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Title of Paper:	Derivation of Standard Quantum Theory via State Discrimination
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Dear Referees and Editor

We sincerely appreciate the valuable comments of the referees on our manuscript. The comments were very helpful and enabled us to improve the quality of our manuscript significantly. We have carefully considered the comments and suggestions, and we resubmitted our revised manuscript. The manuscript has been revised thoroughly according to the referees' comments. Thus, we hope that the present version of the manuscript is suitable for publication in New Journal of Physics.

Yours sincerely,

Hayato Arai
Masahito Hayashi

List of major changes

(C1) In Section 3, for the purpose of the revised version of Theorem 5, we have additionally introduced the sum of minimum and maximum eigenvalues of measurements (7).

(C2) In the section 3, we have revised the statement of Theorem 5 because the previous statement does not hold in the case of $p \neq 1/2$ by adding an extra term in inequality (8). Due to this revision, we have also revised the proof of Theorem 5 in Appendix. However, here we remark that this revision does not change the main theorem (Theorem 7) and the importance of this manuscript.

(C3) In Section 3, we have revised Theorem 6 by restricting the case $p = 1/2$ because of the revision of Theorem 5. However, here we remark that Theorem 7 still holds in any p even though Theorem 6 is proven only in the case of $p = 1/2$ because we only applied Theorem 6 in the case of $p = 1/2$ in our original proof in the previous version.

(C4) In Section 3 (after Theorem 7), for the purpose of clarifying the novelty of our results, we have given an additional explanation about the relation between our theorem and the results of other norms.

Response to Referee 1

Comment 1-1

This paper claims to give a characterization of quantum theory among GPTs (with certain dimension condition) by an operational condition, namely in terms of error probabilities in state discrimination problems. This is done in the following way: under the dimension condition, the state space of the theory in question can be mapped into the set of quantum states by an affine isomorphism, it is shown that the image is a strict subset (and hence the GPT is different from the quantum theory) if and only if the error probabilities in some state discrimination problem are smaller than the quantum ones. I have the following comments:

Response

Thank you for the good summary of our manuscript.

Comment 1-2

Inequality Eq. (7) stated in Theorem 5 is easily seen to be wrong unless $p = 1/2$, in fact, the equality denoted as (d) in the proof of this theorem does not hold. To find a counterexample, consider the state discrimination problem for two quantum states ρ_0, ρ_1, with $p = 1/2$. Let $P_\pm$ denote the projection onto the support of $(p\rho_0 - (1-p)\rho_1)_\pm$ and put $M_0 = sP_+ + rP_-$ with $0 \le r < s \le 1$. Then $\{M_0, I - M_0\}$ is a valid quantum measurement and it is easy to compute that

$$\mathrm{Err}(\rho_0, \rho_1; p; \boldsymbol{M}) = \frac{1}{2} - \frac{1}{2}\|p\rho_0 - (1-p)\rho_1\|_1 - \frac{1}{2}(2p-1)(s+r-1).$$

We clearly may choose $s + r > 1$ if $p > 1/2$ and $s + r < 1$ if $p < 1/2$, to violate the inequality Eq.(7). Nevertheless, Theorem 7 in fact holds (note that only the case $p = 1/2$ in Theorem 5 is used in the proof).

Response

Thank you for the crucial comment and a kind counterexample. Your comment is definitely correct. Our previous statement is false in the case of $p \neq 1/2$ because of the failure reduction (d) in (A2) in the proof of Theorem 5. Therefore, as the major change (C2), we revised Theorem 5 by adding an extra term to satisfy (d) in (A2). Due to the revision of Teorem 5, we state Theorem 6 only in the case of $p = 1/2$. However, as you mentioned, Theorem 7 still holds for any p because our original proof only applied Theorem 6 in the case of $p = 1/2$.

Comment 1-3

The inequality Eq.(2) holds in any GPT, with the trace norm replaced by the base norm corresponding to the state space. The main result of the paper just gives the fact that the base norm increases if it is computed with respect to a smaller base, this is well known and easy to see.

Response

Thank you for the important comment. As the major change (C4), we added an explanation about the relation between our results and the results that you pointed out.

As we mentioned in the remark after Theorem 7 in the revised version, the condition of isomorphic embedding of the state space does not trivially imply that the base norm is strictly smaller than trace norm. It is the only trivial thing that the base norm is not larger than trace norm for a pair of any states under the condition of isomorphic embedding ((18) in the revised version). In order to show Theorem 7, we need the equality condition of (18). We rather give the equality condition of (18) as Corollary 9 by applying Theorem 7 to the results of the preceding studies, which is the main contribution of our manuscript for derivation of quantum theory.

Therefore, our results have the above clear novelty for foundation of quantum theory.

Comment 1-4

An operational characterization of quantum theory would be an intrinsic property of states and measurements of a GPT that singles out the quantum theory. This is not given in the present paper. The main result is just an easy and well-known step away from stating that a state space is not quantum if and only if it is not affinely isomorphic to a set of quantum states, which is trivial.

Response

Thank you for the important comment. As the major changes [C4], we have emphasized the importance of our results. At first, it is always possible to affinely embed the state space of a given model to the set of quantum states when the dimensions of background vector spaces are the same. Our result implies the equivalence between the two conditions; (i) Such embedding is an isomorphism between the two state spaces, i.e., the model is isomorphic to quantum theory. (ii) The isomorphism preserves the quantum bound of state discrimination with trace norm. In other words, we clarify that no model satisfies the quantum bound with trace norm other than quantum theory under the isomorphic embedding of state space. This is our non-trivial contribution.

Comment 1-5

There are a lot of misspellings, incomplete or grammatically incorrect sentences, etc. I do not point them out, since it would not improve the paper in any significant way.

Response

Thank you for the kind comment. We have carefully checked the paper, and we have revised the writing of the paper with Grammarly.

Response to Referee 2

Comment 2-1

The framework of General Probabilistic Theories (GPTs) is a new information theoretical approach to single out standard quantum theory and quantum state discrimination is a fundamental task in quantum theory. The authors proposed to characterize the model of standard quantum theory out of general models via quantum state discrimination. The manuscript introduce an equivalent condition for outperforming the minimum discrimination error probability under the standard quantum theory. The result is interesting and important in quantum theory.

Response

Thank you for the good summary and positive comment on our manuscript.

Comment 2-2

To prove the main result, the authors claim that Eq.(A.3) holds, namely

$$\mathrm{Tr}(p\rho_0 - (1-p)\rho_1)M_i \geq \lambda_{\max}(M_i)\,\mathrm{Tr}(p\rho_0 - (1-p)\rho_1)_+ + \lambda_{\min}(M_i)\,\mathrm{Tr}(p\rho_0 - (1-p)\rho_1)_-$$

for any measurement M_0, M_1, where $(p\rho_0 - (1-p)\rho_1)_+$ and $(p\rho_0 - (1-p)\rho_1)_-$ are the positive and negative part of Hermitian matrix $(p\rho_0 - (1-p)\rho_1)$.

(1) However, the following example implies that it is not true.

Example 1. *Let* $\rho_0 = \frac{2}{3}|0\rangle\langle 0| + \frac{1}{9}(|1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3|)$, $\rho_1 = \frac{1}{3}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| + \frac{1}{12}(|2\rangle\langle 2| + |3\rangle\langle 3|)$. $p = \frac{3}{7}$, *and measurement* $M_0 = \frac{7}{8}|2\rangle\langle 2| + \frac{1}{8}|3\rangle\langle 3|$. *Then, it holds that*

$$p\rho_0 - (1-p)\rho_1 = \frac{2}{21}|0\rangle\langle 0| - \frac{5}{21}|1\rangle\langle 1|,$$

that is orthogonal with M_0. *As a result, one has*

$$\begin{aligned}&\lambda_{\max}(M_i)\,\mathrm{Tr}(p\rho_0 - (1-p)\rho_1)_+ + \lambda_{\min}(M_i)\,\mathrm{Tr}(p\rho_0 - (1-p)\rho_1)_- \\ =&\frac{7}{8}\times\frac{2}{21} + \frac{1}{8}\times\frac{-5}{21} = \frac{3}{56} > 0 = \mathrm{Tr}(p\rho_0 - (1-p)\rho_1)M_0\end{aligned}$$

Response

Thank you for the critical comment and a kind counterexample. In the previous version, we miswrote the inequality that you pointed out as the opposite inequality. The correct inequality is

$$\mathrm{Tr}(p\rho_0 - (1-p)\rho_1)M_i \leq \lambda_{\max}(M_i)\,\mathrm{Tr}(p\rho_0 - (1-p)\rho_1)_+ + \lambda_{\min}(M_i)\,\mathrm{Tr}(p\rho_0 - (1-p)\rho_1)_-.$$

This inequality holds because any Hermitian matrix X and M satisfies

$$\mathrm{Tr}\,XM = \sum_{i\in I}\lambda_i\,\mathrm{Tr}(X_+ + X_i)E_i \leq \lambda_{\max}X_+ + \lambda_{\min}X_-, \tag{1}$$

where $M = \sum_{i\in I}\lambda_i E_i$ is the spectral decomposition of M. This was just a typo, and we applied the opposite inequality to the original proof. We have just revised the inequality. (However, we had a wrong induction in the previous proof, and therefore, we have updated the proof.)

Comment 2-3

(2) Moreover, in the proof of Theorem 5, why the step (c) is correct?

Response

As we mentioned in the Response of Comment 2-2, the correct inequality is opposite. Then, the induction (c) is the inequality itself that you mentioned.

Comment 2-4

In conclusion, before making a further decision, I would like the author to clarify the above two points and read and examine the manuscript carefully.

Response

We apologize for the mistake of the inequality. We have revised the mistake and have checked the proof carefully again.