A geometric view on quantum incompatibility

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Outline

- Introduction
- GPT: basic definitions and examples
- Incompatibility:
  - characterization
  - incompatibility witnesses and degree
  - maximal incompatibility
- Incompatibility and Bell non-locality
- Steering

General probabilistic theories: basic notions

states: preparation procedures of a given system

- convex structure: probabilistic mixtures of states

> **Assumption**: Any state space is a compact convex subset $K \subset \mathbb{R}^m$.

effects: yes/no experiments

- determined by outcome probabilities in each state
- respect the convex structure of states: affine maps $K \to [0,1]$

> **Assumption**: All affine maps $K \to [0,1]$ correspond to effects.

For the more general framework, see e.g (G. Chiribella, G. D'Ariano, P. Perinotti, PRA 2010)

General probabilistic theories: basic notions

measurements: (with finite number of outcomes)

- described by outcome statistics in each state
- affine maps $K \to \Delta_n$
  Δ_n: simplex of probabilities over $\{0, \ldots, n\}$
- given by effects:

$$f_i(x) = f(x)_i, \ i = 0, \ldots, n, \qquad \sum_i f_i = 1$$

Assumption: All affine maps $K \to \Delta_n$ correspond to measurements.

General probabilistic theories: basic examples

Classical systems:

- state spaces: Δ_m
- effects: vectors in $\mathbb{R}^{m+1}$ with entries in $[0,1]$
- measurements: classical channels $T : \Delta_m \to \Delta_n$

The measurements are identified with $(m+1) \times (n+1)$ stochastic matrices (conditional probabilities) $\{T(j|i)\}_{i,j}$:

$$T(j|i) = f(\delta_i^m)_j, \quad \delta_i^m \; = \; \text{vertices of } \Delta_m$$

General probabilistic theories: basic examples

Quantum systems

- state spaces: $\mathfrak{S}(\mathcal{H})$ = density operators on a Hilbert space $\mathcal{H}$, $\dim(\mathcal{H}) < \infty$
- effects: $E(\mathcal{H})$ = quantum effects,

$$0 \le E \le I, \qquad E \in B(\mathcal{H})$$

- measurements: POVMs on $\mathcal{H}$

$$M_0, \ldots, M_n \in E(\mathcal{H}), \quad \sum_i M_i = I$$

General probabilistic theories: basic examples

Spaces of quantum channels

- state spaces: $\mathcal{C}_{A,A'}$ = set of all quantum channels (CPTP maps) $B(\mathcal{H}_A) \to B(\mathcal{H}_{A'})$
- effects: $f \in E(\mathcal{C}_{A,A'})$,

$$f(\Phi) = \mathrm{Tr}\, M(\Phi \otimes id_R)(\rho_{AR}), \quad \Phi \in \mathcal{C}_{A,A'},$$

  for some state $\rho_{AR} \in \mathfrak{S}(\mathcal{H}_{AR})$ and effect $M \in E(\mathcal{H}_{A'R})$
- measurements: $f_0, \dots, f_n$,

$$f_i(\Phi) = \mathrm{Tr}\, M_i(\Phi \otimes id_R)(\rho_{AR}), \quad \Phi \in \mathcal{C}_{A,A'},$$

  for some $\rho_{AR} \in \mathfrak{S}(\mathcal{H}_{AR})$ and a POVM $\{M_0, \dots, M_n\}$ on $\mathcal{H}_{A'R}$.

GPT and ordered vector spaces

Ordered vector space: (V, V^+)

- a real vector space V ($\dim(V) < \infty$)
- a closed convex cone $V^+ \subset V$, generating in V, $V^+ \cap -V^+ = \{0\}$

Dual OVP: an ordered vector space $(V^*, (V^+)^*)$

- vector space dual V^*
- dual cone

$$(V^+)^* = \{\varphi \in V^*, \langle \varphi, x \rangle \geq 0, \forall x \in V\}$$

We have $V^{**} = V$, $(V^+)^{**} = V^+$.

GPT and ordered vector spaces

Any state space K determines an OVP:

- $A(K)$ = all affine functions $K \to \mathbb{R}$
- $A(K)^+$ = positive affine functions
- $E(K) = \{f \in A(K), 0 \leq f \leq 1_K\}$, 1_K is the constant unit function

Then $(A(K), A(K)^+)$ is an OVP, $E(K)$ is the set of all effects.

A norm in $A(K)$:

$$\|f\|_{max} = \max_{x \in K} |f(x)|$$

GPT and ordered vector spaces

Let $(V(K), V(K)^+)$ be the dual OVP.

- $K \simeq \{\varphi \in V(K)^+, \langle \varphi, 1_K \rangle = 1\}$ a base of $V(K)^+$
- $V(K)^+ \simeq \cup_{\lambda \geq 0} \lambda K$ the cone generated by K
- $V(K) \simeq$ the vector space generated by K

Base norm:

$$\|\psi\|_K = \inf\{a + b,\ \psi = ax - by,\ a, b \geq 0, x, y \in K\}, \psi \in V(K)$$

- the dual norm to $\| \cdot \|_{max}$.

GPT and ordered vector spaces: self-duality

We say that the cone V^+ is (weakly) self-dual if $V^+ \simeq (V^+)^*$

- classical: $V(\Delta_n)^+ \simeq A(\Delta_n)^+ (\simeq (\mathbb{R}^{n+1})^+)$
- quantum: $V(\mathfrak{S}(\mathcal{H}))^+ \simeq A(\mathfrak{S}(\mathcal{H}))^+ (\simeq B(\mathcal{H})^+)$
- not true for spaces of quantum channels
- not true for all spaces of classical channels

Composition of state spaces: tensor products

Assumption: For state spaces K_A and K_B, the joint state space $K_A \tilde{\otimes} K_B$ is a subset in $V(K_A) \otimes V(K_B)$.

We have:

$$K_A \otimes_{min} K_B \subseteq K_A \tilde{\otimes} K_B \subseteq K_A \otimes_{max} K_B$$

minimal tensor product: separable states

$$K_A \otimes_{min} K_B = co\{x_A \otimes x_B,\ x_A \in K_A,\ x_B \in K_B\}$$

maximal tensor product: no-signalling

$$K_A \otimes_{max} K_B := \{y \in V(K_A) \otimes V(K_B), \langle f_A \otimes f_B, y\rangle \geq 0, \\ \langle 1_A \otimes 1_B, y\rangle = 1\}$$

Composition of state spaces: tensor products

classical:

- $\Delta_{n_A} \otimes_{min} \Delta_{n_B} = \Delta_{n_A} \otimes_{max} \Delta_{n_B} = \Delta_{n_{AB}}$
- the probability simplex on $\{0, \ldots, n_A\} \times \{0, \ldots, n_B\}$

quantum:

- $\mathfrak{S}(\mathcal{H}_A) \widetilde{\otimes} \mathfrak{S}(\mathcal{H}_B) = \mathfrak{S}(\mathcal{H}_{AB})$
- $\mathfrak{S}(\mathcal{H}_A) \otimes_{min} \mathfrak{S}(\mathcal{H}_B)$ separable states
- $\mathfrak{S}(\mathcal{H}_A) \otimes_{max} \mathfrak{S}(\mathcal{H}_B)$ normalized entanglement witnesses

quantum channels:

- $\mathcal{C}_{A,A'} \widetilde{\otimes} \mathcal{C}_{B,B'} = \mathcal{C}^{caus}_{AB,A'B'}$ causal bipartite channels
- $\mathcal{C}_{A,A'} \otimes_{min} \mathcal{C}_{B,B'} = \mathcal{C}^{loc}_{AB,A'B'}$ local bipartite channels
- $\mathcal{C}_{A,A'} \otimes_{max} \mathcal{C}_{B,B'}$ causal, not necessarily CP

Channels and positive maps

Channels: transformations of the systems allowed in the theory

- affine maps between state spaces $K \to K'$
- affine maps $K \to V(K')^+$ extend to positive maps of the ordered vector spaces

$$(V(K), V(K)^+) \to (V(K'), V(K')^+)$$

not all affine maps are allowed in general:

- $\Delta_n \to \Delta_m$: all classical channels
- $\mathfrak{S}(\mathcal{H}) \to \mathfrak{S}(\mathcal{H}')$: must be completely positive

Entanglement breaking maps

A positive map $T_A : K_A \to V(K'_A)^+$ is entanglement breaking (ETB) if

$$(T_A \otimes id_B)(K_A \otimes_{max} K_B) \subseteq V(K'_A \otimes_{min} K_B)^+$$

for all state spaces K_B.

> T_A is ETB iff it factorizes through a simplex:
>
> $$T_A : K \xrightarrow{g} \Delta_n \xrightarrow{T_0} V(K')^+$$
>
> (measure (g) and "prepare" (T_0))

Duality

The space of all linear maps $V(K) \to V(K')$, with the cone of positive maps is an ordered vector space.

Its dual is the space of linear maps $V(K') \to V(K)$, with the cone of positive ETB maps, duality:

$$\langle T, T' \rangle = \mathrm{Tr}\, TT'$$

Polysimplices

A **polysimplex** is a Cartesian product of simplices

$$S_{l_o,\ldots,l_k} := \Delta_{l_0} \times \cdots \times \Delta_{l_k}$$

with pointwise defined convex structure.

- states of a device specified by inputs and allowed outputs
- theories exhibiting super-quantum correlations

(S. Popescu, D. Rohrlich, Found. Phys. 1994; J. Barrett, PRA 2007; P. Janotta, R. Lal, PRA 2013)

Polysimplices

$\mathsf{S} = \mathsf{S}_{l_o,\dots,l_k}$:

- convex polytope, with vertices

$$\mathsf{s}_{n_0,\dots,n_k} = (\delta^{l_0}_{n_0}, \dots, \delta^{l_k}_{n_k})$$

  δ^i_j is the j-th vertex of Δ_{l_i}
- $A(\mathsf{S})^+$: generated by effects of the projections

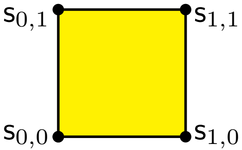
$$\mathsf{m}^i : \mathsf{S}_{l_0,\dots,l_k} \to \Delta_{l_i},\ \mathsf{m}^i_0, \dots, \mathsf{m}^i_{l_i} \in E(\mathsf{S}),$$

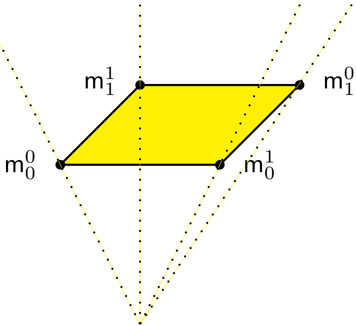
  The base of $A(\mathsf{S})^+$ is the dual polytope.

Polysimplices: examples

Square (gbit, square-bit): $\square = \Delta_1 \times \Delta_1$

- $V(\square)^+ \simeq A(\square)^+$ - weakly self-dual
- the only polysimplex with this property



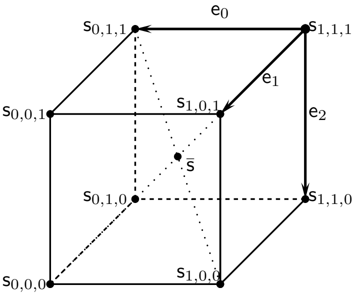


the cone $A(\square)^+$

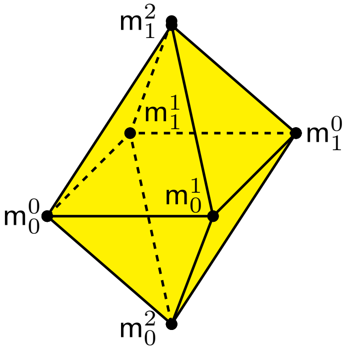
Polysimplices: examples

Hypercube: $\square_n = \Delta_1 \times \cdots \times \Delta_1$

- base of $A(\square_n)^+$: a cross-polytope



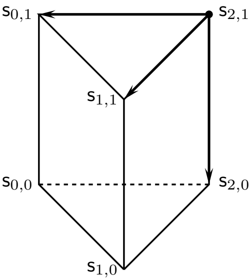
the cube $\square_3$



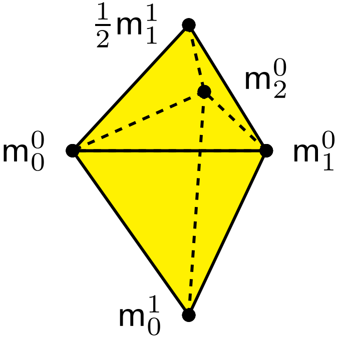
octahedron

Polysimplices: examples

Prism:



$S_{2,1} = \Delta_2 \times \Delta_1$



a base of $A(S_{2,1})^+$

Polysimplices and classical channels

Let $\mathsf{S} = \Delta_n^{k+1}$.
A correspondence between $s \in \Delta_n^{k+1}$ and stochastic matrices T:

$$T(j|i) = \mathsf{m}_j^i(s), \qquad s = (T(\cdot|0), \dots, T(\cdot|k))$$

Δ_n^{k+1} is isomorphic to the set of all classical channels

$$\Delta_k \to \Delta_n.$$

Any polysimplex is isomorphic to a face in a set of classical channels.

Polysimplices and quantum channels

There are channels $R : C_{A,A'} \to \Delta_n^m$ and $R' : \Delta_n^m \to C_{A,A'}$, such that

$$RR' = id.$$

The maps are determined by ONBs $\{|i_A\rangle\}$, $\{|j_{A'}\rangle\}$ as

$$R(\Phi)(j|i) = \langle j, \Phi(|i\rangle\langle i|_A)|j\rangle_{A'}, \quad \forall i,j; \Phi \in \mathcal{C}_{A,A'}$$

$$R'(s)(\rho) = \sum_{i,j} \mathrm{m}_j^i(s)\langle i, \rho|i\rangle_A |j\rangle\langle j|_{A'}, \quad \rho \in \mathfrak{S}(\mathcal{H}_A);\ s \in \Delta_n^m.$$

Such maps are called: R - retraction, R' - section. Note that $R'R$ is a projection (onto a set of c-c channels).

Incompatible measurements in GPT

A collection of measurements $f^0, \dots, f^k$, $f^i : K \to \Delta_{l_i}$, is the same as a channel $F = (f^0, \dots, f^k) : K \to \mathsf{S}_{l_0,\dots,l_k}$:

$$F(x) = (f^0(x), \dots, f^k(x)), \quad f^i = \mathrm{m}^i F,\ i = 0, \dots, k$$

- compatible: marginals of a single joint measurement

$$g : K \to \Delta_L = \Delta_{l_0} \otimes \cdots \otimes \Delta_{l_k}$$

- that is, $(f^0, \dots, f^k) : K \xrightarrow{g} \Delta_L \xrightarrow{J} \mathsf{S}$

$f^0, \dots, f^k$ are compatible if and only if $(f^0, \dots, f^k)$ is ETB.

Incompatibility witnesses

By duality of the spaces of maps:

$F = (f^0, \ldots, f^k) : K \to \mathsf{S}$ is incompatible if and only if there is an incompatibility witness: a map $W : \mathsf{S} \to V(K)^+$ such that

$$\operatorname{Tr} FW < 0$$

Incompatibility witnesses

Any $W : \mathsf{S} \to V(K)^+$ is determined by images of vertices:

$$w_{n_0,\dots,n_k} = W(\mathsf{s}_{n_0,\dots,n_k})$$

W is ETB iff there are $\psi^i_j \in V(K)^+$ such that

$$w_{n_0,\dots,n_k} = \sum_i \psi^i_{n_i}$$

Incompatibility witnesses

A witness must be non-ETB, but this is not enough

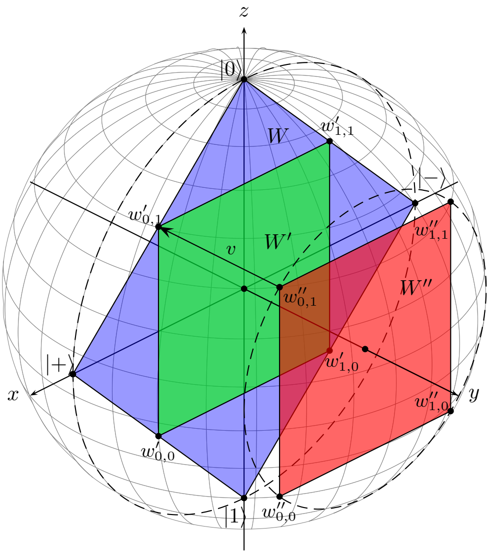
Characterization of witnesses: $W : S \to V(K)^+$ is a witness iff no translation of W along K is ETB.

Translation along K: $\tilde{W} : S \to V(K)^+$, such that

$$\tilde{W}(s) = W(s) + v,$$

for some $\langle 1_K, v \rangle = 0$.

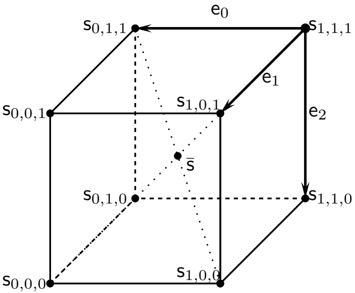
Incompatibility witnesses in Bloch ball



Incompatibility witnesses for two-outcome measurements

We have another characterization if S is a hypercube $\Box_{k+1}$:
Let $W : \Box_{k+1} \to V(K)^+$

pick a vertex: $\mathsf{s}_{n_0,\ldots,n_k}$
all adjacent edges: $e_0, \ldots, e_k$



$$\sum_{i=0}^{k} \|W(e_i)\|_K > 2\langle 1_K, W(\bar{\mathsf{s}})\rangle$$

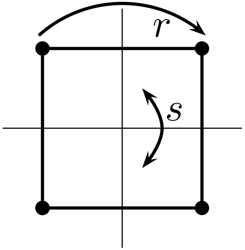
Examples of extremal witnesses for pairs of effects

It is enough to use extremal incompatibility witnesses:
extremal as maps $S \to V(K)^+$

Some examples for $S = \square$:

Square-bit: $K = \square$

- extremal non-ETB maps = symmetries of the square



dihedral group D_4:

- group of order 8
- 2 generators: r, s

- non-ETB, no nontrivial translations: witnesses

Examples of extremal witnesses for pairs of effects

Quantum states: $K = \mathfrak{S}(\mathcal{H})$

extremal non-ETB maps: parallelograms in $B(\mathcal{H})^+$ with rank one vertices:

$$|x_{00}\rangle\langle x_{00}| + |x_{11}\rangle\langle x_{11}| = \rho = |x_{01}\rangle\langle x_{01}| + |x_{10}\rangle\langle x_{10}|,$$

- incompatibility witness if perimeter (in trace norm) $> 2\mathrm{Tr}\,\rho$
- for compatibility of pairs of effects, it is enough to consider restrictions to 2-dimensional subspaces

Incompatibility degree

Can we quantify incompatibility?

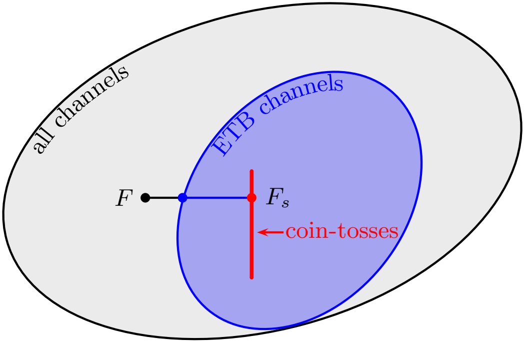
(M.M. Wolf et al., PRL 2009; P. Busch et al., EPL 2013; T. Heinosaari et al., J. Phys. A 2016; D. Cavalcanti, P. Szkrzypczyk, PRA 2016)

- Incompatibility degree: the least amount of noise that has to be added to obtain a compatible collection.
- different definitions by the choice of noise
- we choose coin-toss measurements = constant maps $f_p(x) \equiv p \in \Delta$
- Collection of coin-tosses = constant map

$$F_s : K \to s = (p^0, \ldots, p^k) \in S$$

always compatible (ETB)

Incompatibility degree



Let $F, F_s : K \to \mathsf{S}$, $s \in \mathsf{S}$.

We put

$$ID_s(F) = \min\{\lambda,\ (1-\lambda)F + \lambda F_s \text{ is ETB}\},$$

$$ID(F) := \inf_{s \in \mathsf{S}} ID_s(F)$$

(T. Heinosaari et al. PLA 2014)

Incompatibility degree by incompatibility witnesses

For $s \in int(\mathsf{S})$, let us denote

$$\mathcal{W}_s := \{W : \mathsf{S} \to V(K)^+,\ W(s) \in K\}$$

and

$$q_s(F) := \min_{W \in \mathcal{W}_s} \mathrm{Tr}\, FW.$$

Then

$$ID_s(F) = \begin{cases} 0 & \text{if } q_s(F) > 0 \\ \frac{-q_s(F)}{1-q_s(F)} & \text{otherwise.} \end{cases}$$

This expression is related to (dual) linear programs for incompatibility degree

e.g. (M. Wolf, D. Perez-Garcia, C. Fernandez, PRL 2009)

ID attainable for pairs of quantum effects

Using extremal witnesses $\square \to B(\mathcal{H})^+$, we can prove:

For quantum state spaces, we have

$$\max_{F:\mathfrak{S}(\mathcal{H})\to\square} ID(F) = 1 - \frac{1}{\sqrt{2}}$$

- for $ID_{\bar{s}}$, $\bar{s}$ the barycenter of $\square$, proved already in

(M. Banik et al., PRA 2013)

Maximal incompatibility in GPT

For any $s \in \mathsf{S}$, it is known that

$$ID_s(F) \leq \frac{k}{k+1}$$

The joint measurement for $\frac{1}{k+1}F + \frac{k}{k+1}F_s$:

- choose one measurement in F uniformly at random
- replace all others by coin-tosses

We say that F is maximally incompatible if $ID(F) = \frac{k}{k+1}$.

Maximal incompatibility for effects

For two-outcome measurements, we have a nice characterization:

Let $F : K \to \square_k$:

F is maximally incompatible if and only if F is a retraction. The corresponding section is the witness $W : \square_k \to K$ such that $ID(F)$ is attained.

There exist k maximally incompatible effects on K if and only if there exists a projection $K \to K$ whose range is affinely isomorphic to the hypercube $\square_k$.

Maximal incompatibility: examples

- Polysimplices: Let $M : \mathsf{S} \to \square_{k+1}$,

$$M = (\mathrm{m}^0_{n_0}, \ldots, \mathrm{m}^k_{n_k}), \qquad n_i \in \{0, \ldots, l_i\}$$

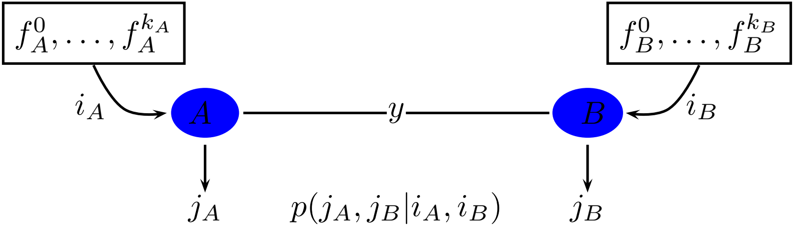
  Then M is maximally incompatible.
- Quantum channels: There are $m = \dim(\mathcal{H}_A)$ maximally incompatible effects on $\mathcal{C}_{A,A'}$

  compose the retraction $R : \mathcal{C}_{A,A'} \to \Delta^m_n$ with M as above.

  (cf. M. Sedlák et al., PRA 2016; AJ, M. Plávala, PRA 2017)

Bell non-locality in GPT

Bell scenario:



The conditional probabilities satisfy the no-signalling conditions:

$$\sum_{j_A} p(j_A, j_B | i_A, i_B) = p_B(j_B | i_B), \quad \forall i_A$$

$$\sum_{j_B} p(j_A, j_B | i_A, i_B) = p_A(j_A | i_A), \quad \forall i_B$$

Bell non-locality in GPT

In our setting:
$F_A = (f_A^0, \ldots, f_A^{k_A})$, $F_B = (f_B^0, \ldots, f_B^{k_B})$, $y \in K_A \widetilde{\otimes} K_B$

$$(F_A \otimes F_B)(y) \in \mathrm{S}_A \otimes_{max} \mathrm{S}_B$$

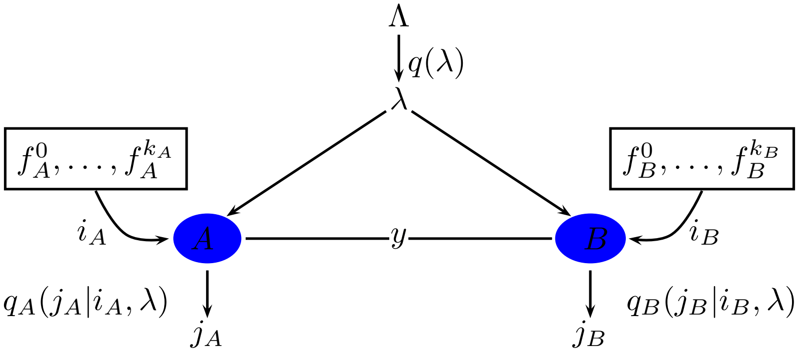
There is a correspondence $\mathrm{S}_A \otimes_{max} \mathrm{S}_B \equiv$ no-signalling conditional probabilities:

$$s \leftrightarrow p(j_A, j_B | i_A, i_B) := (\mathrm{m}_{j_A}^{i_A} \otimes \mathrm{m}_{j_B}^{i_B})(s)$$

- the no-signalling polytope

Bell non-locality in GPT

Local hidden variable model:



$$p(j_A, j_B | i_A, i_B) = \sum_{\lambda} q(\lambda) q_A(j_A | i_A, \lambda) q_B(j_B | i_B, \lambda)$$

(H. M. Wiseman, S. J. Jones, A. C. Doherty, PRL 2007)

Bell witnesses and Bell inequalities

- $p(j_A, j_B | i_A, i_B)$ admit LHV iff $s \in \mathsf{S}_A \otimes_{min} \mathsf{S}_B$:
  - the local polytope
- Bell witnesses: entangled elements in $A(\mathsf{S}_A \otimes_{min} \mathsf{S}_B)^+$
- Extremal: finitely many $\mu_1, \dots, \mu_N$
- Bell inequalities:

$$s \in \mathsf{S}_A \otimes_{min} \mathsf{S}_B \iff \langle \mu_i, s \rangle \geq 0, \quad i = 1, \dots, N$$

- $\mu_i \equiv M_i$ extremal affine maps $\mathsf{S}_A \to A(\mathsf{S}_B)^+$

Let $s = (F_A \otimes F_B)(y)$. If F_A or F_B is compatible or y separable, then $s \in \mathsf{S}_A \otimes_{min} \mathsf{S}_B$.

The CHSH inequality

If $\mathsf{S}_A = \mathsf{S}_B = \square$:

- the CHSH witnesses: $\mu_\square \equiv$ isomorphisms

$$M_\square : V(\square)^+ \to A(\square)^+$$

- the CHSH inequality:

$$\begin{aligned} 0 &\leq \langle \mu_\square, (F_A \otimes F_B)(y) \rangle \\ &= \frac{1}{2}\left(1 - \frac{1}{2}\langle a_0 \otimes (b_0 + b_1) + a_1 \otimes (b_0 - b_1), y \rangle\right) \end{aligned}$$

$$a_i = 1 - 2(f_A^i)_0, \qquad b_i = 1 - 2(f_B^i)_0$$

Bell inequalities and the incompatibility degree

Relation of violation of Bell inequalities to incompatibility degree:

If F_A is incompatible, then for any $y \in K_A \widetilde{\otimes} K_B$, any Bell witness μ and $s \in int(S_A)$, we have

$$\langle \mu, F_A \otimes F_B(y) \rangle \geq \|\mu\|_{max} q_s(F_A).$$

Bell inequalities and the incompatibility degree

- Maximal violation of CHSH inequality: CHSH bound
- Quantum case: Tsirelson bound

Equality case for the CHSH bound: If $K = \mathfrak{S}(\mathcal{H})$ and $\mathsf{S}_A = \Box$, then there is some $\mathcal{H}_B \simeq \mathcal{H}_A$, $F_B : \mathfrak{S}(\mathcal{H}_B) \to \Box$ and $y \in \mathfrak{S}(\mathcal{H}_{AB})$ such that

$$\langle \mu_\Box, F_A \otimes F_B(y) \rangle = \frac{1}{2} q_{\bar{\mathsf{s}}}(F_A)$$

$\bar{\mathsf{s}}$ is the barycenter of $\Box$

(cf. M. M. Wolf et al., PRL 2009; P. Busch, N. Stevens, PRA 2014)

Bell inequalities and the incompatibility degree

Sketch of a proof using incompatibility witnesses:

- $\langle \mu, F_A \otimes F_B(y) \rangle = \mathrm{Tr}\, F_A W \geq (\mathrm{Tr}\, F_s W) q_s(F_A)$, with

$$\begin{tikzcd} S_A \arrow[r, "M"] \arrow[rrr, bend right, "W"'] & A(S_B)^+ \arrow[r, "F_B^*"] & A(K_B)^+ \arrow[r, "T"] & V(K_A)^+ \end{tikzcd}$$

W is an incompatibility witness if Bell inequality is violated.
(M is a map related to μ and T to y.)

> Bell inequalities are obtained from special incompatibility witnesses.

Bell inequalities and the incompatibility degree

- For the equality:

$$\begin{tikzcd} \square_A \arrow[r, "M_\square", "\simeq"'] \arrow[rrr, bend right, "W"'] & A(\square_B)^+ \arrow[r, "F_B^*"] & B(\mathcal{H}_B)^+ \arrow[r, "\simeq"] & B(\mathcal{H}_A)^+ \end{tikzcd}$$

All incompatibility witnesses are obtained from CHSH inequalities.

Bell inequalities and the incompatibility degree

In general:

$$\begin{tikzcd} \mathsf{S}_A \arrow[r, "M"] \arrow[rrr, bend right, "W"'] & A(\mathsf{S}_B)^+ \arrow[r, "F_B^*"] & A(K_B)^+ \arrow[r, "T"] & V(K_A)^+ \end{tikzcd}$$

if $\mathsf{S}_A \neq \Box$ of $\mathsf{S}_B \neq \Box$, $M : V(\mathsf{S}_A)^+ \to A(\mathsf{S}_B)^+$ is never an isomorphism: weaker witnesses

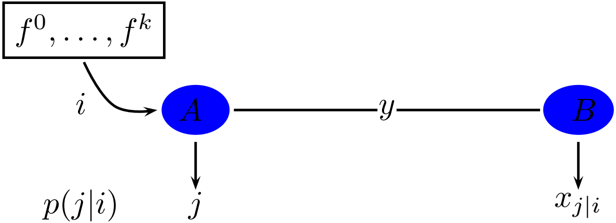
$\Longrightarrow$

there exists incompatible collections that do not violate Bell inequalities

(M. T. Quintino, T. Vértesi, N. Brunner, PRL 2014)

Steering in GPT

- Quantum steering: (E. Schrödinger, Proc. Camb. Phil. Soc. 1936)
- Rigorous definition (in GPT setting):

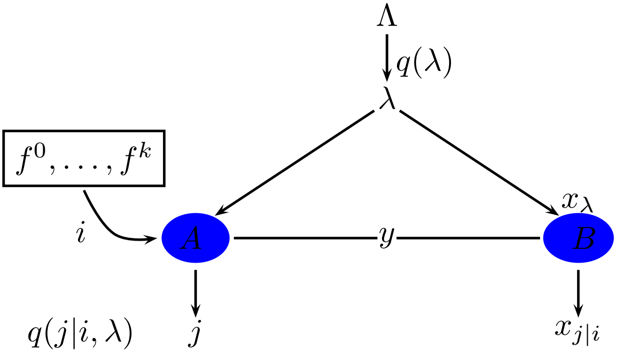


- assemblage: $\{p(j|i), x_{j|i}\}$, $x_{j|i} \in K_B$, $p(j|i)$ probabilities

$$\sum_j p(j|i)x(j|i) = y_B \in K_B, \qquad \forall i$$

Steering in GPT

Local hidden state (LHS) model:



$$p(j|i)x_{j|i} = \sum_\lambda q(\lambda)q(j|i,\lambda)x_\lambda.$$

(cf. H. M. Wiseman, S. J. Jones, A. C. Doherty, PRL 2007)

Assemblages and tensor products

Let $F_A = (f^0, \dots, f^k)$.

- $(F_A \otimes id_B)(y) \in \mathrm{S} \otimes_{max} K_B$
- assemblages $\equiv$ elements $\beta \in \mathrm{S} \otimes_{max} K_B$:

$$p(j|i)x_{j|i} = \langle \mathrm{m}_j^i \otimes id_B, \beta \rangle$$

- admits LHS model if and only if β is separable
- for $\beta = (F_A \otimes id_B)(y)$:
  no steering if y is separable or F_A are compatible.
- steering witnesses: all entangled elements in $A(\mathrm{S} \otimes_{min} K_B)^+$
- steering degree