Notes on incompatibility witnesses in GPTs

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What is done here

After some basics, these notes contain:

- the definition of incompatibility witness for k effects as a k-tuple with some properties;
- "maximal incompatibility" attainable for a state space K, denoted by $\beta_k(K)$;
- a description of the witnesses by tensor norms and an expression of $\beta_k(K)$ as the ratio of the projective norm and some cross norm on $V \otimes \ell_1^k$;
- symmetric state spaces are defined as those isomorphic to a unit ball in some Banach space (finite dim) X, it is proved that then $\beta_k(K)$ is the projective/injective ratio for $X \otimes \ell_1^k$, this gives some bounds;
- examples: hypercubes are solved completely, spherical (unit balls of ℓ_2^n-norms) for $k \leq n$.
- a dimension-dependent bound is conjectured for all state spaces.

1 Basic notations

Let $K \subset \mathbb{R}^N$ be a compact convex set. We will use the notations $A(K) = \{f : K \to \mathbb{R}, \quad f \text{ is affine}\}$, $A(K)^+ = \{f \in A(K),\ f(x) \geq 0, x \in K\}$, $1_K \in A(K)$, $1_K(x) = 1$, $\forall x \in K$. Then $(A(K), A(K)^+, 1_K)$ is an order unit space and it is easy to see that

$$\|f\|_{max} = \max_{x \in K} |f(k)|$$

is the order unit norm. Let $V(K) = A(K)^*$ be the dual space and $V(K)^+ = (A(K)^+)^* = \{v \in V(K), \langle v, f\rangle \geq 0,\ \forall f \in A(K)^+\}$ the dual cone. Then $K \simeq \{v \in V(K)^+, \langle v, 1_K\rangle = 1\}$ and with this identification we have $V(K) = \mathrm{span}(K)$. The dual norm to $\|\cdot\|_{max}$ is the base norm in $V(K)$, which will be denoted by $\|\cdot\|_K$. Below, we will often drop the set K from the notation if it is clear, so that $V = V(K)$, $V^+ = V(K)^+$, $A = A(K)$, etc.

Example 1. (*Hypercubes*) Let $K = H_k := [-1, 1]^k$, then $A(K) \simeq \mathbb{R}^{k+1}$, with the basis consisting of the unit and coordinate maps:

$$e_0 = 1_K, \quad e_i(x_1, \dots, x_k) = x_i,\ i = 1, \dots, k.$$

Then

$$A(H_k)^+ = \{(a_0, a_1, \ldots, a_k),\ \sum_i |a_i| \leq a_0\}, \quad V(H_k)^+ = \{(v_0, v_1, \ldots, v_k),\ |v_i| \leq v_0\}$$

and H_k is identified as $H_k \simeq \{(1, x_1, \ldots, x_k),\ |x_i| \leq 1\}$.

1.1 Measurements and compatibility

Let $E = E(K) = \{0 \leq f \leq 1 = 1_K\}$ denote the set of effects. A measurement on K with outcomes in a finite set $\{0, 1, \ldots, l-1\}$ is defined as an affine map from K into the probability simplex $\Delta_l = \{(p_0, \ldots, p_{l-1},\ p_i \geq 0, \sum_i p_i = 1\}$. Any measurement $f : K \to \Delta_l$ is described by the l-tuple of effects $f_i \in E$, $\sum_i f_i = 1$, defined as

$$f_i(x) = (f(x))_i, \quad i = 0, \ldots l-1.$$

In particular, any effect $e \in E$ determines a two-outcome measurement $\{e, 1-e\}$.

A collection of measurements $\{f^1, \ldots, f^k\}$ with l outcomes is identified with the map

$$F : K \to \Delta_l^k, \quad F(x) = (f^1(x), \ldots, f^k(x)).$$

This map extends uniquely to a positive map $(V, V^+) \to (V(\Delta_l^k), V(\Delta_l^k)^+)$. The collection is compatible iff the map F is separable, that is, factorizes through a simplex.

2 Incompatibility witnesses

We will assume from now on that $l = 2$, that is, we have a collection of effects $f_1, \ldots, f_k \in E$ and $F(x) = (f_1(x), \ldots, f_k(x)) \in H_k$. Let $v_1, \ldots, v_n$ be a basis of V and let $f_1, \ldots, f_n \in A$ be a dual basis, that is

$$\langle f_i, v_j \rangle = \delta_{ij}.$$

Let $\chi_V = \sum_i v_i \otimes f_i$, then $\chi_V \in V^+ \otimes_{max} A^+$ and F is separable iff $(F \otimes id)(\chi_V) \in V(H_k)^+ \otimes_{min} A^+$. Hence if $f_1, \ldots, f_k$ is not compatible, there must be some element $z \in A(H_k)^+ \otimes_{max} V^+$ that witnesses the incompatibility, that is,

$$\langle (F \otimes id)(\chi_V), z \rangle = \langle \chi_V, (F^* \otimes id)(z) \rangle < 0.$$

Using the basis of $A(H_k)$ from Example 1, any $z \in A(H_k)^+ \otimes_{max} V^+$ has the form

$$z = e_0 \otimes z_0 + \sum_{i=1}^k e_i \otimes z_i,\ z_i \in V,\ i = 0, \ldots, k,$$

such that

$$z_0 + \sum_{i=1}^k \epsilon_i z_i \in V^+, \quad \forall \epsilon \in \{\pm 1\}^k. \tag{1}$$

Since z is a witness, we have

$$0 > \inf_F \langle \chi_V, (F^* \otimes id)(z)\rangle = \inf_F (\langle \chi_V, 1 \otimes z_0 + \sum_{i=1}^k (2f_i - 1) \otimes z_i \rangle)$$
$$= \langle 1, z_0 \rangle + \inf_F \sum_{i=1}^k \langle 2f_i - 1, z_i \rangle = \langle 1, z_0 \rangle - \sum_{i=1}^k \|z_i\|_K$$

Under the normalization $\langle 1, z_0 \rangle = 1$, we obtain the condition $\sum_{i=1}^k \|z_i\|_K > 1$. It is easy to see that $z_0 \in V^+$, so the normalization means that $z_0 \in K$.

Note that there is some ambiguity in z: suppose that $z_0' \in K$, $z_0 \neq z_0'$ is a state such that (4) is satisfied, then $z' = e_0 \otimes z_0' + \sum_{i=1}^k e_i \otimes z_i$ is another element of $A(H_k)^+ \otimes_{max} V^+$ such that

$$\langle \chi_V, (F^* \otimes id)(z') \rangle = \langle \chi_V, (F^* \otimes id)(z) \rangle, \qquad \forall F.$$

It is therefore reasonable to introduce the following definition.

Definition 1. *Let us denote*

$$\Gamma_k(K) := \{(z_1, \ldots, z_k) \in V^k,\ \exists z_0 \in K \text{ such that } z_0 + \sum_{i=1}^k \epsilon_i z_i \in V^+, \forall \epsilon \in \{\pm 1\}^k\}.$$

An incompatibility witness for k effects on K is a tuple $(z_1, \ldots, z_k) \in \Gamma_k(K)$ such that $\sum_{i=1}^k \|z_i\|_K > 1$.

The largest value attainable in the above definition by witnesses for k effects on K will be denoted by $\beta_k(K)$, more precisely,

$$\beta_k(K) = \sup\{\sum_{i=1}^k \|z_i\|_K,\ (z_1, \ldots, z_k) \in \Gamma_k(K)\}.$$

There are some easily obtained bounds on this quantity: for any state space K, we have

$$1 \leq \beta_k(K) \leq k. \tag{2}$$

Indeed, the lower bound is obtained by the tuple $(z_1, 0, \ldots, 0)$ with $z_1 \in K$. For the upper bound, put $z_\epsilon = \sum_i \epsilon_i z_i$ and note that for all i,

$$\|z_i\|_K = \frac{1}{2^{k-1}} \| \sum_{\epsilon, \epsilon_i = 1} z_\epsilon \|_K \leq 1.$$

Moreover, these bounds are tight. It follows from [3] that $\beta_k(K) = 1$ iff K is a simplex and by [2, Coro. 6], $\beta_k(K) = k$ iff there exists a projection on K with range isomorphic to H_k. We conjecture a dimension-dependent bound below.

We clearly also have the following characterization of compatible effects.

Proposition 1. *The effects $f_1, \ldots, f_k \in E$ are compatible if and only if*

$$\sum_i \langle 2f_i - 1, z_i \rangle \leq 1$$

for all $(z_1, \ldots, z_k) \in \Gamma_k(K)$.

2.1 Witnesses and cross norms

Let us consider the Banach space $V = (V, \|\cdot\|_K)$ and let $\ell_1^k = (\mathbb{R}^k, \|\cdot\|_1)$. Clearly, we may identify $V \otimes \ell_1^k \simeq V^k$, so that in particular any incompatibility witness can be seen as an element of $V \otimes \ell_1^k$. We next describe the witnesses in terms of cross norms on $V \otimes \ell_1^k$.

Proposition 2. *Let $(z_1, \dots, z_k) \in V \otimes \ell_1^k$. Then*

(a) $\Gamma_k(K)$ is the unit ball of a cross norm γ in $V \otimes \ell_1^k$

(b) $\pi(z_1, \dots, z_k) = \sum_i \|z_i\|_K$ is the projective cross norm in $V \otimes \ell_1^k$.

(c) $\varepsilon(z_1, \dots, z_k) = \sup_{\epsilon \in \{\pm 1\}^k} \|z_\epsilon\|_K$ is the injective cross norm in $V \otimes \ell_1^k$.

Proof. The statements (b) and (c) are well known and easy to prove. For (a), it is easily seen that $\Gamma_k(K)$ is absolutely convex, closed and bounded. Moreover, $\Gamma_k(K)$ contains the unit ball of π. To see this, let $\sum_i \|z_i\|_K \leq 1$ and put $\lambda_i := \|z_i\|_K$. Then by definition of the base norm, $\pm z_i \leq \lambda_i z_i^0$ for some $z_i^0 \in K$ and hence

$$\sum_i \epsilon_i z_i \leq \sum_i \lambda_i z_i^0 + (1 - \sum_i \lambda_i) z_0, \qquad \forall \epsilon,$$

where z_0 is an arbitrary element of K. It follows that Γ_k is also absorbing, hence the unit ball of a norm γ, obtained as

$$\gamma(z_1, \dots, z_k) = \min\{\lambda > 0, (z_1, \dots, z_k) \leq \lambda \Gamma_k\} = \min_{z_0 \in K} \min\{\lambda > 0, \sum_i \epsilon_i z_i \leq \lambda z_0, \ \forall \epsilon\}.$$

Let now $z \in V$ and $t = (t_1, \dots, t_k) \in \mathbb{R}^k$, then $z \otimes t = (t_1 z, \dots, t_k z)$. If $\lambda > 0$ and $z_0 \in K$ are such that

$$\sum_i \epsilon_i t_i z \leq \lambda z_0, \quad \forall \epsilon,$$

then also $\pm \|t\|_1 z \leq \lambda z_0$, so that $\|t\|_1 \|z\|_K \leq \lambda$. For the converse inequality, let $z_0 \in K$ be such that $\pm z \leq \|z\|_K z_0$ and let $\epsilon \in \{\pm 1\}^k$. Then $\sum_i \epsilon_i t_i = t_+ - t_-$, where $t_+, t_- \geq 0$ are such that $t_+ + t_- = \|t\|_1$. Then

$$\sum_i \epsilon_i t_i z = (t_+ - t_-) z \leq (t_+ + t_-) \|z\|_K z_0 = \|t\|_1 \|z\|_K z_0.$$

□

Now we see that we may express β_k in terms of cross norms:

$$\beta_k(K) = \sup_{z \in V \otimes \mathbb{R}^k} \frac{\pi(z)}{\gamma(z)}.$$

3 Symmetric state spaces

We will say that a state space K is symmetric if it is isomorphic to the unit ball of $(\mathbb{R}^n, \|\cdot\|)$ for some norm $\|\cdot\|$. Let $\|\cdot\|_*$ denote the dual norm. We have

$$K \simeq \{(1, x) \in \mathbb{R}^{n+1}, \ \|x\| \leq 1\}, \tag{3}$$

which is a compact convex subset of a hyperplane in $\mathbb{R}^{n+1}$ that does not contain 0. Now we have $V(K) \simeq \mathbb{R}^{n+1} \simeq A(K)$ and it is easy to see that

$$V(K)^+ = \{(\lambda, x),\ \|x\| \leq \lambda\}, \quad A(K)^+ = \{(c, f),\ \|f\|_* \leq c\}.$$

The unit functional $1_K \in A(K)$ is represented as $1_K = (1, 0)$.

Lemma 1. *Let K be as in* (3)*. The order unit norm and base norm are determined as*

$$\|(c, f)\|_{max} = |c| + \|f\|_*, \qquad \|(\lambda, x)\|_K = \max\{|\lambda|, \|x\|\}$$

Proof. By definition,

$$\|(c, f)\|_{max} = \min\{\lambda \geq 0,\ (\lambda \pm c, f) \geq 0\}$$

and by the expression for the positive, cone

$$(\lambda \pm c, f) \geq 0 \iff \|f\|_* \leq \lambda \pm c \iff \|f\|_* \leq \lambda - |c|,$$

this proves the first part of the statement. For the second part, we have by duality

$$\begin{aligned}\|(\lambda, x)\|_K &= \sup_{\|(c,f)\|_{max} \leq 1} \lambda c + \langle f, x\rangle = \sup_{\substack{t\in[0,1], |c|=t,\\ \|f\|_* \leq 1-t}} \lambda c + \langle f, x\rangle \\ &= \sup_{t\in[0,1]} t|\lambda| + (1-t)\|x\| = \max\{|\lambda|, \|x\|\}.\end{aligned}$$

□

We now express the norm γ and the value of β_k in this case. So let $z_i = (\lambda_i, x_i)$, $i = 1, \ldots, k$. Then $(z_1, \ldots, z_k) \in \Gamma_k$ means that there is some $z_0 = (1, x_0)$ such that $\|x_0\| \leq 1$ and $(1 + \lambda_\epsilon, x_0 + x_\epsilon) \geq 0$, for all ϵ, that is

$$\|x_0 + x_\epsilon\| \leq 1 + \lambda_\epsilon, \qquad \forall \epsilon \tag{4}$$

where $x_\epsilon = \sum_i \epsilon_i x_i$ and similarly λ_ϵ. Note that we in fact have

$$\|x_0 \pm x_\epsilon\| \leq 1 \pm \lambda_\epsilon,$$

so that for any ϵ,

$$\|x_\epsilon\| = \frac{1}{2}\|(x_\epsilon + x_0) + (x_\epsilon - x_0)\| \leq \frac{1}{2}((1 + \lambda_\epsilon) + (1 - \lambda_\epsilon)) = 1$$

If all $\lambda_i = 0$, then the condition (4) is equivalent to $\|x_\epsilon\| \leq 1$ for all ϵ, indeed, we can put $x_0 = 0$ so that $z_0 = (1, 0)$. Note that as in Prop. 2, this means that $\varepsilon(x_1, \ldots, x_k) \leq 1$, where ε is the injective cross norm over $(\mathbb{R}^n, \|\cdot\|) \otimes \ell_1^k$. The proof of the following proposition shows that a k-tuple in Γ_k attaining β_k is necessarily of the form $\{z_i = (0, x_i)\}_{i=1}^k$.

Proposition 3. *Let ε and π denote the injective and projective cross norms over $(\mathbb{R}^n, \|\cdot\|) \otimes \ell_1^k$. Then*

$$\beta_k = \max_{x\in\mathbb{R}^n\otimes\ell_1^k} \frac{\pi(x)}{\varepsilon(x)} = \rho(\ell_1^k, (\mathbb{R}^n, \|\cdot\|)).$$

Proof. Let us denote the value on the right hand side by β'_k. It is then quite clear that we have $\beta_k \geq \beta'_k$, since β'_k is obtained by maximizing over all witnesses with $\lambda_i = 0$.

To prove the converse, let $\{z_i = (\lambda_i, x_i)\}$ be a witness and let us assume, without loss of generality, that $\|x_i\| < |\lambda_i|$ for $i = 1, \ldots, l$ and $\|x_i\| \geq |\lambda_i|$ otherwise. We may clearly also assume that $\lambda_i \geq 0$, by changing the sign of z_i if necessary.

Note that by (4),

$$\lambda := \sum_{i=1}^{l} \lambda_i \leq \sum_{i=1}^{k} |\lambda_i| \leq 1$$

Let us denote $y_j := x_{l+j}$, $\mu_j := \lambda_{l+j}$, $j = 1, \ldots, k-l$ and let $y_\eta = \sum_j \eta_j y_j$, $\mu_\eta = \sum_j \eta_j \mu_j$ for $\eta \in \{\pm 1\}^{k-1}$. Then (4) implies that for all η,

$$\|\sum_{i=1}^{l} x_i \pm y_\eta - x_0\| \leq 1 - \lambda \pm \mu_\eta$$

so that

$$\|y_\eta\| = \frac{1}{2}\|(\sum_{i=1}^{l} x_i + y_\eta - x_0) - (\sum_{i=1}^{l} x_i - y_\eta - x_0)\| \leq 1 - \lambda, \quad \forall \eta.$$

We then have by Lemma 1 that

$$\begin{aligned}\sum_i \|z_i\|_K &= \sum_{i=1}^{l} \lambda_i + \sum_{j=1}^{k-l} \|y_j\| \leq \lambda + \max\{\sum_{j=1}^{k-1} \|y_j\|,\ \|y_\eta\| \leq 1 - \lambda, \forall \eta\} \\ &= \lambda + (1-\lambda)\beta'_{k-l} = \beta'_{k-l} - \lambda(\beta'_{k-l} - 1) \leq \beta'_{k-l} \leq \beta'_k.\end{aligned}$$

□

In [1], the value of $\rho(X, Y)$ is considered at some length, in particular, we have the following tighter bounds.

Corollary 1. *For a symmetric state space K, we have*

$$\sqrt{2} \leq \beta_k(K) \leq \min\{\dim(K), k\}.$$

Proof. The upper bound is obtained by the inequality $\rho(X, Y) \leq \min\{\dim(X), \dim(Y))$ in [1, Prop. 11]. The lower bound follows by [1, Prop. 11, Eq. (58)] and [1, Prop. 14].

□

This leads to the following conjecture.

Conjecture 1. *We have $\beta_k(K) \leq \min\{\dim(K), k\}$ for all state spaces K.*

3.1 Examples

Example 2. (*Hypercubes*) Let $\|x\| = \|x\|_\infty = \max_j |x_j|$, so that $K \simeq H_n$. Let $x_i = (x_i^1, \dots, x_i^n)$, $i = 1, \dots, k$. Then

$$\|x_\epsilon\|_\infty \le 1,\ \forall \epsilon \iff |\sum_i \epsilon_i x_i^j| \le 1,\ \forall \epsilon, \forall j \iff \sum_i |x_i^j| \le 1,\ \forall j.$$

We therefore have

$$\beta_k(H_n) = \max\{\sum_i \max_j \{x_i^j\},\ \sum_i |x_i^j| \le 1\}.$$

Put $x_i^j = \delta_{ij}$, then $\sum_i |x_j^i| = 1$ for all j and $\sum_{i=1}^k \max_j\{x_i^j\} = \min\{n, k\}$. By Corollary 1, we conclude that $\beta_k(H_n) = \min\{n, k\}$.

Example 3. (l_2*-norms*) Let $\|x\| = \|x\|_2 = (\sum_j x_j^2)^{1/2}$, we will denote this state space by S_n. Let $E_1 = \{\epsilon \in \{\pm 1\}^k, \epsilon_1 = 1\}$. We have

$$\sum_{i=1}^k \|x_i\|_2^2 = \frac{1}{2^{k-1}} \sum_{\epsilon \in E_1} \|x_\epsilon\|_2^2,$$

this follows easily by induction. If $\{(0, x_i)\}$ is a witness, it follows that we must have $\sum_i \|x_i\|_2^2 \le 1$, which entails that $\sum_i \|x_i\|_2 \le \sqrt{k}$. Let $n \ge k$ and let $x_1, \dots, x_k$ be mutually orthogonal elements, with $\|x_i\|_2 = k^{-1/2}$. Then $\|x_\epsilon\|_2^2 = \sum_i \|x_i\|_2^2 = 1$ and $\sum_i \|x_i\|_2 = \sqrt{k}$, so we conclude that in this case, $\beta_k(S_n) = \sqrt{k}$, see also [1, Eq. (56)].

What if $n < k$? As an example, assume that $n = 2$, $k = 3$. Let $x_1 = (1/2, 0)$, $x_2 = (-1/4, \sqrt{3}/4)$, $x_3 = (-1/4, -\sqrt{3}/4)$. Then $\|x_i\|_2 = 1/2$ and $\|x_\epsilon\|_2 \le 1$ for all ϵ. It follows that

$$\sqrt{2} < \frac{3}{2} = \sum_i \|x_i\|_2 \le \beta_3(S_2) \le \sqrt{3}.$$

This value seems to be optimal (at least it is if all $\|x_i\|_2$ are equal).

References

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