Geometric conditions for saturating the data processing inequality

Referee report

The data processing inequality (DPI) is one of the fundamental properties of distinguishability measures on the set of quantum states, such as the relative entropy. It means that distinguishability of two states cannot be increased after a quantum channel is applied. The present paper is concerned with the question of equality in DPI, in which case some distinguishability measure on a given pair of states is preserved under a given channel. This is an important question, since it is known that for some (but not all) such measures the equality implies that the pair of states can be recovered by another channel, which is referred to as "sufficiency", "reversibility" or "recoverability" in the literature. In many other cases, it was shown that the equality implies (or is equivalent to) certain operator equalities which have consequences on the structure of both the states and the channel.

The author(s) propose a geometric approach, expressing equality in DPI as a minimum of certain differentiable scalar function, which can be characterized by its gradient. In this way, a necessary condition for equality is obtained for a quite general class of distinguishability measures and it is shown that in some cases it is also sufficient. This is then applied to some distinguishability measures and the results are compared to previously obtained conditions.

This is a simple idea, but to my knowledge not perceived before. The advantage is its general applicability (albeit only in finite dimensions) and rather simple derivation of the resulting conditions. On the other hand, as admitted by the author(s), the obtained conditions are either the same or more complicated and less clear than those already known. This is not surprising, since those conditions were obtained using specific properties of particular divergence measures, such as a variational formula or an integral representation. Perhaps it would be possible to obtain better results by taking such specifics into account when computing the gradient, or studying its properties.

In fact, no advantages of the proposed methods were demonstrated: for all the studied examples of distinguishability measures some equality conditions are already known that are either identical or seem preferable to those obtained here. Nevertheless, the main idea is nice and new and worth pursuing, the paper might serve as a starting point of further investigation.

Some small remarks and suggestions

1. It is written at the end of the abstract that connections to Petz recovery and approximate DPI are to be commented on, but, in fact, it is only mentioned in the discussion that (more or less) such connections are possible. I find this misleading and suggest to remove the last sentence from the abstract.

2. p. 2: Λ is recoverable - it would be good to mention also the other names (sufficiency, reversibility) used in this context in some papers
3. p. 2, line 4 from below: in Ref. [16] the transition probability was studied, not the relative entropy
4. p.5, Eq. (2.4) f -> f_j
5. better do not use the same symbol $\times$ for the Cartesian product and the product of functions, this is confusing
6. p. 15, Eq. (4.6): does this mean that Λ^* is required to map the gradient in $T_{\Lambda(\rho)}PSD$ into $T_\rho PSD$?
7. p. 16, Sec. 4.3: the obtained condition should be compared to Theorem 5.1 (ix) in [10].
8. Many of the references are incomplete: only arxiv reference is given for some papers already published in a journal