Tensor product of dimension effect algebras

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Abstract

Key Words:

AMS Classification

1 Introduction

2 Preliminaries

The notion of an effect algebra was introduced by D.J. Foulis and M.K. Bennett in [6]. An alternative definition of so called *D-poset* was introduced in [10]. Effect algebras and D-posets are categorically equivalent structures [4].

2.1 Definition. An *effect algebra* is an algebraic system $(E; 0, 1, \oplus)$, where $\oplus$ is a partial binary operation and 0 and 1 are constants, such that the following axioms are satisfied for $a, b, c \in E$:

(i) if $a \oplus b$ is defined the $b \oplus a$ is defined and $a \oplus b = b \oplus a$ (commutativity);

(ii) if $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, the $a \oplus (b \oplus c)$ is defined and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (associativity);

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(iii) for every $a \in E$ there is a unique $a' \in E$ such that $a \oplus a' = 1$;

(iv) if $a \oplus 1$ is defined then $a = 0$.

In what follows, if we write $a \oplus b$, $a, b \in E$, we tacitly assume that $a \oplus b$ is defined in E. The operation $\oplus$ can be extended to the $\oplus$-sum of finitely many elements by recurrence in an obvious way. Owing to commutativity and associativity, the element $a_1 \oplus a_2 \oplus \cdots \oplus a_n$ is unambigously defined. In any effect algebra a partial order can be defined as follows: $a \leq b$ if there is $c \in E$ with $a \oplus c = b$. In this partial order, 0 is the smallest and 1 is the greatest element in E. Moreover, if $a \oplus c_1 = a \oplus c_2$, then $c_1 = c_2$, and we define $c = b \ominus a$ iff $a \oplus c = b$. In particular, $1 \ominus a = a'$ is called the *orthosupplement* of a. We say that $a, b \in E$ are *orthogonal*, written $a \perp b$, iff $a \oplus b$ exists in E. It can be shown that $a \perp b$ iff $a \leq b^{\perp}$. An effect algebra which is a lattice with respect to the above ordering is called a *lattice effect algebra*.

Important examples of effect algebras are obtained in the following way. Let $(G, G^+, 0)$ be a (additively written) partially ordered abelian group with a positive cone G^+ and neutral element 0. For $a \in G^+$ define the interval $G[0, a] := \{x \in G : 0 \leq x \leq a\}$. Then $G[0, a]$ can be endowed with a structure of an effect algebra by defining $x \perp y$ iff $x + y \leq a$, and then putting $a \oplus b := a + b$. Effect algebras obtained in this way are called *interval effect algebras*. We note that a prototype of effect algebras is the interval $[0, I]$ in the group of self-adjoint operators on a Hilbert space, so-called algebra of Hilbert space effects. Hilbert space effects play an important role in quantum measurement theory, and the abstract definition was motivated by this example.

Let E and F be effect algebras. A mapping $\phi : E \to F$ is an *effect algebra morphism* iff $\phi(1) = 1$ and $\phi(e \oplus f) = \phi(e) \oplus \phi(f)$ whenever $e \oplus f$ is defined in E. A morphism from E into the effect algebra $\mathbb{R}[0, 1]$ is called a *state* on E.

In what follows, we need some elements from the theory of partially ordered abelian groups.

G is said to have the *Riesz interpolation property* (RIP), or to be an *interpolation group*, if given a_i, b_j $(1 \leq i \leq m, 1 \leq j \leq n)$ with $a_i \leq b_j$ for all i, j, there exists $c \in G$ such that $a_i \leq c \leq b_j$ for all i, j. The Riesz interpolation property is equivalent to the *Riesz decomposition property* (RDP): given $a_i, b_j \in G^+$, $(1 \leq i \leq m, 1 \leq j \leq n)$ with $\sum a_i = \sum b_j$, there exist

$c_{ij} \in G^+$ with $a_i = \sum_j c_{ij}, b_j = \sum_i c_{ij}$. An equivalent definition of the RDP is as follows: given a, b_i in G^+, $i \leq n$ with $a \leq \sum_{i \leq n} b_i$, there exist $a_i \in G^+$ with $a_i \leq b_i, i \leq n$, and $a = \sum_{i \leq n} a_i$. To verify these properties, it is only necessary to consider the case $m = n = 2$ (cf. [**?**, 7]).

Let (G, G^+) be an abelian partially ordered group. Then G is *directed* iff $G = G^+ - G^+$. An element $u \in G^+$ is an *order unit* iff for all $a \in G$, $a \leq nu$ for some $n \in \mathbb{N}$. If G has an order unit u, it is directed, indeed, if $g \leq nu$, then $g = nu - (nu - g)$. An element $u \in G^+$ is called a *generating unit* iff every $a \in G^+$ is a finite sum of (not necessarily different) elements of the interval $G[0, u]$. Clearly, a generating unit is order unit, and if G is an interpolation group, then the RDP implies that conversely, every order unit is generating.

G is called *unperforated* iff given $n \in \mathbb{N}$ and $a \in G$, then $na \in G^+$ implies $a \in G^+$. G is *archimedean* iff whenever $x, y \in G$ such that $nx \leq y \forall n \in \mathbb{N}$, then $x \leq 0$. Every archimedean, directed abelian group is unperforated [7, Prposition 1.24], and also every lattice ordered abelian group is unperforated [7, Proposition 1.22].

If G and H are partially ordered abelian groups, then a group homomorphism $\phi : G \to H$ is *positive* iff $\phi(G^+) \subseteq H^+$. An isomorphism $\phi : G \to H$ is an *order isomorphsim* iff $\phi(G^+) = H^+$. If G and H have order units u and v, respectively, then a positive homomorphism $\phi : G \to H$ is called *unital* iff $\phi(u) = v$. For a fixed order unit $u \in G^+$, a positive homomorphism $f : G \to \mathbb{R}$ is called a *state* iff $f(u) = 1$.

In a similar way as in partially ordered abelian groups, RIP and RDP can be defined in effect algebras. We say that an effect algebra E has the *Riesz decomposition property* (RDP) if one of the following equivalent properties is satisfied:

(R1) $a \leq b_1 \oplus b_2 \oplus \cdots \oplus b_n$ implies $a = a_1 \oplus a_2 \oplus \cdots \oplus a_n$ with $a_i \leq b_i, i \leq n$;

(R2) $\oplus_{i \leq m} a_i = \oplus_{j \leq m} b_j$, $m, n \in \mathbb{N}$, implies $a_i = \oplus_j c_{ij}, i \leq m$, and $b_j = \oplus_i c_{ij}, j \leq n$, where $c_{ij} \in E$.

Similarly as for partially ordered groups, it suffices to consider the case $m = n = 2$.

An effect algebra E has RIP iff $a_1, a_2, b_1, b_2 \in E$ and $a_i \leq b_j$ for all $i, j = 1, 2$, there is an element $c \in E$ such that $a_i \leq c \leq b_j$ for all $i, j = 1, 2$. To the difference with partially ordered abelian groups, RIP and RDP in effect algebras are not equivalent: RDP implies RIP, but there are examples of effect algebras with RIP which do not have RDP (e.g., the "diamond" is lattice ordered effect algebra that does not satisfy RDP, [4]).

Relations between interval effect algebras and partially ordered abelian groups are described in the following theorem [?]. Recall that a mapping $\phi : E \to K$, where E is an effect algebra and K is any abelian group, is called a *K-valued measure* on E iff $\phi(a \oplus b) = \phi(a) + \phi(b)$ whenever $a \oplus b$ is defined in E.

2.2 Theorem. *Let E be an interval effect algebra. Then there exists a unique (up to isomorphism) partially ordered directed abelian group (G, G^+) and an element $u \in G^+$ such that the following conditions are satisfied:*

(i) *E is isomorphic to the interval effect algebra $G^+[0, u]$.*

(ii) *u is a generating unit.*

(iii) *Every K-valued measure $\phi : E \to K$ can be extended uniquely to a group homomorphism $\phi^* : G \to K$.*

The group G in the preceding theorem is called a *universal group* for E, and will be denoted by G_E. There are effect algebras that are not interval effect algebras, see e.g. [?]. Moreover, not every unital group is the universal group for its unit interval (see [?, Example 11.3, 11.5] or [9, Example 3.6]). But for interpolation groups we have the following theorem [?], [9, Theorem3.5].

2.3 Theorem. *Let G be an interpolation group with order unit u. Put $E := G^+[0, u]$. Then (G, u) is the universal group for E.*

It was proved by Ravindran [?], that every effect algebra with RDP is an interval effect algebra, and its universal group is an interpolation group. We have the following lemma [9, Lemma 3.7].

2.4 Lemma. *Let E and F be effect algebras with RDP and let (G_E, u) and (G_F, v) be their universal groups respectively. Every effect algebra morphism $\Phi : E \to F$ uniquely extends to a unital group homomorphism $\phi^* : G_E \to G_F$.*

Ravindran's result can be extended to a categorical equivalence between the category **E** of effect algebras with RDP with effect algebra morphisms and the category **IG** of interpolation groups with order unit with positive unital group homomorphisms.

2.5 Theorem. [9, Theorem 3.8] *Denote by* $S : \mathbf{E} \to \mathbf{IG}$ *the map that to every effect algebra* $E \in \mathbf{E}$ *assigns its universal group* G_E*, and let* $T : \mathbf{IG} \to \mathbf{E}$ *denote the map that to every* $(G, u) \in \mathbf{IG}$ *assigns its unit interval* $G^+[0, u]$.

(i) $S : \mathbf{E} \to \mathbf{IG}$ *and* $T : \mathbf{IG} \to \mathbf{E}$ *are functors;*

(ii) *there is a categorical equivalence between the category* $\mathbf{E}$ *of effect algebras with RDP and the category* $\mathbf{IG}$ *of interpolation groups with order units.*

Notice that the preceding theorem extends the well known categorical equivalence between MV-algebras and lattice ordered groups with order unit proved in [**?**]. We recall that lattice ordered effect algebras with RDP are called *MV-effect algebras*, and MV-affect algebras are equivalent to MV-algebras introduced by Chang [1].

3 Dimension groups and dimension effect algebras

3.1 Definition. [7, Definition p. 183], [8] A *simplicial group* is any partially ordered abelian group that is isomorphic (as partially ordered abelian group) to $\mathbb{Z}^n$ (with the product ordering) for some nonnegative integer n. A *simplicial basis* for a simplicial group G is any basis $(x_1, \ldots, x_n)$ for G as a free abelian group such that $G^+ = \mathbb{Z}^+ x_1 + \cdots + \mathbb{Z}^+ x_n$.

3.2 Definition. [7] A partially ordered group G is a *dimension group* (or a *Riesz group*) if it is directed, unperforated and has the interpolation property.

The following result was proved by Effros, Handelman and Shen [5].

3.3 Theorem. *Let* (G, u) *be a nonzero partially ordered dimension group with order unit. Then* (G, u) *is isomorphic (as partially ordered abelian groups with order unit) to a direct limit of a directed system of simplicial groups with order unit and unital positive homomorphisms.*

An element $v \in \mathbb{Z}^r$ is an order unit if and only if all of its coordinates are strictly positive. If $\mathbf{v} = (v_1, v_2. \ldots, v_r)$ is an order unit, then the interval $(\mathbb{Z}^+)^r[0, v]$ is the direct product of finite chains $(0, 1, \ldots, v_i), i = 1, 2, \ldots, r$. It is a finite effect algebra with RDP, therefore it is a lattice, a hence an MV-algebra. Moreover, every finite effect algebra with RDP is a unit interval in a simplicial group.

By Theorem 3.3, the dimension groups are direct limits of directed systems of simplicial groups. In analogy, in [9], direct limits of directed systems of finite effect algebras with RDP have been called *dimension effect algebras*. It was shown in [9], that an effect algebra is a dimension effect algebra iff its universal group is a dimension group. It is clear that an effect algebra with RDP is a dimension effect algebra iff its universal group is unperforated. A characterization of effect algebras with RDP whose universal groups are unperforated was given in [9, Theorem 4.2]. This yields the following intrinsic characterization of dimension effect algebras.

3.4 Theorem. *An effect algebra E is a dimension effect algebra iff the following conditions are satisfied:*

(i) *E has RDP;*

(ii) *if $c_{i,j}^{k,\ell} \in E$, $i,j = 1,\dots,n, k = 1,\dots,m, \ell = 1,\dots p$, where $n,m,p \in \mathbb{N}$, are such that*

$$\oplus_{j,k} c_{i,j}^{k,\ell} = a_\ell^i = a_\ell, \text{ for all } i,\ell, \tag{1}$$
$$\oplus_{i,\ell} c_{i,j}^{k,\ell} = b_k^j \leq b_k, \text{ for all } j,k, \tag{2}$$

for some elements $a_\ell, b_k \in E$, then there is some $q \in \mathbb{N}$ and some elements $d_{k,\ell} \in E$, $k = 1,\dots,m, \ell = 1,\dots,p+q$, such that

$$\oplus_k d_{k,\ell} = a_\ell, \ \ell = 1,\dots,p, \tag{3}$$
$$\oplus_\ell d_{k,\ell} \leq b_k, \text{ for all } k = 1,\dots,m. \tag{4}$$

For the convenience of the readers, we give a short description of the directed system and direct limit of effect algebras [4, Definition 1.9.36]..

A *directed system of effect algebras* is a family $A_I := (A_i; (f_{ij} : A_i \to A_j, i,j \in I, i \leq j)$ where $(I, \leq)$ is a directed set, A_i is an effect algebra for each $i \in I$, and f_{ij} is a morphism $(i \leq j)$ such that

(i1) $f_{ii} = id_{A_i}$ fir every $i \in I$,

(i2) if $i \leq j \leq m$ in I, then $f_{jm} f_{ij} = f_{im}$.

Let A_I be a directed system of effect algebras, then $\underline{f} := (A; (f_i : A_i \to A; i \in I))$ is called the *direct limit* of A_I iff the following conditions hold:

(ii1) A is an effect algebra; f_i is a morphism for eah $i \in I$;

(ii2) if $i \leq j$ in I, then $f_j f_{ij} = f_i$ (i.e., $\underline{f}$ is compatible with A_I).

(ii3) If $\underline{g} := (B; (g_i : A_i \to B, i \in I))$ is any system compatible with A_I (i.e., $g_j f_{ij} = g_i$ for all $i \leq j$ in I), then there exists exactly one morphism $g : A \to B$ such that $g f_i = g_i$, for every $i \in I$.

It was proved (cf. [4, Thorem 1.9.27]) that the direct limit in the category of effect algebras exists. A sketch of the construction of the direct limit is as follows. Let $A = \dot\bigcup_{i\in I} A_i$ be the disjoint union of $A_i, i \in I$. Define a relation $\equiv$ on A as follows. Put $a \equiv b$ ($a \in A_i, b \in A_j$) if there exists a $k \in I$ with $i, j \leq k$ such that $f_{ik}(a) = f_{jk}(b)$ in A_k. Then $\equiv$ is an equivalence relation, and the quotient $\bar{A} := A/\equiv$ can be organized into an effect algebra with the operation $\oplus$ defined as follows: let $\bar{a}$ denotes the equivalence class corresponding to a. For $a \in A_I, b \in A_k$, $\bar{a} \oplus \bar{b}$ is defined iff there is $k \in I$, $i, j \leq k$ such that $f_{ik}(a) \oplus f_{jk}(b)$ exists in A_k, and then $\bar{a} \oplus \bar{b} = \overline{(f_{ik}(a) \oplus f_{jk}(b))}$ in A_k. For every $i \in I$, define $f_i : A_i \to A/\equiv$ as the natural projection $f_i(a) = \bar{a}$. Then $\lim_{\to} A := (\bar{A}; f_i : A_i \to \bar{A}, i \in I)$ is the desired direct limit.

From the construction of the direct limit it is easy to derive that direct limits of directed systems of effect algebras with RDP are effect algebras with RDP.

4 Tensor product of dimension effect algebras

Tensor product of effect algebras (resp. D-posets) was introduced in [?, ?], see also [4, Chap. 4.2].

Let P, Q, L be effect algebras. A mapping $\beta : P \times Q \to L$ is called a *bimorphism* iff

(i) $a, b \in P$ with $a \perp b$, $q \in Q$ imply $\beta(a, q) \perp \beta(b, q)$ and $\beta(a \oplus b, q) = \beta(a, q) \oplus \beta(b, q)$;

(ii) $c, d \in Q$ with $c \perp d$, $p \in P$ imply $\beta(p, c) \perp \beta(p, d)$ and $\beta(p, (c \oplus d)) = \beta(p, c) \oplus \beta(p, d)$;

(iii) $\beta(1, 1) = 1$.

4.1 Definition. Let P and Q be effect algebras. A pair (T, τ) consisting of an effect algebra T and a bimorphism $\tau : P \times Q \to T$ is said to be the *tensor product* of P and Q iff the following conditions are satisfied:

(i) If L is an effect algebra and $\beta : P \times Q \to L$ is a bimorphism, then there exists a morphism $\phi : T \to L$ such that $\beta = \phi \circ \tau$.

(ii) Every element of T is a finite orthogonal sum of elements of the form $\tau(p, q)$ with $p \in P, q \in Q$.

It is clear that if the tensor product exists, it is unique up to isomorphism. The following theorem was proved in [**?**, Theorem 7.2], see also [4, Theorem 4.2.2]..

4.2 Theorem. *Effect algebras P and Q admit a tensor product if and only if there is at least one effect algebra L and bimorphism $\beta : P \times Q \to L$.*

There are examples of effect algebras for which the tensor product in the category of effect algebras does not exist, see e.g., [4, Example 4.2.4]. On the other hand, if the effect algebras P and Q both have at least one state, then the the tensor product of P and Q exists [4, Theorem 4.2.3].

Let E and F be finite MV-effect algebras, and let

$$\tilde{e}_1 = (e_1, 0, \dots, 0), \tilde{e}_2 = (0, e_2,) \dots, 0), \dots \tilde{e}_n = (0, 0, \dots, e_n),$$
$$\tilde{f}_1 = (f_1, 0, \dots, 0), \tilde{f}_2 = (0, f_2, 0 \dots, 0), \dots, \tilde{f}_m) = (0, 0, \dots, f_m),$$

with units $u = \sum_{i=1}^n u_i \tilde{e}_i$, and $v = \sum_{j=1}^m v_j \tilde{f}_j$ be the bases for E and F.

Since E and F have states, their tensor product in the category of effect algebras exists. Let $\otimes : E \times F \to E \otimes F$ be the tensor product. If $x \in E, y \in F$, $x = \sum_{i=1}^n x_i \tilde{e}_i$, $y = \sum_{j=1}^m y_j \tilde{f}_j$, since $\otimes$ is a bimorphism, we have

$$x \otimes y = \sum_{i=1}^n \sum_{j=1}^m x_i y_j \tilde{e}_i \otimes \tilde{f}_j, 0 \leq x_i \leq u_i, 0 \leq y_j \leq v_j \; \forall i, j$$

and $u \otimes v = \sum_{i=1}^n \sum_{j=1}^m u_i v_j \tilde{e}_i \otimes \tilde{f}_j$ is the unit element in $E \otimes F$.

Every element in the tensor product is the orthosum of simple tensors, i.e., it is of the form

$$z = \sum_{\ell=1}^k x^\ell y^\ell = \sum_{\ell=1} \sum_{i=1}^n \sum_{j=1}^m x_i^\ell y_j^\ell \tilde{e}_i \otimes \tilde{f}_j, \tag{5}$$

with $0 \leq x_i^\ell \leq u_i,\ 0 \leq y_j^\ell \leq v_j$, for all i, j.

Put $g_{ij} := \tilde{e}_i \otimes \tilde{f}_j$, then we may write $x \otimes y = \sum_{i=1}^n \sum_{j=1}^m (x \otimes y)_{ij} g_{ij}$, where $(x \otimes y)_{ij} = \sum_{\ell=1}^k x_i^\ell y_j^\ell$. Clearly, $(x \otimes y)_{ij} \leq (u \otimes v)_{ij} = u_i v_j$.

In what follows, we need the following lemma.

4.3 Lemma. *Let $z, m, n \in \mathbb{Z}$ and $z \leq mn$. Then there are m_i, n_i, $i = 1, 2, \dots k$ such that $z = \sum_{i=1}^k n_i m_i$, $n_i \leq n, m_i \leq n$.*

Proof. If $z \leq m$ or $z \leq n$, we may put $m_1 = z, n_1 = 1$, or $m_1 = 1, n_1 = z$. Assume $m \leq z$ and $n \leq z$. Then $1 \leq \frac{z}{m}$. Let $r \in \mathbb{Z}$ be maximal such that $r \leq \frac{z}{m}$. Then

$$mr \leq z \leq mn \implies r \leq n,$$

and we may put

$$z = rm + (z - rm), \ z - rm \leq z \leq mn \implies \frac{z}{m} - r \leq n.$$

We have $r \leq n$ and $z - rm \leq m$. Indeed, if $m < z - rm$, then $0 \leq z - (r+1)m$, which contradicts maximality of r. $\square$

4.4 Theorem. *Tensor product of finite MV-effect algebras in the category of effect algebras is a finite MV-effect algebra.*

Proof. If $z_{ij} \leq u_i v_j$, $0 \leq i \leq n$, $0 \leq j \leq m$, then $\sum_{i=1}^n \sum_{j=1}^m z_{ij} \tilde{e_i} \otimes \tilde{f_j} \leq \sum_{i=i}^n \sum_{j=1}^m u_i v_j \tilde{e_i} \otimes \tilde{f_j} = u \otimes v$. By Lemma 4.3, every such element can be rewritten in the form (5), so it is an element of the tensor product $E \otimes F$. From this it follows that $E \otimes F$ is a direct product of finite chains with the basis $(\tilde{e_i} \otimes \tilde{f_j}), 0 \leq i \leq n, 0 \leq j \leq m$, and with the unit $u \otimes v$. $\square$

4.5 Theorem. *Let* $A_I = (A_i; f_{ij} : A_i \to A_j); i, j \in I, i \leq j)$ *and* $B_J = (B_k; (g_{k\ell} : B_k \to B_\ell; k, \ell \in J, k \leq \ell)$ *be directed systems of finite MV-effect algebras. Define*

$$A_I \otimes B_J := (A_i \otimes B_k; (f_{ij} \otimes g_{k\ell} : A_i \otimes B_k \to A_j \otimes B_\ell), \\ i, j \in I, k, \ell \in J, i \leq j, k \leq \ell).$$

Then $A_I \otimes B_J$ *is a directed system of finite MV-algebras.*

Proof. Each $A_i \otimes B_k$, $i \in I, k \in J$ is a finite MV-effect algebra, and $f_{ij} \otimes g_{k\ell}$ is a tensor product of morphisms. On simple tensors we have $f_{ij} \otimes g_{k,\ell}(a \otimes b) = f_{ij}(a) \otimes g_{k\ell}(b), a \in A_i, b \in B_k$, which is a bimorphism from $A_i \times B_k$ to $A_j \otimes B_\ell$. By properties of tensor product, it extends to a morphism (also denoted by $f_{ij} \otimes g_{k\ell}$) from $A_i \otimes B_k$ to $A_j \otimes B_\ell$

We have to check properties (i1) and (i2).

(i1): $f_{ii} = id_{A_i}$, $g_{kk} = id_{B_k}$ impliy $f_{ii} \otimes g_{kk} = id_{A_i \otimes B_k}$.

(i2): $(i, k) \leq (j, \ell) \leq (m, n)$. Then

$$i \leq j \leq m \implies f_{jm} f_{ij} = f_{im}$$

$$k \le \ell \le n \implies g_{\ell n} g_{k\ell} = g_{kn}$$

and for $a_i \in A_i, b_k \in B_k$,

$$\begin{aligned}
f_{jm} \otimes g_{\ell n} f_{ij} \otimes g_{k\ell}(a_i \otimes b_k) &= f_{jm} \otimes g_{\ell n} f_{ij}(a_i) \otimes g_{k\ell}(b_k) \\
&= f_{jm} f_{ij}(a_i) \otimes g_{\ell n} g_{k\ell}(b_k) \\
&= f_{im}(a_i) \otimes g_{kn}(b_k) \\
&= f_{im} \otimes g_{kn}(a_i \otimes b_k).
\end{aligned}$$

Since this holds on simple tensors, it extends to whole $A_i \otimes B_k$.

□

4.6 Theorem. *Let A_I, B_J be directed systems of finite MV-effect algebras, and let $(\bar{A}; (f_i : A_i \to \bar{A}, i \in I))$ and $(\bar{B}; (g_j : B_j \to \bar{B}, j \in J))$ be their corresponding direct limits. Then $(\bar{A} \otimes \bar{B}; (f_i \otimes g_j : A_i \otimes B_J \to \bar{A} \otimes \bar{B}, i \in I, j \in J))$ is the direct limit of $A_I \otimes B_J$.*

Proof. We have to cheque properties (ii1), (ii2) and (ii3).

(ii1): Since $\bar{A}$, $\bar{B}$ are effect algebras, $\bar{A} \otimes \bar{B}$ is an effect algebra as well.

(ii2) (compatibility): $(i, k) \le (j, \ell)$ iff $i \le j, k \le \ell$. Then $(f_i \otimes g_k)(f_{ij} \otimes g_{k\ell}) = f_i f_{ij} \otimes g_k g_{k\ell} = f_j \otimes g_\ell$.

(ii3): Let $(C; (h_{ij} : A_i \otimes B_j \to C, i \in I, j \in J))$ be another system compatible with $A_I \otimes B_J$ (i.e., $h_{mn}(f_{im} \otimes g_{j\ell} = h_{ij}, i \le m, j \le n)$. Define, on simple tensors, $h(f_i \otimes g_j)(a, b) = h(f_i(a) \otimes g_j(b)) := h_{ij}(a \otimes b)$, $a \in A_i, b \in B_j$. Then $h : \bar{A} \times \bar{B} \to C$ is a bimorphism, which extends to a morphism $\bar{h} : \bar{A} \otimes \bar{B} \to C$. □

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