Complex Interpolation Spaces, a Discrete Definition and Reiteration

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We shall use definitions and notation in accordance with those of A. P. Calderón's paper [2] on complex interpolation spaces, (see also [1] Chapter 4). In the first part of this note we consider the reiteration formula (paragraph 12.3 [2], 4.6 [1])

$$[[A_0, A_1]_{\theta_0}, [A_0, A_1]_{\theta_1}]_\sigma = [A_0, A_1]_s \tag{1}$$

with equality of norms, where θ_0, θ_1 and σ are in $[0, 1]$ and $s = (1 - \sigma)\theta_0 + \sigma\theta_1$. Calderón proved this formula ([2], 32.3) for the case when $A_0 \cap A_1$ is a dense subset of the space $[A_0, A_1]_{\theta_0} \cap [A, A_1]_{\theta_1}$. We show here that this density requirement is unnecessary and the result is true for any interpolation pair (A_0, A_1) of Banach spaces.

In the second part of the note we give an equivalent "periodic" or "discrete" definition of the spaces $[A_0, A_1]_\theta$.

I. The reiteration formula. To abbreviate the notation let $B_0 = [A_0, A_1]_{\theta_0}$ and $B_1 = [A_0, A_1]_{\theta_1}$. It suffices of course to consider the case $\theta_0 \neq \theta_1$. Let A_j^* denote the dual of $\bar{A}_j$, the closed subspace of A_j generated by $A_0 \cap A_1, j = 0$, 1. From [2] 9.3, we have $[A_0, A_1]_j = \bar{A}_j$. The inclusion $[A_0, A_1]_s \subset [B_0, B_1]_\sigma$ with

$$\|a\|_{[B_0,B_1]_\sigma} \leq \|a\|_{[A_0,A_1]_s}$$

for all $a \in [A_0, A_1]_s$ was established in [2] 32.3. It remains to prove the reverse inclusion and the reverse norm inequality.

We first observe that

$$B_0 \cap B_1 \subset [A_0, A_1]_s. \tag{2}$$

If $s = (1 - \sigma)\theta_0 + \sigma\theta_1$ coincides with θ_0 or θ_1 this is immediate and if $\theta_0 < s < \theta_1$, using elementary properties of real interpolation spaces (4.7 [1], [3] p. 29) we have

$$B_0 \cap B_1 \subset (A_0, A_1)_{\theta_0,\infty} \cap (A_0, A_1)_{\theta_1,\infty} \subset (A_0, A_1)_{s,1} \subset [A_0, A_1]_s.$$

It is now evident that $B_0 \cap B_1$, which contains $A_0 \cap A_1$, is a dense subspace of both $[A_0, A_1]_s$ and $[B_0, B_1]_\sigma$ so to complete the proof of (1) we have only to show

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that

(3) $$\|b\|_{[A_0,A_1]_s} \leq \|b\|_{[B_0,B_1]_\sigma} \qquad \text{for all} \quad b \in B_0 \cap B_1.$$

If $s = 0$ or 1 then σ also equals 0 or 1 and one can readily check that $[B_0, B_1]_\sigma = \bar{A}_s = [A_0, A_1]_s$ with equality of norms. Henceforth we assume that $0 < s < 1$. Fix $b \in B_0 \cap B_1$ and let ℓ be a continuous linear functional of norm 1 acting on $[A_0, A_1]_s$ such that $\langle b, \ell \rangle = \|b\|_{[A_0,A_1]_s}$. The duality theory of [2] 12.1 permits us to identify ℓ with $h'(s)$, the derivative at $z = s$ of a function $h(z) \in \bar{\mathscr{F}}(A_0^*, A_1^*)$ with

$$\|h\|_{\bar{\mathscr{F}}(A_0^*,A_1^*)} = 1.$$

(Recall that here A_j^* is the dual of $\bar{A}_j$ rather than of A_j.)

At this point it would be natural to use the result ([2] 32.3) that if $h(z) \in \bar{\mathscr{F}}(A_0^*, A_1^*)$ then

$$\tilde{h}(z) = \frac{1}{\theta_1 - \theta_0} h((1 - z)\theta_0 + z\theta_1)$$

is in $\bar{\mathscr{F}}(B_0^*, B_1^*)$. In fact strictly speaking one should consider $\tilde{h}(z) + c$ rather than $\tilde{h}(z)$, where c is a suitable constant element of $A_0^* + A_1^*$ chosen to ensure that $\tilde{h}(z) + c$ takes values in $B_0^* + B_1^*$.

Alternatively we may proceed *via* a representation which may also be useful in other contexts, of ℓ as a weak limit of elements in $[A_0^*, A_1^*]_s$ (*cf.* [2], 29.5). For each $a \in A_0 \cap A_1$ we have by definition that

(4) $$\langle a, \ell \rangle = \langle a, h'(s) \rangle = \lim_{\eta \to 0} \left\langle a, e^{\eta s^2} \frac{(h(s + i\eta) - h(s))}{i\eta} \right\rangle$$

where η ranges through positive values. Since $A_0 \cap A_1$ is dense in $[A_0, A_1]_s$ and since

$$h_\eta(z) = e^{\eta z^2}(h(z + i\eta) - h(z))/i\eta$$

is an element of $\mathscr{F}(A_0^*, A_1^*)$ with

$$\|h_\eta\|_{\mathscr{F}(A_0^*,A_1^*)} \leq e^\eta \|h\|_{\bar{\mathscr{F}}(A_0^*,A_1^*)} = e^\eta$$

we may readily show that (4) holds for all $a \in [A_0, A_1]_s$. Now we note that $[A_0^*, A_1^*]_{\theta_j} \subset [A_0^*, A_1^*]^{\theta_j} = B_j^*$ for $j = 0, 1$ where the inclusion is a continuous embedding of norm 1 and B_j^* denotes the dual of B_j. Applying this and the inclusion established in ([2] 32.3) to the pair (A_0^*, A_1^*) we see that $h_\eta(s) \in [B_0^*, B_1^*]_\sigma \subset [B_0^*, B_1^*]^\sigma$ with

$$\|h_\eta(s)\|_{[B_0^*,B_1^*]^\sigma} \leq e^\eta.$$

Then, for the element b chosen above,

$$\|b\|_{[A_0,A_1]_s} = \langle b, \ell \rangle \leq \limsup_{\eta \to 0} |\langle b, h_\eta(s) \rangle| \leq \|b\|_{[B_0,B_1]_\sigma} \limsup_{\eta \to 0} e^\eta = \|b\|_{[B_0,B_1]_\sigma}.$$

This completes the proof.

II. A discrete definition of $[A_0, A_1]_\theta$. It is convenient for our purposes here to use a slightly modified though obviously equivalent form of the customary definition of the spaces $[A_0, A_1]_\theta$. In place of the space $\mathscr{F}(A_0, A_1)$ we use $\mathscr{F}_\infty(A_0, A_1)$ which consists of all bounded and continuous $A_0 + A_1$-valued functions $f(z)$ on $0 \leq \operatorname{re} z \leq 1$ which are analytic in the interior $0 < \operatorname{re} z < 1$, for which $f(j + iy)$ is a continuous A_j-valued function of the real variable y, $j = 0, 1$ and for which the norm

$$\|f\|_{\mathscr{F}} = \sup \{\|f(j + iy)\|_{A_j} : j = 0, 1, -\infty < y < \infty\}$$

is finite. ($\mathscr{F}(A_0, A_1)$ is of course a closed subspace of $\mathscr{F}_\infty(A_0, A_1)$ defined by the additional conditions $\lim_{|y| \to \infty} \|f(j + iy)\|_{A_j} = 0, j = 0, 1$). It is immediately clear that replacing $\mathscr{F}$ by $\mathscr{F}_\infty$ in Calderón's definition of $[A_0, A_1]_\theta$ changes neither the space defined nor the norm of any of its elements.

For some positive number λ let us take $\mathscr{F}_\lambda = \mathscr{F}_\lambda(A_0, A_1)$ to be the closed subspace of $\mathscr{F}_\infty$ defined by the additional condition $f(z + i\lambda) = f(z)$ for all z in the strip $0 \leq \operatorname{re} z \leq 1$. We may then define the "periodic" interpolation space $[A_0, A_1]_\theta^\lambda$ analogously to $[A_0, A_1]_\theta$.

$$[A_0, A_1]_\theta^\lambda = \{f(\theta) : f(z) \in \mathscr{F}_\lambda(A_0, A_1)\}$$

$$\|a\|_{[A_0,A_1]_\theta^\lambda} = \inf \{\|f\|_{\mathscr{F}} : f(\theta) = a,\ f \in \mathscr{F}_\lambda(A_0, A_1)\}.$$

Obviously

$$[A_0, A_1]_\theta^\lambda \subset [A_0, A_1]_\theta$$

and

$$\|a\|_{[A_0,A_1]_\theta} \leq \|a\|_{[A_0,A_1]_\theta^\lambda}$$

for each $a \in [A_0, A_1]_\theta^\lambda$. We shall show here that the reverse inclusion holds and so for each positive λ the space $[A_0, A_1]_\theta^\lambda$ coincides with $[A_0, A_1]_\theta$ and their norms are equivalent. This amounts to giving a discrete definition of $[A_0, A_1]_\theta$ and so resolves a question posed by J. Peetre [4, 5], who introduced spaces of the form $[A_0, A_1]_\theta^\lambda$ in [4]. Let us note (*cf.* [4] pp. 175–176, [2], p. 133) that any f in $\mathscr{F}_\lambda(A_0, A_1)$ has a Fourier series representation

$$f(z) \sim \sum_{k=-\infty}^{\infty} a_k e^{2\pi k z/\lambda}$$

where the coefficients a_k are in $A_0 \cap A_1$ and $f(z)$ may be recovered as the limit of the $(C, 1)$ means of the series. Let us here take $\lambda = 2\pi$ and write $a_k = u_k e^{-k\theta}$ so that

$$f(z) \sim \sum_{k=-\infty}^{\infty} u_k e^{k(z-\theta)}.$$

For $j = 0, 1$ let $C(A_j)$ denote the space of continuous A_j-valued functions $g(y)$ on the real line with period 2π equipped with the norm

$$\|g\|_{C(A_j)} = \sup_{0 \le y \le 2\pi} \|g(y)\|_{A_j}.$$

Let $\hat{C}(A_j)$ denote the space of A_j-valued sequences $(g_k)_{k=-\infty}^{\infty}$ which arise as the Fourier coefficients of functions g in $C(A_j)$ with norm $\|(g_k)\|_{\hat{C}(A_j)} = \|g\|_{C(A_j)}$ where

$$g(y) \sim \sum_{k=-\infty}^{\infty} g_k e^{iky}.$$

We can see then that $[A_0, A_1]_\theta = [A_0, A_1]_\theta^{2\pi}$ consists of all elements a in $A_0 + A_1$ which have a representation of the form $a = \sum_{k=-\infty}^{\infty} u_k$ where the sequence $(u_k)_{k=-\infty}^{\infty}$ consists of elements in $A_0 \cap A_1$ and satisfies the conditions $(e^{k(j-\theta)}u_k) \in \hat{C}(A_j)$ $j = 0, 1$, and a is the limit in $A_0 + A_1$ norm of the $(C, 1)$ means of the series. Furthermore the norm of a in $[A_0, A_1]_\theta$ will be the infimum of

$$\max_{j=0,1} [\|(e^{k(j-\theta)}u_k)\|_{\hat{C}(A_j)}]$$

over all such representations $\sum_{k=-\infty}^{\infty} u_k$ of a. This gives a close analogue of the discrete definition of the real interpolation spaces $(A_0, A_1)_{\theta,p}$ (see [1, 4]). In fact the elements of $(A_0, A_1)_{\theta,p}$ may be defined as sums of $A_0 \cap A_1$-valued sequences $\{u_k\}_{k=-\infty}^{\infty}$ which satisfy the same conditions as those just given except that $\hat{C}(A_j)$ must be replaced by $\ell^p(A_j)$.

Proof of the Equivalence. As already noted, it suffices to show that $[A_0, A_1]_\theta \subset [A_0, A_1]_\theta^\lambda$. The proof turns out to be very simple. We need a scalar-valued function $w(z)$ which is continuous and bounded on $0 \le \operatorname{re} z \le 1$ and analytic in the interior with $w(\theta) = 1$ and $w(\theta + in\lambda) = 0$ for each nonzero integer n. For example we may take

$$w(z) = \lambda(e^{2\pi(z-\theta)/\lambda} - 1)/2\pi(z - \theta).$$

Now for any $a \in [A_0, A_1]_\theta$ choose $f(z) \in \mathscr{F}(A_0, A_1)$ with $a = f(\theta)$, and $\|f\|_{\mathscr{F}}$ close to $\|a\|_{[A_0,A_1]_\theta}$. Then it follows that

$$F(z) = e^{\delta(z-\theta)^2} w(z) f(z) \in \mathscr{F}(A_0, A_1)$$

for any choice of $\delta > 0$ and

$$G(z) = \sum_{k=-\infty}^{\infty} F(z + ik\lambda) \in \mathscr{F}_\lambda(A_0, A_1) \qquad (cf.\ [2]\ 29.2).$$

Since $G(\theta) = a$ and $\|G\|_{\mathscr{F}} \le C\|f\|_{\mathscr{F}}$, where the constant C depends only on θ and λ, the inclusion is proved and of course we also have that

$$\|a\|_{[A_0,A_1]_\theta^\lambda} \le C\|a\|_{[A_0,A_1]_\theta}.$$

Remarks (1). Using the obvious one-to-one correspondence between functions in $\mathscr{F}_\lambda(A_0, A_1)$ and analytic $A_0 + A_1$-valued functions on the annulus $1 \le |z| \le e^{2\pi/\lambda}$ which take values in A_0 on the inner boundary and in A_1 on the outer boundary, we see that the strip $0 \le \operatorname{re} z \le 1$ can be replaced by an annulus in the definition of $[A_0, A_1]_\theta$. Note that annuli of different dimensions, which are not conformally equivalent, nevertheless give rise to the same interpolation spaces.

(2). From the proof given above we are unable to give a uniform upper bound for the constant C as λ approaches zero. The following argument shows that in general no such bound can exist.

Proposition. *Let $\theta \in (0, 1)$ and suppose that for some $\alpha \in (0, 1)$ $[A_0, A_1]_\theta$ is not continuously embedded in $[A_0, A_1]_\alpha$. Let*

$$C(\theta, \lambda) = \sup \{\|a\|_{[A_0,A_1]^\lambda_\theta} : a \in [A_0, A_1]_\theta, \|a\|_{(A_0,A_1]_\theta} \leq 1\}.$$

Then $\limsup_{\lambda \to 0} C(\theta, \lambda) = \infty$.

Proof. Suppose the result is untrue. Then $\sup \{C(\theta, 2^{-n}), n = 1, 2, \cdots\} = M$ is finite. Let us choose $a \in A_0 \cap A_1$ with

$$\|a\|_{[A_0,A_1]_\theta} = 1$$

and $\|a\|_{[A_0,A_1]_\alpha} \geq M + 4$. We choose $g(z) \in \bar{\mathscr{F}}(A_0^*, A_1^*)$ with

$$\|g\|_{\bar{\mathscr{F}}(A_0^*,A_1^*)} = 1$$

and $\langle a, g'(\alpha)\rangle \geq M + 3$. For each positive integer n there must exist $f_n(z) \in \mathscr{F}_{2^{-n}}(A_0,A_1)$ with $f_n(\theta) = a$ and $\|f_n\|_{\mathscr{F}(A_0,A_1)} \leq M + 1$. The functions $\phi_n(z) = \langle f_n(z), g'(z)\rangle$ are uniform limits of analytic functions and so are analytic in $0 < \operatorname{re} z < 1$ and $|\phi_n(z)| \leq M + 1$ for all z in the open strip. Since we have a normal family we may select a subsequence of $(\phi_n(z))$ which converges uniformly to a limit function $\phi(z)$ on a closed rectangle $|\operatorname{im} z| \leq 1, \epsilon \leq \operatorname{re} z \leq 1 - \epsilon$ which contains the points θ and α. Clearly $\phi_n(\theta + i2^{-m}) = \langle a, g'(\theta + i2^{-m})\rangle$ for all $n \geq m$ and so $\phi(\theta + i2^{-m}) = \langle a, g'(\theta + i2^{-m})\rangle$ for all positive integers m. Consequently $\phi(z) = \langle a, g'(z)\rangle$ on the whole rectangle. But here we have a contradiction since $\langle a, g'(\alpha)\rangle \geq M + 3$ although $|\phi(z)| \leq M + 1$, and this completes the proof.

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REFERENCES

1. J. BERGH & J. LÖFSTRÖM, *Interpolation Spaces. An Introduction*, Springer Verlag, Berlin, 1976.
2. A. P. CALDERÓN, *Intermediate spaces and interpolation, the complex method,* Studia Math. **24** (1964), 113–190.
3. J. L. LIONS & J. PEETRE, *Sur une classe d'espaces d'interpolation,* Publ. Math. I.H.E.S. **19** (1964), 5–68.
4. J. PEETRE, *Sur l'utilisation des suites inconditionellement sommables dans la théorie des espaces d'interpolation*, Rend. Sem. Mat. Univ. Padova, **46** (1971), 173–190.
5. J. PEETRE, Notices Amer. Math. Soc. **22** (1975), 124.

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