Quantum exponential Orlicz space in information geometry

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1 Introduction - information geometry

2 Quantum relative entropy and exponential families

2.1 Basic setting and notations

- $\mathcal{M}$ a von Neumann algebra (σ-finite),
- $\mathcal{M}^s = \{h = h^* \in \mathcal{M}\}$ self-adjoint part,
- $\mathcal{M}_*$ the predual, $\mathcal{M}_*^+$ the positive cone in $\mathcal{M}_*$,
- $\mathfrak{S}_*(\mathcal{M})$ the set of normal states,
- $\mathcal{M}_*^s = \{\psi(a) \in \mathbb{R},\ \forall a \in \mathcal{M}^s\}$ hermitian normal functionals.

We will also fix a faithful normal state $\rho \in \mathfrak{S}_*(\mathcal{M})$.

Haagerup L_p-spaces: $L_p(\mathcal{M})$, $1 \le p \le \infty$, the norm: $\|\cdot\|_p$

- $\mathcal{M} \simeq L_\infty(\mathcal{M})$,
- $\mathcal{M}_* \simeq L_1(\mathcal{M})$:
$$\psi \mapsto h_\psi, \qquad \mathrm{Tr}\,[h_\psi] = \psi(1),$$
- $L_2(\mathcal{M})$ a Hilbert space
$$(\xi, \eta) = \mathrm{Tr}\,[\xi^* \eta], \qquad \xi, \eta \in L_2(\mathcal{M})$$

(In the case $\mathcal{M} = B(\mathcal{H})$, isomorphic to Schatten classes)

Standard form: $(\lambda, L_2(\mathcal{M}), J, L_2(\mathcal{M})^+)$

$$\lambda(x)\xi = x\xi, \quad J\xi = \xi^*, \qquad x \in \mathcal{M},\ \xi \in L_2(\mathcal{M}).$$

$h_\omega^{1/2}$ - (unique) vector representative of $\omega \in \mathcal{M}_*^+$ in $L_2(\mathcal{M})^+$.

Kosaki L_p-spaces: with respect to ρ, $\eta \in [0,1]$

$$L_p^\eta(\mathcal{M},\rho) = \{h_\rho^{\eta/q} k h_\rho^{(1-\eta)/q},\ k \in L_p(\mathcal{M})\} \subseteq L_1(\mathcal{M}),$$

$$\text{the norm: } \|h_\rho^{\eta/q} k h_\rho^{(1-\eta)/q}\|_{p,\sigma}^\eta = \|k\|_p$$

Symmetric L_p-spaces: $L_p(\mathcal{M},\rho) := L_p^{1/2}(\mathcal{M},\rho)$

2.2 Relative entropy

Araki relative entropy: for $\omega \in \mathcal{M}_*^+$:

$$S(\omega\|\rho) = -\langle \log(\Delta_{\rho,\xi_\omega})\xi_\omega, \xi_\omega \rangle$$

in finite dimensions, this is the same as the **Umegaki relative entropy**: ω, ρ density operators,

$$S(\omega\|\rho) = \operatorname{Tr}\omega(\log\omega - \log\rho)$$

Properties of the relative entropy

- The map $\omega \mapsto S(\omega\|\rho)$ is strictly convex, lower semicontinuous,
- $S(\omega\|\rho) \geq \omega(1)\log\omega(1)$, with equality iff $\omega = \lambda\rho$, $\lambda \geq 0$,
- For a positive unital normal map $T : \mathcal{M} \to \mathcal{N}$, with predual $T_* : \mathcal{N}_* \to \mathcal{M}_*$,
$$S(T_*(\omega)\|T_*(\rho)) \leq S(\omega\|\rho), \qquad \omega \in \mathcal{M}_*^+$$

Relation to Kosaki (symmetric) L_p-spaces
Let $h_\omega \in L_p(\mathcal{M},\rho)$, $p > 1$, and put

$$f(\alpha) = \frac{1}{\alpha - 1}\log\frac{\|h_\omega\|_{\alpha,\rho}}{\omega(1)}, \qquad \alpha \in (0,p].$$

(the Sandwiched Rényi relative entropy $\tilde{D}_\alpha(\omega\|\rho)$)

- the function f is nondecreasing on $(0,p]$
- $\lim_{\alpha\downarrow 1} f(\alpha) = \frac{S(\omega\|\rho)}{\omega(1)}$

In particular, $S(\omega\|\rho) < \infty$.

Sets with finite relative entropy
We define

$$\mathcal{P}_\rho = \{\omega \in \mathcal{M}_*^+,\ S(\omega\|\rho) < \infty\}$$

$\mathcal{S}_\rho = \{\omega \in \mathfrak{S}_*(\mathcal{M}),\ S(\omega\|\rho) < \infty\}$
$K_\rho = \{\omega \in \mathfrak{S}_*(\mathcal{M}),\ S(\omega\|\rho) \leq 1\}$
Donald's identity: for $\omega_i \in \mathcal{M}_*^+$, $\omega = \sum_i \omega_i$

$$S(\omega\|\rho) + \sum_i S(\omega_i\|\omega) = \sum_i S(\omega_i\|\rho)$$

- $\mathcal{P}_\rho$ is a convex cone, face of $\mathcal{M}_*^+$,
- $L_p(\mathcal{M}, \rho)^+ \subseteq \mathcal{P}_\rho$ for $p > 1$,
- $\mathcal{S}_\rho$ is a base of $\mathcal{P}_\rho$,
- K_ρ is convex and weakly compact, generating $\mathcal{P}_\rho$.

Perturbation of states and relative entropy
Let $h \in \mathcal{M}^s$ and

$$c_\rho(h) = \sup_{\omega \in \mathfrak{S}_*(\mathcal{M})} \omega(h) - S(\omega\|\rho)$$

- $c_\rho(h) < \infty$ for all $h \in \mathcal{M}^s$,
- c_ρ is convex and continuous.

The perturbed state is the unique state $[\rho^h] \in \mathcal{S}_\rho$ such that the supremum is attained:

$$c_\rho(h) = [\rho^h](h) - S([\rho^h]\|\rho).$$

For all $\omega \in \mathfrak{S}_*(\mathcal{M})$, we have

$$\omega(h) - S(\omega\|\rho) = c_\rho(h) - S(\omega\|[\rho^h])$$

h defines an affine function on $\mathcal{S}_\rho$:

$$h(\omega) = S(\omega\|\rho) - S(\omega\|[\rho^h]) + c_\rho(h).$$

In finite dimensions, we have

$$[\rho^h] = \exp(\log\rho + h - c_\rho(h)), \quad c_\rho(h) = \log \operatorname{Tr}[\exp(\log\rho + h)].$$

Quantum exponential family: $\{[\rho^{th}], t \in I \subseteq \mathbb{R}\}$

3 The quantum exponential Orlicz space

3.1 The exponential Young function and its dual

The exponential Young function
For the dual pair, we choose $X = \mathcal{M}^s$, $V = \mathcal{M}_*^s$. For $h \in \mathcal{M}^s$, put

$$\Phi_\rho(h) := \frac{\exp(c_\rho(h)) + \exp(c_\rho(-h))}{2} - 1$$

- Φ_ρ is a Young function $X \to [0,\infty)$,
- (X,V) and Φ_ρ satisfy the additional assumptions.

The quantum exponential Orlicz space

Assume that $\mathcal{M}$ is commutative: $\mathcal{M} \simeq L_\infty(\Omega,\Sigma,\rho)$ a probability space. Then for $u \in \mathcal{M}^s$,

$$\Phi_\rho(u) = \int_\Omega \big(\cosh(|u|) - 1\big) d\rho$$

and B_{Φ_ρ} is the closure $E_{\exp}(\Omega,\Sigma,\rho)$ of L_∞ in $L_{\exp}(\Omega,\Sigma,\rho)$. We have

- $u \in E_{\exp}$ if and only if $\Phi_\rho(tu) < \infty$ for all $t \in \mathbb{R}$
- $E_{\exp}^{**} = L_{\exp}$.

We define: $E_{\exp}(\mathcal{M},\rho) := B_{\Phi_\rho}$ and $L_{\exp}(\mathcal{M},\rho) := B^{**}_{\Phi_\rho}$ - the quantum exponential Orlicz space. We will mostly work with $E_{\exp}$.

The dual space

The conjugate function: $\Phi^*_\rho : V \to [0,\infty]$

$$\Phi^*_\rho(v) = \frac{1}{2} \inf_{\substack{\omega_1,\omega_2 \in \mathcal{M}^+_* \\ 2v = \omega_1 - \omega_2}} [S(\omega_1\|\rho) - \omega_1(1) + S(\omega_2\|\rho) - \omega_2(1)] + 1$$

We know that $E^*_{\exp}(\mathcal{M},\rho) = B^*_{\Phi_\rho} \simeq B_{\Phi^*_\rho}$ and $B_{\Phi^*_\rho} \sqsubseteq V = \mathcal{M}^s_*$

- $E^*_{\exp} = \mathcal{P}_\rho - \mathcal{P}_\rho$, $E^*_{\exp} \cap \mathcal{M}^+_* = \mathcal{P}_\rho$
- The unit ball in $E^*_{\exp}$:

$$U_\rho := \{v \in \mathcal{M}^s_*,\ v = \frac{1}{2}(\omega_1 - \omega_2),\ S(\omega_1\|\rho) + S(\omega_2\|\rho) \le \omega_1(1) + \omega_2(1)\}$$

An alternative definition

Let $\tilde{K}_\rho = \{\omega \in \mathcal{M}^+_*,\ S(\omega\|\rho) \le \omega(1)\}$. Then $\tilde{K}_\rho \subset \mathcal{P}_\rho$ is convex and compact in the $\sigma(\mathcal{M}_*,\mathcal{M})$ topology. Indeed, since $S(\omega\|\rho)$ is weakly lsc, $\tilde{K}_\rho$ is weakly closed. Moreover, let $\omega \in \tilde{K}_\rho$ and $\omega(1) = \lambda$, then

$$S(\omega\|\rho) = S(\lambda\omega_0\|\rho) = \lambda S(\omega_0\|\rho) + \lambda \log\lambda \le \lambda$$

for some $\omega_0 \in \mathcal{S}_\rho$ and this entails that $0 \le S(\omega_0\|\rho) \le 1 - \log\lambda$, so that $\|\omega\|_1 = \omega(1)$ is bounded. Let $A(\tilde{K}_\rho)$ be the set of continuous affine functions over $\tilde{K}_\rho$. Then $A(\tilde{K}_\rho)$ is an order unit space, with the natural positive cone $A)\tilde{K}_\rho)^+$ and the constant unit functional as order unit. The order unit norm coincides with the supremum norm. The set of states of $A(\tilde{K}_\rho)$ (with its weak*-topology) is homeomorphic to $\tilde{K}_\rho$ with the inherited $\sigma(\mathcal{M}_*,\mathcal{M})$ topology. The dual space $A(\tilde{K}_\rho)^*$ is the real linear span of $\tilde{K}_\rho$ in $\mathcal{M}_*$ and the unit ball is

$$\tilde{U}_\rho = \mathrm{co}(\tilde{K}_\rho \cup -\tilde{K}_\rho).$$

Observe that any element $h \in \mathcal{M}^s$ defines an element of $A(\tilde{K}_\rho)$ and since $\mathcal{M}^s$ separates points of $\tilde{K}_\rho$ and contains all the constats, we see that $\mathcal{M}^s$ is a norm-dense subspace in $A(\tilde{K}_\rho)$.

Now note that $\frac{1}{2}\tilde{K}_\rho$ and $-\frac{1}{2}\tilde{K}_\rho$ are included in the unit ball U_ρ in $E^*_{\exp}$, hence we have $\frac{1}{2}\tilde{U}_\rho \subseteq U_\rho$. Conversely, let $v = \frac{1}{2}(\omega_1 - \omega)$ such that $S(\omega_1\|\rho) + S(\omega_2\|\rho) \leq \omega_1(1) + \omega_2(1)$. Note that then

$$S(\omega_1\|\rho) \leq \omega_1(1) + \omega_2(1) - S(\omega_2\|\rho) \leq \omega_1(1) + \omega_2(1) - \omega_2(1)\log\omega_2(1) \leq \omega_1(1) + 1$$

and therefore by Donald's identity,

$$S(\omega_1 + \rho\|\rho) \leq S(\omega_1\|\rho) \leq \omega_1(1) + 1 = (\omega_1 + \rho)(1),$$

so that $\omega_1 + \rho \in \tilde{K}_\rho$, similarly also $\omega_2 + \rho \in \tilde{K}_\rho$. It follows that

$$v = \frac{1}{2}(\omega_1 - \omega_2) = \frac{1}{2}(\omega_1 + \rho - (\omega_2 + \rho)) \in \tilde{U}_\rho,$$

hence $\frac{1}{2}\tilde{U}_\rho \subseteq U_\rho \subseteq \tilde{U}_\rho$. Consequently, we the two norms are equivalent.

It follows that the two corresponding norms on $\mathcal{M}^s$ are also equivalent and since $\mathcal{M}^s$ is norm-dense in both $A(\tilde{K}_\rho)$ and $E_{\exp}$, we conclude that $A(\tilde{K}_\rho) = E_{\exp}$, with equivalent norms. Note that the second dual is $A^{**}(\tilde{K}_\rho) = A_b(\tilde{K}_\rho)$, the set of bounded affine functions $\tilde{K}_\rho \to \mathbb{R}$.

Properties of the quantum exponential Orlicz space

We have the following continuous inclusions: for $p > p' > 1$

$$\mathcal{M}^s \sqsubseteq E_{\exp}(\mathcal{M},\rho) \sqsubseteq L_{\exp}(\mathcal{M},\rho) \sqsubseteq L_p(\mathcal{M},\rho) \sqsubseteq L_{p'}(\mathcal{M},\rho) \sqsubseteq E^*_{\exp}(\mathcal{M},\rho) \sqsubseteq \mathcal{M}^s_*$$

Proof. We first note that Φ^*_ρ is finite valued on $L_p(\mathcal{M},\rho)_s$ (the self-adjoint part). Indeed, we have $L_p(\mathcal{M},\rho)_s = L_p(\mathcal{M},\rho)^+ - L_p(\mathcal{M},\rho)^+ \subseteq \mathcal{P}_\rho - \mathcal{P}_\rho$, so that for $v \in L_p(\mathcal{M},\rho)_s$ there are some $\omega_i \in \mathcal{P}_\rho$ such that $v = \frac{1}{2}(\omega_1 - \omega_2)$. Therefore

$$\Phi^*_\rho(v) \leq 1 + \frac{1}{2}(S(\omega_1\|\rho) - \omega_1(1) + S(\omega_2\|\rho) - \omega_2(1)) < \infty.$$

Since we have the continuous embedding $L_p(\mathcal{M},\rho) \sqsubseteq L_1(\mathcal{M})$, the restriction of Φ^*_ρ to $L_p(\mathcal{M},\rho)_s$ is convex and lsc. Since a finite valued convex lsc function on a Banach space is continuous, we conclude that Φ^*_ρ is continuous as a function on $L_p(\mathcal{M},\rho)_s$. It follows that the Minkowski functional of $C_{\Phi^*_\rho}$ is continuous on $L_p(\mathcal{M},\rho)_s$, hence the continuous embedding $L_p(\mathcal{M},\rho)_s \sqsubseteq E^*_{\exp}$. By duality, $L_{\exp} = E^{**}_{\exp} \sqsubseteq L_p(\mathcal{M},\rho)^* \simeq L_q(\mathcal{M},\rho)$ for $1/p + 1/q = 1$. The inclusion $\mathcal{M}^s \sqsubseteq E_{\exp}$ is by continuity of Φ_ρ, $E_{\exp} \sqsubseteq L_{\exp}$ is clear. $\square$

Positive unital normal maps

Let $T : \mathcal{N} \to \mathcal{M}$ be a positive unital normal map, $T_* : \mathcal{M}_* \to \mathcal{N}_*$ its predual. Since $S(\omega\|\rho)$ is monotone under such maps, we have for any $h \in \mathcal{N}^s$

$$\begin{aligned} c_\rho(Th) &= \sup_{\omega \in \mathfrak{S}_*(\mathcal{M})} \omega(Th) - S(\omega\|\rho) \\ &\leq \sup_{\omega \in \mathfrak{S}_*(\mathcal{M})} T_*\omega(h) - S(T_*\omega\|T_*\rho) \\ &\leq \sup_{\sigma \in \mathfrak{S}_*(\mathcal{N})} \sigma(h) - S(\sigma\|T_*\rho) = c_{T_*\rho}(h). \end{aligned}$$

It follows that $\Phi_\rho(Th) \leq \Phi_{T_*\rho}(h)$, so that $\|Th\|_{\exp,\rho} \leq \|h\|_{\exp,T_*\rho}$.

Hence T extends to a contraction $E_{\exp}(\mathcal{N}, T_*\rho) \to E_{\exp}(\mathcal{M}, \rho)$.

4 The quantum information manifold

The extended functional

For $h \in E_{\exp}(\mathcal{M}, \rho)$,

$$c_\rho(h) := \sup_{\omega \in \mathcal{S}_\rho} \omega(h) - S(\omega\|\rho)$$

is

- finite valued
- attained at a unique state $[\rho^h] \in \mathcal{S}_\rho$
- for all $h \in E_{\exp}(\mathcal{M}, \rho)$:

$$\omega(h) - S(\omega\|\rho) = c_\rho(h) - S(\omega\|[\rho^h])$$

The chain rule

For $h, k \in E_{\exp}(\mathcal{M}, \rho)$:

- $\mathcal{P}_\rho = \mathcal{P}_{[\rho^h]}$
- $E_{\exp}(\mathcal{M}, \rho) = E_{\exp}(\mathcal{M}, [\rho^h])$ (equivalent norms)
- $c_\rho(h + k) = c_{[\rho^h]}(k) + c_\rho(k)$
- $[\rho^{h+k}] = [[\rho^h]^k]$
- $[\rho^h] = [\rho^k]$ if and only if $h - k = \rho(h - k)$.

A C^∞-atlas on the set of all faithful normal states

definition construction parallel transport??