Report on the PhD thesis by Gergely Bunth: On quantum Rényi divergences

The topic of the thesis is centered around quantum extensions of Rényi divergences, which can be defined as certain quantities on pairs (or larger sets) of quantum states that coincide with the classical (multivariate) Rényi divergence in mutually commuting cases. Of course, such extensions cannot be arbitrary and the quantities must satisfy certain properties to be potentially useful. To construct such extensions, essentially two different approaches are taken in the present thesis.

The first approach, described in Section 3, is inspired by a variational formula that relates the classical Rényi divergence to the weighted (left) radius of two probability functions with respect to the relative entropy (Kullback-Leibler divergence). This formula can be extended to multivariate cases and coincides with the Hellinger transform if the weights are given by a probability distribution. In this part of the thesis, an analogous formula is used to define barycentric Rényi divergences for pairs or families of quantum states (or positive semidefinite operators). Since there exist many versions of the quantum relative entropy, a different version may be used, even for each state, so that the obtained class of quantities is quite large. These quantities inherit some useful properties of the defining relative entropies, such as the scaling property or the data processing inequality. It is also proved that the previously known quantum versions of the Rényi divergences, that is, the $\alpha - z$-divergences, measured or maximal Rényi divergences, are not contained in this class.

Another approach, described in Section 4, is axiomatic, based on the fact that the elements in the spectrum of certain preordered semirings constructed from relative (sub)majorization of families of probability vectors (or vectors of nonnegative numbers) are obtained from Rényi divergences. The importance of the spectrum lies in the fact that its elements characterize asymptotic or catalytic majorization in preordered semirings. This idea is applied to ordered semirings consisting of pairs of continuous functions $\rho : X \to B(\mathcal{H})_{>0}$ and $\sigma : Y \to B(\mathcal{H})_{>0}$ from compact Hausdorff spaces to positive definite operators on a finite dimensional Hilbert space and with preorder given by relative submajorization with respect to completely positive trace nonincreasing maps. In the case when all elements in the range of σ commute, the spectrum is fully characterized in terms of sandwiched Rényi divergences with $\alpha > 1$, which leads to a characterization of asymptotic submajorization and a sufficient condition for catalytic submajorization. In the general case, some further elements of the spectrum are constructed by composing the sandwiched Rényi divergence with a quantum version of the geometric mean, thus obtaining a new set of necessary conditions.

Applications discussed in the thesis include characterization of the error exponents in the strong converse regime in the composite hypothesis testing, equivariant relative submajorization and a description of asymptotic transformations of by thermal processes, hypothesis testing with group symmetry or with reference frames and approximate joint transformations.

The thesis is based on three papers published in to quality journals. Two of the papers are written with the coauthor Péter Vrana, the third paper including also the thesis supervisor is currently in press. Given the established importance of the quantum Rényi divergences in quantum information theory, including quite recent results on the use of related quantities in the characterization of (asymptotic, approximate or catalytic) convertibility between sets of classical or quantum states, the topic of the thesis is timely and important. The results of the thesis are based on original ideas and are rather strong, especially in Sec. 4. The approach of the first part gives a very reasonable extension to the case of more than two variables (hence a quantum version of the classical Hellinger transform) and even in the two-variable case, the proposed barycentric Rényi divergences introduce a potentially rich supply of promising quantities to be explored further. In any case, the thesis brings an important contribution to the field, with many possible further applications and follow up research directions. Putting all together, the thesis clearly demonstrates the capability of Gergely Bunth for research in his field and I am happy to recommend him to be awarded the PhD title.