Monotonicity of $\alpha \mapsto D_{\alpha,z}$ (3)

1 Finite-dimensional case

Let $\mathbb{M}_n$ be the $n \times n$ matrix algebra. For each $A \in \mathbb{M}_n$ we write $\lambda(A) = (\lambda_1(A), \dots, \lambda_n(A))$ for the eigenvalues of A in decreasing order (with multiplicities).

Proposition 1.1. *Let $A_j, B_j \in \mathbb{M}_n^+$, $j = 1, 2$, be such that $A_1A_2 = A_2A_1$ and $B_1B_2 = B_2B_1$. Then for every $\theta \in (0, 1)$,*

$$\lambda\big((A_1^\theta A_2^{1-\theta})^{1/2}(B_1^\theta B_2^{1-\theta})(A_1^\theta A_2^{1-\theta})^{1/2}\big) \prec_{\log} \lambda\big(A_1^{1/2}B_1A_1^{1/2}\big)^\theta \lambda\big(A_2^{1/2}B_2A_2^{1/2}\big)^{1-\theta}, \tag{1.1}$$

that is,

$$\prod_{i=1}^k \lambda_i\big((A_1^\theta A_2^{1-\theta})^{1/2}(B_1^\theta B_2^{1-\theta})(A_1^\theta A_2^{1-\theta})^{1/2}\big) \le \prod_{i=1}^k \lambda_i\big(A_1^{1/2}B_1A_1^{1/2}\big)^\theta \lambda_i\big(A_2^{1/2}B_2A_2^{1/2}\big)^{1-\theta}$$

for all $k = 1, \dots, n$ with equality for $k = n$.

Proof. Since equality for $k = n$ is immediate from a simple computation of determinants, it suffices to prove the case $k = 1$, by the familiar technique using antisymmetric tensor powers. Moreover, by continuity we may assume that A_j, B_j are invertible. Then it suffices to prove that if $\|A_1^{1/2}B_1A_1^{1/2}\| = \|A_2^{1/2}B_2A_2^{1/2}\| = 1$ then $\|(A_1^\theta A_2^{1-\theta})^{1/2}(B_1^\theta B_2^{1-\theta})(A_1^\theta A_2^{1-\theta})^{1/2}\| \le 1$, that is, if $B_j \le A_j^{-1}$ for $j = 1, 2$ then $B_1^\theta B_2^{1-\theta} \le (A_1^\theta A_2^{1-\theta})^{-1}$. But this is immediate as $B_1^\theta B_2^{1-\theta} = B_1 \#_{1-\theta} B_2 \le A_1^{-1} \#_{1-\theta} A_2^{-1} = (A_1^\theta A_2^{1-\theta})^{-1}$. □

Corollary 1.2. *Let A_j, B_j, θ be as in Proposition 1.1. Then for every $z > 0$ and $k = 1, \dots, n$,*

$$\begin{aligned} &\log \sum_{i=1}^k \big(\lambda_i\big((A_1^\theta A_2^{1-\theta})^{1/2}(B_1^\theta B_2^{1-\theta})(A_1^\theta A_2^{1-\theta})^{1/2}\big)\big)^z \\ &\qquad \le \theta \log \sum_{i=1}^k \big(\lambda_i\big(A_1^{1/2}B_1A_1^{1/2}\big)\big)^z + (1-\theta) \log \sum_{i=1}^k \big(\lambda_i\big(A_2^{1/2}B_2A_2^{1/2}\big)\big)^z. \end{aligned} \tag{1.2}$$

In particular,

$$\begin{aligned} &\log \mathrm{Tr}\,\big((A_1^\theta A_2^{1-\theta})^{1/2}(B_1^\theta B_2^{1-\theta})(A_1^\theta A_2^{1-\theta})^{1/2}\big)^z \\ &\qquad \le \theta \log \mathrm{Tr}\,\big(A_1^{1/2}B_1A_1^{1/2}\big)^z + (1-\theta) \log \mathrm{Tr}\,\big(A_2^{1/2}B_2A_2^{1/2}\big)^z. \end{aligned} \tag{1.3}$$

Proof. Since log-majorization $\prec_{\log}$ implies weak majorization $\prec_w$ (see [2, Prop. 4.1.6]), it follows from (1.1) that

$$\begin{aligned} &\sum_{i=1}^k \lambda_i\big((A_1^\theta A_2^{1-\theta})^{1/2}(B_1^\theta B_2^{1-\theta})(A_1^\theta A_2^{1-\theta})^{1/2}\big)^z \\ &\qquad \le \sum_{i=1}^k \lambda_i\big(A_1^{1/2}B_1A_1^{1/2}\big)^{z\theta} \lambda_i\big(A_2^{1/2}B_2A_2^{1/2}\big)^{z(1-\theta)} \end{aligned}$$

$$\le \left[\sum_{i=1}^{k} \lambda_i \big(A_1^{1/2} B_1 A_1^{1/2} \big)^z \right]^{\theta} \left[\sum_{i=1}^{k} \lambda_i \big(A_2^{1/2} B_2 A_2^{1/2} \big)^z \right]^{1-\theta},$$

which gives (1.2). $\square$

Proposition 1.3. *For every $\rho, \sigma \in \mathbb{M}_n^+$ with $\rho \neq 0$ and for evert $z > 0$, the function $\alpha \mapsto D_{\alpha,z}(\rho\|\sigma)$ is monotone increasing on $(0, \infty)$, where $D_{1,z}(\rho\|\sigma) := D_1(\rho\|\sigma) = D(\rho\|\sigma)/\operatorname{Tr}\rho$.*

Proof. Since $\lim_{\alpha\to1} D_{\alpha,z}(\rho\|\sigma) = D_1(\rho\|\sigma)$ for any $z > 0$ [6, Prop. III.36], it suffices to show the monotone increasing of $\alpha \mapsto D_{\alpha,z}(\rho\|\sigma)$ on $(0, 1)$ and on $(1, \infty)$ separately. First, let $z > 0$, $\alpha_1, \alpha_2 \in (0, 1)$ and $\theta \in (0, 1)$. Applying (1.3) with $A_j := \rho^{\alpha_j/z}$ and $B_j := \sigma^{(1-\alpha_j)/z}$ we have

$$\begin{aligned}
&\log Q_{\theta\alpha_1+(1-\theta)\alpha_2,z}(\rho\|\sigma)\\
&\quad= \log \operatorname{Tr} \Big(\rho^{\frac{\theta\alpha_1+(1-\theta)\alpha_2}{2z}} \sigma^{\frac{1-\theta\alpha_1-(1-\theta)\alpha_2}{z}} \rho^{\frac{\theta\alpha_1+(1-\theta)\alpha_2}{2z}} \Big)^z\\
&\quad= \log \operatorname{Tr} \Big(\Big[\big(\rho^{\frac{\alpha_1}{z}}\big)^{\theta} \big(\rho^{\frac{\alpha_2}{z}}\big)^{1-\theta} \Big]^{1/2} \Big[\big(\sigma^{\frac{1-\alpha_1}{z}}\big)^{\theta} \big(\sigma^{\frac{1-\alpha_2}{z}}\big)^{1-\theta} \Big] \Big[\big(\rho^{\frac{\alpha_1}{z}}\big)^{\theta} \big(\rho^{\frac{\alpha_2}{z}}\big)^{1-\theta} \Big] \Big)^z\\
&\quad\le \theta \log \operatorname{Tr} \big(\rho^{\frac{\alpha_1}{z}} \sigma^{\frac{1-\alpha_1}{z}} \rho^{\frac{\alpha_1}{z}} \big)^z + (1-\theta) \log \operatorname{Tr} \big(\rho^{\frac{\alpha_2}{z}} \sigma^{\frac{1-\alpha_2}{z}} \rho^{\frac{\alpha_2}{z}} \big)^z\\
&\quad= \theta \log Q_{\alpha_1,z}(\rho\|\sigma) + (1-\theta) \log Q_{\alpha_2,z}(\rho\|\sigma).
\end{aligned}$$

This implies that $\alpha \mapsto \log Q_{\alpha,z}(\rho\|\sigma)$ is convex on $(0, 1)$. Here we may assume that $s(\rho) \not\perp s(\sigma)$, since otherwise $D_{\alpha,z}(\rho\|\sigma) = \infty$ for all $\alpha > 0$. Then $Q_{\alpha,z}(\rho\|\sigma) > 0$ for all $\alpha \in (0, 1)$ and $\lim_{\alpha\nearrow1} Q_{\alpha,z}(\rho\|\sigma) \le \operatorname{Tr}\rho$. Therefore,

$$\alpha \mapsto D_{\alpha,z}(\rho\|\sigma) = \frac{\log Q_{\alpha,z}(\rho\|\sigma) - \log \operatorname{Tr}\rho}{\alpha - 1}$$

is increasing on $(0, 1)$.

Next, let $\alpha_1, \alpha_2 \in (1, \infty)$ and $\theta \in (0, 1)$. We may assume that $s(\rho) \le s(\sigma)$, since otherwise $D_{\alpha,z}(\rho\|\sigma) = \infty$ for all $\alpha \ge 1$. Apply (1.3) with $A_j := \rho^{\alpha/z}$ and $B_j := (\sigma^{-1})^{\alpha_j - 1}$ (where σ^{-1} is the generalized inverse of σ); then we have

$$\begin{aligned}
&\log Q_{\theta\alpha_1+(1-\theta)\alpha_2,z}(\rho\|\sigma)\\
&\quad= \log \operatorname{Tr} \Big(\rho^{\frac{\theta\alpha_1+(1-\theta)\alpha_2}{2z}} (\sigma^{-1})^{\frac{\theta\alpha_1+(1-\theta)\alpha_2-1}{z}} \rho^{\frac{\theta\alpha_1+(1-\theta)\alpha_2}{2z}} \Big)^z\\
&\quad\le \theta \log \operatorname{Tr} \big(\rho^{\frac{\alpha_1}{z}} (\sigma^{-1})^{\frac{\alpha_1-1}{z}} \rho^{\frac{\alpha_1}{z}} \big)^z + (1-\theta) \log \operatorname{Tr} \big(\rho^{\frac{\alpha_2}{z}} (\sigma^{-1})^{\frac{\alpha_2-1}{z}} \rho^{\frac{\alpha_2}{z}} \big)^z\\
&\quad= \theta \log Q_{\alpha_1,z}(\rho\|\sigma) + (1-\theta) \log Q_{\alpha_2,z}(\rho\|\sigma).
\end{aligned}$$

Since $\lim_{\alpha\searrow1} Q_{\alpha,z}(\rho\|\sigma) = \operatorname{Tr}\rho$, the function $\alpha \mapsto D_{\alpha,z}(\rho\|\sigma)$ is increasing on $(1, \infty)$. $\square$

2 von Neumann algebra case

Let $\mathcal{M}$ be a semi-finite von Neumann algebra with a faithful normal semi-finite trace τ. Let $\widetilde{\mathcal{M}}$ denote the space of τ-measurable operators affiliated with $\mathcal{M}$. For each $a \in \widetilde{\mathcal{M}}$ we write $\mu_t(a)$ for the (tth) generalized s-number of a (see [1]). Below we consider operators $a \in \widetilde{\mathcal{M}}$ satisfying

$$(*) \quad a \in \mathcal{M} \quad \text{or} \quad \mu_t(a) \le C t^{-\gamma} \ \ (t > 0) \ \ \text{for some } C, \gamma > 0.$$

For each $a \in \widetilde{\mathcal{M}}$ with $(*)$ we define [1]

$$\Lambda_t(a) := \exp \int_0^t \log \mu_s(a)\, ds, \qquad t > 0.$$

Note that $\Lambda_t(a)$, $t > 0$, are well defined in $[0, \infty)$ whenever a satisfies $(*)$.

Lemma 2.1. *If $a, b \in \widetilde{\mathcal{M}}$ satisfies $(*)$, then $|a|^p$ $(p > 0)$ and ab satisfy $(*)$ too.*

Proof. Easy since $\mu_t(ab) \le \|a\|\mu_t(b)$ if $a \in \mathcal{M}$, $\mu_t(|a|^p) = \mu_t(a)^p$, and $\mu_t(ab) \le \mu_{t/2}(a)\mu_{t/2}(b)$ (see [1, Lemma 2.5]). $\square$

Proposition 2.2. *Let $a_j, b_j \in \widetilde{\mathcal{M}}_+$, $j = 1, 2$, be such that a_j, b_j satisfy $(*)$ and $a_1a_2 = a_2a_1$, $b_1b_2 = b_2b_1$. Then for every $\theta \in (0, 1)$ and any $t > 0$,*

$$\Lambda_t\big((a_1^\theta a_2^{1-\theta})^{1/2}(b_1^\theta b_2^{1-\theta})(a_1^\theta a_2^{1-\theta})^{1/2}\big) \le \Lambda_t(a_1^\theta b_1^\theta)\Lambda_t(a_2^{1-\theta}b_2^{1-\theta}). \tag{2.1}$$

In particular,

$$\Lambda_t\big((a_1^{1/2}a_2^{1/2})^{1/2}(b_1^{1/2}b_2^{1/2})(a_1^{1/2}a_2^{1/2})^{1/2}\big) \le \Lambda_t(a_1^{1/2}b_1a_1^{1/2})^{1/2}\Lambda_t(a_2^{1/2}b_2a_2^{1/2})^{1/2}. \tag{2.2}$$

Proof. For any $k \in \mathbb{N}$, since

$$\begin{aligned}
&\big((a_1^\theta a_2^{1-\theta})^{1/2}(b_1^\theta b_2^{1-\theta})(a_1^\theta a_2^{1-\theta})^{1/2}\big)^k \\
&\quad = (a_1^\theta a_2^{1-\theta})^{1/2}(b_1^\theta b_2^{1-\theta})(a_1^\theta a_2^{1-\theta})(b_1^\theta b_2^{1-\theta}) \cdots (a_1^\theta a_2^{1-\theta})(b_1^\theta b_2^{1-\theta})(a_1^\theta a_2^{1-\theta})^{1/2} \\
&\quad = (a_1^\theta a_2^{1-\theta})^{1/2}b_2^{1-\theta}(b_1^\theta a_1^\theta)(a_2^{1-\theta}b_2^{1-\theta}) \cdots (b_1^\theta a_1^\theta)(a_2^{1-\theta}b_2^{1-\theta})b_1^\theta(a_1^\theta a_2^{1-\theta})^{1/2},
\end{aligned}$$

we have, by [1, Theorem 4.2] with Lemma 2.1,

$$\begin{aligned}
&\Lambda_t\big((a_1^\theta a_2^{1-\theta})^{1/2}(b_1^\theta b_2^{1-\theta})(a_1^\theta a_2^{1-\theta})^{1/2}\big)^k \\
&\quad \le \Lambda_t\big((a_1^\theta a_2^{1-\theta})^{1/2}b_2^{1-\theta}\big)\Lambda_t(b_1^\theta a_1^\theta)^{k-1}\Lambda_t(a_2^{1-\theta}b_2^{1-\theta})^{k-1}\Lambda_t\big(b_1^\theta(a_1^\theta a_2^{1-\theta})^{1/2}\big)
\end{aligned}$$

so that

$$\begin{aligned}
&\Lambda_t\big((a_1^\theta a_2^{1-\theta})^{1/2}(b_1^\theta b_2^{1-\theta})(a_1^\theta a_2^{1-\theta})^{1/2}\big) \\
&\quad \le \Lambda_t\big((a_1^\theta a_2^{1-\theta})^{1/2}b_2^{1-\theta}\big)^{1/k}\Lambda_t(b_1^\theta a_1^\theta)^{1-\frac{1}{k}}\Lambda_t(a_2^{1-\theta}b_2^{1-\theta})^{1-\frac{1}{k}}\Lambda_t\big(b_1^\theta(a_1^\theta a_2^{1-\theta})^{1/2}\big)^{1/k}.
\end{aligned}$$

Letting $k \to \infty$ gives (2.1). When $\theta = 1/2$, (2.1) is rewritten as (2.2). $\square$

Remark 2.3. In view of Proposition 1.1 what we would like to obtain is

$$\Lambda_t\big((a_1^\theta a_2^{1-\theta})^{1/2}(b_1^\theta b_2^{1-\theta})(a_1^\theta a_2^{1-\theta})^{1/2}\big) \le \Lambda_t\big(a_1^{1/2}b_1a_1^{1/2}\big)^\theta\Lambda_t\big(a_2^{1/2}b_2a_2^{1/2}\big)^{1-\theta}, \qquad t > 0. \tag{2.3}$$

Since $\Lambda_t(a_j^rb_j^r) \le \Lambda_t(a_jb_j)^r$ for any $r \in (0, 1)$ by [5], we have from (2.1)

$$\Lambda_t\big((a_1^\theta a_2^{1-\theta})^{1/2}(b_1^\theta b_2^{1-\theta})(a_1^\theta a_2^{1-\theta})^{1/2}\big) \le \Lambda_t\big(a_1b_1^2a_1\big)^{\frac{\theta}{2}}\Lambda_t\big(a_2b_2^2a_2\big)^{\frac{1-\theta}{2}},$$

which is weaker than (2.3)

From now on, let $\mathcal{M}$ be a general (σ-finite) von Neumann algebra.

Lemma 2.4. *Let $\psi,\varphi\in\mathcal{M}_*^+$ with $\psi\ne 0$, and assume that $s(\psi)\not\perp s(\varphi)$. Then for every $z>0$, $Q_{\alpha,z}(\psi\|\varphi)>0$ for all $\alpha\in(0,1)$, and $\alpha\mapsto Q_{\alpha,z}(\psi\|\varphi)$ is continuous on $(0,1)$.*

Proof. Assume that $Q_{\alpha,z}(\psi\|\varphi)=0$ for some $z>0$ and $\alpha\in(0,1)$. Then $h_\psi^{\alpha/2z}h_\varphi^{(1-\alpha)/2z}=0$ as a τ-measurable operator affiliated with $\mathcal{N}:=\mathcal{M}\rtimes_\sigma\mathbb{R}$, where τ is the canonical trace on $\mathcal{N}$. Since $s(\psi)=s(h_\psi^{\alpha/2z})$ and $s(\varphi)=s(h_\varphi^{(1-\alpha)/2z})$, it is easy to see that $s(\psi)\perp s(\varphi)$. Hence the first assertion follows.

Next, since $p>0\mapsto a^p\in\widetilde{\mathcal{N}}$ is differentiable in the measure topology for any $a\in\widetilde{\mathcal{N}}_+$ (see, e.g., [3, Lemma 9.19]), we see that $\alpha\mapsto h_\psi^{\alpha/2z}h_\varphi^{(1-\alpha)/z}h_\psi^{\alpha/2z}$ is differentiable (hence continuous) on $(0,1)$ in the measure topology. Hence by [3, Lemma 9.14], the function $\alpha\mapsto Q_{\alpha,z}(\psi\|\varphi)=\|h_\psi^{\alpha/2z}h_\varphi^{(1-\alpha)/z}h_\psi^{\alpha/2z}\|_z^z$ is continuous. Here, when $z<1$, note [1, Theorem 4.9(iii)] that $|\,\|a\|_z^z-\|b\|_z^z|\le\|a-b\|_z^z$ for $a,b\in L^z(\mathcal{M})$. ☐

Proposition 2.5. *For every $\psi,\varphi\in\mathcal{M}_*^+$ with $\psi\ne0$ and for any $z>0$, the function $\alpha\mapsto D_{\alpha,z}(\psi\|\varphi)$ is monotone increasing and continuous on $(0,1)$.*

Proof. We may assume that $s(\psi)\not\perp s(\varphi)$; otherwise, $D_{\alpha,z}(\psi\|\varphi)=0$ for all $\alpha\in(0,1)$. Then by Lemma 2.4, $Q_{\alpha,z}(\psi\|\varphi)\in(0,\infty)$ for all $\alpha\in(0,1)$, and $\alpha\mapsto Q_{\alpha,z}(\psi\|\varphi)$ is continuous on $(0,1)$. Hence $\alpha\mapsto D_{\alpha,z}(\psi\|\varphi)$ is continuous on $(0,1)$ too.

Let $\alpha_1,\alpha_2\in(0,1)$ and $z>0$. Consider $\mathcal{N}:=\mathcal{M}\rtimes_\sigma\mathbb{R}$ with the canonical trace τ. Apply (2.2) to $a_j:=h_\psi^{\alpha_j/z}$ and $b_j:=h_\varphi^{(1-\alpha_j)/z}$ in $\widetilde{\mathcal{N}}_+$ with $t=1$; we then have

$$\begin{aligned}&\int_0^1\log\mu_s\Big(h_\psi^{\frac{\alpha_1+\alpha_2}{4z}}h_\varphi^{\frac{2-\alpha_1-\alpha_2}{2z}}h_\psi^{\frac{\alpha_1+\alpha_2}{4z}}\Big)\,ds\\&\quad\le\frac12\bigg[\int_0^1\log\mu_s\Big(h_\psi^{\frac{\alpha_1}{2z}}h_\varphi^{\frac{1-\alpha_1}{z}}h_\psi^{\frac{\alpha_1}{2z}}\Big)\,ds+\int_0^1\log\mu_s\Big(h_\psi^{\frac{\alpha_2}{2z}}h_\varphi^{\frac{1-\alpha_2}{z}}h_\psi^{\frac{\alpha_2}{2z}}\Big)\,ds\bigg].\end{aligned}\tag{2.4}$$

Since $h_\psi^{\frac{\alpha_1+\alpha_2}{4z}}h_\varphi^{\frac{2-\alpha_1-\alpha_2}{2z}}h_\psi^{\frac{\alpha_1+\alpha_2}{4z}}$ is in $L^z(\mathcal{M})$, note [1, Lemma 4.8] that

$$\mu_s\Big(h_\psi^{\frac{\alpha_1+\alpha_2}{4z}}h_\varphi^{\frac{2-\alpha_1-\alpha_2}{2z}}h_\psi^{\frac{\alpha_1+\alpha_2}{4z}}\Big)=s^{-1/z}\Big\|h_\psi^{\frac{\alpha_1+\alpha_2}{4z}}h_\varphi^{\frac{2-\alpha_1-\alpha_2}{2z}}h_\psi^{\frac{\alpha_1+\alpha_2}{4z}}\Big\|_z$$

so that

$$\log\mu_s\Big(h_\psi^{\frac{\alpha_1+\alpha_2}{4z}}h_\varphi^{\frac{2-\alpha_1-\alpha_2}{2z}}h_\psi^{\frac{\alpha_1+\alpha_2}{4z}}\Big)^z=-\log s+\log Q_{\frac{\alpha_1+\alpha_2}{2},z}(\psi\|\varphi).\tag{2.5}$$

Similarly,

$$\log\mu_s\Big(h_\psi^{\frac{\alpha_j}{2z}}h_\varphi^{\frac{1-\alpha_j}{z}}h_\psi^{\frac{\alpha_j}{2z}}\Big)^z=-\log s+\log Q_{\alpha_j,z}(\psi\|\varphi),\qquad j=1,2.\tag{2.6}$$

Multiply z to both sides of (2.4) and insert (2.5) and (2.6) into it. Since $\int_0^1(-\log s)\,ds=1$, we then arrive at

$$1+Q_{\frac{\alpha_1+\alpha_2}{2},z}(\psi\|\varphi)\le\frac12\big[2+\log Q_{\alpha_1,z}(\psi\|\varphi)+\log Q_{\alpha_2,z}(\psi\|\varphi)\big],$$

which implies that $\alpha\mapsto\log Q_{\alpha,z}(\psi\|\varphi)$ is midpoint convex on $(0,1)$. Since midpoint convexity implies convexity for continuous functions, it follows that $\alpha\mapsto\log Q_{\alpha,z}(\psi\|\varphi)$ is convex on $(0,1)$. Moreover, by [4, Theorem 1(vii)] we find that $\lim_{\alpha\nearrow1}Q_{\alpha,z}(\psi\|\varphi)\le\psi(1)$. Therefore, the monotone increasing of $\alpha\mapsto D_{\alpha,z}(\psi\|\varphi)$ on $(0,1)$ follows. ☐

References

[1] T. Fack and H. Kosaki, Generalized s-numbers of σ-measurable operators, *Pacific J. Math.* **123** (1986), 269–300.

[2] F. Hiai, Matrix analysis: matrix monotone functions, matrix means, and majorization, *Interdisciplinary Information Sciences* **16** (2010), 139–248.

[3] F. Hiai, *Lectures on Selected Topics in von Neumann Algebras*, EMS Press, Berlin, 2021.

[4] S. Kato, On α-z-Rényi divergence in the von Neumann algebra setting, Preprint, 2023.

[5] H. Kosaki, An inequality of Araki-Lieb-Thirring (von Neumann algebra case), *Proc. Amer. Math. Soc.* **114** (1992), 477–481.

[6] M. Mosonyi and F. Hiai, Some continuity properties of quantum Rényi divergences, *IEEE Trans. Inform. Theory*, to appear, DOI 10.1109/TIT.2023.3324758.