On the properties $\alpha - z$ Rényi divergences on general von Neumann algebras

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1 Introduction

2 Preliminaries

2.1 Basic definitions

Let $\mathcal{M}$ be a von Neumann algebra and let $\mathcal{M}^+$ be the cone of positive elements in $\mathcal{M}$. We denote the predual by $\mathcal{M}_*$, its positive part by $\mathcal{M}_*^+$ and the set of normal states by $\mathfrak{S}_*(\mathcal{M})$. For $\psi \in \mathcal{M}_*^+$, we will denote by $s(\psi)$ the support projection of ψ.

For $0 < p \leq \infty$, let $L_p(\mathcal{M})$ be the Haagerup L_p-space over $\mathcal{M}$ and let $L_p(\mathcal{M})$ its positive cone, [?]. We will use the identifications $\mathcal{M} \simeq L_\infty(\mathcal{M})$, $\mathcal{M}_* \ni \psi \leftrightarrow h_\psi \in L_1(\mathcal{M})$ and the notation $\operatorname{Tr} h_\psi = \psi(1)$ for the trace in $L_1(\mathcal{M})$. It this way, $\mathcal{M}_*^+$ is identified with the positive cone $L_1(\mathcal{M})^+$ and $\mathfrak{S}_*(\mathcal{M})$ with subset of elements in $L_1(\mathcal{M})^+$ with unit trace. Precise definitions and further details on the spaces $L_p(\mathcal{M})$ can be found in the notes [?].

2.2 The $\alpha - z$-Rényi divergences

In [? ?], the $\alpha - z$-Rényi divergence for $\psi, \varphi \in \mathcal{M}_*^+$ was defined as follows:

Definition 1. Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$ and let $\alpha, z > 0$, $\alpha \neq 1$. The $\alpha - z$-Rényi divergence is defined as

$$D_{\alpha,z}(\psi\|\varphi) := \frac{1}{\alpha - 1} \log \frac{Q_{\alpha,z}(\psi\|\varphi)}{\psi(1)},$$

where

$$Q_{\alpha,z} = \begin{cases} \operatorname{Tr}\left(h_\varphi^{(1-\alpha)/2z} h_\psi^{\alpha/z} h_\varphi^{(1-\alpha)/2z}\right)^z, & \text{if } 0 < \alpha < 1 \\ \|x\|_z^z, & \text{if } \alpha > 1 \text{ and} \\ & h_\psi^{\alpha/z} = h_\varphi^{(\alpha-1)/2z} x h_\varphi^{(\alpha-1)/2z}, \text{ with } x \in s(\varphi)L_z(\mathcal{M})s(\varphi) \\ \infty & \text{otherwise.} \end{cases}$$

In the case $\alpha > 1$, the following alternative form will be useful.

Lemma 1. *[?] Let $\alpha > 1$ and $\psi, \varphi \in \mathcal{M}_*^+$. Then $Q_{\alpha,z}(\psi\|\varphi) < \infty$ if and only if there is some $y \in L_{2z}(\mathcal{M})s(\varphi)$ such that*

$$h_\psi^{\alpha/2z} = y h_\varphi^{(\alpha-1)/2z}.$$

Moreover, in this case, such y is unique and we have $Q_{\alpha,z}(\psi\|\varphi) = \|y\|_{2z}^{2z}$.

The standard Rényi divergence [? ? ?] is contained in this range as $D_\alpha(\psi\|\varphi) = D_{\alpha,1}(\psi\|\varphi)$. The sandwiched Rényi divergence is obtained as $\tilde{D}_\alpha(\psi\|\varphi) = D_{\alpha,\alpha}(\psi\|\varphi)$, see [? ? ? ?] for some alternative definitions and properties of $\tilde{D}_\alpha$. The definition in [?] and [?] is based on the Kosaki interpolation spaces $L_p(\mathcal{M},\varphi)$ with respect to a state [?]. These spaces and complex interpolation method will be used frequently also in the present work.

Many of the properties of $D_{\alpha,z}(\psi\|\varphi)$ were extended from the finite dimensional case in [?]. In particular, the following variational expressions will be an important tool for our work.

Theorem 1. *Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$.*

(i) Let $0 < \alpha < 1$ and $\max\{\alpha, 1-\alpha\} \leq z$. Then

$$Q_{\alpha,z}(\psi\|\varphi) = \inf_{a \in \mathcal{M}^{++}} \left\{ \alpha \mathrm{Tr}\left((a^{1/2} h_\psi^{\alpha/z} a^{1/2})^{z/\alpha} \right) + (1-\alpha) \mathrm{Tr}\left((a^{-1/2} h\varphi^{(1-\alpha)/z} a^{-1/2})^{z/(1-\alpha)} \right) \right\}.$$

Moreover, if $\lambda^{-1}\varphi \leq \psi \leq \lambda\varphi$ for some $\lambda > 0$, then the infimum is attained and...

(ii) $\alpha > 1$

Proof. [?] for (i).

□

3 Data processing inequality and reversibility of quantum channels