Resource quantification of the no-programming thm.

arxiv:1809.00750 (Kubicki)

1. No-programming theorem, UPQP
2. approximate (deterministic / probabilistic) UPQP
3. ε-UPQP
4. Main result
5. Proof: ε-UPQP and embeddings $S_1^d \hookrightarrow B(\mathcal{H})$
6. embeddings $S_1^d \to B(\mathcal{H}_m)$ and type 2-constants (Banach spaces)

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1) UPQP

$$\psi \longrightarrow \boxed{V} \longrightarrow U\psi U^\dagger = \mathrm{tr}_{\mathcal{H}_m}\left(V(\psi\otimes\phi_u)V^\dagger\right)$$

ψ: density $\in \mathcal{D}(\mathcal{H})$ matrices; V: unitary; U: unitary

$\phi_u \in \mathcal{D}(\mathcal{H}_m)$ — memory

$\dim(\mathcal{H}) = d$

$\dim(\mathcal{H}_m) = m$

Def: $V \in \mathcal{U}(\mathcal{H}\otimes\mathcal{H}_m)$ is a UPQP (universal programmable q. processor)

if for $\forall\, U \in \mathcal{U}(\mathcal{H})\ \exists\, \phi_u \in \mathcal{D}(\mathcal{H}_m)$ s.t.

$\forall \psi \in \mathcal{D}(\mathcal{H})$, $\mathrm{tr}_m V(\psi\otimes\phi_m)V^\dagger = U\psi U^\dagger$.

(Nielsen & Chuang, 1997)

<u>No-programming theorem</u>:

$V \in \mathcal{U}(\mathcal{H}\otimes\mathcal{H}_m)$ is a UPQP, then $m = \infty$.

(1) What if $m < \infty$?

2. Approximate UPQP, deterministic & probabilistic

$$U\psi U^\dagger \overset{\epsilon}{\approx} \mathrm{tr}_{\mathcal{H}_m}\left[V(\psi\otimes\phi_U)V^\dagger\right]$$

deterministic:

$$\psi \to \boxed{V} \to \psi' \underset{\epsilon}{\approx} U\psi U^\dagger$$

$$\uparrow \phi_U$$

- close in fidelity
- $\| \ \|_\diamond$

probabilistic:

$$\psi \to \boxed{V} \to \text{success / fail}$$

$$\uparrow \phi_U \qquad \text{success} \downarrow U\psi U^\dagger$$

figure of merit : success probability

$$1-\epsilon \le P_{succ} < 1 \to \text{no-prog}$$

$m \to \epsilon$, $m \to d$

$\epsilon = 0 \Rightarrow m = \infty$

$\epsilon > 0 \overset{?}{\Rightarrow}$

Constructions of UPQPs :

upper bounds (port-based teleportation) $m \sim \exp\left(\frac{d^2 \log d}{\epsilon}\right)$

fix d, $m(\epsilon) = \left(\frac{1}{\epsilon}\right)^{d^2}$ (probabilistic) [Sedlák, et al '19]

deterministic [Hus Kol]

gap

lower bounds : fix ϵ : $m(d) \ge d^2/\epsilon$

fix d : $m(\epsilon) \ge (1/\epsilon)^d$

known: $d^2_{1/\varepsilon} \leq m(d_\varepsilon) \leq \exp(d_\varepsilon^2 \log d)$ → PBT

main result: improve lower bounds: $\exp[(1-\varepsilon)d^2] \leq$

3. ε - UPQPs: $\nu \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H}_m)$ is an ε-UPQP if

$\forall u \in \mathcal{U}(\mathcal{H})\ \exists\ \phi_u \in D(\mathcal{H}_m)$ s.t.

$$\frac{1}{2} \| \mathrm{tr}_m \nu (\cdot \otimes \phi_u) \nu^\dagger - U \cdot U^\dagger \|_\diamond \leq \varepsilon$$

Comment: (i) ε-aPQPs are deterministic approx UPQPs

(ii) if ν is prob UPQP with $p_{succ} \geq 1-\varepsilon$

then ν is also 2ε-UPQP

→ both kinds

$\boxed{\text{Thm 1}}$ $\nu \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H}_m)$ is an ε-UPQP$_d$, then

$$m \geq 2^{\frac{(1-\varepsilon)d^2}{c}}, \quad c \text{ some constant}$$

5. Proof: First step: Understanding ε-UPQPs as embeddings

$$S_1(\mathcal{H}) \to B(\mathcal{H}_m) \quad \text{→ Banach space}$$

$\{$ Op on $\mathcal{H}_m$ with trace norm $\}$

Second step: understanding isometric embeddings $S_1(\mathcal{H}) \to B(\mathcal{H}_m)$

ε - UPQPs and embeddings:

ν - unitary in $\mathcal{U}(\mathcal{H} \otimes \mathcal{H}_m)$

$$M_{dm} \simeq M_d \otimes M_m, \quad \nu = \sum_i A_i \otimes B_i$$

$\mathcal{L}(M_d^{12+}, M_m)$:

$$\Phi_\nu : \sigma \longmapsto \sum_i (\mathrm{tr}\, \sigma A_i)\, B_i$$

$$S(\mathcal{H}) \longrightarrow B(\mathcal{H}_m)$$

Fact: $\| \Phi_\nu : S_1(\mathcal{H}) \to B(\mathcal{H}_m) \| \leq \| \nu \|_{B(\mathcal{H} \otimes \mathcal{H}_m)} = 1$

trace norm, op. norm

contractive map

Th. 2: If ν is an ε-URQP, then Φ_ν is an approx. isometry:

$$\forall\, \sigma \in S(\mathcal{H}) \quad \boxed{\underset{\|\sigma\|_1}{(1-\varepsilon)} \leq \|\Phi_\nu(\sigma)\|_{B(\mathcal{H}_m)} \leq \|\sigma\|_{S_1(\mathcal{H})}}$$

$$\Downarrow$$

Rem: $\|\Phi_\nu\| \cdot \|\Phi_\nu^{-1}\| \leq \dfrac{1}{(1-\varepsilon)^{1/2}}$

$\varepsilon = 0 \Rightarrow$ perfect URQP $\Rightarrow \Phi_\nu$ - isometry $\rightarrow$ not possible

Thm 1 Embeddings $S_1(\mathcal{H}) \to B(\mathcal{H}_m)$ and type 2 constants

ν ε-URQP $\rightarrow \Phi_\nu$ embedding (contraction, approx isometry)

$$\Phi_\nu : S_1(\mathcal{H}) \to \Phi_\nu(S_1(\mathcal{H})) \subseteq B(\mathcal{H}_m)$$

$$\|\Phi_\nu\| \|\Phi_\nu^{-1}\| \leq \frac{1}{(1-\varepsilon)^{1/2}}$$

- We look for a metric invariant ~~[illegible]~~ of $S_1(\mathcal{H}), B(\mathcal{H}_m)$

(property of $\rightarrow$ invariant when Φ_ν is applied)

Def: A ~~a~~ normed space X is ~~a~~ called of type 2 if for every finite sequence $\{x_i\} \subset X$

$$\mathbb{E} \left\| \sum_i \varepsilon_i x_i \right\|_X^2 \leq T \left(\sum_i \|x_i\|^2 \right)^{1/2}$$

$\varepsilon_i \in \{\pm 1\}$ random variable

for some constant T

The inf. of constants T is called the type 2 constant of $X \equiv T_2(X)$

Comments: (i) $X \equiv$ Hilbert space, then $T_2(\mathcal{H}) = 1$ (parallelogram law)

(ii): For $S_1(\mathcal{H})$: $T_2(S_1(\mathcal{H})) \geq \sqrt{d}$

$T_2(B(\mathcal{H}_m)) \leq c\sqrt{\log m}$

(iii): For subspaces: $T_2(S) \leq T_2(X)$
$S \subseteq X$

(iv) If $\phi : X \to Y$ ~~isomorphism~~ linear isomorphism

$T_2(X) \leq \|\phi\| \|\phi^{-1}\| T_2(Y)$

$\boxed{\text{Proof of thm 1}}$ $T_2(S_1(\mathcal{H})) \leq \frac{1}{(1-\varepsilon)^{1/2}} T_2(B(\mathcal{H}_m))$

$\sqrt{d} \leq$ $\leq \frac{C\sqrt{\log m}}{(1-\varepsilon)^{1/2}}$

$\Rightarrow m \geq 2^{\frac{(1-\varepsilon)m}{C}}$ ~~d~~

$\exp((1-\varepsilon)d) \leq m \leq \exp(d^2 C_\varepsilon d)$

$\boxed{\text{OPTIMALITY (in Thm 1)}}$ Fix ε, Standard Q-teleportation is also an ε-UPQP, $(1-\varepsilon)^2 = 1/d^2$, $m = d^2$

<u>Rem</u> the bound is true for ε-PQP ~~some~~ which implement the diagonal unitaries