INCOMPATIBILITY IN GENERAL PROBABILISTIC THEORIES, GENERALIZED SPECTRAHEDRA, AND TENSOR NORMS

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Many of the phenomena that distinguish quantum mechanics from a classical theory can be traced back to the incompatibility of measurements [1, 2]. Two measurements are *incompatible* if there does not exist a third one which implements them simultaneously. This non-classicality present in a collection of measurements is necessary for many tasks in quantum information theory, because compatible measurements cannot exhibit non-locality in terms of violation of a Bell inequality [3, 4] or steering [5]. In this sense, incompatibility is a resource for quantum processing tasks similar to entanglement [6]. This resource can be quantified using *noise robustness* [7]. This leads to the *compatibility regions* studied in [8, 9, 10, 11].

In fact, the incompatibility of measurements is not restricted to quantum mechanics, but is present in all non-classical theories [12]. These theories can conveniently be described in the framework of *general probabilistic theories* (GPTs). Important examples of GPTs are classical probability theory, quantum mechanics and the GPT of quantum channels.

Any GPT is built on basic operational notions of *states* (or preparation procedures) and *effects* (or dichotomic measurements) of the theory, which are identified with certain positive elements in a pair of dual ordered vector spaces, (V, V^+) and (A, A^+), respectively. The space of effects, (A, A^+), contains a distinguished order unit $\mathbb{1}$, corresponding to the trace in quantum mechanics. Elements $f \in A$ for which $0 \leqslant f \leqslant \mathbb{1}$ correspond to effects. In order to treat multipartite systems, one needs to specify a tensor product of the cones describing the single systems. In general, there is a minimal tensor cone (corresponding to separable states in quantum mechanics) and a maximal tensor cone (corresponding to block-positive matrices, i.e. entanglement witnesses, in quantum mechanics).

Compatibility of GPT measurements

In our work, we introduce and develop several conceptual frameworks to study measurement compatibility in GPTs. Our first main contribution can be informally stated as follows.

We provide equivalent characterizations *of the compatibility of a given set of GPT measurements in terms of:*

- *the* positivity properties *of a linear map encoding the measurement results*
- *the inclusion of a* universal generalized spectrahedron*, dubbed the* GPT jewel*, inside a generalized spectrahedron associated to the measurements*
- *the* norm of a tensor *associated to the measurements, corresponding to a tensor product of* Banach spaces *(for dichotomic measurements).*

Let $\mathbf{k} = (k_1, \ldots, k_g) \in \mathbb{N}^g$ be a g-tuple determining the number of outcomes of a set of g measurements. We first define an ordered vector space $(E_\mathbf{k}, E_\mathbf{k}^+)$ which plays a *universal role* in the sense that it only depends on the number of measurement outcomes, but not on the GPT under study. To a given g-tuple of measurements $f = (f^{(1)}, \ldots, f^{(g)})$ with outcomes determined by $\mathbf{k}$, we relate a linear map $\Phi^{(f)} : E_\mathbf{k} \to A$. By a standard identification, there is a related tensor $\varphi^{(f)} \in E_\mathbf{k}^* \otimes A$. It is then shown that such a map is positive if and only if it corresponds to a g-tuple of measurements, or equivalently, if $\varphi^{(f)}$ is in the maximal tensor product of A^+ and the dual cone $(E_\mathbf{k}^+)^*$.

We show in Theorem 7.3 that compatibility of f is characterized by either of the following equivalent conditions, which are closely related to the results of [13]:

(a) $\Phi^{(f)}$ has a *positive extension* to a map $(\mathbb{R}^{\mathbf{k}}, \mathbb{R}^{\mathbf{k}}_+) \to (A, A^+)$;
(b) $\Phi^{(f)}$ is *entanglement breaking*, in very much the same sense as the entanglement breaking channels in quantum theory;
(c) $\varphi^{(f)}$ is in the *minimal tensor product* of A^+ and $(E^+_{\mathbf{k}})^*$.

The next part of the paper is motivated by the effort to extend the results of [10, 11] to GPTs. These works showed that measurement compatibility can be phrased as the *inclusion of free spectrahedra*, which led to significantly improved bounds on the compatibility regions for quantum mechanics. Thus, we need some generalization of the free spectrahedra considered e.g. in [14, 15, 16], which have found various applications in optimization theory. While a free spectrahedron is the solution set of a linear matrix inequality, we replace the vector space of Hermitian matrices and the cone of semidefinite matrices by arbitrary ordered vector spaces and suitable tensor cones to define *generalized spectrahedra*. The generalized spectrahedron $\mathcal{D}_{\mathrm{GPT}\diamond}(\mathbf{k}; V, V^+)$, called the *GPT jewel*, is determined by $(E_{\mathbf{k}}, E^+_{\mathbf{k}})$ and plays the same universal role as the matrix jewel in [11]. We also define the generalized spectrahedron $\mathcal{D}_f(\mathbf{k}; V, V^+)$ determined by the measurements under study. We hence find the following geometric condition for compatibility of f, see Theorem 8.10:

(d) $\mathcal{D}_{\mathrm{GPT}\diamond}(\mathbf{k}; V, V^+) \subseteq \mathcal{D}_f(\mathbf{k}; V, V^+)$.

This shows that the conditions (a) - (c) can be expressed by inclusion of generalized spectrahedra and gives a connection between the results of [10, 11] and [13].

The last part of the characterization is obtained if we restrict to dichotomic measurements. It establishes a connection to the well-developed field of Banach space theory, which makes a new set of tools available to us to study measurement compatibility. We use the symmetry of the cone $E^+_{(2,2,\ldots,2)}$ to relate compatibility of effects to *reasonable crossnorms* on the tensor product $\ell^g_\infty \otimes A$ (where we always endow A with its order unit norm). Namely, we can study the tensor $\bar{\varphi}^{(f)} \in \ell^g_\infty \otimes A$ associated to the measurements in question. Then g-tuples of effects are determined by the condition $\|\bar{\varphi}^{(f)}\|_\varepsilon \leqslant 1$, where $\|\cdot\|_\varepsilon$ is the injective crossnorm in $\ell^g_\infty \otimes A$. Moreover, we find a reasonable crossnorm $\|\cdot\|_\rho$ in $\ell^g_\infty \otimes A$, such that compatibility of a g-tuple f of dichotomic measurements is equivalent to

(e) $\|\bar{\varphi}^{(f)}\|_\rho \leqslant 1$,

see Theorem 9.2. A connection to tensor norms on Banach spaces has been used successfully in the previous work [17] to study non-local games.

We expect that our fivefold characterization (a)-(e) of compatibility in GPTs will enable cross-fertilization of ideas across the different mathematical and physical disciplines mentioned above, allowing for a deep and multi-faceted picture of the fundamental notion of measurement compatibility.

The compatibility degree of a GPT

The second main result we obtain is measurement-independent: it concerns structural properties of a GPT in relation to measurement incompatibility. With the help of the equivalent characterizations, we provide several results on the incompatibility regions of GPTs.

We characterize the minimal quantity of noise necessary to render any set of GPT measurements compatible in terms of the inclusion constants of the GPT jewel. In the case of centrally symmetric GPTs and dichotomic measurements, this value is related to the ratio between the minimal and the maximal reasonable norms one can put on a tensor product of Banach spaces.

The minimal amount of noise such that any collection of g measurements with k_i outcomes in the i-th measurement is compatible is a measure of the amount of incompatibility available in the GPT.

We consider robustness to *white noise*, i.e. the effects of the noisy measurements are

$$\tilde{f}_j^{(i)} = s f_j^{(i)} + (1-s)\frac{\mathbb{1}}{k_i}.$$

The largest $s \in [0,1]$ such that the noisy $\tilde{f}^{(i)}$ are compatible for *all* collections of measurements f with outcomes given by $\mathbf{k}$ is the *compatibility degree* $\gamma(\mathbf{k}; V, V^+)$ of the GPT. Allowing for different noise levels in the measurements leads to the *compatibility region* $\Gamma(\mathbf{k}; V, V^+) \subseteq [0,1]^g$ of the GPT.

Consider the set of vectors $s \in [0,1]^g$ that can be used to scale the GPT jewel $\mathcal{D}_{\mathrm{GPT}\diamondsuit}$ such that it is contained in $\mathcal{D}_f(\mathbf{k}; V, V^+)$ for any collection of g measurements $f^{(i)}$ with k_i outcomes. This set is the set of *inclusion constants* of $\mathcal{D}_{\mathrm{GPT}\diamondsuit}$, denoted by $\Delta(\mathbf{k}; V, V^+)$. Following along the lines of [10, 11], we prove that (see Theorem 8.17):

$$\Gamma(\mathbf{k}; V, V^+) = \Delta(\mathbf{k}; V, V^+).$$

We can get more results for dichotomic measurements. In this case, we write $\gamma(g; V, V^+)$ for the compatibility degree. Using the compatibility characterization (e) we immediately obtain a direct relation of the compatibility measures to the norm $\|\cdot\|_\rho$: We find in Theorem 9.6 that

$$\gamma(g; V, V^+) = 1/\max_{\|\varphi\|_\varepsilon \leqslant 1} \|\varphi\|_\rho \geqslant 1/\rho(\ell_\infty^g, A) \geqslant 1/\min\{g, \dim(V)\}, \tag{1}$$

where the quantity $\rho(X, Y)$ for a pair of Banach spaces X, Y was introduced in [17] as the maximum ratio of the projective over the injective tensor norm. The compatibility region can be described in a similar way. Moreover, a characterization which is dual to the one above can be obtained using *incompatibility witnesses*, see Theorem 9.18.

Finally, we consider a special class of GPTs for which our results have a simpler form: the *centrally symmetric* GPTs [18]. The state spaces of these GPTs are the unit balls of some norm on a vector space $\bar{V}$ and we have $V \cong \mathbb{R} \oplus \bar{V}$. Important examples are the Bloch ball describing 2-level systems in quantum mechanics, or the hypercubic GPT used to model theories with PR boxes [19]. In Theorem 10.5, we show that for this class of GPTs, we can replace in Equation (1) the space A by $\bar{A}$ and the norm $\|\cdot\|_\rho$ by the projective norm in $\ell_\infty^g \otimes_\pi \bar{A}$, where $\bar{A}$ is the dual Banach space to $\bar{V}$:

$$\gamma(g; V, V^+) = 1/\rho(\ell_\infty^g, \bar{A}) \geqslant 1/\min\{g, \dim(\bar{A})\}.$$

We also put forward a connection to 1-summing norms: $\lim_{g\to\infty} \gamma(g; V, V^+) = 1/\pi_1(\bar{V})$, where π_1 is the 1-summing norm of the Banach space $(\bar{V}, \|\cdot\|_{\bar{V}})$. This allows us to leverage powerful results from Banach space theory and to compute the compatibility degree for many GPTs of interest for the first time. In particular, we prove new lower bounds for the compatibility degree of qubit effects ($\bar{A} = \ell_2^3$), namely $0.58 \approx 1/\sqrt{3} \geqslant \gamma(g; \mathrm{QM}_2) \geqslant 1/2$ for $g \geqslant 4$. This is the first bound on the compatibility degree of more than 3 qubit effects, whereas the case for $g \leqslant 3$ was already known (see [10] and references therein).

In conclusion, in this submission we connect the crucial problem of measurement compatibility in general probabilistic theories to several yet unrelated branches of mathematics:

- the study of linear maps that are positive with respect to tensor cones; map extension (*functional analysis*)
- inclusion of generalized spectrahedra (*algebraic convexity*)
- ratio of tensor norms in Banach spaces (*Banach space theory*)

These newly established bridges allow us to compute many explicit constants and compatibility degrees for various GPTs, and also to obtain new results for quantum mechanics in the most important case of several dichotomic qubit measurements. Our work unearths yet another fundamental connection between quantum information theory and functional analysis. The interaction between these two fields has produced many remarkable outcomes, from the study of non-locality [17] to the resolution of longstanding conjectures in operator algebra [20, 21]. We expect our results to be of similar importance in the foundations of operational theories.

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