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Environment induced decoherence for Markovian evolutions

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We study environmental decoherence for a quantum Markov semigroup $\mathcal{T}$ acting on an arbitrary von Neumann algebra $\mathcal{M}$. In particular, we analyze the relationships between the decomposition of the algebra induced by decoherence and a sort of "isometric-sweeping decomposition" for the space of states. Moreover, when the semigroup has a faithful normal invariant state, we embed the algebra $\mathcal{M}$ in its completion $\widehat{\mathcal{M}}$ with respect to the scalar product induced by the faithful state, and we compare the decomposition induced by decoherence with some other kind of isometric-sweeping decompositions of $\mathcal{M}$ and $\widehat{\mathcal{M}}$. © *2015 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4931479]

I. INTRODUCTION

Environmental decoherence is a very promising and discussed theory that have been proposed in the literature in order to give a mathematical description of some decoherence phenomena which are very popular in quantum theory (see, for instance, Refs. 1, 3, 26, 34, and 39): it aims to better understand the frontiers among classical and quantum theory, and, in general, it is seen as a phenomenon due to the interaction of a quantum system with its environment (see Refs. 19, 22, 24, and 40).

The first mathematical formulation of environmental decoherence has been suggested by Blanchard and Olkiewicz in Ref. 5, and their definition is the starting point of our study. We consider a Quantum Markov Semigroup $\mathcal{T}$ (QMS), i.e., a weakly* continuous semigroup of identity preserving, normal and completely positive maps $(\mathcal{T}_t)_{t\geq 0}$ on an arbitrary von Neumann algebra $\mathcal{M}$ acting on a complex Hilbert space h. The idea at the basis of environmental decoherence is then that $\mathcal{M}$ can be decomposed as direct sum of a subalgebra $\mathcal{M}_1$ undergoing unitary evolutions, and a remaining part $\mathcal{M}_2$ on which the semigroup vanishes on time. In particular, the algebra $\mathcal{M}_1$ may possess central elements, which are classical observables, and so this decomposition provides a mathematical description of appearance of classical properties in the system, but, in environmental decoherence, this is not necessarily requested. Maybe it is important to underline that the idea of environmental decoherence does not correspond to loss of coherences, but, more in general, it wants to capture other effects of the interaction between system and environment. In fact, the word "decoherence" is often used to indicate many different consequences of the openness of the system (see also Refs. 12, 23, 27, 28, 31, and 32): this feature is manifested by the loss of the $*$-automorphic property of the evolution maps and environmental decoherence is a strengthened form of this phenomenon because it demands that the non-automorphic part of the evolution vanishes in time. For finite-dimensional Hilbert spaces, there is already a quite exhaustive characterization of environmental decoherence (see Refs. 8 and 9), while, for the general case, the situation is more complicated and it is very difficult to realize when such a decomposition occurs, also because it is not clear whether the spaces $\mathcal{M}_1$ and $\mathcal{M}_2$ can be univocally determined.

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In this paper, in order to better understand the problem, since the splitting $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ is clearly related to the asymptotic properties of the semigroup, we are also interested in the comparison between this decomposition and other kinds of asymptotic splittings of QMSs on von Neumann algebras. More precisely, the proposal we want to develop here is considering environmental decoherence and its relations with different forms of "isometric-sweeping" decompositions. The term "isometric-sweeping" was used by Olkiewicz in different papers[4,25,29] and seems to be appropriate to describe decompositions in different contexts since it indicates the fact that the domain of the evolution is seen as the direct sum of two components such that the evolution on the first component is isometric, with respect to a suitable norm, while the evolution on the second component is vanishing in time, in a sense which can be made more precise in different ways, according to the context we are approaching.

The first isometric-sweeping decomposition we consider is in the context of the space of states (see Section III), proving that it is equivalent to the splitting induced by decoherence under some natural assumptions. In a second stage, we assume the existence of a faithful invariant state and we study two isometric-sweeping decompositions of the algebra: that one introduced in Ref. 25 when the semigroup commutes with the modular group associated with the faithful state, and the Jacobs-de Leeuw-Glicksberg (JDG) splitting. These two kinds of decompositions, together with the decoherence one, are very close to each other, and in several cases they agree (see Theorem 28 and Proposition 31). However, the splitting induced by environmental decoherence is more general in the sense that it may exist also in situations in which other decompositions fail to hold (for example, when the QMS does not have a faithful invariant state, see Proposition 2.5 in Ref. 10).

The organization of the paper is as follows. In Section II, we detail and discuss the definition of environment induced decoherence (EID), we recall its basic properties, and explore some features of key spaces which are naturally connected to the problem. Most of the results aim to give indications about how to identify the spaces $\mathcal{M}_1$ and $\mathcal{M}_2$ and to give some sufficient conditions for the unicity of the decomposition. An important ingredient is the example closing the section, which displays, in our opinion, many interesting attributes related to the study of EID. In Section III, we study how to translate EID to the predual level. We can determine an equivalent splitting of the states' space: the idea is always to have a form of isometric-sweeping decomposition, where we have to add a kind of automorphic property on the isometric part and we use weak topology for the "sweeping" component. Finally, in Section IV, we analyze Markovian evolutions with a faithful normal invariant state. We consider the Hilbert space $\widehat{\mathcal{M}}$ obtained by the completion of the algebra $\mathcal{M}$ induced by the scalar product associated with the invariant state. We embed the algebra $\mathcal{N}(\mathcal{T})$ (i.e., the biggest von Neumann algebra on which the evolution maps are $*$-morphisms) in the Hilbert space $\widehat{\mathcal{M}}$ and we study the parallel among EID decomposition and the natural decomposition of $\widehat{\mathcal{M}}$ seen as the sum of the closure of $\mathcal{N}(\mathcal{T})$ and its orthogonal space. In particular, we show that, when $\mathcal{T}$ commutes with the modular group, this splitting agrees with the unitary-completely non-unitary decomposition of $\widehat{\mathcal{M}}$ induced by the semigroup of contractions $\widehat{\mathcal{T}}$ which extends $\mathcal{T}$ on the completion. Moreover, in this case, the EID decomposition is exactly the generalization of the isometric-sweeping decomposition given in Refs. 4 and 25 for a single normal completely positive map possessing a subinvariant semifinite weight.

Finally, in Section IV B, we study the Jacobs-de Leeuw-Glicksberg splitting, which originally arose in the theory of weakly almost periodic semigroups (see, e.g., Ref. 14), and has recently been extended to weakly* almost periodic QMSs.[2,21] In this decomposition, the algebra splits into the direct sum of the so-called reversible and flights parts, i.e., the space generated by eigenvectors of the generator associated with the purely imaginary eigenvalues, and the vectors x for which 0 is a weak* cluster point for the trajectory $\{\mathcal{T}_t x\}_{t\geq 0}$. We can find some sufficient conditions to make this splitting coincide with the EID decomposition (see Proposition 31).

II. THE ENVIRONMENT INDUCES DECOHERENCE

In this section, we shall introduce the precise definition of environmental decoherence, consistently with our previous papers on the subject, and we shall study some basic properties of the spaces involved in the related decomposition of the algebra $\mathcal{M}$.

Let $\mathcal{M}$ be a von Neumann algebra acting on a complex Hilbert space h and consider $\mathcal{T}$ a QMS on $\mathcal{M}$. We say that there is EID on the system described by $\mathcal{T}$ if there exist a $\mathcal{T}_t$-invariant von Neumann subalgebra $\mathcal{M}_1$ of $\mathcal{M}$ and a $\mathcal{T}_t$-invariant and $*$-invariant weak* closed subspace $\mathcal{M}_2$ of $\mathcal{M}$ such that

(EID1) $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ with $\mathcal{M}_2 \neq \{0\}$,
(EID2) $\mathcal{M}_1$ is a maximal von Neumann subalgebra of $\mathcal{M}$ on which the restriction of every $\mathcal{T}_t$ is a $*$-automorphism,
(EID3) $w^* - \lim_{t\to\infty} \mathcal{T}_t(x) = 0$ for all $x \in \mathcal{M}_2$.

$\mathcal{M}_1$ is called the algebra of *effective observables* or *decoherence-free algebra*, while $\mathcal{M}_2$ is the space of *non-detectable observables*.

This definition, that we have already introduced in Ref. 8, is simply a slight elaboration of the one given by Blanchard and Olkiewicz in Ref. 5. For a more detailed discussion on how this definition descends from the original one, we refer to Section 2 in Ref. 8. Here, we simply recall some fundamental features. First, the two definitions (the one above and the one given in Ref. 5) are equivalent when $\mathcal{T}$ is a QMS with a faithful normal invariant state (see Theorem 3 and following comments in Ref. 8). We underline that, in Ref. 5, the existence of a faithful invariant semifinite weight is required for the quantum evolution, which is not necessarily Markovian; on the other hand, we are interested only in Markovian evolutions and drop any assumption about invariant weights for the definition. However, the existence of a faithful invariant state is always helpful in studying decoherence and it can be a central point in the discussion about this and similar properties. Another important element emerging in the EID definition given in Ref. 5 is the existence of a normal conditional expectation (i.e., a weak* continuous norm one projection) on the decoherence-free algebra. As one can see, this aspect does not appear in the definition above, but it can be a crucial support in the study of EID: this projection is considered and discussed in our previous papers[9,10] and its importance will be evident in Secs. III and IV.

Finally, it is important to highlight that, even if we have repeatedly mentioned[5] for a formal definition of environmental decoherence, the same authors, Blanchard and Olkiewicz, discussed analogous problems in many other occasions (see Refs. 4, 6, 25, 29, and 30) and we shall discuss the connections with some of these papers in Secs. III and IV.

Now, going back to the definition of EID given above, in general, it is very difficult to understand when such a decomposition occurs and then to determine the spaces $\mathcal{M}_1$ and $\mathcal{M}_2$. In particular cases, however, there are some natural characterizations of these two spaces.

Preserving the same notations as in some previous papers, we introduce here two sets, $\mathcal{N}(\mathcal{T})$ and $\mathcal{M}_0$ (defined in (2) and (1)) which clearly have some relations with the spaces $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, and we discuss some properties about these spaces. One of the main reasons why we consider the spaces $\mathcal{N}(\mathcal{T})$ and $\mathcal{M}_0$ is that they always exist for a QMS (even if EID does not hold!) and moreover, when EID holds, they contain the spaces $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively; really, they result to coincide with $\mathcal{M}_1$ and $\mathcal{M}_2$ in many relevant cases and we shall recall some of them below. So they are, in our opinion, a good starting point that we can always consider. Then, a difficult point is understanding whether EID occurs and to eventually determine the decoherence-free algebra and the decoherent observables. Here, another important question arises, which is still open and we are unfortunately unable to solve, in the general case, also in this article: is the EID decomposition unique? We have proved this only under some additional assumptions (see Refs. 9–11 and here, for instance, Proposition 5).

We start by introducing the space $\mathcal{M}_0$ of observables which are, in some sense, vanishing in time. More precisely, when there is decoherence, by definition, $\mathcal{M}_2$ is always contained in

$$\mathcal{M}_0 := \{\, x \in \mathcal{M} \,:\, w^* - \lim_{t\to\infty} \mathcal{T}_t(x) = 0 \,\}. \tag{1}$$

In particular, $\mathcal{M}_2$ and $\mathcal{M}_0$ coincide, for instance, when $\mathcal{M}$ is the algebra of linear operators on a finite dimensional Hilbert space h, and also for ergodic semigroups and other remarkable classes of evolutions (see Refs. 8–11). However, this result cannot be true in general, as we will show in

Example 9, even if it is clear that $\mathcal{M}_0$ is a good starting point to search for a possible candidate of $\mathcal{M}_2$.

Proposition 1. $\mathcal{M}_0$ is a norm-closed (and weak-closed) subspace of $\mathcal{M}$ which is $*$-invariant and $\mathcal{T}_t$-invariant for all $t \geq 0$.

Proof. Let $x \in \mathcal{M}_0$, then obviously $x^* \in \mathcal{M}_0$ since $T_t(x^*) = T_t(x)^*$ and, given $s \geq 0$, we have

$$w^* - \lim_t \mathcal{T}_t(\mathcal{T}_s(x)) = w^* - \lim_t \mathcal{T}_{t+s}(x) = 0,$$

so that $\mathcal{T}_s(x)$ belong to $\mathcal{M}_0$ too. This proves that $\mathcal{M}_0$ is invariant with respect to the involution $*$ and $\mathcal{T}_s$.

Finally, if $(x_n)_n$ is a sequence in $\mathcal{M}_0$ such that $\lim_n x_n = x$ in norm and $\varphi \in \mathcal{M}_*$, we have to prove that $\varphi(\mathcal{T}_t(x)) \to_t 0$. Assume $\|\varphi\|_1 = 1$ (if $\varphi = 0$ the thesis is trivial) and let $\epsilon > 0$. Then, there exists $n_0 > 0$ and $t_0 > 0$ such that

$$\|x_{n_0} - x\| < \epsilon/2, \qquad |\varphi(\mathcal{T}_t(x_{n_0}))| < \epsilon/2 \qquad \forall\, t > t_0.$$

Therefore, we have, for all $t > t_0$,

$$|\varphi(\mathcal{T}_t(x))| \leq |\varphi(\mathcal{T}_t(x - x_{n_0}))| + |\varphi(\mathcal{T}_t(x_{n_0}))| < \|x - x_{n_0}\| + |\varphi(\mathcal{T}_t(x_{n_0}))| < \epsilon,$$

so that $x \in \mathcal{M}_0$. This allows to conclude that $\mathcal{M}_0$ is norm-closed. Then, it will be also weak-closed since it is a subspace, and so it is convex. □

Remark 2. $\mathcal{M}_0$ is not necessarily w^*-closed in general. An example can be easily constructed taking $\mathcal{T}$ as the semigroup associated with a transient (classical) birth and death process. Then, $\mathcal{M}$ can be chosen as the set of bounded complex functions defined on the set $\mathbb{N}$ of natural numbers and $\mathcal{M}_0$ is w^*-dense in it since it contains, for instance, all the indicator functions of a single state $n \in \mathbb{N}$. But the constant 1 is not in $\mathcal{M}_0$ since the semigroup is Markov, while it is an element of its w^*-closure $\mathcal{M}$.

Moving now our attention to the algebra $\mathcal{M}_1$ in the decoherence decomposition, it is quite natural to consider the following set $\mathcal{N}(\mathcal{T})$,

$$\mathcal{N}(\mathcal{T}) := \{\, a \in \mathcal{M} \,:\, \mathcal{T}_t(a^*a) = \mathcal{T}_t(a^*)\mathcal{T}_t(a),\ \mathcal{T}_t(aa^*) = \mathcal{T}_t(a)\mathcal{T}_t(a^*)\ \ \forall\, t \geq 0 \,\} \tag{2}$$

which is the biggest von Neumann subalgebra of $\mathcal{M}$ on which the action of every $\mathcal{T}_t$ is a $*$-homomorphism. This space was already known by many authors in the 1970s and used for instance in order to study ergodic properties (see Refs. 13, 15, 17, 18, and 33). In particular, $\mathcal{N}(\mathcal{T})$ is $\mathcal{T}_t$-invariant and we have that

$$\mathcal{T}_t(x^*y) = \mathcal{T}_t(x^*)\mathcal{T}_t(y) \quad \forall\, t \geq 0, \qquad \text{if either } x \text{ or } y \text{ belongs to } \mathcal{N}(\mathcal{T}). \tag{3}$$

An almost complete proof of the previous result can be essentially obtained combining different pieces in Refs. 15, 18, and 33 and it is written in detail in Ref. 8. Therefore, $\mathcal{N}(\mathcal{T})$ always contains the decoherence-free space $\mathcal{M}_1$ and, in many cases, coincides with it, as the following proposition states.

Proposition 3. Assume that one of the following conditions holds:

1. *$\mathcal{T}$ possesses a faithful normal invariant state;*
2. *$\mathcal{T}$ is uniformly continuous on $\mathcal{B}(\mathsf{h})$;*
3. *$\mathcal{N}(\mathcal{T})$ is contained in $\mathcal{F}(\mathcal{T})$, the set of the fixed points of the semigroup.*

Then, $\mathcal{N}(\mathcal{T})$ is the biggest von Neumann subalgebra of $\mathcal{M}$ on which the restriction of every $\mathcal{T}_t$ is a $*$-automorphism. In particular, if EID holds, we have $\mathcal{M}_1 = \mathcal{N}(\mathcal{T})$.

Proof. We have seen that the restriction of each $\mathcal{T}_t$ to $\mathcal{N}(\mathcal{T})$ is a $*$-homomorphism. So we simply have to prove that, when EID takes place, these restrictions are also bijective.

The thesis follows by Section 3 in Ref. 33 under assumption 1 and is obvious under assumption 3.

When condition 2 holds, we can easily observe that, since the generator of the semigroup has to be a bounded operator (so everywhere defined) and $\mathcal{N}(\mathcal{T})$ is preserved by the semigroup, the maps $\mathcal{T}_t$ are invertible and also their restrictions to $\mathcal{N}(\mathcal{T})$, since we can define $(\mathcal{T}_{t|\mathcal{N}(\mathcal{T})})^{-1}$ as $\exp(-t\mathcal{L}_{|\mathcal{N}(\mathcal{T})})$, where $\mathcal{L}$ is the generator of $\mathcal{T}$. □

Remark 4. The case when $\mathcal{N}(\mathcal{T})$ consists only of fixed points (point 3 in previous proposition) is studied in Ref. 10. Under this condition, we showed that any QMS displaying decoherence is necessarily ergodic (in the sense that, for any operator x, there exists $w^* - \lim_{t\to\infty}\mathcal{T}_t(x)$) and $\mathcal{F}(\mathcal{T})$ has to be an algebra. More precisely, when EID holds, we obtain that $\mathcal{N}(\mathcal{T}) = \mathcal{F}(\mathcal{T})$ coincides with the biggest subalgebra on which $\mathcal{T}$ acts in a unitary way, the decomposition induced by decoherence is unique (with $\mathcal{M}_1 = \mathcal{F}(\mathcal{T})$ and $\mathcal{M}_2 = \mathcal{M}_0$), and there exists a normal conditional expectation on $\mathcal{M}_1$. In addition, if $\mathcal{F}(\mathcal{T})$ is an algebra, ergodicity and decoherence are equivalent.

In the cases of Proposition 3, the effective algebra $\mathcal{M}_1$ is then univocally determined and it coincides with $\mathcal{N}(\mathcal{T})$, while the space $\mathcal{M}_2$ in the decomposition $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ could not be unique.

However, if $\mathcal{M}$ acts on a finite-dimensional space, then $\mathcal{N}(\mathcal{T}) \cap \mathcal{M}_0 = \{0\}$, and so EID holds if and only if $\mathcal{M} = \mathcal{N}(\mathcal{T}) \oplus \mathcal{M}_0$ (see Ref. 9, Proposition 2). Unfortunately, when h is infinite-dimensional, the spaces $\mathcal{N}(\mathcal{T})$ and $\mathcal{M}_0$ can have a non-zero intersection even if EID holds (see Example 9). However, when this intersection is trivial, the space $\mathcal{M}_2$ is univocally determined, as the following proposition shows.

Proposition 5. Assume there exists the biggest von Neumann subalgebra $\mathcal{N}$ of $\mathcal{M}$ on which every $\mathcal{T}_t$ acts as a $*$-automorphism. If EID holds and $\mathcal{N} \cap \mathcal{M}_0 = \{0\}$, then the decomposition is uniquely determined: $\mathcal{M}_1 = \mathcal{N}$ and $\mathcal{M}_2 = \mathcal{M}_0$.

Proof. We have already noticed that, if there exists the biggest von Neumann subalgebra $\mathcal{N}$ of $\mathcal{M}$ on which every $\mathcal{T}_t$ acts as a $*$-automorphism, then $\mathcal{M}_1 = \mathcal{N}$.

Now suppose we have a decoherence decomposition $\mathcal{M} = \mathcal{N} \oplus \mathcal{M}_2$. We know that $\mathcal{M}_2 \subseteq \mathcal{M}_0$ and we want to prove that equality holds. Then, consider $x \in \mathcal{M}_0$, $x = x_1 + x_2$ with $x_1 \in \mathcal{N}$ and $x_2 \in \mathcal{M}_2$. We get $x_1 = x - x_2 \in \mathcal{N} \cap \mathcal{M}_0 = \{0\}$, so $x = x_2$ is in $\mathcal{M}_2$ and the proof of the second inclusion is complete. □

Assumptions of Proposition 5 are satisfied in the following case.

Proposition 6. Assume $\mathcal{T}$ is uniformly continuous on $\mathcal{M} = \mathcal{B}(\mathsf{h})$. Let

$$\mathcal{L}(x) = i[H,x] - \frac{1}{2}\sum_k \left(L_k^* L_k x - 2L_k^* x L_k + x L_k^* L_k\right) \qquad x \in \mathcal{B}(\mathsf{h})$$

be the generator of $\mathcal{T}$, with $H = H^* \in \mathcal{B}(\mathsf{h})$ and $(L_k)_k$ a family of bounded operators such that $\sum_k L_k^* L_k$ is strongly convergent. Then, the $*$-automorphic action of the semigroup on $\mathcal{N}(\mathcal{T})$ can be more precisely expressed by

$$\mathcal{T}_t(x) = e^{itH} x e^{-itH} \qquad \forall\, x \in \mathcal{N}(\mathcal{T}).$$

Moreover, if H has pure point spectrum, then $\mathcal{N}(\mathcal{T}) \cap \mathcal{M}_0 = \{0\}$.

Proof. The first statement follows by Corollary 2.1 in Ref. 17. Moreover, by Proposition 3, we have $\mathcal{M}_1 = \mathcal{N}(\mathcal{T})$.

Suppose by contradiction $x \in \mathcal{N}(\mathcal{T}) \cap \mathcal{M}_0$, $x \neq 0$ and let $(e_n)_{n\geq 0}$ be an onb of h given by eigenvectors of H, $He_n = \lambda_n e_n$ for all n. Then there exist $n, m \geq 0$ such that $\langle e_n, x e_m\rangle \neq 0$, so that

$$|\langle e_n, \mathcal{T}_t(x) e_m\rangle| = |\langle e^{-itH} e_n, x e^{-itH} e_m\rangle| = |e^{it(\lambda_n - \lambda_m)}| \cdot |\langle e_n, x e_m\rangle| = |\langle e_n, x e_m\rangle|$$

cannot converge to 0 as $t \to \infty$. This contradicts the assumption $x \in \mathcal{M}_0$. □

In Propositions 3 and 6, we have seen that, for uniformly continuous semigroups, the decoherence-free algebra always coincides with $\mathcal{N}(\mathcal{T})$ and the action of the semigroup can be made explicit through a Hamiltonian of the system. We would like to tell something more about the action of the semigroup on the algebra $\mathcal{M}_1$ in a more general case.

Theorem 7. *Let $\mathcal{N}$ be a von Neumann algebra acting on h and possessing in it a cyclic and separating vector. If $(\alpha_t)_{t\in\mathbb{R}}$ is a σ-weakly continuous one-parameter group of $*$-automorphism on $\mathcal{N}$, then the following facts hold.*

1. *There exists a strongly continuous one-parameter group $(U_t)_{t\in\mathbb{R}}$ of unitary operators on h such that*

$$\alpha_t(x) = U_t\, x U_t^* \qquad \forall\, t \in \mathbb{R},\ x \in \mathcal{N}; \tag{4}$$

moreover, $U_t = e^{itA}$ for some (not necessarily bounded) selfadjoint operator A on h.

2. *The unitary operators U_t may be chosen in $\mathcal{N}$ when A has a non-negative spectrum.*

Proof. 1. By Corollary 2.5.32 of Ref. 7, we have that there exists a unitary representation

$$Aut(\mathcal{N}) \ni \alpha \mapsto U(\alpha) \tag{5}$$

of the group $Aut(\mathcal{N})$ of all $*$-automorphisms of $\mathcal{N}$ on h such that $\alpha(x) = U(\alpha)xU(\alpha)^*$ for all $x \in \mathcal{N}$. Moreover, if α and β are $*$-automorphism on $\mathcal{N}$ such that $\omega \circ \alpha \to \omega \circ \beta$ in norm for all $\omega \in \mathcal{N}_*$, then $U(\alpha) \to U(\beta)$ in the strong operator topology. Therefore, for all $t \in \mathbb{R}$, there exists a unitary operator U_t on h satisfying Equation (4). We want to prove that $(U_t)_t$ is a strongly continuous one-parameter group.

First of all note that, if $t \to t_0$, the strongly continuity of the predual group $(\alpha_{*t})_t$ on $\mathcal{N}_*$ gives $\alpha_{*t}(\omega) \to \alpha_{*t_0}(\omega)$ in norm for all $\omega \in \mathcal{N}_*$, i.e.,

$$\|\omega \circ \alpha_t - \omega \circ \alpha_{t_0}\| \to_{t\to t_0} 0 \qquad \forall\, \omega \in \mathcal{N}_*.$$

As a consequence, $U_t \to U_{t_0}$ in the strong operator topology, i.e., $t \mapsto U_t$ is strongly continuous.

Further, since U is a representation of the automorphism group, we clearly have

$$\begin{aligned} U_{t+s} &= U(\alpha_{t+s}) = U(\alpha_t\alpha_s) = U(\alpha_t)U(\alpha_s) = U_tU_s \qquad \forall\, t, s \in \mathbb{R},\\ U_0 &= U(\alpha_0) = U(\mathbb{1}) = \mathbb{1}, \end{aligned}$$

i.e., the family $(U_t)_t$ satisfies the semigroup property. This proves that $(U_t)_t$ is a strongly continuous one-parameter group of unitary operators, and so, by the Stone's theorem, there exists a selfadjoint operator A on h such that $U_t = e^{itA}$ for all $t \in \mathbb{R}$.

2. It is a consequence of Borchers-Arveson theorem (see, e.g., Theorem 3.2.46 in Ref. 7), since the spectrum of a strongly continuous one-parameter group of unitary operators coincides with the spectrum of its generator (see page 246 in Ref. 7). □

Remark 8. Recall that, if there exist a normal faithful state ω on the algebra $\mathcal{N}$ and $(\mathcal{H}_\omega, \pi_\omega)$ is the GNS representation associated with ω, then $\mathcal{N}$ is isomorphic to the von Neumann algebra $\pi_\omega(\mathcal{N})$ having on $\mathcal{H}_\omega$ a cyclic and separating vector (see Proposition 2.5.6 of Ref. 7). Unfortunately, this is not sufficient to conclude the existence of a cyclic and separating vector in h for $\mathcal{N}$.

We conclude this section with an example that we regard as meaningful for many reasons. Maybe the main feature which distinguishes it is the fact that the spaces $\mathcal{N}(\mathcal{T})$ and $\mathcal{M}_0$ have a non-zero intersection. Nonetheless, we can detail many properties about it: EID holds and the spaces $\mathcal{M}_1$ and $\mathcal{M}_2$ are explicitly determined; moreover, $\mathcal{M}_1$ is the image of a normal conditional expectation, which is also specified, and the action of the semigroup on $\mathcal{M}_1$ can be expressed through the Hamiltonian, even if the semigroup is not uniformly continuous.

Example 9. Assume $\mathcal{M} = \mathcal{B}(\mathsf{h})$ with $\mathsf{h} = L^2(\mathbb{R}^d; \mathrm{d}s)$, $d \geq 1$. Let L be the orthogonal projection (in h) onto the space of symmetric functions with respect to the origin, i.e.,

$$(Lf)(s) = \frac{f(s) + f(-s)}{2}, \qquad f \in \mathsf{h},\ s \in \mathbb{R}^d,$$

and put $Hf = -\Delta f$ for all f in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$.

We consider the minimal semigroup $\mathcal{T}$ associated with the operators L and H (see Ref. 16) with domains h (L is bounded and self-adjoint) and $\mathcal{S}(\mathbb{R}^d)$, respectively, i.e., the semigroup with generator $\mathcal{L}$ defined by the generalized Lindblad form

$$\mathcal{L}(x) = i[H,x] - \frac{1}{2}\left(L^*Lx - 2L^*xL + xL^*L\right)$$

to be read in the usual way (see also Section 2 in Ref. 10 for a summary of the results allowing to construct the minimal dynamical semigroup and the related decoherence-free algebra).

Since $[H,L] = 0$, we have $e^{-itH}Le^{itH} = L$, and then $\mathcal{N}(\mathcal{T}) \subseteq \{L\}'$ by Theorem 3.2 in Ref. 13. Moreover, we can apply Theorem 4.1 in the same paper taking $D = C_c^\infty(\mathbb{R}^d)$, $\Phi = L^2$, and $C = (L+\mathbb{1})^2$, and so

$$\mathcal{N}(\mathcal{T}) = \{L\}', \qquad \mathcal{T}_t(x) = e^{-it\Delta}xe^{it\Delta} \quad \forall\, x \in \mathcal{N}(\mathcal{T}).$$

Here, the assumptions of Proposition 3 are not satisfied, but we can again prove that $\mathcal{N}(\mathcal{T})$ is the biggest von Neumann subalgebra of $\mathcal{B}(\mathsf{h})$ on which $\mathcal{T}$ acts as a $*$-automorphism (see Refs. 10 (Section 2) and 13).

We show that $\mathcal{N}(\mathcal{T})$ *is the image of a normal conditional expectation and it has a nontrivial intersection with* $\mathcal{M}_0$*. Moreover, EID holds with*

$$\mathcal{M}_1 = \mathcal{N}(\mathcal{T}) = \{LxL + L^\perp xL^\perp \;:\; x \in \mathcal{B}(\mathsf{h})\}, \tag{6}$$
$$\mathcal{M}_2 = \{LxL^\perp + L^\perp xL \;:\; x \in \mathcal{B}(\mathsf{h})\}. \tag{7}$$

Indeed, if we define

$$\mathcal{E}(x) = LxL + L^\perp xL^\perp, \qquad x \in \mathcal{B}(\mathsf{h}),$$

we obtain a w*-continuous and positive operator such that $\mathcal{E}^2 = \mathcal{E}$, and then $\mathcal{B}(\mathsf{h}) = \operatorname{Ker}\mathcal{E} \oplus \operatorname{Ran}\mathcal{E}$, with $\operatorname{Ker}\mathcal{E}$ and $\operatorname{Ran}\mathcal{E}$ weak*-closed subspaces of $\mathcal{B}(\mathsf{h})$. In particular, since every element $x \in \mathcal{B}(\mathsf{h})$ can be written as

$$x = (LxL + L^\perp xL^\perp) + (LxL^\perp + L^\perp xL) = \mathcal{E}(x) + (\mathbb{1} - \mathcal{E})(x) = \mathcal{E}(x) + \mathcal{E}^\perp(x)$$

and $\mathcal{E}(x)$ commutes with L, we have that $\operatorname{Ran}\mathcal{E} = \{LxL + L^\perp xL^\perp \;:\; x \in \mathcal{B}(\mathsf{h})\} = \mathcal{N}(\mathcal{T})$ and $\operatorname{Ker}\mathcal{E} = \{LxL^\perp + L^\perp xL \;:\; x \in \mathcal{B}(\mathsf{h})\}$.

Since $\operatorname{Ker}\mathcal{E}$ is clearly $*$-invariant, to conclude we have to show that it is also $\mathcal{T}_t$-invariant, and $w^* - \lim_t \mathcal{T}_t(y) = 0$ for all $y \in \operatorname{Ker}\mathcal{E}$. To this end, define

$$\mathcal{L}_0(x) = -\frac{1}{2}L^2x + LxL - \frac{1}{2}xL^2, \qquad x \in \mathcal{B}(\mathsf{h}),$$

so that $\mathcal{L}(x) = \mathcal{L}_0(x) + i[H,x]$ for all $x \in \mathrm{D}(\mathcal{L})$. Since

$$\mathcal{L}_0(\mathcal{E}(x)) = 0, \qquad \mathcal{L}_0(\mathcal{E}^\perp(x)) = -\frac{1}{2}(\mathcal{E}^\perp(x)),$$

and $[H,L] = 0$, for all $x \in \mathcal{B}(\mathsf{h})$ we have

$$\mathcal{T}_t(x) = e^{-it\Delta}\mathcal{E}(x)e^{it\Delta} + e^{-t/2}e^{-it\Delta}(\mathcal{E}^\perp(x))e^{it\Delta}.$$

So, given $y = \mathcal{E}^\perp(x) \in \operatorname{Ker}\mathcal{E}$, we get

$$\mathcal{T}_t(y) = e^{-t/2}e^{-it\Delta}ye^{it\Delta} = e^{-t/2}\mathcal{E}^\perp(e^{-it\Delta}xe^{it\Delta}) \in \operatorname{Ker}\mathcal{E}$$

and

$$\|\mathcal{T}_t(y)\| \le e^{-t/2}\|e^{-it\Delta}xe^{it\Delta}\| \le e^{-t/2}\|x\| \to_t 0,$$

so that the claim is proved, i.e., EID holds with $\mathcal{M}_1, \mathcal{M}_2$ given by (6) and (7).

Finally, we want to show there exists a non-zero $x \in \mathcal{N}(\mathcal{T})$ satisfying $\mathrm{w}^* - \lim_t \mathcal{T}_t(x) = 0$, i.e., $\mathcal{N}(\mathcal{T}) \cap \mathcal{M}_0 \neq \{0\}$.

Let $u \in L^2(\mathbb{R}^d; \mathrm{d}s) \cap L^1(\mathbb{R}^d; \mathrm{d}s)$ be a symmetric function. Then, $|u\rangle\langle u|$ clearly belongs to $\mathcal{N}(\mathcal{T})$, so that $\mathcal{T}_t(|u\rangle\langle u|) = |e^{-it\Delta}u\rangle\langle e^{-it\Delta}u|$, with

$$(e^{-it\Delta}u)(s) = \frac{1}{(4\pi it)^{d/2}}\int_{\mathbb{R}^d} u(r)e^{i|s-r|^2/4t}\mathrm{d}r \qquad \forall s \in \mathbb{R}^d.$$

We prove that $\langle v, \mathcal{T}_t(|u\rangle\langle u|)w\rangle \to_t 0$ for all $v, w \in \mathsf{h}$, which implies the weak*-convergence of the net $\{\mathcal{T}_t(|u\rangle\langle u|)\}_t$. Since

$$\langle v, \mathcal{T}_t(|u\rangle\langle u|)w\rangle = \langle v, e^{-it\Delta}u\rangle\langle e^{-it\Delta}u, w\rangle,$$

it is enough to show that

$$\lim_t \langle v, e^{-it\Delta}u\rangle = 0 \qquad \forall\, v \in \mathsf{h}. \tag{8}$$

If we take $v \in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{supp} v \subseteq S(0,R)$ (where $S(0,R)$ denotes the d-dimensional ball of radius R), we have

$$\begin{aligned}
|\langle v, e^{-it\Delta}u\rangle| &= \frac{1}{(4\pi t)^{d/2}}\left|\int_{\mathbb{R}^d}\overline{v(s)}\left(\int_{\mathbb{R}^d}u(r)e^{i|s-r|^2/4t}\mathrm{d}r\right)\mathrm{d}s\right| \\
&\le \frac{1}{(4\pi t)^{d/2}}\int_{\mathbb{R}^d}|v(s)|\left(\int_{\mathbb{R}^d}|u(r)|\mathrm{d}r\right)\mathrm{d}s \\
&\le \frac{1}{(4\pi t)^{d/2}}\,\|u\|_1\,\|v\|_\infty\,|S(0,R)|_d.
\end{aligned} \tag{9}$$

Hence, given $v \in \mathsf{h}$ and $\epsilon > 0$, let $v_\epsilon \in C_c^\infty(\mathbb{R}^d)$ such that $\|v - v_\epsilon\|_2 < \epsilon$. If we assume $\operatorname{supp} v_\epsilon \subseteq S(0,R_\epsilon)$, inequality (9) gives

$$\begin{aligned}
|\langle v, e^{-it\Delta}u\rangle| &\le |\langle v - v_\epsilon, e^{-it\Delta}u\rangle| + |\langle v_\epsilon, e^{-it\Delta}u\rangle| \\
&\le \epsilon\,\|u\|_2 + \frac{1}{(4\pi t)^{d/2}}\,\|u\|_1\,\|v_\epsilon\|_\infty\,|S(0,R_\epsilon)|_d,
\end{aligned}$$

and so thesis (8) follows since $(4\pi t)^{-d/2}$ goes to 0 as $t \to \infty$.

Thus, we found a non-zero $|u\rangle\langle u| \in \mathcal{N}(\mathcal{T}) \cap \mathcal{M}_0$.

III. SPLITTING OF $\mathcal{M}_*$

We are interested in studying a version of EID decomposition on the predual space $\mathcal{M}_*$ of $\mathcal{M}$. More precisely, we wonder about a suitable predual form of the definition of environmental decoherence and we study a decomposition of $\mathcal{M}_*$ into the direct sum of a part on which every predual map $\mathcal{T}_{*t}$ acts as a bijective isometry, and a remaining part vanishing in time (with respect to the weak topology on $\mathcal{M}_*$). We will see when this decomposition is equivalent with EID.

It will be immediately clear that, if decoherence takes place, the existence of a normal conditional expectation onto $\mathcal{M}_1$ allows to easily "transport" the EID decomposition to the predual level. Moreover, in this context, the use of orthogonal subspaces arises in a natural way. In the following, we will adopt the usual notations for orthogonals: given a subset A of $\mathcal{M}$, we denote by ${}^\perp A$ the space of normal functionals whose action on A is zero, i.e.,

$${}^\perp A := \{\sigma \in \mathcal{M}_* \,:\, \sigma(x) = 0\ \forall\, x \in A\};$$

similarly, for $B \subset \mathcal{M}_*$,

$$B^\perp := \{x \in \mathcal{M} \,:\, \sigma(x) = 0\ \forall\, \sigma \in B\}.$$

Theorem 10. *Suppose that EID displays with decomposition $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ and there exists a normal conditional expectation $\mathcal{E}$ such that $\mathcal{M}_1 = \mathrm{Ran}\mathcal{E}$, $\mathcal{M}_2 = \mathrm{Ker}\mathcal{E}$. Then, we have a decomposition $\mathcal{M}_* = \mathcal{M}_{*1} \oplus \mathcal{M}_{*2}$ such that*

1. *$\mathcal{E}_*$ (defined as $\mathcal{E}_*(\sigma) = \sigma \circ \mathcal{E}$) is a positive norm one $w - w$ continuous projection onto $\mathcal{M}_{*1}$ commuting with every $\mathcal{T}_{*t}$, and*
$$\mathcal{M}_{*1} = \mathrm{Ran}\mathcal{E}_* = {}^\perp\mathcal{M}_2 \simeq (\mathcal{M}_1)_*, \qquad \mathcal{M}_{*2} = \mathrm{Ker}\mathcal{E}_* = {}^\perp\mathcal{M}_1 \simeq (\mathcal{M}_2)_*,$$
where $\simeq$ means "$*$-isomorphic";
2. *$\mathcal{M}_{*1}$ is a Banach space, $\mathcal{M}_{*2}$ is a w-closed space, both are $\mathcal{T}_{*t}$- invariant and $*$-invariant (where $\omega^*(x) := \overline{\omega(x^*)}$, for ω in $\mathcal{M}_*$);*

3. *$\mathcal{T}_{*t}$ is a bijective isometry on $\mathcal{M}_{*1}$ for any $t \geq 0$ and $\mathcal{M}_1$ is maximal in the set of $\mathcal{T}_t$-invariant von Neumann subalgebras $\widetilde{\mathcal{M}}$ verifying*

$$(\mathcal{T}_{*t}(\sigma))(x^*x) = \sigma(\mathcal{T}_t(x)^*\mathcal{T}_t(x)) \qquad \forall\, \sigma \in \widetilde{\mathcal{M}}_*, \quad x \in \widetilde{\mathcal{M}}; \tag{10}$$

4. *$w - \lim_t \mathcal{T}_{*t}(\sigma) = 0$ for all σ in $\mathcal{M}_{*2}$.*

Proof. 1. We consider the predual map $\mathcal{E}_* : \mathcal{M}_* \to \mathcal{M}_*$, given by $\mathcal{E}_*(\sigma) = \sigma \circ \mathcal{E}$. $\mathcal{E}_*$ is clearly a $w - w$ continuous, norm one (since $\|\mathcal{E}_*\| = \|\mathcal{E}\| = 1$) projection such that

$$\mathcal{M}_{*1} := \operatorname{Ran}\mathcal{E}_* = \{\sigma \in \mathcal{M}_* \,:\, \sigma(x) = 0\,\forall\, x \in \mathcal{M}_2\} = {}^{\perp}\mathcal{M}_2,$$

$$\mathcal{M}_{*2} := \operatorname{Ker}\mathcal{E}_* = \{\sigma \in \mathcal{M}_* \,:\, \sigma(x) = 0\,\forall\, x \in \mathcal{M}_1\} = {}^{\perp}\mathcal{M}_1.$$

In particular, $\mathcal{M}_{*2}$ is a weak closed subspace of $\mathcal{M}_*$. $\mathcal{E}_*$ commutes with $\mathcal{T}_{*t}$ since $\mathcal{E}$ and $\mathcal{T}_t$ commute for all $t \geq 0$.

Also, it is immediate that $\mathcal{M}_* = \mathcal{M}_{*1} \oplus \mathcal{M}_{*2}$ since $\mathcal{E}$ is normal and each σ in $\mathcal{M}_*$ can be uniquely written as $\sigma = \sigma_1 + \sigma_2$ with $\sigma_1 = \sigma \circ \mathcal{E} \in \mathcal{M}_{*1}$ and $\sigma_2 = \sigma \circ \mathcal{E}^{\perp} \in \mathcal{M}_{*2}$, where $\mathcal{E}^{\perp}$ denote the projection $I - \mathcal{E}$, being I the identity operator on $\mathcal{M}$. Moreover, $\mathcal{M}_{*1}$ ($\mathcal{M}_{*2}$, respectively) is $*$-isomorphic with the predual $\mathcal{M}_{1*}$ ($\mathcal{M}_{2*}$, respectively) of $\mathcal{M}_1$ through the correspondence

$$\mathcal{M}_{1*} \ni \sigma \mapsto \sigma \circ \mathcal{E} \in \mathcal{M}_{*1} \qquad (\mathcal{M}_{2*} \ni \sigma \mapsto \sigma \circ \mathcal{E}^{\perp} \in \mathcal{M}_{*2}, \text{respectively});$$

hence, $\mathcal{M}_{*1}$ is a Banach space, being $\mathcal{M}_1$ a von Neumann algebra (see Ref. 7, Proposition 2.4.18).

Moreover, every $\mathcal{T}_{*t}$ preserves $\mathcal{M}_{*k}$, $k = 1, 2$, since $\mathcal{M}_k$ is $\mathcal{T}_t$-invariant and $\mathcal{T}_{*t}(\sigma) = \sigma \circ \mathcal{T}_t$ for all $\sigma \in \mathcal{M}_*$. Therefore, item 1 follows.

3. First notice that, for σ in $\mathcal{M}_{*1}$, since $\sigma = \mathcal{E}_*(\sigma) = \sigma \circ \mathcal{E}$ and $\operatorname{Ran}\mathcal{E} = \mathcal{M}_1$, we have

$$\|\sigma\| = \sup_{\|x\|\leq 1} \|\sigma(x)\| = \sup_{\|x\|\leq 1} \|\sigma(\mathcal{E}(x))\| \leq \sup_{x\in\mathcal{M}_1, \|x\|\leq 1} \|\sigma(x)\| \leq \|\sigma\|.$$

So all the quantities above coincide and, in particular, the norm of σ is realized on $\mathcal{M}_1$ in the sense that $\|\sigma\| = \sup_{x\in\mathcal{M}_1,\|x\|\leq 1} \|\sigma(x)\|$. Then, since $\mathcal{T}_{*t}(\sigma)$ is in $\mathcal{M}_{*1}$,

$$\begin{aligned}\|\mathcal{T}_{*t}(\sigma)\| &= \sup_{x\in\mathcal{M}_1,\|x\|\leq 1} \|\sigma(\mathcal{T}_t(x))\| \\ &\quad \text{and, since } \mathcal{T}_t \text{ is a bijective isometry on } \mathcal{M}_1, \\ &= \sup_{x\in\mathcal{M}_1,\|x\|\leq 1} \|\sigma(x)\| = \|\sigma\|.\end{aligned}$$

This shows that $\mathcal{T}_{*t}$ is an isometry on $\mathcal{M}_{*1}$. Moreover, $\mathcal{T}_{*t}$ is surjective on $\mathcal{M}_{*1}$ since, for all σ in $\mathcal{M}_{*1}$, $\sigma = \mathcal{T}_{*t}(\sigma \circ \mathcal{T}^{-1}_{t|\mathcal{M}_1})$ and $\sigma \circ \mathcal{T}^{-1}_{t|\mathcal{M}_1} \in \mathcal{M}_{*1}$. Finally, relation (10) is surely verified by definition of EID with $\widetilde{\mathcal{M}} = \mathcal{M}_1$. We still have to prove the maximality of $\mathcal{M}_1$. So suppose $\widetilde{\mathcal{M}}$ is a $\mathcal{T}_t$-invariant von Neumann subalgebra verifying (10). Then, by the separating property of the predual, we deduce that $\mathcal{T}_t(x^*x) = \mathcal{T}_t(x)^*\mathcal{T}_t(x)$ for all x in $\widetilde{M}$ (otherwise there would exist a $\sigma \in \widetilde{\mathcal{M}}_*$ such that $\sigma(\mathcal{T}_t(x^*x)) = (\mathcal{T}_{*t}(\sigma))(x^*x) \neq \sigma(\mathcal{T}_t(x)^*\mathcal{T}_t(x))$); so $\widetilde{M}$ is in the set of $\mathcal{T}_t$-invariant von Neumann algebras on which $\mathcal{T}$ acts as a $*$-automorphism and $\mathcal{M}_1$ is maximal in this set by (EID2).

4. For $\sigma \in \mathcal{M}_{*2}$ and $x \in \mathcal{M}$, we have

$$\mathcal{T}_{*t}(\sigma)(x) = \sigma(\mathcal{T}_t(x)) = \sigma(\mathcal{T}_t(\mathcal{E}^{\perp}(x))) \to_t 0,$$

since $\mathcal{E}^{\perp}(x) \in \mathcal{M}_2$ and EID holds. Therefore, $w - \lim \mathcal{T}_{*t}(\sigma) = 0$. □

Remark 11. If ρ is an invariant state for $\mathcal{T}$, then $\rho \in {}^{\perp}\mathcal{M}_0$. Indeed, for $x \in \mathcal{M}_0$,

$$\rho(x) = \rho(\mathcal{T}_t x) \to 0.$$

In particular, if EID holds, then $\rho \in {}^{\perp}\mathcal{M}_2 \simeq \mathcal{M}_{*1}$.

Inspired by some papers authored by Blanchard, Lugiewicz, and Olkiewicz (see, in particular, Theorem 5 in Ref. 25 and Theorem 8 in Ref. 29, Theorem 9 in Ref. 6), we introduce the following:

Definition 12. We call isometric-sweeping decomposition of $\mathcal{M}_*$ every splitting $\mathcal{M}_* = \mathcal{M}_{*1} \oplus \mathcal{M}_{*2}$ such that

1. *$\mathcal{M}_{*1}$ is a Banach space, $\mathcal{M}_{*2}$ is a w-closed space, both are $\mathcal{T}_{*t}$- invariant and $*$-invariant;*
2. *$\mathcal{T}_{*t}$ is a bijective isometry on $\mathcal{M}_{*1}$ for any $t \geq 0$ and $w - \lim_t \mathcal{T}_{*t}(\sigma) = 0$ for all σ in $\mathcal{M}_{*2}$.*

When $\mathcal{M} = \mathcal{B}(\mathsf{h})$, Theorem 7 helps writing the action of the semigroup and of its predual in a more explicit way. Indeed, as a direct byproduct, identifying any state on $\mathcal{B}(\mathsf{h})$ with its density, we obtain

Corollary 13. Assume the same hypotheses as in Theorem 10 and consider $\mathcal{M} = \mathcal{B}(\mathsf{h})$. Assume there exists a strongly continuous one-parameter group $(U_t)_{t\in\mathbb{R}}$ of unitary operators on h such that

$$\mathcal{T}_t(x) = U_t x U_t^*, \qquad \forall\, x \in \mathcal{M}_1,\ t \geq 0. \tag{11}$$

1. *We have $\mathcal{T}_{*t}(\rho) = \mathcal{E}_*(U_t^* \rho U_t)$ for all $\rho \in \mathcal{M}_{*1}$. Moreover, the equality $\mathcal{T}_{*t}(\rho) = U_t^* \rho U_t$ for all $\rho \in \mathcal{M}_{*1}$ holds if and only if $U_t \mathcal{M}_2 U_t^* \subseteq \mathcal{M}_2$.*
2. *Denote by S the QMS on $\mathcal{M}_1$ obtained by restriction of $\mathcal{T}$ to this von Neumann subalgebra, and let σ be a state on $\mathcal{M}_1$. If ρ is a density of σ (i.e., $\sigma(x) = \mathrm{tr}(\rho x)$ for all $x \in \mathcal{M}_1$), then $U_t^* \rho U_t$ is a density for $S_{*t}(\sigma)$, $t \geq 0$.*

Proof. 1. Given $\rho \in \mathcal{M}_{*1} = {}^{\perp}\mathcal{M}_2$ and x in $\mathcal{M}$, $x = x_1 + x_2$ as usual, we have

$$\mathrm{tr}((\mathcal{T}_{*t}(\rho))x) = \mathrm{tr}(\rho \mathcal{T}_t(x_1)) = \mathrm{tr}(\rho U_t x_1 U_t^*)) = \mathrm{tr}(U_t^* \rho U_t x_1) = \mathrm{tr}(\mathcal{E}_*(U_t^* \rho U_t) x),$$

being $x_1 = \mathcal{E}(x)$. Therefore, the equality $\mathcal{T}_{*t}(\rho) = \mathcal{E}_*(U_t^* \rho U_t)$ follows.

Moreover, we have $\mathcal{T}_{*t}(\rho) = U_t^* \rho U_t$ for all $\rho \in \mathcal{M}_1$ if and only if $\mathrm{tr}(U_t^* \rho U_t \mathcal{E}^{\perp}(x)) = 0$ for all $x \in \mathcal{M}$, and this happens if and only if $U_t \mathcal{M}_2 U_t^* \subseteq \mathcal{M}_{*1}^{\perp} = \mathcal{M}_2$, being $\mathrm{Ran}\, \mathcal{E}^{\perp} = \mathrm{Ker}\, \mathcal{E} = \mathcal{M}_2$.

2. Given $x \in \mathcal{M}_1$, we clearly have

$$(S_{*t}(\sigma))(x) = \sigma(\mathcal{T}_t(x)) = \mathrm{tr}(\rho \mathcal{T}_t(x)) = \mathrm{tr}(U_t^* \rho U_t x)$$

so that $U_t^* \rho U_t$ is a density for $S_{*t}(\sigma)$. □

Remarks 14. 1. Note that, when EID takes place, sufficient conditions to have a strongly continuous one-parameter group of unitary operators satisfying Equation (11) are $\mathcal{T}$ uniformly continuous or h with a cyclic and separating vector for $\mathcal{M}_1$. Indeed, in the first case, it follows from Proposition 6, while in the second one it is enough to apply Theorem 7 taking $\mathcal{N} = \mathcal{M}_1$ and $\alpha_t = \mathcal{T}_{t|\mathcal{M}_1}$ with the proper extension for negative times t.

2. Recall that, by the general theory of von Neumann algebras, the predual $(\mathcal{M}_1)_*$ of $\mathcal{M}_1$ can be identified with the Banach space $L^1(\mathsf{h})/{}^{\perp}\mathcal{M}_1$ through the map $\sigma \mapsto \rho + {}^{\perp}\mathcal{M}_1$, where ρ is a trace class operator such that $\sigma(x) = \mathrm{tr}(\rho x)$ for all $x \in \mathcal{M}_1$.

Therefore, item 2 of the previous corollary tells us that

$$S_{*t}(\sigma) = U_t^* \rho U_t + {}^{\perp}\mathcal{M}_1 \qquad \forall\, t \geq 0, \tag{12}$$

i.e., states on $\mathcal{M}_1$ evolve in a unitary way, but this is not true in general for their representing densities.

Note that, in particular, $\mathcal{E}_*(U_t^* \rho U_t)$ belongs to $U_t^* \rho U_t + {}^{\perp}\mathcal{M}_1$.

Also, the converse of Theorem 10 states as shown:

Theorem 15. *Assume there exist an isometric-sweeping decomposition $\mathcal{M}_* = \mathcal{M}_{*1} \oplus \mathcal{M}_{*2}$ and a contractive $w - w$ continuous positive projection $\mathcal{E}_*$ onto $\mathcal{M}_{*1}$ which transforms states into states and such that $\mathcal{M}_{*2} = Ker \mathcal{E}_*$. Then there is a decomposition $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ such that*

1. *$\mathcal{M}_1$ is a von Neumann algebra, $\mathcal{M}_2$ is a weak* closed subspace, both are $\mathcal{T}_t$-invariant and $*$-invariant;*
2. *the restriction of every $\mathcal{T}_t$ to $\mathcal{M}_1$ is a surjective isometry;*
3. *$w^* - \lim_t \mathcal{T}_t(x) = 0$ for all $x \in \mathcal{M}_2$;*
4. *$\mathcal{M}_1 = \mathcal{M}_{*2}^{\perp} \simeq (\mathcal{M}_{*1})^*$ and $\mathcal{M}_2 = \mathcal{M}_{*1}^{\perp} \simeq (\mathcal{M}_{*1})^*$.*

Moreover, $\mathcal{E} = (\mathcal{E}_*)^*$ is a normal conditional expectation onto $\mathcal{M}_1$ such that $Ker\mathcal{E} = \mathcal{M}_2$.
Finally, if $\mathcal{M}_1$ is a maximal $\mathcal{T}_t$-invariant von Neumann algebra such that relation (10) holds, then every $\mathcal{T}_t$ acts as a $*$-automorphism on $\mathcal{M}_1$, and EID holds.

Proof. $\mathcal{E}$ is normal as adjoint of a $w-w$ continuous operator, and it is clearly a contractive projection. Since $\mathcal{E}_*$ maps states in states, for every state $\sigma \in \mathcal{M}_*$ we have $\sigma(\mathcal{E}(\mathbb{1})) = (\mathcal{E}_*(\sigma))(\mathbb{1}) = 1$, so that $\mathcal{E}(\mathbb{1}) = \mathbb{1}$. Moreover, $\mathcal{E}$ is positive since, given $x \geq 0$ in $\mathcal{M}$, then

$$\langle u, \mathcal{E}(x)u\rangle = (\mathcal{E}_*(|u\rangle\langle u|))(x) \geq 0 \qquad \forall\, u \in \mathsf{h}$$

by the positivity of $\mathcal{E}_*$. Hence, $\mathcal{E}$ is a normal conditional expectation with

$$\mathcal{M}_1 := \operatorname{Ran}\mathcal{E} = \mathcal{M}_{*2}^{\perp}, \qquad \mathcal{M}_2 := \operatorname{Ker}\mathcal{E} = \mathcal{M}_{*1}^{\perp},$$

which are clearly isomorphic to the dual spaces of $\mathcal{M}_{*1}$ and $\mathcal{M}_{*2}$, respectively. In particular, $\mathcal{M}_1$ is a von Neumann algebra as dual of a Banach space (by Sakai theorem), and $\mathcal{M}_2$ is weak* closed by the normality of $\mathcal{E}$. The decomposition $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 = \operatorname{Ran}\mathcal{E} \oplus \operatorname{Ker}\mathcal{E}$ easily follows. The $\mathcal{T}_{*t}$-invariance of $\mathcal{M}_{*k}$ implies the $\mathcal{T}_t$-invariance of $\mathcal{M}_k$ for $k = 1, 2$. In the similar way, every $\mathcal{M}_k$ is also $*$-invariant.

The proof of conditions 1–3 is analogue to the corresponding ones in Theorem 10. The last statement also easily follows (see the proof of point 2 in Theorem 10). □

We conclude the section by analyzing what happens when h has finite dimension. First of all note that, in this case, when EID holds, we always have a normal conditional expectation $\mathcal{E}$ such that $\mathcal{M}_1 = \operatorname{Ran}\mathcal{E}$, $\mathcal{M}_2 = \operatorname{Ker}\mathcal{E}$.

Indeed, for any $x = x_1 + x_2$ we can define $\mathcal{E}(x) = x_1$. Then, $\mathcal{E}^2 = \mathcal{E}$ trivially and

$$\|\mathcal{E}(x)\| = \|x_1\| = \lim_t \|\mathcal{T}_t(x_1)\| \leq \lim_t \|\mathcal{T}_t(x)\| \leq \|x\|,$$

since $\|\mathcal{T}_t(x_2)\| \to_t 0$. Moreover, $\mathcal{M}_1$ always contains $\mathbb{1}$, so $\mathcal{E}(\mathbb{1}) = \mathbb{1}$ and $\|\mathcal{E}\| = 1$; consequently, $\mathcal{E}$ is positive (Corollary 3.2.6 in Ref. 7) and normal.

Corollary 16. Assume $\mathcal{M} = \mathcal{B}(\mathsf{h})$ with h finite-dimensional.

1. *If EID holds, then we have an isometric-sweeping decomposition of the predual $L^1(\mathsf{h}) = \mathcal{B}(\mathsf{h})_*$ which satisfies properties 1–4 of Theorem 10, with $\mathcal{M}_{*1} = {}^{\perp}\mathcal{M}_0$ and $\mathcal{M}_{*2} = {}^{\perp}\mathcal{N}(\mathcal{T})$.*
   Moreover, ${}^{\perp}\mathcal{M}_0$ is the biggest subspace of $L^1(\mathsf{h})$ such that the restriction of each $\mathcal{T}_{*t}$ to it is a surjective isometry.
2. *Conversely, if there exists an isometric-sweeping decomposition $L^1(\mathsf{h}) = \mathcal{M}_{*1} \oplus \mathcal{M}_{*2}$, then we have $\mathcal{N}(\mathcal{T}) = \mathcal{M}_{*1}^{\perp}$ and $\mathcal{M}_0 = \mathcal{M}_{*2}^{\perp}$.*
   In particular, EID holds and there is a unique isometric-sweeping decomposition of $L^1(\mathsf{h})$.

Proof. 1. Assume EID. Due to Theorem 10, we have the desired decomposition of $L^1(\mathsf{h})$: the conditional expectation $\mathcal{E}$ clearly exists and is normal by the previous discussion. Moreover, if there exists a subspace T of $L^1(\mathsf{h})$ on which every map $\mathcal{T}_{*t}$ acts as a surjective isometry, given $\sigma \in T$ and $x \in \mathcal{M}_0$ we have $\sigma = \mathcal{T}_{*t}(\sigma'_t)$ for some σ'_t in T, and then

$$\operatorname{tr}(\sigma x) = \operatorname{tr}(\sigma'_t \mathcal{T}_t(x)) \leq \|\sigma'_t\|_1 \cdot \|\mathcal{T}_t(x)\| = \|\sigma\|_1 \cdot \|\mathcal{T}_t(x)\|.$$

Since the last term goes to 0 as $t \to \infty$ (by definition of $\mathcal{M}_0$), we can then conclude that σ belongs to $\mathcal{M}_0^{\perp}$. This shows the inclusion $T \subseteq \mathcal{M}_0^{\perp}$, and so we easily conclude.

2. Assume now to have an isometric-sweeping decomposition $L^1(\mathsf{h}) = \mathcal{M}_{*1} \oplus \mathcal{M}_{*2}$. As above, we can easily show that $\mathcal{M}_{*1} \subseteq \mathcal{M}_0^{\perp}$. On the other hand, given $\sigma \in \mathcal{M}_{*2}$ and $x \in \mathcal{N}(\mathcal{T})$, we have, by Proposition 6

$$\begin{aligned}\operatorname{tr}(\sigma x) &= \operatorname{tr}(\sigma e^{itH}\left(e^{-itH} x e^{itH}\right) e^{-itH}) = \operatorname{tr}(\sigma \mathcal{T}_t(e^{-itH} x e^{itH})) = \operatorname{tr}(\mathcal{T}_{*t}(\sigma) e^{-itH} x e^{itH}) \\ &\leq \|\mathcal{T}_{*t}(\sigma)\|_1 \cdot \|e^{-itH} x e^{itH}\| = \|\mathcal{T}_{*t}(\sigma)\|_1 \cdot \|x\|.\end{aligned}$$

Since $\|\mathcal{T}_{*t}(\sigma)\|_1 \to 0$ by definition of $\mathcal{M}_{*2}$, we obtain the inclusion $\mathcal{M}_{*2} \subseteq \mathcal{N}(\mathcal{T})^{\perp}$. Therefore, by the decomposition $L^1(\mathsf{h}) = \mathcal{M}_{*1} \oplus \mathcal{M}_{*2}$, we immediately obtain the equalities $\mathcal{M}_{*1} = \mathcal{M}_0^{\perp}$ and $\mathcal{M}_{*2} = \mathcal{N}(\mathcal{T})^{\perp}$.

Now, by defining the map $\mathcal{E}_* : L^1(\mathsf{h}) \to \mathcal{M}_{*1}$ through $\mathcal{E}_*(\sigma) = \sigma_1$ for $\sigma = \sigma_1 + \sigma_2$ with $\sigma_i \in \mathcal{M}_{*i}$, $i = 1,2$, we obtain a norm 1 projection whose dual map $\mathcal{E} : \mathcal{B}(\mathsf{h}) \to \mathcal{B}(\mathsf{h})$ induces the splitting $\mathcal{B}(\mathsf{h}) = \mathrm{Ran}\,\mathcal{E} \oplus \mathrm{Ker}\,\mathcal{E}$. In particular, we have $\mathrm{Ran}\,\mathcal{E} = \mathcal{M}_{*2}^{\perp} = \mathcal{N}(\mathcal{T})$, and $\mathrm{Ker}\,\mathcal{E} = \mathcal{M}_{*1}^{\perp} = \mathcal{M}_0$, so that EID holds by Theorem 15. □

IV. SEMIGROUPS WITH A FAITHFUL NORMAL INVARIANT STATE

In this section, we study environmental decoherence assuming that <u>$\mathcal{T}$ has a faithful normal invariant state ω</u>. This hypothesis was initially introduced by Blanchard and Olkiewicz in the definition of EID for quantum evolutions which were not necessarily Markovian (see Ref. 5): really, it is not necessary in the definition but it assures some nice properties which can be a technical advantage for the proof of some results, as can be appreciated in Ref. 5 or also in Ref. 9 (see Section 2.1). In particular, the existence of a faithful normal invariant state allows to identify the effective algebra $\mathcal{M}_1$ with $\mathcal{N}(\mathcal{T})$ (see Proposition 3) and to define a scalar product on $\mathcal{M}$. So, in the finite-dimensional case or when $\mathcal{N}(\mathcal{T}) \subseteq \mathcal{F}(\mathcal{T})$, we proved in Ref. 9 and in Ref. 10 that *every* QMS with this property displays decoherence with $\mathcal{M}_1 = \mathcal{N}(\mathcal{T})$ and $\mathcal{M}_2$ its orthogonal complement; moreover, in these cases, there always exists a normal conditional expectation onto $\mathcal{N}(\mathcal{T})$ having $\mathcal{M}_2$ as kernel.

Therefore, also in the general case, it is natural to consider with particular attention the case when a faithful normal invariant state ω exists. As we have said, the faithfulness of ω allows us to define a scalar product $\langle\cdot,\cdot\rangle_\omega$ on $\mathcal{M}$ by setting

$$\langle x, y\rangle_\omega := \omega(x^* y) \qquad \forall\, x, y \in \mathcal{M},$$

but, unfortunately, in general $\mathcal{M}$ will not be complete with respect to it. As a consequence, in contrast with the finite-dimensional case, $\mathcal{N}(\mathcal{T})$ could be not closed in $\mathcal{M}$ and we cannot define a conditional expectation onto $\mathcal{N}(\mathcal{T})$ in a canonical way (i.e., as the orthogonal projection onto this space).

We shall denote by $\widehat{\mathcal{M}}$ the completion of $\mathcal{M}$ with respect to the induced norm: $\widehat{\mathcal{M}}$ is a Hilbert space with norm $\|\cdot\|_\omega$, and the canonical embedding $\Lambda : \mathcal{M} \hookrightarrow \widehat{\mathcal{M}}$ has dense range. Moreover, we can consider the splitting

$$\widehat{\mathcal{M}} = \overline{\mathcal{N}(\mathcal{T})}^{\omega} \oplus \mathcal{N}(\mathcal{T})^{\perp,\omega},$$

where $\overline{\mathcal{N}(\mathcal{T})}^{\omega}$ and $\mathcal{N}(\mathcal{T})^{\perp,\omega}$ are the closure of $\mathcal{N}(\mathcal{T})$ and its orthogonal complement in $(\widehat{\mathcal{M}}, \langle\cdot,\cdot\rangle_\omega)$, respectively. Now, since in the finite-dimensional case the space

$$\mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M} = \{x \in \mathcal{M} \,:\, \omega(y^* x) = 0 \;\forall\, y \in \mathcal{N}(\mathcal{T})\}$$

plays a fundamental role in the study of decoherence (see Ref. 9), we analyze here some of its properties.

Lemma 17. Let ω be a faithful normal invariant state. If $x \in \mathcal{N}(\mathcal{T})$ and $y \in \mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$, then xy and yx belong to $\mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$.

As a consequence, $xyz \in \mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$ for all $x, z \in \mathcal{N}(\mathcal{T})$ and $y \in \mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$.

Proof. Let $z \in \mathcal{N}(\mathcal{T})$. Then $\langle z, xy\rangle_\omega = \omega(z^* xy) = 0$ since $x^* z \in \mathcal{N}(\mathcal{T})$. This means that $xy \in \mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$. As a consequence, also $yx = (x^* y^*)^*$ belongs to $\mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$, since this space is $*$-invariant, $x^* \in \mathcal{N}(\mathcal{T})$ and $y^* \in \mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$. □

Proposition 18. Let ω be a faithful normal invariant state. Then,

1. *$\mathcal{N}(\mathcal{T}) = \{x \in \mathcal{M} : \|\mathcal{T}_t(x)\|_\omega = \|x\|_\omega,\ \|\mathcal{T}_t(x^*)\|_\omega = \|x^*\|_\omega\ \forall\, t \geq 0\} = \overline{\mathcal{N}(\mathcal{T})}^{\omega} \cap \mathcal{M}$;*
2. *$\mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$ is a weakly* closed $\mathcal{T}_t$-invariant and $*$-invariant subspace of $\mathcal{M}_0$.*

Proof. 1. Fix $t \geq 0$. Since ω is a faithful invariant state and $\mathcal{T}$ is Schwarz,

$$\|\mathcal{T}_t(x)\|_\omega^2 = \|x\|_\omega^2 \Leftrightarrow \omega(\mathcal{T}_t(x)^*\mathcal{T}_t(x)) = \omega(\mathcal{T}_t(x^*x)) \Leftrightarrow \mathcal{T}_t(x)^*\mathcal{T}_t(x) = \mathcal{T}_t(x^*x);$$

the same can be written with x^* in place of x. So the first equality of point 1 follows.

The inclusion $\mathcal{N}(\mathcal{T}) \subseteq \overline{\mathcal{N}(\mathcal{T})}^\omega \cap \mathcal{M}$ is clear. Conversely, if $x \in \overline{\mathcal{N}(\mathcal{T})}^\omega \cap \mathcal{M}$, then there exists a sequence $(x_n)_n \subseteq \mathcal{N}(\mathcal{T})$ such that $\omega((x_n - x)^*(x_n - x)) = \|x_n - x\|_\omega^2 \to_n 0$. Therefore, the invariance of ω, Equation (3), and the Cauchy-Schwarz inequality give

$$\begin{aligned}\|x_n - x\|_\omega^2 &= \omega(\mathcal{T}_t((x_n - x)^*(x_n - x))) = \omega(\mathcal{T}_t(x_n^*x_n - x_n^*x - x^*x_n + x^*x))\\ &= \omega(\mathcal{T}_t(x_n^*)\mathcal{T}_t(x_n) - \mathcal{T}_t(x_n^*)\mathcal{T}_t(x) - \mathcal{T}_t(x^*)\mathcal{T}_t(x_n) + \mathcal{T}_t(x^*x))\\ &\geq \omega(\mathcal{T}_t(x_n^*)\mathcal{T}_t(x_n) - \mathcal{T}_t(x_n^*)\mathcal{T}_t(x) - \mathcal{T}_t(x^*)\mathcal{T}_t(x_n) + \mathcal{T}_t(x^*)\mathcal{T}_t(x))\\ &= \|\mathcal{T}_t(x_n) - \mathcal{T}_t(x)\|_\omega^2,\end{aligned}$$

so that $\|\mathcal{T}_t(x_n) - \mathcal{T}_t(x)\|_\omega \to_n 0$. This implies that $\|\mathcal{T}_t(x_n)\|_\omega \to_n \|\mathcal{T}_t(x)\|_\omega$. Now, since $\|\mathcal{T}_t(x_n)\|_\omega = \|x_n\|_\omega \to_n \|x\|_\omega$, we can conclude that $\|\mathcal{T}_t(x)\|_\omega = \|x\|_\omega$. Similarly we can prove that $\|\mathcal{T}_t(x^*)\|_\omega = \|x^*\|_\omega$, and so $x \in \mathcal{N}(\mathcal{T})$.

2. Let $x \in \mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$ and fix $y \in \mathcal{N}(\mathcal{T})$, $t \geq 0$. Since the restriction of $\mathcal{T}_t$ to $\mathcal{N}(\mathcal{T})$ is a $*$-automorphism, we can write $y = \mathcal{T}_t(z)$ for some $z \in \mathcal{N}(\mathcal{T})$; therefore, Equation (3) and the invariance of ω imply

$$\omega(y^*\mathcal{T}_t(x)) = \omega(\mathcal{T}_t(z^*)\mathcal{T}_t(x)) = \omega(\mathcal{T}_t(z^*x)) = \omega(z^*x) = 0,$$

since $z^* \in \mathcal{N}(\mathcal{T})$ and $x \in \mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$. This proves that $\mathcal{T}_t(x)$ belongs to $\mathcal{N}(\mathcal{T})^{\perp,\omega}$, i.e., $\mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$ is $\mathcal{T}_t$-invariant.

Similarly, given $x \in \mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$ and $y \in \mathcal{N}(\mathcal{T})$, we have $\omega(y^*x^*) = \overline{\omega(xy)} = 0$, so that also x^* belongs to $\mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$, i.e., the space is $*$-invariant.

Now we show that $\mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$ is weakly* closed in $\mathcal{M}$. Let $(x_\alpha)_\alpha$ be a net in $\mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$ such that $w^* - \lim_\alpha x_\alpha = x$ for some $x \in \mathcal{M}$. Given $y \in \mathcal{N}(\mathcal{T})$, we have to prove that $\omega(y^*x) = 0$. But $\omega(y^*x) = \lim_\alpha \omega(y^*x_\alpha)$, and every term $\omega(y^*x_\alpha)$ is 0, for $x_\alpha \in \mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$. Therefore, we can conclude that $x \in \mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$.

Finally, given $x \in \mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$, let x_0 be a weak* cluster point of the net $\{\mathcal{T}_t(x)\}_{t\geq 0}$. Since when there exists a faithful invariant state $\mathcal{N}(\mathcal{T})$ contains any weak* cluster point of the net $(\mathcal{T}_t(a))_{t\geq 0}$ for every $a \in \mathcal{M}$ (see Ref. 18, proof of Theorem 3.1), we immediately have $x_0 \in \mathcal{N}(\mathcal{T})$. On the other hand, since $\mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$ is weakly* closed and $\mathcal{T}_t$-invariant by item 1, we have $x_0 \in \mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$, so that $x_0 = 0$ and so $\mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M} \subseteq \mathcal{M}_0$. □

Proposition 18, together with the results we obtained in the finite-dimensional case treated in Ref. 9, suggests $\mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$ as a good candidate to be the space of non-detectable observables $\mathcal{M}_2$. Unfortunately, things seem to be more complex in this case, since we cannot expect that, in general, the equality $\mathcal{M} = \mathcal{N}(\mathcal{T}) \oplus (\mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M})$ holds. However, the existence of a normal conditional expectation onto $\mathcal{N}(\mathcal{T})$ seems to simplify the situation.

Theorem 19. *Assume there exists a faithful invariant state ω. If $\mathcal{N}(\mathcal{T})$ is the image of a normal conditional expectation $\mathcal{E}$, then* Ker $\mathcal{E}$ *is a weak* closed and $*$-invariant subspace of $\mathcal{M}$ such that*

$$\mathcal{M} = \mathcal{N}(\mathcal{T}) \oplus \operatorname{Ker}\mathcal{E}.$$

Moreover, the following facts are equivalent:

(a) Ker $\mathcal{E}$ *is $\mathcal{T}_t$-invariant,*
(b) *each $\mathcal{T}_t$ commutes with $\mathcal{E}$,*
(c) $\mathrm{w}^* - \lim_t \mathcal{T}_t(x) = 0$ *for all* $x \in \operatorname{Ker}\mathcal{E}$,
(d) $\omega \circ \mathcal{E} = \omega$, *i.e., $\mathcal{E}$ is compatible with ω,*
(e) $\operatorname{Ker}\mathcal{E} = \mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$.

If one of the previous conditions is verified, then EID holds if and only if $\mathcal{E} \neq I$ and, in this case, we have $\mathcal{M}_2 = \operatorname{Ker}\mathcal{E}$.

Proof. Ker $\mathcal{E}$ is clearly a closed subspace of $\mathcal{M}$ and it is also weakly* closed by the normality of $\mathcal{E}$. Moreover, it is $*$-invariant as a consequence of the relation $\mathcal{E}(x^*) = \mathcal{E}(x)^*$, being any conditional expectation a positive operator by Theorem 1 in Ref. 38.

Prove now the equivalence of conditions (a)–(e).

$(a) \Rightarrow (b)$ Fix $t \geq 0$, assume Ker $\mathcal{E}$ $\mathcal{T}_t$-invariant and let $x \in \mathcal{M}$; since $\mathcal{T}_t(x) = \mathcal{T}_t(\mathcal{E}(x)) + \mathcal{T}_t(\mathcal{E}^\perp(x))$ with $\mathcal{T}_t(\mathcal{E}(x)) \in \mathcal{N}(\mathcal{T}) = \mathrm{Ran}\,\mathcal{E}$ and $\mathcal{T}_t(\mathcal{E}^\perp(x)) \in \mathrm{Ker}\,\mathcal{E} = \mathrm{Ran}\,\mathcal{E}^\perp$, applying $\mathcal{E}$ to the both sides of the previous equation we have $\mathcal{E}(\mathcal{T}_t(x)) = \mathcal{T}_t(\mathcal{E}(x))$, i.e., $\mathcal{T}_t$ and $\mathcal{E}$ commute.

$(b) \Rightarrow (c)$ If each $\mathcal{T}_t$ commutes with $\mathcal{E}$, it also commutes with $\mathcal{E}^\perp$, and so Ker $\mathcal{E}$ is $\mathcal{T}_t$-invariant. Therefore, given $x \in \mathrm{Ker}\,\mathcal{E}$, the boundedness of the net $(\mathcal{T}_t(x))_{t\geq 0}$ implies that it admits a weak* cluster point x_0 belonging to Ker $\mathcal{E}$. But x_0 also belongs to $\mathcal{N}(\mathcal{T})$ by Theorem 3.1 in Ref. 18 (since ω is a faithful invariant state), so that $x_0 = 0$. This proves that there exists $\mathrm{w}^* - \lim_t \mathcal{T}_t(x)$ and that it is zero.

$(c) \Rightarrow (d)$ The invariance of ω gives $\omega(\mathcal{E}^\perp(x)) = \omega(\mathcal{T}_t(\mathcal{E}^\perp(x)))$ for all $x \in \mathcal{M}$, and this last term goes to 0 as $t \to \infty$, since $\mathcal{E}^\perp(x)$ belongs to Ker $\mathcal{E}$. As a consequence, $\omega \circ \mathcal{E}^\perp = 0$, i.e., $\omega \circ \mathcal{E} = \omega$.

$(d) \Rightarrow (e)$ The properties of conditional expectation and the equality $\omega \circ \mathcal{E} = \omega$ imply that $\mathcal{E}$ is selfadjoint with respect to the scalar product $\langle \cdot, \cdot \rangle_\omega$: indeed,

$$\begin{aligned}\langle x, \mathcal{E}(y)\rangle_\omega &= \omega(x^*\mathcal{E}(y)) = \omega(\mathcal{E}(x^*\mathcal{E}(y)) = \omega(\mathcal{E}(x^*)\mathcal{E}(y)) \\ &= \omega(\mathcal{E}(\mathcal{E}(x^*)y)) = \omega(\mathcal{E}(x^*)y) = \langle \mathcal{E}(x), y\rangle_\omega\end{aligned}$$

for all $x, y \in \mathcal{M}$. Consequently, we claim that $\mathrm{Ker}\,\mathcal{E} = \mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$.

If $x \in \mathrm{Ker}\,\mathcal{E}$, we can write $x = \mathcal{E}^\perp(z)$ for some $z \in \mathcal{M}$. Therefore, given an arbitrary $y \in \mathcal{N}(\mathcal{T}) = \mathrm{Ran}\,\mathcal{E}$, we have

$$\langle x, y\rangle_\omega = \langle z, \mathcal{E}^\perp(y)\rangle_\omega = \langle z, y\rangle_\omega - \langle z, \mathcal{E}(y)\rangle_\omega = 0,$$

since $\mathcal{E}(y) = y$. This means that $x \in \mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$. Conversely, given $x \in \mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$, for all $z \in \mathcal{M}$ we have

$$\langle x, z\rangle_\omega = \langle x, z\rangle_\omega - \langle x, \mathcal{E}(z)\rangle_\omega = \langle x, \mathcal{E}^\perp(z)\rangle_\omega = \langle \mathcal{E}^\perp(x), z\rangle_\omega,$$

since $\langle x, \mathcal{E}(z)\rangle_\omega = 0$. Therefore, $x = \mathcal{E}^\perp(x)$ by the density of $\mathcal{M}$ in its completion, and so $x \in \mathrm{Ker}\,\mathcal{E}$. Thus we have established item (e).

$(e) \Rightarrow (a)$ and the last assertion are immediate due to point 2 of Proposition 18.

Assume now that equivalent conditions (a)–(e) are satisfied.

The decomposition $\mathcal{M} = \mathcal{N}(\mathcal{T}) \oplus \mathrm{Ker}\,\mathcal{E}$ is trivial by definition of $\mathcal{E}$, being $\mathcal{N}(\mathcal{T}) = \mathrm{Ran}\,\mathcal{E}$. Therefore, EID holds if and only if $\mathcal{M}_2 \neq \{0\}$, i.e., $\mathcal{E} \neq I$. □

Remark 20. Theorem 19 gives sufficient conditions to have the desired decomposition of the algebra $\mathcal{M}$. Takesaki theorem[36] provides a criterion to see when such conditions are satisfied: more precisely, there exists a normal conditional expectation onto $\mathcal{N}(\mathcal{T})$ compatible with the faithful state ω if and only if $\mathcal{N}(\mathcal{T})$ is invariant under the modular group $(\sigma_t^\omega)_{t\geq 0}$ associated with ω.

In particular, this happens when each $\mathcal{T}_t$ commutes with the modular group $(\sigma_s)_{s\geq 0}$: indeed, given $x \in \mathcal{N}(\mathcal{T})$ and $s \geq 0$, we have

$$\begin{aligned}\mathcal{T}_t(\sigma_s(x)^*\sigma_s(x)) &= \mathcal{T}_t(\sigma_s(x^*x)) = \sigma_s(\mathcal{T}_t(x^*x)) = \sigma_s(\mathcal{T}_t(x^*)\mathcal{T}_t(x)) \\ &= \sigma_s(\mathcal{T}_t(x^*))\sigma_s(\mathcal{T}_t(x)) = \mathcal{T}_t(\sigma_s(x)^*)\mathcal{T}_t(\sigma_s(x)) \qquad \forall\, t \geq 0,\end{aligned}$$

since also x^* and x^*x belong to $\mathcal{N}(\mathcal{T})$. Similarly, one can prove that

$$\mathcal{T}_t(\sigma_s(x)\sigma_s(x)^*) = \mathcal{T}_t(\sigma_s(x))\mathcal{T}_t(\sigma_s(x)^*) \qquad \forall\, t \geq 0,$$

and so $\sigma_s(x) \in \mathcal{N}(\mathcal{T})$.

Proposition 21. Let ω be a faithful normal invariant state and assume EID. Then, $\mathcal{M}_2 = \mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$ if and only if $\mathcal{N}(\mathcal{T})$ is the image of a unique normal conditional expectation $\mathcal{E}$ compatible with ω.

Proof. Assume EID with $\mathcal{M} = \mathcal{N}(\mathcal{T}) \oplus (\mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M})$. First we prove that $\omega(x) = \omega(x_1)$ for $x = x_1 + x_2 \in \mathcal{M}$ with $x_1 \in \mathcal{N}(\mathcal{T})$, $x_2 \in \mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$. Indeed,

$$\omega(x) = \omega(x_1) + \omega(x_2) = \omega(x_1) + \omega(\mathcal{T}_t(x_2)) \qquad \forall\, t \geq 0,$$

so that $\omega(x) = \omega(x_1) + \lim_t \omega(\mathcal{T}_t(x_2)) = \omega(x_1)$, by definition of $\mathcal{M}_2 \ni x_2$.

Now, define $\mathcal{E} : \mathcal{M} \to \mathcal{N}(\mathcal{T})$ by setting

$$\mathcal{E}(x) = x_1 \qquad \text{for all } x = x_1 + x_2 \in \mathcal{M},\ x_1 \in \mathcal{N}(\mathcal{T}),\ x_2 \in \mathcal{M}_2,$$

so that $\omega(\mathcal{E}(x)) = \omega(x)$ for all $x \in \mathcal{M}$. We have to show that $\mathcal{E}$ is a normal norm one projection. It is clear that $\mathcal{E}^2 = \mathcal{E}$, $\mathcal{E}(\mathbb{1}) = \mathbb{1}$, $\operatorname{Ran}\mathcal{E} = \mathcal{N}(\mathcal{T})$, and $\operatorname{Ker}\mathcal{E} = \mathcal{M}_2 = \mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$. If we prove that $\mathcal{E}$ is a positive map, then it is automatically bounded and $\|\mathcal{E}\| = \|\mathcal{E}(\mathbb{1})\| = 1$, and so we can conclude that it is a norm one projection, i.e., a conditional expectation.

So, let $x \geq 0$ in $\mathcal{M}$. Since $\mathcal{E}(x)^* = \mathcal{E}(x^*) = \mathcal{E}(x)$, by spectral theorem we have $\mathcal{E}(x) = \int_{-\infty}^{+\infty} \lambda\, \mathrm{d}E(\lambda)$, where E denotes the spectral measure associated with $\mathcal{E}(x)$. Now, denoted by $\mathcal{B}(\mathbb{R})$ the σ-algebra of Borel sets on $\mathbb{R}$, since $\mathcal{N}(\mathcal{T})$ is a von Neumann algebra, for every $S \in \mathcal{B}(\mathbb{R})$ the projection $E(S)$ belongs to $\mathcal{N}(\mathcal{T})$; hence, we get

$$E(S)xE(S) = E(S)\mathcal{E}(x)E(S) + E(S)\mathcal{E}^{\perp}(x)E(S), \tag{13}$$

with

$$E(S)\mathcal{E}(x)E(S) = \int_S \lambda\, \mathrm{d}E(\lambda) \in \mathcal{N}(\mathcal{T}),$$
$$E(S)\mathcal{E}^{\perp}(x)E(S) \in \mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M} \tag{14}$$

by Lemma 17, for $\mathcal{E}^{\perp}(x) \in \mathcal{M}_2 = \mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$. In particular, since $\mathbb{1} \in \mathcal{N}(\mathcal{T})$, Equation (14) gives $\omega(E(S)\mathcal{E}^{\perp}(x)E(S)) = 0$, so that $\omega(\mathcal{E}(E(S)xE(S))) = \omega(E(S)\mathcal{E}(x)E(S))$ by Equation (13). Therefore, we have

$$0 \leq \omega(E(S)xE(S)) = \omega(\mathcal{E}(E(S)xE(S))) = \int_S \lambda\, \mathrm{d}\omega(E(\lambda)),$$

and so $\omega(E(S)) = 0$ for all $S \subseteq (-\infty, 0)$. Now, the faithfulness of ω and $E(S) \geq 0$ imply $E(S) = 0$ for all borel set $S \subseteq (-\infty, 0)$, i.e., $\mathcal{E}(x) \geq 0$.

Finally, we show that $\mathcal{E}$ is normal. Let $(x_\alpha)_\alpha$ be an increasing net of positive elements of $\mathcal{M}$ such that $x = \sup_\alpha x_\alpha$. Since $\mathcal{E}$ and $I - \mathcal{E}$ are positive operators, $(\mathcal{E}(x_\alpha))_\alpha$ and $((I - \mathcal{E})x_\alpha)_\alpha$ are increasing sequences of positive elements, and so they converge weakly*. Let $y = w^* - \lim_\alpha \mathcal{E}(x_\alpha) = \sup_\alpha \mathcal{E}(x_\alpha)$ and $z = w^* - \lim_\alpha (I - \mathcal{E})(x_\alpha) = \sup_\alpha (I - \mathcal{E})(x_\alpha)$. Hence, we get

$$x = \sup_\alpha x_\alpha = w^* - \lim_\alpha x_\alpha = y + z$$

with $y \in \mathcal{N}(\mathcal{T})$ and $z \in \mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$, since $\mathcal{N}(\mathcal{T})$ and $\mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$ are weakly* closed. It follows that $y = \mathcal{E}(x) = \sup_\alpha \mathcal{E}(x_\alpha)$ and $z = (I - \mathcal{E})(x)$ by the uniqueness of the decomposition. In particular, this means that $\mathcal{E}$ is normal.

The opposite implication is a trivial consequence of Theorem 19.

Finally, we prove that, if $\mathcal{E}'$ is another normal conditional expectation onto $\mathcal{N}(\mathcal{T})$ which is compatible with ω, then $\mathcal{E} = \mathcal{E}'$.

Given $x \in \mathcal{M}$, we have

$$\mathcal{E}(x) = \mathcal{E}(\mathcal{E}'(x)) + \mathcal{E}((I - \mathcal{E}')(x)) = \mathcal{E}'(x)$$

since $\mathcal{E}'(x) \in \operatorname{Ran}\mathcal{E}' = \mathcal{N}(\mathcal{T}) = \operatorname{Ran}\mathcal{E}$, while $(I - \mathcal{E}')(x)$ belongs to $\operatorname{Ker}\mathcal{E}' = \mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M} = \operatorname{Ker}\mathcal{E}$ by Theorem 19. We can conclude that $\mathcal{E} = \mathcal{E}'$. □

A. Splitting of $\widehat{\mathcal{M}}$

After some introductory results, we can here better detail the properties of the orthogonal decomposition of $\widehat{\mathcal{M}}$ induced by ω (so we always suppose that $\underline{\mathcal{T} \text{ has a faithful normal invariant state } \omega}$, even if we shall not repeat it in the statements). Assuming the existence of a normal conditional expectation, we shall arrive to a kind of isometric-sweeping decomposition.

Let $\Lambda : \mathcal{M} \to \widehat{\mathcal{M}}$ be the canonical inclusion. Define the linear operator $\widehat{\mathcal{T}_t} : \Lambda(\mathcal{M}) \to \Lambda(\mathcal{M})$ by setting $\widehat{\mathcal{T}_t}(\Lambda(x)) = \Lambda(\mathcal{T}_t(x))$ for all $x \in \mathcal{M}$ and $t \geq 0$. Since

$$\|\widehat{\mathcal{T}_t}(\Lambda(x))\|_\omega^2 = \omega(\mathcal{T}_t(x^*)\mathcal{T}_t(x)) \leq \omega(\mathcal{T}_t(x^*x)) = \|x\|_\omega^2 \qquad \forall\, x \in \mathcal{M},$$

every $\widehat{\mathcal{T}_t}$ can be extended to a contraction to the whole space $\widehat{\mathcal{M}}$. We denote also this extension by $\widehat{\mathcal{T}_t}$. Moreover, for $x \in \mathcal{M}$, we clearly have

$$\begin{aligned}\|\widehat{\mathcal{T}_t}(\Lambda(x)) - \Lambda(x)\|_\omega^2 &= \omega(\mathcal{T}_t(x^*)\mathcal{T}_t(x) - \mathcal{T}_t(x^*)x - x^*\mathcal{T}_t(x) + x^*x)\\ &\leq 2\omega(x^*x) - 2\mathfrak{R}\,\omega(\mathcal{T}_t(x^*)x),\end{aligned}$$

and this last part goes to 0 as $t \to 0$ by the weak* continuity of the map $t \mapsto \mathcal{T}_t(x)$. Since $\Lambda(\mathcal{M})$ is dense in $\widehat{\mathcal{M}}$ and every $\widehat{\mathcal{T}_t}$ is a contraction, this proves the strong continuity of the semigroup $(\widehat{\mathcal{T}_t})_t$.

We prove now that the splitting $\widehat{\mathcal{M}} = \overline{\mathcal{N}(\mathcal{T})}^\omega \oplus \mathcal{N}(\mathcal{T})^{\perp,\omega}$ is a sort of isometric-sweeping decomposition associated with the semigroup $\widehat{\mathcal{T}}$.

Proposition 22. The following facts hold:

1. *Each $\widehat{\mathcal{T}_t}$ acts in a unitary way on $\overline{\mathcal{N}(\mathcal{T})}^\omega$.*
2. *$\mathcal{N}(\mathcal{T})^{\perp,\omega}$ is $\widehat{\mathcal{T}_t}$-invariant for all $t \geq 0$.*

Moreover, if there exists a normal conditional expectation $\mathcal{E}$ onto $\mathcal{N}(\mathcal{T})$ compatible with ω, then

3. *$\Lambda \circ \mathcal{E} = P \circ \Lambda$, where $P : \widehat{\mathcal{M}} \to \overline{\mathcal{N}(\mathcal{T})}^\omega$ is the orthogonal projection onto $\overline{\mathcal{N}(\mathcal{T})}^\omega$.*
4. *$\overline{\mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}}^\omega = \mathcal{N}(\mathcal{T})^{\perp,\omega}$.*
5. *$(\widehat{\mathcal{T}_t}\xi)_t$ weakly converges to 0 for all $\xi \in \mathcal{N}(\mathcal{T})^{\perp,\omega}$.*

Proof. 1. Let $x \in \mathcal{N}(\mathcal{T})$ and $t \geq 0$. Then,

$$\|\widehat{\mathcal{T}_t}(\Lambda(x))\|_\omega^2 = \omega(\mathcal{T}_t(x^*)\mathcal{T}_t(x)) = \omega(\mathcal{T}_t(x^*x)) = \|\Lambda(x)\|_\omega^2,$$

so that, by density, $\widehat{\mathcal{T}_t}$ is an isometry on $\overline{\mathcal{N}(\mathcal{T})}^\omega$.

Now, given $\xi \in \overline{\mathcal{N}(\mathcal{T})}^\omega$, let $(x_n)_n \subseteq \mathcal{N}(\mathcal{T})$ such that $\lim_n \Lambda(x_n) = \xi$. Since the restriction of $\mathcal{T}_t$ to $\mathcal{N}(\mathcal{T})$ is $*$-automorphism, in particular, it is surjective, there exists a sequence $(y_n)_n \subseteq \mathcal{N}(\mathcal{T})$ satisfying $\widehat{\mathcal{T}_t}(\Lambda(y_n)) = \Lambda(\mathcal{T}_t(y_n)) = \Lambda(x_n)$. Moreover, $(\Lambda(y_n))_n$ is a Cauchy sequence in $\overline{\mathcal{N}(\mathcal{T})}^\omega$, being $\widehat{\mathcal{T}_t}$ an isometry on $\overline{\mathcal{N}(\mathcal{T})}^\omega$. Therefore, there exists $\eta \in \overline{\mathcal{N}(\mathcal{T})}^\omega$ such that $\eta = \lim_n \Lambda(y_n)$ and so

$$\widehat{\mathcal{T}_t}(\eta) = \lim_n \widehat{\mathcal{T}_t}(\Lambda(y_n)) = \lim_n \Lambda(x_n) = \xi,$$

i.e., $\widehat{\mathcal{T}_t}$ is surjective.

2. Given $\xi \in \mathcal{N}(\mathcal{T})^{\perp,\omega}$ and $t \geq 0$, it is enough to prove that $\langle \Lambda(x), \widehat{\mathcal{T}_t}(\xi)\rangle_\omega = 0$ for all $x \in \mathcal{N}(\mathcal{T})$.

Since $\mathcal{T}_t : \mathcal{N}(\mathcal{T}) \to \mathcal{N}(\mathcal{T})$ is surjective, we have $x = \mathcal{T}_t(y)$ for some $y \in \mathcal{N}(\mathcal{T})$. Therefore, using that $\widehat{\mathcal{T}_t}^*\widehat{\mathcal{T}_t}$ is the identity operator on $\overline{\mathcal{N}(\mathcal{T})}^\omega$ (by item 1), we get

$$\langle \Lambda(x), \widehat{\mathcal{T}_t}(\xi)\rangle_\omega = \langle \widehat{\mathcal{T}_t}(\Lambda(y)), \widehat{\mathcal{T}_t}(\xi)\rangle_\omega = \langle \Lambda(y), \xi\rangle_\omega = 0,$$

for $\Lambda(y) \in \overline{\mathcal{N}(\mathcal{T})}^\omega$ and $\xi \in \mathcal{N}(\mathcal{T})^{\perp,\omega}$.

Assume now there exists a normal conditional expectation $\mathcal{E}$ onto $\mathcal{N}(\mathcal{T})$ compatible with ω.

3. Let $x \in \mathcal{M}$; since $\Lambda(\mathcal{E}(x)) \in \mathcal{N}(\mathcal{T}) \subseteq \overline{\mathcal{N}(\mathcal{T})}$, we clearly have $P\Lambda(\mathcal{E}(x)) = \Lambda(\mathcal{E}(x))$. Therefore,

$$P\Lambda(x) = P\Lambda(\mathcal{E}(x)) + P\Lambda((I - \mathcal{E})(x)) = P\Lambda(\mathcal{E}(x)) = \Lambda(\mathcal{E}(x)),$$

for $(I - \mathcal{E})(x) \in \operatorname{Ker}\mathcal{E} = \mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M} \subseteq \mathcal{N}(\mathcal{T})^{\perp,\omega}$.

4. The inclusion $\overline{\mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}}^\omega \subseteq \mathcal{N}(\mathcal{T})^{\perp,\omega}$ is trivial. Conversely, if $\xi \in \mathcal{N}(\mathcal{T})^{\perp,\omega}$, then there exists a sequence $(x_n)_n \subseteq \mathcal{M}$ such that $\xi = \lim_n \Lambda(x_n)$. Therefore, $P\xi = \lim_n P\Lambda(x_n) = \lim_n \Lambda$

$(\mathcal{E}(x_n))$ by item 1. But P is the orthogonal projection onto $\overline{\mathcal{N}(\mathcal{T})}^{\omega}$, whereas ξ belongs to $\mathcal{N}(\mathcal{T})^{\perp,\omega} = \mathrm{Ker}\, P$, and so we have $P\xi = 0$, i.e., $(\mathcal{E}(x_n))_n$ is norm convergent to 0 in $\widehat{\mathcal{M}}$. As a consequence, also $((I-\mathcal{E})(x_n))_n$ is convergent in $\|\cdot\|_\omega$ and it holds

$$\xi = \lim_n(\mathcal{E}(x_n) + (I-\mathcal{E})(x_n))) = \lim_n(I-\mathcal{E})(x_n)),$$

i.e., $\xi \in \overline{\mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}}^{\omega}$.

5. By Theorem 19, it follows that $w^* - \lim_t \mathcal{T}_t(x) = 0$ for all $x \in \mathcal{M} \cap \mathcal{N}(\mathcal{T})^{\perp,\omega}$. Therefore, we have $\langle \Lambda(y), \widehat{\mathcal{T}_t}(\Lambda(x))\rangle_\omega \to 0$ for $x \in \mathcal{M} \cap \mathcal{N}(\mathcal{T})^{\perp,\omega}$ and $y \in \mathcal{M}$, since

$$\langle \Lambda(y), \widehat{\mathcal{T}_t}(\Lambda(x))\rangle_\omega = \langle y, \mathcal{T}_t(x)\rangle_\omega = \omega(y^*\mathcal{T}_t(x)).$$

Now, given $\xi \in \mathcal{N}(\mathcal{T})^{\perp,\omega}$, $\eta \in \widehat{\mathcal{M}}$, and $\epsilon > 0$, by item 4 there exist $x_\epsilon \in \mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$ and $y_\epsilon \in \mathcal{M}$ such that

$$\|\xi - \Lambda(x_\epsilon)\|_\omega < \epsilon \qquad \|\eta - \Lambda(y_\epsilon)\|_\omega < \epsilon.$$

Hence,

$$\begin{aligned} |\langle \eta, \widehat{\mathcal{T}_t}(\xi)\rangle_\omega| &\le |\langle \eta - \Lambda(y_\epsilon), \widehat{\mathcal{T}_t}(\xi)\rangle_\omega| + |\langle \Lambda(y_\epsilon), \widehat{\mathcal{T}_t}(\xi - \Lambda(x_\epsilon))\rangle_\omega| + |\langle \Lambda(y_\epsilon), \widehat{\mathcal{T}_t}(\Lambda(x_\epsilon))\rangle_\omega| \\ &\le \epsilon\,\|\xi\|_\omega + \epsilon\,\|y_\epsilon\|_\omega + |\langle \Lambda(y_\epsilon), \widehat{\mathcal{T}_t}(\Lambda(x_\epsilon))\rangle_\omega|. \end{aligned}$$

Since $\langle \Lambda(y_\epsilon), \widehat{\mathcal{T}_t}(\Lambda(x_\epsilon))\rangle_\omega \to_t 0$, there exists $t_0 > 0$ such that $|\langle \Lambda(y_\epsilon), \widehat{\mathcal{T}_t}(\Lambda(x_\epsilon)\rangle_\omega| < \epsilon$ for all $t \ge t_0$. This proves that $\langle \eta, \widehat{\mathcal{T}_t}(\xi)\rangle_\omega \to_t 0$, i.e., $(\widehat{\mathcal{T}_t}(\xi))_t$ weakly converges to 0 in $\widehat{\mathcal{M}}$. □

Remark 23. Under the assumption of the existence of a normal conditional expectation onto $\mathcal{N}(\mathcal{T})$ compatible with ω, the splitting $\widehat{\mathcal{M}} = \overline{\mathcal{N}(\mathcal{T})}^{\omega} \oplus \mathcal{N}(\mathcal{T})^{\perp,\omega}$ provides an isometric-sweeping decomposition of the space $\widehat{\mathcal{M}}$, with respect to the semigroup $\widehat{\mathcal{T}}$: indeed, the action of every $\widehat{\mathcal{T}_t}$ is unitary on $\overline{\mathcal{N}(\mathcal{T})}^{\omega}$, while it goes to 0 in the weak topology on $\mathcal{N}(\mathcal{T})^{\perp,\omega}$ (see items 1 and 5 of Proposition 22).

Moreover, we can establish a correspondence between this splitting and the EID decomposition $\mathcal{M} = \mathcal{N}(\mathcal{T}) \oplus (\mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M})$ (whose existence is guaranteed by Theorem 19) . More precisely, by considering the closure of $\mathcal{N}(\mathcal{T})$ and $\mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$, we obtain $\overline{\mathcal{N}(\mathcal{T})}^{\omega}$ and $\mathcal{N}(\mathcal{T})^{\perp,\omega}$, respectively (see item 4 in Proposition 22). Vice versa, the intersection of $\overline{\mathcal{N}(\mathcal{T})}^{\omega}$ and $\mathcal{N}(\mathcal{T})^{\perp,\omega}$ with the algebra $\mathcal{M}$ gives spaces $\mathcal{N}(\mathcal{T})$ and $\mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$ in the EID decomposition (see item 1 in Proposition 18).

We want now to prove that, under some additional hypothesis, the splitting $\widehat{\mathcal{M}} = \overline{\mathcal{N}(\mathcal{T})}^{\omega} \oplus \mathcal{N}(\mathcal{T})^{\perp,\omega}$ corresponds to the canonical decomposition of the semigroup $\widehat{\mathcal{T}}$ (Theorem 28), while the splitting induced by EID is the extension of the isometric-sweeping decomposition proved by Lugiewicz and Olkiewicz in Ref. 25 for a single map $\mathcal{T}_t$ to the whole QMS $\mathcal{T}$.

To this end, we first recall some results about the canonical decomposition of a semigroup of contractions in unitary and completely non-unitary parts (see Proposition 8.3 of Chapter 3 in Ref. 35 and Theorem 15.16 in Ref. 20).

Theorem 24. *If $(V_t)_{t\ge 0}$ is a strongly continuous semigroup of contractions on a Hilbert space $\mathcal{H}$, there exists a maximal closed subspace $\mathcal{H}_0 \subseteq \mathcal{H}$ such that the restriction of $(V_t)_t$ to $\mathcal{H}_0$ is a unitary semigroup. The restriction of $(V_t)_t$ to $\mathcal{H}_0^\perp$ is completely non-unitary (i.e., there exists no nontrivial subspaces of $\mathcal{H}_0^\perp$ on which the action of the semigroup is unitary). Moreover, this decomposition is uniquely determined and*

$$\mathcal{H}_0 = \{x \in \mathcal{H} \ : \ \|V_t x\| = \|x\| = \|V_t^* x\| \text{ for all } t \ge 0\}$$

is called the unitary space of $(V_t)_t$. Finally, we have $\|V_t x\| \to_t 0$ for all $x \in \mathcal{H}_0^\perp$ if and only if $P = Q$ is a projection, where

$$Px := \lim_t V_t^* V_t x, \quad Qx := \lim_t V_t V_t^* x \qquad \forall\, x \in \mathcal{H}.$$

In this case, the range of $P = Q$ is $\mathcal{H}_0$.

The decomposition provided by this result is called *canonical decomposition* of $(V_t)_t$, and we clearly have

$$x \in \mathcal{H}_0 \Longleftrightarrow V_t V_t^* x = x = V_t^* V_t x \quad \forall\, t \geq 0. \tag{15}$$

We denote by K the unitary space of $\widehat{\mathcal{T}}$, i.e.,

$$K := \{\xi \in \widehat{\mathcal{M}} \, : \, \|\widehat{\mathcal{T}_t}\xi\|_\omega = \|\xi\|_\omega = \|\widehat{\mathcal{T}_t}^*\xi\|_\omega \quad \forall\, t \geq 0\}. \tag{16}$$

Note that, in particular, $K = \cap_{t\geq 0} K_t$, where any K_t is the unitary space of $\widehat{\mathcal{T}_t}$, i.e.,

$$K_t := \{\xi \in \widehat{\mathcal{M}} \, : \, \|\widehat{\mathcal{T}_t}^n\xi\|_\omega = \|\xi\|_\omega = \|\widehat{\mathcal{T}_t}^{*n}\xi\|_\omega \quad \forall\, n \geq 1\}. \tag{17}$$

Remark 25. By item 1 of Proposition 22, we clearly have the inclusion $\overline{\mathcal{N}(\mathcal{T})}^{\omega} \subseteq K$, and one can wonder whether the equality holds. Since a necessary condition is to have $K \cap \mathcal{M} = \mathcal{N}(\mathcal{T})$ (see item 1 of Proposition 18), a problem is understanding whether $K \cap \mathcal{M}$ is closed under the adjoint. For example, this happens when $w\mathcal{T}^*_{t(\mathcal{M})} \subseteq \mathcal{M}$ and $\widehat{\mathcal{T}}^*$ preserves self-adjointness, but it is not always true. However, it is easy to show that the previous equality $K \cap \mathcal{M} = \mathcal{N}(\mathcal{T})$ holds when $\mathcal{M}$ is a matrix algebra (see Ref. 9).

Assume now *that $\mathcal{T}$ commutes with the modular group associated with the faithful invariant state ω*: using the isometric-sweeping decomposition proved by Lugiewicz and Olkiewicz for a single map $\mathcal{T}_t$ (see Proposition 3, Theorems 4 and 6 in Ref. 25), we want to show that the splitting $\widehat{\mathcal{M}} = \overline{\mathcal{N}(\mathcal{T})}^{\omega} \oplus \mathcal{N}(\mathcal{T})^{\perp,\omega}$ corresponds to the canonical decomposition of the semigroup $\widehat{\mathcal{T}}$, $\widehat{\mathcal{M}} = K \oplus K^\perp$ (Theorem 28).

Recall that, under this assumption, they prove there exists a normal conditional expectation $Q_t : \mathcal{M} \to \mathcal{M}$ such that

- $\mathcal{M} = \mathrm{Ran}\, Q_t \oplus \mathrm{Ker}\, Q_t$;
- $\mathrm{Ran}\, Q_t$ is a weakly* closed $*$-subalgebra of $\mathcal{M}$ on which $\mathcal{T}_t$ acts as a $*$-automorphism, while $\mathrm{Ker}\, Q_t$ is a $\mathcal{T}_t$ and $*$-invariant weakly* closed subspace such that $w^* - \lim_n \mathcal{T}_t{}^n(x) = 0$ for all $x \in \mathrm{Ker}\, Q_t$;
- $Q_{2,t}\Lambda(x) = \Lambda(Q_t x)$ for all $x \in \mathcal{M}$, where $Q_{2,t} : \widehat{\mathcal{M}} \to \widehat{\mathcal{M}}$ denotes the orthogonal projection onto the unitary space K_t of the contraction $\widehat{\mathcal{T}_t}$;
- Q_t is compatible with ω if and only if $\mathbb{1} \in \mathrm{Ran}\, Q_t$. Moreover, in this case, $\mathrm{Ran}\, Q_t$ is a von Neumann subalgebra and $Q_t(\mathbb{1}) = \mathbb{1}$.

In particular, we have $\Lambda(\mathrm{Ran}\, Q_t) = K_t \cap \Lambda(\mathcal{M})$ and $\Lambda(\mathrm{Ker}\, Q_t) = K_t^\perp \cap \Lambda(\mathcal{M})$, i.e.,

$$\mathrm{Ran}\, Q_t = K_t \cap \mathcal{M}, \qquad \mathrm{Ker}\, Q_t = K_t^\perp \cap \mathcal{M}. \tag{18}$$

In other words, the isometric-sweeping decomposition $\mathcal{M} = \mathrm{Ran}\, Q_t \oplus \mathrm{Ker}\, Q_t$ is obtained from the canonical decomposition of $\widehat{\mathcal{T}_t}$ (associated with the splitting $\widehat{\mathcal{M}} = K_t \oplus K_t^\perp$) simply by intersecting $\mathcal{M}$ with the unitary and completely non-unitary space, respectively, of this contraction.

We can now extend this result in this way:

Proposition 26. Assume that every $\mathcal{T}_t$ commutes with the modular group associated with ω. Then,

1. $\mathcal{N}(\mathcal{T}) = \cap_{t\geq 0}\mathrm{Ran}\, Q_t = K \cap \mathcal{M}$*;*
2. *there exists a normal conditional expectation Q onto $K \cap \mathcal{M}$ such that* $\mathrm{Ker}\, Q = K^\perp \cap \mathcal{M}$ *and $\Lambda \circ Q = Q_2 \circ \Lambda$, where Q_2 is the orthogonal projection onto K. Moreover, Q is compatible with ω and*

$$Qx = w^* - \lim_t Q_t x \qquad \forall\, x \in \mathcal{M}. \tag{19}$$

Proof. 1. The equality $\cap_{t\geq 0}\mathrm{Ran}\, Q_t = K \cap \mathcal{M}$ follows from Equation (18), being $\cap_t K_t = K$.

We have $\mathcal{N}(\mathcal{T}) \subseteq \cap_{t\geq 0}\mathrm{Ran}\, Q_t$ by item 1 in Proposition 18; in particular, this means that $\mathbb{1}$ belongs to every subalgebra $\mathrm{Ran}\, Q_t$, which consequently becomes a von Neumann subalgebra. On the other hand, since $\mathcal{N}(\mathcal{T})$ is the biggest von Neumann subalgebra of $\mathcal{M}$ on which every $\mathcal{T}_t$ acts as a $*$-automorphism, we immediately obtain $\mathcal{N}(\mathcal{T}) \supseteq \mathrm{Ran}\, Q_t$ for all $t \geq 0$, and so the equality $\mathcal{N}(\mathcal{T}) = \cap_{t\geq 0}\mathrm{Ran}\, Q_t$ follows.

2. Since the maps $\widehat{\mathcal{T}_t}$ and $\widehat{\mathcal{T}_t}^*$ are contractions, the spaces $(K_t)_t$ give a decreasing net: indeed, given $t, s \geq 0$, and $\xi \in K_{t+s}$, we have

$$\|\xi\|_\omega = \|\widehat{\mathcal{T}}_{t+s}^n \xi\|_\omega = \|\widehat{\mathcal{T}_s}^n(\widehat{\mathcal{T}_t}^n \xi)\|_\omega \leq \|\widehat{\mathcal{T}_t}^n \xi\|_\omega \leq \|\xi\|_\omega \qquad \forall\, n \geq 0,$$

and so the equalities $\|\widehat{\mathcal{T}_t}^n \xi\|_\omega = \|\xi\|_\omega$ hold for every n. Since the same result is true for $\widehat{\mathcal{T}_t}^*$, we can conclude that ξ belongs to K_t, i.e., $K_{t+s} \subseteq K_t$.

As a consequence $(Q_{2,t})_t$ is a decreasing net of orthogonal projections and then, by Vigier theorem, it is convergent in the strong operator topology (on $\mathcal{B}(\widehat{\mathcal{M}})$) to the orthogonal projection Q_2 onto $\cap_t K_t = K$. This means, in particular, that $(Q_{2,t}(\Lambda x))_t \subseteq \Lambda(\mathcal{M})$ is convergent with respect to the $\|\ \|_\omega$-topology for all $x \in \mathcal{M}$; since on the unit ball of $\mathcal{M}$ this topology is equivalent to the σ-strong topology (see Ref. 37, Proposition III.5.3) which is finer than the weak* topology of $\mathcal{M}$, we also obtain that $(Q_{2,t}(\Lambda x))_t$ is weakly* convergent in $\mathcal{M}$. Therefore, the equality $\Lambda \circ Q_t = Q_{2,t} \circ \Lambda$ ensures the existence of the weak* limit of $(Q_t x)_t$ for all $x \in \mathcal{M}$; denoting by Qx such a limit, we clearly obtain a contractive projection $Q : \mathcal{M} \to \mathcal{M}$ (and so a norm one projection, i.e., a conditional expectation) satisfying $\Lambda \circ Q = Q_2 \circ \Lambda$, so that $\mathrm{Ran}\, Q = \mathrm{Ran}\, Q_2 \cap \mathcal{M} = K \cap \mathcal{M}$ and $\mathrm{Ker}\, Q = \mathrm{Ker}\, Q_2 \cap \mathcal{M} = K^\perp \cap \mathcal{M}$.

Now, we should prove that Q is compatible with ω. Recall that Q_t is compatible with ω, since $\mathbb{1} \in \cap_{t\geq 0}\mathrm{Ran}\, Q_t$ by item 1. Therefore, for every $x \in \mathcal{M}$, we have

$$\omega \circ Q(x) = \lim_t \omega \circ Q_t(x) = \omega(x).$$

Finally, in order to show the normality of Q, we prove that $Q^\perp$ satisfies the same property. First remark that, since $(\mathrm{Ran}\, Q_t^\perp)$ is increasing in t, for any $s, t > 0$ and any positive operator x we have

$$Q_{t+s}^\perp(x) = Q_{t+s}^\perp(Q_t(x) + Q_t^\perp(x)) = Q_{t+s}^\perp(Q_t(x)) + Q_t^\perp(x) \geq Q_t^\perp(x),$$

where the last inequality is due to the fact that $Q_{t+s}^\perp$ and Q_t are positive. Then $(Q_t^\perp(x))_t$ is an increasing net in $\mathcal{M}$ for any positive operator x, bounded by $Q^\perp(x)$; so its least upper bound exists and coincides with $Q^\perp(x)$, by construction of Q.

Let us now consider an increasing net $(x_\lambda)_\lambda$ of positive elements in $\mathcal{M}$ such that there exists $x = \sup_\lambda x_\lambda$. Since any Q_t is normal, also $Q_t^\perp$ is so, and then we have $\lim_\lambda Q_t^\perp x_\lambda = \sup_\lambda Q_t^\perp x_\lambda$; moreover, by the previous remark, for any λ, $(Q_t^\perp(x_\lambda))_t$ is an increasing net with least upper bound $\sup_t Q_t^\perp(x_\lambda) = Q^\perp(x_\lambda)$. Consequently,

$$Q^\perp x = \sup_t Q_t^\perp x = \sup_t \sup_\lambda Q_t^\perp x_\lambda = \sup_\lambda \sup_t Q_t^\perp x_\lambda = \sup_\lambda Q^\perp x_\lambda,$$

being $Q_t^\perp x_\lambda$ a positive operator for every λ and t. This proves that $Q^\perp$ is normal. □

Remark 27. The normal conditional expectation Q defined by (19) clearly induces the splitting $\mathcal{M} = \mathrm{Ran}\, Q \oplus \mathrm{Ker}\, Q$ which generalizes, by definition, the isometric-sweeping decomposition provided by Lugiewicz and Olkiewicz for a single map $\mathcal{T}_t$.

We can then show that $\widehat{\mathcal{M}} = \overline{\mathcal{N}(\mathcal{T})}^\omega \oplus \mathcal{N}(\mathcal{T})^{\perp,\omega}$ is the canonical decomposition of $(\widehat{\mathcal{T}_t})_t$.

Theorem 28. *Assume that every $\mathcal{T}_t$ commutes with the modular group associated with ω. Then, we have $K = \overline{\mathcal{N}(\mathcal{T})}^\omega$, i.e., $\overline{\mathcal{N}(\mathcal{T})}^\omega$ and $\mathcal{N}(\mathcal{T})^{\perp,\omega}$ are the unitary space and the completely non-unitary space, respectively, of the semigroup $\widehat{\mathcal{T}}$.*

Moreover, the isometric-sweeping decomposition induced by the normal conditional expectation Q given in Proposition 26 is an EID decomposition.

Proof. We just have to prove that $K = \overline{K \cap \mathcal{M}}^\omega$, so that the equality $\overline{\mathcal{N}(\mathcal{T})}^\omega = K$ follows from item 1 of Proposition 26.

Given $\xi \in K$, $\xi = \lim \Lambda(x_n)$ with $(x_n)_n \subseteq \mathcal{M}$, we have

$$0 = Q_2^\perp \xi = \lim_n Q_2^\perp \Lambda(x_n) = \lim_n \Lambda(Q^\perp x_n),$$

since $Q_2^{\perp}$ is the orthogonal projection onto $K^{\perp}$ and $Q_2 \circ \Lambda = \Lambda \circ Q$. Consequently, since $\operatorname{Ran} Q = K \cap \mathcal{M}$ by item 2 of Proposition 26, we get

$$\xi = \lim_n \left(\Lambda(Qx_n) + \Lambda(Q^{\perp}x_n)\right) = \lim_n \Lambda(Qx_n) \in \overline{K \cap \mathcal{M}}^{\omega},$$

and so $K \subseteq \overline{K \cap \mathcal{M}}^{\omega}$. The opposite inclusion is trivial.

Last statement immediately follows by Theorem 19, being Q a normal conditional expectation onto $\mathcal{N}(\mathcal{T})$ which is compatible with ω. □

Note that, this theorem shows, in particular, that the spaces $\mathcal{M}_1$ and $\mathcal{M}_2$ in the EID decomposition are obtained simply by considering the intersection of the unitary and the completely non-unitary subspaces of $\widehat{\mathcal{T}}$, respectively, with the algebra $\mathcal{M}$.

B. Jacobs-deLeeuw-Glicksberg splitting

In this section, we clarify the relationships between the decomposition induced by EID and the weak* version of the JDG splitting for one-parameter semigroups on dual Banach spaces. This last splitting has been proved by Hellmich in Ref. 21 under the assumption of a faithful normal invariant state, and by Batkai *et al.* in Ref. 2 for QMS possessing a faithful family of invariant states. As usual, we shall denote by $\mathcal{L}$ the infinitesimal generator and by $\mathcal{T}$ the generated semigroup.

In order to introduce the Jacobs-de Leeuw-Glicksberg splitting, we recall the definition of the *reversible subspace* $\mathfrak{M}_r$ involved in such a decomposition. So,

$$\begin{aligned}\mathfrak{M}_r &:= \overline{\operatorname{span}}^{w^*}\{x \in \mathcal{M} \mid \mathcal{L}(x) = i\lambda x \text{ for some } \lambda \in \mathbb{R}\}\\ &= \overline{\operatorname{span}}^{w^*}\{x \in \mathcal{M} \mid \exists\, \lambda \in \mathbb{R} \text{ such that } \mathcal{T}_t(x) = e^{i\lambda t}\ \forall\, t \geq 0\}.\end{aligned}$$

Theorem 29 (Jacobs-deLeeuw-Glicksberg splitting). *If there exists a faithful normal invariant state ω, then $\mathfrak{M}_r$ is a von Neumann subalgebra of $\mathcal{M}$ and we have $\mathcal{M} = \mathfrak{M}_r \oplus \mathfrak{M}_s$ with*

$$\mathfrak{M}_s := \{x \in \mathcal{M} \,:\, 0 \in \overline{\{\mathcal{T}_t(x)\}_{t\geq 0}}^{w^*}\}$$

a weak Banach subspace which is $*$-invariant and $\mathcal{T}_t$-invariant.*

Moreover, the action of every $\mathcal{T}_t$ on $\mathfrak{M}_r$ is a $*$-automorphism, and $\mathfrak{M}_r$ is the image of a normal conditional expectation Q compatible with ω.

Finally, if one of the following conditions holds

(i) $\mathfrak{M}_r = \mathcal{N}(\mathcal{T})$,
(ii) $\sigma(\mathcal{L}) \cap i\mathbb{R}$ *is at most countable,*
(iii) $\{\mathcal{T}_{*t}\}_{t\geq 0}$ *is strongly relatively compact,*

then, $\mathfrak{M}_s = \mathcal{M}_0$.

Proof. The first part of the statement follows by Corollary 3.3 and Proposition 3.3 in Ref. 21. In particular, we find that the conditional expectation Q commutes with every $\mathcal{T}_t$, $\operatorname{Ran} Q = \mathfrak{M}_r$, and $\operatorname{Ker} Q = \mathfrak{M}_s$.

Condition (i) ((ii), respectively) implies $\mathfrak{M}_s = \mathcal{M}_0$ by Proposition 3.6 (Proposition 2.3, respectively) in the same paper. Finally, if $\{\mathcal{T}_{*t}\}_{t\geq 0}$ is strongly relatively compact, then by Theorem 2.14 in Ref. 14 we have that

$$(\mathfrak{M}_s)_* = \operatorname{Ker} Q_* = \{\varphi \in \mathcal{M}_* \,:\, 0 \in \overline{\{\mathcal{T}_{*t}\varphi\}_{t\geq 0}}^{w}\} = \{\varphi \in \mathcal{M}_* \,:\, \|\mathcal{T}_{*t}\varphi\|_1 \to_t 0\};$$

hence, for all $x \in \mathfrak{M}_s$ and $\varphi \in \mathcal{M}_*$, by setting $Q^{\perp} = I - Q$, we get

$$|\varphi(\mathcal{T}_t(x))| = |\varphi(\mathcal{T}_t(Q^{\perp}x))| = |(\mathcal{T}_{*t}(Q_*^{\perp}\varphi))(x)| \leq \|x\|\|\mathcal{T}_{*t}(Q_*^{\perp}\varphi)\| \to_t 0,$$

since $Q_*^{\perp}\varphi \in (\mathfrak{M}_s)_*$. Therefore, $(\mathcal{T}_t(x))_t$ weakly* converges to 0, i.e., $x \in \mathcal{M}_0$. □

Remark 30. As a consequence of Theorem 29, when there exists a faithful normal invariant state the inclusion $\mathfrak{M}_r \subseteq \mathcal{N}(\mathcal{T})$ always holds, since $\mathfrak{M}_r$ is a von Neumann subalgebra of $\mathcal{M}$ on which every $\mathcal{T}_t$ acts as a $*$-automorphism.

The relationships between the spaces $\mathfrak{M}_r$ and $\mathcal{N}(\mathcal{T})$, $\mathfrak{M}_s$, $\mathcal{M}_2$, and $\mathcal{M}_0$ are given in the following Proposition.

Proposition 31. If the QMS $\mathcal{T}$ possesses a faithful normal invariant state ω, then the following conditions are equivalent:

1. $\mathfrak{M}_r = \mathcal{N}(\mathcal{T})$;
2. $\mathcal{N}(\mathcal{T}) \cap \mathfrak{M}_s = \{0\}$;
3. *EID holds with $\mathcal{M}_2 = \mathcal{M}_0$ and the induced decomposition coincides with the JDG splitting.*

Moreover, if one of the previous conditions holds, then $\mathcal{N}(\mathcal{T})$ is the image of a normal conditional expectation $\mathcal{E}$ compatible with ω and $\mathcal{M}_2 = \mathrm{Ker}\,\mathcal{E} = \mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M} = \mathcal{M}_0 = \mathfrak{M}_s$.

Proof. The implication $1 \Rightarrow 2$ is clear since $\mathfrak{M}_r \cap \mathfrak{M}_s = \{0\}$. Conversely, assume $\mathcal{N}(\mathcal{T}) \cap \mathfrak{M}_s = \{0\}$ and let $x \in \mathcal{N}(\mathcal{T})$. Since JDG holds by Theorem 29, we have $x = x_1 + x_2$ with $x_1 \in \mathfrak{M}_r \subseteq \mathcal{N}(\mathcal{T})$ and $x_2 \in \mathfrak{M}_s$. It follows that also x_2 belongs to $\mathcal{N}(\mathcal{T})$, and then $x_2 = 0$. Therefore, $x = x_1 \in \mathfrak{M}_r$. This proves the equivalence of statements 1 and 2.

$1 \Rightarrow 3$: If $\mathfrak{M}_r = \mathcal{N}(\mathcal{T})$, by Theorem 29, we also have $\mathfrak{M}_s = \mathcal{M}_0$, and so EID holds with $\mathcal{M}_2 = \mathfrak{M}_s = \mathcal{M}_0$, i.e., the decomposition induced by decoherence coincides with the JDG splitting. The opposite implication is trivial.

If one of the conditions 1–3 holds, then by Theorem 29, $\mathcal{N}(\mathcal{T})$ is the image of a normal conditional expectation $\mathcal{E}$ compatible with ω. Moreover, we have $\mathcal{M}_2 = \mathrm{Ker}\,\mathcal{E} = \mathcal{N}(\mathcal{T})^{\perp,\omega} \cap \mathcal{M}$ by Theorem 19. □

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