Math. Z. 180, 275–286 (1982)

Mathematische Zeitschrift

Long-Time Asymptotic Properties of Dynamical Semigroups on W^*-algebras

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Introduction

The theory of quantum dynamical semigroups provides a convenient mathematical description of the irreversible dynamics of an open quantum system. This motivates an interest in the study of conditions for a dynamical semigroup to induce approach to a stationary state [7, 8, 13, 14] and of related subjects such as irreducibility [2, 4], ergodic theorems [8, 10, 15] and Perron-Frobenius type results [1, 5, 16]. Most of the results have been shown so far in the finite-dimensional case [5, 13, 14] or when there exists a faithful (family of) stationary state(s) [1, 7, 8, 10, 15]. Here we prove more general theorems of ergodic type and on approach to equilibrium, by suitably modifying some of the techniques of the above quoted papers and using a result of [12]. We also give some applications to dynamical semigroups of Lindblad type [6, 11] and asymptotically finite-dimensional.

1. Preliminaries

Let $T=\{T_t\colon t\in\mathbb{R}^+\}$ be a *dynamical semigroup* on a W^*-algebra $\mathcal{M}$, i.e. a weakly* continuous one-parameter semigroup of completely positive identity preserving normal maps of $\mathcal{M}$ into itself, with T_0 the identity map of $\mathcal{M}$. Denote by T_{t*} the preadjoint map of T_t on the predual space $\mathcal{M}_*$ of normal linear functionals on $\mathcal{M}$: then $T_*=\{T_{t*}\colon t\in\mathbb{R}^+\}$ is a strongly continuous one-parameter semigroup of positive contractions on $\mathcal{M}_*$. Let also $\mathcal{F}(T)$, resp. $\mathcal{F}(T_*)$, be the fixed point set of T in $\mathcal{M}$, respectively of T_* in $\mathcal{M}_*$. If there exists a faithful family of T-invariant normal states on $\mathcal{M}$, then $\mathcal{F}(T)$ is a W^*-subalgebra of $\mathcal{M}$ [7, 14]. If P is a projection in $\mathcal{M}$ and φ is in $\mathcal{M}_*$, we denote by $P\varphi P$ the element of $\mathcal{M}_*$ defined as $P\varphi P(A)=\varphi(PAP)$ for all A in $\mathcal{M}$, and by $P\mathcal{M}_*P$ the set of such elements as φ spans $\mathcal{M}_*$. Then the hereditary W^*-subalgebra $P\mathcal{M}P$ of $\mathcal{M}$ is (canonically isomorphic to) the dual space of $P\mathcal{M}_*P$. A non-zero projection P in $\mathcal{M}$ is said to *reduce* T_* [2] if $P\mathcal{M}_*P$ is

0025-5874/82/0180/0275/$02.40

globally invariant under T_* or, equivalently, if

$$P\,T_t(A)\,P = P\,T_t(P\,A\,P)\,P \qquad \text{for all } A \text{ in } \mathscr{M},\ t \text{ in } \mathbb{R}^+. \tag{1.1}$$

Then

$$T_t^P(A) = P\,T_t(A)\,P \qquad \text{for all } A \text{ in } P\,\mathscr{M}\,P \tag{1.2}$$

defines a dynamical semigroup $T^P = \{T_t^P\colon t \in \mathbb{R}^+\}$ on $P\mathscr{M}P$. The support projection $S(\omega)$ of any state ω in $\mathscr{F}(T_*)$ reduces T_* since $S(\omega)\,\mathscr{M}_*S(\omega)$ is the norm closure of the set of linear combinations of the normal states on $\mathscr{M}$ which are majorized by a scalar multiple of ω, and the same is true for the *recurrent subspace projection* (cf. [5])

$$R = \sup\{S(\omega)\colon \omega \text{ is a state in } \mathscr{F}(T_*)\} \tag{1.3}$$

since $R\mathscr{M}_*R$ is the norm closure of $\bigcup\{S(\omega)\,\mathscr{M}_*S(\omega)\colon \omega \text{ is a state in } \mathscr{F}(T_*)\}$. We have $R=0$ when the semigroup T has no normal stationary state (see examples in [2, 4]), and $R=\mathbb{1}$ when T has a faithful family of normal stationary states.

The application of the mean ergodic theorems of [8, 10, 15] to T^R leads to the following.

Theorem 1.1. *For any dynamical semigroup $T=\{T_t\colon t\in\mathbb{R}^+\}$ on a W^*-algebra $\mathscr{M}$,*

$$E(A) = w^*\text{-}\lim_{t\to\infty} \frac{1}{t}\int_0^t R\,T_s(A)\,R\,\mathrm{d}s \tag{1.4}$$

exists for all A in $\mathscr{M}$, and defines a T-invariant normal conditional expectation E onto the W^*-subalgebra $\mathscr{F}(T^R)$ of $R\mathscr{M}R$; and a normal state ω on $\mathscr{M}$ is T-invariant if and only if $\omega\circ E=\omega$.

As a consequence, the study of normal T-invariant states on $\mathscr{M}$ is reduced to the study of normal states on $\mathscr{F}(T^R)$ (cf. [8]).

It should be remarked that the above theorem and all results of Sect. 2 and 3 are also valid if complete positivity is replaced by the requirement that the maps T_t satisfy the Kadison-Schwarz inequality $T_t(A^*A) \geqq T_t(A)^*\,T_t(A)$ for all A in $\mathscr{M}$. As is well known, this property is stronger than positivity, but weaker than 2-positivity [6].

2. The Ergodic Theorem

In this Section we extend to the general case the mean ergodic theorems which were proved in [8, 10, 15] under the assumption $R \doteq \mathbb{1}$.

Theorem 2.1. *For a dynamical semigroup $T=\{T_t\colon t\in\mathbb{R}^+\}$ on a W^*-algebra $\mathscr{M}$ the following are equivalent:*

(i) *there exists a normal T-invariant norm one projection F of $\mathscr{M}$ onto $\mathscr{F}(T)$;*

(ii) $w\text{-}\lim\limits_{t\to\infty} \frac{1}{t}\int_0^t T_{s*}(\varphi)\,\mathrm{d}s$ *exists in $\mathscr{M}_*$ for all φ in $\mathscr{M}_*$;*

(iii) $w^*\text{-}\lim_{t\to\infty} \frac{1}{t}\int_0^t T_s(R)\,\mathrm{d}s = \mathbb{1}$;

(iv) $\mathscr{F}(T_*)$ *separates* $\mathscr{F}(T)$.

If the above conditions are satisfied, then

$$F(A) = w^*\text{-}\lim_{t\to\infty} \frac{1}{t}\int_0^t T_s(A)\,\mathrm{d}s \quad \textit{for all } A \textit{ in } \mathscr{M}. \tag{2.1}$$

Proof. (i)⇒(ii): For each A in $\mathscr{M}$, the net $\left\{\frac{1}{t}\int_0^t T_s(A)\,\mathrm{d}s\right\}_{t>0}$ is compact in the weak* topology of $\mathscr{M}$. Any of its limit points, say A_∞, is in $\mathscr{F}(T)$, hence, if (i) holds, $A_\infty = F(A_\infty)$; on the other hand, $F \circ T_s = F$ for all s and F is normal, hence $F(A_\infty) = F(A)$. So, $\frac{1}{t}\int_0^t T_s(A)\,\mathrm{d}s$ converges in the weak* topology to $F(A)$ as $t\to\infty$. This proves (2.1). As a consequence, for each φ in $\mathscr{M}_*$, the net $\left\{\frac{1}{t}\int_0^t T_{s*}(\varphi)\,\mathrm{d}s\right\}_{t>0}$ converges to $\varphi\circ F$ in the $\sigma(\mathscr{M}^*, \mathscr{M})$ topology of $\mathscr{M}^*$. Both the net and its limit point are in $\mathscr{M}_*$, since F is assumed to be normal, and the $\sigma(\mathscr{M}^*, \mathscr{M})$ topology of $\mathscr{M}^*$, restricted to $\mathscr{M}_*$, is the weak topology of $\mathscr{M}_*$.

(ii)⇒(iii): For any state φ in $\mathscr{M}_*$, the support projection of $w\text{-}\lim_{t\to\infty} \frac{1}{t}\int_0^t T_{s*}(\varphi)\,\mathrm{d}s$ $=\varphi_\infty$ satisfies $S(\varphi_\infty) \leqq R$, hence $\lim_{t\to\infty} \frac{1}{t}\int_0^t \varphi(T_s(R))\,\mathrm{d}s = \varphi(\mathbb{1})$ for all φ in $\mathscr{M}_*$, which is (iii).

(iii)⇒(iv): If A is in $\mathscr{F}(T)$, then $RAR = E(A)$ is in $\mathscr{F}(T^R)$, where E and T^R have been defined in Section 1. We know from Theorem 1.1 that $\mathscr{F}(T_*)$ is isomorphic to the predual space of $\mathscr{F}(T^R)$, hence it separates $\mathscr{F}(T)$ if and only if $RAR = 0$, $A \in \mathscr{F}(T)$, implies $A = 0$. Now, from (iii) it follows that $w^*\text{-}\lim_{t\to\infty} \frac{1}{t}\int_0^t T_s((\mathbb{1} - R)A)\,\mathrm{d}s = w^*\text{-}\lim_{t\to\infty} \frac{1}{t}\int_0^t T_s(A(\mathbb{1} - R))\,\mathrm{d}s = 0$ for all A in $\mathscr{M}$, by application of the Kadison-Schwarz inequality to the maps $\frac{1}{t}\int_0^t T_s(.)\,\mathrm{d}s$. Hence, if A is in $\mathscr{F}(T)$,

$$A = w^*\text{-}\lim_{t\to\infty} \frac{1}{t}\int_0^t T_s(RAR)\,\mathrm{d}s$$

which proves that $A = 0$ if $RAR = 0$.

(iv)⇒(i): This is a special case of a theorem due to Nagel [12].

Remark 2.2. It should be noticed that for *any* dynamical semigroup T on a W^*-algebra $\mathscr{M}$ there exists a unique T-invariant norm one projection F, which satisfies (2.1); however, F need not be normal. The proof of this fact may be obtained by using Theorem 5.1 of [3], where the weak topology should be replaced by the weak* topology. It follows also from [3] that the Cesaro limit or the Abel limit may be used equivalently in (2.1).

Remark 2.3. When $\mathscr{M}$ is finite-dimensional, the map F must be normal, so that (iii) always holds. Actually, more is true in this case: Evans and Høegh-Krohn have shown that $\lim_{t\to\infty} T_t(R)=\mathbb{1}$ [5].

In the general, infinite-dimensional case, the problem of finding R and of ascertaining whether $w^*\text{-}\lim_{t\to\infty} \frac{1}{t}\int_0^t T_s(R)\,\mathrm{d}s=\mathbb{1}$ seems hard and a detailed analysis of concrete cases would be necessary. There is, however, a class of dynamical semigroups for which the above problem can be reduced to the analogous problem for a Markov semigroup on a discrete state space. Suppose there exist a totally atomic abelian W^*-subalgebra $\mathscr{Z}$ of $\mathscr{M}$ and a normal conditional expectation N of $\mathscr{M}$ onto $\mathscr{Z}$, commuting with the dynamical semigroup T. (For instance, this is the case when $\mathscr{M}$ is the algebra $\mathscr{B}(\mathscr{H})$ of all bounded linear operators on a separable Hilbert space $\mathscr{H}$, $\mathscr{Z}=\{P_i\colon i\in I\}''$ where $\{P_i\colon i\in I\}$ is a family of mutually orthogonal one-dimensional projections with $\sum_{i\in I} P_i=\mathbb{1}$, and T commutes with a group $\{\alpha_t\colon t\in\mathbb{R}\}$ of $*$-automorphisms of $\mathscr{M}$ determined by a Hamiltonian $H=\sum_{i\in I}\varepsilon_i P_i$ where the ε_i are distinct real numbers). Then, for all t in $\mathbb{R}^+$, T_t may be restricted to a map $\tilde{T}_t$ on $\mathscr{Z}$, and $\tilde{T}=\{\tilde{T}_t\colon t\in\mathbb{R}^+\}$ is a Markov semigroup on a discrete state space, the "states" corresponding to the atoms of $\mathscr{Z}$. Now, we claim that $R=\tilde{R}$, where R and $\tilde{R}$ are the recurrent subspace projections of T_* and $\tilde{T}_*$, respectively. Indeed, each state φ in $\mathscr{F}(\tilde{T}_*)$ can be extended to a state $\varphi\circ N$ in $\mathscr{F}(T_*)$ and $\tilde{R}$ is just $\sup\{S(\omega)\colon \omega$ is an N-invariant state in $\mathscr{F}(T_*)\}$. This proves $R\geqq\tilde{R}$. On the other hand, we have $\tilde{R}=N(\tilde{R})$ and, for any state ω in $\mathscr{F}(T_*)$, $\omega\circ N\restriction\mathscr{Z}$ is a state in $\mathscr{F}(\tilde{T}_*)$. Then, $\omega(\tilde{R})=(\omega\circ N)(\tilde{R})=\mathbb{1}$, so that $\tilde{R}\geqq R$.

Finally, since $R=\tilde{R}=N(\tilde{R})$ and N is normal, for each φ in $\mathscr{M}_*$ we have

$$\varphi\left(\frac{1}{t}\int_0^t T_s(R)\,\mathrm{d}s\right)=(\varphi\circ N)\left(\frac{1}{t}\int_0^t T_s(\tilde{R})\,\mathrm{d}s\right),$$

hence the existence of $\sigma(\mathscr{M},\mathscr{M}_*)\text{-}\lim_{t\to\infty}\frac{1}{t}\int_0^t T_s(R)\,\mathrm{d}s$ is equivalent to the existence of $\sigma(\mathscr{Z},\mathscr{Z}_*)\text{-}\lim_{t\to\infty}\frac{1}{t}\int_0^t T_s(\tilde{R})\,\mathrm{d}s$, and the former is $\mathbb{1}$ if and only if the latter is.

3. Approach to Equilibrium

In this Section we study conditions under which any normal state on $\mathscr{M}$ approaches a normal state under the action of T_{t*} as $t\to\infty$. We reduce this problem to the corresponding problem for T^R, which has a faithful family of normal stationary states, plus the condition that $w^*\text{-}\lim_{t\to\infty} T_t(R)=\mathbb{1}$. Then, by restricting considerations to T^R, we may assume $R=\mathbb{1}$; for this case we give some extension of the results of [8].

The following Lemma is needed repeatedly.

Lemma 3.1. *Let $T=\{T_t\colon t\in\mathbb{R}^+\}$ be a dynamical semigroup on a W^*-algebra $\mathcal{M}$, and P a projection in $\mathcal{M}$ reducing T_*. Then the following are equivalent:*

(i) $w^*\text{-}\lim_{t\to\infty} T_t(P)=\mathbb{1}$;

(ii) $\lim_{t\to\infty}\|T_{t*}\varphi-P(T_{t*}\varphi)P\|=0$ *for all φ in $\mathcal{M}_*$;*

(iii) *for all φ in $\mathcal{M}_*$ and $\varepsilon>0$, there exists ψ in $P\mathcal{M}_*P$ and $t>0$ such that $\|T_{t*}\varphi-\psi\|<\varepsilon$.*

Proof. (i)$\Rightarrow$(ii): We can assume $\mathcal{M}$ to be concretely represented as a von Neumann algebra of bounded linear operators on a Hilbert space $\mathcal{H}$. For all A in $\mathcal{M}$, ξ in $\mathcal{H}$, and $t\geqq 0$, we have, using the Kadison-Schwarz inequality for T_t,

$$\begin{aligned}\|T_t(A(\mathbb{1}-P))\,\xi\|^2&\leqq(\xi,\,T_t((\mathbb{1}-P)\,A^*A(\mathbb{1}-P))\,\xi)\\&\leqq\|A\|^2(\xi,\,T_t(\mathbb{1}-P)\,\xi)\end{aligned}$$

so that $T_t(A(\mathbb{1}-P))$ tends to zero as $t\to\infty$ in the strong operator topology, uniformly in A. Thus, $T_t(A-PAP)$ tends to zero as $t\to\infty$ in the weak (hence ultraweak, being a bounded net) operator topology, uniformly in A. This proves (ii).

(ii)$\Rightarrow$(i): For any state φ in $\mathcal{M}_*$, (ii) gives

$$0\leqq\varphi(T_t(\mathbb{1}-P))=(T_{t*}\varphi-P(T_{t*}\varphi)P)(\mathbb{1})\to 0$$

as $t\to\infty$, hence (i) follows.

(ii)$\Rightarrow$(iii): Given φ in $\mathcal{M}_*$ and $\varepsilon>0$, choose t large enough and $\psi=P(T_{t*}\varphi)P$.

(iii)$\Rightarrow$(ii): Let φ, ε, ψ, t be given such that (iii) holds. For all s in $\mathbb{R}^+$ we have

$$\begin{aligned}\|P(T_{t+s*}\varphi)P-P(T_{s*}\psi)P\|&\leqq\|T_{t+s*}\varphi-T_{s*}\psi\|\\&\leqq\|T_{t*}\varphi-\psi\|<\varepsilon.\end{aligned}$$

But ψ is in $P\mathcal{M}_*P$ and P reduces T_*, hence $P(T_{s*}\psi)P=T_{s*}\psi$. Then, $\|T_{t+s*}\varphi-P(T_{t+s*}\varphi)P\|<2\varepsilon$ for all s in $\mathbb{R}^+$ and a suitable t, which is (ii).

Theorem 3.2. *Let $T=\{T_t\colon t\in\mathbb{R}^+\}$ be a dynamical semigroup on a W^*-algebra $\mathcal{M}$, and let T^R and E be defined as in Section 1. Then the following conditions are equivalent:*

(i) $w^*\text{-}\lim_{t\to\infty} T_t(R)=\mathbb{1}$ *and* $w\text{-}\lim_{t\to\infty}\psi\circ T_t^R=\psi\circ E$ *for all ψ in $R\mathcal{M}_*R$;*

(ii) $w\text{-}\lim_{t\to\infty} T_{t*}\varphi$ *exists in $\mathcal{M}_*$ for all φ in $\mathcal{M}_*$.*

Similarly, the following conditions are equivalent:

(i′) $w^*\text{-}\lim_{t\to\infty} T_t(R)=\mathbb{1}$ *and* $\lim_{t\to\infty}\|\psi\circ T_t^R-\psi\circ E\|=0$ *for all ψ in $R\mathcal{M}_*R$;*

(ii′) $n\text{-}\lim_{t\to\infty} T_{t*}\varphi$ *exists in $\mathcal{M}_*$ for all φ in $\mathcal{M}_*$.*

Furthermore, conditions (i′), (ii′) *imply* (i), (ii), *and* (i) *implies the equivalent conditions of Theorem* 2.1.

Proof. (ii)$\Rightarrow$(i) and (ii$'$)$\Rightarrow$(i$'$): If the limit in (ii) (or in (ii$'$)) exists, it coincides with $\varphi \circ F$, where F is defined by (2.1) and is normal since the limit is assumed to exist in $\mathcal{M}_*$. For any state φ in $\mathcal{M}_*$, the support projection of $\varphi \circ F$ is not greater than R, so that $\lim_{t\to\infty} \varphi(T_t(R)) = \varphi(\mathbb{1})$ for all φ in $\mathcal{M}_*$, i.e. $w^*\text{-}\lim_{t\to\infty} T_t(R) = \mathbb{1}$. Moreover, it is clearly $RF(.)R = E$, hence the conditions $w\text{-}\lim_{t\to\infty} \psi \circ T_t^R = \psi \circ E$ and $\lim_{t\to\infty} \|\psi \circ T_t^R - \psi \circ E\| = 0$ for all ψ in $R\mathcal{M}_*R$ are the specializations of (ii) and (ii$'$), respectively, to ψ in $R\mathcal{M}_*R$.

(i)$\Rightarrow$(ii) and (i$'$)$\Rightarrow$(ii$'$): Notice, first of all, that (i$'$)$\Rightarrow$(i), which in turn implies (iii) of Theorem 2.1, hence F exists, given by (2.1), and is normal. For all A in $\mathcal{M}$ with $\|A\| \leqq 1$, φ in $\mathcal{M}_*$ and t, s in $\mathbb{R}^+$, we have

$$\begin{aligned}
&|\varphi(T_{t+s}(A)) - \varphi(F(A))| \\
&\quad \leqq |\varphi \cdot T_t(T_s(A) - R\,T_s(A)\,R)| + |\varphi \circ T_t(F(A) - R\,F(A)\,R)| + |\varphi \circ T_t(T_s^R(A) - E(A))| \\
&\quad \leqq \|T_{t*}\varphi - R(T_{t*}\varphi)R\| \, [\|T_s(A)\| + \|F(A)\|] + |T_{t*}\varphi(T_s^R(A) - E(A))| \\
&\quad \leqq 2\|T_{t*}\varphi - R(T_{t*}\varphi)R\| + |T_{t*}\varphi(T_s^R(A) - E(A))|,
\end{aligned}$$

where $T_t \circ F = F$ and $RF(.)R = E$ have been used.

Take $\varepsilon > 0$. Because of Lemma 3.1, there exists $t(\varepsilon, \varphi)$ such that

$$\|T_{t*}\varphi - R(T_{t*}\varphi)R\| < \varepsilon/3 \qquad \text{for all } t \geqq t(\varepsilon, \varphi).$$

Under condition (i), there exists $s(t, \varepsilon, A, \varphi)$ such that

$$|T_{t*}\varphi(T_s^R(A) - E(A))| < \varepsilon/3 \qquad \text{for all } s \geqq s(t, \varepsilon, A, \varphi)$$

where we have taken into account that $T_{t*}\varphi(T_s^R(A) - E(A)) = (R\,T_{t*}\varphi R)\,(T_t^R(A) - E(A))$. If condition (i$'$) holds, $s(t, \varepsilon, A, \varphi)$ may be chosen independent of A for $\|A\| \leqq 1$. So, if $u \geqq u(\varepsilon, A, \varphi) = t(\varepsilon, \varphi) + s(t(\varepsilon, \varphi), \varepsilon, A, \varphi)$, we may decompose u as $t(\varepsilon, \varphi) + s$ with $s \geqq s(t(\varepsilon, \varphi), \varepsilon, A, \varphi)$ and we have

$$|\varphi(T_u(A)) - \varphi(F(A))| < \varepsilon \qquad \text{for all } u \geqq u(\varepsilon, A, \varphi).$$

This proves (ii) from (i). If (i$'$) holds, $u(\varepsilon, A, \varphi)$ may be chosen independent of A for $\|A\| \leqq 1$ and (ii$'$) follows.

In the rest of the Section, we assume that $R = \mathbb{1}$ and we given some extension of the results of [8] concerning the weak approach to equilibrium. These considerations could, in principle, be applied to T^R. As regards the condition $w^*\text{-}\lim_{t\to\infty} T_t(R) = \mathbb{1}$, we shall discuss it in some special case in the next Section.

Let $\mathcal{M}_\Gamma(T)$ be the weak closure of the linear span of the eigenvectors of T_t corresponding to eigenvalues of modulus one, and define

$$\begin{aligned}
\mathcal{N}(T) = \{A \in \mathcal{M}:\ & T_t(A^*A) = T_t(A^*)\,T_t(A), \\
& T_t(AA^*) = T_t(A)\,T_t(A^*) \text{ for all } t \text{ in } \mathbb{R}^+\}.
\end{aligned}$$

It has been shown by Evans [4, Theorem 3.1] that $\mathcal{N}(T)$ is a W^*-subalgebra of $\mathcal{M}$. When there exists a faithful family of normal T-invariant states, $\mathcal{M}_\Gamma(T)$ is a

W^*-subalgebra of $\mathcal{M}$ and is contained in $\mathcal{N}(T)$, as shown by Albeverio and Høegh-Krohn in [1]; furthermore, when $R=\mathbb{1}$, $\mathcal{N}(T)$ can be equivalently characterized as

$$\mathcal{N}(T)=\{A\in\mathcal{M}:\ \omega(A^*A)=\omega(T_t(A^*)\,T_t(A)),\ \omega(AA^*) =\omega(T_t(A)\,T_t(A^*)) \text{ for all states } \omega \text{ in } \mathcal{F}(T_*) \text{ and } t \text{ in } \mathbb{R}^+\},$$

which shows that $\mathcal{N}(T)$ is globally invariant under T when $R=\mathbb{1}$.

Theorem 3.3. *Let $T=\{T_t:\ t\in\mathbb{R}^+\}$ be a dynamical semigroup on a W^*-algebra $\mathcal{M}$. If there is a faithful family of normal T-invariant states, then $\mathcal{N}(T)=\mathcal{F}(T)$ implies that*

$$w^*\text{-}\lim_{t\to\infty} T_t(A)=E(A) \qquad \textit{for all } A \textit{ in } \mathcal{M}, \tag{3.1}$$

which in turn implies that $\mathcal{M}_\Gamma(T)=\mathcal{F}(T)$.

Proof. Under the above assumptions, we have

$$E(A)=F(A)=w^*\text{-}\lim_{t\to\infty}\frac{1}{t}\int_0^t T_s(A)\,\mathrm{d}s \qquad \text{for all } A \text{ in } \mathcal{M}.$$

Then the first implication is a straightforward generalization of Theorem 3.1 of [8] (we warn the reader that the set $\mathcal{N}(T)$ defined here is $\mathcal{N}(T)\cap\mathcal{N}(T)^*$ in the notation of [8]), and the second one is obvious.

Theorem 3.4. *Under the assumptions of Theorem* 3.3, *suppose in addition that either*

(a) *$\mathcal{M}$ is finite-dimensional,*

or

(b) *$\mathcal{M}$ is the algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on a separable Hilbert space $\mathcal{H}$, and T is norm continuous and satisfies detailed balance, in the sense of* [9], *with respect to a faithful normal state ω.*

Then $\mathcal{N}(T)=\mathcal{F}(T)$ is a necessary and sufficient condition for the validity of (3.1).

Proof. We show that in both cases (a) and (b) $\mathcal{M}_\Gamma(T)=\mathcal{N}(T)$ and then we apply Theorem 3.3. Notice that, under the present assumptions, $\mathcal{N}(T)$ is the largest T-invariant W^*-subalgebra of $\mathcal{M}$ such that the restriction of T to it is a semigroup of *-homomorphisms.

When $\mathcal{M}$, hence $\mathcal{N}(T)$, is finite-dimensional, a semigroup of *-homomorphisms can always be extended to a group of *-automorphisms with pure point spectrum. This proves $\mathcal{N}(T)=\mathcal{M}_\Gamma(T)$ in case (a).

In case (b), the detailed balance condition of [9] means that the infinitesimal generator L of T can be decomposed as $L=L_h+L_s$, where $L_h(A)=[iH,A]$ for all A in $\mathcal{M}$, H being a self-adjoint operator on $\mathcal{H}$ commuting with the density matrix ρ determining ω, and where L_s satisfies $\omega(A\,L_s(B))=\omega(L_s(A)\,B)$ for all A, B in $\mathcal{M}$. Recalling Theorem 3.1 of [4], we see that, if A is in $\mathcal{N}(T)$, we have $\omega(A\,L(B))=-\omega(L(A)\,B)$ for all B in $\mathcal{M}$, so that $L(A)=L_h(A)$ for all A in $\mathcal{N}(T)$. Then, $T_t(A)=e^{iHt}A\,e^{-iHt}$ for all A in $\mathcal{N}(T)$, t in $\mathbb{R}^+$, and H has pure point spectrum since it commutes with the strictly positive density matrix ρ. This proves $\mathcal{N}(T)=\mathcal{M}_\Gamma(T)$ in case (b).

4. Applications

Let $\mathscr{M}$ be the algebra $\mathscr{B}(\mathscr{H})$ of all bounded linear operators on a separable Hilbert space $\mathscr{H}$ and let $T=\{T_t\colon t\in\mathbb{R}^+\}$ be a norm continuous dynamical semigroup on $\mathscr{M}$ with infinitesimal generator L. Then L is of the form (Lindblad [11])

$$L(A)=K^*A+AK+W(A) \qquad \text{for all } A \text{ in } \mathscr{B}(\mathscr{H}) \tag{4.1}$$

where K is in $\mathscr{B}(\mathscr{H})$ and W is a completely positive normal map of $\mathscr{B}(\mathscr{H})$ into itself, satisfying $K^*+K+W(\mathbb{1})=0$. By Stinespring's theorem, W can be written as

$$W(A)=\sum_{i=1}^{\infty} V_i^* A V_i \qquad \text{for all } A \text{ in } \mathscr{B}(\mathscr{H}) \tag{4.2}$$

where the V_i are in $\mathscr{B}(\mathscr{H})$ and the convergence is ultraweak. We can allow for an unbounded K in (4.1) in the following way: if $C_t=\exp(Kt)$ is a strongly continuous contraction semigroup on $\mathscr{H}$ such that $(K\xi,\ \eta)+(\xi,\ K\eta)+(\xi,\ W(\mathbb{1})\eta)=0$ for all ξ, η in $\operatorname{dom}(K)$, and $S_t(A)=C_t^*AC_t$, $A\in\mathscr{B}(\mathscr{H})$, $t\in\mathbb{R}^+$, then there exists a dynamical semigroup $T=\{T_t\colon t\in\mathbb{R}^+\}$ of $\mathscr{B}(\mathscr{H})$ satisfying

$$T_t(A)=S_t(A)+\int_0^t S_{t-s}WT_s(A)\,\mathrm{d}s \qquad \text{for all } A \text{ in } \mathscr{B}(\mathscr{H}). \tag{4.3}$$

Such dynamical semigroups will be called *of Lindblad type* [6].

A projection P reduces the preadjoint of a dynamical semigroup T of Lindblad type if and only if [2]

$$(\mathbb{1}-P)\,V_iP=0 \text{ for all } i \text{ and } (\mathbb{1}-P)\,C_tP=0 \text{ for all } t \text{ in } \mathbb{R}^+, \tag{4.4}$$

and if T is norm continuous with generator of the form (4.1), T^P is norm continuous with a generator of the same form, in which K is replaced by PKP and V_i by PV_iP. Moreover, $\mathscr{N}(T)\subseteq\{V_i^*, V_i\}'$, and if T has a faithful family of normal stationary states, then $\mathscr{M}(T)=\{V_i^*,\ V_i,\ K,\ K^*\}'$ [4, 7]. If $\operatorname{lin}\{V_i\}$ is self-adjoint, a projection reducing T is also a fixed point of T, hence in this case the equivalent conditions of Theorem 2.1 give $R=\mathbb{1}$.

For some semigroup of Lindblad type, it is possible to check concretely the equivalent conditions of Lemma 3.1.

Proposition 4.1. *Let $T=\{T_t\colon t\in\mathbb{R}^+\}$ be a dynamical semigroup of Lindblad type on $\mathscr{M}=\mathscr{B}(\mathscr{H})$ and let $\{P_n\colon n=0,1,2,\ldots\}$ be an increasing (finite or infinite) sequence of projections in $\mathscr{B}(\mathscr{H})$ satisfying*

(i) $P_0=P_1=P$, $\sup_n P_n=\mathbb{1}$;

(ii) $(\mathbb{1}-P_n)\,V_iP_{n+1}=0$ *for all n and for all i;*

(iii) $(\mathbb{1}-P_n)\,C_tP_n=0$ *for all n and for all t in $\mathbb{R}^+$;*

(iv) $\lim_{t\to\infty} C_t(\mathbb{1}-P)\,\xi=0$ *for all ξ in $\mathscr{H}$.*

Then P satisfies the equivalent conditions of Lemma 3.1.

Proof. Note, first of all, that all P_n reduce T_* by (ii), (iii) and (4.4). Then it follows from (ii), (iii), (4.3) and (4.2) that $(T_{t*}-S_{t*})(\varphi)$ is in $\mathscr{T}(P_{n-1}\mathscr{H})$

$=P_{n-1}\mathcal{M}_* P_{n-1}$ if φ is in $\mathcal{T}(P_n\mathcal{H})=P_n\mathcal{M}_* P_n$ (here we denote by $\mathcal{T}(\mathcal{K})$ the space of all trace class operators on a Hilbert space $\mathcal{K}$). By (iv), $\lim_{t\to\infty}\|S_{t*}(\varphi-P\varphi P)\| =0$ for all φ in $\mathcal{M}_*$. Thus, for all $n=2,3,\ldots$, for all φ_n in $\mathcal{T}(P_n\mathcal{H})$ and all $\varepsilon>0$, it is possible to find φ_{n-1} in $\mathcal{T}(P_{n-1}\mathcal{H})$ and $t=t(\varphi_n,\varepsilon)>0$ such that $\|T_{t*}\varphi_n -\varphi_{n-1}\|<\varepsilon$; take, e.g., $\varphi_{n-1}=S_{t*}(P\varphi_n P)+\int_0^t T_{s*}W_* S_{t-s*}(\varphi_n)\,\mathrm{d}s$ with t large enough: then, $\|T_{t*}\varphi_n-\varphi_{n-1}\|=\|S_{t*}(\varphi_n-P\varphi_n P)\|$. By iterating this procedure, given φ_n in $\mathcal{T}(P_n\mathcal{H})$ and $\varepsilon>0$, it is possible to find ψ in $\mathcal{T}(P\mathcal{H})$ and $s =s(n,\varphi_n,\varepsilon)>0$ such that

$$\|T_{s*}\varphi_n-\psi\|<\varepsilon;$$

here, $s(n,\varphi_n,\varepsilon)=\sum_{j=2}^{n} t(\varphi_j,\varepsilon/(n-1))$ and $\psi=\varphi_1$. Then, by Lemma 3.1, we have for all n

$$\lim_{t\to\infty}\|T_{t*}\varphi_n-P(T_{t*}\varphi_n)P\|=0 \qquad \text{for all } \varphi_n \text{ in } \mathcal{T}(P_n\mathcal{H}). \tag{4.5}$$

On the other hand, by (i) we have that

$$\lim_{n\to\infty}\|\varphi-P_n\varphi P_n\|=0 \qquad \text{for all } \varphi \text{ in } \mathcal{T}(\mathcal{H}). \tag{4.6}$$

Combining (4.5) and (4.6), with $\varphi_n=P_n\varphi P_n$, we conclude by an $\varepsilon/3$ argument that

$$\lim_{t\to\infty}\|T_{t*}\varphi-P(T_{t*}\varphi)P\|=0 \qquad \text{for all } \varphi \text{ in } \mathcal{T}(\mathcal{H}),$$

which is (ii) of Lemma 3.1.

Remark 4.2. Conditions (ii) to (iv) of Proposition (4.1) are most easily checked when there is a family $\{Q_k\colon k\in I\}$ of mutually orthogonal projections with $\sum_{k\in I} Q_k =\mathbb{1}$ such that $\{Q_k\colon k\in I\}''$ contains all P_n and C_t, and is mapped into itself by W. Explicitly, let

$$P_n=\sum_{k\in I_n} Q_k \qquad \text{with } I_n\subseteq I_{n+1} (n=0,1,\ldots), \qquad \sup_n I_n=I;$$

$$C_t=\sum_{k\in I}\exp[-(i\omega_k+\gamma_k)t]\,Q_k \qquad \text{with } \omega_k\in\mathbb{R},\ \gamma_k\in\mathbb{R}^+;$$

$$W(Q_k)=\sum_{j\in I} W_{kj}Q_j \qquad \text{with } W_{kj}\in\mathbb{R}^+.$$

Then (iii) is satisfied and (iv) becomes $\gamma_k>0$ for all $k\notin I_1$; it is easily shown that (ii) is equivalent to $W(\mathcal{B}(P_{n+1}\mathcal{H}))\subseteq\mathcal{B}(P_n\mathcal{H})$ for all n, which in turn is equivalent to $W_{kj}=0$ whenever $k\notin I_n$, $j\in I_{n+1}$.

Note that, when $\operatorname{tr}Q_k<\infty$ for all $k\in I$, the W_{kj} are just the transition rates of the Pauli master equation

$$\dot{n}_k=\sum_j W_{kj}n_j-W_{jk}n_k$$

governing the evolution under T_* of the diagonal density matrices of the form $\sum_{k\in I} n_k Q_k$, $\sum_{k\in I} n_k \operatorname{tr} Q_k < \infty$.

The following Proposition is the generalization of Theorem 3.2 of [8].

Proposition 4.3. *Under the assumptions of Proposition 4.1, suppose in addition that*

(a) *there exists a T-invariant normal state ω, and*

(b) $\operatorname{lin}\{PV_iP\}$ *is self-adjoint and its commutant in $\mathscr{B}(P\mathscr{H})$ is reduced to the multiples of P.*

Then $S(\omega)=P$, $R=P$ and $w\text{-}\lim_{t\to\infty} \varphi\circ T_t = \varphi(\mathbb{1})\,\omega$ for all φ in $\mathscr{M}_*$.

Proof. Apply Theorem 3.2 of [8] to T^P to prove that $S(\omega)=P$ and that $w\text{-}\lim_{t\to\infty} \varphi\circ T_t^P = \varphi(\mathbb{1})\,\omega$ for all φ in $P\mathscr{M}_*P$. Then $R\geqq P$. On the other hand, by Proposition 4.1, $w^*\text{-}\lim_{t\to\infty} T_t(P)=\mathbb{1}$, hence $\varphi(P)=\mathbb{1}$ for each state φ in $\mathscr{F}(T_*)$ and $P\geqq R$. It follows that $R=P$ and, by Theorem 3.2, $\varphi\circ T_t$ tends weakly as $t\to\infty$ to $\varphi(\mathbb{1})\,\omega$ for all φ in $\mathscr{M}_*$.

Let $\mathscr{M}$ be any W^*-algebra. We say that the preadjoint semigroup T_* of a dynamical semigroup T on $\mathscr{M}$ is *asymptotically finite-dimensional* if there exists a projection P in $\mathscr{M}$ reducing T_* and satisfying the conditions of Lemma 3.1 such that $P\mathscr{M}_*P$ is finite-dimensional.

Proposition 4.4. *Let T_* be asymptotically finite-dimensional Then $w^*\text{-}\lim_{t\to\infty} T_t(R) = \mathbb{1}$.*

Proof. Let R^P be the recurrent subspace projection of T_*^P. Clearly, $R^P\leqq R$. It is also easy to show that R^P reduces T_* since it reduces T_*^P and P reduces T_*. Next, we show that $w^*\text{-}\lim_{t\to\infty} T_t(R^P)=\mathbb{1}$. Since, by assumption, P satisfies the conditions of Lemma 3.1, for all φ in $\mathscr{M}_*$ and $\varepsilon>0$ there exist ψ in $P\mathscr{M}_*P$ and $t>0$ such that

$$\|T_{t*}\varphi-\psi\|<\varepsilon/2.$$

Similarly, by Lemma 3.1 and [5], for all ψ in $P\mathscr{M}_*P$ and $\varepsilon>0$ there exist ω in $R^P\mathscr{M}_*R^P$ and $s>0$ such that

$$\|T_{s*}^P\,\psi-\omega\|<\varepsilon/2.$$

But P reduces T, hence $T_{s*}^P\psi = T_{s*}\psi$ for all ψ in $P\mathscr{M}_*P$ and $s>0$, so that

$$\|T_{t+s*}\varphi-\omega\|<\varepsilon.$$

Therefore, since R^P reduces T_*, $w^*\text{-}\lim_{t\to\infty} T_t(R^P)=\mathbb{1}$ by Lemma 3.1. Finally, using the same argument as in Proposition 4.3, we find that $R^P\geqq R$, so $R^P=R$ and $w^*\text{-}\lim_{t\to\infty} T_t(R)=\mathbb{1}$.

Theorem 4.5. *Let $T=\{T_t\colon t\in\mathbb{R}^+\}$ be a dynamical semigroup on a W^*-algebra $\mathscr{M}$ such that T_* is asymptotically finite-dimensional. Then the following are equivalent:*

(i) *there is a state* ω *in* $\mathcal{M}_*$ *such that*

$$\lim_{t\to\infty} \|T_{t*}\varphi - \varphi(\mathbb{1})\,\omega\| = 0 \qquad \textit{for all } \varphi \textit{ in } \mathcal{M}_*;$$

(ii) *there is exactly one* T*-invariant state* ω *in* $\mathcal{M}_*$*;*

(iii) *the set* $\mathcal{P}(T_*)$ *of all projections in* $\mathcal{M}$ *reducing* T_* *has a non-zero smallest element.*

If these conditions hold, then $\inf \mathcal{P}(T_*) = S(\omega) = R$.

Proof. (i)⇒(ii): Obvious.

(i)⇒(iii): The support $S(\omega)$ of ω reduces T_* and, if Q is in $\mathcal{P}(T_*)$, it follows from (i) that ω is in $Q\mathcal{M}_*Q$, so that $S(\omega) \leqq Q$ for all Q in $\mathcal{P}(T_*)$.

(iii)⇒(ii): Let ψ be an extremal T-invariant normal state on $\mathcal{M}$, and consider the dynamical semigroup $T^{S(\psi)}$ on $S(\psi)\mathcal{M}S(\psi)$. For all φ in $(\mathcal{M}_*)_+$ with $S(\varphi) \leqq S(\psi)$, we have $T_{t*}\varphi = T_{t*}^{S(\psi)}\varphi$ for all t in $\mathbb{R}^+$. Hence, ψ is also extremal $T^{S(\psi)}$-invariant, and since it is faithful on $S(\psi)\mathcal{M}S(\psi)$, it follows from [7] that no projection in $S(\psi)\mathcal{M}S(\psi)$ reduces $T^{S(\psi)}$. Then, no projection Q in $\mathcal{M}$ with $Q \leqq S(\psi)$ can reduce T. As a consequence, the support of any extremal T-invariant normal state is a minimal element of $\mathcal{P}(T_*)$. By (iii), $\mathcal{P}(T_*)$ has a unique minimal element, hence there exists just one extremal T-invariant normal state. Since T_* is asymptotically finite-dimensional, $\mathcal{F}(T_*)$ is non-zero and finite-dimensional, hence any state in $\mathcal{F}(T_*)$ can be decomposed (not uniquely, in general) into extremal T-invariant normal states. We can conclude that there exists exactly one T-invariant state ω in $\mathcal{M}_*$, with support $S(\omega) = \inf \mathcal{P}(T_*)$.

(ii)⇒(i): If ω is the unique T-invariant state in $\mathcal{M}_*$ we have $R = S(\omega)$. Since T_* is asymptotically finite-dimensional, $w^*\text{-}\lim_{t\to\infty} T_t(R) = \mathbb{1}$ by Proposition 4.4. Hence, by Theorem 3.2, it suffices to show that

$$\lim_{t\to\infty} \|\varphi \circ T_t - \varphi(\mathbb{1})\,\omega\| = 0 \qquad \text{for all } \varphi \text{ in } S(\omega)\mathcal{M}_*S(\omega). \tag{4.7}$$

Let P be the projection satisfying $w^*\text{-}\lim_{t\to\infty} T_t(P) = \mathbb{1}$ such that $P\mathcal{M}_*P$ is finite-dimensional, whose existence has been assumed. Then $\omega(P) = 1$ and $S(\omega) \leqq P$. It follows that $S(\omega)\mathcal{M}_*S(\omega)$ is finite-dimensional, and we get (4.7) by the uniqueness of the invariant state, as shown by Evans and Høegh-Krohn [5].

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Received March 31, 1981; received in final form January 15, 1982