December 12, 2019

Dear Editors,

We are very grateful to you and the Referees for your time in carefully assessing our manuscript. In particular, we thank the Referees for their appreciation of the importance and interest of our work, and we thank you for judging its acceptance likely, once revised.

The main criticisms of both Referees focus on the presentation of the proofs of our theorems in the previous revision. In particular:

1. The proofs were relegated to the Supplemental Material.
2. The proofs were formulated in a language, that of bilinear theories, that differ from the standard formalism of quantum information theory adopted in the main text, and that was not properly introduced.
3. No cohesive presentation and no intuitive insight was provided for the various lemmas that comprise each proof.

Accordingly, these are the main changes that we have implemented in the new revision of our manuscript (same numbering as above):

1. We have reorganized our manuscript in two main sections, the former containing our main results, and the latter containing our technical proofs. The new layout of the paper is described in a new paragraph at the end of the introduction, beginning with "The paper is structured as follows."
2. At the beginning of the Section "Proofs of Theorems 1 and 2", we have added an extensive discussion on the formalism of bilinear theories. This should provide a smooth transition from the standard formalism of quantum information theory adopted in the previous Section "Main results".
3. We have provided a cohesive and, hopefully, insightful discussion of our proofs by adding, before each of the lemmas that comprise our proofs, an explanation of the role played by the lemma for the proof of the theorem.

The other criticisms that are common to the reports of both Referees are some typos and unclear notation that made it difficult to understand our results. Following the recommendation of the Referees, we have improved the notation and fixed several typos.

Additionally, we reformatted our manuscript using the Quantum Journal class.

We sincerely hope to have addressed all of the Referees criticisms in the new revision of our manuscript, and we believe that, as a result, the new presentation is significantly clearer. We therefore hope that the new revision of our manuscript will be considered suitable for publication in Quantum Journal.

In the following, we address the comments of each Referee, and we list in detail the changes we made accordingly.

Referee 1

> **Summary: what are the main questions posed by the manuscript and how does it answer them?**
>
> The paper aims to extend the Alberti-Ulhmann criterion beyond that of qubit dichotomies. The original work by Alberti and Ulhmann showed a result for the transformation of a pair of qubits, which was an extension of the work by Blackwell to the quantum regime. This work still deals with qubits (mainly) but goes beyond just pairs. The methodology is more general than that of Alberti-Ulhmann and thus may pave the way to a more general result as well.

We are very grateful to the Referee for his time in assessing our manuscript and for their appreciation of the generality of our methodology.

> **What is your assessment of the paper? If you recommend acceptance, make a case that this work does indeed make a significant technical or conceptual contribution to scholarship (including experimental methods and/or mathematical tools).**
>
> The mathematical methodology used by the authors is more amenable to being extended. I recommend that the paper be revised and resubmitted for a few reasons. While the results are important, there are several typos and steps with insufficient explanation.

We thank the Referee for their appreciation of the extensibility of our methodology and the importance of our results, and for pointing out typos and shortcomings in our presentation, that greatly helped us improving the presentation of our results.

> **To what extent have you checked the technical correctness of the paper?**
>
> I have checked the paper for technical correctness by going through the proofs and supplemental material thoroughly. Firstly, there are typos in some results (E.g. Lemma 3 and Lemma 4). Furthermore, there are several steps in the supplemental material that need further explanation (E.g. the fact that S_1 can be written as a transposition map implies that it is purely real, etc.).

We have fixed the following typos in the Lemmas mentioned by the Referee. In Lemma 3, we replaced the superscript k with i in the right hand side of both equations. Towards the end of the proof of Lemma 4, we replaced the subscript k with superscript k in the right hand side of both equations, and we replaced the equation $S_0 = CS_1$ with $S_0 = S_1C$.

As detailed towards the beginning of this reply letter, we have clarified the connection between the formalism of quantum information theory adopted in the Section "Main results" and the formalism of bilinear theories adopted in the Section "Proofs of Theorems 1 and 2" (e.g. the equivalence mentioned by

the Referee between $\{\rho_x\}$ being real and $S_1 T = S_1$), by adding an extensive discussion in the latter section.

> **Comment on the presentation of the paper. Is it well written? Are the main results clearly laid out? Does the manuscript clearly describe assumptions and limitations? Is the literature review adequate?**
>
> The paper needs to be expanded. The main results, Theorem 1 and Theorem 2, should be proved in one cohesive unit each instead of being split. The definitions of Family of states, statistical morphisms, etc. need further explanation and an indication of their relevance to standard quantum theory. Assumption and limitations are laid out well.

As recommended by the Referee, we have significantly expanded the discussion around Theorem 1 and 2. This includes additional explanations among the various Lemmas that comprise each proof, in order to provide a cohesive presentation of the proofs, and further explanations of the formalism of bilinear theories (e.g. state and statistical morphisms) at the beginning of Section "Proofs of Theorems 1 and 2". More details are provided towards the beginning of this reply letter.

> **If the submission includes numerical or physical experiments, does it provide sufficient details such that they could be reproduced by readers? This includes for example source code, documentation, experimental data, experimental setup specifications, etc.**
>
> —

> **Suggested changes, corrections, and general comments.**
>
> To prevent typos, like the ones in Lemma 3 and 4, the paper needs to be checked thoroughly. If each line of the various proofs is offered with an explanation, it will prove beneficial to any reader. For example, in Lemma 4, the line with "$S_1^k = Lambda^k s_1^0 + (1 - Lambda^k) s_1^1$", the second term as subscript k instead of superscript k, and the line is given no explanation. Furthermore, the last line has $S_0 = CS_1$, where it should be $S_1 C$.
>
> Overall, I recommend a deeper review of the minutia of the manuscript before resubmission.

Regarding the specific criticisms about the proof of Lemma 4, we have fixed both the typos reported by the Referee and added an explanation of the equation reported by the Referee immediately after the equation itself. We have also fixed some other typos, as reported in this reply letter.

More generally, we have provided extensive explanation of the intuition behind each proof by adding several paragraphs in between the Lemmas that constitute each proof, as detailed towards the beginning of this reply letter.

We are very grateful to the Referee for their insightful comments, that allowed us to significantly improve the presentation of our results. We sincerely hope

that the new revision of our work will be considered suitable for publication in Quantum Journal.

Referee 2

> **Summary: what are the main questions posed by the manuscript and how does it answer them?**
>
> Given two indexed families of quantum states, is there a quantum channel transforming one to the other? The Alberti-Uhlmann theorem gives a criterion for the case of pairs of qubit states, based solely on outcome probabilities of tests. The authors present some conditions under which this criterion can be extended to larger families of qubit states (the target states are sometimes allowed to be qutrits). A similar question is solved for qubit measurements. The results are then applied to semi device independent simulability of families of states and measurements, which can be described as the problem of certifying that some set of states (measurements) can be simulated by another set (measurement) of known dimension, based solely on the input-output correlations obtained by applying an unknown device to the latter set (measurement).

We are very grateful to the Referee for their time in assessing our manuscript, and we are happy to see that our presentation clearly conveyed the results of our work.

> **What is your assessment of the paper? If you recommend acceptance, make a case that this work does indeed make a significant technical or conceptual contribution to scholarship (including experimental methods and/or mathematical tools).**
>
> An extension of the Alberti-Uhlmann theorem to higher dimensions and/or larger families of states was studied by many authors from various perspectives, but the simple criterion is in general not sufficient already for qutrits or triples of states, and more complicated conditions have to be checked, which can become very difficult. The paper contributes to this study in the quite particular case of real qubit families. If the target states are also assumed to be qubits, this statement is rather straightforward: it can be seen from the fact that the linear span of a real family of qubits has dimension at most 2, which allows us to apply the Alberti-Uhlmann criterion to any pair that spans it and extend the resulting map over the rest. The rest of the results is, besides the decomposability properties of positive maps on qubits, based on simple linear algebra: namely the well known fact that a surjective map has a one-sided inverse. Nevertheless, to my knowledge, this extension has not been observed before and is an interesting contribution to the important problem of comparison and simulability for sets of quantum states and measurements. However, I had some difficulties in reading the paper and a careful revision is needed.

We are very grateful to the Referee for their appreciation of the importance of the problem we address and of the novelty and interest of our results, and for pointing our various shortcomings in our presentation, which greatly helped us improving the readability of our work.

> **To what extent have you checked the technical correctness of the paper?**
> I have checked all the proofs and found them correct.

We thank the Referee for their time in verifying the correctness of our results.

> **Comment on the presentation of the paper. Is it well written? Are the main results clearly laid out? Does the manuscript clearly describe assumptions and limitations? Is the literature review adequate?**
> The paper is not very well written. The main results are quite well explained, but the proofs are in a Supplemental material which uses a setting that is not explicitly related to the studied problem, and are, in my opinion, not as clear and efficient as they should be. There are also some confusing notations, repetitions, careless copy-pasting, etc., see the comments below. The authors could also make a stronger case for the significance of their results in the introduction. The bibliography seems adequate.

We thank the Referee for their criticisms about our presentation, that focus on the presentation of the proofs of our results, and can be summarized as:

1. the proofs being relegated to the Supplemental Material;
2. the proofs being based on a formalism different from that adopted in the main text;
3. the proofs not being as clear as they should be.

To address such criticisms, we have implemented the following changes (numbering follows that of the Referee's criticisms):

1. we moved the proofs of our results to the Section "Proofs of Theorems 1 and 2" within the main text;
2. at the beginning of Section "Proofs of Theorems 1 and 2", we provide an extensive introduction of the formalism of bilinear theories therein adopted and we justify its adoption;
3. we have added several paragraphs among the lemmas that constitute the proofs of our results, with the aim of providing insight on the proof techniques and cohesion to the presentation.

These changes are detailed toward the beginning of this reply letter.

We have also fixed the notation, repetitions, and typos reported by the Referee, as detailed below.

If the submission includes numerical or physical experiments, does it provide sufficient details such that they could be reproduced by readers? This includes for example source code, documentation, experimental data, experimental setup specifications, etc.

The paper is entirely theoretical, no experiments are included.

Suggested changes, corrections, and general comments.

1. I personally dislike the habit of just stating the theorems in the main text and banishing the proofs into a "Supplemental material". Without proofs, there are no theorems, and it can be annoying for the readers to go back and forth reading the statements and searching for the proofs. This can be justified for some really technical parts, but this is not the case here.
2. The setting of the Supplemental material is not clearly related to the problem studied in the main text. Better spend a short paragraph or two to explain this. Also notations such as S^+ are not explained. I would suggest to put the results into the general setting in the main text and include the proofs, leaving perhaps some technicalities about linear maps for the SM.
3. The proofs should be made clearer and more efficient: For example, the result of Thm. 1 (proved in Lemmas 3 and 4) follows by a straightforward argument (described above). This argument is in fact used here, but it is not clear from the exposition of the two lemmas. Also, both Lemma 2 and Lemma 5 are based on simple facts about linear maps with including ranges, which form a large part of both proofs. I think this simplicity is one of the appealing aspects of the paper and the authors should emphasise it.
4. confusing notations: for example, the vectors p_y with entries indexed by states, and p_x with entries indexed by effects are designed to get confused (in fact, I think it should be p_y in Coro. 1).
5. Further remarks:
   - p. 3, column 1, "Also the convex hull..." there is something wrong with this sentence
   - on p.3, column 2, instead of showing that the free energy is monotone in the area of the range, the argument here seems to show that both the free energy and the area are monotone in the variable a
   - p. 3, column 2: what is $\mathcal{E}$ here?
   - p. 6, column 2, paragraph before Lemma 2 (some copy-pasting?)
   - proof of Lemma 2: $M_0 = M_1 C ... M_i$ instead of S_i

We thank the Referee for their insightful criticisms of the presentation of our results. To address them, we made the following changes to the new version of

our manuscript (the numbering follows that of the Referee's criticisms):

1. We moved the proofs of our theorems to the Section "Proofs of Theorems 1 and 2" within the main text of our manuscript, and we smoothly integrated them by adding an explanation of the formalism therein adopted, as detailed towards the beginning of this reply letter.
2. As the Referee correctly pointed out, the formalism used in the new Section "Proofs of Theorems 1 and 2" is that of bilinear theories, that generalizes that of quantum information theory adopted in the Section "Main results". We explained the former formalism and justified its adoption extensively at the beginning of the Section "Proofs of Theorems 1 and 2".
3. We have included, before each lemma, a paragraph or two providing insight on the proof techniques adopted. Also, to emphasize that our results follow naturally within the formalism of bilinear theories, we have added the following sentence to the introduction: "Our results follow as a natural consequence of the Woronowicz decomposition [23] of linear maps, once families of states and measurements are regarded as linear transformations."
4. In the new version of our manuscript we adopt different notations for the two types of vectors. In particular, while discussing the simulability of families of states we use the notation $\mathbf{q}_y$ for vectors indexed by the choice of an effect, and while discussing the simulability of measurements we use the notation $\mathbf{p}_x$ for vectors indexed by the choice of a state. We also replaced $\mathbf{p}_x$ with $\mathbf{q}_y$ in Corollary 1, and we thank the Referee for pointing out this typo.
5. - For clarity, we rephrased the sentence reported by the Referee as follows: "Conversely, for any (possibly degenerate) ellipsoid centered in $\mathbf{u}/2$ and contained in the hypercube $[0,1]^m$, its convex hull with 0 and $\mathbf{u}$ is the testing region of a qubit family of states."
   - We have replaced the sentence " Since $\lambda_\pm = 1/2 \pm a$, where a is the non-null semi-axis of $\mathcal{E}(\{\rho_x\})$, and the volume of the range of $\{\rho_0, \mathbb{1}/2\}$ is of course a monotone in a, the statement remains proved." with the sentence "By setting $\lambda_\pm = 1/2 \pm a$, by explicit computation it immediately follows that the volume of the range of $\{\rho_0, \mathbb{1}/2\}$ is proportional to a, hence the statement remains proved."
   - This criticism should be solved by the change reported in the previous point.
   - We fixed the typo reported by the Referee before Lemma 2, and also extended the discussion before such lemma to provide insight on the proof technique.
   - We fixed the typo reported by the Referee.

We are very grateful to the Referee for their insightful comments, that allowed us to significantly improve the presentation of our results. We sincerely hope that the new revision of our work will be considered suitable for publication in

Quantum Journal.

Yours sincerely,

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Francesco Buscemi
Valerio Scarani