Characterization of sufficient channels by a Rényi relative entropy

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1. Quantum versions of the Rényi relative entropy

In [10], it was suggested that the quantum generalization of the Rényi relative entropy should be

$$D_\alpha(\rho\|\sigma) = \begin{cases} D_\alpha^{(old)}(\rho,\sigma), & \alpha \in (0,1) \\ D_\alpha^{(new)}(\rho\|\sigma), & \alpha > 1 \end{cases}$$

for pairs of states ρ, σ on a finite dimensional Hilbert space. The "old" entropies

$$D_\alpha^{(old)}(\rho\|\sigma) = \frac{1}{\alpha-1}\log \mathrm{Tr}\,\rho^\alpha\sigma^{1-\alpha}$$

for $\alpha \in (0,1)$ have a direct operational interpretation as error exponents and cutoff rates in binary state discrimination [1, 9]. The "new", or "sandwiched" version [14, 11]

$$D_\alpha^{(new)}(\rho\|\sigma) = \begin{cases} \frac{1}{\alpha-1}\log \mathrm{Tr}\,\left(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha, & \mathrm{supp}(\rho)\subseteq \mathrm{supp}(\sigma) \\ \infty, & \text{otherwise} \end{cases}$$

has similar applications in the strong converse domain, [10, 14]. These quantities have some important properties ([4, 12, 2, 3]):

(i) **positivity**: $D_\alpha(\rho,\sigma) \ge 0$ and equality holds if and only if $\rho = \sigma$.

(ii) **data processing inequality (DPI)**: If Φ is a channel (i.e. a completely positive trace preserving map), then

$$D_\alpha(\rho\|\sigma) \ge D_\alpha(\Phi(\rho)\|\Phi(\sigma))$$

2. Sufficient channels

Let Φ be a channel and let $\mathcal{S}$ be a set of states. We say that Φ is **sufficient** with respect to $\mathcal{S}$ if there is some channel Ψ, called a **recovery map**, such that

$$\Psi\circ\Phi(\rho) = \rho \qquad \rho \in \mathcal{S}.$$

Sufficient channels can be characterized by equality in the data processing inequality for a large class of information theoretic quantities, see e.g. [4, 5]. In particular, if $\mathcal{S}$ contains an invertible element σ, then Φ is sufficient with respect to $\mathcal{S}$ if and only if

$$D_\alpha(\rho\|\sigma) = D_\alpha(\Phi(\rho)\|\Phi(\sigma)), \qquad \rho \in \mathcal{S} \tag{1}$$

for some $\alpha \in (0,1)$, [4, 7]. We show that this is true also for $\alpha > 1$ and in infinite dimensions. Similarly to [2], we use an interpolating family of non-commutative L_p-spaces with respect to a state.

3. Noncommutative L_p-spaces and interpolation

Let $\mathcal{H}$ be a separable Hilbert space. For $p \le 1$, let

$$\mathcal{L}_p(\mathcal{H}) = \{X \in B(\mathcal{H}), \mathrm{Tr}\,|X|^p < \infty\}$$

be the Schatten class, with the norm $\|X\|_p = (\mathrm{Tr}\,|X|^p)^{1/p}$. Let

$$\mathcal{L}_\infty(\mathcal{H}) = B(\mathcal{H}),$$

with $\|\cdot\|_\infty = \|\cdot\|$, the operator norm. Let also

$$\mathfrak{S}(\mathcal{H}) = \{\rho \in \mathcal{L}_1(\mathcal{H})^+, \mathrm{Tr}\,\rho = 1\}$$

be the set of normal states. Fix a faithful $\sigma \in \mathfrak{S}(\mathcal{H})$ and $p \in [1,\infty]$, $1/p + 1/q = 1$. The **noncommutative L_p-space with respect to** σ is defined as

$$L_p(\mathcal{H},\sigma) := \{\sigma^{1/2q}X\sigma^{1/2q},\ X \in \mathcal{L}_p(\mathcal{H})\},$$

with the norm

$$\|\sigma^{1/2q}X\sigma^{1/2q}\|_{p,\sigma} := \|X\|_p.$$

Then $L_p(\mathcal{H},\sigma) \subseteq \mathcal{L}_1(\mathcal{H}) = L_1(\mathcal{H},\sigma)$ and

$$L_p(\mathcal{H},\sigma) = C_{1/p}(L_\infty(\mathcal{H},\sigma), \mathcal{L}_1(\mathcal{H})),$$

where C_θ is given by complex interpolation [8, 15].
This has some consequences:

- For $1 \le p \le p' \le \infty$, $L_{p'}(\mathcal{H},\sigma) \subseteq L_p(\mathcal{H},\sigma)$ and
$$\|X\|_{p,\sigma} \le \|X\|_{p',\sigma}, \quad \forall X \in L_{p'}(\mathcal{H},\sigma).$$
- For $1 \le p < \infty$, $L_q(\mathcal{H},\sigma)$ is the dual space of $L_p(\mathcal{H},\sigma)$, with duality $\langle\cdot,\cdot\rangle : L_q(\mathcal{H},\sigma)\times L_p(\mathcal{H},\sigma) \to \mathbb{C}$ given by
$$\langle\, \sigma^{1/2p}X\sigma^{1/2p}, \sigma^{1/2q}Y\sigma^{1/2q}\,\rangle := \mathrm{Tr}\,XY,$$
for any $X \in \mathcal{L}_q(\mathcal{H})$, $Y \in \mathcal{L}_p(\mathcal{H})$.
- $L_2(\mathcal{H},\sigma)$ is a Hilbert space, with inner product
$$(S,T) \mapsto \langle\, S^*, T\,\rangle.$$

Let $\Phi : \mathcal{L}_1(\mathcal{H}) \to \mathcal{L}_1(\mathcal{K})$ be a channel. Then

- for $1 \le p \le \infty$, Φ restricts to a contraction $L_p(\mathcal{H},\sigma) \to L_p(\mathrm{supp}(\Phi(\sigma)), \Phi(\sigma))$.
- The equality
$$\langle\, \Phi_\sigma(X), \sigma^{1/2}Y\sigma^{1/2}\,\rangle = \langle\, X, \Phi(\sigma^{1/2}Y\sigma^{1/2})\,\rangle,$$
for all $X \in \mathcal{L}_1(\mathcal{K})$ and $Y \in B(\mathcal{H})$, defines a channel $\Phi_\sigma : \mathcal{L}_1(\mathcal{K}) \to \mathcal{L}_1(\mathcal{H})$ - the **Petz recovery map**.

4. Quantum Rényi entropies and L_p-spaces

Let $\rho,\sigma \in \mathfrak{S}(\mathcal{H})$, $\alpha \in (0,1)$. Assume that $\tau \in \mathfrak{S}(\mathcal{H})$ is faithful. Then

$$\ell_\alpha : \omega \mapsto \tau^{(1-\alpha)/2}\omega^\alpha\tau^{(1-\alpha)/2}$$

defines a homeomorpism of the positive cone $\mathcal{L}_1(\mathcal{H})^+$ onto $L_{1/\alpha}(\mathcal{H},\tau)^+$. Put

$$D_\alpha(\rho\|\sigma) := \frac{1}{\alpha-1}\log\langle\, \ell_\alpha(\rho), \ell_{1-\alpha}(\sigma)\,\rangle$$

If $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$, then we may put $\tau = \sigma$ and

$$D_\alpha(\rho\|\sigma) = \frac{1}{\alpha-1}\log\|\ell_\alpha(\rho)\|_1.$$

For $\alpha > 1$, let $1/\alpha + 1/\beta = 1$. Put

$$L_\alpha(\sigma) := L_\alpha(\mathrm{supp}(\sigma),\sigma).$$

Then $\rho \in L_\alpha(\sigma)$ if $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$ and there is some $\omega \in \mathcal{L}_1(\mathcal{H})^+$ such that

$$\rho = \sigma^{1/2\beta}\omega^{1/\alpha}\sigma^{1/2\beta}.$$

We have $\|\rho\|_{\alpha,\sigma} = (\mathrm{Tr}\,\omega)^{1/\alpha}$. Define

$$D_\alpha(\rho\|\sigma) := \begin{cases} \frac{\alpha}{\alpha-1}\log(\|\rho\|_{\alpha,\sigma}), & \text{if } \rho \in L_\alpha(\sigma) \\ \infty, & \text{otherwise} \end{cases}$$

If $\dim(\mathcal{H}) < \infty$, $D_\alpha(\rho\|\sigma)$ coincides with the original definition. The following properties extend to infinite dimensional case:

(i) positivity

(ii) DPI

(iii) $\alpha \mapsto D_\alpha(\rho\|\sigma)$ is non-decreasing.

(iv) $(\rho,\sigma) \mapsto D_\alpha(\rho\|\sigma)$ is jointly lower semicontinuous.

5. Characterization of sufficient channels by factorization

We may assume that $\sigma \in \mathfrak{S}(\mathcal{H})$ is faithful.

Theorem 1. *[13] Let $\Phi : \mathcal{L}_1(\mathcal{H}) \to \mathcal{L}_1(\mathcal{K})$ be a channel. Then Φ is sufficient with respect to $\{\rho,\sigma\}$ if and only if ρ is an invariant state of the channel $\Omega := \Phi_\sigma\circ\Phi$.*

Since Ω possesses a faithful normal invariant state σ, the structure of its invariant states is known.

Theorem 2. *There is a unitary U and factorizations*

$$U\mathcal{H} = \bigoplus_n \mathcal{H}_n^L \otimes \mathcal{H}_n^R, \qquad \sigma = U^*\left(\bigoplus_n A_n^L \otimes \sigma_n^R\right)U,$$

where $A_n^L \in \mathcal{L}_1(\mathcal{H}_n^L)^+$ and $\sigma_n^R \in \mathfrak{S}(\mathcal{H}_n^R)$, such that for any $\rho \in \mathfrak{S}(\mathcal{H})$, Φ is sufficient with respect to $\{\sigma,\rho\}$ if and only if ρ can be factorized as

$$\rho = U^*\left(\bigoplus_n B_n^L \otimes \sigma_n^R\right)U$$

for some $B_n^L \in \mathcal{L}_1(\mathcal{H}_n^L)^+$.

6. Characterization of sufficient channels by D_α

Theorem 3. *Assume that $D_\alpha(\rho\|\sigma) < \infty$ for some $\alpha \in (0,1)\cup(1,\infty)$. Then the channel Φ is sufficient with respect to $\{\rho,\sigma\}$ if and only if*

$$D_\alpha(\Phi(\rho)\|\Phi(\sigma)) = D_\alpha(\rho\|\sigma).$$

- For $\alpha \in (0,1)$, this was proved in [7].
- For $\alpha = 2$, it is easy: equality in DPI implies:
$$\|\rho\|_{2,\sigma}^2 = \|\Phi(\rho)\|_{2,\Phi(\sigma)}^2 = \langle\, \rho, \Phi_\sigma\circ\Phi(\rho)\,\rangle \le \|\rho\|_{2,\sigma}^2$$
so that $\rho = \Phi_\sigma\circ\Phi(\rho)$ by equality condition for Schwarz inequality.
- Assume the equality holds for some $1 < \alpha < \infty$ and let
$$\rho = \sigma^{1/2\beta}\omega^{1/\alpha}\sigma^{1/2\beta}$$
for some $\omega \in \mathcal{L}_1(\mathcal{H})^+$. Let $S = \{z \in \mathbb{C}, 0 \le \Re(z) \le 1\}$ and
$$\rho(z) = \|\omega\|_1^{(1/\alpha - z)}\sigma^{(1-z)/2}\omega^z\sigma^{(1-z)/2}, \qquad z \in S$$
Then $z \mapsto \rho(z)$ is bounded and continuous on S, holomorphic in the interior, $\rho = \rho(1/\alpha)$ and we have
$$\|\Phi(\rho(\theta))\|_{1/\theta,\Phi(\sigma)} = \|\rho(\theta)\|_{1/\theta,\sigma}, \qquad \forall\theta \in (0,1)$$
Let $\theta = 1/2$ and put
$$\xi := c\rho(1/2) = c_1\sigma^{1/4}\omega^{1/2}\sigma^{1/4} \in \mathfrak{S}(\mathcal{H}),$$
$c, c_1 > 0$ are normalization constants. Then
$$\|\Phi(\xi)\|_{2,\Phi(\sigma)} = \|\xi\|_{2,\sigma}$$
It follows that Φ is sufficient with respect to $\{\xi,\sigma\}$, which implies that ξ has a factorization as in Theorem 2. But then also ω, and, consequently, ρ has such a factorization. Hence Φ is sufficient with respect to $\{\rho,\sigma\}$.
- The converse is clear from the data processing inequality.

Remark 1. For $\dim(\mathcal{H}) < \infty$, see [6]. Similar results can be obtained for pairs of normal states on arbitrary von Neumann algebras and normal unital completely positive maps.

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