# Equality in DPI for sandwiched Rényi divergence

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Below, $\gamma : \mathcal{N} \to \mathcal{M}$ is a normal positive unital map, $\psi, \varphi \in \mathcal{M}_*^+$ and we put $\psi_0 = \psi \circ \gamma$, $\varphi_0 = \varphi \circ \gamma$. We consider the following equality

$$D_{\alpha,z}(\psi\|\varphi) = D_{\alpha,z}(\psi_0\|\varphi_0) \tag{1}$$

where $\alpha > 1$ and $\max\{\alpha/2, \alpha - 1\} \le z \le \alpha$. We put $q = \frac{z}{\alpha-1}$. In the case $\alpha = z$ we have $D_{\alpha,\alpha} = \tilde{D}_\alpha$ is the sandwiched Rényi divergence. We assume $D_{\alpha,z}(\psi\|\varphi) < \infty$, then by DPI we also have $D_{\alpha,z}(\psi_0\|\varphi_0) < \infty$. Hence there are $y \in L_{2z}(\mathcal{M})$ and $y_0 \in L_{2z}(\mathcal{N})$ such that

$$h_\psi^{\frac{\alpha}{2z}} = y h_\varphi^{\frac{\alpha-1}{2z}}, \qquad h_{\psi_0}^{\frac{\alpha}{2z}} = y_0 h_{\varphi_0}^{\frac{\alpha-1}{2z}}.$$

**Lemma 1.** *[1, Lemma 3.10] Let $\gamma : \mathcal{N} \to \mathcal{M}$ be a normal positive unital map. Let $\gamma^*_{\varphi,q} : L_q(\mathcal{N}) \to L_q(\mathcal{M})$ be the contraction as in [1, Lemma 3.1]. Let $\bar{w} : (y^*y)^{\alpha-1} \in L_q(\mathcal{M})$ and $\bar{w}_0 := (y_0^* y_0)^{\alpha-1} \in L_q(\mathcal{N})$. Then (1) holds if and only if*

$$\bar{w} = \gamma^*_{\varphi,q}(\bar{w}_0) \quad \textit{and} \quad \mathrm{Tr}\left(\bar{w}_0^q\right) = \mathrm{Tr}\left(\gamma^*_{\varphi,q}(\bar{w}_0)^q\right). \tag{2}$$

We show that (1) implies reversibility of $\gamma$ for the sandwiched Rényi divergence.

**Theorem 1.** *Let $\gamma : \mathcal{N} \to \mathcal{M}$ be a normal positive unital map. Let $\psi, \varphi \in \mathcal{M}_*^+$ be such that $\tilde{D}_\alpha(\psi\|\varphi) < \infty$ for some $\alpha > 1$. Then $\tilde{D}_\alpha(\psi\|\varphi) = \tilde{D}_\alpha(\psi_0\|\varphi_0)$ if and only if $\psi \circ \gamma \circ \gamma^*_\varphi = \psi$.*

*Proof.* Let $\bar{w}$ and $\bar{w}_0$ be as in Lemma 1. Let $\omega \in \mathcal{M}_*^+$ and $\omega_0 \in \mathcal{N}_*^+$ be such that

$$h_\omega = h_\varphi^{\frac{1}{2\alpha}} \bar{w} h_\varphi^{\frac{1}{2\alpha}}, \qquad h_{\omega_0} = h_{\varphi_0}^{\frac{1}{2\alpha}} \bar{w}_0 h_{\varphi_0}^{\frac{1}{2\alpha}}.$$

Note that in this case $q = \frac{\alpha}{\alpha-1}$. Then $h_\omega \in L_q(\mathcal{M}, \varphi)$ and $h_{\omega_0} \in L_q(\mathcal{N}, \varphi_0)$. Assume the equality in DPI holds, then by Lemma 1 we have

$$\|h_\omega\|_{q,\varphi} = \|\bar{w}\|_q = \|\gamma^*_{\varphi,q}(\bar{w}_0)\|_q = \|\bar{w}_0\|_q = \|h_{\omega_0}\|_{q,\varphi_0} =: l$$

and using also [1, Lemma 3.1], we have

$$(\gamma^*_\varphi)_*(h_{\omega_0}) = h_\varphi^{\frac{1}{2\alpha}} \gamma^*_{\varphi,q}(\bar{w}_0) h_\varphi^{\frac{1}{2\alpha}} = h_\omega.$$

Let $f_{h_{\omega_0},q}$ and $f_{h_\omega,q}$ be the functions as in [2, Eq. (9)], so that

$$f_{h_{\omega_0},q}(s) = l^{1-sq} h_{\varphi_0}^{\frac{1-s}{2}} \bar{w}_0^{qs} h_{\varphi_0}^{\frac{1-s}{2}}, \qquad f_{h_\omega,q}(s) = l^{1-sq} h_\varphi^{\frac{1-s}{2}} \bar{w}^{qs} h_\varphi^{\frac{1-s}{2}}, \qquad s \in S,$$

here $S$ is the strip $S=\{s\in\mathbb{C},\ 0\le \mathrm{Re}(s)\le 1\}$. Note that we have $f_{h_\omega,q}(1/q)=h_\omega$ and

$$\sup_t \|f_{h_{\omega_0},q}(it)\|_{\infty,\varphi_0}=\sup_t \|f_{h_{\omega_0},q}(1+it)\|_1=l=\|h_{\omega_0}\|_{q,\varphi},$$

similarly for $f_{h_\omega,q}$.

Put $g(s):=(\gamma^*_\varphi)_*(f_{h_{\omega_0},q}(s))$, $s\in S$, then $g$ is a bounded continuous function $S\to L_1(\mathcal{M})$, analytic in the interior and such such that $g(1/q)=h_\omega$. Since $(\gamma^*_\varphi)_*$ is a contraction $L_r(\mathcal{N},\varphi_0)\to L_r(\mathcal{M},\varphi)$ for any $1\le r\le\infty$, we have by the Hadamard three lines theorem (see e.g. [2, Thm. 2.10] in this context)

$$\begin{aligned}\|g(1/q)\|_{q,\varphi}&\le \big(\sup_t\|g(it)\|_{\infty,\varphi}\big)^{1-1/q}\big(\sup_t\|g(1+it)\|_1\big)^{1/q}\\ &\le \big(\sup_t\|f_{h_{\omega_0},q}(it)\|_{\infty,\varphi}\big)^{1-1/q}\big(\sup_t\|f_{h_{\omega_0},q}(1+it)\|_1\big)^{1/q}=\|h_{\omega_0}\|_{q,\varphi_0}=\|g(1/q)\|_{q,\varphi}.\end{aligned}$$

It follows that $g$ satisfies equality in the Hadamard three lines theorem and we must have

$$\sup_t\|g(it)\|_{\infty,\varphi}=\sup_t\|f_{h_{\omega_0},q}(it)\|_{\infty,\varphi_0}=\sup_t\|f_{h_{\omega_0},q}(1+it)\|_1=\sup_t\|g(1+it)\|_1.$$

By [2, Thm. 2.10], this implies that $g(s)=f_{h_\omega,q}(s)$ for all $s\in S$. For $s=1/\alpha$ this implies

$$h_\psi=h_\varphi^{\frac{q-1}{2q}}y^*yh_\varphi^{\frac{q-1}{2q}}=(\gamma^*_\varphi)_*(h_{\varphi_0}^{\frac{q-1}{2q}}y_0^*y_0h_{\varphi_0}^{\frac{q-1}{2q}})=(\gamma^*_\varphi)_*(h_{\psi_0}),$$

that is, $\psi\circ\gamma\circ\gamma^*_\varphi=\psi$. The converse is clear from DPI.

$\square$

For $\alpha\neq z$, we still need 2-positivity of $\gamma$ for the proof that (1) implies sufficiency. Using Theorem 1 and similar arguments as in its proof, we can prove equivalent conditions for (1) for positive $\gamma$, which are of the form

$$\gamma_*((y^*y)^z)=(y_0^*y_0)^z$$

for $\alpha>1$ and

$$\gamma^*((h_\psi^{\frac{\alpha}{2z}}h_\varphi^{\frac{1-\alpha}{z}}h_\psi^{\frac{\alpha}{2z}})^z)=(h_{\psi_0}^{\frac{\alpha}{2z}}h_{\varphi_0}^{\frac{1-\alpha}{z}}h_{\psi_0}^{\frac{\alpha}{2z}})^z$$

for $\alpha\in(0,1)$, $z>\alpha$,

$$\gamma^*((h_\varphi^{\frac{1-\alpha}{2z}}h_\psi^{\frac{\alpha}{z}}h_\varphi^{\frac{1-\alpha}{2z}})^z)=(h_{\varphi_0}^{\frac{1-\alpha}{2z}}h_{\psi_0}^{\frac{\alpha}{z}}h_{\varphi_0}^{\frac{1-\alpha}{2z}})^z$$

for $\alpha\in(0,1)$, $z>1-\alpha$ (all within the DPI bounds). This is related but not quite the same as the conditions by Zhang [3]. For example, if $\psi\sim\varphi$ the equality in the last case becomes

$$\gamma_*((h_\varphi^{\frac{1-\alpha}{2z}}\bar a^{-1}h_\varphi^{\frac{1-\alpha}{2z}})^{\frac{z}{1-\alpha}})=(h_{\varphi_0}^{\frac{1-\alpha}{2z}}\bar a_0^{-1}h_{\varphi_0}^{\frac{1-\alpha}{2z}})^{\frac{z}{1-\alpha}},$$

whereas the corresponding Zhang's condition is

$$\gamma_*((\bar a^{-\frac12}h_\varphi^{\frac{1-\alpha}{z}}\bar a^{-\frac12})^{\frac{z}{1-\alpha}})=(\bar a_0^{-\frac12}h_{\varphi_0}^{\frac{1-\alpha}{z}}\bar a_0^{-\frac12})^{\frac{z}{1-\alpha}}.$$

# References


[1] F. Hiai, A. Jenčová. On the properties of $\alpha - z$-Rényi divergences on von Neumann algebras, (version 2). In preparation.

[2] A. Jenčová. Rényi relative entropies and noncommutative $L_p$-spaces. *Annales Henri Poincaré*, 19:2513–2542, 2018. doi:10.1007/s00023-018-0683-5.

[3] H. Zhang. Equality conditions of data processing inequality for $\alpha$-z Rényi relative entropies. *Journal of Mathematical Physics*, 61(10), 2020. doi:10.1063/5.0022787.