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Quantum Bayes' rule affirming consistency in measurement inferences in quantum mechanics
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Quantum Bayes' Rule rms Consistency in Measurement Inferences in Quantum Mechanics

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Traditional Bayes' rule lays the foundation for causal reasoning and finding relation between cause (input) and effect (output). This causal reasoning is universally applied to all physical processes to establish causal relations. Here, we show that it does not establish correct causal correspondence between quantum causes and effects in general. In fact, there are instances within the framework of quantum mechanics where the use of traditional Bayes' rule leads to inconsistencies in quantum measurement inferences. We consider two such cases, inspired by Frauchiger-Renner's and Hardy's setups, where traditional Bayes' rule results in paradoxical situations even after assuming quantum mechanics as a non-local theory. As a remedy, we introduce an input-output causal relation using the reasoning based on quantum Bayes' rule. It applies to general quantum processes even when a cause (or effect) is in coherent superposition with other causes (or effects), involves non-local correlations as allowed by quantum mechanics, and in the cases where causes belonging to one system induce effects in some other system as it happens in quantum measurement processes. This enables us to propose a resolution to the contradictions that appear in the context of Frauchiger-Renner's and Hardy's setups. Our results thereby affirm that quantum mechanics, equipped with quantum Bayes' rule, can indeed consistently describe the use of itself.

I. INTRODUCTION

In any process, an observed effect (output) can be attributed to the cause combining the initial state (input) and the evolution it has undergone. In an arbitrary stochastic process, the traditional Bayes' rule (TBR) allows one to deterministically predict an effect for a given cause if the conditional probability (or transition probability encoding the evolution) is unity. Similarly, an observed effect can be used to infer a cause with certainty if the transition probability corresponding to the "inverse" process, also known as the retrodicted conditional probability, is unity. The transition probabilities for the "retrodicted" process are derived using traditional Bayes' rule [1]. In this sense, TBR, in general, provides a logical line of reasoning with which one makes predictions and inferences in a process. Say an event (cause or input) a occurs with probability $P(a)$ and undergoes a stochastic evolution to give rise to another event b (effect or output) with probability $P(b)$. The conditional probability encoding the evolution is given by $P(b|a)$. It means, although a occurs with probability $P(a)$, any occurrence of a can predict the observation of b with the probability $P(b|a)$. In the case with $P(b|a) = 1$, the event a deterministically predicts the occurrence of b through the process. To make an inference, i.e., finding out a cause of observing b, the process has to be inverted, and this is done using the Bayesian retrodiction rule given by

$$P(a|b) = \frac{P(b|a)\, P(a)}{P(b)}.$$

It implies that whenever we observe the event b, we can connect it to the cause a with probability $P(a|b)$. With $P(a|b) = 1$, an observation of b is used to deterministically infer the cause a.

The TBR is applied to a wide range physical theories [1, 2]. While its successful application in classical mechanics is not surprising, as all classical stochastic processes involve probability distributions and conditional probabilities, it is widely used in quantum mechanics, where states and evolution cannot be expressed in terms of probabilities and conditional probabilities in general. For instance, the TBR finds applications in quantum parameter estimation [3–9], state estimation [10–15], process tomography [16, 17], etc. Nevertheless, quantum mechanics is fundamentally different from its classical analog in many aspects. First, the former allows superpositions of orthogonal (perfectly distinguishable) states. It also allows superposition in evolutions [18]. Thus, quantum mechanics allows a superposition of causes, that may represent a different cause, leading to a superposition of effects that can again be seen as a different effect. Second, there are non-deterministic processes, such as quantum measurements, where the system's original state collapses to some other state with a probability. Third, quantum mechanics allows non-local correlations that lead to "spooky action at a distance" [19]. Consequently, for a correlated composite, a measurement made on one subsystem may induce collapse on the other. Can these peculiarities of quantum mechanics be captured by the TBR as the latter, a priory, does not take into account any such quantum features? It is now known that, in certain situations, TBR leads to paradoxical results in quantum mechanics. One prominent example is Frauchiger-Renner's [20] paradox. There, the prediction based on a given past directly contradicts with the inferences made about the same past for observations made at the present. Therefore, revisiting the applicability of TBR in the quantum domain is important and may find fundamental implications in the foundation of quantum mechanics.

Here, we show that the TBR is inadequate to establish a consistent causal correspondence between quantum causes and effects, particularly when quantum entanglement and selective measurements are involved. We introduce a quantum (input-output) causal relation based on quantum Bayes' rule (QBR) applicable to general quantum processes. The QBR is expressed in terms of density matrices and causal condi-

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tional states instead of probabilities and conditional probabilities. With two examples involving quantum entanglement and selective (local) measurements, we demonstrate how TBR and QBR lead to contradictory causal inferences for the same observations or effects and why TBR may not be adequate in general. We also provide conditions for deterministic causal correspondence between quantum causes and effects. The QBR accounts for the situation where a cause is in a superposition of other causes and, similarly, for the effects. Beyond that, it correctly describes the causal correspondence between causes belonging to one quantum system and effects belonging to the others for a global process. We propose a resolution to the contradictions that appear in the case of Frauchiger-Renner's paradox with the help of QBR. Furthermore, we revisit Hardy's setup leading to bipartite nonlocality without a Bell inequality [21, 22] and show that, within the framework of quantum mechanics (a non-local theory), there also appears contradiction in prediction and inferences. Yet again, it is resolved with the help of QBR. Thus our results advocate that, while deriving correct correspondence between cause and effect in quantum mechanics, one must resort to quantum Bayes' rule.

The article is organized as follows. In section II, we briefly discuss quantum conditional states and quantum Bayes' rule and introduce deterministic quantum causal relations. In section III, we consider quantum processes in which the predictions and inferences based on TBR and QBR drastically differ. Sections IV and V revisit Frauchiger-Renner's and Hardy's setups, respectively. We demonstrate that the contradictions that appear there are due to TBR, and these may be resolved by exploiting QBR. Finally, we conclude in section VI.

II. QUANTUM CONDITIONAL STATES, BAYES' RULE, AND CAUSAL RELATIONS

For classical systems, the states are described by the probabilities, and (stochastic) processes are expressed in terms of conditional probabilities. However, for a quantum system, probabilities are not sufficient. One needs to express the states in terms of density matrices which, in addition to probabilities, carry the information about the quantum superposition they may have. For the same reason, conditional probabilities should be upgraded to conditional states capable of encoding information about quantum evolution and possible superpositions in causes and effects. Below we define the conditional states and quantum Bayes' rule and provide conditions for deterministic causal relations for quantum processes.

Consider a quantum evolution by a completely positive trace preserving (CPTP) map $\mathcal{E} : \mathcal{L}(\mathcal{H}_S) \mapsto \mathcal{L}(\mathcal{H}_R)$ where $\mathcal{H}_S$ and $\mathcal{H}_R$ are the corresponding Hilbert spaces of the systems S and R respectively. The quantum causal conditional state, encoding the evolution that causally relates S and R, is then given by [23, 24]

$$\mathcal{P}_{R|S} = \sum_{m,n} |n_S\rangle\langle m_S| \otimes \mathcal{E}(|m_{S'}\rangle\langle n_{S'}|)\,, \tag{1}$$

where $\{|m_S\rangle\}$ and $\{|m_{S'}\rangle\}$ are the complete set of orthonormal bases spanning $\mathcal{H}_S$ and $\mathcal{H}_{S'}$ respectively. Here $\mathcal{H}_{S'}$ is a copy of $\mathcal{H}_S$ and $\mathcal{E} : \mathcal{L}(\mathcal{H}_{S'}) \mapsto \mathcal{L}(\mathcal{H}_R)$ as well. A state transformation $\rho_R = \mathcal{E}(\rho_S)$, where ρ_S and ρ_R are the density operators representing the states of R and S respectively, is equivalently expressed as $\rho_R = \mathrm{Tr}_S\left[\mathcal{P}_{R|S} \star \rho_S\right]$ with $X \star Y = Y^{\frac{1}{2}} X Y^{\frac{1}{2}}$. Now the quantum Bayes' rule [23] can be cast as

$$\mathcal{P}^{\bar{\mathcal{E}}}_{S|R} = \mathcal{P}_{R|S} \star (\rho_S \otimes \rho_R^{-1}), \tag{2}$$

and it satisfies $\rho_S = \mathrm{Tr}_S\left[\mathcal{P}^{\bar{\mathcal{E}}}_{S|R} \star \rho_R\right]$. Note, $\mathcal{P}^{\bar{\mathcal{E}}}_{S|R}$ is the causal conditional state corresponding to Petz recovery channel [25–27] or the retrodicted process

$$\bar{\mathcal{E}}(\cdot) := \rho_S^{\frac{1}{2}}\, \mathcal{E}^{\dagger}\left(\rho_R^{-\frac{1}{2}} (\cdot)\, \rho_R^{-\frac{1}{2}}\right) \rho_S^{\frac{1}{2}}, \tag{3}$$

where $\mathcal{E}^{\dagger}$ is the trace dual of $\mathcal{E}$, satisfying the relation $\mathrm{Tr}\,[Y\,\mathcal{E}(X)] = \mathrm{Tr}\,[\mathcal{E}^{\dagger}(Y)\,X]$ for all operators X and Y. Note in the situations when the quantum state and dynamics can be simulated by probability distribution and classical stochastic dynamics, the Petz recovery map reduces to traditional Bayes' rule [28, 29].

The causal conditional state $\mathcal{P}^{\bar{\mathcal{E}}}_{S|R}$ corresponding to the retrodicted process depends on the reference prior ρ_S. However, while making inferences, this prior is often unknown. Then, there are two possible choices. One choice is to consider a known steady state $\rho_S = \gamma_S$ as prior, satisfying $\mathcal{E}(\gamma_S) = \gamma_S$. Another is the uniform prior $\rho_S = \frac{\mathbb{I}}{d}$. The latter is obviously the viable choice in 'inverting' a process when no prior information is available. We note that without having correct knowledge of the prior reference state, in general, it is not possible to provide an exact retrodicted process. Interestingly, the retrodiction of a deterministic (or unitary) process is independent of the reference prior [30]. For any isometric (or unitary) evolution $U : \mathcal{H}_S \mapsto \mathcal{H}_R$, where $U|m_S\rangle = |m_R\rangle$ for a complete set of orthonormal bases $\{|m_S\rangle\}$, the causal conditional states assume simpler forms

$$\mathcal{P}^{U}_{R|S} = \sum_{m,n} |n_S\rangle\langle m_S| \otimes U|m_{S'}\rangle\langle n_{S'}|U^{\dagger}, \tag{4}$$

$$\mathcal{P}^{U^{\dagger}}_{S|R} = \sum_{m,n} U^{\dagger}|n_{R'}\rangle\langle m_{R'}|U \otimes |m_R\rangle\langle n_R|, \tag{5}$$

where R' is the second copy of R and $\bar{U} = U^{\dagger}$. Note, $\mathcal{P}^{U^{\dagger}}_{S|R}$ represents the evolution $U^{\dagger} : \mathcal{H}_R \mapsto \mathcal{H}_S$, and $\mathcal{P}^{U^{\dagger}}_{S|R} = \mathcal{P}^{U}_{R|S}$.

We can now establish deterministic causal relations between quantum cause and effect. For a general evolution $\mathcal{E} : \mathcal{L}(\mathcal{H}_S) \mapsto \mathcal{L}(\mathcal{H}_R)$, the causal conditional states can be found. Here for the inverse or retrodicted process, we shall exploit uniform prior (or a steady state, whenever it is known). Say, after the process, one observes an effect by selectively measuring R to find σ_R and wants to infer the cause corresponding to it or vice versa. Then the conditions to draw deterministic causal relation using QBR are given in the following definition.

Definition 1 (Deterministic quantum causal relation). *A cause τ_S deterministically predicts the effect τ_R due to the evolution* *if*

$$\tau_R = \mathrm{Tr}_S \left[\mathcal{P}_{R|S} \star \tau_S \right]. \tag{6}$$

In reverse, an observed e ect σ_R after the evolution by in-fers the cause σ_S with certainty if

$$\sigma_S = \mathrm{Tr}_R \left[\bar{\mathcal{P}}_{S|R} \star \sigma_R \right]. \tag{7}$$

Consider the earlier isometry U leading to a (pure) state transformation $|\psi_S\rangle \mapsto |\psi_R\rangle$. An effect $|\phi_R\rangle = \sum_m a_m |m_R\rangle$ observed in the final state $|\psi_R\rangle$, upon a projective measurement using $|\phi_R\rangle\langle\phi_R|$, has one-to-one causal correspondence with the cause $|\phi_S\rangle = \sum_m b_m |m_S\rangle$ if they respect the relations

$$|\phi_R\rangle = U|\phi_S\rangle, \tag{8}$$

$$|\phi_S\rangle = U^\dagger|\phi_R\rangle, \tag{9}$$

which are equivalently the conditions (6) and (7) in Definition 1. Consequently, $|\phi_S\rangle$ and $|\phi_R\rangle$ have deterministic causal correspondence if $a_m = b_m$, $\forall m$.

Now we turn to a situation where two quantum systems are evolved with a known global (i.e., non-local or entangling) evolution, and causes belonging to one system induce effects in the other. In particular, we focus on quantum measurement processes involving a system (S) and an apparatus ( ), in which observations in the latter are used to infer about the former. Consider an evolution by a CPTP map $_M : \mathcal{L}(\mathcal{H}_S \otimes \mathcal{H}\ ) \to \mathcal{L}(\mathcal{H}_R \otimes \mathcal{H}_B)$. Without loss of generality, we assume $\mathcal{H}_R$ and $\mathcal{H}_B$ are the copies of $\mathcal{H}_S$ and $\mathcal{H}\ $, respectively. The composite S is initially in an uncorrelated state, say $\rho_S\ = \rho_S \otimes |0\ \rangle\langle 0\ |$, where $|0\ \rangle$ is a known state of . The composite may become strongly correlated after the global evolution by $_M$. Because of that, a local measurement on B may induce a change in R. Say, a local measurement on B after applying a rank-1 projector results in an observation of a state σ_B, where the overall updated state is σ_{RB} with $\sigma_B = \mathrm{Tr}_R[\sigma_{RB}]$. Then, the effect σ_B in B establishes one-to-one causal correspondence with a cause σ_S in S if they satisfy the conditions involving the causal conditional states $\mathcal{P}^{M}_{RB|S}$ and $\bar{\mathcal{P}}^{M}_{S\ |RB}$ given in the definition below.

Definition 2 (Deterministic quantum causal relation between cause and effect belonging to different systems)**.** *cause τ_S in S deterministically predicts the e ect $\tau_B = \mathrm{Tr}_R[\tau_{RB}]$ in B via the evolution by $_M$ if*

$$\tau_{RB} = \mathrm{Tr}_S \left[\mathcal{P}^{M}_{RB|S} \star \tau_S \otimes |0\ \rangle\langle 0\ | \right], \tag{10}$$

where $\tau_S \equiv \mathrm{Tr}_B[\tau_{RB}]$. In reverse, an observed e ect $\sigma_B = \mathrm{Tr}_R[\sigma_{RB}]$ in B deterministically infers the cause σ_S in S if

$$\sigma_S \otimes |0\ \rangle\langle 0\ | = \mathrm{Tr}_{RB} \left[\bar{\mathcal{P}}^{M}_{S\ |RB} \star \sigma_{RB} \right], \tag{11}$$

where $\sigma_S \equiv \mathrm{Tr}_B[\sigma_{RB}]$.

The condition for deterministic prediction of effects, in Eq. (10), cannot be satisfied for arbitrary input state of S in general. The channel $_M$ leads to the state transformation $\tau_S \otimes |0\ \rangle\langle 0\ | \to\ _M(\tau_S \otimes |0\ \rangle\langle 0\ |) = \tau_{RB}$, where the input state of S does not get disturbed, i.e., $\mathrm{Tr}_B[\tau_{RB}] = \tau_S$. Here $|0\rangle$ is a fixed state. This implies that the reduced channel applied on , $\Phi\ = \mathrm{Tr}_R \circ\ _M : \mathcal{L}(\mathcal{H}\ ) \to \mathcal{L}(\mathcal{H}_B)$ would have to be a replacement channel. This restricts the set of causes to be a mutually orthogonal set of states or their convex mixtures, in general, which will enable us to make deterministic predictions. For deterministic inference, the first step is to guess a prior state to find the correct retrodicted process and the corresponding causal conditional state. Even if one guesses a correct prior state, the reduced channel corresponding to the retrodicted process applied on B would have to be a replacement channel. This again restricts the set of effects to be a mutually orthogonal set of states or their convex mixtures, in general, and then only deterministic inferences can be made. These restrictions may be reasonable as we provide conditions for deterministic predictions and inferences.

At least for unitary processes, the knowledge of the prior state is not required. We now consider a case of quantum measurement where the system and apparatus interact via a unitary process. An ideal quantum measurement process involves coherent copying (i.e., generalized C-NOT) operation $U_{mes} : \mathcal{H}_S \otimes \mathcal{H}\ \mapsto \mathcal{H}_R \otimes \mathcal{H}_B$. Then, an arbitrary state $|\psi_S\rangle = \sum_i c_i |i_S\rangle$ of S leads to

$$|\psi_S\rangle|0\ \rangle \xrightarrow{U_{mes}} \sum_i c_i |i_R\rangle|i_B\rangle, \tag{12}$$

where $|k\rangle|l\rangle = |k\rangle \otimes |l\rangle$. Unlike in classical cases, in this quantum evolution, S and both may causally influence R and B [31, 32]. Say, one observes an effect $|\phi_B\rangle = \sum_i k_i |i_B\rangle$ in B after implementing the (rank-1) projector $|\phi_B\rangle\langle\phi_B|$ on B and, consequently the updated RB state becomes $|\phi'_{RB}\rangle = \frac{1}{N} \sum_i c_i k_i^* |i_R\rangle|\phi_B\rangle = |\phi_R\rangle|\phi_B\rangle$, where $N = (\sum_i |c_i k_i^*|^2)^{1/2}$. Note the collapse induced in R due to the observation on B. Now, a cause $|\phi_S\rangle$ belonging to S has one-to-one causal correspondence with the effect $|\phi_B\rangle$ in B if

$$U_{mes}|\phi_S\rangle|0\ \rangle = |\phi_R\rangle|\phi_B\rangle, \tag{13}$$

$$U^\dagger_{mes}|\phi_R\rangle|\phi_B\rangle = |\phi_S\rangle|0\ \rangle, \tag{14}$$

which are exactly the conditions (10) and (11) in Definition 2. Here $|\phi_S\rangle = |\phi_R\rangle$. Hence, only the effects $\{|i_B\rangle\}$ establishes one-to-one correspondence with the causes $\{|i_S\rangle\}$ respectively. Any other effect in B will not establish such a correspondence with a cause in S and vice versa.

In general, the situation for unitary evolution is more straightforward than the non-unitary ones. This is because unitary evolutions preserve all information and are invertible (without a need for a reference prior). Thus it is sufficient to check conditions (8) and (13) for deterministic causal predictions and conditions (9) and (14) for deterministic causal inferences. In the rest of the article, we restrict ourselves to the cases that involve unitary evolution and measurements using rank-1 projectors.

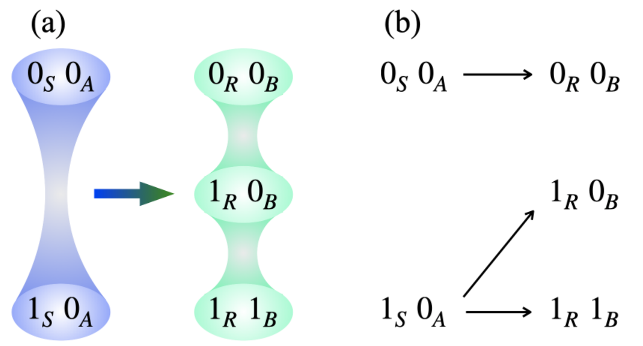

Figure 1. **Example-1.** (a) The quantum process (15)-(17) allows superposition among orthogonal causes and, as a result, can result in an effect that is in a superposition between orthogonal effects. (b) The figure displays the stochastic process analogous to the quantum one in Eq. (15). Here, the classical bit $0_S0\ $ deterministically evolves to 0_R0_B, as the conditional probability is $P(0_R0_B|0_S0\ ) = 1$. But, bits $1_S0\ $ results in incoherent mixtures of 1_R0_B and 1_R1_B, with conditional probabilities $P(1_R0_B|1_S0\ ) = \frac{1}{2}$ and $P(1_R1_B|1_S0\ ) = \frac{1}{2}$. See text for more details.

III. TR DITION L VS QU NTUM B YES' RULE

In this section, we shall study the situations that will help reveal the inadequacy of TBR is some situations. In particular, we consider two examples where causal inferences based on TBR differ from those based on QBR.

Example-1: This example involves an initially uncorrelated system-apparatus composite and a global unitary evolution (see Figure 1). Say both system (S) and apparatus ( ) are qubits, and they are evolved together with an isometry $V : \mathcal{H}_S \otimes \mathcal{H}\ \ \mapsto \mathcal{H}_R \otimes \mathcal{H}_B$, given by

$$|0_S\rangle|0\ \rangle \xrightarrow{V} |0_R\rangle|0_B\rangle, \quad |1_S\rangle|0\ \rangle \xrightarrow{V} |1_R\rangle|+_B\rangle, \tag{15}$$

where $\{|0_X\rangle, |1_X\rangle\}$ are the orthonormal bases of $\mathcal{H}_X$ and $|\pm_X\rangle = 1/\sqrt{2}(|0_X\rangle \pm |1_X\rangle)$. The initial state is $|\phi_S\rangle|0\ \rangle$ with $|\phi_S\rangle = \sqrt{1\ \ r}|0_S\rangle + \sqrt{r}|1_S\rangle$ for $0 < r < 1$. After the evolution by V, the final state $|\phi_{RB}\rangle = V|\phi_S\rangle|0\ \rangle$ becomes

$$|\phi_{RB}\rangle = \sqrt{1\ \ r}\,|0_R\rangle|0_B\rangle + \sqrt{r}\,|1_R\rangle|+_B\rangle, \tag{16}$$
$$= \sqrt{1\ \ r}\,|0_R\rangle + \sqrt{r/2}\,|1_R\rangle\Big)|0_B\rangle + \sqrt{r/2}\,|1_R\rangle|1_B\rangle. \tag{17}$$

The evolution by V does not establish deterministic causal relations for arbitrary causes and effects. Some causes like $|0_S\rangle$ and $|1_S\rangle$ can predict the effects $|0\ \rangle$ and $|+\ \rangle$ with certainty. However, the observations of $|0\ \rangle$ and $|+\ \rangle$ cannot be used to deterministically infer the causes $|0_S\rangle$ and $|1_S\rangle$ respectively.

To highlight how the logical reasoning based on TBR and QBR differs, let us analyze the observation of the effect $|1_B\rangle$ in B and infer its cause in S. Due to the presence of entanglement in the state (17), a local observation of $|1_B\rangle$ induces a collapse in R to the state $|1_R\rangle$. Because of that, we need to consider the causal correspondence between global causes and effects. Thus, the task is now to find the cause in $S\ \ $ for the observed effect $|1_R\rangle|1_B\rangle$ in RB. We start with inference using TBR. There, the conditional probability for $|1_S\rangle|0\ \rangle \mapsto |1_R\rangle|1_B\rangle$ is $P(1_R1_B|1_S0\ ) = |\langle 1_R|\langle 1_B|V|1_S\rangle|0_B\rangle|^2 = \frac{1}{2}$, which is nothing but the transition probability. Following Born rule, the probabilities of finding $|1_S\rangle|0\ \rangle$ and $|1_R\rangle|1_B\rangle$ before and after the global evolution are $P(1_S0\ ) = r$ and $P(1_R1_B) = \frac{r}{2}$ respectively. Then the conditional probability $P(1_S0\ |1_R1_B)$ corresponding to the retrodicted evolution can be derived using TBR, and that is

$$P(1_S0\ |1_R1_B) = P(1_R1_B|1_S0\ )\,P(1_S0\ )/P(1_R1_B) = 1. \tag{18}$$

Note the conditional probability $P(1_S0\ |1_R1_B) = 1$ can also be found if one considers an analogous stochastic evolution given in Figure 1(b), along with the probabilities $P(1_S0\ ) = r$ and $P(1_R1_B) = \frac{r}{2}$. The causal reasoning based on TBR, at least in this case, cannot differentiate whether the evolution has occurred via a unitary or stochastic process. Nevertheless, following the reasoning based on TBR, this unit conditional probability implies that the effect $|1_R\rangle|1_B\rangle$ can deterministically infer the cause $|1_S\rangle|0\ \rangle$. This, in turn, means that $|1_S\rangle$ in S is the cause for the effect $|1_B\rangle$ observed in B. But this inference cannot be true because it does not satisfy the condition (11) or (14) for deterministic quantum causal inference based on QBR, as

$$\mathrm{Tr}_{RB}\left[\mathcal{P}^{V^\dagger}_{S\ \ |RB} \star |1_R\rangle\langle 1_R| \otimes |1_B\rangle\langle 1_B|\right]\ \ \ |1_S\rangle\langle 1_S| \otimes |0\ \rangle\langle 0\ |,$$

or $V^\dagger|1_R\rangle|1_B\rangle = |1_S\rangle|\ \ \rangle\ \ \ |1_S\rangle|0\ \rangle$. Thus, according to QBR, the effect $|1_B\rangle$ cannot deterministically infer the cause $|1_S\rangle$. It can at most be claimed that the cause $|1\rangle_S$ may result in the effect $|1\rangle_B$ with probability $\frac{1}{2}$. This is in direct contradiction with the inference made using TBR.

Example 2 – This example assumes a situation where the initial state of the system (S) and apparatus ( ) composite is entangled and evolves via local unitary operations (see Figure 2). Say a two-qubit composite $S\ \ $ is in an initially entangled state

$$|\psi_{S\ }\rangle = \frac{1}{\sqrt{3}}\left(i|0_S\rangle|1\ \rangle + i|1_S\rangle|0\ \rangle + |1_S\rangle|1\ \rangle\right), \tag{19}$$

where $\{|0_X\rangle, |1_X\rangle\}$ are the orthonornal bases of $\mathcal{H}_X$. The qubit undergoes an evolution by the isometry $U\ \ : \mathcal{H}\ \ \mapsto \mathcal{H}_B$, with

$$|0\ \rangle \mapsto \frac{1}{\sqrt{2}}\left(|0_B\rangle + i|1_B\rangle\right),\quad |1\ \rangle \mapsto \frac{1}{\sqrt{2}}\left(i|0_B\rangle + |1_B\rangle\right), \tag{20}$$

to result in the final state $|\phi_{SB}\rangle = \ _S \otimes U\ \ |\psi_{S\ }\rangle$ given by

$$|\phi_{SB}\rangle = \frac{1}{\sqrt{6}}\left(2i|1_S\rangle|0_B\rangle\ \ \ |0_S\rangle|0_B\rangle + i|0_S\rangle|1_B\rangle\right). \tag{21}$$

Here the evolution implemented is local in nature and thus cannot establish causal relations between S and B in general. But, initial entanglement may result in a causal correspondence between S and B. Thus the causal relation must involve the global causes in $S\ \ $ and global effects in SB.

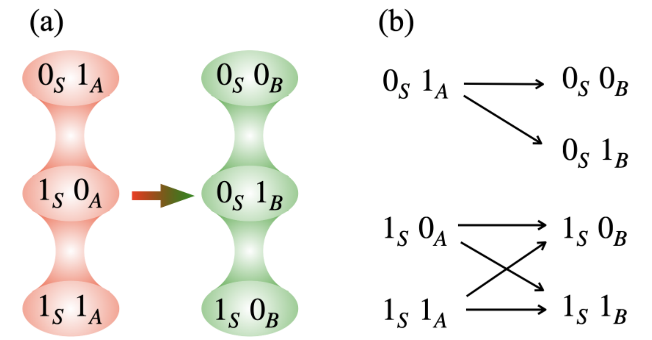

Figure 2. **Example-2.** (a) The quantum evolution is depicted. It allows superposition causes in S to result in a superposition of effects in SB. (b) The figure represents analogous stochastic evolution where classical bits S act as the cause to induce incoherent mixtures of effects in SB, with all classical conditional probabilities $P(0_S 0_B|0_S 1\ )$, $P(0_S 1_B|0_S 1\ )$, $P(1_S 0_B|1_S 0\ )$, $P(1_S 1_B|1_S 0\ )$, $P(1_S 0_B|1_S 1\ )$, and $P(1_S 1_B|1_S 1\ )$ equal to $\frac{1}{2}$. See text for more details.

To highlight how inferences based on TBR and QBR differ, let us now find out the cause (in S) corresponding to the effect $|0_S\rangle|1_B\rangle$ observed in the final state $|\phi_{SB}\rangle$. With the conditional probability, i.e., the transition probability, $P(0_S 1_B|0_S 1\ ) = |\langle 0_S|\langle 1_B|\ _S \otimes U\ \ |0_S\rangle|1\ \rangle|^2 = \frac{1}{2}$ for the transition $|0_S\rangle|1\ \rangle \mapsto |0_S\rangle|1_B\rangle$ and probabilities of finding $|0_S\rangle|1\ \rangle$ and $|0_S\rangle|1_B\rangle$ in the initial and final states $P(0_S 1\ ) = \frac{1}{3}$ and $P(0_S 1_B) = \frac{1}{6}$ respectively, the TBR leads to

$$P(0_S 1\ \ |0_S 1_B) = P(0_S 1_B|0_S 1\ )\ P(0_S 1\ )/P(0_S 1_B) = 1. \quad (22)$$

Again, this unit conditional probability can be derived if we consider an analogous stochastic evolution depicted in Figure 2(b) with the probabilities $P(0_S 1\ ) = \frac{1}{3}$ and $P(0_S 1_B) = \frac{1}{6}$. Thus the TBR cannot recognize the difference between a unitary process and an analogous stochastic process in this example. Nevertheless, as per TBR, $P(0_S 1\ \ |0_S 1_B) = 1$ signifies that an observation of the effect $|0_S\rangle|1_B\rangle$ deterministically infers the cause $|0_S\rangle|1\ \rangle$. This may be further argued by the facts that the observation of $|1_B\rangle$ happens together with $|0_S\rangle$ in $|\phi_{SB}\rangle$, and $|0_S\rangle$ in $|\phi_{SB}\rangle$ implies $|0_S\rangle$ in $|\psi_S\ \ \rangle$ because in the transformation $|\psi_S\ \ \rangle \mapsto |\phi_{SB}\rangle$ the qubit S did not evolve. Altogether, the observation of the effect $|1_B\rangle$ in $|\phi_{SB}\rangle$ demands the cause $|0_S\rangle$ to be present in $|\psi_S\ \ \rangle$. But, this inference cannot be true. According to QBR

$$\mathrm{Tr}_{SB}\left[\mathcal{P}_{S\ \ |SB}^{U^\dagger} \star |0_S\rangle\langle 0_S| \otimes |1_B\rangle\langle 1_B|\right] \quad |0_S\rangle\langle 0_S| \otimes |1\ \rangle\langle 1\ |,$$

or $U^\dagger|0_S\rangle|1_B\rangle = \frac{1}{\sqrt{2}}(|0_S\rangle|1\ \rangle \quad i|0_S\rangle|0\ \rangle) \quad |0_S\rangle|1\ \rangle$, and it does not satisfy the condition (7) or (9) for deterministic causal inference. One may, at most, claim that the cause $|0_S\rangle|1\ \rangle$ is responsible for the effect $|0_S\rangle|1_B\rangle$ with probability $\frac{1}{2}$.

A similar problem also appears in case of prediction. For instance, consider the prediction of the effect $|1_S\rangle|1_B\rangle$ in the final state $|\phi_{SB}\rangle$. The probabilities of finding $|1_S\rangle|0\ \rangle$ and $|1_S\rangle|1\ \rangle$ in $|\phi_S\ \ \rangle$ are $P(1_S 0\ ) = \frac{1}{3}$ and $P(1_S 1\ ) = \frac{1}{3}$ respectively. The transition probabilities for $|1_S\rangle|0\ \rangle \mapsto |1_S\rangle|1_B\rangle$ and $|1_S\rangle|1\ \rangle \mapsto |1_S\rangle|1_B\rangle$ are $P(1_S 1_B|1_S 0\ ) = \frac{1}{2}$ and $P(1_S 1_B|1_S 1\ ) = \frac{1}{2}$ respectively. Then, according to TBR, the effect $|1_S\rangle|1_B\rangle$ should be observed with a probability

$$P(1_S 1_B|1_S 0\ )P(1_S 0\ ) + P(1_S 1_B|1_S 1\ )P(1_S 1\ ) = \frac{1}{3}.$$

However, according to QBR, the effect $|1_S\rangle|1_B\rangle$ can never be observed in the final state, which is indeed the case.

Therefore, the inferences and predictions based on the reasoning following TBR differ from the ones following on QBR. The use of TBR, in fact, leads to various contradictions in quantum mechanics, and that can again be resolved with the help of QBR. Two such cases are considered in the following sections.

IV. FR UCHIGER-RENNER'S P R DOX ND QBR

We now consider a case based on the setup assumed in Frauchiger-Renner's paradox [20]. The paradox is an extension of Wigner's friend paradox [33] and introduces a no-go theorem between various interpretations of quantum mechanics and claims that "quantum mechanics cannot consistently describe the use of itself".

While this paradox delineates the contradictions between the knowledge of the observers and super-observers about a system after performing 'measurements' at different stages of evolution, here we re-visit the paradox purely from the point of view of causal relation. In this setup, the initial states of the systems involved and the measurement (unitary) evolutions are known a priory. Without bringing in technicalities, the paradox is demonstrated using the steps below.

(F0) At first, a qubit R is prepared in the initial state

$$|\psi_R\rangle = \frac{1}{\sqrt{3}}|h_R\rangle + \sqrt{\frac{2}{3}}|t_R\rangle. \quad (23)$$

(F1) Then, R is attached with a spin-$\frac{1}{2}$ system S and evolved using the isometry V_1 given by

$$|h_R\rangle \mapsto |\bar{h}_{\bar{L}}\rangle|\downarrow_S\rangle, \quad |t_R\rangle \mapsto |\bar{t}_{\bar{L}}\rangle|\rightarrow_S\rangle, \quad (24)$$

where $|\ \ \ _S\rangle = \frac{1}{\sqrt{2}}(|\downarrow_S\rangle \pm |\uparrow_S\rangle)$. The isometry V_1 updates the initial state $|\psi_R\rangle$ to

$$|\psi_{\bar{L}S}\rangle = \frac{1}{\sqrt{3}}\ \ |\bar{h}_{\bar{L}}\rangle|\downarrow_S\rangle + |\bar{t}_{\bar{L}}\rangle|\downarrow_S\rangle + |\bar{t}_{\bar{L}}\rangle|\uparrow_S\rangle\Big), \quad (25)$$

$$= \sqrt{\frac{2}{3}}|\bar{f}_{\bar{L}}\rangle|\downarrow_S\rangle + \frac{1}{\sqrt{6}}|\bar{f}_{\bar{L}}\rangle|\uparrow_S\rangle \quad \frac{1}{\sqrt{6}}|\bar{o}_{\bar{L}}\rangle|\uparrow_S\rangle, \quad (26)$$

where in the second step we have used $|\bar{t}_{\bar{L}}\rangle = \frac{1}{\sqrt{2}}(|\bar{f}_{\bar{L}}\rangle \quad |\bar{o}_{\bar{L}}\rangle)$ and $|\bar{h}_{\bar{L}}\rangle = \frac{1}{\sqrt{2}}(|\bar{f}_{\bar{L}}\rangle + |\bar{o}_{\bar{L}}\rangle)$.

(F2) Now another isometry V_2 is applied on S, given by

$$|\downarrow_S\rangle \mapsto \frac{1}{\sqrt{2}}(|f_L\rangle + |o_L\rangle), \quad |\uparrow\rangle_S \mapsto \frac{1}{\sqrt{2}}(|f_L\rangle \quad |o_L\rangle), \quad (27)$$

and as consequence, the state $|\psi_{\bar{L}S}\rangle$ modifies to

$$|\psi_{\bar{L}L}\rangle = \frac{1}{\sqrt{12}}\Big(|\bar{o}_{\bar{L}}\rangle(|o_L\rangle \quad |f_L\rangle) + |\bar{f}_{\bar{L}}\rangle(|o_L\rangle + 3|f_L\rangle)\Big). \tag{28}$$

The contradiction leading to the paradox can be understood by noting inconsistencies in the chain of arguments based on traditional Bayes' theorem below, similar to the ones considered in [20]. (A1) An observation of $|\bar{o}_{\bar{L}}\rangle$ in the state $|\psi_{\bar{L}S}\rangle$ ensures the observation of $|\uparrow_S\rangle$ (see Eq. (26)). (A2) Again from $|\psi_{\bar{L}S}\rangle$, it is guaranteed that the observation of $|\uparrow_S\rangle$ always occurs together with the observation of $|\bar{t}_{\bar{L}}\rangle$ (see Eq. (25)). (A3) From the action of the isometry V_1 in step (F1), it is "inferred" that the observation of $|\bar{t}_{\bar{L}}\rangle$ has the underlying cause $|t_R\rangle$. (A4) The cause $|t_R\rangle$ guarantees that the state of S is $|\rightarrow_S\rangle$ after the evolution by V_1. (A5) With the evolution by V_2, the cause $|\rightarrow_S\rangle$ in S leads to the effect $|f_L\rangle$ in L. (A6) As seen from the state $|\psi_{\bar{L}L}\rangle$ (see Eq. (28)) in step (F2), the joint state $|\bar{o}_{\bar{L}}\rangle|o_L\rangle$ is observed with the probability $\frac{1}{12}$.

Using arguments (A1)-(A3), it is "inferred" that the cause of the effect $|\bar{o}_{\bar{L}}\rangle$ in $\bar{L}$ is $|t_R\rangle$ and, following the arguments (A4)-(A5), this cause predicts the effect $|f_L\rangle$ in L. Therefore, each observation of $|\bar{o}_{\bar{L}}\rangle$ is associated with the observation of $|f_L\rangle$. This is equivalent to say that the joint state $|\bar{o}_{\bar{L}}\rangle|o_L\rangle$ should never be observed. However, the argument (A6) claims that $|\bar{o}_{\bar{L}}\rangle|o_L\rangle$ will be observed with a non-zero probability. This leads to a contradiction and the paradox.

The root of this apparent inconsistency lies in the ignorance of the role of quantum evolution, measurement induced collapse while making inferences and predictions, and that quantum causes (effects) may coherently superpose to represent another cause (effects). Let us start by re-analyzing step (F1). The isometry V_1 can be implemented in two stages. First, an isometry $V_1^{(1)}$ that maps R as $|h_R\rangle \mapsto |\bar{h}_{\bar{L}}\rangle$ and $|t_R\rangle \mapsto |\bar{t}_{\bar{L}}\rangle$. Then the system S in a state $|\downarrow_S\rangle$ is clubbed with the $\bar{L}$ and a global unitary $V_1^{(2)}$ is applied on $\bar{L}S$, where

$$|\bar{h}_{\bar{L}}\rangle|\downarrow_S\rangle \xrightarrow{V_1^{(2)}} |\bar{h}_{\bar{L}}\rangle|\downarrow_S\rangle, \quad |\bar{t}_{\bar{L}}\rangle|\downarrow_S\rangle \xrightarrow{V_1^{(2)}} |\bar{t}_{\bar{L}}\rangle|\rightarrow_S\rangle. \tag{29}$$

The overall isometry becomes $V_1 = V_1^{(2)} \circ V_1^{(1)}$ as required. The unitary $V_1^{(2)}$ has properties similar to the unitary V considered in Example-1 (see Eq. (15)). It is true that the state $|\uparrow_S\rangle$ of S always appears with the state $|\bar{t}_{\bar{L}}\rangle$ of $\bar{L}$ (argument (A2)). The corresponding probability of observing $|\bar{t}_{\bar{L}}\rangle|\uparrow_S\rangle$ in $|\psi_{\bar{L}S}\rangle$ is $P(\bar{t}_{\bar{L}} \uparrow_S) = \frac{1}{3}$. The probability of finding $|\bar{t}_{\bar{L}}\rangle|\downarrow_S\rangle$, before the application of $V_1^{(2)}$, i.e., in the state $V_1^{(1)}|\psi_R\rangle|\downarrow_S\rangle$, is $P(\bar{t}_{\bar{L}} \downarrow_S) = \frac{2}{3}$. Now with the transitional or conditional probability $P(\bar{t}_{\bar{L}} \uparrow_S |\bar{t}_{\bar{L}} \downarrow_S) = \frac{1}{2}$ for the transition $|\bar{t}_{\bar{L}}\rangle|\downarrow_S\rangle \to |\bar{t}_{\bar{L}}\rangle|\uparrow_S\rangle$ and using TBR, one finds

$$P(\bar{t}_{\bar{L}} \downarrow_S |\bar{t}_{\bar{L}} \uparrow_S) = P(\bar{t}_{\bar{L}} \uparrow_S |\bar{t}_{\bar{L}} \downarrow_S)P(\bar{t}_{\bar{L}} \downarrow_S)/P(\bar{t}_{\bar{L}} \uparrow_S) = 1. \tag{30}$$

With the unit (retrodicted) conditional probability and following TBR, the observation of the effect $|\bar{t}_{\bar{L}}\rangle|\uparrow_S\rangle$ deterministically infers the cause $|\bar{t}_{\bar{L}}\rangle|\downarrow_S\rangle$. The cause $|\uparrow_S\rangle$ is always observed together with $|\bar{t}_{\bar{L}}\rangle$. As the evolution by $V_1^{(1)}$ only changes the label of Hilbert space from R to $\bar{L}$, the effect $|\bar{t}_{\bar{L}}\rangle$ in $\bar{L}$ is attributed to the cause $|t_R\rangle$ in R, which is the basis for the argument (A3). Note, this is precisely the reasoning considered to arrive at the *Statement* $F^{n:12}$ in [20], which in our opinion, is the root cause of the inconsistency leading to the Frauchiger-Renner's paradox.

However, as we have discussed in Example-1, the argument (A3) cannot be true. Because it does not respect the conditions for a deterministic quantum causal inference. That is, according to QBR,

$$\mathrm{Tr}_{\bar{L}S}\left[\mathcal{P}^{V_1^{(2)\dagger}}_{\bar{L}S|\bar{L}S} \star |\bar{t}_{\bar{L}}\rangle\langle\bar{t}_{\bar{L}}| \otimes |\uparrow_S\rangle\langle\uparrow_S|\right] \quad |\bar{t}_{\bar{L}}\rangle\langle\bar{t}_{\bar{L}}| \otimes |\downarrow_S\rangle\langle\downarrow_S|,$$

or equivalently, $V_1^{(2)\dagger}|\bar{t}_{\bar{L}}\rangle|\uparrow_S\rangle = |\bar{t}_{\bar{L}}\rangle|\leftarrow_S\rangle \quad |\bar{t}_{\bar{L}}\rangle|\downarrow_S\rangle$. Thus, the inference drawn from the arguments (A2) and (A3) is incomplete. It can at most be said that the observed global effect $|\bar{t}_{\bar{L}}\rangle|\uparrow_S\rangle$ is the result of a global cause $|\bar{t}_{\bar{L}}\rangle|\leftarrow_S\rangle$, where the cause $|\bar{t}_{\bar{L}}\rangle|\downarrow_S\rangle$ is present with probability $\frac{1}{2}$. Given this global cause, the isometry V_2 in step (F2) guarantees that the effect $|\bar{o}_{\bar{L}}\rangle|o_L\rangle$ is observed with a non-zero probability, since

$$|\bar{t}_{\bar{L}}\rangle|\leftarrow_S\rangle \mapsto |\bar{t}_{\bar{L}}\rangle|o_L\rangle = \frac{1}{\sqrt{2}}(|\bar{f}_{\bar{L}}\rangle|o_L\rangle \quad |\bar{o}_{\bar{L}}\rangle|o_L\rangle). \tag{31}$$

Clearly, the conclusion drawn from the arguments (A1)-(A5) earlier is untrue. Furthermore, we can easily see that the $|\psi_{\bar{L}S}\rangle$ represents a superposition of other global causes, where $|\bar{t}_{\bar{L}}\rangle|\uparrow_S\rangle$ being one of them. All these causes together in superposition, i.e., the cause $|\psi_{\bar{L}S}\rangle$, lead to a the observation of the effect $|\bar{o}_{\bar{L}}\rangle|o_L\rangle$ with the probability $\frac{1}{12}$ upon application of $V_2 \circ V_1^{(2)\dagger}$ following QBR. This agrees with the argument (A6). Therefore, there is no contradiction (or paradox) once one uses QBR while making deterministic predictions or inferences.

V. H RDY'S SETUP ND QBR

In 1993, Hardy introduced nonlocality in a two-particle system without needing a Bell inequality [22], where his paradox [21] becomes a special case of this nonlocal feature. It highlights that if one assumes quantum mechanics is a local-realistic theory, then the information gained through local measurements on a bipartite system is not sufficient to characterize the global state.

Below, we reconsider Hardy's exposition of nonlocality in a bipartite system purely from the perspective of causal relation. We show that there is still a contradiction even after assuming that quantum mechanics is a non-local theory. And this is exclusively due to the use of traditional Bayes' rule. Consider a bipartite system composed of two qubits M and N in a state

$$|\psi_{MN}\rangle = \quad |0_M\rangle|0_N\rangle + \beta|1_M\rangle|1_N\rangle, \tag{32}$$

where $\;, \beta \in \mathbb{R}, | \quad | \quad |\beta|$, and satisfy $| \quad |^2 + |\beta|^2 = 1$. Here $\{|0_X\rangle, |1_X\rangle\}$ is the orthonormal basis set spanning the Hilbert space $\mathcal{H}_X$ of the qubit X. The state can be re-expressed in a new orthonormal basis set

$$|\psi_{MN}\rangle = \sqrt{\quad\beta}\,(|u_M\rangle|v_N\rangle + |v_M\rangle|u_N\rangle) + (| \quad | \quad |\beta|)\,|v_M\rangle|v_N\rangle \tag{33}$$

after dropping the overall factor 1, where $|0_X\rangle = B|u_X\rangle + i\ \ ^*|v_X\rangle$ and $|1_X\rangle = i\ \ |u_X\rangle + B^*|v_X\rangle$ with $\ \ = \sqrt{\ \ }/\sqrt{|\ \ | + |\beta|}$ and $B = i\sqrt{\beta}/\sqrt{|\ \ | + |\beta|}$. Now the state is evolved with two local unitaries (isometries) $U_M : \mathcal{H}_M \mapsto \mathcal{H}_R$ and $U_N : \mathcal{H}_N \mapsto \mathcal{H}_S$ given by

$$|u_{M/N}\rangle \mapsto a^*|c_{R/S}\rangle \quad b|d_{R/S}\rangle, \ \ |v_{M/N}\rangle \mapsto b^*|c_{R/S}\rangle + a|d_{R/S}\rangle,$$

where $a = \sqrt{\ \ \beta}/\sqrt{1 \quad |\ \ \beta|}$ and $b = (|\ \ | \quad |\beta|)/\sqrt{1 \quad |\ \ \beta|}$. Two sequences of unitaries are applied on the initial state leading to the same end state, as

$$|\psi_{MN}\rangle \xrightarrow{U_M\otimes} |\psi_{RN}\rangle \xrightarrow{\otimes U_N} |\psi_{RS}\rangle,$$
$$\text{and } |\psi_{MN}\rangle \xrightarrow{\otimes U_N} |\psi_{MS}\rangle \xrightarrow{U_M\otimes} |\psi_{RS}\rangle.$$

Then, the updated states are

$$|\psi_{RN}\rangle = n\Big[|c_R\rangle\,(a|u_N\rangle + b|v_N\rangle) \quad a^2\,(a^*|c_R\rangle \quad b|d_R\rangle)\,|u_N\rangle\Big],$$
$$|\psi_{MS}\rangle = n\Big[(a|u_M\rangle + b|v_M\rangle)\,|c_S\rangle \quad a^2|u_M\rangle\,(a^*|c_S\rangle \quad b|d_S\rangle)\Big],$$
$$|\psi_{RS}\rangle = n\Big[|c_R\rangle|c_S\rangle \quad a^2\,(a^*|c_R\rangle \quad b|d_R\rangle)\,(a^*|c_S\rangle \quad b|d_S\rangle)\Big],$$

where the normalization constant $n = (1 \quad |\ \ \beta|)/(|\ \ | \quad |\beta|)$. Below we expose an irreconcilable contradiction in this general setting, even after accepting quantum mechanics as a nonlocal theory, in terms of the statements based on traditional Bayes' rule.

For instance, (H0) we never observe $|u_M\rangle|u_N\rangle$ in the state $|\psi_{MN}\rangle$ because former is absent in the latter. (H1) From $|\psi_{RS}\rangle$ is it evident that the effect $|d_R\rangle|d_S\rangle$ is found upon a "global" observation with a probability given by $|na^2b^2| \quad 0$. (H2) Each observation of $|d_R\rangle|d_S\rangle$ in $|\psi_{RS}\rangle$ deterministically infers the cause $|d_R\rangle|u_N\rangle$ in $|\psi_{RN}\rangle$. (H3) Again, the observation of $|u_N\rangle$ in $|\psi_{RN}\rangle$ implies the presence of $|u_N\rangle$ in ψ_{MN} as the evolution $|\psi_{MN}\rangle \mapsto |\psi_{RN}\rangle$ only locally updates the qubit M without altering N. (H4) Each observation of $|d_R\rangle|d_S\rangle$ in $|\psi_{RS}\rangle$ deterministically infers the cause $|u_M\rangle|d_S\rangle$ in $|\psi_{MS}\rangle$. (H5) And, the observation of $|u_M\rangle$ in $|\psi_{MS}\rangle$ demands the presence of $|u_M\rangle$ in $|\psi_{MN}\rangle$, as the evolution $|\psi_{MN}\rangle \mapsto |\psi_{MS}\rangle$ only locally updates the qubit N. Now, the statements (H1)-(H5) together imply that the observation of the effect $|d_R\rangle|d_S\rangle$ in $|\psi_{RS}\rangle$ must has the cause $|u_M\rangle|u_N\rangle$ in $|\psi_{MN}\rangle$. However, this directly contradicts with the statement (H0).

Now we re-investigate the conclusions drawn from the arguments above and their contradiction in light of quantum causal relation. It is worth noting that, contrary to Frauchiger-Renner's paradox, the local nature of the evolutions here does not necessarily establish a correspondence between the cause belonging to one system with the effect resulting in the other and vice versa. However, the presence of initial entanglement may establish some correspondence. Therefore, the causal analysis of the arguments requires simultaneously considering global cause and effect belonging to both systems.

What we demonstrate now is that the statements leading to the contradiction rely on TBR, and how the contradiction disappears once QBR is used for making inferences and predictions. Here the situation is similar to the case considered in Example-2. Let us reanalyze the statement (H2) and identify the cause in RN corresponding to the effect $|d_R\rangle|d_S\rangle$ observed in RS and, in particular, what TBR and QBR infer. The conditional probability $P(d_Rd_S|d_Ru_N) = |b|^2$ for the transition $|d_R\rangle|u_N\rangle \mapsto |d_R\rangle|d_S\rangle$. The probabilities of finding $|d_R\rangle|u_N\rangle$ in $|\psi_{RN}\rangle$ and $|d_R\rangle|d_S\rangle$ in $|\psi_{RS}\rangle$ are respectively $P(d_Ru_N) = |na^2b|^2$ and $P(d_Rd_S) = |na^2b^2|^2$. Using the traditional Bayes' rule, we have

$$P(d_Ru_N|d_Rd_S) = P(d_Rd_S|d_Ru_N)P(d_Ru_N)/P(d_Rd_S) = 1. \quad (34)$$

Thus, the observation of $|d_R\rangle|d_S\rangle$ deterministically infers the cause $|d_R\rangle|u_N\rangle$, as exploited to construct the statement (H2). However, this inference cannot be true because the quantum Bayes' rule implies

$$\mathrm{Tr}_{RS}\left[\mathcal{P}^{U_N^\dagger}_{RN|RS} \star |d_R\rangle\langle d_R| \otimes |d_S\rangle\langle d_S|\right] \quad |d_R\rangle\langle d_R| \otimes |u_N\rangle\langle u_N|,$$

or equivalently, $\ \ \otimes U_N^\dagger|d_R\rangle|d_S\rangle = |d_R\rangle(\ \ b^*|u_N\rangle + a^*|v_N\rangle) \quad |d_R\rangle|u_N\rangle$. Here the inverse transformations are $U_M^\dagger : \mathcal{H}_R \mapsto \mathcal{H}_M$ and $U_N^\dagger : \mathcal{H}_S \mapsto \mathcal{H}_N$, where

$$|c_{R/S}\rangle \mapsto a|u_{M/N}\rangle + b|v_{M/N}\rangle, \ \ |d_{R/S}\rangle \mapsto \quad b^*|u_{M/N}\rangle + a^*|v_{M/N}\rangle.$$

It does not satisfy the condition (7) for deterministic causal inference. One may, at most, claim that the cause $|d_R\rangle|u_N\rangle$ results in the effect $|d_R\rangle|d_S\rangle$ with a probability $|b|^2$. Therefore, the statement made in (H2) is only true probabilistically, and the same applies to the statements (H3)-(H5). Thus, the contradiction as s result of the statements (H0)-(H5) is flawed.

In fact, following QBR, the observation of the effect $|d_R\rangle|d_S\rangle$ in RS implies that MN should contain the cause $|u_M\rangle|u_N\rangle$ with the probability $|b|^4$, as

$$U_M^\dagger \otimes U_N^\dagger|d_R\rangle|d_S\rangle = (\ \ b^*|u_M\rangle + a^*|v_M\rangle)(\ \ b^*|u_N\rangle + a^*|v_N\rangle).$$

However, it is clear that the cause $|u_M\rangle|u_N\rangle$ is not present in the state $|\psi_{MN}\rangle$, and it does not lead to a contradiction as such. Because, there are other effects present in $|\psi_{RS}\rangle$ that are in coherent superposition with $|d_R\rangle|d_S\rangle$. Once we consider all these effects in superposition, i.e., the overall effect $|\psi_{RS}\rangle$, and infer the cause by applying $U_M^\dagger \otimes U_N^\dagger$ or QBR, we see that the overall cause (initial state) does not include $|u_M\rangle|u_N\rangle$, and this is exclusively due to the fact that quantum causes can coherently superpose. Therefore, there is no contradiction once we use quantum Bayes' rule.

VI. CONCLUSION

We have demonstrated that the inferences (similarly, predictions) following causal relation based on traditional Bayes' rule significantly differ from those based on quantum Bayes' rule. The differences in the inferences and predictions using traditional and quantum Bayes' rules are due to three main reasons. First, quantum causes, as well as effects, can coherently superpose. Second, the act of observation of an effect or cause leads to a collapse in the observed system. Third, for bipartite systems, the local effects (also causes) can have strong quantum correlations (like entanglement). Because of that, observing a local effect (cause) on one system may induce a collapse of the other system. But traditional Bayes' rule is based on conditional probability, and correspondingly the causal relations, completely ignores these aspects that are very particular to quantum mechanics, making it inadequate in constituting correct causal correspondence between quantum cause and effects in some cases where the process involves quantum superposition, entanglement, and measurement induced collapse. The use of the traditional Bayes' rule, in fact, leads to various contradictions in quantum mechanics in some situations. To demonstrate that, we have considered two cases. One is based on a paradox by Frauchiger and Renner [20], which claims that quantum mechanics cannot consistently explain the use of itself. Apart from that, it also introduces a no-go theorem between various interpretations of quantum mechanics. The paradox initiates fresh discussions and debates on the foundations of quantum mechanics within the scientific community leading to studies to unveil the reasons behind the incompatibilities of various interpretations and explore possible extensions of quantum mechanics that may resolve this paradox, see for example [34–42]. Here, we have shown that there is no inconsistency in predictions and measurement inferences if one uses the quantum Bayes' rule. The proposed resolution to the Frauchiger-Renner paradox does not require some new rule of reasoning, and it is based on what has always been known to obtain via the Petz Recovery or, more generally, via quantum Bayes' rule. The other case we have considered is based on the paradox by Hardy [21, 22]. However, unlike the original approach to Hardy's paradox, we have assumed that quantum mechanics is non-local and made predictions and inferences based on global measurements. Even in that case, the traditional Bayes' rule leads to an irreconcilable contradiction, and it, again, is resolved with the use of quantum Bayes' rule.

Therefore, we conclude that causal reasoning using traditional Bayes' rule is not universally applicable to quantum mechanics. To have consistent predictions and inferences (or causal relation), one must rely on quantum Bayes' rule. In quantum mechanics, traditional Bayes' rule is often applied in numerous contexts, e.g., causal inferences and predictions [1, 2], parameter estimations [3–9], state tomography [10–15], process tomography [13, 16, 17], etc. We anticipate that our findings will have important implications in these research areas.

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