Math. Nachr. 147 (1990) 123—138

Noncommutative Orlicz Spaces and Generalized Arens Algebras

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(Received July 5, 1988)

0. Introduction

The noncommutative integration over a semifinite VON NEUMANN algebra generalizes the usual LEBESGUE integration and the L_p-spaces $L_p(\Omega, \mu)$ over a localizable measure space (Ω, μ) (see [18, 21, 25]). We deal with the noncommutative analogue of the ORLICZ space. In the usual ORLICZ space theory there are two equivalent startpoints. The classical definition of ORLICZ is given by the norm $\|x\|'_\phi = \sup\left\{\int\limits_\Omega |x(t)\, y(t)|\, d\mu(t)\colon y \in \mathcal{M}(\Omega, \mu) \text{ and } \int\limits_\Omega \psi(|y(t)|)\, d\mu(t) \leqq 1\right\}$ for a pair (ϕ, ψ) of complementary YOUNG functions. From the viewpoint of locally convex spaces we can start with the MINKOWSKI functional

$$\|x\|_\phi = \inf\left\{\lambda > 0\colon \int\limits_\Omega \phi\left(\frac{1}{\lambda}\, |x|\right) d\mu(t) \leqq 1\right\}$$

to the absolutely convex set

$$K_\phi = \left\{x \in \mathcal{M}(\Omega, \mu)\colon \int\limits_\Omega \phi(|x(t)|)\, d\mu(t) \leqq 1\right\}$$

as norm, the so called LUXEMBURG norm.

Some results to noncommutative ORLICZ spaces is due to M. A. MURATOV ([16, 17]) generalizing the first definition.

We start with the second definition and show in section 2 that

$$K_\phi = \{a \in \mathcal{M}(\mathcal{A})\colon \tau(\phi(|a|)) \leqq 1\}$$

is an absolutely convex set for a semifinite VON NEUMANN algebra $\mathcal{A}$ with a faithful semifinite normal trace τ and a positive increasing convex function ϕ on $[0, \infty)$. We call the linear space $L_\phi(\mathcal{A}, \tau)$ generated by K_ϕ ORLICZ space and furnish $L_\phi(\mathcal{A}, \tau)$ with the MINKOWSKI functional to K_ϕ as norm. $L_\phi(\mathcal{A}, \tau)$ is a BANACH space. We generalize the YOUNG inequality for a pair (ϕ, ψ) of complementary YOUNG functions:

$$\tau(|ab|) \leqq \tau(\phi(|a|)) + \tau(\psi(|b|)).$$

Using the YOUNG inequality we can prove the equivalence of the ORLICZ and the LUXEMBURG norm. For $\phi \in \Delta_2$, $L_\psi(\mathcal{A}, \tau)$ is isomorphic to the dual space of $L_\phi(\mathcal{A}, \tau)$. In section 3 we compare the ORLICZ spaces to different YOUNG functions and are able to construct "noncommutative" locally convex spaces. This we use in section 4 and introduce the generalized ARENS algebras

$$L^\Lambda(\mathcal{A}, \tau) = \bigcap_{\phi \in \Lambda} L_\phi(\mathcal{A}, \tau)$$

for quadratic families Λ of YOUNG functions. We furnish $L^\Lambda(\mathcal{A}, \tau)$ with the topology of the projective limit τ^Λ with respect to the identical imbettings of $L^\Lambda(\mathcal{A}, \tau)$ in $L_\phi(\mathcal{A}, \tau)$. $L^\Lambda(\mathcal{A}, \tau)[\tau^\Lambda]$ is a locally convex *-algebra with jointly continuous multiplication. Using the comparable proposition and regarding the great possibilities for choosing YOUNG functions we get a new wide class of *-algebras of unbounded operators.

Acknowledgement. I am grateful to Prof. K. SCHMÜDGEN and Dr. K. KÜRSTEN for reading the manuscript.

1. Preliminaries

Throughout this paper, let $\mathcal{A}$ be a semifinite VON NEUMANN algebra on a HILBERT space $\mathcal{H}$ with a faithful normal semifinite trace τ. A densely-defined closed operator x affiliated with $\mathcal{A}$ is said to be τ-measurable if, for each $\varepsilon > 0$, there exists a projection p in $\mathcal{A}$ such that $p\mathcal{H} \subseteqq \mathcal{D}(x)$ and $\tau(1 - p) < \varepsilon$. The set of all τ-measurable operators we denote by $\mathcal{M}(\mathcal{A})$. It is a topological *-algebra equipped with the measure topology (see [18]). The measure topology on $\mathcal{M}(\mathcal{A})$ is by definition the linear topology whose fundamental system of neighborhoods of zero is given by $V(\varepsilon, \delta) = \{a \in \mathcal{M}(\mathcal{A})$: there exists a projection $f \in \mathcal{A}$ with

$$\|af\| < \varepsilon \quad \text{and} \quad \tau(1 - f) < \delta\}.$$

The linear space $L_p(\mathcal{A}, \tau) = \{a \in \mathcal{M}(\mathcal{A}) \colon \tau(|a|^p) < \infty\}$, $p \in [1, \infty)$, is a BANACH space in the norm $\|a\|_p = \tau(|a|^p)^{1/p}$ (see [25]). Let x be a densely-defined closed operator affiliated with $\mathcal{A}$. Let $x = u\,|x|$ be the polar decomposition and $|x| = \int_0^\infty \lambda \, de_\lambda$ be the spectral decomposition. The rearrangement $\tilde{x}$ of x is the function $\tilde{x} \colon (0, \infty) \to (0, \infty)$ defined by

$$\tilde{x}(t) = \inf \{\lambda > 0 \colon \tau(e_{(\lambda, \infty)}) \leqq t\},$$

where $e_{(\lambda,\infty)}$ is the spectral projection of $|x|$ corresponding to the interval (λ, ∞). The rearrangement $\tilde{x}(t)$ is the noncommutative analogue of the distribution function in the classical analysis.

1.1. Lemma [6]. *Let $0 \leqq a \in \mathcal{M}(\mathcal{A})$. Then we have $\varphi(\tilde{a}(t)) = \widetilde{\varphi(a)}\,(t)$, $t > 0$ for each continuous increasing function ϕ on $[0, \infty)$ with $\phi(0) \geqq 0$ and $\tau(\phi(a)) = \int_0^\infty \phi(\tilde{a}(t))\, dt$.*

1.2. Proposition [1]. *For any $a, b \in \mathcal{M}(\mathcal{A})$ there are partial isometries $u, v \in \mathcal{A}$ with $uu^* = vv^* = 1$ and*

$$|a + b| \leqq u^* |a| u + v^* |b| v.$$

The proof for measurable operators is straightforward like in [1] using the fact that the square root function is operator monoton.

1.3. Proposition [6]. *Let $0 \leqq a, b \in \mathcal{M}(\mathcal{A})$. Then we have:*

(i) $\tau(\phi(a)) \leqq \tau(\phi(b))$ *for $a \leqq b$ and for any continuous increasing function ϕ on $[0, \infty)$ with $\phi(0) = 0$.*

(ii) $\tau\left(\phi(\alpha a + (1 - \alpha) b)\right) \leqq \alpha\tau(\phi(a)) + (1 - \alpha) \tau(\phi(b))$ *for $0 \leqq \alpha \leqq 1$ and for any convex increasing function ϕ with $\phi(0) = 0$.*

We shall say that ϕ is a Young function, if

$$\phi(t) = \int_0^t \varphi(s)\, ds, \qquad t \geqq 0,$$

where the real-valued function φ defined on $[0, \infty)$ has the following properties:

(i) $\varphi(0) = 0$, $\varphi(s) > 0$ for $s > 0$ and $\lim\limits_{s\to\infty} \varphi(s) = \infty$,
(ii) φ is right continuous,
(iii) φ is nondecreasing on $(0, \infty)$.

Every Young function is a continuous, convex and strictly increasing function. For every Young function there is a complementary Young function ψ given by the density

$$\tilde{\psi}(t) = \sup \{s : \varphi(s) \leqq t\}.$$

The complement of ψ is ϕ again. A pair of complementary Young functions (ϕ, ψ) fulfils the Young inequality:

$$st \leqq \phi(s) + \psi(t), \qquad s, t \in [0, \infty)$$

and the equality holds if and only if $t = \varphi(s)$ or $s = \tilde{\psi}(t)$. Further a Young function is said to satisfy the Δ_2-condition, shortly $\phi \in \Delta_2$, if there exists a $k > 0$ and $T \geqq 0$ such that:

$$\phi(2t) \leqq k\phi(t) \quad \text{for all} \quad t \geqq T.$$

For the background of Young functions and Orlicz spaces see [11] or [12].

We use the topological notions as in [19].

2. Noncommutative Orlicz spaces

We shall define the noncommutative Orlicz spaces by the Luxemburg norm. We start with the proof of the absolute convexity of the set:

$$K_\phi = \left\{a \in \mathcal{M}(\mathcal{A}) : \tau(\phi(|a|)) \leqq 1\right\}.$$

2.1. Proposition. *Let φ be an increasing convex function over $[0, \infty)$ with $\varphi(0) = 0$. Then*

$$K_{\mathrm{n}} = \{a \in \mathcal{M}(\mathcal{A}) : \tau(\phi(|a|)) \leqq 1\}$$

is absolutely convex.

Proof. Let $a, b \in K_\phi$ and $0 \leqq \alpha \leqq 1$. By Proposition 1.2 there are isometries u, v with

$$|\alpha a + (1 - \alpha) b| \leqq \alpha u^* |a| u + (1 - \alpha) v^* |b| v.$$

Since u, v are partial isometries with $uu^* = vv^* = 1$, we have

$$\phi(u^* |a| u) = u^* \phi(|a|) u \quad \text{and} \quad \phi(v^* |b| v) = v^* \phi(|b|) v.$$

Hence,

$$\begin{aligned} \tau\left(\phi(|\alpha a + (1 - \alpha) b|)\right) &\leqq \tau\left(\phi(\alpha u^* |a| u + (1 - \alpha) v^* |b| v)\right) \\ &\qquad \text{(Proposition 1.3(i))} \\ &\leqq \alpha\tau(\phi(u^* |a| u)) + (1 - \alpha) \tau(\phi(v^* |b| v)) \\ &\qquad \text{(Proposition 1.3(ii))} \\ &= \alpha\tau(\phi(|a|)) + (1 - \alpha) \tau(\phi(|b|)) \leqq 1. \end{aligned}$$

We call $L_\phi(\mathcal{A}, \tau) = \bigcup\limits_{n=1}^{\infty} nK_\phi$ an ORLICZ *space* and equip $L_\phi(\mathcal{A}, \tau)$ with the MINKOWSKI functional

$$\|a\|_\phi = \inf\left\{\lambda > 0 : \frac{1}{\lambda} a \in K_\phi\right\}, \qquad a \in L_\phi(\mathcal{A}, \tau),$$

to K_ϕ. $\|\cdot\|_\phi$ is a norm. The proof is the same like for commutative VON NEUMANN algebras. For commutative VON NEUMANN algebras we obtain usual ORLICZ spaces $L_\phi(\Omega, \mu)$ over a localizable measure space (Ω, μ). If $\mathcal{B}$ is a VON NEUMANN subalgebra of $\mathcal{A}$, then:

$$L_\phi(\mathcal{B}, \tau) = \mathcal{M}(\mathcal{B}) \cap L_\phi(\mathcal{A}, \tau).$$

2.2. Proposition (Young inequality). *For a pair (ϕ, ψ) of complementary* YOUNG *functions we have:*

$$\tau(|ab|) \leqq \tau(\phi(|a|)) + \tau(\psi(|b|)) \quad \textit{for all} \quad a, b \in \mathcal{M}(\mathcal{A}).$$

Moreover, if $0 \leqq a \in \mathcal{M}(\mathcal{A})$ with $\tau(\phi(a)) < \infty$, then there is a $0 \leqq b \in \mathcal{M}(\mathcal{A})$ with

$$\tau(ab) = \tau(\phi(a)) + \tau(\psi(b)) \quad \textit{and} \quad \tau(\psi(b)) \leqq 1.$$

Proof. By [25], theorem 3.3, we have

$$\tau(|ab|) \leqq \int\limits_0^\infty \tilde{a}(t)\, \tilde{b}(t)\, dt.$$

We use the classical YOUNG inequality

$$\int\limits_0^\infty \tilde{a}(t)\, \tilde{b}(t)\, dt \leqq \int\limits_0^\infty \phi(\tilde{a}(t))\, dt + \int\limits_0^\infty \psi(\tilde{b}(t))\, dt$$

and obtain

$$\tau(|ab|) \leqq \int_0^\infty \tilde{a}(t)\,\tilde{b}(t)\,dt \leqq \int_0^\infty \phi(\tilde{a}(t))\,dt + \int_0^\infty \psi(\tilde{b}(t))\,dt$$
$$= \tau(\phi(|a|)) + \tau(\psi(|b|)) \qquad \text{(Lemma 1.1)}.$$

For the second statement we choose b commuting with a and get so a problem for functions. This we can solve by the classical YOUNG inequality (see [11, or 12]).

Let (ϕ, ψ) be a pair of complementary YOUNG functions. The following norms are generalizations of the ORLICZ norm:

$$\|a\|_\phi' = \sup\{\tau(|ab|) : b \in \mathcal{M}(\mathcal{A}) \text{ and } \tau(\psi(|b|)) \leqq 1\}$$
$$\|a\|_\phi'' = \sup\{|\tau(ab)| : b \in \mathcal{M}(\mathcal{A}) \text{ and } \tau(\psi(|b|)) \leqq 1\}.$$

2.3. Proposition. *The norms $\|\cdot\|_\phi$, $\|\cdot\|_\phi'$ and $\|\cdot\|_\phi''$ are equivalent on $L_\phi(\mathcal{A}, \tau)$, i.e. more precisely it is:*

$$\|a\|_\phi \leqq \|a\|_\phi'' \leqq \|a\|_\phi' \leqq 2\,\|a\|_\phi \quad \textit{for all} \quad a \in L_\phi(\mathcal{A}, \tau).$$

Proof. 1. Let $a \in L_\phi(\mathcal{A}, \tau)$ and $\|a\|_\phi'' = 1$. We can choose a $0 \leqq b \in \mathcal{M}(\mathcal{A})$ with $\tau(|a|\,b) = \tau(\phi(|a|)) + \tau(\psi(b))$ and $\tau(\psi(b)) \leqq 1$ by the foregoing Proposition. Hence,

$$\tau(\phi(|a|)) \leqq \tau(\phi(|a|)) + \tau(\psi(b)) = \tau(|a|\,b).$$

By the polar decomposition $a = v\,|a|$ we obtain

$$\tau(|a|\,b) = \tau(v^*ab) = \tau(abv^*)$$
$$\leqq \sup\{|\tau(ac)| : c \in \mathcal{M}(\mathcal{A}) \text{ and } \tau(\psi(|c|)) \leqq 1\} = 1.$$

Therefore,

$$\|a\|_\phi = \inf\left\{\lambda > 0 : \tau\left(\phi\left(\frac{1}{\lambda}\,|a|\right)\right) \leqq 1\right\} \leqq \|a\|_\phi''.$$

2. By the polar decomposition we get $|\tau(ab)| \leqq \tau(|ab|)$ and consequently

$$\|a\|_\phi'' \leqq \|a\|_\phi'.$$

3. Using Proposition 2.2 we obtain for $a, b \in \mathcal{M}(\mathcal{A})$ with $\|a\|_\phi = 1$, $\tau(\psi(|b|)) \leqq 1$:

$$\tau(|ab|) \leqq \tau(\phi(|a|)) + \tau(\psi(|b|)) \leqq 2$$

so that $\|a\|_\phi' \leqq 2\,\|a\|_\phi$ for all $a \in L_\phi(\mathcal{A}, \tau)$.

2.4. Lemma (Hoelder inequality).

$$|\tau(ab)| \leqq \tau(|ab|) \leqq \|a\|_\phi'\,\|b\|_\psi \leqq 2\,\|a\|_\phi\,\|b\|_\psi$$

for $a \in L_\phi(\mathcal{A}, \tau)$, $b \in L_\psi(\mathcal{A}, \tau)$.

Proof. By the definition of $\|\cdot\|_\phi'$ we obtain

$$|\tau(ab)| \leqq \tau(|ab|) \leqq \tau\left(\left|ab\,\frac{1}{\|b\|_\psi}\right|\right)\|b\|_\psi \leqq \|a\|_\phi'\,\|b\|_\psi$$

for $a \in L_\phi(\mathcal{A}, \tau)$, $b \in L_\psi(\mathcal{A}, \tau)$.

2.5. Proposition. *$L_\phi(\mathcal{A}, \tau)$ $[\|\cdot\|_\phi]$ is a* BANACH *space.*

Proof. Let $\{a_n\}$ be a CAUCHY sequence in $L_\phi(\mathcal{A}, \tau)$ and let $|a_n - a_{n+1}| = \int\limits_0^\infty \lambda \, de_{\lambda,n}$. Given $\varepsilon > 0$, there is an $n_0 \in \mathbb{N}$ with $\|a_n - a_{n+1}\|_\phi < \varepsilon$ for all $n \geqq n_0$.

Then we have

$$\phi\left(\frac{c}{\varepsilon}\right)\tau(1 - e_{c,n}) \leqq \int\limits^\infty \phi\left(\frac{\lambda}{\varepsilon}\right) d\tau(e_{\lambda,n}) = \tau\left(\phi\left(\frac{|a_n - a_{n+1}|}{\varepsilon}\right)\right) \leqq 1$$

and

$$\|(a_n - a_{n+1})\, e_{c,n}\| < c\,.$$

Let $c = \dfrac{1}{2^n}$. There exists a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ and $\phi\left(\dfrac{1}{2^n\varepsilon_n}\right) \to \infty$ for $n \to \infty$.

Hence, there is a subsequence $\{a_{n'}\}$ of $\{a_n\}$ and projections $p_{n'}$ with

$$\|(a_{n'} - a_{n'+1})\, p_{n'}\| < \frac{1}{2^{n'}}$$

$$\tau(1 - p_{n'}) < \frac{1}{2^{n'}}.$$

As in the proof of [24], theorem 3.4, $\{a_{n'}\}$ converges in measure to $a \in \mathcal{M}(\mathcal{A})$. For arbitrary $c \in \mathcal{M}(\mathcal{A})$ with $\tau(\psi(|c|)) \leqq 1$ the sequence $\{a_{n'}\, c\}$ converges in measure to ac by [24], Theorem 3.3., $\{a_{n'}c\}$ is a CAUCHY sequence in $L_1(\mathcal{A}, \tau)$ by Proposition 2.3, so $L_1(\mathcal{A}, \tau)$ is complete. Hence, $a_{n'}c \to ac$ in $L_1(\mathcal{A}, \tau)$ by $n' \to \infty$. By Proposition 2.3 we have $a \in L_\phi(\mathcal{A}, \tau)$ and $a_n \to a$ in $L_\phi(\mathcal{A}, \tau)$ $[\|\cdot\|_\phi]$ by $n \to \infty$.

2.6. Proposition. *Let ϕ be a* YOUNG *function. Then the following is true:*

(i) *If $\phi \in \Delta_2$, then we have*

$$L_\phi(\mathcal{A}, \tau) = \{a \in \mathcal{M}(\mathcal{A}) : \tau(\phi(|a|)) < \infty\}.$$

(ii) *If there exists a projection z in the center $\mathcal{Z}$ of $\mathcal{A}$ such that $\mathcal{A}z$ has no minimal projection, then $L_\phi(\mathcal{A}, \tau) = \{a \in \mathcal{M}(\mathcal{A}) : \tau(\phi(|a|)) < \infty\}$ implies $\phi \in \Delta_2$.*

If $\mathcal{A}$ is generated by a countable set of minimal projections p_k with $0 < \liminf \tau(p_k) \leqq \limsup \tau(p_k) < \infty$, then we obtain from $L_\phi(\mathcal{A}, \tau) = \{a \in \mathcal{M}(\mathcal{A}) : \tau(\phi(|a|)) < \infty\}$ the δ_2-condition for ϕ, i.e. there are $c > 0$, $T > 0$ with $\phi(2t) \leqq c\phi(t)$ for all $0 \leqq t \leqq T$.

Remark. The reverse implication in Proposition 2.6 is not complete. If $\mathcal{A}$ is generated by a finite number of minimal projections, we get no condition on ϕ. Moreover, if $\mathcal{A}$ is generated by a countable set $\{p_k\}$ of minimal projections and $\liminf \tau(p_k) = 0$, then there examples for both, $\phi \in \Delta_2$ and $\phi \notin \Delta_2$. It is possible to separate the cases farther (see [8]).

Proof. Ad (i). The Δ_2-condition for ϕ is equivalent to the following:

For all $l \in (0, \infty)$ there exists a $k(l) \in (0, \infty)$ with

$$\phi(lt) \leqq k(l)\, \phi(t) \quad \text{for all } t \geqq T.$$

Let $a \in L_\phi(\mathcal{A}, \tau)$ with $\|a\|_\phi \leqq n$ and let $|a| = \int\limits_0^\infty \lambda \, de_\lambda$ be the spectral decomposition of $|a|$. Then

$$\tau\left(\phi(|a|\,(1 - e_T))\right) \leqq k(n)\, \tau\left(\phi\left(\frac{1}{n}\,|a|\,(1 - e_T)\right)\right) \leqq k(n).$$

If $\tau(1) < \infty$ then $\tau(\phi(|a|\, e_T)) \leqq \phi(T)\, \tau(1)$ and

$$\begin{aligned}\tau(\phi(|a|)) &= \tau\left(\phi(|a|\,(1 - e_T))\right) + \tau(\phi(|a|\, e_T)) \\ &\leqq \phi(T)\, \tau(1) + k(n) < \infty.\end{aligned}$$

Hence $L_\phi(\mathcal{A}, \tau) \subseteqq \{a \in \mathcal{M}(\mathcal{A}) \colon \tau(\phi(|a|)) < \infty\}$. If $a \in \mathcal{M}(\mathcal{A})$ and $\tau(\phi(|a|)) = k_a \in [1, \infty)$, then $\tau\left(\phi\left(\frac{1}{k_a}\,|a|\right)\right) \leqq \frac{1}{k_a}\, \tau(\phi(|a|)) \leqq 1$ by the spectral theorem and the convexity of ϕ. Therefore,

$$L_\phi(\mathcal{A}, \tau) \supseteqq \{a \in \mathcal{M}(\mathcal{A}) \colon \tau(\phi(|a|)) < \infty\}.$$

Ad (ii). We restrict us to the Orlicz space over a commutative von Neumann subalgebra of $\mathcal{A}$ and get the assertion as in [8, 12].

As for usual Orlicz spaces we define

$$E_\phi(\mathcal{A}, \tau) = \overline{\mathcal{A} \cap L_\phi(\mathcal{A}, \tau)}^{\|\cdot\|_\phi}.$$

Evidently, $E_\phi(\mathcal{A}, \tau)$ is a linear space.

2.7. Proposition. *We have:*

(i) *For $\phi \in \Delta_2$ we get $L_\phi(\mathcal{A}, \tau) = E_\phi(\mathcal{A}, \tau)$.*

(ii) *If there is a projection $z \in \mathcal{Z}$ with $\tau(z) < \infty$ and $\mathcal{A}z$ has no minimal projection, then $L_\phi(\mathcal{A}, \tau) = E_\phi(\mathcal{A}, \tau)$ implies $\phi \in \Delta_2$.*

Proof. Ad (i). If $\phi \in \Delta_2$, we have

$$L_\phi(\mathcal{A}, \tau) = \{a \in \mathcal{M}(\mathcal{A}) \colon \tau(\phi(|a|)) < \infty\},$$

i.e. if $a \in L_\phi(\mathcal{A}, \tau)$, then

$$\tau(\phi(|a|)) = \int\limits_0^\infty \phi(\lambda)\, d\tau(e_\lambda) \quad \text{for } |a| = \int\limits_0^\infty \lambda\, de_\lambda.$$

Hence, for every $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ with

$$\int\limits_{n_0}^\infty \phi(\lambda)\, d\tau(e_\lambda) < \varepsilon.$$

We choose for $\varepsilon > 0$ $a_\varepsilon = \int\limits_0^{n_0} \phi(\lambda)\, de_\lambda \in \mathcal{A} \cap L_\phi(\mathcal{A}, \tau)$ and get the assertion.

Ad (ii). If $a \in E_\phi(\mathcal{A}z, \tau)$, then there is an element $b \in \mathcal{A}z$ with $\|a - b\|_\phi < \frac{1}{2}$. By Proposition 1.2 there exists partial isometries $u, v \in \mathcal{A}z$ with $uu^* = vv^* = 1$ and

$$|a| \leqq u^*\, |a - b|\, u + v^*\, |b|\, v.$$

Also, by Proposition 1.3 we have

$$\tau(\phi(|a|)) \leqq \tau(\phi(u^* |a - b| u + v^* |b| v))$$

$$\leqq \frac{1}{2} \tau(\phi(2 |a - b|)) + \frac{1}{2} \tau(\phi(2 |b|))$$

$$\leqq \frac{1}{2} + \frac{1}{2} \tau(z) \phi(2 \|b\|)$$

i.e. $E_\phi(\mathcal{A}z, \tau) \subseteqq \{a \in \mathcal{M}(\mathcal{A}z): \tau(\phi(|a|)) < \infty\}$. We get the assertion from $L_\phi(\mathcal{A}z, \tau) = E_\phi(\mathcal{A}z, \tau) = \{a \in \mathcal{M}(\mathcal{A}z): \tau\phi((|a|)) < \infty\}$ by Proposition 2.6.

2.8. Proposition. *Let (ϕ, ψ) be a complementary pair of* YOUNG *functions. Then:*

(i) *For $\phi \in \Delta_2$ the dual space of $L_\phi(\mathcal{A}, \tau)$ is isomorphic to $L_\psi(\mathcal{A}, \tau)$.*

(ii) *We suppose that the dual space of $L_\phi(\mathcal{A}, \tau)$ is isomorphic to $L_\psi(\mathcal{A}, \tau)$. If there is a projection $z \in \mathcal{Z}$ with $\tau(z) < \infty$ and $\mathcal{A}z$ has no minimal projection, then we have $\phi \in \Delta_2$.*

Remark. For $b \in L_\psi(\mathcal{A}, \tau)$ let $f_b(a) = \tau(ab)$, $a \in L_\phi(\mathcal{A}, \tau)$. Then f_b is a continuous linear functional on $L_\phi(\mathcal{A}, \tau)$ and $\|f_b\| = \|b\|_\psi''$.

Proof. Ad (i). ϕ is a convex function with $\phi(0) = 0$. If we have $0 \leqq x \leqq 1$, then $0 \leqq \phi(x) \leqq \phi(1)\, x$. As in the proof of Theorem 4.4 in [25] there exists a $x \in \mathcal{M}(\mathcal{A})$ with $f(a) = \tau(ax)$. Let $a \in L_\phi(\mathcal{A}, \tau)$. Since we have $L_\phi(\mathcal{A}, \tau) = E_\phi(\mathcal{A}, \tau)$, by Proposition 2.6 there is a sequence $\{a_n\} \subseteqq \mathcal{A} \cap L_\phi(\mathcal{A}, \tau)$ with $a_n \to a$ in $L_\phi(\mathcal{A}, \tau)$ by $n \to \infty$. Hence,

$$|\tau(xa_n - xa_m)| = |f(a_n - a_m)| \leqq \|f\| \, \|a_n - a_m\|_\phi$$

and $\{xa_n\}$ is a CAUCHY sequence in $L_1(\mathcal{A}, \tau)$. Therefore $xa \in L_1(\mathcal{A}, \tau)$, since $a_n \to a$ in measure. Consequently,

$$f(a) = \tau(xa) \quad \text{for all} \quad a \in L_\phi(\mathcal{A}, \tau).$$

Since $f \in L_\phi(\mathcal{A}, \tau)'$, we obtain

$$|\tau(xa)| \leqq c \, \|a\|_\phi \quad \text{for all} \quad a \in L_\phi(\mathcal{A}, \tau).$$

Hence,

$$\|x\|_\psi'' \leqq c$$

and the equivalence of the norms gives $\|x\|_\psi \leqq c$, i.e. $x \in L_\phi(\mathcal{A}, \tau)$.

Ad (ii). Suppose, $\phi \notin \Delta_2$. Then $E_\phi(\mathcal{A}z, \tau)$ is a proper closed linear subspace of $L_\phi(\mathcal{A}z, \tau)$ and there exists an element $0 \neq a_0$ in $L_\phi(\mathcal{A}z, \tau)$ with $a_0 \notin E_\phi(\mathcal{A}z, \tau)$. By the HAHN-BANACH theorem the existence of a continuous linear functional f on $L_\phi(\mathcal{A}z, \tau)$ with $f(a_0) = 1$ and $f(b) = 0$ for all $b \in E_\phi(\mathcal{A}z, \tau)$ follows. Suppose that there is a $x \in L_1(\mathcal{A}z, \tau)$ with $f(a) = \tau(xa)$. We choose $x_n = |x_n| \, u^*$, where $x = u\,|x|$, $|x| = \int_0^\infty \lambda \, de_\lambda$ and $|x_n|$

$= \int_0^n \lambda \, de_\lambda$. Then $x_n \in \mathcal{A}z \cap L_\phi(\mathcal{A}z, \tau)$ for all $n \in \mathbb{N}$, since $\tau(z) < \infty$. But

$$0 = f(x_n) = \tau(x x_n) = \int_0^n \lambda^2 \, d\tau(e_\lambda) \quad \text{for all} \quad n \in \mathbb{N},$$

so that $x = 0$. This is a contradiction to $f(a_0) = 1$.

2.9. Corollary. *If there is a projection $z \in \mathcal{Z}$ with the properties that $\tau(z) < \infty$ and $\mathcal{A}z$ has no minimal projection, then $L_\phi(\mathcal{A}, \tau)$ is reflexive if and only of $\phi, \psi \in \Delta_2$.*

3. Noncommutative locally convex spaces

We start with the comparison of the noncommutative Orlicz spaces. For the Young functions the partially orderings "$\prec$" and "$\prec\!\!\!\prec$" are introduced. We say that $\phi_1 \prec \phi_2$, if there exists two nonnegative constants c and T such that $\phi_1(t) \leqq \phi_2(ct)$ for all $t \geqq T$. If we have for any $\varepsilon > 0$ that $\lim\limits_{t \to 0} \dfrac{\phi_1(t)}{\phi_2(\varepsilon t)} = 0$, then we write that $\phi_1 \prec\!\!\!\prec \phi_2$. The Young functions are in general not comparable regarding the partially ordering "$\prec$". Further, if $\phi_1 < \phi_2$, then we get for the complementary Young function $\psi_2 \prec \psi_1$. If $\phi_1 \prec\!\!\!\prec \phi_2$, then there is a Young function ϕ with $\phi_1 \circ \phi = \phi_2$. (See [12].)

3.1. Proposition. *Let ϕ_1 and ϕ_2 be Young functions. Then:*

(i) $L_{\phi_2}(\mathcal{A}, \tau) \subseteqq L_{\phi_1}(\mathcal{A}, \tau)$ *if and only if* $\phi_1 \prec \phi_2$ *(with $T = 0$ for $\tau(1) = \infty$). The validity of one side implies the continuity of the imbedding i.e.*

$$L_{\phi_2}(\mathcal{A}, \tau)\,[\|\cdot\|_{\phi_2}] \hookrightarrow L_{\phi_1}(\mathcal{A}, \tau)\,[\|\cdot\|_{\phi_1}].$$

(ii) $L_{\phi_2}(\mathcal{A}, \tau) \subseteqq E_{\phi_1}(\mathcal{A}, \tau) = \overline{\mathcal{A} \cap L_{\phi_1}(\mathcal{A}, \tau)}^{\|\cdot\|_{\phi_1}}$; *if* $\phi_1 \prec\!\!\!\prec \phi_2$.

The proof can be obtained straightforward as for commutative Orlicz spaces (see [11]) by the spectral decomposition.

In the theory of locally convex spaces one often uses function spaces over a messure space, i.e. "commutative" linear spaces. Proposition 2.1 allows us to introduce a new class of locally convex spaces with a "noncommutative" structure. We construct locally convex spaces on a semifinite von Neumann algebra $\mathcal{A}$ by Young functions, more precisely by their corresponding Orlicz spaces $L_\phi(\mathcal{A}, \tau)$. Let Λ be a generating family of Young function, i.e. for $\phi_1, \phi_2 \in \Lambda$ there is a $\psi \in \Lambda$ with $\phi_1, \phi_2 \prec \psi$. Then we have

$$cK_\psi \subseteqq K_{\phi_1} \cap K_{\phi_2} \quad \text{for a} \quad c > 0,$$

where $K_\phi = \{a \in \mathcal{M}(\mathcal{A}) \colon \ \tau(\phi(|a|)) \leqq 1\}$, by the preceding proposition. Therefore, $\{K_\phi \colon \phi \in \Lambda\}$ is a family of absolutely convex absorbing sets on a linear space $F_\Lambda(\mathcal{A}, \tau) \subseteqq \bigcap\limits_{\phi \in \Lambda} L_\phi(\mathcal{A}, \tau)$ and generates a locally convex topology τ_Λ on $F_\Lambda(\mathcal{A}, \tau)$. The topology is separated.

Further, we can construct inductive and projective limits, products and direct sums of noncommutative Orlicz spaces by families of Young functions.

Regarding the result of A. KATAVOLOUS [10] that noncommutative L_p-spaces are really a new class of BANACH spaces for non-commutative VON NEUMANN algebras we conjecture that the class of noncommutative locally convex spaces introduced above is a new class of locally convex spaces. This is supported by the following.

In F_{Ah} we have a natural ordering given by the positive wedge

$$P(F_A) = \{x \in F_A : x \geqq 0 \text{ in } \mathcal{M}(\mathcal{A})\}.$$

3.2. Proposition. *Let* $F_A(\mathcal{A}, \tau) = \bigcap_{\phi \in A} L_\phi(\mathcal{A}, \tau)$ *for a generating family of* YOUNG *functions and let* F_{Ah} *be the hermitian part of* F_A. *Then we have:*

(i) $F_{Ah}(\mathcal{A}, \tau)$ *is a lattice if and only if* $\mathcal{A}$ *is commutative. In this case the lattice is continuous and order complete.*

(ii) $F_{Ah}(\mathcal{A}, \tau)$ *is an antilattice, i.e. if there exists* $x \wedge y$, $x, y \in F_{Ah}$, *then we have* $x \leqq y$ *or* $y \leqq x$, *if and only if* $\mathcal{A}$ *is a factor.*

Proof. (ii) follows as in [9].

Ad (i). Let F_{Ah} be a lattice and let $\mathcal{P}_{\mathcal{A}}$ be the set of all finite projections in F_{Ah}. Then the set of all linear combinations of $\mathcal{P}_{\mathcal{A}}$ is dense in F_{Ah} in the topology of measure. For arbitrary $p, q \in \mathcal{P}_{\mathcal{A}}$ there exists $p \wedge q$. Hence, by [9], Theorem 1, p and q commute. Therefore all projections of $\mathcal{P}_{\mathcal{A}}$ commute. The multiplication is in the measure topology jointly continuous, such that the products of operators in F_{Ah} commute.

If all products of operators in F_{Ah} commute, then all projections of $\mathcal{P}_{\mathcal{A}}$ commute. Hence, $\mathcal{M}(\mathcal{A})$ is commutative. By functional calculus there exists

$$x \vee y = \frac{1}{2}(x + y + |x - y|)$$

for $x, y \in \mathcal{M}(\mathcal{A})_h$ and $x \vee y \in L_\phi(\mathcal{A}, \tau)$ for all $\phi \in A$. Also, $x \vee y \in F_{Ah}(\mathcal{A}, \tau)$.

The basis of neighbourhoods of zero $\{K_\phi : \phi \in A\}$ is a basis of solid sets, i.e. for $y \in K_\phi$ and $|x| \leqq |y|$, we have $x \in K_\phi$ such that by [19] ch. 7.1, the positive cone $P(F_{Ah})$ is normal in F_{Ah} and the lattice operations are continuous.

Let $\emptyset \neq S$ be an order bounded set in F_{Ah}, i.e. there are $x, y \in F_{Ah}$ with $S \subseteqq [x, y]$, the order interval of x and y. We can suppose that $x = 0$. If $S \subseteqq [0, y]$, then

$$(1 + y^2)^{-1} S(1 + y^2)^{-1} \subseteqq [0, 1].$$

The hermitian part $\mathcal{A}_h$ of the VON NEUMANN algebra $\mathcal{A}$ is order complete and hence there exists

$$w = \sup \{(1 + y^2)^{-1} z(1 + y^2)^{-1} : z \in S\}.$$

We have $\sup \{z : z \in S\} \leqq y$ and therefore $\sup \{z : z \in S\} \in L_\phi(\mathcal{A}, \tau)$ for all $\phi \in A$. F_{Ah} is order complete.

4. Generalized Arens algebras

We generalize the ARENS algebra

$$L^\omega(\mathcal{A}, \tau) = \bigcap_{p \geqq 2} L_p(\mathcal{A}, \tau)$$

by families of ORLICZ spaces, where $\mathcal{A}$ is a semifinite VON NEUMANN algebra with trace τ.

We call a generating family Λ of YOUNG functions quadratic, if for any $\phi \in \Lambda$ there is a $\psi \in \Lambda$ such that the composition of ϕ and the square as YOUNG function is smaller than ψ regarding the partially ordering "$\prec$", i.e. there are a $c > 0$ and a $T \geqq 0$ with $\phi(t^2) \leqq \psi(ct)$ for all $t \geqq T$ (with $T = 0$ for $\tau(1) = \infty$). For a quadratic family Λ of YOUNG functions we define:

$$L^\Lambda(\mathcal{A}, \tau) = \bigcap_{\phi \in \Lambda} L_\phi(\mathcal{A}, \tau)$$

and call this linear locally convex space *generalized* ARENS *algebra*. We provide $L^\Lambda(\mathcal{A}, \tau)$ with the projective limit topology τ^Λ regarding the identical imbettings id_ϕ: $L^\Lambda(\mathcal{A}, \tau) \hookrightarrow L_\phi(\mathcal{A}, \tau)\,[\|\cdot\|_\phi]$, $\phi \in \Lambda$, as locally convex topology.

4.1. Proposition. *If Λ is a quadratic family of* YOUNG *functions, then $L^\Lambda(\mathcal{A}, \tau)\,[\tau^\Lambda]$ is a complete locally convex topological *-algebra with jointly continuous multiplication.*

Proof. The generalized ARENS algebra $L^\Lambda(\mathcal{A}, \tau)\,[\tau^\Lambda]$ is the projective limit of BANACH spaces and hence a complete locally convex space. The set $\{\|\cdot\|_\phi : \phi \in \Lambda\}$ generates the topology τ^Λ on $L^\Lambda(\mathcal{A}, \tau)$, such that by $\|x\|_\phi = \|x^*\|_\phi$ for all $\phi \in \Lambda$ and $x \in L^\Lambda(\mathcal{A}, \tau)$ we get a continuous involution. It remains to show the existence and the joint continuity of the multiplication. By the CAUCHY-SCHWARZ inequality and the polar decomposition we conclude from $a \in \mathcal{M}(\mathcal{A})$ with $\tau\big(\chi(|a|)\big) \leqq 1$, where χ is the complementary YOUNG function to ϕ, that

$$\begin{aligned}|\tau(axy)|^2 &= |\tau(v\,|a|\,xy)|^2 = |\tau(|a|^{1/2}\,xyv\,|a|^{1/2})|^2 \\ &\leqq \tau(|a|\,xx^*)\,\tau(v\,|a|\,v^*y^*y) \\ &= \tau(|a|\,xx^*)\,\tau(|a^*|\,y^*y) \quad \text{for all } x, y \in L_\phi(\mathcal{A}, \tau).\end{aligned}$$

We obtain from Proposition 2.3

$$\begin{aligned}\|xy\|_\phi^2 &\leqq \|xy\|_\phi''^2 = \sup\big\{|\tau(axy)| : a \in \mathcal{M}(\mathcal{A}) \text{ and } \tau\big(\psi(|a|)\big) \leqq 1\big\}^2 \\ &\leqq \|xx^*\|_\phi''\,\|y^*y\|_\phi'' \\ &\leqq 4\,\|xx^*\|_\phi\,\|y^*y\|_\phi \leqq 4\,\||x|^2\|_\phi\,\||y|^2\|_\phi\end{aligned}$$

for all $x, y \in L_\phi(\mathcal{A}, \tau)$.

Λ is a quadratic family of YOUNG functions. For any $\phi \in \Lambda$ there are a $\psi \in \Lambda$ and a $c > 0$, $T \geqq 0$ (resp. $T = 0$ for $\tau(1) = \infty$) with $\phi(t^2) \leqq \psi(ct)$ for all $t \geqq T$. By Proposition 3.1 there is a $k > 0$ such that

$$\||x|^2\|_\phi \leqq k\,\|x\|_\psi^2$$

and hence,

$$\|xy\|_\phi^2 \leqq 4\, \||x|^2\|_\phi\, \||y|^2\|_\phi \leqq 4k^2\, \|x\|_\psi^2\, \|y\|_\psi^2$$

for all $x, y \in L^\Lambda(\mathcal{A}, \tau)$.

Let $\mathcal{A}$ be a semifinite VON NEUMANN algebra and Λ a quadratic family of YOUNG functions. There exists a projection $z \in \mathcal{Z}$, the center of $\mathcal{A}$, such that $\mathcal{A}z$ is a VON NEUMANN algebra of type I and $\mathcal{A}(1-z)$ is a VON NEUMANN algebra of type II. Relative to this decomposition we get

$$L^\Lambda(\mathcal{A}, \tau) = L^\Lambda(\mathcal{A}_1, \tau) + L^\Lambda(\mathcal{A}_2, \tau),$$

where $\mathcal{A}_1 = \mathcal{A}z$, $\mathcal{A}_2 = \mathcal{A}(1-z)$ and $xy = 0$, if $x \in L^\Lambda(\mathcal{A}_1, \tau)$, $y \in L^\Lambda(\mathcal{A}_2, \tau)$. Moreover, disregarding the center we have the following relations between $L^\Lambda(\mathcal{A}, \tau)$ and $\mathcal{A}$.

4.2. Proposition. *Let Λ be a quadratic family of* YOUNG *functions. If $\mathcal{A}$ is a factor and there is a* YOUNG *function ψ with $\phi \prec \psi$ for all $\phi \in \Lambda$, then*

(i) *$\mathcal{A}$ is of type* I *if and only if $L^\Lambda(\mathcal{A}, \tau) \subseteqq \mathcal{A}$, moreover, $L^\Lambda(\mathcal{A}, \tau) = \mathcal{A}$ if and only if $\mathcal{A}$ is of type I_n.*

(ii) *$\mathcal{A}$ is of type* II *if and only if $L^\Lambda(\mathcal{A}, \tau) \cap \mathcal{A} \subsetneqq L^\Lambda(\mathcal{A}, \tau)$. Moreover, $\mathcal{A} \subset L^\Lambda(\mathcal{A}, \tau)$ if and only if $\mathcal{A}$ is of type II_1. If there is no* YOUNG *function ψ with $\phi \prec \psi$ for all $\phi \in \Lambda$, then $L^\Lambda(\mathcal{A}, \tau) \subseteqq \mathcal{A}$ and for $\tau(1) < \infty$ we have $\mathcal{A} = L^\Lambda(\mathcal{A}, \tau)$.*

Proof. Ad (i). If $\mathcal{A}$ is of type I, then we get by proposition 4.4 of [5] $\mathcal{M}(\mathcal{A}) \subseteqq \mathcal{A}$ and $L^\Lambda(\mathcal{A}, \tau) \subseteqq \mathcal{A}$. For $\mathcal{A} = L^\Lambda(\mathcal{A}, \tau)$ we have especially $\tau(1) < \infty$ and $L^\Lambda(\mathcal{A}, \tau) = \mathcal{A}$ is of type I_n.

Ad (ii). If $\mathcal{A}$ is a factor of type II, then there is a sequence of mutually orthogonal projections $\{p_n\}$ with $\tau(p_n) \leqq \frac{1}{4^n}$. There is a YOUNG function ψ with $\phi \prec \psi$ for all $\phi \in \Lambda$. Since ψ is an increasing continuous function with $\psi(0) = 0$ and $\lim\limits_{x\to\infty} \psi(x) = \infty$, we can find a sequence $\{\lambda_i\}$ with $0 < \lambda_i \to \infty$, $i \to \infty$, and $\psi(\lambda_i) = 2^i$. Then we have

$$\sum_{i=1}^{\infty} \psi(\lambda_i)\, \tau(p_i) \leqq 1.$$

We take

$$x = \sum_{i=1}^{\infty} \lambda_{ipi}$$

and have $x \in L_\psi(\mathcal{A}, \tau)$, since

$$\|x\|_\psi = \inf \left\{\mu > 0 : \tau\left(\psi\left(\frac{|x|}{\mu}\right)\right) = \sum_{i=1}^{\infty} \psi\left(\frac{\lambda_i}{\mu}\right) \tau(p_i) \leqq 1\right\} \leqq 1.$$

The linear operator x is unbounded and

$$x \in L_\psi(\mathcal{A}, \tau) \subseteqq L^\Lambda(\mathcal{A}, \tau)$$

by Proposition 3.1.

From $\mathcal{A} \cap L^\Lambda(\mathcal{A}, \tau) \subsetneqq L^\Lambda(\mathcal{A}, \tau)$ it follows that the factor $\mathcal{A}$ is not of type I by (i). If $\mathcal{A} \subseteqq L^\Lambda(\mathcal{A}, \tau)$, then $1 \in L^\Lambda(\mathcal{A}, \tau)$ and $\tau(1) < \infty$. If the VON NEUMANN algebra $\mathcal{A}$ is of type II_1 and $|x| = \int\limits_0^\infty \lambda \, de_\lambda$, then

$$\tau\left(\phi\left(\frac{|x|}{\mu}\right)\right) = \int\limits_0^\infty \phi\left(\frac{\lambda}{\mu}\right) d\tau(e_\lambda) \leqq \phi\left(\frac{\|x\|}{\mu}\right) \tau(1) < \infty$$

and $\|x\|_\phi \leqq c \|x\|$ for all $x \in \mathcal{A}$.

If Λ is maximal in the family of all YOUNG functions, $\tau(1) < \infty$ and there is an unbounded operator x with $x \in L^\Lambda(\mathcal{A}, \tau)$, then $|x| = \int\limits_0^\infty \lambda \, de_\lambda \in L^\Lambda(\mathcal{A}, \tau)$. There exists a sequence of mutually orthogonal projections $\{f_{\lambda i}\}$ with $\lambda_i \to \infty$ and $f_{\lambda_i} = e_{\lambda_i} - e_{\lambda_{i-1}} \neq 0$ for all $i \in \mathbb{N}$. For a YOUNG function ϕ and an arbitrary sequence $\{\chi_i\}$ with $\chi_i > 0$ for all $i \in \mathbb{N}$ there is a YOUNG function $\psi \in \Lambda$ with $\phi(\chi_i\lambda_i) \leqq \psi(\lambda_i)$ for all $i \geqq i_0$ such that there is a $c_\psi > 0$ with

$$\sum_{i=1}^\infty \phi\left(\frac{\chi_i\lambda_i}{c_\psi}\right) \tau(f_i) \leqq \sum_{i=1}^\infty \psi\left(\frac{\lambda_i}{c_\psi}\right) \tau(f_i) \leqq 1\,.$$

But $\{\chi_i\}$ is arbitrary, also $\tau(f_i) = 0$ for all $i \geqq i'$. Therefore, $x \in \mathcal{A}$.

If we have $\tau(1) = \infty$, we can find a λ_0 with $\tau(1 - e_{\lambda_0}) < \infty$. The conclusion is like above.

By help of the great possibilities for the construction of YOUNG functions we obtain a new wide class of noncommutative *-algebras of unbounded operators in some sense far from the LMC*-algebras. We introduce the partially ordering "$\leqq$" in the quadratic families of YOUNG functions by:

$\Lambda \leqq \Lambda'$ if and only if for every $\phi \in \Lambda$ there is a $\psi \in \Lambda'$ with $\phi \prec \psi$ then we have $L^{\Lambda'}(\mathcal{A}, \tau) \subseteqq L^\Lambda(\mathcal{A}, \tau)$ for $\Lambda \leqq \Lambda'$ and $L^\Lambda(\mathcal{A}, \tau) = L^{\Lambda'}(\mathcal{A}, \tau)$ if $\Lambda \leqq \Lambda'$ and $\Lambda' \leqq \Lambda$, i.e. $\Lambda = \Lambda'$.

Further, if $\tau(1) < \infty$, then $t \prec \phi(t)$ for every YOUNG function and we get for all quadratic families of YOUNG functions

$$L^\Lambda(\mathcal{A}, \tau) \leqq L^\omega(\mathcal{A}, \tau) \leqq L_1(\mathcal{A}, \tau)\,.$$

For $\tau(1) = \infty$ the situation is not so simple. It is indicated by ARENS algebras. Here we have

$$L^\Lambda(\mathcal{A}, \tau) \leqq L_2(\mathcal{A}, \tau) \text{ resp. } L^\Lambda(\mathcal{A}, \tau) \leqq L^\omega(\mathcal{A}, \tau)$$

not in general. But if $L^\Lambda(\mathcal{A}, \tau) \leqq L_2(\mathcal{A}, \tau)$, or what is the same by Proposition 3.1. $t^2 \prec \phi(t)$ for any $\phi \in \Lambda$, then it follows

$$L^\Lambda(\mathcal{A}, \tau) \leqq L^\omega(\mathcal{A}, \tau)\,.$$

4.3. Proposition. *Let Λ be a quadratic family of* YOUNG *functions and there exists a family Λ' of* YOUNG *functions with $\Lambda \leqq \Lambda'$ and every $\phi \in \Lambda'$ fulfils the Δ_2-condition. Then*

$$L^\omega(\mathcal{A}, \tau) \leqq L^\Lambda(\mathcal{A}, \tau)\,.$$

Proof. For any $\phi \in \Lambda$ there is a $\psi \in \Lambda'$ with $\phi \prec \psi$. ψ fulfils the Δ_2-condition such that there is an $n \in \mathbb{N}$ with $\psi \prec t^n$ (see [11], 3.4). By Proposition 3.1 we get $L_n(\mathcal{A}, \tau) \leqq L_\phi(\mathcal{A}, \tau)$ and hence:

$$L^\omega(\mathcal{A}, \tau) \leqq L^\Lambda(\mathcal{A}, \tau).$$

We introduce the notion of the GC^*-algebra (see [2], [4], [13]). A topological $*$-algebra $\mathcal{A}[\tau]$ is called GC^*-algebra, if $\mathcal{A}$ has a C^*-subalgebra $\mathcal{B}$ with $(1 + x^*x)^{-1} \in \mathcal{B}$ for all $x \in \mathcal{A}$ and the unit ball of $\mathcal{B}$ is τ-bounded. If the bounded part $\mathcal{B}$ of $\mathcal{A}$ is a W^*-algebra, we call $\mathcal{A}[\tau]$ GW^*-algebra.

4.4. Proposition. *The generalized* Arens *algebra* $L^\Lambda(\mathcal{A}, \tau)\,[\tau^\Lambda]$ *is a* GW^**-algebra if and only if* $\tau(1) < \infty$.

Proof. ($\Rightarrow$) If it is $\tau(1) = \infty$, then $1 \notin L^\Lambda(\mathcal{A}, \tau)$, also $L^\Lambda(\mathcal{A}, \tau)$ is not a GC^*-algebra.

($\Leftarrow$) If it is $\tau(1) < \infty$, then $\mathcal{A}$ is a finite von Neumann algebra and we have $\mathcal{A} \subseteqq L^\Lambda(\mathcal{A}, \tau)$. For all $x \in L^\Lambda(\mathcal{A}, \tau)$ it is $(1 + x^*x)^{-1} \in \mathcal{B}(\mathcal{H})$. x is affiliated to $\mathcal{A}$ and hence $(1 + x^*x)^{-1}$ is affiliated to $\mathcal{A}$, also $(1 + x^*x)^{-1} \in \mathcal{A}$.

Let

$$c_\phi = \begin{cases} 1, & \text{if } \tau(1)\,\phi(1) \leqq 1 \\ \tau(1)\,\phi(1), & \text{if } \tau(1)\,\phi(1) > 1 \end{cases} \qquad \text{for } \phi \in \Lambda.$$

Then

$$\tau\left(\phi\left(\frac{|a|}{c_\phi\,\|a\|}\right)\right) \leqq \tau\left(\phi\left(\frac{1}{c_\phi}\right)\right) \leqq 1$$

by Proposition 1.3 for all $a \in \mathcal{A}$.

Hence, $\|a\|_\phi \leqq c_\phi\,\|a\|$ for all $a \in \mathcal{A}$.

Therefore the unit ball of $\mathcal{A}$ is τ^Λ-bounded.

We get with the generalized Arens algebras a new rich class of GW^*-algebras. In general the topology τ^Λ is not barreled, i.e. $\tau^\Lambda \neq T$, the greatest GC^*-topology on $L^\Lambda(\mathcal{A}, \tau)$.

A further class of $*$-algebras of unbounded operators are the Op^*-algebras (see [15]). Let $\mathcal{D}$ be a dense linear subspace of the Hilbert space $\mathcal{H}$. By $\mathcal{L}_+(\mathcal{D})$ we denote the set of all linear operators a with

1. $\mathcal{D}(a) = \mathcal{D}$, $a\mathcal{D} \subseteqq \mathcal{D}$.

2. The adjoint operator a^* exists and $\mathcal{D}(a^*) \supseteqq \mathcal{D}$, $a^*\mathcal{D} \subseteqq \mathcal{D}$. Let a^+ be the restriction of a^* to $\mathcal{D}$. $\mathcal{L}_+(\mathcal{D})$ becomes a $*$-algebra with the multiplication $(ab)\,(\phi) = a(b\phi)$, $\phi \in \mathcal{D}$, and the involution $a \mapsto a^+$. A $*$-subalgebra will be called an Op^*-algebra. We say that the Op^*-algebra $\mathcal{A}$ is closed (essentially selfadjoint, selfadjoint), if $\mathcal{D} = \bigcap_{a\in\mathcal{A}} \mathcal{D}(\bar{a})$ $\left(\bigcap_{a\in\mathcal{A}} \mathcal{D}(\bar{a}) = \bigcap_{a\in\mathcal{A}} \mathcal{D}(a^*),\ \mathcal{D} = \bigcap_{a\in\mathcal{A}} \mathcal{D}(a^*)\right)$, where $\bar{a}$ is the closure of a.

By $t_\mathcal{A}$ we denote the weakest locally convex topology on $\mathcal{D}$ with respect to which each operator $a \in \mathcal{A}$ is continuous from $\mathcal{D}[t_\mathcal{A}]$ into $\mathcal{H}$. $t_\mathcal{A}$ is given by the system of seminorms $\|\phi\|_a := \|a\phi\|$, $a \in \mathcal{A}$. The system of all bounded sets of $\mathcal{D}[t_\mathcal{A}]$ we denote by $\mathfrak{S}$. The seminorms

$$p_M(a) = \sup_{\varphi,\psi \in M} |(a\phi, \psi)|, \qquad M \in \mathfrak{S}$$

define a locally convex topology $\tau_{\mathfrak{D}}$ on $\mathcal{A}$ the uniform topology on $\mathcal{A}$. A topological *-algebra $\mathcal{A}[\tau]$ is called $A\hat{O}^*$-algebra, if it is topologically and algebraically isomorphic to an Op^*-algebra $\mathcal{B}[\tau_{\mathfrak{D}}]$. If $\mathcal{A}[\tau]$ is moreover complete, then we say that $\mathcal{A}[\tau]$ is an $A\hat{O}^*$-algebra.

4.5. Proposition. *$L^A(\mathcal{A}, \tau)$ $[\tau^A]$ is an AO^*-algebra, if the topology τ^A is barreled and $\tau(1) < \infty$.*

Proof. By [20], theorem 5.1, we must only show that the cone $P(L^A) = \left\{\sum_{i=1}^{n} x_i^* x_i, x_i \in L^A(\mathcal{A}, \tau)\right\}$ is normal in the ordered topological space $L_h^A(\mathcal{A}, \tau)$, i.e. for every $\phi \in A$ we have $\|x\|_\phi \leqq \|y\|_\phi$ for $0 \leqq x \leqq y$. Let $0 \leqq x \leqq y$. Then for any $\lambda > 0$

$$\tau\left(\phi\left(\frac{|x|}{\lambda}\right)\right) \leqq \tau\left(\phi\left(\frac{|y|}{\lambda}\right)\right)$$

follows by Proposition 1.3. Hence,

$$\|x\|_\phi = \inf\left\{\lambda > 0 : \tau\left(\phi\left(\frac{|x|}{\lambda}\right)\right) \leqq 1\right\} \leqq \inf\left\{\lambda > 0 : \tau\left(\phi\left(\frac{|y|}{\lambda}\right)\right) \leqq 1\right\} = \|y\|_\phi .$$

This implies that the cone $P(L^A)$ is normal. $L^A[\tau^A]$ is complete, since τ^A is the projective limit of the Banach spaces $L_\phi[\|\cdot\|_\phi]$.

Remark. If $t^2 < \phi(t)$ for any $\phi \in A$, then $L^A(\mathcal{A}, \tau)$ is an unbounded Hilbert algebra, since $L^A(\mathcal{A}, \tau) \leqq L_2(\mathcal{A}, \tau)$. We have the scalar product $\langle a, b\rangle = \tau(b^*a)$, $a, b \in L^A(\mathcal{A}, \tau)$, on $L^A(\mathcal{A}, \tau)$. $L^A(\mathcal{A}, \tau)$ is a *-algebra, such that we get a representation of $L^A(\mathcal{A}, \tau)$ as Op^*-algebra on $\mathfrak{D} = L^A(\mathcal{A}, \tau)$:

$$L^A(\mathcal{A}, \tau) \ni a \mapsto \pi(a) \quad \text{where} \quad \langle \pi(a)\, b, c\rangle = \tau(c^*ab),$$
$$b, c \in L^A(\mathcal{A}, \tau).$$

The domain $\mathfrak{D}$ is essential selfadjoint, since the image of every hermitian operator of $L^A(\mathcal{A}, \tau)$ is essential selfadjoint.

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