On the structure of higher order quantum maps

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1 Affine subspaces and higher order maps

1.1 The category FinVect

Let FinVect be the category of finite dimensional real vector spaces with linear maps. We will denote the usual tensor product by $\otimes$, then $(\mathrm{FinVect}, \otimes, I = \mathbb{R})$ is a symmetric monoidal category, with the associators, unitors and symmetries given by the obvious isomorphisms

$$\begin{aligned} \alpha_{U,V,W} &: (U \otimes V) \otimes W \simeq U \otimes (V \otimes W), \\ \lambda_V &: I \otimes V \simeq V, \qquad \rho_V : V \otimes I \simeq V, \\ \sigma_{U,V} &: U \otimes V \simeq V \otimes U. \end{aligned}$$

Let $(-)^* : V \mapsto V^*$ be the usual vector space dual, with duality denoted by $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}$. We will use the canonical identification $V^{**} = V$ and $(V_1 \otimes V_2)^* = V_1^* \otimes V_2^*$. With this duality, FinVect is compact closed. This means that for each object V, there are morphisms $\eta_V : I \to V^* \otimes V$ (the "cup") and $\epsilon_V : V \otimes V^* \to I$ (the "cap") such that the following snake identities hold:

$$(\epsilon_V \otimes V) \circ (V \otimes \eta_V) = V, \qquad (V^* \otimes \epsilon_V) \circ (\eta_V \otimes V^*) = V^*, \tag{1}$$

here we denote the identity map on the object V by V. Let us identify these morphisms. First, η_V is a linear map $\mathbb{R} \to V^* \otimes V$, which can be identified with the element $\eta_V(1) \in V^* \otimes V$ and $\epsilon_V \in (V \otimes V^*)^* = V^* \otimes V$ is again an element of the same space. Choose a basis $\{e_i\}$ of V, let $\{e_i^*\}$ be the dual basis of V^*, that is, $\langle e_i^*, e_j \rangle = \delta_{i,j}$. Let us define

$$\eta_V(1) = \epsilon_V := \sum_i e_i^* \otimes e_i.$$

It is easy to see that this definition does not depend on the choice of the basis, indeed, ϵ_V is the linear functional on $V \otimes V^*$ defined by

$$\langle \epsilon_V, x \otimes x^* \rangle = \langle x^*, x \rangle, \qquad x \in V,\ x^* \in V^*.$$

It is also easily checked that the snake identities (1) hold.

For two objects V and W in FinVect, we will denote the set of all morphisms (i.e. linear maps) $V \to W$ by $\mathrm{FinVect}(V, W)$. Then $\mathrm{FinVect}(V, W)$ is itself a real linear space and we have the well-known identification $\mathrm{FinVect}(V, W) \simeq V^* \otimes W$. This can be given as follows: for each $f \in \mathrm{FinVect}(V, W)$, we have $C_f := (V^* \otimes f)(\epsilon_V) = \sum_i e_i^* \otimes f(e_i) \in V^* \otimes W$. Conversely, since $\{e_i^*\}$

is a basis of V^*, any element $w \in V^* \otimes W$ can be uniquely written as $w = \sum_i e_i^* \otimes w_i$ for $w_i \in W$, and since $\{e_i\}$ is a basis of V, the assignment $f(e_i) := w_i$ determines a unique map $f : V \to W$. The relations between $f \in \mathrm{FinVect}(V, W)$ and $C_f \in V^* \otimes W$ can be also written as

$$\langle C_f, x \otimes y^* \rangle = \langle \epsilon_V, x \otimes f^*(y^*) \rangle = \langle f^*(y^*), x \rangle = \langle y^*, f(x) \rangle, \qquad x \in V,\ y^* \in W^*,$$

here $f^* : W^* \to V^*$ is the adjoint of f. Note that by compactness, the internal hom in FinVect satisfies $[V, W] \simeq V^* \otimes W$, so that in the case of FinVect, the object $[V, W]$ can be identified with the space of linear maps $\mathrm{FinVect}(V, W)$.

We now present two examples that are most important for us.

Example 1. Let $V = \mathbb{R}^N$. In this case, we fix the canonical basis $\{|i\rangle,\ i = 1, \dots, N\}$. We will identify $(\mathbb{R}^N)^* = \mathbb{R}^N$, with duality $\langle x, y \rangle = \sum_i x_i y_i$, in particular, we identify $I = I^*$. We then have $\epsilon_V = \sum_i |i\rangle \otimes |i\rangle$ and if $f : \mathbb{R}^N \to \mathbb{R}^M$ is given by the matrix A in the two canonical bases, then $C_f = \sum_i |i\rangle \otimes A|i\rangle$ is the vectorization of A.

Example 2. Let $V = M_n^h$ be the space of $n \times n$ complex hermitian matrices. We again identify $(M_n^h)^* = M_n^h$, with duality $\langle A, B \rangle = \mathrm{Tr}\, A^T B$, where A^T is the usual transpose of the matrix A. Let us choose the basis in M_n^h, given as

$$\left\{ |j\rangle\langle k| + |k\rangle\langle j|,\ j \le k,\ i\Big(|j\rangle\langle k| - |k\rangle\langle j| \Big),\ j < k \right\}.$$

Then one can check that

$$\left\{ \frac{1}{2}\Big(|j\rangle\langle k| + |k\rangle\langle j| \Big),\ j \le k,\ \frac{i}{2}\Big(|k\rangle\langle j| - |j\rangle\langle k| \Big),\ j < k \right\}$$

is the dual basis and we have

$$\epsilon_V = \sum_{j,k} |j\rangle\langle k| \otimes |j\rangle\langle k|.$$

For any $f : M_n^h \to M_m^h$,

$$C_f = \sum_{j,k} |j\rangle\langle k| \otimes f(|j\rangle\langle k|)$$

is the Choi matrix of f.

1.2 The category Af

We now introduce the category Af, whose objects are of the form $X = (V_X, A_X)$, where V_X is an object in FinVect and $A_X \subsetneq V_X$ is a proper affine subspace, see Appendix B for definitions and basic properties. Morphisms $X \xrightarrow{f} Y$ in Af are linear maps $f : V_X \to V_Y$ such that $f(A_X) \subseteq A_Y$. For any object X, we put

$$L_X := \mathrm{Lin}(A_X), \quad S_X := \mathrm{Span}(A_X), \quad D_X = \dim(V_X), \quad d_X = \dim(L_X).$$

We have

$$A_X = a + L_X = S_X \cap \{\tilde{a}\}^{\sim}, \tag{2}$$

for any choice of elements $a \in A_X$ and $\tilde{a} \in \tilde{A}_X$. We now introduce a tensor product and duality that endow Af with the structure of a *-autonomous category.

By Corollary 3, the dual $\tilde{A}_X$ is a proper affine subspace in V_X^*, so that $X^* := (V_X^*, \tilde{A}_X)$ is an object in Af. We have $X^{**} = X$ and the corresponding subspaces are related as

$$L_{X^*} = S_X^\perp, \qquad S_{X^*} = L_X^\perp. \tag{3}$$

It is easily seen that for any $X \xrightarrow{f} Y$, the adjoint map satisfies $f^*(\tilde{A}_Y) \subseteq \tilde{A}_X$, so that $Y^* \xrightarrow{f^*} X^*$ and the duality $(-)^*$ is a full and faithful functor $\mathrm{Af}^{op} \to \mathrm{Af}$.

We will next introduce a monoidal structure $\otimes$ as follows. For two objects X and Y, we put $V_{X\otimes Y} = V_X \otimes V_Y$ and construct the affine subspace $A_{X\otimes Y}$ as the affine span of

$$A_X \otimes A_Y = \{a \otimes b,\ a \in A_X,\ b \in A_Y\}.$$

Fix any $\tilde{a}_X \in \tilde{A}_X$ and $\tilde{a}_Y \in \tilde{A}_Y$. Since $A_X \otimes A_Y \subseteq \{\tilde{a}_X \otimes \tilde{a}_Y\}^\sim$, the affine span of $A_X \otimes A_Y$ is a proper affine subspace and we have by Lemma 24

$$A_{X\otimes Y} := \mathrm{Aff}(A_X \otimes A_Y) = \{A_X \otimes A_Y\}^{\approx}.$$

Lemma 1. *For any $a_X \in A_X$, $a_Y \in A_Y$, we have*

$$\begin{aligned} L_{X\otimes Y} &= \mathrm{Lin}(A_X \otimes A_Y) = \mathrm{Span}(\{x \otimes y - a_X \otimes a_Y,\ x \in A_X,\ y \in A_Y\}) && (4)\\ &= (a_X \otimes L_Y) + (L_X \otimes a_Y) + (L_X \otimes L_Y) && (5) \end{aligned}$$

(here + denotes the direct sum of subspaces). We also have

$$S_{X\otimes Y} = S_X \otimes S_Y.$$

Proof. The equality (4) follows from Lemma 24. For any $x \in A_X$, $y \in A_Y$ we have

$$x \otimes y - a_X \otimes a_Y = a_X \otimes (y - a_Y) + (x - a_X) \otimes a_Y + (x - a_X) \otimes (y - a_Y),$$

so that $L_{X\otimes Y} = \mathrm{Lin}(A_X \otimes A_Y)$ is contained in the subspace on the RHS of (5). Let d be the dimension of this subspace, then clearly

$$d_{X\otimes Y} \leq d \leq d_X + d_Y + d_X d_Y.$$

On the other hand, any element of S_X has the form tx for some $t \in \mathbb{R}$ and $x \in A_X$, so that it is easily seen that $S_X \otimes S_Y = S_{X\otimes Y}$. Hence

$$\begin{aligned} d_{X\otimes Y} &= \dim(L_{X\otimes Y}) = \dim(S_{X\otimes Y}) - 1 = \dim(S_X)\dim(S_Y) - 1 = (d_X + 1)(d_Y + 1) - 1\\ &= d_X + d_Y + d_X d_Y. \end{aligned}$$

This completes the proof.

□

Lemma 2. *Let $I = (\mathbb{R}, \{1\})$. Then $(\mathrm{Af}, \otimes, I)$ is a symmetric monoidal category.*

Proof. Note that this structure is inherited from the symmetric monoidal structure in FinVect. To show that $\otimes$ is a functor, we have to check that for $X_1 \xrightarrow{f} Y_1$ and $X_2 \xrightarrow{g} Y_2$ in Af, we have $X_1 \otimes Y_1 \xrightarrow{f\otimes g} X_2 \otimes Y_2$ which amounts to showing that

$$(f \otimes g)(A_{X_1\otimes Y_1}) \subseteq A_{X_2\otimes Y_2}.$$

Let $x \in A_{X_1}$, $y \in A_{Y_1}$, then $f(x) \otimes g(y) \in A_{X_2} \otimes A_{Y_2} \subseteq A_{X_2 \otimes Y_2}$. Since $A_{X_1 \otimes Y_1}$ is the affine subspace generated by $A_{X_1} \otimes A_{Y_1}$, the above inclusion follows by linearity of $f \otimes g$.

It only remains to prove that the associators, unitors and symmetries from FinVect are morphisms in Af. We will prove this for the associators $\alpha_{X,Y,Z} : V_X \otimes (V_Y \otimes V_Z) \to (V_X \otimes V_Y) \otimes V_Z$, the other proofs are similar. We need to check that $\alpha_{X,Y,Z}(A_{X \otimes (Y \otimes Z)}) \subseteq A_{(X \otimes Y) \otimes Z}$. It is easily checked that $A_{X \otimes (Y \otimes Z)}$ is the affine span of elements of the form $x \otimes (y \otimes z)$, $x \in A_X$, $y \in A_Y$ and $z \in A_Z$, and we have

$$\alpha_{X,Y,Z}(x \otimes (y \otimes z)) = (x \otimes y) \otimes z \in A_{(X \otimes Y) \otimes Z}$$

for all such elements. The desired inclusion follows by linearity.

□

Theorem 1. $(\mathrm{Af}, \otimes, I)$ *is a *-autonomous category, with duality* $(-)^*$, *such that* $I^* = I$.

Proof. By Lemma 2, we have that $(\mathrm{Af}, \otimes, I)$ is a symmetric monoidal category. We have also seen that the duality $(-)^*$ is a full and faithful contravariant functor. We only need to check the natural isomorphisms

$$\mathrm{Af}(X \otimes Y, Z^*) \simeq \mathrm{Af}(X, (Y \otimes Z)^*).$$

Since FinVect is compact, we have the natural isomorphisms

$$\mathrm{FinVect}(V_X \otimes V_Y, V_Z^*) \simeq \mathrm{FinVect}(V_X, V_Y^* \otimes V_Z^*),$$

determined by the equalities

$$\langle\, f(x \otimes y), z \,\rangle = \langle\, \hat{f}(x), y \otimes z \,\rangle, \qquad x \in V_X,\ y \in V_Y,\ z \in V_Z,$$

for $f \in \mathrm{FinVect}(V_X \otimes V_Y, V_Z^*)$ and the corresponding morphism $\hat{f} \in \mathrm{FinVect}(V_Z, V_Y^* \otimes V_Z^*)$. Since $A_{X \otimes Y}$ is an affine span of $A_X \otimes A_Y$, we see that f is in $\mathrm{Af}(X \otimes Y, Z^*)$ if and only if $f(x \otimes y) \in \tilde{A}_Z$ for all $x \in A_X$, $y \in A_Y$, that is,

$$1 = \langle\, f(x \otimes y), z \,\rangle = \langle\, \hat{f}(x), y \otimes z \,\rangle \qquad \forall x \in A_X,\ \forall y \in A_Y,\ \forall z \in A_Z.$$

But this is equivalent to

$$\hat{f}(x) \in (A_Y \otimes A_Z)^{\sim} = \tilde{A}_{Y \otimes Z}, \qquad \forall x \in A_X,$$

which means that $\hat{f} \in \mathrm{Af}(X, (Y \otimes Z)^*)$.

□

A *-autonomous category is compact closed if it satisfies $(X \otimes Y)^* = X^* \otimes Y^*$. In general, $X \odot Y = (X^* \otimes Y^*)^*$ defines a dual symmetric monoidal structure that is different from $\otimes$. We next show that Af is not compact.

Proposition 1. *For objects in* Af, *we have* $(X \otimes Y)^* = X^* \otimes Y^*$ *exactly in one of the following situations:*

(i) $X \simeq I$ *or* $Y \simeq I$,

(ii) $d_X = d_Y = 0$,

(iii) $d_{X^*} = d_{Y^*} = 0$.

Proof. Since FinVect is compact, we have $V_{(X\otimes Y)^*}=(V_X\otimes V_Y)^*=V_X^*\otimes V_Y^*=V_{X^*\otimes Y^*}$. It is also easily seen by definition that $A_{X^*}\otimes A_{Y^*}=\tilde{A}_X\otimes\tilde{A}_Y\subseteq\tilde{A}_{X\otimes Y}=A_{(X\otimes Y)^*}$, so that we always have $A_{X^*\otimes Y^*}\subseteq A_{(X\otimes Y)^*}$. Hence the equality holds if and only if $d_{X^*\otimes Y^*}=d_{(X\otimes Y)^*}$. From Lemma 1, we see that

$$d_{X^*\otimes Y^*}=d_{X^*}+d_{Y^*}+d_{X^*}d_{Y^*}.$$

On the other hand, we have using (3) that $L_{(X\otimes Y)^*}=S_{X\otimes Y}^{\perp}=(S_X\otimes S_Y)^{\perp}$, so that

$$d_{(X\otimes Y)^*}=D_XD_Y-\dim(S_X)\dim(S_Y)=D_XD_Y-(d_X+1)(d_Y+1).$$

Taking into account that by (3) we have $d_{X^*}=D_X-d_X-1$, similarly for d_{Y^*}, we obtain

$$d_{(X\otimes Y)^*}-d_{X^*\otimes Y^*}=d_Xd_{Y^*}+d_{X^*}d_Y.$$

This is equal to 0 iff $d_Xd_{Y^*}=d_Yd_{X^*}=0$, which amounts to the conditions in the lemma.

□

In a *-autonomous category, the internal hom can be identified as $[X,Y]=(X\otimes Y^*)^*$. The underlying vector space is $V_{[X,Y]}=(V_X\otimes V_Y^*)^*=V_X^*\otimes V_Y$ and we have seen in Section 1.1 that we may identify this space with $\mathrm{FinVect}(V_X,V_Y)$, by $f\leftrightarrow C_f$. This property is extended to Af, in the following sense.

Proposition 2. *For any objects X,Y in* Af*, the map $f\mapsto C_f$ is a bijection of* $\mathrm{Af}(X,Y)$ *onto $A_{[X,Y]}$. In particular, $\tilde{A}_X$ can be identified with* $\mathrm{Af}(X,I)$*.*

Proof. Let $f\in\mathrm{FinVect}(V_X,V_Y)$. Since by definition $A_{[X,Y]}=\tilde{A}_{X\otimes Y^*}=(A_X\otimes A_Y^*)^{\sim}$, we see that $C_f\in A_{[X,Y]}$ if and only if for all $x\in A_X$ and $y^*\in\tilde{A}_Y$, we have

$$1=\langle\, C_f,x\otimes y^*\,\rangle=\langle\, y^*,f(x)\,\rangle.$$

This latter statement is clearly equivalent to $f(A_X)\subseteq A_Y$, so that $f\in\mathrm{Af}(X,Y)$.

□

In the next result, we restrict the objects to spaces of hermitian matrices, as in Example 2 and morphisms to completely positive maps. We show that this restriction amounts to taking an intersection of $A_{[X,Y]}$ with the cone of positive semidefinite matrices. This, and subsequent examples, shows that for characterization of sets of quantum objects such as states, channels, combs and transformations between them, it is enough to work with the category Af.

An object X of Af will be called quantum if $V_X=M_n^h$ for some n and A_X is an affine subspace such that both A_X and $\tilde{A}_X$ contain a positive multiple of the identity matrix E_n[1]. (recall that we identify $(M_n^h)^*=M_n^h$).

Proposition 3. *Let X, Y be quantum objects in* Af*. Then*

(i) X^* and $X\otimes Y$ are quantum objects as well.

(ii) Let $V_X=M_n^h$, $V_Y=M_m^h$. Then for any $f\in\mathrm{FinVect}(M_n^h,M_m^h)$, we have $C_f\in A_{[X,Y]}\cap M_{mn}^+$ if and only if f is completely positive and

$$f(A_X\cap M_n^+)\subseteq A_Y\cap M_m^+.$$

[1]We use the notation E_n, and not I_n, to avoid the slight chance that it might be confused with the monoidal unit.

Proof. The statement (i) is easily seen from $A_X \otimes A_Y \subseteq A_{X\otimes Y}$ and $\tilde{A}_X \otimes \tilde{A}_Y \subseteq \tilde{A}_{X\otimes Y}$. To show (ii), let $C_f \in A_{[X,Y]} \cap M^+_{mn}$. By the properties of the Choi isomorphism f is completely positive and by Proposition 2, $f(A_X) \subseteq A_Y$, this proves one implication. For the converse, note that we only need to prove that under the given assumptions, $f(A_X) \subseteq A_Y$, for which it is enough to show that $A_X \subseteq \mathrm{Aff}(A_X \cap M^+_n)$. To see this, let $c_X E_n \in A_X$ for $c_X > 0$. Any element in A_X can be written in the form $c_X E_n + v$ for some $v \in L_X$. Since $c_X E_n \in int(M^+_n)$, there is some $s > 0$ such that $a_\pm := c_X E_n \pm sv \in M^+_n$, and since $\pm sv \in L_X$, we see that $a_\pm \in A_X \cap M^+_n$. It is now easily checked that

$$c_X E_n + v = \frac{1+s}{2s} a_+ + \frac{s-1}{2s} a_- \in \mathrm{Aff}(A_X \cap M^+_n).$$

□

We can define classical objects in Af in a similar way, replacing M^h_n by $\mathbb{R}^N$ and the positive cone by $\mathbb{R}^N_+$, and we require that both A_X and $\tilde{A}_X$ contains a positive multiple of the unit vector $e_N = (1, \ldots, 1) \in \mathbb{R}^N$. A similar statement holds in this case, with complete positivity replaced by positivity. We can similarly treat classical-to-quantum and quantum-to-classical maps as morphisms between these types of objects, satisfying appropriate positivity assumptions.

Example 3 (Higher order quantum maps). The basic example of a quantum object corresponds to the set of quantum states. Let

$$\mathcal{A}_n := \{T \in M^h_n,\ \mathrm{Tr}\,[T] = 1\}.$$

Then $\mathcal{S}_n := (M^h_n, \mathcal{A}_n)$ is an object in Af, and it is a quantum object, since we have $E_n \in \tilde{\mathcal{A}}_n = \{E_n\}$ and $\frac{1}{n} E_n \in \mathcal{A}_n$. The set $\mathcal{A}_n \cap M^+_n$ is the set of quantum states. By Proposition 2, $\mathcal{C}_{m,n} := [\mathcal{S}_m, \mathcal{S}_n]$ is a quantum object as well, such that the corresponding vector space is M^h_{mn} and $A_{\mathcal{C}_{m,n}} \cap M^+_{mn}$ is the set of Choi matrices of quantum channels $M_m \to M_n$. Note that the dual object $\mathcal{C}^*_{m,n} = \mathcal{S}_m \otimes \mathcal{S}^*_n$ represents the set of Choi matrices of replacement channels $\mathcal{S}_n \to \mathcal{S}_m$, that is, channels that map any state in M_n to a fixed state in M_m. We also have $\mathcal{S}_n = [\mathcal{S}_n, I] = \mathcal{C}_{n,1}$.

The set of higher order quantum maps is constructed inductively from the channel objects $\mathcal{C}_{m,n}$ by applying the internal hom $[\cdot,\cdot]$. For example, $[\mathcal{C}_{m,n}, \mathcal{C}_{k,l}]$ is a quantum object, corresponding to the set of Choi matrices of quantum superchannels, or 2-combs, etc. Note that in this way we always obtain quantum objects. The corresponding affine subspaces are identified using (2) with a and $\tilde{a}$ replaced by the appropriate multiple of the identity, together with (3) and Lemma 1.

Example 4 (Partially classical maps). We may similarly define the basic classical object as

$$\mathcal{P}_N := (\mathbb{R}^N, \{x,\ \sum_i x_i = 1\}).$$

In this case, $\mathcal{A}_{\mathcal{P}_N} \cap \mathbb{R}^N_+$ is the probability simplex. We then obtain further useful objects by combining with the quantum objects. For example, it can be easily seen that $[\mathcal{S}_n, \mathcal{P}_N]$ corresponds to N-outcome measurements, $[\mathcal{S}_m, \mathcal{S}_n \otimes \mathcal{P}_N]$ to N-outcome quantum instruments, $[\mathcal{S}_m \otimes \mathcal{P}_N, \mathcal{P}_M]$ to quantum multimeters, etc.

1.3 First order and higher order objects

We say that an object X in Af is first order if $d_X = D_X - 1$, equivalently, $S_X = V_X$. Another equivalent condition is $d_{X^*} = 0$, which means that A_X is determined by a single element $\tilde{a}_X \in V^*_X$ as

$$A_X = \{\tilde{a}_X\}^\sim, \qquad \tilde{A}_X = \{\tilde{a}_X\}.$$

Note that first order objects, resp. their duals, are exactly those satisfying condition (iii), resp. condition (ii), in Proposition 1, in particular, $(X \otimes Y)^* = X^* \otimes Y^*$ for first order objects X and Y.

Higher order objects in Af are objects obtained from a finite set $\{X_1, \ldots, X_n\}$ of first order objects by taking tensor products and duals. The above is indeed a set, so that all the objects are different (though they may be isomorphic) and the ordering is not essential. We will also assume that the monoidal unit I is not contained in this set. By definition of $[X, Y]$, and since we may dentify $[X, I]$ with X^*, we see that higher order objects are also generated by applying the internal hom inductively on $\{X_1, \ldots, X_n\}$ if we allow $X_i = I$ for some i. It follows that the "higher order quantum maps" in Example 3 are indeed higher order objects in Af according to the above definition.

Of course, any first order object is also higher order with $n = 1$. Note that we cannot say that a higher order object generated from $\{X_1, \ldots, X_n\}$ is automatically "of order n", as the following lemma shows.

Lemma 3. *Let X, Y be first order, then $X \otimes Y$ is first order as well.*

Proof. We have $S_{X\otimes Y} = S_X \otimes S_Y = V_X \otimes V_Y = V_{X\otimes Y}$.

□

As we have seen, higher order objects are obtained by applying the internal hom iteratively. The following properties of such iterations are easily seen from the definition of $[\cdot, \cdot]$.

Lemma 4. *Let X, Y, Z be any objects in* Af. *Then we have*

(i) $[Z, [X, Y]] \simeq [X, [Z, Y]]$.

(ii) If $X = (V_X, \{\tilde{a}_X\}^\sim)$ and $Y = (V_Y, \{\tilde{a}_Y\}^\sim)$ are first order, then $[Z, [X, Y]]$ is determined as

$$A_{[Z,[X,Y]]} = \{w \in V_Z^* \otimes V_X^* \otimes V_Y, (id \otimes \tilde{a}_Y)(w) \in A_Z^* \otimes \tilde{a}_X\}.$$

Example 5 (Combs). The higher order objects called n-combs are constructed inductively as follows. 1-combs, or channels, are objects of the form $[X, Y]$, with first order objects X and Y. An n-comb is an object of the form $[C_{n-1}, [X, Y]]$, where C_{n-1} is an $n-1$-comb and X, Y are first order objects. Using Lemma 4, we see that an n-comb has the form

$$[X_{2n-1}, [[X_{2n-3}, \ldots, [[X_1, X_2], X_4]], \ldots, X_{2n}]]$$

for first order objects $X_1, X_2, \ldots, X_{2n}$. For quantum objects, we see that an n-comb describes the sets of n-combs introduced in , see Example 3 (We slightly abuse the terminology here).

Since $[X, Y] \simeq [I, [X, Y]]$, we can determine the affine subspace of the channel object $[X, Y]$ using Lemma 4(ii) as

$$A_{[X,Y]} = \{w \in V_X^* \otimes V_Y,\ (id \otimes \tilde{a}_Y)(w) = \tilde{a}_X\}.$$

The subspace A_{C_n} for an n-comb C_n can be found inductively.

2 Combinatorial description of higher order objects

In this section, we discuss a combinatiorial description of higher order objects similar to that of [Pavia]. We will use the definitions and results given in Appendix A.4.

For a first order object $X = (V_X, \{\tilde a_X\}^\sim)$, let us pick an element $a_X \in A_X$. We have a direct sum decomposition

$$V_X = L_{X,0} \oplus L_{X,1},$$

where $L_{X,0} := \mathbb{R}\{a_X\}$, $L_{X,1} := \{\tilde a_X\}^\perp = L_X$. We define the *conjugate object* as $\tilde X = (V_X^*, \{a_X\}^\sim)$. Note that we always have $\tilde a_X \in A_{\tilde X}$ and with the choice $a_{\tilde X} = \tilde a_X$, we obtain $\tilde{\tilde X} = X$ and

$$L_{\tilde X,u} = L^\perp_{X,1-u}, \qquad u \in \{0,1\}. \tag{6}$$

These definitions depend on the choice of a_X, but we will assume below that this choice is fixed and that we choose $a_{\tilde X} = \tilde a_X$. Since we will always work with a finite set of objects at a time, this will not create any problems.

A first order quantum or classical object is the set of states $\mathcal{S}_n$ or the set of probability distributions $\mathcal{P}_N$, see Examples 3, 4. In these cases, a_X will be chosen as the appropriate multiple of the identity. Note that then

$$L_{X,0} = L_{\tilde X,0} = \mathbb{R}\{E_n\}, \qquad L_{X,1} = L_{\tilde X,1} = \mathcal{T}_n := \{T \in M_n^h, \mathrm{Tr}\,[T] = 0\}$$

(similarly for $\mathcal{P}_N$).

Let $X_1, \dots, X_n$ be first order objects in Af. Let $a_{X_i} \in A_{X_i}$ be fixed and let $\tilde X_i$ be the conjugate first order objects. Let us denote $V_i = V_{X_i}$ and

$$L_{i,u} := L_{X_i,u}, \qquad \tilde L_{i,u} := L_{\tilde X_i,u} \qquad u \in \{0,1\},\ i \in [n].$$

For a string $s \in \{0,1\}^n$, we define

$$L_s := L_{1,s_1} \otimes \cdots \otimes L_{n,s_n}, \qquad \tilde L_s := \tilde L_{1,s_1} \otimes \cdots \otimes \tilde L_{n,s_n},$$

then we have the direct sum decompositions

$$V := V_1 \otimes \cdots \otimes V_n = \sum_{s\in\{0,1\}^n} L_s, \qquad V^* = V_1^* \otimes \cdots \otimes V_n^* = \sum_{s\in\{0,1\}^n} \tilde L_s$$

(here $\sum$ denotes the direct sum).

Lemma 5. *For any $s \in \{0,1\}^n$, we have*

$$L_s^\perp = \sum_{t\in\{0,1\}^n} (1-\chi_s(t))\tilde L_t, \qquad \tilde L_s^\perp = \sum_{t\in\{0,1\}^n} (1-\chi_s(t)) L_t.$$

Here $\chi_s : \{0,1\}^n \to \{0,1\}$ is the characteristic function of s.

Proof. Using (6) and the direct sum decomposition of V_i^*, we get

$$
\begin{aligned}
(L_{1,s_1}\otimes\cdots\otimes L_{n,s_n})^\perp &= \bigvee_j \left(V_1^*\otimes\cdots\otimes V_{j-1}^*\otimes \tilde L_{j,1-s_j}\otimes V_{j+1}^*\otimes\cdots\otimes V_n^*\right)\\
&= \bigvee_j \left(\sum_{\substack{t\in\{0,1\}^n\\ t_j\neq s_j}} \tilde L_{1,t_1}\otimes\cdots\otimes\tilde L_{n,t_n}\right)\\
&= \sum_{\substack{t\in\{0,1\}^n\\ t\neq s}} \left(\tilde L_{1,t_1}\otimes\cdots\otimes\tilde L_{n,t_n}\right).
\end{aligned}
$$

The proof of the other equality is the same. □

Lemma 6. *Put $a := a_1\otimes\cdots\otimes a_n$, $\tilde a := \tilde a_1\otimes\cdots\otimes\tilde a_n$. For $f\in\mathcal F_n$ define*

$$
S_f = S_f(X_1,\dots,X_n) := \sum_{s\in\{0,1\}^n} f(s)L_s, \qquad A_f = A_f(X_1,\dots,X_n) := S_f\cap\{\tilde a\}^\sim.
$$

Then A_f is a proper affine subspace in V containing a. Moreover,

$$
\operatorname{Lin}(A_f) = \sum_{s\in\{0,1\}^n\setminus\{\theta_n\}} f(s)L_s, \qquad \operatorname{Span}(A_f) = S_f.
$$

The map $f\mapsto A_f$ is injective and has the following further properties.

(i) The dual affine subspace satisfies

$$
\tilde A_f(X_1,\dots,X_n) = A_{f^*}(\tilde X_1,\dots,\tilde X_n).
$$

(ii) Let $\sigma\in\mathscr S_n$ and let the corresponding symmetry $\otimes_i V_i\to\otimes_i V_{\sigma^{-1}(i)}$ be also denoted by σ. Then we have

$$
A_f(X_{\sigma(1)},\dots,X_{\sigma(n)}) = \sigma^{-1}(A_{f\circ\sigma}(X_1,\dots,X_n)).
$$

(iii) Let $f_1\in\mathcal F_{n_1}$, $f_2\in\mathcal F_{n_2}$, $n_1+n_2=n$. Then

$$
S_{f_1\otimes f_2}(X_1,\dots,X_n) = S_{f_1}(X_1,\dots,X_{n_1})\otimes S_{f_2}(X_{n_1+1},\dots,X_n)
$$

Proof. It is clear from definition that A_f is an affine subspace. Since $f(\theta_n)=1$, the space S_f always contains the subspace $L_0 = L_{1,0}\otimes\cdots\otimes L_{n,0} = \mathbb R\{a\}$ and it is clear that $L_s\subseteq\{\tilde a\}^\perp$ for any $s\neq\theta_n$. It follows that $a\in A_f$, so that $A_f\neq\emptyset$, and since $A_f\subseteq\{\tilde a\}^\sim$, we see that A_f is proper and $\tilde a\in\tilde A_f$. The expressions for $\operatorname{Lin}(A_f)$ and $\operatorname{Span}(A_f)$ are immediate from the definition and (16).

Injectivity of the map $f\mapsto A_f$ is clear from the fact that L_s, $s\in\{0,1\}$ is an independent decomposition. To prove (i), we compute using Lemma 5 and the fact that the subspaces form an independent decomposition,

$$
\begin{aligned}
\operatorname{Span}(\tilde A_f) = \operatorname{Lin}(A_f)^\perp = \left(\sum_{s\in\{0,1\}^n\setminus\{0\}} f(s)L_s\right)^\perp = \bigwedge_{\substack{s\in\{0,1\}^n\\ s\neq 0, f(s)=1}} L_s^\perp = \bigwedge_{\substack{s\in\{0,1\}^n\\ s\neq 0, f(s)=1}} \left(\sum_{t\in\{0,1\}^n} (1-\chi_s(t))\tilde L_t\right)\\
= \sum_{t\in\{0,1\}^n}\left(\bigwedge_{\substack{s\in\{0,1\}^n\\ s\neq 0, f(s)=1}} (1-\chi_s(t))\tilde L_t\right) = \sum_{t\in\{0,1\}^n} f^*(t)\tilde L_t.
\end{aligned}
$$

To see the last equality, note that

$$\bigwedge_{\substack{s\in\{0,1\}^n\\ s\neq 0, f(s)=1}} (1-\chi_s(t)) = \begin{cases} 1 & \text{if } t=\theta_n \\ 1-f(t) & \text{if } t\neq\theta_n \end{cases} = f^*(t).$$

To show (ii), compute

$$\begin{aligned}\sigma(S_f(X_{\sigma(1)},\dots,X_{\sigma(n)})) &= \sigma(\sum_s f(s) L_{\sigma(1),s_1}\otimes\cdots\otimes L_{\sigma(n),s_n}) \\ &= \sum_s f(s) L_{1,s_{\sigma^{-1}(1)}}\otimes\cdots\otimes L_{n,s_{\sigma^{-1}(n)}} = S_{f\circ\sigma}(X_1,\dots,X_n).\end{aligned}$$

It folows that

$$A_f(X_{\sigma(1)},\dots,X_{\sigma(n)}) = S_f(X_{\sigma(1)},\dots,X_{\sigma(n)})\cap\{\sigma^{-1}(\tilde{a})\}^{\sim} = \sigma^{-1}(A_{f\circ\sigma}(X_1,\dots,X_n)).$$

The statement (iii) is easily seen from the definitions. □

Since all the affince subspaces $A_f\subseteq V$ are proper, we may form the objects

$$X_f = X_f(X_1,\dots,X_n) := (V, A_f(X_1,\dots,X_n))$$

in Af. The following properties follow easily from the above Lemma.

Proposition 4. *Let $X_1,\dots,X_n$ be first order objects. The map $\mathcal{F}_n\ni f\mapsto X_f(X_1,\dots,X_n)$ is injective and we have*

(i) For the least and the largest element in $\mathcal{F}_n$,

$$X_{p_{[n]}} = \tilde{X}_1^*\otimes\cdots\otimes\tilde{X}_n^*, \qquad X_{1_n} = X_1\otimes\cdots\otimes X_n,$$

(ii) $X_f^*(X_1,\dots,X_n) = X_{f^*}(\tilde{X}_1,\dots,\tilde{X}_n)$,

(iii) $X_{f_1\otimes f_2}(X_1,\dots,X_n) = X_{f_1}(X_1,\dots,X_{n_1})\otimes X_{f_2}(X_{n_1+1},\dots,X_n)$,

(iv) the symmetry $\sigma\in\mathscr{S}_n$ is an isomorphism $X_f(X_{\sigma(1)},\dots,X_{\sigma(n)})\xrightarrow{\sigma} X_{f\circ\sigma}(X_1,\dots,X_n)$.

It follows from the independence of L_s, $s\in\{0,1\}^n$, that the subspaces S_f form a distributive sublatice in the lattice of subspaces of V and we clearly have $f\leq g$ if and only if $S_f\subseteq S_g$ and $S_{f\wedge g}=S_f\cap S_g$, $S_{f\vee g}=S_f\vee S_g$. The following proposition shows the corresponding properties of X_f, in categorical terms. We skip the easy proof.

Proposition 5. *Let $f,g,h\in\mathcal{F}_n$.*

(i) $f\leq g$ if and only if $X_f\xrightarrow{id_V} X_g$ in Af.

(ii) Let $k\leq f,g\leq h$ then the following diagram is a pullback resp. pushout:

$$\begin{tikzcd} X_{f\wedge g} \arrow[r, "id_V"] \arrow[d, "id_V"'] & X_f \arrow[d, "id_V"] \\ X_g \arrow[r, "id_V"'] & X_h \end{tikzcd} \qquad \begin{tikzcd} X_h \arrow[r, "id_V"] \arrow[d, "id_V"'] & X_f \arrow[d, "id_V"] \\ X_g \arrow[r, "id_V"'] & X_{f\vee g} \end{tikzcd}$$

Our goal is to show that the higher order objects are precisely those of the form $Y = X_f(X_1, \ldots, X_n)$ for some choice of the first order objects $X_1, \ldots, X_n$ and a function f that belongs to a special subclass $\mathcal{T}_n \subseteq \mathcal{F}_n$. The elements of this subclass will be called the *type functions*, or *types*, and are defined as those functions in $\mathcal{F}_n$ that can be obtained by taking the constant function 1_1 in each coordinate and then repeatedly applying duals and tensor products of such functions in any order. The set of indices for which the corresponding coordinate was subjected to taking the dual an even number of times will be called the *outputs* (of f) and denoted by $O = O_f$, indexes in $I = I_f := [n] \setminus O_f$ will be called *inputs*. The reason for this terminology will become clear later. It is easy to observe that if $f \in \mathcal{T}_n$, then $O_{f^*} = I_f$ and $I_{f^*} = O_f$. Further, for $f_1 \in \mathcal{T}_{n_1}$, $f_2 \in \mathcal{T}_{n_2}$, we have $O_{f_1 \otimes f_2} = O_{f_1} \oplus O_{f_2}$ and $I_{f_1 \otimes f_2} = I_{f_1} \oplus I_{f_2}$, see (13) for the definition.

We have the following description of the sets of type functions.

Proposition 6. *The set $\mathcal{T}_n$ is the smallest subset in $\mathcal{F}_n$ such that:*

1. *$\mathcal{T}_1 = \mathcal{F}_1$,*
2. *For $n_1 + n_2 = n$, $\mathcal{T}_{n_1} \otimes \mathcal{T}_{n_2} \subseteq \mathcal{T}_n$,*
3. *$\mathcal{T}_n$ is invariant under permutations: if $f \in \mathcal{T}_n$, then $f \circ \sigma \in \mathcal{T}_n$ for any $\sigma \in \mathscr{S}_n$,*
4. *$\mathcal{T}_n$ is invariant under complementation: if $f \in \mathcal{T}_n$ then $f^* \in \mathcal{T}_n$.*

Proof. It is clear by construction that any system of subsets $\{\mathcal{S}_n\}_n$ with these properties must contain the type functions and that $\{\mathcal{T}_n\}_n$ itself has these properties.

$\square$

Assume that Y is a higher order object constructed from a set of distinct first order objects $Y_1, \ldots, Y_n$, $Y_i = (V_{Y_i}, \{\tilde{a}_{Y_i}\}^\sim)$. Let us fix elements $a_{Y_i} \in A_{Y_i}$ and construct the conjugate objects $\tilde{Y}_i$. By compactness of FinVect, we may assume (relabeling the objects if necessary) that the vector space of Y has the form

$$V_Y = V := V_1 \otimes \cdots \otimes V_n,$$

where V_i is either V_{Y_i} or $V_{Y_i}^*$, according to whether Y_i was subjected to taking duals an even or odd number of times. Similarly as for the type functions, the indices such that the first case is true will be called the outputs and the subset of outputs in $[n]$ will be denoted by O, or O_Y, when we need to specify the object. The set $I = I_Y := [n] \setminus O_Y$ is the set of inputs.

Theorem 2. *Let Y be a higher order object, constructed from first order objects $Y_1, \ldots, Y_n$. For $i \in [n]$, let $X_i = Y_i$ if $i \in O_Y$ and $X_i = \tilde{Y}_i$ for $i \in I_Y$. There is a unique function $f \in \mathcal{T}_n$, with $O_f = O_Y$, such that*

$$Y = X_f = (V, A_f(X_1, \ldots, X_n)).$$

Conversely, let $X_1, \ldots, X_n$ be first order objects and let $f \in \mathcal{T}_n$. Then $Y = X_f$ is a higher order object with $O_Y = O_f$, with underlying first order objects $Y_1, \ldots, Y_n$, where $Y_i = X_i$ for $i \in O_f$ and $Y_i = \tilde{X}_i$ for $i \in I_f$.

Proof. Since the map $f \mapsto X_f$ is injective, uniqueness is clear. To show existence of this function, we will proceed by induction on n. For $n = 1$, the assertion is easily seen to be true, since in this case, we we have either $Y = Y_1$ or $Y = Y_1^*$. In the first case, $O = [1]$, $X_1 = Y_1$ and

$$S_Y = V_Y = V_1 = L_{1,0} \oplus L_{1,1} = 1(0)L_{1,0} \oplus 1(1)L_{1,1},$$

so in this case $f \in \mathcal{T}_1$ is the constant 1. If $Y = Y_1^*$, we have $O = \emptyset$, $X_1 = \tilde{Y}_1$, and then

$$S_Y = \mathbb{R}\{\tilde{a}_{Y_1}\} = L_{1,0} = 1^*(0)L_{1,0} \oplus 1^*(1)L_{1,1},$$

so that $f = 1^* = p_{[1]} \in \mathcal{T}_1$. It is clear that $_f = O_Y$ in both cases.

Assume now that the assertion is true for all $m < n$. By construction, Y is either the tensor product $Y = Z_1 \otimes Z_2$, with Z_1 constructed from $Y_1, \dots, Y_m$ and Z_2 from $Y_{m+1}, \dots, Y_n$, or Y is the dual of such a product. Let us assume the first case. It is clear that $O_{Z_1} \oplus O_{Z_2} = O_Y$, and similarly for I, so that the corresponding objects $X_1, \dots, X_m$ and $X_{m+1}, \dots, X_n$ remain the same. By the induction assumption, there are functions $f_1 \in \mathcal{T}_m$ and $f_2 \in \mathcal{T}_{n-m}$ such that $O_{f_1} = O_{Z_1}$, $O_{f_2} = O_{Z_2}$ and, by Proposition 4(iii),

$$Y = Z_1 \otimes Z_2 = X_{f_1}(X_1, \dots, X_m) \otimes X_{f_2}(X_{m+1}, \dots, X_n) = X_{f_1 \otimes f_2}(X_1, \dots, X_n)$$

This implies the assertion, with $f = f_1 \otimes f_2 \in \mathcal{T}_n$ and $O_f = O_{f_1} \oplus O_{f_2} = O_Y$. To finish the proof, it is now enough to observe that if the assertion holds for Y then it also holds for Y^*. So assume that $Y = X_f(X_1, \dots, X_n)$ for some $f \in \mathcal{T}_n$, then by Proposition 4(ii), $Y^* = X_f^* = \tilde{X}_{f^*}(\tilde{X}_1, \dots, \tilde{X}_n)$. By the construction of conjugate objects, we have $\tilde{X}_i = \tilde{\tilde{Y}}_i = Y_i$ if $i \in I_Y$ and $\tilde{X}_i = \tilde{Y}_i$ if $i \in O_Y$. Since by definition and the assumption, $O_{Y^*} = I_Y = I_f = O_{f^*}$, this proves the statement.

The converse is proved by a similar induction argument, using Proposition 4.

$\square$

Let us stress that in general, the objects X_f depend on the choice of the elements a_{X_i}. From the above proof, it is clear that the description in Theorem 2 does not depend on the choice of the elements $a_{Y_i} \in A_{Y_i}$. Furthermore, if all the first order objects are quantum, we have $S_f(X_1, \dots, X_n) = S_f(\tilde{X}_1, \dots, \tilde{X}_n)$ and both a and $\tilde{a}$ are some, possibly different, multiple of the identity. The spaces $A_f(X_1, \dots, X_n)$ and $A_f(\tilde{X}_1, \dots, \tilde{X}_n)$ differ only by this multiple.

3 The type functions

The aim of this section is to gain some understanding into the structure and properties of the set of types. We start by an important example.

Example 6. Let $T \subseteq [n]$. It is easily seen that the function p_T (see Appendix A.4) is a type function, since we have

$$p_T(s) = \Pi_{j \in T}(1 - s_j) = \Pi_{j \in T} 1^*(s_j).$$

By definition, T is the set of inputs for p_T. Let $\mathcal{S} = \{X_1, \dots, X_n\}$ be a set of first order objects. Let $k = |T|$ and let $\sigma \in \mathscr{S}_n$ bw such that $p_T \circ \sigma = p_{[k]} \otimes 1_{n-k}$. By Proposition 4, it follows that we have the isomorphism

$$X_{p_T}(X_1, \dots, X_n) \xrightarrow{\sigma} X_{p_{[k]} \otimes 1_{n-k}}(X_{\sigma^{-1}(1)}, \dots, X_{\sigma^{-1}(n)}) = \tilde{X}_T^* \otimes X_{[n] \setminus T},$$

here $\tilde{X}_T = \otimes_{j \in T} \tilde{X}_j$ and $X_{[n] \setminus T} = \otimes_{j \in [n] \setminus T} X_j$ are first order object by Lemma 3. It follows that p_T describes replacement channels with set of input indices T. By duality, we obtain the isomorphisms

$$X_{p_T^*}(X_1, \dots, X_n) = X_{p_T}^*(\tilde{X}_1, \dots, \tilde{X}_n) \xrightarrow{\sigma} (X_T^* \otimes \tilde{X}_{[n] \setminus T})^* \xrightarrow{\rho} [\tilde{X}_{[n] \setminus T}, X_T],$$

where ρ denotes the symmetry given by the transposition in $\mathscr{S}_2$. It follows that $p_T^* = 1 - p_T + p_{[n]}$ corresponds to general channels with output indices T.

Lemma 7. *Let $f \in \mathcal{T}_n$ and let $O_f = O$, $I = I_f$. Then*

$$p_I \leq f \leq p_O^*.$$

Proof. This is obviously true for $n = 1$. Indeed, in this case, $\mathcal{T}_1 = \mathcal{F}_1 = \{1 = p_\emptyset, 1^* = p_{[1]}\}$. If $f = 1$, then $O = [1]$, $I = \emptyset$ and

$$p_I = p_\emptyset = 1 = p_O^*,$$

the case $f = 1^*$ is obtained by taking complements. Assume that the assertion holds for $m < n$. Let $f \in \mathcal{T}_n$ and assume that $f = g \otimes h$ for some $g \in \mathcal{T}_m$, $h \in \mathcal{T}_{n-m}$. By the assumption,

$$p_{I_g} \otimes p_{I_h} \leq g \otimes h \leq p_{O_g}^* \otimes p_{O_h}^* \leq (p_{O_g} \otimes p_{O_h})^*,$$

the last inequality follows from Lemma 22. With the decomposition $[n] = [m][m+1, n]$, we have $O_f = O_g \oplus O_h$, $I_f = I_g \oplus I_h$, so that by Lemma 23, $p_{O_f} = p_{O_g} \otimes p_{O_h}$ and similarly for p_{I_f}. Now notice that any $f \in \mathcal{T}_n$ is either of the form $(f \otimes g) \circ \sigma$ or of the form $(f \otimes g)^* \circ \sigma$, for some permutation σ. Since the inequality is easily seen to be preserved by permutations, and reversed by duality which also swiches the input and output sets, the assertion is proved.

□

Combining this with Proposition 4, we get the following result (cf. cite).

Corollary 1. *Let Y be a higher order objects constructed from first order objects $Y_1, \dots, Y_n$, $O_Y = O$, $I_Y = I$. Then there are $\sigma_1, \sigma_2 \in \mathscr{S}_n$ such that we have the morphisms*

$$Y_I^* \otimes Y_O \xrightarrow{\sigma_1} Y \xrightarrow{\sigma_2} [Y_I, Y_O].$$

We also obtain a simple way to identify the output indices of a type function.

Proposition 7. *For $f \in \mathcal{T}_n$, $j \in O_f$ if and only if $f(e^j) = 1$, here $e^j = \delta_{1,j} \dots \delta_{n,j}$.*

Proof. Let $i \in O_f$, then by Lemma 7, $p_{I_f}(e^i) = 1 \leq f(e^i)$, so that $f(e^i) = 1$. Conversely, if $f(e^i) = 1$, then by the other inequality in lemma 7, $p_{O_f}(e^i) = 0$, whence $i \in O_f$.

□

We now look at some examples and non-examples.

Example 7. The type functions for $n = 2$ are given as (writing $\bar{u} = 1 - u$ for $u \in \{0, 1\}$)

$$1_2(s) = 1, \quad p_{[2]}(s) = \bar{s}_1 \bar{s}_2, \quad p_{\{1\}}(s) = \bar{s}_1, \quad p_{\{1\}}^*(s) = 1 - \bar{s}_1 + \bar{s}_1 \bar{s}_2,$$

and functions obtained from these by permutation, which gives 6 elements. We have seen that $\mathcal{F}_n$ has $2^{2^n - 1}$ elements, so that $\mathcal{F}_2$ has 8 elements in total. The two of them that are not type functions are

$$g(s) = 1 - \bar{s}_1 - \bar{s}_2 + 2\bar{s}_1 s_2, \qquad g^*(s) = \bar{s}_1 + \bar{s}_2 - \bar{s}_1 \bar{s}_2.$$

This can be checked directly from Lemma 7 and Proposition 7. Indeed, if $g \in \mathcal{T}_2$, we would have $O_g = \emptyset$, so that $p_{[2]} \leq g \leq p_\emptyset^* = p_{[2]}$, which is obviously not the case. Clearly, also the complement $g^* \notin \mathcal{T}_2$. Notice also that $g^* = p_{\{1\}} \vee p_{\{2\}}$, so that $\mathcal{T}_2$ is not a lattice.

Since $\mathcal{F}_2$ can be identified as a sublattice in $\mathcal{F}_n$ for all $n \geq 2$ as $\mathcal{F}_2 \ni f \mapsto f \otimes 1_{n-2} \in \mathcal{F}_n$, the above example shows that $\mathcal{T}_n$, $n \geq 2$ is a subposet in the distributive lattice $\mathcal{F}_n$ but itself not a lattice, so that for $f_1, f_2 \in \mathcal{T}_n$, none of $f_1 \wedge f_2$ or $f_1 \vee f_2$ has to be a type function. Nevertheless, we have by the above results that all type functions with the same output indices are contained in the interval $p_I \leq f \leq p_O^*$, which is a distributive lattice. Elements of such an interval will be called *subtypes*. It is easily seen that for $n = 2$ all subtypes are type functions, but it is not difficult to find a subtype for $n = 3$ which is not in $\mathcal{T}_3$. The objects corresponding to subtypes are not necessarily higher order objects, but are embedded in $[Y_I, Y_O]$ and contain the replacement channels. If f_1 and f_2 have the same output set, then $f_1 \vee f_2$ and $f_1 \wedge f_2$ are subtypes. By Proposition **??**, the corresponding objects can be obtained by pushouts resp. pullbacks of the higher order objects corresponding to f_1 and f_2.

3.1 The poset $\mathcal{P}_f$

Here we introduce the basic tools for describing and visualising the structure of type functions. We will need the notions and results of Appendix **??**.

By Theorem 6, any boolean function has a unique expression of the form

$$f = \sum_{T \subseteq [n]} \hat{f}_T p_T,$$

where $\hat{f}$ is the Möbius transform of f. Let $\mathcal{P}_f$ be the subposet in the distributive lattice 2^n, of elements such that $\hat{f}_T \neq 0$. We show that any type function $f \in \mathcal{T}_n$ is fully determined by $\mathcal{P}_f$.

Proposition 8. *Let $f \in \mathcal{T}_n$, then $\mathcal{P}_f$ is a graded poset with even rank $k \leq n$ and rank function ρ_f. Moreover, we have*

$$f = \sum_{S \in \mathcal{P}_f} (-1)^{\rho_f(S)} p_S.$$

Then rank of $\mathcal{P}_f$ will be denoted by $r(f)$ and called the *rank* of f. Note that the assertion means that for $f \in \mathcal{T}_n$,

$$\hat{f}_S = \begin{cases} (-1)^{\rho_f(S)}, & \text{if } S \in \mathcal{P}_f \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We first note that the property in the statement is invariant under permutations and complements. Assume the statement holds for f and let us take any $\sigma \in \mathscr{S}_n$. From Proposition 14 we have that $\widehat{f \circ \sigma}_S = \hat{f}_{\sigma(S)}$ so that $S \mapsto \sigma(S)$ is an isomprphism of $\mathcal{P}_{f\circ\sigma}$ onto $\mathcal{P}_f$. Hence if $\mathcal{P}_f$ is graded with rank function ρ_f, then $\mathcal{P}_{f\circ\sigma}$ is graded with the same rank and has rank function $\rho_{f\circ\sigma} = \rho_f \circ \sigma$. By the assumption we have

$$f \circ \sigma = \sum_{S \in \mathcal{P}_f} (-1)^{\rho_f(S)} p_S \circ \sigma = \sum_{S \in \mathcal{P}_f} (-1)^{\rho_f(S)} p_{\sigma^{-1}(S)} = \sum_{S \in \mathcal{P}_{f\circ\sigma}} (-1)^{\rho_{f\circ\sigma}(S)} p_S.$$

For the complement, we have from the assumption and Proposition 14(ii) that

$$f^* = (1 - \hat{f}_\emptyset)1 - \sum_{\substack{S \in \mathcal{P}_f \\ \emptyset \neq S, [n] \neq S}} (-1)^{\rho_f(S)} p_S + (1 - \hat{f}_{[n]}) p_n. \tag{7}$$

If $\emptyset \in \mathcal{P}_f$, then $\emptyset$ is the least element of $\mathcal{P}_f$, so that $\rho_f(\emptyset) = 0$ and therefore $\hat{f}_\emptyset = (-1)^0 = 1$. Similarly, if $[n] \in \mathcal{P}_f$, then $[n]$ is the largest element in $\mathcal{P}_f$, hence it is the last element in any maximal chain. It follows that $\rho_f([n]) = k$ and hence $\hat{f}_{[n]} = (-1)^k = 1$ (since k is even). Therefore the equality (7) implies that $\mathcal{P}_{f^*}$ differs from $\mathcal{P}_f$ only in the bottom and top elements: $\emptyset \in \mathcal{P}_f$ iff $\emptyset \notin \mathcal{P}_{f^*}$ and $p_n \in \mathcal{P}_f$ iff $p_n \notin \mathcal{P}_{f^*}$. It follows that $\mathcal{P}_{f^*}$ is graded as well, with rank equal to $k-2$, k or $k+2$, which in any case is even. Furthermore, this also implies that for all $S \in \mathcal{P}_f$, $S \notin \{\emptyset, [n]\}$, we have $\rho_{f^*}(S) = \rho_f(S) \pm 1$, according to whether $\emptyset$ was added or removed. The statement now follows from (7).

We now proceed by induction on n. For $n = 1$, we have $2 = \{\emptyset, [1]\}$ and $\mathcal{T}_1 = \{1, 1^*\}$. For $f = 1$, $\mathcal{P}_f = \{\emptyset\}$ is a singleton, which is clearly a graded poset, with rank $k = 0$ and trivial rank function ρ_f. We have

$$f = 1 = p_\emptyset = (-1)^{\rho(\emptyset)} p_\emptyset.$$

The statement for $f = 1^*$ follows by duality. Assume that the statement is true for $m < n$ and let $f \in \mathcal{T}_n$. By the first part of the proof, we only need to prove that the statement holds for $f = f_1 \otimes f_2$, where $f_1 \in \mathcal{T}_{n_1}$, $f_2 \in \mathcal{T}_{n_2}$, $n_1 + n_2 = n$. By the induction assumption, $\mathcal{P}_{f_i}$ is graded with even rank. We also have using Proposition 14

$$f = f_1 \otimes f_2 = \sum_{S \subseteq [m], T \subseteq [n-m]} (\widehat{f_1})_S (\widehat{f_2})_T p_S \otimes p_T = \sum_{S \subseteq [m], T \subseteq [n-m]} (-1)^{\rho_{f_1}(S) + \rho_{f_2}(T)} p_{S \oplus T}.$$

It follows that $\mathcal{P}_f \simeq \mathcal{P}_{f_1} \times \mathcal{P}_{f_2}$, where the direct product of posets $\mathcal{P}_f$ and $\mathcal{P}_g$ is defined as the set of all pairs (S,T), $S \in \mathcal{P}_f$, $T \in \mathcal{P}_g$, and $(S,T) \leq (S',T')$ iff $S \leq S'$ and $T \leq T'$. By [Stanley], the direct product of graded posets is a graded poset with rank $r(f) = r(f_1) + r(f_2)$ and rank function $\rho_f = \rho_{f_1} + \rho_{f_2}$. This proves the statement.

$\square$

Remark 1. Notice that we need to assume n to be known. Indeed, for any m and f, $\mathcal{P}_f$ and $\mathcal{P}_{f \otimes 1_m}$ are the same, but the corresponding constructions of higher order objects are different.

We introduce *labels* for the elements of $\mathcal{P}_f$ in the following way. For $S \in \mathcal{P}_f$, put

$$L_S := \{i \in [n] \ : \ i \in S, \ \forall S' \subsetneq S, i \notin S'\}.$$

In other words, i is a label for S if S is a minimal element in the subposet of elements containing i in $\mathcal{P}_f$. We also denote

$$\mathcal{M}_i := \{S, \ i \in L_S\}.$$

We will use the notation $L_{S,f}$ and $\mathcal{M}_{i,f}$ if the function f has to be specified. It is clear from this definition that $\mathcal{M}_i = \emptyset$ if and only if $i \notin \mathcal{P}_f$ and if $\mathcal{M}_i$ is nonempty, it is an antichain in $\mathcal{P}_f$.

It is easily seen that

$$\forall S \in \mathcal{P}_f, \quad S = \cup_{S' \subseteq S} L_{S'} \quad \text{and } \cup \{L_S, \ S \in \mathcal{P}_f\} = \cup \mathcal{P}_f.$$

It follows that $f \in \mathcal{T}_n$ (with known n) is fully determined by the order relation on $\mathcal{P}_f$ and the label sets. We will therefore treat $\mathcal{P}_f$ as an abstract poset in which some elements are labelled by subsets of $[n]$.

The two distinguished elements $\emptyset$ and $[n]$, if present in $\mathcal{P}_f$, can be easily recognized from its structure as a labelled poset. Indeed, $\emptyset \in \mathcal{P}_f$ if and only if it is the smallest element in $\mathcal{P}_f$ and has an empty label. Similarly, $[n] \in \mathcal{P}_f$ if and only if it is the largest element and $\cup_{S \in \mathcal{P}_f} L_S = [n]$.

Let us denote

$$O_f^F := [n] \setminus \cup_{S\in\mathcal{P}_f} L_S, \qquad I_f^F := \cap_{S\in \mathrm{Min}(\mathcal{P}_f)} L_S.$$

It is easily checked by Proposition 7 that any $i \in O_f^F$ is an output index, since in this case we have $f(e^i) = f(\theta_n) = 1$. Such elements will be called the *free outputs* of f. If f has some free outputs, then necessarily $[n] \notin \mathcal{P}_f$. Similarly, any $j \in I_f^F$ is an input of f, since j must be contained in any $T \in \mathcal{P}_f$, so that $p_T(e^j) = 0$ for all $T \in \mathcal{P}_f$ and consequently $f(e^j) = 0$. Such elements will be called *free inputs* of f. It is clear that if f has free inputs, then $\mathcal{P}_f$ has a smallest element and its label is I_f^F. Moreover, up to a permutation, $f = p_k \otimes g \otimes 1_l$, where $k = |I_f^F|$, $l = |O_f^F|$ and $g \in \mathcal{T}_{n-k-l}$ has no free inputs or outputs.

Examples (n=2,3,4). Hasse diagrams

Example 8 ($\mathcal{T}_3$). As we can see from Example 7, all elements in $\mathcal{T}_2$ are chains. This is also true for $n = 3$. Indeed, up to a permutation that does not change the chain structure, any $f \in \mathcal{T}_3$ is either a product of two elements $g \in \mathcal{T}_2$ and $h \in \mathcal{T}_1$, or the dual of such a product. Since g must be a chain and $|\mathcal{P}_h| = 1$, their product must be a chain as well. Taking the dual of a chain only adds/removes the least/largest elements, so the dual of a chain must be a chain as well.

We next characterize the basic operations on type functions in terms of the labelled posets $\mathcal{P}_f$.

Corollary 2. *Let $f \in \mathcal{T}_n$, $g \in \mathcal{T}_m$. Then*

(i) For $\sigma \in \mathscr{S}_n$, $\mathcal{P}_{f\circ\sigma}$ is the same poset as $\mathcal{P}_f$, with the labels changed as $L_S \mapsto \sigma^{-1}(L_S)$.

(ii) $\mathcal{P}_{f^*}$ is obtained from from $\mathcal{P}_f$ by adding/removing $\emptyset$ and $[n]$. If $[n]$ is added, then $L_{[n],f^*} = O_f^F$. All other elements and labels remain the same.

(iii) Assume the decomposition $[n+m] = [n] \oplus [m]$. Then $\mathcal{P}_{f\otimes g} = \mathcal{P}_f \times \mathcal{P}_g$, with label sets

$$L_{(S,T)} = \begin{cases} L_S \cup L_T, & \text{if } S \in \mathrm{Min}(\mathcal{P}_f),\ T \in \mathrm{Min}(\mathcal{P}_g) \\ L_S, & \text{if } S \notin \mathrm{Min}(\mathcal{P}_f),\ T \in \mathrm{Min}(\mathcal{P}_g) \\ n + L_T = \{n+i,\ i \in L_T\}, & \text{if } S \in \mathrm{Min}(\mathcal{P}_f),\ T \notin \mathrm{Min}(\mathcal{P}_g) \\ \emptyset, & \text{otherwise.} \end{cases}$$

Proof. (i) and (ii) are easily seen from the proof of Proposition 8 and the definition of label sets. It can be also seen that $\mathcal{P}_{f\otimes g} = \mathcal{P}_f \times \mathcal{P}_g$, identifying the disjoint union $(T \oplus S)$ with (S,T). For the label sets, assume that $i \in L_{(S,T)}$, then $S \oplus T$ must be a minimal element in $\mathcal{P}_{f\otimes g}$ containing i. Hence, either $i \in S$ or $i \in n + T$. In the first case, $i \in (S' \oplus T') \leq (S \oplus T)$ whenever $i \in S' \leq S$ and $T' \leq T$, so we must have $i \in L_S$ and $T \in \mathrm{Min}(\mathcal{P}_g)$. Similarly, for $i \in n+T$, we get $i \in n + L_T$ and $S \in \mathrm{Min}(\mathcal{P}_f)$.

□

We next show that the input and output sets of $f \in \mathcal{T}_n$ can be easily recognized from the labels in $\mathcal{P}_f$.

Proposition 9. *Let $f \in \mathcal{T}_n$ and $i \in [n]$. Then*

(i) If $\mathcal{M}_{i,f} \neq \emptyset$, then all elements in $\mathcal{M}_{i,f}$ have the same rank, which will be denoted by $r_f(i)$.

(ii) $\mathcal{M}_{i,f} = \emptyset$ if and only if $\mathcal{M}_{i,f^*} = \{[n]\}$. In this case, we put $r_f(i) := r(f) + 1$.

(iii) $i \in O_f$ if and only if $r_f(i)$ is odd.

Proof. The statement (ii) follows from the equivalence $\mathcal{M}_{i,f} = \emptyset \iff i \in O_f^F = L_{[n],f^*}$ and the fact that $\mathcal{M}_{i,f^*}$ is an antichain, so it cannot contain other elements beside $[n]$. We also see that i is an output, and $r_f(i) = r(f) + 1$ is odd.

The statement (i) clearly holds if $\mathcal{M}_{i,f} = \{[n]\}$. Since $[n]$ is the largest element of $\mathcal{P}_f$, we have $r_f(i) = \rho_f([n]) = r(f)$ by definition of the rank, hence $r_f(i)$ is even. As we have seen in the first paragraph of this proof, $i \in O_{f^*}^F \subseteq O_{f^*} = I_f$.

By Corollary 2(i), it is quite clear that the properties (i) and (iii) are preserved by permutations. To show that they are preserved by complementation, we may assume that $\mathcal{M}_{i,f}$ is not equal to $\emptyset$ or $\{[n]\}$, since these cases were already shown. Then we must have $\mathcal{M}_{i,f} = \mathcal{M}_{i,f^*}$ and by the proof of Proposition 8 we have $\rho_{f^*}(S) = \rho_f(S) \pm 1$ for any S, depending only on the fact whether $\emptyset \in \mathcal{P}_f$. This implies that the properties are preserved by complementation.

We will now proceed by induction on n as before. Both assertions are quite trivial for $n = 1$, so assume the statements hold for $m < n$. It is enough to assume that $f = g \otimes h$ for some $g \in \mathcal{T}_m$ and $h \in \mathcal{T}_{n-m}$. Suppose without loss of generality that $i \in [m]$, then all elements of $\mathcal{M}_{i,f}$ have the form $S \oplus T$, with $S \in \mathcal{M}_{i,g}$ and T a minimal element in $\mathcal{P}_h$. Since $\rho_h(T) = 0$ for any minimal element $T \in \mathcal{P}_h$, we have by the induction assumption

$$\rho_f(S \oplus T) = \rho_g(S) + \rho_h(T) = \rho_g(S) = r_g(i).$$

The statement (ii) follows from the fact that $i \in O_f$ if and only if $i \in O_g$.

□

3.1.1 Chains and combs

We have seen that for some type functions the poset $\mathcal{P}_f$ is a chain, which is also a basic example of a graded poset. A chain in 2^n has the form $\mathcal{P} = \{S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_N\}$, $S_i \subseteq [n]$. Note that the length of the chain $\mathcal{P}$ is $N - 1$. It is clear that $\mathcal{P}$ is graded with rank $N - 1$ and rank function $\rho(S_i) = i - 1$.

Proposition 10. *For a chain $\mathcal{P} = \{S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_N\}$, the function*

$$f = f_{\mathcal{P}} := \sum_{i=1}^{N} (-1)^{i-1} p_{S_i}$$

is a type function if and only if N odd. In this case, we say that f is a chain type.

Proof. By Proposition 8, if $f \in \mathcal{T}_n$, then the rank of f must be even, so that N must be odd. We will show that the converse is true. We proceed by induction on N. For $N = 1$, we have $f = p_{S_1} \in \mathcal{T}_n$. Assume that the statement holds for all odd numbers $M < N$ and let $\mathcal{P}$ be a chain as above. Up to a permutation $\sigma \in \mathscr{S}_n$, we may assume that $S_j = [n_j]$ for some $0 \leq n_1 < \cdots < n_N \leq n$. Then we have

$$f = p_{[n_1]} \sum_{i=1}^{N} (-1)^{i-1} p_{[n_1+1,n_i]} = p_{n_1} \otimes g \otimes 1_{n-n_N},$$

where $g = f_{\mathcal{P}'} \in \mathcal{F}_{n_N - n_1}$ is the function for the chain $\mathcal{P}' := \{\emptyset \subsetneq [n_2 - n_1] \subsetneq \cdots \subsetneq [n_N - n_1]\}$. Since f is a type function if g is, this shows that we may assume that the chain contains $\emptyset$ and $[n]$. But then

$$f = 1 + \sum_{j=2}^{N-1} (-1)^{j-1} p_{S_j} + p_n$$

and

$$f^* = 1 - f + p_n = \sum_{j=1}^{N-2} (-1)^{j-1} p_{S_{j+1}},$$

By the induction assumption $f^* \in \mathcal{T}_n$, hence also $f = f^{**} \in \mathcal{T}_n$. $\square$

Let $f \in \mathcal{T}_n$ be a chain type and let $\mathcal{P}_f = \{S_1 \subsetneq \cdots \subsetneq S_N\}$ be the corresponding chain. There is a decomposition of $[n]$ given as

$$T_0 := S_1, \quad T_j := S_{j+1} - S_j,\ j = 1, \dots, N-1, \quad T_N := [n] \setminus S_N.$$

It is clear that the label sets are given as $L_{S_j} = T_{j-1}$, $j = 1, \dots, N$ and it can be easily seen from Proposition 9 that

$$I_f = \bigcup_{j=0}^{(N-1)/2} T_{2j}, \qquad O_f = \bigcup_{j=0}^{(N-1)/2} T_{2j+1} \cup O_f^F, \qquad I_f^F = T_0, \qquad O_f^F = T_N \tag{8}$$

(note that N must be odd). As before, up to a permutation, we may assume that there are some $0 \le n_1 < n_2 < \cdots < n_N \le n$ such that $S_j = [n_j]$, $j = 1, \dots, N$, and $T_0 = [n_1]$, $T_j = [n_j + 1, n_{j+1}]$, $j = 1, \dots, N-1$ and $T_N = [n_N + 1, n]$.

We will assume below that $T_0 = T_N = \emptyset$ and show that such chain types correspond to important higher order objects.

Proposition 11. *Let $f \in \mathcal{T}_n$ be a chain type, with corresponding decomposition $T_0, \dots, T_N$ of $[n]$, such that $T_0, T_N = \emptyset$. Let $Y = X_f(X_1, \dots, X_n)$ for some first order objects $X_1, \dots, X_n$. Then for $N \ge 3$, Y is an $(N-1)/2$-comb. More precisely, let $Y_1, \dots, Y_n$ be such that $Y_i = X_i$ for $i \in O_f$ and $Y_i = \tilde{X}_i$ for $i \in I_f$. Then*

$$Y \simeq [Y_{T_{N-1}}, [[Y_{T_{N-2}}, [[\dots, [[Y_{T_{\frac{N+1}{2}}}, Y_{T_{\frac{N-1}{2}}}], Y_{T_{\frac{N-3}{2}}}]], \dots, Y_{T_2}]], Y_{T_1}]]$$

where we put $Y_T = \otimes_{j \in T} Y_j$, and the isomorphism is given by a symmetry.

Proof. Let $Y_1, \dots, Y_n$ be as assumed, then by (8),

$$Y_{T_i} = \begin{cases} \otimes_{j \in T_i} X_j, & \text{if } i \text{ is odd,} \\ \otimes_{j \in T_i} \tilde{X}_i, & \text{if } i \text{ is even.} \end{cases}$$

As before, up to a permutation we may assume that $S_i = [n_i]$, $i = 1, \dots, N$, where by the assumptions $0 = n_1 < n_2 < \cdots < n_N = n$, so that $T_1 = [n_2]$ and $T_j = [n_j + 1, n_{j+1}]$, $j = 1, \dots, N-1$.

We will proceed by induction on N. Let $N = 3$, then $f = 1 - p_{[n_2]} + p_n$, and we see from Example 6 that $Y \simeq [Y_{T_2}, Y_{T_1}]$, where the isomorphism is a symmetry. Assume the assertion is true for $N-2$. As in the proof of Proposition 10, we see that

$$f^* = \sum_{i=1}^{N-2} (-1)^{i+1} p_{[n_{i+1}]} = p_{n_2} \otimes g \otimes 1_{n - n_{N-1}}$$

where g is the chain type for a chain of intervals given by $0 = m_1 < m_2 < \cdots < m_{N-2} = m$ in $\mathcal{T}_m$, here $m_i = n_{i+1} - n_2$. By Proposition 4, we see that

$$X_f(X_1, \dots, X_n) = X^*_{f^*}(\tilde{X}_1, \dots, \tilde{X}_n) \simeq (Y_{T_{N-1}} \otimes \tilde{X}_g \otimes Y^*_{T_1})^* \simeq [Y_{T_{N-1}}, [\tilde{X}_g, Y_{T_1}]]$$

where $\tilde{X}_g = X_g(\tilde{X}_{n_2+1}, \dots, \tilde{X}_{n_{N-1}})$ and the isomorphisms are symmetries. Since g satisfies the induction assumption, using that $\tilde{\tilde{X}}_i = X_i$, we obtain

$$\tilde{X}_g \simeq [Y_{T_{N-2}}, [[\dots, [[Y_{T_{\frac{N+1}{2}}}, Y_{T_{\frac{N-1}{2}}}], Y_{T_{\frac{N-3}{2}}}]], \dots, Y_{T_2}]].$$

This implies the result.

□

Using Proposition 4, we can see that for a general chain type, we obtain $X_f(X_1, \dots, X_n) \simeq \tilde{X}_{T_0} X_g(X_{n_1+1}, \dots, X_{n-n_N}) \otimes X_{T_N}$, where g is a chain type with no free inputs or outputs, so it is a comb as i Proposition 11.

Combs, picture

Other examples

3.2 Connecting chains: the causal product

We now introduce further operations of boolean functions. For a fixed decomposition $[n] = [n_1] \oplus [n_2]$ and functions $f_1 : \{0,1\}^{n_1} \to \{0,1\}$, $f_2 : \{0,1\}^{n_2} \to \{0,1\}$, we define their *causal product* as

$$f_1 \lhd f_2 := f_1 \otimes 1_{n_2} + p_{n_1} \otimes (f_2 - 1_{n_2}).$$

For $s^1 \in \{0,1\}^{n_1}$ and $s^2 \in \{0,1\}^{n_2}$, this function acts as

$$(f_1 \lhd f_2)(s^1 s^2) = f_1(s^1) + p_{n_1}(s^1)(f_2(s^2) - 1) = \begin{cases} f_1(s^1), & \text{if } s^1 \neq \theta_{n_1}, \\ f_2(s^2), & \text{if } s^1 = \theta_{n_1}. \end{cases} \tag{9}$$

The following properties are immediate from (9).

Lemma 8. *Let $f_1, g_1 \in \mathcal{F}_{n_1}$, $f_2, g_2 \in \mathcal{F}_{n_2}$. Then $f_1 \lhd f_2 \in \mathcal{F}_{n_1+n_2}$ and we have*

(i) $(f_1 \lhd f_2)^* = f_1^* \lhd f_2^*$,

(ii) $(f_1 \vee g_1) \lhd (f_2 \vee g_2) = (f_1 \lhd f_2) \vee (g_1 \lhd g_2) = (f_1 \lhd g_2) \vee (g_1 \lhd f_2)$,

(iii) $(f_1 \wedge g_1) \lhd (f_2 \wedge g_2) = (f_1 \lhd f_2) \wedge (g_1 \lhd g_2) = (f_1 \lhd g_2) \wedge (g_1 \lhd f_2)$.

Moreover, for any $f_3 \in \mathcal{F}_{n_3}$, and for the decomposition $[n] = [n_1] \oplus [n_2] \oplus [n_3]$, we have

$$(f_1 \lhd f_2) \lhd f_3 = f_1 \lhd (f_2 \lhd f_3).$$

We can also combine f_1 and f_2 in the opposite order:

$$f_2 \lhd f_1 := 1_{n_1} \otimes f_2 + (f_1 - 1_n) \otimes p_{n_2},$$

so that

$$(f_2 \lhd f_1)(s^1 s^2) = f_2(s^2) + p_{n_2}(s^2)(f_1(s^1) - 1_{n_1}) = \begin{cases} f_2(s^2), & \text{if } s^2 \neq \theta_{n_2}, \\ f_1(s^1), & \text{if } s^2 = \theta_{n_2}. \end{cases} \tag{10}$$

Of course, this product has similar properties as listed in the above lemma. To avoid any confusion, we have to bear in mind the fixed decomposition $[n] = [n_1] \oplus [n_2]$ and that for the concatenation $s = s^1 s^2$, f_i acts on s^i.

Lemma 9. *In the situation as above, we have*

$$f_1 \otimes f_2 = (f_1 \rhd f_2) \wedge (f_2 \rhd f_1).$$

Proof. This is again by straightforward computation from (9) and (10): let $s^1 \in \{0,1\}^{n_1}$, $s^2 \in \{0,1\}^{n_2}$ and compute

$$\begin{aligned}(f_1 \lhd f_2) \wedge (f_2 \lhd f_1)(s^1 s^2) &= \left(f_1(s^1) + p_{n_1}(s^1)(f_2(s^2) - 1)\right)\left(f_2(s^2) + p_{n_2}(s^2)(f_1(s^1) - 1)\right)\\ &= f_1(s^1) f_2(s^2),\end{aligned}$$

the last equality follows from the fact that $f_i(s^i)(1 - f_i(s^i)) = 0$ (since $f_i(s^i) \in \{0,1\}$) and the fact that p_{n_1} is the least element in $\mathcal{F}_{n_1}$, so that $p_{n_1}(s^1)(f_1(s^1) - 1) = p_{n_1}(s^1) - p_{n_1}(s^1) = 0$.

□

For the smallest and the largest element in $\mathcal{F}_n$, the causal product behaves as follows.

Lemma 10. *Let $f \in \mathcal{F}_{n_1}$ and let $1_{n_2}, p_{n_2} \in \mathcal{F}_{n_2}$, we have*

$$f \lhd 1_{n_2} = f \otimes 1_{n_2} \leq 1_{n_2} \lhd f = 1 - (1 - \hat{f}_\emptyset) p_{[n_2]} + \sum_{\emptyset \neq S \subseteq [n_1]} \hat{f}_S p_{S \oplus [n_2]}$$

and

$$p_{n_2} \lhd f = f \otimes p_{n_2} \leq f \lhd p_{n_2} = \sum_{S \subsetneq [n_1]} \hat{f}_S p_S + (\hat{f}_{[n_1]} - 1) p_{[n_1]} + p_{n_1 + n_2}.$$

In particular,

$$(p_{n_1} \otimes 1_{n_2})^* = 1_{n_1} \lhd p_{n_2} = 1 - p_{[n_1]} + p_{n_1 + n_2}$$

is the chain type for $\{\emptyset \subsetneq [n_1] \subsetneq [n_1 + n_2]\}$. Similar properties hold for p_{n_1}, 1_{n_1} and $f \in \mathcal{T}_{n_2}$.

Proof. Immediate from the definition of the causal product and Lemma 9.

□

Using the last part of Lemma 8, for a decomposition $[n] = \oplus_i [n_i]$ and $f_i \in \mathcal{F}_{n_i}$, we may define the function $f_1 \lhd \ldots \lhd f_k \in \mathcal{F}_n$. Note that we have for $s = s^1 \ldots s^k$,

$$\begin{aligned}(f_1 \lhd \ldots \lhd f_k)(s) &= f_1(s^1) + p_{n_1}(s^1)(f_2(s^2) - 1) + \cdots + p_{n_1}(s^1) \ldots p_{n_{k-1}}(s^{k-1})(f_k(s^k) - 1)\\ &= \begin{cases} f_1(s_1) & \text{if } s^1 \neq \theta_{n_1} \\ f_2(s^2) & \text{if } s^1 = \theta_{n_1}, s^2 \neq \theta_{n_2} \\ \ldots & \\ f_k(s^k) & \text{if } s^1 = \theta_{n_1}, \ldots, s^{k-1} = \theta_{n_{k-1}}. \end{cases}\end{aligned}$$

For any permutation $\pi \in \mathscr{S}_k$, we define $f_{\pi^{-1}(1)} \lhd \ldots \lhd f_{\pi^{-1}(k)} \in \mathcal{F}_n$ in an obvious way.

We will show that the causal product is related to the ordinal sum $\star$ of the corresponding posets.

Proposition 12. *Let $f \in \mathcal{T}_n$, $g \in \mathcal{T}_m$ and consider the decomposition $[n+m] = [n] \oplus [m]$. Replace the labels of $\mathcal{P}_g$ by their translations $L_S \mapsto n + L_S = \{n + i,\ i \in L_S\}$. Then*

(a) If $[n] \in \mathcal{P}_f$ and $\emptyset \in \mathcal{P}_g$, then $\mathcal{P}_{f \lhd g} = \mathcal{P}_f \star (\mathcal{P}_g \setminus \{\emptyset\})$, with all labels remaining the same.

(b) If $[n] \in \mathcal{P}_f$ and $\emptyset \notin \mathcal{P}_g$, then $\mathcal{P}_{f \lhd g} = (\mathcal{P} \setminus \{[n]\}) \star \mathcal{P}_g$, where the labels of $[n]$ are added to the labels of elements in $\mathrm{Min}(\mathcal{P}_g)$.

(c) If $[n] \notin \mathcal{P}_f$ and $\emptyset \in \mathcal{P}_g$, then $\mathcal{P}_{f \lhd g} = \mathcal{P}_f \star (\mathcal{P}_g \setminus \{\emptyset\})$, where the free outputs of f are added to the label sets of elements in $\mathrm{Min}(\mathcal{P}_g \setminus \emptyset)$.

(d) If $[n] \notin \mathcal{P}_f$ and $\emptyset \notin \mathcal{P}_g$, then $\mathcal{P}_{f \lhd g} = \mathcal{P}_f \star \{\bullet\} \star \mathcal{P}_g$, where $\{\bullet\}$ is a one-element poset whose label is given by the free outputs of f.

Proof. By definition of the causal product, we have

$$f \lhd g = \sum_{S \in \mathcal{P}_f \setminus \{[n]\}} \hat{f}_S p_S + (\hat{f}_{[n]} - 1 + \hat{g}_\emptyset) p_{[n]} + \sum_{T \in \mathcal{P}_g \setminus \{\emptyset\}} \hat{g}_T p_{[n] \oplus T}.$$

The terms in brackets can be equal to 1, -1, or 0, depending on whether $[n] \in \mathcal{P}_f$ and $\emptyset \in \mathcal{P}_g$. The statement is now immediate.

□

It is not clear that if f and g are type functions, then $f \lhd g$ is a type function as well. Nevertheless, it can be seen from the above proposition that if both f and g are chain types with chains of N and M elements, respectively, then $f \lhd g$ is a chain type for a chain with $M + N \pm 1$ elements. Note also that that this construction can be interpreted as appending the two chains in the respective order. Our next result shows that if f or g is a chain type, we always obtain a type function.

Proposition 13. *Let $f \in \mathcal{T}_{n_1}$ and let $\beta \in \mathcal{T}_{n_2}$ be a chain type. Then both $f \lhd \beta$ and $\beta \lhd f$ are types, with outputs $O = O_f \oplus O_\beta$ and inputs $I = I_f \oplus I_\beta$.*

Proof. Let $\beta = \sum_{k=1}^N (-1)^{k-1} p_{S_k}$ for some odd N and $S_1 \subsetneq \cdots \subsetneq S_N \subseteq [n_2]$. We will proceed by induction on N. Suppose $N = 1$. If $S_1 = \emptyset$, then $\beta = 1_{n_2}$ and we have by Lemma 10

$$f \lhd 1_{n_2} = f \otimes 1_{n_2} \in \mathcal{T}_{n_1+n_2}$$

and

$$1_{n_2} \lhd f = (p_{n_2} \lhd f^*)^* = (f \otimes p_{n_2})^* \in \mathcal{T}_{n_1+n_2}.$$

Assume that $S_1 = [n_2]$, then $\beta = p_{n_2}$ and the assertion follows by duality. If $\emptyset \neq S_1 \subsetneq [n_2]$, then up to a permutation, we have $\beta = p_{m_1} \otimes 1_{m_2} = p_{m_1} \lhd 1_{m_2}$ for $m_1 = |S_1|$, $m_1 + m_2 = n_2$. Then

$$\beta \lhd f = p_{m_1} \lhd (1_{m_2} \lhd f), \quad f \lhd \beta = (f \lhd p_{m_1}) \lhd 1_{m_2} \in \mathcal{T}_{n_1+n_2},$$

by the first part of the proof and Lemma 8.

Assume next that the assertion holds for all odd numbers $M < N$. As before, up to a permutation, we may assume that $S_k = [l_k]$ for some $l_1 < \cdots < l_N$. Let β_1 be the chain type for the chain given by $l_1 < \cdots < l_{N-2}$ in $[l_{N-1}]$ and put $\beta_2 := p_{[l_N - l_{N-1}]}$, the 1-element chain type in $\mathcal{T}_{n_2 - l_{N-1}}$. By Proposition 12(d), we see that $\beta = \beta_1 \lhd \beta_2$, so that

$$\beta \lhd f = \beta_1 \lhd (\beta_2 \lhd f), \qquad f \lhd \beta = (f \lhd \beta_1) \lhd \beta_2$$

are type functions, by the induction assumption.

To prove the statement on the output and input indices, note that for any $i \in [n_1] \oplus [n_2]$, we have $e^i_{n_1+n_2} = e^j_{n_1} \theta_{n_2}$ or $e^i_{n_1+n_2} = \theta_{n_1} e^k_{n_2}$ for some $j \in [n_1]$, $k \in [n_2]$. Then

$$f \lhd \beta(e^i) = f(e^j_{n_1}) \text{ or } f \lhd \beta(e^i) = \beta(e^k_{n_2}).$$

The statement on input/output indices follow from Lemma 7. The proof for $\beta \lhd f$ is similar.

□

3.3 The structure of type functions

Our main result here is the following structure theorem for the type functions.

Theorem 3. *Let $f \in \mathcal{T}_n$. Then there is a permutation $\rho \in \mathscr{S}_n$, a decomposition $[n] = \oplus_{i=1}^k [n_i]$, chain types $\beta_1 \in \mathcal{T}_{n_1}, \dots, \beta_k \in \mathcal{T}_{n_k}$ such that $O_f = \oplus_j O_{\beta_j}$, $I_f = \oplus_j I_{\beta_j}$, finite index sets A, B and permutations $\pi_{a,b} \in \mathscr{S}_k$, $a \in A$, $b \in B$ such that*

$$f \circ \rho = \bigvee_{a \in A} \bigwedge_{b \in B} (\beta_{\pi_{a,b}^{-1}(1)} \lhd \dots \lhd \beta_{\pi_{a,b}^{-1}(k)}) = \bigwedge_{b \in B} \bigvee_{a \in A} (\beta_{\pi_{a,b}^{-1}(1)} \lhd \dots \lhd \beta_{\pi_{a,b}^{-1}(k)}).$$

Proof. It is obvious that the condition is invariant under permutations. Since any element in $\mathcal{T}_n$ for $n \leq 3$ is a chain type, the statement clearly holds in this case. Assume f can be written in the given form, then

$$f^* \circ \rho = \bigwedge_{a \in A} \bigvee_{b \in B} (\beta^*_{\pi_{a,b}^{-1}(1)} \lhd \dots \lhd \beta^*_{\pi_{a,b}^{-1}(k)}) = \bigvee_{b \in b} \bigwedge_{a \in A} (\beta^*_{\pi_{a,b}^{-1}(1)} \lhd \dots \lhd \beta^*_{\pi_{a,b}^{-1}(k)}).$$

Since β_j^* is a chain type for each j, this proves the statement for f^*. It is now enough to show this form for $f = f_1 \otimes f_2$, where $f_1 \in \mathcal{T}_m$, $f_2 \in \mathcal{T}_{n-m}$ satisfy the conditions, so that

$$f_1 \circ \rho_1 = \bigvee_{a \in A} \bigwedge_{b \in B} (\beta^1_{\pi_{a,b}^{-1}(1)} \lhd \dots \lhd \beta^1_{\pi_{a,b}^{-1}(k_1)}) = \bigwedge_{b \in B} \bigvee_{a \in A} (\beta^1_{\pi_{a,b}^{-1}(1)} \lhd \dots \lhd \beta^1_{\pi_{a,b}^{-1}(k_1)}),$$

$$f_2 \circ \rho_2 = \bigvee_{c \in C} \bigwedge_{d \in D} (\beta^2_{\tau_{c,d}^{-1}(1)} \lhd \dots \lhd \beta^2_{\tau_{c,d}^{-1}(k_2)}) = \bigwedge_{d \in D} \bigvee_{c \in C} (\beta^2_{\tau_{c,d}^{-1}(1)} \lhd \dots \lhd \beta^2_{\tau_{c,d}^{-1}(k_2)})$$

for some chain types $\beta^1_j \in \mathcal{T}_{m_j}$, $[m] = \oplus_{j=1}^{k_1} [m_j]$, and $\beta^2_j \in \mathcal{T}_{l_j}$, $[n-m] = \oplus_{j=1}^{k_2} [l_j]$ and permutations $\pi_{a,b} \in \mathscr{S}_{k_1}$, $\tau_{c,d} \in \mathscr{S}_{k_2}$, $\rho_1 \in \mathscr{S}_m$, $\rho_2 \in \mathscr{S}_{n-m}$. Let

$$\beta_1^{a,b} := \beta^1_{\pi_{a,b}^{-1}(1)} \lhd \dots \lhd \beta^1_{\pi_{a,b}^{-1}(k_1)}, \qquad \beta_2^{c,d} := \beta^2_{\tau_{c,d}^{-1}(1)} \lhd \dots \lhd \beta^2_{\tau_{c,d}^{-1}(k_2)}.$$

Using the properties of the tensor product (Lemma 22)ii), we get from Lemma 9

$$(\bigvee_{a \in A} \bigwedge_{b \in B} \beta_1^{a,b}) \otimes (\bigvee_{c \in C} \bigwedge_{d \in D} \beta_2^{c,d}) = \bigvee_{a,c} \bigwedge_{b,d} (\beta_1^{a,b} \otimes \beta_2^{c,d}) = \bigvee_{a,c} \bigwedge_{b,d} (\beta_1^{a,b} \lhd \beta_2^{c,d}) \wedge (\beta_2^{c,d} \lhd \beta_1^{a,b})$$

On the other hand, using Lemma 8, we get

$$\begin{aligned} &(\bigwedge_{b \in B} \bigvee_{a \in A} \beta_1^{a,b}) \otimes (\bigwedge_{d \in D} \bigvee_{c \in C} \beta_2^{c,d}) \\ &\quad = \Big[(\bigwedge_{b \in B} \bigvee_{a \in A} \beta_1^{a,b}) \lhd (\bigwedge_{d \in D} \bigvee_{c \in C} \beta_2^{c,d}) \Big] \wedge \Big[(\bigwedge_{d \in D} \bigvee_{c \in C} \beta_2^{c,d}) \lhd (\bigwedge_{b \in B} \bigvee_{a \in A} \beta_1^{a,b}) \Big] \\ &\quad = (\bigwedge_{b,d} \bigvee_{a,c} \beta_1^{a,b} \lhd \beta_2^{c,d}) \wedge (\bigwedge_{b,d} \bigvee_{a,c} \beta_2^{c,d} \lhd \beta_1^{a,b}). \end{aligned}$$

We have the decomposition $[n] = \oplus_{j=1}^k [n_j]$, with $k = k_1 + k_2$ and $n_j = m_j$, $j = 1, \dots, k_1$, $n_j = l_{j-k_1}$, $j = k_1 + 1, \dots, k$, and chain types $\beta_j \in \mathcal{T}_{n_j}$, $\beta_j = \beta_j^1$ for $j = 1, \dots, k_1$ and $\beta_j = \beta^2_{j-k_1}$ for $j = k_1 + 1, \dots, k$. To get the permutation sets, let $A' = A \times C$, $B' = B \times D \times \mathscr{S}_2$ and define $\pi_{a',b'}$ in $\mathscr{S}_k$ as the block permutation with respect to the decomposition $[k] = [k_1] \oplus [k_2]$ (see Appendix A.1)

$$\pi_{(a,c),(b,d,\lambda)} = \rho_\lambda \circ (\pi_{a,b} \oplus \tau_{c,d}).$$

Finally, putting $\rho = \rho_1 \otimes \rho_2$ finishes the proof.

$\square$

Remark 2. Not always types, but subtypes.

It is clear that the chains in Theorem 3 are not given uniquely, for example, some chains can be decomposed as a concatenation of chains. We will show that some choice of the chains $\beta_1, \dots, \beta_k$ can be found from the labelled poset $\mathcal{P}_f$. For this, let $\mathcal{P}^0_f \subseteq \mathcal{P}_f$ be the subposet consisting of all labelled elements and possibly $\emptyset$, that is

$$\mathcal{P}^0_f = \{S \in \mathcal{P}_f,\ L_{S,f} \neq \emptyset \text{ or } S = \emptyset\}.$$

**** For a finite poset $\mathcal{P}$, we denote by $\mathrm{Min}(\mathcal{P})$ the set of minimal elements of $\mathcal{P}$ and $\mathrm{Max}(\mathcal{P})$ the set of maximal elements of $\mathcal{P}$. ****

The following lemma collects some basic properties of $\mathcal{P}^0_f$.

Lemma 11. *Let $f \in \mathcal{T}_n$.*

(i) $\mathcal{P}^0_f$ is a chain $\iff$ $\mathcal{P}_f$ is a chain $\iff$ $\mathcal{P}^0_f = \mathcal{P}_f$.

(ii) $\mathcal{P}^0_f$ has a largest element if and only if f or f^* has a free output. In this case the corresponding set is the largest element in $\mathcal{P}_f$.

(iii) If $\mathcal{P}^0_f$ is an antichain, then $\mathcal{P}^0_f = \{\bullet\}$.

Proof. (i) Assume that $\mathcal{P}^0_f$ is a chain, so that, in particular, $\mathcal{M}_{i,f}$ is a singleton or empty for any $i \in [n]$. In the first case, we put $\mathcal{M}_{i,f} = \{M_i\}$, by defintion of $\mathcal{M}_{i,f}$, M_i is contained in any $S \in \mathcal{P}_f$ containing i. Since $\mathcal{P}^0_f$ is a chain, we have $M_{i_1} \subseteq M_{i_2} \subseteq \dots \subseteq M_{i_k}$, where $\{i_1, \dots, i_k\} = \cup\mathcal{P}_f$. Let S, T be two elements in $\mathcal{P}_f$ and assume that $i \in S \setminus T$. For any $j \in T$, we have either $M_i \subseteq M_j$ or $M_j \subseteq M_i$. In the first case, we get $i \in M_i \subseteq M_j \subseteq T$, which is not possible. Hence $M_j \subseteq M_i$ for all $j \in T$. It follows that $T = \cup_{j\in T} M_j \subseteq M_i \subseteq S$, so that $\mathcal{P}_f$ is a chain, and it is clear that then $\mathcal{P}_f = \mathcal{P}^0_f$.

If $\mathcal{P}_f$ is not a chain, then there are some type functions f_1, f_2 such that $f = f_1 \otimes f_2$ or $f = (f_1 \otimes f_2)^*$. Moreover, the ranks of f_1 and f_2 are at least 2. It follows that both $\mathcal{P}_{f_1\otimes f_2}$ and $\mathcal{P}_{(f_1\otimes f_2)^*}$ contain an element $S \oplus T$, where $S \in \mathcal{P}_{f_1}$, $T \in \mathcal{P}_{f_2}$ but none of the two elements is minimal. Then there is some $S' \in \mathcal{P}_{f_1}$ and $T' \in \mathcal{P}_{f_2}$ such that $S' \oplus T$, $S \oplus T' \subsetneq S \oplus T$, so that no element of $S \oplus T$ is a label. Hence $S \oplus T \notin \mathcal{P}^0_f$, so that $\mathcal{P}_f \neq \mathcal{P}^0_f$.

(ii) The largest element $T \in P^0_f$ must satisfy

$$\cup\mathcal{P}_f = \cup L_S \subseteq T \subseteq \cup\mathcal{P}_f.$$

If follows that $T = \cup\mathcal{P}_f$ is the largest element in $\mathcal{P}_f$. If $T \neq [n]$, then clearly, f has some free outputs. If $T = [n]$, then for any index $i \in L_T$, we have by the proof of Proposition 9 that $M_{i,f^*} = \emptyset$, so that i is a free output.

(iii) Assume that $\mathcal{P}^0_f$ is an antichain with $k > 1$ elements. Since then f is not a chain, up to a permutation we either have $f = f_1 \otimes f_2$ or $f^* = f_1 \otimes f_2$ for some type functions f_1 and f_2. Since $\emptyset \notin \mathcal{P}_f$ and by (ii), f nor f^* have any free outputs, we have $\mathcal{P}^0_{f^*} = \{\emptyset\} \star \mathcal{P}^0_f$. Assume that $f^* = f_1 \otimes f_2$, then it follows by Corollary 2(iii) that both $\mathcal{P}^0_{f_1}$ and $\mathcal{P}^0_{f_2}$ have the same form as $\mathcal{P}_{f^*}$. Hence both $\mathcal{P}^0_{f_i^*}$ are antichains with k_i elements, $k_1 + k_2 = k$. If $f = f_1 \otimes f_2$, then again using Corollary 2(iii) we see that both $\mathcal{P}_{f_i}$ are antichains with k_i elements, $k = k_1 k_2$. Disregarding the trivial case $k_i = 1$ for some i, we see that we can reduce to the case when $k = 2$. Then we must have $f^* = f_1 \otimes f_2$, where each $\mathcal{P}_{f_i}$ is a 2-element chain, which is not possible.

□

We now identify certain chains in $\mathcal{P}_f^0$.

Lemma 12. *Assume that $\mathcal{P}_f^0$ has a largest element. Then either f is a chain type, or there is some $h \in \mathcal{T}_m$ such that $\mathcal{P}_h^0$ has no largest element, and a chain type $\beta \in \mathcal{T}_{n-m}$ such that up to a permutation, $f = h \lhd \beta$.*

Proof. Let T be the largest element in $\mathcal{P}_f^0$. If $T = [n]$, then by Lemma 11(ii) and its proof, f^* has some free outputs. Hence, up to a permutation, $f^* = h_1 \otimes 1_{k_1}$ for some $h_1 \in \mathcal{T}_{n-k_1}$ with no free outputs. It follows that $f = (h_1 \otimes 1_{k_1})^* = (h_1 \lhd 1_{k_1})^* = h_1^* \lhd p_{k_1}$.

If $T \neq [n]$, then, since T is also the largest element in $\mathcal{P}_f$, we see that f has free outputs, so that (up to a permutation) $f = h_1 \otimes 1_{k_1} = h_1 \lhd 1_1$, where $h_1 \in \mathcal{T}_{n-k_1}$ is a type function with no free outputs. Clearly, $\mathcal{P}_{h_1} = \mathcal{P}_f$ and $\mathcal{P}_{h_1}^0 = \mathcal{P}_f^0$, so that T is the largest element in $\mathcal{P}_{h_1}^0$, but this time $T = [n - k_1]$, so that we may use the first part of the proof. We obtain that there is some k_2 and a type function $h_2 \in \mathcal{T}_{n-k_1-k_2}$ with no free outputs such that

$$f = h_1 \lhd 1_{k_1} = h_2^* \lhd p_{k_2} \lhd 1_{k_1}$$

So far, we have written f in the form $f = h \lhd \beta$, where β is a chain type and $h \in \mathcal{T}_{n-k}$ for $k > 0$ is such that h^* has no free outputs. It follows that if $\mathcal{P}_h^0$ has a largest element, then it cannot be equal to $[n-k]$. Hence h must have some free outputs, and we may proceed as above, replacing f by h. Since n is decreasing at each step, we either get to $n - k \leq 3$, in which case h must be a chain and therefore also $f = h \lhd \beta$ is a chain, or $\mathcal{P}_h^0$ has no largest element.

□

In the situation of the above lemma, if f is not a chain, $\mathcal{P}_h^0$ and β can be found as follows. Since h has no free outputs and does not contain $[m]$, we obtain from Proposition 12 that $\mathcal{P}_f^0 = \mathcal{P}_h^0 \star (\mathcal{P}_\beta \setminus \{\emptyset\})$, with the same sets of labels. It follows that there exists a largest element in $\mathcal{P}_f^0$ with the property that it covers more than one element. Let this element be S and let $T_1, \dots, T_k$ be the elements covered by S. There is a chain $S = S_1 \lneq \cdots \lneq S_K$, where S_K is the largest element in $\mathcal{P}_f^0$. If K is even, add an element S_0 with empty label at the bottom of the chain. Let $\sigma \in \mathscr{S}_n$ be such that $\sigma^{-1}(\cup_j L_{S_j}) = [m+1, n]$ and apply σ^{-1} to the label sets. Then we have $f \circ \sigma = h \lhd \beta$, where $\mathcal{P}_h^0$ is obtained as $\cup_j T_j^{\downarrow} = \cup_j \{T \in \mathcal{P}_f^0,\ T \leq T_j\}$ and β is the chain type for the obtained labelled chain.

Lemma 13. *Let $f \in \mathcal{T}_n$ be such that either f or f^* has a free input. Then either f is a chain type, or there is some chain type $\beta \in \mathcal{T}_k$ and some $h \in \mathcal{T}_{n-k}$ such that h nor h^* has a free input and up to a permutation, $f = \beta \lhd h$.*

Proof. Assume f has a free input, then up to a permutation, $f = p_k \otimes h = p_k \lhd h$, where $k = |S|$ and $h \in \mathcal{P}_{n-k}$ has no free inputs. On the other hand, if f^* has a free input, then, similarly, $f^* = p_k \lhd g$ for some $g \in \mathcal{T}_{n-k}$ with no free inputs, hence $f = (p_k \lhd g)^* = 1_k \lhd g^*$. Repeating the process, we get the result after finitely many steps.

□

Under the assumptions of the lemma, assume that f is not a chain type. Suppose that $\mathcal{P}_f^0$ has a least element S_1 and let S be the smallest elemet in $\mathcal{P}_f^0$ with the property that it is covered by more than one element. Let these elements be $T_1, \dots, T_l$ and let $S_1 \leq \cdots \leq S_K$ be a chain such that $S_K = S$. Put $L := \cap_j L_{T_j}$. Using Proposition 12 as before, we have the following situations. Assume that $L = \emptyset$, then put $k = |S_K|$. If K is odd, then the chain corresponds to a chain type

$\beta \in \mathcal{T}_k$ and $\mathcal{P}^0_g = \cup_j T_j^{\uparrow} \cup \{\emptyset\}$. If K is even, then $\mathcal{P}^0_g = \cup_j T_j^{\uparrow}$ and β is the chain type for the chain $S_1 \subsetneq \cdots \subsetneq S_{K-1}$ (with free outputs in L_{S_K}). If $L \neq \emptyset$, then add an element S_{K+1} at the end of the chain, with label $L_{S_{K+1}} = L$ and put $k = |S_{K+1}|$. If K is odd, then $\beta \in \mathcal{T}_k$ is the chain type for the chain $S_1 \lneq \cdots \lneq S_K$ (with free outputs in L) and $\mathcal{P}^0_g = \cup_j T_j^{\uparrow} \cup \{\emptyset\}$, with labels of T_j replaced by $L_{T_j} \setminus L$. If K is even, then $\mathcal{P}^0_g = \cup_j T_j^{\uparrow}$ and β is the chain type for $S_1 \subsetneq \cdots \subsetneq S_{K+1}$.

If $\mathcal{P}^0_f$ has no least element, then by the assumptions f must have free inputs (since $\mathcal{P}^0_{f^*}$ has least element $\emptyset$). Hence there is a symmetry σ such that $f \circ \sigma = p_k \lhd g$, where $[k] = \sigma^{-1}(I^F_f)$. Note also that we have $\emptyset \notin \mathcal{P}_g$, so that g^* has no free input as well.

Lemma 14. *Any element $T \in \mathcal{P}^0_f$ can cover at most one minimal element.*

Proof. We will proceed by induction on n. Since the assertion is trivial for chain types, it holds for $n \leq 3$. Assume it is true for $m < n$ and let $f \in \mathcal{T}_n$. Let T be an element that covers $T_1, \dots, T_k \in \mathrm{Min}(\mathcal{P}_f)$, $k > 1$, so that we must have $\emptyset \notin \mathcal{P}_f$. We may assume that T is not the largest element, since then by Lemma 12 and the remarks below it, up to a permutation, $f = h \lhd \beta$ and $\mathcal{P}^0_h = \cup_j T_j^{\downarrow} = \{T_1, \dots, T_k\}$. This means that $\mathcal{P}^0_h$ is an antichain, which is impossible by Lemma 11. It follows that $T \neq [n]$, therefore T and $T_1, \dots, T_k$ are all contained also in $\mathcal{P}^0_{f^*}$.

As usual, we have either $f^* = f_1 \otimes f_2$ or $f = f_1 \otimes f_2$, for some type functions f_1 and f_2. In the first case, any element in $\mathcal{P}_{f^*}$ is of the form $(S, \emptyset)$ for $S \in \mathcal{P}^0_{f_1}$ or $(\emptyset, S')$ for $S' \in \mathcal{P}^0_{f_2}$. Since $T_i \leq T$ for all i, they all must have one of these forms. Hence we may assume that there are $S, S_1, \dots, S_k \in \mathcal{P}^0_{f_1}$ such that $\emptyset \ll S_i \ll S$. As before, S cannot be the largest element in $\mathcal{P}^0_{f_1}$, so that $S_1, \dots, S_k \in \mathrm{Min}(\mathcal{P}_{f_1^*})$ are all covered by $S \in \mathcal{P}^0_{f_1^*}$. By induction assumption, this is not possible.

If $f = f_1 \otimes f_2$, then we can similarly use Corollary 2 to obtain the same situation for f_1 or f_2. □

Lemma 15. *If $\mathcal{P}_f \neq \{\bullet\}$, then any minimal element is covered by at least one $T \in \mathcal{P}^0_f$.*

Proof. We will proceed by induction on n. The assertion is clearly true for chains, so for $n \leq 3$. Assume it holds for all $m < n$ and let $f \in \mathcal{T}_n$. If $\mathcal{P}_f \neq \{\bullet\}$ and $\mathcal{P}_f$ has a smallest element, the assertion is clearly true. Assume that $S \in \mathrm{Min}(\mathcal{P}_f)$ is not covered by any element. Then, in particular, $\mathcal{P}^0_f$ has no largest element and also does not contain $\emptyset$. Similarly as before, if $f^* = f_1 \otimes f_2$, then we may assume that, say, $\mathcal{P}^0_{f_1}$ contains an element S that covers $\emptyset$ but is not covered by any element. Since then $\mathcal{P}^0_{f_1}$ has no largest element, we have that $\mathcal{P}^0_{f_1^*} = \mathcal{P}^0_{f_1} \setminus \{\emptyset\}$ and $S \in \mathrm{Min}(\mathcal{P}_{f_1^*})$ is not covered by any element. By the induction assumption, $f_1^* = p_S$, but then $\mathcal{P}_{f_1}$ is a 3-element chain, which is not possible. If $f = f_1 \otimes f_2$, then it similarly follows that both $\mathcal{P}_{f_1}$ and $\mathcal{P}_{f_2}$ must be 1-element chains, which is not possible.

□

Let $\mathcal{P}$ be a poset with labels in $[n]$ and let $\mathcal{P}_,\mathcal{P}_2 \subseteq \mathcal{P}$. We will say that $\mathcal{P}_1$ and $\mathcal{P}_2$ are *independent components* of $\mathcal{P}$ if $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$ (direct sum of posets) and $L_S \cap L_T = \emptyset$ for any $S \in \mathcal{P}_1$ and $T \in \mathcal{P}_2$. In this case, we will write

$$\mathcal{P} = \mathcal{P}_1 +^L \mathcal{P}_2.$$

The independent components are nontrivial if they are nonemty or not equal to $\mathcal{P}$.

Lemma 16. *Let $f \in \mathcal{T}_n$ be such that f nor f^* have any free inputs or outputs and assume that $\emptyset \notin \mathcal{P}_f$.*

(i) If (up to a permutation) $f^* = f_1 \otimes f_2$ for some type functions f_1 and f_2, then $\mathcal{P}^0_f = (\mathcal{P}^0_{f_1} \setminus \emptyset) +^L (\mathcal{P}^0_{f_2} \setminus \emptyset)$.

(ii) If $\mathcal{P}^0_f = \mathcal{P}_1 +^L \mathcal{P}_2$ for some nontrivial labelled subposets $\mathcal{P}_1$ and $\mathcal{P}_2$, then there are some type functions f_1 and f_2 such that $\mathcal{P}_1 = (\mathcal{P}^0_{f_1} \setminus \emptyset)$, $\mathcal{P}_2 = (\mathcal{P}^0_{f_2} \setminus \emptyset)$ and (up to a permutation) $f^* = f_1 \otimes f_2$.

(iii) If (up to a permutation) $f = f_1 \otimes f_2$ for type functions f_1 and f_2, then no nontrivial decomposition of $\mathcal{P}^0_f$ into independent components exists.

Proof. Note that under the assumptions, $\emptyset \in \mathcal{P}_{f^*}$ and we have $\mathcal{P}^0_f = \mathcal{P}^0_{f^*} \setminus \{\emptyset\}$. Further, $\mathcal{P}^0_f$ has no least element. Note that applying a permutation does not change the structure of $\mathcal{P}^0_f$ or the decomposition into independend components, so we may skip "up to a permutation" from the statements.

Assume that $f_1 \in \mathcal{T}_{n_1}$ and $f_2 \in \mathcal{T}_{n_2}$ are such that $f^* = f_1 \otimes f_2$ for the decomposition $[n] = [n_1] \oplus [n_2]$. Since $\emptyset \in \mathcal{P}_{f^*} = \mathcal{P}_{f_1} \times \mathcal{P}_{f_2}$, we see that both $\mathcal{P}_{f_1}$ and $\mathcal{P}_{f_2}$ must contain $\emptyset$. By Corollary 2(iii), we see that $\mathcal{P}^0_{f^*}$ consists of $\mathcal{P}^0_{f_1}$ and $\mathcal{P}^0_{f_2}$ glued at $\emptyset$, with labels of $\mathcal{P}^0_{f_2}$ translated by n_1. Since $\mathcal{P}^0_f = \mathcal{P}^0_{f^*} \setminus \{\emptyset\}$, the assertion (i) follows.

For (iii), assume that $f = f_1 \otimes f_2$ and $\mathcal{P}^0_f = \mathcal{P}_1 +^L \mathcal{P}_2$. Note that none of the functions can be a 1-element chain, since then f would have free inputs or outputs. Let $\mathrm{Min}(\mathcal{P}_{f_1}) = \{U_1, \dots, U_k\}$, $\mathrm{Min}(\mathcal{P}_{f_2}) = \{V_1, \dots, V_l\}$. By Corollary 2(iii), we have $\mathrm{Min}(\mathcal{P}_f) = \mathrm{Min}(\mathcal{P}_{f_1}) \times \mathrm{Min}(\mathcal{P}_{f_2})$. For some i and j, let $(U_i, V_j) \in \mathcal{P}_1$. Since (U_i, V_j) and (T, V_j) are comparable for any $T \in \mathcal{P}_{f_1}$, $U_i \leq T$, we must have $(T, V_j) \in \mathcal{P}_1$ for all such T. By Lemma 15, there is some T that covers U_i. But then $L_{(T,V_j)} = L_{T,V_{j'}} = L_T$ for all j', so that $(T, V_{j'}) \in \mathcal{P}_1$ for all j'. Since $(U_i, V_{j'}) \leq (T, V_{j'})$ for all j', this implies that $(U_i, V_{j'}) \in \mathcal{P}_1$ for all j'. By the same reasoning with V_j, we get that all $(U_i, V_j) \in \mathcal{P}_1$, so that $\mathcal{P}_2 = \emptyset$.

For (ii), assume that $\mathcal{P}^0_f = \mathcal{P}_1 +^L \mathcal{P}_2$. Since either f or f^* is a tensor product of type functions and we cannot have $f = f_1 \otimes f_2$ by (iii), it must hold that $f^* = f_1 \otimes f_2$. But then by (i) $\mathcal{P}^0_f = \mathcal{P}'_1 +^L \mathcal{P}'_2$, with $\mathcal{P}'_i = \mathcal{P}_{f_i} \setminus \{\emptyset\}$. Let

$$\mathcal{P}^0_f = \mathcal{Q}_1 +^L \cdots +^L \mathcal{Q}_M$$

be the finest decomposition into nontrivial independent components. Then there are some $C, D \subset [M]$ such that

$$\mathcal{P}_1 = +^L_{i \in C} \mathcal{Q}_i, \quad \mathcal{P}_2 = +^L_{i \in [M] \setminus C} \mathcal{Q}_i, \quad \mathcal{P}'_1 = +^L_{i \in D} \mathcal{Q}_i, \quad \mathcal{P}'_2 = +^L_{i \in [M] \setminus D} \mathcal{Q}_i.$$

Assume that $M \geq 3$, otherwise each $\mathcal{P}_i$ is one of $\mathcal{P}'_j$ and we are done. Then D or $[M] \setminus D$ has at least two elements. So assume $|D| \geq 2$. Then $\mathcal{P}'_1$ has no largest element, which implies that $\mathcal{P}'_1 = \mathcal{P}^0_{f_1^*}$ and f_1^* satisfies the assumptions in (iii). Hence $f_1 = g_1 \otimes g_2$ for some type functions g_1 and g_2 and we obtain that

$$\mathcal{P}^0_{g_1} \setminus \{\emptyset\} = +^L_{i \in D'} \mathcal{Q}_i, \quad \mathcal{P}^0_{g_2} \setminus \{\emptyset\} = +^L_{i \in D \setminus D'}$$

for some $D' \subset D$. Continuing in this way, we obtain that for any $i \in [M]$ there is some type function g_i such that $\emptyset \in \mathcal{P}^0_{g_i}$ and $\mathcal{Q}_i = \mathcal{P}^0_{g_i} \setminus \{\emptyset\}$. It follows that

$$\mathcal{P}_1 = +^L_{i \in C} (\mathcal{P}^0_{g_i} \setminus \{\emptyset\}) = \mathcal{P}^0_{\otimes_{i \in D} g_i} \setminus \{\emptyset\},$$

similarly, $\mathcal{P}_2 = \mathcal{P}^0_{\otimes_{i \in [M] \setminus D} g_i} \setminus \{\emptyset\}$.

□

Assume now that $f \in \mathcal{T}_n$ has no free indices and $\emptyset \notin \mathcal{P}_f$. Assume that $\mathcal{P}_f^0$ has no independent component, so that $f = f_1 \otimes \cdots \otimes f_k$ for some type functions $f_1, \ldots, f_k$. We will now show how to extract $\mathcal{P}_{f_i}^0$ for the finest decomposition of this form from $\mathcal{P}_f^0$. Note that f_i cannot have any free inputs or outputs.

Since f has no free outputs, any element of $[n]$ is contained in some label set. Further, any element has the form $T = (T_1, \ldots, T_k)$ with all $T_l \in \mathrm{Min}(\mathcal{P}_{f_l}^0)$ except possibly one index l_0, in which case there is a bijection of $L_{T_{l_0}}$ onto L_T. To ease the subsequent notations, we will identify the labels in $\mathcal{P}_{f_{l_0}}^0$ with the corresponding label of L_T in this case. It follows that there is a decomposition $C_1, \ldots, C_k$ of $[n]$ such that C_l consists of all labels of elements in $\mathcal{P}_{f_l}^0$. We will refer to the inclusion of an index i in C_l as coloring i by a color $l \in \{1, \ldots, k\}$.

We first note that it is enough to know the coloring of label indices of minimal elements and all elements that cover them (we will say that $T \in \mathcal{P}_f^0$ is *minimal covering* if it covers some minimal element). Indeed, assume this is known and let $U \in \mathrm{Min}(\mathcal{P}_f^0)$. Then $U = (Z_1, \ldots, Z_k)$, with $Z_l \in \mathrm{Min}(\mathcal{P}_{f_l})$, so that $Z_l \subseteq C_l$. In this way, we get all minimal elements in $\mathcal{P}_{f_l}$. (Note that we can identify minimal elements with their labels as $L_{(Z_1,\ldots,Z_k)} = \cup_l Z_l$).

To obtain $\mathcal{P}_{f_1}^0$, fix $Z_l \in C_l$, $l = 2, \ldots, k$ and consider the subposet

$$\mathcal{P} := \{(Z, Z_2, \ldots, Z_k),\ Z \in \mathrm{Min}(\mathcal{P}_{f_1}^0)\}^{\uparrow}.$$

From Corollary 2(iii), we see that after removing the minimal elements of $\mathcal{P}$, the poset decomposes into independent components, one of which corresponds to $\mathcal{P}_{f_1}^0$ with removed minimal elements. This component can be recognized by the labels of minimal covering elements, colored by $l = 1$. We then add all $Z \in \mathrm{Min}(\mathcal{P}_{f_1})$, and for $T \in \mathcal{P}_{f_i}^0$, $Z \leq T$ in $\mathcal{P}_{f_1}^0$ iff $(Z, Z_2, \ldots, Z_k) \leq T$ in $\mathcal{P}_f^0$. Other $\mathcal{P}_{f_l}^0$ are obtained similarly.

We now show how to obtain the coloring of labels of minimal and minimal covering elements. So let $L_1, \ldots, L_M$ be all the different label sets for minimal covering elements. Let $U_1, \ldots, U_N$ be the minimal elements of $\mathcal{P}_f^0$. Let $\mathcal{U} := \cup_j U_j$. For any label set L, define

$$\mathcal{V}_L^f := \cap\{U \ll_f L,\, U \in \mathrm{Min}(\mathcal{P}_f)\}, \qquad \mathcal{W}_L^f := (\cup\{U, U \in \mathrm{Min}(\mathcal{P}_f)\}) \setminus (\cup\{U \ll_f L,\, U \in \mathrm{Min}(\mathcal{P}_f)\}).$$

where we write $U \ll_f L$ if $U \ll T$ in $\mathcal{P}_f^0$ with $L_T = L$. Put $\mathcal{V}_i := \mathcal{W}_{L_i}^f$ and $\mathcal{W}_i := \mathcal{W}_{L_i}^f$. Then there is some l such that $\mathcal{V}_i, \mathcal{W}_i, L_i \subseteq C_l$. Indeed, let T be a minimal covering element such that $L_T = L_i$. Then $T = (T_1, \ldots, T_k)$, where all $T_{l'}$ are minimal in $\mathcal{P}_{f_{l'}}$ except a single index l, for which T_l is minimal covering in $\mathcal{P}_{f_l}^0$. We then have $L_i = L_T = L_{T_l} \subseteq C_l$. If $U_j = (V_1^j, \ldots, V_k^j) \ll T$, then necessarily $V_{l'}^j = T_{l'}$ for $l' \neq l$ and $V_l^j \ll T_l$, so that $V_l^j \ll_{f_l} L_i$. We obtain

$$\mathcal{V}_i = \cap\{(S_1, \ldots, S_k),\ S_{l'} \in \mathrm{Min}(\mathcal{P}_{f_{l'}}),\ S_l \ll_{f_l} L_i\}.$$

The above intersection is a disjoint union of elements of the form $\cap \mathrm{Min}(\mathcal{P}_{f_{l'}})$ for $l' \neq l$ and $\mathcal{V}_{L_i}^{f_l}$. Since no $f_{l'}$ can have free inputs, the above intersections are empty and we obtain $\mathcal{V}_i = \mathcal{V}_{L_i}^{f_l}$. Similarly, we obtain that $\mathcal{W}_i = \mathcal{W}_{L_i}^{f_l}$. It is clear from this that $\mathcal{V}_i, \mathcal{W}_i, L_i$ must all be colored by the same color. For all i, let $C_i' = \mathcal{V}_i \cup \mathcal{W}_i \cup L_i$ and define $i \sim j$ if there are some T, S that have a common upper bound in $\mathcal{P}_f^0$ and $L_T = L_i$, $L_S = L_j$, or if $C_i' \cap C_j' \neq \emptyset$. Color the elements in $\cup_{j \in [i]_\sim} C_j'$ by the same color. We next prove several claims:

Claim 1. For any $p \in \mathcal{U}$, there is some $i \in [M]$ such that $p \in \mathcal{V}_i$, so that all labels of mi nimal elements are colored. For this, let $p \in U_j = (V_1^j, \ldots, V_l^j)$, then $p \in V_l^j$ for exactly one l. By Lemma 15, V_j^j is covered by at least one $S \in \mathcal{P}_{f_l}^0$, so that $U_j \ll L_T$ for $T = (T_1, \ldots, T_k)$, with

$T_{l'} = V_{l'}^j$ for $l' \neq l$ and $T_l = S$. Hence $L_S = L_i$ for some $i \in [M]$ and in this case, $\mathcal{V}_i = \mathcal{V}_{L_S}^{f_l}$. Hence it is enough to show that $p \in \mathcal{V}_L^{f_l}$ for some label set L of a minimal covering element in $\mathcal{P}_{f_l}^0$.

By the assumptions f_l cannot have free inputs, and since $\emptyset \neq V_l^j \in \mathrm{Min}(\mathcal{P}_{f_l})$, we see that $\emptyset \in \mathcal{P}_{f_l^*}$, so that f_l^* has no free inputs as well. If $\mathcal{P}_{f_l}^0$ has a largest element, then it is the largest element in $\mathcal{P}_{f_l}$ by Lemma 11, so it cannot be a minimal covering element, since then the rank of $\mathcal{P}_{f_l}$ would be 1 which is not possible. Hence removing a chain on top of $\mathcal{P}_{f_l}^0$ does not touch the minimal covering or minimal elements, so using Lemma 12, we may assume that $f_l = h \lhd \beta$ for a type function h with no free indices, with a minimal element V_l^j and the same minimal covering label sets as f_l. Then $\mathcal{V}_L^{f_l} = \mathcal{V}_L^h$ for all minimal covering label set L.

If h is again a product, we continue this process, until we get to a situation such that $\mathcal{P}_h^0$ has independent components. In this case, the minimal set containing p and all label sets that cover it are contained in one component. If this component has a least element, we are done. If not, let g be a type function such that this component is equal to $\mathcal{P}_g^0 \setminus \{\emptyset\}$. Arguing as above about the top elements, we obtain that all minimal and minimal covering elements in the component are contained in $\mathcal{P}_{g^*}^0$. Therefore, we have $\mathcal{V}_L^h = \mathcal{V}_L^{g^*}$. If p is a free input of g^*, then $p \in \mathcal{V}_L^{g^*}$ for any minimal covering label L. Otherwise, let q be a type function with no free indices such that $g^* = p_r \lhd q$, then $\mathcal{P}_{g^*}^0$ and $\mathcal{P}_q^0$ are the same as posets, with the same label sets except that the free indices of g^* are added to the labels of minimal elements in $\mathcal{P}_q$. We may therefore continue the same process with $\mathcal{P}_q^0$. Since the number of minimal elements is decreasing, we get to a situation when all components have a least element. Hence $p \in \mathcal{V}_i$ for some i.

Claim 2. If $\emptyset \notin \mathcal{P}_f$ and $L_i, L_j \subseteq C_l$, then $i \sim j$. Since $\mathcal{V}_i$ and $\mathcal{W}_i$ have the same color as L_i, it then follows using Claim 1. that all indices in label sets of minimal and minimal covering elements in C_l will have the same color. If $\mathcal{P}_{f_l}^0$ has a largest element, then its label is an upper bound of both L_i and L_j, so that $i \sim j$. Assmue that $\mathcal{P}_{f_l}$ has no largest element, then f_l has no free indices. Since the decomposition $f = f_1 \otimes \cdots \otimes f_k$ is the finest decomposition of f as a product, f_l cannot be a product. By Lemma 16, we obtain that $\mathcal{P}_{f_l}^0$ must have independent components $\mathcal{P}_{f_l}^0 = \mathcal{P}_1 +^L \mathcal{P}_2$. Assume that $L_i \in \mathcal{P}_1$, then any minimal element it covers must be in $\mathcal{P}_1$ too. Hence $\{V \ll_{f_l} L_i,\ V \in \mathrm{Min}(\mathcal{P}_{f_l})\} \subseteq \mathcal{P}_1$, so that $\mathcal{W}_i$ contains all indices of minimal elements in $\mathcal{P}_2$. In particular, $\mathcal{V}_{i'} \subseteq \mathcal{W}_i$ for all $L_{i'}$ in $\mathcal{P}_2$, so that $i \sim i'$ for all such i'. It is easily concluded that this proves the claim.

Claim 3. If $i \sim j$, then $L_i, L_j \in C_l$ for some l. Similarly as before, this implies that if the indices in $\mathcal{U} \cup (\cup_i L_i)$ have the same color, then they belong to the same function f_l. It is quite clear that if $L_i \in C_l$ and $L_j \in C_{l'}$ with $l \neq l'$, then they cannot have a common upper bound. In this situation, we have seen that the sets C_i' and C_j' are contained in separated sets of labels, so we cannot have $C_i' \cap C_j' \neq \emptyset$. Hence $i \not\sim j$.

The only question left is the case when $\emptyset \in \mathcal{P}_{f_l}$. In this case we have $\mathcal{V}_i = \mathcal{W}_i = \emptyset$. If $L_i, L_j \subseteq C_l$ and $\mathcal{P}_{f_l}$ has a largest element, then $i \sim j$ and we are done. But otherwise $C_i' \cap C_j' = L_i \cap L_j$ which may be empty, so then $i \not\sim j$. We have to use a different procedure to obtain $\mathcal{P}_{f_l}^0$. Note that in this case, all posets o the form $U_j^{\uparrow} \setminus \{U_j\}$ contain $\mathcal{P}_{f_l}^0 \setminus \{\emptyset\}$ as one of its components, so we may look for such repeating components with label sets that have no common colors with the minimal elements. If $\mathcal{P}_{f_l}^0$ has a largest element, then $\mathcal{P}_{f_l}^0 \setminus \{\emptyset\}$ cannot have independent components. Otherwise we have $\mathcal{P}_{f_l}^0 \setminus \{\emptyset\} = \mathcal{P}_{f_l^*}^0$ and since f_l is not a product, $\mathcal{P}_{f_l^*}^0$ cannot have any independent components as well, by Lemma 16. Hence every component of $U_j^{\uparrow} \setminus \{U_j\}$ such that its minimal covering labels have no common colors with any minimal elements must correspond to a function f_l, with $\emptyset \in \mathcal{P}_{f_l}$.

Theorem 4. *Every type function $f \in \mathcal{T}_n$ is fully determined by the labelled poset $\mathcal{P}_f^0$.*

Proof. We will proceed by induction on n. If f is a chain type, the assertion follows from Lemma 11 and Proposition 8, so that the assertion holds for $n \leq 3$. Assume it is true for all $m < n$ and let $f \in \mathcal{T}_n$. By Lemmas 12 and 13 and remarks below them, if f or f^* has some free indices, then we have $f = \beta_1 \lhd h \lhd \beta_2$, where β_1, β_2 are chain types and h nor h^* do not have any free indices, moreover, the chain types and $\mathcal{P}_h^0$ can be obtained from $\mathcal{P}_f^0$. In this case, $h \in \mathcal{T}_m$ for $m < n$, so h is determined by $\mathcal{P}_h^0$ by the induction assumtions, so we are done.

If both f and f^* have no free indices, we may assume that $\emptyset \notin \mathcal{P}_f$, otherwise we replace f by f^*. Then if $\mathcal{P}_f^0$ has independent components, we have $f^* = f_1 \otimes f_2$ for some type functions $f_i \in \mathcal{T}_{n_i}$, $i = 1, 2$ and $n = n_1 + n_2$, such that the components have the form $\mathcal{P}_{f_i}^0 \setminus \{\emptyset\}$. Since $n_1, n_2 < n$, we are done. If $\mathcal{P}_f^0$ has no independent components, then $f = f_1 \otimes \cdots \otimes f_k$ for some $f_i \in \mathcal{T}_{m_i}$, $i = 1, \dots, k$, $m_1 + \cdots + m_k = n$, and we may recognize $\mathcal{P}_{f_l}^0$ for all $l = 1, \dots, k$ by the procedure described above. Again, the assertion follows by induction assumption.

□

It also folows that we can obtain a choice of set of chain types appearing in the decomposition of Theorem 3 from $\mathcal{P}_f^0$. If $f = \beta_1 \lhd h \lhd \beta_2$ for chain types β_1 and β_2 and a type function h, then we can construct such a set by adding β_1, β_2 to a collection of chains for h. If h and h^* have no free indices, then if $\mathcal{P}_h^0$ has independent components, then $h^* = h_1 \otimes \cdots \otimes h_k$, so the set of chain types of h is obtained by putting together the complements of all chain types for the functions h_i. Similarly, if $h = g_1 \otimes \cdots \otimes g_k$, we collect all chain types for all the components g_j (without taking complements this time). Each h_i or g_j can be decomposed as above, until we necessarily get to the situation when all the components are chain types themselves.

4 Conclusions

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Lemma 17. *Let $S, T \in \mathcal{P}_f^0$ be such that $S \leq T$, then $L_S \cap L_T = \emptyset$.*

Proof. Obvious from the definition of the labels.

□

Lemma 18. *Let $S, T \in \mathcal{P}_f^0$, $S \neq T$ both cover the same minimal element. Then $L_S \neq L_T$.*

Proof. Let U be the minimal element covered by both S and T. By Lemma 14, TS and T cannot cover any other minimal element and since $\mathcal{P}_f$ is graded, there can be no other element under S or T. It follows that $S = L_S \cup L_U$ and $T = L_T \cup L_U$. This proves the statement.

□

Let us introduce the following relation in $\mathcal{P}_f^0$. Denote $\mathcal{P}' = \mathcal{P}_f^0 \setminus \mathrm{Min}(\mathcal{P}_f^0)$ and for $S \in \mathcal{P}'$, let $S^\downarrow$ be the downset of S in $\mathcal{P}'$. For $S, T \in \mathcal{P}'$, we define $S \sim T$ if $S^\downarrow \simeq T^\downarrow$ (as posets), with equal labels. The minimal elements are unrelated with any other element. It is easy to see that $\sim$ is an equivalence.

Lemma 19. *Assume that $\mathcal{P}_f$ has a smallest element. Then $S \sim T$ implies $S = T$.*

Proof. Let S_0 be the least element in $\mathcal{P}_f$ and let φ be the isomorphism $S^\downarrow$ onto $T^\downarrow$ such that $L_{\varphi(U)} = L_U$, $U \in S^\downarrow$. We then have

$$T = \bigcup_{U \in \mathcal{P}_0^f, U \leq T} L_U = L_{S_0} \cup \bigcup_{U' \in S^\downarrow, U' \leq S} L_{\varphi(U')} = \bigcup_{U' \in \mathcal{P}_0^f, U' \leq S} L_{U'} = S.$$

$\square$

Lemma 20. *Let $f \in \mathcal{T}_n$ and let $T, T' \in \mathcal{P}^0_f$, $T \sim T'$. Then if $T \leq R$, then there is some $R' \sim R$ such that $T' \leq R'$.*

Proof. We will proceed by induction on n. The property is trivial for chains, hence for $n \leq 3$. Assume the property holds for all $m < n$. If $\mathcal{P}^0_f$ has a smallest element, then $T = T'$ by Lemma 19. Also, the desired property is unchanged by adding/removing a chain at the top of $\mathcal{P}^0_f$. We may therefore assume that $\mathcal{P}^0_f$ has no smallest or largest element.

If T is minimal, then $T = T'$ so the assertion is trivial. Assume that T is not minimal and $T \sim T'$. If $\mathcal{P}^0_f = \mathcal{P}_1 +^L \mathcal{P}_2$, then since $T^\downarrow$ and $(T')^\downarrow$ have the same labels, they must be contained in the same component, say $\mathcal{P}_1$. If $T \leq R$, then also $R \in \mathcal{P}_1$. By Lemma 16 there is a type function $h \in \mathcal{T}_m$ such that $\mathcal{P}_1 = \mathcal{P}^0_h \setminus \{\emptyset\}$, so that all $T^\downarrow$, $(T')^\downarrow$ and R are contained in $\mathcal{P}^0_h$. Then $T \sim T'$ also in $\mathcal{P}^0_h$ and since $m < n$, the assertion holds by induction assumption.

If $\mathcal{P}^0_f$ has no independent components, then by Lemma 16 we see that $f = f_1 \otimes f_2$ for some $f \in \mathcal{T}_m$, $f_1 \in \mathcal{T}_{n-m}$ and decomposition $n = [m] \oplus [n-m]$. In this case, we have by Corollary 2 that $T = (T_1, T_2)$, $T' = (T'_1, T'_2)$ for some $T_i, T'_i \in \mathcal{P}^0_{f_i}$, and we may assume that T_2 is minimal (the case when T_1 is minimal is treated similarly). Since T is not minimal, T_1 is not minimal as well. We have $L_{T'} = L_T = L_{(T_1,T_2)} = L_{T_1} \subseteq [m]$, so that also T'_2 is minimal, since $L_{(T'_1,T'_2)} = L_{T'_2} \subseteq [m+1, n]$ in the other case. It follows that $T^\downarrow = (T_1^\downarrow, T_2)$ and $(T')^\downarrow = ((T'_1)^\downarrow, T'_2)$, so that $T_1 \sim T'_1$. If $T \leq R$, then $R = (R_1, T_2)$ for some $R_1 \in \mathcal{P}^0_{f_1}$, $T_1 \leq R_1$ and by the induction assumption, there is some $R'_1 \in \mathcal{P}^0_{f_1}$ such that $R'_1 \sim R_1$ and $T'_1 \leq R'_1$. It is now easy to see that $R' = (R'_1, T'_2)$ satisies $T' \leq R'$ and $R \sim R'$.

$\square$

Let us denote $\mathcal{P} := \mathcal{P}^0_f|_\sim$ and let us introduce a relation $\leq$ in $\mathcal{P}$ as $[S]_\sim \leq [T]_\sim$ if $S \leq T'$ for some $T' \in [T]_\sim$. Then $\leq$ is an order relation in $\mathcal{P}$. Indeed, $\leq$ is obviously reflexive. To see that it is antisymmetric, let $T \leq S$ and $S' \leq T'$ for $T \sim T'$ and $S \sim S'$. By Lemma 20, there is some $T'' \sim T'$ such that $T \leq S \leq T''$. Since $T \sim T''$, they have the same label, so that $T = T'' = S$. The proof of transitivity is similar. We also define a label set of each $[T]_\sim \in \mathcal{P}$ as $L_{[T]_\sim} = L_T$. In this way, $\mathcal{P}$ becomes a labelled poset, with minimal elements $[U]_\sim = \{U\}$, $U \in \mathrm{Min}(\mathcal{P}^0_f)$.

We now "disentangle" the minimal elements. For any $[T]_\sim \in \mathcal{P}$ that covers a minimal element of $\mathcal{P}$ put

$$C_{[T]_\sim} := \{U \in \mathrm{Min}(\mathcal{P}^0_f), [U]_\sim \leq [T]_\sim\}, \qquad \hat{L}_{[T]_\sim} := \bigcap_{U \in C_{[T]_\sim}} L_U.$$

Let $\hat{L}_1, \ldots, \hat{L}_t$ be such that $\hat{L}_{[T]_\sim} = \hat{L}_j$ for some j and $\hat{L}_i \neq \hat{L}_j$ for $i \neq j$. We construct a new labelled poset $\hat{\mathcal{P}}$ as follows. Remove the minimal points of $\mathcal{P}$ and add elements $\mathbf{U}_1, \ldots, \mathbf{U}_t$ with labels $L_{\mathbf{U}_j} = \hat{L}_j$ and such that $\mathbf{U}_j \leq [S]_\sim$ if and only if there is some $[T]_\sim$ covering a minimal element in $\mathcal{P}$, such that $[T]_\sim \leq [S]_\sim$ and $\hat{L}_{[T]_\sim} = \hat{L}_j$.

Lemma 21. *Let $f \in \mathcal{T}_n$ be such that f nor f^* have any free inputs or outputs and let $\emptyset \notin \mathcal{P}^0_f$. Assume that $\mathcal{P}^0_f$ has no independent components. Then there is some $\sigma \mathscr{S}_n$ such that by replacing f by $f \circ \sigma$, we get $\hat{\mathcal{P}} = +^L_j \mathcal{P}_j$, where each labelled component has the form $\mathcal{P}_j = \mathcal{P}^0_{f_j}$ for some $f_j \in \mathcal{T}_{n_j}$ such that $f = \otimes_j f_j$.*

Proof. By the assumptions and Lemma 16, we must have $f = f_1 \otimes f_2$ up to a permutation $\sigma \in \mathscr{S}_n$. Let $T_1 \in \mathcal{P}^0_{f_1}$ and $T_2 \in \mathcal{P}^0_{f_2}$ be not minimal elements, then for any $U_1 \in \mathrm{Min}(\mathcal{P}_{f_1})$ and

$U_2 \in \mathrm{Min}(\mathcal{P}_{f_2})$, (T_1, U_2) and (U_1, T_2) are elements in $\mathcal{P}_f^0$, with label sets $L_{(T_1,U_2)} = L_{T_1} \subseteq [n_1]$ and $L_{(U_1,T_2)} = n_1 + L_{T_2} \subseteq [n_1+1, n]$. Such elements are incomaprable in $\mathcal{P}_f^0$ and it follows from the separation of the label sets that we have $(T_1, U_2) \not\sim (U_1, T_2)$ for such elements. It follows that $[(T_1, U_2)]_\sim$ and $[(U_1, T_2)]_\sim$ are incomparable in $\mathcal{P}$ and hence also in $\hat{\mathcal{P}}$.

Let $[(T_1, U_2)]_\sim$ be an element that covers a minimal element $[U] = \{U\}$ in $\mathcal{P}$. Let $U = (V_1, V_2)$ for some $V_i \in \mathrm{Min}(\mathcal{P}_{f_i}^0)$, then there must be some $T \in [(T_1, U_2)]_\sim$ such that $U \ll T$. As we have seen above, all elements in $[(T_1, U_2)]_\sim$ must be of the form $[(S_1, W_2)]_\sim$ for $S_1 \in \mathcal{P}_{f_1}^0$ and $W_2 \in \mathrm{Min}(\mathcal{P}_{f_2})$. It follows that we must have $W_2 = V_2$ and $V_1 \ll S_1$.

Then all $T \in [(T_1, U_2)] \in \mathcal{P}_f^0$ must cover $U = (V_1, V_2)$ for some $V_i \in \mathrm{Min}(\mathcal{P}_{f_i}^0)$. It follows that $V_2 = U_2$ and $V_1 \ll T_1$. Since all $T \in [(T_1, U_2)]_\sim$ must be of the form (S_1, W_2) for $S_1 \in \mathcal{P}_{f_1}^0$ and $W_2 \in \mathrm{Min}(\mathcal{P}_{f_2})$, we see that $V_1 \ll S_1$ and $W_2 = U_2$.

and consequently such elements are incomparable

Therefore any $T \in \mathcal{P}_f^0$ has the form (T_1, T_2) such that $T_i \in \mathcal{P}_{f_i}^0$ and at least one of T_1, T_2 is minimal.

□

In this way, using the above two lemmas, if f is not a chain type, we obtain a decomposition $f = \beta_1 \lhd h \lhd \beta_2$, with chains β_1 and β_2 and a type function h such that $I_h^F = I_{h^*}^F = O_h^F = O_{h^*}^F = \emptyset$. It is clear that f can be composed as in Theorem 3, using some set of chains for h, together with β_1 and β_2.

Let now $f \in \mathcal{T}_n$ be such that f nor f^* have any free inputs or outputs. Then $\mathcal{P}_f^0$ and $\mathcal{P}_{f^*}^0$ have no largest element, moreover, exactly one of them, say $\mathcal{P}_f^0$, contains a least element that then must be $\emptyset$ and we have $\mathcal{P}_{f^*} = \mathcal{P}_f^0 \setminus \{\emptyset\}$, with the same sets of labels. The case that $\emptyset \in \mathcal{P}_{f^*}^0$ is obtained by duality.

By construction, up to a permutation, we have either $f = f_1 \otimes f_2$ or $f^* = f_1 \otimes f_2$ for some $f_1 \in \mathcal{T}_{n_1}$ and $f_2 \in \mathcal{T}_{n_2}$ and a decomposition $[n] = [n_1] \oplus [n_2]$. Then both $\mathcal{P}_{f_1}$ and $\mathcal{P}_{f_2}$ must contain $\emptyset$. By Corollary 2, we see that $\mathcal{P}_f^0$ consits of $\mathcal{P}_{f_1}^0$ and $\mathcal{P}_{f_2}^0$ glued at the least element $\emptyset$, with labels of f_2 translated by n_1. In other words, $\mathcal{P}_{f^*}^0$ is the direct sum

$$\mathcal{P}_{f^*}^0 = (\mathcal{P}_{f_1}^0 \setminus \{\emptyset\}) + (\mathcal{P}_{f_2}^0 \setminus \{\emptyset\})$$

(with translated labels for $\mathcal{P}_{f_2}$). If $f^* = f_1 \otimes f_2$, then $\mathcal{P}_{f^*}^0$ is obtained by taking copies of $\mathcal{P}_{f_1}^0$ indexed by minimal elements in $\mathcal{P}_{f_2}$, copies of $\mathcal{P}_{f_2}^0$ indexed by minimal elements in $\mathcal{P}_{f_1}$, and glueing them at the respective minimal elements.

These considerations suggest that if $\mathcal{P}_{f^*}^0$ is a direct sum of posets with independent sets of labels, then f is a product of some f_1 and f_2, such that $\mathcal{P}_{f_i}^0$ can be obtained by adding $\emptyset$ to each of the two parts. If we can obtain some sets of chains for f_1 and f_2, then their union is a set of chains for f, as can be seen from the proof of Theorem 3. On the other hand, if $\mathcal{P}_{f^*}^0$ is a direct set of posets but the labels are repeating, then $f^* = f_1 \otimes f_2$ and the posets $\mathcal{P}_{f_i}^0$ are recognizable from the structure of $\mathcal{P}_{f^*}^0$. We then can obtain a set of chains for f^* from those for f_1 and f_2 as before. Taking duals, we obtain a set of chains for f.

Repeating the process with f_1 and f_2, we get to type functions for $n_1, n_2 < n$, so after a finite number of steps, we get to the situation when f is itself a chain.

Examples /////

A Some basic definitions

For $m \leq n \in \mathbb{N}$, we will denote the corresponding interval $\{m, m+1, \dots, n\}$ by $[m, n]$. For $m = 1$, we will simplify to $[n] := [1, n]$. Let $\mathscr{S}_n$ denote the set of all permutations of $[n]$.

A.1 Block permutations

For $n_1, n_2 \in \mathbb{N}$, $n_1 + n_2 = n$, we will denote by $[n] = [n_1] \oplus [n_2]$ the decomposition of $[n]$ as a concatenation of two intervals

$$[n] = [n_1][n_1 + 1, n_1 + n_2].$$

Similarly, for $n = \sum_{j=1}^{k} n_j$, we have the decomposition

$$[n] = \oplus_{j=1}^{k} [n_j] = [m_1, m_1 + n_1][m_2, m_2 + n_2] \dots [m_k, m_k + n_k],$$

where $m_j := \sum_{l=1}^{j-1} n_j$ (so $m_1 = 0$). Note that the order of $n_1, \dots, n_k$ in this decomposition is fixed.

We have two kinds of special permutations related to the above decomposition. For $\sigma_j \in \mathscr{S}_{n_j}$, we denote by $\oplus_j \sigma_j \in \mathscr{S}_n$ the permutation that acts as

$$m_j + l \mapsto m_j + \sigma_j(l), \qquad l = 1, \dots, n_j,\ j = 1, \dots, k.$$

On the other hand, we have for any $\lambda \in \mathscr{S}_k$ a unique permutation $\rho_\lambda \in \mathscr{S}_n$ such that ρ_λ^{-1} acts as

$$[m_1, m_1 + n_1][m_2, m_2 + n_2]...[m_k, m_k + n_k] \mapsto [m_{\lambda(1)} + n_{\lambda(1)}][m_{\lambda(2)} + n_{\lambda(2)}] \dots [m_{\lambda(k)} + n_{\lambda(k)}]$$

Note that we have

$$\rho_\lambda \circ (\oplus_j \sigma_j) = (\oplus_j \sigma_{\lambda(j)}) \circ \rho_\lambda.$$

(These permutations come from the operadic structure on the set of all permutations $\mathscr{S}_*$.)

A.2 Partially ordered sets

A partially ordered set, or a poset, is a set $\mathcal{P}$ endowed with a reflexive, antisymmetric and transitive relation $\leq$, called the partial order. We will only encounter the situation when $\mathcal{P}$ is finite. A basic example of a poset is the set $\mathcal{P}(X)$ of all subsets of a finite set X, ordered by inclusion. If $X = [n]$, we will denote $\mathcal{P}(X)$ by 2^n.

The set of minimal elements in $\mathcal{P}$ will be denoted by $\mathrm{Min}(\mathcal{P})$. For elements $p, q \in \mathcal{P}$, we say that q covers p, in notation $p \ll q$, if $p \leq q$ and for any r such that $p \leq r \leq q$ we have $r = p$ or $r = q$. If p covers a minimal element, we will say that p is a minimal covering element.

A totally ordered subposet $\mathcal{C} \subseteq \mathcal{P}$ is called a chain in $\mathcal{P}$. Such a chain is maximal if it is not contained in any other chain in $\mathcal{P}$. The length of a chain $\mathcal{C}$ is defined as $|\mathcal{C}| - 1$.

We say that a poset $\mathcal{P}$ is graded of rank k if every maximal chain in $\mathcal{P}$ has the same length equal to k. Equivalently, there is a unique rank function $\rho : \mathcal{P} \to \{0, 1, \dots, k\}$ such that $\rho(p) = 0$ if p is a minimal element of $\mathcal{P}$ and $\rho(q) = \rho(p) + 1$ if $p \ll q$. Basic examples of graded posets are chains, antichains and 2^n.

If $\mathcal{P}$ and $\mathcal{Q}$ are posets with disjoint sets, their direct sum $\mathcal{P} + \mathcal{Q}$ is a poset defined as the disjoint union $\mathcal{P} \cup \mathcal{Q}$, such that the order is preserved in each component and elements in different components are incomparable. Another way to compose $\mathcal{P}$ and $\mathcal{Q}$ is the ordinal sum $\mathcal{P} \star \mathcal{Q}$, where the underlying set is again the union $\mathcal{P} \cup \mathcal{Q}$ and the order in each component is preserved, but for $p \in \mathcal{P}$ and $q \in \mathcal{Q}$ we have $p \leq q$. A third way to compose posets that we will use is the direct product $\mathcal{P} \times \mathcal{Q}$, where the underlying set is the cartesian product $\mathcal{P} \times \mathcal{Q}$, with $(p_1, q_1) \leq (p_2, q_2)$ if $p_1 \leq p_2$ in $\mathcal{P}$ and $q_1 \leq q_2$ in $\mathcal{Q}$.

The Hasse diagram of a finite poset $\mathcal{P}$ is a graph whose vertices are elements of $\mathcal{P}$ and there is an edge between p and q if $p \ll q$, and if $p \lneq r$, then r is drawn above p.

A.3 Binary strings

A binary string of length n is a sequence $s = s_1 \dots s_n$, where $s_i \in \{0,1\}$. Such a string can be interpreted as an element $\{0,1\}^n$, but also as a map $[n] \to \{0,1\}$, or a subset in $[n] := \{1, \dots, n\}$. It will be convenient to use all these interpretations, but we will distinguish between them. The strings in $\{0,1\}^n$ will be denoted by small letters, whereas the corresponding subsets of $[n]$ will be denoted by the corresponding capital letters. More specifically, for $s \in \{0,1\}^n$ and $T \subseteq [n]$, we denote

$$S := \{i \in [n],\ s_i = 0\}, \qquad t := t_1 \dots t_n,\ t_j = 0 \iff j \in T. \tag{11}$$

As usual, the set of all subsets of $[n]$ will be denoted by 2^n. With the inclusion ordering and complementation $S^c := [n] \setminus S$, 2^n is a boolean algebra, with the smallest element $\emptyset$ and largest element $[n]$.

The group $\mathscr{S}_n$ has an obvious action on $\{0,1\}^n$. Indeed, for a string s interpreted as a map $[n] \to 2$, we may define the action of $\sigma \in \mathscr{S}_n$ by precomposition as

$$\sigma(s) := s \circ \sigma^{-1} = s_{\sigma^{-1}(1)} \dots s_{\sigma^{-1}(n)}.$$

Note that in this way we have $\rho(\sigma(s)) = (\rho \circ \sigma)(s)$. For a decomposition $[n] = \oplus_{j=1}^k [n_j]$, we have a corresponding decomposition of any string $s \in \{0,1\}^n$ as a concatenation of strings

$$s = s^1 \dots s^k, \qquad s^j \in \{0,1\}^{n_j}.$$

For permutations $\sigma_j \in \mathscr{S}_{n_j}$ and $\lambda \in \mathscr{S}_k$, we have

$$\rho_\lambda \circ (\oplus_j \sigma_j)(s^1 \dots s^k) = \rho_\lambda(\sigma_1(s^1) \dots \sigma_k(s^k)) = \sigma_{\lambda(1)}(s^{\lambda(1)}) \sigma_{\lambda(2)}(s^{\lambda(2)}) \dots \sigma_{\lambda(k)}(s^{\lambda(k)}).$$

A.4 Boolean functions and the Möbius transform

A function $f : \{0,1\}^n \to \{0,1\}$ is called a boolean function. The set of boolean functions, with pointwise ordering and complementation given by the negation $\bar{f} = 1 - f$, is a boolean algebra that can be identified with 2^{2^n}. We will denote the maximal element (the constant 1 function) by 1_n. Similarly, we denote the constant zero function by 0_n. For boolean functions f, g, the pointwise minima and maxima will be denoted by $f \wedge g$ and $f \vee g$. It is easily seen that we have

$$f \vee g = f + g - fg, \qquad f \wedge g = fg, \tag{12}$$

all the operations are pointwise. We now introduce and important example.

Example 9. For $S \subseteq [n]$, we define

$$p_S(t) = \Pi_{j \in S}(1 - t_j), \qquad t \in \{0,1\}^n.$$

That is, $p_S(t) = 1$ if and only if $S \subseteq T$. In particular, $p_\emptyset = 1_n$ and $p_{[n]}$ is the characteristic function of the zero string. Clearly, for $S, T \subseteq [n]$ we have $p_{S \cup T} = p_S p_T = p_S \wedge p_T$, in particular, $p_S = \Pi_j p_{\{j\}}$.

By the Möbius transform, all boolean functions can be expressed as combinations of the functions p_S, $S \subseteq [n]$ from the previous example.

Theorem 5. *Any $f : \{0,1\}^n \to 2$ can be expressed in the form*

$$f = \sum_{S \subseteq [n]} \hat{f}_S p_S$$

in a unique way. The coefficients $\hat{f}_S \in \mathbb{R}$ obtained as

$$\hat{f}_S = \sum_{\substack{t \in \{0,1\}^n \\ t_j = 1, \forall j \in S^c}} (-1)^{\sum_{j \in S} t_j} f(t).$$

Proof. By the Möbius inversion formula (see [Stanley, Sec. 3.7] for details), functions $f, g : 2^n \to \mathbb{R}$ satisfy

$$f(S) = \sum_{T \subseteq S} g(T), \qquad S \in 2^n$$

if and only if

$$g(S) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} f(T).$$

We now express this in terms of the corresponding strings s and t. It is easily seen that $T \subseteq S$ if and only if $s_j = 0$ for all $j \in T$, equivalently, $t_j = 1$ for all $j \in S^c$. Moreover, in this case we have $|S \setminus T| = \sum_{j \in S} t_j$. This shows that $g(S) = \hat{f}_S$, as defined in the statement. The first equality now gives

$$f(s) = f(S) = \sum_{T \subseteq S} g(T) = \sum_{T : s_i = 0, \forall i \in T} \hat{f}_T = \sum_{T : p_T(s) = 1} \hat{f}_T = \sum_{T \subseteq [n]} \hat{f}_T p_T.$$

For uniqueness, assume that $f = \sum_{T \subseteq [n]} c_T p_T$ for some coefficients $c_T \in \mathbb{R}$. Then

$$f(s) = \sum_{T : p_T(s) = 1} c_T = \sum_{T \subseteq S} c_T.$$

Uniqueness now follows by uniqueness in the Möbius inversion formula.

□

A.5 The boolean algebra $\mathcal{F}_n$

Let us introduce the subset of boolean functions

$$\mathcal{F}_n := \{f : \{0,1\}^n \to 2,\ f(\theta_n) = 1\},$$

where we use θ_n to denote the zero string $00\ldots0$. In other words, $\mathcal{F}_n$ is the interval of all elements greater than $p_{[n]}$ in the boolean algebra 2^{2^n} of all boolean functions. With the pointwise ordering, $\mathcal{F}_n$ is a distributive lattice, with top element 1_n and bottom element $p_{[n]}$. We also define complementation in $\mathcal{F}_n$ as

$$f^* := 1_n - f + p_{[n]}.$$

It can be easily checked that with these structures $\mathcal{F}_n$ is a boolean algebra, though it is not a subalgebra of 2^{2^n}.

We now introduce some more operations in $\mathcal{F}_n$. For $f \in \mathcal{F}_n$ and any permutation $\sigma \in \mathscr{S}_n$, we clearly have $f \circ \sigma \in \mathcal{F}_n$. Further, let $f \in \mathcal{F}_{n_1}$ and $g \in \mathcal{F}_{n_2}$. With the decomposition $[n_1 + n_2] = [n_1] \oplus [n_2]$ and the corresponding concatenation of strings $s = s^1 s^2$, we define the function $f \otimes g \in \mathcal{F}_{n_1+n_2}$ as

$$(f \otimes g)(s^1 s^2) = f(s^1)g(s^2), \qquad s^1 \in \{0,1\}^{n_1},\ s^2 \in \{0,1\}^{n_2}.$$

Let $\lambda \in \mathscr{S}_2$ be the transposition, then we have for any $f \in \mathcal{F}_{n_1}$ and $g \in \mathcal{F}_{n_2}$

$$(g \otimes f) = (f \otimes g) \circ \rho_\lambda,$$

where ρ_λ is the block permutation defined in Section A.1. We now show some important properties of these operations.

Lemma 22. *For $f \in \mathcal{F}_{n_1}$ and $g, h \in \mathcal{F}_{n_2}$, we have*

(i) $f \otimes g \leq (f^* \otimes g^*)^*$, with equality if and only if either $f = 1_{n_1}$ and $g = 1_{n_2}$, or $f = p_{[n_1]}$ and $g = p_{[n_2]}$.

(ii) $f \otimes (g \vee h) = (f \otimes g) \vee (f \otimes h)$, $f \otimes (g \wedge h) = (f \otimes g) \wedge (f \otimes h)$.

Proof. The inequality in (i) is easily checked, since $(f \otimes g)(s^1 s^2)$ can be 1 only if $f(s^1) = g(s^2) = 1$. If both s^1 and s^2 are the zero strings, then $s^1 s^2 = \theta_{n_1+n_2}$ and both sides are equal to 1. Otherwise, the condition $f(s^1) = g(s^2) = 1$ implies that $(f^* \otimes g^*)(s^1 s^2) = 0$, so that the right hand side must be 1. If f and g are both constant 1, then $(1_{n_1} \otimes 1_{n_2})^* = 1^*_{n_1+n_2} = p_{[n_1+n_2]} = p_{[n_1]} \otimes p_{[n_2]} = 1^*_{n_1} \otimes 1^*_{n_2}$, in the case when both f and g are the minimal elements equality follows by duality. Finally, asume the equality holds and that $f \neq 1_{n_1}$, so that there is some s^1 such that $f(s^1) = 0$. But then $s^1 \neq \theta_{n_1}$, so that $f^*(s_1) = 1$ and for any s^2,

$$0 = (f \otimes g)(s^1 s^2) = (f^* \otimes g^*)^*(s^1 s^2) = 1 - f^*(s^1) g^*(s^2) + p_{[n_1+n_2]}(s^1 s^2) = 1 - g^*(s^2),$$

which implies that $g(s^2) = 0$ for all $s^2 \neq \theta_{n_2}$, that is, $g = p_{[n_2]}$. By the same argument, $f = p_{[n_1]}$ if $g \neq 1_{n_2}$, which implies that either $f = 1_{n_1}$ and $g = 1_{n_2}$, or $f = p_{[n_1]}$ and $g = p_{[n_2]}$.

The statement (ii) is easily proved from (12).

□

Consider the decomposition $[n] = [n_1] \oplus [n_2]$ and let $S \subseteq [n_1]$, $T \subseteq [n_2]$. We then denote by $S \oplus T$ the disjoint union

$$S \oplus T := S \cup (n_1 + T) = S \cup \{n_1 + j,\ j \in T\}. \tag{13}$$

We summarize some easy properties of the basic functions p_S, $S \subseteq [n]$.

Lemma 23. *(i) For $S, T \subseteq [n]$, we have $S \subseteq T \iff p_T \leq p_S \iff p_S p_T = p_S$.*

(ii) For $S \subseteq [n]$, $\sigma \in \mathscr{S}_n$, $p_S \circ \sigma = p_{\sigma^{-1}(S)}$.

(iii) For $S \subseteq [n_1]$ and $T \subseteq [n_2]$, $p_S \otimes p_T = p_{S \oplus T}$.

Let $f \in \mathcal{F}_n$ and let $\hat{f}$ be the Möbius transform. Note that since f has values in $\{0,1\}$, we have by the proof of Theorem 6

$$\forall S \in 2^n, \quad \sum_{T \subseteq S} \hat{f}_T = f(s) \in \{0,1\}; \qquad \sum_{T \in 2^n} \hat{f}_T = f(\theta_n) = 1.$$

Proposition 14. *(i) For $f \in \mathcal{F}_n$ and $\sigma \in \mathscr{S}_n$, $\widehat{(f \circ \sigma)}_S = \hat{f}_{\sigma(S)}$,* $\qquad S \subseteq [n]$.

(ii) For $f \in \mathcal{F}_n$, $\widehat{f^*}_S = \begin{cases} 1 - \hat{f}_S & S = \emptyset \text{ or } S = [n], \\ -\hat{f}_S & \text{otherwise.} \end{cases}$

(iii) For $f \in \mathcal{F}_{n_1}$, $g \in \mathcal{F}_{n_2}$, we have $\widehat{(f \otimes g)}_{S \oplus T} = \hat{f}_S \hat{g}_T$, $S \subseteq [n_1]$, $T \subseteq [n_2]$.

Proof. All statements follow easily from Lemma 23 and the uniqueness part in Theorem 6. □

B Affine subspaces

Let V be a finite dimensional real vector space. A subset $A \subseteq V$ is an affine subspace in V if for any choice of $a_1, \dots, a_k \in A$ and $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ such that $\sum_i \alpha_i = 1$, we have $\sum_i \alpha_i a_i \in A$. It is clear that $A = \emptyset$ is trivially an affine subspace. Moreover, any linear subspace in V is an affine subspace, and an affine subspace A is linear if and only if $0 \in A$. If $A \neq \emptyset$ and also $0 \notin A$, we say that A is proper.

A proper affine subspace $A \subseteq V$ can be determined in two ways. Let

$$\mathrm{Lin}(A) := \{a_1 - a_2,\ a_1, a_2 \in A\}.$$

It is easily verified that $\mathrm{Lin}(A)$ is a linear subspace, moreover, for any $a \in A$, we have

$$\mathrm{Lin}(A) = \{a_1 - a,\ a_1 \in A\}, \qquad A = a + \mathrm{Lin}(A). \tag{14}$$

We put $\dim(A) := \dim(\mathrm{Lin}(A))$, the dimension of A.

Let V^* be the vector space dual of V and let $\langle \cdot, \cdot \rangle$ be the duality. For a subset $C \subseteq V$, put

$$\tilde{C} := \{v^* \in V^*,\ \langle v^*, a \rangle = 1,\ \forall a \in A\}.$$

Let $\tilde{a} \in \tilde{A}$ be any element and let $\mathrm{Span}(A)$ be the linear span of A in V. We then have

$$A = \mathrm{Span}(A) \cap \{\tilde{a}\}^\sim, \tag{15}$$

independently of $\tilde{a}$. The relation between the two expressions for A, given by (14) and (15) is obtained as

$$\mathrm{Span}(A) = \mathrm{Lin}(A) + \mathbb{R}\{a\}, \qquad \mathrm{Lin}(A) = \mathrm{Span}(A) \cap \{\tilde{a}\}^\perp, \tag{16}$$

independently of $a \in A$ or $\tilde{a} \in \tilde{A}$. Here $+$ denotes the direct sum of the vector spaces and $C^\perp$ denotes the annihilator of a set C. The following lemma is easily proven.

Lemma 24. *Let $C \subseteq V$ be any subset. Then $\tilde{C}$ is an affine subspace in V^* and we have*

$$0 \in \tilde{C} \iff C = \emptyset, \qquad \tilde{C} = \emptyset \iff 0 \in \mathrm{Aff}(C).$$

Assume $C \neq \emptyset$ and $0 \notin \mathrm{Aff}(C)$. Then

(i) $\tilde{C}$ is proper and we have $\mathrm{Lin}(\tilde{C}) = C^\perp = \mathrm{Span}(C)^\perp$,

(ii) $\mathrm{Aff}(C) = \tilde{\tilde{C}}$ *and for any* $c_0 \in C$*, we have*

$$\mathrm{Lin}(C) := \mathrm{Span}\{c_1 - c_2,\ c_1, c_2 \in C\} = \mathrm{Span}\{c - c_0,\ c \in C\} = \mathrm{Lin}(\tilde{\tilde{C}}).$$

Corollary 3. *Let* $A \subseteq V$ *be a proper affine subspace. Then*

(i) $\tilde{A}$ *is a proper affine subspace in* V^* *and* $\tilde{\tilde{A}} = A$*.*

(ii) $\mathrm{Lin}(\tilde{A}) = \mathrm{Span}(A)^\perp$, $\mathrm{Span}(\tilde{A}) = \mathrm{Lin}(A)^\perp$.

(iii) $\dim(\tilde{A}) = \dim(V) - \dim(A) - 1$.

The proper affine subspace $\tilde{A}$ in the above Corollary will be called the affine dual of A. Note that the dual depends on the choice of the ambient vector space V.

C Properties of $\mathcal{P}_f^0$