A. K. Karn, On the geometry of an order unit space

Referee report

The aim of the paper is to give a geometric characterization of order unit spaces, similar to the characterization of a base of the base-normed space as a radially compact convex set. Here, a subset of a real vector space, called a skeleton with a head, is described by its geometric properties, such that it generates an order unit space (V, V^+, e), in which it corresponds precisely to the set of elements $\{u \in V, \|u\| = \|e - u\| = 1\} \cup \{0, e\}$. The author also discusses some conditions under which an order unit space contains a copy of ℓ_∞^n.

Overall evaluation

The results of the paper seem rather interesting to me (though I am not entirely sure about their novelty). Recently, there is an increased interest in order unit spaces and their geometric structure coming from their relation to physics, since they serve as a basis for a description of general probabilistic, or convex operational, theories. The description obtained in the present paper might prove useful also in that context. However, in the present form, the paper is not written well enough to be published.

First of all, the paper introduces a lot of terminology: "peripheral element", "skeleton with a head", "periphery", "canopy" and its "summit", "lead point", "periphery", "semi-peripheral". This does not seem standard to me, and I do not think that all of these notions are necessary or useful. There are also unclear formulations, not properly defined or inconsistent notations, obscure or even plainly wrong arguments in some of the proofs (though the main results seem to be correct). I also found a relatively large number of typos. Some of these problems are listed in the comments below.

Specific comments

1. absolutely ∞-orthogonal - does not seem to be used anywhere
2. p. 5, last line of the proof of Prop. 2.3: "$K(e-u) = K(u)$" should be $K(e-u) = e-K(u)$
3. p. 6, line 5: why is $\kappa \leq \frac{1}{2}$?
4. p. 6: What is the meaning of "K has a representation in C"?
5. p. 7, line 6: "Thus by Lemma 2.1 (3)..." how does the inequality follow from that statement?
6. p. 8, line 9 from below: "...by the definition of a canopy..." it is not clear how $\alpha, \beta \leq 1$ follows from the definition given here.
7. p. 9: "Thus by Proposition 2.7..." it is not clear how the statement follows from Prop. 2.7
8. p. 9, line 2 from below: $\alpha = 1 = \beta$ also follows from Lemma 2.1 directly.
9. p. 10: statement (1)(b) is immediate from the definition of the order unit norm
10. p. 11, Lemma 3.3: better remind the definition of $(S_V)_0$

11. p. 11, Lemma 3.3 (3): "u has an ∞-orthogonal pair": A pair always consists of two things. Better write: there is an element in C_V which is ∞-orthogonal to u

12. p. 11. line 4 from below: better repeat the reference to Ref. [11]

13. p. 12: is it clear that S_V generates (V, V^+, e)?

14. Sec. 4: "$[0, e] = \{v \in V^+ : \|v\| \leq 1\}$ this notation interferes with previously used notation of $[u, v]$ as a line segment

15. From Sec. 4 onward, the notation R_V is used instead of $(S_V)_0$. This should be unified.

16. p. 13, proof of Thm. 4.1 does not work: a weak*-compact subset is not necessarily sequentially compact. The dual unit ball of ℓ_∞, with the sequence of states $f_n(x_1, x_2, \ldots) = x_n$, is a notorious counterexample.

17. Corollary 4.8: This is quite obvious.

18. Remark 5.2: Since χ is a unital order isomorphism, it is obvious that it must be an isometry.

19. Theorem 5.3: what is the meaning of " $\perp_\infty$ is additive"?

20. Prop. 5.5: This is the same as Coro. 2.4.

21. Remark 5.6: I do not understand this remark at all.

Typos:

1. p.1: "in stead" instead

2. p. 2: "Kadison prove" proved

3. p. 3: "role model for a non-commutative ordered spaces" either skip "a" or "spaces"-> space. Also better skip "role".

4. p. 6, line 8 from below: "$1 - \alpha\gamma \geq 1$" should be probably ≥ 0

5. p. 7: "and and"

6. p.7, line 4 from below: e is missing in the equality after "Now"

7. p. 8, last displayed equation: "$K(w)$" should be $K(e - w)$ (?)

8. p. 9: "$\alpha_n + \beta$" -> $\alpha_n + \beta_n$, "$u_1 := u$" -> $u := u_1$

9. p. 9, line 3 from below: R should probably be S (?)

10. p. 13: "Then Then"

11. p.14, Lemma 4.6: $V_!$, also "e_2" should be e_1 at several places in the lemma and its proof.

12. p. 17, last line of the second displayed equation: should be $\sum_i \alpha_i u_i$.

Notations used without definition:

1. $[u, v]$ or (u, v): line segment connecting u and v
2. $S(V)$: set of states of the order unit space
3. $[0, e]_o$ order interval(?)