Conditions on the existence of maximally incompatible two-outcome measurements in general probabilistic theory

Anna Jenčová* and Martin Plávala†

Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, Bratislava, Slovakia

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We formulate the necessary and sufficient conditions for the existence of a pair of maximally incompatible two-outcome measurements in a finite-dimensional general probabilistic theory. The conditions are on the geometry of the state space; they require the existence of two pairs of parallel exposed faces with an additional condition on their intersections. We introduce the notion of discrimination measurement and show that the conditions for a pair of two-outcome measurements to be maximally incompatible are equivalent to requiring that a (potential, yet nonexisting) joint measurement of the maximally incompatible measurements would have to discriminate affinely dependent points. We present several examples to demonstrate our results.

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I. INTRODUCTION

General probabilistic theories (GPTs) form a general framework that provides a unified description of all physical systems known today. In such theories, the central object is the state space represented by a convex set and the measurements (or more general devices) are seen as certain maps on the state space (see, e.g., [1]). The study of such theories began some time ago, related to mathematical foundations of quantum mechanics, but has gained a great deal of attention recently in connection with information theory. It was identified that several nonclassical effects that we know from quantum mechanics, such as steering and Bell nonlocality [2], can be found in this broader framework.

It has been known since the beginning of quantum theory that some quantum mechanical measurements cannot be implemented simultaneously; this phenomenon is referred to as incompatibility of measurements. It was shown that such measurements appear in any nonclassical GPT [3]. Moreover, one can violate even the bounds that hold in quantum mechanics: In finite dimensions, the minimal degree of compatibility of quantum measurements is bounded from below by a dimension-dependent constant [4], while a GPT may admit pairs of maximally incompatible two-outcome measurements [5], i.e., two-outcome measurements such that their degree of compatibility attains the minimal value of $\frac{1}{2}$.

A long-standing question is what the properties that single out quantum mechanics in the framework of GPTs are. To answer this question, it is important to know the relation of the manifestations of nonclassical effects to the geometry of state spaces. One step in this direction was made in [3], where it was proved that the nonexistence of incompatible pairs of two-outcome measurements characterizes classical theories, that is, such GPTs where the state space is a simplex. In the present work we aim at the other extreme, namely, we find necessary and sufficient conditions for a pair of maximally incompatible measurements to exist in a given GPT.

This question is of interest also because of the relation of incompatibility to other nonclassical features: It was shown that one would need maximally incompatible measurements for maximal violation of Bell inequalities [6,7]. We obtain conditions that restrain the geometry of the state spaces for which this is possible. The essence of what is required was already captured in [5] in the example of the square state space. On the other hand, we prove that maximally incompatible measurements exist if the state space is the set of quantum channels; this is related to the results recently found in [8], where a somewhat different notion of the compatibility of measurements on quantum channels and combs was studied. We also introduce the notion of discrimination two-outcome measurement and show how the concept of discrimination measurements is connected to maximally incompatible measurements. Our results are demonstrated on some examples.

The paper is organized as follows. In Sec. II we provide a quick review of GPTs and of the notation we will use. Section III deals with measurements and their compatibility, especially the two-outcome measurements. In Sec. IV we introduce the degree of compatibility and show its relation to a linear program for the compatibility of two-outcome measurements. In Sec. V we formulate and prove the necessary and sufficient conditions for the existence of maximally incompatible two-outcome measurements. In Sec. VI we introduce the concept of discrimination measurements and study their (in)compatibility. We summarize in Sec. VII.

II. STRUCTURE OF A GENERAL PROBABILISTIC THEORY

We present the standard definition of a finite-dimensional GPT in a quick review just to settle the notation. See, e.g., [9] for more information.

In a GPT, a state represents a mathematical description of a procedure for preparation of a physical system. To express the possibility of forming probabilistic mixtures of such procedures, it is assumed that the state space is a convex subset of a vector space $\mathcal{V}$. The convex combinations are interpreted operationally (see, e.g., [10], Pt. 2). We will assume below that $\mathcal{V}$ is finite dimensional and the state space is a compact convex subset $K \subset \mathcal{V}$.

Measurements on the system are represented by maps assigning to each state the corresponding outcome probabilities.

*jenca@mat.savba.sk

†martin.plavala@mat.savba.sk

These maps are assumed affine, that is, mapping a convex mixture of states to the corresponding convex mixture of probability distributions on the set of outcomes. The two-outcome measurements (yes-no experiments) are represented by affine maps $f : K \to [0,1]$, mapping each state $x \in K$ to the corresponding probability of success (the "yes" outcome). Such maps are called effects.

We next list some basic definitions and briefly present the framework of ordered vector spaces of affine functions, which will be useful in the following. A good handbook for some of the standard notions from convex analysis is [11].

Let $\mathcal{V}$ be a finite-dimensional real vector space. For any $X \subset \mathcal{V}$, conv(X) will denote the convex hull of X and aff(X) the affine hull of X. If X is convex, then a face of X is a convex subset $F \subseteq X$ such that $\lambda x + (1-\lambda) \in F$ for some $x,y \in X$ and $\lambda \in (0,1)$ implies that $x,y \in F$.

Let $K \subset \mathcal{V}$ be a compact convex subset and let $A(K)$ denote the set of affine functions $f : K \to \mathbb{R}$. Then $A(K)$ is a finite-dimensional real vector space. The partial order on $A(K)$ is introduced in a natural way: Let $f,g \in A(K)$; then $f \geqslant g$ if and only if $f(x) \geqslant g(x)$ for all $x \in K$. The corresponding positive cone is the convex cone of positive affine functions, which will be denoted by $A(K)^+$. The constant functions are denoted by the value they attain, i.e., $1(x) = 1$ and $0(x) = 0$ for all $x \in K$.

The set of effects on K will be denoted by $E(K)$, that is, $E(K) = \{f \in A(K) : 1 \geqslant f \geqslant 0\}$. For any $f \in E(K)$, the set

$$f^{-1}(0) = \{x \in K,\ f(x) = 0\}$$

is a face of K. A face of this form is called an exposed face. Two exposed faces F_0 and F_1 of $E(K)$ are parallel if $F_0 = f^{-1}(0)$ and $F_1 = f^{-1}(1) = (1-f)^{-1}(0)$.

Let $A(K)^*$ be the vector space dual to $A(K)$ and let $\langle \psi, f \rangle$ denote the value of the functional $\psi \in A(K)^*$ on $f \in A(K)$. The positive cone $A(K)^+$ defines the dual order on $A(K)^*$ as follows: For $\psi_1,\psi_2 \in A(K)^*$, $\psi_1 \geqslant \psi_2$ if and only if $\langle \psi_1, f \rangle \geqslant \langle \psi_2, f \rangle$ for every $f \in A(K)^+$. Here the positive cone is the dual cone $A(K)^{*+} = \{\psi \in A(K)^* : \psi \geqslant 0\}$ of positive functionals.

For any $x \in K$ let ϕ_x denote the functional in $A(K)^*$, given by the evaluation $\langle \phi_x, f \rangle = f(x)$. Then clearly ϕ_x is positive and normalized: $\langle \phi_x, 1 \rangle = 1$. On the other hand, it can be seen that every positive normalized functional $\psi \in A(K)^*$ is of the form $\psi = \phi_x$ for some $x \in K$ (see, e.g., [12]). This implies that the set $\mathfrak{S}_K = \{\phi_x : x \in K\}$ is a base of the cone $A(K)^{*+}$, i.e., for every $\psi \in A(K)^{*+}$, $\psi \neq 0$, there is a unique $x \in K$ and unique $\alpha \in \mathbb{R}$, $\alpha > 0$, such that $\psi = \alpha \phi_x$.

The simplest example of a state space is an $(n-1)$-dimensional simplex Δ_n; this describes the state space of a classical system. We have $A(\Delta_n) \simeq \mathbb{R}^n$; $A(\Delta_n)^+$ is identified with the set of vectors with non-negative entries and the effects are given by vectors with entries in $[0,1]$.

A quantum state space is the set of density operators $\mathfrak{S}(\mathcal{H})$ on a finite-dimensional Hilbert space $\mathcal{H}$. The space $A(\mathfrak{S}(\mathcal{H}))$ is identified with the space $B_h(\mathcal{H})$ of self-adjoint operators on $\mathcal{H}$ by $A(\rho) \equiv \operatorname{Tr} A\rho$. Here the positive cone is the cone of positive-semidefinite operators $B(\mathcal{H})^+$, the constant 1 is the identity operator $\mathbb{1}$, and the effects satisfy $0 \leqslant E \leqslant \mathbb{1}$.

The ensuing two further examples will be important in the following.

Example 1. The square state space is defined as the convex hull $S := \text{conv}\{x_{00},x_{01},x_{10},x_{11}\}$ of points $x_{ij} \in \mathcal{V}$, satisfying $x_{00} + x_{11} = x_{01} + x_{10}$. The space $A(S)$ can be identified with $\mathbb{R}^3$, with a positive cone $A(S)^+$ whose base is again a square. The vertices of this square correspond to the extremal effects, determined by

$$f_{k,l}(x_{n_1,n_2}) = \begin{cases} 1 & \text{for } n_k = l \\ 0 & \text{otherwise,} \end{cases}$$

where $n_1,n_2,k,l \in \{0,1\}$. The set of effects $E(S)$ is a double pyramid, with base formed by the above square and the two apexes given by the 0 and 1 functionals.

Example 2. Let $\mathcal{C}(\mathcal{H})$ denote the set of completely positive trace-preserving maps $B(\mathcal{H}) \to B(\mathcal{H})$; such maps are often called quantum channels. We will use the standard and well-known definitions that may be found in [10].

Let $\mathcal{H} \otimes \mathcal{H}$ denote the tensor product of $\mathcal{H}$ with itself and let Tr_1 denote the partial trace. Using the Choi representation of quantum channels, we have the identification

$$\mathcal{C}(\mathcal{H}) \equiv \{C \in B_h(\mathcal{H} \otimes \mathcal{H}) : \operatorname{Tr}_1(C) = \mathbb{1}, C \geqslant 0\}.$$

This is clearly a finite-dimensional state space. Affine functions on $\mathcal{C}(\mathcal{H})$ have the form $C \mapsto \operatorname{Tr} CA$ for some $A \in B_h(\mathcal{H} \otimes \mathcal{H})$, but note that A and $A + \mathbb{1} \otimes X$ define the same function if $X \in B_h(\mathcal{H})$ and $\operatorname{Tr} X = 0$. Moreover, all elements of $A(\mathcal{C}(\mathcal{H}))^+$ are given by some positive operator A and all effects have the form $C \mapsto \operatorname{Tr} CM$, where $M \in B_h(\mathcal{H} \otimes \mathcal{H})$ is such that

$$0 \leqslant M \leqslant \mathbb{1} \otimes \sigma$$

for some $\sigma \in \mathfrak{S}(\mathcal{H})$ [1,13], so that effects are given by two-outcome process positive-operator-valued measures defined in [14].

III. COMPATIBLE MEASUREMENTS IN GPT

Let K be a state space and let Ω be the set of all possible outcomes of some measurement. In the most general case, Ω carries the structure of a measurable space, but since we are mostly interested in two-outcome measurements, we will assume that Ω is a finite set. Let $\mathcal{P}(\Omega)$ be the set of all probability measures on Ω.

Definition 1. A measurement m on K with sample space Ω is an affine map

$$m : K \to \mathcal{P}(\Omega).$$

Let $\Omega = \{\omega_1, \ldots, \omega_k\}$ and let m be a measurement on K with sample space Ω. Then $x \mapsto m(x;\omega_i) := m(x)(\omega_i)$ is clearly an effect on K with $\sum_i m(\cdot;\omega_i) = 1$ and any such k-tuple of effects determines a measurement on K. In particular, the general form of a two-outcome measurement is

$$m_f(x) := f(x)\delta_1 + (1-f)(x)\delta_2,$$

where $f \in E(K)$ and $\delta_1 = \delta_{\omega_1}$ and $\delta_2 = \delta_{\omega_2}$ are the two Dirac measures on $\Omega = \{\omega_1,\omega_2\}$. In other words, this means that $f(x)$ is the probability of getting the outcome ω_1 by the measurement m_f if the system is prepared in the state $x \in K$.

We will present the standard definition of compatibility of measurements [15].

Definition 2. Let m_1 and m_2 be measurements on K with sample spaces Ω_1 and Ω_2, respectively. We say that the measurements m_1 and m_2 are compatible if there exists a measurement m on K with sample space $\Omega_1 \times \Omega_2$ such that m_1 and m_2 are marginals of m: For all $x \in K$ and $A_1 \subset \Omega_1$ and $A_2 \subset \Omega_2$ we have

$$m_1(x;A_1) = m(x;A_1 \times \Omega_2),$$
$$m_2(x;A_2) = m(x;\Omega_1 \times A_2).$$

In this case, m is called a joint measurement of m_1 and m_2.

In the following, we deal with the compatibility of two-outcome measurements, given by two effects $f,g \in E(K)$.

Proposition 1. The two-outcome measurements m_f and m_g are compatible if and only if there exists a function $p \in E(K)$ such that

$$f \geqslant p, \tag{1}$$

$$g \geqslant p, \tag{2}$$

$$1 + p \geqslant f + g. \tag{3}$$

Moreover, any joint measurement of m_f and m_g is of the form

$$m = p\delta_{(1,1)} + (f-p)\delta_{(1,2)} + (g-p)\delta_{(2,1)} + (1+p-f-g)\delta_{(2,2)}, \tag{4}$$

where $\delta_{(i,j)} = \delta_{(\omega_i,\omega_j)}$ and $p \in E(K)$ satisfies (1)–(3).

Proof. Let $\Omega = \{\omega_1,\omega_2\}$ and let $m : K \to \mathcal{P}(\Omega \times \Omega)$ be a joint measurement of m_f and m_g. Let us define $h_{ij} := m(\cdot,(\omega_i,\omega_j)) \in E(K)$, $i,j = 1,2$. Then we must have

$$f = h_{11} + h_{12},\ 1 - f = h_{21} + h_{22},$$
$$g = h_{11} + h_{21},\ 1 - g = h_{12} + h_{22},$$

which follows from Definition 2. Denoting $h_{11} = p$, one can show that this is equivalent to the conditions (1)–(3) and m is given by (4). See [3] for a more throughout derivation of these conditions. ■

Proposition 2. m_f and m_g are compatible if and only if $m_{(1-f)}$ and m_g are compatible.

Proof. Assume that m_f and m_g are compatible and let $p \in E(K)$ satisfy (1)–(3). If we let $p' = g - p$, then Eq. (2) implies $p' \geqslant 0$, Eq. (3) implies $1 - f \geqslant p'$, $p \geqslant 0$ implies $g \geqslant p'$, and Eq. (1) implies $1 + p' \geqslant (1-f) + g$. Since $1 - (1-f) = f$ it is clear that the compatibility of $m_{(1-f)}$ and m_g implies the compatibility of m_f and m_g in the same manner. ■

IV. DEGREE OF COMPATIBILITY

A degree of compatibility gives a way to quantify the (in)compatibility of a pair of measurements. One possibility to introduce such a degree in any GPT is to use the least amount of noise needed to make the measurements compatible [see [16,17] for some different (but related) definitions]. We first introduce the coin-toss measurements that will represent the noise.

Definition 3. A coin-toss measurement on K is a constant map

$$\tau(x) = \mu \in \mathcal{P}(\Omega), \quad x \in K.$$

Such a measurement ignores the input state and just returns the outcomes according to some fixed probability distribution. It is straightforward that a coin-toss measurement is compatible with any other measurement. Observe also that any pair of measurements m_1 and m_2 can be made compatible by mixing with a coin toss. Indeed, let τ be a coin toss and let

$$m_i' = \tfrac{1}{2}m_i + \tfrac{1}{2}\tau.$$

The measurements m_1' and m_2' are compatible: The joint measurement consists of choosing one of the measurements at random (by flipping a coin) and replacing the other by τ. This observation leads to the following definition of the degree of compatibility, introduced in [4].

Definition 4. Let m_1 and m_2 be two measurements on K with sample space Ω. The degree of compatibility of m_1 and m_2 is defined as

$$\mathrm{DegCom}(m_1,m_2) = \sup_{\substack{0 \leqslant \lambda \leqslant 1 \\ \tau_1,\tau_2}} \{\lambda : \lambda m_1 + (1-\lambda)\tau_1, \lambda m_2 + (1-\lambda)\tau_2 \text{ are compatible}\},$$

where the supremum is taken over all coin-toss measurements τ_1 and τ_2.

The interpretation of this measure of compatibility is clear: The convex combination is a mathematical representation of making the measurements m_1 and m_2 less sharp by adding noise in the form of the coin tosses. As the value of λ decreases, the measurements get less and less sharp, until at some point they become compatible. If for a pair of measurements this happens at a larger value of λ than for another pair, we may say that the first pair is more compatible.

It can be seen from the remarks after Definition 3 that the lowest possible value of degree of compatibility is $\frac{1}{2}$. If this happens for a pair of measurements, it means that the only way to make them compatible is to discard one of them completely and replace it by a coin toss. It is known that such pairs of measurements exist for some state spaces [5], but not in finite-dimensional quantum mechanics [4].

Definition 5. We will say that two measurements are maximally incompatible if $\mathrm{DegCom}(m_f,m_g) = \frac{1}{2}$.

We will now turn to the study of the degree of compatibility of two-outcome measurements m_f and m_g. The following statement follows immediately from Proposition 2.

Corollary 1. Let m_f and m_g be two-outcome measurements. Then

$$\begin{aligned}\mathrm{DegCom}(m_f,m_g) &= \mathrm{DegCom}(m_{(1-f)},m_g)\\ &= \mathrm{DegCom}(m_f,m_{(1-g)})\\ &= \mathrm{DegCom}(m_{(1-f)},m_{(1-g)}).\end{aligned}$$

It was shown in [17] that compatibility of m_f and m_g can be formulated as a problem of linear programming. A similar linear program was introduced in [3] and it was shown that the dual program is of the form

$$\sup (a_3[f(z_3)+g(z_3)-1]-a_1f(z_1)-a_2g(z_2)),$$
$$a_1+a_2 \leqslant 2,\ a_3\phi_{z_3} \leqslant a_1\phi_{z_1}+a_2\phi_{z_2},$$

where $z_1,z_2,z_3 \in K$ and a_1,a_2,a_3 are non-negative numbers. Let β denote the supremum. Then we have

$$\beta = \frac{1-\mathrm{DegCom}_{1/2}(m_f,m_g)}{\mathrm{DegCom}_{1/2}(m_f,m_g)},$$

where

$$\mathrm{DegCom}_{1/2}(m_f,m_g) := \sup_{0\leqslant\lambda\leqslant 1}\{\lambda : \lambda m_f+(1-\lambda)\tau,\lambda m_g + (1-\lambda)\tau \text{ are compatible}\}$$

is the degree of compatibility provided by mixing the measurements m_f and m_g with the fixed coin-toss measurement $\tau = \frac{1}{2}(\delta_1+\delta_2)$. The measurements m_f and m_g are compatible if and only if $\beta = 0$. We clearly have

$$\mathrm{DegCom}_{1/2}(m_1,m_2) \leqslant \mathrm{DegCom}(m_f,m_g),$$

so $\mathrm{DegCom}_{1/2}(m_f,m_g)=1$ implies $\mathrm{DegCom}(m_f,m_g)=1$ and $\mathrm{DegCom}(m_f,m_g)=\frac{1}{2}$ implies $\mathrm{DegCom}_{1/2}(m_f,m_g)=\frac{1}{2}$.

We next show that if m_f and m_g are incompatible, the supremum in the above program is reached with $a_1+a_2=2$, which allows us to rewrite the program in a more convenient way. So assume that the measurements m_f and m_g are incompatible. Then we have $\beta>0$, which implies $a_1+a_2>0$. Assume that the supremum is reached for some a_1,a_2,a_3 and z_1,z_2,z_3 such that $a_1+a_2<2$. Define

$$a_1' = \frac{2}{a_1+a_2}a_1,$$
$$a_2' = \frac{2}{a_1+a_2}a_2,$$
$$a_3' = \frac{2}{a_1+a_2}a_3.$$

It is straightforward to see that $a_3'\phi_{z_3} \leqslant a_1'\phi_{z_1}+a_2'\phi_{z_2}$. Moreover,

$$\beta < a_3'[f(z_3)+g(z_3)-1]-a_1'f(z_1)-a_2'g(z_2),$$

which is a contradiction. It follows that in the case when the measurements m_f and m_g are incompatible we can write the linear program as

$$\sup 2(\eta[f(z_3)+g(z_3)-1]-\nu f(z_1)-(1-\nu)g(z_2)),$$
$$\eta\phi_{z_3} \leqslant \nu\phi_{z_1}+(1-\nu)\phi_{z_2}, \tag{5}$$

where we have set $a_1+a_2=2$ and used the substitutions $2\nu=a_1$ and $2\eta=a_3$. Also note that $\eta\phi_{z_3} \leqslant \nu\phi_{z_1}+(1-\nu)\phi_{z_2}$ implies that there exists $z_4\in K$ such that

$$\nu z_1+(1-\nu)z_2 = \eta z_3+(1-\eta)z_4.$$

V. MAXIMALLY INCOMPATIBLE TWO-OUTCOME MEASUREMENTS

In this section we find necessary and sufficient conditions for the existence of maximally incompatible measurements m_f and m_g on a given state space K. A sufficient condition was proved in [5]: A pair of maximally incompatible two-outcome measurements exists if K is the square state space of Example 1 or, more generally, there are two pairs of parallel hyperplanes tangent to K such that the corresponding exposed faces contain the edges of a square. Besides the square, such state spaces include the cube, pyramid, double pyramid, cylinder, etc. We will show that this condition is also necessary so that it characterizes state spaces admitting a pair of maximally incompatible two-outcome measurements.

The following notation will be used throughout:

$$F_0 = \{z\in K : f(z)=0\},$$
$$F_1 = \{z\in K : f(z)=1\},$$
$$G_0 = \{z\in K : g(z)=0\},$$
$$G_1 = \{z\in K : g(z)=1\}.$$

We begin by rephrasing the above sufficient condition. For completeness, we add a proof along the lines of [5].

Proposition 3. Assume there are some points $x_{00}\in F_0\cap G_0$, $x_{10}\in F_1\cap G_0$, $x_{01}\in F_0\cap G_1$, and $x_{11}\in F_1\cap G_1$ such that

$$\tfrac{1}{2}(x_{00}+x_{11}) = \tfrac{1}{2}(x_{10}+x_{01}).$$

Then $\mathrm{DegCom}(m_f,m_g)=\frac{1}{2}$.

Proof. Let p be any positive affine function on K. Then we have

$$p(x_{11})+p(x_{00}) = p(x_{10})+p(x_{01})$$

and

$$p(x_{11}) \leqslant p(x_{10})+p(x_{01})$$

follows. Let $\tau_1 = \mu_1\delta_{\omega_1}+(1-\mu_1)\delta_{\omega_2}$ and $\tau_2=\mu_2\delta_{\omega_1}+(1-\mu_2)\delta_{\omega_2}$ be coin-toss measurements. Then the conditions (1)–(3) for $\lambda m_f+(1-\lambda)\tau_1$ and $\lambda m_g+(1-\lambda)\tau_2$ take the form

$$\lambda f+(1-\lambda)\mu_1 \geqslant p,$$
$$\lambda g+(1-\lambda)\mu_2 \geqslant p,$$
$$1+p \geqslant \lambda(f+g)+(1-\lambda)(\mu_1+\mu_2).$$

Expressing some of these conditions at the points x_{10},x_{01},x_{11}, we get

$$1+p(x_{11}) \geqslant 2\lambda+(1-\lambda)(\mu_1+\mu_2), \tag{6}$$
$$(1-\lambda)\mu_1 \geqslant p(x_{01}), \tag{7}$$
$$(1-\lambda)\mu_2 \geqslant p(x_{10}). \tag{8}$$

From (6) we obtain

$$2\lambda \leqslant 1+p(x_{11})-(1-\lambda)(\mu_1+\mu_2)$$

and since from (7) and (8) we have

$$p(x_{11}) \leqslant p(x_{10})+p(x_{01}) \leqslant (1-\lambda)(\mu_1+\mu_2),$$

it follows that $\lambda \leqslant \frac{1}{2}$. ■

At this point we can demonstrate that maximally incompatible two-outcome measurements exist for the set $\mathcal{C}(\mathcal{H})$ of quantum channels (see Example 2).

Example 3. Let $K=\mathcal{C}(\mathcal{H})$, with $\dim(\mathcal{H})=2$. Let $|0\rangle,|1\rangle$ be an orthonormal basis of $\mathcal{H}$ and let $M,N\in B_h(\mathcal{H}\otimes\mathcal{H})$ be given as

$$M = |0\rangle\langle 0| \otimes |0\rangle\langle 0|,$$
$$N = |0\rangle\langle 0| \otimes |1\rangle\langle 1|.$$

Then $0 \leqslant M \leqslant \mathbb{1} \otimes |0\rangle\langle 0|$ and $0 \leqslant N \leqslant \mathbb{1} \otimes |1\rangle\langle 1|$, so $f(C) = \operatorname{Tr} CM$ and $g(C) = \operatorname{Tr} CN$ define effects on $\mathcal{C}(\mathcal{H})$. Let

$$\begin{aligned}
C_{00} &= |1\rangle\langle 1| \otimes \mathbb{1},\\
C_{10} &= |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|,\\
C_{01} &= |0\rangle\langle 0| \otimes |1\rangle\langle 1| + |1\rangle\langle 1| \otimes |0\rangle\langle 0|,\\
C_{11} &= |0\rangle\langle 0| \otimes \mathbb{1}.
\end{aligned}$$

It is easy to check that $C_{00},C_{10},C_{01},C_{11} \in \mathcal{C}(\mathcal{H})$. Moreover,

$$C_{00} + C_{11} = \mathbb{1} \otimes \mathbb{1} = C_{10} + C_{01}$$

and

$$\begin{aligned}
&\operatorname{Tr}(C_{00}M) = \operatorname{Tr}(C_{00}N) = 0,\\
&\operatorname{Tr}(C_{10}M) = 1, \quad \operatorname{Tr}(C_{10}N) = 0,\\
&\operatorname{Tr}(C_{01}M) = 0, \quad \operatorname{Tr}(C_{01}N) = 1,\\
&\operatorname{Tr}(C_{11}M) = \operatorname{Tr}(C_{11}N) = 1.
\end{aligned}$$

In conclusion, C_{00}, C_{10}, C_{01}, and C_{11} satisfy the properties in Proposition 3, so the two-outcome measurements m_f and m_g are maximally incompatible. An analogical fact was also observed in [1,8] in different circumstances.

We proceed to prove some necessary conditions.

Proposition 4. $\operatorname{DegCom}(m_f,m_g) = \frac{1}{2}$ only if

$$\begin{aligned}
F_0 \cap G_0 &\neq \emptyset,\\
F_0 \cap G_1 &\neq \emptyset,\\
F_1 \cap G_0 &\neq \emptyset,\\
F_1 \cap G_1 &\neq \emptyset.
\end{aligned}$$

Proof. Let $F_1 \cap G_1 = \emptyset$. Then $f + g < 2$. Let $\tau = \delta_{\omega_2}$ and consider the measurements $\lambda m_f + (1-\lambda)\tau = m_{\lambda f}$ and $\lambda m_g + (1-\lambda)\tau = m_{\lambda g}$, $\lambda \in [0,1]$. Since $f + g < 2$, we can choose $\lambda > \frac{1}{2}$ such that $1 \geqslant \lambda(f+g)$, so $p = 0$ satisfies Eqs. (1)–(3) for $m_{\lambda f}$ and $m_{\lambda g}$.

The result for the other sets follows by using Corollary 1. ■

The conditions given by Proposition 4 are not sufficient, as we will demonstrate in the following example.

Example 4. Let K be a simplex with vertices x_1,x_2,x_3,x_4 and let b_1,b_2,b_3,b_4 denote positive affine functions such that

$$b_i(x_j) = \delta_{ij}.$$

Such functions exist because K is a simplex. Let

$$f = b_1 + b_2, \quad g = b_1 + b_3.$$

Then we have

$$\begin{aligned}
F_1 \cap G_1 &= \{x_1\},\\
F_1 \cap G_0 &= \{x_2\},\\
F_0 \cap G_1 &= \{x_3\},\\
F_0 \cap G_0 &= \{x_4\},
\end{aligned}$$

but clearly the measurements m_f and m_g must be compatible as K is a simplex. As a matter of fact, Eqs. (1)–(3) are satisfied with $p = b_1$.

Proposition 5. $\operatorname{DegCom}(m_f,m_g) = \frac{1}{2}$ if and only if there exist points $x_{00},x_{01},x_{10},x_{11}$ such that $x_{00} \in F_0 \cap G_0$, $x_{10} \in F_1 \cap G_0$, $x_{01} \in F_0 \cap G_1$, $x_{11} \in F_1 \cap G_1$, and

$$\tfrac{1}{2}(x_{00} + x_{11}) = \tfrac{1}{2}(x_{10} + x_{01}).$$

Proof. The "if" part is proved in Proposition 3. Conversely, if $\operatorname{DegCom}(m_f,m_g) = \frac{1}{2}$ then according to the results of Sec. IV, the supremum in (5) must be equal to 1, so we must have

$$\eta[f(z_3) + g(z_3) - 1] - \nu f(z_1) - (1-\nu)g(z_2) = \tfrac{1}{2} \tag{9}$$

for some $\eta,\nu \in [0,1]$ and $z_1,z_2,z_3 \in K$ such that

$$\nu\phi_{z_1} + (1-\nu)\phi_{z_2} \geqslant \eta\phi_{z_3}.$$

It follows that

$$\nu\phi_{z_1} \geqslant \eta\phi_{z_3} - (1-\nu)\phi_{z_2}, \tag{10}$$
$$(1-\nu)\phi_{z_2} \geqslant \eta\phi_{z_3} - \nu\phi_{z_1}. \tag{11}$$

Rewriting Eq. (9) we get

$$\langle \eta\phi_{z_3} - \nu\phi_{z_1}, f\rangle + \langle \eta\phi_{z_3} - (1-\nu)\phi_{z_2}, g\rangle - \eta = \tfrac{1}{2}.$$

We clearly have $\langle \eta\phi_{z_3} - \nu\phi_{z_1}, f\rangle \leqslant \eta$, but Eq. (11) implies $\langle \eta\phi_{z_3} - \nu\phi_{z_1}, f\rangle \leqslant 1 - \nu$ and thus we must have $\langle \eta\phi_{z_3} - \nu\phi_{z_1}, f\rangle \leqslant \min(\eta, 1-\nu)$. Similarly, we get $\langle \eta\phi_{z_3} - (1-\nu)\phi_{z_2}, g\rangle \leqslant \min(\eta,\nu)$ and

$$\tfrac{1}{2} \leqslant \min(\eta,\nu) + \min(\eta, 1-\nu) - \eta = \min(\nu, 1-\nu, \eta, 1-\eta),$$

which implies $\nu = \eta = \frac{1}{2}$. Moreover, there must be some $z_4 \in K$ such that

$$\tfrac{1}{2}(z_1 + z_2) = \tfrac{1}{2}(z_3 + z_4). \tag{12}$$

Equation (9) now becomes

$$f(z_3) + g(z_3) - f(z_1) - g(z_2) = 2,$$

which implies $f(z_3) = g(z_3) = 1$ and $f(z_1) = g(z_2) = 0$ as $f,g \in E(K)$. From Eq. (12) we get

$$f(z_2) = 1 + f(z_4),$$

which implies $f(z_2) = 1$, $f(z_4) = 0$, and

$$g(z_1) = 1 + g(z_4),$$

which implies $g(z_1) = 1$ and $g(z_4) = 0$. Together we get

$$\begin{aligned}
z_3 &\in F_1 \cap G_1,\\
z_2 &\in F_1 \cap G_0,\\
z_1 &\in F_0 \cap G_1,\\
z_4 &\in F_0 \cap G_0.
\end{aligned}$$

■

We now give a characterization of state spaces admitting a pair of maximally incompatible two-outcome measurements.

Theorem 1. Let K be a state space. A pair of maximally incompatible two-outcome measurements on K exists if and only if K contains a square (as defined in Example 1) whose opposite edges lie in parallel exposed faces of K. The effects

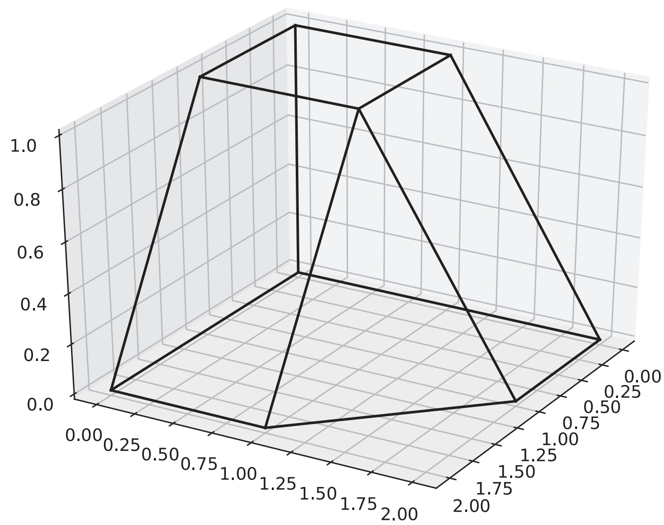

FIG. 1. State space K used in Example 5.

determining these exposed faces then constitute the pair of maximally incompatible measurements.

Proof. Let m_f and m_g be maximally incompatible. Then F_0,F_1 and G_0,G_1 are parallel exposed faces that contain the edges of a square by Proposition 5. Conversely, assume that the condition holds. Let $x_{00},x_{01},x_{10},x_{11} \in K$ be the vertices of the square and let F_0,F_1 and G_0,G_1 be pairs of parallel exposed faces containing the opposite edges, so that $x_{ij} \in F_i \cap G_j,\ i,j \in \{1,2\}$. Then there are some $f,g \in E(K)$ such that $F_0 = f^{-1}(0), F_1 = f^{-1}(1)$ and $G_0 = g^{-1}(0), G_1 = g^{-1}(1)$. By Proposition 3, m_f and m_g are maximally incompatible. ■

In the remainder of this section, we aim to give some geometric insight into the above condition.

Corollary 2. Let K be a state space. If $\dim(\mathrm{aff}(K)) = 2$, then maximally incompatible two-outcome measurements exist on K if and only if K is a square. In general, such measurements exist only if there is an affine subspace $V \subset \mathrm{aff}(K)$, $\dim(V) = 2$, such that $S = V \cap K$ is a square.

Proof. The first statement is immediate from Theorem 1. The principal idea for the second statement is that $V = \mathrm{aff}(S)$, where S is the square in question. Let us assume that there exist maximally incompatible measurements m_f and m_g on K and let S be the square as in Theorem 1, with vertices $x_{00},x_{01},x_{10},x_{11}$. Let $V = \mathrm{aff}(x_{00},x_{10},x_{01})$ and let F_0,F_1 and G_0,G_1 be the parallel exposed faces of K, containing the edges of S. It is easy to see that $F_i \cap V$ and $G_i \cap V$ are faces of $K \cap V$ and they coincide with the edges of S. It is now obvious that $S = V \cap K$. ■

We will present an example to show that the condition in Corollary 2 is not sufficient, even if the square $V \cap K$ is an exposed face of K.

Example 5. Let $K \subset \mathbb{R}^3$ be defined as

$$K = \mathrm{conv}\{(0,0,0),(2,0,0),(0,2,0),(2,1,0),(1,2,0),(1,1,1), \times (1,0,1),(0,1,1),(0,0,1)\}$$

(see Fig. 1). Let

$$V = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_3 = 1\}.$$

Then $K \cap V = S$, where

$$S = \mathrm{conv}\{(1,1,1),(1,0,1),(0,1,1),(0,0,1)\}$$

is an exposed face and a square.

To see that there is not a pair of maximally incompatible measurements m_f and m_g, corresponding to S, it is enough to realize that the effects f and g would have to reach the values 0 and 1 on maximal faces that are not parallel, i.e., we would have to have $\mathrm{aff}(F_0) \cap \mathrm{aff}(F_1) \neq \emptyset$ and $\mathrm{aff}(G_0) \cap \mathrm{aff}(G_1) \neq \emptyset$, which is impossible. On the other hand, the examples of a double pyramid or a cylinder show that maximally incompatible two-outcome measurements may exist on K even if the square $V \cap K$ described in Corollary 2 is not a face of K.

VI. DISCRIMINATION MEASUREMENTS

We now introduce a type of measurement that will allow us to formulate the conditions for existence of maximally incompatible two-outcome measurements in a clearer way.

Definition 6. We say that a two-outcome measurement m_f discriminates the sets $E_0,E_1 \subset K$ if

$$E_0 \subset f^{-1}(0), \quad E_1 \subset f^{-1}(1).$$

We call such measurement a discrimination measurement.

The idea of the definition is simple: Assume that a system is in an unknown state, but we know that it belongs to either E_0 or E_1. By performing the discrimination measurement m_f we can tell with 100% accuracy whether the state of the system belongs to E_0 or E_1. Clearly, m_f is a discrimination measurement if and only if both $f^{-1}(0)$ and $f^{-1}(1)$ are nonempty.

The definition can be generalized in an obvious way to general measurements that can discriminate more than two sets. The most well-known discrimination measurements used in quantum mechanics are projective measurements consisting of rank-1 projections that discriminate the states corresponding to the projections.

We are ready to reformulate the necessary and sufficient condition for maximal incompatibility of two-outcome measurements.

Proposition 6. The measurements m_f and m_g are maximally incompatible if and only if there is a square $S \subseteq K$ such that m_f and m_g discriminate the opposite edges of S.

Proof. First assume that there is a square S whose opposite edges can be discriminated by measurements m_f and m_g. Denote the vertices of S by x_{00}, x_{10}, x_{01}, and x_{11}. Then it is clear that the requirements of Proposition 3 are satisfied and thus we must have $\mathrm{DegCom}(m_f,m_g) = \frac{1}{2}$.

Conversely, assume that $\mathrm{DegCom}(m_f,m_g) = \frac{1}{2}$. Then by Proposition 5 there is a square S with vertices x_{00}, x_{10}, x_{01}, and x_{11} such that $x_{i0},x_{i1} \in F_i$ and $x_{0j},x_{1j} \in G_j$, $i,j \in \{0,1\}$. By convexity, m_f discriminates the edges between x_{00},x_{01} and x_{10},x_{11} and similarly m_g discriminates the other parallel pair. ■

We finish with a necessary condition for compatibility of two-outcome discrimination measurements. Assume that m_f and m_g are such that $F_i \cap G_j \neq \emptyset$ (obviously, m_f and m_g are discrimination measurements in this case) and let $x_{ij} \in F_i \cap G_j$, $i,j \in \{0,1\}$. Assume also that m_f and m_g are compatible, so that (1)–(3) hold. Inserting $x_{11} \in F_1 \cap G_1$ into Eq. (3), we get

$$p(x_{11}) \geqslant 1,$$

which together with $p \in E(K)$ implies $p(x_{11}) = 1$. Equations (1) and (2) and the positivity of p imply

$$p(x_{00}) = p(x_{10}) = p(x_{01}) = 0.$$

Expressing also the functions $(f - p)$, $(g - p)$, and $(1 + p - f - g)$ on the points x_{00}, x_{10}, x_{01}, and x_{11}, we get

$$\begin{aligned}
(f - p)(x_{00}) = (f - p)(x_{01}) = (f - p)(x_{11}) &= 0, \\
(f - p)(x_{10}) &= 1, \\
(g - p)(x_{00}) = (g - p)(x_{10}) = (g - p)(x_{11}) &= 0, \\
(g - p)(x_{01}) &= 1, \\
(1 + p - f - g)(x_{00}) = (1 + p - f - g)(x_{10}) &= 0, \\
(1 + p - f - g)(x_{01}) &= 0, \\
(1 + p - f - g)(x_{11}) &= 1.
\end{aligned}$$

This shows that the joint measurement m given by (4) discriminates the points x_{00}, x_{10}, x_{01}, and x_{11}. In particular, this implies that these points must be affinely independent. We have proved the following.

Proposition 7. Let m_f and m_g be compatible discrimination measurements such that $F_i \cap G_j \neq \emptyset$ for all $i,j \in \{0,1\}$. Then any joint measurement of m_f and m_g must discriminate $F_i \cap G_j$.

VII. CONCLUSION

We have shown that the existence of maximally incompatible two-outcome measurements in GPT is equivalent to a geometric condition on the state space K. The essence of this condition is covered by the example of square state space in [5]. The importance of this result lies in its connection to maximal violation of Bell's inequalities and therefore to possible realizations of Popescu-Rorlich boxes [18], which are studied as potentially powerful resources in information theory. The example of the state space $\mathcal{C}(\mathcal{H})$ (Example 3) is particularly interesting in this respect, since it shows that maximal incompatibility can be achieved by devices existing in quantum theory.

The geometric interpretation of the minimal degree of compatibility that can be attained on a state space K is an interesting question for future research. It would be also of interest whether the connection between discriminating certain sets and compatibility of measurements could be fruitful from an information-theoretic viewpoint. This area of research might also yield some insight into why there exist maximally incompatible measurements on quantum channels although they do not exist on quantum states.

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[1] T. Heinosaari, T. Miyadera, and M. Ziman, J. Phys. A: Math. Theor. **49**, 123001 (2015).
[2] H. M. Wiseman, S. J. Jones, and A. C. Doherty, Phys. Rev. Lett. **98**, 140402 (2007).
[3] M. Plávala, Phys. Rev. A **94**, 042108 (2016).
[4] T. Heinosaari, J. Schultz, A. Toigo, and M. Ziman, Phys. Lett. A **378**, 1695 (2014).
[5] P. Busch, T. Heinosaari, J. Schultz, and N. Stevens, Europhys. Lett. **103**, 10002 (2013).
[6] N. Stevens and P. Busch, Phys. Rev. A **89**, 022123 (2014).
[7] M. Banik, M. R. Gazi, S. Ghosh, and G. Kar, Phys. Rev. A **87**, 052125 (2013).
[8] M. Sedlák, D. Reitzner, G. Chiribella, and M. Ziman, Phys. Rev. A **93**, 052323 (2016).
[9] G. Chiribella, G. M. D'Ariano, and P. Perinotti, Phys. Rev. A **81**, 062348 (2010).
[10] T. Heinosaari and M. Ziman, *The Mathematical Language of Quantum Theory: From Uncertainty to Entanglement* (Cambridge University Press, Cambridge, 2012).
[11] R. T. Rockafellar, *Convex Analysis* (Princeton University Press, Princeton, 1997).
[12] L. Asimow and A. J. Ellis, *Convexity Theory and its Applications in Functional Analysis* (Academic, New York, 1980), Theorem 4.3.
[13] A. Jenčová, J. Math. Phys. **53**, 012201 (2012).
[14] M. Ziman, Phys. Rev. A **77**, 062112 (2008).
[15] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (Scuola Normale Superiore, Pisa, 2011), Chap. 2.
[16] E. Haapasalo, J. Phys. A: Math. Theor. **48**, 255303 (2015).
[17] M. M. Wolf, D. Perez-Garcia, and C. Fernandez, Phys. Rev. Lett. **103**, 230402 (2009).
[18] S. Popescu and D. Rohrlich, Found. Phys. **24**, 379 (1994).