1 On the unit interval as a DEA

Let $A = (x_1, \dots, x_n)$ be an n-tuple of elements in $\mathbb{R}^+$. Let f_A denote the positive group homomorphism

$$f_A : \mathbb{Z}^n \to \mathbb{R}, \quad (z_1, \dots, z_n) \mapsto \sum_i z_i x_i$$

and let

$$L(A) := f_A(\mathbb{Z}^n), \quad L(A)^+ := f_A((\mathbb{Z}^n)^+), \quad L_0(A)^+ := f_A((Z_0^n)^+),$$

where $(Z_0^n)^+ := \{\sum_i z_i e_i^n$ with $z_i > 0$ for all $i = 1, \dots, n$.

We first need to prove some lemmas. We also use the notations $Q(A) := Lin_{\mathbb{Q}}(A)$ and $Q(A)^+ := Q(A) \cap \mathbb{R}^+$.

1.1 Lemma. *Let $B = (y_1, \dots, y_k)$, $y_i \in \mathbb{R}^+$ be such that, for some $1 \le N < k$,*

$$\sum_{i=1}^{N} y_i = \sum_{i=N+1}^{k} y_i.$$

Then there is $A = (x_1, \dots, x_l)$, $x_j \in Q(B)^+$, $l < k$, such that $y_i \in L(A)^+$, $i = 1, \dots, k$.

Proof. We proceed by induction on k. By the assumptions, we see that k is at least 2, in which case we have $y_1 = y_2$. Put $A := \{y_2\}$ and we are done.

Now let $k > 2$ and assume that the assertion be true for $k - 1$. By reindexing and rearranging the sums, we may assume that $y_k = \min\{y_1, \dots, y_k\}$. Put $y_1' := y_1 - y_k$, then $y_1' \in Q(B)^+$ and we have the equality

$$y_1' + y_2 + \dots + y_N = y_{N+1} + \dots + y_{k-1}$$

containing only $k - 1$ elements. By the induction hypothesis, there is some tuple $A' = (x_1, \dots, x_{l'})$ with elements in $\in Q(B)^+$ and $l' < k - 1$, and some $(k-1) \times l'$ matrix Z' with values in nonnegative integers such that

$$y_1' = f_{A'}(z_{1\cdot}'), \quad y_i = f_{A'}(z_{i\cdot}'), \ i = 2, \dots, k-1,$$

here $z_{i\cdot}'$ denotes the i-th row of Z'. Let $A = (x_1, \dots, x_{l'}, y_k)$ and

$$Z = \begin{pmatrix} & 1 \\ & 0 \\ Z' & \vdots \\ & 0 \\ 0 & 1 \end{pmatrix}.$$

Then A is an l-tuple of elements in $Q(B)^+$, $l = l' + 1 < k$ and $y_i = f_A(z_{i\cdot}) \in L(A)^+$ for all i.

$\square$

1.2 Lemma. *Let $B = \{y_1, \dots, y_k\} \subset \mathbb{R}^+$. Then there is a basis $A = (x_1, \dots, x_n)$ in $Q(B)$ such that $A \subset \mathbb{R}^+$ and $B \subset L(A)^+$, $i = 1, \dots, k$.*

Proof. If B is $\mathbb{Q}$-linearly independent, there is nothing to do. Otherwise, there are some $r_i \in \mathbb{Q}$ such that $\sum_i r_i y_i = 0$ with some $r_i \neq 0$. Clearly, by multiplying by a common denominator, we may assume that $r_i \in \mathbb{Z}$. Assume that the elements are arranged in such a way that

$$r_i \begin{cases} > 0 & \text{for } i = 1, \dots, N \\ < 0 & \text{for } i = N+1, \dots M \\ = 0 & \text{for } i = M+1, \dots k. \end{cases}$$

Put also $p_i = \Pi_{i \neq j \leq M} |r_j|$ and let $y_i' = \frac{y_i}{p_i}$ for $i = 1, \dots, M$. Clearly, these are positive elements in $Q(B)$. Then by multiplying the equality by $\Pi_{j=1}^M |r_i|^{-1}$, we obtain

$$\sum_{i=1}^N y_i' = \sum_{i=N+1}^M y_i'.$$

Applying Lemma 1.1, there is some l-tuple $A' = (x_1', \dots, x_l') \in Q(B)^+$ with $l < M$ such that $y_i' \in L(A')^+$ for $i = 1, \dots, M$, so that also $y_i = p_i y_i' \in L(A')^+$.

We now repeat the same process with $B' = \{x_1', \dots, x_l', y_{M+1}, \dots, y_k\}$. Since $Q(B') = Q(B)$ and $|B'| < k$, after a finite number of steps we obtain a linearly independent set $A = \{x_1, \dots, x_n\}$ with the required properties.

$\square$

We now obtain the effect algebra $[0,1]$ as a direct limit of finite MV-algebras. Define the index set as

$$\mathcal{I} := \{A \subset [0,1], \mathbb{Q} - \text{linearly independent}, 1 \in L_0(A)^+\}.$$

Any $A \in \mathcal{I}$, $|A| = n$ is identified with the n-tuple of its elements $(x_1, \dots, x_n)$, indexed so that $x_1 < \dots < x_n$. For $A, B \in \mathcal{I}$, write $B \leq A$ if $B \subset L(A)^+$. It is easy to see that $\leq$ is a preorder in $\mathcal{I}$.

We next prove that $\mathcal{I}_1$ is directed. Let $B, C \in \mathcal{I}$, then by Lemma 1.2 there is some $\mathbb{Q}$-linearly independent $A = \{x_1, \dots, x_n\} \subset Q(B \cup C)$ such that $B \cup C \subset L(A)^+$. By assumptions, $1 \in L_0(B)^+ \subset L(A)^+$, so that $1 = \sum_i z_i x_i$ for unique coefficients $z_1, \dots, z_n \in \mathbb{Z}^+$. Assume that $z_{i_0} = 0$ for some i_0. Let $B = \{y_1, \dots, y_k\}$. There are some positive integers $v_1, \dots, v_k$ such that $1 = \sum_{j=1}^k v_j y_j$ and some nonnegative integers $w_1^j, \dots, w_n^j$ such that $y_j = \sum_j w_i^j x_i$. It follows that

$$1 = \sum_{j=1}^k v_j y_j = \sum_i (\sum_j v_j w_i^j) x_i = \sum_i z_i x_i,$$

so that $\sum_j v_j w_i^j = z_i$, in particular, $\sum_j v_j w_{i_0}^j = 0$. Since all v_j are positive, this implies that $w_{i_0}^j = 0$ for all j and we have

$$y_j = \sum_{i \neq i_0} w_i^j x_i.$$

Hence $B \subset L(A \setminus \{x_0\})^+$, similarly also $C \subset L(A \setminus \{x_0\})^+$. It follows that we may assume that $1 \in L_0(A)^+$, so that $A \in \mathcal{I}$ and $\mathcal{I}$ is directed.

For $A, B \in \mathcal{I}$, we define:

- E_A as the interval $[0, f_A^{-1}(1)]$ in $\mathbb{Z}^{|A|}$;
- $g_A : E_A \to [0,1]$ as the restriction of f_A;
- if $B = \{y_1, \dots, y_k\} \leq A$, $g_{AB} : E_B \to E_A$ is defined by $g_{AB}(e_i^B) = g_A^{-1}(y_i)$ (note that $B \subset E_A$).

1.3 Lemma. *(i) For any $A \in \mathcal{I}$, g_A is an isomorphism onto its range.*

(ii) If $B \leq A$, then $g_{AB} g_B^{-1} = g_A^{-1}|_{g_B(E_B)}$.

(iii) Any $x \in [0,1]$ is contained in the range of g_A for some $A \in \mathcal{I}$.

Proof. The statement (i) follows from the fact that A is $\mathbb{Q}$-linearly independent. The map g_A^{-1} asigns to each x in $g_A(E_A)$ its coordinates in the basis A. If $B \leq A$, then $g_B(E_B) \subseteq g_A(E_A)$ and g_{AB} is the transition matrix between the two coordinate systems. The statement (ii) follows easily from these observations. For (iii), let $x \in \mathbb{Q}$. Then $x = \frac{m}{n}$ for some $n \in \mathbb{N}$, $n \geq m \in \mathbb{Z}^+$ and clearly $A = \{\frac{1}{n}\} \in \mathcal{I}_1$. We have $E_A = [0, n](\mathbb{Z})$ and $x = g_A(m)$. If $x \notin \mathbb{Q}$, then $A = \{x, 1-x\} \in \mathcal{I}_1$ and $x \in A \subset g_A(E_A)$.

$\square$

It is easy to see that $\mathcal{E} = \{E_A, A\imath\mathcal{I}; g_{AB}, B \leq A\}$ is a directed system of finite MV-algebras and $\{[0,1]; g_A, A \in \mathcal{I}\}$ is compatible. We will show that it is the direct limit of $\mathcal{E}$.

Let $\{E; k_A, A \in \mathcal{I}_1\}$ be compatible with $\mathcal{G}_1$. By Lemma 1.3, any $x \in [0,1]$ is in the range of some g_A, $A \in \mathcal{I}_1$. In this case, we put

$$\psi(x) = k_A(g_A^{-1}(x)).$$

1.4 Proposition. ψ *defines the unique morphism* $[0,1] \to E$ *such that* $k_A = \psi g_A$ *for any* $A \in \mathcal{I}$*.*

Proof. Let $B, C \in \mathcal{I}$ be such that x is in the range of both g_B and g_C. Let $A \in \mathcal{I}$ be such that $B, C \leq A$. Then $g_B(E_B) \subseteq g_A(E_A)$ and by compatibility and Lemma 1.3,

$$k_B(g_B^{-1}(x)) = k_A g_{AB}(g_B^{-1}(x)) = k_A(g_A^{-1}(x)).$$

Similarly we obtain that $k_C(g_C^{-1}(x)) = k_A(g_A^{-1}(x)) = k_B(g_B^{-1}(x))$, hence ψ is a well defined map.

Let $I = \{1\}$, then clearly $I \in \mathcal{I}$, $E_I = \{0_{\mathbb{Z}}, 1_{\mathbb{Z}}\}$ and we have

$$\psi(0) = k_I(0_{\mathbb{Z}}) = 0, \qquad \psi(1) = k_I(1_{\mathbb{Z}}) = 1.$$

Further, let $x_1, x_2, x \in [0,1]$ be such that $x = x_1 + x_2$. Let $A \in \mathcal{I}$ be such that $x_1, x_2 \in g_A(E_A)$, then clearly also $x \in g_A(E_A)$ and we have $g_A^{-1}(x_1) + g_A^{-1}(x_2) = g_A^{-1}(x)$, since g_A is an isomorphism onto its range. Hence

$$\psi(x) = k_A(g_A^{-1}(x)) = k_A(g_A^{-1}(x_1) + g_A^{-1}(x_2)) = \psi(x_1) + \psi(x_2).$$

This proves that ψ is an effect algebra morphism $[0,1] \to E$. Further, for any $z \in E_A$,

$$k_A(z) = k_A(g_A^{-1} g_A(z)) = \psi g_A(z),$$

so that $k_A = \psi k_A$. Uniqueness is clear.

$\square$