Monotonicity of $\alpha \mapsto D_{\alpha,z}$

Assume that $\psi,\varphi\in\mathcal{M}_*^+$ are faithful. For each $\eta\in[0,1]$ consider the embedding

$$x\in\mathcal{M}\mapsto h_\psi^\eta x h_\varphi^{1-\eta}\in L^1(\mathcal{M})\ (\cong\mathcal{M}_*).$$

Let $p\in(1,\infty)$ and $1/p+1/q=1$. In [10] Kosaki introduced the interpolation L^p-space (with respect to ψ,φ) as the complex interpolation space

$$C_{1/p}(h_\psi^\eta\mathcal{M}h_\varphi^{1-\eta},L^1(\mathcal{M})),$$

and proved that it is exactly $h_\psi^{\eta/q}L^p(\mathcal{M})h_\varphi^{(1-\eta)/q}$ ($\subset L^1(\mathcal{M})$) with the norm

$$\|h_\psi^{\eta/q}ah_\varphi^{(1-\eta)/q}\|_{p,\psi,\varphi,\eta}=\|a\|_p,\qquad a\in L^p(\mathcal{M}).$$

In particular, when $\eta=0,1$, the "left" and the "right" interpolation L^p-spaces are given as

$$\begin{aligned}L^p(\mathcal{M};\varphi)_L&:=C_{1/p}(\mathcal{M}h_\varphi,L^1(\mathcal{M}))=L^p(\mathcal{M})h_\varphi^{1/q},\\ L^p(\mathcal{M};\psi)_R&:=C_{1/p}(h_\psi\mathcal{M},L^1(\mathcal{M}))=h_\psi^{1/q}L^p(\mathcal{M}).\end{aligned}$$

The non-commutative Stein–Weiss interpolation theorem proved in [10, Theorem 11.1] says that, for each $0<\eta<1$ and $1<p<\infty$,

$$C_{1/p}(h_\psi^\eta\mathcal{M}h_\varphi^{1-\eta},L^1(\mathcal{M}))=C_\eta(L^p(\mathcal{M};\varphi)_L,L^p(\mathcal{M};\psi)_R),\tag{0.1}$$

Now, let $\alpha>1$ and $z\ge\alpha/2$. Set

$$p:=2z,\qquad q:=\frac{2z}{2z-1},\qquad \eta:=\frac{2z-\alpha}{2z-1};\tag{0.2}$$

then $1<p<\infty$, $1/p+1/q=1$ and $0\le\eta<1$. Assume that $Q_{\alpha,z}(\psi\|\varphi)<\infty$ so that there exists a (unique) $y\in L^{2z}(\mathcal{M})$ such that $h_\psi^{\alpha/2z}=yh_\varphi^{(\alpha-1)/2z}$. Since

$$h_\psi=h_\psi^{\frac{2z-\alpha}{2z}}yh_\varphi^{\frac{\alpha-1}{2z}}=h_\psi^{\eta/q}yh_\varphi^{(1-\eta)/q}\in h_\psi^{\eta/q}L^p(\mathcal{M})h_\varphi^{(1-\eta)/q},$$

we see that h_ψ belongs to $C_{1/p}(h_\psi^\eta\mathcal{M}h_\varphi^{1-\eta},L^1(\mathcal{M}))$ and hence

$$Q_{\alpha,z}(\psi\|\varphi)=\|y\|_{2z}^{2z}=\|h_\psi\|_{p,\psi,\varphi,\eta}^p,\tag{0.3}$$

as observed by Jenčová in [6, Sec. 2.3].

Proposition 0.1. *Let $1/2<z\le1$ and $\psi,\varphi\in\mathcal{M}_*^+$ with $\psi\ne0$. Then we have:*

1. *(1) The function $\alpha\mapsto\log Q_{\alpha,z}(\psi\|\varphi)\in(-\infty,\infty]$ is convex on $(1,2z]$.*
2. *(2) The function $\alpha\mapsto D_{\alpha,z}(\psi\|\varphi)$ is monotone increasing on $(1,2z]$.*
3. *(3) If $Q_{\alpha,z}(\psi\|\varphi)<\infty$ for some $\alpha\in(1,2z]$, then $\lim_{\alpha\searrow1}Q_{\alpha,z}(\psi\|\varphi)=\psi(1)$.*

Proof. (1) Set p, q, η be as in (0.2) and $r := \frac{z}{1-z}$; then $1/p + 1/q = 1$, $1/r + 1/q = 1/p$ and $1/r + 2/q = 1$. Let $\alpha_1, \alpha_2 \in (1, 2z]$ and $\eta_k := \frac{2z-\alpha_k}{2z-1} \in [0,1)$, $k = 1, 2$. For each $\theta \in (0,1)$ set $\alpha := (1-\theta)\alpha_1 + \theta\alpha_2$; then $\eta = (1-\theta)\eta_1 + \theta\eta_2$. To show (i), it suffices to prove that

$$Q_{\alpha,z}(\psi\|\varphi) \le Q_{\alpha_1,z}(\psi\|\varphi)^{1-\theta} Q_{\alpha_2,z}(\psi\|\varphi)^{\theta}. \tag{0.4}$$

To do so, we may and do assume that $Q_{\alpha_k,z}(\psi\|\varphi) < \infty$, $k = 1, 2$. Assume first that ψ, φ are faithful. Then the above observation shows that h_ψ belongs to $C_{1/p}(h_\psi^{\eta_k}\mathcal{M}h_\varphi^{1-\eta_k}, L^1(\mathcal{M}))$, $k = 1, 2$. Hence by (0.1) we have

$$h_\psi \in C_{\eta_k}(L^p(\mathcal{M};\varphi)_L, L^p(\mathcal{M};\psi)_R), \qquad k = 1, 2, \tag{0.5}$$

where the above RHS is $L^p(\mathcal{M};\varphi)_L$ when $\eta_k = 0$.

Now, note that

$$\mathcal{A} := h_\psi^{1/q} L^r(\mathcal{M}) h_\varphi^{1/q} \subset L^p(\mathcal{M}) h_\varphi^{1/q} \cap h_\psi^{1/q} L^p(\mathcal{M}) \ (\subset L^1(\mathcal{M})),$$

and $\mathcal{A}$ is dense both in $L^p(\mathcal{M};\varphi)_L = L^p(\mathcal{M})h_\varphi^{1/q}$ and $L^p(\mathcal{M};\psi)_R = h_\psi^{1/q}L^p(\mathcal{M})$, as immediately seen since $h_\psi^{1/q}L^r(\mathcal{M})$ and $L^r(\mathcal{M})h_\varphi^{1/q}$ are dense in $L^p(\mathcal{M})$. Since $L^p(\mathcal{M};\varphi)_L$ and $L^p(\mathcal{M};\psi)_R$ are reflexive Banach spaces, it follows from the reiteration theorem (see [1, Theorems 4.6.1 and 4.3.1] and [11, pp. 60–61, Remark 1]) that

$$\begin{aligned} h_\psi^{\eta/q} L^p(\mathcal{M}) h_\varphi^{(1-\eta)/q} &= C_\eta(L^p(\mathcal{M};\varphi)_L, L^p(\mathcal{M};\psi)_R) \\ &= C_\theta(C_{\eta_1}(L^p(\mathcal{M};\varphi)_L, L^p(\mathcal{M};\psi)_R), C_{\eta_2}(L^p(\mathcal{M};\varphi)_L, L^p(\mathcal{M};\psi)_R)). \end{aligned}$$

[It seems that the proof of [1, Theorem 4.6.1] with [1, Theorem 4.3.1] gives the result, though my understanding is not complete. The result is also mentioned in [11, pp. 60–61, Remark 1] without the proof.] From this and (0.5) we have $h_\psi \in h_\psi^{\eta/q} L^p(\mathcal{M}) h_\varphi^{(1-\eta)/q}$ and

$$\|h_\psi\|_{p,\psi,\varphi,\eta} \le \|h_\psi\|_{p,\psi,\varphi,\eta_1}^{1-\theta} \|h_\psi\|_{p,\psi,\varphi,\eta_2}^{\theta}.$$

Indeed, this is a special case of the Riesz–Thorin theorem applied to the map $T(z) := zh_\psi$, $z \in \mathbb{C}$. Therefore, by (0.3), $Q_{\alpha,z}(\psi\|\varphi) < \infty$ and (0.4) is obtained, in the case when ψ, φ are faithful.

Next, let ψ, φ be general. Since $s(\psi) \le s(\varphi)$ holds due to the assumption $D_{\alpha_k,z}(\psi\|\varphi) < \infty$, it suffices to assume that φ is faithful. Then from the above case it follows that

$$Q_{\alpha,z}((1-\varepsilon)\psi + \varepsilon\varphi\|\varphi) \le Q_{\alpha_1,z}((1-\varepsilon)\psi + \varepsilon\varphi\|\varphi)^{1-\theta} Q_{\alpha_2,z}((1-\varepsilon)\psi + \varepsilon\varphi\|\varphi)^{\theta} \tag{0.6}$$

for all $\varepsilon \in (0,1)$. Since $(\psi,\varphi) \mapsto Q_{\alpha,z}(\psi\|\varphi)$ is jointly convex when $1/2 < \alpha \le 1$ and $1 < \alpha \le 2z$ ([8, 9, 7]), one has

$$\begin{aligned} Q_{\alpha,z}((1-\varepsilon)\psi + \varepsilon\varphi\|\varphi) &\le (1-\varepsilon)Q_{\alpha,z}(\psi\|\varphi) + \varepsilon Q_{\alpha,z}(\varphi\|\varphi) \\ &= (1-\varepsilon)Q_{\alpha,z}(\psi\|\varphi) + \varepsilon\varphi(1). \end{aligned} \tag{0.7}$$

By the lower semi-continuity of $(\psi,\varphi) \mapsto Q_{\alpha,z}(\psi\|\varphi)$ ([8, Theorem 2 (iv)]), one moreover has

$$Q_{\alpha,z}(\psi\|\varphi) \le \liminf_{\varepsilon \searrow 0} Q_{\alpha,z}((1-\varepsilon)\psi + \varepsilon\varphi\|\varphi). \tag{0.8}$$

From (0.7) and (0.8) it follows that

$$Q_{\alpha,z}(\psi\|\varphi) = \lim_{\varepsilon\searrow 0} Q_{\alpha,z}((1-\varepsilon)\psi + \varepsilon\varphi\|\varphi)$$

and similarly

$$Q_{\alpha_k,z}(\psi\|\varphi) = \lim_{\varepsilon\searrow 0} Q_{\alpha_k,z}((1-\varepsilon)\psi + \varepsilon\varphi\|\varphi), \qquad k = 1.2.$$

Hence (0.4) follows by taking the limit of (0.6) as $\varepsilon \searrow 0$.

(2) Since $D_{\alpha,z}(\psi\|\varphi) = \lim_{\varepsilon\searrow 0} D_{\alpha,z}(\psi\|\varphi + \varepsilon\psi)$ by [8, Theorem 2 (v)], we may assume that $\psi \le \lambda\varphi$ for some $\lambda > 0$. Set $f(\alpha) := \log Q_{\alpha,z}(\psi\|\varphi)$, that is a $(-\infty,\infty]$-valued convex function on $(1,2z]$ by (1). By assumption $\psi \le \lambda\varphi$, $[D\psi : D\varphi]_t$ extends a strongly continuous ($\mathcal{M}$-valued) function $[D\psi : D\varphi]_z$ on $-1/2 \le \operatorname{Im} z \le 0$. According to the argument in [4, (0.5)] we have

$$Q_{\alpha,z}(\psi\|\varphi) = \big\|h_\psi^{1/2z}[D\psi : D\varphi]_{-ip}\big\|_{2z}^{2z}, \quad \text{where } p := \tfrac{\alpha-1}{2z} \in (0,1/2).$$

Since $h_\psi^\delta \le \lambda^\delta h_\varphi^\delta$ for any $\delta \in (0,1)$ (see [3, Lemma B.7], [5, Lemma 3.2]), we note (see [2, Lemma A.1]) that

$$\|[D\psi : D\varphi]_{-i\delta/2}\| \le \lambda^{\delta/2}, \qquad \delta \in (0,1).$$

Hence we have

$$Q_{\alpha,z}(\psi\|\varphi) = \operatorname{tr}\big(h_\psi^{1/2z}[D\psi : D\varphi]_{-ip}[D\psi : D\varphi]_{-ip}^* h_\psi^{1/2z}\big)^z \le \lambda^{2pz}\operatorname{tr} h_\psi = \lambda^{\alpha-1}\psi(1),$$

so that

$$f(\alpha) \le \log\psi(1) + (\alpha-1)\log\lambda, \qquad \alpha \in (1,2z]. \tag{0.9}$$

On the other hand, by [8, Theorem 2 (vii)],

$$f(\alpha) \ge \log(\psi(1)^\alpha\varphi(1)^{1-\alpha}) = \alpha\log\psi(1) + (1-\alpha)\log\varphi(1), \qquad \alpha \in (1,2z]. \tag{0.10}$$

Combining (0.9) and (0.10) gives $f(1^+) := \lim_{\alpha\searrow 1} f(\alpha)$ exists and $f(1^+) = \log\psi(1)$. Therefore,

$$D_{\alpha,z}(\psi\|\varphi) = \frac{f(\alpha) - f(1^+)}{\alpha - 1}, \qquad \alpha \in (1,2z],$$

and the assertion follows from the convexity of f on $(1,2z]$.

(3) Let $f(\alpha)$ be as in the proof of (2). By (1) note that $f(1^+) := \lim_{\alpha\searrow 1} f(\alpha)$ exists in $(-\infty,\infty]$. Assume that $\lim_{\alpha\searrow 1} Q_{\alpha,z}(\psi\|\varphi) = \psi(1)$ does not hold; then $f(1^+) \ne \log\psi(1)$. Since (0.10) holds for general ψ,φ, we must have $f(1^+) > \log\psi(1)$ so that

$$D_{\alpha,z}(\psi\|\varphi) = \frac{f(\alpha) - \log f(1)}{\alpha - 1} \to \infty \quad \text{as } \alpha \searrow 1.$$

By (2) this implies that $D_{\alpha,z}(\psi\|\varphi) = \infty$, i.e., $Q_{\alpha,z}(\psi\|\varphi) = \infty$ for all $\alpha \in (1,2z]$. Hence (3) is shown. $\square$

Remark 0.2. To the best of my knowledge, the monotone increasing of $D_{\alpha,z}$ such as (2) of the proposition is new even in the finite-dimensional case.

Remark 0.3. The proof of the proposition is based on Jenčová's observation, where $z > 1/2$ and $1 < \alpha \le 2z$ seems essential. The condition $z \le 1$ is used only to show in the proof of (1) that $\mathcal{A}$ is dense both in $L^p(\mathcal{M};\varphi)_L = L^p(\mathcal{M})h_\varphi^{1/q}$ and $L^p(\mathcal{M};\psi)_R = h_\psi^{1/q}L^p(\mathcal{M})$. We need this to utilize the reiteration theorem for complex interpolation method. Since $\mathcal{A} = L^p(\mathcal{M})h_\varphi^{1/q} = h_\psi^{1/q}L^p(\mathcal{M}) = \mathcal{M}$ if $\mathcal{M}$ is finite-dimensional, we note that the proposition holds for any $z > 1/2$ (without the restriction $z \le 1$) in the finite-dimensional case. It seems natural for the proposition to hold for other range of α, z, for instance, for $\alpha < 1$ and $z \ge 1$.

Remark 0.4. An interesting open question is to prove that if $1/2 < z \le 1$ and $D_{\alpha,z}(\psi\|\varphi) < \infty$ for some $\alpha \in (1, 2z]$, then $\lim_{\alpha\searrow 1} D_{\alpha,z}(\psi\|\varphi) = D_1(\psi\|\varphi)$. If this question is affirmative, it obviously implies (3) of the proposition.

References

[1] J. Bergh and J. Löfström, *Interpolation Spaces: An Introduction*, Springer-Verlag, Berlin, 1976.

[2] F. Hiai, Quantum f-divergences in von Neumann algebras I. Standard f-divergences, *J. Math. Phys.* **59** (2018), 102202, 27 pp.

[3] F. Hiai, *Quantum f-Divergences in von Neumann Algebras: Reversibility of Quantum Operations*, Mathematical Physics Studies, Springer, Singapore, 2021.

[4] F. Hiai, $\lim_{\alpha\searrow 1} D_{\alpha,z} = D_1$ when $1/2 < z < 1$, Notes (12/27/2023).

[5] F. Hiai and H. Kosaki, Connections of unbounded operators and some related topics: von Neumann algebra case, *Internat. J. Math.* **32** (2021), 2150024, 88 pp.

[6] A. Jenčová, Notes for α-z Rényi divergence, Notes, December 7, 2023.

[7] A. Jenčová, DPI for α-z-Rényi divergence, Notes, Nov. 23, 2023.

[8] S. Kato, On α-z-Rényi divergence in the von Neumann algebra setting, Preprint, 2023.

[9] S. Kato, variational expression for $\alpha > 1$ (short proof), Notes, November 26, 2023.

[10] H. Kosaki, Applications of the complex interpolation method to a von Neumann algebra: non-commutative L^p-spaces, *J. Funct. Anal.* **56** (1984), 29–78.

[11] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland Mathematical Library, Vol. 18, North-Holland, Amsterdam, 1978.