Equality conditions for the sandwiched Renyi relative entropy

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Abstract

summary of discussion between Nilanjana Datta, Anna Jencova, and Mark M. Wilde

1 From discussion

Let ρ be a density operator and σ be a positive semi-definite operator. Recall that the sandwiched Renyi relative entropy is defined as follows:

$$\widetilde{D}_\alpha(\rho\|\sigma) \equiv \frac{1}{\alpha-1}\log \operatorname{Tr}\left\{\left(\sigma^{(1-\alpha)/2\alpha}\rho\sigma^{(1-\alpha)/2\alpha}\right)^\alpha\right\}. \tag{1}$$

This can be rewritten in several different ways:

$$\widetilde{D}_\alpha(\rho\|\sigma) \equiv \frac{\alpha}{\alpha-1}\log \left\|\sigma^{(1-\alpha)/2\alpha}\rho\sigma^{(1-\alpha)/2\alpha}\right\|_\alpha \tag{2}$$

$$= \frac{\alpha}{\alpha-1}\log \left\|\rho^{1/2}\sigma^{(1-\alpha)/\alpha}\rho^{1/2}\right\|_\alpha \tag{3}$$

$$= \frac{2\alpha}{\alpha-1}\log \left\|\sigma^{(1-\alpha)/2\alpha}\rho^{1/2}\right\|_{2\alpha}. \tag{4}$$

We can rewrite it in an even different way which will be useful for our purposes here. Let $\|A\|_{\alpha,\sigma}$ denote the weighted α-norm of an operator A, defined as

$$\|A\|_{\alpha,\sigma}^\alpha \equiv \operatorname{Tr}\left\{\left|\sigma^{1/2\alpha}A\sigma^{1/2\alpha}\right|^\alpha\right\}, \tag{5}$$

where $\alpha \geq 1$. We also define a weighted inner product $\langle A, B\rangle_\sigma$ for two operators A and B as follows:

$$\langle A, B\rangle_\sigma \equiv \operatorname{Tr}\left\{A^\dagger\sigma^{1/2}B\sigma^{1/2}\right\}. \tag{6}$$

We define the Radon-Nikodym derivative d for ρ and σ as follows:

$$d \equiv \sigma^{-1/2}\rho\sigma^{-1/2}, \tag{7}$$

and for the noisy versions of these states as d_0:

$$d_0 \equiv \mathcal{N}(\sigma)^{-1/2}\mathcal{N}(\rho)\mathcal{N}(\sigma)^{-1/2}, \tag{8}$$

where $\mathcal{N}$ is a linear, positive, trace-preserving map. Then we can define the sandwiched Renyi relative entropy as

$$\widetilde{D}_\alpha(\rho\|\sigma) \equiv \frac{\alpha}{\alpha-1}\log\|d\|_{\alpha,\sigma}. \tag{9}$$

Let α' denote the Holder conjugate of α (i.e., α' is such that $1/\alpha + 1/\alpha' = 1$). Consider that

$$\begin{aligned}
\|d_0\|_{\alpha,\mathcal{N}(\sigma)} &= \sup_{\|A_0\|_{\alpha',\mathcal{N}(\sigma)}\leq 1} \langle A_0, d_0\rangle_{\mathcal{N}(\sigma)} && (10)\\
&= \sup_{\|A_0\|_{\alpha',\mathcal{N}(\sigma)}\leq 1} \operatorname{Tr}\{A_0\mathcal{N}(\rho)\} && (11)\\
&= \sup_{\|A_0\|_{\alpha',\mathcal{N}(\sigma)}\leq 1} \operatorname{Tr}\left\{\mathcal{N}^\dagger(A_0)\,\rho\right\} && (12)\\
&\leq \sup_{\|A\|_{\alpha',\sigma}\leq 1} \operatorname{Tr}\{A\rho\} && (13)\\
&= \sup_{\|A\|_{\alpha',\sigma}\leq 1} \langle A, d\rangle_\sigma && (14)\\
&= \|d\|_{\alpha,\sigma}\,. && (15)
\end{aligned}$$

In the second line, it suffices to optimize the quantity $\operatorname{Tr}\{A_0\mathcal{N}(\rho)\}$ because the norm $\|A_0\|_{\alpha',\mathcal{N}(\sigma)}$ is invariant with respect to taking the adjoint of A_0. The inequality follows because

$$\|\phi(B)\|_{\alpha,\sigma} \leq \|B\|_{\alpha,\phi^\dagger(\sigma)} \tag{16}$$

for ϕ unital and positive (here we take ϕ to be $\mathcal{N}^\dagger$ and note that the above implies $\left\|\mathcal{N}^\dagger(A_0)\right\|_{\alpha',\sigma} \leq \|A_0\|_{\alpha',\mathcal{N}(\sigma)} \leq 1$, so that an optimization over all A satisfying $\|A\|_{\alpha',\sigma} \leq 1$ can only give a larger value). So this is an alternate proof that the sandwiched Renyi relative entropy is monotone with respect to positive maps for $\alpha > 1$ (an original proof for this available in Beigi's paper on sandwiched Renyi relative entropy). The main inequality follows because

$$\begin{aligned}
\|\phi(B)\|_{1,\sigma} &= \sup_{\|C\|_{\infty,\sigma}\leq 1} \langle C, \phi(B)\rangle_\sigma && (17)\\
&= \sup_{\|C\|_\infty\leq 1} \langle C, \phi(B)\rangle_\sigma && (18)\\
&= \sup_{\|C\|_\infty\leq 1} \left\langle \phi^\dagger_\sigma(C), B\right\rangle_{\phi^\dagger(\sigma)} && (19)\\
&\leq \left\|\phi^\dagger_\sigma(C)\right\|_\infty \|B\|_{1,\phi^\dagger(\sigma)} && (20)\\
&\leq \|B\|_{1,\phi^\dagger(\sigma)}\,. && (21)
\end{aligned}$$

where

$$\phi^\dagger_\sigma(\cdot) \equiv \phi^\dagger(\sigma)^{-1/2}\,\phi^\dagger\left[\sigma^{1/2}(\cdot)\,\sigma^{1/2}\right]\phi^\dagger(\sigma)^{-1/2} \tag{22}$$

(NEED TO JUSTIFY FOR ∞ and apply Riesz-Thorin)

Remark 1 *Contractivity of unital positive maps for $\alpha = \infty$: Note that $\|\cdot\|_\infty$ is the operator norm. Contractivity follows by the Russo-Dye theorem, [1, Corollary 2.9].*

For the moment we focus on the special case of the collision relative entropy (i.e., when $\alpha = 2$). Consider the equality case for the sandwiched Renyi relative entropy when $\alpha = 2$. Then we have that

$$\begin{aligned}
\langle d, d\rangle_\sigma &= \|d\|_{2,\sigma} &(23)\\
&= \|d_0\|_{2,\mathcal{N}(\sigma)} &(24)\\
&= \langle d_0, d_0\rangle_{\mathcal{N}(\sigma)} &(25)\\
&= \mathrm{Tr}\left\{\mathcal{N}\left(\rho\right) d_0\right\} &(26)\\
&= \mathrm{Tr}\left\{\rho\mathcal{N}^\dagger\left(d_0\right)\right\} &(27)\\
&= \left\langle \mathcal{N}^\dagger\left(d_0\right), d\right\rangle_\sigma . &(28)
\end{aligned}$$

We know generally from the above and the assumption that

$$\left\|\mathcal{N}^\dagger\left(d_0\right)\right\|_{2,\sigma} \leq \|d_0\|_{2,\mathcal{N}(\sigma)} = \|d\|_{2,\sigma} . \tag{29}$$

It is then the case that

$$\mathcal{N}^\dagger\left(d_0\right) = d, \tag{30}$$

which is the same as

$$\mathcal{N}^\dagger\left[\mathcal{N}\left(\sigma\right)^{-1/2}\mathcal{N}\left(\rho\right)\mathcal{N}\left(\sigma\right)^{-1/2}\right] = \sigma^{-1/2}\rho\sigma^{-1/2}, \tag{31}$$

which in turn is the same as

$$\sigma^{1/2}\mathcal{N}^\dagger\left[\mathcal{N}\left(\sigma\right)^{-1/2}\mathcal{N}\left(\rho\right)\mathcal{N}\left(\sigma\right)^{-1/2}\right]\sigma^{1/2} = \rho. \tag{32}$$

This means that the Petz recovery map perfectly recovers ρ from $\mathcal{N}\left(\rho\right)$.

2 Speculation

Shouldn't this same reasoning work for more general values of $\alpha > 1$? Here we start out with the assumption that

$$\|d\|_{\alpha,\sigma} = \|d_0\|_{\alpha,\mathcal{N}(\sigma)} . \tag{33}$$

Consider from the above that the following equality holds for some A_0 such that $\|A_0\|_{\alpha',\mathcal{N}(\sigma)} \leq 1$:

$$\begin{aligned}
\|d_0\|_{\alpha,\mathcal{N}(\sigma)} &= \langle A_0, d_0\rangle_{\mathcal{N}(\sigma)} &(34)\\
&= \mathrm{Tr}\left\{A_0\mathcal{N}\left(\rho\right)\right\} &(35)\\
&= \mathrm{Tr}\left\{\mathcal{N}^\dagger\left(A_0\right)\rho\right\} &(36)\\
&= \left\langle \mathcal{N}^\dagger\left(A_0\right), d\right\rangle_\sigma &(37)\\
&\leq \left\|\mathcal{N}^\dagger\left(A_0\right)\right\|_{\alpha',\sigma} \|d\|_{\alpha,\sigma} &(38)\\
&\leq \|A_0\|_{\alpha',\mathcal{N}(\sigma)} \|d\|_{\alpha,\sigma} &(39)\\
&\leq \|d\|_{\alpha,\sigma} . &(40)
\end{aligned}$$

This is essentially Holder's inequality and we see that equality is achieved due to the assumption. We can then infer that the following relation holds

$$\left[\sigma^{1/2\alpha'}\mathcal{N}^{\dagger}\left(A_0\right)\sigma^{1/2\alpha'}\right]^{\alpha'} = \left[\sigma^{1/2\alpha}d\sigma^{1/2\alpha}\right]^{\alpha}. \tag{41}$$

(IS THIS CORRECT?) If A_0 could be taken as d_0 (or a scaled d_0?), we would have that

$$\left[\sigma^{1/2\alpha'}\mathcal{N}^{\dagger}\left(\mathcal{N}\left(\sigma\right)^{1/2\alpha'}d_0\mathcal{N}\left(\sigma\right)^{1/2\alpha'}\right)\sigma^{1/2\alpha'}\right]^{\alpha'} = \left[\sigma^{1/2\alpha}d\sigma^{1/2\alpha}\right]^{\alpha}, \tag{42}$$

which is the same as

$$\left[\sigma^{1/2\alpha'}\mathcal{N}^{\dagger}\left(\mathcal{N}\left(\sigma\right)^{(1-\alpha')/2\alpha'}\mathcal{N}\left(\rho\right)\mathcal{N}\left(\sigma\right)^{(1-\alpha')/2\alpha'}\right)\sigma^{1/2\alpha'}\right]^{\alpha'} = \left[\sigma^{(1-\alpha)/2\alpha}\rho\sigma^{(1-\alpha)/2\alpha}\right]^{\alpha}. \tag{43}$$

This part is not clear to me, but it seems that we might be able to get sufficiency for all $\alpha > 1$ with some kind of argument that starts from equality in weighted Holder inequality.

3 Equality conditions for $\alpha > 1$

Let $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$. For any element A, there is a unique $\tilde{A}$ with $\|\tilde{A}\|_{\alpha',\sigma} \leq 1$ and $\langle \tilde{A}, A\rangle_\sigma = \|A\|_{\alpha,\sigma}$. (Existence of such element is clear by duality of the norms. Uniqueness is a consequence of strict convexity of L_p spaces). If A is positive, then $\tilde{A} = \|A\|_{\alpha,\sigma}^{1-\alpha}\phi_\sigma^\alpha(A)$, where

$$\phi_\sigma^\alpha(A) = \sigma^{-\frac{1}{2\alpha'}}\left(\sigma^{\frac{1}{2\alpha}}A\sigma^{\frac{1}{2\alpha}}\right)^{\frac{\alpha}{\alpha'}}\sigma^{-\frac{1}{2\alpha'}}$$

The equality $\tilde{D}_\alpha(\rho\|\sigma) = \tilde{D}_\alpha(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$ implies that

$$\|d\|_{\alpha,\sigma} = \|d_0\|_{\alpha,\mathcal{N}(\sigma)} = \langle \tilde{d}_0, d_0\rangle_{\mathcal{N}(\sigma)} = \mathrm{Tr}\mathcal{N}(\rho)\tilde{d}_0 = \mathrm{Tr}\rho\mathcal{N}^{\dagger}(\tilde{d}_0) = \langle \mathcal{N}^{\dagger}(\tilde{d}_0), d\rangle_\sigma \tag{44}$$

Since $\|\mathcal{N}^{\dagger}(\tilde{d}_0)\|_{\alpha',\sigma} \leq \|\tilde{d}_0\|_{\alpha',\mathcal{N}(\sigma)} = 1$, we have $\tilde{d} = \mathcal{N}^{\dagger}(\tilde{d}_0)$ by uniqueness, so that

$$\phi_\sigma^\alpha(d) = \mathcal{N}^{\dagger}(\phi^\alpha_{\mathcal{N}(\sigma)}(d_0)), \tag{45}$$

that is

$$\begin{aligned}&\sigma^{-\frac{1}{2\alpha'}}\left(\sigma^{-\frac{1}{2\alpha'}}\rho\sigma^{-\frac{1}{2\alpha'}}\right)^{\frac{\alpha}{\alpha'}}\sigma^{-\frac{1}{2\alpha'}}\\ &= \mathcal{N}^{\dagger}\left(\mathcal{N}(\sigma)^{-\frac{1}{2\alpha'}}\left(\mathcal{N}(\sigma)^{-\frac{1}{2\alpha'}}\mathcal{N}(\rho)\mathcal{N}(\sigma)^{-\frac{1}{2\alpha'}}\right)^{\frac{\alpha}{\alpha'}}\mathcal{N}(\sigma)^{-\frac{1}{2\alpha'}}\right)\end{aligned}$$

It is not clear if this implies sufficiency. By (45) it seems that some relation between $\mathcal{N}^{\dagger}(\phi^\alpha_{\mathcal{N}(\sigma)}(d_0))$ and $\phi^\alpha_\sigma(\mathcal{N}^{\dagger}(d_0))$ would be helpful. (Maybe by some sort of weighted Jensen's operator inequality?)

References

[1] V. Paulsen, Completely bounded maps and operator algebras, Cambridge Univ. Press, 2002