# Randomization theorems for quantum channels

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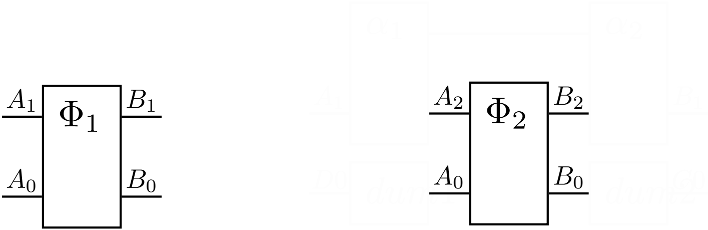
# The problem

Given two channels



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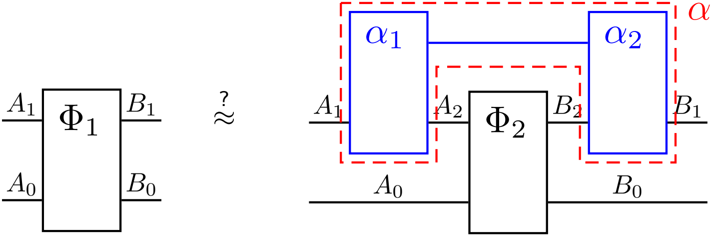


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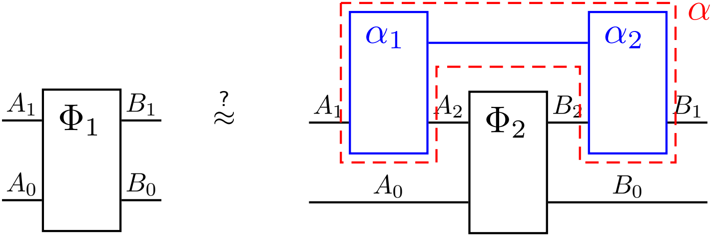


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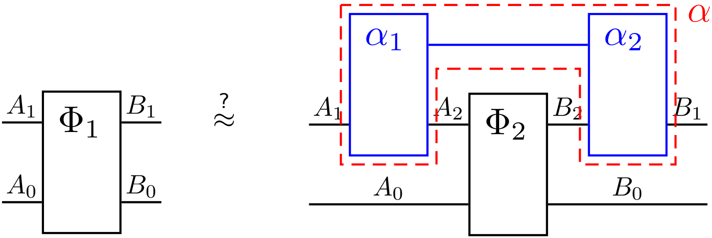


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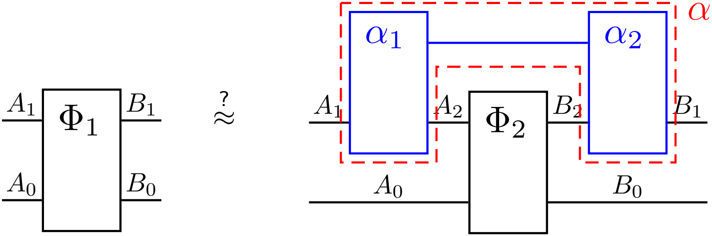


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- all superchannels or some restrictions on $\alpha$

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  - comparison of classical experiments: (Blackwell 1953, Torgersen 1970)
  - more general comparison of classical channels: (Shannon 1958)

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# Deficiency

Let us return to the general case:

- we define the deficiency as

$$\delta_{\mathcal{T}}(\Phi_1 \| \Phi_2) := \min_{\alpha \in \mathcal{T}} \|\Phi_1 - \alpha(\Phi_2)\|_{\diamond}$$

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- equivalence relation:

$$\Phi_1 \sim_{\mathcal{T}} \Phi_2 \iff \Delta_{\mathcal{T}}(\Phi_1, \Phi_2) = 0$$

# Deficiency

- These are extensions of Le Cam deficiency/distance for classical statistical experiments: $\mathcal{F}_1 = \{p_\theta, \theta \in \Theta\}$, $\mathcal{F}_2 = \{q_\theta, \theta \in \Theta\}$

$$\delta(\mathcal{F}_1 \| \mathcal{F}_2) = \min_{\alpha} \sup_{\theta} \|p_\theta - \alpha(q_\theta)\|_1$$

- Randomization theorem (Le Cam 1964): deficiency is characterized by comparing risks in decision problems: informativity

# Randomization theorem for classical channels

$\Phi_1$, $\Phi_2$ - classical channels with equal input spaces: $A_1 = A_2 = A$,
$\mathcal{T} = post$ := set of post-processings

**Theorem**

*Let $\epsilon \geq 0$, Then $\delta_{post}(\Phi_1 \| \Phi_2) \leq \epsilon$ if and only if:*
*for any ensemble $\mathcal{E} = \{\lambda_x, p_x\}$ of classical states $p_x \in \mathcal{S}(A)$,*

$$P_{succ}(\Phi_1(\mathcal{E})) \leq P_{succ}(\Phi_2(\mathcal{E})) + \frac{\epsilon}{2} P_{succ}(\mathcal{E}),$$

*here $\Phi_i(\mathcal{E}) = \{\lambda_x, \Phi_i(p_x)\}$.*

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for any ensemble on $AR \iff$
for any $\rho \in \mathcal{S}(AR)$:

$$2^{-H_{min}(R|B_1)_{\rho_1}} \leq 2^{-H_{min}(R|B_2)_{\rho_2}} + \frac{\epsilon}{2} 2^{-H_{min}(R|A)_\rho}$$

$\rho_i = (\Phi_i \otimes id_R)(\rho)$, $H_{min}$ - conditional min-entropy

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- minimax theorem (Le Cam)

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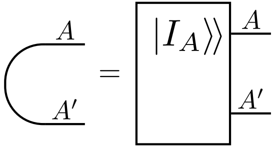

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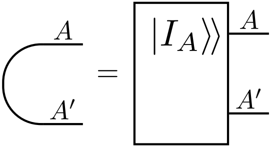


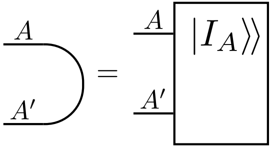

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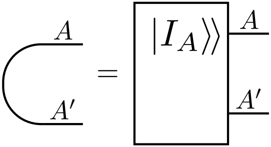


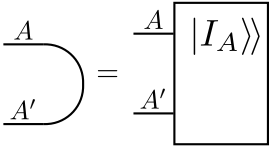


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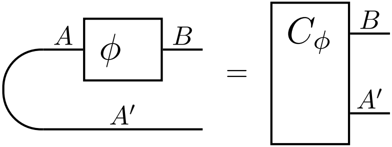

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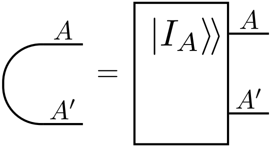
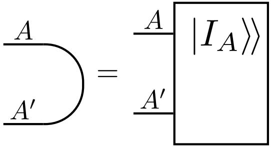
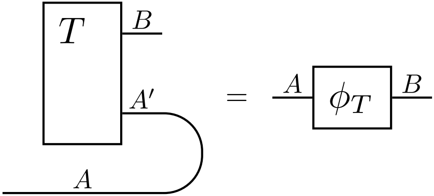



the inverse:



# Ordered spaces of hermitian maps

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- $(\mathcal{L}, \mathcal{L}^+)$ is an ordered vector space
- Choi isomorphism:

$$(\mathcal{L}, \mathcal{L}^+) \simeq (\mathcal{B}_h(BA'), \mathcal{B}(BA')^+)$$

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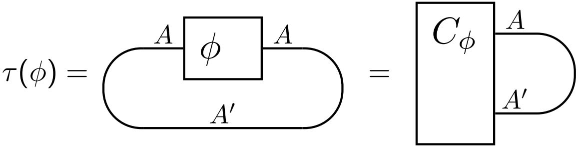

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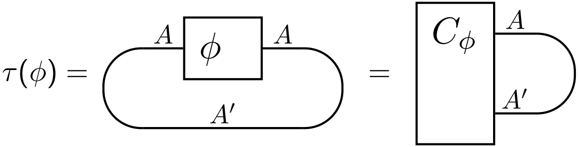


the trace of $\phi$ as a linear map

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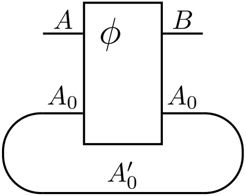

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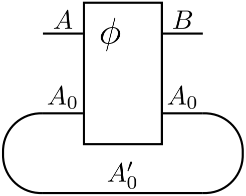


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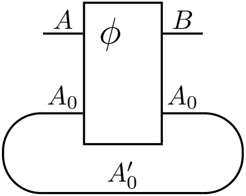


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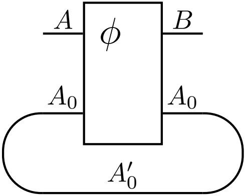


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- $\tau(\phi \circ (\alpha \otimes id)) = \tau(\tau_{A_0}(\phi) \circ \alpha)$

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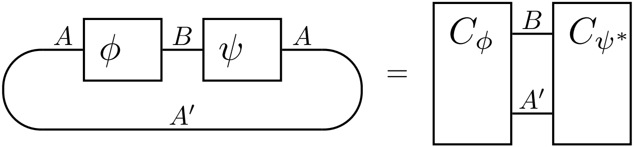

# Diamond norm and conditional min-entropy

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- distinguishability norm for channels $\mathcal{B}(A) \to \mathcal{B}(B)$
- constructed from the convex structure of the set of channels

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Dual expressions:

$$\begin{aligned} \|\phi\|_\diamond &= \min_{\alpha \in \mathcal{C}} \min\{\lambda > 0, -\lambda\alpha \le \phi \le \lambda\alpha\} \\ &= \max_{\gamma \in \tilde{\mathcal{C}}} \max_{-\gamma \le \eta \le \gamma} \langle \eta, \phi \rangle \end{aligned}$$

# Diamond norm and conditional min-entropy

The dual norm: for $\psi \in \mathcal{L}^*$,

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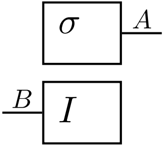


$\sigma \in \mathcal{S}(A)$

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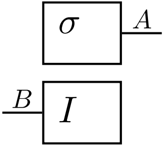


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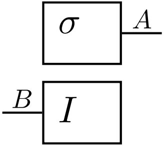


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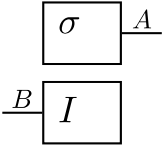


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- conditional min-entropy: $H_{min}(B|A)_\rho$

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Dual expression:

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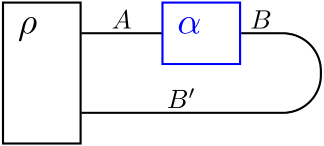

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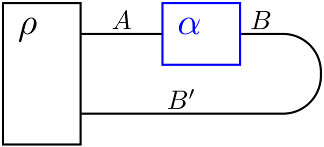


operational interpretation of $H_{min}(B|A)_\rho$ (König et al., 2009):

(up to $d_B$) the largest fidelity with maximally entangled state, that can be obtained by applying a channel on $A$

# The 2-diamond norm and conditional 2-min-entropy

Let now

- $\mathcal{L} = \mathcal{L}(A_1 A_2 \to B_1 B_2)$

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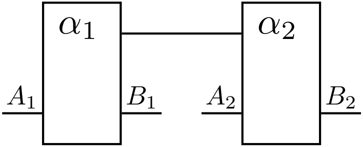


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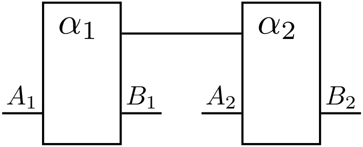


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- $\mathcal{C}_2$ is an affine section of $\mathcal{L}^+$

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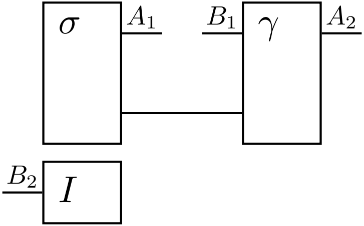


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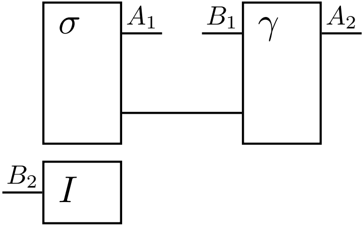


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$\sigma$ is a state, $\gamma$ a channel

# The 2-diamond norm and conditional 2-min-entropy

- we can define a pair of dual norms in $\mathcal{L}$, $\mathcal{L}^*$ as before:

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\|\phi\|_{2\diamond} &= \min_{\alpha\in\mathcal{C}_2} \min\{\lambda > 0, -\lambda\alpha \le \phi \le \lambda\alpha\} \\
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- $\|\cdot\|_{2\diamond}$ is a distinguishability norm for superchannels
- from $\|\cdot\|^{2\diamond}$, we obtain the conditional 2-min-entropy

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For $\psi \in (\mathcal{L}^+)^*$, $\rho = C_\psi$:

$$\|\psi\|^{2\diamond} = \min_{\sigma,\gamma} \min\{\lambda > 0,\ \psi \leq \lambda\sigma * \gamma \otimes \mathrm{Tr}_{B_2}\} = 2^{-H_{min}^{(2)}(B|A)_\rho}$$

$$= \max_{\alpha \in \mathcal{C}_2} \langle\, \alpha, \psi \,\rangle = \max_{\alpha \in \mathcal{C}_2} \langle\langle\, I_{B_1 B_2} | (\alpha \otimes id)(\rho) | I_{B_1 B_2} \rangle\rangle$$

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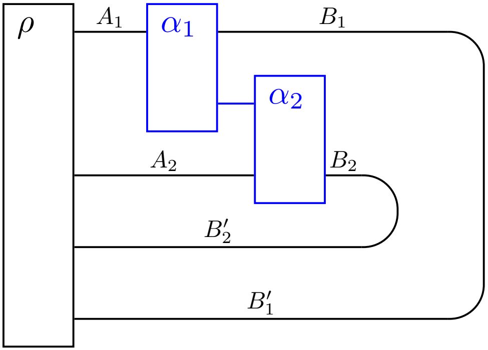

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Useful properties of $H_{min}^{(2)}$: (Gour, 2018)

- monotonicity for any superchannel $\Theta \in \mathcal{C}_2(B_3 \to B_1, A_1 \to A_3)$:

$$\|\phi\|^{2\diamond} \geq \|\Theta(\phi)\|^{2\diamond}$$

- additivity

$$\|\phi \otimes \psi\|^{2\diamond} = \|\phi\|^{2\diamond} \|\psi\|^{2\diamond}$$

# Comparison of bipartite channels

We compute the deficiency $\delta_{sc}(\Phi_1 \| \Phi_2)$:

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(\text{minimax thm.}) &= 2 \max_\gamma \min_\alpha \langle\, \gamma, \Phi_1 - \alpha(\Phi_2)\,\rangle \\
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\text{(minimax thm.)} &= 2 \max_{\gamma} \min_{\alpha} \langle\, \gamma, \Phi_1 - \alpha(\Phi_2)\,\rangle \\
&\leq 2 \max_{\gamma} \left( \max_{\beta \in \mathcal{C}_2} \langle\, \gamma, \beta(\Phi_1)\,\rangle - \max_{\alpha \in \mathcal{C}_2} \langle\, \gamma, \alpha(\Phi_2)\,\rangle \right)
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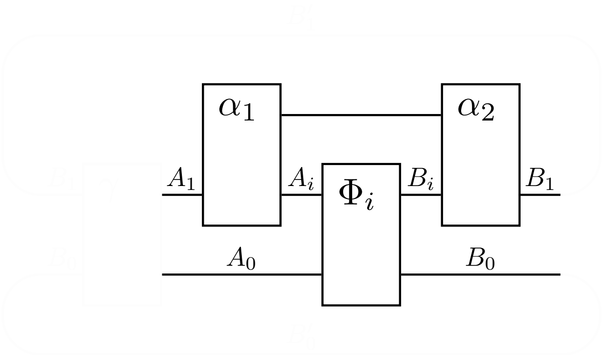

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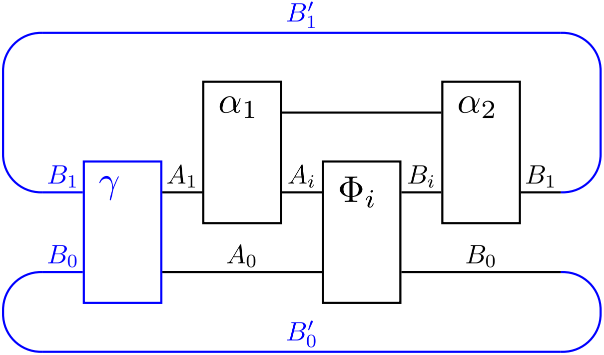

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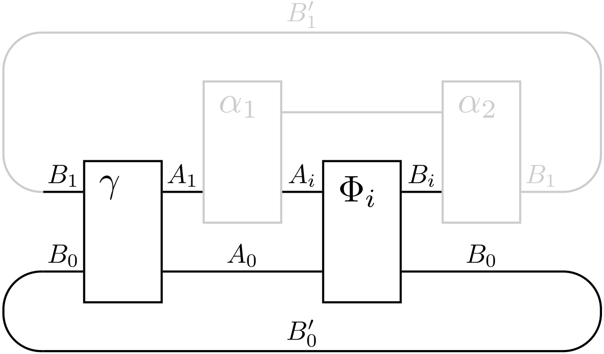


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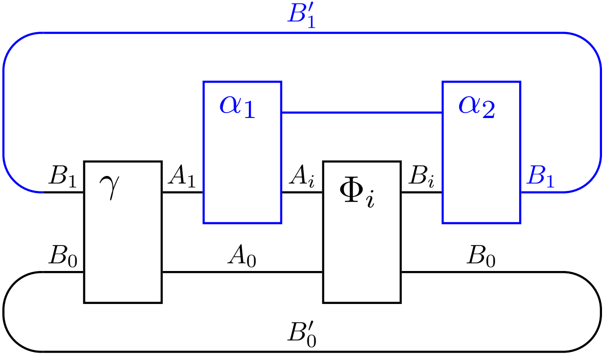

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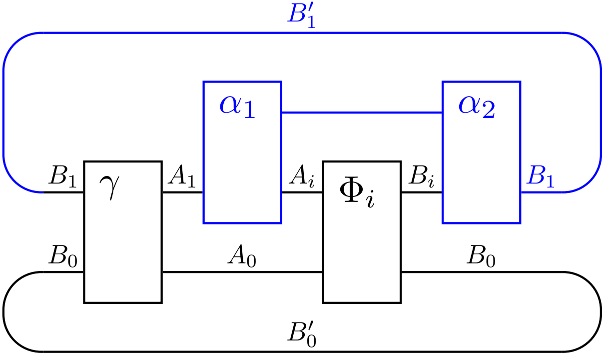
$$\max_{\alpha \in \mathcal{C}_2} \langle\, \gamma, \alpha(\Phi_i)\, \rangle = \max_{\alpha \in \mathcal{C}_2} \langle\, \tau_{B_0}(\gamma * \Phi_i), \alpha\, \rangle = \|\tau_{B_0}(\gamma * \Phi_i)\|^{2\diamond}$$



# Comparison of bipartite channels

## Theorem

*Let $\epsilon \geq 0$. Then $\delta_{sc}(\Phi_1 \| \Phi_2) \leq \epsilon$ if and only if for all systems $A_3, B_3$ and all $\gamma \in \mathcal{L}^+(B_3 B_0 \to A_3 A_0)$ we have*

$$\|\tau_{B_0}(\gamma * \Phi_1)\|^{2\diamond} \leq \|\tau_{B_0}(\gamma * \Phi_2)\|^{2\diamond} + \frac{\epsilon}{2}\|\gamma\|^{\diamond}.$$

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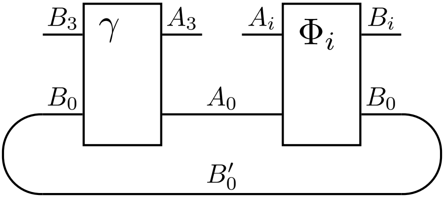

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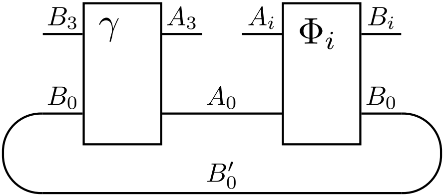


We can restrict to $A_3 \simeq A_1$ and $B_3 \simeq B_1$.

# Conditional min-entropy and guessing probabilities

For an ensemble $\mathcal{E} = \{\lambda_x, \rho_x\}$, $\rho_x \in \mathcal{S}(A)$, let

$$\phi_{\mathcal{E}} \in \mathcal{L}^{+}(B \to A), \quad C_{\phi_{\mathcal{E}}} = \rho_{\mathcal{E}} := \sum_{x} |x\rangle\langle x| \otimes \lambda_x \rho_x$$

$$\phi_{\mathcal{E}} =$$


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Optimal success probability (König et al. 2009)

$$P_{succ}(\mathcal{E}) = \max_M \sum_x \lambda_x \mathrm{Tr}\left[\rho_x M_x\right] = \|\phi_{\mathcal{E}}\|^{\diamond} = 2^{-H_{min}(X|A)_{\rho_{\mathcal{E}}}}$$

# Conditional min-entropy and guessing probabilities

Let $\gamma \in \mathcal{L}^+(R \to A)$, $\rho = C_\gamma \in \mathcal{S}(AR)$. We produce an ensemble

$$\mathcal{E}_\rho = \{\frac{1}{d_R^2}, \rho_x\}, \quad \rho_x = (id_A \otimes \mathcal{U}_x^R)(\rho) \in \mathcal{S}(AR),$$

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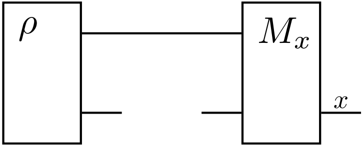
Then we have

$$P_{succ}(\mathcal{E}_\rho) = \frac{1}{d_R} \|\gamma\|^\diamond = \frac{1}{d_R} 2^{-H_{min}(R|A)_\rho}$$

# Conditional min-entropy and guessing probabilities

## Channel discrimination problem:

- an ensemble of channels $\mathcal{E}_R = \{\frac{1}{d_R^2}, \mathcal{U}_x^R\}$
- testers (PPOVMs) with input state $\rho$:




- success probability:

$$P_{succ}(\mathcal{E}_R, \rho) := \max_M \sum_x \mathrm{Tr}\left[(id \otimes \mathcal{U}_x^R)(\rho) M_x\right] = \frac{1}{d_R} 2^{-H_{min}(R|A)_\rho}$$

# Conditional min-entropy and guessing probabilities

For any ensemble $\mathcal{E} = \{\frac{1}{d_R^2}, \Psi_x\}_{x=1}^{d_R^2}$ of unital channels:

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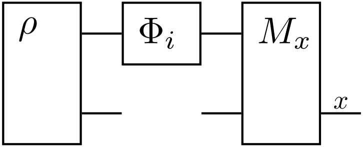
For a pair of quantum channels $\Phi_i : A \to B_i$, $\delta_{post}(\Phi_1 \| \Phi_2)$ can be characterized by comparing testers of the form




for this type of tasks, for any $\rho \in \mathcal{S}(AR)$.

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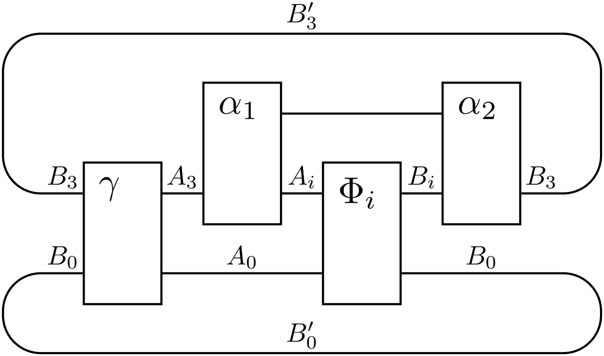

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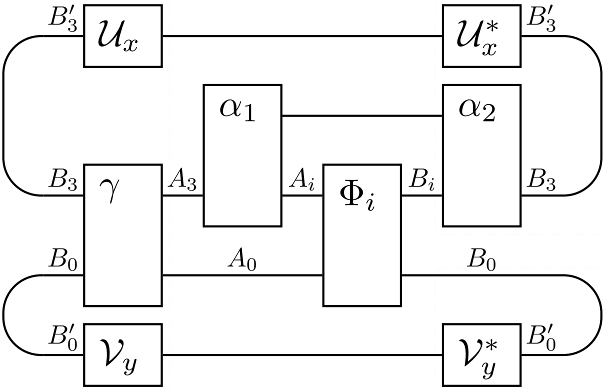

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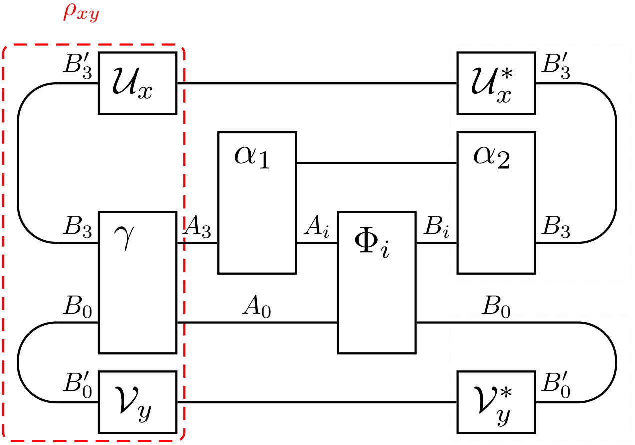

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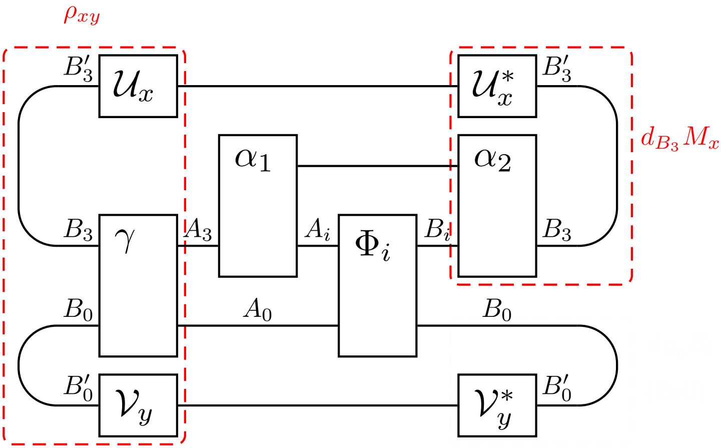

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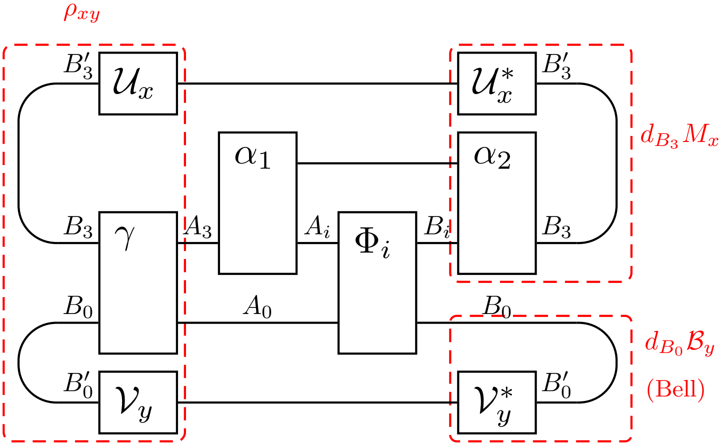

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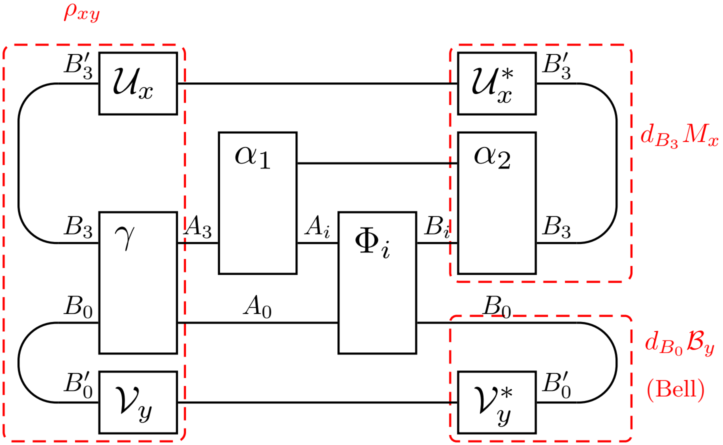


$$= \frac{1}{d_{B_3} d_{B_0}} \sum_{x,y} \mathrm{Tr}\left[\rho_{xy}(\alpha_1 * \Phi_i)^*(M_x \otimes \mathcal{B}_y)\right]$$

# Comparison of bipartite channels by guessing probabilities

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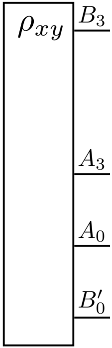

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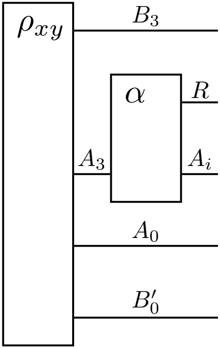

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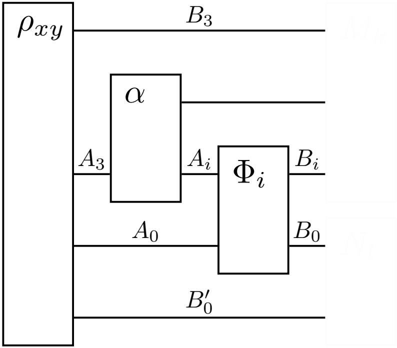

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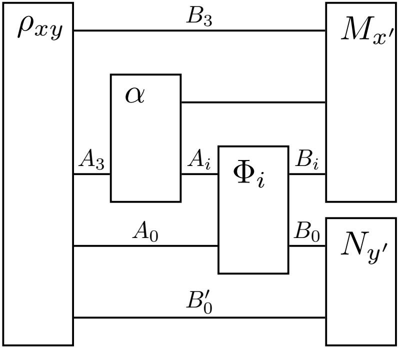

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The optimal success probability is

$$P_{succ}(\mathcal{E}, \Phi_i, N) := \max_{\alpha, M} P_{succ}(\mathcal{E}, (\Phi_i * \alpha)^*(M \otimes N))$$

# Comparison of bipartite channels: guessing probabilities

**Theorem**

*$\delta_{sc}(\Phi_1\|\Phi_2)\le\epsilon$ if and only if for any $A_3$, $B_3$, any POVM $N$ on $B_0B_0'$ and any ensemble $\mathcal{E}=\{\lambda_{xy},\rho_{xy}\}$, on $B_3A_3A_0B_0'$, we have*

$$P_{succ}(\mathcal{E},\Phi_1,N)\le P_{succ}(\mathcal{E},\Phi_2,N)+\frac{\epsilon}{2}P_{succ}(\mathcal{E})$$

*We may restrict to $A_3\simeq A_1$, $B_3\simeq B_1$ and $N=\mathcal{B}$ the Bell measurement.*

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For characterization by guessing probabilities: restrictions on allowed pairs $(\alpha, M)$ of pre-processing and measurement.