

# A note on equality in DPI for the BS relative entropy

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Let  $\rho, \sigma \in B(\mathcal{H})^+$ . The Belavkin-Staszewski relative entropy is defined as

$$\hat{D}(\rho\|\sigma) := \text{Tr } \rho \log(\rho^{1/2} \sigma^{-1} \rho^{1/2}) = \text{Tr } \sigma f(\sigma^{-1/2} \rho \sigma^{-1/2}),$$

with  $f(t) = t \log t$ . By [?, Cor. 3.31],  $\hat{D}$  is nonincreasing under positive trace preserving maps  $\Phi : B(\mathcal{H}) \rightarrow B(\mathcal{K})$ , and by [?, Thm. 3.34 (h)], the equality

$$\hat{D}(\Phi(\rho)\|\Phi(\sigma)) = \hat{D}(\rho\|\sigma) \quad (1)$$

holds if and only if  $d := \sigma^{-1/2} \rho \sigma^{-1/2}$  satisfies  $\Phi_\sigma(d^2) = \Phi_\sigma(d)^2$ , where

$$\Phi_\sigma(X) = \Phi(\sigma)^{-1/2} \Phi(\sigma^{1/2} X \sigma^{1/2}) \Phi(\sigma)^{-1/2}, \quad X \in B(\mathcal{H})$$

is the Petz dual of  $\Phi$  with respect to  $\sigma$ . Note that  $\Phi_\sigma$  is positive and unital. If  $\Phi$  is completely positive, we may use the following fact.

**Lemma 1.** *Let  $\Psi : B(\mathcal{H}) \rightarrow B(\mathcal{K})$  be a completely positive unital map with Kraus representation  $\Psi(\cdot) = \sum_i K_i^*(\cdot) K_i$ . Then the multiplicative domain of  $\Psi$  has the form*

$$\mathcal{M}_\Psi = \{K_i K_j^*, i, j\}',$$

(here  $C'$  denotes the commutant of a subset  $C \subseteq B(\mathcal{H})$ ).

This implies the following result. Assume that  $\Phi : B(\mathcal{H}) \rightarrow B(\mathcal{K})$  has the form  $\Phi(\cdot) = \sum_{i=1}^n L_i^*(\cdot) L_i$ , for some  $L_i : \mathcal{K} \rightarrow \mathcal{H}$  such that  $\sum_i L_i L_i^* = I_{\mathcal{H}}$ . Then the equality (1) holds if and only if  $d$  commutes with all elements of the form

$$\sigma^{1/2} L_i \Phi(\sigma)^{-1} L_j^* \sigma^{1/2}, \quad i, j = 1, \dots, n.$$

Let us apply this in the special case when  $\rho = \rho_{ABC} \in B(\mathcal{H}_{ABC})^+$ ,  $\sigma = \rho_{AB} \otimes \tau_C$  and  $\Phi = \text{Tr}_A$ , here  $\tau_C = \dim(\mathcal{H}_C)^{-1} I_C$  is the maximally mixed state. The condition then becomes that  $d$  must commute with all elements of the form

$$\rho_{AB}^{1/2} (|i\rangle\langle j|_A \otimes \rho_B^{-1}) \rho_{AB}^{1/2} \otimes I_C, \quad i, j = 1, \dots, \dim(\mathcal{H}_A).$$

We may clearly replace  $d$  by  $\tilde{d} = (\rho_{AB}^{-1/2} \otimes I_C) \rho_{ABC} (\rho_{AB}^{-1/2} \otimes I_C)$ . The condition is then

$$\tilde{d} \in \mathcal{R} \otimes B(\mathcal{H}_C),$$

where  $\mathcal{R} = \Gamma(B(\mathcal{H}_A))'$ , with  $\Gamma : B(\mathcal{H}_A) \rightarrow B(\mathcal{H}_{AB})$  is a completely positive map given as

$$\Gamma(X_A) = V^*(X_A \otimes I_B)V, \quad X_A \in B(\mathcal{H}_A),$$

here  $V = (I_A \otimes \rho_B^{-1/2})\rho_{AB}^{1/2}$ . Assume that  $\rho_{AB}$  is invertible. By Arveson's commutant lifting theorem [?, Thm. 1.3.1], for every  $T \in \mathcal{R}$  there is a unique  $T_1 \in B(\mathcal{H}_B)$  such that  $(I_A \otimes T_1)V = VT$  and the map  $T \mapsto T_1$  is a \*-isomorphism of  $\mathcal{R}$  onto  $(I_A \otimes B(\mathcal{H}_B)) \cap \{VV^*\}'$ .

Note that  $VV^* = (I_A \otimes \rho_B^{-1/2})\rho_{AB}(I_A \otimes \rho_B^{-1/2})$ , so that the map  $T \mapsto T_1$  is given as

$$\text{Tr}_A[VT V^*] = \text{Tr}_A[(I_A \otimes T_1)V V^*] = T_1 \text{Tr}_A[V V^*] = T_1.$$

The inverse map  $T_1 \mapsto T$  is obtained from the polar decomposition of  $V$ :  $V = HW$ , where  $H = (VV^*)^{1/2}$  and  $W$  is a unitary. Then  $T_1$  commutes with  $H$  and we have

$$VW^*(I_A \otimes T_1)W = H(I_A \otimes T_1)W = (I_A \otimes T_1)HW = (I_A \otimes T_1)V,$$

so that  $T = W^*(I_A \otimes T_1)W$ .

This leads to the following construction of states  $\rho_{ABC}$  such that

$$\hat{D}(\rho_{ABC} \| \rho_{AB} \otimes \tau_C) = \hat{D}(\rho_{BC} \| \rho_B \otimes \tau_C).$$

Let  $M \in B(\mathcal{H}_{AB})^{++}$  be any element such that  $\text{Tr}_A[M] = I_B$ . Let  $\mathcal{T} \subseteq B(\mathcal{H}_B)$  be such that  $I_A \otimes \mathcal{T} = \{M\}' \cap I_A \otimes B(\mathcal{H}_B)$ . Take any state  $\rho_B \in B(\mathcal{H}_B)^{++}$  and put

$$\rho_{AB} = (I_A \otimes \rho_B^{1/2})M(I_A \otimes \rho_B^{1/2}),$$

then clearly  $\rho_{AB}$  is a state and  $\text{Tr}_A \rho_{AB} = \rho_B$ . Let  $W$  be the unitary such that

$$(I_A \otimes \rho_B)^{1/2}M^{1/2} = \rho_{AB}^{1/2}W^*.$$

Now choose any positive element  $\tilde{d} \in W^*(I_A \otimes \mathcal{T})W \otimes B(\mathcal{H}_C)$  such that  $\text{Tr}_C \tilde{d} = I_{AB}$  and put

$$\rho_{ABC} = (\rho_{AB}^{1/2} \otimes I_C)\tilde{d}(\rho_{AB}^{1/2} \otimes I_C).$$

Then we have  $\text{Tr}_C \rho_{ABC} = \rho_{AB}$ ,  $\text{Tr}_A \rho_{ABC} = \rho_B$ .

Since  $\mathcal{T}$  is a subalgebra in  $B(\mathcal{H}_B)$ , there is a unitary  $U_B$  and a decomposition

$$\mathcal{T} = U_B \left( \bigoplus_n I_{B_n^L} \otimes B(\mathcal{H}_{B_n^R}) \right) U_B^*,$$

so that

$$(W \otimes I_C)\tilde{d}(W^* \otimes I_C) = (I_A \otimes U_B \otimes I_C) \left( \bigoplus_n I_{AB_n^L} \otimes d_{B_n^R C} \right) (I_A \otimes U_B^* \otimes I_C)$$

for some positive elements  $d_{B_n^R C} \in B(\mathcal{H}_{B_n^R} \otimes \mathcal{H}_C)^+$ . Moreover, from  $M \subseteq (I \otimes \mathcal{T})'$ , we get

$$M = (I_A \otimes U_B) \left( \bigoplus_n M_{AB_n^L} \otimes I_{B_n^R} \right) (I_A \otimes U_B^*)$$

for  $M_{AB_n^L} \in B(\mathcal{H}_A \otimes \mathcal{H}_{B_n^L})^+$ . Putting all together, we get

$$\begin{aligned} \rho_{ABC} &= (\rho_{AB}^{1/2} W^* W \otimes I_C) \tilde{d} (W^* W \rho_{AB}^{1/2} \otimes I_C) \\ &= ((I_A \otimes \rho_B^{1/2}) M^{1/2} \otimes I_C) (I_A \otimes U_B \otimes I_C) \left( \bigoplus_n I_{AB_n^L} \otimes d_{B_n^R C} \right) (I_A \otimes U_B^* \otimes I_C) (M^{1/2} (I_A \otimes \rho_B^{1/2}) \otimes I_C) \\ &= (I_A \otimes \rho_B^{1/2} U_B \otimes I_C) \left( \bigoplus_n M_{AB_n^L} \otimes d_{B_n^R C} \right) (I_A \otimes U_B^* \rho_B^{1/2} \otimes I_C). \end{aligned}$$

# 1 The structure of $\rho_{ABC}$

**Proposition 1.** *Let  $\rho_{ABC}$  be a state (such that  $\rho_{AB}$  is invertible). The equality*

$$\hat{D}(\rho_{ABC} \parallel \rho_{AB} \otimes \tau_C) = \hat{D}(\rho_{BC} \parallel \rho_B \otimes \tau_C) \quad (2)$$

*holds if and only if there are:*

- (i) Hilbert spaces  $\mathcal{H}_{B_n^L}, \mathcal{H}_{B_n^R}$  such that  $\mathcal{H}_B \simeq \oplus_n (\mathcal{H}_{B_n^L} \otimes \mathcal{H}_{B_n^R})$ ,
- (ii) positive (invertible) elements  $M_n \in B(\mathcal{H}_A \otimes \mathcal{H}_{B_n^L})$  such that  $\text{Tr}_A M_n = I_{B_n^L}$ ,
- (iii) positive elements  $d_n \in B(\mathcal{H}_{B_n^R} \otimes \mathcal{H}_C)$  such that  $\text{Tr}_C d_n = I_{B_n^R}$ ,
- (iv) an (invertible) operator  $S_B : \oplus_n (\mathcal{H}_{B_n^L} \otimes \mathcal{H}_{B_n^R}) \rightarrow \mathcal{H}_B$  such that  $\text{Tr}[S_B S_B^*] = 1$

*such that*

$$\rho_{ABC} = (I_A \otimes S_B \otimes I_C) (\oplus_n M_n \otimes d_n) (I_A \otimes S_B^* \otimes I_C)$$

*Proof.* Assume that  $\rho_{ABC}$  has this form. Then

$$\rho_{AB} = \text{Tr}_C \rho_{ABC} = (I_A \otimes S_B) (\oplus_n M_n \otimes I_{B_n^R}) (I_A \otimes S_B^*), \quad \rho_B = S_B S_B^*.$$

Let us denote  $M := \oplus_n M_n \otimes I_{B_n^R}$ ,  $d := \oplus_n I_{B_n^L} \otimes d_n$ . Using polar decompositions, there is some unitary  $W \in B(\mathcal{H}_{AB})$  such that

$$(I_A \otimes S_B) M^{1/2} W^* = \rho_{AB}^{1/2} = W M^{1/2} (I_A \otimes S_B^*).$$

It follows that

$$(\rho_{AB}^{-1/2} \otimes I_C) \rho_{ABC} (\rho_{AB}^{-1/2} \otimes I_C) = (W \otimes I_C) (I_A \otimes d) (W^* \otimes I_C)$$

and

$$\rho_{AB} = W M^{1/2} (I_A \otimes S_B^* S_B) M^{1/2} W^*.$$

We may clearly replace  $\tau_C$  by  $I_C$  in the equality (2), since this only adds a constant to both sides. We get

$$\begin{aligned} \hat{D}(\rho_{ABC} \parallel \rho_{AB} \otimes I_C) &= \text{Tr}(\rho_{AB} \otimes I_C) f((W \otimes I_C)(I_A \otimes d)(W^* \otimes I_C)) \\ &= \text{Tr}[(M^{1/2}(I_A \otimes S_B^* S_B) M^{1/2} \otimes I_C) f(I_A \otimes d)] \\ &= \text{Tr}[(M(I_A \otimes S_B^* S_B) \otimes I_C) f(I_A \otimes d)] = \text{Tr}[(S_B^* S_B \otimes I_C) f(d)], \end{aligned}$$

here  $f(t) = t \log t$  and we have used the fact that  $M \otimes I_C$  commutes with  $I_A \otimes d$ .

We also have

$$\rho_{BC} = (S_B \otimes I_C) d (S_B^* \otimes I_C)$$

and with the polar decomposition  $S_B = \rho_B^{1/2} U_B$ , we get

$$(\rho_B^{-1/2} \otimes I_C) \rho_{BC} (\rho_B^{-1/2} \otimes I_C) = (U_B \otimes I_C) d (U_B^* \otimes I_C).$$

It follows that

$$\hat{D}(\rho_{BC} \parallel \rho_B \otimes I_C) = \text{Tr}[(\rho_B \otimes I_C) f((U_B \otimes I_C) d (U_B^* \otimes I_C))] = \text{Tr}[(S_B^* S_B \otimes I_C) f(d)] = \hat{D}(\rho_{ABC} \parallel \rho_{AB} \otimes I_C).$$

□