Note on the limit $\alpha \searrow 1$

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1 Conditional expectations and L_p-spaces

Let $\mathcal{N}$ be a von Neumann algebra and let $\mathcal{M} \subseteq \mathcal{N}$ be a von Neumann subalgebra such that there is a conditional expectation $\mathcal{E}$ onto $\mathcal{M}$ preserving a faithful normal state ϕ. Then the modular group σ^ϕ of ϕ preserves $\mathcal{M}$ by the Takesaki theorem and we have $\sigma^{\phi|_\mathcal{M}} = \sigma^\phi|_\mathcal{M}$. It follows that the crossed product $\mathcal{M} \rtimes_{\sigma^{\phi|_\mathcal{M}}} \mathbb{R}$ can be identified with a subalgebra in $\mathcal{N} \rtimes_{\sigma^\phi} \mathbb{R}$. By [1, Thm. 4.1], the map

$$\hat{\mathcal{E}} := (\mathcal{E} \otimes id_{B(L_2(G))})|_{\mathcal{N} \rtimes_{\sigma^\phi} \mathbb{R}}$$

is a faithful normal conditional expectation of $\mathcal{N} \rtimes_{\sigma^\phi} \mathbb{R}$ onto $\mathcal{M} \rtimes_{\sigma^{\phi|_\mathcal{M}}} \mathbb{R}$, moreover, we have

$$\hat{\sigma} \circ \hat{\mathcal{E}} = \hat{\mathcal{E}} \circ \hat{\sigma}$$

and clearly also

$$\hat{\mathcal{E}}(\pi(x)\lambda(s)) = \pi(\mathcal{E}(x))\lambda(s), \qquad x\mathcal{M},\ s \in \mathbb{R}.$$

For the dual weight $\hat{\phi}$, we obtain

$$\hat{\mathcal{E}} \circ \sigma^{\hat{\phi}} = \sigma^{\hat{\phi}} \circ \hat{\mathcal{E}}$$

and $\hat{\phi} = \hat{\phi} \circ \hat{\mathcal{E}}$. Clearly, the dual weight for $\mathcal{M}$ is the restriction of $\hat{\phi}$. Let τ be the canonical trace and let us denote the canonical trace for $\mathcal{M}$ by $\tau_\mathcal{M}$, then we have by [6, Cor. 4.22]

$$[D\hat{\phi} \circ \hat{\mathcal{E}} : \tau_\mathcal{M} \circ \hat{\mathcal{E}}]_t = [D\hat{\phi}|_{\mathcal{M} \rtimes_{\sigma^{\phi|_\mathcal{M}}} \mathbb{R}} : D\tau_\mathcal{M}]_t = \lambda(t) = [D\hat{\phi} \circ \hat{\mathcal{E}} : D\tau]_t,$$

it follows that $\tau = \tau_\mathcal{M} \circ \hat{\mathcal{E}} = \tau \circ \hat{\mathcal{E}}$. Consequently, we can see that the space of $\tau_\mathcal{M}$-measurable elements $L_0(\mathcal{M})$ can be identified with a *-subalgebra in $L_0(\mathcal{N})$ and therefore also $L_p(\mathcal{M}) \subseteq L_p(\mathcal{N})$, $0 < p \leq \infty$. In particular, for $p = 1$ we obtain the identification $\mathcal{M}_* \subseteq \mathcal{N}_*$, given as

$$\omega \equiv \omega \circ \mathcal{E}, \qquad \omega \in \mathcal{M}_*.$$

By [5, Prop. 2.3], for $1 \leq p \leq \infty$, $\mathcal{E}$ can be extended to a contractive projection $\mathcal{E}_p$ of $L_p(\mathcal{N})$ onto $L_p(\mathcal{M})$. We have

$$\mathcal{E}_1(h_\omega) = h_{\omega\circ\mathcal{E}}, \qquad h_\omega \in L_1(\mathcal{N})$$

and $\mathcal{E}_q$ is the adjoint of $\mathcal{E}_p$ for $1/p + 1/q = 1$. The index p is often dropped, so we just write $\mathcal{E}$ instead of $\mathcal{E}_p$. For $1 \leq p, q, r \leq \infty$ such that $1/p + 1/q + 1/r \leq 1$, we have

$$\mathcal{E}(hxk) = h\mathcal{E}(x)k, \qquad h \in L_p(\mathcal{M}),\ k \in L_q(\mathcal{M}),\ x \in L_r(\mathcal{N}). \tag{1}$$

Lemma 1. *In the above situation, let $\psi, \varphi \in \mathcal{M}_*^+$ and let $\tilde{\psi} = \psi \circ \mathcal{E}$, $\tilde{\varphi} = \varphi \circ \mathcal{E}$. Then for $1/2 < \alpha/2 \le z$ we have*

$$D_{\alpha,z}(\psi\|\varphi) = D_{\alpha,z}(\tilde{\psi}\|\tilde{\varphi}).$$

Proof. Using the above identifications, we see that $h_\psi = h_{\tilde{\psi}}$, $h_\varphi = h_{\tilde{\varphi}}$. Assume that $D_{\alpha,z}(\psi\|\varphi) < \infty$, then there is some $y \in L_{2z}(\mathcal{M})s(\varphi)$ such that

$$h_\psi^{\frac{\alpha}{2z}} = y h_\varphi^{\frac{\alpha-1}{2}z}. \tag{2}$$

Since $L_{2z}(\mathcal{M}) \subseteq L_{2z}(\mathcal{N})$ and $s(\varphi) = s(\tilde{\varphi})$, we see that $y \in L_{2z}(\mathcal{N})s(\tilde{\varphi})$, so that

$$Q_{\alpha,z}(\tilde{\psi}\|\tilde{\varphi}) = \|y\|_{2z}^{2z} = Q_{\alpha,z}(\psi\|\varphi).$$

This implies that $D_{\alpha,z}(\tilde{\psi}\|\tilde{\varphi}) \le D_{\alpha,z}(\psi\|\varphi)$ in general. Assume next that $1/2 < \alpha/2 \le z$ and let $D_{\alpha,z}(\tilde{\psi}\|\tilde{\varphi}) < \infty$, so that (2) is satisfied with some $y \in L_{2z}(\mathcal{N})s(\tilde{\varphi})$. Using the assumption on α, z and (1), we have

$$h_\psi^{\frac{\alpha}{2z}} = \mathcal{E}(h_\psi^{\frac{\alpha}{2z}}) = \mathcal{E}(y) h_\varphi^{\frac{\alpha-1}{2z}}.$$

By uniqueness of y and the fact that $s(\tilde{\varphi}) = s(\varphi) \in \mathcal{M}$, we obtain $y = \mathcal{E}(y) \in L_{2z}(\mathcal{M})s(\varphi)$. This finishes the proof.

□

2 The limit $\alpha \searrow 1$

Haagerup reduction theorem [1, Thm. 2.1] says that there is a von Neumann algebra $\mathcal{R}$ with a faithful normal state ϕ and a sequence of von Neumann algebras $(\mathcal{R}_n)_{n\ge1}$ such that

(i) $\mathcal{M} \subseteq \mathcal{R}$ and there is a conditional expectation $\mathcal{E}$ on $\mathcal{R}$ onto $\mathcal{M}$ such that $\phi \circ \mathcal{E} = \phi$,

(ii) $\mathcal{R}_n \subseteq \mathcal{R}_{n+1}$ and each $\mathcal{R}_n$ is finite,

(iii) $\bigcup_n \mathcal{R}_n$ is w*-dense in $\mathcal{R}$,

(iv) for each n there is a conditional expectation on $\mathcal{R}$ onto $\mathcal{R}_n$ such that $\phi \circ \mathcal{E}_n = \phi$.

For any $\psi \in \mathcal{M}_*^+$, let us denote $\hat{\psi} := \psi \circ \mathcal{E}$ and $\psi_n := \hat{\psi} \circ \mathcal{E}_n$. Then $\psi_n \to \hat{\psi}$ in norm. By DPI and martingale convergence (or DPI + LS), we have

$$D_{\alpha,z}(\psi\|\varphi) = D_{\alpha,z}(\hat{\psi}\|\hat{\varphi}) = \lim_n D_{\alpha,z}(\psi_n\|\varphi_n), \qquad \text{if } \max\{\frac{\alpha}{2}, \alpha-1\} \le z \le \alpha. \tag{3}$$

Using Lemma 1 and LS, we get

$$D_{\alpha,z}(\psi\|\varphi) = D_{\alpha,z}(\hat{\psi}\|\hat{\varphi}) \le \liminf_n D_{\alpha,z}(\psi_n\|\varphi_n), \qquad \text{if } \frac{\alpha}{2} \le z. \tag{4}$$

Proposition 1. *Let $\max\{\alpha/2, \alpha-1\} \le z \le \alpha$. Then for any $z' \ge z$,*

$$D_{\alpha,z'}(\psi\|\varphi) \le D_{\alpha,z}(\psi\|\varphi).$$

Proof. By [3, Lemma 1.3], we have $D_{\alpha,z'}(\psi_n\|\varphi_n) \le D_{\alpha,z}(\psi_n\|\varphi_n)$ for all n. The statement is proved by using (3) for z and (4) for z'.

□

Corollary 1. *Let $1 < \alpha \leq 2$. Then for any $z \geq 1$, we have*

$$D_{\alpha,z}(\psi\|\varphi) \leq D_{\alpha,1}(\psi\|\varphi).$$

Proof. This follows by putting $z = 1$ and $z' = z$ is Proposition 1. $\square$

The next statement is an extension of [4, Lemma 2].

Lemma 2. *Let $\alpha > 1$ and $z \geq 1$. Then*

$$D_{\beta,1}(\psi\|\varphi) \leq D_{\alpha,z}(\psi\|\varphi),$$

where $\beta := \frac{\alpha+z-1}{z} > 1$.

Proof. Using the scaling property of $D_{\alpha,z}$, we may assume that $\psi(1) = 1$. We will also suppose that $D_{\alpha,z}(\psi\|\varphi) < \infty$, otherwise there is nothing to prove. In this case,

$$h_\psi^{\frac{\alpha}{2z}} = y h_\varphi^{\frac{\alpha-1}{2z}}$$

for some $y \in L_{2z}(\mathcal{M})s(\varphi)$. We then get

$$h_\psi^{\frac{\beta}{2}} = h_\psi^{\frac{z-1}{2z}} h_\psi^{\frac{\alpha}{2z}} = h_\psi^{\frac{z-1}{2z}} y h_\varphi^{\frac{\alpha-1}{2z}} = \eta h_\varphi^{\frac{\beta-1}{2}}$$

with $\eta = h_\psi^{\frac{z-1}{2z}} y \in L_2(\mathcal{M})$. By [2, Thm. 3.6], it follows that

$$Q_{\beta,1}(\psi\|\varphi) = \|\Delta_{\psi,\varphi}^{\frac{\beta}{2}}(h_\varphi^{1/2})\|_2^2 = \|\eta\|_2^2 \leq \|y\|_{2z}^2 = Q_{\alpha,z}(\psi\|\varphi)^{1/z},$$

this implies the statement. $\square$

Corollary 2. *Assume that $D_{\alpha_0,z_0}(\psi\|\varphi) < \infty$ for some $1 < \alpha_0$ and either $z_0 \geq 1$ or $\alpha_0/2 \leq z_0 \leq 1$. Then for any $z > 1/2$ we have*

$$\lim_{\alpha \searrow 1} D_{\alpha,z}(\psi\|\varphi) = D_1(\psi\|\varphi) < \infty.$$

Proof. We first note that under the above assumptions, $D_{\beta,1}(\psi\|\varphi) < \infty$ for some $\beta > 1$. Indeed, this follows from Lemma 2 in the first case, or by [3, Prop. 2.3] in the second case (note that we necessarily have $\alpha_0 - 1 \leq \alpha_0/2 \leq z_0 \leq 1 < \alpha_0$).

For $z \geq 1$, the statement now follows by using Lemma 2 and Corollary 1 for α close enough to 1. If $1/2 < z \leq 1$, we may use [3, Prop. 2.3] or Proposition 1 for α close enough to 1. $\square$

References

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