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Observables on synaptic algebras ☆

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Abstract

Synaptic algebras, introduced by D. Foulis, generalize different algebraic structures used so far as mathematical models of quantum mechanics: the traditional Hilbert space approach, order unit spaces, Jordan algebras, effect algebras, MV-algebras, orthomodular lattices. We study sharp and fuzzy observables on two special classes of synaptic algebras: on the so called generalized Hermitian algebras and on synaptic algebras which are Banach space duals. Relations between fuzzy and sharp observables on these two types of synaptic algebras are shown.

Keywords: Synaptic algebras; Observables; Smearings; Effect algebras; GH-algebras

1. Introduction

In the traditional Hilbert space approach to quantum mechanics, physical quantities (also called observables) are modelled by PV-measures, that is, measures on a measurable space with ranges in the orthomodular lattice of projection operators. It is now recognized that a more proper model is obtained by measures with ranges in the algebra of Hilbert space effects (self-adjoint operators between the zero and identity operator), or POV-measures. This approach also provides a frame to investigate imprecise measurements. The observables described by PV-measures are often called sharp.

The notion of fuzzy (or unsharp) observable has been formulated in the literature [29] as a smearing of a sharp observable (PV-measure) by means of a (weak) Markov kernel. This can be interpreted as a post-processing of the sharp observable by some classical methods. An unsharp observable is always a POV-measure but, unlike the classical case, not all observables can be obtained in this way. In fact, they form the subclass of observables consisting of all POV-measures with commuting ranges [1,30,32,34].

A special kind of smearings are the functions of PV-measures. It is well known that the PV-measures are in one-to-one correspondence with self-adjoint operators and there is a well-developed functional calculus for commuting

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self-adjoint operators. Functions of PV-measures are provided by functions of the corresponding self-adjoint operators and can be seen as smearings by deterministic Markov kernels (with values in $\{0, 1\}$).

Effect algebras and some of their special subclasses were proposed as abstract mathematical models of quantum mechanics, [13]. Some of the above results have been generalized in the literature for effect algebras or their subclasses, MV-algebras and orthomodular lattices [32,33,38,45]. As observables are usually defined as σ-homomorphisms from a σ-field of sets to a given algebraic structure, we need to restrict to σ-orthocomplete effect algebras. It also turns out that for a proper definition of smearings we need existence of σ-additive states. But in general, on an arbitrary σ-orthocomplete effect algebra, a smearing need not always exist.

In this paper, we study observables in the frame of synaptic algebras introduced by D. Foulis [12] as possible models for quantum mechanics. The aim was to build a mathematical description of quantum theory based on a few relatively simple and physically relevant axioms. Synaptic algebras put together several algebraic structures used so far as models of quantum systems: the traditional Hilbert space model, Jordan algebras, order unit spaces, effect algebras, orthomodular lattices. In this paper, we show how some results on observables and their smearings, obtained in the latter models, can be adopted and enhanced for synaptic algebras.

Sections 2 and 3 introduce the synaptic algebras and effect algebras and some of their properties needed later. In section 4, we describe observables and their smearings on σ-orthocomplete effect algebras. We prove that if the effect algebra is also convex and has an ordering set of σ-additive states, all smearings by (weak) Markov kernels are well defined and unique.

In section 5, we turn to observables on a synaptic algebra A. Such observables have ranges in the unit interval of A, which is a convex effect algebra. As before, we need this effect algebra to be σ-orthocomplete, which implies additional σ-property on A. Therefore we study sharp observables and observables with commuting ranges on a special kind of synaptic algebras, namely on GH-algebras (generalized Hermitian algebras), in which the set of projections is a σ-OML and every commutative sub-GH-algebra is monotone σ-complete. Using a version of Loomis-Sikorski theorem for commutative GH-algebras, it was proved in [15], that to every element of any GH-algebra there exists a unique sharp real observable. In this paper we show that also conversely, every sharp real observable on a GH-algebra A is determined by an element of A. Moreover, a functional calculus for several commuting sharp real observables is defined. It is also shown that every observable with commuting range is defined by a smearing of a special sharp observable, whose existence follows by the Loomis-Sikorski representation. If A has an ordering set of σ-normal states, then the converse is also true, so any smearing of a sharp observable is uniquely defined and has a commuting range. This result is similar to that for POV-measures with commuting ranges [32].

In the last section, we consider synaptic algebras which are duals of Banach spaces. By [4] and [43], this happens if and only if the synaptic algebra is isomorphic to a JW-algebra, hence a weakly closed Jordan algebra of Hilbert space operators [44]. Using the results of [43] and [11], we collect there some criteria under which a synaptic algebra is a Banach space dual, hence a JW-algebra. In this case, for every observable ξ and every weak Markov kernel ν there is a unique observable η defined by a smearing of ξ by ν. We also prove that an observable is a smearing of a sharp observable if and only if it has a commuting range. In the case that the Hilbert space is separable, the sharp observable may be chosen real, which is a property shared with the POV-measures on a separable Hilbert space.

2. Synaptic algebras

In this section, we review some basic notions and facts pertaining to synaptic algebras. For exact definitions, proofs and more details see [12,17,22]. Further results and examples can be found in [14,20,21,39], see also the review paper [40].

A prototype example of a synaptic algebra is the set $\mathcal{A}$ of all self-adjoint operators in the algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on the Hilbert space $\mathcal{H}$. In general, a synaptic algebra A is defined by a set of axioms proposed in [12]. We will not reproduce the whole definition, but rather list some of the properties that are important in the sequel.

First of all, a synaptic algebra A is an *order unit space* [2] embedded in an enveloping algebra R. The order unit will be denoted by 1 and it will be assumed that $1 \neq 0$. Recall that an order unit space carries a norm defined as $\|a\| := \inf\{0 < \lambda \in \mathbb{R} : -\lambda \leq a \leq \lambda\}$.

If $a, b \in A$, then the product ab, calculated in the enveloping algebra R, may or may not belong to A. However, if $ab = ba$, i.e., if a commutes with b (in symbols aCb), then $ab \in A$. In particular, A is closed under squaring, moreover, the positive elements in A are precisely those of the form $a = b^2$ for some $b \in A$, and every positive element has a unique positive square root. For $a \in A$, the *absolute value* of a is defined by $|a| := (a^2)^{1/2}$, the *positive* and *negative* parts of a are defined by $a^+ := \frac{1}{2}(|a| + a)$ and $a^- := \frac{1}{2}(|a| - a)$. Then $a = a^+ - a^-$, $|a| = a^+ + a^-$, $a^+a^- = 0 = a^-a^+$.

It also follows that A is a special *Jordan algebra* [36] under the Jordan product $a \circ b := \frac{1}{2}(ab + ba) = \frac{1}{2}((a + b)^2 - a^2 - b^2) \in A$ for $a, b \in A$. If $ab = 0$, then aCb and $ba = 0$. For $a \in A$, the set $C(a) := \{b \in A : bCa\}$ is the *commutant* of a in A. If $B \subseteq A$, then $C(B) := \bigcap_{b \in B} C(b)$ is the commutant of B in A, $CC(B) := C(C(B))$ is the *bicommutant* of B, and $CC(a) := CC(\{a\})$. For every $a \in A$, $C(a)$ is a norm-closed subset of A. In particular, for every $a \in A$, $|a|, a^+, a^- \in CC(a)$.

Elements of the set $P := \{p \in A : p = p^2\}$ are called *projections* and it is understood that P is partially ordered by the restriction of the ordering $\leq$ in A. It turns out that P is a lattice, moreover, $p \leq q$ iff $pq = qp = p$. The projections p and q are said to be *orthogonal*, in symbols $p \perp q$, iff $p \leq 1 - q$. The *orthosum* $p \oplus q$ is defined iff $p \perp q$, in which case $p \oplus q := p + q$. It turns out that $p \perp q \Leftrightarrow pCq$ with $pq = qp = 0$ and $p \oplus q = p + q = p \vee q \in P$. The lattice P, equipped with the orthocomplementation $p \mapsto p^\perp = 1 - p$, is an *orthomodular lattice* (OML) [5,35].

A subset B of a synaptic algebra A is *commutative* iff $ab = ba$ for all $a, b \in B$. A synaptic algebra A is commutative iff it is a vector lattice iff P is a Boolean algebra. If B is a commutative subset of A, then $CC(B)$ is a commutative sub-synaptic algebra of A.

There is a mapping ${}^o : A \to P$ such that $ab = 0 \Leftrightarrow a^o b = 0$. The element a^o is called the *carrier* of a. It turns out that $a^o \in CC(a)$ and a^o is the smallest projection $p \in P$ such that $a = ap$ (equivalently, $a = pa$). If $n \in \mathbb{N}$, then $(a^n)^o = a^o$. This mapping is closely related to the Rickart mapping, [18,19].

A *bounded resolution of identity* in A is a system $(p_\lambda)_{\lambda \in \mathbb{R}}$ of projections in A satisfying the following conditions for $\lambda, \lambda' \in \mathbb{R}$ (see [25, Definition 4.1] and [12, Theorem 8.4]):

(1) There exists $0 \leq K \in \mathbb{R}$ such that $p_\lambda = 0$ if $\lambda < -K$ and $p_\lambda = 1$ if $K \leq \lambda$.

(2) $p_\lambda \leq p_{\lambda'}$ if $\lambda \leq \lambda'$.

(3) $p_\lambda = \bigwedge_{\lambda' > \lambda} p_{\lambda'}$.

Notice that condition (2) implies that the projections in a bounded resolution of identity pairwise commute. Every $a \in A$ determines and is determined by a bounded resolution of identity, so called *spectral resolution* of a, $(p_{a,\lambda})_{\lambda \in \mathbb{R}}$ in $P \cap CC(a)$, where $p_{a,\lambda} := 1 - ((a - \lambda)^+))^o = (((a - \lambda)^+)^o)^\perp$ for $\lambda \in \mathbb{R}$. Put $L_a := \sup\{\lambda \in \mathbb{R} : \lambda \leq a\} = \sup\{\lambda \in \mathbb{R} : p_{a,\lambda} = 0\}$ and $U_a := \inf\{\lambda \in \mathbb{R} : a \leq \lambda\} = \inf\{\lambda \in \mathbb{R} : p_{a,\lambda} = 1\}$, we will also use the notation $spec(a) := [L_a, U_a]$. Then

$$a = \int_{L_a - 0}^{U_a} \lambda d p_{a,\lambda},$$

where the Riemann-Stieltjes type sums converge to the integral in norm. Two elements in A commute iff their respective spectral resolutions commute pairwise [12, §8]. In general it is not clear whether a bounded resolution of identity is the spectral resolution of some element in A, but by [25, Theorem 4.2] it is true for Banach (norm-complete) synaptic algebras.

A *morphism* of synaptic algebras (or a synaptic morphism) is a linear mapping $\phi : A_1 \to A_2$, where A_1, A_2 are synaptic algebras, with the following properties for all $a, b \in A_1$:

(1) $\phi(1) = 1$;
(2) $\phi(a^2) = \phi(a)^2$;
(3) $aCb \implies \phi(a)C\phi(b)$;
(4) $\phi(a^o) = \phi(a)^o$.

A *state* on the synaptic algebra A is defined just as it is for any ordered vector space with an order unit, namely as a linear functional $\rho : A \to \mathbb{R}$ that is positive ($a \in A^+ \Rightarrow \rho(a) \in \mathbb{R}^+$) and normalized ($\rho(1) = 1$), [16]. The state space of A is denoted by $\mathcal{S}(A)$. A set $S \subseteq \mathcal{S}(A)$ of states is *separating* iff $\rho(a) = 0$ for all $\rho \in S$ implies $a = 0$, and S is *ordering* if $\rho(a) \leq \rho(b) \, \forall \rho \in S$ implies $a \leq b$. Since A is an order unit space, the set of all states $\mathcal{S}(A)$ is ordering, [2].

A state ρ on A is a *normal state* iff, for every monotone increasing net (a_α), $0 \le a_\alpha \nearrow a \implies \rho(a_\alpha) \nearrow \rho(a)$, and σ*-normal* iff for every monotone increasing sequence (a_n) of positive elements, $a_n \nearrow a \implies \rho(a_n) \nearrow \rho(a)$.

We say that A is a *Banach synaptic algebra* if it is norm complete. It was proved in [18,19] that a Banach synaptic algebra is isomorphic to Rickart JC-algebra, that is, a norm-closed subspace of self-adjoint operators on a complex Hilbert space, closed under the Jordan product $a \circ b := \frac{1}{2}(ab + ba)$ [4], and with the Rickart property, [6].

3. Effect algebras

An *effect algebra* [13] is a set L with two distinguished elements 0, 1 and with a partial binary operation $\oplus : L \to L$ such that for all $a, b, c \in L$ the following holds:

(EA1) if $a \oplus b$ is defined then $b \oplus a$ is defined and $a \oplus b = b \oplus a$ (commutativity);
(EA2) if $b \oplus c$ and $a \oplus (b \oplus c)$ are defined then $a \oplus b$ and $(a \oplus b) \oplus c$ are defined and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ (associativity);
(EA3) for every $a \in L$ there is a unique $a' \in L$ such that $a \oplus a' = 1$ (orthosupplementation);
(EA4) if $1 \oplus a$ is defined then $a = 0$ (zero-one law).

We will write $L = (L, \oplus, 0, 1)$ for effect algebra. Elements $a, b \in L$ are *orthogonal* (written $a \perp b$) iff $a \oplus b$ is defined in L. In what follows, we often write $a \oplus b$ tacitly assuming that $a \perp b$. A partial ordering is defined on L as follows: $a \le b$ iff there is $c \in L$ such that $a \oplus c = b$. The element c is uniquely defined, and we write $c = b \ominus a$. It is easy to check that $a \perp b$ iff $a \le b'$.

The operation $\oplus$ can be extended to finite number of elements by recurrence in an obvious way. Owing to (EA2) we may omit parentheses in the expressions of the form $a_1 \oplus a_2 \oplus \cdots \oplus a_n$.

A family $\{a_i : i \in I\}$, where I is an arbitrary set, is called *orthogonal* iff every finite subfamily of it admits an $\oplus$-sum (or *orthosum*) in L. If the element $a = \bigvee_{F \subseteq I} \oplus_{i \in F} a_i$ exists in L, where the supremum is taken over all finite subsets F of I, then a is called the *orthosum* of the orthogonal family $\{a_i : i \in I\}$, and is denoted by $a := \bigoplus_{i \in I} a_i$.

An effect algebra L is called *orthocomplete* iff the orthosum exists for any orthogonal family of its elements, and L is called σ*-orthocomplete* iff the orthosum exists for every countable orthogonal family of its elements. By standard arguments, L is σ-orthocomplete iff it is monotone σ-complete, i.e., every ascending (descending) sequence has a supremum (infimum) in L.

Two elements a, b of an effect algebra L are called compatible if there are elements a_1, b_1 and c such that $a = a_1 + c$, $b = b_1 + c$, and $a_1 + b_1 + c \in L$. An effect algebra that forms a lattice and in which every two elements a, b are compatible (equivalently, $(a \vee b) \ominus a = b \ominus (a \wedge b)$ for all $a, b \in L$) is called an *MV-effect algebra*. As a lattice, an MV-effect algebra is distributive. MV-effect algebras are closely related to MV-algebras introduced by Chang [9]. Every MV-effect algebra can be organized into an MV-algebra, and reciprocally, an MV-algebra can be organized into an MV-effect algebra (see e.g. [10]). By a result of Mundici [37], there is a categorical equivalence between MV-algebras and lattice ordered groups.

A mapping $s : L \to [0, 1]$ from L to the interval $[0, 1]$ of real numbers is a *state* on L if (i) $s(1) = 1$; (ii) $s(a \oplus b) = s(a) + s(b)$ whenever $a \oplus b$ exists in L. A state s is said to be σ*-additive* or *completely additive* iff $s(\bigoplus_{i \in I} a_i) = \sum_{i \in I} s(a_i)$ holds for any countable or arbitrary index set I such that $\bigoplus_{i \in I} a_i$ exists in L.

A nonempty set S of states on L is *ordering* iff, for $a, b \in L$, $a \le b$ iff $s(a) \le s(b)$ for all $s \in S$: S is *separating* iff $s(a) = s(b)$ for all $s \in S$ implies $a = b$.

A mapping $\phi : L_1 \to L_2$, where L_1, L_2 are effect algebras, is a *morphism* if (i) $\phi(1) = 1$, (ii) $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$ whenever $a \oplus b$ exists in L_1. A morphism ϕ is a σ*-morphism* (*complete morphism*) iff it preserves all existing countable (arbitrary) $\oplus$-sums. A bijective morphism such that $a \perp b$ iff $\phi(a) \perp \phi(b)$ is an *isomorphism*. A σ-isomorphism, resp. complete isomorphism is defined in an obvious way.

A subset M of an effect algebra L is a *sub-effect algebra* iff (i) $0 \in M$; (ii) $a, b \in M$, $a \perp b$ implies $a \oplus b \in M$; (iii) $a \in M$ implies $a' \in M$.

A prototype example of an effect algebra is the *effect algebra* $\mathcal{E}(\mathcal{H})$ *of Hilbert space effects*, that is the operators on a complex Hilbert space between 0 and I. More generally, if A is a synaptic algebra, the elements of the unit interval

$$E := A[0, 1] = \{e \in A : 0 \le e \le 1\}$$

are called *effects*, and E is an effect algebra. Note that E has a convex structure inherited from A, so that it is a *convex effect algebra*. Convex effect algebras were studied in [26,27,7] and it was proved that any such effect algebra is affinely isomorphic to a generating interval $V[0,u]$ in an ordered vector space (V, V^+), with $u \in V^+$.

The set P is a subset of the convex set E, in fact, P is the extreme boundary of E ([12, Theorem 2.6]). Evidently, $0, 1 \in P$ and $0 \le p \le 1$ for all $p \in P$. If $p, q \in P$, then they are compatible in E iff they are compatible in P (as a sub-effect algebra) iff pCq [12].

Notice that a convex effect algebra is an MV-effect algebra iff it is lattice ordered [27]. It follows that the synaptic algebra A is commutative iff the unit interval E is an MV-effect algebra.

Let $\mathcal{S}(E)$ be the state space of the convex effect algebra $E \subseteq A$. Recall that there is an affine bijection $\rho \leftrightarrow s$ between states $\rho \in \mathcal{S}(A)$ and states $s \in \mathcal{S}(E)$ via extension and restriction. As we have seen, A and hence also E possesses an ordering set of states.

4. Observables and their smearings

The notion of a quantum-mechanical observable can be easily extended to a more general effect algebra L, provided that L is σ-orthocomplete. This will be assumed throughout this section.

Definition 4.1. Let $(X, \mathcal{A})$ be a measurable space. By an $(X, \mathcal{A})$-*observable* on L we mean a mapping $\xi : \mathcal{A} \to L$ such that

(i) $\xi(X) = 1$;
(ii) the system $(\xi(A_i))_{i\in\mathbb{N}}$ is orthogonal and $\xi(\cup_{i=1}^\infty A_i) = \bigoplus_{i=1}^\infty \xi(A_i)$ whenever $A_i \in \mathcal{A}$ for $i \ge 1$ and $A_i \cap A_j = \emptyset$, $i \ne j$.

If $(X, \mathcal{B}) \equiv (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, then $\xi : \mathcal{B}(\mathbb{R}) \to L$ is a *real* observable.

Let $\mathcal{R}(\xi) := \{\xi(A) : A \in \mathcal{A}\}$ denote the range of the observable ξ. Then ξ is called *sharp* if $\mathcal{R}(\xi)$ consists of sharp elements (recall that an element $a \in L$ is sharp if $a \wedge a' = 0$). For example, in the effect algebra $\mathcal{E}(H)$ of Hilbert space effects (and also for more general synaptic algebras), sharp elements are precisely the projections, so that sharp observables on $\mathcal{E}(\mathcal{H})$ coincide with PV-measures.

Let $\mathcal{S}_\sigma(L)$ denote the set of σ-additive states on L and let $M_1^+(X, \mathcal{A})$ denote the set of all probability measures on $(X, \mathcal{A})$. Let ξ be and $(X, \mathcal{A})$-observable on L, then $m \circ \xi \in M_1^+(X, \mathcal{A})$ for all $m \in \mathcal{S}_\sigma(L)$. The map

$$\Phi_\xi : \mathcal{S}_\sigma(L) \to M_1^+(X, \mathcal{A}), \qquad m \mapsto m \circ \xi$$

is called the *probability distribution* of ξ. Note that in the case that $\mathcal{S}_\sigma(L)$ is ordering for L, any observable ξ on L is characterized by Φ_ξ. If ξ is a real observable, we denote by

$$m(\xi) := \int_{\mathbb{R}} t(m \circ \xi)(dt)$$

the *expectation* of ξ in $m \in \mathcal{S}_\sigma(L)$ whenever the right-hand side of the above equation exists and is finite.

Let $(X_1, \mathcal{A}_1)$ be another measurable space and let $f : (X, \mathcal{A}) \to (X_1, \mathcal{A}_1)$ be a measurable function. If $\xi : \mathcal{A} \to L$ is an observable, then

$$f(\xi) : A \mapsto \xi(f^{-1}(A)), \qquad A \in \mathcal{A}_1$$

is an $(X_1, \mathcal{A}_1)$-observable on L, which is called the *f-function* of ξ. If $(X_1, \mathcal{A}_1) = (\mathbb{R}, \mathcal{B}(\mathcal{R}))$, then $f(\xi)$ is a real observable and we obtain using the integral transformation theorem that

$$\begin{aligned} m(f(\xi)) &= \int_{\mathbb{R}} u(m \circ f(\xi))(du) \\ &= \int_{\mathbb{R}} um(\xi(f^{-1}(du))) = \int_X f(t)(m \circ \xi)(dt) \end{aligned}$$

for any $m \in \mathcal{S}_\sigma(L)$.

We will consider more general transformations of observables obtained from transformations of probability measures in $M_1^+(X, \mathcal{A})$ by (weak) Markov kernels as defined below.

Let $(X, \mathcal{A})$ and $(X_1, \mathcal{A}_1)$ be measurable spaces. A *Markov kernel* is a mapping $\nu : X \times \mathcal{A}_1 \to [0, 1]$ such that the following properties are satisfied:

(i) for any fixed $x \in X$, $\nu_x(.) := \nu(x, .) : \mathcal{A}_1 \to [0, 1]$ is a probability measure;
(ii) for any fixed $\Delta \in \mathcal{A}_1$, the mapping $x \mapsto \nu_\Delta(x) := \nu(x, \Delta)$ is $\mathcal{A}$-measurable.

Any measurable function $f : (X, \mathcal{A}) \to (X_1, \mathcal{A}_1)$ defines a Markov kernel ν^f with values in $\{0, 1\}$, given as

$$\nu^f(x, \Delta) := \chi_{f^{-1}(\Delta)}(x).$$

The notion of Markov kernel has been weakened in [32] to so-called *weak Markov kernel* as follows. Let $\mathcal{P} \subseteq M_1^+(X, \mathcal{A})$ be any subset. We say that ν is a *weak Markov kernel with respect to* $\mathcal{P}$ if

(i) $x \mapsto \nu(x, \Delta)$ is $\mathcal{A}$-measurable for all $\Delta \in \mathcal{A}_1$;
(ii) for every $\Delta \in \mathcal{A}_1$, $0 \le \nu(x, \Delta) \le 1$ $\mathcal{P}$-a.e.;
(iii) $\nu(x, X_1) = 1$ $\mathcal{P}$-a.e., and $\nu(x, \emptyset) = 0$ $\mathcal{P}$-a.e.;
(iv) if $(\Delta_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{A}_1$ such that $\Delta_n \cap \Delta_m = \emptyset$, $n \neq m$, then

$$\nu(x, \bigcup \Delta_n) = \sum_n \nu(x, \Delta_n),\ \mathcal{P} - a.e.$$

Note that ν defines a map $\mathcal{P} \to M_1^+(X_1, \mathcal{A}_1)$, given as

$$\nu(P)(\Delta) := \int_X \nu(x, \Delta) P(dx),\ \Delta \in \mathcal{A}_1, \quad P \in \mathcal{P}. \tag{1}$$

It is easy to see that a weak Markov kernel with respect to $M_1^+(X, \mathcal{A})$ is a Markov kernel.

If $\nu, \mu : X \times \mathcal{A}_1 \to \mathbb{R}$ are weak Markov kernels with respect to $\mathcal{P}$, then we say that $\nu \sim_{\mathcal{P}} \mu$ if for all $\Delta \in \mathcal{A}_1$, $P(\{x : \nu(x, \Delta) \neq \mu(x, \Delta)\}) = 0$ for all $P \in \mathcal{P}$. Clearly, $\sim_{\mathcal{P}}$ is an equivalence relation.

Let now $\xi : (X, \mathcal{A}) \to L$ be an observable. In general, to define a smearing of ξ by a (weak) Markov kernel, we will need to assume that L has at least some σ-additive states, so $\mathcal{S}_\sigma(L) \neq \emptyset$. In this case, we will say that $X \times \mathcal{A}_1 \to \mathbb{R}$ is a weak Markov kernel with respect to ξ if it is a weak Markov kernel with respect to the range of the probability distribution Φ_ξ.

Definition 4.2. Let L be a σ-orthocomplete effect algebra with a nonempty set of σ-additive states $\mathcal{S}_\sigma(L)$. Let ξ be a $(X, \mathcal{A})$-observable on L. Let $(X_1, \mathcal{A}_1)$ be a measurable space and let $\nu : X \times \mathcal{A}_1 \to \mathbb{R}$ be a weak Markov kernel with respect to ξ. An $(X_1, \mathcal{A}_1)$-observable η is called a *smearing* (or a *fuzzy version*) of ξ (with respect to ν) if $\Phi_\eta = \nu \circ \Phi_\xi$, that is,

$$(m \circ \eta)(\Delta) = m(\eta(\Delta)) = \int_X \nu(x, \Delta)(m \circ \xi)(dx), \quad \forall \Delta \in \mathcal{A}_1,\ m \in \mathcal{S}_\sigma(L).$$

It is clear that if μ and ν are equivalent weak Markov kernels with respect to ξ, then the smearings with respect to μ are the same as those with respect to ν. Note also that the function $f(\xi)$ is a smearing of ξ with respect to the Markov kernel ν^f. The difference here is that while the function is uniquely defined for any observable on a σ-complete effect algebra, the definition of a smearing depends on existence of σ-additive states. In particular, a smearing of an observable is not defined if $\mathcal{S}_\sigma(L) = \emptyset$ and need not be unique if the set $\mathcal{S}_\sigma(L)$ is small. So in the latter case, there might be several smearings of ξ by the Markov kernel ν^f but a unique f-function of ξ. On the other hand, even if $\mathcal{S}_\sigma(L)$ is ordering, a smearing by a general weak Markov kernel might not exist at all. The next result shows that the situation is different if L is also convex.

Theorem 4.3. *Let L be a convex σ-orthocomplete effect algebra with an ordering set $\mathcal{S}_\sigma(L)$ of σ-additive states and let ξ be an $(X, \mathcal{A})$-observable on L. Let $(X_1, \mathcal{A}_1)$ be a measurable space and let $\nu : X \times \mathcal{A}_1 \to \mathbb{R}$ be a weak Markov kernel with respect to ξ. Then there is a unique smearing of ξ with respect to ν.*

Proof. We first define integrals with respect to ξ, in the following sense. Let $f : X \to [0, 1]$ be $\mathcal{A}$-measurable. We will show that there is an element $\xi(f) \in L$ such that for all $m \in \mathcal{S}_\sigma(L)$, we have

$$m(\xi(f)) = \int_X f(x)(m \circ \xi)(dx).$$

Since $\mathcal{S}_\sigma(L)$ is ordering, it is clear that such an element must be unique. First, let $f = \chi_\Delta$ for $\Delta \in A$, in this case, we put $\xi(f) := \xi(\Delta)$. Next, let $f = \sum_i c_i \Delta_i$ be a simple function, then by standard arguments, we may suppose that Δ_i are pairwise disjoint and $c_i \in [0, 1]$. Put $\xi(f) := \sum_i c_i \xi(\Delta_i)$. Since L is convex and $\oplus_i \xi(\Delta_i)$ exists in L, we see that $\xi(f) \in L$, moreover, for $m \in \mathcal{S}_\sigma(L)$,

$$m(\xi(f)) = \sum_i c_i m(\xi(\Delta_i)) = \int_X f(x)(m \circ \xi)(dx).$$

If $f : X \to [0, 1]$ is a measurable function, then there is an increasing sequence of simple functions $f_n : X \to [0, 1]$ converging pointwise to f. Since $\mathcal{S}_\sigma(L)$ is ordering and we have

$$m(\xi(f_n)) = \int f_n(x)(m \circ \xi)(dx) \leq \int f_{n+1}(x)(m \circ \xi)(dx) = m(\xi(f_{n+1})), \qquad \forall m \in \mathcal{S}_\sigma(L),$$

it follows that $\xi(f_n) \leq \xi(f_{n+1})$. Since L is σ-orthocomplete, there is some element $\xi(f) \in L$ such that $\vee_n \xi(f_n) = \xi(f)$. Using Lebesgue monotone convergence theorem, we have for $m \in \mathcal{S}_\sigma(L)$,

$$m(\xi(f)) = \bigvee_n m(\xi(f_n)) = \lim_n m(\xi(f_n)) = \int_X f(x)(m \circ \xi)(dx).$$

By uniqueness, it is clear that $\xi(f)$ does not depend on the choice of the sequence f_n. Note also that if $g : X \to [0, 1]$ is such that $\xi(\{x \in X, f(x) \neq g(x)\}) = 0$, then $\xi(f) = \xi(g)$.

We now define $\eta(\Delta) = \xi(\nu_\Delta)$, $\Delta \in \mathcal{A}_1$, where $\nu_\Delta = \nu(\cdot, \Delta)$. We now show that η is an observable, it is then clear by definition that η must be the unique smearing of ξ with respect to ν. So let $\{\Delta_i\}$ be such that $\Delta_i \cap \Delta_j = \emptyset$ for $i \neq j$ and let $\Delta = \cup_i \Delta_i$. Then by the definition of weak Markov kernel, $\nu_\Delta = \sum_i \nu_{\Delta_i}$ up to some set Δ_0 with $(m \circ \xi)(\Delta_0) = 0$ for all $m \in \mathcal{S}_\sigma(L)$. Hence $\xi(\Delta_0) = 0$ and

$$\eta(\Delta) = \xi(\nu_\Delta) = \xi(\sum_i \nu_{\Delta_i}) = \sum_i \eta(\Delta_i),$$

the last equality holds because

$$m(\xi(\sum_i \nu_{\Delta_i})) = \int_X \sum_i \nu_{\Delta_i}(x)(m \circ \xi)(dx) = \sum_i \int_X \nu_{\Delta_i}(x)(m \circ \xi)(dx) = \sum_i m(\xi(\nu_{\Delta_i})).$$

The facts that $\eta(\emptyset) = 0$ and $\eta(X_1) = 1$ are proved similarly. □

Remark 4.4. Note that the element $\xi(f)$ defined in the above proof is such that for each σ-additive state m, $m(\xi(f))$ is the expectation of the observable $f(\xi)$ in m.

5. Observables on GH-algebras

For a measurable space $(X, \mathcal{A})$, an $(X, \mathcal{A})$-observable on a synaptic algebra A is defined as an observable on the effect algebra $E = A[0, 1]$ or, in the case of sharp observables, on the OML of projections P of A. Since observables are usually studied on σ-orthocomplete effect algebras resp. σ-complete OMLs, we will need to assume that E, or at least P, is σ-orthocomplete.

It can be easily seen that the effect algebra E of a synaptic algebra A is σ-orthocomplete iff A is monotone σ-complete (i.e., every ascending above bounded sequence has a supremum). Indeed, assume that E is σ-orthocomplete (and hence monotone σ-complete), and let $(a_n)_n$ be an ascending sequence of elements in A bounded above by an element $b \in A$. Then

$$0 \leq \frac{b - a_n}{\|b - a_1\|} \leq 1,$$

and

$$(\frac{b - a_n}{\|b - a_1\|})_n$$

is descending, so it has an infimum in E, hence $(a_n)_n$ has a supremum in A. The converse is clear.

This leads to a quite strong restriction on the synaptic algebra A: by [28, Proposition 3.9], every monotone σ-complete synaptic algebra is a Banach synaptic algebra, hence isomorphic to a subspace of $B(\mathcal{H})$ for a Hilbert space $\mathcal{H}$.

We also need to consider the set of σ-additive states $\mathcal{S}_\sigma(E)$. It is easy to see that a state on E is σ-additive iff its (unique) extension to A is σ-normal. So the assumption that E is σ-orthocomplete and $\mathcal{S}_\sigma(E)$ is ordering leads to a monotone σ-complete synaptic algebra A with an ordering set of σ-normal states.

In the present section, we will restrict our attention to observables with commutative range $\mathcal{R}$. Since the commutative subset $\mathcal{R}$ is contained in a commutative subalgebra $CC(\mathcal{R}) \subseteq A$, it is enough to assume that all commutative subalgebras in A are monotone σ-complete. This leads to a special kind of a synaptic algebra called a *generalized Hermitian (GH-) algebra*, which was introduced and studied in [23,24]. The following characterization was found in [18, Theorem 9.1].

Theorem 5.1. *A GH-algebra is the same thing as a synaptic algebra A such that every bounded monotone increasing sequence $a_1 \leq a_2 \leq \cdots$ of pairwise commuting elements in A has a supremum in A.*

Clearly, if A is monotone σ-complete synaptic algebra, then it is a GH-algebra, and a commutative synaptic algebra is a GH-algebra if and only if it is monotone σ-complete. By [23, Lemma 5.4], if A is a GH-algebra, then P is a σ-complete OML, so we may consider sharp observables on a GH-algebra. Moreover, for any synaptic algebra A, if $T \subseteq A$ and T has a supremum b in A, then $b \in CC(T)$ [18, Theorem 6.2]. Applying this to a commutative subset T of a GH-algebra A, we obtain that the commutative subalgebra $CC(T)$ is monotone σ-complete (this, in particular, is true for the subalgebra $CC(a)$ for any element $a \in A$). Therefore, we may consider observables with a commutative range on a GH-algebra A and for this, we may assume that A is commutative.

An important example of a commutative GH-algebra is obtained as follows. Let X be a basically disconnected compact topological space and let $A = C(X, \mathbb{R})$ be the set of all continuous functions $f : X \to \mathbb{R}$. Then A is a commutative GH-algebra and the corresponding set of projections is the σ-complete Boolean algebra $P = P(X, \mathbb{R})$ of all characteristic functions of clopen subsets of X. The following representation theorem, proved in [16, Theorem 6.6], states that any commutative GH-algebra is isomorphic to some GH-algebra of this form. Recall that a *morphism of GH-algebras* $\phi : A_1 \to A_2$ is defined as a synaptic morphism with the additional property that given a sequence of pairwise commuting elements $(a_n)_n$ such that $a_n \nearrow a$ in A_1, then $\phi(a_n) \nearrow \phi(a)$ in A_2.

Theorem 5.2. *Suppose A is a commutative GH-algebra and let X be the basically disconnected Stone space of the σ-complete Boolean algebra P. Then there is an isomorphism of GH-algebras $\Psi : A \to C(X, \mathbb{R})$ of A onto $C(X, \mathbb{R})$, such that the restriction ψ of Ψ to P is a boolean isomorphism of P onto $P(X, \mathbb{R})$ as per Stone's representation theorem.*

The representation theorem can be used to define a functional calculus for continuous functions on GH-algebras. Let $a \in A$ and let $f \in C(spec(a), \mathbb{R})$ (recall the spectral resolution of a in Sec. 2). Let Ψ be the isomorphism of Theorem 5.2 for the commutative GH-algebra $CC(a)$ and let $g := \Psi(a) \in C(X, \mathbb{R})$. Then $spec(a) = \{g(x) : x \in X\}$, $f \circ g \in C(X, \mathbb{R})$, and we define the element $f(a) \in CC(a)$ by $f(a) := \Psi^{-1}(f \circ g)$. We have $f(a) = \int_{L_a - 0}^{U_a} f(\lambda) dp_{a,\lambda}$, $f \in C(spec(a), \mathbb{R})$, [15, Theorem 7.7].

Notice that countable suprema in $C(X,\mathbb{R})$ are not the pointwise suprema of functions. This can be improved by the following version of the Loomis-Sikorski theorem [15, Theorem 6.6], which is an extension of the Loomis-Sikorski theorem for σ-MV algebras (cf. [10]). For the definition of a gh-tribe see [15]. In short, a gh-tribe $\mathcal{T}$ is a commutative GH-algebra consisting of bounded real-valued functions on a nonempty set X with pointwise ordering and supremum norm. The characteristic functions in $\mathcal{T}$ are identified with the corresponding subsets of X and these form a σ-field denoted by $\mathcal{B}(\mathcal{T})$. By [8], every function $f \in \mathcal{T}$ is $\mathcal{B}(\mathcal{T})$-measurable. Moreover, for every σ-normal state m on $\mathcal{T}$, we have $m(f) = \int_X f(t)P(dt)$, where $P := m/\mathcal{B}(\mathcal{T})$ is a probability measure on $\mathcal{B}(\mathcal{T})$.

Theorem 5.3 *(Loomis-Sikorski theorem). Let A be a commutative GH-algebra and let X be the basically disconnected Stone space for the σ-complete Boolean algebra P of projections in A. Then there exists a gh-tribe $\mathcal{T}$ on X such that $C(X,\mathbb{R}) \subseteq \mathcal{T}$ and there exists a surjective morphism h of GH-algebras from $\mathcal{T}$ onto A. The triple $(X,\mathcal{T},h)$ is called the Loomis-Sikorski representation of A.*

Using Theorem 5.3, it was shown that each element a in a GH-algebra A corresponds to a sharp real observable ξ_a on the σ-OML P of projections on A [15, Theorem 7.4]. Let $(X,\mathcal{T},h)$ be the Loomis-Sikorski representation of $CC(a)$ by Theorem 5.3 and let f_a be a function in $\mathcal{T}$ such that $h(f_a) = a$. We define the observable ξ_a by

$$\xi_a(B) = h(f_a^{-1}(B)), \quad B \in \mathcal{B}(\mathbb{R}).$$

This definition is independent on the choice of the function f_a, and is the unique real observable on P such that $\xi_a((-\infty,\lambda]) = p_{a,\lambda}$ for all $\lambda \in \mathbb{R}$, where $\{p_{a,\lambda}\}$ is the spectral resolution of a. Since every element in A determines and is uniquely determined by its spectral resolution, the observable ξ_a is uniquely determined by a, and also determines a. The observable ξ_a is bounded, in the sense that there is some $K \geq 0$ such that $\xi_a((-K,K)) = 1$.

Since elements a and b in A commute iff their respective spectral resolutions pairwise commute, we obtain that aCb iff the ranges of ξ_a and ξ_b pairwise commute, i.e., iff ξ_a and ξ_b are compatible observables on the OML P (cf. [45]). Moreover, for every σ-normal state m on A,

$$m(a) = m(\xi_a) = \int\limits_{\mathbb{R}} \lambda(m \circ \xi_a)(d\lambda), \ a \in A.$$

The following result shows a one-to-one correspondence between bounded sharp real observables and elements of a GH-algebra.

Theorem 5.4. *For any bounded sharp real observable ξ on a GH-algebra A, there is a unique element $a \in A$ such that $\xi = \xi_a$.*

Proof. Define $p_\lambda = \xi((-\infty,\lambda])$, $\lambda \in \mathbb{R}$. Then $\{p_\lambda\}$ is a bounded resolution of identity. Indeed, we have

$$p_\lambda = \bigwedge_{\lambda'>\lambda, \lambda' \in \mathbb{Q}} p_{\lambda'} \geq \bigwedge_{\lambda'>\lambda} p_{\lambda'} \geq p_\lambda$$

where the equality follows by σ-additivity of ξ. The other two properties of a bounded resolution of identity are clear. Since all p_λ commute, they are contained in a commutative GH-subalgebra B in A. Since B is monotone σ-complete, it is a Banach GH-algebra. By [25, Theorem 4.2], $\{p_\lambda\}$ is the spectral resolution of an element $a \in B$. It follows that $\xi_a((-\infty,\lambda]) = p_\lambda = \xi((-\infty,\lambda])$, hence $\xi = \xi_a$. □

The functional calculus on GH-algebras can be extended to bounded Borel measurable functions as follows. Let $a \in A$ and let f be a real-valued bounded Borel function defined on $spec(a)$. Let $\Psi : CC(a) \to C(X,\mathbb{R})$ be the isomorphism from Theorem 5.2 and let $(\mathcal{T}, X, h)$ be the Loomis-Sikorski representation of $CC(a)$ of Theorem 5.3. By the proof of [15, Theorem 6.6], $\mathcal{T}$ is the gh-tribe on X generated by $C(X,\mathbb{R})$, and by [10, Proposition 7.1.11] and [10, Proposition 7.1.25], $\mathcal{T}$ coincides with the set of bounded Baire measurable functions on X. Let $g = \Psi(a) \in C(X,\mathbb{R})$, then clearly $f \circ g$ belongs to $\mathcal{T}$ and we may put $f(a) = h(f \circ g) \in CC(a)$. Notice also that instead of g we may use any function $f_a \in \mathcal{T}$ such that $h(f_a) = a$. Indeed, we have $h(f_a) = h(g)$ iff the set $\{x \in X : f_a(x) \neq g(x)\}$ is meager. But then $\{x \in X : f \circ f_a(x) \neq f \circ g(x)\}$ is meager too, and so $h(f \circ g) = h(f \circ f_a)$. For the corresponding observables, we have

$$\xi_{f(a)}(B) = h((f \circ f_a)^{-1}(B)) = h(f_a^{-1}(f^{-1}(B)) = \xi_a(f^{-1}(B)), \qquad B \in \mathcal{B}(\mathbb{R}),$$

so that $\xi_{f(a)} = f(\xi_a)$.

Following [45], we can also define functions of several commuting elements. Let $a_1, a_2, \ldots, a_n \in A$ be pairwise commuting, and let $\xi_1, \xi_2, \ldots, \xi_n$ be their corresponding sharp real observables. Then these observables are compatible observables on the OML P. Let A_1 be the smallest commutative sub-synaptic algebra containing all $a_1, a_2, \ldots, a_n$. Then the ranges of $\xi_i, i = 1, 2, \ldots, n$ are contained in $P_1 := A_1 \cap P$, which is the smallest Boolean subalgebra of the OML P containing them. Let $(X, \mathcal{T}, h)$ be the Loomis-Sikorski representation of A_1 according Theorem 5.3. Then h maps $\mathcal{B}(\mathcal{T})$ onto P_1. We may consider observables $\xi_i, i = 1, 2, \ldots, n$ as observables from $\mathcal{B}(\mathbb{R})$ to P_1. Let $f_i, i = 1, 2, \ldots, n$ be functions in $\mathcal{T}$ such that $h(f_i) = a_i$, so that $\xi_i(B) = h(f_i^{-1}(B)), i = 1, 2, \ldots, n$, for all $B \in \mathcal{B}(\mathbb{R})$.

We will follow the construction in the proof of [45, Theorem 1.6 (ii)]. Define $F : X \to \mathbb{R}^n$ by $F(x) = (f_1(x), f_2(x), \ldots, f_n(x))$. Then F is $\mathcal{B}(\mathcal{T})$-measurable. Let $u := h \circ F^{-1}$. Then $u : \mathcal{B}(\mathbb{R}^n) \to P$ is a σ-morphism such that $\xi_i(B) = u(\pi_i^{-1}(B)), i = 1, 2, \ldots, n$, for all $B \in \mathcal{B}(\mathbb{R})$. Here $\pi_i(t_1, t_2, \ldots, t_n) \to t_i$ is a projection of $\mathbb{R}^n \to \mathbb{R}^1$. Since $\mathcal{B}(\mathbb{R}^n)$ is the smallest σ-algebra of subsets of $\mathbb{R}^n$ containing all $\pi_i^{-1}(B), i = 1, 2, \ldots, n$, the range of u is in P_1. The uniqueness of u is obvious. For any Borel function $G : \mathbb{R}^n \to \mathbb{R}$, the mapping $u \circ G^{-1}$ is an observable on P whose range is contained in P_1. We then define the function G of the observables $\xi_1, \xi_2, \ldots, \xi_n$ as the observable $u \circ G^{-1}$, i.e., $G(\xi_1, \xi_2, \ldots, \xi_n) := u \circ G^{-1}$. We have the following.

Theorem 5.5. *Let a, b be commuting elements in a GH-algebra A. The following statements hold (cf. also [41,42]):*

(i) *The observable ξ_{a+b} is the G-function of the observables ξ_a and ξ_b, where $G(t_1, t_2) = t_1 + t_2$.*
(ii) *The observable ξ_{ab} is the G-function of the observables ξ_a and ξ_b, where $G(t_1, t_2) = t_1.t_2$.*

Proof. (i) We have $\xi_a = h \circ f_a^{-1}$, where $f_a \in \mathcal{T}$ is any function such that $h(f_a) = a$. Since h is a morphism of GH-algebras, we have $h(f_a + f_b) = h(f_a) + h(f_b) = a + b = h(f_{a+b})$. For every $B \in \mathcal{B}(\mathbb{R})$ we have,

$$\begin{aligned} G(\xi_a, \xi_b)(B) &= u \circ G^{-1}(B) = h \circ F^{-1}(G^{-1}(B)) \\ &= h((G \circ F)^{-1}(B)) \\ &= h((f_a + f_b)^{-1}(B)) \\ &= \xi_{a+b}(B). \end{aligned}$$

The proof of (ii) is similar. $\square$

Let us now turn to general observables with a commuting range. We first observe that the Loomis-Sikorski theorem provides a special sharp observable for any commutative GH-algebra A. Let $(X, \mathcal{T}, h)$ be the Loomis-Sikorski representation of A, then it is easy to see that the restriction of h to $\mathcal{B}(\mathcal{T})$ is a sharp $(X, \mathcal{B}(\mathcal{T}))$-observable on A. By the proof of the next theorem, every observable with range in A is a smearing of h.

Theorem 5.6. *Let A be a GH-algebra with a nonempty set of σ-normal states. Then every observable with commuting range on A is a smearing of a sharp observable.*

Proof. This proof is analogous to that of [33, Theorem 4.4]. Let ξ be an $(\Omega, \mathcal{B})$-observable on a GH-algebra such that the range $\mathcal{R}(\xi) = \{\xi(B) : B \in \mathcal{B}\}$ consists of pairwise commuting elements. Then $CC(\mathcal{R}(\xi))$ is a commutative GH-algebra. Let $(X, \mathcal{T}, h)$ be the Loomis-Sikorski representation. By Theorem 5.3, for every $B \in \mathcal{B}$ there is an $f_B \in \mathcal{T}[0, 1]$ with $h(f_B) = \xi(B)$, where f_B is $\mathcal{B}(\mathcal{T})$-measurable and is unique up to h-null sets. Define $\nu : X \times \mathcal{B} \to [0, 1]$ by $\nu(x, B) = f_B(x)$. It can be proved that $\nu(X, B)$ is a weak Markov kernel with respect to h. Indeed, let $(B_i)_i$ be a disjoint sequence of elements of $\mathcal{B}$, and put $B = \bigcup_i B_i$. Then

$$h(f_B) = \xi(B) = \oplus_i \xi(B_i) = \sum_i h(f_{B_i}) = h(\sum_i f_{B_i})$$

and hence $\nu(x, B) = f_B(x) = \sum_i f_{B_i}(x) = \sum_i \nu(x, B_i)$, $m \circ h$-a.e. for any σ-normal state m. This proves property (iv) in the definition of a weak Markov kernel, the remaining properties are obvious.

Owing to [8] we have for every σ-normal state m

$$m(\xi(B)) = m(h(f_B)) = \int_X f_B(x)(m \circ h)(dx) = \int_X \nu(x, B)(m \circ \xi)(dx).$$

By Definition 4.2, the observable ξ is a smearing of the $(X, \mathcal{B}(\mathcal{T}))$-observable h. □

Note that for $a \in A$, the corresponding sharp real observable ξ_a is the function $f_a(h)$ of the observable h and hence a smearing of h by the Markov kernel $\nu(x, B) = \chi_{f_a^{-1}(B)}(x)$. By Theorem 5.4, any bounded sharp real observable has this form for some function $f \in \mathcal{T}$.

Remark 5.7. Let A be a commutative GH-algebra with the Loomis-Sikorski representation $(X, \mathcal{T}, h)$. Let $(Y, \mathcal{B})$ be a measurable space and let $\nu : X \times \mathcal{B} \to [0, 1]$ be a weak Markov kernel with respect to h. Then $\xi(B) := h(\nu_B)$ is an observable and if A has some σ-normal states, then ξ is a smearing of h with respect to ν.

We will next assume that A has an ordering set of σ-normal states. By Theorem 4.3, for any observable ξ on A and any weak Markov kernel with respect to ξ there exists a unique smearing.

In the next theorem we prove that in this case, also the converse of Theorem 5.6 holds true.

Theorem 5.8. *Let A be a GH-algebra with an ordering set of σ-normal states. Then an observable η is a smearing of a sharp observable ξ if and only if the range $\mathcal{R}(\eta)$ consists of pairwise commuting effects.*

Proof. If the range of η is pairwise commuting, the result follows by Theorem 5.6.

For the converse, let $\eta : (X_1, \mathcal{A}_1) \to E$ be an observable on E that is a smearing of a sharp observable $\xi : (X, \mathcal{A}) \to P$. This means that for every σ-additive state $m \in S(E)$ and every set $\Delta \in \mathcal{A}_1$,

$$m(\eta(\Delta)) = \int_X \nu(x, \Delta)(m \circ \xi)(dx) = m(\nu_\Delta(\xi)),$$

where $\nu : X \times \mathcal{A}_1 \to [0, 1]$ is a weak Markov kernel with respect to ξ and $\nu_\Delta(\xi)$ is a sharp real observable that is a function of ξ, hence its range is contained in $\mathcal{R}(\xi)$. Since the range $\mathcal{R}(\xi)$ consists of mutually commuting projections, it is contained in a commutative sub-GH-algebra A_0. By Theorem 5.4, there exists an element of A_0 corresponding to $\nu_\Delta(\xi)$, and since the σ-normal states are ordering, this element must be equal to $\eta(\Delta)$. It follows that the range of η is contained in A_0, hence is commutative. □

6. Observables on synaptic algebras which are dual Banach spaces

In what follows, we will consider a synaptic algebra A which is the dual of a Banach space. In this case, A is itself a Banach space, hence a Banach synaptic algebra. As it was proved in [18,19], A is then isomorphic to a JC-algebra and by [43, Corollary 2.4], a JC-algebra is a dual Banach space iff it is a JW-algebra (that is, a weakly closed Jordan operator algebra, see [44]), equivalently, A is monotone complete and has a separating set of normal states.

Since an effect algebra is monotone complete iff it is orthocomplete [31], it can be seen that a synaptic algebra A is the dual of a Banach space iff its unit interval E is orthocomplete and has an ordering set of completely additive states.

To describe the predual of A, we will need the notion of a base norm space. Let (V, V^+) be an ordered vector space and let K be a base of V^+, that is, a convex subset of V^+ such that every nonzero $v \in V^+$ can be uniquely written in the form $v = \lambda x$ for $\lambda > 0$ and $x \in K$. Let

$$\|v\|_K := \inf\{\lambda + \mu, v = \lambda x - \mu y,\ \lambda, \mu \geq 0, x, y \in K\}.$$

If $\| \cdot \|_K$ defines a norm in V, we say that V is a base norm space, with distinguished base K. Let us remark that the dual of an order unit space is a base norm space with the base given by the set of states. Conversely, the dual of a base norm space is an order unit space such that the order unit has constant value 1 on the distinguished base, [3].

We will make use of the following theorem [3, Proposition 1.11].

Theorem 6.1. *If V is a base-norm space with distinguished base K, then the restriction map $f \mapsto f/K$ is an order and norm preserving isomorphism of V^* onto the space $A_b(K)$ of all real valued bounded affine functions on K equipped with pointwise ordering and supremum norm.*

The next result is based on [11, Theorem 6] and characterizes the convex effect algebras such that the corresponding ordered vector space is an order unit space which is the dual of a Banach space. In particular, we obtain an alternative characterization of synaptic algebras which are JW-algebras.

Let E be a convex effect algebra and let $S \subseteq S(E)$ be a set of states. The $\sigma(E, S)$-topology on E is given by the neighbourhoods basis consisting of the sets

$$V(a; s_1, \dots, s_n, \epsilon) := \{b \in E : |s_i(a) - s_i(b)| < \epsilon\}, \quad s_i \in S, i = 1, 2, \dots, n, \epsilon > 0.$$

Theorem 6.2. *Let E be a convex effect algebra and let (V, V^+, u) be the ordered vector space with order unit u such that E is isomorphic to the unit interval $V[0, u]$ of V. Then (V, V^+, u) is an order unit space which is the dual of a Banach space if and only if E is compact with respect to the $\sigma(E, S)$-topology for a separating set $S \subseteq S(E)$. The predual is a base norm space, whose base is an ordering set of completely additive states on E.*

Proof. Since $[0, u]$ generates V, every state $\rho \in S$ can be uniquely extended to a state $\hat{\rho}$ on V. Let τ be the locally convex topology defined by the seminorms $x \mapsto |\hat{\rho}(x)|$, $x \in V$ for $\rho \in S$. Since S is separating, this family of seminorms is separated. Indeed, let $x \in V$ be such that $\hat{\rho}(x) = 0$ for all $\rho \in S$. Let $\lambda, \mu \geq 0$ and $a, b \in E$ be such that $x = \lambda a - \mu b$, so that

$$\lambda\rho(a) = \mu\rho(b), \quad \forall \rho \in S.$$

We may assume that $\mu \leq \lambda$ and $\lambda > 0$. Then for $\rho \in S$,

$$\rho(a) = \frac{\mu}{\lambda}\rho(b) = \rho(\frac{\mu}{\lambda}b).$$

It follows that $a = \frac{\mu}{\lambda}b$, hence $x = 0$.

Since the restriction of τ to E coincides with $\sigma(E, S)$, E is compact with respect to τ. By [11, Theorem 6], (V, V^+, u) is an order unit space which is a dual of the Banach space N of all τ-continuous functionals on V, moreover, the $\sigma(V, N)$-topology agrees with τ on norm-bounded sets. Let N^+ be the cone of positive functionals in N, then N^+ has a base B consisting of states in N. By the proof of [11, Theorem 6], N is base normed with respect to B. By Theorem 6.1, (V, V^+, u) is order isomorphic to $(A_b(B), A_b(B)^+, 1_B)$ with pointwise ordering, which implies that B is an ordering set of completely additive states on $E \simeq V[0, u] \simeq A_b(B)[0, 1_B]$.

For the converse, assume that (V, V^+, u) is an order unit space which is a Banach space dual. By the Banach-Alaoglu theorem, the unit ball in a dual of a Banach space is w^*-compact. In our case, this implies that $[-u, u]$ is w^*-compact and using homogeneity and translation invariance, we obtain that $[0, u] = \frac{1}{2}([-u, u] + u)$ is w^*-compact as well. Let $S \subset S(E)$ be the set of (restrictions of) normal states, then S is separating. The topology $\sigma(E, S)$ is coarser than the w^*-topology, hence E is compact in $\sigma(E, S)$. □

To summarize, we obtain the following statement.

Corollary 6.3. *Let A be a synaptic algebra. The following statements are equivalent:*

(i) A is a dual of a Banach space.
(ii) A is monotone complete and has a separating set of normal states.
(iii) E is orthocomplete and has a separating set of completely additive states.
(iv) E is compact in the topology defined by a separating set of states.
(v) A is a JW-algebra.
(vi) A is the dual of a base norm space whose base is the set of normal states of A.

Notice that every JW-algebra A is a synaptic algebra, and being monotone complete, it is a GH-algebra. We can therefore discuss observables on A and apply the results of the previous section. Moreover, it follows by Theorem 4.3 that for observables on A we have well defined and unique smearings. We give here a more direct proof based on Theorem 6.1.

Theorem 6.4. *Let A be a JW-algebra and $(X, \mathcal{A})$, $(X_1, \mathcal{A}_1)$ be measurable spaces. Let ξ be an $(X, \mathcal{A})$-observable. For every weak Markov kernel $\nu : X \times \mathcal{A}_1 \to [0, 1]$ with respect to ξ, there exists a unique observable η on E which is the smearing of ξ with respect to ν.*

Proof. Let K be the set of normal states on A. By Corollary 6.3 and Theorem 6.1, E is isomorphic to the set of all affine functions from K to $\mathbb{R}[0, 1]$. For every $m \in K$, $m \circ \xi$ is a probability measure on $(X, \mathcal{A})$, and for every $\Delta \in \mathcal{A}_1$, $\nu_\Delta : X \to \mathbb{R}[0, 1]$ is a measurable function on X. Put

$$\eta(\Delta)(m) := (m \circ \xi)(\nu_\Delta) = \int_X \nu_\Delta(x)(m \circ \xi)(dx).$$

Then $m \mapsto \eta(\Delta)(m)$ is an affine function on K with values in $\mathbb{R}[0, 1]$, therefore $\eta(\Delta) \in E$. To prove that $\Delta \mapsto \eta(\Delta)$ is an observable, let $(\Delta_i)_{i=1}^{\infty}$ be a sequence of pairwise disjoint sets with $\cup_i \Delta_i = \Delta$. By the properties of a weak Markov kernel we have $\nu(x, \Delta) = \sum_i \nu(x, \Delta_i)$ ξ-a.e. Hence for every $m \in K$ we have

$$\begin{aligned}\eta(\Delta)(m) &= \int_X \nu_\Delta(x)(m \circ \xi)(dx) = \int_X \sum_i \nu_{\Delta_i}(x)(m \circ \xi)(dx) \\ &= \sum_i \int_X \nu_{\Delta_i}(x)(m \circ \xi)(dx) = \sum_i \eta(\Delta_i)(m),\end{aligned}$$

so that $\eta(\Delta) = \sum_i \eta(\Delta_i)$. It then follows that η is an observable defined by a smearing of ξ. Uniqueness is clear. □

Since the set of normal states is ordering for A, we may apply Theorem 5.8 to show that observables with a commuting range are precisely the smearings of sharp observables. We next prove that if A is a JW-algebra of operators on a separable Hilbert space, we may assume that the sharp observable in *real*. The corresponding statement for the effect algebra $\mathcal{E}(H)$ on the separable Hilbert space H (i.e. that a POV-measure is a smearing of a real PV-measure iff it has a commutative range) was proved in [32, Theorem 4.4].

Corollary 6.5. *Let A be a JW-algebra of operators on a separable Hilbert space. Let η be an observable on A whose range $\mathcal{R}(\eta)$ consists of pairwise commuting effects in E. Then η is a smearing of a sharp real observable.*

Proof. Since A is a GH-algebra, for every $\eta(\Delta)$, $\Delta \in \mathcal{A}$, there is a sharp real observable $\xi_\Delta = \xi_{\eta(\Delta)}$ such that $m(\eta(\Delta)) = m(\xi_\Delta)$ for every normal state m on A. So we have a system $\{\xi_\Delta : \Delta \in \mathcal{A}\}$ of compatible observables on the OML P. By [45, Theorem 3.9], there exists an observable ξ and real valued Borel measurable functions $f_\Delta : X \to \mathbb{R}$ for all $\Delta \in \mathcal{A}$ such that $\xi_\Delta = f_\Delta(\xi)$. Since the OML of projections on a separable Hilbert space is separable (in the sense that every Boolean subalgebra of it is countably generated), by [45, Theorem 3.9], we may assume that ξ is a real observable. Put $\nu(\Delta, x) := f_\Delta(x)$. Similarly as in the proof of [32, Theorem 4.4], we prove that ν is a weak Markov kernel with respect to ξ and that η is a smearing of ξ. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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