P. Lipka-Bartosik, C. T. Chubb, J. M. Renes, et al.: Quantum dichotomies and coherent thermodynamics beyond first-order asymptotics

Referee report

This paper concerns the problem of exact or approximate transformation of quantum dichotomies in the asymptotic regime. In the case when the target dichotomy is commutative, explicit expressions for second-order transformation rates are given under different assumptions on the behaviour of the transformation errors, in the general case only the optimality part is proved. The results are based on the characterization of approximate transformations of classical dichotomies in terms of hypothesis testing errors, together with the properties of the pinching map and monotonicity of various relative entropy quantities. These results are applied to obtain second-order rates for thermodynamic transformations and for LOCC transformations between pure bipartite entangled states.

Overall evaluation

The results of the paper are timely and important, since the problem of conversions between sets of states under various regimes is fundamental in many resource theories. The present paper gives answers for dichotomies (that is, pairs of states) in the second-order asymptotic limit and provides significant extensions of the existing results. Moreover, in the scase of commuting targets, it explores and succesfully utilises connections of convertibility problems to asymtotic hypothesis testing.

However, the paper in the present form is not very well written. While the basic ideas are clearly explained and the main results are well described, the actual proofs are quite difficult to read. This is partly due to the fact that the arguments are sometimes given in a confusing and descriptive way, and partly due to a relatively high number of mistakes. These are mostly just typos, but since most of the proofs consist of complicated manipulations, this makes reading exceedingly difficult.

Some of these issues are listed in the specific comments below. I suggest that the authors carefully read the manuscript, identifying also other places where some confusion may occur. Some further suggestions are added at the end, that the authors might find useful.

Some specific comments

1. p. 15, eq. (70a): ϵ should be $\epsilon - 2\delta$ (?)

2. p. 15, eq. (73): the assumption $\lambda > 0$ is needed

3. p. 15, paragraph above eq. (74): $\gamma_\lambda(\rho\|\sigma)$: λ or x?

4. p.16-17, Lemma 17: In the results in [106,107] and [108,109] used in the proof, the role of type-I and type-II errors are exchanged, this should reflect in the exchange $\rho \leftrightarrow \sigma$ in the expressions of the lemma (the bounds for λ seem to be given correctly).

5. p. 17, Eq. (86): $D(\sigma, \rho)$ should be $D(\sigma\|\rho)$

6. p.17: the arguments of the proof between the last paragraph of the left column, finishing the proof for $\lambda \in \mathcal{R}_R$ are quite unclear and confusing, in particular see the three points below.

7. Are the functions in eq. (87) invertible? It is better to explain, prove or give a reference for this.

8. p. 17, line under eq. (89): the swapping of ρ and σ should also affect the bound in $\mathcal{R}_L$

9. p.17, looking at eq. (91), I would say that the inequalities are opposite...

10. p. 19, Lemma 19: As far as I can see, the limit values $\check{D}_{\pm\infty}$ were not specified. (In case I missed it: it is better to point to the place in the text where it is done)

11. p.19, proof of Lemma 19: this is a bit confusing. Q_t were defined to be projections, but the test given in eq. (106) is not a projection in general (unless Π is rank one).

12. p. 33, paragraph below eq. (A8): $A \leq \Phi$ or $A \geq \Phi$?

13. p.34, line below eq. (A16): $y \in (0, \epsilon)$ or $y \in (\epsilon, 1)$?

14. p. 35, line below eq. (A21): what is "our expression for ϵ"?

Some additional remarks and suggestions

1. p.3, last line above Sec. II.B: There are some conditions that characterize transformations between general quantum dichotomies (and more generally families of states), given in terms of expressions related to state discrimination, in the spirit of classical Blackwell and Le Cam theorems, e.g. [74], Prop. 5 (exact case) and (A. Jenčová, Comparison of quantum channels and quantum statistical experiments, 2016 IEEE International Symposium on Information Theory (ISIT), 2249 - 2253, IEEE Conference Publications, 2016), Thm. 4 (approximate case). Usefulness is not clear, however.

2. In the proof of Lemma 25, to show that the limit in eq. (B16) exists, one can use the following argument: by Lemma 23, there is some $C_0 >$ such that the sequence $y_n := nC_0 + D_\alpha(\mathcal{P}_{\sigma^{\otimes n}}(\rho^{\otimes n})\|\sigma^{\otimes n})$ is nonnegative and eq. (B19) shows that it is subadditive: $y_{n+m} \leq y_n + y_m$. By the Fekete Lemma, this implies that $\lim_n n^{-1}y_n = \inf_n n^{-1}y_n$ exists.