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The exponential Orlicz space in quantum information geometry

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The exponential Orlicz space in quantum information geometry

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Abstract

We review the construction of a quantum version of the exponential statistical manifold over the set of all faithful normal positive functionals on a von Neumann algebra. The construction is based on the relative entropy approach to state perturbation. We construct a quantum version of the exponential Orlicz space and discuss the properties of this space and its dual with respect to Kosaki L_p-spaces. We show that the constructed manifold admits a canonical divergence satisfying a Pythagorean relation. We also prove that the manifold structure is invariant under sufficient channels.

Keywords: quantum exponential manifold, quantum relative entropy, perturbation of states, canonical divergence

1 Introduction

One of the fundamental achievements of Information geometry is the rigorous extension from parametric statistical models to the nonparametric case by Pistone and Sempi, [1], who constructed a Banach manifold structure on the set of probability measures equivalent to a given probability measure. The manifold structure is based on an Orlicz space associated to an exponential Young function $\Phi(x) = \cosh(x) - 1$. This theory has been subsequently developed in a number of works, see e.g. [2, 3]. In this construction, the properties of the moment generating function and its conjugate, the Kullback Leibler divergence (relative entropy), play a central role.

To obtain a full quantum version of the Pistone-Sempi construction would mean to introduce an analogous Banach manifold structure on the set of faithful normal states of a general (σ-finite) von Neumann algebra. The problem is that known versions of quantum Orlicz spaces are either restricted to the semifinite case (e.g. [4, 5]) or are technically quite involved ([6]) and it is unclear how to introduce an exponential structure on the set of states, based on these spaces.

Another approach using perturbation of states on the algebra $B(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$, was taken in [7, 8]. Here the manifold is modelled on the space of certain (unbounded) perturbation operators, which are given the structure of a Banach space. In [8], the Banach space is constructed from the free energy functional, which can be seen as the counterpart of the classical cumulant generating function.

This idea inspired the works [9, 10], where a definition of the exponential Orlicz space over a von Neumann algebra $\mathcal{M}$ with respect to a faithful normal state ρ is proposed, using the relative entropy approach to state perturbation. This definition has the advantage that it is based directly on the relative entropy $S(\cdot\|\cdot)$ and its properties. In particular, the set $\mathcal{P}_\rho$ of all normal positive linear functionals such that $S(\omega\|\rho) < \infty$ is identified with a generating cone in the dual of the constructed exponential Orlicz space, so that a manifold structure on normal states of $\mathcal{M}$, respecting the relative entropy, can be introduced by immersion into a Banach space. Moreover, an exponential manifold structure is obtained using perturbations of the state ρ by elements of the exponential Orlicz space and the connected components of this manifold are contained in $\mathcal{P}_\rho$.

In the present paper, we review the construction of the exponential Orlicz space and its dual, as defined in [9, 10]. We present the proofs in a more streamlined and precise form. The dual space is found explicitly as an Orlicz space, using the conjugate Young function. We show the relation to of the constructed spaces to the Kosaki L_p-spaces. The manifold structure is introduced over the positive cone of faithful positive linear functionals, rather than states, similarly to the approach in [11]. We define a canonical divergence on the manifold, satisfying a generalized Pythagorean relation. Finally, we prove the invariance of our structures under sufficient channels, which is the counterpart of the important invariance property of the classical information geometry.

2 The exponential Orlicz space

In this section, we review the definition of the exponential Orlicz space from [9], construct its dual as an Orlicz space and study some of the properties of these spaces.

2.1 A general construction of an Orlicz space

A function $\Phi : X \to [0, \infty]$ is called a Young function if it satisfies:

(i) Φ is convex,
(ii) $\Phi(x) = \Phi(-x)$ for all $x \in X$ and $\Phi(0) = 0$,
(iii) If $x \neq 0$ then $\lim_{t\to\infty} \Phi(tx) = \infty$.

For a Young function Φ, put $C_\Phi := \{x \in X, \Phi(x) \le 1\}$ and $V_\Phi := \{x \in \Phi(x),\ \exists s > 0,\ \Phi(sx) < \infty\}$. The set C_Φ is absolutely convex and $V_\Phi = \bigcup_n nC_\Phi$ is a linear span of the effective domain $\mathrm{Dom}(\Phi) = \{x \in X,\ \Phi(x) < \infty\}$. We can define a norm in V_Φ as the Minkowski functional of C_Φ:

$$\|x\|_\Phi := \inf\{\lambda > 0,\ \Phi(\frac{x}{\lambda}) \le 1\}, \qquad x \in V_\Phi.$$

The completion of V_Φ with respect to this norm will be denoted by B_Φ.

Let (Ω, Σ, μ) be a measure space and let X to be the vector space of measurable functions $f : \Omega \to \mathbb{R}$. Let φ be a Young function on $\mathbb{R}$ and put

$$\Phi_\varphi(f) := \int_\Omega \varphi(\mathsf{f})d\mu.$$

Then B_{Φ_φ} is the classical Orlicz space $L^\varphi(\Omega, \Sigma, \mu)$ and $\|\cdot\|_{\Phi_\varphi}$ is the Luxemburg-Nakano norm, [12]. As another example, let $\mathcal{M}$ be semifinite von Neumann algebra with a faithful normal semifinite trace τ. Let X be the vector space of self-adjoint τ-measurable operators and $\Phi_\varphi(a) = \tau(\varphi(\mathsf{a}))$, then we obtain the Orlicz space $L^\varphi(\mathcal{M}, \tau)$, [4]. In [5], a version of the Orlicz space with respect to a faithful normal state ρ was also defined by a construction of this type. See also [6] in the case of a general von Neumann algebra with a faithful normal weight.

It was shown in [9, Prop. 2] that if X is a Banach space and Φ is continuous, then X is continuously embedded in B_Φ. The conjugate function Φ^* is again a Young function such that $V_{\Phi^*} = \mathrm{Dom}(\Phi^*) = B_{\Phi^*}$ and we have a continuous embedding $B_{\Phi^*} \sqsubseteq X^*$. Moreover, $B_{\Phi^*} = B_\Phi^*$, with equivalent norms.

2.2 Basic setting and notations

We briefly describe the setting of von Neumann algebras and noncommutative L_p-spaces. For a quick overview of these topics, see [13].

Let $\mathcal{M}$ be a (σ-finite) von Neumann algebra. We will denote by $\mathcal{M}^*$ the dual space of $\mathcal{M}$ and by $\mathcal{M}_*$ the predual, consisting of normal functionals in $\mathcal{M}^*$. The positive cones in these spaces will be denoted by $\mathcal{M}^+$, $(\mathcal{M}^*)^+$ and $\mathcal{M}_*^+$. An element $\rho \in \mathcal{M}_*^+$ is faithful if $\rho(a) = 0$ implies $a = 0$, for any $a \in \mathcal{M}^+$.

For $1 \le p \le \infty$, we denote the Haagerup L_p-space over $\mathcal{M}$ by $L_p(\mathcal{M})$ and its norm by $\|\cdot\|_p$. We will often use the identification of $L_\infty(\mathcal{M})$ with $\mathcal{M}$ and $L_1(\mathcal{M})$ with $\mathcal{M}_*$. Let $h_\psi \in L_1(\mathcal{M})$ be the element corresponding to $\psi \in \mathcal{M}_*$, then we can define the trace in $L_1(\mathcal{M})$ by $\mathrm{Tr}[h_\psi] = \psi(1)$.

For $p, q, r \ge 1$ such that $1/p + 1/q = 1/r$ and $h \in L_p(\mathcal{M})$, $k \in L_q(\mathcal{M})$, we have $hk \in L_r(\mathcal{M})$ and the Hölder inequality holds:

$$\|hk\|_r \le \|h\|_p \|k\|_q.$$

For $1 \leq p < \infty$ and $1/p + 1/q = 1$, the space $L_q(\mathcal{M})$ can be identified with the dual space $L_p(\mathcal{M})^*$, with duality given by

$$\langle h, k \rangle = \mathrm{Tr}[hk], \qquad h \in L_p(\mathcal{M}),\ k \in L_q(\mathcal{M}).$$

The space $L_2(\mathcal{M})$ is a Hilbert space with inner product

$$(h, k) = \mathrm{Tr} h^* k, \qquad h, k \in L_2(\mathcal{M}).$$

We will assume the standard form $(\lambda(\mathcal{M}), L_2(\mathcal{M}), L_2(\mathcal{M})^+, J)$ ([14, Thm. 3.6]), where λ is the left action $\lambda(a) : h \mapsto ah$ for $a \in \mathcal{M}$ and $h \in L_2(\mathcal{M})$ and J is defined by $Jh = h^*$. For more on the standard form see [15]. Any positive normal functional $\varphi \in \mathcal{M}_*^+$ has a unique vector representative $h_\psi^{1/2}$ in the cone $L_2(\mathcal{M})^+$, that is,

$$\varphi(a) = (h_\varphi^{1/2}, a h_\varphi^{1/2}), \qquad a \in \mathcal{M}.$$

See the notes [14] for more details.

Let us now fix a faithful positive normal functional $\rho \in \mathcal{M}_*^+$. The (symmetric) Kosaki L_p-space with respect to ρ [16] is defined via complex interpolation, using the continuous embedding

$$i_{\infty,\rho} : \mathcal{M} \to L_1(\mathcal{M}), \qquad a \mapsto h_\rho^{1/2} a h_\rho^{1/2}. \tag{1}$$

Let us denote the image $i_{\infty,\rho}(\mathcal{M})$ by $L_\infty(\mathcal{M}, \rho)$, with the norm $\|i_{\infty,\rho}(a)\|_{\infty,\rho} = \|a\|$. The interpolation space $C_{1/p}(L_\infty(\mathcal{M}, \rho), L_1(\mathcal{M}))$ [17] will be denoted by $L_p(\mathcal{M}, \rho)$ and the norm by $\|\cdot\|_{p,\rho}$. The map

$$i_{p,\rho} : L_p(\mathcal{M}) \to L_1(\mathcal{M}), \qquad k \mapsto h_\rho^{1/2q} k h_\rho^{1/2q} \tag{2}$$

with $1/p + 1/q = 1$ is an isometric isomorphism of $L_p(\mathcal{M})$ onto $L_p(\mathcal{M}, \rho)$ for $1 \leq p \leq \infty$. From the properties of complex interpolation spaces, we have for $1 \leq p' \leq p \leq \infty$ the continuous embeddings $L_p(\mathcal{M}, \rho) \sqsubseteq L_{p'}(\mathcal{M}, \rho) \sqsubseteq L_1(\mathcal{M})$. We have $L_q(\mathcal{M}, \rho) \simeq L_p(\mathcal{M}, \rho)^*$ for $1 \leq p < \infty$ and $1/p + 1/q = 1$, with duality given by

$$\langle i_{p,\rho}(k), i_{q,\rho}(l) \rangle = \mathrm{Tr}[kl], \qquad k \in L_p(\mathcal{M}),\ l \in L_q(\mathcal{M}).$$

Note also that the Kosaki L_p-spaces can be constructed as in Section 2.1, where $X = \mathcal{M}^s := \{a = a^* \in \mathcal{M}\}$ and $\Phi(a) = \|h_\rho^{1/2p} a h_\rho^{1/2p}\|_p$, [18].

Let $\mathcal{N}$ be another von Neumann algebra and let $T : L_1(\mathcal{M}) \to L_1(\mathcal{N})$ be a positive linear map that preserves trace. Such a map will be called a channel. The adjoint of T is a positive unital normal map $T^* : \mathcal{N} \to \mathcal{M}$.

Let ρ be a faithful element in $\mathcal{M}_+^*$. One can see ([19, Sec. 3.3] that if $e = s(T(\rho))$ is the support projection of $T(\rho)$, then we have $T(\omega) = eT(\omega)e$, hence we may suppose that $T(\rho)$ is faithful by replacing $\mathcal{N}$ by $e\mathcal{N}e$.

Proposition 1 *[19] A channel T restricts to a contraction $L_p(\mathcal{M},\rho) \to L_p(\mathcal{N}, T(\rho))$ for any $1 \le p \le \infty$.*

In the case $p = \infty$, there is a positive linear map $T^*_\rho : \mathcal{M} \to \mathcal{N}$, defined by

$$T(h_\rho^{1/2} a h_\rho^{1/2}) = T(\rho)^{1/2} T^*_\rho(a) T(\rho)^{1/2}, \qquad a \in \mathcal{M}.$$

The map T^*_ρ was introduced in [20] and is called the Petz dual of T (with respect to ρ). It was also proved that T^*_ρ is unital and normal, moreover, it is n-positive if and only if T is n-positive, for any n. Let $T_\rho : L_1(\mathcal{N}) \to L_1(\mathcal{M})$ be the preadjoint of T^*_ρ. Then $T_\rho \circ T(\rho) = \rho$ and the Petz dual of T_ρ is T^*.

2.3 Relative entropy and related functionals

The Araki relative entropy for $\omega, \rho \in \mathcal{M}_*^+$ [21, 22] is defined using the relative modular operator $\Delta_{\rho,\omega} (= \Delta_{\rho, h_\omega^{1/2}})$ as

$$S(\omega\|\rho) = \begin{cases} -\langle \log(\Delta_{\rho,\omega}) h_\omega^{1/2}, h_\omega^{1/2}\rangle & \text{if } s(\omega) \le s(\rho) \\ \infty & \text{otherwise.} \end{cases}$$

Here $s(\rho)$ denotes the support projection of ρ. Alternatively, we have the following variational formula due to Kosaki [23]:

$$S(\omega\|\rho) = \sup_n \sup \left\{ \omega(1) \log n - \int_{1/n}^\infty (\omega(y(t)^* y(t)) + t^{-1} \rho(x(t) x(t)^*)) \frac{dt}{t} \right\},$$

here the second supremum is taken over all step functions $x : (1/n, \infty) \to L$ with finite range, $y(t) = 1 - x(t)$ and L is a subspace in $\mathcal{M}$ containing 1 which is dense in the strong*-operator topology.

The relative entropy S is a jointly convex function $S : \mathcal{M}_*^+ \times \mathcal{M}_*^+ \to \mathbb{R} \cup \{\infty\}$, lower semicontinuous with respect to the $\sigma(\mathcal{M}_*, \mathcal{M})$-topology. Moreover, S is strictly convex in the first variable, which can be inferred from the identity [22, Prop. 5.22]

$$S(\omega\|\rho) + \sum_i S(\omega_i\|\omega) = \sum_i S(\omega_i\|\rho), \quad \omega = \sum_{i=1}^k \omega_i,\ \omega \in \mathcal{M}_*^+. \tag{3}$$

Note also that since $\omega_i \le \omega$ in (3), we have $S(\omega_i\|\omega) \le S(\omega_i\|\omega_i) < \infty$ for all i. See [22, Sec. 5] for details and a list of further important properties of S. The next statement shows the relation to the Kosaki L_p-space.

Proposition 2 *[19, 24] Let $\omega, \rho \in \mathcal{M}_*^+$ be such that $h_\omega \in L_p(\mathcal{M}, \rho)$ for some $p > 1$. Then the function $f : (1, p] \to \mathbb{R}$, defined as*

$$f(\alpha) := \frac{1}{\alpha - 1} \log \frac{\|h_\omega\|_{\alpha,\rho}^\alpha}{\omega(1)}$$

is increasing and $\lim_{\alpha \downarrow 1} f(\alpha) = \frac{1}{\omega(1)} S(\omega\|\rho)$.

Using the above variational formula, the relative entropy can be readily extended to a function $S : (\mathcal{M}^*)^+ \times (\mathcal{M}^*)^+ \to \mathbb{R} \cup \{\infty\}$.

Proposition 3 *Let $\rho \in \mathcal{M}_*^+$ and let $\omega \in (\mathcal{M}^*)^+$. If $\omega \notin \mathcal{M}_*^+$, then $S(\omega\|\rho) = \infty$.*

Proof [22] There is another way to define the relative entropy for elements in $(\mathcal{M}^*)^+$. Let $(\pi_u, \mathcal{H}_u)$ be the universal representation of $\mathcal{M}$ and let $\bar{\mathcal{M}} = (\pi_u(\mathcal{M})'' \cong \mathcal{M}^{**}$ be the universal enveloping von Neumann algebra of $\mathcal{M}$ [15]. Then each element of the dual space $\omega \in \mathcal{M}^*$ has an extension to a normal functional $\bar{\omega}$ on $\bar{\mathcal{M}}$ and $\mathcal{M}^*$ is the predual of $\bar{\mathcal{M}}$. Moreover, there is a central projection $z_0 \in \bar{\mathcal{M}}$ such that $\mathcal{M}_* = \mathcal{M}^* z_0$. We can define for $\omega, \rho \in (\mathcal{M}^*)^+$ the relative entropy $\bar{S} : (\mathcal{M}^*)^+ \times (\mathcal{M}^*)^+ \to \mathbb{R}$ as

$$\bar{S}(\omega\|\rho) := S_{\bar{\mathcal{M}}}(\bar{\omega}\|\bar{\rho})$$

(here $S_{\bar{\mathcal{M}}}$ is computed with respect to the von Neumann algebra $\bar{\mathcal{M}}$). Now note that we may use $L = \pi_u(\mathcal{M})$ in the variational formula for $S_{\bar{\mathcal{M}}}$ and $L = \mathcal{M}$ for S, which implies that $\bar{S} = S$. Let $\rho \in \mathcal{M}_*^+$, $\omega \in (\mathcal{M}^*)^+$ and assume that ω is not normal. Then we must have $\bar{\omega}(1 - z_0) > 0$ but $\bar{\rho}(1 - z_0) = 0$, so that $s(\bar{\omega}) \not\leq s(\bar{\rho})$. By definition of the relative entropy, this implies that $\bar{S}(\omega\|\rho) = \infty$.

□

From now on, let us fix a faithful normal functional $\rho \in \mathcal{M}_*^+$. Let us define the function $F_\rho : \mathcal{M}^* \to \mathbb{R}$ by

$$F_\rho(\omega) := \begin{cases} S(\omega\|\rho) - \omega(1) & \text{if } \omega \in (\mathcal{M}^*)^+ \\ \infty & \text{otherwise.} \end{cases}$$

We also define the sets

$$\mathcal{S}_C := \{\omega \in \mathcal{M}^*,\ F_\rho(\omega) \leq C\},\ C \in \mathbb{R}, \qquad \mathcal{P}_\rho := \{\omega \in \mathcal{M}^*,\ F_\rho(\omega) < \infty\}.$$

In other words, $\mathcal{P}_\rho$ is the effective domain of F_ρ. Note that we have $\mathcal{S}_C \subseteq \mathcal{P}_\rho \subseteq \mathcal{M}_*^+$, by Proposition 3. The next proposition lists some important properties of F_ρ and these sets.

Proposition 4 *(i) F_ρ is strictly convex and weak*-lower semicontinuous.*

(ii) We have the inequalities

$$F_\rho(\omega) \geq \omega(1)(\log \frac{\omega(1)}{\rho(1)} - 1) \geq -\rho(1).$$

The first inequality becomes an equality if and only if $\omega = \lambda\rho$ for some $\lambda \geq 0$. In particular, $F_\rho(\omega) = -\rho(1)$ if and only if $\omega = \rho$.

(iii) For any $C \in \mathbb{R}$, $\mathcal{S}_C$ is convex and compact in both the weak- and the $\sigma(\mathcal{M}_*, \mathcal{M})$-topology.*

(iv) The set $\mathcal{P}_\rho$ is a face of the cone $\mathcal{M}_*^+$, containing $L_p(\mathcal{M}, \rho)^+$ for any $1 < p \leq \infty$.

Proof The proof of (i)-(ii) follows from the variational formula and properties of S. For the proof of (iii), let $\omega \in \mathcal{S}_C$, then by (ii),

$$\omega(1)(\log \frac{\omega(1)}{\rho(1)} - 1) \leq F_\rho(\omega) \leq C.$$

This implies that $\omega(1) = \|\omega\|$ must be bounded over $\mathcal{S}_C$. Since $\mathcal{S}_C$ is weak*-closed by (i), this implies that $\mathcal{S}_C$ is weak*-compact. But $\mathcal{S}_C \subseteq \mathcal{M}_*$, so that it is also compact in the $\sigma(\mathcal{M}_*, \mathcal{M})$-topology.

To prove the last statement (iv), let $\omega = \sum_i \omega_i$ for some $\omega_i \in \mathcal{M}_*^+$. Then by (3)

$$F_\rho(\omega) + \sum_i S(\omega_i\|\omega) = \sum_i F_\rho(\omega_i).$$

Since $S(\omega_i\|\omega) < \infty$, $\omega \in \mathcal{P}_\rho$ if and only if all $\omega_i \in \mathcal{P}_\rho$, so that $\mathcal{P}_\rho$ is a face of $\mathcal{M}_*^+$. The fact that $L_p(\mathcal{M}, \rho)^+ \subseteq \mathcal{P}_\rho$ for $1 < p \leq \infty$ follows from Proposition 2. □

We also have the following important monotonicity property.

Proposition 5 *Let $T : L_1(\mathcal{M}) \to L_1(\mathcal{N})$ be a channel. Then*

$$F_{T(\rho)}(T(\omega)) \leq F_\rho(\omega), \qquad \omega \in \mathcal{M}^*.$$

Proof The statement follows from [25, Sec. 3.1 and Thm. 4.1]. □

Let $\mathcal{M}^s$ denote the real vector subspace of self-adjoint elements of $\mathcal{M}$. Then $\mathcal{M}^s$ is closed in $\mathcal{M}$ and the dual space $(\mathcal{M}^s)^*$ is the space of all linear functionals $\varphi \in \mathcal{M}^*$ satisfying $\varphi(a^*) = \overline{\varphi(a)}$, $a \in \mathcal{M}$. Similarly, $\mathcal{M}_*^s = (\mathcal{M}^s)^* \cap \mathcal{M}_*$ is the predual of $\mathcal{M}^s$.

We now consider F_ρ as a function over $(\mathcal{M}^*)^s$ and study its Legendre-Fenchel conjugate with respect to the dual pair $((\mathcal{M}^s)^*, \mathcal{M}^s)$. Namely, put

$$C_\rho(a) := F_\rho^*(a) = \sup_{\omega \in (\mathcal{M}^s)^*} \omega(a) - F_\rho(\omega), \qquad a \in \mathcal{M}^s. \tag{4}$$

The proof of the following result can be obtained from [22, Sec. 12]. See also [26, 27]. We collect the arguments for convenience of the reader.

Theorem 6 *The supremum in (4) is attained at a unique functional $\rho^a \in \mathcal{M}_*^+$. The element ρ^a is faithful and $C_\rho(a) = \rho^a(1)$. Moreover, we have the equality*

$$\omega(a) + S(\omega\|\rho^a) = S(\omega\|\rho), \qquad \omega \in \mathcal{M}_*^+ \tag{5}$$

and the chain rule

$$\rho^{a+b} = (\rho^a)^b, \quad C_\rho(a+b) = C_{\rho^a}(b), \qquad a, b \in \mathcal{M}^s. \tag{6}$$

Proof Let $a \in \mathcal{M}^s$ and let $\xi(a)$ denote the perturbed vector [28]

$$\xi(a) = \sum_{n=0}^{\infty} \int_0^{1/2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_n} dt_n \Delta_\rho^{t_n} a \Delta_\rho^{t_{n-1}-t_n} a \ldots \Delta_\rho^{t_1-t_2} a h_\rho^{1/2}.$$

Then $\xi(a) \in L_2(\mathcal{M})^+$ and the functional $\rho^a \in \mathcal{M}_*^+$ given by $(\xi(a), \cdot\xi(a))$ is faithful. By [21, Thm. 3.10], ρ^a satisfies (5). It follows that for $\omega \in \mathcal{P}_\rho$,

$$\omega(a) - F_\rho(\omega) = -F_{\rho^a}(\omega) \leq \rho^a(1),$$

with equality if and only if $\omega = \rho^a$ (Proposition 4 (iii)). By replacing ρ by ρ^b in (5), we obtain

$$\omega(a+b) + S(\omega\|(\rho^b)^a) = \omega(b) + S(\omega\|\rho^b) = S(\omega\|\rho),$$

which implies the chain rule (6).

□

Example 1 Let $\mathcal{M} = B(\mathcal{H})$, the algebra of bounded operators on a Hilbert space $\mathcal{H}$. The functional ρ is represented as a density operator $\rho \in B(\mathcal{H})^+$ with finite trace, such that $\rho(a) = \mathrm{tr}[\rho a]$, (tr being the usual trace on $B(\mathcal{H})$). One can see that in this case,

$$\rho^a = \exp(\log \rho + a).$$

We will also need the following result.

Lemma 7 *For $a, b \in \mathcal{M}^s$, we have*

$$C_\rho(a) - C_\rho(b) \geq \rho^b(a-b).$$

Proof This follows by the fact that ρ^b is the Gateaux derivative of C_ρ at b and convexity of C_ρ (see [29, Prop. 5.3 and 5.4]).

□

2.4 The exponential Young function and its dual

We now introduce a conjugate pair of Young functions on the Banach spaces $\mathcal{M}^s$ and $(\mathcal{M}^*)^s$. Define

$$\Phi_\rho(a) := \frac{1}{2}(C_\rho(a) + C_\rho(-a)) - \rho(1), \qquad a \in \mathcal{M}^s.$$

$$\Psi_\rho(\psi) := \frac{1}{2} \inf_{\substack{\omega_\pm \in (\mathcal{M}^*)^+ \\ 2\psi = \omega_+ - \omega_-}} [F_\rho(\omega_+) + F_\rho(\omega_-)] + \rho(1), \qquad \psi \in (\mathcal{M}^*)^s.$$

It was proved in [9] that Φ_ρ is a strictly convex and continuous Young function $\mathcal{M}^s \to \mathbb{R}$. We now look at the properties of Ψ_ρ.

Lemma 8 Ψ_ρ *is a strictly convex and weak*-lower semicontinuous function on* $(\mathcal{M}^*)^s$*, with effective domain*

$$\mathrm{Dom}(\Psi_\rho) = \{\psi \in (\mathcal{M}^*)^s,\ \Psi_\rho(\psi) < \infty\} = \mathcal{P}_\rho - \mathcal{P}_\rho \subseteq \mathcal{M}_*^s.$$

Proof It is quite clear that $\Psi_\rho(\psi)$ is finite if and only if $\psi = \omega_+ - \omega_-$ for some $\omega_\pm \in \mathcal{P}_\rho$. Further, strict convexity of F_ρ implies that Ψ_ρ is strictly convex as well. For the last statement we have to show that for any $c > 0$, the set $\{\psi \in (\mathcal{M}^*)^s,\ \Psi_\rho(\psi) \leq c\}$ is weak*-closed. So assume that (ψ_i) is a net in $(\mathcal{M}^*)^s$ such that $\Psi_\rho(\psi_i) \leq c$ and let $\psi_i \to \psi$ in the weak*-topology. For each $\varepsilon > 0$ and for all i there are some functionals $\psi_{i,\pm}^\varepsilon \in \mathcal{M}_*^+$ such that $2\psi_i = \psi_{i,+}^\varepsilon - \psi_{i,-}^\varepsilon$ and

$$\frac{1}{2}[F_\rho(\psi_{i,+}^\varepsilon) + F_\rho(\psi_{i,-}^\varepsilon)] + \rho(1) \leq c + \varepsilon.$$

By Proposition 4 (iii), we obtain that $\psi_{i,\pm}^\varepsilon \in \mathcal{S}_{K_\epsilon}$ with $K_\epsilon = 2(c+\varepsilon) - \rho(1)$. By Proposition 4 (iii), $\mathcal{S}_{K_\varepsilon}$ is weak*-compact, so that there is a subnet (ψ_j) and some $\psi_\pm^\varepsilon \in \mathcal{M}_*^+$ such that $\psi_{j,\pm}^\varepsilon \to \psi_\pm^\varepsilon$. We therefore have $\psi_+^\varepsilon - \psi_-^\varepsilon = \lim \psi_{j,+}^\varepsilon - \psi_{j,-}^\varepsilon = 2\psi$ and by weak*-lower semicontinuity of F_ρ,

$$\Psi_\rho(\psi) \leq \frac{1}{2}[F_\rho(\psi_+^\varepsilon) + F_\rho(\psi_-^\varepsilon)] + \rho(1) \leq \liminf_j \frac{1}{2}[F_\rho(\psi_{j,+}^\varepsilon) + F_\rho(\psi_{j,-}^\varepsilon)] + \rho(1) \leq c + \epsilon.$$

Since this holds for all $\varepsilon > 0$, we have $\Psi_\rho(\psi) \leq c$.

□

Proposition 9 Ψ_ρ *is the Legendre-Fenchel conjugate of* Φ_ρ*, with respect to the dual pair* $(\mathcal{M}^s, (\mathcal{M}^*)^s)$*. In particular,* Ψ_ρ *is a Young function on* $(\mathcal{M}^s)^*$*.*

Proof Since F_ρ is weak*-lower semicontinuous, we see that $C_\rho^* = F_\rho^{**} = F_\rho$. Let D_ρ be given by $D_\rho(a) = C_\rho(-a)$ for $a \in \mathcal{M}^s$, then $D_\rho^*(\psi) = C_\rho^*(-\psi)$ for $\psi \in (\mathcal{M}^s)^*$. By [30, Cor. 2.3.5] and the fact that Ψ_ρ is weak*-lower semicontinuous, we obtain $\Psi_\rho = \Phi_\rho^*$, so that Ψ_ρ is a Young function on $(\mathcal{M}^*)^s$ by [9, Lemma 3.4].

□

2.5 The spaces $E_{\exp}(\mathcal{M},\rho)$ and $L_{\log}(\mathcal{M},\rho)$

Using the Young functions Φ_ρ and Ψ_ρ, we construct the corresponding Banach spaces B_{Φ_ρ} and B_{Ψ_ρ} as in Section 2.1. The following is a consequence of [9, Prop.2] and the above results.

Proposition 10 *We have $V_{\Phi_\rho} = \mathcal{M}^s$ and $B_{\Psi_\rho} = V_{\Psi_\rho} = \mathcal{P}_\rho - \mathcal{P}_\rho$. Moreover, $B_{\Psi_\rho} = B^*_{\Phi_\rho}$ (with equivalent norms) and we have the continuous embeddings*

$$\mathcal{M}^s \sqsubseteq B_{\Phi_\rho}, \qquad B_{\Psi_\rho} \sqsubseteq \mathcal{M}^s_*.$$

Let us now look at the case when $\mathcal{M}$ is commutative. Since ρ is faithful, $\mathcal{M}$ can be identified with the space $L_\infty(\Omega,\Sigma,\rho)$ where ρ is a finite measure on (Ω,Σ). Let $\phi:\mathbb{R}\to\mathbb{R}$, $\phi(x)=\cosh(x)-1$ and let ψ be its conjugate, then ψ satisfies the Δ_2 condition $\psi(2u)\le K\psi(u)$ for some $K>0$. The exponential Orlicz space $L^\phi(\Omega,\Sigma,\rho)$ is the dual space of $L^\psi(\Omega,\Sigma,\rho)$. Since the measure ρ is finite, we have $L_\infty(X,\Sigma,\rho)\subseteq L^\phi(X,\Sigma,\rho)$ and one can see that the norm obtained from our construction coincides with the Luxemburg-Nakano norm in $L^\phi(X,\Sigma,\rho)$. Hence B_{Φ_ρ} coincides with the closure $E^\phi(X,\Sigma,\rho)$ of $L_\infty(X,\Sigma,\rho)$ in $L^\phi(X,\Sigma,\rho)$. We then have

$$L^\psi(X,\Sigma,\rho) = E^\phi(X,\Sigma,\rho)^* = B_{\Psi_\rho}$$

and $L^\phi(X,\Sigma,\rho)$ coincides with the second dual $B^{**}_{\Phi_\rho}$, see [12] for details. These facts were also pointed out in [31]. It is therefore reasonable to identify the noncommutative counterpart of L^ψ with the space B_{Ψ_ρ}, while the noncommutative exponential Orlicz space should be identified with $B^{**}_{\Phi_\rho} = B^*_{\Psi_\rho}$. Nevertheless, we will work with the more tractable space B_{Φ_ρ}.

Let us denote $E_{\exp}(\mathcal{M},\rho) := B_{\Phi_\rho}$, with the norm $\|\cdot\|_{\exp,\rho} := \|\cdot\|_{\Phi_\rho}$ and $L_{\log}(\mathcal{M},\rho) := B_{\Psi_\rho}$, with the norm $\|\cdot\|_{\log,\rho} := \|\cdot\|_{\Psi_\rho}$. In the rest of this section, we will identify $\mathcal{M}_*$ with $L_1(\mathcal{M})$, so that $\mathcal{M}^s_*$ is identified with the space $L_1(\mathcal{M})^s$ of self-adjoint elements and $\mathcal{M}^+_*$ with the cone $L_1(\mathcal{M})^+$ of positive elements in $L_1(\mathcal{M})$.

Theorem 11 *(i) $L_{\log}(\mathcal{M},\rho) = \mathcal{P}_\rho - \mathcal{P}_\rho$ is dense in $L_1(\mathcal{M})^s$ and we have*

$$L_p(\mathcal{M},\rho)^s \sqsubseteq L_{\log}(\mathcal{M},\rho) \sqsubseteq L_1(\mathcal{M})^s, \qquad 1<p\le\infty.$$

(ii) $L_{\log}(\mathcal{M},\rho)^+ := L_{\log}(\mathcal{M},\rho)\cap L_1(\mathcal{M})^+ = \mathcal{P}_\rho$ is a closed convex cone in $L_{\log}(\mathcal{M},\rho)$.

(iii) Let $\psi\in L_{\log}(\mathcal{M},\rho)$. Then $\|\psi\|_{\log,\rho}\le 1$ if and only if there are some $\omega_\pm\in\mathcal{P}_\rho$ such that $\psi=\frac{1}{2}(\omega_+-\omega_-)$ and

$$F_\rho(\omega_+) + F_\rho(\omega_-) \le 2 - 2\rho(1).$$

Proof By Proposition 4 (iv), we see that $\mathrm{Dom}(\Psi_\rho) = \mathcal{P}_\rho - \mathcal{P}_\rho$ contains the self-adjoint part of $L_p(\mathcal{M}, \rho)$ for any $p > 1$, so $L_{\log}(\mathcal{M}, \rho)$ is dense in $L_1(\mathcal{M})^s$. This also shows that Ψ_ρ is finite valued on $L_p(\mathcal{M}, \rho)^s$. Since $L_p(\mathcal{M}, \rho)^s \sqsubseteq L_1(\mathcal{M})^s$ and Ψ_ρ is lower semicontinuous on $L_1(\mathcal{M})^s$, the restriction of Ψ_ρ defines a convex and lower semicontinuous Young function $L_p(\mathcal{M}, \rho)^s \to \mathbb{R}$, which then must be continuous by [29, Cor. 2.5]. Let B be the corresponding Banach space, then by construction, B is a closed subspace in $L_{\log}(\mathcal{M}, \rho)$ and using again [9, Prop. 2], we have

$$L_p(\mathcal{M}, \rho)^s \sqsubseteq B \sqsubseteq L_{\log}(\mathcal{M}, \rho).$$

Let now $\omega \in L_{\log}(\mathcal{M}, \rho)^+$, then there are some $\omega_\pm \in \mathcal{P}_\rho$ such that $2\omega = \omega_+ - \omega_-$. It follows that $2\omega + \omega_- = \omega_+ \in \mathcal{P}_\rho$. By Proposition 4 (iv), this implies that we must have $\omega \in \mathcal{P}_\rho$ as well. The fact that the cone is closed in $L_{\log}(\mathcal{M}, \rho)$ follows by the continuous embedding in $L_1(\mathcal{M})^s$.

Assume that $\|\psi\|_{\log,\rho} \leq 1$, equivalently, $\Psi_\rho(\psi) \leq 1$. Then for any $n \in \mathbb{N}$, there are some $\omega_{\pm,n} \in \mathcal{P}_\rho$ such that $\psi = \frac{1}{2}(\omega_{+,n} - \omega_{-,n})$ and $F_\rho(\omega_{+,n}) + F_\rho(\omega_{-,n}) \leq 2(1+1/n-\rho(1))$. It then follows that $\omega_{\pm,n} \in \mathcal{S}_C$ for some C and all n. By Proposition 4 (iii), there is some subsequence such that $\omega_{\pm,n_k} \to \omega_\pm$ in the $\sigma(\mathcal{M}_*, \mathcal{M})$-topology. It follows that $\psi = \frac{1}{2}(\omega_+ - \omega_-)$ and by lower semicontinuity,

$$F_\rho(\omega_+) + F_\rho(\omega_-) \leq \liminf F_\rho(\omega_{+,n_k}) + F_\rho(\omega_{-,n_k}) \leq 2 - 2\rho(1).$$

The converse is obvious.

□

Let us now recall the embedding $i_{\infty,\rho} : \mathcal{M}^s \to L_1(\mathcal{M})^s$, given by (1). Note that $i_{\infty,\rho}(\mathcal{M}^s) = L_\infty(\mathcal{M}, \rho)^s \sqsubseteq L_p(\mathcal{M}, \rho)^s$, for all $1 \leq p \leq \infty$.

Theorem 12 *For any $1 \leq p < \infty$, $i_{\infty,\rho}$ extends to a continuous embedding*

$$i_{\exp,\rho} : E_{\exp}(\mathcal{M}, \rho) \to L_p(\mathcal{M}, \rho)$$

and $i_{\exp,\rho}(E_{\exp}(\mathcal{M}, \rho))$ is dense in $L_p(\mathcal{M}, \rho)^s$.

Proof Let $a \in \mathcal{M}^s$, $1 \leq p < \infty$ and let $1/p + 1/q = 1$. By Theorem 11, we have $L_q(\mathcal{M}, \rho)^s \sqsubseteq L_{\log}(\mathcal{M}, \rho)$. It follows that for any $k \in L_q(\mathcal{M}, \rho)$, we have

$$\langle i_{\infty,\rho}(a), k\rangle = \mathrm{Tr}[ak] \leq \|a\|_{\exp,\rho}\|k\|_{\log,\rho}.$$

Since $\|k\|_{\log,\rho} \leq M\|k\|_{q,\rho}$ for some $M > 0$, this shows that $i_{\infty,\rho} : \mathcal{M}^s \to L_p(\mathcal{M}, \rho)^s$ is continuous with respect to the norm $\|\cdot\|_{\exp,\rho}$ in $\mathcal{M}^s$ and therefore has a unique continuous extension $i_{\exp,\rho}$. The rest follows from the fact that $i_{\infty,\rho}(\mathcal{M}^s) = L_\infty(\mathcal{M}, \rho)^s$ is dense in $L_p(\mathcal{M}, \rho)^s$ for any p.

□

To summarize, we have for $1 < p < \infty$:

$$L_\infty(\mathcal{M}, \rho) \sqsubseteq E_{\exp}(\mathcal{M}, \rho) \sqsubseteq L_p(\mathcal{M}, \rho) \sqsubseteq L_{\log}(\mathcal{M}, \rho) \sqsubseteq L_1(\mathcal{M}). \tag{7}$$

Note that we have analogous properties for the classical exponential Orlicz spaces, [2, Prop. 8].

Proposition 13 *Let $T : L_1(\mathcal{M}) \to L_1(\mathcal{N})$ be a channel. Then T restricts to a contraction $L_{\log}(\mathcal{M},\rho) \to L_{\log}(\mathcal{N},T(\rho))$ and T^* extends to a contraction $E_{\exp}(\mathcal{N},T(\rho)) \to E_{\exp}(\mathcal{M},\rho)$.*

Proof By Proposition 5, we have $F_{T(\rho)}(T(\omega)) \le F_\rho(\omega)$. Let $\psi \in L_{\log}(\mathcal{M},\rho)$, then

$$\begin{aligned}\Psi_{T(\rho)}(T(\psi)) &\le \frac{1}{2}\inf_{2\psi=\psi_+-\psi_-}(F_{T(\rho)}(T(\psi_+)) + F_{T(\rho)}(T(\psi_-))) + T(\rho)(1)\\ &\le \frac{1}{2}\inf_{2\psi=\psi_+-\psi_-}(F_\rho(\psi_+) + F_\rho(\psi_-)) + \rho(1) = \Psi_\rho(\psi).\end{aligned}$$

By definition of $L_{\log}(\mathcal{M},\rho)$ and duality, this implies the statement.

□

3 The quantum exponential manifold

Let $\tilde{\mathcal{F}}$ denote the set of all faithful functionals in the cone $\mathcal{M}_*^+$. In this section we will construct a Banach manifold structure on $\tilde{\mathcal{F}}$, using an extension of Theorem 6 to perturbations in $E_{\exp}(\mathcal{M},\rho)$.

3.1 Extended perturbations

Obviously, the restriction of F_ρ to $L_{\log}(\mathcal{M},\rho)$ is strictly convex and its effective domain is the positive cone $L_{\log}(\mathcal{M},\rho)^+ = \mathcal{P}_\rho$. Let us investigate this function and its conjugate with respect to the dual pair $(L_{\log}(\mathcal{M},\rho), E_{\exp}(\mathcal{M},\rho))$.

We first note $F_\rho|_{L_{\log}(\mathcal{M},\rho)}$ is weak*-lower semicontinuous. Indeed, since $\mathcal{M}^s$ is norm dense in B_ρ, the weak*-topology on $L_{\log}(\mathcal{M},\rho)$ coincides with the restriction of the $\sigma(\mathcal{M}_*^s, \mathcal{M}^s)$-topology on bounded subsets. By Proposition 4 (iii), the claim will follow by the next Lemma.

Lemma 14 *For each $C \in \mathbb{R}$, $\mathcal{S}_C$ is norm-bounded in $L_{\log}(\mathcal{M},\rho)$.*

Proof We may assume that $C \ge -\rho(1)$, otherwise $\mathcal{S}_C$ is empty. If $\omega \in \mathcal{S}_C$, then $\omega \in \mathcal{M}_*^+$ and we have (using the decomposition $2(\frac{1}{2}\omega) = \omega - 0$)

$$\Psi_\rho(\frac{1}{2}\omega) \le \frac{1}{2}F_\rho(\omega) + \rho(1) \le \frac{1}{2}C + \rho(1).$$

If $\Psi_\rho(\frac{1}{2}\omega) \le 1$, then $\|\omega\|_{\log,\rho} \le 2$, otherwise we have by [9, Lemma 3.3] that $\|\frac{1}{2}\omega\|_{\log,\rho} \le \Psi_\rho(\frac{1}{2}\omega) \le \frac{1}{2}C + \rho(1)$. Hence $\|\omega\|_{\log,\rho} \le \max\{2, C + 2\rho(1)\}$.

□

Let us now turn to the function C_ρ. It is easily seen that C_ρ is bounded over the unit ball with respect to $\|\cdot\|_{\exp,\rho}$ in $\mathcal{M}^s$. By [29, Cor. 2.4], this implies that C_ρ is continuous (in fact, locally Lipschitz) with respect to this norm. It follows that C_ρ extends uniquely to a continuous function $C_\rho : E_{\exp}(\mathcal{M},\rho) \to \mathbb{R}$.

Theorem 15 *For $h \in E_{\exp}(\mathcal{M}, \rho)$, we have*

$$C_\rho(h) = \sup_{\omega \in L_{\log}(\mathcal{M},\rho)} \omega(h) - F_\rho(\omega).$$

The supremum is attained at a unique functional $\rho^h \in \mathcal{P}_\rho$. Moreover, ρ^h is faithful, $C_\rho(h) = \rho^h(1)$ and the map $B_{\Phi_\rho} \ni h \mapsto \rho^h \in \mathcal{M}_*$ is norm-to-norm continuous.

Proof Let $a_n \in \mathcal{M}^s$ be a sequence such that $\|h - a_n\|_\rho \to 0$. By putting $a = 2a_n$ and $b = a_n$ in Lemma 7, we obtain the inequality

$$C_\rho(2a_n) - C_\rho(a_n) \geq \rho^{a_n}(a_n), \qquad \forall n.$$

By continuity of C_ρ, this implies that $\{\rho^{a_n}(a_n)\}_n$ is a bounded sequence, so that also

$$\{F_\rho(\rho^{a_n}) = \rho^{a_n}(a_n) - C_\rho(a_n)\}_n$$

is bounded and therefore $\rho^{a_n} \in \mathcal{S}_K$ for some K. By Proposition 4 (iii) we may assume (by restricting to a subsequence) that there is some $\sigma \in \mathcal{S}_K$ such that $\rho^{a_n} \to \sigma$ in the $\sigma(\mathcal{M}_*, \mathcal{M})$-topology. Since $\mathcal{S}_K$ is norm bounded in $L_{\log}(\mathcal{M}, \rho)$ (Lemma 14) and the weak*-topology coincides with the $\sigma(\mathcal{M}_*, \mathcal{M})$-topology on $\mathcal{S}_K$, it can be seen that $\rho^{a_n}(a_n) \to \sigma(h)$. For $\omega \in \mathcal{P}_\rho$, we get by definition of ρ^{a_n} and lower semicontinuity

$$\begin{aligned}\omega(h) - F_\rho(\omega) &= \lim_n(\omega(a_n) - F_\rho(\omega)) \leq \lim_n C_\rho(a_n) = \lim_n(\rho^{a_n}(a_n) - F_\rho(\rho^{a_n})) \\ &\leq \sigma(h) - F_\rho(\sigma).\end{aligned}$$

It follows that σ is a maximizer of $\omega(h) - F_\rho(\omega)$ and by strict convexity of F_ρ such maximizer is unique. Let us denote $\rho^h := \sigma$. Note that we have $C_\rho(h) = \lim_n C_\rho(a_n) = \lim_n \rho^{a_n}(1) = \rho^h(1)$ and the above computation also implies that

$$C_\rho(h) \leq \rho^h(h) - F_\rho(\rho^h) = \sup_{\omega \in \mathcal{P}_\rho} \omega(h) - F_\rho(\omega).$$

On the other hand, we obtain using (5) and lower semicontinuity of S:

$$S(\omega\|\rho) = \lim_n(\omega(a_n) + S(\omega\|\rho^{a_n})) \geq \omega(h) + S(\omega\|\rho^h). \tag{8}$$

Putting $\omega = \rho$, we see that $S(\rho\|\rho^h)$ is finite, so that ρ^h must be faithful. Further, putting $\omega = \rho^h$ it follows that $\rho^h(h) - F_\rho(\rho^h) = \rho^h(1) = C_\rho(h)$. We also get $\lim_n S(\sigma\|\rho^{a_n}) = 0$, so that $\rho^{a_n} \to \rho^h$ strongly in $\mathcal{M}_*$, this is easily extended to all sequences $h_n \to h$ in $E_{\exp}(\mathcal{M}, \rho)$.

□

We now extend the equalities in Theorem 6 to all elements in B_{Φ_ρ}.

Lemma 16 *Let $h \in E_{\exp}(\mathcal{M}, \rho)$.*

(i) For $a \in \mathcal{M}^s$, we have the chain rule

$$\rho^{h+a} = (\rho^h)^a, \qquad C_\rho(h + a) = C_{\rho^h}(a).$$

(ii) The norm $\|\cdot\|_{\exp,\rho^h}$ is continuous with respect to $\|\cdot\|_{\exp,\rho}$ on $\mathcal{M}^s$.

Proof Let $a_n \in \mathcal{M}^s$ be a sequence such that $\|h - a_n\|_{\exp,\rho} \to 0$. For $a \in \mathcal{M}^s$ we have $C_\rho(a_n + a) \to C_\rho(h + a)$ and $\rho^{a_n+a} \to \rho^{h+a}$ strongly, by Theorem 15. Since also $\rho^{a_n} \to \rho^h$ strongly, we have $(\rho^{a_n})^a \to (\rho^h)^a$ strongly, by [32, Thm. 1.1]. By the chain rule (6), we obtain

$$\rho^{h+a} = (\rho^h)^a, \qquad C_\rho(h + a) = \rho^{h+a}(1) = (\rho^h)^a(1) = C_{\rho^h}(a).$$

To prove (ii), note that (i) implies

$$\Phi_{\rho^h}(a) = \frac{1}{2}(C_\rho(h + a) - C_\rho(h) + C_\rho(h - a) - C_\rho(h)), \qquad a \in \mathcal{M}^s.$$

By continuity of C_ρ, this shows that there is some $\delta > 0$ such that $\Phi_{\rho^h}(a) < 1$ whenever $\|a\|_{\exp,\rho} < \delta$, this proves (ii).

□

Theorem 17 *Let $h \in E_{\exp}(\mathcal{M}, \rho)$. Then*

$$\omega(h) + S(\omega\|\rho^h) = S(\omega\|\rho), \qquad \omega \in \mathcal{P}_\rho. \tag{9}$$

Moreover, $E_{\exp}(\mathcal{M}, \rho) = E_{\exp}(\mathcal{M}, \rho^h)$ (equivalent norms) and we have the chain rule

$$\rho^{h+k} = (\rho^h)^k, \qquad C_\rho(h + k) = C_{\rho^h}(k), \qquad h, k \in E_{\exp}(\mathcal{M}, \rho). \tag{10}$$

Proof Let $a_n \in \mathcal{M}^s$, $\|a_n - h\|_{\exp,\rho} \to 0$. By Lemma 16 and Theorem 15, we obtain that also $h \in E_{\exp}(\mathcal{M}, \rho^h)$ and $\|a_n - h\|_{\exp,\rho^h} \to 0$. Moreover,

$$(\rho^h)^{-h} = \lim_n (\rho^h)^{-a_n} = \lim_n \rho^{h-a_n} = \rho^0 = \rho.$$

Replacing ρ by ρ^h and h by $-h$ in (8), we obtain

$$S(\omega\|\rho^h) \geq -\omega(h) + S(\omega\|\rho).$$

Together with (8), this implies (9). Similarly, using this replacement in Lemma 16 (ii), we obtain that $E_{\exp}(\mathcal{M}, \rho) = E_{\exp}(\mathcal{M}, \rho^h)$ with equivalent norms. The chain rule (10) is now proved from (9) exactly as in the proof of Theorem 6.

□

Corollary 18 *With respect to the dual pair $(L_{\log}(\mathcal{M}, \rho), E_{\exp}(\mathcal{M}, \rho))$, we have $C_\rho = F_\rho^*$ and $F_\rho = C_\rho^*$. Moreover, C_ρ is Gateaux differentiable on $E_{\exp}(\mathcal{M}, \rho)$, with the Gateaux derivative $C'_\rho(h) = \rho^h$, and $h \mapsto \rho^h$ defines an injective and norm-to-weak*-continuous map $E_{\exp}(\mathcal{M}, \rho) \to L_{\log}(\mathcal{M}, \rho)$.*

Proof The first part is clear from Theorem 15 and weak*-lower semicontinuity of F_ρ. Injectivity of the map $h \mapsto \rho^h$ follows by Theorem 17, for continuity, see e.g. [30].

□

3.2 Exponential families in $\mathcal{M}_*^+$

Let $E \subseteq E_{\exp}(\mathcal{M}, \rho)$ be a closed subspace. The set

$$\mathcal{E}_\rho(E) := \{\rho^h,\ h \in E\}$$

will be called an exponential family (at ρ). The set $\mathcal{E}_\rho := \mathcal{E}_\rho(E_{\exp}(\mathcal{M}, \rho))$ will be called the full exponential family (at ρ). For the following characterization of elements of $\mathcal{E}_\rho$, note that by (3)

$$\omega \mapsto S(\omega\|\rho) - S(\omega\|\sigma)$$

defines an affine map $h_{\sigma,\rho} : \mathcal{P}_\rho \to [-\infty, \infty)$ such that $h_{\sigma,\rho}(0) = 0$.

Corollary 19 *Let $\sigma \in \mathcal{M}_*^+$. Then $\sigma = \rho^h$ for some $h \in E_{\exp}(\mathcal{M}, \rho)$ if and only if there is some $C > -\rho(1)$ such that $h_{\sigma,\rho}$ is bounded and $\sigma(\mathcal{M}_*, \mathcal{M})$-continuous on the set $\mathcal{S}_C$. In this case h coincides with $h_{\sigma,\rho}$ on $\mathcal{P}_\rho$.*

Proof Assume that $\sigma = \rho^h$ for $h \in E_{\exp}(\mathcal{M}, \rho)$, then by Theorem 17, we see that $h(\omega) = h_{\sigma,\rho}(\omega)$ for $\omega \in \mathcal{P}_\rho$. Since the $\sigma(\mathcal{M}_*, \mathcal{M})$-topology coincides with the weak*-topology on $\mathcal{S}_C$, the assertion follows from $L_{\log}(\mathcal{M}, \rho) = E_{\exp}(\mathcal{M}, \rho)^*$.

Assume conversely that $h_{\sigma,\rho}$ has the stated properties on $\mathcal{S}_C$ for some $C > -\rho(1)$. Then the same is true for any $C' \in \mathbb{R}$, since for $C' > C$, there is some $t \in [0, 1]$ such that $F_\rho(t\omega + (1-t)\rho) \le tC' - (1-t)\rho(1) \le C$ for any $\omega \in \mathcal{S}_{C'}$.

Now note that by Theorem 11 (iii) and Proposition 4 (ii), the unit ball in $L_{\log}(\mathcal{M}, \rho)$ is a subset of $\mathcal{S}_C - \mathcal{S}_C$ for $C = 2\rho(1) - 1$, so that $h_{\sigma,\rho}$ extends to a bounded linear map on $L_{\log}(\mathcal{M}, \rho)$, moreover, since the weak*-topology coincides with the $\sigma(\mathcal{M}_*, \mathcal{M})$-topology on bounded subsets in $L_{\log}(\mathcal{M}, \rho)$, this extension is weak*-continuous and hence defines an element $h \in E_{\exp}(\mathcal{M}, \rho)$. For $\omega \in \mathcal{P}_\rho$, we get

$$\omega(h) - F_\rho(\omega) = -F_\sigma(\omega) \le \sigma(1) = \sigma(h) - F_\rho(\sigma),$$

so that $\sigma = \rho^h$.

□

We are now ready to introduce a Banach manifold structure on $\tilde{\mathcal{F}}$ using the parametrization $h \mapsto \rho^h$, similarly as in [9] for the set of faithful states. For $\rho \in \tilde{\mathcal{F}}$, let V_ρ be the open unit ball in $E_{\exp}(\mathcal{M}, \rho)$ and $s_\rho : V_\rho \to \tilde{\mathcal{F}}$ the map $h \mapsto \rho^h$. We construct a C^∞-atlas on $\tilde{\mathcal{F}}$ as

$$\{(U_\rho, e_\rho),\ \rho \in \tilde{\mathcal{F}}\}$$

where $U_\rho = s_\rho(V_\rho)$ and $e_\rho = s_\rho^{-1}|_{U_\rho}$. To show that this is indeed a C^∞-atlas, it is enough to notice that if $U_{\rho_1} \cap U_{\rho_2} \neq \emptyset$, then we must have $\rho_1 = \rho_2^k$ for some $k = E_{\exp}(\mathcal{M}, \rho_2)$, and

$$e_{\phi_1}(U_{\rho_1} \cap U_{\rho_2}) = \{h_1 \in E_{\exp}(\mathcal{M}, \rho_1),\ \|h_1\|_{\exp,\rho_1} < 1,\ \|h_1 + k\|_{\exp,\rho_2} < 1\}.$$

The proof is finished similarly as in [9], using the equivalence of the two norms $\|\cdot\|_{\exp,\rho_1}$ and $\|\cdot\|_{\exp,\rho_2}$. It is also clear that the connected components of $\tilde{\mathcal{F}}$ are exactly the full exponential families $\mathcal{E}_\rho$, $\rho \in \tilde{\mathcal{F}}$.

3.3 The canonical divergence

The function C_ρ can be used to introduce a canonical divergence in the connected component $\mathcal{E}_\rho$.

Theorem 20 *For $\rho \in \tilde{\mathcal{F}}$ and $h, k \in E_{\exp}(\mathcal{M}, \rho)$, put*

$$D(h\|k) = C_\rho(h) + F_\rho(\rho^k) - C'_\rho(h).$$

(i) $D : E_{\exp}(\mathcal{M}, \rho) \times E_{\exp}(\mathcal{M}, \rho) \to \mathbb{R}$ is jointly continuous, and it is strictly convex and Gateaux differentiable in the first variable.

(ii) We have

$$D(h\|k) = S(\rho^k\|\rho^h) - (\rho^k - \rho^h)(1).$$

(iii) For $h, k, l \in E_{\exp}(\mathcal{M}, \rho)$, D satisfies the generalized Pythagorean relation

$$D(h\|k) + D(k\|l) = D(h\|l) + (\rho^k - \rho^l)(k - h).$$

Proof By Corollary 19, we have $C'_\rho(h) = \rho^h$. To prove joint continuity, let h_n and k_n be two sequences such that $h_n \to h$, $k_n \to k$ in $E_{\exp}(\mathcal{M}, \rho)$. Since $h \mapsto \rho^h$ is norm to weak*-continuous, it follows that ρ^{k_n} is a norm-bounded sequence in $L_{\log}(\mathcal{M}, \rho)$, this implies that $\rho^{k_n}(h_n) \to \rho^k(h)$. By definition, we have $F_\rho(\rho^{k_n}) = \rho^{k_n}(k_n) - C_\rho(k_n)$, so the result just proved and continuity of C_ρ implies that $F_\rho(\rho^{k_n}) \to F_\rho(\rho^k)$. It follows that $D(\rho^{k_n}\|\rho^{h_n}) \to D(\rho^h\|\rho^k)$. The rest of (i) is straightforward from properties of C_ρ. The statement (ii) follows from Theorem 17. The Pythagorean relation (iii) is also follows from the definition and the fact that $C_\rho(h) = \rho^h(h) - F_\rho(\rho^h)$. □

3.4 Sufficient channels and invariance

Let $\mathcal{E}$ be a subset of nonzero elements in $\mathcal{M}_*^+$ and let $T : L_1(\mathcal{M}) \to L_1(\mathcal{N})$ be a channel. We say that T is sufficient for $\mathcal{E}$ if there is a channel $S : L_1(\mathcal{N}) \to L_1(\mathcal{M})$ such that

$$S \circ T(\sigma) = \sigma, \qquad \forall \sigma \in \mathcal{E}.$$

In this situation, S will be called a recovery channel for T on $\mathcal{E}$.

The notion of a sufficient channel was introduced by Petz [20, 33] in the situation when $\mathcal{E}$ is a set of states. Since the channels are trace preserving, the extension to positive functionals is straightforward.

Theorem 21 ([20, 33]) *Let T be a 2-positive channel and assume that there is some faithful element $\rho \in \mathcal{E}$ such that $\mathcal{E} \subseteq \mathcal{P}_\rho$. The following are equivalent.*

(i) T is sufficient for $\mathcal{E}$;

(ii) $S(T(\sigma)\|T(\rho)) = S(\sigma\|\rho)$;
(iii) $T_\rho \circ T(\sigma) = \sigma$, *for all* $\sigma \in \mathcal{E}$.

We will study the case when $\mathcal{E} = \mathcal{E}_\rho(E)$ is an exponential family at some $\rho \in \tilde{\mathcal{F}}$. Then the conditions of the above theorem are fulfilled.

Theorem 22 *Let* $\rho \in \tilde{\mathcal{F}}$, $h \in E_{\exp}(\mathcal{M}, \rho)$. *Let* $T : L_1(\mathcal{M}) \to L_1(\mathcal{N})$ *be a channel and let* T_ρ *be the Petz dual of* T *with respect to* ρ. *The following are equivalent.*

(i) T *is sufficient with respect to* $\{\rho, \rho^h\}$.
(ii) $T(\rho^h) = T(\rho)^{h_0}$ *for some* $h_0 \in E_{\exp}(\mathcal{N}, T(\rho))$ *and* $h = T^*(h_0)$
(iii) $T^* \circ T^*_\rho(h) = h$.

Proof Since T^*_ρ defines a map $E_{\exp}(\mathcal{M}, \rho) \to E_{\exp}(\mathcal{N}, T(\rho))$, we have for $\omega_0 \in \mathcal{P}_{T(\rho)}$,

$$\begin{aligned} \omega_0(T^*_\rho(h)) - F_{T(\rho)}(\omega_0) &= F_\rho(T_\rho(\omega_0)) - F_{\rho^h}(T_\rho(\omega_0)) - F_{T(\rho)}(\omega_0) \\ &\le -F_{\rho^h}(T_\rho(\omega_0)) \le \rho^h(1) = C_\rho(h). \end{aligned} \tag{11}$$

Here we have used Theorem 17, monotonicity of relative entropy together with the fact that $\rho = T_\rho \circ T(\rho)$, and Proposition 4 (ii). Assume (i) and put $\omega_0 = T(\rho^h)$ in (11). By Theorem 22 (ii) and (iii) we get

$$T(\rho^h)(T^*_\rho(h)) - F_{T(\rho)}(T(\rho^h)) = -F_{\rho^h}(\rho^h) = \rho^h(1) = C_\rho(h).$$

It follows that $T(\rho^h) = T(\rho)^{h_0}$, with $h_0 = T^*_\rho(h)$. By a similar computation, we obtain

$$\omega(T^*(h_0)) - F_\rho(\omega) \le C_{T(\rho)}(h_0), \qquad \forall \omega \in \mathcal{P}_\rho$$

and equality is attained for $\omega = \rho^h$. Hence $\rho^h = \rho^{T^*(h_0)}$, so that $h = T^*(h_0)$ by objectivity of the map $h \mapsto \rho^h$. Assume (ii), then we have

$$\begin{aligned} C_\rho(h) = \rho^h(1) = T(\rho^h)(1) = T(\rho)^{h_0}(1) = C_{T(\rho)}(h_0) = T(\rho^h)(h_0) - F_{T(\rho)}(T(\rho^h)) \\ = \rho^(h) - F_{T(\rho)}(T(\rho^h)) \ge \rho^h(h) - F_\rho(\rho^h) = C_\rho(h). \end{aligned}$$

This implies (i) by Theorem 22 (ii). We then obtain that $T(\rho^h) = T(\rho)^{T^*_\rho(h)}$, exactly as in the proof of (ii). This implies that $h_0 = T^*_\rho(h)$ and (iii) is proved. Finally, from (iii) and $T_\rho \circ T(\rho) = \rho$, we have

$$\begin{aligned} C_\rho(h) &\ge \sup_{\omega_0 \in \mathcal{P}_{T(\rho)}} T_\rho(\omega_0)(h) - F_\rho(T_\rho(\omega_0)) \ge \sup_{\omega_0} \omega_0(T^*_\rho(h)) - F_{T(\rho)}(\omega_0) \\ &\ge T(\rho^h)(T^*_\rho(h)) - F_{T(\rho)}(T(\rho^h)) \ge \rho^h(T^* \circ T\rho^*(h)) - F_\rho(\rho^h) = C_\rho(h). \end{aligned}$$

This implies $F_\rho(\rho^h) = F_{T(\rho)}(T(\rho^h))$, which implies (i) by Theorem 22.

□

Corollary 23 *Let* $\rho \in \tilde{\mathcal{F}}$ *and let* $\mathcal{E} = \{\rho^h,\ h \in E_0\}$ *for some subset* $E_0 \subseteq E_{\exp}(\mathcal{M}, \rho)$ *and let* $E \in E_{\exp}(\mathcal{M}, \rho)$ *be the closed linear span of* E_0. *Let* $T : L_1(\mathcal{M}) \to L_1(\mathcal{N})$ *be a channel sufficient with respect to* $\mathcal{E}$. *Then*

(i) T *is sufficient for the exponential family* $\mathcal{E}_\rho(E)$.
(ii) $T^*_\rho|_E$ *is an isometric isomorphism of* E *onto* $T^*_\rho(E)$ *and we have* $T(\mathcal{E}_\rho(E)) = \mathcal{E}_{T(\rho)}(T^*_\rho(E))$, *and* $T(\rho^h) = T(\rho)^{T^*_\rho(h)}$, *for* $h \in E$.

4 Conclusions

We have constructed an exponential manifold structure over the set $\tilde{\mathcal{F}}$ of faithful positive functionals on a von Neumann algebra, which in the commutative case coincides with a restriction of the Pistone-Sempi construction. The manifold is based on the Araki relative entropy and its conjugate C_ρ, playing the role of the moment generating function from the classical theory. We showed the relation of the obtained structures to Kosaki L_p spaces and proved an invariance property of the exponential manifold. Note that the function C_ρ was only proved to be Gateaux differentiable, so we do not get the full power of the Pistone-Sempi construction. Nevertheless, the manifold admits a canonical divergence satisfying a generalized Pythagorean relation.

Declarations

Data Availability: Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.
Conflict of interest: There is no conflict of interests.

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