F. B. Maciejewski, Z. Puchała, and M. Oszmaniec: Exploring quantum average-case distances: proofs, properties and examples

Referee report

The paper introduces and studies a distance measure on quantum objects (states, POVMs and channels), that the authors call the average-case distance. It is shown that these distances approximate the average total variation distance between probability measures that are obtained by a random choice of a measurement procedure applied to a pair of objects of a given type. Here the random choice is provided by applying a random unitary, chosen by (approximate) 4-designs, to a pure state, or a measurement in computational basis, or both, according to the type of the objects. Several important properties of the average-case distances are proved and illustrative examples are discussed.

Defining a distinguishability measure based on average behaviour under random circuits is a very interesting, and also very natural idea, though the authors themselves point out some disadvantages, e.g. it does not work well for large samples of states, or the measures are not monotone under channels that are not unital. It also provides an important operational interpretation for the Hilbert-Schmidt distance.

Some minor comments and questions are listed below.

1. Reference [1] is not available.
2. The authors deal with states, measurements and channels separately, but all of these objects can be seen as a special type of channels, e.g. a state ρ can be related to a preparation channel $\sigma \mapsto \mathrm{Tr}\,[\sigma]\rho$. What is obtained from the results for channels if the channels in question are in fact both preparation channels (or measurements)?
3. There are more general measurement procedures for channels, using an ancilla (as in Fig. 1). It is a natural question what happens if the random choice is over bipartite input states and POVMs. Lemma 20 seems to suggest that for unital channels this would lead to the same results. What happens in general? Remark 8 seems also relevant.
4. $\mathcal{H} \simeq \mathcal{H}$ (in Lemma 4 and in Appendix A) seems superfluous
5. Eq.(19): a factor $C^{-1/2}$ seems missing
6. I think it would be a benefit for the readers if the authors add some explanations/more explicit computations/references for some of the formulas involving $\mathbb{P}^{(k)}_{sym}$ or expectations over the Haar measure (k-designs), e.g. for Eq. (38) or Eq. (A.3).
7. Eq. (46): $\frac{1}{2}$ seems to be missing in the second line (?)
8. Table 1, last line: some confusion of $d_{av}/d^s_{av}/d_{tr}$.
9. Table 2, last line, column 2: should probably be d_{op}
10. page 19, proof of Lemma 22: As also stated in Ref. [66], the inequality (99) for Hermiticity preserving operations was first proved in arXiv:1603.01437. It would be appropriate to cite this source.
11. Page 20, first paragraph of Sec. VI.1.: "AC-distances" appear only in this paragraph (twice), called "average-case distances" elsewhere

There are also are some more typos:

1. page 2, line 5 of the "Summary of results": only be -> only by
2. page 2, line 11 from below: ... "and the fidelity is a standard state..." something seems to be missing here
3. page 3: "Trave-preserving"
4. page 6, line 1 in Sec. IIIA: "bounds average" bounds *for* average (?)

5. Proof of Thm. 2, line 1: The proof *is* in fact...
6. page 15, just below Eq.(75): “Uhlamnn’s”
7. Lemma 18: “convextiy”
8. Page 21, line under Example 8: “can be view twofold”