H. Podsędkowska: Strong subadditivity of quantum mechanical entropy for semifinite von Neumann algebras

Referee report

In this paper, strong subadditivity of Segal entropy on semifinite von Neumann algebras is proved, along with some further properties (subadditivity, Araki-Lieb inequality, concavity of the conditional entropy). The proofs use similar ideas as can be found in the finite dimensional case, basically using the expression of SSA through the relative entropy (or the information I), but are technically more involved. The paper is quite well written, giving a nice historical overview of SSA and other entropy inequalities, but it seems that, also from the references in this paper, there has been little interest around these properties for Segal entropy recently. Nevertheless, I think it is worthwhile to have these questions settled and the proofs explicitly written out, which, to my knowledge, has not been done before.

Further comments and suggestions

1. It is claimed in the abstract that the triangle inequality is proved. This usually indicates the inequality $|S(\rho_1) - S(\rho_2)| \leq S(\rho_{12})$, which is clearly not true here, since Segal entropy can be negative. The statement of Theorem 10, described as general Araki-Lieb inequality, was proved for density matrices in H. Araki, E. Lieb, Commun. math. Phys. 18, 160-170 (1970) as a weaker version of SSA, which was then applied to the proof of the triangle inequality. (But their proof involves purification of density operators, which is not available here.)
2. page 2: it is *proved* that SSA is equivalent to monotonicity of the relative entropy in $B(\mathcal{H})$. Calling it a "widespread belief" does not seem appropriate.
3. the word "classical" seems to refer to both commutative systems and $B(\mathcal{H})$, this might be confusing.
4. page 2: some property is missing in the definition of τ-measurable operators
5. page 4, 2. displayed equation after "The proper formula...": parentheses mismatch
6. page 4: reference to unavailable (?) preprint [8], here the fact that the information I is equivalent to the Araki relative entropy (and hence is monotone) is crucial, a better reference should be given, or the preprint made accessible
7. page 6, last equation: missing parentheses
8. page 7, proof of Prop. 5: there should be x_ϵ instead of x_ϵ^2 in the first displayed equation, (anyway, $h^{1/2}x_\epsilon \in L_2(\mathcal{M}, \tau)$ is immediate by Hölder)