Advances in Operator Theory

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--Manuscript Draft--

Manuscript Number:	AIOT-D-23-00203
Full Title:	On the geometry of an order unit space
Article Type:	Original Paper
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ON THE GEOMETRY OF AN ORDER UNIT SPACE

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ABSTRACT. We introduce the notion of *skeleton* with a head in a non-zero real vector space. We prove that skeletons with heads describe order unit spaces geometrically. Next, we consider the notion of *periphery* corresponding to an order unit space which is a part of the skeleton. We note that periphery consists of boundary elements of the positive cone with unit norms. We discuss some elementary properties of the periphery. We also find a condition under which V would contain a copy of ℓ_∞^n for some $n \in \mathbb{N}$ as an order unit subspace.

1. INTRODUCTION

Let X be a normed linear space and let $x, y \in X$. We say that x is ∞-*orthogonal* to y, (we write, $x \perp_\infty y$), if $\|x + ky\| = \max\{\|x\|, \|ky\|\}$ for all $k \in \mathbb{R}$. It was proved in [11] that if (V, e) is an order unit space and if $u, v \in V^+ \setminus \{0\}$, then $u \perp_\infty v$ if and only if $\| \|u\|^{-1} u + \|v\|^{-1} v \| = 1$. For $u, v \in V^+$, we say that u is *absolutely* ∞-*orthogonal* to v (we write $u \perp_\infty^a v$) if $u_1 \perp_\infty v_1$ whenever $0 \leq u_1 \leq u$ and $0 \leq v_1 \leq v$.

Let A be a unital C*-algebra. Then $p \in A$ is a projection if $p^2 = p = p^*$ or equivalently, $p, 1 - p \in A^+$ and $p(1 - p) = 0$. Following [12, Theorem 2.1], we note that p is a projection if and only if $p, 1 - p \in A^+$ and $p \perp_\infty^a (1 - p)$. In this paper, we weaken the notion of projections and consider in stead the notion of *peripheral elements*. Let (V, e) be an order unit space. An element $u \in V$ is said to be a peripheral element if $u, e - u \in V^+$ and we have $u \perp_\infty (e - u)$. The set of all peripheral elements together with 0 and e form the notion of *skeleton with a head* in an order unit space in the following sense.

Definition 1.1. *Let X be a non-zero real vector space and let $S \subset X$ containing 0 and $e \neq 0$. We say that S is a* skeleton *with e as its* head, *if the following properties hold.*

2020 *Mathematics Subject Classification.* Primary: 46B40; Secondary: 46B20, 46L05.

Key words and phrases. Order unit space, matrix order unit space, canopy, periphery, ∞-orthogonality.

(1) If $u \in S$, then $e - u \in S$;
Put $S_0 = S \setminus \{0, e\}$.
(2) If $u, v \in S$ and $\lambda \in [0,1]$, then there exist $w \in S_0$ and $\alpha, \beta \in \mathbb{R}^+$ such that $\lambda u + (1-\lambda)v = \alpha e + \beta w$; and
(3) If $e = \sum_{i=1}^{n} \alpha_i u_i$ for some $u_1, \ldots, u_n \in S_0$ and $\alpha_1, \ldots, \alpha_n > 0$ with $n \geq 2$, then $\sum_{i \neq j} \alpha_i \geq 1$ for all $j = 1, \ldots, n$.

The members of S_0 are called the peripheral *elements of S and S_0 is called the* periphery *of S.*

Order unit spaces dominate the interface of commutative and non-commutative C*-algebras. In early 1940's, Stone, Kakutani, Krein and Yosida proved independently that if an order unit space (V, e) is a vector lattice in its order structure, then it is unitally lattice isomorphic to a dense lattice subspace of $C_{\mathbb{R}}(X)$ for some suitable compact Hausdorff space X [1, Theorem II.1.10]. (see the notes after Section1, Chapter II of [1] for the details.) In 1951, Sherman proved that the self-adjoint part of a C*-algebra A is a vector lattice in its order structure if and only if A is commutative [16]. The same year, Kadison prove that the infimum of a pair of self-adjoint operators on a complex Hilbert space exists if and only if they are comparable [9]. The same year in another paper, he proved that any unital self-adjoint subspace of a unital C*-algebra is an order unit space [10]. (Much later in 1977, Choi and Effros proved that a unital self-adjoint subspace of a unital C*-algebra is precisely a matrix order unit space [4].)

Soon after Kadison underscored the importance of order unit spaces as a possible role model for a non-commutative ordered spaces, there was a flux of research in the study of order unit spaces and their duals. Some early prominent references are Bonsall, Edwards, Ellis, Asimov and Ng, besides many others. (See [2, 3, 5, 6, 15]. We refer to [1, 8] for more references and details.)

The dual of an order unit space is a base normed space which is defined through the geometric notion of a *base* in an ordered vector space. On the other hand, the notion of an order unit is order theoretic. In this paper we propose to study a set of geometric properties that determine order unit spaces. More precisely, in the following result we show that *skeletons* describe order unit spaces geometrically.

Theorem 1.2. *Let X be a non-zero real vector space and let S be a skeleton in X with e as its head for some $e \in X$ with $e \neq 0$. Let V be the linear span of S and let V^+ be the cone generated by S. Then (V, V^+, e) is an order unit space such that*

$$S_0 := S \setminus \{0, e\} = \{v \in V^+ : \|v\| = \|e - v\| = 1\}.$$

(Here $\|\cdot\|$ is the order unit norm on V.)

We also prove the converse of this result.

Theorem 1.3. *Let (V, e) be an order unit space. Put*

$$(S_V)_0 := C_V \cap (e - C_V) = \{u \in V : \|u\| = \|e - u\| = 1\}.$$

Then $S_V := (S_V)_0 \bigcup \{0, e\}$ is a skeleton in V with e as its head such that (V, e) is the order unit space generated by S_V.

Next, we discuss the *periphery* corresponding to an order unit space. We find that the periphery is consists of maximal elements of a canopy in a certain sense. The periphery includes projections whenever they exist. We discuss some elementary properties of the periphery. Using these properties, we prove that any order unit space (V, e) of dimension more than 1 contains a copy of ℓ^2_∞ as an order unit subspace. We also prove that V is a union these copies in such a way any two such subspace meet at the *axis* $\mathbb{R}e$. Further we find a condition under which V would contain a copy of ℓ^n_∞ for some $n \in \mathbb{N}$ as an order unit subspace.

The scheme of the paper is as follows. In Section 2, we discuss some of the properties of *skeleton* in a non-zero real vector space and prove Theorem 1.2. In Section 3, we prove Theorem 1.3. In Section 4, we study some elementary properties of the periphery corresponding to an order unit space. In Section 5, we find a condition under which an order unit space would contain a copy of ℓ^n_∞ for some $n \in \mathbb{N}$ as an order unit subspace besides some other results.

2. The skeleton

In this section we shall prove Theorem 1.2. We begin with some preliminary results. Throughout in this section, we shall assume that X is a non-zero real vector space and S is a skeleton in X with $e \neq 0$ as its head (see Definition 1.1). First of all, we prove some of the easy consequences of Definition 1.1.

Lemma 2.1. *(1) $S_0 \bigcap [0, 1]e = \emptyset$.*

(2) If $u, v \in S_0$ and $\alpha \in [0, 1]$, then there exist $w \in S_0$ and $\lambda, \mu \in \mathbb{R}^+$ with $\lambda \leq \min\{\alpha, 1 - \alpha\}$ and $\lambda + \mu \leq 1$ such that $\alpha u + (1 - \alpha)v = \lambda e + \mu w$.

(3) Let $u, v \in S_0$ be such that $e = \alpha u + \beta v$ for some $\alpha, \beta \in \mathbb{R}^+$. Then $\alpha = 1 = \beta$.

(4) Let $u, v \in S_0$ such that $\alpha u = \beta v$ for some $\alpha, \beta \in \mathbb{R}$.

(a) Then $\alpha\beta \geq 0$;

(b) $\alpha = 0$ if and only if $\beta = 0$;

(c) If $\alpha\beta > 0$. Then $u = v$.

(5) *Let $u_1, \dots, u_n$ be distinct elements of S_0 and $\alpha_0, \alpha_1, \dots, \alpha_n \in \mathbb{R}^+$ such that $\alpha_0 e + \sum_{i=1}^n \alpha_i u_i = 0$. Then $\alpha_i = 0$ for each $i = 0, 1, \dots, n$.*

Proof. (1) Let $\alpha e \in S_0$ for some $\alpha \in (0,1)$. Then by 1.1(1), $(1-\alpha)e \in S_0$. Since $e = \alpha e + (1-\alpha)e$, by 1.1(3), we get $\alpha \geq 1$ and $1 - \alpha \geq 1$ which is absurd.

(2) By 1.1(2), we have $\alpha u + (1-\alpha)v = \lambda e + \mu w$ for some $w \in$ and $\lambda, \mu \in \mathbb{R}^+$. Thus $(\lambda + \mu)e = \alpha u + (1-\alpha)v + \mu(e - w)$. Now by 1.1(3), we get $\lambda + \mu \leq 1$, $\lambda + \mu \leq \alpha + \mu$ and $\lambda + \mu \leq 1 - \alpha + \mu$. Thus $\lambda \leq \min\{\alpha, 1-\alpha\}$ and $\lambda + \mu \leq 1$.

(3) By 1.1(3), we have $\alpha \geq 1$ and $\beta \geq 1$. Now, $\alpha(e-u) + \beta(e-v) = (\alpha + \beta - 1)e$ and by 1.1(1), $e - u, e - v \in S_0$. Thus invoking 1.1(3) once again, we get $\alpha \geq \alpha + \beta - 1$ and $\beta \geq \alpha + \beta - 1$. Thus $\alpha \leq 1$ and $\beta \leq 1$ so that $\alpha = 1 = \beta$.

(4)(a) Let $\alpha\beta < 0$. For definiteness, we assume that $\alpha > 0$ and $\beta < 0$. Then $\alpha(e-u) - \beta(e-v) = (\alpha - \beta)e$ with $\alpha - \beta > 0$. Thus by condition 1.1(3), we get $\alpha, -\beta \geq \alpha - \beta$. But then $\alpha = 0 = \beta$ which contradicts the assumption. Thus $\alpha\beta \geq 0$.

(4)(b) follows immediately as $0 \notin S_0$.

(4)(c) Now assume that $\alpha\beta > 0$. For definiteness, we assume that $\alpha > 0$ and $\beta > 0$. Further, without any loss of generality, we may assume that $\alpha \leq \beta$. Put $\frac{\alpha}{\beta} = \lambda$. Then $0 < \lambda \leq 1$ and $\lambda u = v$. Thus $e = \lambda u + (e - v)$. Now by 1.1(3), we get $\lambda \geq 1$ so that $\lambda = 1$. Thus $u = v$.

(5) We have $\sum_{i=1}^n \alpha_i(e - u_i) = (\sum_{i=0}^n \alpha_i)e$. Assume, if possible that $\sum_{i=0}^n \alpha_i > 0$. Then by 1.1(3), we get $\sum_{i=0}^n \alpha_i \leq (\sum_{i=1}^n \alpha_i) - \alpha_j$ for all $j = 1, \dots, n$. In other words, $\alpha_0 + \alpha_j = 0$ for all $j = 1, \dots, n$. Therefore, $\alpha_j = 0$ for every $j = 0, 1, \dots, n$. □

Lemma 2.2. *Let $u, v \in S_0$ be such that $\alpha e + \beta u = \gamma e + \delta v$ for some $\alpha, \beta, \delta, \gamma \in \mathbb{R}$ with $\beta, \delta \geq 0$. Then $\alpha = \gamma$ and we have either $\beta = 0 = \delta$ or $u = v$.*

Proof. First, we show that $\alpha = \gamma$. If $\alpha < \gamma$, then $(\gamma - \alpha + \delta)e = \beta u + \delta(e - v)$. By Lemma 2.1(5), we must have $\gamma - \alpha + \delta > 0$ as $\delta \geq 0$. Thus by 1.1(3), we have $\gamma - \alpha + \delta \leq \delta$. But then we arrive at a contradiction, $\gamma \leq \alpha$. Thus $\alpha \geq \gamma$. Now, by symmetry, we have $\gamma \geq \alpha$ so that $\alpha = \gamma$. Thus $\beta u = \delta v$. The rest of the proof follows from Lemma 2.1(4). □

Proposition 2.3. *For $u \in S_0$ we consider*

$$K(u) := \text{co}\{0, e, u\} = \bigcup_{\alpha \in [0,1]} \alpha[e, u].$$

(a) For $u, v \in S_0$ with $u \neq v$, we have $K(u) \cap K(v) = [0,1]e$.
(b) For $u \in S_0$, we have $K(e-u) = e - K(u)$.

Proof. (a) Let $w \in K(u) \cap K(v)$. Then there exist $\alpha, \beta, \gamma, \delta \in [0,1]$ with $\alpha + \beta \leq 1$ and $\gamma + \delta \leq 1$ such that $w = \alpha e + \beta u = \gamma e + \delta v$. If possible, assume that $\alpha \neq \gamma$. For definiteness, we let $\alpha > \gamma$. Then $(\alpha - \gamma + \beta)e = \beta(e - v) + \delta v$. Thus by 1.1(3), we get $\beta \geq \alpha - \gamma + \beta$ so that $\alpha \leq \gamma$. This contradicts the assumption. Hence $\alpha = \gamma$ so that $\beta u = \delta v$. Now by Lemma 2.1(4), we have $\beta = 0 = \delta$ as $u \neq v$. Thus $w \in [0,1]e$, that is, $K(u) \cap K(v) \subset [0,1]e$. As $[0,1]e \subset K(x)$ for any $x \in S_0$, the proof is complete.

(b) Let $w \in K(e-u)$. Then $w = \alpha e + \beta(e-u)$ for some $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$. Thus $e - w = (1 - \alpha - \beta)e + \beta u$. Since $1 - \alpha - \beta, \beta \in [0,1]$ and $1 - \alpha - \beta + \beta = 1 - \alpha \leq 1$, we get that $e - w \in K(u)$. Thus $K(e-u) \subset K(u)$ for all $u \in S_0$. So for any $u \in S_0$, we also have $K(u) = K(e - (e-u)) \subset K(e-u)$. Therefore, $K(e-u) = K(u)$ for all $u \in S_0$. □

Corollary 2.4. *For $u, v \in S_0$ with $u \neq v$, we have $[e,u] \bigcap [e,v] = \{e\}$.*

Proof. Note that if $\alpha e \in [e,u]$ for some $u \in S_0$ and $\alpha \in [0,1]$, say, $\alpha e = (1-\lambda)e + \lambda u$, then $\lambda(e-u) = (1-\alpha)e$. As $e - u \in S_0$ by 1.1(1) and as $S_0 \bigcap \mathbb{R}e = \emptyset$, we must have $\lambda = 0 = 1 - \alpha$. Thus $[e,u] \bigcap [e,v] = \{e\}$. Now as $[e,u] \subset K(u)$, the result follows from Proposition 2.3. □

Let E be a convex subset of a real vector space X with $0 \in E$. An element $x \in E$ is called a *lead point* of E, if for any $y \in E$ and $\lambda \in [0,1]$ with $x = \lambda y$, we have $\lambda = 1$ and $y = x$. The set of all lead points of E is denoted by $Lead(E)$.

A non-empty set E of a real vector space V is said to be *linearly compact*, if for any $x, y \in E$ with $x \neq y$, we have, the intersection of E with the line through x and y, $\{\lambda \in \mathbb{R} : (1-\lambda)x + \lambda y \in E\}$, is compact (in $\mathbb{R}$). Note that if E is convex, the above intersection is an interval. Following [7, Proposition 3.2], we may conclude that if E is a linearly compact convex set with $0 \in E$, then $Lead(E)$ is non-empty and for each $x \in E, x \neq 0$, there exist a unique $u \in Lead(E)$ and a unique $0 < \alpha \leq 1$ such that $x = \alpha u$.

Theorem 2.5. *Let X be a non-zero real vector space and let S be a skeleton in X with $e \neq 0$ as its head. Consider $K = \bigcup_{u \in S_0} K(u)$ and $C = \bigcup_{u \in S_0} [e,u]$. Then K is convex set containing 0 and e such that $Lead(K) = C$. Moreover, e is an extreme point of K.*

Proof. Let $x, y \in K$. Then there are $u, v \in S_0$ and $\alpha, \beta, \gamma, \delta \in [0,1]$ with $\alpha + \beta \leq 1$ and $\gamma + \delta \leq 1$ such that $x = \alpha e + \beta u$ and $y = \gamma e + \delta v$.

Then for $\lambda \in [0,1]$, we have

$$\begin{aligned}\lambda x + (1-\lambda)y &= (\lambda\alpha + (1-\lambda)\gamma)e + \lambda\beta u + (1-\lambda)\delta v \\ &= (\lambda\alpha + (1-\lambda)\gamma)e + k(\lambda_1 u + (1-\lambda_1)v)\end{aligned}$$

where $k := \lambda\beta + (1-\lambda)\delta$ and $\lambda_1 := \frac{\lambda\beta}{\lambda\beta+(1-\lambda)\delta} \in [0,1]$. By Lemma 2.1(2), we can find $w \in S_0$ and $\eta, \kappa \in [0, \frac{1}{2}]$ with $\eta \le \min\{\lambda_1, 1-\lambda_1\}$ and $\eta + \kappa \le 1$ such that $\lambda_1 u + (1-\lambda_1)v = \eta e + \kappa w$. Thus

$$\lambda x + (1-\lambda)y = (\lambda\alpha + (1-\lambda)\gamma)e + k(\eta e + \kappa w) = \alpha_1 e + \beta_1 w$$

where

$$\alpha_1 = \lambda\alpha + (1-\lambda)\gamma + k\eta = \lambda\alpha + (1-\lambda)\gamma + (\lambda\beta + (1-\lambda)\delta)\eta$$

and

$$\beta_1 = k\kappa = (\lambda\beta + (1-\lambda)\delta)\kappa.$$

Now $\alpha_1 \ge 0$, $\beta_1 \ge 0$ and

$$\begin{aligned}\alpha_1 + \beta_1 &= \lambda\alpha + (1-\lambda)\gamma + (\lambda\beta + (1-\lambda)\delta)(\eta + \kappa) \\ &\le \lambda\alpha + (1-\lambda)\gamma + \lambda\beta + (1-\lambda)\delta \\ &= \lambda(\alpha+\beta) + (1-\lambda)(\gamma+\delta) \\ &\le 1\end{aligned}$$

for $\alpha+\beta \le 1$, $\gamma+\delta \le 1$ and $\eta+\kappa \le 1$. Thus $\lambda x+(1-\lambda)y \in K(w) \subset K$. Hence K is a convex set containing 0 and e.

Next, we show that $Lead(K) = C$. Let $v \in K$ $v \ne 0$. If $v = \alpha e$ then $0 < \alpha \le 1$ and $e \in C$. Now assume that $v \notin [0,1]e$. Then there exists a unique $u \in S_0$ such that $v \in K(u)$. In other words, $v = \alpha e + \beta u$ for some $\alpha, \beta \in [0,1]$ with $\alpha + \beta \le 1$. Since $v \ne 0$, we have $\beta > 0$. Set $w = (\alpha+\beta)^{-1}v$. Then $w \in C$ and $v = (\alpha+\beta)w$. Thus K has a representation in C. We show that $C = Lead(K)$.

Let $u \in C$ and assume that $u = \alpha w$ for some $w \in K$ and $\alpha \in [0,1]$. Let $u \in [e, v]$ for some $v \in S_0$, say $u = \lambda v + (1-\lambda)e = e - \lambda(e-v)$. As $w \in K$, we have $w = \gamma e + \delta x$ for some $x \in S_0$ and $\gamma, \delta \in [0,1]$ with $\gamma + \delta \le 1$. Then $e - \lambda(e-v) = \alpha(\gamma e + \delta x)$ or equivalently, $(1-\alpha\gamma)e = \lambda(e-v) + \alpha\delta x$. Since $\alpha, \gamma \in [0,1]$, we have $1 - \alpha\gamma \ge 1$. If $\alpha\gamma = 1$, then $\alpha = 1$ and we have $u = w$. So we assume that $1-\alpha\gamma > 0$. Thus by 1.1(3), we get $1 - \alpha\gamma \le \alpha\delta$. Therefore, $1 \le \alpha(\gamma+\delta) \le \alpha \le 1$ for $\gamma + \delta \le 1$. So we have $\alpha = 1$ and $u = w$ once again. Hence $C \subset Lead(K)$.

Conversely, let $u \in Lead(K)$. If $u = \alpha e$ for some $\alpha \in [0,1]$, then by the definition of Lead, we have $\alpha = 1$ and $u = e \in C$. So we assume that $u \notin [0,1]e$. Then as above, there exists $x \in C$ and $\lambda \in [0,1]$ such that $u = \lambda x$. As $x \in Lead(K)$, we must have $\lambda = 1$ and $u = x \in C$. Hence $Lead(K) \subset C$ and consequently, $Lead(K) = C$.

Finally, we show that e is an extreme point of K. Let $e = \alpha u + (1-\alpha)v$ for some $u, v \in K$ and and for $0 < \alpha < 1$. Find $u_1, v_1 \in C$ and $\lambda_1, \mu_1 \in [0,1]$ such that $u = \lambda_1 u_1$ and $v = \mu_1 v_1$. Then $e = \alpha\lambda_1 u_1 + (1-\alpha)\mu_1 v_1$. Thus by Lemma 2.1(3), we have $1 \leq \alpha\lambda_1 + (1-\alpha)\mu_1 \leq 1$, that is, $\lambda_1 = 1 = \mu_1$. Therefore, $u, v \in C$. Find $x, y \in S_0$ and $\lambda, \mu \in [0,1]$ such that $u = e - \lambda x$ and $v = e - \mu y$. Then $e = e - \alpha\lambda x - (1-\alpha)\mu y$ whence $\alpha\lambda x + (1-\alpha)\mu y = 0$. Now, by Lemma 2.1(4), we must have $\alpha\lambda = 0 = (1-\alpha)\mu$. Since $0 < \alpha < 1$, we conclude that $\lambda = 0 = \mu$. Thus $u = e = v$ and consequently, we conclude that e is an extreme point of K. □

We call C the *canopy* of K with e its *summit*.

Remark 2.6. *It is easy to note that $K = \mathrm{co}(S)$. Thus $K \cap \mathbb{R}e = [0,1]e$. Also, by Proposition 2.3, we have $e - K = K$. Thus 0 is also an extreme point of K.*

Proposition 2.7. *Let X be a non-zero real vector space and let S be a skeleton in X with $e \neq 0$ as its head and let $u \in S_0$. Then $\alpha e + \beta u \in K$ if and only if $\alpha, \alpha + \beta \in \mathbb{R}^+$ and $\max\{\alpha, \alpha+\beta\} \leq 1$.*

Proof. Let $\alpha e + \beta u \in K$. If $\alpha e + \beta u = \lambda e$ for some $\lambda \in [0,1]$, then $(\lambda - \alpha)e = \beta u$. As $u \in S_0$, we must have $\lambda = \alpha$ and $\beta = 0$. Thus $0 \leq \alpha + \beta = \alpha \leq 1$. So we assume that $\alpha e + \beta u \notin \mathbb{R}e$. Then by Theorem 2.5, there exists a unique $x \in C$ and $0 < \lambda \leq 1$ such that $\alpha e + \beta u = \lambda x$. Consequently, we can also find $w \in S_0$ and $1 \leq \theta < 1$ such that $x = \theta e + (1-\theta)w$. Thus $\alpha e + \beta u = \lambda\theta e + \lambda(1-\theta)w$. Now we show that $\alpha \geq 0$.

Assume, if possible, that $\alpha < 0$. Then $\lambda - \alpha > 0$ and we have

$$(\lambda - \alpha)e = (\lambda\theta + \lambda(1-\theta) - \alpha)e = \beta u + \lambda(1-\theta)(e - w).$$

If $\beta \geq 0$, then by Lemma 2.1(3), we have $\lambda - \alpha = \beta = \lambda(1-\theta)$. But then $\alpha = \lambda\theta \geq 0$ which is a contradiction. Thus $\beta < 0$ and we have

$$(\lambda - \alpha - \beta)e = -\beta(e - u) + \lambda(1-\theta)(e - w).$$

Again invoking Lemma 2.1(3), we conclude that $\lambda - \alpha - \beta = -\beta = \lambda(1-\theta)$. This leads to another contradiction $\alpha = \lambda \geq 0$. Hence $\alpha \geq 0$.

Next, we aim to prove that $\alpha + \beta \geq 0$ and assume to the contrary that $\alpha + \beta < 0$, that is, $0 \leq \alpha < -\beta$. Now $(\alpha - \lambda\theta) = -\beta u + (1-\lambda)\theta w$ so by Lemma 2.1(5), we must have $\alpha - \lambda\theta > 0$ as $\beta < 0$. By Lemma 2.1(3), we have $\alpha - \lambda\theta = -\beta = (1-\lambda)\theta$. Thus $\alpha + \beta = \lambda\theta \geq 0$ contradicting the assumption $\alpha + \beta < 0$. Thus $\alpha + \beta \geq 0$.

Since $\alpha e + \beta u \in K \setminus \mathbb{R}e$, there exists a unique $w \in S_0$ such that $\alpha e + \beta u = \gamma e + \delta w$ where $\gamma, \delta \in \mathbb{R}^+$ with $\gamma + \delta \leq 1$.

Let $\beta \leq 0$. Then $(\alpha - \gamma)e = -\beta u + \delta w$ so by Lemma 2.1(5), we must have $\alpha - \gamma \geq 0$. If $\alpha = \gamma$, then we further get $-\beta = 0 = \delta$ so that

$$0 \leq \alpha + \beta = \alpha = \gamma = \gamma + \delta \leq 1.$$

If $\alpha > \gamma$, then by Lemma 2.1(3), we get $\alpha - \gamma = -\beta = \delta$. Thus $\alpha = \gamma + \delta \leq 1$ and $\alpha + \beta = \gamma \leq \gamma + \delta \leq 1$.

Next, let $\beta > 0$. Then $(\alpha + \beta - \gamma)e = \beta(e - u) + \delta w$. Thus by Lemma 2.1(5), we get $\alpha + \beta - \gamma \geq 0$. If $\alpha + \beta = \gamma$, we further get $\beta = 0 = \delta$, contradicting $\beta > 0$. Thus $\alpha + \beta > \gamma$. Invoking Lemma 2.1(3), we have $\alpha + \beta - \gamma = \beta = \delta$. Hence $0 \leq \alpha \leq \alpha + \beta = \gamma + \delta \leq 1$.

Conversely, we assume that $0 \leq \alpha, \alpha + \beta \leq 1$. When $\beta \geq 1$, we have $\alpha e + \beta u \in K(u) \subset K$ for $\alpha, \beta \geq 0$ and $\alpha + \beta \leq 1$. When $\beta < 0$, we can write $\alpha e + \beta u = (\alpha + \beta)e - \beta(e - u) \in K(e - u) \subset K$ for $\alpha + \beta, -\beta \geq 0$ and $\alpha = (\alpha + \beta) - \beta = \alpha \leq 1$. $\square$

Now we prove the characterization of order unit spaces.

Proof of Theorem 1.2. Put $C = \bigcup_{u \in S_0} [e, u]$ and $K = \mathrm{co}(S)$. Then by Theorem 2.5, K is a convex set containing 0 with $C = \mathit{Lead}(K)$. Also, by Proposition 2.3, $K = \bigcup_{\alpha \in [0,1]} \alpha C$. Thus $V^+ = \bigcup_{\lambda \in \mathbb{R}^+} \lambda C = \bigcup_{n \in \mathbb{N}} nK$ and $V = V^+ - V^+$. We prove that

$$K = \{v \in V^+ : 0 \leq v \leq e\}. \qquad (*)$$

Let $v \in K$. Then $v \in V^+$ so that $0 \leq v$. Let $v = \alpha u$ for some $u \in C$ and $\alpha \in [0, 1]$. If $u = e$, then $v = \alpha e \leq e$. So let $u \neq e$. Then there exists $w \in S_0$ and $\lambda \in [0, 1]$ such that $u = (1 - \lambda)e + \lambda w$. Thus

$$e - v = (1 - \alpha)e + \alpha\lambda(e - w) \in K(w) \subset V^+$$

for $e - w \in S_0$ and $1 - \alpha, \alpha\lambda \geq 0$ with $1 - \alpha + \alpha\lambda \leq 1$. Therefore, $K \subset \{v \in V^+ : 0 \leq v \leq e\}$.

Conversely, assume that $0 \leq u \leq e$. Then $u, e - u \in V^+$. Thus there exist $v, w \in C$ and $\alpha, \beta \geq 0$ such that $u = \alpha v$ and $e - u = \beta w$. Then $e = \alpha v + \beta w$ so by the definition of a canopy, we must have $\alpha, \beta \leq 1$. Therefore, $u = \alpha v \in K$. Hence $(*)$ is proved.

Since $V = V^+ - V^+$, it follows from $(*)$ that e is an order unit for V. We prove that V^+ is proper. Let $\pm u \in V^+$. Then there exist $v, w \in C$ and $\alpha, \beta \geq 0$ such that $u = \alpha v$ and $-u = \beta w$. Thus $\alpha v + \beta w = 0$. We show that $\alpha = 0 = \beta$. Assume, if possible, that $\alpha > 0$. Then $\beta > 0$ too, for $v \neq 0$. Put $k = \frac{\alpha}{\alpha + \beta}$. Then $0 < k < 1$ and we have $ku + (1 - k)v = 0$ so that $e = k(e - u) + (1 - k)(e - v)$. As $u, v \in C$, we have $e - u, e - v \in K$. As e is an extreme point of K by Theorem 2.5, we deduce that $e - u = e = e - v$, that is, $u = 0 = v$ which is absurd. Thus $\alpha = 0$. Therefore, V^+ is proper.

Next, we show that V^+ is Archimedean. Let $v \in V$ be such that $ke + v \in V^+$ for all $k > 0$. Then $v_n := \left(\frac{1}{1+\|v\|}\right)\left(\frac{1}{n+1}e + \frac{n}{n+1}v\right) \in V^+$ for all $n \in \mathbb{N}$. Since

$$\|v_n\| \leq \left(\frac{1}{1+\|v\|}\right)\left(\frac{1+n\|v\|}{1+n}\right) < 1,$$

we have $v_n \in K$ for every n. Thus by Proposition 2.7, for each n, there exists a unique $u_n \in S_0$ and $\alpha_n, \beta_n \in \mathbb{R}^+$ with $\alpha_n + \beta \leq 1$ such that $v_n = \alpha_n e + \beta_n u_n$. Set $u_1 := u, \alpha = 2\alpha_1(1+\|v\|)$ and $\beta = 2\beta_1(1+\|v\|)$. Then $v_1 = \frac{e+v}{2(1+\|v\|)}$. Thus $e + v = \alpha e + \beta u$, or equivalently, $v = (\alpha - 1)e + \beta u$ where $\alpha, \beta \in \mathbb{R}^+$ with $\alpha + \beta \leq 2(1+\|v\|)$. It follows that

$$\begin{aligned} \alpha_n e + \beta_n u_n = v_n &= \frac{1}{(1+\|v\|)}\left(\frac{e+nv}{n+1}\right) \\ &= \frac{(n\alpha - n + 1)e + n\beta u}{(n+1)(1+\|v\|)}, \end{aligned}$$

that is, $\alpha_n e + \beta_n u_n = \gamma_n e + \delta_n u$ for all n where $\gamma_n = \frac{(n\alpha - n+1)}{(n+1)(1+\|v\|)}$ and $\delta_n = \frac{+n\beta}{(n+1)(1+\|v\|)}$. Let $n \in \mathbb{N}$. Since $\beta_n \geq 0$ and $\delta_n \geq 0$, by Lemma 2.2 we have $\frac{(n\alpha - n+1)}{(n+1)(1+\|v\|)} = \gamma_n = \alpha_n \in [0,1]$. Taking limit as $n \to \infty$, we may conclude that $0 \leq \alpha - 1 \leq 1 + \|v\|$. Thus $v = (\alpha-1)e + \beta u \in V^+$. Therefore V^+ is Archimedean.

Now it follows that (V, V^+, e) is an order unit space. We also note that $K = \{v \in V^+ : \|v\| \leq 1\}$. Thus

$$C = Lead(K) = \{v \in V^+ : \|v\| = 1\}.$$

We show that $S_0 = C \cap (e - C)$. Since $S_0 \subset C$ and since $e - S_0 = S_0$ by condition 1.1(1), we have $S_0 \subset C \bigcap (e - C)$. Now let $w \in C \bigcap (e - C)$. Then $w, e - w \in C$. Thus there exist $u, v \in S_0$ and $\alpha, \beta \in [0,1]$ such that

$$w = \alpha u + (1-\alpha)e = e - \alpha(e - u)$$

and

$$e - w = \beta v + (1-\beta)e = e - \beta(e - v).$$

Thus $e = \alpha(e-u) + \beta(e-v)$. Since $e - u, e - v \in R$, by condition 1.1(3), we get $\alpha \geq 1$ and $\beta \geq 1$. Therefore, $\alpha = 1 = \beta$ so that $w = u \in S_0$. Hence $S_0 = \{v \in V^+ : \|v\| = 1 = \|e - v\|\}$. $\square$

3. The positive part of the closed unit ball

We now discuss some properties of the positive elements with norm one in an order unit space.

Proposition 3.1. *Let (V, e) be an order unit space of dimension ≥ 2. Consider $C_V = \{u \in V^+ : \|u\| = 1\}$.*

(1) Fix $u \in C_V$ with $u \neq e$ and consider the one dimensional affine subspace

$$L(u) = \{u_\lambda := e - \lambda(e - u) : \lambda \in \mathbb{R}\}$$

of V. Then
(a) $\{u_\lambda : \lambda \in \mathbb{R}\}$ is decreasing;
(b) $u_\lambda \in V^+$ if and only if $\lambda\|e - u\| \leq 1$;
(c) $\|u_\lambda\| = \max\{1, |\lambda\|e - u\| - 1|\}$ for every $\lambda \in \mathbb{R}$;
(d) there exists a unique $\bar{u} \in L(u)$ such that $\bar{u}, e - \bar{u} \in C_V$.

(2) For $u, v \in C_V$, we have either $L(u) \bigcap L(v) = \{e\}$ or $L(u) = L(v)$.

Proof. (1)(a) follows from the construction of u_λ.

(1)(b): Since $e - u \in V^+ \setminus \{0\}$, there exist $f_u \in S(V)$ such that $\|e - u\| = f_u(e - u) = 1 - f_u(u)$. Also then, $f_u(u) \leq f(u)$ for all $f \in S(V)$ with $f_u(u) < 1$. Set $\bar{\lambda} := \|e - u\|^{-1} = \left(\frac{1}{1 - f_u(u)}\right)$ and $\bar{u} := u_{\bar{\lambda}} = e - \bar{\lambda}(e - u)$. For $f \in S(V)$, we have

$$f(\bar{u}) = f(e) - \left(\frac{f(e) - f(u)}{1 - f_u(u)}\right) = \frac{f(u) - f_u(u)}{1 - f_u(u)} \geq 0$$

so that $\bar{u} \in V^+$. Now by (1), $u_\lambda \in V^+$ if $\lambda \leq \bar{\lambda}$. Also, if $u_\lambda \in V^+$ for some $\lambda \in \mathbb{R}$, then

$$0 \leq f_u(u_\lambda) = f_u(e) - \lambda(f_u(e) - f_u(u)) = 1 - \lambda(1 - f_u(u)).$$

Thus $\lambda\|e - u\| \leq 1$.

(1)(c): Fix $\lambda \in \mathbb{R}$.

Case 1. $\lambda \geq 0$. Then for $k \in \mathbb{R}$, we have $u_\lambda \leq ke$, that is, $\lambda u \leq (k - 1 + \lambda)e$ if and only if $k \geq 1$. Next, for $l \in \mathbb{R}$, we have $le + u_\lambda \in V^+$, that is, $(l + 1 - \lambda)e + \lambda u \in V^+$ if and only if $l + 1 - \lambda + \lambda f_u(u) \geq 0$ as $f_u(u) \leq f(u)$ for all $f \in S(V)$. In other words, $le + u_\lambda \in V^+$ if and only if $l \geq \lambda\|e - u\| - 1$. Thus for $\lambda \geq 0$, we have

$$\|u_\lambda\| = \inf\{\alpha > 0 : \alpha e \pm u_\lambda \in V^+\} = \max\{1, \lambda\|e - u\| - 1\}.$$

Case 2. $\lambda < 0$. Then $le + u_\lambda \in V^+$ for all $l \geq 0$. Next, for $k \in \mathbb{R}$, we have $u_\lambda \leq ke$, that is, $(k - 1 + \lambda)e - \lambda u \geq 0$ if and only $k - 1 + \lambda - \lambda f_u(u) \geq 0$ for $f_u(u) \leq f(u)$ for all $f \in S(V)$ and $-\lambda > 0$.

Thus $u_\lambda \le ke$ if and only if $k \ge -\lambda \|e-u\| + 1$. Therefore, for $\lambda < 0$, we have

$$\|u_\lambda\| = \inf\{\alpha > 0 : \alpha e \pm u_\lambda \in V^+\} = 1 - \lambda\|e-u\|.$$

Summing up, for any $\lambda \in \mathbb{R}$, we have

$$\|u_\lambda\| = \max\{1, |\lambda\|e-u\| - 1|\}.$$

(1)(d): Put $\bar{u} = e - \|e-u\|^{-1}(e-u)$. Then by (c), $\|\bar{u}\| = 1$. Also by construction, $\|e-\bar{u}\| = 1$. Next, assume that $u_\lambda \in L(u)$ is such that $u_\lambda, e-u_\lambda \in C_V$. Then, as $\|e-u_\lambda\| = 1$, we get $|\lambda|\|e-u\| = 1$. If $\lambda = -\|e-u\|^{-1}$, then $\|u_\lambda\| = 2$ so we must have $\lambda = \|e-u\|^{-1}$. Thus $u_\lambda = \bar{u}$.

(2): Let $w \in L(u) \bigcap L(v)$ with $w \ne e$. Then there are $\lambda, \mu \in \mathbb{R} \setminus \{0\}$ such that $w = e - \lambda(e-u) = e - \mu(e-v)$. Thus $\lambda(e-u) = \mu(e-v)$. Let $\alpha \in \mathbb{R}$. Then

$$e - \alpha(e-u) = e - \alpha\lambda\mu^{-1}(e-v)$$

so that $L(u) \subset L(v)$. Now by symmetry, we have $L(u) = L(v)$. $\square$

The following statements can be verified easily.

Corollary 3.2. *Under the assumptions of Lemma 3.1, we have*

(1) $\|u_\lambda\| = 1$ *if and only if* $0 \le \lambda\|e-v\| \le 2$*;*
(2) $\{\|u_\lambda\| : \lambda \in (-\infty, 0]\}$ *is strictly decreasing;*
(3) $\left\{\|u_\lambda\| : \lambda \in \left[\frac{2}{\|e-u\|}, \infty\right)\right\}$ *is strictly increasing*
(4) $C_V \bigcap (e - C_V) = \{\bar{u} : u \in C_V\}$*;*
(5) $C(u) := L(u) \bigcap C_V = [e, \bar{u}]$*;*
(6) $L(u) \bigcap (e - C_V) = \{\bar{u}\}$*.*

Lemma 3.3. *Let* (V, e) *be an order unit space and let* $u \in C_V$*. Then the following statements are equivalent:*

(1) $u \in (S_V)_0$*;*
(2) $u \perp_\infty (e-u)$*;*
(3) u *has an* ∞*-orthogonal pair in* C_V*;*
(4) there exists $v \in C_V$ *such that* $u + v \in C_V$*;*
(5) there exists a state f *of* V *such that* $f(u) = 0$*.*

Proof. If $u \in (S_V)_0$, then $u, e-u \in V^+$ with $\|u\| = 1 = \|e-u\|$. Also then $\|u + e - u\| = \|e\| = 1$ so that $u \perp_\infty (e-u)$. Thus (1) implies (2). Also, (2) implies (3) trivially.

Next, let $u \perp_\infty v$ for some $v \in C_V$. Then $\|u+v\| = 1$ so that $u + v \in C_V$. That is, (3) implies (4).

Now, if $u+v \in C_V$, then $u+v \le e$. Thus $v \le e-u \le e$ and we have $1 = \|v\| \le \|e-u\| \le \|e\| = 1$. Therefore, $u \in (S_V)_0$ so that (4) implies (1).

Again, if $u \in (S_V)_0$, then $e-u \in C_V$. Thus there exists a state f of V such that $1 = f(e-u) = 1 - f(u)$, or equivalently, $f(u) = 0$. Therefore, (1) implies (5).

Conversely, if $f(u) = 0$ for some state f on V, then $f(e-u) = 1$ so that $\|e-u\| \ge 1$. Also, as $0 \le u \le e$, we have $0 \le e-u \le e$ so that $\|e-u\| \le 1$. Thus $e-u \in C_V$ whence $u \in (S_V)_0$. Hence (5) implies (1). □

Proof of Theorem 1.3. It is enough to prove that S_V is a skeleton in V with e as its head. By construction, we have $e-u \in S_V$ whenever $u \in S_V$. Also we note that $S_V \subset V^+$. In fact, if $u \in (S_V)_0$, then $u, e-u \le e$.

Let $u,v \in S_V$ and $\lambda \in (0,1)$. Without any loss of generality, we may assume that $u,v \notin \{0,e\}$. Then $x := \lambda u + (1-\lambda)v \neq 0$. Put $x_1 = \|x\|^{-1}x$. Then $x_1 \in C_V$. Thus by Corollary 3.2(5), there exists a $w \in (S_V)_0$ and $\alpha_1 \in [0,1]$ such that $x_1 = \alpha_1 e + (1-\alpha_1)w$. Now it follows that $\lambda u + (1-\lambda)v = \alpha e + \beta w$ where $\alpha = \alpha_1\|\lambda u + (1-\lambda)v\|$ and $\beta = (1-\alpha_1)\|\lambda u + (1-\lambda)v\|$. Here $\alpha, \beta \in \mathbb{R}^+$ with $\alpha + \beta = \|\lambda u + (1-\lambda)v\| \le 1$.

Next, let $e = \sum_{i=1}^n \alpha_i$ for some $u_1, \dots, u_n \in (S_V)_0$ and $\alpha_1, \dots, \alpha_n > 0$ with $n \ge 2$. By Lemma 3.3, There exist $f_1, \dots, f_n \in S(V)$ such that $f_i(u_i) = 0$ for $i = 1, \dots, n$. Thus

$$1 = f_j(e) = f_j\left(\sum_{i=1}^n \alpha_i u_i\right) = \sum_{i \neq j} \alpha_i f_i(u_i) \le \sum_{i \neq j} \alpha_i.$$

□

4. The periphery

Recall that if (V,e) is an order unit space, then (the linearly compact set) $K_V := [0,e] = \{v \in V^+ : \|v\| \le 1\}$ has a canopy $C_V := \{v \in V^+ : \|v\| = 1\}$ with its summit at the order unit e. The periphery of C_V, denoted by R_V, is given by

$$R_V := C_V \cap (e - C_V) = \{v \in V^+ : \|v\| = 1 = \|e-v\|\}.$$

In this section, we discuss some of the properties and examples of the periphery corresponding to an order unit space.

Theorem 4.1. *Let (V,e) be an order unit space and let $Bd(V^+)$ denote the $\|\cdot\|$-boundary of V^+. Then $R_V = C_V \bigcap Bd(V^+)$.*

Proof. Let $u \in R_V$. Then by Lemma 3.3, $u \in C_V$ and there exists $f \in S(V)$ such that $f(u) = 0$. Thus for any $\epsilon > 0$, we have $f(u - \epsilon e) = -\epsilon < 0$ so that $u - \epsilon e \notin V^+$. Hence $u \in \overline{(V \setminus V^+)}$ and consequently, $u \in Bd(V^+)$.

Conversely, let $v \in Bd(V^+)$. Then $v \in V^+$ and there exists a sequence $\langle v_n \rangle$ in $V \setminus V^+$ such that $v_n \to v$. Let $n \in \mathbb{N}$. As $v_n \notin V^+$, we can find $f_n \in S(V)$ such that $f_n(v_n) < 0$. Since $S(V)$ is weak*-compact, by passing to a subsequence, we may deduce that $f_n \to f_0 \in S(V)$ in the weak*-topology. Now, as

$$\begin{aligned} |f_n(v_n) - f_0(v)| &\leq |f_n(v_n) - f_0(v_n)| + |f_0(v_n - v)| \\ &\leq |f_n(v_n) - f_0(v_n)| + \|v_n - v\|, \end{aligned}$$

we conclude that $f_0(v) = \lim f_n(v_n) \leq 0$. Since $v \in V^+$ and since $f_0 \in S(V)$, we also have $f_0(v) \geq 0$. Thus $f_0(v) = 0$. Therefore, by Lemma 3.3, we have $C_V \bigcap Bd(V^+) \subset R_V$ which completes the proof. $\square$

Remark 4.2. *We have $Bd(V^+) = \bigcup_{u \in R_V} [0, \infty)u$. Also, for $u, v \in R_V$ with $u \neq v$, we have $[0, \infty)u \bigcap [0, \infty)v = \{0\}$. Thus the $\| \cdot \|$-boundary of V^+ is the disjoint union of the rays passing through R_V.*

Proposition 4.3. *Let (V, e) be an order unit space and let $u, v \in R_V$ with $u \neq v$. Then the following statements are equivalent:*

(1) *$(u, v) \bigcap R_V \neq \emptyset$;*
(2) *there are states f and g of V such that $f(u) = 1 = f(v)$ and $g(u) = 0 = g(v)$;*
(3) *$[u, v] \subset R_V$.*

Proof. Let $w \in (u, v) \bigcap R_V$. Then $w = (1 - \alpha)u + \alpha v \in (S_V)_0$ for some $\alpha \in (0, 1)$. Find $f, g \in S(V)$ such that $f((1 - \alpha)u + \alpha v) = 1$ and $g((1 - \alpha)u + \alpha v) = 0$. Thus

$$1 = f((1 - \alpha)u + \alpha v) = (1 - \alpha)f(u) + \alpha f(v) \leq (1 - \alpha) + \alpha = 1$$

so that $f(u) = 1 = f(v)$. Again

$$0 = g((1 - \alpha)u + \alpha v) = (1 - \alpha)g(u) + \alpha g(v)$$

so that $(1 - \alpha)g(u) = 0 = \alpha g(v)$. Since $0 < \alpha < 1$, we get $g(u) = 0 = g(v)$.

Let $\lambda \in [0, 1]$ and consider $x = (1 - \lambda)u + \lambda v$. Then Then $x \in V^+$ with $\|x\| \leq 1$. Also as

$$f(x) = (1 - \lambda)f(u) + \lambda f(v) = 1 - \lambda + \lambda = 1,$$

we note that $x \in C_V$. Further,

$$g(x) = (1 - \lambda)g(u) + \lambda g(v) = 0$$

so that $x \in R_V$. $\square$

Corollary 4.4. *Let $u, v \in R_V$ with $u \le v$. Then $[u, v] \subset R_V$.*

Proof. Put $w = \frac{1}{2}(u + v)$. Then $u \le w \le v$ and $e - v \le e - w \le e - u$. Since $u, v, e - u, e - v \in C_V$, we get that $w, e - w \in C_V$. Thus $w \in R_V$. Now by Proposition 4.3, we may conclude that $[u, v] \subset R_V$. $\square$

Now the next result follows immediately.

Corollary 4.5. *Let (V, e) be an order unit space and assume that $u_1, u_2 \in R_V$ with $u_1 \perp_\infty u_2$. Then*

(1) $[u_1, e - u_2] \bigcup [u_2, e - u_1] \subset R_V$;
(2) $((u_1, u_2) \bigcup (e - u_1, e - u_2)) \bigcap R_V = \emptyset$.

4.1. **Direct sum of order unit spaces.** Next we turn to describe $C_{\ell_\infty^n}$ and $R_{\ell_\infty^n}$.

Lemma 4.6. *Let (V_1, e_1) and (V_2, e_2) be any two order unit spaces. Consider $V = V_1 \times V_2$, $V^+ = V_!^+ \times V_2^+$ and $e = (e_1, e_2)$. Then (V, e) is also an order unit space and we have*

(1) $C_V = (C_{V_1} \times [0, e_2]_o) \bigcup ([0, e_2]_o \times C_{V_2})$ and
(2) $R_V = (R_{V_1} \times [0, e_2]_o) \bigcup ([0, e_2]_o \times R_{V_2}) \bigcup (C_{V_1} \times (e_2 - C_{V_2})) \bigcup ((e_1 - C_{V_1}) \times C_{V_2})$.

Proof. For $(v_1, v_2) \in V$, we have $\|(v_1, v_2)\| = \max\{\|v_1\|, \|v_2\|\}$. Thus $(u_1, u_2) \in C_V$ if and only if $u_1 \in V_1^+$, $u_2 \in V_2^+$ and $\max\{\|u_1\|, \|u_2\|\} = 1$. Therefore, $C_V = (C_{V_1} \times [0, e_2]_o) \bigcup ([0, e_2]_o \times C_{V_2})$. Now, as $(u_1, u_2) \in R_V$ if and only $(u_1, u_2), (e_1 - u_1, e_2 - u_2) \in C_V$, we may deduce that $R_V = (R_{V_1} \times [0, e_2]_o) \bigcup ([0, e_2]_o \times R_{V_2}) \bigcup (C_{V_1} \times (e_2 - C_{V_2})) \bigcup ((e_1 - C_{V_1}) \times C_{V_2})$. $\square$

Replace V_2 by $\mathbb{R}$. As $C_{\mathbb{R}} = \{1\}$ and $R_{\mathbb{R}} = \emptyset$, we may conclude the following:

Corollary 4.7. *Let (V, e) be an order unit space. Consider $\hat{V} = V \times \mathbb{R}$, $\hat{V}^+ = V^+ \times \mathbb{R}^+$ and $\hat{e} = (e, 1)$. Then $(\hat{V}, \hat{e})$ is an order unit space and we have*

(1) $C_{\hat{V}} = (C_V \times [0, 1]) \bigcup ([0, e_2]_o \times \{1\})$ and
(2) $R_{\hat{V}} = (R_V \times [0, 1]) \bigcup (C_V \times \{0\}) \bigcup ((e - C_V) \times \{1\})$.

Again using $C_1 = \{1\}$, $R_1 = \emptyset$ and following the induction on n, we can easily obtain the canopy and its periphery of ℓ_∞^n with the help of Corollary 4.7.

Corollary 4.8. *Fix $n \in \mathbb{N}$, $n \ge 2$. Put $C_n := C_{\ell_\infty^n}$ and $R_n := R_{\ell_\infty^n}$. Then*

(1) $C_n = \{(\alpha_1, \dots, \alpha_n) : \min\{\alpha_i\} \geq 0$ *and* $\max\{\alpha_i\} = 1\}$ *and*
(2) $R_n = \{(\alpha_1, \dots, \alpha_n) : \min\{\alpha_i\} = 0$ *and* $\max\{\alpha_i\} = 1\}$.

4.2. **Adjoining a normed linear space to an order unit space.** Let (V, e) be an order unit space and let X be a real normed linear space. Consider $V_X := V \oplus_1 X$ and put $V_X^+ = \{(v, x) : \|x\|e \leq v\}$ and $e_X = (e, 0)$. It was shown in [14] that (V_X, V_X^+, e_X) is an order unit space in such a way that the order unit norm coincides with the ℓ_1-norm on V_X. Here we describe the canopy and its periphery corresponding to V_X. For this purpose, we introduce the following notion.

Definition 4.9. *Let (V, e) be an order unit space. Then $u \in V^+$ is said to be a* semi-peripheral *element if $u = \alpha(e-w)+(1-\alpha)w$ for some $w \in R_V$ and $\alpha \in [0, 1]$. When $\alpha = \frac{1}{2}$, then $u = \frac{1}{2}e$ which is called the* central semi-peripheral *element. The set of all semi-peripheral elements of V is denoted by R_V^S.*

Theorem 4.10. *Let (V, V^+, e) be an order unit space and let X a real normed linear space. Consider the corresponding order unit space $(V \oplus_1 X, V_X^+, e_X)$. Then the canopy and the periphery of $V \oplus_1 X$ are given by*

$$C_{V \oplus_1 X} = \{(u, x) \in V_X^+ : \|x\|e \leq u \text{ and } \|u\| + \|x\| = 1\}$$

and

$$R_{V \oplus_1 X} = \{(u, x) : u \in R_V^S \text{ and } \|x\| + \|u\| = 1\}.$$

Proof. Let $w \in R_V$, $\alpha \in [0, 1]$ and put $u = \alpha(e - w) + (1 - \alpha)w$. Then $u \in V^+$ and $\|u\| = \max\{\alpha, 1 - \alpha\}$. Let $x \in X$ with $\|x\| = 1 - \|u\|$. For definiteness, we assume that $\alpha \geq \frac{1}{2}$. Then $\|u\| = \alpha$ and $\|x\| = 1 - \alpha$. Now, as

$$\begin{aligned} u - \|x\|e &= \alpha(e - w) + (1 - \alpha)w - (1 - \alpha)e \\ &= (2\alpha - 1)(e - w) \in V^+ \end{aligned}$$

and $\|(u, x)\|_1 = 1$, we have $(u, x) \in C_{V \oplus_1 X}$. Further, as $e - u = (1 - \alpha)(e - w) + \alpha w$, we have $\|e - u\| = \alpha$ so that $\|e_X - (u, x)\|_1 = 1$. Thus $(u, x) \in R_{V \oplus_1 X}$.

Conversely, we assume that $(u, x) \in R_{V \oplus_1 X}$. Then $(u, x), (e-u, -x) \in V_X^+$ and we have $\|(u, x)\|_1 = 1 = \|(e - u, -x)\|_1$. Thus $\|x\|e \leq u$, $\| - x\|e \leq e - u$ and

$$\|u\| + \|x\| = 1 = \|e - u\| + \| - x\|.$$

Now, it follows that $\|u\| = \|e-u\| = 1 - \|x\|$ whence $(1-\|u\|)e \le u$. Thus

$$\begin{aligned} \|u-(1-\|u\|)e\| &= \sup\{\phi\,(u-(1-\|u\|)e) : \phi \in S(V)\} \\ &= \sup\{\phi(u) : \phi \in S(V)\} - (1-\|u\|) \\ &= 2\|u\| - 1. \end{aligned}$$

If $2\|u\| - 1 = 0$, that is $\|u\| = \frac{1}{2}$, then $u = \frac{1}{2}e$ and is a (centrally) semi-peripheral element, for $(1-\|u\|)e \le u \le \|u\|e$. So we may assume that $2\|u\| - 1 > 0$. Put $w = \frac{u-(1-\|u\|)e}{2\|u\|-1}$. Then $w \in V^+$ and consequently, $w \in C_V$.

Find $\phi \in S(V)$ such that $f\phi(e-u) = \|e-u\| = \|u\|$. Then $\phi(w) = 0$ so that $w \in R_V$. Further, we have $u = \|x\|(e-w) + (1-\|x\|)w$ for $\|x\| = 1 - \|u\|$. Thus u is again a semi-peripheral element. □

Remark 4.11. *We can deduce from the proof of Theorem 4.10 that*

$$R_V^S = \{u \in [0,e] : \|u\| = \|e-u\|\} = \{u \in V : \|u\| = \|e-u\| \le 1\}.$$

5. Some applications

Lemma 5.1. *Let (V,e) be an order unit space of dimension ≥ 2 (so that $R_V \ne \emptyset$). Let $u \in R_V$ and consider*

$$P_u := \{\alpha e + \beta u : \alpha, \beta \in \mathbb{R}\}.$$

Then P_u is a unitally order isomorphic to ℓ_∞^2.

Proof. Consider the mapping $\chi : P_u \to \ell_\infty^2$ given by $\chi(\alpha e + \beta u) = (\alpha, \alpha+\beta)$ for all $\alpha, \beta \in \mathbb{R}$. Then χ is a unital bijection. We show that χ is an order isomorphism. We first assume that $\alpha e + \beta u \in V^+$. Since $u \in R_V$, we can find $f, g \in S(V)$ such that $f(u) = 1$ and $g(u) = 0$. Thus $0 \le f(\alpha e + \beta u) = \alpha + \beta$ and $0 \le g(\alpha e + \beta u) = \alpha$. Thus $(\alpha, \alpha+\beta) \in \ell_\infty^{2+}$. Conversely, we assume that $(\alpha, \alpha+\beta) \in \ell_\infty^{2+}$. Then $\alpha e + \beta u = \alpha(e-u) + (\alpha+\beta)u \in V^+$. □

Remark 5.2. *As $u \in R_V$, we have $u \perp_\infty (e-u)$. Thus, it is simple to show that χ is an isometry. In fact, if $v \in P_u$, say $v = \alpha e + \beta u = \alpha(e-u) + (\alpha+\beta)u$, then*

$$\|v\| = \|\alpha(e-u) + (\alpha+\beta)u\| = \max\{|\alpha|, |\alpha+\beta|\}.$$

Theorem 5.3. *Let (V,e) be an order unit space with $\dim(V) \ge 2$. Then V contains a copy of ℓ_∞^2 as an order unit subspace. Moreover, V contains a copy of ℓ_∞^n $(n \ge 2)$ as an order unit subspace if and only if there exist $u_1, \dots, u_n \in R_V$ such that $u_i \perp_\infty u_j$ for all $i, j \in \{1, \dots, n\}$ with $i \ne j$, $\sum_{i=1}^n u_i = e$ and $\perp_\infty$ is additive in the linear span of $u_1, \dots, u_n$.*

Proof. It follows from Lemma 5.1 that V contains a copy of ℓ_∞^2 as an order unit subspace.

Next, we assume that W is an order unit subspace of V and $\Gamma : \ell_\infty^n \to W$ is a surjective unital order isomorphism. Put $\gamma(e_i) = u_i$ for $i = 1, \dots, n$ where $\{e_1, \dots, e_n\}$ is the standard unit basis of ℓ_∞^n. Then $u_1, \dots, u_n \in C_V$ with $\sum_{i=1}^n u_i = e$. Consider the biorthonormal system $\{f_1, \dots, f_n\}$ in ℓ_1^n so that $f_i(e_j) = \delta_{ij}$. Then $\{f_1 \circ \Gamma^{-1}, \dots, f_n \circ \Gamma^{-1}\}$ is the set of pure states of W. We can extend $f_i \circ \Gamma^{-1}$ to a pure state g_i of V for each $i = 1, \dots, n$. Then $g_i(u_j) = \delta_{ij}$ so that $u_1, \dots, u_n \in R_V$ and we have $u_i \perp_\infty u_j$ for all $i, j \in \{1, \dots, n\}$ with $i \neq j$. Also, if $\alpha_1, \dots, \alpha_n \in \mathbb{R}$, then

$$\begin{aligned} \left\| \sum_{i=1}^n \alpha_i u_i \right\| &= \left\| \sum_{i=1}^n \alpha_i e_i \right\|_\infty \\ &= \max\{|\alpha_i| : 1 \leq i \leq n\} \\ &= \max\{\|\alpha_i u_i\| : 1 \leq i \leq n\}. \end{aligned}$$

Thus $\perp_\infty$ is additive in the linear span of $u_1, \dots, u_n$.

Conversely, we assume that there exist $u_1, \dots, u_n \in R_V$ such that $u_i \perp_\infty u_j$ for all $i, j \in \{1, \dots, n\}$ with $i \neq j$, $\sum_{i=1}^n u_i = e$ and $\perp_\infty$ is additive in the linear span U of $u_1, \dots, u_n$. Define $\Phi : U \to \ell_\infty^n$ by $\Phi(\sum_{i=1}^n \alpha_i u_i) = (\alpha_i)$. Then Φ is a unital linear bijection. Also

$$\begin{aligned} \left\| \Phi\left(\sum_{i=1}^n \alpha_i u_i \right) \right\| &= \|(\alpha_i)\|_\infty \\ &= \max\{|\alpha_i| : 1 \leq i \leq n\} \\ &= \max\{\|\alpha_i u_i\| : 1 \leq i \leq n\} \\ &= \left\| \sum_{i=1}^n u_i e_i \right\| \end{aligned}$$

as $\perp_\infty$ is additive on U. Thus that Φ is an isometry. Now, being a unital linear surjective isometry, Φ is a unital order isomorphism. $\square$

Remark 5.4. *An order unit space of dimension* 2 *is unitally isometric to* ℓ_∞^2. *However, we shall show in a forthcoming paper that for the dimension greater than* 2, *there exist non-isometric order unit spaces of the same dimension.*

Proposition 5.5. *Let* (V, e) *be an order unit space and let* $u, v \in R_V$. *Then* $u \neq v$ *if and only if* $[e, u] \bigcap [e, v] = \{e\}$.

Proof. First, let $w \in [e, u] \bigcap [e, v]$ with $w \neq e$.As $w \in [e, u] \bigcap [e, v]$, we can find $0 < \lambda, \mu \leq 1$ such that $w = e - \lambda(e - u) = e - \mu(e - v)$.

Then $\lambda(e-u) = \mu(e-v)$. Since $u, v \in R_V$, we have $e-u, e-v \in C_V$. Thus $\|e-u\| = 1 = \|e-v\|$ so that $\lambda = \mu$. As $\lambda, \mu > 0$, we get $u = v$. Thus $u \neq v$ implies $[e,u]\bigcap[e,v] = \{e\}$. Evidently, $u = v$ implies $[e,u]\bigcap[e,v] = [e,u] \neq \{e\}$. □

Remark 5.6. *We note that* $C_V = \bigcup_{u \in R_V}[0,u]$ *is a disjoint union of* untangled strings $[0,u]$*'s of* K_V *attached to* e*. Consequently,* $e - C_V = \bigcup_{u \in R_V}[0,u]$ *is a disjoint union of* untangled strings $[0,u]$*'s of* K_V *attached to* 0 *as well.*

Proposition 5.7. *Let* (V,e) *be an order unit space and let* $u, v \in R_V$*. Then, either* $P_u = P_v$ *or* $P_u \bigcap P_v = \mathbb{R}e$*.*

Proof. Let $w \in P_u \bigcap P_v$ be such that $w \notin \mathbb{R}e$. Without any loss of generality, we assume that $\|w\| = 1$. Consider $w_1 := \frac{1}{2}(e+w)$. Then $w_1 \in V^+$. Also then $w_1 \in P_u \bigcap P_v$. So, without any loss of generality again, we further assume that $w \in V^+$, that is, $w \in C_V$.

Since $w \in P_u$, we have $w = \alpha e + \beta u$ for some $\alpha, \beta \in \mathbb{R}$. As $w \in C_V$, we have $\alpha \geq 0$, $\alpha + \beta \geq 0$ and $\max\{\alpha, \alpha+\beta\} = 1$. Since $w \notin \mathbb{R}e$, we have $\beta \neq 0$. If $\beta > 0$, then we have $1 = \alpha + \beta > \alpha \geq 0$. Thus $w = \alpha e + (1-\alpha)u \in [e,u]$. Next if $\beta < 0$, then $1 = \alpha > \alpha + \beta \geq 0$ so that $-1 \leq \beta < 0$. Thus $w = e + \beta u = e - (-\beta)(e-(e-u)) \in [e, e-u]$. Summing up, we have $w \in [e,u]\bigcup[e,e-u]$. Similarly, as $w \in P_v$, we also have $w \in [e,v]\bigcup[e,e-v]$. Thus $w \in ([e,u]\bigcup[e,e-u]) \bigcap ([e,v]\bigcup[e,e-v])$. Since $w \neq e$, using Proposition 5.5, we conclude that one of the equalities $[e,u] = [e,v]$ or $[e,u] = [e,e-v]$, or $[e,e-u] = [e,v]$, or $[e,e-u] = [e,e-v]$ hold. In other words, either $u = v$ or $u = e - v$. In both the situations, we have $P_u = P_v$. □

Theorem 5.8. *Let* (V,e) *be an order unit space with* $\dim V \geq 2$*. Then* $V = \bigcup\{P_u : u \in R_V\}$ *in such a way that* $\bigcap\{P_u : u \in R_V\} = \mathbb{R}e$ *and if* $v \in V$ *with* $v \notin \mathbb{R}e$*, then there exists a unique* $w \in R_V$ *such that* $v \in P_w = P_{(e-w)}$*.*

Proof. Let $v \in V$ with $v \notin \mathbb{R}e$. For simplicity, we assume that $\|v\| = 1$. Put $v_1 = \frac{1}{2}(e+v)$ and $v_2 = \frac{1}{2}(e-v)$. Then $v_1, v_2 \in V^+$ and $v_1, v_2 \notin \mathbb{R}e$. Also then $v = v_1 - v_2$ and $1 = \|v\| = \max\{\|v_1\|, \|v_2\|\}$. Replacing v by $-v$, if required, we further assume that $0 < \|v_2\| \leq \|v_1\| = 1$, that is, $\|e+v\| = 2$. Thus we can find $f \in S(V)$ such that $1 + f(v) = f(e+v) = 2$. Therefore, $f(v) = 1$. Put $w = \|e-v\|^{-1}(e-v)$. Then $w \in C_V$. Further, $f(e-w) = 1 - \|e-v\|^{-1}(1-f(v)) = 1$ so that $e - w \in C_V$. Thus $w \in R_V$. Now $v = e - \|e-v\|w$ so that $v \in P_w = P_{(e-w)}$.

Uniqueness of w follows from Proposition 5.7. □

Remark 5.9. *Let* $v \in V$ *with* $v \notin \mathbb{R}e$*. Find* $f \in S(V)$ *such that* $\|v\| = |f(v)|$*. If* $\|v\| = f(v)$*, then* $\|e + \|v\|^{-1}v\| = 2$*. Thus* v *has a unique* *representation* $v = \lambda e + \mu w$ *in* P_w *where* $w = \| \|v\|e - v\|^{-1}(\|v\|e - v) \in R_V$ *and* $\lambda = \|v\|$ *and* $\mu = -\| \|v\|e - v\|$. *When* $\|v\| = -f(v)$, *we replace* v *by* $-v$.

Acknowledgements: The author was partially supported by Science and Engineering Research Board, Department of Science and Technology, Government of India sponsored Mathematical Research Impact Centric Support project (reference no. MTR/2020/000017).

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