Tom Bullock, Teiko Heinosaari: Quantum state discrimination via repeated measurements and the rule of three.

Referee report

Perfectly distinguishable states are mutually commuting, therefore the problem of their discrimination is purely classical. The results of the paper do not express any "quantum" property, but just some behaviour of the classical binomial (or perhaps multinomial) distribution. The whole problem can be formulated and solved by solely classical means and, in fact, becomes a nice exercise in elementary probability theory.

The proofs provided by the authors seem lengthy and involved and I did not read them. In fact, at least for $N = 2$, it is much easier for readers to find their own proof. To show what I have in mind, I will sketch the case $N = 2$.

So let $N = 2$. The classical formulation can be as follows: Assume that an event occurs with probability $1/2$ and we have a test for it such that the probability of positive outcome is λ if the event occurred and $1 - \lambda$ if it didn't. What is the best probability of success if we repeat the test n-times?

As the authors correctly observe, it is enough to base our guess on the number of positive outcomes $p = 0, \dots, n$, that behaves according to the binomial distribution

$$f(n, \lambda, p) := \left(\begin{array}{c} n \\ p \end{array} \right) \lambda^p (1-\lambda)^{n-p}, \qquad p = 0, \dots n$$

(these are the probabilities of attaining p successes in an independent sequence of Bernoulli trials with probability λ). The optimal success probabilities are easily found, the equalities (20) and (21) in the paper can be written as

$$\begin{aligned} P_{succ}^{(n)} &= \frac{1}{2}\left(1 + \sum_{p=r}^{n} (f(n,\lambda,p) - f(n,1-\lambda,p))\right) \\ &= \frac{1}{2}\left(\sum_{p=r}^{n} f(n,\lambda,p) + \sum_{p=n+1-r}^{n} f(n,\lambda,p)\right) \end{aligned}$$

where $r = \frac{n+1}{2}$ if n is odd and $r = \frac{n}{2} + 1$ if n is even, here we used the obvious equality $f(n, 1-\lambda, p) = f(n, \lambda, n-p)$. For odd n, this readily implies

$$P_{succ}^{(n)} = \sum_{p=\frac{n+1}{2}}^{n} f(n,\lambda,p) = 1 - F_{n,\lambda}(\frac{n+1}{2}),$$

where $F_{n,\lambda}$ is the distribution function of the binomial distribution $b(n, \lambda)$, and

$$P_{succ}^{(n+1)} = \sum_{p=\frac{n+1}{2}}^{n} f(n+1,\lambda,p+1) + (1-\lambda) f(n,\lambda,\frac{n+1}{2}),$$

where we have used the equality

$$\frac{1}{2} f(n+1, \lambda, \frac{n+1}{2}) = (1-\lambda) f(n, \lambda, \frac{n+1}{2}).$$

To obtain the equality $P_{succ}^{(n)} = P_{succ}^{(n+1)}$, it is enough to note that we have for $p < n$:

$$f(n+1, \lambda, p+1) = \lambda f(n, \lambda, p) + (1-\lambda) f(n, \lambda, p+1),$$

which is easily seen, since we can decompose the event of obtaining $p+1$ successes in $n+1$ trials by conditioning on the outcome of the last (or any other) trial. We obtain

$$\begin{aligned} P_{succ}^{(n+1)} &= \sum_{p=\frac{n+1}{2}}^{n-1} (\lambda f(n, \lambda, p) + (1-\lambda) f(n, \lambda, p+1)) + \lambda^{n+1} + (1-\lambda) f(n, \lambda, \frac{n+1}{2}) \\ &= \lambda P_{succ}^{(n)} + (1-\lambda) \left(\sum_{p=\frac{n+1}{2}+1}^{n} f(n, \lambda, p) + f(n, \lambda, \frac{n+1}{2}) \right) = P_{succ}^{(n)} \end{aligned}$$

With a little more effort, one can treat the case of more outcomes along similar lines.

To conclude, in my opinion, the results of the paper do not bring much new insight into the problem of quantum state discrimination and are not significant enough to warrant publication in Journal of Physics A.