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Fuzzy observables: from weak Markov kernels to Markov kernels

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Abstract We provide a proof based on transfinite induction that every weak Markov kernel is equivalent to a Markov kernel. We only assume the space where the weak Markov kernel is defined to be second countable and metrizable. That generalizes some previous results where the kernel is required to be defined on a standard Borel space (which is second countable and completely metrizable) and the framework is the theory of stochastic operators. This property of weak Markov kernels is at the root of the characterization of a commutative POVM as the fuzzification of a spectral measure through a Markov kernel. As a consequence the characterization of commutative POVMs is generalized as well. We then revisit the relationships between weak Markov kernels, Markov kernels, commutative POVMs and fuzzy observables.

Keywords Weak Markov kernels, Markov kernels, Positive Operator valued Measures, Fuzzy Observables, Transfinite Induction, Borel Hierarchy

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1 Introduction

Fuzzy observables play a very important role in quantum physics and in the analysis of its foundations. Moreover, they are connected to fuzzy sets since they can be interpreted as fuzzification of sharp observables. From the mathematical viewpoint, they are described by commutative positive operator valued measures (POVMs). A POVM is a map $F:\mathscr{B}(X)\to\mathscr{L}_s^+(\mathscr{H})$ from the Borel σ-algebra of a topological space X to the space of linear, self-adjoint, positive operators in the Hilbert space $\mathscr{H}$

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satisfying the following properties: 1) $F(X) = \mathbf{1}$ where $\mathbf{1}$ is the identity operator, 2) for every countable family $\{\Delta_n\}$ of disjoint sets in $\mathscr{B}(X)$,

$$F\big(\bigcup_{n=1}^{\infty}\Delta_n\big) = \sum_{n=1}^{\infty} F(\Delta_n).$$

where the series converges in the weak operator topology.
The POVM is said to be commutative if $\big[F(\Delta_1), F(\Delta_2)\big] = \mathbf{0}$ for all $\Delta_1, \Delta_2 \in \mathscr{B}(X)$. A spectral measure is a real POVM ($X = \mathbb{R}$) such that $F(\Delta)$ is a projection operator for every $\Delta \in \mathscr{B}(\mathbb{R})$. As it is well known, spectral measures are in a one-to-one correspondence with self-adjoint operators, the latter representing standard (or sharp) quantum observables. By analyzing the process of measurement in quantum physics, it can be shown [23, 19, 14] that POVMs provide the right mathematical representation for quantum observable; spectral measures being a too restrictive mathematical tool. Quantum observables represented by POVMs that are not sharp are called **generalised** or **unsharp** observables.

We recall that $\langle\psi, F(\Delta)\psi\rangle$ (where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathscr{H}$) is interpreted as the probability that a measurement of the observable represented by F gives a result in Δ.

Before we proceed with the proof of the main results it is helpful to illustrate the connections between Markov kernels, POVMs and fuzzy observables. We can consider, for example, the joint measurement of position and momentum observables. If the position and momentum observables are represented by the spectral measures E^Q and E^P respectively, the mathematical formalism does not allow us to describe their joint measurement which should be represented by a joint POVM of which E^Q and E^P are the marginals. Such a POVM exists if and only if the spectral measures E^Q and E^P commute and it is well known that they do not. On the contrary, there are couples of non commuting POVMs that are the marginals of a joint POVM. An example is provided by the fuzzification of E^Q and E^P by means of two Markov kernels μ^Q and μ^P,

$$\begin{aligned} F^Q(\Delta) &= \int_{\mathbb{R}} \mu^Q_\Delta(\lambda)\, dE^Q_\lambda = \mu^Q_\Delta(Q), \\ F^P(\Delta) &= \int_{\mathbb{R}} \mu^P_\Delta(\lambda)\, dE^P_\lambda = \mu^P_\Delta(P). \end{aligned} \tag{1}$$

where $\mu^Q_{(\cdot)}(\lambda)$ is a probability measure for every $\lambda \in \mathbb{R}$ and $\mu^Q_\Delta(\cdot)$ is measurable for every Δ and the same is true for $\mu^P_{(\cdot)}(\lambda)$ and $\mu^P_\Delta(\cdot)$ (those are the conditions defining a Markov kernel).
The POVMs F^Q and F^P can be interpreted [24, 12] as fuzzification of E^Q and E^P respectively. Indeed, for every $\Delta \in \mathscr{B}(\mathbb{R})$, $(\mathbb{R}, \mu^Q_\Delta(\cdot))$ defines a fuzzy set [29] (see below). A very relevant feature of F^Q and F^P is that there is a third POVM $F(\Delta_1 \times \Delta_2)$ of which F^Q and F^P are the marginals, i.e., $F^Q(\Delta_1) = F(\Delta_1 \times \mathbb{R})$, $F^P(\Delta_2) = F(\mathbb{R} \times \Delta_2)$. The POVM F represents the joint measurement of position and momentum [15, 25, 3]. The property that, in some cases, two not commuting POVMs can have a joint

POVM is at the root of the formulation of quantum mechanics on phase space [25, 26, 10, 11].

Note that F^Q and F^P are commutative POVMs. Indeed, every operator $F^Q(\Delta)$ is a function of the self-adjoint operator Q. As a consequence the family of operators $\{F^Q(\Delta)\}_{\Delta\in\mathscr{B}(\mathbb{R})}$ is commutative. The same is true for F^P. That is true in general, i.e., a POVM which is the fuzzification of a spectral measure is commutative.

The fundamental result about commutative POVMs is that all of them are the fuzzification of a spectral measure [2, 4, 20, 18]: let X be Hausdorff, second countable, and locally compact. Every commutative POVM $F:\mathscr{B}(X)\to\mathscr{L}_s^+(\mathscr{H})$ is the fuzzy version of a spectral measure E^F (the sharp version of F) with the fuzzification represented by a Markov kernel μ,

$$\langle\psi, F(\Delta)\psi\rangle := \int \mu_\Delta(\lambda)\, d\langle\psi, E^F_\lambda\psi\rangle, \quad \Delta\in\mathscr{B}(X), \quad \psi\in\mathscr{H}. \tag{2}$$

The quantity $\langle\psi, E^F(\Delta)\psi\rangle$ can be interpreted as the probability that a perfectly accurate measurement (sharp measurement) of the observable represented by the spectral measure E gives a result in Δ. A possible interpretation of equation (2) is that [1, 25, 2, 4], due to measurement imprecision[1], the outcomes of the measurement of E^F are randomized: if the sharp value of the outcome of the measurement of E^F is λ then the apparatus produces with probability $\mu_\Delta(\lambda)$ a reading in Δ. As a result, the probability of an outcome in Δ is given by $\langle\psi, F(\Delta)\psi\rangle$ so that F represents an unsharp measurement of E.

In the framework of fuzzy sets theory, equation (2) can be interpreted as follows (see [12] for more details). The Markov kernel μ provides a family of fuzzy events $\{(\mathbb{R},\mu_\Delta)\}_{\Delta\in\mathscr{B}(\mathbb{R})}$. For every $\psi\in\mathscr{H}$, the expression

$$\langle\psi, F(\Delta)\psi\rangle = \int_{\mathbb{R}} \mu_\Delta(\lambda)\, d\langle\psi, E^F_\lambda\psi\rangle$$

can then be interpreted as the probability of the fuzzy event $(\mathbb{R},\mu_\Delta)$ with respect to the probability measure $\langle\psi, E^F(\cdot)\psi\rangle$. In other words, the unsharp observable F gives the probabilities of the fuzzy events $(\mathbb{R},\mu_\Delta)$ with respect to the probability measures corresponding to E^F (they are $\langle\psi, E^F(\cdot)\psi\rangle$, $\psi\in\mathscr{H}$).

It is worth remarking that starting from the fuzzy observable F it is possible to obtain the sharp observable E^F of which F is a fuzzy version (see Ref.s [5–8]).

Several proofs of (2) have been provided [2, 4, 20, 21, 18]. All of them are based on the existence, for every weak Markov kernel γ, of a Markov kernel μ which is equivalent to γ. This equivalence proof is provided in [28, 20, 21] where the space X is required to be second countable and complete metrizable (e.g., a standard Borel space) and the proof is given in the framework of stochastic operators. In section 2 we generalize this result to the case of a second countable, metrizable space (not necessarily completely metrizable) by giving an independent proof which is based on transfinite induction. Then, in section 3, we use this result in order to generalize some previous results about commutative POVMs.

First we recall the definitions of Markov and weak Markov kernel.

[1] Which can be thought to be intrinsic to the quantum measurement process and then to be unavoidable

Definition 1 Let $(\Lambda, \mathscr{A})$ be a measurable space and $\mathscr{B}(X)$ the Borel σ-algebra of a topological space X. A Markov kernel is a map $\mu : \Lambda \times \mathscr{B}(X) \to [0,1]$ such that,

1. $\mu_\Delta(\cdot)$ is a measurable function for each $\Delta \in \mathscr{B}(X)$,
2. $\mu_{(\cdot)}(\lambda)$ is a probability measure for each $\lambda \in \Lambda$.

Definition 2 Let ν be a measure on Λ. A map $\mu : \Lambda \times \mathscr{B}(X) \to [0,1]$ is a weak Markov kernel with respect to ν if:

1. $\mu_\Delta(\cdot)$ is a measurable function for each $\Delta \in \mathscr{B}(X)$,
2. for every $\Delta \in \mathscr{B}(X)$, $0 \leq \mu_\Delta(\lambda) \leq 1, \quad \nu - a.e.$,
3. $\mu_\emptyset(\lambda) = 0, \mu_\Lambda(\lambda) = I \quad \nu - a.e.$,
4. for any sequence $\{\Delta_i\}_{i\in\mathbb{N}}$, $\Delta_i \cap \Delta_j = \emptyset$,

$$\sum_i \mu_{\Delta_i}(\lambda) = \mu_{(\cup_i \Delta_i)}(\lambda), \quad \nu - a.e. \tag{3}$$

Definition 3 Two weak Markov kernels $\gamma : (\Lambda, \nu) \times \mathscr{B}(X) \to [0,1]$ and $\beta : (\Lambda, \nu) \times \mathscr{B}(X) \to [0,1]$ are said to be equivalent if, for every $\Delta \in \mathscr{B}(X)$, $\gamma_\Delta(\lambda) = \beta_\Delta(\lambda)$, $\nu - a.e.$.

2 From weak Markov kernels to Markov kernels

In what follows, $\mathscr{B}(X)$ denotes the Borel σ-algebra of a topological space X, Λ a compact subsets of $[0,1]$, ν a probability measure on Λ and $L^\infty(\Lambda, \nu)$ the space of essentially bounded measurable functions (with two functions identified if they coincide up to ν-null sets). Finally I denotes a closed subset of $[0,1]$

Lemma 1 *Let X be a second countable topological space, $\mathscr{S}$ a basis for its topology and $\mathscr{R}(\mathscr{S})$ the ring generated by $\mathscr{S}$. Let $\gamma : (\Lambda, \nu) \times \mathscr{B}(X) \to [0,1]$ be a weak Markov kernel. Then, there is a weak Markov kernel $\beta : (I, \nu_0) \times \mathscr{B}(X) \to [0,1]$ with β_Δ continuous for every $\Delta \in \mathscr{R}(\mathscr{S})$, a function $g : \Lambda \to I$ and a set $N \subset \Lambda$, $\nu(N) = 1$, such that, for every $\Delta \in \mathscr{R}(\mathscr{S})$ and for all $x \in N$, $\beta_\Delta(g(x)) = \gamma_\Delta(x)$.*

Proof Without loss of generality, we can assume Λ to be the support of ν. Let $\mathscr{A}_\nu$ be the von Neumann algebra of multiplication operators in $\mathscr{H} = L^2(\Lambda, \nu)$ which is isometrically *-isomorphic [16] to $L^\infty(\Lambda, \nu)$. In particular, for every function $f \in L^\infty(\Lambda, \nu)$ there is a multiplication operator

$$M_f : L^2(\Lambda, \nu) \to L^2(\Lambda, \nu)$$

$$[M_f(h)](x) = f(x)h(x), \quad h \in L^2(\Lambda, \nu).$$

The self-adjoint operator $B := M_x$, $[Bh](x) = [M_x(h)](x) = xh(x)$, $x \in \Lambda$ generates $\mathscr{A}_\nu$. The spectrum of B, $\sigma(B)$, coincides with the support, Λ, of ν and the spectral measure corresponding to B is $E^B(\Delta) = M_{\chi_\Delta}$. Moreover, ν is a scalar-valued spectral measure for B, i.e., ν and E^B are mutually absolutely continuous (see [16], page 133).

Now, we define the commutative POVM,

$$F(\Delta) = M_{\gamma_\Delta} = \int \gamma_\Delta(x) M_{\chi_{dx}}, \quad \Delta \in \mathscr{B}(X). \tag{4}$$

Let us consider the von Neumann algebra $\mathscr{A}^W(F)$ generated by $\{F(\Delta)\}_{\Delta\in\mathscr{B}(X)}$. It coincides with the von Neumann algebra generated by the set $O_2 := \{F(\Delta)\}_{\Delta\in\mathscr{R}(\mathscr{S})}$ (see proposition 1 in appendix A). We recall that both $\mathscr{S}$ and $\mathscr{R}(\mathscr{S})$ are countable. It can be proved that there is a generator A of $\mathscr{A}^W(F)$ with spectrum $I \subset [0,1]$ and scalar valued spectral measure ν_0. Moreover, there is a weak Markov kernel $\beta : (I, \nu_0) \times \mathscr{B}(X) \to [0,1]$ such that, for every $\Delta \in \mathscr{R}(\mathscr{S})$, $M_{\gamma_\Delta} = \beta_\Delta(A)$ with β_Δ continuous (see theorem 6 in the appendix for the details). Since $A \in \mathscr{A}^W(F) \subset \mathscr{A}_\nu$, there must be a measurable function $g : \Lambda \to I$ such that $A = M_g = \int_\Lambda g(x) M_{\chi_{dx}} = g(B)$. Then, the spectral measure E^A corresponding to A is such that $E^A(D) = E^B(g^{-1}(D)) = M_{\chi_{g^{-1}(D)}}$, $D \in \mathscr{B}(I)$. Hence, $\forall \Delta \in \mathscr{B}(X)$,

$$\begin{aligned}\int_\Lambda \gamma_\Delta(\lambda) M_{\chi_{d\lambda}} &= M_{\gamma_\Delta} = \int_I \beta_\Delta(x) E^A_{dx} \\ &= \int_I \beta_\Delta(x) M_{\chi_{g^{-1}(dx)}} = \int_\Lambda \beta_\Delta(g(\lambda)) M_{\chi_{d\lambda}}\end{aligned}$$

so that, $\beta_\Delta(g(\lambda)) = \gamma_\Delta(\lambda)$, ν-a.e.. In particular, being $\mathscr{R}(\mathscr{S})$ countable, there is a set $N \subset \Lambda$, $\nu(N) = 1$, such that, for every $\Delta \in \mathscr{R}(\mathscr{S})$ and for all $\lambda \in N$, $\beta_\Delta(g(\lambda)) = \gamma_\Delta(\lambda)$.

Theorem 1 *Let $\beta : (I, \nu_0) \times \mathscr{R}(\mathscr{S}) \to [0,1]$ be a weak Markov kernel such that β_Δ is continuous. Then there is a subset $M \subset I$, $\nu_0(M) = 1$, such that $\beta : M \times \mathscr{R}(\mathscr{S}) \to [0,1]$ is a Markov kernel.*

Proof Let $\Delta \in \mathscr{R}(\mathscr{S})$. A partition, $\mathfrak{p}^\Delta = \{\Delta_i\}_{i\in J}$, $J \subset \mathbb{N}$, of Δ is a collection of disjoint sets $\Delta_i \in \mathscr{R}(\mathscr{S})$ such that $\Delta = \cup_{i\in J} \Delta_i$.

Since β is a weak Markov kernel, to every partition $\mathfrak{p}^\Delta = \{\Delta_i\}_{i\in\mathbb{N}}$ there corresponds a set $N_{\mathfrak{p}^\Delta} \subset I$ such that $\nu_0(N_{\mathfrak{p}^\Delta}) = 1$ and

$$\beta_\Delta(\lambda) - \sum_i \beta_{\Delta_i}(\lambda) = 0, \quad \lambda \in N_{\mathfrak{p}^\Delta}. \tag{5}$$

Consider the collection $\mathfrak{P}_\Delta$ of all the partitions of Δ. The power of $\mathfrak{P}_\Delta$ is at most $\mathfrak{c}$, the power of the continuum. Let Z be the collection of all countable ordinals. Let $\mathfrak{P}_\Delta(Z)$ be a well ordering of $\mathfrak{P}_\Delta$ through Z. Then, $\mathfrak{P}_\Delta = \{\mathfrak{p}^\Delta_\alpha\}_{\alpha\in Z}$, $\mathfrak{p}^\Delta_\alpha = \{\Delta^\alpha_i\}_{i\in J^\alpha}$, $J^\alpha \subset \mathbb{N}$. Now we use transfinite recursion in order to define a new family of sets. Let $N^\Delta_\alpha := N_{\mathfrak{p}^\Delta_\alpha}$ denote the set corresponding to the partition $\mathfrak{p}^\Delta_\alpha$ as in equation (5). Let $M^\Delta_1 = N^\Delta_1$, $M^\Delta_{\alpha+1} = M^\Delta_\alpha \cap N^\Delta_{\alpha+1}$ and $M^\Delta_\eta := \cap_{\alpha<\eta} M^\Delta_\alpha$ if η is a limit ordinal.

Since every $\eta \in Z$ is an enumerable ordinal, $\nu(M^\Delta_\eta) = 1$ for every $\eta \in Z$. Moreover, $\{M^\Delta_\alpha\}_{\alpha\in Z}$ is non increasing. Now, take the closure $\overline{M^\Delta_\alpha}$ of M^Δ_α for every $\alpha \in Z$. By the continuity of β_Δ,

$$\beta_\Delta(\lambda) - \sum_i \beta_{\Delta^\alpha_i}(\lambda) = 0, \quad \lambda \in \overline{M^\Delta_\alpha}. \tag{6}$$

Since $\{\overline{M^{\Delta}_{\alpha}}\}_{\alpha\in Z}$, is a non-increasing family of closed sets, there must be an index $\eta < \omega_1$ (ω_1 denotes the first uncountable ordinal) such that $\cap_{\alpha<\omega_1}\overline{M^{\Delta}_{\alpha}} = \cap_{\alpha<\eta}\overline{M^{\Delta}_{\alpha}} = M_{\Delta}$. Since η is countable, $\nu_0(M_{\Delta}) = \nu_0(\cap_{\alpha<\eta}\overline{M^{\Delta}_{\alpha}}) = 1$. Then, for every partition $p^{\Delta}_{\alpha} = \{\Delta^{\alpha}_i\}_{i\in J^{\alpha}} \in \mathfrak{P}_{\Delta}$.,

$$\beta_{\Delta}(\lambda) - \sum_i \beta_{\Delta^{\alpha}_i}(\lambda) = 0, \quad \forall\lambda \in M_{\Delta}. \tag{7}$$

Since $\mathscr{R}(\mathscr{S})$ is countable, $M := \cap_{\Delta\in\mathscr{R}(\mathscr{S})} M_{\Delta}$ is such that $\nu_0(M) = 1$. Then, for every $\Delta \in \mathscr{R}(\mathscr{S})$ and $p^{\Delta}_{\alpha} = \{\Delta^{\alpha}_i\}_{i\in I^{\alpha}} \in \mathfrak{P}_{\Delta}$,

$$\beta_{\Delta}(\lambda) - \sum_i \beta_{\Delta^{\alpha}_i}(\lambda) = 0, \quad \lambda \in M. \tag{8}$$

Define

$$h_{\Delta}(\lambda) = \begin{cases} \beta_{\Delta}(\lambda) & \lambda \in M \\ \varphi(\Delta) & \lambda \in I/M \end{cases} \tag{9}$$

where φ is an arbitrary probability measure on $\mathscr{R}(\mathscr{S})$. The map $h : \Lambda \times \mathscr{R}(\mathscr{S}) \to [0,1]$ is a Markov kernel and coincides with β on M.

Corollary 1 *Let $\beta : (I, \nu_0) \times \mathscr{R}(\mathscr{S}) \to [0,1]$ be a weak Markov kernel such that β_{Δ} is continuous. Then there is a set $M \subset I$, $\nu_0(M) = 1$, and a Markov kernel $h : I \times \mathscr{R}(\mathscr{S}) \to [0,1]$ such that $\beta_{\Delta}(\lambda) = h_{\Delta}(\lambda)$ for every $\lambda \in M$. In particular, β and h are ν_0-equivalent.*

Corollary 2 *Let γ, β, g and h be as in lemma 1 and theorem 1. Then, $h\circ g : \Lambda \times \mathscr{R}(\mathscr{S}) \to [0,1]$ is a Markov kernel. Moreover, there is a set $\Lambda_1 \subset \Lambda$, $\nu(\Lambda_1) = 1$, such that $h_{\Delta}(g(\lambda)) = \gamma_{\Delta}(\lambda)$, $\lambda \in \Lambda_1$.*

Proof Since $h : I \times \mathscr{R}(\mathscr{S}) \to [0,1]$ is a Markov kernel, $h\circ g : \Lambda \times \mathscr{R}(\mathscr{S}) \to [0,1]$ is a Markov kernel as well. Let $\Lambda_0 := g^{-1}(M)$ where M has been defined in theorem 1. Then, $E^B(\Lambda_0) = E^B[g^{-1}(M)] = E^A(M) = \mathbf{1}$, where $\mathbf{1}$ denotes the identity operator. Therefore, $\nu(\Lambda_0) = 1$. Let $\Lambda_1 := \Lambda_0 \cap N$ where N has been defined in lemma 1. Then, by lemma 1, $h_{\Delta}(g(\lambda)) = \beta_{\Delta}(g(\lambda)) = \gamma_{\Delta}(\lambda)$ for every $\Delta \in \mathscr{R}(\mathscr{S})$ and $\lambda \in \Lambda_1$, $\nu(\Lambda_1) = 1$.

Theorem 2 *The Markov kernel $h\circ g : \Lambda \times \mathscr{R}(\mathscr{S}) \to [0,1]$ in corollary 2 can be extended to a Markov kernel $\mu : \Lambda \times \mathscr{B}(X) \to [0,1]$ such that, for every $\Delta \in \mathscr{B}(X)$, $\mu_{\Delta}(\lambda) = \gamma_{\Delta}(\lambda)$, ν-a.e..*

Proof For every $\lambda \in \Lambda$, the measure $h_{(\cdot)}(g(\lambda)) : \mathscr{R}(\mathscr{S}) \to [0,1]$ can be extended to the Borel σ-algebra $\mathscr{B}(X)$. Let $\mu_{(\cdot)}(\lambda) : \mathscr{B}(X) \to [0,1]$ denotes such an extension. We want to show that, for each $\Delta \in \mathscr{B}(X)$, $\mu_{\Delta} : \Lambda \to [0,1]$ is measurable and $\mu_{\Delta} = \gamma_{\Delta}$, ν-a.e.. That can be proved by using transfinite induction. We start by recalling the definition of Borel Hierarchy in a second countable metrizable space [22,27]. Let ω_1 be the first uncountable ordinal. Let $\mathscr{K}$ be a family of subsets of X. The Borel classes, $\mathscr{B}(\mathscr{K})$, generated by $\mathscr{K}$ are defined inductively as follows. $\Sigma^0_0 = \emptyset$, $\Sigma^0_1 = \mathscr{K}$, Σ^0_2 is

the class of countable unions of sets in $\Pi_1^0 = \{X - \Delta, \Delta \in \mathscr{K}\}$, Σ_α^0, $2 < \alpha < \omega_1$, is the class of countable unions of sets in $\cup_{\beta<\alpha}\Pi_\beta^0$ where $\Pi_\beta^0 = \{X - \Delta \,|\, \Delta \in \Sigma_\beta^0\}$. Then $\mathscr{B}(\mathscr{K}) = \cup_{\alpha<\omega_1}\Sigma_\alpha^0$. If $\mathscr{K} = \mathscr{R}(\mathscr{S})$ then $\cup_{\alpha<\omega_1}\Sigma_\alpha^0 = \mathscr{B}(X)$ (see [27], proposition 3.6.1, page 116).
Let $G \in \mathscr{R}(\mathscr{S})$. Then $\mu_G = h_G \circ g$ is measurable and, by corollary 2, it is such that $\mu_G(\lambda) = \gamma_G(\lambda)$, $\lambda \in \Lambda_1$. Suppose that for every $\Delta \in \Sigma_\beta^0$, $\beta < \alpha$, the function μ_Δ is measurable and such that $\mu_\Delta = \gamma_\Delta$, ν-a.e.. As a consequence, μ_Δ is measurable and such that $\mu_\Delta = \gamma_\Delta$, ν-a.e. for every $\Delta \in \Pi_\beta^0$. Let $\Delta \in \Sigma_\alpha^0$. Then, $\Delta = \cup_{i=1}^\infty \Delta_i$, $\Delta_i \in \Pi_\beta^0$. Setting $\widetilde{\Delta}_n = \sum_{i=1}^n \Delta_i$, we obtain a non-decreasing family of sets such that $\widetilde{\Delta}_n \uparrow \Delta$. Note that, due to the fact that the pi-classes Π_β^0 are closed with respect to finite unions, for every $n \in \mathbb{N}$ there is an index $\beta < \alpha$ such that $\widetilde{\Delta}_n \in \Pi_\beta^0$. Hence, $\mu_{\widetilde{\Delta}_n}$ is measurable and, for every $\lambda \in \Lambda$,

$$\mu_\Delta(\lambda) = \lim_{n\to\infty} \mu_{\widetilde{\Delta}_n}(\lambda)$$

which prove the measurability of μ_Δ. Hence, $\mu : \Lambda \times \mathscr{B}(X) \to [0,1]$ is a Markov kernel. Moreover, by the inductive hypothesis, $\mu_{\widetilde{\Delta}_n}(\lambda) = \gamma_{\widetilde{\Delta}_n}$, ν-a.e.. Then,

$$\begin{aligned}\int \gamma_\Delta(\lambda)\, dE_\lambda^B = F(\Delta) &= \lim_{n\to\infty} F(\widetilde{\Delta}_n) \\ &= \lim_{n\to\infty} \int_\Lambda \gamma_{\widetilde{\Delta}_n}(\lambda)\, dE_\lambda^B = \lim_{n\to\infty} \int_\Lambda \mu_{\widetilde{\Delta}_n}(\lambda)\, dE_\lambda^B = \int_\Lambda \mu_\Delta(\lambda)\, dE_\lambda^B\end{aligned}$$

so that $\mu_\Delta(\lambda) = \gamma_\Delta(\lambda)$, ν-a.e.

The following theorem is a straightforward consequence of lemma 1, theorem 1, corollary 2 and theorem 2.

Theorem 3 *Let X be a second countable topological space. Let* $\gamma : (\Lambda, \nu) \times \mathscr{B}(X) \to [0,1]$ *be a weak Markov kernel. Then, there is a Markov kernel* $\mu : \Lambda \times \mathscr{B}(X) \to [0,1]$ *such that, for every* $\Delta \in \mathscr{B}(X)$, $\mu_\Delta = \gamma_\Delta$, *$\nu$-a.e..*

3 Fuzzy Observables

The previous result can be used to generalize the connection between commutative POVMs and Markov kernels (see equation (2)) to the case of POVMs defined on a second countable, metrizable space. In its turn that leads to the interpretation of commutative POVMs as the fuzzification of a spectral measure as we have discussed in the introduction.

The following theorem gives a characterization of commutative POVMs as smearings of spectral measures with the smearing realized by means of Markov kernels and generalizes some previous results [2,4,20,21]. The possible extension of the results connecting Naimark's operators and sharp reconstructions [6,7,13,9,5] to the case of a metrizable second countable space can be analyzed as well.

In the following, the symbol μ is used to denote both Markov kernels and weak Markov kernels. The symbol $\mathscr{A}^W(F)$ denotes the von Neumann algebra generated by the POVM F, i.e., the von Neumann algebra generated by the set $\{F(\Delta)\}_{\mathscr{B}(X)}$. Analogously $\mathscr{A}(B)$ denotes the von Neumann algebra generated by the self-adjoint operator B.

Definition 4 Whenever F, A, and μ are such that $F(\Delta) = \mu_\Delta(A)$, $\Delta \in \mathscr{B}(X)$, we say that (F, A, μ) is a von Neumann triplet.

Theorem 4 *Let X be a second countable, metrizable space. A POVM $F : \mathscr{B}(X) \to \mathscr{L}_s^+(\mathscr{H})$ is commutative if and only if, there exists a bounded self-adjoint operator $A = \int \lambda \, dE_\lambda$ with spectrum $\sigma(A) \subset [0,1]$ and a Markov Kernel μ such that:*

$$F(\Delta) = \int_{\sigma(A)} \mu_\Delta(\lambda) \, dE_\lambda, \quad \Delta \in \mathscr{B}(X). \tag{10}$$

Proof The von Neumann algebra $\mathscr{A}^W(F)$ is commutative. Then, it is singly generated by a self-adjoint operator A with compact spectrum $\sigma(A) \subset [0,1]$. Hence, for every $\Delta \in \mathscr{B}(X)$, there is a measurable function $\gamma_\Delta : \sigma(A) \to [0,1]$ such that

$$F(\Delta) = \int_{\sigma(A)} \gamma_\Delta(\lambda) \, dE^A_\lambda \tag{11}$$

where E^A is the spectral measure corresponding to A. Let Δ be the disjoint union of the sets $\{\Delta_i\}_{i \in \mathbb{N}}$. Then,

$$\int_{\sigma(A)} \gamma_\Delta(\lambda) \, dE^A_\lambda = F(\Delta) = \sum_{i=1}^{\infty} F(\Delta_i) = \sum_{i=1}^{\infty} \int_{\sigma(A)} \gamma_{\Delta_i}(\lambda) \, dE^A_\lambda \tag{12}$$

For each $\lambda \in \sigma(A)$, $\sum_{i=1}^n \gamma_{\Delta_i}(\lambda)$ is an nondecreasing family of measurable functions. Then, it converges to a measurable function f_Δ and, by the Lebesgue convergence theorem,

$$\int_{\sigma(A)} \gamma_\Delta(\lambda) \, dE^A_\lambda = \sum_{i=1}^{\infty} \int_{\sigma(A)} \gamma_{\Delta_i}(\lambda) \, dE^A_\lambda = \int_{\sigma(A)} f_\Delta(\lambda) \, dE^A_\lambda \tag{13}$$

Hence,

$$\gamma_\Delta(\lambda) = f_\Delta(\lambda) = \sum_{i=1}^{\infty} \gamma_{\Delta_i}(\lambda), \quad E^A - a.e.$$

In other words, $\gamma : (\sigma(A), \nu^A) \times \mathscr{B}(X) \to [0,1]$ is a weak Markov kernel. Here $\nu^A(\Delta) := \langle \psi_0, E^A(\cdot)\psi_0 \rangle$ where ψ_0 is a separating vector for $\mathscr{A}^W(F) = \mathscr{A}^W(A)$ and $E^A(\Delta) = \mathbf{0}$ if and only if $\nu^A(\Delta) = 0$.

By theorem 3 there is a Markov kernel $\mu : \sigma(A) \times \mathscr{B}(X) \to [0,1]$ which is equivalent to γ with respect to ν^A. Therefore, μ is such that

$$F(\Delta) = \int_{\sigma(A)} \mu_\Delta(\lambda) \, dE^A_\lambda.$$

In Ref. [4] it has been proved that if X is Hausdorff, locally compact and second countable, the Markov kernel can be replaced by a Feller Markov kernel (theorem 4.3 in [4]). Thanks to theorems 3 and to proposition 1 in the appendix, the proof can be generalized to the case of an arbitrary POVM defined on a second countable metrizable space. We limit ourselves to restate the theorem since the only difference in the proof is in the use of transfinite induction (see proposition 1 in the appendix) in order to show that $\mathscr{A}^W(F) = \mathscr{A}^W(O_2)$ avoiding to require that F is regular and in the use of theorem 3 in order to replace a weak Markov kernel by an equivalent Markov kernel.

Definition 5 Let $E : \mathscr{B}(\Lambda) \to \mathscr{E}(\mathscr{H})$ be a PVM. A map $\mu_{(\cdot)}(\cdot) : \Lambda \times \mathscr{B}(X) \to [0,1]$ is a strong Markov kernel with respect to E if it is a weak Markov kernel with respect to E and there exists a set $\Gamma \subset \Lambda$, $E(\Gamma) = \mathbf{1}$, such that $\mu_{(\cdot)}(\cdot) : \Gamma \times \mathscr{B}(X) \to [0,1]$ is a Markov kernel. A strong Markov kernel is denoted by the symbol $(\mu, E, \Gamma \subset \Lambda)$.

Definition 6 A Feller Markov kernel is a Markov kernel $\mu_{(\cdot)}(\cdot) : \Lambda \times \mathscr{B}(X) \to [0,1]$ such that the function

$$G(\lambda) = \int_X f(x)\, \mu_{dx}(\lambda), \quad \lambda \in \Lambda$$

is continuous and bounded whenever f is continuous and bounded.

Theorem 5 *Let X be a second countable, metrizable space. Let $F : \mathscr{B}(X) \to \mathscr{F}(\mathscr{H})$ be a regular POVM. Then, F is commutative if and only if, there exists a bounded self-adjoint operator $A = \int \lambda\, dE_\lambda$ with spectrum $\sigma(A) \subset [0,1]$ and a strong Markov Kernel $(\mu, E, \Gamma \subset \sigma(A))$ such that:*

1) $\mu_\Delta(\cdot) : \sigma(A) \to [0,1]$ is continuous for each $\Delta \in \mathscr{R}(\mathscr{S})$,
2) $F(\Delta) = \int_\Gamma \mu_\Delta(\lambda)\, dE_\lambda, \quad \Delta \in \mathscr{B}(X)$.
3) $\mathscr{A}^W(F) = \mathscr{A}^W(A)$.
4) $\mu : \Gamma \times \mathscr{B}(X) \to [0,1]$ is a Feller Markov kernel.

Appendix

The following proposition has been proved in reference [12].

Proposition 1 ([12]) *Let X be second countable. Let $\mathscr{S}$ be a basis for the topology of X. Let $\mathscr{R}(\mathscr{S})$ be the ring generated by $\mathscr{S}$. Let $F : \mathscr{B}(X) \to \mathscr{L}_s^+(\mathscr{H})$ be a POVM. Then, the von Neumann algebra $\mathscr{A}^W(\mathscr{R}(\mathscr{S}))$ generated by $\{F(\Delta)\}_{\Delta \in \mathscr{R}(\mathscr{S})}$ coincides with the von Neumann algebra $\mathscr{A}^W(F)$.*

The following theorem has been proved in Ref. [4] where the POVM F was required to be normal. Such an assumption can be relaxed thanks to proposition 1. The rest of the proof is unchanged and is repeated here for the readers convenience.

Theorem 6 *Let X be second countable and metrizable. Let $F : \mathscr{B}(X) \to \mathscr{L}_s^+(\mathscr{H})$ be a POVM and $\mathscr{A}^W(F)$ the von Neumann algebra generated by F. Then, there is a generator A and a weak Markov kernel $\beta : \sigma(A) \times \mathscr{B}(X) \to [0,1]$ such that*

$$F(\Delta) = \int_{\sigma(A)} \beta_\Delta(\lambda)\, dE^A_\lambda, \quad \Delta \in \mathscr{B}(X),$$

and β_Δ is continuous for every $\Delta \in \mathscr{R}(\mathscr{S})$.

Proof By proposition 1, the von Neumann algebra $\mathscr{A}^W(F)$ coincides with the von Neumann algebra generated by the set $O_2 := \{F(\Delta)\}_{\Delta \in \mathscr{R}(\mathscr{S})}$. We recall that both $\mathscr{S}$ and $\mathscr{R}(\mathscr{S})$ are countable. Now, let $\{\Delta_i\}_{i\in\mathbb{N}}$ be an enumeration of the set $\mathscr{R}(\mathscr{S})$. Let $E^{(i)}$ denote the spectral measure corresponding to $F(\Delta_i) \in O_2$. We have $F(\Delta_i) = \int x\, dE^{(i)}_x$. Therefore, for each $i,k \in \mathbb{N}$ there exists a division $\{\Delta^{(i,k)}_j\}_{j=1,\dots,m_{i,k}}$ of $[0,1]$ such that

$$\Big\| \sum_{j=1}^{m_{i,k}} x^{(i,k)}_j E^{(i)}(\Delta^{(i,k)}_j) - F(\Delta_i) \Big\| \le \frac{1}{k}. \tag{14}$$

where, $x^{(i,k)}_j \in \Delta^{(i,k)}_j$ for any $i,k \in \mathbb{N}$ and $j = 1,\dots,m_{i,k}$. By the spectral theorem, $\{E^{(i)}(\Delta^{i,k}_j)\}_{j \le m_{i,k}} \subset \mathscr{A}^W(F)$ for any $i,k \in \mathbb{N}$. Therefore, the von Neumann algebra $\mathscr{A}^W(D)$ generated by the set $D := \{E^{(i)}(\Delta^{i,k}_j),\ j \le m_{i,k},\ i,k \in \mathbb{N}\}$ is contained in $\mathscr{A}^W(F)$

$$\mathscr{A}^W(D) \subset \mathscr{A}^W(F) = \mathscr{A}^W(O_2). \tag{15}$$

Moreover, by (14)

$$\mathscr{A}^C(O_2) \subset \mathscr{A}^C(D) \subset \mathscr{A}^W(F).$$

where $\mathscr{A}^C(O_2)$ and $\mathscr{A}^C(D)$ are the C^*-algebras generated by O_2 and D respectively. By the double commutant theorem,

$$\mathscr{A}^W(F) = [\mathscr{A}^C(O_2)]'' \subset [\mathscr{A}^C(D)]'' = \mathscr{A}^W(D)$$

so that (see equation 15),

$$\mathscr{A}^W(D) = \mathscr{A}^W(F). \tag{16}$$

By theorem 11, page 871 in Ref. [17], there is a homeomorphism $\pi : \Lambda \to \pi(\Lambda) \subset \prod_{i=1}^\infty \{0,1\}$ which identifies the spectrum Λ of $\mathscr{A}^C(D)$ with a closed subset of $\prod_{i=1}^\infty \{0,1\}$. Moreover, the function $f : \Lambda \to [0,1]$,

$$f(\lambda) := \sum_{i=1}^\infty \frac{x_i}{3^i}; \quad (x_1,\dots,x_n,\dots) = \pi(\lambda)$$

is continuous and injective and then it distinguishes the points of Λ. Since Λ and $[0,1]$ are Hausdorff, $f : \Lambda \to f(\Lambda)$ is a homeomorphism.
By the Gelfand-Naimark theorem and the spectral theorem for representations of commutative C^*-algebras, there is an isometric *-isomorphism between $\mathscr{A}^C(D)$ and $\mathscr{C}(\Lambda)$

$$\begin{aligned} T : \mathscr{C}(\Lambda) &\to \mathscr{A}^C(D) \subset B(\mathscr{H}) \\ g &\mapsto T(g) = \int_\Lambda g(\lambda)\, d\widetilde{E}_\lambda. \end{aligned} \tag{17}$$

where $\widetilde{E}$ is the spectral measure from $\mathscr{B}(\Lambda)$ to $\mathscr{E}(\mathscr{H})$ corresponding to T.
Since f distinguishes the points of Λ, it generates $\mathscr{C}(\Lambda)$ and then

$$A = \int_{\Lambda} f(\lambda)\, d\widetilde{E}_{\lambda}$$

generates both $\mathscr{A}^C(D)$ and $\mathscr{A}^W(F)$.

Now, we proceed to the proof of the existence of the weak Markov kernel β.

By (17), for each $\Delta \in \mathscr{R}(\mathscr{S})$, there exists a continuous function $\gamma_\Delta \in \mathscr{C}(\Lambda)$ such that

$$F(\Delta) = \int_{\Lambda} \gamma_\Delta(\lambda)\, d\widetilde{E}_{\lambda}.$$

Let us consider the continuous function

$$\nu_\Delta(t) := (\gamma_\Delta \circ f^{-1})(t), \quad \Delta \in \mathscr{R}(\mathscr{S}).$$

By the change of measure principle, we have,

$$\begin{aligned} F(\Delta) &= \int_{\Lambda} \gamma_\Delta(\lambda)\, d\widetilde{E}_{\lambda} = \int_{\Lambda} \gamma_\Delta(f^{-1}(f(\lambda)))\, d\widetilde{E}_{\lambda} \\ &= \int_{I} \gamma_\Delta(f^{-1}(t))\, dE^A_t = \int_{I} \nu_\Delta(t)\, dE^A_t = \nu_\Delta(A) \end{aligned}$$

where $I = f(\Lambda)$ and E^A is the spectral measure corresponding to A and defined by $E^A(\Delta) = \widetilde{E}(f^{-1}(\Delta))$, $\Delta \in \mathscr{B}(I)$. Therefore, for each $\Delta \in \mathscr{R}(\mathscr{S})$, $\nu_\Delta(f(\lambda)) = \gamma_\Delta(\lambda)$, $\lambda \in \Lambda$, and $F(\Delta) = \nu_\Delta(A)$.
Now, we extend ν to all $\mathscr{B}(X)$. Since A is the generator of $\mathscr{A}^W(F)$, for each $\Delta \in \mathscr{B}(X)$, there exists a Borel function ω_Δ such that.

$$F(\Delta) = \int_{I} \omega_\Delta(t)\, dE^A_t$$

Then, we can consider the map $\beta : \sigma(A) \times \mathscr{B}(X) \to [0,1]$ defined as follows

$$\beta_\Delta(\lambda) = \begin{cases} \nu_\Delta(\lambda) & if \quad \Delta \in \mathscr{R}(\mathscr{S}) \\ \omega_\Delta(\lambda) & if \quad \Delta \notin \mathscr{R}(\mathscr{S}). \end{cases} \tag{18}$$

which coincides with ν on $\mathscr{R}(\mathscr{S})$ and is such that $\beta_\Delta(A) = F(\Delta)$. Now, let $\psi_0 \in \mathscr{H}$ be a separating vector for $\mathscr{A}^W(A)$ and $\nu_0(\cdot) := \langle \psi_0, E^A(\cdot)\psi_0 \rangle$. In order to prove that $\beta : (I, \nu_0) \times \mathscr{B}(X) \to [0,1]$ is a weak Markov kernel, we proceed as in the proof of theorem 4 (see equation (13)). Note that $\nu_0(\Delta) = 0$ if and only if $E^A(\Delta) = \mathbf{0}$.

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