Supplemental Material for "Quantum process discrimination with restricted strategies"

Kenji Nakahira

Quantum Information Science Research Center, Quantum ICT Research Institute, Tamagawa University, Machida, Tokyo 194-8610, Japan

I. PRELIMINARIES

A. Notation

Let N_V be the dimension of a system V. $\mathbb{0}$ stands for a zero matrix. Let $\mathbb{C}$ and $\mathbb{R}_+$ be, respectively, the sets of all complex and nonnegative real numbers. Also, let Her_V, Pos_V, Den_V, $\mathsf{Den}^{\mathsf{P}}_V$, and Meas_V be, respectively, the sets of all Hermitian matrices, positive semidefinite matrices, states (i.e., density matrices), pure states, and measurements of a system V. Let I_V and $\mathbb{1}_V$ be, respectively, the identity matrix on V and the identity map on Her_V. We call a quantum operation, which corresponds to a completely positive map, a single-step process. Let $\mathsf{Pos}(V, W)$ and $\mathsf{Chn}(V, W)$ denote, respectively, the sets of all single-step processes and channels from a system V to a system W. In this manuscript, a one-dimensional system, which we call a trivial system, is identified with $\mathbb{C}$. Also, $\mathsf{Pos}(\mathbb{C}, V)$ and $\mathsf{Pos}(\mathbb{C}, \mathbb{C})$ are identified with Pos_V and $\mathbb{R}_+$, respectively. $H_1 \geq H_2$ with Hermitian matrices H_1 and H_2 denotes that $H_1 - H_2$ is positive semidefinite. Given a set $\mathcal{X}$ in a real Hilbert space, we denote its interior by $\mathrm{int}(\mathcal{X})$, its closure by $\overline{\mathcal{X}}$, its convex hull by $\mathsf{co}\,\mathcal{X}$, its (convex) conical hull by $\mathsf{coni}\,\mathcal{X}$, and its dual cone by $\mathcal{X}^*$. $\overline{\mathsf{co}\,\mathcal{X}}$ and $\overline{\mathsf{coni}\,\mathcal{X}}$ are, respectively, denoted by $\overline{\mathsf{co}}\,\mathcal{X}$ and $\overline{\mathsf{coni}}\,\mathcal{X}$. x^T denotes the transpose of a matrix x. Uni_V denotes the set of all unitary matrices on a system V. For a unitary matrix $U \in \mathsf{Uni}_V$, let Ad_U be the unitary channel defined as $\mathrm{Ad}_U(\rho) = U\rho U^\dagger$ $(\rho \in \mathsf{Pos}_V)$. Let $\tilde{V} := W_T \otimes V_T \otimes \cdots \otimes W_1 \otimes V_1$, where T is some positive integer.

As well as the notation in the main paper, we will denote the Choi-Jamiołkowski representations of processes as the same letter without the hat symbol, e.g., the Choi-Jamiołkowski representation of a process $\hat{\mathcal{E}}$ is denoted by $\mathcal{E}$. Note that the Choi-Jamiołkowski representation, ρ, of a state $\hat{\rho}$ is equal to $\hat{\rho}$ itself. In this paper, for each tester element $\hat{\Phi}_m$, Φ_m denotes the Choi-Jamiołkowski representation of the process $\hat{\Phi}^\dagger_m$, where † is the adjoint operator. $\mathsf{Comb}_{W_T,V_T,\dots,W_1,V_1}$ denotes the set of all $\tau \in \mathsf{Pos}_{W_T \otimes V_T \otimes \cdots \otimes W_1 \otimes V_1}$ such that there exists $\{\tau^{(t)} \in \mathsf{Pos}_{W_t \otimes V_t \otimes \cdots \otimes W_1 \otimes V_1}\}_{t=1}^{T-1}$ satisfying

$$\mathrm{Tr}_{W_t}\, \tau^{(t)} = I_{V_t} \otimes \tau^{(t-1)}, \quad \forall 1 \leq t \leq T,$$

where $\tau^{(0)} := 1$ and $\tau^{(T)} := \tau$. In particular, $\mathsf{Comb}_{\mathbb{C},W_T,V_T,\dots,W_1,V_1,\mathbb{C}}$ is denoted by $\mathsf{Comb}^*_{W_T,V_T,\dots,W_1,V_1}$, which is the set of all $\tau \in \mathsf{Pos}_{W_T \otimes V_T \otimes \cdots \otimes W_1 \otimes V_1}$ such that there exist $\tau^{(1)} \in \mathsf{Den}_{V_1}$ and $\{\tau^{(t)} \in \mathsf{Pos}_{V_t \otimes W_{t-1} \otimes V_{t-1} \otimes \cdots \otimes W_1 \otimes V_1}\}_{t=2}^{T}$ satisfying

$$\begin{aligned} \tau &= I_{W_T} \otimes \tau^{(T)}, \\ \mathrm{Tr}_{V_t}\, \tau^{(t)} &= I_{W_{t-1}} \otimes \tau^{(t-1)}, \quad \forall 2 \leq t \leq T. \end{aligned} \tag{S1}$$

We have

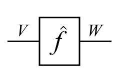
$$\langle \sigma, \tau \rangle = 1, \quad \forall \tau \in \mathsf{Comb}_{W_T,V_T,\dots,W_1,V_1},\ \sigma \in \mathsf{Comb}^*_{W_T,V_T,\dots,W_1,V_1}. \tag{S2}$$

Let $\mathcal{C}_\mathrm{G} := \mathsf{Pos}^M_{\tilde{V}}$ and $\mathcal{S}_\mathrm{G} := \mathsf{Comb}^*_{W_T,V_T,\dots,W_1,V_1}$. We refer to a feasible solution, χ, to Problem (D) as proportional to some quantum comb if χ is expressed in the form $\chi = \lambda\tilde{\chi}$, with $\lambda \in \mathbb{R}_+$ and $\tilde{\chi} \in \mathsf{Comb}_{W_T,V_T,\dots,W_1,V_1}$.

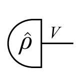
Figure and equation numbers without the prefix 'S' refer to those given in the main paper.

B. Diagrammatic representations

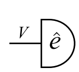
In this manuscript, we often use diagrammatic representations to provide an intuitive understanding. We here briefly review the diagrammatic representation of quantum processes. A single-step process $\hat{f} \in \mathsf{Pos}(V, W)$ is depicted by



The trivial system $\mathbb{C}$ is represented by 'no wire'. For example, $\hat{\rho} \in \mathsf{Pos}_V$ and $\hat{e} \in \mathsf{Pos}(V, \mathbb{C})$ are diagrammatically represented as

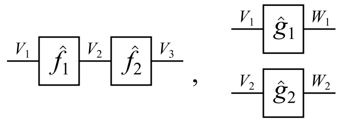


,

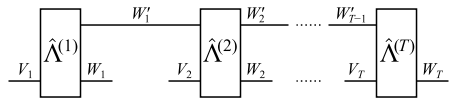


.

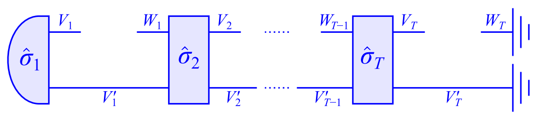
Single-step processes can be linked sequentially or in parallel. The sequential concatenation of $f_1 \in \mathsf{Pos}(V_1, V_2)$ and $f_2 \in \mathsf{Pos}(V_2, V_3)$ is a single-step process in $\mathsf{Pos}(V_1, V_3)$, denoted as $f_2 \circ f_1$. Also, the parallel concatenation of $g_1 \in \mathsf{Pos}(V_1, W_1)$ and $g_2 \in \mathsf{Pos}(V_2, W_2)$ is a single-step process in $\mathsf{Pos}(V_1 \otimes V_2, W_1 \otimes W_2)$, denoted as $g_1 \otimes g_2$. In diagrammatic terms, they are depicted as



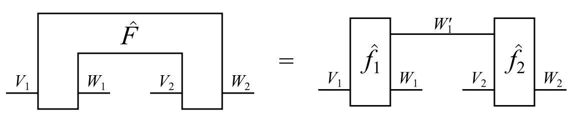
A quantum process is a concatenation of single-step processes; in particular, a quantum comb is a concatenation of channels. Each element of $\mathsf{Comb}_{W_T,V_T,\ldots,W_1,V_1}$ corresponds to a comb expressed in the form



where $\hat{\Lambda}^{(1)}, \ldots, \hat{\Lambda}^{(T)}$ are channels. Each element of $\mathsf{Comb}^*_{W_T,V_T,\ldots,W_1,V_1}$ corresponds to a comb expressed in the form



where $\hat{\sigma}_1, \ldots, \hat{\sigma}_T$ are channels (in particular, $\hat{\sigma}_1$ is a state) and "⊣|⊢" denotes the trace. Any tester for a process in $\mathsf{Comb}_{W_T,V_T,\ldots,W_1,V_1}$ is represented by $\{\Phi_m\}_{m=1}^M \in \mathsf{Pos}_{\tilde{V}}^M =: C_{\mathrm{G}}$ satisfying $\sum_{m=1}^M \Phi_m \in \mathsf{Comb}^*_{W_T,V_T,\ldots,W_1,V_1} =: \mathcal{S}_{\mathrm{G}}$. In our manuscript, processes belonging to $\mathsf{Comb}^*_{W_T,V_T,\ldots,W_1,V_1}$ and tester elements are diagrammatically depicted in blue. The concatenation of two single-step processes $\hat{f}_1 \in \mathsf{Pos}(V_1, W'_1 \otimes W_1)$ and $\hat{f}_2 \in \mathsf{Pos}(W'_1 \otimes V_2, W_2)$, denoted by the process $\hat{F} := \hat{f}_1 \circledast \hat{f}_2$ (where $\circledast$ denotes the concatenation), is often depicted as



II. PROOF OF THEOREM 1

We consider the following Lagrangian associated with Problem (P):

$$L(\Phi, \varphi, \chi) := \sum_{m=1}^M \langle \Phi_m, \tilde{\mathcal{E}}_m \rangle + \left\langle \varphi - \sum_{m=1}^M \Phi_m, \chi \right\rangle = \langle \varphi, \chi \rangle - \sum_{m=1}^M \langle \Phi_m, \chi - \tilde{\mathcal{E}}_m \rangle, \tag{S3}$$

where $\Phi \in C$, $\varphi \in \mathcal{S}$, $\chi \in \mathsf{Her}_{\tilde{V}}$, and $\tilde{\mathcal{E}}_m := p_m \mathcal{E}_m$. From Eq. (S3), we have

$$\inf_\chi L(\Phi, \varphi, \chi) = \begin{cases} \sum_{m=1}^M \langle \Phi_m, \tilde{\mathcal{E}}_m \rangle, & \varphi = \sum_{m=1}^M \Phi_m, \\ -\infty, & \text{otherwise}, \end{cases}$$

$$\sup_\Phi L(\Phi, \varphi, \chi) = \begin{cases} \langle \varphi, \chi \rangle, & \chi \in \mathcal{D}_C, \\ \infty, & \text{otherwise}. \end{cases}$$

Thus, the left- and right-hand sides of the max-min inequality

$$\sup_{\Phi,\varphi} \inf_\chi L(\Phi, \varphi, \chi) \le \inf_\chi \sup_{\Phi,\varphi} L(\Phi, \varphi, \chi)$$

equal the optimal values of Problems (P) and (D), respectively.

To show the strong duality, it suffices to show that there exists $\Phi^\star \in \overline{\mathrm{co}}\,\mathcal{P}$ such that $P(\Phi^\star) \geq D^\star$, where $D^\star$ is the optimal value of Problem (D). Let us consider the following set:

$$\mathcal{Z} := \left\{\left(\{y_m + \tilde{\mathcal{E}}_m - \chi\}_{m=1}^M, D_{\mathcal{S}}(\chi) - d\right) : (\chi, y, d) \in \mathcal{Z}_0\right\} \subset \mathsf{Her}_{\tilde{V}}^M \times \mathbb{R},$$

where $y := \{y_m\}_{m=1}^M$ and

$$\mathcal{Z}_0 := \left\{(\chi, y, d) \in \mathsf{Her}_{\tilde{V}} \times C^* \times \mathbb{R} : d < D^\star\right\}.$$

It is easily seen that $\mathcal{Z}$ is a nonempty convex set. Arbitrarily choose $(\chi, y, d) \in \mathcal{Z}_0$ such that $y_m + \tilde{\mathcal{E}}_m - \chi = \mathbb{0}$ $(\forall m)$; then, $D_{\mathcal{S}}(\chi) \geq D^\star$ holds from $\{\chi - \tilde{\mathcal{E}}_m\}_m \in C^*$, which yields $D_{\mathcal{S}}(\chi) - d \geq D^\star - d > 0$. Thus, we have $(\{\mathbb{0}\}, 0) \notin \mathcal{Z}$. From separating hyperplane theorem [S1], there exists $(\{\Psi_m\}_{m=1}^M, \alpha) \neq (\{\mathbb{0}\}, 0)$ such that

$$\sum_{m=1}^M \langle \Psi_m, y_m + \tilde{\mathcal{E}}_m - \chi \rangle + \alpha[D_{\mathcal{S}}(\chi) - d] \geq 0, \quad \forall (\chi, y, d) \in \mathcal{Z}_0. \tag{S4}$$

Substituting $y_m = c y'_m$ $(c \in \mathbb{R}_+, \{y'_m\}_m \in C^*)$ into Eq. (S4) and taking the limit $c \to \infty$ yields $\{\Psi_m\}_m \in C$. Taking the limit $d \to -\infty$ gives $\alpha \geq 0$. To show $\alpha > 0$, assume by contradiction $\alpha = 0$. Substituting $\chi = c I_{\tilde{V}}$ $(c \in \mathbb{R}_+)$ and taking the limit $c \to \infty$ yields $\sum_{m=1}^M \mathrm{Tr}\,\Psi_m \leq 0$. From $\{\Psi_m\}_m \in C \subseteq \mathsf{Pos}_{\tilde{V}}^M$, $\Psi_m = \mathbb{0}$ $(\forall m)$ holds. This contradicts $(\{\Psi_m\}_m, \alpha) \neq (\{\mathbb{0}\}, 0)$, and thus $\alpha > 0$ holds. Let $\Phi_m^\star := \Psi_m / \alpha$; then, Eq. (S4) yields

$$\sum_{m=1}^M \langle \Phi_m^\star, y_m + \tilde{\mathcal{E}}_m - \chi \rangle + D_{\mathcal{S}}(\chi) - d \geq 0, \quad \forall (\chi, y, d) \in \mathcal{Z}_0. \tag{S5}$$

By substituting $\chi = c\chi'$ $(c \in \mathbb{R}_+, \chi' \in \mathsf{Her}_{\tilde{V}})$ into Eq. (S5) and taking the limit $c \to \infty$, we have $D_{\mathcal{S}}(\chi') \geq \sum_{m=1}^M \langle \Phi_m^\star, \chi' \rangle$ $(\forall \chi' \in \mathsf{Her}_{\tilde{V}})$. This implies $\sum_{m=1}^M \Phi_m^\star \in \mathcal{S}$, i.e., $\Phi^\star \in \overline{\mathrm{co}}\,\mathcal{P}$. Indeed, assume by contradiction $\sum_{m=1}^M \Phi_m^\star \notin \mathcal{S}$; then, since $\mathcal{S}$ is a closed convex set, from separating hyperplane theorem, there exists $\chi' \in \mathsf{Her}_{\tilde{V}}$ such that $\langle \phi, \chi' \rangle < \langle \sum_{m=1}^M \Phi_m^\star, \chi' \rangle$ $(\forall \phi \in \mathcal{S})$, which contradicts $D_{\mathcal{S}}(\chi') \geq \sum_{m=1}^M \langle \Phi_m^\star, \chi' \rangle$. Substituting $y_m = \mathbb{0}$ and $\chi = \mathbb{0}$ into Eq. (S5) and taking the limit $d \to D^\star$ yields $P(\Phi^\star) = \sum_{m=1}^M \langle \Phi_m^\star, \tilde{\mathcal{E}}_m \rangle \geq D^\star$. ■

Theorem 1 can be generalized to the following corollary.

Corollary S1 Given $\mathcal{P}$, let us arbitrarily choose a subset C of C_{G} and a bounded subset $\mathcal{S}$ of $\mathcal{S}_{\mathrm{G}}$ such that

$$\overline{\mathrm{co}}\,\mathcal{P} = \left\{\Phi \in \overline{\mathrm{coni}}\,C : \sum_{m=1}^M \Phi_m \in \overline{\mathrm{co}}\,\mathcal{S}\right\}.$$

Then, the problem

$$\begin{array}{ll} \text{minimize} & \sup_{\varphi \in \mathcal{S}} \langle \varphi, \chi \rangle \\ \text{subject to} & \chi \in \mathcal{D}_C \end{array}$$

has the same optimal value as Problem (P).

Proof From Theorem 1, the following problem

$$\begin{array}{ll} \text{minimize} & D_{\overline{\mathrm{co}}\,\mathcal{S}}(\chi) \\ \text{subject to} & \chi \in \mathcal{D}_{\overline{\mathrm{coni}}\,C} \end{array}$$

has the same optimal value as Problem (P). Also, it is easily seen that $D_{\overline{\mathrm{co}}\,\mathcal{S}}(\chi) = D_{\mathrm{co}\,\overline{\mathcal{S}}}(\chi) = \sup_{\varphi \in \mathcal{S}} \langle \varphi, \chi \rangle$ and $\mathcal{D}_{\overline{\mathrm{coni}}\,C} = \mathcal{D}_C$ hold. ■

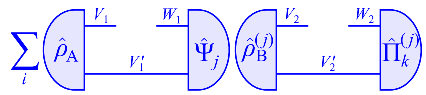
III. SUPPLEMENT OF THE EXAMPLE OF SEQUENTIAL STRATEGIES

A. Formulation of $\mathcal{P}_{\rm seq}$

We show that the set of all sequential testers in $\mathcal{P}_{\rm G}$ is expressed as

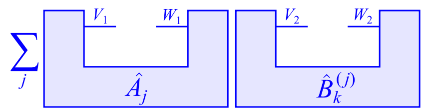
$$\mathcal{P}_{\rm seq} := \left\{ \left\{ \sum_j B_m^{(j)} \otimes A_j \right\}_{m=1}^{3} : \{A_j\} \in \mathsf{Test},\ \{B_m^{(j)}\}_m \in \mathsf{Test}_3 \right\}.$$

From Fig. 3, $\Phi \in \mathcal{P}_{\rm seq}$ holds if and only if $\hat{\Phi}_k$ is expressed in the form



, (S6)

where $\hat{\rho}_{\rm A} \in \mathsf{Den}_{V_1 \otimes V_1'}$, $\{\hat{\Psi}_j\}_j \in \mathsf{Meas}_{W_1 \otimes V_1'}$, $\hat{\rho}_{\rm B}^{(j)} \in \mathsf{Den}_{V_2 \otimes V_2'}$ $(\forall j)$, and $\{\hat{\Pi}_m^{(j)}\}_{m=1}^{M} \in \mathsf{Meas}_{W_2 \otimes V_2'}$ $(\forall j)$. It follows that $\{\hat{A}_j := \hat{\Psi}_j \circledast \hat{\rho}_{\rm A}\}_{j \in \mathcal{J}}$ and $\{\hat{B}_k^{(j)} := \hat{\Pi}_k^{(j)} \circledast \hat{\rho}_{\rm B}^{(j)}\}_{k=1}^{3}$ are testers and Eq. (S6) can be rewritten as



. (S7)

This gives that $\Phi \in \mathcal{P}_{\rm seq}$ holds if and only if Φ is expressed in the form $\Phi = \left\{ \sum_j B_m^{(j)} \otimes A_j \right\}_{m=1}^{3}$ with

$$\{A_j\}_{j \in \mathcal{J}} \subset \mathsf{Pos}_{W_1 \otimes V_1}, \quad \sum_{j \in \mathcal{J}} A_j \in \mathsf{Comb}^{*}_{W_1, V_1} \tag{S8}$$

and

$$\{B_m^{(j)}\}_{m=1}^{3} \subset \mathsf{Pos}_{W_2 \otimes V_2}, \quad \sum_{m=1}^{3} B_m^{(j)} \in \mathsf{Comb}^{*}_{W_2, V_2}, \quad \forall j \in \mathcal{J}. \tag{S9}$$

From Eq. (S1), Eqs. (S8) and (S9) are, respectively, equivalent to $\{A_j\} \in \mathsf{Test}$ and $\{B_m^{(j)}\}_m \in \mathsf{Test}_3$.

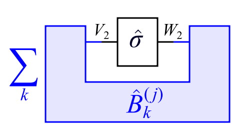
B. Derivation of Eq. (2)

Let $\mathcal{P}'$ be the right-hand side of Eq.(2), i.e.,

$$\mathcal{P}' := \left\{ \Phi \in \mathcal{C} : \sum_{m=1}^{M} \Phi_m \in \mathcal{S} \right\}. \tag{S10}$$

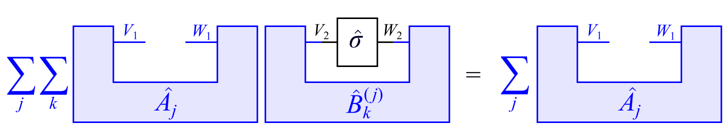
Since we can easily obtain $\overline{\mathsf{co}}\,\mathcal{P} = \mathcal{P}$ and $\mathcal{P} \subseteq \mathcal{P}'$ (i.e., $\Phi \in \mathcal{C}$ and $\sum_{m=1}^{M} \Phi_m \in \mathcal{S}$ hold for any $\Phi \in \mathcal{P}$), it suffices to show $\mathcal{P}' \subseteq \mathcal{P}$.

Let us consider $\mathcal{P}'$ with Eq. (4). Arbitrarily choose $\Phi' \in \mathcal{P}'$. From $\Phi' \in \mathcal{C}$, $\hat{\Phi}_k'$ is expressed in the form of Eq. (S7) with $A_j \in \mathsf{Pos}_{W_1 \otimes V_1}$ and $\{B_m^{(j)}\}_m \in \mathsf{Test}_{W_2, V_2}$ $(\forall j)$. Arbitrarily choose $\hat{\sigma} \in \mathsf{Chn}(V_2, W_2)$; then, from



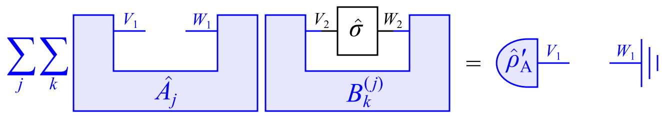
$= 1,$

we have



. (S11)

Also, from $\sum_{k=1}^{M} \Phi'_k \in \mathcal{S}_{\mathrm{G}}$, we have



(S12)

with some $\hat{\rho}'_{\mathrm{A}} \in \mathsf{Den}_{V_1}$. Equations (S11) and (S12) yield that $\{\hat{A}_j\}$ is a tester. Since $\{\hat{A}_j\}$ and $\{\hat{B}_m^{(j)}\}_m$ are testers, $\hat{\Phi}'_k$ is expressed in the form of Eq. (S6), i.e., $\mathcal{P}' \subseteq \mathcal{P}$.

C. Derivation of Eq. (6)

There exists an optimal solution χ to Problem (5) expressed in the form $\chi = \lambda\tilde{\chi}$ with $\lambda \in \mathbb{R}_+$ and $\tilde{\chi} \in \mathsf{Comb}_{W_2,V_2,W_1,V_1}$ (see [S2]). Note that from Eq. (S2), $D_{\mathcal{S}}(\tilde{\chi}) = 1$ holds for any $\tilde{\chi} \in \mathsf{Comb}_{W_2,V_2,W_1,V_1}$. Since $\tilde{\chi}$ is a comb, we can see that

$$X := \sum_{m=1}^{3} \mathrm{Tr}_{W_2 \otimes V_2}\left[\left[B_m^{(j)} \otimes I_2\right]\tilde{\chi}\right] \in \mathsf{Comb}_{W_1,V_1} \tag{S13}$$

is independent of the measurement $\{B_m^{(j)}\}_{m=1}^{3}$. Conversely, for any $X \in \mathsf{Comb}_{W_1,V_1}$, there exists $\tilde{\chi} \in \mathsf{Comb}_{W_2,V_2,W_1,V_1}$ satisfying Eq. (S13). Thus, Problem (5) is rewritable as

$$\begin{aligned} &\text{minimize} \quad \lambda \\ &\text{subject to} \quad \lambda X \geq \frac{1}{3}\sum_{m=1}^{3} \langle B_m, \Lambda_m \rangle \, \Lambda_m \quad (\forall \{B_m\} \in \mathsf{Test}_3) \end{aligned} \tag{S14}$$

with $\lambda \in \mathbb{R}_+$ and $X \in \mathsf{Comb}_{W_1,V_1}$. Due to the symmetry of

$$\Lambda_m = \begin{bmatrix} 1 & 0 & 0 & \omega^{-m} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \omega^m & 0 & 0 & 1 \end{bmatrix},$$

we can assume without loss of generality that

$$X := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

holds. This reduces Problem (S14) to Eq. (6).

D. Perfect distinguishability

We here show that $\{\mathcal{E}_m := \Lambda_m \otimes \Lambda_m\}_{m=1}^{3}$ can be perfectly distinguished if any physically allowed discrimination strategy can be performed. Assume that the prior probabilities are equal; then, the maximum success probability is equal to the optimal value of the following problem [S3]:

$$\begin{aligned} &\text{minimize} \quad \lambda \\ &\text{subject to} \quad \lambda\tilde{\chi} \geq \mathcal{E}_m/3 \end{aligned} \tag{S15}$$

with $\lambda \in \mathbb{R}_+$ and $\tilde{\chi} \in \mathsf{Comb}_{W_2,V_2,W_1,V_1}$. Note that $\mathcal{E}_m$ is given by

$$\mathcal{E}_m = \begin{bmatrix} 1 & 0 & 0 & \omega^{-m} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega^{-m} & 0 & 0 & \omega^{m} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \omega^{m} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \omega^{-m} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \omega^{m} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \omega^{-m} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \omega^{-m} & 0 & 0 & \omega^{m} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega^{m} & 0 & 0 & 1 \end{bmatrix}.$$

Due to the symmetry of $\mathcal{E}_m$, we can assume, without loss of generality, that $\lambda\tilde{\chi}$ is expressed in the form

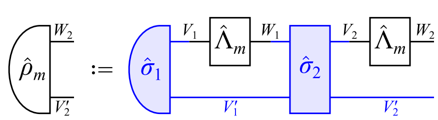
$$\lambda\tilde{\chi} = \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{bmatrix}$$

with $y \in \mathbb{C}$. Note that since $\mathcal{E}_m = \mathrm{Ad}_{U^{m-n} \otimes I_2 \otimes U^{m-n} \otimes I_2}(\mathcal{E}_n)$ $(\forall m, n \in \{1,2,3\})$ holds, there exists an optimal solution $(\lambda, \tilde{\chi})$ to Problem (S15) satisfying $\tilde{\chi} = \mathrm{Ad}_{U^k \otimes I_2 \otimes U^k \otimes I_2}(\tilde{\chi})$ $(\forall k \in \{1,2,3\})$, as will be shown in Lemma S3. After some simple algebra, we can see that $\lambda\tilde{\chi} \geq \mathcal{E}_m/3$ is equivalent to

$$3\lambda^2 - 4\lambda + 3y\lambda - 2y \geq 0, \ \lambda \geq y \geq \frac{4}{3} - 2\lambda.$$

We obtain the minimal λ satisfying this inequalities, which is the optimal value of Problem (S15), to be 1. Thus, $\{\mathcal{E}_m := \Lambda_m \otimes \Lambda_m\}_{m=1}^3$ are perfectly distinguishable.

We here give another proof. Let us consider an ensemble of three states $\{\hat{\rho}_m\}_{m=1}^3$ expressed as



[i.e., $\hat{\rho}_m := (\hat{\Lambda}_m \circledast \hat{\sigma}_2 \circledast \hat{\Lambda}_m)(\hat{\sigma}_1)$] with $\hat{\sigma}_1 \in \mathsf{Den}_{V_1 \otimes V_1'}$ and $\hat{\sigma}_2 \in \mathsf{Chn}(W_1 \otimes V_1', V_2 \otimes W_2')$. We choose

$$\hat{\sigma}_1 := \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad \hat{\sigma}_2 := \mathrm{Ad}_{U'}, \quad U' := \frac{1}{\sqrt{3}}\begin{bmatrix} \sqrt{2} & 0 & 1 & 0 \\ 0 & \sqrt{2} & 0 & 1 \\ -1 & 0 & \sqrt{2} & 0 \\ 0 & -1 & 0 & \sqrt{2} \end{bmatrix};$$

then, it is easily seen that $\{\hat{\rho}_m\}_{m=1}^3$ are orthogonal. Therefore, $\{\mathcal{E}_m\}_{m=1}^3$ are perfectly distinguishable.

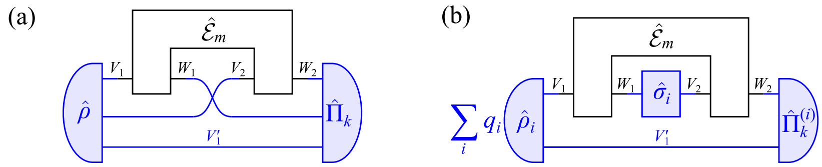


FIG. S1. Examples of two types of restricted testers. (a) A nonadaptive tester, in which first a state $\hat{\rho} \in \mathsf{Den}_{V_1 \otimes V_2 \otimes V_1'}$ is prepared, then the parts V_1 and V_2 are transmitted through a given process $\hat{\mathcal{E}}_m$, and finally a measurement $\{\hat{\Pi}_k\}_{k=1}^M \in \mathsf{Meas}_{W_2 \otimes W_1 \otimes V_1'}$ is performed. (b) A tester performed by two parties, Alice and Bob. In this tester, Alice first prepares a state $\hat{\rho}_i \in \mathsf{Den}_{V_1 \otimes V_1'}$ with a probability q_i and sends its one part V_1 to a given process $\hat{\mathcal{E}}_m$. She also sends the classical information i to Bob, who performs a channel $\hat{\sigma}_i \in \mathsf{Chn}(W_1, V_2)$ depending on i. Alice finally performs a measurement $\{\hat{\Pi}_k^{(i)}\}_{k=1}^M \in \mathsf{Meas}_{W_2 \otimes V_1'}$. Only one-way classical communication from Alice to Bob is allowed.

IV. APPLICATIONS OF THEOREM 1

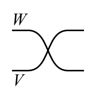
In addition to the example given in the main paper, we will provide two other examples demonstrating the utility of Theorem 1. In this section, we consider the case $T = 2$.

A. First example

The first example is the restriction to nonadaptive testers [see Fig. S1(a)]. Let

$$\begin{aligned} \mathcal{C} &:= \mathcal{C}_{\mathrm{G}}, \\ \mathcal{S} &:= \{\times_{W_1,V_2}(I_{W_2 \otimes W_1} \otimes \rho') : \rho' \in \mathsf{Den}_{V_2 \otimes V_1}\}, \end{aligned} \tag{S16}$$

where $\times_{W,V} \in \mathsf{Chn}(W \otimes V, V \otimes W)$ is the channel that swaps two systems W and V, which is depicted by

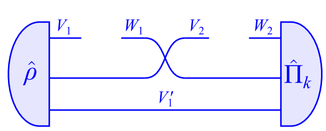


We will show that $\mathcal{P}'$ of Eq. (S10) is equal to $\overline{\mathsf{co}}\,\mathcal{P}$ [i.e., Eq. (2) holds] in the next paragraph. Substituting Eq. (S16) into Problem (D) yields the following dual problem

$$\begin{aligned} &\text{minimize} \quad \max_{\rho' \in \mathsf{Den}_{V_2 \otimes V_1}} \langle \times_{W_1,V_2}(I_{W_2 \otimes W_1} \otimes \rho'), \chi \rangle \\ &\text{subject to} \quad \chi \in \mathcal{D}_{\mathcal{C}_{\mathrm{G}}} = \{\chi \in \mathsf{Pos}_{\tilde{V}} : \chi \ge p_m \mathcal{E}_m \ (\forall m)\}. \end{aligned} \tag{S17}$$

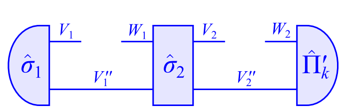
Note that although this problem is also formulated as the task of discriminating $\{\times_{V_2,W_1}(\mathcal{E}_m) \in \mathsf{Comb}_{W_2 \otimes W_1, V_2 \otimes V_1}\}_m$ with a single use, the expression of Problem (S17) is useful for verifying the global optimality of nonadaptive testers (see Secs. V A and VI B).

We here show $\mathcal{P}' = \overline{\mathsf{co}}\,\mathcal{P}$. It is easily seen $\overline{\mathsf{co}}\,\mathcal{P} = \mathcal{P}$ and $\mathcal{P} \subseteq \mathcal{P}'$ (i.e., $\Phi \in \mathcal{C}$ and $\sum_{m=1}^M \Phi_m \in \mathcal{S}$ hold for any $\Phi \in \mathcal{P}$). Thus, it suffices to show $\mathcal{P}' \subseteq \mathcal{P}$. From Fig. S1(a), $\Phi \in \mathcal{P}$ holds if and only if $\hat{\Phi}_k$ is expressed in the form



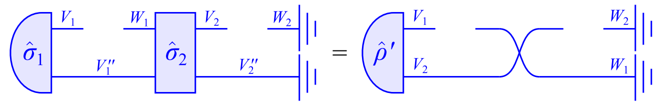
, (S18)

where $\hat{\rho} \in \mathsf{Den}_{V_1 \otimes V_2 \otimes V_1'}$ and $\{\hat{\Pi}_m\}_{m=1}^M \in \mathsf{Meas}_{W_2 \otimes W_1 \otimes V_1'}$. Arbitrarily choose $\Phi' \in \mathcal{P}'$. One can easily verify $\mathcal{S} \subseteq \mathcal{S}_{\mathrm{G}}$, i.e., $\mathcal{P}' \subseteq \mathcal{P}_{\mathrm{G}}$. Thus, $\hat{\Phi}_k'$ is expressed in the form



,

where $\hat{\sigma}_1 \in \mathsf{Den}_{V_1 \otimes V_1''}$, $\hat{\sigma}_2 \in \mathsf{Chn}(W_1 \otimes V_1'', V_2 \otimes V_2'')$, and $\{\hat{\Pi}'_m\}_{m=1}^M \in \mathsf{Meas}_{W_2 \otimes V_2''}$. V_1'' and V_2'' are some ancillary systems. Also, from $\sum_{m=1}^M \Phi'_m \in \mathcal{S}$, there exists $\hat{\rho}' \in \mathsf{Den}_{V_1 \otimes V_2}$ such that



Thus, $\hat{\Phi}'_k$ is expressed in the form of Eq. (S18), where $\hat{\rho}$ is a purification of $\hat{\rho}'$. Therefore, $\mathcal{P}' \subseteq \mathcal{P}$ holds.

B. Second example

The second example is described in Fig. S1(b). We want to find C and $\mathcal{S}$ such that Problem (D) is easy to solve; however, it is hard to find such C and $\mathcal{S}$ satisfying Eq. (2). Instead, we consider relaxing this equation to $\overline{\mathsf{co}}\,\mathcal{P} \subseteq \mathcal{P}'$. We here choose

$$C := \left\{ \left\{ \times_{V_2 \otimes W_1, W_2} \left(\sum_i \sigma_i \otimes A_m^{(i)} \right) \right\}_{m=1}^{M} : \sigma_i \in \mathsf{Comb}_{V_2, W_1},\ A_m^{(i)} \in \mathsf{Pos}_{W_2 \otimes V_1} \right\}$$

and $\mathcal{S} := \mathcal{S}_{\mathrm{G}}$. This allows Problem (D) to be rewritten in this situation as

$$\begin{aligned} &\text{minimize} \quad D_{\mathcal{S}_{\mathrm{G}}}(\chi) \\ &\text{subject to} \quad \mathrm{Tr}_{V_2 \otimes W_1}[\sigma(\chi - p_m \mathcal{E}_m)] \geq \mathbb{0} \quad (\forall 1 \leq m \leq M,\ \sigma \in \mathsf{Comb}_{V_2, W_1}) \end{aligned}$$

with $\chi \in \mathsf{Her}_{\tilde{V}}$. We relatively easily obtain the optimal value, denoted by $D^\star$, of this problem. The optimal value of Problem (P) is upper bounded by $D^\star$, since $D^\star$ coincides with the optimal value of Problem (P) where the feasible set is relaxed from $\mathcal{P}$ to $\mathcal{P}'$.

V. GLOBAL OPTIMALITY

A. Necessary and sufficient condition for global optimality

Given a feasible set $\mathcal{P}$, we now ask the question whether the optimal values of Problems (P) and ($\mathrm{P_G}$) coincide. We can derive a necessary and sufficient condition for global optimality by considering Problem (D) with $\mathcal{P} = \mathcal{P}_{\mathrm{G}}$ (i.e., $C = C_{\mathrm{G}}$ and $\mathcal{S} = \mathcal{S}_{\mathrm{G}}$), which is written as

$$\begin{aligned} &\text{minimize} \quad D_{\mathcal{S}_{\mathrm{G}}}(\chi) \\ &\text{subject to} \quad \chi \in \mathcal{D}_{C_{\mathrm{G}}}. \end{aligned} \tag{$\mathrm{D_G}$}$$

Since Theorem 1 guarantees that Problems (D) and ($\mathrm{D_G}$), respectively, have the same optimal values as Problems (P) and ($\mathrm{P_G}$), the task is to obtain a necessary and sufficient condition for the optimal values of Problems (D) and ($\mathrm{D_G}$) to coincide. To this end, we have the following statement:

Proposition S2 Let us arbitrarily choose a closed convex cone C and a closed convex set $\mathcal{S}$ satisfying Eq. (2). Then, the following statements are all equivalent.

(1) The optimal values of Problems (P) and ($\mathrm{P_G}$) are the same.
(2) Any optimal solution to Problem ($\mathrm{D_G}$) is optimal for Problem (D).
(3) There exists an optimal solution $\chi^\star$ to Problem (D) such that $\chi^\star$ is in $\mathcal{D}_{C_{\mathrm{G}}}$ and is proportional to some quantum comb.
(4) There exists an optimal solution $\chi^\star$ to Problem (D) such that $\chi^\star \in \mathcal{D}_{C_{\mathrm{G}}}$ and $D_{\mathcal{S}}(\chi^\star) = D_{\mathcal{S}_{\mathrm{G}}}(\chi^\star)$ hold.

Proof Let $D^\star$ and $D^\star_{\mathrm{G}}$ be, respectively, the optimal values of Problems (D) and ($\mathrm{D_G}$) [or, equivalently, the optimal values of Problems (P) and ($\mathrm{P_G}$)]. We show (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4), and (4) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2) : Let us arbitrarily choose an optimal solution $\chi^\star$ to Problem ($\mathrm{D_G}$). Since $\mathcal{D}_{C_{\mathrm{G}}} \subseteq \mathcal{D}_C$ holds from $C \subseteq C_{\mathrm{G}}$, $\chi^\star \in \mathcal{D}_C$ holds. Also, from $\mathcal{S} \subseteq \mathcal{S}_{\mathrm{G}}$, we have $D^\star \leq D_{\mathcal{S}}(\chi^\star) \leq D_{\mathcal{S}_{\mathrm{G}}}(\chi^\star)$. Since $D^\star = D^\star_{\mathrm{G}} = D_{\mathcal{S}_{\mathrm{G}}}(\chi^\star)$ holds, we have $D^\star = D_{\mathcal{S}}(\chi^\star)$. Thus, $\chi^\star$ is optimal for Problem (D).

(2) $\Rightarrow$ (3) : It is known that there exists an optimal solution, $\chi^\star \in \mathcal{D}_{C_{\mathrm{G}}}$, to Problem ($\mathrm{D_G}$) such that $\chi^\star$ is proportional to some quantum comb [S2, S3]. From Statement (2), $\chi^\star$ is optimal for Problem (D).

(3) $\Rightarrow$ (4) : $\chi^\star$ can be expressed as $\chi^\star = qT$ with $q \in \mathbb{R}_+$ and a quantum comb T. Since $\langle \varphi, \chi^\star \rangle = q \langle \varphi, T \rangle = q$ holds for any $\varphi \in \mathcal{S}_{\mathrm{G}}$, we have $D_{\mathcal{S}}(\chi^\star) = D_{\mathcal{S}_{\mathrm{G}}}(\chi^\star)$.

(4) $\Rightarrow$ (1) : We have $D^\star = D_{\mathcal{S}}(\chi^\star) = D_{\mathcal{S}_{\mathrm{G}}}(\chi^\star) \geq D^\star_{\mathrm{G}}$. Since $D^\star \leq D^\star_{\mathrm{G}}$ holds, $D^\star = D^\star_{\mathrm{G}}$ must hold. ■

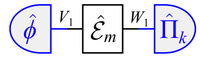


FIG. S2. Channel discrimination problems with a single use in which a state input to the channel is restricted to be separable. We can assume that the input state is a pure state of the system V_1, i.e., a tester consists of $\hat{\phi} \in \mathsf{Den}^{\mathsf{P}}_{V_1}$ and $\{\hat{\Pi}_m\}_{m=1}^M \in \mathsf{Meas}_{W_1}$.

B. Another example of necessary and sufficient optimality condition

In some individual cases, necessary and sufficient conditions for global optimality can be derived from Theorem 1. To give an example, let us consider the problem of discriminating quantum channels $\{\hat{\mathcal{E}}_m\}_{m=1}^M \in \mathsf{Chn}(V_1, W_1)$ with a single use in which a state input to the channel is restricted to be separable (see Fig. S2). Since we can assume, without loss of generality, that the input state is a pure state of the system V_1, the optimal value $P^\star$ of Problem (P) is written as

$$P^\star := \max_{\phi \in \mathsf{Den}^{\mathsf{P}}_{V_1}} \max_{\Pi \in \mathsf{Meas}_{W_1}} \sum_{m=1}^M p_m \langle \Pi_m, \hat{\mathcal{E}}_m(\phi) \rangle .$$

Since the dual of the discrimination problem in which an input state is fixed to ϕ is formulated as Problem (D) with $C = C_{\mathrm{G}}$ and $\mathcal{S} = \left\{ I_{W_1} \otimes \phi^{\mathsf{T}} \right\}$, Theorem 1 gives

$$\max_{\Pi \in \mathsf{Meas}_{W_1}} \sum_{m=1}^M p_m \langle \Pi_m, \hat{\mathcal{E}}_m(\phi) \rangle = \min_{\chi \in \mathcal{D}_{C_{\mathrm{G}}}} \langle I_{W_1} \otimes \phi^{\mathsf{T}}, \chi \rangle, \quad \forall \phi \in \mathsf{Den}^{\mathsf{P}}_{V_1},$$

and thus

$$P^\star = \max_{\phi' \in \mathsf{Den}^{\mathsf{P}}_{V_1}} \min_{\chi \in \mathcal{D}_{C_{\mathrm{G}}}} \langle I_{W_1} \otimes \phi', \chi \rangle .$$

Also, the optimal value of Problem ($\mathrm{D_G}$) is expressed by

$$\min_{\chi \in \mathcal{D}_{C_{\mathrm{G}}}} \max_{\rho \in \mathsf{Den}_{V_1}} \langle I_{W_1} \otimes \rho^{\mathsf{T}}, \chi \rangle = \min_{\chi \in \mathcal{D}_{C_{\mathrm{G}}}} \max_{\phi \in \mathsf{Den}^{\mathsf{P}}_{V_1}} \langle I_{W_1} \otimes \phi^{\mathsf{T}}, \chi \rangle = \min_{\chi \in \mathcal{D}_{C_{\mathrm{G}}}} \max_{\phi' \in \mathsf{Den}^{\mathsf{P}}_{V_1}} \langle I_{W_1} \otimes \phi', \chi \rangle .$$

Thus, globally optimal discrimination is achieved without entanglement if and only if the following max-min inequality holds as an equality:

$$\max_{\phi' \in \mathsf{Den}^{\mathsf{P}}_{V_1}} \min_{\chi \in \mathcal{D}_{C_{\mathrm{G}}}} \langle I_{W_1} \otimes \phi', \chi \rangle \le \min_{\chi \in \mathcal{D}_{C_{\mathrm{G}}}} \max_{\phi' \in \mathsf{Den}^{\mathsf{P}}_{V_1}} \langle I_{W_1} \otimes \phi', \chi \rangle .$$

VI. SYMMETRIC PROBLEMS

Given a process discrimination problem that has a certain symmetry, we present a sufficient condition for a nonadaptive tester to be globally optimal. We here limit our discussion to a specific type of symmetries (see [S2] for a more general case). Note that several related results in particular cases have been reported [S3–S5].

A. Lemmas

Let $\mathcal{G}$ be a group with the identity element e. Let $\varpi := \{\varpi_g\}_{g \in \mathcal{G}}$ be a group action of $\mathcal{G}$ on $\{1, \dots, M\}$, i.e., a set of maps on $\{1, \dots, M\}$ satisfying $\varpi_{gh}(m) = \varpi_g[\varpi_h(m)]$ and $\varpi_e(m) = m$ for any $g, h \in \mathcal{G}$ and $m \in \{1, \dots, M\}$. We consider a set

$$\mathcal{U} := \left\{ \mathcal{U}_g := \mathrm{Ad}_{U_g^{(T)} \otimes \tilde{U}_g^{(T)} \otimes \cdots \otimes U_g^{(1)} \otimes \tilde{U}_g^{(1)}} \right\}_{g \in \mathcal{G}},$$

where, for each $t \in \{1, \dots, T\}$, $\mathcal{G} \ni g \mapsto U_g^{(t)} \in \mathsf{Uni}_{W_t}$ and $\mathcal{G} \ni g \mapsto \tilde{U}_g^{(t)} \in \mathsf{Uni}_{V_t}$ are projective unitary representations of $\mathcal{G}$. We will refer to an ensemble of M combs $\{\mathcal{E}_m\}_{m=1}^M \subset \mathsf{Comb}_{W_T, V_T, \dots, W_1, V_1}$ as $(\mathcal{G}, \mathcal{U}, \varpi)$-*covariant* if

$$\mathcal{U}_g(\mathcal{E}_m) = \mathcal{E}_{\varpi_g(m)}, \quad \forall g \in \mathcal{G} \tag{S19}$$

holds.

To give our results in the next subsection, we first prove the following two lemmas.

Lemma S3 Assume that $\{\mathcal{E}_m\}_{m=1}^M$ is $(\mathcal{G},\mathcal{U},\varpi)$-covariant. If

$$\begin{aligned} C &= C_{\mathrm{G}}, \\ \mathcal{U}_g(\varphi) &\in \mathcal{S}, \quad \forall g \in \mathcal{G},\ \varphi \in \mathcal{S} \end{aligned} \tag{S20}$$

holds, then there exists an optimal solution, $\chi^\star \in \mathsf{Pos}_{\tilde{V}}$, to Problem (D) such that

$$\mathcal{U}_g(\chi^\star) = \chi^\star, \quad \forall g \in \mathcal{G}. \tag{S21}$$

Proof Let χ be an optimal solution to Problem (D). From $C = C_{\mathrm{G}}$, we can easily see $\chi \in \mathsf{Pos}_{\tilde{V}}$. Also, let

$$\chi^\star := \frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} \mathcal{U}_g(\chi) \in \mathsf{Pos}_{\tilde{V}},$$

where $|\mathcal{G}|$ is the order of $\mathcal{G}$. Since $\mathcal{U}_g \circ \mathcal{U}_{g'} = \mathcal{U}_{gg'}$ holds for any $g, g' \in \mathcal{G}$, one can easily see Eq. (S21). We have that from Eq. (S19) and $\chi \geq \mathcal{E}_m$,

$$\chi^\star - \mathcal{E}_m = \frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} \mathcal{U}_g(\chi - \mathcal{E}_{\varpi_{\bar{g}}(m)}) \geq \mathbb{0}, \quad \forall m \in \{1,\ldots,M\},$$

i.e., $\chi^\star \in \mathcal{D}_C$, where $\bar{g}$ is the inverse of g. Moreover, we have

$$\begin{aligned} D_{\mathcal{S}}(\chi^\star) &= \max_{\varphi\in\mathcal{S}} \langle \varphi, \chi^\star \rangle = \frac{1}{|\mathcal{G}|}\max_{\varphi\in\mathcal{S}} \sum_{g\in\mathcal{G}} \langle \varphi, \mathcal{U}_g(\chi)\rangle \leq \frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} \max_{\varphi\in\mathcal{S}} \langle \varphi, \mathcal{U}_g(\chi)\rangle \\ &= \frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} \max_{\varphi\in\mathcal{S}} \langle \mathcal{U}_{\bar{g}}(\varphi), \chi\rangle \leq \frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} D_{\mathcal{S}}(\chi) = D_{\mathcal{S}}(\chi), \end{aligned}$$

where the last inequality follows from $\mathcal{U}_{\bar{g}}(\varphi) \in \mathcal{S}$ for each $\varphi \in \mathcal{S}$ and $g \in \mathcal{G}$, which follows from the second line of Eq. (S20). Therefore, $\chi^\star$ is optimal for Problem (D). ■

The proof of this lemma also shows that if there exists an optimal solution that is proportional to some quantum comb, then there also exists an optimal solution, $\chi^\star$, that is proportional to some quantum comb and that satisfies Eq. (S21).

Lemma S4 Assume that, for each $t \in \{1,\ldots,T\}$, there exists a group $\mathcal{G}^{(t)}$ that has two projective unitary representations $\mathcal{G}^{(t)} \ni g \mapsto U_g^{(t)} \in \mathsf{Uni}_{W_t}$ and $\mathcal{G}^{(t)} \ni g \mapsto \tilde{U}_g^{(t)} \in \mathsf{Uni}_{V_t}$. If $g \mapsto \tilde{U}_g^{(t)}$ is irreducible for each $t \in \{1,\ldots,T\}$, then any $\chi^\star \in \mathsf{Pos}_{\tilde{V}}$ satisfying

$$(\mathbb{1}_{W_T\otimes V_T\otimes\cdots\otimes W_{t+1}\otimes V_{t+1}} \otimes \mathrm{Ad}_{U_g^{(t)}\otimes\tilde{U}_g^{(t)}} \otimes \mathbb{1}_{W_{t-1}\otimes V_{t-1}\otimes\cdots\otimes W_1\otimes V_1})(\chi^\star) = \chi^\star, \qquad \forall t \in \{1,\ldots,T\},\ g \in \mathcal{G}^{(t)} \tag{S22}$$

is proportional to some quantum comb.

Proof It suffices to show that, for each $t \in \{1,\ldots,T\}$, $\mathrm{Tr}_{W_t}\chi^\star$ is expressed in the form $\mathrm{Tr}_{W_t}\chi^\star = I_{V_t} \otimes \chi_t^\star$ with $\chi_t^\star \in \mathsf{Pos}_{X_t}$, where X_t is the tensor product of all $W_{t'} \otimes V_{t'}$ with $t' \in \{1,\ldots,t-1,t+1,\ldots,T\}$. Indeed, in this case, one can easily verify $\chi^\star$ is proportional to some quantum comb. Let us fix $t \in \{1,\ldots,T\}$. Also, let $\chi_s^\star := \mathrm{Tr}_{X_t}[(I_{V_t} \otimes s)\,\mathrm{Tr}_{W_t}\chi^\star] \in \mathsf{Pos}_{V_t}$, where $s \in \mathsf{Pos}_{X_t}$ is arbitrarily chosen; then, we have

$$\mathrm{Tr}\,\chi_s^\star = \langle s, \chi' \rangle, \quad \chi' := \mathrm{Tr}_{W_t\otimes V_t}\chi^\star. \tag{S23}$$

Equation (S22) gives $\mathrm{Ad}_{\tilde{U}_g^{(t)}}(\chi_s^\star) = \chi_s^\star$ $[\forall g \in \mathcal{G}^{(t)}]$. From Schur's lemma, $\chi_s^\star$ must be proportional to I_{V_t}. Thus, from Eq. (S23), $\chi_s^\star = \langle s, \chi' \rangle I_{V_t}/N_{V_t}$ holds. We have that for any $s' \in \mathsf{Pos}_{V_t}$,

$$\langle s' \otimes s, \mathrm{Tr}_{W_t}\chi^\star \rangle = \langle s', \chi_s^\star \rangle = \langle s, \chi' \rangle \langle s', I_{V_t}/N_{V_t} \rangle = \langle s' \otimes s, I_{V_t} \otimes \chi'/N_{V_t} \rangle. \tag{S24}$$

Since Eq. (S24) holds for any s and s', we have $\mathrm{Tr}_{W_t}\chi^\star = I_{V_t} \otimes \chi_t^\star$ with $\chi_t^\star := \chi'/N_{V_t} \in \mathsf{Pos}_{X_t}$. ■

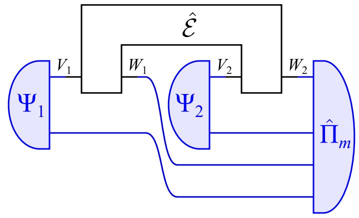


FIG. S3. Tester with maximally entangled pure states (in the case of $T = 2$), which consists of maximally entangled pure states Ψ_t and a measurement $\{\hat{\Pi}_m\}$. We can assume, without loss of generality, that each Ψ_t is a generalized Bell state $|I_{V_t}\rangle\!\rangle\langle\!\langle I_{V_t}|/N_{V_t}$.

B. Sufficient condition for a nonadaptive tester to be globally optimal

We will call a tester each of whose output systems is one part of a bipartite system in a maximally entangled pure state (see Fig. S3) a *tester with maximally entangled pure states*. Such a tester is obviously nonadaptive. From Lemmas S3 and S4, we obtain a sufficient condition that there exists a tester with maximally entangled pure states that is globally optimal. Let $\mathcal{P}$ be the set of testers with maximally entangled pure states; then, it follows that Eq. (2) holds with

$$C := C_{\mathrm{G}}, \quad \mathcal{S} := \left\{ I_{\tilde{V}} / \prod_{t=1}^{T} N_{V_t} \right\}. \tag{S25}$$

Note that $\overline{\mathsf{co}}\,\mathcal{P} = \mathcal{P}$ holds in this case. It is easily seen that Eq. (S20) holds. Thus, we immediately obtain the following proposition.

Proposition S5 Assume that, for each $t \in \{1, \dots, T\}$, there exists a group $\mathcal{G}^{(t)}$ that has a projective unitary representation $\mathcal{G}^{(t)} \ni g \mapsto U_g^{(t)} \in \mathsf{Uni}_{W_t}$ and an irreducible projective unitary representation $\mathcal{G}^{(t)} \ni g \mapsto \tilde{U}_g^{(t)} \in \mathsf{Uni}_{V_t}$. Let $\mathcal{G} := \mathcal{G}^{(T)} \times \mathcal{G}^{(T-1)} \times \cdots \times \mathcal{G}^{(1)}$, $\mathcal{U} := \{\mathrm{Ad}_{U_{g_T}^{(T)} \otimes \tilde{U}_{g_T}^{(T)} \otimes \dots \otimes U_{g_1}^{(1)} \otimes \tilde{U}_{g_1}^{(1)}}\}_{(g_T,\dots,g_1)\in\mathcal{G}}$, and ϖ be some group action on $\mathcal{G}$ on $\{1, \dots, M\}$. If $\{\mathcal{E}_m\}_{m=1}^M$ is $(\mathcal{G}, \mathcal{U}, \varpi)$-covariant, then there exists a globally optimal tester with maximally entangled pure states.

Proof Let C and $\mathcal{S}$ be defined as Eq. (S25). Let e_t be the identity element of $\mathcal{G}^{(t)}$. From Lemma S3, there exists an optimal solution $\chi^\star \in \mathsf{Pos}_{\tilde{V}}$ to Problem (D) satisfying Eq. (S21). Thus, $\mathcal{U}_{(e_T,\dots,e_{t+1},g_t,e_{t-1},\dots,e_1)}(\chi^\star) = \chi^\star$ holds for any $t \in \{1, \dots, T\}$ and $g_t \in \mathcal{G}^{(t)}$, i.e., Eq. (S22) holds, which implies from Lemma S4 that $\chi^\star$ is proportional to some quantum comb. Therefore, Proposition S2 concludes the proof. ∎

See Ref. [S2] for some examples and for more general results.

VII. RELATIONSHIP BETWEEN ROBUSTNESSES AND PROCESS DISCRIMINATION PROBLEMS

Let us consider the robustness of $\mathcal{E} \in \mathsf{Her}_{\tilde{V}}$ defined by

$$R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E}) := \inf \left\{ \lambda \in \mathbb{R}_+ : \frac{\mathcal{E} + \lambda \mathcal{E}'}{1 + \lambda} \in \mathcal{F}, \ \mathcal{E}' \in \mathcal{K} \right\}, \quad \mathcal{E} \in \mathsf{Her}_{\tilde{V}}, \tag{S26}$$

where $\mathcal{K}$ $(\subset \mathsf{Her}_{\tilde{V}})$ is a proper convex cone [or, equivalently, $\mathcal{K}$ is a closed convex cone that is pointed (i.e., $\mathcal{K} \cap -\mathcal{K} = \{\mathbb{0}\}$) and has nonempty interior] and $\mathcal{F}$ $(\subset \mathsf{Her}_{\tilde{V}})$ is a compact set. Assume that the set $\{\lambda Z : \lambda \in \mathbb{R}, \ 0 \le \lambda \le 1, \ Z \in \mathcal{F}\}$ is convex. In order for this value to be well-defined, we assume that $\mathcal{F} \cap \mathrm{int}(\mathcal{K})$ is not empty. The so-called global (or generalized) robustness of a state $\rho \in \mathsf{Den}_V$ with respect to $\mathcal{F} \subseteq \mathsf{Den}_V$, defined as [S6]

$$R_{\mathcal{F}}(\rho) := \min \left\{ \lambda \in \mathbb{R}_+ : \frac{\rho + \lambda \rho'}{1 + \lambda} \in \mathcal{F}, \ \rho' \in \mathsf{Den}_V \right\}, \quad \rho \in \mathsf{Den}_V,$$

is equal to $R_{\mathsf{Pos}_V}^{\mathcal{F}}(\rho)$. In other words, $R_{\mathcal{F}} : \mathsf{Den}_V \to \mathbb{R}_+$ is the same function as $R_{\mathsf{Pos}_V}^{\mathcal{F}} : \mathsf{Her}_V \to \mathbb{R}_+$, but is only defined on Den_V. As an example of $R_{\mathcal{F}}$, if $\mathcal{F}$ is the set of all bipartite separable states, then $R_{\mathcal{F}}(\rho)$ can be understood as a measure of entanglement. The robustness $R_{\mathcal{F}}(\rho)$ is known to represent the maximum advantage that ρ provides in a certain subchannel discrimination problem (e.g., [S7, S8]). Similarly, as will be seen in Proposition S8, $R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E})$ has a close relationship with the maximum advantage that $\mathcal{E}$ provides in a certain discrimination problem.

By letting $Z := (\mathcal{E} + \lambda\mathcal{E}')/(1+\lambda)$, we can rewrite $R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E})$ of Eq. (S26) as

$$R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E}) = \min\{\lambda \in \mathbb{R}_+ : (1+\lambda)Z - \mathcal{E} \in \mathcal{K},\ Z \in \mathcal{F}\}.$$

Let

$$\mathcal{N} := \{\mathcal{E} \in \mathsf{Her}_{\tilde{V}} : \delta Z - \mathcal{E} \notin \mathcal{K}\ (\forall \delta < 1, Z \in \mathcal{F})\};$$

then, it follows that

$$R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E}) = \min\{\lambda \in \mathbb{R} : (1+\lambda)Z - \mathcal{E} \in \mathcal{K},\ Z \in \mathcal{F}\}, \quad \forall \mathcal{E} \in \mathcal{N}. \tag{S27}$$

We first prove the following two lemmas.

Lemma S6 If $\mathcal{E} \in \mathcal{N}$ holds, then

$$1 + R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E}) = \max\{\langle \varphi, \mathcal{E}\rangle : \varphi \in \mathcal{K}^*, \langle \varphi, Z\rangle \le 1\ (\forall Z \in \mathcal{F})\} \tag{S28}$$

holds.

Proof Let $\tilde{\mathcal{F}} := \mathsf{coni}\,\mathcal{F}$. $\eta : \tilde{\mathcal{F}} \to \mathbb{R}_+$ denotes the gauge function of $\mathcal{F}$, which is defined as

$$\eta(Y) := \min\{\lambda \in \mathbb{R}_+ : Y = \lambda Z,\ Z \in \mathcal{F}\}, \quad Y \in \tilde{\mathcal{F}}. \tag{S29}$$

Note that η is a convex function. Equation (S27) gives

$$1 + R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E}) = \min\{\eta(Y) : Y - \mathcal{E} \in \mathcal{K},\ Y \in \tilde{\mathcal{F}}\}.$$

Let us consider the following Lagrangian

$$L(Y, \varphi) := \eta(Y) - \langle \varphi, Y - \mathcal{E}\rangle = \langle \varphi, \mathcal{E}\rangle + \eta(Y) - \langle \varphi, Y\rangle$$

with $Y \in \tilde{\mathcal{F}}$ and $\varphi \in \mathcal{K}^*$. We can easily verify

$$\sup_{\varphi \in \mathcal{K}^*} L(Y, \varphi) = \begin{cases} \eta(Y), & Y - \mathcal{E} \in \mathcal{K}, \\ \infty, & \text{otherwise}, \end{cases}$$

$$\inf_{Y \in \tilde{\mathcal{F}}} L(Y, \varphi) = \begin{cases} \langle \varphi, \mathcal{E}\rangle, & \langle \varphi, Y'\rangle \le \eta(Y')\ (\forall Y' \in \tilde{\mathcal{F}}), \\ -\infty, & \text{otherwise}. \end{cases}$$

Thus, the max-min inequality

$$\inf_{Y \in \tilde{\mathcal{F}}} \sup_{\varphi \in \mathcal{K}^*} L(Y, \varphi) \ge \sup_{\varphi \in \mathcal{K}^*} \inf_{Y \in \tilde{\mathcal{F}}} L(Y, \varphi)$$

yields

$$1 + R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E}) \ge \max\{\langle \varphi, \mathcal{E}\rangle : \varphi \in \mathcal{K}^*,\ \langle \varphi, Y\rangle \le \eta(Y)\ (\forall Y \in \tilde{\mathcal{F}})\}. \tag{S30}$$

We now prove the equality of Eq. (S30). To this end, it suffices to show that there exists $\varphi \in \mathrm{int}(\mathcal{K}^*)$ such that $\langle \varphi, Y\rangle \le \eta(Y)$ $(\forall Y \in \tilde{\mathcal{F}})$; indeed, in this case, the equality of Eq. (S30) follows from Slater's condition. Arbitrarily choose $\varphi' \in \mathrm{int}(\mathcal{K}^*)$ and let $\gamma := \sup_{Y \in \tilde{\mathcal{F}} \setminus \{\mathbb{0}\}}[\langle \varphi', Y\rangle / \eta(Y)]$ [note that $\eta(Y) > 0$ holds for any $Y \in \tilde{\mathcal{F}} \setminus \{\mathbb{0}\}$]. Since $\mathcal{F} \cap \mathrm{int}(\mathcal{K})$ is not empty, there exists $Y \in \mathcal{F} \cap \mathrm{int}(\mathcal{K})$ such that $\langle \varphi', Y\rangle > 0$, which yields $\gamma > 0$. Let $\varphi := \gamma^{-1}\varphi' \in \mathrm{int}(\mathcal{K}^*)$; then, we can easily verify $\langle \varphi, Y\rangle \le \eta(Y)$ $(\forall Y \in \tilde{\mathcal{F}})$.

It remains to show

$$\langle \varphi, Z\rangle \le 1\ (\forall Z \in \mathcal{F}) \quad \Leftrightarrow \quad \langle \varphi, Y\rangle \le \eta(Y)\ (\forall Y \in \tilde{\mathcal{F}}).$$

We first prove "$\Rightarrow$". Arbitrarily choose $Y \in \tilde{\mathcal{F}}$; then, from Eq. (S29), there exists $Z \in \mathcal{F}$ such that $Y = \eta(Y)Z$. Thus, $\langle \varphi, Y\rangle = \eta(Y)\langle \varphi, Z\rangle \le \eta(Y)$ holds. We next prove "$\Leftarrow$". Arbitrarily choose $Z \in \mathcal{F}$. Since the case $\langle \varphi, Z\rangle \le 0$ is obvious, we may assume $\langle \varphi, Z\rangle > 0$. Let $Z' := Z/\eta(Z)$; then, from $\eta(Z) \le 1$, $Z' \in \tilde{\mathcal{F}}$, and $\eta(Z') = 1$, we have $\langle \varphi, Z\rangle \le \langle \varphi, Z'\rangle \le \eta(Z') = 1$. ■

Lemma S7 If $\mathcal{E} \in \mathcal{N}$ holds, then we have

$$\max_{\varphi \in \mathcal{X} \setminus \{\mathbb{0}\}} \frac{\langle \varphi, \mathcal{E} \rangle}{\max_{Z \in \mathcal{F}} \langle \varphi, Z \rangle} = 1 + R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E}),$$

where $\mathcal{X}$ is any set such that the cone generated by $\mathcal{X}$ is $\mathcal{K}^*$, i.e., $\{\lambda\varphi : \lambda \in \mathbb{R}_+,\ \varphi \in \mathcal{X}\} = \mathcal{K}^*$.

Proof Let

$$\varphi^\star \coloneqq \operatorname*{argmax}_{\varphi \in \mathcal{K}^*, \Gamma(\varphi) \le 1} \langle \varphi, \mathcal{E} \rangle, \quad \Gamma(\varphi) \coloneqq \max_{Z \in \mathcal{F}} \langle \varphi, Z \rangle;$$

then, from Eq. (S28), we have $\langle \varphi^\star, \mathcal{E} \rangle = 1 + R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E})$. Since $\mathcal{F} \cap \mathrm{int}(\mathcal{K})$ is not empty, $\Gamma(\varphi) > 0$ holds for any $\varphi \in \mathcal{K}^* \setminus \{\mathbb{0}\}$. It follows that $\Gamma(\varphi^\star) = 1$ must hold [otherwise, $\tilde{\varphi} \coloneqq \varphi^\star / \Gamma(\varphi^\star)$ satisfies $\langle \tilde{\varphi}, \mathcal{E} \rangle > \langle \varphi^\star, \mathcal{E} \rangle$, $\tilde{\varphi} \in \mathcal{K}^*$, and $\Gamma(\tilde{\varphi}) = 1$, which contradicts the definition of $\varphi^\star$]. For any $\varphi \in \mathcal{X} \setminus \{\mathbb{0}\}$, $\varphi' \coloneqq \varphi / \Gamma(\varphi)$ satisfies $\langle \varphi, \mathcal{E} \rangle / \Gamma(\varphi) = \langle \varphi', \mathcal{E} \rangle$ and $\Gamma(\varphi') = 1$. Thus, we have

$$\max_{\varphi \in \mathcal{X} \setminus \{\mathbb{0}\}} \frac{\langle \varphi, \mathcal{E} \rangle}{\Gamma(\varphi)} = \max_{\substack{\varphi' \in \mathcal{X} \setminus \{\mathbb{0}\}, \\ \Gamma(\varphi') = 1}} \langle \varphi', \mathcal{E} \rangle = \langle \varphi^\star, \mathcal{E} \rangle = 1 + R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E}).$$

■

We should note that, in practical situations, many physically interesting processes belong to $\mathcal{N}$. As an example, if $\mathcal{F}$ is a subset of all combs in $\mathsf{Her}_{\tilde{V}}$ and $\mathrm{int}(\mathcal{K}^*) \cap \mathsf{Comb}^*_{W_T, V_T, \dots, W_1, V_1}$ is not empty, then any comb in $\mathsf{Her}_{\tilde{V}}$ belongs to $\mathcal{N}$. [Indeed, arbitrarily choose $\phi \in \mathrm{int}(\mathcal{K}^*) \cap \mathsf{Comb}^*_{W_T, V_T, \dots, W_1, V_1}$ and a comb $\mathcal{E} \in \mathsf{Her}_{\tilde{V}}$; then, from $\phi \in \mathsf{Comb}^*_{W_T, V_T, \dots, W_1, V_1}$, $\langle \phi, Z \rangle = \langle \phi, \mathcal{E} \rangle = 1\ \ (\forall Z \in \mathcal{F})$ holds. Thus, for any $\delta < 1$, $\langle \phi, \delta Z - \mathcal{E} \rangle = \delta - 1 < 0$ holds, which yields $\delta Z - \mathcal{E} \notin \mathcal{K}$. Therefore, we have $\mathcal{E} \in \mathcal{N}$.] For instance, if $\mathcal{F} \subseteq \mathsf{Den}_V$ and $\mathrm{Tr}\, x > 0\ \ (\forall x \in \mathcal{K} \setminus \{\mathbb{0}\})$ hold, then any $\rho \in \mathsf{Den}_V$ is in $\mathcal{N}$. As another example, if $R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E}) > 0$ holds, then $\mathcal{E} \in \mathcal{N}$ always holds. [Indeed, by contraposition, assume $\mathcal{E} \notin \mathcal{N}$; then, there exists $\delta < 1$ and $Z \in \mathcal{F}$ such that $\delta Z - \mathcal{E} \in \mathcal{K}$. Let $\lambda^\star \coloneqq R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E})$. It is easily seen from $R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E}) \ge 0$ that there exists $Z^\star \in \mathcal{F}$ such that $(1 + \lambda^\star) Z^\star - \mathcal{E} \in \mathcal{K}$. Let $p \coloneqq (1 - \delta)/(\lambda^\star + 1 - \delta)$ and $Z' \coloneqq p(1 + \lambda^\star) Z^\star + (1 - p)\delta Z$; then, we have $0 \le p \le 1$, $Z' \in \mathcal{F}$, and $Z' - \mathcal{E} \in \mathcal{K}$. This implies $R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E}) = 0$.]

We obtain the following proposition.

Proposition S8 Let us consider $\mathcal{E} \in \mathcal{N}$. Let $\tilde{V}'$ be an arbitrary system. We consider a set of pairs $\mathcal{L} \coloneqq \{(\{\hat{\mathcal{J}}_m\}_{m=1}^M, \{\Phi_m\}_{m=1}^M)\}$, where $\{\hat{\mathcal{J}}_m : \mathsf{Her}_{\tilde{V}} \to \mathsf{Her}_{\tilde{V}'}\}_{m=1}^M$ is a collection of linear maps and $\Phi_1, \dots, \Phi_M \in \mathsf{Her}_{\tilde{V}'}$. Assume that the cone generated by

$$\mathcal{X} \coloneqq \left\{ \sum_{m=1}^M \hat{\mathcal{J}}_m^\dagger(\Phi_m) : \ (\{\hat{\mathcal{J}}_m\}, \{\Phi_m\}) \in \mathcal{L} \right\}$$

is $\mathcal{K}^*$, where $\hat{\mathcal{J}}_m^\dagger$ is the adjoint of $\hat{\mathcal{J}}_m$, which is defined as $\langle \hat{\mathcal{J}}_m^\dagger(\Phi'), \mathcal{E}' \rangle = \langle \Phi', \hat{\mathcal{J}}_m(\mathcal{E}') \rangle\ \ (\forall \mathcal{E}' \in \mathsf{Her}_{\tilde{V}}, \Phi' \in \mathsf{Her}_{\tilde{V}'})$. Then, we have

$$\max_{(\{\hat{\mathcal{J}}_m\}, \{\Phi_m\}) \in \mathcal{L}'} \frac{\sum_{m=1}^M \langle \Phi_m, \hat{\mathcal{J}}_m(\mathcal{E}) \rangle}{\max_{Z \in \mathcal{F}} \sum_{m=1}^M \langle \Phi_m, \hat{\mathcal{J}}_m(Z) \rangle} = 1 + R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E}), \tag{S31}$$

where

$$\mathcal{L}' \coloneqq \left\{ (\{\hat{\mathcal{J}}_m\}, \{\Phi_m\}) \in \mathcal{L} : \sum_{m=1}^M \hat{\mathcal{J}}_m^\dagger(\Phi_m) \neq \mathbb{0} \right\}.$$

Proof The left-hand side of Eq. (S31) is rewritten by

$$\max_{\varphi \in \mathcal{X} \setminus \{\mathbb{0}\}} \frac{\langle \varphi, \mathcal{E} \rangle}{\max_{Z \in \mathcal{F}} \langle \varphi, Z \rangle}.$$

Thus, an application of Lemma S7 completes the proof. ■

The operational meaning of Eq. (S31) is as follows. Suppose that $\{\hat{\mathcal{J}}_m\}_m$ is a collection of (unnormalized) processes such that $\sum_{m=1}^M \hat{\mathcal{J}}_m$ is a comb from $\mathsf{Pos}_{\tilde{V}}$ to $\mathsf{Pos}_{\tilde{V}'}$ and that $\{\Phi_k\}_k$ is a tester, where the pair $(\{\hat{\mathcal{J}}_m\}_m, \{\Phi_k\}_k)$ is restricted to belong to $\mathcal{L}$. We consider the situation that a party, Alice, applies a process $\hat{\mathcal{J}}_m$ to a comb $\mathcal{E} \in \mathsf{Pos}_{\tilde{V}} \cap \mathcal{N}$, and then another party, Bob, applies a tester $\{\Phi_k\}_k$ to $\hat{\mathcal{J}}_m(\mathcal{E})$. The probability of Bob correctly guessing which of the processes $\hat{\mathcal{J}}_1, \dots, \hat{\mathcal{J}}_M$ Alice applies is expressed by $\sum_{m=1}^M \langle \Phi_m, \hat{\mathcal{J}}_m(\mathcal{E}) \rangle$ [note that $\sum_{k=1}^M \sum_{m=1}^M \langle \Phi_k, \hat{\mathcal{J}}_m(\mathcal{E}) \rangle = 1$ holds]. Equation (S31) implies that the advantage of $\mathcal{E}$ over all

$Z \in \mathcal{F}$ in such a discrimination problem can be exactly quantified by the robustness $R^{\mathcal{F}}_{\mathcal{K}}(\mathcal{E})$. In this situation, $\mathcal{K}^* \subseteq \mathsf{Pos}^*_{\tilde{V}}$, i.e., $\mathcal{K} \supseteq \mathsf{Pos}_{\tilde{V}}$, holds.

We give two examples of the application of Proposition S8. The first example is the case $\mathcal{K} = \mathsf{Pos}_{\tilde{V}}$. Let us consider the case where $\sum_{m=1}^{M} \hat{\mathcal{J}}_m$ can be any comb from $\mathsf{Pos}_{\tilde{V}}$ to $\mathsf{Pos}_{\tilde{V}'}$ and $\{\Phi_k\}_k$ can be any tester. We can easily see that Eq. (S31) with $\mathcal{K} = \mathsf{Pos}_{\tilde{V}}$ holds. Note that, for example, Theorem 2 of Ref. [S7] and Theorems 1 and 2 of Ref. [S8] can be understood as special cases of Proposition S8 with $\mathcal{K} = \mathsf{Pos}_{\tilde{V}}$. The second example is the case $\mathcal{K} \neq \mathsf{Pos}_{\tilde{V}}$. For instance, for a given channel $\hat{\mathcal{E}}$ from a system V to a system W, assume that $\hat{\mathcal{J}}_m$ is the process that applies $\hat{\mathcal{E}}$ to a state $\rho_m \in \mathsf{Den}_V$ with probability p_m [i.e., $\hat{\mathcal{J}}_m(\mathcal{E}) = p_m \operatorname{Tr}_V[(I_W \otimes \rho_m^{\mathsf{T}})\mathcal{E}]$ holds] and $\{\Phi_k\}_k$ is a measurement of W. Then, we have $\langle \Phi_k, \hat{\mathcal{J}}_m(-)\rangle = p_m \langle \Phi_k \otimes \rho_m^{\mathsf{T}}, -\rangle$. It is easily seen that Eq. (S31) with $\mathcal{K}^* = \mathsf{Sep}_{W,V}$ (or, equivalently, $\mathcal{K} = \mathsf{Sep}^*_{W,V}$) holds, where $\mathsf{Sep}_{W,V}$ is the set of all bipartite separable elements in $\mathsf{Pos}_{W \otimes V}$. Note that, for a linear map $\hat{\Psi}$ from V to W, $\Psi \in \mathsf{Sep}^*_{W,V}$ holds if and only if $\hat{\Psi}$ is a positive map.

[S1] A. Dhara and J. Dutta, *Optimality conditions in convex optimization: a finite-dimensional view* (CRC Press, 2011).
[S2] K. Nakahira and K. Kato, Phys. Rev. A **103**, 062606 (2021).
[S3] G. Chiribella, New J. Phys. **14**, 125008 (2012).
[S4] A. Jenčová and M. Plávala, J. Math. Phys. **57**, 122203 (2016).
[S5] Q. Zhuang and S. Pirandola, Phys. Rev. Lett. **125**, 080505 (2020).
[S6] A. W. Harrow and M. A. Nielsen, Phys. Rev. A **68**, 012308 (2003).
[S7] R. Takagi, B. Regula, K. Bu, Z.-W. Liu, and G. Adesso, Phys. Rev. Lett. **122**, 140402 (2019).
[S8] R. Uola, T. Kraft, J. Shang, X.-D. Yu, and O. Gühne, Phys. Rev. Lett. **122**, 130404 (2019).