RIGHT RÉNYI MEAN AND TENSOR PRODUCT

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Abstract. We study in this paper the right Rényi mean for a quantum divergence induced from the $\alpha - z$ Rényi relative entropy. Many properties including homogeneity, invariance under permutation, repetition and unitary congruence transformation, and determinantal inequality have been presented. Moreover, we give the identity of two right Rényi means with respect to tensor product.

Mathematics Subject Classification (2020): 81P17, 15B48
Keywords: Rényi relative entropy, quantum divergence, right mean, determinantal inequality, tensor product

1. Introduction

A divergence is a generalization of squared distance. Indeed, it is almost a distance function that may not be symmetric or not satisfy the triangle inequality. It is originated in statistics, probability theory and information theory, and recently plays important roles in many practical areas such as signal processing [13], medical image analysis [11], econometrics [12], and clustering algorithms [1, 3, 8]. In particular, the notion of mean is essential in clustering algorithms based on $k-$means.

Analogous to the least squares mean in a metric space, we consider in this paper the mean minimizing the weighted sum of divergences. On the other hand, since the divergence D on a set X is not symmetric in general, we have two kinds of weighted means

$$\underset{x \in X}{\arg\min} \sum_{i=1}^{n} w_i D(a_i, x) \quad \text{and} \quad \underset{x \in X}{\arg\min} \sum_{i=1}^{n} w_i D(x, a_i)$$

for given $a_1, a_2, \dots, a_n \in X$, where $(w_1, w_2, \dots, w_n)$ is a positive probability vector in $\mathbb{R}^n$. We call these the right mean and left mean, respectively. Chebbi and Moakher [7] have introduced the right mean and left mean of positive definite Hermitian matrices for the log-determinant $\alpha-$divergence:

$$D_{LD}^{\alpha}(A,B) = \frac{4}{1-\alpha^2}\left[\log\det\left(\frac{1-\alpha}{2}A + \frac{1+\alpha}{2}B\right) - \log(\det A)^{\frac{1-\alpha}{2}}(\det B)^{\frac{1+\alpha}{2}}\right]$$

for $\alpha \in (-1,1)$ and $A, B \in \mathbb{P}_m$, where $\mathbb{P}_m$ is the open convex cone of all $m \times m$ positive definite Hermitian matrices.

Recently, a new quantum divergence has been recently introduced [9]: for $0 < \alpha \leq z < 1$

$$\Phi_{\alpha,z}(A,B) = \mathrm{tr}((1-\alpha)A + \alpha B) - \mathrm{tr}\left(A^{\frac{1-\alpha}{2z}} B^{\frac{\alpha}{z}} A^{\frac{1-\alpha}{2z}}\right)^z. \tag{1.1}$$

For $\alpha = z = \frac{1}{2}$, $\Phi_{\alpha,z}(A,B)$ coincides with the Bures-Wasserstein metric of A and B [6]. Moreover, it has been shown that the right mean for the quantum divergence $\Phi_{\alpha,z}$ exists uniquely and coincides with the unique positive definite solution of the equation

$$X = \sum_{i=1}^{n} w_i \left(X^{\frac{\alpha}{2z}} A_i^{\frac{1-\alpha}{z}} X^{\frac{\alpha}{2z}}\right)^z.$$

In this paper we verify many interesting properties such as homogeneity, invariance under permutation, repetition and unitary congruence transformation, and determinantal inequality. Furthermore, we see the identity of two right Rényi means for the quantum divergence $\Phi_{\alpha,z}$ with respect to tensor product.

2. Right Rényi mean

Let $\mathbb{M}_{m,k}$ be the set of all $m \times k$ matrices with complex entries. We simply denote as $\mathbb{M}_m := \mathbb{M}_{m,m}$. Let $\mathbb{H}_m \subset \mathbb{M}_m$ be the real vector space of all $m \times m$ Hermitian matrices. For any $A, B \in \mathbb{H}_m$ we denote as $A \leq (<) B$ if $B - A$ is positive semi-definite (positive definite, respectively). This is truly a partial order on $\mathbb{H}_m$, known as the Loewner order. Let $\mathbb{P}_m \subset \mathbb{H}_m$ be the open convex cone of all positive definite matrices.

For $A, B \in \mathbb{P}_m$, $0 \leq \alpha \leq 1$ and $z > 0$

$$Q_{\alpha,z}(A,B) = \left(A^{\frac{1-\alpha}{2z}} B^{\frac{\alpha}{z}} A^{\frac{1-\alpha}{2z}}\right)^z$$

is the matrix version of the $\alpha-z$ Rényi relative operator entropy [2]. Especially, $Q_{\alpha,\alpha}(A,B)$ is known as the sandwiched quasi-relative operator entropy [14].

Let Δ_n be the simplex of positive probability vectors in $\mathbb{R}^n$, spanned by coordinate vectors. For an $n-$tuple $\mathbb{A} = (A_1, \dots, A_n) \in \mathbb{P}_m^n$ a positive probability vector $\omega =$

$(w_1, \ldots, w_n) \in \Delta_n$, and $0 < \alpha \le z < 1$, we consider the minimization problem

$$\underset{X \in \mathbb{P}_m}{\arg\min} \sum_{i=1}^{n} w_i \Phi_{\alpha,z}(A_i, X), \tag{2.2}$$

where the quantum divergence $\Phi_{\alpha,z}$ is given by (1.1). To show that the above minimization has a unique solution in $\mathbb{P}_m$, we deduce the following.

Lemma 2.1. *For $0 < \alpha \le z < 1$ and $A \in \mathbb{P}_m$, the map $\mathbb{P}_m \ni X \mapsto \operatorname{tr} Q_{\alpha,z}(A, X)$ is strictly concave.*

Proof. Given $A \in \mathbb{P}_m$, let $G(X) = \operatorname{tr} Q_{\alpha,z}(A, X)$ for $X \in \mathbb{P}_m$. It is enough to show that for any $X, Y \in \mathbb{P}_m$

$$G\left(\frac{X+Y}{2}\right) \ge \frac{G(X) + G(Y)}{2}, \tag{2.3}$$

where equality holds if and only if $X = Y$. Since the map $\mathbb{P}_m \ni A \to A^t$ for $t \in (0, 1)$ is concave [4, Theorem 4.2.3] and $0 < \frac{\alpha}{z} \le 1$, we have

$$\left(\frac{X+Y}{2}\right)^{\frac{\alpha}{z}} \ge \frac{X^{\frac{\alpha}{z}} + Y^{\frac{\alpha}{z}}}{2}.$$

Applying the congruence transformation by $A^{\frac{1-\alpha}{2z}}$ on both sides,

$$A^{\frac{1-\alpha}{2z}} \left(\frac{X+Y}{2}\right)^{\frac{\alpha}{z}} A^{\frac{1-\alpha}{2z}} \ge \frac{A^{\frac{1-\alpha}{2z}} X^{\frac{\alpha}{z}} A^{\frac{1-\alpha}{2z}} + A^{\frac{1-\alpha}{2z}} Y^{\frac{\alpha}{z}} A^{\frac{1-\alpha}{2z}}}{2}.$$

This implies that

$$\begin{aligned} Q_{\alpha,z}\left(A, \frac{X+Y}{2}\right) &\ge \left(\frac{A^{\frac{1-\alpha}{2z}} X^{\frac{\alpha}{z}} A^{\frac{1-\alpha}{2z}} + A^{\frac{1-\alpha}{2z}} Y^{\frac{\alpha}{z}} A^{\frac{1-\alpha}{2z}}}{2}\right)^{z} \\ &\ge \frac{Q_{\alpha,z}(A, X) + Q_{\alpha,z}(A, Y)}{2}, \end{aligned}$$

where the last inequality follows from [4, Theorem 4.2.3]. Therefore, it is clear that

$$\operatorname{tr}\left[Q_{\alpha,z}\left(A, \frac{X+Y}{2}\right)\right] \ge \operatorname{tr}\left[\frac{Q_{\alpha,z}(A, X) + Q_{\alpha,z}(A, Y)}{2}\right].$$

Now, we prove that $G\left(\frac{X+Y}{2}\right) = \frac{G(X)+G(Y)}{2}$ if and only if $X = Y$. Obviously, $X = Y$ implies the equality of (2.3). Conversely, if the equality of (2.3) holds then

$$\begin{aligned} \operatorname{tr}\left[Q_{\alpha,z}\left(A, \frac{X+Y}{2}\right)\right] &= \operatorname{tr}\left[\left(\frac{A^{\frac{1-\alpha}{2z}} X^{\frac{\alpha}{z}} A^{\frac{1-\alpha}{2z}} + A^{\frac{1-\alpha}{2z}} Y^{\frac{\alpha}{z}} A^{\frac{1-\alpha}{2z}}}{2}\right)^z\right] \\ &= \operatorname{tr}\left[\frac{Q_{\alpha,z}(A,X) + Q_{\alpha,z}(A,Y)}{2}\right]. \end{aligned}$$

Thus we have

$$\left(\frac{A^{\frac{1-\alpha}{2z}} X^{\frac{\alpha}{z}} A^{\frac{1-\alpha}{2z}} + A^{\frac{1-\alpha}{2z}} Y^{\frac{\alpha}{z}} A^{\frac{1-\alpha}{2z}}}{2}\right)^z = \frac{(A^{\frac{1-\alpha}{2z}} X^{\frac{\alpha}{z}} A^{\frac{1-\alpha}{2z}})^z + (A^{\frac{1-\alpha}{2z}} Y^{\frac{\alpha}{z}} A^{\frac{1-\alpha}{2z}})^z}{2}.$$

Since the map $\mathbb{P}_m \ni A \to \operatorname{tr} A^t$ for $t \in (0,1)$ is strictly concave from [6], we obtain $A^{\frac{1-\alpha}{2z}} X^{\frac{\alpha}{z}} A^{\frac{1-\alpha}{2z}} = A^{\frac{1-\alpha}{2z}} Y^{\frac{\alpha}{z}} A^{\frac{1-\alpha}{2z}}$. Therefore, $X = Y$ as desired. □

From Lemma 2.1, we know that the objective function $F(X) = \sum_{i=1}^n w_i \Phi_{\alpha,z}(A_i, X)$ is strictly convex for $0 < \alpha \leq z < 1$, so the minimization (2.2) has a unique solution in $\mathbb{P}_m$. By vanishing the gradient, we obtain in [9] that the unique solution coincides with the unique positive definite solution of the matrix equation

$$X = \sum_{i=1}^n w_i \left(X^{\frac{\alpha}{2z}} A_i^{\frac{1-\alpha}{z}} X^{\frac{\alpha}{2z}}\right)^z. \tag{2.4}$$

We write such a unique minimizer of (2.2) as $\mathcal{R}_{\alpha,z}(\omega; \mathbb{A})$ and call the $\alpha - z$ *weighted right Rényi mean.*

Let $\mathbb{A} = (A_1, \ldots, A_n) \in \mathbb{P}_m^n$, $\omega = (w_1, \ldots, w_n) \in \Delta_n$, $\sigma \in S_n$ a permutation on a $n-$letters, and $M \in \mathrm{GL}_m$, the general linear group. For convenience, we denote as

$$\begin{aligned} \omega_\sigma &:= (w_{\sigma(1)}, \ldots, w_{\sigma(n)}) \in \Delta_n, \\ \mathbb{A}_\sigma &:= (A_{\sigma(1)}, \ldots, A_{\sigma(n)}) \in \mathbb{P}_m^n, \\ M\mathbb{A}M^* &:= (MA_1M^*, \ldots, MA_nM^*) \in \mathbb{P}_m^n, \end{aligned}$$

and

$$\begin{aligned} \omega^k &:= \frac{1}{k}(\underbrace{w_1, \ldots, w_n}, \ldots, \underbrace{w_1, \ldots, w_n}) \in \Delta_{nk}, \\ \mathbb{A}^k &:= (\underbrace{A_1, \ldots, A_n}, \ldots, \underbrace{A_1, \ldots, A_n}) \in \mathbb{P}_m^{nk} \end{aligned}$$

of which number of tuples is $k \in \mathbb{N}$.

It is known from [10, Theorem 7.6.6] that the map $f : \mathbb{P}_m \to \mathbb{R}, f(A) = \log\det A$ is strictly concave: for any $A, B \in \mathbb{P}_m$ and $t \in [0,1]$

$$\log\det((1-t)A + tB) \geq (1-t)\log\det A + t\log\det B,$$

where equality holds if and only if $A = B$. By induction together with this property, we have

Lemma 2.2. *Let $A_1, \dots, A_n \in \mathbb{P}_m$ and $\omega = (w_1, \dots, w_n) \in \Delta_n$. Then*

$$\log\det\left(\sum_{i=1}^n w_i A_i\right) \geq \sum_{i=1}^n w_i \log\det A_i,$$

where equality holds if and only if $A_1 = \cdots = A_n$.

In the following, we provide several fundamental properties of the right Rényi mean.

Lemma 2.3. *The right Rényi mean satisfies the following:*

(1) $\mathcal{R}_{\alpha,z}(\omega; \mathbb{A}) = \left(\sum_{i=1}^n w_i A_i^{1-\alpha}\right)^{\frac{1}{1-\alpha}}$ *if the A_i's commute;*
(2) $\mathcal{R}_{\alpha,z}(\omega; c\mathbb{A}) = c\mathcal{R}_{\alpha,z}(\omega; \mathbb{A})$ *for any $c > 0$;*
(3) $\mathcal{R}_{\alpha,z}(\omega_\sigma; \mathbb{A}_\sigma) = \mathcal{R}_{\alpha,z}(\omega; \mathbb{A})$ *for any permutation σ on $\{1, \dots, n\}$;*
(4) $\mathcal{R}_{\alpha,z}(\omega^k; \mathbb{A}^k) = \mathcal{R}_{\alpha,z}(\omega; \mathbb{A})$ *for any $k \in \mathbb{N}$;*
(5) $\mathcal{R}_{\alpha,z}(\omega; U\mathbb{A}U^*) = U\mathcal{R}_{\alpha,z}(\omega; \mathbb{A})U^*$ *for any unitary matrix U;*
(6) $\det(\mathcal{R}_{\alpha,z}(\omega; \mathbb{A})) \geq \prod_{i=1}^n (\det A_i)^{w_i}$, *and equality holds if and only if $A_1 = \cdots = A_n$.*

Proof. By definition of the right Rényi mean, the invariance properties under permutation (3) and repetition (4) are obvious. Let $\mathbb{A} = (A_1, \dots, A_n) \in \mathbb{P}_m^n$ and $\omega = (w_1, \dots, w_n) \in \Delta_n$.

(1) If the A_i's commute, then

$$X = \left(\sum_{i=1}^n w_i A_i^{1-\alpha}\right)^{\frac{1}{1-\alpha}}$$

satisfies (2.4). By uniqueness of the positive definite solution of (2.4), it is proved.

(2) For $c > 0$, set $X = \mathcal{R}_{\alpha,z}(\omega; c\mathbb{A})$. Then

$$X = \sum_{i=1}^n w_i \left(X^{\frac{\alpha}{2z}} (cA_i)^{\frac{1-\alpha}{z}} X^{\frac{\alpha}{2z}}\right)^z = c^{1-\alpha} \sum_{i=1}^n w_i \left(X^{\frac{\alpha}{2z}} A_i^{\frac{1-\alpha}{z}} X^{\frac{\alpha}{2z}}\right)^z.$$

Equivalently,

$$\frac{1}{c}X = \sum_{i=1}^{n} w_i \left(\left(\frac{1}{c}X\right)^{\frac{\alpha}{2z}} A_i^{\frac{1-\alpha}{z}} \left(\frac{1}{c}X\right)^{\frac{\alpha}{2z}} \right)^{z}.$$

Thus we have $\frac{1}{c}X = \mathcal{R}_{\alpha,z}(\omega; \mathbb{A})$, which is proved.

(5) Let $X = \mathcal{R}_{\alpha,z}(\omega; \mathbb{A})$. Applying the congruence transformation by a unitary matrix U,

$$\begin{aligned} UXU^* &= U \left(\sum_{i=1}^{n} w_i \left(X^{\frac{\alpha}{2z}} A_i^{\frac{1-\alpha}{z}} X^{\frac{\alpha}{2z}} \right)^{z} \right) U^* \\ &= \sum_{i=1}^{n} w_i \left((UXU^*)^{\frac{\alpha}{2z}} (UA_iU^*)^{\frac{1-\alpha}{z}} (UXU^*)^{\frac{\alpha}{2z}} \right)^{z}. \end{aligned}$$

So we obtain the desired property.

(6) Applying Lemma 2.2 to the matrix equation (2.4),

$$\begin{aligned} \log\det X &\geq \sum_{i=1}^{n} w_i \log\det \left(X^{\frac{\alpha}{2z}} A_i^{\frac{1-\alpha}{z}} X^{\frac{\alpha}{2z}} \right)^{z} \\ &= \alpha \log\det X + (1-\alpha) \sum_{i=1}^{n} w_i \log\det A_i. \end{aligned}$$

Thus we see that

$$\log\det X \geq \sum_{i=1}^{n} w_i \log\det A_i = \log \prod_{i=1}^{n} (\det A_i)^{w_i}.$$

Since the exponential map is increasing, it is proved.

□

For given $\omega = (w_1, \ldots, w_n) \in \Delta_n$, we denote as

$$\hat{\omega} = \frac{1}{1-w_n}(w_1, \ldots, w_{n-1}) \in \Delta_{n-1}.$$

Proposition 2.4. *The right Rényi mean satisfies the following:*

(1) $\mathcal{R}_{\alpha,z}(\omega; A_1, \ldots, A_{n-1}, X) = \mathcal{R}_{\alpha,z}(\hat{\omega}; A_1, \ldots, A_{n-1})$;

(2) $\mathcal{R}_{\alpha,z}(\omega; \mathbb{A}) = \mathcal{R}_{\alpha,z}\left(\sum_{i=1}^{k} w_i, w_{k+1}, \ldots, w_n; A_1, A_{k+1}, \ldots, A_n\right)$ *if* $A_1 = \cdots = A_k$ *for* $1 \leq k < n$.

Proof. (1) Let $X = \mathcal{R}_{\alpha,z}(\omega; A_1, \dots, A_{n-1}, X)$. Then we have

$$X = \sum_{i=1}^{n-1} w_i \left(X^{\frac{\alpha}{2z}} A_i^{\frac{1-\alpha}{z}} X^{\frac{\alpha}{2z}} \right)^z + w_n X.$$

Solving for X, we get

$$X = \sum_{i=1}^{n-1} \frac{w_i}{1 - w_n} \left(X^{\frac{\alpha}{2z}} A_i^{\frac{1-\alpha}{z}} X^{\frac{\alpha}{2z}} \right)^z.$$

Therefore, $X = \mathcal{R}_{\alpha,z}(\hat{\omega}; A_1, \dots, A_{n-1})$.

(2) Let $X = \mathcal{R}_{\alpha,z}(\omega; \mathbb{A})$. If $A_1 = \cdots = A_k$ for $1 \leq k < n$, then we have

$$\begin{aligned} X &= \sum_{i=1}^{n} w_i \left(X^{\frac{\alpha}{2z}} A_i^{\frac{1-\alpha}{z}} X^{\frac{\alpha}{2z}} \right)^z \\ &= \left(\sum_{i=1}^{k} w_i \right) \left(X^{\frac{\alpha}{2z}} A_1^{\frac{1-\alpha}{z}} X^{\frac{\alpha}{2z}} \right)^z + \sum_{i=k+1}^{n} w_i \left(X^{\frac{\alpha}{2z}} A_i^{\frac{1-\alpha}{z}} X^{\frac{\alpha}{2z}} \right)^z. \end{aligned}$$

Equivalently,

$$X = \mathcal{R}_{\alpha,z} \left(\sum_{i=1}^{k} w_i, w_{k+1}, \dots, w_n; A_1, A_{k+1}, \dots, A_n \right).$$

□

3. Tensor product

We provide in this section the identity of two right Rényi means with respect to tensor product. The *tensor product*(or the *Kronecker product*) $A \otimes B$ of $A = [a_{ij}] \in M_{m,k}$ and $B = [b_{ij}] \in M_{s,t}$ is the $ms \times kt$ matrix given by

$$A \otimes B := \begin{bmatrix} a_{11}B & \cdots & a_{1k}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mk}B \end{bmatrix}.$$

Simply one can see that the tensor product is associative and bilinear, but not commutative. Moreover, the tensor product preserves positivity; the tensor product of two positive definite (positive semi-definite) matrices is positive definite (positive semi-definite, respectively) [4, 15]. We list a few properties of the tensor product that we will use in the following.

Lemma 3.1. [15, Section 4.3] *The tensor product satisfies the following.*

(1) *For $A \in M_{m,k}, B \in M_{r,s}, C \in M_{k,l}$ and $D \in M_{s,t}$*

$$(A \otimes B)(C \otimes D) = AC \otimes BD.$$

(2) *For $A, B \in \mathbb{P}_m$ and any real number t*

$$(A \otimes B)^t = A^t \otimes B^t.$$

Theorem 3.2. *Let $\mathbb{A} = (A_1, \ldots, A_n), \mathbb{B} = (B_1, \ldots, B_n) \in \mathbb{P}_m^n$, and let $\omega = (w_1, \ldots, w_n), \mu = (\mu_1, \ldots, \mu_n) \in \Delta_n$. Then*

$$\mathcal{R}_{\alpha,z}(\omega; \mathbb{A}) \otimes \mathcal{R}_{\alpha,z}(\mu; \mathbb{B}) = \mathcal{R}_{\alpha,z}(\omega \otimes \mu; \underbrace{A_1 \otimes B_1, \ldots, A_1 \otimes B_n}, \ldots, \underbrace{A_n \otimes B_1, \ldots, A_n \otimes B_n})$$

where $\omega \otimes \mu := (\underbrace{w_1\mu_1, \ldots, w_1\mu_n}, \ldots, \underbrace{w_n\mu_1, \ldots, w_n\mu_n}) \in \Delta_{n^2}$.

Proof. Let $X = \mathcal{R}_{\alpha,z}(\omega; \mathbb{A})$ and $Y = \mathcal{R}_{\alpha,z}(\mu; \mathbb{B})$. By the linearity of tensor product and Lemma 3.1, we have

$$\begin{aligned} X \otimes Y &= \left(\sum_{i=1}^n w_i \left(X^{\frac{\alpha}{2z}} A_i^{\frac{1-\alpha}{z}} X^{\frac{\alpha}{2z}} \right)^z \right) \otimes \left(\sum_{j=1}^n \mu_j \left(Y^{\frac{\alpha}{2z}} B_j^{\frac{1-\alpha}{z}} Y^{\frac{\alpha}{2z}} \right)^z \right) \\ &= \sum_{i,j=1}^n \omega_i \mu_j \left((X \otimes Y)^{\frac{\alpha}{2z}} (A_i \otimes B_j)^{\frac{1-\alpha}{z}} (X \otimes Y)^{\frac{\alpha}{2z}} \right)^z. \end{aligned}$$

Note that $\omega \otimes \mu \in \Delta_{n^2}$, and hence, we obtain that

$$X \otimes Y = \mathcal{R}_{\alpha,z}(\omega \otimes \mu; \underbrace{A_1 \otimes B_1, \ldots, A_1 \otimes B_n}, \ldots, \underbrace{A_n \otimes B_1, \ldots, A_n \otimes B_n}).$$

$\square$

Corollary 3.3. *Let $\mathbb{A} = (A_1, \ldots, A_n) \in \mathbb{P}_m^n$ and $\omega = (w_1, \ldots, w_n) \in \Delta_n$. Then for any $X \in \mathbb{P}_m$*

$$\mathcal{R}_{\alpha,z}(\omega; \mathbb{A}) \otimes X = \mathcal{R}_{\alpha,z}(\omega; A_1 \otimes X, \ldots, A_n \otimes X).$$

Proof. By Lemma 2.3 (1), $X = \mathcal{R}_{\alpha,z}(\mu; X, \ldots, X)$ for any $\mu = (\mu_1, \ldots, \mu_n) \in \Delta_n$. Then

$$\begin{aligned} \mathcal{R}_{\alpha,z}(\omega; \mathbb{A}) \otimes X &= \mathcal{R}_{\alpha,z}(\omega; \mathbb{A}) \otimes \mathcal{R}_{\alpha,z}(\mu; X, \ldots, X) \\ &= \mathcal{R}_{\alpha,z}(\omega \otimes \mu; \underbrace{A_1 \otimes X, \ldots, A_1 \otimes X}, \ldots, \underbrace{A_n \otimes X, \ldots, A_n \otimes X}) \\ &= \mathcal{R}_{\alpha,z}(w_1, \ldots, \underbrace{w_n\mu_1, \ldots, w_n\mu_n}; A_1 \otimes X, \ldots, \underbrace{A_n \otimes X, \ldots, A_n \otimes X}) \\ &= \mathcal{R}_{\alpha,z}(\omega; A_1 \otimes X, \ldots, A_n \otimes X). \end{aligned}$$

The second equality follows from Theorem 3.2, the third follows from Proposition 2.4 (2), and the last follows from Lemma 2.3 (3). □

4. Further research

We have studied the right Rényi mean $\mathcal{R}_{\alpha,z}(\omega;\mathbb{A})$ for $0 < \alpha \leq z < 1$ as a unique minimizer of the objective function $\sum_{i=1}^{n} w_i \Phi_{\alpha,z}(A_i, X)$. On the other hand, the quantum divergence $\Phi_{\alpha,z}$ is not symmetric. So one may be interested in the left mean

$$\underset{X \in \mathbb{P}_m}{\arg\min} \sum_{i=1}^{n} w_i \Phi_{\alpha,z}(X, A_i).$$

It has been known neither the divergence function $\Phi_{\alpha,z}(X, A_i)$ is strictly convex nor the above minimization can be solved.

Moreover, the extension of the right Rényi mean to the positive operators would be an interesting problem. Since the quantum divergence $\Phi_{\alpha,z}$ can not be defined in general, (2.2) is not valid. Instead, one can define the right Renyi mean of positive operators by solving the equation (2.4).

Acknowledgement

All authors contributed equally to this paper. This research was supported by Chungbuk National University Korea National University Development Project (2020).

References

[1] M. R. Ackermann and J. Blömer, Coresets and approximate clustering for Bregman divergences, in: Proceedings of the Twentieth Annual ACM-SIAM Symposium on Discete Algorithms, SIAM, Philadelphia, New York, 2009, 1088-1097.

[2] K. Audenaert and N. Datta, $\alpha - z-$Rényi relative entropies, J. Math. Phys. **56** (2015), 022202.

[3] A. Banerjee, S. Merugu, I. Dhillon and J. Ghosh, Clustering with Bregman divergences, J. Mach. Learn. Res. **6** (2005), 1705-1749.

[4] R. Bhatia, Positive Definite Matrices, Princeton Series in Applied Mathematics, Princeton University Press, 2007.

[5] R. Bhatia, T. Jain and Y. Lim, On the Bures-Wasserstein distance between positive definite matrices, Expo. Math. **37** (2) (2019), 165-191.

[6] R. Bhatia, T. Jain and Y. Lim, Strong convexity of sandwiched entropies and related optimization problems, Rev. Math. Phys. **30** (2018), no. 9, 1850014.

[7] Z. Chebbi and M. Moakher, Means of Hermitian positive-definite matrices based on the log-determinant α-divergence function, Linear Algebra Appl. **436** (2012), 1872-1889.

[8] I. S. Dhillon and J. A. Tropp, Matrix nearness problems with Bregman divergences, SIAM J. Matrix Anal. Appl. **29** (4) (2007), 1120-1146.

[9] T.-H. Dinh, C.-T. Le, B.-K. Vo and T.-D. Vuong, The $\alpha-z$-Bures Wasserstein divergence, submitted.

[10] R. A. Horn and C. R. Johnson, Matrix Analysis, 2nd edition, Cambridge University Press, 2013.

[11] J. P. W. Pluim, J. B. A. Maintz and M. A. Viergever, f-Information measures in medical image registration, IEEE Trans. Med. Imaging **23** (12) (2004), 1508-1516.

[12] A. Ullah, Uses of entropy and divergence measures for evaluating econometric approximations and inference, J. Economet. **107** (1-2) (2002), 313-326.

[13] R. Vergin and D. O'Shaughnessy, On the use of some divergence measures in speaker recognition, IEEE Int. Conf. Acoust. Speech Signal Process, **1** (1999), 309-312.

[14] M. Wilde, A. Winter and D. Yang, Strong converse for the classical capacity of entanglement-breaking and Hadamard channels via a sandwiched Rényi relative entropy, Comm. Math. Phys. textbf331 (2014), 593-622.

[15] F. Zhang, Matrix Theory: Basic Results and Techniques, 2nd edition, Springer, 2011.

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