On the structure of higher order quantum maps

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1 Affine subspaces and higher order maps

1.1 The category FinVect

Let FinVect be the category of finite dimensional real vector spaces with linear maps. We will denote the usual tensor product by $\otimes$, then $(\mathrm{FinVect}, \otimes, I = \mathbb{R})$ is a symmetric monoidal category, with the associators, unitors and symmetries given by the obvious isomorphisms

$$\begin{aligned} \alpha_{U,V,W} &: (U \otimes V) \otimes W \simeq U \otimes (V \otimes W), \\ \lambda_V &: I \otimes V \simeq V, \qquad \rho_V : V \otimes I \simeq V, \\ \sigma_{U,V} &: U \otimes V \simeq V \otimes U. \end{aligned}$$

Let $(-)^* : V \mapsto V^*$ be the usual vector space dual, with duality denoted by $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}$. We will use the canonical identification $V^{**} = V$ and $(V_1 \otimes V_2)^* = V_1^* \otimes V_2^*$. With this duality, FinVect is compact closed. This means that for each object V, there are morphisms $\eta_V : I \to V^* \otimes V$ (the "cup") and $\epsilon_V : V \otimes V^* \to I$ (the "cap") such that the following snake identities hold:

$$(\epsilon_V \otimes V) \circ (V \otimes \eta_V) = V, \qquad (V^* \otimes \epsilon_V) \circ (\eta_V \otimes V^*) = V^*, \tag{1}$$

here we denote the identity map on the object V by V. Let us identify these morphisms. First, η_V is a linear map $\mathbb{R} \to V^* \otimes V$, which can be identified with the element $\eta_V(1) \in V^* \otimes V$ and $\epsilon_V \in (V \otimes V^*)^* = V^* \otimes V$ is again an element of the same space. Choose a basis $\{e_i\}$ of V, let $\{e_i^*\}$ be the dual basis of V^*, that is, $\langle e_i^*, e_j \rangle = \delta_{i,j}$. Let us define

$$\eta_V(1) = \epsilon_V := \sum_i e_i^* \otimes e_i.$$

It is easy to see that this definition does not depend on the choice of the basis, indeed, ϵ_V is the linear functional on $V \otimes V^*$ defined by

$$\langle \epsilon_V, x \otimes x^* \rangle = \langle x^*, x \rangle, \qquad x \in V,\ x^* \in V^*.$$

It is also easily checked that the snake identities (1) hold.

For two objects V and W in FinVect, we will denote the set of all morphisms (i.e. linear maps) $V \to W$ by $\mathrm{FinVect}(V, W)$. Then $\mathrm{FinVect}(V, W)$ is itself a real linear space and we have the well-known identification $\mathrm{FinVect}(V, W) \simeq V^* \otimes W$. This can be given as follows: for each $f \in \mathrm{FinVect}(V, W)$, we have $C_f := (V^* \otimes f)(\epsilon_V) = \sum_i e_i^* \otimes f(e_i) \in V^* \otimes W$. Conversely, since $\{e_i^*\}$

is a basis of V^*, any element $w \in V^* \otimes W$ can be uniquely written as $w = \sum_i e_i^* \otimes w_i$ for $w_i \in W$, and since $\{e_i\}$ is a basis of V, the assignment $f(e_i) := w_i$ determines a unique map $f : V \to W$. The relations between $f \in \mathrm{FinVect}(V, W)$ and $C_f \in V^* \otimes W$ can be also written as

$$\langle C_f, x \otimes y^* \rangle = \langle \epsilon_V, x \otimes f^*(y^*) \rangle = \langle f^*(y^*), x \rangle = \langle y^*, f(x) \rangle, \qquad x \in V,\ y^* \in W^*,$$

here $f^* : W^* \to V^*$ is the adjoint of f. Note that by compactness, the internal hom in FinVect satisfies $[V, W] \simeq V^* \otimes W$, so that in the case of FinVect, the object $[V, W]$ can be identified with the space of linear maps $\mathrm{FinVect}(V, W)$.

We now present two examples that are most important for us.

Example 1. Let $V = \mathbb{R}^N$. In this case, we fix the canonical basis $\{|i\rangle,\ i = 1, \ldots, N\}$. We will identify $(\mathbb{R}^N)^* = \mathbb{R}^N$, with duality $\langle x, y \rangle = \sum_i x_i y_i$, in particular, we identify $I = I^*$. We then have $\epsilon_V = \sum_i |i\rangle \otimes |i\rangle$ and if $f : \mathbb{R}^N \to \mathbb{R}^M$ is given by the matrix A in the two canonical bases, then $C_f = \sum_i |i\rangle \otimes A|i\rangle$ is the vectorization of A.

Example 2. Let $V = M_n^h$ be the space of $n \times n$ complex hermitian matrices. We again identify $(M_n^h)^* = M_n^h$, with duality $\langle A, B \rangle = \mathrm{Tr}\, A^T B$, where A^T is the usual transpose of the matrix A. Let us choose the basis in M_n^h, given as

$$\Big\{ |j\rangle\langle k| + |k\rangle\langle j|,\ j \le k,\ i\Big(|j\rangle\langle k| - |k\rangle\langle j|\Big),\ j < k \Big\}.$$

Then one can check that

$$\Big\{ \frac{1}{2}\Big(|j\rangle\langle k| + |k\rangle\langle j|\Big),\ j \le k,\ \frac{i}{2}\Big(|k\rangle\langle j| - |j\rangle\langle k|\Big),\ j < k \Big\}$$

is the dual basis and we have

$$\epsilon_V = \sum_{j,k} |j\rangle\langle k| \otimes |j\rangle\langle k|.$$

For any $f : M_n^h \to M_m^h$,

$$C_f = \sum_{j,k} |j\rangle\langle k| \otimes f(|j\rangle\langle k|)$$

is the Choi matrix of f.

1.2 The category Af

We now introduce the category Af, whose objects are of the form $X = (V_X, A_X)$, where V_X is an object in FinVect and $A_X \subseteq V_X$ is a proper affine subspace, see Appendix B for definitions and basic properties. Morphisms $X \xrightarrow{f} Y$ in Af are linear maps $f : V_X \to V_Y$ such that $f(A_X) \subseteq A_Y$. For any object X, we put

$$L_X := \mathrm{Lin}(A_X), \quad S_X := \mathrm{Span}(A_X), \quad D_X = \dim(V_X), \quad d_X = \dim(L_X).$$

We have

$$A_X = a + L_X = S_X \cap \{\tilde{a}\}^{\sim}, \tag{2}$$

for any choice of elements $a \in A_X$ and $\tilde{a} \in \tilde{A}_X$. We now introduce a tensor product and duality that endow Af with the structure of a *-autonomous category.

By Corollary 4, the dual $\tilde{A}_X$ is a proper affine subspace in V_X^*, so that $X^* := (V_X^*, \tilde{A}_X)$ is an object in Af. We have $X^{**} = X$ and the corresponding subspaces are related as

$$L_{X^*} = S_X^\perp, \qquad S_{X^*} = L_X^\perp. \tag{3}$$

It is easily seen that for any $X \xrightarrow{f} Y$, the adjoint map satisfies $f^*(\tilde{A}_Y) \subseteq \tilde{A}_X$, so that $Y^* \xrightarrow{f^*} X^*$ and the duality $(-)^*$ is a full and faithful functor $\mathrm{Af}^{op} \to \mathrm{Af}$.

We will next introduce a monoidal structure $\otimes$ as follows. For two objects X and Y, we put $V_{X\otimes Y} = V_X \otimes V_Y$ and construct the affine subspace $A_{X\otimes Y}$ as the affine span of

$$A_X \otimes A_Y = \{a \otimes b,\ a \in A_X,\ b \in A_Y\}.$$

Fix any $\tilde{a}_X \in \tilde{A}_X$ and $\tilde{a}_Y \in \tilde{A}_Y$. Since $A_X \otimes A_Y \subseteq \{\tilde{a}_X \otimes \tilde{a}_Y\}^\sim$, the affine span of $A_X \otimes A_Y$ is a proper affine subspace and we have by Lemma 16

$$A_{X\otimes Y} := \mathrm{Aff}(A_X \otimes A_Y) = \{A_X \otimes A_Y\}^{\approx}.$$

Lemma 1. *For any $a_X \in A_X$, $a_Y \in A_Y$, we have*

$$\begin{aligned} L_{X\otimes Y} &= \mathrm{Lin}(A_X \otimes A_Y) = \mathrm{Span}(\{x \otimes y - a_X \otimes a_Y,\ x \in A_X,\ y \in A_Y\}) && (4)\\ &= (a_X \otimes L_Y) + (L_X \otimes a_Y) + (L_X \otimes L_Y) && (5)\end{aligned}$$

(here + denotes the direct sum of subspaces). We also have

$$S_{X\otimes Y} = S_X \otimes S_Y.$$

Proof. The equality (4) follows from Lemma 16. For any $x \in A_X$, $y \in A_Y$ we have

$$x \otimes y - a_X \otimes a_Y = a_X \otimes (y - a_Y) + (x - a_X) \otimes a_Y + (x - a_X) \otimes (y - a_Y),$$

so that $L_{X\otimes Y} = \mathrm{Lin}(A_X \otimes A_Y)$ is contained in the subspace on the RHS of (5). Let d be the dimension of this subspace, then clearly

$$d_{X\otimes Y} \le d \le d_X + d_Y + d_X d_Y.$$

On the other hand, any element of S_X has the form tx for some $t \in \mathbb{R}$ and $x \in A_X$, so that it is easily seen that $S_X \otimes S_Y = S_{X\otimes Y}$. Hence

$$\begin{aligned} d_{X\otimes Y} &= \dim(L_{X\otimes Y}) = \dim(S_{X\otimes Y}) - 1 = \dim(S_X)\dim(S_Y) - 1 = (d_X + 1)(d_Y + 1) - 1\\ &= d_X + d_Y + d_X d_Y.\end{aligned}$$

This completes the proof.

□

Lemma 2. *Let $I = (\mathbb{R}, \{1\})$. Then $(\mathrm{Af}, \otimes, I)$ is a symmetric monoidal category.*

Proof. Note that this structure is inherited from the symmetric monoidal structure in FinVect. To show that $\otimes$ is a functor, we have to check that for $X_1 \xrightarrow{f} Y_1$ and $X_2 \xrightarrow{g} Y_2$ in Af, we have $X_1 \otimes Y_1 \xrightarrow{f\otimes g} X_2 \otimes Y_2$ which amounts to showing that

$$(f \otimes g)(A_{X_1\otimes Y_1}) \subseteq A_{X_2\otimes Y_2}.$$

Let $x \in A_{X_1}$, $y \in A_{Y_1}$, then $f(x) \otimes g(y) \in A_{X_2} \otimes A_{Y_2} \subseteq A_{X_2 \otimes Y_2}$. Since $A_{X_1 \otimes Y_1}$ is the affine subspace generated by $A_{X_1} \otimes A_{Y_1}$, the above inclusion follows by linearity of $f \otimes g$.

It only remains to prove that the associators, unitors and symmetries from FinVect are morphisms in Af. We will prove this for the associators $\alpha_{X,Y,Z} : V_X \otimes (V_Y \otimes V_Z) \to (V_X \otimes V_Y) \otimes V_Z$, the other proofs are similar. We need to check that $\alpha_{X,Y,Z}(A_{X \otimes (Y \otimes Z)}) \subseteq A_{(X \otimes Y) \otimes Z}$. It is easily checked that $A_{X \otimes (Y \otimes Z)}$ is the affine span of elements of the form $x \otimes (y \otimes z)$, $x \in A_X$, $y \in A_Y$ and $z \in A_Z$, and we have

$$\alpha_{X,Y,Z}(x \otimes (y \otimes z)) = (x \otimes y) \otimes z \in A_{(X \otimes Y) \otimes Z}$$

for all such elements. The desired inclusion follows by linearity.

□

Theorem 1. $(\mathrm{Af}, \otimes, I)$ *is a *-autonomous category, with duality* $(-)^*$*, such that* $I^* = I$*.*

Proof. By Lemma 2, we have that $(\mathrm{Af}, \otimes, I)$ is a symmetric monoidal category. We have also seen that the duality $(-)^*$ is a full and faithful contravariant functor. We only need to check the natural isomorphisms

$$\mathrm{Af}(X \otimes Y, Z^*) \simeq \mathrm{Af}(X, (Y \otimes Z)^*).$$

Since FinVect is compact, we have the natural isomorphisms

$$\mathrm{FinVect}(V_X \otimes V_Y, V_Z^*) \simeq \mathrm{FinVect}(V_X, V_Y^* \otimes V_Z^*),$$

determined by the equalities

$$\langle\, f(x \otimes y), z \,\rangle = \langle\, \hat{f}(x), y \otimes z \,\rangle, \qquad x \in V_X,\ y \in V_Y,\ z \in V_Z,$$

for $f \in \mathrm{FinVect}(V_X \otimes V_Y, V_Z^*)$ and the corresponding morphism $\hat{f} \in \mathrm{FinVect}(V_Z, V_Y^* \otimes V_Z^*)$. Since $A_{X \otimes Y}$ is an affine span of $A_X \otimes A_Y$, we see that f is in $\mathrm{Af}(X \otimes Y, Z^*)$ if and only if $f(x \otimes y) \in \tilde{A}_Z$ for all $x \in A_X$, $y \in A_Y$, that is,

$$1 = \langle\, f(x \otimes y), z \,\rangle = \langle\, \hat{f}(x), y \otimes z \,\rangle \qquad \forall x \in A_X,\ \forall y \in A_Y,\ \forall z \in A_Z.$$

But this is equivalent to

$$\hat{f}(x) \in (A_Y \otimes A_Z)^{\sim} = \tilde{A}_{Y \otimes Z}, \qquad \forall x \in A_X,$$

which means that $\hat{f} \in \mathrm{Af}(X, (Y \otimes Z)^*)$.

□

A *-autonomous category is compact closed if it satisfies $(X \otimes Y)^* = X^* \otimes Y^*$. In general, $X \odot Y = (X^* \otimes Y^*)^*$ defines a dual symmetric monoidal structure that is different from $\otimes$. We next show that Af is not compact.

Proposition 1. *For objects in* Af*, we have* $(X \otimes Y)^* = X^* \otimes Y^*$ *exactly in one of the following situations:*

(i) $X \simeq I$ *or* $Y \simeq I$*,*

(ii) $d_X = d_Y = 0$*,*

(iii) $d_{X^*} = d_{Y^*} = 0$*.*

Proof. Since FinVect is compact, we have $V_{(X\otimes Y)^*}=(V_X\otimes V_Y)^*=V_X^*\otimes V_Y^*=V_{X^*\otimes Y^*}$. It is also easily seen by definition that $A_{X^*}\otimes A_{Y^*}=\tilde{A}_X\otimes\tilde{A}_Y\subseteq\tilde{A}_{X\otimes Y}=A_{(X\otimes Y)^*}$, so that we always have $A_{X^*\otimes Y^*}\subseteq A_{(X\otimes Y)^*}$. Hence the equality holds if and only if $d_{X^*\otimes Y^*}=d_{(X\otimes Y)^*}$. From Lemma 1, we see that

$$d_{X^*\otimes Y^*}=d_{X^*}+d_{Y^*}+d_{X^*}d_{Y^*}.$$

On the other hand, we have using (3) that $L_{(X\otimes Y)^*}=S_{X\otimes Y}^{\perp}=(S_X\otimes S_Y)^{\perp}$, so that

$$d_{(X\otimes Y)^*}=D_XD_Y-\dim(S_X)\dim(S_Y)=D_XD_Y-(d_X+1)(d_Y+1).$$

Taking into account that by (3) we have $d_{X^*}=D_X-d_X-1$, similarly for d_{Y^*}, we obtain

$$d_{(X\otimes Y)^*}-d_{X^*\otimes Y^*}=d_Xd_{Y^*}+d_{X^*}d_Y.$$

This is equal to 0 iff $d_Xd_{Y^*}=d_Yd_{X^*}=0$, which amounts to the conditions in the lemma.

□

In a *-autonomous category, the internal hom can be identified as $[X,Y]=(X\otimes Y^*)^*$. The underlying vector space is $V_{[X,Y]}=(V_X\otimes V_Y^*)^*=V_X^*\otimes V_Y$ and we have seen in Section 1.1 that we may identify this space with $\mathrm{FinVect}(V_X,V_Y)$, by $f\leftrightarrow C_f$. This property is extended to Af, in the following sense.

Proposition 2. *For any objects X,Y in* Af*, the map $f\mapsto C_f$ is a bijection of* $\mathrm{Af}(X,Y)$ *onto $A_{[X,Y]}$. In particular, $\tilde{A}_X$ can be identified with* $\mathrm{Af}(X,I)$*.*

Proof. Let $f\in\mathrm{FinVect}(V_X,V_Y)$. Since by definition $A_{[X,Y]}=\tilde{A}_{X\otimes Y^*}=(A_X\otimes A_Y^*)^{\sim}$, we see that $C_f\in A_{[X,Y]}$ if and only if for all $x\in A_X$ and $y^*\in\tilde{A}_Y$, we have

$$1=\langle\, C_f,x\otimes y^*\,\rangle=\langle\, y^*,f(x)\,\rangle.$$

This latter statement is clearly equivalent to $f(A_X)\subseteq A_Y$, so that $f\in\mathrm{Af}(X,Y)$.

□

In the next result, we restrict the objects to spaces of hermitian matrices, as in Example 2 and morphisms to completely positive maps. We show that this restriction amounts to taking an intersection of $A_{[X,Y]}$ with the cone of positive semidefinite matrices. This, and subsequent examples, shows that for characterization of sets of quantum objects such as states, channels, combs and transformations between them, it is enough to work with the category Af.

An object X of Af will be called quantum if $V_X=M_n^h$ for some n and A_X is an affine subspace such that both A_X and $\tilde{A}_X$ contain a positive multiple of the identity matrix E_n[1]. (recall that we identify $(M_n^h)^*=M_n^h$).

Proposition 3. *Let X, Y be quantum objects in* Af*. Then*

(i) X^* and $X\otimes Y$ are quantum objects as well.

(ii) Let $V_X=M_n^h$, $V_Y=M_m^h$. Then for any $f\in\mathrm{FinVect}(M_n^h,M_m^h)$, we have $C_f\in A_{[X,Y]}\cap M_{mn}^+$ if and only if f is completely positive and

$$f(A_X\cap M_n^+)\subseteq A_Y\cap M_m^+.$$

[1]We use the notation E_n, and not I_n, to avoid the slight chance that it might be confused with the monoidal unit.

Proof. The statement (i) is easily seen from $A_X \otimes A_Y \subseteq A_{X\otimes Y}$ and $\tilde{A}_X \otimes \tilde{A}_Y \subseteq \tilde{A}_{X\otimes Y}$. To show (ii), let $C_f \in A_{[X,Y]} \cap M^+_{mn}$. By the properties of the Choi isomorphism f is completely positive and by Proposition 2, $f(A_X) \subseteq A_Y$, this proves one implication. For the converse, note that we only need to prove that under the given assumptions, $f(A_X) \subseteq A_Y$, for which it is enough to show that $A_X \subseteq \mathrm{Aff}(A_X \cap M^+_n)$. To see this, let $c_X E_n \in A_X$ for $c_X > 0$. Any element in A_X can be written in the form $c_X E_n + v$ for some $v \in L_X$. Since $c_X E_n \in int(M^+_n)$, there is some $s > 0$ such that $a_\pm := c_X E_n \pm sv \in M^+_n$, and since $\pm sv \in L_X$, we see that $a_\pm \in A_X \cap M^+_n$. It is now easily checked that

$$c_X E_n + v = \frac{1+s}{2s} a_+ + \frac{s-1}{2s} a_- \in \mathrm{Aff}(A_X \cap M^+_n).$$

□

We can define classical objects in Af in a similar way, replacing M^h_n by $\mathbb{R}^N$ and the positive cone by $\mathbb{R}^N_+$, and we require that both A_X and $\tilde{A}_X$ contains a positive multiple of the unit vector $e_N = (1, \dots, 1) \in \mathbb{R}^N$. A similar statement holds in this case, with complete positivity replaced by positivity. We can similarly treat classical-to-quantum and quantum-to-classical maps as morphisms between these types of objects, satisfying appropriate positivity assumptions.

Example 3 (Higher order quantum maps). The basic example of a quantum object corresponds to the set of quantum states. Let

$$\mathcal{A}_n := \{T \in M^h_n,\ \mathrm{Tr}\,[T] = 1\}.$$

Then $\mathcal{S}_n := (M^h_n, \mathcal{A}_n)$ is an object in Af, and it is a quantum object, since we have $E_n \in \tilde{\mathcal{A}}_n = \{E_n\}$ and $\frac{1}{n} E_n \in \mathcal{A}_n$. The set $\mathcal{A}_n \cap M^+_n$ is the set of quantum states. By Proposition 2, $\mathcal{C}_{m,n} := [\mathcal{S}_m, \mathcal{S}_n]$ is a quantum object as well, such that the corresponding vector space is M^h_{mn} and $A_{\mathcal{C}_{m,n}} \cap M^+_{mn}$ is the set of Choi matrices of quantum channels $M_m \to M_n$. Note that the dual object $\mathcal{C}^*_{m,n} = \mathcal{S}_m \otimes \mathcal{S}^*_n$ represents the set of Choi matrices of replacement channels $\mathcal{S}_n \to \mathcal{S}_m$, that is, channels that map any state in M_n to a fixed state in M_m. We also have $\mathcal{S}_n = [\mathcal{S}_n, I] = \mathcal{C}_{n,1}$.

The set of higher order quantum maps is constructed inductively from the channel objects $\mathcal{C}_{m,n}$ by applying the internal hom $[\cdot,\cdot]$. For example, $[\mathcal{C}_{m,n}, \mathcal{C}_{k,l}]$ is a quantum object, corresponding to the set of Choi matrices of quantum superchannels, or 2-combs, etc. Note that in this way we always obtain quantum objects. The corresponding affine subspaces are identified using (2) with a and $\tilde{a}$ replaced by the appropriate multiple of the identity, together with (3) and Lemma 1.

Example 4 (Partially classical maps). We may similarly define the basic classical object as

$$\mathcal{P}_N := (\mathbb{R}^N, \{x,\ \sum_i x_i = 1\}).$$

In this case, $\mathcal{A}_{\mathcal{P}_N} \cap \mathbb{R}^N_+$ is the probability simplex. We then obtain further useful objects by combining with the quantum objects. For example, it can be easily seen that $[\mathcal{S}_n, \mathcal{P}_N]$ corresponds to N-outcome measurements, $[\mathcal{S}_m, \mathcal{S}_n \otimes \mathcal{P}_N]$ to N-outcome quantum instruments, $[\mathcal{S}_m \otimes \mathcal{P}_N, \mathcal{P}_M]$ to quantum multimeters, etc.

1.3 First order and higher order objects

We say that an object X in Af is first order if $d_X = D_X - 1$, equivalently, $S_X = V_X$. Another equivalent condition is $d_{X^*} = 0$, which means that A_X is determined by a single element $\tilde{a}_X \in V^*_X$ as

$$A_X = \{\tilde{a}_X\}^\sim, \qquad \tilde{A}_X = \{\tilde{a}_X\}.$$

Note that first order objects, resp. their duals, are exactly those satisfying condition (iii), resp. condition (ii), in Proposition 1, in particular, $(X \otimes Y)^* = X^* \otimes Y^*$ for first order objects X and Y.

Higher order objects in Af are objects obtained from a finite set $\{X_1, \dots, X_n\}$ of first order objects by taking tensor products and duals. The above is indeed a set, so that all the objects are different (though they may be isomorphic) and the ordering is not essential. We will also assume that the monoidal unit I is not contained in this set. By definition of $[X, Y]$, and since we may dentify $[X, I]$ with X^*, we see that higher order objects are also generated by applying the internal hom inductively on $\{X_1, \dots, X_n\}$ if we allow $X_i = I$ for some i. It follows that the "higher order quantum maps" in Example 3 are indeed higher order objects in Af according to the above definition.

Of course, any first order object is also higher order with $n = 1$. Note that we cannot say that a higher order object generated from $\{X_1, \dots, X_n\}$ is automatically "of order n", as the following lemma shows.

Lemma 3. *Let X, Y be first order, then $X \otimes Y$ is first order as well.*

Proof. We have $S_{X\otimes Y} = S_X \otimes S_Y = V_X \otimes V_Y = V_{X\otimes Y}$.

□

As we have seen, higher order objects are obtained by applying the internal hom iteratively. The following properties of such iterations are easily seen from the definition of $[\cdot, \cdot]$.

Lemma 4. *Let X, Y, Z be any objects in* Af*. Then we have*

(i) $[Z, [X, Y]] \simeq [X, [Z, Y]]$.

(ii) If $X = (V_X, \{\tilde{a}_X\}^\sim)$ and $Y = (V_Y, \{\tilde{a}_Y\}^\sim)$ are first order, then $[Z, [X, Y]]$ is determined as

$$A_{[Z,[X,Y]]} = \{w \in V_Z^* \otimes V_X^* \otimes V_Y, (id \otimes \tilde{a}_Y)(w) \in A_Z^* \otimes \tilde{a}_X\}.$$

Example 5 (Combs). The higher order objects called n-combs are constructed inductively as follows. 1-combs, or channels, are objects of the form $[X, Y]$, with first order objects X and Y. An n-comb is an object of the form $[C_{n-1}, [X, Y]]$, where C_{n-1} is an $n-1$-comb and X, Y are first order objects. Using Lemma 4, we see that an n-comb has the form

$$[X_{2n-1}, [[X_{2n-3}, \dots, [[X_1, X_2], X_4]], \dots, X_{2n}]]$$

for first order objects $X_1, X_2, \dots, X_{2n}$. For quantum objects, we see that an n-comb describes the sets of n-combs introduced in , see Example 3 (We slightly abuse the terminology here).

Since $[X, Y] \simeq [I, [X, Y]]$, we can determine the affine subspace of the channel object $[X, Y]$ using Lemma 4(ii) as

$$A_{[X,Y]} = \{w \in V_X^* \otimes V_Y,\ (id \otimes \tilde{a}_Y)(w) = \tilde{a}_X\}.$$

The subspace A_{C_n} for an n-comb C_n can be found inductively.

2 Combinatorial description of higher order objects

In this section, we discuss a combinatiorial description of higher order objects similar to that of [Pavia]. We will use the definitions and results given in Appendix A.3.

For a first order object $X = (V_X, \{\tilde{a}_X\}^\sim)$, let us pick an element $a_X \in A_X$. We have a direct sum decomposition

$$V_X = L_{X,0} \oplus L_{X,1},$$

where $L_{X,0} := \mathbb{R}\{a_X\}$, $L_{X,1} := \{\tilde{a}_X\}^\perp = L_X$. We define the *conjugate object* as $\tilde{X} = (V_X^*, \{a_X\}^\sim)$. Note that we always have $\tilde{a}_X \in A_{\tilde{X}}$ and with the choice $a_{\tilde{X}} = \tilde{a}_X$, we obtain $\tilde{\tilde{X}} = X$ and

$$L_{\tilde{X},u} = L_{X,1-u}^\perp, \qquad u \in \{0,1\}. \tag{6}$$

These definitions depend on the choice of a_X, but we will assume below that this choice is fixed and that we choose $a_{\tilde{X}} = \tilde{a}_X$. Since we will always work with a finite set of objects at a time, this will not create any problems.

A first order quantum or classical object is the set of states $\mathcal{S}_n$ or the set of probability distributions $\mathcal{P}_N$, see Examples 3, 4. In these cases, a_X will be chosen as the appropriate multiple of the identity. Note that then

$$L_{X,0} = L_{\tilde{X},0} = \mathbb{R}\{E_n\}, \qquad L_{X,1} = L_{\tilde{X},1} = \mathcal{T}_n := \{T \in M_n^h, \mathrm{Tr}\,[T] = 0\}$$

(similarly for $\mathcal{P}_N$).

Let $X_1, \dots, X_n$ be first order objects in Af. Let $a_{X_i} \in A_{X_i}$ be fixed and let $\tilde{X}_i$ be the conjugate first order objects. Let us denote $V_i = V_{X_i}$ and

$$L_{i,u} := L_{X_i,u}, \qquad \tilde{L}_{i,u} := L_{\tilde{X}_i,u} \qquad u \in \{0,1\},\ i \in [n].$$

For a string $s \in \{0,1\}^n$, we define

$$L_s := L_{1,s_1} \otimes \cdots \otimes L_{n,s_n}, \qquad \tilde{L}_s := \tilde{L}_{1,s_1} \otimes \cdots \otimes \tilde{L}_{n,s_n},$$

then we have the direct sum decompositions

$$V := V_1 \otimes \cdots \otimes V_n = \sum_{s\in\{0,1\}^n} L_s, \qquad V^* = V_1^* \otimes \cdots \otimes V_n^* = \sum_{s\in\{0,1\}^n} \tilde{L}_s$$

(here $\sum$ denotes the direct sum).

Lemma 5. *For any $s \in \{0,1\}^n$, we have*

$$L_s^\perp = \sum_{t\in\{0,1\}^n} (1-\chi_s(t))\tilde{L}_t, \qquad \tilde{L}_s^\perp = \sum_{t\in\{0,1\}^n} (1-\chi_s(t))L_t.$$

Here $\chi_s : \{0,1\}^n \to \{0,1\}$ is the characteristic function of s.

Proof. Using (6) and the direct sum decomposition of V_i^*, we get

$$(L_{1,s_1}\otimes\cdots\otimes L_{n,s_n})^\perp = \bigvee_j \left(V_1^*\otimes\cdots\otimes V_{j-1}^*\otimes\tilde{L}_{j,1-s_j}\otimes V_{j+1}^*\otimes\cdots\otimes V_n^*\right)$$
$$= \bigvee_j \left(\sum_{\substack{t\in\{0,1\}^n\\ t_j\neq s_j}} \tilde{L}_{1,t_1}\otimes\cdots\otimes\tilde{L}_{n,t_n}\right)$$
$$= \sum_{\substack{t\in\{0,1\}^n\\ t\neq s}} \left(\tilde{L}_{1,t_1}\otimes\cdots\otimes\tilde{L}_{n,t_n}\right).$$

The proof of the other equality is the same.

□

Lemma 6. *Put $a := a_1\otimes\cdots\otimes a_n$, $\tilde{a} := \tilde{a}_1\otimes\cdots\otimes\tilde{a}_n$. For $f\in\mathcal{F}_n$ define*

$$S_f = S_f(X_1,\ldots,X_n) := \sum_{s\in\{0,1\}^n} f(s)L_s, \qquad A_f = A_f(X_1,\ldots,X_n) := S_f\cap\{\tilde{a}\}^\sim.$$

Then A_f is a proper affine subspace in V containing a. Moreover,

$$\mathrm{Lin}(A_f) = \sum_{s\in\{0,1\}^n\setminus\{\theta_n\}} f(s)L_s, \qquad \mathrm{Span}(A_f) = S_f.$$

The map $f\mapsto A_f$ is injective and has the following further properties.

(i) The dual affine subspace satisfies

$$\tilde{A}_f(X_1,\ldots,X_n) = A_{f^*}(\tilde{X}_1,\ldots,\tilde{X}_n).$$

(ii) Let $\sigma\in\mathscr{S}_n$ and let the corresponding symmetry $\otimes_i V_i\to\otimes_i V_{\sigma^{-1}(i)}$ be also denoted by σ. Then we have

$$A_f(X_{\sigma(1)},\ldots,X_{\sigma(n)}) = \sigma^{-1}(A_{f\circ\sigma}(X_1,\ldots,X_n)).$$

(iii) Let $f_1\in\mathcal{F}_{n_1}$, $f_2\in\mathcal{F}_{n_2}$, $n_1+n_2=n$. Then

$$S_{f_1\otimes f_2}(X_1,\ldots,X_n) = S_{f_1}(X_1,\ldots,X_{n_1})\otimes S_{f_2}(X_{n_1+1},\ldots,X_n)$$

Proof. It is clear from definition that A_f is an affine subspace. Since $f(\theta_n)=1$, the space S_f always contains the subspace $L_0 = L_{1,0}\otimes\cdots\otimes L_{n,0} = \mathbb{R}\{a\}$ and it is clear that $L_s\subseteq\{\tilde{a}\}^\perp$ for any $s\neq\theta_n$. It follows that $a\in A_f$, so that $A_f\neq\emptyset$, and since $A_f\subseteq\{\tilde{a}\}^\sim$, we see that A_f is proper and $\tilde{a}\in\tilde{A}_f$. The expressions for $\mathrm{Lin}(A_f)$ and $\mathrm{Span}(A_f)$ are immediate from the definition and (16).

Injectivity of the map $f\mapsto A_f$ is clear from the fact that L_s, $s\in\{0,1\}$ is an independent decomposition. To prove (i), we compute using Lemma 5 and the fact that the subspaces form an independent decomposition,

$$\mathrm{Span}(\tilde{A}_f) = \mathrm{Lin}(A_f)^\perp = \left(\sum_{s\in\{0,1\}^n\setminus\{0\}} f(s)L_s\right)^\perp = \bigwedge_{\substack{s\in\{0,1\}^n\\ s\neq0,f(s)=1}} L_s^\perp = \bigwedge_{\substack{s\in\{0,1\}^n\\ s\neq0,f(s)=1}} \left(\sum_{t\in\{0,1\}^n}(1-\chi_s(t))\tilde{L}_t\right)$$
$$= \sum_{t\in\{0,1\}^n}\left(\bigwedge_{\substack{s\in\{0,1\}^n\\ s\neq0,f(s)=1}}(1-\chi_s(t))\tilde{L}_t\right) = \sum_{t\in\{0,1\}^n} f^*(t)\tilde{L}_t.$$

To see the last equality, note that

$$\bigwedge_{\substack{s\in\{0,1\}^n\\ s\neq 0, f(s)=1}} (1-\chi_s(t)) = \begin{cases} 1 & \text{if } t=\theta_n \\ 1-f(t) & \text{if } t\neq\theta_n \end{cases} = f^*(t).$$

To show (ii), compute

$$\begin{aligned}\sigma(S_f(X_{\sigma(1)},\dots,X_{\sigma(n)})) &= \sigma(\sum_s f(s)L_{\sigma(1),s_1}\otimes\cdots\otimes L_{\sigma(n),s_n}) \\ &= \sum_s f(s)L_{1,s_{\sigma^{-1}(1)}}\otimes\cdots\otimes L_{n,s_{\sigma^{-1}(n)}} = S_{f\circ\sigma}(X_1,\dots,X_n).\end{aligned}$$

It folows that

$$A_f(X_{\sigma(1)},\dots,X_{\sigma(n)}) = S_f(X_{\sigma(1)},\dots,X_{\sigma(n)})\cap\{\sigma^{-1}(\tilde a)\}^{\sim} = \sigma^{-1}(A_{f\circ\sigma}(X_1,\dots,X_n)).$$

The statement (iii) is easily seen from the definitions. $\square$

Since all the affince subspaces $A_f\subseteq V$ are proper, we may form the objects

$$X_f = X_f(X_1,\dots,X_n) := (V, A_f(X_1,\dots,X_n))$$

in Af. The following properties follow easily from the above Lemma.

Proposition 4. *Let $X_1,\dots,X_n$ be first order objects. The map $\mathcal{F}_n\ni f\mapsto X_f(X_1,\dots,X_n)$ is injective and we have*

(i) For the least and the largest element in $\mathcal{F}_n$,

$$X_{p_{[n]}} = \tilde X_1^*\otimes\cdots\otimes\tilde X_n^*, \qquad X_{1_n} = X_1\otimes\cdots\otimes X_n,$$

(ii) $X_f^*(X_1,\dots,X_n) = X_{f^*}(\tilde X_1,\dots,\tilde X_n)$,

(iii) $X_{f_1\otimes f_2}(X_1,\dots,X_n) = X_{f_1}(X_1,\dots,X_{n_1})\otimes X_{f_2}(X_{n_1+1},\dots,X_n)$,

(iv) the symmetry $\sigma\in\mathscr{S}_n$ is an isomorphism $X_f(X_{\sigma(1)},\dots,X_{\sigma(n)})\xrightarrow{\sigma} X_{f\circ\sigma}(X_1,\dots,X_n)$.

It follows from the independence of L_s, $s\in\{0,1\}^n$, that the subspaces S_f form a distributive sublatice in the lattice of subspaces of V and we clearly have $f\leq g$ if and only if $S_f\subseteq S_g$ and $S_{f\wedge g} = S_f\cap S_g$, $S_{f\vee g} = S_f\vee S_g$. The following proposition shows the corresponding properties of X_f, in categorical terms. We skip the easy proof.

Proposition 5. *Let $f,g,h\in\mathcal{F}_n$.*

(i) $f\leq g$ if and only if $X_f\xrightarrow{id_V} X_g$ in Af.

(ii) Let $k\leq f,g\leq h$ then the following diagram is a pullback resp. pushout:

$$\begin{tikzcd} X_{f\wedge g} \arrow[r, "id_V"] \arrow[d, "id_V"'] & X_f \arrow[d, "id_V"] \\ X_g \arrow[r, "id_V"'] & X_h \end{tikzcd} \qquad \begin{tikzcd} X_h \arrow[r, "id_V"] \arrow[d, "id_V"'] & X_f \arrow[d, "id_V"] \\ X_g \arrow[r, "id_V"'] & X_{f\vee g} \end{tikzcd}$$

Our goal is to show that the higher order objects are precisely those of the form $Y = X_f(X_1, \dots, X_n)$ for some choice of the first order objects $X_1, \dots, X_n$ and a function f that belongs to a special subclass $\mathcal{T}_n \subseteq \mathcal{F}_n$. The elements of this subclass will be called the *type functions*, or *types*, and are defined as those functions in $\mathcal{F}_n$ that can be obtained by taking the constant function 1_1 in each coordinate and then repeatedly applying duals and tensor products of such functions in any order. The set of indices for which the corresponding coordinate was subjected to taking the dual an even number of times will be called the *outputs* (of f) and denoted by $O = O_f$, indexes in $I = I_f := [n] \setminus O_f$ will be called *inputs*. The reason for this terminology will become clear later. It is easy to observe that if $f \in \mathcal{T}_n$, then $O_{f^*} = I_f$ and $I_{f^*} = O_f$. Further, for $f_1 \in \mathcal{T}_{n_1}$, $f_2 \in \mathcal{T}_{n_2}$, we have $O_{f_1 \otimes f_2} = O_{f_1} \oplus O_{f_2}$ and $I_{f_1 \otimes f_2} = I_{f_1} \oplus I_{f_2}$, see (13) for the definition.

We have the following description of the sets of type functions.

Proposition 6. *The set $\mathcal{T}_n$ is the smallest subset in $\mathcal{F}_n$ such that:*

1. $\mathcal{T}_1 = \mathcal{F}_1$,
2. *For $n_1 + n_2 = n$, $\mathcal{T}_{n_1} \otimes \mathcal{T}_{n_2} \subseteq \mathcal{T}_n$,*
3. *$\mathcal{T}_n$ is invariant under permutations: if $f \in \mathcal{T}_n$, then $f \circ \sigma \in \mathcal{T}_n$ for any $\sigma \in \mathscr{S}_n$,*
4. *$\mathcal{T}_n$ is invariant under complementation: if $f \in \mathcal{T}_n$ then $f^* \in \mathcal{T}_n$.*

Proof. It is clear by construction that any system of subsets $\{\mathcal{S}_n\}_n$ with these properties must contain the type functions and that $\{\mathcal{T}_n\}_n$ itself has these properties.

$\square$

Assume that Y is a higher order object constructed from a set of distinct first order objects $Y_1, \dots, Y_n$, $Y_i = (V_{Y_i}, \{\tilde{a}_{Y_i}\}^{\sim})$. Let us fix elements $a_{Y_i} \in A_{Y_i}$ and construct the conjugate objects $\tilde{Y}_i$. By compactness of FinVect, we may assume (relabeling the objects if necessary) that the vector space of Y has the form

$$V_Y = V := V_1 \otimes \cdots \otimes V_n,$$

where V_i is either V_{Y_i} or $V_{Y_i}^*$, according to whether Y_i was subjected to taking duals an even or odd number of times. Similarly as for the type functions, the indices such that the first case is true will be called the outputs and the subset of outputs in $[n]$ will be denoted by O, or O_Y, when we need to specify the object. The set $I = I_Y := [n] \setminus O_Y$ is the set of inputs.

Theorem 2. *Let Y be a higher order object, constructed from first order objects $Y_1, \dots, Y_n$. For $i \in [n]$, let $X_i = Y_i$ if $i \in O_Y$ and $X_i = \tilde{Y}_i$ for $i \in I_Y$. There is a unique function $f \in \mathcal{T}_n$, with $O_f = O_Y$, such that*

$$Y = X_f = (V, A_f(X_1, \dots, X_n)).$$

Conversely, let $X_1, \dots, X_n$ be first order objects and let $f \in \mathcal{T}_n$. Then $Y = X_f$ is a higher order object with $O_Y = O_f$, with underlying first order objects $Y_1, \dots, Y_n$, where $Y_i = X_i$ for $i \in O_f$ and $Y_i = \tilde{X}_i$ for $i \in I_f$.

Proof. Since the map $f \mapsto X_f$ is injective, uniqueness is clear. To show existence of this function, we will proceed by induction on n. For $n = 1$, the assertion is easily seen to be true, since in this case, we we have either $Y = Y_1$ or $Y = Y_1^*$. In the first case, $O = [1]$, $X_1 = Y_1$ and

$$S_Y = V_Y = V_1 = L_{1,0} \oplus L_{1,1} = 1(0)L_{1,0} \oplus 1(1)L_{1,1},$$

so in this case $f \in \mathcal{T}_1$ is the constant 1. If $Y = Y_1^*$, we have $O = \emptyset$, $X_1 = \tilde{Y}_1$, and then

$$S_Y = \mathbb{R}\{\tilde{a}_{Y_1}\} = L_{1,0} = 1^*(0)L_{1,0} \oplus 1^*(1)L_{1,1},$$

so that $f = 1^* = p_{[1]} \in \mathcal{T}_1$. It is clear that $_f = O_Y$ in both cases.

Assume now that the assertion is true for all $m < n$. By construction, Y is either the tensor product $Y = Z_1 \otimes Z_2$, with Z_1 constructed from $Y_1, \dots, Y_m$ and Z_2 from $Y_{m+1}, \dots, Y_n$, or Y is the dual of such a product. Let us assume the first case. It is clear that $O_{Z_1} \oplus O_{Z_2} = O_Y$, and similarly for I, so that the corresponding objects $X_1, \dots, X_m$ and $X_{m+1}, \dots, X_n$ remain the same. By the induction assumption, there are functions $f_1 \in \mathcal{T}_m$ and $f_2 \in \mathcal{T}_{n-m}$ such that $O_{f_1} = O_{Z_1}$, $O_{f_2} = O_{Z_2}$ and, by Proposition 4(iii),

$$Y = Z_1 \otimes Z_2 = X_{f_1}(X_1, \dots, X_m) \otimes X_{f_2}(X_{m+1}, \dots, X_n) = X_{f_1 \otimes f_2}(X_1, \dots, X_n)$$

This implies the assertion, with $f = f_1 \otimes f_2 \in \mathcal{T}_n$ and $O_f = O_{f_1} \oplus O_{f_2} = O_Y$. To finish the proof, it is now enough to observe that if the assertion holds for Y then it also holds for Y^*. So assume that $Y = X_f(X_1, \dots, X_n)$ for some $f \in \mathcal{T}_n$, then by Proposition 4(ii), $Y^* = X_f^* = \tilde{X}_{f^*}(\tilde{X}_1, \dots, \tilde{X}_n)$. By the construction of conjugate objects, we have $\tilde{X}_i = \tilde{\tilde{Y}}_i = Y_i$ if $i \in I_Y$ and $\tilde{X}_i = \tilde{Y}_i$ if $i \in O_Y$. Since by definition and the assumption, $O_{Y^*} = I_Y = I_f = O_{f^*}$, this proves the statement.

The converse is proved by a similar induction argument, using Proposition 4.

□

Let us stress that in general, the objects X_f depend on the choice of the elements a_{X_i}. From the above proof, it is clear that the description in Theorem 2 does not depend on the choice of the elements $a_{Y_i} \in A_{Y_i}$. Furthermore, if all the first order objects are quantum, we have $S_f(X_1, \dots, X_n) = S_f(\tilde{X}_1, \dots, \tilde{X}_n)$ and both a and $\tilde{a}$ are some, possibly different, multiple of the identity. The spaces $A_f(X_1, \dots, X_n)$ and $A_f(\tilde{X}_1, \dots, \tilde{X}_n)$ differ only by this multiple.

3 The type functions

The aim of this section is to gain some understanding into the structure and properties of the set of types. We start by an important example.

Example 6. Let $T \subseteq [n]$. It is easily seen that the function p_T (see Appendix A.3) is a type function, since we have

$$p_T(s) = \Pi_{j \in T}(1 - s_j) = \Pi_{j \in T} 1^*(s_j).$$

By definition, T is the set of inputs for p_T. Let $\mathcal{S} = \{X_1, \dots, X_n\}$ be a set of first order objects. Let $k = |T|$ and let $\sigma \in \mathscr{S}_n$ bw such that $p_T \circ \sigma = p_{[k]} \otimes 1_{n-k}$. By Proposition 4, it follows that we have the isomorphism

$$X_{p_T}(X_1, \dots, X_n) \xrightarrow{\sigma} X_{p_{[k]} \otimes 1_{n-k}}(X_{\sigma^{-1}(1)}, \dots, X_{\sigma^{-1}(n)}) = \tilde{X}_T^* \otimes X_{[n]\setminus T},$$

here $\tilde{X}_T = \otimes_{j \in T} \tilde{X}_j$ and $X_{[n]\setminus T} = \otimes_{j \in [n]\setminus T} X_j$ are first order object by Lemma 3. It follows that p_T describes replacement channels with set of input indices T. By duality, we obtain the isomorphisms

$$X_{p_T^*}(X_1, \dots, X_n) = X_{p_T}^*(\tilde{X}_1, \dots, \tilde{X}_n) \xrightarrow{\sigma} (X_T^* \otimes \tilde{X}_{[n]\setminus T})^* \xrightarrow{\rho} [\tilde{X}_{[n]\setminus T}, X_T],$$

where ρ denotes the symmetry given by the transposition in $\mathscr{S}_2$. It follows that $p_T^* = 1 - p_T + p_{[n]}$ corresponds to general channels with output indices T.

Lemma 7. *Let $f \in \mathcal{T}_n$ and let $O_f = O$, $I = I_f$. Then*

$$p_I \leq f \leq p_O^*.$$

Proof. This is obviously true for $n = 1$. Indeed, in this case, $\mathcal{T}_1 = \mathcal{F}_1 = \{1 = p_\emptyset, 1^* = p_{[1]}\}$. If $f = 1$, then $O = [1]$, $I = \emptyset$ and

$$p_I = p_\emptyset = 1 = p_O^*,$$

the case $f = 1^*$ is obtained by taking complements. Assume that the assertion holds for $m < n$. Let $f \in \mathcal{T}_n$ and assume that $f = g \otimes h$ for some $g \in \mathcal{T}_m$, $h \in \mathcal{T}_{n-m}$. By the assumption,

$$p_{I_g} \otimes p_{I_h} \leq g \otimes h \leq p_{O_g}^* \otimes p_{O_h}^* \leq (p_{O_g} \otimes p_{O_h})^*,$$

the last inequality follows from Lemma 14. With the decomposition $[n] = [m][m+1, n]$, we have $O_f = O_g \oplus O_h$, $I_f = I_g \oplus I_h$, so that by Lemma 15, $p_{O_f} = p_{O_g} \otimes p_{O_h}$ and similarly for p_{I_f}. Now notice that any $f \in \mathcal{T}_n$ is either of the form $(f \otimes g) \circ \sigma$ or of the form $(f \otimes g)^* \circ \sigma$, for some permutation σ. Since the inequality is easily seen to be preserved by permutations, and reversed by duality which also swiches the input and output sets, the assertion is proved.

□

Combining this with Proposition 4, we get the following result (cf. cite).

Corollary 1. *Let Y be a higher order objects constructed from first order objects $Y_1, \dots, Y_n$, $O_Y = O$, $I_Y = I$. Then there are $\sigma_1, \sigma_2 \in \mathscr{S}_n$ such that we have the morphisms*

$$Y_I^* \otimes Y_O \xrightarrow{\sigma_1} Y \xrightarrow{\sigma_2} [Y_I, Y_O].$$

We also obtain a simple way to identify the output indices of a type function.

Proposition 7. *For $f \in \mathcal{T}_n$, $j \in O_f$ if and only if $f(e^j) = 1$, here $e^j = \delta_{1,j} \dots \delta_{n,j}$.*

Proof. Let $i \in O_f$, then by Lemma 7, $p_{I_f}(e^i) = 1 \leq f(e^i)$, so that $f(e^i) = 1$. Conversely, if $f(e^i) = 1$, then by the other inequality in lemma 7, $p_{O_f}(e^i) = 0$, whence $i \in O_f$.

□

We now look at some examples and non-examples.

Example 7. The type functions for $n = 2$ are given as (writing $\bar{u} = 1 - u$ for $u \in \{0, 1\}$)

$$1_2(s) = 1, \quad p_{[2]}(s) = \bar{s}_1 \bar{s}_2, \quad p_{\{1\}}(s) = \bar{s}_1, \quad p_{\{1\}}^*(s) = 1 - \bar{s}_1 + \bar{s}_1 \bar{s}_2,$$

and functions obtained from these by permutation, which gives 6 elements. We have seen that $\mathcal{F}_n$ has $2^{2^n - 1}$ elements, so that $\mathcal{F}_2$ has 8 elements in total. The two of them that are not type functions are

$$g(s) = 1 - \bar{s}_1 - \bar{s}_2 + 2\bar{s}_1 s_2, \qquad g^*(s) = \bar{s}_1 + \bar{s}_2 - \bar{s}_1 \bar{s}_2.$$

This can be checked directly from Lemma 7 and Proposition 7. Indeed, if $g \in \mathcal{T}_2$, we would have $O_g = \emptyset$, so that $p_{[2]} \leq g \leq p_\emptyset^* = p_{[2]}$, which is obviously not the case. Clearly, also the complement $g^* \notin \mathcal{T}_2$. Notice also that $g^* = p_{\{1\}} \vee p_{\{2\}}$, so that $\mathcal{T}_2$ is not a lattice.

Since $\mathcal{F}_2$ can be identified as a sublattice in $\mathcal{F}_n$ for all $n \geq 2$ as $\mathcal{F}_2 \ni f \mapsto f \otimes 1_{n-2} \in \mathcal{F}_n$, the above example shows that $\mathcal{T}_n$, $n \geq 2$ is a subposet in the distributive lattice $\mathcal{F}_n$ but itself not a lattice, so that for $f_1, f_2 \in \mathcal{T}_n$, none of $f_1 \wedge f_2$ or $f_1 \vee f_2$ has to be a type function. Nevertheless, we have by the above results that all type functions with the same output indices are contained in the interval $p_I \leq f \leq p_O^*$, which is a distributive lattice. Elements of such an interval will be called *subtypes*. It is easily seen that for $n = 2$ all subtypes are type functions, but it is not difficult to find a subtype for $n = 3$ which is not in $\mathcal{T}_3$. The objects corresponding to subtypes are not necessarily higher order objects, but are embedded in $[Y_I, Y_O]$ and contain the replacement channels. If f_1 and f_2 have the same output set, then $f_1 \vee f_2$ and $f_1 \wedge f_2$ are subtypes. By Proposition **??**, the corresponding objects can be obtained by pushouts resp. pullbacks of the higher order objects corresponding to f_1 and f_2.

3.1 The poset $\mathcal{P}_f$

By Theorem 4, any boolean function has a unique expression of the form

$$f = \sum_{T \subseteq [n]} \hat{f}_T p_T,$$

where $\hat{f}$ is the Möbius transform of f. Let $\mathcal{P}_f$ be the subposet in the distributive lattice 2^n, of elements such that $\hat{f}_T \neq 0$. We will show below that any type function $f \in \mathcal{T}_n$ is fully determined by $\mathcal{P}_f$.

We say that a poset $\mathcal{P}$ is graded of rank k if every maximal chain in $\mathcal{P}$ has the same length equal to k (recall that the length of a chain is defined as the number of its elements -1). Equivalently, there is a unique rank function $\rho : \mathcal{P} \to \{0, 1, \ldots, k\}$ such that $\rho(S) = 0$ if S is a minimal element of $\mathcal{P}$ and $\rho(T) = \rho(S) + 1$ if T covers S, that is, $S \leq T$ and for any R such that $S \leq R \leq T$ we have $R = T$ or $R = S$. See [Stanley] for details.

Proposition 8. *Let $f \in \mathcal{T}_n$, then $\mathcal{P}_f$ is a graded poset with even rank $k \leq n$ and rank function ρ_f. Moreover, we have*

$$f = \sum_{S \in \mathcal{P}_f} (-1)^{\rho_f(S)} p_S.$$

Then rank of $\mathcal{P}_f$ will be denoted by $r(f)$ and called the rank of f. Note that the assertion means that for $f \in \mathcal{T}_n$,

$$\hat{f}_S = \begin{cases} (-1)^{\rho_f(S)}, & \text{if } S \in \mathcal{P}_f \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We first note that the property in the statement is invariant under permutations and complements. Assume the statement holds for f and let us take any $\sigma \in \mathscr{S}_n$. From Proposition 14 we have that $\widehat{f \circ \sigma}_S = \hat{f}_{\sigma(S)}$ so that $S \mapsto \sigma(S)$ is an isomprphism of $\mathcal{P}_{f\circ\sigma}$ onto $\mathcal{P}_f$. Hence if $\mathcal{P}_f$ is graded with rank function ρ_f, then $\mathcal{P}_{f\circ\sigma}$ is graded with the same rank and has rank function $\rho_{f\circ\sigma} = \rho_f \circ \sigma$. By the assumption we have

$$f \circ \sigma = \sum_{S \in \mathcal{P}_f} (-1)^{\rho_f(S)} p_S \circ \sigma = \sum_{S \in \mathcal{P}_f} (-1)^{\rho_f(S)} p_{\sigma^{-1}(S)} = \sum_{S \in \mathcal{P}_{f\circ\sigma}} (-1)^{\rho_f \circ \sigma(S)} p_S.$$

For the complement, we have from the assumption and Proposition 14(ii) that

$$f^* = (1-\hat{f}_\emptyset)1 - \sum_{\substack{S\in\mathcal{P}_f\\ \emptyset\neq S,[n]\neq S}} (-1)^{\rho_f(S)} p_S + (1-\hat{f}_{[n]})p_n. \tag{7}$$

If $\emptyset \in \mathcal{P}_f$, then $\emptyset$ is the least element of $\mathcal{P}_f$, so that $\rho_f(\emptyset) = 0$ and therefore $\hat{f}_\emptyset = (-1)^0 = 1$. Similarly, if $[n] \in \mathcal{P}_f$, then $[n]$ is the largest element in $\mathcal{P}_f$, hence it is the last element in any maximal chain. It follows that $\rho_f([n]) = k$ and hence $\hat{f}_{[n]} = (-1)^k = 1$ (since k is even). Therefore the equality (7) implies that $\mathcal{P}_{f^*}$ differs from $\mathcal{P}_f$ only in the bottom and top elements: $\emptyset \in \mathcal{P}_f$ iff $\emptyset \notin \mathcal{P}_{f^*}$ and $p_n \in \mathcal{P}_f$ iff $p_n \notin \mathcal{P}_{f^*}$. It follows that $\mathcal{P}_{f^*}$ is graded as well, with rank equal to $k-2$, k or $k+2$, which in any case is even. Furthermore, this also implies that for all $S \in \mathcal{P}_f$, $S \notin \{\emptyset, [n]\}$, we have $\rho_{f^*}(S) = \rho_f(S) \pm 1$, according to whether $\emptyset$ was added or removed. The statement now follows from (7).

We now proceed by induction on n. For $n = 1$, we have $2 = \{\emptyset, [1]\}$ and $\mathcal{T}_1 = \{1, 1^*\}$. For $f = 1$, $\mathcal{P}_f = \{\emptyset\}$ is a singleton, which is clearly a graded poset, with rank $k = 0$ and trivial rank function ρ_f. We have

$$f = 1 = p_\emptyset = (-1)^{\rho(\emptyset)} p_\emptyset.$$

The statement for $f = 1^*$ follows by duality. Assume that the statement is true for $m < n$ and let $f \in \mathcal{T}_n$. By the first part of the proof, we only need to prove that the statement holds for $f = f_1 \otimes f_2$, where $f_1 \in \mathcal{T}_{n_1}$, $f_2 \in \mathcal{T}_{n_2}$, $n_1 + n_2 = n$. By the induction assumption, $\mathcal{P}_{f_i}$ is graded with even rank. We also have using Proposition 14

$$f = f_1 \otimes f_2 = \sum_{S\subseteq[m],T\subseteq[n-m]} (\widehat{f_1})_S (\widehat{f_2})_T p_S \otimes p_T = \sum_{S\subseteq[m],T\subseteq[n-m]} (-1)^{\rho_{f_1}(S)+\rho_{f_2}(T)} p_{S\oplus T}.$$

It follows that $\mathcal{P}_f \simeq \mathcal{P}_{f_1} \times \mathcal{P}_{f_2}$, where the direct product of posets $\mathcal{P}_f$ and $\mathcal{P}_g$ is defined as the set of all pairs (S,T), $S \in \mathcal{P}_f$, $T \in \mathcal{P}_g$, and $(S,T) \le (S',T')$ iff $S \le S'$ and $T \le T'$. By [Stanley], the direct product of graded posets is a graded poset with rank $r(f) = r(f_1) + r(f_2)$ and rank function $\rho_f = \rho_{f_1} + \rho_{f_2}$. This proves the statement.

□

In the above proof, we also proved the following.

Corollary 2. *Let $f \in \mathcal{T}_n$, $g \in \mathcal{T}_m$. Then*

(i) For $\sigma \in \mathscr{S}_n$, $S \mapsto \sigma(S)$ is an isomprphism of $\mathcal{P}_{f\circ\sigma}$ onto $\mathcal{P}_f$.

(ii) $\mathcal{P}_f^*$ is obtained from from $\mathcal{P}^f$ by adding/removing the top and bottom elements $\emptyset$ and $[n]$.

(iii) $\mathcal{P}_{f\otimes g} \simeq \mathcal{P}_f \times \mathcal{P}_g$.

We next show that the input and output sets of $f \in \mathcal{T}_n$ can be easily recognized from $\mathcal{P}_f$. For an index $i \in [n]$, let $M_{f,i}$ be the set of minimal elements of the subposet $\{S \in \mathcal{P}_f,\ i \in S\}$. Note that $M_{f,i}$ is empty if and only if for any $S \in \mathcal{P}_f$, $i \notin S$.

Proposition 9. *Let $f \in \mathcal{T}_n$ and $i \in [n]$. Then*

1. *If $M_{f,i} \neq \emptyset$, then all elements in $M_{f,i}$ have the same rank, which will be denoted by $r_f(i)$. If $M_{f,i} = \emptyset$, we put $r_f(i) := r(f) + 1$.*

2. $i \in O_f$ if and only if $r_f(i)$ is odd.

Proof. Since $\mathcal{P}_f \simeq \mathcal{P}_{f\circ\sigma}$, it is quite clear that the two properties are preserved by permutations. We will show that they are preserved by complementation. Observe first that $M_{f,i} = \emptyset$ if and only if $M_{f^*,i} = \{[n]\}$, since $\mathcal{P}_{f^*}$ differs from $\mathcal{P}_f$ only up to adding/removing the least element $\emptyset$ and the greatest element $[n]$. If $M_{f,i}$ is empty, then $p_S(e^i) = 1$ for all $S \in \mathcal{P}_f$, so that $f(e^i) = f(\theta_n) = 1$ and $i \in O_f$, we also see that $r_f(i) = r(f) + 1$ is odd. If $M_{f,i} = \{[n]\}$, then $r_f(i) = \rho_f([n]) = r(f)$ by definition of the rank, hence $r_f(i)$ is even. As we have seen, $i \in O_{f^*} = I_f$.

Let us assume that $M_{f,i}$ is not equal to $\emptyset$ or $\{[n]\}$. Then we must have $M_{f,i} = M_{f^*,i}$ and by the proof of Proposition 8 we have $\rho_{f^*}(S) = \rho_f(S) \pm 1$ for any S, depending only on the fact whether $\emptyset \in \mathcal{P}_f$. This implies that the properties are preserved by complementation.

We will now proceed by induction on n as before. Both assertions are quite trivial for $n = 1$, so assume the statements hold for $m < n$. It is enough to assume that $f = g \otimes h$ for some $g \in \mathcal{T}_m$ and $h \in \mathcal{T}_{n-m}$. Suppose without loss of generality that $i \in [m]$, then all elements of $M_{f,i}$ have the form $S \oplus T$, with $S \in M_{g,i}$ and T a minimal element in $\mathcal{P}_h$. Since $\rho_h(T) = 0$ for any minimal element $T \in \mathcal{P}_h$, we have by the induction assumption

$$\rho_f(S \oplus T) = \rho_g(S) + \rho_h(T) = \rho_g(S) = r_g(i).$$

The statement (ii) follows from the fact that $i \in O_f$ if and only if $i \in O_g$.

□

Corollary 3. *We have $\cap\mathcal{P}_f \in I_f$, $[n] \setminus \cup\mathcal{P}_f \in O_f$.*

Proof. If $i \in \cap\mathcal{P}_f$, then clearly $M_{f,i}$ is the set of minimal elements in $\mathcal{P}_f$, so that $r_f(i) = 0$ and $i \in I_f$ by Proposition 9. If $i \notin S$ for any $S \in \mathcal{P}_f$, then $M_{f,i} = \emptyset$ and $r_f(i) = r(f) + 1$ is odd. Hence $i \in O_f$.

□

The elements in $I_f^F := \cap\mathcal{P}_f$ resp. $O_f^F := (\cup\mathcal{P}_f)^c$ will be called free inputs, resp. outputs.

3.2 Chains and combs

A basic example of a graded poset is a chain $\mathcal{P} = \{S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_N\}$. It is clear that $\mathcal{P}$ is graded with rank $N - 1$ and rank function $\rho(S_i) = i - 1$.

Proposition 10. *For a chain $\mathcal{P} = \{S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_N\}$, the function*

$$f = f_{\mathcal{P}} := \sum_{i=1}^{N} (-1)^{i-1} p_{S_i}$$

is a type function if and only if N odd. In this case, we say that f is a chain type.

Proof. By Proposition 8, if $f \in \mathcal{T}_n$, then the rank of f must be even, so that N must be odd. We will show that the converse is true. We proceed by induction on N. For $N = 1$, we have $f = p_{S_1} \in \mathcal{T}_n$. Assume that the statement holds for all odd numbers $M < N$ and let $\mathcal{P}$ be a chain as above. Up to a permutation $\sigma \in \mathscr{S}_n$, we may assume that $S_j = [n_j]$ for some $0 \le n_1 < \cdots < n_N \le n$. Then we have

$$f = p_{[n_1]} \sum_{i=1}^{N} (-1)^{i-1} p_{[n_1+1, n_i]} = p_{[n_1]} \otimes g \otimes 1_{[n-n_N]},$$

where $g \in \mathcal{F}_{n_N - n_1}$ is the function for the chain $\emptyset \subsetneq [n_2 - n_1] \subsetneq \cdots \subsetneq [n_N - n_1]$. Since f is a type function if g is, this shows that we may assume that the chain contains $\emptyset$ and $[n]$. But then

$$f = 1 + \sum_{j=2}^{N-1} (-1)^{j-1} p_{S_j} + p_{[n]}$$

and

$$f^* = 1 - f + p_{[n]} = \sum_{j=1}^{N-2} (-1)^{j-1} p_{S_{j+1}}.$$

By the induction assumption $f^* \in \mathcal{T}_n$, hence also $f = f^{**} \in \mathcal{T}_n$. □

Example 8 ($\mathcal{T}_3$). As we can see from Example 7, all elements in $\mathcal{T}_2$ are chains. This is also true for $n = 3$. Indeed, up to a permutation that does not change the chain structure, any $f \in \mathcal{T}_3$ is either a product of two elements $g \in \mathcal{T}_2$ and $h \in \mathcal{T}_1$, or the dual of such a product. Since g must be a chain and $|\mathcal{P}_h| = 1$, their product must be a chain as well. Taking the dual of a chain only adds/removes the least/largest elements, so the dual of a chain must be a chain as well.

Let $f \in \mathcal{T}_n$ be a chain type and let $\mathcal{P} = \mathcal{P}_f = \{S_1 \subsetneq \cdots \subsetneq S_N\}$ be the corresponding chain. Put $T_0 := S_1$ and $T_j := S_{j+1} \setminus S_j$, $j = 1, \dots, N-1$, $T_N := [n] \setminus S_N$. It can be easily seen from Proposition 9 that

$$I_f = \bigcup_{l=0}^{(N-1)/2} T_{2j}, \qquad O_f = \bigcup_{l=0}^{(N-1)/2} T_{2j+1} \tag{8}$$

(note that N must be odd). Clearly, $T_0 = I_f^F$, $T_N = O_f^F$ are the free inputs resp. outputs. As before, up to a permutation, we may assume that there are some $0 \leq n_1 < n_2 < \cdots < n_N \leq n$ such that $S_j = [n_j]$, $j = 1, \dots, N$, and $T_0 = [n_1]$, $T_j = [n_j + 1, n_{j+1}]$, $j = 1, \dots, N-1$ and $T_N = [n_N + 1, n]$. We have

$$f = p_{[n_1]} \otimes g \otimes 1_{n - n_N}$$

where $g \in \mathcal{T}_{n_N - n_1}$ is a chain type with no free inputs or outputs. We will assume below that $T_0 = T_N = \emptyset$ and show that such chains correspond to important higher order objects.

Proposition 11. *Let N be odd and let $\mathcal{P} = \{\emptyset = S_1 \subsetneq \cdots \subsetneq S_N = [n]\}$. Let $f = f_{\mathcal{P}}$ be the corresponding chain type and let $Y = X_f(X_1, \dots, X_n)$ for some first order objects $X_1, \dots, X_n$. Then for $N \geq 3$, Y is an $(N-1)/2$-comb. More precisely, let $Y_1, \dots, Y_n$ be such that $Y_i = X_i$ for $i \in O_f$ and $Y_i = \tilde{X}_i$ for $i \in I_f$. Then*

$$Y \simeq [Y_{T_{N-1}}, [[Y_{T_{N-2}}, [[\dots, [[Y_{T_{\frac{N+1}{2}}}, Y_{T_{\frac{N-1}{2}}}], Y_{T_{\frac{N-3}{2}}}]], \dots, Y_{T_2}]], Y_{T_1}]]$$

where we put $Y_T = \otimes_{j \in T} Y_j$, and the isomorphism is given by a symmetry.

Proof. Let $Y_1, \dots, Y_n$ be as assumed, then by (8),

$$Y_{T_i} = \begin{cases} \otimes_{j \in T_i} X_j, & \text{if } i \text{ is odd,} \\ \otimes_{j \in T_i} \tilde{X}_i, & \text{if } i \text{ is even.} \end{cases}$$

As before, up to a permutation we may assume that $S_i = [n_i]$, $i = 1, \dots, N$, where by the assumptions $0 = n_1 < n_2 < \cdots < n_N = n$, so that $T_1 = [n_2]$ and $T_j = [n_j + 1, n_{j+1}]$, $j = 1, \dots, N-1$.

Let $N=3$, then $f=1-p_{[n_2]}+p_{[n]}$, and we see from Example 6 that $Y\simeq[Y_{T_2},Y_{T_1}]$, where the isomorphism is a symmetry. Assume the assertion is true for $N-2$. As in the proof of Proposition 10, we see that

$$f^*=\sum_{i=1}^{N-2}(-1)^{i+1}p_{[n_{i+1}]}=p_{[n_2]}\otimes g\otimes 1_{n-n_{N-1}}$$

where g is the chain type for a chain of intervals given by $0=m_1<m_2<\cdots<m_{N-2}=m$ in $\mathcal{T}_m$, here $m_i=n_{i+1}-n_2$. By Proposition 4, we see that

$$X_f(X_1,\dots,X_n)=X^*_{f^*}(\tilde X_1,\dots,\tilde X_n)\simeq(Y_{T_{N-1}}\otimes\tilde X_g\otimes Y^*_{T_1})^*\simeq[Y_{T_{N-1}},[\tilde X_g,Y_{T_1}]]$$

where $\tilde X_g=X_g(\tilde X_{n_2+1},\dots,\tilde X_{n_{N-1}})$ and the isomorphisms are symmetries. Since g satisfies the induction assumption, using that $\tilde{\tilde X}_i=X_i$, we obtain

$$\tilde X_g\simeq[Y_{T_{N-2}},[[\dots,[[Y_{T_{\frac{N+1}{2}}},Y_{T_{\frac{N-1}{2}}}],Y_{T_{\frac{N-3}{2}}}]],\dots,Y_{T_2}]].$$

This implies the result.

$\square$

Combs, picture
Other examples

3.3 The structure of $\mathcal{P}_f$

We will visualise the poset $\mathcal{P}_f$ by its Hasse diagram, where we only put labels to vertices corresponding to elements in $M_{i,f}$ for some i. More precisely, for $S\in\mathcal{P}_f$, the set of labels of S is defined as

$$L_S=\{i\in[n],\ S\in M_{i,f}\},\qquad L_\emptyset=\{0\}.$$

In the Hasse diagram, we label any element $S\in\mathcal{P}_f$ by L_S, with no label if $L_S=\emptyset$. Below, we give examples for $f\in\mathcal{T}_n$, $n=4$ and $n=5$ which are not chain types.

Let $f\in\mathcal{T}_n$ and let $\mathcal{P}=\mathcal{P}_f$. Let $\mathcal{P}^0_f=\mathcal{P}^0\subseteq\mathcal{P}$ be the subposed consisting of all labelled elements, that is

$$\mathcal{P}^0=\{S\in\mathcal{P},\ L_S\neq\emptyset\}.$$

For a finite poset $\mathcal{P}$, we denote by $\mathrm{Min}(\mathcal{P})$ the set of minimal elements of $\mathcal{P}$ and $\mathrm{Max}(\mathcal{P})$ the set of maximal elements of $\mathcal{P}$.

Lemma 8. *Let $f\in\mathcal{T}_n$, $g\in\mathcal{T}_m$.*

(i) $\mathrm{Min}(\mathcal{P}^0_f)=\mathrm{Min}(\mathcal{P}_f)$.

(ii) $\mathcal{P}^0_f$ *is a chain* $\iff$ $\mathcal{P}_f$ *is a chain* $\iff$ $\mathcal{P}^0_f=\mathcal{P}_f$.

(iii) If $S\in\mathcal{P}^0_f$ *and* $S\neq\emptyset$, $S\neq[n]$, *then* $S\in\mathcal{P}^0_{f^*}$.

(iv) $\emptyset\in\mathcal{P}^0_f\iff\emptyset\notin\mathcal{P}^0_{f^*}$ *and* $[n]\in\mathcal{P}^0_f\iff\cup\{L_S,S\in\mathcal{P}^0_{f^*}\neq[n]\}$.

(v) $\mathcal{P}_{f\otimes g}=\{S\oplus T:\ S\in\mathcal{P}^0_f,\ T\in\mathcal{P}^0_g,\ S\in\mathrm{Min}(\mathcal{P}_f)$ *or* $T\in\mathrm{Min}(\mathcal{P}_g)\}$.

(vi) If $\mathcal{P}^0_f$ *has a largest element, then it is the largest element in* $\mathcal{P}_f$.

Proof. (i) Obvious. (iii)–(v) Follow from the proof of Proposition 9.

(ii) Assume that $\mathcal{P}_f^0$ is a chain, so that, in particular, any $M_{i,f}$ is a singleton or empty. In the first case, we put $M_{i,f} = \{M_i\}$. Let S, T be two elements in $\mathcal{P}_f$ and assume that $i \in S \setminus T$. For any $j \in T$, we have either $M_i \subseteq M_j$ or $M_j \subseteq M_i$. In the first case, we get $i \in M_i \subseteq M_j \subseteq T$, which is not possible. Hence $M_j \subseteq M_i$ for all $j \in T$. It follows that $T = \cup_{j\in T} M_j \subseteq M_i \subseteq S$, so that $\mathcal{P}_f$ is a chain, and it is clear that $\mathcal{P}_f = \mathcal{P}_f^0$. If $\mathcal{P}_f$ is not a chain, then there are some type functions f_1, f_2 such that $f = f_1 \otimes f_2$ or $f = (f_1 \otimes f_2)^*$. Moreover, the ranks of f_1 and f_2 are at least 2. It follows that both $\mathcal{P}_{f_1\otimes f_2}$ and $\mathcal{P}_{(f_1\otimes f_2)^*}$ contain an element $S \oplus T$, where $S \in \mathcal{P}_{f_1}$, $T \in \mathcal{P}_{f_2}$ but none of the two elements is minimal. It follows by part (v) that $S \oplus T \notin \mathcal{P}_f^0$, so that $\mathcal{P}_f \neq \mathcal{P}_f^0$. It is quite clear that $\mathcal{P}_f^0$ is a chain if $\mathcal{P}_f$ is.

(vi) The largest element $T \in P_f^0$ must contain all subsets that are in $M_{i,f}$ for all $i \in [n]$, so that $\cup \mathcal{P}_f \subseteq T$. If follows that $T = \cup \mathcal{P}_f$ is the largest element in $\mathcal{P}_f$.

□

Lemma 9. *Assume that $\mathcal{P}_f^0$ has a largest element. Then either f is a chain type, or there is some $h \in \mathcal{T}_m$ such that $\mathcal{P}_h^0$ has no largest element, and a chain type $\beta \in \mathcal{T}_{n-m}$ such that $f = g \lhd \beta$.*

Proof. Let T be the largest element in $\mathcal{P}_f^0$. If $T = [n]$, then by Lemma 8(iv), f^* has some free outputs. Hence, up to a permutation, $f^* = h_1 \otimes 1_{k_1}$ for some $h_1 \in \mathcal{T}_{n-k_1}$ with no free outputs. It follows that $f = (h_1 \otimes 1_{k_1})^* = (h_1 \lhd 1_{k_1})^* = h_1^* \lhd p_{k_1}$.

If $T \neq [n]$, then, since T is also the largest element in $\mathcal{P}_f$, we see that f has free outputs, so that (up to a permutation) $f = h_\otimes 1_{k_1} = h_1 \lhd 1_1$, where $h_1 \in \mathcal{T}_{n-k_1}$ is a type function with no free outputs. Clearly, $\mathcal{P}_{h_1} = \mathcal{P}_f$ and $\mathcal{P}_{h_1}^0 = \mathcal{P}_f^0$, so that T is the largest element in $\mathcal{P}_{h_1}^0$, but this time $T = [n - k_1]$, so that we may use the first part of the proof. We obtain that there is some k_2 and a type function $h_2 \in \mathcal{T}_{n-k_1-k_2}$ with no free outputs such that

$$f = h_1 \lhd 1_{k_1} = h_2^* \lhd p_{k_2} \lhd 1_{k_1}$$

So far, we have written f in the form $f = h \lhd \beta$, where β is a chain type and $h \in \mathcal{T}_{n-k}$ for $k > 0$ is such that h^* has no free outputs. It follows that if $\mathcal{P}_h^0$ has a largest element, then it cannot be equal to $[n - k]$. Hence h must have some free outputs, and we may proceed as above, replacing f by h. Since n is decreasing at each step, we either get to $n - k \leq 3$, in which case h must be a chain and therefore also $f = h \lhd \beta$ is a chain, or h has no largest element.

□

Lemma 10. *Let $T \in \mathrm{Max}(\mathcal{P}_f^0)$ and let $T^{\downarrow,f} := \{S \in \mathcal{P}_f, S \leq T\}$. Put*

$$\bar{\mathcal{P}}_{T,f} := \begin{cases} T^{\downarrow,f} & \text{if } \rho_f(T) \text{ is even} \\ T^{\downarrow,f} \setminus T & \text{otherwise.} \end{cases}$$

Then there is some type function $g \in \mathcal{T}_{|T|}$ and a permutation $\sigma \in \mathscr{S}n$ such that $\mathcal{P}_g = \bar{\mathcal{P}}_{\sigma(T),f}$.

Proof. We will proceed by induction on n. The assertion is trivial if f is a chain, so it holds for $n \leq 3$. So assume it holds for all $m < n$ and let $f \in \mathcal{T}_n$. If $f = f_1 \otimes f_2$, then $T = T_1 \oplus T_2$, where, say, $T_1 \in \mathrm{Min}(\mathcal{P}_{f_1})$ and $T_2 \in \mathrm{Max}(\mathcal{P}_{f_2}^0)$. It follows that

$$\bar{\mathcal{P}}_{T,f} = T_1 \oplus \bar{\mathcal{P}}_{T_2,f_2}.$$

By the induction assumption, $\bar{\mathcal{P}}_{T_2,f_2} = \mathcal{P}_h$ for some type function $h \in \mathcal{T}_{|T_2|}$ and hence $\bar{\mathcal{P}}_{T,f} = \mathcal{P}_g$ with $g \in \mathcal{T}_{|T|}$, $g = p_{T_1} \otimes h$.

Assume next that $f = (f_1 \otimes f_2)^*$. We may assume that $T \neq \emptyset$ and $T \neq [n]$, since otherwise the assertion is rather trivial. We may also assume that f has no free outputs, since otherwise $f = h \otimes 1$ and we may replace f by h. Then $T \in \mathrm{Max}(\mathcal{P}_{f^*})$.

Let $\rho_f(T)$ be even, so that the labels of T correspond to inputs. Then $\rho_{f^*}(T)$ is odd. Since f^* has the desired property by the first part of the proof, there is some $g \in \mathcal{T}_{|T|}$ such that (up to a permutation)

$$\mathcal{P}_g = \bar{\mathcal{P}}_{T,f^*} = T^{\downarrow,f^*}$$

and it is easy to see that then $\mathcal{P}_{g^*} = \bar{\mathcal{P}}_{T,f}$. The case when $\rho_f(T)$ is odd is proved similarly.

$\square$

3.4 Connecting chains: the causal product

We will introduce further operations of boolean functions. For a fixed decomposition $[n] = [n_1] \oplus [n_2]$ and functions $f_1 : \{0,1\}^{n_1} \to \{0,1\}$, $f_2 : \{0,1\}^{n_2} \to \{0,1\}$, we define their causal product as

$$f_1 \lhd f_2 := f_1 \otimes 1_{n_2} + p_{[n_2]} \otimes (f_2 - 1_{n_2}).$$

For $s^1 \in \{0,1\}^{n_1}$ and $s^2 \in \{0,1\}^{n_2}$, this function acts as

$$(f_1 \lhd f_2)(s^1 s^2) = f_1(s^1) + p_{[n_1]}(s^1)(f_2(s^2) - 1) = \begin{cases} f_1(s^1), & \text{if } s^1 \neq \theta_{n_1}, \\ f_2(s^2), & \text{if } s^1 = \theta_{n_1}. \end{cases} \tag{9}$$

Remark 1. Causal: can be interpreted as "f_1 before f_2" (actually after).

The following properties are immediate from (9).

Lemma 11. *Let $f_1, g_1 \in \mathcal{F}_{n_1}$, $f_2, g_2 \in \mathcal{F}_{n_2}$. Then $f_1 \lhd f_2 \in \mathcal{F}_{n_1+n_2}$ and we have*

(i) $(f_1 \lhd f_2)^* = f_1^* \lhd f_2^*$,

(ii) $(f_1 \vee g_1) \lhd (f_2 \vee g_2) = (f_1 \lhd f_2) \vee (g_1 \lhd g_2) = (f_1 \lhd g_2) \vee (g_1 \lhd f_2)$,

(iii) $(f_1 \wedge g_1) \lhd (f_2 \wedge g_2) = (f_1 \lhd f_2) \wedge (g_1 \lhd g_2) = (f_1 \lhd g_2) \wedge (g_1 \lhd f_2)$.

Moreover, for any $f_3 \in \mathcal{F}_{n_3}$, and for the decomposition $[n] = [n_1] \oplus [n_2] \oplus [n_3]$, we have

$$(f_1 \lhd f_2) \lhd f_3 = f_1 \lhd (f_2 \lhd f_3).$$

We can also combine f_1 and f_2 in the opposite order:

$$f_2 \lhd f_1 := 1_{n_1} \otimes f_2 + (f_1 - 1_n) \otimes p_{[n_2]},$$

so that

$$(f_2 \lhd f_1)(s^1 s^2) = f_2(s^2) + p_{[n_2]}(s^2)(f_1(s^1) - 1_{n_1}) = \begin{cases} f_2(s^2), & \text{if } s^2 \neq \theta_{n_2}, \\ f_1(s^1), & \text{if } s^2 = \theta_{n_2}. \end{cases} \tag{10}$$

Of course, this product has similar properties as listed in the above lemma. To avoid any confusion, we have to bear in mind the fixed decomposition $[n] = [n_1] \oplus [n_2]$ and that for the concatenation $s = s^1 s^2$, f_i acts on s^i.

Lemma 12. *In the situation as above, we have*

$$f_1 \otimes f_2 = (f_1 \rhd f_2) \wedge (f_2 \rhd f_1).$$

Proof. This is again by straightforward computation from (9) and (10): let $s^1 \in \{0,1\}^{n_1}$, $s^2 \in \{0,1\}^{n_2}$ and compute

$$\begin{aligned}(f_1 \lhd f_2) \wedge (f_2 \lhd f_1)(s^1 s^2) &= \left(f_1(s^1) + p_{[n_1]}(s^1)(f_2(s^2) - 1)\right)\left(f_2(s^2) + p_{[n_2]}(s^2)(f_1(s^1) - 1)\right) \\ &= f_1(s^1) f_2(s^2),\end{aligned}$$

the last equality follows from the fact that $f_i(s^i)(1 - f_i(s^i)) = 0$ (since $f_i(s^i) \in \{0,1\}$) and the fact that $p_{[n_1]}$ is the least element in $\mathcal{F}_{[n_1]}$, so that $p_{[n_1]}(s^1)(f_1(s^1) - 1) = p_{[n_1]}(s^1) - p_{[n_1]}(s^1) = 0$. □

For the smallest and the largest element in $\mathcal{T}_n$, the causal product behaves as follows.

Lemma 13. *For the $f \in \mathcal{T}_{n_1}$ and let $1_{n_2}, p_{[n_2]} \in \mathcal{T}_{n_2}$, we have*

$$f \lhd 1_{n_2} = f \otimes 1_{n_2} \leq 1_{n_2} \lhd f = 1 - (1 - \hat{f}_\emptyset) p_{[n_2]} + \sum_{\emptyset \neq S \subseteq [n_1]} \hat{f}_S p_{S \oplus [n_2]}$$

and

$$p_{[n_2]} \lhd f = p_{[n_2]} \otimes f \leq f \lhd p_{[n_2]} = \sum_{S \subsetneq [n]} \hat{f}_S p_S + (\hat{f}_{[n_1]} - 1) p_{[n_1]} + p_{[n_1 + n_2]}.$$

In particular,

$$(p_{n_1} \otimes 1_{n_2})^* = 1_{n_1} \lhd p_{n_2} = 1 - p_{[n_1]} + p_{n_1 + n_2}$$

is the chain type for $\{\emptyset \subsetneq [n_1] \subsetneq [n_1 + n_2]\}$.

Using the last part of Lemma 11, for a decomposition $[n] = \oplus_i [n_i]$ and $f_i \in \mathcal{F}_{n_i}$, we may define the function $f_1 \lhd \ldots \lhd f_k \in \mathcal{F}_n$. Note that we have for $s = s^1 \ldots s^k$,

$$\begin{aligned}(f_1 \lhd \ldots \lhd f_k)(s) &= f_1(s^1) + p_{n_1}(s^1)(f_2(s^2) - 1) + \cdots + p_{n_1}(s^1) \ldots p_{n_{k-1}}(s^{k-1})(f_k(s^k) - 1) \\ &= \begin{cases} f_1(s_1) & \text{if } s^1 \neq \theta_{n_1} \\ f_2(s^2) & \text{if } s^1 = \theta_{n_1}, s^2 \neq \theta_{n_2} \\ \ldots & \\ f_k(s^k) & \text{if } s^1 = \theta_{n_1}, \ldots, s^{k-1} = \theta_{n_{k-1}}. \end{cases}\end{aligned}$$

For any permutation $\pi \in \mathscr{S}_k$, we define $f_{\pi^{-1}(1)} \lhd \ldots \lhd f_{\pi^{-1}(k)} \in \mathcal{F}_n$ in an obvious way.

It is not clear that if f_1 and f_2 are type functions, then $f_1 \lhd f_2$ or $f_2 \lhd f_1$ are type functions as well. Nevertheless, we next show that this is true for chains, and that the result can be seen as appending the two chains.

Proposition 12. *Let N_1 and N_2 be odd and let $\mathcal{P}_1 = \{S_1 \subsetneq \cdots \subsetneq S_{N_1}\}$ be a chain in 2^{n_1} and $\mathcal{P}_2 = \{T_1 \subsetneq \cdots \subsetneq T_{N_2}\}$ a chain in 2^{n_2}, with corresponding chain type functions β_1 and β_2. Then $\beta_1 \lhd \beta_2$ and $\beta_2 \lhd \beta_1$ are type functions corresponding to chains of $N_1 + N_2 \pm 1$ elements in $2^{n_1 + n_2}$.*

Proof. We have

$$\beta_1 = \sum_{j=1}^{N_1} (-1)^{j-1} p_{S_j}, \qquad \beta_2 = \sum_{k=1}^{N_2} (-1)^{k-1} p_{T_k},$$

so that

$$\beta := \beta_1 \lhd \beta_2 = \sum_{j=1}^{N_1-1} (-1)^{j-1} p_{S_j} + (p_{S_{N_1}} - p_{[n_1]} + p_{[n_1]\oplus T_1}) + \sum_{k=2}^{N_2} (-1)^{k-1} p_{[n_1]\oplus T_k}.$$

The resulting funnction depends on whether $S_{N_1} = [n_1]$ and $T_1 = \emptyset$. If at least one of the equalities is true, then the expression in brackets is equal to $p_{[n_1]}$, $p_{S_{N_1}}$ or $p_{[n_1]\oplus T_1}$, and β corresponds to a chain of $N_1 + N_2 - 1$ elements. If both $S_{N_1} \neq [n_1]$ and $T_1 \neq \emptyset$, then $S_{N_1} \subsetneq [n_1] \subsetneq [n_1] \oplus T_1$ and β corresponds to a chain of $N_1 + N_2 + 1$ elements.

$\square$

Proposition 13. *Let $f \in \mathcal{T}_{n_1}$ and let $\beta \in \mathcal{T}_{n_2}$ be a chain type. Then both $f \lhd \beta$ and $\beta \lhd f$ are types, with outputs $O = O_f \oplus O_\beta$ and inputs $I = I_f \oplus I_\beta$.*

Proof. Let $\beta = \sum_{k=1}^{N} (-1)^{k-1} p_{T_k}$ for some odd N. We will proceed by induction on N. Suppose $N = 1$. If $T_1 = \emptyset$, then $\beta = 1_{n_2}$ and we have by Lemma 13

$$f \lhd 1_{n_2} = f \otimes 1_{n_2} \in \mathcal{T}_{n_1+n_2}$$

and

$$1_{n_2} \lhd f = (p_{n_2} \lhd f^*)^* = (f \otimes p_{n_2})^* \in \mathcal{T}_{n_1+n_2}.$$

Assume that $T_1 = [n_2]$, then $\beta = p_{n_2}$ and the assertion follows by duality. In general, up to a permutation, we have $\beta = p_{m_1} \otimes 1_{m_2} = p_{m_1} \lhd 1_{m_2}$ for $m_1 = |T_1|$, $m_1 + m_2 = n_2$. Then

$$\beta \lhd f = p_{m_1} \lhd (1_{m_2} \lhd f), \quad f \lhd \beta = (f \lhd p_{m_1}) \lhd 1_{m_2} \in \mathcal{T}_{n_1+n_2},$$

by the first part of the proof and Lemma 11.

Assume next that the assertion holds for all odd numbers $M < N$. As before, up to a permutation, we may assume that $T_k = [l_k]$ for some $l_1 < \cdots < l_N$. Let β_1 be the chain type for the chain given by $l_1 < \cdots < l_{k-2}$ in $[l_{k-1}]$ and put $\beta_2 := p_{[l_k - l_{k-1}]}$, the 1-element chain type in $\mathcal{T}_{n_2 - l_{k-1}}$. By the proof of Proposition 12, we see that $\beta = \beta_1 \lhd \beta_2$, so that

$$\beta \lhd f = \beta_1 \lhd (\beta_2 \lhd f), \qquad f \lhd \beta = (f \lhd \beta_1) \lhd \beta_2$$

are type functions, by the induction assumption.

For any $i \in [n_1] \oplus [n_2]$, we have $e^i_{n_1+n_2} = e^j_{n_1}\theta_{n_2}$ or $e^i_{n_1+n_2} = \theta_{n_1} e^k_{n_2}$ for some $j \in [n_1]$, $k \in [n_2]$. Then

$$f \lhd \beta(e^i) = f(e^j_{n_1}) \ \text{ or } f \lhd \beta(e^i) = \beta(e^k_{n_2}).$$

The statement on input/output indices follow from Lemma 7. The proof for $\beta \lhd f$ is similar.

$\square$

We now get to prove the following structure theorem for the type functions.

Theorem 3. *Let $f \in \mathcal{T}_n$. Then there is a permutation $\rho \in \mathscr{S}_n$, a decomposition $[n] = \oplus_{i=1}^k [n_i]$, chain types $\beta_1 \in \mathcal{T}_{n_1}, \ldots, \beta_k \in \mathcal{T}_{n_k}$ such that $O_f = \oplus_j O_{\beta_j}$, $I_f = \oplus_j I_{\beta_j}$, finite index sets A, B and permutations $\pi_{a,b} \in \mathscr{S}_k$, $a \in A$, $b \in B$ such that*

$$f = \bigvee_{a\in A}\bigwedge_{b\in B}(\beta_{\pi^{-1}_{a,b}(1)} \lhd \ldots \lhd \beta_{\pi^{-1}_{a,b}(k)}) \circ \rho = \bigwedge_{b\in B}\bigvee_{a\in A}(\beta_{\pi^{-1}_{a,b}(1)} \lhd \ldots \lhd \beta_{\pi^{-1}_{a,b}(k)}) \circ \rho.$$

Proof. It is obvious that the condition is invariant under permutations. Since any element in $\mathcal{T}_n$ for $n \leq 3$ is a chain type, the statement clearly holds in this case. Assume f can be written in the given form, then

$$f^* = \bigwedge_{a\in A}\bigvee_{b\in B}(\beta^*_{\pi^{-1}_{a,b}(1)} \lhd \ldots \lhd \beta^*_{\pi^{-1}_{a,b}(k)}) \circ \rho = \bigvee_{b\in b}\bigwedge_{a\in A}(\beta^*_{\pi^{-1}_{a,b}(1)} \lhd \ldots \lhd \beta^*_{\pi^{-1}_{a,b}(k)}) \circ \rho.$$

Since β^*_j is a chain type for each j, this proves the statement for f^*. It is now enough to show this form for $f = f_1 \otimes f_2$, where $f_1 \in \mathcal{T}_m$, $f_2 \in \mathcal{T}_{n-m}$ satisfy the conditions, so that

$$f_1 = \bigvee_{a\in A}\bigwedge_{b\in B}(\beta^1_{\pi^{-1}_{a,b}(1)} \lhd \ldots \lhd \beta^1_{\pi^{-1}_{a,b}(k_1)}) \circ \rho_1 = \bigwedge_{b\in B}\bigvee_{a\in A}(\beta^1_{\pi^{-1}_{a,b}(1)} \lhd \ldots \lhd \beta^1_{\pi^{-1}_{a,b}(k_1)}) \circ \rho_1,$$
$$f_2 = \bigvee_{c\in C}\bigwedge_{d\in D}(\beta^2_{\tau^{-1}_{c,d}(1)} \lhd \ldots \lhd \beta^2_{\tau^{-1}_{c,d}(k_2)}) \circ \rho_2 = \bigwedge_{d\in D}\bigvee_{c\in C}(\beta^2_{\tau^{-1}_{c,d}(1)} \lhd \ldots \lhd \beta^2_{\tau^{-1}_{c,d}(k_2)}) \circ \rho_2$$

for some chain types $\beta^1_j \in \mathcal{T}_{m_j}$, $[m] = \oplus_{j=1}^{k_1}[m_j]$, and $\beta^2_j \in \mathcal{T}_{l_j}$, $[n-m] = \oplus_{j=1}^{k_2}[l_j]$ and permutations $\pi_{a,b} \in \mathscr{S}_{k_1}$, $\tau_{c,d} \in \mathscr{S}_{k_2}$, $\rho_1 \in \mathscr{S}_m$, $\rho_2 \in \mathscr{S}_{n-m}$. Let

$$\beta_1^{a,b} := \beta^1_{\pi^{-1}_{a,b}(1)} \lhd \ldots \lhd \beta^1_{\pi^{-1}_{a,b}(k_1)}, \qquad \beta_2^{c,d} := \beta^2_{\tau^{-1}_{c,d}(1)} \lhd \ldots \lhd \beta^2_{\tau^{-1}_{c,d}(k_2)}.$$

Using the properties of the tensor product (Lemma 14)ii), we get from Lemma 12

$$(\bigvee_{a\in A}\bigwedge_{b\in B}\beta_1^{a,b}) \otimes (\bigvee_{c\in C}\bigwedge_{d\in D}\beta_2^{c,d}) = \bigvee_{a,c}\bigwedge_{b,d}(\beta_1^{a,b} \otimes \beta_2^{c,d}) = \bigvee_{a,c}\bigwedge_{b,d}(\beta_1^{a,b} \lhd \beta_2^{c,d}) \wedge (\beta_2^{c,d} \lhd \beta_1^{a,b})$$

On the other hand, using Lemma 11, we get

$$\begin{aligned}
&(\bigwedge_{b\in B}\bigvee_{a\in A}\beta_1^{a,b}) \otimes (\bigwedge_{d\in D}\bigvee_{c\in C}\beta_2^{c,d}) \\
&\quad = \left[(\bigwedge_{b\in B}\bigvee_{a\in A}\beta_1^{a,b}) \lhd (\bigwedge_{d\in D}\bigvee_{c\in C}\beta_2^{c,d})\right] \wedge \left[(\bigwedge_{d\in D}\bigvee_{c\in C}\beta_2^{c,d}) \lhd (\bigwedge_{b\in B}\bigvee_{a\in A}\beta_1^{a,b})\right] \\
&= (\bigwedge_{b,d}\bigvee_{a,c}\beta_1^{a,b} \lhd \beta_2^{c,d}) \wedge (\bigwedge_{b,d}\bigvee_{a,c}\beta_2^{c,d} \lhd \beta_1^{a,b}).
\end{aligned}$$

We have the decomposition $[n] = \oplus_{j=1}^k [n_j]$, with $k = k_1 + k_2$ and $n_j = m_j$, $j = 1, \ldots, k_1$, $n_j = l_{j-k_1}$, $j = k_1+1, \ldots, k$, and chain types $\beta_j \in \mathcal{T}_{n_j}$, $\beta_j = \beta^1_j$ for $j = 1, \ldots, k_1$ and $\beta_j = \beta^2_{j-k_1}$ for $j = k_1+1, \ldots, k$. To get the permutation sets, let $A' = A \times C$, $B' = B \times D \times \mathscr{S}_2$ and define $\pi_{a',b'}$ in $\mathscr{S}_k$ as the block permutation with respect to the decomposition $[k] = [k_1] \oplus [k_2]$ (see Appendix A.1)

$$\pi_{(a,c),(b,d,\lambda)} = \rho_\lambda \circ (\pi_{a,b} \oplus \tau_{c,d}).$$

Finally, putting $\rho = \rho_1 \otimes \rho_2$ finishes the proof.

□

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Let $f \in \mathcal{T}_n$ and let $\mathcal{P} = \mathcal{P}_f$. Let $\mathcal{P}_f^0 = \mathcal{P}^0 \subseteq \mathcal{P}$ be the subposet of elements $S \in \mathcal{P}_f$ such that either $S = \emptyset$ or $S \in M_{i,f}$ (so the set of all elements in $\mathcal{P}$ with labels). Let T be a maximal element in $\mathcal{P}^0$ and consider the downset of T in $\mathcal{P}$, that is, $\mathcal{P}_T = \{S \in \mathcal{P}, S \subseteq T\}$. If T is labeled by an output index, that is, if the rank $\rho_f(T)$ is odd, then $\bar{\mathcal{P}}_T = \mathcal{P}_T \setminus \{T\}$, otherwise we put $\bar{\mathcal{P}}_T = \mathcal{P}_T$.

Assume that $T = [n_T]$ (we may always do that). Then there is some $f_T \in \mathcal{T}_{n_T}$ such that $\bar{\mathcal{P}}_T = \mathcal{P}_{f_T}$.

We first show that $\bar{\mathcal{P}}_T$ is a graded poset with even rank.

A Permutations, binary strings and boolean functions

For $m \le n \in \mathbb{N}$, we will denote the corresponding interval $\{m, m+1, \dots, n\}$ by $[m, n]$. For $m = 1$, we will simplify to $[n] := [1, n]$. Let $\mathscr{S}_n$ denote the set of all permutations of $[n]$.

A.1 Block permutations

For $n_1, n_2 \in \mathbb{N}$, $n_1 + n_2 = n$, we will denote by $[n] = [n_1] \oplus [n_2]$ the decomposition of $[n]$ as a concatenation of two intervals

$$[n] = [n_1][n_1 + 1, n_1 + n_2].$$

Similarly, for $n = \sum_{j=1}^k n_j$, we have the decomposition

$$[n] = \oplus_{j=1}^k [n_j] = [m_1, m_1 + n_1][m_2, m_2 + n_2] \dots [m_k, m_k + n_k],$$

where $m_j := \sum_{l=1}^{j-1} n_j$ (so $m_1 = 0$). Note that the order of $n_1, \dots, n_k$ in this decomposition is fixed.

We have two kinds of special permutations related to the above decomposition. For $\sigma_j \in \mathscr{S}_{n_j}$, we denote by $\oplus_j \sigma_j \in \mathscr{S}_n$ the permutation that acts as

$$m_j + l \mapsto m_j + \sigma_j(l), \qquad l = 1, \dots, n_j,\ j = 1, \dots, k.$$

On the other hand, we have for any $\lambda \in \mathscr{S}_k$ a unique permutation $\rho_\lambda \in \mathscr{S}_n$ such that ρ_λ^{-1} acts as

$$[m_1, m_1 + n_1][m_2, m_2 + n_2]...[m_k, m_k + n_k] \mapsto [m_{\lambda(1)} + n_{\lambda(1)}][m_{\lambda(2)} + n_{\lambda(2)}] \dots [m_{\lambda(k)} + n_{\lambda(k)}]$$

Note that we have

$$\rho_\lambda \circ (\oplus_j \sigma_j) = (\oplus_j \sigma_{\lambda(j)}) \circ \rho_\lambda.$$

(These permutations come from the operadic structure on the set of all permutations $\mathscr{S}_*$.)

A.2 Binary strings

A binary string of length n is a sequence $s = s_1 \dots s_n$, where $s_i \in \{0, 1\}$. Such a string can be interpreted as an element $\{0, 1\}^n$, but also as a map $[n] \to \{0, 1\}$, or a subset in $[n] := \{1, \dots, n\}$. It will be convenient to use all these interpretations, but we will distinguish between them. The strings in $\{0, 1\}^n$ will be denoted by small letters, whereas the corresponding subsets of $[n]$ will be denoted by the corresponding capital letters. More specifically, for $s \in \{0, 1\}^n$ and $T \subseteq [n]$, we denote

$$S := \{i \in [n],\ s_i = 0\}, \qquad t := t_1 \dots t_n,\ t_j = 0 \iff j \in T. \tag{11}$$

As usual, the set of all subsets of $[n]$ will be denoted by 2^n. With the inclusion ordering and complementation $S^c := [n] \setminus S$, 2^n is a boolean algebra, with the smallest element $\emptyset$ and largest element $[n]$.

The group $\mathscr{S}_n$ has an obvious action on $\{0,1\}^n$. Indeed, for a string s interpreted as a map $[n] \to 2$, we may define the action of $\sigma \in \mathscr{S}_n$ by precomposition as

$$\sigma(s) := s \circ \sigma^{-1} = s_{\sigma^{-1}(1)} \dots s_{\sigma^{-1}(n)}.$$

Note that in this way we have $\rho(\sigma(s)) = (\rho \circ \sigma)(s)$. For a decomposition $[n] = \oplus_{j=1}^k [n_j]$, we have a corresponding decomposition of any string $s \in \{0,1\}^n$ as a concatenation of strings

$$s = s^1 \dots s^k, \qquad s^j \in \{0,1\}^{n_j}.$$

For permutations $\sigma_j \in \mathscr{S}_{n_j}$ and $\lambda \in \mathscr{S}_k$, we have

$$\rho_\lambda \circ (\oplus_j \sigma_j)(s^1 \dots s^k) = \rho_\lambda(\sigma_1(s^1) \dots \sigma_k(s^k)) = \sigma_{\lambda(1)}(s^{\lambda(1)})\sigma_{\lambda(2)}(s^{\lambda(2)}) \dots \sigma_{\lambda(k)}(s^{\lambda(k)}).$$

A.3 Boolean functions and the Möbius transform

A function $f : \{0,1\}^n \to \{0,1\}$ is called a boolean function. The set of boolean functions, with pointwise ordering and complementation given by the negation $\bar{f} = 1 - f$, is a boolean algebra that can be identified with 2^{2^n}. We will denote the maximal element (the constant 1 function) by 1_n. Similarly, we denote the constant zero function by 0_n. For boolean functions f, g, the pointwise minima and maxima will be denoted by $f \wedge g$ and $f \vee g$. It is easily seen that we have

$$f \vee g = f + g - fg, \qquad f \wedge g = fg, \tag{12}$$

all the operations are pointwise. We now introduce and important example.

Example 9. For $S \subseteq [n]$, we define

$$p_S(t) = \Pi_{j \in S}(1 - t_j), \qquad t \in \{0,1\}^n.$$

That is, $p_S(t) = 1$ if and only if $S \subseteq T$. In particular, $p_\emptyset = 1_n$ and $p_{[n]}$ is the characteristic function of the zero string. Clearly, for $S, T \subseteq [n]$ we have $p_{S \cup T} = p_S p_T = p_S \wedge p_T$, in particular, $p_S = \Pi_j p_{\{j\}}$.

By the Möbius transform, all boolean functions can be expressed as combinations of the functions p_S, $S \subseteq [n]$ from the previous example.

Theorem 4. *Any $f : \{0,1\}^n \to 2$ can be expressed in the form*

$$f = \sum_{S \subseteq [n]} \hat{f}_S p_S$$

in a unique way. The coefficients $\hat{f}_S \in \mathbb{R}$ obtained as

$$\hat{f}_S = \sum_{\substack{t \in \{0,1\}^n \\ t_j = 1, \forall j \in S^c}} (-1)^{\sum_{j \in S} t_j} f(t).$$

Proof. By the Möbius inversion formula (see [Stanley, Sec. 3.7] for details), functions $f, g : 2^n \to \mathbb{R}$ satisfy

$$f(S) = \sum_{T \subseteq S} g(T), \qquad S \in 2^n$$

if and only if

$$g(S) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} f(T).$$

We now express this in terms of the corresponding strings s and t. It is easily seen that $T \subseteq S$ if and only if $s_j = 0$ for all $j \in T$, equivalently, $t_j = 1$ for all $j \in S^c$. Moreover, in this case we have $|S \setminus T| = \sum_{j \in S} t_j$. This shows that $g(S) = \hat{f}_S$, as defined in the statement. The first equality now gives

$$f(s) = f(S) = \sum_{T \subseteq S} g(T) = \sum_{T : s_i = 0, \forall i \in T} \hat{f}_T = \sum_{T : p_T(s) = 1} \hat{f}_T = \sum_{T \subseteq [n]} \hat{f}_T p_T.$$

For uniqueness, assume that $f = \sum_{T \subseteq [n]} c_T p_T$ for some coefficients $c_T \in \mathbb{R}$. Then

$$f(s) = \sum_{T : p_T(s) = 1} c_T = \sum_{T \subseteq S} c_T.$$

Uniqueness now follows by uniqueness in the Möbius inversion formula.

□

A.4 The boolean algebra $\mathcal{F}_n$

Let us introduce the subset of boolean functions

$$\mathcal{F}_n := \{f : \{0,1\}^n \to 2,\ f(\theta_n) = 1\},$$

where we use θ_n to denote the zero string $00\ldots0$. In other words, $\mathcal{F}_n$ is the interval of all elements greater than $p_{[n]}$ in the boolean algebra 2^{2^n} of all boolean functions. With the pointwise ordering, $\mathcal{F}_n$ is a distributive lattice, with top element 1_n and bottom element $p_{[n]}$. We also define complementation in $\mathcal{F}_n$ as

$$f^* := 1_n - f + p_{[n]}.$$

It can be easily checked that with these structures $\mathcal{F}_n$ is a boolean algebra, though it is not a subalgebra of 2^{2^n}.

We now introduce some more operations in $\mathcal{F}_n$. For $f \in \mathcal{F}_n$ and any permutation $\sigma \in \mathscr{S}_n$, we clearly have $f \circ \sigma \in \mathcal{F}_n$. Further, let $f \in \mathcal{F}_{n_1}$ and $g \in \mathcal{F}_{n_2}$. With the decomposition $[n_1 + n_2] = [n_1] \oplus [n_2]$ and the corresponding concatenation of strings $s = s^1 s^2$, we define the function $f \otimes g \in \mathcal{F}_{n_1 + n_2}$ as

$$(f \otimes g)(s^1 s^2) = f(s^1) g(s^2), \qquad s^1 \in \{0,1\}^{n_1},\ s^2 \in \{0,1\}^{n_2}.$$

Let $\lambda \in \mathscr{S}_2$ be the transposition, then we have for any $f \in \mathcal{F}_{n_1}$ and $g \in \mathcal{F}_{n_2}$

$$(g \otimes f) = (f \otimes g) \circ \rho_\lambda,$$

where ρ_λ is the block permutation defined in Section A.1. We now show some important properties of these operations.

Lemma 14. *For $f \in \mathcal{F}_{n_1}$ and $g, h \in \mathcal{F}_{n_2}$, we have*

(i) $f \otimes g \leq (f^* \otimes g^*)^*$, *with equality if and only if either $f = 1_{n_1}$ and $g = 1_{n_2}$, or $f = p_{[n_1]}$ and $g = p_{[n_2]}$.*

(ii) $f \otimes (g \vee h) = (f \otimes g) \vee (f \otimes h)$, $f \otimes (g \wedge h) = (f \otimes g) \wedge (f \otimes h)$.

Proof. The inequality in (i) is easily checked, since $(f \otimes g)(s^1 s^2)$ can be 1 only if $f(s^1) = g(s^2) = 1$. If both s^1 and s^2 are the zero strings, then $s^1 s^2 = \theta_{n_1+n_2}$ and both sides are equal to 1. Otherwise, the condition $f(s^1) = g(s^2) = 1$ implies that $(f^* \otimes g^*)(s^1 s^2) = 0$, so that the right hand side must be 1. If f and g are both constant 1, then $(1_{n_1} \otimes 1_{n_2})^* = 1^*_{n_1+n_2} = p_{[n_1+n_2]} = p_{[n_1]} \otimes p_{[n_2]} = 1^*_{n_1} \otimes 1^*_{n_2}$, in the case when both f and g are the minimal elements equality follows by duality. Finally, asume the equality holds and that $f \neq 1_{n_1}$, so that there is some s^1 such that $f(s^1) = 0$. But then $s^1 \neq \theta_{n_1}$, so that $f^*(s_1) = 1$ and for any s^2,

$$0 = (f \otimes g)(s^1 s^2) = (f^* \otimes g^*)^*(s^1 s^2) = 1 - f^*(s^1) g^*(s^2) + p_{[n_1+n_2]}(s^1 s^2) = 1 - g^*(s^2),$$

which implies that $g(s^2) = 0$ for all $s^2 \neq \theta_{n_2}$, that is, $g = p_{[n_2]}$. By the same argument, $f = p_{[n_1]}$ if $g \neq 1_{n_2}$, which implies that either $f = 1_{n_1}$ and $g = 1_{n_2}$, or $f = p_{[n_1]}$ and $g = p_{[n_2]}$.

The statement (ii) is easily proved from (12).

□

Consider the decomposition $[n] = [n_1] \oplus [n_2]$ and let $S \subseteq [n_1]$, $T \subseteq [n_2]$. We then denote by $S \oplus T$ the disjoint union

$$S \oplus T := S \cup (n_1 + T) = S \cup \{n_1 + j,\ j \in T\}. \tag{13}$$

We summarize some easy properties of the basic functions p_S, $S \subseteq [n]$.

Lemma 15. *(i) For $S, T \subseteq [n]$, we have $S \subseteq T \iff p_T \leq p_S \iff p_S p_T = p_S$.*

(ii) For $S \subseteq [n]$, $\sigma \in \mathscr{S}_n$, $p_S \circ \sigma = p_{\sigma^{-1}(S)}$.

(iii) For $S \subseteq [n_1]$ and $T \subseteq [n_2]$, $p_S \otimes p_T = p_{S \oplus T}$.

Let $f \in \mathcal{F}_n$ and let $\hat{f}$ be the Möbius transform. Note that since f has values in $\{0, 1\}$, we have by the proof of Theorem 4

$$\forall S \in 2^n, \quad \sum_{T \subseteq S} \hat{f}_T = f(s) \in \{0, 1\}; \qquad \sum_{T \in 2^n} \hat{f}_T = f(\theta_n) = 1.$$

Proposition 14. *(i) For $f \in \mathcal{F}_n$ and $\sigma \in \mathscr{S}_n$, $\widehat{(f \circ \sigma)}_S = \hat{f}_{\sigma(S)}$, $\qquad S \subseteq [n]$.*

(ii) For $f \in \mathcal{F}_n$, $\widehat{f^*}_S = \begin{cases} 1 - \hat{f}_S & S = \emptyset \text{ or } S = [n], \\ -\hat{f}_S & \text{otherwise.} \end{cases}$

(iii) For $f \in \mathcal{F}_{n_1}$, $g \in \mathcal{F}_{n_2}$, we have $\widehat{(f \otimes g)}_{S \oplus T} = \hat{f}_S \hat{g}_T$, $S \subseteq [n_1]$, $T \subseteq [n_2]$.

Proof. All statements follow easily from Lemma 15 and the uniqueness part in Theorem 4.

□

B Affine subspaces

Let V be a finite dimensional real vector space. A subset $A \subseteq V$ is an affine subspace in V if for any choice of $a_1, \dots, a_k \in A$ and $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ such that $\sum_i \alpha_i = 1$, we have $\sum_i \alpha_i a_i \in A$. It is clear that $A = \emptyset$ is trivially an affine subspace. Moreover, any linear subspace in V is an affine subspace, and an affine subspace A is linear if and only if $0 \in A$. If $A \neq \emptyset$ and also $0 \notin A$, we say that A is proper.

A proper affine subspace $A \subseteq V$ can be determined in two ways. Let

$$\mathrm{Lin}(A) := \{a_1 - a_2,\ a_1, a_2 \in A\}.$$

It is easily verified that $\mathrm{Lin}(A)$ is a linear subspace, moreover, for any $a \in A$, we have

$$\mathrm{Lin}(A) = \{a_1 - a,\ a_1 \in A\}, \qquad A = a + \mathrm{Lin}(A). \tag{14}$$

We put $\dim(A) := \dim(\mathrm{Lin}(A))$, the dimension of A.

Let V^* be the vector space dual of V and let $\langle \cdot, \cdot \rangle$ be the duality. For a subset $C \subseteq V$, put

$$\tilde{C} := \{v^* \in V^*,\ \langle v^*, a \rangle = 1,\ \forall a \in A\}.$$

Let $\tilde{a} \in \tilde{A}$ be any element and let $\mathrm{Span}(A)$ be the linear span of A in V. We then have

$$A = \mathrm{Span}(A) \cap \{\tilde{a}\}^{\sim}, \tag{15}$$

independently of $\tilde{a}$. The relation between the two expressions for A, given by (14) and (15) is obtained as

$$\mathrm{Span}(A) = \mathrm{Lin}(A) + \mathbb{R}\{a\}, \qquad \mathrm{Lin}(A) = \mathrm{Span}(A) \cap \{\tilde{a}\}^{\perp}, \tag{16}$$

independently of $a \in A$ or $\tilde{a} \in \tilde{A}$. Here $+$ denotes the direct sum of the vector spaces and $C^{\perp}$ denotes the annihilator of a set C. The following lemma is easily proven.

Lemma 16. *Let $C \subseteq V$ be any subset. Then $\tilde{C}$ is an affine subspace in V^* and we have*

$$0 \in \tilde{C} \iff C = \emptyset, \qquad \tilde{C} = \emptyset \iff 0 \in \mathrm{Aff}(C).$$

Assume $C \neq \emptyset$ and $0 \notin \mathrm{Aff}(C)$. Then

(i) *$\tilde{C}$ is proper and we have* $\mathrm{Lin}(\tilde{C}) = C^{\perp} = \mathrm{Span}(C)^{\perp}$,

(ii) $\mathrm{Aff}(C) = \tilde{\tilde{C}}$ *and for any $c_0 \in C$, we have*

$$\mathrm{Lin}(C) := \mathrm{Span}\{c_1 - c_2,\ c_1, c_2 \in C\} = \mathrm{Span}\{c - c_0,\ c \in C\} = \mathrm{Lin}(\tilde{\tilde{C}}).$$

Corollary 4. *Let $A \subseteq V$ be a proper affine subspace. Then*

(i) *$\tilde{A}$ is a proper affine subspace in V^* and $\tilde{\tilde{A}} = A$.*

(ii) $\mathrm{Lin}(\tilde{A}) = \mathrm{Span}(A)^{\perp}$, $\mathrm{Span}(\tilde{A}) = \mathrm{Lin}(A)^{\perp}$.

(iii) $\dim(\tilde{A}) = \dim(V) - \dim(A) - 1$.

The proper affine subspace $\tilde{A}$ in the above Corollary will be called the affine dual of A. Note that the dual depends on the choice of the ambient vector space V.