FREE SPECTRAHEDRA AND GPTS

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Contents

1. Purpose of these notes

These notes build on the "Notes on incompatibility witnesses and free spectrahedra" by A. Jenčová in which the equivalence of joint measurability to an inclusion problem of generalized spectrahedra is proven using the characterization of joint measurability in terms of entanglement breaking maps from [Jen18]. We give a similar proof of this fact which is more along the lines of [BN18], using the extension of positive maps. The aim is to further understand the relation between these two points of view.

2. The tensor product of a simplicial and a closed convex cone is unique

For the cones arising in GPTs, the uniqueness of the tensor product of a closed convex cone and a simplicial cone is proven in [NP69], which is referenced in [Jen18]. For general cones in finite dimensional vector spaces, we find in Proposition 4 (iv) of [Mul97]:

Proposition 2.1. *Let C, D be convex cones in finite dimensional real vector spaces E and F respectively. Then, if C or D is simplicial,*

$$\overline{C \otimes_{\min} D} = C \otimes_{\max} D$$

Here, a cone is simplicial if it arises as the conic hull of some set of elements with cardinality equal to the dimension of the surrounding vector space. In Exercise 4.14 of [AS17], we furthermore find:

Proposition 2.2. *If C and D are closed convex cones, then $C \otimes_{\min} D$ is closed.*

Combining the two, we get the desired statement:

Proposition 2.3. *Let C and D be closed convex cones in finite dimensional real vector spaces E and F respectively. If one of them is simplicial, it holds that*

$$C \otimes_{\min} D = C \otimes_{\max} D,$$

i.e. the conic tensor product is unique.

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Remark 2.4. [Mul97] *writes before Proposition 4 that $C \otimes_{\min} D$ is not always closed even if both cones are proper (i.e. closed, pointed and solid). As no counterexample are provided and in light of Proposition 2.2, we think that these counterexamples only exit in infinite dimensions.*

3. An extension theorem

We prove an extension theorem which will be the main technical tool in establishing the equivalence between spectrahedral inclusion and compatibility in the framework of GPTs.

Let us set up some notation and introduce the setting for this section. Consider two order vector spaces (V, V^+) and (A, A^+) which we assume to be in duality: $A = V^*$ as vector spaces and the closed cones V^+, A^+ are polar to each other. We shall assume that both cones V^+, A^+ are generating inside the respective vector spaces. Moreover, for some dimension d, we shall consider the ordered vector space $(\mathbb{R}^d, \mathbb{R}^d_+)$, where the cone is the non-negative orthant; note that $\mathbb{R}^d_+$ is a simplicial cone. We shall also consider a subspace $E \subseteq \mathbb{R}^d$, which we shall assume to contain a vector with strictly positive coordinates:

$$E \cap \mathrm{ri}(\mathbb{R}^d_+) \neq \emptyset. \tag{1}$$

We set $E^+ := E \cap \mathbb{R}^d_+$. Note that the condition above implies that the same holds for the dual of E: $E^* \cap \mathrm{ri}(\mathbb{R}^d_+) \neq \emptyset$; to show this, one needs to use the fact that the ordered vector space $(\mathbb{R}^d, \mathbb{R}^d_+)$ is self-dual.

We shall use the following key extension theorem (see e.g. [Cas05, Theorem 1]):

Theorem 3.1 (M. Riesz extension theorem)**.** *Let (X, X^+) be an ordered vector space, $Y \subseteq X$ a linear subspace, and $\varphi : Y \to \mathbb{R}$ a positive linear form on $(Y, Y^+ := Y \cap X^+)$. Assume that for every $x \in X$, there exists $y \in Y$ such that $x \leq y$. Then, there exists a positive linear form $\tilde{\varphi} : X \to \mathbb{R}$ such that $\tilde{\varphi}|_Y = \varphi$.*

We now prove the main result of this section; see also Remark 3.4 for an equivalent formulation.

Proposition 3.2. *Any positive linear form*

$$\varphi : (E \otimes V, (E \otimes V) \cap (\mathbb{R}^d_+ \otimes V^+)) \to \mathbb{R} \tag{2}$$

can be extended to a positive linear form $\tilde{\varphi} : \mathbb{R}^d \otimes V \to \mathbb{R}$ (note that we do not specify the tensor cone we put on the domain of $\tilde{\varphi}$ since $\mathbb{R}^d_+$ is simplicial).

Proof. We shall use Theorem 3.1 with $X = \mathbb{R}^d \otimes V$, $X^+ = \mathbb{R}^d_+ \otimes V^+$ and $Y = E \otimes V$. We have to show that for any $x \in \mathbb{R}^d \otimes V$, there is a $y \in E \otimes V$ such that $y - x \in \mathbb{R}^d_+ \otimes V^+$. It is enough to consider simple tensors of the form $x = r \otimes v$, where $r \in \mathbb{R}^d$ and $v \in V$; the general case will follow by linearity. Since V^+ is generating, there are v_+, $v_- \in V^+$ such that $v = v_+ - v_-$. Furthermore, from the assumption (1), E contains an vector with strictly positive coordinates e, hence there exist $\lambda_\pm > 0$ such that $\lambda_+ e - r \geq 0$ and $\lambda_- e + r \geq 0$. Then,

$$\lambda_+ e \otimes v_+ + \lambda_- e \otimes v_- - r \otimes v = \lambda_+ e \otimes v_+ + \lambda_- e \otimes v_- - r \otimes v_+ + r \otimes v_- \in \mathbb{R}^d_+ \otimes V^+$$

and $\lambda_\pm e \otimes v_\pm \in E \otimes V$. Thus, we can choose $y = \lambda_+ e \otimes v_+ + \lambda_- e \otimes v_-$. □

We now provide a useful characterization of the maximal tensor product of E^+ with V^+, identifying at the same time the cone appearing in (2).

Proposition 3.3. *For E, V as above, it holds that*

$$E^+ \otimes_{\max} V^+ = (E \otimes V) \cap (\mathbb{R}^d_+ \otimes V^+).$$

Proof. The inclusion "$\subseteq$" follows from the monotonicity of the max tensor product with respect to each factor. To show the reverse inclusion "$\supseteq$", we have to prove that for any $z \in (E \otimes V) \cap (\mathbb{R}^d_+ \otimes V^+)$, and for any $\varepsilon \in (E^+)^*$, $\alpha \in (V^+)^* = A^+$, we have that $\langle \varepsilon \otimes \alpha, z \rangle \geq 0$.

By Proposition 3.2, we can extend the form $\varepsilon : E^* \to \mathbb{R}$ to a positive form $\tilde{\varepsilon} : \mathbb{R}^d \to \mathbb{R}$. Indeed, one needs to apply Proposition 3.2 with $(E, V) \leftarrow (E^*, \mathbb{R})$; in this case, the proposition is just the usual Arveson extension theorem if $e = (1, \dots, 1)$, because any positive map into $\mathbb{R}$ is completely positive and can hence be extended. Since $z \in \mathbb{R}^d_+ \otimes V^+$, we have a decomposition $z = \sum_{i=1}^d r_i \otimes v_i$, where $r_i \in \mathbb{R}^d_+$ and $v_i \in V^+$. This yields

$$\langle \varepsilon \otimes \alpha, z \rangle = \langle \tilde{\varepsilon} \otimes \alpha, z \rangle = \sum_{i=1}^d \tilde{\varepsilon}(r_i)\alpha(v_i) \geq 0,$$

finishing the proof. □

Remark 3.4. *Using the result above, one can restate Proposition 3.2 as follows: Any positive linear form $\varphi : (E \otimes V, E^+ \otimes_{\max} V^+) \to \mathbb{R}$ can be extended to a positive linear form $\tilde{\varphi} : \mathbb{R}^d \otimes V \to \mathbb{R}$.*

4. Inclusion and GPTs

We relate here the problems of spectrahedral inclusion (for cones) and compatibility of effects in GPTs. We shall consider a GPT defined by dual ordered cones (V, V^+) and (A, A^+) as in the previous section, with A^+ containing a distinguished element $\mathbb{1}_A$. We shall also specialize $d = 2^g$ for some positive number $g \geq 1$ and consider

$$E := \operatorname{span}\{1_{2^g}, c_i : i \in [g]\} \subseteq \mathbb{R}^{2^g}, \tag{3}$$

with $c_i = (1,1) \otimes \cdots \otimes (1,1) \otimes (+1,-1) \otimes (1,1) \otimes \cdots \otimes (1,1) \in \mathbb{R}^{2^g}$, where $(+1,-1)$ is the i-th tensor factor.

Lemma 4.1. *Let $(z_0, \dots, z_g) \in L^{g+1}$ and c_i as in Eq. (3). Moreover, let (L, L^+) be an ordered vector space, with L^+ closed and generating. Then*

$$z_0 + \sum_{i=1}^g \varepsilon_i z_i \in L^+ \qquad \forall \varepsilon \in \{\pm 1\}^g$$

if and only if

$$1_{2^g} \otimes z_0 + \sum_{i=1}^g c_i \otimes z_i \in E^+ \otimes_{\max} L^+.$$

Proof. To begin, fix $\{e_j\}_{j=1}^{2^g}$, the standard basis of $\mathbb{R}^{2^g}$. One has the following decompositions

$$\forall i \in [g], \qquad c_i = \sum_{j=1}^{2^g} \varepsilon_i^{(j)} e_j,$$

having the property that $\{\varepsilon^{(j)}\}_{j=1}^{2^g} = \{\pm 1\}^g$. Let δ_j, $j \in [2^g]$ be the standard basis of $(\mathbb{R}^{2^g})^*$, dual to the basis e_j. Then, δ_j is positive. Let $z \in L^{g+1}$ such that

$$z_0 + \sum_{i=1}^g \varepsilon_i z_i \in L^+ \qquad \forall \varepsilon \in \{\pm 1\}^g.$$

Let furthermore

$$y := I \otimes z_0 + \sum_{i=1}^g c_i \otimes z_i = \sum_{j=1}^{2^g} e_j \otimes (z_0 + \sum_{i=1}^g \varepsilon_i^{(j)} z_i).$$

Then,

$$\langle \delta_j \otimes \beta, y \rangle = \beta(z_0 + \sum_{i=1}^g \varepsilon_i^{(j)} z_i) \geq 0$$

for all $\beta \in (L^+)^*$ and $j \in [2^g]$. As any $\alpha \in (\mathbb{R}^d_+)^*$ can be written as $\alpha = \alpha_1\delta_1 + \ldots \alpha_{2^g}\delta_{2^g}$ with $\alpha_j \geq 0$ for all $j \in [2^g]$, using Proposition 3.3 we have shown that $y \in E^+ \otimes_{\max} L^+$.

Now let $y \in E^+ \otimes_{\max} L^+$. Then, again

$$0 \leq \langle \delta_j \otimes \beta, y \rangle = \beta(z_0 + \sum_{i=1}^g \varepsilon_i^{(j)} z_i)$$

for any $\beta \in (L^+)^*$. Therefore, $z_0 + \sum_{i=1}^g \varepsilon_i^{(j)} z_i \in ((L^+)^*)^*$ for all $j \in [2^g]$. Since L is finite dimensional and L^+ closed, we have that $((L^+)^*)^* = L^+$. □

Definition 4.2. *We define the following two generalized spectrahedra:*

$$\mathcal{D}_{\mathrm{GPT}\diamond,g}(L, L^+) := \left\{ (z_0, \ldots, z_g) \in L^{g+1} : 1_{2^g} \otimes z_0 + \sum_{i=1}^g c_i \otimes z_i \in E^+ \otimes_{\max} L^+ \right\} \tag{4}$$

and for $f_i \in A$, $i \in [g]$,

$$\mathcal{D}_f(L, L^+) := \left\{ (z_0, \ldots, z_g) \in L^{g+1} : \mathbb{1} \otimes z_0 + \sum_{i=1}^g (2f_i - \mathbb{1}) \otimes z_i \in A^+ \otimes_{\min} L^+ \right\}. \tag{5}$$

Proposition 4.3. *Let L^+ be generating. Then, $\mathcal{D}_{\mathrm{GPT}\diamond,g}(L, L^+) \subseteq \mathcal{D}_f(L, L^+)$ if and only if $\Phi \otimes \mathrm{id}_L : (E \otimes L, E^+ \otimes_{\max} L^+) \to (A \otimes L, A^+ \otimes_{\min} L^+)$ is positive, where Φ is defined as*

$$\begin{aligned} \Phi : E &\to A \\ 1_{2^g} &\mapsto \mathbb{1} \\ c_i &\mapsto 2f_i - \mathbb{1}. \end{aligned}$$

Proof. Let $\Phi \otimes \mathrm{id}_L$ be positive (in the sense of the statement) and $z \in \mathcal{D}_{\mathrm{GPT}\diamond,g}$. Then

$$(\Phi \otimes \mathrm{id}_L)(1_{2^g} \otimes z_0 + \sum_{i=1}^g c_i \otimes z_i) = \mathbb{1} \otimes z_0 + \sum_{i=1}^g (2f_i - \mathbb{1}) \otimes z_i \in A^+ \otimes_{\min} L^+.$$

Hence, $z \in \mathcal{D}_f(L, L^+)$. Conversely, let $\mathcal{D}_{\mathrm{GPT}\diamond,g}(L, L^+) \subseteq \mathcal{D}_f(L, L^+)$. Then, any element $y \in E^+ \otimes_{\max} L^+$ can be written as

$$y = I \otimes z_0 + \sum_{j=1}^g c_j \otimes z_j$$

with $z \in D_{\mathrm{GPT}\diamond,g}(L, L^+)$. By inclusion, we have that $z \in \mathcal{D}_f(L, L^+)$ as well and therefore, $\Phi \otimes \mathrm{id}_L$ is positive. □

Proposition 4.4. *$\mathcal{D}_{\mathrm{GPT}\diamond,g}(\mathbb{R}, \mathbb{R}_+) \subseteq \mathcal{D}_f(\mathbb{R}, \mathbb{R}^+)$ if and only if f_j are effects for all $j \in [g]$.*

Proof. Using Proposition 4.3, the condition in the statement is equivalent to the positivity of the map $\Phi \otimes \mathrm{id}_{\mathbb{R}} = \Phi$ from the statement of the aforementioned proposition. We have then, for all $i \in [g]$,

$$\mathbb{1} \pm (2f_i - \mathbb{1}) \in A^+ \iff f_i \in A^+ \quad \text{and} \quad \mathbb{1} - f_i \in A^+.$$

□

Now we show that inclusion with $L = V$ is equivalent to compatibility of the g binary measurements $\{f_i\}$ in the GPT $(V, V^+, \mathbb{1})$. The proof technique is inspired by the finite dimensional version of Arveson's extension theorem [Pau03, Theorem 6.2]. Define

$$\chi := \sum_{i=1}^{\dim V} v_i \otimes a_i \in V \otimes A \cong (A \otimes V)^* \tag{6}$$

for v_i a basis of V and a_i the corresponding dual basis in A (we have $a_i(v_j) = \delta_{ij}$). The vector χ has the following remarkable property:

$$\forall v \in V, \forall \alpha \in A, \qquad \langle \chi, \alpha \otimes v \rangle = \alpha(v).$$

Proposition 4.5. *The linear map* $\Phi \otimes \mathrm{id}_V : (E \otimes V, E^+ \otimes_{\max} V^+) \to (A \otimes V, A^+ \otimes_{\min} V^+)$ *is positive if and only if there exists a positive extension* $\tilde{\Phi} : (\mathbb{R}^{2^g}, \mathbb{R}^{2^g}_+) \to (A, A^+)$ *of* Φ.

Proof. The existence of the positive extension obviously implies the positivity of the map $\Phi \otimes \mathrm{id}_V$. Let us prove the non-trivial implication. Let $\Phi \otimes \mathrm{id}_V : (E \otimes V, E^+ \otimes_{\max} V^+) \to (A \otimes V, A^+ \otimes_{\min} V^+)$ be a positive linear map. We claim that the form

$$\begin{aligned} s_\Phi : (E \otimes V, E^+ \otimes_{\max} V^+) &\to \mathbb{R} \\ z &\mapsto \langle \chi, [\Phi \otimes \mathrm{id}_V](z) \rangle \end{aligned}$$

is positive. Indeed, let $z \in E^+ \otimes_{\max} V^+$ and

$$A^+ \otimes_{\min} V^+ \ni [\Phi \otimes \mathrm{id}_V](z) = \sum_i \alpha_i \otimes x_i,$$

for some $\alpha_i \in A^+$ and $x_i \in V^+$. Then, $\langle \chi, [\Phi \otimes \mathrm{id}_V](z) \rangle = \sum_i \alpha_i(x_i) \geq 0$, proving the claim. Using Proposition 3.2 and Remark 3.4, we extend the form s_Φ to $\tilde{s}_\Phi : \mathbb{R}^{2^g} \otimes V \to \mathbb{R}$. We now go back to linear maps by defining

$$\begin{aligned} \tilde{\Phi} : (\mathbb{R}^{2^g}, \mathbb{R}^{2^g}_+) &\to (A, A^+) \\ r &\mapsto \sum_{i=1}^{\dim V} \tilde{s}_\Phi(r \otimes v_i) a_i, \end{aligned}$$

where the dual bases $\{v_i\}$, $\{a_i\}$ are the ones from (6). One can directly check that the definition above is tailored to have the following relation:

$$\forall (r, v) \in \mathbb{R}^{2^g} \times V, \qquad \langle \tilde{\Phi}(r), v \rangle = \tilde{s}_\Phi(r \otimes v).$$

The positivity of the form $\tilde{s}_\Phi$ implies the positivity of $\tilde{\Phi}$, as it was the case for the pair (s, Φ). To conclude, it remains to check that $\tilde{\Phi}$ is indeed an extension of Φ. For any $e \in E$, we compute

$$\begin{aligned} \tilde{\Phi}(e) &= \sum_{i=1}^{\dim V} \tilde{s}_\Phi(e \otimes v_i) a_i = \sum_{i=1}^{\dim V} s_\Phi(e \otimes v_i) a_i \\ &= \sum_{i=1}^{\dim V} \langle \chi, [\Phi \otimes \mathrm{id}_V](e \otimes v_i) \rangle a_i = \sum_{i=1}^{\dim V} \langle \chi, \Phi(e) \otimes v_i \rangle a_i \\ &= \sum_{i=1}^{\dim V} [\Phi(e)](v_i) a_i = \Phi(e), \end{aligned}$$

finishing the proof. $\square$

Theorem 4.6. $\mathcal{D}_{\mathrm{GPT}\diamond,g}(V, V^+) \subseteq \mathcal{D}_f(V, V^+)$ *if and only if* f_i, $i \in [g]$ *are jointly measurable effects.*

Proof. By Propositions 4.3 and 4.5, the inclusion holds if and only if the corresponding map Φ has a positive extension $\tilde{\Phi}$ to $\mathbb{R}^{2^g}$. We index the coordinates of the vectors c_i defining the GPT-diamond by sign vectors $\varepsilon \in \{-1, +1\}^g$ such that

$$\begin{aligned} c_0 := 1_{2^g}(\varepsilon) &= 1 \\ c_i(\varepsilon) &= \varepsilon_i \qquad \forall i \in [g]. \end{aligned}$$

We choose a basis g_η of $\mathbb{R}^{2^g}$ such that $g_\eta = \mathbf{1}_{\varepsilon=\eta} \geq 0$. Let $G_\eta := \tilde{\Phi}(g_\eta)$. Then, $G_\eta \in A^+$ by positivity of $\tilde{\Phi}$. Furthermore,

$$\mathbb{1} = \tilde{\Phi}(c_0) = \sum_\eta \tilde{\Phi}(g_\eta) = \sum_\eta G_\eta.$$

Finally,

$$c_i(\varepsilon) = \varepsilon_i = 2\mathbf{1}_{\varepsilon_i=1} - 1 = 2\sum_{\eta:\eta_i=1} \mathbf{1}_{\eta=\varepsilon} - 1 = 2\sum_{\eta:\eta_i=1} g_\eta(\varepsilon) - 1_{2^g}.$$

Therefore,

$$2f_i - \mathbb{1} = \tilde{\Phi}(c_i) = 2\sum_{\eta:\eta_i=1} G_\eta - \mathbb{1},$$

thus

$$f_i = \sum_{\eta:\eta_i=1} G_\eta.$$

Hence, the G_η form a joint measurement if and only if $\tilde{\Phi}$ is positive.

□

5. Open questions

We finish this note with a few questions we think would be interesting to pursue:

(1) How do inclusion constants look like for these generalized spectrahedra? Since they are unbounded in general (unlike the free spectrahedra), it is not immediately clear how one would define the inclusion constants necessary for noise robustness. Moreover, in the generalized setting of GPTs, there is usually no preferred ray around which to shrink/dilate the spectrahedral cones (4) and (5).

(2) What do the intermediate levels of the free spectrahedron correspond to? Are these restrictions to subspaces of V of fixed dimension ? We would like to recover the third statement of [BN18, Theorem 5.3] for GPTs.

(3) Can we say anything about the uniqueness of the joint POVM?

(4) Can we formulate the map extension as a conic program? For free spectrahedra, the existence of a completely positive extension can be checked with an SDP [HJRW12].

(5) Can one formalize the duality between the two approaches (extension vs entanglement breaking)?

(6) What does it mean that the elements in the free spectrahedron defined by the effects in [BN18, Theorem 5.3] need not only be positive, but even separable?

(7) Can we characterize the theories admitting maximally incompatible effects from looking at their generalized spectrahedra?

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