Response Letter to the Editors and Referees
(Article Reference: JPhysA-115814)

Dear Phil Brown,

We hope this letter finds you well. We apologize for taking this long to get back to you.

Without any shadow of doubt, the referees' comments were extremely useful. They have made us rethink not only about our style of presentation and how we were conveying our paper's main message but, more importantly, they gave us opportunity and time to think deeply about our own results. Among other things, we have concluded that beyond a merely mathematical curiosity, our findings do give a physical (re)interpretation to degradability – a concept that to the best of our knowledge had only been useful in the very specific framework of estimating quantum channels' capacities. We firmly believe that the output of this first round of interaction with the referee is an enhanced version of the original manuscript.

Below you will find our point-by-point answers to the questions brought in by the referees. We would like to bring to your attention three major modifications: (i) to create a more concise text, we have opened three appendices. They are just portions of the older manuscript that had been negatively impacting the reading. The main text is shorter, drier, and goes directly to the point. No major additions have been made, for the new appendices, we have only changed the text to make them fit together. (ii) We have added new operational interpretations of our findings. Figure 3 summarizes this new discussion. (iii) We had missed one funding source in the older manuscript, and we have added this new source in the latest version. Finally, unless expressly mentioned, all the amendments we are commenting on here reflect the alterations we have made in the main text.

Respectfully,
The Authors.

Response to the Referee's Comments

Referee A

The paper is devoted to exploring whether a connection can be found between a notion of compatibility and of divisibility of quantum channels. The authors define two channels to be divisible if one can be obtained by composing the other with a third channel. They further define two channels to be compatible if they can be both obtained as marginals of a third channel. Relying on these definitions they point to the fact that the two conditions are generally not related to one another. They further consider a special class of channels for which there is an implication between the two notions, namely so-called degradable and self-degradable channels.

It appears to me that the original result of the paper appears only at the end, on page 8, where the authors state th. IV.5 which is actually a quite direct application of theorem IV.2 proven in ref.[13], and corollary IV.6 which is a direct consequence (by the way it looks wrongly formulated, I guess the authors mean "The following statements are equivalent"). Given the very special setting in which a relation between the two notions can be found, it is not obvious insofar such a relation can be relevant. Still, it can be interesting to put it into evidence. Overall however I find the paper is definitely too long and in many places confusing. I think the authors should more plainly present

the actual result, postponing or much more concisely treating various digressions. I address below some specific points that I suggest the authors should consider in this direction.

We want to thank Referee A for pointing out that our original manuscript was too long and confusing in many places. The older manuscript came into existence out of a set of notes where the authors debated the relationship between compatibility and divisibility. When we reached the point of writing the version to be submitted, we decided to keep up with that writing style – conjectures, counter-examples, new conjectures, and finally, some sort of main result or theorem. We even thought that would be pedagogical. We are terribly sorry that our intended style has not met your expectations. In fact, we agree that a shorter and more direct text can be more efficient to communicate our findings. We are thankful that the Referee has read the paper in such great detail and with such great attention - their comments and suggestions have positively impacted our text. We hope that the latest version of our manuscript (shorter and straighter to point) meets your expectations.

As you will see, after your comments, we have greatly revamped our text. The reader now will find our main results over the first few pages. We have diverted all the comments and counter-examples to later in the text. We have also shortened many obscure and pointless passages. A few discussions have been kept though, but we have diverted them to the appendices to cause no harm to the reading flow. We are confident that this has added to enhance the quality of the manuscript.

Your other questions and comments are point-by-point responded below. We have decided to implement almost all of them. We had only one point or two of disagreement, and we hope you can understand our motives. Again, we appreciate the time and energy you have dedicated to our work.

In the Introduction the sentence "Although we do not rule out the natural parallel with joint measurability [10, 19, 20], we do not define the compatibility of two compatible channels as the fact that they can be realized simultaneously. Rather than saying that two compatible channels can be jointly realizable, we define compatible quantum channels as those channels that can be recovered from a third quantum map via marginalization ..." is not useful and a bit obscure. It indicates a path the authors do not follow, and actually the path itself is unclear, what would be the definition of "jointly realizable"?

We thank Referee A for raising this issue and for bringing it up to our attention. We agree with the Referee that this discussion was too verbose and unnecessarily vague. We wanted to make it short, as it was in the introductory section, but the text ended up being too obscure and pointing out to a path we aren't taking. We have decided to remove and rewrite the said sentence. Our latest version contains a more direct, although still in words, introduction to compatibility.

We just want to mention that "jointly realizable" was our way to express what could be seen as a parallel to "joint measurability", where it is usually assumed that two (joint measurable) POVMs can be measured at the same time. This common sense also lingers to quantum channels. There are authors who interpret compatibility as if compatibility meant some sort of possibility of condensing two channels into a larger one, which action encapsulated at the same time the action of the two original channels – see a discussion in [Girard, M., Plávala, M. & Sikora, J. Jordan products of quantum channels and their compatibility. Nat Comm. 12, 2129 (2021)]. The now-removed sentence was solely to explain that we do not subscribe to this idea. That said, we firmly believe that this discussion might cause more harm than benefit to the reader – to the point of throwing them off the scent.

Please remove or fully clarify the sentence. Here and elsewhere the authors mention paths or conjectures that do not hold. It can rarely be useful, but often pointless.

We have greatly reviewed our text. Paths and conjectures we aren't pursuing were either totally removed or pull down to the appendices. The new appendices come as a supplementary material to help less familiarized readers to grasp all of our definitions. In the appendices we decided to keep the examples and counter-examples as we had before, but we are avoiding the terms "conjectures" and the like. For instance, we now motivate some of the

counter-examples (in the appendix) with a question “why did we have to ask for (anti)degradability? Wouldn’t plain divisibility and/or compatibility be enough?”.

We feel this re-framing makes our results more explicit and easier to access, while keeping some of the old discussion alive – although in a new and less harmful format. For example, in our latest version, we are not burying our main findings in the eighth page of the manuscript. We do think with this new format enhanced the quality of our exposition/paper.

With respect to divisibility the authors actually simply consider the composition of two channels. They nevertheless make reference to pieces of literature related to divisibility in the context of non-Markovian quantum processes, while in the definitions they restrict to a discrete-time evolution. I think this discussion is too long and a bit confusing. In particular, they should distinguish the idea of divisibility when the time dependence is there and when this is not the case (for a fair comparison in this respect they miss some key references M. M. Wolf & al. PRL 101 (2008) 150402; CMP 279 (2008) 147; Rivas & al. PRL 105 (2010) 050403; Vacchini & al. NJP 13 (2011) 093004; Breuer & al. RMP 88 (2016) 021002)

We agree with Referee A. Our previous discussion was too long and confusing. Too long to be part of the main text and too confusing text-wise. We have changed the text accordingly. We have pulled down this discussion about Markovianity to the respective appendix and have made an effort to increase the quality of the text. Plus, we have also added a sentence and some references to make the reader aware that there are also subtleties involving the time-dependent and the time-independent cases – we thank the referee for this important addition we had left out.

We respect and agree with Referee A’s comment that we are simply considering the composition of two channels, and as such, we should be more direct and focus only on this case. We hope Referee A can understand that we decided to keep this discussion for a pedagogical reason. We think there might be readers who are not very well familiarized with these concepts and might use our paper as an entrance door. Our solution was an intermediate one. We have delegated expanded definitions, general comments, examples and the like to the appendices. Readers that need supplementary information might find there what they want. The main text is shorter and soberer

What is the use of introducing the Jamiolkowski matrix as in eq.(2)?

The isomorphism is of no use at all. This part is reminiscent of a much older version where we thought of establishing a proper connection between compatibility and conditional independence. Even though we mention conditional independence in Example 2 (now in the appendix), we did not need to use the Jamiolkowski isomorphism to do so. In sum, the Jamiolkowski isomorphism has been removed and is not mentioned in the latest version.

We thank the Referee for having spotted this daft mistake. Very much appreciated!

Fig.3 is of course an addition within the digression, but to make it useful it would appear more natural to associated longer arrows to longer times.

We have modified Fig. 3 accordingly. Now it depicts longer times as longer arrows. We thank the referee for this suggestion.

On page 3 the authors mention "an asymmetry which is always hidden away in the usual definition of divisibility"? What is it? Please specify or remove.

Firstly, we want to point out that this commentary was moved to the appendix. More precisely, this comment now sits in sub-section A.2 and is right before def. A2. That said, in order to clarify what we wanted to say in the original manuscript, we have added a short remark right after def. A.2. We think this “remark” style may catch the reader’s attention and clarify the point raised by Referee A.

Here is what we added to the latest version of the manuscript:

[…] \textbf{Remark:} In general, divisibility is an asymmetric notion. In other words, with no extra assumptions, we cannot say that $\phi_{C|A}$ divides $\psi_{B|A}$ whenever the latter divides the former. There are many ways to get this symmetry back, for instance one could ask for quotient maps that are invertible and whose inverse is cptp. In this rather restrictive case, divisibility becomes an equivalence relation (reflexive, transitive and symmetric). Back to the general case, it is exactly the this lack of symmetry that renders the comparison between divisibility (asymmetric) and compatibility (symmetric) so tricky.[…]

Page 4 example 2: it is a straightforward consequence of trace preservation. Why not simply mentioning this fact?

In addition to the calculation, we are now mentioning this fact. We still think that showing the calculation may be more didactical than simply mention "trace preservation". We are sorry for making the paper not so direct, but we truly hope the referee can understand this (pedagogical) choice.

Is Sec.III necessary? I would definitely tighten or remove it altogether.

We agree with the referee. As the paper was structured before, Sec. III was too long and did not contribute to our exposition. Still, we think that the discussions present there are important and sum up what has been said before. Because of that, we have decided to move it down to the appendix section, and now it is part of the very last appendix – we have also adapted the text accordingly, as we have removed (from the main text) the older structure: conjecture -> counter-example -> conjecture -> example -> theorem.

In Definition IV.1 and right after the author speak about "the" complementary channel. As it appears much later on p.7, this uniqueness is not there and it would be important to clarify this fact from the very beginning and better commenting later on the consequences (in case building again on ref.[13])

The referee is correct, and we should have made this point more explicit from the very beginning. We thank them for bringing this subject up and for allowing us to amend the paper accordingly.

The latest version of our manuscript has a shortened section of definition in the main text. Even though we have opened a remark addressing exactly the point of the uniqueness right after the def.II4. Now, a short discussion about the uniqueness of the complementary map comes right after we define for the first time this class of maps. Plus, we also point out that a more in-depth discussion can be found in Appendix A.2.

The style is quite redundant even in the Conclusions, see e.g. last paragraph.

We are sorry that our text ended up so redundant. We hope our modifications have made the new version of our manuscript less redundant.

Trivial remark: is "thm." The reasonable abbreviations for theorem?

We have changed this abbreviation across the text. We thank the referee for having pointed this out.

Referee B

As the title suggests, the aim of the paper is to connect two important relations of a pair of quantum channels, namely their compatibility and, secondly, the fact that one channel divides the other (in other words, the latter channel is a concatenation of the former with another channel).

Two conjectures are stated: conjecture 1 says that divisibility implies compatibility, while conjecture 2 states the converse. It is shown that the conjectures are not true in general, but hold under some additional assumptions of

one of the channel being (anti)degradable and both conjectures hold if one of the channels is self-degradable. Frankly, I do not understand why these "conjectures" would be of interest. Both are quite obviously false, as the authors themselves explain and demonstrate on very simple examples. The main results for (anti)degradable channels follow straightforwardly from the results of [13] (Prop. 4), stated in the present paper as Theorem IV.2, giving the characterization of compatibility in terms of divisibility by a conjugate channel. In particular, Theorem IV.5 (ii) is a direct consequence of the observation in [13], Sec. 4.3, that the antidegradable channels are self compatible (in fact, these are exactly the self-compatible channels). The paper is written very clearly and is easy to read and understand. On the other hand, the value of the results is questionable. I agree that the (anti)degradable channels are very interesting for various reasons, but the authors do not sufficiently explain why the relation to the above conjectures is of importance. In fact, very little is gained from the paper which is not already contained in [13]. Therefore I cannot recommend the paper for publication.

We sincerely thank the referee for her/his valuable comments. They have allowed us to think more deeply about our results. As a consequence of this, we have provided extra explanation of why the relation of (anti)degradable channels with compatibility and divisibility is of importance. This is contained in the second last paragraph in the introduction and the third paragraph in the conclusion.

Our main argument is that our contribution provides physical insights into the meaning of (anti)degradable channels, by virtue of identifying them with the channels for which two physically meaningful properties – compatibility and divisibility – are related. To the best of our knowledge, (anti)degradable channels are mostly important because they have a suitable mathematical form that allow one to compute the channel's capacity. In this sense, they just arise as a mathematical construct, an auxiliary object in Quantum Information Theory. Our result says that they are also the channels for which the physical notions of compatibility and divisibility are related. We believe that this insight is of importance.

Moreover, our result is potentially useful, as checking the compatibility or the divisibility of two channels is not always straightforward. In the case of self-degradable channels, if one finds that compatibility holds, then she/he already knows that divisibility holds too, and vice versa. For (anti)degradable channels only one of these two implications hold.

Referee C
(Advisory Board Member Adjudication Report)

I have looked at the two referee reports and the original manuscript. This topic is relevant to the journal, and while the results are not necessarily surprising to specialists the journal is more broadly aimed. My opinion is that for a broader audience the relations between these concepts considered would be of interest. If the authors can justify (or perhaps re-frame their work in a better form) then I think they should have the opportunity to reply to the referee comments.