Extension of the Alberti-Ulhmann criterion beyond qubit dichotomies

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The Alberti-Ulhmann criterion states that any given qubit dichotomy can be transformed into any other given qubit dichotomy by a quantum channel if and only if the testing region of the former dichotomy includes the testing region of the latter dichotomy. Here, we generalize the Alberti-Ulhmann criterion to the case of arbitrary number of qubit or qutrit states. We also derive an analogous result for the case of qubit or qutrit measurements with arbitrary number of elements. We demonstrate the possibility of applying our criterion in a semi-device independent way.

1 Introduction

When quantum states are looked at as resources, it is natural to study which states can be transformed into which others by means of an allowed set of operations. This question has been declined in many ways: entanglement processing, thermal operations... In this paper, we consider generalizations of the following task: given a pair of quantum states (ρ_0, ρ_1), called a *dichotomy*, determine which other dichotomies (σ_0, σ_1) can be obtained from it by application of a completely positive trace preserving (CPTP) map. The simplicity of the problem is only apparent: very few results are known about this problem. Before reviewing them, and stating our contribution, let us take a detour to consider the analogous task in classical statistics.

A classical dichotomy is a pair of probability distributions (p_0, p_1). It appears naturally in the simplest formulation of hypothesis testing, in which there are two inputs (the *null* and the *alternative* hypotheses) and two outputs (*accept* or *reject*). In this case, any test is represented by a point in the dichotomy's *hypothesis testing region*, defined as the region $\{(p_0, p_1)\} \subset \mathbb{R}^2$ where p_0 is the probability of correctly accepting the null hypothesis and p_1 is the probability of wrongly accepting the alternative hypothesis with the given test [1]. Tests can be then designed, for instance, to maximize p_0 while keeping p_1 under a certain threshold. In particular, the wider the testing region, the more "testable", that is, the more "distinguishable" the pair of hypotheses is.

That the testing region is all that matters when dealing with pairs of hypotheses is made particularly clear by the celebrated Blackwell's theorem for dichotomies [2]: given two dichotomies (p_0, p_1) and (q_0, q_1), possibly on different sample spaces, there exists a stochastic transformation that transforms p_0 into q_0 and p_1 into q_1 simultaneously ("statistical sufficiency") if and only if the testing region for (p_0, p_1) contains the testing region for (q_0, q_1). In other words, the former dichotomy can be deterministically processed into (or, can deterministically simulate) the latter. In the special case in which $p_1 = q_1 = u$, the uniform distribution, the ordering induced by comparing the testing region coincides with the ubiquitous *majorization ordering*: indeed, the Lorenz curve corresponding to a probability distribution p is nothing but the boundary of the testing region corresponding to the dichotomy (p, u) [2, 4, 3, 1].

Such a compact characterization is not known in the quantum case that concerns us [8, 9, 5, 6, 7]: quantum statistical sufficiency is in general expressed in terms of an infinite number of

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conditions [10, 11, 12] that are, therefore, very difficult to check in practice [13]. Some results, based on [14], are known when the conversion is relaxed to be approximate [15, 16], but the problem remains hard in general. A notable exception is the case in which both quantum dichotomies, (ρ_0, ρ_1) and (σ_0, σ_1), only comprise two-dimensional (i.e., qubit) states. Then, as a consequence of a well-known result by Alberti and Uhlmann, there exists a CPTP map transforming (ρ_0, ρ_1) into (σ_0, σ_1) if and only if the testing region of the former contains the testing region of the latter [17]. This is the perfect analog of Blackwell's theorem; but counterexamples are known as soon as (ρ_0, ρ_1) is a qutrit dichotomy [5].

In this paper, building upon previous works of some of the present authors [18, 19, 20, 21, 22], we derive the following results. First, we show that any family of n qubit states which can all become simultaneously real under a single unitary transformation can be transformed into any other family of n qubit (or, under some conditions, qutrit) states by a CPTP map if the testing region of the former includes the testing region of the latter (the Alberti-Uhlmann case is recovered for $n = 2$, since any pair of qubit states can be made simultaneously real). Second, we show that an analogous result holds for qubit or qutrit measurements with n elements which can all become simultaneously real under a single unitary transformation. Our results follow as a natural consequence of the Woronowicz decomposition [23] of linear maps, once families of states and measurements are regarded as linear transformations. We demonstrate the possibility of witnessing statistical sufficiency in a semi-device independent way, that is, without any assumption on the devices except their Hilbert space dimension.

The paper is structured as follows. In Section 2 we present our main results. We first introduce our extensions of the Alberti-Ulhmann criterion, first for families of states (in Section 2.1) and then for measurements (in Section 2.2). We then discuss semi-device independent applications, first for families of states (in Section 2.3), and then for measurements (in Section 2.4). In Section 3, we provide technical proofs for our results. In particular, Sections 3.1 and 3.2 prove the results of Sections 2.1 and 2.2, respectively. We conclude by summarizing our results in Section 4.

2 Main results

We will make use of standard definitions in quantum information theory [24]. A quantum state is represented by a density matrix, that is, a positive semi-definite operator ρ such that $\mathrm{Tr}[\rho] = 1$. A quantum measurement is represented by a positive operator-valued measure, that is, a family $\{\pi_a\}$ of positive semi-definite operators that satisfy the completeness condition $\sum_a \pi_a = \mathbb{1}$, where $\mathbb{1}$ denotes the identity operator.

A channel is represented by a completely positive trace preserving map, that is, a map $\mathcal{C}$ such that for any state ρ one has $\mathrm{Tr}[\mathcal{C}(\rho)] = \mathrm{Tr}[\rho]$ and $(\mathcal{I} \otimes \mathcal{C})(\rho) \geq 0$. In the Heisenberg picture, a channel is represented by a completely positive unit preserving map, that is, a map $\mathcal{C}$ such that for any $\pi \geq 0$ one has $\mathcal{C}(\mathbb{1}) = \mathbb{1}$ and $(\mathcal{I} \otimes \mathcal{C})(\pi) \geq 0$.

2.1 Simulability of families of states

We say that a family $\{\rho_x\}$ of m states simulates another (possibly different dimensional) family $\{\sigma_x\}$ of m states, in formula

$$\{\sigma_x\} \preceq \{\rho_x\}, \tag{1}$$

if and only if there exists a channel $\mathcal{C}$ such that $\sigma_x = \mathcal{C}(\rho_x)$ for any x.

If condition (1) is verified, it immediately follows that

$$\mathcal{R}\left(\{\sigma_x\}\right) \subseteq \mathcal{R}\left(\{\rho_x\}\right), \tag{2}$$

where $\mathcal{R}(\{\rho_x\})$ denotes the testing region of family $\{\rho_x\}$, defined as the set of all vectors whose x-th entry is the probability $\mathrm{Tr}[\rho_x \pi]$ for any measurement element π, in formula

$$\mathcal{R}\left(\{\rho_x\}\right) := \left\{ \mathbf{q} \,\middle|\, \exists\, 0 \leq \pi \leq \mathbb{1} \text{ s.t. } \mathbf{q}_x = \mathrm{Tr}\left[\rho_x \pi\right] \ \forall x \right\}.$$

In other words, for any measurement element τ there exists a measurement element π such that $\mathrm{Tr}[\rho_x \pi] = \mathrm{Tr}[\sigma_x \tau]$ for any x.

Here, we derive conditions under which the reverse implication is also true, that is Eq. (2) implies Eq. (1):

Theorem 1. *For any family $\{\sigma_x\}$ of qubit states and any real family $\{\rho_x\}$ of qubit states (that is, states that have only real entries in some basis), the following are equivalent:*

- $\{\sigma_x\} \preceq \{\rho_x\}$.
- $\mathcal{R}(\{\sigma_x\}) \subseteq \mathcal{R}(\{\rho_x\})$.

If $\{\rho_x\}$ contains the identity operator $\mathbb{1}$ in its linear span, the statement holds even if $\{\sigma_x\}$ is a family of qutrit states.

The proof is given in Section 3.1.

Notice that the assumption that the family $\{\rho_x\}$ of states is real cannot be relaxed. As a counterexample, take $\{\sigma_x\}$ and $\{\rho_x\}$ to be symmetric informationally complete (or tetrahedral) families of states with $\rho_0 = \sigma_1$ and $\rho_1 = \sigma_0$, while $\rho_k = \sigma_k$ for $k = 2, 3$. A family $\{\rho_x\}$ of four pure qubit states is tetrahedral if and only if $\mathrm{Tr}[\rho_x \rho_z]$ is constant for any $x \neq z$. It immediately follows that there exists a transposition map $\mathcal{T}$ (with respect to some basis) such that $\sigma_x = \mathcal{T}(\rho_x)$ for any x. Due to the informational completeness of $\{\sigma_x\}$ and $\{\rho_x\}$, map $\mathcal{T}$ is the only map such that this is the case. However, map $\mathcal{T}$ is not a channel as it is not completely positive.

2.2 Simulability of measurements

We say that an n-outcome measurement $\{\pi_a\}$ simulates another (possibly, different dimensional) n-outcome measurement $\{\tau_a\}$, in formula

$$\{\tau_a\} \preceq \{\pi_a\}, \tag{3}$$

if and only if there exists a channel $\mathcal{C}$ such that $\tau_a = \mathcal{C}^\dagger(\pi_a)$ for any a, where $\mathcal{C}^\dagger$ denotes channel $\mathcal{C}$ in the Heisenberg picture.

If condition (3) is verified, it follows immediately that

$$\mathcal{R}\left(\{\tau_a\}\right) \subseteq \mathcal{R}\left(\{\pi_a\}\right), \tag{4}$$

where the range $\mathcal{R}(\{\pi_a\})$ of measurement $\{\pi_a\}$ is defined as the set of all probability distributions $\mathrm{Tr}[\rho\pi_a]$ on the outcomes a for any state ρ, in formula

$$\mathcal{R}\left(\{\pi_a\}\right) := \left\{ \mathbf{p} \;\middle|\; \exists\, \rho \geq 0, \mathrm{Tr}\,\rho = 1, \text{ s.t. } \mathbf{p}_a = \mathrm{Tr}\left[\rho\pi_a\right] \;\forall a \right\}.$$

In other words, for any state σ there exists a state ρ such that $\mathrm{Tr}[\rho\pi_a] = \mathrm{Tr}[\sigma\tau_a]$ for any a.

Similarly to what we did before, we derive conditions under which the reverse implication is also true, that is Eq. (4) implies Eq. (3):

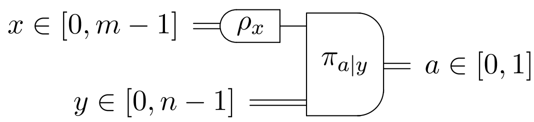
Theorem 2. *For any qubit or qutrit measurement $\{\tau_a\}$ and any real qubit measurement $\{\pi_a\}$ (that is, one whose elements are all real in some basis), the following are equivalent:*

- $\{\tau_a\} \preceq \{\pi_a\}$.
- $\mathcal{R}(\{\tau_a\}) \subseteq \mathcal{R}(\{\pi_a\})$.

The proof is given in Section 3.2. As before, the assumption that measurement $\{\pi_a\}$ is real cannot be relaxed.

2.3 Semi-device independent simulability of families of states

Suppose that a black box preparator with m buttons is given, and let us denote with ρ_x the unknown state prepared upon the pressure of button x. Consider the setup where a black box tester with n buttons is connected to the black box preparator, and let us denote with $\{\pi_{0|y}, \pi_{1|y} := \mathbb{1} - \pi_{0|y}\}$ the test performed upon the pressure of button y. One has



(5)

For each y, by running the experiment asymptotically many times one collects the vectors $\mathbf{q}_y$ and $\mathbf{u} - \mathbf{q}_y$ ($\mathbf{u}$ denotes the vector with unit entries) whose x-th entry are the probabilities $\mathrm{Tr}[\rho_x\pi_{0|y}]$ and $\mathrm{Tr}[\rho_x\pi_{1|y}]$, respectively, that is

$$\left[\mathbf{q}_y\right]_x := \mathrm{Tr}\left[\rho_x\pi_{0|y}\right].$$

We call semi-device independent simulability the problem of characterizing the class of all families of states that can be simulated by the black box $\{\rho_x\}$, for which simulability can be certified based on distributions $\{\mathbf{q}_y\}$ and $\{\mathbf{u} - \mathbf{q}_y\}$ without any characterisation of the tests $\{\pi_{a|y}\}$, under an assumption on the Hilbert space dimension.

Here, we will address the semi-device independent simulability problem under the promise that $\{\rho_x\}$ is a family of qubit states. In this case, the testing region [19, 20] is the convex hull of the isolated points 0 and $\mathbf{u}$ with a (possibly degenerate) ellipsoid centered in $\mathbf{u}/2$. Conversely, for any (possibly degenerate) ellipsoid centered in $\mathbf{u}/2$ and contained in the hypercube $[0, 1]^m$, its convex hull with 0 and $\mathbf{u}$ is the testing region of a qubit family of states. In general, such a testing region identifies the family of states up to unitaries and anti-unitaries.

We will further make the restriction that the black box $\{\rho_x\}$ has $m = 2$ buttons, that is, $\{\rho_x\}$ is a dichotomy. Notice that any qubit dichotomy is necessarily real. Hence, in the discussion above the (possibly degenerate) ellipsoid becomes a (possibly degenerate) ellipse. Additionally, since two anti-unitarily related qubit dichotomies are also unitarily-related, a qubit dichotomy is identified by its range up to unitaries only.

Due to Theorem 1, we have the following result:

Corollary 1. *If the convex hull of points* 0 *and* $\mathbf{u}$ *with any given ellipse centered in* $\mathbf{u}/2$ *is a subset of* $\operatorname{conv}(0, \mathbf{u}, \{\mathbf{q}_y\}, \{\mathbf{u} - \mathbf{q}_y\})$ *it is the testing region of some qubit dichotomy that can be simulated by* $\{\rho_x\}$.

Notice that, on the one hand, the hypothesis of Corollary 1 represents only a sufficient condition for a qubit dichotomy to be simulable by $\{\rho_x\}$. On the other hand, for any other qubit dichotomy that can be simulated by $\{\rho_x\}$ (if any), simulability cannot be certified in a semi-device independent way unless further data is collected.

As an application, consider the case when one of the states prepared by the dichotomy (say ρ_1) is the thermal state at infinite temperature, that is $\rho_1 = \mathbb{1}/2$ (for this example we are assuming more than just the Hilbert space dimension, although no knowledge of ρ_0 is assumed). Consider the problem of finding the dichotomy with maximal free energy among those that can be simulated by $\{\rho_0, \mathbb{1}/2\}$ through a Gibbs-preserving channel (in this case, a unit-preserving channel).

In this case, it immediately follows that the free energy is monotone in the area of the range. This can be seen as follows. First, notice that the free energy in this case is equal to the neg-entropy $-S(\rho_0)$, since the free energy is equal to the relative entropy $S(\rho_0||\mathbb{1}/2)$ and by definition one has $S(\rho_0||\mathbb{1}/2) = -S(\rho_0)$. In turn, $S(\rho_0) = h(\lambda_\pm)$, where $h(\cdot)$ denotes the binary entropy and $\lambda_\pm$ the eigenvalues of ρ_0. By setting $\lambda_\pm = 1/2 \pm a$, by explicit computation it immediately follows that the volume of the range of $\{\rho_0, \mathbb{1}/2\}$ is proportional to a, hence the statement remains proved.

Suppose one test is performed on the black box dichotomy and the following probability vectors are observed:

$$\mathbf{q}_0 = \frac{1}{2}\begin{pmatrix} 1-\epsilon \\ 1 \end{pmatrix}, \qquad \mathbf{u} - \mathbf{q}_0 = \frac{1}{2}\begin{pmatrix} 1+\epsilon \\ 1 \end{pmatrix}. \tag{6}$$

for some value of parameter $0 \leq \epsilon \leq 1$. The situation is illustrated in Fig. 1. We assume that

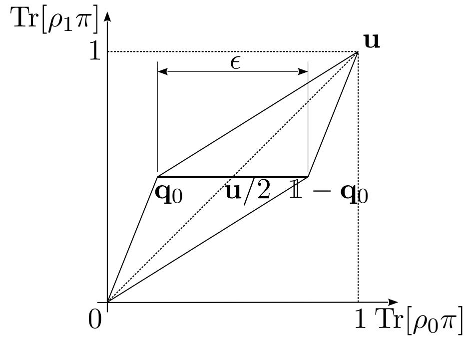


Figure 1: Probability vectors $\mathbf{q}_0$ and $\mathbf{u} - \mathbf{q}_0$ as given by Eq. (6) lie at the vertices of a a line segment of length ϵ and centered in $\mathbf{u}/2$. The maximally committal testing region for qubit dichotomy $\{\sigma_0, \mathbb{1}/2\}$ enclosed in $\operatorname{conv}(0, \mathbf{u}, \mathbf{q}_0, \mathbf{u} - \mathbf{q}_0)$ is given by $\operatorname{conv}(0, \mathbf{u}, \mathbf{q}_0, \mathbf{u} - \mathbf{q}_0)$ itself.

the black box implements a qubit dichotomy, a justified assumption since the probability vector in Eq. (6) belongs, for example, to the range of any qubit dichotomy $\{\phi, \mathbb{1}/2\}$, for any pure state ϕ. It is easy to derive the maximum volume range enclosed in $\operatorname{conv}(0, \mathbf{u}, \mathbf{q}_0, \mathbf{u} - \mathbf{q}_0)$, and to verify using Ref. [18] that it correspond to the range of any ϵ-depolarized dichotomy $\{\mathcal{D}_\epsilon(\phi), \mathbb{1}/2\}$, for any pure state ϕ.

2.4 Semi-device independent simulability of measurements

Suppose that a black box measurement with n outcomes is given, and let us denote with π_a the unknown measurement element corresponding to outcome a. Consider the setup where a black box preparator with m buttons is connected to the black box measurement, and let us denote with ρ_x the unknown state prepared upon the pressure of button x. One has

$$x \in [0, m-1] \Rightarrow \boxed{\rho_x} - \boxed{\pi_a} \Rightarrow a \in [0, n-1] \tag{7}$$

For each x, by running the experiment asymptotically many times one collects the probability distribution $\mathbf{p}_x$ of outcome a, that is

$$[\mathbf{p}_x]_a := \operatorname{Tr}\left[\rho_x \pi_a\right].$$

We call semi-device independent simulability the problem of characterizing the class of all measurements that can be simulated by the black box $\{\pi_a\}$, for which simulability can be certified based on distributions $\{\mathbf{p}_x\}$ without any characterisation of the states $\{\rho_x\}$, under an assumption on the Hilbert space dimension.

Here, we will address the semi-device independent simulability problem under the promise that $\{\pi_y\}$ is a qubit measurement. In this case, the range [18, 20] is a (possibly degenerate) ellipsoid. Conversely, any (possibly degenerate) ellipsoid subset of the probability simplex is the range of a qubit measurement. In general, such a range identifies the measurement up to unitaries and anti-unitaries.

We will further make the restriction that the black box $\{\pi_a\}$ has $n = 3$ outcomes. Notice that any three-outcome qubit measurement is necessarily real due to the completeness condition. Hence, in the discussion above the (possibly degenerate) ellipsoid becomes a (possibly degenerate) ellipse. Additionally, since three-outcome anti-unitarily related qubit measurements are also unitarily related, a three-outcome measurement is identified by its range up to unitaries only.

Due to Theorem 2, we have the following result:

Corollary 2. *Any ellipse subset of* $\mathrm{conv}(\{\mathbf{p}_x\})$ *is the range of some qubit three-outcome measurement that can be simulated by* $\{\pi_a\}$.

Notice that, on the one hand, the hypothesis of Corollary 2 represents only a sufficient condition for a qubit three-outcome measurement to be simulable by $\{\pi_a\}$. On the other hand, for any other qubit three-outcome measurement that can be simulated by $\{\pi_a\}$ (if any), simulability cannot be certified in a semi-device independent way unless further data is collected.

As an application, consider the problem of finding, among the measurements that can be simulated by the black box $\{\pi_a\}$, the one with maximal simulation power, quantified according to Theorem 2 by the volume of its range. Suppose m states are fed into the black-box measurement and the following distributions are observed:

$$\mathbf{p}_x = \begin{pmatrix} 2 - 2\cos\theta_x \\ 2 + cos\theta_x - \sqrt{3}\sin\theta_x \\ 2 + cos\theta_x + \sqrt{3}\sin\theta_x \end{pmatrix}, \tag{8}$$

where $\theta_x := 2\pi x/m$ and $x \in [0, m-1]$. This situation is depicted in Fig. 2. We assume

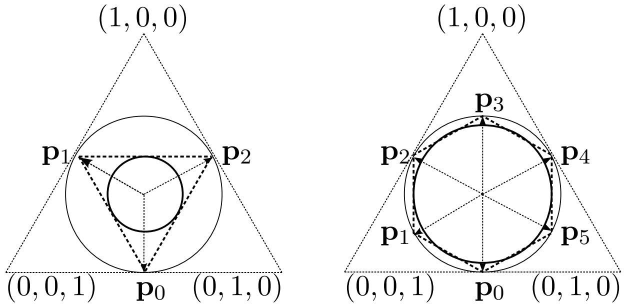


Figure 2: In both left and right figures, the outer dashed triangle represents the simplex of three-outcome probability distributions. The distributions $\{\mathbf{p}_x\}$ given by Eq. (8) lie on the vertices of regular polygons ($m = 3$ and $m = 6$ in left and right figures, respectively). The maximum volume ellipsoid enclosed in $\mathrm{conv}(\{\mathbf{p}_x\})$ is the inner circle, which is the range of a ϵ-depolarized trine qubit measurement ($\epsilon = 1/2$ and $\epsilon = \sqrt{3}/2$ in left and right figures, respectively).

that the black box implements a qubit measurement, a justified assumption since such distributions belong to the range of, for example, a trine qubit measurement, that is, a measurement whose elements lie on the vertices of a regular triangle in the Bloch sphere representation. It is easy to derive the maximum volume ellipse [26, 27, 28, 29] enclosed in $\mathrm{conv}(\{\mathbf{p}_x\})$, and to verify using Ref. [18] that it corresponds to the range of any $[\cos(\pi/m)]$-depolarized trine measurement.

3 Proofs of Theorems 1 and 2

The formalism of quantum information theory, used to present our results in Section 2, is not the most efficient to prove such statements. Here, we introduce a more efficient formalism, that provides the additional benefit of holding for any bilinear physical theory, not just quantum theory.

Each system is associated with a dimension ℓ, and states and measurement elements are represented by vectors in $\mathbb{R}^\ell$. Let us denote with $\mathbb{S}_\ell \subseteq \mathbb{R}^\ell$ and $\mathbb{E}_\ell \subseteq \mathbb{R}^\ell$ the set of all states and the set of all measurement elements, respectively. The probability that measurement element $\mathbf{e} \in \mathbb{E}_\ell$ clicks upon the input of state $\mathbf{s} \in \mathbb{S}_\ell$ is given by the inner product $\mathbf{s}\cdot\mathbf{e}$. Let $\mathbf{u}_n \in \mathbb{R}^n$ denote the vector with unit entries and let us choose a basis in which $\mathbf{u}_\ell \in \mathbb{E}_\ell$ is the measurement element that has unit probability of click over any state. For example, for quantum systems ℓ is the squared Hilbert-space dimension, states and measurement elements can be represented by (generalized) Pauli vectors, their inner product reduces to the Born rule, and $\mathbf{u}_\ell$ corresponds to the identity operator.

A family of n states $\{\mathbf{s}^k\}$ can be conveniently represented by arranging the states as the rows of an $n \times \ell$ matrix S. This way, the corresponding linear map $S : \mathbb{R}^\ell \to \mathbb{R}^n$ maps any effect $\mathbf{e}$ into the vector whose k-th entry is the probability $\mathbf{s}^k \cdot \mathbf{e}$. It immediately follows that, for any family S of states, one has $S\mathbf{u}_\ell = \mathbf{u}_n$. Analogously, an n outcome measurement can be conveniently represented by arranging its elements $\{\mathbf{e}^k\}$ as the rows of an $n\times\ell$ matrix M. This way, the corresponding linear map $M\mathbb{R}^\ell \to \mathbb{R}^n$ maps any state $\mathbf{s}$ into the probability distribution whose k-th entry is the probability $\mathbf{e}^k \cdot \mathbf{s}$. It immediately follows that, for any measurement M, one has $M^T\mathbf{u}_n = \mathbf{u}_\ell$.

Finally, we discuss maps from states to states and from effects to effects.

Definition 1 (State morphism)**.** A linear map $C : \mathbb{R}^{\ell_0} \to \mathbb{R}^{\ell_1}$ is a state morphism if and only if $C\mathbb{S}_0 \subseteq \mathbb{S}_1$.

Definition 2 (Statistical morphism)**.** A linear map $C : \mathbb{R}^{\ell_0} \to \mathbb{R}^{\ell_1}$ is a statistical morphism if and only if $C\mathbb{E}_0 \subseteq \mathbb{E}_1$ and $C\mathbf{u}_{\ell_0} = \mathbf{u}_{\ell_1}$.

In standard quantum theory, a state morphism is a positive (not necessarily completely positive) trace-preserving (PTP) map, that is, a transformation of states in the Schrödinger picture. Analogously, a statistical morphism is a positive (not necessarily completely positive) unit-preserving (PUP) map, that is, a transformation of measurement elements in the Heisenberg picture. For our proofs, we do not need an analogous of complete positivity for arbitrary bilinear theories.

3.1 Simulability of families of states

Here we prove Theorem 1, that we report here for the reader's convenience.

Theorem 1. *For any family $\{\sigma_x\}$ of qubit states and any real family $\{\rho_x\}$ of qubit states, the following are equivalent:*

- $\{\sigma_x\} \preceq \{\rho_x\}$.
- $\mathcal{R}(\{\sigma_x\}) \subseteq \mathcal{R}(\{\rho_x\})$.

If $\{\rho_x\}$ contains the identity operator $\mathbb{1}$ in its linear span, the statement holds even if $\{\sigma_x\}$ is a family of qutrit states.

By adopting the formalism of bilinear theories, we denote with S_0 and S_1 the families of states corresponding to $\{\sigma_x\}$ and $\{\rho_x\}$, respectively. To prove the theorem, we need to distinguish two cases. First, let us consider the case when the linear span of $\{\rho_x\}$ contains the identity operator $\mathbb{1}$, that is, $S_1^+ S_1 \mathbf{u}_{\ell_1} = \mathbf{u}_{\ell_1}$, where $(\cdot)^+$ denotes the Moore-Penrose pseudoinverse. In the following Lemma we show that, under the hypothesis $S_1^+ S_1 \mathbb{E}_1 \subseteq \mathbb{E}_1$, range inclusion between two families of states is equivalent to the existence of a statistical morphism between them.

Lemma 1. *For any families of states $S_0 : \mathbb{R}^{\ell_0} \to \mathbb{R}^n$ and $S_1 : \mathbb{R}^{\ell_1} \to \mathbb{R}^n$ such that $S_1^+ S_1 \mathbb{E}_1 \subseteq \mathbb{E}_1$ and $S_1^+ S_1 \mathbf{u}_{\ell_1} = \mathbf{u}_{\ell_1}$, the following are equivalent:*

1. *$S_0\mathbb{E}_0 \subseteq S_1\mathbb{E}_1$,*
2. *there exists a statistical morphism $C : \mathbb{R}^{\ell_0} \to \mathbb{R}^{\ell_1}$ such that $S_0 = S_1 C$.*

Proof. Implication 1 $\Leftarrow$ 2 is trivial.

Implication 1 $\Rightarrow$ 2 can be shown as follows. Let

$$C := S_1^+ S_0. \tag{9}$$

Let us first show that map C is a statistical morphism. One has

$$C\mathbb{E}_0 = S_1^+ S_0 \mathbb{E}_0 \subseteq S_1^+ S_1 \mathbb{E}_1 \subseteq \mathbb{E}_1,$$

where the equality follows from Eq. (9) and the inclusions follow from the hypothesis $S_0\mathbb{E}_0 \subseteq S_1\mathbb{E}_1$ and $S_1^+ S_1 \mathbb{E}_1 \subseteq \mathbb{E}_1$, respectively. Moreover,

$$C\mathbf{u}_{\ell_0} = S_1^+ S_0 \mathbf{u}_{\ell_0} = S_1^+ \mathbf{u}_n,$$

where the equalities follow from Eq. (9) and from the hypothesis $S_0\mathbf{u}_{\ell_0} = \mathbf{u}_n$, respectively. Since by hypothesis $S_1\mathbf{u}_{\ell_1} = \mathbf{u}_n$, one also has $S_1^+ S_1 \mathbf{u}_{\ell_1} = S_1^+ \mathbf{u}_n$, and hence

$$C\mathbf{u}_{\ell_0} = S_1^+ S_1 \mathbf{u}_{\ell_1} = \mathbf{u}_{\ell_1},$$

where the second inequality follows by hypothesis. Hence map C is a statistical morphism.

Let us now show that $S_0 = S_1 C$. By multiplying Eq. (9) from the left by S_1 one has

$$S_1 C = S_1 S_1^+ S_0.$$

Since $\operatorname{span}\mathbb{E}_0 = \mathbb{R}^{\ell_0}$ and $\operatorname{span}\mathbb{E}_1 = \mathbb{R}^{\ell_1}$, from $S_0\mathbb{E}_0 \subseteq S_1\mathbb{E}_1$ one has $\operatorname{rng} S_0 \subseteq \operatorname{rng} S_1$. Since $S_1 S_1^+$ is the projector on $\operatorname{rng} S_1$, one has $S_0 = S_1 C$. □

It is easy to see that the hypothesis $S_1^+ S_1 \mathbb{E}_1 \subseteq \mathbb{E}_1$ in Lemma 1 is satisfied for any family S_1 of states if $\mathbb{E}_1$ is a ℓ_1–dimensional (hyper)–cone with $(\ell_1 - 1)$–dimensional (hyper)–spherical section with axis along $\mathbf{u}_{\ell_1}$, as is the case for the qubit system, where $\ell_1 = 4$. In this case, by additionally assuming that the family $\{\rho_x\}$ of states is real, that is $S_1 T = S_1$ where T denotes the transposition map with respect to some basis, the following Lemma completes the first part of the proof of Theorem 1.

Lemma 2. *For any qubit or qutrit family $S_0 : \mathbb{R}^{\ell_0} \to \mathbb{R}^n$ of states, with $\ell_0 = 4$ or $\ell_0 = 9$, and any qubit family $S_1 : \mathbb{R}^4 \to \mathbb{R}^n$ of states such that $S_1^+ S_1 \mathbf{u}_{\ell_1} = \mathbf{u}_{\ell_1}$ and $S_1 T = S_1$, the following are equivalent:*

1. *$S_0\mathbb{E}_0 \subseteq S_1\mathbb{E}_1$,*
2. *there exists CPTP map $C : \mathbb{R}^{\ell_0} \to \mathbb{R}^4$ such that $S_0 = S_1 C$.*

Proof. Implication 1 ⇐ 2 is trivial.

Implication 1 ⇒ 2 can be shown as follows. Due to Lemma 1, there exists a statistical morphism $C' : \mathbb{R}^{\ell_0} \to \mathbb{R}^4$, hence a PUP map, such that $S_0 = S_1 C'$. Let us prove that there exists a CPUP map $C : \mathbb{R}^{\ell_0} \to \mathbb{R}^4$ such that $S_0 = S_1 C$.

For any qubit PUP map $C' : \mathbb{R}^{\ell_0} \to \mathbb{R}^4$, Woronowicz [23] proved that there exist $0 \le p \le 1$ and CPUP maps $C_0 : \mathbb{R}^{\ell_0} \to \mathbb{R}^4$ and $C_1 : \mathbb{R}^{\ell_0} \to \mathbb{R}^4$ such that

$$C' = pC_0 + (1-p)\,TC_1. \tag{10}$$

One has

$$\begin{aligned} S_1 C' &= S_1\left[pC_0 + (1-p)\,TC_1\right] \\ &= S_1\left[pC_0 + (1-p)\,C_1\right], \end{aligned}$$

where the equalities follow from Eq. (10) and from the hypothesis $S_1 T = S_1$, respectively. Since the convex combination of CPUP maps is CPUP, map $C := pC_0 + (1-p)\,C_1$ is CPUP. □

Second, let us consider the case when the linear span of $\{\rho_x\}$ does not contain the identity operator $\mathbb{1}$. Since by linearity it immediately follows that any linear dependency in the states $\{\rho_x\}$ must be present also in the states $\{\sigma_x\}$, as formally show in Lemma 3, the remaining part of the proof directly follows from the Alberti-Ulhmann criterion, as formally shown in Lemma 4.

Lemma 3. *For any families $S_0 : \mathbb{R}^{\ell_0} \to \mathbb{R}^n$ and $S_1 : \mathbb{R}^{\ell_1} \to \mathbb{R}^n$ of states such that $S_0\mathbb{E}_0 \subseteq S_1\mathbb{E}_1$, if for some k there exists $\{\lambda_i \in \mathbb{R}\}$ such that*

$$s_1^k = \sum_{i \neq k} \lambda_i s_1^i,$$

then one has

$$s_0^k = \sum_{i \neq k} \lambda_i s_0^i.$$

Proof. By hypothesis, for any $e_0 \in \mathbb{E}_0$ there exists $e_1 \in \mathbb{E}_1$ such that

$$s_0^k \cdot e_0 = s_1^k \cdot e_1.$$

Hence, for any set $\{e_0^j\} \subseteq \mathbb{E}_0$ one has

$$s_0^k \cdot e_0^j = s_1^k \cdot e_1^j = \sum_{i \neq k} \lambda_i s_1^i \cdot e_1^j = \sum_{i \neq k} \lambda_i s_0^i \cdot e_0^j.$$

Since $\operatorname{span}\mathbb{E}_0 = \mathbb{R}^{\ell_0}$, it is possible to take set $\{e_0^j \in \mathbb{E}_0\}$ a spanning set. Hence the thesis. □

Lemma 4. *For any qubit families $S_0 : \mathbb{R}^4 \to \mathbb{R}^n$ and $S_1 : \mathbb{R}^4 \to \mathbb{R}^n$ of states such that $S_1^+ S_1 \mathbf{u}_{\ell_1} \neq \mathbf{u}_{\ell_1}$ and $S_1 T = S_1$ where T denotes the transposition map with respect to some basis, the following are equivalent:*

1. *$S_0\mathbb{E}_0 \subseteq S_1\mathbb{E}_1$,*
2. *there exists CPTP map $C : \mathbb{R}^4 \to \mathbb{R}^4$ such that $S_0 = S_1 C$.*

Proof. Implication 1 ⇐ 2 is trivial.

Implication 1 ⇒ 2 can be shown as follows.

By the hypothesis $S_0\mathbb{E}_0 \subseteq S_1\mathbb{E}_1$, one has that for any $e_0 \in \mathbb{E}_0$ there exists a $e_1 \in \mathbb{E}_1$ such that $s_0^k \cdot e_0 = s_1^k e_1$ for any k. Since in particular this holds for $k = 0, 1$, by denoting with S_0' and S_1' the families of states whose rows are (s_0^0, s_0^1) and (s_1^0, s_1^1), respectively, one has

$$S_0'\mathbb{E}_0 \subseteq S_1'\mathbb{E}_1.$$

Hence, due to a result [7] by Buscemi and Gour, in turn based on a result [17] by Alberti and Ulhmann, there exists a CPTP map $C : \mathbb{R}^4 \to \mathbb{R}^4$ such that $S_0' = S_1' C$.

Due to the hypotheses $S_1^+ S_1 \mathbf{u}_{\ell_1} \neq \mathbf{u}_{\ell_1}$ and $S_1 T = S_1$, for any k there exists $\lambda^k \in \mathbb{R}$ such that

$$s_1^k = \lambda^k s_1^0 + \left(1 - \lambda^k\right) s_1^1,$$

that is, state s_1^k is a convex combination of states s_1^0 and s_1^1. Hence, due to Lemma 3, one also has

$$s_0^k = \lambda^k s_0^0 + \left(1 - \lambda^k\right) s_0^1,$$

that is, state s_0^k is a convex combination of states s_0^0 and s_0^1. Hence, by linearity, $S_0 = S_1 C$. □

This concludes the proof of Theorem 1.

3.2 Simulability of measurements

Here we prove Theorem 2, that we report here for the reader's convenience.

Theorem 2. *For any qubit or qutrit measurement $\{\tau_a\}$ and any real qubit measurement $\{\pi_a\}$, the following are equivalent:*

- $\{\tau_a\} \preceq \{\pi_a\}$.
- $\mathcal{R}(\{\tau_a\}) \subseteq \mathcal{R}(\{\pi_a\})$.

By adopting the formalism of bilinear theories, we denote with M_0 and M_1 the measurements corresponding to $\{\tau_a\}$ and $\{\pi_a\}$, respectively. In contrast to Theorem 1, for whose proof we needed to distinguish two cases based on whether the linear span of $\{\rho_x\}$ contained the identity operator $\mathbb{1}$ or not, due to completeness for any measurement $\{\pi_a\}$ one has $\sum_a \pi_a = \mathbb{1}$, that is, $M_1^+ M_1 \mathbf{u}_{\ell_1} = \mathbf{u}_{\ell_1}$, where $(\cdot)^+$ denotes the Moore-Penrose pseudoinverse. Hence, the proof proceeds along the lines of Lemmas 1 and 2, that are in this case replaced by Lemmas 5 and 6, respectively. In the following Lemma we show that, under the hypothesis $M_1^+ M_1 \mathbb{S}_1 \subseteq \mathbb{S}_1$, range inclusion between two measurements is equivalent to the existence of a state morphism between them.

Lemma 5. *For any measurements $M_0 : \mathbb{R}^{\ell_0} \to \mathbb{R}^n$ and $M_1 : \mathbb{R}^{\ell_1} \to \mathbb{R}^n$ such that $M_1^+ M_1 \mathbb{S}_1 \subseteq \mathbb{S}_1$, the following are equivalent:*

1. $M_0 \mathbb{S}_0 \subseteq M_1 \mathbb{S}_1$,
2. *there exists a state morphism $C : \mathbb{R}^{\ell_0} \to \mathbb{R}^{\ell_1}$ such that $M_0 = M_1 C$.*

Proof. Implication 1 ⇐ 2 is trivial.

Implication 1 ⇒ 2 can be shown as follows. Let

$$C := M_1^+ M_0. \tag{11}$$

Let us first show that map C is a state morphism. One has

$$C\mathbb{S}_0 = M_1^+ M_0 \mathbb{S}_0 \subseteq M_1^+ M_1 \mathbb{S}_1 \subseteq \mathbb{S}_1,$$

where the equality follows from Eq. (11) and the inclusions follow from the hypothesis $M_0 \mathbb{S}_0 \subseteq M_1 \mathbb{S}_1$ and $M_1^+ M_1 \mathbb{S}_1 \subseteq \mathbb{S}_1$, respectively. Hence map C is a state morphism.

Let us now show that $M_0 = M_1 C$. By multiplying Eq. (11) from the left by M_1 one has

$$M_1 C = M_1 M_1^+ M_0.$$

Since $\operatorname{span} \mathbb{S}_0 = \mathbb{R}^{\ell_0}$ and $\operatorname{span} \mathbb{S}_1 = \mathbb{R}^{\ell_1}$, from $M_0 \mathbb{S}_0 \subseteq M_1 \mathbb{S}_1$ one has $\operatorname{rng} M_0 \subseteq \operatorname{rng} M_1$. Since $M_1 M_1^+$ is the projector on $\operatorname{rng} M_1$, one has $M_0 = M_1 C$. □

It is easy to see that the hypothesis $M_1^+ M_1 \mathbb{S}_1 \subseteq \mathbb{S}_1$ in Lemma 5 is satisfied for any measurement M_1 if and only if $\mathbb{S}_1$ is a $(\ell - 1)$–dimensional (hyper)–sphere with center along $\mathbf{u}_{\ell_1}$, as is the case for the qubit system, where $\ell_1 = 4$.

To see this, notice that the (hyper)–sphere is the only body for which there exists a point (the center) such that any line through the point is orthogonal to the surface of the body. Hence, the (hyper)–sphere is also the only body for which the projection of the body on any subspace containing such a point is a subset of the body. Finally, notice that by multiplying condition $M_1^T \mathbf{u}_n = \mathbf{u}_{\ell_1}$ on the left by $M_1^+ M_1$ and using the elementary property of pseudoinverse that $M_1^+ M_1 M_1^T = M_1^T$, one immediately has

$$M_1^+ M_1 \mathbf{u}_{\ell_1} = \mathbf{u}_{\ell_1},$$

that is, $M_1^+ M_1$ is the projector on a subspace that contains the center of the (hyper)–sphere $\mathbb{S}_1$.

In this case, by additionally assuming that the measurement $\{\pi_a\}$ is real, that is $M_1 T = M_1$ where T denotes the transposition map with respect to some basis, the following Lemma completes the proof of Theorem 2.

Lemma 6. *For any qubit or qutrit measurement $M_0 : \mathbb{R}^{\ell_0} \to \mathbb{R}^n$, with $\ell_0 = 4$ or $\ell_0 = 9$, and* *any qubit measurement $M_1 : \mathbb{R}^4 \to \mathbb{R}^n$ such that $M_1 T = M_1$ where T denotes the transposition map with respect to some basis, the following are equivalent:*

1. $M_0 \mathbb{S}_0 \subseteq M_1 \mathbb{S}_1$,
2. *there exists CPTP map $C : \mathbb{R}^{\ell_0} \to \mathbb{R}^4$ such that $M_0 = M_1 C$.*

Proof. Implication 1 $\Leftarrow$ 2 is trivial.

Implication 1 $\Rightarrow$ 2 can be shown as follows. Due to Lemma 5, there exists a state morphism $C' : \mathbb{R}^{\ell_0} \to \mathbb{R}^4$, hence a PTP map, such that $M_0 = M_1 C'$. Let us prove that there exists a CPTP map $C : \mathbb{R}^{\ell_0} \to \mathbb{R}^4$ such that $M_0 = M_1 C$.

For any PTP map $C' : \mathbb{R}^{\ell_0} \to \mathbb{R}^4$, there exists [23] $0 \leq p \leq 1$ and CPTP maps $C_0 : \mathbb{R}^{\ell_0} \to \mathbb{R}^4$ and $C_1 : \mathbb{R}^{\ell_0} \to \mathbb{R}^4$ such that

$$C' = pC_0 + (1-p)\, TC_1. \tag{12}$$

One has

$$\begin{aligned} M_1 C' =& M_1 \left[pC_0 + (1-p)\, TC_1 \right] \\ =& M_1 \left[pC_0 + (1-p)\, C_1 \right], \end{aligned}$$

where the equalities follow from Eq. (12) and from the hypothesis $M_1 T = M_1$, respectively. Since the convex combination of CPTP maps is CPTP, map $C := pC_0 + (1-p)\, C_1$ is CPTP. □

This concludes the proof of Theorem 2.

4 Conclusion

In this work we addressed the problem of quantum simulability, that is, the existence of a quantum channel transforming a given device into another. We considered the cases of families of n qubit or qutrit states and of qubit or qutrit measurements with n elements, thus extending the Alberti-Ulhmann criterion for qubit dichotomies. Based on these results, we demonstrated the possibility of certifying the simulability in a semi-device independent way, that is, without any assumption of the devices except their Hilbert space dimension.

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