteq E_S$. Let $p \in E_S$. By bGC, there is $q \in P$, $q \in PCP(p, p')$, $p, p' \in C(q)$ and $J_q(p) \leq J_q(p')$, $J_{q'}(p') \leq J_{q'}(p)$. Therefore, the suppositions of Lemma 6.2 are satisfied, and the proof of (i) follows.

(ii) We have to prove that all direct compressions belong to the compression base. Let J be a direct compression with focus p. Then p is sharp, hence by (i), $p \in P$. We have to show that $J = J_p$. Define $J'(e) = e \ominus J(e)$, $e \in E$. It is easy to check that J' is a direct compression which is supplementary to J. As $0 \leq J(e) \leq J(1) = p$, we have $J_p(J(e)) = J(e)$. Also, $0 \leq J_p(J'(e)) \leq J_p(J'(1)) = J_p(1 \ominus J(1)) = J_p(p') = 0$, hence $J_p(J'(e) = 0$. Consequently, $J_p(e) = J_p(J(e) \oplus J'(e)) = J(e)$. □

An alternative possibility to introduce a version of general comparability in BC-effect algebras is the following.

DEFINITION 6.4. *We will say that E has the strong general comparability (sGC) if $P(e, f) \neq \emptyset$ for every $e, f \in E$.*

The following result shows that strong general comparability really is an excessively strong property.

PROPOSITION 6.2. *If E satisfies strong general comparability, then $e \leftrightarrow f$ for every $e, f \in E$. Consequently, P is a Boolean algebra, and $E = C(P)$ is an MV-effect algebra.*

Proof: Let $p \in P(e, f)$. Then $J_p(e) \leq J_p(f)$, $J_{p'}(f) \leq J_{p'}(e)$, and in addition, $e = J_p(e) \oplus J_{p'}(e)$, $f = J_p(f) \oplus J_{p'}(f)$. Let $e_1 := J_{p'}(e) \ominus J_{p'}(f)$ and $f_1 := J_p(f) \ominus J_p(e)$. As $J_p(e) \leq p$, $J_{p'}(f) \leq p'$, we have $J_p(e) \perp J_{p'}(f)$, so we can define $c := J_p(e) \oplus J_{p'}(f)$. Then $e = e_1 \oplus c$, $f = f_1 \oplus c$, and $e_1 \oplus f_1 \oplus c = J_p(f) \oplus J_{p'}(e)$, whence $e \leftrightarrow f$. In particular, $p \leftrightarrow q$ for all $p, q \in P$, hence P is a Boolean algebra. Moreover, $E = C(P)$. Using the same method as in the proof of Theorem 7.1 below, we can prove that $E = C(P)$ is an MV-effect algebra. □

COROLLARY 6.1. *Strong general comparability implies* b-*general comparability. If P is a Boolean algebra and $E = C(P)$, then sGC and bGC are equivalent.*

Proof: If sGC is satisfied, then P is a block of itself and $E = C(B)$. Then every element $e \in E$ is a b-element with respect to $B(e) = P$.

Let P be a Boolean algebra and $E = C(P)$. Assume that bGC holds. Then for every $e \in E$, $B(e) \subseteq P$, whence eCf for every $e, f \in E$, so that sGC holds. □

7. Blocks and C-blocks

We maintain our convention that E is a CB-effect algebra with compression base $(J_p)_{p \in P}$.

DEFINITION 7.1. (i) A subset B of P is called a *block* in P if B is a maximal set of pairwise compatible elements of P. (ii) For every block B, the set $C(B)$ is called a *C-block* of E.

We recall that by Corollary 4.2, P is a regular orthomodular poset. Therefore maximal sets of pairwise compatible elements coincide with maximal Boolean subalgebras of P [28].

LEMMA 7.1. *Let $B \subseteq P$ be a block in P.*

(i) *B is a normal sub-effect algebra of E.*

(ii) *B is a Boolean algebra and for $p, q \in B$, $p' = 1 \ominus p$ is the Boolean complement of p in B, $p \wedge_B q = p \wedge q$, $p \vee_B q = p \vee q$.*

(iii) *If E has the projection cover property, then B is sup/inf closed in P and in E.*

(iv) *If E is monotone σ-complete and has the projection cover property, then P is a σ-complete OML and B is a σ-complete Boolean algebra.*

Proof: (i) Clearly, $0, 1 \in B$. If $p \in B$, then $p \leftrightarrow q$ for all $q \in B$, hence also $p' \leftrightarrow q$ for all $q \in B$ and by maximality of B, $p' \in B$. If $p, q \in B$ and $p \oplus q \leq 1$, then $p \oplus q \in P$, and by Theorem 4.4 and maximality of B, $p \oplus q \in B$. This proves that B is a sub-effect algebra of E. Let $e, f, d \in E$, $e \oplus f \oplus d \leq 1$, $p = e \oplus d$, $q = f \oplus d$, and $p, q \in B$. Then $p, q \in P$, and since P is a normal sub-effect algebra of E, we have $d \in P$. Then $e = p \ominus d \in P$, $f = q \ominus d \in P$. Since P is an OMP, we have $d = p \wedge q$, and by regularity of P and maximality of B, we have $d \in B$. This proves that B is a normal sub-effect algebra of E.

(ii) Follows from regularity of P, see Corollary 4.2, and [28].

(iii) Let $M \subseteq B$, $s = \bigvee\{b : b \in B\} \in P$. Since P is an OML, and for every $q \in B$, $q \leftrightarrow p$ for all $p \in M$, by Theorem 5.2 (iv), we get $q \leftrightarrow s$ for all $q \in B$, and maximality of B implies that $s \in B$. Hence B is sup/inf closed in P, and since by Theorem 5.2 (ii), P is sup/inf closed in E, B is sup/inf closed in E.

(iv) Since by (iii) B is sup/inf closed in E, and E is monotone σ-complete, it follows that B is monotone σ-complete, and a monotone σ-complete Boolean algebra is a σ-complete Boolean algebra. □

LEMMA 7.2. (i) *For every block B, $C(B)$ is a sub-effect algebra of E.*

(ii) $C(B) \cap P = B$.

(iii) $p \in B \;\Rightarrow\; J_p(C(B)) \subseteq C(B)$.

(iv) $e \in C(B) \;\Rightarrow\; P \cap CPC(e) \subseteq B$.

Proof: (i) $0, 1 \in B \subseteq C(B)$. Let $e \in C(B)$, then $e \leftrightarrow p$ $\forall p \in B$ implies $e' \leftrightarrow p$ $\forall p \in B$, whence $e' \in C(B)$. Let $e, f \in C(B)$, $e \perp f$. Then for every $p \in B$, $J_p(e \oplus f) \oplus J_{p'}(e \oplus f) = J_p(e) \oplus J_p(f) \oplus J_{p'}(e) \oplus J_{p'}(f) = (J_p(e) \oplus J_{p'}(e)) \oplus (J_p(f) \oplus J_{p'}(f)) = e \oplus f$, whence $e \oplus f \in C(B)$. This proves that $C(B)$ is a sub-effect algebra of E.

(ii) Clearly, $B \subseteq B \cap C(B)$. Let $p \in P \cap C(B)$. Then $p \in C(q)$ $\forall q \in B$, and maximality of B implies $p \in B$.

(iii) Let $p \in B$, $e \in C(B)$, then for every $q \in B$, $J_q(J_p(e)) \oplus J_{q'}(J_p(e)) = J_p(J_q(e) \oplus J_{q'}(e)) = J_p(e)$. Hence $J_p(e) \in C(B)$. Since $e \in C(B)$ was arbitrary, we get $J_p(C(B)) \subseteq C(B)$.

(iv) If $e \in C(B)$ and $q \in P \cap CPC(e)$, then $q \leftrightarrow p$ for all $p \in B$, and by maximality of B, $q \in B$. □

PROPOSITION 7.1. (i) *If B is a block of P, then $B \cap C_P(a)$ is contained in a block B_1 of $C_P(a)$.*
(ii) *To every block B_1 of $C_P(a)$, there is a block B of P such that $B_1 = C_P(a) \cap B$.*

Proof: (i) Let B be a block of P. By 4.4, $C_P(a)$ is a sub-OMP of P, whence also $C_P(a) \cap B$ is a sub-OMP of P. Let $p, q \in C_P(a) \cap B$. Then $a \in C(p, q)$ and $p \leftrightarrow q$ in P. We claim that $p \leftrightarrow q$ in $C_P(a)$. Indeed, let $p = r \vee p_1$, $q = r \vee q_1$, with p_1, q_1, r mutually orthogonal elements in P. We recall that $r = p \wedge q$. By Theorem 4.5, we have $C(p, q) \subseteq C(p \wedge q)$. So we have $a \in C(r)$, which entails $r \in C_P(a)$. Now $p, q, r \in C_P(a)$ implies that $p \ominus r = p_1 \in C_P(a)$, $q \ominus r = q_1 \in C_P(a)$, and hence p, q are compatible in $C_P(a)$. It follows that $C_P(a) \cap B$ is a pairwise compatible subset of $C_P(a)$, therefore it is contained in a block B_1 of $C_P(a)$. So we have $C_P(a) \cap B \subseteq B_1$.

(ii) Let B_1 be a block in $C_P(a)$. Then elements of B_1 are pairwise compatible in $C_P(a)$, hence also in P. Therefore there is a block B of P such that $B_1 \subseteq B \cap C_P(a)$. By (i), there is a block B_2 of $C_P(a)$ such that $B_1 \subseteq B \cap C_P(a) \subseteq B_2$, and since B_1 and B_2 are blocks, it follows that $B_1 = B_2$. □

LEMMA 7.3. *If e and f are* b-*elements, then eCf if and only if $e, f \in C(B)$ for a block B of P.*

Proof: Let $e, f \in C(B)$. As e, f are b-elements, we have $B(e) \subseteq C(B)$, $B(f) \subseteq C(B)$, and since $B(e), B(f) \subseteq P$, we get by Lemma 7.2 (ii) that $B(e), B(f) \subseteq B$. This entails that $B(e) \leftrightarrow B(f)$, that is, eCf. Conversely, let eCf, then $B(e) \leftrightarrow B(f)$, hence there is a block B such that $B(e) \cup B(f) \subseteq B$, and this entails that $e, f \in C(B)$. □

THEOREM 7.1. *Let E be a* b-*comparability effect algebra.*
(i) *For every block B of P, $C(B)$ is an MV-effect algebra.*
(ii) *$C(B)$ is a CB-effect algebra with the compression base $(\bar{J}_p)_{p \in B}$, where $\bar{J}_p = J_p/C(B)$ is the restriction of J_p to $C(B)$.*
(iii) *$C(B)$ is a b-comparability effect algebra with respect to $(\bar{J}_p)_{p \in B}$.*
(iv) *$(\bar{J}_p)_{p \in B}$ is the total direct compression base for the MV-effect algebra $C(B)$.*
(v) *If E has the projection cover property, then $C(B)$ has the projection cover property.*
(vi) *If E is monotone σ-complete, then $C(B)$ is a σ-complete MV-effect algebra.*

Proof: (i) By Lemma 7.2 (i), $C(B)$ is a sub-effect algebra of E. Let $e, f \in C(B)$. By Lemma 7.3, eCf. By b-general comparability, there is $p \in P \cap CPC(e, f)$ with $e, f \in C(p)$, such that $J_p(e) \leq J_p(f)$, and $J_{p'}(f) \leq J_{p'}(e)$. Put $e \sqcap f := J_p(e) \oplus J_{p'}(f)$. From $e, f \in C(p)$, it follows that $e = J_p(e) \oplus J_{p'}(e) \geq e \sqcap f$, $f = J_p(f) \oplus J_{p'}(f) \geq e \sqcap f$. Assume that for $d \in C(B)$ we have $d \leq e, f$. By Lemma 7.2 (iv), $P \cap CPC(e, f) \subseteq B$, hence $p \in B$, and therefore $d =$

$J_p(d) \oplus J_{p'}(d) \leq J_p(e) \oplus J_{p'}(f) = e \sqcap f$. This entails that $e \sqcap f$ is the g.l.b. of e, f in $C(B)$. Similarly we prove that $e \sqcup f := J_p(f) \oplus J_{p'}(e)$ is the l.u.b. of e, f in $C(B)$, hence $C(B)$ is a lattice. Moreover, $e \ominus e \sqcap f = J_{p'}(e) \ominus J_{p'}(f) = e \sqcup f \ominus f$. By [6, Theorem 1.8.12], $C(B)$ is an MV-effect algebra.

(ii) By Lemma 7.2 (iii), $\bar{J}_p : C(B) \to C(B)$ for any $p \in B$. Since J_p is a compression, $\bar{J}_p = J_p/C(B)$ is a compression as well. By Lemma 7.1 (i), B is a normal sub-effect algebra of E, which entails that B is also a normal sub-effect algebra of $C(B)$. If $e, f, d \in C(B)$, $e \oplus f \oplus d \leq 1$, and $p = e \oplus d \in B$, $q = f \oplus d \in B$, then $d \in B$, and for any $a \in C(B)$, $\bar{J}_p \circ \bar{J}_p(a) = J_p \circ J_q(a) = J_d(a) = \bar{J}_d(a)$. Therefore, $(\bar{J}_p)_{p \in B}$ is a compression base for $C(B)$.

(iii) Since E has b-general comparability, for every $e, f \in C(B)$, there is $p \in P \cap CPC(e, f)$ such that $e, f \in C(p)$ and $J_p(e) \leq J_p(f)$, $J_{p'}(f) \leq J_{p'}(e)$. By Lemma 7.2 (iv), $P \cap CPC(e, f) \subseteq B$, and by Lemma 7.2 (ii), $P \cap C(B) = B$. From this it follows that $C(B)$ has the strong general comparability with respect to the compression base $(\bar{J}_p)_{p \in B}$.

(iv) By (iii) and Theorem 6.1, every direct compression on $C(B)$ belongs to the compression base $(\bar{J}_p)_{p \in B}$. By (i) and Theorem 3.1, every retraction on $C(B)$ is a direct compression, hence the compression base is total. We will show that every compression $\bar{J}_p$, $p \in B$, is direct. Indeed, if $e \in C(B)$ and $p \in B$, then $e \in C(p)$, whence $e = J_p(e) \oplus J_{p'}(e)$, hence $\bar{J}_p(e) = J_p(e) \leq e$.

(v) Let $e \in C(B)$, and let $\gamma(e)$ be the projection cover of e in E. By Theorem 5.2, $e \in C(p)$ implies $\gamma(e) \in C(p)$ for all $p \in B$, and hence $\gamma(e) \in C(B)$.

(vi) Assume that E is monotone σ-complete. Theorem 4.6 implies that $C(B)$ is monotone σ-complete, and since $C(B)$ is a lattice, it is a σ-lattice. □

COROLLARY 7.1. *Let E be a* b-*comparability effect algebra.*

(i) *E can be covered by its C-blocks, which are MV-algebras.*

(ii) *For every subset $C \subseteq E$ which is maximal with respect the property eCf for all $e, f \in C$, there is a block B of P such that $C = C(B)$.*

Proof: (i) Follows by Proposition 6.1. (ii) Let $e, f \in C$, then $B(e) \leftrightarrow B(f)$ implies that $\bigcup\{B(e) : e \in C\}$ consists of mutually compatible elements, and therefore it is contained in a block B of P. For every $e \in C$ and every $p \in B$, we have $B(e) \subseteq C(p)$, which implies that $e \in C(p)$. This implies that $C \subseteq C(B)$. On the other hand, let $e \in C(B)$. Then $e \in C(p)$, $p \in B$ implies $e \in C(B(f))$ for all $f \in C$, that is, eCf for all $f \in C$. By maximality of C then $e \in C$, whence $C = C(B)$. □

REMARK 7.1. Notice that the proof of Theorem 7.1(i) suggests that for any $p, q \in P(e, f)$, it is $J_p(e) \oplus J_{p'}(f) = J_q(e) \oplus J_{q'}(f)$. In the following lemma we give an independent proof.

LEMMA 7.4. *For every $e, f \in E$ and every $p, q \in P(e, f)$, we have $J_p(e) \oplus J_{p'}(f) = J_q(e) \oplus J_{q'}(f)$.*

Proof: Let $p, q \in P(e, f)$. From $p \in CPC(e, f)$ and $e, f \in C(q)$ it follows that $p \leftrightarrow q$. Therefore $J_p \circ J_q = J_q \circ J_p = J_{p \wedge q}$. We have

$$J_p(e) \le J_p(f), \qquad J_q(e) \le J_q(f),$$
$$J_{p'}(f) \le J_{p'}(e), \qquad J_{q'}(f) \le J_{q'}(e).$$

Then

$$\begin{aligned} J_q(J_{p'}(f)) &\le J_q(J_{p'}(e)) = J_{p'}(J_q(e)) \\ &\le J_{p'}(J_q(f)) = J_q(J_{p'}(f)), \end{aligned}$$

which entails $J_q(J_{p'}(f)) = J_q(J_{p'}(e))$, and by symmetry, $J_p(J_{q'}(f)) = J_p(J_{q'}(e))$. Taking into account that $e, f \in C(p, q)$, we obtain

$$\begin{aligned} J_p(e) \oplus J_{p'}(f) &= J_p(J_q(e)) \oplus J_p(J_{q'}(e)) \oplus J_{p'}(J_q(f)) \oplus J_{p'}(J_{q'}(f)) \\ &= J_q(J_p(e)) \oplus J_q(J_{p'}(e)) \oplus J_{q'}(J_p(f)) \oplus J_{q'}(J_{p'}(f)) \\ &= J_q(e) \oplus J_{q'}(f). \end{aligned}$$

□

REMARK 7.2. Observe that from the proof of Theorem 7.1 (i) we can derive that $aCb \implies a \leftrightarrow b$. Indeed, we can write $a = a \sqcap b \oplus (a \ominus a \sqcap b)$, $b = a \sqcap b \oplus (b \ominus a \sqcap b)$, and $a \oplus (b \ominus a \sqcap b) = (J_p(a) \oplus J_{p'}(a)) \oplus (J_p(b) \ominus J_p(a)) = J_{p'}(a) \oplus J_p(b) = a \sqcup b \in E$. It is well known that such an implication holds for the Hilbert space effects, namely $AB = BA$ implies $A \leftrightarrow B$, while the converse implication need not hold. As a counterexample, consider unit vectors $x, y \in H$, and let P_x, P_y be the corresponding one-dimensional projections. Then there are real numbers $0 < \lambda, \mu < 1$ such that $\lambda P_x + \mu P_y \le I$, whence $\lambda P_x, \mu P_y$ are coexistent effects, while λP_x and μP_y commute iff either $x \perp y$, or $y = rx$, $|r| = 1$.

EXAMPLE 7.1. Notice that an MV-effect algebra (see Example 3.2) is monotone σ-complete iff it is a σ-lattice. Using the categorical equivalence between MV-algebras and Dedekind σ-complete unital ℓ-groups, we can derive from [18, Lemma 9.8] and [18, Theorem 9.9] that a σ-complete MV-effect algebra M with the compression base $(J_p)_{p \in B(M)}$ has the projection cover property and strong general comparability.

EXAMPLE 7.2. In what follows, G is a unital group with unit u, a compression base $(J_p)_{p \in P}$, and the unit interval $E = [0, u]$. For $g \in G$ and $p \in P$, define $C(p) := \{g \in G : g = J_p(g) + J_{u-p}(g)\}$ (see Example 3.3). Let $\tilde{J}_p, p \in P$, denote the restriction of J_p to E. Recall that by Theorem 3.2, $(\tilde{J}_p)_{p \in P}$ is a compression base for E.

DEFINITION 7.2 ([12]). G has the *Rickart projection property* if there is a mapping $^* : G \to P$, called the *Rickart mapping*, such that, for all $g \in G$ and all $p \in P$, $p \le g^* \iff g \in C(p)$ with $J_p(g) = 0$.

DEFINITION 7.3 ([12, Definition 4.1]). Let $g \in G$.
(i) $CPC(g) := C(\{p \in P : g \in C(p)\})$.

(ii) $P^{\pm}(g) := \{p \in P \cap CPC(g) : g \in C(p)$ and $J_{u-p}(g) \leq 0 \leq J_p(g)\}$.
(iii) G has the *general comparability property* or, for short, is a *comparability group* if $P^{\pm}(g) \neq \emptyset$ for all $g \in G$. If G is a comparability group and also an ℓ-group, then we call G a comparability ℓ-group.
(iv) G is an *RC-group* iff it has both the Rickart property and the general comparability property. If G is an RC-group and also an ℓ-group, then we call G an RCℓ-group.

THEOREM 7.2. (i) *If (G, u) with the compression base $(J_p)_{p \in P}$ has the Rickart projection property, then E with the compression base $(\tilde{J}_p)_{p \in P}$ has the Rickart projection property.*
(ii) *If (G, u) with the compression base $(J_p)_{p \in P}$ has general comparability, then for E with the compression base $(\tilde{J}_p)_{p \in P}$ the following two conditions are satisfied:*
(a) *For every $e \in E$, there is $p \in CPC(e)$, such that $e \in C(p)$ and $\tilde{J}_p(e) \leq \tilde{J}_p(e')$ and $\tilde{J}_{p'}(e') \leq \tilde{J}_{p'}(e)$.*
(b) *Let $e, f \in E$ and let B be a block in P. If $e, f \in C(B)$ then there is $p \in C(B)$ such that $\tilde{J}_p(e) \leq \tilde{J}_p(f)$ and $\tilde{J}_{p'}(f) \leq \tilde{J}_{p'}(e)$.*
(iii) *If G is an RC-group, then E has the* b-*general comparability.*

Proof: (i) is straightforward. (ii) (a) Let $e \in E$. Then $e - e' \in G$, and by general comparability in G, there is a $p \in CPC(e - e')$ such that $e - e' \in C(p)$ and $J_p(e - e') \leq 0 \leq J_{p'}(e - e')$. Now $e - e' = e - (u - e) = 2e - u$, hence $e - e' \in C(p)$ implies $e \in C(p)$. Similarly, $CPC(e - e') = CPC(e)$. Finally, $J_p(e - e') \leq 0$ implies $\tilde{J}_p(e) = J_p(e) \leq J_p(e') = \tilde{J}_p(e')$, $0 \leq J_{p'}(e - e')$ implies $\tilde{J}_{p'}(e') = J_{p'}(e') \leq J_{p'}(e) = \tilde{J}_{p'}(e)$. (b) Let $e, f \in C(B)$, then $e - f \in C(B)$. By general comparability in G, there is $p \in CPC(e - f)$, $e - f \in C(p)$ and $J_p(e - f) \leq 0 \leq J_{p'}(e - f)$. Recall that $CPC(e - f) = C(\{p \in P : e - f \in C(p)\})$. Since $e - f \in C(B)$, we have $B \subseteq \{p \in P : e - f \in C(p)\}$, which entails that $CPC(e - f) \subseteq C(B)$. Therefore $p \in C(B) \cap P = B$. The rest follows analogously as in the proof of (j).

(iii) If G is an RC-group, then by [9], to every $a \in E$ there is a rational spectral resolution $(p_{\lambda,a})_{\lambda \in \mathbb{Q}}$, that is, to every rational number λ, there is a projection $p_{\lambda,a} \in P$, such that $p_{\lambda,a} \leftrightarrow p_{\mu,a}$ for all $\lambda, \mu \in \mathbb{Q}$, and for every $q \in P$, $a \in C(q)$ iff $(p_{\lambda,a})_{\lambda \in \mathbb{Q}} \subseteq C(q)$. Let $B(a)$ denote the smallest sub-OML of P that contains the elements $(p_{\lambda,a})_{\lambda \in \mathbb{Q}}$. As, for every $p, q \in P$, $p \in C(q)$ iff $p \leftrightarrow q$, $B(a)$ is a Boolean subalgebra of P. Moreover, for every $p \in P$, the set $C(p) \cap P = \{q \in P : q \leftrightarrow p\}$ is a sub-OML of P ([28]). From this we obtain, by the minimality of $B(a)$, that $(p_{\lambda,a})_{\lambda \in \mathbb{Q}} \subseteq C(p)$ iff $B(a) \subseteq C(p)$. This implies that E has the b-property with respect to the compression base $(\tilde{J}_p)_{p \in P}$. We have eCf iff there is a block B of P such that $e, f \in C(B)$ (Lemma 7.3). Property (b) then implies that the b-general comparability is satisfied. $\square$

8. Sequential effect algebras

A *sequential effect algebra* (SEA) [22] is a system $(E; 0, 1, \oplus, \circ)$ where $(E; 0, 1, \oplus)$ is an effect algebra and $\circ : E \times E \to E$ is a binary operation that satisfies the following conditions (we write $a \mid b$ if $a \circ b = b \circ a$):

(S1) $b \mapsto a \circ b$ is additive for all $a \in E$,
(S2) $1 \circ a = a$ for all $a \in E$,
(S3) If $a \circ b = 0$ then $a \mid b$,
(S4) If $a \mid b$ then $a \mid b'$ and $a \circ (b \circ c) = (a \circ b) \circ c$ for all $c \in E$,
(S5) If $c \mid a$ and $c \mid b$ then $c \mid a \circ b$ and $c \mid (a \oplus b)$ if in addition $a \perp b$.

An operation that satisfies (S1)–(S5) is called a *sequential product* on E. If $a \mid b$ for all $a, b \in E$ we call E a *commutative SEA*.

A prototype of a SEA is the Hilbert space effect algebra $\mathcal{E}(H)$, where the sequential product is defined by $A \circ B = A^{1/2} B A^{1/2}$, $A^{1/2}$ being the unique positive square root of A. It was shown that $A \mid B$ iff $AB = BA$ [23].

The following lemma summarizes some properties of a SEA E.

LEMMA 8.1. (i) $a \circ 0 = 0 \circ a = 0$ *and* $a \circ 1 = 1 \circ a = a$.
(ii) $a \circ b \leq a$ *for all* $a, b \in E$.
(iii) *If* $a \leq b$, *then* $c \circ a \leq c \circ b$ *for all* $c \in E$.
(iv) *If* $a \leq b$ *then* $c \circ (b \ominus a) = c \circ b \ominus c \circ a$.
(v) *If* $a \leq b$ *and* $c \mid a$, $c \mid b$, *then* $c \mid b \ominus a$.

Recall that an element a of an effect algebra E is sharp if $a \wedge a' = 0$. The set of all sharp elements of E is denoted by E_S. In a SEA, we have the following characterizations of sharp elements [22].

LEMMA 8.2. *In a SEA, the following statements are equivalent.*
(i) $a \in E_S$,
(ii) $a \circ a' = 0$,
(iii) $a \circ a = a$.

The following important theorem was proved in [22, Corollary 3.5].

THEOREM 8.1. *The set E_S of sharp elements of a SEA E is a sub-effect algebra of E that is an orthomodular poset.*

In addition, we have the following results in a SEA E (see [22]).

PROPOSITION 8.1. *An element $a \in E$ is principal if and only if $a \in E_S$.*

PROPOSITION 8.2. *Let $a \in E$, $b \in E_S$, then*
(i) $a \circ b = b \circ a = a$ *if and only if* $a \leq b$.
(ii) $a \perp b$ *if and only if* $a \circ b = 0$.

The following statements give relations between the commutativity of $\circ$ and the coexistence relation $\leftrightarrow$ in E.

PROPOSITION 8.3. *Let E be a SEA.*
(i) *For $a, b \in E$, $a \mid b \implies a \leftrightarrow b$.*
(ii) *For $a \in E$, $b \in E_S$, $a \mid b \Leftrightarrow a \leftrightarrow b$. Moreover, if $a \leftrightarrow b$ then $a \circ b = a \wedge b$.*

Compressions on SEAs were studied in [20, 21]. It is easy to check that for any $p \in E_S$, the mapping $J(a) = p \circ a$ is a compression.

THEOREM 8.2 ([21]). *Let E be a SEA. The family $\{J_p : p \in E_S\}$, where $J_p(a) = p \circ a$, is a compression base for E.*

Proof: (1) First we prove that the set E_S is a normal sub effect algebra of E. Let $e, f, d \in E$ be such that $e \oplus f \oplus d \in E$ and $p = e \oplus d$, $q = f \oplus d$, $p, q \in E_S$. Then $p \leftrightarrow q$, and by Proposition 8.3, $p \circ q = q \circ p = p \wedge q = d$. Then $d \circ d = (p \circ q) \circ d = p \circ (q \circ d) = p \circ d = d$. Hence $d \in E_S$.
(2) Let $p, q, r \in E_S$ and $p \oplus q \oplus r \in E$. Then $p \oplus r, q \oplus r \in E_S$, and $p \oplus r \leftrightarrow q \oplus r$, whence $p \oplus r \mid q \oplus r$. By (S4), for any $a \in E$, $J_{p\oplus r} \circ J_{q\oplus r}(a) = (p \oplus r) \circ ((q \oplus r) \circ a) = ((p \oplus r) \circ (q \oplus r)) \circ a = r \circ a = J_r(a)$, hence $J_{p\oplus r} \circ J_{q\oplus r} = J_r$.
This concludes the proof that $(J_p)_{p \in E_S}$ is a compression base for E. $\square$

In [21, Theorem 3.4] it was proved, in addition, that $\{J_p : p \in E_S\}$ is a maximal compression base for E.
For $n \in \mathbb{N}$, $a \in E$ we define $a^n = a \circ a \circ \cdots \circ a$ (n-factors). The smallest $n \in \mathbb{N}$ such that $a^n \in E_S$ (if it exists) is called the *sharpness index* of a, and is denoted by $s(a)$. If no such n exists, then the sharpness index of a is ∞ [22].

LEMMA 8.3. [22] *If $n = s(a) < \infty$, then a^n is the largest sharp element below a.*

A σ-SEA is a SEA that is a monotone σ-complete effect algebra satisfying
1. if $a_1 \geq a_2 \geq \cdots$, then $b \circ (\bigwedge a_i) = \bigwedge (b \circ a_i)$ for every $b \in E$,
2. if $a_1 \geq a_2 \geq \cdots$ and $b \mid a_i$, $i = 1, 2, \ldots$, then $b \mid \bigwedge a_i$.
In particular, $\mathcal{E}(H)$ is a σ-SEA.

THEOREM 8.3 ([22]). *Let E be a σ-SEA. If $a \in E$, then there exist $b, c \in E_S$ such that b is the largest sharp element below a and c is the smallest sharp element above a.*

From Theorems 8.2 and 8.3 we conclude that a σ-SEA has the projection cover property. Consequently, the set E_S of all projections (equivalently, sharp elements) in E is a σ-orthomodular sublattice of E.
A counterexample in [22] shows that a commutative SEA need not be an MV-algebra, in general.

THEOREM 8.4. *Let E be a commutative SEA with the* b*-general comparability. Then E is an MV-algebra.*

Proof: As E is commutative, we have $p \mid e$ for every $e \in E$ and $p \in E_S$. By Proposition 8.3(i) this implies that $e \leftrightarrow p$ for all $e \in E$, $p \in E_S$, and by Lemma 4.1 it follows that $e \in C(p)$ for all $e \in E$ and all $p \in E_S$. Since an orthomodular poset in which any two elements are compatible is a Boolean algebra, we obtain that $B = E_S$ is a Boolean algebra, hence the unique block of itself. In addition, $E = C(E_S)$. By Theorem 7.1 (i) we conclude that E is an MV-algebra. $\square$

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REFERENCES

[1] M. K. Bennett and D. J. Foulis: Interval and scale effect algebras, *Advances in Appl. Math.* **19** (1997), 200.

[2] P. Busch, P. J. Lahti and P. Mittelstaedt: *The Quantum Theory of Measurement*, Lecture Notes in Physics, **m2**, Springer, Berlin/Heidelberg/New York 1991.

[3] P. Busch, M. Grabowski and J. P. Lahti: *Operational Quantum Physics*, Springer, Berlin 1995.

[4] C. C. Chang: Algebraic analysis of many-valued logics, *Trans. Amer. Math. Soc.* **88** (1957), 467.

[5] F. Chovanec and F. Kôpka: Boolean D-posets, *Tatra Mt. Math. Publ.* **10** (1997), 183.

[6] A. Dvurečenskij and S. Pulmannová: *New Trends in Quanum Structures*, Kluwer Academic Publ. Dordrecht, and Ister Science, Bratislava, 2000.

[7] D. J. Foulis: Compressible groups, *Math. Slovaca* **53** (2003), 433.

[8] D. J. Foulis: Compressions on partially ordered abelian groups, *Proc. Amer. Math. Soc.* **132** (2004), 3581.

[9] D. J. Foulis: Spectral resolution in a Rickart comgroup, *Rep. Math. Phys.* **54** (2004), 319.

[10] D. J. Foulis: Compressible groups with general comparability, *Math. Slovaca* **55** (2005), 409.

[11] D. J. Foulis: Compression bases in unital groups, *Internat. J. Theoret. Phys.*, to appear.

[12] D. J. Foulis: Comparability groups, *Demonstratio Math.* **39** (2006), 15.

[13] D. J. Foulis: Sharp and fuzzy elements of an RC-group, *Math. Slovaca*, to appear.

[14] D. J. Foulis and M. K. Bennett: Effect algebras and unsharp quantum logics, *Found. Phys.* **24** (1994), 1325.

[15] D. J. Foulis and S. Pulmannová: Monotone σ-complete RC-groups, *J. London Math. Soc.* **73** (2) (2006) 304.

[16] R. J. Greechie, D. J. Foulis and S. Pulmannová: The center of an effect algebra, *Order* **12** (1995), 91.

[17] R. Giuntini and H. Greuling: Toward a forma language for unsharp properties, *Found. Phys.* **19** (1989), 931.

[18] K. R. Goodearl: *Partially Ordered Abelian Groups with Interpolation*, A.M.S. Mathematical Surveys and Monographs, **No. 20**, American Mathematical Society, Providence, RI, 1986.

[19] S. Gudder: Sharply dominating effect algebras, *Tatra Mt. Math. Publ.* **15** (1998), 23.

[20] S. Gudder: Compressible effect algebras, *Rep. Math. Phys.* **54** (2004), 93.

[21] S. Gudder: Compression bases in effect algebras, *Demonstratio Math.* **39** (2006), 43.

[22] S. Gudder and R. Greechie: Sequential products on effect algebras, *Rep. Math. Phys.* **54** (2002), 87.

[23] S. Gudder and G. Nagy: Sequential quantum measurements, *J. Math. Phys.* **42** (2001), 5212.

[24] G. Jenča: Sharp and meager elements in orthocomplete homogeneous effect algebras, Preprint (2004), available from http://www.elf.stuba.sk/'jenca/preprint

[25] F. Kôpkaand F. Chovanec: D-posets, *Math. Slovaca* **44** (1994), 21.

[26] K. Kraus: *States, Effects, and Operations*, Springer, Berlin 1983.

[27] D. Mundici: Interpretation of AF C*-algebras in Łukasiewicz sentential calculus, *J. Funct. Anal.* **65** (1986), 15.

[28] P. Pták and S. Pulmannová: *Orthomodular Structures as Quantum Logics*, Kluwer Academic Publ. Dordrecht, and VEDA, Bratislava 1991.

[29] S. Pulmannová: A spectral theorem for σ-MV-algebras, *Kybernetika* **41** (2005), 361.

[30] S. Pulmannová: Spectral resolutions in Dedekind-σ-complete ℓ-groups, *J. Math. Anal. Appl.* **309** (2005), 322.