# INCOMPATIBILITY IN GPTS AND GENERALIZED SPECTRAHEDRA

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Contents



## 1. Purpose of these notes

The aim of these notes is to study the incompatibility of measurements in GPTs. These notes unify the results of A. Jenčová, building on [Jen18] and us, building on [BN18b]. In particular, we show how these two frameworks are related.

## 2. Preliminaries

In this paper, we will assume all vector spaces to be finite dimensional and over the real field.

2.1. **Convex cones.** Let $V$ be a finite-dimensional real vector space. A subset $V^+ \subset V$ is a *convex cone* if $\lambda x + \mu y \in V^+$ for all $x$, $y \in V^+$ and all $\lambda$, $\mu \in \mathbb{R}_+$. Often, we will drop "convex" and talk simply about "cones". A cone will be called *generating* if $V = V^+ - V^+$. Moreover, it is *pointed* if $V^+ \cap (-V^+) = \{0\}$. A *proper* cone is a cone which is closed, pointed and generating. The *dual cone* of $V^+$ is the cone

$$(V^+)^* := \{f \in V^* : f(x) \geqslant 0 \;\forall x \in V\},$$

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where $V^*$ is the dual vector space of $V$. Two cones $V_1^+$ and $V_2^+$ living in vector spaces $V_1$ and $V_2$, respectively, are *isomorphic* if there is a linear bijection $\Gamma : V_1 \to V_2$ such that $\Gamma(V_1^+) = V_2^+$. A cone $V^+$ is called *simplicial* if it is isomorphic to $\mathbb{R}_+^d$, where $d = \dim(V^+)$. An *(*ordered vector space) is a tuple $(V, V^+)$, where $V$ is a vector space and $V^+$ is a pointed convex cone. We write $y \geqslant x$ for $x$, $y \in V$ to mean that $y - x \in V^+$.

2.2. **Tensor products of cones.** There are in general infinitely many ways to define the tensor product of two cones $V_1^+ \subset V_1$ and $V_2^+ \subset V_2$. Among these, there is a minimal and a maximal choice: The *minimal tensor product* of $V_1^+$ and $V_2^+$ is the cone

$$V_1^+ \otimes_{\min} V_2^+ := \operatorname{conv}\{x \otimes y : x \in V_1^+, y \in V_2^+\},$$

whereas the *maximal tensor product* of $V_1^+$ and $V_2^+$ is defined as

$$V_1^+ \otimes_{\max} V_2^+ := ((V_1^+)^* \otimes_{\min} (V_2^+)^*)^*.$$

It can be seen that if $V_1^+$ and $V_2^+$ are proper, $V_1^+ \otimes_{\min} V_2^+$ and $V_1^+ \otimes_{\max} V_2^+$ are proper as well [ALP19, Fact S23]. We call $C$ a *tensor cone* for $V_1^+$ and $V_2^+$ if

$$V_1^+ \otimes_{\min} V_2^+ \subseteq C \subseteq V_1^+ \otimes_{\max} V_2^+.$$

From the recent work [ALPP19], we know that the tensor product of two cones is unique if and only one of the cones is simplicial. This solves a longstanding open problem from [NP69, Bar81]:

**Theorem 2.1** ([ALPP19, Theorem A])**.** *Let $V_1^+$ and $V_2^+$ be proper cones. Then, $V_1^+ \otimes_{\min} V_2^+ = V_1^+ \otimes_{\max} V_2^+$ if and only if $V_1^+$ or $V_2^+$ is simplicial.*

2.3. **Base norms.** In GPTs, we have natural norms induced by the cones in the state space and the space of effects. For more details about the following, see the excellent [Lam18, Chapter 1.6].

**Definition 2.2.** *Given a GPT $(V, V^+, \mathbb{1})$, define the following norm on $V$, called a* base norm

$$\|x\|_V = \inf\{\mathbb{1}(y) + \mathbb{1}(z) \,:\, y, z \in V^+ \text{ s.t. } \; x = y - z\}$$

*as well as a norm on $A := V^*$, called an* order unit norm

$$\|\alpha\|_A = \inf\{t \geqslant 0 \,:\, \alpha \in t[-\mathbb{1}, \mathbb{1}]\}.$$

Using the above definition, one can characterize positivity in $V$ using metric properties:

$$x \in V^+ \iff \|x\|_V = \mathbb{1}(x).$$

**Question 2.3.** *Can one do the same for $A^+$?*

2.4. **Centrally symmetric GPTs.** We review in this section what it means for a GPT $(V, V^+, \mathbb{1})$ to be centrally symmetric in the sense of [LPW18, Definition 25]. In this case, the vector space $V$ admits a decomposition $V = \mathbb{R}v_0 \oplus \bar{V}$, and we shall write $x = (x_0, \bar{x})$ for a vector $x = x_0 v_0 \oplus \bar{x}$. We then have

$$x = (x_0, \bar{x}) \in V^+ \iff \|\bar{x}\|_{\bar{V}} \leqslant x_0,$$

where $\|\cdot\|_{\bar{V}}$ denotes the norm on $\bar{V}$. The decomposition $V = \mathbb{R}v_0 \oplus \bar{V}$ is such that the form $\mathbb{1}$ is given by $\mathbb{1}(x) = x_0$. The base norm is given as $\|x\|_V = \max(|x_0|, \|\bar{x}\|_{\bar{V}})$, where the absolute value symbol is overloaded. The dual vector space $A = V^*$ admits a similar description in terms of the dual norm $\|\cdot\|_{\bar{A}}$: $A = \mathbb{R}\mathbb{1} \oplus \bar{A}$ and $a = (a_0, \bar{a}) \in A^+ \iff \|\bar{a}\|_{\bar{A}} \leqslant a_0$.

**Lemma 2.4.** *Let $(V, V^+, \mathbb{1})$ be a centrally symmetric GPT. The order unit norm on $A$ satisfies*

$$\|\varphi\|_A = |\varphi_0| + \|(\varphi_1, \ldots, \varphi_g)\|_{\bar{A}}$$

*for all $\varphi \in A$ with coordinates $\varphi = \varphi_0 \mathbb{1} + \sum_{i=1}^g \varphi_i \delta_i$, where $\varphi_0$, $\varphi_i \in \mathbb{R}$ for all $i \in [g]$ and $\{\mathbb{1}, \delta_i\}_{i \in [g]}$ is a basis of $A$.*

*Proof.* By definition of the order unit norm of a vector $\varphi \in A$, $\|\varphi\|_A$ is the minimal $t \geqslant 0$ such that $\varphi \in t[-\mathbb{1}, \mathbb{1}]$.

$$\begin{aligned}\varphi \in t[-\mathbb{1}, \mathbb{1}] &\iff t\mathbb{1} - \varphi \in A^+ \wedge t\mathbb{1} + \varphi \in A^+ \\ &\iff (t-\varphi_0)\mathbb{1} - \sum_{i=1}^{g} \varphi_i \delta_i \in A^+ \wedge (t+\varphi_0)\mathbb{1} + \sum_{i=1}^{g} \varphi_i \delta_i \in A^+ \\ &\iff (t-\varphi_0) \geqslant \|(\varphi_1, \ldots, \varphi_g)\|_{\bar{A}} \wedge (t+\varphi_0) \geqslant \|(\varphi_1, \ldots, \varphi_g)\|_{\bar{A}} \\ &\iff t \geqslant |\varphi_0| + \|(\varphi_1, \ldots, \varphi_g)\|_{\bar{A}}.\end{aligned}$$

This proves the assertion. □

## 3. An extension theorem

We prove an extension theorem which will be used to relate the compatibility of measurements in a GPT to properties of an associated map.

$$E \cap \mathrm{ri}(\mathbb{R}^d_+) \neq \varnothing. \tag{1}$$

We set $E^+ := E \cap \mathbb{R}^d_+$. Note that the condition above implies that the same holds for the dual of $E$: $E^* \cap \mathrm{ri}(\mathbb{R}^d_+) \neq \varnothing$; to show this, one needs to use the fact that the ordered vector space $(\mathbb{R}^d, \mathbb{R}^d_+)$ is self-dual.

We shall use the following key extension theorem (see e.g. [Cas05, Theorem 1]):

**Theorem 3.1** (M. Riesz extension theorem)**.** *Let $(X, X^+)$ be an ordered vector space, $Y \subseteq X$ a linear subspace, and $\varphi : Y \to \mathbb{R}$ a positive linear form on $(Y, Y^+)$, where $Y^+ := Y \cap X^+$. Assume that for every $x \in X$, there exists $y \in Y$ such that $x \leqslant y$. Then, there exists a positive linear form $\tilde{\varphi} : X \to \mathbb{R}$ such that $\tilde{\varphi}|_Y = \varphi$.*

We now prove the main result of this section; see also Remark 3.5 for an equivalent formulation.

**Proposition 3.2.** *Let $(V, v^+)$ be an ordered vector space and let $V^+$ be generating. Any positive linear form*

$$\varphi : (E \otimes V, (E \otimes V) \cap (\mathbb{R}^d_+ \otimes V^+)) \to \mathbb{R} \tag{2}$$

*can be extended to a positive linear form $\tilde{\varphi} : \mathbb{R}^d \otimes V \to \mathbb{R}$ (note that we do not specify the tensor cone we put on the domain of $\tilde{\varphi}$ since $\mathbb{R}^d_+$ is simplicial).*

*Proof.* We shall use Theorem 3.1 with $X = \mathbb{R}^d \otimes V$, $X^+ = \mathbb{R}^d_+ \otimes V^+$ and $Y = E \otimes V$. We have to show that for any $x \in \mathbb{R}^d \otimes V$, there is a $y \in E \otimes V$ such that $y - x \in \mathbb{R}^d_+ \otimes V^+$. It is enough to consider simple tensors of the form $x = r \otimes v$, where $r \in \mathbb{R}^d$ and $v \in V$; the general case will follow by linearity. Since $V^+$ is generating, there are $v_+$, $v_- \in V^+$ such that $v = v_+ - v_-$. Furthermore, from the assumption (1), $E$ contains a vector with strictly positive coordinates $e$, hence there exist $\lambda_\pm > 0$ such that $\lambda_+ e - r \geqslant 0$ and $\lambda_- e + r \geqslant 0$. Then,

$$\lambda_+ e \otimes v_+ + \lambda_- e \otimes v_- - r \otimes v = \lambda_+ e \otimes v_+ + \lambda_- e \otimes v_- - r \otimes v_+ + r \otimes v_- \in \mathbb{R}^d_+ \otimes V^+$$

and $\lambda_\pm e \otimes v_\pm \in E \otimes V$. Thus, we can choose $y = \lambda_+ e \otimes v_+ + \lambda_- e \otimes v_-$. □

We now provide a useful characterization of the maximal tensor product of $E^+$ with $V^+$, identifying at the same time the cone appearing in (2).

**Proposition 3.3.** *For $E, V$ as above, it holds that*

$$E^+ \otimes_{\max} V^+ = (E \otimes V) \cap (\mathbb{R}^d_+ \otimes V^+).$$

*Proof.* The inclusion "$\subseteq$" follows from the monotonicity of the max tensor product with respect to each factor. To show the reverse inclusion "$\supseteq$", we have to prove that for any $z \in (E \otimes V) \cap (\mathbb{R}^d_+ \otimes V^+)$, and for any $\varepsilon \in (E^+)^*$, $\alpha \in (V^+)^* = A^+$, we have that $\langle \varepsilon \otimes \alpha, z\rangle \geqslant 0$.

By Proposition 3.2, we can extend the positive form $\varepsilon : E \to \mathbb{R}$ to a positive form $\tilde{\varepsilon} : \mathbb{R}^d \to \mathbb{R}$. Indeed, one needs to apply Proposition 3.2 with $(V, V^+) = (\mathbb{R}, \mathbb{R}_+)$. Since $z \in \mathbb{R}^d_+ \otimes V^+$, we have a decomposition $z = \sum_{i=1}^d r_i \otimes v_i$, where $r_i \in \mathbb{R}^d_+$ and $v_i \in V^+$. This yields

$$\langle \varepsilon \otimes \alpha, z\rangle = \langle \tilde{\varepsilon} \otimes \alpha, z\rangle = \sum_{i=1}^d \tilde{\varepsilon}(r_i)\alpha(v_i) \geqslant 0,$$

finishing the proof. □

**Corollary 3.4.** *For $E_1$, $E_2$ satisfying the assumptions on $E$ (but not necessarily of the same dimension), it holds that*

$$E_1^+ \otimes_{\max} E_2^+ = (E_1 \otimes E_2) \cap (\mathbb{R}^{d_1}_+ \otimes \mathbb{R}^{d_2}_+).$$

**Remark 3.5.** *Using Proposition 3.3, one can restate Proposition 3.2 as follows: Any positive linear form $\varphi : (E \otimes V, E^+ \otimes_{\max} V^+) \to \mathbb{R}$ can be extended to a positive linear form $\tilde{\varphi} : \mathbb{R}^d \otimes V \to \mathbb{R}$.*

## 4. Generalized spectrahedra

4.1. **Generalized spectrahedra.** In this section, we will generalize some of the theory of (free) spectrahedra to the setting of ordered vector spaces, i.e. we will allow for more general cones than the cone of positive semidefinite matrices.

Recall that a spectrahedron is a convex set that can be represented by a linear matrix inequality, that is by positive semidefinite constraint. We define a generalized spectrahedron as a convex subset of some vector space which can be represented by positivity conditions with respect to some abstract cone.

**Definition 4.1.** *Let $V, A$ be two finite-dimensional vector spaces and consider a cone $C \subseteq A \otimes V$. For a $g$-tuple of elements $a = (a_1, \ldots, a_g) \in A^g$, we define the* generalized spectrahedron

$$\mathcal{D}_a(V, C) := \{(v_1, \ldots, v_g) \in V^g \; : \; \sum_{i=1}^g a_i \otimes v_i \in C\}.$$

**Remark 4.2.** *It is easy to see that any generalized spectrahedron is a cone.*

Note that usual spectrahedra correspond to the choice $V = \mathbb{R}$, $A = M_n(\mathbb{C})$, and $C$ being the positive semidefinite cone. Free spectrahedra are the union over $d \geqslant 1$ of generalized spectrahedra for $V = M_d(\mathbb{C})$ and $C$ begin the PSD cone of $dn \times dn$ matrices.

As for free spectrahedra, we can connect the inclusion of generalized spectrahedra to the positivity of an associated map [HKM13, HKMS14, DDOSS17].

**Proposition 4.3.** *Let $V$, $A$ and $B$ be finite-dimensional vector spaces and let $C_A \subset A \otimes V$, $C_B \subset B \otimes V$ be two cones. Moreover, let $a \in A^g$, $b \in B^g$ be two tuples, where $a$ consists of linearly independent elements which span the subspace $A' \subseteq A$. We define a map $\Phi : A' \to B$, $\Phi(a_i) = b_i$ for all $i \in [g]$. Then,*

$$\mathcal{D}_a(V, C_A) \subseteq \mathcal{D}_b(V, C_B)$$

*if and only if $\Phi \otimes \mathrm{id}_V : (A' \otimes V, A' \otimes V \cap C_A) \to (B \otimes V, C_B)$ is positive.*

*Proof.* Let us assume the inclusion. Consider $z \in A' \otimes V$. Thus, we can write

$$z = \sum_{i=1}^g a_i \otimes z_i,$$

where $z_i \in V$ for all $i \in [g]$. If $z \in C_A$, then $(z_1, \ldots, z_g) \in \mathcal{D}_a(V, C_A)$ and hence

$$(\Phi \otimes \mathrm{id}_V)(z) = \sum_{i=1}^{g} b_i \otimes z_i.$$

The right hand side is in $C_B$ since $\mathcal{D}_a(V, C_A) \subseteq \mathcal{D}_b(V, C_B)$. Conversely, let $\Phi \otimes \mathrm{id}_V$ be positive. Let $(v_1, \ldots, v_g) \in \mathcal{D}_a(V, C_A)$. Then,

$$\sum_{i=1}^{g} a_i \otimes v_i \in (A' \otimes V) \cap C_A$$

and the assertion follows from an application of $\Phi \otimes \mathrm{id}_V$ to this element. □

As for free spectrahedra, it is possible to look at what inclusion for some $(V, V^+)$ implies for the inclusion with $(\mathbb{R}, \mathbb{R}_+)$.

**Proposition 4.4.** *Let $(A, A^+)$, $(B, B^+)$ and $(V, V^+)$ be ordered vector spaces where $B^+$ is closed and $V^+$ contains at least one non-zero element. Moreover, let $C_A \subset A \otimes V$ and $C_B \subset B \otimes V$ be tensor cones and $a \in A^g$, $b \in B^g$ for some $g \in \mathbb{N}$. Then,*

$$\mathcal{D}_a(V, C_A) \subseteq \mathcal{D}_b(V, C_B) \implies \mathcal{D}_a(\mathbb{R}, A^+) \subseteq \mathcal{D}_b(\mathbb{R}, B^+)$$

*Proof.* Let $v \in V^+$, $v \neq 0$ and $x \in \mathcal{D}_a(V, C_A)$. Then, $(x_1 v, \ldots, x_g v) \in \mathcal{D}_a(V, C_A)$, since $C_A$ contains in particular $A^+ \otimes_{\min} V^+$. Thus,

$$\left(\sum_{i=1}^{g} x_i b_i\right) \otimes v \in C_B.$$

This implies that $\sum_{i=1}^{g} x_i b_i \in (B^+)^{**} \cong B^+$ and hence $x \in \mathcal{D}_b(\mathbb{R}, B^+)$. □

In general, $\mathcal{D}_a(\mathbb{R}, A^+) \subseteq \mathcal{D}_b(\mathbb{R}, B^+)$ does not imply $\mathcal{D}_a(V, C_A) \subseteq \mathcal{D}_b(V, C_B)$. However, if $A$ and $B$ contain an order unit, the implication can be made true by shrinking the left hand side.

**Definition 4.5.** *Let $(A, A^+)$, $(B, B^+)$ and $(V, V^+)$ be ordered vector spaces where $A^+$ and $B^+$ contain order units $\mathbb{1}_A$ and $\mathbb{1}_B$, respectively. Moreover, let $C_A \subset A \otimes V$ and $C_B \subset B \otimes V$ be tensor cones and $a \in A^g$, $b \in B^g$ for some $g \in \mathbb{N}$. The* set of inclusion constants for $\mathcal{D}_a(V, C_A)$ and $C_B$ *is defined as*

$$\begin{aligned}\Delta_a(V, C_A, C_B) :=& \{s \in [0,1]^g : \forall b \in B^g,\ \mathcal{D}_{(\mathbb{1}_A, a)}(\mathbb{R}, A^+) \subseteq \mathcal{D}_{(\mathbb{1}_B, b)}(\mathbb{R}, B^+) \\ &\implies (1, s) \cdot \mathcal{D}_{(\mathbb{1}_A, a)}(V, C_A) \subseteq \mathcal{D}_{(\mathbb{1}_B, b)}(V, C_B)\}.\end{aligned}$$

*Here, $(1, s) \cdot \mathcal{D}_{(\mathbb{1}_A, a)}(V, C_A) := \{(v_0, s_1 v_1, \ldots, s_g v_g) : v \in \mathcal{D}_{(\mathbb{1}_A, a)}(V, C_A)\}$.*

**Question 4.6.** *Can we impose some symmetry constraints here to show $\{s : \sum_i s_i \leq 1\} \subseteq \Delta_a(V, C_A, C_B)$?*

The following definition is motivated by the matrix range introduced in [Arv72] and generalized in [DDOSS17].

**Definition 4.7.** *Let $(A, A^+)$ be an ordered vector space with order unit $\mathbb{1}_A$ and let $a \in A^g$. Then,*

$$\mathcal{W}(a) = \{(\varphi(a_1), \ldots, \varphi(a_g)) : \varphi \in (A^+)^*, \varphi(\mathbb{1}_A) = 1\}.$$

**Proposition 4.8.** *Let $(A, A^+)$ be an ordered vector space with order unit $\mathbb{1}_A$ and let $a \in A^g$. Then, $\mathcal{W}(a)$ is compact and convex.*

*Proof.* For any $a_i$, $i \in [g]$, there is a $t_i \geq 0$ such that $a_i \in t_i[-\mathbb{1}_A, \mathbb{1}_A]$. Thus, $|\varphi(a_i)| \leq t_i$ for all $\varphi \in (A^+)^*, \varphi(\mathbb{1}_A) = 1$ and $\mathcal{W}(a)$ follows. Let $a^{(n)}$ be a sequence in $\mathcal{W}(a)$ converging to $x$. With any $a^{(n)}$, we can associate an $\varphi_n \in (A^+)^*, \varphi_n(\mathbb{1}_A) = 1$. Since $(A^+)^*$ is closed by definition, there is a map $\varphi \in (A^+)^*, \varphi(\mathbb{1}_A) = 1$ such that $x = (\varphi(a_1), \ldots, \varphi(a_n))$ and $x \in \mathcal{W}(a)$. Convexity follows from the fact that the set of $\varphi$ as in the statement is convex. □

**Proposition 4.9.** *Let $(A, A^+)$ be an ordered vector space with order unit $\mathbb{1}_A$ and let $A^+$ be closed. Furthermore, let $a \in A^g$. Let $\mathcal{C}_a := \{x \in \mathbb{R}^g : (1, -x) \in \mathcal{D}_{(\mathbb{1}_A, a)}(\mathbb{R}, A^+)$. Then, $\mathcal{W}(a)^\circ = \mathcal{C}_a$. If $0 \in \mathcal{W}(a)$, then $\mathcal{C}_a^\circ = \mathcal{W}(a)$.*

*Proof.* Since $A^+$ is closed,

$$\mathbb{1} + \sum_i x_i a_i \in A^+ \iff 1 + \sum_i x_i \varphi(a_i) \geqslant 0 \qquad \forall \varphi \in (A^+)^*, \varphi(\mathbb{1}_A) = 1.$$

Note that the only map in $(A^+)^*$ with $\varphi(\mathbb{1}_A) = 0$ is the constant map, since $\mathbb{1}_A$ is an order unit. This proves the first assertion. The second assertion follows from the bipolar theorem since $\mathcal{W}(a)$ is closed. □

**Proposition 4.10.** *Let $(A, A^+)$ be an ordered vector space with order unit $\mathbb{1}_A$ and let $A^+$ be closed. Furthermore, let $a \in A^g$. Then, $\mathcal{C}_a$ is bounded if and only if $0 \in \operatorname{int} \mathcal{W}(a)$.*

*Proof.* This follows from the fact that for convex sets $K \subseteq \mathbb{R}^n$, $K^\circ$ is bounded if and only if $0 \in \operatorname{int} K$ [AS17, Exercise 1.14], combined with Proposition 4.9. □

**Proposition 4.11.** *Let $(A, A^+)$ and $(B, B^+)$ be two ordered vector spaces with closed cones containing order units $\mathbb{1}_A$ and $\mathbb{1}_B$. Let $a \in A^k$ and $b \in B^l$ be such that $\mathcal{C}_a$, $\mathcal{C}_b$ are polytopes for $k$, $l \in \mathbb{N}$. Then, for any closed tensor cone $C_{AB}$,*

$$\mathcal{D}_{(\mathbb{1}_A \otimes \mathbb{1}_B, a \otimes \mathbb{1}_B, \mathbb{1}_A \otimes b)}(\mathbb{R}, C_{AB}) = \mathbb{R}_+\{(1, -z) : z \in \mathcal{C}_a \oplus \mathcal{C}_b\}.$$

*Proof.* Let $-x \in C_a$, $-y \in \mathcal{C}_b$. Then, $(1, x, 0) \in \mathcal{D}_{(\mathbb{1}_A \otimes \mathbb{1}_B, a \otimes \mathbb{1}_B, \mathbb{1}_A \otimes b)}(\mathbb{R}, C_{AB})$, because $(\mathbb{1}_A + \sum_i x_i a_i) \otimes \mathbb{1}_B \in \mathcal{A}^+ \otimes_{\min} B^+$, and likewise $(1, 0, y) \in \mathcal{D}_{(\mathbb{1}_A \otimes \mathbb{1}_B, a \otimes \mathbb{1}_B, \mathbb{1}_A \otimes b)}(\mathbb{R}, C_{AB})$, such that "$\supset$" holds in the assertion. Conversely, let $(c, x, y) \in \mathcal{D}_{(\mathbb{1}_A \otimes \mathbb{1}_B, a \otimes \mathbb{1}_B, \mathbb{1}_A \otimes b)}(\mathbb{R}, C_{AB})$, where $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^l$. Then,

$$c\mathbb{1}_A \otimes \mathbb{1}_B + (\sum_i x_i a_i) \otimes \mathbb{1}_B + \mathbb{1}_A \otimes (\sum_j y_j b_j) \in C_{AB}. \tag{3}$$

Boundedness of $\mathcal{C}_a$ and $\mathcal{C}_b$ implies by Proposition 4.10 that $0 \in \operatorname{int} \mathcal{W}(a)$, $0 \in \operatorname{int} \mathcal{W}(b)$. Let $\varphi$, $\psi$ be the maps which send $a$ and $b$ to zero, respectively. Then, $\varphi \otimes \varphi \in (C_{AB})^*$. An application of this map to Equation (3) implies $c > 0$. Thus, we can set $c = 1$ without loss of generality. Let now for $\varphi' \in (A^+)^*$, $\psi' \in (B^+)^*$, $\varphi'(\mathbb{1}_A) = 1 = \psi'(\mathbb{1}_B)$. An application of $\varphi' \otimes \psi'$ to Equation (3) implies $\mathcal{C}^* \supset W(a) \times \mathcal{W}(b)$, where $\mathcal{C} := \{z : (1, -z) \in \mathcal{D}_{(\mathbb{1}_A \otimes \mathbb{1}_B, a \otimes \mathbb{1}_B, \mathbb{1}_A \otimes b)}(\mathbb{R}, C_{AB})\}^\circ$. Since $\mathcal{C}$ is closed and contains 0, we obtain $\mathcal{C} \subset (\mathcal{W}(a) \times \mathcal{W}(b))^\circ = (\mathcal{C}_a^\circ \times \mathcal{C}_b^\circ)^\circ = \mathcal{C}_a \oplus \mathcal{C}_b$ with Proposition 4.9. □

**Corollary 4.12.** *Let $(A, A^+)$ and $(B, B^+)$ be two ordered vector spaces with closed cones containing order units $\mathbb{1}_A$ and $\mathbb{1}_B$. Let $a \in A^k$ and $b \in B^l$ be such that $\mathcal{C}_a$, $\mathcal{C}_b$ are polytopes for $k$, $l \in \mathbb{N}$. Let moreover $(F, F^+)$ be another ordered vector space with order unit $\mathbb{1}_F$ and $f_1 \in F^k$, $f_2 \in F^l$. Then, for any closed tensor cone $C_{AB}$,*

$$\mathcal{D}_{(\mathbb{1}_A \otimes \mathbb{1}_B, a \otimes \mathbb{1}_B, \mathbb{1}_A \otimes b)}(\mathbb{R}, C_{AB}) \subseteq \mathcal{D}_{(\mathbb{1}_F, f_1, f_2)}(\mathbb{R}, F^+)$$

*if and only if*

$$\mathcal{D}_{(\mathbb{1}_A, a)}(\mathbb{R}, A^+) \subseteq \mathcal{D}_{(\mathbb{1}_F, f_1)}(\mathbb{R}, F^+) \quad \wedge \quad \mathcal{D}_{(\mathbb{1}_B, b)}(\mathbb{R}, B^+) \subseteq \mathcal{D}_{(\mathbb{1}_F, f_2)}(\mathbb{R}, F^+).$$

4.2. **The jewel for ordered vector spaces.** In this section, we define the universal free spectrahedra we will consider to relate the compatibility of measurements in GPTs to inclusion problems of generalized spectrahedra.

Let us consider the vectors $v_1^{(k)}, \ldots, v_{k-1}^{(k)} \in \mathbb{R}^k$ defined as

$$v_j^{(k)}(\varepsilon) := -\frac{2}{k} + 2\delta_{\varepsilon,j}, \qquad \forall j \in [k-1], \forall \varepsilon \in [k].$$

**Definition 4.13.** *Let $g \in \mathbb{N}$ and $\mathbf{k} \in \mathbb{N}^g$. Then, for $i \in [g]$ and $j \in [k_i - 1]$, let us define*

$$w_j^{(i)} := \underbrace{1^{\times k_1} \otimes \cdots \otimes 1^{\times k_{i-1}}}_{i-1 \text{ factors}} \otimes v_j^{(k_i)} \otimes \underbrace{1^{\times k_{i+1}} \otimes \cdots \otimes 1^{\times k_g}}_{g-i \text{ times}} \in \mathbb{R}^{k_1 \cdots k_g}$$

*and $\mathbb{1}_E = 1^{\times k_1} \otimes \ldots \otimes 1^{\times k_g} = 1^{\times k_1 \cdots k_g}$, where by $1^{\times n} \in \mathbb{R}^n$ we denote the all-ones vector. We denote by $E_{\mathbf{k}} \subseteq \mathbb{R}^{k_1 \cdots k_g}$ the range of these vectors,*

$$E_{\mathbf{k}} = \operatorname{span}\{\mathbb{1}_E, w_j^{(i)} : i \in [g], j \in [k_i - 1]\} \tag{4}$$

*and write $w$ for the vector containing $\mathbb{1}_E$ and the $w_j^{(i)}$. We also define the cone $E_{\mathbf{k}}^+ := E_{\mathbf{k}} \cap \mathbb{R}_+^{k_1 \cdots k_g}$. Then, the $(\mathbf{k}; V, V^+)$-jewel is defined as*

$$\mathcal{D}_{\mathrm{GPT}\heartsuit}(\mathbf{k}; V, V^+) := \mathcal{D}_w(V, E_{\mathbf{k}}^+ \otimes_{\max} V^+).$$

For $k_1 = \ldots = k_g = 2$, the above has an especially easy form and we write

$$\mathcal{D}_{\mathrm{GPT}\diamondsuit}(g; V, V^+) := \mathcal{D}_{\mathrm{GPT}\heartsuit}(2^{\times g}; V, V^+)$$

for short. In this case, $w_1^{(i)} = c_i$ for all $i \in [g]$, with $c_i = (1,1) \otimes \cdots \otimes (1,1) \otimes (+1,-1) \otimes (1,1) \otimes \cdots \otimes (1,1) \in \mathbb{R}^{2^g}$. We call this object the $(\mathbf{g}; V, V^+)$-diamond.

## 5. Inclusion of generalized spectrahedra and compatibility of measurements in GPTs

We relate here the problems of spectrahedral inclusion (for cones) and compatibility of effects in GPTs. We shall consider a GPT defined by dual ordered cones $(V, V^+)$ and $(A, A^+)$ as in the previous section, with $A^+$ containing a distinguished element $\mathbb{1}_A$. Below, $g$ will be a positive integer that corresponds to the number of measurements we want to assess.

**Lemma 5.1.** *Let $(L, L^+)$ be an ordered vector space, with $L^+$ proper. Moreover, let $\mathbf{k} \in \mathbb{N}^g$. $z_0$, $z_j^{(i)} \in L$, where $i \in [g]$, $j \in [k_i - 1]$, and $E_{\mathbf{k}}$ as in Eq. (4). Then*

$$z_0 + \sum_{i=1}^{g} \sum_{j=1}^{k_i - 1} w_j^{(i)}(\varepsilon) z_j^{(i)} \in L^+ \qquad \forall \varepsilon \in [k_1] \times \ldots \times [k_g] \tag{5}$$

*if and only if*

$$\mathbb{1}_E \otimes z_0 + \sum_{i=1}^{g} \sum_{j=1}^{k_i - 1} w_j^{(i)} \otimes z_j^{(i)} \in E_{\mathbf{k}}^+ \otimes_{\max} L^+.$$

*In particular, for $\mathbf{k} = 2^{\times g}$, eq. (5) has the form*

$$z_0 + \sum_{i=1}^{g} \varepsilon_i z_i \in L^+ \qquad \forall \varepsilon \in \{\pm 1\}^g.$$

*Proof.* To begin, fix $\{e_\varepsilon\}_{\varepsilon \in [k_1] \times \ldots \times [k_g]}$, the standard basis of $\mathbb{R}^{k_1 \cdot \ldots \cdot k_g}$. One has the following decompositions

$$\forall i \in [g], j \in [k_i - 1], \qquad w_j^{(i)} = \sum_{\varepsilon \in [k_1] \times \ldots \times [k_g]} w_j^{(i)}(\varepsilon) e_\varepsilon.$$

Let $\{\delta_\varepsilon\}_{\varepsilon \in [k_1] \times \ldots \times [k_g]}$ be the standard basis of $(\mathbb{R}^{k_1 \cdots k_g})^*$, dual to the basis $e_\varepsilon$. Then, $\delta_\varepsilon$ is positive. Let $z_0, z_j^{(i)} \in L$, $i \in [g]$, $j \in [k_i - 1]$ such that

$$z_0 + \sum_{i=1}^{g} \sum_{j=1}^{k_i - 1} w_j^{(i)}(\varepsilon) z_j^{(i)} \in L^+ \qquad \forall \varepsilon \in [k_1] \times \ldots \times [k_g].$$

Let furthermore

$$y := \mathbb{1}_E \otimes z_0 + \sum_{i=1}^{g} \sum_{j=1}^{k_i - 1} w_j^{(i)} \otimes z_j^{(i)} = \sum_{\varepsilon \in [k_1] \times \ldots \times [k_g]} e_\varepsilon \otimes (z_0 + \sum_{i=1}^{g} \sum_{j=1}^{k_i - 1} w_j^{(i)}(\varepsilon) z_j^{(i)}).$$

Then, for all $\beta \in (L^+)^*$ and $\varepsilon \in [k_1] \times \ldots \times [k_g]$,

$$\langle \delta_\varepsilon \otimes \beta, y \rangle = \langle \beta, z_0 + \sum_{i=1}^{g} \sum_{j=1}^{k_i - 1} w_j^{(i)}(\varepsilon) z_j^{(i)} \rangle \geqslant 0.$$

As any $\alpha \in (\mathbb{R}_+^{k_1 \cdots k_g})^*$ can be written as $\alpha = \sum_\varepsilon \alpha_\varepsilon \delta_\varepsilon$ with $\alpha_\varepsilon \geqslant 0$ for all $\varepsilon \in [k_1] \times \ldots \times [k_g]$, using Proposition 3.3 we have shown that $y \in E_{\mathbf{k}}^+ \otimes_{\max} L^+$.

Now let $y \in E_{\mathbf{k}}^+ \otimes_{\max} L^+$. Then, again, for any $\beta \in (L^+)^*$

$$0 \leqslant \langle \delta_\varepsilon \otimes \beta, y \rangle = \langle \beta, z_0 + \sum_{i=1}^{g} \sum_{j=1}^{k_i - 1} w_j^{(i)}(\varepsilon) z_j^{(i)} \rangle.$$

Therefore, $z_0 + \sum_{i=1}^{g} \sum_{j=1}^{k_i-1} w_j^{(i)}(\varepsilon) z_j^{(i)} \in ((L^+)^*)^*$ for all $\varepsilon \in [k_1] \times \ldots \times [k_g]$. Since $L$ is finite dimensional and $L^+$ closed, we have that $((L^+)^*)^* = L^+$. □

**Remark 5.2.** *The above lemma yields a more appealing form of the $(L, L^+)$-diamond, namely*

$$\mathcal{D}_{\mathrm{GPT}\diamondsuit}(g; L, L^+) = \left\{ (z_0, \ldots, z_g) \in L^{g+1} : \forall \varepsilon \in \{\pm 1\}^g,\ z_0 + \sum_{i=1}^{g} \varepsilon_i z_i \in L^+ \right\}.$$

**Definition 5.3.** *We define the following generalized spectrahedron: for the ordered vector space $(L, L^+)$ and $\mathbf{k} \in \mathbb{N}^g$, $f_j^{(i)} \in A$, $i \in [g]$, $j \in [k_i - 1]$,*

$$\mathcal{D}_f(\mathbf{k}; L, L^+) := \left\{ (z_0, z_j^{(i)})_{ij} \in L : \mathbb{1} \otimes z_0 + \sum_{i=1}^{g} \sum_{j=1}^{k_i - 1} \left( 2 f_j^{(i)} - \frac{2}{k_i} \mathbb{1} \right) \otimes z_j^{(i)} \in A^+ \otimes_{\min} L^+ \right\}. \tag{6}$$

*In the case where $\mathbf{k} = 2^{\times g}$, we will write $\mathcal{D}_f(g; L, L^+)$ for simplicity.*

**Proposition 5.4.** *Let $\mathbf{k} \in \mathbb{N}^g$ and let $(L, L^+)$ be an ordered vector space. Then, $\mathcal{D}_{\mathrm{GPT}\diamondsuit}(\mathbf{k}; L, L^+) \subseteq \mathcal{D}_f(\mathbf{k}; L, L^+)$ if and only if $\Phi^{(f)} \otimes \mathrm{id}_L : (E_{\mathbf{k}} \otimes L, E_{\mathbf{k}}^+ \otimes_{\max} L^+) \to (A \otimes L, A^+ \otimes_{\min} L^+)$ is positive, where $\Phi^{(f)}$ is defined as*

$$\begin{aligned} \Phi^{(f)} : E_{\mathbf{k}} &\to A \\ \mathbb{1}_E &\mapsto \mathbb{1} \\ w_j^{(i)} &\mapsto 2 f_j^{(i)} - \frac{2}{k_i} \mathbb{1}. \end{aligned} \tag{7}$$

*Proof.* This follows directly from Propositions 3.3 and 4.3. □

**Proposition 5.5.** *Let $\mathbf{k} \in \mathbb{N}^g$ and let $(V, V^+, \mathbb{1})$ be a GPT. Then, $\mathcal{D}_{\mathrm{GPT}\diamondsuit}(\mathbf{k}; \mathbb{R}, \mathbb{R}_+) \subseteq \mathcal{D}_f(\mathbf{k}; \mathbb{R}, \mathbb{R}_+)$ if and only if $\{f_1^{(i)}, \ldots, f_{k_i}^{(i)}\}$ form measurements for all $i \in [g]$, where $f_{k_i}^{(i)} := \mathbb{1} - \sum_{j=1}^{k_i - 1} f_j^{(i)}$.*

*Proof.* Corollaries 3.4 and 4.12 yield that the inclusion is true if and only if

$$D_{(\mathbb{1}_{E_i}, w^{(i)})}(\mathbb{R}, E_i^+) \subseteq \mathcal{D}_{(\mathbb{1}, f^{(i)})}(\mathbb{R}, A^+) \qquad \forall i \in [g].$$

From [BN18a, Lemma 4.3], it follows that the extreme rays of $D_{(\mathbb{1}_{E_i}, w^{(i)})}(\mathbb{R}, E_i^+)$ are

$$\mathbb{R}_+(1, \frac{k_i}{2} e_j) \quad \forall j \in [k_i - 1], \qquad \mathbb{R}_+(1, -\frac{k_i}{2}(1, \ldots 1)).$$

Here, $\{e_j\}_{j\in[k_i-1]}$ is the standard basis in $\mathbb{R}^{k_i-1}$. Thus, the inclusion is equivalent to

$$k_i f_j^{(i)} \in A^+ \quad \forall j \in [k_i - 1] \quad \wedge \quad k_i \mathbb{1} - \sum_{j=1}^{k_i-1} k_i f_j^{(i)} \in A^+$$

for all $i \in [g]$. Dividing by $k_i$ proves the assertion. □

Now we show that inclusion with $L = V$ is equivalent to compatibility of the $g$ measurements $\{f^{(i)}\}$ in the GPT $(V, V^+, \mathbb{1})$. The proof technique is inspired by the finite dimensional version of Arveson's extension theorem [Pau03, Theorem 6.2]. Consider the identity map $\mathrm{id} : V \to V$ and the associated canonical evaluation tensor $\chi \in V \otimes A$, where $A = V^*$. Using coordinates, we have

$$\chi = \sum_{i=1}^{\dim V} v_i \otimes a_i \in V \otimes A \cong (A \otimes V)^* \tag{8}$$

for $v_i$ a basis of $V$ and $a_i$ the corresponding dual basis in $A$ (we have $a_i(v_j) = \delta_{ij}$). The tensor $\chi$ has the following remarkable property:

$$\forall v \in V, \forall \alpha \in A, \qquad \langle \chi, \alpha \otimes v\rangle = \alpha(v). \tag{9}$$

**Lemma 5.6.** *The identity map* $\mathrm{id} : (V, V^+) \to (V, V^+)$ *is positive. Equivalently,* $\chi \in V^+ \otimes_{\max} A^+$*.*

*Proof.* The first assertion is obvious. The second one follows from Equation (9). □

**Proposition 5.7.** *Let* $E \subseteq \mathbb{R}^d$ *be a subspace such that* $E \cap \mathrm{ri}(\mathbb{R}^d_+) \neq \varnothing$ *and* $E^+ = E \cap \mathbb{R}^d_+$*. Let* $(V, V^+)$ *be an ordered vector space and let* $V^+$ *be generating. Finally, let* $\Phi : E \to A$ *be a linear map. The following are equivalent:*

(1) *The linear map* $\Phi \otimes \mathrm{id}_V : (E \otimes V, E^+ \otimes_{\max} V^+) \to (A \otimes V, A^+ \otimes_{\min} V^+)$ *is positive.*
(2) *There exists a positive extension* $\tilde{\Phi} : (\mathbb{R}^d, \mathbb{R}^d_+) \to (A, A^+)$ *of* $\Phi$*.*
(3) *The form*

$$\begin{aligned} s_\Phi : (E \otimes V, E^+ \otimes_{\max} V^+) &\to \mathbb{R} \\ z &\mapsto \langle \chi, [\Phi \otimes \mathrm{id}_V](z)\rangle \end{aligned}$$

*is positive.*

*Proof.* We start by showing that the existence of the positive extension (2) implies the positivity of the map $\Phi \otimes \mathrm{id}_V$ (1). Both $\tilde{\Phi}$ and $\mathrm{id}_V$ are positive maps. Therefore, $\tilde{\Phi} \otimes \mathrm{id}_V : (\mathbb{R}^d \otimes V, \mathbb{R}^d_+ \otimes_{\min} V^+ = \mathbb{R}^d_+ \otimes_{\max} V^+) \to (A \otimes V, A^+ \otimes_{\min} V^+)$ is positive. The claim follows since $\Phi \otimes \mathrm{id}_V$ is a restriction of this map to $E \otimes V$ by Proposition 3.3.

We now prove the implication (1) $\implies$ (3). Indeed, let $z \in E^+ \otimes_{\max} V^+$ and

$$A^+ \otimes_{\min} V^+ \ni [\Phi \otimes \mathrm{id}_V](z) = \sum_i \alpha_i \otimes x_i,$$

for some $\alpha_i \in A^+$ and $x_i \in V^+$. Then, $\langle \chi, [\Phi \otimes \mathrm{id}_V](z)\rangle = \sum_i \alpha_i(x_i) \geqslant 0$, proving the implication.

It remains to show that (3) $\implies$ (2). Using Proposition 3.2 and Remark 3.5, we extend the form $s_\Phi$ to $\tilde{s}_\Phi : \mathbb{R}^d \otimes V \to \mathbb{R}$. We now go back to linear maps by defining

$$\begin{aligned} \tilde{\Phi} : (\mathbb{R}^d, \mathbb{R}^d_+) &\to (A, A^+) \\ r &\mapsto \sum_{i=1}^{\dim V} \tilde{s}_\Phi(r \otimes v_i) a_i, \end{aligned}$$

where the dual bases $\{v_i\}$, $\{a_i\}$ are the ones from (8). One can directly check that the definition above is tailored to have the following relation:

$$\forall (r, v) \in \mathbb{R}^d \times V, \qquad \langle \tilde{\Phi}(r), v\rangle = \tilde{s}_\Phi(r \otimes v).$$

The positivity of the form $\tilde{s}_\Phi$ implies the positivity of $\tilde{\Phi}$, as $\langle \tilde{\Phi}(r), v\rangle \geqslant 0$ for all $r \in \mathbb{R}^d_+$ and all $v \in V^+$. To conclude, it remains to check that $\tilde{\Phi}$ is indeed an extension of $\Phi$. For any $e \in E$, we compute

$$\begin{aligned}\tilde{\Phi}(e) &= \sum_{i=1}^{\dim V} \tilde{s}_\Phi(e \otimes v_i) a_i = \sum_{i=1}^{\dim V} s_\Phi(e \otimes v_i) a_i \\ &= \sum_{i=1}^{\dim V} \langle \chi, [\Phi \otimes \mathrm{id}_V](e \otimes v_i)\rangle a_i = \sum_{i=1}^{\dim V} \langle \chi, \Phi(e) \otimes v_i\rangle a_i \\ &= \sum_{i=1}^{\dim V} [\Phi(e)](v_i) a_i = \Phi(e),\end{aligned}$$

finishing the proof. □

**Theorem 5.8.** *Let $\mathbf{k} \in \mathbb{N}^g$ and let $(V, V^+, \mathbb{1})$ be a GPT. Then, $\mathcal{D}_{\mathrm{GPT}\heartsuit}(\mathbf{k}; V, V^+) \subseteq \mathcal{D}_f(\mathbf{k}; V, V^+)$ if and only if $\{f^{(i)}_1, \ldots, f^{(i)}_{k_i}\}_{i \in [g]}$ are compatible measurements, where $f^{(i)}_{k_i} := \mathbb{1} - \sum_{j=1}^{k_i - 1} f^{(i)}_j$ for all $i \in [g]$.*

*Proof.* By Propositions 5.4 and 5.7, the inclusion holds if and only if the corresponding map $\Phi^{(f)}$ has a positive extension $\tilde{\Phi}$ to $\mathbb{R}^{k_1 \cdots k_g}$. Let $\varepsilon \in [\mathbf{k}] := [k_1] \times \ldots [k_g]$. Then,

$$w^{(i)}_j(\varepsilon) = -\frac{2}{k_i} + 2\delta_{\varepsilon(i), j}. \tag{10}$$

Let $g_\eta \in \mathbb{R}^{k_1 \cdots k_g}$, $\eta \in [\mathbf{k}]$ such that $g_\eta(\varepsilon) = \delta_{\varepsilon, \eta}$. These vectors form a basis of $\mathbb{R}^{k_1 \cdots k_g}$. Thus, we can rewrite Equation (10) as

$$w^{(i)}_j(\varepsilon) = -\frac{2}{k_i} \sum_{\eta \in [\mathbf{k}]} g_\eta(\varepsilon) + 2 \sum_{\substack{\eta \in [\mathbf{k}] \\ \eta(i) = j}} g_\eta(\varepsilon). \tag{11}$$

Let $\tilde{\Phi}$ be any extension of $\Phi$ and let $G_\eta := \tilde{\Phi}(g_\eta)$. For $\tilde{\Phi}$ to be chosen to be positive, we need $G_\eta \in A^+$. Since

$$\mathbb{1} = \tilde{\Phi}(\mathbb{1}_E) = \sum_{\eta \in [\mathbf{k}]} \tilde{\Phi}(g_\eta) = \sum_\eta G_\eta,$$

equation (11) implies that

$$2f^{(i)}_j - \frac{2}{k_i}\mathbb{1} = \tilde{\Phi}(w^{(i)}_j) = 2 \sum_{\substack{\eta \in [\mathbf{k}] \\ \eta(i) = j}} G_\eta - \frac{2}{k_i}\mathbb{1}$$

for $j \in [k_i - 1]$, $i \in [g]$. By definition

$$f^{(i)}_{k_i} = \mathbb{1} - f^{(i)}_1 - \ldots - f^{(i)}_{k_i - 1} = \sum_{\substack{\eta \in [\mathbf{k}] \\ \eta(i) = k_i}} G_\eta.$$

Thus, $\tilde{\Phi}$ is a positive extension of $\Phi$ if and only if

$$\begin{aligned} G_\eta &\in A^+ \qquad \forall \eta \in [\mathbf{k}], \\ \sum_{\eta \in [\mathbf{k}]} G_\eta &= 1, \\ f^{(i)}_j &= \sum_{\substack{\eta \in [\mathbf{k}] \\ \eta(i) = j}} G_\eta \qquad \forall j \in [k_i], i \in [g]. \end{aligned}$$

These conditions are met if and only if $\{G_\eta\}_{\eta\in[\mathbf{k}]}$ is a joint measurement for $\{f_1^{(i)}, \ldots, f_{k_i}^{(i)}\}_{i\in[g]}$. □

**Corollary 5.9.** *Let $(V, V^+, \mathbb{1})$ be a GPT. Elements $\{f_1^{(i)}, \ldots, f_{k_i}^{(i)}\}$ form compatible measurements for all $i \in [g]$ if and only if*

$$\forall z \in \mathcal{D}_{\mathrm{GPT}\diamondsuit}(\mathbf{k}; V, V^+), \qquad \mathbb{1}(z_0) \geqslant \sum_{i=1}^{g} \sum_{j=1}^{k_i-1} \langle 2f_1^{(i)} - \frac{2}{k_i}\mathbb{1}, z_j^{(i)}\rangle. \tag{12}$$

*Proof.* The equation above is equivalent to the positivity of the linear form $s_\Phi$ from Proposition 5.7. □

**Remark 5.10.** *Note that the condition in (12) is much weaker than the requirement from Theorem 5.8, which reads*

$$\mathbb{1} \otimes z_0 + \sum_{i=1}^{g} \sum_{j=1}^{k_i-1} \left[2f_1^{(i)} - \frac{2}{k_i}\right] \mathbb{1} \otimes z_j^{(i)} \in A^+ \otimes_{\min} V^+.$$

*Indeed, the condition in (12) is the evaluation of the tensor above against the tensor $\chi$ from (8).*

We can now relate the notion of compatibility to the inclusion constants defined in Section 4.

**Definition 5.11.** *Given a GPT $(V, V^+, \mathbb{1})$, $g \in \mathbb{N}$ and $\mathbf{k} \in \mathbb{N}^g$, we define the* compatibility region for $\mathbf{k}$ outcomes *as*

$$\Gamma(\mathbf{k}; V, V^+) := \{s \in [0,1]^g \, : \, s_i f^{(i)} + (1 - s_i)\mathbb{1}/k_i \text{ are compatible measurements for all collections } f^{(i)} \in A^{k_i}, i \in [g], \text{ of } g \text{ measurements with } \mathbf{k} \text{ outcomes}\}.$$

*If $\mathbf{k} = 2^{\times g}$, we will just write $\Gamma(g; V, V^+)$.*

The definition above corresponds to the *noise robustness of incompatibility* with respect to *white noise*, see [HKR15].

The following is a restriction of the inclusion constants in Definition 4.5, where we require the coefficients by which we scale to be the same on some elements.

**Definition 5.12.** *Given a GPT $(V, V^+, \mathbb{1})$, $g \in \mathbb{N}$ and $\mathbf{k} \in \mathbb{N}^g$, we define the set of* inclusion constants for the $(\mathbf{k}; V, V^+)$-jewel *as*

$$\Delta(\mathbf{k}; V, V^+) := \{s \in [0,1]^g \, : \, \forall f_j^{(i)} \in A, \; j \in [k_i - 1], \; i \in [g], \; \mathcal{D}_{\mathrm{GPT}\diamondsuit}(\mathbf{k}; \mathbb{R}, \mathbb{R}^+) \subseteq \mathcal{D}_f(\mathbf{k}; \mathbb{R}, \mathbb{R}^+) \implies (1, s_1^{\times(k_1-1)}, \ldots, s_g^{\times(k_g-1)}) \cdot \mathcal{D}_{\mathrm{GPT}\diamondsuit}(\mathbf{k}; V, V^+) \subseteq \mathcal{D}_f(\mathbf{k}; V, V^+)\}.$$

*If $\mathbf{k} = 2^{\times g}$, we will just write $\Delta(g; V, V^+)$.*

Now, we can prove that the two previous definitions actually coincide.

**Theorem 5.13.** *We have $\Gamma(\mathbf{k}; V, V^+) = \Delta(\mathbf{k}; V, V^+)$.*

*Proof.* From Proposition 5.5 we infer that the inclusion $\mathcal{D}_{\mathrm{GPT}\diamondsuit}(\mathbf{k}; \mathbb{R}, \mathbb{R}^+) \subseteq \mathcal{D}_f(\mathbf{k}; \mathbb{R}, \mathbb{R}^+)$ holds if and only if the $\{f_1^{(i)}, \ldots f_{k_i}^{(i)}\}$ are measurements for all $i \in [g]$. Here, $f_{k_i}^{(i)} = \mathbb{1} - \sum_{j=1}^{k_i-1} f_j^{(i)}$ for all $i \in [g]$. The statement then is an easy consequence of Theorem 5.8 and the following equivalence:

$$\begin{aligned}
&(1, s_1^{\times(k_1-1)}, \ldots, s_g^{\times(k_g-1)}) \cdot \mathcal{D}_{\mathrm{GPT}\diamondsuit}(\mathbf{k}; V, V^+) \subseteq \mathcal{D}_f(\mathbf{k}; V, V^+) \\
&\quad \iff \mathbb{1} \otimes z_0 + \sum_{i=1}^{g} \sum_{j=1}^{k_i-1} s_i \left(2f_j^{(i)} - \frac{2}{k_i}\mathbb{1}\right) \otimes z_j^{(i)} \in A^+ \otimes_{\min} V^+ \quad \forall z \in \mathcal{D}_{\mathrm{GPT}\diamondsuit}(\mathbf{k}; V, V^+) \\
&\quad \iff \mathcal{D}_{\mathrm{GPT}\diamondsuit}(\mathbf{k}; V, V^+) \subseteq \mathcal{D}_{f'}(\mathbf{k}; V, V^+),
\end{aligned}$$

where $(f')_j^{(i)} = s_i f_j^{(i)} + (1 - s_i)\frac{2}{k_i}\mathbb{1}$ for $j \in [k_i - 1]$, $i \in [g]$. □

## 6. Map extension via conic programming

To begin, let us briefly recapitulate the theory of conic programming. We follow [GM12, Section 4]. All vector spaces we consider will be finite dimensional. [GM12] assumes that there exists a scalar product on the vector spaces. We should be able to use functional evaluation though, choosing a basis and a dual basis.

**Definition 6.1** (Conic program [GM12, Definition 4.6.1])**.** *Let $L^+ \subseteq L$, $M^+ \subseteq M$ be closed convex cones, let $b \in M$, $c \in L^*$ and let $A : L \to M$ be a linear operator. A* conic program *is an optimization problem of the form*

$$\begin{aligned} \text{maximize} \qquad & \langle c, x \rangle \\ \text{subject to} \qquad & b - A(x) \in M^+ \\ & x \in L^+ \end{aligned}$$

The dual problem is then given by [GM12, Section 4.7]

$$\begin{aligned} \text{minimize} \qquad & \langle b, y \rangle \\ \text{subject to} \qquad & A^*(y) - c \in (L^+)^* \\ & y \in (M^+)^* \end{aligned}$$

Weak duality always hold, i.e. the value of the primal problem is upper bounded by the value of the dual program if the dual program is feasible. A sufficient condition for strong duality to hold is the following version of Slater's condition:

**Theorem 6.2** ([GM12, Theorem 4.7.1])**.** *If the conic program in Definition 6.1 is feasible, has finite value $\gamma$ and has an interior point $\tilde{x}$, then the dual program is also feasible and has the same value $\gamma$.*

If $M^+ \neq \{0\}$, $\tilde{x}$ is an interior point if $\tilde{x} \in \operatorname{int}(L^+)$ and $b - A(\tilde{x}) \in \operatorname{int}(M^+)$ [GM12, Definition 4.6.4]. We can give the following generalization of the results in [HJRW12].

**Theorem 6.3.** *Let $E \subseteq \mathbb{R}^d$ be a $g$-dimensional subspace such that $E \cap \operatorname{ri}(\mathbb{R}^d_+) \neq \emptyset$ and $E^+ = E \cap \mathbb{R}^d_+$. Moreover, let $(V, V^+, \mathbb{1})$ be a GPT. Finally, let $\Phi : E \to A$ be given by a basis $\{e_i\}_{i \in [g]} \subset E$ of $E$, $g \in \mathbb{N}$, and $\{f_i\}_{i \in [g]} \subset A$ such that $\Phi(e_i) = f_i$ for all $i \in [g]$. Then, there exists a positive extension $\tilde{\Phi} : (\mathbb{R}^d, \mathbb{R}^d_+) \to (A, A^+)$ of $\Phi$ if and only if the conic program*

$$\begin{aligned} \text{maximize} \qquad & -\langle s, (h_1^+ - h_1^-, \ldots, h_g^+ - h_g^-) \rangle \\ \text{subject to} \qquad & \sum_{i \in [g]} e_i \otimes (h_i^+ - h_i^-) \in E^+ \otimes_{\max} V^+ \\ & 1 - \langle \mathbb{1}, h_i^\pm \rangle \geqslant 0 \\ & h_i^\pm \in V^+ \qquad \forall i \in [g] \end{aligned}$$

*has value 0. Here, $s : V^g \to \mathbb{R}$ is given as*

$$s(h_1, \ldots, h_g) = \langle \chi, \sum_{i \in g} f_i \otimes (h_i^+ - h_i^-) \rangle.$$

*Proof.* Let $z \in E \otimes V$. Using that the $e_i$ form a basis, we can write

$$z = \sum_{i \in [g]} e_i \otimes z_i,$$

where $z_i \in V$ for all $i \in [g]$. Since $V^+$ is proper, we can decompose each $z_i = z_i^+ - z_i^-$, where $z_i^\pm \in V^+$. Then, $\langle s, (z_1, \ldots, z_g) \rangle = s_\Phi(z)$, where $s_\Phi$ is defined as in Proposition 5.7. By linearity, it suffices to restrict to $z$ such that $\langle \mathbb{1}, z_i^{\pm} \rangle \leqslant 1$ for all $i \in [g]$ in order to check positivity of $s_\Phi$. Thus, the conic program has value 0 if and only if $s_\Phi$ is positive. The assertion then follows from Proposition 5.7. ☐

**Theorem 6.4.** *The conic program in Theorem 6.3 is feasible and satisfies strong duality.*

*Proof.* In the following, we identify $z = \sum_{i \in [2g]} z_i \otimes \varepsilon_i \in L \otimes \mathbb{R}^{2g}$ with the vector $(z_1, \ldots z_{2g})$, where $\{\varepsilon_i\}_{i \in [2g]}$ is an orthonormal basis of $\mathbb{R}^{2g}$. Comparing the conic program to Definition 6.1, we identify

$$\begin{aligned}
M &= (E \otimes V) \times \mathbb{R}^{2g} \\
M^+ &= (E^+ \otimes_{\max} V^+) \times \mathbb{R}^{2g}_+ \\
L &= V \otimes \mathbb{R}^{2g} \\
L^+ &= (V^+) \otimes \mathbb{R}^{2g}_+ \\
c &= -s \\
b &= (0, 1, \ldots, 1) \in (E \otimes V) \times \mathbb{R}^{2g} \\
A(h_1^{\pm}, \ldots, h_g^{\pm}) &= \left( -\sum_{i \in [g]} e_i \otimes (h_i^+ - h_i^-), \langle \mathbb{1}, h_1^{\pm} \rangle, \ldots, \langle \mathbb{1}, h_g^{\pm} \rangle \right).
\end{aligned}$$

It can be verified that dual conic program is thus given by

$$\begin{aligned}
\text{minimize} \qquad & \sum_{i \in [2g]} y_i \\
\text{subject to} \qquad & s + \sum_{i \in [2g]} y_i \mathbb{1} \otimes \delta_i + B(z) \in A^+ \otimes \mathbb{R}^{2g}_+ \\
& z \in (E^+)^* \otimes_{\min} A^+ \\
& y_i \in \mathbb{R}_+ \qquad \forall i \in [2g].
\end{aligned}$$

Here, $B(z) \in A \otimes \mathbb{R}^{2g}$ is given as $B(z)(h_1, \ldots, h_g) = \langle z, \sum_{i \in [g]} e_i \otimes h_i \rangle$ and $\delta_i$ is the dual basis of $\varepsilon_i$. Letting $y_1 = \ldots = y_{2g}$ and realizing that $\mathbb{1} \otimes (1, \ldots 1)$ is an order unit in $A^+ \otimes \mathbb{R}^{2g}_+$, for any $z \in (E^+)^* \otimes_{\min} A^+$ we can find a $y_1 > 0$ such that

$$s + y_1 \mathbb{1} \otimes (1, \ldots, 1) + B(z) \in \operatorname{int} \left( A^+ \otimes \mathbb{R}^{2g}_+ \right).$$

This is true, since the order unit is an interior point of $A^+ \otimes \mathbb{R}^{2g}_+$, hence there is a $y_1$ such that

$$\frac{1}{y_1}(s + B(z)) + \mathbb{1} \otimes (1, \ldots, 1) \in \operatorname{int} \left( A^+ \otimes \mathbb{R}^{g}_+ \right).$$

Since the interior points of $A^+ \otimes \mathbb{R}^g_+$ are those points $w$ such that $\langle w, x \rangle > 0$ for all $x \in V^+ \otimes \mathbb{R}^{2g}_+$, multiplication by $y_1$ preserves the fact that the point is in $\operatorname{int} \left( A^+ \otimes \mathbb{R}^{2g}_+ \right)$. Therefore, the dual problem has an interior point. This also implies that the value of the dual program is finite, since it is at lower bounded by 0. The remarks at the beginning of [GM12, Section 4.7] imply that Theorem 6.2 still applies if we interchange the primal and the dual problem. Thus, the assertion follows. ☐

## 7. Linking map extension to the entanglement breaking property

The aim of this Section is to clarify the link between the approach motivated by [BN18b, BN18a] and the one motivated by [Jen18]. The following lemma clarifies what happens under reordering of tensor products of cones.

**Lemma 7.1.** *Let $A, B, C, D$ be cones. Then*

$$(A^+ \otimes_{\min} B^+) \otimes_{\min} (C^+ \otimes_{\max} D^+) \subseteq (A^+ \otimes_{\min} C^+) \otimes_{\max} (B^+ \otimes_{\min} D^+).$$

*Proof.* Consider arbitrary $a \in A^+$, $b \in B^+$, and $e \in C^+ \otimes_{\max} D^+$ such that $a \otimes b \otimes e$ lies on an extreme ray of the one on the LHS. We have to check that for any $\varphi \in (A^*)^+ \otimes_{\max} (C^*)^+$ and $\psi \in (B^*)^+ \otimes_{\max} (D^*)^+$. We have to show that

$$\langle a \otimes b \otimes e, \varphi \otimes \psi \rangle \geqslant 0.$$

We shall use (twice) the following fact: given cones $X^+, Y^+$ in vector spaces $X$, $Y$, $z \in X^+ \otimes_{\max} Y^+$ and $\sigma \in (X^*)^+$, the vector $y = \langle \sigma, z \rangle \in Y$ defined by

$$\langle \tau, y \rangle = \langle \sigma \otimes \tau, z \rangle, \qquad \forall \tau \in Y^*$$

is positive (i.e. $y \in \overline{Y^+} \cong (Y^+)^{**}$). This corresponds to the fact that the evaluation of a positive map at a positive element is positive, and we leave its proof as an exercise for the reader.

Using the fact above, and writing $\gamma := \langle a, \varphi \rangle \in (C^+)^*$ and $\delta := \langle b, \psi \rangle \in (D^+)^*$, we have

$$\langle a \otimes b \otimes e, \varphi \otimes \psi \rangle = \langle e, \gamma \otimes \delta \rangle \geqslant 0,$$

proving the statement of the lemma. □

For a map $\Phi : E \to F$, we denote by $\varphi \in E^* \otimes F$ the corresponding tensor. We have the following obvious equivalence

$$\Phi \text{ is a positive map } \iff \varphi \in (E^*)^+ \otimes_{\max} F^+$$

which is to be compared to the point (3) in the definition below.

**Proposition 7.2.** *Let $(E, E^+)$, $(F, F^+)$ be two ordered vector spaces. A map $\Phi : (E, E^+) \to (F, F^+)$ is called* entanglement breaking *if any of the following equivalent conditions is satisfied:*

1. *The map $\Phi \otimes \mathrm{id} : (E \otimes F^*, E^+ \otimes_{\max} (F^*)^+) \to (F \otimes F^*, F^+ \otimes_{\min} (F^*)^+)$ is positive*
2. *The map $s_\Phi : (E \otimes F^*, E^+ \otimes_{\max} (F^*)^+) \to \mathbb{R}$ is positive, where*
$$\begin{aligned} s_\Phi : (E \otimes F, E^+ \otimes_{\max} F^+) &\to \mathbb{R} \\ z &\mapsto \langle \chi, [\Phi \otimes \mathrm{id}_F](z) \rangle \end{aligned}$$
*and $\chi$ was defined in (8).*
3. $\varphi \in (E^*)^+ \otimes_{\min} F^+$
4. *The map $\Phi \otimes \mathrm{id} : (E \otimes L, E^+ \otimes_{\max} L^+) \to (F \otimes L, F^+ \otimes_{\min} L^+)$ is positive, for any ordered vector space $(L, L^+)$.*

*Proof.* Since both (2) $\iff$ (3) and (4) $\implies$ (1) are trivial, we only prove (1) $\implies$ (2) and (3) $\implies$ (4). For the first implication, note that $s_\Phi = (z \mapsto \langle \chi_F, z \rangle) \circ (\Phi \otimes \mathrm{id})$. Since $\chi_F \in (F^*)^+ \otimes_{\max} F^+$ (see Lemma 5.6), the claim follows. For the second implication, use Lemma 7.1 to prove that

$$\varphi \otimes \chi_L \in ((E^*)^+ \otimes_{\min} F^+) \otimes_{\min} ((L^*)^+ \otimes_{\max} L^+) \subseteq (E^+ \otimes_{\max} L^+)^* \otimes_{\max} (F^+ \otimes_{\min} L^+),$$

which is precisely the desired conclusion, since $\varphi \otimes \chi_L$ is the tensor corresponding to the map $\Phi \otimes \mathrm{id}_L$. □

The following corresponds to [Jen18, Proposition 1]:

**Lemma 7.3.** *The above conditions are equivalent to: there exist an $n \in \mathbb{N}$ and positive maps $g : (E, E^+) \to (\mathbb{R}^n, \mathbb{R}^n_+)$ and $J : (\mathbb{R}^n, \mathbb{R}^n_+) \to (F, F^+)$ such that $\Phi = J \circ g$.*

*Proof.* Note that point (3) in Proposition 7.2 is equivalent to the existence of a positive integer $n$ and of positive elements $\varepsilon_i \in (E^*)^+$, $f_i \in F^+$, $i \in [n]$, such that $\varphi = \sum_{i=1}^n \varepsilon_i \otimes f_i$. From the positive elements $\varepsilon_i$ and $f_i$, one can construct the positive maps $g$ and $J$:

$$g(e) = (\varepsilon_1(e), \dots, \varepsilon_n(e)) \in \mathbb{R}^n$$

$$J(x_1, \dots, x_n) = \sum_{i=1}^n x_i f_i \in F.$$

The reverse direction is proven in the same way, since any positive maps $g, J$ are as above. For the latter claim, we use the fact that $(\mathbb{R}^n, \mathbb{R}^n_+)$ is simplicial:

$$(\mathbb{R}^n, \mathbb{R}^n_+) \otimes (V, V^+) \cong \bigtimes_{i=1}^n (V, V^+).$$

□

Let us now come back to GPTs. Let $(V, V^+, \mathbb{1})$ be a GPT and $g \in \mathbb{N}$, $\mathbf{k} \in \mathbb{N}^g$. We consider a collection $f_j^{(i)} \in A$, where $j \in [k_i]$ and $i \in [g]$. Moreover, let $E_\mathbf{k}$ be as in Equation (4) and $\Phi^{(f)}$ as in Equation (7). Let $F : (V, V^+) \to (V(S_\mathbf{k}), V(S_\mathbf{k})^+)$ be the measurement map defined in [Jen18] as

$$F = (f_1^{(1)}, f_2^{(1)}, \dots, f_{k_g}^{(g)}).$$

Here, $V(S_\mathbf{k})^+$ is the cone generated by the polysimplex $S_{k_1-1,\dots,k_g-1}$ defined in [Jen18, Section III]. Theorem 1 of [Jen18] states that $F$ is entanglement breaking if and only if the measurements $f_i$ are compatible. We can verify that its dual map $F^* : (A(S_\mathbf{k}), A(S_\mathbf{k})^+) \to (A, A^+)$ is given on basis elements by

$$F^* : 1_S \mapsto \mathbb{1}, \qquad m_j^i \mapsto f_j^{(i)} \qquad \forall j \in [k_i - 1],\ \forall i \in [g].$$

Our aim is to connect $F^*$ and $\Phi^{(f)}$.

**Lemma 7.4.** *The cones $A(S_\mathbf{k})^+$ and $E_\mathbf{k}$ are isomorphic.*

*Proof.* Let $\iota : A(S_\mathbf{k}) \to E_\mathbf{k}$ be given by

$$\iota : 1_S \mapsto \mathbb{1}, \qquad m_j^i \mapsto \frac{1}{k_i}\mathbb{1} + \frac{1}{2} w_j^{(i)} \qquad \forall j \in [k_i - 1],\ \forall i \in [g].$$

The considerations of [BN18a, Lemma 4.3] and Proposition 4.11 let us infer that the extreme rays of $E_\mathbf{k}$ are generated by

$$\mathbb{1} + \frac{k_i}{2} w_j^{(i)}, \qquad \mathbb{1} - \frac{k_i}{2} \sum_{j=1}^{k_i - 1} w_j^{(i)}, \qquad \forall j \in [k_i - 1], \forall i \in [g].$$

The extreme rays of $A(S_\mathbf{k})^+$ are the $m_j^i$ for all $j \in [k_i]$, $i \in [g]$, where $\mathbb{1}_S = \sum_{j=1}^{k_i}$ for all $i \in [g]$. Thus, we can verify that both $\iota$ and its inverse map extreme rays to extreme rays, which proves the assertion since both cones are polyhedral. □

With this lemma, we can prove that the map $F$ studied in [Jen18] and the map $\Phi^{(f)}$ studied here play the same role.

**Proposition 7.5.** *Let $\Phi^{(f)}$ and $F$ be as above. Then, $F$ is entanglement breaking if and only if $\Phi^{(f)}$ is.*

*Proof.* It is easy to see that $F$ is entanglement breaking if and only if $F$ is (since a map is positive if and only if its adjoint map is). Since $A(S_\mathbf{k})^+$ and $E_\mathbf{k}$ are isomorphic by Lemma 7.4, the isomorphism $\iota$ between them is in particular a positive map. This means, that also $\iota \otimes \mathrm{id} : (A(S_\mathbf{k}) \otimes L, A(S_\mathbf{k})^+ \otimes_{\max} L^+) \to (E_\mathbf{k} \otimes L, E_\mathbf{k}^+ \otimes_{\min} L^+)$ is positive and has a positive inverse. From this, the assertion follows from point (1) or (4) in Proposition 7.2. □

**Example 7.6.** *To make explicit the role of $F$, let us consider the example of quantum mechanics and $k_i = 2$ for all $i \in [g]$. Here, $f_i(\rho) = \mathrm{tr}[E_i \rho]$ for some effect operators $E_i$, $i \in [g]$. Let us assume that the measurements are compatible and let $\{R_{i_1,\ldots,i_g}\}$ be the corresponding joint POVM, where $i_j \in [2]$ for all $j \in [g]$. Then,*

$$F(\rho) = (\mathrm{tr}[E_1\rho]\,|1\rangle\langle 1| + \mathrm{tr}[(\mathbb{1} - E_1)\rho]\,|2\rangle\langle 2|, \ldots, \mathrm{tr}[E_g\rho]\,|1\rangle\langle 1| + \mathrm{tr}[(\mathbb{1} - E_g)\rho]\,|2\rangle\langle 2|) \qquad \forall \rho \in \mathcal{M}_d^{sa}.$$

*Now, we can define*

$$g(\rho) = \sum_{i_j \in [2], j \in [g]} \mathrm{tr}\big[R_{i_1,\ldots,i_g}\rho\big]\,|i_1,\ldots,i_g\rangle\langle i_1,\ldots,i_g|\,.$$

*Here, we have identified $\mathbb{R}^d$ with the diagonal matrices in $\mathcal{M}_d^{sa}$. Moreover, we can choose*

$$J : |i_1,\ldots,i_g\rangle\langle i_1,\ldots,i_g| \mapsto (|i_1\rangle\langle i_1|, \ldots, |i_g\rangle\langle i_g|)$$

*and extend it by linearity. It is then straightforward to check that $F = J \circ g$. The reverse implication follows from the fact that joint measurability is equivalent to obtaining the compatible measurement from classical post-processing of a joint POVM* [HKR15] *(see also* [BN18a, Lemma 3.19]*.*

**Remark 7.7.** *To sum up, we have found several reformulation of the compatibility problem. Indeed, we have:*

$$\begin{aligned}
\{f^{(i)}\} \text{ compatible} &\iff \mathcal{D}_{\mathrm{GPT}\diamondsuit}(\mathbf{k}; V, V^+) \subseteq \mathcal{D}_f(\mathbf{k}; V, V^+) \\
&\iff \forall z \in \mathcal{D}_{\mathrm{GPT}\diamondsuit}(\mathbf{k}; V, V^+), \qquad \mathbb{1}(z_0) \geqslant \sum_{i=1}^{g} \sum_{j=1}^{k_i - 1} \langle 2 f_1^{(i)} - \frac{2}{k_i}\mathbb{1}, z_j^{(i)} \rangle \\
&\iff \Phi^{(f)} \text{ } EB \\
&\iff \varphi^{(f)} \in (E_{\mathbf{k}}^*)^+ \otimes_{\min} A^+ \\
&\iff F \text{ } EB,
\end{aligned}$$

*where in the last equivalence we have used Proposition 7.5.*

## 8. Tensors and cross norms

In this section, we will only consider binary measurements. For simplicity, we will write $E_g$ for $E_{2^{\times g}}$.

8.1. $E_g$ **as a centrally symmetric GPT.** Recall the definition of the $(g; V, V^+)$-diamond from Section 4.2. Define the vectors $\mathbb{1}_E, c_i \in \mathbb{R}^{2^g}$ by

$$\begin{aligned}
\mathbb{1}_E &:= (1, 1, \ldots 1) \\
c_i &:= (1,1)^{\otimes(i-1)} \otimes (1,-1) \otimes (1,1)^{\otimes(g-i)}.
\end{aligned}$$

These vectors are linearly independent and span a $(g+1)$-dimensional space $E_g \cong \mathbb{R}^{g+1}$ on which we put the positivity structure inherited from the canonical positive orthant in $\mathbb{R}^{2^g}$:

$$E_g^+ := E_g \cap \mathbb{R}_+^{2^g}.$$

We also consider the dual space $E_g^*$, spanned by a dual basis $\check{\mathbb{1}}_E, \gamma_i$. For example, one could take

$$\begin{aligned}
\check{\mathbb{1}}_E &:= 2^{-g}\langle \cdot, \mathbb{1}_E \rangle \\
\gamma_i &:= 2^{-g}\langle \cdot, c_i \rangle
\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the canonical scalar product in $\mathbb{R}^{2^g}$. Let us point out that $(E_g, E_g^+, \check{\mathbb{1}}_E)$ is a *centrally symmetric GPT* in the sense of [LPW18, Definition 25] corresponding to the $\ell_1$ norm in $\mathbb{R}^g$ (see also Section 2.4).

Note that the positivity in $(E, E^+, \check{\mathbb{1}}_E)$ is characterized by the norm:

$$x_0 \mathbb{1}_E + \sum_{i=1}^{g} x_i c_i \in E_g^+ \iff \|\bar{x}\|_1 \leqslant x_0,$$

where $\bar{x} = (x_1, \ldots, x_g) \in \mathbb{R}^g$. Since the $\ell_1$ norm is invariant under sign changes of the coordinates, for any sign vector $\varepsilon \in \{\pm 1\}^g$, the map

$$\begin{aligned} \sigma_\varepsilon : E_g &\to E_g \\ \mathbb{1}_E &\mapsto \mathbb{1}_E \\ c_i &\mapsto \varepsilon_i c_i \end{aligned}$$

is positive. In particular, if we denote, for some ordered vector space $(L, L^+)$

$$\begin{aligned} z &= \mathbb{1}_E \otimes z_0 + \sum_{i=1}^{g} c_i \otimes z_i \in E_g \otimes L \\ z_\varepsilon &= \mathbb{1}_E \otimes z_0 + \sum_{i=1}^{g} \varepsilon_i c_i \otimes z_i \in E_g \otimes L, \end{aligned}$$

we have

$$\begin{aligned} z \in E_g^+ \otimes_{\min} L^+ &\iff z_\varepsilon \in E_g^+ \otimes_{\min} L^+ \\ z \in E_g^+ \otimes_{\max} L^+ &\iff z_\varepsilon \in E_g^+ \otimes_{\max} L^+. \end{aligned}$$

Given a $g$-tuple of effects $f = (f_1, \ldots, f_g)$, $0 \leqslant f_i \leqslant \mathbb{1}$, let us define the map

$$\begin{aligned} \Phi^{(f)} : E_g &\to A \\ \mathbb{1}_E &\mapsto \mathbb{1} \\ c_i &\mapsto 2 f_i - \mathbb{1}. \end{aligned}$$

Equivalently, we consider the tensor

$$\varphi^{(f)} = \check{\mathbb{1}}_E \otimes \mathbb{1} + \underbrace{\sum_{1=1}^{g} \gamma_i \otimes (2 f_i - \mathbb{1})}_{-\bar{\varphi}^{(f)}} \in E_g^* \otimes A. \tag{13}$$

Since positivity in $(E_g^*, (E_g^*)^+)$ is characterized by

$$x_0 \check{\mathbb{1}}_E + \sum_{i=1}^{g} x_i \gamma_i \in (E_g^*)^+ \iff \|\bar{x}\|_\infty \leqslant x_0,$$

the map

$$\begin{aligned} \sigma_\varepsilon : E_g^* &\to E_g^* \\ \check{\mathbb{1}}_E &\mapsto \check{\mathbb{1}}_E \\ \gamma_i &\mapsto \varepsilon_i \gamma_i \end{aligned}$$

also is positive for any $\varepsilon \in \{\pm 1\}^g$. Thus, in particular

$$\begin{aligned} \varphi^{(f)} \in (E_g^*)^+ \otimes_{\min} A^+ &\iff \varphi^{(f_\varepsilon)} \in (E_g^*)^+ \otimes_{\min} A^+ \\ \varphi^{(f)} \in (E_g^*)^+ \otimes_{\max} A^+ &\iff \varphi^{(f_\varepsilon)} \in (E_g^*)^+ \otimes_{\max} A^+, \end{aligned} \tag{14}$$

where $f_\varepsilon(i) = f_i$ if $\varepsilon_i = 1$ and $f_\varepsilon(i) = \mathbb{1} - f_i$ if $\varepsilon_i = -1$, because $(2(\mathbb{1} - f_i) - \mathbb{1}) = -(2 f_i - \mathbb{1})$. As we will see below, this is a manifestation of the fact that we could relabel the outcomes of $f_i$ and $\mathbb{1} - f_i$ without changing the compatibility of the effects under study.

8.2. **Tensor cross norms and compatibility.**

**Proposition 8.1.** *The following are equivalent*

1. *The elements $f_1, \ldots, f_g$ are quantum effects*
2. *The map $\Phi^{(f)} : E_g^+ \to A^+$ is positive*
3. $\varphi^{(f)} \in (E_g^*)^+ \otimes_{\max} A^+$
4. $\|\bar\varphi^{(f)}\|_\varepsilon \leqslant 1$, *where $E_g^*$ and $A$ are endowed with their order unit norms, and $\varepsilon$ denotes the injective tensor product of Banach spaces.*

*Proof.* We only need to show (3) $\iff$ (4). Let us first prove (4) $\implies$ (3). Consider an arbitrary $x \in E_g^+ \otimes_{\min} V^+$. From the hypothesis, we have $\langle \bar\varphi^{(f)}, x\rangle \leqslant \|x\|_\pi = (\check{\mathbb{1}}_E \otimes \mathbb{1})(x)$, where we have used [LPW18, Proposition 22]: for the GPTs $(E_g, E_g^+, \check{\mathbb{1}}_E)$ and $(V, V^+, \mathbb{1})$, we have $\|\cdot\|_\pi = \|\cdot\|_{E_g^+ \otimes_{\min} V^+}$. Thus, we have $\langle \check{\mathbb{1}}_E \otimes \mathbb{1} - \bar\varphi^{(f)}, x\rangle \geqslant 0$, which is (3).

For the converse, we need to show that for all $x \in E_g \otimes V$, $\langle \bar\varphi^{(f)}, x\rangle \leqslant \|x\|_\pi$. Using again the fact that the projective norm is equal to the base norm, we want: $\forall x_\pm \in E_g^+ \otimes_{\min} V^+$,

$$\langle 2 \cdot \check{\mathbb{1}}_E \otimes \mathbb{1} - \varphi^{(f)}, x_-\rangle + \langle \varphi^{(f)}, x_+\rangle \geqslant 0.$$

From the symmetry of the dual GPT $(E_g^*, (E_g^*)^+, \mathbb{1}_E)$, we have that $\varphi^{(f)} \in (E_g^*)^+ \otimes_{\max} A^+$ implies

$$2 \cdot \check{1} \otimes \mathbb{1} - \varphi^{(f)} = \check{1} \otimes \mathbb{1} - \sum_{1=1}^{g} \gamma_i \otimes (2f_i - \mathbb{1}) \in (E_g^*)^+ \otimes_{\max} A^+,$$

and the conclusion follows. □

**Question 8.2.** *Can we extend this to more outcomes? Point (4) uses the symmetry of $E_g$, which is not present in general, so the proof does not immediately extend.*

**Proposition 8.3.** *The following are equivalent*

1. *The elements $f_1, \ldots, f_g$ are* compatible *quantum effects*
2. *The map $\Phi^{(f)} : E_g^+ \to A^+$ is entanglement breaking*
3. $\varphi^{(f)} \in (E_g^*)^+ \otimes_{\min} A^+$
4. $\|\bar\varphi^{(f)}\|_\rho \leqslant 1$, *where $\|\cdot\|_\rho$ is the dual norm to the base norm in the GPT $(E_g \otimes V, E_g^+ \otimes_{\max} V^+, \check{\mathbb{1}}_E \otimes \mathbb{1})$.*

*Proof.* The proof is identical to that of the previous result, with the only difference that we cannot identify in this case the norm which is dual to the base norm in $(E_g \otimes V, E_g^+ \otimes_{\max} V^+, \mathbb{1}_E \otimes \mathbb{1})$ with some Banach space tensor norm. □

The above Proposition motivates us to understand better the properties of $\|\cdot\|_\rho$. We will show now that $\|\cdot\|_\rho$ is a reasonable crossnorm.

**Definition 8.4** ([Rya02])**.** *Let $X$ and $Y$ be two Banach spaces. We say that a norm $\|\cdot\|_\alpha$ on $X \otimes Y$ is a* reasonable crossnorm *if it has the following properties:*

1. $\|x \otimes y\|_\alpha \leqslant \|x\|_X \|y\|_Y$ *for all $x \in X$, $y \in Y$,*
2. *For all $\varphi \in X^*$, for all $\psi \in Y^*$, $\varphi \otimes \psi$ is bounded on $X \otimes Y$ and $\|\varphi \otimes \psi\|_{\alpha^*} \leqslant \|\varphi\|_{X^*}\|\psi\|_{Y^*}$,*

*where $\|\cdot\|_{\alpha^*}$ is the dual norm to $\|\cdot\|_\alpha$.*

The reason why we are interested in reasonable crossnorms becomes apparent from the next proposition:

**Proposition 8.5** ([Rya02, Proposition 6.1])**.** *Let $X$ and $Y$ be Banach spaces.*

(a) *A norm* $\|\cdot\|_\alpha$ *on* $X \otimes Y$ *is a reasonable crossnorm if and only if*

$$\|z\|_\varepsilon \leqslant \|z\|_\alpha \leqslant \|z\|_\pi$$

*for all* $z \in X \otimes Y$

(b) *If* $\|\cdot\|_\alpha$ *is a reasonable crossnorm on* $X \otimes Y$*, then* $\|x \otimes y\|_\alpha = \|x\|_X \|y\|_Y$ *for every* $x \in X$ *and every* $y \in Y$*. Furthermore, for all* $\varphi \in X^*$ *and all* $\psi \in Y^*$*, the norm* $\|\cdot\|_{\alpha^*}$ *satisfies* $\|\varphi \otimes \psi\|_{\alpha^*} = \|\varphi\|_{X^*} \|\psi\|_{Y^*}$*.*

The results of [LPW18] imply that any base norm on a bipartite GPT is a reasonable crossnorm.

**Proposition 8.6.** *Consider the GPT* $(V_A \otimes V_B, V_A^+ \otimes V_B^+, \mathbb{1}_A \otimes \mathbb{1}_B)$*, where* $V_A^+ \otimes V_B^+$ *is some tensor product of cones. Then, the base norm* $\|\cdot\|_{V_A^+ \otimes V_B^+}$ *is a reasonable cross norm.*

*Proof.* Let $x \in V_A \otimes V_B$. From [LPW18, Equation (26)], it holds that $\|x\|_{V_A^+ \otimes V_B^+} \leqslant \|x\|_{V_A^+ \otimes_{\min} V_B^+}$. Together with [LPW18, Proposition 22], this implies that $\|x\|_{V_A^+ \otimes V_B^+} \leqslant \|x\|_\pi$. Moreover, [LPW18, Proposition 22] implies $\|x\|_\varepsilon \leqslant \|x\|_{\mathrm{LO}}$, where we refer to [LPW18] for the definition of the latter norm. From the discussion after [LPW18, Definition 6], it follows that $\|x\|_{\mathrm{LO}} \leqslant \|x\|_{V_A^+ \otimes V_B^+}$. Thus,

$$\|x\|_\varepsilon \leqslant \|x\|_{V_A^+ \otimes V_B^+} \leqslant \|x\|_\pi.$$

Therefore, the assertion follows from point of (a) of Proposition 8.5. □

**Corollary 8.7.** *The norm* $\|\cdot\|_\rho$ *is a reasonable crossnorm.*

*Proof.* This follows from Proposition 8.6 since $\|\cdot\|_\rho$ is the dual norm of $\|\cdot\|_{E_g^+ \otimes_{\max} V^+}$. □

8.3. **Inclusion constants.** Given a $g$-tuple of elements $f_i \in A$ encoded tensor $\varphi$ defined in (13), as well as a tuple $s \in [0,1]^g$, define the "shrank" tensor

$$\begin{aligned} s.\varphi^{(f)} = \varphi_s^{(f)} &:= \mathbb{1}_E \otimes \mathbb{1} + \sum_{1=1}^{g} s_i \gamma_i \otimes (2f_i - \mathbb{1}) \\ &= \mathbb{1}_E \otimes \mathbb{1} + \sum_{1=1}^{g} \gamma_i \otimes (2(s_i f_i + (1-s_i)\mathbb{1}/2) - \mathbb{1}) \in E_g^* \otimes A \end{aligned}$$

corresponding to the noisy effects $s_i f_i + (1-s_i)\mathbb{1}/2$. With this, we can reformulate the compatibility region for $g$-effects in terms of the corresponding tensors.

**Proposition 8.8.** *We have*

$$s \in \Gamma(g; V, V^+) \iff \left(\forall f \in A^g, \varphi^{(f)} \in (E_g^*)^+ \otimes_{\max} A^+ \implies s.\varphi^{(f)} \in (E^*)^+ \otimes_{\min} A^+\right).$$

*Proof.* This follows from Propositions 8.1 and 8.3 together with the definition of $\Gamma(g; V, V^+)$. □

## 9. Incompatibility witnesses and cross norms

We will restrict to binary measurements in this section. In this section, we will consider different notions of witnesses. We will start with objects certifying that the elements of $A$ under study are effects before considering objects certifying compatibility.

Motivated by Corollary 5.9, we introduce the set of *effect witnesses*

$$\mathcal{Q}_{\mathrm{GPT}\diamond}(g; V, V^+) := \{z \in V^g : \sum_{i=1}^{g} \|z_i\|_V \leqslant 1\}. \tag{15}$$

Note that the set $\mathcal{Q}$ is the unit ball of $\ell_1^g \otimes_\pi \|\cdot\|_V$.

**Proposition 9.1.** *Elements $f_1, \ldots, f_g \in A$ are effects (i.e. $f_i, \mathbb{1} - f_i \in A^+$ for all $i \in [g]$) if and only if*

$$\sum_{i=1}^{g} \langle 2f_i - \mathbb{1}, z_i \rangle \leqslant 1 \qquad \forall (z_1, \ldots, z_g) \in \mathcal{Q}_{\mathrm{GPT}\diamondsuit}(g; V, V^+).$$

*Proof.* If the $f_i$ are effects, then $2f_i - \mathbb{1} \in [-\mathbb{1}, \mathbb{1}]$, and thus

$$\|z_i\|_V = \sup_{a_i \in [-\mathbb{1}, \mathbb{1}]} \langle a_i, z_i \rangle \geqslant \langle 2f_i - \mathbb{1}, z_i \rangle.$$

Conversely, let $v \in K$. Since $(0, \ldots, 0, -v, 0, \ldots, 0) \in \mathcal{Q}_{\mathrm{GPT}\diamondsuit, g}(V, V^+)$ (where $-v$ appears on the $i$-th position), we have

$$1 \geqslant \langle 2f_i - \mathbb{1}, -v \rangle = -2\langle f_i, v \rangle + 1 \implies \langle f_i, v \rangle \geqslant 0.$$

A similar reasoning can be used to show that $\langle \mathbb{1} - f_i, v \rangle \geqslant 0$, finishing the proof. □

In [BN18a], we introduced the notion of an incompatibility witness. Translated to the GPT setting, an incompatibility witness is an element from $\mathcal{D}_{\mathrm{GPT}\diamondsuit}(g; V, V^+)$. A refined notion of incompatibility witness could be to consider only elements $z \in \mathcal{D}_{\mathrm{GPT}\diamondsuit}(g; V, V^+)$ which detect incompatibility for some choice of effects, i.e. for which $z \notin \mathcal{D}_f(g; V, V^+)$ for some collection of effects $f_i$, $i \in g$.

**Definition 9.2** (Strict incompatibility witnesses)**.** *Let $z \in \mathcal{D}_{\mathrm{GPT}\diamondsuit}(g; V, V^+)$ such that*

$$z \notin \mathcal{D}_f(V, V^+)$$

*for some collection of effects $f_i$, $i \in [g]$. Then, we call $z$ a* strict incompatibility witness*.*

There is a useful characterization of strict incompatibility witnesses:

**Proposition 9.3.** *Let $(z_0, \ldots, z_g) \in \mathcal{D}_{\mathrm{GPT}\diamondsuit}(g; V, V^+)$. Then, it is a strict incompatibility witness if and only if*

$$\sum_{i=1}^{g} \|z_i\|_V > \mathbb{1}(z_0).$$

*Proof.* From Theorem 5.8 and Proposition 5.7, we infer that $(z_0, \ldots, z_g)$ is a strict incompatibility witness if and only if

$$s_{\Phi(f)}\left(\mathbb{1}_E \otimes z_0 + \sum_{i=1}^{g} c_i \otimes z_i\right) < 0$$

for some $\Phi^{(f)}$ defined by effects $f_i$, $i \in [g]$. Let us write $a_i := 2f_i - \mathbb{1}$. Then, $f_i$ is an effect if and only if $a_i \in [-\mathbb{1}, \mathbb{1}]$. Thus, we are looking for $(z_0, \ldots, z_g)$ such that

$$\begin{aligned} 0 &> \min_{a_i \in [-\mathbb{1}, \mathbb{1}], i \in [g]} \langle \chi, \mathbb{1} \otimes z_0 + \sum_{i=1}^{g} a_i \otimes z_i \rangle \\ &= \mathbb{1}(z_0) - \max_{a_i \in [-\mathbb{1}, \mathbb{1}], i \in [g]} \sum_{i=1}^{g} a_i(z_i) \\ &= \mathbb{1}(z_0) - \sum_{i=1}^{g} \|z_i\|_V. \end{aligned}$$

The last equality follows since the base norm is the dual of the supremum norm. □

**Example 9.4.** *In quantum mechanics $(V, V^+) = (\mathrm{M}_d, \mathrm{PSD}_d)$, $\mathbb{1}$ is the trace operator, and the base norm is the trace norm. Note that this does not imply that we can restrict to elements $z$ in the matrix diamond with $\sum_i \|z_i\|_1 > d$, because for that, we would need to restrict to elements such that $z_0 = I_d$. For inclusion, it is enough to consider such elements, because conjugation with $z_0^{-1/2}$ preserves positivity. It changes, however, the norm such that $\left\| z_0^{-1/2} z_i z_0^{-1/2} \right\|_1$ is in general different from $\|z_i\|_1$.*

Recall that $\mathcal{D}_{\mathrm{GPT}\diamond}(g; V, V^+)$ is a cone and that $(z_0, z_1, \ldots, z_g) \in \mathcal{D}_{\mathrm{GPT}\diamond,g}$ if and only if $(z_0, -z_1, \ldots, -z_g) \in \mathcal{D}_{\mathrm{GPT}\diamond}(g; V, V^+)$. Thus, $z_0 \in V^+$. We can now restrict to $z_0$ such that $\mathbb{1}(z_0) = 1$. This is the case if and only if $z_0 \in K$.

**Definition 9.5.** *The set of* projected incompatibility witnesses *is defined as*

$$\mathcal{P}_{\mathrm{GPT}\diamond}(g; V, V^+) := \{(z_1, \ldots, z_g) \in V^g : \exists z_0 \in K \text{ s.t. } (z_0, \ldots, z_g) \in \mathcal{D}_{\mathrm{GPT}\diamond}(g; V, V^+)\}. \tag{16}$$

*Here, $K$ is the state space of the GPT, $K := \{v \in V^+ : \mathbb{1}(v) = 1\}$.*

The following result is an easy consequence of Corollary 5.9:

**Corollary 9.6.** *Let $f_1, \ldots, f_g \in A$. Then, the $f_i$ are compatible effects if and only if*

$$\sum_{i=1}^{g} \langle 2f_i - \mathbb{1}, z_i \rangle \leqslant 1 \qquad \forall (z_1, \ldots, z_g) \in \mathcal{P}_{\mathrm{GPT}\diamond,g}.$$

**Definition 9.7.** *The set*

$$\Pi(g; V, V^+) := \left\{ (s_1, \ldots, s_g) \in [0,1]^g : \sum_{i=1}^{g} s_i \|z_i\|_V \leqslant 1 \ \forall (z_1, \ldots, z_g) \in \mathcal{P}_{\mathrm{GPT}\diamond,g} \right\} \tag{17}$$

*is called the $(g; V, V^+)$-*blind region.

**Proposition 9.8.** *It holds that*

$$\Pi(g; V, V^+) = \Gamma(g; V, V^+) = \Delta(g; V, V^+)$$

*Proof.* The statement follows basically from Corollary 9.6. We have:

$$\begin{aligned} \sup_{f_i, i \in [g]} \sum_{i=1}^{g} \langle 2(s_i f_i + (1 - s_i)\mathbb{1}/2) - \mathbb{1}, z_i \rangle &= \sum_{i=1}^{g} s_i \sup_{f_i} \langle 2f_i - \mathbb{1}, z_i \rangle \\ &= \sum_{i=1}^{g} s_i \|z_i\|_V. \end{aligned}$$

Here, the supremum is over all collection of effects $f_i$, $i \in [g]$. This establishes $\Pi(g; V, V^+) = \Gamma(g; V, V^+)$. The other assertion follows from Theorem 5.13. □

## 10. Centrally symmetric GPTs and cross norms

We shall discuss now the concept of compatibility witnesses in the special case of centrally symmetric GPTs (see Section 2.4). In this setting, the vector space of un-normalized states decomposes as $V = \mathbb{R}v_0 \oplus \bar{V}$, and we shall write $V \ni z = (z^\circ, \bar{z})$ to denote the vector $z = z^\circ v_0 + \bar{z}$, with $z^\circ \in \mathbb{R}$ and $\bar{z} \in \bar{V}$.

Similarly to equations (16),(17), let us consider the following sets

$$\begin{aligned}\mathcal{P}'_{\mathrm{GPT}\diamondsuit}(g;V,V^+) &:= \{(z_1,\ldots,z_g)\in V^g : z_i^\circ = 0\ \ \forall i\in[g] \text{ and } \exists z_0\in K \text{ s.t. } (z_0,\ldots,z_g)\in\mathcal{D}_{\mathrm{GPT}\diamondsuit}(g;V,V^+)\}\\ &= (0\oplus\bar V)^g \cap \mathcal{P}_{\mathrm{GPT}\diamondsuit,g}(V,V^+) \qquad (18)\end{aligned}$$

$$\begin{aligned}\Pi'(g;V,V^+) &:= \{(s_1,\ldots,s_g)\in[0,1]^g : \sum_{i=1}^g s_i\|z_i\|_V \leqslant 1\,\forall (z_1,\ldots,z_g)\in\mathcal{P}'_{\mathrm{GPT}\diamondsuit,g}\}\\ &= \{(s_1,\ldots,s_g)\in[0,1]^g : \sum_{i=1}^g s_i\|\bar z_i\|_{\bar V} \leqslant 1\,\forall ((0,\bar z_1),\ldots,(0,\bar z_g))\in\mathcal{P}'_{\mathrm{GPT}\diamondsuit,g}\}.\end{aligned}$$

The set $\mathcal{P}'_{\mathrm{GPT}\diamondsuit}(g;V,V^+)$ can be characterized in the following way.

**Lemma 10.1.** *For a centrally symmetric GPT $(V,V^+,\mathbb{1})$, we have*

$$\begin{aligned} z\in\mathcal{P}'_{\mathrm{GPT}\diamondsuit}(g;V,V^+) &\iff z_i=(0,\bar z_i) \text{ and } \sum_{i=1}^g \varepsilon_i\bar\alpha(z_i)\leqslant 1,\ \forall\varepsilon\in\{\pm1\}^g,\ \forall\bar\alpha\in\bar A^g \text{ s.t. } \|\bar\alpha\|_{\bar A}\leqslant 1\\ &\iff z_i=(0,\bar z_i) \text{ and } (v_0,z_1,\ldots,z_g)\in\mathcal{D}_{\mathrm{GPT}\diamondsuit}(g;V,V^+).\end{aligned}$$

*Proof.* Let us prove the first equivalence. For the " $\Longrightarrow$ " direction, consider $z\in\mathcal{P}'_{\mathrm{GPT}\diamondsuit}(g;V,V^+)$ and $\bar\alpha\in\bar A^g$ such that $\|\bar\alpha\|_{\bar A}\leqslant 1$. Let $z_0\in K$ be such that $(z_0,\ldots,z_g)\in\mathcal{D}_{\mathrm{GPT}\diamondsuit}(g;V,V^+)$. Since $\mathbb{1}+\bar\alpha\in A^+$, we have, for all $\varepsilon\in\{\pm1\}^g$,

$$\langle \mathbb{1}+\bar\alpha, z_0-\sum_{i=1}^g\varepsilon_i z_i\rangle \geqslant 0.$$

Recall that in the centrally symmetric setting, the map $(v^\circ,\bar v)\mapsto(v^\circ,-\bar v)$ leaves the GPT diamond invariant. Hence, we also have

$$\langle \mathbb{1}+\bar\alpha, (z_0^\circ,-\bar z_0)+\sum_{i=1}^g\varepsilon_i(-z_i)\rangle \geqslant 0,$$

where we have also changed the signs of $\varepsilon$. Summing the two previous equations gives the desired conclusion, $\sum_{i=1}^g\varepsilon_i\bar\alpha(z_i)\leqslant 1$.

For the reverse implication " $\Longleftarrow$ ", we show that $(v_0,z_1,\ldots,z_g)\in\mathcal{D}_{\mathrm{GPT}\diamondsuit}(g;V,V^+)$. To this end, let us take any (normalized) $(1,\bar\alpha)\in A^+$, and evaluate

$$\langle \mathbb{1}+\bar\alpha, v_0-\sum_{i=1}^g\varepsilon_i(0,\bar z_i)\rangle = 1-\sum_{i=1}^g\varepsilon_i\bar\alpha(\bar z_i)\geqslant 0,$$

proving the claim.

Note that the considerations above also show the second equivalence in the statement, that is the fact that one can always chose $z_0=v_0$ in (18). $\square$

Remarkably, we have the following result.

**Theorem 10.2.** *For a centrally symmetric GPT $(V,V^+,\mathbb{1})$, we have $\Pi(g;V,V^+)=\Pi'(g;V,V^+)$.*

*Proof.* We have $\mathcal{P}'_{\mathrm{GPT}\diamondsuit}(g;V,V^+)\subseteq\mathcal{P}_{\mathrm{GPT}\diamondsuit}(g;V,V^+)$, hence $\Pi(g;V,V^+)\subseteq\Pi'(g;V,V^+)$. To prove the reverse inclusion, let us define, for a tuple $s\in[0,1]^g$,

$$\begin{aligned}\beta_s &:= \max_{z\in\mathcal{P}_{\mathrm{GPT}\diamondsuit}(g;V,V^+)}\sum_{i=1}^g s_i\|z_i\|_V\\ \beta'_s &:= \max_{z\in\mathcal{P}'_{\mathrm{GPT}\diamondsuit}(g;V,V^+)}\sum_{i=1}^g s_i\|z_i\|_V.\end{aligned}$$

Obviously, $s \in \Pi(g; V, V^+)$ if and only if $\beta_s \leqslant 1$, the same being true for the primed variants. We shall prove that $\beta_s \leqslant \beta'_s$, for all $s \in [0,1]^g$. To this end, fix some $s$ and let $z \in \mathcal{P}_{\mathrm{GPT}\diamondsuit}(g; V, V^+)$ achieve the maximum in $\beta_s$. We consider the set partition $[g] = I \sqcup J$, where $i \in I \iff z_i^\circ > \|\bar z_i\|_{\bar V}$. We have thus

$$\beta_s = \sum_{i\in I} s_i |z_i^\circ| + \sum_{j \in J} s_j \|\bar z_i\|_{\bar V}. \tag{19}$$

Let us denote by $s_I$, resp. $s_J$, the restriction of the $g$-tuple $s$ to the index set $I$, resp. $J$. Putting

$$\lambda := \sum_{i\in I} |z_i^\circ|,$$

we have $\sum_{i \in I} s_i |z_i^\circ| \leqslant \lambda \|s_I\|_\infty \leqslant \lambda \beta'_{s_I}$, where the last inequality follows from the fact that by Corollary 5.9,

$$(v_0, 0, \ldots, 0, \bar v, 0, \ldots, 0) \in \mathcal{D}_{\mathrm{GPT}\diamondsuit}(g; V, V^+),$$

for a vector $\bar v \in \bar V$ such that $\|\bar v\|_{\bar V} = 1$. Note that $\lambda \leqslant 1$, since $z \in \mathcal{P}_{\mathrm{GPT}\diamondsuit}(g; V, V^+)$ implies that $z_0 + \sum_{i=1}^g \varepsilon_i z_i \in V^+$ for some $z_0 \in K$ and all $\varepsilon \in \{\pm 1\}^g$. Therefore, an application of $\mathbb{1}$ yields $\sum_{i=1}^g |z_i^\circ| \leqslant 1$.

Let us now focus on the second term in the RHS of (19). First, note that if the set $J$ is empty we are done: $\beta_s \leqslant \lambda \beta'_s \leqslant \beta'_s$. Assume now $\lambda \in [0,1)$. Using Lemma 10.3, we have that

$$\frac{1}{1-\lambda} \left((0, \bar z_j)\right)_{j \in J} \in \mathcal{P}'_{\mathrm{GPT}\diamondsuit}(|J|; V, V^+),$$

and thus

$$\sum_{j \in J} s_j \|\bar z_i\|_{\bar V} \leqslant (1-\lambda) \beta'_{s_J}.$$

We have proven, up to this point, that $\beta_s \leqslant \lambda \beta'_{s_I} + (1-\lambda)\beta'_{s_J}$. To conclude, we need to show that the function $\beta'_\cdot$ is "concave", which we do next. Let $\bar x_0, \bar x_1, \ldots, \bar x_{|I|} \in \bar V$ be such that $\|\bar x_0\|_{\bar V} \leqslant 1$ and

$$\big((1, \bar x_0), (0, \bar x_1), \ldots, (0, \bar x_{|I|})\big) \in \mathcal{D}_{\mathrm{GPT}\diamondsuit}(|I|; V, V^+) \qquad \text{and}$$

$$\beta'_{s_I} = \sum_{i=1}^{|I|} s_i \|\bar x_i\|_{\bar V}.$$

Consider similar elements $\bar y_0, \bar y_1, \ldots, \bar y_{|J|} \in \bar V$. The claim follows from the following fact:

$$\big((1, \lambda \bar x_0 + (1-\lambda)\bar y_0), \lambda(0, \bar x_1), \ldots, \lambda(0, \bar x_{|I|}), (1-\lambda)(0, \bar y_1), \ldots, (1-\lambda)(0, \bar y_{|J|})\big) \in \mathcal{D}_{\mathrm{GPT}\diamondsuit}(g; V, V^+).$$

□

**Lemma 10.3.** *Let $z \in \mathcal{P}_{\mathrm{GPT}\diamondsuit}(g; V, V^+)$, and consider a subset $J \subseteq [g]$ such that $\sum_{i \notin J} |z_i^\circ| < 1$. Then,*

$$\frac{1}{1 - \sum_{i \notin J} |z_i^\circ|} \left((0, \bar z_j)\right)_{j\in J} \in \mathcal{P}'_{\mathrm{GPT}\diamondsuit, |J|}(V, V^+).$$

*Proof.* Let us put $I := [g] \backslash J$. From the hypothesis, we have that, for any signs $\varepsilon_i$, $\eta_j$,

$$1 + \sum_{i \in I} \varepsilon_i z_i^\circ + \sum_{j\in J} \eta_j z_j^\circ \geqslant \left\| \bar z_0 + \sum_{i \in I} \varepsilon_i \bar z_i + \sum_{j \in J} \eta_j \bar z_j \right\|_{\bar V},$$

where $\bar z_0 \in \bar V$ is such that $(z_0, z_1, \ldots, z_g) \in \mathcal{D}_{\mathrm{GPT}\diamondsuit}(g; V, V^+)$. Writing

$$\sum_{j \in J} \eta_j \bar z_j = \frac{1}{2}\left[ \left( \bar z_0 + \sum_{i \in I} \varepsilon_i \bar z_i + \sum_{j\in J} \eta_j \bar z_j \right) - \left( \bar z_0 + \sum_{i\in I} \varepsilon_i \bar z_i + \sum_{j \in J} (-\eta_j) \bar z_j \right) \right],$$

we get

$$\left\| \sum_{j \in J} \eta_j \bar{z}_j \right\|_{\bar{V}} \leqslant \frac{1}{2} \left[ \left( 1 + \sum_{i \in I} \varepsilon_i z_i^\circ + \sum_{j \in J} \eta_j z_j^\circ \right) + \left( 1 + \sum_{i \in I} \varepsilon_i z_i^\circ + \sum_{j \in J} (-\eta_j) z_j^\circ \right) \right]$$
$$= 1 + \sum_{i \in I} \varepsilon_i z_i^\circ.$$

The conclusion follows by setting, for all $i \in I$, $\varepsilon_i = -\operatorname{sign}(z_i^\circ)$. □

The set $\Pi'$ can be described in terms of tensor norms, as follows.

**Proposition 10.4.** *For a centrally symmetric GPT $(V, V^+, \mathbb{1})$ we have*

$$\Pi'(g; V, V^+) = \{s \in [0,1]^g \,:\, \|z\|_{\ell_1^g \otimes_\varepsilon \|\cdot\|_{\bar{V}}} \leqslant 1 \implies \|s.z\|_{\ell_1^g \otimes_\pi \|\cdot\|_{\bar{V}}} \leqslant 1 \ \forall z \in \bar{V}^g\} \tag{20}$$
$$= \{s \in [0,1]^g \,:\, \|y\|_{\ell_\infty^g \otimes_\varepsilon \|\cdot\|_{\bar{A}}} \leqslant 1 \implies \|s.y\|_{\ell_\infty^g \otimes_\pi \|\cdot\|_{\bar{A}}} \leqslant 1 \ \forall y \in \bar{A}^g\}. \tag{21}$$

*The diagonal value is related to the quantity $\rho$ from* [ALP+18, Eq. (14)]*:*

$$\rho(\ell_1^g, (\bar{V}, \|\cdot\|_{\bar{V}})) = 1/\max\{s \in [0,1] \,:\, (s, s, \ldots, s) \in \Delta(g, V, V^+)\}.$$

*Proof.* The statement follows from the following two facts. First, we have

$$z \in \mathcal{P}'_{\mathrm{GPT}\diamondsuit}(g; V, V^+) \iff \|z\|_{\ell_1^g \otimes_\varepsilon \|\cdot\|_{\bar{V}}} \leqslant 1. \tag{22}$$

To show this, use the characterization from Lemma 10.1 and the fact that the extremal points of the unit ball of the dual norm $\ell_\infty^g \otimes_\pi \|\cdot\|_{\bar{A}}$ are of the form $\sum_{i=1}^g \varepsilon_i e_i \otimes \bar{\alpha}$. Secondly, we have

$$\|z\|_{\ell_1^g \otimes_\pi \|\cdot\|_{\bar{V}}} = \sum_{i=1}^g \|z_i\|_{\bar{V}}, \tag{23}$$

which is a general result in the theory of Banach spaces. The first claim, Equation (20), follows from Equations (22) and (23) together with Proposition 9.8 and 10.2.

The second claim in the statement, Equation (21), follows from duality, using Proposition 9.1, Corollary 5.9 and Theorem 10.2. □

We have the following important corollary, relating the inclusion sets with tensors norms.

**Corollary 10.5.** *For a centrally symmetric GPT $(V, V^+, \mathbb{1})$ we have*

$$\Gamma(g; V, V^+) = \Delta(g; V, V^+) = \Pi(g; V, V^+) = \Pi'(g; V, V^+)$$
$$= \{s \in [0,1]^g \,:\, \|z\|_{\ell_1^g \otimes_\varepsilon \|\cdot\|_{\bar{V}}} \leqslant 1 \implies \|s.z\|_{\ell_1^g \otimes_\pi \|\cdot\|_{\bar{V}}} \leqslant 1 \ \forall z \in \bar{V}^g\}$$
$$= \{s \in [0,1]^g \,:\, \|y\|_{\ell_\infty^g \otimes_\varepsilon \|\cdot\|_{\bar{A}}} \leqslant 1 \implies \|s.y\|_{\ell_\infty^g \otimes_\pi \|\cdot\|_{\bar{A}}} \leqslant 1 \ \forall z \in \bar{A}^g\}.$$

Importantly, in the case of the symmetric GPT $(E_g^*, (E_g^*)^+, \mathbb{1}_E)$ from Section 8.1, the base norm is the $\ell_\infty^g$ norm. Consider now the tensor $\varphi^{(f)}$ from (13). We shall write

$$\varphi^{(f)} = \check{\mathbb{1}}_E \otimes \mathbb{1} + \left[ \sum_{1=1}^g (2f_i(v_0) - 1)\gamma_i \right] \otimes \mathbb{1} + \underbrace{\sum_{1=1}^g \gamma_i \otimes [2f_i - 2f_i(v_0)\mathbb{1}]}_{\hat{\varphi}^{(f)}}$$

with $\hat{\varphi}^{(f)} \in \mathbb{R}^g \otimes \bar{A}$ playing the role of $\hat{M}$ in [LPW18, Proposition 27].

**Proposition 10.6.** *We have, for a tuple $f \in A^g$:*

$$f \text{ effects} \implies \|\hat{\varphi}^{(f)}\|_{\ell_\infty^g \otimes_\varepsilon \|\cdot\|_{\bar{A}}} = 2 \max_{i=1}^g \|f_i - f_i(v_0)\mathbb{1}\|_{\bar{A}} \leqslant 1 \tag{24}$$

*and*

$$f \text{ compatible effects} \implies \|\hat{\varphi}^{(f)}\|_{\ell_\infty^g \otimes_\pi \|\cdot\|_{\bar{A}}} \leqslant 1.$$

*If, moreover, the effects $f_i$ are* unbiased, *i.e.* $f_i(v_0) = 1/2$ *for all* $i \in [g]$, *the reverse implications hold.*

*Proof.* The results follow from Proposition 8.1 (resp. Proposition 8.3), together with [Lam18, Proposition 2.25] or [LPW18, Proposition 27]. □

**Remark 10.7.** *Note that the reverse implication in eq.* (24) *is true if and only if the effect* $f$ *is unbiased (we are considering here* $g = 1$*). Indeed, an element* $f = f_0\mathbb{1} + \bar{f} \in A$ *is an effect if and only if*

$$\begin{aligned} f \in A^+ &\iff \|\bar{f}\|_{\bar{A}} \leqslant f_0 \qquad \text{and} \\ \mathbb{1} - f \in A^+ &\iff \|\bar{f}\|_{\bar{A}} \leqslant 1 - f_0, \end{aligned}$$

*which give together* $\|\bar{f}\|_{\bar{A}} \leqslant \min(f_0, 1 - f_0)$*. However, from* (24) *gives* $\|\bar{f}\|_{\bar{A}} \leqslant 1/2$ *which is identical to the previous condition if and only if* $f_0 = \langle f, v_0 \rangle = 1/2$*, i.e. if the effect is unbiased.*

## 11. Open questions

We finish this note with a few questions we think would be interesting to pursue:

(1) What do the intermediate levels of the free spectrahedron correspond to? Are these restrictions to subspaces of $V$ of fixed dimension? We would like to recover the third statement of [BN18b, Theorem 5.3] for GPTs.
(2) Can we say anything about the uniqueness of the joint POVM?
(3) What does it mean that the elements in the free spectrahedron defined by the effects in [BN18b, Theorem 5.3] need not only be positive, but even separable?
(4) Can we characterize the theories admitting maximally incompatible effects from looking at their generalized spectrahedra?
(5) State all the results in these notes for QM, and analyze their consequences
(6) Spherical model is the same as qubits in QM
(7) Compare our definition of EB maps with the usual one in QM (write down)
(8) Connect inclusion of generalized spectrahedra to map extension
(9) Compute inclusion constants for generalized spectrahedra under generic requirements

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