On period, cycles and fixed points of a quantum channel

This article concerns the structure of quantum channels on the von Neumann algebras $\mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is *infinite* dimensional. There are two main results explained below, which considerably improved the current knowledge and help to fill an important gap in the literature. The proof are technical and fit the level of excellence required by AHP. The results are well-explained and illustrated by inspiring examples. Consequently, I highly recommand this article for publication in this journal.

For finite dimensional Hilbert spaces the theory was already well understood, notably with [1, 4], where the Schrödinger picture of time evolution is favored. A common assumption is the existence of a faithful invariant state, which is also used a lot in infinite dimension. For infinite dimensional spaces, it was already understood that recovering the classical results would require that the decoherence-free algebra (DF algebra) should be atomic (or equivalently, there exists a normal conditional expectation on it compatible with the evolution, see in particular [2]), which in continuous time was proved to imply similar properties as the finite-dimensional case. Two important remaining questions were therefore:

1. first, is the DF algebra always atomic (assuming the existence of the faithful invariant state)? If yes, this would imply numerous and most wanted properties for quantum channels, such as environmental-induced decoherence which described its asymptotic behavior.
2. Secondly, from this can we recover the finite dimensional results on the structure of discrete time quantum channels? As in the classical case, discrete time is more difficult to deal with, because of the existence of peripheric eigenvalues. The novelty here is that the Schrödinger picture is not anymore fit to answer this question and the authors instead address this problem from the Heisenberg picture using tools from von Neumann algebra theory.

Both questions are answered positively in this article. Then a third part of the article concerns a class of examples of quantum channels, namely open quantum walks, where the authors illustrate the different situations with different examples.

As a suggestion for the authors, I would recommand to look at the situation where there is only an invariant weight. In continuous-time and when the semigroup commutes with the modular group of this weight, it was already proved in [3] that the DF algebra is again atomic. Can we lift this last assumption and still get the same result?

Comments for the authors:

- Proposition 6 last statement: "positive recurrent" is not introduced before (for a quantum channel).
- Proof of the last statement in Proposition 6: I am confused, I think positive recurrent is a property of *irreducible* channels, so it makes no sense to prove it before irreducibility.
- Proof of the last statement in Proposition 6: I think the proof can be simplified: one cannot have $Q \in \mathcal{N}$ and $0 < Q < Q_j$ for some j as

$$\mathcal{N} = \{Q_0, ..., Q_{d-1}\}''$$

  and $Q \in \mathcal{F}(\Phi^d) = \mathcal{N}$ by point 2.
- Idem for the proof of aperiodicity, any $R \in \mathcal{N}_k$ belongs to $\mathcal{N}$ so $R = 0$ or $R = Q_k$ and $\mathcal{N}_k = \mathbb{C}\, Q_k$.

- Remark 4: the second equation is missing a dot at the end.
- There is a problem in the parentheses in the second equation of Remark 4.
- Proposition 7 is the proof that Definition 2 makes sense. I would put it before the definition (I was myself confused, not understanding why the definition makes sense before reading proposition 7).
- Proof of Proposition 7: it would be nice to explain why the d_i's are finite if there exists a faithful invariant state, as I think it is one of the important features of the case of study. The authors could also give an example where it is not the case but where Φ_i is still a $*$-automorphism.
- p23: I think there is a normalization $1/d$ missing in the invariant normal states ρ.
- p28: the notation $\|$ for colinear is not clear. Also, according to the notations, 1 should be I_2 in

$$\Phi^2(P)(1 \otimes |0\rangle\langle 0|)\,.$$

References

[1] R. Carbone and Y. Pautrat. Irreducible decompositions and stationary states of quantum channels. ArXiv e-prints, July 2015.

[2] J. Deschamps, F. Fagnola, E. Sasso, and V. Umanità. Structure of uniformly continuous quantum Markov semigroups. Reviews in Mathematical Physics, 28(01):1650003, 2016.

[3] P. Ługiewicz and R. Olkiewicz. Classical Properties of Infinite Quantum Open Systems. Communications in Mathematical Physics, 239(1):241–259, 2003.

[4] M. M. Wolf. Quantum channels & operations: Guided tour. `http://www-m5.ma.tum.de/foswiki/pub/M5/Allgemeines/MichaelWolf/QChannelLecture.pdf`, 2012. Lecture notes based on a course given at the Niels-Bohr Institute.