Variational expression when $\alpha > 1$

The next lemmas may be well known; proofs are given for completeness. [I am not sure whether or not Lemma 0.1 is true even for $p \in (0,1)$, where $\|\cdot\|_p$ is a quasi-norm and the L^p-L^q-duality is not available.]

Lemma 0.1. *For any $p \in [1,\infty)$, $h_\varphi^{1/2p}\mathcal{M}^+ h_\varphi^{1/2p}$ is dense in $L_p(\mathcal{M})^+$ in the norm $\|\cdot\|_p$.*

Proof. We may assume that φ is faithful. Suppose that there is an $a \in L_p(\mathcal{M})^+$ which is not in the closure of $h_\varphi^{1/2p}\mathcal{M}^+ h_\varphi^{1/2p}$. Then by the L^p-L^q-duality (see [5]) and the Hahn–Banach separation theorem, one can choose a selfadjoinf element $b \in L_q(\mathcal{M})$ such that

$$\mathrm{tr}(ab) < 0 \le \mathrm{tr}\left((h_\varphi^{1/2p} x h_\varphi^{1/2p})b\right) = \mathrm{tr}\left(x(h_\varphi^{1/2p} b h_\varphi^{1/2p})\right), \qquad x \in \mathcal{M}^+.$$

This gives that $h_\varphi^{1/2p} b h_\varphi^{1/2p} \ge 0$ and hence $b \ge 0$. Therefore, one has $\mathrm{tr}(ab) \ge 0$, a contradiction. □

Lemma 0.2. *Let $\psi, \varphi \in \mathcal{M}_*^+$ with $\psi \le \varphi$. Then for any $a \in \mathcal{M}$ and $p \in [1,\infty)$,*

$$\mathrm{tr}\left((a^* h_\psi^{1/p} a)^p\right) \le \mathrm{tr}\left((a^* h_\varphi^{1/p} a)^p\right).$$

Proof. Since $1/p \in (0,1]$, it follows (see [2, Lemma B.7] and [3, Lemma 3.2]) that $h_\psi^{1/p} \le h_\varphi^{1/p}$ as τ-measurable positive operators affiliated with $\mathcal{M} \rtimes_{\sigma^{\varphi_0}} \mathbb{R}$ (in which $L_p(\mathcal{M})$ lives). Hence $a^* h_\psi^{1/p} a \le a^* h_\varphi^{1/p} a$ in the same sense. Therefore, by [1, Lemma 2.5 (iii)] and [1, Lemma 4.8] we have $\|a^* h_\psi^{1/p} a\|_p \le \|a^* h_\varphi^{1/p} a\|_p$. □

Theorem 0.3. *Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \ne 0$. Let $\alpha > 1$ and $z \ge \max\{\alpha/2, \alpha - 1\}$. Then*

$$Q_{\alpha,z}(\psi\|\varphi) = \sup_{a \in \mathcal{M}_+} \left\{\alpha \mathrm{Tr}\left((a^{\frac{1}{2}} h_\psi^{\frac{\alpha}{z}} a^{\frac{1}{2}})^{\frac{z}{\alpha}}\right) - (\alpha-1)\mathrm{Tr}\left((a^{\frac{1}{2}} h_\varphi^{\frac{\alpha-1}{z}} a^{\frac{1}{2}})^{\frac{z}{\alpha-1}}\right)\right\}.$$

Proof. The inequality $\ge$ holds for all α and z and was proved in [4, Theorem 2 (vi)]. We now prove the opposite inequality.

Assume first that $Q_{\alpha,z}(\psi\|\varphi) < \infty$, so that there is some $x \in s(\varphi)L_z(\mathcal{M})^+ s(\varphi)$ such that $h_\psi^{\frac{\alpha}{z}} = h_\varphi^{\frac{\alpha-1}{2z}} x h_\varphi^{\frac{\alpha-1}{2z}}$. Plugging this into the right hand side, we obtain

$$\begin{aligned}
&\sup_{a \in \mathcal{M}_+} \left\{\alpha \mathrm{Tr}\left((a^{\frac{1}{2}} h_\psi^{\frac{\alpha}{z}} a^{\frac{1}{2}})^{\frac{z}{\alpha}}\right) - (\alpha-1)\mathrm{Tr}\left((a^{\frac{1}{2}} h_\varphi^{\frac{\alpha-1}{z}} a^{\frac{1}{2}})^{\frac{z}{\alpha-1}}\right)\right\} \\
&= \sup_{a \in \mathcal{M}_+} \left\{\alpha \mathrm{Tr}\left((a^{\frac{1}{2}} h_\varphi^{\frac{\alpha-1}{z}} x h_\varphi^{\frac{\alpha-1}{2z}} a^{\frac{1}{2}})^{\frac{z}{\alpha}}\right) - (\alpha-1)\mathrm{Tr}\left((a^{\frac{1}{2}} h_\varphi^{\frac{\alpha-1}{z}} a^{\frac{1}{2}})^{\frac{z}{\alpha-1}}\right)\right\} \\
&= \sup_{a \in \mathcal{M}_+} \left\{\alpha \mathrm{Tr}\left((x^{\frac{1}{2}} h_\varphi^{\frac{\alpha-1}{2z}} a h_\varphi^{\frac{\alpha-1}{2z}} x^{\frac{1}{2}})^{\frac{z}{\alpha}}\right) - (\alpha-1)\mathrm{Tr}\left((h_\varphi^{\frac{\alpha-1}{2z}} a h_\varphi^{\frac{\alpha-1}{2z}})^{\frac{z}{\alpha-1}}\right)\right\} \\
&= \sup_{w \in L_{\frac{z}{\alpha-1}}(\mathcal{M})^+} \left\{\alpha \mathrm{Tr}\left((x^{\frac{1}{2}} w x^{\frac{1}{2}})^{\frac{z}{\alpha}}\right) - (\alpha-1)\mathrm{Tr}\left(w^{\frac{z}{\alpha-1}}\right)\right\},
\end{aligned}$$

where we used the fact that $\mathrm{Tr}\left((a^*a)^p\right) = \mathrm{Tr}\left((aa^*)^p\right)$ for $p > 0$ and $a \in L_{\frac{p}{2}}(\mathcal{M})$ and Lemma 0.1 since $\frac{z}{\alpha-1} \geq 1$ by the assumption. Putting $w = x^{\alpha-1}$ we get

$$\sup_{w \in L_{\frac{z}{\alpha-1}}(\mathcal{M})^+} \left\{ \alpha \mathrm{Tr}\left((x^{\frac{1}{2}} w x^{\frac{1}{2}})^{\frac{z}{\alpha}} \right) - (\alpha-1)\mathrm{Tr}\left(w^{\frac{z}{\alpha-1}} \right) \right\} \geq \mathrm{Tr}(x^z) = \|x\|_z^z = Q_{\alpha,z}(\psi\|\varphi).$$

This finishes the proof in the case that $Q_{\alpha,z}(\psi\|\varphi) < \infty$. Note that this holds if $\psi \leq \lambda\varphi$ for some $\lambda > 0$. Indeed, since $\frac{\alpha}{2z} \in (0,1]$ by the assumption, we then have

$$h_\psi^{\frac{\alpha}{2z}} \leq \lambda^{\frac{\alpha}{2z}} h_\varphi^{\frac{\alpha}{2z}},$$

hence by [2, Lemma A.58] there is some $b \in \mathcal{M}$ such that

$$h_\psi^{\frac{\alpha}{2z}} = b h_\varphi^{\frac{\alpha}{2z}} = y h_\varphi^{\frac{\alpha-1}{2z}},$$

where $y = b h_\varphi^{\frac{1}{2z}} \in L_{2z}(\mathcal{M})$. By Lemma 1 we get $Q_{\alpha,z}(\psi\|\varphi) = \|y\|_{2z}^{2z} < \infty$.

In the general case, for every $\epsilon > 0$, the variational expression holds for $Q_{\alpha,z}(\psi\|\varphi + \epsilon\psi)$, so that we have

$$\begin{aligned} Q_{\alpha,z}(\psi\|\varphi+\epsilon\psi) &= \sup_{a \in \mathcal{M}_+} \left\{ \alpha\mathrm{Tr}\left((a^{\frac{1}{2}} h_\psi^{\frac{\alpha}{z}} a^{\frac{1}{2}})^{\frac{z}{\alpha}} \right) - (\alpha-1)\mathrm{Tr}\left((a^{\frac{1}{2}} (h_\varphi + \epsilon\psi)^{\frac{\alpha-1}{z}} a^{\frac{1}{2}})^{\frac{z}{\alpha-1}} \right) \right\} \\ &\leq \sup_{a \in \mathcal{M}_+} \left\{ \alpha\mathrm{Tr}\left((a^{\frac{1}{2}} h_\psi^{\frac{\alpha}{z}} a^{\frac{1}{2}})^{\frac{z}{\alpha}} \right) - (\alpha-1)\mathrm{Tr}\left((a^{\frac{1}{2}} h_\varphi^{\frac{\alpha-1}{z}} a^{\frac{1}{2}})^{\frac{z}{\alpha-1}} \right) \right\}, \end{aligned}$$

where the inequality above follows since Lemma 0.2 gives that

$$\mathrm{Tr}\left((a^{\frac{1}{2}} (h_\varphi + \epsilon\psi)^{\frac{\alpha-1}{z}} a^{\frac{1}{2}})^{\frac{z}{\alpha-1}} \right) \geq \mathrm{Tr}\left((a^{\frac{1}{2}} h_\varphi^{\frac{\alpha-1}{z}} a^{\frac{1}{2}})^{\frac{z}{\alpha-1}} \right)$$

for every $a \in \mathcal{M}^+$. Therefore, since lower semicontinuity [4, Theorem 2 (iv)] gives

$$Q_{\alpha,z}(\psi\|\varphi) \leq \liminf_{\epsilon \searrow 0} Q_{\alpha,z}(\psi\|\varphi + \epsilon\psi)$$

the desired inequality follows. □

References

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