E. Haapasalo, J. P. Pellonpää, Optimal covariant quantum measurements

Referee report

The paper studies the structure of covariant quantum instruments, in the finite dimensional case with a finite group and in infinite dimensions with respect to a transitive action of a locally compact group with a compact stability subgroup. In these cases, a detailed structure of covariant instruments is obtained, which is then used to characterize the extreme points of the set of covariant instruments. In finite dimensions, extreme rank-1 covariant POVMs, which are PVM or information complete, are studied as an application. In infinite dimensions, covariant phase space observables are investigated as an example.

The results of the paper seem interesting and worth publication, but a substantial revision is needed before a recommendation is possible.

Overall remarks

1. The paper lacks a proper introduction, clarifying the questions asked and their motivation, describing (in an understandable way, without much technical details) the results obtained and putting them into context of previous works. The authors should state precisely what is new here compared to other works also cited here, including the previous works of the authors. A description of the structure of the paper is also advisable.

2. The paper is rather technical and the form of presentation chosen by the authors makes it difficult to read. The authors should provide some guidance for the reader, on what is going on and why. Moreover, I find some parts of the paper rather confusing and it seems that some reorganization is needed. Overall, changes are necessary to increase readability of the paper. Some suggestions are in the specific comments below, but these do not cover all of the problematic places.

3. There is a lot of symbols introduced in the paper, either plain or calligraphic, adorned with upper and lower indices, which are difficult to keep track of. In this situation a typo can cause severe problems for the reader. The authors should **thoroughly check the paper for typos**. A few are suggested below.

Specific comments

1. p. 3, l. 40/41 - A strange statement: "...there is necessarily no nontrivial solution M for (1)."

2. p. 6/7 - It is not clear what exactly is constructed in the steps (i) - (iv). Namely, it is not clear what "the above type covariant rank -1 PVM" or "rank-1 PVM as above" in the last paragraph on p. 5 refers to.

3. p. 7 - the steps (i)-(iv) are referred to as (1)-(4).

4. p. 7 - As I understand, Example 1 gives a way to construct a one-parameter family of covariant rank-1 extreme POVMs, containing both a PVM and IC POVMs. The relation between this example and the construction (i) - (iv) is not very clearly explained. Especially, I find the paragraph before Example 1 rather confusing.

5. p. 11 - It is suggested that a conjecture in [16] is resolved, if so, this should be better explained.

6. p. 12 - proof of Thm. 3 uses the representation of the imprimitivity system given in Appendix B. I strongly suggest to give its form and the introduced notations explicitly before the statement of Thm. 3.

7. p. 12 - There seems to be a typo: the computations from the second paragraph of the proof of Thm 3 onward make sense only if one replaces $\mathcal{M}^\Omega$ by $\mathcal{H}^\Omega$.

8. p. 12, 1 line before last - e_{η_i} should be $e_{\eta,i}$.

9. p. 13 - ζ should be ζ^η

10. Better also explain somewhere that one may put $M_\eta = 0$ if $L^\Omega_{\eta,i,m}$ would have to be 0

11. p. 18 - there is some discrepancy in the definition of $\mathcal{M}^\Omega$ in the proof of Thm. 4 and in Appendix B (similar to the above).

12. p. 25 - $\mathcal{I}(\Omega, \cdot)$ should probably be $\mathcal{I}(\mathbb{X}, \cdot)$

13. p. 27 - I find this page quite confusing. What is the very particular form of the minimal covariant dilation mentioned in the first paragraph? What is the role of the assumptions (a)-(c)? What was proved in [4,12] and how it is related to the results of Thm. 5? I would also suggest to give a more comprehensible description of some of the examples at the bottom of p. 27 (and perhaps only mention the others)

14. p. 27, assumption (c) - it seems that $\mathcal{H}^g_{\pi_0}$ should be $L^2_\mu \otimes \mathcal{H}_{\pi_0}$? Also clarify that $x = gH$

15. p. 29, proof of Thm. 5 - the existence of the minimal covariant Naimark dilation of M at the beginning of the proof should follow by assumption (c). Why the reference to [5] and [12]?

16. p. 29, proof of Thm. 5 - here P_0 should perhaps be $\mathsf{P}^G_{\pi_0}$