# Remarks on positive projections on JB-algebras

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## 1 Existence of positive projections

**1.1 Lemma.** *$A$ is a finite dimensional JB-factor, $B \subseteq A$ a subalgebra, $1 \in B$. Then there exists a positive unital projection $P : A \to B$.*

*Proof.* $A$ is an EJA (=Euclidean Jordan algebra) - Recall that a finite dimensional JB-algebra $A$ is an EJA iff it is formally real [5]. Then there is an inner product $< .,. >: A \times A \to \mathbb{R}$ such that

$$< a \circ b, c >=< b, a \circ c >$$

with it $A$ is a real Hilbert space.

• $a \in A$ $a \geq 0 \Leftrightarrow < a, b >\geq 0$ $\forall b \geq 0$.

To prove it we use spectral decomposition ($A$ is a JBW algebra, hence spectral):

$\forall a \in A$, $a = \sum_i \lambda_i p_i$, $\lambda_i \in \mathbb{R}$, $p_i^2 = p_i$, $p_i \circ p_j = 0, i \neq j$, $a \geq 0 \Leftrightarrow \forall i \lambda_i \geq 0$.

Assume first that $p^2 = p$, $b \geq 0$. Then

$$< p, b >=< 1, p \circ b >, < p^2, b >=< p, p \circ b >=, 1, p \circ (p \circ b) >$$

this implies

$$< p, b >= 2 < 1, p \circ (p \circ b) > - < 1, p \circ b >=< 1, U_p(b) >$$

Since $b$ is positive, $U_p b \geq 0$, hence $U_p(b) = c^2$ for some $c \in A$. This yields

$$< 1, c^2 >=< c, c >\geq 0$$

Let $a \geq 0 \Rightarrow \sum_i \lambda_i p_i, \lambda_i \geq 0$,

$$< a, b >= \sum_i \lambda_< p_i, b >\geq 0.$$

Conversely: Let $< a, b >\geq 0$ $\forall b \geq 0$, $a = \sum_i \lambda_i p_i$, where $a = \sum_i \lambda_i p_i$ is the spectral decomposition of $a$.

Then $0 \leq< a, p_j >= \lambda_j < p_j, p_j >\to \lambda_j \geq 0 \forall j \implies a \geq 0$.

• We now define the projection $P$:

$$\forall a \in A, \forall b \in B, < P(a), b >=< a, b > .$$

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Then $P$ is clearly linear, idempotent, unital and is also positive since $a \geq 0$ implies $< P(a), b >=< a, b >\geq 0 \forall b \in B$, hence $P(a) \in B^+$. $\square$

Remark: By the proof of Lemma 1.1, it seems that this should hold also for finite JBW factors.

# 2 Support projection

Let $A$ be a JBW algebra, $P : A \to A$ unital positive normal map. Support of $P$:

$$e = \wedge\{p : P(p) = 1\}$$

(the smallest projection $p$ such that $P(p) = 1$).

**2.1 Lemma.** *For all $a \in A$, $P(a) = P(U_e(a)$.*

*Proof.* Let $a = a_1 + a_2 + a_3$ be the Peirce decomposition of $a \in A$ with respect to $e$, so that

$$a_1 = U_e(a), a_2 = U_{e,f}(a) = \{eaf\}, f = 1 - e, a_3 = U_f(a)$$

We show that $P(a) = P(a_1)$.

First, let $0 \leq a \leq 1$, then $0 \leq U_f(a) \leq U_f(1) \leq f$ which implies $0 \leq P(U_f(a)) \leq P(f) = 1 - P(e) = 0$. This implies that for $a \geq 0$ it is $P(a_3) = P(U_f(a)) = 0$. Since for every $a$, $a = a^+ - a^-$, $a^+ a^- \geq 0$, this yields $P(a_3) = P(U_f(a)) = 0$ for all $a \in A$.

For $a_2$, we have $f \circ a_2 = (1 - e) \circ a_2 = a_2 - \frac{1}{2}a_2 = \frac{1}{2}a_2$.

Let $\rho$ be any state on $A$ and put $\omega := \rho \circ P$. Then by Schwarz inequality [5, Lemma 3.6.2],

$$\omega(a_2)^2 = (2\omega(f \circ a_2))^2 = 4\omega(f \circ a_2)^2 \leq 4\omega(f^2)\omega(a_2^2) = 0$$

hence $\rho(P(a_2) = 0 \forall \rho$, that is $P(a_2) = 0$. $\square$

**2.2 Lemma.** *Let $A$ be JBW algebra, $P : A \to A$ unital, positive, normal map. If $a \geq 0$, then $P(a) = 0 iff U_e(a) = 0$.*

*Proof.* If $U_e(a) = 0$, then $P(a) = P(U_e(a)) = 0$. Conversely, let $P(a) = 0$, $a \geq 0$. We may assume $0 \leq a \leq 1$, then $U_e(a) \leq e$. Let $q$ be the range projecion of $U_e(a)$, that is, $q$ is the smallest projection majorizing $U_(a)$. Then we have $0 \leq U_e(a) \leq q \leq e$. By [1, Theorem 2.15], for every state $\rho$ on $A$, $\rho(P(U_e(a)) = 0$ implies $\rho(P(q)) = 0$. Since this holds for every state $\rho$, we get $P(q) = 0$. But then we have $e - q \geq 0$, and $P(e - q) = 1$. By definition of $e$ this yields $q = 0$, which entails $U_e(a) = 0$. $\square$

**2.3 Lemma.** *If $A$ is a JB algebra, $P : A \to A$ unital positive projection, then*

$$P(P(a) \circ P(B)) = P(a \circ P(b)).$$

*Proof.* The same proof as that of [2, Lemma 1.1]. $\square$

**2.4 Lemma.** *Let $M$ be a JBW algebra, $P : M \to M$ a normal, unital, positive projection. Let $a \in M$, $r \in P(M)$ $x, y \in [P(M)]$, where $[P(M)]$ denotes the JBW algebra generated by $P(M)$. Then:*

*(1) $P(r \circ a) = P(U_e(r) \circ U_e(a))$.*

*(2) $r$ and $e$ operator commute.*

*(3) $x \in [P(M)]$, $U_e(P(x)) = U_e(x)$.*

*(4) $P(x \circ y) = P(P(x) \circ P(y)$.*

*Proof.* (1) Since $r = P(r)$, we have by 2.3 and 2.1

$P(r \circ a) = P(r \circ P(a)) = P(r \circ P(U_e(a))) = P(r \circ U_e(a)) = P(U_e(r \circ U_e(a)))$.

Recall that $U_e(b) = 2e \circ (e \circ b) - (e \circ b)$. From this $U_e(r \circ U_e(a)) = 2e \circ (e \circ (r \circ U_e(a) - e \circ (r \circ U_e(a))$.

By MacDonald's theorem subalgebra generated by $e$ and $a$ is special, hence $U_e(a) = eae$. From this

$$\begin{aligned} U_e(r \circ U_e(a)) &= e(r \circ eae)e = \frac{1}{2}e(reae + eaer)e \\ &= \frac{1}{2}((ere)(eae) + (eae)(ere)) \\ &= (ere \circ eae) = U_e(r) \circ U_e(a). \end{aligned}$$

(2) $r$ and $e$ operator commute: Subalgebra generated by $e$ and $r$ is special, so

$$\begin{aligned} P(r^2) = P(r \circ r) &= P(U_e(r) \circ U_e(r)) = P((U_e(r))^2) = P(U_e(U_e(r)^2)) \\ (U_e(r))^2 &= \{ere\}^2 = U_{ere}(1) = U_e U_r U_e(1) = U_e U_r(e) \\ P(r^2) &= P((U_e(r))^2) = P(U_e U_r(e)) = P(U_r(e)) \end{aligned}$$

From $P(r^2 - U_r(e)) = 0$ we get $U_e(r^2 - U_r(e)) = 0$ since $r^2 - U_r(e) \geq 0$. From this we have

$$U_e(r^2 - U_r(e)) = 0 \implies U_e U_r(1) - U_e U_r U_e(1) = 0.$$

It follows that $\{er(1-e)\} = 0$, hence in Peirce decomposition by $e$, $r = (U_e + U_{(1-e)})(r)$.

$$U_e(P(x)) = U_e(x) \tag{1}$$

Put $A_1 = P(M)$, $A_{n+1} = span\{a \circ b : a, b \in A_n\}$. Then $\mathcal{A} = \bigcup A_n$ is the Jordan subalgebra generated by $P(M)$.

(i)If $n = 1$, we have $r = P(r)$, so (1) holds trivially. We prove (1) for $r^2$. We have by Schwarz inequality $U_e(P(r^2)) \geq U_e((P(r)^2) = U_e(r^2)$, hence $U_e(P(r^2)) - U_e(r^2) \geq 0$, while $P(U_e(P(r^2)) - U_e(r^2)) = P(P(r^2) - r^2) = 0$, which implies (1). Recall that $r$ and $e$ operator commute iff $T_e(r) = U_e(r)$ iff $r = (U_e + U_{(1-e)}(r)$. We get

$$0 = U_e U_r(1) - U_e U_r U_e(1) = U_e U_r(1) - U_e U_r(e) = U_e U_r(1-e) = U_e U_r U_{(1-e)}(1) = (\{er(1-r)\})^2.$$

(1).

(ii) Assume that (1) holds for $x \in A_n$, we prove that (1) holds for $x^2$: $U_e(P(x^2)) = U_e(x^2)$.

Observe that $e$ operator commutes with $x \in \bigcup\{A_n\}$: we know it holds for $x \in A_1$. Assume it holds for $x \in A_n$ and let $x, y \in A_n$. We then have $x = U_e(x) + U_{(1-e)}(x), y = U_e(y) + U_{(1-e)}(y)$, and $x \circ y = U_e(x) \circ U_e(y) + U_{(1-e)} x U_{(1-e)}(y) = U_e(x \circ y) = U_{(1-e)}(x \circ y)$. Hence $e$ operator commutes with $x \circ y$. Notice that by weak continuity of $U_e$ and $T_e$, $e$ operato commutes with all $x \in [P(M)]$.

By operator commutativity of $e$ and $x$ we then have $U_e(x^2) = U_e(x)^2$, $x \in A_n$. We have

$$U_e(P(x^2)) \geq U_e(P(x)^2) = U_e(P(x))^2 = U_e(x)^2 \implies U_e(P(x^2) - x^2) \geq 0,$$

so that

$$P(P(x^2) - x^2) = 0 \implies U_e(P(x^2) = U_e(x^2).$$

For $x \circ y$, $x, y \in A_n$, we use the identity

$$x \circ y = \frac{1}{4}((x = y)^2 - x^2 - y^2).$$

(4) Since $e$ commutes with $x, y$,

$$P(x \circ y = P(U_e(x \circ y)) = P(U_e(x) \circ U_e(y)) = P(U_e(P(x) \circ U_e(P(y)) = P(P(x) \circ P(y)).$$

□

**2.5 Lemma.** *Let $M$ be a JBW algebra, $P : M \to M$ normal unital positive idempotent mapping. Then $P(M)$ is a Jordan algebra under the product*

$$r \star s = P(r \circ s), r, s \in P(M).$$

*Proof.* • $1 \circ r = P(1 \circ r) = P(r) = r$.

• Jordan identity:

$$\begin{aligned} (r \star r) \star (s \star r) &= P((r \star r) \circ (s \star r)) = P(P(r \circ r) \circ P(s \circ r)) \\ &= P((r \circ r) \circ (s \circ r) = P(((r \circ r) \circ s) \circ r) = P(P((r \circ r) \circ s) \circ P(r)) \\ &= P(P(P(r \circ r) \circ P(s)) \circ P(r)) = P(P(P(P(r) \circ P(r))) \circ P(s))) \circ P(r)) \\ &= ((r \star r) \star s) \star r. \end{aligned}$$

□

**2.6 Theorem.** *Let $A$ be a JB algebra, $P : A \to A$ unital positive projection. Let $N := \{n \in A : P(n^2) = 0\}$. Then .*

*(1)$P(A)$ is a JB subalgebra under the given vector operation and the product $r \star s = P(r \circ s)$.*

*(2) $P(A + N)$ is a JB subalgebra of $A$.*

*(3) $P$ restricts to a Jordan homomorphism of $P(A) + N$ onto $P(A)$ with kernel $N$.*

*(4) $P(A) + N$ consists of all $a \in A$ for which $P(a^2) = P(P(a^2)$.*

*Proof.* $A^{\star\star} = M$ is a JBW algebra, $A \subseteq M$ is weakly dense. $P$ extends to $P^{\star\star} : M \to M$, and $P^{\star\star}(M)$ is a Jodan subalgebra under the product $x \star y = P(x \circ y)$ (by Lemma **??**).

(2) We first show that $P(A) + N$ is a JB subalgebra of $A$, that is, is closed under $\circ$ and norm closed.

Let $r \in P(A)$, $n \in N$, we have to show that $(r + n)^2 \in P(a) + N$.

$$(r + n)^2 = r^2 + 2r \circ n + n^2$$

• $n^2 \in N : 0 \le n^4 \le \|n^2\|.n^2 \implies P(n^4) = 0 \implies n^2 \in N$.

• Let $n \in N$, $0 = P(n^2) \le P(n)^2 \implies P(n) = 0$.

• $r^2 \in P(A) + N \Leftrightarrow m := r^2 - P(r^2) \in N$.

$$\begin{aligned} P(m^2) = P((r^2 - P(r^2))^2) = P(r^4 - 2r^2 \circ P(r^2) + P(r^2)^2) \\ = P(r^4) - 2P(r^2 \circ P(r^2)) + P(P(r^2)^2) = P(r^4) - 2P(P(r^2) \circ P(r^2)) + P(P(r^2)^2) \\ = P(r^4) - P(P(r^2)^2 0 = P(P(r^2))^2 - P(P(r^2)^2) = 0 \end{aligned}$$

the last but one equality holds by Lemma 2.5 (4). This shows that $r^2 \in P(A) + N$.

• $r \circ n \in N \leftrightarrow \{e(r \circ n)^2 e\} = 0$ (Lemma 2.2).

Let $A = A_0 + A_{\frac{1}{2}} + A_1$ be the Peirce decomposition of $A$ with respect to $e$. Since $e$ operator commutes with $r$,

$$r = r_0 + r_1.$$

• $P(n^2) = 0 \leftrightarrow U_e(n^2) = 0$. Subalgebra generated by $e, n$ is special (Shirshov-Cohn theorem), so $\{en^2e\} = en^2e$. Then

$$en^2e = 0 \ \Leftrightarrow \ en = 0 \ \Leftrightarrow \ ne = 0 \ \Leftrightarrow \ ene = 0$$

$$en(1-e) + (1-e)ne = 0$$

It follows $n = (1-e)n(1-e$ i.e., $n \in A_0$.

From $r \in A_0 + A_1, n \in A_0$ we get $r \circ n \in A_0$, the also $(r \circ n)^2 \in A_0$. This implies

$$\{e(r \circ n)^2 e\} = 0 \implies P(r \circ n)^2 0 = 0 \implies r \circ n \in N.$$

We have proved that $(r+n)^2 \in P(A) + N$, and (since $x \circ y = (x+y)^2 - x^2 - y^2$ this implies that $P(A) + N$ is a Jordan subalgebra. We still have to prove that $P(A) + N$ is norm closed:

Let $(a_k) \subseteq P(M) + N$, $a_k = r_k + n_k \to a \in A$ in norm. Then $(a_k)$ is Cauchy so that

$$r_k - r_l + n_k - n_l \to 0.$$

Applying $P$ yields

$$r_k - r_l \to 0,$$

so $(r_k)$ is Cauchy and has a limit $r$. Then

$$r_k \to r \implies P(r_k) \to P(r), \text{ and } r_k = P(r_k) \implies r = P(r).$$

Moreover, $(n_k)$ is also Cauchy, and

$$n_k \to n \implies n_k^2 \to n^2 \implies 0 = P(n_k^2) \to P(n^2) = 0$$

So we obtained $r_k + n_k \to r + n \in P(a) + N$. We have proved that $P(A) + N$ is a JB subalgebra of $A$.

(3) We next show that $P : P(A) + N \to P(A)$ is a Jordan homomorphism with kernel $N$:

$$x, y \in P(A) + N \implies P(x \circ y) = P(x) \star P(y) = P(P(x) \circ P(y)).$$

It is enough to show that for $x \in P(A) + N$, $P(x^2) = P(x) \star P(x) = P(P(x)^2))$.

So let $x = r + n, r \in P(A), n \in N$, then

$$\begin{aligned} P(x^2) &= P(r+n)^2) = P(r^2 + 2r \circ n = n^2) = P(r^2) \\ &= P(r \circ r) = P(P(r) \circ P(n) = P(r) \star P(r) = P(x) \star P(x). \end{aligned}$$

We show that $N$ is the kernel of $P|_{P(A)+N}$:

$P(r+n) = 0$ iff $r + n \in N$, indeed, we already proved that $r + n \in N$ implies $P(r+n) = 0$. Conversely, $P(r+n) = 0 \implies r = P(r) = 0 \in N$.

(4) $P(A) + N = \{a]inA : P(a^2) = P(P(a)^2)$.

Let $x \in A$, $a := P(x) + n, n \in N$,

$$(a^2 = (P(x) + n)^2 = P(x)^2 + n^2 + 2P() \circ n, P(a^2) = P(P(x)^2) + p(n^2) + 2P(P(x) \circ n).$$

For a state $\rho$ put $\omega = \rho \circ p$. By Schwarz inequality,

$$\rho(P(P(x) + n))^2 = \omega(P(x) \circ n)^2 \leq \omega(P(x)^2)\omega(n^2) = 0, \text{ since } \omega(n^2) = \rho(P(n^2)) = 0.$$

Since this holds for every state $rho$, we have $P(a^2) = P(P(x)^2) = P(P(a)^2)$.

Conversely, let $P(a^2) = P(P(a)^2)$. Pu $n := a - P(a)$.

$$\begin{aligned} P(n^2) &= P((a - P(a)^2) = P(a^2 - 2a \circ P(a) + P(a)^2) \\ &= P(a^2) - 2P(a \circ P(a)) + P(P(a)^2) = P(a^2) - 2P(P(a) \circ P(a)) + P(P(a)^2) = 0. \end{aligned}$$

Thus $a = P(a) + (a - P(a)) \in P(A) + N$.

□

**2.7 Corollary.** *Let $M$ be a JBW algebra, $P : M \to M$ a normal unital positive projection with suppert $e$. Let $N = \{a \in M : P(a^2) = 0\}$. Then we have:*

*(1) $P(M)$ is a JBW algebra under the given vector operatons and the product $r \star s = P(r \circ s)$.*

*(2) $P(M) + N = eP(M)e + (1 - e)M(1 - e)$.*

*(3) $P(M) + N$is a JBW subalgebra of $M$.*

*(4) $P$ restricts to a normal Jordan homomorphism of $P(M) + N$ onto $P(M)$ with kernel $N$.*

*Proof.* We only prove (2): $P(M) + N = \{eP(M)e\} + \{(1 - e)M(1 - e)\}$:

Clearly

$$\{(1 - e)M(1 - e)\} \subseteq N, (1 - e) \in N, e = 1 - (1 - e) \in P(M) + N.$$

Moreover,

$$P(M) \subseteq P(M) + N, \ \{eP(M)e\} \subseteq P(M) + N \ (subalgebra)$$

this yields

$$\{eP(M)e\} + \{(1 - e)M(1 - e)\} \subseteq P(M) + N.$$

Conversely, in Peirce decomposition with respect to $e$,

$$P(M) \subseteq M_1 + M_0, \ r \in P(M) \implies$$

$$r = \{ere\} + \{(1 - e)r(1 - e)\} \in \{eP(M)e\} + \{(1 - e)M(1 - e)\}.$$

□

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