Some notes

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1 QHT

Let $\mathcal{M}$ be a von Neumann algebra and let ρ, σ be a pair of normal states. We consider the problem of testing the hypothesis $H_0 = \rho$ againts alternative $H_1 = \sigma$. Any test is represented by an operator $M \in \mathcal{M}$, $0 \leq M \leq I$, such that the value $\omega(M)$ is interpreted as the probability of accepting the hypothesis H_0 in the state ω.

The two error probabilities related to a test M are given as

$$\alpha(M) = \rho(1 - M), \qquad \beta(M) = \sigma(M).$$

In the Bayes approach, we choose a prior probability $\pi \in (0, 1)$ and minimize the average error probability over all tests, obtaining the optimal value

$$\begin{aligned} b_\pi(\rho\|\sigma) &:= \inf_M \pi\alpha(M) + (1-\pi)\beta(M) \\ &= \inf_M \pi\rho(1-M) + (1-\pi)\sigma(M) \\ &= \pi - \sup_M (\pi\rho - (1-\pi)\sigma)(M) = \pi(1 - (\rho - s\sigma)_+(1)) \\ &= \frac{1}{2}(1 - \pi\|\rho - s\sigma\|_1), \qquad s = \frac{1-\pi}{\pi}. \end{aligned}$$

The optimal test for $\pi \in (0, 1)$ is the quantum Neyman-Pearson test M_π, which is of the form

$$M_\pi = \{\rho > s\sigma\} + X, \qquad 0 \leq X \leq \{\rho = s\sigma\}, \qquad s = \frac{1-\pi}{\pi},$$

here for $\varphi, \psi \in \mathcal{M}_*^+$ we define $\{\varphi > \psi\}$ as the projection ionto the support of $(\varphi - \psi)_+$, similarly $\{\varphi < \psi\}$ is the projectio onto the support of $(\psi - \varphi)_+ = (\varphi - \psi)_-$. We then have $\{\varphi \leq \psi\} = 1 - \{\varphi > \psi\}$, $\{\varphi \geq \psi\} = 1 - \{\varphi < \psi\}$ and $\{\varphi = \psi\} = 1 - \{\varphi < \psi\} - \{\varphi > \psi\}$.

It is clear from these expressions that the function $\pi \mapsto b_\pi(\rho\|\sigma)$ is continuous. Furthermore, let us look at the error probabilities for the QNP test $\{\rho > s\sigma\}$. Put $\alpha(s) := \rho(\{\rho \leq s\sigma\})$ and $\beta(s) := \sigma(\{\rho > s\sigma\})$. Then By the QNP lemma, we have for any $s, s' \in (0, \infty)$ that

$$\alpha(s) + s\beta(s) \leq \alpha(s') + s\beta(s') \tag{1}$$

hence

$$s(\beta(s) - \beta(s')) \leq \alpha(s') - \alpha(s) \leq s'(\beta(s) - \beta(s')),$$

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the second inequality is obtained by exchanging s and s' in (**??**). It follows that if $s' > s$, then $\beta(s) \geq \beta(s')$ and therefore also $\alpha(s) \leq \alpha(s')$.

In the asymmetric QHT approach, we fix the maximum acceptable value ϵ of the error $\alpha(M) \leq \epsilon$ and minimize the error $\beta(M)$:

$$d_H^{\epsilon}(\rho\|\sigma) := \inf\{\sigma(M),\ 0 \leq M \leq I,\ \rho(1-M) \leq \epsilon\}.$$

The following result is proved similarly as in the classical case (cf. [**?**]).

Lemma 1. *We have*

$$b_\pi(\rho\|\sigma) = \inf_{0<\epsilon<1}[\pi\epsilon + (1-\pi)d_H^{\epsilon}(\rho\|\sigma)], \qquad \pi \in (0,1)$$

Proof. Let $\pi, \epsilon \in (0,1)$ and let M be any test such that $\rho(1-M) \leq \epsilon$, then we have

$$b_\pi(\rho\|\sigma) \leq \pi\rho(1-M) + (1-\pi)\sigma(M) \leq \pi\epsilon + (1-\pi)\sigma(M).$$

Taking infimum over all such tests, we obtain

$$b_\pi(\rho\|\sigma) \leq \pi\epsilon + (1-\pi)d_H^{\epsilon}(\rho\|\sigma), \qquad \forall \epsilon, \pi \in (0,1). \tag{2}$$

For a fixed $\pi \in (0,1)$ and a Neyman-Pearson test M_π, let $\epsilon_\pi = \rho(1-M_\pi)$, then

$$b_\pi(\rho\|\sigma) = \pi\epsilon_\pi + (1-\pi)\sigma(M_\pi) \geq \pi\epsilon_\pi + (1-\pi)d_H^{\epsilon_\pi}(\rho\|\sigma),$$

this proves equality.

□

1.1 In finite dimensions...

Lemma 2. *Let $s(\rho) \leq s(\sigma)$. Then for any $\epsilon \in (0,1)$ there is some $\pi_\epsilon \in (0,1)$ and a Neyman-Pearson test $M_\epsilon = M_{\pi_\epsilon}$ such that $\rho(1-M_\epsilon) = \epsilon$.*

Proof. Let

$$s_\epsilon := \sup\{s \geq 0, \mathrm{Tr}\,[\rho P_{s,-}] < \epsilon\}.$$

Under the assumption, there is some $\lambda > 0$ such that $\rho \leq \lambda\sigma$, and then $P_{s,-} = I$ for all $s \geq \lambda$. It follows that we must have $c_\epsilon < \lambda$, in particular, c_ϵ is finite. Further, by [**?** , Lemma], we have for $s \in [0,\infty)$:

$$\lim_{t\to s^-} P_{t,-} = P_{s,-}, \qquad \lim_{t\to s+} P_{t,-} = P_{s,-} + P_{s,0}$$

and $P_{s,0} \neq 0$ for a finite number of values of s. If follows that

$$\mathrm{Tr}\,[\rho P_{c_\epsilon,-}] = \lim_{t\to c_\epsilon -} \mathrm{Tr}\,[\rho P_{t,-}] \leq \epsilon \leq \lim_{t\to c_\epsilon +} \mathrm{Tr}\,[\rho P_{t,-}] = \mathrm{Tr}\,[\rho(P_{c_\epsilon,-} + P_{c_\epsilon,0})]$$

and

$$M_\epsilon = P_{c_\epsilon,-} + \frac{\epsilon - \mathrm{Tr}\,[\rho P_{c_\epsilon,-}]}{\mathrm{Tr}\,[\rho P_{c_\epsilon,0}]} P_{c_\epsilon,0}$$

is a Neyman-Pearson test for $\pi_\epsilon := (c_\epsilon + 1)^{-1}$ such that $\mathrm{Tr}\,[\rho M_\epsilon] = \epsilon$.

□

Lemma 3. *If $s(\rho) \le s(\sigma)$ then also*

$$d_H^\epsilon(\rho\|\sigma) = \sup_{0<\pi<1} \frac{1}{1-\pi}[b_\pi(\rho\|\sigma) - \pi\epsilon], \qquad \epsilon \in (0,1).$$

Proof. We see that (**??**) implies

$$d_H^\epsilon(\rho\|\sigma) \ge \frac{1}{1-\pi}[b_\pi(\rho\|\sigma) - \pi\epsilon]$$

for all ϵ and π. If $s(\rho) \le s(\sigma)$, then the second equality follows by Lemma **??**.

□

2 Measured Rényi relative entropy

Let $\mathcal{M}$ be a von Neumann algebra. A measurement on $\mathcal{M}$ (with values in a finite set Ω) is defined as a positive linear map $M : L_\infty(\Omega) \to \mathcal{M}$ such that $M(1) = 1$. Equivalently, a measurement is given by a set of positive operators $\{a_\omega\}_{\omega\in\Omega}$ such that $\sum_\omega a_\omega = 1$. More generally, a measurement can be defined as a unital positive map from a commutative von Neumann algebra to $\mathcal{M}$.

Let ρ, σ be normal states on $\mathcal{M}$. The measured Rényi relative entropy is defined as

$$D_\alpha^M(\rho\|\sigma) = \sup\{D_\alpha(\rho \circ M\|\sigma \circ M),\ M \text{ is a measurement}\}.$$

By [**?** , Thm. 5.2], we may restrict to finite valued measurements.

It is easily seen that D_α^M is monotone under positive unital normal maps. Indeed, let $\Phi : \mathcal{N} \to \mathcal{M}$ be such a map and let $M : L_\infty(\Omega) \to \mathcal{N}$ be a measurement on $\mathcal{N}$, then $\Phi \circ M$ is a measurement on $\mathcal{M}$ and we have

$$\begin{aligned} D_\alpha^M(\rho\circ\Phi\|\sigma\circ\Phi) &= \sup\{D_\alpha(\rho\circ\Phi\circ M\|\sigma\circ\Phi\circ M),\ M \text{ is a measurement on } \mathcal{N}\} \\ &\le \sup\{D_\alpha(\rho\circ M\|\sigma\circ M),\ M \text{ is a measurement on } \mathcal{M}\} \\ &= D_\alpha^M(\rho\|\sigma). \end{aligned}$$

Further, by monotonicity of the sandwiched Rényi relative entropy $\tilde{D}_\alpha$ for $\alpha \in [1/2, 1) \cup (1, \infty]$, and by additivity with respect to tensor products, we have

$$\frac{1}{n} D_\alpha^M(\rho^{\otimes n}\|\sigma^{\otimes n}) \le \frac{1}{n}\tilde{D}_\alpha(\rho^{\otimes n}\|\sigma^{\otimes n}) = \tilde{D}_\alpha(\rho\|\sigma).$$

We are interested in the equality

$$\lim_n \frac{1}{n} D_\alpha^M(\rho^{\otimes n}\|\sigma^{\otimes n}) = \tilde{D}_\alpha(\rho\|\sigma), \qquad \alpha > 1. \tag{3}$$

This equality was proved in [**?**] in the finite dimensional case, in [**?**] in the approximately finite (injective) case and in [**?**] it was extended to semifinite von Neumann algebras. We will use the Haagerup reduction to show that this equality holds for all von Neumann algebras.

2.1 Haagerup reduction

We just give an outline:

- There exists a von Neumann algebra $\hat{\mathcal{M}}$ (a crossed product) such that we may identify $\mathcal{M}$ as a subalgebra in $\hat{\mathcal{M}}$ and there is a canonical normal conditional expectation $E_{\mathcal{M}}$ of $\hat{\mathcal{M}}$ onto $\mathcal{M}$,
- There is an increasing family of subalgebras $\mathcal{M}_n \subseteq \hat{\mathcal{M}}$ such that
  - Each $\mathcal{M}_n$ is finite (equipped with a faithful normal tracial state τ_n),
  - $\bigcup_{n\geq 1} \mathcal{M}_n$ is weak*-dense in $\hat{\mathcal{M}}$
  - For each n, there exists a faithful normal conditional expectation E_n of $\hat{\mathcal{M}}$ onto $\mathcal{M}_n$ and for all $x \in \hat{\mathcal{M}}$, $E_n(x) \to x$ in the σ-strong topology.
- Any normal state ρ on $\mathcal{M}$ extends to a normal state on $\hat{\mathcal{M}}$ as $\hat{\rho} := \rho \circ E_{\mathcal{M}}$. Put $\rho_n := \hat{\rho}|_{\mathcal{M}_n}$, then ρ_n is a normal state on $\mathcal{M}_n$. Using the conditional expectation E_n, ρ_n extends to a state $\hat{\rho}_n = \rho_n \circ E_n = \hat{\rho} \circ E_n$. We have $\|\hat{\rho}_n - \hat{\rho}\|_1 \to 0$.

We will use this result to prove (**??**) for any von Neumann algebra $\mathcal{M}$. First, note that since $\rho = \hat{\rho}|_{\mathcal{M}}$ and $\hat{\rho} = \rho \circ E_{\mathcal{M}}$, we have by DPI that

$$\tilde{D}_\alpha(\rho\|\sigma) = \tilde{D}_\alpha(\hat{\rho}\|\hat{\sigma}), \qquad D_\alpha^M(\rho^{\otimes n}\|\sigma^{\otimes n}) = D_\alpha^M(\hat{\rho}^{\otimes n}\|\hat{\sigma}^{\otimes n}).$$

Hence it is enough to work with $\hat{\rho}$ and $\hat{\sigma}$. Notice that by th DPI for D_α^M, we have for any m and $\alpha > 1$

$$\lim_n \frac{1}{n} D_\alpha^M(\hat{\rho}^{\otimes n}\|\sigma^{\otimes n}) \geq \lim_n \frac{1}{n} D_\alpha^M(\hat{\rho}_m^{\otimes n}\|\hat{\sigma}_m^{\otimes n}) = \tilde{D}_\alpha(\hat{\rho}_m\|\hat{\sigma}_m),$$

since $\mathcal{M}_m$ is finite. We therefore have

$$\lim_n \frac{1}{n} D_\alpha^M(\hat{\rho}^{\otimes n}\|\hat{\sigma}^{\otimes n}) \geq \lim_m \tilde{D}_\alpha(\hat{\rho}_m\|\hat{\sigma}_m) = \tilde{D}_\alpha(\hat{\rho}\|\hat{\sigma}).$$

The last equality holds by DPI and lower semicontinuity of $\tilde{D}_\alpha$. Since the opposite inequality always holds, this proves (**??**).

2.2 Strong converse exponents