A note on monotonicity of $z \mapsto D_{\alpha,z}$ and $\alpha \mapsto D_{\alpha,z}(\psi\|\varphi)$ for $1 < \alpha \leq 2z$

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We will assume trouhgout that $Q_{\alpha,z}(\psi\|\varphi) < \infty$ for some $1 < \alpha \leq 2z$, in which case there is some $y \in L^{2z}(\mathcal{M})$ such that

$$h_\psi^{\frac{\alpha}{2z}} = y h_\varphi^{\frac{\alpha-1}{2z}}, \qquad Q_{\alpha,z}(\psi\|\varphi) = \|y\|_{2z}^{2z}.$$

In particular, $e := s(\psi) \leq s(\varphi)$, so that we may assume that φ is faithful. Let $\sigma \in \mathcal{M}_*^+$ be such that $s(\sigma) = 1 - e$ and put $\psi_0 := \psi + \sigma$, so that ψ_0 is faithful as well. We will use the notation $L_L^p := L^p(\mathcal{M};\varphi)_L$, $1 \leq p \leq \infty$.

Consider the function

$$f(w) = h_{\psi_0}^{\frac{\alpha}{2z}w} e h_\varphi^{1-\frac{\alpha}{2z}w}, \qquad w \in S,$$

where $S := \{w \in \mathbb{C},\ 0 \leq \operatorname{Re} w \leq 1\}$. Then f is a bounded continuous function $S \to L^1(\mathcal{M})$, analytic in the interior. Further,

$$f(it) = h_{\psi_0}^{\frac{\alpha}{2z}it} e h_\varphi^{-\frac{\alpha}{2z}it} h_\varphi \in L_L^\infty, \qquad t \in \mathbb{R},$$

and $\|f(it)\|_{L_L^\infty} = 1$ for all t. We also have

$$f(1+it) = h_{\psi_0}^{\frac{\alpha}{2z}it} h_\psi^{\frac{\alpha}{2z}} h_\varphi^{1-\frac{\alpha}{2z}} h_\varphi^{-\frac{\alpha}{2z}it} = (h_{\psi_0}^{\frac{\alpha}{2z}it} y h_\varphi^{-\frac{\alpha}{2z}it}) h_\varphi^{\frac{2z-1}{2z}} \in L_L^{2z}, \qquad t \in \mathbb{R}.$$

By [1, Lemmas 10.1 and 10.2],

$$\|f(1+it)\|_{L_L^{2z}} = \|h_{\psi_0}^{\frac{\alpha}{2z}it} y h_\varphi^{-\frac{\alpha}{2z}it}\|_{2z} = \|y\|_{2z}$$

and the functions $t \mapsto f(it)$ and $t \mapsto f(1+it)$ are continuous in L_L^{2z}. It follows that $f \in \mathcal{F}'(L_L^\infty, L_L^{2z})$, that is, f is a function $S \to L_L^{2z}$, bounded and continuous on S and analytic in the interior of S, such that the boundary values define bounded functions to L_L^∞ resp. L_L^{2z}, see [1, Definition 1.4].

1 Monotonicity in z

Let $z < z'$, we will prove that $Q_{\alpha,z}(\psi\|\varphi) \geq Q_{\alpha,z'}(\psi\|\varphi)$. By [1, Remark 3.4], the set of functions $\mathcal{F}'(L_L^\infty, L_L^{2z})$ defines the interpolation spaces $C_\theta = C_\theta(L_L^\infty, L_L^{2z})$, so that for any $\theta \in (0,1)$, $f(\theta) \in C_\theta$ and

$$\|f(\theta)\|_{C_\theta} \leq (\max_t \|f(it)\|_{L_L^\infty})^{1-\theta} (\max_t \|f(1+it)\|_{L_L^{2z}})^\theta = \|y\|_{2z}^\theta.$$

By the reiteration theorem, $C_\theta = L_L^{2z/\theta}$. Putting $\theta = z/z'$, we get

$$f(z/z') = h_\psi^{\frac{\alpha}{2z'}} h_\varphi^{1-\frac{\alpha}{2z'}} = y' h_\varphi^{\frac{2z'-1}{2z'}}$$

for some $y' \in L^{2z'}(\mathcal{M})$, and $\|y'\|_{2z'} \le \|y\|_{2z}^{z/z'}$. It follows that $h_\psi^{\frac{\alpha}{2z'}} = y' h_\varphi^{\frac{\alpha-1}{2z'}}$, so that

$$Q_{\alpha,z'}(\psi\|\varphi) = \|y'\|_{2z'}^{2z'} = \|f(z/z')\|_{L_L^{2z'}}^{2z'},$$

this proves the result.

2 Monotonicity in α

The above function allows us also to prove monotonicity in α. Indeed, let $1 < \alpha' < \alpha$. For any $t \in \mathbb{R}$,

$$f(\frac{1}{\alpha} + it) = h_{\psi_0}^{\frac{\alpha}{2z}it} h_\psi^{\frac{1}{2z}} h_\varphi^{-\frac{\alpha}{2z}it} h_\varphi^{\frac{2z-1}{2z}},$$

so that $\|f(\frac{1}{\alpha} + it)\|_{L_L^{2z}} \le \psi(1)^{\frac{1}{2z}}$. Further, since $\frac{\alpha'}{\alpha} < 1$, we get $f(\frac{\alpha'}{\alpha}) \in L_L^{2z}$, so that there is some $y' \in L^{2z}(\mathcal{M})$ such that

$$f(\frac{\alpha'}{\alpha}) = h_\psi^{\frac{\alpha'}{2z}} h_\varphi^{1-\frac{\alpha'}{2z}} = y' h_\varphi^{\frac{2z-1}{2z}}$$

so that $h_\psi^{\frac{\alpha'}{2z}} = y' h_\varphi^{\frac{\alpha'-1}{2z}}$ and $Q_{\alpha',z}(\psi\|\varphi) = \|y'\|_{2z}^{2z} = \|f(\frac{\alpha'}{\alpha})\|_{L_L^{2z}}^{2z}$. Now let λ be such that $(1-\lambda)+\lambda\alpha = \alpha'$, so that $\lambda = \frac{\alpha'-1}{\alpha-1}$, then by the Hadamard three lines theorem, we get

$$Q_{\alpha',z}(\psi\|\varphi) = \|f(\frac{\alpha'}{\alpha})\|_{L_L^{2z}}^{2z} \le \left(\max_t \|f(\frac{1}{\alpha} + it)\|_{L_L^{2z}}^{1-\lambda} \max_t \|f(1+it)\|_{L_L^{2z}}^{\lambda}\right)^{2z} \le \psi(1)^{1-\lambda} Q_{\alpha,z}(\psi\|\varphi)^\lambda,$$

this proves that $D_{\alpha',z}(\psi\|\varphi) \le D_{\alpha,z}(\psi\|\varphi)$.

3 The limit $\alpha \searrow 1$

We now try to prove the limit $\lim_{\alpha\searrow 1} D_{\alpha,z}(\psi\|\varphi)$, using the same ideas as in [**?**] but applying analyticity of the function f instead of the Connes cocycle.

The function f is analytic in a neighborhood of $\frac{1}{\alpha}$. Therefore, we have the expansion

$$f(w) = f(\frac{1}{\alpha}) + (w - \frac{1}{\alpha})h + o(w - \frac{1}{\alpha})$$

where $h \in L_L^{2z}$ is the derivative of f at $w = \frac{1}{\alpha}$ and $\frac{\|o(\omega)\|_{L_L^{2z}}}{|\omega|} \to 0$ as $|\omega| \to 0$. It follows that for $1 < \alpha' < \alpha$,

$$f(\frac{\alpha'}{\alpha}) = f(\frac{1}{\alpha}) + \frac{\alpha'-1}{\alpha}h + o(\frac{\alpha'-1}{\alpha})$$

Using the fact that the L_L^p spaces are uniformly Fréchet differentiable, we can prove similarly as in [**?**] that

$$\lim_{\alpha'\to 1} \frac{\|f(\frac{\alpha'}{\alpha})\|_{L_L^{2z}} - \|f(\frac{1}{\alpha})\|_{L_L^{2z}}}{\frac{\alpha'-1}{\alpha}} = \langle a_0, h\rangle$$

where $\langle \cdot, \cdot \rangle$ is the duality between L_L^{2z} and $L_L^{\frac{2z}{2z-1}}$ and a_0 is the element in $L_L^{\frac{2z}{2z-1}}$ with unit norm such that $\mathrm{Tr}\, a_0 f(\frac{1}{\alpha}) = \|f(\frac{1}{\alpha})\|_{L_L^{2z}}$, that is,

$$a_0 = \left(\frac{h_\psi}{\psi(1)} \right)^{\frac{2z-1}{2z}} h_\varphi^{\frac{1}{2z}}.$$

Since f is uniformly differentiable and h is the derivative of f at $\frac{1}{\alpha}$, we have

$$\begin{aligned}
\langle a_0, h \rangle &= \lim_{t \to 0} (it)^{-1} \langle a_0, f(\frac{1}{\alpha} + it) - f(\frac{1}{\alpha}) \rangle = \lim_{t \to 0} (it)^{-1} \langle a_0, \left(h_\psi^{\frac{1}{2z}} h_{\psi_0}^{\frac{\alpha}{2z} it} h_\varphi^{-\frac{\alpha}{2z} it} - h_\psi^{\frac{1}{2z}} \right) h_\varphi^{\frac{2z-1}{2z}} \rangle \\
&= \psi(1)^{-\frac{2z-1}{2z}} \frac{\alpha}{2z} \lim_{t \to 0} (it)^{-1} \langle h_\psi^{\frac{2z-1}{2z}} h_\varphi^{\frac{1}{2z}}, \left(h_\psi^{\frac{1}{2z}} h_{\psi_0}^{it} h_\varphi^{-it} - h_\psi^{\frac{1}{2z}} \right) h_\varphi^{\frac{2z-1}{2z}} \rangle \\
&= \psi(1)^{-\frac{2z-1}{2z}} \frac{\alpha}{2z} \lim_{t \to 0} (it)^{-1} \mathrm{Tr}\, h_\psi \left(h_{\psi_0}^{it} h_\varphi^{-it} - 1 \right) = \psi(1)^{-\frac{2z-1}{2z}} \frac{\alpha}{2z} D(\psi \| \varphi),
\end{aligned}$$

where we use [**?** , Thm.5.7] in the last equality. We therefore have

$$\begin{aligned}
\lim_{\alpha' \searrow 1} D_{\alpha', z}(\psi \| \varphi) &= \lim_{\alpha' \searrow 1} \frac{\log Q_{\alpha', z}(\psi \| \varphi) - \log \psi(1)}{\alpha' - 1} = \lim_{\alpha' \searrow 1} \frac{2z \log \|f(\frac{\alpha'}{\alpha})\|_{L_L^{2z}} - 2z \log \|f(\frac{1}{\alpha})\|_{L_L^{2z}}}{\alpha' - 1} \\
&= \lim_{\alpha' \to 1} \left(\frac{\log \|f(\frac{\alpha'}{\alpha})\|_{L_L^{2z}} - \log \|f(\frac{1}{\alpha})\|_{L_L^{2z}}}{\|f(\frac{\alpha'}{\alpha})\|_{L_L^{2z}} - \|f(\frac{1}{\alpha})\|_{L_L^{2z}}} \right) \frac{2z}{\alpha} \left(\frac{\|f(\frac{\alpha'}{\alpha})\|_{L_L^{2z}} - \|f(\frac{1}{\alpha}\|_{L_L^{2z}}}{\frac{\alpha'-1}{\alpha}} \right) \\
&= \psi(1)^{-1} D(\psi \| \varphi).
\end{aligned}$$

References

[1] H. Kosaki. Applications of the complex interpolation method to a von Neumann algebra: Non-commutative L_p-spaces. *J. Funct. Anal.*, 56:26–78, 1984. doi:https://doi.org/10.1016/0022-1236(84)90025-9.