



Geometric and algebraic aspects of spectrality in order unit spaces: A comparison



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ABSTRACT

Two approaches to spectral theory of order unit spaces are compared: the spectral duality of Alfsen and Shultz and the spectral compression bases due to Foulis. While the former approach uses the geometric properties of an order unit space in duality with a base norm space, the latter notion is purely algebraic. It is shown that the Foulis approach is strictly more general and contains the Alfsen-Shultz approach as a special case. This is demonstrated on two types of examples: the JB-algebras which are Foulis spectral if and only if they are Rickart and the centrally symmetric state spaces, which may be Foulis spectral while not necessarily Alfsen-Shultz spectral.

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1. Introduction

The operational approach to foundations of quantum mechanics works with an order unit space A , in duality with a base normed space V , where the distinguished base of the positive cone represents states of a physical system, whereas elements of the unit interval in A are interpreted as effects, or dichotomic measurements. This theory goes back to the works of Mackey [28] and Ludwig [27] and many others, see e.g. [25] for a historical account and further references. More recent approaches include the convex operational theories or general probabilistic theories (GPT) (e.g. [8,9]), working mainly in finite dimension, see also [25] for an introduction and overview or [32] for other approaches.

An important property of any mathematical theory describing quantum mechanics is spectrality, that is, existence of spectral resolutions of effects (and hence all elements in the space A) that allows an integral expression in terms of some special elements called projections. Perhaps the best well known extension of spectrality to order unit spaces is due to Alfsen and Shultz [3,5], motivated by characterization of state spaces of operator algebras and JB-algebras. This theory is based on the notion of compressions on an order unit space A which employs a separating order and norm duality with a base norm space V . The compressions are in one-to-one correspondence with special elements of the unit interval in A called projective units

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and also with so-called projective faces of the set of states. The duality of A and V is called spectral if there are “sufficiently many” compressions on A , it is also assumed that the order unit space is monotone σ -complete [3] or that $A = V^*$ [5]. Under these conditions, any element has a spectral resolution in terms of the projective units and there is a well defined functional calculus on A .

There are other approaches to spectrality based solely on order theoretic or algebraic properties of the order unit space, see e.g. [1,17], or the unit interval in A e.g. [18,19,30], also more recently [20] or [23]. The relations of these various definitions are not clear and it is a natural question whether the algebraic approach could lead to the same results as the Alfsen-Shultz theory (if similar conditions are assumed).

The aim of the present paper is to compare the Alfsen-Shultz approach to that of [17]. The latter notion of spectrality follows the works by Foulis [12–15] in the more general setting of ordered groups with order unit. This definition also uses a notion of a compression, but since no duality is used, the definition is different from that of Alfsen-Shultz. In this case, A is spectral if there are enough compressions to form a spectral compression base: a set of compressions with given properties. It is proved in [17] that this implies existence of spectral resolutions for elements in A .

In order to compare the two theories, we will assume that A is in a separating order and norm duality with a base norm space V and that the (Foulis) compressions are continuous with respect to this duality, note that we can always take $V = A^*$ and then continuity always holds.

It is noted in [17] that the spectral theory of Alfsen-Shultz is a special case of the spectral theory of Foulis and examples are given showing that the latter notion works also in the case that the order unit space is not σ -monotone complete. However, the exact relation of the two theories is unknown. In particular, it is a question whether these two notions of spectrality are equivalent if $A = V^*$ for some base norm space V . We prove that the Foulis spectrality is strictly more general than Alfsen-Shultz spectrality, in the sense that if $A = V^*$, the Alfsen-Shultz spectrality implies Foulis spectrality, but we show examples that the opposite implication does not hold. We compare the two notions of compressions and show that these are not the same. We also analyze the case of JB-algebras and show that JB-algebras are Foulis spectral if and only if they are Rickart, but the Rickart JB-algebras are not Alfsen-Shultz spectral in general, although the compressions are the same in both cases.

The outline of the paper is as follows. In the next section, we introduce some basic definitions related to order unit and base norm spaces and dualities between them. In Section 3 we describe the two basic notions of spectrality that we are going to compare. While the Alfsen-Shultz theory is developed in detail in [5] and is therefore introduced only briefly in Section 3.1, the theory proposed in [17] is much less well known, with many proofs scattered about several papers and in different settings. For this reason, the exposition of this theory in Section 3.2 is more detailed and quite self-contained, with many results reproved. In Section 4, we proceed to comparison of the two theories, here the main result is Theorem 4.7. The last two sections contain examples showing the difference between the two theories. In Section 5, we study the Rickart JB-algebras in which case we may have $A \neq V^*$. We work with compressions in Alfsen-Shultz sense and show that they form a spectral compression base, while Alfsen-Shultz spectrality cannot hold. Section 6 is devoted to centrally symmetric theories, where $A = V^*$ and the distinguished base in V is (isomorphic to) the unit ball in a (reflexive) Banach space $(X, \|\cdot\|)$. We show that Alfsen-Shultz spectrality holds if and only if the space X is strictly convex and smooth, whereas smoothness is equivalent to Foulis spectrality.

2. Basic notions

For more details for this section, we refer the reader to [2,4,5].

Recall that an *order unit space* is an archimedean partially ordered real vector space with a distinguished order unit. In the sequel, we denote the positive cone by $A^+ := \{a \in A : 0 \leq a\}$ and the order unit by 1, the order unit space will be denoted by the triple $(A, A^+, 1)$. The unit interval in A will be denoted by

$E := \{e \in A : 0 \leq e \leq 1\}$. Note that E endowed with the partial operation inherited from $+$ in A is an effect algebra, see [11] for the definition and [10] for more information on effect algebras and related structures.

The order unit norm $\|\cdot\|_1$ on A is defined by

$$\|a\|_1 = \inf\{\lambda \in \mathbb{R}^+ : -\lambda \leq a \leq \lambda\}, \quad a \in A.$$

A *base norm space* is a partially ordered normed real vector space V with a generating cone V^+ and a distinguished base K of V^+ such that K lies in a hyperplane $H \not\equiv 0$ in V and $\text{co}(K \cup -K)$ is the unit ball in V . The base norm space will be denoted by (V, K) . The norm of V is the base norm with respect to K denoted by $\|\cdot\|_K$ and is given by

$$\|v\|_K = \inf\{\lambda + \mu : v = \lambda x - \mu y, \lambda, \mu \geq 0, x, y \in K\}, \quad v \in V.$$

An order unit space $(A, A^+, 1)$ and a base norm space (V, K) are in *separating order and norm duality* if there is a duality $\langle \cdot, \cdot \rangle : A \times V \rightarrow \mathbb{R}$ such that

- (i) $a \in A^+ \Leftrightarrow \langle a, \varphi \rangle \geq 0$ for all $\varphi \in V^+$,
- (ii) $\varphi \in V^+ \Leftrightarrow \langle a, \varphi \rangle \geq 0$ for all $a \in A^+$,
- (iii) for all $a \in A$ and $\varphi \in V$:

$$\|a\|_1 = \sup_{\|\sigma\|_K \leq 1} |\langle a, \sigma \rangle|, \quad \|\varphi\|_K = \sup_{\|a\|_1 \leq 1} |\langle a, \varphi \rangle|.$$

In this case, $\|\varphi\|_K = \langle 1, \varphi \rangle$ for $\varphi \in V^+$ and $K = \{\rho \in V^+ : \langle 1, \rho \rangle = 1\}$.

2.1 Example. Let $(A, A^+, 1)$ be an order unit space and let $V := A^*$, $V^+ := \{\varphi \in V, \langle a, \varphi \rangle \geq 0 \forall a \in A^+\}$ and $K := \{\rho \in V^+ : \langle 1, \rho \rangle = 1\}$. Then A and V are in separating order and norm duality.

Similarly, let (V, K) be a base norm space and let $A := V^*$, $A^+ := \{a \in A, \langle a, \varphi \rangle \geq 0, \forall \varphi \in V^+\}$ and $1 \in A^+$ be the functional determining the hyperplane H containing K . Then A and V are in separating order and norm duality.

2.2 Example. Let $A = \mathcal{A}^{sa}$ be the self-adjoint part of the C^* -algebra \mathcal{A} . Let $A^+ = \mathcal{A}^+ = \{b^*b, b \in \mathcal{A}\}$ be the positive cone in \mathcal{A} and let 1 be the identity in \mathcal{A} . Then $(A, A^+, 1)$ is an order unit space. If $V = \mathcal{A}^*$ and $K = \mathfrak{S}(\mathcal{A})$ is the set of states on \mathcal{A} , then (V, K) is a base norm space in separating order and norm duality with A . If \mathcal{A} is a von Neumann algebra, then the predual $V = \mathcal{A}_*$ with the set $K = \mathfrak{S}_*(\mathcal{A})$ of normal states is a base norm space in separating order and norm duality with A .

More generally, we may take A to be a JB-algebra resp. a JBW-algebra and obtain similar results.

Let $(A, A^+, 1)$ and (V, K) be an order unit space and a base norm space in separating order and norm duality. We denote by $\sigma(A, V)$ the locally convex topology in A or V , given by the duality of A and V . Unless otherwise stated, we will always consider A and V endowed with these topologies.

Recall that a *face* of the positive cone A^+ (or similarly of V^+) is a hereditary subcone $F \subseteq A^+ : a \in F$ and $b \leq a$ implies $b \in F$. The face of A^+ generated by $a \in A^+$ will be denoted as

$$\text{face}_{A^+}(a) := \{b \in A^+ : b \leq \lambda a, \lambda \geq 0\}.$$

A subcone $F \subseteq V^+$ is a face of V^+ if and only if $F \cap K$ is a face of K : $\rho \in F \cap K$ and $\rho = \lambda \sigma_1 + (1 - \lambda) \sigma_2$ with $\lambda \in (0, 1)$ and $\sigma_1, \sigma_2 \in K$ implies $\sigma_1, \sigma_2 \in F \cap K$.

Let $C \subseteq A$, $W \subseteq V$ be any subsets. We will use the notations

$$C^\circ := \{\varphi \in V : \langle a, \varphi \rangle = 0, \forall a \in C\}, \quad W^\circ := \{a \in A : \langle a, \varphi \rangle = 0, \forall \varphi \in W\}.$$

Note that $C^{\circ\circ}$ is the closed linear span of $C \cup \{0\}$, similarly for $W^{\circ\circ}$.

If $C \subseteq A^+$ and $W \subseteq V^+$, we will denote

$$C^\bullet := \{\varphi \in V^+ : \langle a, \varphi \rangle = 0, \forall a \in C\}, \quad W^\bullet := \{a \in A^+ : \langle a, \varphi \rangle = 0, \forall \varphi \in W\}.$$

Note that C^\bullet is a face of V^+ , similarly for W^\bullet . A face $F \subseteq V^+$ is called *exposed* if there is some $a \in A^+$ such that $F = \{a\}^\bullet$ and *semiexposed* if $F = C^\bullet$ for some $C \subseteq A^+$, equivalently, $F^{\bullet\bullet} = F$. We have similar definitions for faces of A^+ . Note that a face $F \subseteq V^+$ is semiexposed iff $F \cap K$ is semiexposed, which means that there is a subset of effects $D \subseteq E$ such that

$$F \cap K = \{\rho \in K : \langle a, \rho \rangle = 1, \forall a \in D\},$$

and F is exposed iff we may take a singleton $D = \{a\}$.

3. Spectrality in order unit spaces

In this section, we will recall the definitions and basic results of the two approaches to spectrality in order unit spaces that we are going to compare. Throughout the rest of the paper, we will work with a pair (A, V) of an order unit space $(A, A^+, 1)$ and a base norm space (V, K) in separating order and norm duality.

3.1. Spectral duality of Alfsen and Shultz

We first introduce the theory of spectral duality due to Alfsen and Shultz. We refer to [3,5] for details.

Let $J : A \rightarrow A$ be a positive idempotent linear map, continuous in the $\sigma(A, V)$ -topology. We will say that J is *normalized* if $\|J\| \leq 1$, equivalently $p := J(1) \leq 1$. Let us denote

$$\text{Ker}^+ J := \{a \in A^+ : J(a) = 0\} \quad \text{Im}^+ J := \{a \in A^+ : J(a) = a\}. \quad (1)$$

We say that two such maps J and J' are *complementary* if

$$\text{Ker}^+ J = \text{Im}^+ J' \quad \text{and} \quad \text{Ker}^+ J' = \text{Im}^+ J.$$

In this case we say that J' is a *complement* of J .

Notice that J has an adjoint map $J^* : V \rightarrow V$, which is positive and idempotent as well and we have similar definitions of $\text{Ker}^+(J^*)$, $\text{Im}^+(J^*)$ and complementarity. We will say that the maps J and J' are *bicomplementary* if they are complementary and $(J')^*$ is a complement of J^* . In this case, we say that J (or J') is *bicomplemented*. Note that then J has a unique complement J' [5, Corollary 7.11].

3.1 Definition. A *compression* on A is a bicomplemented normalized $\sigma(A, V)$ -continuous positive idempotent linear map $A \rightarrow A$.

If J is a compression, the element $p = J(1)$ is called a *projective unit*. If p is a projective unit, then there is a unique compression J such that $J(1) = p$ and then $J'(1) = 1 - p$ for the complementary compression.

3.2 Definition. [5, Def. 8.42] We say that the spaces A and V are in *spectral duality* if $A = V^*$ and for every $a \in A$, there is a least compression J such that $J(a) \geq a$ and $J(a) \geq 0$.

Note that if $a \in A$ and J is a compression as in the above definition with complement J' , then we have

$$a = J(a) + J'(a) = a^+ - a^-, \quad a^+ := J(a), \quad a^- := -J'(a) \geq 0.$$

In general, for $a \in A$, $b, c \in A^+$, we say that $a = b - c$ is an *orthogonal decomposition* of a if there is a compression J such that $J(b) = b$, $J(c) = 0$.

There are several equivalent characterizations of spectral duality. Here an important role is played by the following assumption:

3.3 Standing Hypothesis. [5] Every exposed face $F \subseteq K$ is projective, which means that it is determined by a projective unit. Since K is a base of V^+ , this means that for every $a \in A^+$ there is a projective unit p such that

$$\{a\}^\bullet = \{p\}^\bullet.$$

By [5, Thms. 8.52 and 8.55], A and V are in spectral duality iff the Standing Hypothesis holds and every element $a \in A$ has a unique orthogonal decomposition: $a = a^+ - a^-$.

The importance of the notion of spectral duality is the fact that it allows us to define spectral resolutions of elements in A , as we briefly describe next. For $a \in A$, let

$$e_\lambda := 1 - J_{\lambda,a}(1),$$

where $J_{\lambda,a}$ is the compression corresponding to $a - \lambda$ (by Definition 3.2). The family $\{e_\lambda\}_{\lambda \in \mathbb{R}}$ has exactly the properties of the spectral resolution of a self-adjoint element in a von Neumann algebra. In particular, we have

$$a = \int \lambda de_\lambda,$$

where the integral is defined as the norm-limit of approximating Riemann-Stieltjes sums. The family is uniquely determined by a and called the *spectral resolution* of a .

The following is a prototypical example of a spectral duality.

3.4 Example. Let $A = \mathcal{M}^{sa}$ be the self-adjoint part of a von Neumann algebra \mathcal{M} . For any projection $p \in A$, the map

$$U_p : a \mapsto pap, \quad a \in A$$

is a compression on A and $p = U_p(1)$. Moreover, any compression on A is necessarily of this form for some projection p , [5, Theorem 7.23]. In this way, projective units generalize the projections in von Neumann algebras. Furthermore, $A = V^*$ where V is the hermitian part of the predual of \mathcal{M} and A and V are in spectral duality. For $a \in A$, let $a = a^+ - a^-$ be the decomposition into the positive and negative part and let p be the support projection of a^+ , then $U_p(a) = a^+ \geq 0$, $a \leq a^+ = U_p(a)$ and U_p is the compression corresponding to a in Definition 3.2. It is easily seen that in this case, $\{e_\lambda\}$ for $a \in A$ is the usual spectral resolution for a . Similar results hold in the case of a general JBW algebra.

3.2. Compression bases and spectrality

In this section, we discuss the notion of spectrality in an order unit space along the lines of [17]. This notion is based on the works [12–15] on compressions on unital ordered abelian groups, that were applied

to order unit spaces in [17]. We stress that the original definitions are of algebraic nature and no duality with a base norm space is assumed. Since we want to compare this to the theory of Alfsen and Shultz, we have to admit such duality here and the maps are assumed continuous. Note that we can always put $V = A^*$, then all the involved mappings are automatically continuous (with respect to the duality of A and V). To distinguish the maps used here from the compressions in the previous paragraph, we use the term F-compressions.

The following definition is based on [17, Definition 1.1 and Lemma 1.1].

3.5 Definition. Let $J : A \rightarrow A$ be a $\sigma(A, V)$ -continuous positive linear mapping. Then J is an *F-compression with focus p on A* if for all $e \in E$,

(F1) $J(1) = p \in E$ (that is, J is normalized),

(F2) $e \leq p \implies J(e) = e$,

(F3) $J(e) = 0 \implies e \leq 1 - p$.

If J satisfies (F1) and (F2) but not necessarily (F3), we say that J is a *retraction*.

It is easily seen from this definition that any retraction is idempotent. Let J be an F-compression and let $\text{Ker}^+ J$ and $\text{Im}^+ J$ be as in (1). The following equalities follow easily from the properties (F2) and (F3):

$$\text{Ker}^+(J) = \{a \in A^+, \exists t > 0, a \leq t(1 - p)\} = \text{face}_{A^+}(1 - p) \quad (2)$$

$$\text{Im}^+(J) = \text{face}_{A^+}(p). \quad (3)$$

We next collect some easy observations on F-compressions and their foci for later use.

3.6 Lemma. Let J and J' be F-compressions, $J(1) = p$ and $J'(1) = p'$. Then J and J' are complementary if and only if $p' = 1 - p$.

Proof. Let J and J' be complementary. Then $1 - p \in \text{Ker}^+ J = \text{Im}^+ J'$ and $p \in \text{Im}^+ J = \text{Ker}^+ J'$, so that $J'(p) = 0$ and $1 - p = J'(1 - p) = J'(1) = p'$. The converse is clear from the equalities (2) and (3). \square

3.7 Lemma. Let p be the focus of a retraction J . Then p is a principal element in E : for any $a, b \in E$ such that $a + b \leq 1$ and $a, b \leq p$ we have $a + b \leq p$. This property is equivalent to the following facial property (cf. [5, Def. 8.45]):

$$\text{face}_{A^+}(p) \cap E = [0, p].$$

If J is an F-compression, the same is true for $1 - p$.

Proof. Let $a, b \in [0, p]$ be such that $a + b \leq 1$. Then by property (F2) we have $a = J(a)$, $b = J(b)$, so that

$$a + b = J(a + b) \leq J(1) = p.$$

Next, assume that an element $q \in E$ has the facial property and let $a, b \leq q$, $a + b \in E$. Then $a + b \leq 2q$, so that $a + b \in \text{face}_{A^+}(q) \cap E = [0, q]$. Conversely, assume that q is principal, and let $0 \leq a \leq \lambda q$ for some $\lambda > 0$. If $\lambda \leq 1$ there is nothing to prove, so assume that $\lambda > 1$. Then $(1/\lambda)a \leq q$ and $a = (1/\lambda)a + (1 - 1/\lambda)a$. We can decompose

$$1 - 1/\lambda = \sum_i \beta_i, \quad i = 1, \dots, n, \quad 0 \leq \beta_i \leq 1/\lambda, \quad \forall i.$$

Then $\beta_i a \leq (1/\lambda)a \leq q$ for all i and since q is principal, we must have $a = (1/\lambda)a + \sum_i \beta_i a \leq q$. The last statement follows easily by (2) and the property (F3). \square

3.8 Lemma. For $p \in E$, consider the following statements:

- (a) p is principal;
- (b) p is extremal in E ;
- (c) p is sharp: if $a \in E$ is such that $a \leq p$ and $a \leq 1 - p$, then $a = 0$.

Then $(a) \implies (b) \implies (c)$.

Proof. Assume (a) and let $a, b \in E$, $\lambda \in (0, 1)$ be such that $\lambda a + (1 - \lambda)b = p$. By the facial property of p , we must have $a, b \leq p$, but this is possible only if $a = b = p$, so that p is extremal in E . Assume (b) and let $a \in E$, $a \leq p$, $a \leq 1 - p$. Then $p \pm a \in E$ and $p = \frac{1}{2}((p - a) + (p + a))$, so that $p - a = p + a = p$ and $a = 0$. Hence p is sharp. \square

We say that the elements $a, b \in E$ are *Mackey compatible*, if there are some elements $c, a_1, b_1 \in E$ such that $c + a_1 + b_1 \leq 1$ and $a = c + a_1$, $b = c + b_1$. A sub-effect algebra P in E is *normal* if for all $d, e, f \in E$ such that $d + e + f \leq 1$ and $d + e, d + f \in P$ we have $d \in P$ [15, Definition 1]. Note that this implies that elements of P are compatible in E iff they are compatible in P .

3.9 Definition. [14, Definition 2] A *compression base* for A is a family $(J_p)_{p \in P}$ of F-compressions on A , indexed by their own foci, such that P is a normal subalgebra of E and whenever $p, q, r \in P$ and $p + q + r \leq 1$, then $J_{p+r} \circ J_{q+r} = J_r$. Elements of P will be called *projections*.

3.10 Example. Let X be a compact Hausdorff space and let $A = C(X, \mathbb{R})$ be the set of continuous functions $X \rightarrow \mathbb{R}$. With the usual ordering of functions and $1 = 1_X$ the constant unit, A is an order unit space and the order unit norm coincides with the maximum norm on $C(X, \mathbb{R})$. Since A is the self-adjoint part of the C^* -algebra $C(X, \mathbb{C})$, we know by [15] (see also Theorem 5.10 below) that the retractions on A are precisely of the form $U_p(f) = pf$, $f \in A$, where $p \in A$ is a projection, which means that p is the characteristic function of a clopen subset of X . By a similar reasoning as in the proof of Corollary 5.11, we have that $(U_p)_{p \in \mathcal{P}(X)}$ is a compression base, here $\mathcal{P}(X)$ is the set of all projections in A .

We will encounter this example repeatedly below and we will always assume that $C(X, \mathbb{R})$ is endowed with the compression base $\{U_p\}_{p \in \mathcal{P}(X)}$.

In the sequel, we fix a compression base $(J_p)_{p \in P}$ for A . Note that since P is a subalgebra, any J_p has a (fixed) complementary F-compression J_{1-p} . Observe also that using the fact that all projections are principal elements, it is easily seen that if $p, q \in P$ and $p + q \leq 1$, then $p + q = p \vee q$ in P , so that P with the orthocomplementation $p \mapsto 1 - p$ is an *orthomodular poset* (OMP) [29, 22]. It follows that if $p, q \in P$ are Mackey compatible, then $p \wedge q$ and $p \vee q$ exists in P . It can be shown as in [14, Thm. 2.5] that P is a *regular* OMP, which means that any pairwise Mackey compatible subset in P is contained in a Boolean subalgebra of P ([22]). A maximal subset of pairwise Mackey compatible elements is called a *block* of P . Every element $p \in P$ is contained in some block B of P .

3.2.1. Compatibility

The following notion for $p \in P$ and $a \in A$ was introduced in [17, Def. 1.5], see also [5, Def. 7.41] for projective units p .

3.11 Definition. For $p \in P$ and $a \in A$, we say that a is *compatible with p* if

$$a = J_p(a) + J_{1-p}(a).$$

The set of all a compatible with $p \in P$ will be denoted by $C(p)$. For a subset $Q \subseteq P$, we denote $C(Q) := \bigcap_{p \in Q} C(p)$. If $B \subseteq P$ is a block of P , the set $C(B)$ is called a *C-block* of A .

The following facts are easily checked (see also [17, Lemma 1.3]).

3.12 Lemma. Let $a \in A$, $p \in P$. Then

- (i) If $J_p(a) \leq a$, then $a \in C(p)$.
- (ii) If $a \in A^+$, then $a \in C(p)$ if and only if $J_p(a) \leq a$.
- (iii) If $a \in E$, then $a \in C(p)$ if and only if a and p are Mackey compatible.
- (iv) If $q \in P$, then $q \in C(p)$ if and only if $p \in C(q)$ if and only if

$$J_p J_q = J_q J_p = J_{p \wedge q}.$$

We denote the set of all projections compatible with $a \in A$ by $PC(a)$ and for $B \subseteq A$, $PC(B) := \bigcap_{a \in B} PC(a)$.

3.13 Lemma. For $B \subseteq A$ we have:

- (i) If $p_1, \dots, p_n \in PC(B)$ are such that $\sum_i p_i \leq 1$, then

$$J_{\sum_i p_i}(a) = \sum_i J_{p_i}(a) \quad \forall a \in B.$$

- (ii) For compatible $p, q \in PC(B)$, we have $p \vee q, p \wedge q \in PC(B)$.

Consequently, $PC(B)$ is a normal subalgebra in E and a regular OMP.

Proof. Let $p, q \in PC(a)$, $p + q \leq 1$ and let $r = 1 - p - q$. Then since $p \leq p + q$ and $1 - p = r + q$, we have for $a \in B$

$$J_{p+q}(a) = J_{p+q}(J_p(a) + J_{1-p}(a)) = J_p(a) + J_{p+q}J_{r+q}(a) = J_p(a) + J_q(a).$$

By induction, this proves (i).

Assume that $p, q \in PC(B)$ are compatible and put $p' = 1 - p$, $q' = 1 - q$. Then for all $a \in B$

$$\begin{aligned} a &= J_p(a) + J_{p'}(a) = J_p(J_q(a) + J_{q'}(a)) + J_{p'}(J_q(a) + J_{q'}(a)) \\ &= J_{p \wedge q}(a) + J_{p \wedge q'}(a) + J_{p' \wedge q}(a) + J_{p' \wedge q'}(a). \end{aligned}$$

Since $1 - p \wedge q = p \wedge q' + q' \wedge p + q' \wedge p'$, all the involved elements are compatible and we obtain by Lemma 3.12

$$\begin{aligned} J_{1-p \wedge q}(a) &= J_{1-p \wedge q}(J_{p \wedge q'}(a) + J_{p' \wedge q}(a) + J_{p' \wedge q'}(a)) \\ &= J_{p \wedge q'}(a) + J_{p' \wedge q}(a) + J_{p' \wedge q'}(a) \end{aligned}$$

so that

$$a = J_{p \wedge q}(a) + J_{1-p \wedge q}(a).$$

Hence $p \wedge q \in PC(B)$, (ii) follows from $p \vee q = 1 - p' \wedge q'$. The last statement follows by [22, Cor. 2.1.14]. \square

3.14 Lemma. For $Q \subseteq P$, $C = C(Q)$ is a norm-closed subspace and an order unit space, with order unit 1. Let $\tilde{J}_p := J_p|_C$, then $\{\tilde{J}_p\}_{p \in P \cap C}$ is a compression base in C . If Q is a block, then $P \cap C = Q$.

Proof. Let $q \in P$, then $C(q) = \{a \in A, a = J_q(a) + J_{1-q}(a)\} = J^{-1}(0)$, where $J = id - J_q - J_{1-q}$ is a bounded linear map. Hence $C(q)$ is a norm-closed subspace and since $1 \in C(q)$, it is an order unit space with order unit 1. Further, let $p \in C(q) \cap P$, then since $J_p J_q = J_q J_p$, we have for any $a \in C(q)$

$$J_p(a) = J_p(J_q(a) + J_{1-q}(a)) = J_q(J_p(a)) + J_{1-q}(J_p(a)),$$

so that $J_p(C(q)) \subseteq C(q)$. Since $C = \bigcap_{q \in Q} C(q)$, we see that C is a norm closed subspace in A and an order unit space with order unit 1. Moreover, for $p \in C \cap P$, $\tilde{J}_p(C) \subseteq C$ and it is straightforward that \tilde{J}_p is an F-compression. By Lemma 3.13, $P \cap C(Q) = PC(Q)$ is a normal subalgebra in E and hence also in $E \cap C$. It is also easily checked that $\{\tilde{J}_p\}_{p \in C \cap P}$ is a compression base in C .

If Q is a block, then clearly $Q \subseteq C(Q) \cap P$, the other inclusion follows by maximality of Q . \square

The following subset can be seen as the *bicommutant* of a in P :

$$P(a) := PC(PC(a) \cup \{a\}),$$

i.e. the set of all projections compatible with a and with all projections compatible with a . Note that all elements in $P(a)$ are pairwise compatible, so that by Lemma 3.13, $P(a)$ is a boolean subalgebra in P . If $q \in P$, then $P(q) = \{0, 1, q, 1 - q\}$.

3.2.2. Projection cover property

3.15 Definition. We say that the compression base $(J_p)_{p \in P}$ (or A) has the *projection cover property* if for every effect $e \in E$ there is an element $p \in P$ such that for $q \in P$, we have $e \leq q$ iff $p \leq q$. Such an element p is necessarily unique and is called the *projection cover* of e , denoted by e^0 .

In this section, we assume that $(J_p)_{p \in P}$ has the projection cover property.

3.16 Lemma. Assume that $(J_p)_{p \in P}$ has the projection cover property. For $a \in E$, $a^0 \in P(a)$.

Proof. Let $q \in PC(a)$, then $a = b + c$, with $b = J_q(a)$, $c = J_{1-q}(a)$. Since $b \leq q$, $c \leq 1 - q$, we have $b^0 \leq q$ and $c^0 \leq 1 - q$, so that $b^0, c^0 \in PC(q)$. We will show that $a^0 = b^0 + c^0 = b^0 \vee c^0 \in PC(q)$. Indeed, since $a = b + c \leq b^0 + c^0$, we have $a^0 \leq b^0 + c^0$. On the other hand, $b, c \leq a \leq a^0$, so that $b^0, c^0 \leq a^0$ and $b^0 + c^0 = b^0 \vee c^0 \leq a^0$. The fact that $a^0 \in PC(a)$ is clear from $a \leq a^0$. \square

The following theorem was proved in [14, Theorem 6.4 (iii)] in a more general context. We give a simpler proof here.

3.17 Theorem. Suppose that $(J_p)_{p \in P}$ has the projection cover property. Then

- (i) P is an orthomodular lattice (OML);
- (ii) P is sup/inf closed in E , in the sense that for every subset $M \subseteq P$, whenever the supremum $\vee M$ exists in E , then $\vee M \in P$, and similarly for the infimum $\wedge M$.

Proof. Since, as noted above, P is an OMP, we only need to prove that it is a lattice, in fact, it is enough to show that any two elements $p, q \in P$ have a supremum in P . Let us denote $r_\lambda := (\lambda p + (1 - \lambda)q)^0$ for some $\lambda \in (0, 1)$. Then since $\lambda p \leq r_\lambda$, we have $p \leq r_\lambda$ by the facial property, similarly $q \leq r_\lambda$. Let $r \in P$ be such that $p, q \leq r$, then $\lambda p + (1 - \lambda)q \leq r$, so that $r_\lambda \leq r$ (since r_λ is the projection cover). Hence we may put $p \vee q = r_\lambda$, for any $\lambda \in (0, 1)$.

Let $M \subseteq P$ and $\wedge M =: a$ exist in E . For all $b \in M$, $a \leq b \implies a^0 \leq b$, hence $a^0 \leq \wedge M = a$, which entails $a = a^0 \in P$. \square

3.2.3. Comparability

3.18 Definition. [17, Def. 1.6] We say that the compression base $(J_p)_{p \in P}$ in A has the *comparability property* if, for every $a \in A$, $P^\pm(a) \neq \emptyset$, where

$$P^\pm(a) := \{p \in P(a) \text{ and } J_{1-p}(a) \leq 0 \leq J_p(a)\}.$$

Let $p \in P^\pm(a)$ and put $b := J_p(a)$, $c := -J_{1-p}(a)$, then we have

$$a = b - c, \quad b, c \in A^+, \quad J_p(b) = b, \quad J_p(c) = 0. \quad (4)$$

Any decomposition of a of the form (4) for some $p \in P$ is called a *P-orthogonal decomposition* of a .

3.19 Proposition. If $P^\pm(a) \neq \emptyset$, then $a \in A$ has a unique *P-orthogonal decomposition*, which will be written as

$$a = a^+ - a^-.$$

Moreover we have $a^+, a^-, |a| := a^+ + a^- \in C(PC(a))$.

Proof. Let $p \in P^\pm(a)$, we have seen that a has a *P-orthogonal decomposition* $a = J_p(a) - (-J_{1-p}(a))$. To prove uniqueness, let $a = b - c$ with $b, c \in A^+$ and $J_q(b) = b$, $J_q(c) = 0$ for some $q \in P$. Then $a \in C(q)$ and hence p and q are compatible, by definition of $P^\pm(a)$. Since $J_{1-p}(a) \leq 0$ and J_q is positive, we have $J_q J_{1-p}(a) \leq 0$, similarly, we also have $J_{1-p} J_q(a) \geq 0$. But q and $1 - p$ are compatible, so that

$$0 \leq J_{1-p} J_q(a) = J_q J_{1-p}(a) \leq 0,$$

hence $J_{1-p} J_q(a) = 0$ and $J_p J_{1-q}(a) = 0$ follows by a similar argument. We obtain

$$J_p(a) = J_p J_q(a) = J_q J_p(a) = J_q(a),$$

this proves that the *P-orthogonal decomposition* is unique. For the last statement, let $q \in PC(a)$, then

$$a^+ = J_p(a) = J_p(J_q(a) + J_{1-q}(a)) = J_q(a^+) + J_{1-q}(a^+),$$

so that $a^+ \in C(PC(a))$. The proof for a^- and $|a|$ is similar. \square

We will assume below that $(J_p)_{p \in P}$ is a compression base with the comparability property. We will show that in this case any $a \in A$ is contained in some C-block of A and that the C-blocks are isomorphic to (sub)spaces of functions as in Example 3.10. Moreover, we will extend the notion of compatibility to all pairs of elements in A . We start by showing that in this case all sharp elements are projections (cf. [14,30]).

3.20 Lemma. Suppose that $(J_p)_{p \in P}$ has the comparability property and let $q \in E$. Then $q \in P$ if and only if q is sharp.

Proof. We know by Lemmas 3.7 and 3.8 that all elements in P are sharp. Conversely, let $p \in E$ be sharp and let $a = 2p - 1$. Let $q \in P^\pm(a)$, then $J_q(a) \geq 0$, so that $J_q(1 - p) \leq J_q(p)$. On the other hand, since $a \in C(q)$, it is easily seen that q is compatible with p , so that $J_q(p) \leq p$ and also $J_q(1 - p) \leq 1 - p$. Hence $J_q(1 - p) \leq p, 1 - p$ and we must have $J_q(1 - p) = 0$. It follows that

$$a^+ = J_q(a) = J_q(1 - 2(1 - p)) = q.$$

Similarly, we obtain that $a^- = 1 - q$, so that $a = a^+ - a^- = 2q - 1$ and $p = q \in P$. \square

3.21 Lemma. (cf. [14, Thm. 3.7]) Assume that $(J_p)_{p \in P}$ has the comparability property. Let $a \in A$ and let $\lambda_1 \leq \lambda_2 \leq \dots$ be a sequence. Then there are projections $q_i \in P^\pm(a - \lambda_i)$ such that $q_i \geq q_{i+1}$, $i = 1, 2, \dots$

Proof. We take any $q_1 \in P^\pm(a - \lambda_1)$ and construct the rest of the sequence by induction. So assume that for some $k \geq 1$ we already have a sequence $q_1 \geq \dots \geq q_k$ with the required properties. Choose some $q \in P^\pm(a - \lambda_{k+1})$ and put $q_{k+1} := q \wedge q_k$. Clearly, $q_{k+1} \in P(a) = P(a - \lambda_{k+1})$ and

$$J_{q_{k+1}}(a - \lambda_{k+1}) = J_{q_k} J_q(a - \lambda_{k+1}) \geq 0.$$

Let $q' = 1 - q$, $q'_k = 1 - q_k$. Since q, q_k are elements of the boolean algebra $P(a) = P(a - \lambda_{k+1})$, so are q', q'_k and we have $1 - q \wedge q_k = q' \wedge q_k + q \wedge q'_k + q' \wedge q'_k$. By Lemma 3.13,

$$J_{1-q \wedge q_k}(a - \lambda_{k+1}) = (J_{q' \wedge q_k} + J_{q \wedge q'_k} + J_{q' \wedge q'_k})(a - \lambda_{k+1})$$

and we have

$$\begin{aligned} J_{q' \wedge q_k}(a - \lambda_{k+1}) &= J_{q_k} J_{q'}(a - \lambda_{k+1}) \leq 0, \\ J_{q \wedge q'_k}(a - \lambda_{k+1}) &\leq J_{q \wedge q'_k}(a - \lambda_k) = J_q J_{q'_k}(a - \lambda_k) \leq 0 \\ J_{q' \wedge q'_k}(a - \lambda_{k+1}) &\leq J_{q'} J_{q'_k}(a - \lambda_k) \leq 0. \end{aligned}$$

Hence $J_{1-q_{k+1}}(a - \lambda_{k+1}) \leq 0$ and so $q_{k+1} \in P^\pm(a - \lambda_{k+1})$. \square

3.22 Theorem. Assume that $(J_p)_{p \in P}$ has the comparability property. Then any $a \in A$ is in the norm-closed linear span of $P(a)$: $a \in \overline{\text{span}}(P(a))$, the same is true for $a^+, a^-, |a|$.

Proof. For $\epsilon > 0$, let us choose a sequence

$$-\|a\|_1 =: \lambda_0 < \lambda_1 < \dots < \lambda_n < \lambda_{n+1} := \|a\|_1, \quad \lambda_{i+1} - \lambda_i \leq \epsilon, \quad i = 0, \dots, n.$$

By Lemma 3.21, there is a sequence $q_i \in P^\pm(a - \lambda_i)$, $i = 0, \dots, n+1$ such that $q_i \geq q_{i+1}$, note that we may put $q_0 := 1$ and $q_{n+1} := 0$. Put

$$p_i := q_{i-1} - q_i = q_{i-1} \wedge q'_i, \quad i = 1, \dots, n+1.$$

Then $p_i \in P(a)$, $\sum_i p_i = 1$, so by Lemma 3.13, $a = \sum_i J_{p_i}(a)$. We also have

$$\lambda_{i-1} p_i \leq J_{p_i}(a) \leq \lambda_i p_i, \quad i = 1, \dots, n+1$$

Indeed, $J_{p_i}(a - \lambda_i) = J_{q_{i-1}}J_{q'_i}(a - \lambda_i) \leq 0$, so that $J_{p_i}(a) \leq \lambda_i p_i$. The other inequality follows from $J_{p_i}(a - \lambda_{i-1}) = J_{q'_i}J_{q_{i-1}}(a - \lambda_{i-1}) \geq 0$. For any choice of $\xi_i \in (\lambda_{i-1}, \lambda_i)$, we obtain

$$-\epsilon p_i \leq J_{p_i}(a) - \xi_i p_i \leq \epsilon p_i, \quad i = 1, \dots, n+1,$$

so that, summing up over i , we obtain $-\epsilon \leq a - \sum_i \xi_i p_i \leq \epsilon$ which means that $\|a - \sum_i \xi_i p_i\|_1 \leq \epsilon$.

Let $p \in P^\pm(a)$, so that $a^+ = J_p(a)$, and let $a_n \in \overline{\text{span}}(P(a))$ be such that $a_n \rightarrow a$ in norm, then also $J_p(a_n) \in \overline{\text{span}}(P(a))$ and $J_p(a_n) \rightarrow J_p(a) = a^+$. The proof for a^- and $|a|$ is similar. \square

3.23 Corollary. Assume that $(J_p)_{p \in P}$ has the comparability property.

- (i) For any $a \in A$ and $p \in P$, $a \in C(p)$ if and only if $P(a) \subseteq C(p)$.
- (ii) For any $a \in A$, there is some block $B \subseteq P$ such that $a \in C(B)$.
- (iii) For any block $B \subseteq P$, $C(B) = \overline{\text{span}}(B)$.

Proof. For (i), let $a \in A$ and $p \in P$. It is clear that if $a \in C(p)$ then $P(a) \subseteq C(p)$. Conversely, assume that $P(a) \subseteq C(p)$, then $q = J_p(q) + J_{1-p}(q)$ for all $q \in P(a)$, hence for all elements of $\overline{\text{span}}(P(a))$, the statement (i) now follows from Theorem 3.22.

Since $P(a)$ is a boolean subalgebra in P , it is contained in some block $B \subseteq P$. Then $P(a) \subseteq C(B)$ and by Lemma 3.14 also $\overline{\text{span}}(P(a)) \subseteq C(B)$, this shows (ii).

For (iii) let $a \in C(B)$, then $B \subseteq PC(a)$, so that $P(a) \subseteq PC(B) = B$, the last equality follows by maximality of B . We obtain

$$a \in \overline{\text{span}}(P(a)) \subseteq \overline{\text{span}}(B),$$

so that $C(B) \subseteq \overline{\text{span}}(B)$. The converse inclusion is obvious. \square

We will now look at the structure of the C-blocks of A in the comparability case.

3.24 Lemma. Assume that $(J_p)_{p \in P}$ has the comparability property and let B be a block of P . Then $C(B)$ is an order unit space and $(\tilde{J}_p)_{p \in B}$ where $\tilde{J}_p = J_p|_{C(B)}$ is a compression base in $C(B)$ with the comparability property.

Proof. By Lemma 3.14, $C(B)$ is an order unit space and $(\tilde{J}_p)_{p \in B}$ is a compression base in $C(B)$. By Corollary 3.23 (i), we see that $a \in C(B)$ if and only if $P(a) \subseteq C(B)$, where $P(a)$ is computed with respect to all of P . By maximality, this means that $P(a) \subseteq B$. On the other hand, since all elements in $C(B)$ are compatible with all projections in B , the set $P_B(a)$, computed in $C(B)$ with respect to B , is all of B . Hence $\emptyset \neq P^\pm(a) \subseteq B$ has the required properties also with respect to B . It follows that $(\tilde{J}_p)_{p \in B}$ has the comparability property. \square

3.25 Theorem. Let $\{J_p\}_{p \in P}$ be a compression base in A with the comparability property. Let $B \subseteq P$ be a block. Then there is a totally disconnected compact Hausdorff space X such that

- (i) B is isomorphic (as a Boolean algebra) to the Boolean algebra $\mathcal{P}(X)$ of all clopen subsets in X .
- (ii) $C(B)$ is isomorphic (as an order unit space) to a norm-dense order unit subspace in $C(X, \mathbb{R})$.
- (iii) If A is norm-complete, then $C(B) \simeq C(X, \mathbb{R})$ (as an order unit space).

Proof. By Corollary 3.23, $C(B)$ is the norm-closed linear span of B . Let X be the Stone space of the block B of P , so that X is a totally disconnected compact Hausdorff space such that (i) is satisfied. Let

$\phi : B \rightarrow \mathcal{P}(X)$ be the boolean algebra isomorphism. Since B is a Boolean algebra, any element $a \in \text{span}(B)$ can be (uniquely) written as $a = \sum_{i=1}^k a_i p_i$, where $p_i \in B$ and $\sum_i p_i = 1$. In this case, $a \in A^+$ iff $a_i \geq 0$ for all i and $\|a\|_1 = \sum_i |a_i|$. Similarly, let $F(X, \mathbb{R})$ be the set of all simple functions in $C(X, \mathbb{R})$. Since the clopen sets separate points of X , $F(X, \mathbb{R})$ is norm-dense in $C(X, \mathbb{R})$ by Stone-Weierstrass theorem. Any $f \in F(X, \mathbb{R})$ can be written as $f = \sum_i a_i \chi_{X_i}$, where $\{X_i\}_{i=1}^k$ is a decomposition of X and χ_{X_i} is the characteristic function of X_i . So the map ϕ extends to a bijection $\phi : \text{span}(B) \rightarrow F(X, \mathbb{R})$, given as

$$\phi\left(\sum_i a_i p_i\right) = \sum_i a_i \chi_{\phi(p_i)}.$$

Let $a = \sum_i a_i p_i$, $b = \sum_j b_j q_j$, where $p_i, q_j \in B$, $\sum_i p_i = \sum_j q_j = 1$. Then $p_i = \bigvee_j (p_i \wedge q_j) = \sum_j p_i \wedge q_j$, $q_j = \sum_i p_i \wedge q_j$ and we have

$$a + b = \sum_{i,j} (a_i + b_j) p_i \wedge q_j, \quad p_i \wedge q_j \in B, \quad \sum_{i,j} p_i \wedge q_j = 1.$$

Hence

$$\phi(a + b) = \sum_{i,j} (a_i + b_j) \chi_{\phi(p_i \wedge q_j)} = \sum_{i,j} (a_i + b_j) \chi_{\phi(p_i)} \chi_{\phi(q_j)} = \phi(a) + \phi(b),$$

so that ϕ is a linear isomorphism of $\text{span}(B)$ onto $F(X, \mathbb{R})$. It is also clear that ϕ is a unital order and norm isomorphism. Extending ϕ to $\overline{\text{span}}(B)$, we obtain (ii), (iii) is clear. \square

In the comparability case, we can extend the notion of compatibility to all pairs of elements in A .

3.26 Definition. Let $(J_p)_{p \in P}$ be a compression base in A with the comparability property. We say that $a, b \in A$ are *compatible* if p and q are compatible for all $p \in P(a)$ and $q \in P(b)$.

Note that by Corollary 3.23 (i), a and b are compatible iff $a \in C(P(b))$ or, equivalently, $b \in C(P(a))$. This also shows that if a or b is in P , we obtain the same notion of compatibility as in Definition 3.11.

3.27 Proposition. Let $(J_p)_{p \in P}$ be a compression base in A with the comparability property. Then:

- (i) Two elements in A are compatible if and only if they are in the same C-block.
- (ii) The C-blocks of A are precisely the maximal sets of mutually compatible elements.

Proof. By Corollary 3.23 (i) and the definition, we see that a and b are compatible iff $P(a)$ and $P(b)$ are contained in the same block B , which is equivalent to $a, b \in C(B)$. This proves (i). To prove (ii), note that by (i), the elements on any C-block are mutually compatible. On the other hand, if $c \in A$ is compatible with all elements of $C(B)$, then clearly $c \in C(B)$. \square

3.28 Example. Let $A = \mathcal{M}^{sa}$ be the self-adjoint part of a von Neumann algebra \mathcal{M} . For any projection $p \in \mathcal{M}$, let $U_p : A \rightarrow A$ be defined as $U_p(a) = pap$, $a \in A$. Then $(U_p)_{p \in \mathcal{P}}$ is a compression base, where \mathcal{P} is the set of all projections in \mathcal{M} . It is easily seen that this compression base has the comparability property.

The elements $a, b \in A$ are compatible in the sense of Definition 3.26 if and only if they commute. Blocks of \mathcal{P} are the maximal sets of mutually commuting projections in \mathcal{M} and the C-blocks are precisely the maximal abelian von Neumann subalgebras in \mathcal{M} .

3.2.4. Spectrality

3.29 Definition. [17, Def. 1.7] The compression base $(J_p)_{p \in P}$ in an order unit space is *spectral* if it has both the projection cover and the comparability property. A *spectral order unit space* is an order unit space with a spectral compression base.

Recall that we say that P is monotone σ -complete if suprema (infima) of ascending (descending) sequences in P exist and belong to P .

3.30 Theorem. Assume that P is monotone σ -complete. Then $(J_p)_{p \in P}$ is spectral if and only if it has the comparability property.

Proof. Assume that P is monotone σ -complete and let $(J_p)_{p \in P}$ have the comparability property. We will show that $(J_p)_{p \in P}$ also has the projection cover property. So let $a \in E$. By considering the sequence $\lambda_n = 1 - 1/n$ and $1 - a$ in Lemma 3.21, we obtain a descending sequence $q_n \geq q_{n+1}$ such that $q_n \in P^\pm(1/n - a)$. Then $J_{q_n}(1/n - a) \geq 0$, so that $J_{q_n}(a) \leq 1/nq_n$. Put $q = \wedge_n q_n$, then

$$J_q(a) = J_q J_{q_n}(a) \leq 1/n J_q(q_n) = 1/nq, \quad \forall n \in \mathbb{N},$$

it follows that $J_q(a) = 0$ and $a \leq 1 - q$. Assume that $p \in P$ is such that $a \leq p$, then $a \in C(p)$ so that by Corollary 3.23 (i), $q_n \in C(p)$ for all n . Since $J_{q'_n}(a) \geq 1/nq'_n$, we obtain

$$1/n J_{p'}(q'_n) \leq J_{p'} J_{q'_n}(a) = J_{q'_n} J_{p'}(a) = 0,$$

hence $J_{p'}(q'_n) = 0$, so that $q'_n \leq p$ for all n . It follows that $1 - q = \vee_n q'_n \leq p$, so that $1 - q$ is the projection cover of a . This shows that $(J_p)_{p \in P}$ is spectral. \square

3.31 Lemma. Let X be a compact Hausdorff space. The following are equivalent.

- (i) $C(X, \mathbb{R})$ is spectral.
- (ii) $C(X, \mathbb{R})$ is monotone σ -complete.
- (iii) X is basically disconnected.
- (iv) The Boolean algebra $\mathcal{P}(X)$ is monotone σ -complete.

Proof. The equivalences (i) - (iii) were proved in [16, Thm. 4.8]. The equivalence (iii) \iff (iv) is well known. \square

3.32 Example. (cf. [17, Ex. 1.7]) Let X be a totally disconnected compact Hausdorff space which is not basically disconnected and let $A = F(X, \mathbb{R})$ be the algebra of continuous simple functions, with the compression base $(J_p)_{p \in \mathcal{P}(X)}$, where $J_p = U_p|_A$, see Example 3.10. Then $\mathcal{P}(X)$ is not monotone σ -complete, but it is easily checked that the compression base is spectral.

As we will see next, the property that A is not norm complete was crucial in the above example.

3.33 Theorem. Assume that A is norm complete and let $(J_p)_{p \in P}$ be a compression base with the comparability property. The following are equivalent.

- (i) $(J_p)_{p \in P}$ is spectral.
- (ii) For any block B of P , $(\tilde{J}_p = J_p|_{C(B)})_{p \in B}$ is a spectral compression base in $C(B)$.

- (iii) Any C -block in A is monotone σ -complete.
- (iv) Any C -block in A is isomorphic to $C(X, \mathbb{R})$ for some basically disconnected compact Hausdorff space X .
- (v) P is monotone σ -complete.

Proof. Assume that (i) holds and let B be a block of P , then by Corollary 3.23 (i), $P(a) \subseteq B$ if and only if $a \in C(B)$. By Lemma 3.16, $a^0 \in P(a) \subseteq B$, so that $(\tilde{J}_p)_{p \in B}$ has the projection cover property. We have the comparability property by Lemma 3.24. This proves (ii).

By Theorem 3.25 (iii), any C -block $C(B)$ is isomorphic to $C(X, \mathbb{R})$, where X is the Stone space of B . It is clear that in this isomorphism, $(\tilde{J}_p)_{p \in B}$ is identified with the compression base $(U_p)_{p \in \mathcal{P}(X)}$, see Example 3.10. The equivalence of (ii), (iii) and (iv) now follows by Lemma 3.31.

Assume (iii), we will prove that then every descending sequence in P has an infimum in P . We use a method similar to the one used in [31, Lemma 1.3]. So let (p_n) be such a sequence and let $L \subseteq P$ be the set of all lower bounds of (p_n) in P . Notice that L is upward directed. Indeed, let $p, q \in L$ and let $a = \frac{1}{2}(p+q)$. Then a is compatible with each p_n , so that there is a block B in P such that $(p_n) \cup \{a\} \subseteq C(B)$. Since $C(B)$ is spectral, a has a projection cover $r = a^0 \in B$. Since $a \leq p_n$, we have $r \leq p_n$ for all n , so that $r \in L$. Also $p, q \leq 2a \leq 2r$, we have $p, q \leq r$ by the facial property of r . Hence L is upward directed.

Further, let L_0 be an increasing chain in L , then again $L_0 \cup (p_n)$ is contained in some block B_0 of P . Since B_0 is monotone σ -complete, (p_n) has an infimum p_0 in B_0 . Clearly, $p_0 \in L$ and $p_0 \geq q$ for any $q \in L_0$. It follows that every increasing chain in L has an upper bound, so by Zorn's Lemma, L has a maximal element, but since L is upward directed, L has a greatest element, which is the greatest lower bound for (p_n) . This proves (v).

The implication (v) \implies (i) is proved in Theorem 3.30. \square

In the rest of this section we assume that $(J_p)_{p \in P}$ is a spectral compression base in A . In such a case, there is a unique mapping $*$: $A \rightarrow P$, called the *Rickart mapping*, such that for all $a \in A$ and $p \in P$

$$p \leq a^* \Leftrightarrow a \in C(p) \text{ and } J_p(a) = 0.$$

Indeed, put

$$a^* := 1 - (\|a\|_1^{-1}|a|)^0, \quad a \in A.$$

If $p \leq a^*$, then $(\|a\|_1^{-1}|a|)^0 \leq 1 - p$. It follows that $a^+, a^- \leq \|a\|_1(1 - p)$ and hence $J_p(a) = 0$, $J_{1-p}(a) = a$, so that $a \in C(p)$. For the converse, assume that $a \in C(p)$ and $J_p(a) = 0$. Let $q \in P^\pm(a)$, then $q \in C(p)$ and

$$0 = J_p(a) = J_{p \wedge q}(a) + J_{p \wedge q'}(a).$$

Since $0 \leq J_{p \wedge q}(a) \leq q$ and $0 \leq -J_{p \wedge q'}(a) \leq 1 - q$, we must have $J_{p \wedge q}(a) = J_{p \wedge q'}(a) = 0$. Hence $J_p(a^+) = J_p(a^-) = J_p(|a|) = 0$, so that $|a| \in \text{Ker}^+ J_p$. By (2) and the facial property, we see that $|a| \leq \|a\|_1(1 - p)$, which entails that $\|a\|_1^{-1}|a| \leq 1 - p$. This implies that $p \leq 1 - (\|a\|_1^{-1}|a|)^0 = a^*$. Uniqueness is clear.

We note that if an order unit space has the comparability property, then the projection cover property is equivalent to the existence of the Rickart mapping [17, Thm 2.1][14, Thm. 6.5].

Some important properties of the Rickart mapping are collected in the following lemma.

3.34 Lemma. [17, Lemma 2.1]

- (i) For $0 \leq a \leq b$, we have $b^* \leq a^*$;

- (ii) For $a \in A$, $a^* \in P(a)$;
 (iii) For $a \in A$, $a^{**} := (a^*)^* = 1 - a^*$ is determined by the condition

$$J_p(a) = a \Leftrightarrow a^{**} \leq p;$$

- (iv) For $e \in E$, e^{**} is the projection cover of e .

3.35 Proposition. For $a \in A$, $(a^+)^{**}$ is the least element in $P^\pm(a)$.

Proof. Let $p = (a^+)^{**}$ and $q \in P^\pm(a)$, so that $a^+ = J_q(a)$, $a^- = -J_{1-q}(a)$. Then $J_q(a^+) = a^+$, so that $p \leq q$. Hence $1 - q \leq 1 - p$, so that $J_p(a^-) = 0$. It follows that $J_p(a) = J_p(a^+) = a^+ \geq 0$ and $J_{1-p}(a) = J_{1-p}(-a^-) = -a^- \leq 0$. In particular, $p \in PC(a)$. We next show that $p \in PC(PC(a))$.

Assume that $q \in PC(a)$ and let $r \in P^\pm(a)$. Then r is compatible with q and $r, q \in PC(a)$, so that by Lemma 3.13, we obtain

$$a = (J_{q \wedge r} + J_{q' \wedge r} + J_{q \wedge r'} + J_{q' \wedge r'})(a),$$

so that

$$a^+ = J_r(a) = J_{q \wedge r}(a) + J_{q' \wedge r}(a) = J_q(a^+) + J_{q'}(a^+).$$

It follows that $PC(a) \subseteq PC(a^+)$. Using Lemma 3.34, we obtain

$$p \in P(a^+) \subseteq PC(PC(a^+)) \subseteq PC(PC(a)).$$

Now we see that $p \in P^\pm(a)$ and we have already shown that $p \leq q$ for all $q \in P^\pm(a)$. \square

Similarly as in the case of spectral duality, we may now define the *spectral resolution* of a as a family $(p_{a,\lambda})_{\lambda \in \mathbb{R}} \subseteq P$ defined by [17, Def. 3.2]

$$p_{a,\lambda} := ((a - \lambda)^+)^*.$$

Note that by Lemma 3.34, $p_{a,\lambda} \in P(a)$ for all $\lambda \in \mathbb{R}$.

Let $\lambda_1 \leq \dots \leq \lambda_n$ be any sequence. Using Proposition 3.35 and the construction in the proof of Lemma 3.21, we see that $q_i := ((a - \lambda_i)^+)^{**}$ is a descending sequence in P , which by Lemma 3.34 (iii) gives an ascending sequence of spectral projections

$$p_{a,\lambda_1} \leq \dots \leq p_{a,\lambda_n}.$$

Proceeding as in the proof of Theorem 3.22, we can see that every $a \in A$ can be written as a norm-convergent integral of Riemann-Stieltjes type

$$a = \int_{L_a - 0}^{U_a} \lambda dp(\lambda),$$

where the *spectral lower and upper bounds* for a are defined by $L_a := \sup\{\lambda \in \mathbb{R} : \lambda \leq a\}$ and $U_a := \inf\{\lambda \in \mathbb{R} : a \leq \lambda\}$, respectively.

Properties of spectral resolutions are described in [17, Section 3]. In particular, we have

3.36 Corollary. *If $p \in P$, then $a \in C(p) \Leftrightarrow p_{a,\lambda} \in C(p)$ for all $\lambda \in \mathbb{R}$.*

3.37 Corollary. *There exists an ascending sequence $a_1 \leq a_2 \leq \dots$ in $C(PC(a))$ such that each a_n is a finite linear combination of projections in the family $(p_{a,\lambda})_{\lambda \in \mathbb{R}}$ and $\|a - a_n\|_1 \rightarrow 0$.*

4. A comparison of the two spectrality theories

In this section we compare the spectral theory in order unit spaces presented in [17] with the theory of Alfsen and Shultz [5]. We show in more details that the Alfsen and Shultz theory may be considered as a special case of the theory presented in [17].

4.1. Compressions and F-compressions

We start by describing more properties of F-compressions that can be obtained taking the $\sigma(A, V)$ -continuity into account.

Let $J : A \rightarrow A$ be an F-compression with focus p and let $J^* : V \rightarrow V$ be its adjoint:

$$J^* : V \rightarrow V, \quad J^*\varphi = \varphi \circ J, \quad \varphi \in V.$$

It is easy to see that J^* is $\sigma(A, V)$ -continuous, positive and idempotent. Moreover, we have

$$\text{Ker}^+ J^* = \{p\}^\bullet, \quad \text{Im}^+ J^* \subseteq \{1 - p\}^\bullet = (\text{Im}^+ J^*)^{\bullet\bullet}. \quad (5)$$

Indeed, we have

$$\begin{aligned} \text{Ker}^+ J^* &= \{\varphi \in V^+ : J^*\varphi = 0\} = \{\varphi \in V^+ : \langle J(a), \varphi \rangle = 0, \forall a \in A^+\} \\ &= (\text{Im}^+ J)^\bullet = \{p\}^\bullet \end{aligned}$$

and similarly

$$\begin{aligned} \text{Ker}^+ J &= \{a \in A^+ : J(a) = 0\} = \{a \in A^+ : \langle J(a), \varphi \rangle = 0, \forall \varphi \in V^+\} \\ &= \{a \in A^+ : \langle a, J^*\varphi \rangle = 0, \forall \varphi \in V^+\} = (\text{Im}^+ J^*)^\bullet, \end{aligned}$$

so that

$$\text{Im}^+ J^* \subseteq (\text{Im}^+ J^*)^{\bullet\bullet} = (\text{Ker}^+ J)^\bullet = \{1 - p\}^\bullet.$$

4.1 Lemma. *Let p be the focus of an F-compression. Then*

$$\{1 - p\}^{\bullet\bullet} \cap E = [0, 1 - p].$$

Proof. As we have seen,

$$\text{face}_{A^+}(1 - p) = \text{Ker}^+ J = (\text{Im}^+ J^*)^\bullet = \{1 - p\}^{\bullet\bullet}.$$

The result follows by Lemma 3.7. \square

For an F-compression J with focus p , we denote

$$K_p := \{1 - p\}^\bullet \cap K = \{\rho \in K, \langle p, \rho \rangle = 1\},$$

$$\mathcal{S}(J) := \text{Im}(J^*) \cap K.$$

Then $\mathcal{S}(J)$ is a base of the subcone $\text{Im}^+ J^* \subseteq V^+$ and by (5),

$$\mathcal{S}(J) \subseteq K_p = \mathcal{S}(J)^{\bullet\bullet} \cap K. \quad (6)$$

4.2 Lemma. *Let J be an F-compression. Then J is uniquely determined by its focus p and the subset $\mathcal{S}(J)$.*

Proof. Any idempotent map J is uniquely determined by its image and kernel. Since $\text{Im}(J)$ is positively generated and $\text{Im}^+ J = \text{face}_{A^+}(p)$, we see that

$$\text{Im}(J) = \{a \in A : \exists \lambda \geq 0, -\lambda p \leq a \leq \lambda p\}$$

is the order ideal in A generated by p . Further, it is easy to see that

$$\text{Ker}(J) = \text{Im}(J^*)^\circ = (\text{Im}^+ J^*)^\circ = \mathcal{S}(J)^\circ,$$

since $\text{Im}(J^*)$ is positively generated as well and $\mathcal{S}(J)$ is a base of $\text{Im}^+ J^*$. \square

Following [5], we introduce an important property. A continuous positive idempotent map $J : A \rightarrow A$ is *smooth* if

$$\text{Ker}(J) = \text{Tan}_{A^+}(\text{Ker}^+ J),$$

where $\text{Tan}_{A^+}(\text{Ker}^+ J)$ is the *tangent space* of A^+ at $\text{Ker}^+ J$. If J is an F-compression with focus p , then we have (see [5, Prop. 7.3])

$$\text{Tan}_{A^+}(\text{Ker}^+ J) = \text{Tan}_{A^+}(1 - p) = \{1 - p\}^{\bullet\circ}.$$

We say that the dual map $J^* : V \rightarrow V$ is *neutral* if

$$\rho \in K_p \implies \langle J(a), \rho \rangle = \langle a, \rho \rangle, \quad \forall a \in A.$$

4.3 Lemma. *Let J be an F-compression with focus p . The following are equivalent.*

- (i) J is smooth;
- (ii) $\mathcal{S}(J) = K_p$;
- (iii) $\text{Im}^+ J^* = \{1 - p\}^\bullet$;
- (iv) $\text{Im}^+ J^*$ is a semi-exposed face of V^+ ;
- (v) J^* is neutral.

Proof. It is clear that (ii) is equivalent to (iii), the equivalence of (iii) and (iv) is clear from (5). The fact that (i) is equivalent to (iv) and (v) was proved in [5, Chap. 7] for more general maps, we give an easy proof for F-compressions. Note that (v) can be reformulated as:

$$\varphi \in V^+, \langle 1 - p, \varphi \rangle = 0 \implies J^* \varphi = \varphi,$$

in other words, $\{1 - p\}^\bullet \subseteq \text{Im}^+ J^*$. Again by (5), this is equivalent to (iii). Observe that

$$\{1 - p\}^{\bullet\circ} \subseteq (\text{Im}^+ J^*)^\circ = \{a \in A : \langle J(a), \varphi \rangle = 0, \forall \varphi \in V^+\} = \text{Ker}(J) \quad (7)$$

and (i) means that equality holds in (7). It is now clear (iii) implies (i). Conversely, assume (i), then

$$\{1 - p\}^\bullet \subseteq \{1 - p\}^{\bullet\circ} \cap V^+ = (\text{Im}^+ J^*)^{\circ\circ} \cap V^+ = \text{Im}(J^*) \cap V^+ = \text{Im}^+ J^*,$$

here we used the fact that $\text{Im}(J^*)$ is a closed linear subspace generated by $\text{Im}^+ J^*$, so that $(\text{Im}^+ J^*)^{\circ\circ} = \text{Im}(J^*)$. The proof now follows by (5). \square

4.4 Lemma. *Let $p \in E$ be the focus of a smooth F -compression J . Then J is the unique F -compression with focus p .*

Proof. Let \tilde{J} be an F -compression with focus p . Then $\text{Im}(\tilde{J}) = \text{Im}(J)$ and

$$\text{Ker}(J) = K_p^\circ \subseteq \mathcal{S}(\tilde{J})^\circ = \text{Ker}(\tilde{J}).$$

But then for any $a \in A$,

$$\tilde{J}(a) = \tilde{J}(J(a) + a - J(a)) = \tilde{J}(J(a)) = J(a). \quad \square$$

We can now compare compressions (Definition 3.1) and F -compressions (Definition 3.5).

4.5 Theorem. *Any compression is an F -compression. An F -compression is a compression if and only if it is smooth and has a smooth complementary F -compression.*

Proof. Let $J : A \rightarrow A$ be a compression. Since J is a normalized idempotent $\sigma(A, V)$ -continuous linear mapping, we only need to check the properties (F2) and (F3) of Definition 3.5.

Let J' be the complement of J , then $J'(1) = 1 - p$, this can be seen similarly as in Lemma 3.6. To show (F2), let $0 \leq e \leq p$, then $J'(e) \leq J'(p) = 0$, so $e \in \text{Ker}^+ J' = \text{Im}^+ J$, whence $J(e) = e$. For (F3), observe that $J(e) = 0$ implies $e \in \text{Ker}^+ J = \text{Im}^+ J'$. Hence $J'(e) = e \leq 1$ so that $e = J'(e) \leq J'(1) = 1 - p$.

By Definition 3.1, a continuous positive idempotent map J is a compression if and only if it is bicomplemented, which means that there is a complementary continuous positive idempotent map J' such that the dual maps J^* and $(J')^*$ are also complementary. In this case, J' is a compression as well. In particular, J and J' are complementary F -compressions such that by (5)

$$\{p\}^\bullet = \text{Ker}^+ J^* = \text{Im}^+(J')^*, \quad \{1 - p\}^\bullet = \text{Ker}^+(J')^* = \text{Im}^+ J^*.$$

By Lemma 4.3, this means that both J and J' are smooth (cf. [5, Prop. 7.20]). \square

4.2. Compression bases and spectrality

Recall that the focus of a compression is called a projective unit. Let \mathcal{P} denote the set of all projective units in A and for each $p \in \mathcal{P}$, let J_p be the unique compression with focus p . The set $\mathcal{J} = (J_p)_{p \in \mathcal{P}}$ of compressions has a natural ordering given as $J_p \preceq J_q$ if $J_q J_p = J_p$ and the map $J_p \mapsto p$ is an order isomorphism onto the set \mathcal{P} with the ordering induced from A , [5, Prop. 7.32].

In general, it is not clear whether $(J_p)_{p \in \mathcal{P}}$ is a compression base in A . Therefore, although all J_p are F -compressions, we cannot directly use the properties of the map $p \mapsto J_p$ proved for compression bases. We

will next show that it is the case if the Standing Hypothesis 3.3 is true, recall that this means that for any $a \in A^+$ there is a projective unit denoted by $r(a)$ such that the exposed face of K (or, equivalently, of V^+) determined by a is also determined by $r(a)$, that is

$$\{a\}^\bullet = \{r(a)\}^\bullet.$$

It is proved in [5, Prop. 8.1] that under this assumption, \mathcal{P} is an orthomodular lattice and by the above order isomorphism, \mathcal{J} (with complementation) is an OML as well, with $J_p \vee J_q = J_{p \vee q}$ and $J_{p \wedge q} = J_p \wedge J_q$. Moreover, two compressions J_p and J_q commute if and only if $J_p(q) \leq q$ and then $J_p J_q = J_p \wedge J_q = J_{p \wedge q}$, [5, Thm. 8.3].

4.6 Theorem. *If the Standing Hypothesis is satisfied, then $\{J_p\}_{p \in \mathcal{P}}$ is a compression base in A , with the projection cover property.*

Proof. We first prove that \mathcal{P} is a normal subalgebra in E . If $p + q \in E$, then $p \leq 1 - q$ and $p + q = p \vee q = (J_p \vee J_q)(1) \in \mathcal{P}$, so that \mathcal{P} is a subalgebra. Further, let $e, f, d \in E$, $e + d + f \leq 1$, $p := e + d, q := f + d \in \mathcal{P}$. Then $f \leq 1 - p$, $d \leq p$ implies that $J_p(q) = J_p(f) + J_p(d) = d \leq q$ so that J_p and J_q commute and we obtain

$$d = J_p J_q(1) = J_q \wedge J_p(1) \in \mathcal{P}.$$

Similarly, if $p, q, r \in \mathcal{P}$ and $p + q + r \leq 1$, then $J_{p+r} J_{q+r} = J_{q+r} J_{p+r} = J_r$. Hence $(J_p)_{p \in \mathcal{P}}$ is a compression base. Let now $a \in E$ and let $p = r(a)$. This implies that

$$a \in \{a\}^{\bullet\bullet} \cap E = \{p\}^{\bullet\bullet} \cap E = [0, p],$$

hence $a \leq p$, here we have used Lemma 4.1. If $a \leq q \in \mathcal{P}$, then clearly $\{q\}^\bullet \subseteq \{a\}^\bullet = \{p\}^\bullet$, so that

$$p \in \{p\}^{\bullet\bullet} \cap E \subseteq \{q\}^{\bullet\bullet} \cap E = [0, q],$$

hence $p \leq q$. It follows that $p = a^0$. (Also [5, Prop. 8.24]: p is the least projective unit such that $a \in \text{face}_{A^+}(p)$.) \square

4.7 Theorem. *Assume that $A = V^*$. Then the following are equivalent:*

- (i) A and V are in spectral duality;
- (ii) $(J_p)_{p \in \mathcal{P}}$ is a spectral compression base;
- (iii) There is a spectral compression base $(J_p)_{p \in P}$ such that all J_p are smooth.

Proof. By [5, Thm. 8.52, Thm. 8.55], A and V are in spectral duality if and only if the Standing Hypothesis holds and every element $a \in A$ has a unique orthogonal decomposition $a = a^+ - a^-$. So assume (i). We already know by Theorem 4.6 that $(J_p)_{p \in \mathcal{P}}$ is a compression base with the projection cover property. To show comparability, let $a \in A$ and let $a = a^+ - a^-$ be the unique orthogonal decomposition. By [5, Thm. 8.57], J_r with $r = r(a^+)$ is the least compression such that $J_r(a) \geq 0$ and $J_r(a) \geq a$, which implies that $J_{1-r}(a) \leq 0$. Moreover, by [5, Thm. 8.62], $r \in C(PC(a))$. Hence $r \in P^\pm(a)$ and $(J_p)_{p \in \mathcal{P}}$ has the comparability property, so (ii) is true.

The implication (ii) \implies (iii) follows by Theorem 4.5, so assume (iii). Then by Theorem 4.5, all J_p are compressions and so $\mathcal{P} \supseteq P$. Moreover, by Lemma 3.20, all sharp elements are in P , so that $\mathcal{P} = P$ and $(J_p)_{p \in P}$ is the set of all compressions. Let $a \in A$. By Proposition 3.35, $r = (a^+)^{**}$ is the least element in

$P^\pm(a)$ and since $p \mapsto J_p$ is an order isomorphism, J_r is the least compression J such that $J'(a) \leq 0 \leq J(a)$, equivalently, $J(a) \geq a$ and $J(a) \geq 0$. This implies (i) (see Definition 3.2). \square

4.8 Corollary. *Let A and V be in spectral duality. Then F -compressions coincide with compressions.*

Proof. Let A and V be in spectral duality. We already know that every compression is an F -compression and that $(J_p)_{p \in \mathcal{P}}$ is a spectral compression base. By Lemma 3.20, every sharp element in E is in \mathcal{P} and therefore is a projective unit. Since the focus of an F -compression is sharp, it is a projective unit. The statement follows by Lemma 4.4. \square

5. Spectrality for JB-algebras

We describe an important example of (spectral) compression bases on JB-algebras. An overall reference we use for JB-algebras is [5, Chap. 1].

5.1 Definition. [5, Definition 1.1] A *Jordan algebra* A over \mathbb{R} is a vector space equipped with a commutative bilinear product \circ that satisfies the identity

$$(a^2 \circ b) \circ a = a^2 \circ (b \circ a) \text{ for all } a, b \in A. \quad (8)$$

5.2 Definition. [5, Definition 1.5] A JB-algebra is a Jordan algebra A over \mathbb{R} with identity element 1 equipped with a complete norm satisfying the following requirements for $a, b \in A$:

$$\begin{aligned} \|a \circ b\| &\leq \|a\| \|b\|, \\ \|a^2\| &= \|a\|^2, \\ \|a^2\| &\leq \|a^2 + b^2\|. \end{aligned}$$

5.3 Example. The self-adjoint part A^{sa} of a C^* -algebra A is a JB-algebra with Jordan product $a \circ b = \frac{1}{2}(ab + ba)$. Any norm closed Jordan subalgebra of $\mathcal{B}(H)^{sa}$ is a JB-algebra.

Let A be a JB-algebra. The positive cone in A is defined as

$$A^+ = A^2 = \{a^2 : a \in A\}.$$

Then $(A, A^+, 1)$ is a norm-complete order unit space and its order unit norm $\|\cdot\|_1$ coincides with the norm $\|\cdot\|$ in A , [5, Theorem 1.11]. An element $p \in A$ is called a *projection* if $p = p^2$. The set of all projections in A will be denoted by P . Clearly, we then have $p \in A^+$ and $(1 - p)^2 = 1 - p$, so that $1 - p \in P$ and $p \leq 1$.

The Jordan triple product in A is defined as

$$\{abc\} := (a \circ b) \circ c + (c \circ b) \circ a - (a \circ c) \circ b, \quad a, b, c \in A.$$

If $a = c$ we have

$$\{aba\} = 2(a \circ b) \circ a - a^2 \circ b.$$

5.4 Lemma. [5, Lemmas 1.26 and 1.37] *If $a, b \in A^+$, then (i) and (ii) are equivalent and imply (iii), where*

- (i) $\{aba\} = 0$,
- (ii) $\{bab\} = 0$,

(iii) $a \circ b = 0$.

If a is a projection, then all (i) - (iii) are equivalent.

For any $a, c \in A$, we define a linear map $U_{a,c} : A \rightarrow A$ by

$$U_{a,c}(b) = \{abc\}, \quad b \in A.$$

If $a = c$, we put $U_a(b) := U_{a,a}(b)$. For any $a \in A$, the map U_a is positive [5, Theorem 1.25]. If $p \in P$, we get

$$U_p(a) = 2p \circ (p \circ a) - p \circ a, \quad a \in A.$$

5.5 Lemma. Let $a \in E$, $p \in P$. Then $a \leq p$ if and only if $a \circ p = a$ if and only if $U_p(a) = a$ if and only if $U_{1-p}(a) = 0$.

Proof. Using Lemma 5.4, we obtain the following equivalences:

$$a \circ p = a \iff a \circ (1 - p) = 0 \iff U_{1-p}(a) = 0.$$

On the other hand, it is clear that $a \circ p = a$ implies $U_p(a) = a$ and since U_p is positive and $U_p(1) = p$, we obtain that $a = U_p(a) \leq U_p(1) = p$. From $a \leq p$, it similarly follows that $0 \leq U_{1-p}(a) \leq U_{1-p}(p) = 0$, which implies $a \circ p = a$, closing the chain of implications. \square

Elements $a, b \in A^+$ are *orthogonal* ($a \perp b$) if $\{aba\} = \{bab\} = 0$. Note that if $p, q \in P$, then $p \perp q$ if and only if $p \leq 1 - q$ (by Lemma 5.5). By [5, Proposition 1.28], each $a \in A$ can be expressed uniquely as a difference of positive orthogonal elements:

$$a = a^+ - a^-, \quad a^+, a^- \in A^+, \quad a^+ \perp a^-. \quad (9)$$

Both a^+, a^- are contained in the norm-closed subalgebra $A(a, 1) \subseteq A$ generated by a and 1. The decomposition in (9) is called the *orthogonal decomposition* of a .

From now on, we will consider A in the separating order and norm duality with $V = A^*$.

5.6 Proposition. For any $p \in P$, U_p is a compression on A with focus p .

Proof. We will first show that U_p is an F-compression. As we have seen, U_p is linear and positive and it is clear that $U_p(1) = p \leq 1$, so (F1) holds. The conditions (F2) and (F3) follow from Lemma 5.5. Let $U_p^* : V \rightarrow V$ be the dual map. By [5, Prop. 1.41], U_p^* is neutral, hence U_p is smooth (Lemma 4.3). Since U_{1-p} has the same properties and is complementary to U_p (Lemma 3.6), the proof is finished by Theorem 4.5. \square

We have the following characterization of projections.

5.7 Lemma. For $p \in E$, the following are equivalent.

- (i) p is a projection;
- (ii) p is principal;
- (iii) p is sharp.

Proof. Assume that $p \in P$, then by Proposition 5.6, p is the focus of the compression U_p . Hence p is principal, so (i) implies (ii). The implication (ii) \implies (iii) is by Lemma 3.8. Finally, let p be sharp. Since $p \in E$, we have

by the functional calculus [5, Prop. 1.21] that $p^2 \leq p$. It follows that $0 \leq p - p^2 = (1 - p) - (1 - p)^2 \leq p, 1 - p$, so that $p - p^2 = 0$ and $p \in P$. \square

It follows by Proposition 5.6, Lemmas 3.8 and 4.4 that any F-compression on A is of the form U_p for some projection p . We will next show that also any retraction on A is of this form. By [15], this means that the additive group of A is a compressible group, which will show that $(U_p)_{p \in P}$ is a compression base. We will need some preparation first.

Let $T_a : A \rightarrow A$ be the map obtained by Jordan multiplication by a . We say that $a, b \in A$ *operator commute* if $T_a T_b = T_b T_a$. This will be denoted by $a \leftrightarrow b$. The next result shows that if b or a is a projection, then this relation is the same as compatibility.

5.8 Theorem. [5, Proposition 1.47] *Let A be a JB-algebra, $a \in A$, and let p be a projection in A . The following are equivalent:*

- (i) *a and p operator commute;*
- (ii) *$T_p a = U_p a$;*
- (iii) *$(U_p + U_{1-p})(a) = a$ (that is, $a \in C(p)$);*
- (iv) *a and p are contained in an associative subalgebra in A .*

It also follows that any element in the subalgebra $A(a, 1)$ generated by a and 1 commutes with any projection that commutes with a , that is,

$$A(a, 1) \subseteq C(PC(a)). \quad (10)$$

Indeed, let $a \in C(p)$, then a and p are contained in an associative subalgebra $A_0 \subseteq A$, which clearly also contains $A(a, 1)$, so that $A(a, 1) \subseteq C(p)$.

We will also need the following *Peirce decomposition* of elements in A .

5.9 Theorem. [21] *Let A be a Jordan algebra with unit 1 , $p \in A$ be a projection. Then A decomposes into the direct sum*

$$A = U_p(A) \oplus U_{p, 1-p}(A) \oplus U_{1-p}(A),$$

where

$$\begin{aligned} U_p(A) &= \{x \in A : p \circ x = x\}, \\ U_{p, 1-p}(A) &= \{x \in A : p \circ x = \frac{1}{2}x\}, \\ U_{1-p}(A) &= \{x \in A : p \circ x = 0\}. \end{aligned}$$

If $x \in U_p(A)$ or $x \in U_{1-p}(A)$ then $x \in C(p)$.

The proof of the following result is a modification of the proof of [13, Theorem 4.5].

5.10 Theorem. *Let A be a JB-algebra and let $J : A \rightarrow A$ be a retraction with focus $p = J(1)$. Then $p \in P$ and $J = U_p$.*

Proof. As the focus of a retraction, p is principal and hence $p \in P$ by Lemma 5.7. Let $p' = 1 - p$. If $e \in A$, $0 \leq e \leq 1$ then $0 \leq \{p'ep'\} \leq p'$, whence

$$0 \leq e \leq 1 \implies J(\{p'ep'\}) = 0.$$

If $0 \leq e \leq 1$ then $0 \leq pep \leq p$, whence

$$0 \leq e \leq 1 \implies J(\{pep\}) = \{pep\}.$$

We claim that for $a \in A$, $J(\{pap\}) = \{pap\} = U_p(a)$. Indeed, this is easily seen to be true for all positive elements by normalization. Since we can always write $a = x - y$ with $x, y \in A^+$, we have $J(\{pap\}) = \{pap\}$ for all A . By the same argument, $J(\{p'ap'\}) = 0$ for $a \in A$. For the projection p we have the Peirce decomposition as in Theorem 5.9:

$$A = U_p(A) \oplus U_{p,1-p}(A) \oplus U_{1-p}(A).$$

Let $a \in A$, then $a = a_1 + a_2 + a_3$ where $a_1 \in U_p(A)$, $a_2 \in U_{p,1-p}(A)$, $a_3 \in U_{1-p}(A)$. Then we have

$$a_1 = \{pa_1p\}, \quad a_2 \circ p = \frac{1}{2}a_2, \quad a_3 = \{p'a_3p'\}.$$

For $n \in \mathbb{Z}$, consider

$$\begin{aligned} 0 \leq (a_2 - np')^2 &= a_2^2 - 2na_2 \circ p' + n^2p' \\ &= a_2^2 - na_2 + n^2p'. \end{aligned}$$

Applying J and taking into account that $J(p') = 0$ we get

$$nJ(a_2) \leq J(a_2^2), \quad \forall n \in \mathbb{Z}.$$

For positive n we obtain $J(a_2) \leq 0$, for n negative we obtain $-J(a_2) \leq 0$, hence $J(a_2) = 0$. Consequently,

$$J(a) = J(a_1) + J(a_2) + J(a_3) = \{pa_1p\} = \{pap\}. \quad \square$$

5.11 Corollary. *The set $\{U_p\}_{p \in P}$ is a compression base in A .*

Proof. The statement is a special case of [15, Theorem 2.3]. We rewrite the proof in the present situation for the convenience of the reader.

Assume that $p, q \in P$. It is well known that $p + q \leq 1$ if and only if $p + q \in P$, so that P is a subalgebra in E . Let us prove that P is normal. So let $e, f, d \in E$, $e + f + d \leq 1$ and assume that $p := e + d \in P$ and $q := f + d \in P$. Put $J := U_p \circ U_q$, then $J : A \rightarrow A$ is a positive linear map and since $f \leq 1 - p$ and $d \leq p$, we have

$$J(1) = U_p(q) = U_p(f + d) = U_p(d) = d \in E.$$

Further, let $0 \leq a \leq d$, then since $a \leq p$ and $a \leq q$, we have

$$J(a) = U_p(U_q(a)) = U_p(a) = a.$$

Hence J is a retraction and by Theorem 5.10, $d \in P$ and $J = U_d$. This also shows that for $p, q, r \in P$, $p + q + r \leq 1$, we have $U_{p+r} \circ U_{q+r} = U_r$. \square

In [6] the following symbols are defined for a subset S of A :

$$\begin{aligned} S^\perp &= \{a \in A : U_a(x) = 0, \forall x \in S\}, \\ {}^\perp S &= \{x \in A : U_a(x) = 0, \forall a \in S\}, \\ {}^\perp S^+ &= {}^\perp S \cap A^+. \end{aligned}$$

The following notion of a Rickart Jordan algebra was introduced by Ayupov and Arzikulov.

5.12 Definition. [7, Definition], [7, Theorem 1.6, Theorem 1.7] A Jordan algebra A is *Rickart* if one of the following equivalent statements is true

(A1) For every element $x \in A^+$ there is a projection $p \in A$ such that

$$\{x\}^\perp = U_p(A)$$

(A2) For every element $x \in A$ there is a projection $p \in A$ such that

$${}^\perp \{x\}^+ = U_p(A)^+.$$

It was proved in [7] that the self-adjoint part of a C^* -algebra \mathcal{A} is Rickart JB-algebra iff \mathcal{A} is a Rickart C^* -algebra.

5.13 Definition. An element $p \in P$ is called a *carrier* of $a \in A$ if it is the smallest projection such that $p \circ a = a$. The carrier of a will be denoted by $s(a)$.

The following result is straightforward from Lemma 5.5.

5.14 Lemma. Let $a \in E$. Then $s(a)$ is the smallest projection such that $a \leq s(a)$.

5.15 Lemma. [6, Lemma 2.7] Let A be a Rickart JB-algebra and let $a \in A^+$. Then ${}^\perp \{a\}^+ = U_{1-s(a)}(A)^+$.

Proof. Note that using Lemma 5.5, we have for $q \in P$ and $a \in A^+$ that $q \circ a = a$ iff $(1 - q) \circ a = 0$ iff $1 - q \in {}^\perp \{a\}^+$. It follows that $1 - s(a)$ is the largest projection in ${}^\perp \{a\}^+$. If $p \in P$ is such that ${}^\perp \{a\}^+ = U_{1-p}(A)^+$, then $1 - p = U_{1-p}(1 - p) \in {}^\perp \{a\}^+$ and for any projection $r \in {}^\perp \{a\}^+$ we have $U_{1-p}(r) = r$, so that $r \leq 1 - p$ by Lemma 5.5. Hence $p = s(a)$. \square

We are now ready to prove the main result of this section.

5.16 Theorem. Let A be a JB-algebra. Then A is Rickart if and only if A is spectral.

Proof. The family $(U_p)_{p \in P}$ is a compression base by Corollary 5.11. If A is Rickart, then it follows by Lemma 5.15 that any $a \in A^+$ has a carrier $s(a)$ (this was proved in [6, Theorem 2.1, Theorem 2.8] for all $a \in A$). By Lemma 5.14, if $a \in E$, then $s(a)$ is a projection cover of a . Hence $(U_p)_{p \in P}$ has the projection cover property. We have to prove the comparability property.

So let $a \in A$ and let $a = a^+ - a^-$ as in (9). We will prove that $p = s(a^+) \in P^\pm(a)$. Note first that if $a \in E$, then $s(a) = a^0 \in P(a)$, by Lemma 3.16. By normalization, we have $s(a) \in P(a)$ for all $a \in A^+$. In the general case, the elements a^+, a^- are contained in the subalgebra $A(a, 1)$, therefore $a^+, a^- \in C(PC(a))$ by (10). Hence $PC(a) \subseteq PC(\{a^+, a^-\}) \subseteq PC(a^+)$ so that

$$p = s(a^+) \in P(a^+) \subseteq C(PC(a^+)) \subseteq C(PC(a)).$$

By the definition of the carrier and Lemma 5.5, we see that $U_p(a^+) = a^+$. Since $a^- \in {}^\perp\{a^+\}^+$, we have by Lemma 5.15 that $U_{1-p}(a^-) = a^-$, so that $U_p(a^-) = 0$. This implies that $a \in C(p)$, so that $p \in P(a)$. Further,

$$U_p(a) = U_p(a^+) - U_p(a^-) = a^+ - 0 = a^+ \geq 0,$$

and

$$U_{1-p}(a) = U_{1-p}(a^+) - U_{1-p}(a^-) = 0 - a^- = -a^- \leq 0.$$

Hence $p \in P^\pm(a)$ and the comparability property is satisfied.

Conversely, assume that A is spectral. Since the sharp elements are precisely the projections, we have by Lemma 3.20 and Theorem 5.10 that $(U_p)_{p \in P}$ is a spectral compression base. We will check that the condition (A2) of Definition 5.12 is satisfied, so that A is Rickart. Let $a \in A$ and let C be a C-block of A such that $a \in C$ (Corollary 3.23 (ii)). Then by (10) we have $A(a, 1) \subseteq C$. By Theorem 3.33 (iv), there is a basically disconnected compact Hausdorff space X such that $C \simeq C(X, \mathbb{R})$. We will denote the elements of C and their representing functions in $C(X, \mathbb{R})$ by the same symbol. Let $Y := \{x \in X : a(x) \neq 0\}$, then the support of a is the closure \bar{Y} , which is clopen in X .

Let p be the projection in C corresponding to \bar{Y} , then observe that $p = a^{**}$. Indeed, we have $U_p(a) = \tilde{U}_p(a) = pa = a$, hence $a^{**} \leq p$ by Lemma 3.34. Conversely, $a^{**} \in P(a) \subseteq C$, so that $a = U_{a^{**}}(a) = \tilde{U}_{a^{**}}(a) = a^{**}a$. Since \bar{Y} is the support of a , this implies that $p \leq a^{**}$. Note that it also follows that $a^{**} = (a^2)^{**}$.

We will next prove that ${}^\perp\{a\}^+ = U_{1-p}(A)^+$, so that (A2) holds. This amounts to proving that

$$\{b \in A^+ : U_a(b) = 0\} = \text{Ker}^+ U_p.$$

Assume first that $b \in \text{Ker}^+ U_p$. Since $a = U_p(a)$, we have by [5, Eq. (1.16)]

$$U_a(b) = U_{\{pap\}}(b) = U_p U_a U_p(b) = 0.$$

Conversely, let $b \in A^+$ and let $U_a(b) = 0$. Then we also have $U_{a^2}(b) = U_a U_a(b) = 0$, so that $a^2 \perp b$. Let

$$c := a^2 - b,$$

then by the uniqueness of the orthogonal decomposition, we get $a^2 = c^+$ and $b = c^-$. Consequently, if now C' is a block of A containing c , then also $a^2, b \in C'$. Considering again the representing functions in $C(X', \mathbb{R})$ for a basically disconnected compact Hausdorff space X' , we see that

$$U_{a^2}(b) = 0 \iff 0 = U_b(a^2) = b^2 a^2 \iff b a^2 = 0.$$

It follows that b must be 0 on the support of a^2 , and we have seen above that the projection corresponding to the support is $(a^2)^{**} = a^{**} = p$. This implies that $U_p(b) = 0$, so that we must have $b \in \text{Ker}^+ U_p$. \square

6. Centrally symmetric state spaces

In this section, we obtain a class of order unit spaces such that $A = V^*$ for a base norm space V , where a spectral compression base exists but the duality of A and V is not spectral.

Let $(X, \|\cdot\|)$ be a normed space and let B be its closed unit ball. We will construct a dual pair (A, V) with $A = V^*$ such that the state space $K \cong B$, such state spaces are sometimes called centrally symmetric (cf. [26]).

Put $V := \mathbb{R} \times X$ and

$$V^+ := \{(\alpha, x) : \|x\| \leq \alpha\}, \quad K := \{(1, x) : x \in B\}.$$

It is easy to see that V^+ is a generating cone: indeed, any element $(\alpha, x) \in V$ can be written as

$$(\alpha, x) = (\max\{\alpha, \|x\|\}, x) - (\max\{\|x\| - \alpha, 0\}, 0) \in V^+ - V^+.$$

Moreover, K is a base of V^+ and is located on the hyperplane

$$H_1 := \{(\alpha, x) \in V : \alpha = 1\} \neq \emptyset.$$

Note that $\text{co}(K \cup -K) = \{(\alpha, x), |\alpha| \leq 1, x \in B\}$. This set is clearly radially compact, that is, its intersection with any line L through the origin of V is a closed segment. By [2, Prop. II.1.12], (V, K) is a base normed space and $\text{co}(K \cup -K)$ is the unit ball of the base norm $\|\cdot\|_K$. It follows that

$$\|(\alpha, x)\|_K = \max\{|\alpha|, \|x\|\}.$$

Let $A = V^*$, then A is isomorphic to the space $A_b(K)$ of bounded affine functions $K \rightarrow \mathbb{R}$ ([5]). With the cone $A^+ \cong A_b(K)^+$ of positive functions on K and with $1 \cong 1_K$ the constant unit functional, $(A, A^+, 1)$ is an order unit space, in separating order and norm duality with V .

It is easily seen that we may put $A = \mathbb{R} \times X^*$ where X^* is the dual normed space of X , with the cone

$$A^+ = \{(a_0, y) : y \in X^*, \|y\| \leq a_0\}$$

and the unit $1 = (1, 0)$. The order unit norm is

$$\|(a_0, y)\|_1 = \|y\| + |a_0|$$

and the unit interval E has the form

$$E = \{(a_0, y) : \|y\| \leq \min\{a_0, 1 - a_0\}\},$$

note that this implies

$$(a_0, y) \in E \implies a_0 \in [0, 1] \text{ and } \|y\| \leq 1/2. \quad (11)$$

We next characterize some special elements and subsets of E and show their relation to the structure of the dual unit ball, which will be denoted by B^* .

6.1 Lemma. *Let $a \in E$, $a \neq 0$, $a \neq 1$. The following are equivalent.*

- (i) a is sharp;
- (ii) $\|a\|_1 = \|1 - a\|_1 = 1$;
- (iii) $a = (1/2, y)$ with $\|y\| = 1/2$.

Proof. Assume that $\|a\|_1 < 1$, then there is some $s \in (0, 1)$ such that $(1 + s)a \leq 1$. But then

$$sa \leq a, \quad sa \leq 1 - a,$$

so that a is not sharp. Since a is sharp iff $1 - a$ is sharp, we obtain that (i) implies (ii).

Assume the equality $\|a\|_1 = \|1 - a\|_1 = 1$, that is

$$\|y\| + a_0 = 1 = \|-y\| + 1 - a_0$$

Hence $a_0 = 1 - a_0 = 1/2 = \|y\|$ and (ii) implies (iii).

To show that (iii) implies (i), let $a = (1/2, y)$ with $\|y\| = 1/2$ and let $b = (b_0, w) \in E$, $b \leq a$. Then $(1/2 - b_0, y - w) \in A^+$, so that $\|y - w\| \leq 1/2 - b_0$. We obtain

$$1/2 - b_0 \leq 1/2 - \|w\| \leq \|\|y\| - \|w\|\| \leq \|y - w\| \leq 1/2 - b_0.$$

This entails that $b_0 = \|w\|$ and $1/2 - \|w\| = \|y - w\|$. If also $b \leq 1 - a$ we similarly obtain $1/2 - \|w\| = \|y + w\|$. But then

$$1/2 = \|y\| \leq 1/2\|y - w\| + 1/2\|y + w\| = 1/2 - \|w\| \leq 1/2,$$

which implies that $b_0 = \|w\| = 0$. Hence $b = 0$ and a is sharp. \square

6.2 Lemma. *A subset $F \subseteq E$ is a face of E not containing 0 or 1 if and only if there is a face F_0 of $1/2B^*$ such that*

$$F = \{(1/2, y) : y \in F_0\}.$$

In particular, $a \in E$ is an extreme point of E other than 0 or 1 if and only if $a = (1/2, y)$ for some extreme point y of $1/2B^$.*

Proof. Let F be of the given form and let $a = (a_0, y), b = (b_0, w) \in E$ be such that $\lambda a + (1 - \lambda)b \in F$. Then $\lambda a_0 + (1 - \lambda)b_0 = 1/2$ and $\lambda y + (1 - \lambda)w \in F_0$. Since by (11) $\|y\|, \|w\| \leq 1/2$, it follows that $y, w \in F_0$ as well. In particular, $\|y\| = \|w\| = 1/2$, this and (11) imply that $a_0, b_0 = 1/2$. This shows that F is a face of E , obviously not containing 0 or 1.

Conversely, let F be a face of E , $0, 1 \notin F$, and let $a = (a_0, y) \in F$. Note that if $ta \leq 1$ for $t > 1$, then $a = t^{-1}ta + (1 - t^{-1})0$, so that $0 \in F$, similarly, $\|1 - a\|_1 < 1$ implies $1 \in F$. So we must have $\|a\|_1 = \|1 - a\|_1 = 1$ and by Lemma 6.1, this implies that $a_0 = \|y\| = 1/2$. Let

$$F_0 := \{y \in 1/2B^* : (1/2, y) \in F\},$$

then it is easy to see that F_0 is a face of $1/2B^*$. The last statement is now obvious. \square

6.3 Corollary. *Any extremal element $p \in E$ is one dimensional, that is, $0 \leq a \leq p$ implies $a = tp$ for some $t \in [0, 1]$.*

Proof. By Lemma 6.2 an extremal element in E is of the form $p = (1/2, y)$ for some y extremal in $1/2B^*$. Let $0 \leq b = (b_0, w) \leq p$. Since $\|y\| = 1/2$, we obtain similarly as in the proof of Lemma 6.1 that we must have $\|w\| = b_0$ and $\|y - w\| = 1/2 - \|w\| = \|y\| - \|w\|$. Put $\lambda := 2\|y - w\| = 1 - 2\|w\|$, then we get

$$y = \lambda \left(\frac{1}{2\|y - w\|} (y - w) \right) + (1 - \lambda) \left(\frac{1}{2\|w\|} w \right).$$

As y is extremal, we get $w = 2\|w\|y = 2b_0y$, so that $b = (b_0, 2b_0y) = 2b_0p$, with $2b_0 \in [0, 1]$. \square

We next describe retractions, F-compressions and compressions on A . For this, we need to introduce the following notations for $x \in X$ and $y \in X^*$:

$$\partial_x := \{y \in B^* : \langle y, x \rangle = \|x\|\}, \quad \partial_y^* := \{x \in B : \langle y, x \rangle = \|y\|\}.$$

Then ∂_x, ∂_y^* are faces of the respective unit ball and $\partial_x \neq \emptyset$.

6.4 Proposition. *An element $p \in E$ is the focus of a retraction if and only if $p = (1/2, y)$, where y is an extremal element in $1/2B^*$ attaining its norm on B (i.e. $\partial_y^* \neq \emptyset$). A map $J : A \rightarrow A$ is a retraction with focus p if and only if*

$$J(a) = (a_0 + \langle x, w \rangle)p, \quad a = (a_0, w) \in A$$

for some $x \in \partial_y^*$. Moreover

- (i) The map J is an F-compression if and only if ∂_x is a singleton, that is, $\partial_x = \{2y\}$.
- (ii) The map J is a compression if and only if $\partial_y^* = \{x\}$ and $\partial_x = \{2y\}$.

Proof. Assume that $J : A \rightarrow A$ is a retraction with focus p . By Lemma 3.8, p must be extremal, so that $p = (1/2, y)$ for an extremal element $y \in 1/2B^*$. Note that $1 - p = (1/2, -y)$, so that, by symmetry of B^* , $1 - p$ is an extremal element in E as well. It follows that both p and $1 - p$ are one dimensional. By (3), we obtain $\text{Im}^+ J = \mathbb{R}^+ p$ and since $\text{Im}(J)$ is positively generated, $\text{Im}(J) = \mathbb{R}p$. Consequently, J must be of the form

$$J(a) = \langle a, \rho \rangle p, \quad a \in A,$$

for some linear functional ρ on A . Since J is continuous, positive and $J(1) = p$, we must have $\rho \in K$ and since J is idempotent, $\langle p, \rho \rangle = 1$, so that

$$\rho \in K_p = \{(1, x) : x \in B, \langle (1/2, y), (1, x) \rangle = 1\} = \{(1, x) : x \in \partial_y^*\}.$$

Let $0 \leq a \leq p$, then by Corollary 6.3, $a = tp$ for some $t \in [0, 1]$ so that

$$J(a) = tJ(p) = tp = a.$$

Hence a map of this form is indeed a retraction. This proves the first part of the statement.

A retraction is an F-compression if and only if it satisfies (F3) of Definition 3.5. So let $a = (a_0, w) \in E$, then clearly $J(a) = 0$ if and only if $a_0 + \langle x, w \rangle = 0$, which means that $-a_0^{-1}w \in \partial_x$. So (F3) is true if and only if such a is in $[0, 1 - p]$, but since $1 - p$ is one-dimensional, this means that $(a_0, w) = t(1 - p) = (t/2, -ty)$, which implies $-a_0^{-1}w = 2y$. Since we always have $2y \in \partial_x$, this proves (i).

By the symmetry of the unit ball, we see from (i) that if J is an F-compression, then it has a complementary F-compression J' with focus $1 - p$. Indeed, if $x \in \partial_y^*$ and $\partial_x = \{2y\}$, then clearly $-x \in \partial_{-y}^*$ and $\partial_{-x} = \{-2y\}$. By Theorem 4.5, J is a compression if and only if both J and J' are smooth. By Lemma 4.3, J is smooth if and only if

$$K_p = \mathcal{S}(J) = \text{Im}(J^*) \cap K.$$

Since $J^*\varphi = \langle p, \varphi \rangle \rho$, we see that

$$\mathcal{S}(J) = \{\omega \in K : J^*\omega = \omega\} = \{\omega \in K : \langle p, \omega \rangle \rho = \omega\} = \{\rho\}.$$

This means that K_p and consequently also ∂_y^* must be a singleton, and $\partial_y^* = \{x\}$. If this is true, J is smooth and again by symmetry, J' is smooth as well. This proves (ii). \square

A normed space X is called strictly convex if its closed unit ball B is strictly convex, that is, every boundary point of B is an extreme point of B . A normed space is called smooth if any nonzero element attains its norm at a unique point of the dual unit ball. If X^* is strictly convex (smooth), then X is smooth (strictly convex). If X is reflexive, then these properties are mutually dual. More precisely, X is smooth iff X^* is strictly convex iff ∂_x is a singleton for all $0 \leq x \in X$. For more details, see e.g. [24, §26].

6.5 Theorem. *Let $(X, \|\cdot\|)$ be a Banach space and let the pair (A, V) be constructed as above. Then there is a spectral compression base $(J_p)_{p \in P}$ on A if and only if X is reflexive and smooth.*

Proof. Let $(J_p)_{p \in P}$ be a spectral compression base. By Lemma 3.20, all sharp elements are in P . By Lemma 6.1 and Proposition 6.4, this means that $\partial_y^* \neq \emptyset$ for all $y \in X^*$ with $\|y\| = 1/2$, and hence for all nonzero y in X^* . By James' theorem, this means that X is reflexive. Moreover, since for any y with $\|y\| = 1/2$, $\{2y\} = \partial_x$ is a face of B^* , we see that X^* must be strictly convex, so that X is smooth.

Conversely, if X is a reflexive smooth Banach space, then $\partial_y^* \neq \emptyset$ for all $y \neq 0$ and ∂_x is a singleton for all $x \neq 0$. Therefore, any $p = (1/2, y)$ with $\|y\| = 1/2$ is a focus of an F-compression J_p of the form

$$J_p(a) = (a_0 + \langle x_p, w \rangle)p, \quad a = (a_0, w),$$

for some choice of $x_p \in \partial_y^*$.

Let $P = \{(1/2, y) : \|y\| = 1/2\} \cup \{0, 1\}$ be the set of all sharp elements in E . Since B^* is strictly convex, all $p \in P$, $p \neq 0, 1$ are extremal, hence one dimensional. Using this fact, it is easily seen that P is a normal subalgebra in E and that $(J_p)_{p \in P}$ is a compression base with the projection cover property (we put $J_1 = id$ and $J_0 = 0$). To prove comparability, let $a = (a_0, w) \in A$. If $\pm a \in A^+$, then clearly 0 or 1 is in $P^\pm(a)$. So assume that $\pm a \notin A^+$, which amounts to $\|w\| > |a_0|$. Put $y := (2\|w\|)^{-1}w$, then $p = (1/2, y) \in P$ and we have

$$(a_0, w) = (\|w\| + a_0)p - (\|w\| - a_0)(1 - p),$$

hence $p \in P^\pm(a)$. So $(J_p)_{p \in P}$ is a spectral compression base. \square

6.6 Theorem. *Let $(X, \|\cdot\|)$ be a Banach space and let the pair (A, V) be constructed as above. Then (A, V) is in spectral duality if and only if X is reflexive, smooth and strictly convex.*

Proof. Assume that (A, V) is in spectral duality. Let \mathcal{P} be the set of all projective units and let J_p be the corresponding compressions. By Theorem 4.7, $(J_p)_{p \in \mathcal{P}}$ is a spectral compression base, so that by Theorem 6.5, X is reflexive and smooth. As before, all sharp elements are in \mathcal{P} , so that any $(1/2, y)$ with $\|y\| = 1/2$ is a projective unit. By Proposition 6.4, ∂_y^* must be a singleton, which implies that X^* is smooth. Hence X is strictly convex.

Conversely, let X be a strictly convex and smooth reflexive Banach space. Then we may construct a spectral compression base $(J_p)_{p \in P}$ as in the proof of Theorem 6.5, where P is the set of all sharp elements. Since ∂_y^* is a singleton for all $y \neq 0$, it follows by Proposition 6.4 that all J_p are compressions. The proof now follows by Theorem 4.7. \square

7. Conclusions

We studied in some detail the notion of spectrality in order unit spaces in the sense of Foulis [17]. This notion is based on a particular set of idempotent mappings, called a compression base. Spectrality is characterized by two properties of the compression base: projection cover property and comparability. While the projection cover property can be trivial in some cases (consider for example the compression base consisting only of the zero and identity maps), we proved that the comparability property is already rather strong. If comparability holds, the order unit space is covered by subsets called C-blocks that admit a functional representation. Moreover, this property is equivalent to spectrality if the set P of projections is monotone σ -complete, in particular, in the finite dimensional case.

We compared this notion of spectrality with the spectral theory of Alfsen and Shultz [5], where it is assumed that the order unit space is the dual of a Banach space. We proved that this theory is a special case of the Foulis spectrality and found conditions under which they coincide, as well as examples when they are different. We studied the spectral theory for JB-algebras and proved that Rickart JB-algebras are exactly those that are spectral in the Foulis sense. In this case, the compression base is formed by compressions in the sense of Alfsen-Shultz, but the Alfsen-Shultz theory is not applicable since a Rickart JB-algebra does not have a predual in general.

There is also a more general version of Alfsen-Shultz theory [3], where A is only assumed to be monotone σ -complete. For C^* -algebras, it is known that a C^* -algebra is Rickart if and only if the self-adjoint part of any maximal abelian subalgebra is monotone σ -complete [31], but it is not known if this implies monotone σ -completeness for the self-adjoint part of the whole algebra. A similar characterization of Rickart JB-algebras is unknown. Here the maximal abelian subalgebras should be replaced by maximal associative subalgebras. It is an interesting question whether the maximal associative subalgebras of Rickart JB-algebras coincide with the C-blocks.

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