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A Relation Between Completely Bounded Norms and Conjugate Channels

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Abstract: We show a relation between a quantum channel Φ and its conjugate Φ^C, which implies that the $p \to p$ Schatten norm of the channel is the same as the $1 \to p$ completely bounded norm of the conjugate. This relation is used to give an alternative proof of the multiplicativity of both norms.

1. Introduction

A quantum channel is a completely positive trace preserving (CPT) map $\Phi : M_d \to M_{d'}$, M_d is the set of $d \times d$ complex matrices. Any channel can be viewed as a map $L_q(M_d) \to L_p(M_{d'})$, where $L_q(M_d)$ denotes the space M_d with the Schatten norm $\|A\|_q = \mathrm{T}r(|A|^q)^{1/q}$, $1 \le q \le \infty$. Let $\|\Phi\|_{q\to p}$ be the corresponding norm of Φ,

$$\|\Phi\|_{q\to p} = \sup_{A\in M_d} \frac{\|\Phi(A)\|_p}{\|A\|_q} = \sup_{A\in M_d, A\ge 0} \frac{\|\Phi(A)\|_p}{\|A\|_q},$$

the second equality was proved in [1, 9]. We say that norms of this type are multiplicative if

$$\|\Phi_1 \otimes \Phi_2\|_{q\to p} = \|\Phi_1\|_{q\to p}\|\Phi_2\|_{q\to p}$$

for any channels Φ_1 and Φ_2.

The spaces $L_q(M_d)$ and $L_p(M_{d'})$ can be endowed with an operator space structure as in [8], then Φ is a completely bounded map. Multiplicativity of the corresponding completely bounded norms $\|\Phi\|_{CB,q\to p}$ for all $1 \le p, q \le \infty$ was proved in [3]. In particular, this implies multiplicativity of $\|\Phi\|_{q\to p}$ for $q \ge p$, since this is equal to $\|\Phi\|_{CB,q\to p}$ for CPT maps. It was shown that the norm $\|\Phi\|_{CB,1\to p}$ is equal to the quantity

$$\omega_p(\Phi) = \sup_{\psi\in\mathbb{C}^d\otimes\mathbb{C}^d} \frac{\|(\mathcal{I}\otimes\Phi)(|\psi\rangle\langle\psi|)\|_p}{\|\mathrm{T}r_2(|\psi\rangle\langle\psi|)\|_p}.$$

Multiplicativity of ω_p then yields the additivity for the CB minimal conditional entropy, defined as

$$S_{CB,\min} = \inf_{\psi \in \mathbb{C}^d \otimes \mathbb{C}^d} \left(S\left[(\mathcal{I} \otimes \Phi)(|\psi\rangle\langle\psi|) \right] - S\left[\mathrm{T}r_2(|\psi\rangle\langle\psi|) \right] \right).$$

In the present note, we show that there is a relation between $\omega_p(\Phi)$ and the norm $\|\Phi^C\|_{p\to p}$ of the conjugate map Φ^C. This relation is then used for an alternative proof of multiplicativity of both quantities, avoiding the use of the deep results of the theory of operator spaces and CB norms, involved in the proofs in [3].

2. Representations of CPT Maps and Conjugate Channels

Let $e_1^d, \ldots, e_d^d$ be the standard basis in $\mathbb{C}^d$ and let $\beta_0 = \frac{1}{\sqrt{d}} \sum_i e_i^d \otimes e_i^d$ be a maximally entangled vector. Let $\Phi : M_d \to M_{d'}$ be a CPT map. Then Φ is uniquely represented by its Choi-Jamiolkowski matrix $X_\phi \in M_d \otimes M_{d'}$, defined by

$$X_\Phi = d(\mathcal{I} \otimes \Phi)(|\beta_0\rangle\langle\beta_0|) = \sum_{i,j} \left| e_i^d \rangle\langle e_j^d \right| \otimes \Phi\left(\left| e_i^d \rangle\langle e_j^d \right| \right). \tag{1}$$

Other representations of Φ can be obtained from the Stinespring representation, which in the case of matrices has the form [7]

$$\Phi(\rho) = V^\dagger(\rho \otimes I_\kappa)V, \quad V : \mathbb{C}^{d'} \to \mathbb{C}^d \otimes \mathcal{H}, \quad \mathrm{T}r_2 V V^\dagger = I_d, \tag{2}$$

where $\mathcal{H}$ is an auxiliary Hilbert space, $\kappa = \dim \mathcal{H} \leq dd'$. The Lindblad-Stinespring representation of Φ is

$$\Phi(\rho) = \mathrm{T}r_2 U(\rho \otimes |\phi\rangle\langle\phi|)U^\dagger, \tag{3}$$

where ϕ is a unit vector in $\mathcal{H}$, and $U : \mathbb{C}^d \otimes \mathcal{H} \to \mathbb{C}^{d'} \otimes \mathcal{H}$ is a partial isometry. This can be obtained from the Stinespring representation of the dual map $\hat{\Phi}$. The Kraus representation

$$\Phi(\rho) = \sum_{k=1}^{\kappa} F_k \rho F_k^\dagger, \quad F_k : \mathbb{C}^d \to \mathbb{C}^{d'}, \quad \sum_k F_k^\dagger F_k = I_d \tag{4}$$

is related to (2) and (3) by

$$V = \sum_{k=1}^{\kappa} F_k^\dagger \otimes |e_k^\kappa\rangle,$$

$$F_k = \mathrm{T}r_2 U \left(I \otimes |\phi\rangle\langle e_k^\kappa| \right), \quad k = 1, \ldots, \kappa,$$

where $e_1^\kappa, \ldots, e_\kappa^\kappa$ is an orthonormal basis in $\mathcal{H}$.

Let Φ be given by (3). The conjugate channel to Φ is the map $\Phi^C : M_d \to B(\mathcal{H})$, defined as

$$\Phi^C(\rho) = \mathrm{T}r_1 U(\rho \otimes |\phi\rangle\langle\phi|)U^\dagger = \sum_{j,k} \mathrm{T}r\left(F_j \rho F_k^\dagger \right) |e_j^\kappa\rangle\langle e_k^\kappa|. \tag{5}$$

This definition appeared in [4] (under the name “complementary channels”) and was used in [5, 6] in the context of multiplicativity and additivity problems.

The next lemma shows a relation between the Stinespring representation (2) of Φ and the Choi-Jamiolkowski matrix (1) of its conjugate.

Lemma 1. *Let Φ be a CPT map, such that $\Phi(\rho) = V^{\dagger}(\rho \otimes I_{\kappa})V$ is the Stinespring representation. Then*

$$X_{\Phi^C} = (VV^{\dagger})^T,$$

where B^T is the transpose of the matrix B, $B^T_{ij} = B_{ji}$.

Proof. Let $V = \sum_{k=1}^{\kappa} F_k^{\dagger} \otimes |e_k^{\kappa}\rangle$, then using (5), we get

$$\begin{aligned} VV^{\dagger} &= \sum_{i,j=1}^{\kappa} F_i^{\dagger} F_j \otimes |e_i^{\kappa}\rangle\langle e_j^{\kappa}| = \sum_{i,j=1}^{\kappa} \sum_{k,l=1}^{d} \langle e_k^d | F_i^{\dagger} F_j | e_l^d\rangle \, |e_k^d\rangle\langle e_l^d| \otimes |e_i^{\kappa}\rangle\langle e_j^{\kappa}| \\ &= \sum_{k,l=1}^{d} |e_k^d\rangle\langle e_l^d| \otimes \sum_{i,j=1}^{\kappa} \mathrm{Tr}\left(F_j |e_l^d\rangle\langle e_k^d| F_i^{\dagger}\right) |e_i^{\kappa}\rangle\langle e_j^{\kappa}| \\ &= \sum_{k,l=1}^{d} |e_k^d\rangle\langle e_l^d| \otimes \left[\Phi^C(|e_l^d\rangle\langle e_k^d|)\right]^T = X^T_{\Phi^C}. \end{aligned}$$

□

Theorem 1. *For a CPT map Φ and $1 \le p \le \infty$,*

$$\|\Phi\|_{p\to p} = \omega_p(\Phi^C).$$

Proof. Note first that for any CPT map, we have ([3])

$$\omega_p(\Phi) = \sup_{A\ge 0, \|A\|_{2p}\le 1} \|(A \otimes I_{d'}) X_{\Phi} (A \otimes I_{d'})\|_p. \tag{6}$$

Let the Stinespring representation (2) of Φ be $\Phi(\rho) = V^{\dagger}(\rho \otimes I_{\kappa})V$. Then by Lemma 1,

$$\begin{aligned} \|\Phi\|_{p\to p} &= \sup_{A\ge 0, \|A\|_p \le 1} \|\Phi(A)\|_p = \sup_{B\ge 0, \|B\|_{2p}\le 1} \|V^{\dagger}(B^2 \otimes I_{\kappa})V\|_p = \\ &= \sup_{B\ge 0, \|B\|_{2p}\le 1} \|(B \otimes I_{\kappa}) X^T_{\Phi^C} (B \otimes I_{\kappa})\|_p = \omega_p(\Phi^C), \end{aligned}$$

the last equality follows from the fact that $B^T \ge 0$ if $B \ge 0$ and $\|B^T\|_p = \|B\|_p$. □

Remark. Let $q > p$. Exactly as in the above proof, we get that

$$\|\Phi\|_{q\to p} = \sup_{A\ge 0, \|A\|_{2q}\le 1} \|(A \otimes I_{d'}) X_{\Phi^C} (A \otimes I_{d'})\|_p.$$

The last expression is equal to the $L_r(M_d, L_p(M_{d'}))$ norm $\|X_{\Phi^C}\|_{(r,p)}$ for $1/q + 1/r = 1/p$, see Eq. (3.18) in [3]. This is an operator space type of norm, but not a CB norm, in general.

3. Multiplicativity

To prove multiplicativity, we need the following observation:

$$\omega_p(\Phi \otimes \mathrm{T}r) = \omega_p(\mathrm{T}r \otimes \Phi) = \omega_p(\Phi). \tag{7}$$

This follows from Lemma 2, proved in the Appendix. We remark that this equality implies that the supremum in the definition of ω_p can be taken over all $M_d \otimes M_d$, that is,

$$\omega_p(\Phi) = \sup_{X \in M_d \otimes M_d} \frac{\|(\mathcal{I} \otimes \Phi)(X)\|_p}{\|\mathrm{T}r_2(X)\|_p}. \tag{8}$$

To show this, we first note that the supremum in (8) may be restricted to positive X. Let $X \geq 0$ and let $|\psi_{123}\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^{d^2}$ be a purification of X, $X = \mathrm{T}r_3(|\psi_{123}\rangle\langle\psi_{123}|)$. Then

$$\begin{aligned}\frac{\|(\mathcal{I} \otimes \Phi)(X)\|_p}{\|\mathrm{T}r_2(X)\|_p} &= \frac{\|(\mathcal{I}_1 \otimes \Phi)(\mathrm{T}r_3(|\psi_{123}\rangle\langle\psi_{123}|))\|_p}{\|\mathrm{T}r_{23}\,(|\psi_{123}\rangle\langle\psi_{123}|)\,\|_p} \\ &= \frac{\|(\mathcal{I}_1 \otimes \Phi \otimes \mathrm{T}r)(|\psi_{123}\rangle\langle\psi_{123}|))\|_p}{\|\mathrm{T}r_{23}(|\psi_{123}\rangle\langle\psi_{123}|)\|_p}.\end{aligned}$$

Consequently,

$$\begin{aligned}\omega_p(\Phi) &\leq \sup_{X \in M_d \otimes M_d} \frac{\|(\mathcal{I} \otimes \Phi)(X)\|_p}{\|\mathrm{T}r_2(X)\|_p} \\ &\leq \sup_{\psi \in (\mathbb{C}^d \otimes \mathbb{C}^{d^2})^{\otimes 2}} \frac{\|(\mathcal{I}_{12} \otimes \Phi \otimes \mathrm{T}r)(|\psi\rangle\langle\psi|)\|_p}{\|\mathrm{T}r_{34}(|\psi\rangle\langle\psi|)\|_p} = \omega_p(\Phi \otimes \mathrm{T}r) = \omega_p(\Phi),\end{aligned}$$

hence the assertion.

We now obtain an alternative proof of multiplicativity of $\|\cdot\|_{p\to p}$ and ω_p.

Theorem 2. *For CPT maps* $\Phi_1 : M_{d_1} \to M_{d_1'}$ *and* $\Phi_2 : M_{d_2} \to M_{d_2'}$ *and for* $1 \leq p \leq \infty$,

$$\begin{aligned}\|\Phi_1 \otimes \Phi_2\|_{p\to p} &= \|\Phi_1\|_{p\to p}\|\Phi_2\|_{p\to p} \\ \omega_p(\Phi_1 \otimes \Phi_2) &= \omega_p(\Phi_1)\omega_p(\Phi_2).\end{aligned}$$

Proof. We first show that the $p \to p$ norm of a channel Φ is not changed by tensoring with identity. Indeed, by Theorem 1 and (7),

$$\|\Phi \otimes \mathcal{I}\|_{p\to p} = \omega_p((\Phi \otimes \mathcal{I})^C) = \omega_p(\Phi^C \otimes \mathrm{T}r) = \omega_p(\Phi^C) = \|\Phi\|_{p\to p}.$$

Similarly, $\|\mathcal{I} \otimes \Phi\|_{p\to p} = \|\Phi\|_{p\to p}$.

Let now $A \in M_{d_1} \otimes M_{d_2}$, $B = (\mathcal{I} \otimes \Phi_2)(A)$ and compute

$$\begin{aligned}\sup_A \frac{\|(\Phi_1 \otimes \Phi_2)(A)\|_p}{\|A\|_p} &= \sup_A \frac{\|(\Phi_1 \otimes \mathcal{I})(B)\|_p}{\|B\|_p} \frac{\|(\mathcal{I} \otimes \Phi_2)(A)\|_p}{\|A\|_p} \\ &\leq \|\Phi_1\|_{p\to p}\|\Phi_2\|_{p\to p}.\end{aligned}$$

Since the opposite inequality is easy, we get $\|\Phi_1 \otimes \Phi_2\|_{p\to p} = \|\Phi_1\|_{p\to p}\|\Phi_2\|_{p\to p}$, which in turn implies the multiplicativity of ω_p. □

Remark. Note that the fact that Φ preserves trace was never used in the paper, so that the results are valid for all completely positive maps on matrices.

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Appendix

The following lemma is due to C. King.

Lemma 2. *Let $\Omega : M_n \to M_m$ be a channel with the covariance property $\Omega(U\rho U^\dagger) = U'\Omega(\rho)(U')^\dagger$, where U' is a unitary in M_m, for any unitary $U \in M_n$. Then for any CPT map $\Phi : M_d \to M_{d'}$, we have*

$$\omega_p(\Omega \otimes \Phi) = \omega_p(\Phi \otimes \Omega) = \omega_p(\Phi)\omega_p(\Omega).$$

Proof. The proof uses the fact that there are n^2 unitary operators in M_n, such that $\sum_{k=0}^{n^2-1} U_k A U_k^\dagger = n(\mathrm{T}r A) I_n$ for any $n \times n$ matrix A, and therefore

$$\sum_k (U_k \otimes I_d) A_{12} \left(U_k^\dagger \otimes I_d\right) = n I_n \otimes A_2$$

for $A_{12} \in M_n \otimes M_d$, $A_2 = \mathrm{T}r_1 A_{12}$. Let us define

$$g_p(\rho, \Phi) = \mathrm{T}r\left[\left(\rho^{1/2p} \otimes I_{d'}\right) X_\Phi \left(\rho^{1/2p} \otimes I_{d'}\right)\right]^p = \mathrm{T}r\left[X_\Phi^{1/2}\left(\rho^{1/p} \otimes I_{d'}\right) X_\Phi^{1/2}\right]^p$$

so that $\omega_p(\Phi)^p = \sup_{\rho \geq 0, \mathrm{T}r\rho \leq 1} g_p(\rho, \Phi)$. Then by [2], $\rho \mapsto g_p(\rho, \Phi)$ is concave. It is easy to see that $g_p(\rho, \Omega) = g_p(U\rho U^\dagger, \Omega)$ for any unitary operator U on $\mathbb{C}^n$. It follows that for any $\rho \geq 0$, $\mathrm{T}r\rho = 1$,

$$\begin{aligned} g_p(\rho, \Omega) &= \frac{1}{n^2}\sum_k g_p(U_k \rho U_k^\dagger, \Omega) \leq g_p\left(\frac{1}{n^2}\sum_k U_k \rho U_k^\dagger, \Omega\right) \\ &= g_p\left(\frac{1}{n} I_n, \Omega\right) = \frac{1}{n}\|X_\Omega\|_p^p = \omega_p(\Omega)^p. \end{aligned}$$

Similarly, we have

$$\begin{aligned} g_p(\rho_{12}, \Omega \otimes \Phi) &= \frac{1}{n^2}\sum_k g_p\left((U_k \otimes I_d)\,\rho_{12}\left(U_k^\dagger \otimes I_d\right), \Omega \otimes \Phi\right) \\ &\leq g_p\left(\frac{1}{n^2}\sum_k (U_k \otimes I_d)\rho_{12}(U_k^\dagger \otimes I_d), \Omega \otimes \Phi\right) \\ &= g_p\left(\frac{1}{n} I_n \otimes \rho_2, \Omega \otimes \Phi\right) = g_p\left(\frac{1}{n} I_n, \Omega\right) g_p(\rho_2, \Phi). \end{aligned}$$

The easy inequality $\omega_p(\Omega)\omega_p(\Phi) \leq \omega_p(\Omega \otimes \Phi)$ now finishes the proof. The equality $\omega_p(\Phi \otimes \Omega) = \omega_p(\Phi)\omega_p(\Omega)$ is proved similarly. $\square$

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