Rényi relative entropies and sufficiency of quantum channels

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Dedicated to the memory of Dénes Petz

Classical Rényi relative entropies

For p, q probability measures over a finite set X, $0 < \alpha \neq 1$:

$$D_\alpha(p\|q) = \frac{1}{\alpha - 1} \log \sum_x p(x)^\alpha q(x)^{1-\alpha}$$

In the limit $\alpha \to 1$: relative entropy

$$S(p\|q) = \sum_x p(x) \log(p(x)/q(x))$$

- introduced as the unique family of divergences satisfying a set of postulates
- fundamental quantities appearing in many information - theoretic tasks

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A. Rényi, *Proc. Symp. on Math., Stat. and Probability*, 1961.

Standard quantum Rényi relative entropies

For density matrices ρ, σ, $0 < \alpha \neq 1$,

$$D_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1} \log \mathrm{Tr}\, \rho^\alpha \sigma^{1-\alpha}$$

In the limit $\alpha \to 1$: quantum (Umegaki) relative entropy

$$S(\rho\|\sigma) = \mathrm{Tr}\, \rho(\log(\rho) - \log(\sigma))$$

- obtained from Petz quasi-entropies[1,2]
- defined for normal states of a von Neumann algebra, using the relative modular operator

[1] D. Petz, *Rep. Math. Phys., 1986*

[2] D. Petz, Publ. RIMS, Kyoto Univ., 1985

Standard quantum Rényi relative entropies

It follows from the properties of quasi-entropies that: if $\alpha \in (0, 2]$

- strict positivity: $D_\alpha(\rho\|\sigma) \geq 0$ with equality iff $\rho = \sigma$;
- data processing inequality:

  $$D_\alpha(\rho\|\sigma) \geq D_\alpha(\Phi(\rho)\|\Phi(\sigma))$$

  for any quantum channel Φ
- joint lower semicontinuity
- joint (quasi)-convexity: the map

  $$(\rho, \sigma) \mapsto \exp\{(\alpha - 1)D_\alpha(\rho\|\sigma)\}$$

  is jointly convex.

Equality in DPI: sufficient quantum channels

> Let the quantum states ρ, σ and a channel Φ be such that
>
> $$S(\Phi(\rho)\|\Phi(\sigma)) = S(\rho\|\sigma) < \infty$$

This condition was introduced as a quantum extension of classical sufficient statistics:

A statistic T is sufficient with respect to a pair of probability distributons $\{p, q\}$ if

- the conditional expectation satisfies $E_p[\cdot|T] = E_q[\cdot|T]$
- an equivalent Kullback-Leibler characterization by the classical relative entropy:

$$S(p^T\|q^T) = S(p\|q) \text{ (if } < \infty)$$

D. Petz, CMP, 1986

D. Petz, *Quart. J. Math. Oxford*, 1988

Sufficient quantum channels

Let ρ, σ be quantum states (normal states of a von Neumann algebra), σ faithful. Assume that $S(\rho\|\sigma) < \infty$

Theorem

The following are equivalent.

- $S(\rho\|\sigma) = S(\Phi(\rho)\|\Phi(\sigma))$;
- There is a quantum channel Ψ such that

$$\Psi \circ \Phi(\rho) = \rho, \qquad \Psi \circ \Phi(\sigma) = \sigma$$

We say in this case that Φ is sufficient (reversible) with respect to $\{\rho, \sigma\}$.

Sufficient quantum channels: divergences

A divergence D characterizes sufficiency if

$$D(\Phi(\rho)\|\Phi(\sigma)) = D(\rho\|\sigma) < \infty$$

implies that Φ is sufficient with respect to $\{\rho, \sigma\}$.

The following divergences characterize sufficiency:

- relative entropy
- D_α, with $\alpha \in (0, 2)$[3,4]
- a class of f-divergences (in finite dimension)[5,6]

[3] D. Petz, *Quart. J. Math. Oxford*, 1988
[4] AJ and D. Petz, 2006
[5] Hiai, Mosonyi, Petz, Bény, 2011
[6] F. Hiai and M. Mosonyi, 2017

Sufficient quantum channels: universal recovery channel

The Petz recovery channel is defined as

$$\Phi_\sigma(Y) = \sigma^{1/2}\Phi^*(\Phi(\sigma)^{-1/2}Y\Phi(\sigma)^{-1/2})\sigma^{1/2}$$

Note that we always have $\Phi_\sigma \circ \Phi(\sigma) = \sigma$.

Theorem

Φ is sufficient with respect to $\{\rho, \sigma\}$ if and only if

$$\Phi_\sigma \circ \Phi(\rho) = \rho.$$

Sufficient quantum channels: a conditional expectation

Note that by the last condition, ρ (and σ) must be invariant states of the channel $\Phi_\sigma \circ \Phi$.

> There is a conditional expectation E, $\sigma \circ E = \sigma$, such that Φ is sufficient with respect to $\{\rho, \sigma\}$ if and only if $\rho \circ E = \rho$.

Structure ot the states[7]: in finite dimensions (or on $B(\mathcal{H})$), there is a decomposition

$$U\sigma U^* = \bigoplus_n \lambda_n \sigma_n^L \otimes \sigma_n^R, \qquad \sigma_n^L, \sigma_n^R \text{ states, } \lambda_n \text{ probabilities}$$

such that Φ is sufficient with respect to $\{\rho, \sigma\}$ iff

$$U\rho U^* = \bigoplus_n \mu_n \rho_n^L \otimes \sigma_n^R, \qquad \rho_n^L \text{ states, } \mu_n \text{ probabilities}$$

[7] M. Mosonyi and D. Petz, 2004; AJ and D. Petz, 2006

Sufficient quantum channels: applications

Characterization of equality in various entropic inequalities:

- strong subadditivity: characterization of quantum Markov states[8]
- monotonicity of quantum Fisher information, Holevo quantity, etc.

Approximate version - recoverability[9]:

$$S(\rho\|\sigma) - S(\Phi(\rho)\|\Phi(\sigma)) \geq d(\rho\|\tilde{\Phi}_\sigma \circ \Phi(\rho))$$

d some divergence measure, $\tilde{\Phi}_\sigma$ a modification of Petz recovery channel

[8] Hayden, Josza, Petz, Winter, 2004

[9] O. Fawzi and R. Renner, 2014; M. M. Wilde, 2015; Junge et. al, 2015; ...

Sandwiched Rényi relative entropy

Another version:

for density matrices ρ, σ:

$$\tilde{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1} \log \mathrm{Tr}\left[\left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha\right]$$

M. Müller-Lennert et al., *J. Math. Phys.*, 2013
M. M. Wilde et al., *Commun. Math. Phys.*, 2014

- satisfies DPI (+ other properties) if $\alpha \in [1/2, 1) \cup (1, \infty)$
- $\lim_{\alpha \to 1} \tilde{D}_\alpha(\rho\|\sigma) = S(\rho\|\sigma)$
- operational interpretation for $\alpha > 1$: strong converse exponents in quantum hypothesis testing[10]

[10] M. Mosonyi, and T. Ogawa, *Commun. Math. Phys.*, 2017

The purpose of the rest of this talk

Extend the sandwiched Rényi relative entropies to normal states of von Neumann algebras and show some properties

- the standard version D_α (quasi-entropies) is defined in this setting and has an operational interpretation in hypothesis testing as in finite dimensions[11]

In this general setting, prove that $\tilde{D}_\alpha$ characterize sufficiency of channels, for $\alpha \in (1/2, 1) \cup (1, \infty)$.

[11] V. Jaksic et al., *Rev. Math. Phys., 2012*

Extensions of $\tilde{D}_\alpha$ to von Neumann algebras

Let ρ, σ be normal states on a von Neumann algebra $\mathcal{M}$.

Two constructions, using noncommutative L_p-spaces:

- Araki-Masuda divergences, defined for $\alpha \in [1/2, 1) \cup (1, \infty]$, uses Araki-Masuda L_p-spaces[12]
- sandwiched Rényi relative entropies, defined for $\alpha > 1$, uses Kosaki L_p-spaces[13]

[12] M. Berta, V. B. Scholz, and M. Tomamichel, Ann. H. Poincaré, 2018

[13] AJ, Ann. H. Poincaré, 2018

Haagerup L_p-spaces

For $1 \leq p \leq \infty$, $L_p(\mathcal{M})$ - Haagerup L_p-space, with norm $\| \cdot \|_p$

- $\mathcal{M} \simeq L_\infty(\mathcal{M})$;
- the predual $\mathcal{M}_* \simeq L_1(\mathcal{M})$: $\rho \mapsto h_\rho$, $\mathrm{Tr}\, h_\rho = \rho(1)$;
- $L_2(\mathcal{M})$ a Hilbert space: $\langle\, h, k \,\rangle = \mathrm{Tr}\, k^* h$

If $\mathcal{M} = B(\mathcal{H})$, we can use the Schatten classes:

$$L_p(\mathcal{M}) = \{X \in B(\mathcal{H}), \mathrm{Tr}\, |X|^p < \infty\}, \quad \|X\|_p = (\mathrm{Tr}\, |X|^p)^{1/p}$$

The standard form

Standard form: $(\lambda(\mathcal{M}), L_2(\mathcal{M}), J, L_2(\mathcal{M})^+)$

a representation of $\mathcal{M}$ on $L_2(\mathcal{M})$ by left multiplication:

$$\lambda(x)h = xh, \qquad x \in \mathcal{M},\ h \in L_2(\mathcal{M})$$

Any $\rho \in \mathcal{M}_*^+$ has a unique vector representative $h_\rho^{1/2}$ in $L_2(\mathcal{M})^+$:

$$\rho(a) = \langle\, a h_\rho^{1/2}, h_\rho^{1/2} \,\rangle$$

Kosaki L_p-spaces with respect to a faithful normal state

Let σ be a faithful normal state:

- continuous embedding

$$\mathcal{M} \to L_1(\mathcal{M}), \quad x \mapsto h_\sigma^{1/2} x h_\sigma^{1/2}$$

- Put

$$L_\infty(\mathcal{M}, \sigma) := h_\sigma^{1/2} \mathcal{M} h_\sigma^{1/2}, \qquad \|h_\sigma^{1/2} x h_\sigma^{1/2}\|_{\infty,\sigma} = \|x\|_\infty$$

- let $\rho \in \mathcal{M}_*^+$, then $h_\rho \in L_\infty(\mathcal{M}, \sigma)$ iff $\rho \leq \lambda\sigma$ and

$$\|h_\rho\|_{\infty,\sigma} = \inf\{\lambda > 0, \rho \leq \lambda\sigma\}.$$

Kosaki L_p-spaces with respect to a faithful normal state

Let $1 < p < \infty$:

- $(L_\infty(\mathcal{M}, \sigma), L_1(\mathcal{M}))$ compatible pair of Banach spaces
- interpolation space

$$L_p(\mathcal{M}, \sigma) := C_{1/p}(L_\infty(\mathcal{M}, \sigma), L_1(\mathcal{M})), \text{ with norm } \|\cdot\|_{p,\sigma}$$

- Let $1/p + 1/q = 1$, then

$$L_p(\mathcal{M}, \sigma) = \{h_\sigma^{1/2q} k h_\sigma^{1/2q},\ k \in L_p(\mathcal{M})\},$$
$$\|h_\sigma^{1/2q} k h_\sigma^{1/2q}\|_{p,\sigma} = \|k\|_p$$

A definition of $\tilde{D}_\alpha$, $\alpha > 1$

Extension to non-faithful σ: by restriction to support $s(\sigma) = e$

$$L_p(\mathcal{M}, \sigma) = \{h \in L_1(\mathcal{M}),\ h = ehe \in L_p(e\mathcal{M}e, \sigma|_{e\mathcal{M}e})\}.$$

For normal states ρ, σ and $1 < \alpha < \infty$:

$$\tilde{D}_\alpha(\rho\|\sigma) = \begin{cases} \frac{\alpha}{\alpha-1}\log(\|h_\rho\|_{\alpha,\sigma}) & \text{if } h_\rho \in L_\alpha(\mathcal{M}, \sigma) \\ \infty & \text{otherwise.} \end{cases}$$

Some properties of $\tilde{D}_\alpha$

Using complex interpolation, we can prove

- strict positivity: $\tilde{D}_\alpha(\rho\|\sigma) \geq 0$, with equality iff $\rho = \sigma$.
- joint lower semicontinuity (on $L_1(\mathcal{M})^+ \times L_1(\mathcal{M})^+$)
- if $\rho \neq \sigma$ and $\tilde{D}_\alpha(\rho\|\sigma) < \infty$, then

$$\alpha' \mapsto \tilde{D}_{\alpha'}(\rho\|\sigma) \text{ is strictly increasing for } \alpha' \in (1, \alpha].$$

- quasi-convexity

Relation to the standard version D_α, limit values

For normal states ρ, σ, $\alpha > 1$:

$$D_{2-1/\alpha}(\rho\|\sigma) \leq \tilde{D}_\alpha(\rho\|\sigma) \leq D_\alpha(\rho\|\sigma)$$

$$\Downarrow$$

Limit values:

$$\lim_{\alpha\to1} \tilde{D}_\alpha(\rho\|\sigma) = S(\rho\|\sigma)$$

$$\lim_{\alpha\to\infty} \tilde{D}_\alpha(\rho\|\sigma) = \tilde{D}_\infty(\rho\|\sigma) \text{ relative max entropy}$$

Data processing inequality

$\Phi : L_1(\mathcal{M}) \to L_1(\mathcal{N})$ positive, trace-preserving. Let $\sigma_0 = \Phi(\sigma)$.

- Φ is a contraction $L_1(\mathcal{M}) \to L_1(\mathcal{N})$.
- if $0 \leq \rho \leq \lambda\sigma$, then also $0 \leq \Phi(\rho) \leq \lambda\sigma_0$, hence

$$\Phi(L_\infty(\mathcal{M}, \sigma)^+) \to L_\infty(\mathcal{N}, \sigma_0)^+$$

- this extends to a map $x \mapsto y$, $\Phi(h_\sigma^{1/2} x h_\sigma^{1/2}) = h_{\sigma_0}^{1/2} y h_{\sigma_0}^{1/2}$ (adjoint of) the Petz recovery channel[14] Φ_σ
- Φ defines a contraction $L_\infty(\mathcal{M}, \sigma) \to L_\infty(\mathcal{N}, \sigma_0)$.

> Φ defines a contraction $L_p(\mathcal{M}, \sigma) \to L_p(\mathcal{N}, \sigma_0)$ for all $1 \leq p \leq \infty$.

[14] D. Petz, Quart. J. Math. Oxford, 1984

Data processing inequality

For $\alpha > 1$, normal states ρ, σ, positive, trace-preserving Φ:

$$\tilde{D}_\alpha(\rho\|\sigma) \geq \tilde{D}_\alpha(\Phi(\rho)\|\Phi(\sigma))$$

Consequently, by the limit $\alpha \to 1$:

For normal states ρ, σ,

$$S(\rho\|\sigma) \geq S(\Phi(\rho)\|\Phi(\sigma))$$

holds for any positive trace-preserving map Φ.

A. Müller-Hermes, D. Reeb, Ann. H. Poincaré, 2017 (for $\mathcal{M} = B(\mathcal{H})$)

The Araki-Masuda divergences

The Araki-Masuda L_p-norm: defined on $L_2(\mathcal{M})$

- for $2 \leq p \leq \infty$, $\xi \in L_2(\mathcal{M})^+$,

$$\|\xi\|_{p,\sigma}^{AM} = \sup_{\omega \in \mathcal{M}_*^+, \omega(1)=1} \|\Delta_{\omega,\sigma}^{1/2-1/p}\xi\|_2$$

  if $s(\omega_\xi) \leq s(\sigma)$ and is infinite otherwise

- for $1 \leq p < 2$,

$$\|\xi\|_{p,\sigma}^{AM} = \inf_{\omega \in \mathcal{M}_*^+, \omega(1)=1, s(\omega) \geq s(\omega_\xi)} \|\Delta_{\omega,\sigma}^{1/2-1/p}\xi\|_2$$

M. Berta, V. B. Scholz, and M. Tomamichel, Ann. H. Poincaré, 2018

The Araki-Masuda divergences

For normal states ρ, σ and $\alpha \in [1/2, 1) \cup (1, \infty)$:

$$\tilde{D}_\alpha^{AM}(\rho\|\sigma) = \frac{2\alpha}{\alpha - 1}\log(\|h_\rho^{1/2}\|_{2\alpha,\sigma}^{AM})$$

- can be defined using any *-representation of $\mathcal{M}$ on a Hilbert space $\mathcal{H}$ and any vector $\xi \in \mathcal{H}$ representing ρ
- duality relation: for $1/p + 1/q = 1$

$$|\langle\, \eta, \xi \,\rangle| \leq \|\eta\|_{p,\sigma}^{AM}\|\xi\|_{q,\sigma}^{AM}, \qquad \xi, \eta \in \mathcal{H}$$

- if $1 < p \leq 2$, there is a (unique) unit vector $\eta_0 \in \mathcal{H}$ such that

$$\langle\, \eta, \eta_0 \,\rangle = \|\eta\|_{p,\sigma}^{AM}\|\eta_0\|_{q,\sigma}^{AM}$$

M. Berta, V. B. Scholz, and M. Tomamichel, Ann. H. Poincaré, 2018

$\tilde{D}_\alpha^{AM}$ and $\tilde{D}_\alpha$

Araki-Masuda L_p-norms can be introduced by interpolation:

- For $2 \leq p \leq \infty$: a continuous embedding $\mathcal{M} \to L_2(\mathcal{M})$

$$x \mapsto h_\sigma^{1/2} x, \qquad x \in \mathcal{M}$$

  the interpolation norm $\|\cdot\|_{p,\sigma}^{AM}$ in $C_{1/p}(\mathcal{M}, L_2(\mathcal{M}))$.

- For $1 \leq p \leq 2$: a continuous embedding $L_2(\mathcal{M}) \to L_1(\mathcal{M})$

$$k \mapsto k h_\sigma^{1/2}, \qquad k \in L_2(\mathcal{M})$$

  the interpolation norm $\|\cdot\|_{p,\sigma}^{AM}$ in $C_{1/p}(L_2(\mathcal{M}), L_1(\mathcal{M}))$.

$\tilde{D}_\alpha^{AM}$ and $\tilde{D}_\alpha$

For $1 < \alpha < \infty$,

$$\tilde{D}_\alpha(\rho\|\sigma) = \tilde{D}_\alpha^{AM}(\rho\|\sigma)$$

For $1/2 < \alpha < 1$: $h_\sigma^{\frac{1-\alpha}{2\alpha}} h_\rho^{1/2} \in L_{2\alpha}(\mathcal{M})$ and

$$\tilde{D}_\alpha(\rho\|\sigma) := \tilde{D}_\alpha^{AM}(\rho\|\sigma) = \frac{2\alpha}{\alpha - 1} \log \| h_\sigma^{\frac{1-\alpha}{2\alpha}} h_\rho^{1/2} \|_{2\alpha}$$

Limit values:

- $\lim_{\alpha \nearrow 1} \tilde{D}_\alpha(\rho\|\sigma) = S(\rho\|\sigma)$
- $\lim_{\alpha \to 1/2} \tilde{D}_\alpha(\rho\|\sigma) = -\log F(\rho, \sigma)$ Uhlmann's fidelity

Data processing inequality for $\tilde{D}_\alpha$, $\alpha \in (1/2, 1)$

We have to assume that $\Phi : L_1(\mathcal{M}) \to L_1(\mathcal{N})$ is trace preserving and completely positive, with Stinespring representation:

$$\Phi^* = T^* \pi(\cdot) T$$

π a *-representation, T an isometry

- Let $p = 2\alpha$, $1/p + 1/q = 1$. Let $\rho = \omega_\eta$ and let $\omega := \omega_{\eta_0}$ be such that $\langle \eta, \eta_0 \rangle = \|\eta\|^{AM}_{p,\sigma} \|\eta_0\|^{AM}_{q,\sigma}$. Then

$$\langle \eta, \eta_0 \rangle = \langle T\eta, T\eta_0 \rangle \leq \|T\eta\|^{AM}_{p,\Phi(\sigma)} \|T\eta_0\|^{AM}_{q,\Phi(\sigma)}$$

- we obtain, with $\alpha^* = \alpha/(2\alpha - 1) > 1$:

$$\begin{aligned} \tilde{D}_\alpha(\rho\|\sigma) &\geq \tilde{D}_\alpha(\Phi(\rho)\|\Phi(\sigma)) + \tilde{D}_{\alpha^*}(\omega\|\sigma) - \tilde{D}_{\alpha^*}(\Phi(\omega)\|\Phi(\sigma)) \\ &\geq \tilde{D}_\alpha(\Phi(\rho)\|\Phi(\sigma)) \end{aligned}$$

M. Berta, V. B. Scholz, and M. Tomamichel, Ann. H. Poincaré, 2018

Characterizations of sufficient channels by $\tilde{D}_\alpha$

Theorem

The sandwiched Rényi relative entropies $\tilde{D}_\alpha$ characterize sufficiency for $\alpha \in (1/2, 1) \cup (1, \infty)$.

AJ, Ann. H. Poincaré, 2018; AJ, arXiv:1707.00047

That is:

$$\tilde{D}_\alpha(\Phi(\rho)\|\Phi(\sigma)) = \tilde{D}_\alpha(\rho\|\sigma) < \infty$$

implies that

$$\Psi \circ \Phi(\rho), \qquad \Psi \circ \Phi(\sigma).$$

for some channel Ψ.

The proof in case $\alpha > 1$

Let $\alpha > 1$.

- the assumption

$$\tilde{D}_\alpha(\Phi(\rho)\|\Phi(\sigma)) = \tilde{D}_\alpha(\rho\|\sigma) < \infty$$

  implies that $h_\rho \in L_\alpha(\mathcal{M}, \sigma)$ and Φ is a contraction preserving its norm.

- An easy proof for $\alpha = 2$:
  $L_2(\mathcal{M}, \sigma)$ is a Hilbert space, Φ_σ is the adjoint of Φ

  By properties of contractions on Hilbert spaces:

$$\Phi_\sigma \circ \Phi(h_\rho) = h_\rho$$

  (note that positivity of Φ is enough for this)

The proof in case $\alpha > 1$

For general $\alpha > 1$, use interpolation:

Let τ be a normal state, $s(\tau) \leq s(\sigma)$. Put

$$h_\tau(z) = h_\sigma^{(1-z)/2} h_\tau^z h_\sigma^{(1-z)/2} \in L_1(\mathcal{M}), \quad 0 \leq Re(z) \leq 1,$$

continuous function, analytic in $0 < Re(z) < 1$.

If the equality

$$\|\Phi(h_\tau(1/\alpha))\|_{\alpha,\Phi(\sigma)} = \|h_\tau(1/\alpha)\|_{\alpha,\sigma}$$

holds for some $\alpha > 1$, then it holds for all $\alpha > 1$.

The proof in case $\alpha > 1$

By assumptions $h_\rho = th_\tau(1/\alpha)$ for some state τ, $t > 0$ - normalization, and we have

$$\|\Phi(h_\tau(1/\alpha)\|_{\alpha,\Phi(\sigma)} = \|h_\tau(1/\alpha)\|_{\alpha,\sigma}.$$

Then the equality holds also for $\alpha = 2$, so that

Φ is sufficient with respect to $\{\omega, \sigma\}$, where

$$h_\omega = sh_\tau(1/2) = sh_\sigma^{1/4} h_\tau^{1/2} h_\sigma^{1/4}, \qquad s > 0.$$

Ok, but this is not what we wanted to prove!

The proof in case $\alpha > 1$

Assume $\mathcal{M} = B(\mathcal{H})$, replace h_ρ by the density operator ρ.

There is a decomposition $U\sigma U^* = \bigoplus_n \lambda_n \sigma_n^L \otimes \sigma_n^R$ such that Φ is sufficient with respect to $\{\rho, \sigma\}$ iff

$$U\rho U^* = \bigoplus_n \mu_n \rho_n^L \otimes \sigma_n^R$$

But we have such a decomposition for $\omega = s\sigma^{1/4}\tau^{1/2}\sigma^{1/4}$, hence also for $\rho = t\sigma^{1/2\beta}\tau^{1/\alpha}\sigma^{1/2\beta}$, which imlies the result.

(In the general case, we use the characterization by conditional expectations.)

The case $\alpha \in (1/2, 1)$

Let $\alpha \in (1/2, 1)$. Then

$$h_\sigma^{\frac{1-\alpha}{2\alpha}} h_\rho^{1/2} = h_\tau^{1/2\alpha} u \in L_{2\alpha}(\mathcal{M})$$

for some $\tau \in \mathcal{M}_*^+$ and partial isometry $u \in \mathcal{M}$.

- We recall the inequality

$$\begin{aligned}\tilde{D}_\alpha(\rho\|\sigma) &\geq \tilde{D}_\alpha(\Phi(\rho)\|\Phi(\sigma)) + \tilde{D}_{\alpha^*}(\omega\|\sigma) - \tilde{D}_{\alpha^*}(\Phi(\omega)\|\Phi(\sigma)) \\ &\geq \tilde{D}_\alpha(\Phi(\rho)\|\Phi(\sigma))\end{aligned}$$

  for $\alpha^* > 1$ and some state ω.
- Actually, we have $h_\omega = th_\tau(1/\alpha^*)$.

The case $\alpha \in (1/2, 1)$

- By assumptions, we obtain

$$\tilde{D}_{\alpha^*}(\omega\|\sigma) = \tilde{D}_{\alpha^*}(\Phi(\omega)\|\Phi(\sigma))$$

- since $\alpha^* > 1$, this implies that Φ is sufficient with respect to $\{\omega, \sigma\}$.
- use the decompositions (conditional expectations) as before.