T. Faulkner, S. Hollands: Approximate recoverability and relative entropy II: 2-positive channels of general von Neumann algebras

Referee report

The paper deals with the problem of recoverability of channels between von Neumann algebras. More precisely, the usual data processing inequality (DPI) shows that the relative entropy of two states is decreased under the action of a quantum channel and it is known [38,39] that preservation of the relative entropy is equivalent to existence of a channel that recovers both states perfectly. In finite dimensions, more detailed DPI's are proved, showing that the decrease in the relative entropy is related to the precision (usually given by fidelity) with which the states can be recovered. The present work brings similar results in the case of normal states and channels between von Neumann algebras.

An extension of the recoverability results to this general framework is necessary for systems that cannot be represented by finite dimensional algebras, as often happens in applications beyond quantum information theory, such as in quantum field theory. The results of the present paper are definitely worth publication.

As the title suggests, the paper is a continuation of a previous work [16] dealing with the case when the channel is the inclusion of a subalgebra. Here the results are extended for an arbitrary 2-positive unital normal map. Both papers follow a strategy similar to the one taken in finite dimensions in [26], dealing with the rather nontrivial technical problems appearing in the von Neumann algebra framework. As in [26], the main method relies on interpolation techniques of certain L_p-norms, more precisely the Araki-Masuda norms. The proof methods are almost the same as those in [16], in fact, almost the same properties of the map V_ψ are used and the proofs can be extended rather easily. Nevertheless, the authors use slightly different techniques to obtain the results. Although such technicalities may be of limited interest for readers in general, they could be helpful for further research on recoverability and related questions in the von Neumann algebra framework. In conclusion, I recommend the paper for publication, after addressing some minor points listed below.

Some comments, suggestions (and typos)

1. paragraph before Eq. (2): it seems that the Heisenberg and Schrödinger picture is somehow mixed, better choose one of them
2. p. 2, line -5: a connection with the Jones index is promised, but it is not mentioned in Sec. 4
3. p. 4, line 2 below Eq. (8): remove one "to"
4. p. 8, Eq.(27): it is not clear how $\pi^{\mathcal{B}}(\psi)$ appeared in the first of these equations (it seems that it can be just removed)

5. p. 8, Remark 8: the monotonicity of the norms was proved in [20, Thm. 3.16 (7)]. Similarly, the joint convexity Eq. (33) is proved in [20, Thm. 3.16 (5)]

6. p. 9, last line before Sec. 4: in [35], the operational meaning of sandwiched Rényi divergences is given only for the case $\alpha > 1$, it is not known to me whether an operational interpretation exists for the case $\alpha < 1$ which appears in Eq. (34)

7. p. 6: "flowed Petz map": not defined

8. p. 10: What is the point of switching between notations like $\Delta_{\eta_{\mathcal{A}},\psi_{\mathcal{A}}}$ and $\Delta_{\eta,\psi;\mathcal{A}}$? Why not choose one and stick to it?

9. p. 12, paragraph under Eq. (45): some careless copy-pasting?

10. p. 12, Lemma 5: this result is well known, it would be enough to just give a reference, or it could be moved to an Appendix. As given here it just distracts from the main line of the proof and impedes the reading

11. p. 13, last paragraph of the proof of Lemma 5: what is $\mathcal{M}$ here?

12. p. 14, Lemma 6: the first inequality follows by duality of the L_p and L_q norm (with $1/p+1/q=1$) and the fact that $\|\psi\|_{q,\psi}=1$ (see [7, Eq. (18)], note that we may replace ζ by $\pi(\psi)\zeta$):

$$|\langle\zeta,\psi\rangle| \leq \|\zeta\|_{p,\psi}\|\psi\|_{q,\psi} \leq \|\zeta\|_{p,\psi}$$

The second inequality can be seen from the fact [25] that the $\|\cdot\|_{p,\psi}$ norm can be defined as an interpolation norm via the Kosaki right L_p-spaces, which directly implies that for $p\in[1,2]$, $\|\zeta\|_{p,\psi}\leq\|\zeta\|_{2,\psi}$.

13. p. 14 Lemma 6 and p. 15, Lemma 7: what is the relation of $\mathcal{M}$ to the norm $\|\cdot\|_{p,\psi}$?

14. p. 15: there should be $\psi_{\mathcal{B}}$ instead of $\psi_{\mathcal{A}}$ at the very end of Eq. (57)

15. p. 17, Eq. (69): in the last expression, "p" should be "q"

16. p. 17, "Hilbertspace".