Report on the PhD thesis by Gergely Bunth:
On quantum Rényi divergences

The thesis is concerned with quantum extensions of Rényi divergences, which can be defined as certain quantities on pairs of quantum states that coincide with the classical Rényi divergence in mutually commuting cases. Of course, such extensions cannot be arbitrary and the quantities must satisfy certain properties to be potentially useful. To construct such extensions, essentially two different approaches are taken in the present thesis.

The first approach, described in Section 3, is inspired by a variational formula that relates the classical Rényi divergence to the weighted (left) radius of two probability functions with respect to the relative entropy (Kullback-Leibler divergence). This formula can be extended to multivariate cases and coincides with the Hellinger transform if the weights are given by a probability distribution. In this part of the thesis, an analogical formula is used to define barycentric Rényi divergences for pairs or families of quantum states (or positive semidefinite operators). Since there exist many versions of the quantum relative entropy, a different version may be used, even for each state, so that the obtained class of quantities is quite large. These quantities inherit some useful properties of the defining relative entropies, such as the scaling property or the data processing inequality. It is also proved that the previously known quantum versions of the Rényi divergences, that is, the $\alpha - z$-divergences, measured or maximal Rényi divergences, are not contained in this class.

Another approach, described in Section 4, is axiomatic, based on the fact that the elements in the spectrum of certain preordered semirings constructed from relative (sub)majorization of families of probability vectors (or vectors of nonnegative numbers) are obtained from Rényi divergences. The importance of the spectrum lies in the fact that its elements characterize asymptotic or catalytic majorization in preordered semirings. This idea is applied to ordered semirings consisting of pairs of continuous functions $\rho : X \to B(\mathcal{H})_{>0}$ and $\sigma : Y \to B(\mathcal{H})_{>0}$ from compact Hausdorff spaces to positive definite operators on a finite dimensional Hilbert space and with preorder given by relative submajorization with respect to completely positive trace nonincreasing maps. In the case when all elements in the range of σ commute, the spectrum is fully characterized in terms of sandwiched Rényi divergences with $\alpha > 1$, which leads to a characterization of asymptotic submajorization and a sufficient condition for catalytic submajorization. In the general case, some further elements of the spectrum are constructed by composing the sandwiched Rényi divergence with a quantum version of the geometric mean, thus obtaining a set of necessary conditions.

Applications discussed in the thesis include characterization of the error exponents in the strong converse regime in the composite hypothesis testing, equivariant relative submajorization and a description of asymptotic transformations of by thermal processes, hypothesis testing with group symmetry or with reference frames and approximate joint transformations.

The thesis is based on three papers, two of which with the coauthor Péter Vrana are published in high quality journals, and a third paper including also the thesis supervisor, which is essentially accepted for publication. Given the established importance of the quantum Rényi divergences in quantum information theory, including quite recent results on the use of related quantities in the characterization of (asymptotic, approximate or catalytic) convertibility between sets of classical or quantum states, the topic of the thesis is timely and important. The results of the thesis are based on original ideas and are rather strong, especially in Sec. 4. The approach of the first part gives a very reasonable extension to the case of more than two variables (hence a quantum version of the classical Hellinger transform) and even in the two-variable case, the proposed barycentric Rényi divergences introduce a potentially rich supply of promising quantities to be explored further. In any case, the thesis brings an important contribution to the field, with many possible further applications and follow up research directions. Therefore, I am happy to **recommend the thesis to be presented at the departmental defense**.

There are a few points where I think the thesis could be improved, these are listed below. I would like to stress that the suggested improvements concern solely the thesis presentation.

1. The part of the thesis on barycentric Rényi divergences is based on the paper [MBV23], which starts from a general definition of a (many valued) quantum divergence and discusses more material than described in the thesis. Some parts of this paper are contained in the thesis and some are not, as a consequence, some important definitions seem to be missing. Some instances are listed below, but it is recommended to check whether there are more:

   - a quantum divergence is defined in [MBV23] to be invariant under isometries, but this property is missing in Definition 2.2.6 of the quantum Rényi divergence. Isometric invariance is a very basic property that any reasonable quantity should have, so this omission should definitely be amended. Besides, it is repeatedly used in proofs. A further example is Lemma 3.2.1 whose statement makes little sense without a proper definition of a quantum divergence.
   - γ-weighted versions of some relative entropies (D^{meas}, D^{Um}) are mentioned at some places, see Remark 3.1.13, also p. 40, first paragraph of Sec. 3.2. These quantities are not defined here and the corresponding places are taken from the paper [MBV23], where the weighted divergences are introduced.
   - p. 43, Remark 3.2.10: $Q_P^{G_P^{D^q}}$ seems to refer to a weighted divergence, again not defined in the thesis.

2. In Sec. 2.2.2, properties of the quantum Rényi divergences are often referred to before their definition in Section 2.2.4. This is confusing, especially when it concerns some commonly used expressions like "nonnegative", which is not easy to guess that this is actually some special property to be defined later. I strongly suggest to rearrange this part.

3. Example 4.1.4, proof of Prop. 4.2.3, etc: It would be better to define relative majorization and explain its relation to relative submajorization for states and/or include a reference to [BV1] .

4. I think a concluding section, with a discussion of possible future directions and open questions would be appropriate at the end.

Some minor comments and typos:

- page 6, line 6 from below: the sentence starting with "In a resource theory..." is rather strange. Also "sates" -> states

- p. 7: "...target operators are only bounds" this is difficult to understand
- $\mathcal{P}_f$ is used to denote two very different things: $\mathcal{P}_f(\mathcal{I})$ on p. 10 and the operator perspective function on p. 13. Though it should not cause confusion, a different notation would be better.
- p.17, first displayed equation: $D_{\alpha,+\infty}$... in the second line, $(1-\alpha)$ is missing
- p. 17, paragraph above Ex. 2.2.11: I think that strict positivity of $D_{\alpha,z}$ is treated in [Mos23], Corollary III.28 only in the case $\alpha > 1$
- p. 20: "realation"
- p. 25, Remark 2.2.25: Actually, it is the Hellinger transform of a set of probability distributions (P is a parameter)
- p. 28, last paragraph: $\mathcal{D}_{\mathcal{H}}(D_P)$ etc, does not seem to be defined before.
- p. 29, displayed equations on the top of the page: V and V^* seem missing in line 3 and 4 of the equations
- p. 29: "isomeric"
- p. 29, definition of "regular" D_α: here $\kappa_{\rho,\sigma} > 0$ has no meaning
- p. 34, Theorem 2.3.36, part (ii)(c): $x \succcurlyeq_c y$ (instead of $ax \succcurlyeq_c ay$)
- p. 36., Remark 3.1.3: I think $P(x) \leq 0$, $x \in \mathcal{X}$ is not possible, since $\sum_x P(x) = 1$
- p. 36., Remark 3.1.7, equality for $R_{D^q,left}$: ω (instead of τ)
- p. 37, Lemma 3.1.9: Reference to Eq.(3.12) is used repeatedly, but I think it should be Eq. (3.3)
- p. 41, line 6 from below: S_+^0 should be S_+
- p. 42, last line: "Lemma 3.2.1..." perhaps should be "Lemma 3.2.2"
- p. 57, line above Def. 4.1.8: "to subsemiring" -> two subsemirings (?)
- top of p. 60, end of the proof of Lemma 4.2.4: better write out that $k = 1$ in the real case and $k = 0$ in the tropical case.
- p. 64, line 2: "realative". Also line 3 from below: in one case $\tilde{f}(c_1, d_1)$ should be $\tilde{f}(c_2, d_2)$.
- p. 65, Eq.(4.8) and the displayed equation above it: ρ should be $\rho(x)$ in some places
- p. 66, Prop 4.3.4: $\operatorname{Tr}\sigma(y')$ -> $\operatorname{Tr}\sigma(y') = 1$
- p. 67, Def. 4.3.5: "...a collection of maps $M : \bigtimes_{\mathcal{H}} C(Y, \mathcal{B}(\mathcal{H})_{>0} \to \bigtimes_{\mathcal{H}} C(Y', \mathcal{B}(\mathcal{H})_{>0}$... this makes no sense. I think what you mean is $M = \{M_{\mathcal{H}}\}$, where for each $\mathcal{H}$, $M_{\mathcal{H}} : C(Y, \mathcal{B}(\mathcal{H})_{>0}) \to C(Y', \mathcal{B}(\mathcal{H})_{>0})$ (but this is not a map between the products...).
- p. 69, displayed equations on the top: the inequality sign should be opposite

- p. 84, Corollary 4.4.21: $\alpha \geq 0$ -> $\alpha \geq 1$ (?)
- p. 85, Prop. 4.4.23 (i): ”trace-nonincreasing” - according to the proof, maybe ”trace-preserving”?
- p. 88: Example 4.4.27: ”Lemma 4.3.16” is actually a Proposition
- p. 88 Example 4.4.28 ”...arbitrary an suppose..,” -> ”and”
- Ref. [JV18], [FF20] I think these were already published
- Incomplete references: [MBV23], [MH23a], [FFHT23], [HT23]