Notes on two kinds of incompatibility witnesses

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Here the two definitions of incompatibility witnesses are briefly described. I call them the d-witness (d as "discrimination") and the g-witness (g as "geometric"), but perhaps better names can be found.

1 Some notations

Let us first fix some notations. Let $\mathcal{H}$, $\dim\mathcal{H} < \infty$ be a Hilbert space, $\mathcal{L}_s(\mathcal{H})$ the set of self-adjoint operators, $\mathcal{L}(\mathcal{H})^+$ the cone of positive operators. For a finite set X, let $\mathcal{O}(X)$ denote the set of observables with values in X.

Let $X_1, \dots X_k$ be finite sets and let $\mathcal{O}_i = (\mathcal{O})(X_i)$. Further, let Δ_i be the simplex with vertices $\{\delta^i_j,\ j \in X_i\}$. Let $\mathsf{S} = \Pi_i \Delta_i$ denote the Cartesian product with pointwise convex structure and let

$$\mathsf{s}_{n_1,\dots,n_k} = (\delta^1_{n_1}, \dots, \delta^k_{n_k})$$

be the vertices of S. We put

$$\begin{aligned} A(\mathsf{S}) &= \{\text{affine functions } f : \mathsf{S} \to \mathbb{R}\} \\ A(\mathsf{S})^+ &= \text{ positive elements in } A(\mathsf{S}) \\ V(\mathsf{S}) &= A(\mathsf{S})^* \\ V(\mathsf{S})^+ &= \{\psi \in V(\mathsf{S}),\ \langle \psi, f\rangle \geq 0,\ \forall f \geq 0\} \end{aligned}$$

Note that $V(\mathsf{S})^+$ is a closed convex cone with base (isomorphic to) S. Let

$$\mathsf{m}^i : \mathsf{S} \to \Delta_i$$

be the projection map and let $\mathsf{m}^i_j \in A(\mathsf{S})^+$ be such that

$$\mathsf{m}^i(\mathsf{s}) = \sum_{j \in X_i} \mathsf{m}^i_j(\mathsf{s})\delta^i_j.$$

Note that m^i_j generate the extremal rays of $A(\mathsf{S})^+$ (cf. [2, Sec. III])).

2 Collections of observables and compatibility

Let $\mathcal{O} = \mathcal{O}_1 \times \cdots \times \mathcal{O}_k$ be the set of collections of observables. We can identify collections $(O^1, \dots, O^k) \in \mathcal{O}$, with affine maps $F : \mathcal{S}(\mathcal{H}) \to \mathsf{S}$ in an obvious way: we identify O^i with the affine map

$$O^i : \mathcal{S}(\mathcal{H}) \in \rho \mapsto \sum_{j \in X_i} \mathrm{Tr}\,[O^i_j \rho]\delta^i_j$$

and put

$$F(\rho) = (O^1(\rho), \dots O^k(\rho)).$$

Conversely, any affine map $F : \mathcal{S}(\mathcal{H}) \to \mathsf{S}$ determines a collection of observables as

$$\mathrm{Tr}\,[O^i_j \rho] = \langle \mathsf{m}^i_j, F(\rho) \rangle, \qquad \rho \in \mathfrak{S}(\mathcal{H}). \tag{1}$$

We can also determine the map F as

$$F(\rho) = \sum_{n_1,\dots,n_k} \mathrm{Tr}\,[F_{n_1,\dots,n_k}\rho]\mathsf{s}_{n_1,\dots,n_k}$$

for some elements $F_{n_1,\dots,n_k} \in \mathcal{L}_s(\mathcal{H})$. Note that these elements are not unique, but by (1) we must have

$$O^i_j = \sum_{n_1,\dots,n_{i-1},n_{i+1},\dots,n_k} F_{n_1,\dots,n_{i-1},j,n_{i+1},\dots,n_k}, \qquad \forall i,j$$

and therefore the elements $F_{n_1,\dots,n_k}$ must sum to identity. The collection is compatible if and only if $F_{n_1,\dots,n_k}$ may be chosen positive, equivalently, the map F is entanglement breaking, since it can be written as a composition of a measurement and preparation.

3 The g-witnesses

Let $\mathcal{A}$ be the vector space of all linear maps $\mathcal{L}_s(\mathcal{H}) \to V(\mathsf{S})$. Since any affine map on $\mathfrak{S}(\mathcal{H})$ can be extended to a linear map on $\mathcal{L}_s(\mathcal{H})$, the set $\mathcal{O}$ can be identified with a subset of positive maps in $\mathcal{A}$. Let $\mathcal{P} \subset \mathcal{O}$ be the set of compatible collections, which corresponds to etb elements in $\mathcal{O}$. We now want to witness the set $\mathcal{P}$ by affine functionals on $\mathcal{O}$. Any such affine functional is given in the form

$$p = \psi + t,$$

where $\psi \in \mathcal{A}^*$ and $t \in \mathbb{R}$. The dual space $\mathcal{A}^*$ can be identified with the space of linear maps $\{W : V(\mathsf{S}) \to \mathcal{L}_s(\mathcal{H})\}$, with duality

$$\langle F, W \rangle = \mathrm{tr}(F \circ W),$$

here tr denotes the usual trace of a linear map on a vector space.

Any linear map $W : V(\mathsf{S}) \to \mathcal{L}_s(\mathcal{H})$ is determined by the values

$$\psi_{n_1,\dots,n_k} := W(\mathsf{s}_{n_1,\dots,n_k}) \tag{2}$$

of W on the extremal points $\mathsf{s}_{n_1,\dots,n_k}$ (since S generates $V(\mathsf{S})$). Moreover, the map W is positive if and only if $\psi_{n_1,\dots,n_k} \in \mathcal{L}(\mathcal{H})^+$. Alternatively, since the elements m^i_j generate $A(\mathsf{S})$, W is determined as

$$W(\mathsf{s}) = \sum_{i,j} \psi^i_j \mathsf{m}^i_j(\mathsf{s}), \tag{3}$$

where $\psi^i_j \in \mathcal{L}_s(\mathcal{H})$ are some (nonunique) elements satisfying

$$\psi_{n_1,\dots,n_k} = \sum_i \psi^i_{n_i}.$$

We then have

$$\langle F, W\rangle = \mathrm{tr}(F\circ W) = \sum_{i,j} \mathrm{Tr}\,[\psi^i_j O^i_j]$$
$$= \sum_{n_1,\dots,n_k} \mathrm{Tr}\,[F_{n_1,\dots,n_k}\psi_{n_1,\dots,n_k}]$$

Further, let $u_i := \sum_j |X_i|^{-1}\delta^i_j \in \Delta_i$ be the uniform probability distributions in Δ_i. Then $\bar{\mathsf{s}} = (u_1,\dots,u_k)$ is the barycenter of S. We have

$$\bar\psi := W(\bar{\mathsf{s}}) = \frac{1}{\Pi_i|X_i|}\sum_{n_1,\dots,n_k}\psi_{n_1,\dots,n_k} = \sum_{i,j}\frac{1}{|X_i|}\psi^i_j,$$

this element will be called the barycenter of W.

It now follows from known properties of cones of positive maps and their duals that $\{O^1,\dots,O^k\}$ is incompatible (that is, the corresponding map F is non-etb) if and only if there is a positive map W such that $\langle F, W\rangle < 0$. Such maps are called the g-witnesses.

Note that if all ψ^i_j are positive, W is etb and $\langle F, W\rangle$ is positive for all F. However, not every non-etb map is a witness (this is because the collections of observables generate only a section of the cone of positive maps, see [2, Thm. 2]).

4 g-witnesses and d-witnesses

A d-witness can be defined as an ensemble $\rho^i_j \in \mathcal{L}(\mathcal{H})^+$ such that $\sum_{i,j}\mathrm{Tr}\,[\rho^i_j] = 1$ and

$$\pi(\{\rho^i_j\}) := \sup_{(O^1,\dots,O^k)\in\mathcal{O}}\sum_{i,j}\mathrm{Tr}\,[O^i_j\rho^i_j] > \sup_{(O^1,\dots,O^k)\in\mathcal{P}}\sum_{i,j}\mathrm{Tr}\,[O^i_j\rho^i_j] =: \pi_c(\{\rho^i_j\}).$$

This has some operational interpretation, see [3].

We now show the relation of these two types of witnesses. First, let $\{\rho^i_j\}$ be a d-witness. Define the map $\tilde W : \mathsf{S}\to\mathcal{L}_s(\mathcal{H})$ by

$$\tilde W(\mathsf{s}) = \sum_{i,j}\rho^i_j\mathsf{m}^i_j(\mathsf{s}),$$

then for $(O^1,\dots,O^k)\in\mathcal{O}$ and the corresponding map F, we have

$$\sum_{i,j}\mathrm{Tr}\,[O^i_j\rho^i_j] = \mathrm{tr}(F\circ\tilde W).$$

As remarked before, the map $\tilde W$ is positive and etb, so cannot be a g-witness itself. Let us denote $\pi_c := \pi_c(\{\rho^i_j\})$. Note that

$$\pi_c = \sup_{O\in\mathcal{O}(X_1\times\cdots\times X_k)}\sum_{n_1,\dots,n_k}\mathrm{Tr}\,[O_{n_1,\dots,n_k}\tilde\psi_{n_1,\dots,n_k}],$$

where $\tilde\psi_{n_1,\dots,n_k} := \tilde W(\mathsf{s}_{n_1,\dots,n_k}) = \sum_{i=1}^k \rho^i_{n_i}$. This was observed in [1] and follows also from the above considerations. Using the dual SDP program, we obtain that there is some state ρ_c such that $\tilde\psi_{n_1,\dots,n_k}\le\pi_c\rho_c$ for all $n_i\in X_i$.

Let $\rho \in \mathfrak{S}(\mathcal{H})$ be arbitrary and let W_ρ be the constant map $\mathsf{S} \to \rho$. We have $\mathrm{tr}(W_\rho \circ F) = 1$ for any $F \in \mathcal{O}$. Put

$$W := \pi_c W_{\rho_c} - \tilde{W}.$$

Then W is a g-witness: W is a positive map, since all images of the vertices are positive. If $F = (O^1, \dots, O^k)$ represents a collection of measurements such that $\mathrm{Tr}\,[O^i_j \rho^i_j] > \pi_c$, then

$$\mathrm{tr}(F \circ W) = \pi_c - \mathrm{Tr}\,[O^i_j \rho^i_j] < 0.$$

In fact, $\{\rho^i_j\}$ is a d-witness if and only if $\tilde{W}$ is a g-witness.

Conversely, let W be a g-witness, with the corresponding expressions given by Eqs. (2) and (3). For each i, let $\psi^i \in \mathcal{L}_s(\mathcal{H})$ be such that $\psi^i_j \leq \psi^i$ for all $j \in X_i$. Put

$$\tilde{\rho}^i_j := \psi^i - \psi^i_j, \qquad c := \sum_{i,j} \mathrm{Tr}\,\tilde{\rho}^i_j, \qquad \rho^i_j := c^{-1}\tilde{\rho}^i_j.$$

Then $\{\rho^i_j\}$ is an ensemble, moreover, we have for $F = (O^1, \dots, O^k) \in \mathcal{P}$,

$$\begin{aligned}\sum_{i,j} \mathrm{Tr}\,[O^i_j \rho^i_j] &= c^{-1}(\sum_i \mathrm{Tr}\,\psi^i - \mathrm{Tr}\,(F \circ W)) \leq c^{-1} \sum_i \mathrm{Tr}\,\psi^i \\ &< c^{-1}(\sum_i \mathrm{Tr}\,\psi^i - \inf_{F \in \mathcal{O}} \mathrm{Tr}\,(F \circ W)) = \sup_{F \in \mathcal{O}} \mathrm{Tr}\,[O^i_j \rho^i_j].\end{aligned}$$

5 Two-outcome measurements

An advantage of g-witnesses may be that, at least in the case of two-outcome measurements, it is not difficult to find witnesses of this form, or to establish whether a given positive map is a witness or not:

If $|X_i| = 2$ for all i, we have the following characterization of g-witnesses (cf. [2, Coro 3]): Let $X_i = \{0, 1\}$ for all i and let $W : \mathsf{S} \to \mathcal{L}(\mathcal{H})^+$. Let $\psi_{n_1,\dots,n_k}$, $n_i \in \{0, 1\}$ and ψ^i_j be determined by (3) and (2). Put

$$\begin{aligned} e^i &:= \psi_{1,\dots,1,0,1,\dots,1} - \psi_{1,\dots,1} \qquad 0 \text{ in } i\text{-th place} \\ &= \psi^i_0 - \psi^i_1 \end{aligned}$$

Note that e^i are the images under W of vectors given by edges of the hypercube S, adjacent to a fixed vertex. Further, the barycenter of W is

$$\bar{\psi} = \frac{1}{2} \sum_{i,j} \psi^i_j.$$

Then W is a witness if and only if

$$\sum_{i=1}^{k} \|e^i\|_1 > 2\mathrm{Tr}\,\bar{\psi}$$

which means that

$$\sum_i \|\psi^i_0 - \psi^i_1\|_1 > \sum_{i,j} \mathrm{Tr}\,[\psi^i_j].$$

(Note that the ψ^i_j cannot be all positive). The two-outcome measurements $O^1, \dots, O^k$ attaining $\inf \mathrm{tr}(F \circ W)$ are given by support projections of $(\psi^i_0 - \psi^i_1)_+$.

Example 1. (Pairs and triples of two-outcome qubit measurements) Any incompatibility witness for pairs of two-outcome measurements is described by a parallelogram whose vertices $\psi_{i,j}$ are positive operators. We now look at a special case of squares such that the vertices are pure qubit states. Let

$$|y_k(\theta)\rangle = \cos(\frac{\theta}{2})|0\rangle + i^k \sin(\frac{\theta}{2})|1\rangle, \qquad k = 0,\dots,3,\ \theta \in (0,\pi)$$

The witness $W(\theta)$ has vertices $\psi_{i,j}(\theta)$ given by

$$\psi_{0,0}(\theta) = y_0(\theta),\ \psi_{0,1}(\theta) = y_1(\theta),\ \psi_{1,1}(\theta) = y_2(\theta), \psi_{1,0}(\theta) = y_3(\theta)$$

By the above condition, W is a witness iff

$$\|\psi_{1,1} - \psi_{0,1}\|_1 + \|\psi_{1,1} - \psi_{1,0}\|_1 > 2$$

which gives the condition

$$\sin(\theta) > \frac{1}{\sqrt{2}}.$$

Similarly, consider cubes with vertices on the Bloch sphere, given by

$$\begin{aligned} |y_{k,1}(\theta) &= \cos(\frac{\theta}{2})|0\rangle + i^k \sin(\frac{\theta}{2})|1\rangle \\ |y_{k,0}(\theta) &= \sin(\frac{\theta}{2})|0\rangle + i^k \cos(\frac{\theta}{2})|1\rangle, \qquad k = 0,\dots,3,\ \theta \in (0,\pi/2) \end{aligned}$$

(the elements $y_{k,0}$ are vertices of the base of the cube). Here we get the condition

$$\sqrt{2}\sin(\theta) + \cos(\theta) > 1$$

References

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[3] A. Toigo, Preliminary notes on incompatibility witnesses