AIP | Journal of Mathematical Physics

# Geodesic distances on density matrices

Anna Jenčová

Citation: Journal of Mathematical Physics **45**, 1787 (2004); doi: 10.1063/1.1689000
View online: http://dx.doi.org/10.1063/1.1689000
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/45/5?ver=pdfcov
Published by the AIP Publishing

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 45, NUMBER 5 MAY 2004

# Geodesic distances on density matrices

Anna Jenčová[a]
*Mathematical Institute, Slovak Academy of Sciences,*
*Štefánikova 49 SK-814 73 Bratislava, Slovakia*

(Received 11 December 2003; accepted 17 January 2004; published 1 April 2004)

We find an upper bound for geodesic distances associated to monotone Riemannian metrics on positive definite matrices and density matrices. © *2004 American Institute of Physics.* [DOI: 10.1063/1.1689000]

## I. INTRODUCTION

The notion and importance of Fisher information is well established in statistics and probability theory. As a measure of distinguishability of probability densities, the Fisher information was used by Rao to define a Riemannian metric on probability spaces. On the simplex of probability vectors $\mathcal{P}_n=\{p=(p_1,\ldots,p_n),\ \Sigma_i p_i=1, p_i>0, i=1,\ldots,n\}$, this is the unique metric contracting under Markovian mappings, by the Chentsov uniqueness theorem. On $\mathcal{P}_n$, the Fisher metric is

$$\lambda_p(x,y)=\sum_i p_i^{-1}x_i y_i, \quad x,y\in T_p\mathcal{P}_n.$$

The geometry of $\mathcal{P}_n$ with this metric is quite simple. By

$$p\mapsto 2(\sqrt{p_1},\ldots,\sqrt{p_n}), \tag{1}$$

it is isometric with an open subset in the sphere of radius 2 in $\mathbb{R}^n$.[5] The metric can be extended to the set $\mathcal{M}_n=\{p=(p_1,\ldots,p_n),\ p_i>0\}$ of all finite (strictly positive) measures on the set $\{1,\ldots,n\}$. Using the isometry (1) and elementary geometry in $R^n$, we may compute the geodesic distance for the Fisher metric in $\mathcal{P}_n$ and $\mathcal{M}_n$:

$$D(p,q)=2\arccos\left(\sum_i \sqrt{p_i}\sqrt{q_i}\right), \quad p,q\in\mathcal{P}_n$$

(the Bhattacharya distance) and

$$d(p,q)=2\left(\sum_i (\sqrt{p_i}-\sqrt{q_i})^2\right)^{1/2}, \quad p,q\in\mathcal{M}_n.$$

The last expression is related to the Hellinger distance $H(p,q)$ by $d(p,q)=\sqrt{2H(p,q)}$. The Hellinger distance belongs to the family of Cziszár's $f$-divergences

$$D_f(p,q)=\int f(q/p)\,dp.$$

Here $f$ is a convex function. As it was shown in Ref. 1, the metric given by the Hessian of $f$-divergence is a constant multiple of the Fisher metric.

In the case of a quantum system, the situation becomes more complicated. In the simplest case, the states of the system are represented by density matrices. In analogy with manifolds of classical probability densities, a quantum version of the Fisher information metric must be de-

[a]Electronic mail: jenca@mat.savba.sk

creasing under stochastic maps. Contrary to the classical case, this monotonicity condition does not specify the metric uniquely. In fact, it was shown by Petz that the monotone metrics can be labeled by operator-monotone functions.

As it was mentioned in Ref. 5, there is no general formula for geodesic path and distance for monotone metrics. Explicit expressions are known only in two particular cases, namely the Bures metric and the Wigner–Yanase metric. In the present paper, we find an upper bound for the geodesic distances for all monotone metrics. This is done in a simple way: Following Uhlmann,[18,19] we obtain the Bures geodesics from certain purifying lifts of curves of density matrices and then make use of a duality relation between the smallest (Bures) and the largest (RLD) of monotone metrics. It is also shown that this upper bound is related to a particular noncommutative version of the Hellinger distance.

## II. THE MANIFOLD AND MONOTONE METRICS

Let $M_n$ be the algebra of $n$ by $n$ complex matrices. The set of faithful positive linear functionals on $M_n$ is identified with the cone of positive definite matrices. This set, with the differentiable manifold structure inherited from $M_n$, will be denoted by $\mathcal{M}$. Let $\mathcal{D}\subset\mathcal{M}$ denote the submanifold of density matrices in $\mathcal{M}$, that is,

$$\mathcal{D}=\{\rho\in\mathcal{M}:\mathrm{Tr}\,\rho=1\}.$$

The tangent space to $\mathcal{M}$ at $\rho\in\mathcal{M}$ is $T_\rho\mathcal{M}=\{x\in M_n:x=x^*\}$. If $\rho\in\mathcal{D}$, then the tangent space $T_\rho\mathcal{D}$ is the subspace of traceless matrices in $T_\rho\mathcal{M}$.

Let $\lambda$ be a Riemannian metric on $\mathcal{M}$. Then we will say that $\lambda$ is a monotone metric if

$$\lambda_{T(\rho)}(T(h),T(h))\leqslant\lambda_\rho(h,h),\quad \rho\in\mathcal{M},\ \ h\in T_\rho\mathcal{M},$$

for all completely positive trace preserving maps $T$. It is an important result of Petz[16] that a Riemannian metric is monotone if and only if it has the form

$$\lambda_\rho(h,k)=\mathrm{Tr}\,hJ_\rho(k),$$

where $J_\rho$ is given by the operator mean

$$J_\rho=R_\rho^{-1}[f(L_\rho/R_\rho)]^{-1}. \tag{2}$$

Here $L_\rho$ and $R_\rho$ are the left and the right multiplication operators and $f:(0,\infty)\to\mathbb{R}$ is an operator monotone function which is symmetric, that is, $f(t)=tf(t^{-1})$. It is immediate from (2) that under the normalization $f(1)=1$, any monotone metric is equal to the Fisher metric on commutative submanifolds. Moreover, we have

$$\frac{2t}{1+t}\leqslant f(t)\leqslant\frac{1+t}{2}$$

for all symmetric normalized operator monotone functions.[12] Accordingly, there is a greatest and a smallest element in the set of monotone metrics.

The smallest monotone metric is obtained for $f(t)=(1+t)/2$. It is called the Bures metric because it is related to the Bures distance (see also Sec. IV). The operator

$$J_\rho(h)=g,\quad \rho g+g\rho=2h,$$

is the symmetric logarithmic derivative (see Refs. 9, 18, and 2).

The greatest monotone metric corresponds to the function $f(t)=2t/(1+t)$. In this case $J_\rho$ is the right logarithmic derivative (RLD)

$$J_\rho(h) = \tfrac{1}{2}(\rho^{-1}h + h\rho^{-1})$$

(see Refs. 9, 16, and 17). More examples of monotone metrics can be found in Sec. V.

## III. STANDARD REPRESENTATION AND MONOTONE METRICS

The standard representation of the algebra $M_n$ is obtained if $M_n$ is endowed with the Hilbert–Schmidt inner product

$$\langle x,y\rangle = \mathrm{Tr}\, x^* y.$$

Let us denote the resulting Hilbert space by $H$. Then $M_n$ is represented on $H$ by

$$\phi: M_n \to \mathcal{B}(H), \quad a \mapsto L_a,$$

where $L_a$ is the left multiplication operator $L_a w = aw$, $w \in H$. Each element $\rho$ in $\mathcal{M}$ has a vector representative, or purification, $w$ in $H$, such that

$$\mathrm{Tr}\,\rho a = \langle w, L_a w\rangle \quad \forall a \in M_n.$$

Then $w \in H$ is a vector representative of $\rho \in \mathcal{M}$ if and only if $\rho = ww^*$.

Let $\rho_t$, $t \in I$, be a smooth curve in $\mathcal{M}$. A curve $w_t$ in $H$, such that $w_t$ is a vector representative of $\rho_t$ for all $t \in I$, is called a lift of $\rho_t$. In this case, the tangent vectors are related by

$$\dot\rho_t = \dot w_t w_t^* + w_t \dot w_t^*. \tag{3}$$

Let us denote the corresponding projection of the tangent spaces $T_w H \to T_{ww^*}\mathcal{M}$ by $\Pi$.

Let $w_0 w_0^* = \rho_0$. There are many lifts of $\rho_t$ through $w_0$. Among such lifts, there is a unique lift with minimal Hilbert space length

$$l_H(w_t) = \int_I \sqrt{\langle \dot w_t, \dot w_t\rangle}\, dt.$$

It will be called the horizontal lift.

The horizontal lift was introduced in Refs. 18 and 19, where the geometric phase was extended to mixed states. It was shown that the above minimalization problem leads to the condition

$$w_t^* \dot w_t = \dot w_t^* w_t \tag{4}$$

for all $t$. The curves $w_t$ in $H$, satisfying this condition, are called horizontal curves. The tangent vectors to horizontal curves at $w \in H$ form a real vector subspace $H_w = \{gw,\ g = g^*\}$. Let $H_w$ be endowed with the inner product $\mathrm{Re}\langle\cdot,\cdot\rangle$. Then it is a real Hilbert space, called the horizontal subspace. For each $h \in T_{ww^*}\mathcal{M}$, there is a unique element $\hat h$ in $H_w$ satisfying $h = \Pi(\hat h)$. It follows that the inner product in $H_w$ can be projected onto $T_{ww^*}\mathcal{M}$. As it turns out, this projection defines a Riemannian metric on $\mathcal{M}$. Moreover,

$$4\,\mathrm{Re}\langle \hat h, \hat k\rangle = 2\,\mathrm{Tr}\, h(L_\rho + R_\rho)^{-1}(k), \quad h,k \in T_\rho\mathcal{M}, \tag{5}$$

is exactly the Bures metric.

The commutant of $\phi(M_n)$ is the algebra of right multiplication operators $R_a w = wa$, $a \in M_n$, on $H$. For each $\sigma \in \mathcal{M}$, there is an element $w \in H$ such that

$$\langle w, R_a w\rangle = \mathrm{Tr}\,\sigma a.$$

This element is given by $\sigma = w^* w$. For each curve $\sigma_t$ in $\mathcal{M}$, let us consider the curves $w_t$ in $H$ satisfying $w_t^* w_t = \sigma_t$. The tangent vectors of such curves satisfy $\dot{\sigma}_t = \tilde{\Pi}(\dot{w}_t)$, where $\tilde{\Pi}: T_w H \to T_{w^*w}\mathcal{M}$ is given by

$$\tilde{\Pi}(x) = x^* w + w^* x.$$

We may now proceed exactly as before, choosing for each $\sigma_t$ the shortest of these curves. It is quite clear that $w_t$ is the shortest curve if and only if $w_t^*$ is horizontal; equivalently, $\dot{w}_t \in \tilde{H}_{w_t} := \{w_t g,\ g = g^*\}$ for all $t$. Moreover, we have

$$x \in H_w \Leftrightarrow x^* \in \tilde{H}_{w^*}. \tag{6}$$

If we now project the real Hilbert space structure from $\tilde{H}_w$ to $T_{w^*w}\mathcal{M}$, using the projection $\tilde{\Pi}$, we will, of course, get the Bures metric again. On the other hand, it is easy to see that for each $\rho = ww^*$ and $h \in T_\rho \mathcal{M}$, $\tilde{h} := \frac{1}{2} h (w^*)^{-1}$ is the unique element in $\tilde{H}_w$ satisfying $h = \Pi(\tilde{h})$. We may therefore define

$$\lambda_\rho(h,k) := 4\,\mathrm{Re}\langle \tilde{h}, \tilde{k}\rangle = \tfrac{1}{2}\mathrm{Tr}\,\rho^{-1}(hk + kh), \tag{7}$$

which is the RLD metric. This shows that there is a duality relation between the Bures metric and RLD (see also Refs. 14 and 10).

## IV. THE GEODESIC DISTANCES

Let $\lambda$ be a Riemannian metric on $\mathcal{M}$. A curve $\rho_t$, $t \in [0,1]$, is a geodesic path in $\mathcal{M}$ if its length

$$l_\lambda(\rho_t) = \int_0^1 \sqrt{\lambda_{\rho_t}(\dot{\rho}_t, \dot{\rho}_t)}\, dt$$

is the minimum of lengths of all curves connecting $\rho_0$ and $\rho_1$. This length is then the geodesic distance of $\rho_0$ and $\rho_1$. Let us denote by $d_\lambda$ the geodesic distance for the metric $\lambda$ in $\mathcal{M}$ and by $D_\lambda$ the geodesic distance in $\mathcal{D}$.

For the Bures metric, the geodesic paths and distances were obtained by Uhlmann[18,19] as follows. Let $\rho_0$ and $\rho_1$ be two elements in $\mathcal{M}$ and let $\rho_t$ be a curve connecting them. If $w_t$ is the horizontal lift of $\rho_t$, then by (5)

$$l_{\mathrm{Bures}}(\rho_t) = 2 l_H(w_t),$$

hence minimizing the Bures length means minimizing the Hilbert space length of horizontal lifts of curves connecting $\rho_0$ and $\rho_1$. From the definition of horizontality, this minimum is attained at the line segment $w_t = t w_1 + (1-t) w_0$, such that $\|w_0 - w_1\|$ is minimal over $w_0 w_0^* = \rho_0$, $w_1 w_1^* = \rho_1$. This happens if and only if $w_1$ and $w_0$ are parallel amplitudes, that is, these satisfy Uhlmann's parallelity condition

$$w_1^* w_0 \geqslant 0. \tag{8}$$

For each $w_0$ there is a unique $w_1$ parallel to $w_0$, given by[19]

$$w_1 = \rho_0^{-1/2} (\rho_0^{1/2} \rho_1 \rho_0^{1/2})^{1/2} \rho_0^{-1/2} w_0.$$

The geodesic path in $\mathcal{M}$, connecting $\rho_0$ and $\rho_1$, is then

$$\rho_t = (t w_1 + (1-t) w_0)(t w_1 + (1-t) w_0)^*,$$

and the geodesic distance is

$$d_{\mathrm{Bures}}(\rho_0,\rho_1)=2\|w_0-w_1\|=2\sqrt{\mathrm{Tr}\,\rho_0+\mathrm{Tr}\,\rho_1-2\,\mathrm{Tr}\,(\rho_0^{1/2}\rho_1\rho_0^{1/2})^{1/2}}.$$

This is called the Bures distance.

Let $\rho_t$ now be a curve in $\mathcal{D}$. Then all lifts of $\rho_t$ are curves on the unit sphere $S$ in $H$. If $w_0$, $w_1\in S$, the shortest curve connecting them lies on the large circle in $S$ through them. The length of such arcs for $w_0w_0^*=\rho_0$ and $w_1w_1^*=\rho_1$ is minimal if $w_0$ and $w_1$ are parallel amplitudes and, by definition, in this case the arc is also horizontal. Hence, the Bures geodesic in $\mathcal{D}$ is

$$\rho_t=\frac{(w_0+(1-t)w_1)(tw_0+(1-t)w_1)^*}{\|tw_0+(1-t)w_1)\|^2}$$

for parallel amplitudes $w_0$ and $w_1$ and the Bures distance

$$D_{\mathrm{Bures}}(\rho_0,\rho_1)=2\ \arccos\,\mathrm{Tr}\,w_0w_1^*=2\ \arccos\,\mathrm{Tr}\,(\rho_0^{1/2}\rho_1\rho_0^{1/2})^{1/2}.$$

The duality of the Bures and RLD metrics leads to the following upper bound for the RLD geodesic distance.

*Proposition 4.1:* Let $\rho_0$, $\rho_1\in\mathcal{M}$. *Then*

$$d_{\mathrm{RLD}}(\rho_0,\rho_1)\leqslant d_{\mathrm{Bures}}(\rho_0,\rho_0^{-1/2}(\rho_0\#\rho_1)^2\rho_0^{-1/2}),$$

*where*

$$\rho_0\#\rho_1=\rho_0^{1/2}(\rho_0^{-1/2}\rho_1\rho_0^{-1/2})^{1/2}\rho_0^{1/2}$$

*is the geometric mean. If $\rho_0$ and $\rho_1$ are in $\mathcal{D}$, the same holds for geodesic distances $D_{\mathrm{RLD}}$ and $D_{\mathrm{Bures}}$.*

*Proof:* Let $w_0=\rho_0^{1/2}$ and let $w\in H$ be such that $w_0$ and $w$ satisfy the parallelity condition (8). Then the curve $w_t=tw+(1-t)w_0$ is the horizontal lift of the Bures geodesic connecting $\rho_0$ and $ww^*$, in particular, $\dot{w}_t\in H_{w_t}$ for all $t$. Then $w_t^*$ is a lift of a curve $\rho_t$ in $\mathcal{M}$, connecting $\rho_0$ and $w^*w$ and by (6), $\dot{w}_t^*\in\tilde{H}_{w_t^*}$. Consequently, by (7),

$$d_{\mathrm{RLD}}(\rho_0,w^*w)\leqslant l_{\mathrm{RLD}}(\rho_t)=2\|w^*-w_0^*\|=2\|w-w_0\|=d_{\mathrm{Bures}}(\rho_0,ww^*).$$

From the parallelity condition, $w=qw_0$ for some $q=q^*>0$. Let us choose $w$ such that

$$\rho_1=w^*w=\rho_0^{1/2}q^2\rho_0^{1/2}.$$

Then $q=(\rho_0^{-1/2}\rho_1\rho_0^{-1/2})^{1/2}$ and

$$ww^*=\rho_0^{-1/2}(\rho_0\#\rho_1)^2\rho_0^{-1/2}.$$

The statement for distances in $\mathcal{D}$ is proved exactly the same way. □

*Remark 4.1:* Let $w_0$, $w$ and $q$ be as in the proof of the previous proposition. Then we have

$$\|w_0-w\|^2=\mathrm{Tr}\,\rho_0+\mathrm{Tr}\,\rho_1-2\,\mathrm{Tr}\,\rho_0q$$

and

$$d_{\mathrm{Bures}}(\rho_0,\rho_0^{-1/2}(\rho_1\#\rho_0)^2\rho_0^{-1/2})=2\sqrt{\mathrm{Tr}\,\rho_0+\mathrm{Tr}\,\rho_1-2\,\mathrm{Tr}\,\rho_0\#\rho_1}\tag{9}$$

so that $\rho_0$ and $\rho_1$ can be exchanged.

*Remark 4.2:* Let $\rho_t = w_t^* w_t$ and $q$ be as in the proof of Proposition 4.1. Then, in general, $\rho_t$ is not the RLD geodesic. Indeed, it can be easily computed that for the RLD metric, the geodesic equation reads

$$\ddot{\rho}_t + \frac{1}{L_{\rho_t} + R_{\rho_t}}(\dot{\rho}_t^2) - \dot{\rho}_t \rho_t^{-1} \dot{\rho}_t = a(t)\dot{\rho}_t,$$

where $a$ is a smooth function $a: I \to \mathbb{R}$ (see also Ref. 3). We have

$$\rho_t = w_t^* w_t = \rho_0^{1/2}(1 + t(q-1))^2 \rho_0^{1/2}.$$

It can be shown by direct computation that the geodesic equation is satisfied if and only if

$$q(\rho_0 q - q\rho_0) = (\rho_0 q - q\rho_0) q,$$

which, for self-adjoint operators, implies $q\rho_0 = \rho_0 q$. It follows that the inequality in Proposition 4.1 is strict, unless $\rho_0$ and $\rho_1$ commute. In that case, the geodesic distances are the same for all monotone metrics.

In Ref. 17, a class of generalized relative entropies

$$H_g(\rho_0, \rho_1) = \mathrm{Tr}\, \rho_0 g(\rho_0^{-1/2} \rho_1 \rho_0^{-1/2})$$

was introduced; here $g$ is an operator convex function. This is a noncommutative version of the $f$-divergence. It was shown in Ref. 17 that the generalized entropy $H_g$ leads to a constant multiple of the RLD metric for infinitesimaly close elements in $\mathcal{D}$.

It is easy to see that the right hand side of (9) is equal to $\sqrt{2H_{g_0}(\rho_0, \rho_1)}$, where

$$g_0(t) = 2 + 2t - 4t^{1/2}. \tag{10}$$

Note that on commuting elements, $H_{g_0}$ is equal to the Hellinger distance.

By maximality of the RLD metric, we obtain the following.

*Corollary 4.1: Let $\rho_0, \rho_1 \in \mathcal{M}$ and let $\lambda$ be a monotone metric. Then*

$$d_{\mathrm{Bures}}(\rho_0, \rho_1) \leqslant d_\lambda(\rho_0, \rho_1) \leqslant \sqrt{2H_{g_0}(\rho_0, \rho_1)} < 2\sqrt{\mathrm{Tr}\, \rho_0 + \mathrm{Tr}\, \rho_1}.$$

*If $\rho_0$, $\rho_1 \in \mathcal{D}$, then*

$$2 \arccos \mathrm{Tr}(\rho_0^{1/2} \rho_1 \rho_0^{1/2})^{1/2} \leqslant D_\lambda(\rho_0, \rho_1) \leqslant 2 \arccos \mathrm{Tr}\, \rho_0 \# \rho_1 < \pi.$$

## V. THE WYD METRICS

The Wigner–Yanase–Dyson (WYD) metrics are defined by

$$\lambda_\rho^\alpha(h,k) = \frac{\partial^2}{\partial t \partial s} \mathrm{Tr}\, f_\alpha(\rho + th) f_{-\alpha}(\rho + sk)\big|_{s,t=0},$$

where

$$f_\alpha(x) = \begin{cases} \dfrac{2}{1-\alpha} x^{(1-\alpha)/2}, & \alpha \neq 1, \\ \log(x), & \alpha = 1. \end{cases}$$

As it was shown in Ref. 8, these metrics are monotone for $\alpha \in [-3,3]$. The family of WYD metrics is important in quantum information geometry (see Refs. 7, 11, and 6). As special cases, for $\alpha=\pm 1$, we get the well known Bogoljubov–Kubo–Mori metric and for $\alpha=\pm 3$ we get the RLD metric.

The smallest in this family is the Wigner–Yanase (WY) metric, obtained for $\alpha=0$. The WY metric has the form

$$\lambda_\rho(h,k)=4\,\mathrm{Tr}\,h(\sqrt{L_\rho}+\sqrt{R_\rho})^{-2}(k).$$

The corresponding geodesic path and distance was computed in Ref. 5, using a noncommutative version of the square root map (1) and a pullback technique. We will show that these can also be easily obtained using a similar method as in the Bures case.

Let $\rho_t$ be a curve in $\mathcal{M}$. Among its lifts $w_t w_t^*=\rho_t$, we will again choose a horizontal one. In this case, the lift $w_t$ is horizontal if it is contained in the natural positive cone at $w_0$, that is, if $w_t=\rho_t^{1/2}u_0$ for all $t$. In this case, the horizontal subspace is $H_w^0=\{gu,\ g=g^*\}$, where $w=\rho^{1/2}u$ is the polar decomposition of $w$. Each tangent vector $h \in T_{ww^*}\mathcal{M}$ has a unique horizontal lift $h^0=gu \in H_w^0$, such that $h=\Pi(h^0)=g\rho^{1/2}+\rho^{1/2}g$. The induced metric

$$\lambda_\rho(h,k)=4\,\mathrm{Re}\langle h^0,k^0\rangle=4\,\mathrm{Tr}\,h(L_\rho^{1/2}+R_\rho^{1/2})^{-2}(k)$$

is the WY metric. Note that in this case $x \in H_w^0$ if and only if $x^* \in H_{w^*}^0$, so that the WY metric is self-dual, in the sense mentioned in Sec. III. Let us also remark that it is possible to obtain all the monotone metrics in a similar manner (see Refs. 4 and 10).

Now let $\rho_0$ and $\rho_1$ be in $\mathcal{M}$ and let $\rho_t$ be a curve connecting them. Again, the WY length of $\rho_t$ is twice the Hilbert space length of its horizontal lift $w_t=\rho_t^{1/2}u_0$. Therefore, $\rho_t$ is the geodesic path if $w_t=t\rho_1^{1/2}u_0+(1-t)\rho_0^{1/2}u_0$, that is,

$$\rho_t=(t\rho_1^{1/2}+(1-t)\rho_0^{1/2})^2$$

and the geodesic distance is

$$d_{\mathrm{WY}}(\rho_0,\rho_1)=2\|\rho_0^{1/2}-\rho_1^{1/2}\|=2\sqrt{\mathrm{Tr}\,\rho_0+\mathrm{Tr}\,\rho_1-2\,\mathrm{Tr}\,\rho_0^{1/2}\rho_1^{1/2}}.$$

Similarly, if $\rho_0,\ \rho_1 \in \mathcal{D}$, then $\rho_t$ is a geodesic path if and only if $w_t$ lies on the large circle connecting $\rho_0^{1/2}u_0$ and $\rho_1^{1/2}u_0$. Hence

$$\rho_t=\frac{(t\rho_1^{1/2}+(1-t)\rho_0^{1/2})^2}{\|t\rho_1^{1/2}+(1-t)\rho_0^{1/2}\|^2}$$

and

$$D_{\mathrm{WY}}(\rho_0,\rho_1)=2\arccos\mathrm{Tr}\,\rho_0^{1/2}\rho_1^{1/2}.$$

Let us denote by $\Delta_{\sigma,\rho}=L_\sigma R_\rho^{-1}$ the relative modular operator. In Ref. 15, a class of quasi-entropies was introduced by

$$S_g(\rho,\sigma)=\mathrm{Tr}\,\rho^{1/2}g(\Delta_{\sigma,\rho})(\rho^{1/2}),$$

where $g$ is an operator convex function. This is another quantum version of the $f$-divergences. It is easy to see that

$$d_{WY}(\rho_0,\rho_1)=\sqrt{2S_{g_0}(\rho_0,\rho_1)}, \tag{11}$$

where $g_0$ is given by (10). It was proved in Ref. 13 that each monotone metric can be obtained as the Hessian of $S_g$ for a suitable operator convex function $g$. The choice $g = g_0$ leads to the WY metric.

From the previous section and the fact that the WY metric is the least element in the family of WYD metrics, we obtain the following.

*Corollary 5.1: Let* $\lambda$ *be a WYD metric and* $\rho_0, \rho_1 \in \mathcal{M}$*. Then*

$$\sqrt{2S_{g_0}(\rho_0,\rho_1)} = d_{\mathrm{WY}}(\rho_0,\rho_1) \leqslant d_\lambda(\rho_0,\rho_1) \leqslant \sqrt{2H_{g_0}(\rho_0,\rho_1)},$$

*where* $g_0(t) = 2 + 2t - 4t^{1/2}$. If $\rho_0, \rho_1 \in \mathcal{D}$, *then*

$$2 \arccos \mathrm{Tr}\, \rho_0^{1/2}\rho_1^{1/2} \leqslant D_\lambda(\rho_0,\rho_1) \leqslant 2 \arccos \mathrm{Tr}\, \rho_0 \# \rho_1 .$$

## ACKNOWLEDGEMENT

The research was supported by the grant VEGA 1/0264/03.


[1] Amari, S., *Differential-Geometrical Methods in Statistics*, Lecture Notes in Statistics 28 (Springer, Berlin, 1985).
[2] Braunstein, S. L., and Caves, C. M., in *Quantum Communication, Computing and Measurement*, edited by V. P. Belavkin, O. Hirota, and R. L. Hudson (Plenum, Press, New York, 1995).
[3] Dittmann, J., ''On the curvature of monotone metrics and a conjecture concerning the Kubo-Mori metric,'' Linear Algebr. Appl. **315**, 83–112 (2000).
[4] Dittmann, J., and Uhlmann, A., ''Connections and metrics respecting standard purification,'' J. Math. Phys. **40**, 3246–3267 (1999).
[5] Gibilisco, P., and Isola, T., ''Wigner-Yanase information on quantum state space: The geometric approach,'' J. Math. Phys. **44**, 3752–3762 (2003).
[6] Grasselli, M. R., ''Monotonicity, Duality and Uniqueness of the WYD Metrics,'' submitted to IDAQP, math-ph/0212022.
[7] Hasegawa, H., ''Dual geometry of Wigner-Yanase-Dyson information contents,'' IDAQP **6**, 413–431 (2003).
[8] Hasegawa, H., and Petz, D., ''Non-commutative extension of information geometry II,'' in *Quantum Communication, Computing and Measurement*, edited by O. Hirota, A. S. Holevo, and C. M. Caves (Plenum, New York, 1997).
[9] Holevo, A. S., *Probabilistic and Statistical Aspects of Quantum Theory* (Nauka, Moscow, 1980) (in Russian).
[10] Jenčová, A., ''Quantum information geometry and standard purification,'' J. Math. Phys. **43**, 2187–2201 (2002).
[11] Jenčová, A., ''Flat connections and Wigner-Yanase-Dyson metrics,'' Rep. Math. Phys. **52**, 331–351 (2003).
[12] Kubo, F., and Ando, T., ''Means of positive linear operators,'' Math. Ann. **246**, 205–224 (1980).
[13] Lesniewski, A., and Ruskai, M. B., ''Monotone Riemannian metrics and relative entropy on noncommutative probability spaces,'' J. Math. Phys. **40**, 5702–5724 (1999).
[14] Matsumoto, K., ''Uhlmann's parallelism and Nagaoka's quantum information geometry,'' METR 97–09 (1997).
[15] Petz, D., ''Quasi-entropies for finite quantum systems,'' Rep. Math. Phys. **23**, 57–65 (1986).
[16] Petz, D., ''Monotone metrics on matrix spaces,'' Linear Algebr. Appl. **244**, 81–96 (1996).
[17] Petz, D., and Ruskai, M. B., ''Contraction of generalized relative entropy under stochastic mappings on matrices,'' IDAQP **1**, 83–89 (1998).
[18] Uhlmann, A., ''Density operators as an arena for differential geometry,'' Rep. Math. Phys. **33**, 253–263 (1993).
[19] Uhlmann, A., ''Geometric phases and related structures,'' Rep. Math. Phys. **36**, 461–481 (1995).