$\mathcal{A}$-extreme points of generalized state spaces

Anna Jenčová
Mathematical Institute, Slovak Academy of Sciences
email:jenca@mat.savba.sk

Draft.

1 Introduction

This work is inspired by the works [3, 4] on C*-extreme generalized states and motivated by the results in [5] on extreme process POVMs.

2 Preliminaries

Let $\mathcal{B}$ be a unital C*-algebra and let $B(\mathcal{K})$ be the algebra of bounded linear operators on a finite dimensional Hilbert space $\mathcal{K}$. Let us denote by $S_{\mathcal{K}}(\mathcal{B})$ the generalized state space of $\mathcal{B}$, that is, the set of unital completely positive (ucp) maps $\mathcal{B} \to B(\mathcal{K})$, and let $\Phi \in S_{\mathcal{K}}(\mathcal{B})$.

Let $\Phi \in S_{\mathcal{K}}(\mathcal{B})$ and let $\Phi = V^*\pi V$ be a minimal Stinespring representation of Φ, that is, $\pi : \mathcal{B} \to B(\tilde{\mathcal{H}})$ is a representation of $\mathcal{B}$ on some Hilbert space $\tilde{\mathcal{H}}$ and $V : \mathcal{K} \to \tilde{\mathcal{H}}$ is an isometry, $V^*V = I_{\mathcal{K}}$, such that $\tilde{\mathcal{H}} = [\pi(\mathcal{B})V\mathcal{K}]$. Moreover, we will denote the commutant $\pi(\mathcal{B})'$ in $B(\tilde{\mathcal{H}})$ by $\mathcal{B}_0$, then $\mathcal{B}_0$ is a von Neumann subalgebra in $B(\tilde{\mathcal{H}})$. In the sequel, we will use the following three important theorems due to Arveson.

Theorem 1. *[1, Theorem 1.4.2] Let $\Psi : \mathcal{B} \to B(\mathcal{K})$ be a completely positive map. Then $\Psi \leq \Phi$ (in the sense that $\Phi - \Psi$ is completely positive) if and only if there is a (unique) positive contraction $T \in \mathcal{B}_0$, such that $\Psi = \Phi_T := V^*\pi(\cdot)TV$.*

Theorem 2. *[1, Corollary 1.3.2] There is a canonical *-isomorphism between the von Neumann algebras $\Phi(\mathcal{B})' \subseteq B(\mathcal{K})$ and $\mathcal{B}_0 \cap \{VV^*\}'$. This isomorphism is given by the restriction of the map $\Phi^C : \mathcal{B}_0 \to B(\mathcal{K})$, $T \mapsto V^*TV$.*

Theorem 3. *[1, Theorem 1.4.6] The map Φ is an extreme point of $S_{\mathcal{K}}(\mathcal{B})$ if and only if the map Φ^C is injective.*

Note that the uniqueness in Theorem 1 implies that the map Φ^C is faithful, that is, $\Phi^C(T) = 0$ for some $T \geq 0$ implies $T = 0$. The *right multiplicative domain* of Φ^C is defined as

$$\mathcal{M}_R := \{T \in \mathcal{B}_0, \Phi^C(T^*T) = \Phi^C(T)^*\Phi^C(T)\}$$

Then $\mathcal{M}_R$ is a subalgebra in $\mathcal{B}_0$ (not necessarily self-adjoint) and

$$\mathcal{M}_R = \{T \in \mathcal{B}_0, \Phi^C(ST) = \Phi^C(S)\Phi^C(T),\ \forall S \in \mathcal{B}_0\},$$

see e.g. [6]. Consequently, the restriction $\Phi^C|_{\mathcal{M}_R}$ is a homomorphism. In fact, since Φ^C is faithful, $\Phi^C|_{\mathcal{M}_R}$ is an isomorphism onto its range, so that $\mathcal{M}_R$ is finite dimensional.

Lemma 1. *Let $T \in \mathcal{B}_0$. The following are equivalent.*

(i) The subspace $V\mathcal{K}$ is invariant under T

(ii) $TV = VA$ for some $A \in B(\mathcal{K})$

(iii) $T \in \mathcal{M}_R$

Proof. Suppose (i) and let $\xi \in \mathcal{K}$. Since $V\mathcal{K}$ is finite dimensional, there is some $\eta \in \mathcal{K}$ such that $TV\xi = V\eta$ and since V is an isometry, we must have $\eta = V^*TV\xi = \Phi^C(T)\xi$. Hence

$$TV\xi = V\eta = V\Phi^C(T)\xi.$$

Since this is true for all $\xi \in \mathcal{K}$, we have (ii), with $A = \Phi^C(T)$.

Suppose (ii), then we must have $A = \Phi^C(T)$ and

$$\Phi^C(T^*T) = V^*T^*TV = A^*V^*VA = A^*A = \Phi^C(T)^*\Phi^C(T),$$

so that $T \in \mathcal{M}_R$.

Finally, suppose (iii) and let $P_V = VV^*$ be the projection onto $V\mathcal{K}$. From $\Phi^C(T^*T) = \Phi^C(T)^*\Phi^C(T)$, it is easy to see that $P_VT^*(I - P_V)TP_V = 0$, hence $TP_V = P_VTP_V$. This clearly implies (i). □

The C*-subalgebra $\mathcal{M} := \mathcal{M}_R^* \cap \mathcal{M}_R \subseteq \mathcal{B}_0$ is called the *multiplicative domain* of Φ^C. Since the elements of $\mathcal{M}$ are reduced by P_V, $\mathcal{M}$ is exactly the set of operators in $\mathcal{B}_0$ that commute with P_V. Theorem 2 now becomes $\Phi(\mathcal{B})' = \Phi^C(\mathcal{M})$.

3 $\mathcal{A}$-convexity and $\mathcal{A}$-extreme maps

Let $\mathcal{A} \subseteq B(\mathcal{K})$ be a C*-subalgebra and let $\Phi, \Psi \in S_{\mathcal{K}}(\mathcal{B})$. Then Φ and Ψ are $\mathcal{A}$-*equivalent*, $\Phi \sim_{\mathcal{A}} \Psi$, if there is a unitary $U \in \mathcal{A}$ such that $\Phi = U^*\Psi U$. We say that Φ is $\mathcal{A}$-*irreducible* if the only projections in $\mathcal{A}$ commuting with all operators in the range of Φ are 0 and I. If $\Phi_1, \dots, \Phi_m \in S_{\mathcal{K}}(\mathcal{B})$, then Φ is an $\mathcal{A}$-*convex combination* of $\Phi_1, \dots, \Phi_m$ if there are some $X_1, \dots, X_m \in \mathcal{A}$, such that $\sum_i X_i^* X_i = I$ and

$$\Phi(B) = \sum_i X_i^* \Phi_i(B) X_i, \quad \forall B \in \mathcal{B}.$$

An $\mathcal{A}$-convex combination is called *proper* if all the coefficients are invertible. The set $S_{\mathcal{K}}(\mathcal{B})$ is obviously $\mathcal{A}$-*convex*, in the sense that it contains all $\mathcal{A}$-convex combinations of its elements. Note that the notion of $\mathcal{A}$-convexity contains both the usual convexity when $\mathcal{A} = \mathbb{C}I$, and C*-convexity when $\mathcal{A} = B(\mathcal{K})$. The $\mathcal{A}$-*extreme* elements in $S_{\mathcal{K}}(\mathcal{B})$ are defined similarly as C*-extreme elements. Namely, whenever Φ is a proper $\mathcal{A}$-convex combination of $\Phi_1, \dots, \Phi_m$, then we must have $\Phi \sim_{\mathcal{A}} \Phi_i$ for all i. Similarly to Theorem 3, we are going to characterize the $\mathcal{A}$-extreme points by some properties of the map Φ^C.

Let us denote $\mathcal{T}_{\mathcal{A}} := (\Phi^C)^{-1}(\mathcal{A})$, $\mathcal{T}_{\mathcal{A}}^+ = \mathcal{T}_{\mathcal{A}} \cap \mathcal{B}_0^+$ and let

$$\mathcal{M}_{R,\mathcal{A}} := \mathcal{M}_R \cap \mathcal{T}_{\mathcal{A}}.$$

Since the restriction of Φ^C to $\mathcal{M}_R$ is a homomorphism and $\mathcal{A}$ is a subalgebra, $\mathcal{M}_{R,\mathcal{A}}$ is a subalgebra in $\mathcal{M}_R$ and $\mathcal{M}_{\mathcal{A}} = \mathcal{M}_{R,\mathcal{A}}^* \cap \mathcal{M}_{R,\mathcal{A}}$ is a C*-subalgebra in $\mathcal{M}$, such that

$$\Phi^C(\mathcal{M}_{\mathcal{A}}) = \Phi^C(\mathcal{M}) \cap \mathcal{A} = \Phi(\mathcal{B})' \cap \mathcal{A}.$$

The following is now immediate.

Corollary 1. *Φ is $\mathcal{A}$-irreducible if and only if $\mathcal{M}_{\mathcal{A}} = \mathbb{C}I$.*

Lemma 2. *The map Φ^C restricted to $\mathcal{M}_{R,\mathcal{A}}$ is an isomorphism onto the subalgebra*

$$\{L \in \mathcal{A},\ \exists t \geq 0\ :\ L^*\Phi L \preceq t\Phi\}$$

Proof. We already noted that Φ^C restricted to $\mathcal{M}_{R,\mathcal{A}}$ is an isomorphism. Let now $T \in \mathcal{M}_{R,\mathcal{A}}$ and $L = \Phi^C(T)$. Then by Lemma 1 (ii), $TV = VL$ and we have

$$L^*\Phi(B)L = L^*V^*\pi(B)VL = V^*T^*T\pi(B)V \leq \|T\|^2\Phi(B).$$

Conversely, let $L \in \mathcal{A}$ and $t \geq 0$ be such that $L^*\Phi L \preceq t\Phi$. By Theorem 1, there is some $0 \leq T \leq tI$ in $\mathcal{B}_0$ such that $L^*\Phi L = \Phi_T$. Let us define the map $U : \tilde{H} \to \tilde{H}$ by

$$U\pi(B)T^{1/2}V\xi = \pi(B)VL\xi, \quad B \in \mathcal{B},\ \xi \in \mathcal{K}$$

and put $U\eta = 0$ for $\eta \in [T\tilde{H}]^\perp$. Then since

$$\begin{aligned}\|U(\pi(B)T^{1/2}V\xi + \eta)\|^2 = \|\pi(B)VL\xi\|^2 = \langle \xi, L^*\Phi(B)L\xi\rangle = \langle \xi, \Phi_T(B)\xi\rangle \\ = \|\pi(B)T^{1/2}V\xi\|^2 \leq \|\pi(B)T^{1/2}V\xi + \eta\|^2,\end{aligned}$$

U extends to a partial isometry on $\tilde{H}$, with initial space the range of T. Moreover, let $s(T) \in \mathcal{B}_0$ be the range projection of T, then since we have

$$U\pi(B)[\pi(B')T^{1/2}V\xi] = \pi(B)U[\pi(B')T^{1/2}V\xi]$$

for all $B, B' \in \mathcal{B}$ and $\xi \in \mathcal{K}$, we obtain

$$U\pi(B) = Us(T)\pi(B) = U\pi(B)s(T) = \pi(B)Us(T) = \pi(B)U$$

for all $B \in \mathcal{B}$, so that $U \in \mathcal{B}_0$. Put now $T_0 = UT^{1/2}$, then $T_0 \in \mathcal{B}_0$ and we have

$$T_0V\xi = UT^{1/2}V\xi = VL\xi, \quad \xi \in \mathcal{K},$$

so that $T_0 \in \mathcal{M}_R$ and $\Phi^C(T_0) = V^*T_0V = L \in \mathcal{A}$.

□

We now obtain a first characterization of $\mathcal{A}$-extreme generalized states (cf. [4, Corollary 3.3]).

Proposition 1. Φ *is $\mathcal{A}$-extreme if and only if any $T \in \mathcal{T}_{\mathcal{A}}^+$ has the form $T = T_0^*T_0$ for some $T_0 \in \mathcal{M}_{R,\mathcal{A}}$.*

Proof. Suppose that Φ is $\mathcal{A}$-extreme. Note that then Φ is also extreme in the usual sense, this is proved exactly as [3, Proposition 1.1] in the C*-extreme case. By Theorem 3, Φ^C is injective and therefore $\mathcal{B}_0$ is finite dimensional.

Let $T \in \mathcal{T}_{\mathcal{A}}^+$, we may assume that $0 \leq T \leq I$. Let $\Phi^C(T) = K \in \mathcal{A}$ and suppose first that K is invertible. Then for any $\lambda \in (0,1)$, both λK and $I - \lambda K$ are positive and invertible. Let

$$\Phi_1 = (\lambda K)^{-1/2}\Phi_{\lambda T}(\lambda K)^{-1/2}, \quad \Phi_2 = (I-\lambda K)^{-1/2}\Phi_{I-\lambda T}(I-\lambda K)^{-1/2}$$

Then both Φ_1, Φ_2 are ucp maps and

$$\Phi = (\lambda K)^{1/2}\Phi_1(\lambda K)^{1/2} + (I-\lambda K)^{1/2}\Phi_2(I-\lambda K)^{1/2}$$

is a proper $\mathcal{A}$-convex combination. Hence there is a unitary $U \in \mathcal{A}$ such that $\Phi_1 = U^*\Phi U$. It follows that

$$\Phi_{\lambda T} = (\lambda K)^{1/2}\Phi_1(\lambda K)^{1/2} = (\lambda K)^{1/2}U^*\Phi U(\lambda K)^{1/2} = L^*\Phi L,$$

where $L = U(\lambda K)^{1/2} \in \mathcal{A}$. By Lemma 2, there is some $S \in \mathcal{M}_{R,\mathcal{A}}$, such that $\Phi^C(S) = L$. Put $T_0 := \lambda^{-1/2}S$, then

$$\Phi_{T_0^*T_0} = \lambda^{-1}L^*\Phi L = \Phi_T$$

and the uniqueness part of Theorem 1 implies that $T_0^*T_0 = T$.

In the general case, for any $\epsilon > 0$, $K + \epsilon I$ is invertible, $T_\epsilon := T + \epsilon I \in \mathcal{T}_{\mathcal{A}}^+$ and $\Phi^C(T_\epsilon) = K + \epsilon I$. By the first part of the proof, there is some $T_{0,\epsilon} \in \mathcal{M}_{R,\mathcal{A}}$ such that $T_{0,\epsilon}^*T_{0,\epsilon} = T_\epsilon$. Since $\|T_{0,\epsilon}\|^2 = \|T_\epsilon\| \leq 1 + \epsilon$ and $\mathcal{B}_0$ is finite dimensional, there is some sequence $\epsilon_n \to 0$ such that $T_{0,n} := T_{0,\epsilon_n}$ converges to some operator $T_0 \in \mathcal{M}_{R,\mathcal{A}}$ and

$$T = \lim_n T_{\epsilon_n} = \lim_n T_{0,n}^*T_{0,n} = T_0^*T_0.$$

For the converse, let $\Phi = \sum_i X_i^*\Phi_i X_i$ be a proper $\mathcal{A}$-convex combination of $\Phi_i \in S_{\mathcal{K}}(\mathcal{B})$. Fix any i, then $X_i^*\Phi_i X_i \preceq \Phi$, so that by Theorem 1, there is some $T \in \mathcal{B}_0^+$ such that $\Phi_T = X_i^*\Phi_i X_i$. Since

$$\Phi^C(T) = \Phi_T(I) = X_i^*X_i \in \mathcal{A},$$

we have $T \in \mathcal{T}_{\mathcal{A}}^+$ and by the assumption, $T = T_0^*T_0$ for some $T_0 \in \mathcal{M}_{R,\mathcal{A}}$, so that

$$X_i^*\Phi_i X_i = \Phi_{T_0^*T_0} = L^*\Phi L,$$

where $L = \Phi^C(T_0)$. We have $L^*L = \Phi^C(T_0)^*\Phi^C(T_0) = \Phi^C(T) = X_i^*X_i$. It follows that $U_i := LX_i^{-1}$ is a unitary element in $\mathcal{A}$ and $\Phi_i = U_i^*\Phi U_i$.

□

4 $\mathcal{A}$-extreme and $\mathcal{A}$-pure maps.

We will say that Φ is $\mathcal{A}$-*pure* if it is $\mathcal{A}$-extreme and $\mathcal{A}$-irreducible.

Proposition 2. *Φ is $\mathcal{A}$-pure if and only if $\mathcal{T}_\mathcal{A} = \mathbb{C}I$.*

Proof. Assume that Φ is $\mathcal{A}$-pure and let $T \in \mathcal{T}_\mathcal{A}$. Since $\mathcal{T}_\mathcal{A}$ is a self-adjoint subspace containing the unit, it is clear that we may suppose that $0 \le T \le I$. By Proposition 1, there are some $T_0, T_1 \in \mathcal{M}_{R,\mathcal{A}}$ such that $T_0^*T_0 = T$ and $T_1^*T_1 = I - T$. Let $L_i = \Phi^C(T_i)$, then $\Phi_{T_i^*T_i} = L_i^*\Phi L_i$ and therefore

$$\Phi = L_0^*\Phi L_0 + L_1^*\Phi L_1.$$

Let $\phi_L : B(\mathcal{K}) \to B(\mathcal{K})$ be defined by $A \mapsto L_0^*AL_0 + L_1^*AL_1$, then ϕ_L is a ucp map and $\Phi(\mathcal{B})$ is contained in the set $\mathcal{F}$ of its fixed points. It is clear that $\mathcal{A}' \subseteq \mathcal{F}$, so that $\mathcal{F}' \subseteq \mathcal{A}$, but since Φ is $\mathcal{A}$-irreducible, this implies that $\mathcal{F}' = \mathbb{C}I$. Using [2] (see the proof of Theorem 2.1.1 and Remark 2), it can be shown that this implies $\phi_L = id$, so that there are some $z_i \in \mathbb{C}$, $|z_0|^2 + |z_1|^2 = 1$, and unitaries U_i such that $L_i = z_iU_i$. Hence $\Phi^C(T) = L_0^*L_0 = |z_0|^2 I$ and since Φ^C is injective, $T \in \mathbb{C}I$.

Conversely, if $\mathcal{T}_\mathcal{A} = \mathbb{C}I$, then Φ is $\mathcal{A}$-pure by Proposition 1 and Corollary 1.

□

We will now show that any $\mathcal{A}$-extreme map can be decomposed to a direct sum of $\mathcal{A}$-pure maps.

Lemma 3. *Let Φ be $\mathcal{A}$-extreme and let $T \in \mathcal{T}_\mathcal{A}^+$ be such that $s(T) \le P \in \mathcal{M}_\mathcal{A}$. Then there is some $T_0 \in \mathcal{M}_{R,\mathcal{A}} \cap P\mathcal{B}_0P$, such that $T = T_0^*T_0$.*

Proof. Since $T + P^\perp \in \mathcal{T}_\mathcal{A}^+$, there is some $S_0 \in \mathcal{M}_{R,\mathcal{A}}$ such that $S_0^*S_0 = T + P^\perp$. Let

$$S_0 = U(T + P^\perp)^{1/2} = U(T^{1/2} + P^\perp)$$

be the polar decomposition. Then $UT^{1/2} = S_0P$ and $UP^\perp = S_0P^\perp$ are in $\mathcal{M}_{R,\mathcal{A}}$.

Let $P' = \Phi^C(P)$, then P' is a projection in $\mathcal{A}$, that commutes with all operators in the range of Φ. Put $K := \Phi^C(T)$, then K is a positive element in $\mathcal{A}$ with support $s(K) \le P'$. We have $\Phi^C(S_0) = V(K^{1/2} + (P')^\perp)$, with V a unitary element in $\mathcal{A}$. Then

$$\Phi^C(UP^\perp) = \Phi^C(S_0P^\perp) = \Phi^C(S_0)(P')^\perp = V(P')^\perp$$

and similarly

$$\Phi^C(UT^{1/2}) = VK^{1/2}.$$

From

$$\Phi_T + \Phi_{P^\perp} = \Phi_{S_0^*S_0} = (K^{1/2} + (P')^\perp)V^*\Phi V(K^{1/2} + (P')^\perp),$$

it is easy to see that $(P')^\perp$ commutes with the range of $V^*\Phi V$, equivalently, $VP'V^*$ commutes with the range of Φ. Hence there is some projection $Q \in \mathcal{M}_\mathcal{A}$ such that $\Phi^C(Q) = VP'V^*$. Now observe that

$$\Phi^C(QUP^\perp) = \Phi^C(Q)\Phi^C(UP^\perp) = VP'V^*V(P')^\perp = 0,$$

and since Φ^C is injective, $QUP^\perp = 0$. This implies $QUP^\perp U^* = 0$, that is, $UP^\perp U^* \leq Q^\perp$. Using this and Schwarz inequality, we obtain

$$V(P')^\perp V^* = \Phi^C(Q^\perp) \geq \Phi^C(UP^\perp U^*) \geq \Phi^C(UP^\perp)\Phi^C(UP^\perp)^* = V^*(P')^\perp V$$

It follows that $UP^\perp \in \mathcal{M}_\mathcal{A}$, so that $P^\perp$ and $UP^\perp U^*$ are equivalent projections in $\mathcal{M}_\mathcal{A}$. Since $\mathcal{M}_\mathcal{A}$ is finite dimensional, there is a unitary $U_0 \in \mathcal{M}_\mathcal{A}$, such that $U_0PU_0^* = UPU^*$. Putting $T_0 = U_0^*UT^{1/2}$ now gives the result.

□

The following result is now easy to prove.

Proposition 3. *Let Φ be $\mathcal{A}$-extreme and let $P \in \mathcal{M}_\mathcal{A}$ be a projection. Then $\Phi_P : \mathcal{B} \to B(P'\mathcal{K})$ is $P'\mathcal{A}P'$-extreme, where $P' = \Phi^C(P)$.*

Proof. Notice that if $\tilde{\pi} : \mathcal{B} \to B(P\tilde{H})$, $\tilde{\pi} := \pi P$, then $\Phi_P = (PV)^*\tilde{\pi}(PV)$ is a minimal Stinespring representation and $\Phi_P^C = \Phi^C(P \cdot P) = P'\Phi^C P'$. We have $\tilde{\pi}(\mathcal{B})' = P\mathcal{B}_0P$ and it is easy to see that for $T \in P\mathcal{B}_0P$, $\Phi_P^C(T) \in P'\mathcal{A}P'$ if and only if $T = PSP$ for some $S \in \mathcal{T}_\mathcal{A}$ and that $\Phi_P^C(T^*T) = \Phi_P^C(T)^*\Phi_P^C(T)$ if and only if $T = PSP$ with $S \in \mathcal{M}_R$. Consequently, the corresponding subsets for Φ_P^C are

$$\mathcal{T}_{P'\mathcal{A}P'} = P\mathcal{T}_\mathcal{A}P, \quad \mathcal{T}^+_{P'\mathcal{A}P'} = P\mathcal{T}^+_\mathcal{A}P \quad \text{and} \quad \mathcal{M}_{R,P'\mathcal{A}P'} = P\mathcal{M}_{R,\mathcal{A}}P.$$

The statement now follows by Proposition 1 and Lemma 3.

□

Theorem 4. *Let Φ be $\mathcal{A}$-extreme, then there is a maximal orthogonal family of projections $P'_1, \ldots, P'_N \in \mathcal{A}$ and $P'_i\mathcal{A}P'_i$-pure ucp maps $\Phi_i : \mathcal{B} \to B(P'_i\mathcal{K})$, such that*

$$\Phi(B) = \oplus_i \Phi_i(B), \quad B \in \mathcal{B}.$$

Proof. Let $P_1, \dots, P_N$ be a maximal orthogonal family of minimal projections in $\mathcal{M}_\mathcal{A}$ and let $P_i' = \Phi^C(P_i)$, $\Phi_i = \Phi_{P_i}$. Then it is clear that Φ_i are ucp maps $\mathcal{B} \to B(P_i'\mathcal{K})$ and $\Phi = \oplus_i \Phi_i$. By Proposition 3, Φ_i is $P_i'\mathcal{A}P_i'$-extreme and since P_i is a minimal projection, $\mathcal{M}_{P_i'\mathcal{A}P_i'} = P_i \mathcal{M}_\mathcal{A} P_i = \mathbb{C}P_i$. By Corollary 1, Φ_i is $P_i'\mathcal{A}P_i'$-irreducible.

□

5 A characterization of $\mathcal{A}$-extreme maps

In this section, we further investigate the structure of $\mathcal{M}_{R,\mathcal{A}}$, $\mathcal{M}_\mathcal{A}$ and $\mathcal{T}_\mathcal{A}$ for $\mathcal{A}$-extreme maps.

5.1 Some special cases

We first find a characterization of $\mathcal{A}$-extreme maps for abelian $\mathcal{A}$.

Lemma 4. *Let Φ be $\mathcal{A}$-extreme. Then $\mathcal{T}_\mathcal{A} \cap \mathcal{M}_\mathcal{A}' = \mathcal{Z}(\mathcal{M}_\mathcal{A})$, where $\mathcal{Z}(\mathcal{M}_\mathcal{A})$ is the center of $\mathcal{M}_\mathcal{A}$.*

Proof. Let $\{P_1, \dots, P_N\}$ be a maximal orthogonal family of minimal projections in $\mathcal{M}_\mathcal{A}$ and let $P_i' = \Phi^C(P_i)$, $i = 1, \dots, N$. Since Φ_{P_i} is $P_i'\mathcal{A}P_i'$-pure, we have $P_i \mathcal{T}_\mathcal{A} P_i = \mathbb{C}P_i$. Hence if $T \in \mathcal{T}_\mathcal{A} \cap \mathcal{M}_\mathcal{A}'$, we have

$$T = \sum_i P_i T P_i = \sum_i z_i P_i \in \mathcal{M}_\mathcal{A}.$$

It follows that $\mathcal{T}_\mathcal{A} \cap \mathcal{M}_\mathcal{A}' \subseteq \mathcal{Z}(\mathcal{M}_\mathcal{A}) \subseteq \mathcal{T}_\mathcal{A} \cap \mathcal{M}_\mathcal{A}'$.

□

Lemma 5. *Assume that Φ is extreme (in the usual sense) in $S_\mathcal{K}(\mathcal{B})$. Let $\mathcal{A}'$ be the commutant of $\mathcal{A}$. Then $\mathcal{T}_{\mathcal{A}'} \subseteq \mathcal{M}_\mathcal{A}'$.*

Proof. Let $T \in \mathcal{T}_{\mathcal{A}'}$, $S \in \mathcal{M}_\mathcal{A}$. Then

$$\Phi^C(TS) = \Phi^C(T)\Phi^C(S) = \Phi^C(S)\Phi^C(T) = \Phi^C(ST).$$

Since Φ is extreme, Φ^C is injective, so that $ST = TS$ and $T \in \mathcal{M}_\mathcal{A}'$.

□

Proposition 4. *Let $\mathcal{A} \subseteq B(\mathcal{K})$ be abelian. Then Φ is $\mathcal{A}$-extreme if and only if $\mathcal{T}_\mathcal{A} = \mathcal{M}_\mathcal{A}$.*

Proof. Assume that Φ is $\mathcal{A}$-extreme. Since $\mathcal{A} \subseteq \mathcal{A}'$, we have $\mathcal{T}_{\mathcal{A}} \subseteq \mathcal{T}_{\mathcal{A}'}$ and by Lemma 5, $\mathcal{T}_{\mathcal{A}'} \subseteq \mathcal{M}'_{\mathcal{A}}$. Using Lemma 4, we obtain

$$\mathcal{M}_{\mathcal{A}} \subseteq \mathcal{T}_{\mathcal{A}} = \mathcal{T}_{\mathcal{A}} \cap \mathcal{M}'_{\mathcal{A}} = \mathcal{Z}(\mathcal{M}_{\mathcal{A}}) \subseteq \mathcal{M}_{\mathcal{A}}.$$

The converse is clear from Proposition 1.

□

Lemma 6. *Let Φ be $\mathcal{A}$-extreme and let $T_0 \in \mathcal{M}_{R,\mathcal{A}}$. Then there are partial isometries $W_1, W_2 \in \mathcal{M}_{R,\mathcal{A}}$ such that $W_1 W_1^* = \mathcal{R}(T_0)$, $W_2 W_2^* = Ker(T_0)$ and $W_1^* W_1 = (W_2^* W_2)^\perp \in \mathcal{M}_{\mathcal{A}}$.*

Proof. Let $T = T_0^* T_0$ and let $T_0 = UT^{1/2}$ be the polar decomposition. For any $\lambda \in (0,1)$, $(1-\lambda)T + \lambda I$ is an invertible element in $\mathcal{T}_{\mathcal{A}}^+$. By Proposition 1, there is some (invertible) $T_\lambda \in \mathcal{M}_{R,\mathcal{A}}$ such that $T_\lambda^* T_\lambda = (1-\lambda)T + \lambda I$. We have

$$\lambda^{1/2} T_\lambda^{-1}, (1-\lambda)^{1/2} T_0 T_\lambda^{-1} \in \mathcal{M}_{R,\mathcal{A}}, \quad \forall \lambda \in (0,1).$$

Let $T_\lambda = U_\lambda [(1-\lambda)T + \lambda I]^{1/2}$ be the polar decomposition. Since $\mathcal{B}_0$ is finite dimensional, the set of all unitaries is compact and there is some subsequence $\lambda_n \to 0$ and a unitary U_0 such that $U_{\lambda_n} \to U_0$. Moreover,

$$\lim_{\lambda \to 0^+} (1-\lambda)^{1/2} T^{1/2} \left[(1-\lambda)T + \lambda I\right]^{-1/2} = Q^\perp$$

and

$$\lim_{\lambda \to 0^+} \lambda^{1/2} [(1-\lambda)T + \lambda I]^{-1/2} = Q,$$

where $Q = Ker(T_0)$. Since $\mathcal{M}_{R,\mathcal{A}}$ is closed, this implies that $UQ^\perp U_0^*, QU_0^* \in \mathcal{M}_{R,\mathcal{A}}$. Therefore

$$\Phi^C(U_0 Q U_0^*) = \Phi^C(U_0 Q) \Phi^C(Q U_0^*).$$

Note that

$$\Phi^C(QU_0^*) = \lim_{n \to \infty} \Phi^C(\lambda_n^{1/2} T_{\lambda_n}^{-1}) = \lim_{n \to \infty} \lambda_n^{1/2} \Phi_C(T_{\lambda_n})^{-1}$$

Let now

$$\Phi^C(T_\lambda) = V_\lambda [(1-\lambda)\Phi^C(T) + \lambda I]^{1/2}$$

for some unitary $V_\lambda \in \mathcal{A}$ be the polar decomposition. Exactly as above, there is some subsequence n_k such that $V_{\lambda_{n_k}} \to V_0$ for a unitary operator $V_0 \in \mathcal{A}$. We obtain

$$\Phi^C(QU_0^*) = PV_0^*,$$

where $P = Ker(\Phi^C(T_0))$. It follows that $\Phi^C(U_0 Q U_0^*) = V_0 P V_0^*$ is a projection and it is easy to see that then $U_0 Q U_0^* \in \mathcal{M}_\mathcal{A}$. Putting $W_1 = U Q^\perp U_0^*$ and $W_2 = Q U_0^*$, we obtain the result.

□

We now obtain a characterization of C*-extreme points of ucp maps between matrix algebras. Note that if $\mathcal{B} = B(\mathcal{H})$ for a finite dimensional Hilbert space, then we may assume that $\tilde{H} = \mathcal{H} \otimes \mathcal{H}_0$ for some finite dimensional Hilbert space $\mathcal{H}_0$ and the Stinespring representation has the form $\Phi(B) = V^*(B \otimes I)V$. The commutant $\mathcal{B}_0$ can be identified with the algebra $B(\mathcal{H}_0)$. The equivalence (i) $\iff$ (ii) was already obtained in [3, 4].

Theorem 5. *The following are equivalent.*

(i) There are partial isometries $V_1, \dots, V_k : \mathcal{K} \to \mathcal{H}$, such that $\sum_i V_i^* V_i = I$, $V_1 V_1^* \geq \dots \geq V_k V_k^*$ and

$$\Phi(B) = \sum_{i=1}^k V_i^* B V_i, \qquad B \in B(\mathcal{H}).$$

(ii) $\mathcal{M}_R$ contains the subalgebra of upper triangular elements with respect to some ONB of $\mathcal{H}_0$.

(iii) Φ is C-extreme.*

Proof. Let Φ be of the form as in (i) and let $|e_1\rangle, \dots, |e_k\rangle$ be an ONB in $\mathbb{C}^k$. Let $\tilde{V} = \sum_i V_i \otimes |e_i\rangle$ be the operator $\mathcal{K} \to \mathcal{H} \otimes \mathbb{C}^k$ such that $\tilde{V}\xi = \sum_i V_i \xi \otimes |e_i\rangle$ for any $\xi \in \mathcal{K}$. It is the easy to see that $\Phi(B) = \tilde{V}^*(B \otimes I_k)\tilde{V}$ is a minimal Stinespring representation of Φ. It follows that there is a unitary $U : \mathbb{C}^k \to \mathcal{H}_0$ such that $(I \otimes U)\tilde{V} = V$, so that

$$V = \sum_{i=1}^k V_i \otimes |x_i\rangle, \qquad |x_i\rangle = U|e_i\rangle,\ i = 1, \dots, k.$$

For any $j = 1, \dots, k$ and $\xi \in \mathcal{K}$,

$$(I \otimes |x_j\rangle\langle x_j|)V\xi = (I \otimes |x_j\rangle\langle e_j|)\tilde{V}\xi = V_j \xi \otimes |x_j\rangle = V R_j \xi,$$

where $R_j = V_j^* V_j$. It follows that $|x_j\rangle\langle x_j| \in \mathcal{M}$ and $\Phi^C(|x_j\rangle\langle x_j|) = R_j$. Moreover, it follows from the asumptions that $W_j := V_j^* V_{j+1}$ is a partial isometry and $V_k W_j = \delta_{kj} V_{j+1}$. Hence we have

$$VW_j = V_{j+1} \otimes |x_j\rangle = (I \otimes |x_j\rangle\langle x_{j+1}|)V,$$

so that $|x_j\rangle\langle x_{j+1}| \in \mathcal{M}_R$, for $j = 1, \ldots, k-1$. Since $\mathcal{M}_R$ is a subalgebra, (ii) follows.

Suppose (ii) is true and let $T \in B(\mathcal{H}_0)^+$. Then T can be written in the form $T = T_0^* T_0$, where T_0 is upper triangular with respect to the ONB $|x_i\rangle$ (the Cholesky decomposition of T). Thus $T_0 \in \mathcal{M}_R$ and Φ is C*-extreme by Proposition 1.

Finally, assume that Φ is C*-extreme and let $P \in B(\mathcal{H}_0)$ be any 1-dimensional projection. By Lemma 6, P is equivalent with a projection $Q \in \mathcal{M}$, which must be again 1-dimensional. Let x_1 be a corresponding unit vector in $\mathcal{H}_0$, so that $Q = |x_1\rangle\langle x_1|$, and put $R_1 = \Phi^C(Q)$. Then $\Phi_{Q^\perp}$ is a ucp map $B(\mathcal{H}) \to B(R_1^\perp \mathcal{K})$, which is again C*-extreme, by Proposition 3. Repeating k times, we obtain an ONB $|x_1\rangle, \ldots, |x_k\rangle$, such that $|x_i\rangle\langle x_i| \in \mathcal{M}$ for all i. Put $R_i := \Phi^C(|x_i\rangle\langle x_i|)$, then R_i are projections in $B(\mathcal{K})$ and $\sum_i R_i = I$.

Let $V_i : \mathcal{K} \to \mathcal{H}$ be the linear operator given by

$$\langle \eta, V_i \xi \rangle = \langle \eta \otimes x_i, V\xi \rangle, \quad \eta \in \mathcal{H},\ \xi \in \mathcal{K},$$

so that $V = \sum_i V_i \otimes |x_i\rangle$ and $\Phi(B) = \sum_i V_i^* B V_i$ is a minimal Kraus representation of Φ. Then we have for any $\xi \in \mathcal{K}$,

$$V_i \xi \otimes |x_i\rangle = |x_i\rangle\langle x_i| V\xi = VR_i\xi = \sum_j V_j R_i \xi \otimes |x_j\rangle,$$

hence $V_j R_i = \delta_{ij} V_i$. It follows that $V_i^* V_i \leq R_i$ and since $\sum_i V_i^* V_i = V^* V = I$, we must have $V_i^* V_i = R_i$ for all i, so that the Kraus operators are partial isometries.

Choose any pair of indices i, j, $i \neq j$, and let $P_{ij} = |x_i\rangle\langle x_i| + |x_j\rangle\langle x_j| \in \mathcal{M}$. By Lemma 3, any positive operator $T \in B(P_{ij}\mathcal{H}_0)^+$ has the form $T = T_0^* T_0$ for some $T_0 \in P_{ij}\mathcal{M}_R P_{ij} \subseteq \mathcal{M}_R$. Clearly, we may chose some T not commuting with $|x_i\rangle\langle x_i|$ and it is easy to see that then we must have $T_0 = \sum_{a,b \in \{i,j\}} t_{ab} |x_a\rangle\langle x_b|$, with at least one of t_{ij} or t_{ji} nonzero. Since $\mathcal{M}_R$ is a subalgebra containing $|x_i\rangle\langle x_i|$ and $|x_j\rangle\langle x_j|$, it follows that it must contain

$|x_i\rangle\langle x_j|$ or $|x_j\rangle\langle x_i|$ (or both). Assume $|x_i\rangle\langle x_j| \in \mathcal{M}_R$, then there is some $L_{ij} \in B(\mathcal{K})$ such that

$$|x_i\rangle\langle x_j|V = V_j \otimes |x_i\rangle = VL_{ij} = \sum_k V_k L_{ij} \otimes |x_k\rangle$$

This implies $V_k L_{ij} = \delta_{ik} V_j$ and

$$L_{ij} = \sum_k V_k^* V_k L_{ij} = V_i^* V_j,$$

so that

$$V_j V_j^* = V_i L_{ij} V_j^* = V_i V_i^* V_j V_j^*.$$

Hence we have proved that for any pair of indices i, j, we have either $V_i V_i^* \geq V_j V_j^*$ or $V_j V_j^* \geq V_i V_i^*$ (or both), in other words, the set of projections $\{V_1 V_1^*, \dots, V_k V_k^*\}$ is linearly ordered. By permuting the operators $V_1, \dots, V_k$ if necessary, we obtain (i).

□

5.2 The general case

Let $P \in \mathcal{M}_{R,\mathcal{A}}$ be a projection. We will say that P is *minimal* in $\mathcal{M}_{R,\mathcal{A}}$ if $P\mathcal{M}_{R,\mathcal{A}}P = \mathbb{C}P$. The aim of this section is to prove the following characterization of $\mathcal{A}$-extreme maps.

Theorem 6. *Let $\Phi \in S_{\mathcal{K}}(\mathcal{B})$ and let $\mathcal{A} \subseteq B(\mathcal{K})$ be a C^*-subalgebra. Then Φ is $\mathcal{A}$-extreme in $S_{\mathcal{K}}(\mathcal{B})$ if and only if*

(i) there is a family $\{P_1, \dots, P_N\}$ of minimal projections in $\mathcal{M}_{R,\mathcal{A}}$ such that $\sum_i P_i = I$.

(ii) $\mathcal{T}_{\mathcal{A}} = \mathcal{M}_{R,\mathcal{A}} + \mathcal{M}^*_{R,\mathcal{A}}$.

We divide the proof into several lemmas.

Lemma 7. *Let Φ be $\mathcal{A}$-extreme and let P_1, P_2 be minimal projections in $\mathcal{M}_{\mathcal{A}}$. Then P_i are minimal in $\mathcal{M}_{R,\mathcal{A}}$ and for all $0 \neq T \in P_1\mathcal{M}_{R,\mathcal{A}}P_2$, we have $T = zW$ for some $z \in \mathbb{C}$ and a partial isometry $W \in \mathcal{M}_{R,\mathcal{A}}$, such that $W^*W = P_2$ and $WW^* \leq P_1$.*

Proof. As it was shown in the proof of Lemma 4, $P_i\mathcal{T}_\mathcal{A}P_i = \mathbb{C}P_i$, so that P_i are minimal in $\mathcal{M}_{R,\mathcal{A}}$, $i = 1, 2$. If $0 \neq T \in P_1\mathcal{M}_{R,\mathcal{A}}P_2$, then $T^*T = P_2T^*TP_2 \in P_2\mathcal{T}_\mathcal{A}P_2 = \mathbb{C}P_2$, so that there is some $t > 0$ such that $T^*T = tP_2$. Put $W = t^{-1/2}T$, then $W \in \mathcal{M}_{R,\mathcal{A}}$, $W^*W = P_2$ and it is easy to see that $WW^* \leq P_1$.

□

Lemma 8. *Let Φ be $\mathcal{A}$-extreme, then (i) and (ii) of Theorem 6 hold.*

Proof. Let $\{P_1, \dots, P_N\}$ be a maximal orthogonal family of minimal projections in $\mathcal{M}_\mathcal{A}$. By Lemma 7, each P_i is minimal in $\mathcal{M}_{R,\mathcal{A}}$ and (i) holds.

To prove (ii), note that if $N = 1$, then $\mathcal{M}_\mathcal{A} = \mathbb{C}I$ and by Corollary 1, Φ is $\mathcal{A}$-irreducible, hence $\mathcal{A}$-pure. By Proposition 2, $\mathcal{T}_\mathcal{A} = \mathcal{M}_\mathcal{A} = \mathcal{M}_{R,\mathcal{A}} = \mathbb{C}I$, so the equality trivially holds. So let $N > 1$ and let $T \in \mathcal{T}_\mathcal{A}^+$. Let $i \neq j \in \{1, \dots, N\}$ and put $P_{ij} := P_i + P_j$, $P'_{ij} = \Phi^C(P_{ij})$. By Lemma 3, $\Phi_{P_{ij}}$ is $P'_{ij}\mathcal{A}P'_{ij}$-extreme and $T_{ij} := P_{ij}TP_{ij} \in \mathcal{T}^+_{P'_{ij}\mathcal{A}P'_{ij}}$. By Lemma 3, there is some $S_{ij} \in \mathcal{M}_{R,P'_{ij}\mathcal{A}P'_{ij}} = P_{ij}\mathcal{M}_{R,\mathcal{A}}P_{ij}$ such that $T_{ij} = S_{ij}^*S_{ij}$. We clearly have $P_iS_{ij}P_i = s_iP_i$ and similarly $P_jS_{ij}P_j = s_jP_j$ for some $s_i, s_j \in \mathbb{C}$. Lemma 7 now implies that

$$S_{ij} = s_iP_i + s_jP_j + s_{ij}W_{ij} + s_{ji}W_{ji},$$

where $s_{ij}, s_{ji} \in \mathbb{C}$ and whenever $s_{ij} \neq 0$, W_{ij} is a partial isometry in $\mathcal{M}_{R,\mathcal{A}}$ such that $W_{ij}^*W_{ij} = P_j$, $W_{ij}W_{ij}^* \leq P_i$, similarly for s_{ji} and W_{ji}. It follows that

$$T_{ij} = t_iP_i + t_jP_j + t_{ij}W_{ij} + \bar{t}_{ij}W_{ij}^* + t_{ji}W_{ji} + \bar{t}_{ji}W_{ji}^*.$$

This implies that for all $i \neq j$

$$P_iTP_j = P_iT_{ij}P_j = t_{ij}W_{ij} + \bar{t}_{ji}W_{ji}^*$$

and hence

$$T = \sum_i t_iP_i + \sum_{i\neq j}(t_{ij}W_{ij} + \bar{t}_{ji}W_{ji}^*),$$

which clearly implies (ii).

□

Let now $P, Q \in \mathcal{M}_{R,\mathcal{A}}$ be projections. We will write $P \preceq Q$ if there exists a partial isometry $W \in \mathcal{M}_{R,\mathcal{A}}$ such that $W^*W = P$ and $WW^* \leq Q$. Note that P and Q necessarily belong to $\mathcal{M}_\mathcal{A}$, but is is possible that $W \notin \mathcal{M}_\mathcal{A}$.

It is easy to see that $\preceq$ is a preorder on the set of projections in $\mathcal{M}_{\mathcal{A}}$. Let us denote by $\sim$ the associated equivalence relation. Note that in general, this is not (?) the same as equivalence of projections with respect to $\mathcal{M}_{\mathcal{A}}$.

Lemma 9. *Assume that the conditions in Theorem 6 are satisfied and let $\{P_1, \dots, P_N\}$ be a family of projections as in (i). Then*

1. *$P_i \mathcal{T}_{\mathcal{A}} P_i = \mathbb{C} P_i$, for all i.*

2. *$P_i \preceq P_j$ if and only if $P_j \mathcal{M}_{R,\mathcal{A}} P_i \neq \{0\}$.*

3. *$P_i \sim P_j$ if and only if P_i and P_j are equivalent projections in $\mathcal{M}_{\mathcal{A}}$. In this case, $P_i \mathcal{M}_{R,\mathcal{A}} P_j = (P_j \mathcal{M}_{R,\mathcal{A}} P_i)^* \subseteq \mathcal{M}_{\mathcal{A}}$.*

Proof. By the condition (ii), any element $T \in \mathcal{T}_{\mathcal{A}}$ has the form $T = T_1 + T_2^*$, with $T_1, T_2 \in \mathcal{M}_{R,\mathcal{A}}$. Using the condition (i), 1. follows. Next, let $P_i \preceq P_j$ and let $W \in \mathcal{M}_{R,\mathcal{A}}$ be a corresponding partial isometry, then clearly $0 \neq W \in P_j \mathcal{M}_{R,\mathcal{A}} P_i$. Conversely, let us assume that T is a nonzero element in $P_j \mathcal{M}_{R,\mathcal{A}} P_i$, then $T^*T \in P_i \mathcal{T}_{\mathcal{A}} P_i = \mathbb{C} P_i$, so that $T^*T = tP_i$ for some $t > 0$. Put $W := t^{-1/2}T$, then $W \in \mathcal{M}_{R,\mathcal{A}}$, $W^*W = P_i$, $WW^* \leq P_j$, so that $P_i \preceq P_j$.

If $P_i \sim P_j$, then there are partial isometries $U, W \in \mathcal{M}_{R,\mathcal{A}}$ such that $W^*W = P_j \geq UU^*$ and $WW^* \leq P_i = U^*U$. Put $Z = WU$, then $Z \in P_i \mathcal{M}_{R,\mathcal{A}} P_i = \mathbb{C} P_i$, so that $Z = zP_i$ for some $z \in \mathbb{C}$. It follows that

$$U = P_j U = W^*WU = W^*Z = zW^*P_i = zW^*$$

and since $UU^* = |z|^2 W^*W = |z|^2 P_j$ is a projection, we have $|z| = 1$ and $UU^* = P_j$, similarly $WW^* = P_i$. This also implies that $W \in \mathcal{M}_{R,\mathcal{A}} \cap \mathcal{M}_{R,\mathcal{A}}^* = \mathcal{M}_{\mathcal{A}}$ and P_i and P_j are equivalent in $\mathcal{M}_{\mathcal{A}}$. By the first part of the proof, any nonzero element in $P_j \mathcal{M}_{R,\mathcal{A}} P_i$ is a multiple of such a partial isometry, this implies $P_i \mathcal{M}_{R,\mathcal{A}} P_j \subseteq \mathcal{M}_{\mathcal{A}}$.

□

Lemma 10. *Let $P \in \mathcal{M}_{R,\mathcal{A}}$ be a projection such that $P\mathcal{T}_{\mathcal{A}}P = \mathbb{C}P$ and $P\mathcal{T}_{\mathcal{A}}P^{\perp} \subseteq \mathcal{M}_{R,\mathcal{A}}$. Then Φ is $\mathcal{A}$-extreme if and only if $\Phi_{P^{\perp}}$ is $Q^{\perp}\mathcal{A}Q^{\perp}$-extreme, where $Q = \Phi^C(P)$.*

Proof. The 'only if' part follows by Proposition 3. For the converse, assume that $\Phi_{P^{\perp}}$ is $Q^{\perp}\mathcal{A}Q^{\perp}$-extreme and let $T \in \mathcal{T}_{\mathcal{A}}^+$ be any element. Then $S := P^{\perp}TP^{\perp} \in \mathcal{T}^+_{Q^{\perp}\mathcal{A}Q^{\perp}}$, so that there is some $S_0 \in \mathcal{M}_{R,Q^{\perp}\mathcal{A}Q^{\perp}} = P^{\perp}\mathcal{M}_{R,\mathcal{A}}P^{\perp}$

such that $S = S_0^* S_0$, by Proposition 1. By the assumptions on P, the operator-matrix decomposition of T with respect to $P^\perp$ has the form

$$T = \begin{pmatrix} S & X^* \\ X & tP \end{pmatrix}$$

where $t \geq 0$ and $X \in \mathcal{M}_{R,\mathcal{A}}$. Since T is positive, $t = 0$ implies $X = 0$ and then $T = S = S_0^* S_0$, with $S_0 \in \mathcal{M}_{R,\mathcal{A}}$. So assume that $t > 0$, then $T \geq 0$ implies that $S - t^{-1}X^*X \geq 0$. Since $S - t^{-1}X^*X \in \mathcal{T}^+_{Q^\perp \mathcal{A} Q^\perp}$, there is some $S_1 \in P^\perp \mathcal{M}_{R,\mathcal{A}} P^\perp$ such that $S - t^{-1}X^*X = S_1^* S_1$. Put

$$T_0 := (S_1 + t^{1/2}P)(I + t^{-1}X).$$

Then $T_0 \in \mathcal{M}_{R,\mathcal{A}}$ and it is easy to check that $T = T_0^* T_0$. By Proposition 1, this implies that Φ is $\mathcal{A}$-extreme.

□

We are now ready to prove Theorem 6.

Proof of Theorem 6. We proceed by induction on N. So let $N = 1$, then by the assumptions, $\mathcal{T}_\mathcal{A} = \mathcal{M}_{R,\mathcal{A}} = \mathbb{C}I$. By proposition 2, Φ is $\mathcal{A}$-pure and hence also $\mathcal{A}$-extreme.

Next, suppose that the statement holds whenever $N = k - 1$ and let $N = k$. We may assume that P_k is maximal in the set $\{P_1, \ldots, P_k\}$ with respect to the preorder $\preceq$, that is, if $P_k \preceq P_i$ for some i then $P_i \sim P_k$. This means that for all $j = 1, \ldots, k$, we either have $P_j \mathcal{M}_{R,\mathcal{A}} P_k = \{0\}$ or $P_j \mathcal{M}_{R,\mathcal{A}} P_k \subseteq \mathcal{M}_\mathcal{A}$.

Let $T \in \mathcal{T}_\mathcal{A}$ be any element, then $T = T_1 + T_2^*$ for some $T_1, T_2 \in \mathcal{M}_{R,\mathcal{A}}$. We have

$$P_k T P_k = P_k T_1 P_k + (P_k T_2 P_k)^* = z P_k$$

for some $z \in \mathbb{C}$, and

$$P_k T P_j = P_k T_1 P_j + (P_j T_2 P_k)^* \in \mathcal{M}_{R,\mathcal{A}}$$

for all j. Since $\Phi_{P_k^\perp}$ is $(P_k')^\perp \mathcal{A} (P_k')^\perp$-extreme by the induction hypothesis, the assumptions of Lemma 10 are satisfied, so that Φ is $\mathcal{A}$-extreme.

□

References

[1] W. Arveson, Subalgebras of C*-algebras, Acta Math. 123 (1969), 141224

[2] W. Arveson, Subalgebras of C-algebras II, Acta Math 128 (1972), 271308

[3] D. Farenick and P. Morenz, C*-extreme points in the generalized state space of a C*-algebra, Trans. Amer. Math. Soc. 349(1997), 1725-1748

[4] D. Farenick and H. Zhou, The structure of C*-extreme points in spaces of completely positive linear maps on C*-algebras, Proc.Amer. Math. Soc.126 (1998), 1467-1477

[5] A. Jenčová, On the convex structure of process POVMs, in preparation

[6] V. Paulsen, Completely Bounded Maps and Operator Algebras, Cambridge University Press, 2002