



Rényi Relative Entropies and Noncommutative L_p -Spaces II

Anna Jenčová 

Abstract. We study an extension of the sandwiched Rényi relative entropies for normal positive functionals on a von Neumann algebra, for parameter values $\alpha \in [1/2, 1)$. This work is intended as a continuation of Jenčová (Ann Henri Poincaré 19:2513–2542, 2018), where the values $\alpha > 1$ were studied. We use the Araki–Masuda divergences of Berta et al. (Ann Henri Poincaré 9:1843–1867, 2018) and treat them in the framework of Kosaki’s noncommutative L_p -spaces. Using the variational formula, recently obtained by F. Hiai, for $\alpha \in [1/2, 1)$, we prove the data processing inequality with respect to positive trace preserving maps and show that for $\alpha \in (1/2, 1)$, equality characterizes sufficiency (reversibility) for any 2-positive trace preserving map.

1. Introduction

In [12], we introduced and studied an extension \tilde{D}_α of the sandwiched Rényi relative entropy for $\alpha > 1$, from density matrices to positive normal functionals on a von Neumann algebra \mathcal{M} . These quantities were defined using the noncommutative L_p -spaces due to Kosaki [14]. We proved a number of properties of \tilde{D}_α , in particular the data processing inequality (DPI) with respect to any positive trace preserving map $\Phi: L_1(\mathcal{M}) \rightarrow L_1(\mathcal{N})$. Moreover, we proved that if Φ is 2-positive, then equality in DPI is equivalent to the fact that Φ is sufficient with respect to $\{\psi, \varphi\}$.

A similar extension was obtained in [3] for the larger interval of parameters $\alpha \in [1/2, 1) \cup (1, \infty]$. These quantities were called the Araki–Masuda divergences because the definition is based on the noncommutative L_p -spaces due to Araki and Masuda [1]. We showed in [12] that for $\alpha > 1$, these quantities coincide with \tilde{D}_α .

More recently, an extension to all $\alpha \in (0, 1) \cup (1, \infty)$ was proposed in [9], using interpolation of quasi Banach spaces. Recall that as shown in [16], DPI cannot hold for $\alpha < 1/2$.

The aim of the present work is to continue [12] by the study of \tilde{D}_α for the parameter values $\alpha \in [1/2, 1)$. We use the Araki–Masuda divergences of [3], but we show that these can be obtained using Kosaki’s right L_p -spaces, for all $\alpha \in [1/2, \infty] \setminus \{1\}$. Most of the properties denoted by (a)–(h) in [12] for all these values, or some weaker versions for $\alpha \in [1/2, 1)$, were proved in [3, 10] (see [10, Theorem 3.16]). In particular, the DPI with respect to quantum channels was proved for $\alpha \in [1/2, 1)$. Also, a variational expression was obtained for $\alpha \in (0, 1)$ in [10, Lemma 3.19], generalizing the expression obtained in [8] to the setting of von Neumann algebras.

We complete these results as follows. We add the lower bound in property (d) in [12] (describing the relation to the standard Rényi relative entropy). We complete the variational formula of [10, Lemma 3.19] to $\alpha > 1$. Moreover, using the variational formula, we prove that also for $\alpha \in [1/2, 1)$, DPI holds for all *positive* trace preserving maps, which seems new even in the finite-dimensional case.

We also study equality in DPI. We prove that as in the case $\alpha > 1$ ([12, Theorem 4.6]), equality in DPI for \tilde{D}_α , $\alpha \in (1/2, 1)$ implies sufficiency for any 2-positive trace preserving map.

This paper is intended as a continuation of [12] and the notations, definitions and results from there will be used repeatedly. We begin in Sect. 2 by showing how the weighted L_p -norms of [3] can be introduced in the standard form $(\lambda, L_2(\mathcal{M}), J, L_2(\mathcal{M})^+)$ by using Kosaki’s right L_p -spaces with respect to a faithful positive normal functional. In Sect. 3, we introduce \tilde{D}_α for $\alpha \in [1/2, 1)$ and extend the variational formula to $\alpha > 1$. In Sect. 4, we prove the DPI with respect to positive trace preserving maps. Sect. 5 is devoted to DPI equality conditions.

2. Interpolation L_p -Spaces for von a Neumann Algebra

Let \mathcal{M} be a (σ -finite) von Neumann algebra. We will denote the predual of \mathcal{M} by \mathcal{M}_* , the positive cone in \mathcal{M}_* by \mathcal{M}_*^+ and the set of normal states by $\mathfrak{S}_*(\mathcal{M})$. For $\varphi \in \mathcal{M}_*^+$, $s(\varphi)$ denotes the support of φ . The Haagerup L_p -spaces with $1 \leq p \leq \infty$ will be denoted by $L_p(\mathcal{M})$. The isomorphism $\mathcal{M}_* \simeq L_1(\mathcal{M})$ is determined as $\varphi \mapsto h_\varphi$ and we define $\text{Tr } h_\varphi = \varphi(1)$. The inner product in the Hilbert space $L_2(\mathcal{M})$ is defined as

$$(\xi, \eta) = \text{Tr } \eta^* \xi, \quad \xi, \eta \in L_2(\mathcal{M})$$

and we use the notation ω_η for the functional $\omega_\eta(a) = (a\eta, \eta)$, $a \in \mathcal{M}$. We will also assume the standard form $(\lambda(\mathcal{M}), L_2(\mathcal{M}), J, L_2(\mathcal{M})^+)$ as described at the beginning of [12, Sec. 2].

Let $\varphi_0 \in \mathcal{M}_*^+$ be faithful. We begin with the definition of the family of L_p -spaces with respect to φ_0 as introduced by Kosaki [14]. For $0 \leq \eta \leq 1$,

consider the continuous embedding $\mathcal{M} \rightarrow L_1(\mathcal{M})$, defined by

$$x \mapsto h_{\varphi_0}^\eta x h_{\varphi_0}^{1-\eta}, \quad x \in \mathcal{M}.$$

The range of this embedding, endowed with the norm

$$\|h_{\varphi_0}^\eta x h_{\varphi_0}^{1-\eta}\|_{\infty, \varphi_0}^\eta := \|x\|,$$

will be denoted by \mathcal{M}^η . For $1 \leq p \leq \infty$, $L_p^\eta(\mathcal{M}, \varphi_0)$ is defined as the interpolation space [2]

$$L_p^\eta(\mathcal{M}, \varphi_0) := C_{1/p}(\mathcal{M}^\eta, L_1(\mathcal{M})),$$

see also [12, Appendix B] for a brief description of the interpolation space C_θ , $\theta \in [0, 1]$. According to [14, Thm. 9.1], the map

$$i_{p, \varphi_0}^\eta: L_p(\mathcal{M}) \ni k \mapsto h_{\varphi_0}^{\eta/q} k h_{\varphi_0}^{(1-\eta)/q}, \quad 1/p + 1/q = 1$$

is an isometric isomorphism of $L_p(\mathcal{M})$ onto $L_p^\eta(\mathcal{M}, \varphi_0)$.

In [12, Sec. 2], we used the *symmetric* L_p -space with $\eta = 1/2$, which was there denoted by $L_p(\mathcal{M}, \varphi_0)$. In this section, we will concentrate on the *right* L_p -space, where $\eta = 1$. We will denote this space by $L_p^R(\mathcal{M}, \varphi_0)$ and its norm by $\|\cdot\|_{p, \varphi_0}^R$. Note that the above isometric isomorphism implies that $L_p^R(\mathcal{M}, \varphi_0) \subseteq L_1(\mathcal{M})$ is the (dense) subspace of operators of the form

$$h = h_{\varphi_0}^{1/q} k, \quad k \in L_p(\mathcal{M}), \quad \|h\|_{p, \varphi_0}^R = \|k\|_p.$$

In particular, $L_\infty^R(\mathcal{M}, \varphi_0) \simeq \mathcal{M}$ and $L_1^R(\mathcal{M}, \varphi_0) = L_1(\mathcal{M})$. Moreover, if $1 < p < \infty$, $h = h_{\varphi_0}^{1/q} k$ and $k = h_\mu^{1/p} u$ is the polar decomposition of k in $L_p(\mathcal{M})$, then

$$f_{p, h, \varphi_0}^R(z) = \mu(1)^{1/p-z} h_{\varphi_0}^{1-z} h_\mu^z u, \quad z \in S, \quad (1)$$

is a function in $\mathcal{F}(\mathcal{M}^1, L_1(\mathcal{M}))$ such that

$$f_{p, h, \varphi_0}^R(1/p) = h, \quad \|h\|_{p, \varphi_0}^R = \|f\|_{\mathcal{F}},$$

see [12, Appendix B].

For $1 < p \leq \infty$ and $1/p + 1/q = 1$, we have the Banach space dual $L_p^R(\mathcal{M}, \varphi_0)^* \simeq L_q^R(\mathcal{M}, \varphi_0)$, with duality given by

$$\langle h_{\varphi_0}^{1/q} k, h_{\varphi_0}^{1/p} l \rangle_{\varphi_0}^R := \text{Tr } kl, \quad k \in L_p(\mathcal{M}), \quad l \in L_q(\mathcal{M}).$$

2.1. Interpolation Norms in $L_2(\mathcal{M})$

We next introduce a family of interpolation (semi)norms in $L_2(\mathcal{M})$. Let $\varphi \in \mathcal{M}_*^+$ with $e := s(\varphi)$ be arbitrary but fixed throughout. Let us also fix some $\sigma \in \mathcal{M}_*^+$ such that $s(\sigma) = 1 - e$, so that $\varphi_0 := \varphi + \sigma$ is faithful. For $\xi \in L_2(\mathcal{M})$ and $1 \leq p \leq \infty$, we define:

$$\|\xi\|_{p, \varphi}^{(2)} = \begin{cases} \|h_\varphi^{1/2} \xi\|_{p, \varphi_0}^R & \text{if } 1 \leq p < 2 \text{ or } \xi = e\xi \\ +\infty & \text{otherwise} \end{cases}$$

(with the understanding that $\|h_\varphi^{1/2} \xi\|_{p, \varphi_0}^R = \infty$ if $h_\varphi^{1/2} \xi \notin L_p^R(\mathcal{M}, \varphi_0)$).

It is immediate from this definition that if $\xi, \eta \in L_2(\mathcal{M})$ are such that $\omega_\eta = \omega_\xi$, then we have $\|\xi\|_{p, \varphi}^{(2)} = \|\eta\|_{p, \varphi}^{(2)}$, for any $1 \leq p \leq \infty$. Indeed, the

condition means that there is a partial isometry $u \in \mathcal{M}$ such that $\eta = \xi u$ and $\xi = \eta u^*$. If $h_\varphi^{1/2} \xi = h_{\varphi_0}^{1/q} k$ with $k \in L_p(\mathcal{M})$, then $h_\varphi^{1/2} \eta = h_{\varphi_0}^{1/q} k u$ with $k u \in L_p(\mathcal{M})$, so that

$$\|\eta\|_{p,\varphi}^{(2)} = \|k u\|_p \leq \|k\|_p \|u\| \leq \|k\|_p = \|\xi\|_{p,\varphi}^{(2)}.$$

Similarly, we obtain $\|\xi\|_{p,\varphi}^{(2)} \leq \|\eta\|_{p,\varphi}^{(2)}$.

Notice that for $p \geq 2$, $\|\cdot\|_{p,\varphi}^{(2)}$ is a norm in the subspace where it is finite and the proposition below shows that this subspace is dense in $eL_2(\mathcal{M})$ and complete with respect to $\|\cdot\|_{p,\varphi}$. For $1 \leq p < 2$, we will see that $\|\cdot\|_{p,\varphi}^{(2)}$ is always finite and defines a seminorm in $L_2(\mathcal{M})$ which is a norm if and only if φ is faithful.

Proposition 2.1. (i) *Let $2 \leq p \leq \infty$. Then, $\|\xi\|_{p,\varphi}^{(2)} < \infty$ if and only if $\xi = h_\varphi^{1/2-1/p} k$ for some $k \in L_p(\mathcal{M})$ with $k = ek$. Moreover, such k is unique and we have $\|\xi\|_{p,\varphi}^{(2)} = \|k\|_p$.*

(ii) *Let $1 \leq p < 2$ and let $k = h_\varphi^{1/p-1/2} \xi$. Then, $k \in L_p(\mathcal{M})$ and $\|\xi\|_{p,\varphi}^{(2)} = \|k\|_p < \infty$.*

Proof. (i) Assume $\|\xi\|_{p,\varphi}^{(2)} < \infty$, then we must have $e\xi = \xi$ and $h_\varphi^{1/2} \xi \in L_p^R(\mathcal{M}, \varphi_0)$. By [14, Thm. 9.1], there is a unique element $k \in L_p(\mathcal{M})$ such that $h_\varphi^{1/2} \xi = h_{\varphi_0}^{1/q} k$, and we have $\|h_\varphi^{1/2} \xi\|_{p,\varphi_0}^R = \|k\|_p$. Since $h_{\varphi_0}^z = h_\varphi^z + h_\sigma^z$ for all $z \in \mathbb{C}$, we have $eh_{\varphi_0}^z = h_{\varphi_0}^z e = h_\varphi^z$, so that $h_\varphi^{1/2} \xi = h_{\varphi_0}^{1/2} \xi$. Since φ_0 is faithful, it follows that

$$\xi = h_{\varphi_0}^{1/2-1/p} k.$$

Similarly, we obtain

$$h_{\varphi_0}^{1/2-1/p} k = \xi = e\xi = h_\varphi^{1/2-1/p} k = h_{\varphi_0}^{1/2-1/p} ek$$

and this implies $k = ek$. Conversely, assume that ξ has a decomposition as required, then clearly $e\xi = \xi$ and

$$h_\varphi^{1/2} \xi = h_{\varphi_0}^{1/q} k = h_{\varphi_0}^{1/q} k.$$

Uniqueness follows by the isometric isomorphism in [14, Thm. 9.1].

(ii) Assume $1 \leq p < 2$, then $1/p - 1/2 = 1/2 - 1/q > 0$ and since $\xi \in L_2(\mathcal{M})$, we have $k = h_\varphi^{1/p-1/2} \xi \in L_p(\mathcal{M})$. Further,

$$h_\varphi^{1/2} \xi = h_{\varphi_0}^{1/2} e\xi = h_{\varphi_0}^{1/q} h_{\varphi_0}^{1/2-1/q} e\xi = h_{\varphi_0}^{1/q} k$$

so that $h_\varphi^{1/2} \xi \in L_p^R(\mathcal{M}, \varphi_0)$ and $\|h_\varphi^{1/2} \xi\|_{p,\varphi_0}^R = \|k\|_p$. \square

From the (right) polar decomposition in $L_p(\mathcal{M})$, we obtain:

Proposition 2.2 (Polar decomposition). *Let $\xi \in L_2(\mathcal{M})$ and $1 < p < \infty$. Then, $\|\xi\|_{p,\varphi}^{(2)} < \infty$ if and only if there is some $\mu \in \mathcal{M}_*^+$ and a partial isometry $u \in \mathcal{M}$, where $u u^* = s(\mu) \leq e$ and $u^* u = \text{supp}(e\xi)$ (the support projection of $e\xi$), such that*

(i) *if $2 \leq p < \infty$, $\xi = h_\varphi^{1/2-1/p} h_\mu^{1/p} u$;*

(ii) if $1 \leq p < 2$, $h_\varphi^{1/p-1/2}\xi = h_\mu^{1/p}u$.

Moreover, such μ and u are unique and we have $\|\xi\|_{p,\varphi}^{(2)} = \mu(1)^{1/p}$.

In the situation of the above proposition, we will say that ξ has the p -polar decomposition $\xi = \mu^{1/p}u$ (with respect to φ).

Proposition 2.3 (Duality). *Let $1 \leq p \leq \infty$, $1/p + 1/q = 1$, $\xi, \eta \in L_2(\mathcal{M})$. Then:*

- (i) $|(\xi, \eta)| \leq \|\xi\|_{p,\varphi}^{(2)} \|\eta\|_{q,\varphi}^{(2)}$.
- (ii) If $\xi = e\xi$ or $1 \leq p < 2$:

$$\|\xi\|_{p,\varphi}^{(2)} = \sup_{\eta \in L_2(\mathcal{M}), \|\eta\|_{q,\varphi}^{(2)} \leq 1} |(\xi, \eta)|.$$

- (iii) If $1 \leq p < 2$ and $\xi = \mu^{1/p}u$ is the p -polar decomposition of ξ , then there is a unique element $\tilde{\xi} \in L_2(\mathcal{M})$ with $\|\tilde{\xi}\|_{q,\varphi}^{(2)} = 1$ such that $\|\xi\|_{p,\varphi}^{(2)} = (\tilde{\xi}, \xi)$. This element has the q -polar decomposition $\tilde{\xi} = (\mu(1)^{-1}\mu)^{1/q}u$.

Proof. We clearly may suppose that both norms on the right-hand side of (i) are finite. Assume that, say $p \geq 2$ and let $k = ek \in L_p(\mathcal{M})$ be such that $\xi = h_\varphi^{1/2-1/p}k$. Then,

$$|(\xi, \eta)| = |\text{Tr } \eta^* h_\varphi^{1/2-1/p}k| \leq \|k\|_p \|h_\varphi^{1/q-1/2}\eta\|_q = \|\xi\|_{p,\varphi}^{(2)} \|\eta\|_{q,\varphi}^{(2)}.$$

From $k = ek$, we also have

$$|(\xi, \eta)| = |\text{Tr } \eta^* h_\varphi^{1/2-1/p}k| = |\text{Tr } \eta^* h_{\varphi_0}^{1/2-1/p}ek|.$$

Since the elements $\zeta h_{\varphi_0}^{1/2-1/p}$ with $\zeta \in L_2(\mathcal{M})$ are dense in $L_q(\mathcal{M})$ and

$$\|k\|_p = \sup_{l=le \in L_q(\mathcal{M}), \|l\|_q \leq 1} |\text{Tr } lk|,$$

we obtain (ii) in the case that $p \geq 2$ and $\|\xi\|_{p,\varphi}^{(2)} < \infty$. Similarly, one can see that if $\xi = e\xi$ and the supremum in (ii) is finite, then the map $h_{\varphi_0}^{1/2}\eta \mapsto (\xi, \eta)$ extends to a bounded linear functional on $L_q^R(\mathcal{M}, \varphi_0)$. Note that we have by Proposition 2.1 (ii) and the Hölder inequality that

$$\|\eta\|_{q,\varphi}^{(2)} = \|eh_{\varphi_0}^{1/q-1/2}\eta\|_q \leq \|h_{\varphi_0}^{1/q-1/2}\eta\|_q = \|h_{\varphi_0}^{1/2}\eta\|_{q,\varphi_0}^R$$

for all $\eta \in L_2(\mathcal{M})$. Hence, there is some element $h = h_{\varphi_0}^{1/q}k \in L_p^R(\mathcal{M}, \varphi_0)$ such that for $\eta \in L_2(\mathcal{M})$,

$$(\xi, \eta) = (\xi, e\eta) = \langle h_{\varphi_0}^{1/q}k, h_{\varphi_0}^{1/2}e\eta \rangle_{\varphi_0}^R = \text{Tr } kh_{\varphi_0}^{1/2-1/p}\eta = (h_{\varphi_0}^{1/2-1/p}k^*, \eta)$$

so that $\xi = h_{\varphi_0}^{1/2-1/p}k^*$ and $\|\xi\|_{p,\varphi}^{(2)} < \infty$. This finishes the proof of (ii) in the case $p \geq 2$ and $\xi = e\xi$.

Assume next that $p < 2$ and let $k = h_\varphi^{1/p-1/2}\xi$. Then, since $k = ek$,

$$\begin{aligned} \|\xi\|_{p,\varphi}^{(2)} &= \|k\|_p = \sup_{l \in L_q(\mathcal{M}), \|l\|_q \leq 1} |\text{Tr } lk| = \sup_{l=le \in L_q(\mathcal{M}), \|l\|_q \leq 1} |\text{Tr } lk| \\ &= \sup_{l=le \in L_q(\mathcal{M}), \|l\|_q \leq 1} |(\xi, h_\varphi^{1/p-1/2}l^*)| = \sup_{\eta \in L_2(\mathcal{M}), \|\eta\|_{q,\varphi}^{(2)} \leq 1} |(\xi, \eta)|. \end{aligned}$$

The statement (iii) follows by the duality of the $L_p(\mathcal{M})$ spaces. \square

2.2. Relation to the Weighted L_p -Norms of Berta–Scholz–Tomamichel

We next recall the definition of the weighted L_p -norms in [3] and show that these are equal to the $\|\cdot\|_{p,\varphi}^{(2)}$ for the standard representation on $L_2(\mathcal{M})$.

Let $\varphi \in \mathcal{M}_*^+$ and let $\pi: \mathcal{M} \rightarrow B(\mathcal{H})$ be any $*$ -representation on a complex Hilbert space \mathcal{H} with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. For $\xi \in \mathcal{H}$, let ω_ξ be the functional given by ξ , that is $\omega_\xi(a) = \langle \pi(a)\xi, \xi \rangle_{\mathcal{H}}$. We also denote by ω'_ξ the corresponding functional on the commutant: $\omega'_\xi(a') = \langle a'\xi, \xi \rangle_{\mathcal{H}}$, $a' \in \pi(\mathcal{M})'$. Let $\Delta(\xi/\varphi)$ denote the spatial derivative introduced in [5] (see also [3, Section 2.2], [12, Appendix A.2]). The φ -weighted p -norm of $\xi \in \mathcal{H}$ is defined as:

1. for $2 \leq p \leq \infty$, we define

$$\|\xi\|_{p,\varphi}^{BST} := \sup_{\zeta \in \mathcal{H}, \|\zeta\|=1} \|\Delta(\zeta/\varphi)^{1/2-1/p}\xi\|$$

if $s(\omega_\xi) \leq s(\varphi)$ and $+\infty$ otherwise. Note that the supremum can be infinite also when the condition on the supports holds.

2. for $1 \leq p < 2$, we define

$$\|\xi\|_{p,\varphi}^{BST} := \inf_{\zeta \in \mathcal{H}, \|\zeta\|=1, s(\omega'_\zeta) \geq s(\omega'_\xi)} \|\Delta(\zeta/\varphi)^{1/2-1/p}\xi\|.$$

According to [3], this quantity depends only on the functionals φ and ω_ξ and not on the representation π or the representing vector ξ . Moreover, similar duality relation holds as those in Proposition 2.3, in particular for any representation $\pi: \mathcal{M} \rightarrow B(\mathcal{H})$ and any $\xi, \eta \in \mathcal{H}$,

$$|\langle \xi, \eta \rangle_{\mathcal{H}}| \leq \|\xi\|_{p,\varphi}^{BST} \|\eta\|_{q,\varphi}^{BST}. \quad (2)$$

Let us now assume that $\mathcal{H} = L_2(\mathcal{M})$ and $\pi = \lambda: \mathcal{M} \rightarrow B(L_2(\mathcal{M}))$ is the representation by left multiplication. By [12, Appendix A.2] (notice a small mistake there), we have for $\eta \in L_2(\mathcal{M})$

$$\Delta(\eta/\varphi) = F_{\eta, h_\varphi^{1/2}}^* \bar{F}_{\eta, h_\varphi^{1/2}} = J \Delta_{\omega, \varphi} J,$$

where $\omega = \omega_{\eta^*}$. It follows that for all $\xi \in L_2(\mathcal{M})$, we have

$$\begin{aligned} \|\xi\|_{p,\varphi}^{BST} &= \begin{cases} \sup_{\omega \in \mathfrak{S}_*(\mathcal{M})} \|\Delta_{\omega, \varphi}^{1/2-1/p} \xi^*\|_2 & \text{if } s(\omega_\xi) \leq s(\varphi) \\ +\infty & \text{otherwise} \end{cases}, \quad 2 \leq p \leq \infty, \\ \|\xi\|_{p,\varphi}^{BST} &= \inf_{\omega \in \mathfrak{S}_*(\mathcal{M}), s(\omega) \geq s(\omega_{\xi^*})} \|\Delta_{\omega, \varphi}^{1/2-1/p} \xi^*\|_2, \quad 1 \leq p < 2. \end{aligned}$$

Proposition 2.4. *Let $\xi \in L_2(\mathcal{M})$, $\omega = \omega_\xi$. We have*

$$\|\xi\|_{p,\varphi}^{BST} = \|\xi\|_{p,\varphi}^{(2)} = \|h_\omega^{1/2}\|_{p,\varphi}^{(2)}.$$

Proof. Since both norms depend only on ω , we may suppose that $\xi = h_\omega^{1/2}$. Assume first that $1 \leq p < 2$. Using the properties of the relative modular

operator [12, Appendix A.1], we see that

$$\begin{aligned}\|h_\omega^{1/2}\|_{p,\varphi}^{BST} &= \inf_{\psi \in \mathfrak{S}_*(\mathcal{M}), s(\psi) \geq s(\omega)} \|\Delta_{\psi,\varphi}^{1/2-1/p} h_\omega^{1/2}\|_2 \\ &= \inf_{\psi \in \mathfrak{S}_*(\mathcal{M}), s(\psi) \geq s(\omega)} \|J\Delta_{\varphi,\psi}^{1/p-1/2} Jh_\omega^{1/2}\|_2 \\ &= \inf_{\psi \in \mathfrak{S}_*(\mathcal{M}), s(\psi) \geq s(\omega)} \|\Delta_{\varphi,\psi}^{1/p-1/2} h_\omega^{1/2}\|_2.\end{aligned}$$

Assume that $\psi \in \mathfrak{S}_*(\mathcal{M})$ is such that $h_\omega^{1/2} \in \mathcal{D}(\Delta_{\varphi,\psi}^{1/p-1/2})$, which by [12, Eq. (A.3)] means that there is some $\eta \in L_2(\mathcal{M})$ such that $h_\varphi^{1/p-1/2} h_\omega^{1/2} = \eta h_\psi^{1/p-1/2}$ and then $\Delta_{\varphi,\psi}^{1/p-1/2} h_\omega^{1/2} = \eta$. By Proposition 2.1 (ii) and the Hölder inequality, we obtain

$$\|h_\omega^{1/2}\|_{p,\varphi}^{(2)} = \|h_\varphi^{1/p-1/2} h_\omega^{1/2}\|_p = \|\eta h_\psi^{1/p-1/2}\|_p \leq \|\eta\|_2 = \|\Delta_{\varphi,\psi}^{1/p-1/2} h_\omega^{1/2}\|_2.$$

This shows that $\|h_\omega^{1/2}\|_{p,\varphi}^{(2)} \leq \|h_\omega^{1/2}\|_{p,\varphi}^{BST}$. Conversely, let $h_\omega^{1/2} = \mu^{1/p} u$ be the p -polar decomposition with respect to φ . Put $\psi = \mu(1)^{-1} \mu(u \cdot u^*)$, then $\psi \in \mathfrak{S}_*(\mathcal{M})$, but note that in general we have $s(\psi) = u^* u \leq s(\omega)$. Here the inequality follows from the fact that $u^* u$ is the right support of the operator $k = h_\varphi^{1/p-1/2} h_\omega^{1/2}$ and $ks(\omega) = k$. Let $\psi_0 \in \mathfrak{S}_*(\mathcal{M})$ be any state with $s(\psi_0) = s(\omega) - s(\psi)$ and put

$$\psi_\epsilon := \epsilon\psi + (1-\epsilon)\psi_0, \quad \epsilon \in (0, 1).$$

Then, we have $s(\psi_\epsilon) = s(\omega)$. Moreover,

$$h_{\psi_\epsilon}^{1/p-1/2} = \epsilon^{1/p-1/2} \mu(1)^{1/2-1/p} u^* h_\mu^{1/p-1/2} u + (1-\epsilon)^{1/p-1/2} h_{\psi_0}^{1/p-1/2}$$

and

$$h_\varphi^{1/p-1/2} h_\omega^{1/2} = h_\mu^{1/p} u = k h_{\psi_\epsilon}^{1/p-1/2}$$

with $k = \epsilon^{1/2-1/p} \mu(1)^{1/p-1/2} h_\mu^{1/2} u$. Hence, $h_\omega^{1/2} \in \mathcal{D}(\Delta_{\varphi,\psi_\epsilon}^{1/p-1/2})$ and

$$\|\xi\|_{p,\varphi}^{BST} \leq \|\Delta_{\varphi,\psi_\epsilon}^{1/p-1/2} h_\omega^{1/2}\|_2 = \|k\|_2 = \epsilon^{1/2-1/p} \mu(1)^{1/p} = \epsilon^{1/2-1/p} \|h_\omega^{1/2}\|_{p,\varphi}^{(2)}.$$

Letting $\epsilon \rightarrow 1$, we obtain the result.

Let $2 \leq p \leq \infty$. Assume that $\|h_\omega^{1/2}\|_{p,\varphi}^{(2)} < \infty$, so that there is some $k = ek \in L_p(\mathcal{M})$ such that $h_\omega^{1/2} = h_\varphi^{1/2-1/p} k$ and $\|h_\omega^{1/2}\|_{p,\varphi}^{(2)} = \|k\|_p$. Let $\psi \in \mathfrak{S}_*(\mathcal{M})$, then since $h_\psi^{1/2-1/p} h_\omega^{1/2} = h_\psi^{1/2-1/p} k^* h_\varphi^{1/2-1/p}$, we have by [12, (A.3)]

$$\|\Delta_{\psi,\varphi}^{1/2-1/p} h_\omega^{1/2}\|_2 = \|h_\psi^{1/2-1/p} k^*\|_2 = \|k h_\psi^{1/2-1/p}\|_2 \leq \|k\|_p,$$

which means that $\|h_\omega^{1/2}\|_{p,\varphi}^{BST} \leq \|h_\omega^{1/2}\|_{p,\varphi}^{(2)}$. For the opposite inequality, note that for any $\eta \in L_2(\mathcal{M})$ with $\|\eta\|_{q,\varphi}^{(2)} = \|\eta\|_{q,\varphi}^{BST} \leq 1$, we have by (2)

$$|(h_\omega^{1/2}, \eta)| \leq \|h_\omega^{1/2}\|_{p,\varphi}^{BST} \|\eta\|_{q,\varphi}^{BST} \leq \|h_\omega^{1/2}\|_{p,\varphi}^{BST}.$$

If $\|h_\omega^{1/2}\|_{p,\varphi}^{BST} < \infty$, then we must have $s(\omega) \leq s(\varphi)$ and consequently $h_\omega^{1/2} = e h_\omega^{1/2}$. By Proposition 2.3 (ii) this implies the result. \square

3. Rényi Relative Entropies

In accordance with [3], we introduce the following version of the sandwiched Rényi relative entropy.

Definition 1. For $\psi, \varphi \in \mathcal{M}_*^+$ and $\alpha \in [1/2, 1) \cup (1, \infty)$, we define

$$\tilde{D}_\alpha(\psi\|\varphi) := \frac{1}{\alpha - 1} \log \tilde{Q}_\alpha(\psi\|\varphi) \quad (3)$$

where

$$\tilde{Q}_\alpha(\psi\|\varphi) := (\|h_\psi^{1/2}\|_{2\alpha, \varphi}^{(2)})^{2\alpha}.$$

By Proposition 2.4, \tilde{D}_α coincide with the Araki–Masuda divergences defined in [3]. Moreover, it was proved in [12, Theorem 3.3] that for $\alpha > 1$, the Araki–Masuda divergences coincide with the sandwiched Rényi relative entropies defined in [12], so the notation is justified. The following expression follows easily from Proposition 2.1 (ii).

Theorem 3.1. Let $\psi \in \mathcal{M}_*^+$, $\alpha \in [1/2, 1)$. Then,

$$\tilde{Q}_\alpha(\psi\|\varphi) = \|h_\varphi^{\frac{1-\alpha}{2\alpha}} h_\psi^{1/2}\|_{2\alpha}^{2\alpha} = \text{Tr} (h_\varphi^{\frac{1-\alpha}{2\alpha}} h_\psi h_\varphi^{\frac{1-\alpha}{2\alpha}})^\alpha.$$

It is immediate from the above expression and [12, Example 2.4] that \tilde{D}_α extends the sandwiched Rényi relative entropies on density matrices also for $\alpha \in [1/2, 1)$. We also have that for $\lambda, \mu > 0$ and all $\alpha \in [1/2, 1) \cup (1, \infty)$,

$$\tilde{D}_\alpha(\mu\psi\|\lambda\varphi) = \tilde{D}_\alpha(\psi\|\varphi) + \frac{\alpha}{\alpha - 1} \log \mu - \log \lambda.$$

3.1. Relation to Standard Rényi Relative Entropy

Recall that the standard Rényi relative entropy for $\alpha \in (0, 1)$ can be written as

$$D_\alpha(\psi\|\varphi) = \frac{1}{\alpha - 1} \log(\text{Tr} h_\psi^\alpha h_\varphi^{1-\alpha}) = \frac{1}{\alpha - 1} \log \|h_\varphi^{\frac{1-\alpha}{2}} h_\psi^{\frac{\alpha}{2}}\|_2^2.$$

A detailed account on D_α and their properties was recently given in [11].

Proposition 3.2. Let $\varphi, \psi \in \mathcal{M}_*^+$, $\alpha \in (1/2, 1)$. Then,

$$\|h_\varphi^{\frac{1-\alpha}{2}} h_\psi^{\frac{\alpha}{2}}\|_2^2 \leq \|h_\varphi^{\frac{1-\alpha}{2\alpha}} h_\psi^{1/2}\|_{2\alpha}^{2\alpha} \leq \psi(1)^{1-\alpha} \|h_\varphi^{\frac{1-\alpha}{2\alpha}} h_\psi^{1-\frac{1}{2\alpha}}\|_2^{2\alpha}$$

Proof. By Hölder,

$$\|h_\varphi^{\frac{1-\alpha}{2\alpha}} h_\psi^{1/2}\|_{2\alpha} = \|h_\varphi^{\frac{1-\alpha}{2\alpha}} h_\psi^{1-\frac{1}{2\alpha}} h_\psi^{\frac{1-\alpha}{2\alpha}}\|_{2\alpha} \leq \psi(1)^{\frac{1-\alpha}{2\alpha}} \|h_\varphi^{\frac{1-\alpha}{2\alpha}} h_\psi^{1-\frac{1}{2\alpha}}\|_2,$$

this implies the second inequality. The first inequality was proved in [3], we add a proof in our setting. Let us define the function

$$f(z) = h_\varphi^{1-\alpha z} h_\psi^{\alpha z} = h_{\varphi_0}^{1-\alpha z} e h_\psi^{\alpha z} \in L_1(\mathcal{M}), \quad z \in S.$$

Then, $f \in \mathcal{F}(\mathcal{M}^1, L_1(\mathcal{M}))$, so that we can use the properties of the interpolation spaces $L_p^R(\mathcal{M}, \varphi_0)$. Note that $\|f(1/2)\|_{2, \varphi_0}^R = \|h_\varphi^{\frac{1-\alpha}{2}} h_\psi^{\frac{\alpha}{2}}\|_2$. Since $1/2 = \alpha \frac{1}{2\alpha} + (1-\alpha)0$, we obtain by Hadamard three lines that

$$\|f(1/2)\|_{2, \varphi_0}^R \leq \left(\sup_{t \in \mathbb{R}} \|f(it)\|_{\infty, \varphi_0}^R \right)^{1-\alpha} \left(\sup_{t \in \mathbb{R}} \|f\left(\frac{1}{2\alpha} + it\right)\|_{2\alpha, \varphi_0}^R \right)^\alpha.$$

Let $u_t = h_\varphi^{-i\alpha t} h_\psi^{i\alpha t}$, then $u_t \in \mathcal{M}$ is a contraction, so that

$$\|f(it)\|_{\infty, \varphi_0}^R = \|h_{\varphi_0} u_t\|_{\infty, \varphi_0}^R = \|u_t\| \leq 1.$$

Let ψ_0 be a faithful state obtained from ψ similarly as φ_0 from φ . Then, for $t \in \mathbb{R}$,

$$\|f\left(\frac{1}{2\alpha} + it\right)\|_{2\alpha, \varphi_0}^R = \|h_{\varphi_0}^{-i\alpha t} h_\varphi^{\frac{1-\alpha}{2\alpha}} h_\psi^{1/2} h_\psi^{i\alpha t}\|_{2\alpha} = \|h_\varphi^{\frac{1-\alpha}{2\alpha}} h_\psi^{1/2}\|_{2\alpha},$$

where the last equality holds by [14, Lemma 10.1]. \square

The next statement is an extension of [12, Corollary 3.6] to all values of α . Note that the first inequality for states of a finite-dimensional algebra was proved in [17, Proposition 11]. The proof follows easily from Proposition 3.2 and [12, Corollary 3.6].

Theorem 3.3. *Let $\psi, \varphi \in \mathfrak{S}_*(\mathcal{M})$ and let $\alpha \in [1/2, 1) \cup (1, \infty]$. Then,*

$$D_{2-1/\alpha}(\psi\|\varphi) \leq \tilde{D}_\alpha(\psi\|\varphi) \leq D_\alpha(\psi\|\varphi).$$

These inequalities and the limit values for the standard Rényi relative entropies immediately imply that

$$\lim_{\alpha \uparrow 1} \tilde{D}_\alpha(\psi\|\varphi) = D_1(\psi\|\varphi),$$

the Araki relative entropy.

3.2. A Variational Formula for \tilde{Q}_α

In the rest of the paper, the symmetric L_p -spaces $L_p(\mathcal{M}, \varphi)$ and its norm $\|\cdot\|_{p, \varphi}$ will be frequently used, see [12, Sec. 2]. Recall that if $\varphi \in \mathcal{M}_*^+$ is not faithful, with $s(\varphi) = e$, we put [12, Sec. 3]

$$L_p(\mathcal{M}, \varphi) = \{h \in L_1(\mathcal{M}), h = ehe \in L_p(e\mathcal{M}e, \varphi|_{e\mathcal{M}e})\}.$$

The set $\{h_x = h_\varphi^{1/2} x h_\varphi^{1/2}, x \in \mathcal{M}^+\}$ is dense in the positive cone $L_p(\mathcal{M}, \varphi)^+$ for all $p > 1$ and we have $\|h_x\|_{p, \varphi} = \|h_\varphi^{1/2p} x h_\varphi^{1/2p}\|_p$. Recall also the duality pairing $\langle \cdot, \cdot \rangle$ between $L_p(\mathcal{M}, \varphi)$ and $L_q(\mathcal{M}, \varphi)$, in particular, we have for $x \in \mathcal{M}$ and $\psi \in \mathcal{M}_*$

$$\langle h_x, h_\psi \rangle = \text{Tr } eh_\psi ex = \psi(exe).$$

The next result is an extension of [8, Lemma 4], obtained in [10] for $\alpha \in (0, 1)$. We will use the notation \mathcal{M}^+ for the set of positive operators and \mathcal{M}^{++} for the set of positive invertible operators in \mathcal{M} .

Proposition 3.4 (Variational formula). *Let $\psi, \varphi \in \mathcal{M}_*^+$. Then,*

(i) For $\alpha \in (1, \infty)$, we have

$$\tilde{Q}_\alpha(\psi\|\varphi) = \sup_{x \in \mathcal{M}^+} \left(\alpha \operatorname{Tr} h_\psi x - (\alpha - 1) \operatorname{Tr} \left(h_\varphi^{\frac{\alpha-1}{2\alpha}} x h_\varphi^{\frac{\alpha-1}{2\alpha}} \right)^{\frac{\alpha}{\alpha-1}} \right).$$

(ii) For $\alpha \in [1/2, 1)$, we have

$$\tilde{Q}_\alpha(\psi\|\varphi) = \inf_{x \in \mathcal{M}^{++}} \left(\alpha \operatorname{Tr} h_\psi x + (1 - \alpha) \operatorname{Tr} \left(h_\varphi^{\frac{1-\alpha}{2\alpha}} x^{-1} h_\varphi^{\frac{1-\alpha}{2\alpha}} \right)^{\frac{\alpha}{1-\alpha}} \right).$$

Proof. Let $\alpha \in (1, \infty)$ and let $\beta = \frac{\alpha}{\alpha-1}$. Assume that $h_\psi \in L_\alpha(\mathcal{M}, \varphi)^+$. We have

$$\begin{aligned} & \sup_{x \in \mathcal{M}^+} \left(\alpha \operatorname{Tr} h_\psi x - (\alpha - 1) \|h_x\|_{\beta, \varphi}^\beta \right) \\ &= \sup_{t \geq 0} \sup_{\substack{x \in \mathcal{M}^+ \\ \|h_x\|_{\beta, \varphi} = t}} \left(\alpha \operatorname{Tr} h_\psi x - (\alpha - 1) \|h_x\|_{\beta, \varphi}^\beta \right) \\ &= \sup_{t \geq 0} \sup_{\substack{x \in \mathcal{M}^+ \\ \|h_x\|_{\beta, \varphi} = t}} \left(\alpha \langle h_x, h_\psi \rangle - (\alpha - 1) t^\beta \right) \\ &= \sup_{t \geq 0} \left(\alpha t \|h_\psi\|_{\alpha, \varphi} - (\alpha - 1) t^\beta \right) = \|h_\psi\|_{\alpha, \varphi}^\alpha. \end{aligned}$$

Note also that if $h_\psi \notin L_\alpha(\mathcal{M}, \varphi)$, then $\operatorname{Tr} h_\psi x$ is unbounded over $\{x \in \mathcal{M}^+, \|h_x\|_{\beta, \varphi} = t\}$ for any $t > 0$, so the right-hand side in (i) is infinite in this case. This proves (i). The statement (ii) was proved in [10, Lemma 3.19]. \square

It will be useful to introduce the following notations. For $\alpha \in [1/2, 1)$ and $\psi, \varphi \in \mathcal{M}_*^+$, let $\mu_\alpha(\psi\|\varphi) \in \mathcal{M}_*^+$ be given by

$$h_{\mu_\alpha(\psi\|\varphi)} := |h_\psi^{1/2} h_\varphi^{1/2\alpha-1/2}|^{2\alpha}. \quad (4)$$

Clearly, if $\mu = \mu_\alpha(\psi\|\varphi)$, then for some partial isometry $u \in \mathcal{M}$, $h_\psi^{1/2} = \mu^{1/2\alpha} u$ is the 2α -polar decomposition with respect to φ and by Theorem 3.1,

$$\tilde{Q}_\alpha(\psi\|\varphi) = \mu_\alpha(\psi\|\varphi)(1).$$

By [15, Theorem 4.2], the map $h \mapsto h^{1/p}$ is a homeomorphism of the positive cones $L_1(\mathcal{M})^+ \rightarrow L_p(\mathcal{M})^+$ (we will use this result repeatedly below). By this and the Hölder inequality, the map $(\psi, \varphi) \mapsto h_\varphi^{1/2\alpha-1/2} h_\psi^{1/2} \in L_{2\alpha}(\mathcal{M})$ is jointly continuous. The continuity of the absolute value $L_p(\mathcal{M}) \rightarrow L_p(\mathcal{M})^+$ [15, Theorem 4.4] now implies that the map $\mathcal{M}_*^+ \times \mathcal{M}_*^+ \ni (\psi, \varphi) \mapsto \mu_\alpha(\psi\|\varphi) \in \mathcal{M}_*^+$ is jointly (norm) continuous.

We further put

$$\xi_{\alpha, \varphi}(x) := h_\varphi^{1/2\gamma} x^{-1} h_\varphi^{1/2\gamma}, \quad x \in \mathcal{M}^{++}, \quad \gamma = \frac{\alpha}{1-\alpha} \geq 1 \quad (5)$$

and

$$f_{\alpha, \psi\|\varphi}(x) := \alpha \operatorname{Tr} h_\psi x + (1 - \alpha) \|\xi_{\alpha, \varphi}(x)\|_\gamma^\gamma, \quad x \in \mathcal{M}^{++}. \quad (6)$$

Then, since the function $t \mapsto t^{-1}$ is operator convex, we have

$$\xi_{\alpha,\varphi}((1-s)x + sy) \leq (1-s)\xi_{\alpha,\varphi}(x) + s\xi_{\alpha,\varphi}(y), \quad x, y \in \mathcal{M}^{++}, \quad s \in [0, 1]. \quad (7)$$

By the properties of the L_p -norms, $x \mapsto f_{\alpha,\psi\|\varphi}(x)$ defines a convex and Fréchet differentiable function $\mathcal{M}^{++} \rightarrow \mathbb{R}^+$, strictly convex if φ is faithful, and by (ii), $\tilde{Q}_\alpha(\psi\|\varphi) = \inf_{x \in \mathcal{M}^{++}} f_{\alpha,\psi\|\varphi}(x)$.

By the proof of [10, Lemma 3.19], the infimum in (ii) is attained in the special case when there is some $\lambda > 0$ such that $\lambda^{-1}\varphi \leq \psi \leq \lambda\varphi$, this situation will be denoted as $\psi \sim \varphi$. The next lemma also follows.

Lemma 3.5. *Let $\psi \sim \varphi$. Then, there is some $\bar{x} \in \mathcal{M}^{++}$ such that*

$$\tilde{Q}_\alpha(\psi\|\varphi) = f_{\alpha,\psi\|\varphi}(\bar{x}).$$

Moreover, we have

$$\xi_{\alpha,\varphi}(\bar{x}) = h_{\mu_\alpha(\psi\|\varphi)}^{1/\gamma}, \quad \text{Tr } h_\psi \bar{x} = \|h_{\mu_\alpha(\psi\|\varphi)}^{1/\gamma}\|_\gamma^\gamma = \tilde{Q}_\alpha(\psi\|\varphi).$$

4. Data Processing Inequality

The aim of this section is to prove the following general data processing inequality for \tilde{D}_α with $\alpha \in [1/2, 1)$. For $\alpha > 1$, the DPI was proved in [12].

Theorem 4.1 (Data processing inequality). *Let $\Phi: L_1(\mathcal{M}) \rightarrow L_1(\mathcal{N})$ be positive and trace preserving. Then, for $\alpha \in [1/2, 1)$, the DPI holds:*

$$\tilde{D}_\alpha(\Phi(\psi)\|\Phi(\varphi)) \leq \tilde{D}_\alpha(\psi\|\varphi).$$

In the case that Φ is a quantum channel (that is, completely positive and trace preserving), the statement was proved in [3]. We will first give a similar proof here, since it will be used later. The proof for the general case follows a different strategy, using the variational expression in Proposition 3.4. It is presented in Sect. 4.2.

4.1. DPI with Respect to Quantum Channels

Let $\Phi: L_1(\mathcal{M}) \rightarrow L_1(\mathcal{N})$ be a quantum channel. Then, the dual map $\Phi^*: \mathcal{N} \rightarrow \mathcal{M}$ is a completely positive unital normal map. Any such map has a Stinespring representation (\mathcal{K}, π, T) , consisting of a Hilbert space \mathcal{K} , a normal *-representation $\pi: \mathcal{N} \rightarrow B(\mathcal{K})$ and an isometry $T: L_2(\mathcal{M}) \rightarrow \mathcal{K}$ such that

$$\Phi^*(a) = T^* \pi(a) T, \quad a \in \mathcal{N}.$$

Let $\xi \in L_2(\mathcal{M})$ be a representing vector for $\psi \in \mathcal{M}_*^+$, then $T\xi \in \mathcal{K}$ is a representing vector for $\Phi(\psi)$; hence, we have

$$\tilde{D}_\alpha(\Phi(\psi)\|\Phi(\varphi)) = \frac{2\alpha}{\alpha-1} \log \|T\xi\|_{2\alpha, \Phi(\varphi)}^{BST}.$$

For $\alpha \in [1/2, 1)$, let $\alpha^* > 1$ be such that $\frac{1}{2\alpha} + \frac{1}{2\alpha^*} = 1$. Then, the dual parameter is

$$\frac{\alpha^*}{\alpha^* - 1} = \frac{\alpha}{1 - \alpha} = \gamma.$$

Theorem 4.2. *Let $\alpha \in [1/2, 1)$ and put $\mu = \mu_\alpha(\psi\|\varphi)$. Let $\omega \in \mathcal{M}_*^+$ be such that*

$$h_\omega = h_\varphi^{1/2\gamma} h_\mu^{1/\alpha^*} h_\varphi^{1/2\gamma}. \quad (8)$$

Then, $\tilde{D}_{\alpha^}(\omega\|\varphi) < \infty$, and for any quantum channel $\Phi: L_1(\mathcal{M}) \rightarrow L_1(\mathcal{N})$, we have*

$$\tilde{D}_\alpha(\psi\|\varphi) \geq \tilde{D}_\alpha(\Phi(\psi)\|\Phi(\varphi)) + \tilde{D}_{\alpha^*}(\omega\|\varphi) - \tilde{D}_{\alpha^*}(\Phi(\omega)\|\Phi(\varphi)) \geq \tilde{D}_\alpha(\Phi(\psi)\|\Phi(\varphi)).$$

Proof. Put $p = 2\alpha$, then $p \in [1, 2)$ and the dual parameter $q = 2\alpha^* \geq 2$. Let $h_\psi^{1/2} = \mu^{1/p} u$ be the p -polar decomposition and let $\eta = h_\varphi^{1/2-1/q} h_\mu^{1/q} u$, then $\|\eta\|_{q,\varphi}^{(2)} = \mu(1)^{1/q}$ and by Proposition 2.3 (iii),

$$\|h_\psi^{1/2}\|_{p,\varphi}^{(2)} = \mu(1)^{-1/q} (h_\psi^{1/2}, \eta).$$

Let $\Phi: L_1(\mathcal{M}) \rightarrow L_1(\mathcal{N})$ be a quantum channel and let (\mathcal{K}, π, T) be a Stinespring representation of Φ^* . Since T is an isometry, we obtain using (2)

$$\|h_\psi^{1/2}\|_{p,\varphi}^{(2)} \|\eta\|_{q,\varphi}^{(2)} = (h_\psi^{1/2}, \eta) = \langle Th_\psi^{1/2}, T\eta \rangle_{\mathcal{K}} \leq \|Th_\psi^{1/2}\|_{p,\Phi(\varphi)}^{BST} \|T\eta\|_{q,\Phi(\varphi)}^{BST}.$$

Put $\omega := \omega_\eta \in \mathcal{M}_*^+$, then $h_\omega = \eta\eta^*$ and (8) holds. Note that both $h_\omega^{1/2}$ and η are vector representatives of ω , so that $\|\eta\|_{q,\varphi}^{(2)} = \|h_\omega^{1/2}\|_{q,\varphi}^{(2)}$. Moreover, $T\eta$ is a vector representative of $\Phi(\omega)$. The statement is now obtained by taking the logarithm of the last inequality, observing that $\frac{2\alpha}{\alpha-1} = -\frac{2\alpha^*}{\alpha^*-1}$, and using DPI for $\alpha^* > 1$. \square

4.2. The Proof of General DPI

In this paragraph, we will prove Theorem 4.1, using the variational formula in Proposition 3.4 (ii).

Let $\Phi: L_1(\mathcal{M}) \rightarrow L_1(\mathcal{N})$ be a trace preserving positive (not necessarily completely positive) map. By [12, Proposition 3.12], Φ restricts to a contraction $L_p(\mathcal{M}, \varphi) \rightarrow L_p(\mathcal{N}, \Phi(\varphi))$ for any $1 \leq p \leq \infty$. In [12, Section 3.3], several maps related to Φ and a state φ were considered: the adjoint map $\Phi^*: \mathcal{N} \rightarrow \mathcal{M}$, the Petz dual $\Phi_\varphi^*: e\mathcal{M}e \rightarrow e'\mathcal{N}e'$ [here $e' = s(\Phi(\varphi))$], determined by the condition

$$\Phi(h_\varphi^{1/2} x h_\varphi^{1/2}) = \Phi(h_\varphi)^{1/2} \Phi_\varphi^*(x) \Phi(h_\varphi^{1/2}),$$

and its preadjoint $\Phi_\varphi: e'L_1(\mathcal{N})e' \rightarrow eL_1(\mathcal{M})e$, which also restricts to the adjoint of Φ with respect to the duality pairing $\langle \cdot, \cdot \rangle$ between $L_p(\mathcal{M}, \varphi)$ and $L_q(\mathcal{M}, \varphi)$ [12, Eq. (17)]. We now extend these definitions to obtain a family of contractions $L_p(\mathcal{M}) \rightarrow L_p(\mathcal{N})$.

Let $i_{p,\varphi}: eL_p(\mathcal{M})e \rightarrow L_p(\mathcal{M}, \varphi)$ be defined as

$$i_{p,\varphi}(k) = h_\varphi^{1/2q} k h_\varphi^{1/2q} \quad (1/p + 1/q = 1),$$

then $i_{p,\varphi}$ is an isometry and we see that

$$\Phi_{p,\varphi} = i_{p,\Phi(\varphi)}^{-1} \circ \Phi \circ i_{p,\varphi}$$

defines a linear contraction $\Phi_{p,\varphi}: eL_p(\mathcal{M})e \rightarrow e'L_p(\mathcal{N})e'$. Note that we have $\Phi_{1,\varphi} = \Phi$ and $\Phi_{\infty,\varphi} = \Phi_\varphi^*$. To obtain the adjoint map $\Phi_{p,\varphi}^*: e'L_q(\mathcal{N})e' \rightarrow eL_q(\mathcal{M})e$, we have for any $k \in eL_p(\mathcal{M})e$ and $l \in eL_q(\mathcal{M})e$

$$\langle i_{p,\varphi}(k), h_\varphi^{1/2p} l h_\varphi^{1/2p} \rangle = \text{Tr } kl = \text{Tr } k i_{q,\varphi}^{-1} (h_\varphi^{1/2p} l h_\varphi^{1/2p}),$$

so that $i_{p,\varphi}^* = i_{q,\varphi}^{-1}$ and therefore

$$\Phi_{p,\varphi}^* = i_{q,\varphi}^{-1} \circ \Phi_\varphi \circ i_{q,\Phi(\varphi)}. \quad (9)$$

The map $\Phi_{p,\varphi}$ can be easily extended to a contraction $L_p(\mathcal{M}) \rightarrow L_p(\mathcal{N})$ and its adjoint is a similar extension of $\Phi_{p,\varphi}^*$. The following lemma will be useful later.

Lemma 4.3. *Let $\Phi: L_1(\mathcal{M}) \rightarrow L_1(\mathcal{N})$ be positive and trace preserving. Assume that $\varphi_n, \varphi \in \mathcal{M}_+^+$ are such that $\varphi_n \rightarrow \varphi$ (in norm) and $\Phi(\varphi)$ is faithful. Then, for any $1 \leq p \leq \infty$ and $k \in L_q(\mathcal{N})$, we have $\Phi_{p,\varphi_n}^*(k) \rightarrow \Phi_{p,\varphi}^*(k)$ in $L_q(\mathcal{N})$.*

Proof. Let $k \in L_q(\mathcal{N})$. We may assume that $k = \Phi(h_\varphi)^{1/2q} x \Phi(h_\varphi)^{1/2q}$ for some $x \in \mathcal{N}$, since the set of such elements is dense in $L_q(\mathcal{N})$ and all the maps are contractions. In this case, we have

$$\begin{aligned} \Phi_{p,\varphi}^*(k) &= i_{q,\varphi}^{-1} \Phi_\varphi(\Phi(h_\varphi)^{1/2} x \Phi(h_\varphi)^{1/2}) = i_{q,\varphi}^{-1} (h_\varphi^{1/2} \Phi^*(x) h_\varphi^{1/2}) \\ &= h_\varphi^{1/2q} \Phi^*(x) h_\varphi^{1/2q}. \end{aligned} \quad (10)$$

Let $k_n = \Phi(h_{\varphi_n})^{1/2q} x \Phi(h_{\varphi_n})^{1/2q}$, then $k_n \rightarrow k$ and we similarly have

$$\Phi_{p,\varphi_n}^*(k_n) = h_{\varphi_n}^{1/2q} \Phi^*(x) h_{\varphi_n}^{1/2q}.$$

Hence,

$$\begin{aligned} \|\Phi_{p,\varphi_n}^*(k) - \Phi_{p,\varphi}^*(k)\|_q &\leq \|k - k_n\|_q + \|\Phi_{p,\varphi_n}^*(k_n) - \Phi_{p,\varphi}^*(k)\|_q \\ &\leq \|k - k_n\|_q + \|h_{\varphi_n}^{1/2q} \Phi^*(x) h_{\varphi_n}^{1/2q} - h_\varphi^{1/2q} \Phi^*(x) h_\varphi^{1/2q}\|_q \\ &\rightarrow 0. \end{aligned}$$

□

We will also need the following inequality due to Fack and Kosaki, see [7, Lemma 5.1]:

$$\|k - l\|_p^p \leq \|k\|_p^p - \|l\|_p^p, \quad 0 \leq l \leq k \in L_p(\mathcal{M}). \quad (11)$$

Proof of Theorem 4.1. Clearly, the DPI is equivalent to $\tilde{Q}_\alpha(\Phi(\psi)\|\Phi(\varphi)) \geq \tilde{Q}_\alpha(\psi\|\varphi)$. Let $y \in \mathcal{N}^{++}$. Since Φ^* is positive and unital, we have by the Choi inequality (see [4, Corollary 2.3]) that $\Phi^*(y)^{-1} \leq \Phi^*(y^{-1})$, so that

$$\begin{aligned} \xi_{\alpha,\varphi}(\Phi^*(y)) &= h_\varphi^{1/2\gamma} \Phi^*(y)^{-1} h_\varphi^{1/2\gamma} \leq h_\varphi^{1/2\gamma} \Phi^*(y^{-1}) h_\varphi^{1/2\gamma} \\ &= \Phi_{\alpha^*,\varphi}^*(\Phi(h_\varphi)^{1/2\gamma} y^{-1} \Phi(h_\varphi)^{1/2\gamma}) = \Phi_{\alpha^*,\varphi}^*(\xi_{\alpha,\Phi(\varphi)}(y)), \end{aligned} \quad (12)$$

here the second equality is obtained as in (10). By (11), this implies

$$\|\xi_{\alpha,\varphi}(\Phi^*(y))\|_\gamma^\gamma \leq \|\Phi_{\alpha^*,\varphi}^*(\xi_{\alpha,\Phi(\varphi)}(y))\|_\gamma^\gamma \leq \|\xi_{\alpha,\Phi(\varphi)}(y)\|_\gamma^\gamma, \quad (13)$$

the last inequality follows since $\Phi_{\alpha^*, \varphi}^*$ is a contraction. Putting all together, we obtain for any $y \in \mathcal{N}^{++}$ that

$$\tilde{Q}_\alpha(\psi \| \varphi) = \inf_{x \in \mathcal{M}^{++}} (\alpha \operatorname{Tr} h_\psi x + (1 - \alpha) \|\xi_{\alpha, \varphi}(x)\|_\gamma^\gamma) \quad (14)$$

$$\leq \alpha \operatorname{Tr} h_\psi \Phi^*(y) + (1 - \alpha) \|\xi_{\alpha, \varphi}(\Phi^*(y))\|_\gamma^\gamma \quad (15)$$

$$\leq \alpha \operatorname{Tr} \Phi(h_\psi) y + (1 - \alpha) \|\xi_{\alpha, \Phi(\varphi)}(y)\|_\gamma^\gamma \quad (16)$$

$$= f_{\alpha, \Phi(\psi) \| \Phi(\varphi)}(y), \quad (17)$$

which implies that

$$\tilde{Q}_\alpha(\psi \| \varphi) \leq \tilde{Q}_\alpha(\Phi(\psi) \| \Phi(\varphi)).$$

□

5. Equality Conditions and Sufficiency

Recall that a quantum channel (or more generally a 2-positive trace preserving map) $\Phi: L_1(\mathcal{M}) \rightarrow L_1(\mathcal{N})$ is sufficient with respect to a pair $\{\psi, \varphi\}$ with $\psi, \varphi \in \mathcal{M}_*^+$ if there is a recovery map $\Psi: L_1(\mathcal{N}) \rightarrow L_1(\mathcal{M})$ such that $\Psi \circ \Phi(\psi) = \psi$ and $\Psi \circ \Phi(\varphi) = \varphi$. The map Ψ is assumed trace preserving and 2-positive, but note that the result of [12, Theorem 4.6] and the DPI in [12, Theorem 3.14] imply that positive trace-preserving recovery maps would lead to an equivalent notion.

In this section, we prove the following extension of [12, Theorem 4.6].

Theorem 5.1. *Let $\alpha \in (1/2, 1)$ and assume that the linear map $\Phi: L_1(\mathcal{M}) \rightarrow L_1(\mathcal{N})$ is 2-positive and trace preserving. Let $\psi, \varphi \in \mathcal{M}_*^+$ be such that $s(\psi) \leq s(\varphi)$. Then, the equality*

$$\tilde{D}_\alpha(\psi \| \varphi) = \tilde{D}_\alpha(\Phi(\psi) \| \Phi(\varphi))$$

holds if and only if Φ is sufficient with respect to $\{\psi, \varphi\}$.

Because of the assumption on the supports, we may and will suppose below as in the proof of [12, Theorem 4.6] that both φ and $\Phi(\varphi)$ are faithful.

The proof in the most general case is somewhat complicated and technical. We therefore treat separately two special cases where the proof is simpler: first in the case when Φ is a quantum channel and then under the assumption that $\psi \sim \varphi$. A crucial observation is the following.

Lemma 5.2. *Let $\alpha \in (1/2, 1)$ and let $\mu = \mu_\alpha(\psi \| \varphi)$. Then, Φ is sufficient with respect to $\{\psi, \varphi\}$ if and only if it is sufficient with respect to $\{\mu, \varphi\}$.*

Proof. By [12, Theorem 4.2] and [12, Lemma 4.3], there is a faithful normal conditional expectation $E: \mathcal{M} \rightarrow \mathcal{M}$ such that $\varphi \circ E = \varphi$ and for any $\rho \in \mathcal{M}_*^+$, Φ is sufficient with respect to $\{\rho, \varphi\}$ if and only if $\rho \circ E = \rho$. Let $p = 2\alpha$ and let E_p be the extension of E to $L_p(\mathcal{M})$ [13], [12, Appendix A.3]. Let $u \in \mathcal{M}$

be the partial isometry such that $h_\psi^{1/2} = \mu^{1/p}u$ is the p -polar decomposition of $h_\psi^{1/2}$. Using [13, Proposition 2.3 (ii)], [12, Eq. (A.7)], we have

$$E_p(h_\mu^{1/p}u) = E_p(h_\varphi^{1/p-1/2}h_\psi^{1/2}) = h_\varphi^{1/p-1/2}E_2(h_\psi^{1/2}).$$

If Φ is sufficient with respect to $\{\psi, \varphi\}$, then $\psi \circ E = \psi$, so that $E_2(h_\psi^{1/2}) = h_\psi^{1/2}$. Consequently, $E_p(h_\mu^{1/p}u) = h_\mu^{1/p}u$, so that $h_\mu^{1/p}u \in L_p(\mathcal{M}_0)$, where $\mathcal{M}_0 \subseteq \mathcal{M}$ is the range of E . By uniqueness of the polar decomposition in $L_p(\mathcal{M})$, we have $\mu \in L_1(\mathcal{M}_0)$, so that $\mu = \mu \circ E$ and Φ is sufficient with respect to $\{\mu, \varphi\}$.

For the converse, assuming that $\mu \circ E = \mu$, we similarly obtain

$$h_\varphi^{1/p-1/2}h_\psi^{1/2}u^* = h_\mu^{1/p} = E_p(h_\mu^{1/p}) = h_\varphi^{1/p-1/2}E_2(h_\psi^{1/2}u^*).$$

Since φ is faithful, we have $u^*u = s(\psi)$ and the above equalities imply that $h_\psi^{1/2}u^* = E_2(h_\psi^{1/2}u^*)$. Hence, (using [13, Proposition 2.3 (ii)] again)

$$h_{\psi \circ E} = E_1(h_\psi) = h_\psi^{1/2}u^*uh_\psi^{1/2} = h_\psi$$

so that Φ is sufficient for $\{\psi, \varphi\}$. \square

Proof of Theorem 5.1 for quantum channels Let α^* , μ and ω be as in Theorem 4.2. Assume that the equality holds, then we must have

$$\tilde{D}_{\alpha^*}(\omega \parallel \varphi) = \tilde{D}_{\alpha^*}(\Phi(\omega) \parallel \Phi(\varphi)).$$

Since $1 < \alpha^* < \infty$, this equality implies that Φ is sufficient with respect to $\{\omega, \varphi\}$ [12, Theorem 4.6]. By (8) and [12, Lemma 4.4], Φ is sufficient with respect to $\{\mu, \varphi\}$. By Lemma 5.2, Φ is sufficient with respect to $\{\psi, \varphi\}$. The converse is obvious from DPI. \square

In the case that Φ is not a quantum channel but only 2-positive, we cannot use Theorem 4.2. Instead, we will apply the variational expression. We will do this first in the case when $\psi \sim \varphi$, where we can exploit the fact that the infimum in the variational expression is attained at a unique point. It will be convenient to prove the following lemma, showing that sufficiency holds if a condition on $\mu_\alpha(\psi \parallel \varphi)$ and $\mu_\alpha(\Phi(\psi) \parallel \Phi(\varphi))$ is added to the equality in DPI.

Lemma 5.3. *Let $\mu = \mu_\alpha(\psi \parallel \varphi)$, $\nu = \mu_\alpha(\Phi(\psi) \parallel \Phi(\varphi))$ and let $\gamma = \alpha/(1 - \alpha)$. Assume that*

$$\mu(1) = \nu(1) \quad \text{and} \quad h_\mu^{1/\gamma} = \Phi_{\alpha^*, \varphi}^*(h_\nu^{1/\gamma}),$$

where $1/\alpha^* + 1/\gamma = 1$. Then, Φ is sufficient with respect to $\{\psi, \varphi\}$.

Proof. Let $\sigma \in \mathcal{M}_*^+$ be such that $h_\sigma = h_\varphi^{1/2\alpha^*}h_\mu^{1/\gamma}h_\varphi^{1/2\alpha^*}$ and $\rho \in \mathcal{N}_*^+$ be such that $h_\rho = \Phi(h_\varphi)^{1/2\alpha^*}h_\nu^{1/\gamma}\Phi(h_\varphi)^{1/2\alpha^*}$. By (9) and the assumptions, we obtain

$$\Phi_\varphi(h_\rho) = h_\varphi^{1/2\alpha^*}\Phi_{\alpha^*, \varphi}^*(h_\nu^{1/\gamma})h_\varphi^{1/2\alpha^*} = h_\varphi^{1/2\alpha^*}h_\mu^{1/\gamma}h_\varphi^{1/2\alpha^*} = h_\sigma$$

and

$$\|h_\rho\|_{\gamma, \Phi(\varphi)} = \nu(1)^{1/\gamma} = \mu(1)^{1/\gamma} = \|h_\sigma\|_{\gamma, \varphi} = \|\Phi_\varphi(h_\rho)\|_{\gamma, \varphi}.$$

By [12, Theorem 4.6], this implies that the 2-positive trace preserving map $\Phi_\varphi: L_1(\mathcal{N}) \rightarrow L_1(\mathcal{M})$ is sufficient with respect to $\{\rho, \Phi(\varphi)\}$. Since the Petz

dual of Φ_φ^* with respect to $\Phi(\varphi)$ is Φ^* , we obtain that $\Phi \circ \Phi_\varphi(\rho) = \rho$. Putting all together, we get

$$\Phi_\varphi \circ \Phi(\sigma) = \Phi_\varphi \circ \Phi \circ \Phi_\varphi(\rho) = \Phi_\varphi(\rho) = \sigma.$$

Hence, Φ is sufficient with respect to $\{\sigma, \varphi\}$ and again by [12, Lemma 4.4], it is also sufficient with respect to $\{\mu, \varphi\}$. The proof now follows by Lemma 5.2. \square

We are now ready to prove Theorem 5.1 in the case $\psi \sim \varphi$. Note that this holds, e.g., if \mathcal{M} is finite-dimensional and $s(\psi) = s(\varphi)$.

Proof of Theorem 5.1 for $\psi \sim \varphi$ Note that $\psi \sim \varphi$ implies $\Phi(\psi) \sim \Phi(\varphi)$. By Lemma 3.5, there are some $\bar{x} \in \mathcal{M}^{++}$ and $\bar{y} \in \mathcal{N}^{++}$ such that $\tilde{Q}_\alpha(\psi\|\varphi) = f_{\alpha, \psi\|\varphi}(\bar{x})$ and $\tilde{Q}_\alpha(\Phi(\psi)\|\Phi(\varphi)) = f_{\alpha, \Phi(\psi)\|\Phi(\varphi)}(\bar{y})$. The equality in DPI implies that with $y = \bar{y}$, all the inequalities between (14) and (17), and hence also in (13), must be equalities. Since $f_{\alpha, \psi\|\varphi}$ is strictly convex, the infimum is attained at a unique point and hence $\bar{x} = \Phi^*(\bar{y})$. Using (12), equality in the first inequality of (13) and (11), we obtain

$$\xi_{\alpha, \varphi}(\bar{x}) = \xi_{\alpha, \varphi}(\Phi^*(\bar{y})) = \Phi_{\alpha^*, \varphi}^*(\xi_{\alpha, \Phi(\varphi)}(\bar{y})).$$

By Lemma 3.5, this means that $h_\mu^{1/\gamma} = \Phi_{\alpha^*, \varphi}^*(h_\nu^{1/\gamma})$, where $\mu = \mu_\alpha(\psi\|\varphi)$ and $\nu = \mu_\alpha(\Phi(\psi)\|\Phi(\varphi))$. By the equality in DPI, we obtain

$$\nu(1) = \tilde{Q}_\alpha(\Phi(\psi)\|\Phi(\varphi)) = \tilde{Q}_\alpha(\psi\|\varphi) = \mu(1),$$

so that the conditions of Lemma 5.3 are fulfilled and Φ is sufficient with respect to $\{\psi, \varphi\}$. The converse is clear from DPI. \square

5.1. The General Case

In the general case, we follow the same strategy as in the case $\psi \sim \varphi$, by proving that equality in DPI implies that the conditions of Lemma 5.3 are fulfilled. Since we cannot rely on Lemma 3.5, we will need some refinement of the inequalities (14)–(17). For this, we will use the fact that the spaces $L_p(\mathcal{M})$, $1 < p < \infty$ are uniformly convex. Throughout this section, we fix the notations

$$\mu = \mu_\alpha(\psi\|\varphi), \quad \nu = \mu_\alpha(\Phi(\psi)\|\Phi(\varphi)), \quad f = f_{\alpha, \psi\|\varphi}, \quad \xi = \xi_{\alpha, \varphi}.$$

Lemma 5.4. *Let $\psi \sim \varphi$. Then, for any $y \in \mathcal{M}^{++}$, we have*

$$f(y) - \tilde{Q}_\alpha(\psi\|\varphi) \geq 2(1 - \alpha) \left[\frac{1}{2} \|h_\mu^{1/\gamma}\|_\gamma^\gamma + \frac{1}{2} \|\xi(y)\|_\gamma^\gamma - \frac{1}{2} \|(h_\mu^{1/\gamma} + \xi(y))\|_\gamma^\gamma \right] \geq 0.$$

In particular, for any $\epsilon > 0$, there is some $\delta > 0$ (depending only on the value of $\max\{\|\varphi\|_1, \|\psi\|_1, \|\xi(y)\|_\gamma^\gamma\}$) such that if $\|\xi(y) - h_\mu^{1/\gamma}\|_\gamma \geq \epsilon$, then

$$f(y) - \tilde{Q}_\alpha(\psi\|\varphi) \geq \delta(\|h_\mu^{1/\gamma}\|_\gamma^\gamma + \|\xi(y)\|_\gamma^\gamma).$$

Proof. For $x, y \in \mathcal{M}^{++}$, we have

$$\begin{aligned} \langle \nabla f(x), y - x \rangle &= \lim_{s \rightarrow 0^+} s^{-1} [f((1-s)x + sy) - f(x)] \\ &= \alpha \operatorname{Tr}(h_\psi(y - x)) \\ &\quad + (1 - \alpha) \lim_{s \rightarrow 0^+} s^{-1} [\|\xi((1-s)x + sy)\|_\gamma^\gamma - \|\xi(x)\|_\gamma^\gamma]. \end{aligned}$$

For $s \in (0, 1/2)$, we have by (7) and (11)

$$\begin{aligned} \|\xi((1-s)x + sy)\|_\gamma^\gamma &\leq \|(1-s)\xi(x) + s\xi(y)\|_\gamma^\gamma \\ &= \|(1-2s)\xi(x) + 2s\frac{1}{2}(\xi(x) + \xi(y))\|_\gamma^\gamma \\ &\leq (1-2s)\|\xi(x)\|_\gamma^\gamma + 2s\|\frac{1}{2}(\xi(x) + \xi(y))\|_\gamma^\gamma. \end{aligned}$$

Using this in the above computation, we get

$$\begin{aligned} \langle \nabla f(x), y - x \rangle &\leq f(y) - f(x) \\ &\quad - 2(1 - \alpha) \left[\frac{1}{2} \|\xi(x)\|_\gamma^\gamma + \frac{1}{2} \|\xi(y)\|_\gamma^\gamma - \left\| \frac{1}{2}(\xi(x) + \xi(y)) \right\|_\gamma^\gamma \right]. \end{aligned}$$

By Lemma 3.5, there is some $\bar{x} \in \mathcal{M}^{++}$ such that $\tilde{Q}_\alpha(\psi\|\varphi) = f(\bar{x})$, in this case we have $\nabla f(\bar{x}) = 0$ and $\xi(\bar{x}) = h_\mu^{1/\gamma}$, this yields the first part of the lemma. For the second part, note that if $c := \max\{\|\varphi\|_1, \|\psi\|_1, \|\xi(y)\|_\gamma^\gamma\}$, then $\|h_\mu^{1/\gamma}\|_\gamma, \|\xi(y)\|_\gamma \leq c^{1/\gamma}$. The statement now follows by uniform convexity of $L_\gamma(\mathcal{M})$ (see, e.g., [6]). \square

Lemma 5.5. *Let $\psi \sim \varphi$ and let $\Phi : L_1(\mathcal{M}) \rightarrow L_1(\mathcal{N})$ be positive and trace preserving. Let $\bar{y} \in \mathcal{N}^{++}$ be such that $\tilde{Q}_\alpha(\Phi(\psi)\|\Phi(\varphi)) = f_{\alpha, \Phi(\psi)\|\Phi(\varphi)}(\bar{y})$. Then, we have*

$$\tilde{Q}_\alpha(\Phi(\psi)\|\Phi(\varphi)) \geq f(\Phi^*(\bar{y})) + (1 - \alpha) \|\Phi_{\alpha^*, \varphi}^*(h_\nu^{1/\gamma}) - \xi(\Phi^*(\bar{y}))\|_\gamma^\gamma.$$

Proof. Since $\xi_{\alpha, \Phi(\varphi)}(\bar{y}) = h_\nu^{1/\gamma}$, we have

$$\begin{aligned} \tilde{Q}_\alpha(\Phi(\psi)\|\Phi(\varphi)) &= f(\Phi^*(\bar{y})) + (1 - \alpha) \left[\|h_\nu^{1/\gamma}\|_\gamma^\gamma - \|\xi(\Phi^*(\bar{y}))\|_\gamma^\gamma \right] \\ &\geq f(\Phi^*(\bar{y})) + (1 - \alpha) \left[\|\Phi_{\alpha^*, \varphi}^*(h_\nu^{1/\gamma})\|_\gamma^\gamma - \|\xi(\Phi^*(\bar{y}))\|_\gamma^\gamma \right] \\ &\geq f(\Phi^*(\bar{y})) + (1 - \alpha) \|\Phi_{\alpha^*, \varphi}^*(h_\nu^{1/\gamma}) - \xi(\Phi^*(\bar{y}))\|_\gamma^\gamma, \end{aligned}$$

where we used Lemma 3.5 for the equality, the first inequality follows from the fact that $\Phi_{\alpha^*, \varphi}^*$ is a contraction and the second inequality is obtained by (12) and (11). \square

Now we can prove Theorem 5.1 in the general case.

Proof of Theorem 5.1. Fix some sequences $\varphi_n \rightarrow \varphi$ and $\psi_n \rightarrow \psi$ in \mathcal{M}_*^+ such that $\psi_n \sim \varphi_n$. By joint continuity of \tilde{Q}_α [10, Theorem 3.16 (3)] and the assumption, we have

$$\lim_n \tilde{Q}_\alpha(\psi_n\|\varphi_n) = \lim_n Q_\alpha(\Phi(\psi_n)\|\Phi(\varphi_n)) = \tilde{Q}_\alpha(\psi\|\varphi)$$

Put

$$\mu_n = \mu_\alpha(\psi_n \| \varphi_n), \quad \nu_n = \mu_\alpha(\Phi(\psi_n) \| \Phi(\varphi_n)), \quad f_n = f_{\alpha, \psi_n \| \varphi_n}.$$

We have $\mu_n \rightarrow \mu$ and $\nu_n \rightarrow \nu$, so that $h_{\mu_n}^{1/\gamma} \rightarrow h_\mu^{1/\gamma}$ and $h_{\nu_n}^{1/\gamma} \rightarrow h_\nu^{1/\gamma}$. By Lemma 4.3, we obtain $\Phi_{\alpha^*, \varphi_n}^*(h_{\nu_n}^{1/\gamma}) \rightarrow \Phi_{\alpha^*, \varphi}^*(h_\nu^{1/\gamma})$.

Let $y_n \in \mathcal{N}^{++}$ be such that $\tilde{Q}_\alpha(\Phi(\psi_n) \| \Phi(\varphi_n)) = f_{\alpha, \Phi(\psi_n) \| \Phi(\varphi_n)}(y_n)$ and let us denote $\xi_n = \xi_{\alpha, \varphi_n}(\Phi^*(y_n))$. Using Lemma 5.5, we get

$$\tilde{Q}_\alpha(\Phi(\psi_n) \| \Phi(\varphi_n)) - \tilde{Q}_\alpha(\psi_n \| \varphi_n) \geq f_n(\Phi^*(y_n)) - \tilde{Q}_\alpha(\psi_n \| \varphi_n) \quad (18)$$

$$+ (1 - \alpha) \|\Phi_{\alpha^*, \varphi_n}^*(h_{\nu_n}^{1/\beta}) - \xi_n\|_\beta^\beta. \quad (19)$$

Since both quantities on the right-hand side of (18) and (19) are nonnegative, we immediately obtain that $\xi_n \rightarrow \Phi_{\alpha^*, \varphi}^*(h_\nu^{1/\gamma})$ and by Lemma 5.4, we also get $\xi_n \rightarrow h_\mu^{1/\gamma}$. This proves $\Phi_{\alpha^*, \varphi}^*(h_\nu^{1/\gamma}) = h_\mu^{1/\gamma}$. The proof now can be finished by Lemma 5.3. \square

Remark 5.6. In the case $\gamma \geq 2$ (that is, $2/3 \leq \alpha < 1$), we may use the Clarkson inequality [7, Theorem 5.2] in Lemma 5.4 and obtain

$$f(y) - \tilde{Q}_\alpha(\psi \| \varphi) \geq (1 - \alpha) 2^{1-\gamma} \|\xi(y) - h_\mu^{1/\gamma}\|_\gamma^\gamma.$$

Using this with $y = \Phi^*(\bar{y})$, we get from Lemma 5.5 that

$$\begin{aligned} & \tilde{Q}_\alpha(\Phi(\psi) \| \Phi(\varphi)) - \tilde{Q}_\alpha(\psi \| \varphi) \\ & \geq (1 - \alpha) 2^{1-\gamma} \left[\|\xi(\Phi^*(\bar{y})) - h_\mu^{1/\gamma}\|_\gamma^\gamma + \|\Phi_{\alpha^*, \varphi}^*(h_\nu^{1/\gamma}) - \xi(\Phi^*(\bar{y}))\|_\gamma^\gamma \right] \\ & \geq (1 - \alpha) 4^{1-\gamma} \|\Phi_{\alpha^*, \varphi}^*(h_\nu^{1/\gamma}) - h_\mu^{1/\gamma}\|_\gamma^\gamma. \end{aligned}$$

By a limit argument, this inequality holds for all ψ, φ with $s(\psi) \leq s(\varphi)$ and positive trace preserving maps Φ . Another DPI lower bound (for all values of $\alpha \in [1/2, 1)$) is in Proposition 5.7.

Proposition 5.7. *Let $\psi, \varphi \in \mathcal{M}_*^+$ with $s(\psi) \leq s(\varphi)$ and let $\Phi: L_1(\mathcal{M}) \rightarrow L_1(\mathcal{N})$ be positive and trace preserving. Then, we have*

$$\tilde{Q}_\alpha(\Phi(\psi) \| \Phi(\varphi)) - \tilde{Q}_\alpha(\psi \| \varphi) \geq (1 - \alpha) \left[\|h_\nu^{1/\gamma}\|_\gamma^\gamma - \|\Phi_{\alpha^*, \varphi}^*(h_\nu^{1/\gamma})\|_\gamma^\gamma \right].$$

Proof. Assume first that $\psi \sim \varphi$ and let $\bar{y} \in \mathcal{N}^{++}$ be chosen as before. Then, by (15) and (13), we have

$$\begin{aligned} \tilde{Q}_\alpha(\psi \| \varphi) & \leq \alpha \operatorname{Tr} \Phi(\psi) \bar{y} + (1 - \alpha) \|\Phi_{\alpha^*, \varphi}^*(\xi_{\alpha, \Phi(\varphi)}(\bar{y}))\|_\gamma^\gamma \\ & = \alpha \operatorname{Tr} \Phi(\psi) \bar{y} + (1 - \alpha) \|\Phi_{\alpha^*, \varphi}^*(h_\nu^{1/\gamma})\|_\gamma^\gamma. \end{aligned}$$

The proof follows from $\tilde{Q}_\alpha(\Phi(\psi) \| \Phi(\varphi)) = f_{\alpha, \Phi(\psi) \| \Phi(\varphi)}(\bar{y})$. The general case is proved by a limit argument. \square

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Anna Jenčová
Mathematical Institute
Slovak Academy of Sciences
Štefánikova 49
814 73 Bratislava
Slovakia
e-mail: jenca@mat.savba.sk

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