Let V be a finite dimensional vector space with the standard Euclidean topology and let $C \subset V$ be a convex, closed, pointed and generating cone. The cone C gives rise to a natural order on V given for $x, y \in V$ as follows: $x \geq y$ if and only if $x - y \in C$.

Let $J \subset V$ be a subspace such that $J \cap \operatorname{int}(C) \neq \emptyset$ where $\operatorname{int}(C)$ denotes the interior of C. Let K be a base of the cone $C \cap J$. In what follows we will be interested in the intervals of the form $[0, K]$ where

$$[0, K] = \{x \in C : \exists y \in K : y \geq x\}. \tag{1}$$

In a non-causal theory the interval $[0, K]$ would represent the set of states of some theory and the set K would represent the set of deterministic states.

Proposition 1. *V, C and J are uniquely given by $[0, K]$ as follows: $V = \operatorname{span}([0, K])$, $C = \operatorname{cone}([0, K])$ and $J = \operatorname{span}(K)$ where $\operatorname{span}([0, K])$ and $\operatorname{span}(K)$ represents the linear hull of $[0, K]$ and K respectively and $\operatorname{cone}([0, K])$ represents the cone generated by $[0, K]$.*

Proof. Since $K \cap \operatorname{int}(C) \neq \emptyset$ follows from $J \cap \operatorname{int}(C) \neq \emptyset$, there is $x \in K \cap \operatorname{int}(C)$, i.e. x in an order unit in C. We know that for every $y \in C$ there is $\lambda \in \mathbb{R}$, $\lambda > 0$ such that $x \geq \lambda y$. It follows that $\lambda y \in [0, K]$ so we have $y \in \operatorname{cone}([0, K])$ as $y = \frac{1}{\lambda}(\lambda y)$. It follows that $V = \operatorname{span}([0, K])$ from the fact that C is generating.

To show that $J = \operatorname{span}(K)$ first note that since K is a base of $J \cap C$ it follows that $\operatorname{cone}(K) = J \cap C$. Let $x \in K \cap \operatorname{int}(C)$ and let $v \in J$ then there is $\lambda \in \mathbb{R}$ such that $\lambda x \geq v$. It follows that there is $y \in C$ such that $\lambda x = v + y$ and from $y = \lambda x - v$ it follows that $y \in J$. It follows that $\lambda x - v = y \in C \cap J$ and $v = \lambda x - y \in \operatorname{span}(K)$. □

This result shows why we can assume the cone C to be generating; if it would not then we could simply take $\operatorname{span}(C)$ instead of V. In a similar fashion if $J \cap \operatorname{int}(C) = \emptyset$ then we can replace C with $\operatorname{cone}([0, K])$.

Definition 1. Let V be a finite-dimensional vector space and let $C \subset V$ be a convex, closed, pointed and generating cone. We will call the interval $[0, K] \subset C$ a generating interval if $\operatorname{cone}([0, K]) = C$, $\operatorname{cone}(K) = \operatorname{span}(K) \cap C$ and K is a base of $\operatorname{cone}(K)$.

Maybe a better name than generating interval could be considered.

Note that similar concept of base section was already introduced in the appendix of [1] where also some of the results on duality were obtained.

Definition 2. We call the generating interval $[0, K]$ causal if $K \neq \{0\}$ and $\operatorname{cone}(K) = \operatorname{cone}([0, K])$, i.e. if K is a base of $\operatorname{cone}([0, K])$.

Definition 3. We call the generating interval $[0, K]$ effect algebra if $K = \{x\}$, i.e. if K contains only a single point. We will use the shorthand $[0, \{x\}] = [0, x]$.

Proposition 2. *Let $C \subset V$ be a convex, close, pointed and generating cone and let $\emptyset \neq B \subset C$, then*

$$\operatorname{cone}(B) \subset \operatorname{span}(B) \cap C. \tag{2}$$

Proof. The result follows from $\operatorname{cone}(B) \subset \operatorname{span}(B)$ and $\operatorname{cone}(B) \subset C$. □

Proposition 3. *Let $C \subset V$ be a convex, close, pointed and generating cone and let $\emptyset \neq B \subset C$ be a compact convex set, such that B is the base of $\operatorname{cone}(B)$, then*

$$(\operatorname{cone}(B) = \operatorname{span}(B) \cap C) \Leftrightarrow (B = \operatorname{aff}(B) \cap C). \tag{3}$$

Proof. Note that since B is a base of $\operatorname{cone}(B)$ then there is a unique $u \in \operatorname{cone}(B)^*$ such that $\langle u, x \rangle = 1$ for all $x \in B$.

Assume that $\operatorname{cone}(B) = \operatorname{span}(B) \cap C$ and denote

$$U_1 = \{v \in \operatorname{span}(B) : \langle u, v \rangle = 1\} \tag{4}$$

then $B = \operatorname{cone}(B) \cap U_1$, $\operatorname{aff}(B) = \operatorname{span}(B) \cap U_1$ and we have

$$B = \operatorname{cone}(B) \cap U_1 = \operatorname{span}(B) \cap C \cap U_1 = \operatorname{aff}(B) \cap C. \tag{5}$$

Now assume that $B = \operatorname{aff}(B) \cap C$ and remember that according to Prop. 2 we have $\operatorname{cone}(B) \subset \operatorname{span}(B) \cap C$. Let $y \in \operatorname{span}(B) \cap C$ and assume that $\langle u, y \rangle = 0$, then we also must have $\langle \varphi, u \rangle = 0$ for all $\varphi \in \operatorname{cone}(B)^*$ and since $\operatorname{cone}(B)^*$ is generating in $\operatorname{span}(B)^*$ we have $y = 0$. It follows that we can assume $\langle u, y \rangle > 0$. Denote $y' = \frac{y}{\langle u, y \rangle}$ then $y \in \operatorname{aff}(B) \cap C = B$ and since $y = \langle u, y \rangle y'$ it follows that $y \in \operatorname{cone}(B)$. □

Proposition 4. *Let $x \in [0, K]$ then $x \in K$ if and only if for every $y \in K$ such that $y \geq x$ we have that $y = x$.*

Proof. If $x \notin K$ then there must be some $y \in K$ such that $y \geq x$ by construction. Assume that $x \in K$ and that there is $y \in K$ such that $y \geq x$ then there is $w \in K$ and $\lambda \in \mathbb{R}^+$ such that $y = x + \lambda w$. Let $\varphi \in \mathrm{cone}(K)^*$ be the unique functional such that $\langle \varphi, z \rangle = 1$ for all $z \in K$. We have

$$1 = \langle \varphi, y \rangle = \langle \varphi, x \rangle + \langle \varphi, \lambda w \rangle = 1 + \langle \varphi, \lambda w \rangle \tag{6}$$

which implies

$$0 = \langle \varphi, \lambda w \rangle = \lambda \tag{7}$$

so we have $x = y$. □

Proposition 5. *Let $B \subset C$ be a convex set and denote*

$$[0, B] = \{x \in C : \exists y \in B, y \geq x\}. \tag{8}$$

We have $B \cap \mathrm{ri}(C) \neq \emptyset$ if and only if $\mathrm{cone}([0, B]) = C$.

Proof. Assume that $\mathrm{cone}([0, B]) = C$, then for every $y \in C$ there is some $z \in [0, B]$ and $\lambda \in \mathbb{R}^+$ such that $y = \lambda z$. It follows that it suffices to prove that there is $x \in B$ such that for some $\mu \in \mathbb{R}^+$ such that $\mu x \geq z$ because then $\lambda \mu x \geq y$. Now let $x \in \mathrm{ri}(B)$, we are going to show that for every $x' \in B$ there is some $\mu' \in \mathbb{R}^+$ such that $\mu' x \geq x'$. This will be sufficient to prove the proposition as by construction for every $w \in [0, B]$ there is some $x' \in B$ such that $w \leq x'$. Since $x \in \mathrm{ri}(B)$ then there is $\alpha < 0$ such that $(1 - \alpha)x + \alpha x' = x'' \in K$. Now we have

$$x = \frac{1}{1 - \alpha} \left(-\alpha x' + x'' \right) \tag{9}$$

and it follows that

$$x \geq \frac{-\alpha}{1 - \alpha} x'. \tag{10}$$

To show the converse, let $x \in B \cap \mathrm{ri}(C)$ and let $y \in C$, then there is $\lambda \in \mathbb{R}^+$ such that $\lambda x \geq y$ and so $\frac{1}{\lambda} y \in [0, B]$. □

Proposition 6. *Let $[0, K] \subset C$ be a generating interval, then $\mathrm{cone}(K)$ has the positive extension property (PEP), i.e. let $\varphi \in \mathrm{cone}(K)^*$ then there is $\varphi' \in C^*$ such that for every $v \in \mathrm{span}(K)$ we have*

$$\langle \varphi', v \rangle = \langle \varphi, v \rangle. \tag{11}$$

Proof. We are going to use the Riesz theorem [2, Theorem 1] which says that $\mathrm{cone}(K)$ has PEP if for every $v \in V$ there is some $x \in \mathrm{cone}(K)$ such that $y \leq x$. According to Prop. 5 we can simply take $x \in K \cap \mathrm{ri}(C)$ and then we already know that for some $\lambda \in \mathbb{R}^+$ we have $v \leq \lambda x$ for the given $v \in V$. □

Example 1 (Classical cone with non-classical set of deterministic states). Let $e_1, e_2, e_3, e_4 \subset \mathbb{R}^4$ be the standard base given as

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad e_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{12}$$

then $S_4 = \mathrm{conv}(\{e_1, e_2, e_3, e_4\})$ is a simplex and we denote $C(S_4) = \mathrm{cone}(S_4)$ which is the positive orthant in $\mathbb{R}^4$. Let

$$s_1 = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \qquad s_2 = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} \qquad s_3 = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \qquad e_4 = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \tag{13}$$

and denote $S = \mathrm{conv}(\{s_1, s_2, s_3, s_4\}) \subset S_4$ and $C(S) = \mathrm{cone}(S)$, it is easy to see that S is a base of $C(S)$. Both sets are depicted in Fig. 1. We will show that $[0, S]$ is a generating interval. Let $b_1, b_2, b_3, b_4 \in C(S_4)^*$ be the base dual to e_1, e_2, e_3, e_4, i.e. functionals such that we have

$$\langle b_j, s_i \rangle = \delta_{ij} \tag{14}$$

where δ_{ij} is the Kronecker delta; $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ otherwise. Let $J \subset \mathbb{R}^4$ be a subspace given as

$$J = \{v \in \mathbb{R}^4 : \langle b_1 + b_2 - (b_3 + b_4), v \rangle = 0\} \tag{15}$$

then $S = S_4 \cap J$ and so $C(S) = C(S_4) \cap J$. It remains to show that $\mathrm{cone}([0, S]) = C(S_4)$, but this follows from $s_1 + s_3 = s_2 + s_4 \in \mathrm{ri}(C(S_4))$ and Prop. 5.

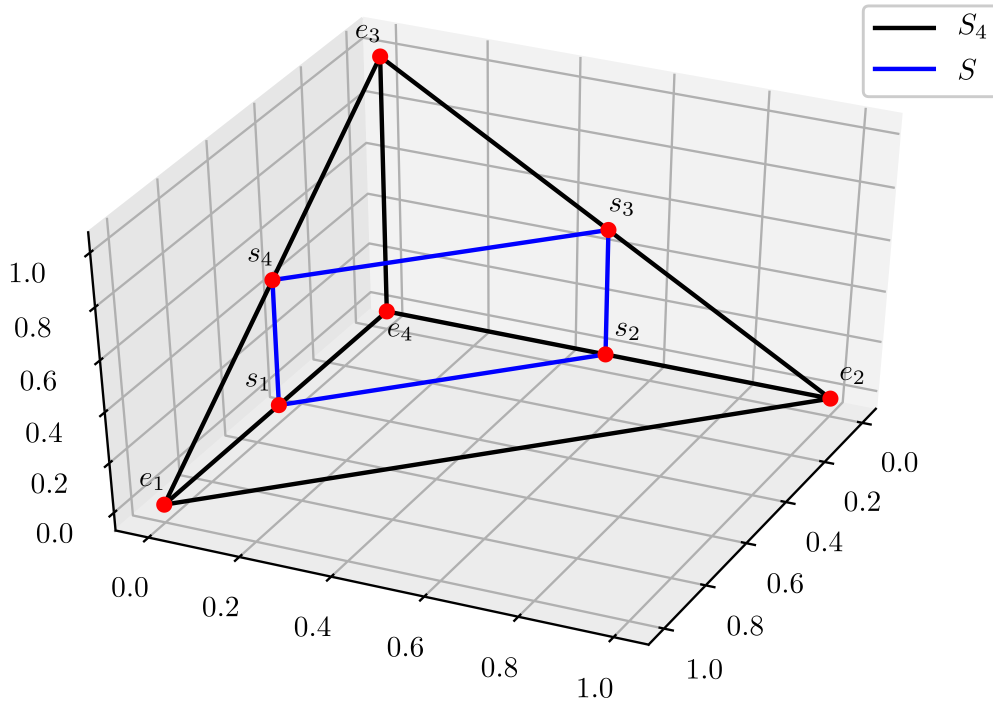

Figure 1. The simplex S_4 and square $S = S_4 \cap J$.

I. DUALITY

We will denote V^* the algebraic dual of V and for $v \in V$ and $\varphi \in V^*$ we will denote $\langle \varphi, v \rangle$ the value of the functional φ on the vector v. We will denote C^* the dual cone to C, i.e.

$$C^* = \{\varphi \in V^* : \langle \varphi, x \rangle \geq 0, \forall x \in C\} \tag{16}$$

Definition 4. Let $[0, K] \subset C$ then

$$U(K) = \{\varphi \in C^* : \langle \varphi, x \rangle = 1, \forall x \in K\}. \tag{17}$$

Proposition 7. *Let $K \neq \{0\}$, then*

$$\dim(\mathrm{span}(U(K)) = \dim(V) - \dim(\mathrm{span}(K)) + 1. \tag{18}$$

Proof. Let $x_1, \ldots, x_k$ be a basis of $\mathrm{span}(K)$ such that $x_i \in K$ for all $i \in \{1, \ldots, k\}$; we know that such basis exists as $K \neq \{0\}$ and so $\dim(\mathrm{span}(K)) \geq 1$. Now let $v_{k+1}, \ldots, v_N \in V$ be vectors such that $x_1, \ldots, x_k, v_{k+1}, \ldots, v_N$ is a basis of V. Let $\psi_1, \ldots \psi_N \in V^*$ be the dual base. Let $\varphi \in \mathrm{span}(U(K))$ then we have

$$\varphi = \sum_{j=1}^{N} \alpha_j \psi_j. \tag{19}$$

We will show that we must have $\alpha_j = \alpha_{j'}$ for $j, j' \in \{1, \ldots, k\}$. Since $\varphi \in \mathrm{span}(U(K))$ there must be $\varphi_i \in U(K)$ and $\beta_i \in \mathbb{R}$, $i \in \{1, \ldots, M\}$ such that

$$\varphi = \sum_{i=1}^{M} \beta_i \varphi_i. \tag{20}$$

Now we have

$$\alpha_j = \langle \varphi, x_j \rangle = \sum_{i=1}^{M} \beta_i \langle \varphi_i, x_j \rangle = \sum_{i=1}^{M} \beta_i \langle \varphi_i, x_{j'} \rangle = \langle \varphi, x_{j'} \rangle = \alpha_{j'}. \tag{21}$$

Let us denote

$$\Psi = \sum_{j=1}^{k} \psi_k \tag{22}$$

then for any $\varphi \in \mathrm{span}(U(K))$ we have

$$\varphi = \alpha \sum_{j=1}^{k} \psi_j + \sum_{j=k+1}^{N} \alpha_j \psi_j = \sum_{j=k+1}^{N} \alpha_j (\Psi + \psi_j) + (\alpha - \sum_{j=k+1}^{N} \alpha_j)\Psi. \tag{23}$$

It follows that if the vectors $\Psi, \Psi + \psi_{k+1}, \dots, \Psi + \psi_N$ are linearly independent, then they form a basis of $\mathrm{span}(U(K))$. So assume there are numbers $\gamma \in \mathbb{R}$ and $\gamma_i \in \mathbb{R}$ for $i \in \{k+1, \dots, N\}$ such that

$$\gamma\Psi + \sum_{i=k+1}^{N} \gamma_i (\Psi + \psi_i) = 0. \tag{24}$$

It follows that we have

$$(\gamma + \sum_{i=k+1}^{N} \gamma_i)\Psi + \sum_{i=k+1}^{N} \gamma_i \psi_i = 0 \tag{25}$$

from where it follows that $\gamma_i = 0$ for all $i \in \{k+1, \dots, N\}$ and $\gamma = 0$.

To finish the proof we have that

$$\dim(\mathrm{span}(U(K)) = 1 + N - k = \dim(V) - \dim(\mathrm{span}(K)) + 1. \tag{26}$$

□

Corollary 1. *Let $K \neq \{0\}$, then $U(K) \neq \emptyset$.*

Proof. If $U(K) = \emptyset$ then $\dim(\mathrm{span}(U(K)) = 0$ and as a result of Prop. 7 we must have $\dim(\mathrm{span}(K)) = \dim(V) + 1$ which is a contradiction. □

Proposition 8. *Let $K \neq \{0\}$, then $U(U(K)) = K$.*

Proof. We will use the shorthand

$$U^2(K) = U(U(K)). \tag{27}$$

We clearly have $K \subset U^2(K)$. According to Prop. 7 we have

$$\begin{aligned} \dim(\mathrm{span}(U^2(K)) &= \dim(V) - \dim(\mathrm{span}(U(K))) + 1 && (28) \\ &= \dim(V) + 1 - (\dim(V) - \dim(\mathrm{span}(K)) + 1) = \dim(\mathrm{span}(K)) && (29) \end{aligned}$$

and as a result we must have

$$\mathrm{span}(U^2(K)) = \mathrm{span}(K) \tag{30}$$

because if there would be $v \in \mathrm{span}(U^2(K))$ such that $v \notin \mathrm{span}(K)$ then we would have $\dim(\mathrm{span}(U^2(K)) > \dim(\mathrm{span}(K))$.

Assume there is $y \in U^2(K)$ such that $y \notin K$ and let$x_1, \dots, x_k \in K$ be a basis of $\mathrm{span}(K)$, then there must be numbers $\alpha_i \in \mathbb{R}$, $i \in \{1, \dots, k\}$ such that

$$y = \sum_{i=1}^{k} \alpha_i x_i. \tag{31}$$

Let $\varphi \in U(K)$ then we must have

$$1 = \langle \varphi, y \rangle = \langle \varphi, \sum_{i=1}^{k} \alpha_i x_i \rangle = \sum_{i=1}^{k} \alpha_i \tag{32}$$

hence $y \in \operatorname{aff}(K)$. From here the result follows from the separation theorem in the standard way [3, Theorem 4.3]. $\square$

Proposition 9. *Let $x \in \operatorname{cone}(K)$ be such that there is $u \in U(K)$ such that $\langle u, x \rangle = 0$, then $x = 0$.*

Proof. If $x \in \operatorname{cone}(K)$ then there is $y \in K$ such that $x = \lambda y$ for some $\lambda \in \mathbb{R}^+$ and we have

$$0 = \langle u, x \rangle = \lambda \langle u, y \rangle = \lambda. \tag{33}$$

$\square$

Proposition 10. *Let $K \neq \{0\}$, then $U(K) \cap \operatorname{int}(C^*) \neq \emptyset$.*

Proof. Assume that $U(K) \cap \operatorname{int}(C^*) = \emptyset$, then since $U(K)$ is a convex set there must be an exposed face $F \subset C^*$ such that $U(K) \subset F$. It follows that there must be an $x \in [0, K]$, $x \neq 0$ such that

$$\langle u, x \rangle = 0 \tag{34}$$

for all $u \in U(K)$. By construction there are $y \in [0, K]$ and $z \in K$ such that

$$x + y = z \tag{35}$$

and for all $\varphi \in U(K)$ we have

$$1 = \langle \varphi, z \rangle = \langle \varphi, y \rangle \tag{36}$$

hence $y \in K$ by Prop. 8. Since $x = z - y$ we must have $x \in \operatorname{span}(K)$ and since $x \in [0, K]$ we must also have $x \in \operatorname{cone}(K)$. It follows by Prop. 9 that $x = 0$, which is a contradiction. $\square$

Definition 5. The set of effects on the interval $[0, K]$ is defined as

$$E([0, K]) = \{\varphi \in V^* : \langle \varphi, x \rangle \leq 1, \forall x \in [0, K]\}. \tag{37}$$

Proposition 11. *Let $K \neq \{0\}$, then $E([0, K]) = [0, U(K)]$*

Proof. We have $[0, U(K)] \subset E([0, K])$ because let $x \in [0, K]$, then there is $x' \in [0, K]$ such that $x + x' \in K$. Now if $\varphi \in [0, U(K)]$ then there is $\varphi' \in [0, U(K)]$ such that $\varphi + \varphi' \in U(K)$ and we have

$$\langle \varphi, x \rangle \leq \langle \varphi + \varphi', x + x' \rangle = 1. \tag{38}$$

Let $f \in E(K)$; we will show that $f \in [0, U(K)]$. Let $u \in \operatorname{cone}(K)^*$ be the unique functional defined as $\langle u, x \rangle = 1$ for all $x \in K$. Then also $u - f \in \operatorname{cone}(K)*$ and according to Prop. 6 there is $\psi \in C^*$ such that

$$\langle \psi, x \rangle = \langle u - f, x \rangle \tag{39}$$

for all $x \in K$. Now we have $f \leq f + \psi$ and $f + \psi \in U(K)$ as

$$\langle f + \psi, x \rangle = \langle f, x \rangle + \langle u - f, x \rangle = 1 \tag{40}$$

for all $x \in K$. $\square$

Proposition 12. *Let $K \neq \{0\}$, then $[0, U(K)]$ is a generating interval.*

Proof. We need to show that $\operatorname{cone}([0, U(K)]) = C^*$, $\operatorname{cone}(U(K)) = \operatorname{span}(U(K)) \cap C^*$ and that $U(K)$ is a base of $\operatorname{cone}(U(K))$. So let $\varphi \in C^*$ and let

$$m = \sup_{x \in K} \langle \varphi, x \rangle. \tag{41}$$

It follows that we have $\frac{1}{m}\varphi \in E([0, K]) = [0, U(K)]$ and so we must have $\operatorname{cone}([0, U(K)]) = C^*$.

Now we will show that $\mathrm{cone}(U(K)) = \mathrm{span}(U(K)) \cap C^*$. We clearly have $\mathrm{cone}(U(K)) \subset \mathrm{span}(U(K)) \cap C^*$ so let us assume that there is $\varphi \in \mathrm{span}(U(K)) \cap C^*$, then there must be $u_i \in U(K)$ and $\alpha_i \in \mathbb{R}$, $i \in \{1, \dots, n\}$ such that

$$\varphi = \sum_{i=1}^{n} \alpha_i u_i. \tag{42}$$

Moreover let $x \in K$ then from $\langle \varphi, x \rangle \geq 0$ we get

$$\sum_{i=1}^{n} \alpha_i \geq 0. \tag{43}$$

If $\sum_{i=1}^{n} \alpha_i = 0$ then we have $\langle \varphi, x \rangle = 0$ for all $x \in K$, which implies $\langle \varphi, y \rangle = 0$ for all $y \in [0, K]$ and that implies $\varphi = 0$. Hence we will assume $\sum_{i=1}^{n} \alpha_i = \alpha > 0$. Now consider $\frac{1}{\alpha}\varphi \in C^*$. Clearly for every $x \in K$ we must have

$$\langle \frac{1}{\alpha}\varphi, x \rangle = 1 \tag{44}$$

so we must have $\frac{1}{\alpha}\varphi \in U(K)$ and $\varphi \in \mathrm{cone}(U(K))$. Note that this also shows that $U(K)$ is a base of $\mathrm{cone}(U(K))$, which concludes the proof. □

Proposition 13. $E(E([0, K]) = [0, K]$.

Proof. Let $K \neq \{0\}$, then $E(E([0, K]) = E([0, U(K)]) = [0, U(U(K))] = [0, K]$. If $K = \{0\}$ then also $[0, K] = \{0\}$ and $E([0, K]) = \{0\} = [0, K]$. □

II. CLOSURE

Proposition 14. *Let $C \subset V$ be a convex, closed, pointed and generating cone and let $B \subset C$ be a convex set such that $B \cap \mathrm{ri}(C) \neq \emptyset$ and B is a base of* $\mathrm{cone}(B)$*. Let*

$$\overline{B} = \mathrm{aff}(B) \cap C \tag{45}$$

then $[0, \overline{B}] \subset C$ is the smallest generating interval containing B.

Proof. XXX □

Proposition 15. *Let $C \subset V$ be a convex, closed, pointed and generating cone and let $B \subset C$ be a convex set such that $B \cap \mathrm{ri}(C) \neq \emptyset$ and B is a base of* $\mathrm{cone}(B)$*. Then we have*

$$\overline{B} = U^2(B) \tag{46}$$

where

$$U(B) = \{\varphi \in C^* : \langle \varphi, x \rangle = 1, \forall x \in K\}. \tag{47}$$

Proof. XXX □

Proposition 16. *Let $C \subset V$ be a convex, closed, pointed and generating cone and let $B \subset C$ be a convex set such that $B \cap \mathrm{ri}(C) \neq \emptyset$ and B is a base of* $\mathrm{cone}(B)$*, then the map $B \mapsto \overline{B}$ is a closure operator.*

Proof. XXX □

Proposition 17. *Let $B_i \subset C$ for $i \in \{1, 2\}$ be convex sets such that $B_i \cap \mathrm{ri}(C_i) \neq \emptyset$, B_i is a base of* $\mathrm{cone}(B_i)$ *and* $\mathrm{span}(B_1) = \mathrm{span}(B_2)$*, then $\overline{B}_1 = \overline{B}_2$.*

Proof. $\mathrm{span}(B_1) = \mathrm{span}(B_2)$ implies $\mathrm{aff}(B_1) = \mathrm{aff}(B_2)$ and we have

$$\overline{B}_1 = \mathrm{aff}(B_1) \cap C = \mathrm{aff}(B_2) \cap C = \overline{B}_2. \tag{48}$$

□

III. NORMS

Proposition 18. *Let $\varphi \in V^*$ and let*

$$\|\varphi\| = \sup_{x\in[0,K]} |\langle \varphi, x\rangle|. \tag{49}$$

$\|\cdot\|$ is a norm on V^*.

Proof. Let $\varphi_1, \varphi_2 \in V^*$ then we have

$$\begin{aligned} \|\varphi_1 + \varphi_2\| &= \sup_{x\in[0,K]} |\langle \varphi_1, x\rangle + \langle \varphi_2, x\rangle| &(50)\\ &\leq \sup_{x\in[0,K]} (|\langle \varphi_1, x\rangle| + |\langle \varphi_2, x\rangle|) &(51)\\ &\leq \sup_{x_1\in[0,K]} |\langle \varphi_1, x_1\rangle| + \sup_{x_1\in[0,K]} |\langle \varphi_2, x_2\rangle| = \|\varphi_1\| + \|\varphi_2\|. &(52)\end{aligned}$$

Now let $\varphi \in V^*$ and $\alpha \in \mathbb{R}$ then we have

$$\|\alpha\varphi\| = \sup_{x\in[0,K]} |\alpha\langle \varphi, x\rangle| = |\alpha|\|\varphi\|. \tag{53}$$

Finally assume that for some $\varphi \in V^*$ we have $\|\varphi\| = 0$. Then also we have $\langle \varphi, x\rangle = 0$ for all $x \in [0, K]$ and since $\mathrm{cone}([0, K]) = C$ is generating we have $\langle \varphi, v\rangle = 0$ for all $v \in V$. It follows that $\varphi = 0$. □

Proposition 19. *Let $\varphi \in C^*$ then*

$$\|\varphi\| = \sup_{x\in K} \langle \varphi, x\rangle. \tag{54}$$

Proof. Let $x \in [0, K]$ be such that $\|\varphi\| = \langle \varphi, x\rangle$; we know that such $x \in [0, K]$ exists because $[0, K]$ is a closed set. By construction there are $y \in [0, K]$ such that $x + y \in K$ and we have

$$\|\varphi\| \geq \langle \varphi, x + y\rangle \geq \langle \varphi, x\rangle = \|\varphi\| \tag{55}$$

so we must have $\langle \varphi, x + y\rangle = \|\varphi\|$. □

Proposition 20. *Let $\varphi \in V^*$ and let*

$$\|\varphi\|_B = \sup_{x\in[-K,K]} |\langle \varphi, x\rangle|. \tag{56}$$

$\|\cdot\|_B$ is a norm on V^*

Proof. XXX □

Proposition 21. *Let $\varphi \in V^*$, then*

$$\|\varphi\| \leq \|\varphi\|_B. \tag{57}$$

Proof. XXX □

Proposition 22. *Let $\varphi \in C^*$ then*

$$\|\varphi\| = \|\varphi\|_B. \tag{58}$$

Proposition 23. *We have $\|\cdot\| = \|\cdot\|_B$ if and only if $[-K, K] = \mathrm{conv}([-K, 0] \cup [0, K])$.*

Proof. If $[-K, K] = \mathrm{conv}([-K, 0] \cup [0, K])$ then let $\varphi \in V^*$ and we have

$$\|\varphi\|_B = \sup_{x\in[-K,0]\cup[0,K]} |\langle \varphi, x\rangle| = \|\varphi\|. \tag{59}$$

Now assume there is $v \in [-K, K] \setminus \mathrm{conv}([-K, 0] \cup [0, K])$. Note that since $x \in \mathrm{conv}([-K, 0] \cup [0, K])$ if and only if $x \in \mathrm{conv}([-K, 0] \cup [0, K])$ we have

$$\sup_{x \in \mathrm{conv}([-K,0] \cup [0,K])} \langle \varphi, x \rangle = \|\varphi\|. \tag{60}$$

By separation theorem there is a functional $0 \neq \varphi \in V^*$ such that

$$\langle \varphi, v \rangle > \sup_{x \in \mathrm{conv}([-K,0] \cup [0,K])} \langle \varphi, x \rangle = \|\varphi\|. \tag{61}$$

□

Example 2. We can find $\varphi \in V^*$ such that

$$\|\varphi\| \neq \|\varphi\|_B. \tag{62}$$

Simply take the classical bit and its effect algebra.

IV. MORPHISMS

Definition 6. Let $[0, K_A] \subset C_A$ and $[0, K_B] \subset C_B$ then a morphism $\Phi : [0, K_A] \to [0, K_B]$ is a positive linear map $\Phi : V_A \to V_B$ such that $\Phi(K_A) \subset K_B$.

Definition 7. Let $\Phi : [0, K_A] \to [0, K_B]$ be a morphism ten we define the adjoint map $\Phi^* : C_B^* \to C_A^*$ as follows: let $x \in [0, K_A]$ and $\varphi \in C_B^*$, then

$$\langle \varphi, \Phi(x) \rangle = \langle \Phi^*(\varphi), x \rangle. \tag{63}$$

Proposition 24. *Let $\Phi : [0, K_A] \to [0, K_B]$ be a morphism, then $\Phi^*(U(K_B)) \subset U(K_A)$ and $\Phi^* : [0, U(K_B)] \to [0, U(K_A)]$ is a morphism.*

Proof. It follows from the definition that Φ^* is a positive linear map, to show that Φ^* is a morphism we only need to show that $\Phi^*(U(K_B) \subset U(K_A)$. So let $x \in K_A$ and $\varphi \in U(K_B)$ then we have

$$\langle \Phi^*(\varphi), x \rangle = \langle \varphi, \Phi(x) \rangle = 1 \tag{64}$$

and $\Phi^*(\varphi) \in U(K_A)$ follows. □

Proposition 25. *Let $[0, K]$ be a generating interval such that $K \neq \{0\}$ then there is a surjective morphism*

$$\Phi : [0, K] \to [0, 1] \tag{65}$$

where $[0, 1] \subset \mathbb{R}$.

Proof. Simply take any $\Phi = u \in U(K)$; note that we have already proved in Coro. 1 that $U(K) \neq \emptyset$ if $K \neq \{0\}$. The morphism is surjective as $\langle u, 0 \rangle = 0$ and $\langle u, x \rangle = 1$ for any $x \in K$. □

Corollary 2. *For any generating interval $[0, K]$ there is a morphism*

$$\Phi : [0, 1] \to [0, K]. \tag{66}$$

If $K \neq \{0\}$ then Φ is injective.

Proof. If $K = \{0\}$ then simply take $\Phi(1) = 0$. If $K \neq \{0\}$ then we know that there is a morphism

$$\Phi^* : E([0, 1]) \to [0, K] \tag{67}$$

which is adjoint to the morphism

$$\Phi : [0, U(K)] \to [0, 1] \tag{68}$$

existence of which was proved in Prop. 25. At last note that $E([0, 1]) = [0, 1]$. Note that Φ is injective simply because $\Phi^*(1) \neq 0$. □

V. PRODUCTS AND COPRODUCTS

Definition 8. We say that a generating interval $[0, K_{A\times B}]$ is a product of the generating intervals $[0, K_A]$ and $[0, K_B]$ if there are morphisms

$$\Pi_A : [0, K_{A\times B}] \to [0, K_A] \tag{69}$$
$$\Pi_B : [0, K_{A\times B}] \to [0, K_B] \tag{70}$$

such that for any other generating interval $[0, K_C]$ and morphisms

$$\Phi_A : [0, K_C] \to [0, K_A] \tag{71}$$
$$\Phi_B : [0, K_C] \to [0, K_B] \tag{72}$$

there is a unique morphism

$$\Phi : [0, K_C] \to [0, K_{A\times B}] \tag{73}$$

such that the following diagram commutes

$$\begin{tikzcd} {[0,K_A]} & {[0,K_{A\times B}]} \arrow[l, "\Pi_A"'] \arrow[r, "\Pi_B"] & {[0,K_B]} \\ & {[0,K_C]} \arrow[ul, "\Phi_A"] \arrow[u, dashed, "\Phi"] \arrow[ur, "\Phi_B"'] & \end{tikzcd} \tag{74}$$

i.e. we have

$$\Phi_A = \Pi_A \circ \Phi \tag{75}$$
$$\Phi_B = \Pi_B \circ \Phi. \tag{76}$$

Definition 9. We say that a generating interval $[0, K_{A\oplus B}]$ is a coproduct of the generating intervals $[0, K_A]$ and $[0, K_B]$ if there are morphisms

$$\iota_A : [0, K_A] \to [0, K_{A\oplus B}] \tag{77}$$
$$\iota_B : [0, K_B] \to [0, K_{A\oplus B}] \tag{78}$$

such that for any other generating interval $[0, K_C]$ and morphisms

$$\Phi_A : [0, K_A] \to [0, K_C] \tag{79}$$
$$\Phi_B : [0, K_B] \to [0, K_C] \tag{80}$$

there is a unique morphism

$$\Phi : [0, K_{A\oplus B}] \to [0, K_C] \tag{81}$$

such that the following diagram commutes

$$\begin{tikzcd} {[0,K_A]} \arrow[r, "\iota_A"] \arrow[dr, "\Phi_A"'] & {[0,K_{A\oplus B}]} \arrow[d, dashed, "\Phi"] & {[0,K_B]} \arrow[l, "\iota_B"'] \arrow[dl, "\Phi_B"] \\ & {[0,K_C]} & \end{tikzcd} \tag{82}$$

i.e. we have

$$\Phi_A = \Phi \circ \iota_A \tag{83}$$
$$\Phi_B = \Phi \circ \iota_B. \tag{84}$$

Proposition 26. *Let $[0, K_A]$ and $[0, K_B]$ be generating intervals and let $[0, K_{A\times B}]$ be their product and let $[0, K_{A\oplus B}]$ be their coproduct. Then $[0, U(K_{A\times B})]$ is the coproduct of $[0, U(K_A)]$ and $[0, U(K_B)]$ and $[0, U(K_{A\oplus B})]$ is the product of $[0, U(K_A)]$ and $[0, U(K_B)]$.*

Proof. This should be cited from somewhere. □

Proposition 27. *The coproduct of effect algebras is not effect algebra.*

Proof. Consider the effect algebra $[0,1] \subset \mathbb{R}$ and assume that the coproduct of two copies of $[0,1]$ is an effect algebra $[0,u]$. Let

$$C_2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : 0 \leq x \leq y \right\}, \tag{85}$$

denote

$$s_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{86}$$

$$s_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{87}$$

and let

$$S_2 = \operatorname{conv}(\{s_0, s_1\}) \subset C_2. \tag{88}$$

It is straightforward to see that $[0, S_2]$ is a causal generating interval. Now consider the morphisms

$$\Phi_A : [0,1] \to [0, S_2] \tag{89}$$

$$\Phi_B : [0,1] \to [0, S_2] \tag{90}$$

such that

$$\Phi_A(1) = s_0 \tag{91}$$

$$\Phi_B(1) = s_1. \tag{92}$$

Now let $\iota_A : [0,1] \to [0,u]$, $\iota_B : [0,1] \to [0,u]$, $\Phi : [0,u] \to [0,S_2]$ be the morphisms corresponding to the construction of the coproduct, then we must have

$$\iota_A(1) = u = \iota_B(1) \tag{93}$$

which implies

$$\Phi_A(1) = (\Phi \circ \iota_A)(1) = (\Phi \circ \iota_A)(1) = \Phi_B(1) \tag{94}$$

which is a contradiction.

Now let $[0, u_A]$ and $[0, u_B]$ be effect algebras and assume that their coproduct is the effect algebra $[0, u_{A \oplus B}]$ and let

$$\Upsilon_A : [0, u_A] \to [0,1] \tag{95}$$

$$\Upsilon_B : [0, u_B] \to [0,1] \tag{96}$$

be morphisms, we know that such morphisms exist as a result of Prop. 25 and take

$$\Psi_A = \Phi_A \circ \Upsilon_A \tag{97}$$

$$\Psi_B = \Phi_B \circ \Upsilon_B. \tag{98}$$

We can again see from the properties of the coproduct that we should have

$$\Psi_A(u_A) = \Psi_B(u_B) \tag{99}$$

which is a contradiction. □

Corollary 3. *The product of causal generating intervals is not causal.*

Proof. It follows from Prop. 7 that if $[0, K]$ is a generating interval, then $[0, U(K)]$ is an effect algebra. Now the result follows from Prop. 27 as if product of causal generating intervals would be causal, then the coproduct of their respective effect algebras would again be an effect algebra. □

Note that the result of Prop. 27 does not violate any operational understanding of the theories as the product of state spaces does not have any operational interpretation. it is only the coproduct of state spaces that has an operational interpretation via superselection rules.

Proposition 28. *Let* $[0, K_A]$ *and* $[0, K_B]$ *be generating intervals and denote*

$$C_A = \text{cone}([0, K_A]) \tag{100}$$
$$V_A = \text{span}([0, K_A]) \tag{101}$$
$$C_B = \text{cone}([0, K_B]) \tag{102}$$
$$V_B = \text{span}([0, K_B]). \tag{103}$$

The product of $[0, K_A]$ *and* $[0, K_B]$ *is the generating interval*

$$[0, K_A \times K_B] \subset C_A \times C_B \tag{104}$$

where

$$C_A \times C_B = \{(x, y) \in V_A \times V_B : x \in C_A, y \in C_B\}. \tag{105}$$

Proof. As first we have to show that $[0, K_A \times K_B]$ is a generating interval. Let $u_A \in U(K_A)$ and $u_B \in U(K_B)$ then

$$\text{span}(K_A \times K_B) = \{(x, y) \in \text{span}(K_A) \times \text{span}(K_B) : \langle u_A, x\rangle = \langle u_B, y\rangle\} \tag{106}$$
$$\text{cone}(K_A \times K_B) = \{(x, y) \in \text{cone}(K_A) \times \text{cone}(K_B) : \langle u_A, x\rangle = \langle u_B, y\rangle\} \tag{107}$$

and note that the above results are independent of the choice of u_A and u_B. Now let $(x, y) \in (C_A \times C_B) \cap \text{span}(K_A \times K_B)$, then from $x \in C_A$ and $x \in \text{span}(K_A)$ it follows that $x \in \text{cone}(K_A)$ and analogically we get $y \in \text{cone}(K_B)$. Moreover since $\langle u_A, x\rangle = \langle u_B, y\rangle$ we have $(x, y) \in \text{cone}(K_A \times K_B)$, i.e. we have shown that

$$\text{cone}(K_A \times K_B) = (C_A \times C_B) \cap \text{span}(K_A \times K_B). \tag{108}$$

Note that $(x, y) \in [0, K_A \times K_B]$ if and only if there is $(z_A, z_B) \in K_A \times K_B$ such that $(x, y) \leq (z_A, z_B)$, which is equivalent to $x \leq z_A$ and $y \leq z_B$, so we have $(x, y) \in [0, K_A] \times [0, K_B]$. It follows that

$$[0, K_A \times K_B] = [0, K_A] \times [0, K_B]. \tag{109}$$

Now let $(\lambda x, \mu y) \in C_A \times C_B$ where $\lambda, \mu \in \mathbb{R}^+$ and $x \in [0, K_A]$, $y \in [0, K_B]$ and without loss of generality assume that $\lambda \leq \mu$, then

$$(\lambda x, \mu y) = \lambda \left(x, \frac{\mu}{\lambda} y\right) \in \text{cone}([0, K_A] \times [0, K_B]) \tag{110}$$

so we get

$$\text{cone}([0, K_A] \times [0, K_B]) = C_A \times C_B. \tag{111}$$

Finally we get

$$\text{cone}([0, K_A \times K_B]) = \text{cone}([0, K_A] \times [0, K_B]) = C_A \times C_B \tag{112}$$

so $[0, K_A \times K_B]$ is a generating interval.

We can easily construct the projections $\Pi_A : [0, K_A \times K_B] \to [0, K_A]$ and $\Pi_B : [0, K_A \times K_B] \to [0, K_B]$ as follows: let $(x, y) \in [0, K_A \times K_B]$, then

$$\Pi_A(x, y) = x \tag{113}$$
$$\Pi_B(x, y) = y. \tag{114}$$

Now we will show that $[0, K_A \times K_B]$ is a product of $[0, K_A]$ and $[0, K_B]$. So assume there is $[0, K_C]$ and morphisms $\Phi_A : [0, K_C] \to [0, K_A]$ and $\Phi_B : [0, K_C] \to [0, K_B]$. Let $(\Phi_A, \Phi_B) : [0, K_C] \to [0, K_A \times K_B]$ be a morphism given for $x \in K_C$ as

$$(\Phi_A, \Phi_B)(x) = (\Phi_A(x), \Phi_B(x)) \tag{115}$$

then clearly $\Phi_A = \Pi_A \circ (\Phi_A, \Phi_B)$, $\Phi_B = \Pi_B \circ (\Phi_A, \Phi_B)$, moreover one can see that (Φ_A, Φ_B) is the unique morphism with such properties as follows: let $(y, z) \in [0, K_A \times K_B]$ be such that $\Pi_A(y, z) = \Phi_A(x)$, then clearly $y = \Phi_A(x)$ and in a similar fashion we get $z = \Phi_B(x)$. It follows that we are looking for a morphism $\Phi : [0, K_C] \to [0, K_A \times K_B]$ such that $\Phi(x) = (\Phi_A(x), \Phi_B(x))$ so we must have $\Phi = (\Phi_A, \Phi_B)$. $\square$

Definition 10. Let $[0, K_A]$ and $[0, K_B]$ be generating intervals, then we define

$$K_A \oplus K_B = \{(\lambda x, (1-\lambda)y) : x \in K_A, y \in K_B, \lambda \in [0,1]\}. \tag{116}$$

Proposition 29. *Let $[0, K_A]$ and $[0, K_B]$ be generating intervals, then their coproduct is given as*

$$[0, K_{A\oplus B}] = [0, K_A \oplus K_B]. \tag{117}$$

Proof. According to Prop. 26 we have

$$[0, K_{A\oplus B}] = [0, U(U(K_A) \times U(K_B))] \tag{118}$$

so it only remains to show that

$$U(U(K_A) \times U(K_B)) = K_A \oplus K_B. \tag{119}$$

It is easy to see that $K_A \oplus K_B \subset U(U(K_A) \times U(K_B))$.

So let $u_A \in U(K_A)$ and $u_B \in U(K_B)$ and let $(x, y) \in U(U(K_A) \times U(K_B))$ so we must have

$$\langle (u_A, u_B), (x, y) \rangle = 1. \tag{120}$$

It follows that for any other $u'_A \in U(K_A)$ we must have

$$\langle u_A, x \rangle = \langle u'_A, x \rangle \tag{121}$$

and so x is constant on $U(K_A)$ and positive, which implies $x \in \mathrm{cone}(K_A)$. In a similar way we can prove $y \in \mathrm{cone}(K_B)$ and there must be $x' \in K_A$, $y' \in K_B$ and $\lambda, \mu \in \mathbb{R}^+$ such that $x = \lambda x'$, $y = \mu y'$ and we have

$$\langle (u_A, u_B), (x, y) \rangle = \lambda + \mu = 1 \tag{122}$$

and so $\mu = 1 - \lambda$ and $(x, y) \in K_A \oplus K_B$. □

Corollary 4. *The direct sum of causal generating intervals is again causal.*

Proof. Let $[0, K_A]$ and $[0, K_B]$ be causal generating intervals, and let $(\lambda x, \mu y) \in \mathrm{cone}([0, K_A]) \times \mathrm{cone}([0, K_B])$, where $x \in K_A$, $y \in K_B$ and $\lambda, \mu \in \mathbb{R}^+$, note that this is the general form of every element of $\mathrm{cone}([0, K_A]) \times \mathrm{cone}([0, K_B])$ because $[0, K_A]$ and $[0, K_B]$ are causal. We have

$$(\lambda x, \mu y) = (\lambda + \mu) \left(\frac{\lambda}{\lambda + \mu}(x, 0) + \frac{\mu}{\lambda + \mu}(0, y) \right) \in \mathrm{cone}(K_A \oplus K_B). \tag{123}$$

□

Note that the result we have obtain is in line with operational interpretation of direct sum as forming superselection sectors [4, 5].

VI. MONOIDAL STRUCTURE

Definition 11. Let V_A, V_B be finite-dimensional vector spaces and let $C_A \subset V_A$, $C_B \subset V_B$ be convex, closed, pointed and generating cones. Let $[0, K_A] \subset C_A$, $[0, K_B] \subset C_B$ be generating intervals. We will denote

$$C_A \dot{\otimes} C_B = \mathrm{cone}(\mathrm{conv}(\{x \otimes y : x \in C_A, y \in C_B\})) \subset V_A \times V_B \tag{124}$$

$$K_A \dot{\otimes} K_B = \mathrm{conv}(\{x \otimes y : x \in K_A, y \in K_B\}) \subset \mathrm{span}(K_A) \times \mathrm{span}(K_B). \tag{125}$$

We define the minimal tensor product of $[0, K_A]$ and $[0, K_B]$ as the smallest generating interval in $C_A \dot{\otimes} C_B$ containing $K_A \dot{\otimes} K_B$, i.e.

$$[0, K_A] \dot{\otimes} [0, K_B] = [0, \overline{K_A \dot{\otimes} K_B}]. \tag{126}$$

Proposition 30. *Let V_A, V_B be finite-dimensional vector spaces and let $C_A \subset V_A$, $C_B \subset V_B$ be convex, closed, pointed and generating cones. Let $[0, K_A] \subset C_A$, $[0, K_B] \subset C_B$ be generating intervals. We will denote*

$$C_A \hat{\otimes} C_B = \{v \in V_A \otimes V_B : \langle \varphi_A \otimes \varphi_B, v \rangle \geq 0, \forall \varphi_A \in C_A, \forall \varphi_B \in C_B\} \tag{127}$$

$$\begin{aligned} K_A \hat{\otimes} K_B = \{v \in \operatorname{span}(K_A) \otimes \operatorname{span}(K_B) : &\langle \varphi_A \otimes \varphi_B, v \rangle \geq 0, \forall \varphi_A \in C_A, \varphi_B \in C_B, \\ &\langle u_A \otimes u_B, v \rangle = 1, \forall u_A \in U(K_A), \forall u_B \in U(K_B)\}. \end{aligned} \tag{128}$$

Then $[0, K_A \hat{\otimes} K_B] \subset C_A \hat{\otimes} C_B$ is a generating interval.

Proof. Note that $K_A \dot{\otimes} K_B \subset K_A \hat{\otimes} K_B$, $C_A \dot{\otimes} C_B \subset C_A \hat{\otimes} C_B$ and $\operatorname{span}(C_A \dot{\otimes} C_B) = \operatorname{span}(C_A \hat{\otimes} C_B)$ follows easily. It is straightforward to see that $K_A \hat{\otimes} K_B$ is the base of $\operatorname{cone}(K_A \hat{\otimes} K_B)$, moreover let $r_A \in K_A \cap \operatorname{ri}(C_A)$ and $r_B \in K_B \cap \operatorname{ri}(C_B)$, then

$$r_A \otimes r_B \in (K_A \hat{\otimes} K_B) \cap \operatorname{ri}(C_A \hat{\otimes} C_B) \tag{129}$$

so $\operatorname{cone}[0, K_A \hat{\otimes} K_B] = C_A \hat{\otimes} C_B$ follows from Prop. 5. Finally, we have that

$$\begin{aligned} \operatorname{span}(K_A \hat{\otimes} K_B) \cap (C_A \hat{\otimes} C_B) &= \{v \in \operatorname{span}(K_A \hat{\otimes} K_B) : \langle \varphi_A \otimes \varphi_B, v \rangle \geq 0, \forall \varphi_A \in C_A, \forall \varphi_B \in C_B\} && (130) \\ &= \operatorname{cone}(K_A \hat{\otimes} K_B). && (131) \end{aligned}$$

□

Definition 12. Let V_A, V_B be finite-dimensional vector spaces and let $C_A \subset V_A$, $C_B \subset V_B$ be convex, closed, pointed and generating cones and let $[0, K_A] \subset C_A$, $[0, K_B] \subset C_B$ be generating intervals. We define the maximal tensor product of $[0, K_A]$ and $[0, K_B]$ to be the generating interval $[0, K_A \hat{\otimes} K_B] \subset C_A \hat{\otimes} C_B$.

Example 3. Let $[0, S] \subset C(S_4)$ be the generating interval introduced in Example 1, we will show that we have

$$[0, S] \dot{\otimes} [0, S] = [0, S] \hat{\otimes} [0, S] \tag{132}$$

which will demonstrate that the closure in Def. 11 is necessary and not always trivial.

Since $C(S_4) \dot{\otimes} C(S_4) = C(S_4) \hat{\otimes} C(S_4)$ it follows that both of the generating intervals $[0, S] \dot{\otimes} [0, S]$ and $[0, S] \hat{\otimes} [0, S]$ generate the same cone. Moreover since $\operatorname{span}(S \dot{\otimes} S) = \operatorname{span}(S \hat{\otimes} S)$ it follows from Prop. 17 that

$$\overline{S \dot{\otimes} S} = \overline{S \hat{\otimes} S} = S \hat{\otimes} S \tag{133}$$

and so

$$[0, S] \dot{\otimes} [0, S] = [0, S \hat{\otimes} S] = [0, S] \hat{\otimes} [0, S]. \tag{134}$$

Proposition 31. *We have*

$$U(K_A) \hat{\otimes} U(K_B) = U(K_A \dot{\otimes} K_B) \cap \operatorname{span}(U(K_A)) \otimes \operatorname{span}(U(K_B)). \tag{135}$$

Proof. XXX □

Proposition 32. *We have*

$$\begin{aligned} &\dim(\operatorname{span}(U(K_A \dot{\otimes} K_B))) - \dim(\operatorname{span}(U(K_A) \dot{\otimes} U(K_B))) = \\ &(\dim(\operatorname{span}(K_A)) - 1)(\dim(V_B) - \dim(\operatorname{span}(K_B))) + (\dim(\operatorname{span}(K_B)) - 1)(\dim(V_A) - \dim(\operatorname{span}(K_A))). \end{aligned} \tag{136}$$

Moreover

$$\dim(\operatorname{span}(U(K_A \dot{\otimes} K_B))) = \dim(\operatorname{span}(U(K_A) \dot{\otimes} U(K_B))) \tag{137}$$

if and only if either both $[0, K_A]$ and $[0, K_B]$ are causal or effect algebras.

Proof. Eq. (136) follows from Prop. 7 and $\dim(\operatorname{span}(U(K_A) \dot{\otimes} U(K_B))) = \dim(\operatorname{span}(U(K_A))) \dim(\operatorname{span}(U(K_B)))$. Since $\dim(\operatorname{span}(K_A)) \geq 1$, $\dim(V_A) \geq \dim(\operatorname{span}(K_A))$ and similar inequalities also hold for K_B, we get that Eq. (137) holds if and only if

$$\begin{aligned} (\dim(\operatorname{span}(K_A)) - 1)(\dim(V_B) - \dim(\operatorname{span}(K_B))) = 0 && (138) \\ (\dim(\operatorname{span}(K_B)) - 1)(\dim(V_A) - \dim(\operatorname{span}(K_A))) = 0 && (139) \end{aligned}$$

from which it follows that either $\dim(\operatorname{span}(K_A)) = \dim(\operatorname{span}(K_B)) = 1$ or $\dim(V_A) = \dim(\operatorname{span}(K_A))$ and $\dim(V_B) = \dim(\operatorname{span}(K_B))$, i.e. either both $[0, K_A]$ and $[0, K_B]$ are effect algebras or causal. □

Example 4. Let V be a real vector space, $\dim(V) = 3$ and let

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{140}$$

be the standard basis. Let $S_3 = \operatorname{conv}(\{e_1, e_2, e_3\})$ be a simplex and let $C(S_3) = \operatorname{cone}(S_3)$. Let b_1, b_2, b_3 denote the dual basis to e_1, e_2, e_3 and let

$$J = \{v \in V : \langle b_1 - b_2, v\rangle = 0\}. \tag{141}$$

Let

$$s = \frac{1}{2}(e_1 + e_2) \tag{142}$$

and let $S_2 = \operatorname{conv}(\{s, e_3\})$ be the line segment such that

$$S_2 = S_3 \cap J. \tag{143}$$

We are going to be interested in the generating interval $[0, S_2] \subset C(S_3)$.

Let $\varphi \in U(S_2)$, then it is easy to show that there must be $\lambda \in [0, 1]$ such that

$$\varphi = 2(\lambda b_1 + (1 - \lambda)b_2) + b_3 \tag{144}$$

and so

$$U(S_2) = \operatorname{conv}(\{2b_1 + b_3, 2b_2 + b_3\}). \tag{145}$$

Since we have

$$[0, S_2]\dot{\otimes}[0, S_2] = [0, S_2\hat{\otimes}S_2] = [0, S_2\dot{\otimes}S_2] \tag{146}$$

we can explicitly find $U(S_2\dot{\otimes}S_2)$ to characterize $E([0, S_2]\dot{\otimes}[0, S_2])$. Let $\varphi \in V^* \otimes V^*$, then it is of the form

$$\varphi = \sum_{i,j=1}^{3} \alpha_{ij} b_i \otimes b_j \tag{147}$$

and we will identify φ with the matrix α given by the numbers α_{ij}.

$U(S_2)\dot{\otimes}U(S_2) = U(S_2)\hat{\otimes}U(S_2)$ has four extreme points of the form

$$(2b_1 + b_3) \otimes (2b_1 + b_3) = \begin{pmatrix} 4 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix} \qquad (2b_1 + b_3) \otimes (2b_2 + b_3) = \begin{pmatrix} 0 & 4 & 2 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix} \tag{148}$$

$$(2b_2 + b_3) \otimes (2b_1 + b_3) = \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 2 \\ 2 & 0 & 1 \end{pmatrix} \qquad (2b_2 + b_3) \otimes (2b_2 + b_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 1 \end{pmatrix} \tag{149}$$

while the general element of $U(S_2\dot{\otimes}S_2)$ is given by $\lambda, \mu, \sigma_1, \sigma_2, \sigma_3 \in [0, 1]$ such that $\sigma_1 + \sigma_2 + \sigma_3 \geq 0$ and the corresponding matrix α is given as

$$\alpha = \begin{pmatrix} 4\sigma_1 & 4\sigma_2 & 2\mu \\ 4\sigma_3 & 4(\sigma_1 + \sigma_2 + \sigma_3) & 2\mu \\ 2\lambda & 2(1 - \lambda) & 1 \end{pmatrix}. \tag{150}$$

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