S. Plosker, C. Ramsey, Bistochastic operators and quantum random variables

Referee report

It this paper, an L_1 space structure is constructed on a set of quantum random variables integrable with respect to a positive operator valued measure (POVM). On this space, bistochastic operators and majorization of quantum random variables is considered and some classical results are generalized to this context.

Overall remarks

1. The main problem I have with this paper is the lack of motivation for this study. The authors mention connections with quantum information theory, but while the positive operator valued measures are well established as representing quantum measurements, an interpretation of the quantum random variables, the integral with respect to a POVM and the obtained results is not given. As far as I can see, there are several other works on the topic, some also cited in the paper, but all are by the present authors and their collaborators. So an introduction of the ideas behind these construction would be appropriate, especially given the declared connection to quantum information theory.

2. Related to the above: the definition of the integral is not even given here (Def. 2.2 contains only integrability), although it is not difficult to guess. I would also expect some discussion of the properties of the integral and more examples in specific cases (e.g. if X is finite etc), to give the reader some context. If this is done in some (available) previous work, at least some precise references should be given.

3. Still related to the above: Most of the results are obtained in the special case when the POVM is purely classical (the identity operator multiplied by a measure μ) and/or a restricted set of bistochastic operators. Again, what would be an interpretation of the proposed ordering obtained on the quantum random variables? What is the relation to the classical results on (multivariate) majorization? As it is, it looks like lifting the proof in [16] to some more general context, with unclear meaning or motivation.

To conclude, in my opinion, the paper needs a revision along the above lines before it can be published. Some minor comments:

1. Lemma 3.8: "...$d\nu_\rho I_{\mathcal{H}}$" perhaps some brackets would be helpful.

2. page 8, line 6 from below: "derivitive"

3. Lemma 3.11: the norm $\|\cdot\|_\infty$ on $L^\infty_{\mathcal{H}}$ should be given explicitly before this Prop.

4. Prop. 4.6: some inverses are missing in the statement and proof
5. p. 17: "echos" echoes?
6. p.17, the displayed equation on l. 8 from below: It is not clear what is L^1_ν. Is it a set of functions? Or quantum random variables? This is also a problem in the definition of $\mathfrak{B}(L^1_\nu)$ on p. 18 (and further results on this set). In the case $\nu = \mu I_{\mathcal{H}}$, is it just the set of bistochastic operators on $L_1(X, \mu)$?
7. p.19, first displayed equation: some $I_{\mathcal{H}}$ seems to be missing
8. Prop. 6.5: does this mean that the topology induced from the WOT topology by the inclusion $B \mapsto B \otimes id_{\mathcal{H}} \in \mathfrak{B}(X, \nu)$, is the same as the weak topology on bistochastic operators (as in [2])?
9. Proof of Prop. 6.5: at several places, f_j should be h_j
10. p. 21, "weakly convergent" weakly continuous
11. Def.6.6: P_{inv} should be $\mathcal{P}_{inv}$