A note on monotonicity of $z \mapsto D_{\alpha,z}$ for $1 < \alpha \le 2z$

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Assume that $1 < \alpha \le 2z \le 2z'$. We will prove that $Q_{\alpha,z}(\psi\|\varphi) \ge Q_{\alpha,z'}(\psi\|\varphi)$. We may clearly assume that $Q_{\alpha,z}(\psi\|\varphi) < \infty$, in which case there is some $y \in L^{2z}(\mathcal{M})$ such that

$$h_\psi^{\frac{\alpha}{2z}} = y h_\varphi^{\frac{\alpha-1}{2z}}, \qquad Q_{\alpha,z}(\psi\|\varphi) = \|y\|_{2z}^{2z}.$$

In particular, $e := s(\psi) \le s(\varphi)$, so that we may assume that φ is faithful. Let $\sigma \in \mathcal{M}_*^+$ be such that $s(\sigma) = 1 - e$ and put $\psi_0 := \psi + \sigma$, so that ψ_0 is faithful as well. We will use the notation $L^p_L := L^p(\mathcal{M};\varphi)_L$, $1 \le p \le \infty$.

Consider the function

$$f(w) = h_{\psi_0}^{\frac{\alpha}{2z}w} e h_\varphi^{1-\frac{\alpha}{2z}w}, \qquad w \in S,$$

where $S := \{w \in \mathbb{C},\ 0 \le \operatorname{Re} w \le 1\}$. Then clearly

$$f(it) = h_{\psi_0}^{\frac{\alpha}{2z}it} e h_\varphi^{-\frac{\alpha}{2z}it} h_\varphi \in L^\infty_L, \qquad t \in \mathbb{R},$$

$t \mapsto f(it)$ is continuous in L^∞_L and $\|f(it)\|_{L^\infty_L} = 1$ for all t. We also have by [1, Lemmas 10.1 and 10.2]

$$f(1+it) = h_{\psi_0}^{\frac{\alpha}{2z}it} h_\psi^{\frac{\alpha}{2z}} h_\varphi^{1-\frac{\alpha}{2z}} h_\varphi^{-\frac{\alpha}{2z}it} = (h_{\psi_0}^{\frac{\alpha}{2z}it} y h_\varphi^{-\frac{\alpha}{2z}it}) h_\varphi^{\frac{2z-1}{2z}} \in L^{2z}_L, \qquad t \in \mathbb{R},$$

$t \mapsto f(1+it)$ is continuous in L^{2z}_L and

$$\|f(1+it)\|_{L^{2z}_L} = \|h_{\psi_0}^{\frac{\alpha}{2z}it} y h_\varphi^{-\frac{\alpha}{2z}it}\|_{2z} = \|y\|_{2z}.$$

Therefore $f \in \mathcal{F}(L^\infty_L, L^{2z}_L)$, so that for any $\theta \in (0,1)$, $f(\theta) \in C_\theta(L^\infty_L, L^{2z}_L)$ and

$$\|f(\theta)\|_{C_\theta} \le \|y\|_{2z}^\theta.$$

By the reiteration theorem, $C_\theta = L_L^{2z/\theta}$. Putting $\theta = z/z'$, we get

$$f(z/z') = h_\psi^{\frac{\alpha}{2z'}} h_\varphi^{1-\frac{\alpha}{2z'}} = y' h_\varphi^{\frac{2z'-1}{2z'}}$$

for some $y' \in L^{2z'}(\mathcal{M})$, and $\|y'\|_{2z'} \le \|y\|_{2z}^{z/z'}$, this proves the result.

References

[1] H. Kosaki. Applications of the complex interpolation method to a von Neumann algebra: Non-commutative L_p-spaces. *J. Funct. Anal.*, 56:26–78, 1984. doi:https://doi.org/10.1016/0022-1236(84)90025-9.