Fuzzy Sets and Systems

Fuzzy observables and the universal family of fuzzy events

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1. Abstract

We prove the existence of a universal family of fuzzy events $\mathcal{F}$ which realizes every commutative fuzzy observable in the sense that the connection between an arbitrary fuzzy observable and the sharp observable of which it is a fuzzification is given by the events in $\mathcal{F}$. That is a consequence of the existence of a universal Markov kernel of which we provide a general proof in the case of Hausdorff, locally compact, second countable topological spaces; the proof is based on a modified version of the transferring principle introduced by Lebesgue, Riesz, de la Vallée-Poussin and Jessen and introduces a bijective map between the infinite dimensional cube and the unit interval.

Then, we show that every weak Markov kernel is functionally subordinated to the universal Markov kernel; in some sense the fuzziness encoded in the latter includes any possible fuzziness. As a consequence the universal family of fuzzy events $\mathcal{F}$ includes (up to functional subordination) all the possible fuzzy events of which a positive operator valued measure can represent the probabilities.

Cover Letter

Dear Editor, we are sending you the revised version of the paper ''Fuzzy observables and the universal family of fuzzy events". We thanks the referee for the helpful observations and for the interesting question. We have made all the changes (in blue color to simplify their individuation) suggested by the referee. We also modified the abstract as suggested by the Editor. Here is a detailed list of the changes we have done.

Editor Comment

Note that an abstract should be a purely verbal description of the main results and cannot contain symbols or references.

Corrected

Referee comments

1) *the notion of a "fuzzy observable" is often mentioned, but as far as I can see, never really defined. This term may be familiar to readers of FSS but perhaps not for all researchers in the field.*

   We added an explicit analysis of the concept of fuzzy observable and the corresponding definition (see pages 8 and 9 of the new version). It has been necessary to add definition 7 and two new references (Ref.s [14], [21]). Moreover, we modified the last section (section 3) in order to take into account what we added in pages 8 and 9 and in order to eliminate some redundancies. As a consequence we also made some small changes in the introduction.

2) - *p. 4 (and also p. 20): "proof... has been proved" it is better to write "proof...*

has been given" or "proof ... can be found" etc.

Page 20 has been corrected but we did not find the expression "proof... has been proved" in page 4. It compares in page 4 of the Revision letter but not in page 4 of the paper.

3) - *p. 5, line 3 from below: "than" → "then"*

Corrected

4) - *Def. 3 and below: note that an orthogonal POVM may be not projection valued if it is not normalized. Usually a POVM is assumed normalized, but according to your Def. 1 it is not necessarily so.*

Corrected: we modified the definition. In the new definition, POVMs are assumed to be normalized.

5) - *p. 12, line 9 from below: here "*$\times[0,1]^\omega$*" is missing in the expression for* $I_l^{(n)}$

Corrected

6) - *p. 12, line 3 from below: the lower indices in* $i_1^{(n)}$ *and* $i_2^{(n)}$ *should probably be* j *and* l

Corrected

7) - *p. 13, line 4:* $D_k \to d_k$ *and* $E^F \to E_0^F$

Corrected

8) - *p. 16, first displayed equation: better specify the limit. (It seems to me this is actually a supremum of an increasing sequence of projections (?))*

Yes. We specified that it is a limit in the strong operator topology and that the limit is the least upper bound of the non-decreasing sequence of projection operators.

9) - *p. 22, line 8 from below: "denotes"* $\rightarrow$ *denote*

Corrected

10) - *p. 24, line 3 from below: better give some explanation or reference for the existence of the Borel function* f

We added a reference and gave some explanations.

11) - *just a question: The (weak) Markov kernels can be seen as POVMs, so it is natural to ask whether the functional subordination can be interpreted as some of the usual preorders of POVMs, such as post- or (more likely) pre-processing.*

This is a very interesting question deserving some work to be fully answered. Here are some tentative preliminary observations. At first sight, functional subordination seems not to correspond to post-processing. Indeed, consider two commutative POVMs F_1 and F_2 whose ranges are contained in the same abelian von Neumann algebra $\mathcal{A}$ and suppose that $F_1(\Delta) = \int \mu^1_\Delta(\lambda)\, dE_\lambda$ and $F_2(\Delta) = \int \mu^2_\Delta(\lambda)\, dE_\lambda$ where E is a spectral measure which generates $\mathcal{A}$ and μ^1, μ^2 are Markov kernels. Then, F_1 and F_2 are both post-processings of E but, in general, F_1 is not a post-processing of F_2 and vice versa, even if there is a measurable function f such that $\mu^1_\Delta(\lambda) = \mu^2_\Delta(f(\lambda))$. This latter equation implies that the post-processing described by μ_1 can be decomposed into two steps: the first step is encoded into the function f and corresponds to the deterministic Markov kernel $\chi_{f^{-1}(\Delta)}$, the second step is exactly the post processing μ_2. In our opinion (to be checked further) that does not imply a connection between F_1 and F_2 of the kind $F_1(\Delta) = \int \gamma_\Delta(t)\, F_2(dt)$. A comparison with the preorder introduced in Ref. [1] (which is weaker than smearing) could be of interest.

More likely, as the referee suggests, functional subordination could be related to pre-processing. Indeed, let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a commutative von Neumann algebra and B one of its generators. We can assume B to be bounded with spectrum in $[0,1]$ without loss of generality. Let $f : \sigma(B) \to \sigma(B)$ be a measurable function and $A = f(B)$. Then, we can define a linear map $\Phi_f : \mathcal{A}(B) \to \mathcal{A}(A)$ from the abelian von Neumann algebra $\mathcal{A}(B)$ generated by B to the abelian von Neumann algebra $\mathcal{A}(A)$ generated by A. In particular, for every operator $g(B) = \int g(\lambda)\, dE_\lambda \in \mathcal{A}(B)$ we define the operator $\Phi_f(g(B)) = g(A) = g(f(B)) = \int g(f(\lambda))\, dE_\lambda \in \mathcal{A}(A)$. By the functional calculus for normal operators, it follows that,

1) Suppose $B_1 = g_1(B)$ and $B_2 = g_2(B)$. Then,

$$\begin{aligned}
\Phi_f[a_1 B_1 + a_2 B_2] &= \Phi_f\Big[\int \big(a_1 g_1 + a_2 g_2\big)(\lambda)\, dE_\lambda\Big] \\
&= \int \big(a_1 g_1 + a_2 g_2\big)(f(\lambda))\, dE_\lambda \\
&= a_1 \int g_1(f(\lambda))\, dE_\lambda + a_2 \int g_2(f(\lambda))\, dE_\lambda \\
&= a_1 \Phi_f(g_1(B)) + a_2 \Phi_f(g_2(B)) \\
&= a_1 \Phi_f(B_1) + a_2 \Phi_f(B_2).
\end{aligned}$$

,

2) $\Phi_f(g(B)^\dagger) = \Phi_f\Big[\int \bar{g}_1(\lambda)\,dE_\lambda\Big] = \int \bar{g}_1(f(\lambda))\,dE_\lambda = g(A)^\dagger = \big[\Phi_f(g(B))\big]^\dagger$

3) $\Phi_f(g_1(B)g_2(B)) = \Phi_f(g_1(B))\Phi_f(g_2(B))$

4) $\Phi_f(I) = I$.

Then, Φ is a $*$-homomorphism between C^*-algebras. Hence, it is completely positive and preserves the identity. We than have a pre-processing (a channel in the Heisenberg picture). Now, let γ and B be as in theorem 2.4 of the manuscript and assume the spectrum of B to be contained in I (see definition 8 in the manuscript). Consider the POVMs $F^U(\Delta) = \mu^U_\Delta(B)$ and $F(\Delta) = \gamma_\Delta(B)$ where μ^U is the universal Markov kernel. Then, $\Phi_f(F^U(\Delta)) = F(\Delta)$ (F can be obtained from F^U by a pre-processing, $F < F^U$). That is also true for all the commutative POVMs with range in $\mathcal{A}(B)$; they are obtained by a preprocessing of F^U. Indeed they can be represented in the form $F(\Delta) = \int \mu^U_\Delta(h(\lambda))\,dE^B_\lambda = \Phi_h(F^U(\Delta))$. In other words, F^U is minimal in the space of POVMs with range in $\mathcal{A}(B)$ (in the language of Ref. [2], it is cleaner than any other POVM with range in $\mathcal{A}(B)$).

Possible weak points of the previous reasoning: 1) In the infinite dimensional case, the σ-weak continuity of Φ_f should be proved. 2) Rigor, correctness and generality (for example, considerations about the spectrum of A and B should be added) must be checked.

Anyway, it seems to provide a first tentative path in order to answer the referee's question.

12) Some typos have been corrected.

References

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Sincerely yours,

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Fuzzy observables and the universal family of fuzzy events

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Abstract

We prove the existence of a universal family of fuzzy events $\mathcal{F}$ which realizes every commutative fuzzy observable in the sense that the connection between an arbitrary fuzzy observable and the sharp observable of which it is a fuzzification is given by the events in $\mathcal{F}$. That is a consequence of the existence of a universal Markov kernel of which we provide a general proof in the case of Hausdorff, locally compact, second countable topological spaces; the proof is based on a modified version of the transferring principle introduced by Lebesgue, Riesz, de la Vallée-Poussin and Jessen and introduces a bijective map between the infinite dimensional cube and the unit interval.

Then, we show that every weak Markov kernel is functionally subordinated to the universal Markov kernel; in some sense the fuzziness encoded in the latter includes any possible fuzziness. As a consequence the universal family of fuzzy events $\mathcal{F}$ includes (up to functional subordination) all the possible fuzzy events of which a positive operator valued measure can represent the probabilities.

Key words: `Fuzzy Observables, Fuzzy sets, Positive Operator Valued Measures, Markov Kernels, Foundations of Quantum mechanics, Quantum Structures`

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1. Introduction

In the present section we briefly recall the concept of fuzzy observable and its connections with fuzzy sets. As a consequence of the statistical nature of the measurement process [20], quantum observables are represented by positive operator valued measures [12, 16, 20, 29, 31, 35]. Spectral measures are particular examples of POVMs and, by the spectral theorem, they are in a one-to-one correspondence with self-adjoint operators. Therefore, the analysis of the measurement process reveals that the set of self-adjoint operators is too small in order to represent quantum observables. A very relevant feature of positive operator valued measures (POVMs) is that there are couples of non-commuting POVMs which are the marginals of a joint POVM (they are jointly measurable). We know that in the particular case of self-adjoint operators joint measurability and commutativity coincide [12, 27].

As a relevant example one can consider the spectral measures of the position and momentum operators, E^Q and E^P for which a joint spectral measure does not exist since they do not commute. Nevertheless, it is possible to randomize E^Q and E^F by means of Markov kernels μ^Q and μ^P. That provides two POVMs F^Q and F^P for which a joint POVM does exist [6, 9, 16, 31]. In particular, for every Δ in the Borel σ-algebra of the reals, $\mathcal{B}(\mathbb{R})$,

$$\begin{aligned} F^Q(\Delta) &= \int_{\mathbb{R}} \mu^Q_\Delta(\lambda)\, dE^Q_\lambda, \\ F^P(\Delta) &= \int_{\mathbb{R}} \mu^P_\Delta(\lambda)\, dE^P_\lambda. \end{aligned} \tag{1}$$

The POVMs F^Q and F^P are called the unsharp or fuzzy version of E^Q and E^P respectively [2, 18, 31] and are commutative since they are contained in the commutative von Neumann algebras generated by E^Q and E^P respectively. It is relevant that there is a third POVM, F, of which F^Q and F^P are the marginals [6, 16, 31, 35],

$$F(\Delta_q \times \mathbb{R}) = F^Q(\Delta_q)$$
$$F(\mathbb{R} \times \Delta_p) = F^P(\Delta_p).$$

That allows a representation of quantum mechanics on a phase space which should be interpreted as a stochastic phase space whose points are fuzzy points [31].

There are other examples of the power of POVMs as the mathematical representative of quantum observables [1, 2, 12, 16, 20, 29, 31, 35].

The previous example provides some insight about the relevance of commutative POVMs to quantum physics. We add that they model certain standard forms of noise in quantum measurements and provide optimal approximators as marginals in joint measurements of incompatible observables (e.g., Position and Momentum) [13].

In (1) we obtained the fuzzy position and momentum POVMs (that are commutative POVMs) as the randomization of the sharp position and momentum operators with the randomization realized through Markov kernels. That is a general property of commutative POVMs, i.e., every commutative POVM $F : \mathcal{B}(X) \to \mathcal{L}_s^+(\mathcal{H})$ from the σ-algebra of a topological space X to the space of linear positive self-adjoint operators on a Hilbert space $\mathcal{H}$ is the random version of a spectral measure E^F (the sharp version of F); the randomization being represented by a Markov kernel μ [8, 10, 17, 23, 32],

$$\langle \psi, F(\Delta)\psi \rangle := \int \mu_\Delta(\lambda)\, d\langle \psi, E_\lambda \psi \rangle, \quad \Delta \in \mathcal{B}(X), \quad \psi \in \mathcal{H}. \tag{2}$$

Consider for example the unsharp position observable in equation (1). It can be represented as follows [16],

$$\langle \psi, F^Q(\Delta)\psi \rangle := \int_{\mathbb{R}} \mu_\Delta(\lambda)\, d\langle \psi, E^Q_\lambda \psi \rangle, \quad \Delta \in \mathcal{B}(\mathbb{R}), \quad \psi \in L^2(\mathbb{R}), \tag{3}$$
$$\mu_\Delta(\lambda) := \int_{\mathbb{R}} \chi_\Delta(\lambda - y)\, f(y)\, dy, \quad \lambda \in \mathbb{R}$$

where, f is a positive, bounded, Borel function such that $\int_{\mathbb{R}} f(y)dy = 1$, while E^Q is the spectral measure corresponding to the position operator

$$Q : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$
$$(Q\psi)(x) := x\psi(x), \quad a.a. \quad x \in \mathbb{R}.$$

The quantity $\langle\psi, E^Q(\Delta)\psi\rangle$ can be interpreted as the probability that a perfectly accurate measurement (sharp measurement) of the position gives a result in Δ. A possible interpretation of equation (3) is that, due to measurement imprecision[1], the outcomes of the measurement of E^Q are randomized: if the sharp value of the outcome of the measurement of E^Q is λ then the apparatus produces with probability $\mu_\Delta(\lambda)$ a reading in Δ. As a result, the probability of an outcome in Δ is given by $\langle\psi, F^Q(\Delta)\psi\rangle$ so that F^Q represents an unsharp measurement of E^Q.

An interpretation of equations (2), (3) in the framework of fuzzy sets theory [38, 39] has been suggested in Ref. [18]. A fuzzy set is a pair $A = (\mathbb{R}, \mu_A)$ where $\mu_A : \mathbb{R} \to [0, 1]$ is a membership function. The value $\mu_A(x)$ is interpreted as the membership degree of x in A. A fuzzy event is a fuzzy set such that the membership function μ_A is Borel measurable. If ν is a probability measure on $\mathbb{R}$, the probability of a fuzzy event with respect to ν is defined as

$$P(A) = \int \mu_A(x)\, d\nu(x).$$

Going back to equation (3), we have the following interpretation in terms of fuzzy sets (see also the end of the present section). The Markov kernel μ provides a family of fuzzy events $\{(\mathbb{R}, \mu_\Delta)\}_{\Delta\in\mathcal{B}(\mathbb{R})}$. For every point $x \in \mathbb{R}$, the family $\{\mu_\Delta(x)\}_{\Delta\in\mathcal{B}(\mathbb{R})}$ defines a probability measure. For every $\psi \in \mathcal{H}$, the expression

$$\langle\psi, F^Q(\Delta)\psi\rangle = \int_{\mathbb{R}} \mu_\Delta(x)\, d\langle\psi, E^Q_x\psi\rangle$$

[1]Which can be thought to be intrinsic to the quantum measurement process and then to be unavoidable

can then be interpreted as the probability of the fuzzy event $(\mathbb{R}, \mu_\Delta)$ with respect to the probability measure $\langle\psi, E^Q(\cdot)\psi\rangle$. In other words, the unsharp observable F^Q gives the probabilities of the fuzzy events $(\mathbb{R}, \mu_\Delta)$ with respect to the probability measures corresponding to E^Q. On the other hand $\langle\psi, F^Q(\Delta)\psi\rangle$ gives the probability of the event $\Delta \equiv (\mathbb{R}, \chi_\Delta)$ (which is not a fuzzy event) with respect to the probability measure $\langle\psi, F^Q(\cdot)\psi\rangle$: $\langle\psi, F^Q(\Delta)\psi\rangle = \int_{\mathbb{R}} \chi_\Delta(x)\, d\langle\psi, F^Q_x\psi\rangle$. That is a very peculiar situation with respect to the general definition of fuzzy events.

Equation (3) can also be interpreted as the fuzzification of the spectral measure E^Q through the family of fuzzy events $(\mathbb{R}, \mu_\Delta)_{\Delta\in\mathcal{B}(\mathbb{R})}$ (see below for more details).

The present paper focuses on the analysis of the fuzzification process that connects E and F. In particular, we prove that there is a universal Markov kernel μ^U that connects every commutative POVM F (and then every fuzzy observable) to its sharp version E^F (we just require the POVM to be defined on a Hausdorff, locally compact topological space whose topology is countably generated) and that every Markov kernel is functionally subordinated to the universal Markov kernel. That generalizes some previous results [4, 6] where the existence of a universal Markov kernel is proved in the case $X = \mathbb{R}$. Moreover, we provide a general procedure for the construction of both the sharp version and the universal Markov kernel μ^U which is based on a modified version of Jessen's transfering principle [24]. The transferring principle was introduced by Lebesgue [28], Riesz [33] and de la Vallée Poussin [37] in the n-dimensional case. Later Jessen generalized the transferring principle in order to define integration of functions with a countable number of variables [24].

Concerning the possible interpretations of the universal Markov kernel we have that, if we interpret a Markov kernel as a measure of the randomization due to the measurement imprecision, then the existence of the universal Markov kernel means that there is a randomization which includes all the others (see sections 2.4). On the other hand, if we interpret a Markov kernel as a family of fuzzy events, then the existence of the universal Markov kernel μ^U means that every POVM F gives the probabilities of the fuzzy events, $\{(\mathbb{R}, \mu^U_\Delta)\}_{\Delta\in\mathcal{B}(X)}$, with respect to E^F and that $\{(\mathbb{R}, \mu^U_\Delta)\}_{\Delta\in\mathcal{B}(X)}$ can be interpreted as a universal family of fuzzy events which includes (up to functional subordination) all the possible fuzzy events of which a POVM can represent the probabilities (see section 3).

Now, we briefly recall the main definitions and properties of POVMs. In what follows, $\mathcal{B}(X)$ denotes the Borel σ-algebra of a topological space X and $\mathcal{L}^+_s(\mathcal{H})$ the space of all bounded positive self-adjoint linear operators acting in a Hilbert space $\mathcal{H}$ with scalar product $\langle\cdot,\cdot\rangle$.

Definition 1. A Positive Operator Valued measure (for short, POVM) is a map $F:\mathcal{B}(X)\to\mathcal{L}^+_s(\mathcal{H})$ such that:

$$F(X) = \mathbf{1}$$
$$F(\bigcup_{n=1}^{\infty}\Delta_n) = \sum_{n=1}^{\infty} F(\Delta_n).$$

where, $\mathbf{1}$ is the identity operator, $\{\Delta_n\}$ is a countable family of disjoint sets in $\mathcal{B}(X)$ and the series converges in the weak operator topology.

Definition 2. A POVM is said to be commutative if

$$\big[F(\Delta_1), F(\Delta_2)\big] = \mathbf{0}, \quad \forall\, \Delta_1\,, \Delta_2 \in \mathcal{B}(X). \tag{4}$$

Definition 3. A POVM is said to be orthogonal if $\Delta_1\cap\Delta_2=\emptyset$ implies

$$F(\Delta_1)F(\Delta_2) = \mathbf{0} \tag{5}$$

where $\mathbf{0}$ is the null operator. An orthogonal POVM is called projection valued measure (for short, PVM).

Note that if F is orthogonal, the operators $F(\Delta)$ are projection operators.

Definition 4. A Spectral measure is a real PVM.

In quantum mechanics, non-orthogonal POVMs represent **generalised** or **unsharp** or **fuzzy** observables while PVMs represent **standard** or **sharp** observables.

We recall that $\langle \psi, F(\Delta)\psi \rangle$ is interpreted as the probability that a measurement of the observable represented by F gives a result in Δ.

The following theorem gives a characterization of commutative POVMs as the randomization of spectral measures with the randomization realized by means of Markov kernels.

Definition 5. Let Λ be a topological space. A Markov kernel is a map $\mu : \Lambda \times \mathcal{B}(X) \to [0,1]$ such that,

1. $\mu_\Delta(\cdot)$ is a measurable function for each $\Delta \in \mathcal{B}(X)$,
2. $\mu_{(\cdot)}(\lambda)$ is a probability measure for each $\lambda \in \Lambda$.

Definition 6. Let ν be a measure on Λ. A map $\mu : (\Lambda, \nu) \times \mathcal{B}(X) \to [0,1]$ is a weak Markov kernel with respect to the measure ν if:

1. $\mu_\Delta(\cdot)$ is a measurable function for each $\Delta \in \mathcal{B}(X)$,
2. for every $\Delta \in \mathcal{B}(X)$, $0 \leq \mu_\Delta(\lambda) \leq 1, \quad \nu - a.e.$,
3. $\mu_\emptyset(\lambda) = 0, \mu_\Lambda(\lambda) = I \quad \nu - a.e.$,
4. for any sequence $\{\Delta_i\}_{i\in\mathbb{N}}$, $\Delta_i \cap \Delta_j = \emptyset$,

$$\sum_i \mu_{(\Delta_i)}(\lambda) = \mu_{(\cup_i \Delta_i)}(\lambda), \quad \nu - a.e.$$

Definition 7. A Markov kernel $\mu : \Lambda \times \mathcal{B}(X) \to [0,1]$ such that, for every $\Delta \in \mathcal{B}(X)$, and $\lambda \in \Lambda$, $[\mu_\Delta(\lambda)]^2 = \mu_\Delta(\lambda)$, is said deterministic.

In the following the symbol $\mathcal{A}(F)$ denotes the von Neumann algebra generated by the POVM F, i.e., the von Neumann algebra generated by the set $\{F(\Delta)\}_{\mathcal{B}(X)}$. Analogously $\mathcal{A}(B)$ denotes the von Neumann algebra generated by the self-adjoint operator B. Hereafter, we assume that X is a Hausdorff, locally compact, second countable topological space. The connection between commutative POVMs and randomization of spectral measures has been pointed out by several authors [7, 8, 10, 15, 17, 23, 32]. One of the possible formulation of their equivalence is provided in the following theorem.

Theorem 1.1 ([8, 10])**.** *A POVM $F : \mathcal{B}(X) \to \mathcal{L}_s^+(\mathcal{H})$ is commutative if and only if there exists a bounded self-adjoint operator $A = \int \lambda \, dE_\lambda$ with spectrum $\sigma(A) \subset [0, 1]$, and a Markov Kernel $\mu : \sigma(A) \times \mathcal{B}(X) \to [0, 1]$ such that*

1) $F(\Delta) = \int_{\sigma(A)} \mu_\Delta(\lambda) \, dE_\lambda, \quad \Delta \in \mathcal{B}(X)$.

2) $\mathcal{A}(F) = \mathcal{A}(A)$.

3) there are a ring $\mathcal{R}(\mathcal{S}) \subset \mathcal{B}(X)$ and a set $\Gamma \subset \sigma(A)$, $E(\Gamma) = \mathbf{1}$, such that $\mu_\Delta(\lambda)$ is continuous for every $\Delta \in \mathcal{R}(\mathcal{S})$ and $\lambda \in \Gamma$. In particular, $\mu : \Gamma \times \mathcal{B}(X) \to [0, 1]$ is a Feller Markov kernel.

The operator A (or equivalently, the spectral measure E) introduced in theorem 1.1 is called the **sharp version** of F and is unique up to almost everywhere bijections [7, 8]. The POVM F is said to be a smearing of the spectral measure E or equivalently a smearing of A. It can be interpreted as a fuzzy version of E [10].

Other equivalent versions of these result exist [3, 5, 19, 23]. In [23] and [5] their equivalence has been proved.

In order to explain why and in what sense a commutative POVM can be interpreted as a fuzzy observable we first recall the general definition of smearing.

Definition 8. A POVM F^1 is the smearing of a POVM F^2 if there is a Markov kernel μ such that

$$F^1(\Delta) = \int \mu_\Delta(t) dF_t^2, \quad \Delta \in \mathcal{B}(X).$$

Now, we resort to theorem 1.1 according to which, a commutative POVM F is obtained by a smearing procedure which is realized through a family of fuzzy events (the Markov kernel). The smearing corresponds to the fuzzification a spectral measure. Moreover, the quantity $\langle \psi, F(\Delta)\psi \rangle$ is the probability of the fuzzy event $(\sigma(A), \mu_\Delta)$ with respect to the probability measure $\langle \psi, E(\Delta)\psi \rangle$.

What said above must be detailed since the space of commutative POVMs includes the PVMs as a particular case and the latter cannot be interpreted as representing fuzzy observables. Indeed: a) By applying theorem 1.1 to a PVM F and considering that $F(\Delta)^2 = F(\Delta)$, we find that $[\mu_\Delta(\lambda)]^2 = \mu_\Delta(\lambda)$, E-a.e.. Moreover, μ can be replaced by a deterministic Markov kernel [21]. Also the converse is true; the smearing of a spectral measure through a deterministic Markov kernel is a PVM. b) In the case of a Hausdorff, locally compact, second countable topological space (which is metrizable and σ-compact), a deterministic Markov kernel μ can be represented [21] in the form $\mu_\Delta(\lambda) = \chi_\Delta(f(\lambda))$, $\Delta \in \mathcal{B}(X)$, which corresponds to a family of crisp events. c) Then in the case of a PVM F, $F(\Delta)$ corresponds to the crisp event $(\sigma(A), \chi_\Delta \circ f)$ and $\langle F(\Delta)\psi, \psi\rangle = \int \chi_\Delta(f(\lambda))\, d\langle F_\lambda \psi, \psi\rangle$ is its probability. Equivalently, F is the deterministic smearing of a spectral measure through a family of **crisp** events. In this sense it is not the fuzzification of a spectral measure, d) One could ask if a PVM can be the fuzzification of a general POVM (not generally sharp) but the answer is in the negative since, also in this case, the Markov kernel which realizes the smearing must be deterministic [32]. All of that suggests that the sharp observables (which are represented by PVMs) cannot be interpreted as fuzzy observables and agrees with the interpretation in Ref.s [14]. Therefore, the nature of the Markov kernel (deterministic vs non-deterministic) is crucial in order for the smearing procedure to be a fuzzification and for the resulting observable to be fuzzy.

Definition 9. A POVM F^1 is the fuzzification of a POVM F^2 if there is a non-deterministic Markov kernel μ such that

$$F^1(\Delta) = \int \mu_\Delta(t) dF^2_t, \quad \Delta \in \mathcal{B}(X).$$

In this case F^1 is said to represent a fuzzy observable.

Theorem 1.1 can be restated as follows: every commutative non-orthogonal POVM represents a fuzzy observable.

2. Universal Markov kernel

In the present section we prove the existence of a universal Markov kernel for commutative POVMs (see subsection 2.4). That generalizes the result in [4, 6] where the existence of a universal Markov kernel has been proved for real commutative POVMs.

The proof is based on a modified version of the algorithm developed by Lebesgue, Riesz, Jessen and Sz-Nagy [24, 28, 30, 34] that they used in order to prove the transferring principle. In particular it was used by Jessen in order to introduce integration in infinite dimensional spaces. Later, the algorithm was used by Sz-Nagy in order to prove a theorem previously proved by von Neumann, i.e., that to an arbitrary family $\{A_i\}_{i\in I}$ of bounded commuting self-adjoint operators there corresponds a self-adjoint operator A and a family of measurable functions $\{f_i\}_{i\in I}$ such that $A_i = f_i(A)$, for all $i \in I$ (see section 130 in Ref. [33]). A similar algorithm has been used in Ref. [15] in order to show that every real commutative POVM is the smearing of a spectral measure (see also [7]).

In particular, Jessen [24] proposed a quite general procedure to define a bijective function from a subset of the infinite dimensional torus to a subset of the unit interval (transferring principle). Here we modify Jessen's algorithm in order to obtain a bijective function $f : [0,1]^\omega \to [0,1]$ from the infinite dimensional cube to the unit interval and make use of such a function in order to define the sharp version of a commutative POVM $F : \mathcal{B}(X) \to \mathcal{L}_s(\mathcal{H})$ where X is a second countable topological space X. That generalizes some previous results, uses explicitly and in its full generality the transferring principle and provides a more powerful and compact formulation based on the bijective function $f : [0,1]^\omega \to [0,1]$. Then, we proceed to prove the existence of the universal Markov kernel on a Hausdorff, locally compact, second countable space.

2.1. Transferring Principle

We diverge from Jessen's construction in order to obtain a one-to-one function between $[0,\,1]^\omega$ and $[0,\,1]$. In particular, we use left-closed subintervals and provide two related nets of dissections of $[0,\,1]^\omega$ and $[0,\,1]$ in the following recursive way.

Set $D_1 := \{[0,\,1]^\omega\}$ and $d_1 := \{[0,\,1]\}$.

Suppose now that the dissections D_n and d_n are given. We write for them $D_n = \{I_1^{(n)}, \ldots, I_{k_n}^{(n)}\}$ and $d_n = \{i_1^{(n)}, \ldots, i_{h_n}^{(n)}\}$, where $I_l^{(n)} = b_{l,\,1}^{(n)} \times b_{l,\,2}^{(n)} \times \cdots \times b_{l,\,\lambda(l,\,n)}^{(n)} \times [0,\,1]^\omega$. We assume that the following conditions hold:

1. each set $b_{l,\,j}^{(n)}$ is a left-closed interval with the exception of those sets with right endpoint equal to 1;

2. $b_{l,\,\lambda(l,\,n)-1}^{(n)} \neq [0,\,1]$;

3. $b_{l,\,\lambda(l,\,n)}^{(n)} = [0,\,1]$.

At step $n+1$ the dissections are defined as follows. The dissection D_{n+1} is obtained by subdividing each $b_{l,\,j}^{j}$ into 2 subintervals of the same size and the dissection d_{n+1} is obtained by subdividing each interval $i_l^{(n)}$ into 2^n subintervals of the same size.

In order to clarify our construction we will give explicitly the first terms of the two nets. We write $D_1 = \{[0,\,1] \times [0,\,1]^\omega\}$, so

$$\begin{aligned} D_2 &= \left\{\left[0,\,\frac{1}{2}\right) \times [0,\,1]^\omega\,,\ \left[\frac{1}{2},\,1\right] \times [0,\,1]^\omega\right\} = \\ &= \left\{\left[0,\,\frac{1}{2}\right) \times [0,\,1] \times [0,\,1]^\omega\,,\ \left[\frac{1}{2},\,1\right] \times [0,\,1] \times [0,\,1]^\omega\right\} \end{aligned}$$

and

$$d_2 = \left\{\left[0,\,\frac{1}{2}\right)\,,\ \left[\frac{1}{2},\,1\right]\right\}$$

Thus we have $k_2 = h_2 = 2$, $\lambda(l,\,2) = 2$ for each l, $b_{1,\,1}^{(2)} = \left[0,\,\frac{1}{2}\right)$, $b_{1,\,2}^{(2)} = [0,\,1]$, $b_{2,\,1}^{(2)} = \left[\frac{1}{2},\,1\right]$ and $b_{2,\,2}^{(2)} = [0,\,1]$.

The dissection D_3 is the following.

$$D_3 = \left\{ \left[0,\, \frac{1}{4}\right) \times \left[0,\, \frac{1}{2}\right) \times [0,\, 1]^\omega,\ \left[0,\, \frac{1}{4}\right) \times \left[\frac{1}{2},\, 1\right] \times [0,\, 1]^\omega, \right.$$
$$\left[\frac{1}{4},\, \frac{1}{2}\right) \times \left[0,\, \frac{1}{2}\right) \times [0,\, 1]^\omega,\ \left[\frac{1}{4},\, \frac{1}{2}\right) \times \left[\frac{1}{2},\, 1\right] \times [0,\, 1]^\omega,$$
$$\left[\frac{1}{2},\, \frac{3}{4}\right) \times \left[0,\, \frac{1}{2}\right) \times [0,\, 1]^\omega,\ \left[\frac{1}{2},\, \frac{3}{4}\right) \times \left[\frac{1}{2},\, 1\right] \times [0,\, 1]^\omega,$$
$$\left. \left[\frac{3}{4},\, 1\right] \times \left[0,\, \frac{1}{2}\right) \times [0,\, 1]^\omega,\ \left[\frac{3}{4},\, 1\right] \times \left[\frac{1}{2},\, 1\right] \times [0,\, 1]^\omega \right\}$$

The corresponding dissection d_3 is the following

$$d_3 = \left\{ \left[0,\, \frac{1}{8}\right),\, \left[\frac{1}{8},\, \frac{1}{4}\right),\, \left[\frac{1}{4},\, \frac{3}{8}\right),\, \left[\frac{3}{8},\, \frac{1}{2}\right),\, \left[\frac{1}{2},\, \frac{5}{8}\right),\, \left[\frac{5}{8},\, \frac{3}{4}\right),\, \left[\frac{3}{4},\, \frac{7}{8}\right),\, \left[\frac{7}{8},\, 1\right] \right\},$$

so $k_3 = h_3 = 8$, $\lambda(l,\, 3) = 3$ for each l. Moreover notice that each $I_1^{(3)}$ has the same Lebesgue measure, equal to $\frac{1}{8}$, and the same holds for each $i_1^{(3)}$.

It is easy to prove that, for each $n \in \mathbb{N}$ and for each $l = 1,\, 2,\, \ldots,\, k_n$, $k_n = h_n = 2^{\frac{n(n-1)}{2}}$, $\lambda(l,\, n) = n$ and the intervals $I_l^{(n)}$ and $i_l^{(n)}$ have the same Lebesgue measure.

In D_n we can order the intervals $I_l^{(n)}$ according to the lexicographical order in $[0,\, 1]^\omega$ and, at the same time, the intervals $i_l^{(n)}$ are taken according to the total order in $[0,\, 1]$. With such a choice we have the following useful property, that can be easily proved by using an induction. Let $x_l^{(k)}$ be the point in $[0,\, 1]^\omega$ whose first k coordinates are the left endpoints of the intervals $b_{l,\, j}^{(k)}$ and the others are equal to 0, and let m be a non negative integer. Then it holds that

$$x_l^{(k)} = x^{(k+m)}_{2^{mk+\binom{m}{2}}l+1}, \forall m \in \mathbb{N}$$

More in general the two sequences of dissections satisfy the following property. Let $\{I_n\}$ be a sequence of subintervals of $[0,\, 1]^\omega$ such that $I_n \in D_n$ and $I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$. Then the corresponding intervals i_n are such that $i_1 \supset i_2 \supset \cdots \supset i_n \supset \cdots$ and the cardinality of their intersections are

$$\left| \bigcap_n i_n \right| = \left| \bigcap_n I_n \right| \leq 1.$$

Nevertheless, for every point $x \in [0,\, 1]^\omega$, there is exactly one sequence of nested intervals $\{I_n\}$, $I_n \in D_n$, whose intersection is $\{x\}$. Analogously, every point in

[0, 1] is the intersection of a unique sequence of nested intervals, $\{i_n\}$, $i_n \in d_n$. Notice that to the point $x_l^{(k)}$ correspond the sequence $\{I_n\}$ such that, for each $m \in \mathbb{N}$, $I_{k+m} = I^{(k+m)}_{2^{mk+\binom{m}{2}}l+1}$.

Now, let $\{I_n\}$, $I_n \in D_n$, be the descending chain of intervals corresponding to the point $x \in [0,\,1]^\omega$. Let $\{i_n\}$, $i_n \in d_n$ be the corresponding chain of descending intervals contained in [0, 1]. The previous property ensures the existence of a point $y_x \in [0,\,1]$ such that $y_x \in i_n$, for all $n \in \mathbb{N}$. We then have the following theorem which slightly modify the transferring principle introduced by Jessen (see [24] for more details).

Theorem 2.1 (Transferring Principle)**.** *There exists a one-to-one function*

$$f : [0,\,1]^\omega \to [0,\,1],$$

with the property that corresponding sets have always the same Lebesgue measure. Moreover, trivially, $f^{-1}(i_l^{(n)}) = I_l^{(n)}$.

At variance with Jessen's construction we have used half-open intervals and added more constraints in the definition of the dissections. That ensured the bijectivity of the function $f : [0,\,1]^\omega \to [0,\,1]$ that in Jessen's version is bijectvie on a subset of $[0,\,1]^\omega$.

2.2. The Sharp Version

Now, we can use the transferring principle in order to define the spectral measure (sharp version) E^F associated to a commutative POVM F. Here we diverge from Ref. [15] since we do not limit ourselves to introduce, for every n, a map between D_n and d_n but we use the transferring principle in its full generality and meaning, i.e., we show that there is a function $f : [0,1]^\omega \to [0,1]$ which epitomizes the link between E^F and F. Moreover, we extend the results in [15] to the case of a Hausdorff, locally compact, second countable space. Let X be a second countable space and $\mathcal{S}$ a countable basis for the topology of X (we suppose $\emptyset, X \in \mathcal{S}$). Let $\mathcal{R}(\mathcal{S})$ be the ring generated by $\mathcal{S}$ and $\mathcal{B}(X)$ the Borel σ-algebra generated by $\mathcal{R}(\mathcal{S})$.

Let $\{\Delta_n\}_{n\in\mathbb{N}}$ be an enumeration of $\mathcal{R}(\mathcal{S})$. Let F be a commutative POVM on $\mathcal{B}(X)$ and E_n the spectral resolution of $F(\Delta_n)$. Let $f_j^{-1}(x)$ be the j-th coordinate of $f^{-1}(x)$, $x\in[0,1]$. In particular $f_j^{-1}(i_l^{(n)})=b_{l,\,j}^{(n)}$ where $b_{l,\,j}^{(n)}$ is the j-th edge of the subinterval $I_l^{(n)}=b_{l,\,1}^{(n)}\times b_{l\,2}^{(n)}\times\cdots\times b_{l,\,n}^{(n)}\times[0,1]^\omega$.

We can define a projection valued map on the set of subintervals of [0, 1] as follows. Note that $\mathcal{I}:=\cup_{n=1}^\infty d_n$ is a semiring. Then, thanks to the commutativity of the POVM F,

$$
\begin{aligned}
&E_0^F:\mathcal{I}\to\mathcal{L}_s^+(\mathcal{H})\\
&E_0^F(i_l^{(n)}):=\prod_{j=1}^n E_j[f_j^{-1}(i_l^{(n)})]=\prod_{j=1}^n E_j(b_{l,\,j}^{(n)})
\end{aligned}
\tag{6}
$$

defines a projection valued map. It is straightforward to show that $E_0^F(i_l^{(n)})E_0^F(i_j^{(n)})=\mathbf{0}$ if $l\neq j$ and that $E_0^F([0,\,1])=\mathbf{1}$ (concerning this last property, note that $f_j^{-1}([0,\,1])=[0,\,1]$).

Moreover, E_0^F is additive on $\mathcal{I}$. Indeed, assume $i_l^{(n)}=\bigsqcup_{j=1}^m i_{r_j}^{(k_j)}$, with $k_j>n$, for all j. This case can be reduced to one in which the intervals are all in the same dissection as follows. Let $k=max\{k_j\}$. Then, each set $i_{r_j}^{(k_j)}$ can be decomposed as the disjoint union of sets $i_p^{(k)}$ from d_k and the additivity of E_0^F on $i_p^{(k)}$ implies the additivity on the family of sets $\{i_{r_j}^{(k_j)}\}_j$. Finally, by using induction on $k-n$ one can prove that the additivity in the case $i_l^{(n)}=\bigsqcup_{j=1} i_j^{(k)}$ is equivalent to the additivity in the case $k-n=1$. Let us prove the additivity in this last case. Let $I_l^{(n)}=b_{l,\,1}^{(n)}\times b_{l,\,2}^{(n)}\times\cdots\times b_{l,\,n}^{(n)}\times[0,\,1]^\omega$. Then

$$
i_l^{(n)}=\bigsqcup_{s=2^n(l-1)+1}^{l\,2^n} i_s^{(n+1)}
$$

For each s, we write, as usual,

$$
I_s^{(n+1)}=b_{s,\,1}^{(n+1)}\times b_{s,\,2}^{(n+1)}\times\cdots\times b_{s,\,n+1}^{(n+1)}\times[0,\,1]^\omega
$$

Moreover we notice that

$$
b_{l,\,j}^{(n)}=b_{2^n(l-1)+1,\,j}^{(n+1)}\sqcup b_{2^n\,l,\,j}^{(n+1)}=b_{2^n(l-1)+1,\,j}^{(n+1)}\sqcup b_{2^n(l-1)+2^{n-j+1},\,j}^{(n+1)},\ \forall j.
$$

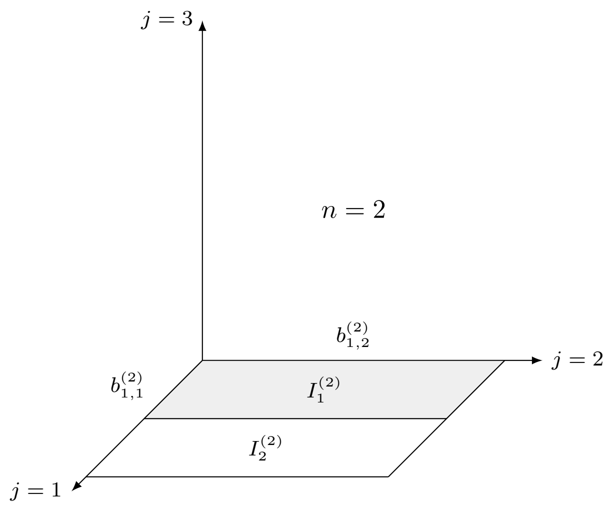
Thus, by definition of E_0^F,

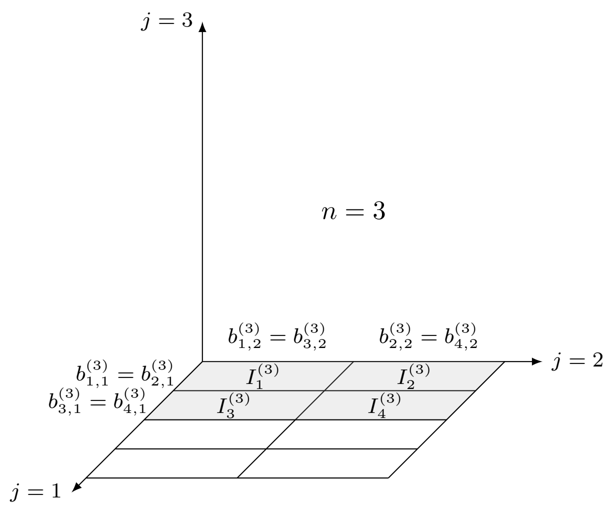
$$E_0^F(i_l^{(n)}) = \prod_{j=1}^{n} E_j(b_{l,\,j}^{(n)})$$

and

$$E_0^F(i_s^{(n+1)}) = \prod_{k=1}^{n+1} E_k(b_{s,\,k}^{(n+1)})$$

For example, in the case $n = 2$ we have,





$$I_1^{(2)} = I_1^{(3)} + I_2^{(3)} + I_3^{(3)} + I_4^{(3)}$$

and

$$b^{(2)}_{1,1} = b^{(3)}_{1,1} + b^{(3)}_{4,1}, \;\; b^{(2)}_{1,2} = b^{(3)}_{1,2} + b^{(3)}_{2,2} = b^{(3)}_{3,2} + b^{(3)}_{4,2}$$
$$b^{(3)}_{1,1} = b^{(3)}_{2,1}, \;\; b^{(3)}_{3,1} = b^{(3)}_{4,1}, \;\; b^{(3)}_{3,2} = b^{(3)}_{1,2}$$

so that,

$$\begin{aligned}
&E_1(b^{(3)}_{1,1})E_2(b^{(3)}_{1,2}) + E_1(b^{(3)}_{2,1})E_2(b^{(3)}_{2,2}) + E_1(b^{(3)}_{3,1})E_2(b^{(3)}_{3,2}) + E_1(b^{(3)}_{4,1})E_2(b^{(3)}_{4,2}) \\
&= E_1(b^{(3)}_{1,1})[E_2(b^{(3)}_{1,2}) + E_2(b^{(3)}_{2,2})] + E_1(b^{(3)}_{4,1})[E_2(b^{(3)}_{3,2}) + E_2(b^{(3)}_{4,2})] \\
&= E_1(b^{(3)}_{1,1})E_2(b^{(2)}_{1,2}) + E_1(b^{(3)}_{4,1})E_2(b^{(2)}_{1,2}) \\
&= [E_1(b^{(3)}_{1,1}) + E_1(b^{(3)}_{4,1})]E_2(b^{(2)}_{1,2}) = E_1(b^{(2)}_{1,1})E_2(b^{(2)}_{1,2}).
\end{aligned}$$

In the general case,

$$\begin{aligned}
\sum_{s=2^n(l-1)+1}^{2^n l} E^F_0(i^{(n+1)}_s) &= \sum_{s=2^n(l-1)+1}^{2^n l} \prod_{k=1}^{n+1} E_k(b^{(n+1)}_{s,k}) \\
&= \sum_{s=2^n(l-1)+1}^{2^n l} \prod_{k=1}^{n} E_k(b^{(n+1)}_{s,k}) \\
&= \Big[E_1(b^{(n+1)}_{2^n(l-1)+1,1}) + E_1(b^{(n+1)}_{2^n l,1})\Big] \sum_{s=2^n(l-1)+1}^{2^n(l-1)+2^{n-1}} \prod_{k=2}^{n} E_k(b^{(n+1)}_{s,k}) \\
&= E_1(b^{(n)}_{l,1}) \sum_{s=2^n(l-1)+1}^{2^n(l-1)+2^{n-1}} \prod_{k=2}^{n} E_k(b^{(n+1)}_{s,k}) \\
&= E_1(b^{(n)}_{l,1}) \Big[E_2(b_{2^n(l-1)+1,2}) + E_2(b_{2^n(l-1)+2^{n-1},2})\Big] \sum_{s=2^n(l-1)+1}^{2^n(l-1)+2^{n-2}} \prod_{k=3}^{n} E_k(b^{(n+1)}_{s,k}) \\
&= E_1(b^{(n)}_{l,1})E_2(b^{(n)}_{l,2}) \sum_{s=2^n(l-1)+1}^{2^n(l-1)+2^{n-2}} \prod_{k=3}^{n} E_k(b^{(n+1)}_{s,k}) \\
&= \cdots \\
&= E_1(b^{(n)}_{l,1})E_2(b^{(n)}_{l,2}) \cdots E_{n-1}(b^{(n)}_{l,n-1}) \sum_{s=2^n(l-1)+1}^{2^n(s-1)+1} \prod_{k=n}^{n} E_k(b^{(n+1)}_{s,k}) \\
&= E_1(b^{(n)}_{l,1})E_2(b^{(n)}_{l,2}) \cdots E_n(b^{(n)}_{l,n}) = E^F_0(i^{(n)}_l)
\end{aligned} \tag{7}$$

Now, we can define a spectral measure as follows. Let $\lambda \in [0,1]$. For every $n \in \mathbb{N}$, let $C_\lambda^n := \{l \,|\, i_l^{(n)} \subset [0,\lambda)\}$ and define

$$\widetilde{E}_0^F(\lambda) := \lim_{n\to\infty} \sum_{l \in C_\lambda^n} E_0^F(i_l^{(n)}). \tag{8}$$

where the limit is in the strong operator topology and coincides with the least upper bound of the non-decreasing family of projection operators $\sum_{l\in C_\lambda^n} E_0^F(i_l^{(n)})$. Note that, for every $\lambda \in [0,1]$, $\widetilde{E}_0^F(\lambda)$ is a projection operator. Moreover, $\widetilde{E}_0^F(0) = \mathbf{0}$, $\widetilde{E}_0^F(1) = \mathbf{1}$ and $\widetilde{E}_0^F(\lambda_1) \leq \widetilde{E}_0^F(\lambda_2)$ whenever $\lambda_1 < \lambda_2$. We also define $\widetilde{E}_0^F(\lambda) = \mathbf{0}$, $\lambda < 0$ and $\widetilde{E}_0^F(\lambda) = \mathbf{1}$, $\lambda > 1$. Note also that, by the additivity of E_0^F, $\widetilde{E}_0^F(\lambda) = \sum_{l\in C_\lambda^n} E_0^F(i_l^{(n)})$ if λ is the right extreme of an interval in d_n. Now, let λ be the right extreme of an interval in d_j and, for every $n > j$, let β_n be the right extreme of an interval in d_n such that there is an index l_n for which $[0,\beta_n) \cup i_{l_n}^{(n)} = [0,\lambda)$. Then, $\widetilde{E}_0^F(\beta_n) + E_0^F(i_{l_n}^{(n)}) = \widetilde{E}_0^F(\lambda)$. Moreover, $\lim_{n\to\infty} E_0^F(i_{l_n}^{(n)}) = \lim_{n\to\infty} \prod_{k=1}^n E_k[f_k^{-1}(i_{l_n}^{(n)})] = \mathbf{0}$ since $i_{l_n}^{(n)} \downarrow \emptyset$ and, by construction, $f_j^{-1}(i_{l_n}^{(n)}) \downarrow \emptyset$ for every $j \in \mathbb{N}$. Therefore,

$$\widetilde{E}_0^F(\lambda) = \lim_{n\to\infty} \widetilde{E}_0^F(\beta_n). \tag{9}$$

Note also that if $i_l^{(r)} = [\lambda, \lambda')$ then,

$$\widetilde{E}_0^F(\lambda') - \widetilde{E}_0^F(\lambda) = E_0^F(i_l^{(r)}). \tag{10}$$

Now, the family of projections $\{E^F_\lambda\}$,

$$E^F_\lambda := \lim_{\beta\to\lambda^-} \widetilde{E}^F_0(\beta), \quad \lambda\in\mathbb{R}, \tag{11}$$

defines a spectral family such that

$$E^F(i^{(r)}_l) = E^F_{\lambda'} - E^F_\lambda = E^F_0(i^{(r)}_l), \qquad i^{(r)}_l = [\lambda,\lambda'). \tag{12}$$

In order to prove (12), for every $n>r$, let β_n be the right extreme of an interval in d_n such that there is an index l_n for which $[0,\beta_n)\cup i^{(n)}_{l_n} = [0,\lambda)$ and γ_n the right extreme of an interval in d_n such that there is an index j_n for which $[0,\gamma_n)\cup i^{(n)}_{j_n} = [0,\lambda')$. Then, by (9) and (10),

$$E^F_{\lambda'} - E^F_\lambda = \lim_{n\to\infty}\widetilde{E}^F_0(\gamma_n) - \lim_{n\to\infty}\widetilde{E}^F_0(\beta_n) = \widetilde{E}^F_0(\lambda') - \widetilde{E}^F_0(\lambda) = E^F_0(i^{(n)}_l).$$

To the spectral family E^F_λ, there corresponds a spectral measure $E^F:\mathcal{B}([0,1])\to\mathcal{L}^+_s(\mathcal{H})$. Indeed, $E^F([\lambda,\lambda') := E^F_{\lambda'} - E^F_\lambda$ is σ-additive on $\mathcal{I}$. Then it can be extended to the ring $\mathcal{R}(\mathcal{I})$ and then to $\mathcal{B}([0,1])$ (see theorem 7 in Ref. [11]).
By (6), (8) and (11), for every $B\in\mathcal{R}(\mathcal{I})$, $E^F(B)\in\mathcal{A}^W(F)$ where $\mathcal{A}^W(F)$ denotes the von Neumann algebra generated by $\{F(\Delta)\}_{\Delta\in\mathcal{B}(X)}$. Actually we can prove the following lemma.

Lemma 2.1. $\mathcal{A}^W(E^F)\subset\mathcal{A}^W(F)$.

Proof. We have, $\mathcal{A}^W\big(\{E^F(\Delta)\}_{\Delta\in\mathcal{R}(\mathcal{I})}\big)\subset\mathcal{A}^W(F)$. Moreover, $\mathcal{I}$ is a basis for the topology of $[0,1]$ and by proposition 3.1 in the appendix, $\mathcal{A}^W\big(\{E^F(\Delta)\}_{\mathcal{R}(\mathcal{I})}\big) = \mathcal{A}^W(E^F)$. □

2.3. The kernel

In order to define the universal Markov kernel we first define a family of functions $\omega^U_{\Delta_j}$, $\Delta_j\in\mathcal{R}(\mathcal{S})$, whose integral with respect to E^F gives $F(\Delta_j)$. For each $j\in\{1,2,\ldots,n\}$, let $\nu_j(i^{(n)}_l)) = \sup f^{-1}_j(i^{(n)}_l)$, and $\chi_{i^{(n)}_l}(\lambda)$ the characteristic function of the interval $i^{(n)}_l$. The sequence of non-increasing functions

$$\omega^{(n)}_{\Delta_j}(\lambda) := \sum_{l=1}^{2^{\binom{n}{2}}} \nu_j(i^{(n)}_l)\, \chi_{i^{(n)}_l}(\lambda) \geq 0$$

converges uniformly to a function $\omega^U_{\Delta_j}(\lambda) := \lim_{n\to\infty} \omega^{(n)}_{\Delta_j}(\lambda)$. Let $B^n_{k,j} := \{l \,|\, \nu_j(i^{(n)}_l) = \frac{k}{2^{n-j}}\}$. We have,

$$\begin{aligned}
\int_0^1 \omega^U_{\Delta_j}(\lambda)\, dE^F_\lambda &= \lim_{n\to\infty} \int_0^1 \omega^{(n)}_{\Delta_j}(\lambda)\, dE^F_\lambda = \lim_{n\to\infty} \sum_{j=1}^{2^{\binom{n}{2}}} \nu_j((i^{(n)}_j) E^F(i^{(n)}_l) = \\
&= \lim_{n\to\infty} \sum_{k=1}^{2^{n-j}} \frac{k}{2^{n-j}} \sum_{l\in B^n_{k,j}} \prod_r E_r(f_r^{-1}(i^{(n)}_l)) = \\
&= \lim_{n\to\infty} \sum_{k=1}^{2^{n-j}} \frac{k}{2^{n-j}}\, E_j\left(\left[\frac{k-1}{2^{n-j}}, \frac{k}{2^{n-j}}\right)\right) \sum_{l\in B^n_{k,j}} \prod_{r\neq j} E_r(f_r^{-1}(i^{(n)}_l)) \\
&= \lim_{n\to\infty} \sum_{k=1}^{2^{n-j}} \frac{k}{2^{n-j}}\, E_j\left(\left[\frac{k-1}{2^{n-j}}, \frac{k}{2^{n-j}}\right)\right) \\
&= \int_0^1 \lambda\, E_j(d\lambda) = F(\Delta_j) \qquad (13)
\end{aligned}$$

where we have used equation (12) and the identity $\sum_{j\in B^n_{k,j}} \prod_{r\neq j} E_r(f_r^{-1}(i^{(n)}_l)) = \mathbf{1}$ which can be derived by noting that the set $B^n_{k,j}$ includes all the indexes l such that the j-th edge of $f^{-1}(i^{(n)}_l)$ is fixed to be $[\frac{k-1}{2^{n-j}}, \frac{k}{2^{n-j}})$. Then, the other edges of $\{f^{-1}(i^{(n)}_l)\}_{l\in B^n_{k,j}}$ are arbitrary. Hence,

$$\left\{ \bigtimes_{i\neq j} f_i^{-1}(i^{(n)}_l) \right\}_{l\in B^n_{k,j}} = \left\{ \bigtimes_{i\neq j} \left[\frac{k_i-1}{2^{n-i}}, \frac{k_i}{2^{n-i}}, \right)\right\}_{k_i=1,\dots,2^{n-i}}.$$

and (supposing $j\neq n-1$)

$$\begin{aligned}
\left\{ \bigtimes_{i\neq j} \left[\frac{k_i-1}{2^{n-i}}, \frac{k_i}{2^{n-i}}\right)\right\}_{k_i=1,\dots,2^{n-i}} &= \left\{ \bigtimes_{i\neq j,n-1} \left[\frac{k_i-1}{2^{n-i}}, \frac{k_i}{2^{n-i}}\right)\right\}_{k_i=1,\dots,2^{n-i}} \bigtimes \left[0, \frac{1}{2}\right) \\
&\bigcup \left\{ \bigtimes_{i\neq j,n-1} \left[\frac{k_i-1}{2^{n-i}}, \frac{k_i}{2^{n-i}}\right)\right\}_{k_i=1,\dots,2^{n-i}} \bigtimes \left[\frac{1}{2}, 1\right] \\
&= \left\{ \bigtimes_{i\neq j,n-1} \left[\frac{k_i-1}{2^{n-i}}, \frac{k_i}{2^{n-i}}\right)\right\}_{k_i=1,\dots,2^{n-i}} \bigtimes [0,\, 1]
\end{aligned}$$

so that

$$\sum_{l\in B^n_{k,j}} \prod_{r\neq j} E_r(f_r^{-1}(i_l^{(n)})) = \sum_{l\in B^n_{k,j}} \prod_{r\neq j,n-1} E_r(f_r^{-1}(i_l^{(n)})).$$

In the case $j = n-1$,

$$\begin{aligned}
\left\{ \bigtimes_{i\neq j} \left[\frac{k_i-1}{2^{n-i}}, \frac{k_i}{2^{n-i}}\right)\right\}_{k_i=1,\dots,2^{n-i}} &= \left\{ \bigtimes_{i\neq j,n-2} \left[\frac{k_i-1}{2^{n-i}}, \frac{k_i}{2^{n-i}}\right)\right\}_{k_i=1,\dots,2^{n-i}} \times \left[0, \frac{1}{4}\right) \\
&\bigcup \left\{ \bigtimes_{i\neq j,n-2} \left[\frac{k_i-1}{2^{n-i}}, \frac{k_i}{2^{n-i}}\right)\right\}_{k_i=1,\dots,2^{n-i}} \times \left[\frac{1}{4}, \frac{1}{2}\right) \\
&\bigcup \left\{ \bigtimes_{i\neq j,n-2} \left[\frac{k_i-1}{2^{n-i}}, \frac{k_i}{2^{n-i}}\right)\right\}_{k_i=1,\dots,2^{n-i}} \times \left[\frac{1}{2}, \frac{3}{4}\right) \\
&\bigcup \left\{ \bigtimes_{i\neq j,n-2} \left[\frac{k_i-1}{2^{n-i}}, \frac{k_i}{2^{n-i}}\right)\right\}_{k_i=1,\dots,2^{n-i}} \times \left[\frac{3}{4}, 1\right] \\
&= \left\{ \bigtimes_{i\neq j,n-2} \left[\frac{k_i-1}{2^{n-i}}, \frac{k_i}{2^{n-i}}\right)\right\}_{k_i=1,\dots,2^{n-i}} \times [0,1]
\end{aligned}$$

so that

$$\sum_{l\in B^n_{k,j}} \prod_{r\neq j} E_r(f_r^{-1}(i_l^{(n)})) = \sum_{l\in B^n_{k,j}} \prod_{r\neq j,n-2} E_r(f_r^{-1}(i_l^{(n)})).$$

By iterating the procedure, one arrives at $\sum_{l\in B^n_{k,j}} \prod_{r\neq j} E_r(f_r^{-1}(i_l^{(n)})) = \mathbf{1}$.
It is worth remarking that the functions $\omega^U_{\Delta_j}$ do not depend on the POVM F. That is analogous to what has been observed for the construction in Ref. [15] and is at the root of the proof of the existence of a universal Markov kernel (see below).

In the following, we use the symbol $\mathcal{D}(X)$ to denote the set of POVMs from the Borel σ-algebra $\mathcal{B}(X)$ to $\mathcal{L}^+_s(\mathcal{H})$. We then have a map $\omega^U : [0,1]\times\mathcal{R}(\mathcal{S}) \to [0,1]$ such that, for every $F\in\mathcal{D}(X)$, $F(\Delta) = \int \omega^U_\Delta(\lambda)\, dE^F_\lambda$, $\Delta\in\mathcal{R}(\mathcal{S})$.

Lemma 2.2. $\mathcal{A}^W(E^F) = \mathcal{A}^W(F)$.

Proof. By lemma 2.1, $\mathcal{A}^W(E^F)\subset\mathcal{A}^W(F)$. By equation (13) $\mathcal{A}^W(\{F(\Delta)\}_{\Delta\in\mathcal{R}(\mathcal{S})})\subset \mathcal{A}^W(E^F)$. By proposition 3.1 in the appendix $\mathcal{A}^W(\{F(\Delta)\}_{\Delta\in\mathcal{R}(\mathcal{S})}) = \mathcal{A}^W(F)$. $\square$

We have proved the following proposition.

Proposition 2.2. *There is a map $\omega^U : [0,1] \times \mathcal{R}(\mathcal{S}) \to [0,1]$ with the following property. For every $F \in \mathcal{D}(X)$, there is a spectral measure E^F with spectrum in $[0,1]$ which generates $\mathcal{A}^W(F)$ and is such that*

$$F(\Delta) = \int \omega^U_\Delta(\lambda)\, dE^F_\lambda, \quad \Delta \in \mathcal{R}(\mathcal{S}).$$

2.4. The Universal Markov Kernel

We are now ready to prove the existence of the universal Markov kernel. That includes the proof of the σ-additivity of the set function $\omega^U_{(\cdot)}(\lambda)$ in 2.3. Note that in [15] the problem of the σ-additivity of $\omega_{(\cdot)}(\lambda) : \mathcal{B}(\mathbb{R}) \to [0,1]$ (which correspond to the function $\omega^U_{(\cdot)}(\lambda) : \mathcal{B}(X) \to [0,1]$ of the present paper in the particular case $X = \mathbb{R}$) is not taken into consideration and only the measurability of $\omega_\Delta(\cdot)$, for every $\Delta \in \mathcal{B}(\mathbb{R})$, is proved. Furtheremore, in Ref. [7] only the finite additivity of $\omega_{(\cdot)}(\lambda)$ is proved; the algorithmic definition of σ-additive set functions $\omega_{(\cdot)}(\lambda)$ is left as an open problem (see section III, part C in [7]). The algorithmic construction we provide in the present paper puts an end to this question. Actually we prove a quite stronger result since we show that $\omega^U_{(\cdot)}(\lambda)$ is σ-additive for every λ in a set of measure one with respect to E^F for every POVM F. That is crucial in order to show the existence of a universal Markov kernel. All of that requires a new self-contained and more general algorithmic proof of the existence of the functions ω^U_Δ. Concerning this last point, note that the proof of the existence of the universal Markov kernel in the real case, $X = \mathbb{R}$, has been given in [6] and is based both on the additivity of $\omega_\Delta(\lambda)$ (proved in [7]) and on corollary 1 in [8] which uses the ordered structure of $\mathbb{R}$. While an extension of the proof of the additivity to the general case seems to be straightforward, it seems to us that the extension of corollary 1 is not so easy. We circumvented the problem by proving that there is a subset $I \subset [0,1]$, $E^F(I) = \mathbf{1}$, such that, for every $\lambda \in I$, $\omega^U_{(\cdot)}(\lambda)$ is σ-additive on a ring $\mathcal{R}(\mathcal{S})$ which generates $\mathcal{B}(X)$, the Borel σ-algebra of X. Then, transfinite induction is used in order to prove that $\omega^U_{(\cdot)}(\lambda)$ can be extended to $\mathcal{B}(X)$.

Theorem 2.3. *There are a subset $I \subset [0,1]$ and a Markov kernel $\mu^U : \mathcal{B}(X) \times I \to [0,1]$ such that, for every $F \in \mathcal{D}(X)$, $E^F(I) = \mathbf{1}$ and*

$$F(\Delta) = \int \mu^U_\Delta(\lambda)\, dE^F_\lambda$$

where E^F is the spectral measure whose existence has been proved in subsection 2.2

Proof. Let $F \in \mathcal{D}(X)$. By theorem 1.1 there are a self-adjoint operator B with spectrum in $[0,1]$ and a Markov kernel μ such that B generates $\mathcal{A}^W(F)$ and $\int \mu_\Delta(\lambda)\, dE^B_\lambda = F(\Delta)$ for every $\Delta \in \mathcal{B}(X)$. Let $\nu(\cdot) = \langle \psi_0, E^B(\cdot)\psi_0 \rangle$ where E^B is the spectral measure corresponding to B and ψ_0 is a separating vector for $\mathcal{A}^W(F)$.
By lemma 2.2 there is a spectral measure E^F which generates $\mathcal{A}^W(F)$. Let A^F be the corresponding self-adjoint operator and $\sigma(A^F) \subset [0,1]$ its spectrum. We have

$$\int_{\sigma(A^F)} \omega^U_{\Delta_i}(\lambda)\, dE^F_\lambda = F(\Delta_i), \quad i \in \mathbb{N}.$$

Since B and A^F generate $\mathcal{A}^W(F)$, there are two Borel functions $f_F : \sigma(B) \to \sigma(A^F)$ and $h_F : \sigma(A^F) \to \sigma(B)$ such that $E^F(G) = E^B(f_F^{-1}(G))$, $G \in \mathcal{B}([0,1])$ and $E^B(D) = E^F(h_F^{-1}(D))$, $D \in \mathcal{B}([0,1])$. Then, there is a set N such that $E^B(N) = \mathbf{1}$ and $(h_F \circ f_F)(\lambda) = \lambda$ for every $\lambda \in N$. In other words $f_F : N \to f_F(N)$ is injective. By Souslin's theorem ([26], page 440-442) $f_F(N)$ is a Borel set and $E^F(f_F(N)) = E^B[f_F^{-1}(f_F(N))] = \mathbf{1}$.

Then, by the change of measure principle,

$$\int_{\sigma(B)} \omega^U_{\Delta_i}(f_F(\lambda))\, dE^B_\lambda = \int_{\sigma(A^F)} \omega^U_{\Delta_i}(\lambda)\, dE^F_\lambda = F(\Delta_i) = \int_{\sigma(B)} \mu_{\Delta_i}(\lambda)\, dE^B_\lambda$$

Therefore, $\omega^U_{\Delta_i}(f_F(\lambda)) = \mu_{\Delta_i}(\lambda)$, $E^B - a.e.$. Since, E^B and ν are mutually absolutely continuous,

$$\omega^U_{\Delta_i}(f_F(\lambda)) = \mu_{\Delta_i}(\lambda), \quad \nu - a.e. \tag{14}$$

Let $M^F \subset \sigma(B)$, $\nu(M^F) = 1$ be such that

$$\omega^U_{\Delta_i}(f_F(\lambda)) = \mu_{\Delta_i}(\lambda), \quad \lambda \in M^F, i \in \mathbb{N}. \tag{15}$$

Thus, thanks to the σ-additivity of μ, $(\omega^U_{(\cdot)} \circ f_F)(\lambda))$ is σ-additive on $\mathcal{R}(\mathcal{S})$ for every $\lambda \in N^F := M^F \cap N$. As a consequence, $\omega^U_{(\cdot)}(\lambda)$ is σ-additive on $\mathcal{R}(\mathcal{S})$ for every $\lambda \in f_F(N^F)$. Note that $E^F[f_F(N^F)] = E^B[f_F^{-1}(f_F(N^F))] = \mathbf{1}$.

By repeating the reasoning for every $F \in \mathcal{D}(X)$, one proves that the set $I := \cup_{F\in\mathcal{D}(X)} f_F(N^F)$ is such that $\omega^U_{(\cdot)}(\lambda)$ is σ-additive on $\mathcal{R}(\mathcal{S})$ for every $\lambda \in I$.

In what follows we need the set I to be measurable. Thus, if I is not a Borel set we enlarge the Borel σ-algebra in order to include I. In particular, we consider the σ-algebra $\mathfrak{S}$ generated by I and $\mathcal{B}([0,1])$.

Since, $\forall F \in \mathcal{D}(X)$, $[0,1] \setminus I \subset [0,1] \setminus f_F(N^F)$ and $E^F\big([0,1] \setminus f_F(N^F)\big) = \mathbf{0}$, the set $[0,1] \setminus I$ is a subset of a E^F-null set for any $F \in \mathcal{D}(X)$. Then, each PVM E^F can be extended to $\mathfrak{S}$. The extension $\widetilde{E}^F : \mathfrak{S} \to \mathcal{L}_s(\mathcal{H})$ satisfies the following relations:

$$\widetilde{E}^F(\sigma(A^F)) = \widetilde{E}^F(I) = \mathbf{1}$$
$$\widetilde{E}^F(I \cap \Delta) = E^F(\Delta), \quad \forall \Delta \in \mathcal{B}[0,1]$$
$$A^F = \int_{[0,1]} \lambda \, d\widetilde{E}^F_\lambda.$$

The space $([0,1], \mathfrak{S})$ is a measurable space and I is a measurable subset of $\mathfrak{S}$. Moreover, for each $\Delta \in \mathcal{R}(\mathcal{S})$, the function $\omega^U_\Delta : ([0,1], \mathfrak{S}) \to ([0,1], \mathcal{B}[0,1])$ is $\mathfrak{S}$-measurable and

$$\int_{[0,1]} \omega^U_\Delta(\lambda) \, d\widetilde{E}^F_\lambda = \int_{[0,1]} \omega^U_\Delta(\lambda) \, dE^F_\lambda = F(\Delta), \quad \forall F \in \mathcal{D}(X). \tag{16}$$

Now, for every $\lambda \in I$, the measure $\omega^U_{(\cdot)}(\lambda) : \mathcal{R}(\mathcal{S}) \to [0,1]$ can be extended to the Borel σ-algebra $\mathcal{B}(X)$. Let $\mu_{(\cdot)}(\lambda) : \mathcal{B}(X) \to [0,1]$ denote such an extension. We want to show that, for each $\Delta \in \mathcal{B}(X)$, μ_Δ is $\mathfrak{S}$-measurable and $\int \mu^U_\Delta(\lambda) \, dE_\lambda = F(\Delta)$. That can be proved by using transfinite induction. Let Δ be an open set. Then, there is an increasing sequence of open sets $\Delta_{k_i} \in \mathcal{S}$

such that $\Delta_{k_i} \uparrow \Delta$. Then, for every λ, $\omega^U_{\Delta_{k_i}}(\lambda) = \mu^U_{\Delta_{k_i}}(\lambda) \uparrow \mu^U_\Delta(\lambda)$ so that μ^U_Δ is $\mathfrak{S}$-measurable. Let Δ be a G_δ set. Then there is a decreasing sequence of open sets G_i such that $G_i \downarrow \Delta$. Moreover, for every λ, $\mu^U_{G_i}(\lambda) \downarrow \mu^U_\Delta(\lambda)$ so that μ^U_Δ is $\mathfrak{S}$-measurable.

Let G_0 be the family of open subsets of X, ω_1 the first uncountable ordinal and G_α, $\alpha < \omega_1$ the Borel hierarchy [26]. In particular, $G_1 = G_\delta$, $G_2 = G_{\delta\sigma}$, $G_3 = G_{\delta\sigma\delta}, \dots$ and $G_\alpha = (\cup_{\beta<\alpha} G_\beta)_\sigma$ for each limit ordinal α. By means of the same reasoning that we used in the case of open and G_δ sets, one can prove the $\mathfrak{S}$-measurability of μ_Δ for every Δ of the kind $G_{\delta,\sigma}, G_{\delta\sigma\delta} \dots$. Analogously, if μ^U_Δ is $\mathfrak{S}$-measurable for each $\Delta \in G_\alpha$ then, μ^U_Δ is $\mathfrak{S}$-measurable for each Δ in $G_{\alpha+1}$. Indeed, each set in $G_{\alpha+1}$ is either the countable union or the countable intersection of sets in G_α and the previous reasoning can be used. If α is a limit ordinal and μ^U_Δ is $\mathfrak{S}$-measurable for each $\Delta \in G_\beta$, $\beta < \alpha$, then, μ^U_Δ is $\mathfrak{S}$-measurable for each $\Delta \in G_\alpha = (\cup_{\beta<\alpha} G_\beta)_\sigma$. Indeed, each set in G_α is the countable union of sets in $\cup_{\beta<\alpha} G_\beta$ and the previous reasoning can be used. Therefore, by transfinite induction, $\mu^U_\Delta(\cdot) : I \to [0,1]$ is $\mathfrak{S}$-measurable for each $\Delta \in \cup_{\alpha<\omega_1} G_\alpha = \mathcal{B}(X)$.

Moreover, since $\mu^U_\Delta = \omega^U_\Delta$, $\Delta \in \mathcal{R}(\mathcal{S})$, the POVM $F'(\Delta) = \mu^U_\Delta(A^F)$, $\Delta \in \mathcal{B}(X)$, coincides with $F(\Delta)$ for every $\Delta \in \mathcal{R}(\mathcal{S})$. Since $F : \mathcal{R}(\mathcal{S}) \to \mathcal{E}(\mathcal{H})$ has a unique extension [11] to the Borel σ-algebra $\mathcal{B}(X)$, it must be $F(\Delta) = F'(\Delta) = \mu^U_\Delta(A^F)$.

□

Definition 10. The Markov kernel $\mu^U : \mathcal{B}(X) \times I \to [0,1]$ whose existence has been proved in theorem 2.3 is called the universal Markov kernel.

2.5. Functional subordination

The following definition establishes a weak functional relationship between weak Markov kernels. In the following Λ and Γ denote compact subsets of $[0,1]$ while ν denotes a probability measure.

Definition 11. A weak Markov kernel $\gamma : (\Lambda, \nu) \times \mathcal{B}(X) \to [0, 1]$ is functionally subordinated to a weak Markov kernel $\mu : \big(\Gamma, \nu\big) \times \mathcal{B}((X) \to [0, 1]$ if there is a measurable function $f : \Lambda \to \Gamma$ such that, $\gamma_\Delta(\lambda) = \mu_\Delta(f(\lambda))$, ν-a.e.

The following theorem has been established in references [4, 6] in the case of real POVMs. Once theorem 2.3 has been proved, the proof can be straightforwardly extended to the case of an arbitrary POVM. For completeness we provide the proof below.

Theorem 2.4. *Every weak Markov kernel* $\gamma : \big(\Lambda, \nu\big) \times \mathcal{B}(X) \to [0, 1]$ *is functionally subordinated to the universal Markov kernel.*

Proof. We proceed as in the proof of theorem 6 in Ref. [4]. Without loss of generality, we can assume Λ to be the support of ν. Let $L_\infty(\Lambda, \nu)$ be the space of essentially bounded measurable functions (with two functions identified if they coincide up to ν-null sets) and $\mathcal{A}_\nu$ the von Neumann algebra of multiplication operators on $\mathcal{H} = L_2(\Lambda, \nu)$ which corresponds to $L_\infty(\Lambda, \nu)$. In particular, for every function $f \in L_\infty(\Lambda, \nu)$ there is a multiplication operator

$$M_f : L_2(\Lambda, \nu) \to L_2(\Lambda, \nu)$$

$$[M_f(h)](x) = f(x)h(x).$$

The generator of $\mathcal{A}_\nu$ is $B := M_x$, $[Bh](x) = [M_x(h)](x) = xh(x)$, $x \in \Lambda$. The spectrum of B, $\sigma(B)$, coincides with the support, Λ, of ν and the spectral measure corresponding to B is $E^B(\Delta) = M_{\chi_\Delta}$. Moreover, ν is a scalar-valued spectral measure for B, i.e., ν and E^B are mutually absolutely continuous.

Now, we can define a commutative POVM,

$$F(\Delta) := \gamma_\Delta(B) = \int_{\sigma(B)} \gamma_\Delta(\lambda)\, dE^B_\lambda = M_{\gamma_\Delta}, \quad \Delta \in \mathcal{B}(X). \tag{17}$$

By lemma 2.2, there is a generator A^F of $\mathcal{A}^W(F)$ with spectral resolution E^F and a Markov kernel μ^U such that

$$\int_{\sigma(A^F)} \mu^U_\Delta(\lambda)\, dE^F_\lambda = F(\Delta), \quad \Delta \in \mathcal{B}(X).$$

Since B generates $\mathcal{A}_\nu \supset \mathcal{A}^W(F)$, there is[2] a Borel function $f:\sigma(B)\to\sigma(A^F)$ such that $E^F(G)=E^B(f^{-1}(G))$, $G\in\mathcal{B}([0,1])$. Then,

$$\int_{\sigma(B)} \mu_\Delta^U(f(\lambda))\,dE_\lambda^B = \int_{\sigma(A^F)} \mu_\Delta^U(\lambda)\,dE_\lambda^F = F(\Delta) = \int_{\sigma(B)} \gamma_\Delta(\lambda)\,dE_\lambda^B$$

Therefore, $\mu_\Delta^U(f(\lambda)) = \gamma_\Delta(\lambda)$, $E^B - a.e.$. Since, E^B and ν are mutually absolutely continuous,

$$\mu_\Delta^U(f(\lambda)) = \gamma_\Delta(\lambda), \quad \nu - a.e.$$

$\square$

3. The universal family of fuzzy events

In section 2, we proved that for every commutative POVM F,

$$F(\Delta) = \int_I \mu_\Delta^U(\lambda)\,dE_\lambda^F = \mu_\Delta^U(A).$$

where E^F is the sharp version of F (corresponding to the self-adjoint operator A) and μ^U is the universal Markov kernel. Therefore, every commutative POVM F corresponds to the same family of fuzzy events $\{(I,\mu_\Delta^U)\}$. Moreover,

$$\langle\psi, F(\Delta)\psi\rangle = \int_I \mu_\Delta^U(\lambda)\,d\langle\psi, E_\lambda^F\psi\rangle, \quad \psi\in\mathcal{H}$$

so that every POVM F gives the probabilities of the fuzzy events (I,μ_Δ^U), $\Delta\in\mathcal{B}(X)$, with respect to the probability measures corresponding to E^F. In other words, given two commutative POVMs F_1 and F_2, the operators $F_1(\Delta)$ and $F_2(\Delta)$ correspond to the **same** fuzzy event (I,μ_Δ^U) and the quantities $\langle\psi, F_1(\Delta)\psi\rangle$ and $\langle\psi, F_2(\Delta)\psi\rangle$ are the probabilities of (I,μ_Δ^U), with respect to the probability measures $\langle\psi, E^{F_1}(\cdot)\psi\rangle$ and $\langle\psi, E^{F_2}\psi\rangle$ respectively.

[2] By lemma 1 in Ref. [36], the von Neumann algebra $\mathcal{A}_\nu$ coincides with the set of Borel functions of B (which is the generator of $\mathcal{A}_\nu$). In particular, there is a function f such that $A^F = f(B)$ and then $E^F(G) = E^B(f^{-1}(G))$.

If we now recall that every Markov kernel γ is functionally subordinated to the universal Markov kernel, $\gamma_\Delta = \mu^U_\Delta \circ f$, we see that, in a certain sense, $\{(I, \mu^U_\Delta)\}_{\Delta\in\mathcal{B}(X)}$ is a universal family of fuzzy events since it includes (up to functional subordination) all the possible fuzzy events of which a POVM can represent the probabilities. Indeed, suppose

$$F(\Delta) = \int_{\mathbb{R}} \gamma_\Delta(\lambda)\, dE_\lambda.$$

Then, $F(\Delta)$ corresponds to the fuzzy event $\{(\mathbb{R}, \gamma_\Delta)\}$. By functional subordination, there is a function f such that $\gamma_\Delta = \mu^U_\Delta \circ f$. Therefore, $\{(\mathbb{R}, \gamma_\Delta)\}_{\Delta\in\mathcal{B}(X)}$ coincides with $\{(\mathbb{R}, \mu^U_\Delta \circ f)\}_{\Delta\in\mathcal{B}(X)}$ and the probability of the fuzzy event $(\mathbb{R}, \gamma_\Delta)$ with respect to E, coincides with the probability of the fuzzy event $(\mathbb{R}, \mu^U_\Delta \circ f)$ with respect to E and with the probability of the fuzzy event (I, μ^U_Δ) with respect to E^F,

$$\langle\psi, F(\Delta)\psi\rangle = \int_{\mathbb{R}} \mu^U_\Delta(f(\lambda))\, d\langle\psi, E_\lambda\psi\rangle = \int_I \mu^U_\Delta(\lambda)\, d\langle\psi, E^F_\lambda\psi\rangle.$$

Appendix

Proposition 3.1. *Let X be second countable. Let $\mathcal{S}$ be a basis for the topology of X. Let $\mathcal{R}(\mathcal{S})$ be the ring generated by $\mathcal{S}$. Let $F : \mathcal{B}(X) \to \mathcal{L}^+_s(\mathcal{H})$ be a POVM. Then, the von Neumann algebra $\mathcal{A}^W(\mathcal{R}(\mathcal{S}))$ generated by $\{F(\Delta)\}_{\Delta\in\mathcal{R}(\mathcal{S})}$ coincides with the von Neumann algebra $\mathcal{A}^W(F)$.*

Proof. Let $G \subset X$ be an open set. Then, there is an increasing sequence of sets $\Delta_k \in \mathcal{S}$ such that $\Delta_k \uparrow G$. By the continuity of F, $F(\Delta_k) \uparrow F(G)$. Since $F(\Delta_k) \in \mathcal{A}^W(\mathcal{R}(\mathcal{S}))$ and $\mathcal{A}^W(\mathcal{R}(\mathcal{S}))$ is weakly closed, $F(G) \in \mathcal{A}^W(\mathcal{R}(\mathcal{S}))$. Let $\Delta \subset X$ be a G_δ set. Then there is a decreasing sequence of open sets G_k such that $G_k \downarrow \Delta$ and, by the continuity of F and the weak closure of $\mathcal{A}^W(\mathcal{R}(\mathcal{S}))$, $F(G_k) \downarrow F(\Delta) \in \mathcal{A}^W(\mathcal{R}(\mathcal{S}))$. Let G_0 be the family of open subsets of X, ω_1 the first uncountable ordinal and G_α, $\alpha < \omega_1$ the Borel hierarchy [26]. In particular, $G_1 = G_\delta$, $G_2 = G_{\delta\sigma}$, $G_3 = G_{\delta\sigma\delta}, \ldots$ and $G_\alpha = (\cup_{\beta<\alpha} G_\beta)_\sigma$ for each limit ordinal α. Suppose $F(\Delta) \in \mathcal{A}^W(\mathcal{R}(\mathcal{S}))$ for each $\Delta \in G_\alpha$. Then, $F(\Delta) \in \mathcal{A}^W(\mathcal{R}(\mathcal{S}))$ for each $\Delta \in G_{\alpha+1}$. Indeed, each set in $G_{\alpha+1}$ is either the countable union or the countable intersection of sets in G_α and the previous reasoning can be used. If α is a limit ordinal and $F(\Delta) \in \mathcal{A}^W(\mathcal{R}(\mathcal{S}))$ for each $\Delta \in G_\beta$, $\beta < \alpha$, then $F(\Delta) \in \mathcal{A}^W(\mathcal{R}(\mathcal{S}))$ for each $\Delta \in G_\alpha = (\cup_{\beta<\alpha} G_\beta)_\sigma$. Indeed, each set in G_α is the countable union of sets in $\cup_{\beta<\alpha} G_\beta$ and the previous reasoning can be used. Therefore, by transfinite induction, $F(\Delta) \in \mathcal{A}^W(\mathcal{R}(\mathcal{S}))$ for each $\Delta \in \cup_{\alpha<\omega_1} G_\alpha = \mathcal{B}(X)$ so that $\mathcal{A}^W(\mathcal{R}(\mathcal{S})) = \mathcal{A}^W(F)$.

□

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