On $\alpha - z$-Rényi divergences in von Neumann algebras

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The α-z-Rényi divergences

For density operators ρ, σ on a finite dimensional Hilbert space:

$$D_{\alpha,z}(\rho\|\sigma) = \frac{1}{\alpha - 1} \log \frac{\operatorname{Tr}\big(\rho^{\frac{\alpha}{2z}} \sigma^{\frac{1-\alpha}{z}} \rho^{\frac{\alpha}{2z}}\big)^z}{\operatorname{Tr}\rho},$$

where $0 < \alpha \neq 1$ and $z > 0$.

For each $z > 0$: a quantum extension of classical Rényi α-divergences for probability vectors p, q:

$$D_\alpha(p\|q) = \frac{1}{\alpha} \log\Big(\sum_i p_i^\alpha q_i^{1-\alpha}\Big).$$

V. Jaksić et al, 2011
K. Audenaert, N. Datta, 2015

The α-z-Rényi divergences

Important special cases:

- Relative entropy:

$$\lim_{\alpha\to 1} D_{\alpha,z}(\rho\|\sigma) = D_1(\rho\|\sigma) = \frac{\operatorname{Tr}\big(\rho(\log\rho - \log\sigma)\big)}{\operatorname{Tr}\rho}$$

- Petz-type (standard) Rényi divergence: $z = 1$, $0 < \alpha \neq 1$

$$D_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1}\log\frac{\operatorname{Tr}\big(\rho^\alpha\sigma^{1-\alpha}\big)}{\operatorname{Tr}\rho}$$

- Sandwiched Rényi divergence: $0 < z = \alpha \neq 1$

$$\tilde{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1}\log\frac{\operatorname{Tr}\big(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}}\big)}{\operatorname{Tr}\rho}$$

Data processing inequality (DPI)

For a quantum channel (CPTP map) Φ and any ρ, σ:

$$D_{\alpha,z}(\Phi(\rho)\|\Phi(\sigma)) \leq D_{\alpha,z}(\rho\|\sigma)$$

- not true for all values of α, z:

- Petz-type:[1] $\alpha \in (0,1) \cup (1,2]$;
- sandwiched:[2] $\alpha \in [1/2,1) \cup (1,\infty]$;
- general case:[3]

$$0 < \alpha < 1, \quad \max\{\alpha, 1-\alpha\} \leq z$$

or

$$\alpha > 1, \quad \max\{\alpha/2, \alpha - 1\} \leq z \leq \alpha.$$

[1] Ando's convexity theorem, 1979
[2] S. Beigi, 2013; Frank and Lieb, 2013
[3] Carlen, Frank and Lieb, 2018; Zhang, 2020

Outline of this talk

- extension of $D_{\alpha,z}$ to the setting of von Neumann algebras
- DPI with respect to positive trace preserving maps (within the same bounds on parameters as in finite dimensions)
- equality in DPI implies sufficiency (reversibility) for 2-positive trace preserving maps

Our tools

- variational formula for $D_{\alpha,z}$
- known results in the sandwiched case
- properties of conditional expectations

von Neumann algebra extensions

The Rényi divergences were defined for normal positive functionals ψ, φ on a von Neumann algebra, using some technical tools:

- Araki relative entropy[4]: relative modular operator $\Delta_{\psi,\varphi}$
- Petz-type (Petz quasi divergence)[5]: $\Delta_{\psi,\varphi}$
- sandwiched Rényi divergence:[6] Araki-Masuda or Kosaki L^p-spaces
- general α-z Rényi divergences:[7] Haagerup L^p-spaces

[4] Araki, 1976
[5] Petz, 1985
[6] Berta, Scholtz and Tomamichel, 2018; AJ, 2018; 2021
[7] Kato and Ueda, 2023; Kato, 2024

von Neumann algebras and Haagerup L^p-spaces

Let $\mathcal{M}$ be a von Neumann algebra $\mathcal{M}$, with predual $\mathcal{M}_*$.

- Haagerup L^p-space $L^p(\mathcal{M})$, $0 < p \leq \infty$
- $\mathcal{M} = L^\infty(\mathcal{M})$, $\mathcal{M}_* \simeq L^1(\mathcal{M})$, $\varphi \mapsto h_\varphi$, $\mathrm{tr}\,(h_\varphi) = \varphi(1)$
- order isomorphism: $\mathcal{M}_*^+ \ni \varphi \mapsto h_\varphi \in L^1(\mathcal{M})^+$
- polar decomposition: for $0 < p < \infty$, $k \in L^p(\mathcal{M})$, $k = u|k|$:

$$u \in \mathcal{M} \text{ partial isometry}, \ |k| = h_\varphi^{1/p} \in L^p(\mathcal{M})^+, \ \varphi \in \mathcal{M}_*^+$$

von Neumann algebras and Haagerup L^p-spaces

For $0 < p < \infty$, $k \in L^p(\mathcal{M})$, put $\|k\|_p = (\mathrm{tr}\ |k|^p)^{1/p}$.

- For $1 < p < \infty$, $\|k\|_p$ is a norm in $L^p(\mathcal{M})$, which is a reflexive Banach space, with dual $L^p(\mathcal{M})^* \simeq L^q(\mathcal{M})$, $1/p + 1/q = 1$
- $\|k\|_p$ is a quasi norm for $0 < p < 1$
- Hölder inequality: for $1/p + 1/q = 1/r$, $0 < p, q, r \leq \infty$, $h \in L^p(\mathcal{M})$, $k \in L^q(\mathcal{M})$:

$$hk \in L^r(\mathcal{M}) \quad \text{and} \quad \|hk\|_r \leq \|h\|_p \|k\|_q$$

$D_{\alpha,z}$ for von Neumann algebras

Let $0 < \alpha \neq 1$, $0 < z$. For $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$, we define[8]

$$D_{\alpha,z}(\psi\|\varphi) = \frac{1}{\alpha - 1} \log \frac{Q_{\alpha,z}(\psi\|\varphi)}{\psi(1)}$$

where

$$Q_{\alpha,z}(\psi\|\varphi) := \begin{cases} \mathrm{tr}\left(h_\varphi^{\frac{1-\alpha}{2z}} h_\psi^{\frac{\alpha}{z}} h_\varphi^{\frac{1-\alpha}{2z}} \right)^z, & \text{if } 0 < \alpha < 1, \\ \|x\|_z^z, & \text{if } \alpha > 1 \text{ and } h_\psi^{\frac{\alpha}{z}} = h_\varphi^{\frac{\alpha-1}{2z}} x h_\varphi^{\frac{\alpha-1}{2z}} \\ & \text{with } x \in s(\varphi) L^z(\mathcal{M}) s(\varphi), \\ \infty, & \text{otherwise.} \end{cases}$$

[8] Kato and Ueda, 2023; Kato 2024

Positive maps and the Petz dual

Let $\mathcal{M}, \mathcal{N}$ be von Neumann algebras, $\gamma : \mathcal{N} \to \mathcal{M}$ positive unital normal map.

- The predual map: $\gamma_* : L^1(\mathcal{M}) \to L^1(\mathcal{N})$,

$$\gamma_*(h_\omega) := h_{\omega\circ\gamma}, \quad \text{positive, trace preserving}$$

- Let $\rho \in \mathcal{M}_*^+$, $e := s(\rho)$, $e_0 := s(\rho\circ\gamma)$. The Petz dual $\gamma_\rho^* : e\mathcal{M}e \to e_0\mathcal{N}e_0$ is determined by

$$(\gamma_\rho^*)_*\big(h_{\rho\circ\gamma}^{1/2} b h_{\rho\circ\gamma}^{1/2}\big) = h_\rho^{1/2}\gamma(b)h_\rho^{1/2}, \qquad b \in \mathcal{N}^+.$$

- positive, unital and normal,
- n-positive whenever γ is.

DPI in von Neumann algebra setting

For any $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$, and a positive unital normal map $\gamma : \mathcal{N} \to \mathcal{M}$:

$$D_{\alpha,z}(\psi \circ \gamma \| \varphi \circ \gamma) \leq D_{\alpha,z}(\psi \| \varphi).$$

This was already proved for:

- Petz type: $\alpha \in (0,1) \cup (1,2]$, γ a Schwarz map[9],
- sandwiched: $\alpha \in [1/2, 1) \cup (1, \infty]$, γ completely positive[10], γ positive[11]
- $D_{\alpha,z}$ with $0 < \alpha < 1$, $\max\{\alpha, 1-\alpha\} \leq z$, γ positive[12]

[9] Petz, 1985
[10] Berta, Scholz and Tomamichel, 2018
[11] AJ, 2018, 2021
[12] Kato, 2024

Variational expressions

Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$.

(i) Let $0 < \alpha < 1$, $\max\{\alpha, 1-\alpha\} \le z$. Then

$$Q_{\alpha,z}(\psi\|\varphi) = \inf_{a\in\mathcal{M}^{++}} \left\{\alpha \operatorname{tr}\left(\left(h_\psi^{\frac{\alpha}{2z}} a h_\psi^{\frac{\alpha}{2z}}\right)^{\frac{z}{\alpha}}\right)\right.$$
$$\left. +(1-\alpha)\operatorname{tr}\left(\left(h_\varphi^{\frac{1-\alpha}{2z}} a^{-1} h_\varphi^{\frac{1-\alpha}{2z}}\right)^{\frac{z}{1-\alpha}}\right)\right\}.$$

(ii) Let $\alpha > 1$, $\max\{\alpha/2, \alpha-1\} \le z$. Then

$$Q_{\alpha,z}(\psi\|\varphi) = \sup_{a\in\mathcal{M}^{+}} \left\{\alpha \operatorname{tr}\left(\left(h_\psi^{\frac{\alpha}{2z}} a h_\psi^{\frac{\alpha}{2z}}\right)^{\frac{z}{\alpha}}\right)\right.$$
$$\left. -(\alpha-1)\operatorname{tr}\left(\left(h_\varphi^{\frac{\alpha-1}{2z}} a h_\varphi^{\frac{\alpha-1}{2z}}\right)^{\frac{z}{\alpha-1}}\right)\right\}.$$

Finite dimensions: Zhang, 2020

Useful inequalities

$\gamma : \mathcal{N} \to \mathcal{M}$ a normal positive unital map, $\rho \in \mathcal{M}_*^+$, $b \in \mathcal{N}^+$.

(1) If $p \in [1/2, 1)$, then

$$\left\| h_{\rho\circ\gamma}^{\frac{1}{2p}} b h_{\rho\circ\gamma}^{\frac{1}{2p}} \right\|_p \leq \left\| h_\rho^{\frac{1}{2p}} \gamma(b) h_\rho^{\frac{1}{2p}} \right\|_p.$$

Proof.

Let $\omega \in \mathcal{N}_*^+$, $h_\omega = h_{\rho\circ\gamma}^{\frac{1}{2}} b h_{\rho\circ\gamma}^{\frac{1}{2}}$.

$$\begin{aligned} \left\| h_\rho^{\frac{1}{2p}} \gamma(b) h_\rho^{\frac{1}{2p}} \right\|_p^p &= Q_{p,p}(\omega \circ \gamma_\rho^* \| \rho) = Q_{p,p}(\omega \circ \gamma_\rho^* \| \rho \circ \gamma \circ \gamma_\rho^*) \\ &\geq {}^{13} Q_{p,p}(\omega \| \rho \circ \gamma) = \left\| h_{\rho\circ\gamma}^{\frac{1}{2p}} b h_{\rho\circ\gamma}^{\frac{1}{2p}} \right\|_p^p \end{aligned}$$

□

[13] Sandwiched case: AJ, 2021

Useful inequalities

$\gamma : \mathcal{N} \to \mathcal{M}$ a normal positive unital map, $\rho \in \mathcal{M}_*^+$, $b \in \mathcal{N}^+$.

(2) If $p \in [1, \infty]$, then

$$\left\| h_{\rho\circ\gamma}^{\frac{1}{2p}} b h_{\rho\circ\gamma}^{\frac{1}{2p}} \right\|_p \geq \left\| h_\rho^{\frac{1}{2p}} \gamma(b) h_\rho^{\frac{1}{2p}} \right\|_p.$$

Proof.

Let $\omega \in \mathcal{N}_*^+$, $h_\omega = h_{\rho\circ\gamma}^{\frac{1}{2}} b h_{\rho\circ\gamma}^{\frac{1}{2}}$.

$$\begin{aligned} \left\| h_\rho^{\frac{1}{2p}} \gamma(b) h_\rho^{\frac{1}{2p}} \right\|_p^p &= Q_{p,p}(\omega \circ \gamma_\rho^* \| \rho) = Q_{p,p}(\omega \circ \gamma_\rho^* \| \rho \circ \gamma \circ \gamma_\rho^*) \\ &\leq {}^{14} Q_{p,p}(\omega \| \rho \circ \gamma) = \left\| h_{\rho\circ\gamma}^{\frac{1}{2p}} b h_{\rho\circ\gamma}^{\frac{1}{2p}} \right\|_p^p \end{aligned}$$

□

[14] Sandwiched case: AJ, 2018

DPI in the von Neumann algebra setting

Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$ and let $\gamma : \mathcal{N} \to \mathcal{M}$ be a normal positive unital map.

Assume either of the following conditions:

(i) $0 < \alpha < 1$, $\max\{\alpha, 1-\alpha\} \leq z$,

(ii) $\alpha > 1$, $\max\{\alpha/2, \alpha - 1\} \leq z \leq \alpha$.

Then we have

$$D_{\alpha,z}(\psi \circ \gamma \| \varphi \circ \gamma) \leq D_{\alpha,z}(\psi \| \varphi).$$

DPI in the von Neumann algebra setting

Let $0 < \alpha < 1$, $\max\{\alpha, 1-\alpha\} \leq z$. We have

$$Q_{\alpha,z}(\psi\|\varphi) = \inf_{a \in \mathcal{M}^{++}} \left\{ \alpha \left\| h_\psi^{\frac{1}{2p}} a h_\psi^{\frac{1}{2p}} \right\|_p^p + (1-\alpha) \left\| h_\varphi^{\frac{1}{2r}} a^{-1} h_\varphi^{\frac{1}{2r}} \right\|_r^r \right\},$$

with $p := \frac{z}{\alpha}$, $r := \frac{z}{1-\alpha}$. In the above bounds, $p, r \geq 1$.

By the inequality (2) and the Choi inequality:

$$\gamma(b)^{-1} \leq \gamma(b^{-1}),$$

we get

$$Q_{\alpha,z}(\psi \circ \gamma \| \varphi \circ \gamma) \geq Q_{\alpha,z}(\psi\|\varphi).$$

DPI in the von Neumann algebra setting

Let $\alpha > 1$, $\max\{\alpha/2, \alpha - 1\} \leq z \leq \alpha$. We have

$$Q_{\alpha,z}(\psi\|\varphi) = \sup_{a\in\mathcal{M}^+} \left\{\alpha\left\|h_\psi^{\frac{1}{2p}} a h_\psi^{\frac{1}{2p}}\right\|_p^p - (\alpha-1)\left\|h_\varphi^{\frac{1}{2q}} a h_\varphi^{\frac{1}{2q}}\right\|_q^q\right\},$$

with $p := \frac{z}{\alpha}$, $q := \frac{z}{\alpha-1}$. In the above bounds, $p \in [1/2, 1)$, $q \geq 1$.

By the inequalities (1) and (2) we get

$$Q_{\alpha,z}(\psi\circ\gamma\|\varphi\circ\gamma) \leq Q_{\alpha,z}(\psi\|\varphi).$$

Sufficient channels and equality in DPI

A channel is a 2-positive unital normal map $\gamma : \mathcal{N} \to \mathcal{M}$.

Let $\psi, \varphi \in \mathcal{M}_*^+$. We say that γ is sufficient with respect to $\{\psi, \varphi\}$ if there exists a recovery channel $\beta : \mathcal{M} \to \mathcal{N}$ such that

$$\psi \circ \gamma \circ \beta = \psi, \qquad \varphi \circ \gamma \circ \beta = \varphi.$$

Petz theorem: Assume that $D_1(\psi\|\varphi) < \infty$. Then γ is sufficient with respect to $\{\psi, \varphi\}$ if and only if

$$D_1(\psi \circ \gamma \| \varphi \circ \gamma) = D_1(\psi\|\varphi).$$

A similar result holds for the transition probability ($D_{\frac{1}{2},1}$).

Petz, 1986, 1988

Known results on equality in DPI

Characterization of sufficient channels:

- Petz-type: $D_{\alpha,1}$, $\alpha \in (0,1) \cup (1,2)$[15]
- sandwiched: $D_{\alpha,\alpha}$, $\alpha \in (1/2,1) \cup (1,\infty)$[16]

Other equality conditions for $D_{\alpha,z}$ were found in finite dimensions[17]

- no clear relation to sufficiency of channels (apart from some special cases).

[15] AJ and Petz, 2006; Hiai et al, 2011; Hiai and Mosonyi 2017; Hiai, 2018
[16] AJ, 2018, 2021
[17] Leditzky, Rouzé and Datta, 2017; Hiai and Mosonyi, 2017; Zhang 2020

Universal recovery channel

The Petz dual γ_φ^* is a universal recovery channel:
Let $\psi \in \mathcal{M}_*^+$ be such that $s(\psi) \leq s(\varphi)$. Then γ is sufficient with respect to $\{\psi, \varphi\}$ if and only if

$$\psi \circ \gamma \circ \gamma_\varphi^* = \psi.$$

Equality in DPI

What are the conditions for

$$D_{\alpha,z}(\psi \circ \gamma \| \varphi \circ \gamma) = D_{\alpha,z}(\psi \| \varphi)?$$

If γ is 2-positive, the equality is in some cases equivalent to existence of a recovery map: 2-positive, unital, normal map $\beta : \mathcal{M} \to \mathcal{N}$ such that

$$\psi \circ \gamma \circ \beta = \psi, \qquad \varphi \circ \gamma \circ \beta = \varphi.$$

$\equiv \gamma$ is sufficient with respect to $\{\psi, \varphi\}$[18].

[18] Petz, 1986, 1988

Equality in DPI and sufficiency