Y. Kuramochi: Infinite dimensionality of the post-processing order of measurements on a general state space

Referee report

The paper studies the order dimension, resp. the order monotone dimension of the postprocessing (pre)order on the set of finite outcome measurements on the state space of arbitrary general probabilistic theory. It is proved that apart from the trivial case of the singleton state space, the order (monotone) dimension is infinite. If the state space is norm-separable, then the order (monotone) dimension is countably infinite. A similar result is shown for the postprocessing preorder of quantum channels with fixed separable input space.

The postprocessing preorder on the set of measurements or more general channels is important in quantum information theory, for example, by the quantum version of the BSS theorem (Prop. 1 and 2) it is directly related to optimal guessing probabilities in discrimination problems. (Pre)orders of this type also play an important role in resource theories. The main result of the paper shows that the preorder on measurement can never be characterized by a finite number of order monotones, in any general probabilistic theory, including the classical and quantum theory. In particular, in the quantum case, a countable number of inequalities have to be checked. This is in contrast with some other orderings studied in quantum information, such as adiabatic accessibility in thermodynamics or LOCC convertibility of bipartite states.

I find the results important and interesting, bringing new insight into the postprocessing preorder on quantum measurements and channels. I only have a few minor comments listed below.

1. One could also mention that the postprocessing preorder is also used for characterization of compatible measurements (e.g. arXiv:2202.00725, where different order monotones are used).
2. p.4, Definition 1.(i): better move the reference to [21] at the end of the last sentence in this paragraph
3. p. 8, Prop. 1: "..for all ensemble.." ensembles
4. p. 12: the notation $\bigotimes$ for the direct product seems strange to me, the notation $\times$ or Π is more usual.
5. Is the notion of dpc really needed for the proof of Lemma 2? If f is order monotone, then $x \leq_f y$ if $f(x) \leq f(y)$ defines a linear extension of $\preceq$ and the inequality is a direct consequence of the definition of the orders.