

# Equality in DPI for sandwiched Rényi divergence

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Below,  $\gamma : \mathcal{N} \rightarrow \mathcal{M}$  is a normal positive unital map,  $\psi, \varphi \in \mathcal{M}_*^+$  and we put  $\psi_0 = \psi \circ \gamma$ ,  $\varphi_0 = \varphi \circ \gamma$ . We consider the following equality

$$D_{\alpha,z}(\psi\|\varphi) = D_{\alpha,z}(\psi_0\|\varphi_0) \quad (1)$$

where  $\alpha > 1$  and  $\max\{\alpha/2, \alpha - 1\} \leq z \leq \alpha$ . We put  $q = \frac{z}{\alpha-1}$ . In the case  $\alpha = z$  we have  $D_{\alpha,\alpha} = \tilde{D}_\alpha$  is the sandwiched Rényi divergence. We assume  $D_{\alpha,z}(\psi\|\varphi) < \infty$ , then by DPI we also have  $D_{\alpha,z}(\psi_0\|\varphi_0) < \infty$ . Hence there are  $y \in L_{2z}(\mathcal{M})$  and  $y_0 \in L_{2z}(\mathcal{N})$  such that

$$h_\psi^{\frac{\alpha}{2z}} = y h_\varphi^{\frac{\alpha-1}{2z}}, \quad h_{\psi_0}^{\frac{\alpha}{2z}} = y_0 h_{\varphi_0}^{\frac{\alpha-1}{2z}}.$$

**Lemma 1.** *[1, Lemma 3.10] Let  $\gamma : \mathcal{N} \rightarrow \mathcal{M}$  be a normal positive unital map. Let  $\gamma_{\varphi,q}^* : L_q(\mathcal{N}) \rightarrow L_q(\mathcal{M})$  be the contraction as in [1, Lemma 3.1]. Let  $\bar{w} : (y^*y)^{\alpha-1} \in L_q(\mathcal{M})$  and  $\bar{w}_0 := (y_0^*y_0)^{\alpha-1} \in L_q(\mathcal{N})$ . Then (1) holds if and only if*

$$\bar{w} = \gamma_{\varphi,q}^*(\bar{w}_0) \quad \text{and} \quad \text{Tr}(\bar{w}_0^q) = \text{Tr}(\gamma_{\varphi,q}^*(\bar{w}_0)^q). \quad (2)$$

We show that (1) implies reversibility of  $\gamma$  for the sandwiched Rényi divergence.

**Theorem 1.** *Let  $\gamma : \mathcal{N} \rightarrow \mathcal{M}$  be a normal positive unital map. Let  $\psi, \varphi \in \mathcal{M}_*^+$  be such that  $\tilde{D}_\alpha(\psi\|\varphi) < \infty$  for some  $\alpha > 1$ . Then  $\tilde{D}_\alpha(\psi\|\varphi) = \tilde{D}_\alpha(\psi_0\|\varphi_0)$  if and only if  $\psi \circ \gamma \circ \gamma_\varphi^* = \psi$ .*

*Proof.* Let  $\bar{w}$  and  $\bar{w}_0$  be as in Lemma 1. Let  $\omega \in \mathcal{M}_*^+$  and  $\omega_0 \in \mathcal{N}_*^+$  be such that

$$h_\omega = h_\varphi^{\frac{1}{2\alpha}} \bar{w} h_\varphi^{\frac{1}{2\alpha}}, \quad h_{\omega_0} = h_{\varphi_0}^{\frac{1}{2\alpha}} \bar{w}_0 h_{\varphi_0}^{\frac{1}{2\alpha}}.$$

Note that in this case  $q = \frac{\alpha}{\alpha-1}$ . Then  $h_\omega \in L_q(\mathcal{M}, \varphi)$  and  $h_{\omega_0} \in L_q(\mathcal{N}, \varphi_0)$ . Assume the equality in DPI holds, then by Lemma 1 we have

$$\|h_\omega\|_{q,\varphi} = \|\bar{w}\|_q = \|\gamma_{\varphi,q}^*(\bar{w}_0)\|_q = \|\bar{w}_0\|_q = \|h_{\omega_0}\|_{q,\varphi_0} =: l$$

and using also [1, Lemma 3.1], we have

$$(\gamma_\varphi^*)_*(h_{\omega_0}) = h_\varphi^{\frac{1}{2\alpha}} \gamma_{\varphi,q}^*(\bar{w}_0) h_\varphi^{\frac{1}{2\alpha}} = h_\omega.$$

Let  $f_{h_{\omega_0},q}$  and  $f_{h_\omega,q}$  be the functions as in [2, Eq. (9)], so that

$$f_{h_{\omega_0},q}(s) = l^{1-sq} h_{\varphi_0}^{\frac{1-s}{2}} \bar{w}_0^{qs} h_{\varphi_0}^{\frac{1-s}{2}}, \quad f_{h_\omega,q}(s) = l^{1-sq} h_\varphi^{\frac{1-s}{2}} \bar{w}^{qs} h_\varphi^{\frac{1-s}{2}}, \quad s \in S,$$

here  $S$  is the strip  $S = \{s \in \mathbb{C}, 0 \leq \operatorname{Re}(s) \leq 1\}$ . Note that we have  $f_{h_\omega, q}(1/q) = h_\omega$  and

$$\sup_t \|f_{h_{\omega_0}, q}(it)\|_{\infty, \varphi_0} = \sup_t \|f_{h_{\omega_0}, q}(1+it)\|_1 = l = \|h_{\omega_0}\|_{q, \varphi},$$

similarly for  $f_{h_\omega, q}$ .

Put  $g(s) := (\gamma_\varphi^*)_*(f_{h_{\omega_0}, q}(s))$ ,  $s \in S$ , then  $g$  is a bounded continuous function  $S \rightarrow L_1(\mathcal{M})$ , analytic in the interior and such such that  $g(1/q) = h_\omega$ . Since  $(\gamma_\varphi^*)_*$  is a contraction  $L_r(\mathcal{N}, \varphi_0) \rightarrow L_r(\mathcal{M}, \varphi)$  for any  $1 \leq r \leq \infty$ , we have by the Hadamard three lines theorem (see e.g. [2, Thm. 2.10] in this context)

$$\begin{aligned} \|g(1/q)\|_{q, \varphi} &\leq \left(\sup_t \|g(it)\|_{\infty, \varphi}\right)^{1-1/q} \left(\sup_t \|g(1+it)\|_1\right)^{1/q} \\ &\leq \left(\sup_t \|f_{h_{\omega_0}, q}(it)\|_{\infty, \varphi}\right)^{1-1/q} \left(\sup_t \|f_{h_{\omega_0}, q}(1+it)\|_1\right)^{1/q} = \|h_{\omega_0}\|_{q, \varphi_0} = \|g(1/q)\|_{q, \varphi}. \end{aligned}$$

It follows that  $g$  satisfies equality in the Hadamard three lines theorem and we must have

$$\sup_t \|g(it)\|_{\infty, \varphi} = \sup_t \|f_{h_{\omega_0}, q}(it)\|_{\infty, \varphi_0} = \sup_t \|f_{h_{\omega_0}, q}(1+it)\|_1 = \sup_t \|g(1+it)\|_1.$$

By [2, Thm. 2.10], this implies that  $g(s) = f_{h_\omega, q}(s)$  for all  $s \in S$ . For  $s = 1/\alpha$  this implies

$$h_\psi = h_\varphi^{\frac{q-1}{2q}} y^* y h_\varphi^{\frac{q-1}{2q}} = (\gamma_\varphi^*)_*(h_{\varphi_0}^{\frac{q-1}{2q}} y_0^* y_0 h_{\varphi_0}^{\frac{q-1}{2q}}) = (\gamma_\varphi^*)_*(h_{\psi_0}),$$

that is,  $\psi \circ \gamma \circ \gamma_\varphi^* = \psi$ . The converse is clear from DPI. □

For  $\alpha \neq z$ , we still need 2-positivity of  $\gamma$  for the proof that (1) implies sufficiency. Using Theorem 1 and similar arguments as in its proof, we can prove equivalent conditions for (1) of the form

$$\gamma_*((y^* y)^z) = (y_0^* y_0)^z$$

for  $\alpha > 1$  and

$$\gamma^*((h_\psi^{\frac{\alpha}{2z}} h_\varphi^{\frac{1-\alpha}{z}} h_\psi^{\frac{\alpha}{2z}})^z) = (h_{\psi_0}^{\frac{\alpha}{2z}} h_{\varphi_0}^{\frac{1-\alpha}{z}} h_{\psi_0}^{\frac{\alpha}{2z}})^z$$

for  $\alpha \in (0, 1)$ ,  $z > \alpha$ ,

$$\gamma^*((h_\varphi^{\frac{1-\alpha}{2z}} h_\psi^{\frac{\alpha}{z}} h_\varphi^{\frac{1-\alpha}{2z}})^z) = (h_{\varphi_0}^{\frac{1-\alpha}{2z}} h_{\psi_0}^{\frac{\alpha}{z}} h_{\varphi_0}^{\frac{1-\alpha}{2z}})^z$$

for  $\alpha \in (0, 1)$ ,  $z > 1 - \alpha$  (all within the DPI bounds). This is related but not quite the same as the conditions by Zhang [3]. For example, if  $\psi \sim \varphi$  the equality in the last case becomes

$$\gamma_*((h_\varphi^{\frac{1-\alpha}{2z}} \bar{a}^{-1} h_\varphi^{\frac{1-\alpha}{2z}})^{\frac{z}{1-\alpha}}) = (h_{\varphi_0}^{\frac{1-\alpha}{2z}} \bar{a}_0^{-1} h_{\varphi_0}^{\frac{1-\alpha}{2z}})^{\frac{z}{1-\alpha}},$$

whereas the corresponding Zhang's condition is

$$\gamma_*((\bar{a}^{-\frac{1}{2}} h_\varphi^{\frac{1-\alpha}{z}} \bar{a}^{-\frac{1}{2}})^{\frac{z}{1-\alpha}}) = (\bar{a}_0^{-\frac{1}{2}} h_{\varphi_0}^{\frac{1-\alpha}{z}} \bar{a}_0^{-\frac{1}{2}})^{\frac{z}{1-\alpha}}.$$

## References

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