On the properties $\alpha - z$ Rényi divergences on general von Neumann algebras

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1 Introduction

2 Preliminaries

2.1 Basic definitions

Let $\mathcal{M}$ be a von Neumann algebra and let $\mathcal{M}^+$ be the cone of positive elements in $\mathcal{M}$. We denote the predual by $\mathcal{M}_*$, its positive part by $\mathcal{M}_*^+$ and the set of normal states by $\mathfrak{S}_*(\mathcal{M})$. For $\psi \in \mathcal{M}_*^+$, we will denote by $s(\psi)$ the support projection of ψ.

For $0 < p \leq \infty$, let $L_p(\mathcal{M})$ be the Haagerup L_p-space over $\mathcal{M}$ and let $L_p(\mathcal{M})$ its positive cone, [4]. We will use the identifications $\mathcal{M} \simeq L_\infty(\mathcal{M})$, $\mathcal{M}_* \ni \psi \leftrightarrow h_\psi \in L_1(\mathcal{M})$ and the notation $\operatorname{Tr} h_\psi = \psi(1)$ for the trace in $L_1(\mathcal{M})$. It this way, $\mathcal{M}_*^+$ is identified with the positive cone $L_1(\mathcal{M})^+$ and $\mathfrak{S}_*(\mathcal{M})$ with subset of elements in $L_1(\mathcal{M})^+$ with unit trace. Precise definitions and further details on the spaces $L_p(\mathcal{M})$ can be found in the notes [19].

2.2 The $\alpha - z$-Rényi divergences

In [11, 12], the $\alpha - z$-Rényi divergence for $\psi, \varphi \in \mathcal{M}_*^+$ was defined as follows:

Definition 2.1. Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$ and let $\alpha, z > 0$, $\alpha \neq 1$. The $\alpha - z$-Rényi divergence is defined as

$$D_{\alpha,z}(\psi\|\varphi) := \frac{1}{\alpha - 1} \log \frac{Q_{\alpha,z}(\psi\|\varphi)}{\psi(1)},$$

where

$$Q_{\alpha,z} = \begin{cases} \operatorname{Tr}\left(h_\varphi^{\frac{1-\alpha}{2z}} h_\psi^{\frac{\alpha}{z}} h_\varphi^{\frac{1-\alpha}{2z}}\right)^z, & \text{if } 0 < \alpha < 1 \\ \|x\|_z^z, & \text{if } \alpha > 1 \text{ and} \\ & h_\psi^{\frac{\alpha}{z}} = h_\varphi^{\frac{\alpha-1}{2z}} x h_\varphi^{\frac{\alpha-1}{2z}}, \text{ with } x \in s(\varphi)L_z(\mathcal{M})s(\varphi) \\ \infty & \text{otherwise.} \end{cases}$$

In the case $\alpha > 1$, the following alternative form will be useful.

Lemma 2.2. *[11, Lemma 7] Let $\alpha > 1$ and $\psi, \varphi \in \mathcal{M}_*^+$. Then $Q_{\alpha,z}(\psi\|\varphi) < \infty$ if and only if there is some $y \in L_{2z}(\mathcal{M})s(\varphi)$ such that*

$$h_\psi^{\frac{\alpha}{2z}} = y h_\varphi^{\frac{\alpha-1}{2z}}.$$

Moreover, in this case, such y is unique and we have $Q_{\alpha,z}(\psi\|\varphi) = \|y\|_{2z}^{2z}$.

The standard Rényi divergence [5, 6, 16] is contained in this range as $D_\alpha(\psi\|\varphi) = D_{\alpha,1}(\psi\|\varphi)$. The sandwiched Rényi divergence is obtained as $\tilde{D}_\alpha(\psi\|\varphi) = D_{\alpha,\alpha}(\psi\|\varphi)$, see [1, 6, 8, 9] for some alternative definitions and properties of $\tilde{D}_\alpha$. The definition in [8] and [9] is based on the Kosaki interpolation spaces $L_p(\mathcal{M}, \varphi)$ with respect to a state [13]. These spaces and complex interpolation method will be used frequently also in the present work.

Many of the properties of $D_{\alpha,z}(\psi\|\varphi)$ were extended from the finite dimensional case in [11]. In particular, the following variational expressions will be an important tool for our work.

Theorem 2.3 (Variational expressions)**.** *Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$.*

(i) Let $0 < \alpha < 1$ and $\max\{\alpha, 1-\alpha\} \le z$. Then

$$Q_{\alpha,z}(\psi\|\varphi) = \inf_{a \in \mathcal{M}^{++}} \left\{ \alpha \mathrm{Tr}\left((a^{\frac12} h_\psi^{\frac{\alpha}{z}} a^{\frac12})^{\frac{z}{\alpha}} \right) + (1-\alpha)\mathrm{Tr}\left((a^{\frac12} h_\varphi^{\frac{1-\alpha}{z}} a^{\frac12})^{\frac{z}{1-\alpha}} \right) \right\}.$$

(ii) Let $1 < \alpha$, $\max\{\frac{\alpha}{2}, \alpha - 1\} \le z$. Then

$$Q_{\alpha,z}(\psi\|\varphi) = \sup_{a \in \mathcal{M}_+} \left\{ \alpha \mathrm{Tr}\left((a^{\frac12} h_\psi^{\frac{\alpha}{z}} a^{\frac12})^{\frac{z}{\alpha}} \right) - (\alpha-1)\mathrm{Tr}\left((a^{\frac12} h_\varphi^{\frac{\alpha-1}{2z}} a^{\frac12})^{\frac{z}{\alpha-1}} \right) \right\}.$$

Proof. For part (i) see [11, Theorem 1 (vi)]. The inequality $\ge$ in part (ii) holds for all α and z and was proved in [11, Theorem 2 (vi)]. We now prove the opposite inequality.

Assume first that $Q_{\alpha,z}(\psi\|\varphi) < \infty$, so that there is some $x \in s(\varphi)L_z(\mathcal{M})^+ s(\varphi)$ such that $h_\psi^{\frac{\alpha}{z}} = h_\varphi^{\frac{\alpha-1}{2z}} x h_\varphi^{\frac{\alpha-1}{2z}}$. Plugging this into the right hand side, we obtain

$$\begin{aligned}
&\sup_{a \in \mathcal{M}_+} \left\{ \alpha \mathrm{Tr}\left((a^{\frac12} h_\psi^{\frac{\alpha}{z}} a^{\frac12})^{\frac{z}{\alpha}} \right) - (\alpha-1)\mathrm{Tr}\left((a^{\frac12} h_\varphi^{\frac{\alpha-1}{2z}} a^{\frac12})^{\frac{z}{\alpha-1}} \right) \right\} \\
&= \sup_{a \in \mathcal{M}_+} \left\{ \alpha \mathrm{Tr}\left((a^{\frac12} h_\varphi^{\frac{\alpha-1}{2z}} x h_\varphi^{\frac{\alpha-1}{2z}} a^{\frac12})^{\frac{z}{\alpha}} \right) - (\alpha-1)\mathrm{Tr}\left((a^{\frac12} h_\varphi^{\frac{\alpha-1}{2z}} a^{\frac12})^{\frac{z}{\alpha-1}} \right) \right\} \\
&= \sup_{a \in \mathcal{M}_+} \left\{ \alpha \mathrm{Tr}\left((x^{\frac12} h_\varphi^{\frac{\alpha-1}{2z}} a h_\varphi^{\frac{\alpha-1}{2z}} x^{\frac12})^{\frac{z}{\alpha}} \right) - (\alpha-1)\mathrm{Tr}\left((h_\varphi^{\frac{\alpha-1}{2z}} a h_\varphi^{\frac{\alpha-1}{2z}})^{\frac{z}{\alpha-1}} \right) \right\} \\
&= \sup_{w \in L_{\frac{z}{\alpha-1}}(\mathcal{M})^+} \left\{ \alpha \mathrm{Tr}\left((x^{\frac12} w x^{\frac12})^{\frac{z}{\alpha}} \right) - (\alpha-1)\mathrm{Tr}\left(w^{\frac{z}{\alpha-1}} \right) \right\},
\end{aligned}$$

where we used the fact that $\mathrm{Tr}\left((h^*h)^p\right) = \mathrm{Tr}\left((hh^*)^p\right)$ for $p > 0$ and $h \in L_{\frac{p}{2}}(\mathcal{M})$ and Lemma A.1. Putting $w = x^{\alpha-1}$ we get

$$\sup_{w \in L_{\frac{z}{\alpha-1}}(\mathcal{M})^+} \left\{ \alpha \mathrm{Tr}\left((x^{\frac12} w x^{\frac12})^{\frac{z}{\alpha}} \right) - (\alpha-1)\mathrm{Tr}\left(w^{\frac{z}{\alpha-1}} \right) \right\} \ge \mathrm{Tr}\left(x^z\right) = \|x\|_z^z = Q_{\alpha,z}(\psi\|\varphi).$$

This finishes the proof in the case that $Q_{\alpha,z}(\psi\|\varphi) < \infty$. Note that this holds if $\psi \le \lambda\varphi$ for some $\lambda > 0$. Indeed, since $\frac{\alpha}{2z} \in (0,1]$ by the assumption, we then have

$$h_\psi^{\frac{\alpha}{2z}} \le \lambda^{\frac{\alpha}{2z}} h_\varphi^{\frac{\alpha}{2z}},$$

hence by [6, Lemma A.58] there is some $b \in \mathcal{M}$ such that

$$h_\psi^{\frac{\alpha}{2z}} = b h_\varphi^{\frac{\alpha}{2z}} = y h_\varphi^{\frac{\alpha-1}{2z}},$$

where $y = b h_\varphi^{\frac{1}{2z}} \in L_{2z}(\mathcal{M})$. By Lemma 2.2 we get $Q_{\alpha,z}(\psi\|\varphi) = \|y\|_{2z}^{2z} < \infty$.

In the general case, the variational expression holds for $Q_{\alpha,z}(\psi\|\varphi + \epsilon\psi)$ for all $\epsilon > 0$, so that we have

$$\begin{aligned} Q_{\alpha,z}(\psi\|\varphi+\epsilon\psi) &= \sup_{a\in\mathcal{M}_+} \left\{\alpha \mathrm{Tr}\left((a^{\frac12} h_\psi^{\frac{\alpha}{z}} a^{\frac12})^{\frac{z}{\alpha}}\right) - (\alpha-1)\mathrm{Tr}\left((a^{\frac12} h_{\varphi+\epsilon\psi}^{\frac{\alpha-1}{2z}} a^{\frac12})^{\frac{z}{\alpha-1}}\right)\right\} \\ &\le \sup_{a\in\mathcal{M}_+} \left\{\alpha \mathrm{Tr}\left((a^{\frac12} h_\psi^{\frac{\alpha}{z}} a^{\frac12})^{\frac{z}{\alpha}}\right) - (\alpha-1)\mathrm{Tr}\left((a^{\frac12} h_{\varphi}^{\frac{\alpha-1}{2z}} a^{\frac12})^{\frac{z}{\alpha-1}}\right)\right\}, \end{aligned}$$

where the inequality above follows by Lemma A.3. Therefore, since lower semicontinuity [11, Theorem 2 (iv)] gives

$$Q_{\alpha,z}(\psi\|\varphi) \le \liminf_{\epsilon\searrow 0} Q_{\alpha,z}(\psi\|\varphi+\epsilon\psi)$$

the desired inequality follows.

□

We finish this section by investigation of the properties of the variatonal expression for $0 < \alpha < 1$, in the case when $\lambda^{-1}\varphi \le \psi \le \lambda\varphi$ for some $\lambda > 0$. This will be denoted as $\psi \sim \varphi$.

Lemma 2.4. *Assume that $\psi \sim \varphi$. Then the infimum in the variational expression in Theorem 2.3 (i) is attained at a unique element $\bar{a} \in \mathcal{M}^{++}$. This element satisfies*

$$h_\psi^{\frac{\alpha}{2z}} \bar{a} h_\psi^{\frac{\alpha}{2z}} = \left(h_\psi^{\frac{\alpha}{2z}} h_\varphi^{\frac{1-\alpha}{z}} h_\psi^{\frac{\alpha}{2z}}\right)^{\alpha} \tag{2.1}$$

$$h_\varphi^{\frac{1-\alpha}{2z}} \bar{a}^{-1} h_\varphi^{\frac{1-\alpha}{2z}} = \left(h_\varphi^{\frac{1-\alpha}{2z}} h_\psi^{\frac{\alpha}{z}} h_\varphi^{\frac{1-\alpha}{2z}}\right)^{1-\alpha}. \tag{2.2}$$

Proof. We may assume that φ and hence also ψ is faithful. Following the proof of [11, Theorem 1 (vi)], we may use the assumptions and [6, Lemma A.58] to show that there are $b, c \in \mathcal{M}$ such that

$$h_\varphi^{\frac{1-\alpha}{2z}} = b\left(h_\varphi^{\frac{1-\alpha}{2z}} h_\psi^{\frac{\alpha}{z}} h_\varphi^{\frac{1-\alpha}{2z}}\right)^{\frac{1-\alpha}{2}}, \qquad \left(h_\varphi^{\frac{1-\alpha}{2z}} h_\psi^{\frac{\alpha}{z}} h_\varphi^{\frac{1-\alpha}{2z}}\right)^{\frac{1-\alpha}{2}} = c h_\varphi^{\frac{1-\alpha}{2z}}. \tag{2.3}$$

With $\bar{a} := bb^* \in \mathcal{M}^{++}$ we have $\bar{a}^{-1} = c^*c$ and $\bar{a}$ is indeed a minimizer of

$$Q_{\alpha,z}(\psi\|\varphi) = \inf_{a\in\mathcal{M}^{++}} \left\{\alpha \left\|h_\psi^{\frac{\alpha}{2z}} a h_\psi^{\frac{\alpha}{2z}}\right\|_{\frac{z}{\alpha}}^{\frac{z}{\alpha}} + (1-\alpha)\left\|h_\varphi^{\frac{1-\alpha}{2z}} a^{-1} h_\varphi^{\frac{1-\alpha}{2z}}\right\|_{\frac{z}{1-\alpha}}^{\frac{z}{1-\alpha}}\right\}. \tag{2.4}$$

We next observe that such minimizer is unique. Indeed, suppose that the infimum is attained at some $a_1, a_2 \in \mathcal{M}^{++}$. Let $a_0 := (a_1 + a_2)/2$. Since the map $L^p(\mathcal{M}) \ni k \mapsto \|k\|_p^p$ is convex for any $p \ge 1$ and $a_0^{-1} \le (a_1^{-1} + a_2^{-1})/2$, we have using Lemma A.2 in the second inequality that

$$\left\|h_\psi^{\frac{\alpha}{2z}} a_0 h_\psi^{\frac{\alpha}{2z}}\right\|_{\frac{z}{\alpha}}^{\frac{z}{\alpha}} \le \frac12\left\{\left\|h_\psi^{\frac{\alpha}{2z}} a_1 h_\psi^{\frac{\alpha}{2z}}\right\|_{\frac{z}{\alpha}}^{\frac{z}{\alpha}} + \left\|h_\psi^{\frac{\alpha}{2z}} a_2 h_\psi^{\frac{\alpha}{2z}}\right\|_{\frac{z}{\alpha}}^{\frac{z}{\alpha}}\right\},$$

$$\begin{aligned} \left\|h_\varphi^{\frac{1-\alpha}{2z}} a_0^{-1} h_\varphi^{\frac{1-\alpha}{2z}}\right\|_{\frac{z}{1-\alpha}}^{\frac{z}{1-\alpha}} &\le \left\|h_\varphi^{\frac{1-\alpha}{2z}} \left(\frac{a_1^{-1}+a_2^{-1}}{2}\right) h_\varphi^{\frac{1-\alpha}{2z}}\right\|_{\frac{z}{1-\alpha}}^{\frac{z}{1-\alpha}} \\ &\le \frac12\left\{\left\|h_\varphi^{\frac{1-\alpha}{2z}} a_1^{-1} h_\varphi^{\frac{1-\alpha}{2z}}\right\|_{\frac{z}{1-\alpha}}^{\frac{z}{1-\alpha}} + \left\|h_\varphi^{\frac{1-\alpha}{2z}} a_2^{-1} h_\varphi^{\frac{1-\alpha}{2z}}\right\|_{\frac{z}{1-\alpha}}^{\frac{z}{1-\alpha}}\right\}. \end{aligned}$$

Hence we have

$$\left\| h_\varphi^{\frac{1-\alpha}{2z}} a_0^{-1} h_\varphi^{\frac{1-\alpha}{2z}} \right\|_{\frac{z}{1-\alpha}} = \left\| h_\varphi^{\frac{1-\alpha}{2z}} \left(\frac{a_1^{-1} + a_2^{-1}}{2} \right) h_\varphi^{\frac{1-\alpha}{2z}} \right\|_{\frac{z}{1-\alpha}},$$

which implies that $a_0^{-1} = \frac{a_1^{-1}+a_2^{-1}}{2}$, as easily verified by Lemma A.2. From this we easily have $a_1 = a_2$.

The equality (2.2) is obvious from the second equality in (2.3) and $\bar{a}^{-1} = c^*c$. Since $Q_{\alpha,z}(\psi\|\varphi) = Q_{1-\alpha,z}(\varphi\|\psi)$, we see by uniqueness that the minimizer of the infimum expression for $Q_{1-\alpha,z}(\varphi\|\psi)$ (instead of (2.4)) is $\bar{a}^{-1}$ (instead of $\bar{a}$). This says that (2.1) is the equality corresponding to (2.2) when ψ, φ, α are replaced with $\varphi, \psi, 1-\alpha$, respectively.

$\square$

In the next lemma, we will use the following notations:

$$p := \frac{z}{\alpha}, \quad r := \frac{z}{1-\alpha}, \quad \xi_p(a) := h_\psi^{\frac{1}{2p}} a h_\psi^{\frac{1}{2p}}, \quad \eta_r(a) = h_\varphi^{\frac{1}{2r}} a^{-1} h_\varphi^{\frac{1}{2r}}.$$

We will also denote the function under the infimum in the variational expression in Theorem 2.3 (i) by f, that is,

$$f(a) = \alpha\|\xi_p(a)\|_p^p + (1-\alpha)\|\eta_r(a)\|_r^r, \qquad a \in \mathcal{M}^{++}.$$

Lemma 2.5. *Assume that $\psi \sim \varphi$ and let $0 < \alpha < 1$, $\max\{\alpha, 1-\alpha\} \le z$. If $p > 1$, then for every $C \ge Q_{\alpha,z}(\psi\|\varphi)$ and $\varepsilon > 0$ there is some $\delta > 0$ such that whenever $\|\xi_p(b)\|_p^p \le C$ and $\|\xi_p(b) - \xi_p(\bar{a})\|_p \ge \varepsilon$, we have*

$$f(b) - Q_{\alpha,z}(\psi\|\varphi) \ge \delta.$$

A similar statement holds if $r > 1$.

Proof. By assumptions, $p, r \ge 1$. For $a, b \in \mathcal{M}^{++}$ and $s \in (1/2, 0)$, we have

$$\begin{aligned}\|\xi_p(sb + (1-s)a)\|_p^p = \|s\xi_p(b) + (1-s)\xi_p(a)\|_p^p &= \|(1-2s)\xi_p(a) + 2s\frac{1}{2}(\xi_p(a) + \xi_p(b))\|_p^p \\ &\le (1-2s)\|\xi_p(a)\|_p^p + 2s\|\frac{1}{2}(\xi_p(a) + \xi_p(b))\|_p^p.\end{aligned}$$

Similarly,

$$\|\eta_r(sb + (1-s)a)\|_r^r \le (1-2s)\|\eta_r(a)\|_r^r + 2s\|\frac{1}{2}(\eta_r(a) + \eta_r(b))\|_r^r,$$

here we also used the fact that $(ta + (1-t)b)^{-1} \le ta^{-1} + (1-t)b^{-1}$ for $t \in (0,1)$ and Lemma A.2. It follows that

$$\begin{aligned}\langle \nabla f(a), b - a\rangle &= \lim_{s\to 0^+} s^{-1}[f(sb + (1-s)a) - f(a)] \\ &\le 2\alpha\Big(\|\frac{1}{2}(\xi_p(a) + \xi_p(b))\|_p^p - \|\xi_p(a)\|_p^p \Big) + 2(1-\alpha)\Big(\|\frac{1}{2}(\eta_r(a) + \eta_r(b))\|_r^r - \|\eta_r(a)\|_r^r \Big) \\ &= f(b) - f(a) - 2\Big(\alpha\big(\frac{1}{2}\|\xi_p(a)\|_p^p + \frac{1}{2}\|\xi_p(b)\|_p^p - \|\frac{1}{2}(\xi_p(a) + \xi_p(b))\|_p^p\big) \\ &\qquad + (1-\alpha)\big(\frac{1}{2}\|\eta_r(a)\|_r^r + \frac{1}{2}\|\eta_r(b)\|_r^r - \|\frac{1}{2}(\eta_r(a) + \eta_r(b))\|_r^r\big) \Big).\end{aligned}$$

Since $p, r \geq 1$, both terms in brackets in the last expression above are nonnegative. Assume now that $p > 1$. Let $\bar{a} \in \mathcal{M}^{++}$ be the minimizer as in Lemma 2.4, then $f(\bar{a}) = Q_{\alpha,z}(\psi\|\varphi)$ and $\nabla f(\bar{a}) = 0$, so that we get

$$f(b) - Q_{\alpha,z}(\psi\|\varphi) \geq 2\alpha\Big(\frac{1}{2}\|\xi_p(a)\|_p^p + \frac{1}{2}\|\xi_p(b)\|_p^p - \|\frac{1}{2}(\xi_p(a) + \xi_p(b))\|_p^p\Big).$$

The space $L_p(\mathcal{M})$ is uniformly convex, so that the function $h \mapsto \|h\|_p^p$ is uniformly convex on each set where it is bounded ([**?** , Thm. 3.7.7. and p. 288]). Hence for each $C > 0$ and $\epsilon > 0$ there is some $\delta > 0$ such that for every h, k with $\|h\|_p^p, \|k\|_p^p \leq C$ and $\|h - k\|_p \geq \epsilon$, we have

$$\frac{1}{2}\|h\|_p^p + \frac{1}{2}\|k\|_p^p - \|\frac{1}{2}(h+k)\|_p^p \geq \delta,$$

([**?** , Exercise 3.3]). The proof in the case $r > 1$ is similar.

□

3 Data processing inequality and reversibility of channels

Let $\gamma : \mathcal{N} \to \mathcal{M}$ be a normal positive unital map. Then the predual of γ defines a positive linear map $\gamma_* : L_1(\mathcal{M}) \to L_1(\mathcal{N})$ that preserves the trace, acting as

$$L_1(\mathcal{M}) \ni h_\rho \mapsto h_{\rho\circ\gamma} \in L_1(\mathcal{N}).$$

The support of γ will be denoted by $s(\gamma)$, recall that this is defined as the smallest projection $e \in \mathcal{N}$ such that $\gamma(e) = 1$ and in this case, $\gamma(a) = \gamma(eae)$ for any $a \in \mathcal{N}$. For any $\rho \in \mathcal{M}_*^+$ we clearly have $s(\rho \circ \gamma) \leq s(\gamma)$, with equality if ρ is faithful. It follows that γ_* maps $L_1(\mathcal{M})$ to $s(\gamma)L_1(\mathcal{N})s(\gamma) \equiv L_1(s(\gamma)\mathcal{N}s(\gamma))$. For any $\rho \in \mathcal{M}_+^*$, $\rho \neq 0$, the map

$$\gamma_0 : s(\gamma)\mathcal{N}s(\gamma) \to s(\rho)\mathcal{M}s(\rho), \qquad a \mapsto s(\rho)\gamma(a)s(\rho)$$

is a faithful normal positive unital map. Moreover, for any $\sigma \in \mathcal{M}_*^+$ such that $s(\sigma) \leq s(\rho)$, we have for any $a \in \mathcal{N}$,

$$\sigma(\gamma_0(s(\gamma)as(\gamma))) = \sigma(s(\rho)\gamma(a)s(\rho)) = \sigma(a).$$

Replacing γ by γ_0 and ρ by the restriction $\rho|_{s(\rho)\mathcal{M}s(\rho)}$, we may assume that both ρ and $\rho \circ \gamma$ are faithful.

The Petz dual of γ with respect to a faithful $\rho \in \mathcal{M}_*^+$ is a map $\gamma_\rho^* : \mathcal{M} \to \mathcal{N}$, introduced in [18]. It was proved that it is again normal, positive and unital, in addition, it is n-positive whenever γ is. As explained in [8] γ_ρ^* is determined by the equality

$$(\gamma_\rho^*)_*(h_{\rho\circ\gamma}^{\frac{1}{2}} b h_{\rho\circ\gamma}^{\frac{1}{2}}) = h_\rho^{\frac{1}{2}}\gamma(b)h_\rho^{\frac{1}{2}}, \tag{3.1}$$

for all $b \in \mathcal{N}^+$, here $(\gamma_\rho^*)_*$ is the predual map of γ_ρ^*. We also have

$$(\gamma_\rho^*)_*(h_{\rho\circ\gamma}) = (\gamma_\rho^*)_* \circ \gamma_*(h_\rho) = h_\rho$$

and $(\gamma_\rho^*)_{\rho\circ\gamma}^* = \gamma$. In the special case that γ is the inclusion map $\gamma : \mathcal{N} \hookrightarrow \mathcal{M}$ for a subalgebra $\mathcal{N} \subseteq \mathcal{M}$, the Petz dual is the generalized conditional expectation $\mathcal{E}_{\mathcal{N},\varphi} : \mathcal{M} \to \mathcal{N}$, as introduced in [**?**]; see e.g. [6, Proposition 6.5]. Hence $\mathcal{E}_{\mathcal{N},\varphi}$ is a normal completely positive unital with range in $\mathcal{N}$ and such that

$$\varphi \circ \mathcal{E}_{\mathcal{N},\varphi} = \varphi.$$

3.1 Data processing inequality

In this paragraph we prove the data processing inequality (DPI) for $D_{\alpha,z}$ with respect to normal positive unital maps. In the case of the sandwiched divergences $\tilde{D}_\alpha$ with $1/2 \le \alpha \neq 1$, DPI was proved in [8, 9], see also [1] for an alternative proof in the case when the maps are also completely positive.

Lemma 3.1. *Let $\gamma : \mathcal{N} \to \mathcal{M}$ be a normal positive unital map and let $\rho \in \mathcal{M}_*^+$, $b \in \mathcal{N}^+$.*

(i) If $p \in [1/2, 1)$, then

$$\|h_{\rho\circ\gamma}^{\frac{1}{2p}} b h_{\rho\circ\gamma}^{\frac{1}{2p}}\|_p \le \|h_\rho^{\frac{1}{2p}} \gamma(b) h_\rho^{\frac{1}{2p}}\|_p.$$

(ii) If $p \in [1, \infty]$, the inequality reverses.

Proof. Let us denote $\beta := \gamma_\rho^*$ and let $\omega \in \mathcal{M}_*^+$ be such that $h_\omega := h_{\rho\circ\gamma}^{\frac{1}{2}} b h_{\rho\circ\gamma}^{\frac{1}{2}} \in L_1(\mathcal{N})^+$. Then β is a normal positive unital map and we have

$$\beta_*(h_\omega) = h_\rho^{\frac{1}{2}} \gamma(b) h_\rho^{\frac{1}{2}}, \qquad \beta_*(h_{\rho\circ\gamma}) = h_\rho.$$

Let $p \in [1/2, 1)$, then

$$\begin{aligned} \|h_\rho^{\frac{1}{2p}} \gamma(b) h_\rho^{\frac{1}{2p}}\|_p^p &= \|h_\rho^{\frac{1-p}{2p}} \beta_*(h_\omega) h_\rho^{\frac{1-p}{2p}}\|_p^p = Q_{p,p}(\beta_*(h_\omega)\|h_\rho) = Q_{p,p}(\beta_*(h_\omega)\|\beta_*(h_{\rho\circ\gamma})) \\ &\ge Q_{p,p}(h_\omega\|h_{\rho\circ\gamma}) = \|h_{\rho\circ\gamma}^{\frac{1-p}{2p}} h_\omega h_{\rho\circ\gamma}^{\frac{1-p}{2p}}\|_p^p = \|h_{\rho\circ\gamma}^{\frac{1}{2p}} b h_{\rho\circ\gamma}^{\frac{1}{2p}}\|_p^p. \end{aligned}$$

Here we have used the DPI for the sandwiched Rényi divergence $D_{\alpha,\alpha}$ for $\alpha \in [1/2, 1)$, [9, Theorem 4.1]. This proves (i). The case (ii) was proved in [11] (see Eq. (22) therein), using the relation of the sandwiched Rényi divergence to the Kosaki L_p norms. In our setting, the proof can be written as

$$\begin{aligned} \|h_\rho^{\frac{1}{2p}} \gamma(b) h_\rho^{\frac{1}{2p}}\|_p^p &= Q_{p,p}(h_\rho^{\frac{1}{2}} \gamma(b) h_\rho^{\frac{1}{2}}\|h_\rho) = Q_{p,p}(\beta_*(h_\omega)\|\beta_*(h_{\rho\circ\gamma})) \\ &\le Q_{p,p}(h_\omega\|h_{\rho\circ\gamma}) = \|h_{\rho\circ\gamma}^{\frac{1}{2p}} b h_{\rho\circ\gamma}^{\frac{1}{2p}}\|_p^p, \end{aligned}$$

here the inequality follows from the DPI for sandwiched Rényi divergence $D_{\alpha,\alpha}$ with $\alpha > 1$, [8]. □

Remark 3.2. Using the fact that $(\gamma_\rho^*)_*$ is a contraction for the Kosaki L_p spaces $L_p(\mathcal{N}, \rho\circ\gamma) \to L_p(\mathcal{M}, \rho)$ for all $p \ge 1$ (see [8, Thm.] and properties of the Petz dual), we can see that the map $\gamma_{\rho,p} : L_p(\mathcal{N}) \to L_p(\mathcal{M})$ determined by

$$\gamma_{\rho,p} : h_{\rho\circ\gamma}^{\frac{1}{2p}} a h_{\rho\circ\gamma}^{\frac{1}{2p}} \mapsto h_\rho^{\frac{1}{2p}} \gamma(a) h_\rho^{\frac{1}{2p}}, \qquad a \in \mathcal{N}$$

is a contraction. Notice that $\gamma_{\rho,1} = (\gamma_\rho^*)_*$, $\gamma_{\rho,\infty} = \gamma$ and for any $p \ge 1$,

$$(\gamma_\rho^*)_*(h_{\rho\circ\gamma}^{\frac{p-1}{2p}} w h_{\rho\circ\gamma}^{\frac{p-1}{2p}}) = h_\rho^{\frac{p-1}{2p}} \gamma_{\rho,p}^*(w) h_\rho^{\frac{p-1}{2p}}, \qquad w \in L_p(\mathcal{N}).$$

Theorem 3.3 (DPI)**.** *Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$ and let $\gamma : \mathcal{N} \to \mathcal{M}$ be a normal positive unital map. Assume either of the following conditions:*

(i) $0 < \alpha < 1$, $\max\{\alpha, 1-\alpha\} \le z$

(ii) $\alpha > 1$, $\max\{\alpha/2, \alpha - 1\} \le z \le \alpha$.

Then we have

$$D_{\alpha,z}(\psi \circ \gamma \| \varphi \circ \gamma) \le D_{\alpha,z}(\psi \| \varphi).$$

Proof. Under the conditions (i), the DPI was proved in [11, Theorem 1 (viii)]. Since parts of this proof will be used below, we repeat it here.

Assume the conditions in (i) and put $p := \frac{z}{\alpha}$, $r := \frac{z}{1-\alpha}$, so that $p, r \ge 1$. For any $b \in \mathcal{N}^{++}$, we have by the Choi inequality [2] that $\gamma(b)^{-1} \le \gamma(b^{-1})$, so that by Lemma A.2 and 3.1 (ii), we have

$$\|h_\varphi^{\frac{1}{2r}} \gamma(b)^{-1} \varphi^{\frac{1}{2r}}\|_r \le \|h_\varphi^{\frac{1}{2r}} \gamma(b^{-1}) \varphi^{\frac{1}{2r}}\|_r \le \|h_{\varphi\circ\gamma}^{\frac{1}{2r}} b^{-1} h_{\varphi\circ\gamma}^{\frac{1}{2r}}\|_r^r. \tag{3.2}$$

Using the variational expression in Theorem 2.3 (i), we have

$$\begin{aligned}
Q_{\alpha,z}(\psi\|\varphi) &\le \alpha \|h_\psi^{\frac{1}{2p}} \gamma(b) h_\psi^{\frac{1}{2p}}\|_p^p + (1-\alpha)\|h_\varphi^{\frac{1}{2r}} \gamma(b)^{-1} h_\varphi^{\frac{1}{2r}}\|_r^r &(3.3)\\
&\le \alpha \|h_\psi^{\frac{1}{2p}} \gamma(b) h_\psi^{\frac{1}{2p}}\|_p^p + (1-\alpha)\|h_\varphi^{\frac{1}{2r}} \gamma(b^{-1}) h_\varphi^{\frac{1}{2r}}\|_r^r &(3.4)\\
&\le \alpha \|h_{\psi\circ\gamma}^{\frac{1}{2p}} b h_{\psi\circ\gamma}^{\frac{1}{2p}}\|_p^p + (1-\alpha)\|h_{\varphi\circ\gamma}^{\frac{1}{2r}} b^{-1} h_{\varphi\circ\gamma}^{\frac{1}{2r}}\|_r^r. &(3.5)
\end{aligned}$$

Since this holds for all $b \in \mathcal{N}^{++}$, it follows that $Q_{\alpha,z}(\psi\|\varphi) \le Q_\alpha(\psi \circ \gamma \| \varphi \circ \gamma)$, which proves the DPI in this case.

Assume next the condition (ii), and put $p := \frac{z}{\alpha}$, $q := \frac{z}{\alpha-1}$, so that $p \in [1/2, 1)$ and $q \ge 1$. Using Theorem 2.3 (ii), we get for any $b \in \mathcal{N}^+$,

$$\begin{aligned}
Q_{\alpha,z}(\psi\|\varphi) &\ge \alpha \|h_\psi^{\frac{1}{2p}} \gamma(b) h_\psi^{\frac{1}{2p}}\|_p^p - (\alpha-1)\|h_\varphi^{\frac{1}{2q}} \gamma(b) h_\varphi^{\frac{1}{2q}}\|_q^q\\
&\ge \alpha \|h_{\psi\circ\gamma}^{\frac{1}{2p}} b h_{\psi\circ\gamma}^{\frac{1}{2p}}\|_p^p - (\alpha-1)\|h_{\varphi\circ\gamma}^{\frac{1}{2r}} b h_{\varphi\circ\gamma}^{\frac{1}{2q}}\|_q^q,
\end{aligned}$$

here we used both (i) and (ii) in Lemma 3.1. Again, since this holds for all $b \in \mathcal{N}^+$, we get the desired inequality.

□

3.2 Martingale convergence

An important consequence of DPI is the martingale convergence property that will be proved in this paragraph.

Let $\mathcal{M}$ be a σ-finite von Neumann algebra. Let $\{\mathcal{M}_i\}$ be an increasing net of von Neumann subalgebras of $\mathcal{M}$ containing the unit of $\mathcal{M}$ such that $\mathcal{M} = \left(\bigcup_i \mathcal{M}_i\right)''$.

Theorem 3.4. *Assume that α and z satisfy the DPI bounds (that is, conditions (i) or (ii) in Theorem 3.3). Then we have*

$$D_{\alpha,z}(\psi\|\varphi) = \lim_i D_{\alpha,z}(\psi|_{\mathcal{M}_i} \| \varphi|_{\mathcal{M}_i}) \quad \textit{increasingly}. \tag{3.6}$$

Proof. Let $\varphi_i := \varphi|_{\mathcal{M}_i}$ and $\psi_i := \psi|_{\mathcal{M}_i}$. From Theorem 3.3, it follows that $D_{\alpha,z}(\psi\|\varphi) \geq D_{\alpha,z}(\psi_i\|\varphi_i)$ for all i and $i \mapsto D_{\alpha,z}(\psi_i\|\varphi_i)$ is increasing. Hence, to show (3.6), it suffices to prove that

$$D_{\alpha,z}(\psi\|\varphi) \leq \sup_i D_{\alpha,z}(\psi_i\|\varphi_i). \tag{3.7}$$

To do this, we may assume that φ is faithful. Indeed, assume that (3.7) has been sbown when φ is faithful. For general $\varphi \in \mathcal{M}_*^+$, from the assumption of $\mathcal{M}$ being σ-finite, there exists a $\varphi_0 \in \mathcal{M}_*^+$ with $s(\varphi_0) = \mathbf{1} - s(\varphi)$. Let $\varphi^{(n)} := \varphi + n^{-1}\varphi_0$ and $\varphi_i^{(n)} := \varphi^{(n)}|_{\mathcal{M}_i}$ for each $n \in \mathbb{N}$. Thanks to the lower semi-continuity [11, Theorem 1(iv) and Theorem 2(iv)] and the order relation [11, Theorem 1(iii) and Theorem 2(iii)] we have

$$\begin{aligned} D_{\alpha,z}(\psi\|\varphi) &\leq \liminf_{n\to\infty} D_{\alpha,z}(\psi\|\varphi^{(n)}) \\ &\leq \liminf_{n\to\infty} \sup_i D_{\alpha,z}(\psi_i\|\varphi_i^{(n)}) \\ &\leq \sup_i D_{\alpha,z}(\psi_i\|\varphi_i), \end{aligned}$$

proving (3.7) for general φ. Below we assume the faithfulness of φ and write $\mathcal{E}_{\mathcal{M}_i,\varphi}$ for the generalized conditional expectation from $\mathcal{M}$ to $\mathcal{M}_i$ with respect to φ. Then we note that we have by [**?** , Theorem 3],

$$\psi_i \circ \mathcal{E}_{\mathcal{M}_i,\varphi} = \psi \circ \mathcal{E}_{\mathcal{M}_i,\varphi} \to \psi \qquad \text{in the norm,} \tag{3.8}$$

as well as

$$\varphi_i \circ \mathcal{E}_{\mathcal{M}_i,\varphi} = \varphi \circ \mathcal{E}_{\mathcal{M}_i,\varphi} = \varphi. \tag{3.9}$$

Using lower semicontinuity and DPI, we obtain

$$D_{\alpha,z}(\psi\|\varphi) \leq \liminf_i D_{\alpha,z}(\psi_i \circ \mathcal{E}_{\mathcal{M}_i,\varphi}\|\varphi) \leq \liminf_i D_{\alpha,z}(\psi_i\|\varphi) \leq \sup_i D_{\alpha,z}(\psi_i\|\varphi).$$

□

3.3 Equality in DPI and reversibility of channels

In what follows, a channel is a normal 2-positive unital map $\gamma : \mathcal{N} \to \mathcal{M}$.

Definition 3.5. Let $\gamma : \mathcal{N} \to \mathcal{M}$ be a channel and let $\mathcal{S} \subset \mathcal{M}_*^+$. We say that γ is reversible (or sufficient) with respect to $\mathcal{S}$ if there exists a channel $\beta : \mathcal{M} \to \mathcal{N}$ such that

$$\rho \circ \gamma \circ \beta = \rho, \qquad \forall \rho \in \mathcal{S}.$$

The notion of sufficient channels was introduced by Petz [17, 18], who also obtained a number of conditions characterizing this situation. It particular, it was proved in [18] that sufficient channels can be characterized by equality in DPI for the relative entropy $D(\psi\|\varphi)$: if $D(\psi\|\varphi) < \infty$, then a channel γ is sufficient with respect to $\{\psi, \varphi\}$ if and only if

$$D(\psi \circ \gamma\|\varphi \circ \gamma) = D(\psi\|\varphi).$$

This result has been proved for a number of other divergence measures, including the standard Rényi divergences $D_{\alpha,1}$ with $0<\alpha<2$ ([]) and the sandwiched Rényi divergences $D_{\alpha,\alpha}$ for $\alpha>1/2$ ([8, 9]). Our aim in this section is to prove that a similar statement holds for $D_{\alpha,z}$ for values of the parameters strictly contained in the DPI bounds of Theorem 3.3.

Another important result of [18] shows that the Petz dual γ^*_φ is a universal recovery map, in the sense given in the proposition below.

Proposition 3.6. *Let $\gamma:\mathcal{N}\to\mathcal{M}$ be a channel and let $\varphi\in\mathcal{M}_*^+$ be such that both φ and $\varphi\circ\gamma$ are faithful. Then for any $\psi\in\mathcal{M}_*^+$, γ is sufficient with respect to $\{\psi,\varphi\}$ if and only if $\psi\circ\gamma\circ\gamma^*_\varphi=\psi$.*

Consequently, there is a faithful normal conditional expectation $\mathcal{E}$ on $\mathcal{M}$ such that $\varphi\circ\mathcal{E}=\varphi$ and γ is sufficient with respect to $\{\psi,\varphi\}$ if and only if also $\psi\circ\mathcal{E}=\psi$.

Note that the range of the conditional expectation $\mathcal{E}$ in the above proposition is the set of fixed points of the channel $\gamma\circ\gamma^*_\varphi$.

3.3.1 The case $\alpha\in(0,1)$

We first prove some equivalent conditions for equality in DPI, in the case $\psi\sim\varphi$. We will use the notations $\psi_0:=\psi\circ\gamma$, $\varphi_0:=\varphi\circ\gamma$. Note that these conditions do not require γ to be 2-positive.

Proposition 3.7. *Let $0<\alpha<1$, $\max\{\alpha,1-\alpha\}\le z$ and assume that $\psi\sim\varphi$. Let $\gamma:\mathcal{N}\to\mathcal{M}$ be a normal positive unital map and put $\psi_0=\psi\circ\gamma$, $\varphi_0=\varphi\circ\gamma$. Let $\bar a\in\mathcal{M}^{++}$ be the unique minimizer as in Lemma 2.4 for $Q_{\alpha,z}(\psi\|\varphi)$ and let $\bar a_0\in\mathcal{N}^{++}$ be the minimizer for $Q_{\alpha,z}(\psi_0\|\varphi_0)$. The following conditions are equivalent:*

- *(i)* $D_{\alpha,z}(\psi_0\|\varphi_0)=D_{\alpha,z}(\psi\|\varphi)$, *i.e.,* $Q_{\alpha,z}(\psi_0\|\varphi_0)=Q_{\alpha,z}(\psi\|\varphi)$.
- *(ii)* $\gamma(\bar a_0)=\bar a$ *and* $\big\|h_\psi^{\frac{\alpha}{2z}}\gamma(\bar a_0)h_\psi^{\frac{\alpha}{2z}}\big\|_{z/\alpha}=\big\|h_{\psi_0}^{\frac{\alpha}{2z}}\bar a_0h_{\psi_0}^{\frac{\alpha}{2z}}\big\|_{z/\alpha}$.
- *(iii)* $\big\|h_\psi^{\frac{\alpha}{2z}}\bar a h_\psi^{\frac{\alpha}{2z}}\big\|_{z/\alpha}=\big\|h_{\psi_0}^{\frac{\alpha}{2z}}\bar a_0h_{\psi_0}^{\frac{\alpha}{2z}}\big\|_{z/\alpha}$.
- *(iv)* $\gamma(\bar a_0^{-1})=\bar a^{-1}$ *and* $\big\|h_\varphi^{\frac{1-\alpha}{2z}}\gamma(\bar a_0^{-1})h_\varphi^{\frac{1-\alpha}{2z}}\big\|_{z/(1-\alpha)}=\big\|h_{\varphi_0}^{\frac{1-\alpha}{2z}}\bar a_0^{-1}h_{\varphi_0}^{\frac{1-\alpha}{2z}}\big\|_{z/(1-\alpha)}$.
- *(v)* $\big\|h_\varphi^{\frac{1-\alpha}{2z}}\bar a^{-1}h_\varphi^{\frac{1-\alpha}{2z}}\big\|_{z/(1-\alpha)}=\big\|h_{\varphi_0}^{\frac{1-\alpha}{2z}}\bar a_0^{-1}h_{\varphi_0}^{\frac{1-\alpha}{2z}}\big\|_{z/(1-\alpha)}$.

Proof. By the assumptions, $s(\psi)=s(\varphi)$ and since also $\psi_0\sim\varphi_0$, we have $s(\psi_0)=s(\varphi_0)$. Using restrictions, we may assume that all $\psi,\varphi,\psi_0,\varphi_0$ are faithful.

(i) $\Longrightarrow$ (ii) & (iv). By Lemma 3.1 (ii)

$$\Big\|h_\psi^{\frac{\alpha}{2z}}\gamma(\bar a_0)h_\psi^{\frac{\alpha}{2z}}\Big\|_{z/\alpha}\le\Big\|h_{\psi_0}^{\frac{\alpha}{2z}}\bar a_0h_{\psi_0}^{\frac{\alpha}{2z}}\Big\|_{z/\alpha},\tag{3.10}$$

and by (3.2) one has

$$\Big\|h_\varphi^{\frac{1-\alpha}{2z}}\gamma(\bar a_0)^{-1}h_\varphi^{\frac{1-\alpha}{2z}}\Big\|_{z/(1-\alpha)}\le\Big\|h_\varphi^{\frac{1-\alpha}{2z}}\gamma(\bar a_0^{-1})h_\varphi^{\frac{1-\alpha}{2z}}\Big\|_{z/(1-\alpha)}\le\Big\|h_{\varphi_0}^{\frac{1-\alpha}{2z}}\bar a_0^{-1}h_{\varphi_0}^{\frac{1-\alpha}{2z}}\Big\|_{z/(1-\alpha)}.\tag{3.11}$$

From (3.10) and (3.11) it follows that

$$\begin{aligned}Q_{\alpha,z}(\psi\|\varphi)&\le\alpha\big\|h_\psi^{\frac{\alpha}{2z}}\gamma(\bar a_0)h_\psi^{\frac{\alpha}{2z}}\big\|_{z/\alpha}+(1-\alpha)\big\|h_\varphi^{\frac{1-\alpha}{2z}}\gamma(\bar a_0)^{-1}h_\varphi^{\frac{1-\alpha}{2z}}\big\|_{z/(1-\alpha)}\\&\le\alpha\big\|h_{\psi_0}^{\frac{\alpha}{2z}}\bar a_0h_{\psi_0}^{\frac{\alpha}{2z}}\big\|_{z/\alpha}+(1-\alpha)\big\|h_{\varphi_0}^{\frac{1-\alpha}{2z}}\bar a_0^{-1}h_{\varphi_0}^{\frac{1-\alpha}{2z}}\big\|_{z/(1-\alpha)}=Q_{\alpha,z}(\psi_0\|\varphi_0)=Q_{\alpha,z}(\psi\|\varphi).\end{aligned}$$

By uniqueness in Lemma 2.4 we find that $\gamma(\bar a_0)=\bar a$ and all the inequalities in (3.10) and (3.11) must become equalities. Since $\gamma(\bar a_0^{-1})\ge\gamma(\bar a_0)^{-1}$, we verify by Lemma A.2 that the equality in (3.11) yields $\gamma(\bar a_0^{-1})=\gamma(\bar a_0)^{-1}=\bar a^{-1}$. Therefore, (ii) and (iv) hold.

The implications (ii) $\Longrightarrow$ (iii) and (iv) $\Longrightarrow$ (v) are obvious.

(iii) $\Longrightarrow$ (i). By (iii) with the equality (2.1) in Lemma 2.4 we have

$$\begin{aligned}Q_{\alpha,z}(\psi\|\varphi)&=\operatorname{Tr}\left(h_\psi^{\frac{\alpha}{2z}}h_\varphi^{\frac{1-\alpha}{z}}h_\psi^{\frac{\alpha}{2z}}\right)^z=\operatorname{Tr}\left(h_\psi^{\frac{\alpha}{2z}}\bar a h_\psi^{\frac{\alpha}{2z}}\right)^{z/\alpha}\\&=\operatorname{Tr}\left(h_{\psi_0}^{\frac{\alpha}{2z}}\bar a_0 h_{\psi_0}^{\frac{\alpha}{2z}}\right)^{z/\alpha}=\operatorname{Tr}\left(h_{\psi_0}^{\frac{\alpha}{2z}}h_{\varphi_0}^{\frac{1-\alpha}{z}}h_{\psi_0}^{\frac{\alpha}{2z}}\right)^z\\&=Q_{\alpha,z}(\psi_0\|\varphi_0).\end{aligned}$$

(v) $\Longrightarrow$ (i). Similarly, by (v) with the equality (2.2) in Lemma 2.4 we have

$$\begin{aligned}Q_{\alpha,z}(\psi\|\varphi)&=\operatorname{Tr}\left(h_\varphi^{\frac{1-\alpha}{2z}}h_\psi^{\frac{\alpha}{z}}h_\varphi^{\frac{1-\alpha}{2z}}\right)^z=\operatorname{Tr}\left(h_\varphi^{\frac{1-\alpha}{2z}}\bar a^{-1}h_\varphi^{\frac{1-\alpha}{2z}}\right)^{z/(1-\alpha)}\\&=\operatorname{Tr}\left(h_{\varphi_0}^{\frac{1-\alpha}{2z}}\bar a_0^{-1}h_{\varphi_0}^{\frac{1-\alpha}{2z}}\right)^{z/(1-\alpha)}=\operatorname{Tr}\left(h_{\varphi_0}^{\frac{1-\alpha}{2z}}h_{\psi_0}^{\frac{\alpha}{z}}h_{\varphi_0}^{\frac{1-\alpha}{2z}}\right)^z\\&=Q_{\alpha,z}(\psi_0\|\varphi_0).\end{aligned}$$

□

Remark 3.8. Note that the above conditions extend the results obtained in [**?**] and [**?**] in the finite dimensional case. Indeed, the condition (ii) with $\alpha=z$ is equivalent to the condition in [**?** , Thm. 1], note here that in this case the second condition in (ii) is automatic. Moreover, (ii) extends the necessary condition in [**?** , Thm. 1.2 (2)] to a necessary and sufficient one. While in both these works γ was required to be completely positive, we have shown that only positivity is enough. See also the related condition in Corollary 3.10 below.

Theorem 3.9. *Let $0<\alpha<1$ and $\alpha,1-\alpha\le z$. Let $\psi,\varphi\in\mathcal M_*^+$ and assume that $\alpha<z$ and $s(\varphi)\le s(\psi)$ or $1-\alpha<z$ and $s(\psi)\le s(\varphi)$. Then γ is reversible with respect to $\{\psi,\varphi\}$ if and only if*

$$D_{\alpha,z}(\psi\|\varphi)=D_{\alpha,z}(\psi\circ\gamma\|\varphi\circ\gamma).$$

Proof. This proof is a modification of the proof of [9, Thm. 5.1]. Let us denote $\psi_0:=\psi\circ\gamma$, $\varphi_0:=\varphi\circ\gamma$ and put $p=\frac{z}{\alpha}$, $r=\frac{z}{1-\alpha}$. We will assume that $p>1$ and $s(\varphi)\le s(\psi)$, otherwise we may exchange the role of p,r and ψ, φ by the equality $Q_{\alpha,z}(\psi\|\varphi)=Q_{1-\alpha,z}(\varphi\|\psi)$. As before, we may assume that both ψ and ψ_0 are faithful.

The strategy of the proof is to use known results for the sandwiched Rényi divergence $D_{p,p}$ with $p>1$, [8]. For this, notice that

$$Q_{z,\alpha}(\psi\|\varphi)=Q_{p,p}(\omega\|\psi),$$

where $\omega\in\mathcal M_*^+$ is such that

$$h_\omega=h_\psi^{\frac{p-1}{2p}}h_\mu^{\frac1p}h_\psi^{\frac{p-1}{2p}},\qquad h_\mu=|h_\varphi^{\frac{1}{2r}}h_\psi^{\frac{1}{2p}}|^{2z}. \tag{3.12}$$

Let $\omega_0,\mu_0\in\mathcal N_*^+$ be similar functionals obtained from ψ_0,φ_0. Then we have the equality

$$Q_{p,p}(\omega_0\|\psi_0)=Q_{\alpha,z}(\psi_0\|\varphi_0)=Q_{\alpha,z}(\psi\|\varphi)=Q_{p,p}(\omega\|\psi).$$

Our first goal is to show that $\omega_0 = \omega\circ\gamma$, which implies by [8] that γ is sufficient with respect to $\{\omega,\psi\}$.

Let $\psi_n\to\psi$ and $\varphi_n\to\varphi$ in $\mathcal{M}_*^+$ be some sequences such that $\psi_n\sim\varphi_n$ for all n. Then $\psi_n\circ\gamma\sim\varphi_n\circ\gamma$, $\psi_n\circ\gamma\to\psi_0$, $\varphi_n\circ\gamma\to\varphi_0$ and by joint continuity of $Q_{\alpha,z}$ ([11, Thm. 1 (iv)]), we have

$$\lim_n Q_{\alpha,z}(\psi_n\circ\gamma\|\varphi_n\circ\gamma) = Q_{\alpha,z}(\psi_0\|\varphi_0) = Q_{\alpha,z}(\psi\|\varphi) = \lim_n Q_{\alpha,z}(\psi_n\|\varphi_n).$$

Let $\bar b_n\in\mathcal{N}^{++}$ be the minimizer for the variational expression for $Q_{\alpha,z}(\psi_n\circ\gamma\|\varphi_n\circ\gamma)$. Let also $\bar a_n$ be the minimizer for $Q_{\alpha,z}(\psi_n\|\varphi_n)$ and let $f_n:\mathcal{M}^{++}\to\mathbb{R}^+$ be the function minimized in the expression for $Q_{\alpha,z}(\psi_n\|\varphi_n)$. We then have

$$\begin{aligned}Q_{\alpha,z}(\psi_n\circ\gamma\|\varphi_n\circ\gamma)-f_n(\gamma(\bar b_n)) =&\alpha\left(\|h_{\psi_n\circ\gamma}^{\frac{1}{2p}}\bar b_n h_{\psi_n\circ\gamma}^{\frac{1}{2p}}\|_p^p-\|h_{\psi_n}^{\frac{1}{2p}}\gamma(\bar b_n)h_{\psi_n}^{\frac{1}{2p}}\|_p^p\right)\\ &+(1-\alpha)\left(\|h_{\varphi_n\circ\gamma}^{\frac{1}{2r}}\bar b_n^{-1}h_{\varphi_n\circ\gamma}^{\frac{1}{2r}}\|_r^r-\|h_{\varphi_n}^{\frac{1}{2r}}\gamma(\bar b_n)^{-1}h_{\varphi_n}^{\frac{1}{2r}}\|_r^r\right)\geq 0,\end{aligned}$$

where the inequality follows from Lemma 3.1 (ii) and (3.2). We obtain

$$Q_{\alpha,z}(\psi_n\circ\gamma\|\varphi_n\circ\gamma)-Q_{\alpha,z}(\psi_n\|\varphi_n)\geq f_n(\gamma(\bar b_n))-Q_{\alpha,z}(\psi_n\|\varphi_n)\geq 0. \tag{3.13}$$

Now let $\mu_{n,0}\in\mathcal{N}_*^+$ and $\mu_n\in\mathcal{M}_*^+$ be such that by (2.1) in Lemma 2.4

$$h_{\mu_{n,0}}^{\frac1p}=|h_{\varphi_n\circ\gamma}^{\frac{1}{2r}}h_{\psi_n\circ\gamma}^{\frac{1}{2p}}|^{2\alpha}=h_{\psi_n\circ\gamma}^{\frac{1}{2p}}\bar b_n h_{\psi_n\circ\gamma}^{\frac{1}{2p}},\qquad h_{\mu_n}^{\frac1p}=|h_{\varphi_n}^{\frac{1}{2r}}h_{\psi_n}^{\frac{1}{2p}}|^{2\alpha}=h_{\psi_n}^{\frac{1}{2p}}\bar a_n h_{\psi_n}^{\frac{1}{2p}}.$$

Then $h_{\mu_{n,0}}^{\frac1p}\to h_{\mu_0}^{\frac1p}$ in $L_p(\mathcal{N})$, this follows by the Hölder inequality and the fact that the map $L_{2z}(\mathcal{N})\to L_p(\mathcal{N})$, given as $h\mapsto|h|^{2\alpha}$ is norm to norm continuous, [**?**]. Similarly, $h_{\mu_n}^{\frac1p}\to h_\mu^{\frac1p}$ in $L_p(\mathcal{M})$. Next, note that since $Q_{\alpha,z}(\psi_n\circ\gamma\|\varphi_n\circ\gamma)$ and $Q_{\alpha,z}(\psi_n\|\varphi_n)$ have the same limit, we see from (3.13) and Lemma 2.5 that $h_{\psi_n}^{\frac{1}{2p}}\gamma(\bar b_n)h_{\psi_n}^{\frac{1}{2p}}-h_{\mu_n}^{\frac1p}\to 0$ in $L_p(\mathcal{M})$. On the other hand, let $\gamma^*_{\psi_n,p},\gamma^*_{\psi,p}:L_p(\mathcal{N})\to L_p(\mathcal{M})$ be the contractions as in Remark 3.2. We then have

$$h_{\psi_n}^{\frac{1}{2p}}\gamma(\bar b_n)h_{\psi_n}^{\frac{1}{2p}}=\gamma^*_{\psi_n,p}(h_{\mu_{n,0}}^{\frac1p})$$

and since $\gamma^*_{\psi_n,p}(k)\to\gamma^*_{\psi,p}(k)$ in $L_p(\mathcal{M})$ for any $k\in L_p(\mathcal{N})$ by [8, Lemma 4.3], we have

$$\|\gamma^*_{\psi,p}(h_{\mu_0}^{\frac1p})-\gamma^*_{\psi_n,p}(h_{\mu_{n,0}}^{\frac1p})\|_p\leq\|(\gamma^*_{\psi,p}-\gamma^*_{\psi_n,p})(h_{\mu_0}^{\frac1p})\|_p+\|\gamma^*_{\psi_n,p}(h_{\mu_0}^{\frac1p}-h_{\mu_{n,0}}^{\frac1p})\|_p\to 0.$$

Putting all together, we obtain that

$$h_\mu^{\frac1p}=\lim_n h_{\mu_n}^{\frac1p}=\lim_n\gamma^*_{\psi_n,p}(h_{\mu_{n,0}}^{\frac1p})=\gamma^*_{\psi,p}(h_{\mu_0}^{\frac1p}).$$

It follows that

$$(\gamma^*_\psi)_*(h_{\omega_0})=h_\psi^{\frac{p-1}{2p}}\gamma^*_{\psi,p}(h_{\mu_0}^{\frac1p})h_\psi^{\frac{p-1}{2p}}=h_\psi^{\frac{p-1}{2p}}h_\mu^{\frac1p}h_\psi^{\frac{p-1}{2p}}=h_\omega,$$

so that we have

$$Q_{p,p}(\omega_0\|\psi_0)=Q_{p,p}(\omega\|\psi)=Q_{p,p}(\omega_0\circ\gamma^*_\psi\|\psi_0\circ\gamma^*_\psi).$$

Let us remark here that in the situation of Proposition 3.7, we have $h_\omega = h_\psi^{\frac{1}{2}}\bar{a}h_\psi^{\frac{1}{2}}$ and $h_{\omega_0} = h_{\psi_0}^{\frac{1}{2}}\bar{a}_0 h_{\psi_0}^{\frac{1}{2}}$, so that the equality

$$(\gamma_\psi^*)_*(h_{\omega_0}) = h_\psi^{\frac{1}{2}}\gamma(\bar{a}_0)h_\psi^{\frac{1}{2}} = h_\omega$$

is immediate from the condition (iii) in Proposition 3.7.

By the properties of the sandwiched Rényi divergence $D_{p,p}$, it follows that γ_ψ^* is sufficient with respect to $\{\omega_0, \psi_0\}$. By Proposition 3.6 and the fact that the Petz dual $(\gamma_\psi^*)_{\psi_0}^*$ is γ itself, this implies

$$\omega \circ \gamma = \omega_0 \circ \gamma_\psi^* \circ \gamma = \omega_0.$$

Next, let $\mathcal{E}$ be the faithful normal conditional expectation onto the set of fixed points of $\gamma \circ \gamma_\psi^*$, as in Proposition 3.6. Then $\mathcal{E}$ preserves both ψ and ω, which by [10] ...!! implies that

$$h_\psi^{\frac{p-1}{2p}} h_\mu^{\frac{1}{p}} h_\psi^{\frac{p-1}{2p}} = h_\omega = \mathcal{E}_*(h_\omega) = h_\psi^{\frac{p-1}{2p}} \mathcal{E}_p(h_\mu^{\frac{1}{p}}) h_\psi^{\frac{p-1}{2p}},$$

so that $|h_\varphi^{\frac{1}{2r}} h_\psi^{\frac{1}{2p}}|^{2\alpha} = h_\mu^{\frac{1}{p}} \in L_p(\mathcal{E}(\mathcal{M}))$ and consequently $|h_\varphi^{\frac{1}{2r}} h_\psi^{\frac{1}{2p}}| = h_\mu^{\frac{1}{2z}} \in L_{2z}(\mathcal{E}(\mathcal{M}))$. Note that by the assumptions $2z > 1$, so that we may use the multiplicativity properties of the extension of $\mathcal{E}$ [10]. Let

$$h_\varphi^{\frac{1}{2r}} h_\psi^{\frac{1}{2p}} = u|h_\varphi^{\frac{1}{2r}} h_\psi^{\frac{1}{2p}}|$$

be the polar decomposition in $L_{2z}(\mathcal{M})$, then we have

$$u^* h_\varphi^{\frac{1}{2r}} h_\psi^{\frac{1}{2p}} = \mathcal{E}_{2z}(u^* h_\varphi^{\frac{1}{2r}} h_\psi^{\frac{1}{2p}}) = \mathcal{E}_{2r}(u^* h_\varphi^{\frac{1}{2r}}) h_\psi^{\frac{1}{2p}},$$

which implies that $u^* h_\varphi^{\frac{1}{2r}} \in L_{2r}(\mathcal{E}(\mathcal{M}))$. Since ψ is faithful, we have $uu^* = r(h_\varphi^{\frac{1}{2r}} h_\psi^{\frac{1}{2p}}) = s(\varphi)$, so that by uniqueness of the polar decomposition in $L_{2r}(\mathcal{M})$ and $L_{2r}(\mathcal{E}(\mathcal{M}))$, we must have $h_\varphi^{\frac{1}{2r}} \in L_{2r}(\mathcal{E}(\mathcal{M}))$ and $u \in \mathcal{E}(\mathcal{M})$. Hence $\varphi \circ \mathcal{E} = \varphi$ and γ is sufficient with respect to $\{\psi, \varphi\}$.

□

We give further equality conditions related to those by Zhang (?) [**?**] in finite dimensions.

Corollary 3.10. *Let $0 < \alpha < 1$, $\max\{\alpha, 1-\alpha\} \le z$ and let γ be a normal positive unital map.*

(i) Let $s(\varphi) \le s(\psi)$ and $p > 1$. Then equality in DPI for $D_{\alpha,z}$ is equivalent to

$$\gamma_*\big((h_\psi^{\frac{1}{2p}} h_\varphi^{\frac{1}{r}} h_\psi^{\frac{1}{2p}})^z\big) = (h_{\psi_0}^{\frac{1}{2p}} h_{\varphi_0}^{\frac{1}{r}} h_{\psi_0}^{\frac{1}{2p}})^z.$$

(ii) Let $s(\psi) \le s(\varphi)$ and $r > 1$. Then equality in DPI for $D_{\alpha,z}$ is equivalent to

$$\gamma_*\big((h_\varphi^{\frac{1}{2r}} h_\psi^{\frac{1}{p}} h_\varphi^{\frac{1}{2r}})^z\big) = (h_{\varphi_0}^{\frac{1}{2r}} h_{\psi_0}^{\frac{1}{p}} h_{\varphi_0}^{\frac{1}{2r}})^z.$$

Note that in the case $\psi \sim \varphi$, the condition in (i) can be written as

$$\gamma_*\big((h_\psi^{\frac{1}{2p}} \bar{a} h_\psi^{\frac{1}{2p}})^p\big) = (h_{\psi_0}^{\frac{1}{2p}} \bar{a}_0 h_{\psi_0}^{\frac{1}{2p}})^p$$

and the condition in (ii) as

$$\gamma_*\big((h_\varphi^{\frac{1}{2r}}\bar{a}^{-1}h_\varphi^{\frac{1}{2r}})^r\big) = (h_{\varphi_0}^{\frac{1}{2r}}\bar{a}_0^{-1}h_{\varphi_0}^{\frac{1}{2r}})^r.$$

This is similar to the condition by Zhang, but not the same, Zhang's condition is

$$\gamma_*\big((\bar{a}^{\frac{1}{2}}h_\psi^{\frac{1}{p}}\bar{a}^{\frac{1}{2}})^p\big) = (\bar{a}_0^{\frac{1}{2}}h_{\psi_0}^{\frac{1}{p}}\bar{a}_0^{\frac{1}{2}})^p \qquad \text{or} \qquad \gamma_*\big((\bar{a}^{-\frac{1}{2}}h_\varphi^{\frac{1}{r}}\bar{a}^{-\frac{1}{2}})^r\big) = (\bar{a}_0^{-\frac{1}{2}}h_{\psi_0}^{\frac{1}{r}}\bar{a}_0^{-\frac{1}{2}})^r.$$

Note that by the proof of [11, Thm. 1 (vi)], we have

$$b^*h_\psi^{\frac{1}{p}}b = (h_\varphi^{\frac{1}{2r}}h_\psi^{\frac{1}{p}}h_\varphi^{\frac{1}{2r}})^\alpha,$$

where $bb^* = \bar{a}$, so that the first Zhang's equality above seems to be related to part (ii) rather than part (i), although the requirement for this in [? , Thm. 1.2 (iii)] is that $z \neq \alpha$, which is the condition in (i). I am a bit confused about this.

Proof. We prove (i), the statement (ii) is proved by exchanging the roles of ψ and φ as before. As one of the steps in the proof of Theorem 3.9, we have shown that if equality in DPI holds, γ is sufficient with respect to $\{\omega, \psi\}$, where $\omega \in \mathcal{M}_*^+$ is given by

$$h_\omega = h_\psi^{\frac{p-1}{2p}}h_\mu^{\frac{1}{p}}h_\psi^{\frac{p-1}{2p}}, \qquad h_\mu = (h_\psi^{\frac{1}{2p}}h_\varphi^{\frac{1}{r}}h_\psi^{\frac{1}{2p}})^z.$$

We also proved that $\omega\circ\gamma = \omega_0$ where $\omega_0 \in \mathcal{N}_*^+$ is similarly obtained from ψ_0, φ_0. Notice that $h_\omega \in L_p(\mathcal{M}, \psi)$ and we have

$$f_{\omega,p}(s) = \mu(1)^{\frac{1}{p}-s}h_\psi^{\frac{1-s}{2}}h_\mu^s h_\psi^{\frac{1-s}{2}}, \qquad s \in S,$$

see Appendix **??**. Similarly,

$$f_{\omega_0,p}(s) = \mu_0(1)^{\frac{1}{p}-s}h_{\psi_0}^{\frac{1-s}{2}}h_{\mu_0}^s h_{\psi_0}^{\frac{1-s}{2}}, \qquad s \in S.$$

Equality in DPI implies that $\mu(1) = \mu_0(1)$ and by the Hadamard three lines theorem and the fact that γ_* is a contraction on $L_q(\mathcal{M}, \psi)$ for each $q \geq 1$,

$$\begin{aligned}\|h_{\omega_0}\|_{p,\psi_0} = \|\gamma_*(f_{\omega,p}(\frac{1}{p}))\|_{p,\psi_0} &\leq \left(\sup_t \|\gamma_*(f_{\omega,p}(it))\|_{\infty,\psi}\right)^{1-\frac{1}{p}}\left(\sup_t \|\gamma_*(f_{\omega,p}(1+it))\|_1\right)^{\frac{1}{p}} \\ &\leq \left(\sup_t \|f_{\omega,p}(it)\|_{\infty,\psi}\right)^{1-\frac{1}{p}}\left(\sup_t \|f_{\omega,p}(1+it)\|_1\right)^{\frac{1}{p}} = \|h_\omega\|_{p,\psi} = \|h_{\omega_0}\|_{p,\psi_0}.\end{aligned}$$

We see that the function $S \ni s \mapsto \gamma_*(f_{\omega,p}(s))$ satisfies equality in Hadamard three lines at $\theta = 1/p \in (0,1)$, whence by [8, Thm. 2.10] we must have $\gamma_*(f_{\omega,p}(s)) = f_{\omega_0,p}(s)M^{s-\frac{1}{p}}$ for all $s \in S$, where $M = M_1/M_0$ with $M_0 = \sup_t \|f_{\omega,p}(it)\|_{\infty,\psi}$, $M_1 = \sup_t \|f_{\omega,p}(1+it)\|_1$. The equality above implies that we have $M_0 = M_1 = \|h_\omega\|_{p,\psi}$, so that $M = 1$. Putting $s = 1$ implies that $\gamma_*(h_\mu) = h_{\mu_0}$, which is the equality in (i). The converse implication is clear. $\square$

3.3.2 The case $\alpha > 1$

We now turn to the case $\alpha > 1$. We will put $p := \frac{z}{\alpha}$ and $q := \frac{z}{\alpha-1}$, then within the DPI bounds, we have $p \in [1/2, 1)$ and $q \geq 1$. Here we need to assume that $D_{\alpha,z}(\psi\|\varphi) < \infty$, so that by Lemma 2.2 there is some (unique) $y \in L_{2z}(\mathcal{M})$ such that

$$h_\psi^{\frac{1}{2p}} = yh_\varphi^{\frac{1}{2q}}.$$

By the proof of Theorem 2.3, we have the following variational expression

$$Q_{\alpha,z}(\psi\|\varphi) = \sup_{w\in L_q(\mathcal{M})^+} \alpha\mathrm{Tr}\left((ywy^*)^p\right) - (\alpha-1)\mathrm{Tr}\left(w^q\right). \tag{3.14}$$

The supremum is attained at a unique point $\bar{w} = (y^*y)^{\alpha-1} \in L_q(\mathcal{M})^+$, uniqueness follows from strict concavity of the function $w \mapsto \alpha\mathrm{Tr}\left((ywy^*)^p\right) - (\alpha-1)\mathrm{Tr}\left(w^q\right)$.

By DPI, we have $D_{\alpha,z}(\psi_0\|\varphi_0) \leq D_{\alpha,z}(\psi\|\varphi) < \infty$, so that there is some (unique) $y_0 \in L_{2z}(\mathcal{N})$ such that

$$h_{\psi_0}^{\frac{1}{2p}} = y_0 h_{\varphi_0}^{\frac{1}{2q}}.$$

Lemma 3.11. *Let us assume that both φ and φ_0 are faithful. Let $\gamma^*_{\varphi,q}$ be the contraction as in Remark 3.2. Keeping the above assumptions and notations, we have for any $w_0 \in L_q(\mathcal{N})^+$*

$$\mathrm{Tr}\left((y\gamma^*_{\varphi,q}(w_0)y^*)^p\right) \geq \mathrm{Tr}\left((y_0w_0y_0^*)^p\right).$$

Proof. Let us first assume that $w_0 = h_{\varphi_0}^{\frac{1}{2q}} b h_{\varphi_0}^{\frac{1}{2q}}$ for some $b \in \mathcal{N}_+$. Then $\gamma^*_{\varphi,q}(w_0) = h_\varphi^{\frac{1}{2q}}\gamma(b)h_\varphi^{\frac{1}{2q}}$. Therefore

$$\begin{aligned}\mathrm{Tr}\left((y\gamma^*_{\varphi,q}(w_0)y^*)^p\right) &= \mathrm{Tr}\left((yh_\varphi^{\frac{1}{2q}}\gamma(b)h_\varphi^{\frac{1}{2q}}y^*)^p\right) = \mathrm{Tr}\left((h_\psi^{\frac{1}{2p}}\gamma(b)h_\psi^{\frac{1}{2p}})^p\right) \geq \mathrm{Tr}\left((h_{\psi_0}^{\frac{1}{2p}}bh_{\psi_0}^{\frac{1}{2p}})^p\right)\\ &= \mathrm{Tr}\left((y_0h_{\varphi_0}^{\frac{1}{2q}}bh_{\varphi_0}^{\frac{1}{2q}}y_0^*)^p\right) = \mathrm{Tr}\left((y_0w_0y_0^*)^p\right),\end{aligned}$$

here the inequality is from Lemma 3.1 (i). The statement follows by Lemma A.1.

□

Theorem 3.12. *Let $\gamma : \mathcal{N} \to \mathcal{M}$ be a channel and let $\psi, \varphi \in \mathcal{M}_*^+$ be such that $s(\psi) \leq s(\varphi)$ and $D_{\alpha,z}(\psi\|\varphi) < \infty$. Then $D_{\alpha,z}(\psi\circ\gamma\|\varphi\circ\gamma) = D_{\alpha,z}(\psi\|\varphi)$ if and only if γ is sufficient with respect to $\{\psi, \varphi\}$.*

Proof. Supose that $D_{\alpha,z}(\psi_0\|\varphi_0) = D_{\alpha,z}(\psi\|\varphi)$. As before, we may assume that both φ and φ_0 are faithful. Let $\bar{w} \in L_q(\mathcal{M})^+$ and $\bar{w}_0 \in L_q(\mathcal{N})^+$ be the unique elements such that the suprema in the variational expression (3.14) for $D_{\alpha,z}(\psi\|\varphi)$ resp. $D_{\alpha,z}(\psi_0\|\varphi_0)$ are attained. We have by Lemma 3.11 and the fact that $\gamma^*_{\varphi,q}$ is a contraction,

$$\begin{aligned}D_{\alpha,z}(\psi\|\varphi) &\geq \alpha\mathrm{Tr}\left((y\gamma^*_{\varphi,q}(\bar{w}_0)y^*)^p\right) - (\alpha-1)\mathrm{Tr}\left(\gamma^*_{\varphi,q}(\bar{w}_0)^q\right)\\ &\geq \alpha\mathrm{Tr}\left((y_0\bar{w}_0y_0^*)^p\right) - (\alpha-1)\mathrm{Tr}\left(\bar{w}_0^q\right) = D_{\alpha,z}(\psi_0\|\varphi_0) = D_{\alpha,z}(\psi\|\varphi),\end{aligned}$$

so that both inequalities must be equalities. This implies that in particular

$$\mathrm{Tr}\left(\bar{w}_0^q\right) = \mathrm{Tr}\left(\gamma^*_{\varphi,q}(\bar{w}_0)^q\right).$$

By uniqueness, we must also have $\bar{w} = \gamma^*_{\varphi,q}(\bar{w}_0)$. Let now $\omega \in \mathcal{M}^+_*,\ \omega_0 \in \mathcal{N}^+_*$ be given by

$$h_\omega = h_\varphi^{\frac{q-1}{2q}} \bar{w} h_\varphi^{\frac{q-1}{2q}}, \qquad h_{\omega_0} = h_{\varphi_0}^{\frac{q-1}{2q}} \bar{w}_0 h_{\varphi_0}^{\frac{q-1}{2q}}, \tag{3.15}$$

then we get $(\gamma^*_\varphi)_*(\omega_0) = \omega$ and also by definition of the sandwiched Rényi divergence,

$$D_{q,q}(\omega_0\|\varphi_0) = \operatorname{Tr}\left(\bar{w}_0^q\right) = \operatorname{Tr}\left(\gamma^*_{\varphi,q}(\bar{w}_0)^q\right) = D_{q,q}(\omega_0 \circ \gamma^*_\varphi\|\varphi_0 \circ \gamma^*_\varphi).$$

Similarly as in the proof of Theorem 3.12, this shows that γ is sufficient with respect to $\{\omega, \varphi\}$. Hence $\omega \circ \mathcal{E} = \omega$, where $\mathcal{E}$ is the conditional expectation onto the fixed points of $\gamma \circ \gamma^*_\varphi$. Using the extensions of $\mathcal{E}$ and their properties, we get

$$h_\varphi^{\frac{q-1}{2q}} \bar{w} h_\varphi^{\frac{q-1}{2q}} = h_\omega = \mathcal{E}(h_\omega) = h_\varphi^{\frac{q-1}{2q}} \mathcal{E}(\bar{w}) h_\varphi^{\frac{q-1}{2q}},$$

which implies that $\bar{w} = \mathcal{E}(\bar{w}) \in L_q(\mathcal{E}(\mathcal{M}))^+$. But then also

$$|y| = \bar{w}^{\frac{1}{2(\alpha-1)}} = \bar{w}^{\frac{q}{2z}} \in L_{2z}(\mathcal{E}(\mathcal{M}))^+$$

Let $y = u|y|$ be the polar decomposition of y, then we obtain from the definition of y that $uu^* = s(|y|) = s(\psi)$. Further,

$$u^* h_\psi^{\frac{1}{2p}} = |y| h_\varphi^{\frac{1}{2q}} \in L_{2p}(\mathcal{E}(\mathcal{M}))$$

and by uniqueness of the polar decomposition in $L_{2p}(\mathcal{M})$ and $L_{2p}(\mathcal{E}(\mathcal{M}))$, we obtain that $h_\psi^{\frac{1}{2p}} \in L_{2p}(\mathcal{E}(\mathcal{M}))^+$, $u \in \mathcal{E}(\mathcal{M})$. Hence we must have $h_\psi \in L_1(\mathcal{E}(\mathcal{M}))$ so that $\psi \circ \mathcal{E} = \psi$.

□

Corollary 3.13. *Let $\alpha > 1$ and $\max\{\alpha/2, \alpha - 1\} \le z \le \alpha$. Let $\psi, \varphi \in \mathcal{M}^+_*$ be such that $s(\psi) \le s(\varphi)$. Let $\gamma : \mathcal{N} \to \mathcal{M}$ be a normal positive unital map. Assume that $D_{\alpha,z}(\psi\|\varphi) < \infty$ and let $x \in L_z(\mathcal{M})^+$, $x_0 \in L_z(\mathcal{N})^+$ be such that*

$$h_\psi^{\frac{\alpha}{z}} = h_\varphi^{\frac{\alpha-1}{2z}} x h_\varphi^{\frac{\alpha-1}{2z}}, \qquad h_{\psi_0}^{\frac{\alpha}{z}} = h_{\varphi_0}^{\frac{\alpha-1}{2z}} x_0 h_{\varphi_0}^{\frac{\alpha-1}{2z}}.$$

Then equality in DPI holds if and only if $\gamma_*(x^z) = x_0^z$.

Proof. This can be proved the same way as Corollary 3.10, using h_ω and h_{ω_0} given by (3.15). Note that here $\bar{w} = (y^*y)^{\alpha-1} = x^{\alpha-1}$ and $\bar{w}_0 = (y_0^*y_0)^{\alpha-1} = x_0^{\alpha-1}$ and the equality $\omega \circ \gamma = \omega_0$ only uses positivity of γ.

□

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4 Monotonicity in the parameter z

It is well known [1, 5, 8] that the standard Rényi divergence $D_{\alpha,1}(\psi\|\varphi)$ is monotone increasing in $\alpha \in (0,1) \cup (1,\infty)$ and the sandwiched Rényi divergence $D_{\alpha,\alpha}(\psi\|\varphi)$ is monotone increasing in $\alpha \in [1/2, 1) \cup (1, \infty)$. It is also known [1, 5, 8] that

$$\lim_{\alpha \nearrow 1} D_{\alpha,1}(\psi\|\varphi) = \lim_{\alpha \nearrow 1} D_{\alpha,\alpha}(\psi\|\varphi) = D_1(\psi\|\varphi),$$

and if $D_{\alpha,1}(\psi\|\varphi)<\infty$ (resp., $D_{\alpha,\alpha}(\psi\|\varphi)<\infty$) for some $\alpha>1$, then

$$\lim_{\alpha\searrow 1} D_{\alpha,1}(\psi\|\varphi)=D_1(\psi\|\varphi)\quad\Bigl(\text{resp., }\lim_{\alpha\searrow 1} D_{\alpha,\alpha}(\psi\|\varphi)=D_1(\psi\|\varphi)\Bigr).$$

In the rest of the paper we will discuss similar monotonicity properties and limits for $D_{\alpha,z}(\psi\|\varphi)$. We consider monotonicity in the parameter z in Sec. 4 and monotonicity in the parameter α in Sec. 5.

4.1 The finite von Neumann algebra case

Assume that $(\mathcal{M},\tau)$ is a semi-finite von Neumann algebra $\mathcal{M}$ with a faithful normal semi-finite trace τ. Then the Haagerup L_p-space $L_p(\mathcal{M})$ is identified with the L_p-space $L_p(\mathcal{M},\tau)$ with respect to τ [7, Example 9.11]. Hence one can define $Q_{\alpha,z}(\psi\|\varphi)$ for $\psi,\varphi\in\mathcal{M}_*^+$ by replacing, in Definition 2.1, $L_p(\mathcal{M})$ with $L_p(\mathcal{M},\tau)$ and $h_\psi\in L_1(\mathcal{M})_+$ with the Radon–Nikodym derivative $d\psi/d\tau\in L_1(\mathcal{M},\tau)^+$. Below we use the symbol h_ψ to denote $d\psi/d\tau$ as well. Note that τ on $\mathcal{M}_+$ is naturally extended to the positive part $\widetilde{\mathcal{M}}^+$ of the space $\widetilde{\mathcal{M}}$ of τ-measurable operators. We then have [7, Proposition 4.20]

$$\tau(a)=\int_0^\infty \mu_s(a)\,ds,\qquad a\in\widetilde{\mathcal{M}}^+,\tag{4.1}$$

where $\mu_s(a)$ is the generalized s-number of a [3].

Throughout this subsection we assume that $\mathcal{M}$ is a finite von Neumann algebra with a faithful normal finite trace τ; then $\widetilde{\mathcal{M}}^+$ consists of all positive self-adjoint operators affiliated with $\mathcal{M}$.

Lemma 4.1. *For every $\psi,\varphi\in\mathcal{M}_*^+$ with $\psi\neq 0$ and for any $\alpha,z>0$ with $\alpha\neq 1$,*

$$D_{\alpha,z}(\psi\|\varphi)=\lim_{\varepsilon\searrow 0} D_{\alpha,z}(\psi\|\varphi+\varepsilon\tau)\quad\textit{increasingly},\tag{4.2}$$

and hence $D_{\alpha,z}(\psi\|\varphi)=\sup_{\varepsilon>0} D_{\alpha,z}(\psi\|\varphi+\varepsilon\tau)$.

Proof. Case $0<\alpha<1$. We need to prove that

$$Q_{\alpha,z}(\psi\|\varphi)=\lim_{\varepsilon\searrow 0} Q_{\alpha,z}(\psi\|\varphi+\varepsilon\tau)\quad\text{decreasingly.}\tag{4.3}$$

In the present setting we have by (4.1)

$$Q_{\alpha,z}(\psi\|\varphi)=\tau\Bigl(\bigl(h_\psi^{\alpha/2z}h_\varphi^{\frac{1-\alpha}{z}}h_\psi^{\alpha/2z}\bigr)^z\Bigr)=\int_0^\infty \mu_s\bigl(h_\psi^{\alpha/2z}h_\varphi^{\frac{1-\alpha}{z}}h_\psi^{\alpha/2z}\bigr)^z\,ds,\tag{4.4}$$

and similarly

$$Q_{\alpha,z}(\psi\|\varphi+\varepsilon\tau)=\int_0^\infty \mu_s\bigl(h_\psi^{\alpha/2z}h_{\varphi+\varepsilon\tau}^{\frac{1-\alpha}{z}}h_\psi^{\alpha/2z}\bigr)^z\,ds.$$

Since $h_{\varphi+\varepsilon\tau}^{\frac{1-\alpha}{z}}=(h_\varphi+\varepsilon\mathbf{1})^{\frac{1-\alpha}{z}}$ decreases to $h_\varphi^{\frac{1-\alpha}{z}}$ in the measure topology as $\varepsilon\searrow 0$, it follows that $h_\psi^{\alpha/2z}h_{\varphi+\varepsilon\tau}^{\frac{1-\alpha}{z}}h_\psi^{\alpha/2z}$ decreases to $h_\psi^{\alpha/2z}h_\varphi^{\frac{1-\alpha}{z}}h_\psi^{\alpha/2z}$ in the measure topology. Hence by [3, Lemma 3.4] we have $\mu_s\bigl(h_\psi^{\alpha/2z}h_{\varphi+\varepsilon\tau}^{\frac{1-\alpha}{z}}h_\psi^{\alpha/2z}\bigr)\searrow\mu_s\bigl(h_\psi^{\alpha/2z}h_\varphi^{\frac{1-\alpha}{z}}h_\psi^{\alpha/2z}\bigr)$ as $\varepsilon\searrow 0$ for almost every $s>0$. Since $s\mapsto\mu_s\bigl(h_\psi^{\alpha/2z}h_{\varphi+\tau}^{\frac{1-\alpha}{z}}h_\psi^{\alpha/2z}\bigr)$ is integrable on $(0,\infty)$, the Lebesgue convergence theorem gives (4.3).

Case $\alpha > 1$. We need to prove that

$$Q_{\alpha,z}(\psi\|\varphi) = \lim_{\varepsilon\searrow 0} Q_{\alpha,z}(\psi\|\varphi+\varepsilon\tau) \quad \text{increasingly.} \tag{4.5}$$

For any $\varepsilon > 0$, since $h_{\varphi+\varepsilon\tau} = h_\psi + \varepsilon\mathbf{1}$ has the bounded inverse $h_{\varphi+\varepsilon\tau}^{-1} = (h_\varphi + \varepsilon\mathbf{1})^{-1} \in \mathcal{M}^+$, one can define $x_\varepsilon := (h_\varphi+\varepsilon\mathbf{1})^{-\frac{\alpha-1}{2z}} h_\psi^{\alpha/z} (h_\varphi+\varepsilon\mathbf{1})^{-\frac{\alpha-1}{2z}} \in \widetilde{\mathcal{M}}^+$ so that

$$h_\psi^{\alpha/z} = (h_\varphi+\varepsilon\mathbf{1})^{\frac{\alpha-1}{2z}} x_\varepsilon (h_\varphi+\varepsilon\mathbf{1})^{\frac{\alpha-1}{2z}}.$$

In the present setting one can write by (4.1)

$$Q_{\alpha,z}(\psi\|\varphi+\varepsilon\tau) = \tau(x_\varepsilon^z) = \int_0^\infty \mu_s(x_\varepsilon)^z\,ds \ (\in [0,\infty]). \tag{4.6}$$

Let $0 < \varepsilon \le \varepsilon'$. Since $(h_\varphi+\varepsilon\mathbf{1})^{-\frac{\alpha-1}{z}} \ge (h_\varphi+\varepsilon'\mathbf{1})^{-\frac{\alpha-1}{z}}$, one has $\mu_s(x_\varepsilon) \ge \mu_s(x_{\varepsilon'})$ for all $s > 0$, so that

$$Q_{\alpha,z}(\psi\|\varphi+\varepsilon\tau) \ge Q_{\alpha,z}(\psi\|\varphi+\varepsilon'\tau).$$

Hence $\varepsilon > 0 \mapsto D_{\alpha,z}(\psi\|\varphi+\varepsilon\tau)$ is decreasing.

First, assume that $s(\psi) \not\le s(\varphi)$. Then $\mu_{s_0}(h_\psi^{\alpha/2z} s(\varphi)^\perp h_\psi^{\alpha/2z}) > 0$ for some $s_0 > 0$; indeed, otherwise, $h_\psi^{\alpha/2z} s(\varphi)^\perp h_\psi^{\alpha/2z} = 0$ so that $s(\psi) \le s(\varphi)$. Hence we have

$$\mu_s(x_\varepsilon) = \mu_s\big(h_\psi^{\alpha/2z}(h_\varphi+\varepsilon\mathbf{1})^{-\frac{\alpha-1}{z}} h_\psi^{\alpha/2z}\big) \ge \varepsilon^{-\frac{\alpha-1}{z}} \mu_s(h_\psi^{\alpha/2z} s(\varphi)^\perp h_\psi^{\alpha/2z}) \nearrow \infty \quad \text{as } \varepsilon \searrow 0$$

for all $s \in (0, s_0]$. Therefore, it follows from (4.6) that $Q_{\alpha,z}(\psi\|\varphi+\varepsilon\tau) \nearrow \infty = Q_{\alpha,z}(\psi\|\varphi)$.

Next, assume that $s(\psi) \le s(\varphi)$. Take the spectral decomposition $h_\varphi = \int_0^\infty t\,de_t$ and define $y, x \in \widetilde{\mathcal{M}}_+$ by

$$y := h_\varphi^{-\frac{\alpha-1}{z}} s(\varphi) = \int_{(0,\infty)} t^{-\frac{\alpha-1}{z}}\,de_t, \qquad x := y^{1/2} h_\psi^{\alpha/z} y^{1/2}.$$

Since

$$h_\psi^{\alpha/z} = s(\varphi) h_\psi^{\alpha/z} s(\varphi) = h_\varphi^{\frac{\alpha-1}{2z}} y^{1/2} h_\psi^{\alpha/z} y^{1/2} h_\varphi^{\frac{\alpha-1}{2z}} = h_\varphi^{\frac{\alpha-1}{2z}} x h_\varphi^{\frac{\alpha-1}{2z}},$$

one has, similarly to 4.6,

$$Q_{\alpha,z}(\psi\|\varphi) = \tau(x^z) = \int_0^\infty \mu_s(x)^z\,ds. \tag{4.7}$$

We write $(h_\varphi+\varepsilon\mathbf{1})^{-\frac{\alpha-1}{z}} s(\varphi) = \int_{(0,\infty)} (t+\varepsilon)^{-\frac{\alpha-1}{z}}\,de_t$, and for any $\delta > 0$ choose a $t_0 > 0$ such that $\tau(e_{(0,t_0)}) < \delta$. Then, since $\int_{[t_0,\infty)} (t+\varepsilon)^{-\frac{\alpha-1}{z}}\,de_t \to \int_{[t_0,\infty)} t^{-\frac{\alpha-1}{z}}\,de_t$ in the operator norm as $\varepsilon \searrow 0$, we obtain $(h_\varphi+\varepsilon\mathbf{1})^{-\frac{\alpha-1}{z}} s(\varphi) \nearrow y$ in the measure topology (see [3, 1.5]), so that $h_\psi^{\alpha/2z}(h_\varphi+\varepsilon\mathbf{1})^{-\frac{\alpha-1}{z}} h_\psi^{\alpha/2z} \nearrow h_\psi^{\alpha/2z} y h_\psi^{\alpha/2z}$ in the measure topology as $\varepsilon \searrow 0$. Hence we have by [3, Lemma 3.4]

$$\mu_s(x_\varepsilon) = \mu_s\big(h_\psi^{\alpha/2z}(h_\varphi+\varepsilon\mathbf{1})^{-\frac{\alpha-1}{z}} h_\psi^{\alpha/2z}\big) \nearrow \mu_s(h_\psi^{\alpha/2z} y h_\psi^{\alpha/2z}) = \mu_s(x) \tag{4.8}$$

for all $s > 0$. Therefore, by (4.6) and (4.7) the monotone convergence theorem gives (4.5). $\square$

Lemma 4.2. *Let $(\mathcal{M},\tau)$ and ψ,φ be as above, and let $0<z\le z'$. Then*

$$\begin{cases}D_{\alpha,z}(\psi\|\varphi)\le D_{\alpha,z'}(\psi\|\varphi), & 0<\alpha<1,\\ D_{\alpha,z}(\psi\|\varphi)\ge D_{\alpha,z'}(\psi\|\varphi), & \alpha>1.\end{cases}$$

Proof. The case $0<\alpha<1$ was shown in [11, Theorem 1(x)] for general von Neumann algebras. For the case $\alpha>1$, by Lemma 4.1 it suffices to show that, for every $\varepsilon>0$,

$$\tau\Bigl(\Bigl(y_\varepsilon^{\frac{\alpha-1}{2z}}h_\psi^{\alpha/z}y_\varepsilon^{\frac{\alpha-1}{2z}}\Bigr)^z\Bigr)\ge\tau\Bigl(\Bigl(y_\varepsilon^{\frac{\alpha-1}{2z'}}h_\psi^{\alpha/z'}y_\varepsilon^{\frac{\alpha-1}{2z'}}\Bigr)^z\Bigr),$$

where $y_\varepsilon:=(h_\varphi+\varepsilon\mathbf{1})^{-1}\in\mathcal{M}_+$. The above is equivalently written as

$$\tau\Bigl(\bigl|(h_\psi^{\alpha/2z'})^r(y^{(\alpha-1)/2z'})^r\bigr|^{2z}\Bigr)\ge\tau\Bigl(\bigl|h_\psi^{\alpha/2z'}y^{(\alpha-1)/2z'}\bigr|^{2zr}\Bigr),$$

where $r:=z'/z\ge1$. Hence the desired inequality follows from Kosaki's ALT inequality [14, Corollary 3]. $\square$

When $(\mathcal{M},\tau)$ and ψ,φ are as in Lemma 4.1, one can define, thanks to Lemma 4.2, for any $\alpha\in(0,\infty)\setminus\{1\}$,

$$\begin{aligned}Q_{\alpha,\infty}(\psi\|\varphi)&:=\lim_{z\to\infty}Q_{\alpha,\infty}(\psi\|\varphi)=\inf_{z>0}Q_{\alpha,z}(\psi\|\varphi),\\ D_{\alpha,\infty}(\psi\|\varphi)&:=\frac{1}{\alpha-1}\log\frac{Q_{\alpha,\infty}(\psi\|\varphi)}{\psi(\mathbf{1})}\\ &=\lim_{z\to\infty}D_{\alpha,z}(\psi\|\varphi)=\begin{cases}\sup_{z>0}D_{\alpha,z}(\psi\|\varphi), & 0<\alpha<1,\\ \inf_{z>0}D_{\alpha,z}(\psi\|\varphi), & \alpha>1.\end{cases}\end{aligned}\tag{4.9}$$

If $h_\psi,h_\varphi\in\mathcal{M}^{++}$ (i.e., $\delta\tau\le\psi,\varphi\le\delta^{-1}\tau$ for some $\delta\in(0,1)$), then the Lie–Trotter formula gives

$$Q_{\alpha,\infty}(\psi\|\varphi)=\tau\bigl(\exp(\alpha\log h_\psi+(1-\alpha)\log h_\varphi)\bigr).\tag{4.10}$$

Lemma 4.3. *Let $(\mathcal{M},\tau)$ and ψ,φ be as above. Then for any $z>0$,*

$$\begin{cases}D_{\alpha,z}(\psi\|\varphi)\le D_1(\psi\|\varphi), & 0<\alpha<1,\\ D_{\alpha,z}(\psi\|\varphi)\ge D_1(\psi\|\varphi), & \alpha>1.\end{cases}$$

Proof. First, assume that $h_\psi,h_\varphi\in\mathcal{M}^{++}$. Set self-adjoint $H:=\log h_\psi$ and $K:=\log h_\varphi$ in $\mathcal{M}$ and define $F(\alpha):=\log\tau\bigl(e^{\alpha H+(1-\alpha)K}\bigr)$ for $\alpha>0$. Then by (4.10), $F(\alpha)=\log Q_{\alpha,\infty}(\psi\|\varphi)$ for all $\alpha\in(0,\infty)\setminus\{1\}$, and we compute

$$\begin{aligned}F'(\alpha)&=\frac{\tau\bigl(e^{\alpha H+(1-\alpha)K}(H-K)\bigr)}{\tau\bigl(e^{\alpha H+(1-\alpha)K}\bigr)},\\ F''(\alpha)&=\frac{\bigl\{\tau\bigl(e^{\alpha H+(1-\alpha)K}(H-K)\bigr)\bigr\}^2-\tau\bigl(e^{\alpha H+(1-\alpha)K}(H-K)^2\bigr)}{\bigl\{\tau\bigl(e^{\alpha H+(1-\alpha)K}\bigr)\bigr\}^2}.\end{aligned}$$

Since $F''(\alpha)\ge0$ on $(0,\infty)$ thanks to the Schwarz inequality, we see that $F(\alpha)$ is convex on $(0,\infty)$ and hence

$$D_{\alpha,\infty}(\psi\|\varphi)=\frac{F(\alpha)-F(1)}{\alpha-1}$$

is increasing in $\alpha \in (0,\infty)$, where for $\alpha = 1$ the above RHS is understood as

$$F'(1) = \frac{\tau(e^H(H-K))}{\tau(e^H)} = \frac{\tau\big(h_\psi(\log h_\psi - \log h_\varphi)\big)}{\tau(h_\psi)} = D_1(\psi\|\varphi).$$

Hence by (4.9) the assertion holds when $h_\psi, h_\varphi \in \mathcal{M}^{++}$. Below we extend it to general $\psi,\varphi \in \mathcal{M}_*^+$.

Case $0 < \alpha < 1$. Let $\psi,\varphi \in \mathcal{M}_*^+$ and $z > 0$. From [11, Theorem 1(iv)] and [6, Corollary 2.8(3)] we have

$$\begin{aligned} D_{\alpha,z}(\psi\|\varphi) &= \lim_{\varepsilon\searrow 0} D_{\alpha,z}(\psi+\varepsilon\tau\|\varphi+\varepsilon\tau), \\ D_1(\psi\|\varphi) &= \lim_{\varepsilon\searrow 0} D_1(\psi+\varepsilon\tau\|\varphi+\varepsilon\tau), \end{aligned}$$

so that we may assume that $\psi,\varphi \ge \varepsilon\tau$ for some $\varepsilon > 0$. Take the spectral decompositions $h_\psi = \int_0^\infty t\,de_t^\psi$ and $h_\varphi = \int_0^\infty t\,de_t^\varphi$, and define $e_n := e_n^\psi \wedge e_n^\varphi$ for each $n \in \mathbb{N}$. Then $\tau(e_n^\perp) \le \tau((e_n^\psi)^\perp) + \tau((e_n^\varphi)^\perp) \to 0$ as $n \to \infty$, so that $e_n \nearrow \mathbf{1}$. We set $\psi_n := \psi(e_n \cdot e_n)$ and $\varphi_n := \varphi(e_n \cdot e_n)$; then $h_{\psi_n} = e_n h_\psi e_n$ and $h_{\varphi_n} = e_n h_\varphi e_n$ are in $(e_n\mathcal{M}e_n)^{++}$. Note that

$$\begin{aligned} \|h_\psi - e_n h_\psi e_n\|_1 &\le \|(\mathbf{1}-e_n)h_\psi\|_1 + \|e_n h_\psi(\mathbf{1}-e_n)\|_1 \\ &\le \|(\mathbf{1}-e_n)h_\psi^{1/2}\|_2\|h_\psi^{1/2}\|_2 + \|e_n h_\psi^{1/2}\|_2\|h_\psi^{1/2}(\mathbf{1}-e_n)\|_2 \\ &= \psi(\mathbf{1}-e_n)^{1/2}\psi(\mathbf{1})^{1/2} + \psi(e_n)^{1/2}\psi(\mathbf{1}-e_n)^{1/2} \to 0 \quad \text{as } n\to\infty, \end{aligned}$$

and similarly $\|h_\varphi - e_n h_\varphi e_n\|_1 \to 0$. Hence by [11, Theorem 1(iv)] one has $D_{\alpha,z}(e_n\psi e_n\|e_n\varphi e_n) \to D_{\alpha,z}(\psi\|\varphi)$. On the other hand, one has $D_1(e_n\psi e_n\|e_n\varphi e_n) \to D_1(\psi\|\varphi)$ by [6, Proposition 2.10]. Since $D_{\alpha,z}(e_n\psi e_n\|e_n\varphi e_n) \le D_1(e_n\psi e_n\|e_n\varphi e_n)$ holds by regarding $e_n\psi e_n, e_n\varphi e_n$ as functionals on the reduced von Neumann algebra $e_n\mathcal{M}e_n$, we obtain the desired inequality for general $\psi,\varphi \in \mathcal{M}_*^+$.

Case $\alpha > 1$. We show the extension to general $\psi,\varphi \in \mathcal{M}_*^+$ by dividing four steps as follows, where $h_\psi = \int_0^\infty t\,e_t^\psi$ and $h_\varphi = \int_0^\infty t\,de_t^\varphi$ are the spectral decompositions.

(1) Assume that $h_\psi \in \mathcal{M}^+$ and $h_\varphi \in \mathcal{M}^{++}$. Set $\psi_n \in \mathcal{M}_*^+$ by $h_{\psi_n} = (1/n)e_{[0,1/n]}^\psi + \int_{(1/n,\infty)} t\,de_t^\psi$ $(\in \mathcal{M}^{++})$. Since $h_{\psi_n}^{\alpha/z} \searrow h_\psi^{\alpha/z}$ in the operator norm, we have by (4.4) and [3, Lemma 3.4]

$$\begin{aligned} Q_{\alpha,z}(\psi\|\varphi) &= \int_0^\infty \mu_s\big((h_\varphi^{-1})^{\frac{\alpha-1}{2z}} h_\psi^{\alpha/z} (h_\varphi^{-1})^{\frac{\alpha-1}{2z}}\big)^z\,ds \\ &= \lim_{n\to\infty} \int_0^\infty \mu_s\big((h_\varphi^{-1})^{\frac{\alpha-1}{2z}} h_{\psi_n}^{\alpha/z} (h_\varphi^{-1})^{\frac{\alpha-1}{2z}}\big)^z\,ds = \lim_{n\to\infty} Q_{\alpha,z}(\psi_n\|\varphi). \end{aligned} \tag{4.11}$$

From this and the lower semicontinuity of D_1 the extension holds in this case.

(2) Assume that $h_\psi \in \mathcal{M}^+$ and $h_\varphi \ge \delta\mathbf{1}$ for some $\delta > 0$. Set $\varphi_n \in \mathcal{M}_*^+$ by $h_{\varphi_n} = \int_{[\delta,n]} t\,de_t^\varphi + ne_{(n,\infty)}^\varphi$ $(\in \mathcal{M}^{++})$. Since $h_{\varphi_n}^{-\frac{\alpha-1}{z}} \searrow h_\varphi^{-\frac{\alpha-1}{z}}$ in the operator norm, we have by (4.4) and [3, Lemma 3.4] again

$$\begin{aligned} Q_{\alpha,z}(\psi\|\varphi) &= \int_0^\infty \mu_s\big(h_\psi^{\alpha/2z} h_\varphi^{-\frac{\alpha-1}{z}} h_\psi^{\alpha/2z}\big)^z\,ds \\ &= \lim_{n\to\infty} \int_0^\infty \mu_s\big(h_\psi^{\alpha/2z} h_{\varphi_n}^{-\frac{\alpha-1}{z}} h_\psi^{\alpha/2z}\big)^z\,ds = \lim_{n\to\infty} Q_{\alpha,z}(\psi,\varphi_n). \end{aligned}$$

From this and (1) above the extension holds in this case too.

(3) Assume that ψ is general and $\varphi \ge \delta\tau$ for some $\delta > 0$. Set $\psi_n \in \mathcal{M}_*^+$ by $h_{\psi_n} = \int_{[0,n]} t\, de_t^\psi + n e_{(n,\infty)}^\varphi$ $(\in \mathcal{M}_+)$. Since $h_{\psi_n}^{\alpha/z} \nearrow h_\psi^{\alpha/z}$ in the measure topology, one can argue as in (4.11) with use of the monotone convergence theorem to see from (2) that the extension holds in this case too.

(4) Finally, from (3) with Lemma 4.1 and [6, Corollary 2.8(3)] it follows that the desired extension hods for general $\psi, \varphi \in \mathcal{M}_*^+$. □

In the next proposition, we summarize inequalities for $D_{\alpha,z}$ obtained so far in Lemmas 4.2 and 4.3.

Proposition 4.4. *Assume that $\mathcal{M}$ is a finite von Neumann algebra with a faithful normal finite trace τ. Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \ne 0$. If $0 < \alpha < 1 < \alpha'$ and $0 < z \le z' \le \infty$, then*

$$D_{\alpha,z}(\psi\|\varphi) \le D_{\alpha,z'}(\psi\|\varphi) \le D_1(\psi\|\varphi) \le D_{\alpha',z'}(\psi\|\varphi) \le D_{\alpha',z}(\psi\|\varphi).$$

Corollary 4.5. *Let $(\mathcal{M}, \tau)$ and ψ, φ be as in Proposition 4.4. Then for any $z \in [1, \infty]$,*

$$\lim_{\alpha \nearrow 1} D_{\alpha,z}(\psi\|\varphi) = D_1(\psi\|\varphi). \tag{4.12}$$

Moreover, if $D_{\alpha,\alpha}(\psi\|\varphi) < \infty$ for some $\alpha > 1$ then for any $z \in (1, \infty]$,

$$\lim_{\alpha \searrow 1} D_{\alpha,z}(\psi\|\varphi) = D_1(\psi\|\varphi). \tag{4.13}$$

Proof. Let $z \ge 1$. For every $\alpha \in (0,1)$, Proposition 4.4 gives

$$D_{\alpha,1}(\psi\|\varphi) \le D_{\alpha,z}(\psi\|\varphi) \le D_1(\psi\|\varphi).$$

Hence (4.12) follows since it holds for $D_{\alpha,1}$ [5, Proposition 5.3(3)].

Next, assume that $D_{\alpha,\alpha}(\psi\|\varphi) < \infty$ for some $\alpha > 1$. Let $z > 1$. For every $\alpha \in (1, z]$, Proposition 4.4 gives

$$D_1(\psi\|\varphi) \le D_{\alpha,z}(\psi\|\varphi) \le D_{\alpha,\alpha}(\psi\|\varphi).$$

Hence (4.13) follows since it holds for $D_{\alpha,\alpha}$ [8, Proposition 3.8(ii)]. □

In this subsection, in the specialized setting of finite von Neumann algebras, we have given monotonicity of $D_{\alpha,z}$ in the parameter z in an essentially similar way to the finite-dimensional case [15]. In the next subsection we will extend it to general von Neumann algebras under certain restrictions of α, z.

4.2 The general von Neumann algebra case

Theorem 4.6. *For every $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \ne 0$, and $0 < \alpha < 1$, we have:*

(1) If $0 < \alpha < 1$ and $\max\{\alpha, 1-\alpha\} \le z \le z'$, then

$$D_{\alpha,z}(\psi\|\varphi) \le D_{\alpha,z'}(\psi\|\varphi) \le D_1(\psi\|\varphi).$$

(2) If $\alpha > 1$ and $\max\{\alpha/2, \alpha-1\} \le z \le z' \le \alpha$, then

$$D_1(\psi\|\varphi) \le D_{\alpha,z'}(\psi\|\varphi) \le D_{\alpha,z}(\psi\|\varphi).$$

Hiai (12/8/2023) In fact, (2) is improved in Theorem 6.

Theorem 4.7. *For every $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \ne 0$, and $\alpha > 1$, the function $z \mapsto D_{\alpha,z}(\psi\|\varphi)$ is monotone decreasing on $[\alpha/2, \infty)$.*

Anna (Jan. 23, 2024)

5 Monotonicity in the parameter α

5.1 The case $\alpha < 1$ and all $z > 0$

Theorem 5.1. *Let $\psi, \varphi \in \mathcal{M}_*^+$ and $z > 0$. Then we have*

(1) $\alpha \mapsto \log Q_{\alpha,z}(\psi\|\varphi)$ *is convex on* $(0,1)$*,*

(2) $\alpha \mapsto D_{\alpha,z}(\psi\|\varphi)$ *is monotone increasing on* $(0,1)$*.*

Anna (Jan. 10, 2024), Hiai (1/16/2024)

5.2 The case $1 < \alpha \le 2z$

Theorem 5.2. *Let $\psi, \varphi \in \mathcal{M}_*^+$ and $z > 1/2$. Then we have*

(1) $\alpha \mapsto \log Q_{\alpha,z}(\psi\|\varphi)$ *is convex on* $(1,2z]$*,*

(2) $\alpha \mapsto D_{\alpha,z}(\psi\|\varphi)$ *is monotone increasing on* $(1,2z]$*.*

Anna (Jan. 23, 2024), Hiai (12/31/2023)

5.3 Limits as $\alpha \nearrow 1$ and $\alpha \searrow 1$

Theorem 5.3. *Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$. For every $z \in (0,1]$ we have*

$$\lim_{\alpha \nearrow 1} D_{\alpha,z}(\psi\|\varphi) = D_1(\psi\|\varphi).$$

Anna (Dec. 7, 2023)

Theorem 5.4. *Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$, and $z > 1/2$. Assume that $D_{\alpha,z}(\psi\|\varphi) < \infty$ for some $\alpha \in (1,2z]$. Then we have*

$$\lim_{\alpha \searrow 1} D_{\alpha,z}(\psi\|\varphi) = D_1(\psi\|\varphi).$$

Anna (Jan. 23, 2024)

A Haagerup L_p-spaces

The following lemmas are well known, proofs are given for completeness.

Lemma A.1. *For any $0 < p < \infty$ and $\varphi \in \mathcal{M}_*^+$, $h_\varphi^{\frac{1}{2p}} \mathcal{M}^+ h_\varphi^{\frac{1}{2p}}$ is dense in $L_p(\mathcal{M})^+$ with respect to the (quasi)-norm $\|\cdot\|_p$.*

Proof. We may assume that φ is faithful. By [10, Lemma 1.1], $\mathcal{M}h_\varphi^{\frac{1}{2p}}$ is dense in $L_{2p}(\mathcal{M})$ for any $0 < p < \infty$. Let $y \in L_p(\mathcal{M})^+$, then $y^{\frac{1}{2}} \in L_{2p}(\mathcal{M})$, hence there is a sequence $a_n \in \mathcal{M}$ such that $\|a_n h_\varphi^{\frac{1}{2p}} - y^{\frac{1}{2}}\|_{2p} \to 0$. Then also

$$\|h_\varphi^{\frac{1}{2p}} a_n^* - y^{\frac{1}{2}}\|_p = \|(a_n h_\varphi^{\frac{1}{2p}} - y^{\frac{1}{2}})^*\|_p = \|a_n h_\varphi^{\frac{1}{2p}} - y^{\frac{1}{2}}\|_p \to 0$$

and

$$\|h_\varphi^{\frac{1}{2p}} a_n^* a_n h_\varphi^{\frac{1}{2p}} - y\|_p = \|(h_\varphi^{\frac{1}{2p}} a_n^* - y^{\frac{1}{2}}) a_n h_\varphi^{\frac{1}{2p}} + y^{\frac{1}{2}} (a_n h_\varphi^{\frac{1}{2p}} - y^{\frac{1}{2}})\|_p$$

Since $\|\cdot\|_p$ is a (quasi)-norm, the above expression goes to 0 by the Hölder inequality.

□

Lemma A.2. *Let $0 < p \leq \infty$ and let $h, k \in L_p(\mathcal{M})^+$ be such that $h \leq k$. Then $\|h\|_p \leq \|k\|_p$. Moreover, if $1 \leq p < \infty$, then*

$$\|k - h\|_p^p \leq \|k\|_p^p - \|h\|_p^p.$$

Proof. The first statement follows from [3, Lemma 2.5 (iii) and Lemma 4.8]. The second statement is from [3, Lemma 5.1].

□

Lemma A.3. *Let $\psi, \varphi \in \mathcal{M}_*^+$ with $\psi \leq \varphi$. Then for any $a \in \mathcal{M}$ and $p \in [1, \infty)$,*

$$\mathrm{Tr}\left((a^* h_\psi^{\frac{1}{p}} a)^p\right) \leq Tr\left((a^* h_\varphi^{\frac{1}{p}} a)^p\right)$$

Proof. Since $1/p \in (p, 1]$, it follows (see [6, Lemma B.7] and [**?** , Lemma 3.2]) that $h_\psi^{1/p} \leq h_\varphi^{1/p}$. Hence $a^* h_\psi^{1/p} a \leq a^* h_\varphi^{1/p} a$. Therefore, by Lemma A.2, we have the statement.

□

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