IRREDUCIBLE DECOMPOSITIONS AND STATIONARY STATES OF QUANTUM CHANNELS

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For a quantum channel (completely positive, trace-preserving map), we prove a generalization to the infinite-dimensional case of a result by Baumgartner and Narnhofer [3]: this result is, in a probabilistic language, a decomposition of a general quantum channel into its irreducible recurrent components. More precisely, we prove that the positive recurrent subspace (i.e. the space supporting the invariant states) can be decomposed as the direct sum of supports of extremal invariant states; this decomposition is not unique, in general, but we can determine all the possible decompositions. This allows us to describe the full structure of invariant states.

Keywords: quantum channel, (extremal) invariant state, enclosure.

1. Introduction

The time evolution of states of a closed quantum system is usually described as the conjugation by a group of unitary operators on the Hilbert space representing the state space of the system. When the system is open, that is, interacts with its surroundings, the situation is more complicated and rigorous treatment usually requires approximations. The most standard approach was put on solid mathematical ground by Davies in the seventies (in [12], see also [13]), and leads to describe the system's evolution by a semigroup $(\Phi_t)_{t \in \mathbb{R}_+}$ of linear maps on the set of states (i.e. positive, normalized functionals acting on the set of operators on the Hilbert space) with specific algebraic properties (see Section 2). Many features of these continuous parameter semigroups are already contained in the case of discrete semigroups $(\Phi^n)_{n \in \mathbb{N}}$. In addition, the interest in the discrete case was renewed by quantum computation theory (where the maps Φ model quantum gates, see [21]) and by quantum repeated interaction systems, see [7]. We therefore restrict ourselves

to the discrete case, and focus on the study of $\Phi = \Phi^1$, a linear map which is completely positive and trace-preserving. Such a map is called a quantum channel.

The study of ergodic properties of an open quantum system is related to the study of invariants of Φ, and of the associated spectrum. Analogies with operators associated with Markov chains (see Example 2.1) inspired the development of a notion of irreducible quantum channel by various authors in the seventies and eighties (see [1,15,16,19]), with different (and sometimes conflicting) definitions and implications. Associated with this notion is the possibility to decompose a reducible quantum channel into a sum of irreducible ones. A vision of this decomposition as related to an intuitive notion of trajectories (as for Markov chains), however, was not developed explicitly before the work of Baumgartner and Narnhofer in [3], where it is done in the case of a finite-dimensional Hilbert space.

In [11], we studied open quantum random walks, a special class of evolutions belonging to the above case. This led us to restate and extend the results of [3] to the case of open quantum random walks, which required in particular an extension to the infinite-dimensional case. Our proofs, however, apply to a wider class of evolutions than just quantum random walks. We therefore describe our results in full generality here. Even if the statements of the results remain the same, we underline that the infinite-dimensional context forces to have various modifications with respect to the original proofs presented in [3]: the existence of a null recurrent subspace has to be considered here and this affects the ergodic properties of the process (see e.g. Theorem 2.1 in [18]); also compactness arguments and spectral properties need a little more attention; the characterization of minimal enclosures (Proposition 5.3), for instance, requires completely new elements here.

The structure of this article is as follows. In Section 2, we describe our framework and in particular the evolutions Φ of interest, the so-called quantum channels. In Section 3, we recall the different notions of irreducibility. In Section 4, we define enclosures, our key tool, which originated in [3]. In Section 5, we describe the relation between enclosures and supports of invariant states. In Section 6 we discuss the structure of invariant states of a simple reducible evolution. In Section 7, we state our general decomposition theorem, that describes irreducible decompositions of evolutions and the general structure of the set of invariant states. In Section 8, we apply these results to a number of examples.

2. States and quantum channels

In this section we give a short summary of the theory of quantum channels, i.e. completely positive, trace-preserving maps on an ideal of trace-class operators. We fix a separable Hilbert space $\mathcal{H}$, which is supposed to play the role of a state space for a quantum system. We denote by $\mathcal{I}_1(\mathcal{H})$ the set of trace-class operators on $\mathcal{H}$ (see [22]), and equip it with the topology induced by the trace norm. We recall that the topological dual $\mathcal{I}_1(\mathcal{H})^*$ can be identified with the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators through the Schatten duality $(\rho, X) \mapsto \mathrm{Tr}(\rho X)$. Therefore, the topology of $\mathcal{I}_1(\mathcal{H})$ is the same as the weak topology induced by $\mathcal{B}(\mathcal{H})$. We

also recall that an operator X on $\mathcal{H}$ is called nonnegative (respectively positive or positive-definite), denoted $X \geq 0$ (resp. $X > 0$), if for $\varphi \in \mathcal{H} \setminus \{0\}$, one has $\langle \varphi, X \varphi \rangle \geq 0$ (resp. $\langle \varphi, X \varphi \rangle > 0$).

The states of a system will be represented by an operator belonging to a specific class:

DEFINITION 2.1. An operator ρ is called a *state* if it is self-adjoint (i.e. $\rho = \rho^*$), nonnegative, and is trace-class with trace one. We denote by $\mathcal{S}(\mathcal{H})$ the set of states on $\mathcal{H}$. A state is called *faithful* if it is positive definite.

REMARK 2.1. In the literature, a state is sometimes defined as a positive linear form on $\mathcal{B}(\mathcal{H})$ mapping Id to 1, i.e. as an element of the set

$$\mathcal{B}(\mathcal{H})^*_{+,1} = \{\eta \in B(\mathcal{H})^* \text{ s.t. } \eta(X) \geq 0 \text{ for } X \geq 0 \text{ and } \eta(\mathrm{Id}) = 1\}$$

equipped with the weak-* topology. The objects defined in Definition 2.1 are then called normal states. Obviously $\mathcal{S}(\mathcal{H})$ is homeomorphic to a subset of $\mathcal{B}(\mathcal{H})^*_{+,1}$.

Consider now a linear map Φ on $\mathcal{I}_1(\mathcal{H})$. We say that this map is positive if it maps nonnegative elements of $\mathcal{I}_1(\mathcal{H})$ to nonnegative elements of $\mathcal{I}_1(\mathcal{H})$. We say that it is *n*-positive, for $n \in \mathbb{N}$, if the map $\Phi \otimes \mathrm{Id}_{\mathcal{M}_n(\mathbb{C})}$ is positive as a map on $\mathcal{I}_1(\mathcal{H} \otimes \mathbb{C}^n)$; and completely positive if it is *n*-positive for any n in $\mathbb{N}$. We say that it is trace-preserving if, for any $\rho \in \mathcal{I}_1(\mathcal{H})$, one has $\mathrm{Tr}(\Phi(\rho)) = \mathrm{Tr}(\rho)$; in particular a positive trace-preserving map induces a map on $\mathcal{S}(\mathcal{H})$. Our main objects of interest will be maps that are completely positive and trace-preserving.

DEFINITION 2.2. A completely positive, trace-preserving map on a space $\mathcal{I}_1(\mathcal{H})$ is called *a quantum channel on* $\mathcal{H}$.

REMARK 2.2. A positive linear map on $\mathcal{I}_1(\mathcal{H})$ is automatically bounded (see Lemma 2.2 in [24]), so that it is weak-continuous.

The following theorem states a well-known fact about quantum channels (see [20, 21]).

THEOREM 2.1. *A linear map* Φ *on* $\mathcal{I}_1(\mathcal{H})$ *is completely positive if and only if there exists a family* $(V_i)_{i \in I}$ *of operators on* $\mathcal{H}$ *such that for any* ρ *in* $\mathcal{I}_1(\mathcal{H})$,

$$\Phi(\rho) = \sum_{i \in I} V_i \rho V_i^*. \tag{2.1}$$

If in addition Φ *is trace-preserving, then the operators* V_i *satisfy the relation*

$$\sum_{i \in I} V_i^* V_i = \mathrm{Id}_{\mathcal{H}}.$$

The decomposition (2.1) is called the Kraus form of Φ, and the family $(V_i)_{i \in I}$ an unravelling. Note that an unravelling of Φ is not unique (see [21] for more details).

We have mentioned that a source of inspiration is the analogy between quantum channels and Markov chains. In the following example we point out that Markov

chains are a special case of quantum channels. Note that, for any two vectors x and y in a Hilbert space $\mathcal{H}$ with scalar product $\langle \cdot, \cdot \rangle$ (which we assume is antilinear in the left variable), we denote by $|x\rangle\langle y|$ the map $z \mapsto \langle y, z \rangle \, x$.

EXAMPLE 2.1. Consider a Markov chain $(X_n)_n$ on a countable set E with transitions $p_{i,j} = \mathbb{P}(X_{n+1} = i \mid X_n = j)$. If we let $\mathcal{H}$ be $\ell^2(E)$, the set of (complex-valued) square-summable sequences indexed by E, denote by $(e_i)_{i \in E}$ the canonical orthonormal basis, and consider $V_{i,j} = \sqrt{p_{i,j}} \, |e_i\rangle\langle e_j|$ for i, j in E, then (2.1) defines a quantum channel which maps $\sum_{i \in E} \pi_n(i) \otimes |e_i\rangle\langle e_i|$ to $\sum_{i \in E} \pi_{n+1}(i) \otimes |e_i\rangle\langle e_i|$ where $\pi_{n+1} = p \pi_n$. In addition, any invariant state is of the form $\rho = \sum_{i \in E} \pi(i) |e_i\rangle\langle e_i|$ with $(\pi(i))_{i \in E}$ an invariant probability measure for the Markov chain.

REMARK 2.3. Trace-preservation of a map Φ is equivalent to $\Phi^*(\mathrm{Id}) = \mathrm{Id}$. The adjoint Φ^* is then a positive, unital (i.e. $\Phi^*(\mathrm{Id}) = \mathrm{Id}$) map on $\mathcal{B}(\mathcal{H})$, and by the Russo–Dye theorem [23] one has $\|\Phi^*\| = \|\Phi^*(\mathrm{Id})\|$ so that $\|\Phi\| = \|\Phi^*\| = 1$.

A quantum channel represents the (discrete) dynamics of an open quantum system in the Schrödinger picture (see [21] for more details). We denote by $\mathcal{F}(\Phi)$ the subset of $\mathcal{I}_1(\mathcal{H})$ of invariant elements of Φ and we will be specifically interested in the set $\mathcal{S}(\mathcal{H}) \cap \mathcal{F}(\Phi)$ of invariant states, i.e. elements of $\mathcal{S}(\mathcal{H})$ that are invariant by Φ.

For a state ρ we will consider its support, which is defined as the range of the projection $\mathrm{Id} - P_0(\rho)$, where

$$P_0(\rho) = \sup\{P \text{ orthogonal projection s.t. } \rho(P) = 0\}.$$

The supremum taken above is considered with respect to the order induced by the relation $\geq$ for operators, and always exists in the present situation. Following [18], we denote

$$\mathcal{R} = \sup\{\operatorname{supp} \rho \mid \rho \text{ an invariant state}\}$$

so that by definition, $\operatorname{supp} \rho \subset \mathcal{R}$ if ρ is an invariant state. This space is often called the fast recurrent space, in parallel with the classical case, where the fast recurrent configurations are the ones which support the invariant probability laws. The orthogonal complement of $\mathcal{R}$ is

$$\mathcal{D} = \{x \in \mathcal{H} \mid \langle x, \rho \, x \rangle = 0 \quad \text{for any invariant state } \rho\}.$$

REMARK 2.4. In [3], the states $\mathcal{R}$ and $\mathcal{D}$ are defined without reference to the set of invariant states as

$$\mathcal{D} = \{x \in \mathcal{H} \mid \langle x, \Phi^n(\rho) \, x \rangle \underset{n \to \infty}{\longrightarrow} 0 \text{ for any state } \rho\}$$

and $\mathcal{R} = \mathcal{D}^\perp$. These different definitions of $\mathcal{R}$ and $\mathcal{D}$ are equivalent in finite dimension.

REMARK 2.5. The space $\mathcal{D}$ is the sum of the transient and slow recurrent subspaces, as defined in [26].

3.   Irreducibility

Before we discuss decompositions of quantum channels, we need to discuss the relevant reducing components of the decomposition, i.e. irreducible quantum channels. As we will see in Proposition 3.2, irreducibility is strongly connected with the uniqueness of the invariant state.

As we already mentioned in Introduction, however, different definitions of irreducibility of quantum channels can be found in the literature. We will briefly recall them here. First we need to define some relevant concepts.

DEFINITION 3.1. Let Φ be a quantum channel on $\mathcal{S}(\mathcal{H})$. We say that an orthogonal projection P:
- *reduces* Φ if we have $\Phi(P\mathcal{I}_1(\mathcal{H})P) \subset P\mathcal{I}_1(\mathcal{H})P$,
- is *subharmonic* for Φ^* if $\Phi^*(P) \geq P$.

The complete proof of the following Proposition is given in [11].

PROPOSITION 3.1. *Let Φ be a quantum channel on $\mathcal{I}_1(\mathcal{H})$. The following properties are equivalent:*
- *Φ is Davies-irreducible: the only orthogonal projections reducing Φ are $P = 0$ and* Id*;*
- *the only orthogonal projections that are subharmonic for Φ^* are $P = 0$ and* Id*;*
- *ergodicity: for any state ρ, the operator $(\exp t\Phi)(\rho)$ is definite-positive for any $t > 0$.*

We say that Φ is irreducible if and only if any of the above properties holds.

REMARK 3.1. Regarding the above concepts and their interrelations:
- The equivalence between the first two properties follows from the simple observation that an orthogonal projection reduces Φ if and only if it is subharmonic for Φ^* (see [11, Proposition 3.3]).
- The definition of ergodicity given here originates in [24], and extends the definition given in [16] to infinite-dimensional $\mathcal{H}$.
- There exists yet another notion of irreducibility: one says that Φ is Evans-irreducible if the only orthogonal projections that are harmonic for Φ, i.e. such that $\Phi^*(P) = P$, are $P = 0$ and Id. Clearly Davies-irreducibility implies Evans-irreducibility, but the converse is not true in general.

In the same fashion as for Markov semigroups, there exists a Perron–Frobenius theorem related to the property of irreducibility. We state it in the next proposition, in a form essentially due to Schrader in [24].

PROPOSITION 3.2. *Let Φ be a quantum channel on $\mathcal{I}_1(\mathcal{H})$, and assume it has an eigenvalue λ of modulus 1, with eigenvector ρ. Then:*
- *1 is also an eigenvalue, with eigenvector $|\rho| = (\rho^*\rho)^{1/2}$,*
- *if Φ is irreducible, then λ is a simple eigenvalue and $|\rho| > 0$.*

REMARK 3.2. Proposition 3.2 still holds if Φ is not completely positive and trace-invariant, but simply 2-positive. For this reason, the same statement holds when the map Φ on $\mathcal{I}_1(\mathcal{H})$ is replaced with the map Φ^* on $\mathcal{B}(\mathcal{H})$, and all subsequent results about quantum channels will hold for 2-positive and trace-invariant maps on $\mathcal{I}_1(\mathcal{H})$, as long as they do not involve the Kraus form or unravelling of Φ.

An immediate consequence of this proposition is that an irreducible quantum channel on $\mathcal{I}_1(\mathcal{H})$ has at most one invariant state. In Sections 6 and 7 we will study the relations between the invariant states of a reducible quantum channel and the invariant states of its irreducible components.

4. Enclosures and communicating classes

For Markov chains, it is well known that irreducibility is related with the notion of communication within the induced graph. In addition, communicating classes have an explicit description as orbits of points, and are the relevant objects to break down a reducible Markov chain into irreducible ones. In this section we introduce the notion of enclosure, that will parallel the notion of closed set for Markov chains, and allows us to study irreducible decompositions of quantum channels.

DEFINITION 4.1. Let Φ be a quantum channel. A closed subspace $\mathcal{V}$ is an *enclosure* for Φ if, for any state ρ, $\operatorname{supp}\rho \subset \mathcal{V}$ implies $\operatorname{supp}\Phi(\rho) \subset \mathcal{V}$.

We will call nontrivial any enclosure which is neither $\{0\}$ nor $\mathcal{H}$. Clearly, a subspace $\mathcal{V}$ is an enclosure if and only if it is the range of a reducing orthogonal projector. Therefore, a quantum channel Φ is irreducible if and only if it has no nontrivial enclosures. Enclosures are relevant to reducibility properties and consequently to many other features of the channel; indeed, the notion of enclosure has been used in literature with different names (see for instance [4, 5, 8]).

We now prove a simpler characterization of enclosures.

LEMMA 4.1. *A closed vector subspace $\mathcal{V}$ of $\mathcal{H}$ is an enclosure if and only if, for any x in $\mathcal{V}$ with $\|x\| = 1$, the state $\Phi(|x\rangle\langle x|)$ has support in $\mathcal{V}$.*

Proof: Let ρ be a state with support in $\mathcal{V}$. The spectral decomposition of ρ is of the form $\sum_{i\in I} \lambda_i |e_i\rangle\langle e_i|$ with $\lambda_i > 0$, $\sum_{i\in I} \lambda_i = 1$ and $e_i \in \mathcal{V}$. Therefore, $\operatorname{supp}\Phi(|e_i\rangle\langle e_i|) \subset \operatorname{supp}\Phi(\rho)$, which shows the direct implication; in addition, the support of $\Phi(\rho)$ is the supremum of the projectors on the ranges of $\Phi(|e_i\rangle\langle e_i|)$ and this shows the converse. $\square$

This has the following useful corollary. Note that, for $(\mathcal{V}_i)_{i\in I}$ a family of closed subspaces of $\mathcal{H}$, we denote by e.g. $\mathcal{V}_1 + \mathcal{V}_2 + \ldots$ or $\sum_{i\in I} \mathcal{V}_i$ the closed vector space generated by $\bigcup_{i\in I} \mathcal{V}_i$.

COROLLARY 4.1. *Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two enclosures. The closed subspace $\mathcal{V}_1 + \mathcal{V}_2$ is also an enclosure.*

Proof: By a direct computation, $|x_1+x_2\rangle\langle x_1+x_2| \leq 2\,|x_1\rangle\langle x_1| + 2\,|x_2\rangle\langle x_2|$ for x_1, x_2 in $\mathcal{V}_1, \mathcal{V}_2$ respectively. Applying Lemma 4.1 shows that $\mathcal{V}_1 + \mathcal{V}_2$ is an enclosure. $\square$

This allows us to obtain an explicit characterization of enclosures in terms of unravellings of Φ, and connect them to a notion of orbit under the action of possible transitions of Φ.

PROPOSITION 4.1. *Consider a quantum channel Φ with unravelling $(V_i)_{i \in I}$. A subspace $\mathcal{V}$ of $\mathcal{H}$ is an enclosure if and only if $V_i \mathcal{V} \subset \mathcal{V}$ for any i.*

Proof: The proposition follows from Lemma 4.1 and the fact that, by the trace norm continuity of Φ one has for any $x \in \mathcal{V}$,

$$\Phi(|x\rangle\langle x|) = \sum_{i \in I} |V_i x\rangle\langle V_i x|. \qquad \square \tag{4.1}$$

Our goal is to consider enclosures defined as the set of points accessible from a given initial $x \in \mathcal{H}$. Proposition 4.1 suggests a natural definition and proposition.

PROPOSITION 4.2. *Let Φ be a quantum channel on $\mathcal{I}_1(\mathcal{H})$. Let $(V_i)_{i \in I}$ be an unravelling of Φ. For x in $\mathcal{H} \setminus \{0\}$, we call enclosure generated by x the closed vector space*

$$\mathrm{Enc}(x) = \mathbb{C}x + \overline{\mathrm{span}\{V_{i_1} \cdots V_{i_n} x, \,|\, n \in \mathbb{N}^*, \, i_1, ..i_n \in I\}}. \tag{4.2}$$

With this definition, the space $\mathrm{Enc}(x)$ is the smallest enclosure containing x.

Proof: It follows from (4.1) that Definition (4.2) also satisfies

$$\mathrm{Enc}(x) = \overline{\mathrm{span}\{\mathrm{supp}\,\Phi^n(|x\rangle\langle x|), \, n \geq 0\}}. \tag{4.3}$$

This shows that Definition (4.2) is independent of the choice of unravelling. The fact that $\mathrm{Enc}(x)$ is an enclosure then follows from Proposition 4.1. $\square$

REMARK 4.1. This implies in particular that a quantum channel Φ is irreducible if and only if $\mathcal{H} = \mathrm{Enc}(x)$ for any x in $\mathcal{H} \setminus \{0\}$.

We can define a notion of accessibility among vectors in $\mathcal{H}$, related to the notion of enclosure, and consider an equivalence relation. We will argue, however, that this will not immediately provide us with an interesting decomposition of a quantum channel.

DEFINITION 4.2. For x, y in $\mathcal{H}$, we say that:
- y is *accessible* from x (and denote it by $x \to y$) if $y \in \mathrm{Enc}(x)$;
- y *and* x *communicate* (and denote it by $x \leftrightarrow y$) if $\mathrm{Enc}(x) = \mathrm{Enc}(y)$.

One can immediately observe that accessibility is a transitive relation, and communication is an equivalence relation. We denote by $\mathcal{C}(x)$ the equivalence class of a vector x in $\mathcal{H}$ for the relation $\leftrightarrow$,

$$\mathcal{C}(x) = \{y \in \mathrm{Enc}(x) \text{ s.t. } x \in \mathrm{Enc}(y)\}.$$

An equivalence class of a vector x by $\leftrightarrow$ is a subset of $\mathrm{Enc}(x)$ but it is not a vector space since, for $x \neq 0$, $\mathcal{C}(x)$ cannot contain 0. Even adding the point 0 may fail to make $\mathcal{C}(x)$ a vector space, as the next example shows.

EXAMPLE 4.1. Take $\mathcal{H} = \mathbb{C}^2$ and denote by e_1, e_2 its canonical basis. Consider a quantum channel Φ on $\mathcal{I}_1(\mathcal{H})$ with unravelling (V_1, V_2) given by $V_1 = \sqrt{p} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $V_2 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}$ for some $p \in (0, 1)$ so that, for $\rho = \begin{pmatrix} \rho_{1,1} & \rho_{1,2} \\ \rho_{2,1} & \rho_{2,2} \end{pmatrix}$ in $\mathcal{I}_1(\mathcal{H})$, we have

$$\Phi(\rho) = \begin{pmatrix} p\rho_{2,2} + \rho_{1,1} & \sqrt{1-p}\,\rho_{1,2} \\ \sqrt{1-p}\,\rho_{2,1} & (1-p)\rho_{2,2} \end{pmatrix}.$$

By an immediate direct computation, the state $|e_1\rangle\langle e_1|$ is the only invariant state of this map. We want to describe the equivalence classes and the enclosures of the map Φ. We notice that, for any vector $u = {}^t(u_1, u_2)$ in $\mathbb{C}^2$,

$$|u\rangle\langle u| = \begin{pmatrix} |u_1|^2 & u_1\bar{u}_2 \\ \bar{u}_1 u_2 & |u_2|^2 \end{pmatrix} \quad \text{so that} \quad \Phi(|u\rangle\langle u|) = \begin{pmatrix} p|u_2|^2 + |u_1|^2 & \sqrt{1-p}\,u_1\bar{u}_2 \\ \sqrt{1-p}\,\bar{u}_1 u_2 & (1-p)|u_2|^2 \end{pmatrix}.$$

It is immediate that $\Phi(|u\rangle\langle u|)$ is a positive-definite matrix whenever $u_2 \neq 0$, so that
- $\mathrm{supp}\,\Phi^n(|e_1\rangle\langle e_1|) = \mathbb{C}\,e_1$ for all $n \geq 0$,
- for $u_2 \neq 0$, $\mathrm{supp}\,\Phi^n(|u\rangle\langle u|) = \mathbb{C}^2$ for all $n \geq 1$.

Identity (4.3) allows us to determine all the enclosures and equivalence classes:
- $\mathrm{Enc}(0) = \mathcal{C}(0) = \{0\}$,
- $\mathrm{Enc}(e_1) = \mathbb{C}\,e_1$ and $\mathcal{C}(e_1) = \mathrm{Enc}(e_1) \setminus \{0\}$,
- for all $u \in \mathbb{C}^2 \setminus \mathbb{C}\,e_1$, $\mathrm{Enc}(u) = \mathbb{C}^2$ and $\mathcal{C}(u) = \mathbb{C}^2 \setminus \mathbb{C}\,e_1$.

Supports of invariant states, on the other hand, are always vector spaces. Therefore, the naïve approach of considering the partition of $\mathcal{H}$ induced by the relation $\leftrightarrow$ to obtain a relevant decomposition of a quantum channel into irreducible such maps fails, as it does not seem to involve the vector space structure. A natural idea, derived from the study of Markov chains, is to consider specifically minimal objects. We therefore give the following definition of a minimal enclosure:

DEFINITION 4.3. Let $\mathcal{V}$ be an enclosure. We say that $\mathcal{V}$ is a *minimal enclosure* if any enclosure $\mathcal{V}'$ satisfying $\mathcal{V}' \subset \mathcal{V}$ is either $\{0\}$ or $\mathcal{V}$. We say that $\mathcal{V}$ is a *minimal nontrivial enclosure* if in addition $\mathcal{V} \neq \{0\}$.

The following easy proposition shows that this notion is indeed relevant.

PROPOSITION 4.3. $\mathcal{C}(x) = \mathrm{Enc}(x) \setminus \{0\}$ *if and only if* $\mathrm{Enc}(x)$ *is a minimal nontrivial enclosure.*

Proof: If $\mathcal{C}(x) = \mathrm{Enc}(x) \setminus \{0\}$, then, for all y in $\mathrm{Enc}(x) \setminus \{0\}$, we have $\mathrm{Enc}(x) = \mathrm{Enc}(y)$ and consequently $\mathrm{Enc}(x)$ is minimal. Conversely, if $\mathcal{V} = \mathrm{Enc}(x)$ is a minimal enclosure, for any y in $\mathcal{V} \setminus \{0\}$, $\mathrm{Enc}(y)$ is a nontrivial enclosure contained in $\mathcal{V}$ so that $\mathrm{Enc}(y) = \mathcal{V}$. Therefore $x \leftrightarrow y$ and $\mathcal{V} = \mathcal{C}(x) \cup \{0\}$. □

5.   Enclosures and invariant states

Baumgartner and Narnhofer in [3] studied a decomposition of a quantum channel related to the supports of extremal invariant states, in the case of a finite-dimensional space $\mathcal{H}$. In the present paper, we extend this analysis to the infinite-dimensional case. For this we will need to relate extremal invariant states to minimal enclosures. We will see that the form of invariant states for the quantum channel is dictated by the uniqueness or nonuniqueness of the decompositions into minimal enclosures and that this is related to the existence of mutually nonorthogonal minimal enclosures. The first result is as follows.

PROPOSITION 5.1. *Let Φ be a quantum channel on $\mathcal{H}$.*
1. *The support of an invariant state is an enclosure.*
2. *The fast recurrent subspace $\mathcal{R}$ is an enclosure.*

Proof: To prove the first point, fix an invariant state ρ_0, and let ρ be another state with support contained in $\operatorname{supp} \rho_0$. Fix an orthonormal family of eigenvectors for ρ_0 generating $\operatorname{supp} \rho_0$, and let X_0 be the set of finite linear combinations of these vectors. This set X_0 is dense in $\operatorname{supp} \rho_0$ and for every x in X_0 there exists λ such that $|x\rangle\langle x| \leq \lambda \rho_0$. Therefore there exists an approximation of ρ in the $\mathcal{I}_1(\mathcal{H})$ norm sense by an increasing sequence of finite-rank operators $(\rho_p)_p$ such that for every p there exists a λ_p with $\rho_p \leq \lambda_p \rho_0$, so that $\Phi(\rho_p) \leq \lambda_p \Phi(\rho_0)$ and therefore $\operatorname{supp} \Phi(\rho_p) \subset \operatorname{supp} \rho_0$. The sequence $\Phi(\rho_p)$ is increasing and weakly convergent to $\Phi(\rho)$ so that $\operatorname{supp} \Phi(\rho) \subset \operatorname{supp} \rho_0$, which proves that $\operatorname{supp} \rho_0$ is an enclosure.

To prove the second point, associate with every invariant state ρ the orthogonal projector P_ρ on its support. Then the orthogonal projector P on $\mathcal{R}$ is the supremum of the family $(P_\rho)_\rho$. For any invariant state ρ, P_ρ is subharmonic, i.e. $\Phi^*(P_\rho) \geq P_\rho$ and moreover, $\Phi^*(P) \geq \Phi^*(P_\rho) \geq P_\rho$, so that $\Phi^*(P) \geq P$ and the conclusion follows. □

REMARK 5.1. The first point of the previous proposition has already been proven in [17] and [26] in the dual setting, *i.e.* considering reducing projections for Φ^*. If $\mathcal{H}$ is separable, the second point can also be derived from a result in [26] which proves that there exists an invariant state with support equal to $\mathcal{R}$.

REMARK 5.2. The converse of point 1 of Proposition 5.1 is not true. Consider Example 2.1 associated with the symmetric random walk on $\mathbb{Z}$. Then $\mathcal{H} = \ell^2(\mathbb{Z})$ is an enclosure but the quantum channel Φ has no invariant state.

PROPOSITION 5.2. *Let $\mathcal{V}$ be an enclosure, $\mathcal{W}$ be a subspace of $\mathcal{H}$ which is in direct sum with $\mathcal{V}$, and $P_\mathcal{V}$ and $P_\mathcal{W}$ be the projections associated with the decomposition $\mathcal{V} \oplus \mathcal{W}$. Consider a state ρ with support in $\mathcal{V} \oplus \mathcal{W}$ and denote*

$$\rho_\mathcal{V} = P_\mathcal{V} \, \rho \, P_\mathcal{V}, \qquad \rho_\mathcal{W} = P_\mathcal{W} \, \rho \, P_\mathcal{W}, \qquad \rho_\mathcal{C} = P_\mathcal{V} \, \rho \, P_\mathcal{W}, \qquad \rho'_\mathcal{C} = P_\mathcal{W} \, \rho \, P_\mathcal{V}.$$

Similarly, decompose $\Phi(\rho)$ into $\Phi(\rho)_\mathcal{V} + \Phi(\rho)_\mathcal{W} + \Phi(\rho)_\mathcal{C} + \Phi(\rho)'_\mathcal{C}$. Then

1. $P_{\mathcal{W}}\left(\Phi(\rho_{\mathcal{C}}) + \Phi(\rho_{\mathcal{C}}')\right) P_{\mathcal{W}} = 0$,
2. if $\mathcal{Z}$ is another enclosure with $\mathcal{V} \subset \mathcal{Z} \subset \mathcal{R}$, then $\mathcal{Z} \cap \mathcal{V}^{\perp}$ is an enclosure,
3. if $\mathcal{W}$ is also an enclosure, then

$$\Phi(\rho)_{\mathcal{V}} = \Phi(\rho_{\mathcal{V}}), \quad \Phi(\rho)_{\mathcal{W}} = \Phi(\rho_{\mathcal{W}}), \quad \Phi(\rho)_{\mathcal{C}} = \Phi(\rho_{\mathcal{C}}), \quad \Phi(\rho)_{\mathcal{C}}' = \Phi(\rho_{\mathcal{C}}').$$

Proof:
1. Let $\kappa_{\pm\varepsilon} = \frac{1}{\varepsilon}\rho_{\mathcal{V}} \pm (\rho_{\mathcal{C}} + \rho_{\mathcal{C}'}) + \varepsilon\,\rho_{\mathcal{W}}$. We have $\kappa_{\pm\varepsilon} \geq 0$ (as can be checked from $\langle u, \kappa_{\pm\varepsilon} u\rangle = \langle u_{\pm\sqrt{\varepsilon}}, \kappa\,u_{\pm\sqrt{\varepsilon}}\rangle$, where $u_{\pm\sqrt{\varepsilon}} = \frac{1}{\sqrt{\varepsilon}} P_{\mathcal{V}} u \pm \sqrt{\varepsilon}\, P_{\mathcal{W}} u$), so that $\Phi(\kappa_{\pm\varepsilon}) \geq 0$, and, because $\mathcal{V}$ is an enclosure, the support of $\Phi(\rho_{\mathcal{V}})$ is contained in $\mathcal{V}$, so that

$$P_{\mathcal{W}}\,\Phi(\kappa_{\pm\varepsilon})\,P_{\mathcal{W}} = \pm P_{\mathcal{W}}\left(\Phi(\rho_{\mathcal{C}}) + \Phi(\rho_{\mathcal{C}}')\right)P_{\mathcal{W}} + \varepsilon\,P_{\mathcal{W}}\,\Phi(\rho_{\mathcal{W}})\,P_{\mathcal{W}} \geq 0,$$

   and by necessity $P_{\mathcal{W}}\left(\Phi(\rho_{\mathcal{C}}) + \Phi(\rho_{\mathcal{C}}')\right)P_{\mathcal{W}} = 0$.

2. Consider $\mathcal{W} = \mathcal{Z} \cap \mathcal{V}^{\perp}$ and ρ any invariant state; then

$$\rho_{\mathcal{V}} + \rho_{\mathcal{W}} + \rho_{\mathcal{C}} + \rho_{\mathcal{C}}' = \Phi(\rho_{\mathcal{V}}) + \Phi(\rho_{\mathcal{W}}) + \Phi(\rho_{\mathcal{C}}) + \Phi(\rho_{\mathcal{C}}').$$

   Considering $P_{\mathcal{W}} \cdot P_{\mathcal{W}}$ this yields $\rho_{\mathcal{W}} = P_{\mathcal{W}}\Phi(\rho_{\mathcal{W}})P_{\mathcal{W}}$, so that $P_{\mathcal{V}}\,\Phi(\rho_{\mathcal{W}})\,P_{\mathcal{V}}$ is positive with zero trace. Therefore $P_{\mathcal{V}}\,\Phi(\rho_{\mathcal{W}})\,P_{\mathcal{V}} = 0$ which implies $P_{\mathcal{V}}\,\Phi(\rho_{\mathcal{W}}) = \Phi(\rho_{\mathcal{W}})\,P_{\mathcal{V}} = 0$ and so $\rho_{\mathcal{W}} = \Phi(\rho_{\mathcal{W}})$. As the support of a stationary state, $\operatorname{supp}\rho_{\mathcal{W}} = \operatorname{supp}\rho \cap \mathcal{Z} \cap \mathcal{V}^{\perp}$ is an enclosure. By the same argument used to prove point 2 of Proposition 5.1, the supremum of $\operatorname{supp}\rho \cap \mathcal{Z} \cap \mathcal{V}^{\perp}$ over all possible invariant states ρ is still an enclosure, and this is $\mathcal{Z} \cap \mathcal{V}^{\perp}$.

3. If $\mathcal{V}$ and $\mathcal{W}$ are enclosures, then by definition $\operatorname{supp}\Phi(\rho_{\mathcal{V}}) \subset \mathcal{V}$ and $\operatorname{supp}\Phi(\rho_{\mathcal{W}}) \subset \mathcal{W}$. The equality

$$\Phi(\rho_{\mathcal{V}}) + \Phi(\rho_{\mathcal{W}}) + \Phi(\rho_{\mathcal{C}}) + \Phi(\rho_{\mathcal{C}}') = \Phi(\rho)_{\mathcal{V}} + \Phi(\rho)_{\mathcal{W}} + \Phi(\rho)_{\mathcal{C}} + \Phi(\rho)_{\mathcal{C}}'$$

   implies $\Phi(\rho_{\mathcal{V}}) = \Phi(\rho)_{\mathcal{V}}$ and $\Phi(\rho_{\mathcal{W}}) = \Phi(\rho)_{\mathcal{W}}$. We then have $\Phi(\rho)_{\mathcal{C}} + \Phi(\rho)_{\mathcal{C}}' = \Phi(\rho_{\mathcal{C}}) + \Phi(\rho_{\mathcal{C}}')$. Since $\operatorname{Ran}\rho_{\mathcal{C}} \subset \mathcal{V} \subset \operatorname{Ker}\rho_{\mathcal{C}}$ and $V_i\mathcal{V} \subset \mathcal{V}$ for all i by Proposition 4.1, one has $\operatorname{Ran}\Phi(\rho_{\mathcal{C}}) \subset \mathcal{V} \subset \operatorname{Ker}\Phi(\rho_{\mathcal{C}})$ and therefore $P_{\mathcal{W}}\Phi(\rho_{\mathcal{C}})P_{\mathcal{V}} = 0$. This implies $\Phi(\rho_{\mathcal{C}}') = \Phi(\rho)_{\mathcal{C}}'$ and similarly $\Phi(\rho_{\mathcal{C}}) = \Phi(\rho)_{\mathcal{C}}$. $\qquad\square$

COROLLARY 5.1. *For any enclosure $\mathcal{V}$ contained in $\mathcal{R}$, there exists an invariant state ρ such that* $\operatorname{supp}\rho \subset \mathcal{V}$.

Proof: By definition of $\mathcal{R}$, there exists an invariant state ρ with $\operatorname{supp}\rho \cap \mathcal{V} \neq \{0\}$. By Proposition 5.2, $P_{\mathcal{V}}\rho\,P_{\mathcal{V}}$ is (up to normalization) an invariant state with support in $\mathcal{V}$. $\qquad\square$

We will now discuss the connection between minimal enclosures and extremal invariant states, i.e. states ρ such that $\rho = t\,\rho_1 + (1-t)\,\rho_2$, with ρ_1, ρ_2 in $\mathcal{S}(\mathcal{H}) \cap \mathcal{F}(\Phi)$ and $t \in (0, 1)$, implies $\rho_1 = \rho_2 = \rho$.

REMARK 5.3. The distinction between states and normal states mentioned in Remark 2.1 does not lead to an ambiguity: by Example 4.1.35 in [6], the set $\mathcal{S}(\mathcal{H})$,

when viewed as a subspace of $\mathcal{B}(\mathcal{H})^*_{+,1}$, is a face, so that $\rho \in \mathcal{S}(\mathcal{H})$ is extremal regarding convex decompositions in $\mathcal{S}(\mathcal{H}) \cap \mathcal{F}(\Phi)$ if and only if it is extremal regarding convex decompositions in $\mathcal{B}(\mathcal{H})^*_{+,1} \cap \mathcal{F}(\Phi)$.

The following proposition is the main result in this section.

PROPOSITION 5.3. *A subspace of $\mathcal{R}$ is a minimal enclosure if and only if it is the support of an extremal invariant state. Moreover, any enclosure included in $\mathcal{R}$ contains a (nontrivial) minimal enclosure. Equivalently, for any invariant state ρ, there exists an extremal invariant state ρ_{ex} with $\operatorname{supp}\rho_{ex} \subset \operatorname{supp}\rho$.*

Proof: If $\mathcal{V}$ is a minimal enclosure contained in $\mathcal{R}$, then by Corollary 5.1, there exists a Φ-invariant state $\rho_\mathcal{V}$ with support in $\mathcal{V}$. By the discussion following Definition 4.1, the restriction of Φ to $\mathcal{I}_1(\mathcal{V})$ is irreducible. Proposition 3.2 shows that $\rho_\mathcal{V}$ is the unique Φ-invariant state with support in $\mathcal{V}$, and $\operatorname{supp}\rho_\mathcal{V} = \mathcal{V}$. This $\rho_\mathcal{V}$ must be extremal since $\rho_\mathcal{V} = t\,\rho_1 + (1-t)\,\rho_2$ with ρ_1, ρ_2 invariant states and $t \in (0, 1)$ would imply that ρ_1, ρ_2 are invariant states with support in $\mathcal{V}$ but then by uniqueness, $\rho_\mathcal{V} = \rho_1 = \rho_2$.

Conversely, if $\mathcal{V} = \operatorname{supp}\rho$ with ρ an extremal invariant state, then by Proposition 5.1, $\mathcal{V}$ is an enclosure. If we suppose, by contradiction, that it is not minimal, then there exists an enclosure $\mathcal{W}$ with $\mathcal{W} \subsetneq \mathcal{V} \subset \mathcal{R}$ and, by Corollary 5.1, an invariant state ρ' with $\operatorname{supp}\rho' \subset \mathcal{W}$. Since ρ is faithful on $\mathcal{V}$, by the same argument as in the proof of Proposition 5.1, we can approximate ρ' in the $\mathcal{I}_1(\mathcal{V})$ norm sense by a sequence $(\rho'_p)_p$ of finite-dimensional operators such that for every p, there exists λ_p with $\rho'_p \leq \lambda_p \rho$. If we let $\Psi_n = \frac{1}{n}\sum_{k=0}^{n-1} \Phi^k$ then by a standard compactness argument, $(\Psi_n(\rho'_p))_n$ converges weakly to a Φ-invariant nonnegative trace-class operator ρ_p^{inv} which therefore satisfies $\rho_p^{inv} \leq \lambda_p\,\rho$. The extremality of ρ implies that ρ_p^{inv} is proportional to ρ. This in turn implies that $(\Psi_n(\rho'))_n$ converges weakly to ρ, but $\Psi_n(\rho') = \rho'$ by the Φ-invariance of ρ'. Therefore, $\rho' = \rho$, a contradiction.

By Proposition 5.1 and Corollary 5.1, the second and third claims are equivalent. To prove the second one, consider the maps $\Phi^*_\mathcal{R}$ on the set $\mathcal{B}(\mathcal{R})$ of bounded operators acting on $\mathcal{R}$ defined by

$$\Phi^*_\mathcal{R}(P_\mathcal{R} x P_\mathcal{R}) = P_\mathcal{R}\Phi^*(x)P_\mathcal{R},$$

and denote by $\mathcal{F}(\Phi^*_\mathcal{R})$ the vector space of the fixed points for $\Phi^*_\mathcal{R}$, i.e. $\mathcal{F}(\Phi^*_\mathcal{R}) = \{X \in P_\mathcal{R}\mathcal{B}(\mathcal{H})P_\mathcal{R} : \Phi^*_\mathcal{R}(X) = X\}$. We know that $\mathcal{F}(\Phi^*_\mathcal{R})$ is the image of a normal conditional expectation by Theorem 2.1 of [18]. The proof of Theorem 5 of [25] shows then that $\mathcal{F}(\Phi^*_\mathcal{R})$ is an atomic subalgebra. It is trivial to verify that the projections contained in $\mathcal{F}(\Phi^*_\mathcal{R})$ are exactly the projections on enclosures contained in $\mathcal{R}$. So, for any enclosure $\mathcal{V}$, we consider the corresponding projection $P_\mathcal{V} \in \mathcal{F}(\Phi^*_\mathcal{R})$; but since $\mathcal{F}(\Phi^*_\mathcal{R})$ is atomic, it contains a minimal projection $P' \leq P$ and the range of P' is then a minimal enclosure contained in $\mathcal{V}$. $\qquad\square$

REMARK 5.4. The proof of point 3 of Proposition 5.3 can be given in a more constructive way: consider an invariant state ρ, which by restriction one can assume is faithful, i.e. with support $\mathcal{H}$. By the Banach–Alaoglu theorem, the set $\mathcal{B}(\mathcal{H})^*_{+,1} \cap \mathcal{F}(\Phi)$ is a compact, convex, metrizable subset of the locally convex space $\mathcal{B}(\mathcal{H})^*$ equipped with the weak-* topology. By Theorem 4.1.11 and Proposition 4.1.3 in [6], and the fact that affine maps on $\mathcal{B}(\mathcal{H})^*$ are exactly the maps $\eta \mapsto \eta(X)$ for $X \in \mathcal{B}(\mathcal{H})$, there exists a Borel probability measure μ in $\mathcal{B}(\mathcal{H})^*$, such that $\rho(X) = \int \eta(X) \mathrm{d}\mu(\eta)$ for any X, and μ has support in the set of extremal states of $\mathcal{B}(\mathcal{H})^*_{+,1} \cap \mathcal{F}(\Phi)$. Since in addition the set $\mathcal{S}(\mathcal{H}) \cap \mathcal{F}(\Phi)$ is a face, μ has support in the set of extremal states of $\mathcal{S}(\mathcal{H}) \cap \mathcal{F}(\Phi)$. For any Borel set B of $\mathcal{B}(\mathcal{H})^*$ with $\mu(B) > 0$ one can define $\rho_B = \frac{1}{\mu(B)} \int_B \eta(X) \mathrm{d}\mu(\eta)$. This ρ_B is a state with $\mathrm{supp}\,\rho_B \subset \mathrm{supp}\,\rho$. By considering a sequence of Borel sets that are balls $B(\rho_0, \frac{1}{n})$ for the metric compatible with the weak-* topology restricted to the unit sphere of $\mathcal{B}(\mathcal{H})^*$, one has for μ-almost all ρ_0 that $\rho_{B(\rho_0, \frac{1}{n})} \to \rho_0$ in the topology of $\mathcal{S}(\mathcal{H})$, so that $\mathrm{supp}\,\rho_0 \subset \mathrm{supp}\,\rho$.

For any quantum channel Φ, point 2 of Proposition 5.2, together with Proposition 5.3, will allow us to decompose the space $\mathcal{R}$ associated with Φ into a direct sum of minimal enclosures, and each of them is the support of an extremal invariant state. We give the following sequel to the two results quoted above, that essentially shows that the procedure of taking orthogonal complements is efficient in terms of decomposition into minimal enclosures.

LEMMA 5.1. *Let* $\mathcal{V} = \mathcal{V}_1 + \cdots + \mathcal{V}_n + \mathcal{V}_{n+1}$, *where the* $\mathcal{V}_i$, $i = 1, \ldots, n+1$, *are distinct minimal enclosures contained in* $\mathcal{R}$, *and* $\mathcal{V}_i \perp \mathcal{V}_j$ *for* $i \neq j$ *in* $1, \ldots, n$. *Then there exists a minimal enclosure* $\mathcal{V}'_{n+1}$, *orthogonal to* $\mathcal{V}_1, \ldots, \mathcal{V}_n$ *and such that* $\mathcal{V} = \mathcal{V}_1 + \ldots + \mathcal{V}_n + \mathcal{V}'_{n+1}$. *If* $n = 1$ *then one can take* $\mathcal{V}'_2 = \mathcal{V} \cap \mathcal{V}_1^\perp$. *In particular, if a subspace of* $\mathcal{R}$ *can be written as a sum of minimal enclosures, then it can be written as a sum of mutually orthogonal minimal enclosures.*

Proof: Let us first prove the claim for $n = 1$. We know that $\mathcal{V}$ is an enclosure as direct sum of two enclosures and so by Proposition 5.2, $\mathcal{V}'_2$ is an enclosure. If $\mathcal{V}_2 \perp \mathcal{V}_1$ then $\mathcal{V}'_2 = \mathcal{V}_2$ and there is nothing to prove. Assume therefore that $\mathcal{V}_2 \not\perp \mathcal{V}_1$. Proposition 5.3 provides us with a nontrivial minimal enclosure $\mathcal{W} \subseteq \mathcal{V}'_2$. Then $\mathcal{W} \not\subset \mathcal{V}_2$ for otherwise $\mathcal{W} = \mathcal{V}_2 \subset \mathcal{V}'_2$ and $\mathcal{V}_2 \perp \mathcal{V}_1$, a contradiction. Since $\mathcal{W} \subset \mathcal{V}_1 + \mathcal{V}_2$, there exists $w = v_1 + v_2$ in $\mathcal{W}$ with $v_i \in \mathcal{V}_i$ and $v_i \neq 0$ for $i = 1, 2$. Then $0 \neq v_2 = w - v_1 \in \mathcal{V}_2 \cap (\mathcal{W} + \mathcal{V}_1)$. Since $\mathcal{V}_2$ is a minimal enclosure one must have $\mathcal{V}_2 \subset \mathcal{W} + \mathcal{V}_1$ so that $\mathcal{V}_1 + \mathcal{V}_2 = \mathcal{V}_1 + \mathcal{W}$ and necessarily $\mathcal{W} = \mathcal{V}'_2$. This proves the minimality of $\mathcal{V}'_2$.

Now if $n > 1$, define $\mathcal{V}'_{n+1,1} = (\mathcal{V}_1 + \mathcal{V}_{n+1}) \cap \mathcal{V}_1^\perp$. By the preceding discussion, $\mathcal{V}'_{n+1,1}$ is orthogonal to $\mathcal{V}_1$ and $\mathcal{V}_1 + \mathcal{V}_{n+1} = \mathcal{V}_1 + \mathcal{V}'_{n+1,1}$. Then define $\mathcal{V}'_{n+1,2} = (\mathcal{V}_2 + \mathcal{V}'_{n+1,1}) \cap \mathcal{V}_2^\perp$. This $\mathcal{V}'_{n+1,2}$ is now orthogonal to $\mathcal{V}_1$ and $\mathcal{V}_2$ and $\mathcal{V}_2 + \mathcal{V}'_{n+1,1} = \mathcal{V}_2 + \mathcal{V}'_{n+1,2}$ so that $\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_{n+1} = \mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}'_{n+1,2}$. Iterating this process gives the desired $\mathcal{V}'_{n+1}$ in the form of $\mathcal{V}'_{n+1,n}$. $\square$

We therefore have our main tool for decompositions of quantum channels into irreducible ones. We wish to relate these decompositions to the structure of invariant states of Φ. In the case of Markov chains, it is well known that these are all convex combinations of the extremal invariant states associated with irreducible parts in the decomposition. We will see in the next section, however, that this is not the case for general quantum channels.

6.   Invariant states of nonirreducible quantum channels

In this section we study the last ingredient of our decomposition, that is, how the invariant states of a quantum channel on a sum $\mathcal{V}_1 + \mathcal{V}_2$ of two minimal enclosures relate to the extremal invariant states associated with these two minimal enclosures. We will see that this relation will depend on the uniqueness of the decomposition $\mathcal{V}_1 + \mathcal{V}_2$.

Let us define what we mean by this uniqueness. We say that the decomposition of a subspace $\mathcal{Z}$ of $\mathcal{R}$ in a direct sum of minimal enclosures is unique, if, whenever $(\mathcal{V}_\alpha)_{\alpha \in A}$ and $(\mathcal{W}_\beta)_{\beta \in B}$ are two families of minimal enclosures with

$$\mathcal{V}_\alpha \cap \mathcal{V}_{\alpha'} = \{0\} \text{ for any } \alpha \neq \alpha', \qquad \mathcal{W}_\beta \cap \mathcal{W}_{\beta'} = \{0\} \text{ for any } \beta \neq \beta',$$

and $\mathcal{Z} = \sum_{\alpha \in A} \mathcal{V}_\alpha = \sum_{\beta \in B} \mathcal{W}_\beta$, then the sets $\{\mathcal{V}_\alpha, \alpha \in A\}$ and $\{\mathcal{W}_\beta, \beta \in B\}$ coincide, and in particular A and B have the same cardinality.

The following lemma characterizes the situations when the decomposition of a subspace as the direct sum of two enclosures is unique. First remark that, by point 2 in Proposition 5.2, if x and y are in $\mathcal{R}$ then
- either $\mathrm{Enc}(x) \perp \mathrm{Enc}(y)$,
- or $x \notin \mathrm{Enc}(y)^\perp$ and $y \notin \mathrm{Enc}(x)^\perp$.

Indeed, if $y \in \mathrm{Enc}(x)^\perp \cap \mathcal{R}$ then $\mathrm{Enc}(y) \perp \mathrm{Enc}(x)$.

LEMMA 6.1. *Let $\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2$, where $\mathcal{V}_1$ and $\mathcal{V}_2$ are minimal enclosures contained in $\mathcal{R}$. The decomposition of $\mathcal{V}$ in a direct sum of minimal enclosures is unique if and only if any enclosure $\mathcal{W}$ such that $\mathcal{W} \not\perp \mathcal{V}_1$ and $\mathcal{W} \not\perp \mathcal{V}_2$ satisfies $\mathcal{W} \cap \mathcal{V} = \{0\}$. If the latter statement holds, then the two enclosures are orthogonal.*

Proof: Assume the decomposition of $\mathcal{V}$ as a direct sum of minimal enclosures is unique. Then $\mathcal{V}_1 \perp \mathcal{V}_2$, otherwise by Proposition 5.2, $\mathcal{V} \cap \mathcal{V}_1^\perp$ would be an enclosure that does not contain $\mathcal{V}_2$, leading to a different decomposition of $\mathcal{V}$. Now consider a minimal enclosure $\mathcal{W}$ with $\mathcal{W} \not\perp \mathcal{V}_1$ and $\mathcal{W} \not\perp \mathcal{V}_2$. This implies $\mathcal{W} \neq \mathcal{V}_1$ so that $\mathcal{W} \cap \mathcal{V}_1 = \{0\}$. If $\mathcal{W} \cap \mathcal{V} \neq \{0\}$ then it is an enclosure in $\mathcal{W}$ so by minimality, $\mathcal{W} \subset \mathcal{V}$. Then $\mathcal{W} \oplus \mathcal{V}_1$ is a direct sum of minimal enclosures contained in $\mathcal{V}$, so, by Proposition 5.3, one can complete this as a decomposition of $\mathcal{V}$ into a direct sum of minimal enclosures. This is a contradiction, leading to $\mathcal{W} \cap \mathcal{V} = \{0\}$.

Now assume that any enclosure $\mathcal{W}$ such that $\mathcal{W} \not\perp \mathcal{V}_1$ and $\mathcal{W} \not\perp \mathcal{V}_2$ satisfies $\mathcal{W} \cap \mathcal{V} = \{0\}$. Taking first $\mathcal{W} = \mathcal{V}_2$, which obviously has a nontrivial intersection with $\mathcal{V}$, we obtain that $\mathcal{V}_1 \perp \mathcal{V}_2$. Now consider some minimal enclosure $\mathcal{V}_3$ contained

in $\mathcal{V}$. Then, by assumption, one has e.g. $\mathcal{V}_3 \perp \mathcal{V}_1$ and $\mathcal{V}_3 \not\perp \mathcal{V}_2$ and so $\mathcal{V}_3 \subset \mathcal{V}_1^\perp \cap \mathcal{V}$, which, as proved above, is $\mathcal{V}_2$. This proves the uniqueness of the decomposition. $\square$

Next we need to strengthen Proposition 5.2 to distinguish between the situations where the decomposition into minimal enclosures is unique or not. The first result treats the situation where the decomposition is unique. To simplify the notation, from now on, when $\mathcal{V}$ is an enclosure, we will denote by $\Phi_{|\mathcal{V}}$ (instead of $\Phi_{|\mathcal{I}_1(\mathcal{V})}$) the restriction of Φ to $\mathcal{I}_1(\mathcal{V})$.

PROPOSITION 6.1. *If ρ is Φ-invariant and $\mathcal{V}$ and $\mathcal{W}$ are two minimal enclosures contained in $\mathcal{R}$, such that the decomposition of $\mathcal{V} + \mathcal{W}$ into a sum of minimal enclosures is unique, then $P_\mathcal{V}\, \rho\, P_\mathcal{W} = P_\mathcal{W}\, \rho\, P_\mathcal{V} = 0$, i.e. with the notation of Proposition 5.2 one has $\rho_\mathcal{C} = \rho'_\mathcal{C} = 0$.*

Proof: If $\mathcal{V}$ and $\mathcal{W}$ are minimal enclosures in $\mathcal{R}$, then, by Proposition 5.3, they are the supports of extremal invariant states $\rho_\mathcal{V}$ and $\rho_\mathcal{W}$. Because the decomposition of $\mathcal{V} + \mathcal{W}$ into minimal enclosures is unique, $\rho_\mathcal{V}$ and $\rho_\mathcal{W}$ are the unique extremal invariant states of $\Phi_{|(\mathcal{V}+\mathcal{W})}$. Since the set of invariant states is convex, then by the Krein–Milman theorem, ρ is a convex combination of $\rho_\mathcal{V}$ and $\rho_\mathcal{W}$, so $\rho_\mathcal{C}$ and $\rho'_\mathcal{C}$ must be zero. $\square$

REMARK 6.1. Consider the quantum channel Φ associated with a Markov chain as in Example 2.1. It is a simple observation that a minimal enclosure for Φ is necessarily of the form $\mathcal{V} = \ell^2(C)$ for C a minimal communication class for the Markov chain (where $\ell^2(C)$ is viewed as a subspace of $\ell^2(E)$). Therefore, two distinct minimal enclosures $\mathcal{V}_1$ and $\mathcal{V}_2$ are necessarily orthogonal, decompositions into sums of minimal enclosures are unique, and any invariant state on $\mathcal{H} = \ell^2(V_1 + V_2)$ is a convex combination of the extremal invariant states ρ_1, ρ_2 with supports $\ell^2(V_1)$, $\ell^2(V_2)$, respectively.

A second result will allow us to describe more explicitly the situation where the decomposition into minimal enclosures is not unique, and describe the associated invariant states.

PROPOSITION 6.2. *Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two minimal enclosures contained in $\mathcal{R}$. Assume that the decomposition of $\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2$ in a direct sum of minimal enclosures is not unique. Then $\dim \mathcal{V}_1 = \dim \mathcal{V}_2$. If, in addition, $\mathcal{V}_1 \perp \mathcal{V}_2$ (as can be chosen by Lemma 5.1) then there exists a partial isometry Q from $\mathcal{V}_1$ to $\mathcal{V}_2$ satisfying*

$$Q^*Q = \mathrm{Id}_{|\mathcal{V}_1}, \qquad Q\,Q^* = \mathrm{Id}_{|\mathcal{V}_2}, \tag{6.1}$$

and for any ρ in $\mathcal{I}_1(\mathcal{H})$, and $R = Q P_{\mathcal{V}_1} + Q^* P_{\mathcal{V}_2}$,

$$R\,\Phi(\rho)\,P_{\mathcal{V}_i} + P_{\mathcal{V}_i}\,\Phi(\rho)\,R = \Phi\big(R\,\rho\,P_{\mathcal{V}_i} + P_{\mathcal{V}_i}\,\rho\,R\big) \qquad \text{for } i = 1, 2. \tag{6.2}$$

Proof: By Lemma 6.1, there exists a minimal enclosure $\mathcal{W}$ in $\mathcal{V}_1 + \mathcal{V}_2$ such that $\mathcal{W} \not\perp \mathcal{V}_i$, $i = 1, 2$. If $\mathcal{V}_1 \cap \mathcal{W}^\perp = \{0\}$ then by minimality of $\mathcal{V}_1$, we have $\mathcal{V}_1 \subset \mathcal{W}^\perp$,

a contradiction. Therefore $\mathcal{V}_1 \subset \mathcal{W}$, and $\dim \mathcal{V}_1 \leq \dim \mathcal{W}$; by symmetry one has $\dim \mathcal{V}_1 = \dim \mathcal{W}$.

Assume now that $\mathcal{V}_1 \perp \mathcal{V}_2$. Define the map $\Phi_{\mathcal{R}}^*$ as in the proof of Proposition 5.3. By Remark 3.2, if $E = \mathcal{V}_1$, $\mathcal{V}_2$ or $\mathcal{W}$, then P_E is (up to multiplication) the unique invariant of the restriction Φ_E^* of $\Phi_{\mathcal{R}}^*$ to $\mathcal{B}(E)$. Consider the decomposition of $P_{\mathcal{W}} = \left(\begin{smallmatrix} A & B^* \\ B & C \end{smallmatrix} \right)$ in the splitting $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$, where necessarily $B \neq 0$. A simple consequence of Proposition 5.2 is that in the same decomposition, $\Phi_{\mathcal{R}}^*(P_{\mathcal{W}}) = \left(\begin{smallmatrix} \Phi_{\mathcal{R}}^*(A) & \Phi_{\mathcal{R}}^*(B)^* \\ \Phi_{\mathcal{R}}^*(B) & \Phi_{\mathcal{R}}^*(C) \end{smallmatrix} \right)$. Therefore A is proportional to $P_{\mathcal{V}_1}$ and C to $P_{\mathcal{V}_2}$. Writing relations $P = P^* = P^2$ satisfied by $P_{\mathcal{W}}$, one sees that B must be proportional to an operator Q satisfying relations (6.1). Fix Q; for $\theta \in [0, \pi]$, the operator defined by

$$P_\theta = \begin{pmatrix} \cos^2 \theta & \sin \theta \cos \theta \, Q^* \\ \sin \theta \cos \theta \, Q & \sin^2 \theta \end{pmatrix}$$

is an orthogonal projection preserved by the map $\Phi_{\mathcal{R}}^*$. So its range is an enclosure and, by point 3 of Proposition 5.2, P_θ will satisfy the relation

$$\Phi(P_\theta \, \rho \, P_\theta) = P_\theta \, \Phi(\rho) \, P_\theta,$$

for any ρ in $\mathcal{I}_1(\mathcal{H})$. Differentiating this relation with respect to θ, we have

$$\Phi\left(\frac{\mathrm{d}P_\theta}{\mathrm{d}\theta} \rho \, P_\theta + P_\theta \, \rho \, \frac{\mathrm{d}P_\theta}{\mathrm{d}\theta} \right) = \frac{\mathrm{d}P_\theta}{\mathrm{d}\theta} \Phi(\rho) \, P_\theta + P_\theta \, \Phi(\rho) \, \frac{\mathrm{d}P_\theta}{\mathrm{d}\theta}.$$

Computing the derivatives at $\theta = 0$ and $\theta = \pi/2$, we obtain relations (6.2).   □

COROLLARY 6.1. *Assume that* $\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2$ *where* $\mathcal{V}_1$ *and* $\mathcal{V}_2$ *are mutually orthogonal minimal enclosures, contained in* $\mathcal{R}$, *but that the decomposition of* $\mathcal{V}$ *into a direct sum of minimal enclosures is nonunique. For* $i = 1, 2$ *let* ρ_i^{inv} *be the unique invariant state with support in* $\mathcal{V}_i$. *Consider* Q *the partial isometry defined in Proposition 6.2. Then* $\rho_2^{\mathrm{inv}} = Q \, \rho_1^{\mathrm{inv}} \, Q^*$.

If ρ *is an invariant state with support in* $\mathcal{V}$, *then:*
- $P_{\mathcal{V}_1} \rho \, P_{\mathcal{V}_1}$ *is proportional to* ρ_1^{inv},
- $P_{\mathcal{V}_2} \rho \, P_{\mathcal{V}_2}$ *is proportional to* ρ_2^{inv},
- $P_{\mathcal{V}_1} \rho \, P_{\mathcal{V}_2}$ *is proportional to* $\rho_1^{\mathrm{inv}} Q^* = Q^* \rho_2^{\mathrm{inv}}$,
- $P_{\mathcal{V}_2} \rho \, P_{\mathcal{V}_1}$ *is proportional to* $\rho_2^{\mathrm{inv}} Q = Q \rho_1^{\mathrm{inv}}$.

Proof: The first identity is obtained by applying relation (6.2) to $\rho = \rho_1^{\mathrm{inv}}$ with P_1, then applying it again to the resulting relation, this time with P_2.

That each $\rho_{i,j} = P_{\mathcal{V}_i} \rho \, P_{\mathcal{V}_j}$ is an invariant is an immediate consequence of Proposition 5.2. The relation satisfied by $\rho_{1,2}$ and $\rho_{2,1}$ is then obtained by applying relation (6.2) to e.g. $\rho_{1,2}$, with P_1 or P_2.   □

7. Irreducible decompositions of quantum channels and invariant states

We are now in a position to state the relevant decomposition associated with Φ.

PROPOSITION 7.1. *Let Φ be a quantum channel on a separable Hilbert space $\mathcal{H}$. There exists a decomposition of $\mathcal{H}$ in the form*

$$\mathcal{H} = \mathcal{D} + \sum_{\alpha \in A} \mathcal{V}_\alpha + \sum_{\beta \in B} \sum_{\gamma \in C_\beta} \mathcal{V}_{\beta,\gamma}, \tag{7.1}$$

where any set A, B, C_β is at most countable, any C_β has cardinality at least two, and:
- *every $\mathcal{V}_\alpha$ or $\mathcal{V}_{\beta,\gamma}$ in this decomposition is a minimal enclosure,*
- *for β in B, any minimal enclosure that is not orthogonal to $\sum_{\gamma \in C_\beta} \mathcal{V}_{\beta,\gamma}$ is contained in $\sum_{\gamma \in C_\beta} \mathcal{V}_{\beta,\gamma}$,*
- *any two distinct subspaces $\mathcal{D}$, $\mathcal{V}_\alpha$, $\mathcal{V}_{\beta,\gamma}$ are mutually orthogonal.*

Notice that the sets A and B above can be empty, but they are simultaneously empty only when Φ does not admit any invariant state.

Proof: We start with the orthogonal decomposition $\mathcal{H} = \mathcal{D} + \mathcal{R}$, and proceed to decompose $\mathcal{R}$, when it is not trivial. Consider the set of all minimal enclosures $\mathcal{V}$ with the property that any minimal enclosure different from $\mathcal{V}$ is orthogonal to $\mathcal{V}$. By separability, this set is at most countable. Then we can denote all such minimal enclosures by $\mathcal{V}_\alpha$, with α in a (countable) set of indices A. Let $\mathcal{O}$ be the direct sum of all these enclosures, $\mathcal{O} = \sum_{\alpha \in A} \mathcal{V}_\alpha$. Then $\mathcal{O}$ is an enclosure, and, by point 2 of Proposition 5.2, $\mathcal{R} \cap \mathcal{O}^\perp$ is also an enclosure.

Assume that $\mathcal{R} \cap \mathcal{O}^\perp$ is nontrivial; we proceed to decompose it. Let $\beta(1) = 1$ and consider a minimal enclosure $\mathcal{V}_{\beta(1),1} \subset \mathcal{R} \cap \mathcal{O}^\perp$. By the definition of $\mathcal{O}$, there exists a minimal enclosure $\mathcal{V}_2$ in $\mathcal{R} \cap \mathcal{O}^\perp$, and by Lemma 5.1 we can choose $\mathcal{V}_{\beta(1),2}$ minimal, orthogonal to $\mathcal{V}_1$, and such that $\mathcal{V}_{\beta(1),1} + \mathcal{V}_{\beta(1),2} = \mathcal{V}_{\beta(1),1} + \mathcal{V}_2$. If all minimal enclosures are either included in $\mathcal{V}_{\beta(1),1} + \mathcal{V}_{\beta(1),2}$ or orthogonal to $\mathcal{V}_{\beta(1),1} + \mathcal{V}_{\beta(1),2}$, we set $C_{\beta(1)} = \{1, 2\}$. Otherwise, we call $\mathcal{V}_3$ a minimal enclosure not included in and not orthogonal to $\mathcal{V}_{\beta(1),1} + \mathcal{V}_{\beta(1),2}$. By Lemma 5.1 we can choose $\mathcal{V}_{\beta(1),3}$ minimal, orthogonal to $\mathcal{V}_{\beta(1),1} + \mathcal{V}_{\beta(1),2}$ and such that

$$\mathcal{V}_{\beta(1),1} + \mathcal{V}_{\beta(1),2} + \mathcal{V}_{\beta(1),3} = \mathcal{V}_{\beta(1),1} + \mathcal{V}_{\beta(1),2} + \mathcal{V}_3,$$

and we proceed again with the same method for a denumerable number of steps so that we construct $C_{\beta(1)}$. If $\mathcal{R} \cap \mathcal{O}^\perp \cap \left(\sum_{\gamma \in C_{\beta(1)}} E_{\beta(1),\gamma} \right)^\perp \neq \{0\}$, we can iterate the procedure. $\qquad \square$

Before we state our next result, let us give some notation. We fix a decomposition (7.1) as considered in Proposition 7.1. We define

$$P_0 = P_{\mathcal{R}^\perp}, \qquad P_i = P_{\mathcal{V}_i} \quad \text{for } i \in A \text{ or } i \in \bigcup_{\beta \in B} \{\beta\} \times C_\beta,$$

and, for a state ρ, and i, j taking the values 0, $\alpha \in A$ or $(\beta, \gamma) \in \bigcup_{\beta \in B} \{\beta\} \times C_\beta$,

$$\rho_i = P_i \rho P_i, \qquad \rho_{i,j} = P_i \rho P_j. \tag{7.2}$$

As before, we denote by ρ_i^{inv} the unique invariant state of $\Phi_{|\mathcal{V}_i}$.

We can now state the following result.

THEOREM 7.1. *Let ρ be a Φ-invariant state and consider a related orthogonal decomposition of the form (7.1). With the notation (7.2), we have*

1. $\rho_0 = 0$,
2. *every ρ_i is proportional to ρ_i^{inv}, for all indices $i \in A \cup \bigcup_{\beta \in B} \{\beta\} \times C_\beta$,*
3. *for $\gamma \neq \gamma'$ in C_β, the off-diagonal term $\rho_{((\beta,\gamma),(\beta,\gamma'))}$, which we simply denote by $\rho_{(\beta,\gamma,\gamma')}$, may be nonzero, and is Φ-invariant. In addition, there exists a partial isometry $Q_{(\beta,\gamma,\gamma')}$ from $\mathcal{V}_{\beta,\gamma}$ to $\mathcal{V}_{\beta,\gamma'}$ such that:*

   - $\rho_{(\beta,\gamma')}^{\mathrm{inv}} = Q_{(\beta,\gamma,\gamma')} \, \rho_{(\beta,\gamma)}^{\mathrm{inv}} \, Q_{(\beta,\gamma,\gamma')}^*$,
   - $\rho_{(\beta,\gamma,\gamma')}$ *is proportional to* $Q_{(\beta,\gamma,\gamma')}^* \, \rho_{(\beta,\gamma')}^{\mathrm{inv}} = \rho_{(\beta,\gamma)}^{\mathrm{inv}} \, Q_{(\beta,\gamma,\gamma')}^*$,

4. *all other $\rho_{i,j}$ (for i, j taking all possible values in $\{0\} \cup A \cup \bigcup_{\beta \in B} \{\beta\} \times C_\beta$) are zero.*

Proof: This follows from a repeated application of Propositions 5.2 and 7.1, and Corollary 6.1. ☐

REMARK 7.1. The decomposition of an invariant state ρ given by Theorem 7.1 can be rewritten in the same form as in formula (12) of Theorem 7 in [3], or as in Theorem 22 of [14], by simple algebraic manipulations. The key object is an isomorphism between $\mathcal{V}_{\beta,1} \otimes \mathbb{C}^{C_\beta}$ and $\sum_{\gamma \in C_\beta} \mathcal{V}_{\beta,\gamma}$ for each β, given by

$$\mathcal{E}(u \otimes x) = \sum_{\gamma \in C_\beta} u_\gamma \, Q_{(\beta,1,\gamma)} x \quad \text{for} \quad u = (u_\gamma)_{\gamma \in C_\beta}.$$

REMARK 7.2. The representation of invariant states appearing in Theorem 7.1 has recently been studied in [14], where an analogous result is proven in infinite dimension (and in the continuous time setting, but this point is not crucial). Our techniques and starting points are completely different and essentially replicate the approach used in [3] and [11]. Concerning the orthogonal decomposition and the representation of invariant states, however, our result is more general than the one in [14, Theorem 2.1], since we do not need to assume the atomicity of the decoherence free algebra (notice that the existence of a faithful normal invariant state assumed in [14] is not a restriction, since our decomposition is anyway only for the fast recurrent subspace $\mathcal{R}$, and by Remark 5.1, the restriction of Φ to $\mathcal{R}$ has a faithful invariant state). The key step which allows us to avoid this additional assumption is that we can prove that the fixed point algebra $\mathcal{F}(\Phi_\mathcal{R}^*)$ is atomic. When there exists a faithful invariant state, this means that $\mathcal{F}(\Phi^*)$ is atomic. However, we do not know whether the decoherence free algebra (see [14]), usually denoted by $\mathcal{N}(\Phi^*)$, is atomic, neither can we so far deduce other generalizations of the results on the structure of this algebra studied in [14].

8. Examples

Even though our proofs do carry over to infinite dimension, the mechanisms behind the decompositions of quantum channels are the same for finite- and infinite-dimensional systems. Due to the lack of space, we consider mostly finite-dimensional examples.

EXAMPLE 8.1 (classical Markov chains). Consider as in Example 2.1 a Markov chain on a countable set E. Denote by $(C_\alpha)_{\alpha \in A}$ the family of minimal communication classes C_α such that the Markov chain has an invariant probability $\pi^{(\alpha)}$ with support C_α, by $R = \cup_{\alpha \in A} C_\alpha$ the (disjoint) union of these classes, and by D the complement $D = E \setminus R$. Then, according to the discussion in Remark 6.1, the decomposition (7.1) of $\mathcal{H} = \ell^2(E)$ is given by

$$\mathcal{H} = \mathcal{D} + \sum_{\alpha \in A} \mathcal{V}_\alpha \qquad \text{where } \mathcal{D} = \ell^2(D), \ \mathcal{V}_\alpha = \ell^2(C_\alpha)$$

and any invariant state on $\mathcal{H}$ is a convex combination of the extremal states, which are of the form $\sum_{i \in C_\alpha} \pi_i^{(\alpha)} |e_i\rangle \langle e_i|$.

EXAMPLE 8.2. Consider the quantum channel defined in Example 4.1. From the computations in Example 4.1, one has $\mathcal{R} = \mathbb{C} e_1$ and therefore $\mathcal{D} = \mathbb{C} e_2$.

EXAMPLE 8.3 (2×2 matrices). Consider $\mathcal{H} = \mathbb{C}^2$ and Φ a positive quantum map on the algebra $\mathcal{B}(\mathbb{C}^2)$, which we identify with the set $M_2(\mathbb{C})$ of 2×2 matrices and equiped with the scalar product $\langle x, y \rangle_{M_2} = \operatorname{tr}(x^* y)$. The Pauli matrices

$$\sigma_0 = \frac{1}{\sqrt{2}} \operatorname{Id}_{\mathbb{C}^2}, \quad \sigma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

form an orthonormal basis of $M_2(\mathbb{C})$ and satisfy

$$\sigma_k^2 = \sigma_0^2, \qquad \sigma_k \sigma_j = -\sigma_j \sigma_k, \qquad \sigma_j \sigma_k = i \sigma_\ell,$$

if $(j, k, \ell) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$.

It is easy to see that, since Φ is trace preserving and positive, its matrix in the basis $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ is of the form

$$\Phi = \begin{pmatrix} 1 & {}^t 0 \\ b & A \end{pmatrix} \tag{8.1}$$

where $b \in \mathbb{R}^3$, ${}^t 0 = (0, 0, 0)$, A is a 3×3 matrix with real coefficients. The map Φ is positive if and only if $\|b + Ax\| \leq 1$ for all x such that $\|x\| \leq 1$ (see [9] for more details, even if in the continuous time setting).

It is well known that states on $\mathbb{C}^2$ are all operators of the form $\rho = \sigma_0 + u \cdot \sigma$ with u in $\mathbb{R}^3$, $\|u\| \leq 1$ (here we use the standard notation $u \cdot \sigma = \sum_{i=1,2,3} u_i \sigma_i$). This is called the Bloch sphere representation. In addition, it is easy to see that a state $\rho = \sigma_0 + u \cdot \sigma$ is invariant for Φ if and only if $b + Au - u = 0$.

Essentially, the problem of decomposing $\mathcal{R}$ into minimal enclosures is reduced to solving the linear system $b + Au - u = 0$, and then considering if there exist solutions with $\|u\| = 1$ and how many they are. However, by the Markov–Kakutani theorem (see [22]), an invariant state always exists. For the decomposition of the fast recurrent space $\mathcal{R}$, only 3 different cases are possible.

- There exists a unique invariant state ρ. Then the only minimal enclosure in $\mathcal{R}$ is $\mathcal{R}$ itself and it has dimension 2 when ρ is faithful and 1 otherwise.
- There exist infinitely many invariant states, and they are given by all convex combinations of two extremal invariant states ρ_1 and ρ_2. Then $\mathcal{R}$ can be written in a unique way as the direct sum of two minimal enclosures, which are the supports of ρ_1 and ρ_2.
- There exist infinitely many extremal invariant states. Then any state is invariant, any one dimensional subspace is an enclosure, and $\mathcal{R}$ can be written as $\mathcal{R} = \mathbb{C}\, e_1 \oplus \mathbb{C}\, e_2$, for any two linearly independent vectors e_1, e_2 in $\mathbb{C}^2$. This third case is possible if and only if Φ is the identity operator.

EXAMPLE 8.4. Define $V = \mathbb{N} \cup \{0\}$, $\mathfrak{h} = \mathbb{C}^3$, $\mathcal{H} = \mathfrak{h} \otimes \ell^2(V)$ and fix a canonical basis $(e_k)_{k=1,2,3}$ of $\mathfrak{h}$ so that we represent matrices and vectors in this basis. Choose $p, q > 0$ such that $p < 1/2$, $p + q < 1$ and a family of operators $(L_{i,j})_{i,j \in V}$ on $\mathfrak{h}$ such that $L_{ij} = 0$ when $|i - j| \geq 2$, $L_{00} = \sqrt{1 - p}\, \mathrm{Id}_{\mathfrak{h}}$, $L_{j+1,j} = \sqrt{p}\, \mathrm{Id}_{\mathfrak{h}}$ for $j \geq 0$ and

$$L_{j-1,j} = \begin{pmatrix} \sqrt{1-p} & 0 & 0 \\ 0 & \sqrt{1-p} & 0 \\ 0 & 0 & \sqrt{q} \end{pmatrix} \qquad \text{for } j \geq 1,$$

$$L_{j,j} = \sqrt{\frac{(1-p-q)}{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{for } j \geq 1.$$

We have $\sum_{i \in V} L_{i,j}^* L_{ij} = \mathrm{Id}$ for all j in V, so that the map Φ acting on $\mathcal{I}_1(\mathcal{H})$ defined by

$$\Phi(\rho) = \sum_{i,j \in V} (L_{i,j} \otimes |i\rangle\langle j|)\, \rho\, (L_{i,j}^* \otimes |j\rangle\langle i|),$$

is a quantum channel. This map Φ is an open quantum random walk with transition operators $(L_{i,j})_{i,j \in V}$ as defined in [2]. Denote by $(|j\rangle)_{j \in V}$ the canonical basis of $\ell^2(V)$. It was proved in [11] that minimal enclosures for open quantum random walks are generated by vectors of the form $u \otimes |i\rangle$. Consider therefore $u = {}^t(u_1, u_2, u_3)$ in $\mathfrak{h}$, then

$$\mathrm{Enc}(u \otimes |i\rangle) = \begin{cases} \mathrm{span}\{u \otimes |j\rangle, \, j \geq 0\}, & \text{if } u_3 = 0, \\ \mathrm{span}\{e_3 \otimes |j\rangle, (e_1 + e_2) \otimes |j\rangle, \, j \geq 0\}, & \text{if } u_3 \neq 0, u_1 = u_2 \\ \mathcal{H}, & \text{if } u_3 \neq 0, u_1 \neq u_2. \end{cases}$$

The enclosures described in the first case ($u_3 = 0$) are the minimal ones and so they support the extremal invariant states of the evolution. Using finite difference equations as for similar classical Markov chains, one can compute these extremal invariant states,

$$\rho(u) = c \sum_{j \geq 0} \left(\frac{p}{1-p} \right)^j |u\rangle\langle u| \otimes |j\rangle\langle j|,$$

for $u = {}^t(u_1, u_2, 0) \neq 0$ and a normalizing constant c.

Then we have $\mathcal{R} = \text{span}\{e_1, e_2\} \otimes \ell^2(V)$, $\mathcal{D} = \text{span}\{e_3\} \otimes \ell^2(V)$ and the decomposition (7.1) can be written with A empty, B consisting of only one element β, $C_\beta = \{1, 2\}$,

$$\mathcal{V}_{\beta,1} = \text{span}\{v^1 \otimes |j\rangle, j \geq 0\}, \qquad \mathcal{V}_{\beta,2} = \text{span}\{v^2 \otimes |j\rangle, j \geq 0\},$$

for any linearly independent vectors v^1 and v^2 orthogonal to e_3. We observe that $\rho_{\beta,1}^{\text{inv}} = \rho(e_1)$ is the only invariant state with support $E = \text{span}\{e_1 \otimes |j\rangle, j \geq 0\}$ and, defining Q as $|e_2\rangle\langle e_1|$, that any invariant state has a decomposition

$$\rho = \begin{pmatrix} t\,\rho(e_1) & \lambda\,\rho(e_1)Q^* \\ \bar{\lambda}\,Q\rho(e_1) & (1-t)\,Q\rho(e_1)Q^* \end{pmatrix} \qquad \text{with } t \in [0, 1].$$

Using the previous expressions for enclosures, one can also deduce the communication classes, in particular for the vectors of the form $u \otimes |j\rangle$, which are the most interesting in the special case of open quantum random walks.

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