Some remarks on the **NC** category

October 22, 2019

Abstract

Some remarks that appeared from the discussion at our seminar.

1 Definition and duality in NC

The category NC

It seems more natural to determine the objects of the category as triples (V, C, K), where (V, C) is a finite dimensional ordered vector space and $K \subset C$ is such that

(i) K is the base of the cone $span(K) \cap C$ in $span(K)$,

(ii) $cone([0, K]) = C$.

This is better than to use the generating intervals $[0, K]$ as objects, since we do not have an intrinsic description of the intervals, so we always need the triple to describe the morphisms or to prove anything. This is similar to the case of convex effect algebras: the category is equivalent to the category of order unit spaces (at least in the finite dim. case), whose objects are triples (V, C, u).

The morphisms $(V_1, C_1, K_1) \to (V_2, C_2, K_2)$ in the category are positive maps $f : (V_1, C_1) \to (V_2, C_2)$ such that $f(K_1) \subseteq f(K_2)$.

The sets K

The sets satisfying (i) and (ii) above can be seen to coincide with base sections studied in [**?**]. A better name and/or set of defining conditions can be considered. Such sets should definitely be given a name, I will refer to them as BS's for the time being. Another name that appeared at the seminar were the lid (dekel) of the interval, or an order lid in (V, C) (everything can be pushed beneath).

Properties of BS's

Any BS in (V, C) is a compact convex subset of C and lies in some base of C. Any such base is determined by an order unit $u \in int(C^*)$ as

$$B_u := \{c \in C,\ \langle u, c\rangle = 1\}$$

and K is the section $K = span(K) \cap B_u$ for u such that $\langle u, x\rangle = 1$ for all $x \in K$. Moreover, K is the intersection of all bases of C that contain K. For the dual BS $U(K)$, we have

$$ri(U(K)) = \{u \in int(C^*), K \subseteq B_u\}$$

(ri denotes the relative interior).

For any subset K of some base B of C, $U(K)$ is a BS in (V^*, C^*) and $U^2(K)$ is the smallest BS in (V, C) containing K, equivalently, it is the intersection of all bases of C containing K.

Initial/terminal objects in NC

Assume that $V = \{0\}$ and $C = \{0\}$, this is an ordered vector space. There are two possibilities for a BS: $K = \{0\}$ and $K = \emptyset$. Note that these are indeed BS's: $span(K) = \{0\}$ in both cases and since a base of a cone is defined as a convex subset such that each *nonzero* $c \in C$ can be written as etc..., we see that both $\{0\}$ and $\emptyset$ are bases of $C = \{0\}$. Moreover $[0, 0] = \{0\}$ and $[0, \emptyset] = \emptyset$, so that (ii) is satisfied as well. Note that we have $U(\emptyset) = \{0\}$ and $U(\{0\}) = \emptyset$. We will further denote these objects as $\boldsymbol{\emptyset} := (\{0\}, \{0\}, \emptyset)$ and $\mathbf{0} := (\{0\}, \{0\}, \{0\})$. Note that $\boldsymbol{\emptyset}$ is an initial object in **NC**, while $\mathbf{0}$ is a terminal object.

Duality in NC

The duality in **NC** is a contravariant functor that maps (V, C, K) to $(V^*, C^*, U(K))$ and any morphism $f : (V_1, C_1, K_1) \to (V_2, C_2, K_2)$ to $U(f) : (V_2^*, C_2^*, U(K_2)) \to (V_1^*, C_1^*, U(K_1))$, given by

$$U(f)(\varphi) = \varphi \circ f = f^*(\varphi), \qquad \varphi \in V_2^*.$$

Note that $U(K)$ is the set of all morphisms $(V, C, K) \to \mathbf{1}$, where $\mathbf{1} = (\mathbb{R}, \mathbb{R}^+, 1)$ is another special (dualizing) object. The functor U gives the equivalence of categories **NC** and $\mathbf{NC}^{op}$.

2 Monoidal structures

A natural way to construct a symmetric monoidal structure comes from the category of (finite dimensional) vector spaces, where it can be constructed as a universal bimorphism. Indeed, we can use this structure and just check that all morphisms are indeed morphisms in **NC**.

Bimorphisms in NC

Let (V_1, C_1, K_1), (V_2, C_2, K_2), (V, C, K) be objects in **NC**. A bimorphism in **NC** is a map $b : V_1 \times V_2 \to V$ (in Set) such that

1. for any $v_1 \in V_1$, $v_2 \in V_2$, $b(v_1, \cdot) : V_2 \to V$ and $b(\cdot, v_2) : V_1 \to V$ are linear maps (that is, b is a vector space bimorphism);

2. for $c_1 \in C_1$, $c_2 \in C_2$, $b(c_1, c_2) \in C$ (that is, b is an OVS bimorphism);
3. for $x_1 \in K_1$, $x_2 \in K_2$, $b(x_1, x_2) \in K$.

Universal bimorphisms

A universal bimorphism is an object $(\tilde{V}, \tilde{C}, \tilde{K})$ and a bimorphism $\otimes : V_1 \times V_2 \to \tilde{V}$ such that for any bimorphism b there is a unique morphism $f : (\tilde{V}, \tilde{C}, \tilde{K}) \to (V, C, K)$ such that the following diagram commutes:

$$\begin{tikzcd} V_1 \times V_2 \arrow[d, "\otimes"'] \arrow[dr, "b"] & \\ \tilde{V} \arrow[r, dashed, "f"'] & V \end{tikzcd}$$

As noted above, we will use the universal bimorphisms in Vect. Let us put $\tilde{V} = V_1 \otimes V_2$ and let $\otimes(v_1, v_2) := v_1 \otimes v_2$ (in Vect). Since b is a bimorphism in Vect, we have from universality in Vect that the morphism f exists in Vect and is unique. All we have to do is to chose the cone $\tilde{C}$ and the BS $\tilde{K}$ in such a way that f is a morphism in **NC**. A natural choice is

$$\tilde{C} = cone(\{c_1 \otimes c_2, c_1 \in C_1, c_2 \in C_2\})$$

and

$$\tilde{K} = U^2(\{x_1 \otimes x_2, x_1 \in K_1, x_2 \in K_2\}),$$

that is, the smallest BS containing all $x_1 \otimes x_2$.

Some remarks are in order:

1. we have
$$\tilde{C} = C_1 \otimes C_2 = \{\sum_i c_i \otimes d_i, c_i \in C_1, d_i \in C_2\} \tag{1}$$
(the only perhaps nontrivial question here is whether this is closed). Moreover, with this choice, $\otimes$ is a universal bimorphism in OVS, this should be cited from somewhere, but is straightforward to check from (1).

2. there is the notion of a minimal tensor product of compact convex sets:
$$K_1 \otimes_{min} K_2 = co\{x_1 \otimes x_2,\ x_1 \in K_1, x_2 \in K_2\}.$$
The trouble is that indeed in general this is not a BS in $C_1 \otimes C_2$. (I have a stupid roundabout way to show this, involving non-existence of extensions of positive maps, but perhaps there is something better).

3. Let $u_1 \in int(C_1^*)$, $u_2 \in int(C_2^*)$ be such that $K_1 \subseteq B_{u_1}$ and $K_2 \subseteq B_{u_2}$, then $x_1 \otimes x_2 \in B_{u_1 \otimes u_2}$ for all $x_1 \in K_1$ and $x_2 \in K_2$, so that $\tilde{K}$ is well defined and in fact
$$\tilde{K} = U^2(K_1 \otimes_{min} K_2)$$
is the smallest BS containing $K_1 \otimes_{min} K_2$.

So let us check that f is a morphism in **NC**. We already know that it is linear and positive (by the remark 1. above). Let $y \in \tilde{K}$, we need to show that $f(y) \in K$. By duality, this is equivalent to requiring that $\langle \varphi, f(y) \rangle = 1$ for all $\varphi \in U(K)$. By the properties of bimorphisms, we know that $f(x_1 \otimes x_2) \in K$ for all $x_1 \in K_1$, $x_2 \in K_2$, so that $1 = \langle \varphi, f(x_1 \otimes x_2) \rangle = \langle f^*(\varphi), x_1 \otimes x_2 \rangle$, hence $f^*(\varphi) \in U(K_1 \otimes_{min} K_2)$. Since $y \in \tilde{K} = U^2(K_1 \otimes_{min} K_2)$, it follows that $1 = \langle f^*(\varphi), y \rangle = \langle \varphi, f(y) \rangle$. So $f(\tilde{K}) \subseteq K$ and f is a morphism in **NC**.

We have proved that with this structure, $\otimes$ is a universal bimorphism in **NC**. We will further denote $\tilde{K} = K_1 \otimes K_2$, hoping that this will not cause too much confusion.

Symmetric monoidal structure from universal bimorphisms

A symmetric monoidal structure on **NC** consist of:

- a functor $\otimes : \mathbf{NC} \times \mathbf{NC} \to \mathbf{NC}$
- the unit object $\mathbf{1} \in \mathbf{NC}$
- natural isomorphisms:

$$\begin{aligned} \alpha_{K_1,K_2,K_3} :& (K_1 \otimes K_2) \otimes K_3 \to K_1 \otimes (K_2 \otimes K_3) \qquad \text{(associator)} \\ \lambda_K :& \mathbf{1} \otimes K \to K \qquad \text{(left unitor)} \\ \rho_K :& K \otimes \mathbf{1} \to K \qquad \text{(right unitor)} \\ \sigma_{K_1,K_2} :& K_1 \otimes K_2 \to K_2 \otimes K_1, \qquad \sigma_{K_2,K_1} \circ \sigma_{K_1,K_2} = id_{K_1 \otimes K_2} \end{aligned}$$

satisfying certain coherences. (Note that here we denote the object (V, C, K) just as K, but one always has to keep the ambient OVS in mind.) Since we already have a symmetric monoidal structure in Vect, all we need to do is to set the unit object to $\mathbf{1} = (\mathbb{R}, \mathbb{R}^+, 1)$ and check that all the involved morphisms (in Vect) are morphisms in **NC**. Since these structures are obtained from universality of $\otimes$, this can be done using the results of the previous paragraph. This is easily accomplished for λ, ρ and σ:

Clearly, $\lambda_V : \mathbf{1} \otimes V \to V$, the left unitor for the symmetric monoidal structure in Vect, is determined by

$$\lambda_V(s \otimes v) = sv, \qquad s \in \mathbb{R},\ v \in V.$$

If $(V, C, K) \in \mathbf{NC}$, then obviously $(s, v) \mapsto sv$ is a bimorphism in **NC**, for which the unique morphism $\mathbf{1} \otimes K \to K$ is $\lambda_K \equiv \lambda_V$. One similarly obtains $\rho_K \equiv \rho_V$ and $\sigma_{K_1,K_2} \equiv \sigma_{V_1,V_2}$.

The associator is somewhat more involved. So let α_{V_1,V_2,V_3} be the associator in Vect, we have to prove that it is a morphism in **NC**. Let us fix some $x_3 \in K_3$ and define the bimorphism

$$b_{x_3} : V_1 \times V_2 \to V_1 \otimes (V_2 \otimes V_3), \quad b_{x_3}(v_1, v_2) = v_1 \otimes (v_2 \otimes x_3).$$

Then b_{x_3} is a bimorphism in **NC** and the unique **NC** morphism $f_{x_3} : K_1 \otimes K_2 \to K_1 \otimes (K_2 \otimes K_3)$ must be equal to $f_{x_3} = \alpha_{V_1,V_2,V_3}(\cdot \otimes x_3)$. It follows that $(v, v_3) \mapsto \alpha_{V_1,V_2,V_3}(v \otimes v_3)$ defines a bimorphism $V_1 \otimes V_2 \times V_2 \to V_1 \otimes (V_2 \otimes V_3)$ in **NC**, such that the unique morphism $(K_1 \otimes K_2) \otimes K_3 \to K_1 \otimes (K_2 \otimes K_3)$ is $\alpha_{K_1,K_2,K_3} \equiv \alpha_{V_1,V_2,V_3}$.

NC as a *-autonomous category

This will be done next.