

LL18554 is being considered for publication in Physical Review Letters.

Limitations of classically-simulable measurements for quantum state discrimination
by Chengkai Zhu, Zhiping Liu, Chenghong Zhu, et al.

Dear Dr. Jencova,

We would appreciate your review of this manuscript, which is being considered by Physical Review Letters. The abstract is below.

Please let us know within 3 weekdays whether you can review it. We generally hope for reports within 2 weeks, but if you need more time than that, please let us know. If you cannot review, advice on suitable referees would be welcome.

To download the manuscript, obtain more information, or send a report, please log into our referee server at:
<https://referees.aps.org/reviews/LL18554-6eab1ca-943792>

To accept to review, visit:
<https://referees.aps.org/reviews/LL18554-6eab1ca-943792/promise>

To decline to review, visit:
<https://referees.aps.org/reviews/LL18554-6eab1ca-943792/decline>

A previous referee was unable to review this. We append previous correspondence. Physical Review Letters publishes manuscripts that should not be missed by physicists in the field of the paper (i.e. they represent significant advances over previously published work and have an immediate impact on workers in the field), and that are interesting and important to related fields. Is this submission of this caliber? Please address this in your report.

One or more pieces of previous correspondence that we need to send to you is in PDF format. Please find the individual files attached to this email.

Supplemental Material associated with this manuscript is available via our referee server.

Thank you for your help.

Yours sincerely,

Stojan Rebic, Ph.D.

Senior Associate Editor
Physical Review Letters
Email: prl@aps.org

ADDITIONAL MATERIAL AVAILABLE (SEE FULL REFERRAL LETTER):

- [Supplemental Material intended for publication \(via referee server\)](#)
- Report of Referee A – LL18554/Zhu
- Report of Referee B – LL18554/Zhu
- Resubmission letter from author – LL18554/Zhu
- Marked Manuscript from author – LL18554/Zhu
- Memo: Guidelines to referees for Physical Review Letters

Limitations of Classically-Simulable Measurements for Quantum State Discrimination

Chengkai Zhu,^{1,*} Zhiping Liu,^{2,*} Chenghong Zhu,¹ and Xin Wang^{1,†}

¹*Thrust of Artificial Intelligence, Information Hub,*

The Hong Kong University of Science and Technology (Guangzhou), Guangzhou 511453, China

²*National Laboratory of Solid State Microstructures, School of Physics and Collaborative*

Innovation Center of Advanced Microstructures, Nanjing University, Nanjing 210093, China

(Dated: March 11, 2024)

In the realm of fault-tolerant quantum computing, stabilizer operations play a pivotal role, characterized by their remarkable efficiency in classical simulation. This efficiency sets them apart from non-stabilizer operations within the quantum computational theory. In this paper, we investigate the limitations of classically-simulable measurements in distinguishing quantum states. We demonstrate that any pure magic state and its orthogonal complement of odd prime dimensions cannot be unambiguously distinguished by stabilizer operations, regardless of how many copies of the states are supplied. We also reveal intrinsic similarities and distinctions between the quantum resource theories of magic states and entanglement in quantum state discrimination. The results emphasize the inherent limitations of classically-simulable measurements and contribute to a deeper understanding of the quantum-classical boundary.

Introduction.—The computational power of quantum computers, including a substantial speed-up over their classical counterparts in solving certain number-theoretic problems [1–3] and simulating quantum systems [4, 5], can only be unlocked with a scalable quantum computing solution. Fault-tolerant quantum computation (FTQC) provides a scheme to overcome obstacles of physical implementation such as decoherence and inaccuracies [6–8].

A cornerstone of the FTQC resides in stabilizer circuits, comprised exclusively of the Clifford gates. It is well-known that the stabilizer circuits can be efficiently classically simulated [9], and therefore do not confer any quantum computational advantage. However, the so-called *magic state* can promote the stabilizer circuits to universal quantum computation via state injection [10–12]. In this context, the magic states and non-stabilizer operations characterize the computational power of universal quantum computation.

While extensive research has explored the stabilizerness of quantum states and gates within circuits [13, 14], a crucial yet underexplored facet is the stabilizerness of quantum measurements [15] – a critical process for reliably decoding classical information encoded in quantum states. In general, it is not applicable for one to access the physical properties of a locally interacting quantum many-body system by classical simulation. However, when information is encoded in a stabilizer state, the decoding process via stabilizer measurements remains efficiently classically simulable [16]. This prompts a fundamental question: can stabilizer measurements perfectly decode all tasks, or are there inherent limitations? Investigating the distinction in decoding capabilities between stabilizer measurements, which are classically efficiently simulable, and other measurements becomes paramount for understanding the intricate relationship between classical information encoded in quantum states and the measurement process.

The ability to retrieve classical information from quantum systems varies significantly with different measurements.

One celebrated example is the quantum nonlocality without entanglement [17]. In essence, global measurements can always perfectly distinguish mutually orthogonal quantum states, while there is a set of product states that cannot be distinguished via local quantum operations and classical communications (LOCC). This distinction between global and local measurements has garnered substantial attention, proving to be intricately linked with quantum entanglement theory and the concept of nonlocality [18–25]. This primitive gap between distinct classes of measurements makes quantum state discrimination (QSD) a crucial aspect of fundamental physics [26, 27], where it can be used to test the principles and nature of quantum mechanics. Moreover, QSD has led to fruitful applications in quantum cryptography [28–30], quantum dimension witness [31, 32] and quantum data hiding [33–35].

Inspired by the intrinsic behavior of different measurements in entanglement theory, we raise a natural and important question for understanding the limit and power of the classically-simulable measurements. In particular, is there a sharp gap between the classically-simulable measurements and those that could potentially promote universal quantum computation? If such a gap exists, it will imply considerable advantages that the resource of magic states can provide to the measurement in quantum information processing.

In this Letter, we give an affirmative answer to this question. We show that any pure magic state and its orthogonal complement cannot be unambiguously distinguished via Positive Operator-Valued Measures (POVMs) having positive discrete Wigner functions, which are classically-simulable and strictly include stabilizer measurements [16, 36], no matter how many copies of the states are supplied. We also demonstrate an exponential decay on the asymptotic minimal error probability for distinguishing the Strange state and its orthogonal complement via POVMs having positive discrete Wigner functions, where the Strange state is a representative magic state defined as $|\mathcal{S}\rangle := (|1\rangle - |2\rangle)/\sqrt{2}$ [37].

In addition, we show that every set of orthogonal pure stabilizer states can be unambiguously distinguished via POVMs having positive discrete Wigner functions, indicating there

* Chengkai Zhu and Zhiping Liu contributed equally to this work.

† felixxinwang@hkust-gz.edu.cn

is no similar phenomenon as the unextendible product basis (UPB) in entanglement theory. Moreover, we demonstrate that even with the assistance of one or two copies of any qutrit magic state, the Strange state and its orthogonal complement remain indistinguishable via POVMs having positive discrete Wigner functions. It is different from entanglement theory where a single copy of the Bell state is always sufficient to perfectly distinguish a pure entangled state and its orthogonal complement using PPT POVMs [38].

Preliminaries.— To characterize the stabilizerness of quantum states and operations, we first recall the definition of the discrete Wigner function [39–41]. Throughout the paper, we study the Hilbert space \mathcal{H}_d with an odd dimension d , and if the dimension is not prime, it should be understood as a tensor product of Hilbert spaces each having an odd prime dimension. Let $\mathcal{L}(\mathcal{H}_d)$ be the space of linear operators mapping \mathcal{H}_d to itself and $\mathcal{D}(\mathcal{H}_d)$ be the set of density operators acting on \mathcal{H}_d . It is worth noting that qudit-based quantum computing is gaining increasing significance, as numerous problems in the field are awaiting further exploration [42].

Given a standard computational basis $\{|j\rangle\}_{j=0,\dots,d-1}$, the unitary boost and shift operators $X, Z \in \mathcal{L}(\mathcal{H}_d)$ is defined by $X|j\rangle = |j \oplus 1\rangle$, $Z|j\rangle = w^j|j\rangle$, where $w = e^{2\pi i/d}$. The discrete phase space of a single d -level system is $\mathbb{Z}_d \times \mathbb{Z}_d$. At each point $\mathbf{u} = (a_1, a_2) \in \mathbb{Z}_d \times \mathbb{Z}_d$, the discrete Wigner function of a state ρ is defined as $W_\rho(\mathbf{u}) := \frac{1}{d} \text{Tr}[A_{\mathbf{u}}\rho]$ where $A_{\mathbf{u}}$ is the phase-space point operator given by $A_0 := \frac{1}{d} \sum_{\mathbf{u}} T_{\mathbf{u}}$, $A_{\mathbf{u}} := T_{\mathbf{u}} A_0 T_{\mathbf{u}}^\dagger$ and $T_{\mathbf{u}} = \tau^{-a_1 a_2} Z^{a_1} X^{a_2}$, $\tau = e^{(d+1)\pi i/d}$. We say a state ρ has positive discrete Wigner functions (PWFs) if $W_\rho(\mathbf{u}) \geq 0, \forall \mathbf{u} \in \mathbb{Z}_d \times \mathbb{Z}_d$ and briefly call it PWF state. Let E be a POVM acting on \mathcal{H}_d whose discrete Wigner function is given by $W(E|\mathbf{u}) = \text{Tr}[E A_{\mathbf{u}}]$. E is said to be a PWF POVM if it has PWFs. A detailed introduction of the terminology can be found in the Supplemental Material.

In odd prime dimensions, quantum circuits with initial states and all subsequent quantum operations having PWFs, which strictly include stabilizer (STAB) operations, admit efficient classical simulations [16, 36], extending the Gottesman-Knill theorem. On the contrary, negativity in Wigner functions is usually regarded as an indication of ‘non-classicality’ [43, 44] and identified as a computational resource. Thus, PWF POVMs are recognized as classically-simulable measurements [45]. The exclusive applicability of these results to odd prime dimensions may stem from the unique property that only quantum systems of such dimensions exhibit covariance of the Wigner function w.r.t. Clifford operations [46]. It’s worth noting that there exist mixed magic states with PWFs, rendering them useless for magic state distillation [12]. These states are termed *bound universal states* [47], analogous to states with a positive partial transpose (PPT) in entanglement distillation [48]. Therefore, PWF POVMs strictly include all STAB POVMs as

$$\text{STAB POVMs} \subsetneq \text{PWF POVMs} \subsetneq \text{All POVMs}.$$

Asymptotic limits of PWF POVMs for a pure state and its orthogonal complement.— Our primary aim is to elucidate the constraints inherent in measurements that can be efficiently

classically simulated. QSD describes a general process of extracting classical information from quantum systems via measurements. To distinguish two states, one usually performs two-outcome POVMs on the received state and then determines which state it is according to the measurement outcome.

It is well-known that the asymptotic regime of QSD can unravel the underlying mechanism of entanglement [21, 24, 49]. The limit of local measurements exhibits a fundamental distinction between pure and mixed states [21]. Moreover, the asymptotic error probability in QSD is interlinked with the quantum relative entropy, Petz’s Rényi divergence [50], and the sandwiched Rényi divergence [51, 52]. Notably, in the regime of many copies, a greater degree of flexibility and options exist for the potential POVMs. However, we shall show there is a wide range of quantum states that cannot be unambiguously distinguished via PWF POVMs, including STAB POVMs, no matter how many copies of them are supplied.

Theorem 1 *Let $\rho_0 \in \mathcal{D}(\mathcal{H}_d)$ be a pure magic state and $\rho_1 = (I_d - \rho_0)/(d - 1)$ be its orthogonal complement, where I_d is the identity. Then for any integer $n \in \mathbb{Z}^+$, $\rho_0^{\otimes n}$ and $\rho_1^{\otimes n}$ cannot be unambiguously distinguished by PWF POVMs.*

Theorem 1 reveals a significant disparity in the ability of PWF POVMs and other measurements in QSD. It indicates that the classical information you are allowed to extract from the encoded states is limited when the measurements allowed are restricted to those classically-simulable ones. The limitation of the classically-simulable measurements cannot be overcome even by increasing the number of copies of the states.

From the angle of quantum resource theories (QRTs) [53], this theorem unravels the difficulty of distinguishing a pure resourceful state and its orthogonal complement via free operations in the QRT of magic states. This is an analog to the phenomenon in entanglement theory that any pure entangled state and its orthogonal complement cannot be unambiguously distinguished via PPT POVMs with an arbitrary number of copies provided [38, 49, 54]. However, they can always be perfectly distinguished if global measurements are allowed. In particular, Takagi and Regula introduced a quantifier for the resourcefulness of a measurement [55] and it was proved that resourceful measurements can offer an advantage for some QSD tasks, compared to free measurements [56]. Here, our result further concretely characterizes the limitation of free measurements in the QRT of magic states. That is, free operations cannot distinguish a pure resourceful state and its orthogonal complement, even in the many-copy regime.

The proof of Theorem 1 relies on Lemma 2 which identifies the feature of *PWF unextendible* subspaces, and a fact that the orthogonal complement of any pure magic state is PWF since $-1/d \leq W_\rho(\mathbf{u}) \leq 1/d, \forall \rho \in \mathcal{D}(\mathcal{H}_d), \forall \mathbf{u}$ [46]. We call a subspace $\mathcal{S} \subseteq \mathcal{H}_d$ *PWF unextendible* if there is no PWF state ρ whose support is a subspace of \mathcal{S}^\perp , and *PWF extendible* otherwise. A subspace $\mathcal{S} \subseteq \mathcal{H}_d$ is called *strongly PWF unextendible* if for any positive integer n , $\mathcal{S}^{\otimes n}$ is PWF unextendible. As a simple example, if we let \mathcal{S}^\perp be a one-dimensional subspace spanned by the strange state $|\mathbb{S}\rangle$, then \mathcal{S} is (strongly) PWF unextendible. In fact, the unextendibility of

subspaces indicates the distinguishability of quantum states. It is well-known that a UPB for a multipartite quantum system indicates indistinguishability using LOCC operations [25].

Lemma 2 *For a PWF unextendible subspace $\mathcal{S} \subseteq \mathcal{H}_d$, if there is a PWF state $\rho \in \mathcal{D}(\mathcal{S})$ such that $\text{supp}(\rho) = \mathcal{S}$, then \mathcal{S} is strongly PWF unextendible.*

We note that Lemma 2 implies that for a set of orthogonal quantum states $\{\rho_1, \dots, \rho_n\}$, if there is a ρ_i whose support is strongly PWF unextendible, then $\{\rho_1, \dots, \rho_n\}$ cannot be unambiguously distinguished by PWF POVMs no matter how many copies are used. This leads to and generalizes the result of Theorem 1. The sketch of the proof for Lemma 2 is as follows.

First, we demonstrate that $\mathcal{S}^{\otimes 2}$ is PWF unextendible through a proof by contradiction. Suppose $\rho_s \in \mathcal{D}(\mathcal{S})$ is a PWF state such that $\text{supp}(\rho_s) = \mathcal{S}$. If there is a PWF state σ supporting on $(\mathcal{S}^{\otimes 2})^\perp$, then we have $\text{Tr}[\sigma(\rho_s \otimes \rho_s)] = 0$ which leads to $\text{Tr}[\rho_s \text{Tr}_2[\sigma(I_d \otimes \rho_s)]] = 0$. It is easy to check that $\sigma' = \text{Tr}_2[\sigma(I_d \otimes \rho_s)]$ is a positive semi-definite operator with PWFs if it is non-zero. If it is zero, we can check that $\text{Tr}_1 \sigma$ is a positive semi-definite operator with PWFs. In either case, we will get a PWF state supported on \mathcal{S}^\perp , a contradiction to the PWF unextendibility of \mathcal{S} . Hence, we conclude that $\mathcal{S}^{\otimes 2}$ is PWF unextendible. Using a similar technique, we can conclusively demonstrate that $\mathcal{S}^{\otimes n}$ is PWF unextendible for any positive integer n . The detailed proof can be found in the Supplementary Material.

Asymptotic limits of PWF POVMs for mixed states.— Followed by Lemma 2, we note that Theorem 1 displays a special case of a strongly PWF unextendible subspace. The orthogonal complement of a pure magic state turns out to be a PWF state which lies in a $d - 1$ dimensional PWF unextendible subspace. It is interesting to ask what is the minimum dimension of a PWF unextendible subspace. Notably, we will show there is a much smaller strongly PWF unextendible subspace, indicating there are mixed magic states that cannot be unambiguously distinguished from their orthogonal complements via PWF POVMs in the many-copy scenario.

Proposition 3 *There exists a strongly PWF unextendible subspace $\mathcal{S} \subseteq \mathcal{H}_d$ of dimension $(d + 1)/2$.*

This proposition implies there is a $(d - 1)/2$ -dimensional subspace in which all states are magic states. We defer the detailed proof to the Supplementary Material and give a simple example as follows.

Example 1 Consider a qudit system with $d = 5$. We have the following basis that spans \mathcal{H}_5 .

$$\begin{aligned} |v_0\rangle &= |0\rangle, \\ |v_1\rangle &= (|1\rangle + |2\rangle + |3\rangle + |4\rangle)/2, \\ |v_2\rangle &= (-|1\rangle + |2\rangle + |3\rangle - |4\rangle)/2, \\ |v_3\rangle &= (|1\rangle - |2\rangle + |3\rangle - |4\rangle)/2, \\ |v_4\rangle &= (|1\rangle + |2\rangle - |3\rangle - |4\rangle)/2. \end{aligned} \quad (1)$$

Let $\rho_0 = (|v_0\rangle\langle v_0| + |v_1\rangle\langle v_1| + |v_2\rangle\langle v_2|)/3$, $\rho_1 = (|v_3\rangle\langle v_3| + |v_4\rangle\langle v_4|)/2$, and $\mathcal{S}_0 = \text{supp}(\rho_0)$, $\mathcal{S}_1 = \text{supp}(\rho_1)$. Followed

by the idea in the proof of Proposition 3, one can check that there is no PWF state in \mathcal{S}_1 , and ρ_0 is a PWF state. Thus, \mathcal{S}_0 is a strongly PWF unextendible subspace. ρ_0 and ρ_1 cannot be unambiguously distinguished by PWF POVMs, no matter how many copies of them are supplied.

More generally, we establish an easy-to-compute criterion for identifying the circumstances under which two quantum states cannot be unambiguously distinguished by PWF POVMs in the many-copy scenario. A sufficient condition is formally given as follows.

Theorem 4 *Given $\rho_0, \rho_1 \in \mathcal{D}(\mathcal{H}_d)$, if any of them has strictly positive discrete Wigner functions, i.e., $W_{\rho_i}(\mathbf{u}) > 0, \forall \mathbf{u}$, then for any integer $n \in \mathbb{Z}^+$, $\rho_0^{\otimes n}$ and $\rho_1^{\otimes n}$ cannot be unambiguously distinguished by PWF POVMs.*

Theorem 4 is broad applicability for both pure and mixed states. The indistinguishability can be checked through a simple computation of the discrete Wigner functions, streamlining the conventional method by analyzing exponentially large Hilbert space. We leave the detailed proof in the Supplementary Material.

Minimum error discrimination by PWF POVMs.— After characterizing the limits of PWF POVMs, we further study the minimum error QSD to unveil the capabilities inherent in PWF POVMs. For states ρ_0 and ρ_1 with prior probability p and $1 - p$, respectively, we denote $P_e^{\text{PWF}}(\rho_0, \rho_1, p)$ as the optimal error probability of distinguishing them by PWF POVMs. Mathematically, this optimal error probability can be expressed via semidefinite programming (SDP) [57] as follows.

$$P_e^{\text{PWF}} = \min_{E_0, E_1} (1 - p) \text{Tr}(E_0 \rho_1) + p \text{Tr}(E_1 \rho_0), \quad (2a)$$

$$\text{s.t. } E_0 \geq 0, E_1 \geq 0, E_0 + E_1 = I, \quad (2b)$$

$$W(E_0|\mathbf{u}) \geq 0, W(E_1|\mathbf{u}) \geq 0, \forall \mathbf{u}, \quad (2c)$$

where Eq. (2c) ensures E_0, E_1 are PWF POVMs. We remain the dual SDP in the Supplementary Material. For ρ_0 to be the Strange state and ρ_1 to be its orthogonal complement, we demonstrate the following asymptotic error behavior.

Proposition 5 *Let ρ_0 be the Strange state $|\mathbb{S}\rangle\langle\mathbb{S}|$ and $\rho_1 = (I - |\mathbb{S}\rangle\langle\mathbb{S}|)/2$ be its orthogonal complement. For $n \in \mathbb{Z}^+$, we have*

$$P_e^{\text{PWF}}(\rho_0^{\otimes n}, \rho_1^{\otimes n}, \frac{1}{2}) = \frac{1}{2^{n+1}}. \quad (3)$$

The optimal PWF POVMs are $E_0 = (|\mathbb{K}\rangle\langle\mathbb{K}| + |\mathbb{S}\rangle\langle\mathbb{S}|)^{\otimes n}$ and $E_1 = I - E_0$, where $|\mathbb{K}\rangle = (|1\rangle + |2\rangle)/\sqrt{2}$.

We remark what we obtain here is the optimal error probability using PWF POVMs to distinguish n copies of the Strange state and its orthogonal complement. We first find the protocol above for the desired error probability and then utilize the dual SDP of (2) to establish the optimality of this protocol. The detailed proof is provided in the Supplementary Material. It can be seen that the optimal error probability

will exponentially decay with respect to the number of copies supplied. Nevertheless, it is important to note that the error persists for all finite values of n , aligning with the indistinguishability established in Theorem 1.

We further discuss the relationship between Proposition 5 and the Chernoff exponent in hypothesis testing. The celebrated quantum Chernoff theorem [50, 58, 59] establishes that $\xi_C(\rho_0, \rho_1) := \lim_{n \rightarrow \infty} -\frac{1}{n} \log P_e(\rho_0^{\otimes n}, \rho_1^{\otimes n}, p) = -\min_{0 \leq s \leq 1} \log \text{Tr}[\rho_0^{1-s} \rho_1^s]$, where $P_e(\rho_0^{\otimes n}, \rho_1^{\otimes n}, p)$ is the average error of distinguishing ρ_0 and ρ_1 via global measurements, $\xi_C(\rho_0, \rho_1)$ is the so-called Chernoff exponent. The Chernoff exponent concerning a specific class of measurements, e.g., {LOCC, PPT, SEP}, is defined in [49]. The authors proved that the Chernoff bounds in these cases are indeed faithful by showing an exponential decay of $P_e^X(\rho_0, \rho_1, p)$ where $X \in \{\text{LOCC, PPT, SEP}\}$. Similarly, Proposition 5 may give an insight that the Chernoff bound concerning PWF measurements is also faithful.

Proposition 5 also implies applications in quantum data hiding [33, 60, 61]. Despite the original data-hiding setting where pairs of states of a bipartite system are perfectly distinguishable via general entangled measurements yet almost indistinguishable under LOCC, it is conceivable to extend data-hiding techniques to broader contexts dictated by specific physical circumstances [55]. As discussed in [55], one may consider the scenario that information is encoded in a way that Pauli measurements have less capability of decoding it than arbitrary measurements. Then only the party with the ability to generate magic can reliably retrieve the message. Here, we define $\|\cdot\|_{\text{PWF}}$ and $R(\text{PWF})$ as the *distinguishability norm* and the *data-hiding ratio* [61] associated with PWF POVMs, respectively. Proposition 5 directly gives a lower bound on the *data-hiding ratio* against PWF POVMs as follows.

$$R(\text{PWF}) = \max \frac{\|p\rho - (1-p)\sigma\|_{\text{All}}}{\|p\rho - (1-p)\sigma\|_{\text{PWF}}} \geq \frac{1}{1-2^{-n}}. \quad (4)$$

We also observe that a potential correlation between $R(\text{PWF})$ and the generalized robustness of measurement [55] merits further investigation, with preliminary evidence provided in the Supplemental Material.

Distinctions between QRT of magic states and entanglement in QSD tasks.— The asymptotic limits of PWF POVMs share similarities with LOCC operations, both of which are considered free within their respective resource theories. Whereas, there are fundamental distinctions between the QRT of magic states and entanglement, considering the QSD tasks. In Table I, we display a comparison between the QRT of magic states and entanglement in QSD, including their similarities and the following distinctions.

Recall that in entanglement theory, the UPB is an incomplete orthogonal product basis whose complementary subspace contains no product state [25]. It shows examples of orthogonal product states that cannot be perfectly distinguished by LOCC operations. Correspondingly, we may imagine whether there is a similar "UPB" phenomenon in the QRT of magic states. That is if there is an incomplete orthogonal stabilizer basis whose complementary subspace contains no stabilizer state. We show that this is not the case as follows.

	QRT of magic states	QRT of entanglement
Asymptotic limits of free POVMs	✓	✓
Existence of UPB phenomenon	✗	✓
Perfect discrimination with the aid of one copy of maximal resource	✗	✓

TABLE I. **Comparison between the QRT of magic states and entanglement.** The second row represents if any resourceful pure state and its orthogonal complement are indistinguishable by free measurements in the many-copy scenario while always distinguishable by resourceful measurements. The third row represents whether there is a UPB phenomenon. The last row represents whether the assistance of one copy of the maximally resourceful state is sufficient for perfect discrimination.

Theorem 6 For a subspace $S \in \mathcal{H}_d$, if S has a set of basis $\{|\psi_i\rangle\}_{i=1}^n$ where every $|\psi_i\rangle$ is a stabilizer state, then S is PWF extendible.

The detailed proof is in the Supplementary Material. A direct consequence of this theorem is that any set of orthogonal pure stabilizer states $\{|\psi_i\rangle\}_{i=1}^n$ can be unambiguously distinguished via PWF POVMs as we can choose $E_i = |\psi_i\rangle\langle\psi_i|$ for $i = 1, 2, \dots, n$ and $E_{n+1} = I - \sum_{i=1}^n |\psi_i\rangle\langle\psi_i|$. Therefore, we demonstrate the absence of an analogous UPB phenomenon in the QRT of magic states.

Besides, it was shown that one copy of the Bell state is always sufficient for perfectly distinguishing any pure state ρ_0 and its orthogonal complement ρ_1 via PPT POVMs [38], i.e., distinguishing $\rho_0 \otimes \Phi_2^+$ and $\rho_1 \otimes \Phi_2^+$. However, things are different in the QRT of magic states where we show that the Strange state and its orthogonal complement cannot be perfectly distinguished by PWF POVMs with the assistance of one or two copies of any qutrit magic state.

Proposition 7 Let ρ_0 be the Strange state $|\mathbb{S}\rangle\langle\mathbb{S}|$ and $\rho_1 = (I - |\mathbb{S}\rangle\langle\mathbb{S}|)/2$ be its orthogonal complement. $\rho_0 \otimes \tau^{\otimes k}$ and $\rho_1 \otimes \tau^{\otimes k}$ cannot be perfectly distinguished for any qutrit magic state τ and $k = 1$ or 2 .

We leave the proof in the Supplementary Material. The main idea is to analyze the minimal *mana* [62] $\tau^{\otimes k}$ must have to perfectly distinguish $\rho_0 \otimes \tau^{\otimes k}$ and $\rho_1 \otimes \tau^{\otimes k}$ by PWF POVMs. Proposition 7 reveals that one or two copies of any auxiliary qutrit state are not sufficient to distinguish the Strange state and its orthogonal complement. A similar result can be obtained for the Norell state $|\mathbb{N}\rangle := (-|0\rangle + 2|1\rangle - |2\rangle)/\sqrt{6}$ [37]. Hence, we have witnessed the distinctions of the QRT of magic states and entanglement in regard to the resource cost for perfect discrimination.

Concluding remarks.— We have explored the limitations of PWF POVMs which can be efficiently classically simulated and strictly include all stabilizer measurements. Our results show that the QRT of magic states and entanglement exhibit significant similarities and distinctions in quantum state discrimination.

Our findings reveal the limitations of classically-simulable measurements and highlight the operational advantage real-

ized by the quantum resource of magic states. These results have implications in various fields, including connections between the QRT of magic states and quantum data hiding [35, 55, 60, 61]. It remains interesting to further study the limits of stabilizer measurements or classically-simulable ones in quantum channel discrimination [63–65] and other operational tasks [66–68]. Note that as it is still open whether all operations with negative discrete Wigner functions are useful for magic state distillation [37], a comprehensive characterization of the quantum-classical boundary of measurements is still needed. Additionally, it is interesting to study the limita-

tions of stabilizer measurements in a multi-qubit system [69–72], and recent advances in generalized phase-space simulation methods for qubits [73, 74] offer potential avenues to explore this, which we will leave to future work.

Acknowledgments.— We would like to thank the anonymous referees for their helpful comments. This work was supported by the Start-up Fund (No. G0101000151) from The Hong Kong University of Science and Technology (Guangzhou), the Guangdong Provincial Quantum Science Strategic Initiative (No. GDZX2303007), and the Education Bureau of Guangzhou Municipality.

-
- [1] P. W. Shor, *SIAM Journal on Computing* **26**, 1484 (1997).
 - [2] L. K. Grover, “A fast quantum mechanical algorithm for database search,” (1996), [arXiv:quant-ph/9605043 \[quant-ph\]](#).
 - [3] A. M. Childs and W. van Dam, *Rev. Mod. Phys.* **82**, 1 (2010).
 - [4] S. Lloyd, *Science* **273**, 1073 (1996).
 - [5] A. M. Childs, D. Maslov, Y. Nam, N. J. Ross, and Y. Su, *Proceedings of the National Academy of Sciences* **115**, 9456 (2018).
 - [6] P. W. Shor, “Fault-tolerant quantum computation,” (1997), [arXiv:quant-ph/9605011 \[quant-ph\]](#).
 - [7] E. T. Campbell, B. M. Terhal, and C. Vuillot, *Nature* **549**, 172 (2017).
 - [8] E. Knill, *Nature* **434**, 39 (2005).
 - [9] D. Gottesman, *Stabilizer codes and quantum error correction* (California Institute of Technology, 1997).
 - [10] X. Zhou, D. W. Leung, and I. L. Chuang, *Physical Review A* **62** (2000), [10.1103/physreva.62.052316](#).
 - [11] D. Gottesman and I. L. Chuang, *Nature* **402**, 390 (1999).
 - [12] S. Bravyi and A. Kitaev, *Physical Review A* **71** (2005), [10.1103/physreva.71.022316](#).
 - [13] E. T. Campbell, H. Anwar, and D. E. Browne, *Physical Review X* **2** (2012).
 - [14] M. Yoganathan, R. Jozsa, and S. Strelchuk, *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* **475**, 20180427 (2019).
 - [15] M. A. Nielsen and I. L. Chuang, *Quantum computation and quantum information* (Cambridge university press, 2010).
 - [16] A. Mari and J. Eisert, *Physical Review Letters* **109**, 1 (2012).
 - [17] C. H. Bennett, D. P. DiVincenzo, C. A. Fuchs, T. Mor, E. Rains, P. W. Shor, J. A. Smolin, and W. K. Wootters, *Physical Review A* **59**, 1070 (1999).
 - [18] D. Leung, A. Winter, and N. Yu, *Reviews in Mathematical Physics* **33**, 2150013 (2021).
 - [19] S. Bandyopadhyay, A. Cosentino, N. Johnston, V. Russo, J. Watrous, and N. Yu, *IEEE Transactions on Information Theory* **61**, 3593 (2015).
 - [20] A. M. Childs, D. Leung, L. Mančinska, and M. Ozols, *Communications in Mathematical Physics* **323**, 1121 (2013).
 - [21] S. Bandyopadhyay, *Physical Review Letters* **106**, 1 (2011).
 - [22] J. Calsamiglia, J. I. De Vicente, R. Muñoz-Tapia, and E. Bagan, *Physical Review Letters* **105**, 1 (2010).
 - [23] S. Halder, M. Banik, S. Agrawal, and S. Bandyopadhyay, *Physical Review Letters* **122**, 40403 (2019).
 - [24] J. Walgate, A. J. Short, L. Hardy, and V. Vedral, *Physical Review Letters* **85**, 4972 (2000).
 - [25] C. H. Bennett, D. P. DiVincenzo, T. Mor, P. W. Shor, J. A. Smolin, and B. M. Terhal, *Physical Review Letters* **82**, 5385 (1999).
 - [26] J. Bae and L.-C. Kwek, *Journal of Physics A: Mathematical and Theoretical* **48**, 083001 (2015).
 - [27] W. H. G. Correa, L. Lami, and C. Palazuelos, *IEEE Transactions on Information Theory* **68**, 7306 (2022).
 - [28] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, *Reviews of Modern Physics* **74**, 145 (2002).
 - [29] R. Cleve, D. Gottesman, and H.-K. Lo, *Physical Review Letters* **83**, 648 (1999).
 - [30] A. Leverrier and P. Grangier, *Physical Review Letters* **102** (2009).
 - [31] N. Brunner, M. Navascués, and T. Vértesi, *Physical Review Letters* **110** (2013).
 - [32] M. Hendrych, R. Gallego, M. Mičuda, N. Brunner, A. Acín, and J. P. Torres, *Nature Physics* **8**, 588 (2012).
 - [33] B. M. Terhal, D. P. DiVincenzo, and D. W. Leung, *Phys. Rev. Lett.* **86**, 5807 (2001).
 - [34] T. Eggeling and R. F. Werner, *Phys. Rev. Lett.* **89**, 097905 (2002).
 - [35] W. Matthews, S. Wehner, and A. Winter, *Communications in Mathematical Physics* **291**, 813 (2009).
 - [36] V. Veitch, C. Ferrie, D. Gross, and J. Emerson, *New Journal of Physics* **14**, 1 (2012).
 - [37] V. Veitch, S. A. Hamed Mousavian, D. Gottesman, and J. Emerson, *New Journal of Physics* **16**, 1 (2014).
 - [38] N. Yu, R. Duan, and M. Ying, *IEEE Transactions on Information Theory* **60**, 2069 (2014).
 - [39] W. K. Wootters, *Annals of Physics* **176**, 1 (1987).
 - [40] D. Gross, *Journal of Mathematical Physics* **47** (2006).
 - [41] D. Gross, *Applied Physics B* **86**, 367 (2006).
 - [42] Y. Wang, Z. Hu, B. C. Sanders, and S. Kais, *Frontiers in Physics* **8** (2020).
 - [43] E. F. Galvão, *Physical Review A* **71** (2005).
 - [44] C. Cormick, E. F. Galvão, D. Gottesman, J. P. Paz, and A. O. Pittenger, *Physical Review A* **73** (2006).
 - [45] M. Howard, J. Wallman, V. Veitch, and J. Emerson, *Nature* **510**, 351 (2014).
 - [46] H. Zhu, *Phys. Rev. Lett.* **116**, 040501 (2016).
 - [47] E. T. Campbell and D. E. Browne, *Physical Review Letters* **104** (2010).
 - [48] M. Horodecki, P. Horodecki, and R. Horodecki, *Phys. Rev. Lett.* **80**, 5239 (1998).
 - [49] H.-C. Cheng, A. Winter, and N. Yu, *Communications in Mathematical Physics*, 1 (2023).
 - [50] K. M. R. Audenaert, M. Nussbaum, A. Szkoła, and F. Verstraete, *Communications in Mathematical Physics* **279**, 251 (2008).
 - [51] T. Ogawa and H. Nagaoka, in *Asymptotic Theory of Quantum Statistical Inference* (WORLD SCIENTIFIC, 2005) pp. 28–42.

- [52] M. Mosonyi and T. Ogawa, [Communications in Mathematical Physics](#) **334**, 1617 (2014).
- [53] E. Chitambar and G. Gour, [Reviews of Modern Physics](#) **91** (2019).
- [54] Y. Li, X. Wang, and R. Duan, [Physical Review A](#) **95**, 052346 (2017).
- [55] R. Takagi and B. Regula, [Physical Review X](#) **9** (2019), 10.1103/physrevx.9.031053.
- [56] M. Oszmaniec and T. Biswas, [Quantum](#) **3**, 133 (2019).
- [57] S. P. Boyd and L. Vandenberghe, *Convex optimization* (Cambridge university press, 2004).
- [58] K. M. R. Audenaert, J. Calsamiglia, R. Muñoz-Tapia, E. Bagan, L. Masanes, A. Acín, and F. Verstraete, [Physical Review Letters](#) **98** (2007).
- [59] F. Hiai and D. Petz, [Communications in mathematical physics](#) **143**, 99 (1991).
- [60] D. DiVincenzo, D. Leung, and B. Terhal, [IEEE Transactions on Information Theory](#) **48**, 580 (2002).
- [61] L. Lami, C. Palazuelos, and A. Winter, [Communications in Mathematical Physics](#) **361**, 661 (2018).
- [62] X. Wang, M. M. Wilde, and Y. Su, [Physical Review Letters](#) **124**, 090505 (2018).
- [63] S. Pirandola, R. Laurenza, C. Lupo, and J. L. Pereira, [npj Quantum Information](#) **5** (2019).
- [64] X. Wang and M. M. Wilde, [Physical Review Research](#) **1** (2019).
- [65] R. Takagi, B. Regula, K. Bu, Z.-W. Liu, and G. Adesso, [Physical Review Letters](#) **122** (2019).
- [66] R. Uola, T. Bullock, T. Kraft, J.-P. Pellonpää, and N. Brunner, [Physical Review Letters](#) **125** (2020).
- [67] A. F. Ducuara and P. Skrzypczyk, [Physical Review Letters](#) **125** (2020).
- [68] P. Lipka-Bartosik, A. F. Ducuara, T. Purves, and P. Skrzypczyk, [PRX Quantum](#) **2** (2021).
- [69] M. Howard and E. Campbell, [Physical Review Letters](#) **118** (2017).
- [70] S. Bravyi, G. Smith, and J. A. Smolin, [Physical Review X](#) **6**, 1 (2016).
- [71] J. R. Seddon and E. T. Campbell, [Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences](#) **475**, 20190251 (2019).
- [72] O. Hahn, A. Ferraro, L. Hultquist, G. Ferrini, and L. García-Álvarez, [Physical Review Letters](#) **128** (2022).
- [73] M. Zurel, C. Okay, and R. Raussendorf, [Physical Review Letters](#) **125** (2020), 10.1103/physrevlett.125.260404.
- [74] R. Raussendorf, J. Bermejo-Vega, E. Tyhurst, C. Okay, and M. Zurel, [Physical Review A](#) **101** (2020), 10.1103/physreva.101.012350.