Multiplicative domain and fixed points of a OQRW

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What is here: the section on reducible maps rewritten

- Lemma 1: structure of faithful normal conditional expectations and their invariant states;
- Prop. 1: decomposition of $\mathcal{N}$ to "periodic" blocks;
- Prop. 2: structure of one block of $\mathcal{N}$, Kraus operators for Φ;
- Prop. 3: structure of $\mathcal{F}$, invariant states for Φ and a decomposition of Φ given by $\mathcal{F}$.

0.1 Reducible maps

Let Φ be a nucp map on $B(\mathcal{H})$, fixed throughout. We assume that Φ admitts a faithful normal invariant state ρ. For simplicity of notations, we put $\mathcal{N} := \mathcal{N}(\Phi)$ and $\mathcal{F} := \mathcal{F}(\Phi)$.

By Corollary **??**, $\mathcal{N}$ is the range of a faithful normal conditional expectation F and therefore must be type I with discrete center, [**?**]. On the other hand, it is known [**?**] that the limit

$$E = \lim_n \frac{1}{n} \sum_{k=0}^{n-1} \Phi^k$$

exists in the point-ultraweak topology, moreover, E is a faithful normal conditional expectation onto $\mathcal{F}$, satisfying $E \circ \Phi = \Phi \circ E = E$. Hence $\mathcal{F}$ is an atomic von Neumann subalgebra of $\mathcal{N}$. In this section, we study the structure of the two algebras.

We first describe a general form of a faithful normal conditional expectation on $B(\mathcal{H})$.

Lemma 1. *Let $E : B(\mathcal{H}) \to B(\mathcal{H})$ be a faithful normal conditional expectation and let $\mathcal{R} = E(B(\mathcal{H}))$ be its range. Then*

(i) $\mathcal{R}$ is atomic, so that there is a direct sum decomposition $\mathcal{H} = \bigoplus_j \mathcal{H}_j$, Hilbert spaces $\mathcal{H}_j^L$, $\mathcal{H}_j^R$ and unitaries $U_j : \mathcal{H}_j \to \mathcal{H}_j^L \otimes \mathcal{H}_j^R$ such that

$$\mathcal{R} = \bigoplus_j U_j^*(B(\mathcal{H}_j^L) \otimes I_{\mathcal{H}_j^R})U_j;$$

(ii) the orthogonal projections P_j onto $\mathcal{H}_j$ are minimal central projections in $\mathcal{R}$ and

$$E(A) = \sum_j E(P_j A P_j);$$

(iii) identifying $P_jB(\mathcal{H})P_j$ with $B(\mathcal{H}_j)$, the restriction of E to $P_jB(\mathcal{H})P_j$ is determined by

$$E(U_j^*(A_j \otimes B_j)U_j) = U_j^*(A_j \otimes \mathrm{Tr}\,[\rho_j B_j] I_{\mathcal{H}_j^R})U_j,$$

where each $\rho_j \in \mathfrak{S}(\mathcal{H}_j^R)$ is a (fixed) faithful normal state;

(iv) a normal state $\omega \in \mathfrak{S}(\mathcal{H})$ is invariant under E if and only if

$$\omega = \oplus_j \lambda_j U_j^*(\omega_j^L \otimes \rho_j)U_j,$$

where ρ_j are as in (iii), $\{\lambda_j\}$ are probabilities and $\omega_j^L \in \mathfrak{S}(\mathcal{H}_j^L)$.

Proof. The range $\mathcal{R}$ is atomic by [**?**]. Let $\{P_j\}$ be the minimal central projections in $\mathcal{R}$ and let $\mathcal{H}_j = P_j\mathcal{H}$. Since $\mathcal{R}P_j$ is a type I factor acting on $\mathcal{H}_j$, there are Hilbert spaces $\mathcal{H}_j^L$, $\mathcal{H}_j^R$ and a unitary $U_j : \mathcal{H}_j \to \mathcal{H}_j^L \otimes \mathcal{H}_j^R$ such that

$$\mathcal{R}P_j = U_j^*(B(\mathcal{H}_j^L) \otimes I_{\mathcal{H}_j^R})U_j,$$

this proves (i). By the properties of conditional expectations,

$$E(P_jAP_k) = P_jE(A)P_k = P_jP_kE(A)$$

for any $A \in B(\mathcal{H})$, this proves (ii). It also follows that under the identification in (iii), $E(B(\mathcal{H}_j)) \subseteq B(\mathcal{H}_j)$ for all j and the restriction E_j of E is a faithful normal conditional expectation on $B(\mathcal{H}_j)$, with range $U_j^*(B(\mathcal{H}_j^L) \otimes I_{\mathcal{H}_j^R})U_j$. Let $A_j \in B(\mathcal{H}_j^L)$, $B_j \in B(\mathcal{H}_j^R)$, then we have

$$E(U_j^*(A_j \otimes B_j)U_j) = U_j^*(A_j \otimes I)U_jE(U_j^*(I \otimes B_j)U_j) = E(U_j^*(I \otimes B_j)U_j)U_j^*(A_j \otimes I)U_j, \quad (1)$$

it follows that $E(U_j^*(I \otimes B_j)U_j)$ commutes with all elements in $U_j^*(B(\mathcal{H}_j^L) \otimes I))U_j$, so that there is some $\rho_j(B_j) \in \mathbb{C}$ such that $E(U_j^*(I \otimes B_j)U_j) = \rho_j(B_j)P_j$. It is clear that $B_j \mapsto \rho_j(B_j)$ defines a normal state on $B(\mathcal{H}_j^R)$, which must be faithful since E is. This proves (iii).

Finally, let $\omega \in \mathfrak{S}(\mathcal{H})$. It is clear that if $\omega \circ E = \omega$, then we must have $\omega = \lambda_j\omega_j$ for some $\omega_j \in \mathfrak{S}(\mathcal{H}_j)$ and $\lambda_j = \mathrm{Tr}\,P_j\omega$. Let $\omega_j^L \in \mathfrak{S}(\mathcal{H}_j^L)$ be determined by $\omega_j^L(A_j) = \omega_j(U_j^*(A_j \otimes I)U_j)$. Then ω_j, and consequently also ω, is invariant under E if and only if for all $A_j \in B(\mathcal{H}_j^L)$ and $B_j \in B(\mathcal{H}_j^R)$,

$$\omega_j(U_j^*(A_j \otimes B_j)U_j) = \omega_j \circ E(U_j^*(A_j \otimes B_j)U_j) = \omega_j^L(A_j)\rho_j(B_j) = (\omega_j^L \otimes \rho_j)(A_j \otimes B_j).$$

□

Let us now turn to the algebras $\mathcal{F}$ and $\mathcal{N}$. We begin with the central projections. Let $\mathcal{Z}(\mathcal{F})$ and $\mathcal{Z}(\mathcal{N})$ denote the center of $\mathcal{F}$ and $\mathcal{N}$, and let $\mathcal{Z} = \mathcal{Z}(\mathcal{F}) \cap \mathcal{Z}(\mathcal{N})$. Clearly, $\mathcal{Z}$ is a discrete abelian von Neumann algebra. Let $\{Z_1, Z_2, \dots\}$ be minimal projections in $\mathcal{Z}$ and put $\mathcal{N}_i := Z_i\mathcal{N}$. Note that identifying $Z_iB(\mathcal{H})Z_i$ with $B(Z_i\mathcal{H})$, we have $\Phi(B(Z_i\mathcal{H})) \subseteq B(Z_i\mathcal{H})$, so that $\Phi_i := \Phi|_{B(Z_i\mathcal{H})}$ is a nucp map on $B(Z_i\mathcal{H})$, with $\mathcal{N}(\Phi_i) = \mathcal{N}_i$.

Proposition 1. *For each i, there is some $d_i \in \mathbb{N}$ and minimal projections $Q_0^i, \dots, Q_{d_i-1}^i \in \mathcal{Z}(\mathcal{N})$ forming a cyclic resolution of identity for Φ_i. That is, $Z_i = \sum_{m=0}^{d_i-1} Q_m^i$ and*

$$\Phi(Q_m^i) = Q_{m\oplus_{d_i}1}^i.$$

The number d_i will be called the period of Φ_i.

Proof. Let $Q_0^i, Q_1^i, \dots$ be minimal central projections in $\mathcal{N}_i$, then clearly all Q_m^i are minimal central projections in $\mathcal{N}$ and we have $\sum_m Q_m^i = Z_i$. Since the restriction of Φ_i to $\mathcal{N}_i$ is a *-automorphism, $\Phi(Q_m^i) = \Phi_i(Q_m^i)$ is a minimal central projection as well. Put

$$d_i := \inf\{m, \Phi^m(Q_0^i) = Q_0^i\},$$

then since Φ preserves the faithful state ρ, $d_i < \infty$. Assume that the projections are numbered so that

$$Q_m^i = \Phi^m(Q_0^i), \qquad m = 0, \dots, d_i - 1.$$

Put $Q^i := \sum_{m=0}^{d_i-1} Q_m^i$, then obviously $Q^i \in \mathcal{Z}(\mathcal{N})$ and $\Phi(Q^i) = Q^i$, so that $Q^i \in \mathcal{Z}$. Since also $Q^i \leq Z_i$ and Z_i is minimal in $\mathcal{Z}$, we must have $Q^i = Z_i$.

□

We now describe the action of Φ_i on one component $\mathcal{N}_i$. For simplicity, we drop the index i, this correspond to assuming that there is only one such component, so that $\mathcal{Z}$ is trivial. Let the period of Φ be d. In this case, the center of $\mathcal{N}$ has dimension d and is generated by the minimal cyclic projections $Q_0, \dots, Q_{d-1}$. Let us denote $\mathcal{K}_m := Q_m\mathcal{H}$. By Lemma 1, there are Hilbert spaces $\mathcal{K}_m^L, \mathcal{K}_m^R$, $m = 0, \dots, d-1$ and unitaries $S_m : \mathcal{K}_m \to \mathcal{K}_m^L \otimes \mathcal{K}_m^R$ such that

$$\mathcal{N} = \bigoplus_{m=0}^{d-1} S_m^*(B(\mathcal{K}_m^L) \otimes I_m^R) S_m. \tag{2}$$

Here we put $I_m^R = I_{\mathcal{K}_m^R}$ to simplify notations, we will use a similar notation for $I_{\mathcal{K}_m^L}$. Let also $\rho_m \in \mathfrak{S}(\mathcal{K}_m^R)$ be the states determining the conditional expectation F, as in Lemma 1 (iii).

Proposition 2. *Assume that $\mathcal{Z}$ is trivial and let the period of Φ be d. Let $\oplus = \oplus_d$ denote addition modulo d. Then there are*

(a) unitaries $T_m : K_m^L \to K_{m\oplus 1}^L$, $m = 0, \dots, d-1$;

(b) nucp maps $\Phi_m : B(\mathcal{H}_m^R) \to B(\mathcal{H}_{m\oplus 1}^R)$, $m = 0, \dots, d-1$;

such that for all m,

(i) $\rho_{m\oplus 1} \circ \Phi_m = \rho_m$;

(ii) $\Phi_{m\oplus(d-1)} \circ \cdots \circ \Phi_{m\oplus 1} \circ \Phi_m$ is irreducible and aperiodic;

(iii) the restriction $\Phi|_{B(\mathcal{H}_m)}$ is a nucp map $B(\mathcal{H}_m) \to B(\mathcal{H}_{m\oplus 1})$, determined as

$$\Phi(S_m^*(A_m \otimes B_m)S_m) = S_{m\oplus 1}^*(T_m A_m T_m^* \otimes \Phi_m(B_m))S_{m\oplus 1}.$$

(iv) Φ has a Kraus representation $\Phi(A) = \sum_k V_k^* A V_k$, such that

$$V_k = \sum_m S_m^*(T_m^* \otimes L_{m,k}) S_{m\oplus 1},$$

where $\Phi_m = \sum_k L_{m,k}^* \cdot L_{m,k}$ is a Kraus representation of Φ_m.

Proof. Let $A_m \in B(\mathcal{K}_m^L)$. Since $\Phi(Q_m\mathcal{N}) = Q_{m\oplus 1}\mathcal{N}$, we have

$$\Phi(S_m^*(A_m \otimes I_m^R)S_m) = S_{m\oplus 1}^*(A'_m \otimes I_{m\oplus 1}^R)S_{m\oplus 1}$$

for some $A'_m \in B(\mathcal{K}_{m\oplus 1}^L)$ and the map $A_m \mapsto A'_m$ defines a *-isomorphism of $B(\mathcal{K}_m^L)$ onto $B(\mathcal{K}_{m\oplus 1}^L)$. Hence there is a unitary operator $T_m : \mathcal{K}_m^L \to \mathcal{K}_{m\oplus 1}^L$, such that $A'_m = T_m A_m T_m^*$. Moreover, by the multiplicativity properties of Φ on $\mathcal{N}$, we have $\Phi(Q_m A) = Q_{m\oplus 1}\Phi(A)$ for all $A \in B(\mathcal{H})$, and for all $B_m \in B(\mathcal{K}_m^R)$,

$$\begin{aligned}\Phi(S_m^*(A_m \otimes B_m)S_m) &= \Phi(S_m^*(A_m \otimes I_m^R)S_m)\Phi(S_m^*(I_m^L \otimes B_m)S_m)\\ &= \Phi(S_m^*(I_m^L \otimes B_m)S_m)\Phi(S_m^*(A_m \otimes I_m^R)S_m).\end{aligned}$$

It follows that $\Phi(S_m^*(I_m^L \otimes B_m)S_m)$ is an element in $B(\mathcal{K}_{m\oplus 1})$, commuting with all elements in $S_{m\oplus 1}^*(B(\mathcal{K}_{m\oplus 1}^L) \otimes I_{m\oplus 1}^R)S_{m\oplus 1}$, so that

$$\Phi(S_m^*(I_m^L \otimes B_m)S_m) = S_{m\oplus 1}^*(I_{m\oplus 1}^L \otimes B'_m)S_{m\oplus 1}$$

for some $B'_m \in B(\mathcal{K}_{m\oplus 1}^R)$. It is clear that $B_m \mapsto B'_m$ defines a nucp map $\Phi_m : B(\mathcal{H}_m^R) \to B(\mathcal{H}_{m\oplus 1}^R)$. Putting al together proves (iii).

To see (ii), let $\tilde{\Phi}_m$ be the given composition and let $R_m \in B(\mathcal{K}_m^R)$ be a projection that is fixed or periodic for $\tilde{\Phi}_m$. Then $S_m^*(I_m^L \otimes R_m)S_m$ is in $\mathcal{N}$, so that R_m must be trivial. Finally, note that since $F \in \mathbf{S}$, Φ must commute with F. For $B_m \in B(\mathcal{K}_m^R)$, we have by Lemma 1

$$\Phi \circ F(S_m^*(I_m^L \otimes B_m)S_m) = \rho_m(B_m)\Phi(Q_m) = \rho_m(B_m)Q_{m\oplus 1}$$

and

$$F \circ \Phi(S_m^*(I_m^L \otimes B_m)S_m) = F(S_{m\oplus 1}^*(I_{m\oplus 1}^L \otimes \Phi_m(B_m))S_{m\oplus 1}) = \rho_{m\oplus 1}(\Phi_m(B_m))Q_{m\oplus 1},$$

so that (i) holds.

Finally, let $\Phi = \sum_k V_k^* \cdot V_k$ be any Kraus representation of Φ. Then we have

$$\Phi(A) = \sum_{m,n=0}^{d-1} \Phi(Q_m A Q_n) = \sum_{m,n=0}^{d-1} Q_{m\oplus 1}\Phi(Q_m A Q_n)Q_{n\oplus 1},$$

so that we may assume that each V_k has the form $V_k = \sum_m V_{k,m}$, with $V_{k,m} = Q_m V_k Q_{m\oplus 1}$. Moreover, for each m, $\sum_k V_{k,m}^* \cdot V_{k,m}$ is a Kraus representation of the restriction $\Phi|_{B(\mathcal{K}_m)}$.

Let $\Phi_m = \sum_l K_{m,l}^* \cdot K_{m,l}$ be a minimal Kraus representation. It follows from (iii) that

$$\Phi|_{B(\mathcal{K}_m)} = \sum_l S_{m\oplus 1}^*(T_m \otimes K_{m,l}^*)S_m \cdot S_m^*(T_m^* \otimes K_{m,l})S_{m\oplus 1}$$

is another Kraus representation of $\Phi|_{B(\mathcal{K}_m)}$, hence there are some $\{\eta_{k,l}^j\}$ such that $\sum_i \eta_{i,k}^j \bar{\eta}_{i,l}^j = \delta_{k,l}$ and

$$V_{k,m} = \sum_l \eta_{k,l}^j S_m^*(T_m^* \otimes K_{m,l})S_{m\oplus 1} = S_m^*(T_m^* \otimes L_{m,k})S_{m\oplus 1},$$

where $L_{m,k} := \sum_l \eta_{k,l}^m K_{m,l}$, this proves (iv).

□

Note that by identifying

$$\mathcal{H} = \bigoplus_m \mathcal{K}_m \simeq \sum_m \mathcal{K}_m \otimes |m\rangle$$

and

$$\mathcal{K} := \bigoplus_m \mathcal{K}_m^L \otimes \mathcal{K}_m^R \simeq \sum_m \mathcal{K}_m^L \otimes \mathcal{K}_m^R \otimes |m\rangle,$$

(2) can be written as

$$\mathcal{N} = S^*(\sum_{m=0}^{d-1} B(\mathcal{K}_m^L) \otimes I_m^R \otimes |m\rangle\langle m|)S,$$

where $S : \mathcal{H} \to \mathcal{H}'$ is a unitary given as $S = \sum_m S_m \otimes |m\rangle\langle m|$. We will also use the notation

$$\mathcal{K}^R := \bigoplus_m \mathcal{K}_m^R \simeq \sum_m \mathcal{K}_m^R \otimes |m\rangle\langle m|$$

and put $I^R := I_{\mathcal{K}^R}$. We are now ready to describe the subalgebra $\mathcal{F}$. In the following proposition, we keep the notations of Prop. 2.

Proposition 3. *Let us denote*

$$\tilde{T}_m : \mathcal{K}_0^L \to \mathcal{K}_{m\oplus 1}^L, \quad \tilde{T}_m := T_m \dots T_0,\ m = 0, \dots, d-1; \quad \tilde{T}_{-1} := I_0^L$$

and let $T : \mathcal{K}_0^L \otimes \mathcal{K}^R \to \mathcal{K}$ be the unitary defined as

$$T = \sum_{m=0}^{d-1} \tilde{T}_{m-1} \otimes I_m^R \otimes |m\rangle\langle m|.$$

(i) The operator $\tilde{T}_{d-1} \in \mathcal{U}(\mathcal{K}_0^L)$ has a discrete spectrum. Let R_j be its minimal spectral projections and let $\mathcal{L}_j := R_j \mathcal{K}_0^L$, then

$$\mathcal{F} = S^* T (\bigoplus_j B(\mathcal{L}_j) \otimes I^R) T^* S;$$

(ii) Let $\sigma_j \in \mathfrak{S}(\mathcal{H}^R)$ be the faithful normal states corresponding to E as in Lemma 1 (iii) and (iv). Then

$$\sigma_j \equiv \sigma := \frac{1}{d} \sum_{m=0}^{d-1} \rho_m \otimes |m\rangle\langle m|, \qquad \forall j;$$

(iii) Invariant states $\xi \in \mathfrak{S}(\mathcal{H})$ for Φ are precisely those of the form

$$\xi = S^* T \,(\omega \otimes \sigma)\, T^* S,$$

where $\omega = \sum_j \lambda_j \omega_j \otimes |j\rangle\langle j|$ for some probabilities $\{\lambda_j\}$ and states $\omega_j \in \mathfrak{S}(\mathcal{L}_j)$.

(iv) Let $P_j := S^* T (R_j \otimes I^R) T^* S$ be the minimal central projections in $\mathcal{F}$. The restrictions $\Phi|_{B(P_j \mathcal{H})}$ have the form

$$\Phi|_{B(P_j\mathcal{H})}(S^*T(A_j \otimes B)T^*S) = S^*T(A_j \otimes \Psi_j(B))T^*S, \quad A_j \in B(\mathcal{L}_j), B \in B(\mathcal{K}^R),$$

where Ψ_j are irreducible nucp maps on $B(\mathcal{K}^R)$. Moreover, all Ψ_j coincide on diagonal elements and we have

$$\Psi_j(\sum_m B_{mm} \otimes |m\rangle\langle m|) = \sum_m \Phi_m(B_{mm}) \otimes |m \oplus 1\rangle\langle m \oplus 1|.$$

In particular, for all Ψ_j, $\mathcal{N}(\Psi_j) = span\{I^R_m \otimes |m\rangle\langle m|, m = 0, \dots, d-1\}$ and σ is the unique invariant state.

Proof. Since $\mathcal{F} \subseteq \mathcal{N}$, we may apply Proposition 2. It can be easily checked that an element of $\mathcal{N}$ is in $\mathcal{F}$ if and only if it is of the form

$$S^*T(A \otimes I^R)T^*S$$

with $A \in \mathcal{A} := \{\mathcal{T}_{d-1}\}' \cap B(\mathcal{H}^L_0)$. Note that the commutant $\mathcal{A}' := \{\mathcal{T}_{d-1}\}'' \cap B(\mathcal{H}^L_0) = \mathcal{Z}(\mathcal{A})$ is abelian. Further, we have $\mathcal{F} \simeq \mathcal{A}$ and since $\mathcal{F}$ is atomic, $\mathcal{A}$ must be such as well, so that $\{\mathcal{T}_{d-1}\}'' \cap B(\mathcal{H}^L_0)$ must be discrete. This proves (i).

By Lemma 1, there are some states $\sigma_j \in \mathfrak{S}(\mathcal{K}^R)$ such that

$$E(S^*T(R_j \otimes B)T^*S) = \sigma_j(B)P_j, \tag{3}$$

where $B \in B(\mathcal{K}^R)$ and $P_j := S^*T(R_j \otimes I^R)T^*S$ are the minimal central projections in $\mathcal{F}$. Moreover, any state of the form $\psi = T^*S(\omega_j \otimes \sigma_j)S^*T$ with $\omega_j \in \mathfrak{S}(\mathcal{L}_j)$ is an invariant state for Φ. It follows that for any $m = 0, \dots d-1$,

$$\begin{aligned}\sigma_j(I_m \otimes |m\rangle\langle m|) &= \psi(S^*T(R_j \otimes I_m \otimes |m\rangle\langle m|)T^*S) = \psi \circ \Phi(S^*T(R_j \otimes I_m \otimes |m\rangle\langle m|)T^*S) \\ &= \psi(S^*T(R_j \otimes I_{m\oplus 1} \otimes |m \oplus 1\rangle\langle m \oplus 1|)T^*S) = \sigma_j(I_{m\oplus 1} \otimes |m \oplus 1\rangle\langle m \oplus 1|)\end{aligned}$$

so that $\sigma_j(I_m \otimes |m\rangle\langle m|) = 1/d$. Let now $B = \sum_{m,n} B_{mn} \otimes |m|m\rangle\langle n| \in B(\mathcal{K}^R)$. Since $E \circ \Phi = \Phi \circ E = E$, we see that $E \circ F = F \circ E = E$. Using Lemma 1 for F, we obtain $E(A) = E(F(A)) = \sum_m E(F(Q_m A Q_m))) = \sum_m E(Q_m A Q_m)$, so that

$$\begin{aligned}E(S^*T(R_j \otimes B)T^*S) &= \sum_m E \circ F(S^*(\tilde{T}_{m-1}R_j\tilde{T}^*_{m-1} \otimes B_{mm} \otimes |m\rangle\langle m|)S) \\ &= \sum_m \rho_m(B_{mm})E(S^*(\tilde{T}_{m-1}R_j\tilde{T}^*_{m-1} \otimes I^R_m \otimes |m\rangle\langle m|)S) \\ &= \sum_m \rho_m(B_{mm})E(S^*T(R_j \otimes I^R_m \otimes |m\rangle\langle m|)T^*S) = \frac{1}{d}\sum_m \rho_m(B_{mm})P_j.\end{aligned}$$

This and (3) proves (ii). Since ξ is invariant for Φ if and only if it is invariant for E, (iii) now follows by Lemma 1.

Finally, we prove (iv). We see by the multiplicativity properties of Φ on $\mathcal{N}$ that $\Phi(B(P_j\mathcal{H})) \subseteq B(P_j\mathcal{H})$ and that the restrictions have the given form with some nucp map Ψ_j on $B(\mathcal{K}^R)$. Since any fixed point of Ψ_j is related to a fixed point of Φ, we can see that it must be trivial, so that Ψ_j are irreducible. For any $B_m \in B(\mathcal{K}^R_m)$, we have by Proposition 2,

$$\begin{aligned}\Phi(S^*T(R_j \otimes B_m \otimes |m\rangle\langle m|)T^*S &= \Phi(S^*_m(\tilde{T}_{m-1}R_j\tilde{T}^*_{m-1} \otimes B_m)S_m) \\ &= S^*_{m\oplus 1}(\tilde{T}_m R_j \tilde{T}^*_m \otimes \Phi_m(B_m))S_{m\oplus 1}) \\ &= S^*T(R_j \otimes \Phi_m(B_m) \otimes |m \oplus 1\rangle\langle m \oplus 1|)T^*S.\end{aligned}$$

It follows that $I_m^R \otimes |m\rangle\langle m| \in \mathcal{N}(\Psi_j)$ for all m and j. Hence any minimal projection in $\mathcal{N}(\Psi_j)$ must be of the form $Q \otimes |m\rangle\langle m|$ for some projection $Q \in B(\mathcal{K}_m^R)$. But then it easily follows that $I_m \otimes Q$ is is in $\mathcal{N}$, so that we must have $Q = I_m^R$. Further, observe that from $\Phi(Q_m A Q_n) = Q_{m\oplus 1}\Phi(A)Q_{n\oplus 1}$ we get

$$\Psi_j(\sum_{m,n} B_{mn} \otimes |m\rangle\langle n|) = \sum_{m,n} \Psi_{j,mn}(B_{mn}) \otimes |m \oplus 1\rangle\langle n \oplus 1|,$$

where $\Psi_{j,mm} = \Phi_m$ for all j and m. Hence by Proposition 2 (i)

$$\sigma(\Psi_j(\sum_{m,n} B_{mn} \otimes |m\rangle\langle n|)) = \sum_m \rho_{m\oplus 1}(\Phi_m(B_{mm})) = \sum_m \rho_m(B_{mm}) = \sigma(\sum_{m,n} B_{mn} \otimes |m\rangle\langle n|).$$

$\square$