On the structure of higher order quantum maps

Anna Jenčová

September 26, 2024

1 Affine subspaces and higher order maps

1.1 The category FinVect

sec:fv

Let FinVect be the category of finite dimensional real vector spaces with linear maps. We will denote the usual tensor product by $\otimes$, then $(\mathrm{FinVect}, \otimes, I = \mathbb{R})$ is a symmetric monoidal category, with the associators, unitors and symmetries given by the obvious isomorphisms

$$\begin{aligned}\alpha_{U,V,W} &: (U \otimes V) \otimes W \simeq U \otimes (V \otimes W),\\ \lambda_V &: I \otimes V \simeq V, \qquad \rho_V : V \otimes I \simeq V,\\ \sigma_{U,V} &: U \otimes V \simeq V \otimes U.\end{aligned}$$

Let $(-)^* : V \mapsto V^*$ be the usual vector space dual, with duality denoted by $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}$. We will use the canonical identification $V^{**} = V$ and $(V_1 \otimes V_2)^* = V_1^* \otimes V_2^*$. With this duality, FinVect is compact closed. This means that for each object V, there are morphisms $\eta_V : I \to V^* \otimes V$ (the "cup") and $\epsilon_V : V \otimes V^* \to I$ (the "cap") such that the following snake identities hold:

$$(\epsilon_V \otimes V) \circ (V \otimes \eta_V) = V, \qquad (V^* \otimes \epsilon_V) \circ (\eta_V \otimes V^*) = V^*, \tag{1}$$

eq:snake

here we denote the identity map on the object V by V. Let us identify these morphisms. First, η_V is a linear map $\mathbb{R} \to V^* \otimes V$, which can be identified with the element $\eta_V(1) \in V^* \otimes V$ and $\epsilon_V \in (V \otimes V^*)^* = V^* \otimes V$ is again an element of the same space. Choose a basis $\{e_i\}$ of V, let $\{e_i^*\}$ be the dual basis of V^*, that is, $\langle e_i^*, e_j \rangle = \delta_{i,j}$. Let us define

$$\eta_V(1) = \epsilon_V := \sum_i e_i^* \otimes e_i.$$

It is easy to see that this definition does not depend on the choice of the basis, indeed, ϵ_V is the linear functional on $V \otimes V^*$ defined by

$$\langle \epsilon_V, x \otimes x^* \rangle = \langle x^*, x \rangle, \qquad x \in V,\ x^* \in V^*.$$

It is also easily checked that the snake identities (eq:snake 1) hold.

For two objects V and W in FinVect, we will denote the set of all morphisms (i.e. linear maps) $V \to W$ by $\mathrm{FinVect}(V, W)$. Then $\mathrm{FinVect}(V, W)$ is itself a real linear space and we have the well-known identification $\mathrm{FinVect}(V, W) \simeq V^* \otimes W$. This can be given as follows: for each $f \in \mathrm{FinVect}(V, W)$, we have $C_f := (V^* \otimes f)(\epsilon_V) = \sum_i e_i^* \otimes f(e_i) \in V^* \otimes W$. Conversely, since $\{e_i^*\}$

is a basis of V^*, any element $w \in V^* \otimes W$ can be uniquely written as $w = \sum_i e_i^* \otimes w_i$ for $w_i \in W$, and since $\{e_i\}$ is a basis of V, the assignment $f(e_i) := w_i$ determines a unique map $f : V \to W$. The relations between $f \in \mathrm{FinVect}(V, W)$ and $C_f \in V^* \otimes W$ can be also written as

$$\langle C_f, x \otimes y^* \rangle = \langle \epsilon_V, x \otimes f^*(y^*) \rangle = \langle f^*(y^*), x \rangle = \langle y^*, f(x) \rangle, \qquad x \in V,\ y^* \in W^*,$$

here $f^* : W^* \to V^*$ is the adjoint of f. Note that by compactness, the internal hom in FinVect satisfies $[V, W] \simeq V^* \otimes W$, so that in the case of FinVect, the object $[V, W]$ can be identified with the space of linear maps $\mathrm{FinVect}(V, W)$.

We now present two examples that are most important for us.

lassical *Example* 1. Let $V = \mathbb{R}^N$. In this case, we fix the canonical basis $\{|i\rangle,\ i = 1, \dots, N\}$. We will identify $(\mathbb{R}^N)^* = \mathbb{R}^N$, with duality $\langle x, y \rangle = \sum_i x_i y_i$, in particular, we identify $I = I^*$. We then have $\epsilon_V = \sum_i |i\rangle \otimes |i\rangle$ and if $f : \mathbb{R}^N \to \mathbb{R}^M$ is given by the matrix A in the two canonical bases, then $C_f = \sum_i |i\rangle \otimes A|i\rangle$ is the vectorization of A.

:quantum *Example* 2. Let $V = M_n^h$ be the space of $n \times n$ complex hermitian matrices. We again identify $(M_n^h)^* = M_n^h$, with duality $\langle A, B \rangle = \mathrm{Tr}\, A^T B$, where A^T is the usual transpose of the matrix A. Let us choose the basis in M_n^h, given as

$$\Big\{ |j\rangle\langle k| + |k\rangle\langle j|,\ j \leq k,\ i\Big(|j\rangle\langle k| - |k\rangle\langle j| \Big),\ j < k \Big\}.$$

Then one can check that

$$\Big\{ \frac{1}{2}\Big(|j\rangle\langle k| + |k\rangle\langle j| \Big),\ j \leq k,\ \frac{i}{2}\Big(|k\rangle\langle j| - |j\rangle\langle k| \Big),\ j < k \Big\}$$

is the dual basis and we have

$$\epsilon_V = \sum_{j,k} |j\rangle\langle k| \otimes |j\rangle\langle k|.$$

For any $f : M_n^h \to M_m^h$,

$$C_f = \sum_{j,k} |j\rangle\langle k| \otimes f(|j\rangle\langle k|)$$

is the Choi matrix of f.

1.2 The category Af

We now introduce the category Af, whose objects are of the form $X = (V_X, A_X)$, where V_X is an object in FinVect and $A_X \subseteq V_X$ is a proper affine subspace, see Appendix sec:app_affine ?? for definitions and basic properties. Morphisms $X \xrightarrow{f} Y$ in Af are linear maps $f : V_X \to V_Y$ such that $f(A_X) \subseteq A_Y$. For any object X, we put

$$L_X := \mathrm{Lin}(A_X), \quad S_X := \mathrm{Span}(A_X), \quad D_X = \dim(V_X), \quad d_X = \dim(L_X).$$

We have

$$A_X = a + L_X = S_X \cap \{\tilde{a}\}^{\sim}, \tag{2}$$ eq:ALS

for any choice of elements $a \in A_X$ and $\tilde{a} \in \tilde{A}_X$. We now introduce a tensor product and duality that endow Af with the structure of a *-autonomous category.

By Corollary 2, the dual $\tilde{A}_X$ is a proper affine subspace in V_X^*, so that $X^* := (V_X^*, \tilde{A}_X)$ is an object in Af. We have $X^{**} = X$ and the corresponding subspaces are related as

$$L_{X^*} = S_X^\perp, \qquad S_{X^*} = L_X^\perp. \tag{3}$$

It is easily seen that for any $X \xrightarrow{f} Y$, the adjoint map satisfies $f^*(\tilde{A}_Y) \subseteq \tilde{A}_X$, so that $Y^* \xrightarrow{f^*} X^*$ and the duality $(-)^*$ is a full and faithful functor $\mathrm{Af}^{op} \to \mathrm{Af}$.

We will next introduce a monoidal structure $\otimes$ as follows. For two objects X and Y, we put $V_{X\otimes Y} = V_X \otimes V_Y$ and construct the affine subspace $A_{X\otimes Y}$ as the affine span of

$$A_X \otimes A_Y = \{a \otimes b,\ a \in A_X,\ b \in A_Y\}.$$

Fix any $\tilde{a}_X \in \tilde{A}_X$ and $\tilde{a}_Y \in \tilde{A}_Y$. Since $A_X \otimes A_Y \subset \{\tilde{a}_X \otimes \tilde{a}_Y\}^\sim$, the affine span of $A_X \otimes A_Y$ is a proper affine subspace and we have by Lemma 11

$$A_{X\otimes Y} := \mathrm{Aff}(A_X \otimes A_Y) = \{A_X \otimes A_Y\}^{\approx}.$$

Lemma 1. *For any $a_X \in A_X$, $a_Y \in A_Y$, we have*

$$\begin{aligned} L_{X\otimes Y} &= \mathrm{Lin}(A_X \otimes A_Y) = \mathrm{Span}(\{x \otimes y - a_X \otimes a_Y,\ x \in A_X,\ y \in A_Y\}) && (4) \\ &= (a_X \otimes L_Y) + (L_X \otimes a_Y) + (L_X \otimes L_Y) && (5) \end{aligned}$$

(here $+$ denotes the direct sum of subspaces). We also have

$$S_{X\otimes Y} = S_X \otimes S_Y.$$

Proof. The equality (4) follows from Lemma 11. For any $x \in A_X$, $y \in A_Y$ we have

$$x \otimes y - a_X \otimes a_Y = a_X \otimes (y - a_Y) + (x - a_X) \otimes a_Y + (x - a_X) \otimes (y - a_Y),$$

so that $L_{X\otimes Y} = \mathrm{Lin}(A_X \otimes A_Y)$ is contained in the subspace on the RHS of (5). Let d be the dimension of this subspace, then clearly

$$d_{X\otimes Y} \le d \le d_X + d_Y + d_X d_Y.$$

On the other hand, any element of S_X has the form tx for some $t \in \mathbb{R}$ and $x \in A_X$, so that it is easily seen that $S_X \otimes S_Y = S_{X\otimes Y}$. Hence

$$\begin{aligned} d_{X\otimes Y} &= \dim(L_{X\otimes Y}) = \dim(S_{X\otimes Y}) - 1 = \dim(S_X)\dim(S_Y) - 1 = (d_X+1)(d_Y+1) - 1 \\ &= d_X + d_Y + d_X d_Y. \end{aligned}$$

This completes the proof. □

Lemma 2. *Let $I = (\mathbb{R}, \{1\})$. Then $(\mathrm{Af}, \otimes, I)$ is a symmetric monoidal category.*

Proof. Note that this structure is inherited from the symmetric monoidal structure in FinVect. To show that $\otimes$ is a functor, we have to check that for $X_1 \xrightarrow{f} Y_1$ and $X_2 \xrightarrow{g} Y_2$ in Af, we have $X_1 \otimes Y_1 \xrightarrow{f\otimes g} X_2 \otimes Y_2$ which amounts to showing that

$$(f \otimes g)(A_{X_1 \otimes Y_1}) \subseteq A_{X_2 \otimes Y_2}.$$

Let $x \in A_{X_1}$, $y \in A_{Y_1}$, then $f(x) \otimes g(y) \in A_{X_2} \otimes A_{Y_2} \subseteq A_{X_2 \otimes Y_2}$. Since $A_{X_1 \otimes Y_1}$ is the affine subspace generated by $A_{X_1} \otimes A_{Y_1}$, the above inclusion follows by linearity of $f \otimes g$.

It only remains to prove that the associators, unitors and symmetries from FinVect are morphisms in Af. We will prove this for the associators $\alpha_{X,Y,Z} : V_X \otimes (V_Y \otimes V_Z) \to (V_X \otimes V_Y) \otimes V_Z$, the other proofs are similar. We need to check that $\alpha_{X,Y,Z}(A_{X \otimes (Y \otimes Z)}) \subseteq A_{(X \otimes Y) \otimes Z}$. It is easily checked that $A_{X \otimes (Y \otimes Z)}$ is the affine span of elements of the form $x \otimes (y \otimes z)$, $x \in A_X$, $y \in A_Y$ and $z \in A_Z$, and we have

$$\alpha_{X,Y,Z}(x \otimes (y \otimes z)) = (x \otimes y) \otimes z \in A_{(X \otimes Y) \otimes Z}$$

for all such elements. The desired inclusion follows by linearity.

$\square$

Theorem 1. *$(\mathrm{Af}, \otimes, I)$ is a *-autonomous category, with duality $(-)^*$, such that $I^* = I$.*

Proof. By Lemma lemma:monoidal 2, we have that $(\mathrm{Af}, \otimes, I)$ is a symmetric monoidal category. We have also seen that the duality $(-)^*$ is a full and faithful contravariant functor. We only need to check the natural isomorphisms

$$\mathrm{Af}(X \otimes Y, Z^*) \simeq \mathrm{Af}(X, (Y \otimes Z)^*).$$

Since FinVect is compact, we have the natural isomorphisms

$$\mathrm{FinVect}(V_X \otimes V_Y, V_Z^*) \simeq \mathrm{FinVect}(V_X, V_Y^* \otimes V_Z^*),$$

determined by the equalities

$$\langle\, f(x \otimes y), z \,\rangle = \langle\, \hat{f}(x), y \otimes z \,\rangle, \qquad x \in V_X,\ y \in V_Y,\ z \in V_Z,$$

for $f \in \mathrm{FinVect}(V_X \otimes V_Y, V_Z^*)$ and the corresponding morphism $\hat{f} \in \mathrm{FinVect}(V_Z, V_Y^* \otimes V_Z^*)$. Since $A_{X \otimes Y}$ is an affine span of $A_X \otimes A_Y$, we see that f is in $\mathrm{Af}(X \otimes Y, Z^*)$ if and only if $f(x \otimes y) \in \tilde{A}_Z$ for all $x \in A_X$, $y \in A_Y$, that is,

$$1 = \langle\, f(x \otimes y), z \,\rangle = \langle\, \hat{f}(x), y \otimes z \,\rangle \qquad \forall x \in A_X,\ \forall y \in A_Y,\ \forall z \in A_Z.$$

But this is equivalent to

$$\hat{f}(x) \in (A_Y \otimes A_Z)^{\sim} = \tilde{A}_{Y \otimes Z}, \qquad \forall x \in A_X,$$

which means that $\hat{f} \in \mathrm{Af}(X, (Y \otimes Z)^*)$.

$\square$

A *-autonomous category is compact closed if it satisfies $(X \otimes Y)^* = X^* \otimes Y^*$. In general, $X \odot Y = (X^* \otimes Y^*)^*$ defines a dual symmetric monoidal structure that is different from $\otimes$. We next show that Af is not compact.

ncompact **Proposition 1.** *For objects in* Af*, we have $(X \otimes Y)^* = X^* \otimes Y^*$ exactly in one of the following situations:*

(i) $X \simeq I$ *or* $Y \simeq I$,

(ii) $d_X = d_Y = 0$,

(iii) $d_{X^*} = d_{Y^*} = 0$.

Proof. Since FinVect is compact, we have $V_{(X\otimes Y)^*} = (V_X \otimes V_Y)^* = V_X^* \otimes V_Y^* = V_{X^*\otimes Y^*}$. It is also easily seen by definition that $A_{X^*} \otimes A_{Y^*} = \tilde{A}_X \otimes \tilde{A}_Y \subseteq \tilde{A}_{X\otimes Y} = A_{(X\otimes Y)^*}$, so that we always have $A_{X^*\otimes Y^*} \subseteq A_{(X\otimes Y)^*}$. Hence the equality holds if and only if $d_{X^*\otimes Y^*} = d_{(X\otimes Y)^*}$. From Lemma 1, we see that

$$d_{X^*\otimes Y^*} = d_{X^*} + d_{Y^*} + d_{X^*}d_{Y^*}.$$

On the other hand, we have using (3) that $L_{(X\otimes Y)^*} = S_{X\otimes Y}^{\perp} = (S_X \otimes S_Y)^{\perp}$, so that

$$d_{(X\otimes Y)^*} = D_X D_Y - \dim(S_X)\dim(S_Y) = D_X D_Y - (d_X+1)(d_Y+1).$$

Taking into account that by (3) we have $d_{X^*} = D_X - d_X - 1$, similarly for d_{Y^*}, we obtain

$$d_{(X\otimes Y)^*} - d_{X^*\otimes Y^*} = d_X d_{Y^*} + d_{X^*} d_Y.$$

This is equal to 0 iff $d_X d_{Y^*} = d_Y d_{X^*} = 0$, which amounts to the conditions in the lemma.

□

In a *-autonomous category, the internal hom can be identified as $[X,Y] = (X \otimes Y^*)^*$. The underlying vector space is $V_{[X,Y]} = (V_X \otimes V_Y^*)^* = V_X^* \otimes V_Y$ and we have seen in Section 1.1 that we may identify this space with $\mathrm{FinVect}(V_X, V_Y)$, by $f \leftrightarrow C_f$. This property is extended to Af, in the following sense.

Proposition 2. *For any objects X, Y in* Af*, the map $f \mapsto C_f$ is a bijection of* $\mathrm{Af}(X,Y)$ *onto* $A_{[X,Y]}$.

Proof. Let $f \in \mathrm{FinVect}(V_X, V_Y)$. Since by definition $A_{[X,Y]} = \tilde{A}_{X\otimes Y^*} = (A_X \otimes A_Y^*)^{\sim}$, we see that $C_f \in A_{[X,Y]}$ if and only if for all $x \in A_X$ and $y^* \in \tilde{A}_Y$, we have

$$1 = \langle C_f, x \otimes y^* \rangle = \langle y^*, f(x) \rangle.$$

This latter statement is clearly equivalent to $f(A_X) \subseteq A_Y$, so that $f \in \mathrm{Af}(X,Y)$. □

In the next result, we restrict the objects to spaces of hermitian matrices, as in Example 2 and morphisms to completely positive maps. We show that this restriction amounts to taking an intersection of $A_{[X,Y]}$ with the cone of positive semidefinite matrices. This, and subsequent examples, shows that for characterization of sets of quantum objects such as states, channels, combs and transformations between them, it is enough to work with the category Af.

An object X of Af will be called quantum if $V_X = M_n^h$ for some n and A_X is an affine subspace such that both A_X and $\tilde{A}_X$ have nonempty intersection with the interior of the positive cone $int(M_n^+)$ (recall that we identify $(M_n^h)^* = M_n^h$). A quantum object will be called standart if both A_X and $\tilde{A}_X$ contain a positive multiple of the identity matrix I_n.

Proposition 3. *Let X, Y be quantum objects in* Af*. Then*

(i) X^* and $X \otimes Y$ are quantum objects as well. If X and Y are standart, then so are X^* and $X \otimes Y$.

(ii) Let $V_X = M_n^h$, $V_Y = M_m^h$. Then for any $f \in \mathrm{FinVect}(M_n^h, M_m^h)$, we have $C_f \in A_{[X,Y]} \cap M_{mn}^+$ if and only if f is completely positive and

$$f(A_X \cap M_n^+) \subseteq A_Y \cap M_m^+.$$

Proof. The statement (i) is easily seen from $A_X \otimes A_Y \subseteq A_{X \otimes Y}$ and $\tilde{A}_X \otimes \tilde{A}_Y \subseteq \tilde{A}_{X \otimes Y}$, together with the fact that $int(M_n^+) \otimes int(M_m^+) \subseteq int(M_{mn}^+)$. To show (ii), let $C_f \in A_{[X,Y]} \cap M_{mn}^+$. By the properties of the Choi isomorphism f is completely positive and by Proposition prop:ihom_morphisms 2, $f(A_X) \subseteq A_Y$, this proves one implication. For the converse, note that we only need to prove that under the given assumptions, $f(A_X) \subseteq A_Y$, for which it is enough to show that $A_X \subseteq \mathrm{Aff}(A_X \cap M_n^+)$. To see this, pick some $a_X \in A_X \cap int(M_n^+)$. Any element in A_X can be written in the form $a_X + v$ for some $v \in L_X$. Since $a_X \in int(M_n^+)$, there is some $s > 0$ such that $a_\pm := a_X \pm sv \in M_n^+$, and since $\pm sv \in L_X$, we see that $a_\pm \in A_X \cap M_n^+$. It is now easily checked that

$$a_X + v = \frac{1+s}{2s} a_+ + \frac{s-1}{2s} a_- \in \mathrm{Aff}(A_X \cap M_n^+).$$

□

We can define classical objects in Af in a similar way, replacing M_n^h by $\mathbb{R}^N$ and the positive cone by $\mathbb{R}_+^N$. Classical objects are standart if both A_X and $\tilde{A}_X$ contains a positive multiple of the unit vector $(1, \ldots, 1) \in \mathbb{R}^N$. A similar statement holds in this case, with complete positivity replaced by positivity. We can similarly treat classical-to-quantum and quantum-to-classical maps as morphisms between these types of objects, satisfying appropriate positivity assumptions.

tum_maps *Example* 3. The basic example of a standart quantum object is

$$\mathcal{A}_n := \{T \in M_n^h,\ \mathrm{Tr}\,[T] = 1\}.$$

Then $\mathcal{S}_n := (M_n^h, \mathcal{A}_n)$ is an object in Af, and it is a standart quantum object, since we have $I_n \in \tilde{\mathcal{A}}_n = \{I_n\}$ and $\frac{1}{n} I_n \in \mathcal{A}_n$. The set $\mathcal{A}_n \cap M_n^+$ is the set of quantum states. By Proposition prop:ihom_mor 2, $\mathcal{C}_{m,n} := [\mathcal{S}_m, \mathcal{S}_n]$ is a standart quantum object as well, such that the corresponding vector space is M_{mn}^h and $A_{\mathcal{C}_{m,n}} \cap M_{mn}^+$ is the set of Choi matrices of quantum channels $M_m \to M_n$. Note that the dual object $\mathcal{C}_{m,n}^* = \mathcal{S}_m \otimes \mathcal{S}_n^*$ represents the set of Choi matrices of replacement channels $\mathcal{S}_n \to \mathcal{S}_m$, that is, channels that map any state in M_n to a fixed state in M_m.

Using We will present several well known examples that can be represented by quantum objects, such that the distinguished positive definite elements in A_X and $\tilde{A}_X$ are multiples of the identity matrix I_n.

1. Quantum states are represented by density matrices, that is, elements $\rho \in M_n^+$ such that $\mathrm{Tr}\,[\rho] = 1$. The set of all density matrices is represented by the quantum object

   $$\mathcal{S}_n = (M_n^h, A_{\mathcal{S}_n} = \mathrm{Tr}^{-1}(1)).$$

   In this case, we choose the distinguished elements as $I_n \in \tilde{A}_{\mathcal{S}_n} = \{I_n\})$, $\frac{1}{n} I_n \in A_{\mathcal{S}_n}$.

2. Let us look at the object

   $$\mathcal{C}_{m,n} = [\mathcal{S}_m, \mathcal{S}_n] = (\mathcal{S}_m \otimes \mathcal{S}_n^*)^*.$$

   By Proposition prop:ihom_morphisms 2, $\mathcal{C}_{m,n}$ is a quantum object with underlying vector space $M_m^h \otimes M_n^h = M_{mn}^h$, such that $A_{\mathcal{C}_{m,n}} \cap M_{mn}^+$ is the set of Choi matrices of quantum channels $\mathcal{S}_m \to \mathcal{S}_n$.

   The distinguished elements can be chosen as $\frac{1}{n} I_{mn} \in A_{\mathcal{C}_{m,n}}$ and $\frac{1}{m} I_{mn} \in \tilde{\mathcal{C}}_{m,n} = \mathcal{S}_m \otimes I_n$.

States, channels, combs, nonsignaling, etb, dual, process matrices

exm:qccq *Example* 4. POVMs, instruments, multimeters.

1.3 First order and higher order objects

We say that an object X in Af is first order if $d_X = D_X - 1$, equivalently, $S_X = V_X$. Another equivalent condition is $d_{X^*} = 0$, which means that A_X is determined by a single element $\tilde{a}_X \in V_X^*$ as

$$A_X = \{\tilde{a}_X\}^{\sim}, \qquad \tilde{A}_X = \{\tilde{a}_X\}.$$

In the case of first order quantum objects we additionally require that $\tilde{a}_X \in int(M_n^+)$, similarly for classical first order objects. Note that first order objects, resp. their duals, are exactly those satisfying condition (iii), resp. condition (ii), in Proposition prop:noncompact 1, in particular, $(X \otimes Y)^* = X^* \otimes Y^*$ for first order objects X and Y.

For a first order object $X = (V_X, \{\tilde{a}_X\}^{\sim})$, let us pick an element $a_X \in A_X$, then we have a direct sum decomposition

$$V_X = L_{X,0} \oplus L_{X,1},$$

where $L_{X,0} := \mathbb{R}\{a_X\}$, $L_{X,1} := \{\tilde{a}_X\}^{\perp} = L_X$. We also define the *conjugate object* $\tilde{X} = (V_X^*, \{a_X\}^{\sim})$, note that we always have $\tilde{a}_X \in A_{\tilde{X}}$ and with the choice $a_{\tilde{X}} = \tilde{a}_X$, we have $\tilde{\tilde{X}} = X$ and

$$L_{\tilde{X},u} = L_{X,\bar{u}}^{\perp}, \qquad u \in \{0,1\}. \tag{6}$$

eq:compl

These definitions depend on the choice of a_X, but we will assume below that this choice is fixed and that we choose $a_{\tilde{X}} = \tilde{a}_X$. For quantum objects we will assume that $a_X \in int(M_n^+)$.

Example 5. Example: quantum states, multiple of identity...

Higher order objects are those obtained from a finite set $\{X_1, \dots, X_n\}$ of first order objects by taking tensor products and duals, and applying any permuations of the spaces. The above is indeed a set, so that all the objects are different (though they may be isomorphic) and the ordering is not essential. We will also assume that the tensor unit is not contained in this set. Of course, any first order object is also higher order with $n = 1$. Note that we cannot say that such an object is automatically "of order n", as the following lemma shows.

ertensor **Lemma 3.** *Let X, Y be first order, then $X \otimes Y$ is first order as well.*

Proof. We have

$$S_{X\otimes Y} = S_X \otimes S_Y = V_X \otimes V_Y = V_{X\otimes Y}.$$

□

Example 6. (states quantum first order, channels, supermaps - quantum higher order)

Example 7. replacement $X^* \otimes Y$, quantum

1.4 Description of higher order objects

We start by noticing that there are certain objects in Af that can be constructed from a set of first order objects and functions in $\mathcal{F}_n$.

Let $X_1, \dots, X_n$ be first order objects in Af. Let $a_{X_i} \in A_{X_i}$ be fixed and let $\tilde{X}_i$ be the conjugate first order objects. Let us denote $V_i = V_{X_i}$ and

$$L_{i,u} := L_{X_i,u}, \qquad \tilde{L}_{i,u} := L_{\tilde{X}_i,u} \qquad u \in \{0,1\},\ i \in [n].$$

For $s \in \{0,1\}^n$, we define

$$L_s := L_{1,s_1} \otimes \cdots \otimes L_{n,s_n}, \qquad \tilde{L}_s := \tilde{L}_{1,s_1} \otimes \cdots \otimes \tilde{L}_{n,s_n},$$

then we have the direct sum decompositions

$$V := V_1 \otimes \cdots \otimes V_n = \bigoplus_{s\in\{0,1\}^n} L_s, \qquad V^* = V_1^* \otimes \cdots \otimes V_n^* = \bigoplus_{s\in\{0,1\}^n} \tilde{L}_s.$$

ma:Lperp **Lemma 4.** *For any $s \in \{0,1\}^n$, we have*

$$L_s^\perp = \bigoplus_{t\in\{0,1\}^n} \bar{\chi}_s(t)\tilde{L}_t, \qquad \tilde{L}_s^\perp = \bigoplus_{t\in\{0,1\}^n} \bar{\chi}_s(t)L_t.$$

Here $\chi_s : \{0,1\}^n \to \{0,1\}$ is the characteristic function of s, $\bar{\chi}_s = 1 - \chi_s$.

Proof. Using (eq:complement 6) and the direct sum decomposition of V_i^*, we get

$$\begin{aligned}
(L_{1,s_1} \otimes \cdots \otimes L_{n,s_n})^\perp &= \bigvee_j \left(V_1^* \otimes \cdots \otimes V_{j-1}^* \otimes \tilde{L}_{j,\bar{s}_j} \otimes V_{j+1}^* \otimes \cdots \otimes V_n^*\right) \\
&= \bigvee_j \left(\bigoplus_{\substack{t\in\{0,1\}^n \\ t_j \neq s_j}} \tilde{L}_{1,t_1} \otimes \cdots \otimes \tilde{L}_{n,t_n} \right) \\
&= \bigoplus_{\substack{t\in\{0,1\}^n \\ t\neq s}} \left(\tilde{L}_{1,t_1} \otimes \cdots \otimes \tilde{L}_{n,t_n}\right).
\end{aligned}$$

The proof of the other equality is the same. $\square$

lemma:Xf **Lemma 5.** *Put $a := a_1 \otimes \cdots \otimes a_n$, $\tilde{a} := \tilde{a}_1 \otimes \cdots \otimes \tilde{a}_n$. For $f \in \mathcal{F}_n$ define*

$$S_f := \bigoplus_{s\in\{0,1\}^n} f(s)L_s, \qquad A_f = A_f(X_1, \dots, X_n) := S_f \cap \{\tilde{a}\}^\sim.$$

Then A_f is a proper affine subspace in V containing a. Moreover,

$$L_{A_f} = \bigoplus_{s\in\{0,1\}^n\setminus\{0\}} f(s)L_s, \qquad S_{A_f} = S_f$$

and the dual affine subspace satisfies

$$\tilde{A}_f(X_1, \dots, X_n) = A_{f^*}(\tilde{X}_1, \dots, \tilde{X}_n) = \bigoplus_{s\in\{0,1\}^n} f^*(s)\tilde{L}_s \cap \{a\}^\sim.$$

Proof. It is clear from definition that A_f is an affine subspace. Since $f(0) = 1$, the space S_f always contains the subspace $L_0 = L_{1,0} \otimes \cdots \otimes l_{n,0} = \mathbb{R}\{a\}$ and it is clear that $L_s \subseteq \{\tilde{a}\}^\perp$ for any $s \neq 0$. It follows that $a \in A_f$, so that $A_f \neq \emptyset$, and since $A_f \subseteq \{\tilde{a}\}^\sim$, we see that A_f is proper and $\tilde{a} \in \tilde{A}_f$. The expressions for L_{A_f} and S_{A_f} are immediate from the definition and $L_{A_f} = S_{A_f} \cap \{\tilde{a}\}^\sim$. To obtain the dual affine subspace, we compute using Lemma 4 and the fact that the subspaces form an independent decomposition,

$$S_{\tilde{A}_f} = L_{A_f}^{\perp} = \left(\bigoplus_{s \in \{0,1\}^n \setminus \{0\}} f(s) L_s \right)^{\perp} = \bigwedge_{\substack{s \in \{0,1\}^n \\ s \neq 0, f(s)=1}} L_s^{\perp} = \bigwedge_{\substack{s \in \{0,1\}^n \\ s \neq 0, f(s)=1}} \left(\bigoplus_{t \in \{0,1\}^n} \bar{\chi}_s(t) \tilde{L}_t \right)$$

$$= \bigoplus_{t \in \{0,1\}^n} \left(\bigwedge_{\substack{s \in \{0,1\}^n \\ s \neq 0, f(s)=1}} \bar{\chi}_s(t) \tilde{L}_t \right) = \bigoplus_{t \in \{0,1\}^n} f^*(t) \tilde{L}_t.$$

To see the last equality, note that

$$\bigwedge_{\substack{s \in \{0,1\}^n \\ s \neq 0, f(s)=1}} \bar{\chi}_s(t) = \begin{cases} 1 & \text{if } t = 0 \\ 1 - f(t) & \text{if } t \neq 0 \end{cases} = f^*(t).$$

□

Since L_s, $s \in \{0,1\}$ is an independent decomposition, the map $f \mapsto S_f$, and hence also $f \mapsto A_f$, is injective. This map has the following further properties, which are easily checked:

(i) For the bottom and top elements in $\mathcal{F}_n$ we have

$$A_{p_n} = \{a\}, \qquad A_{1_n} = \{\tilde{a}\}^{\sim},$$

(ii) We have $f \leq g$ if and only if $A_f \subseteq A_g$,

(iii) $A_{f \wedge g} = A_f \cap A_g$,

(iv) $A_{f \vee g} = A_f \vee A_g := \mathrm{Aff}(A_f \cup A_g)$.

It follows that the set $\{A_f,\ f \in \mathcal{F}_n\}$ is a distributive lattice, with respect to the lattice operations $\cap$ and $\vee$.

Since all the affince subspaces $A_f \subseteq V$ are proper, there are objects $X_f := (V, A_f)$ in Af. The above relations can be rephrased as follows:

(i) $X_{1_n} = (V, \{\tilde{a}\}^{\sim})$ is a first order object, $X_{p_n} = (V^*, \{a\}^{\sim})^{\sim}$ is a dual first order object.

(ii) We have $f \leq g$ if and only if id_V is a morphism $X_f \to X_g$ in Af,

(iii) Let $f, g \leq h$. The following is a pullback diagram:

$$\begin{tikzcd} X_{f \wedge g} \arrow[r, "id_V"] \arrow[d, "id_V"'] & X_f \arrow[d, "id_V"] \\ X_g \arrow[r, "id_V"'] & X_h \end{tikzcd}$$

(iv) Let $h \leq f, g$. The following is a pushout diagram:

$$\begin{tikzcd} X_h \arrow[r, "id_V"] \arrow[d, "id_V"'] & X_f \arrow[d, "id_V"] \\ X_g \arrow[r, "id_V"'] & X_{f\vee g} \end{tikzcd}$$

In particular, it follows that $\{X_f,\ f \in \mathcal{F}_n\}$, is a distributive lattice, with pullbacks and pushouts as lattice operations. Furthermore, using the conjugate objects, we may construct

$$\tilde{X}_f := (V^*, A_f(\tilde{X}_1, \dots, \tilde{X}_n))$$

and we see from Lemma lemma:Xf 5 that

$$X_f^* = \tilde{X}_{f^*}, \qquad f \in \mathcal{F}_n. \tag{7}$$ eq:duali

We next observe that the higher order objects are of the form X_f, for some choice of the first order objects $X_1, \dots, X_n$ and a function f that belongs to a special subclass of $\mathcal{F}_n$. So assume that Y is a higher order object constructed from a set of distinct first order objects $Y_1, \dots, Y_n$, $Y_i = (V_{Y_i}, \{\tilde{a}_{Y_i}\}^\sim)$, we will write $Y \sim \{Y_1, \dots, Y_n\}$ in this case. Let us fix elements $a_{Y_i} \in A_{Y_i}$ and construct the objects $\tilde{Y}_i$.

By compactness of FinVect, we may assume (relabeling the objects if necessary) that the vector space of Y has the form

$$V_Y = V := V_1 \otimes \cdots \otimes V_n,$$

where V_i is either V_{Y_i} or $V_{Y_i}^*$, according to whether Y_i was subjected to taking duals an even or odd number of times. The indices such that the first case is true will be called the outputs and the subset of outputs in $[n]$ will be denoted by O, or O_Y, when we need to specify the object. The set $I = I_Y := [n] \setminus O_Y$ is the set of inputs. The reason for this terminology will become clear later.

:boolean **Proposition 4.** *For $i \in [n]$, let $X_i = Y_i$ if $i \in O_Y$ and $X_i = \tilde{Y}_i$ for $i \in I_Y$. There is a unique function $f \in \mathcal{F}_n$ such that*

$$Y = X_f = (V, A_f(X_1, \dots, X_n)).$$

Proof. Since the map $f \mapsto X_f$ is injective, uniqueness is clear. To show existence of this function, we will proceed by induction on n. For $n = 1$, the assertion is easily seen to be true, since in this case, we we have either $Y = Y_1$ or $Y = Y_1^*$. In the first case, $O = [1]$, $X_1 = Y_1$ and

$$S_Y = V_Y = V_1 = L_{1,0} \oplus L_{1,1} = 1(0)L_{1,0} \oplus 1(1)L_{1,1},$$

so in this case f is the constant 1. If $Y = Y_1^*$, we have $O = \emptyset$, $X_1 = \tilde{Y}_1$, and then

$$S_Y = \mathbb{R}\{\tilde{a}_{Y_1}\} = L_{1,0} = 1^*(0)L_{1,0} \oplus 1^*(1)L_{1,1},$$

so that $f = 1^*$.

Assume now that the assertion is true for all $m < n$. By construction, Y is either the tensor product $Y = Z_1 \otimes Z_2$, with

$$Z_1 \sim \{Y_1, \dots, Y_m\}, \qquad Z_2 \sim \{Y_{m+1}, \dots, Y_n\},$$

or Y is the dual of such a product. Let us assume the first case. It is clear that $O_{Z_1} \cup O_{Z_2} = O_Y$, and similarly for I, so that the corresponding objects $X_1, \dots, X_m$ and $X_{m+1}, \dots, X_n$ remain the same. By the induction assumption, there are functions $f_1 \in \mathcal{F}_m$ and $f_2 \in \mathcal{F}_{n-m}$ such that

$$S_Y = S_{Z_1} \otimes S_{Z_2} = \bigoplus_{\substack{s \in \{0,1\}^m \\ t \in \{0,1\}^{n-m}}} f_1(s) f_2(t) L_{1,s_1} \otimes \cdots \otimes L_{m,s_m} \otimes L_{m+1,t_1} \otimes \cdots \otimes L_{n,t_{n-m}}.$$

This implies the assertion, with $f = f_1 \otimes f_2$.

To finish the proof, it is now enough to observe that if the assertion holds for Y then it also holds for Y^*. So assume that $Y = X_f = (V, A_f(X_1, \dots, X_n))$ for some $f \in \mathcal{F}_n$, then $Y^* = X_f^* = \tilde{X}_{f^*} = (V^*, A_{f^*}(\tilde{X}_1, \dots, \tilde{X}_n))$. In is now enough to notice that $\tilde{X}_i = \tilde{\tilde{Y}}_i = Y_i$ if $i \in I_Y$ and $\tilde{X}_i = \tilde{Y}_i$ if $i \in O_Y$. Since by definition $O_{Y^*} = I_Y$, this proves the statement.

□

Let us stress that in general, the objects X_f depend on the choice of the elements a_{X_i}. From the above proof, it is clear that the description in Proposition prop:boolean 4 does not depend on the choice of the elements $a_{Y_i} \in A_{Y_i}$.

1.5 Type funtions and higher order objects

Let $\mathcal{T}_n \subseteq \mathcal{F}_n$ be defined as the subset generated from the constant function 1 on $\{0,1\}$ by taking duals and tensor products. For example, we have

$$\mathcal{T}_1 = \mathcal{F}_1 = \{1, 1^*\}, \quad \mathcal{T}_2 = \{1 \otimes 1, (1 \otimes 1)^*, 1 \otimes 1^*, 1^* \otimes 1, (1^* \otimes 1)^*, (1 \otimes 1^*)^*\},$$

etc. Elements of $\mathcal{T}_n$ will be called *type functions*. Similarly as for the higher order objects, the indexes in $[n]$ such that the corresponding component was subjected to taking the dual an even number of times will be called the outputs (of f) and denoted by $O = O_f$, indexes in $I = I_f := [n] \setminus O_f$ will be called inputs. From the proof of Proposition prop:boolean 4, it is easily seen that a higher order object is of the form $Y = X_f$ for a function $f \in \mathcal{T}_n$ with the same outputs (and of course also inputs) as Y. We next show that the converse is true.

type_hom **Proposition 5.** *Let $\{X_1, \dots, X_n\}$ be first order objects and let $f \in \mathcal{T}_n$. Then $Y = X_f$ is a higher order object with $O_Y = O_f$ and $Y \sim \{Y_1, \dots, Y_n\}$, where $Y_i = X_i$ for $i \in O_f$ and $Y_i = \tilde{X}_i$ for $i \in I_f$.*

Proof. As before, we will proceed by induction on n. For $n = 1$, we only have the possibilities $f = 1$ or $f = 1^*$. In the first case, $O = [1]$ and we get

$$S_f = 1L_{1,0} \oplus 1L_{1,1} = V_1,$$

so that $X_f = (V_1, \{\tilde{a}_1\}^\sim) = X_1$. In the second case, $O = \emptyset$ and

$$S_f = 1L_{1,0} = \mathbb{R}\{a_1\},$$

so that $X_f = (V_1, \{a_1\}) = \tilde{X}_1^*$. Assume next that the statement is true for all $m < n$ and assume that $f = f_1 \otimes f_2$ for some $f_1 \in \mathcal{F}_m$, $f_2 \in \mathcal{F}_{n-m}$, then it is easily seen that $Y = Z_1 \otimes Z_2$ for $Z_1 = X_{f_1}$ and $Z_2 = X_{f_2}$, constructed from $\{X_1, \dots, X_m\}$ resp. $\{X_{m+1}, \dots, X_n\}$. By the induction assumption, Z_1 and Z_2 are higher order objects, with $O_{Z_i} = O_{f_i}$, it follows that Y is a higher order object with $O_Y = O_{Z_1} \cup O_{Z_2} = O_{f_1} \cup O_{f_2} = O_f$.

Finally, assume that the statement is true for $f \in \mathcal{F}_n$, we will show that it holds for f^*. From eq:dualityXf (7), we see that $X_{f^*} = \tilde{X}_f$, which shows that $X_{f^*} \sim \{\tilde{Y}_1, \dots, \tilde{Y}_n\}$. Since taking duals will switch inputs and outputs, this finishes the proof.

□

Let $\{Y_1, \dots, Y_n\}$ be first order objects. The above results show that any of higher order object $Y \sim \{Y_1, \dots, Y_n\}$ with fixed set of outputs $O_Y = O$ satisfies $Y \simeq X_f$ for a unique type function $f \in \mathcal{T}_n$, $O_f = O$, and a fixed set of objects $\{X_1, \dots, X_n\}$, where the isomorphism is given by the action of some permutation in S_n on the space $V_1 \otimes \cdots \otimes V_n$. Conversely, any object of this form has the above properties. A basic example of such a function is (see Section sec:boolean ??)

$$p_I(s) = \Pi_{i \in I} \bar{s}_i = \otimes_{i \in I} 1^*(s_i), \qquad s \in \{0,1\}^n,$$

where $I = [n] \setminus O$. Clearly, $p_I \in \mathcal{T}_n$ and the set of outputs of p_I is O. It is easy to see that $X_{p_I} \simeq Y_I^* \otimes Y_O$, where $Y_I = \otimes_{i \in I} Y_i$ and $Y_O = \otimes_{i \in O} Y_i$ are first order objects, so that the corresponding higher order object can be identified with the set of replacement channels $Y_I \to Y_O$. Similarly, the function $p_O^* \in \mathcal{T}_n$ has output set O and $X_{p_O^*} \simeq (Y_I \otimes Y_O^*)^* = [Y_I, Y_O]$, the set of all channels $Y_I \to Y_O$.

_setting **Lemma 6.** *Let $f \in \mathcal{T}_n$ and let $O_f = O$, $I = I_f$. Then*

$$p_I \leq f \leq p_O^*.$$

Proof. This is obviously true for $n = 1$. Indeed, in this case, $\mathcal{T}_1 = \mathcal{F}_1 = \{1 = p_\emptyset, 1^* = p_{[1]}\}$. If $f = 1$, then $O = [1]$, $I = \emptyset$ and

$$p_I = p_\emptyset = 1 = p_O^*,$$

the case $f = 1^*$ is obtained by taking complements. Assume that the assertion holds for $m < n$. Let $f \in \mathcal{T}_n$ and assume that $f = g \otimes h$ for some $g \in \mathcal{T}_m$, $h \in \mathcal{T}_{n-m}$. By the assumption,

$$p_{I_g} \otimes p_{I_h} \leq g \otimes h \leq p_{O_g}^* \otimes p_{O_h}^* \leq (p_{O_g} \otimes p_{O_f})^*,$$

the last inequality follows from Lemma lemma:fproduct 9. We have $O_f = O_g \cup (m + O_h)$, $I_f = I_g \cup (m + I_h)$, so that $p_{O_f} = p_{O_g} \otimes p_{O_h}$ and similarly for p_{I_f}. Now notice that any $f \in \mathcal{T}_n$ is either of the form $(f \otimes g) \circ \sigma$ or of the form $(f \otimes g)^* \circ \sigma$, for some permutation σ. Since the inequality is easily seen to be preserved by permutations, and reversed by duality which also swiches the input and output sets, the assertion is proved.

□

Combining this with the remarks below Lemma lemma:Xf 5, we get the following result (cf. cite). Recall that a bimorphism in a category is a morphism $X \xrightarrow{\varphi} Y$ which is both mono and epi, that is, such that for any pairs of arrows (with appropriate sources and targets) we have $\psi \circ f = \psi \circ g \iff f = g$ and $k \circ \psi = l \circ \psi \iff k = l$. It can be shown that the bimorphisms in Af are precisely those morphisms that are given by isomorphisms in FinVect.

:setting **Theorem 2.** *Let $Y \sim \{Y_1, \dots, Y_n\}$ be such that $O_Y = O$, $I_Y = I$. Then there exist bimorphisms*

$$Y_I^* \otimes Y_O \xrightarrow{\varphi} Y \xrightarrow{\psi} [Y_I, Y_O].$$

The bimorphisms are given by permutations.

We will see below that $\mathcal{T}_n$ is not a lattice for $n \geq 2$, so that for $f_1, f_2 \in \mathcal{T}_n$, neither of $f_1 \wedge f_2$ or $f_1 \vee f_2$ has to be a type function. Nevertheless, we have by the above results that if $O_{f_1} = O_{f_2}$,

$$Y_I^* \otimes Y_0 \xrightarrow{\varphi} X_{f_1 \wedge f_2} \xrightarrow{id_V} X_{f_1 \vee f_2} \xrightarrow{\psi} [Y_I, Y_O]$$

for some suitable bimorphisms φ, ψ, moreover, the objects $X_{f_1 \wedge f_2}$ and $X_{f_1 \vee f_2}$ are obtained as a pullback resp. pushout. It follows that although these objects may not be higher order objects themselves, they are included in some higher order object (e.g. $[Y_I, Y_O]$) with the same sets of inputs and outputs.

We finish this section by showing a simple way to obtain the output set of a type function.

`_outputs` **Proposition 6.** *For $f \in \mathcal{T}_n$, $i \in O_f$ if and only if $f(e^i) = 1$.*

Proof. Let $i \in O_f$, then by Lemma 6, $p_{I_f}(e^i) = 1 \leq f(e^i)$, so that $f(e^i) = 1$. Conversely, if $f(e^i) = 1$, then by the other inequality in lemma 6, $p_{O_f}(e^i) = 0$, whence $i \in O_f$.

□

2 Characterizations of type functions

We have the following description of the sets of type functions.

`type_min` **Proposition 7.** *The set $\mathcal{T}_n$ is the smallest subset in $\mathcal{F}_n$ such that:*

1. *$\mathcal{T}_n$ is invariant under permutations: if $f \in \mathcal{T}_n$, then $f \circ \sigma \in \mathcal{T}_n$ for any permutation $\sigma \in S_n$,*
2. *$\mathcal{T}_n$ is invariant under taking duals: if $f \in \mathcal{T}_n$ then $f^* \in \mathcal{T}_n$,*
3. *$\mathcal{T}_m \otimes \mathcal{T}_n \subseteq \mathcal{T}_{m+n}$,*
4. *$\mathcal{T}_1 = \{1, p_1\} = \mathcal{F}_1$.*

Proof. It is clear by construction that any system of subsets $\{\mathcal{S}_n\}_n$ with these properties must contain the type functions and that $\{\mathcal{T}_n\}_n$ itself has these properties.

□

Our goal is to find some characterization of the type functions. We start by looking at some examples and non-examples.

`exm:T2` *Example* 8. The type functions for $n = 2$ are given as

$$s \quad \mapsto \quad 1, \quad \bar{s}_1\bar{s}_2, \quad \bar{s}_1, \quad 1 - \bar{s}_1 + \bar{s}_1\bar{s}_2,$$

and functions obtained from these by exchanging $s_1 \leftrightarrow s_2$, which gives 6 elements. It can be seen that $\mathcal{F}_n$ has 2^{2^n-1} elements, so that $\mathcal{F}_2$ has 8 elements in total. The two of them that are not type functions are

$$g(s) = 1 - \bar{s}_1 - \bar{s}_2 + 2\bar{s}_1 s_2, \qquad g^*(s) = \bar{s}_1 + \bar{s}_2 - \bar{s}_1\bar{s}_2.$$

This can be checked directly from Propositions 6 and 6. Indeed, if $g \in \mathcal{T}_2$, we would have $O_g = \emptyset$, so that $p_2 \leq g \leq p_\emptyset^* = p_2$, which is obviously not the case. Clearly, also the complement $g^* \notin \mathcal{T}_2$. Notice also that $g^* = p_{\{1\}} \vee p_{\{2\}}$, so that $\mathcal{T}_2$ is not a lattice. Since $\mathcal{F}_2$ can be identified as a sublattice in $\mathcal{F}_n$ for all $n \geq 2$ as $\mathcal{F}_2 \ni f \mapsto f \otimes 1_{n-2} \in \mathcal{F}_n$, we see that $\mathcal{T}_n$, $n \geq 2$ is a subposet in the distributive lattice $\mathcal{F}_n$ but itself not a lattice.

2.1 The poset $\mathcal{P}_f$

It will be convenient to use the representation

$$f = \sum_{S \subseteq [n]} \hat{f}_S p_S,$$

of a boolean function f obtained in Theorem 4. Let $\mathcal{P}_f$ be the subposet in $\mathcal{L}_n$ of elements such that $\hat{f}_S \neq 0$. We will show that if f is a type function, it is fully determined by $\mathcal{P}_f$.

We say that a poset $\mathcal{P}$ is graded of rank k if every maximal chain in $\mathcal{P}$ has the same length equal to k (recall that the length of a chain is defined as number of its elements -1). Equivalently, there is a unique rank function $\rho : \mathcal{P} \to \{0, 1, \dots, k\}$ such that $\rho(S) = 0$ if S is a minimal element of $\mathcal{P}$ and $\rho(T) = \rho(S) + 1$ if T covers S, that is, $S \leq T$ and for any R such that $S \leq R \leq T$ we have $R = T$ or $R = S$.

Proposition 8. *Let $f \in \mathcal{T}_n$, then $\mathcal{P}_f$ is a graded poset with even rank $k \leq n$. If ρ is the rank function, then we have*

$$f = \sum_{S \in \mathcal{P}_f} (-1)^{\rho(S)} p_S.$$

See [Stanley] for details.

Then rank of $\mathcal{P}_f$ will be denoted by $r(f)$ and called the rank of f. Note that the assertion means that for $f \in \mathcal{T}_n$,

$$\hat{f}_S = \begin{cases} (-1)^{\rho(S)}, & \text{if } S \in \mathcal{P}_f \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We first note that the property in the statement is invariant under permutations and complements. Assume the statement holds for f and let us take any $\sigma \in \mathscr{S}_n$. From Proposition 13 that $\hat{f \circ \sigma}_S = \hat{f}_{\sigma(S)}$ so that $S \mapsto \sigma(S)$ is an isomprphism of $\mathcal{P}_{f\circ\sigma}$ onto $\mathcal{P}_f$. Hence if $\mathcal{P}_f$ is graded with rank function ρ, then $\mathcal{P}_{f\circ\sigma}$ is graded with the same rank and has rank function $\rho \circ \sigma$. By the assumption we have

$$f \circ \sigma = \sum_{S \in \mathcal{P}_f} (-1)^{\rho(S)} p_S \circ \sigma = \sum_{S \in \mathcal{P}_f} (-1)^{\rho(S)} p_{\sigma^{-1}(S)} = \sum_{S \in \mathcal{P}_{f\circ\sigma}} (-1)^{\rho\circ\sigma(S)} p_S.$$

For the complement, we have from the assumption and Proposition 13(ii) that

$$f^* = (1 - \hat{f}_\emptyset)1 - \sum_{\substack{S \in \mathcal{P}_f \\ \emptyset \neq S, [n] \neq S}} (-1)^{\rho(S)} p_S + (1 - \hat{f}_{[n]}) p_n. \tag{8}$$

If $\emptyset \in \mathcal{P}_f$, then $\emptyset$ is the least element of $\mathcal{P}_f$, so that $\rho(\emptyset) = 0$ and therefore $\hat{f}_\emptyset = (-1)^0 = 1$. Similarly, if $[n] \in \mathcal{P}_f$, then $[n]$ is the largest element in $\mathcal{P}_f$, hence it is the last element in any maximal chain. It follows that $\rho([n]) = k$ and hence $\hat{f}_{[n]} = (-1)^k = 1$ (since k is even). Therefore the equality (8) implies that $\mathcal{P}_{f^*}$ differs from $\mathcal{P}_f$ only in the bottom and top elements: $\emptyset \in \mathcal{P}_f$ iff $\emptyset \notin \mathcal{P}_{f^*}$ and $p_n \in \mathcal{P}_f$ iff $p_n \notin \mathcal{P}_{f^*}$. It follows that $\mathcal{P}_{f^*}$ is graded as well, with rank equal to $k - 2$, k or $k + 2$, which in any case is even. Furthermore, let ρ^* be the rank function of f^*, then this also implies that for all $S \in \mathcal{P}_f$, $S \notin \{\emptyset, [n]\}$, we have $\rho^*(S) = \rho(S) \pm 1$, according to whether $\emptyset$ was added or removed. The statement now follows from (8).

We now proceed by induction on n. For $n=1$, we have $\mathcal{L}_1 = \{\emptyset, [1]\}$ and $\mathcal{T}_1 = \{1, 1^*\}$. For $f=1$, $\mathcal{P}_f = \{\emptyset\}$ is a singleton, which is clearly a graded poset, with rank $k=0$ and trivial rank function ρ. We have

$$f = 1 = p_\emptyset = (-1)^{\rho(\emptyset)} p_\emptyset.$$

The proof for $f = 1^*$ is similar, replacing $\emptyset$ by $[1]$.

To finish the proof, assume that the statement is true for $m < n$ and let $f \in \mathcal{T}_n$. Then f is either a permutation of a product of some $f_1 \in \mathcal{F}_m$ and $f_2 \in \mathcal{T}_{n-m}$, or a dual of such an element. By the first part of the proof, we only need to prove that the statement holds for $f = f_1 \otimes f_2$. But in this case, by the induction assumption, $\mathcal{P}_{f_i}$ is graded with even rank k_i and rank function ρ_i. We also have

$$f = f_1 \otimes f_2 = \sum_{S \subseteq [m], T \subseteq [n-m]} (\widehat{f}_1)_S (\widehat{f}_2)_T p_S p_T = \sum_{S \subseteq [m], T \subseteq [n-m]} (-1)^{\rho_1(S)+\rho_2(T)} p_{S \dot\cup T}.$$

It follows that $\mathcal{P}_f = \mathcal{P}_{f_1} \times \mathcal{P}_{f_2}$ is the product of the two posets, which is a graded poset with rank $k = k_1 + k_2$ and rank function $\rho = \rho_1 + \rho_2$. This proves the statement.

$\square$

We next show that the input and output sets of $f \in \mathcal{T}_n$ can be obtained from $\mathcal{P}_f$. For an index $i \in [n]$, let $M_{f,i}$ be the set of minimal elements of the subposet $\{S \in \mathcal{P}_f,\ i \in S\}$. Note that $M_{f,i}$ may be empty.

:pfinput **Proposition 9.** *Let $f \in \mathcal{T}_n$ and $i \in [n]$. Then*

1. *If $M_{f,i} \neq \emptyset$, then all elements in $M_{f,i}$ have the same rank, which will be denoted by $r_f(i)$. If $M_{f,i} = \emptyset$, we put $r_f(i) := r(f) + 1$.*

2. *$i \in O_f$ if and only if $r_f(i)$ is odd.*

Proof. Since $\mathcal{P}_f \simeq \mathcal{P}_{f\circ\sigma}$, it is quite clear that the two properties are preserved by permutations. We will show that they are preserved by complementation. Observe first that $M_{f,i} = \emptyset$ if and only if $M_{f^*,i} = \{[n]\}$, since $\mathcal{P}_{f^*}$ differs from $\mathcal{P}_f$ only up to adding/removing the least element $\emptyset$ and the greatest element $[n]$. If $M_{f,i}$ is empty, then $p_S(e^i) = 1$ for all $S \in \mathcal{P}_f$, so that $f(e^i) = f(0) = 1$ and $i \in O_f$, we also see that $r_f(i) = r(f) + 1$ is odd. If $\mathcal{P}_{f,i} = [n]$, then $r_f(i) = \rho_f([n]) = r(f)$ by definition of the rank, hence $r_f(i)$ is even. As we have seen, $i \in O_{f^*} = I_f$.

Let us assume that $M_{f,i}$ is not equal to $\emptyset$ or $\{[n]\}$. Then we must have $M_{f,i} = M_{f^*,i}$ and by the proof of Proposition prop:graded 8 we have $\rho_{f^*}(S) = \rho_f(S) \pm 1$ for any S, depending only on the fact whether $\emptyset \in \mathcal{P}_f$. This implies that the properties are preserved by complementation.

We will now proceed by induction on n as before. Both assertions are quite trivial for $n=1$, so assume the statements hold for $m < n$. It is enough to assume that $f = g \otimes h$ for some $g \in \mathcal{T}_m$ and $h \in \mathcal{T}_{n-m}$. Suppose without loss of generality that $i \in [m]$, then all elements of $M_{f,i}$ have the form $S \dot\cup T$, with $S \in M_{g,i}$ and T a minimal element in $\mathcal{P}_h$. Since $\rho_h(T) = 0$ for any minimal element $T \in \mathcal{P}_h$, we have by the induction assumption

$$\rho_f(S \dot\cup T) = \rho_g(S) + \rho_h(T) = \rho_g(S) = r_g(i).$$

The statement (ii) follows from the fact that $i \in O_f$ if and only if $i \in O_g$.

$\square$

oro:free **Corollary 1.** *We have $\cap \mathcal{P}_f \in I_f$, $[n] \setminus \cup \mathcal{P}_f \in O_f$.*

Proof. If $i \in \cap \mathcal{P}_f$, then clearly $M_{f,i}$ is the set of minimal elements in $\mathcal{P}_f$, so that $r_f(i) = 0$ and $i \in I_f$ by Proposition 9. If $i \notin S$ for any $S \in \mathcal{P}_f$, then $M_{f,i} = \emptyset$ and $r_f(i) = r(f) + 1$ is odd. Hence $i \in O_f$.

□

Let us denote $F_{f,in} := \cap \mathcal{P}_f$, $F_{f,out} := [n] \setminus \cup \mathcal{P}_f$. Elements of these sets will be called free inputs resp. outputs. It is easily seen that we have

$$f = p_{F_{f,in}} \otimes g \otimes 1_{F_{f,out}}$$

for some type function g with no free inputs or outputs.

Examples/NOnexamples?

Obrazky?

2.2 Chains and combs

Let $\mathcal{P} = \{S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_N\}$ be a chain in $\mathcal{L}_n$. Then $\mathcal{P}$ is graded with rank $N-1$ and rank function $\rho(S_i) = i - 1$.

Proposition 10. *For a chain $\mathcal{P} = \{S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_N\}$, the function*

$$f = f_{\mathcal{P}} := \sum_i (-1)^{i-1} p_{S_i}$$

is a type function if and only if N odd.

Proof. By Proposition 8, if $f \in \mathcal{T}_n$, then the rank of f must be even, so that N must be odd. We will show that the converse is true. We proceed by induction on N. For $N = 1$, we have $f = p_{S_1} \in \mathcal{T}_n$. Assume that the statement holds for all odd numbers $M < N$ and let $\mathcal{P}$ be a chain as above. Then we have

$$f = p_{S_1} \otimes g \otimes 1_{[n] \setminus S_N}$$

where g is the function for the chain $\emptyset = S'_1 \subsetneq S'_2 \subsetneq \cdots \subsetneq S'_N$, with $S'_i := S_i \setminus S_1$. Since f is a type function if g is, this shows that we may assume that the chain contains $\emptyset$ and $[n]$. But then

$$f = 1 + \sum_{j=2}^{N-1} (-1)^{j-1} p_{S_j} + p_n$$

and

$$f^* = 1 - f + p_n = \sum_{j=1}^{N-2} (-1)^{j-1} p_{T_j},$$

where $T_j := S_{j+1}$. By the induction assumption $f^* \in \mathcal{T}_n$, hence also $f = f^{**} \in \mathcal{T}_n$. □

As we can see from Example 8, all elements in $\mathcal{T}_2$ are chains. This is also true for $n = 3$. Indeed, up to a permutation that does not change the chain structure, any $f \in \mathcal{T}_3$ is either a product of two elements $g \in \mathcal{T}_2$ and $h \in \mathcal{T}_1$, or the dual of such a product. Since g must be a chain and $|\mathcal{P}_h| = 1$, their product must be a chain as well. Taking the dual of a chain only adds/removes the least/largest elements, so the dual of a chain must be a chain as well.

We now show that chains correspond to important higher order objects. Let k be odd and let $\mathcal{P} = \{\emptyset \subsetneq S_1 \subsetneq \cdots \subsetneq S_k \subsetneq [n]\}$. Let $f = f_{\mathcal{P}}$, then f is a type function. By Proposition 5, for any first order objects $X_1, \ldots, X_n$, $Y = X_f$ is a higher order object such that $Y \sim \{Y_1, \ldots, Y_n\}$.

ns_combs **Proposition 11.** *Let $T_1 = S_1$, $T_i = S_i \setminus S_{i-1}$ for $i = 1, \dots, k$ and $T_{k+1} = S_k^c$. For $S \subseteq [n]$, we denote $Y_S := \otimes_{i \in S} Y_i$. Then for $k = 1$, $Y \simeq [Y_{T_2}, Y_{T_1}]$ and for any odd $k > 1$,*

$$Y \simeq [Y_{T_{k+1}}, [[Y_{T_k}, [[....], Y_{T_2}]], Y_{T_1}]].$$

Remark (quantum) comb (examples)

Proof. It is easily checked by Proposition prop:fh_outputs 6 that $T_i \subseteq O_f$ if i is odd and $T_i \subseteq I_f$ otherwise. We therefore have

$$Y_{T_i} = \begin{cases} \otimes_{i \in T_i} X_i, & \text{if } i \text{ is odd,} \\ \otimes_{i \in T_i} \tilde{X}_i, & \text{if } i \text{ is even.} \end{cases}$$

Let $k = 1$, then $f = 1 - p_{T_1} + p_n$, and $f^* = p_{T_1} = p_{T_1} \otimes 1_{T_2}$. Then $X_f = \tilde{X}^*_{f^*}$, and we see that $\tilde{X}_{f^*} = X^*_{T_1} \otimes \tilde{X}_{T_2} = Y^*_{T_1} \otimes Y_{T_2}$. It follows that $X_f = (Y^*_{T_1} \otimes Y_{T_2})^* \simeq [Y_{T_2}, Y_{T_1}]$, where the isomorphism is given by swapping the spaces $V^*_{T_1}$ and V_{T_2}. Assume the assertion is true for $k - 2$. As in the proof of Proposition prop:chains 10, we see that

$$f^* = \sum_{i=1}^{k} (-1)^{i-1} p_{S_i} = p_{T_1} \otimes g \otimes 1_{T_{k+1}}$$

where $g = 1 - \sum_{i=2}^{k-1} (-1)^{i-1} p_{S'_i} + p_{S'_k}$ is the function for the chain $\mathcal{P}' = \{\emptyset \subsetneq S'_2 \subsetneq \cdots \subsetneq S'_k\}$ in $S'_k \simeq [n']$ for $n' = |S'_k|$, $S'_i = S_i \setminus S_1 = \cup_{j=2}^{i} T_j$. We have

$$X_f = \tilde{X}^*_{f^*} = (X^*_{Y_1} \otimes \tilde{X}_g \otimes \tilde{X}_{T_{k+1}})^* \simeq (\tilde{X}_{T_{k+1}} \otimes [\tilde{X}_g, X_{T_1}]^*)^* = [\tilde{X}_{T_{k+1}}, [\tilde{X}_g, X_{T_1}]].$$

Here we have used the fact that $\tilde{X}_{f^*}$ is constructed from $\tilde{X}_1, \dots, \tilde{X}_n$. By induction assumption, we get

$$[\tilde{X}_{T_{k+1}}, [\tilde{X}_g, X_{T_1}]] = [\tilde{X}_{T_{k+1}}, [[X_{T_k}, [[....], \tilde{X}_{T_2}]], X_{T_1}],$$

which is as required.

$\square$

Combs, picture, without $\emptyset$ or $[n]$? Free inputs and outputs!

2.3 Connecting chains: the causal product

We will introduce further operations of boolean functions. We will fix a decomposition $\{0,1\}^{m+n} \simeq \{0,1\}^m \times \{0,1\}^n$. For $f_1 : \{0,1\}^m \to \{0,1\}$, $f_2 : \{0,1\}^n \to \{0,1\}$, we define their causal product as

$$f_1 \lhd f_2 := f_1 \otimes 1_n + p_m \otimes (f_2 - 1).$$

For any $s^1 \in \{0,1\}^m$ and $s^2 \in \{0,1\}^n$, this function acts as

$$(f_1 \lhd f_2)(s^1 s^2) = f_1(s^1) + p_m(s^1)(f_2(s^2) - 1) = \begin{cases} f_1(s^1), & \text{if } s^1 \neq 0_m, \\ f_2(s^2), & \text{if } s^1 = 0_m. \end{cases} \tag{9}$$ eq:causa

Remark 1. Causal: can be interpreted as "f_1 before f_2" (actually after).

The following properties are immediate from eq:causal_product (9).

Lemma 7. *Let $f_1 \in \mathcal{F}_m$, $f_2 \in \mathcal{F}_n$. Then $f_1 \lhd f_2 \in \mathcal{F}_{n+m}$ and we have for all $f_1, g_1 \in \mathcal{F}_m$ and $f_2, g_2 \in \mathcal{F}_n$,*

(i) $(f_1 \lhd f_2)^* = f_1^* \lhd f_2^*$,

(ii) $(f_1 \vee g_1) \lhd (f_2 \vee g_2) = (f_1 \lhd f_2) \vee (g_1 \lhd g_2)$,

(iii) $(f_1 \wedge g_1) \lhd (f_2 \wedge g_2) = (f_1 \lhd f_2) \wedge (g_1 \lhd g_2)$.

Moreover, for any $f_3 \in \mathcal{F}_k$, and for the decomposition $s = s^1 s^2 s^3$, we have

$$(f_1 \lhd f_2) \lhd f_3 = f_1 \lhd (f_2 \lhd f_3).$$

We can also combine f_1 and f_2 in the opposite order:

$$f_2 \lhd f_1 := 1_m \otimes f_2 + (f_1 - 1_n) \otimes p_n,$$

so that

$$(f_2 \lhd f_1)(s^1 s^2) = f_2(s^2) + p_n(s^2)(f_1(s^1) - 1) = \begin{cases} f_2(s^2), & \text{if } s^2 \neq 0_n, \\ f_1(s^1), & \text{if } s^2 = 0_n. \end{cases} \tag{10}$$

Of course, this product has similar properties as listed in the above lemma. To avoid any confusion, we have to bear in mind the fixed concatenation decomposition $s = s^1 s^2$ and that f_i acts on s^i.

Lemma 8. *In the situation as above, we have*

$$f_1 \otimes f_2 = (f_1 \rhd f_2) \wedge (f_2 \rhd f_1).$$

Proof. This is again by straightforward computation from (9) and (10): let $s^1 \in \{0,1\}^m$, $s^2 \in \{0,1\}^n$ and compute

$$\begin{aligned} (f_1 \lhd f_2) \wedge (f_2 \lhd f_1)(s^1 s^2) &= \big(f_1(s^1) + p_m(s^1)(f_2(s^2) - 1)\big)\big(f_2(s^2) + p_n(s^2)(f_1(s^1) - 1)\big) \\ &= f_1(s^1) f_2(s^2), \end{aligned}$$

the last equality follows from the fact that $f_i(s^i)(1 - f_i(s^i)) = 0$ (since $f_i(s^i) \in \{0,1\}$) and the fact that p_m is the least element in $\mathcal{F}_m$, so that $p_m(s^1)(f_1(s^1) - 1) = p_m(s^1) - p_m(s^1) = 0$. $\square$

Using the last part of Lemma 7, for a decomposition $s = s^1 \dots s^k$ and $f_i \in \mathcal{F}_{n_i}$, $\sum_i n_i = n$, acting on s^i, $i = 1, \dots, k$, we may define the function $f_1 \lhd \dots \lhd f_k \in \mathcal{F}_n$. Note that we have

$$\begin{aligned} (f_1 \lhd \dots \lhd f_k)(s) &= f_1(s^1) + p_{n_1}(s^1)(f_2(s^2) - 1) + \dots + p_{n_1}(s^1) \dots p_{n_{k-1}}(s^{k-1})(f_k(s^k) - 1) \\ &= \begin{cases} f_1(s_1) & \text{if } s^1 \neq 0_{n_1} \\ f_2(s^2) & \text{if } s^1 = 0_{n_1}, s^2 \neq 0_{n_2} \\ \dots & \\ f_k(s^k) & \text{if } s^1 = 0_{n_1}, \dots, s^{k-1} = 0_{n_{k-1}}. \end{cases} \end{aligned}$$

For any permutation $\pi \in \mathscr{S}_k$, we define $f_{\pi^{-1}(1)} \lhd \dots \lhd f_{\pi^{-1}(k)} \in \mathcal{F}_n$ in an obvious way.

It is not clear that if f_1 and f_2 are type functions, then $f_1 \lhd f_2$ or $f_2 \lhd f_1$ are type functions as well. Nevertheless, we next show that this is true for chains.

d_chains **Proposition 12.** *Let $\mathcal{P}_1 = \{S_1 \subsetneq \cdots \subsetneq S_M\}$ be a chain in $[m]$ and $\mathcal{P}_2 = \{T_1 \subsetneq \cdots \subsetneq T_N\}$ a chain in $[n]$, with corresponding functions β_1 and β_2. Assume that both M and N are odd, so that β_1 and β_2 are type functions. Then $\beta_1 \lhd \beta_2$ and $\beta_2 \lhd \beta_1$ are type functions corresponding to chains of $M + N \pm 1$ elements in $[m+n]$, with output and input indices given by $O_{\beta_1} \dot\cup O_{\beta_2}$ and $I_{\beta_1} \dot\cup I_{\beta_2}$.*

Proof. We have

$$\beta_1 = \sum_{j=1}^{M} (-1)^{j-1} p_{S_j}, \qquad \beta_2 = \sum_{k=1}^{N} (-1)^{k-1} p_{T_k},$$

so that

$$\beta := \beta_1 \lhd \beta_2 = \sum_{j=1}^{M-1} (-1)^{j-1} p_{S_j} + (p_{S_M} - p_{[m]} + p_{[m]\dot\cup T_1}) + \sum_{k=2}^{N} (-1)^{k-1} p_{[m]\dot\cup T_k}.$$

The resulting funnction depends on whether $S_M = [m]$ and $T_1 = \emptyset$. If at least one of the equalities is true, then the expression in brackets is equal to $p_{[m]}$, p_{S_M} or $p_{[m]\dot\cup T_1}$ and β corresponds to a chain of $M + N - 1$ elements. If both $S_M \neq [m]$ and $T_1 \neq \emptyset$, then $S_M \subsetneq [m] \subsetneq [m]\dot\cup T_1$ and β corresponds to a chain of $M + N + 1$ elements.

For any $i \in [m]\dot\cup[n]$, we have $e^i_{m+n} = e^j_m 0_n$ or $e^i_{m+n} = 0_m e^k_n$ for some $j \in [m]$, $k \in [n]$. Then

$$\beta(e^i) = \beta_1(e^j_m) \text{ or } \beta(e^i) = \beta_2(e^k_n).$$

The statement on input/output indices follow from Lemma lemma:fh_setting 6. The proof for $\beta_2 \lhd \beta_1$ is similar. ∎

tructure **Theorem 3.** *Let $f \in \mathcal{T}_n$. Then there is a prrmutation $\rho \in \mathscr{S}_n$, a decomposition $[n] = [n_1] \oplus [n_1 + 1, n_2] \oplus \cdots \oplus [n_{k-1} + 1, n_k]$, chains $\beta_1 \in \mathcal{T}_{n_1}, \dots, \beta_k \in \mathcal{T}_{n_k}$, $n = n_1 + \cdots + n_k$, finite index sets A, B and permutations $\pi_{a,b} \in \mathscr{S}_k$ such that*

$$f = \bigvee_{a \in A} \bigwedge_{b \in B} (\beta_{\pi^{-1}_{a,b}(1)} \lhd \ldots \lhd \beta_{\pi^{-1}_{a,b}(k)}) \circ \rho.$$

Moreover, there is a decomposition $[k] = \oplus_{j=1}^{l} [k_{j-1} + 1, k_j]$, permutations $\pi^j_a \in \mathscr{S}_{k_j}$ and $\tau_b \in \mathscr{S}_l$ such that either $\pi_{a,b}$ or $\pi_{b,a}$ have the form

$$t_1 \ldots t_k \mapsto \pi_a^{\tau_b^{-1}(1)}(t_{\tau_b^{-1}}(1)) \ldots \pi_a^{\tau_b^{-1}(l)}(t_{\tau_b^{-1}}(l))$$

for some finite index sets A and B, where $\pi_{a,b} \in \mathscr{S}_k$ and $\rho_{a,b} = \rho_{\pi_{a,b}} \circ \sigma$ for some $\sigma \in \mathscr{S}_n$ and $\rho_{\pi_{a,b}}(s^1 \ldots s^k) = s^{\pi^{-1}_{a,b}(1)} \ldots s^{\pi^{-1}_{a,b}(k)}$, $a \in A$, $b \in B$.

Proof. It is quite clear that the condition is invariant under permutations. Assume f can be written in the given form, then

$$f^* = \bigwedge_{a \in A} \bigvee_{b \in B} (\beta^*_{\pi^{-1}_{a,b}(1)} \lhd \ldots \lhd \beta^*_{\pi^{-1}_{a,b}(k)}) \circ \rho_{a,b}.$$

Since $\mathcal{F}_n$ is a distributive lattice, this can be rewritten as

$$f^* = \bigvee_{a^* \in B^{|A|}} \bigwedge_{b^* \in A} (\beta^*_{\pi^{-1}_{a^*,b^*}(1)} \lhd \ldots \lhd \beta^*_{\pi^{-1}_{a^*,b^*}(k)}) \circ \rho_{a^*,b^*},$$

where for $a^*=(b_a)_{a\in A}$ and $b^*=a$, we have $\pi_{a^*,b^*}=\pi_{a,b_a}$, and $\rho_{a^*,b^*}=\rho_{\pi_{a^*,b^*}}\circ\sigma$. Since β^*_j are chains in $[n_j]$, the assertion is true also for f.

Since for $n=1$, $f\in\mathcal{T}_n$ is itself a (trivial) chain, the assertion is true in this case (note that it is also trivially true for $n=2$ and $n=3$, since all elements in $\mathcal{T}_2$ and $\mathcal{T}_3$ are chains). Proceeding by induction, it is now enough to show this form for $f=f_1\otimes f_2$, where $f_1\in\mathcal{T}_m$, $f_2\in\mathcal{T}_{n-m}$ satisfy the conditions, so that

$$f_1=\bigvee_{a\in A}\bigwedge_{b\in B}(\beta^1_{\pi^{-1}_{a,b}(1)}\lhd\ldots\lhd\beta^1_{\pi^{-1}_{a,b}(k_1)})\circ\rho_{\pi_{a,b}}\circ\sigma_1,\qquad f_2=\bigvee_{c\in C}\bigwedge_{d\in D}(\beta^2_{\pi^{-1}_{c,d}(1)}\lhd\ldots\lhd\beta^2_{\pi^{-1}_{c,d}(k_2)})\circ\rho_{\pi_{c,d}}\circ\sigma_2.$$

for some chains $\beta^1_j\in\mathcal{T}_{m_j}$, $\sum_j m_j=m$, and $\beta^2_j\in\mathcal{T}_{l_j}$, $\sum_j l_j=n-m$ and permutations $\pi_{a,b}\in\mathscr{S}_{k_1}$, $\pi_{c,d}\in\mathscr{S}_{k_2}$, $\sigma_1\in\mathscr{S}_m$, $\sigma_2\in\mathscr{S}_{n-m}$. Using properties of the tensor product, we get

$$f_1\otimes f_2=\bigvee_{a\in A,c\in C}\bigwedge_{b\in B,d\in D}(\beta^1_{\pi^{-1}_{a,b}(1)}\lhd\ldots\lhd\beta^1_{\pi^{-1}_{a,b}(k_1)})\otimes(\beta^2_{\pi^{-1}_{c,d}(1)}\lhd\ldots\lhd\beta^2_{\pi^{-1}_{c,d}(k_2)})\circ(\rho_{\pi^1_{a,b}}\otimes\rho_{\pi^2_{c,d}})\circ(\sigma_1\otimes\sigma_2).$$

Let

$$\beta_1^{a,b}:=\beta^1_{\pi^{-1}_{a,b}(1)}\lhd\ldots\lhd\beta^1_{\pi^{-1}_{a,b}(k_1)},\qquad \beta_2^{c,d}:=\beta^2_{\pi^{-1}_{c,d}(1)}\lhd\ldots\lhd\beta^2_{\pi^{-1}_{c,d}(k_2)}.$$

Then $\beta_1^{a,b}\in\mathcal{T}_m$ and $\beta_2^{c,d}\in\mathcal{T}_{n-m}$ are chains and we have

$$\beta_1^{a,b}\otimes\beta_2^{c,d}=(\beta_1^{a,b}\lhd\beta_2^{c,d})\wedge(\beta^{c,d}_{\xi^{-1}(1)}\lhd\beta^{a,b}_{\xi^{-1}(2)})\circ\rho_\xi,$$

where ξ is the swap $1\leftrightarrow2$ and ρ_ξ acts on $[n]=[m]\dot\cup[n]$ as $\rho_\xi(s^1s^2)=s^2s^1$. Taking into account the decompositions $[m]=\dot\cup_j[m_j]$ and $[n-m]=\dot\cup_j[l_j]$, we see that $\rho_\xi=\rho_\varpi$ for a permutation $\varpi\in\mathscr{S}_{k_1+k_2}$ that swaps the two blocks $[k_1]$ and $[k_2]$.

Put $\beta_j:=\beta^1_j$, $j=1,\ldots,k_1$ and $\beta_{k_1+j}:=\beta^2_j$, $j=1,\ldots k_2$. Then β_j, $j=1,\ldots,k:=k_1+k_2$ are chains, $\beta_j\in\mathcal{T}_{n_j}$, where $n_j:=m_j$, $j=1,\ldots,k_1$, $n_{k_1+j}:=l_j$, $j=1,\ldots,k_2$ and $\sum_{j=1}^k n_j=n$.

To get the permutation sets, let $A'=A\times C$, $B'=\{0,1\}\times B\times D$. Let $\pi_{(a,c),(0,b,d)}=\pi_{a,b}\dot\cup\pi_{c,d}$ and $\pi_{(a,c),(1,b,d)}=\varpi\circ(\pi_{a,b}\dot\cup\pi_{c,d})$. Put also $\sigma=\sigma_1\otimes\sigma_2$. Then

$$f=\bigvee_{a'\in A'}\bigwedge_{b'\in B'}(\beta_{\pi^{-1}_{a',b'}(1)}\lhd\ldots\lhd\beta_{\pi^{-1}_{a',b'}(k)})\circ\rho_{\pi_{a',b'}}\circ(\sigma_1\otimes\sigma_2).$$

The proof is complete.

□

2.3.1 Some examples

Example 9 (Tensor products). Let $n=n_1+n_2+n_3+n_4$ and let $f_i\in\mathcal{T}_{n_i}$, $i=1,\ldots,4$. Fix the concatenation decompositions $\{0,1\}^n\simeq\Pi_i\{0,1\}^{n_i}$, $\{0,1\}^{n_1+n_2}\simeq\{0,1\}^{n_1}\times\{0,1\}^{n_2}$, $\{0,1\}^{n_3+n_4}\simeq\{0,1\}^{n_3}\times\{0,1\}^{n_4}$. Using Lemmas 7 and 8, we compute

$$f_1\otimes f_2=(f_1\lhd f_2)\wedge(f_2\lhd f_1),\qquad (f_1\otimes f_2)^*=(f_1^*\lhd f_2^*)\vee(f_2^*\lhd f_1^*).$$

We then have using Lemmas 9 and 8,

$$\begin{aligned}(f_1\otimes f_2)\otimes f_3&=[(f_1\lhd f_2)\wedge(f_2\lhd f_1)]\otimes f_3=[(f_1\lhd f_2)\otimes f_3]\wedge[(f_2\lhd f_1)\otimes f_3]\\&=(f_1\lhd f_2\lhd f_3)\wedge(f_3\lhd f_1\lhd f_2)\wedge(f_2\lhd f_1\lhd f_3)\wedge(f_3\lhd f_2\lhd f_1).\end{aligned}$$

On the other hand, note that we also have

$$\begin{aligned} f_1 \otimes (f_2 \otimes f_3) &= f_1 \otimes [(f_2 \lhd f_3) \wedge (f_3 \lhd f_2)] \\ &= (f_1 \lhd f_2 \lhd f_3) \wedge (f_2 \lhd f_3 \lhd f_1) \wedge (f_1 \lhd f_3 \lhd f_2) \wedge (f_3 \lhd f_2 \lhd f_1). \end{aligned}$$

Since the left hand sides are obviously equal, we obtain

$$f_1 \otimes f_2 \otimes f_3 = \bigwedge_{\pi \in \mathscr{S}_3} (f_{\pi^{-1}(1)} \lhd f_{\pi^{-1}(2)} \lhd f_{\pi^{-1}(3)}),$$

but we may restrict to permutations of the two kinds above:

$$\{123,\ 213,\ 312,\ 321\}, \quad \{123,\ 231,\ 132,\ 321\}.$$

Similarly, for $f_1, \ldots, f_k$ we get

$$f_1 \otimes \cdots \otimes f_k = \bigwedge_{\pi \in \mathscr{S}_k} (f_{\pi^{-1}(1)} \lhd \ldots \lhd f_{\pi^{-1}(k)}),$$

but we may restrict to some subsets of permutations.

We look at the case $k = 4$. Then we have

$$\begin{aligned} ((f_1 \otimes f_2) \otimes f_3) \otimes f_4 &= (f_1 \otimes (f_2 \otimes f_3)) \otimes f_4 = f_1 \otimes (f_2 \otimes (f_3 \otimes f_4)) = f_1 \otimes ((f_2 \otimes f_3) \otimes f_4) \\ &= (f_1 \otimes f_2) \otimes (f_3 \otimes f_4) \end{aligned}$$

For the first 4 equalities, we get the sets of permutations:

$$\begin{gathered} \{1234,\ 2134,\ 3124,\ 3214,\ 4123,\ 4213,\ 4312,\ 4321\} \\ \{1234,\ 2314,\ 1324,\ 3214,\ 4123,\ 4231,\ 4132,\ 4321\} \\ \{1234,\ 1324,\ 1423,\ 1432,\ 2341,\ 3241,\ 4231,\ 4321\} \\ \{1234,\ 1342,\ 1243,\ 1432,\ 2341,\ 3421,\ 2431,\ 4321\} \end{gathered}$$

For the last one, we compute:

$$\begin{aligned} (f_1 \otimes f_2) \otimes (f_3 \otimes f_4) &= [(f_1 \lhd f_2) \wedge (f_2 \lhd f_1)] \otimes [(f_3 \lhd f_4) \wedge (f_4 \lhd f_3)] \\ &= \Big([(f_1 \lhd f_2) \wedge (f_2 \lhd f_1)] \lhd [(f_3 \lhd f_4) \wedge (f_4 \lhd f_3)] \Big) \\ &\wedge \Big([(f_3 \lhd f_4) \wedge (f_4 \lhd f_3)] \lhd [(f_1 \lhd f_2) \wedge (f_2 \lhd f_1)] \Big) \\ &= (f_1 \lhd f_2 \lhd f_3 \lhd f_4) \wedge (f_2 \lhd f_1 \lhd f_4 \lhd f_3) \\ &\wedge (f_3 \lhd f_4 \lhd f_1 \lhd f_2) \wedge (f_4 \lhd f_3 \lhd f_2 \lhd f_1), \end{aligned}$$

which gives us any of the sets

$$\begin{gathered} \{1234,\ 2143,\ 3412,\ 4321\} \\ \{1243,\ 2134,\ 3421,\ 4312\}, \end{gathered}$$

note these have only four elements!

:tensor3 *Example* 10. In the situation of the previous example, we compute $(f_1 \otimes f_2)^* \otimes f_3$. By the previous example, we have

$$(f_1 \otimes f_2)^* \otimes f_3 = \big[(f_1^* \lhd f_2^*) \vee (f_2^* \lhd f_1^*)\big] \otimes f_3.$$

This can be written as

$$\begin{aligned}(f_1 \otimes f_2)^* \otimes f_3 &= \big[(f_1^* \lhd f_2^*) \otimes f_3\big] \vee \big[(f_2^* \lhd f_1^*) \otimes f_3\big] \\ &= \Big[\big(f_1^* \lhd f_2^* \lhd f_3\big) \wedge \big(f_3 \lhd f_1^* \lhd f_2^*\big)\Big] \vee \Big[\big(f_2^* \lhd f_1^* \lhd f_3\big) \wedge \big(f_3 \lhd f_2^* \lhd f_1^*\big)\Big]\end{aligned}$$

but also as

$$\begin{aligned}(f_1 \otimes f_2)^* \otimes f_3 &= \Big[\big[(f_1^* \lhd f_2^*) \vee (f_2^* \lhd f_1^*)\big] \lhd f_3\Big] \wedge \Big[f_3 \lhd \big[(f_1^* \lhd f_2^*) \vee (f_2^* \lhd f_1^*)\big]\Big] \\ &= \Big[\big(f_1^* \lhd f_2^* \lhd f_3\big) \vee \big(f_2^* \lhd f_1^* \lhd f_3\big)\Big] \wedge \Big[\big(f_3 \lhd f_1^* \lhd f_2^*\big) \vee \big(f_3 \lhd f_2^* \lhd f_1^*\big)\Big]\end{aligned}$$

tensor34 *Example* 11. Let us next compute $(f_1 \otimes f_1)^* \otimes (f_3 \otimes f_4)$. There are several possibilities how to get this: either plug $(f_3 \otimes f_4)$ instead of f_3 into the two forms in the above example, or compute as tensoring the two forms with f_4. That is,

$$\begin{aligned}(f_1 \otimes f_2)^* \otimes f_3 \otimes f_4 &= \Big[\big(f_1^* \lhd f_2^* \lhd (f_3 \otimes f_4)\big) \wedge \big((f_3 \otimes f_4) \lhd f_1^* \lhd f_2^*\big)\Big] \\ &\quad \vee \Big[\big(f_2^* \lhd f_1^* \lhd (f_3 \otimes f_4)\big) \wedge \big((f_3 \otimes f_4) \lhd f_2^* \lhd f_1^*\big)\Big] \\ &= \Big[(f_1^* \lhd f_2^* \lhd f_3 \lhd f_4) \wedge (f_1^* \lhd f_2^* \lhd f_4 \lhd f_3) \\ &\qquad \wedge (f_3 \lhd f_4 \lhd f_1^* \lhd f_2^*) \wedge (f_4 \lhd f_3 \lhd f_1^* \lhd f_2^*)\Big] \\ &\quad \vee \Big[(f_2^* \lhd f_1^* \lhd f_3 \lhd f_4) \wedge (f_2^* \lhd f_1^* \lhd f_4 \lhd f_3) \\ &\qquad \wedge (f_3 \lhd f_4 \lhd f_2^* \lhd f_1^*) \wedge (f_4 \lhd f_3 \lhd f_2^* \lhd f_1^*)\Big]\end{aligned}$$

or

$$
\begin{aligned}
(f_1 \otimes f_2)^* \otimes f_3 \otimes f_4 &= \Big[\big[(f_1^* \lhd f_2^* \lhd f_3) \vee (f_2^* \lhd f_1^* \lhd f_3)\big] \wedge \big[(f_3 \lhd f_1^* \lhd f_2^*) \vee (f_3 \lhd f_2^* \lhd f_1^*)\big]\Big] \otimes f_4 \\
&= \Big[\Big[\big[(\dots) \vee (\dots)\big] \wedge \big[(\dots) \vee (\dots)\big]\Big] \lhd f_4\Big] \\
&\qquad\quad \wedge \Big[f_4 \lhd \Big[\big[(\dots) \vee (\dots)\big] \wedge \big[(\dots) \vee (\dots)\big]\Big]\Big] \\
&= \Big[(f_1^* \lhd f_2^* \lhd f_3 \lhd f_4) \vee (f_2^* \lhd f_1^* \lhd f_3 \lhd f_4)\Big] \\
&\wedge \Big[(f_3 \lhd f_1^* \lhd f_2^* \lhd f_4) \vee (f_3 \lhd f_2^* \lhd f_1^* \lhd f_4)\Big] \\
&\wedge \Big[(f_4 \lhd f_1^* \lhd f_2^* \lhd f_3) \vee (f_4 \lhd f_2^* \lhd f_1^* \lhd f_3)\Big] \\
&\wedge \Big[(f_4 \lhd f_3 \lhd f_1^* \lhd f_2^*) \vee (f_4 \lhd f_3 \lhd f_2^* \lhd f_1^*)\Big]
\end{aligned}
$$

Example 12. Here we compute $(f_1 \otimes f_2)^* \otimes (f_3 \otimes f_4)^*$. We have

$$
\begin{aligned}
(f_1 \otimes f_2)^* \otimes (f_3 \otimes f_4)^* &= \big[(f_1^* \lhd f_2^*) \vee (f_2^* \lhd f_1^*\big] \otimes \big[(f_3^* \lhd f_4^*) \vee (f_4^* \lhd f_3^*)\big] \\
&= \big[(f_1^* \lhd f_2^*) \otimes (f_3^* \lhd f_4^*)\big] \vee \big[(f_1^* \lhd f_2^*) \otimes (f_4^* \lhd f_3^*)\big] \\
&\quad \vee \big[(f_2^* \lhd f_1^*) \otimes (f_3^* \lhd f_4^*)\big] \vee \big[(f_2^* \lhd f_1^*) \otimes (f_4^* \lhd f_3^*)\big] \\
&= [1234 \wedge 3412] \vee [1243 \wedge 4312] \vee [2134 \wedge 3421] \vee [2143 \wedge 4321]
\end{aligned}
$$

but also

$$
\begin{aligned}
(f_1 \otimes f_2)^* \otimes (f_3 \otimes f_4)^* &= \Big[\big[(f_1^* \lhd f_2^*) \vee (f_2^* \lhd f_1^*)\big] \lhd \big[(f_3^* \lhd f_4^*) \vee (f_4^* \lhd f_3^*)\big]\Big] \\
&\quad \wedge \Big[\big[(f_3^* \lhd f_4^*) \vee (f_4^* \lhd f_3^*)\big] \lhd \big[(f_1^* \lhd f_2^*) \vee (f_2^* \lhd f_1^*)\big]\Big] \\
&= [1234 \vee 2143] \wedge [3412 \vee 4321]
\end{aligned}
$$

///

Examples??

Let f be a function of the form as in Theorem thm:structure 3. Since all the chains $\beta_{a,b} := \beta_{\pi_{a,b}^{-1}(1)} \lhd \dots \lhd \beta_{\pi_{a,b}^{-1}(k)} \circ \rho_{a,b}$ in the decomposition have the same input and output indices, they must satsfy the inequality $p_I \leq \beta_{a,b} \leq p_O^*$. But then the same is true for f. It follows that although we do not know whether f is a type function, the corresponding object X_f is always included in a set of channels.

In particular, in the quantum case, each $X_{\beta_{a,b}}$ describes quantum combs obtained by connecting combs described by $X_{\beta_1}, \dots, X_{\beta_k}$, in different orders according to $\pi_{a,b}$. ... NOT QUITE LIKE THIS!!!

– Quantum combs

A Permutations, binary strings and boolean functions

For $m \leq n \in \mathbb{N}$, we will denote the corresponding interval $\{m, m+1, \ldots, n\}$ by $[m, n]$. For $m = 1$, we will simplify to $[n] := [1, n]$. Let $\mathscr{S}_n$ denote the set of all permutations of $[n]$.

A.1 Block permutations

c:permut

For $n_1, n_2 \in \mathbb{N}$, $n_1 + n_2 = n$, we will denote by $[n] = [n_1] \oplus [n_2]$ the decomposition of $[n]$ as a concatenation of two intervals

$$[n] = [n_1][n_1 + 1, n_1 + n_2].$$

Similarly, for $n = \sum_{j=1}^{k} n_j$, we have the decomposition

$$[n] = \oplus_{j=1}^{k}[n_j] = [m_1, m_1 + n_1][m_2, m_2 + n_2] \ldots [m_k, m_k + n_k],$$

where $m_j := \sum_{l=1}^{j-1} n_j$ (so $m_1 = 0$). Note that the order of $n_1, \ldots, n_k$ in this decomposition is fixed.

We have two kinds of special permutations related to the above decomposition. For $\sigma_j \in \mathscr{S}_{n_j}$, we denote by $\oplus_j \sigma_j \in \mathscr{S}_n$ the permutation that acts as

$$m_j + l \mapsto m_j + \sigma(l), \qquad l = 1, \ldots, n_j,\ j = 1, \ldots, k.$$

On the other hand, we have for any $\lambda \in \mathscr{S}_k$ a unique permutation $\rho_\lambda \in \mathscr{S}_n$ such that ρ_λ^{-1} acts as

$$[m_1, m_1 + n_1][m_2, m_2 + n_2] ... [m_k, m_k + n_k] \mapsto [m_{\lambda(1)} + n_{\lambda(1)}][m_{\lambda(2)} + n_{\lambda(2)}] \ldots [m_{\lambda(k)} + n_{\lambda(k)}]$$

Note that we have

$$\rho_\lambda \circ (\oplus_j \sigma_j) = (\oplus_j \sigma_{\lambda(j)}) \circ \rho_\lambda.$$

(These permutations come from the operadic structure on the set of all permutations $\mathscr{S}_*$.)

A.2 Binary strings

A binary string of length n is a sequence $s = s_1 \ldots s_n$, where $s_i \in \{0, 1\}$. Such a string can be interpreted as an element $\{0, 1\}^n$, but also as a map $[n] \to \{0, 1\}$, or a subset in $[n] := \{1, \ldots, n\}$. It will be convenient to use all these interpretations, but we will distinguish between them. The strings in $\{0, 1\}^n$ will be denoted by small letters, whereas the corresponding subsets of $[n]$ will be denoted by the corresponding capital letters. More specifically, for $s \in \{0, 1\}^n$ and $T \subseteq [n]$, we denote

$$S := \{i \in [n],\ s_i = 0\}, \qquad t := t_1 \ldots t_n,\ t_j = 0 \iff j \in T. \tag{11}$$

eq:strin

As usual, the set of all subsets of $[n]$ will be denoted by 2^n. With the inclusion ordering and complementation $S^c := [n] \setminus S$, 2^n is a boolean algebra, with the smallest element $\emptyset$ and largest element $[n]$.

The group $\mathscr{S}_n$ has an obvious action on $\{0, 1\}^n$. Indeed, for a string s interpreted as a map $[n] \to 2$, we may define the action of $\sigma \in \mathscr{S}_n$ by precomposition as

$$\sigma(s) := s \circ \sigma^{-1} = s_{\sigma^{-1}(1)} \ldots s_{\sigma^{-1}(n)}.$$

Note that in this way we have $\rho(\sigma(s)) = (\rho \circ \sigma)(s)$. For a decomposition $[n] = \oplus_{j=1}^{k}[n_j]$, we have a corresponding decomposition of any string $s \in \{0, 1\}^n$ as a concatenation of strings

$$s = s^1 \ldots s^k, \qquad s^j \in \{0, 1\}^{n_j}.$$

For permutations $\sigma_j \in \mathscr{S}_{n_j}$ and $\lambda \in \mathscr{S}_k$, we have

$$\rho_\lambda \circ (\oplus_j \sigma_j)(s^1 \ldots s^k) = \rho_\lambda(\sigma_1(s^1) \ldots \sigma_k(s^k)) = \sigma_{\lambda(1)}(s^{\lambda(1)}) \sigma_{\lambda(2)}(s^{\lambda(2)}) \ldots \sigma_{\lambda(k)}(s^{\lambda(k)}).$$

A.3 Boolean functions and the Möbius transform

A function $f : \{0,1\}^n \to \{0,1\}$ is called a boolean function. The set of boolean functions, with pointwise ordering and complementation given by the negation $\bar{f} = 1 - f$, is a boolean algebra that can be identified with 2^{2^n}. We will denote the maximal element (the constant 1 function) by 1_n. Similarly, we denote the constant zero function by 0_n. For boolean functions f, g, the pointwise minima and maxima will be denoted by $f \wedge g$ and $f \vee g$. It is easily seen that we have

$$f \vee g = f + g - fg, \qquad f \wedge g = fg, \tag{12}$$ eq:wedge

all the operations are pointwise. We now introduce and important example.

ex:pS *Example* 13. For $S \subseteq [n]$, we define

$$p_S(t) = \Pi_{j\in S}(1 - t_j), \qquad t \in \{0,1\}^n.$$

That is, $p_S(t) = 1$ if and only if $S \subseteq T$. In particular, $p_\emptyset = 1_n$ and $p_{[n]}$ is the characteristic function of the zero string. Clearly, for $S, T \subseteq [n]$ we have $p_{S\cup T} = p_S p_T = p_S \wedge p_T$, in particular, $p_S = \Pi_j p_{\{j\}}$.

By the Möbius transform, all boolean functions can be expressed as combinations of the functions p_S, $S \subseteq [n]$ from the previous example.

hm:basis **Theorem 4.** *Any $f : \{0,1\}^n \to 2$ can be expressed in the form*

$$f = \sum_{S\subseteq[n]} \hat{f}_S p_S$$

in a unique way. The coefficients $\hat{f}_S \in \mathbb{R}$ obtained as

$$\hat{f}_S = \sum_{\substack{t\in\{0,1\}^n \\ t_j=1,\forall j\in S^c}} (-1)^{\sum_{j\in S} t_j} f(t).$$

Proof. By the Möbius inversion formula (see [Stanley, Sec. 3.7] for details), functions $f, g : 2^n \to \mathbb{R}$ satisfy

$$f(S) = \sum_{T\subseteq S} g(T), \qquad S \in 2^n$$

if and only if

$$g(S) = \sum_{T\subseteq S} (-1)^{|S\setminus T|} f(T).$$

We now express this in terms of the corresponding strings s and t. It is easily seen that $T \subseteq S$ if and only if $s_j = 0$ for all $j \in T$, equivalently, $t_j = 1$ for all $j \in S^c$. Moreover, in this case we have $|S \setminus T| = \sum_{j\in S} t_j$. This shows that $g(S) = \hat{f}_S$, as defined in the statement. The first equality now gives

$$f(s) = f(S) = \sum_{T\subseteq S} g(T) = \sum_{T:s_i=0,\forall i\in T} \hat{f}_T = \sum_{T:p_T(s)=1} \hat{f}_T = \sum_{T\subseteq[n]} \hat{f}_T p_T.$$

For uniqueness, assume that $f = \sum_{T\subseteq[n]} c_T p_T$ for some coefficients $c_T \in \mathbb{R}$. Then

$$f(s) = \sum_{T:p_T(s)=1} c_T = \sum_{T\subseteq S} c_T.$$

Uniqueness now follows by uniqueness in the Möbius inversion formula.

□

A.4 The boolean algebra $\mathcal{F}_n$

Let us introduce the subset of boolean functions

$$\mathcal{F}_n := \{f : \{0,1\}^n \to 2,\ f(\theta_n) = 1\},$$

where we use θ_n to denote the zero string $00\ldots0$. In other words, $\mathcal{F}_n$ is the interval of all elements greater than $p_{[n]}$ in the boolean algebra 2^{2^n} of all boolean functions. With the pointwise ordering, $\mathcal{F}_n$ is a distributive lattice, with top element 1_n and bottom element $p_{[n]}$. We also define complementation in $\mathcal{F}_n$ as

$$f^* := 1_n - f + p_{[n]}.$$

It can be easily checked that with these structures $\mathcal{F}_n$ is a boolean algebra, though it is not a subalgebra of 2^{2^n}.

We now introduce some more operations in $\mathcal{F}_n$. For $f \in \mathcal{F}_n$ and any permutation $\sigma \in \mathscr{S}_n$, we clearly have $f \circ \sigma \in \mathcal{F}_n$. Further, let $f \in \mathcal{F}_{n_1}$ and $g \in \mathcal{F}_{n_2}$. With the decomposition $[n_1 + n_2] = [n_1] \oplus [n_2]$ and the corresponding concatenation of strings $s = s^1 s^2$, we define the function $f \otimes g \in \mathcal{F}_{n_1+n_2}$ as

$$(f \otimes g)(s^1 s^2) = f(s^1) g(s^2), \qquad s^1 \in \{0,1\}^{n_1},\ s^2 \in \{0,1\}^{n_2}.$$

Let $\lambda \in \mathscr{S}_2$ be the transposition, then we have for any $f \in \mathcal{F}_{n_1}$ and $g \in \mathcal{F}_{n_2}$

$$(g \otimes f) = (f \otimes g) \circ \rho_\lambda,$$

where ρ_λ is the block permutation defined in Section sec:permut A.1. We now show some important properties of these operations.

fproduct **Lemma 9.** *For $f \in \mathcal{F}_{n_1}$ and $g, h \in \mathcal{F}_{n_2}$, we have*

(i) $f \otimes g \leq (f^* \otimes g^*)^*$, with equality if and only if either $f = 1_{n_1}$ and $g = 1_{n_2}$, or $f = p_{[n_1]}$ and $g = p_{[n_2]}$.

(ii) $f \otimes (g \vee h) = (f \otimes g) \vee (f \otimes h)$, $f \otimes (g \wedge h) = (f \otimes g) \wedge (f \otimes h)$.

Proof. The inequality in (i) is easily checked, since $(f \otimes g)(s^1 s^2)$ can be 1 only if $f(s^1) = g(s^2) = 1$. If both s^1 and s^2 are the zero strings, then $s^1 s^2 = \theta_{n_1+n_2}$ and both sides are equal to 1. Otherwise, the condition $f(s^1) = g(s^2) = 1$ implies that $(f^* \otimes g^*)(s^1 s^2) = 0$, so that the right hand side must be 1. If f and g are both constant 1, then $(1_{n_1} \otimes 1_{n_2})^* = 1^*_{n_1+n_2} = p_{[n_1+n_2]} = p_{[n_1]} \otimes p_{[n_2]} = 1^*_{n_1} \otimes 1^*_{n_2}$, in the case when both f and g are the minimal elements equality follows by duality. Finally, asume the equality holds and that $f \neq 1_{n_1}$, so that there is some s^1 such that $f(s^1) = 0$. But then $s^1 \neq \theta_{n_1}$, so that $f^*(s_1) = 1$ and for any s^2,

$$0 = (f \otimes g)(s^1 s^2) = (f^* \otimes g^*)^*(s^1 s^2) = 1 - f^*(s^1) g^*(s^2) + p_{[n_1+n_2]}(s^1 s^2) = 1 - g^*(s^2),$$

which implies that $g(s^2) = 0$ for all $s^2 \neq \theta_{n_2}$, that is, $g = p_{[n_2]}$. By the same argument, $f = p_{[n_1]}$ if $g \neq 1_{n_2}$, which implies that either $f = 1_{n_1}$ and $g = 1_{n_2}$, or $f = p_{[n_1]}$ and $g = p_{[n_2]}$.

The statement (ii) is easily proved from eq:wedgevee_fun (12). $\square$

Consider the decomposition $[n] = [n_1] \oplus [n_2]$ and let $S \subseteq [n_1]$, $T \subseteq [n_2]$. We then denote by $S \oplus T$ the disjoint union

$$S \oplus T := S \cup (n_1 + T) = S \cup \{n_1 + j,\ j \in T\}.$$

We summarize some easy properties of the basic functions p_S, $S \subseteq [n]$.

Lemma 10. *(i) For $S, T \subseteq [n]$, we have $S \subseteq T \iff p_T \leq p_S \iff p_S p_T = p_S$.*

(ii) For $S \subseteq [n]$, $\sigma \in \mathscr{S}_n$, $p_S \circ \sigma = p_{\sigma^{-1}(S)}$.

(iii) For $S \subseteq [n_1]$ and $T \subseteq [n_2]$, $p_S \otimes p_T = p_{S \oplus T}$.

Let $f \in \mathcal{F}_n$ and let $\hat{f}$ be the Möbius transform. Note that since f has values in $\{0,1\}$, we have by the proof of Theorem 4

$$\forall S \in 2^n, \quad \sum_{T \subseteq S} \hat{f}_T = f(s) \in \{0,1\}; \qquad \sum_{T \in 2^n} \hat{f}_T = f(\theta_n) = 1.$$

Proposition 13. *(i) For $f \in \mathcal{F}_n$ and $\sigma \in \mathscr{S}_n$, $\widehat{(f \circ \sigma)}_S = \hat{f}_{\sigma(S)}, \qquad S \subseteq [n]$.*

(ii) For $f \in \mathcal{F}_n$, $\widehat{f^*}_S = \begin{cases} 1 - \hat{f}_S & S = \emptyset \text{ or } S = [n], \\ -\hat{f}_S & \text{otherwise.} \end{cases}$

(iii) For $f \in \mathcal{F}_{n_1}$, $g \in \mathcal{F}_{n_2}$, we have $\widehat{(f \otimes g)}_{S \oplus T} = \hat{f}_S \hat{g}_T$, $S \subseteq [n_1]$, $T \subseteq [n_2]$.

Proof. All statements follow easily from Lemma 10 and the uniqueness part in Theorem 4. □

B Affine subspaces

Let V be a finite dimensional real vector space. A subset $A \subseteq V$ is an affine subspace in V if for any choice of $a_1, \dots, a_k \in A$ and $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ such that $\sum_i \alpha_i = 1$, we have $\sum_i \alpha_i a_i \in A$. It is clear that $A = \emptyset$ is trivially an affine subspace. Moreover, any linear subspace in V is an affine subspace, and an affine subspace A is linear if and only if $0 \in A$. If $A \neq \emptyset$ and also $0 \notin A$, we say that A is proper.

A proper affine subspace $A \subseteq V$ can be determined in two ways. Let

$$\mathrm{Lin}(A) := \{a_1 - a_2,\ a_1, a_2 \in A\}.$$

It is easily verified that $\mathrm{Lin}(A)$ is a linear subspace, moreover, for any $a \in A$, we have

$$\mathrm{Lin}(A) = \{a_1 - a,\ a_1 \in A\}, \qquad A = a + \mathrm{Lin}(A). \tag{13}$$

We put $\dim(A) := \dim(\mathrm{Lin}(A))$, the dimension of A.

Let V^* be the vector space dual of V and let $\langle \cdot, \cdot \rangle$ be the duality. A subset $C \subseteq V$, put

$$\tilde{C} := \{v^* \in V^*,\ \langle v^*, a \rangle = 1,\ \forall a \in A\}.$$

Let $\tilde{a} \in \tilde{A}$ be any element and let $\mathrm{Span}(A)$ be the linear span of A in V. We then have

$$A = \mathrm{Span}(A) \cap \{\tilde{a}\}^{\sim}, \tag{14}$$ eq:affin

eq:affine eq:affine_s

independently of $\tilde{a}$. The relation between the two expressions for A, given by (13) and (14) is obtained as

$$\mathrm{Span}(A) = \mathrm{Lin}(A) + \mathbb{R}\{a\}, \qquad \mathrm{Lin}(A) = \mathrm{Span}(A) \cap \{\tilde{a}\}^{\perp},$$

independently of $a \in A$ or $\tilde{a} \in \tilde{A}$. Here $+$ denotes the direct sum of the vector spaces and $C^{\perp}$ denotes the annihilator of a set C. The following lemma is easily proven.

mma:dual **Lemma 11.** *Let $C \subseteq V$ be any subset. Then $\tilde{C}$ is an affine subspace in V^* and we have*

$$0 \in \tilde{C} \iff C = \emptyset, \qquad \tilde{C} = \emptyset \iff 0 \in \mathrm{Aff}(C).$$

Assume $C \neq \emptyset$ and $0 \notin \mathrm{Aff}(C)$. Then

(i) $\tilde{C}$ *is proper and we have* $\mathrm{Lin}(\tilde{C}) = C^{\perp} = \mathrm{Span}(C)^{\perp}$,

(ii) $\mathrm{Aff}(C) = \tilde{\tilde{C}}$ *and for any $c_0 \in C$, we have*

$$\mathrm{Lin}(C) := \mathrm{Span}\{c_1 - c_2,\ c_1, c_2 \in C\} = \mathrm{Span}\{c - c_0,\ c \in C\} = \mathrm{Lin}(\tilde{\tilde{C}}).$$

oro:dual **Corollary 2.** *Let $A \subseteq V$ be a proper affine subspace. Then*

(i) $\tilde{A}$ *is a proper affine subspace in V^* and $\tilde{\tilde{A}} = A$.*

(ii) $\mathrm{Lin}(\tilde{A}) = \mathrm{Span}(A)^{\perp}$, $\mathrm{Span}(\tilde{A}) = \mathrm{Lin}(A)^{\perp}$.

(iii) $\dim(\tilde{A}) = \dim(V) - \dim(A) - 1$.

The proper affine subspace $\tilde{A}$ in the above Corollary will be called the affine dual of A. Note that the dual depends on the choice of the ambient vector space V.