On the structure of higher order quantum maps

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1 The category of affine subspaces

1.1 The category FinVect

Let FinVect be the category of finite dimensional real vector spaces with linear maps. We will denote the usual tensor product by $\otimes$, then $(\mathrm{FinVect}, \otimes, I = \mathbb{R})$ is a symmetric monoidal category, with the associators, unitors and symmetries given by the obvious isomorphisms

$$\begin{aligned} \alpha_{U,V,W} &: (U \otimes V) \otimes W \simeq U \otimes (V \otimes W), \\ \lambda_V &: I \otimes V \simeq V, \qquad \rho_V : V \otimes I \simeq V, \\ \sigma_{U,V} &: U \otimes V \simeq V \otimes U. \end{aligned}$$

Let $(-)^* : V \mapsto V^*$ be the usual vector space dual, with duality denoted by $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}$. We will use the canonical identification $V^{**} = V$ and $(V_1 \otimes V_2)^* = V_1^* \otimes V_2^*$. With this duality, FinVect is compact closed. This means that for each object V, there are maps $\eta_V : I \to V^* \otimes V$ (the "cup") and $\epsilon_V : V \otimes V^* \to I$ (the "cap") such that the following snake identities hold:

$$(\epsilon_V \otimes V) \circ (V \otimes \eta_V) = V, \qquad (V^* \otimes \epsilon_V) \circ (\eta_V \otimes V^*) = V^*, \tag{1}$$

here we denote the identity map on the object V by V. Indeed, η_V can be identified with an element $\eta_V(1) \in V^* \otimes V$ and $\epsilon_V \in (V \otimes V^*)^* = V^* \otimes V$ is again an element of the same space. Choose a basis $\{e_i\}$ of V, let $\{e_i^*\}$ be the dual basis of V^*, that is, $\langle e_i^*, e_j \rangle = \delta_{i,j}$. Let us then define

$$\eta_V(1) = \epsilon_V := \sum_i e_i^* \otimes e_i.$$

It is easy to see that this definition does not depend on the choice of the basis, indeed, ϵ_V is the linear functional on $V \otimes V^*$ defined by

$$\langle \epsilon_V, x \otimes x^* \rangle = \langle x^*, x \rangle, \qquad x \in V, \ x^* \in V^*.$$

It is also easily checked that the snake identities (1) hold.

For two objects V and W in FinVect, let $L(V, W)$ be the space of all linear maps $V \to W$. Then $L(V, W)$ is itself an object in FinVect and we have the well-known identification $L(V, W) \simeq V^* \otimes W$. This can be given as follows: for each $f \in L(V, W)$, we have $C_f := (V^* \otimes f)(\epsilon_V) = \sum_i e_i^* \otimes f(e_i) \in V^* \otimes W$. Conversely, since $\{e_i^*\}$ is a basis of V^*, any element $w \in V^* \otimes W$ can be uniquely written as $w = \sum_i e_i^* \otimes w_i$ for $w_i \in W$, and since $\{e_i\}$ is a basis of V, the assignment $f(e_i) := w_i$ determines a unique map $f : V \to W$. The relations between $f \in L(V,W)$ and $C_f \in V^* \otimes W$ can be also written as

$$\langle C_f, x \otimes y^* \rangle = \langle \epsilon_V, x \otimes f^*(y^*) \rangle = \langle f^*(y^*), x \rangle = \langle y^*, f(x) \rangle, \qquad x \in V,\ y^* \in W^*,$$

here $f^* : W^* \to V^*$ is the adjoint of f. Note that by compactness, the internal hom in FinVect satisfies $[V,W] \simeq V^* \otimes W$, so that in the case of FinVect, the object $[V,W]$ can be identified with the space of linear maps $L(V,W)$.

Example 1. Let $V = \mathbb{R}^N$. In this case, we fix the canonical basis $\{|i\rangle,\ i = 1,\dots,N\}$. We will identify $(\mathbb{R}^N)^* = \mathbb{R}^N$, with duality $\langle x, y \rangle = \sum_i x_i y_i$, in particular, we identify $I = I^*$. We then have $\epsilon_V = \sum_i |i\rangle \otimes |i\rangle$ and if $f : \mathbb{R}^N \to \mathbb{R}^M$ is given by the matrix A in the two canonical bases, then $C_f = \sum_i |i\rangle \otimes A|i\rangle$ is the vectorization of A.

Example 2. Let $V = M_n^h$ be the space of $n \times n$ complex hermitian matrices. We again identify $(M_n^h)^* = M_n^h$, with duality $\langle A, B \rangle = \mathrm{Tr}\, A^T B$, where A^T is the usual transpose of the matrix A. Let us choose the basis in M_n^h, given as

$$\Big\{ |j\rangle\langle k| + |k\rangle\langle j|,\ j \le k,\ i\Big(|j\rangle\langle k| - |k\rangle\langle j|\Big),\ j < k \Big\}.$$

Then one can check that

$$\Big\{ \frac{1}{2}\Big(|j\rangle\langle k| + |k\rangle\langle j|\Big),\ j \le k,\ \frac{i}{2}\Big(|k\rangle\langle j| - |j\rangle\langle k|\Big),\ j < k \Big\}$$

is the dual basis and we have

$$\epsilon_V = \sum_{j,k} |j\rangle\langle k| \otimes |j\rangle\langle k|.$$

For any $f : M_n^h \to M_m^h$,

$$C_f = \sum_{j,k} |j\rangle\langle k| \otimes f(|j\rangle\langle k|)$$

is the Choi matrix of f.

1.2 The category Af

We now introduce the category Af, whose objects are of the form $X = (V_X, A_X)$, where V_X is an object in FinVect and $A_X \subseteq V_X$ a proper affine subspace, which means that $0 \notin A_X \neq \emptyset$. Morphisms $X \xrightarrow{f} Y$ in Af are linear maps $f : V_X \to V_Y$ such that $f(A_X) \subseteq A_Y$. For any object X, we put

$$L_X = \mathrm{Lin}(A_X) := \{a - a_X, a \in A_X\}, \qquad S_X := \mathrm{Span}(A_X).$$

Here a_X is any element in A_X and L_X does not depend on this choice. Then L_X and S_X are linear subspaces such that $d_X := \dim(L_X) = \dim(S_X) - 1$. We will also denote $D_X = \dim(V_X)$. For any element $a_X \in A_X$, the affine subspace is determined as

$$A_X = a_X + L_X.$$

Let us now define the duality of affine subspaces as follows. Let V be an object in FinVect and let $C \subseteq V$ be any subset. Let

$$\tilde{C} := \{v^* \in V^*,\ \langle v^*, c \rangle = 1\}.$$

The following lemma collects some properties that are easily proven.

Lemma 1. *(i) $\tilde{C}$ is an affine subspace.*

(ii) $0 \in \tilde{C}$ if and only if $C = \emptyset$ and $\tilde{C} = \emptyset$ if and only if $0 \in \mathrm{Aff}(C)$.

(iii) Let $0 \notin \mathrm{Aff}(C)$, then $\mathrm{Aff}(C) = \tilde{\tilde{C}}$ and we have

$$\mathrm{Lin}(C) = \mathrm{Lin}(\tilde{\tilde{C}}) = \tilde{C}^\perp = \mathrm{Span}(\tilde{C})^\perp, \qquad \mathrm{Lin}(\tilde{C}) = C^\perp = \mathrm{Span}(C)^\perp$$
$$\mathrm{Span}(C) = C^{\perp\perp} = \mathrm{Lin}(\tilde{C})^\perp, \qquad \mathrm{Span}(\tilde{C}) = \mathrm{Lin}(C)^\perp.$$

For any $\tilde{a}_X \in \tilde{A}_X$, the subspace A_X is determined as

$$A_X = S_X \cap \{\tilde{a}_X\}^\sim.$$

The relation between the subspaces L_X and S_X is given as

$$S_X = L_X \oplus \mathbb{R}a_X, \qquad L_X = S_X \cap \{\tilde{a}_X\}^\perp.$$

By Lemma 1 above, $\tilde{A}_X$ is a proper affine subspace in V_X^*, so that $X^* := (V_X^*, \tilde{A}_X)$ is an object in Af. We have $X^{**} = X$ and the corresponding subspaces are related as

$$L_{X^*} = S_X^\perp, \qquad S_{X^*} = L_X^\perp. \tag{2}$$

Note also that for $X \xrightarrow{f} Y$, the adjoint map satisfies $f^*(\tilde{A}_Y) \subseteq \tilde{A}_X$, so that $Y^* \xrightarrow{f^*} X^*$ and the duality $(-)^*$ is a full and faithful functor $\mathrm{Af}^{op} \to \mathrm{Af}$.

We will next introduce a monoidal structure $\otimes$ as follows. For two objects X and Y, we put $V_{X\otimes Y} = V_X \otimes V_Y$ and construct the affine subspace $A_{X\otimes Y}$ as the affine span of

$$A_X \otimes A_Y = \{a \otimes b,\ a \in A_X,\ b \in A_Y\}.$$

Fix any $\tilde{a}_X \in \tilde{A}_X$ and $\tilde{a}_Y \in \tilde{A}_Y$. Since $A_X \otimes A_Y \subseteq \{\tilde{a}_X \otimes \tilde{a}_Y\}^\sim$, the affine span of $A_X \otimes A_Y$ is a proper affine subspace and we have by Lemma 1

$$A_{X\otimes Y} := \mathrm{Aff}(A_X \otimes A_Y) = \{A_X \otimes A_Y\}^{\approx}.$$

Lemma 2. *For any $a_X \in A_X$, $a_Y \in A_Y$, we have*

$$L_{X\otimes Y} = \mathrm{Lin}(A_X \otimes A_Y) = \mathrm{Span}(\{x \otimes y - a_X \otimes a_Y,\ x \in A_X,\ y \in A_Y\}) \tag{3}$$
$$= (a_X \otimes L_Y) + (L_X \otimes a_Y) + (L_X \otimes L_Y) \tag{4}$$

(here $+$ denotes the direct sum of subspaces). We also have

$$S_{X\otimes Y} = S_X \otimes S_Y.$$

Proof. The equality (3) follows from Lemma 1. For any $x \in A_X$, $y \in A_Y$ we have

$$x \otimes y - a_X \otimes a_Y = a_X \otimes (y - a_Y) + (x - a_X) \otimes a_Y + (x - a_X) \otimes (y - a_Y),$$

so that $L_{X\otimes Y} = \mathrm{Lin}(A_X \otimes A_Y)$ is contained in the subspace on the RHS of (4). Let d be the dimension of this subspace, then clearly

$$d_{X\otimes Y} \le d \le d_X + d_Y + d_X d_Y.$$

On the other hand, any element of S_X has the form tx for some $t \in \mathbb{R}$ and $x \in A_X$, so that it is easily seen that $S_X \otimes S_Y = S_{X\otimes Y}$. Hence

$$\begin{aligned} d_{X\otimes Y} &= \dim(L_{X\otimes Y}) = \dim(S_{X\otimes Y}) - 1 = \dim(S_X)\dim(S_Y) - 1 = (d_X+1)(d_Y+1) - 1 \\ &= d_X + d_Y + d_X d_Y. \end{aligned}$$

This completes the proof.

□

Lemma 3. *Let $I = (\mathbb{R}, \{1\})$. Then $(\mathrm{Af}, \otimes, I)$ is a symmetric monoidal category.*

Proof. Note that this structure is inherited from the symmetric monoidal structure in FinVect. To show that $\otimes$ is a functor, we have to check that for $X_1 \xrightarrow{f} Y_1$ and $X_2 \xrightarrow{g} Y_2$ in Af, we have $X_1 \otimes Y_1 \xrightarrow{f\otimes g} X_2 \otimes Y_2$ which amounts to showing that

$$(f\otimes g)(A_{X_1\otimes Y_1}) \subseteq A_{X_2\otimes Y_2}.$$

Let $x \in A_{X_1}$, $y \in A_{Y_1}$, then $f(x)\otimes g(y) \in A_{X_2}\otimes A_{Y_2} \subseteq A_{X_2\otimes Y_2}$. Since $A_{X_1\otimes Y_1}$ is the affine subspace generated by $A(X_1)\otimes A(Y_1)$, the above inclusion follows by linearity of $f\otimes g$.

It only remains to prove that the associators, unitors and symmetries from FinVect are morphisms in Af. We will prove this for the associators $\alpha_{X,Y,Z} : V_X \otimes (V_Y \otimes V_Z) \to (V_X \otimes V_Y) \otimes V_Z$, the other proofs are similar. We need to check that $\alpha_{X,Y,Z}(A_{X\otimes(Y\otimes Z)}) \subseteq A_{(X\otimes Y)\otimes Z}$. It is easily checked that $A_{X\otimes(Y\otimes Z)}$ is the affine span of elements of the form $x\otimes(y\otimes z)$, $x \in A_X$, $y \in A_Y$ and $z \in A_Z$, and we have

$$\alpha_{X,Y,Z}(x\otimes(y\otimes z)) = (x\otimes y)\otimes z \in A_{(X\otimes Y)\otimes Z}$$

for all such elements. The desired inclusion follows by linearity.

□

Theorem 1. $(\mathrm{Af}, \otimes, I)$ *is a *-autonomous category, with duality $(-)^*$, such that $I^* = I$.*

Proof. By Lemma 3, we have that $(\mathrm{Af}, \otimes I)$ is a symmetric monoidal category. We have also seen that the duality $(-)^*$ is a full and faithful contravariant functor. We only need to check the natural isomorphisms

$$\mathrm{Af}(X\otimes Y, Z^*) \simeq \mathrm{Af}(X, (Y\otimes Z)^*).$$

Since FinVect is compact, we have the natural isomorphisms

$$\mathrm{FinVect}(V_X\otimes V_Y, V_Z^*) \simeq \mathrm{FinVect}(V_X, V_Y^*\otimes V_Z^*),$$

determined by the equalities

$$\langle\, f(x\otimes y), z\,\rangle = \langle\, h(x), y\otimes z\,\rangle, \qquad x\in V_X,\ y\in V_Y,\ z\in V_Z,$$

for $f \in \mathrm{FinVect}(V_X\otimes V_Y, V_Z^*)$ and $h \in \mathrm{FinVect}(V_Z, V_Y^*\otimes V_Z^*)$. Since $A_{X\otimes Y}$ is an affine span of $A_X\otimes A_Y$, we see that f is in $\mathrm{Af}(X\otimes Y, Z^*)$ if and only if $f(x\otimes y) \in \tilde{A}_Z$ for all $x\in A_X$, $y \in A_Y$, that is,

$$1 = \langle\, f(x\otimes y), z\,\rangle = \langle\, h(x), y\otimes z\,\rangle \qquad \forall x\in A_X,\ \forall y\in A_Y,\ \forall z\in A_Z.$$

But this is equivalent to

$$h(x) \in (A_Y\otimes A_Z)^{\sim} = \tilde{A}_{Y\otimes Z}, \qquad \forall x \in A_X,$$

which means that $h \in \mathrm{Af}(X, (Y\otimes Z)^*)$.

□

It is also shown that we can introduce a tensor product $\otimes$ and duality $(-)^*$ such that Af with these structures becomes a *-autonomous category. We will show that we may use this category to describe classical and quantum higher order maps.

To this end, we introduce a subcategory in Af, consisting of objects $X = (V_X, A_X)$, where the vector space V_X is restricted to be either $\mathbb{R}^n$ or M_n^h (see Example **??**). Accordingly, let also V_X^+ be either $\mathbb{R}_+^n$ (the cone of elements in $\mathbb{R}^n$ with nonnegative coordinates) or M_n^+ (the cone of positive semi-definite matrices). We then also require that both A_X and $\tilde{A}_X$ contain some interior elements in V_X^+ (note that in this case we identify $V_X = V_X^*$). The morphisms are restricted so that we additionaly require that $(V_X, A_X) \xrightarrow{f} (V_Y, A_Y)$ are completely positive.

and morphisms $(\mathbb{R}^n, A) \xrightarrow{f} (\mathbb{R}^m, B)$ in Af such that we also have $f(\mathbb{R}_+^n) \subseteq \mathbb{R}_+^m$. The category Quant consists of objects of the form $X = (M_n, A)$ and morphisms $(M_n, A) \xrightarrow{f} (M_m, B)$ in Af such that $f(M_n^+) \subseteq M_m^+$.

For any object X, we also put

$$L_X := \mathrm{Lin}(A_X) \qquad S_X := \mathrm{Span}(A_X), \qquad d_X := \dim(L_X), \qquad D_X := \dim(V_X).$$

Note that X is uniquely determined also by the triple (V_X, L_X, a_X) with an element $a_X \in A_X$, or by $(V_X, S_X, \tilde{a}_X)$ with an element $\tilde{a}_X \in \tilde{A}_X$.

We will consider the following special kind of morphisms in Af. A morphism $X \xrightarrow{f} Y$ is a monomorphism if $f \circ h = f \circ g$ implies $h = g$ for any morphisms g, h, and an epimorphism if $h \circ f = g \circ f$ implies $h = g$. A morphism that is both mono and epi is called a bimorphism.

Lemma 4. *A morphism $X \xrightarrow{f} Y$ is a monomorphism if and only if it is injective as a map $f : V_X \to V_Y$. Similarly, f is an epimorphism if and only if it is surjective.*

Consequently, f is a bimorphism if and only if it is an isomorphism of V_X and V_Y. Note that a bimorphism is not necessarily an isomorphism in Af, which would mean that the inverse map satisfies $f^{-1}(A_Y) \subseteq A_X$.

Proof. Let f be a monomorphism in Af and let $K = Ker(f)$. Let $Z = (V_X \times K, A_X \times \{0\})$, then Z is an object in Af. Let $g, h : V_Z \to V_X$ be defined as $g(x, y) = x$, $h(x, y) = x + y$, then $g, h : Z \to X$ are morphisms in Af and we have

$$f \circ g(x, y) = f(x) = f(x) + f(y) = f \circ h(x, y), \qquad \forall (x, y) \in V_Z.$$

Hence $h = g$, so that we must have $K = \{0\}$ and f is injective. The converse is clear.

Similarly, let f be an epimorphism and let $R = f(V_X) \subseteq V_Y$. Let $Z = (V_Y \times V_Y|_R, A_Y \times \{[0]\})$ and let $g, h : V_Y \to V_Z$ be given by $g(y) = (y, [0])$, $h(y) = (y, q(y))$, where $q : V_Y \to V_Y|_R$ is the quotient map. Since $A_Y \subseteq R$, we have $q(A_Y) = \{[0]\}$, so that both g, h are morphisms in Af. Moreover,

$$g \circ f(x) = (f(x), [0]) = (f(x), q(f(x))) = h \circ f,$$

so that $g = h$, but this implies that $R = V_Y$ and f is surjective. The converse is clear.

□

Let X, Y, Z be objects in Af such that there are bimorphisms

$$Z \xrightarrow{f} X, \qquad Z \xrightarrow{g} Y.$$

Note that in particular $\psi := f \circ g^{-1}$ is an isomorphism of V_Y onto V_X.

Let us define $X \sqcup_{f,g} Y := (V_X, A_{X\sqcup_{f,g}Y})$, with

$$A_{X\sqcup_{f,g}Y} = \{sa + (1-s)\psi(b),\ a \in A_X,\ b \in A_Y,\ s \in \mathbb{R}\}.$$

Note first that this is a proper object in Af if and only if

$$\forall b \in A_Y, \quad t\psi(b) \in A_X \implies t = 1. \tag{5}$$

Indeed, we only have to check that $0 \notin A_{X\sqcup_{f,g}Y}$ which is easily seen to be equivalent to (5).

Assume (5), then $X \sqcup_{f,g} Y$ together with the morphisms given by the linear maps $id : V_X \to V_X$ and $\psi : V_Y \to V_X$, is the **pushout** of the above diagram. Indeed, these are clearly bimorphisms $X \to X \sqcup_{f,g} Y$ and $Y \to X \sqcup_{f,g} Y$ in Af, and we have

$$id \circ f = f = \psi \circ g.$$

Also, if W is an object in Af and $X \xrightarrow{i} W$ and $Y \xrightarrow{j} W$ are such that $i \circ f = j \circ g$, then $i = i \circ id$, $j = i \circ \psi$, so the map i defines a morphism $X \sqcup_{f_0,g_0} Y \to W$, obviously unique, with the required properties. We have

$$L_{X\sqcup_{f,g}Y} = L_X \vee \psi(L_Y), \qquad S_{X\sqcup_{f,g}Y} = S_X \vee \psi(S_Y).$$

Let us also note that if (5) is not satisfied, there is some $z \in V_Z$ such that for some $t \neq 1$,

$$tf(z) \in A_X, \qquad g(z) \in A_Y.$$

If there are some $X \xrightarrow{i} W$ and $Y \xrightarrow{j} W$ as above, then $ti \circ f(z) \in A_W$, but also $i \circ f(z) = j \circ g(z) \in A_W$, so that W is not a proper object, in this case the pushout is the terminal object 0, with the unique arrows $X \xrightarrow{!} 0$, $Y \xrightarrow{!} 0$.

Similarly, let

$$X \xrightarrow{f} Z, \qquad Y \xrightarrow{g} Z$$

be bimorphisms and let $\psi = f^{-1} \circ g$. If

$$\phi(A_Y) \cap A_X \neq \emptyset, \tag{6}$$

the **pullback** of f, g is $X \sqcap_{f,g} Y = (V_X, A_X \cap \phi(A_Y))$, with the bimorphisms given by id_X and ϕ^{-1}. In this case

$$L_{X\sqcap_{f,g}Y} = L_X \cap \phi(L_Y), \qquad S_{X\sqcap_{f,g}Y} = S_X \cap \phi(S_Y).$$

Without condition (6), the above is not a proper object and in this case the pulback is the initial object $\emptyset$.

.1 Affine subspaces

A subset $A \subseteq V$ of a finite dimensional vector space V is an affine subspace if $\sum_i \alpha_i a_i \in A$ whenever all $a_i \in A$ and $\sum_i \alpha_i = 1$. We say that A is proper if $0 \neq A$ and $A \neq \emptyset$. We will always mean that an affine subspace is proper (if not explicitly stated otherwise).

.1.1 Description

An affine subspace can be determined in two ways:

(i) Let $L \subseteq V$ be a linear subspace and $a_0 \neq L$. Then

$$A = a_0 + L$$

is a proper affine subspace. Note that $a_0 \in A$ and $A \cap L = \emptyset$. Conversely, any proper affine subspace A can be given in this way, with a_0 an arbitrary element in A and

$$L = \mathrm{Lin}(A) := \{a_1 - a_2,\ a_1, a_2 \in A\} = \{a - a_0,\ a \in A\}.$$

(ii) Let $S \subseteq V$ be a linear subspace and $a_0^* \in V^* \setminus S^\perp$. Then

$$A = \{a \in S, \langle\, a_0^*, a\,\rangle = 1\}$$

is a proper affine subspace. Conversely, any proper affine subspace A is given in this way, with $S = \mathrm{Span}(A)$ and a_0^* an arbitrary element in the dual

$$\tilde{A} = \{a^* \in V^*,\ \langle\, a^*, a\,\rangle = 1,\ \forall a \in A\}.$$

For an affine subspace A, the relation of $L = \mathrm{Lin}(A)$ and $S = \mathrm{Span}(A)$ is as follows:

$$S = L + \mathbb{R}a, \qquad L = S \cap \{\tilde{a}\}^\perp,$$

here $a \in A$ and $\tilde{a} \in \tilde{A}$ are arbitrary elements.

.1.2 Duality

For an affine subspace A, $\tilde{A}$ is an affine subspace as well. If A is proper, then $\tilde{A}$ is proper and we have $\tilde{\tilde{A}} = A$. More generally, if $\emptyset \neq C \subseteq A$ is any subset of a proper affine subspace A, then $\tilde{C}$ is a proper affine subspace and $\tilde{\tilde{C}}$ is the affine hull of C, that is,

$$\tilde{\tilde{C}} = \mathrm{Aff}(C) := \{\sum_i \alpha_i c_i,\ c_i \in C,\ \sum_i \alpha_i = 1\}.$$

In this case, we may write $\tilde{\tilde{C}}$ as

$$\tilde{\tilde{C}} = c_0 + \mathrm{Lin}(C) = c_0 + \mathrm{Span}(\{c_1 - c_2,\ c_1, c_2 \in C\})$$

with an arbitrary element $c_0 \in C$, or as

$$\tilde{\tilde{C}} = \{c \in \mathrm{Span}(C),\ \langle\, a_0^*, c\,\rangle = 1\}$$

for an arbitrary element $a_0^* \in \tilde{A}$.

Lemma 5. *Let A be a proper affine subspace and let $C \subseteq A$ be any subset. Then*

$$\mathrm{Lin}(C) = \mathrm{Lin}(\tilde{\tilde{C}}) = \tilde{C}^\perp = \mathrm{Span}(\tilde{C})^\perp, \qquad \mathrm{Lin}(\tilde{C}) = C^\perp = \mathrm{Span}(C)^\perp$$
$$\mathrm{Span}(C) = C^{\perp\perp} = \mathrm{Lin}(\tilde{C})^\perp, \qquad \mathrm{Span}(\tilde{C}) = \mathrm{Lin}(C)^\perp.$$

.1.3 The lattice of affine subspaces

Let $\mathcal{A}(V)$ be the set of all affine subspaces in a finite dimensional vector space V. Then $\mathcal{A}(V)$ can be ordered by inclusion and it is a complete lattice, with

$$\wedge\mathcal{A} = \cap\mathcal{A}, \qquad \vee\mathcal{A} = \{\sum_i \alpha_i a_i,\ a_i \in A_i \in \mathcal{A}, \sum_i \alpha_i = 1\}$$

for any subset $\mathcal{A} \subseteq \mathcal{A}(V)$. Let us choose any nonzero elements $a \in V$, $\tilde{a} \in V^*$ and put

$$\mathcal{A}_{a,\tilde{a}}(V) = \{A \in \mathcal{A}(V),\ a \in A,\ \tilde{a} \in \tilde{A}\}.$$

Note that any subspace in $\mathcal{A}_{a,\tilde{a}}$ is proper and it is a complete sublattice in $\mathcal{A}(V)$. Moreover, we have

$$\mathrm{Lin}(\wedge\mathcal{A}) = \wedge\{\mathrm{Lin}(A),\ A \in \mathcal{A}\}, \qquad \mathrm{Lin}(\vee\mathcal{A}) = \vee\{\mathrm{Lin}(A),\ A \in \mathcal{A}\}$$

and similarly for Span.

We say that $A, B \in \mathcal{A}_{a,\tilde{a}}(V)$ are independent if $A \cap B = \{a\}$, equivalently, $\mathrm{Lin}(A) \cap \mathrm{Lin}(B) = \{0\}$, that is, $\mathrm{Lin}(A)$ and $\mathrm{Lin}(B)$ are independent linear subspaces. A family $\{A_1, \dots, A_n\} \subseteq \mathcal{A}_{a,\tilde{a}}(V)$ is independent if A_i and $\vee_{j\in I} A_j$ are independent for any $i \in \{1, \dots, n\}$ and $i \neq I \subseteq \{1, \dots, n\}$. Equivalently, $\{\mathrm{Lin}(A_1), \dots, \mathrm{Lin}(A_n)\}$ is an independent family of subspaces in V.

Lemma 6. *Let $\{A_1, \dots, A_n\} \subseteq \mathcal{A}_{a,\tilde{a}}(V)$ be an independent family. Then the sublattice generated by $\{A_1, \dots, A_n\}$ is distributive.*

Proof. Clear from a similar property of linear subspaces.

□

.1.4 Limits and colimits

Limits and colimits should be obtained from those in FinVect, we have to spectify the other structures and check whether the corresponding arrows are in Af.

First, note that $\{0\}$ is both initial and terminal in FinVect. In Af, it is easily seen that $\emptyset$ is initial and 0 is terminal in Af.

Let X, Y be two objects in Af. Assume first that both are proper. We define their product as

$$X \times Y := (V_X \times V_Y, A_X \times A_Y, (a_X, a_Y), \frac{1}{2}(\tilde{a}_X, \tilde{a}_Y)),$$

where

$$A_X \times A_Y := \{(x, y) \in V_X \times V_Y,\ x \in A_X, y \in A_Y\}$$

is the direct product of A_X and A_Y. It is easily verified that this is indeed an affine subspace and the usual projections $\pi_X : V_X \times V_Y \to V_X$ and $\pi_Y : V_X \times V_Y \to V_Y$ are in Af. Moreover, for $f : Z \to X$ and $g : Z \to Y$, the map $f \times g(z) = (f(z), g(z))$ is also clearly a morphism $Z \to X \times Y$ in Af. We have

$$L_{X\times Y} = L_X \times L_Y, \qquad S_{X\times Y} = (L_X \times L_Y) \vee \mathbb{R}(a_X, a_Y) = (S_X \times S_Y) \wedge \{(\tilde{a}_X, -\tilde{a}_Y)\}^\perp$$

for an arbitrary choice $a_X \in A_X$, $a_Y \in A_Y$ and $\tilde{a}_X \in \tilde{A}_X$, $\tilde{a}_Y \in \tilde{A}_Y$.

Next, we put $X \times \emptyset = \emptyset$, with the unique morphisms $\pi_X : \emptyset \to X$ and $\pi_\emptyset : \emptyset \to \emptyset$. If $Y \xrightarrow{f} X$ and $Y \xrightarrow{g} \emptyset$, then it is clear that $Y = \emptyset$, this shows that this is indeed the product. Further, put $X \times 0 = X$, with $\pi_X = id_X$ and $\pi_0 : X \xrightarrow{!} 0$. It is also readily verified that this is the product.

The coproduct for proper objects X, Y is defined as

$$X \oplus Y = (V_X \times V_Y, A_X \oplus A_Y)$$

where

$$A_X \oplus A_Y := \{(tx, (1-t)y),\ x \in A_X, y \in A_Y,\ t \in \mathbb{R}\}$$

is the direct sum. To check that this is an affine subspace, let $x_i \in A_X$, $y_i \in A_Y$, $s_i \in \mathbb{R}$ and let $\sum_i \alpha_i = 1$, then

$$\sum_i \alpha_i(s_i x_i, (1-s_i)y_i) = (\sum_i s_i\alpha_i x_i, \sum_i (1-s_i)\alpha_i y_i) = (sx, (1-s)y) \in A_X \oplus A_Y,$$

where $s = \sum_i s_i\alpha_i$, $x = s^{-1}\sum_i s_i\alpha_i x_i$ if $s \neq 0$ and is arbitrary in A_X otherwise, similarly $y = (1-s)^{-1}\sum_i (1-s_i)\alpha_i y_i$ if $s \neq 1$ and is arbitrary otherwise. The usual embeddings $p_X : V_X \to V_X \times V_Y$ and $p_Y : V_Y \to V_X \times V_Y$ are easily seen to be morphsims in Af.

Let $f : X \to Z$, $g : Y \to Z$ be any morphisms in Af and consider the map $V_X \times V_Y \to V_Z$ given as $f \oplus g(u, v) = f(u) + g(v)$. We need to show that it preserves the affine subspaces. So let $x \in A_X$, $y \in A_Y$, then since $f(x), g(y) \in A_Z$, we have for any $s \in \mathbb{R}$,

$$f \oplus g(sx, (1-s)y) = sf(x) + (1-s)g(y) \in A_Z.$$

We also have

$$L_{X\oplus Y} = (L_X \times L_Y) \vee \mathbb{R}\{(a_X, -a_Y)\}, \qquad S_{X\oplus Y} = S_X \times S_Y$$

for some $a_X \in A_X$, $a_Y \in A_Y$.

Similarly as in the case of products, it is verified that $X \oplus \emptyset = X$ and $X \oplus 0 = 0$. (All the statements for coproducts can be obtained from duality defined below).

One can also discuss equalizers and coequalizers, here we only note that these may be trivial even for proper objects. We will consider pullbacks and pushouts for some special morphisms that will be needed below. (We may also add two special objects: the initial object $\emptyset := (\{0\}, \emptyset)$ and the terminal object $0 := (\{0\}, \{0\})$, here the affine subspaces are obviously not proper.)

.1.5 Pullbacks and pushouts

.1.6 Monoidal structure

Let X, Y be objects in Af. Let us define

$$A_{X\otimes Y} := \{x \otimes y, x \in A_X, y \in A_Y\}^{\approx}.$$

In other words, $A_{X\otimes Y}$ is the smallest affine subspace in $V_X \otimes V_Y$ containing $A_X \otimes A_Y$.

.1.7 Duality

We define $X^* := (V_X^*, \tilde{A}_X)$. Note that we have

$$L_{X^*} = S_X^\perp, \qquad S_{X^*} = L_X^\perp, \qquad d_{X^*} = D_X - d_X - 1.$$

It is easily seen that $(-)^*$ defines a full and faithful functor $\mathrm{Af}^{op} \to \mathrm{Af}$, moreover, $X^{**} = X$ (we will use the canonical identification of any V in FinVect with its second dual).

.1.8 The dual tensor product

Let us define the dual tensor product by $\odot$, that is

$$X \odot Y = (X^* \otimes Y^*)^*.$$

We then have

$$\begin{aligned} L_{X\odot Y} &= S^{\perp}_{X^*\otimes Y^*} = (S_{X^*} \otimes S_{Y^*})^{\perp} = (L_X^{\perp} \otimes L_Y^{\perp})^{\perp} \\ S_{X\odot Y} &= L^{\perp}_{X^*\otimes Y^*} = (\tilde{a}_X \otimes S_Y^{\perp})^{\perp} \wedge (S_X^{\perp} \otimes \tilde{a}_Y)^{\perp} \wedge (S_X^{\perp} \otimes S_Y^{\perp})^{\perp} \end{aligned}$$

In particular,

$$d_{X\odot Y} = D_X d_Y + d_X D_Y - d_X d_Y.$$

Lemma 7. *Let X, Y be nontrivial. Then $X \otimes Y = X \odot Y$ exactly in one of the following situations:*

1. *$X \simeq I$ or $Y \simeq I$,*
2. *$d_X = d_Y = 0$,*
3. *$D_X = d_X + 1$ and $D_Y = d_Y + 1$ (Objects with this property will be called first order).*

Proof. It is easy to see that (when identifying $X = X^{**}$), we have $A_X \otimes A_Y \subseteq \tilde{A}_{X^*\otimes Y^*}$, hence $A_{X\otimes Y} \subseteq A_{X\odot Y}$. We see from the above computatons that

$$d_{X\odot Y} - d_{X\otimes Y} = d_X(D_Y - 1) + (D_X - 1)d_Y - 2d_X d_Y \geq 0,$$

with equality if and only if the conditions of the lemma hold.

□

.1.9 The no signaling product

Lemma 8. *The space $A_{X\otimes Y}$ is precisely the affine subspace of elements $w \in A_{X\odot Y}$, such that $\langle\, w, \cdot \otimes \tilde{a}_Y \,\rangle$ and $\langle\, w, \tilde{a}_X \otimes \cdot \,\rangle$ do not depend on the choice of $\tilde{a}_Y \in \tilde{A}_Y$ and $\tilde{a}_X \in \tilde{A}_X$.*

Proof. Any element $w \in A_{X\otimes Y}$ has the form $w = \sum_i \alpha_i x_i \otimes y_i$, for $x_i \in A_X$, $y_i \in A_Y$ and $\sum_i \alpha_i = 1$. It follows that for any $\tilde{a}_X \in \tilde{A}_X$ and $\tilde{a}_Y \in \tilde{A}_Y$,

$$\langle\, w, \cdot \otimes \tilde{a}_Y \,\rangle = \sum_i \alpha_i x_i, \qquad \langle\, w, \tilde{a}_X \otimes \cdot \,\rangle = \sum_i \alpha_i y_i.$$

Conversely, assume that $w \in A_{X\odot Y}$ has this property, then for any $\tilde{x} \in L_{X^*}$ and $\tilde{y} \in L_{Y^*}$, we have

$$\langle\, w, \cdot \otimes \tilde{y} \,\rangle = 0, \qquad \langle\, w, \tilde{x} \otimes \cdot \,\rangle = 0.$$

It follows that

$$w \in (V_X^* \otimes L_{Y^*})^{\perp} \cap (L_{X^*} \otimes V_Y^*)^{\perp} = (V_X \otimes S_Y) \cap (S_X \otimes V_Y) = S_X \otimes S_Y$$

Since $w \in A_{X\odot Y}$, we have $\langle\, w, \tilde{a}_X \otimes \tilde{a}_Y \,\rangle = 1$ for any choice of $\tilde{a}_X \in \tilde{A}_X$, $\tilde{a}_Y \in \tilde{A}_Y$. Since $\tilde{a}_X \otimes \tilde{a}_Y \in \tilde{A}_{X\otimes Y}$ and $S_{X\otimes Y} = S_X \otimes S_Y$, this implies $w \in A_{X\otimes Y}$.

□

We will now define one-sided variants of this property. Namely, let

$$A_{X\prec Y} := \{w \in A_{X\odot Y},\ \langle w, \cdot \otimes \tilde{a}_Y \rangle \text{ does not depend on } \tilde{a}_Y \in \tilde{A}_Y\}$$
$$A_{X\succ Y} := \{w \in A_{X\odot Y},\ \langle w, \tilde{a}_X \otimes \cdot \rangle \text{ does not depend on } \tilde{a}_X \in \tilde{A}_X\}.$$

We then put $X \prec Y = (V_X \otimes V_Y, A_{X\prec Y})$ and $X \succ Y = (V_X \otimes V_Y, A_{X\succ Y})$.

Lemma 9. *For any choice of $a_Y \in A_Y$, we have*

$$L_{X\prec Y} = V_X \otimes L_Y + L_X \otimes a_Y, \qquad S_{X\prec Y} = V_X \otimes L_Y + S_X \otimes a_Y.$$

Similarly, for any $a_X \in A_X$, we have

$$L_{X\succ Y} = L_X \otimes V_Y + a_X \otimes L_Y, \qquad S_{X\succ Y} = L_X \otimes V_Y + a_X \otimes S_Y.$$

Proof. By the definition, we have

$$L_{X\prec Y} = L_{X\odot Y} \cap (V_X^* \otimes L_{Y^*})^\perp = (S_{X^*} \otimes S_{Y^*})^\perp \cap (V_X \otimes S_Y) = V_X \otimes L_Y + L_X \otimes a_Y,$$

for any element $a_Y \in A_Y$. The proof for $\succ$ is similar.

□

For the rest of this section, we fix some $a_X \in A_X$ and $\tilde{a}_X \in \tilde{A}_X$. We will use the notation $X_{\min} := (V_X, \{a_X\})$ and $X_{\max} = (V_X, \{\tilde{a}_X\}^\sim)$.

We also introduce a decomposition of V_X into an independent family of subspaces L_X^0, L_X^1, L_X^2 as $L_X^0 := \mathbb{R}a_X$, $L_X^1 := L_X$ and L_X^2 is any complement of L_X in the subspace $\{\tilde{a}_X\}^\perp$. We see that the L or S spaces of any of the objects discussed in this paragraph is a union of some of the subspaces $L_X^i \otimes L_Y^j$, $i, j = 0, 1, 2$. We may therefore represent the subspaces in question by 3×3 matrices such that the i,j-th element is 1 if the subspace contains $L_X^i \otimes L_Y^j$ and 0 otherwise. For example, we have

$$L_{X\otimes Y} \equiv \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ L_{X\odot Y} \equiv \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix},\ L_{X\prec Y} \equiv \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ L_{X\succ Y} \equiv \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \tag{7}$$

The S-spaces can be obtained from these by putting the top left element to 1. Notice also that all these objects belong to the set of objects $Z = (V_X \otimes V_Y, A_Z)$ such that

$$X_{\min} \otimes Y_{\min} \hookrightarrow Z \hookrightarrow X_{\max} \otimes Y_{\max} (= X_{\max} \odot Y_{\max})$$

and this set forms a lattice under inclusion ordering with $Z_0 \wedge Z_1 = Z_0 \sqcap Z_1$, $Z_0 \vee Z_1 = Z_0 \sqcup Z_1$, where $\sqcap$ and $\sqcup$ are the pulback and pushout of the inclusions $X_{\min} \otimes Y_{\min} \hookrightarrow Z_i \hookrightarrow X_{\max} \otimes Y_{\max}$. Furthermore, since $\{L_X^i \otimes L_Y^j\}$ is an independent decomposition of $V_X \otimes V_Y$, all the objects in (7) are contained in a distributive sublattice of objects such that $A_Z = a_X \otimes a_Y + L$, where L is a subspace represented by a matrix M_Z with the top left element equal to 0. For such elements Z_1 and Z_2, the representing matrices $M_{Z_1 \sqcap Z_2}$ resp. $M_{Z_1 \sqcup Z_2}$ are given by poinwise minimum resp. maximum of M_{Z_1} and M_{Z_2}.

Some further useful elements of this sublattice are represented as

$$L_{X_{\max}\otimes Y} \equiv \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix},\ L_{X\otimes Y_{\max}} \equiv \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ L_{X_{\min}\odot Y} \equiv \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ L_{X\odot Y_{\min}} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

From these consideration, the following is immediate.

Lemma 10. *We have*

$$\begin{aligned}
X \prec Y &= (X \otimes Y) \sqcup (X_{\min} \odot Y) = (X \odot Y) \sqcap (X_{\max} \otimes Y) \\
X \succ Y &= (X \otimes Y) \sqcup (X \odot Y_{\min}) = (X \odot Y) \sqcap (X \otimes Y_{\max}) \\
X \otimes Y &= (X \prec Y) \sqcap (X \succ Y) \\
X \odot Y &= (X \prec Y) \sqcup (X \succ Y).
\end{aligned}$$

We have the inclusions

$$X_a \to X \to X^{\tilde{a}}.$$

We also fix $b \in A_Y$, $\tilde{b} \in \tilde{A}_Y$. By inclusions, we have the following diagrams

$$X_a \otimes Y_b \to X \otimes Y, \qquad X_a \otimes Y_b \to X_a \odot Y$$

and

$$X \odot Y \to X^{\tilde{a}} \odot Y^{\tilde{b}} = X^{\tilde{a}} \otimes Y^{\tilde{b}}, \qquad X \otimes Y^{\tilde{b}} \to X^{\tilde{a}} \otimes Y^{\tilde{b}}.$$

Lemma 11. $X \prec Y = X \otimes Y \sqcup X \odot Y_b = X \odot Y \sqcap X \otimes Y^{\tilde{b}}$.

We see that the identity map $id_{V_X \otimes V_Y}$ defines bimorphisms

$$X \otimes Y \to X \prec Y \to X \odot Y, \qquad X \otimes Y \to X \succ Y \to X \odot Y.$$

Lemma 12. *The pushout and pullback of the above diagram are*

$$X \otimes Y = (X \prec Y) \sqcap (X \succ Y), \qquad X \odot Y = (X \prec Y) \sqcup (X \succ Y).$$

Proof. We have by Lemma 8 that

$$A_{X \otimes Y} = A_{X \prec Y} \cap A_{X \succ Y} = A_{(X \prec Y) \sqcap (X \succ Y)},$$

we clearly have the last equality since the intersection $A_{X \prec Y} \cap A_{X \succ Y}$ is nonempty. For the second part,

$\square$

To each object $X = (V_X, A_X, a_X, \tilde{a}_X)$ we may define two object

$$X_{\min} := (V_X, \{a_X\}, a_X, \tilde{a}_X), \qquad X_{\max} := (V_X, \{\tilde{a}_X\}^{\sim}, a_X, \tilde{a}_X).$$

It is easily seen that $X_{\min} = (X^*_{\max})^*$ and $X_{\max} = (X^*_{\min})^*$, moreover, $X_{\max}$ and $(X_{\min})^*$ are first order objects. We have the inclusions

$$X_{\min} \xrightarrow{id} X \xrightarrow{id} X_{\max}.$$

We also have the inclusions

$$X \otimes Y \to X \odot Y \to X_{\max} \odot Y_{\max} = X_{\max} \otimes Y_{\max}$$

and

$$X \otimes Y \to X_{\max} \otimes Y \to X_{\max} \otimes Y_{\max}, \quad X \otimes Y \to X \otimes Y_{\max} \to X_{\max} \otimes Y_{\max}.$$

We can therefore define pullbacks and pushouts, which then becomes

$$(X_{\max} \otimes Y) \sqcap (X \otimes Y_{\max}) = X \otimes Y, \qquad (X_{\max} \otimes Y) \sqcup (X \otimes Y_{\max}) = X_{\max} \otimes Y_{\max}.$$

Hence we may decompose $X \odot Y$ into two parts

$$X \prec Y := (X \odot Y) \sqcap (X_{\max} \otimes Y), \qquad X \succ Y := (X \odot Y) \sqcap (X \otimes Y_{\max}).$$

Note that these forms do not depend on the choice of the elements a_X, a_Y...!

Lemma 13. *We have*

$$X \otimes Y = (X \prec Y) \sqcap (X \succ Y), \qquad X \odot Y = (X \prec Y) \sqcup (X \succ Y).$$

Proof. We have

$$\begin{aligned}(X \prec Y) \sqcap (X \succ Y) &= ((X \odot Y) \sqcap (X_{\max} \otimes Y)) \sqcap ((X \odot Y) \sqcap (X \otimes Y_{\max}))\\ &= (X \odot Y) \sqcap ((X_{\max} \otimes Y) \sqcap (X \otimes Y_{\max})) = (X \odot Y) \sqcap (X \otimes Y) = X \otimes Y.\end{aligned}$$

The second equality follows easily from Lemma 9 □

We next show that $- \prec -$ and $- \succ -$ define a functor $\mathrm{Af} \times \mathrm{Af} \to \mathrm{Af}$. Let $X_1 \xrightarrow{f} Y_1$ and $X_2 \xrightarrow{g} Y_2$, we will show that $X_1 \prec Y_1 \xrightarrow{f \otimes g} X_2 \prec Y_2$. For this, we need to prove that $(f \otimes g)(A_{X_1 \prec X_2}) \subseteq A_{X_2 \prec Y_2}$ and $(f \otimes g)(A_{X_1 \succ X_2}) \subseteq A_{X_2 \succ Y_2}$. This is clear from Lemma 9.

Lemma 14. $(\mathrm{Af}, \prec, I)$ *is a monoidal category.*

Proof. It is easily checked from Lemma 9 that $\alpha_{X,Y,X}(L_{(X \prec Y) \prec Z}) = L_{X \prec (Y \prec Z)}$ and clearly also $\alpha_{X,Y,Z}(a_X \otimes a_Y \otimes a_Z) = a_X \otimes a_Y \otimes a_Z$, so that α is the associator. Since $I \otimes X = I \odot X$ and $X \odot I = X \otimes I$, we have $I \prec X = I \otimes X$ and $X \prec I = X \otimes I$, so λ and ρ are the unitors. But note that $\sigma_{X,Y}(A_{X \prec Y}) = A_{Y \succ X}$, so this structure is not symmetric.

□

We have $(X \prec Y)^* = X^* \prec Y^*$. Indeed, by duality,

$$\begin{aligned}(X \prec Y)^* &= ((X \odot Y) \sqcap (X_{\max} \odot Y))^* = (X \odot Y)^* \sqcup (X_{\max} \otimes Y)^*\\ &= (X^* \otimes Y^*) \sqcup (X^*_{\max} \odot Y^*)\end{aligned}$$

.1.10 Internal hom

The internal hom has the form

$$[X, Y] = (X \otimes Y^*)^* = X^* \odot Y. \tag{8}$$

We then have

$$\begin{aligned}&L_{[X,Y]} = (S_X \otimes L_Y^{\perp})^{\perp} = (V_X^* \otimes L_Y) \vee (L_{X^*} \otimes V_Y),\\ &S_{[X,Y]} = (\tilde{a}_X \otimes S_Y^{\perp})^{\perp} \wedge (L_X \otimes \tilde{a}_Y)^{\perp} \wedge (L_X \otimes S_Y^{\perp})^{\perp} = (V_X^* \otimes L_Y) \vee (L_{X^*} \otimes V_Y) \vee (\tilde{a}_X \otimes a_Y).\end{aligned}$$

and

$$d_{[X,Y]} = D_X D_Y - (d_X + 1)(D_Y - d_Y).$$

As we have seen in FinVect, the space $V_{[X,Y]} = V_X^* \otimes V_Y$ is identified with the space of all linear maps $V_X \to V_Y$, by (**??**). We will show that $A_{[X,Y]}$ corresponds to the affine subspace of maps mapping A_X into A_Y, that is, morphisms in Af. Indeed, we see from (**??**) that f is in Af if and only if

$$\langle f(x), y^* \rangle = \langle w, x \otimes y^* \rangle = 1, \qquad x \in A_X,\ y^* \in \tilde{A}_Y,$$

which is equivalent to $w \in (A_X \otimes \tilde{A}_Y)^{\sim} = \tilde{A}_{X \otimes Y^*}$.

.1.11 The no signaling product

For two objects X, Y we define

$$X \prec Y := (V_X \otimes V_Y, A_{X\prec Y}, a_X \otimes a_Y, \tilde{a}_X \otimes \tilde{a}_Y),$$

where $A_{X\prec Y}$ is determined by

$S_{X\prec Y} = V_X \otimes S_Y \cap S_{X\odot Y} = (V_X \otimes S_Y) \cap (L_{X\odot Y} \vee \{a_X \otimes a_Y\}) = (V_X \otimes S_Y) \cap ((V_X \otimes L_Y) \vee (L_X \otimes V_Y) \vee \{a$

Lemma 15.

We may similarly define $X \succ Y$. Setting $f \prec g = f \otimes g$ for $X_1 \xrightarrow{f} X_2$, $Y_1 \xrightarrow{g} Y_2$, we see that $(- \prec -)$ is functorial. Indeed, to show that $X_1 \prec Y_1 \xrightarrow{f\otimes g} X_2 \prec Y_2$, we need to show that $f \otimes g(A_{X_1\prec Y_1}) \subseteq A_{X_2\prec Y_2}$. Assume $w \in A_{X_1\prec Y_1}$, that is, $w \in V_{X_1} \otimes S_{Y_1}$ and $\langle\, w, \tilde{a}_{X_1} \otimes \tilde{a}_{Y_1} \,\rangle = 1$.

.1.12 Dualizable (nuclear) objects

An object in Af is nuclear if the natural map $X^* \otimes X \to [X, X]$ is an isomorphism (santocanale). That is, the inclusion $X^* \otimes X \subseteq X^* \odot X$ that comes from the embedding

$$\tilde{A}_X \otimes A_X \subseteq (A_X \otimes \tilde{A}_X)^{\sim}$$

becomes an equality. As we have seen in Lemma 7, for proper objects we have $X^* \otimes X = X^* \odot X$ if and only if

$$d_X + 1 = D_X = D_{X^*} = d_{X^*} + 1 = D_X - d_X.$$

It follows that $d_X = 0$ and $D_X = 1$, so that $X \simeq I$. Hence the tensor unit is the unique dualizable (or nuclear) object in Af.

.1.13 No signaling

We say that $X \xrightarrow{f} Y$ is no signaling if

$$y^* \circ f = \tilde{a}_Y \circ f, \qquad \forall y \in Y^* = [Y, I],$$

in other words

$$y \circ f = 0, \qquad \forall y^* \in L_{Y^*} = S_Y^{\perp}.$$

Taking $w \in A_{[X,Y]}$ be the corresponding elements, this means that

$$\langle\, w, x \otimes y^* \,\rangle = 0, \qquad \forall x \in V_X,\ y^* \in S_Y^{\perp},$$

in other words

$$w \in (V_X \otimes S_Y^{\perp})^{\perp} = V_X^* \otimes S_Y,$$

so that

$$w \in A_{[X,Y]} \cap (V_X^* \otimes S_Y).$$

Since $a_{[X,Y]} = \tilde{a}_X \otimes a_Y \in V_X^* \otimes S_Y$, we have that

$$a_{[X,Y]} - w \in L_{[X,Y]} \cap V_X^* \otimes S_Y = (S_X \otimes L_Y^{\perp})^{\perp} \cap V_X^* \otimes S_Y = (V_X^* \otimes L_Y) + (L_{X^*} \otimes a_Y).$$

We can also define no signaling in the oposite way, that is,

$$f(x) = f(a_X), \qquad \forall x \in A_X.$$

This is of course the same as

$$f(x) = 0, \qquad \forall x \in L_X,$$

or

$$\langle\, w, x \otimes y^* \,\rangle = 0, \qquad \forall x \in L_X,\ y^* \in V_Y^*,$$

that is,

$$w \in (L_X \otimes V_Y^*)^\perp = L_X^\perp \otimes V_Y = S_{X^*} \otimes V_Y.$$

It follows that

$$\tilde{a}_X \otimes a_Y - w \in L_{[X,Y]} \cap S_{X^*} \otimes V_Y = L_{X^*} \otimes V_Y + \tilde{a}_X \otimes L_Y.$$

.2 Once more on the monoidal structures

.2.1 Tensor product

We have

$$L_{X\otimes Y} = (a_X \otimes L_Y) + (L_X \otimes a_Y) + (L_X \otimes L_Y) = (S_X \otimes L_Y) + (L_X \otimes a_Y) = (L_X \otimes S_Y) + (a_X \otimes L_Y)$$

A closed symmetric monoidal structure. We have

$$L_{X_{\max}\otimes Y} = V_X \otimes L_Y + \{\tilde{a}_X\}^\perp \otimes a_Y, \qquad L_{X_{\min}\otimes Y} = a_X \otimes L_Y.$$

Lemma 16. *We have*

$$X \otimes Y = (X_{\max} \otimes Y) \sqcap (X \otimes Y_{\max}).$$

Proof. This is easy, since

$$S_{X_{\max}\otimes Y} \cap S_{X\otimes Y_{\max}} = (V_X \otimes S_Y) \cap (S_X \otimes V_Y) = S_X \otimes S_Y.$$

□

.2.2 Dual product

By definition, $X \odot Y = (X^* \otimes Y^*)^*$. We have

$$L_{X\odot Y} = (L_X^\perp \otimes L_Y^\perp)^\perp = (V_X \otimes L_Y) \vee (L_X \otimes V_Y).$$

We have

$$L_{X_{\max}\odot Y} = (V_X \otimes L_Y) \vee (\{\tilde{a}_X\}^\perp \otimes V_Y), \qquad L_{X_{\min}\odot Y} = V_X \otimes L_Y.$$

Lemma 17. *We have*

$$\begin{aligned} X_{\max} \otimes Y_{\max} &= (X_{\max} \odot Y) \sqcup (X \odot Y_{\max}) \\ X \odot Y &= (X_{\min} \odot Y) \sqcup (X \odot Y_{\min}) \\ X \otimes Y &= (X_{\min} \otimes Y) \sqcup (X \otimes Y_{\min}) \sqcup (X_{\min} \odot Y \sqcap X \odot Y_{\min}) \end{aligned}$$

Proof. The first is easy, the seconf follows from Lemma 16 by duality, the third is also easy.

□

.3 The no signalling product

Let us define $X \prec Y := (X \odot Y) \sqcap (X_{\max} \otimes Y)$. We have

$$L_{X\prec Y} = V_X \otimes L_Y + L_X \otimes a_Y = L_{X_{\min}\odot Y} + L_{Y\otimes Y_{\min}}.$$

So that

$$X \prec Y := (X \odot Y) \sqcap (X_{\max} \otimes Y) = (X_{\min} \odot Y) + (X \otimes Y_{\min})$$

Lemma 18. *We have*

$$\begin{aligned}(X \otimes Y) \sqcup (X_{\min} \odot Y) = (X \odot Y) \sqcap (X_{\max} \otimes Y)\\ = (X_{\min} \odot Y) + (X \otimes Y_{\min}) = (X_{\max} \otimes Y) \sqcap (X \odot Y_{\max}).\end{aligned}$$

Let us denote the above object by $X \prec Y$. Then $A_{X\prec Y}$ is the set of elements in $V_X \otimes V_Y$ such that $\langle w, \cdot \otimes y^* \rangle$ is a fixed element in A_X, independently of $y^* \in \tilde{A}_Y$.

Blbe uvedenie, definicia!

Proof. We see that $A_{X\prec Y} \subseteq A_{X\odot Y}$, moreover,

$$A_{X\prec Y} = \{w \in A_{X\odot Y},\ \langle w, id_X \otimes y^* \rangle = 0,\ \forall y^* \in L_{Y^*}\}.$$

In other words, since clearly $a_X \otimes a_Y \in A_{X\prec Y}$,

$$\begin{aligned}L_{X\prec Y} = \{w - a_X \otimes a_Y,\ w \in A_{X\prec Y}\} = L_{X\odot Y} \cap (V_X^* \otimes L_Y^*)^\perp = (L_X^\perp \otimes S_{Y^*})^\perp \cap (V_X^* \otimes L_{Y^*})^\perp\\ = ((L_X^\perp \otimes S_{Y^*}) \vee (V_X^* \otimes L_{Y^*}))^\perp = ((V_X^* \otimes L_{Y^*}) + (S_X^* \otimes \tilde{a}_Y))^\perp = S_{(X\prec Y)^*}^\perp\end{aligned}$$

But also

$$\begin{aligned}L_{X\odot Y} \cap (V_X^* \otimes L_Y^*)^\perp &= ((V_X \otimes L_Y) \vee L_X \otimes V_Y) \cap (V_X \otimes S_Y)\\ &= ((V_X \otimes L_Y) + (L_X \otimes a_Y) + (L_X \otimes S_Y')) \cap ((V_X \otimes L_Y) + (L_X \otimes a_Y) + (L_X' \otimes a_Y))\\ &= ((V_X \otimes L_Y) + (L_X \otimes a_Y).\end{aligned}$$

First note that we have

$$L_{X_{\min}\odot Y} = V_X \otimes L_Y, \qquad L_{X\otimes Y_{\min}} = L_X \otimes a_Y$$

and therefore

$$L_{X\otimes Y_{\min}} \cap L_{X_{\min}\odot Y} = \{0\}.$$

Further,

$$L_{X\otimes Y} = (S_X \otimes L_Y) + (L_X \otimes a_Y) = (S_X \otimes L_Y) + L_{X\otimes Y_{\min}}$$

and

$$L_{X\odot Y} = (V_X \otimes L_Y) \vee (L_X \otimes V_Y), \qquad L_{X_{\max}\otimes Y} = (V_X \otimes L_Y) + (\{\tilde{a}_X\}^\perp \otimes a_Y)$$

We have

$$S_{(X\otimes Y)\sqcup(X_{\min}\odot Y)} = S_{X\otimes Y} \vee S_{X_{\min}\odot Y} = S_{X\otimes Y} \vee L_{X_{\min}\odot Y}$$

□

.4 The category AfH

The category AfH will be constructed as a subcategory in Af.

.4.1 First order objects

It is easily seen that the following are equivalent:

1. $D_X = d_X + 1$;
2. $S_X = V_X$;
3. $L_X = \{\tilde{a}_X\}^\perp$;
4. $S_{X^*} = \mathbb{R}\tilde{a}_X$;
5. $L_{X^*} = \{0\}$.

We say that an object X is first order if any of these conditions is fulfilled. We have seen that for proper objects, $X \otimes Y = X \odot Y$ if and only if both X and Y are first order. We also have

Lemma 19. *X is first order if and only if $X \otimes X = X \odot X$.*

‘

Lemma 20. *Let X, Y be first order, then $X \otimes Y$ is first order.*

Proof. We have

$$S_{X\otimes Y} = S_X \otimes S_Y = V_X \otimes V_Y = V_{X\otimes Y}.$$

□

.4.2 Channels

A channel is an object $[X, Y]$ where X and Y are first order. As we have seen,

$$X^* \otimes Y \subseteq X^* \odot Y = [X, Y].$$

If X is first order, $\tilde{A}_X = \{\tilde{a}_X\}$ and the elements of $A_{X^*\otimes Y} = \tilde{a}_X \otimes A_Y$ are identified with channels of the form

$$f(x) = \langle\, \tilde{a}_X, x \,\rangle y, \qquad x \in V_X,$$

for some $y \in A_Y$. Such maps will be called replacement channels.

Lemma 21. *Let X, Y be first order and let $w \in V_X^* \otimes V_Y$. Then $w \in A_{[X,Y]}$ if and only if*

$$\circ_Y : w_{X^*Y} \otimes \tilde{a}_Y \mapsto \tilde{a}_X.$$

Proof. Let $f : V_X \to V_Y$ be the map corresponding to w, then

$$\circ_Y(w \otimes \tilde{a}_Y) = (V_X^* \otimes e_{V_Y})(w \otimes \tilde{a}_Y) = \tilde{a}_Y \circ f,$$

where $\tilde{a}_Y \in V_Y^*$ is seen as a map $V_y \to \mathbb{R}$. So $\tilde{a}_Y \circ f : V_X \to \mathbb{R}$ is an element in V_X^*. We know that $w \in A_{[X,Y]}$ iff $f(A_X) \subseteq A_Y$, which is equivalent to $\tilde{a}_Y \circ f(x) = 1$ for all $x \in A_X$, so that $\tilde{a}_Y \circ f \in \tilde{A}_X = \{\tilde{a}_X\}$, since X is first order. □

Lemma 22. *Let Y be first order and $w \in V_X^* \otimes V_Y$. Then $w \in A_{[X,Y]}$ if and only if*

$$\circ_Y(w_{X^*Y} \otimes \tilde{a}_Y) \in \tilde{A}_X.$$

Moreover,

$$\tilde{A}_{[X,Y]} = A_X \otimes \{\tilde{a}_Y\}.$$

Proof. Since Y is first order, we have $A_{Y^*} = \tilde{A}_Y = \{\tilde{a}_Y\}$ and by (8)

$$\tilde{A}_{[X,Y]} = A_{X \otimes Y^*} = A_X \otimes \{\tilde{a}_Y\}.$$

As in the above proof, let $f : V_X \to V_Y$ be the map corresponding to w. Then $\tilde{a}_Y \circ f \in V_X^*$ and $w \in A_{[X,Y]}$ iff $f(A_X) \subseteq A_Y$. This means that

$$\tilde{a}_Y \circ f(x) = 1, \qquad \forall x \in A_X,$$

which means that $\tilde{a}_Y \circ f \in \tilde{A}_X$.

□

.4.3 AfH

The category AfH is the full subcategory in Af created from first order objects by taking tensor products and duals. We will add more later. We will use the notation V_{XY^*} for $V_X \otimes V_Y^*$, etc.

Any object X in AfH is created from first order objects $X_1, \dots, X_k$, so that $V_X = \tilde{V}_{X_1} \otimes \cdots \otimes \tilde{V}_{X_k}$, where $\tilde{V}_{X_i}$ is either V_{X_i} or $V_{X_i}^*$, $i = 1, \dots, k$. We will next show that any object is a set of channels that contains all replacement channels.

Proposition 1. *Let X be an object in* AfH. *Then there are first order objects Y_I and Y_O and inclusions f, g such that*

$$Y_I^* \otimes Y_O \xrightarrow{f} X \xrightarrow{g} [Y_I, Y_O]. \tag{9}$$

Proof. Let X be first order, then since I is first order,

$$I^* \otimes X = I \otimes X \xrightarrow{\lambda_X} X \xrightarrow{\lambda_X^{-1}} I \otimes X = I \odot X = [I, X].$$

Clearly, $f = \lambda_X$ and $g = \lambda_X^{-1}$ are inclusions. Now assume that Z satisfies (9) and let $X = Z^*$. Taking duals and composing with symmetries, we get

$$Y_O^* \otimes Y_I \xrightarrow{\sigma_{V_{Y,O}^*, V_{Y_I}}} Y_I \otimes Y_0^* = [Y_I \otimes Y_O]^* \xrightarrow{g^*} X \xrightarrow{f^*} (Y_I^* \otimes Y_O)^* \xrightarrow{\sigma_{V_{Y_I}, V_{X_O}^*}} (Y_O \otimes Y_I)^* = [Y_O, Y_I].$$

Since the compositions of f^* and g^* with symmetries are inclusions, we see that X satisfies (9).

Next, let X_1 and X_2 satisfy (9) with some first order objects Y_I^i, Y_O^i and inclusions f^i, g^i, $i = 1, 2$, and let $X = X_1 \otimes X_2$. We then have, using the appropriate symmetries

$$Y_I^1 Y_I^2 \otimes (Y_O^1 Y_O^2)^* \xrightarrow{\sigma_{Y_I^2, Y_O^1}} Y_I^1 \otimes (Y_O^1)^* \otimes Y_I^2 \otimes (Y_O^1)^* \xrightarrow{f^1 \otimes f^2} X \xrightarrow{g^1 \otimes g^2} [Y_I^1, Y_O^1] \otimes [Y_I^2, Y_O^2] \xrightarrow{\sigma_{Y_O^1, Y_O^2}} [Y_I^1 Y_I^2, I_O^1 Y_O^2].$$

Perhaps the last arrow needs some checking, so lt us do it properly. We need to show that for $w \in A_{[Y_I^1, Y_O^1] \otimes [Y_I^2, Y_O^2]}$, we have $\sigma_{Y_O^1, Y_O^2}(w) \in A_{[Y_I^1 Y_I^2, I_O^1 Y_O^2]}$, but this is clear using Lemma 21.

□

The pair (Y_I, Y_O) for an object X will be called the setting of X. For objects of the same setting we may take pullbacks and pushouts of the corresponding inclusions.

Pullbacks are intersections, pushouts the affine mixture.

Channels into (from) products and coproducts

We define AfH as the full subcategory of Af containing all first order objects and closed under (finite products,) duals and tensor products .