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Quantum process discrimination with restricted strategies
by Kenji Nakahira

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Quantum process discrimination with restricted strategies

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(Dated: April 28, 2021)

The discrimination of quantum processes, including quantum states, channels, and superchannels, is a fundamental topic in quantum information theory. It is often of interest to analyze the optimal performance that can be achieved when discrimination strategies are restricted to a given subset of all strategies allowed by quantum mechanics. In this Letter, we show that the task of finding the maximum success probability for discriminating any quantum processes with any restricted strategy can always be formulated as a convex optimization problem whose Lagrange dual problem exhibits zero duality gap. We also derive a necessary and sufficient condition for an optimal restricted strategy to be optimal within the set of all strategies. As an application of this result, it is shown that adaptive strategies are not necessary for optimal discrimination if a problem has a certain symmetry. Moreover, we show that the optimal performance of each restricted process discrimination problem can be written in terms of a certain robustness measure. This finding has the potential to provide a deeper insight into the discrimination performance of various restricted strategies.

Quantum processes are fundamental building blocks of quantum information theory. The tasks of discriminating between quantum processes are of crucial importance in quantum communication, quantum metrology, quantum cryptography, etc. In many situations, it is reasonable to assume that the available discrimination strategies (also known as quantum testers) are restricted to a certain subset of all possible testers in quantum mechanics. For example, in practical situations, we are usually concerned only with discrimination strategies that are readily implementable with current technology. Another example is a setting where discrimination is performed by two or more parties whose communication is limited. In such settings, one may naturally ask (i) how the performance of an optimal restricted tester can be evaluated and (ii) for what kind of ensembles of quantum processes a restricted tester can be optimal within the set of all testers, i.e., globally optimal. To answer these questions, different individual problems of distinguishing quantum states [1–5], measurements [6–9], and channels[10–19] have been investigated.

It is known that if all quantum testers are allowed, then the problem of finding the maximum success probability of guessing which process was applied can be formalized as a semidefinite programming problem, and its Lagrange dual problem has zero duality gap [20]. Many discrimination problems of quantum states, measurements, and channels have been addressed through the analysis of their dual problems [7, 16, 19–31]. However, in a general case where the allowed testers are restricted, the problem cannot be formalized as a semidefinite programming problem.

In this paper, we provide a general method to analyze quantum process discrimination problems in which discrimination testers are restricted to given types of testers. We show that the task of finding the maximum success probability for discriminating any quantum processes with any restricted tester can be formulated as a convex optimization problem and that its Lagrange dual problem has zero duality gap. The dual problem is often easier to solve analytically or numerically than the original problem. Our approach can deal with process discrimination problems in both cases with and without the restriction of testers within a common framework, which makes it easy to compare their optimal values. This allows us to derive a necessary and sufficient condition for an optimal restricted tester to be globally optimal. Note that we use the quantum mechanical notation for convenience, but since our method essentially relies only on convex analysis, our techniques are applicable to a general operational probabilistic theory (including a theory that does not obey the no-restriction hypothesis [32]).

The robustness of a resource, which is a topic closely related to discrimination problems, has been recently extensively investigated. It is known that the robustness of a process can be seen as a measure of its advantage over all resource-free processes in some discrimination tasks [33–40]. Conversely, we show that the optimal performance of any restricted process discrimination problem is characterized by a certain robustness measure.

Quantum process discrimination — We first review quantum process discrimination problems where all possible testers are allowed. Let $\mathbb{C}$ and $\mathbb{R}_+$ denote, respectively, the sets of all complex and nonnegative real numbers. Also, let Her_V, Pos_V, and Den_V be, respectively, the sets of all Hermitian, positive semidefinite, and density matrices on a system V. We here address the problem of discriminating M processes $\hat{\mathcal{E}}_1, \dots, \hat{\mathcal{E}}_M$ with maximum success probability, where each $\hat{\mathcal{E}}_m$ consists of T time steps, which is expressed by the combination of T channels $\hat{\Lambda}_m^{(1)}, \dots, \hat{\Lambda}_m^{(T)}$ (see Fig. 1). $\hat{\mathcal{E}}$ has internal systems $W'_1, \dots, W'_{T-1}$. Such a process is called a quantum comb [41] (also called a quantum supermap or quantum strategy [42]). States, channels, and superchannels are special cases of quantum processes. For example, the T-shot discrimination problem of channels $\hat{\Lambda}_1, \dots, \hat{\Lambda}_M$ is a special case of $\hat{\Lambda}_m^{(t)} = \hat{\Lambda}_m$ $(\forall t)$ and $W'_1 = \cdots = W'_{T-1} = \mathbb{C}$. For simplicity, we restrict the discussion to the case $T = 2$, unless otherwise stated. Let $\tilde{V} \coloneqq W_2 \otimes V_2 \otimes W_1 \otimes V_1$. A process $\hat{\mathcal{E}}_m$ is uniquely specified by its Choi-Jamiołkowski representation [41],

$$\mathcal{E}_m \coloneqq (\hat{\mathcal{E}}_m \otimes \mathbb{1}_{V_2 \otimes V_1})(|I_{V_2 \otimes V_1}\rangle\!\rangle\langle\!\langle I_{V_2 \otimes V_1}|) \in \mathsf{Pos}_{\tilde{V}},$$

where $|I_V\rangle\!\rangle \coloneqq \sum_n |n\rangle|n\rangle \in V \otimes V$ (the order of the systems is often not taken into account). To discriminate between

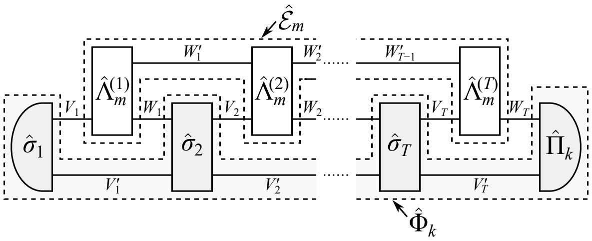

FIG. 1. General protocol of quantum process discrimination.

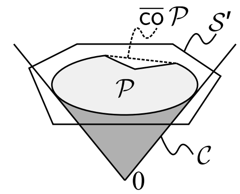

FIG. 2. Schematic diagram of the closed convex hull of $\mathcal{P}$, which is the intersection of a closed convex cone C and a convex set $\mathcal{S}' := \{\Phi : \sum_{m=1}^{M} \Phi_m \in \mathcal{S}\}$.

given processes, we first prepare a bipartite system $V_1 \otimes V_1'$ in an initial state $\hat{\sigma}_1$. One part V_1 is sent through the channel $\hat{\Lambda}_m^{(1)}$, followed by a channel $\hat{\sigma}_2$. After that, we send the system V_2 through the channel $\hat{\Lambda}_m^{(2)}$ and perform a measurement $\{\hat{\Pi}_k\}_{k=1}^{M}$ on the system $W_2 \otimes V_2'$. Such a sequence of processes $\hat{\sigma}_1, \hat{\sigma}_2, \{\hat{\Pi}_k\}_{k=1}^{M}$ allows us to represent any discrimination strategy, i.e., a tester, which includes an entanglement-assisted tester and an adaptive tester. For each tester $\{\hat{\Phi}_k\}_{k=1}^{M}$, $\hat{\Phi}_k$ is uniquely specified by

$$\Phi_k := (\hat{\Phi}_k \otimes \mathbb{1}_{V_2 \otimes V_1})^\dagger (|I_{V_2 \otimes V_1}\rangle\!\rangle\langle\!\langle I_{V_2 \otimes V_1}|) \in \mathsf{Pos}_{\tilde{V}}.$$

When there is no confusion, we simply refer to these representations $\mathcal{E}_m$ and $\{\Phi_k\}$ as a process (or comb) and tester, respectively. The probability that a tester $\Phi := \{\Phi_m\}$ gives the outcome k for the process $\mathcal{E}_m$ is given by

$$\langle \Phi_k, \mathcal{E}_m \rangle := \mathrm{Tr}(\Phi_k \mathcal{E}_m) = \hat{\Pi}_k \circ \hat{\Lambda}_m^{(2)} \circ \hat{\sigma}_2 \circ \hat{\Lambda}_m^{(1)} \circ \hat{\sigma}_1,$$

where $\circ$ denotes the map composition. The task of finding the maximum success probability for discriminating the given quantum processes $\{p_m, \mathcal{E}_m\}_{m=1}^{M}$ (where p_m is the prior probability of $\mathcal{E}_m$) can be formulated by as an optimization problem, namely [20]

$$\begin{aligned} &\text{maximize} \quad P(\Phi) := \sum_{m=1}^{M} p_m \langle \Phi_m, \mathcal{E}_m \rangle \\ &\text{subject to} \quad \Phi \in \mathcal{P}_{\mathrm{G}}. \end{aligned} \tag{P$_\mathrm{G}$}$$

Restricted discrimination — We now consider the situation that the allowed testers are restricted to a nonempty subset $\mathcal{P}$ of $\mathcal{P}_{\mathrm{G}}$; in this case, the problem is formulated as

$$\begin{aligned} &\text{maximize} \quad P(\Phi) \\ &\text{subject to} \quad \Phi \in \mathcal{P}. \end{aligned} \tag{P}$$

Let us interpret each tester as a vector in the real vector space $\mathsf{Her}_{\tilde{V}}^{M}$. This means that one can work with linear combinations of testers $\Phi^{(1)}, \Phi^{(2)}, \dots$; a tester that applies $\Phi^{(i)}$ with probability q_i is represented as $\sum_i q_i \Phi^{(i)}$. Then, one can easily see that the optimal value of Problem (P) remains the same if the feasible set $\mathcal{P}$ is replaced by its closed convex hull $\overline{\text{co}}\,\mathcal{P}$ in the space $\mathsf{Her}_{\tilde{V}}^{M}$. Indeed, an optimal solution, $\Phi^\star \in \overline{\text{co}}\,\mathcal{P}$, to Problem (P) with $\mathcal{P}$ relaxed to $\overline{\text{co}}\,\mathcal{P}$ can be represented as a probabilistic mixture of $\Phi^{(1)}, \Phi^{(2)}, \dots \in \overline{\mathcal{P}}$, i.e., $\Phi^\star = \sum_i \nu_i \Phi^{(i)}$ for some probability distribution $\{\nu_i\}_i$[43]. Since $P(\Phi^\star) \le P[\Phi^{(i)}]$ holds for some i, $\Phi^{(i)} \in \overline{\mathcal{P}}$ must be an optimal solution to the relaxed problem. Thus, Problem (P), whose objective function is convex by construction, is transformed into a convex optimization problem by relaxing $\mathcal{P}$ to $\overline{\text{co}}\,\mathcal{P}$. However, this relaxed problem is often very difficult to solve directly. We became aware that the set of all testers $\mathcal{P}_{\mathrm{G}}$ can be written as

$$\mathcal{P}_{\mathrm{G}} = \left\{ \Phi \in C_{\mathrm{G}} : \sum_{m=1}^{M} \Phi_m \in \mathcal{S}_{\mathrm{G}} \right\},$$

where $C_{\mathrm{G}} := \mathsf{Pos}_{\tilde{V}}^{M}$ and $\mathcal{S}_{\mathrm{G}}$ is the set of all combs in $\mathsf{Pos}_{\tilde{V}}^{M}$, which means that $\mathcal{P}_{\mathrm{G}}$ is fully characterized by the closed convex cone C_{G} and the closed convex set $\mathcal{S}_{\mathrm{G}}$ [44]. Similarly, for any feasible set $\mathcal{P}$, we can choose a closed convex cone C and a closed convex set $\mathcal{S}$ such that (see Fig. 2)

$$\overline{\text{co}}\,\mathcal{P} = \left\{ \Phi \in C : \sum_{m=1}^{M} \Phi_m \in \mathcal{S} \right\}, \quad C \subseteq C_{\mathrm{G}}, \quad \mathcal{S} \subseteq \mathcal{S}_{\mathrm{G}}. \tag{1}$$

Such C and $\mathcal{S}$ always exist [45].

We obtain the following statement [proved in Sec. II of the Supplemental Material (SM)]:

Theorem 1 Let us arbitrarily choose a closed convex cone C and a closed convex set $\mathcal{S}$ satisfying Eq. (1); then, the optimal value of Problem (P) coincides with that of the following Lagrange dual problem:

$$\begin{aligned} &\text{minimize} \quad D_{\mathcal{S}}(\chi) := \max_{\varphi \in \mathcal{S}} \langle \varphi, \chi \rangle \\ &\text{subject to} \quad \chi \in \mathcal{D}_C, \end{aligned} \tag{D}$$

where

$$\mathcal{D}_C := \left\{ \chi \in \mathsf{Her}_{\tilde{V}} : \sum_{m=1}^{M} \langle \Phi_m, \chi - p_m \mathcal{E}_m \rangle \ge 0 \ (\forall \Phi \in C) \right\}.$$

In several problems, Problem (D) provides an efficient way to find the optimal value of Problem (P). Note that the difficulty of solving Problem (D) depends on the choice of C and $\mathcal{S}$.

Unfortunately, for at least some interesting problems, our method does not seem to provide a dual problem that is easy to solve. In such a case, if we can choose appropriate $C \subseteq C_{\mathrm{G}}$ and $\mathcal{S} \subseteq \mathcal{S}_{\mathrm{G}}$ satisfying, instead of Eq. (1),

$$\overline{\text{co}}\,\mathcal{P} \subset \left\{ \Phi \in C : \sum_{m=1}^{M} \Phi_m \in \mathcal{S} \right\} =: \mathcal{P}_{\mathrm{relax}}, \tag{2}$$

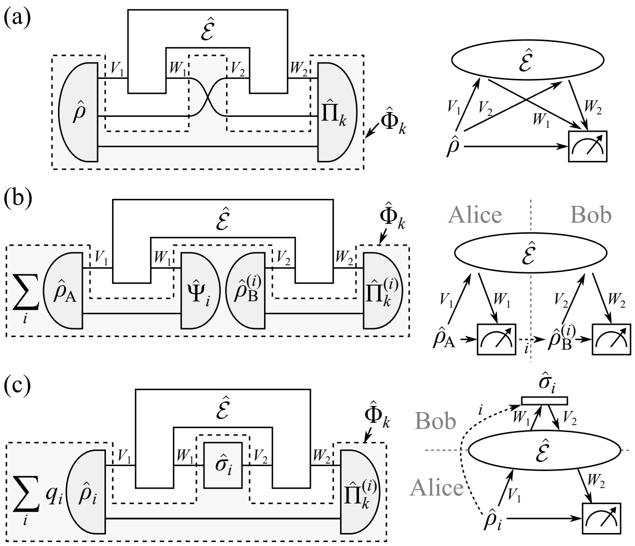

FIG. 3. Examples of three types of restricted testers. The left and right figures, respectively, show the diagrammatic representations of testers and their schematic diagrams. (a) A nonadaptive tester, which consists of a state $\hat{\rho}$ and a measurement $\{\hat{\Pi}_k\}_k$, is applied to a given process $\hat{\mathcal{E}}$. (b) A tester is performed by two parties, Alice and Bob, in which only one-way classical communication from Alice to Bob is allowed. Such a tester consists of two sequentially connected single-shot testers. Alice prepares a state $\hat{\rho}_{\rm A}$, performs a measurement $\{\hat{\Psi}_i\}_i$, and communicates her outcome i to Bob. Then, Bob prepares a state $\hat{\rho}_{\rm B}^{(i)}$ and performs a measurement $\{\hat{\Pi}_k^{(i)}\}_k$. (c) A tester is performed by Alice and Bob, in which only one-way classical communication from Alice to Bob is allowed. Alice prepares a state $\hat{\rho}_i$ with a probability q_i and sends its one part to the process $\hat{\mathcal{E}}$. She also sends i to Bob, who performs a channel $\hat{\sigma}_i$. Alice finally performs a measurement $\{\hat{\Pi}_k^{(i)}\}_k$.

then we easily obtain the optimal value of Problem (D). This optimal value coincides with that of Problem (P) where the feasible set is relaxed from $\mathcal{P}$ to $\mathcal{P}_{\rm relax}$, and thus is an upper bound on that of Problem (P).

Examples — We demonstrate the utility of Theorem 1 in three different situations (see Sec. III of the SM for some additional details).

The first example is the restriction to nonadaptive testers [see Fig. 3(a)]. In this case, it follows that Eq. (1) holds with

$$\begin{aligned} C &:= C_{\rm G}, \\ \mathcal{S} &:= \{\times_{W_1,V_2}(I_{W_2\otimes W_1} \otimes \rho') : \rho' \in \mathsf{Den}_{V_2\otimes V_1}\}, \end{aligned} \tag{3}$$

where $\times_{W,V}$ is the process that swaps two systems W and V. Substituting Eq. (3) into Problem (D) yields the following dual problem

$$\begin{aligned} &\text{minimize} \quad \max_{\rho' \in \mathsf{Den}_{V_2\otimes V_1}} \langle \times_{W_1,V_2}(I_{W_2\otimes W_1} \otimes \rho'), \chi \rangle \\ &\text{subject to} \quad \chi \in \mathcal{D}_{C_{\rm G}} = \{\chi \in \mathsf{Pos}_{\tilde{V}} : \chi \ge p_m \mathcal{E}_m \ (\forall m)\}. \end{aligned} \tag{4}$$

Note that although this problem is also formulated as the task of discriminating $\{\times_{V_2,W_1}(\mathcal{E}_m)\}_m$ in a single-shot scenario, the expression of Problem (4) is useful for verifying the global optimality of nonadaptive testers, which will be discussed later.

The next example concerns the feasible set described in Fig. 3(b), in which the feasible set is described by Eq. (1) with

$$C := \left\{ \left\{ \sum_i B_m^{(i)} \otimes A_i \right\}_m : A_i \in \mathsf{Pos}_{W_1\otimes V_1},\ \{B_m^{(i)}\}_m \in \mathsf{Test}_{W_2,V_2} \right\}$$

and $\mathcal{S} = \mathcal{S}_{\rm G}$, where Test_{W_2,V_2} is a tester with output system V_2 and input system W_2. In this situation, Problem (D) is rewritable as

$$\begin{aligned} &\text{minimize} \quad D_{\mathcal{S}_{\rm G}}(\chi) \\ &\text{subject to} \quad \mathrm{Tr}_{W_2\otimes V_2}\left[\sum_{m=1}^{M} B_m(\chi - p_m\mathcal{E}_m)\right] \ge 0 \\ &\qquad\qquad (\forall \{B_m\} \in \mathsf{Test}_{W_2,V_2}) \end{aligned} \tag{5}$$

with $\chi \in \mathsf{Her}_{\tilde{V}}$. Bipartite state discrimination problems using sequential measurements [46, 47] can be considered as a special case of this problem with $V_1 = V_2 = \mathbb{C}$. Problem (D) is often easier to solve than Problem (P); one of the main reasons is that, in the latter problem, we do not know how many outcomes an optimal tester $\{A_i\}$ has. A numerical example for solving Problem (5) is given in Sec. III of the SM.

The last example is described in Fig. 3(c). In this situation, it is hard to find C and $\mathcal{S}$ such that Eq. (1) holds and Problem (D) is easy to solve. Instead, we can verify that Eq. (2) with

$$C := \left\{ \left\{ \sum_i \sigma_i \otimes A_m^{(i)} \right\}_m : \sigma_i \in \mathsf{Chn}_{V_2,W_1},\ A_m^{(i)} \in \mathsf{Pos}_{W_2\otimes V_1} \right\}$$

and $\mathcal{S} := \mathcal{S}_{\rm G}$ holds, where Chn_{V_2,W_1} is the set of the Choi-Jamiołkowski representations of all channels from W_1 to V_2. This allows Problem (D) to be rewritten in this situation as

$$\begin{aligned} &\text{minimize} \quad D_{\mathcal{S}_{\rm G}}(\chi) \\ &\text{subject to} \quad \mathrm{Tr}_{V_2\otimes W_1}[\sigma(\chi - p_m\mathcal{E}_m)] \ge 0 \\ &\qquad\qquad (\forall 1 \le m \le M,\ \sigma \in \mathsf{Chn}_{V_2,W_1}) \end{aligned}$$

with $\chi \in \mathsf{Her}_{\tilde{V}}$. The optimal value of Problem (P) is upper bounded by that of this problem.

Global optimality — Given a feasible set $\mathcal{P}$, we now ask the question whether there exists a feasible tester that is globally optimal. For ease of discussion, assume that $\mathcal{P}$ is closed. We can derive a necessary and sufficient optimality condition by considering Problem (D) with $\mathcal{P} = \mathcal{P}_{\rm G}$ (i.e., $C = C_{\rm G}$ and $\mathcal{S} = \mathcal{S}_{\rm G}$), which is written as

$$\begin{aligned} &\text{minimize} \quad D_{\mathcal{S}_{\rm G}}(\chi) \\ &\text{subject to} \quad \chi \in \mathcal{D}_{C_{\rm G}}. \end{aligned} \tag{D$_{\rm G}$}$$

Theorem 1 guarantees that Problem (D$_{\rm G}$) has the same optimal value as Problem (P$_{\rm G}$). The task is to obtain a necessary and sufficient condition for the optimal values of Problems (D) and (D$_{\rm G}$) to be the same. To this end, we have the following statement (proved in Sec. IV of the SM):

Proposition 2 Given a closed set $\mathcal{P}$, let us arbitrarily choose a closed convex cone C and a closed convex set $\mathcal{S}$ satisfying

Eq. (1). Then, there exists a tester in $\mathcal{P}$ that is globally optimal if and only if there exists an optimal solution to Problem (D) such that it is a feasible solution to Problem ($\mathrm{D_G}$) and is proportional to some quantum comb.

Note that, in some individual cases, Theorem 1 can also be used to get another necessary and sufficient optimality condition (see Sec. V of the SM).

As an example of the applicability of Proposition 2, we again turn to Problem (4). It follows that since $C = C_{\mathrm{G}}$ holds for this example, a nonadaptive tester can be globally optimal if and only if there exists an optimal solution to Problem (4) that is proportional to some quantum comb. In particular, in the case where processes have a certain type of symmetry, we can easily note the existence of such an optimal solution. For instance, let us consider the case $W'_1 = \mathbb{C}$. Assume that, for each $t = 1, 2$, there exists a group $\mathcal{G}^{(t)}$ having two projective unitary or anti-unitary representations $g \mapsto U_g^{(t)}$ and $g \mapsto \tilde{U}_g^{(t)}$ [$g \in \mathcal{G}^{(t)}$] the latter of which is irreducible and that

$$\hat{\Lambda}_m^{(t)}[\tilde{U}_g^{(t)} \rho \tilde{U}_g^{(t)\dagger}] = U_g^{(t)} \hat{\Lambda}_m^{(t)}(\rho) U_g^{(t)\dagger}, \qquad \forall \rho \in \mathsf{Den}_{V_1}$$

holds for each $1 \le m \le M$ and $g \in \mathcal{G}^{(t)}$. Then, we can show via Proposition 2 that a nonadaptive tester can be globally optimal. Teleportation covariant channels[48, 49] and unital qubit channels are typical examples satisfying this assumption (see Sec. VI of the SM for details).

Relationship with robustnesses — In resource theory, robustness has been used as a measure of the resourcefulness of a quantum process, such as a state, measurement, or channel. For a given closed convex set $\mathcal{F}$, called a free set, and a closed convex cone $\mathcal{K}$ of $\mathsf{Her}_{\tilde{V}}$, the robustness of a process $\mathcal{E} \in \mathsf{Pos}_{\tilde{V}}$ against $\mathcal{K}$ can be defined as [50, 51]

$$R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E}) := \inf \left\{ \lambda \in \mathbb{R}_+ : \frac{\mathcal{E} + \lambda \mathcal{E}'}{1 + \lambda} \in \mathcal{F},\ \mathcal{E}' \in \mathcal{K} \right\}.$$

$R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E})$ can be intuitively interpreted as the minimal amount, λ, of mixing with a process, $\mathcal{E}' \in \mathcal{K}$, necessary in order for the mixed and renormalized process, $(\mathcal{E} + \lambda\mathcal{E}')/(1+\lambda)$, to be in $\mathcal{F}$. As already mentioned in the introduction, the robustness of $\mathcal{E}$ can be interpreted as a measure of the advantage of $\mathcal{E}$ over all the processes in $\mathcal{F}$ in some discrimination tasks (see Sec. VII of the SM). Conversely, we show that the optimal value of Problem (P) is characterized by a robustness measure.

For the problem of discriminating quantum combs $\{\hat{\mathcal{E}}_m\}_{m=1}^M$ with prior probabilities $\{p_m\}_{m=1}^M$, let us suppose that a party, Alice, chooses a state $|m\rangle\langle m|$ with the probability p_m, where $\{|m\rangle\}$ is the standard basis of a classical system W_{A}, and sends the corresponding comb $\hat{\mathcal{E}}_m$ to another party, Bob. The Choi-Jamiołkowski representation of the process shared by Alice and Bob is expressed as

$$\mathcal{E}^{\mathrm{ex}} := \sum_{m=1}^{M} p_m |m\rangle\langle m| \otimes \mathcal{E}_m \in \mathsf{Pos}_{W_{\mathrm{A}} \otimes \tilde{V}}. \tag{6}$$

Bob tries to infer which state Alice has. When he uses a tester $\{\Phi_m\}_m$, the success probability is written as $\sum_{m=1}^M \langle |m\rangle\langle m| \otimes \Phi_m, \mathcal{E}^{\mathrm{ex}} \rangle = P(\Phi)$. Let

$$\mathcal{K} := \left\{ Y \in \mathsf{Her}_{W_{\mathrm{A}} \otimes \tilde{V}} : \sum_{m=1}^{M} \langle |m\rangle\langle m| \otimes \Phi_m, Y \rangle \ge 0 \ (\forall \Phi \in C) \right\},$$
$$\mathcal{F} := \{ I_{W_{\mathrm{A}}} \otimes \chi' : \chi' \in \mathsf{Her}_{\tilde{V}},\ D_{\mathcal{S}}(\chi') \le 1/M \};$$

then, we can easily prove that the optimal value of Problem (P) is equal to $[1 + R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E}^{\mathrm{ex}})]/M$ (see Sec. VIII of the SM). If $\mathcal{E}^{\mathrm{ex}}$ belongs to the free set $\mathcal{F}$, then $p_1 = \cdots = p_M = 1/M$ and $\mathcal{E}_1 = \cdots = \mathcal{E}_M$ must hold, which implies that $\mathcal{F}$ can intuitively be thought of as the set of all $\mathcal{E}^{\mathrm{ex}}$ corresponding to trivial process discrimination problems. This robustness measure indicates how far $\mathcal{E}^{\mathrm{ex}}$ is from $\mathcal{F}$. This interpretation has the potential to provide a deeper insight into optimal discrimination of quantum processes with restricted testers.

Conclusions — We have presented a general approach for solving quantum process discrimination problems with restricted testers based on convex programming. Our analysis indicates that a dual problem exhibiting zero duality gap is obtained regardless of the set of all restricted testers. A necessary and sufficient condition for an optimal restricted tester to be globally optimal is also derived. We have shown that the optimal value of each process discrimination problem can be written in terms of a robustness measure. In comparison to previous theoretical works, our approach would allow a unified analysis for a large class of process discrimination problems in which the allowed testers are restricted. A meaningful direction for subsequent work would be to apply our approach to practical fields, such as quantum communication and quantum metrology.

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[44] In the special case of $T = 1$ and $V_1 = \mathbb{C}$, $\mathcal{P}_{\mathrm{G}}$ is the set of all quantum measurements of a system W_1. One can easily check $\mathcal{P}_{\mathrm{G}} = \{\{\Pi_m\}_{m=1}^M \in \mathcal{C}_{\mathrm{G}} : \sum_{m=1}^M \Pi_m \in \mathcal{S}_{\mathrm{G}}\}$ with $\mathcal{C}_{\mathrm{G}} := \mathsf{Pos}_{W_1}^M$ and $\mathcal{S}_{\mathrm{G}} := \{I_{W_1}\}$.
[45] A trivial choice is $\mathcal{C} := \{t\Phi : t \in \mathbb{R}_+, \Phi \in \overline{\mathsf{co}}\,\mathcal{P}\}$ and $\mathcal{S} := \{\sum_{m=1}^M \Phi_m : \Phi \in \overline{\mathsf{co}}\,\mathcal{P}\}$.
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