Decoherence-free algebra and periodicity for a quantum channel

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What is new:

- a description of $\mathcal{N}(\Phi)$ on pp. 9-10 (maybe should be formulated as a theorem?)
- example with generalized Pauli operators (not finished yet), Sec. 2.3.1
- some old stuff cut off, to be perhaps included later

1 Multiplicative domain and fixed points

Let $\mathcal{H}$ be a separable Hilbert space and let $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$ be a unital normal cp map. Then Φ has a representation of the form

$$\Phi(A) = \sum_{k=1}^{\infty} V_k^* A V_k, \qquad A \in B(\mathcal{H}), \tag{1}$$

where the Kraus operators $V_k \in B(\mathcal{H})$ are such that $\sum_k V_k^* V_k = I$.

We will consider the following sets of operators:

$$\begin{aligned}
\mathcal{M}(\Phi) &:= \{A \in B(\mathcal{H}), \Phi(A^*A) = \Phi(A)^*\Phi(A), \Phi(AA^*) = \Phi(A)\Phi(A)^*\} \\
&= \{A \in B(\mathcal{H}), \Phi(AB) = \Phi(A)\Phi(B), \Phi(BA) = \Phi(B)\Phi(A),\ \forall B \in B(\mathcal{H})\} \\
\mathcal{N}(\Phi) &:= \cap_n \mathcal{M}(\Phi^n) \\
\mathcal{F}(\Phi) &:= \{A \in B(\mathcal{H}), \Phi(A) = A\}
\end{aligned}$$

Then $\mathcal{M}(\Phi)$ is called the multiplicative domain, $\mathcal{N}(\Phi)$ is the decoherence free subalgebra and $\mathcal{F}(\Phi)$ is the fixed point domain of Φ. We now collect some basic facts about these sets. The proofs are included for the convenience of the reader.

Proposition 1. *$\mathcal{M}(\Phi) = \{V_j V_k^*,\ j, k = 1, 2, \dots\}'$, where $\{\ \}'$ denotes the commutant.*

Proof. It will be convenient to use the Stinespring representation of Φ: Let $\mathcal{K}$ be a separable Hilbert space and let $\{e_n\}$ be an orthonormal basis. Put

$$V = \sum_j V_j \otimes |e_j\rangle,$$

then $V \in B(\mathcal{H}, \mathcal{H} \otimes \mathcal{K})$, $V^*V = \sum_j V_j^* V_j = I$ and

$$\Phi(A) = V^*(A \otimes I)V, \qquad A \in B(\mathcal{H}).$$

Let $P = VV^*$, then $P \in B(\mathcal{H} \otimes \mathcal{K})$ is a projection and we have $A \in \mathcal{M}$ if and only if $A \otimes I$ commutes with P. Indeed, suppose $A \in \mathcal{M}$, then

$$V^*(A^*A \otimes I)V = V^*(A^* \otimes I)P(A \otimes I)V.$$

It follows that $P(A^* \otimes I)(1 - P)(A \otimes I)P = 0$, hence $(1 - P)(A \otimes I)P = 0$, so that

$$(A \otimes I)P = P(A \otimes I)P.$$

Similarly, we get the same for A^* and this implies that

$$P(A \otimes I) = P(A \otimes I)P = (A \otimes I)P.$$

The converse is easy. Now notice that $P = \sum_{j,k} V_j V_k^* \otimes |e_j\rangle\langle e_k|$, this implies the statement.

□

By this characterization, we can see that $\mathcal{M}(\Phi)$ is a von Neumann subalgebra in $B(\mathcal{H})$ (see also [?]) and it is clear that the restriction of Φ is a *-homomorphism $\mathcal{M} \to B(\mathcal{H})$. Consequently, $\mathcal{N}(\Phi)$ is a von Neumann subalgebra as well and the restriction of Φ defines a *-endomorphism of $\mathcal{N}(\Phi)$.

Remark 1. *Notice that $\Phi|_{\mathcal{N}(\Phi)}$ is not always a *-automorphism. Indeed, $\mathcal{N}(\Phi)$ can have, for instance, a non-trivial intersection with the kernel of Φ. Since this intersection is a subalgebra, it then contains a nonzero projection $0 \neq P \in Ker(\Phi) \cap \mathcal{N}(\Phi)$. On the other hand, any projection in $Ker(\Phi)$ is necessarily in $\mathcal{N}(\Phi)$, so that this happens if and only if Φ is not faithful. But even if Φ is faithful, $\Phi|_{\mathcal{N}(\Phi)}$ needs not be a *-automorphism, example?*

Proposition 2. *We have the following characterizations of $\mathcal{N}(\Phi)$:*

(i) $\mathcal{N}(\Phi) = \{V_{i_1} \dots V_{i_n} V_{j_1}^* \dots V_{j_n}^*,\ i_k, j_k = 1, 2, \dots;\ n \in \mathbb{N}\}'$.

(ii) $\mathcal{N}(\Phi)$ is the von Neumann algebra generated by the preserved projections, i.e. by the set

$$\{Q \in B(\mathcal{H}) \,:\, \Phi^n(Q) \text{ is a projection } \forall\, n \geq 0\}.$$

Proof. (i) is immediate from Proposition 1. (ii) holds since $P \in \mathcal{M}(\Phi)$ if and only if $\Phi(P)$ is a projection.

□

In contrast, the set of fixed points is in general not a subalgebra. Some example?

Proposition 3. *$\mathcal{F}(\Phi)$ is a von Neumann algebra if and only if it is contained in $\mathcal{N}(\Phi)$. In this case, we have*

$$\mathcal{F}(\Phi) = \{V_j, V_j^*,\ j = 1, 2 \dots\}'$$

Proof. The first statement is quite obvious. Assume now that $\mathcal{F}(\Phi)$ is a von Neumann algebra and let $A \in \mathcal{F}(\Phi)$. Then

$$0 = \Phi(A^*A) - A^*A = (V_jA - AV_j)^*(V_jA - AV_j),$$

this implies $AV_j = V_jA$. Similarly, we obtain $AV_j^* = V_j^*A$. It follows that $\mathcal{F}(\Phi) \subseteq \{V_j, V_j^*,\ j = 1, 2 \dots\}'$ The converse inclusion is clear.

□

2 Maps with a faithful invariant state

In this section, we assume that there is a faithful normal state $\rho \in \mathfrak{S}(\mathcal{H})$ for Φ. The following results are well known.

Proposition 4. *Assume that there is a faithful normal invariant state for Φ. Then*

(i) $\mathcal{F}(\Phi)$ is a von Neumann subalgebra.

*(ii) The restriction $\Phi|_{\mathcal{N}(\Phi)}$ is a *-automorphism.*

Proof. If ρ is a faithful invariant state, then for any $A \in \mathcal{F}(\Phi)$, $\rho(\Phi(A^*A) - A^*A) = 0$. Since $\Phi(A^*A) - A^*A \geq 0$ by the Schwarz inequality for cp maps, this implies that $\Phi(A^*A) = A^*A = \Phi(A)^*\Phi(A)$, hence $\mathcal{F}(\Phi) \subseteq \mathcal{N}(\Phi)$. The statement (i) now follows by Proposition 3. For (ii) see [**?**] ?

□

In the presence of a faithful invariant state, there is another special subalgebra investigated in the literature, e.g. [3, **?**]. More precisely, in this case, the set $\{\Phi^n,\ n \in \mathbb{N}\}$ is relatively compact in the point-ultraweak topology and its closure $\mathbf{S}$ is a compact topological semigroup. It is proved in [3, 15, 16] (see the proof of [15, Proposition 2.2]) that the set

$$\mathcal{M}_r(\Phi) := \overline{\operatorname{span}\{x \in \mathcal{M},\ \Phi(x) = \lambda x,\ |\lambda| = 1\}}^{w*} = \{x \in \mathcal{M}, T(x^*x) = T(x)^*T(x),\ \forall T \in \mathbf{S}\}$$

is a von Neumann subalgebra and there is a conditional expectation $F \in \mathbf{S}$ with range $\mathcal{M}_r(\Phi)$, such that $F_*(\rho) = \rho$.

Theorem 1. *Assume that a uncp map $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$ admits a faithful normal invariant state ρ. Then $\mathcal{N}(\Phi) = \mathcal{M}_r(\Phi)$.*

Proof. Let $\mathcal{B}_1$, $\mathcal{N}_1$ and $\mathcal{R}_1$ be the unit balls of $B(\mathcal{H})$, $\mathcal{N}(\Phi)$ and $\mathcal{M}_r(\Phi)$, respectively. Then

$$\mathcal{R}_1 \subseteq \mathcal{N}_1 \subseteq \bigcap_n \Phi^n(\mathcal{B}_1).$$

Indeed, the first inclusion follows from $\mathcal{M}_r(\Phi) \subseteq \mathcal{N}(\Phi)$ and the second from the fact that the restriction $\Phi|_{\mathcal{N}(\Phi)}$ is an automorphism. We will show that $\mathcal{R}_1 = \bigcap_n \Phi^n(\mathcal{B}_1)$, which implies the statement. (This proof is inspired by [1].)

We will use a Hahn-Banach separation argument. So let $x \in \bigcap_n \Phi^n(\mathcal{B}_1) \setminus \mathcal{R}_1$. Since $\mathcal{R}_1 \subset B(\mathcal{H})$ is convex and compact in the weak*-topology, there exists some $\psi \in B(\mathcal{H})_*$ such that

$$\psi(x) > \sup_{y \in \mathcal{R}_1} \psi(y) = \|\psi|_{\mathcal{M}_r(\Phi)}\|_1 = \|F_*\psi\|_1.$$

For each $n \in \mathbb{N}$, there is some $y_n \in \mathcal{B}_1$ such that $x = \Phi^n(y_n)$ and we have

$$\|\Phi_*^n\psi\|_1 \geq (\Phi_*^n\psi)(y_n) = \psi(x).$$

Note that since Φ_* is a contraction, $\{\|\Phi_*^n\psi\|_1\}_n$ is a bounded nonincreasing sequence and we have

$$\lim_n \|\Phi_*^n\psi\|_1 \geq \psi(x) > \|F_*\psi\|_1.$$

On the other hand, by [**?**, Theorem 2.1], the set $\{\Phi_*^n,\ n \in \mathbb{N}\}$ is relatively compact in the point-weak topology, so that for any $\varphi \in B(\mathcal{H})_*$, the orbit

$$\mathbb{S}_*\varphi := \{S_*\varphi,\ S \in \mathbb{S}\} = \{\Phi_*^n\varphi,\ n \in \mathbb{N}\}^{-w}$$

is weakly compact. Since $F \in \mathbb{S}$, $\mathbb{S}_*\varphi$ contains $F_*\varphi$ and since $B(\mathcal{H})_*$ is a separable Banach space, there is a subsequence of $\Phi_*^n\varphi$ converging weakly to $F_*\varphi$ [10, Theorem V.6.3].

Let Φ^{n_k} be such that $\Phi_*^{n_k}\psi \to F_*\psi$ and let $\varphi_1, \ldots, \varphi_4 \in B(\mathcal{H})_*^+$ be such that $\psi = \sum_i c_i\varphi_i$. Then by [10, Theorem V.6.1] we may assume that $\Phi_*^{n_k}\varphi_i$ are all weakly convergent, restricting to subsequences if necessary. By [18, Corollary III.5.11], $\Phi_*^{n_k}\varphi_i$ are all norm convergent. It follows that $\Phi_*^{n_k}\psi \to F_*\psi$ in norm, so that

$$\lim \|\Phi_*^n\psi\|_1 = \lim \|\Phi_*^{n_k}\psi\|_1 = \|F_*\psi\|_1,$$

a contradiction.

□

Corollary 1. *Let $\mathcal{M} = B(\mathcal{H})$ and let ρ be a faithful normal invariant state. Then $\mathcal{N}(\Phi)$ is the range of a conditional expectation F preserving ρ.*

Remark 2. *Note that by Corollary 1, the subalgebra $\mathcal{N}(\Phi)$ is atomic. Note also that Theorem 1 and Corollary 1 hold for nucp maps on any atomic von Neumann algebra $\mathcal{M}$. The same proof can be used also in continuous time case.*

Remark 3. *In the situation of Theorem 1, we always have environmental decoherence according to [5] with decomposition $\mathcal{N}(\Phi)$ and $\mathcal{M}_s$ maybe we should recall here some definition (?)... This is an almost direct consequence stated for instance in [9, Proposition 31].*

Moreover, in this case, we have some standard tools to study the velocity of decoherence. One is a kind of spectral gap inequality, in order to characterize the situation when the dissipative elements converge to 0 exponentially fast and with uniform rate. Indeed, calling ρ the invariant faithful state, we have that $\|\Phi^n(x) - \Phi^n(Fx)\|_{2,\rho} \leq e^{-\epsilon n}\|x - Fx\|_{2,\rho}$

2.1 Irreducible maps

We say that the map Φ is irreducible if there are no nontrivial subharmonic projections, that is, if $P \in B(\mathcal{H})$ is a projection such that $\Phi(P) \geq P$ then $P = 0$ or $P = I$. If there is faithful normal invariant state, this clearly happens if and only if $\mathcal{F}(\Phi) = \mathbb{C}I$.

Definition 1. *Period of Φ. Let Φ be an irreducible nucp map. Then the period d is the maximal integer d such that there exists a resolution of the identity $Q_0, ...Q_{d-1}$ verifying $\Phi(Q_j) = Q_{j-1}$ for all j (subtraction on indices are modulo d).*

Proposition 5 (Groh [15] and Batkai et al [3, Propositions 6.1 and 6.2])**.** *Let Φ be an irreducible uncp map on $B(\mathcal{H})$ with an invariant faithful state. Then the peripheral point spectrum of Φ is the group of all the d-th roots of unity for some $d \geq 1$ and all the eigenvalues in the peripheral point spectrum are simple. Moreover there exists a unitary operator U such that $U^d = 1$ and $\Phi(U^n) = \exp(i2\pi n/d)U^n$.*

(recall also Evans & Hoegh-Krohn 78, where this result was proven for the finite dimensional case)

Corollary 2. *Let Φ be an irreducible uncp map on $B(\mathcal{H})$ with an invariant faithful state. Then Φ has finite period, the cyclic resolution of Φ is unique and $\mathcal{N}(\Phi)$ is an abelian algebra spanned by the cyclic projections of Φ.*

Proof. Calling ω the primary d-th root of unity and U a pertaining eigenvector, we have that U is a unitary operator satisfying $U^d = 1$ and $\Phi(U^n) = \omega^n U^n$ by previous proposition. It follows that U^n is the unique (up to multiplicative constants) eigenvector associated with the eigenvalue ω^n. By Theorem 1,

$$\mathcal{N}(\Phi) = \mathcal{M}_r(\Phi) = \text{span}\,\{I, U, \ldots U^{d-1}\} = \{U, U^*\}''.$$

In particular, it follows that the abelian subalgebra generated by U is finite dimensional and therefore U admits a spectral representation

$$U = \sum_{j=0}^{d-1} \omega^j Q_j$$

for some orthogonal projections Q_j summing up to 1. We immediately deduce that, since $\Phi(U) = \omega U$, then $\Phi(Q_j) = Q_{j-1}$ for all j, so that $Q_0, ..Q_{d-1}$ is a cyclic decomposition of Φ and we have

$$\mathcal{N}(\Phi) = \{U, U^*\}'' = \text{span}\,\{Q_0, \ldots, Q_{d-1}\}.$$

To prove uniqueness, assume that $P_0, ...P_{d-1}$ is another cyclic resolution of Φ. Then we can construct the unitary operator

$$V = \sum_{j=0}^{d-1} \omega^j P_j,$$

which is an eigenvalue for ω. Since the eigenvalues are simple, we must have $V = zU$ for some $z \in \mathbb{C}$, $|z| = 1$ and it is easy to see that for each n we must have $P_n = Q_j$ for some $j = 0, 1, \ldots, d-1$.

□

Proposition 6. *Suppose Φ is an irreducible uncp map with an invariant faithful state and let $Q_0, \ldots, Q_{d-1}$ be the cyclic resolution for Φ. Then*

1. *$\mathcal{F}(\Phi^m)$ is a subalgebra of $\mathcal{N}(\Phi)$ for any m;*
2. *$\mathcal{F}(\Phi^d) = \mathcal{N}(\Phi)$ and d is the smallest integer with this property;*
3. *$\mathcal{F}(\Phi^m) \neq \mathbb{C}1$ if and only if $GCD(m, d) > 1$.*

Moreover, denote by $\Phi^d_{|k}$ the restriction of Φ^d to the subalgebra $Q_k B(\mathcal{H}) Q_k$, then $\Phi^d_{|k}$ is irreducible, positive recurrent and aperiodic, and consequently ergodic.

Proof. Let $\rho \in \mathfrak{S}(\mathcal{H})$ be the unique faithful invariant state of Φ, then ρ is also invariant for Φ^m, so that by Propositions 3 and 4, $\mathcal{F}(\Phi^m)$ is a subalgebra in $\mathcal{N}(\Phi^m)$. Note that for any $n \in \mathbb{N}$ and $X \in \mathcal{M}(\Phi^n)$, we have from Schwartz inequality for ucp maps that

$$\begin{aligned} \Phi^n(X^*X) &= \Phi(\Phi^{n-1}(X^*X)) \geq \Phi(\Phi^{n-1}(X)\Phi^{n-1}(X)^*) \\ &\geq \Phi^n(X)^*\Phi^n(X) = \Phi^n(X^*X). \end{aligned}$$

Using the fact that $\Phi^{n-1}(X^*X)-\Phi^{n-1}(X)\Phi^{n-1}(X)^* \geq 0$ and that ρ is a faithful invariant state, we obtain that $\Phi^{n-1}(X^*X)=\Phi^{n-1}(X)\Phi^{n-1}(X)^*$. This implies that $\mathcal{M}(\Phi^n) \subseteq \mathcal{M}(\Phi^{n-1})$ for all n and hence

$$\mathcal{N}(\Phi^m)=\cap_n \mathcal{M}(\Phi^{mn})=\cap_n \mathcal{M}(\Phi^n)=\mathcal{N}(\Phi).$$

This proves 1.

By definition of cyclic decomposition, we have $Q_j \in \mathcal{F}(\Phi^d)$ for all j, this implies $\mathcal{N}(\Phi) \subseteq \mathcal{F}(\Phi^d)$. The converse inclusion holds by part 1. If $n<d$, then $\Phi^n(Q_{d-1})=Q_{d-n-1} \neq Q_{d-1}$, so that $Q_{d-1} \notin \mathcal{F}(\Phi^n)$ and hence $\mathcal{F}(\Phi^n) \neq \mathcal{N}(\Phi)$, this proves 2.

Assume now that $\mathcal{F}(\Phi^m) \neq \mathbb{C}1$, then there is some nontrivial minimal projection $P \in \mathcal{F}(\Phi^m)$, which by part 1. must be of the form $P=Q_{j_1}+\cdots+Q_{j_k}$ for some (distinct) indices $0 \leq j_i \leq d-1$ and $k<d$. Let $P_i=\Phi^i(P)$, $i=0,\ldots,m-1$, then all P_i are minimal projections in $\mathcal{F}(\Phi^m)$, so that for $i \neq j$, either $P_iP_j=0$ or $P_i=P_j$. By rearranging the indices if necessary, we may assume that $P_0,\ldots,P_{l-1}$ are mutually orthogonal and all other P_i are contained in $\{P_0,\ldots,P_{l-1}\}$. Then $\sum_{i=0}^{m-1} P_i=\sum_{j=0}^{l-1} n_jP_j$ for some integers n_j. On the other hand, we have $\sum_{i=0}^{m-1} P_i \in \mathcal{F}(\Phi)=\mathbb{C}1$ since Φ is irreducible. It follows that $n_1=\cdots=n_{l-1}=: n$ and

$$\sum_{i=0}^{m-1} P_i=nI=n\sum_{j=0}^{l-1} P_j.$$

This implies $m=nl$. Further, $\sum_{j=0}^{l-1} P_j=I$ implies that $d=kl$ by the definition of P_j. Note also that $l>1$ since otherwise we would have $\Phi(P)=P$, which is not possible. Conversely, assume that $GCD(m,d)=l>1$ and let $d=kl$. Put $P=Q_0+Q_l+\cdots+Q_{(k-1)l}$, then clearly P is a projection, $P \neq 0,1$ and $\Phi^l(P)=P$ and also $\Phi^m(P)=P$, since m is a multiple of l, so that $P \in \mathcal{F}(\Phi^m)$ and $\mathcal{F}(\Phi^m) \neq \mathbb{C}1$.

To prove the last statement, observe that $\Phi^d_{|k}$ is positive recurrent because the restriction of the Φ-invariant state will give a faithful $\Phi^d_{|k}$-invariant state. By contradiction, if $\Phi^d_{|k}$ is reducible, then we have a non trivial $\Phi^d_{|k}$-harmonic projection Q, $0<Q<Q_k$, i.e. such that $\Phi^d(Q)=Q$. But then this Q is in $\mathcal{N}(\Phi)$ and, by positivity $\Phi^j(Q)$ is a projection bounded above by $\Phi^j(Q_k)=Q_{k-j}$. We deduce that $\sum_{j=0}^{d-1}\Phi^j(Q)<\sum_{j=0}^{d-1}Q_{k-j}=1$ is a non trivial projection and a fixed point for Φ and this contradicts the irreducibility of Φ.

Similarly, for the period, we know that $\Phi^d_{|k}$ has finite period by Groh; we call its period d_k, with cyclic decomposition $R_0,\ldots R_{d_k-1}$, $R_0+\cdots+R_{d_k-1}=Q_k$. R_0 is a fixed point for Φ^{dd_k}, so it belongs to $\mathcal{N}(\Phi)$ and $\Phi^j(R_0)$, $j=0,\ldots d\cdot d_k-1$, will give a cyclic decomposition for Φ. So $dd_k=d$ which implies $d_k=1$ and $R_0=Q_k$.

□

2.2 Reducible maps

It is known [12] that the limit

$$E=\lim_n \frac{1}{n}\sum_{k=0}^{n-1}\Phi^k$$

exists in the point-ultraweak topology, moreover, E is a faithful normal conditional expectation onto $\mathcal{F}(\Phi)$, satisfying $E\circ\Phi=\Phi\circ E=E$. It is also easy to see that a normal state is invariant for Φ if and only if it is invariant for E. We have the following decomposition result for $\mathcal{F}(\Phi)$.

Theorem 2. *Assume that ρ is a faithful invariant state for Φ. Then there is a decomposition $\mathcal{H} = \oplus_j \mathcal{H}_j$, Hilbert spaces $\mathcal{H}_j^L$, $\mathcal{H}_j^R$, faithful states $\rho_j \in \mathfrak{S}(\mathcal{H}_j^R)$ and unitaries $U_j : \mathcal{H}_j \to \mathcal{H}_j^L \otimes \mathcal{H}_j^R$ such that*

(i)

$$\mathcal{F}(\Phi) = \bigoplus_n U_n^*(B(\mathcal{H}_n^L) \otimes I_{\mathcal{H}_n^R})U_n$$

(ii) Invariant states $\omega \in \mathfrak{S}(\mathcal{H})$ are precisely those of the form

$$\omega = \oplus_j \lambda_j U_j^*(\omega_j \otimes \rho_j)U_j$$

for some probabilities $\{\lambda_j\}$ and states $\omega_j \in \mathfrak{S}(\mathcal{H}_j^L)$.

(iii) The restriction $\Phi_j := \Phi|_{B(\mathcal{H}_j)}$ is a normal unital cp map on $B(\mathcal{H}_j)$ of the form

$$\Phi_j = \mathrm{Ad}_{U_j} \circ (id_{B(\mathcal{H}_j^L)} \otimes \Psi_j) \circ \mathrm{Ad}_{U_j^*}$$

for some irreducible unital normal cp map Ψ_j on $B(\mathcal{H}_j^R)$, having a faithful invariant state ρ_j.

(iv) There is a Kraus representation $\Phi(A) = \sum_k V_k^* A V_k$, such that

$$V_k = \sum_j U_j^*(I_{\mathcal{H}_j^L} \otimes L_{k,j})U_j,$$

where $L_{k,j} \in B(\mathcal{H}_j^R)$ satisfy $\Psi_j(A_j) = \sum_k L_{k,j}^* A_j L_{k,j}$, $A_j \in B(\mathcal{H}_j^R)$.

Proof. Note first that since $\mathcal{F}(\Phi)$ is the range of the conditional expectation E, it must be type I with discrete center, [19]. Let $\{P_j\}$ be the minimal central projections and let $\mathcal{H}_j = P_j\mathcal{H}$. Since $\mathcal{F}(\Phi)P_j$ is a type I factor acting on $\mathcal{H}_j$, there are Hilbert spaces $\mathcal{H}_j^L$, $\mathcal{H}_j^R$ and a unitary $U_j : \mathcal{H}_j \to \mathcal{H}_j^L \otimes \mathcal{H}_j^R$ such that $\mathcal{F}(\Phi)P_j = U_j^*(B(\mathcal{H}_j^L) \otimes I_{\mathcal{H}_j^R})U_j$, this proves (i).

The conditional expectation E onto $\mathcal{F}(\Phi)$ satisfies $E(P_j A P_k) = P_j E(A) P_k = P_j P_k E(A)$, hence we must have

$$E(A) = \sum_j E(P_j A P_j).$$

It also follows that $E(B(\mathcal{H}_j)) \subseteq B(\mathcal{H}_j)$ for all j. Let $A_j \in B(\mathcal{H}_j^L)$, $B_j \in B(\mathcal{H}_j^R)$, then we have

$$E(U_j^*(A_j \otimes B_j)U_j) = E(U_j^*(A_j \otimes I)U_j U_j^*(I \otimes B_j)U_j) = U_j^*(A_j \otimes I)U_j E(U_j^*(I \otimes B_j)U_j) \quad (2)$$

and similarly

$$E(U_j^*(A_j \otimes B_j)U_j) = E(U_j^*(I \otimes B_j)U_j)U_j^*(A_j \otimes I)U_j$$

It follows that $E(U_j^*(I \otimes B_j)U_j)$ commutes with all elements in $U_j^*(B(\mathcal{H}_j^L) \otimes I))U_j$, so that there is some $\xi(B_j) \in \mathbb{C}$ such that $E(U_j^*(I \otimes B_j)U_j) = \xi(B_j)P_j$. It is clear that $B_j \mapsto \xi(B_j)$ defines a state on $B(\mathcal{H}_j^R)$. Let $\rho_j \in \mathfrak{S}(\mathcal{H}_j^R)$ be the corresponding density operator and let ϕ_{ρ_j} denote the map $B_j \mapsto \mathrm{Tr}\,(\rho_j B_j) I_{\mathcal{H}_j^R}$. We can see from (2) that the restriction $E|_{B(\mathcal{H}_j)}$ has the form

$$E(C_j) = U_j^*(id_{\mathcal{H}_L^j} \otimes \phi_{\rho_j})(U_j C_j U_j^*)U_j,$$

it follows that E must have the form

$$E(A) = \sum_j U_j^*(id_{\mathcal{H}_L^j} \otimes \phi_{\rho_j})(U_j P_j A P_j U_j^*) U_j.$$

Hence we have for $\omega \in \mathfrak{S}(\mathcal{H})$,

$$E^*(\omega) = \sum_j U_j^*(id \otimes \phi_{\rho_j}^*)(U_j P_j \omega P_j U_j^*) U_j = \sum_j U_j^*(\operatorname{Tr}_{\mathcal{H}_j^R}(U_j P_j \omega P_j U_j^*) \otimes \rho_j) U_j,$$

where $\operatorname{Tr}_{\mathcal{H}_j^R}$ denotes the partial trace. Putting $\lambda_j = \operatorname{Tr} P_j \omega$, the statement (ii) now follows from the fact that $\Phi^*(\omega) = \omega$ if and only if $E^*(\omega) = \omega$. Moreover, since E preserves the faithful state ρ, it must be faithful and hence all the states ρ_j have to be faithful as well.

Since $\mathcal{F}(\Phi) \subseteq \mathcal{N}(\Phi) \subseteq \mathcal{M}(\Phi)$, we can see similarly as for E that Φ_j maps $B(\mathcal{H}_j)$ into $B(\mathcal{H}_j)$ and that

$$\Phi_j(U_j^*(A_j \otimes B_j) U_j) = U_j^*(A_j \otimes I)(I \otimes \Psi_j(B_j)) U_j$$

or some normal unital cp map $\Psi_j : B(\mathcal{H}_j^R) \to B(\mathcal{H}_j^R)$, so that

$$\Phi_j(C_j) = U_j^*(id_{B(\mathcal{H}_j^L)} \otimes \Psi_j)(U_j C_j U_j^*) U_j, \quad \forall C_j \in B(\mathcal{H}_j).$$

Since $E \circ \Phi = E$, we must have $\phi_{\rho_j} \circ \Psi_j = \phi_{\rho_j}$, equivalently $\Psi_j^*(\rho_j) = \rho_j$. To prove that Ψ_j is irreducible, let $Q \in B(\mathcal{H}_j^R)$ be a subharmonic projection for Ψ_j, then since Ψ_j has a faithful invariant state, we must have $\Psi_j(Q) = Q$. But then $U_j^*(I \otimes Q) U_j \in \mathcal{F}(\Phi)$, this is possible only if $Q = 0$ or I. This finishes the proof of (iii).

For (iv), let $\Phi(A) = \sum_k V_k^* A V_k$ be any Kraus representation of Φ. We see from

$$\Phi(A) = \sum_{i,j} \Phi(P_i A P_j) = \sum_{i,j} P_i \Phi(P_i A P_j) P_j$$

that we may assume that each V_k is block-diagonal, with blocks $V_{k,j} = P_j V_k P_j$. Moreover, $\Phi_j(A_j) = \sum_k V_{k,j}^* A_j V_{k,j}$ is a Kraus representation of Φ_j. Let $\Psi_j = \sum_l K_{l,j}^* \cdot K_{l,j}$ be a minimal Kraus representation of Ψ_j, then it follows by (iii) that $\Phi_j = \sum_l U_j^*(I_{\mathcal{H}_j^L} \otimes K_{l,j}^*) U_j \cdot U_j^*(I_{\mathcal{H}_j^L} \otimes K_{l,j}) U_j$ is another Kraus representation, hence there are some $\{\mu_{k,l}^j\}$ such that $\sum_i \mu_{i,k}^j \bar{\mu}_{i,l}^j = \delta_{k,l}$ and

$$V_{k,j} = \sum_l \mu_{k,l}^j U_j^*(I_j \otimes K_{l,j}) U_j = U_j^*(I_j \otimes L_{k,j}) U_j,$$

where $L_{k,j} := \sum_l \mu_{k,l}^j K_{l,j}$, this proves (iv).

□

The relation to enclosures could be described here.

We now turn to the decoherence-free subalgebra. By Corollary 1, $\mathcal{N}(\Phi)$ is type I with discrete center. We first describe the central projections in $\mathcal{N}(\Phi)$.

Lemma 1. *Let Φ have the decomposition of Theorem 2. All central projections in $\mathcal{N}(\Phi)$ have the form*

$$Q = \sum_j U_j^*(I_j^L \otimes R_j) U_j$$

where R_j are some central projections in $\mathcal{N}(\Psi_j)$.

Proof. For simplicity, we will assume that $U_j = I_{\mathcal{H}_j^L} =: I_j^L$ for all j. By Proposition 1 and Theorem 2 (iv), we have

$$\mathcal{N}(\Phi) = \left\{ \bigoplus_j I_j^L \otimes L_{k_n,j} \dots L_{k_1,j} L^*_{l_1,j} \dots L^*_{l_n,j},\ n, k_1, \dots, k_n, l_1, \dots, l_n \in \mathbb{N} \right\}'$$

Since $P_j \in \mathcal{N}(\Phi)$, $P_j \mathcal{N}(\Phi) P_j \subseteq \mathcal{N}(\Phi)$ and we have

$$\begin{aligned} P_j \mathcal{N}(\Phi) P_j &= \left\{ (I_j^L \otimes L_{k_n,j} \dots L_{k_1,j} L^*_{l_1,j} \dots L^*_{l_n,j},\ n, k_1, \dots, k_n, l_1, \dots, l_n \in \mathbb{N} \right\}' \cap B(\mathcal{H}_j) \\ &= B(\mathcal{H}_j^L) \otimes \mathcal{N}(\Psi_j). \end{aligned}$$

Let Q be a central projection in $\mathcal{N}(\Phi)$. Then $Q = \sum_j Q_j$, where $Q_j := QP_j$ are central projections in $P_j \mathcal{N}(\Phi) P_j$, so that $Q_j = I_j^L \otimes R_j$ for a central projection R_j in $\mathcal{N}(\Psi_j)$.

□

We next consider the action of Φ on the center of $\mathcal{N}(\Phi)$. Let $\mathcal{I} \subseteq \mathbb{N}$ be such that $\{Q_i, i \in \mathcal{I}\}$ are the minimal central projections in $\mathcal{N}(\Phi)$. Since the restriction of Φ to $\mathcal{N}(\Phi)$ is a *-automorphism, $\Phi(Q_i)$ is a minimal central projection as well. Hence there is a bijective map $\pi : \mathcal{I} \to \mathcal{I}$ such that

$$\Phi(Q_i) = Q_{\pi(i)}, \qquad i \in \mathcal{I}.$$

For each $i \in \mathcal{I}$, let

$$d_i := \inf\{m, \pi^m(i) = i\},$$

then since Φ preserves the faithful state ρ, $d_i < \infty$ for all i. It follows that $\mathcal{I}$ decomposes into finite cycles $[i] := \{i, \pi(i), \dots, \pi^{d_i - 1}(i)\}$. Let $\sim$ be the equivalence relation on $\mathcal{I}$, given by this decomposition and let us denote $Q_{[i]} := \sum_{m=0}^{d_i - 1} \Phi^m(Q_i)$. Then clearly $Q_{[i]} \in \mathcal{F}(\Phi)$. By construction, $\{Q_{[i]},\ [i] \in \mathcal{I}|_\sim\}$ is an orthogonal family of central projections in $\mathcal{F}(\Phi)$, summing up to I. It follows that

$$\mathcal{N}(\Phi) = \bigoplus_{[i] \in \mathcal{I}|_\sim} \mathcal{N}_{[i]},$$

where $\mathcal{N}_{[i]} := Q_{[i]} \mathcal{N}(\Phi)$, and $\Phi(\mathcal{N}_{[i]}) \subseteq \mathcal{N}_{[i]}$ for all $[i]$. We also have $Q_{[i]} = \sum_{j \in \mathcal{J}_{[i]}} P_j$, where $\mathcal{J}_{[i]}$ form a decomposition of the set $\mathcal{J}$ of indices of minimal central projections in $\mathcal{F}(\Phi)$.

We now describe the action of Φ on one component $\mathcal{N}_{[i]}$. For simplicity of notations, let us assume that there is only one such component, $\mathcal{N} = \mathcal{N}(\Phi)$, this can be done by considering the restriction $\Phi_{[i]} = \Phi|_{B(Q_{[i]}\mathcal{H})}$. Then $\mathcal{N}$ has a finite center with minimal projections $\{Q_m := \Phi^m(Q_0),\ m = 0, \dots, d-1\}$. The following considerations are very similar to those used in the proof of Theorem 2. Let $\mathcal{K}_m := Q_m \mathcal{H}$. Since $Q_m \mathcal{N}$ is a factor of type I for each m, there are Hilbert spaces $\mathcal{K}_m^L \otimes \mathcal{K}_m^R$ and unitaries $S_m : \mathcal{K}_m \to \mathcal{K}_m^L \otimes \mathcal{K}_m^R$ such that

$$\mathcal{N} = \bigoplus_{m=0}^{d-1} S_m^* (B(\mathcal{K}_m^L) \otimes I_{\mathcal{K}_m^R}) S_m.$$

Let $A \in B(\mathcal{K}_m^L)$. Since $\Phi(Q_m \mathcal{N}) = Q_{m \oplus 1} \mathcal{N}$ (where $\oplus$ denotes addition modulo d), we have $\Phi(S_m^*(A \otimes I) S_m) = S_{m\oplus 1}^* (A_1 \otimes I) S_{m \oplus 1}$ for some $A_1 \in B(\mathcal{K}_{m\oplus 1}^L)$ and the map $A \mapsto A_1$ defines a *-isomorphism of $B(\mathcal{K}_m^L)$ onto $B(\mathcal{K}_{m \oplus 1}^L)$. Hence there is a unitary operator $T_m : \mathcal{K}_m^L \to \mathcal{K}_{m\oplus 1}^L$, such that

$$\Phi(S_m^*(A \otimes I) S_m) = S_{m \oplus 1}^* (T_m A T_m^* \otimes I) S_{m \oplus 1}.$$

It follows that any element $A \in \mathcal{N}$ can be identified with a tuple $[A_0, \dots, A_{d-1}]$ with $A_m \in B(\mathcal{H}_m^L)$ and

$$\Phi(A) \equiv [T_{d-1}A_{d-1}T_{d-1}^*, T_0A_0T_0^*, \dots, T_{d-2}A_{d-2}T_{d-2}^*]. \tag{3}$$

Let us now determine the elements of the subalgebra $\mathcal{F}(\Phi) \subseteq \mathcal{N}(\Phi)$. Let us denote $\tilde{T}_m := T_m \dots T_0$, then $\tilde{T}_0 = T_0$, each $\tilde{T}_m$ is a unitary and $\tilde{T}_{d-1}$ is a unitary on $\mathcal{K}_0^L$. With these notations, it can be seen from (3) that

$$\mathcal{F}(\Phi) = \{[A_0, \tilde{T}_0A_0\tilde{T}_0^*, \dots, \tilde{T}_{d-2}A_0\tilde{T}_{d-2}^*],\ A_0 \in \{\tilde{T}_{d-1}\}' \cap B(\mathcal{H}_0^L)\}. \tag{4}$$

It follows that $\mathcal{F}(\Phi)$ is isomorphic to the algebra $\{\tilde{T}_{d-1}\}' \cap B(\mathcal{H}_0^L)\}$. Since $\mathcal{F}(\Phi)$ is atomic, $\{\tilde{T}_{d-1}\}' \cap B(\mathcal{H}_0^L)\}$ is atomic as well, so that its center $\{\tilde{T}_{d-1}\}'' \cap B(\mathcal{H}_0^L)$ is discrete. It follows that the unitary $\tilde{T}_{d-1}$ has discrete spectrum. The minimal central projections P_j of $\mathcal{F}(\Phi)$ are obtained from the minimal projections $R_j \in \{\tilde{T}_{d-1}\}'' \cap B(\mathcal{H}_0^L)$, so that

$$P_j \equiv [R_j, T_0R_jT_0^*, \dots, T_{d-1}R_jT_{d-1}^*], \qquad j \in \mathcal{J}_{[i]}$$

are precisely the minimal central projections in $\mathcal{F}(\Phi)$. We now want to specify the decoherence free algebras of the irreducible maps Ψ_j from the decomposition in Theorem 2. It is not difficult to check that for all j,

$$\begin{aligned} U_j^*(I \otimes \mathcal{N}(\Psi_j))U_j &= P_j(\mathcal{F}(\Phi)' \cap \mathcal{N})P_j \\ &= \{[c_0R_j, c_1T_0R_jT_0^*, \dots, c_{d-1}T_{d-1}R_jT_{d-1}^*],\ c_0, \dots, c_{d-1} \in \mathbb{C}\} \\ &= \operatorname{span}\{P_jQ_0, \dots, P_jQ_{d-1}\}. \end{aligned}$$

In particular, any Ψ_j has period equal to d.

3 Open quantum random walks

In this section we discuss an important example of nucp maps.

Let $\mathcal{H} = \oplus_{i \in V}\mathfrak{h}_i$, where V is a countable set of vertices and $\mathfrak{h}_i$ are separable Hilbert spaces. Note that we may express $\mathcal{H}$ as $\mathcal{H} = \sum_{i \in V} \mathfrak{h}_i \otimes |i\rangle$. An open quantum random walk (OQRW) [2]... is a completely positive trace preserving map $\mathfrak{M}$ on the space $\mathcal{I}_1(\mathcal{H})$ of trace-class operators, of the form

$$\mathfrak{M} : \rho \mapsto \sum_{i,j} V_{i,j}\rho V_{i,j}^*,$$

where $V_{i,j} = L_{i,j} \otimes |i\rangle\langle j|$ and $L_{i,j}$ are bounded operators $\mathfrak{h}_j \to \mathfrak{h}_i$ satisfying

$$\sum_{i \in V} L_{i,j}^*L_{i,j} = I_j, \qquad \forall j \in V. \tag{5}$$

Put $\Phi = \mathfrak{M}^*$, then Φ is a nupc map. Note that any operator $A \in B(\mathcal{H})$ can be written as

$$A = \sum_{i,j \in V} A_{i,j} \otimes |i\rangle\langle j|,$$

where $A_{i,j}$ is an operator $\mathfrak{h}_j \to \mathfrak{h}_i$. We next investigate the multiplicative domain and the decoherence-free subalgebra of Φ.

Proposition 7. *Let Φ be an OQRW. Then $A \in \mathcal{M}(\Phi)$ if and only if*

$$A_{i,i}L_{i,j}L_{k,j}^* = L_{i,j}L_{k,j}^*A_{k,k}, \quad \forall i,j,k \tag{6}$$

and

$$A_{l,i}L_{i,j} = 0 = L_{l,j}^*A_{l,i}, \quad \forall i,j,l \in V, \ i \neq l \tag{7}$$

Proof. It is easy to see from Proposition 1 that $A \in \mathcal{M}(\Phi)$ if and only if A commutes with all operators of the form $L_{i,j}L_{k,j}^* \otimes |i\rangle\langle k|$, $i,j,k \in V$. This is equivalent to (6), together with

$$A_{l,i}L_{i,j}L_{k,j}^* = 0 = L_{k,j}L_{l,j}^*A_{l,i}, \quad \forall i,j,k,l \in V, \ l \neq i \tag{8}$$

It is clear that (7) implies (8). For the converse, multiply the first equality of (8) by $L_{k,j}$ from the right and sum over $k \in V$, then (5) implies the first equality of (7). The second equality is proved similarly.

□

To obtain $\mathcal{N}(\Phi)$, we invoke the notation of [7] of the sets $\mathcal{P}_n(i,j)$ of paths from i to j of lenght n and operators L_π for $\pi \in \mathcal{P}_n(i,j)$. Namely, if $\pi = (i_0 = i, i_1, \ldots, i_n = j)$, then

$$L_\pi = L_{i_n,i_{n-1}}L_{i_{n-1},i_{n-2}} \ldots L_{i_1,i_0}$$

Since the Kraus operators of Φ^n are operators of the form $L_\pi \otimes |j\rangle\langle i|$ for $\pi \in \mathcal{P}_n(i,j)$, the next result can be proved exactly as the previous one.

Proposition 8. *$A \in \mathcal{N}(\Phi)$ if and only if for all $i,j,k,l \in V$, $l \neq i$ and $n \in \mathbb{N}$,*

$$A_{i,i}L_\pi L_{\pi'}^* = L_\pi L_{\pi'}^*A_{k,k}, \quad \forall \pi \in \mathcal{P}_n(j,i), \ \pi' \in \mathcal{P}_n(j,k) \tag{9}$$

and

$$A_{l,i}L_\pi = 0 = L_{\pi'}^*A_{l,i}, \quad \forall \pi \in \mathcal{P}_n(j,i), \ \pi' \in \mathcal{P}_n(j,l) \tag{10}$$

Due to the characterization in the previous proposition, we can deduce a decomposition of the decoherence-free algebra in block diagonal and block off-diagonal operators.

Corollary 3. *$\mathcal{N}(\Phi) = \mathcal{N}_D \oplus \mathcal{N}_{OD}$ where:*

$$\begin{aligned}
\mathcal{N}_D &= \{A = \sum_{i \in V} A_{ii} \otimes |i\rangle\langle i|, A \in \mathcal{N}\} \\
&= \{A = \sum_{i \in V} A_{ii} \otimes |i\rangle\langle i| \ : \ A_{i,i}L_\pi L_{\pi'}^* = L_\pi L_{\pi'}^* A_{k,k}, \forall i,k \in V, \forall (\pi,\pi') \in \cup_{j,n}(\mathcal{P}_n(j,i) \times \mathcal{P}_n(j,k))\} \\
\mathcal{N}_{OD} &= \{A = \sum_{i \neq j \in V} A_{ij} \otimes |i\rangle\langle j|, A \in \mathcal{N}\} \\
&= \{A = \sum_{i \neq j \in V} A_{ij} \otimes |i\rangle\langle j| \ : \ A_{l,i}L_\pi = 0 = L_{\pi'}^* A_{l,i}, \ \forall i,l \in V, \forall (\pi,\pi') \in \cup_{j,n}(\mathcal{P}_n(j,i) \times \mathcal{P}_n(j,l))\}.
\end{aligned}$$

When $\mathcal{N}_{OD}$ is non-trivial, it means that $\mathcal{N}(\Phi) \cap \ker \Phi$ is non trivial and, since it is a von Neumann algebra, then it will contain some diagonal projections. Indeed, if x is in $\mathcal{N}(\Phi) \cap \ker \Phi \setminus \{0\}$, then x^*x is a positive element in $\mathcal{N}(\Phi) \cap \ker \Phi \setminus \{0\}$ and its block-diagonal part $(x^*x)_D := \sum_{i \in V} (x^*x)_{ii} \otimes |i\rangle\langle i|$ is also a positive operator in

$$\mathcal{N}(\Phi) \cap \ker \Phi \cap \{\text{block diagonal operators}\} \setminus \{0\} = \mathcal{N}_D \cap \ker \Phi \setminus \{0\}$$

which is also a von Neumann algebra and so it has to contain a non trivial projection. Summing up, we deduce

$$\mathcal{N}_{OD} \neq \{0\} \quad \Rightarrow \quad \mathcal{N}(\Phi) \cap \ker \Phi \neq \{0\} \quad \Leftrightarrow \quad \mathcal{N}_D \cap \ker \Phi \neq \{0\}.$$

A projection P is in $\mathcal{N}_D \cap \ker \Phi$ if and only if $P = \sum_{i \in V} P_i \otimes |i\rangle\langle i|$ with $P_i L_{ij} = 0$ for all i and j. Such a P exists and is non trivial iff there exists an index i such that $W_i := \cap_j \mathrm{Range}(L_{ij})^\perp \neq \{0\}$. Then we take P_i the projection on W_i. Of course, this cannot happen if Φ admits a faithful normal invariant state, since then Φ is faithful and there can be no projections in $\ker \Phi$.

3.1 An example with generalized Pauli operators

Let $V = \{0, 1\}$ and $\mathfrak{h}_0 = \mathfrak{h}_1 = \mathfrak{h}$. Let $L_{00} = L_{11} = \sqrt{p}U_0$, $L_{01} = L_{10} = \sqrt{1-p}U_1$, with $p \in (0, 1)$ and U_0, U_1 unitaries on $\mathfrak{h}$. Explicitly, Φ acts as

$$\Phi(A) = [p(U_0^* A_{00} U_0) + (1-p)U_1^* A_{11} U_1] \otimes |0\rangle\langle 0| + [(1-p)U_1^* A_{00} U_1 + pU_0^* A_{11} U_0] \otimes |1\rangle\langle 1|$$

for $A = \sum_{k,l} A_{k,l} \otimes |k\rangle\langle l| \in \mathfrak{h} \otimes \mathbb{C}^2$. Assume $d := \dim(\mathfrak{h}) < \infty$, then $(2d)^{-1} I_{\mathfrak{h}} \otimes I_2$ is a faithful invariant state for Φ. We next investigate the fixed points and decoherence free subalgebra in the case when U_0 and U_1 are generalized Pauli operators.

Let $\{|j\rangle,\ j = 0, \ldots, d-1\}$ denote an ONB in $\mathfrak{h}$ and let $\oplus$ be addition modulo d. Put $\omega = e^{i2\pi/d}$ and define the operators Z and X as

$$Z|j\rangle = \omega^j |j\rangle$$
$$X|j\rangle = |j \oplus 1\rangle$$

Then Z and X are unitaries satisfying the commutation relation

$$ZX = \omega XZ.$$

Let us also denote

$$W(p) = Z^p X^{-p}, \qquad p \in \mathbb{Z},$$

then $W(p)$ satisfy the relations

$$W(p)W(q) = W(q)W(p) = \omega^{pq} W(p+q). \tag{11}$$

Let Φ be an OQRW as above, with $U_0 = Z$, $U_1 = X$. We first find the fixed point subalgebra of Φ, this can be done using Proposition 3. We see that

$$\mathcal{F}(\Phi) = \{Z \otimes |0\rangle\langle 0|,\ Z \otimes |1\rangle\langle 1|,\ X \otimes |0\rangle\langle 1|, X \otimes |1\rangle\langle 0|\}'$$

and from this, we get

$$\mathcal{F}(\Phi) = \{\begin{pmatrix} A & 0 \\ 0 & XAX^*, \end{pmatrix},\ A \in \{Z, X^2\}'\}.$$

The condition $A \in \{Z, X^2\}'$ implies that A is diagonal in the basis $\{|j\rangle\}$ and

$$A = X^2 A (X^*)^2 \implies \sum_j a_j |j\rangle\langle j| = \sum_j a_j |j \oplus 2\rangle\langle j \oplus 2|,$$

so that $a_j = a_{j\oplus 2}$ for $j = 0, \ldots, d-1$. If d is odd, this means that $a_j = a_0$ for all j, so that $\mathcal{F}(\Phi)$ is trivial. Hence, in this case, Φ is irreducible.

If d is even, $d = 2m$, we obtain $A = a_+P_+ + a_-P_-$, where $a_+, a_- \in \mathbb{C}$ and

$$P_+ = \sum_{k=0}^{m-1} |2k\rangle\langle 2k|, \qquad P_- = \sum_{k=0}^{m-1} |2k+1\rangle\langle 2k+1|$$

So $\mathcal{F}(\Phi)$ is isomorphic to the abelian algebra spanned by these two projections. Note that we have $XP_+X^* = P_-$, $XP_-X^* = P_+$, so that we may write

$$\mathcal{F}(\Phi) = \mathrm{span}\{\begin{pmatrix} P_+ & 0 \\ 0 & P_- \end{pmatrix}, \begin{pmatrix} P_- & 0 \\ 0 & P_+ \end{pmatrix}\}$$

Now for the decoherence-free subalgebra. Let us first assume that d is odd. Put $W = W(1) = ZX^*$, then

$$Z^*WZ = X^*WX = \omega W$$

It follows that $\Phi(W\otimes I_2) = \omega(W\otimes I_2)$, so $\tilde{W} := W\otimes I_2$ is an eigenvector related to the peripheral eigenvalue ω. The eigenvalues of W are ω^k, $k = 0, \ldots, d-1$, each with an eigenvector x_k. Hence the period of Φ is d, the cyclic decomposition is

$$\{Q_k = |x_k\rangle\langle x_k| \otimes I_2,\ k = 0, \ldots, d-1\}$$

and $\mathcal{N}(\Phi)$ is spanned by $\{Q_0, \ldots, Q_{d-1}\}$.

We next turn to the more interesting case $d = 2m$. Let us compute $\mathcal{N}(\Phi)$ using Proposition 8. Note first that for $\pi \in \mathcal{P}_n(i,j)$,

$$L_\pi = xZ^{n-m}X^m,$$

where $|x| = 1$ is some constant obtained from the commutation relations and $m \in \mathbb{N}$ is even if and only if $i = j$. It follows that if $\pi \in \mathcal{P}_n(j,i)$, $\pi' \in \mathcal{P}_n(j,k)$, we have

$$L_\pi L_{\pi'}^* = const.Z^{l-m}X^{m-l} = const.W(l-m) = const.W(p),$$

where $|p| = |m-l|$ is even iff $k = i$. Since all L_π are unitary, we must have $A_{i,j} = 0$ for $i \neq j$. From the conditions on the diagonal blocks, we obtain that $A_{i,i}$ must commute with $W(p)$ for all even $|p|$ and $A_{i,i} = W(p)^*A_{j,j}W(p)$ for all $|p|$ odd if $i \neq j$. Using (11), we obtain that

$$\mathcal{N}(\Phi) = \{\begin{pmatrix} A & 0 \\ 0 & WAW^* \end{pmatrix},\ A \in \{W(2)\}'\}$$

It follows that $\mathcal{N}(\Phi)$ is isomorphic to the algebra $\{W(2)\}'$. Let

$$W = \sum_j \lambda_j |x_j\rangle\langle x_j|$$

be the spectral decomposition of W. Note that in the case $d = 2m$, the eigenvalues are the roots of $\lambda^d = -1$ and satisfy

$$\lambda_j = -\lambda_{j\oplus m}, \qquad j = 0, \ldots, d-1.$$

Further, the eigenvectors $|x_j\rangle$ have the form

$$|x_j\rangle = \frac{1}{\sqrt{d}}\sum_{k=0}^{d-1}\lambda_j^k\bar{\omega}^{n_k}|k\rangle, \qquad n_k = \sum_{l=0}^{k\ominus 1} l,$$

Now, by (11), $W(2) = \bar{\omega}W^2$, so that the eigenvalues of $W(2)$ are $\mu_j := \bar{\omega}\lambda_j^2$. Note that for $j = 0, \dots m-1$, μ_j are the m-th roots of unity and we have $\mu_{j+m} = \mu_j$. It follows that $W(2)$ has the spectral decomposition

$$W(2) = \sum_{j=0}^{m-1}\mu_j Q_j, \qquad Q_j = |x_j\rangle\langle x_j| + |x_{j+m}\rangle\langle x_{j+m}|.$$

Put

$$|j,+\rangle := \sqrt{2}P_+|x_j\rangle, \quad |j,-\rangle := \sqrt{2}P_-|x_j\rangle, \qquad j = 0,\dots,m-1.$$

Then $|j,+\rangle$ and $|j,-\rangle$ are mutually orthogonal unit vectors and we have

$$|x_j\rangle = \frac{1}{\sqrt{2}}|j,+\rangle + \frac{1}{\sqrt{2}}|j,-\rangle, \quad |x_{j+m}\rangle = \frac{1}{\sqrt{2}}|j,+\rangle - \frac{1}{\sqrt{2}}|j,-\rangle,$$

so that

$$Q_j = |j,+\rangle\langle j,+| + |j,-\rangle\langle j,-|.$$

Since Q_j commute with W, we obtain that the center of $\mathcal{N}(\Phi)$ is spanned by the projections

$$Q_j \otimes I_2, \qquad j = 0,\dots,m-1.$$

Further, notice that the action of Φ on elements of $\mathcal{N}(\Phi)$ has the form

$$\Phi\begin{pmatrix} A & 0 \\ 0 & WAW^* \end{pmatrix} = \begin{pmatrix} Z^*AZ & 0 \\ 0 & X^*AX \end{pmatrix}.$$

Since $Z^*|x_j\rangle = |x_{j\ominus 1}\rangle$, we obtain $\Phi(Q_j \otimes I_2) = Q_{j\ominus_m 1}\otimes I_2$, where $\ominus_m$ denotes subtraction modulo m. It follows that there is a unique cycle of length m and consequently only one component $\mathcal{N}_{[1]} = \mathcal{N}(\Phi)$, and we have

$$\mathcal{N}(\Phi) = \oplus_j S_j(B(\mathfrak{h}_j)\otimes I_2)S_j^*$$

where $\mathfrak{h}_j = Q_j\mathfrak{h}$ and $S_j = I_{\mathfrak{h}_j}\otimes|0\rangle\langle 0| + W|_{\mathfrak{h}_j}\otimes|1\rangle\langle 1|$, $j = 0,\dots,m-1$. The unitaries $T_j : \mathfrak{h}_j \to \mathfrak{h}_{j\ominus_m 1}$ are given by the restrictions $T_j = Z^*|\mathfrak{h}_j$, $j = 0,\dots,m-1$ and the unitary $\tilde{T}_{m-1} = Z^{-m}|_{\mathfrak{h}_0}$ has two eigenvalues ± 1, with eigenvectors $|0,\pm\rangle$, so that

$$\tilde{T}_{m-1} = |0,+\rangle\langle 0,+| - |0,-\rangle\langle 0,-|.$$

Now check that the minimal central projections of $\mathcal{F}(\Phi)$ satisfy

$$\sum_{j=0}^{m-1}\tilde{T}_j|0,\pm\rangle\langle 0,\pm|\tilde{T}_j^* = \sum_{j=0}^{m-1}Z^{-j}|0,\pm\rangle\langle 0,\pm|Z^j = \sum_{j=0}^{m-1}|j,\pm\rangle\langle j,\pm| = P_\pm,$$

which corresponds to the results of Section 2.2

3.2 Homogeneous OQRWs

We could also consider the special case of homogeneous nearest neighbor OQWs on $\mathbb{Z}$, $\mathbb{Z}^d$, $\mathbb{Z}_d$ with a finite dimensional local space $\mathfrak{h} = \mathfrak{h}_i$ the homogeneous case, V is a group (I shall concentrate on the cases mentioned above), $\mathfrak{h} = \mathfrak{h}_i$ for all i and the transition operators are translation invariant, i.e. $L_{ij} = L_{i+n,j+n} =: L_n$ for any i, j, n.

We can define the local operator $\mathcal{L}$, acting on $L^1(\mathfrak{h})$

$$\mathcal{L}(\rho) = \sum_n L_n \rho L_n^*$$

This $\mathcal{L}$ has at least one invariant state ρ^{inv}.

For homogeneous OQWs, we have an invariant faithful state only in the case the group is finite (i.e. only $\mathbb{Z}_d$ among the ones mentioned above). In other cases, we shall have an invariant weight ω, that we can define on positive operators in the following way

$$\omega(\sum x_{ij} \otimes |i\rangle\langle j|) = \sum_j \mathrm{Tr}\,(\rho^{\text{inv}} x_{jj}).$$

ω is faithful iff ρ^{inv} is faithful.

When this happens, by Proposition 3, $\mathcal{F}(\mathfrak{M}^*) = \{L_s, s \in S\}'$ will be an algebra included in $\mathcal{N}$.

We consider a set of generators S for the group V: $S = +1, -1$ for $\mathbb{Z}$ or $\mathbb{Z}_d$, $S = \{\pm v_0, ..., \pm v_{d-1}\}$ where $(v_0, ..., v_{d-1})$ is a canonical basis for the case $V = \mathbb{Z}^d$. In the nearest neighbor case we shall have $L_{i-1,i} = L_-$, $L_{i+1,i} = L_+$ and all the other $L_{ij} = 0$.

An immediate application of Proposition .... will give us the following.

Corollary 4. *Characterization of $\mathcal{M}$ for homogeneous OQWs on $\mathbb{Z}$ (or $\mathbb{Z}_d$).*
$A \in \mathcal{M}$ if and only if

$$A_{i,i} L_+ L_-^* = L_+ L_-^* A_{i-2,i-2}, \quad A_{i-2,i-2} L_- L_+^* = L_- L_+^* A_{i,i}, \quad A_{i,i} \in \{|L_+^*|, |L_-^*|\}' \quad \forall i,$$

and

$$A_{ik} L_- = A_{ik} L_+ = L_-^* A_{ik} = L_+^* A_{ik} = 0, \quad \forall i, k \in V, \; i \neq k \tag{12}$$

In particular, when at least one transition operator is invertible, $\mathcal{M}$ contains only block-diagonal operators.

We shall concentrate on the case $V = \mathbb{Z}$ in particular; at least the case $V = \mathbb{Z}_d$ should easily follow being careful to distinguish the cases when d is even or odd.

Let $\mathfrak{M}$ be a OQRW and $\rho \in \mathfrak{S}(\mathcal{H})$. Then ρ is invariant for $\mathfrak{M}$ if and only if $\rho = \sum_i \rho_i \otimes |i\rangle\langle i|$ and

$$\rho = \sum_i \rho_i \otimes |i\rangle\langle i|, \qquad \rho_k = \sum_i L_{k,i} \rho_i L_{k,i}^*, \; \forall k \in V \tag{13}$$

Suppose that there is some faithful normal invariant state ρ. Let $\mathcal{F} = \mathcal{F}_\Phi$. Then it is clear that any $A \in \mathcal{F}$ must be block-diagonal, that is $A = \sum_i A_i \otimes |i\rangle\langle i|$. By Proposition 3, we obtain that $A \in \mathcal{F}$ if and only if

$$A_i L_{i,j} = L_{i,j} A_j, \qquad A_j L_{i,j}^* = L_{i,j}^* A_i, \quad \forall i, j \in V. \tag{14}$$

Moreover, by Proposition ??, we can see that $\mathcal{M} = \mathcal{N} = \mathcal{F}$ if and only if

$$L_{i,j} \in \{\tilde{L}_\pi,\ \pi \in \mathcal{P}_{odd}(j,i)\}'', \qquad \forall i,j \in V, \tag{15}$$

where $\mathcal{P}_{odd}(j,i)$ is the set of paths $j \to i$ with odd lenght and for $\pi = (j, i_1, \dots, i_{n-1}, i) \in \mathcal{P}_{odd}(j,i)$, we define

$$\tilde{L}_\pi := L_{i,i_{n-1}} L^*_{i_{n-2},i_{n-1}} \dots L_{i_2,i_1} L^*_{j,i_1}.$$

Proposition 9. *Consider a homogeneous, irreducible OQRW with local space $\mathbb{C}^2$ and with transition operators which are invertible matrices. Then*

$$\mathcal{N} = span\{P_{odd}, P_{even}\}$$

unless there exists an orthonormal basis $\{f_0, f_1\}$ such that L_- and L_+ are one diagonal and one off-diagonal in this basis.

In the last case, $\mathcal{N}$ is generated by the cyclic projections

$$P_{\epsilon,\delta} = \sum_j (|f_\epsilon\rangle\langle f_\epsilon| \otimes |4j+\delta\rangle\langle 4j+\delta| + |f_{1-\epsilon}\rangle\langle f_{1-\epsilon}| \otimes |4j+2+\delta\rangle\langle 4j+2+\delta|),$$

with $\epsilon, \delta = 0, 1$ and the period is 4. Otherwise the period is 2 with cyclic projections P_{odd}, P_{even}.

The period was already computed in [8].

Proof. By Corollary 3, we know that the decoherence free algebra $\mathcal{N}$ consists only of block-diagonal operators. Then a projection P in $\mathcal{N}$ will have the form

$$P = \sum_j P_j \otimes |j\rangle\langle j|,$$

where, by Corollary 4, satisfy at least the conditions

$$P_j \in \{|L^*_+|, |L^*_-|\}', \qquad P_{j-1} L_- L^*_+ = L_- L^*_+ P_{j+1} \qquad \forall\, j. \tag{16}$$

We can write the action of Φ explicitly, in particular

$$\begin{aligned} \Phi(P) &= \sum_j (L^*_+ P_{j+1} L_+ + L^*_- P_{j-1} L_-) \otimes |j\rangle\langle j|, \\ \Phi^2(P) &= \sum_j (L^{*2}_+ P_{j+2} L^2_+ + L^{*2}_- P_{j-2} L^2_- + L^*_- L^*_+ P_j L_+ L_- + L^*_+ L^*_- P_j L_- L_+) \otimes |j\rangle\langle j|. \end{aligned} \tag{17}$$

By these relations, it is easily deduced that $\Phi^n(P_{odd})$ is equal to P_{odd} for even n and to P_{even} for odd n (and similarly for $\Phi^n(P_{even})$). In particular, $\Phi^n(P_{odd}), \Phi^n(P_{even})$ are always projections and this allows us to conclude that P_{odd} and P_{even} belong to $\mathcal{N}$.... reference? Moreover, they are trivially central, i.e., for any other projection P in $\mathcal{N}$, $PP_{odd} = P_{odd}P$ and $PP_{even} = P_{even}P$.

When there exists an orthonormal basis $\{f_0, f_1\}$ such that L_- and L_+ are one diagonal and one off-diagonal in this basis, it is easy to see that the projections $P_{\epsilon,\delta}$ in the statement are cyclic. It is a little more complicated to see that these cyclic projections can exist only in that case and anyway no other minimal projection can then appear.

So now we want to consider, for a homogeneous irreducible OQRW, whether there exists a projection P in $\mathcal{N} \setminus \text{span}\{P_{odd}, P_{even}\}$. We shall see that this is not possible, unless we are in the special case described in the statement.

If such a P exists, then $P = PP_{odd} + PP_{even}$ and the two addends are both in $\mathcal{N}$, so, by homogeneity, it will be sufficient to search for a projection P in $\mathcal{N}$ such that $P = PP_{even}$ and $0 < P < P_{even}$. Then we consider $P = \sum_j P_{2j} \otimes |2j\rangle\langle 2j|$.

Relations (16) imply that all the P_{2j}'s have the same rank (since the transition operators are invertible). Then, if P is different from 0 and from P_{even}, the only possibility is that P_{2j} is a rank one projection for any j. Calling u a norm one vector such that $P_0 = |u\rangle\langle u|$, and denoting $V := L_- L_+^*$, we deduce

$$P = \sum_j |V^{-j}u\rangle\langle V^{*j}u| \otimes |2j\rangle\langle 2j|,$$

where $V^{-j}u \parallel V^{*j}u$ because any P_{2j} is a projection and, due to the first condition in (16), $V^{*j}u$ is a common eigenvector of $|L_+^*|$ and $|L_-^*|$ for any j.

Similar considerations will hold for $\Phi^n(P)$, but considering only odd vertices instead of even vertices when n is odd. Indeed, starting with $n = 1$ (for $\Phi^n(P)$ we simply proceed inductively),
- $\Phi(P)$ is a projection in $\mathcal{N}$, $\Phi(P) \le P_{odd}$ due to the fact that $0 \le P \le P_{even}$ and Φ is positive,
- moreover , when $P \neq P_{even}$ then $\Phi(P) \neq P_{odd}$ by irreducibility; indeed, if we had for instance $P \neq P_{even}$ and $\Phi(P) = P_{odd}$, then $P_{even} - P$ would be a non-zero projection in the kernel of Φ and this contradicts irreducibility.

Then, using (17), we need that

$$\Phi^2(P)(1 \otimes |0\rangle\langle 0|) = (L_+^{*2} P_2 L_+^2 + L_-^{*2} P_{-2} L_-^2 + L_-^* L_+^* P_0 L_+ L_- + L_+^* L_-^* P_0 L_- L_+)$$

is a one dimensional projection. This implies in particular that $L_-^* L_+^* u \parallel L_+^* L_-^* u$, so that u is an eigenvector for $(L_+^* L_-^*)^{-1} L_-^* L_+^*$.

Also, calling $u^\perp$ a norm one vector orthogonal to u, $P' := P_{even} - P = \sum_j |V^{-j}u^\perp\rangle\langle V^{*j}u^\perp| \otimes |2j\rangle\langle 2j|$, will be a projection in $\mathcal{N}$ and so $u^\perp$ will satisfy the same conditions as u.

Summing up, we have that u and $u^\perp$ should be two distinct eigenvectors for the operators

$$|L_+^*|, \qquad |L_-^*|, \qquad W := (L_+^* L_-^*)^{-1} L_-^* L_+^*. \tag{18}$$

Now, we claim that, due to irreducibility, the previous operators cannot be all proportional to the identity and we postpone of some lines the proof of this claim.

This fact implies that, either such vectors u and $u^\perp$ do not exist, and so $\mathcal{N} =$ ...., or they can be chosen in a unique way, up to multiplicative constants, as the orthonormal basis which diagonalize all the three operators above. In the latter case, we now look at the form of $\Phi(P)$ given in (17) and we see that

$$\Phi(P)(1 \otimes |j\rangle\langle j|) = L_+^* P_{j+1} L_+ + L_-^* P_{j-1} L_-$$

should be a one dimensional projection on a vector v which should be an eigenvector of the same three operators. This implies that

$$L_\epsilon^* u, L_\epsilon^* u^\perp \in \operatorname{span}\{u\} \cup \operatorname{span}\{u^\perp\}, \qquad \epsilon = +, -.$$

This implies that the operators L_+ and L_- should be either diagonal or off-diagonal in the basis $\{u, u^\perp\}$; but they cannot be both diagonal nor both off-diagonal, because this would contradict irreducibility. So the conclusion follows choosing $\{f_0, f_1\} = \{u, u^\perp\}$.

Finally, we go back to prove the claim. By contradiction, we suppose that all the operators in (18) are proportional to the identity, so that

$$L_+ = c_+ U_+, \qquad L_- = c_- U_- \qquad W = c1$$

for some complex numbers c_+, c_-, c and unitary operators U_+, U_-. Then we can rewrite

$$W = c1 = U_- U_+ U_-^* U_+^* \qquad \Rightarrow \qquad U_- = c U_+ U_- U_+^*$$

But now write the diagonal form of the unitary U_+, $U_+^* = \sum_{k=0,1} \lambda_k |v_k\rangle\langle v_k|$, with λ_0, λ_1 in the unit circle and $\{v_0, v_1\}$ orthonormal basis, and consider

$$\langle v_k, U_- v_j\rangle = c\langle v_k, U_+ U_- U_+^* v_j\rangle = c\overline{\lambda}_k \lambda_j \langle v_k, U_- v_j\rangle \qquad \text{for } j,k = 0,1.$$

This implies $c = 1$ and $\lambda_0 = \lambda_1$ which requires that U_+ and so L_+ are proportional to the identity. But this contradicts irreducibility. □

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