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An operational approach to classifying measurement incompatibility

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Measurement incompatibility has proved to be an important resource for information processing. In this work, we study different levels of incompatibility among a set of measurements through an operational approach, utilizing elementary classical operations on the inputs or outputs of these measurements. We provide classifications of measurement incompatibility with respect to two classical operations, *viz.*, coarse-graining of measurement outcomes and convex-mixing of different measurements. We derive analytical criteria for determining when a set of projective measurements is fully incompatible with respect to coarse-graining or convex-mixing. Robustness against white noise is investigated for mutually unbiased bases that can sustain different levels of incompatibility. Furthermore, we study operational witnesses for full incompatibility subject to these classical operations, using the input-output statistics of Bell-type experiments as well as experiments in the prepare-and-measure scenario.

I. INTRODUCTION

Measurement incompatibility is a concept relating observables that cannot be measured jointly with arbitrary accuracy [1]. It is a purely quantum effect of which the most well-known example concerns the position and momentum of a quantum particle. Being a fundamental concept of quantum theory, it takes a pivotal role in explaining several quantum phenomena, such as, Bell-nonlocality [2, 3], Einstein-Podolsky-Rosen steering [4–8], measurement uncertainty relations [9–12], quantum contextuality [13–15], quantum violation of macrorealism [16, 17], and temporal and channel steering [18–20]. Measurement incompatibility is necessary for quantum advantage in any one-way communication task [21].

The significance of measurement incompatibility in various quantum effects calls for its in-depth characterization. Towards this direction, a classification of measurement incompatibility with respect to projection onto subspaces has been recently performed [22]. In the present work, our objective is to classify measurement incompatibility in an operational approach that does not involve the details of a theory. To put it differently, we address how various degrees of incompatibility can be assessed solely by executing basic classical operations on the inputs or outputs of these measurements. Here, we employ two such operations: coarse-graining of measurement outcomes and convex-mixing of measurement settings.

Coarse-graining of measurement outcomes arises naturally [23] in several instances, such as, for example, quite obviously in measurements on continuous variable systems. Though the eigen spectra of the observables are infinite-dimensional, taking continuous values, real-world experimental devices are limited by finite precision, leading to the measurement outcomes taking a finite number of discrete values. This inaccuracy in the recording of measurement outcomes is manifested in the coarse-graining of measurement outcomes, which is inevitable in practice. Coarse-graining has also been employed to study the phenomenon of quantum-to-classical transition. It is observed that quantum phenomena may disappear due to imprecision of measurement outcomes [16, 24–26]. On the other hand, device imperfection may also lead to the measurement device performing probabilistically a set of measurements, instead of always performing the desired particular measurement. In such a case, a convex-mixing of the given set of measurements arises effectively [27]. As incompatibility of measurements is a quantum concept, it is interesting to examine how this property behaves under such elementary classical operations. Our study is motivated by the question of how one may compare the degree of incompatibility between two different sets of measurements. For instance, if the first set remains incompatible for every possible non-trivial coarse-graining of the measurement outcomes, but the second set becomes compatible for a certain coarse-graining, it follows that the first set of measurements exhibits stronger incompatibility compared to the second one. A similar argument holds for the case of convex-mixing of measurements, too.

In this work, we establish analytical criteria for determining when a pair of projective measurements are fully incompatible across all possible coarse-grainings of measurement outcomes. We analyze the full incompatibility of a set of three qubit measurements under all possible convex-mixing. Within the context of our present study, noise is reflected in the degrading of incompatibility properties of various measurement sets [28]. We compute the critical noise threshold below which mutually unbiased bases remain incompatible under the mentioned classical operations.

As incompatible measurements are useful for performing various information processing tasks [29–31], given a measurement device claiming to produce incompatible measurements, we must certify it before using it in an information processing task. Verification of incompatiblity of the measurements is possible from the input-output measurement statistics obtained from the device without knowing its internal functioning. Device-independent protocols are conceptually most powerful, relying only on the input-output statistics [32], with a wide range of applications [33–40]. However, they require shared entanglement, an expensive resource, and prohibition of communication between the involved parties. On the other hand, semi-device-independent protocols are inspired by the standard prepare-and-measure scenario, with an additional constraint on the dimension of the communicated quantum states [41–48]. In the present work, we also investigate the issue of certification of measurement incompatibility subjected to the classical operations of coarse-graining and convex-mixing of measurements from both the device-independent and the semi-device-independent perspective.

Our paper is structured as follows. In Sec. (II), we present the basic mathematical ingredients for classifying measurement incompatibility under the two operations of coarse-graining of outcomes and convex-mixing of measurements, respectively. In Sec. (III), we study the robustness under noise for various levels of incompatibility subjected to the above classical operations. In Sec. (IV), we define operational witnesses of incompatibility and furnish examples to study their performance in both device-independent and semi-device-independent frameworks. Concluding remarks are presented in Sec. (V).

II. CLASSIFYING MEASUREMENT INCOMPATIBILITY UNDER ELEMENTARY CLASSICAL OPERATIONS

In the case of projective measurements, the observables whose corresponding operators commute are jointly measurable. However, this is not true in the case of general quantum measurements. The most general quantum measurement is described by Positive Operator Valued Measures (POVM), which is a set of operators, $\{O_\alpha\}$ with $0 \leq O_\alpha \leq \mathbb{1}$, and $\sum_\alpha O_\alpha = \mathbb{1}$. A set of measurements defined by $\{M_{z_x|x}\}_{z_x,x}$ (here x corresponds to the measurement, z_x denotes the outcomes of the measurement labeled by x) is compatible if there exists a parent POVM, G_λ and classical post-processing for each x given by $\{p(z_x|x,\lambda)\}$ such that

$$\forall z_x, x, \quad M_{z_x|x} = \sum_\lambda p(z_x|x,\lambda) G_\lambda, \tag{1}$$

where $0 \leq p(z_x|x,\lambda) \leq 1, \sum_{z_x} p(z_x|x,\lambda) = 1 \forall x,\lambda$ [1].

A. Coarse-graining of outcomes

Coarse-graining of the outcomes of a measurement is a concept where some of the outcomes are treated on the same footing, *i.e.*, different outcomes are clubbed together to form a single outcome. After the coarse-graining of the outcomes, the effective number of outcomes reduces. Coarse-graining may occur in practice due to measurement errors. Additionally, coarse-graining is used in when one does not require a finer description of the values of the measurement outcomes, but rather a broad description suffices to understand the nature of some physical property.

Let us introduce the notation $[k] := \{0,\cdots,k-1\}$ for any natural number k. Consider a $\overline{d}$ outcome measurement $\{M_{\overline{z}}\}$ and an arbitrary coarse-graining that yields a d outcome measurement $\{M_z\}$, where $\overline{z} \in [\overline{d}]$ and $z \in [d]$. The coarse-graining can be defined by conditional probabilities $\{c(z|\overline{z})\}$ such that

$$M_z = \sum_{\overline{z}} c(z|\overline{z}) M_{\overline{z}}, \tag{2}$$

and $c(z|\overline{z}) \in \{0,1\}$. Trivial coarse-graining is the one where the measurement becomes single outcome, that is, there exists z such that $c(z|\overline{z}) = 1, \forall \overline{z}$. It is worth mentioning that the definition above also considers the permutation of measurement outcomes before undergoing the coarse-graining process. Two incompatible measurements may not necessarily remain incompatible after certain coarse-graining of their outcomes.

Definition 1 (Fully incompatible measurements w.r.t. coarse-graining)**.** *A set of measurements $\{M_{\overline{z}_x|x}\}$ is fully incompatible with respect to (w.r.t.) coarse-graining if they remain incompatible after all possible nontrivial coarse-graining. That is, if the resultant set of measurements given by*

$$M_{z_x|x} = \sum_{\overline{z}_x} c_x(z_x|\overline{z}_x) M_{\overline{z}_x|x}, \tag{3}$$

after all possible sets of nontrivial coarse-graining $\{c_x(z_x|\overline{z}_x)\}$ is incompatible, then we call them fully incompatible. Note that, the coarse-graining $\{c_x(z_x|\overline{z}_x)\}$ can be different for different settings x.

Definition 2 (d-incompatible measurements w.r.t. coarse-graining)**.** *A set of measurements $\{M_{\overline{z}_x|x}\}$ is d-incompatible w.r.t. coarse-graining if they remain incompatible after all possible nontrivial coarse-graining that give rise to at least d outcome measurements. In other words, if the resultant set of measurements*

$$M_{z_x|x} = \sum_{\overline{z}_x} c_x(z_x|\overline{z}_x) M_{\overline{z}_x|x},\ z_x \in [k_x], k_x \geqslant d, \forall x \tag{4}$$

after all possible coarse-graining resulting in at least d outcome measurements is incompatible, then we call them d-incompatible.

For example, the three outcome rank-one projective measurement pair, defined by the following (unnormalized) vectors

$$M = \{|0\rangle, |1\rangle, |2\rangle\} \tag{5}$$

and

$$N = \{|0\rangle + |1\rangle, |0\rangle - |1\rangle, |2\rangle\} \tag{6}$$

is 3-incompatible, but not 2-incompatible since coarse-graining of first two outcomes of these measurements yield compatible measurements.

Observation 1. *A set of fully incompatible measurements is equivalent to 2-incompatible measurements w.r.t. coarse-graining.*

Proof. If a set of measurements is 2-incompatible, then it implies that the set remains incompatible after all possible coarse-graining of the outcomes such that the number of outcomes of each measurement in the newly formed set of measurements is greater than or equal to two. Also, the lowest number of outcomes of measurement is two for a nontrivial coarse-graining. Furthermore, if a set of measurements is d-incompatible, then, by definition, it is n-incompatible as well, where $n > d$, but the reverse is not true. This proves Observation 1. □

Observation 2. *Consider two projective measurements, defined by $\{P_i\}$, and $\{Q_j\}$, where $i \in [d]$ and $j \in [d']$. Let $\{\mathcal{M}_k\}_k$ be the set of all proper subsets of $[d]$, and $\{\mathcal{N}_l\}_l$ be the set of all proper subsets of $[d']$. Then these two measurements are fully incompatible w.r.t. coarse-graining if and only if*

$$\left[\sum_{i\in\mathcal{M}_k} P_i, \sum_{j\in\mathcal{N}_l} Q_j\right] \neq 0, \ \forall k, l. \tag{7}$$

Proof. The result is a direct consequence of the fact that for sharp measurement compatibility and commutativity are equivalent [49]. Suppose $\exists k, l$, such that the left-hand-side of (7) is zero. Then consider the coarse-graining such that the resulting measurements will be $\{\sum_{i\in\mathcal{M}_k} P_i, \mathbb{1} - \sum_{i\in\mathcal{M}_k} P_i\}$ and $\{\sum_{j\in\mathcal{N}_l} Q_j, \mathbb{1} - \sum_{j\in\mathcal{N}_l} Q_j\}$. The resultant measurements will be compatible. The reverse direction holds true from the definition of fully incompatible w.r.t. coarse-graining. □

Theorem 1. *The following condition is necessary but not sufficient for two rank-one projective measurements, defined by $\{|\psi_i\rangle\}$ and $\{|\phi_j\rangle\}$, to be fully incompatible w.r.t. coarse-graining:*

$$\langle\psi_i|\phi_j\rangle \neq 0, \ \forall i, j. \tag{8}$$

The above condition is necessary and sufficient for two 3-dimensional rank-one projective measurements.

Proof. First, note that for sharp measurement compatibility and commutativity are equivalent [49]. Suppose $\exists i, j$, such that $\langle\psi_i|\phi_j\rangle = 0$. Consider coarse-graining of all other outcomes except i and j for the two measurements. Then, the resulting measurements will be $\{|\psi_i\rangle\langle\psi_i|, \mathbb{1} - |\psi_i\rangle\langle\psi_i|\}$ and $\{|\phi_j\rangle\langle\phi_j|, \mathbb{1} - |\phi_j\rangle\langle\phi_j|\}$, which commute with each other (i.e., the resulting measurements are compatible) since $\langle\psi_i|\phi_j\rangle = 0$. Thus, if the measurements are fully incompatible, condition (8) holds.

To show that (8) is not sufficient, consider the following two 4-dimensional rank-one projective measurements (with the normalization factor $1/\sqrt{2}$),

$$\begin{aligned} &\{|0\rangle + |1\rangle, |0\rangle - |1\rangle, |2\rangle + |3\rangle, |2\rangle - |3\rangle\} \\ &\{|0\rangle + |2\rangle, |0\rangle - |2\rangle, |1\rangle + |3\rangle, |1\rangle - |3\rangle\}. \end{aligned} \tag{9}$$

One can check that (8) holds for all $i, j = 0, 1, 2, 3$. But coarse-graining of outcomes $0, 1$ and $2, 3$ for both the measurements leads to

$$\begin{aligned} &\{|0\rangle\langle 0| + |1\rangle\langle 1|, |2\rangle\langle 2| + |3\rangle\langle 3|\}, \\ &\{|0\rangle\langle 0| + |2\rangle\langle 2|, |1\rangle\langle 1| + |3\rangle\langle 3|\}, \end{aligned} \tag{10}$$

which are compatible.

In 3-dimension, say, the measurements are $M = \{|\psi_i\rangle\}$ and $N = \{|\phi_j\rangle\}$ with $i, j \in \{1, 2, 3\}$. Now any non-trivial coarse-graining yields binary-outcome measurements of the form: $\{|\psi_i\rangle\langle\psi_i|, \mathbb{1} - |\psi_i\rangle\langle\psi_i|\}$ and $\{|\phi_j\rangle\langle\phi_j|, \mathbb{1} - |\phi_j\rangle\langle\phi_j|\}$. It is easy to see that these two remain incompatible if and only if $[|\psi_i\rangle\langle\psi_i|, |\phi_j\rangle\langle\phi_j|] \neq 0$, which is equivalent to $\langle\psi_i|\phi_j\rangle \neq 0, 1$. In the case where $\langle\psi_i|\phi_j\rangle = 1$, there exists another pair (i, j') such that $\langle\psi_i|\phi_{j'}\rangle = 0$; thus, (8) implies fully incompatible in dimension 3. □

B. Convex-mixing of measurements

The concept of convex-mixing of measurements may be best understood by considering an example. Suppose we have a measurement device that can implement three different measurements, $M = \{M_z\}_z, N = \{N_z\}_z, R = \{R_z\}_z$, all having same number of outcomes $z \in [d]$. For the first input choice, it performs measurements $\{M_z\}$ with probability q and $\{N_z\}$ with probability $(1-q)$, preceded by applying some permutation on the outcomes of the measurements. For the second input, it always performs measurement $\{R_z\}$. Let $Q_{(M,N,\pi)}$ represents the measurement that is realized by the convex-mixing,

$$Q_{(M,N,\pi)} = \{qM_z + (1-q)N_{\pi(z)}\}_z, \tag{11}$$

where $q \in [0, 1]$ is the weightage of the convex mixture and $\pi(z)$ is a permutation on outcome z. Here $\pi(z)$ is a bijective function from the set $z \in [d]$ to itself. It is important to note that this definition encompasses the most general form of convex-mixing, incorporating the permutation of measurement outcomes. Even if R is incompatible with M and N separately, R is not necessarily incompatible with $Q_{(M,N,\pi)}$ for all values of q.

Definition 3. *Three measurements M, N and R are fully incompatible w.r.t. convex mixtures if each of the pairs, M and $Q_{(N,R,\pi)}$, N and $Q_{(M,R,\pi)}$, R and $Q_{(M,N,\pi)}$, are incompatible for all values of q and all possible permutations π.*

Consider three unbiased qubit binary outcome measurements $\{M_0, M_1\}, \{N_0, N_1\}, \{R_0, R_1\}$,

$$\begin{aligned} M_z &= \tfrac{1}{2}\left(\mathbb{1} + (-1)^z\, \vec{n}_0 \cdot \vec{\sigma}\right), \\ N_z &= \tfrac{1}{2}\left(\mathbb{1} + (-1)^z\, \vec{n}_1 \cdot \vec{\sigma}\right), \\ R_z &= \tfrac{1}{2}\left(\mathbb{1} + (-1)^z\, \vec{n}_2 \cdot \vec{\sigma}\right), \end{aligned} \tag{12}$$

with $z = 0, 1$ and $||\vec{n}_i|| \leqslant 1$ where $i \in \{0, 1, 2\}$. A necessary and sufficient criterion for the incompatibility of two unbiased binary-outcome qubit measurements is widely recognized (see eq.(7) of [1]). By applying this criterion, we find that the above three measurements (12) are fully incompatible w.r.t. convex mixtures if and only if,

$$||\vec{n}_i + q\vec{n}_j \pm (1-q)\vec{n}_k|| + ||\vec{n}_i - q\vec{n}_j \mp (1-q)\vec{n}_k|| > 2, \tag{13}$$

for all combinations of $\pm$, for all q, and for all $(i, j, k) \in \{(0, 1, 2), (1, 2, 0), (2, 0, 1)\}$. Note that the permutation of the outcomes of the measurements in (12) is nothing but taking $-\vec{n}_i$ instead of $\vec{n}_i$.

Theorem 2. *If three qubit measurements (12) are such that $\vec{n}_i$ are in the same plane of the Bloch sphere, then they are not fully incompatible w.r.t. convex mixtures.*

Proof. If the three $\vec{n}_i$ are in the same plane, then there exists at least one triple (i, j, k) and permutations of outcomes such that

$$\vec{n}_0 = \frac{1}{c}(q\vec{n}_1 + (1-q)\vec{n}_2) \tag{14}$$

for some $c \in (0, 1], q \in [0, 1]$. In other words, there exists a triple (i, j, k) and permutation of outcomes such that $\vec{n}_i$ is expressed as a linear combination of $\vec{n}_j$ and $\vec{n}_k$ with non-negative coefficients (here q/c and $(1-q)/c$), such that sum of those two non-negative coefficients is greater than or equal to 1. Substituting this into left hand side of (13) taking $+$ sign, we find

$$||(1+c)\vec{n}_i|| + ||(1-c)\vec{n}_i|| \leqslant |1+c| + |1-c| = 2. \tag{15}$$

This contradicts with (13), implying they are compatible. □

Theorem 3. *Consider three qubit measurements (12) are such that $\vec{n}_0 = \nu_0\hat{x}$, $\vec{n}_1 = \nu_1\hat{y}$, $\vec{n}_2 = \nu_2\hat{z}$ with $0 \leqslant \nu_0, \nu_1, \nu_2 \leqslant 1$, that is, the noisy version of Pauli observables,*

$$\begin{aligned} M_z &= \tfrac{1}{2}\left(\mathbb{1} + (-1)^z \nu_0 \sigma_x\right) = \nu_0 \left(\tfrac{\mathbb{1} + (-1)^z \sigma_x}{2}\right) + (1-\nu_0)\tfrac{\mathbb{1}}{2}, \\ N_z &= \tfrac{1}{2}\left(\mathbb{1} + (-1)^z \nu_1 \sigma_y\right) = \nu_1 \left(\tfrac{\mathbb{1} + (-1)^z \sigma_y}{2}\right) + (1-\nu_1)\tfrac{\mathbb{1}}{2}, \\ R_z &= \tfrac{1}{2}\left(\mathbb{1} + (-1)^z \nu_2 \sigma_z\right) = \nu_2 \left(\tfrac{\mathbb{1} + (-1)^z \sigma_z}{2}\right) + (1-\nu_2)\tfrac{\mathbb{1}}{2}, \end{aligned} \tag{16}$$

with $z = 0, 1$. These measurements are fully incompatible w.r.t. convex mixture if and only if

$$\min\left\{\nu_0^2 + \frac{\nu_1^2\nu_2^2}{\nu_1^2 + \nu_2^2}, \nu_1^2 + \frac{\nu_0^2\nu_2^2}{\nu_0^2 + \nu_2^2}, \nu_2^2 + \frac{\nu_0^2\nu_1^2}{\nu_0^2 + \nu_1^2}\right\} > 1. \tag{17}$$

Proof. In terms of ν_i, the left hand side of (13) becomes $2\sqrt{\nu_i^2 + q^2\nu_j^2 + (1-q)^2\nu_k^2}$. Note that the minimum of $q^2\nu_j^2 + (1-q)^2\nu_k^2$ occurs at $\tilde{q} = \nu_k^2/(\nu_j^2 + \nu_k^2)$. Since $0 \leq \nu_k^2/(\nu_j^2 + \nu_k^2) \leq 1$ for any $\nu_j, \nu_k \in [0, 1]$, the above-mentioned minimum can always be achieved with a suitable choice of q. Hence, we have that

$$\begin{aligned} 2\sqrt{\nu_i^2 + q^2\nu_j^2 + (1-q)^2\nu_k^2} &\geq 2\sqrt{\nu_i^2 + \tilde{q}^2\nu_j^2 + (1-\tilde{q})^2\nu_k^2} \\ &= 2\sqrt{\nu_i^2 + \frac{\nu_j^2\nu_k^2}{\nu_j^2 + \nu_k^2}} \end{aligned} \tag{18}$$

Thus, the right-land-side of (13) is greater than 2 for all values of q if and only if

$$\nu_i^2 + \frac{\nu_j^2\nu_k^2}{\nu_j^2 + \nu_k^2} > 1. \tag{19}$$

Taking all the three possible values of (i, j, k) we get the condition (17). □

Clearly, the three Pauli observables are fully incompatible w.r.t. convex mixture, and moreover, if we take an equal amount of noise $\nu_0 = \nu_1 = \nu_2 = \nu$, then (17) implies $\nu > \sqrt{2/3}$.

We can generalize this notion for a set of n measurements.

Definition 4 (k-incompatible measurements w.r.t. convex mixtures)**.** *Given a set of n measurements, the measurements are k-incompatible w.r.t. convex mixtures if, after taking every possible convex mixtures over all possible k number of partitions with every possible relabelling of outcomes, the resulting k number of measurements are incompatible.*

Definition 5 (Fully incompatible measurements w.r.t. convex mixtures)**.** *A set of n measurements is fully incompatible w.r.t. convex mixtures if it is k-incompatible for all $k = 2, \cdots, n$.*

Observation 3. *Fully incompatible measurements w.r.t. convex mixtures imply that every pair of measurements from that set is incompatible. The reverse implication does not hold.*

Proof. Consider a set of n measurements, in which there is a pair of measurements $\{M_z\}$ and $\{N_z\}$ that are compatible. Now if we make two partitions where $\{M_z\}$ and $\{N_z\}$ are in different partitions and the convex mixture is such that the probabilities of arising all other measurements are zero, then the resultant measurement pair must be compatible. This is true for any compatible pair of measurements. Thus, if the measurements are fully incompatible w.r.t. convex mixture, every pair of measurements must necessarily be incompatible.

The reverse is not true. Consider the three noisy Pauli measurements of Eq.(16). It can be shown by using semi-definite programming that if $0.71 < \nu \leqslant 0.81$, the measurements are pairwise incompatible but it is not fully incompatible w.r.t. convex mixture [40]. □

III. ROBUSTNESS UNDER NOISE FOR DIFFERENT LEVELS OF INCOMPATIBILITY

In this section, we analyze the role of noise on quantum measurements and study how the incompatibility properties depend on it. Due to the ubiquitous nature of noise, it is pertinent to study the extent to which noise could be tolerated by a set of measurements while still retaining their incompatibility. We take a noisy version of mutually unbiased bases measurements (MUBs),

$$M_{i|x} = \nu|\phi_{i|x}\rangle\langle\phi_{i|x}| + \frac{(1-\nu)}{d}\mathbb{1}_d, \tag{20}$$

where $\{|\phi_{i|x}\rangle\}_{i,x}$ form mutually unbiased bases measurements in $\mathbb{C}^d$. Here ν is the robustness parameter (or visibility parameter) and $(1-\nu)$ is the noise parameter, $0 \leq \nu \leq 1$. When the noise parameter is zero (i.e., robustness, $\nu = 1$), the measurements are fully incompatible, and when the noise parameter is one (i.e., the robustness, $\nu = 0$), the measurements are trivial and compatible. Our aim is to obtain the critical value of the robustness parameter above which the measurements remain incompatible.

To check the compatibility, i.e., the existence of a parent POVM, we use the method described in [1]. This can be casted as a semi-definite programming (SDP) problem that takes a set of measurements $\{M_{z_x|x}\}$ and deterministic classical post-processings $p(z_x|x,\lambda)$ as input, and checks whether the measurements are compatible or not, subject to the constraints

$$\textstyle\sum_\lambda p(z_x|x,\lambda)G_\lambda = M_{z_x|x} \forall x, z_x, \tag{21}$$

$$\textstyle\sum_\lambda G_\lambda = \mathbb{1}, \tag{22}$$

$$G_\lambda \geq \mu\mathbb{1}, \tag{23}$$

where μ is the optimization parameter. This method finds the maximum value of μ for each $\{p(z_x|x,\lambda)\}$. If this optimization returns a negative value of μ, then the constraint of Eq.(23) cannot be fulfilled, which implies that the measurements $\{M_{z_x|x}\}$ are incompatible. Otherwise, they are compatible.

A. Coarse-graining of outcomes

Dimension 3. Let $\{M_i\}$ and $\{N_j\}, i, j \in [3]$ be two measurements acting on $\mathbb{C}^3$, where

$$M_i = \nu|i\rangle\langle i| + (1-\nu)\frac{\mathbb{1}}{3},$$
$$N_j = \nu|\psi_j\rangle\langle\psi_j| + (1-\nu)\frac{\mathbb{1}}{3}, \tag{24}$$

with

$$|\psi_0\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle),$$
$$|\psi_1\rangle = \frac{1}{\sqrt{3}}(|0\rangle + \omega|1\rangle + \omega^2|2\rangle),$$
$$|\psi_2\rangle = \frac{1}{\sqrt{3}}(|0\rangle + \omega^2|1\rangle + \omega|2\rangle),$$

ω being the cube roots of unity. The results are summarized in TABLE I.

Dimension 4. For checking incompatibility w.rt coarse-graining in $\mathbb{C}^4$, the same procedure is repeated for two POVM measurements with 4 outcomes each. The corresponding measurements are $\{M'_i\}, \{N'_j\}, i, j \in [4]$:

$$M'_i = \nu|i\rangle\langle i| + (1-\nu)\frac{\mathbb{1}}{4},$$
$$N'_j = \nu|\psi'_j\rangle\langle\psi'_j| + (1-\nu)\frac{\mathbb{1}}{4}, \tag{25}$$

with

$$|\psi'_0\rangle = \frac{1}{2}(|0\rangle + |1\rangle + |2\rangle + |3\rangle),$$
$$|\psi'_1\rangle = \frac{1}{2}(|0\rangle + |1\rangle - |2\rangle - |3\rangle),$$
$$|\psi'_2\rangle = \frac{1}{2}(|0\rangle - |1\rangle - |2\rangle + |3\rangle),$$
$$|\psi'_3\rangle = \frac{1}{2}(|0\rangle - |1\rangle + |2\rangle - |3\rangle). \tag{26}$$

The results are summarized in TABLE I.

B. Convex-mixing of measurements

Dimension 3. For checking incompatibility w.r.t. convex mixtures (CM), a minimum of three measurements are needed. The two measurements are $\{M_i\}$ and $\{N_j\}$,

	In dimension 3	In dimension 4
4-incompatible	N.A.	0.666
3-incompatible	0.683	0.675
2-incompatible	0.711	0.720

TABLE I. Critical values of robustness for MUBs w.r.t. coarse-graining in $\mathbb{C}^3$ and $\mathbb{C}^4$.

as given in Eqs.(24). The third measurement is $\{R_k\}, k \in [3]$ where

$$R_k = \nu|\phi_k\rangle\langle\phi_k| + (1-\nu)\frac{\mathbb{1}}{3}. \tag{27}$$

with

$$\begin{aligned} |\phi_0\rangle &= \frac{\omega|0\rangle + |1\rangle + |2\rangle}{\sqrt{3}}, \ |\phi_1\rangle = \frac{|0\rangle + \omega|1\rangle + |2\rangle}{\sqrt{3}}, \\ |\phi_2\rangle &= \frac{|0\rangle + |1\rangle + \omega|2\rangle}{\sqrt{3}}. \end{aligned} \tag{28}$$

The results are summarized in TABLE II.

Dimension 4. Consider the measurements $\{M'_i\}, \{N'_j\}$ as defined in Eqs. (25) and (26) and another measurement $\{R'_k\}$ where $i, j, k \in [4]$,

$$R'_k = \nu|\phi'_k\rangle\langle\phi'_k| + (1-\nu)\frac{\mathbb{1}}{4} \tag{29}$$

with

$$\begin{aligned} |\phi'_0\rangle &= \frac{1}{2}(|0\rangle - |1\rangle - \mathrm{i}|2\rangle - \mathrm{i}|3\rangle), \\ |\phi'_1\rangle &= \frac{1}{2}(|0\rangle - |1\rangle + \mathrm{i}|2\rangle + \mathrm{i}|3\rangle), \\ |\phi'_2\rangle &= \frac{1}{2}(|0\rangle + |1\rangle + \mathrm{i}|2\rangle - \mathrm{i}|3\rangle), \\ |\phi'_3\rangle &= \frac{1}{2}(|0\rangle + |1\rangle - \mathrm{i}|2\rangle + \mathrm{i}|3\rangle). \end{aligned} \tag{30}$$

The results are summarized in TABLE II.

	In dimension 3	In dimension 4
3-incompatible	0.537	0.692
2-incompatible	0.764	0.705

TABLE II. Critical values of robustness for MUBs w.r.t. convex-mixing in $\mathbb{C}^3$ and $\mathbb{C}^4$.

IV. OPERATIONAL WITNESSES OF INCOMPATIBILITY

In this section, we explore how different levels of incompatibility can be witnessed through the input-output statistics derived from characterized devices.

A. Coarse-graining of outcomes

As we mentioned earlier, measurements that are fully incompatible w.r.t. coarse-graining show stronger incompatibility compared to the measurements that are not fully incompatible w.r.t. coarse-graining. To operationally certify whether a set of measurements is fully incompatible w.r.t. coarse-graining is important from the perspective of determining the practical utility of such a set for revealing phenomena such as Bell inequality, steering and contextuality [2–8, 13–15], as well as for checking the proficiency of such a set for information processing tasks [21, 50]. Certification means that we need to infer the incompatibility of the measurements from the input-output statistics of the measurement device without knowing its internal functioning. Below we study the two classes of certification separately.

Device-independent witness

Violation of Bell inequality provides device-independent witness for incompatible measurements. Here, one requires neither any prior knowledge of the internal functioning of the measurement device nor any idea of the dimension of the system on which the measurements act. However, not all incompatible measurements yield violations of Bell inequalities [51, 52]. Nonetheless, we have the following results.

Observation 4. *Full-incompatibility w.r.t. coarse-graining of any two measurements can always be witnessed in a device-independent way.*

Proof. In [3], it has been proven that any pair of binary-outcome incompatible measurements violate at least one Bell-CHSH inequality by suitable choice of entangled state and measurements on the other subsystem. On the other hand, from Observation 1, we know if two measurements are full-incompatible w.r.t. coarse-graining, then they must be two-incompatible w.r.t. coarse-graining. Combining these two facts, we can conclude this observation. □

Device-independent witness from a single Bell experiment

In the above-mentioned approach, different Bell experiments are probed for different coarse-graining. It is interesting to explore whether different levels of incompatibility w.r.t. coarse-graining can be witnessed universally through a single Bell experiment in a device-independent manner. In this case, the objective is to consider different coarse-grained statistics of a Bell experiment and check whether those statistics have local explanations or not.

To understand how this technique works, we consider a well-known example of two three-outcome rank-one projective measurements in $\mathbb{C}^3$ which give maximum violation of the Collins-Gisin-Linden-Massar-Popescu (CGLMP) inequality [53], a well-studied [54–56] Bell-type inequality. The projective measurements of Alice are: $A_a \equiv \{|\xi\rangle_{A,a}\}$, where

$$|\xi\rangle_{A,a} = \frac{1}{\sqrt{3}}\sum_{j=0}^{2}\exp\left(\mathbb{i}\frac{2\pi}{3}j(\xi+\alpha_a)\right)|j\rangle_A, \tag{31}$$

with $a \in \{1,2\}$ corresponding to two different measurement settings of Alice, and $\xi \in \{0,1,2\}, \alpha_1 = 0, \alpha_2 = \frac{1}{2}$. The projective measurements of Bob are: $B_b \equiv \{|\eta\rangle_{B,b}\}$, where

$$|\eta\rangle_{B,b} = \frac{1}{\sqrt{3}}\sum_{j=0}^{2}\exp\left(\mathbb{i}\frac{2\pi}{3}j(-\eta+\beta_b)\right)|j\rangle_B, \tag{32}$$

with, $b \in \{1,2\}$ corresponding to two different measurement settings of Bob, and $\eta \in \{0,1,2\}, \beta_1 = \frac{1}{4}, \beta_2 = -\frac{1}{4}$.

The incompatibility of the measurements of Eq.(31) before coarse-graining is guaranteed by the CGLMP-inequality violation. Now we investigate the incompatibility status of the measurements of Alice after all possible coarse-graining from the measurement statistics obtained in the CGLMP experiment. Alice's measurements now have two outcomes each and it is a (2,2,2,3) scenario, i.e., two inputs for Alice and two inputs for Bob and each input of Alice has two outcomes and each input of Bob has three outcomes. For this scenario, Collins and Gisin have shown that there are a total of 72 CH-facet inequalities [57]. The CH-inequalities [58] are of the form:

$$\begin{aligned}-1 \leq\ & P(00|A_1,B_1) + P(00|A_1,B_2) + P(00|A_2,B_2) \\ & -P(00|A_2,B_1) - P(0|A_1) - P(0|B_2) \leq 0.\end{aligned} \tag{33}$$

The other CH-inequalities are obtained by (1) interchanging A_1 with A_2, (2) interchanging B_1 with B_2, and (3) interchanging both A_1 with A_2 and B_1 with B_2. Consider the scenario where we coarse-grain (0,1) outcomes for both the inputs of Alice. Lets make the following relabeling $(0,1) \equiv \overline{0}$ and $2 \equiv \overline{1}$ for the outcomes of A_1 and A_2 and also consider the clubbing of $(0,1) \equiv \overline{0}$ outcomes both for B_1 and B_2. Under this relabelling Eq.(33) takes the form:

$$\begin{aligned}-1 \leq\ & P(\overline{0}\,\overline{0}|A_1,B_1) + P(\overline{0}\,\overline{0}|A_1,B_2) + P(\overline{0}\,\overline{0}|A_2,B_2) \\ & -P(\overline{0}\,\overline{0}|A_2,B_1) - P(\overline{0}|A_1) - P(\overline{0}|B_2) \leq 0,\end{aligned} \tag{34}$$

where,

$$\begin{aligned}P(\overline{0}\,\overline{0}|A_i,B_j) = & P(00|A_i,B_j) + P(01|A_i,B_j) \\ & +P(10|A_i,B_j) + P(11|A_i,B_j),\end{aligned} \tag{35}$$

$$\begin{aligned}P(\overline{0}|A_i) &= P(0|A_i) + P(1|A_i), \\ P(\overline{0}|B_i) &= P(0|B_i) + P(1|B_i),\ i,j \in \{1,2\}.\end{aligned} \tag{36}$$

Similarly, there are other 8 possible clubbings for Bob's measurement outcomes and for each clubbing, we have facet inequalities similar to Eq.(34). One can check that when there are the same coarse-graining of outcomes for both measurement inputs of Alice, we get a CH-inequality violation. Thus, under these coarse-grainings, the two measurements of Alice remain incompatible. When there are different coarse-grainings, for some cases we get CH-violation, but for other cases, we do not get CH-violation. CH-violation under a particular coarse-graining signifies that the measurements are incompatible. However, when there is no CH-violation, we can not conclude anything regarding the incompatibility status. This is depicted in TABLE III.

A_1	A_2	Incompatible
(0,1)	(0,1)	yes
(0,1)	(1,2)	?
(0,1)	(0,2)	yes
(1,2)	(0,1)	yes
(1,2)	(1,2)	yes
(1,2)	(0,2)	?
(0,2)	(0,1)	?
(0,2)	(1,2)	yes
(0,2)	(0,2)	yes

TABLE III. The incompatibility status for all possible coarse-graining (CG) of outcomes for Alice's measurements (A_1 and A_2) as inferred from the statistics of CGLMP experiment. Here "yes" denotes that under the particular CG of A_1 and A_2, the measurement remains incompatible. On the other hand, "?" denotes that we can not make any conclusion about their incompatibility.

Semi-device-independent witness

In the semi-device-independent approach in prepare-and-measure experiments, we do not have any prior knowledge of the internal functioning of the measurement device; however, we assume the dimension of the system on which the measurements act. To witness different levels of incompatibility, here we focus on a class of communication tasks, namely, $(2,\overline{d},d)-$RAC. In this task, the sender, Alice, gets two-dit string input message (x_1,x_2) with $x_1,x_2 \in [\overline{d}]$, and can communicate a $d-$dimensional system to the receiver, Bob, who wants to guess the value of any of the two dits, i.e. x_1 or x_2, randomly. The notation of $(2,\overline{d},d)-$RAC is adopted from [21]. If any two POVMs, each with $\overline{d}$ outcomes acting on $\mathbb{C}^d$ are jointly measurable, the average success probability of this task,

$$P(2,\overline{d},d) \leq P_{CB}(2,\overline{d},d) = \frac{1}{2}\left(1+\frac{d}{\overline{d}^2}\right), \quad (37)$$

where $P_{CB}(2,\overline{d},d)$ is an upper bound on average success probability using two compatible measurements [21, 50]. It turns out any two incompatible rank-one projective measurements can be witnessed through $(2,d,d)-$RAC [12]. For full incompatibility of two 3-outcome rank-one projective measurements, we prove the following result.

Theorem 4. *Two rank-one projective measurements, $M = \{|\phi_0\rangle, |\phi_1\rangle, |\phi_2\rangle\}$ and $N = \{|\psi_0\rangle, |\psi_1\rangle, |\psi_2\rangle\}$, can be witnessed to be fully incompatible w.r.t. coarse-graining via RAC if and only if $0 < |\langle\phi_i|\psi_j\rangle| < \frac{4}{5}$, $\forall\ i,j = 0,1,2$.*

Proof. Without loss of generality, any pair of three outcome rank-one projective measurements can be written up to unitary freedom as,

$$\begin{aligned} M &= \{|0\rangle\langle 0|, |1\rangle\langle 1|, |2\rangle\langle 2|\} \equiv \{|\phi_i\rangle\langle\phi_i|\}, \\ N &= \{|\psi_1\rangle\langle\psi_1|, |\psi_2\rangle\langle\psi_2|, |\psi_3\rangle\langle\psi_3|\}, \end{aligned} \quad (38)$$

where $|\psi_j\rangle = \alpha_j|0\rangle + \beta_j|1\rangle + \gamma_j|2\rangle$.

Let us first consider the coarse-graining of the second and third outcomes for both measurements. So, the new measurement pair becomes:

$$\begin{aligned} M^{(2,3)} &= \{|0\rangle\langle 0|, |1\rangle\langle 1| + |2\rangle\langle 2|\}, \\ N^{(2,3)} &= \{|\psi_1\rangle\langle\psi_1|, |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3|\}. \end{aligned} \quad (39)$$

This pair of measurements appears in $(2,2,3)-$RAC and the compatibility bound in this case is,

$$P_{CB}(2,2,3) = \frac{1}{2}\left(1+\frac{3}{2^2}\right) = \frac{7}{8} \quad (40)$$

due to Eq.(37). We have checked that this bound is tight. $M^{(2,3)}$ and $N^{(2,3)}$ will give advantage in $(2,2,3)-$RAC if

$$\sum_{x,y=0}^{1} ||M_x^{(2,3)} + N_y^{(2,3)}|| = \sum_{i=1}^{4} ||A_i|| > 7, \quad (41)$$

with

$$\begin{aligned} A_1 &= M_0^{(2,3)} + N_0^{(2,3)} = |0\rangle\langle 0| + |\psi_1\rangle\langle\psi_1|, \\ A_2 &= M_0^{(2,3)} + N_1^{(2,3)} = \mathbb{1} + |0\rangle\langle 0| - |\psi_1\rangle\langle\psi_1|, \\ A_3 &= M_1^{(2,3)} + N_0^{(2,3)} = \mathbb{1} - |0\rangle\langle 0| + |\psi_1\rangle\langle\psi_1|, \\ A_4 &= M_1^{(2,3)} + N_1^{(2,3)} = 2\mathbb{1} - |0\rangle\langle 0| - |\psi_1\rangle\langle\psi_1|, \end{aligned} \quad (42)$$

and $||\,||$ denotes the maximum eigenvalue of an operator. Now, $|\psi_1\rangle$ can be written as follows

$$\begin{aligned} |\psi_1\rangle &= \langle 0|\psi_1\rangle|0\rangle + \langle u|\psi_1\rangle|u\rangle \text{ with} \\ \langle 0|u\rangle &= 0 \text{ and } |\langle 0|\psi_1\rangle|^2 + |\langle u|\psi_1\rangle|^2 = 1, \end{aligned} \quad (43)$$

for some vector $|u\rangle$. So, A_1 can be written as a (2×2) matrix in the basis $\{|0\rangle, |u\rangle\}$,

$$A_1 = \begin{pmatrix} 1+|\langle 0|\psi_1\rangle|^2 & \langle 0|\psi_1\rangle\langle\psi_1|u\rangle \\ \langle u|\psi_1\rangle\langle\psi_1|0\rangle & |\langle u|\psi_1\rangle|^2 \end{pmatrix}, \quad (44)$$

the maximum eigenvalue of A_1 is $1+|\langle 0|\psi_1\rangle|$. Thus,

$$||A_1|| = 1 + |\langle 0|\psi_1\rangle|. \quad (45)$$

Similarly, A_2 can be expressed in a block diagonal matrix in the ortho-normal basis $\{|0\rangle, |u\rangle, |v\rangle\}$,

$$A_2 = \begin{pmatrix} \Gamma & 0 \\ 0 & 1 \end{pmatrix} \text{ where } \Gamma = \begin{pmatrix} 2+|\langle 0|\psi_1\rangle|^2 & \langle 0|\psi_1\rangle\langle\psi_1|u\rangle \\ \langle u|\psi_1\rangle\langle\psi_1|0\rangle & 1+|\langle u|\psi_1\rangle|^2 \end{pmatrix}, \quad (46)$$

and

$$||A_2|| = 2 + |\langle 0|\psi_1\rangle|. \quad (47)$$

Similarly, A_3 can be expressed in a block diagonal matrix in the basis $\{|0\rangle, |u\rangle, |v\rangle\}$,

$$A_3 = \begin{pmatrix} \Sigma & 0 \\ 0 & 1 \end{pmatrix}, \ \Sigma = \begin{pmatrix} |\langle 0|\psi_1\rangle|^2 & \langle 0|\psi_1\rangle\langle\psi_1|u\rangle \\ \langle u|\psi_1\rangle\langle\psi_1|0\rangle & 1+|\langle u|\psi_1\rangle|^2 \end{pmatrix}, \quad (48)$$

and

$$||A_3|| = 1 + \sqrt{1 - |\langle 0|\psi_1\rangle|^2}. \quad (49)$$

Similarly, A_4 can be expressed in a block diagonal matrix in the basis $\{|0\rangle, |u\rangle, |v\rangle\}$,

$$A_4 = \begin{pmatrix} \Xi & 0 \\ 0 & 2 \end{pmatrix}, \ \Xi = \begin{pmatrix} 1-|\langle 0|\psi_1\rangle|^2 & -\langle 0|\psi_1\rangle\langle\psi_1|u\rangle \\ -\langle u|\psi_1\rangle\langle\psi_1|0\rangle & 2-|\langle u|\psi_1\rangle|^2 \end{pmatrix}, \quad (50)$$

and $||A_4|| = 2$.

Substituting the values of $||A_1||, \ldots, ||A_4||$ in Eq.(41) and after simplification we get, $|\langle 0|\psi_1\rangle| < \frac{4}{5}$. This condition is obtained for a particular coarse-graining. Therefore, to be fully incompatible w.r.t. coarse-graining, $|\langle\phi_i|\psi_j\rangle| < \frac{4}{5}\ \forall\ i,j$, should hold. □

B. Convex-mixing of measurements

We now study the operational witness of incompatibility w.r.t. convex-mixing of measurements.

Device-independent witness

Given a set of n measurements, we can make k partitions of it, where k can be any number from the set $\{2, \cdots, n\}$. If the k number of measurements obtained by the convex mixture of the measurements provides a violation of Bell inequalities for all possible convex-mixing and permutations, then the measurements are witnessed to be $k-$incompatible w.r.t. convex-mixing in a device-independent way. Using the result that any two binary-outcome incompatible measurements violate the Bell-CHSH inequality [3], we can witness 2-incompatibility w.r.t. convex-mixing from any set binary-outcome measurements.

Semi-device-independent witness

For three or more numbers of measurements we can witness whether the measurements are fully-incompatible w.r.t. convex-mixture in a semi-device independent manner by constructing a suitable random access codes task.

Theorem 5. *Three noisy Pauli measurements of Eq. (16) with equal noise ($\nu = \nu_0 = \nu_1 = \nu_2$) are witnessed to be fully incompatible w.r.t. convex-mixing via RAC if and only if $\sqrt{2/3} < \nu \leqslant 1$.*

Proof. Consider the measurements Q^i with $i \in \{1, 2\}$ for two possible permutations of outcomes, to be formed by the convex-mixing of M and N as

$$Q^1 = \{p\ M_0 + (1-p)\ N_0,\ pM_1 + (1-p)N_1\},$$
$$Q^2 = \{p\ M_0 + (1-p)N_1, p\ M_1 + (1-p)N_0\}. \quad (51)$$

To witness the incompatibility status of (Q^i, R), one can construct a $(2,2,2)-$RAC task. Now, when Bob has to guess Alice's 1st bit, he performs the measurement Q^i defined by (51), which is realized by the convex-mixing of measurements M and N. For the second bit, he performs measurement $\mathcal{R}$. The maximum average success probability $P(2,2,2)$ that can be obtained in the RAC task is:

$$\begin{aligned} P(2,2,2) &= \frac{1}{8}\sum_{j,k=0}^{1} ||Q^i_j + R_k|| \\ &= \frac{1}{4}(2 + \sqrt{2\nu^2 - 2p\nu^2 + 2p^2\nu^2}). \end{aligned} \quad (52)$$

It can be shown that for all possible convex-mixing, the maximum average success probability is the same as Eq. (52). Now, to be fully incompatible w.r.t. convex-mixing $P(2,2,2) > 3/4$, which implies $\nu^2(p^2 - p + 1) > \frac{1}{2}$ (by Eq.(52)). Since the minimum value of $(p^2 - p + 1)$ is $\frac{3}{4}$, so for $\nu > \sqrt{2/3}$ the measurements are fully incompatible w.r.t. convex-mixing.

In Observation 3, we have found that this kind of three qubit measurements with noise become fully incompatible w.r.t. convex mixture for $\nu > \sqrt{2/3}$. So, we can conclude that these three measurements are fully incompatible w.r.t. convex-mixing if and only if quantum advantage is obtained in RAC. □

Let us consider another example of three rank-one projective measurements acting on $\mathbb{C}^3$:

$$\begin{aligned} X &= \{|0_x\rangle\langle 0_x|, |1_x\rangle\langle 1_x|, |2_x\rangle\langle 2_x|\}, \\ Y &= \{|0_y\rangle\langle 0_y|, |1_y\rangle\langle 1_y|, |2_y\rangle\langle 2_y|\}, \\ Z &= \{|0_z\rangle\langle 0_z|, |1_z\rangle\langle 1_z|, |2_z\rangle\langle 2_z|\}, \end{aligned} \quad (53)$$

where,

$$\begin{aligned} |0_x\rangle &= \frac{1}{2}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{2}|2\rangle, \\ |1_x\rangle &= -\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|2\rangle, \\ |2_x\rangle &= \frac{1}{2}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{2}|2\rangle, \\ |0_y\rangle &= -\frac{\mathbb{i}}{2}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle + \frac{\mathbb{i}}{2}|2\rangle, \\ |1_y\rangle &= \frac{\mathbb{i}}{\sqrt{2}}|0\rangle + \frac{\mathbb{i}}{\sqrt{2}}|2\rangle, \\ |2_y\rangle &= -\frac{\mathbb{i}}{2}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle + \frac{\mathbb{i}}{2}|2\rangle, \\ |0_z\rangle &\equiv |0\rangle, |1_z\rangle \equiv |1\rangle, |2_z\rangle \equiv |2\rangle. \end{aligned} \quad (54)$$

We take the following convex mixture of the measurements X and Y: $A = \{p|0_x\rangle\langle 0_x| + (1-p)|0_y\rangle\langle 0_y|, p|1_x\rangle\langle 1_x| + (1-p)|1_y\rangle\langle 1_y|, p|2_x\rangle\langle 2_x| + (1-p)|2_y\rangle\langle 2_y|\}$ with $0 \leq p \leq 1$. We now consider the $(2,3,3)$ RAC game involving the two measurements A and Z. The maximum average success probability $P(2,3,3)$ for this RAC task is given by,

$$P(2,3,3) = \frac{1}{18}\sum_{i,j=0}^{2} ||A_i + Z_j||, \quad (55)$$

which turns out to be greater than $\frac{2}{3}$ i.e. $P_{CB}(2,3,3)$ for all $p \in [0,1]$. Hence, the measurements A and Z are incompatible.

Next, consider the following measurement (taking an arbitrary convex mixture of X and Y) $A_{i,j,k,l,m,n} = \{p|i_x\rangle\langle i_x| + (1-p)|j_y\rangle\langle j_y|, p|k_x\rangle\langle k_x| + (1-p)|l_y\rangle\langle l_y|, p|m_x\rangle\langle m_x| + (1-p)|n_y\rangle\langle n_y|\}$ with $0 \leq p \leq 1$, $i, j, k, l, m, n \in \{0, 1, 2\}$, $i \neq k$, $k \neq m$, $m \neq i$ and $j \neq l$, $l \neq n$, $n \neq j$. Note that the aforementioned measurement A is nothing but $A_{0,0,1,1,2,2}$ in the present notation. Following a similar calculation, it can be shown that any such $A_{i,j,k,l,m,n}$ is incompatible with Z for all $p \in [0,1]$.

Similarly, taking an arbitrary convex combination of X and Z (or, Y and Z), it can be shown that the new measurement is incompatible with Y (or, X) for all $p \in [0,1]$.

A generic description for the semi-device independent witness of incompatibility.— Consider a measurement assemblage of n measurements having $\overline{d}$ outcomes, $\mathcal{M} = \{M_{z|x}\}_{z,x}$ where $x \in [n], z \in [\overline{d}]$, and the measurements act on $\mathbb{C}^d$. We make k partitions: $S = \{S_i\}_{i=0}^{k-1}$, $\bigcup_i S_i = \mathcal{M}$, and $S_i \bigcap S_j = \phi$, $\forall\ i,j$ with $i \neq j$. Our purpose is to operationally witness the incompatibility of k measurements produced from the convex-mixing of n measurements from each of the k partitions. We can construct a $(k,\overline{d},d)-$RAC where Alice has an k dit input message, *viz.*, $x = (x_0, x_1, \ldots, x_{k-1})$. Depending upon the message, she encodes it in a qudit and sends it to Bob, who, on the other hand, gets input $y \in [k]$ and accordingly, he has to predict the value of the corresponding bit x_y. He performs the measurement, which is obtained by the convex mixtures of the measurements from the partition S_y and declares the outcome of the measurement. Now, if the success probability $P(k,\overline{d},d)$ is greater than $P_{CB}(k,\overline{d},d)$ [21] for all possible convex mixtures and permutations of measurement outcomes, we can operationally detect $k-$incompatibility under convex-mixing.

V. CONCLUSIONS

Measurement incompatibility is the spectacle in quantum theory that a set of measurements cannot be performed jointly on arbitrary systems. It is one of the fundamental ingredients for non-classical correlations and merits of quantum information science. Incompatibility offers a complex structure with different layers as the number of measurements and the dimension on which the measurements act increases. Thus, understanding the different levels of incompatibility with respect to elementary classical operations is of paramount importance both from the foundational and practical perspectives. In this work, we have considered the two most general classical operations *viz.*, coarse-graining of different outcomes of measurements, and convex-mixing of different measurements.

Through our present analysis we have investigated the different levels of incompatibility arising under the above classical operations. Since environmental effects are ubiquitous in practical scenarios, here we have investigated the tolerance thresholds for maintaining measurement incompatibility against noise under classical operations. Furthermore, we have developed a method to operationally witness different levels of measurement incompatibility in the device-independent framework involving Bell-type experiments, and also in the semi-device-independent framework involving prepare-and-measure experiments.

Several examples have been provided to illustrate the efficacy of the operational witnesses proposed here. Our results are particularly useful for the purpose of comparing measurements in terms of their degree of incompatibility. For example, measurements that remain fully incompatible with respect to coarse-graining of outcomes (or convex-mixing of measurements) show stronger incompatibility compared to those that are not fully incompatible, thus facilitating legitimate choice of measurements in an information-processing task where a high degree of incompatibility is required.

Our present study motivates future work on several open directions. The condition for full incompatibility for two projective measurements with respect to coarse-graining of outcomes can be extended for more number of projective measurements. Similarly, the criterion for full incompatibility of projective measurements with respect to convex-mixing could be generalized for any set of measurements. Further investigation would be required to see if device-independent witnesses could be formulated for more than three outcome measurements with respect to coarse-graining and more than three measurements with repect to convex mixing. It will be interesting to look for examples where the full incompatibility with respect to coarse-graining can be inferred in a device-independent way from a single experimental statistics.

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