Questions

1 Sufficiency (reversibility)

This would be the most interesting question.

2 Martingale convergence

Let $\{\mathcal{M}_i\}$ be an increasing net of unital von Neumann subalgebras of $\mathcal{M}$ such that $\mathcal{M} = \bigl(\bigcup_i \mathcal{M}_i\bigr)''$. Assume that either

$$\alpha \in (0,1), \quad z \ge \max\{\alpha, 1-\alpha\}, \tag{2.1}$$

or

$$\alpha > 1, \quad \max\{\alpha/2, \alpha-1\} \le z \le \alpha. \tag{2.2}$$

Then for every $\varphi, \psi \in \mathcal{M}_*^+$,

$$D_{\alpha,z}(\psi|_{\mathcal{M}_i}\|\varphi|_{\mathcal{M}_i}) \nearrow D_{\alpha,z}(\psi\|\varphi)\,?$$

Maybe, this can be proved in a similar way to the proof of Theorem 3.1 of [1] based on [2].

This can typically be applied in the following situations:

(1) When $\mathcal{M}$ is an injective von Neumann algebra $\mathcal{M} = \bigl(\bigcup_i \mathcal{M}_i\bigr)''$ with an increasing net $\{\mathcal{M}_i\}$ of finite-dimensional subalgebras of $\mathcal{M}$.

(2) According to Haagerup's reduction theory, worked out in [3] (a compact survey is found in [4]), one can define

$$\hat{\mathcal{M}} := \mathcal{M} \rtimes_{\sigma^\phi} G, \qquad G := \bigcup_{n\in\mathbb{N}} 2^{-n}\mathbb{Z},$$

where ϕ is a faithful normal state of $\mathcal{M}$, with a normal conditional expectation $E_{\mathcal{M}} : \hat{\mathcal{M}} \to \mathcal{M}$. Define $\hat\psi := \psi \circ E_{\mathcal{M}}$ for any $\psi \in \mathcal{M}_*^+$. Then we have an increasing sequence $\{\mathcal{M}_n\}$ of unital von Neumann subalgebras of $\hat{\mathcal{M}}$ such that the following hold:

(i) Each $\mathcal{M}_n$ is a type II_1 von Neumann algebra with a faithful normal tracial state τ_n.

(ii) $\bigl(\bigcup_n \mathcal{M}_n\bigr)'' = \hat{\mathcal{M}}$.

(iii) there exist faithful normal conditional expectations $E_{\mathcal{M}_n} : \hat{\mathcal{M}} \to \mathcal{M}_n$ for which

[1] F. Hiai and M. Mosonyi, Quantum Rényi divergences and the strong converse exponent of state discrimination in operator algebras, *Ann. Henri Poincaré* **24** (2023), 1681–1724.

[2] F. Hiai and M. Tsukada, Generalized conditional expectations and martingales in noncommutative L^p-spaces, *J. Operator Theory* **18** (1987), 265–288.

[3] U. Haagerup, M. Junge and Q. Xu. A reduction method for noncommutative L_p-spaces and applications, *Trans. Amer. Math. Soc.* **362** (2010), 2125–2165.

[4] O. Fawzi, L. Gao and M. Rahaman, Asymptotic equipartition theorems in von Neumann algebras, arXiv:2212.14700v2 [quant-ph], 2023.

- $\hat\phi\circ E_{\mathcal{M}_n}=\hat\phi$,
- $E_{\mathcal{M}_n}(x)\to x$ strongly for any $x\in\hat{\mathcal{M}}$,
- $\|\hat\psi\circ E_{\mathcal{M}_n}-\hat\psi\|\to 0$ for any $\psi\in\mathcal{M}_*^+$.

In this situation, by DPI and the above martingale (if proved), we have, for any $\varphi,\psi\in\mathcal{M}_*^+$ and for any (α,z) in either (2.1) or (2.2),

$$D_{\alpha,z}(\psi\|\varphi)=D_{\alpha,z}(\hat\psi\|\hat\varphi)=\lim_{n\to\infty}D_{\alpha,z}(\hat\psi|_{\mathcal{M}_n}\|\hat\varphi|_{\mathcal{M}_n}). \tag{2.3}$$

The property (2.3) holds the relative entropy $D(\alpha\|\varphi)$ (Proposition 2.2 of [4]). Furthermore, there might be a chance for (2.3) to hold for all $\alpha\in(0,\infty)\setminus\{1\}$ and $z>0$ as well. This property enables us to reduce certain questions (e.g., the monotonicity question of $z>0\mapsto D_{\alpha,z}(\psi\|\varphi)$ given below) to the type II_1 case, though questions seems still difficult in the type II_1 case too.

3 Monotonicity of $z>0\mapsto D_{\alpha,z}(\psi\|\varphi)$

The question says that

$$0<z\le z' \implies \begin{cases} D_{\alpha,z}(\psi\|\varphi)\le D_{\alpha,z'}(\psi\|\varphi), & \alpha\in(0,1),\\ D_{\alpha,z}(\psi\|\varphi)\ge D_{\alpha,z'}(\psi\|\varphi), & \alpha>1.\end{cases} \tag{3.1}$$

The case $\alpha\in(0,1)$ has been shown in Theorem 1 (x) of [5] and the case $\alpha>1$ has been raised in Question 5. The question can be reduced to the type II_1 case as far as the property given in (2.3) is affirmative.

Assume that $\mathcal{M}$ is a type II_1 von Neumann algebra with a faithful normal tracial state τ. Then one can identify $L^p(\mathcal{M})$ with $L^p(\mathcal{M},\tau)$, where $h_\psi\in L^1(\mathcal{M})_+$ for $\psi\in\mathcal{M}_*^+$ is the Radon–Nikodym derivative $d\psi/d\tau\in L^1(\mathcal{M},\tau)_+$. For any $\varphi,\psi\in\mathcal{M}*^+$ and for every $\varepsilon>0$ and $z>0$, since $h_{\varphi+\varepsilon\tau}=h_\varphi+\varepsilon 1$ is invertible with $y_\varepsilon:=(h_\varphi+\varepsilon 1)^{-1}\in\mathcal{M}_+$, one has $h_\psi^{\alpha/z}=h_{\varphi+\varepsilon\tau}^{(\alpha-1)/2z}x_\varepsilon h_{\varphi+\varepsilon\tau}^{(\alpha-1)/2z}$ where $x_\varepsilon:=y_\varepsilon^{(\alpha-1)/2z}h_\psi^{\alpha/z}y_\varepsilon^{(\alpha-1)/2z}$, so that

$$Q_{\alpha,z}(\psi\|\varphi+\varepsilon\tau)=\tau\big(x_\varepsilon^z\big)=\tau\big((h_\psi^{\alpha/2z}y_\varepsilon^{(\alpha-1)/z}h_\psi^{\alpha/2z})^z\big).$$

Then, by Kosaki's ALT inequality one can see as in the proof of Theorem 1 (x) of [5] (OK?) that

$$Q_{\alpha,z}(\psi\|\varphi+\varepsilon\tau)\ge Q_{\alpha,z'}(\psi\|\varphi+\varepsilon\tau)\quad \text{if } 0<z\le z'.$$

Hence, the inequality in (3.1) for $\alpha>1$ in this situation follows if we can show that

$$Q_{\alpha,z}(\psi\|\varphi)=\lim_{\varepsilon\searrow 0}Q_{\alpha,z}(\psi\|\varphi+\varepsilon\tau). \tag{3.2}$$

Note that if $0<\varepsilon\le\varepsilon'$ then y_ε and $y_{\varepsilon'}$ commute and $y_\varepsilon\ge y_{\varepsilon'}$, and hence $Q_{\alpha,z}(\psi\|\varphi+\varepsilon\tau)\ge Q_{\alpha,z}(\psi\|\varphi+\varepsilon'\tau)$. Moreover, if $Q_{\alpha,z}(\psi\|\varphi)<+\infty$ with x in (♠), then we have

$$x_\varepsilon:=\bigg(\frac{h_\varphi}{h_\varphi+\varepsilon 1}\bigg)^{(\alpha-1)/2z}x\bigg(\frac{h_\varphi}{h_\varphi+\varepsilon 1}\bigg)^{(\alpha-1)/2z},$$

from which

$$Q_{\alpha,z}(\psi\|\varphi)=\|x\|_z^z\ge\|x_\varepsilon\|_z^z=Q_{\alpha,z}(\psi\|\varphi+\varepsilon\tau),\qquad \varepsilon>0.$$

Therefore, (3.2) follows whenever $\varphi\in\mathcal{M}_*^+\mapsto Q_{\alpha,z}(\psi\|\varphi)$ is lower semi-continuous.

[5] S. Kato, On α-z-Rényi divergence in the von Neumann algebra setting, Preprint, 2023.

4 The convergence of $D_{\alpha,z}(\psi\|\varphi)$ as $\alpha \nearrow 1$ and $\alpha \searrow 1$

Let $\psi, \varphi \in \mathcal{M}_*^+$ with $\psi \neq 0$. The following are known (see [6] [7] [8]):

- We have
$$\lim_{\alpha\nearrow 1} D_{\alpha,1}(\psi\|\varphi) = \lim_{\alpha\nearrow 1} D_{\alpha,\alpha}(\psi\|\varphi) = D_1(\psi\|\varphi) := \frac{D(\psi\|\varphi)}{\psi(1)},$$
where $D(\psi\|\varphi)$ is the relative entropy.

- If $D_{\alpha,1}(\psi\|\varphi) < +\infty$ for some $\alpha \in (1,\infty)$, then
$$\lim_{\alpha\searrow 1} D_{\alpha,1}(\psi\|\varphi) = D_1(\psi\|\varphi).$$

- If $D_{\alpha,\alpha}(\psi\|\varphi) < +\infty$ for some $\alpha \in (1,\infty)$, then
$$\lim_{\alpha\searrow 1} D_{\alpha,\alpha}(\psi\|\varphi) = D_1(\psi\|\varphi).$$

Furthermore, in the finite-dimensional case, it is known (see [9] [10]) that
$$\lim_{\alpha\to 1} D_{\alpha,z}(\rho\|\sigma) = D_1(\rho\|\sigma), \qquad z \in (0,\infty],$$
where $D_{\alpha,\infty}(\rho\|\sigma)$ is defined as
$$\begin{aligned} D_{\alpha,\infty}(\rho\|\sigma) &:= \lim_{z\to\infty} D_{\alpha,z}(\rho\|\sigma) \\ &= \begin{cases} \operatorname{Tr} P \exp\bigl(\alpha P(\log\rho)P + (1-\alpha)P(\log\sigma)P\bigr) \quad \text{with } P := \rho^0 \wedge \sigma^0 & \text{if } \rho^0 \le \sigma^0, \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$
(due to the Lie–Trotter formula $\lim_{z\to\infty}(B^{1/2z}A^{1/z}B^{1/2z})^z = P\exp(P(\log A)P + P(\log B)P)$ where $P := A^0 \wedge B^0$.

It is interesting to obtain convergence properties similar to the above for $D_{\alpha,z}$ in the von Neumann algebra case. In particular, for any $z \in (0,\infty)$,
$$\lim_{\alpha\nearrow 1} D_{\alpha,z}(\psi\|\varphi) = D_1(\psi\|\varphi)\,?$$

5 Epstein type concavity/convexity

By (22) of [5] and (1) of [11], as in the discussion below (19) of [5], we see the following Epstein type concavity/convexity:

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[6] F. Hiai, Quantum f-divergences in von Neumann algebras I. Standard f-divergences, *J. Math. Phys.* **59** (2018), 102202, 27 pp.

[7] M. Berta, V. B. Scholz and M. Tomamichel, Rényi divergences as weighted non-commutative vector valued L_p-spaces, *Ann. Henri Poincaré* **19** (2018), 1843–1867.

[8] A. Jenčová, Rényi relative entropies and noncommutative L_p-spaces, *Ann. Henri Poincaré* **19** (2018), 2513–2542.

[9] S. M. Lin and M. Tomamichel, Investigating properties of a family of quantum Rényi divergences, *Quantum Information Processing* **14** (2015), 1501–1512.

[10] M. Mosonyi and T. Ogawa, Strong converse exponent for classical-quantum channel coding, *Comm. Math. Phy.* **355** (2017), 373–426.

[11] A. Jenčová, DPI for α-z-Rényi divergence, Notes, Nov. 23, 2023.

(i) $\psi \in \mathcal{M}_*^+ \mapsto \mathrm{tr}(a^{1/2} h_\psi^p a^{1/2})^{1/p}$ is convex for any $p \in [1, 2]$ and $a \in \mathcal{M}_+$,

(ii) $\psi \in \mathcal{M}_*^+ \mapsto \mathrm{tr}(a^{1/2} h_\psi^p a^{1/2})^{1/p}$ is concave for any $p \in (0, 1]$ and $a \in \mathcal{M}_+$.

I am interested in discussing these concavity/convexity properties in a more operator theoretic way, though they are not essential in our study of $D_{\alpha,z}$. For instance, let $\widetilde{\mathcal{M}}_+$ be the set of τ-measurable positive operators affiliated with a semi-finite von Neumann algebra $(\mathcal{M}, \tau)$. We then conjecture the following:

- $x \in \widetilde{\mathcal{M}}_+ \mapsto \tau(a^{1/2} x^p a^{1/2})^s$ is convex for any $p \in [1, 2]$, $s \geq 1/p$ and $a \in \widetilde{\mathcal{M}}_+$?
- $x \in \widetilde{\mathcal{M}}_+ \mapsto \tau(a^{1/2} x^p a^{1/2})^s$ is concave for any $p \in (0, 1]$, $0 \leq s \leq 1/p$ and $a \in \widetilde{\mathcal{M}}_+$?

These are well known in the finite-dimensional case. The extension to the $B(\mathcal{H})$ case is probably easy by convergence arguments, but the extension to the semi-finite case might be non-trivial.