Response to Referee A

Referee Point A.1

In this paper, the authors obtain entropic uncertainty relations for a class of measurements consisting of interactions with a quantum process at multiple times and a final measurement. These relations include previously known results for measurements on quantum states and channels. The results are obtained using the framework of quantum combs as well as the techniques of majorization lattice, applied in a previous work concerned with measurements on quantum channels. For specific pairs of measurements, compatible with the common cause resp. direct cause causal structures, it is shown that the obtained bound is tight and the uncertainty trade-off is analogous to the position and momentum measurements.

Reply: We thank Referee A for carefully reading our manuscript, and we agree with Referee A's summary of our work.

Referee Point A.2

The results in this paper are timely and important, since the interactive measurements describe a general class of measurements that can be applied to all quantum processes that have a definite causal structure. The authors also provide a nice and easily understandable operational interpretation of the lower bound, that can be obtained from success probabilities in a guessing game they call a quantum roulette. The main text is written an a concise and accessible style, with illustrative diagrams that are easy to interpret.

Reply: We would like to thank Referee A for his/her thorough reading of our manuscript and believing that the results of our work are "timely and important".

Referee Point A.3

I have only a few minor comments, listed below.

Reply: We thank Referee A again for carefully reading our manuscript and providing us valuable comments. In the following, we address all the comments point-by-point.

Referee Point A.4

Main text:

p. 1, col. 1: "Could such a fundamental..." something seems missing in this sentence.

Reply: In the revised manuscript, we have changed "Could such a fundamental uncertainty principle also exist when multiple preceding interventions" into "Could uncertainty principles also fundamentally constrain such interactive measurements?".

Referee Point A.5

Fig. 1: "the an", "dynamica"

Reply: We have revised Fig. 1 and its caption. Now the problem does not exist anymore.

Referee Point A.6

p. 2. col. 1, line 16 from below: -the state of- H_{2k+1}

Reply: We have revised manuscript. Now the problem does not exist anymore.

Referee Point A.7

p. 2, col. 2, line 6 from below: "$\log(1/c)$" it is not clear what "c" is (define or describe)

Reply: In the revised manuscript, a description of c has been added. In particular, we add "Here c stands for the maximal overlap between measurements".

Referee Point A.8

Theorem 1: "has" $\rightarrow$ have

Reply: In the revised manuscript, we have changed "$C(\mathcal{T}_1, \mathcal{T}_2)$ can be explicitly computed and is strictly non-zero whenever $\mathcal{T}_1$ and $\mathcal{T}_2$ has no common eigencircuit" into "$C(\mathcal{T}_1, \mathcal{T}_2)$ can be explicitly computed and is strictly non-zero whenever $\mathcal{T}_1$ and $\mathcal{T}_2$ have no common eigencircuit".

Referee Point A.9

p. 3, col. 2: "direct-case" $\rightarrow$ cause

Reply: In the revised manuscript, we have changed "Meanwhile, each $\mathcal{T}_2 \in \mathcal{M}_{\text{DC}}$ is a *maximal direct-case indicator*" into "Meanwhile, each $\mathcal{T}_2 \in \mathcal{M}_{\text{DC}}$ is a *maximal direct-cause indicator*".

Referee Point A.10

Eq. (3): it is not clear what $H(T_i)$ is

Reply: In the revised manuscript, we have detailed that "Here $H(\mathcal{T}_i)$ $(i = 1, 2)$ is the Shannon entropy of the probability distribution associated with outcomes when $\mathcal{T}_i$ is measured".

Referee Point A.11

p. 4: last sentence ends in ..

Reply: In the revised manuscript, the additional punctuation mark has been deleted.

Referee Point A.12

Supplemental Material:

p. 1, last line: "player"?

Reply: We have revised supplemental material. Now the problem does not exist anymore.

Referee Point A.13

p. 2: better define nonsignaling

Reply: In the revised supplemental material, we have formally defined non-signaling (Def. I. 2, page 3 of the supplemental material):

Given a bipartite channel $\mathcal{E} : AC \to BD$, it is called non-signaling from $C \to D$ to $A \to B$ if the following condition is satisfied

$$\mathrm{Tr}_D \circ \mathcal{E}_{AC\to BD} = \mathcal{F}_{A\to B} \otimes \mathrm{Tr}_C, \tag{1}$$

for some quantum channel $\mathcal{F}$ from A to B. On the other hand, we say $\mathcal{E}$ is non-signaling from $A \to B$ to $C \to D$, if there exists a quantum channel $\mathcal{G}$ from C to D such that

$$\mathrm{Tr}_B \circ \mathcal{E}_{AC\to BD} = \mathcal{G}_{C\to D} \otimes \mathrm{Tr}_A. \tag{2}$$

Finally, the bipartite channel $\mathcal{E}$ is said to be non-signaling if it meets the condition of both Eq. 1 and Eq. 2.

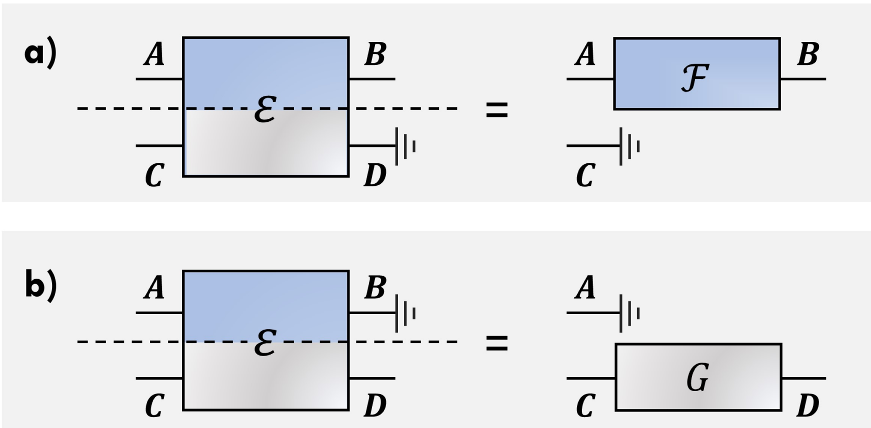

Figure 1: Pictorial demonstrations of non-signaling (NS) bipartite quantum channel $\mathcal{E}$ from systems AC to BD: (a) NS from $C \to D$ to $A \to B$ (see Eq. 1); (b) NS from $A \to B$ to $C \to D$ (see Eq. 2).

An illustration of non-signaling is given in above figure (Fig. 2, page 3 of the supplemental material).

Referee Point A.14

p. 2: better write that $\mathbf{T}_E$ is the partial transpose

Reply: In the revised supplemental material, we have added that "where $\mathbf{T}_E$ stands for the partial transpose over system E" (sentence below Eq. 6 on page 4 of the supplemental material).

Referee Point A.15

p. 4, last sentence: "though" $\rightarrow$ if

Reply: In the revised supplemental material, we have changed "Actually, even though we rearrange the vector $\mathbf{b}_S$ into non-increasing order" into "Actually, even if we rearrange the vector $\mathbf{b}_S$ into non-increasing order" (sentence above Eq. 66 on page 16 of the supplemental material).

Referee Point A.16

p. 5 beginning: Description of flatness process somewhat unclear (what happens if x is already nonincreasing?)

Reply: We thank the Referee for pointing out this concern. On rereading our previous submission, we realized that we have accidentally omitted the cases when the vector is already nonincreasing. To fix this problem, we have revised our representation about "flatness process" (Def. II. 4, page 16 of the supplemental material):

Let $\mathbf{x} \in \mathbb{R}^d_+$ be a non-negative d-dimensional vector, and j be the smallest integer in $\{2, \ldots, d\}$ such that $x_j > x_{j-1}$, and i be the greatest integer in $\{1, \ldots, j-1\}$ such that $x_{i-1} \geqslant (\sum_{k=i}^{j} x_k)/(j-i+1) := a$. Define

$$\mathcal{T}(\mathbf{x}) := (x'_1, \ldots, x'_n) \quad \text{with} \quad x'_k = \begin{cases} a & \text{for} \quad k = i, \ldots, j \\ x_k & \text{otherwise.} \end{cases} \tag{3}$$

and $\mathcal{F}(\mathbf{x}) := \mathcal{T}^{d-1}(\mathbf{x}) = \mathcal{T}(\mathcal{T}^{d-2}(\mathbf{x}))$, i.e. applying $\mathcal{T}$ on the vector $\mathbf{x}$ successively $d-1$ times. We call $\mathcal{F}$ the flatness process of vector $\mathbf{x}$. If the vector $\mathbf{x}$ is already arranged in non-increasing order, or equivalently there does not exist an integer j in $\{2, \ldots, d\}$ such that $x_j > x_{j-1}$, then we simply have $\mathcal{T}(\mathbf{x}) = \mathbf{x}$.

Let us now come back to the question "what happens if $\mathbf{x}$ is already nonincreasing?". In this case, $\mathbf{x}$ remains the same, namely $\mathcal{F}(\mathbf{x}) = \mathbf{x}$. For example, if $\mathbf{x} = (0.6, 0.175, 0.175, 0.05)$, then we have $\mathcal{F}(\mathbf{x}) = \mathbf{x} = (0.6, 0.175, 0.175, 0.05)$.

Referee Point A.17

p. 5, line 2 in Sec. I. C: "proving" $\rightarrow$ probing

Reply: In the revised supplemental material, we have changed "In this game, Alice is an agent that is capable of proving the quantum circuit fragment Φ" into "In this game, Alice is an agent that is capable of probing the quantum circuit fragment Φ" (paragraph 2 in Sec. II. C, page 19 of the supplemental material).

Referee Point A.18

p. 5 (and elsewhere) "successful probability" $\rightarrow$ success probability

Reply: In the revised supplemental material, we have changed all "successful probability" into "success probability" or "winning probability". For example, "Given k chips and a quantum causal map Φ, Bob's maximum winning probability is characterized by ..." (sentence above Eq. 93 on page 19 of the supplemental material).

Referee Point A.19

p. 6: better define $\mathbf{p} \oplus \mathbf{q}$

Reply: In the revised supplemental material, we have formally defined the direct-sum $\oplus$ between vectors (Eq. 86, page 18 of the supplemental material):

Consider the probability vectors $\mathbf{p} := (p_i)_{i=1}^m$ and $\mathbf{q} := (q_j)_{j=1}^n$ obtained by measuring a quantum state ρ with respect to a pair of incompatible measurements M and N, their direct-sum $\oplus$ is defined as the union of vectors. Formally,

$$\mathbf{p} \oplus \mathbf{q} := (p_1, \ldots, p_m, q_1, \ldots, q_n). \tag{4}$$

For example, given probability vectors $\mathbf{p} = (1, 0)$ and $\mathbf{q} = (1/2, 1/2)$, their direct-sum $\mathbf{p} \oplus \mathbf{q}$ is simply $(1, 0, 1/2, 1/2)$.

Referee Point A.20

p. 8: better give some reference for the characterization of quantum combs used in Eq. (49)

Reply: In the revised supplemental material, two original references for characterizing quantum combs have been added, including

- G. Chiribella, G. M. D'Ariano, and P. Perinotti, *Theoretical framework for quantum networks*, Phys. Rev. A **80**, 022339 (2009).
- G. Chiribella, G. M. D'Ariano, and P. Perinotti, *Quantum circuit architecture*, Phys. Rev. Lett. **101**, 060401 (2008).

Now the representation related with Eq. 49 (Eq. 145, page 24 of the supplemental material) has been changed into

Using the language of CJ operators, the maximum winning probability $p_{\text{win},k}$ can be solved by using SDPs [6, 7],

$$
\begin{aligned}
p_{\text{win},k} = \frac{1}{c}\max_{|\mathcal{S}|=k}\max_{\Phi\in\mathfrak{F}_a} J^{\Phi}\star J_{\mathcal{S}} = \frac{1}{c}\max_{|\mathcal{S}|=k}\max \quad & \operatorname{Tr}[J(a)\cdot J_{\mathcal{S}}] \\
\text{s.t.} \quad & J(a)\geqslant 0, \quad \operatorname{Tr}_{\mathcal{H}_1}[J(1)] = 1, \\
& \operatorname{Tr}_{\mathcal{H}_{2i-1}}[J(i)] = \operatorname{Tr}_{\mathcal{H}_{2i-1}\mathcal{H}_{2i-2}}[J(i)]\otimes\frac{\mathbb{1}_{\mathcal{H}_{2i-2}}}{d_{\mathcal{H}_{2i-2}}}, \\
& \text{for } 2\leqslant i\leqslant a,
\end{aligned}
\tag{5}
$$

where $d_{\mathcal{H}_{2i-2}} := \dim\mathcal{H}_{2i-2}$.

In addition to the above changes, we have also highlighted the physical meaning of restrictions formulated in Eq. 5 (sentence below Eq. 145 on page 25 of the supplemental material):

From a causal perspective, the last restriction of Eq. 5 indicates that the quantum circuit fragment Φ is NS from Ψ^i to Ψ^{i-1} for $2\leqslant i\leqslant a$ with $\Psi^1 := \rho$.

Referee Point A.21

p. 9: One should be more precise here. Q is a subset of the unordered $\mathbb{P}_n^d$, here majorization is just a preorder, so it cannot be a lattice. The preorder defines an equivalence relation, the quotient of $\mathbb{P}_n^d$ with respect to this equivalence is isomorphic to the ordered $\mathbb{P}_n^d$ and the preorder becomes a partial order. So, in fact, $(\vee Q)$ is an equivalence class.

Reply: We thank the Referee A for mentioning this problem. In the revised supplemental material, we have solved this problem by defining two sets, the original set $\mathcal{Q}$ (Eq. 200, page 33 of the supplemental material) and the ordered version $\mathcal{Q}_1$ (Eq. 201, page 33 of the supplemental material):

Formally, the set $\mathcal{Q}$ is defined as

$$
\mathcal{Q} := \{\mathbf{p}(\mathcal{U}_1,\mathcal{U}_2)_\Phi \oplus \mathbf{q}(\mathcal{U}_3,\mathcal{U}_4)_\Phi\}_{\mathcal{U}_1,\ldots,\mathcal{U}_4,\Phi}, \tag{6}
$$

and we denote the ordered version of $\mathcal{Q}$ as $\mathcal{Q}_1$, namely

$$
\mathcal{Q}_1 := \mathcal{Q}^{\downarrow} = \{\mathbf{p}(\mathcal{U}_1,\mathcal{U}_2)_\Phi \oplus \mathbf{q}(\mathcal{U}_3,\mathcal{U}_4)_\Phi\}^{\downarrow}_{\mathcal{U}_1,\ldots,\mathcal{U}_4,\Phi}, \tag{7}
$$

where the down-arrow $^{\downarrow}$ indicates that all the elements of $\mathcal{Q}$ are arranged in non-increasing order. In qubit case, $\mathcal{Q}_1$ forms a subset of $\mathbb{P}_2^{8,\downarrow}$, i.e. $\mathcal{Q}_1\subset\mathbb{P}_2^{8,\downarrow}$.

Now, majorization "$\prec$" forms a partial order on the set $\mathcal{Q}_1$. Thanks to the completeness of majorization lattice, there exists a unique "least upper bound" (LUB), denoted by $\vee\mathcal{Q}_1$, for $\mathcal{Q}_1$. In particular, for any vector $\mathbf{x}$ belongs to $\mathcal{Q}_1$, we immediately obtain

$$
\mathbf{x}\prec(\vee\mathcal{Q}_1). \tag{8}
$$

According to the definition of majorization, we know that for any vector $\mathbf{y}$ belongs to the set $\mathcal{Q}$, we also have

$$
\mathbf{y}\prec(\vee\mathcal{Q}_1). \tag{9}
$$

In this case, the LUB is not unique, as

$$\mathbf{y} \prec (\vee \mathcal{Q}_1) \cdot D, \tag{10}$$

holds for any permutation matrix D.

Referee Point A.22

p. 9: The notation $(\vee Q_k)$ is a bit confusing. It took me some time to realize that this is not the LUB of some set Q_k, but the k-th element of the sequence $(\vee Q)$. It would be better to explain this.

Reply: We thank the Referee for pointing out this problem. In the revised supplemental material, we have used $(\vee \mathcal{Q}_1)_k$ to denote the k-th element of vector $(\vee \mathcal{Q}_1)$. For example, as shown in the sentence below Eq. 202 on page 33 of the supplemental material,

Thanks to the completeness of $(\mathbb{P}_2^{8,\downarrow}, \prec, \wedge, \vee)$, there exist a unique LUB, denoted by $\vee \mathcal{Q}_1$, in $\mathbb{P}_2^{8,\downarrow}$; that is

$$(\vee \mathcal{Q}_1)_k \geqslant (\vee \mathcal{Q}_1)_{k+1}, \quad \text{for } 1 \leqslant k \leqslant 7, \tag{11}$$

where $(\vee \mathcal{Q}_1)_k$ denotes the k-th element of $\vee \mathcal{Q}_1$.

Referee Point A.23

p. 11, Eq.(75): Similarly, "$4 \vee Q_2$" looks like the LUB of 4 and some Q_2.

Reply: To minimize ambiguity, instead of using $\vee \mathcal{Q}_k$, we have used $(\vee \mathcal{Q}_1)_k$ to denote the k-th element of vector $(\vee \mathcal{Q}_1)$ in the revised supplemental.

Referee Point A.24

p. 11, below Eq. (77): "families of infinite unitaries"?

Reply: We have revised supplemental material. Now the problem does not exist anymore.

Referee Point A.25

p. 11, Sec. IV. A: sentence ending as "...bipartite unitary." What is the bipartite unitary here?

Reply: We have revised supplemental material. Now the problem does not exist anymore.

Referee Point A.26

p. 11, Eq. (84): why the direct-cause circuit cannot be of the form $\rho_A \otimes E_{B\to FC}$?

Reply: We thank the Referee for pointing out this concern. In principle, the quantum circuit decomposed as $\rho_A \otimes \mathcal{E}_{B\to CF}$ also forms a direct-cause. However, in this work, we focus on the causal structure of system-environment dynamics

$$\Phi_{B\to ACF} = \mathcal{U}_{BE\to CF}(\phi_{AE}), \tag{12}$$

where the dimension of systems (i.e. A, B, C) remains the same, namely $\dim A = \dim B = \dim C$. Hence, the unitarity of $\mathcal{U}_{BE\to CF}$ forces the environment F to be the trivial system $\mathbb{C}$ when E has already been trivialized. To reduce ambiguity, we have added a remark on the possibility of being $\rho_A \otimes \mathcal{E}_{B\to CF}$ in the paragraph below Def. III. 10, page 39 of the supplemental material:

It is worth mentioning that if the system-environment unitary dynamics $\mathcal{U}_{BE\to CF}(\phi_{AE})$ is decomposed as $\rho_A \otimes \mathcal{E}_{B\to CF}$, the causal structure associated with system-environment dynamics can also be defined as purely direct-cause. However, the unitarity of evolution $\mathcal{U}_{BE\to CF}$ and the dimensional restriction on the systems (i.e. $\dim A = \dim B = \dim C$) will force the environment to be a trivial system, i.e. $F = \mathbb{C}$. Hence, the system-environment unitary dynamics can still be described by Eq. 237.

In our revised supplemental material, Eq. 237 on page 39 of the supplemental material reads

$$\mathcal{U}_{BE\to CF}(\phi_{AE}) = \rho_{AF} \otimes \mathcal{E}_{B\to C}. \tag{13}$$

Referee Point A.27

p. 12, just below Eq. (85) "B to C" → B to F.

Reply: In the revised supplemental material, we have changed "$\mathcal{E}_{B\to F}$ is a quantum channel from system B to C" into "$\mathcal{E}_{B\to F}$ is a quantum channel from system B to F" (Def. III. 7, page 38 of the supplemental material).

Referee Point A.28

p. 12, line 4 from below: $\Phi_{AE} \to \phi_{AE}$

Reply: In the revised supplemental material, we have changed Φ_{AE} into ϕ^+_{AE}. More precisely, in Thm. III. 9 (page 38 of the supplemental material), we start with the following sentence:

For a system-environment unitary dynamics $\mathcal{U}_{BE\to CF}(\phi^+_{AE})$ with ϕ^+_{AE} being the maximally entangled state.

Referee Point A.29

p. 13: It seems that the unitary operator U and the corresponding unitary channel are denoted by the same letter. Better change this.

Reply: We thank the Referee for pointing out this concern. In the revised supplemental material, we have used $\mathcal{U}$ and U to represent the unitary channel and the corresponding unitary operator respectively.

Take the definition of maximal common-cause indicator (Def. III. 1, page 32 of the supplemental material) for instance, we see that

An interactive measurement $\mathcal{T}_{\mathsf{CC}}(\mathcal{U}_1,\mathcal{U}_2) := \{\mathcal{T}_{\mathsf{CC},i}(\mathcal{U}_1,\mathcal{U}_2)\}_i \in \mathfrak{T}_2$ is called maximal common-cause indicator, if it has the form of

$$\mathcal{T}_{\mathsf{CC},i}(\mathcal{U}_1,\mathcal{U}_2)(\cdot) = \mathrm{Tr}[\Phi_i \cdot \mathcal{U}_2(\cdot) \otimes \frac{\mathbb{1}_B}{d} \otimes \mathcal{U}_1(\cdot)], \tag{14}$$

where $\mathcal{U}_1$ and $\mathcal{U}_2$ are some local unitary channels acting on systems A and C respectively, namely $\mathcal{U}_b(\cdot) = U_b(\cdot)U_b^\dagger$ $(b = 1, 2)$, with $|\Phi_1\rangle := |\phi^+\rangle = \sum_{k=0}^{d-1} |kk\rangle / \sqrt{d}$ being the maximally entangled state. Measurements are done with respect to a maximally entangling basis $\{\Phi_i\}_i$ with d^2 possible outcomes. Denote the collection of all maximal common-cause indicators as $\mathcal{M}_{\mathsf{CC}}$.

Here we use $\mathcal{U}_b$ $(b = 1, 2)$ to denote unitary channel, and the corresponding unitary operator is described by matrix U_b $(b = 1, 2)$.

Response to Referee B

Referee Point B.1

The manuscript under review derives an uncertainty principle for interactive measurements. An interactive measurement is designed to probe an evolving quantum system, referred to by the authors as a 'dynamical process'. The quantum system undergoes an interaction at a time t_1 with an ancilla system. At a time t_2 a joint measurement is performed on the ancilla and original system. The structure seems to be The manuscript proposes an uncertainty principle which is relevant to measurements of this kind. In the final section, they show how this result can be used to derive a trade-off for distinguishing common-cause from direct-cause processes.

The key result in the paper is the aforementioned uncertainty principle for interactive measurements. The authors initially claim that they are deriving 'the' uncertainty principle for interactive measurements, however acknowledge they are looking at just one among many possible choices. I think, for this reason, that the result is not quite general enough to merit publication in PRL. As an analysis of one specific bound in the field of measurement uncertainty principles, however, it is not uninteresting and I would support a regular article submission to another Physical Review journal, such as Physical Review A

Reply: We see also that the referee's primary concern regarding the manuscript's suitability for PRL due to its lack of generality. We believe this is based on Referee B's assumptions that

1. The manuscript covers only uncertainty relations involve measurements at two different times (interactions at t_1 followed by joint measurement at t_2)
2. The manuscript only covered one of many possible uncertainty principles (i.e., the referee mentioned "as an analysis of one specific bound in the field of measurement uncertainty...")

From this starting point, the referee concludes that 'the result is not quite general enough to merit publication in PRL'. We thank Referee B for his candidate assessment. Indeed, were these restrictions true, we would agree with the Referee! Fortunately, *our results are far more general*, and are not constrained by (1) and (2) above. We wish to assure Referee B that

1. The uncertainty relations covered in this manuscript includes interactive measurements over an *arbitrary number* of time-steps. Such measurements include cases where an ancilla interacts with the system of interest at times t_1, t_2,... t_{a-1}, before a final (possibly) joint measurement of system and ancilla at time t_a. Here a can be an arbitrary integer, and interactions is entirely unconstrained and can include arbitrary adaptive POVMs (See Figure 2).
2. Our work, in fact, implies a infinite family uncertainty relations for interactive measurements via the mathematical framework of majorization. In the previous iteration of the manuscript, we chose to highlight one particular class within this family (entropic uncertainty relations) and referred readers to the supplemental material for sake of conciseness.

We now elaborate on each of the above points in more technical details:

Regarding Applicability Results to General Interactive Measurements - As aforementioned, our

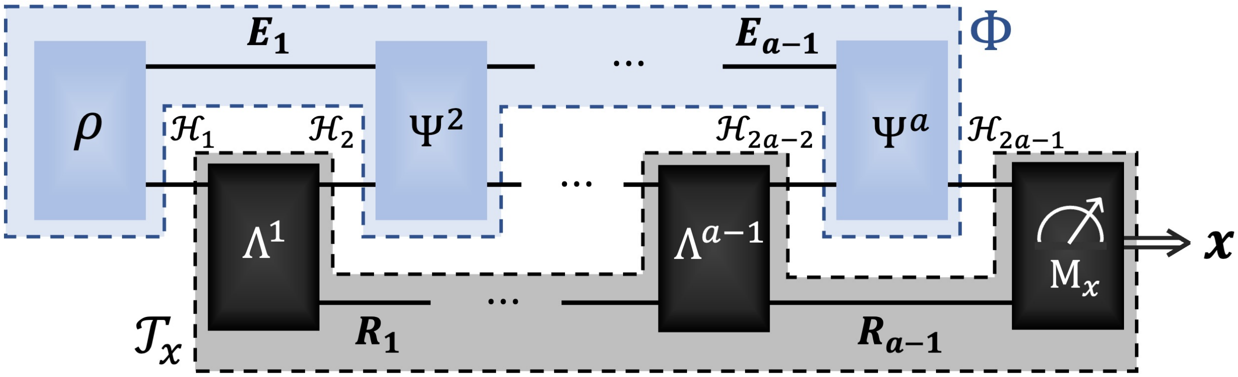

Figure 2: A general interactive measurement $\mathcal{T}_x$ with finite interventions. Here the blue box represents the quantum dynamics Φ being measured. Such a measuring process appears in diverse contexts, including quantum illumination, quantum agents and non-Markovian open systems.

manuscript considers the most general form measurement – where our measurement device is allowed to interact with the system of interest over a rounds.

Let us elaborate on this further to make sure there's no ambiguity. First, the goal of an interactive measurement is to measure properties of a general non-Markovian quantum process. Such quantum processes can be described by a circuit fragment (shaded blue) in Figure 2. Meanwhile an agent conducting a general interactive measurement (shaded black) "interlocks" with circuit fragment. This framework has appeared in a number contexts, including studies of quantum agents and non-Markovian quantum processes.

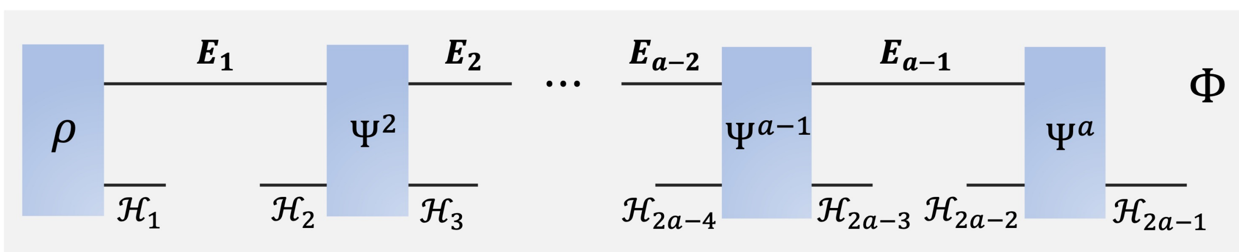

Figure 3: Quantum circuit fragment Φ, including an initial state ρ and $a-1$ multiple quantum processes $\{\Psi^i\}_{i=2}^a$.

To be more precise, if we zoom in the blue circuit fragment (see Figure 3). We see that it consists of accessible register that the agent can interact with (denoted by $\mathcal{H}_{2k-1}$ on the k^{th} round), and some environmental register (denoted by E_k). Between time-steps, the environmental and accessible registers can re-interact (thus the non-Makovianity). Formally we can describe such a general circuit fragment by

- A bipartite quantum state ρ is prepared in systems $\mathcal{H}_1 E_1$

- $a-1$ quantum channels $\{\Psi_2, \ldots, \Psi_a\}$, where

$$\Psi^i : \mathcal{H}_{2i-2}E_{i-1} \to \mathcal{H}_{2i-1}E_i, \quad 2 \leqslant i \leqslant a-1, \tag{15}$$
$$\Psi^a : \mathcal{H}_{2a-2}E_{a-1} \to \mathcal{H}_{2a-1}. \tag{16}$$

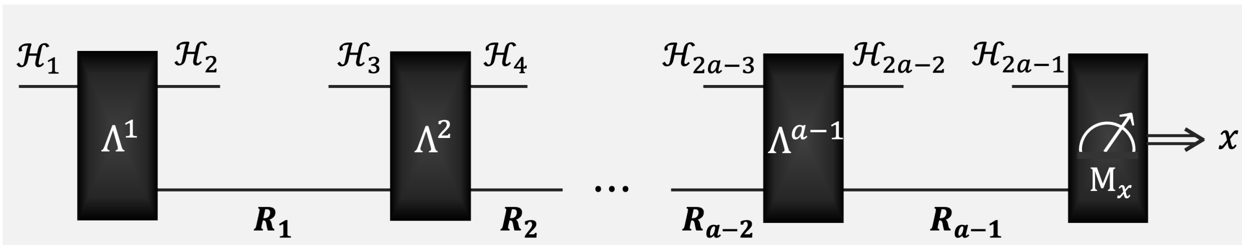

Figure 4: Interactive measurement $\mathcal{T}_x$, including $a-1$ rounds of interventions $\{\Lambda^i\}_{i=1}^{a-1}$ and a joint POVM measurement $M := \{M_x\}_x$.

Meanwhile, a general interactive measurement is illustrated in Figure 4. At each round k, it performs an intervention on the accessible register $\mathcal{H}_{2k-1}$. Generally this involves interacting $\mathcal{H}_{2k-1}$ with some memory register R_{k-1} with some quantum process Λ^k (though for the case $k=1$, the register can be absorbed into Λ^1). On the final round ($k=a$), a final joint measurement between R_{a-1} and $\mathcal{H}_{2a-1}$ is performed to obtain the final measurement result. Mathematically, all such interactive measurements can be full characterized by

- $a-1$ rounds quantum channels $\{\Lambda^i\}_{i=1}^{a-1}$ in the form of

$$\Lambda^1 : \mathcal{H}_1 \to \mathcal{H}_2 R_1, \tag{17}$$
$$\Lambda^i : \mathcal{H}_{2i-1}R_{i-1} \to \mathcal{H}_{2i}R_i, \quad 2 \leqslant i \leqslant a-1, \tag{18}$$

- a joint POVM measurement $M := \{M_x\}_x$ acting on systems $\mathcal{H}_{2a-1}R_{a-1}$, namely

$$M_x : \mathcal{H}_{2a-1}R_{a-1} \to \mathbb{C}. \tag{19}$$

We note that these interactive measurements are completely general. They encompass *everything* the agent can possible do. Notably, the memory R can also store classical information, and can also be of arbitrary dimension. As such, the framework encompasses, for example, making a projective measurements in early rounds, and condition an action on the system based on the result of these measurements. Figure 5 illustrates this for the case of a single intervention.

We hope the referee can see broad generality of our framework in that our uncertainty relations encompasses interactive measurements with an arbitrary number of interventions.

Regarding Generality of Our Uncertainty Relations As aforementioned, our work does not simply provide one of many possible choices of uncertainty relations. In fact our work implies a family of infinite uncertainty relations for interactive measurements. Meanwhile, these uncertainty relations applies to general interactive measurements with a arbitary number of interventions.

Specifically, our generality can be understood in two different ways:

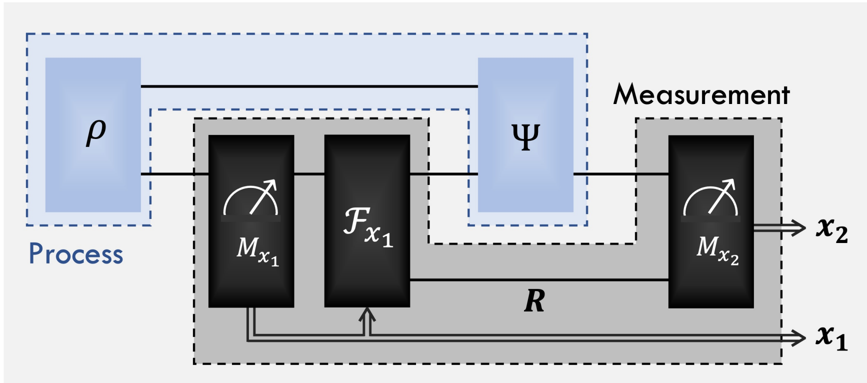

Figure 5: Measuring non-Markovian quantum process with an interactive measurement: Now systems have prior correlations, as shown by the blue dashed box. Based on the measurement outcome x_1 obtained at t_1, observers can then prepare a quantum device, described by quantum channel $\mathcal{F}_{x_1}$, and finalize the measuring process by carrying out a joint measurement M_{x_2} at time point t_2. In this figure, the non-Markovian quantum process is characterized by the blue dashed box. Meanwhile, the interactive measurement is described by the black dashed box. Λ stands for the intervention implemented between systems A and B.

- Mathematically, our framework actually involves developing general uncertainty relations that constrain the joint measurement statistics of two general $\mathcal{T}_1$ and $\mathcal{T}_2$ based on majorization. These constraints can imply an infinite of uncertainty relations, of which the entropic uncertainty relations we choose to highlight in manuscript is a special case.

  Formally, let "$\prec$" stands for majorization: given two probability vectors $\mathbf{x} := (x_k)_k$ and $\mathbf{y} := (y_k)_k$ arranged in non-increasing order, we say $\mathbf{x}$ is majorized by $\mathbf{y}$, written as $\mathbf{x} \prec \mathbf{y}$, if $\sum_{k=1}^{i} x_k \leqslant \sum_{k=1}^{i} y_k$ holds for all index i. Meanwhile, let $\oplus$ represents the union of vectors. Take $(1, 0)$ and $(1/2, 1/2)$ for instance, their direct-sum $(1, 0) \oplus (1/2, 1/2)$ is simply $(1, 0, 1/2, 1/2)$.

  Consider now the probability distributions obtained by implementing interactive measurements $\mathcal{T}_1$ and $\mathcal{T}_2$ on some dynamical process Φ as $\mathbf{p}$ and $\mathbf{q}$, then there exists a probability vector $\mathbf{v}(\mathcal{T}_1, \mathcal{T}_2)$ such that

$$\frac{1}{2}\mathbf{p} \oplus \frac{1}{2}\mathbf{q} \prec \mathbf{v}(\mathcal{T}_1, \mathcal{T}_2). \tag{20}$$

  Here the vector-type bound $\mathbf{v}(\mathcal{T}_1, \mathcal{T}_2)$ is independent of the dynamical process being measured, and hence captures the essential incompatibility between $\mathcal{T}_1$ and $\mathcal{T}_2$.

  In previous studied of uncertainty relations for conventional quantum measurements, analgous majorization relations were often referred to as *universal uncertainty relations*. This is because Eq. 20 implies infinite family of uncertainty relations, namely

$$f(\mathbf{p}/2 \oplus \mathbf{q}/2) \geqslant f(\mathbf{v}(\mathcal{T}_1, \mathcal{T}_2)), \tag{21}$$

  by applying Schur-concave function f. Such Schur-concave functions have been argued to be the most general class of uncertainty quantifiers due to the their monotonicity under random relabeling of measurement outcomes (see, for example, "Shmuel Friedland, Vlad Gheorghiu, and Gilad Gour, *Universal Uncertainty Relations*, Phys. Rev. Lett. **111**, 230401 (2013)").

Entropy, being Schur-concave, is a special case – leading to the entropic uncertainty relations we outlined in Theorem 1. In the previous manuscript, we chose only to highly the entropic variant of our uncertainty relation, as we thought it would keep the manuscript more accessible. We see from the Referee comments that this is an error, and that our general manuscript would greatly benefit from highlighting this more general result.

- Physically, our framework works for all quantum circuit fragments, where the number of system-environment interactions is finite. Take the entropic form of our main result (Thm. 1, page 2 of the main text) for instance, we have

$$H(\mathcal{T}_1)_\Phi + H(\mathcal{T}_2)_\Phi \geqslant C(\mathcal{T}_1, \mathcal{T}_2), \tag{22}$$

holds for any quantum circuit fragment $\Phi \in \mathfrak{F}_a$ with index a. Here we denote the set of all quantum circuit fragments with an initial state $\Lambda^1 := \rho$ and $a-1$ system-environment interactions $\{\Lambda^i\}_{i=2}^a$ as $\mathfrak{F}_a$ (see Figure 3). When $a = 1$, we obtain the result for quantum states. Setting $a = 2$, we derive the uncertainty trade-off for quantum causal maps (see, for example, "Katja Ried, Megan Agnew, Lydia Vermeyden, Dominik Janzing, Robert W. Spekkens, and Kevin J. Resch, *A quantum advantage for inferring causal structure*, Nat. Phys. **11**, 414 (2015)"). For general a, we obtain the fundamental uncertainty principle for all quantum circuit fragments, including quantum causal networks (whose Choi-Jamiołkowski operator is known as "quantum comb" in literature) and process tensor in open quantum systems. A change in the structure of quantum dynamics, i.e. the value of a, will cause a change in our bound $C(\mathcal{T}_1, \mathcal{T}_2)$, leading to entropic uncertainty relations for all quantum circuit fragments. As stated in the main text of our work, "we focus on a choice of $C(\mathcal{T}_1, \mathcal{T}_2)$". But this choice of bound implies an infinite number of uncertainty relations.

Changes to Manuscript We see from referee comments that the previous version of our manuscript did not sufficiently communicate the generality of our results. In response we have made the following major revisions.

- We have significantly restructured the manuscript, which now emphasises the general framework of interactive measurements. Notably, the first three paragraphs of our frameworks section now focus on giving an accessible introduction to general (non-Markovian) quantum processes and the framework of interactive measurements used to measure them. This contrasts with the previous iteration, where the framework section was dominated by an illustrative example involving a single intervention (which we now postpone to the section on causal uncertainty relations).
- We have included a new Lemma within (Lem. 1, page 2 of the main text) that outlines the universal majorization conditions that underlay a broad family of uncertainty relations. Previously, this lemma only appeared in the supplemental material, while the main manuscript focused almost inclusively on the special case of entropy uncertainty relations that is a consequence of this lemma. We believe this addition will make it much clearer to readers that our results include not just a single uncertainty relation, but a broad family of uncertainty relations.

We believe these changes has significant improved the manuscript, making the breadth and generality of our results far more readily apparent. We thank the referee for catalyzing these improvements!

Referee Point B.2

Specific observations and comments:

Reply: We thank Referee B for carefully reading our manuscript, providing us valuable comments and constructive criticism. In the following, we address all the comments point-by-point.

Referee Point B.3

1) The interactive measurements discussed in this paper seem to be a form of 'quantum comb'. I wondered how tightly the two formalisms are connected.

Reply: This is a good question. Indeed, terminologies in this field are quite confusing, as they were independently developed in many different settings – each with its own Jargon. Historically, following the original works of G. Chiribella, G. M. D'Ariano, and P. Perinotti (i.e. *Theoretical framework for quantum networks*, Phys. Rev. A **80**, 022339 (2009) and *Quantum circuit architecture*, Phys. Rev. Lett. **101**, 060401 (2008)), quantum combs were introduced as a means of understanding "fragments" of a quantum circuit such as the ones outlined in Figure 6. See also the work of G. Gutoski and J. Watrous (i.e. *Toward a general theory of quantum games*, STOC'**07**, Pages 565–574 (2007)).

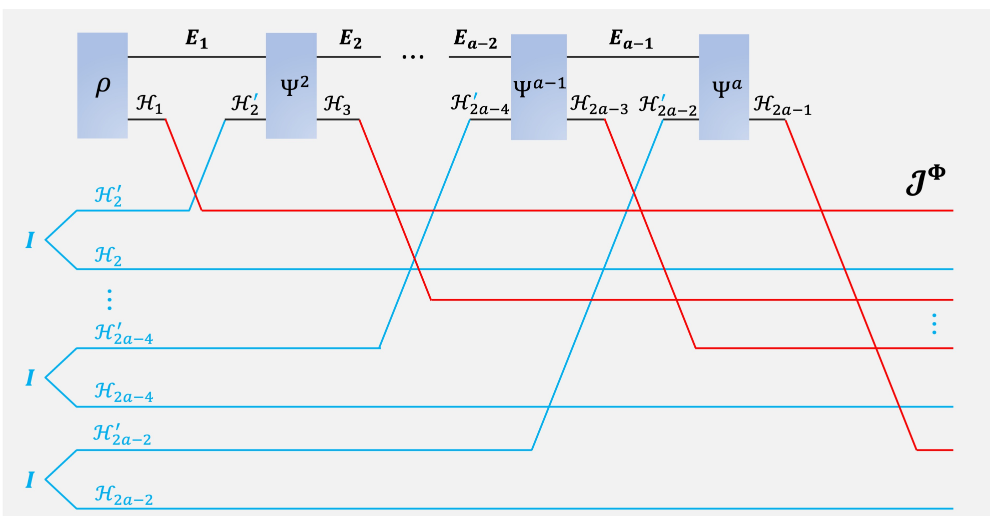

Figure 6: The Choi-Jamiołkowski operator J^Φ of a quantum circuit fragment Φ. Blue lines indicate the systems coming from $|I\rangle\langle I|_{\otimes_{i=2}^{a}\mathcal{H}_{2i-2}\mathcal{H}'_{2i-2}}$, where $|I\rangle := \sum_i |ii\rangle$ is the unnormalized maximally entangled state. Red lines represent the output systems of quantum processes $\{\Psi^i\}_{i=2}^a$.

In the answer to Referee query B.3, we saw that such a circuit fragment is fully characterized by

- A bipartite quantum state ρ prepared in systems $\mathcal{H}_1 E_1$

- $a-1$ quantum channels $\{\Psi_2, \ldots, \Psi_a\}$, where

$$\Psi^i : \mathcal{H}_{2i-2}E_{i-1} \to \mathcal{H}_{2i-1}E_i, \quad 2 \leqslant i \leqslant a-1, \tag{23}$$

$$\Psi^a : \mathcal{H}_{2a-2}E_{a-1} \to \mathcal{H}_{2a-1}. \tag{24}$$

In literature, such a fragment is often referred to as a *quantum stochastic process* or *quantum causal network*. It can be used to describe a quantum system $\mathcal{H}$ that undergoes non-Markovian evolution through constant interaction with an environmental system E (see for example Simon Milz and Kavan Modi, *Quantum Stochastic Processes and Quantum non-Markovian Phenomena*, PRX Quantum **2**, 030201 (2021)).

Quantum combs were then introduced as a mathematical means to characterize these fragments through a generalisation of the Choi-matrix formalism for quantum channels (via the Choi-Jamiołkowski representation). The basic idea is that

1. We prepare a maximally entangled pairs (whose marginals have the same dimension of the system $\mathcal{H}$.
2. At each intervention a, we swap $\mathcal{H}$ with one arm of the a^{th} entanglement pair.

The result state at the end of this process turns out to be one-to-one with the circuit fragment. Quantum combs thus became a particularly useful too for characterizing such circuit fragments [1]. This one-to-one correspondence also meant that many researchers have adopted sense looser definition of a quantum comb. Instead of using it strictly to mean the Choi-Jamiolkowski of a non-Markovian quantum process, they also use to to refer to the non-Markovian quantum process itself. Here we will stick to the more original definition to reduce ambiguity.

We observe also that our interactive measurements have a very similar structure. They are essentially almost equivalent to the blue boxes in Figure 6. Indeed, if we took Figure 4 and relabeled R_k as E_k, the two objects look almost identical. However, there are two small differences.

1. The interactive measurement receives some state acting on $\mathcal{H}_1$ instead of preparing a state ρ.
2. The interactive measure ends with some POVM measurement.

Given these differences, the interactive measurement is similar, but not identical, to a quantum comb (assuming to the looser definition of the word). In particular, by trivializing the Hilbert space $\mathcal{H}_1$ depicted in Figure 3, namely $\mathcal{H}_1 = \mathbb{C}$, we obtain the framework of quantum causal networks. Roughy speaking, interactive measurements can be viewed as such a quantum causal network followed by a POVM.

Just as in the case of quantum causal networks, we can modify the Choi-Jamiołkowski isomorphism to develop a "quantum comb" representation of an interactive measurement (see Figure 7).

In summary,

- A quantum comb historically refers to a specific mathematical representation of a non-Markvovian quantum process (Known as its Choi-Jamiolkowski operator). In our setting, our quantum circuit fragment is the non-Markvovian quantum process that our agent interacts with.

[1] Note that most literature make use of unnormalized entangled states to simplify the mathematics, however, the idea is the same.

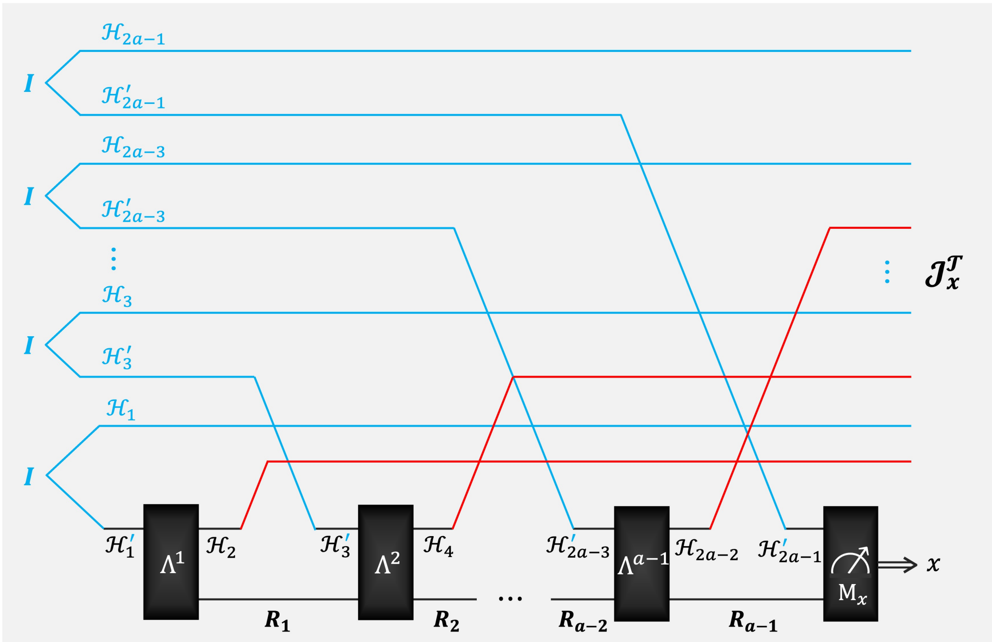

Figure 7: The Choi-Jamiołkowski operator $J_x^{\mathcal{T}}$ of an interactive measurement $\mathcal{T}_x$. Blue lines indicate the systems coming from $|I\rangle\langle I|_{\otimes_{i=1}^a \mathcal{H}_{2i-1}\mathcal{H}'_{2i-1}}$, where $|I\rangle := \sum_i |ii\rangle$ is the unnormalized maximally entangled state. Red lines represent the system after interventions $\{\Lambda^i\}_{i=1}^{a-1}$.

- However, recent years quantum combs are often also used to refer to the quantum process dynamical itself.
- Our interactive measurements share a lot of similarities with a non-Markovian quantum processes, and have their own Choi-Jamiołkowski representation. One could say that they are a form of a non-Markovian quantum process, supplemented by a POVM. Thus they do share some similarities with quantum combs if we use adopted the looser definition of a comb. However, there are still differences.
- To avoid ambiguity, we do not refer to interactive measurements as quantum combs. However, interactive measurements do have their own Choi-Jamiołkowski representation (which is sometimes referred to as the quantum comb representation).

We hope these points help clarify the relationship between quantum combs and interactive measurements.

Changes to Manuscript: Following the referee query, we realized that our manuscript did not spent sufficient time clearly defining a general quantum process Φ or a general interactive measurement $\mathcal{T}$. In the revised version of the manuscript, we have included an explicit discussion of both objects in the first two paragraphs of the framework section:

> The premise of an interactive measurement consists of an agent that wishes to probe the dynamics of some unknown quantum process Φ. Here Φ can be modelled as an open quantum system, consisting of a system accessible to the agent with $\mathcal{H}$, that is coupled with some generally non-

Markovian environment E (see blue shaded region in Fig. 1d). Initially, the $\mathcal{H}$-E system is in some joint state ρ. At each time-step k, the system and environment jointly involves under Ψ^k. Φ is then completely defined by the set $\{\Psi^k\}_{k=2}^{a}$ and the initial state ρ, where a represent to number of time-steps. In literature, Φ offers the most general representations of non-Markovian quantum stochastic processes [22] and is also closely related to concepts of higher-order quantum maps, adaptive agents, and causal networks $[23-28]$.

Interactive measurements then represent the most general means for an agent to ascertain properties of Φ (see black shaded region in Fig. 4d): the agent initializes some internal memory register R; between time-steps (i.e., before Ψ^k with $2 \leqslant k \leqslant a$), the agent performs an *intervention* – some general quantum operation Λ^k that interacts her memory R with the accessible system $\mathcal{H}$; after $a-1$ such interventions, the agent finally makes a joint measurement with respect to some positive operator valued measure (POVM) $M := \{M_x\}$ on the joint $\mathcal{H}$-R system to obtain some outcome x. Thus, each interactive measurement $\mathcal{T}$ is completely defined by set of interventions $\{\Lambda^k\}_{k=1}^{a-1}$ and POVM M. Just as a conventional POVM measurement on a quantum state induces some probability distribution over measurement outcomes, so does an interactive measurement on a quantum process. Analogous to eigenstates, we say Φ is an *eigencircuit* of $\mathcal{T}$ if Φ always yields a definite outcome when measured by $\mathcal{T}$.

We believe these paragraphs will help remove ambiguities regarding the mathematical formalism behind interactive measurement. After careful consideration, we also choose to omit discussion of the Choi-Jamiołkowski representation in the main body, as it is not necessary for understanding the main results of the manuscript. For readers interested in further details, we have included within supplemental material:

- A formal definition of quantum circuit fragments (see Def. I. 4, page 7 of the supplemental material); illustrating how they encompass all quantum dynamics with definite causal order, including quantum states, quantum causal maps and quantum causal networks.
- A formal definition of interactive measurements (see Def. I. 5, page 9 of the supplemental material); demonstrating the most general means for an agent to ascertain properties of quantum circuit fragments.

We believe these changes offer a nice balance of accessibility and completeness, and thank the Referee for instigating the change.

Referee Point B.4

2) Theorem 1 in the main body of the text is introduced as the main result, but isn't proven anywhere. As far as I can tell, it is a more general result from elsewhere which is used as a stepping stone to further calculation, but I'm still unclear. Can the authors be more explicit about where this result comes from?

Reply: We thank the Referee for pointing out this concern. On rereading our submission, we realized there was no explicit statement that proof of Theorem 1 was in the supplementary materials! The statement must have been accidentally removed during revision, and we apologize for the oversight! We realize that this could have given an impression that the result is from elsewhere. As such, we reassure the Referee this is an original contribution of the manuscript – and that *the result has not appeared in scientific literature before*.

In fact, in the process of deriving Theorem 1, we actually first produced a mathematically more powerful result – a universal uncertainty relation for general interactive measurements using majorization [2]. The general entropic uncertainty relations is in factor a corollary of this more general result. In the previous version of the manuscript, we chose to focus only on the entropic uncertainty relation (Theorem 1), but see from referee's comments and concerns that the more general result should be included in the paper main body as well.

In the revised manuscript

- We have added a new lemma (Lem. I, page 2 of the main text) to highlight the importance of universal uncertainty relation for measurements with interventions. From this, our Theorem 1 comes as a corollary.
- Included an explicit sentences in the main body stating where in supplemental material proof of Theorem 1 and the new lemma reside.

We believe these changes should remove all ambiguity on where Theorem 1 came from, and make it clear that it is an original contribution of the current scientific manuscript. We hope this also helps alleviate some of the Referee's concerns regarding the generality of our results.

Referee Point B.5

3) Similarly, Equation 2 in the main body (Theorem 1 of the supplementary materials) is proved with just one line of text, and it's not obvious which assumptions are going in to the calculation. I think this proof could be unpacked a bit more, so that the steps are clear to the reader.

Reply: We thank the Referee for raising this issue. Indeed, we agree that the proof to the Equation 2 could be further unpacked. To reduce ambiguities for readers, we have unpacked the proof of original Eq. 2 and its generalization in Sec. II. D, page 27 of the revised supplemental material. In particular, we first prove a strengthened result for multiple interactive measurements

$$\bigoplus_{b=1}^{c} \frac{1}{c}\mathbf{p}_b \prec \mathbf{w}_{\mathcal{T}_1,\dots,\mathcal{T}_c}. \tag{25}$$

Now it follows that (page 29 of the supplemental material)

[2] The details of this is already outlined above in our response to Referee Point B.1.

Then, by applying Shannon entropy H, it now follows immediately that

$$H(\bigoplus_{b=1}^{c}\frac{1}{c}\mathbf{p}_b) = -\sum_{b,x_b}\frac{p_{x_b}(\Phi,\mathcal{T}_b)}{c}\log\frac{p_{x_b}(\Phi,\mathcal{T}_b)}{c} \tag{26}$$

$$= -\frac{1}{c}\sum_{b,x_b}p_{x_b}(\Phi,\mathcal{T}_b)\log p_{x_b}(\Phi,\mathcal{T}_b) + \frac{\sum_{b,x_b}p_{x_b}(\Phi,\mathcal{T}_b)}{c}\log c \tag{27}$$

$$= \frac{1}{c}\sum_{b=1}^{c}H(\mathcal{T}_b)_\Phi + \log c \tag{28}$$

$$\geqslant H(\mathbf{w}_{\mathcal{T}_1,\ldots,\mathcal{T}_c}), \tag{29}$$

where the second equation comes from the fact that, for each $b\in\{1,2,\ldots,c\}$, we have $\sum_{x_b}p_{x_b}(\Phi,\mathcal{T}_b) = 1$, and thus $\sum_b\sum_{x_b}p_{x_b}(\Phi,\mathcal{T}_b) = c$. The third equation follows from the definition, i.e.

$$H(\mathcal{T}_b)_\Phi := H(\mathbf{p}_b) = -\sum_{x_b}p_{x_b}(\Phi,\mathcal{T}_b)\log p_{x_b}(\Phi,\mathcal{T}_b). \tag{30}$$

Note that throughout this supplemental material, all logarithms are base 2. Now we have

$$\sum_{b=1}^{c}H(\mathcal{T}_b)_\Phi \geqslant c\,(H(\mathbf{w}_{\mathcal{T}_1,\ldots,\mathcal{T}_c}) - \log c) = cH(\mathbf{w}_{\mathcal{T}_1,\ldots,\mathcal{T}_c}) - c\log c. \tag{31}$$

Bu further define

$$C(\mathcal{T}_1,\mathcal{T}_2,\ldots,\mathcal{T}_c) := cH(\mathbf{w}_{\mathcal{T}_1,\ldots,\mathcal{T}_c}) - c\log c, \tag{32}$$

we obtain the entropic uncertainty relation, as required.

Referee Point B.6

4) Physical Review articles should be accessible to a wide audience, so I think a more general overview of uncertainty principles and interactive measurements should be included.

Reply: We thank the Referee for this suggestion. With a fresh eye, we agree that our previous manuscript did not include enough details regarding general uncertainty relations and interactive measurements. In response, we have made significant revisions to the manuscript to address these points. Notably:

- The framework section has undergone significant revision and now includes an introductory overview of interactive measurements and how they can apply to general non-Markovian quantum processes. Figure 1 has also been updated to complement this exposition, giving specific examples of interactive measurements to help readers with their intuition. We believe the resulting presentation should now be accessible to general readers with no previous background in related research areas.
- In regards to uncertainty relations, we have included extra exposition in the introduction and technical sections (specifically the subsection on uncertainty principles) that present a more detailed survey of recent developments in the field. Notable inclusions include a brief discussion on

majorization and its relation to universal uncertainty principles for conventional quantum measurements. This backdrop then allowed us to introduce the aforementioned Lemma 1 of main text, which describes a set of more general majorization constraints that all measurement probabilities of interactive measurements must satisfy.

We believe that these additions have made the manuscript a lot more accessible to a wide audience, and thank the Referee for instigating the change!

Referee Point B.7

5) In general, a lot of material that's required to navigate the paper has been put into the Supplemental Material. I think a reorganization of material would make the paper much more readable.

Reply: We thank the Referee for pointing out this concern. Indeed, what details to include in the main body and what to place in supplemental material was our constant concern when drafting the manuscript. Our previous iteration did not get the balance quite right! Indeed, when reviewing the manuscript in light of referee comments, we also saw too much dependence on supplemental material. In particular, we identified two significant shortcomings.

1. The previous version of the manuscript attempted to introduce general interactive measurements in the final paragraph of the framework's section. Consequently, it left out too many details regarding a general interactive measurement. This naturally made it difficult to absorb the meaning behind our main results and the breadth of their applications.

2. The previous manuscript had omitted a key original result of our research – a key constraint on the measurement probabilities of two interactive measurements based on majorization relations. This constraint is, in fact, an analogue of the universal uncertainty principles for conventional quantum measurement (i.e., Shmuel Friedland, Vlad Gheorghiu, and Gilad Gour, *Universal Uncertainty Relations*, Phys. Rev. Lett. **111**, 230401 (2013)). While more abstract, it is technically a more general result than the entropic uncertainty relations we presented. We see now that this was a major oversight, as the majorization conditions were key to understanding the generality of our results.

In response, we have made very notable revisions. In particular

1. The framework section of the manuscript has been almost completely rewritten. Instead of focusing primarily on the special case of causal inference, it now devotes significant time to introduce general interactive measurements in a more mathematically rigorous manner. We believe that in doing so, readers will no longer need to refer to supplemental material to understand an interactive measurement fully and can thus fully absorb the main results of our manuscript without referring to supplemental material.

2. The subsequent section on uncertainty relations has also undergone a significant rewrite. The section now includes one of our original research results that were previously only in supplementary materials (Lemma 1 of main text), documenting general constraints of the measurement statistics of two interactive measurements based on majorization. Discussions regarding how these relate to an infinite family of uncertainty relations corresponding to most general uncertainty quantifiers (of

which Shannon entropy is a special case) are also included. These changes help illustrate to readers the most general uncertainty relations we've derived without needing to refer to supplemental material.

We feel that both additions have significantly improved the manuscript and again thank the Referee for his/her suggestions.

Referee Point B.9

> 6) Some (not particularly important) typos I found: Page 1 Column 1 "Could such a fundamental uncertainty principle also exist when multiple preceding interventions (see Fig. 1)?" is not a full sentence.

Reply: This is the same with Referee Point A. 4 (page 2 of this reply). In the revised manuscript, we have changed "Could such a fundamental uncertainty principle also exist when multiple preceding interventions" into "Could uncertainty principles also fundamentally constrain such interactive measurements?".

Referee Point B.10

> Page 1 Column 1 "We explore these questions by deriving the uncertainty principle for interactive measurements. These principles...". Change from singular to plural 'principle', leaving the meaning unclear.

Reply: In the revised manuscript, we have changed "We explore these questions by deriving the uncertainty principle for interactive measurements. These principles then pinpoint when two interactive measurements are non-compatible – and quantify the necessary trade-offs in the certainty of their measurement outcomes" into "Here, we explore these questions by deriving a universal uncertainty principle that constrains the joint measurement probabilities of interactive measurements. This principle then pinpoints when two interactive measurements are non-compatible – and quantifies the necessary trade-offs in the certainty of their measurement outcomes.".

Referee Point B.11

> Page 1 Column 1 "Our results... encompasses". Subject/verb disagreement.

Reply: In the revised manuscript, we have changed "Our results make no assumptions on the number of interventions or the causal structure of processes we probe, and encompasses previous uncertainty relations for states and channels" into "Our results make no assumptions on the number of interventions or the causal structure of processes we probe, and encompass previous uncertainty relations for states and channels as special cases".

Referee Point B.12

> Page 4 Column 1 "they saturates" subject/verb disagreement

Reply: In the revised manuscript, we have changed "including cases where they saturates the lower bound of 2" into "including cases where they saturate the lower bound of 2".

Referee Point B.13

Page 3 Column 2 "a specific parametrized quantum circuits". It seems circuit should be singular.

Reply: In the revised manuscript, we have changed "Consider the application of this uncertainty to a specific parameterized quantum circuits $\Phi_{\alpha,\beta}$" into "Consider the application of this uncertainty to a specific parameterized quantum circuit $\Phi_{\alpha,\beta}$".

Referee Point B.14

Page 4 Column 1 "entropic uncertainty relations that governs" subject/verb disagreement

Reply: In the revised manuscript, we have changed "we derive entropic uncertainty relations that governs all interactive measurements" into "we derive entropic uncertainty relations that govern all interactive measurements".

Referee Point B.15

Page 4 Column 1 "In context of" should be "In the context of"

Reply: In the revised manuscript, we have changed 'In context of causal inference" into "In the context of causal inference".

Referee Point B.16

Page 4 Column 2 "merge is other settings" should be "merge in"

Reply: In the revised manuscript, we have changed "Interactive measurements through repeated interventions also merge is other settings" into "Interactive measurements through repeated interventions also merge in other settings".