# DPI and sufficiency, $\alpha > 1$

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Throughout these notes, we will assume that $\alpha > 1$ and $\max\{\alpha/2, \alpha-1\} \leq z \leq \alpha$. Let $\psi, \varphi \in \mathcal{M}_*^+$ be such that $D_{\alpha,z}(\psi\|\varphi) < \infty$, we will also assume that $\varphi$ is faithul.

We put $p := \frac{z}{\alpha}$ and $q := \frac{z}{\alpha-1}$, so that $1/2 \leq p \leq 1 \leq q$. By the assumptions, there is some unique $y \in L_{2z}(\mathcal{M})$ such that

$$h_\psi^{\frac{1}{2p}} = y h_\varphi^{\frac{1}{2q}}. \tag{1}$$

By [5, 6], we have the variational formula

$$\begin{aligned}
Q_{\alpha,z}(\psi\|\varphi) &= \sup_{a\in\mathcal{M}_+} \alpha \mathrm{Tr}\,(h_\psi^{\frac{1}{2p}} a h_\psi^{\frac{1}{2p}})^p - (\alpha-1)\mathrm{Tr}\,(h_\varphi^{\frac{1}{2q}} a h_\varphi^{\frac{1}{2q}})^q \\
&= \sup_{a\in\mathcal{M}_+} \alpha \mathrm{Tr}\,(y h_\varphi^{\frac{1}{2q}} a h_\varphi^{\frac{1}{2q}} y^*)^p - (\alpha-1)\mathrm{Tr}\,(h_\varphi^{\frac{1}{2q}} a h_\varphi^{\frac{1}{2q}})^q \\
&= \sup_{w\in L_q(\mathcal{M})^+} \alpha \mathrm{Tr}\,(y w y^*)^p - (\alpha-1)\mathrm{Tr}\, w^q,
\end{aligned}$$

this follows from the fact that $h_\varphi^{\frac{1}{2q}}\mathcal{M}_+ h_\varphi^{\frac{1}{2q}}$ is dense in $L_q(\mathcal{M})^+$. The supremum is attained at a unique point $\bar w = (y^*y)^{\alpha-1} \in L_1(\mathcal{M})^+$, uniqueness follows from strict concavity of the function $w \mapsto \alpha\mathrm{Tr}\,(ywy^*)^p - (\alpha-1)\mathrm{Tr}\, w^q$.

Let $\Phi : \mathcal{M}_* \to \mathcal{N}_*$ be a 2-positive trace preserving map and let $\varphi_0 := \Phi(\varphi)$, $\psi_0 := \Phi(\psi)$. Assume that also $\varphi_0$ is faithful. Let $\Phi_\varphi : \mathcal{N}_* \to \mathcal{M}_*$ be the Petz dual of $\Phi$ with respect to $\varphi$, then we have

$$\Phi(h_\varphi^{1/2} a h_\varphi^{1/2}) = h_{\varphi_0}^{1/2}\Phi_\varphi^*(a) h_{\varphi_0}^{1/2}, \qquad \Phi_\varphi(h_{\varphi_0}^{1/2} b h_{\varphi_0}^{1/2}) = h_\varphi^{1/2}\Phi^*(b) h_\varphi^{1/2}, \qquad a\in\mathcal{M},\ b\in\mathcal{N},$$

here $\Phi^* : \mathcal{N} \to \mathcal{M}$ and $\Phi_\varphi^* : \mathcal{M} \to \mathcal{N}$ are the 2-positive unital normal maps that are adjoints of $\Phi$ resp. $\Phi_\varphi$. More generally, since for any $r \geq 1$, $\Phi$ is a contraction $L_r(\mathcal{M},\varphi)$ to $L_r(\mathcal{N},\varphi_0)$, and similarly for $\Phi_\varphi$, there are positive contractions $\Phi_{r,\varphi} : L_r(\mathcal{M}) \to L_r(\mathcal{N})$ and $\Phi_{r,\varphi_0} : L_r(\mathcal{N}) \to L_r(\mathcal{M})$ such that

$$\Phi(h_\varphi^{\frac{1}{2r'}} a h_\varphi^{\frac{1}{2r'}}) = h_{\varphi_0}^{\frac{1}{2r'}}\Phi_{r,\varphi}(a) h_{\varphi_0}^{\frac{1}{2r'}}, \qquad \Phi_\varphi(h_{\varphi_0}^{\frac{1}{2r'}} b h_{\varphi_0}^{\frac{1}{2r'}}) = h_\varphi^{\frac{1}{2r'}}\Phi_{r,\varphi_0}(b) h_\varphi^{\frac{1}{2r'}}, \qquad a\in L_r(\mathcal{M}),\ b\in L_r(\mathcal{N})$$

here $r'$ is such that $\frac{1}{r} + \frac{1}{r'} = 1$.

By DPI, we have $D_{\alpha,z}(\psi_0\|\varphi_0) \leq D_{\alpha,z}(\psi\|\varphi) < \infty$, so that there is some unique $y_0 \in L_{2z}(\mathcal{N})$ such that

$$h_{\psi_0}^{\frac{1}{2p}} = y_0 h_{\varphi_0}^{\frac{1}{2q}}.$$

**Lemma 1.** *Keeping the above assumptions and notations, we have for any $w_0 \in L_q(\mathcal{N})^+$*

$$\mathrm{Tr}\,\Phi_{q,\varphi_0}(w_0)^q \leq \mathrm{Tr}\,w_0^q, \qquad \mathrm{Tr}\,(y\Phi_{q,\varphi_0}(w_0)y^*)^p \geq \mathrm{Tr}\,(y_0 w_0 y_0^*)^p.$$

*Proof.* The first inequality is immediate from the fact that $\Phi_{q,\varphi_0}$ is a contraction. For the second inequality, let us first assume that $w_0 = h_{\varphi_0}^{\frac{1}{2q}} b h_{\varphi_0}^{\frac{1}{2q}}$ for some $b \in \mathcal{N}_+$. Then

$$h_\varphi^{\frac{1}{2q'}}\Phi_{q,\varphi_0}(w_0)h_\varphi^{\frac{1}{2q'}} = \Phi_\varphi(h_{\varphi_0}^{\frac{1}{2q'}} w_0 h_{\varphi_0}^{\frac{1}{2q'}}) = \Phi_\varphi(h_{\varphi_0}^{\frac{1}{2}} b h_{\varphi_0}^{\frac{1}{2}}) = h_\varphi^{\frac{1}{2q'}} h_\varphi^{\frac{1}{2q}}\Phi^*(b)h_\varphi^{\frac{1}{2q}} h_\varphi^{\frac{1}{2q'}},$$

so that $\Phi_{q,\varphi_0}(w_0) = h_\varphi^{\frac{1}{2q}}\Phi^*(b)h_\varphi^{\frac{1}{2q}}$. Therefore

$$\begin{aligned}\mathrm{Tr}\,(y\Phi_{q,\varphi_0}(w_0)y^*)^p &= \mathrm{Tr}\,(yh_\varphi^{\frac{1}{2q}}\Phi^*(b)h_\varphi^{\frac{1}{2q}}y^*)^p = \mathrm{Tr}\,(h_\psi^{\frac{1}{2p}}\Phi^*(b)h_\psi^{\frac{1}{2p}})^p \geq \mathrm{Tr}\,(h_{\psi_0}^{\frac{1}{2p}} b h_{\psi_0}^{\frac{1}{2p}})^p\\ &= \mathrm{Tr}\,(y_0 h_{\varphi_0}^{\frac{1}{2q}} b h_{\varphi_0}^{\frac{1}{2q}} y_0^*)^p = \mathrm{Tr}\,(y_0 w_0 y_0^*)^p,\end{aligned}$$

here the inequality was proved in [2]. Since $h_{\varphi_0}^{\frac{1}{2q}}\mathcal{N}_+ h_{\varphi_0}^{\frac{1}{2q}}$ is dense in $L_q(\mathcal{N})^+$, the statement follows. $\square$

**Theorem 1.** *Let $\Phi : \mathcal{M}_* \to \mathcal{N}_*$ be a 2-positive trace preserving map and let $\psi, \varphi \in \mathcal{M}_*^+$ be such that $D_{\alpha,z}(\psi\|\varphi) < \infty$. Then $D_{\alpha,z}(\Phi(\psi)\|\Phi(\varphi)) = D_{\alpha,z}(\psi\|\varphi)$ if and only if $\Phi_\varphi \circ \Phi(\psi) = \psi$.*

*Proof.* By usual arguments, we may assume that both $\varphi$ and $\varphi_0$ are faithful. Then there is a conditional expectation $\mathcal{E}$ onto the set of fixed points of $\Phi^* \circ \Phi_\varphi^*$ such that $\varphi \circ \mathcal{E} = \varphi$ and $\Phi_\varphi \circ \Phi(\psi) = \psi$ if and only if also $\psi \circ \mathcal{E}$. This is what we are going to prove, using the extensions of conditional expectations to the Haagerup $L_p$-spaces in [4], see also [3, Sec. 1].

So assume that $D_{\alpha,z}(\psi_0\|\varphi_0) = D_{\alpha,z}(\psi\|\varphi)$. Let $\bar{w} \in L_q(\mathcal{M})^+$ and $\bar{w}_0 \in L_q(\mathcal{N})^+$ be the unique elements such that the suprema in the variational formulas for $D_{\alpha,z}(\psi\|\varphi)$ resp. $D_{\alpha,z}(\psi_0\|\varphi_0)$ are attained. We have by Lemma 1

$$\begin{aligned}D_{\alpha,z}(\psi\|\varphi) &\geq \alpha\mathrm{Tr}\,(y\Phi_{q,\varphi_0}(\bar{w}_0)y^*)^p - (\alpha-1)\mathrm{Tr}\,\Phi_{q,\varphi_0}(\bar{w}_0)^q\\ &\geq \alpha\mathrm{Tr}\,(y_0\bar{w}_0y_0^*)^p - (\alpha-1)\mathrm{Tr}\,\bar{w}_0^q = D_{\alpha,z}(\psi_0\|\varphi_0) = D_{\alpha,z}(\psi\|\varphi),\end{aligned}$$

so that both inequalities must be equalities. This implies that in particular

$$\mathrm{Tr}\,\bar{w}_0^q = \mathrm{Tr}\,\Phi_{q,\varphi_0}(\bar{w}_0)^q.$$

By uniqueness, we must also have $\bar{w} = \Phi_{q,\varphi_0}(\bar{w}_0)$. Let now $\omega \in \mathcal{M}_*^+$ be given by $h_\omega = h_\varphi^{\frac{1}{2q'}}\bar{w}h_\varphi^{\frac{1}{2q'}}$ and similarly $h_{\omega_0} = h_{\varphi_0}^{\frac{1}{2q'}}\bar{w}_0 h_{\varphi_0}^{\frac{1}{2q'}}$, then we get $\Phi_\varphi(\omega_0) = \omega$ and also by definition of the sandwiched Rényi divergence,

$$\tilde{D}_\alpha(\omega_0\|\varphi_0) = \mathrm{Tr}\,\bar{w}_0^q = \mathrm{Tr}\,\Phi_{q,\varphi_0}(\bar{w}_0)^q = \tilde{D}_\alpha(\Phi_\varphi(\omega_0)\|\Phi_\varphi(\varphi_0)).$$

By [1], this implies that $\Phi_\varphi$ is sufficient with respect to $\{\omega_0, \varphi_0\}$ and hence $\Phi \circ \Phi_\varphi(\omega_0) = \omega_0$. It follows that

$$\Phi_\varphi \circ \Phi(\omega) = \Phi_\varphi \circ \Phi \circ \Phi_\varphi(\omega_0) = \Phi_\varphi(\omega_0) = \omega,$$

which implies that $\omega \circ \mathcal{E} = \omega$. Using the extensions of $\mathcal{E}$ and their properties, we get

$$h_\varphi^{\frac{1}{2q'}}\bar{w}h_\varphi^{\frac{1}{2q'}} = h_\omega = \mathcal{E}(h_\omega) = h_\varphi^{\frac{1}{2q'}}\mathcal{E}(\bar{w})h_\varphi^{\frac{1}{2q'}},$$

which implies that $\bar{w} = \mathcal{E}(\bar{w}) \in L_q(\mathcal{E}(\mathcal{M}))^+$. But then also

$$|y| = \bar{w}^{\frac{1}{2(\alpha-1)}} = \bar{w}^{\frac{q}{2z}} \in L_{2z}(\mathcal{E}(\mathcal{M}))^+$$

Let $y = u|y|$ be the polar decomposition of $y$, then we obtain from (1) that $uu^* = s(\psi)$. Further,

$$u^* h_\psi^{\frac{1}{2p}} = |y| h_\varphi^{\frac{1}{2q}} \in L_{2p}(\mathcal{E}(\mathcal{M}))$$

and by uniqueness of the polar decomposition in $L_{2p}(\mathcal{M})$ and $L_{2p}(\mathcal{E}(\mathcal{M}))$, we obtain that $h_\psi^{\frac{1}{2p}} \in L_{2p}(\mathcal{E}(\mathcal{M}))^+$, $u \in \mathcal{E}(\mathcal{M})$. Hence we must have $h_\psi \in L_1(\mathcal{E}(\mathcal{M}))$ so that $\psi \circ \mathcal{E} = \psi$.

□

# References


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