The AfHom and boolean functions

Anna Jenčová

June 24, 2024

AfH is the subcategory of Af generated from first order objects by taking tensor products and duals. Any first order object X has the form $X = (V_X, \{\tilde{a}_X\}^\sim)$ where $\tilde{a}_X \in V_X^*$. We have

$$S_X = V_X, \qquad L_X = \{\tilde{a}_X\}^\perp.$$

Let us pick any $a_X \in \{\tilde{a}_X\}^\sim$ and let us denote

$$L_{X,0} := \mathbb{R}a_X, \quad L_{X,1} := \{\tilde{a}_X\}^\perp, \quad L_{X^*,0} := \mathbb{R}\tilde{a}_X, \quad L_{X^*,1} := \{a_X\}^\perp.$$

We have the decompositions

$$V_X = L_{X,0} \oplus L_{X,1}, \qquad V_X^* = L_{X^*,0} \oplus L_{X^*,1} \tag{1}$$

and

$$L_{X,0}^\perp = L_{X^*,1}, \quad L_{X,1}^\perp = L_{X^*,0}. \tag{2}$$

Let Y be an object of AfH. Ten Y is constructed from a set of distinct first order object $X_1, \dots, X_n$. In this case, we will write $Y \sim [X_1, \dots, X_n]$. Since FinVect is compact, $(V \otimes W)^* = V^* \otimes W^*$, so that the vector space of Y has the form

$$V_Y = V_{i_1} \otimes \cdots \otimes V_{i_n},$$

where V_i is either V_{X_i} or $V_{X_i}^*$, according to whether X_i was subjected to taking duals an even or odd number of times. The indices such that the first case is true will be called the outputs and the subset of outputs in $[n]$ will be denoted by O, or O_Y, when we need to specify the object. The set $I = I_Y := [n] \setminus O_Y$ is the set of inputs. The following statement shows the reason for this terminology. The proof of Proposition 1 will be given later below.

Proposition 1. *Let $Y \sim [X_1, \dots, X_n]$. Let us denote*

$$X_I := \bigotimes_{i \in I} X_i, \qquad X_O := \bigotimes_{i \in O} X_i.$$

Then there is a permutation $\sigma \in S_n$ such that

$$X_I^* \otimes X_O \xrightarrow{\sigma} Y \xrightarrow{\sigma^{-1}} [X_I, X_O].$$

We introduce the following notations:

$$V_i := V_{X_i}, \quad L_{i,u} := L_{X_i,u}, \quad \tilde{L}_{i,u} = L_{X_i^*,u}, \; u \in \{0,1\}, \qquad i \in O$$

and

$$V_i := V_{X_i}^*, \quad L_{i,u} := L_{X_i^*,u}, \quad \tilde{L}_{i,u} = L_{X_i,u}, \; u \in \{0,1\}, \qquad i \in I.$$

Further,

$$a_i := a_{X_i}, \quad \tilde{a}_i := \tilde{a}_{X_i}, \quad i \in O, \qquad a_i := \tilde{a}_{X_i}, \quad \tilde{a}_i := a_{X_i}, \quad i \in I.$$

We will also denote for $s \in \{0,1\}^n$,

$$L_s := L_{i_1,s_{i_1}} \otimes \cdots \otimes L_{i_n,s_{i_n}}, \qquad \tilde{L}_s := \tilde{L}_{i_1,s_{i_1}} \otimes \cdots \otimes \tilde{L}_{i_n,s_{i_n}}.$$

Notice that the subspaces L_s, $s \in \{0,1\}^n$ form an independent decomposition of the space V_Y, so that they generate a distributive sublattice in the latice of subspaces of V_Y. Similarly, $\tilde{L}_s$, $s \in \{0,1\}^n$ form an independent decomposition of V_Y^*.

We next describe the affine subspace A_Y. We will need the following easy lemmas.

Lemma 1. *We have*

$$a_Y := a_{i_1} \otimes \cdots \otimes a_{i_n} \in A_Y, \quad \tilde{a}_Y := \tilde{a}_{i_1} \otimes \cdots \otimes \tilde{a}_{i_n} \in \tilde{A}_Y.$$

Proof. Easy.

□

Lemma 2. *For any $s \in \{0,1\}^n$, we have*

$$L_s^\perp = \bigoplus_{\substack{t \in \{0,1\}^n \\ t \neq s}} \tilde{L}_t.$$

Proof. Using (1) and (2), we get

$$\begin{aligned}
\left(L_{i_1,s_{i_1}} \otimes \cdots \otimes L_{i_n,s_{i_n}}\right)^\perp &= \bigvee_j \left(V_{i_1}^* \otimes \cdots \otimes V_{i_{j-1}}^* \otimes \tilde{L}_{i_j,1-s_{i_j}} \otimes V_{i_{j+1}}^* \otimes \cdots \otimes V_{i_n}^*\right) \\
&= \bigvee_j \left(\bigoplus_{\substack{t \in \{0,1\}^n \\ t_{i_j} \neq s_{i_j}}} \tilde{L}_{i_1,t_{i_1}} \otimes \cdots \otimes \tilde{L}_{i_n,t_{i_n}} \right) \\
&= \bigoplus_{\substack{t \in \{0,1\}^n \\ t \neq s}} \left(\tilde{L}_{i_1,t_{i_1}} \otimes \cdots \otimes \tilde{L}_{i_n,t_{i_n}}\right).
\end{aligned}$$

□

We next show how the objects are related to some boolean functions, in fact, these functions are related to types of objects in AfH, rather than objects themselves.

Theorem 1. *For any object in* AfH, *there is a function $f = f_Y : \{0,1\}^n \to \{0,1\}$ and a permutation $i_1, \ldots, i_n$ of elements in $[n]$, such that*

$$S_Y = \bigoplus_{s \in \{0,1\}^n} f(s) L_{i_1,s_{i_1}} \otimes \cdots \otimes L_{i_n,s_{i_n}}.$$

Proof. We will proceed by induction on n. For $n = 1$, the assertion is easily seen to be true, since in this case, Y is itself first order and

$$S_Y = V_Y = L_{1,0} \oplus L_{1,1} = f(0)L_{1,0} \oplus f(1)L_{1,1},$$

here $f : \{0,1\} \to \{0,1\}$ is the constant 1. Assume now that the assertion is true for all $m < n$. By construction, Y is either the tensor product of two other objects in AfH or Y is the dual of such a product. Let us assume the first case. Then there is a permutation $i_1, \dots, i_n$ of $[n]$ and $0 < m < n$ such that $Y = Y_1 \otimes Y_2$, with

$$Y_1 \sim [X_{i_1}, \dots, X_{i_m}], \qquad Y_2 \sim [X_{i_{m+1}}, \dots, X_{i_n}].$$

By the assumption, there are functions $f_1 : \{0,1\}^m \to \{0,1\}$ and $f_2 : \{0,1\}^{n-m} \to \{0,1\}$, and permutations $k_1, \dots, k_m$ of $[m]$, $l_1, \dots, l_{n-m}$ of $[n-m]$ such that

$$S_Y = S_{Y_1} \otimes S_{Y_2} = \bigoplus_{\substack{s \in \{0,1\}^m \\ t \in \{0,1\}^{n-m}}} f_1(s) f_2(t) L_{i_{k_1}, s_{k_1}} \otimes \dots \otimes L_{i_{k_m}, s_{k_m}} \otimes L_{i_{m+l_1}, t_{l_1}} \otimes \dots \otimes L_{i_{m+l_{n-m}}, t_{l_{n-m}}}$$

Since $\{0,1\}^n \simeq \{0,1\}^m \times \{0,1\}^{n-m}$, we get the assertion, with $f(s,t) = f_1(s)f_2(t)$ and the permutation $i_{k_1}, \dots, i_{k_m}, i_{m+l_1}, \dots, i_{m+l_{n-m}}$ of $[n]$.

To finish the proof, it is now enough to observe that if the assertion holds for Y then it also holds for Y^*. So assume that Y has the required form. Let a_Y and $\tilde{a}_Y$ be as in Lemma 1, then $\tilde{a}_Y \in \tilde{A}_Y$, so that $L_Y = S_Y \cap \{\tilde{a}_Y\}^\perp$. As before, we use the notation

$$L_s := L_{i_1, s_{i_1}} \otimes \dots \otimes L_{i_n, s_{i_n}}, \qquad \tilde{L}_s := \tilde{L}_{i_1, s_{i_1}} \otimes \dots \otimes \tilde{L}_{i_n, s_{i_n}}.$$

respecting the permutation $i_1, \dots, i_n$ of Y, so that

$$S_Y = \bigoplus_{s \in \{0,1\}^n} f(s) L_s.$$

It is easily seen that for any $s \in \{0,1\}^n$, $L_s \subseteq \{\tilde{a}_Y\}^\perp$ if and only if $s_i = 1$ for at least some $i \in [n]$, that is, $s \neq 00\dots0$. Hence

$$L_Y = \bigoplus_{\substack{s \in \{0,1\}^n \\ s \neq 0}} f(s) L_s.$$

We then have using Lemma 2 and the fact that $\tilde{L}_t$, $t \in \{0,1\}^n$ form an independent decomposition of V_Y^*,

$$S_{Y^*} = L_Y^\perp = \left(\bigoplus_{\substack{s \in \{0,1\}^n \\ s \neq 0}} f(s) L_s \right)^\perp = \bigwedge_{\substack{s \in \{0,1\}^n \\ s \neq 0, f(s)=1}} L_s^\perp = \bigwedge_{\substack{s \in \{0,1\}^n \\ s \neq 0, f(s)=1}} \left(\bigoplus_{\substack{t \in \{0,1\}^n \\ t \neq s}} \tilde{L}_t \right)$$
$$= \bigoplus_{t \in \{0,1\}^n} \left(\bigwedge_{\substack{s \in \{0,1\}^n \\ s \neq 0, f(s)=1}} (1 - \chi_s(t)) \tilde{L}_t \right) = \bigoplus_{t \in \{0,1\}^n} f^*(t) \tilde{L}_t.$$

Here $\chi_s : \{0,1\}^n \to \{0,1\}$ is the characteristic function of s and $f^* : \{0,1\}^n \to \{0,1\}$ is given as

$$f^*(t) := \bigwedge_{\substack{s \in \{0,1\}^n \\ s \neq 0, f(s)=1}} (1 - \chi_s(t)) = \begin{cases} 1 & \text{if } t = 0 \\ 1 - f(t) & \text{if } t \neq 0. \end{cases}$$

Note also that the change L_s to $\tilde{L}_s$ corresponds precisely to swapping the sets of inputs and outputs, which is what happens after taking the dual. This finishes the proof.

□

The above theorem shows that any element $Y \sim [X_1, \dots, X_n]$ is up to a permutation determined by a unique boolean function $f = f_Y : \{0,1\}^n \to \{0,1\}$, uniqueness follows from the fact that $L_s, s \in \{0,1\}^n$ form an independent decomposition of V_Y. It is rather obvious that not all boolean functions give rise to an object of AfH, for example, for $Y \sim [X]$ we only have $Y = X$ and $Y = X^*$, which corresponds to constant 1 and the negation, respectively. In general, one can see from the above proof that we necessarily have

$$f \in \mathcal{F}_n = \{g : \{0,1\}^n \to \{0,1\},\ g(0) = 1\}.$$

Pick any $f \in \mathcal{F}_n$ and any permutation $i_1, \dots, i_n$. Keeping the above notations, in particular the input-output decomposition $[n] = I \cup O$, let

$$S_f := \bigoplus_{s \in \{0,1\}^n} f(s) L_s, \qquad A_f := S_f \cap \{\tilde{a}\}^{\sim}.$$

Then A_f is a proper affinne subspace in $V := \otimes_j V_{i_j}$, this folows from the fact that $f(0) = 1$, so that S contains the subspace $\mathbb{R}a$. Then $Y_f := (\otimes_j V_{i_j}, A_f)$ defines an object in Af such that $a \in A_f$ and $\tilde{a} \in \tilde{A}_f$. Such objects might not belong to AfH in general. With the pointwise ordering, $\mathcal{F}_n$ is a distributive lattice, with the smallest element χ_0 and largest element 1. It is easy to see that for $f, g \in \mathcal{F}_n$ and some corresponding objects Y_f, Y_g, we have $f \leq g$ if and only if there is some permutation $\sigma \in S_n$ such that $Y_f \xrightarrow{\sigma} Y_g$. In particular, since $\chi_0 \leq f \leq 1$ for all $f \in \mathcal{F}_n$, there is some permutation σ such that

$$Y_{\min} \xrightarrow{\sigma} Y_f \xrightarrow{\sigma^{-1}} Y_{\max},$$

where

$$Y_{\min} := (V_1 \otimes \cdots \otimes V_n, \{a_1 \otimes \cdots \otimes a_n\}), \qquad Y_{\max} := (V_1 \otimes \cdots \otimes V_n, \{\tilde{a}_1 \otimes \cdots \otimes \tilde{a}_n\}^{\sim}).$$

If Y_g is an object such that

$$Y_{\min} \xrightarrow{\rho} Y_g \xrightarrow{\rho^{-1}} Y_{\max},$$

for a permutation ρ, then we may define an object corresponding to $f \wedge g$ as the pullback of the two arrows $f \xrightarrow{\sigma^{-1}} Y_{\max}$ and $g \xrightarrow{\rho^{-1}} Y_{\max}$, similarly, $Y_{f \vee g}$ can be found as a pushout.

1 Boolean functions

We will work with certain boolean functions, which are defined as functions from binary strings to $\{0,1\}$. We now list some basic notations used below.

For $s \in \{0,1\}$, we denote $\bar{s} := 1 - s$. For binary strings of fixed length n, that is, elements of $\{0,1\}^n$, we will denote by 0_n or just 0 the string $00\ldots0$ and by e^i the string such that $e^i_j = \delta_{i,j}$. For $m, n \in \mathbb{N}$, the concatenation of strings $s \in \{0,1\}^m$ and $t \in \{0,1\}^n$ will be denoted by st, that is,

$$st = s_1 \ldots s_m t_1 \ldots t_n \in \{0,1\}^{m+n}.$$

For a string $x \in \{0,1\}^n$ and any set of indices $\{i_1, \ldots, i_k\} \subseteq [n]$, we will denote by $x^{i_1 \ldots i_k}$ the string in $\{0,1\}^{n-k}$ obtained from x by removing $x_{i_1}, \ldots, x_{i_k}$.

For any permutation $\sigma \in S_n$, we will denote by the same symbol the obvious action on $\{0,1\}^n$, that is

$$\sigma(s_1 \ldots s_n) = s_{\sigma^{-1}(1)} \ldots s_{\sigma^{-1}(n)}.$$

We start by looking at the set

$$\mathcal{F}_n := \{f : \{0,1\}^n \to \{0,1\},\ f(0) = 1\}.$$

With the poitwise ordering, $\mathcal{F}_n$ is a (finite) distributive lattice, with top element the constant 1 function and the bottom element $p_n := \chi_0$. We may also define complementation in $\mathcal{F}_n$ as

$$f^* := 1 - f + p_n.$$

It can be easily checked that with these structures $\mathcal{F}_n$ is a boolean algebra.

We now introduce some more operations in $\mathcal{F}_n$. For $f \in \mathcal{F}_n$ and any permutation $\sigma \in S_n$, we see that $f \circ \sigma \in \mathcal{F}_n$. For $f \in \mathcal{F}_m$ and $g \in \mathcal{F}_n$, we define the function $f \otimes g \in \mathcal{F}_{m+n}$ as

$$(f \otimes g)(st) = f(s)g(t).$$

As it is, this tensor product is not symmetric, but there is a permutation $\sigma \in S_{m+n}$ such that $(g \otimes f) = (f \otimes g) \circ \sigma$ for any $f \in \mathcal{F}_m$ and $g \in \mathcal{F}_n$.

Lemma 3. *For $f \in \mathcal{F}_m$, $g \in \mathcal{F}_n$, we have*

$$f \otimes g \le (f^* \otimes g^*)^*.$$

Equality holds if and only if f and g are either both the to top or both the bottom elements in $\mathcal{F}_m$ resp. $\mathcal{F}_n$.

Proof. The inequality is easily checked, since $(f \otimes g)(st)$ can be 1 only if $f(s) = g(t) = 1$. If both s and t are the zero strings, then $st = 0_{m+n}$ and both sides are equal to 1. Otherwise, the condition $f(s) = g(t) = 1$ implies that $(f^* \otimes g^*)(st) = 0$, which implies that the right hand side must be 1. If f and g are both constant 1, then $(1 \otimes 1)^* = 1^* = p_{n+m} = 1^* \otimes 1^*$, in the case when both f and g are the minimal elements equality follows by duality. Finally, asume the equality holds and that $f \neq 1$, so that there is some s such that $f(s) = 0$. But then $s \neq 0$ and for any t,

$$0 = (f \otimes g)(st) = (f^* \otimes g^*)^*(st) = 1 - f^*(s)g^*(t) + p_{m+n}(st) = 1 - g^*(t),$$

which implies that $g(t) = 0$ for all $t \neq 0$, that is, $g = p_n$. By the same argument, $f = p_m$ if $g \neq 1$, which implies that either $f = 1$ and $g = 1$, or $f = p_m$ and $g = p_n$. $\square$

We now show an important example.

Example 1. Let $S \subseteq [n]$ be any subset. We will denote

$$p_S(s) := \prod_{i \in S} \bar{s}_i.$$

It is clear that $p_S \in \mathcal{F}_n$, $p_\emptyset = 1$, $p_{[n]} = \chi_0 = p_n$. The folowing properies are also easy to see for $S, T \subseteq [n]$:

(i) if $S \subseteq T$, then $p_T \leq p_S$,

(ii) $p_S \wedge p_T = p_S p_T = p_{S \cup T}$,

(iii) $p_S \vee p_T = p_S + p_T - p_{S \cup T}$.

(iv) let $S \subseteq [m]$ and $T \subseteq [n]$, then

$$p_S \otimes p_T = p_{S \cup (m+T)}.$$

We will use the above functions to introduce a convenient parametrization to $\mathcal{F}_n$. For this, we first include $\mathcal{F}_n$ into a larger set

$$\mathcal{F}_n \subseteq \{f : \{0,1\}^n \to \mathbb{R}\} =: \mathcal{V},$$

which is a 2^n-dimensional real vector space. It becomes a Hilbert space with respect to the inner product

$$\langle f, g \rangle = \sum_{s \in \{0,1\}^n} f(s) g(s).$$

Lemma 4. *The set* $\{p_S,\ S \subseteq [n]\}$ *is a basis of* $\mathcal{V}$. *Any* $f \in \mathcal{V}$ *can be uniquely written as*

$$f = \sum_{S \subseteq [n]} \hat{f}_S p_S,$$

where the coefficients $\hat{f}_S \in \mathbb{R}$ *are obtained as*

$$\hat{f}_S = \sum_{\substack{x \in \{0,1\}^n \\ x_i = 1, \forall i \in S^c}} (-1)^{\sum_{i \in S} x_i} f(x).$$

Proof. For $T \subseteq [n]$, let us define the function $p_T^\perp$ as

$$p_T^\perp(x) := (-1)^{\sum_{i \in T} x_i} \prod_{i \in T^c} x_i.$$

We prove that for $S, T \subseteq [n]$,

$$\langle p_S, p_T^\perp \rangle = \delta_{S,T},$$

which shows that $\{p_S,\ S \subseteq [n]\}$ is a basis and $\{p_T^\perp,\ T \subseteq [n]\}$ is the dual basis. We compute

$$\langle p_S, p_T^\perp \rangle = \sum_x p_S(x) p_T^\perp(x) = \sum_x (-1)^{\sum_{i \in T} x_i} \prod_{i \in S} \bar{x}_i \prod_{j \in T^c} x_j.$$

This expression can be nonzero only if $S \cap T^c = \emptyset$, that is, $S \subseteq T$. In this case, the last sum is equal to

$$\sum_{\substack{x\in\{0,1\}^n \\ x_i=0,\forall i\in S \\ x_i=1,\forall i\in T^c}} (-1)^{\sum_{j\in T\setminus S} x_j} = \begin{cases} 0 & \text{if } S \subsetneq T \\ 1 & \text{if } S = T \end{cases}$$

It is now clear that the coefficients

$$\hat{f}_S = \langle\, f, p_S^{\perp}\,\rangle$$

have the given form.

□

It may be useful to visualise the lattice $\mathcal{L}_n = \{S \subseteq [n]\}$ as a hypercube, and the coefficients of f as labels for its vertices. The fact that the function f has values in $\{0,1\}$ means that for a string $x \in \{0,1\}^n$ such that $x_j = 1$ if and only if $j \in T$, we must have

$$f(x) = \sum_{\substack{S\subseteq[n] \\ S\cap T=\emptyset}} \hat{f}_S \in \{0,1\},$$

that is, the sum of labels $\hat{f}_S$ over any face of the hypercube $\mathcal{L}_n$ containing the vertex $\emptyset$ must be 0 or 1. In particular, $\hat{f}_\emptyset = f(11\dots1) \in \{0,1\}$, which restricts the values of $\hat{f}_{\{i\}} \in \{0,1,-1\}$, etc. The fact that $f \in \mathcal{F}_n$ means that in addition,

$$f(0) = \sum_{S\subseteq[n]} \hat{f}_S = 1.$$

1.1 The type functions

We are now interested in the subset $\mathcal{F}_n^H$ in $\mathcal{F}_n$ of elements that can be obtained from the constant function 1 on $\{0,1\}$ by taking complements, tensor products and precomposing with permutations. This gives the following description.

Definition 1. The set $\mathcal{F}_n^H$ is the smallest subset in $\mathcal{F}_n$ such that:

1. it is invariant under permutations: if $f \in \mathcal{F}_n^H$, then $f \circ \sigma \in \mathcal{F}_n^H$ for any permutation $\sigma \in S_n$,
2. it is invariant under complementation: if $f \in \mathcal{F}_n^H$ then $f^* \in \mathcal{F}_n^H$,
3. if $f \in \mathcal{F}_n^H$ and $g \in \mathcal{F}_m^H$, then $f \otimes g \in \mathcal{F}_{n+m}^H$,
4. $\mathcal{F}_1^H = \{1, \chi_0\} = \mathcal{F}_1$.

Elements of $\mathcal{F}_n^H$ are called type functions of order n.

We can see by the above definiton that a type function can be constructed as

$$f(x_1 \dots x_n) = (((1 \otimes ..1 \otimes 1^* \otimes 1^*)^* \dots) \otimes (1 \otimes ..1 \otimes 1^* \dots.)..)^* ..) \circ \sigma(x_1 \dots x_n)$$

where each 1 or 1^* acts on $\{0,1\}$ and σ is some permutation. This gives a decomposition of the set of indices as follows. We say that $i \in [n]$ is an output index for f if x_i in the above expression is subjected to complementation an even number of times. The subset of such indices in $[n]$ will be denoted by O_f, or just O. We also denote $I = I_f := [n] \setminus O_f$, the elements of I_f will be called input indices for f.

Example 2. Note that $p_S \in \mathcal{F}_n^H$ for any $S \subseteq [n]$. Indeed, we have $p_S = f_1 \otimes \cdots \otimes f_n$, where $f_i = \chi_0 = 1^*$ if $i \in S$ and $f_i = 1$ otherwise. It is clear that in this case $I = O$ is the set of input indices.

Proposition 2. *Let $f \in \mathcal{F}_n^H$ and let $O = O_f$, $I = I_f$. Then*

$$p_I \leq f \leq p_O^*.$$

Proof. This is obviously true for $n = 1$. In this case, $\mathcal{F}_1^H = \mathcal{F}_1 = \{1, 1^* = p_{[1]}\}$. If $f = 1$, then $O = [1]$, $I = \emptyset$ and

$$p_I = p_\emptyset = 1 = p_O^*,$$

the case $f = 1^*$ is obtained by taking complements. Assume that the assertion holds for $m < n$. Let $f \in \mathcal{F}_n^H$ and assume that $f = g \otimes h$ for some $g \in \mathcal{F}_m^H$, $h \in \mathcal{F}_{n-m}^H$. By the assumption,

$$p_{I_g} \otimes p_{I_h} \leq g \otimes h \leq p_{O_g}^* \otimes p_{O_h}^* \leq (p_{O_1} \otimes p_{O_2})^*,$$

the last inequality follows from Lemma 3. It is now enough to notice that $O_f = O_g \cup (m + O_h)$, $I_f = I_g \cup (m + I_h)$, so that $p_{O_f} = p_{O_g} \otimes p_{O_h}$ and similarly for p_{I_f}.

Assume that the inequality holds for $f \in \mathcal{F}_n^H$, we will show that it is preserved by permutations and complements. Indeed, let σ be any permutation of the indices, then clearly

$$p_I \circ \sigma \leq f \circ \sigma \leq p_O^* \circ \sigma.$$

It is enough to note that the outputs of $f \circ \sigma$ are $\sigma^{-1}(O)$, similarly for the inputs, and $p_{\sigma^{-1}(S)} = p_S \circ \sigma$ for any $S \subseteq [n]$. Finally, we have by duality

$$p_O \leq f^* \leq p_I^*,$$

and taking complements exchanges inputs and outputs: $O_{f^*} = I_f$, $I_{f^*} = O_f$.

□

The next result shows that there is a direct way to obtain the input-output decomposition from values of f.

Proposition 3. *For $f \in \mathcal{F}_n^H$, $i \in O_f$ if and only if $f(e^i) = 1$.*

In other words, the outputs of f are precisely those indices for which the sum of the coefficients $\hat{f}_S$ over the facet $\{S \in \mathcal{L}_n,\ i \notin S\}$ is 1.

Proof. Let $i \in O_f$, then by Proposition 2, $p_{I_f}(e^i) = 1 \leq f(e^i)$, so that $f(e^i) = 1$. Conversely, if $f(e^i) = 1$, then by the other inequality in Proposition 2, $p_{O_f}(e^i) = 0$, whence $i \in O_f$.

□

Example 3. Let $n = 2$, then it can be checked that

$$\mathcal{F}_2^H = \{p_2, p_{\{1\}}, p_{\{2\}}, p_{\{1\}}^*, p_{\{2\}}^*, 1\}.$$

Explicitly, the functions are given as

$$x \mapsto \qquad \bar{x}_1\bar{x}_2, \quad \bar{x}_1, \quad \bar{x}_2, \quad 1 - \bar{x}_1 + \bar{x}_1\bar{x}_2, \quad 1 - \bar{x}_2 + \bar{x}_1\bar{x}_2, \quad 1.$$

It can be seen that $\mathcal{F}_n$ has 2^{2^n-1} elements, so that $\mathcal{F}_2$ has 8 elements in total. The two of them that are not type functions are

$$g(x) = 1 - \bar{x}_1 - \bar{x}_2 + 2\bar{x}_1 x_2, \qquad g^*(x) = \bar{x}_1 + \bar{x}_2 - \bar{x}_1\bar{x}_2.$$

This can be checked directly from Propositions 2 and 3. Indeed, if $g \in \mathcal{F}_n^H 2$, we would have $O_g = \emptyset$, so that $p_2 \leq g \leq p_\emptyset^* = p_2$, which is obviously not the case. Clearly, also the complement $g^* \notin \mathcal{F}_2^H$. Notice also that $g^* = p_{\{1\}} \vee p_{\{2\}}$, so that $\mathcal{F}_n^H$ is a subposet in $\mathcal{F}_n$ but not a lattice.

1.1.1 The poset $\mathcal{P}_f$

For $f \in \mathcal{F}_n^H$, let $\mathcal{P}_f$ be the subposet in $\mathcal{L}_n$ of elements such that $\hat{f}_S \neq 0$. Can we characterize such subposets? (we cannot so far.) Can we extract some information about some 'causal structure' on the indices?

We next give some properties such a poset must have. We say that a poset is graded of rank k if every maximal chain in P has the same length equal to k (recall that the length of a chain is defined as number of its elements -1). Equivalently, there is a unique rank function $\rho : P \to \{0, 1, \ldots, k\}$ such that $\rho(s) = 0$ if s is a minimal element of P and $\rho(t) = \rho(s) + 1$ if t covers s.

Proposition 4. *Let $f \in \mathcal{F}_n^H$, then $\mathcal{P}_f$ is a graded poset with even rank $k \leq n$. If ρ is the rank function, then we have*

$$\hat{f}_S = (-1)^{\rho(S)}, \qquad S \in \mathcal{P}_f.$$

Then rank of $\mathcal{P}_f$ will be denoted by $r(f)$ and called the rank of f.

Proof. For $n = 1$, we have $\mathcal{L}_1 = \{\emptyset, [1]\}$ and $\mathcal{F}_1^H = \{1, 1^*\}$. For $f = 1$, $\mathcal{P}_f = \{\emptyset\}$ is a singleton, which is clearly a graded poset, with rank $k = 0$ and trivial rank function ρ. We have

$$\hat{f}_\emptyset = 1 = (-1)^0 = (-1)^{\rho(\emptyset)}.$$

The proof for $f = 1^*$ is similar.

If the statement holds for f, then it holds also for $f \circ \sigma$ for any $\sigma \in S_n$, indeed, $\widehat{f \circ \sigma}_S = \hat{f}_{\sigma^{-1}}(S)$ so that $\mathcal{P}_f$ and $\mathcal{P}_{f \circ \sigma}$ are isomorphic. For f^*, note that

$$f^* = 1 - f + p_n = (1 - \hat{f}_\emptyset)1 - \sum_{\substack{S \in \mathcal{P}_f \\ \emptyset \neq S, [n] \neq S}} (-1)^{\rho(S)} + (1 - \hat{f}_{[n]})p_n. \tag{3}$$

If $\emptyset \in \mathcal{P}_f$, then $\emptyset$ is the least element of $\mathcal{P}_f$, so that $\rho(\emptyset) = 0$ and therefore $\hat{f}_\emptyset = (-1)^0 = 1$. Similarly, if $[n] \in \mathcal{P}_f$, then $[n]$ is the largest element in $\mathcal{P}_f$, hence it is the last element in any maximal chain. It follows that $\rho([n]) = k$ and hence $\hat{f}_{[n]} = (-1)^k = 1$. Therefore the equality (3) implies that $\mathcal{P}_{f^*}$ differs from $\mathcal{P}_f$ only in the bottom and top elements: $\emptyset \in \mathcal{P}_f$ iff $\emptyset \notin \mathcal{P}_{f^*}$ and $p_n \in \mathcal{P}_f$ iff $p_n \notin \mathcal{P}_{f^*}$. It follows that $\mathcal{P}_{f^*}$ is graded as well, with rank equal to $k - 2$, k or $k + 2$, which in any case is even. Furthermore, let ρ^* be the rank function of f^*, then this also implies that for all $S \in \mathcal{P}_f$, $S \notin \{\emptyset, [n]\}$, we have $\rho^*(S) = \rho(S) \pm 1$, according to whether $\emptyset$ was added or removed. The statement now follows from (3).

To finish the proof, assume that the statement is true for $m < n$ and let $f \in \mathcal{F}_n^H$. Then f is either a permutation of a product of some $f_1 \in \mathcal{F}_m^H$ and $f_2 \in \mathcal{F}_{n-m}^H$, or a dual of such an element. By the above discussion, it follows that we only need to prove that the statement holds for $f = f_1 \otimes f_2$. But in this case, by the induction assumption, $\mathcal{P}_{f_i}$ is graded with even rank k_i and rank function ρ_i. We also have

$$f = \sum_{S \subseteq [m], T \subseteq [m-n]} (\widehat{f_1})_S (\widehat{f_2})_T p_S p_T = \sum_{S \subseteq [m], T \subseteq [m-n]} (-1)^{\rho_1(S) + \rho_2(T)} p_{S \cup (m+T)}.$$

It follows that $\mathcal{P}_f = \mathcal{P}_{f_1} \times \mathcal{P}_{f_2}$ is the product of the two posets, which is a graded poset with rank $k = k_1 + k_2$ and rank function $\rho = \rho_1 + \rho_2$. This proves the statement.

□

Example 4 (Chains). Let $\mathcal{P} = \{S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_N\}$ be a chain in $\mathcal{L}_n$. Then $\mathcal{P}$ is graded with rank $N-1$ and rank function $\rho(S_i) = i-1$. Put

$$f = \sum_i (-1)^{i-1} p_{S_i}.$$

By Proposition 4, if $f \in \mathcal{F}_n$, then the rank of f must be even, so that N must be odd. We will show that the converse is true. We proceed by induction on N. For $N=1$, we have $f = p_{S_1} \in \mathcal{F}_n^H$. Assume that the statement holds for all odd numbers $M < N$ and let $\mathcal{P}$ be a chain as above. Then we have

$$f = p_{S_1} \otimes g \otimes 1_{[n]\setminus S_N}$$

where g is the function for the chain $\emptyset = S_1' \subsetneq S_2' \subsetneq \cdots \subsetneq S_N'$, with $S_i' := S_i \setminus S_1$. This shows that we may assume that the chain contains the least and the largest element in $\mathcal{L}_n$. But then

$$f = 1 + \sum_{j=2}^{N-1} (-1)^{j-1} p_{S_j} + p_n$$

and

$$f^* = 1 - f + p_n = \sum_{j=1}^{N-2} (-1)^{j-1} p_{T_j},$$

where $T_j := S_{j+1}$. By the induction assumption $f^* \in \mathcal{F}_n^H$, hence also $f = f^{**} \in \mathcal{F}_n^H$.

As we can see from Example 4, all elements in $\mathcal{F}_2^H$ are chains. This is also true for $n = 3$. Indeed, up to a permutation that does not change the chain structure, any $f \in \mathcal{F}_3^H$ is either a product of two elements $g \in \mathcal{F}_2^H$ and $h \in \mathcal{F}_1^H$, or the dual of such a product. Since g must be a chain and $|\mathcal{P}_h| = 1$, their product must be a chain as well. Taking the dual of a chain only adds/removes the least/largest elements, so the dual of a chain must be a chain as well.

1.1.2 Separating type functions

Let $\mathcal{P} \subseteq \mathcal{L}_n$. We say that $\mathcal{P}$ separates the points of $[n]$, or $\mathcal{P}$ is separating if for any $i, j \in [n]$ there is some $S \in \mathcal{P}$ that contains one of the points but not the other. Note that in this case there can be at most one $i \in [n]$ that is not contained in any $S \in \mathcal{P}$, so that $(\cup\mathcal{P})^c$ is at most a singleton. Similarly, there is at most one i contained in all $S \in \mathcal{P}$, so that $\cap\mathcal{P}$ is at most a singleton. If $(\cup\mathcal{P})^c = \emptyset$, we say that $\mathcal{P}$ is covering. We say that $\mathcal{P}$ is blabla if $\cap\mathcal{P} = \emptyset$. We say that $f \in \mathcal{F}_n$ is separating, covering or blabla if $\mathcal{P}_f$ has these properties.

It is easily seen that up to a permutation, any $f \in \mathcal{F}_n$ can be written as

$$f = p_{n_1} \otimes g \otimes 1_{[n_2]},$$

for some $g \in \mathcal{F}_m$ which is covering and blabla , and that $f \in \mathcal{F}_n^H$ if and only if $g \in \mathcal{F}_m^H$.

Lemma 5. *Let $f \in \mathcal{F}_n$, $g \in \mathcal{F}_m$. Then*

(i) $f \otimes g$ is covering if and only if both f and g are covering.

(ii) $f \otimes g$ is blabla if and only if both f and g are blabla.

(iii) $f \otimes g$ is separating if and only if both f and g are separating, at least one of them is covering and at least one of them is blabla .

Proof. (i) and (ii) are easy to see from the equality

$$\mathcal{P}_{f\otimes g} = \{S \cup (n+T),\ S \in \mathcal{P}_f,\ T \in \mathcal{P}_g\} \simeq \mathcal{P}_f \times \mathcal{P}_g.$$

Indeed, since $S \subseteq [n]$ and $n + T \subseteq [n, n+m]$, we see that

$$(\cup\mathcal{P}_{f\otimes g})^c = (\cup\mathcal{P}_f)^c \cup (n + (\cup\mathcal{P}_g)^c), \quad \cap\mathcal{P}_{f\otimes g} = (\cap\mathcal{P}_f) \cup (n + \cap\mathcal{P}_g) \tag{4}$$

For (iii), assume that $f \otimes g$ is separating and let $i, j \in [n]$. Then there are some $S' \in \mathcal{P}_f$ and $T \in \mathcal{P}_g$ such that $S \cup (n+T)$ separates i and j. But then clearly S must separate i and j, so that f is separating. Similarly, g must be separating as well. The fact that at least one of them must be covering/blabla is immediate from (4).

For the converse, let $i, j \in [n+m]$. If both $i, j \in [n]$, then they msut be separated by $\mathcal{P}_f$, hence there is some S that contains one of them but not the other. Then the same is true for any $S \cup (n+T)$, $T \in \mathcal{P}_g$. We can similarly deal with the case that $i, j \in [n, n+m]$. So assume that $i \in [n]$, $j \in [n, n+m]$. By the assumption, there must be some $S \in \mathcal{P}_f$ and $T \in \mathcal{P}_g$, such that $i \in S$ and $j \notin (n+T)$ (or $i \notin S$ and $j \in (n+T)$). But then $S \cup (n+T)$ separates i and j.

□

Our next goal is to show that any type function $f \in \mathcal{F}_n^H$ of rank k can be obtained from a separating type function $f' \in \mathcal{F}_m^H$ such that $\mathcal{P}_f \simeq \mathcal{P}_{f'}$.

Lemma 6. *Let $f \in \mathcal{F}_n$ and let $f' \in \mathcal{F}_{n+1}$ be such that*

$$f'(x_1 \dots x_{n+1}) = f(x_1 \dots x_{n-1}(x_n \vee x_{n+1})).$$

Then $\mathcal{P}_f \simeq \mathcal{P}_{f'}$ and $f \in \mathcal{F}_n^H$ if and only if $f' \in \mathcal{F}_{n+1}^H$.

Proof. It can be seen using Lemma 4 that for any $S \in \mathcal{P}_{f'}$, we have

$$\hat{f}'_S = \begin{cases} \hat{f}_S & \text{if } \{n, n+1\} \cap S = \emptyset, \\ \hat{f}_{S\setminus\{n+1\}} & \text{if } \{n, n+1\} \subset S, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\mathcal{P}_{f'}$ does not separate n and $n+1$, moreover, from this it is easily seen that $\mathcal{P}_f$ is isomorphic to $\mathcal{P}_{f'}$.

We will show the second assertion by induction. Let $n = 1$, so that $f'(x_1x_2) = f(x_1 \vee x_2)$. If $f \in \mathcal{F}_1 = \{1, 1^*\}$, then either $f' = 1$ or $f'(x_1x_2) = \overline{x_1 \vee x_2} = \bar{x}_1\bar{x}_2 = p_2$. Conversely assume $f' \in \mathcal{F}_2^H$. Since $f'(x_1x_2)$ has the same value for all strings different from 00, it must be either equal to 1 or p_2, in which case f is equal to 1 or p_1, repsectively.

Assume next that the statement holds for $m < n$ and let $f \in \mathcal{F}_n^H$. Assume first that $f = (g \otimes h) \circ \sigma$ for some $g \in \mathcal{F}_m^H$, $h \in \mathcal{F}_{n-m}^H$ and a permutation $\sigma \in \mathcal{S}_n$. Let $\sigma(n) > m$, then we also have $f' = (g \otimes h') \circ \tau$, where $h'(x_1 \dots x_{m+1}) = h(x_1 \dots x_{m-1}(x_m \vee x_{m+1}))$ and τ is a suitable permutaton in $\mathcal{S}_{n+1}$. By the assumption, $h' \in \mathcal{F}_{n+1-m}^H$ so that $f' \in \mathcal{F}_{n+1}^H$. The case $\sigma(n) \leq m$ is treated similarly, replacing h by g. Next, notice that since $p_{n+1}(x_1x_2 \dots x_{n+1}) = p_n(x_1 \dots x_{n-1}(x_n \vee x_{n+1}))$, we get

$$(f')^*(x_1x_2 \dots x_{n+1}) = f^*(x_1 \dots x_{n-1}(x_n \vee x_{n+1})), \tag{5}$$

which finishes the proof in the case that $f = ((g \otimes h) \circ \sigma)^*$.

Conversely, assume that $f' \in \mathcal{F}^H_{n+1}$. By (5), we see that it is enough to show the statement in the case $f' = (g' \otimes h') \circ \sigma$ for some $g' \in \mathcal{F}^H_m$, $h' \in \mathcal{F}^H_{n+1-m}$ and a permutation $\sigma \in \mathcal{S}_{n+1}$. Since $\mathcal{P}_{f'}$ does not separate n and $n+1$, any $S \in \mathcal{P}_{f'}$ either contains both or none of them. Hence if f' has the above product form, then we must have either both $\sigma(n), \sigma(n+1) \leq m$ or both strictly larger than m. Assume the latter, then (by applying a suitable permutation in $\mathcal{S}_{n+1-m}$ if necessary) we get

$$f(x_1 \dots (x_n \vee x_{n+1})) = g'(x_{\sigma^{-1}(1)} \dots x_{\sigma^{-1}(m)})h'(x_{\sigma^{-1}(m+1)} \dots x_{\sigma^{-1}(n-1)}x_n x_{n+1}).$$

It follows that $f = (g' \otimes h) \circ \tau$, where $\tau \in \mathcal{S}_n$ and $h \in \mathcal{F}^H_{n-m}$ is given as

$$h(x_1 \dots x_{n-m}) = h'(x_1 \dots x_{n-m} x_{n-m}).$$

By the induction assumption, $h \in \mathcal{F}^H_{n-m}$ so that $f \in \mathcal{F}^H_n$.

□

Lemma 7. *Let $f \in \mathcal{F}^H_n$ and let $i, j \in [n]$ be such that $\mathcal{P}_f$ does not separate i and j. Then there is a function $f_{ij} \in \mathcal{F}^H_{n-1}$ with $\mathcal{P}_f \simeq \mathcal{P}_{f_{ij}}$, such that*

$$f(x_1 \dots x_n) = f_{ij}(x^{ij}(x_i \vee x_j)).$$

Proof. Without loss of generality we may assume that $i = n-1$, $j = n$. Put

$$f_{(n-1)n}(x_1 \dots x_{n-1}) := f(x_1 \dots x_{n-1} x_{n-1}).$$

By the previous lemma it is enough to prove that

$$f(x_1 x_2 \dots x_n) = f_{(n-1)n}(x_1 \dots (x_{n-1} \vee x_n)) = f(x_1 \dots x_{n-2}(x_{n-1} \vee x_n)(x_{n-1} \vee x_n)).$$

Since $\mathcal{P}_f$ does not separate $n-1$ and n, each $S \in \mathcal{P}_f$ either contains both or none of them. In either case,

$$p_S(x_1 x_2 \dots x_n) = p_S(x_1 \dots (x_{n-1} \vee x_n)(x_{n-1} \vee x_n)).$$

This implies the statement.

□

Clearly, the function f_{ij} above is obtained from f by "glueing" the nonseparable indices into one index.

Proposition 5. *Let $f \in \mathcal{F}^H_n$, $r(f) = k$. Then there is a function $g \in \mathcal{F}^H_m$ such that $\mathcal{P}_g$ separates the points of $[m]$, $\mathcal{P}_g \simeq \mathcal{P}_f$ and there are subsets $I_1, \dots I_l$ of $[n]$ such that*

$$f(x_1 \dots x_n) = g(x^{\cup_j I_j}(\vee I_1) \dots (\vee I_l)).$$

Proof. We can construct the function g by repeated application of Lemma 7, glueing together nonseparable points until, after a finite number of steps, none are left, so that the resulting function g separates the points of m.

□

Example 5 (Separating chains). Clearly, a chain $\mathcal{P} = \{S_1 \subsetneq \cdots \subsetneq S_N\}$ is separating if and only if $S_i \setminus S_{i-1}$ is a singleton for $i = 2, \dots, N$ and both S_1 and S_N^c are at most singletons (that is, these might be also empty). We have seen that the correspnding function is in $\mathcal{F}^H_n$ if and only if $N = r(\mathcal{P}) + 1$ is odd. By the separating property, we must have $n - 1 \leq N \leq n + 1$, depending on whether S_1 or S_N^C are empty or not. It follows that $n-2 \leq r(\mathcal{P}) \leq n$. Since N must be odd, we see that the only possibilities are

$$
\begin{aligned}
&N = n-1 \text{ or } N = n+1 && \text{if } n \text{ is even} \\
&N = n && \text{if } n \text{ is odd.}
\end{aligned}
$$

For even n, put

$$\gamma_n(x) := \sum_{l=0}^{n} (-1)^l p_{[l]}(x) = 1 - \bar{x}_1 + \bar{x}_1\bar{x}_2 - \cdots + \bar{x}_1 \ldots \bar{x}_n.$$

Then γ_n is a separating chain in $\mathcal{F}_n^H$ and the corresponding sets of input and output indices are

$$I = \{2j, j = 1, \ldots n/2\}, \qquad O = \{2j-1,\ j = 1, \ldots n/2\}.$$

For any other separating chain $f \in \mathcal{F}_n^H$, there is some permutation $\sigma \in \mathcal{S}_n$ such that

$$f = (p_1 \otimes \gamma_{n-2} \otimes 1) \circ \sigma \quad \text{or} \quad f = \gamma_n \circ \sigma.$$

These two types are easily seen to be each others complement. Similarly, if n is odd, then any separating chain in $\mathcal{F}_n^H$ must either of the two comlementary forms

$$f = (p_1 \otimes \gamma_{n-1}) \circ \sigma \text{ or } \ (\gamma_{n-1} \otimes 1) \circ \sigma$$

for some permutation $\sigma \in \mathcal{S}_n$. From this and Example 4, we see that there are only two separating elements in $\mathcal{F}_2^H$: $\gamma_2 \circ \sigma$, $\sigma \in \mathcal{S}_2$ that are complementary. Similarly, there are basically two complementary forms of separating elements in $\mathcal{F}_3^H$: $(p_1 \otimes \gamma_2) \circ \sigma$ or $(\gamma_2 \otimes 1) \circ \sigma$, $\sigma \in \mathcal{S}_3$.

We say that $f \in \mathcal{F}_n$ is covering if $\cup \mathcal{P}_f = [n]$.

Lemma 8. *Let $f \in \mathcal{F}_n^H$, $g \in \mathcal{F}_m^H$. Then*

1. *$f \otimes g$ is covering if and only if f and g are both covering.*

2. *$f \otimes g$ is separating if and only if both f and g are separating and at least one of them is covering.*

1.1.3 Paths

Let $f \in \mathcal{F}_n^H$, assume that f separates points of $[n]$. We will study the structure of $\mathcal{P}_f$ in more detail.

For each $i \in [n]$, let $\mathcal{P}_{f,i} := \{S \in \mathcal{P}_f,\ i \in S\}$ and let $\mathcal{M}_{f,i}$ be the set of minimal elements in $\mathcal{P}_{f,i}$. Elements of i will be called minimal sets for i. Note that $\mathcal{P}_{f,i}$ can be empty, there is a unique index with this property, it will be called the free output. Another case is when $\mathcal{P}_{i,f} = \mathcal{M}_{f,i} = \{[n]\}$, there is again a unique index with this property, it will be called the first input of f. Note that taking complements turns the first input into the free output and vice versa.

Lemma 9. *(i) All minimal sets of i have the same rank $r_f(i)$, called the rank of i. If $\mathcal{M}_{f,i} = \emptyset$, we put $r_f(i) = r(f) + 1$.*

(ii) $i \in O_f$ if and only if $r_f(i)$ is odd.

Proof. We will proceed by induction. Both assertions are quite trivial for $n = 1$, assume the statements hold for $m < n$ and let $f \in \mathcal{F}_n^H$. Assume first that the assertions are true for f, then it is easily observed that they hold also for any permutation $f \circ \sigma$. Next, note that $\mathcal{P}_{f^*}$ differs from $\mathcal{P}_f$ only up to adding/removing the least and greatest elements $\emptyset$ and p_n. Let $\mathcal{M}_{f^*,i} = \{[n]\}$, then (i) is trivially satisfied and $r_{f^*}(i) = \rho_{f^*}([n]) = k = r(f^*)$ is even. Since $i \neq S$ for any $S \in \mathcal{P}_{f^*}$ other than $[n]$, we have

$$f^*(e^i) = \sum_{S \in \mathcal{P}_{f^*}, S \neq [n]} \hat{f}^*_S = 1 - p_n = 0,$$

so that $i \in I_{f^*}$ and (ii) holds. If $\mathcal{M}_{f^*,i} = \emptyset$, then by definition $r_{f^*}(i) = r(f^*) + 1$ is odd and clearly $f^*(e^i) = f^*(0) = 1$, so that $i \in O_{f^*}$. Assume now that $\mathcal{M}_{f^*,i}$ is nonempty and not equal to $\{[n]\}$. Then the same is true for $\mathcal{M}_{f,i}$, and by the assumption, $\rho_f(S) = r_f(i)$ for all elements $S \in \mathcal{M}_{f,i}$. Then $\rho_{f^*}(S) = \rho_f(S) \pm 1$, depending only on the fact whether $\emptyset \in \mathcal{P}_f$. It follows that $\rho_{f^*}(S)$ has the same value for all minimal sets for i, and this value is odd if and only if $r_f(i)$ is even. Since complementation exchanges inputs and outputs, the statements are proved for f^*.

Let now $f \in \mathcal{F}_n^H$ be arbitrary and let $i \in [n]$. As we have seen, the statement holds if i is the free output or the first input, so assume that $\emptyset \neq \mathcal{M}_{f,i} \neq \{[n]\}$. By the first part of the proof, it is enough to assume that $f = g \otimes h$ for some $g \in \mathcal{F}_m^H$ and $h \in \mathcal{F}_{n-m}^H$. Suppose without loss of generality that $i \in [m]$, then clearly $\mathcal{M}_{g,i} \neq \emptyset$ and $\mathcal{M}_{f,i}$ consists of $S \cup T$, with $S \in \mathcal{M}_{g,i}$ and T a minimal element in $\mathcal{P}_h$. Since $\rho_h(T) = 0$ for any minimal element T, we have

$$\rho_f(S \cup T) = \rho_g(S) + \rho_h(T) = r_g(i), \qquad \forall S \cup T \in \mathcal{M}_{f,i}.$$

The statement (ii) follows from the fact that $i \in O_f$ if and only if $i \in O_g$.

□