INCOMPATIBILITY IN GPT

Basics of GPT:

- **state space**: $K \subset \mathbb{R}^M$ compact, convex

a pair of ordered vector spaces in duality:

- $A \equiv A(K) = \{ f : K \to \mathbb{R},\ \text{affine} \}$
- $A^+ = A^+(K) = \{ f \in A(K),\ f(x) \geq 0 \}$

(A, A^+) ordered vector space, with order unit $1 = 1_K$

- **the dual**:

$V \equiv V(K) = V^*$, $V^+ \equiv V(K)^+ = (A(K)^+)^*$

$K \cong \mathcal{S}(A, A^+, 1) = \{ v \in V^+,\ \langle v, 1 \rangle = 1 \}$

– states

– construction of a representation of K

- **norms**:

– order unit norm in $(A, A^+, 1)$:

$$\| f \|_{max} = \inf \{ \lambda > 0,\ -\lambda 1 \leq f \leq \lambda 1 \}$$

$$= \sup_{x \in K} |f(x)|$$

– dual norm in (V, V^+): base norm:

$\| v \|_K$

- Examples

- $\boxed{\text{classical}}$: $K = \Delta_n$ probability simplex

$$(V, V^+) \cong (A, A^+) \cong (\mathbb{R}^n, \mathbb{R}^n_+)$$

$1 = (1, \dots, 1)$, $(A, \|\cdot\|_{max}) \cong \ell^n_\infty$, $(V, \|\cdot\|_K)$ ℓ^n_1

- $\boxed{\text{quantum}}$: $K = \mathcal{D}_n$ density matrices

$$(V, V^+) \cong (A, A^+) \cong (M^h_n, M^+_n) \quad , \quad 1 = I$$

- $\boxed{\text{Hypercubes}}$:

$H_k = [0,1]^k$, $k \geq 2$

- $A \equiv \mathbb{R}^{k+1}$: we may choose:

$e_1, \dots, e_k$ coordinate map

e_0 - unit

- but it is better to take the basis

$x_i = 2e_i - e_0$, $i = 0, 1, \dots, k$

- $A^+ = \{ (a_0, a_1, \dots, a_k), \sum_{i=1}^{k} |a_i| \leq a_0 \}$

$$(A, \| \ \|_{max}) \cong \ell_1^{k+1}(\mathbb{R})$$

$V \equiv \mathbb{R}^k$

$V^+ = \{ (v_0, v_1, \dots, v_k), |v_i| \leq v_0 \}$

$K \equiv \{ (1, \lambda_1, \dots, \lambda_k), |\lambda_i| \leq 1 \}$

$$(V, \| \ \|_K) = \ell_\infty^{k+1}(\mathbb{R})$$

$V^+ \not\cong A^+$ unless $k = 2$

• Effects and measurements:

- set of effects
$$E(K) = \{ f \in A(K),\ 0 \leq f \leq 1 \}$$
$$= \{ f : K \to [0,1],\ \text{affine} \}$$

- two-outcome measurements:
  outcomes $\{0, 1\}$
  $f(x) \equiv$ probab. of obtaining '0' in the state $x \in K$

- measurements with finite number of outcomes $\{1, .., e\}$
  $\forall x \in K$, $f_i(x) \equiv$ probab. of outcome 'i'
  - an effect $f_i \in E(K)$
  - $\sum_i f_i = 1_K$ (POVM)
- an affine map $f : K \to \Delta_e$

• (IN) compatibility: $f^1, .., f^k$

measurements $f^i : K \to \Delta_e$

with effects $f^i_1, .., f^i_e$

• compatible $\equiv$ jointly measurable

$\exists$ a joint measurement h, with outcomes in $\{1, .., e\}^k$ such that all f^i are obtained as marginals:

label the effects as

$h_{h_1, .., h_k}$, $h_i \in \{1, .., e\}$

then $\quad f^i_j = \sum \{ h_{n_1 \ldots n_k} ,\ n_i = j \}$

$$\begin{tikzcd} K \arrow[r, "h"] \arrow[dr, "f_i"'] & \Delta_{\ell^k} = \Delta_\ell \otimes \cdots \otimes \Delta_\ell \arrow[d, "1 \otimes \cdots \otimes id \otimes 1 \otimes \cdots \otimes 1 \ \equiv J_i"] \\ & \Delta_\ell \end{tikzcd}$$

i-th marginal

a simple observation (but important):

Put $\quad F(x) = (f^1(x), \ldots, f^k(x)) \in \Delta_\ell^k$

– affine map $\quad F : K \to \Delta_\ell^k$

$$\begin{tikzcd} K \arrow[r, "F"] \arrow[dr, "h"'] & \Delta_\ell^k \\ & \Delta_\ell^{\otimes k} \arrow[u, "J = (J_1, \ldots, J_k)"'] \end{tikzcd}$$

joint $\to$ measurement h

$\boxed{\text{Theorem}}$: $f^1, \ldots, f^k$ are compatible iff the map F factorizes through a simplex: $\exists$ affine maps h, G, such that

$$\begin{tikzcd} K \arrow[rr, "F"] \arrow[dr, "h"'] & & \Delta_\ell^k \\ & \Delta_n \arrow[ur, "G"'] & \end{tikzcd}$$

• Compatibility of effects

two-outcome measurements, given by effects: $f \in E(K)$, $f: K \to [0,1] \cong \Delta_2$

- We say that the effects $f_1, .., f_k \in E(K)$ are compatible if the corresp. measurements are.
- the corresp. map $F = (f_1, .., f_k) : K \to H_k$
- F extends uniquely to a positive map

$$F: V(K) \to V(H_k)$$

• relation of positive maps and tensor products:

$T: (V, V^+) \to (W, W^+)$ positive map if

• $T: V \to W$ linear

• $T(V^+) \subseteq W^+$

(V, V^+), (A, A^+) ovs in duality

- choose a basis $v_1, .., v_N$ of V
  a dual basis $f_1, .., f_N$ of A

$$\langle f_i, v_j \rangle = \delta_{ij}$$

- put

$$\chi_V = \sum_i v_i \otimes f_i \in V \otimes A$$

- does not depend from the choice of basis

max. entangled state $\to \chi_V \in V^+ \otimes_{max} A^+$

- $\chi_V \in V^+ \otimes_{\min} A^+$ iff V is simplicial (K is a simplex)

positive map

- $\langle \chi_V, f \otimes v \rangle = \langle f, v \rangle \quad f \in A, v \in V$

Choi matrix

$$(T \otimes id)(\chi_V) \in W^+ \otimes_{\max} A^+$$

$$\langle (T \otimes id)(\chi_V), g \otimes v \rangle = \langle g, T(v) \rangle \qquad v \in V,\ g \in W^*$$

1-1 correspondence between $W^+ \otimes_{\max} A^+$ and positive maps $V \to W$

$\boxed{\text{Theorem}}$: Equivalent conditions:

(i) $f_1, \dots, f_k \in E(K)$ are compatible effects

(ii) the map $F = (f_1, \dots, f_k)$ is ETB

$$(F \otimes id_W)(V^+ \otimes_{\max} W^+) \subseteq V(H_k)^+ \otimes_{\min} W^+$$

for any OUS (W, W^+)

(iii) $(F \otimes id)(\chi_V) \in V(H_k)^+ \otimes_{\min} A^+$

(iv) $\forall z \in A(H_k)^+ \otimes_{\max} V^+$

$$\langle (F \otimes id)(\chi_V), z \rangle \geq 0$$

- INCOMPATIBILITY WITNESSES:

$f_1, \dots, f_k$ not compatible $\Rightarrow \exists\, \underline{\underline{z}} \in A(H_k)^+ \otimes_{\max} V^+$

and that $\langle (F\otimes \mathrm{id})(\chi_V), z\rangle < 0$

Description: basis $x_i = 2e_i - e_0$ of $A(H_k)$

$$z = \sum_{i=0}^{k} x_i \otimes z_i, \qquad z_i \in V$$

- $z \in A(H_k)^+ \otimes_{max} V^+$?

$\forall s \in H_k, \ \forall f \in A^+ \quad \langle s\otimes f, z\rangle \geq 0$

$$= \sum_{i=0}^{k} \langle s, x_i\rangle \langle f, z_i\rangle =$$

$$= \langle f, \sum_{i=0}^{k} \langle s, x_i\rangle z_i \rangle \geq 0 \quad \forall f \in A^+$$

enough to take extreme points:

$$\Leftrightarrow \quad z_0 + \sum_{i=1}^{k} \varepsilon_i z_i \in V^+, \quad \forall \varepsilon \in \{\pm 1\}^k$$

- z a witness $\quad \exists\ F = (f_1, .., f_k)$ s.t.

$$\langle (F\otimes \mathrm{id})(\chi_V), z\rangle < 0$$

$$\| $$

$$\langle \chi_V, (F^*\otimes \mathrm{id})(z)\rangle = \langle \chi_V, 1_k \otimes z_0 + \sum_{i=1}^{k} (2f_i - 1)\otimes z_i \rangle$$

by the prop of χ_V

$$= \langle 1_k, z_0\rangle + \sum_{i=1}^{k} \langle 2f_i - 1, z_i\rangle$$

$$\inf_{f_1,\dots,f_k \in E(K)} \qquad = \langle 1_k, z_0\rangle - \sum_{i=1}^{k} \|z_i\|_K < 0$$

- Normalized witnesses: $z_0 \in K$

$\boxed{\text{Prop}}$: any normalized IW is given by a k-tuple $z_1, \dots, z_k \in V$, such that

a) $\exists\, z_0 \in K$, s.t.

$$-z_0 \leq \sum_i \varepsilon_i z_i \leq z_0 \quad , \quad \forall \varepsilon \in \{\pm 1\}^k$$

b) $$\sum_{i=1}^{k} \|z_i\|_K > 1$$

A remark: Geometric interpretation:

$z \in A(H_k)^+ \otimes_{\max} V^+$ defines a positive map

$$V(H_k) \longrightarrow V \equiv \text{affine map}$$

$$H_k \longrightarrow V^+$$

! $z_0 \equiv$ image of the barycenter of H_k

$2z_i \equiv$ images of the edges of H_k
$i = 1, \dots, k$

$\sum \|z_i\|_K$ – large

$\to$ large hypercubes in V^+

- Violation of compatibility:

$$\underbrace{\sum_{i=1}^{k} \|z_i\| - 1}$$ → how large this may be for a state space?

— expression in terms of cross-norms:

- we may identify $\{(z_1, \ldots, z_k),\ z_i \in V\} \cong \ell_1^k \otimes V$

  — tensor product of Banach spaces, (V with the base norm)

- $\{(z_1, \ldots, z_k),\ \exists z_0\ \ -z_0 \le \sum \varepsilon_i z_i \le z_0\ \ \forall \varepsilon\}$

  is the unit ball of a cross-norm ! (not the injective / projective)

- $\sum \|z_i\|_K = \|(z_1, \ldots, z_k)\|_{\ell_1^k \otimes_\pi V}$ projective cross-norm

- witness: an element of the unit ball of $\| \ \|_\mu$ s.t. $\|z\|_\pi > 1$

- maximal compatibility violation:

$$\|id : \ell_1^k \otimes_\mu V \to \ell_1^k \otimes_\pi V\| \equiv \beta_k(K)$$

- COMP. DEGREE

- effects $f_1, \ldots, f_k \in E(K)$ become compatible when mixed with noise

- $\tilde{f}_i = s_i f_i + (1-s_i) n_i$ $\rightarrow$ $E(K)$ noise $\quad s_i \in [0,1]$

  $n_i = \frac{1}{2} 1$ for all i (coin-tosses)

  $(s_1, .., s_k)$, $s_i \in [0,1]$, $\tilde{f}_i$ are compatible

- for any witness $(z_0, z_1, \dots, z_k) \equiv z$

$$\langle \tilde{F} \otimes id (x_v), z \rangle = \langle 1, z_0 \rangle + \sum_{i=1}^{k} \langle 2f_i - 1, s_i z \rangle$$

$$= \langle (F \otimes id)(x_v), \tilde{z} \rangle$$

↪ replace z_i by $s_i z_i$

→ we have $\|(s_1 z_1, .., s_k z_k)\|_\lambda \le 1$

– $\|(s_1 z_1, .., s_k z_k)\|_\pi = \sum s_i \|z_i\|_K$

- which tuples $(s_1, .., s_k)$ make all $f_1, .., f_k$ compatible? such that $\sum_i s_i \|z_i\|_K \le 1$

  for all $\|z_1, .., z_k\|_\lambda \le 1$

- in particular: $s_i = s \quad \forall i$

  $\delta_k(F) = \max \{ s \in [0,1],\ s f_i + (1-s) \frac{1}{2} 1 \}$ is called the compatibility degree of $(f_1, .., f_k)$

- smallest compatibility degree for K

$\delta_k(K) = \min\limits_F \delta_k(F) = \max \{ s \in [0,1], s f_i + (1-s)\frac{1}{2} 1$ are comp. for all $f_1, .., f_k \}$

$= \max \{ s \in [0,1], \quad s \| z_1, \dots, z_k \|_\pi \leq 1 \quad \forall$

$\| z_1, \dots, z_k \|_? \leq 1 \}$

$$= \frac{1}{\beta_k(K)}$$

- what is the cross-norm $?$ (hm)

(relevant)

- $\beta_k(K) = ?$ $\boxed{\beta_k(K) = 1 \text{ iff } K \text{ is a simplex}}$

- Two special cases: - quantum

- spaces with minimal $\delta_k(K)$ $(\max \beta_k(K))$

- $\boxed{\text{QUANTUM COMPATIBILITY}}$ → free spectrahedra inclusion

(Bluhm & Nechita, arxiv: 1807.01508)

- $K = \mathcal{D}_n$, $(V, V^+) = (M_n^{h}, M_n^+)$

- Let $z_1, \dots, z_k \in M_n^{h}$ describe an IW:

$\exists\ z_0 \in \mathcal{D}_n$, $\sum \varepsilon_i z_i \leq z_0 \quad \forall \varepsilon$

put $x_i = z_0^{-1/2} z_i z_0^{-1/2}$ (we may assume z_0 is invertible)

We obtain a tuple - Quantum IW:

$x_1, \dots, x_k \in M_n^{h}$ and that:

a) $\sum \varepsilon_i x_i \leq I \quad \forall \varepsilon \in \{\pm 1\}^k$

b) $\sup_{\rho \in \mathcal{D}_n} \sum_i \| \rho^{1/2} x_i \rho^{1/2} \|_1 > 1$

(Relation for Bell's inequalities)

- set of tuples satisfying (a),

matrix diamond - matricial relaxation of the ℓ_1 unit ball

- (in)compatibility described using inclusion of min/max spectrahedra
- this is closely related to operator system structures

We note that:

- a) $\Leftrightarrow$ $\sup_{\varepsilon\in\{\pm1\}^k} \left\| \sum_i \varepsilon_i X_i \right\| \le 1$ ← operator norm in M_n^h

  $(X_1,\dots,X_k)\in \ell_1^k\otimes M_n^h$, $\|X_1,\dots,X_k\|_\varepsilon \le 1$ injective norm

- b) what norm do we obtain?

hermitian map: $\Psi_X : M_k \to M_n$, $A \mapsto \sum_i A_{ii} X_i$

$$\sup_\rho \sum_i \|\rho^{1/2} X_i \rho^{1/2}\|_1 = \|\Psi_X\|_{cb}$$

Operator system norms:

norms on $\ell_1^k\otimes M_n^h = M_n^h(\ell_1^k)$

obtained from isometric embeddings

$$\phi : \ell_1^k \longrightarrow B_h(\mathcal{H})$$

- $\|\cdot\|_\varepsilon$ <u>minimal</u>

  ||

- $\sup_\rho \sum_i \|\rho^{1/2} X_i \rho^{1/2}\|_1$ <u>maximal</u>

- known bounds: $\beta_k(\mathcal{D}_n)\le\sqrt{k}$

  equality holds for a sufficiently large dim.

• Maximal incompatibility in GPT

$z_1,\dots,z_k$ – a witness, then clearly $\|z_i\|_k \le 1$

$$\Rightarrow \quad \sum_i \|z_i\|_k \le k \quad \Rightarrow \beta_k(K) \le k$$

– can equality be attained in GPT? → **yes**

– if this happens, then

$$\delta_k(f_1,\dots,f_k) = \frac{1}{k} \quad \text{for some } f_1,\dots,f_k$$

maximally incompatible

super quantum correlations

– PR boxes

$\boxed{\text{Theorem}}$: $f_1,\dots,f_k \in E(K)$ are maximally incompatible if and only if $F: K \to H_k$ is a retraction: $\exists\ S: H_k \to K$, s.t. $FS = id_{H_k}$

section

• such effects exist for K iff there is a projection $P: K \to K$ (affine, idempotent) such that $P(K) \cong H_k$.

• the corresp. witness: SU, where $U: H_k \to H_k$ is the automorphism that flips all edges

$$U(\lambda_1,\dots,\lambda_k) = (1-\lambda_1,\dots,1-\lambda_k)$$

Examples: 1) H_k, or H_m with $m \ge k$

$f_i = c_i$ - projection onto the i-th component

→ always max. incompatible

2) any product $K_1 \times K_2 \times \dots \times K_k$

of nontrivial state spaces

3) <u>Classical channels</u> : affine maps $T: \Delta_k \to \Delta_m$, $m \geq 2$

$A(\Delta_k, \Delta_m)$ any T is determined by the images of the vertices
$$T(\delta_1), \ldots, T(\delta_k) \in \Delta_m \implies$$
$$A(\Delta_k, \Delta_m) \cong \Delta_m^k$$

max. inc. <u>effects</u>: compose the projections onto ith component with suitable effects (attaining both 0 and 1)

$$f_i(T) = h_i(T(\delta_i)), \quad h_i \in E(\Delta_m)$$

4) <u>Quantum channels</u> $\mathcal{C}(\mathcal{D}_k, \mathcal{D}_m)$, $m \geq 2$
(cptp maps)

- choose ONB's $x_1, \ldots, x_k \in \mathbb{C}^k$, $y_1, \ldots, y_m \in \mathbb{C}^m$
- for $\Phi \in \mathcal{C}(\mathcal{D}_k, \mathcal{D}_m)$, define the map
$$F: \Phi \mapsto \left(\langle y_j, \Phi(|x_i\rangle\langle x_i|) y_j \rangle \right)_{ij} \in \Delta_m^k$$
  - a retraction (onto classical channels)
- max inc. effects: take any $y \in \mathbb{C}^m$
$$f_i = \langle y | \left(\Phi(|x_i\rangle\langle x_i|) \right) | y \rangle$$

section: $(\lambda_1, \ldots, \lambda_k) \mapsto \Phi$
$$\Phi(\rho) = \sum_i \langle x_i | \rho | x_i \rangle \left(\lambda_i |y\rangle\langle y| + (1-\lambda_i) |y^\perp\rangle\langle y^\perp| \right)$$

- realization of PR boxes (Plávala & Ziman)