

A note on monotonicity of $z \mapsto D_{\alpha,z}$ and $\alpha \mapsto D_{\alpha,z}(\psi\|\varphi)$ for $1 < \alpha \leq 2z$

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We will assume throughout that $Q_{\alpha,z}(\psi\|\varphi) < \infty$ for some $1 < \alpha \leq 2z$, in which case there is some $y \in L^{2z}(\mathcal{M})$ such that

$$h_{\psi}^{\frac{\alpha}{2z}} = y h_{\varphi}^{\frac{\alpha-1}{2z}}, \quad Q_{\alpha,z}(\psi\|\varphi) = \|y\|_{2z}^{2z}.$$

In particular, $e := s(\psi) \leq s(\varphi)$, so that we may assume that φ is faithful. Let $\sigma \in \mathcal{M}_*^+$ be such that $s(\sigma) = 1 - e$ and put $\psi_0 := \psi + \sigma$, so that ψ_0 is faithful as well. We will use the notation $L_L^p := L^p(\mathcal{M}; \varphi)_L$, $1 \leq p \leq \infty$.

Consider the function

$$f(w) = h_{\psi_0}^{\frac{\alpha}{2z}w} e h_{\varphi}^{1-\frac{\alpha}{2z}w}, \quad w \in S,$$

where $S := \{w \in \mathbb{C}, 0 \leq \operatorname{Re} w \leq 1\}$. Then f is a bounded continuous function $S \rightarrow L^1(\mathcal{M})$, analytic in the interior. Further,

$$f(it) = h_{\psi_0}^{\frac{\alpha}{2z}it} e h_{\varphi}^{-\frac{\alpha}{2z}it} h_{\varphi} \in L_L^{\infty}, \quad t \in \mathbb{R},$$

and $\|f(it)\|_{L_L^{\infty}} = 1$ for all t . We also have

$$f(1+it) = h_{\psi_0}^{\frac{\alpha}{2z}it} h_{\psi}^{\frac{\alpha}{2z}} h_{\varphi}^{1-\frac{\alpha}{2z}} h_{\varphi}^{-\frac{\alpha}{2z}it} = (h_{\psi_0}^{\frac{\alpha}{2z}it} y h_{\varphi}^{-\frac{\alpha}{2z}it}) h_{\varphi}^{\frac{2z-1}{2z}} \in L_L^{2z}, \quad t \in \mathbb{R}.$$

By [2, Lemmas 10.1 and 10.2],

$$\|f(1+it)\|_{L_L^{2z}} = \|h_{\psi_0}^{\frac{\alpha}{2z}it} y h_{\varphi}^{-\frac{\alpha}{2z}it}\|_{2z} = \|y\|_{2z}$$

and the functions $t \mapsto f(it)$ and $t \mapsto f(1+it)$ are continuous in L_L^{2z} . It follows that $f \in \mathcal{F}'(L_L^{\infty}, L_L^{2z})$, that is, f is a function $S \rightarrow L_L^{2z}$, bounded and continuous on S and analytic in the interior of S , such that the boundary values define bounded functions to L_L^{∞} resp. L_L^{2z} , see [2, Definition 1.4].

1 Monotonicity in z

Let $z < z'$, we will prove that $Q_{\alpha,z}(\psi\|\varphi) \geq Q_{\alpha,z'}(\psi\|\varphi)$. By [2, Remark 3.4], the set of functions $\mathcal{F}'(L_L^{\infty}, L_L^{2z})$ defines the interpolation spaces $C_{\theta} = C_{\theta}(L_L^{\infty}, L_L^{2z})$, so that for any $\theta \in (0, 1)$, $f(\theta) \in C_{\theta}$ and

$$\|f(\theta)\|_{C_{\theta}} \leq (\max_t \|f(it)\|_{L_L^{\infty}})^{1-\theta} (\max_t \|f(1+it)\|_{L_L^{2z}})^{\theta} = \|y\|_{2z}^{\theta}.$$

By the reiteration theorem, $C_\theta = L_L^{2z/\theta}$. Putting $\theta = z/z'$, we get

$$f(z/z') = h_{\psi}^{\frac{\alpha}{2z'}} h_{\varphi}^{1 - \frac{\alpha}{2z'}} = y' h_{\varphi}^{\frac{2z'-1}{2z'}}$$

for some $y' \in L^{2z'}(\mathcal{M})$, and $\|y'\|_{2z'} \leq \|y\|_{2z}^{z/z'}$. It follows that $h_{\psi}^{\frac{\alpha}{2z'}} = y' h_{\varphi}^{\frac{\alpha-1}{2z'}}$, so that

$$Q_{\alpha,z'}(\psi\|\varphi) = \|y'\|_{2z'}^{2z'} = \|f(z/z')\|_{L_L^{2z'}}^{2z'},$$

this proves the result.

2 Monotonicity in α

The above function allows us also to prove monotonicity in α . Indeed, let $1 < \alpha' < \alpha$. For any $t \in \mathbb{R}$,

$$f\left(\frac{1}{\alpha} + it\right) = h_{\psi_0}^{\frac{\alpha}{2z}it} h_{\psi}^{\frac{1}{2z}} h_{\varphi}^{-\frac{\alpha}{2z}it} h_{\varphi}^{\frac{2z-1}{2z}},$$

so that $\|f(\frac{1}{\alpha} + it)\|_{L_L^{2z}} \leq \psi(1)^{\frac{1}{2z}}$. Further, since $\frac{\alpha'}{\alpha} < 1$, we get $f(\frac{\alpha'}{\alpha}) \in L_L^{2z}$, so that there is some $y' \in L^{2z}(\mathcal{M})$ such that

$$f\left(\frac{\alpha'}{\alpha}\right) = h_{\psi}^{\frac{\alpha'}{2z}} h_{\varphi}^{1 - \frac{\alpha'}{2z}} = y' h_{\varphi}^{\frac{2z-1}{2z}}$$

so that $h_{\psi}^{\frac{\alpha'}{2z}} = y' h_{\varphi}^{\frac{\alpha'-1}{2z}}$ and $Q_{\alpha',z}(\psi\|\varphi) = \|y'\|_{2z}^{2z} = \|f(\frac{\alpha'}{\alpha})\|_{L_L^{2z}}^{2z}$. Now let λ be such that $(1-\lambda) + \lambda\alpha = \alpha'$, so that $\lambda = \frac{\alpha'-1}{\alpha-1}$, then by the Hadamard three lines theorem, we get

$$Q_{\alpha',z}(\psi\|\varphi) = \|f(\frac{\alpha'}{\alpha})\|_{L_L^{2z}}^{2z} \leq \left(\max_t \|f(\frac{1}{\alpha} + it)\|_{L_L^{2z}}^{1-\lambda} \max_t \|f(1+it)\|_{L_L^{2z}}^{\lambda} \right)^{2z} \leq \psi(1)^{1-\lambda} Q_{\alpha,z}(\psi\|\varphi)^{\lambda},$$

this proves that $D_{\alpha',z}(\psi\|\varphi) \leq D_{\alpha,z}(\psi\|\varphi)$.

3 The limit $\alpha \searrow 1$

We now try to prove the limit $\lim_{\alpha \searrow 1} D_{\alpha,z}(\psi\|\varphi)$, using the same ideas as in [1] but applying analyticity of the function f instead of the Connes cocycle.

The function f is analytic in a neighborhood of $\frac{1}{\alpha}$. Therefore, we have the expansion

$$f(w) = f\left(\frac{1}{\alpha}\right) + \left(w - \frac{1}{\alpha}\right)h + o\left(w - \frac{1}{\alpha}\right)$$

where $h \in L_L^{2z}$ is the derivative of f at $w = \frac{1}{\alpha}$ and $\frac{\|o(w)\|_{L_L^{2z}}}{|w - \frac{1}{\alpha}|} \rightarrow 0$ as $|w - \frac{1}{\alpha}| \rightarrow 0$. It follows that for $1 < \alpha' < \alpha$,

$$f\left(\frac{\alpha'}{\alpha}\right) = f\left(\frac{1}{\alpha}\right) + \frac{\alpha' - 1}{\alpha}h + o\left(\frac{\alpha' - 1}{\alpha}\right)$$

Using the fact that the L_L^p spaces are uniformly Fréchet differentiable, we can prove similarly as in [1] that

$$\lim_{\alpha' \rightarrow 1} \frac{\|f(\frac{\alpha'}{\alpha})\|_{L_L^{2z}} - \|f(\frac{1}{\alpha})\|_{L_L^{2z}}}{\frac{\alpha'-1}{\alpha}} = \langle a_0, h \rangle$$

where $\langle \cdot, \cdot \rangle$ is the duality between L_L^{2z} and $L_L^{\frac{2z}{2z-1}}$ and a_0 is the element in $L_L^{\frac{2z}{2z-1}}$ with unit norm such that $\text{Tr } a_0 f(\frac{1}{\alpha}) = \|f(\frac{1}{\alpha})\|_{L_L^{2z}}$, that is,

$$a_0 = \left(\frac{h_\psi}{\psi(1)} \right)^{\frac{2z-1}{2z}} h_\varphi^{\frac{1}{2z}}.$$

Since f is uniformly differentiable and h is the derivative of f at $\frac{1}{\alpha}$, we have

$$\begin{aligned} \langle a_0, h \rangle &= \lim_{t \rightarrow 0} (it)^{-1} \langle a_0, f(\frac{1}{\alpha} + it) - f(\frac{1}{\alpha}) \rangle = \lim_{t \rightarrow 0} (it)^{-1} \langle a_0, \left(h_\psi^{\frac{1}{2z}} h_{\psi_0}^{\frac{\alpha}{2z} it} h_\varphi^{-\frac{\alpha}{2z} it} - h_\psi^{\frac{1}{2z}} \right) h_\varphi^{\frac{2z-1}{2z}} \rangle \\ &= \psi(1)^{-\frac{2z-1}{2z}} \frac{\alpha}{2z} \lim_{t \rightarrow 0} (it)^{-1} \langle h_\psi^{\frac{2z-1}{2z}} h_\varphi^{\frac{1}{2z}}, \left(h_\psi^{\frac{1}{2z}} h_{\psi_0}^{it} h_\varphi^{-it} - h_\psi^{\frac{1}{2z}} \right) h_\varphi^{\frac{2z-1}{2z}} \rangle \\ &= \psi(1)^{-\frac{2z-1}{2z}} \frac{\alpha}{2z} \lim_{t \rightarrow 0} (it)^{-1} \text{Tr } h_\psi \left(h_{\psi_0}^{it} h_\varphi^{-it} - 1 \right) = \psi(1)^{-\frac{2z-1}{2z}} \frac{\alpha}{2z} D(\psi \| \varphi), \end{aligned}$$

where we use [3, Thm.5.7] in the last equality. We therefore have

$$\begin{aligned} \lim_{\alpha' \searrow 1} D_{\alpha', z}(\psi \| \varphi) &= \lim_{\alpha' \searrow 1} \frac{\log Q_{\alpha', z}(\psi \| \varphi) - \log \psi(1)}{\alpha' - 1} = \lim_{\alpha' \searrow 1} \frac{2z \log \|f(\frac{\alpha'}{\alpha})\|_{L_L^{2z}} - 2z \log \|f(\frac{1}{\alpha})\|_{L_L^{2z}}}{\alpha' - 1} \\ &= \lim_{\alpha' \rightarrow 1} \left(\frac{\log \|f(\frac{\alpha'}{\alpha})\|_{L_L^{2z}} - \log \|f(\frac{1}{\alpha})\|_{L_L^{2z}}}{\|f(\frac{\alpha'}{\alpha})\|_{L_L^{2z}} - \|f(\frac{1}{\alpha})\|_{L_L^{2z}}} \right) \frac{2z}{\alpha} \left(\frac{\|f(\frac{\alpha'}{\alpha})\|_{L_L^{2z}} - \|f(\frac{1}{\alpha})\|_{L_L^{2z}}}{\frac{\alpha' - 1}{\alpha}} \right) \\ &= \psi(1)^{-1} D(\psi \| \varphi). \end{aligned}$$

References

- [1] F. Hiai, $\lim_{\alpha \searrow 1}$ when $1/2 < z < 1$, (12/27/2023) notes.
- [2] H. Kosaki. Applications of the complex interpolation method to a von Neumann algebra: Non-commutative L_p -spaces. *J. Funct. Anal.*, 56:26–78, 1984. doi:[https://doi.org/10.1016/0022-1236\(84\)90025-9](https://doi.org/10.1016/0022-1236(84)90025-9).
- [3] M. Ohya and D. Petz. *Quantum Entropy and Its Use*. Lecture Notes in Computer Science. Springer-Verlag, 1993. ISBN 9783540548812.