Another note on equality in DPI for the BS relative entropy

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1 Equality conditions in QRE and BS-RE

Let $\mathcal{T}$ be a channel and let ρ, σ be states, σ invertible. According to [**? ?**], we have the following eqivalen conditions for equality in DPI.

QRE	BS-RE
$\sigma^{1/2}\mathcal{T}^*(\mathcal{T}(\sigma)^{-1/2}\mathcal{T}(\rho)\mathcal{T}(\sigma)^{1/2})\sigma^{1/2} = \rho$	$\sigma\mathcal{T}^*(\mathcal{T}(\sigma)^{-1}\mathcal{T}(\rho)) = \rho$
	$\sigma\mathcal{T}^*(\mathcal{T}(\sigma)^{-1}\mathcal{T}(\rho)^2\mathcal{T}(\sigma)^{-1})\sigma = \rho^2$
$\mathrm{Tr}\,\mathcal{T}(\rho)^{1/2}\mathcal{T}(\sigma)^{1/2} = \mathrm{Tr}\,\rho^{1/2}\sigma^{1/2}$	$\mathrm{Tr}\,\mathcal{T}(\rho)^2\mathcal{T}(\sigma)^{-1} = \mathrm{Tr}\,\rho^2\sigma^{-1}$
$\mathcal{T}^*(\mathcal{T}(\sigma)^{-1/2}\mathcal{T}(\rho)\mathcal{T}(\sigma)^{1/2}) = \sigma^{-1/2}\rho\sigma^{-1/2}$	$\mathcal{T}(\rho)\mathcal{T}(\sigma)^{-1}\mathcal{T}(\rho) = \rho\sigma^{-1}\rho$
$\sigma^{-1/2}\rho\sigma^{-1} \in \mathcal{F}_{(\mathcal{T}_\sigma\circ\mathcal{T})^*}$	$\sigma^{-1/2}\rho\sigma^{-1/2} \in \mathcal{M}_{\mathcal{T}^*_\sigma}$
$\sigma^{it-1/2}\rho\sigma^{-it-1/2} \in \mathcal{M}_{\mathcal{T}^*_\sigma}$, $\forall t \in \mathbb{R}$	

Theorem 1. *Assume that ρ_{ABC} is such that ρ_{AB} is invertible. Define the state*

$$\eta_{ABC} := \frac{1}{d_B}\rho_B^{-1/2}\rho_{ABC}\rho_B^{-1/2}.$$

The following are equivalent.

(i) ρ_{ABC} *is a BS-QMC.*

(ii) $\rho_{ABC} = \rho_{AB}\rho_B^{-1}\rho_{BC}$.

(iii) The marginals η_{AB} and η_{BC} commute, and we have $\rho_{ABC} = d_B^2\rho_B^{1/2}\eta_{AB}\eta_{BC}\rho_B^{1/2}$.

(iv) η_{ABC} *is a QMC.*

(v) There are Hilbert spaces $\mathcal{H}_{B_n^L}$, $\mathcal{H}_{B_n^R}$ and a unitary $U_B : \mathcal{H}_B \to \oplus_{n=1}^N \left(\mathcal{H}_{B_n^L} \otimes \mathcal{H}_{B_n^R}\right)$, such that

$$\rho_{ABC} = \rho_B^{1/2}U_B^*\left(\oplus_n d_B p_n \eta_{AB_L^n} \otimes \eta_{B_R^n C}\right)U_B\rho_B^{1/2}$$

for some states $\eta_{AB_L^n}$ on $\mathcal{H}_{AB_L^n}$ and $\eta_{B_R^n C}$ on $\mathcal{H}_{B_R^n C}$ and a probability distribution $\{p_n\}$.

Proof. The equivalence (i) $\iff$ (ii) was proved in [?]. If (ii) holds, then clearly $\rho_{ABC} = d_B^2\rho_B^{1/2}\eta_{AB}\eta_{BC}\rho_B^{1/2} = \rho_{ABC}^*$. Since ρ_B is invertible, this implies that $[\eta_{AB}, \eta_{BC}] = 0$, so that (iii) holds. Assume (iii), then since $\eta_B = d_B^{-1}I_B = \tau_B$, we obtain

$$\eta_{ABC} = d_B\eta_{AB}\eta_{BC} = \eta_{AB}^{1/2}\eta_B^{-1/2}\eta_{BC}\eta_B^{-1/2}\eta_{AB}^{1/2}$$

so that η_{ABC} is a QMC. If (iv) holds, then there are Hilbert spaces $\mathcal{H}_{B_n^L}$, $\mathcal{H}_{B_n^R}$ and a unitary $U_B : \mathcal{H}_B \to \oplus_{n=1}^N \left(\mathcal{H}_{B_n^L} \otimes \mathcal{H}_{B_n^R}\right)$ such that

$$\eta_{ABC} = U_B^*\left(\oplus_n p_n\eta_{AB_L^n} \otimes \eta_{B_R^nC}\right)U_B,$$

this proves (v). Finally, suppose that (v) holds, then from

$$\tau_B = d_B^{-1}\mathrm{Tr}_{AC}\rho_B^{-1/2}\rho_{ABC}\rho_B^{-1/2} = U_B^*\left(\oplus_n p_n\eta_{B_L^n} \otimes \eta_{B_R^n}\right)U_B$$

we infer that $\eta_{B_L^n} = \tau_{B_L^n}$ and $\eta_{B_R^n} = \tau_{B_R^n}$. It follows that $\rho_{AB} = \rho_B^{1/2}U_B^*\left(\oplus_n d_Bp_n\eta_{AB_L^n} \otimes \tau_{B_R^n}\right)U_B\rho_B^{1/2}$ and similarly $\rho_{BC} = \rho_B^{1/2}U_B^*\left(\oplus_n d_Bp_n\tau_{B_L^n} \otimes \eta_{B_R^nC}\right)U_B\rho_B^{1/2}$. The condition (ii) is immediate from this.

□

Remark 1. The condition in Proposition **??** is not clear to me. The decomposition of $\mathcal{H}_B$ for ρ_{ABC} is not unique, so the decomposition for ρ_B should not be fixed to one particular choice of the decomposition. The statement should be more like that there is some decomposition of $\mathcal{H}_B$ that works for both ρ_{ABC} and ρ_B.

It would be nicer to have this condition stated directly in terms of η_{ABC} and ρ_B. One such statement is as follows: A BS-QMC ρ_{ABC} is a QMC if and only if $\rho_B^{it}\eta_{AB}\rho_B^{-it}$ commutes with η_{BC} for all $t \in \mathbb{R}$. Indeed, the condition is easily checked for a QMC, using the decomposition of a QMC. For the converse, one can proceed as follows: since $\rho^{it}(\rho_B^{-1/2}\rho_{AB}\rho_B^{-1/2})\rho^{-it}$ is in the commutant of η_{BC} for all t, one can show that the commutant of η_{BC} is a sufficient subalgebra for $\{\rho_{AB}, \rho_B\}$, and this implies that it contains also $\rho_{AB}^{1/2}\rho_B^{-1/2}$ (there should be a more direct proof of this). Consider the polar decomposition $\rho_{AB}^{1/2}\rho_B^{-1/2} = d_B^{1/2}W_{AB}\eta_{AB}^{1/2}$, then both η_{AB} and the unitary W_{AB} must be contained in the commutant. We then get, using the computations in Corollary 4.7

$$\begin{aligned}\rho_{ABC} &= d_B^2\rho_B^{1/2}\eta_{AB}\eta_{BC}\rho_B^{1/2} = d_B^2\rho_B^{1/2}\eta_{AB}^{1/2}\eta_{BC}\eta_{AB}^{1/2}\rho_B^{1/2} = d_B\rho_{AB}^{1/2}W_{AB}\eta_{BC}W_{AB}^*\rho_{AB}^{1/2}\\ &= d_B\rho_{AB}^{1/2}\eta_{BC}\rho_{AB}^{1/2} = \rho_{AB}^{1/2}\rho_B^{-1/2}\rho_{BC}\rho_B^{-1/2}\rho_{AB}^{1/2}.\end{aligned}$$

This is probably not the most efficient condition, finding equivalent ones might help to understand the relation of the BS-QMC and QMC better.