Supplemental Material for "Quantum process discrimination with restricted strategies"

Kenji Nakahira
Quantum Information Science Research Center, Quantum ICT Research Institute, Tamagawa University, Machida, Tokyo 194-8610, Japan

I. NOTATION

As well as the notation in the main paper, we will denote the Choi-Jamiołkowski representations of processes as the same letter without the hat symbol, e.g., the Choi-Jamiołkowski representation of a process $\hat{\mathcal{E}}$ is denoted by $\mathcal{E}$. Note that the Choi-Jamiołkowski representation, ρ, of a state $\hat{\rho}$ is equal to $\hat{\rho}$ itself. Let N_V be the dimension of a system V. $\mathbb{0}$ stands for a zero matrix. Let $\mathsf{Chn}(V, W)$ be the set of all channels from a system V to a system W. Also, let Her_V, Pos_V, Den_V, $\mathsf{Den}^{\mathsf{P}}_V$, and Meas_V be, respectively, the sets of all Hermitian matrices, positive semidefinite matrices, states (i.e., density matrices), pure states, and measurements of a system V. Given a set $\mathcal{X}$ in a real Hilbert space, we denote its interior by $\mathrm{int}(\mathcal{X})$, its closure by $\overline{\mathcal{X}}$, its convex hull by $\mathsf{co}\,\mathcal{X}$, its (convex) conical hull by $\mathsf{coni}\,\mathcal{X}$, and its dual cone by $\mathcal{X}^*$. $\overline{\mathsf{co}\,\mathcal{X}}$ and $\overline{\mathsf{coni}\,\mathcal{X}}$ are, respectively, denoted by $\overline{\mathsf{co}}\,\mathcal{X}$ and $\overline{\mathsf{coni}}\,\mathcal{X}$. x^T denotes the transpose of a matrix x. Let P_V be the orthogonal projection onto a real Hilbert space V. Let $\tilde{V} \coloneqq W_T \otimes V_T \otimes \cdots \otimes W_1 \otimes V_1$, $\mathsf{Comb}_{W_T,V_T,\dots,W_1,V_1}$ denotes the set of all $\tau \in \mathsf{Pos}_{W_T \otimes V_T \otimes \cdots \otimes W_1 \otimes V_1}$ expressed in the form

$$\begin{aligned} \tau &= I_{W_T} \otimes \tau^{(T)}, \\ \mathrm{Tr}_{V_t} \tau^{(t)} &= I_{W_{t-1}} \otimes \tau^{(t-1)}, \quad \forall 2 \le t \le T \end{aligned}$$

with some $\tau^{(1)} \in \mathsf{Den}_{V_1}$ and $\tau^{(2)}, \dots, \tau^{(T)}$. Note that such τ is a comb. Also, let $\mathsf{Test}_{W_T,V_T,\dots,W_1,V_1}$ be the set of all $\{\Phi_m \in \mathsf{Pos}_{W_T \otimes V_T \otimes \cdots \otimes W_1 \otimes V_1}\}_{m=1}^M$ satisfying $\sum_{m=1}^M \Phi_m \in \mathsf{Comb}_{W_T,V_T,\dots,W_1,V_1}$. $\mathcal{C}_\mathrm{G} \coloneqq \mathsf{Pos}_{\tilde{V}}^M$, and $\mathcal{S}_\mathrm{G} \coloneqq \mathsf{Comb}_{W_T,V_T,\cdots,W_1,V_1}$. Figure and equation numbers without the prefix 'S' refer to those given in the main paper.

II. PROOF OF THEOREM 1

We consider the following Lagrangian associated with Problem (P):

$$L(\Phi, \varphi, \chi) \coloneqq \sum_{m=1}^M \langle \Phi_m, \tilde{\mathcal{E}}_m \rangle + \left\langle \varphi - \sum_{m=1}^M \Phi_m, \chi \right\rangle = \langle \varphi, \chi \rangle - \sum_{m=1}^M \langle \Phi_m, \chi - \tilde{\mathcal{E}}_m \rangle, \tag{S1}$$

where $\Phi \in \mathcal{C}$, $\varphi \in \mathcal{S}$, $\chi \in \mathsf{Her}_{\tilde{V}}$, and $\tilde{\mathcal{E}}_m \coloneqq p_m \mathcal{E}_m$. From Eq. (S1), we have

$$\inf_\chi L(\Phi, \varphi, \chi) = \begin{cases} \displaystyle\sum_{m=1}^M \langle \Phi_m, \tilde{\mathcal{E}}_m \rangle, & \varphi = \displaystyle\sum_{m=1}^M \Phi_m, \\ -\infty, & \text{otherwise}, \end{cases}$$

$$\sup_\Phi L(\Phi, \varphi, \chi) = \begin{cases} \langle \varphi, \chi \rangle, & \chi \in \mathcal{D}_\mathcal{C}, \\ \infty, & \text{otherwise}. \end{cases}$$

Thus, the left- and right-hand sides of the max-min inequality

$$\sup_{\Phi,\varphi} \inf_\chi L(\Phi, \varphi, \chi) \le \inf_\chi \sup_{\Phi,\varphi} L(\Phi, \varphi, \chi)$$

equal the optimal values of Problems (P) and (D), respectively.

To show the strong duality, it suffices to show that there exists $\Phi^\star \in \overline{\mathsf{co}}\,\mathcal{P}$ such that $P(\Phi^\star) \ge D^\star$, where $D^\star$ is the optimal value of Problem (D). Let us consider the following set:

$$\mathcal{Z} \coloneqq \left\{ \left(\{y_m + \tilde{\mathcal{E}}_m - \chi\}_{m=1}^M, D_\mathcal{S}(\chi) - d \right) : (\chi, y, d) \in \mathcal{Z}_0 \right\} \subset \mathsf{Her}_{\tilde{V}}^M \times \mathbb{R},$$

where $y \coloneqq \{y_m\}_{m=1}^M$ and

$$\mathcal{Z}_0 \coloneqq \left\{ (\chi, y, d) \in \mathsf{Her}_{\tilde{V}} \times \mathcal{C}^* \times \mathbb{R} : d < D^\star \right\}.$$

It is easily seen that $\mathcal{Z}$ is a nonempty convex set. Arbitrarily choose $(\chi, y, d) \in \mathcal{Z}_0$ such that $y_m + \tilde{\mathcal{E}}_m - \chi = \mathbb{0}$ $(\forall m)$; then, $D_{\mathcal{S}}(\chi) \geq D^\star$ holds from $\{\chi - \hat{\mathcal{E}}_m\}_m \in C^*$, which yields $D_{\mathcal{S}}(\chi) - d \geq D^\star - d > 0$. Thus, we have $(\{\mathbb{0}\}, 0) \notin \mathcal{Z}$. From separating hyperplane theorem [S1], there exists $(\{\Psi_m\}_{m=1}^M, \alpha) \neq (\{\mathbb{0}\}, 0)$ such that

$$\sum_{m=1}^{M} \langle \Psi_m, y_m + \hat{\mathcal{E}}_m - \chi \rangle + \alpha [D_{\mathcal{S}}(\chi) - d] \geq 0, \quad \forall (\chi, y, d) \in \mathcal{Z}_0. \tag{S2}$$

Substituting $y_m = c y'_m$ $(c \in \mathbb{R}_+, \{y'_m\}_m \in C^*)$ into Eq. (S2) and taking the limit $c \to \infty$ yields $\{\Psi_m\}_m \in C$. Taking the limit $d \to -\infty$ gives $\alpha \geq 0$. To show $\alpha > 0$, assume by contradiction $\alpha = 0$. Substituting $\chi = c I_{\tilde{V}}$ $(c \in \mathbb{R}_+)$ and taking the limit $c \to \infty$ yields $\sum_{m=1}^M \operatorname{Tr} \Psi_m \leq 0$. From $\{\Psi_m\}_m \in C \subseteq \mathsf{Pos}_{\tilde{V}}^M$, $\Psi_m = \mathbb{0}$ $(\forall m)$ holds. This contradicts $(\{\Psi_m\}_m, \alpha) \neq (\{\mathbb{0}\}, 0)$, and thus $\alpha > 0$ holds. Let $\Phi_m^\star := \Psi_m / \alpha$; then, Eq. (S2) yields

$$\sum_{m=1}^{M} \langle \Phi_m^\star, y_m + \hat{\mathcal{E}}_m - \chi \rangle + D_{\mathcal{S}}(\chi) - d \geq 0, \quad \forall (\chi, y, d) \in \mathcal{Z}_0. \tag{S3}$$

By substituting $\chi = c\chi'$ $(c \in \mathbb{R}_+, \chi' \in \mathsf{Her}_{\tilde{V}})$ into Eq. (S3) and taking the limit $c \to \infty$, we have $D_{\mathcal{S}}(\chi') \geq \sum_{m=1}^M \langle \Phi_m^\star, \chi' \rangle$ $(\forall \chi' \in \mathsf{Her}_{\tilde{V}})$. This implies $\sum_{m=1}^M \Phi_m^\star \in \mathcal{S}$, i.e., $\Phi^\star \in \overline{\mathsf{co}}\, \mathcal{P}$. Indeed, assume by contradiction $\sum_{m=1}^M \Phi_m^\star \notin \mathcal{S}$; then, since $\mathcal{S}$ is a closed convex set, from separating hyperplane theorem, there exists $\chi' \in \mathsf{Her}_{\tilde{V}}$ such that $\langle \phi, \chi' \rangle < \langle \sum_{m=1}^M \Phi_m^\star, \chi' \rangle$ $(\forall \phi \in \mathcal{S})$, which contradicts $D_{\mathcal{S}}(\chi') \geq \sum_{m=1}^M \langle \Phi_m^\star, \chi' \rangle$. Substituting $y_m = \mathbb{0}$ and $\chi = \mathbb{0}$ into Eq. (S3) and taking the limit $d \to D^\star$ yields $P(\Phi^\star) = \sum_{m=1}^M \langle \Phi_m^\star, \tilde{\mathcal{E}}_m \rangle \geq D^\star$. ■

Theorem 1 can be generalized to the following corollary.

Corollary S1 Given $\mathcal{P}$, let us arbitrarily choose a subset C of C_{G} and a bounded subset $\mathcal{S}$ of $\mathcal{S}_{\mathrm{G}}$ such that

$$\overline{\mathsf{co}}\, \mathcal{P} = \left\{ \Phi \in \overline{\mathsf{coni}}\, C : \sum_{m=1}^{M} \Phi_m \in \overline{\mathsf{co}}\, \mathcal{S} \right\}.$$

Then, the problem

$$\begin{array}{ll} \text{minimize} & \sup_{\varphi \in \mathcal{S}} \langle \varphi, \chi \rangle \\ \text{subject to} & \chi \in \mathcal{D}_C \end{array}$$

has the same optimal value as Problem (P).

Proof From Theorem 1, the following problem

$$\begin{array}{ll} \text{minimize} & D_{\overline{\mathsf{co}}\, \mathcal{S}}(\chi) \\ \text{subject to} & \chi \in \mathcal{D}_{\overline{\mathsf{coni}}\, C} \end{array}$$

has the same optimal value as Problem (P). Also, it is easily seen that $D_{\overline{\mathsf{co}}\, \mathcal{S}}(\chi) = D_{\mathsf{co}\, \overline{\mathcal{S}}}(\chi) = \sup_{\varphi \in \mathcal{S}} \langle \varphi, \chi \rangle$ and $\mathcal{D}_{\overline{\mathsf{coni}}\, C} = \mathcal{D}_C$ hold. ■

III. SUPPLEMENT OF EXAMPLES OF THEOREM 1

A. Feasible sets

For each of the two situations shown in Figs. 3(a) and (b), we show Eq. (1), where C and $\mathcal{S}$ were given in the main paper. Let $\mathcal{P}'$ be its right-hand side, i.e.,

$$\mathcal{P}' := \left\{ \Phi \in C : \sum_{m=1}^{M} \Phi_m \in \mathcal{S} \right\}. \tag{S4}$$

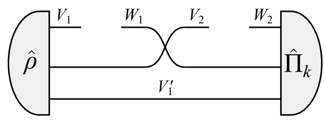
In each of these situations, we can easily obtain $\overline{\mathsf{co}}\, \mathcal{P} = \mathcal{P}$ and $\mathcal{P} \subseteq \mathcal{P}'$ (i.e., $\Phi \in C$ and $\sum_{m=1}^M \Phi_m \in \mathcal{S}$ hold for any $\Phi \in \mathcal{P}$). Thus, it suffices to show $\mathcal{P}' \subseteq \mathcal{P}$. We use diagrammatic representations to provide an intuitive understanding.

1. *The first situation*

Let us consider $\mathcal{P}'$ defined by Eq. (S4) with

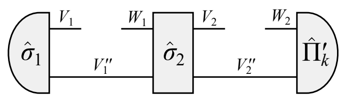
$$\mathcal{C} := \mathcal{C}_{\mathrm{G}},$$
$$\mathcal{S} := \{\times_{W_1,V_2}(I_{W_2\otimes W_1} \otimes \rho') : \rho' \in \mathsf{Den}_{V_2\otimes V_1}\}.$$

From Fig. 3(a), $\Phi \in \mathcal{P}$ holds if and only if $\hat{\Phi}_k$ is expressed in the form



, (S5)

where $\hat{\rho} \in \mathsf{Den}_{V_1\otimes V_2\otimes V_1'}$ and $\{\hat{\Pi}_m\}_{m=1}^M \in \mathsf{Meas}_{W_2\otimes W_1\otimes V_1'}$. Arbitrarily choose $\Phi' \in \mathcal{P}'$. One can easily verify $\mathcal{S} \subseteq \mathcal{S}_{\mathrm{G}}$, i.e., $\mathcal{P}' \subseteq \mathcal{P}_{\mathrm{G}}$. Thus, $\hat{\Phi}_k'$ is expressed in the form



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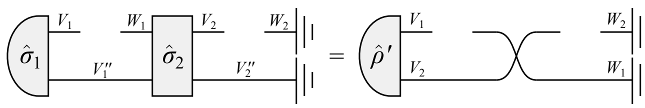
where $\hat{\sigma}_1 \in \mathsf{Den}_{V_1\otimes V_1''}$, $\hat{\sigma}_2 \in \mathsf{Chn}(W_1 \otimes V_1'', V_2 \otimes V_2'')$, and $\{\hat{\Pi}_m'\}_{m=1}^M \in \mathsf{Meas}_{W_2\otimes V_2''}$. Also, from $\sum_{m=1}^M \Phi_m' \in \mathcal{S}$, there exists $\hat{\rho}' \in \mathsf{Den}_{V_1\otimes V_2}$ such that



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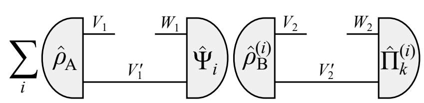
where "$\dashv\!\Vert\,$" denotes the trace. Thus, $\hat{\Phi}_k'$ is expressed in the form of Eq. (S5), where $\hat{\rho}$ is a purification of $\hat{\rho}'$. Therefore, $\mathcal{P}' \subseteq \mathcal{P}$ holds.

2. *The second situation*

Let us consider $\mathcal{P}'$ of Eq. (S4) with

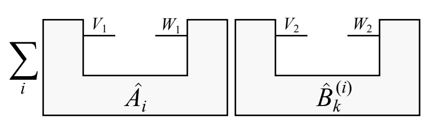
$$\mathcal{C} := \left\{ \left\{ \sum_i B_m^{(i)} \otimes A_i \right\}_m : A_i \in \mathsf{Pos}_{W_1\otimes V_1},\ \{B_m^{(i)}\}_m \in \mathsf{Test}_{W_2,V_2} \right\},$$
$$\mathcal{S} := \mathcal{S}_{\mathrm{G}}.$$

From Fig. 3(b), $\Phi \in \mathcal{P}$ holds if and only if $\hat{\Phi}_k$ is expressed in the form



, (S6)

where $\hat{\rho}_{\mathrm{A}} \in \mathsf{Den}_{V_1\otimes V_1'}$, $\{\hat{\Psi}_i\}_i \in \mathsf{Meas}_{W_1\otimes V_1'}$, $\hat{\rho}_{\mathrm{B}}^{(i)} \in \mathsf{Den}_{V_2\otimes V_2'}$ $(\forall i)$, and $\{\hat{\Pi}_m^{(i)}\}_{m=1}^M \in \mathsf{Meas}_{W_2\otimes V_2'}$ $(\forall i)$. Arbitrarily choose $\Phi' \in \mathcal{P}'$. From $\Phi' \in \mathcal{C}$, $\hat{\Phi}_k'$ is expressed in the form



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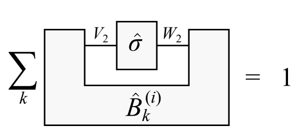
where $A_i \in \mathsf{Pos}_{W_1\otimes V_1}$ and $\{B_m^{(i)}\}_{m=1}^M \in \mathsf{Test}_{W_2,V_2}$ hold for each i. Arbitrarily choose $\hat{\sigma} \in \mathsf{Chn}(V_2, W_2)$; then, from



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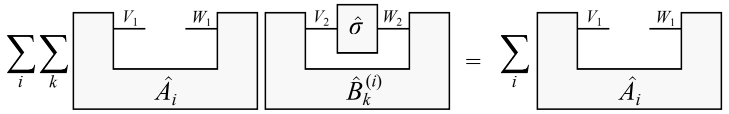
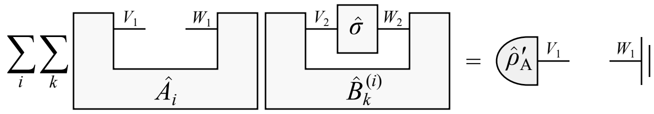
we have



(S7)

Also, from $\sum_{k=1}^{M} \Phi'_k \in \mathcal{S}_{\mathrm{G}}$, we have



(S8)

with some $\hat{\rho}'_{\mathrm{A}} \in \mathsf{Den}_{V_1}$. Equations (S7) and (S8) yield that $\{\hat{A}_i\}$ is a tester. Since $\{\hat{A}_i\}$ and $\{\hat{B}^{(i)}_m\}_m$ are testers, $\hat{\Phi}'_k$ is expressed in the form of Eq. (S6), i.e., $\mathcal{P}' \subseteq \mathcal{P}$.

B. Numerical example of sequential discrimination

We present a numerical example of a discrimination problem when a tester is restricted to be in the form of Fig. 3(b). Such a tester, which we will call a *sequential tester*, consists of two sequentially connected single-shot testers. As a toy problem with $V_2 = \mathbb{C}$, let us consider the discrimination problem of quantum processes $\{\hat{\mathcal{E}}_m\}_{m=1}^M$ with prior probabilities $\{p_m\}_{m=1}^M$. Let $\hat{\mathcal{E}}_m$ be the tensor product of the amplitude damping channel $\hat{A}_{q_m} \in \mathsf{Chn}(V_1, W_1)$ (with the damping parameter q_m) and a state $\hat{\rho}_m \in \mathsf{Den}_{W_2}$, i.e.,

$$\hat{\mathcal{E}}_m \coloneqq \hat{A}_{q_m} \otimes \hat{\rho}_m. \tag{S9}$$

$\hat{A}_{q_m}$ is the qubit channel expressed by

$$\hat{A}_{q_m}(\sigma) = E_0 \sigma E_0^\dagger + E_1 \sigma E_1^\dagger, \quad \forall \sigma \in \mathsf{Den}_{V_1},$$
$$E_0 \coloneqq \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-q_m} \end{bmatrix}, \quad E_1 \coloneqq \begin{bmatrix} 0 & \sqrt{q_m} \\ 0 & 0 \end{bmatrix},$$

whose Choi-Jamiołkowski representation is

$$A_{q_m} = \begin{bmatrix} 1 & 0 & 0 & \sqrt{1-q_m} \\ 0 & q_m & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{1-q_m} & 0 & 0 & 1-q_m \end{bmatrix}.$$

Figure S1 shows process discrimination protocols with any tester and a sequential tester.

The success probability, P_{seq}, of a best sequential tester is the optimal value of Problem (P) with

$$\mathcal{P} = \left\{ \left\{ \sum_i B_m^{(i)} \otimes A_i \right\}_m : \{A_i\} \in \mathsf{Test}^A_{W_1,V_1},\ \{B_m^{(i)}\}_m \in \mathsf{Meas}_{W_2} \right\},$$

where $\mathsf{Test}^A_{W_1,V_1}$ is the set of all testers with any number of outcomes. Since directly solving this problem is hard, we instead solve its dual problem, i.e., Problem (5) with Eq. (S9) (note that $\mathsf{Test}_{W_2,V_2} = \mathsf{Meas}_{W_2}$ holds from $V_2 = \mathbb{C}$). By letting $\tilde{\chi} \coloneqq \mathrm{Tr}_{W_2} \chi \in \mathsf{Her}_{W_1 \otimes V_1}$ and $r_m(\Pi) \coloneqq p_m \langle \Pi_m, \rho_m \rangle$, we can rewrite this problem as

$$\begin{aligned} &\text{minimize} \quad \max_{\rho' \in \mathsf{Den}_{V_1}} \langle I_{W_1} \otimes \rho', \tilde{\chi} \rangle \\ &\text{subject to} \quad \tilde{\chi} \ge \sum_{m=1}^M r_m(\Pi) A_{q_m} = \sum_{m=1}^M r_m(\Pi) \begin{bmatrix} 1 & 0 & 0 & \sqrt{1-q_m} \\ 0 & q_m & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{1-q_m} & 0 & 0 & 1-q_m \end{bmatrix} \quad (\forall \Pi \in \mathsf{Meas}_{W_2}). \end{aligned} \tag{S10}$$

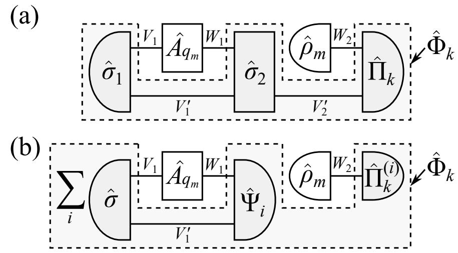


FIG. S1. Discrimination of $\{\hat{\mathcal{E}}_m := \hat{A}_{q_m} \otimes \hat{\rho}_m\}_m$ when (a) all testers are allowed and (b) only sequential testers are allowed, where $\hat{\sigma}_1 \in \mathsf{Den}_{V_1 \otimes V_1'}$, $\hat{\sigma}_2 \in \mathsf{Chn}(W_1 \otimes V_1', V_2')$, $\{\hat{\Pi}_m\}_{m=1}^M \in \mathsf{Meas}_{W_2 \otimes V_2'}$, $\hat{\sigma} \in \mathsf{Den}_{V_1 \otimes V_1'}$, $\{\hat{\Psi}_i\}_i \in \mathsf{Meas}_{W_1 \otimes V_1'}$, and $\{\hat{\Pi}_m^{(i)}\}_{m=1}^M \in \mathsf{Meas}_{W_2}$ $(\forall i)$.

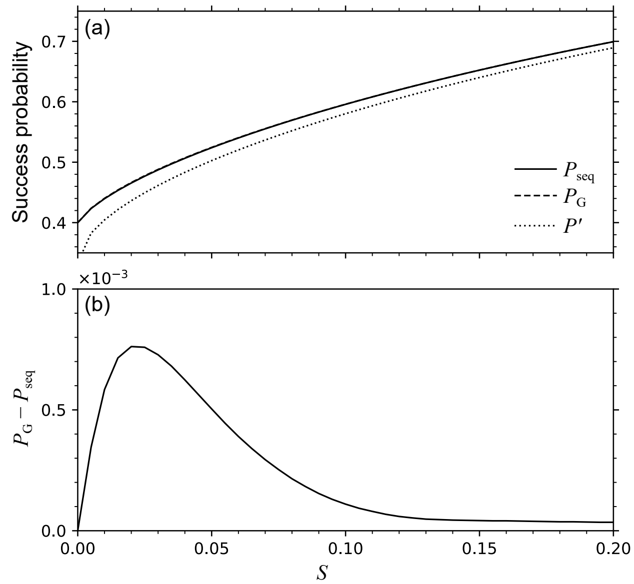


FIG. S2. Maximum success probabilities with and without the restriction of testers: (a) the maximum success probabilities $P_{\rm seq}$, $P_{\rm G}$, and P' (note that $P_{\rm seq}$ and $P_{\rm G}$ are largely overlapped); (b) $P_{\rm G} - P_{\rm seq}$.

Theorem 1 ensures that the optimal value of this problem is $P_{\rm seq}$. It is easily seen that there exists an optimal solution, $\tilde{\chi}^\star$, to Problem (S10) satisfying $\mathrm{Tr}_{W_1} \tilde{\chi}^\star \propto I_{V_1}$. Due to the symmetry of Problem (S10), without loss of generality, we can assume that $\tilde{\chi}^\star$ is in the form

$$\tilde{\chi}^\star = \begin{bmatrix} s & 0 & 0 & y_2 \\ 0 & y_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ y_2 & 0 & 0 & s - y_1 \end{bmatrix}$$

with $s, y_1, y_2 \in \mathbb{R}$. Thus, Problem (S10) is rewritten as

$$\begin{aligned} &\text{minimize} \quad s \\ &\text{subject to} \quad y_1 \ge \sum_{m=1}^M r_m(\Pi) q_m, \quad \begin{bmatrix} s & y_2 \\ y_2 & s - y_1 \end{bmatrix} \ge \sum_{m=1}^M r_m(\Pi) \begin{bmatrix} 1 & \sqrt{1-q_m} \\ \sqrt{1-q_m} & 1-q_m \end{bmatrix} \quad (\forall \Pi \in \mathsf{Meas}_{W_2}). \end{aligned}$$

Figure S2 shows a numerical calculation of $P_{\rm seq}$, where $M = 3$, $(q_1, q_2, q_3) = (0.6, 0.7, 0.8)$, $p_m = 1/3$ (i.e., equal prior probabilities), and $\hat{\rho}_m$ is the optical coherent state with the average photon number S and the phase $2\pi m/3$. $P_{\rm G}$ is the optimal value of Problem ($\mathrm{P_G}$), which is a semidefinite programming problem and thus efficiently solved. P' is the maximum success probability for sequential testers in which Bob discards the result of Alice's tester, i.e., P' is the maximum success probability for discriminating the states $\{\hat{\rho}_m\}_{m=1}^3$ with equal prior probabilities. $P' \le P_{\rm seq} \le P_{\rm G}$ obviously holds. Both $P_{\rm seq}$ and $P_{\rm G}$ approach P' as S increases. One can see that $P_{\rm G} - P_{\rm seq}$ is very small, which implies that an optimal sequential tester is almost globally optimal.

IV. PROOF OF PROPOSITION 2

We prove the following proposition, which immediately implies Proposition 2.

Proposition S2 Let us arbitrarily choose a closed convex cone $\mathcal{C}$ and a closed convex set $\mathcal{S}$ satisfying Eq. (1). Then, the following statements are all equivalent.

(1) The optimal values of Problems (P) and ($\mathrm{P_G}$) are the same.
(2) Any optimal solution to Problem ($\mathrm{D_G}$) is optimal for Problem (D).
(3) There exists an optimal solution $\chi^\star$ to Problem (D) such that $\chi^\star$ is in $\mathcal{D}_{\mathcal{C}_\mathrm{G}}$ and is proportional to some quantum comb.
(4) There exists an optimal solution $\chi^\star$ to Problem (D) such that $\chi^\star \in \mathcal{D}_{\mathcal{C}_\mathrm{G}}$ and $D_\mathcal{S}(\chi^\star) = D_{\mathcal{S}_\mathrm{G}}(\chi^\star)$ hold.

Proof Let $D^\star$ and $D^\star_\mathrm{G}$ be, respectively, the optimal values of Problems (D) and ($\mathrm{D_G}$) [or, equivalently, the optimal values of Problems (P) and ($\mathrm{P_G}$)]. We show (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4), and (4) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2) : Let us arbitrarily choose an optimal solution $\chi^\star$ to Problem ($\mathrm{D_G}$). Since $\mathcal{D}_{\mathcal{C}_\mathrm{G}} \subseteq \mathcal{D}_\mathcal{C}$ holds from $\mathcal{C} \subseteq \mathcal{C}_\mathrm{G}$, $\chi^\star \in \mathcal{D}_\mathcal{C}$ holds. Also, from $\mathcal{S} \subseteq \mathcal{S}_\mathrm{G}$, we have $D^\star \le D_\mathcal{S}(\chi^\star) \le D_{\mathcal{S}_\mathrm{G}}(\chi^\star)$. Since $D^\star = D^\star_\mathrm{G} = D_{\mathcal{S}_\mathrm{G}}(\chi^\star)$ holds, we have $D^\star = D_\mathcal{S}(\chi^\star)$. Thus, $\chi^\star$ is optimal for Problem (D).

(2) $\Rightarrow$ (3) : It is known that there exists an optimal solution, $\chi^\star \in \mathcal{D}_{\mathcal{C}_\mathrm{G}}$, to Problem ($\mathrm{D_G}$) such that $\chi^\star$ is proportional to some quantum comb [S2, S3]. From Statement (2), $\chi^\star$ is optimal for Problem (D).

(3) $\Rightarrow$ (4) : $\chi^\star$ can be expressed as $\chi^\star = qT$ with $q \in \mathbb{R}_+$ and a quantum comb T. Since $\langle \varphi, \chi^\star \rangle = q \langle \varphi, T \rangle = q$ holds for any $\varphi \in \mathcal{S}_\mathrm{G}$, we have $D_\mathcal{S}(\chi^\star) = D_{\mathcal{S}_\mathrm{G}}(\chi^\star)$.

(4) $\Rightarrow$ (1) : We have $D^\star = D_\mathcal{S}(\chi^\star) = D_{\mathcal{S}_\mathrm{G}}(\chi^\star) \ge D^\star_\mathrm{G}$. Since $D^\star \le D^\star_\mathrm{G}$ holds, $D^\star = D^\star_\mathrm{G}$ must hold. ∎

V. NECESSARY AND SUFFICIENT CONDITION FOR GLOBAL OPTIMALITY

In some individual cases, necessary and sufficient conditions for global optimality can be derived from Theorem 1. To give an example, let us consider single-shot channel discrimination problems in which a state input to the channel is restricted to be separable (see Fig. S3). Since we can assume, without loss of generality, that the input state is a pure state of the system V_1, the optimal value $P^\star$ of Problem (P) is written as

$$P^\star \coloneqq \max_{\phi \in \mathsf{Den}^\mathsf{P}_{V_1}} \max_{\Pi \in \mathsf{Meas}_{W_1}} \sum_{m=1}^{M} p_m \langle \Pi_m, \hat{\mathcal{E}}_m(\phi) \rangle .$$

Since the dual of the discrimination problem in which an input state is fixed to ϕ is formulated as Problem (D) with $\mathcal{C} = \mathcal{C}_\mathrm{G}$ and $\mathcal{S} = \left\{ I_{W_1} \otimes \phi^\mathsf{T} \right\}$, Theorem 1 gives

$$\max_{\Pi \in \mathsf{Meas}_{W_1}} \sum_{m=1}^{M} p_m \langle \Pi_m, \hat{\mathcal{E}}_m(\phi) \rangle = \min_{\chi \in \mathcal{D}_{\mathcal{C}_\mathrm{G}}} \langle I_{W_1} \otimes \phi^\mathsf{T}, \chi \rangle , \quad \forall \phi \in \mathsf{Den}^\mathsf{P}_{V_1},$$

and thus

$$P^\star = \max_{\phi' \in \mathsf{Den}^\mathsf{P}_{V_1}} \min_{\chi \in \mathcal{D}_{\mathcal{C}_\mathrm{G}}} \langle I_{W_1} \otimes \phi', \chi \rangle .$$

Also, the optimal value of Problem ($\mathrm{D_G}$) is expressed by

$$\min_{\chi \in \mathcal{D}_{\mathcal{C}_\mathrm{G}}} \max_{\rho \in \mathsf{Den}_{V_1}} \langle I_{W_1} \otimes \rho^\mathsf{T}, \chi \rangle = \min_{\chi \in \mathcal{D}_{\mathcal{C}_\mathrm{G}}} \max_{\phi \in \mathsf{Den}^\mathsf{P}_{V_1}} \langle I_{W_1} \otimes \phi^\mathsf{T}, \chi \rangle = \min_{\chi \in \mathcal{D}_{\mathcal{C}_\mathrm{G}}} \max_{\phi' \in \mathsf{Den}^\mathsf{P}_{V_1}} \langle I_{W_1} \otimes \phi', \chi \rangle .$$

Thus, globally optimal discrimination is achieved without entanglement if and only if the following max-min inequality holds as an equality:

$$\max_{\phi' \in \mathsf{Den}^\mathsf{P}_{V_1}} \min_{\chi \in \mathcal{D}_{\mathcal{C}_\mathrm{G}}} \langle I_{W_1} \otimes \phi', \chi \rangle \le \min_{\chi \in \mathcal{D}_{\mathcal{C}_\mathrm{G}}} \max_{\phi' \in \mathsf{Den}^\mathsf{P}_{V_1}} \langle I_{W_1} \otimes \phi', \chi \rangle .$$

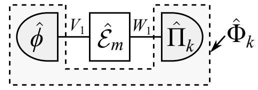


FIG. S3. Single-shot channel discrimination problems in which a state input to the channel is restricted to be separable. We can assume that the input state is a pure state of the system V_1, i.e., a tester consists of $\hat{\phi} \in \mathsf{Den}^{\mathsf{P}}_{V_1}$ and $\{\hat{\Pi}_m\}_{m=1}^M \in \mathsf{Meas}_{W_1}$.

VI. SYMMETRIC PROBLEMS

Given a process discrimination problem that has a certain symmetry, we present a sufficient condition for a nonadaptive tester to be globally optimal. We here limit our discussion to a specific type of symmetries (see [S3] for a more general case).

A. Lemmas

Uni_V denotes the set of all unitary and anti-unitary operators on a system V. Let $\mathcal{G}$ be a group that has two projective unitary or anti-unitary representations (which we simply call projective representations) $\mathcal{G} \ni g \mapsto U_g \in \mathsf{Uni}_W$ and $\mathcal{G} \ni g \mapsto \tilde{U}_g \in \mathsf{Uni}_V$. We will call a channel $\hat{\Lambda} \in \mathsf{Chn}(V, W)$ $\{(U_g, \tilde{U}_g)\}_{g\in\mathcal{G}}$*-covariant* if

$$\mathrm{Ad}_{U_g \otimes \tilde{U}_g}(\Lambda) = \Lambda, \quad \forall g \in \mathcal{G},$$

or, equivalently,

$$\mathrm{Ad}_{U_g} \circ \hat{\Lambda} \circ \mathrm{Ad}_{\tilde{U}_g^{\mathsf{T}}} = \hat{\Lambda}, \quad \forall g \in \mathcal{G},$$

holds, where Ad_U is the unitary or anti-unitary transformation defined as $\mathrm{Ad}_U(\rho) = U\rho U^\dagger$.

To give our results in the next subsection, we first prove the following two lemmas.

Lemma S3 Let us consider Problem (P) with $W_1' = \cdots = W_{T-1}' = \mathbb{C}$, in which case each process $\mathcal{E}_m$ is expressed by the combination of T channels $\{\hat{\Lambda}_m^{(t)} \in \mathsf{Chn}(V_t, W_t)\}_{t=1}^T$. Assume that, for each $t \in \{1, \dots, T\}$, there exists a group $\mathcal{G}^{(t)}$ that has two projective representations $\mathcal{G}^{(t)} \ni g \mapsto U_g^{(t)} \in \mathsf{Uni}_{W_t}$ and $\mathcal{G}^{(t)} \ni g \mapsto \tilde{U}_g^{(t)} \in \mathsf{Uni}_{V_t}$. Also, assume that, for each $t \in \{1, \dots, T\}$ and $m \in \{1, \dots, M\}$, $\hat{\Lambda}_m^{(t)}$ is $\{(U_g^{(t)}, \tilde{U}_g^{(t)})\}_{g\in\mathcal{G}^{(t)}}$-covariant. If

$$\begin{aligned} &C = C_{\mathrm{G}}, \\ &\mathrm{Ad}_{U_g^{(t)} \otimes \tilde{U}_g^{(t)}}(\varphi) \in \mathcal{S}, \quad \forall t \in \{1, \dots, T\},\ g \in \mathcal{G}^{(t)},\ \varphi \in \mathcal{S} \end{aligned} \tag{S11}$$

holds, then there exists an optimal solution, $\chi^\star \in \mathsf{Pos}_{\tilde{V}}$, to Problem (D) such that

$$\mathrm{Ad}_{U_g^{(t)} \otimes \tilde{U}_g^{(t)}}(\chi^\star) = \chi^\star, \quad \forall t \in \{1, \dots, T\},\ g \in \mathcal{G}^{(t)}. \tag{S12}$$

Proof Let χ be an optimal solution to Problem (D). From $C = C_{\mathrm{G}}$, we can easily see $\chi \in \mathsf{Pos}_{\tilde{V}}$. Also, let $\mathcal{G} := \mathcal{G}^{(1)} \times \mathcal{G}^{(2)} \times \cdots \times \mathcal{G}^{(T)}$ and

$$\chi^\star := \frac{1}{|\mathcal{G}|} \sum_{g\in\mathcal{G}} \mathcal{U}_g(\chi) \in \mathsf{Pos}_{\tilde{V}}, \quad \mathcal{U}_{(g_1,\dots,g_T)} := \mathrm{Ad}_{U_{g_T}^{(T)} \otimes \tilde{U}_{g_T}^{(T)} \otimes \cdots \otimes U_{g_1}^{(1)} \otimes \tilde{U}_{g_1}^{(1)}},$$

where $|\mathcal{G}|$ is the order of $\mathcal{G}$. Since $\mathcal{U}_g \circ \mathcal{U}_{g'} = \mathcal{U}_{gg'}$ holds for any $g, g' \in \mathcal{G}$, one can easily see Eq. (S12). We have that from $\mathcal{U}_g(\mathcal{E}_m) = \mathcal{E}_m$ [which follows from $\hat{\Lambda}_m^{(t)}$ being $\{(U_g^{(t)}, \tilde{U}_g^{(t)})\}_{g\in\mathcal{G}^{(t)}}$-covariant] and $\chi \geq \mathcal{E}_m$,

$$\chi^\star - \mathcal{E}_m = \frac{1}{|\mathcal{G}|} \sum_{g\in\mathcal{G}} \mathcal{U}_g(\chi - \mathcal{E}_m) \geq \mathbb{0}, \quad \forall m \in \{1, \dots, M\},$$

i.e., $\chi^\star \in \mathcal{D}_C$. Moreover, we have

$$\begin{aligned} D_{\mathcal{S}}(\chi^\star) &= \max_{\varphi\in\mathcal{S}} \langle \varphi, \chi^\star \rangle = \frac{1}{|\mathcal{G}|} \max_{\varphi\in\mathcal{S}} \sum_{g\in\mathcal{G}} \langle \varphi, \mathcal{U}_g(\chi) \rangle \leq \frac{1}{|\mathcal{G}|} \sum_{g\in\mathcal{G}} \max_{\varphi\in\mathcal{S}} \langle \varphi, \mathcal{U}_g(\chi) \rangle \\ &= \frac{1}{|\mathcal{G}|} \sum_{g\in\mathcal{G}} \max_{\varphi\in\mathcal{S}} \langle \mathcal{U}_{\bar{g}}(\varphi), \chi \rangle \leq \frac{1}{|\mathcal{G}|} \sum_{g\in\mathcal{G}} D_{\mathcal{S}}(\chi) = D_{\mathcal{S}}(\chi), \end{aligned}$$

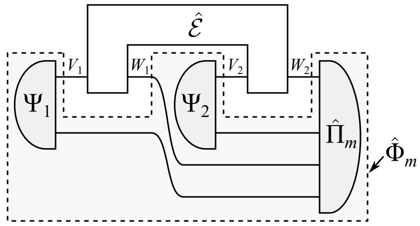


FIG. S4. Tester with maximally entangled pure states (in the case of $T = 2$), which consists of maximally entangled pure states Ψ_t and a measurement $\{\hat{\Pi}_m\}$. We can assume, without loss of generality, that each Ψ_t is a generalized Bell state $|I_{V_t}\rangle\!\rangle\langle\!\langle I_{V_t}|/N_{V_t}$.

where $\bar{g}$ is the inverse of g. The last inequality follows from $\mathcal{U}_{\bar{g}}(\varphi) \in \mathcal{S}$ for each $\varphi \in \mathcal{S}$ and $g \in \mathcal{G}$, which follows from the second line of Eq. (S11). Therefore, $\chi^\star$ is optimal for Problem (D). ■

Lemma S4 Assume that, for each $t \in \{1, \dots, T\}$, there exists a group $\mathcal{G}^{(t)}$ that has two projective representations $\mathcal{G}^{(t)} \ni g \mapsto U_g^{(t)} \in \mathsf{Uni}_{W_t}$ and $\mathcal{G}^{(t)} \ni g \mapsto \tilde{U}_g^{(t)} \in \mathsf{Uni}_{V_t}$. If $g \mapsto \tilde{U}_g^{(t)}$ is irreducible for each $t \in \{1, \dots, T\}$, then any $\chi^\star \in \mathsf{Pos}_{\tilde{V}}$ satisfying Eq. (S12) is proportional to some quantum comb.

Proof It suffices to show that, for each $t \in \{1, \dots, T\}$, $\mathrm{Tr}_{W_t}\chi^\star$ is expressed in the form $\mathrm{Tr}_{W_t}\chi^\star = I_{V_t} \otimes \chi_t^\star$ with $\chi_t^\star \in \mathsf{Pos}_{X_t}$, where X_t is the tensor product of all $W_{t'} \otimes V_{t'}$ with $t' \in \{1, \dots, t-1, t+1, \dots, T\}$. Indeed, in this case, one can easily verify that $\chi^\star$ is proportional to some quantum comb. Let us fix $t \in \{1, \dots, T\}$. Also, let $\chi_s^\star \coloneqq \mathrm{Tr}_{X_t}[(I_{V_t} \otimes s)\,\mathrm{Tr}_{W_t}\chi^\star] \in \mathsf{Pos}_{V_t}$, where $s \in \mathsf{Pos}_{X_t}$ is arbitrarily chosen; then, we have

$$\mathrm{Tr}\,\chi_s^\star = \langle s, \chi' \rangle, \quad \chi' \coloneqq \mathrm{Tr}_{W_t \otimes V_t}\chi^\star. \tag{S13}$$

Equation (S12) gives $\mathrm{Ad}_{\tilde{U}_g^{(t)}}(\chi_s^\star) = \chi_s^\star$ $[\forall g \in \mathcal{G}^{(t)}]$. From Schur's lemma (on anti-unitary groups) [S4], $\chi_s^\star$ must be proportional to I_{V_t}. Thus, from Eq. (S13), $\chi_s^\star = \langle s, \chi' \rangle I_{V_t}/N_{V_t}$ holds. We have that for any $s' \in \mathsf{Pos}_{V_t}$,

$$\langle s' \otimes s, \mathrm{Tr}_{W_t}\chi^\star \rangle = \langle s', \chi_s^\star \rangle = \langle s, \chi' \rangle \langle s', I_{V_t}/N_{V_t} \rangle = \langle s' \otimes s, I_{V_t} \otimes \chi'/N_{V_t} \rangle. \tag{S14}$$

Since Eq. (S14) holds for any s and s', we have $\mathrm{Tr}_{W_t}\chi^\star = I_{V_t} \otimes \chi_t^\star$ with $\chi_t^\star \coloneqq \chi'/N_{V_t} \in \mathsf{Pos}_{X_t}$. ■

B. Sufficient condition for a nonadaptive tester to be globally optimal

We will call a tester each of whose output systems is one part of a bipartite system in a maximally entangled pure state (see Fig. S4) a *tester with maximally entangled pure states*. Such a tester is obviously nonadaptive. From Lemmas S3 and S4, we obtain a sufficient condition that there exists a tester with maximally entangled pure states that is globally optimal. Let $\mathcal{P}$ be the set of testers with maximally entangled pure states; then, it follows that Eq. (1) holds with

$$C \coloneqq C_{\mathrm{G}}, \quad \mathcal{S} \coloneqq \left\{ I_{\tilde{V}} / \prod_{t=1}^{T} N_{V_t} \right\}.$$

Note that $\overline{\mathsf{co}}\,\mathcal{P} = \mathcal{P}$ holds in this case. It is easily seen that Eq. (S11) holds. Thus, we immediately obtain the following proposition.

Proposition S5 Let us consider Problem (P) with $W'_1 = \cdots = W'_{T-1} = \mathbb{C}$, in which case each process $\mathcal{E}_m$ is expressed by the combination of T channels $\{\hat{\Lambda}_m^{(t)} \in \mathsf{Chn}(V_t, W_t)\}_{t=1}^T$. Assume that, for each $t \in \{1, \dots, T\}$, there exists a group $\mathcal{G}^{(t)}$ that has two projective representations $\mathcal{G}^{(t)} \ni g \mapsto U_g^{(t)} \in \mathsf{Uni}_{W_t}$ and $\mathcal{G}^{(t)} \ni g \mapsto \tilde{U}_g^{(t)} \in \mathsf{Uni}_{V_t}$ and the latter of which is irreducible. Also, assume that, for each $t \in \{1, \dots, T\}$, $\hat{\Lambda}_1^{(t)}, \cdots, \hat{\Lambda}_M^{(t)}$ are $\{(U_g^{(t)}, \tilde{U}_g^{(t)})\}_{g \in \mathcal{G}^{(t)}}$-covariant. Then, there exists a globally optimal tester with maximally entangled pure states.

Proof It is obvious from Lemmas S3 and S4 and Proposition 2. ■

The following corollary follows from this proposition in the special case of $\hat{\Lambda}_m^{(1)} = \cdots = \hat{\Lambda}_m^{(T)}$.

Corollary S6 Let us consider the T-shot discrimination problem of channels $\hat{\Lambda}_1, \dots, \hat{\Lambda}_M \in \mathsf{Chn}(V, W)$. Assume that there exists a group $\mathcal{G}$ that has two projective representations $\mathcal{G} \ni g \mapsto U_g \in \mathsf{Uni}_W$ and $\mathcal{G} \ni g \mapsto \tilde{U}_g \in \mathsf{Uni}_V$. Also, assume that $\hat{\Lambda}_1, \cdots, \hat{\Lambda}_M$ are $\{(U_g, \tilde{U}_g)\}_{g \in \mathcal{G}}$-covariant and the representation $g \mapsto \tilde{U}_g$ is irreducible. Then, there exists a globally optimal tester with maximally entangled pure states.

C. Examples

Two examples of $\{(U_g, \tilde{U}_g)\}_{g\in\mathcal{G}}$-covariant channels with an irreducible representation $\mathcal{G} \ni g \mapsto \tilde{U}_g \in \mathsf{Uni}_V$ are given. The first example is teleportation covariant channels [S5, S6]. Let $\mathcal{G} \ni g \mapsto U'_g \in \mathsf{Uni}_V$ be the irreducible projective representation generated by the Bell detection in a quantum teleportation process. A channel $\hat{\Lambda} \in \mathsf{Chn}(V, W)$ is called *teleportation covariant* if there exists a projective representation $\mathcal{G} \ni g \mapsto U_g \in \mathsf{Uni}_W$ such that

$$\mathrm{Ad}_{U_g} \circ \hat{\Lambda} \circ \mathrm{Ad}_{U'_g} = \hat{\Lambda}, \quad \forall g \in \mathcal{G},$$

i.e., $\mathrm{Ad}_{U_g \otimes U_g'^{\mathsf{T}}}(\Lambda) = \Lambda$. This implies that $\hat{\Lambda}$ is $\{(U_g, U_g'^{\mathsf{T}})\}_g$-covariant. The second example is a unital qubit channel, i.e., a channel $\hat{\Lambda} \in \mathsf{Chn}(V, W)$ with $N_V = N_W = 2$ and $\hat{\Lambda}(I_V) = I_W$ (i.e., $\mathrm{Tr}_V \Lambda = I_W$). Let S_{a} be the anti-unitary operator defined by

$$\mathrm{Ad}_{S_{\mathrm{a}}}(\rho) = \mathrm{Ad}_S(\rho^{\mathsf{T}}), \quad \forall \rho \in \mathsf{Den}_V,$$
$$S \coloneqq \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix};$$

then, we can easily verify that Λ satisfies $\mathrm{Ad}_{S_{\mathrm{a}} \otimes S_{\mathrm{a}}}(\Lambda) = \Lambda$. Let $\mathcal{G} \coloneqq \{e, \tilde{g}\}$ be a group whose representation is $g \mapsto U_g$ with $U_e \coloneqq I_V$ and $U_{\tilde{g}} \coloneqq S_{\mathrm{a}}$; then, this representation is irreducible. It follows that $\hat{\Lambda}$ is $\{(U_g, U_g)\}_g$-covariant.

VII. RELATIONSHIP BETWEEN ROBUSTNESSES AND PROCESS DISCRIMINATION PROBLEMS

Let us consider the robustness of $\mathcal{E} \in \mathsf{Her}_{\tilde{V}}$ defined by

$$R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E}) \coloneqq \inf \left\{ \lambda \in \mathbb{R}_+ : \frac{\mathcal{E} + \lambda \mathcal{E}'}{1 + \lambda} \in \mathcal{F},\ \mathcal{E}' \in \mathcal{K} \right\}, \quad \mathcal{E} \in \mathsf{Her}_{\tilde{V}}, \tag{S15}$$

where $\mathcal{K}$ $(\subset \mathsf{Her}_{\tilde{V}})$ is a proper convex cone [or, equivalently, $\mathcal{K}$ is a closed convex cone that is pointed (i.e., $\mathcal{K} \cap -\mathcal{K} = \{\mathbb{0}\}$) and has nonempty interior] and $\mathcal{F}$ $(\subset \mathsf{Her}_{\tilde{V}})$ is a compact convex set. In order for this value to be well-defined, we assume that $\mathcal{F} \cap \mathrm{int}(\mathcal{K})$ is not empty. The so-called global (or generalized) robustness of a state $\rho \in \mathsf{Den}_V$ with respect to $\mathcal{F} \subseteq \mathsf{Den}_V$, defined as [S7]

$$R_{\mathcal{F}}(\rho) \coloneqq \min \left\{ \lambda \in \mathbb{R}_+ : \frac{\rho + \lambda \rho'}{1 + \lambda} \in \mathcal{F},\ \rho' \in \mathsf{Den}_V \right\}, \quad \rho \in \mathsf{Den}_V,$$

is equal to $R_{\mathsf{Pos}_V}^{\mathcal{F}}(\rho)$. In other words, $R_{\mathcal{F}} : \mathsf{Den}_V \to \mathbb{R}_+$ is the same function as $R_{\mathsf{Pos}_V}^{\mathcal{F}} : \mathsf{Her}_V \to \mathbb{R}_+$, but is only defined on Den_V. As an example of $R_{\mathcal{F}}$, if $\mathcal{F}$ is the set of all bipartite separable states, then $R_{\mathcal{F}}(\rho)$ can be understood as a measure of entanglement. The robustness $R_{\mathcal{F}}(\rho)$ is known to represent the maximum advantage that ρ provides in a certain subchannel discrimination problem (e.g., [S8, S9]). Similarly, as will be seen in Proposition S9, $R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E})$ has a close relationship with the maximum advantage that $\mathcal{E}$ provides in a certain discrimination problem.

By letting $Z \coloneqq (\mathcal{E} + \lambda \mathcal{E}')/(1 + \lambda)$, we can rewrite $R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E})$ of Eq. (S15) as

$$R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E}) = \min\{\lambda \in \mathbb{R}_+ : (1 + \lambda)Z - \mathcal{E} \in \mathcal{K},\ Z \in \mathcal{F}\}. \tag{S16}$$

Let

$$\mathcal{N} \coloneqq \{\mathcal{E} \in \mathsf{Her}_{\tilde{V}} : \delta Z - \mathcal{E} \notin \mathcal{K}\ (\forall \delta < 1, Z \in \mathcal{F})\};$$

then, it follows that

$$R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E}) = \min\{\lambda \in \mathbb{R} : (1 + \lambda)Z - \mathcal{E} \in \mathcal{K},\ Z \in \mathcal{F}\}, \quad \forall \mathcal{E} \in \mathcal{N}. \tag{S17}$$

We first prove the following two lemmas.

Lemma S7 If $\mathcal{E} \in \mathcal{N}$ holds, then

$$1 + R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E}) = \max\{\langle \varphi, \mathcal{E} \rangle : \varphi \in \mathcal{K}^*, \langle \varphi, Z \rangle \le 1\ (\forall Z \in \mathcal{F})\} \tag{S18}$$

holds.

Proof Let $\tilde{\mathcal{F}} := \mathsf{coni}\,\mathcal{F}$. $\eta : \tilde{\mathcal{F}} \to \mathbb{R}_+$ denotes the gauge function of $\mathcal{F}$, which is defined as

$$\eta(Y) := \min\{\lambda \in \mathbb{R}_+ : Y = \lambda Z,\ Z \in \mathcal{F}\}, \quad Y \in \tilde{\mathcal{F}}. \tag{S19}$$

Equation (S17) gives

$$1 + R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E}) = \min\{\eta(Y) : Y - \mathcal{E} \in \mathcal{K},\ Y \in \tilde{\mathcal{F}}\}.$$

Let us consider the following Lagrangian

$$L(Y, \varphi) := \eta(Y) - \langle \varphi, Y - \mathcal{E} \rangle = \langle \varphi, \mathcal{E} \rangle + \eta(Y) - \langle \varphi, Y \rangle$$

with $Y \in \tilde{\mathcal{F}}$ and $\varphi \in \mathcal{K}^*$. We can easily verify

$$\sup_{\varphi \in \mathcal{K}^*} L(Y, \varphi) = \begin{cases} \eta(Y), & Y - \mathcal{E} \in \mathcal{K}, \\ \infty, & \text{otherwise}, \end{cases}$$

$$\inf_{Y \in \tilde{\mathcal{F}}} L(Y, \varphi) = \begin{cases} \langle \varphi, \mathcal{E} \rangle, & \langle \varphi, Y' \rangle \le \eta(Y')\ (\forall Y' \in \tilde{\mathcal{F}}), \\ -\infty, & \text{otherwise}. \end{cases}$$

Thus, the max-min inequality

$$\inf_{Y \in \tilde{\mathcal{F}}} \sup_{\varphi \in \mathcal{K}^*} L(Y, \varphi) \ge \sup_{\varphi \in \mathcal{K}^*} \inf_{Y \in \tilde{\mathcal{F}}} L(Y, \varphi)$$

yields

$$1 + R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E}) \ge \max\{\langle \varphi, \mathcal{E} \rangle : \varphi \in \mathcal{K}^*,\ \langle \varphi, Y \rangle \le \eta(Y)\ (\forall Y \in \tilde{\mathcal{F}})\}. \tag{S20}$$

We now prove the equality of Eq. (S20). To this end, it suffices to show that there exists $\varphi \in \mathrm{int}(\mathcal{K}^*)$ such that $\langle \varphi, Y \rangle \le \eta(Y)$ $(\forall Y \in \tilde{\mathcal{F}})$; indeed, in this case, the equality of Eq. (S20) follows from Slater's condition. Arbitrarily choose $\varphi' \in \mathrm{int}(\mathcal{K}^*)$ and let $\gamma := \sup_{Y \in \tilde{\mathcal{F}} \setminus \{\mathbb{0}\}} [\langle \varphi', Y \rangle / \eta(Y)]$ [note that $\eta(Y) > 0$ holds for any $Y \in \tilde{\mathcal{F}} \setminus \{\mathbb{0}\}$]. Since $\mathcal{F} \cap \mathrm{int}(\mathcal{K})$ is not empty, there exists $Y \in \mathcal{F} \cap \mathrm{int}(\mathcal{K})$ such that $\langle \varphi', Y \rangle > 0$, which yields $\gamma > 0$. Let $\varphi := \gamma^{-1} \varphi' \in \mathrm{int}(\mathcal{K}^*)$; then, we can easily verify $\langle \varphi, Y \rangle \le \eta(Y)$ $(\forall Y \in \tilde{\mathcal{F}})$.

It remains to show

$$\langle \varphi, Z \rangle \le 1\ (\forall Z \in \mathcal{F}) \quad \Leftrightarrow \quad \langle \varphi, Y \rangle \le \eta(Y)\ (\forall Y \in \tilde{\mathcal{F}}).$$

We first prove "$\Rightarrow$". Arbitrarily choose $Y \in \tilde{\mathcal{F}}$; then, from Eq. (S19), there exists $Z \in \mathcal{F}$ such that $Y = \eta(Y) Z$. Thus, $\langle \varphi, Y \rangle = \eta(Y) \langle \varphi, Z \rangle \le \eta(Y)$ holds. We next prove "$\Leftarrow$". Arbitrarily choose $Z \in \mathcal{F}$. Since the case $Z = \mathbb{0}$ is obvious, we may assume $Z \ne \mathbb{0}$. Let $Z' := Z/\eta(Z)$; then, from $\eta(Z) \le 1$, $Z' \in \tilde{\mathcal{F}}$, and $\eta(Z') = 1$, we have $\langle \varphi, Z \rangle \le \langle \varphi, Z' \rangle \le \eta(Z') = 1$. ■

Lemma S8 If $\mathcal{E} \in \mathcal{N}$ holds, then we have

$$\max_{\varphi \in \mathcal{X} \setminus \{\mathbb{0}\}} \frac{\langle \varphi, \mathcal{E} \rangle}{\max_{Z \in \mathcal{F}} \langle \varphi, Z \rangle} = 1 + R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E}),$$

where $\mathcal{X}$ is any set such that the cone generated by $\mathcal{X}$ is $\mathcal{K}^*$, i.e., $\{\lambda \varphi : \lambda \in \mathbb{R}_+,\ \varphi \in \mathcal{X}\} = \mathcal{K}^*$.

Proof Let

$$\varphi^\star := \operatorname*{argmax}_{\varphi \in \mathcal{K}^*, \Gamma(\varphi) \le 1} \langle \varphi, \mathcal{E} \rangle, \quad \Gamma(\varphi) := \max_{Z \in \mathcal{F}} \langle \varphi, Z \rangle;$$

then, from Eq. (S18), we have $\langle \varphi^\star, \mathcal{E} \rangle = 1 + R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E})$. Since $\mathcal{F} \cap \mathrm{int}(\mathcal{K})$ is not empty, $\Gamma(\varphi) > 0$ holds for any $\varphi \in \mathcal{K}^* \setminus \{\mathbb{0}\}$. It follows that $\Gamma(\varphi^\star) = 1$ must hold [otherwise, $\tilde{\varphi} := \varphi^\star / \Gamma(\varphi^\star)$ satisfies $\langle \tilde{\varphi}, \mathcal{E} \rangle > \langle \varphi^\star, \mathcal{E} \rangle$, $\tilde{\varphi} \in \mathcal{K}^*$, and $\Gamma(\tilde{\varphi}) = 1$, which contradicts the definition of $\varphi^\star$]. For any $\varphi \in \mathcal{X} \setminus \{\mathbb{0}\}$, $\varphi' := \varphi / \Gamma(\varphi)$ satisfies $\langle \varphi, \mathcal{E} \rangle / \Gamma(\varphi) = \langle \varphi', \mathcal{E} \rangle$ and $\Gamma(\varphi') = 1$. Thus, we have

$$\max_{\varphi \in \mathcal{X} \setminus \{\mathbb{0}\}} \frac{\langle \varphi, \mathcal{E} \rangle}{\Gamma(\varphi)} = \max_{\substack{\varphi' \in \mathcal{X} \setminus \{\mathbb{0}\}, \\ \Gamma(\varphi') = 1}} \langle \varphi', \mathcal{E} \rangle = \langle \varphi^\star, \mathcal{E} \rangle = 1 + R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E}).$$

■

We should note that, in practical situations, many physically interesting processes belong to $\mathcal{N}$. As an example, if $\mathcal{F}$ is a subset of all combs in $\mathsf{Her}_{\tilde{V}}$ and $\mathrm{int}(\mathcal{K}^*) \cap \mathsf{Comb}_{W_T,V_T,\dots,W_1,V_1}$ is not empty, then any comb in $\mathsf{Her}_{\tilde{V}}$ belongs to $\mathcal{N}$. [Indeed, arbitrarily choose $\phi \in \mathrm{int}(\mathcal{K}^*) \cap \mathsf{Comb}_{W_T,V_T,\dots,W_1,V_1}$ and a comb $\mathcal{E} \in \mathsf{Her}_{\tilde{V}}$; then, from $\phi \in \mathsf{Comb}_{W_T,V_T,\dots,W_1,V_1}$, $\langle \phi, Z \rangle = \langle \phi, \mathcal{E} \rangle = 1 \ \ (\forall Z \in \mathcal{F})$ holds. Thus, for any $\delta < 1$, $\langle \phi, \delta Z - \mathcal{E} \rangle = \delta - 1 < 0$ holds, which yields $\delta Z - \mathcal{E} \notin \mathcal{K}$. Therefore, we have $\mathcal{E} \in \mathcal{N}$.] For instance, if $\mathcal{F} \subseteq \mathsf{Den}_V$ and $\mathrm{Tr}\, x > 0 \ \ (\forall x \in \mathcal{K} \setminus \{\mathbb{0}\})$ hold, then any $\rho \in \mathsf{Den}_V$ is in $\mathcal{N}$. As another example, if $R^{\mathcal{F}}_{\mathcal{K}}(\mathcal{E}) > 0$ holds, then $\mathcal{E} \in \mathcal{N}$ always holds. [Indeed, by contraposition, assume $\mathcal{E} \notin \mathcal{N}$; then, there exists $\delta < 1$ and $Z \in \mathcal{F}$ such that $\delta Z - \mathcal{E} \in \mathcal{K}$. Let $\lambda^\star \coloneqq R^{\mathcal{F}}_{\mathcal{K}}(\mathcal{E})$. It is easily seen from $R^{\mathcal{F}}_{\mathcal{K}}(\mathcal{E}) \ge 0$ that there exists $Z^\star \in \mathcal{F}$ such that $(1+\lambda^\star)Z^\star - \mathcal{E} \in \mathcal{K}$. Let $p \coloneqq (1-\delta)/(\lambda^\star + 1 - \delta)$ and $Z' \coloneqq p(1+\lambda^\star)Z^\star + (1-p)\delta Z$; then, we have $0 \le p \le 1$, $Z' \in \mathcal{F}$, and $Z' - \mathcal{E} \in \mathcal{K}$. This implies $R^{\mathcal{F}}_{\mathcal{K}}(\mathcal{E}) = 0$.]

We obtain the following proposition.

Proposition S9 Let us consider $\mathcal{E} \in \mathcal{N}$. Let $\tilde{V}'$ be an arbitrary system. We consider a set of pairs $\mathcal{L} \coloneqq \{(\{\hat{\mathcal{J}}_m\}_{m=1}^M, \{\Phi_m\}_{m=1}^M)\}$, where $\{\hat{\mathcal{J}}_m : \mathsf{Her}_{\tilde{V}} \to \mathsf{Her}_{\tilde{V}'}\}_{m=1}^M$ is a collection of linear maps and $\Phi_1, \dots, \Phi_M \in \mathsf{Her}_{\tilde{V}'}$. Assume that the cone generated by

$$\mathcal{X} \coloneqq \left\{ \sum_{m=1}^M \hat{\mathcal{J}}_m^\dagger(\Phi_m) :\ (\{\hat{\mathcal{J}}_m\}, \{\Phi_m\}) \in \mathcal{L} \right\}$$

is $\mathcal{K}^*$, where $\hat{\mathcal{J}}_m^\dagger$ is the adjoint of $\hat{\mathcal{J}}_m$, which is defined as $\langle \hat{\mathcal{J}}_m^\dagger(\Phi'), \mathcal{E}' \rangle = \langle \Phi', \hat{\mathcal{J}}_m(\mathcal{E}') \rangle \ \ (\forall \mathcal{E}' \in \mathsf{Her}_{\tilde{V}}, \Phi' \in \mathsf{Her}_{\tilde{V}'})$. Then, we have

$$\max_{(\{\hat{\mathcal{J}}_m\},\{\Phi_m\}) \in \mathcal{L}'} \frac{\sum_{m=1}^M \langle \Phi_m, \hat{\mathcal{J}}_m(\mathcal{E}) \rangle}{\max_{Z \in \mathcal{F}} \sum_{m=1}^M \langle \Phi_m, \hat{\mathcal{J}}_m(Z) \rangle} = 1 + R^{\mathcal{F}}_{\mathcal{K}}(\mathcal{E}), \tag{S21}$$

where

$$\mathcal{L}' \coloneqq \left\{ (\{\hat{\mathcal{J}}_m\}, \{\Phi_m\}) \in \mathcal{L} : \sum_{m=1}^M \hat{\mathcal{J}}_m^\dagger(\Phi_m) \neq \mathbb{0} \right\}.$$

Proof The left-hand side of Eq. (S21) is rewritten by

$$\max_{\varphi \in \mathcal{X} \setminus \{\mathbb{0}\}} \frac{\langle \varphi, \mathcal{E} \rangle}{\max_{Z \in \mathcal{F}} \langle \varphi, Z \rangle}.$$

Thus, an application of Lemma S8 completes the proof. ■

The operational meaning of Eq. (S21) is as follows. Suppose that $\{\hat{\mathcal{J}}_m\}_m$ is a collection of (unnormalized) processes such that $\sum_{m=1}^M \hat{\mathcal{J}}_m$ is a comb from $\mathsf{Pos}_{\tilde{V}}$ to $\mathsf{Pos}_{\tilde{V}'}$ and that $\{\Phi_k\}_k$ is a tester, where the pair $(\{\hat{\mathcal{J}}_m\}_m, \{\Phi_k\}_k)$ is restricted to belong to $\mathcal{L}$. We consider the situation that a party, Alice, applies a process $\hat{\mathcal{J}}_m$ to a comb $\mathcal{E} \in \mathsf{Pos}_{\tilde{V}} \cap \mathcal{N}$, and then another party, Bob, applies a tester $\{\Phi_k\}_k$ to $\hat{\mathcal{J}}_m(\mathcal{E})$. The probability of Bob correctly guessing which of the processes $\hat{\mathcal{J}}_1, \dots, \hat{\mathcal{J}}_M$ Alice applies is expressed by $\sum_{m=1}^M \langle \Phi_m, \hat{\mathcal{J}}_m(\mathcal{E}) \rangle$ [note that $\sum_{k=1}^M \sum_{m=1}^M \langle \Phi_k, \hat{\mathcal{J}}_m(\mathcal{E}) \rangle = 1$ holds]. Equation (S21) implies that the advantage of $\mathcal{E}$ over all $Z \in \mathcal{F}$ in such a discrimination problem can be exactly quantified by the robustness $R^{\mathcal{F}}_{\mathcal{K}}(\mathcal{E})$. In this situation, $\mathcal{K}^* \subseteq \mathsf{Pos}^*_{\tilde{V}}$, i.e., $\mathcal{K} \supseteq \mathsf{Pos}_{\tilde{V}}$, holds.

We give two examples of the application of Proposition S9. The first example is the case $\mathcal{K} = \mathsf{Pos}_{\tilde{V}}$. Let us consider the case where $\sum_{m=1}^M \hat{\mathcal{J}}_m$ can be any comb from $\mathsf{Pos}_{\tilde{V}}$ to $\mathsf{Pos}_{\tilde{V}'}$ and $\{\Phi_k\}_k$ can be any tester. We can easily see that Eq. (S21) with $\mathcal{K} = \mathsf{Pos}_{\tilde{V}}$ holds. Note that, for example, Theorem 2 of Ref. [S8] and Theorems 1 and 2 of Ref. [S9] can be understood as special cases of Proposition S9 with $\mathcal{K} = \mathsf{Pos}_{\tilde{V}}$. The second example is the case $\mathcal{K} \neq \mathsf{Pos}_{\tilde{V}}$. For instance, for a given channel $\hat{\mathcal{E}}$ from a system V to a system W, assume that $\hat{\mathcal{J}}_m$ is the process that applies $\hat{\mathcal{E}}$ to a state $\rho_m \in \mathsf{Den}_V$ with probability p_m [i.e., $\hat{\mathcal{J}}_m(\mathcal{E}) = p_m \mathrm{Tr}_V[(I_W \otimes \rho_m^\mathsf{T})\mathcal{E}]$ holds] and $\{\Phi_k\}_k$ is a measurement of W. Then, we have $\langle \Phi_k, \hat{\mathcal{J}}_m(-) \rangle = p_m \langle \Phi_k \otimes \rho_m^\mathsf{T}, - \rangle$. It is easily seen that Eq. (S21) with $\mathcal{K}^* = \mathsf{Sep}_{W,V}$ (or, equivalently, $\mathcal{K} = \mathsf{Sep}^*_{W,V}$) holds, where $\mathsf{Sep}_{W,V}$ is the set of all bipartite separable elements in $\mathsf{Pos}_{W \otimes V}$. Note that, for a linear map $\hat{\Psi}$ from V to W, $\Psi \in \mathsf{Sep}^*_{W,V}$ holds if and only if $\hat{\Psi}$ is a positive map.

VIII. ROBUSTNESSES ASSOCIATED WITH THE OPTIMAL VALUES OF PROCESS DISCRIMINATION PROBLEMS

We prove the following proposition.

Proposition S10 Let

$$\mathcal{K} \coloneqq \left\{ Y \in \mathsf{Her}_{W_A \otimes \tilde{V}} : \sum_{m=1}^M \langle |m\rangle\langle m| \otimes \Phi_m, Y \rangle \ge 0 \ (\forall \Phi \in C) \right\},$$

$$\mathcal{F} \coloneqq \{ I_{W_A} \otimes \chi' : \chi' \in \mathsf{Her}_{\tilde{V}},\ D_{\mathcal{S}}(\chi') \le 1/M \};$$

then, the optimal value of Problem (P) is equal to $[1 + R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E}^{\rm ex})]/M$, where $\mathcal{E}^{\rm ex}$ is defined by Eq. (6).

Proof We have that for any $\chi \in \mathsf{Her}_{\tilde{V}}$,

$$I_{W_{\rm A}} \otimes \chi - \mathcal{E}^{\rm ex} \in \mathcal{K} \quad \Leftrightarrow \quad \sum_{m=1}^{M} \langle \Phi_m, \chi - p_m \mathcal{E}_m \rangle \geq 0 \ (\forall \Phi \in C) \quad \Leftrightarrow \quad \chi \in \mathcal{D}_C.$$

This yields

$$\begin{aligned}
[1 + R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E}^{\rm ex})]/M &= \min\{\lambda/M : \lambda Z - \mathcal{E}^{\rm ex} \in \mathcal{K},\ Z \in \mathcal{F},\ \lambda \geq 1\} \\
&= \min\{\lambda/M : \lambda I_{W_{\rm A}} \otimes \chi' - \mathcal{E}^{\rm ex} \in \mathcal{K},\ \chi' \in \mathsf{Her}_{\tilde{V}},\ D_{\mathcal{S}}(\chi') \leq 1/M,\ \lambda \geq 1\} \\
&= \min\{\lambda/M : I_{W_{\rm A}} \otimes \chi - \mathcal{E}^{\rm ex} \in \mathcal{K},\ \chi \in \mathsf{Her}_{\tilde{V}},\ D_{\mathcal{S}}(\chi) \leq \lambda/M,\ \lambda \geq 1\} \\
&= \min\{\lambda/M : \chi \in \mathcal{D}_C,\ D_{\mathcal{S}}(\chi) \leq \lambda/M,\ \lambda \geq 1\} \\
&= \min\{D_{\mathcal{S}}(\chi) : \chi \in \mathcal{D}_C\},
\end{aligned}$$

where $\chi \coloneqq \lambda\chi'$. The first line follows from Eq. (S16). The last line, which follows from $D_{\mathcal{S}}(\chi) \geq 1/M$ for any $\chi \in \mathcal{D}_C$, is equal to the optimal value of Problem (D), i.e., that of Problem (P). ■

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