On the properties α-z-Rényi divergences on general von Neumann algebras

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α-z-Rényi divergences in von Neumann algebras: data-processing inequality, reversibility and monotonicity properties

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α-z-Rényi divergences in von Neumann algebras via non-commutative L_p-spaces

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Abstract

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Contents

1 Preliminaries

[Since the subsection "Basic definitions" is very short, it seems better to write this without making a subsection.]

Let $\mathcal{M}$ be a von Neumann algebra and let $\mathcal{M}^+$ be the cone of positive elements in $\mathcal{M}$. We denote the predual of $\mathcal{M}$ by $\mathcal{M}_*$, its positive part by $\mathcal{M}_*^+$ and the set of normal states by $\mathfrak{S}_*(\mathcal{M})$. For $\psi \in \mathcal{M}_*^+$, we will denote by $s(\psi)$ the support projection of ψ.

For $0 < p \le \infty$, let $L_p(\mathcal{M})$ be the *Haagerup L_p-space* [10, 35] over $\mathcal{M}$ and let $L_p(\mathcal{M})^+$ its positive cone. We will use the identifications $\mathcal{M} \cong L_\infty(\mathcal{M})$ and $\mathcal{M}_* \ni \psi \leftrightarrow h_\psi \in L_1(\mathcal{M})$, so that the tr-functional on $L_1(\mathcal{M})$ defined by $\operatorname{tr} h_\psi := \psi(1)$ for $\psi \in \mathcal{M}_*$. It this way, $\mathcal{M}_*^+$ is identified with the positive cone $L_1(\mathcal{M})^+$, and $\mathfrak{S}_*(\mathcal{M})$ with the set of elements $h \in L_1(\mathcal{M})^+$ with $\operatorname{tr} h = 1$. Precise definitions and further details on the spaces $L_p(\mathcal{M})$ can be found in [13, Chap. 9], or in the notes [35]. A short summary on the Haagerup L_P-spaces and some technical results that will be used below can be found in Appendix A.

In [21, 22], the α-z-Rényi divergence for $\psi, \varphi \in \mathcal{M}_*^+$ was defined as follows:

Definition 1.1. Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$ and let $\alpha, z > 0$, $\alpha \neq 1$. The *α-z-Rényi divergence* is defined as

$$D_{\alpha,z}(\psi\|\varphi) := \frac{1}{\alpha - 1} \log \frac{Q_{\alpha,z}(\psi\|\varphi)}{\psi(1)},$$

where

$$Q_{\alpha,z}(\psi\|\varphi) := \begin{cases} \mathrm{tr}\left(h_\varphi^{\frac{1-\alpha}{2z}} h_\psi^{\frac{\alpha}{z}} h_\varphi^{\frac{1-\alpha}{2z}}\right)^z, & \text{if } 0<\alpha<1, \\ \|x\|_z^z, & \text{if } \alpha>1 \text{ and } h_\psi^{\frac{\alpha}{z}} = h_\varphi^{\frac{\alpha-1}{2z}} x h_\varphi^{\frac{\alpha-1}{2z}} \text{ with} \\ & x\in s(\varphi)L_z(\mathcal{M})s(\varphi), \\ \infty, & \text{otherwise.} \end{cases}$$

In the case $\alpha>1$, the following alternative form will be useful.

Lemma 1.2 ([21, Lemma 7]). *Let $\alpha>1$ and $\psi,\varphi\in\mathcal{M}_*^+$. Then $Q_{\alpha,z}(\psi\|\varphi)<\infty$ if and only if there is some $y\in L_{2z}(\mathcal{M})s(\varphi)$ such that*

$$h_\psi^{\frac{\alpha}{2z}} = y h_\varphi^{\frac{\alpha-1}{2z}}.$$

Moreover, in this case, such y is unique and we have $Q_{\alpha,z}(\psi\|\varphi)=\|y\|_{2z}^{2z}$.

The *standard* or *Petz-type Rényi divergence* [12, 14, 31] is contained in this range as $D_\alpha(\psi\|\varphi)=D_{\alpha,1}(\psi\|\varphi)$. Also, the *sandwiched Rényi divergence* is obtained as $\tilde{D}_\alpha(\psi\|\varphi)=D_{\alpha,\alpha}(\psi\|\varphi)$; see [4, 14, 18, 19] for some alternative definitions and properties of $\tilde{D}_\alpha$. The definitions in [18, 19] are based on *Kosaki's interpolation L_p-spaces $L_p(\mathcal{M},\varphi)$* [23] with respect to φ. These spaces and the complex interpolation method are briefly summarized in Appendix C, and will be used frequently in the present work.

As have already been done by Kato in [21], many of the properties of $D_{\alpha,z}(\psi\|\varphi)$ are extended from the finite-dimensional case into the general von Neumann algebra case. In particular, the following variational expressions will be an important tool for our work.

Theorem 1.3 (Variational expressions). *Let $\psi,\varphi\in\mathcal{M}_*^+$, $\psi\neq 0$.*

(i) Let $0<\alpha<1$, $\max\{\alpha,1-\alpha\}\le z$. Then

$$Q_{\alpha,z}(\psi\|\varphi) = \inf_{a\in\mathcal{M}^{++}}\left\{\alpha\mathrm{tr}\left((a^{\frac12}h_\psi^{\frac{\alpha}{z}}a^{\frac12})^{\frac{z}{\alpha}}\right)+(1-\alpha)\mathrm{tr}\left((a^{-\frac12}h_\varphi^{\frac{1-\alpha}{z}}a^{-\frac12})^{\frac{z}{1-\alpha}}\right)\right\}. \tag{1.1}$$

(ii) Let $\alpha>1$, $\max\{\alpha/2,\alpha-1\}\le z$. Then

$$Q_{\alpha,z}(\psi\|\varphi) = \sup_{a\in\mathcal{M}_+}\left\{\alpha\mathrm{tr}\left((a^{\frac12}h_\psi^{\frac{\alpha}{z}}a^{\frac12})^{\frac{z}{\alpha}}\right)-(\alpha-1)\mathrm{tr}\left((a^{\frac12}h_\varphi^{\frac{\alpha-1}{z}}a^{\frac12})^{\frac{z}{\alpha-1}}\right)\right\}. \tag{1.2}$$

Moreover, if $\psi\le\lambda\varphi$ for some $\lambda>0$, then (1.2) holds for all $z\ge\alpha-1>0$. [Please check if this addition is OK.]

Proof. For part (i) see [21, Theorem 1(vi)]. The inequality $\ge$ in part (ii) holds for all α and z and was proved in [21, Theorem 2(vi)]. We now prove the opposite inequality.

Assume first that $Q_{\alpha,z}(\psi\|\varphi)<\infty$, so that there is some $x\in s(\varphi)L_z(\mathcal{M})^+s(\varphi)$ such that $h_\psi^{\frac{\alpha}{z}}=h_\varphi^{\frac{\alpha-1}{2z}}xh_\varphi^{\frac{\alpha-1}{2z}}$. Plugging this into the right-hand side of (1.2), we obtain

$$\begin{aligned}
&\sup_{a\in\mathcal{M}_+}\left\{\alpha\mathrm{tr}\left((a^{\frac12}h_\psi^{\frac{\alpha}{z}}a^{\frac12})^{\frac{z}{\alpha}}\right)-(\alpha-1)\mathrm{tr}\left((a^{\frac12}h_\varphi^{\frac{\alpha-1}{2z}}a^{\frac12})^{\frac{z}{\alpha-1}}\right)\right\}\\
&\quad=\sup_{a\in\mathcal{M}_+}\left\{\alpha\mathrm{tr}\left((a^{\frac12}h_\varphi^{\frac{\alpha-1}{2z}}xh_\varphi^{\frac{\alpha-1}{2z}}a^{\frac12})^{\frac{z}{\alpha}}\right)-(\alpha-1)\mathrm{tr}\left((a^{\frac12}h_\varphi^{\frac{\alpha-1}{2z}}a^{\frac12})^{\frac{z}{\alpha-1}}\right)\right\}\\
&\quad=\sup_{a\in\mathcal{M}_+}\left\{\alpha\mathrm{tr}\left((x^{\frac12}h_\varphi^{\frac{\alpha-1}{2z}}ah_\varphi^{\frac{\alpha-1}{2z}}x^{\frac12})^{\frac{z}{\alpha}}\right)-(\alpha-1)\mathrm{tr}\left((h_\varphi^{\frac{\alpha-1}{2z}}ah_\varphi^{\frac{\alpha-1}{2z}})^{\frac{z}{\alpha-1}}\right)\right\}\\
&\quad=\sup_{w\in L_{\frac{z}{\alpha-1}}(\mathcal{M})^+}\left\{\alpha\mathrm{tr}\left((x^{\frac12}wx^{\frac12})^{\frac{z}{\alpha}}\right)-(\alpha-1)\mathrm{tr}\left(w^{\frac{z}{\alpha-1}}\right)\right\},
\end{aligned}\tag{1.3}$$

where we have used the fact that $\mathrm{tr}\left((h^*h)^p\right)=\mathrm{tr}\left((hh^*)^p\right)$ for $p>0$, $h\in L_{\frac{p}{2}}(\mathcal{M})$, and Lemma A.1. Putting $w=x^{\alpha-1}$ we get

$$\sup_{w\in L_{\frac{z}{\alpha-1}}(\mathcal{M})^+}\left\{\alpha\mathrm{tr}\left((x^{\frac12}wx^{\frac12})^{\frac{z}{\alpha}}\right)-(\alpha-1)\mathrm{tr}\left(w^{\frac{z}{\alpha-1}}\right)\right\}\ge\mathrm{tr}(x^z)=\|x\|_z^z=Q_{\alpha,z}(\psi\|\varphi).$$

This finishes the proof of (1.2) in the case that $Q_{\alpha,z}(\psi\|\varphi)<\infty$. Note that this holds if $\psi\le\lambda\varphi$ for some $\lambda>0$. Indeed, since $\frac{\alpha-1}{z}\in(0,1]$ by the assumption, we then have $h_\psi^{\frac{\alpha-1}{z}}\le\lambda^{\frac{\alpha-1}{z}}h_\varphi^{\frac{\alpha-1}{z}}$. Hence by [14, Lemma A.58] there is some $b\in\mathcal{M}$ such that

$$h_\psi^{\frac{\alpha-1}{2z}}=bh_\varphi^{\frac{\alpha-1}{2z}},$$

so that $h_\psi^{\frac{\alpha}{2z}}=yh_\varphi^{\frac{\alpha-1}{2z}}$, where $y:=h_\varphi^{\frac{1}{2z}}b\in L_{2z}(\mathcal{M})$. By Lemma 1.2, $Q_{\alpha,z}(\psi\|\varphi)=\|y\|_{2z}^{2z}<\infty$. This shows the latter assertion too. [My previous proof here is bad. Is this OK?]

In the general case, the variational expression holds for $Q_{\alpha,z}(\psi\|\varphi+\varepsilon\psi)$ for all $\varepsilon>0$, so that we have

$$\begin{aligned}
Q_{\alpha,z}(\psi\|\varphi+\varepsilon\psi)&=\sup_{a\in\mathcal{M}_+}\left\{\alpha\mathrm{tr}\left((a^{\frac12}h_\psi^{\frac{\alpha}{z}}a^{\frac12})^{\frac{z}{\alpha}}\right)-(\alpha-1)\mathrm{tr}\left((a^{\frac12}h_{\varphi+\varepsilon\psi}^{\frac{\alpha-1}{z}}a^{\frac12})^{\frac{z}{\alpha-1}}\right)\right\}\\
&\le\sup_{a\in\mathcal{M}_+}\left\{\alpha\mathrm{tr}\left((a^{\frac12}h_\psi^{\frac{\alpha}{z}}a^{\frac12})^{\frac{z}{\alpha}}\right)-(\alpha-1)\mathrm{tr}\left((a^{\frac12}h_\varphi^{\frac{\alpha-1}{z}}a^{\frac12})^{\frac{z}{\alpha-1}}\right)\right\},
\end{aligned}$$

where the inequality above follows by Lemma A.3. Therefore, since $z\ge\alpha/2$, from lower semicontinuity [21, Theorem 2(iv)] we have

$$Q_{\alpha,z}(\psi\|\varphi)\le\liminf_{\varepsilon\searrow0}Q_{\alpha,z}(\psi\|\varphi+\varepsilon\psi),$$

so that the desired inequality is obtained. □

We finish this section by investigation of the properties of the variational expression for $0<\alpha<1$, in the case when $\lambda^{-1}\varphi\le\psi\le\lambda\varphi$ for some $\lambda>0$. This case will be denoted as $\psi\sim\varphi$.

Lemma 1.4. *Assume that $\psi \sim \varphi$. Then the infimum in* (1.1) *of Theorem 1.3(i) is attained at a unique element $\bar a \in \mathcal{M}^{++}$. This element satisfies*

$$h_\psi^{\frac{\alpha}{2z}} \bar a h_\psi^{\frac{\alpha}{2z}} = \Big(h_\psi^{\frac{\alpha}{2z}} h_\varphi^{\frac{1-\alpha}{z}} h_\psi^{\frac{\alpha}{2z}} \Big)^{\alpha} \tag{1.4}$$

$$h_\varphi^{\frac{1-\alpha}{2z}} \bar a^{-1} h_\varphi^{\frac{1-\alpha}{2z}} = \Big(h_\varphi^{\frac{1-\alpha}{2z}} h_\psi^{\frac{\alpha}{z}} h_\varphi^{\frac{1-\alpha}{2z}} \Big)^{1-\alpha}. \tag{1.5}$$

Proof. We may assume that φ and hence also ψ are(?) faithful. Following the proof of [21, Theorem 1(vi)], we may use the assumptions and [14, Lemma A.58] to show that there are $b, c \in \mathcal{M}$ such that

$$h_\varphi^{\frac{1-\alpha}{2z}} = b\Big(h_\varphi^{\frac{1-\alpha}{2z}} h_\psi^{\frac{\alpha}{z}} h_\varphi^{\frac{1-\alpha}{2z}} \Big)^{\frac{1-\alpha}{2}}, \qquad \Big(h_\varphi^{\frac{1-\alpha}{2z}} h_\psi^{\frac{\alpha}{z}} h_\varphi^{\frac{1-\alpha}{2z}} \Big)^{\frac{1-\alpha}{2}} = c h_\varphi^{\frac{1-\alpha}{2z}}. \tag{1.6}$$

With $\bar a := bb^* \in \mathcal{M}^{++}$ we have $\bar a^{-1} = c^* c$ and $\bar a$ is indeed a minimizer of (1.2), equivalently,

$$Q_{\alpha,z}(\psi\|\varphi) = \inf_{a\in\mathcal{M}^{++}} \Big\{ \alpha \Big\| h_\psi^{\frac{\alpha}{2z}} a h_\psi^{\frac{\alpha}{2z}} \Big\|_{\frac{z}{\alpha}}^{\frac{z}{\alpha}} + (1-\alpha) \Big\| h_\varphi^{\frac{1-\alpha}{2z}} a^{-1} h_\varphi^{\frac{1-\alpha}{2z}} \Big\|_{\frac{z}{1-\alpha}}^{\frac{z}{1-\alpha}} \Big\}. \tag{1.7}$$

We next observe that such a minimizer is unique. Indeed, suppose that the infimum is attained at some $a_1, a_2 \in \mathcal{M}^{++}$. Let $a_0 := (a_1 + a_2)/2$. Since the map $L_p(\mathcal{M}) \ni k \mapsto \|k\|_p^p$ is convex for any $p \geq 1$ and $a_0^{-1} \leq (a_1^{-1} + a_2^{-1})/2$, we have

$$\Big\| h_\psi^{\frac{\alpha}{2z}} a_0 h_\psi^{\frac{\alpha}{2z}} \Big\|_{\frac{z}{\alpha}}^{\frac{z}{\alpha}} \leq \frac{1}{2} \Big\{ \Big\| h_\psi^{\frac{\alpha}{2z}} a_1 h_\psi^{\frac{\alpha}{2z}} \Big\|_{\frac{z}{\alpha}}^{\frac{z}{\alpha}} + \Big\| h_\psi^{\frac{\alpha}{2z}} a_2 h_\psi^{\frac{\alpha}{2z}} \Big\|_{\frac{z}{\alpha}}^{\frac{z}{\alpha}} \Big\}.$$

Moreover, using Lemma A.2 we have

$$\begin{aligned} \Big\| h_\varphi^{\frac{1-\alpha}{2z}} a_0^{-1} h_\varphi^{\frac{1-\alpha}{2z}} \Big\|_{\frac{z}{1-\alpha}}^{\frac{z}{1-\alpha}} &\leq \Big\| h_\varphi^{\frac{1-\alpha}{2z}} \Big(\frac{a_1^{-1} + a_2^{-1}}{2} \Big) h_\varphi^{\frac{1-\alpha}{2z}} \Big\|_{\frac{z}{1-\alpha}}^{\frac{z}{1-\alpha}} \\ &\leq \frac{1}{2} \Big\{ \Big\| h_\varphi^{\frac{1-\alpha}{2z}} a_1^{-1} h_\varphi^{\frac{1-\alpha}{2z}} \Big\|_{\frac{z}{1-\alpha}}^{\frac{z}{1-\alpha}} + \Big\| h_\varphi^{\frac{1-\alpha}{2z}} a_2^{-1} h_\varphi^{\frac{1-\alpha}{2z}} \Big\|_{\frac{z}{1-\alpha}}^{\frac{z}{1-\alpha}} \Big\}. \end{aligned}$$

Hence the assumption of a_1, a_2 being a minimizer gives

$$\Big\| h_\varphi^{\frac{1-\alpha}{2z}} a_0^{-1} h_\varphi^{\frac{1-\alpha}{2z}} \Big\|_{\frac{z}{1-\alpha}} = \Big\| h_\varphi^{\frac{1-\alpha}{2z}} \Big(\frac{a_1^{-1} + a_2^{-1}}{2} \Big) h_\varphi^{\frac{1-\alpha}{2z}} \Big\|_{\frac{z}{1-\alpha}},$$

which implies that $a_0^{-1} = \frac{a_1^{-1} + a_2^{-1}}{2}$, as easily verified by Lemma A.2 again. From this we easily have $a_1 = a_2$.

The equality (1.5) is obvious from the second equality in (1.6) and $\bar a^{-1} = c^* c$. Since $Q_{\alpha,z}(\psi\|\varphi) = Q_{1-\alpha,z}(\varphi\|\psi)$, we see by uniqueness that the minimizer of the infimum expression for $Q_{1-\alpha,z}(\varphi\|\psi)$ (instead of (1.7)) is $\bar a^{-1}$ (instead of $\bar a$). This says that (1.4) is the equality corresponding to (1.5) when ψ, φ, α are replaced with $\varphi, \psi, 1-\alpha$, respectively. □

To make the next lemma more readable, we will use the following notations:

$$p := \frac{z}{\alpha}, \qquad r := \frac{z}{1-\alpha}, \qquad \xi_p(a) := h_\psi^{\frac{1}{2p}} a h_\psi^{\frac{1}{2p}}, \qquad \eta_r(a) = h_\varphi^{\frac{1}{2r}} a^{-1} h_\varphi^{\frac{1}{2r}}.$$

We will also denote the function under the infimum in the variational expression in Theorem 1.3(i) by f, that is,

$$f(a) = \alpha\|\xi_p(a)\|_p^p + (1-\alpha)\|\eta_r(a)\|_r^r, \qquad a \in \mathcal{M}^{++}. \tag{1.8}$$

When $p \in (1,\infty)$, recall that $L_p(\mathcal{M})$ is uniformly convex (see [10], [23, Theorem 4.2]), so that the norm $\|\cdot\|_p$ is uniformly Fréchet differentiable (see, e.g., [2, Part 3, Chap. II]). Hence $a \mapsto \|\xi_p(a)\|_p^p$ and $a \mapsto \|\eta_r(a)\|_r^r$ are Fréchet differentiable on $\mathcal{M}^{++}$. Since differentiability of these functions is obvious when $p=1$ and $r=1$, we see that the function f is Fréchet differentiable on $\mathcal{M}^{++}$ for any $p,r \ge 1$, whose Fréchet derivative at a will be denoted by $\nabla f(a)$.

Lemma 1.5. *Assume that $\psi \sim \varphi$ and let $0<\alpha<1$, $\max\{\alpha, 1-\alpha\} \le z$. Let $\bar a \in \mathcal{M}^{++}$ be as given in Lemma 1.4. If $p>1$, then for every $C \ge Q_{\alpha,z}(\psi\|\varphi)$ and $\varepsilon>0$ there is some $\delta>0$ such that whenever $\|\xi_p(b)\|_p^p \le C$ and $\|\xi_p(b)-\xi_p(\bar a)\|_p \ge \varepsilon$, we have*

$$f(b) - Q_{\alpha,z}(\psi\|\varphi) \ge \delta.$$

A similar statement holds if $r>1$.

Proof. By assumptions, $p,r\ge 1$. For $a,b \in \mathcal{M}^{++}$ and $s \in (0,1/2)$, we have

$$\begin{aligned}\|\xi_p(sb+(1-s)a)\|_p^p &= \|s\xi_p(b)+(1-s)\xi_p(a)\|_p^p \\ &= \left\|(1-2s)\xi_p(a) + 2s\frac{1}{2}(\xi_p(a)+\xi_p(b))\right\|_p^p \\ &\le (1-2s)\|\xi_p(a)\|_p^p + 2s\left\|\frac{1}{2}(\xi_p(a)+\xi_p(b))\right\|_p^p.\end{aligned}$$

Similarly,

$$\|\eta_r(sb+(1-s)a)\|_r^r \le (1-2s)\|\eta_r(a)\|_r^r + 2s\left\|\frac{1}{2}(\eta_r(a)+\eta_r(b))\right\|_r^r,$$

where we have also used the fact that $(ta+(1-t)b)^{-1} \le ta^{-1}+(1-t)b^{-1}$ for $t\in(0,1)$ and Lemma A.2. It follows that

$$\begin{aligned}&\langle \nabla f(a), b-a\rangle \\ &\quad= \lim_{s\to 0^+}\frac{1}{s}[f(sb+(1-s)a)-f(a)] \\ &\quad\le 2\alpha\left[\left\|\frac{1}{2}(\xi_p(a)+\xi_p(b))\right\|_p^p - \|\xi_p(a)\|_p^p\right] + 2(1-\alpha)\left[\left\|\frac{1}{2}(\eta_r(a)+\eta_r(b))\right\|_r^r - \|\eta_r(a)\|_r^r\right] \\ &\quad= f(b)-f(a) - 2\Bigg\{\alpha\left[\frac{1}{2}\|\xi_p(a)\|_p^p + \frac{1}{2}\|\xi_p(b)\|_p^p - \left\|\frac{1}{2}(\xi_p(a)+\xi_p(b))\right\|_p^p\right] \\ &\qquad\qquad + (1-\alpha)\left[\frac{1}{2}\|\eta_r(a)\|_r^r + \frac{1}{2}\|\eta_r(b)\|_r^r - \left\|\frac{1}{2}(\eta_r(a)+\eta_r(b))\right\|_r^r\right]\Bigg\}.\end{aligned}$$

Since $p,r\ge 1$, both terms in brackets [I have changed parentheses $(,)$ into bracket $[,]$ in the above expression.] in the last expression above are nonnegative. Assume now that $p>1$. Let $\bar a \in \mathcal{M}^{++}$ be the minimizer as in Lemma 1.4, then $f(\bar a) = Q_{\alpha,z}(\psi\|\varphi)$ and $\nabla f(\bar a) = 0$, so that we get

$$f(b) - Q_{\alpha,z}(\psi\|\varphi) \ge 2\alpha\left[\frac{1}{2}\|\xi_p(a)\|_p^p + \frac{1}{2}\|\xi_p(b)\|_p^p - \left\|\frac{1}{2}(\xi_p(a)+\xi_p(b))\right\|_p^p\right].$$

Since $L_p(\mathcal{M})$ is uniformly convex, note (see, e.g., [36, Thmeorem 3.7.7]) that the function $h \mapsto \|h\|_p^p$ is uniformly convex on each bounded subset of $L_p(\mathcal{M})$. Hence for each $C > 0$ and $\varepsilon > 0$ there is some $\delta > 0$ such that for every h, k with $\|h\|_p^p, \|k\|_p^p \le C$ and $\|h - k\|_p \ge \varepsilon$, we have

$$\frac{1}{2}\|h\|_p^p + \frac{1}{2}\|k\|_p^p - \left\|\frac{1}{2}(h + k)\right\|_p^p \ge \delta$$

(see [36, p. 288, Exercise 3.3]). The proof in the case $r > 1$ is similar. $\square$

2 Data processing inequality and reversibility of channels

Let $\gamma : \mathcal{N} \to \mathcal{M}$ be a normal positive unital map. Then the predual of γ defines a positive linear map $\gamma_* : L_1(\mathcal{M}) \to L_1(\mathcal{N})$ that preserves the tr-functional, acting as

$$L_1(\mathcal{M}) \ni h_\rho \mapsto h_{\rho\circ\gamma} \in L_1(\mathcal{N}).$$

The support of γ will be denoted by $s(\gamma)$, recall that this is defined as the smallest projection $e \in \mathcal{N}$ such that $\gamma(e) = 1$ and in this case, $\gamma(a) = \gamma(eae)$ for any $a \in \mathcal{N}$. For any $\rho \in \mathcal{M}_*^+$ we clearly have $s(\rho \circ \gamma) \le s(\gamma)$, with equality if ρ is faithful. It follows that γ_* maps $L_1(\mathcal{M})$ to $s(\gamma)L_1(\mathcal{N})s(\gamma) \equiv L_1(s(\gamma)\mathcal{N}s(\gamma))$. For any $\rho \in \mathcal{M}^*_+$, $\rho \neq 0$, the map

$$\gamma_0 : s(\rho\circ\gamma)\mathcal{N}s(\rho\circ\gamma) \to s(\rho)\mathcal{M}s(\rho), \qquad a \mapsto s(\rho)\gamma(a)s(\rho)$$

is a faithful normal positive unital (i.e., $\gamma_0(\rho\circ\gamma) = s(\rho)$) map; see [13, Remark 6.7]. Moreover, for any $\varphi \in \mathcal{M}_*^+$ such that $s(\varphi) \le s(\rho)$, we have for any $a \in \mathcal{N}$,

$$\varphi(\gamma_0(s(\gamma)as(\gamma))) = \varphi(s(\rho)\gamma(a)s(\rho)) = \varphi(\gamma(a)).$$

Replacing γ by γ_0 and ρ by the restriction $\rho|_{s(\rho\circ\gamma)\mathcal{M}s(\rho\circ\gamma)}$, we may assume that both ρ and $\rho\circ\gamma$ are faithful, as far as we are concerned with $\varphi \in \mathcal{M}_*^+$ and $\varphi\circ\gamma \in \mathcal{N}_*^+$ with $s(\varphi) \le s(\rho)$.

The *Petz dual* of γ with respect to $\rho \in \mathcal{M}_*^+$ is a map $\gamma_\rho^* : \mathcal{M} \to \mathcal{N}$, introduced in [33] when ρ and $\rho\circ\gamma$ are faithful (hence so is γ as well). It was proved that γ_ρ^* is again normal, positive and unital, and in addition, it is n-positive whenever γ is. More generally, even though none of ρ, $\rho\circ\gamma$ and γ is faithful, letting $e := s(\rho)$ and $e_0 := s(\rho\circ\gamma)$, we may use the restriction γ_0 as mentioned above to define the Petz dual $\gamma_\rho^* : e\mathcal{M}e \to e_0\mathcal{N}e_0$. As explained in [18], in this general setting, γ_ρ^* is determined by the equality

$$h_{\rho\circ\gamma}^{1/2}\gamma_\rho^*(a)h_{\rho\circ\gamma}^{1/2} = \gamma_*\big(h_\rho^{1/2}ah_\rho^{1/2}\big), \qquad a \in \mathcal{M},$$

equivalently,

$$(\gamma_\rho^*)_*\big(h_{\rho\circ\gamma}^{1/2}bh_{\rho\circ\gamma}^{1/2}\big) = h_\rho^{1/2}\gamma(b)h_\rho^{1/2}, \qquad b \in \mathcal{N}^+, \tag{2.1}$$

where γ_* and $(\gamma_\rho^*)_*$ are the predual maps of γ and γ_ρ^*, respectively. [This seems better (?)] We also have

$$\rho\circ\gamma\circ\gamma_\rho^* = \rho, \qquad (\gamma_\rho^*)_{\rho\circ\gamma}^* = \gamma. \tag{2.2}$$

In the special case where γ is the inclusion map $\gamma : \mathcal{N} \hookrightarrow \mathcal{M}$ for a subalgebra $\mathcal{N} \subseteq \mathcal{M}$, the Petz dual is the *generalized conditional expectation* $\mathcal{E}_{\mathcal{N},\rho} : \mathcal{M} \to \mathcal{N}$, as introduced in [1]; see, e.g., [14, Proposition 6.5]. Hence $\mathcal{E}_{\mathcal{N},\rho}$ is a normal completely positive unital with range in $\mathcal{N}$ and such that

$$\rho\circ\mathcal{E}_{\mathcal{N},\rho} = \rho.$$

2.1 Data processing inequality

In this subsection we prove the *data processing inequality* (*DPI*) for $D_{\alpha,z}$ with respect to normal positive unital maps. For standard Rényi divergence, that is, for $z = 1$, the DPI is known to hold for $\alpha \in (0, 1) \cup (1, 2]$ under stronger positivity assumptions [12]. In the case of the sandwiched divergences $\tilde{D}_\alpha$ with $\alpha \in [1/2, 1) \cup (1, \infty)$, DPI was proved in [18, 19]; see also [4] for an alternative proof in the case when the maps are assumed completely positive. In the finite-dimensional case, the DPI for $D_{\alpha,z}$ under completely positive maps was proved in [37], for α, z in the range specified as in Theorem 2.3 below.

The first part of the next lemma was essentially shown in [18, Proposition 3.12], while we give the proof for the convenience of the reader.

Lemma 2.1. *Let $\gamma : \mathcal{N} \to \mathcal{M}$ be a normal positive unital map. Let $\rho \in \mathcal{M}_*^+$, $\rho \neq 0$, $e := s(\rho)$ and $e_0 := s(\rho \circ \gamma)$. For any $p \geq 1$, the map $\gamma^*_{\rho,p} : L_p(e_0\mathcal{N}e_0) \to L_p(e\mathcal{M}e)$, determined by*

$$\gamma^*_{\rho,p}\Big(h_{\rho\circ\gamma}^{\frac{1}{2p}} b h_{\rho\circ\gamma}^{\frac{1}{2p}}\Big) = h_\rho^{\frac{1}{2p}} \gamma(b) h_\rho^{\frac{1}{2p}}, \qquad b \in \mathcal{N}, \tag{2.3}$$

is a contraction such that

$$(\gamma^*_\rho)_*\Big(h_{\rho\circ\gamma}^{\frac{p-1}{2p}} k h_{\rho\circ\gamma}^{\frac{p-1}{2p}}\Big) = h_\rho^{\frac{p-1}{2p}} \gamma^*_{\rho,p}(k) h_\rho^{\frac{p-1}{2p}}, \qquad k \in L_p(e_0\mathcal{N}e_0). \tag{2.4}$$

Moreover, if $\rho_n \in \mathcal{M}_*^+$ are such that $s(\rho) \leq s(\rho_n)$ and $\|\rho_n - \rho\|_1 \to 0$, then for any $k \in L_p(e_0\mathcal{N}e_0)$ we have $\gamma^*_{\rho_n,p}(k) \to \gamma^*_{\rho,p}(k)$ in $L_p(\mathcal{M})$.

Proof. We use Kosaki's symmetric L_p-spaces $L_p(e_0\mathcal{N}e_0, \rho\circ\gamma)$ and $L_p(e\mathcal{M}e, \rho)$ (see (C.4) in Appendix C). The map $(\gamma^*_\rho)_* : L_1(e_0\mathcal{N}e_0) \to L_1(e\mathcal{M}e)$ is contractive with respect to $\|\cdot\|_1$. Its restriction to $h_{\rho\circ\gamma}^{1/2}\mathcal{N}h_{\rho\circ\gamma}^{1/2}$ ($\subseteq L_1(e_0\mathcal{N}e_0)$) is given by (2.1), which is also contractive with respect to $\|\cdot\|_{\infty,\rho\circ\gamma}$ and $\|\cdot\|_{\infty,\rho}$. Hence it follows from the Riesz–Thorin theorem that $(\gamma^*_\rho)_*$ is a contraction from $L_p(e_0\mathcal{N}e_0, \rho\circ\gamma)$ to $L_p(e\mathcal{M}e, \rho)$ for any $p \in (1, \infty)$. By (C.4) note that we have isometric isomorphisms

$$\begin{aligned} k \in L_p(e_0\mathcal{N}e_0) &\mapsto h_{\rho\circ\gamma}^{\frac{p-1}{2p}} k h_{\rho\circ\gamma}^{\frac{p-1}{2p}} \in L_p(e_0\mathcal{N}e_0, \rho\circ\gamma), \\ h \in L_p(e\mathcal{M}e) &\mapsto h_\rho^{\frac{p-1}{2p}} h h_\rho^{\frac{p-1}{2p}} \in L_p(e\mathcal{M}e, \rho). \end{aligned}$$

Hence we can define a contraction $\gamma^*_{\rho,p} : L_p(e_0\mathcal{N}e_0) \to L_p(e\mathcal{M}e)$ by (2.4). Then, for $k = h_{\rho\circ\gamma}^{\frac{1}{2p}} b h_{\rho\circ\gamma}^{\frac{1}{2p}}$ with $b \in \mathcal{N}$ we have

$$h_\rho^{\frac{p-1}{2p}} \gamma^*_{\rho,p}\Big(h_{\rho\circ\gamma}^{\frac{1}{2p}} b h_{\rho\circ\gamma}^{\frac{1}{2p}}\Big) h_\rho^{\frac{p-1}{2p}} = (\gamma^*_\rho)_*\Big(h_{\rho\circ\gamma}^{\frac{1}{2}} b h_{\rho\circ\gamma}^{\frac{1}{2}}\Big) = h_\rho^{\frac{1}{2}} \gamma(b) h_\rho^{\frac{1}{2}},$$

so that (2.3) holds. Since $h_{\rho\circ\gamma}^{\frac{1}{2p}} \mathcal{N} h_{\rho\circ\gamma}^{\frac{1}{2p}}$ is dense in $L_p(e_0\mathcal{N}e_0)$, this proves the first part of the statement. [I have written here slightly in more detail.]

Let ρ_n be a sequence as required and let $k \in L_p(e_0\mathcal{N}e_0)$. By the assumptions on the supports, $\gamma^*_{\rho_n,p}$ is well defined on k for all n. Further, we may assume that $k = h_{\rho\circ\gamma}^{\frac{1}{2p}} b h_{\rho\circ\gamma}^{\frac{1}{2p}}$ for some $b \in \mathcal{N}$, since the set of such elements is dense in $L_p(e_0\mathcal{N}e_0)$ and all the maps are contractions. Put $k_n := h_{\rho_n\circ\gamma}^{\frac{1}{2p}} b h_{\rho_n\circ\gamma}^{\frac{1}{2p}}$, then we have

$$\gamma_{\rho,p}^*(k) = h_\rho^{\frac{1}{2p}}\gamma(b)h_\rho^{\frac{1}{2p}}, \qquad \gamma_{\rho_n,p}^*(k_n) = h_{\rho_n}^{\frac{1}{2p}}\gamma(b)h_{\rho_n}^{\frac{1}{2p}},$$

and we have $k_n \to k$ in $L_p(\mathcal{N})$ and $\gamma_{\rho_n,p}^*(k_n) \to \gamma_{\rho,p}^*(k)$ in $L_p(\mathcal{M})$. Indeed, this follows by the Hölder inequality and continuity of the map $L_1(\mathcal{M})^+ \ni h \mapsto h^{\frac{1}{2p}} \in L_{2p}(\mathcal{M})^+$; see [24, Lemma 3.4]. Therefore

$$\|\gamma_{\rho_n,p}^*(k) - \gamma_{\rho,p}^*(k)\|_p \le \|\gamma_{\rho_n,p}^*(k-k_n)\|_p + \|\gamma_{\rho_n,p}^*(k_n) - \gamma_{\rho,p}^*(k)\|_p \to 0,$$

showing the latter statement. □

Lemma 2.2. *Let $\gamma : \mathcal{N} \to \mathcal{M}$ be a normal positive unital map, and let $\rho \in \mathcal{M}_*^+$, $\rho \neq 0$, and $b \in \mathcal{N}^+$.*

(i) If $p \in [1/2, 1)$, then

$$\left\| h_{\rho\circ\gamma}^{\frac{1}{2p}} b h_{\rho\circ\gamma}^{\frac{1}{2p}} \right\|_p \le \left\| h_\rho^{\frac{1}{2p}}\gamma(b)h_\rho^{\frac{1}{2p}} \right\|_p.$$

(ii) If $p \in [1, \infty]$, the inequality reverses.

Proof. Let us denote $\beta := \gamma_\rho^*$ and let $\omega \in \mathcal{N}_*^+$ be such that $h_\omega := h_{\rho\circ\gamma}^{\frac{1}{2}} b h_{\rho\circ\gamma}^{\frac{1}{2}} \in L_1(\mathcal{N})^+$. Then β is a normal positive unital map, and by (2.1) we have

$$\beta_*(h_\omega) = h_\rho^{\frac{1}{2}}\gamma(b)h_\rho^{\frac{1}{2}}, \qquad \beta_*(h_{\rho\circ\gamma}) = h_\rho.$$

Let $p \in [1/2, 1)$, then

$$\begin{aligned}\left\| h_\rho^{\frac{1}{2p}}\gamma(b)h_\rho^{\frac{1}{2p}} \right\|_p^p &= \left\| h_\rho^{\frac{1-p}{2p}}\beta_*(h_\omega)h_\rho^{\frac{1-p}{2p}} \right\|_p^p = Q_{p,p}(\beta_*(h_\omega)\|h_\rho) = Q_{p,p}(\beta_*(h_\omega)\|\beta_*(h_{\rho\circ\gamma})) \\ &\ge Q_{p,p}(h_\omega\|h_{\rho\circ\gamma}) = \left\| h_{\rho\circ\gamma}^{\frac{1-p}{2p}} h_\omega h_{\rho\circ\gamma}^{\frac{1-p}{2p}} \right\|_p^p = \left\| h_{\rho\circ\gamma}^{\frac{1}{2p}} b h_{\rho\circ\gamma}^{\frac{1}{2p}} \right\|_p^p.\end{aligned}$$

Here we have used the DPI for the sandwiched Rényi divergence $D_{\alpha,\alpha}$ for $\alpha \in [1/2, 1)$; see [19, Theorem 4.1]. This proves (i). The case (ii) is immediate from Lemma 2.1. This was proved also in [21] (see Eq. (22) therein), by using the same argument. □

Theorem 2.3 (DPI)**.** *Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$ and let $\gamma : \mathcal{N} \to \mathcal{M}$ be a normal positive unital map. Assume either of the following conditions:*

(i) $0 < \alpha < 1$, $\max\{\alpha, 1-\alpha\} \le z$,

(ii) $\alpha > 1$, $\max\{\alpha/2, \alpha - 1\} \le z \le \alpha$.

Then we have

$$D_{\alpha,z}(\psi\circ\gamma\|\varphi\circ\gamma) \le D_{\alpha,z}(\psi\|\varphi).$$

Proof. Under the conditions (i), the DPI was proved in [21, Theorem 1(viii)]. Since parts of this proof will be used below, we repeat it here.

Assume the conditions in (i) and put $p := \frac{z}{\alpha}$, $r := \frac{z}{1-\alpha}$, so that $p, r \geq 1$. For any $b \in \mathcal{N}^{++}$, we have by the Choi inequality [6] that $\gamma(b)^{-1} \leq \gamma(b^{-1})$, so that by Lemmas A.2 and 2.2(ii), we have

$$\left\| h_\varphi^{\frac{1}{2r}} \gamma(b)^{-1} \varphi^{\frac{1}{2r}} \right\|_r \leq \left\| h_\varphi^{\frac{1}{2r}} \gamma(b^{-1}) \varphi^{\frac{1}{2r}} \right\|_r \leq \left\| h_{\varphi\circ\gamma}^{\frac{1}{2r}} b^{-1} h_{\varphi\circ\gamma}^{\frac{1}{2r}} \right\|_r^r. \tag{2.5}$$

Using the variational expression in Theorem 1.3(i), we have

$$\begin{aligned} Q_{\alpha,z}(\psi\|\varphi) &\leq \alpha \| h_\psi^{\frac{1}{2p}} \gamma(b) h_\psi^{\frac{1}{2p}} \|_p^p + (1-\alpha) \| h_\varphi^{\frac{1}{2r}} \gamma(b)^{-1} h_\varphi^{\frac{1}{2r}} \|_r^r \\ &\leq \alpha \| h_{\psi\circ\gamma}^{\frac{1}{2p}} b h_{\psi\circ\gamma}^{\frac{1}{2p}} \|_p^p + (1-\alpha) \| h_{\varphi\circ\gamma}^{\frac{1}{2r}} b^{-1} h_{\varphi\circ\gamma}^{\frac{1}{2r}} \|_r^r. \end{aligned}$$

Since this holds for all $b \in \mathcal{N}^{++}$, it follows that $Q_{\alpha,z}(\psi\|\varphi) \leq Q_\alpha(\psi\circ\gamma\|\varphi\circ\gamma)$, which proves the DPI in this case.

Assume next the condition (ii), and put $p := \frac{z}{\alpha}$, $q := \frac{z}{\alpha-1}$, so that $p \in [1/2, 1)$ and $q \geq 1$. Using Theorem 1.3(ii), we get for any $b \in \mathcal{N}^+$,

$$\begin{aligned} Q_{\alpha,z}(\psi\|\varphi) &\geq \alpha \| h_\psi^{\frac{1}{2p}} \gamma(b) h_\psi^{\frac{1}{2p}} \|_p^p - (\alpha-1) \| h_\varphi^{\frac{1}{2q}} \gamma(b) h_\varphi^{\frac{1}{2q}} \|_q^q \\ &\geq \alpha \| h_{\psi\circ\gamma}^{\frac{1}{2p}} b h_{\psi\circ\gamma}^{\frac{1}{2p}} \|_p^p - (\alpha-1) \| h_{\varphi\circ\gamma}^{\frac{1}{2r}} b h_{\varphi\circ\gamma}^{\frac{1}{2q}} \|_q^q, \end{aligned}$$

here we used both (i) and (ii) in Lemma 2.2. Again, since this holds for all $b \in \mathcal{N}^+$, we get the desired inequality. ☐

2.2 Martingale convergence

An important consequence of DPI is the martingale convergence property that will be proved in this subsection. Here assume that $\mathcal{M}$ is a σ-finite von Neumann algebra.

Theorem 2.4. *Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$, and let $\{\mathcal{M}_i\}$ be an increasing net of von Neumann subalgebras of $\mathcal{M}$ containing the unit of $\mathcal{M}$ such that $\mathcal{M} = \left(\bigcup_i \mathcal{M}_i\right)''$. Assume that α and z satisfy the DPI bounds (that is, conditions (i) or (ii) ["condition" is correct grammatically? If then, "bounds" → "bound"?] in Theorem 2.3). Then we have*

$$D_{\alpha,z}(\psi\|\varphi) = \lim_i D_{\alpha,z}(\psi|_{\mathcal{M}_i}\|\varphi|_{\mathcal{M}_i}) \quad \textit{increasingly.} \tag{2.6}$$

Proof. Let $\varphi_i := \varphi|_{\mathcal{M}_i}$ and $\psi_i := \psi|_{\mathcal{M}_i}$. From Theorem 2.3, it follows that $D_{\alpha,z}(\psi\|\varphi) \geq D_{\alpha,z}(\psi_i\|\varphi_i)$ for all i and $i \mapsto D_{\alpha,z}(\psi_i\|\varphi_i)$ is increasing. Hence, to show (2.6), it suffices to prove that

$$D_{\alpha,z}(\psi\|\varphi) \leq \sup_i D_{\alpha,z}(\psi_i\|\varphi_i). \tag{2.7}$$

To do this, we may assume that φ is faithful. Indeed, assume that (2.7) has been shown when φ is faithful. For general $\varphi \in \mathcal{M}_*^+$, from the assumption of $\mathcal{M}$ being σ-finite, there exists a $\varphi_0 \in \mathcal{M}_*^+$ with $s(\varphi_0) = \mathbf{1} - s(\varphi)$. Let $\varphi^{(n)} := \varphi + n^{-1}\varphi_0$ and $\varphi_i^{(n)} := \varphi^{(n)}|_{\mathcal{M}_i}$ for each $n \in \mathbb{N}$. Thanks to the lower semi-continuity [21, Theorems 1(iv) and 2(iv)] and the order relation [21, Theorems 1(iii) and 2(iii)] we have

$$\begin{aligned} D_{\alpha,z}(\psi\|\varphi) &\le \liminf_{n\to\infty} D_{\alpha,z}(\psi\|\varphi^{(n)}) \\ &\le \liminf_{n\to\infty} \sup_i D_{\alpha,z}(\psi_i\|\varphi_i^{(n)}) \\ &\le \sup_i D_{\alpha,z}(\psi_i\|\varphi_i), \end{aligned}$$

proving (2.7) for general φ. Below we assume the faithfulness of φ and write $\mathcal{E}_{\mathcal{M}_i,\varphi}$ for the generalized conditional expectation from $\mathcal{M}$ to $\mathcal{M}_i$ with respect to φ. Then we note that we have by [17, Theorem 3],

$$\psi_i \circ \mathcal{E}_{\mathcal{M}_i,\varphi} = \psi \circ \mathcal{E}_{\mathcal{M}_i,\varphi} \to \psi \quad \text{in the norm,} \tag{2.8}$$

as well as

$$\varphi_i \circ \mathcal{E}_{\mathcal{M}_i,\varphi} = \varphi \circ \mathcal{E}_{\mathcal{M}_i,\varphi} = \varphi. \tag{2.9}$$

Using lower semicontinuity and DPI, we obtain

$$D_{\alpha,z}(\psi\|\varphi) \le \liminf_i D_{\alpha,z}(\psi_i \circ \mathcal{E}_{\mathcal{M}_i,\varphi}\|\varphi) \le \liminf_i D_{\alpha,z}(\psi_i\|\varphi) \le \sup_i D_{\alpha,z}(\psi_i\|\varphi).$$

□

[I like to add:] The following proposition is another martingale type convergence, which is not included in Theorem 2.4 since $e_i\mathcal{M}e_i$'s do not contain the unit of $\mathcal{M}$.

Proposition 2.5. *Let $\psi,\varphi \in \mathcal{M}_*^+$, $\psi \ne 0$, and let $\{e_i\}$ be an increasing net of projections in $\mathcal{M}$ such that $e_i \nearrow \mathbf{1}$. If α and z satisfy the DPI bounds [bound?], then we have*

$$D_{\alpha,z}(\psi\|\varphi) = \lim_i D_{\alpha,z}(e_i\psi e_i\|e_i\varphi e_i),$$

where $e_i\psi e_i$, $e_i\varphi e_i$ are the restrictions of ψ, φ to the reduced von Neumann subalgebra $e_i\mathcal{M}e_i$.

Proof. It suffices to show that

$$Q_{\alpha,z}(\psi\|\varphi) = \lim_i Q_{\alpha,z}(e_i\psi e_i\|e_i\varphi e_i).$$

Let $\mathcal{M}_i := e_i\mathcal{M}e_i + \mathbb{C}(\mathbf{1}-e_i)$; then $\{\mathcal{M}_i\}$ is an increasing net of von Neumann subalgebras of $\mathcal{M}$ containing the unit of $\mathcal{M}$ with $\mathcal{M} = \left(\bigcup_i \mathcal{M}_i\right)''$. By Theorem 2.4 and [21, Theorems 1(ii) and 2(ii)] we have

$$Q_{\alpha,z}(\psi\|\varphi) = \lim_i \bigl[Q_{\alpha,z}(e_i\psi e_i\|e_i\varphi e_i) + \psi(\mathbf{1}-e_i)^\alpha \varphi(\mathbf{1}-e_i)^{1-\alpha}\bigr].$$

Here, $\psi(\mathbf{1}-e_i)^\alpha \varphi(\mathbf{1}-e_i)^{1-\alpha}$ is defined with the conventions

$$0^{1-\alpha} := \begin{cases} 0 & (0<\alpha<1), \\ \infty & (\alpha>1), \end{cases} \qquad \lambda\cdot\infty := \begin{cases} 0 & (\lambda=0), \\ \infty & (\lambda>0). \end{cases}$$

Then the statement holds if we show the following:

(1) If $Q_{\alpha,z}(\psi\|\varphi) = \infty$, then $\lim_i Q_{\alpha,z}(e_i\psi e_i\|e_i\varphi e_i) = \infty$.

(2) If $Q_{\alpha,z}(\psi\|\varphi) < \infty$, then $\lim_i \psi(\mathbf{1} - e_i)^\alpha \varphi(\mathbf{1} - e_i)^{1-\alpha} = 0$.

These two facts can be shown in the same way as in the proof of [12, Theorem 4.5], whose details are omitted here. □

3 Equality in DPI and reversibility of channels

In what follows, a channel is a normal 2-positive unital map $\gamma : \mathcal{N} \to \mathcal{M}$.

Definition 3.1. Let $\gamma : \mathcal{N} \to \mathcal{M}$ be a channel and let $\mathcal{S} \subset \mathcal{M}_*^+$. We say that γ is reversible (or sufficient) with respect to $\mathcal{S}$ if there exists a channel $\beta : \mathcal{M} \to \mathcal{N}$ such that

$$\rho \circ \gamma \circ \beta = \rho \quad \text{for all } \rho \in \mathcal{S}.$$

The notion of sufficient channels was introduced by Petz [32, 33], who also obtained a number of conditions characterizing this situation. It particular, it was proved in [33] that sufficient channels can be characterized by equality in the DPI for the relative entropy $D(\psi\|\varphi)$: if $D(\psi\|\varphi) < \infty$, then a channel γ is sufficient with respect to $\{\psi, \varphi\}$ if and only if

$$D(\psi \circ \gamma\|\varphi \circ \gamma) = D(\psi\|\varphi).$$

This characterization has been proved for a number of other divergence measures, including the standard Rényi divergences $D_{\alpha,1}$ with $0 < \alpha < 2$ and the sandwiched Rényi divergences $D_{\alpha,\alpha}$ for $\alpha > 1/2$ ([14, 18, 19]). Another important result of [33] shows that the Petz dual γ_φ^* is a universal recovery map, in the sense given in the proposition below.

Proposition 3.2. *Let $\gamma : \mathcal{N} \to \mathcal{M}$ be a channel and let $\varphi \in \mathcal{M}_*^+$ be such that both φ and $\varphi \circ \gamma$ are faithful. Then the following hold:*

(i) For any $\psi \in \mathcal{M}_*^+$, γ is sufficient with respect to $\{\psi, \varphi\}$ if and only if $\psi \circ \gamma \circ \gamma_\varphi^* = \psi$.

(ii) There is a faithful normal conditional expectation $\mathcal{E}$ from $\mathcal{M}$ onto a von Neumann subalgebra of $\mathcal{M}$ such that $\varphi \circ \mathcal{E} = \varphi$, and γ is sufficient with respect to $\{\psi, \varphi\}$ if and only if also $\psi \circ \mathcal{E} = \psi$.

Note (see [33, Theorem 2] and the proof of [33, Theorem 3]) that the range of the conditional expectation $\mathcal{E}$ in statement (ii) above is the set of fixed points of the channel $\gamma \circ \gamma_\varphi^*$. [I write references in detail, since statement (ii) is implicit in [33].]

Our aim in this section is to prove that equality in the DPI for $D_{\alpha,z}$ with values of the parameters (strictly) contained in the DPI bounds of Theorem 2.3 characterizes sufficiency of channels. Throughout this section, we use the notations $\psi_0 := \psi \circ \gamma$ and $\varphi_0 := \varphi \circ \gamma$. We also denote

$$p := \frac{z}{\alpha}, \qquad r := \frac{z}{1 - \alpha}, \qquad q := -r = \frac{z}{\alpha - 1}.$$

3.1 The case $\alpha \in (0,1)$

Here we study equality in the DPI for $D_{\alpha,z}$ with $\alpha \in (0,1)$, for a pair of positive normal functionals $\psi, \varphi \in \mathcal{M}_*^+$ and a normal positive unital map $\gamma : \mathcal{N} \to \mathcal{M}$. We first prove some equality conditions in the case $\psi \sim \varphi$.

Proposition 3.3. *Let $0 < \alpha < 1$, $\max\{\alpha, 1-\alpha\} \le z$. Assume that $\psi \sim \varphi$ and let $\gamma : \mathcal{N} \to \mathcal{M}$ be a normal positive unital map. Let $\bar a \in \mathcal{M}^{++}$ be the unique minimizer as in Lemma 1.4 for $Q_{\alpha,z}(\psi\|\varphi)$ and let $\bar a_0 \in \mathcal{N}^{++}$ be the minimizer for $Q_{\alpha,z}(\psi_0\|\varphi_0)$. The following conditions are equivalent:*

(i) $D_{\alpha,z}(\psi_0\|\varphi_0) = D_{\alpha,z}(\psi\|\varphi)$, *i.e.,* $Q_{\alpha,z}(\psi_0\|\varphi_0) = Q_{\alpha,z}(\psi\|\varphi)$.

(ii) $\gamma(\bar a_0) = \bar a$ *and* $\left\| h_\psi^{\frac{1}{2p}} \gamma(\bar a_0) h_\psi^{\frac{1}{2p}} \right\|_p = \left\| h_{\psi_0}^{\frac{1}{2p}} \bar a_0 h_{\psi_0}^{\frac{1}{2p}} \right\|_p$.

(iii) $\left\| h_\psi^{\frac{1}{2p}} \bar a h_\psi^{\frac{1}{2p}} \right\|_p = \left\| h_{\psi_0}^{\frac{1}{2p}} \bar a_0 h_{\psi_0}^{\frac{1}{2p}} \right\|_p$.

(iv) $\gamma(\bar a_0^{-1}) = \bar a^{-1}$ *and* $\left\| h_\varphi^{\frac{1}{2r}} \gamma(\bar a_0^{-1}) h_\varphi^{\frac{1}{2r}} \right\|_r = \left\| h_{\varphi_0}^{\frac{1}{2r}} \bar a_0^{-1} h_{\varphi_0}^{\frac{1}{2r}} \right\|_r$.

(v) $\left\| h_\varphi^{\frac{1}{2r}} \bar a^{-1} h_\varphi^{\frac{1}{2r}} \right\|_r = \left\| h_{\varphi_0}^{\frac{1}{2r}} \bar a_0^{-1} h_{\varphi_0}^{\frac{1}{2r}} \right\|_r$.

Proof. Since $\psi \sim \varphi$ by assumption and hence $\psi_0 \sim \varphi_0$, we have $s(\psi) = s(\varphi)$ and $s(\psi_0) = s(\varphi_0)$. Using restrictions explained in the beginning of Sec. 2, we may assume that all $\psi, \varphi, \psi_0, \varphi_0$ are faithful.

(i) $\Longrightarrow$ (ii) & (iv). By Lemma 2.2(ii)

$$\left\| h_\psi^{\frac{1}{2p}} \gamma(\bar a_0) h_\psi^{\frac{1}{2p}} \right\|_p \le \left\| h_{\psi_0}^{\frac{1}{2p}} \bar a_0 h_{\psi_0}^{\frac{1}{2p}} \right\|_p, \tag{3.1}$$

and by (2.5) we have

$$\left\| h_\varphi^{\frac{1}{2r}} \gamma(\bar a_0)^{-1} h_\varphi^{\frac{1}{2r}} \right\|_r \le \left\| h_\varphi^{\frac{1}{2r}} \gamma(\bar a_0^{-1}) h_\varphi^{\frac{1}{2r}} \right\|_r \le \left\| h_{\varphi_0}^{\frac{1}{2r}} \bar a_0^{-1} h_{\varphi_0}^{\frac{1}{2r}} \right\|_r. \tag{3.2}$$

From (3.1) and (3.2) it follows that

$$\begin{aligned} Q_{\alpha,z}(\psi\|\varphi) &\le \alpha \left\| h_\psi^{\frac{1}{2p}} \gamma(\bar a_0) h_\psi^{\frac{1}{2p}} \right\|_p^p + (1-\alpha) \left\| h_\varphi^{\frac{1}{2r}} \gamma(\bar a_0)^{-1} h_\varphi^{\frac{1}{2r}} \right\|_r^r \\ &\le \alpha \left\| h_{\psi_0}^{\frac{1}{2p}} \bar a_0 h_{\psi_0}^{\frac{1}{2p}} \right\|_p^p + (1-\alpha) \left\| h_{\varphi_0}^{\frac{1}{2r}} \bar a_0^{-1} h_{\varphi_0}^{\frac{1}{2r}} \right\|_r^r = Q_{\alpha,z}(\psi_0\|\varphi_0) = Q_{\alpha,z}(\psi\|\varphi). \end{aligned}$$

By uniqueness in Lemma 1.4 we find that $\gamma(\bar a_0) = \bar a$ and all the inequalities in (3.1) and (3.2) must become equalities. Since $\gamma(\bar a_0^{-1}) \ge \gamma(\bar a_0)^{-1}$, we see by Lemma A.2 that the equality in (3.2) yields $\gamma(\bar a_0^{-1}) = \gamma(\bar a_0)^{-1} = \bar a^{-1}$. Therefore, (ii) and (iv) hold.

The implications (ii) $\Longrightarrow$ (iii) and (iv) $\Longrightarrow$ (v) are obvious.

(iii) $\implies$ (i). By (iii) with the equality (1.4) in Lemma 1.4 we have

$$\begin{aligned}Q_{\alpha,z}(\psi\|\varphi) &= \mathrm{tr}\Big(h_\psi^{\frac{1}{2p}} h_\varphi^{\frac{1}{r}} h_\psi^{\frac{1}{2p}}\Big)^z = \mathrm{tr}\Big(h_\psi^{\frac{1}{2p}} \bar{a} h_\psi^{\frac{1}{2p}}\Big)^p \\ &= \mathrm{tr}\Big(h_{\psi_0}^{\frac{1}{2p}} \bar{a}_0 h_{\psi_0}^{\frac{1}{2p}}\Big)^p = \mathrm{tr}\Big(h_{\psi_0}^{\frac{1}{2p}} h_{\varphi_0}^{\frac{1}{r}} h_{\psi_0}^{\frac{1}{2p}}\Big)^z = Q_{\alpha,z}(\psi_0\|\varphi_0).\end{aligned}$$

(v) $\implies$ (i). Similarly, by (v) with the equality (1.5) in Lemma 1.4 we have

$$\begin{aligned}Q_{\alpha,z}(\psi\|\varphi) &= \mathrm{tr}\Big(h_\varphi^{\frac{1}{2r}} h_\psi^{\frac{1}{p}} h_\varphi^{\frac{1}{2r}}\Big)^z = \mathrm{tr}\Big(h_\varphi^{\frac{1}{2r}} \bar{a}^{-1} h_\varphi^{\frac{1}{2r}}\Big)^r \\ &= \mathrm{tr}\Big(h_{\varphi_0}^{\frac{1}{2r}} \bar{a}_0^{-1} h_{\varphi_0}^{\frac{1}{2r}}\Big)^r = \mathrm{tr}\Big(h_{\varphi_0}^{\frac{1}{2r}} h_{\psi_0}^{\frac{1}{p}} h_{\varphi_0}^{\frac{1}{2r}}\Big)^z = Q_{\alpha,z}(\psi_0\|\varphi_0).\end{aligned}$$

□

Remark 3.4. Note that the above conditions extend the results obtained in [27] and [38] in the finite-dimensional case. Indeed, the first condition in (ii) with $\alpha = z$ is equivalent to the condition in [27, Theorem 1], as seen from (1.5). [Better to mention that [27, Theorem 1] was shown when $s(\psi) \le s(\varphi)$, while $\psi \sim \varphi$ is assumed in the above proposition.] Note here that in this case the second condition in (ii) is automatic. [Is this directly checked? Or, do you mean this is a consequence of the above proposition?] Moreover, (ii) extends the necessary condition in [38, Theorem 1.2(2)] to a necessary and sufficient one. While in both of these works γ was required to be completely positive, only positivity is enough for our result.

Theorem 3.5. *Let $0 < \alpha < 1$ and $\max\{\alpha, 1-\alpha\} \le z$. Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$, and assume either that $\alpha < z$ and $s(\varphi) \le s(\psi)$, or that $1-\alpha < z$ and $s(\psi) \le s(\varphi)$. Then a channel $\gamma : \mathcal{N} \to \mathcal{M}$ is reversible with respect to $\{\psi, \varphi\}$ if and only if*

$$D_{\alpha,z}(\psi\|\varphi) = D_{\alpha,z}(\psi \circ \gamma\|\varphi \circ \gamma).$$

Proof. This proof is a modification of the proof of [19, Theorem 5.1]. We will assume that $s(\varphi) \le s(\psi)$ and $\alpha < z$, that is, $p > 1$. In the other case we may exchange the roles of p, r and of ψ, φ by the equality $Q_{\alpha,z}(\psi\|\varphi) = Q_{1-\alpha,z}(\varphi\|\psi)$. As before, we may assume that both ψ and ψ_0 are faithful.

The strategy of the proof is to use known results in [18] for the sandwiched Rényi divergence $D_{p,p}$ with $p > 1$. For this, let $\mu, \omega \in \mathcal{M}_*^+$ be such that

$$h_\mu = \Big| h_\varphi^{\frac{1}{2r}} h_\psi^{\frac{1}{2p}} \Big|^{2z}, \qquad h_\omega = h_\psi^{\frac{p-1}{2p}} h_\mu^{\frac{1}{p}} h_\psi^{\frac{p-1}{2p}},$$

and notice that

$$Q_{z,\alpha}(\psi\|\varphi) = Q_{p,p}(\omega\|\psi).$$

Let $\mu_0, \omega_0 \in \mathcal{N}_*^+$ be similar functionals obtained from ψ_0, φ_0. Then we have the equality

$$Q_{p,p}(\omega_0\|\psi_0) = Q_{\alpha,z}(\psi_0\|\varphi_0) = Q_{\alpha,z}(\psi\|\varphi) = Q_{p,p}(\omega\|\psi). \tag{3.3}$$

Our first goal is to show that $\omega_0 = \omega \circ \gamma$, which implies by [18, Theorem 4.6] that γ is sufficient with respect to $\{\omega, \psi\}$. We then apply Proposition 3.2 and the properties of the extensions of the conditional expectation $\mathcal{E}$ to the Haagerup L_p-spaces proved in [20].

Let us remark here that if $\psi \sim \varphi$, it follows from (1.4) that $h_\omega = h_\psi^{\frac{1}{2}} \bar{a} h_\psi^{\frac{1}{2}}$ and $h_{\omega_0} = h_{\psi_0}^{\frac{1}{2}} \bar{a}_0 h_{\psi_0}^{\frac{1}{2}}$. Hence from (2.1) and condition (ii) in Proposition 3.3, we immediately have

$$(\gamma_\psi^*)_*(h_{\omega_0}) = h_\psi^{\frac{1}{2}} \gamma(\bar{a}_0) h_\psi^{\frac{1}{2}} = h_\omega, \quad \text{i.e.,} \quad \omega_0 \circ \gamma_\psi^* = \omega,$$

as well as $\psi_0 \circ \gamma_\psi^* = \psi$ by (2.2). These and (3.3) show that γ_ψ^* is sufficient with respect to $\{\omega_0, \psi_0\}$. By Proposition 3.2 and the fact that the Petz dual $(\gamma_\psi^*)_{\psi_0}^*$ is γ itself, this implies the desired equality

$$\omega \circ \gamma = \omega_0 \circ \gamma_\psi^* \circ \gamma = \omega_0.$$

In the case $\psi \not\sim \varphi$ some more work is required. Let $\psi_n := \psi + \frac{1}{n}\varphi$ and $\varphi_n := \varphi + \frac{1}{n}\psi$. Then all ψ_n, φ_n are faithful, $\psi_n \to \psi$, $\varphi_n \to \varphi$ in $\mathcal{M}_*^+$, and moreover, $\psi_n \sim \varphi_n$ for all n. Then $\psi_n \circ \gamma \sim \varphi_n \circ \gamma$, $\psi_n \circ \gamma \to \psi_0$, $\varphi_n \circ \gamma \to \varphi_0$ and by joint continuity of $Q_{\alpha,z}$ in the norm ([21, Theorem 1(iv)]), we have

$$\lim_n Q_{\alpha,z}(\psi_n \circ \gamma \| \varphi_n \circ \gamma) = Q_{\alpha,z}(\psi_0 \| \varphi_0) = Q_{\alpha,z}(\psi \| \varphi) = \lim_n Q_{\alpha,z}(\psi_n \| \varphi_n).$$

Let $\bar{b}_n \in \mathcal{N}^{++}$ be the minimizer for the variational expression for $Q_{\alpha,z}(\psi_n \circ \gamma \| \varphi_n \circ \gamma)$ given in (1.1). Let also $\bar{a}_n$ be the minimizer for $Q_{\alpha,z}(\psi_n \| \varphi_n)$, and let $f_n : \mathcal{M}^{++} \to \mathbb{R}^+$ be the function minimized in the expression for $Q_{\alpha,z}(\psi_n \| \varphi_n)$ (see (1.8)). We then have

$$\begin{aligned} Q_{\alpha,z}(\psi_n \circ \gamma \| \varphi_n \circ \gamma) - f_n(\gamma(\bar{b}_n)) &= \alpha \left(\left\| h_{\psi_n \circ \gamma}^{\frac{1}{2p}} \bar{b}_n h_{\psi_n \circ \gamma}^{\frac{1}{2p}} \right\|_p^p - \left\| h_{\psi_n}^{\frac{1}{2p}} \gamma(\bar{b}_n) h_{\psi_n}^{\frac{1}{2p}} \right\|_p^p \right) \\ &\quad + (1-\alpha) \left(\left\| h_{\varphi_n \circ \gamma}^{\frac{1}{2r}} \bar{b}_n^{-1} h_{\varphi_n \circ \gamma}^{\frac{1}{2r}} \right\|_r^r - \left\| h_{\varphi_n}^{\frac{1}{2r}} \gamma(\bar{b}_n)^{-1} h_{\varphi_n}^{\frac{1}{2r}} \right\|_r^r \right) \geq 0, \end{aligned}$$

where the inequality follows from Lemma 2.2(ii) and (2.5). We obtain

$$Q_{\alpha,z}(\psi_n \circ \gamma \| \varphi_n \circ \gamma) - Q_{\alpha,z}(\psi_n \| \varphi_n) \geq f_n(\gamma(\bar{b}_n)) - Q_{\alpha,z}(\psi_n \| \varphi_n) \geq 0. \tag{3.4}$$

Now let $\mu_{n,0} \in \mathcal{N}_*^+$ and $\mu_n \in \mathcal{M}_*^+$ be such that (using (1.4) in Lemma 1.4)

$$h_{\mu_{n,0}}^{\frac{1}{p}} = \left| h_{\varphi_n \circ \gamma}^{\frac{1}{2r}} h_{\psi_n \circ \gamma}^{\frac{1}{2p}} \right|^{2\alpha} = h_{\psi_n \circ \gamma}^{\frac{1}{2p}} \bar{b}_n h_{\psi_n \circ \gamma}^{\frac{1}{2p}}, \qquad h_{\mu_n}^{\frac{1}{p}} = \left| h_{\varphi_n}^{\frac{1}{2r}} h_{\psi_n}^{\frac{1}{2p}} \right|^{2\alpha} = h_{\psi_n}^{\frac{1}{2p}} \bar{a}_n h_{\psi_n}^{\frac{1}{2p}}.$$

Then $h_{\mu_{n,0}}^{\frac{1}{p}} \to h_{\mu_0}^{\frac{1}{p}}$ in $L_p(\mathcal{N})$, this follows from the Hölder inequality and the fact [24] that the map $L_{2z}(\mathcal{N}) \ni h \mapsto |h|^{2\alpha} \in L_p(\mathcal{N})$ is continuous in the norm. Similarly, $h_{\mu_n}^{\frac{1}{p}} \to h_\mu^{\frac{1}{p}}$ in $L_p(\mathcal{M})$. Next, note that since $Q_{\alpha,z}(\psi_n \circ \gamma \| \varphi_n \circ \gamma)$ and $Q_{\alpha,z}(\psi_n \| \varphi_n)$ have the same limit, we see from (3.4) that $f_n(\gamma(\bar{b}_n)) - Q_{\alpha,z}(\psi_n \| \varphi_n) \to 0$. Moreover, by Lemma 2.2(ii) note that

$$\sup_n \left\| h_{\psi_n}^{\frac{1}{2p}} \gamma(\bar{b}_n) h_{\psi_n}^{\frac{1}{2p}} \right\|_p^p \leq \sup_n \left\| h_{\psi_n \circ \gamma}^{\frac{1}{2p}} \bar{b}_n h_{\psi_n \circ \gamma}^{\frac{1}{2p}} \right\|_p^p \leq \frac{1}{\alpha} \sup_n D_{\alpha,z}(\psi_n \circ \gamma \| \varphi_n \circ \gamma) < \infty.$$

Therefore, since $\left\| h_{\psi_n}^{\frac{1}{2p}} \gamma(\bar{b}_n) h_{\psi_n}^{\frac{1}{2p}} - h_{\mu_n}^{\frac{1}{p}} \right\|_p$ means $\|\xi_p(\gamma(\bar{b}_n)) - \xi_p(\bar{a}_n)\|_p$ defined for ψ_n (in place of ψ), it follows from Lemma 1.5 that $h_{\psi_n}^{\frac{1}{2p}} \gamma(\bar{b}_n) h_{\psi_n}^{\frac{1}{2p}} - h_{\mu_n}^{\frac{1}{p}} \to 0$ in $L_p(\mathcal{M})$. On the other hand, let $\gamma_{\psi_n,p}^*, \gamma_{\psi,p}^*$ be the contractions defined in Lemma 2.1. We then have

$$h_{\psi_n}^{\frac{1}{2p}} \gamma(\bar{b}_n) h_{\psi_n}^{\frac{1}{2p}} = \gamma_{\psi_n,p}^*(h_{\mu_{n,0}}^{\frac{1}{p}})$$

and since $\gamma^*_{\psi_n,p}(k) \to \gamma^*_{\psi,p}(k)$ in $L_p(\mathcal{M})$ for any $k \in L_p(s(\psi\circ\gamma)\mathcal{N}s(\psi\circ\gamma))$ by Lemma 2.1, we have

$$\left\|\gamma^*_{\psi,p}\big(h_{\mu_0}^{\frac{1}{p}}\big) - \gamma^*_{\psi_n,p}\big(h_{\mu_{n,0}}^{\frac{1}{p}}\big)\right\|_p \le \left\|(\gamma^*_{\psi,p} - \gamma^*_{\psi_n,p})\big(h_{\mu_0}^{\frac{1}{p}}\big)\right\|_p + \left\|\gamma^*_{\psi_n,p}\big(h_{\mu_0}^{\frac{1}{p}} - h_{\mu_{n,0}}^{\frac{1}{p}}\big)\right\|_p \to 0.$$

Putting all together, we obtain that

$$h_\mu^{\frac{1}{p}} = \lim_n h_{\mu_n}^{\frac{1}{p}} = \lim_n \gamma^*_{\psi_n,p}(h_{\mu_{n,0}}^{\frac{1}{p}}) = \gamma^*_{\psi,p}(h_{\mu_0}^{\frac{1}{p}}).$$

It follows from (2.4) that

$$(\gamma^*_\psi)_*(h_{\omega_0}) = h_\psi^{\frac{p-1}{2p}}\gamma^*_{\psi,p}(h_{\mu_0}^{\frac{1}{p}})h_\psi^{\frac{p-1}{2p}} = h_\psi^{\frac{p-1}{2p}}h_\mu^{\frac{1}{p}}h_\psi^{\frac{p-1}{2p}} = h_\omega.$$

As we have seen in the case $\psi\sim\varphi$ above, this and (3.3) imply that

$$\omega\circ\gamma = \omega_0\circ\gamma^*_\psi\circ\gamma = \omega_0.$$

Therefore, we have shown that γ is sufficient with respect to $\{\omega,\psi\}$.

Next, let $\mathcal{E}$ be the faithful normal conditional expectation onto the set of fixed points of $\gamma\circ\gamma^*_\psi$ (see a note after Proposition 3.2). Then by Proposition 3.2(ii), $\mathcal{E}$ preserves both ψ and ω, which by [20] implies that

$$h_\psi^{\frac{p-1}{2p}}h_\mu^{\frac{1}{p}}h_\psi^{\frac{p-1}{2p}} = h_\omega = \mathcal{E}_*(h_\omega) = h_\psi^{\frac{p-1}{2p}}\mathcal{E}_p\big(h_\mu^{\frac{1}{p}}\big)h_\psi^{\frac{p-1}{2p}},$$

so that $\big|h_\varphi^{\frac{1}{2r}}h_\psi^{\frac{1}{2p}}\big|^{2\alpha} = h_\mu^{\frac{1}{p}} \in L_p(\mathcal{E}(\mathcal{M}))$ and consequently $\big|h_\varphi^{\frac{1}{2r}}h_\psi^{\frac{1}{2p}}\big| = h_\mu^{\frac{1}{2z}} \in L_{2z}(\mathcal{E}(\mathcal{M}))^+$. By the assumption $2z>1$ note that we may use the bimodule property of the extension of $\mathcal{E}$; see [20, Proposition 2.3(ii)]. Let $h_\varphi^{\frac{1}{2r}}h_\psi^{\frac{1}{2p}} = u\big|h_\varphi^{\frac{1}{2r}}h_\psi^{\frac{1}{2p}}\big|$ be the polar decomposition in $L_{2z}(\mathcal{M})$; then we have

$$u^*h_\varphi^{\frac{1}{2r}}h_\psi^{\frac{1}{2p}} = \mathcal{E}_{2z}\big(u^*h_\varphi^{\frac{1}{2r}}h_\psi^{\frac{1}{2p}}\big) = \mathcal{E}_{2r}\big(u^*h_\varphi^{\frac{1}{2r}}\big)h_\psi^{\frac{1}{2p}},$$

which implies that $u^*h_\varphi^{\frac{1}{2r}} \in L_{2r}(\mathcal{E}(\mathcal{M}))$. Since ψ is faithful, we have

$$\ker\big(h_\varphi^{\frac{1}{2r}}h_\psi^{\frac{1}{2p}}\big) = \ker\big(h_\psi^{\frac{1}{2p}}h_\varphi^{\frac{1}{2r}}\big) = \ker h_\varphi^{\frac{1}{2r}} = \ker h_\varphi,$$

which implies that $uu^* = s(\varphi)$. [This seems more transparent.] Hence by uniqueness of the polar decomposition in $L_{2r}(\mathcal{M})$ and $L_{2r}(\mathcal{E}(\mathcal{M}))$, we obtain that $h_\varphi^{\frac{1}{2r}} \in L_{2r}(\mathcal{E}(\mathcal{M}))^+$ and $u\in\mathcal{E}(\mathcal{M})$. Therefore, we must have $h_\varphi \in L_1(\mathcal{E}(\mathcal{M}))$, so that $\varphi\circ\mathcal{E} = \varphi$ and γ is sufficient with respect to $\{\psi,\varphi\}$ by Proposition 3.2(ii) again. $\square$

3.2 The case $\alpha > 1$

We now turn to the case $\alpha>1$. Then within the DPI bounds[bound (?)], we have $p := \frac{z}{\alpha} \in [1/2,1]$ and $q := \frac{z}{\alpha-1} \ge 1$, and we note that we always have $p<q$. Here we need to assume that $D_{\alpha,z}(\psi\|\varphi)<\infty$, so that by Lemma 1.2 there is some (unique) $y\in L_{2z}(\mathcal{M})$ such that

$$h_\psi^{\frac{1}{2p}} = yh_\varphi^{\frac{1}{2q}}.$$

By the proof of Theorem 1.3, we have the following variational expression

$$Q_{\alpha,z}(\psi\|\varphi) = \sup_{w\in L_q(\mathcal{M})^+} \left\{\alpha \mathrm{tr}\big((ywy^*)^p\big) - (\alpha-1)\mathrm{tr}\big(w^q\big)\right\}. \tag{3.5}$$

Indeed, we note that x in the proof of Theorem 1.3 is y^*y and $\mathrm{tr}\big((x^{\frac{1}{2}}wx^{\frac{1}{2}})^p\big)$ in expression (1.3) is rewritten as $\mathrm{tr}\big((|y|w|y|)^p\big) = \mathrm{tr}\big((ywy^*)^p\big)$. The supremum is attained at a unique point $\bar{w} = (y^*y)^{\alpha-1} \in L_q(\mathcal{M})^+$, uniqueness follows from strict concavity of the function $w \mapsto \alpha\mathrm{tr}\big((ywy^*)^p\big) - (\alpha-1)\mathrm{tr}\big(w^q\big)$.

By DPI, we have $D_{\alpha,z}(\psi_0\|\varphi_0) \le D_{\alpha,z}(\psi\|\varphi) < \infty$, so that there is some (unique) $y_0 \in L_{2z}(\mathcal{N})$ such that

$$h_{\psi_0}^{\frac{1}{2p}} = y_0 h_{\varphi_0}^{\frac{1}{2q}}.$$

Since $D_{\alpha,z}(\psi\|\varphi) < \infty$ implies that $s(\psi) \le s(\varphi)$, we may assume that both φ and φ_0 are faithful.

Lemma 3.6. *Let $\gamma : \mathcal{N} \to \mathcal{M}$ be a normal positive unital map. Let $\gamma^*_{\varphi,q} : L_q(\mathcal{N}) \to L_q(\mathcal{M})$ be the contraction as in lemma 2.1. Let $\bar{w} := (y^*y)^{\alpha-1} \in L_q(\mathcal{M})$ and $\bar{w}_0 := (y_0^*y_0)^{\alpha-1} \in L_q(\mathcal{N})$. Then equality in the DPI for $D_{\alpha,z}(\psi\|\varphi)$ holds if and only if*

$$\bar{w} = \gamma^*_{\varphi,q}(\bar{w}_0) \quad \textit{and} \quad \mathrm{tr}\big(\bar{w}_0^q\big) = \mathrm{tr}\big(\gamma^*_{\varphi,q}(\bar{w}_0)^q\big). \tag{3.6}$$

Proof. We first show that for any $w_0 \in L_q(\mathcal{N})^+$,

$$\mathrm{tr}\big((y\gamma^*_{\varphi,q}(w_0)y^*)^p\big) \ge \mathrm{tr}\big((y_0w_0y_0^*)^p\big). \tag{3.7}$$

Let us first assume that $w_0 = h_{\varphi_0}^{\frac{1}{2q}} b h_{\varphi_0}^{\frac{1}{2q}}$ for some $b \in \mathcal{N}_+$. Then $\gamma^*_{\varphi,q}(w_0) = h_{\varphi}^{\frac{1}{2q}}\gamma(b)h_{\varphi}^{\frac{1}{2q}}$. Therefore

$$\begin{aligned}
\mathrm{tr}\big((y\gamma^*_{\varphi,q}(w_0)y^*)^p\big) &= \mathrm{tr}\Big(\big(yh_{\varphi}^{\frac{1}{2q}}\gamma(b)h_{\varphi}^{\frac{1}{2q}}y^*\big)^p\Big) = \mathrm{tr}\Big(\big(h_{\psi}^{\frac{1}{2p}}\gamma(b)h_{\psi}^{\frac{1}{2p}}\big)^p\Big) \ge \mathrm{tr}\Big(\big(h_{\psi_0}^{\frac{1}{2p}}bh_{\psi_0}^{\frac{1}{2p}}\big)^p\Big) \\
&= \mathrm{tr}\Big(\big(y_0h_{\varphi_0}^{\frac{1}{2q}}bh_{\varphi_0}^{\frac{1}{2q}}y_0^*\big)^p\Big) = \mathrm{tr}\big((y_0w_0y_0^*)^p\big),
\end{aligned}$$

where the inequality is from Lemma 2.2(i). The proof of inequality (3.7) is finished by Lemma A.1. By using this and the fact that $\gamma^*_{\varphi,q}$ is a contraction, if follows from the variational expression in (3.5) that

$$\begin{aligned}
Q_{\alpha,z}(\psi\|\varphi) &\ge \alpha\mathrm{tr}\big((y\gamma^*_{\varphi,q}(\bar{w}_0)y^*)^p\big) - (\alpha-1)\mathrm{tr}\big(\gamma^*_{\varphi,q}(\bar{w}_0)^q\big) \\
&\ge \alpha\mathrm{tr}\big((y_0\bar{w}_0y_0^*)^p\big) - (\alpha-1)\mathrm{tr}\big(\bar{w}_0^q\big) = Q_{\alpha,z}(\psi_0\|\varphi_0).
\end{aligned}$$

Supose that $Q_{\alpha,z}(\psi_0\|\varphi_0) = Q_{\alpha,z}(\psi\|\varphi)$, then both the inequalities must be equalities. Since $\bar{w} \in L_q(\mathcal{M})^+$ and $\bar{w}_0 \in L_q(\mathcal{N})^+$ are the unique elements such that the suprema in the respective variational expressions in (3.5) for $Q_{\alpha,z}(\psi\|\varphi)$ and $Q_{\alpha,z}(\psi_0\|\varphi_0)$ are attained, this proves (3.6). Conversely, if the equalities in (3.6) hold, then

$$Q_{\alpha,z}(\psi_0\|\varphi_0) = \mathrm{tr}\big((y_0^*y_0)^z\big) = \mathrm{tr}\big(\bar{w}_0^q\big) = \mathrm{tr}\big(\bar{w}^q\big) = \mathrm{tr}\big((y^*y)^z\big) = Q_{\alpha,z}(\psi\|\varphi).$$

□

Theorem 3.7. *Let $\gamma : \mathcal{N} \to \mathcal{M}$ be a channel and let $\psi, \varphi \in \mathcal{M}_*^+$ be such that $s(\psi) \le s(\varphi)$ and $D_{\alpha,z}(\psi\|\varphi) < \infty$. Then $D_{\alpha,z}(\psi \circ \gamma\|\varphi \circ \gamma) = D_{\alpha,z}(\psi\|\varphi)$ if and only if γ is sufficient with respect to $\{\psi, \varphi\}$.*

Proof. Let $\bar{w}$ and $\bar{w}_0$ be as in Lemma 3.6. Let $\omega \in \mathcal{M}_*^+$ and $\omega_0 \in \mathcal{N}_*^+$ be such that

$$h_\omega = h_\varphi^{\frac{q-1}{2q}} \bar{w} h_\varphi^{\frac{q-1}{2q}}, \qquad h_{\omega_0} = h_{\varphi_0}^{\frac{q-1}{2q}} \bar{w}_0 h_{\varphi_0}^{\frac{q-1}{2q}}.$$

Assume that the equality in DPI holds; then by Lemma 3.6 we have

$$Q_{\alpha,z}(\omega_0\|\varphi_0) = \mathrm{tr}\big(\bar{w}_0^q\big) = \mathrm{tr}\big(\bar{w}^q\big) = Q_{\alpha,z}(\omega\|\varphi).$$

and using also Lemma 2.1, we have

$$(\gamma_\varphi^*)_*(h_{\omega_0}) = h_\varphi^{\frac{1}{2\alpha}} \gamma_{\varphi,q}^*(\bar{w}_0) h_\varphi^{\frac{1}{2\alpha}} = h_\omega.$$

Similarly as in the proof of Theorem 3.5, this shows that γ is sufficient with respect to $\{\omega, \varphi\}$. Hence $\omega \circ \mathcal{E} = \omega$, where $\mathcal{E}$ is the conditional expectation onto the fixed points of $\gamma \circ \gamma_\varphi^*$. Using the extensions of $\mathcal{E}$ and their properties in [20], we have

$$h_\varphi^{\frac{q-1}{2q}} \bar{w} h_\varphi^{\frac{q-1}{2q}} = h_\omega = \mathcal{E}(h_\omega) = h_\varphi^{\frac{q-1}{2q}} \mathcal{E}(\bar{w}) h_\varphi^{\frac{q-1}{2q}},$$

which implies that $\bar{w} = \mathcal{E}(\bar{w}) \in L_q(\mathcal{E}(\mathcal{M}))^+$. But then we also have

$$|y| = \bar{w}^{\frac{1}{2(\alpha-1)}} = \bar{w}^{\frac{q}{2z}} \in L_{2z}(\mathcal{E}(\mathcal{M}))^+$$

Let $y = u|y|$ be the polar decomposition of y; then we obtain from the definition of y that $uu^* = s(yy^*) = s(\psi)$. Furthermore, since

$$u^* h_\psi^{\frac{1}{2p}} = |y| h_\varphi^{\frac{1}{2q}} \in L_{2p}(\mathcal{E}(\mathcal{M})),$$

by uniqueness of the polar decomposition in $L_{2p}(\mathcal{M})$ and $L_{2p}(\mathcal{E}(\mathcal{M}))$, we obtain that $h_\psi^{\frac{1}{2p}} \in L_{2p}(\mathcal{E}(\mathcal{M}))^+$ and $u \in \mathcal{E}(\mathcal{M})$. Hence we must have $h_\psi \in L_1(\mathcal{E}(\mathcal{M}))$ so that $\psi \circ \mathcal{E} = \psi$ and γ is sufficient with respect to $\{\psi, \varphi\}$ by Proposition 3.2(ii). The converse is clear from DPI. □

4 Monotonicity in the parameter z

It is well known [4, 12, 18] that the standard Rényi divergence $D_{\alpha,1}(\psi\|\varphi)$ is monotone increasing in $\alpha \in (0,1) \cup (1,\infty)$ and the sandwiched Rényi divergence $D_{\alpha,\alpha}(\psi\|\varphi)$ is monotone increasing in $\alpha \in [1/2, 1) \cup (1, \infty)$. It is also known [4, 12, 18] that

$$\lim_{\alpha \nearrow 1} D_{\alpha,1}(\psi\|\varphi) = \lim_{\alpha \nearrow 1} D_{\alpha,\alpha}(\psi\|\varphi) = D_1(\psi\|\varphi), \tag{4.1}$$

and if $D_{\alpha,1}(\psi\|\varphi) < \infty$ (resp., $D_{\alpha,\alpha}(\psi\|\varphi) < \infty$) for some $\alpha > 1$, then

$$\lim_{\alpha \searrow 1} D_{\alpha,1}(\psi\|\varphi) = D_1(\psi\|\varphi) \quad \Big(\text{resp., } \lim_{\alpha \searrow 1} D_{\alpha,\alpha}(\psi\|\varphi) = D_1(\psi\|\varphi)\Big), \tag{4.2}$$

where $D_1(\psi\|\varphi) := D(\psi\|\varphi)/\psi(1)$, the normalized relative entropy. In the rest of the paper we will discuss similar monotonicity properties and limits for $D_{\alpha,z}(\psi\|\varphi)$. We consider monotonicity in the parameter z in Sec. 4 and monotonicity in the parameter α in Sec. 5.

4.1 The finite von Neumann algebra case

In this subsection we show monotonicity of $D_{\alpha,z}$ in the parameter z in the finite von Neumann algebra setting. Recall that if $(\mathcal{M},\tau)$ is a semi-finite von Neumann algebra $\mathcal{M}$ with a faithful normal semi-finite trace τ, then the Haagerup L_p-space $L_p(\mathcal{M})$ is identified with the L_p-space $L_p(\mathcal{M},\tau)$ with respect to τ [13, Example 9.11]. Hence one can define $Q_{\alpha,z}(\psi\|\varphi)$ for $\psi,\varphi\in\mathcal{M}_*^+$ by replacing, in Definition 1.1, $L_p(\mathcal{M})$ with $L_p(\mathcal{M},\tau)$ and $h_\psi\in L_1(\mathcal{M})_+$ with the Radon–Nikodym derivative $d\psi/d\tau\in L_1(\mathcal{M},\tau)^+$. Below we use the symbol h_ψ to denote $d\psi/d\tau$ as well. Note that τ on $\mathcal{M}^+$ is naturally extended to the positive part $\widetilde{\mathcal{M}}^+$ of the space $\widetilde{\mathcal{M}}$ of τ-measurable operators. We then have [8, Proposition 2.7] (also [13, Proposition 4.20])

$$\tau(a)=\int_0^\infty \mu_s(a)\,ds,\qquad a\in\widetilde{\mathcal{M}}^+,\tag{4.3}$$

where $\mu_s(a)$ is the generalized s-number of a [8].

Now we assume that $(\mathcal{M},\tau)$ is a finite von Neumann algebra with a faithful normal finite trace τ. In this setting note that $\widetilde{\mathcal{M}}^+$ consists of all positive self-adjoint operators affiliated with $\mathcal{M}$. Our discussions below are essentially along the same lines as in the finite-dimensional case [28, 29], where the integral expression in (4.3) is useful.

Lemma 4.1. *For every $\psi,\varphi\in\mathcal{M}_*^+$ with $\psi\neq 0$ and for any $\alpha,z>0$ with $\alpha\neq 1$,*

$$D_{\alpha,z}(\psi\|\varphi)=\lim_{\varepsilon\searrow 0}D_{\alpha,z}(\psi\|\varphi+\varepsilon\tau)\quad\textit{increasingly},\tag{4.4}$$

and hence $D_{\alpha,z}(\psi\|\varphi)=\sup_{\varepsilon>0}D_{\alpha,z}(\psi\|\varphi+\varepsilon\tau)$.

Proof. Case $0<\alpha<1$. We need to prove that

$$Q_{\alpha,z}(\psi\|\varphi)=\lim_{\varepsilon\searrow 0}Q_{\alpha,z}(\psi\|\varphi+\varepsilon\tau)\quad\text{decreasingly.}\tag{4.5}$$

In the present setting we have by (4.3)

$$Q_{\alpha,z}(\psi\|\varphi)=\tau\Big(\big(h_\psi^{\alpha/2z}h_\varphi^{\frac{1-\alpha}{z}}h_\psi^{\alpha/2z}\big)^z\Big)=\int_0^\infty\mu_s\Big(h_\psi^{\alpha/2z}h_\varphi^{\frac{1-\alpha}{z}}h_\psi^{\alpha/2z}\Big)^z\,ds,\tag{4.6}$$

and similarly

$$Q_{\alpha,z}(\psi\|\varphi+\varepsilon\tau)=\int_0^\infty\mu_s\Big(h_\psi^{\alpha/2z}h_{\varphi+\varepsilon\tau}^{\frac{1-\alpha}{z}}h_\psi^{\alpha/2z}\Big)^z\,ds.$$

Since $h_{\varphi+\varepsilon\tau}^{\frac{1-\alpha}{z}}=(h_\varphi+\varepsilon\mathbf{1})^{\frac{1-\alpha}{z}}$ decreases to $h_\varphi^{\frac{1-\alpha}{z}}$ in the measure topology as $\varepsilon\searrow 0$, it follows that $h_\psi^{\alpha/2z}h_{\varphi+\varepsilon\tau}^{\frac{1-\alpha}{z}}h_\psi^{\alpha/2z}$ decreases to $h_\psi^{\alpha/2z}h_\varphi^{\frac{1-\alpha}{z}}h_\psi^{\alpha/2z}$ in the measure topology. Hence by [8, Lemma 3.4] we have $\mu_s\big(h_\psi^{\alpha/2z}h_{\varphi+\varepsilon\tau}^{\frac{1-\alpha}{z}}h_\psi^{\alpha/2z}\big)\searrow\mu_s\big(h_\psi^{\alpha/2z}h_\varphi^{\frac{1-\alpha}{z}}h_\psi^{\alpha/2z}\big)$ as $\varepsilon\searrow 0$ for almost every $s>0$. Since $s\mapsto\mu_s\big(h_\psi^{\alpha/2z}h_{\varphi+\tau}^{\frac{1-\alpha}{z}}h_\psi^{\alpha/2z}\big)$ is integrable on $(0,\infty)$, the Lebesgue convergence theorem gives (4.5).

Case $\alpha>1$. We need to prove that

$$Q_{\alpha,z}(\psi\|\varphi)=\lim_{\varepsilon\searrow 0}Q_{\alpha,z}(\psi\|\varphi+\varepsilon\tau)\quad\text{increasingly.}\tag{4.7}$$

For any $\varepsilon > 0$, since $h_{\varphi+\varepsilon\tau} = h_\varphi + \varepsilon\mathbf{1}$ has the bounded inverse $h_{\varphi+\varepsilon\tau}^{-1} = (h_\varphi + \varepsilon\mathbf{1})^{-1} \in \mathcal{M}^+$, one can define $x_\varepsilon := (h_\varphi + \varepsilon\mathbf{1})^{-\frac{\alpha-1}{2z}} h_\psi^{\alpha/z} (h_\varphi + \varepsilon\mathbf{1})^{-\frac{\alpha-1}{2z}} \in \widetilde{\mathcal{M}}^+$ so that

$$h_\psi^{\alpha/z} = (h_\varphi + \varepsilon\mathbf{1})^{\frac{\alpha-1}{2z}} x_\varepsilon (h_\varphi + \varepsilon\mathbf{1})^{\frac{\alpha-1}{2z}}.$$

In the present setting one can write by (4.3)

$$Q_{\alpha,z}(\psi\|\varphi + \varepsilon\tau) = \tau(x_\varepsilon^z) = \int_0^\infty \mu_s(x_\varepsilon)^z\, ds \ (\in [0,\infty]). \tag{4.8}$$

Let $0 < \varepsilon \le \varepsilon'$. Since $(h_\varphi + \varepsilon\mathbf{1})^{-\frac{\alpha-1}{z}} \ge (h_\varphi + \varepsilon'\mathbf{1})^{-\frac{\alpha-1}{z}}$, one has $\mu_s(x_\varepsilon) \ge \mu_s(x_{\varepsilon'})$ for all $s > 0$, so that

$$Q_{\alpha,z}(\psi\|\varphi + \varepsilon\tau) \ge Q_{\alpha,z}(\psi\|\varphi + \varepsilon'\tau).$$

Hence $\varepsilon > 0 \mapsto D_{\alpha,z}(\psi\|\varphi + \varepsilon\tau)$ is decreasing.

First, assume that $s(\psi) \not\le s(\varphi)$. Then $\mu_{s_0}\Big(h_\psi^{\alpha/2z} s(\varphi)^\perp h_\psi^{\alpha/2z}\Big) > 0$ for some $s_0 > 0$; indeed, otherwise, $h_\psi^{\alpha/2z} s(\varphi)^\perp h_\psi^{\alpha/2z} = 0$ so that $s(\psi) \le s(\varphi)$. Hence we have

$$\mu_s(x_\varepsilon) = \mu_s\Big(h_\psi^{\alpha/2z}(h_\varphi + \varepsilon\mathbf{1})^{-\frac{\alpha-1}{z}} h_\psi^{\alpha/2z}\Big) \ge \varepsilon^{-\frac{\alpha-1}{z}} \mu_s\Big(h_\psi^{\alpha/2z} s(\varphi)^\perp h_\psi^{\alpha/2z}\Big) \nearrow \infty \quad \text{as } \varepsilon \searrow 0$$

for all $s \in (0, s_0]$. Therefore, it follows from (4.8) that $Q_{\alpha,z}(\psi\|\varphi + \varepsilon\tau) \nearrow \infty = Q_{\alpha,z}(\psi\|\varphi)$.

Next, assume that $s(\psi) \le s(\varphi)$. Take the spectral decomposition $h_\varphi = \int_0^\infty t\, de_t$ and define $y, x \in \widetilde{\mathcal{M}}_+$ by

$$y := h_\varphi^{-\frac{\alpha-1}{z}} s(\varphi) = \int_{(0,\infty)} t^{-\frac{\alpha-1}{z}}\, de_t, \qquad x := y^{1/2} h_\psi^{\alpha/z} y^{1/2}.$$

Since

$$h_\psi^{\alpha/z} = s(\varphi) h_\psi^{\alpha/z} s(\varphi) = h_\varphi^{\frac{\alpha-1}{2z}} y^{1/2} h_\psi^{\alpha/z} y^{1/2} h_\varphi^{\frac{\alpha-1}{2z}} = h_\varphi^{\frac{\alpha-1}{2z}} x h_\varphi^{\frac{\alpha-1}{2z}},$$

one has, similarly to 4.8,

$$Q_{\alpha,z}(\psi\|\varphi) = \tau(x^z) = \int_0^\infty \mu_s(x)^z\, ds. \tag{4.9}$$

We write $(h_\varphi + \varepsilon\mathbf{1})^{-\frac{\alpha-1}{z}} s(\varphi) = \int_{(0,\infty)} (t+\varepsilon)^{-\frac{\alpha-1}{z}}\, de_t$, and for any $\delta > 0$ choose a $t_0 > 0$ such that $\tau(e_{(0,t_0)}) < \delta$. Then, since $\int_{[t_0,\infty)} (t+\varepsilon)^{-\frac{\alpha-1}{z}}\, de_t \to \int_{[t_0,\infty)} t^{-\frac{\alpha-1}{z}}\, de_t$ in the operator norm as $\varepsilon \searrow 0$, we obtain $(h_\varphi + \varepsilon\mathbf{1})^{-\frac{\alpha-1}{z}} s(\varphi) \nearrow y$ in the measure topology (see [8, 1.5]), so that $h_\psi^{\alpha/2z}(h_\varphi + \varepsilon\mathbf{1})^{-\frac{\alpha-1}{z}} h_\psi^{\alpha/2z} \nearrow h_\psi^{\alpha/2z} y h_\psi^{\alpha/2z}$ in the measure topology as $\varepsilon \searrow 0$. Hence we have by [8, Lemma 3.4]

$$\mu_s(x_\varepsilon) = \mu_s\Big(h_\psi^{\alpha/2z}(h_\varphi + \varepsilon\mathbf{1})^{-\frac{\alpha-1}{z}} h_\psi^{\alpha/2z}\Big) \nearrow \mu_s\Big(h_\psi^{\alpha/2z} y h_\psi^{\alpha/2z}\Big) = \mu_s(x) \tag{4.10}$$

for all $s > 0$. Therefore, by (4.8) and (4.9) the monotone convergence theorem gives (4.7). $\square$

Lemma 4.2. *Let $(\mathcal{M}, \tau)$ and ψ, φ be as above, and let $0 < z \le z'$. Then*

$$\begin{cases} D_{\alpha,z}(\psi\|\varphi) \le D_{\alpha,z'}(\psi\|\varphi), & 0 < \alpha < 1, \\ D_{\alpha,z}(\psi\|\varphi) \ge D_{\alpha,z'}(\psi\|\varphi), & \alpha > 1. \end{cases}$$

Proof. The case $0 < \alpha < 1$ was shown in [21, Theorem 1(x)] for general von Neumann algebras. For the case $\alpha > 1$, by Lemma 4.1 it suffices to show that, for every $\varepsilon > 0$,

$$\tau\Bigl(\Bigl(y_\varepsilon^{\frac{\alpha-1}{2z}} h_\psi^{\alpha/z} y_\varepsilon^{\frac{\alpha-1}{2z}}\Bigr)^z\Bigr) \ge \tau\Bigl(\Bigl(y_\varepsilon^{\frac{\alpha-1}{2z'}} h_\psi^{\alpha/z'} y_\varepsilon^{\frac{\alpha-1}{2z'}}\Bigr)^{z'}\Bigr),$$

where $y_\varepsilon := (h_\varphi + \varepsilon\mathbf{1})^{-1} \in \mathcal{M}_+$. The above is equivalently written as

$$\tau\Bigl(\bigl|(h_\psi^{\alpha/2z'})^r (y_\varepsilon^{(\alpha-1)/2z'})^r\bigr|^{2z}\Bigr) \ge \tau\Bigl(\bigl|h_\psi^{\alpha/2z'} y_\varepsilon^{(\alpha-1)/2z'}\bigr|^{2zr}\Bigr),$$

where $r := z'/z \ge 1$. Hence the desired inequality follows from Kosaki's ALT inequality [26, Corollary 3]. $\square$

When $(\mathcal{M},\tau)$ and ψ,φ are as in Lemma 4.1, one can define, thanks to Lemma 4.2, for any $\alpha \in (0,\infty)\setminus\{1\}$,

$$\begin{aligned} Q_{\alpha,\infty}(\psi\|\varphi) &:= \lim_{z\to\infty} Q_{\alpha,z}(\psi\|\varphi) = \inf_{z>0} Q_{\alpha,z}(\psi\|\varphi), \\ D_{\alpha,\infty}(\psi\|\varphi) &:= \frac{1}{\alpha-1}\log\frac{Q_{\alpha,\infty}(\psi\|\varphi)}{\psi(\mathbf{1})} \\ &= \lim_{z\to\infty} D_{\alpha,z}(\psi\|\varphi) = \begin{cases} \sup_{z>0} D_{\alpha,z}(\psi\|\varphi), & 0<\alpha<1, \\ \inf_{z>0} D_{\alpha,z}(\psi\|\varphi), & \alpha>1. \end{cases} \end{aligned} \tag{4.11}$$

If $h_\psi, h_\varphi \in \mathcal{M}^{++}$ (i.e., $\delta\tau \le \psi,\varphi \le \delta^{-1}\tau$ for some $\delta \in (0,1)$), then the Lie–Trotter formula gives

$$Q_{\alpha,\infty}(\psi\|\varphi) = \tau\bigl(\exp(\alpha\log h_\psi + (1-\alpha)\log h_\varphi)\bigr). \tag{4.12}$$

Lemma 4.3. *Let $(\mathcal{M},\tau)$ and ψ,φ be as above. Then for any $z > 0$,*

$$\begin{cases} D_{\alpha,z}(\psi\|\varphi) \le D_1(\psi\|\varphi), & 0<\alpha<1, \\ D_{\alpha,z}(\psi\|\varphi) \ge D_1(\psi\|\varphi), & \alpha>1. \end{cases}$$

Proof. First, assume that $h_\psi, h_\varphi \in \mathcal{M}^{++}$. Set self-adjoint $H := \log h_\psi$ and $K := \log h_\varphi$ in $\mathcal{M}$ and define $F(\alpha) := \log\tau\bigl(e^{\alpha H + (1-\alpha)K}\bigr)$ for $\alpha > 0$. Then by (4.12), $F(\alpha) = \log Q_{\alpha,\infty}(\psi\|\varphi)$ for all $\alpha \in (0,\infty)\setminus\{1\}$, and we compute

$$\begin{aligned} F'(\alpha) &= \frac{\tau\bigl(e^{\alpha H+(1-\alpha)K}(H-K)\bigr)}{\tau\bigl(e^{\alpha H+(1-\alpha)K}\bigr)}, \\ F''(\alpha) &= \frac{\tau\bigl(e^{\alpha H+(1-\alpha)K}(H-K)^2\bigr)\tau\bigl(e^{\alpha H+(1-\alpha)K}\bigr) - \bigl\{\tau\bigl(e^{\alpha H+(1-\alpha)K}(H-K)\bigr)\bigr\}^2}{\bigl\{\tau\bigl(e^{\alpha H+(1-\alpha)K}\bigr)\bigr\}^2}. \end{aligned}$$

Since $F''(\alpha) \ge 0$ on $(0,\infty)$ thanks to the Schwarz inequality, we see that $F(\alpha)$ is convex on $(0,\infty)$ and hence

$$D_{\alpha,\infty}(\psi\|\varphi) = \frac{F(\alpha)-F(1)}{\alpha-1}$$

is increasing in $\alpha \in (0,\infty)$, where for $\alpha = 1$ the above RHS is understood as

$$F'(1) = \frac{\tau(e^H(H-K))}{\tau(e^H)} = \frac{\tau\big(h_\psi(\log h_\psi - \log h_\varphi)\big)}{\tau(h_\psi)} = D_1(\psi\|\varphi).$$

Hence by (4.11) the assertion holds when $h_\psi, h_\varphi \in \mathcal{M}^{++}$. Below we extend it to general $\psi,\varphi \in \mathcal{M}_*^+$.

Case $0 < \alpha < 1$. Let $\psi,\varphi \in \mathcal{M}_*^+$ and $z > 0$. From [21, Theorem 1(iv)] and [14, Corollary 2.8(3)] we have

$$D_{\alpha,z}(\psi\|\varphi) = \lim_{\varepsilon\searrow 0} D_{\alpha,z}(\psi+\varepsilon\tau\|\varphi+\varepsilon\tau),$$
$$D_1(\psi\|\varphi) = \lim_{\varepsilon\searrow 0} D_1(\psi+\varepsilon\tau\|\varphi+\varepsilon\tau),$$

so that we may assume that $\psi,\varphi \ge \varepsilon\tau$ for some $\varepsilon > 0$. Take the spectral decompositions $h_\psi = \int_0^\infty t\,de_t^\psi$ and $h_\varphi = \int_0^\infty t\,de_t^\varphi$, and define $e_n := e_n^\psi \wedge e_n^\varphi$ for each $n \in \mathbb{N}$. Then $\tau(e_n^\perp) \le \tau((e_n^\psi)^\perp) + \tau((e_n^\varphi)^\perp) \to 0$ as $n\to\infty$, so that $e_n \nearrow \mathbf{1}$. We set $\psi_n := \psi(e_n\cdot e_n)$ and $\varphi_n := \varphi(e_n\cdot e_n)$; then $h_{\psi_n} = e_n h_\psi e_n$ and $h_{\varphi_n} = e_n h_\varphi e_n$ are in $(e_n\mathcal{M}e_n)^{++}$. Note that

$$\begin{aligned}\|h_\psi - e_nh_\psi e_n\|_1 &\le \|(\mathbf{1}-e_n)h_\psi\|_1 + \|e_nh_\psi(\mathbf{1}-e_n)\|_1\\ &\le \|(\mathbf{1}-e_n)h_\psi^{1/2}\|_2\|h_\psi^{1/2}\|_2 + \|e_nh_\psi^{1/2}\|_2\|h_\psi^{1/2}(\mathbf{1}-e_n)\|_2\\ &= \psi(\mathbf{1}-e_n)^{1/2}\psi(\mathbf{1})^{1/2} + \psi(e_n)^{1/2}\psi(\mathbf{1}-e_n)^{1/2} \to 0 \quad \text{as } n\to\infty,\end{aligned}$$

and similarly $\|h_\varphi - e_nh_\varphi e_n\|_1 \to 0$. Hence by [21, Theorem 1(iv)] one has $D_{\alpha,z}(e_n\psi e_n\|e_n\varphi e_n) \to D_{\alpha,z}(\psi\|\varphi)$. On the other hand, one has $D_1(e_n\psi e_n\|e_n\varphi e_n) \to D_1(\psi\|\varphi)$ by [14, Proposition 2.10]. Since $D_{\alpha,z}(e_n\psi e_n\|e_n\varphi e_n) \le D_1(e_n\psi e_n\|e_n\varphi e_n)$ holds by regarding $e_n\psi e_n, e_n\varphi e_n$ as functionals on the reduced von Neumann algebra $e_n\mathcal{M}e_n$, we obtain the desired inequality for general $\psi,\varphi \in \mathcal{M}_*^+$.

Case $\alpha > 1$. We show the extension to general $\psi,\varphi\in\mathcal{M}_*^+$ by dividing four steps as follows, where $h_\psi = \int_0^\infty t\,e_t^\psi$ and $h_\varphi = \int_0^\infty t\,de_t^\varphi$ are the spectral decompositions.

(1) Assume that $h_\psi \in \mathcal{M}^+$ and $h_\varphi \in \mathcal{M}^{++}$. Set $\psi_n \in \mathcal{M}_*^+$ by $h_{\psi_n} = (1/n)e_{[0,1/n]}^\psi + \int_{(1/n,\infty)} t\,de_t^\psi$ $(\in \mathcal{M}^{++})$. Since $h_{\psi_n}^{\alpha/z} \searrow h_\psi^{\alpha/z}$ in the operator norm, we have by (4.6) and [8, Lemma 3.4]

$$\begin{aligned}Q_{\alpha,z}(\psi\|\varphi) &= \int_0^\infty \mu_s\Big((h_\varphi^{-1})^{\frac{\alpha-1}{2z}} h_\psi^{\alpha/z} (h_\varphi^{-1})^{\frac{\alpha-1}{2z}}\Big)^z\,ds\\ &= \lim_{n\to\infty}\int_0^\infty \mu_s\Big((h_\varphi^{-1})^{\frac{\alpha-1}{2z}} h_{\psi_n}^{\alpha/z} (h_\varphi^{-1})^{\frac{\alpha-1}{2z}}\Big)^z\,ds = \lim_{n\to\infty} Q_{\alpha,z}(\psi_n\|\varphi).\end{aligned} \tag{4.13}$$

From this and the lower semicontinuity of D_1 the extension holds in this case.

(2) Assume that $h_\psi \in \mathcal{M}^+$ and $h_\varphi \ge \delta\mathbf{1}$ for some $\delta > 0$. Set $\varphi_n \in \mathcal{M}_*^+$ by $h_{\varphi_n} = \int_{[\delta,n]} t\,de_t^\varphi + ne_{(n,\infty)}^\varphi$ $(\in\mathcal{M}^{++})$. Since $h_{\varphi_n}^{-\frac{\alpha-1}{z}} \searrow h_\varphi^{-\frac{\alpha-1}{z}}$ in the operator norm, we have by (4.6) and [8, Lemma 3.4] again

$$\begin{aligned}Q_{\alpha,z}(\psi\|\varphi) &= \int_0^\infty \mu_s\Big(h_\psi^{\alpha/2z} h_\varphi^{-\frac{\alpha-1}{z}} h_\psi^{\alpha/2z}\Big)^z\,ds\\ &= \lim_{n\to\infty}\int_0^\infty \mu_s\Big(h_\psi^{\alpha/2z} h_{\varphi_n}^{-\frac{\alpha-1}{z}} h_\psi^{\alpha/2z}\Big)^z\,ds = \lim_{n\to\infty} Q_{\alpha,z}(\psi\|\varphi_n).\end{aligned}$$

From this and (1) above the extension holds in this case too.

(3) Assume that ψ is general and $\varphi \ge \delta\tau$ for some $\delta > 0$. Set $\psi_n \in \mathcal{M}_*^+$ by $h_{\psi_n} = \int_{[0,n]} t\, de_t^\psi + ne_{(n,\infty)}^\varphi$ $(\in \mathcal{M}_+)$. Since $h_{\psi_n}^{\alpha/z} \nearrow h_\psi^{\alpha/z}$ in the measure topology, one can argue as in (4.13) with use of the monotone convergence theorem to see from (2) that the extension holds in this case too.

(4) Finally, from (3) with Lemma 4.1 and [14, Corollary 2.8(3)] it follows that the desired extension hods for general $\psi, \varphi \in \mathcal{M}_*^+$. $\square$

In the next proposition we summarize inequalities for $D_{\alpha,z}$ obtained so far in Lemmas 4.2 and 4.3.

Proposition 4.4. *Assume that $\mathcal{M}$ is a finite von Neumann algebra with a faithful normal finite trace τ. Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \ne 0$. If $0 < \alpha < 1 < \alpha'$ and $0 < z \le z' \le \infty$, then*

$$D_{\alpha,z}(\psi\|\varphi) \le D_{\alpha,z'}(\psi\|\varphi) \le D_1(\psi\|\varphi) \le D_{\alpha',z'}(\psi\|\varphi) \le D_{\alpha',z}(\psi\|\varphi).$$

In view of (4.1) and (4.2), the above proposition gives the next limits in the restrictive situation of this subsection, while in Sec. 5.3 we will obtain similar results in general von Neumann algebras by using the complex interpolation method.

Corollary 4.5. *Let $(\mathcal{M}, \tau)$ and ψ, φ be as in Proposition 4.4. Then for any $z \in [1, \infty]$,*

$$\lim_{\alpha \nearrow 1} D_{\alpha,z}(\psi\|\varphi) = D_1(\psi\|\varphi). \tag{4.14}$$

Moreover, if $D_{\alpha,\alpha}(\psi\|\varphi) < \infty$ for some $\alpha > 1$ then for any $z \in (1, \infty]$,

$$\lim_{\alpha \searrow 1} D_{\alpha,z}(\psi\|\varphi) = D_1(\psi\|\varphi). \tag{4.15}$$

Proof. For every $z \in [1, \infty]$ and $\alpha \in (0, 1)$, Proposition 4.4 gives

$$D_{\alpha,1}(\psi\|\varphi) \le D_{\alpha,z}(\psi\|\varphi) \le D_1(\psi\|\varphi).$$

Hence (4.14) follows since it holds for $D_{\alpha,1}$ as stated in (4.1); see [12, Proposition 5.3(3)].

Next, assume that $D_{\alpha,\alpha}(\psi\|\varphi) < \infty$ for some $\alpha > 1$. For every $z \in (1, \infty]$ and $\alpha \in (1, z]$, Proposition 4.4 gives

$$D_1(\psi\|\varphi) \le D_{\alpha,z}(\psi\|\varphi) \le D_{\alpha,\alpha}(\psi\|\varphi).$$

Hence (4.15) follows since it holds for $D_{\alpha,\alpha}$ as stated in (4.2); see [18, Proposition 3.8(ii)]. $\square$

In this subsection, in the specialized setting of finite von Neumann algebras, we have given monotonicity of $D_{\alpha,z}$ in the parameter z in an essentially similar way to the finite-dimensional case [29]. In the next subsection we will extend it to general von Neumann algebras under certain restrictions of α, z.

4.2 The general von Neumann algebra case

In this subsection we show monotonicity of $D_{\alpha,z}$ in the parameter z in general von Neumann algebras under certain restrictions of α, z. From now on let $\mathcal{M}$ be a general σ-finite von Neumann algebra.

First, we extend Proposition 4.4 to general von Neumann algebras, based on Haagerup's reduction theorem [11] that is briefly explained in Appendix B for the convenience of the reader. Let ω be a faithful normal state of $\mathcal{M}$, and let

$$\hat{\mathcal{M}}, \qquad \hat{\omega}, \qquad E_{\mathcal{M}} : \hat{\mathcal{M}} \to \mathcal{M}, \qquad \mathcal{M}_n, \qquad E_{\mathcal{M}_n} : \hat{\mathcal{M}} \to \mathcal{M}_n \qquad (n \ge 1)$$

be given as in Theorem B.1. We then give the next lemma.

Lemma 4.6. *In the above situation, for any $\psi, \varphi \in \mathcal{M}_*^+$ with $\psi \neq 0$ let $\hat{\psi} := \psi \circ E_{\mathcal{M}}$ and $\hat{\varphi} := \varphi \circ E_{\mathcal{M}}$. If $\alpha, z > 0$ with $\alpha \neq 1$ satisfy either*

$$0 < \alpha < 1, \qquad \max\{\alpha, 1-\alpha\} \le z,$$

or

$$\alpha > 1, \qquad \max\{\alpha/2, \alpha - 1\} \le z \le \alpha,$$

then we have

$$D_{\alpha,z}(\psi\|\varphi) = D_{\alpha,z}(\hat{\psi}\|\hat{\varphi}) = \lim_{n\to\infty} D_{\alpha,z}(\hat{\psi}|_{\mathcal{M}_n}\|\hat{\varphi}|_{\mathcal{M}_n}) \quad \textit{increasingly,} \tag{4.16}$$

$$D_1(\psi\|\varphi) = D_1(\hat{\psi}\|\hat{\varphi}) = \lim_{n\to\infty} D_1(\hat{\psi}|_{\mathcal{M}_n}\|\hat{\varphi}|_{\mathcal{M}_n}) \quad \textit{increasingly.} \tag{4.17}$$

Proof. Apply the DPI for $D_{\alpha,z}$ proved in Theorem 2.3 to the injection $\mathcal{M} \hookrightarrow \hat{\mathcal{M}}$ and to the conditional expectation $E_{\mathcal{M}} : \hat{\mathcal{M}} \to \mathcal{M}$. We then have the first equality in (4.16). By Theorem B.1 we can apply the martingale convergence in Theorem 2.6 to obtain the latter equality in (4.16) with increasing convergence. The assertion of D_1 in (4.17) is included in [9, Proposition 2.2], while this is the well-known martingale convergence of the relative entropy [25]. $\square$

Theorem 4.7. *For every $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$, we have:*

(1) If $0 < \alpha < 1$ and $0 \le z \le z'$, then

$$D_{\alpha,z}(\psi\|\varphi) \le D_{\alpha,z'}(\psi\|\varphi) \le D_1(\psi\|\varphi).$$

(2) If $\alpha > 1$ and $\max\{\alpha/2, \alpha-1\} \le z \le z' \le \alpha$, then

$$D_1(\psi\|\varphi) \le D_{\alpha,z'}(\psi\|\varphi) \le D_{\alpha,z}(\psi\|\varphi).$$

Proof. The first inequality in (1) was shown in [21, Theorem 1(x)]. The other inequalities in (1) and (2) immediately follow from Proposition 4.4 and Lemma 4.6. $\square$

For the proof of Theorem 4.7 using Theorem B.1 it is inevitable to restrict the parameter z to the DPI range for each α. But the next theorem strengthens Theorem 4.7(2) into a wider range of z, whose proof is based on Kosaki's interpolation L_p-spaces [23]. (A brief explanation on Kosaki's interpolation L_p-spaces is given in Appendix C for the convenience of the reader.)

Theorem 4.8. *For every $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$, and $\alpha > 1$, the function $z \mapsto D_{\alpha,z}(\psi\|\varphi)$ is monotone decreasing on $[\alpha/2, \infty)$.*

Proof. Let $\alpha > 1$ and $z, z' \in [\alpha/2, \infty)$ be such that $z < z'$. We need to prove that $Q_{\alpha,z}(\psi\|\varphi) \geq Q_{\alpha,z'}(\psi\|\varphi)$. To do this, we may assume that $Q_{\alpha,z}(\psi\|\varphi) < \infty$. Hence by Lemma 1.2, there is some $y \in L_{2z}(\mathcal{M})$ such that

$$h_\psi^{\frac{\alpha}{2z}} = y h_\varphi^{\frac{\alpha-1}{2z}}, \qquad Q_{\alpha,z}(\psi\|\varphi) = \|y\|_{2z}^{2z}.$$

In particular, $e := s(\psi) \leq s(\varphi)$, so that we may assume that φ is faithful. Let $\sigma \in \mathcal{M}_*^+$ be such that $s(\sigma) = \mathbf{1} - e$, and set $\psi_0 := \psi + \sigma$, so that ψ_0 is faithful too. Let us use for simplicity the notation $L_{p,L}$ for Kosaki's left L_p-space $L_p(\mathcal{M}, \varphi)_L$ for $1 \leq p \leq \infty$; see (C.2) in Appendix C.

Consider the function

$$f(w) := h_{\psi_0}^{\frac{\alpha}{2z}w} e h_\varphi^{1-\frac{\alpha}{2z}w}, \qquad w \in S, \tag{4.18}$$

where $S := \{w \in \mathbb{C} : 0 \leq \operatorname{Re} w \leq 1\}$. Then, for $w = s + it$ with $0 \leq s \leq 1$ and $t \in \mathbb{R}$, since

$$f(s+it) = h_{\psi_0}^{\frac{\alpha}{2z}it} h_{\psi_0}^{\frac{\alpha}{2z}s} e h_\varphi^{1-\frac{\alpha}{2z}s} h_\varphi^{-\frac{\alpha}{2z}it},$$

it is easy to see that f is a bounded continuous function on S into $L_1(\mathcal{M})$ and it is analytic in the interior (see, e.g., [13, Lemma 9.19, Theorem 9.18(c)]). Furthermore, we have

$$f(it) = h_{\psi_0}^{\frac{\alpha}{2z}it} e h_\varphi^{-\frac{\alpha}{2z}it} h_\varphi \in L_{\infty,L},$$
$$\|f(it)\|_{L_{\infty,L}} = \left\| h_{\psi_0}^{\frac{\alpha}{2z}it} e h_\varphi^{-\frac{\alpha}{2z}it} \right\| = 1, \qquad t \in \mathbb{R},$$

and

$$f(1+it) = h_{\psi_0}^{\frac{\alpha}{2z}it} h_\psi^{\frac{\alpha}{2z}} h_\varphi^{1-\frac{\alpha}{2z}} h_\varphi^{-\frac{\alpha}{2z}it} = \left(h_{\psi_0}^{\frac{\alpha}{2z}it} y h_\varphi^{-\frac{\alpha}{2z}it}\right) h_\varphi^{\frac{2z-1}{2z}} \in L_{2z,L},$$
$$\|f(1+it)\|_{L_{2z,L}} = \left\| h_{\psi_0}^{\frac{\alpha}{2z}it} y h_\varphi^{-\frac{\alpha}{2z}it} \right\|_{2z} = \|y\|_{2z}, \qquad t \in \mathbb{R},$$

see [23, Lemmas 10.1 and 10.2] for the last equality. (I think better cite this here to be safe) Therefore, f belongs to the set $\mathcal{F}'(L_{\infty,L}, L_{2z,L})$ of $L_1(\mathcal{M})$-valued functions given in [23, Definition 1.4]. Since $L_{2z,L}$ is reflexive thanks to $1 < \alpha \leq 2z < \infty$, it follows from [23, Theorems 1.5, Remark 3.4] that the set $\mathcal{F}'(L_{\infty,L}, L_{2z,L})$ defines the interpolation space $C_\theta = C_\theta(L_{\infty,L}, L_{2z,L})$ in [23, Definition 1.1]. Hence for any $\theta \in (0,1)$, we have $f(\theta) \in C_\theta$ and

$$\|f(\theta)\|_{C_\theta} \leq \left(\sup_{t\in\mathbb{R}} \|f(it)\|_{L_{\infty,L}}\right)^{1-\theta} \left(\sup_{t\in\mathbb{R}} \|f(1+it)\|_{L_{2z,L}}\right)^{\theta} = \|y\|_{2z}^{\theta}$$

(see [3, Lemma 4.3.2(ii)] for the last inequality). By [23, Theorem 1.9] and the reiteration theorem (see [7]), $C_\theta = L_{2z/\theta,L}$ with equal norms, so that putting $\theta = z/z'$ we have

$$f(z/z') = h_\psi^{\frac{\alpha}{2z'}} h_\varphi^{1-\frac{\alpha}{2z'}} = y' h_\varphi^{\frac{2z'-1}{2z'}}$$

for some $y' \in L^{2z'}(\mathcal{M})$, and $\|y'\|_{2z'} \leq \|y\|_{2z}^{z/z'}$. This implies that $h_\psi^{\frac{\alpha}{2z'}} = y' h_\varphi^{\frac{\alpha-1}{2z'}}$ so that $Q_{\alpha,z'}(\psi\|\varphi) = \|y'\|_{2z'}^{2z'} \leq \|y\|_{2z}^{2z}$, and the assertion follows. $\square$

5 Monotonicity in the parameter α

In this section we show monotonicity of $D_{\alpha,z}$ in the parameter α as well as limits of $D_{\alpha,z}$ as $\alpha \nearrow 1$ and $\alpha \searrow 1$.

5.1 The case $\alpha < 1$ and all $z > 0$

The aim of this subsection is to prove the next theorem.

Theorem 5.1. *Let $\psi, \varphi \in \mathcal{M}_*^+$ and $z > 0$. Then we have*

(1) $\alpha \mapsto \log Q_{\alpha,z}(\psi\|\varphi)$ is convex on $(0,1)$,

(2) $\alpha \mapsto D_{\alpha,z}(\psi\|\varphi)$ is monotone increasing on $(0,1)$.

Below we will supply two different proofs of the theorem. The first proof for all $z > 0$ is given in the real analysis method, and the second one when $z > 1/2$ is in the complex analysis method.

5.1.1 The first proof

To give the first proof, we obtain a certain "log-majorization" result for positive τ-measurable operators, which might be meaningful in its own. Assume that $(\mathcal{N}, \tau)$ be a semi-finite von Neumann algebra $\mathcal{N}$ with a faithful normal semi-finite trace τ. Let $\widetilde{\mathcal{N}}$ denote the space of τ-measurable operators affiliated with $\mathcal{N}$. For each $a \in \widetilde{\mathcal{N}}$ we write $\mu_t(a)$, $t > 0$, for the (tth) *generalized s-number* of a [8]. We consider operators $a \in \widetilde{\mathcal{N}}$ satisfying

$$a \in \mathcal{N} \quad \text{or} \quad \mu_t(a) \le Ct^{-\gamma} \ (t > 0) \text{ for some } C, \gamma > 0. \tag{5.1}$$

For each $a \in \widetilde{\mathcal{N}}$ satisfying (5.1) we define [8]

$$\Lambda_t(a) := \exp \int_0^t \log \mu_s(a)\, ds, \qquad t > 0.$$

Note [8] that $\Lambda_t(a) \in [0, \infty)$, $t > 0$, are well defined whenever a satisfies (5.1). Also, note that if $a, b \in \widetilde{\mathcal{N}}$ satisfy (5.1), then $|a|^p$ ($p > 0$) and ab satisfy (5.1) too, as it is clear since $\mu_t(ab) \le \|a\|\mu_t(b)$ for $a \in \mathcal{N}$, $\mu_t(|a|^p) = \mu_t(a)^p$, and $\mu_t(ab) \le \mu_{t/2}(a)\mu_{t/2}(b)$ (see [8, Lemma 2.5]).

Proposition 5.2. *Let $a_j, b_j \in \widetilde{\mathcal{N}}_+$, $j = 1, 2$, satisfying (5.1) and assume that $a_1a_2 = a_2a_1$ and $b_1b_2 = b_2b_1$. Then for every $\theta \in (0,1)$ and any $t > 0$,*

$$\Lambda_t\Big((a_1^\theta a_2^{1-\theta})^{1/2}(b_1^\theta b_2^{1-\theta})(a_1^\theta a_2^{1-\theta})^{1/2}\Big) \le \Lambda_t\big(a_1^\theta b_1^\theta\big)\Lambda_t\big(a_2^{1-\theta} b_2^{1-\theta}\big). \tag{5.2}$$

In particular,

$$\Lambda_t\Big((a_1^{1/2}a_2^{1/2})^{1/2}(b_1^{1/2}b_2^{1/2})(a_1^{1/2}a_2^{1/2})^{1/2}\Big) \le \Lambda_t\big(a_1^{1/2}b_1a_1^{1/2}\big)^{1/2}\Lambda_t\big(a_2^{1/2}b_2a_2^{1/2}\big)^{1/2}. \tag{5.3}$$

Proof. Let $\theta \in (0,1)$ and $t > 0$ be arbitrary. For any $k \in \mathbb{N}$ we note that

$$
\begin{aligned}
&\big((a_1^\theta a_2^{1-\theta})^{1/2}(b_1^\theta b_2^{1-\theta})(a_1^\theta a_2^{1-\theta})^{1/2}\big)^k \\
&\quad = (a_1^\theta a_2^{1-\theta})^{1/2}(b_1^\theta b_2^{1-\theta})(a_1^\theta a_2^{1-\theta})(b_1^\theta b_2^{1-\theta})\cdots(a_1^\theta a_2^{1-\theta})(b_1^\theta b_2^{1-\theta})(a_1^\theta a_2^{1-\theta})^{1/2} \\
&\quad = (a_1^\theta a_2^{1-\theta})^{1/2} b_2^{1-\theta}(b_1^\theta a_1^\theta)(a_2^{1-\theta} b_2^{1-\theta})\cdots(b_1^\theta a_1^\theta)(a_2^{1-\theta} b_2^{1-\theta}) b_1^\theta (a_1^\theta a_2^{1-\theta})^{1/2}.
\end{aligned}
$$

Since $(a_1^\theta a_2^{1-\theta})^{1/2}(b_1^\theta b_2^{1-\theta})$, $(a_1^\theta a_2^{1-\theta})^{1/2}$, etc. are τ-measurable operators satisfying (5.1), we have by [8, Theorem 4.2(ii)]

$$
\begin{aligned}
&\Lambda_t\big((a_1^\theta a_2^{1-\theta})^{1/2}(b_1^\theta b_2^{1-\theta})(a_1^\theta a_2^{1-\theta})^{1/2}\big)^k \\
&\quad \le \Lambda_t\big((a_1^\theta a_2^{1-\theta})^{1/2} b_2^{1-\theta}\big)\Lambda_t(b_1^\theta a_1^\theta)^{k-1}\Lambda_t\big(a_2^{1-\theta} b_2^{1-\theta}\big)^{k-1}\Lambda_t\big(b_1^\theta (a_1^\theta a_2^{1-\theta})^{1/2}\big),
\end{aligned}
$$

so that

$$
\begin{aligned}
&\Lambda_t\big((a_1^\theta a_2^{1-\theta})^{1/2}(b_1^\theta b_2^{1-\theta})(a_1^\theta a_2^{1-\theta})^{1/2}\big) \\
&\quad \le \Lambda_t\big((a_1^\theta a_2^{1-\theta})^{1/2} b_2^{1-\theta}\big)^{1/k}\Lambda_t(b_1^\theta a_1^\theta)^{1-\frac{1}{k}}\Lambda_t\big(a_2^{1-\theta} b_2^{1-\theta}\big)^{1-\frac{1}{k}}\Lambda_t\big(b_1^\theta (a_1^\theta a_2^{1-\theta})^{1/2}\big)^{1/k}.
\end{aligned}
$$

Letting $k \to \infty$ gives (5.2). When $\theta = 1/2$, (5.2) is rewritten as (5.3). ☐

Remark 5.3. Since $\Lambda_t(a_j^r b_j^r) \le \Lambda_t(a_j b_j)^r$ for any $r \in (0,1)$ by [26], inequality (5.2) implies that

$$
\Lambda_t\big((a_1^\theta a_2^{1-\theta})^{1/2}(b_1^\theta b_2^{1-\theta})(a_1^\theta a_2^{1-\theta})^{1/2}\big) \le \Lambda_t(a_1 b_1)^\theta \Lambda_t(a_2 b_2)^{1-\theta} = \Lambda_t\big(a_1 b_1^2 a_1\big)^{\frac{\theta}{2}}\Lambda_t\big(a_2 b_2^2 a_2\big)^{\frac{1-\theta}{2}}.
$$

We are indeed interested in whether a stronger inequality

$$
\Lambda_t\big((a_1^\theta a_2^{1-\theta})^{1/2}(b_1^\theta b_2^{1-\theta})(a_1^\theta a_2^{1-\theta})^{1/2}\big) \le \Lambda_t\big(a_1^{1/2} b_1 a_1^{1/2}\big)^\theta \Lambda_t\big(a_2^{1/2} b_2 a_2^{1/2}\big)^{1-\theta}
$$

hold or not in the situation of Proposition 5.2. The last inequality is known to hold in the finite-dimensional setting [15, Theorem 2.1].

Lemma 5.4. *Let $\psi, \varphi \in \mathcal{M}_*^+$ with $\psi \ne 0$, and assume that $s(\psi) \not\perp s(\varphi)$. Then for every $z > 0$, $Q_{\alpha,z}(\psi\|\varphi) > 0$ for all $\alpha \in (0,1)$, and $\alpha \mapsto Q_{\alpha,z}(\psi\|\varphi)$ is continuous on $(0,1)$.*

Proof. Assume that $Q_{\alpha,z}(\psi\|\varphi) = 0$ for some $z > 0$ and $\alpha \in (0,1)$. Then $h_\psi^{\alpha/2z} h_\varphi^{(1-\alpha)/2z} = 0$ as a τ-measurable operator affiliated with $\mathcal{N} := \mathcal{M} \rtimes_\sigma \mathbb{R}$ that is a semi-finite von Neumann algebra with the canonical semi-finite trace τ, where σ_t is the modular automorphism group associated with some faithful normal state (or weight) - maybe skip this and just give a reference to Appendix A. Since $s(\psi) = s(h_\psi^{\alpha/2z})$ and $s(\varphi) = s(h_\varphi^{(1-\alpha)/2z})$, it is easy to see that $s(\psi) \perp s(\varphi)$. Hence the first assertion follows.

Next, since $p > 0 \mapsto a^p \in \widetilde{\mathcal{N}}$ is differentiable in the measure topology for any $a \in \widetilde{\mathcal{N}}_+$ (see, e.g., [13, Lemma 9.19]), we see that $\alpha \mapsto h_\psi^{\alpha/2z} h_\varphi^{(1-\alpha)/z} h_\psi^{\alpha/2z}$ is differentiable (hence continuous) on $(0,1)$ in the measure topology. Hence by [13, Lemma 9.14], the function $\alpha \mapsto Q_{\alpha,z}(\psi\|\varphi) = \|h_\psi^{\alpha/2z} h_\varphi^{(1-\alpha)/z} h_\psi^{\alpha/2z}\|_z^z$ is continuous. Here, when $z < 1$, note [8, Theorem 4.9(iii)] that $|\,\|a\|_z^z - \|b\|_z^z| \le \|a - b\|_z^z$ for $a, b \in L_z(\mathcal{M})$. ☐

The first proof of Theorem 5.1. We may assume that $s(\psi) \not\perp s(\varphi)$; otherwise, $D_{\alpha,z}(\psi\|\varphi) = 0$ for all $\alpha \in (0,1)$. Then by Lemma 5.4, $Q_{\alpha,z}(\psi\|\varphi) \in (0,\infty)$ for all $\alpha \in (0,1)$, and $\alpha \mapsto Q_{\alpha,z}(\psi\|\varphi)$ is continuous on $(0,1)$. Hence $\alpha \mapsto D_{\alpha,z}(\psi\|\varphi)$ is continuous on $(0,1)$ too.

Let $\alpha_1, \alpha_2 \in (0,1)$ and $z > 0$. Let $(\mathcal{N}, \tau)$ be as in the proof of Lemma 5.4. Consider $a_j := h_\psi^{\alpha_j/z}$ and $b_j := h_\varphi^{(1-\alpha_j)/z}$ in $\widetilde{\mathcal{N}}_+$. Since $a_j \in L^{z/\alpha_j}(\mathcal{M})$, we note by [8, Lemma 4.8] that $\mu_t(a_j) = t^{-\alpha_j/z}\|a_j\|_{z/\alpha_j}$ $t > 0$, and hence a_j satisfies (5.1). Similarly, b_j does so. Therefore, we can apply (5.3) to a_j, b_j with $t = 1$ to obtain

$$\begin{aligned}
&\int_0^1 \log \mu_s\Big(h_\psi^{\frac{\alpha_1+\alpha_2}{4z}} h_\varphi^{\frac{2-\alpha_1-\alpha_2}{2z}} h_\psi^{\frac{\alpha_1+\alpha_2}{4z}}\Big)\, ds \\
&\qquad \le \frac{1}{2}\bigg[\int_0^1 \log \mu_s\Big(h_\psi^{\frac{\alpha_1}{2z}} h_\varphi^{\frac{1-\alpha_1}{z}} h_\psi^{\frac{\alpha_1}{2z}}\Big)\, ds + \int_0^1 \log \mu_s\Big(h_\psi^{\frac{\alpha_2}{2z}} h_\varphi^{\frac{1-\alpha_2}{z}} h_\psi^{\frac{\alpha_2}{2z}}\Big)\, ds\bigg].
\end{aligned} \tag{5.4}$$

Since $h_\psi^{\frac{\alpha_1+\alpha_2}{4z}} h_\varphi^{\frac{2-\alpha_1-\alpha_2}{2z}} h_\psi^{\frac{\alpha_1+\alpha_2}{4z}}$ is in $L^z(\mathcal{M})$, we have by [8, Lemma 4.8] again

$$\mu_s\Big(h_\psi^{\frac{\alpha_1+\alpha_2}{4z}} h_\varphi^{\frac{2-\alpha_1-\alpha_2}{2z}} h_\psi^{\frac{\alpha_1+\alpha_2}{4z}}\Big) = s^{-1/z}\Big\|h_\psi^{\frac{\alpha_1+\alpha_2}{4z}} h_\varphi^{\frac{2-\alpha_1-\alpha_2}{2z}} h_\psi^{\frac{\alpha_1+\alpha_2}{4z}}\Big\|_z$$

so that

$$\log \mu_s\Big(h_\psi^{\frac{\alpha_1+\alpha_2}{4z}} h_\varphi^{\frac{2-\alpha_1-\alpha_2}{2z}} h_\psi^{\frac{\alpha_1+\alpha_2}{4z}}\Big)^z = -\log s + \log Q_{\frac{\alpha_1+\alpha_2}{2},z}(\psi\|\varphi). \tag{5.5}$$

Similarly,

$$\log \mu_s\Big(h_\psi^{\frac{\alpha_j}{2z}} h_\varphi^{\frac{1-\alpha_j}{z}} h_\psi^{\frac{\alpha_j}{2z}}\Big)^z = -\log s + \log Q_{\alpha_j,z}(\psi\|\varphi), \qquad j = 1,2. \tag{5.6}$$

Multiply z to both sides of (5.4) and insert (5.5) and (5.6) into it. Since $\int_0^1 (-\log s)\, ds = 1$, we then arrive at

$$1 + \log Q_{\frac{\alpha_1+\alpha_2}{2},z}(\psi\|\varphi) \le \frac{1}{2}\big[2 + \log Q_{\alpha_1,z}(\psi\|\varphi) + \log Q_{\alpha_2,z}(\psi\|\varphi)\big],$$

which implies that $\alpha \mapsto \log Q_{\alpha,z}(\psi\|\varphi)$ is midpoint convex on $(0,1)$. Since midpoint convexity implies convexity for continuous functions, (1) holds. Moreover, by [21, Theorem 1(vii)] we find that $\lim_{\alpha\nearrow 1} Q_{\alpha,z}(\psi\|\varphi) \le \psi(1)$. Therefore, (1) implies (2) from the defining formula of $D_{\alpha,z}$ in Definition 1.1. $\square$

5.1.2 The second proof when $z > 1/2$

Assume that $z > 1/2$ and let $p := 2z$ and $q := \frac{2z}{2z-1}$ so that $1/p + 1/q = 1$. Let $\rho, \sigma \in \mathcal{M}_*^+$ be such that $s(\rho) = \mathbf{1} - s(\psi)$ and $s(\sigma) = \mathbf{1} - s(\varphi)$. Put $\psi_0 := \psi + \rho$ and $\varphi_0 := \varphi + \sigma$, which are faithful elements in $\mathcal{M}_*^+$. Below, for each $p \in (1,\infty)$ and $\eta \in (0,1)$, we use for simplicity the notations

$$L_{p,L} := L_p(\mathcal{M}, \varphi_0)_L, \qquad L_{p,R} := L_p(\mathcal{M}, \psi_0)_R, \qquad L_{p,\eta} := L_p(\mathcal{M}, \psi_0, \varphi_0)_\eta,$$

(see (C.1)–(C.3) in Appendix C). Then, by [23, Theorem 11.1] (also see (C.6)) note that

$$L_{p,\eta} = C_\eta(L_{p,L}, L_{p,R}) \tag{5.7}$$

with equal norms. We divide the second proof into the cases $z \ge 1$ and $1/2 < z < 1$.

The second proof of Theorem 5.1 when $z \ge 1$. Assume that $z \ge 1$, and put $h_0 := h_\psi^{1/2} h_\varphi^{1/2} \in L_1(\mathcal{M})$. For each $\alpha \in (0,1)$ put $\eta := \frac{z-\alpha}{2z-1}$; then we have $0 < \eta < 1$ as

$$0 \le 1 - \frac{q}{2} = \frac{z-1}{2z-1} < \eta < \frac{z}{2z-1} = \frac{q}{2} \le 1.$$

Since

$$h_0 = h_\psi^{\frac{\eta}{q}} \Big(h_\psi^{\frac{\alpha}{2z}} h_\varphi^{\frac{1-\alpha}{2z}} \Big) h_\varphi^{\frac{1-\eta}{q}} = h_{\psi_0}^{\frac{\eta}{q}} \Big(h_\psi^{\frac{\alpha}{2z}} h_\varphi^{\frac{1-\alpha}{2z}} \Big) h_{\varphi_0}^{\frac{1-\eta}{q}},$$

we have $h_0 \in L_{p,\eta}$ and

$$\|h_0\|_{p,\psi_0,\varphi_0,\eta}^p = \Big\| h_\psi^{\frac{\alpha}{2z}} h_\varphi^{\frac{1-\alpha}{2z}} \Big\|_p^p = \Big\| h_\psi^{\frac{\alpha}{2z}} h_\varphi^{\frac{1-\alpha}{2z}} \Big\|_{2z}^{2z} = Q_{\alpha,z}(\psi\|\varphi). \tag{5.8}$$

Now let $\alpha_1, \alpha_2 \in (0,1)$ and for each $\theta \in (0,1)$ let $\alpha := (1-\theta)\alpha_1 + \theta\alpha_2$. Put $\eta_j := \frac{z-\alpha_j}{2z-1}$, $j = 1,2$, so that $\eta := \frac{z-\alpha}{2z-1} = (1-\theta)\eta_1 + \theta\eta_2$. To show (1), it suffices to prove that

$$Q_{\alpha,z}(\psi\|\varphi) \le Q_{\alpha_1,z}(\psi\|\varphi)^{1-\theta} Q_{\alpha_2,z}(\psi\|\varphi)^{\theta}. \tag{5.9}$$

From the complex interpolation space in (5.7) and the reiteration theorem [7], we have

$$L_{p,\eta} = C_\theta(L_{p,\eta_1}, L_{p,\eta_2}). \tag{5.10}$$

Since $h_0 \in L_{p,\eta_1} \cap L_{p,\eta_2}$ as shown above, it follows that

$$\|h_0\|_{p,\psi_0,\varphi_0,\eta} \le \|h_0\|_{p,\psi_0,\varphi_0,\eta_1}^{1-\theta} \|h_0\|_{p,\psi_0,\varphi_0,\eta_2}^{\theta}. \tag{5.11}$$

(Indeed, this is a special case of the Riesz–Thorin theorem applied to the map $T(z) := zh_0$, $z \in \mathbb{C}$.) From (5.11) and (5.8) (for η, η_1, η_2) we have (5.9), so that (1) has been shown. Moreover, (1) immediately implies (2) as in the last part of the first proof of Theorem 5.1. [I don't see how to modify Remark 2 in your notes (Jan. 10, 2024) in the case $\alpha < 1$. On the other hand, (2) is immediate from (1) in this case, as in the last part of the first proof of Th. 5.1.] $\square$

Next we turn to the case $1/2 < z < 1$. Here we will need a bit more of the complex interpolation method. Let us denote $\Sigma = \Sigma(L_{p,L}, L_{p,R}) := L_{p,L} + L_{p,R}$, and let $\mathcal{F}'(L_{p,L}, L_{p,R})$ be the set of functions $f : S := \{w \in \mathbb{C} : 0 \le \operatorname{Re} w \le 1\} \to \Sigma$ satisfying

(i) f is bounded, continuous on S and analytic in the interior of S (with respect to the norm in Σ),

(ii) $f(it) \in L_{p,L}$ and $f(1+it) \in L_{p,R}$ for all $t \in \mathbb{R}$,

(iii) the maps $t \in \mathbb{R} \mapsto f(it) \in L_{p,L}$ and $t \in \mathbb{R} \mapsto f(1+it) \in L_{p,R}$ are continuous and

$$\max\Big\{ \sup_{t\in\mathbb{R}} \|f(it)\|_{L_{p,L}}, \sup_{t\in\mathbb{R}} \|f(1+it)\|_{L_{p,R}} \Big\} < \infty.$$

(See [23, Definition 1.4].)

Consider the function $f : S \to L_1(\mathcal{M})$ defined by

$$f(w) := h_\psi^{\frac{w}{q}+\frac{1-w}{p}} h_\varphi^{\frac{1-w}{q}+\frac{w}{p}}, \qquad w \in S. \tag{5.12}$$

The next lemma shows that f has values in Σ.

Lemma 5.5. *We have $f \in \mathcal{F}'(L_{p,L}, L_{p,R})$. Moreover, for each $\eta \in (0,1)$ and $t \in \mathbb{R}$, $f(\eta+it) \in L_{p,\eta}$ and*

$$\|f(\eta+it)\|_{p,\varphi_0,\psi_0,\eta}^p = Q_{1-\eta,z}(\psi\|\varphi).$$

Proof. For any $\eta \in [0,1]$ we have

$$f(\eta+it) = h_\psi^{\frac{\eta}{q}} h_\psi^{i(\frac{1}{q}-\frac{1}{p})t} h_\psi^{\frac{1-\eta}{p}} h_\varphi^{\frac{\eta}{p}} h_\varphi^{i(\frac{1}{p}-\frac{1}{q})t} h_\varphi^{\frac{1-\eta}{q}} = h_{\psi_0}^{\frac{\eta}{q}} \Big(h_{\psi_0}^{i(\frac{1}{q}-\frac{1}{p})t} h_\psi^{\frac{1-\eta}{p}} h_\varphi^{\frac{\eta}{p}} h_{\varphi_0}^{i(\frac{1}{p}-\frac{1}{q})t} \Big) h_{\varphi_0}^{\frac{1-\eta}{q}}.$$

Recall [23, Lemmas 10.1 and 10.2] that $h_{\psi_0}^{it} \cdot h_{\varphi_0}^{-it}$ defines a strongly continuous one-parameter group of isometries on $L_p(\mathcal{M})$ for every $p \in [1,\infty)$. This implies properties (ii) and (iii) in the definition of $\mathcal{F}'(L_{p,L}, L_{p,R})$. Furthermore, for $\eta \in (0,1)$ we see that $f(\eta+it) \in L_{p,\eta}$ and

$$\|f(\eta+it)\|_{p,\varphi_0,\psi_0,\eta}^p = \Big\| h_{\psi_0}^{i(\frac{1}{q}-\frac{1}{p})t} h_\psi^{\frac{1-\eta}{p}} h_\varphi^{\frac{\eta}{p}} h_{\varphi_0}^{i(\frac{1}{p}-\frac{1}{q})t} \Big\|_p^p = \Big\| h_\psi^{\frac{1-\eta}{p}} h_\varphi^{\frac{\eta}{p}} \Big\|_p^p = Q_{1-\eta,z}(\psi\|\varphi).$$

Since $L_{p,\eta}$ for each $\eta \in (0,1)$ is continuouly embedded in Σ, this implies that f is Σ-valued. Since the Hölder inequality gives $\Big\| h_\psi^{\frac{1-\eta}{p}} h_\varphi^{\frac{\eta}{p}} \Big\|_p \le \psi(1)^{\frac{1-\eta}{p}} \varphi(1)^{\frac{\eta}{p}}$ for all $\eta \in (0,1)$, f is also bounded on S. Note that as a function with values in $L_1(\mathcal{M})$, f is bounded, continuous on S and analytic in the interior. To confirm (i), we now prove (similarly to [5, Secs. 9.1 and 29.1] that the continuity and analyticity properties also hold in Σ. Let $\mu_0(w,t)$ and $\mu_1(w,t)$ be the Poisson kernels associated with S. We then have

$$f(w) = \int_{\mathbb{R}} f(it)\mu_0(w,t)\,dt + \int_{\mathbb{R}} f(1+it)\mu_1(w,t)\,dt.$$

The integrals are in $L_1(\mathcal{M})$, but since $t \mapsto f(it) \in L_{p,L}$ and $t \mapsto f(1+it) \in {}_{p,R}$ are continuous and bounded in the respective norms, we see that the integrals also exist in Σ. Hence the above equality holds in Σ since Σ is continuously embedded in $L_1(\mathcal{M})$. This shows that $f : S \to \Sigma$ is continuous. Therefore, for any w, $0 < \operatorname{Re} w < 1$, the expression

$$\frac{1}{2\pi i} \int_\Gamma \frac{f(\xi)}{\xi - w}\, d\xi$$

for a suitable circle Γ around w in the interior of S is defined in Σ. Since f is analytic in $L_1(\mathcal{M})$, the expression is equal to $f(w)$, which shows that f is analytic in Σ in the interior of S. ☐

The second proof of Theorem 5.1 when $1/2 < z < 1$**.** Let $\alpha_1, \alpha_2 \in (0,1)$ and for each $\theta \in (0,1)$ let $\alpha := (1-\theta)\alpha_1 + \theta\alpha_2$. Put $\eta_j := 1 - \alpha_j$, $j = 1,2$, so that $\eta := 1 - \alpha = (1-\theta)\eta_1 + \theta\eta_2$. With the function f given in (5.12) define $f_1(w) := f((1-w)\eta_1 + w\eta_2)$, which belongs to the set

$\mathcal{F}(L_{p,\eta_1}, L_{p,\eta_2})$ (see [23, Definition 1.1]) by Lemma 5.5 and (C.6). Since $L_{p,\eta} = C_\theta(L_{p,\eta_1}, L_{p,\eta_2})$ by the iteration theorem, we have by usual arguments

$$\|f(\eta)\|_{p,\varphi_0,\psi_0,\eta} = \|f_1(\theta)\|_{C_\theta(L_{p,\eta_1},L_{p,\eta_2})} \le \left(\sup_{t\mathbb{R}} \|f_1(it)\|_{L_{p,\eta_1}}\right)^{1-\theta} \left(\sup_{t\in\mathbb{R}} \|f_1(1+it)\|_{L_{p,\eta_2}}\right)^{\theta}.$$

Since $f_1(it) = f(\eta_1 + i(\eta_2 - \eta_1)t)$ and $f_1(1+it) = f(\eta_2 + i(\eta_2 - \eta_1)t)$, it follows from Lemma 5.5 that

$$Q_{1-\eta,z}(\psi\|\varphi) \le Q_{1-\eta_1,z}(\psi\|\varphi)^{1-\theta} Q_{1-\eta_2,z}(\psi\|\varphi)^{\theta}.$$

This shows (1), which implies (2) as in the previous proof. $\square$

5.2 The case $1 < \alpha \le 2z$

In this subsection let us show monotonicity of $D_{\alpha,z}$ in the parameter $\alpha \in (1, 2z]$ when $z > 1/2$, based on the complex interpolation as in Sec. 5.1.2.

Theorem 5.6. *Let $\psi, \varphi \in \mathcal{M}_*^+$ and $z > 1/2$. Then we have*

(1) $\alpha \mapsto \log Q_{\alpha,z}(\psi\|\varphi)$ is convex on $(1, 2z]$,

(2) $\alpha \mapsto D_{\alpha,z}(\psi\|\varphi)$ is monotone increasing on $(1, 2z]$.

Proof. Assume that $z > 1/2$ and let p, q, ψ_0 and φ_0 be defined in the same way as in the beginning of Sec. 5.1.2. For each $\alpha \in (1, 2z]$ put $\eta := \frac{2z-\alpha}{2z-1} \in [0, 1)$. Assume that $Q_{\alpha,z}(\psi\|\varphi) < \infty$ (hence $s(\psi) \le s(\varphi)$), so that there exists a unique $y \in s(\psi)L^p(\mathcal{M})s(\varphi)$ such that $h_\psi^{\frac{\alpha}{2z}} = y h_\varphi^{\frac{\alpha-1}{2z}}$. Since

$$h_\psi = h_\psi^{\frac{2z-\alpha}{2z}} y h_\varphi^{\frac{\alpha-1}{2z}} = h_{\psi_0}^{\frac{\eta}{q}} y h_{\varphi_0}^{\frac{1-\eta}{q}},$$

we have $h_\psi \in L_{p,\eta}$ and

$$\|h_\psi\|_{p,\psi_0,\varphi_0,\eta}^p = \|y\|_p^p = Q_{\alpha,z}(\psi\|\varphi),$$

where for $\eta - 0$ ($\alpha = 2z$) the left-hand side is $\|h_\psi\|_{L_{p,L}}^p$. Now let $\alpha_1, \alpha_2 \in (1, 2z]$ and $\alpha = (1-\theta)\alpha_1 + \theta\alpha_2$ for any $\theta \in (0, 1)$. Put $\eta_j := \frac{2z-\alpha_j}{2z-1}$, $j = 1, 2$, and $\eta := \frac{2z-\alpha}{2z-1} = (1-\theta)\eta + \theta\eta_1$. To show (1), it suffices to prove that (5.9) holds in the present situation. For this, we may assume that $Q_{\alpha_j,z}(\psi\|\varphi) < \infty$, $j = 1, 2$. Then we can use (5.10) similarly to the proof in Sec. 5.1.2 with h_ψ instead of h_0. Hence we have (5.9), and (1) holds.

As for (2), note that $h_\psi = h_{\psi_0}^{1/q} h_\psi^{1/p} \in K_{p,R}^p$ (see (C.3)) and $\|h_\psi\|_{L_{p,R}}^p = \|h_\psi^{1/p}\|_p^p = \psi(1)$. Assume that $1 < \alpha < \alpha_1 \le 2z$ and $Q_{\alpha_1,z}(\psi\|\varphi) < \infty$, so that $\alpha = (1-\theta)\alpha_1 + \theta$ for some $\theta \in (0, 1)$. Let $\eta := \frac{2z-\alpha}{2z-1}$ and $\eta_1 := \frac{2z-\alpha_1}{2z-1}$. Since

$$L_{p,\eta} = C_\theta(L_{p,\eta_1}, L_{p,R})$$

by the reiteration theorem, it follows that

$$Q_{\alpha,z}(\psi\|\varphi) \le Q_{\alpha_1,z}(\psi\|\varphi)^{1-\theta} \psi(1)^{\theta}.$$

Taking the logarithm and noting $\theta = \frac{\alpha_1-\alpha}{\alpha_1-1}$, we obtain $D_{\alpha,z}(\psi\|\varphi) \le D_{\alpha_1,z}(\psi\|\varphi)$, proving (2). $\square$

5.3 Limits as $\alpha \nearrow 1$ and $\alpha \searrow 1$

The aim of this last subsection is to show the limits of $D_{\alpha,z}$ as $\alpha \nearrow 1$ and $\alpha \searrow 1$, extending the limits in (4.1) and (4.2).

Theorem 5.7. *Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$. For every $z > 0$ we have*

$$\lim_{\alpha \nearrow 1} D_{\alpha,z}(\psi\|\varphi) = D_1(\psi\|\varphi).$$

Proof. Assume first that $z \in (0,1]$ and $0 \le 1 - z < \alpha < 1$. Let $\beta := \frac{\alpha-1+z}{z}$; then $0 < \beta < 1$ and $\beta \nearrow 1$ as $\alpha \nearrow 1$. Hence the result follows from Lemma 5.8 below and (4.14) for $D_{\alpha,1}$. On the other hand, for the case $z \in [1,\infty)$ the result follows from Theorem 4.7 as (4.14) does from Proposition 4.4. $\square$

Lemma 5.8. *Assume that $z \in (0,1]$ and $0 \le 1 - z < \alpha < 1$. Let $\beta := \frac{\alpha-1+z}{z}$. Then for any $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$,*

$$D_{\beta,1}(\psi\|\varphi) \le D_{\alpha,z}(\psi\|\varphi) \le D_{\alpha,1}(\psi\|\varphi).$$

Proof. Since the statement is trivial for $z = 1$, we may assume that $z \in (0,1)$. The second inequality follows from Theorem 4.7(1). For the first inequality, noting that $\beta \in (0,1)$ by assumption and usnig by the Hölder inequality with $\frac{1}{2z} = \frac{1-z}{2z} + \frac{1}{2}$, we have

$$\begin{aligned} Q_{\alpha,z}(\psi\|\varphi) &= \left\| h_\psi^{\frac{\alpha}{2z}} h_\varphi^{\frac{1-\alpha}{2z}} \right\|_{2z}^{2z} = \left\| h_\psi^{\frac{1-z}{2z}} h_\psi^{\frac{\beta}{2}} h_\varphi^{\frac{1-\beta}{2}} \right\|_{2z}^{2z} \\ &\le \left\| h_\psi^{\frac{1-z}{2z}} \right\|_{\frac{2z}{1-z}}^{2z} \left\| h_\psi^{\frac{\beta}{2}} h_\varphi^{\frac{1-\beta}{2}} \right\|_{2}^{2z} = \psi(1)^{1-z} Q_{\beta,1}(\psi\|\varphi)^z, \end{aligned}$$

which proves the second inequality since $\alpha - 1 = z(\beta - 1)$. $\square$

Theorem 5.9. *Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$, and $z > 1/2$. Assume that $D_{\alpha,z}(\psi\|\varphi) < \infty$ for some $\alpha \in (1, 2z]$. Then we have*

$$\lim_{\alpha \searrow 1} D_{\alpha,z}(\psi\|\varphi) = D_1(\psi\|\varphi).$$

Proof. Assume that $z > 1/2$ and $D_{\alpha,z}(\psi\|\varphi) < \infty$ for some $\alpha \in (1, 2z]$. We may assume that φ is faithful. We utilize the function f on S given in (4.18), whose values are in $L_{2z,L}$ as seen from the proof of Theorem 4.8. Since f is analytic in a neighborhood of $1/\alpha$, we have the expansion

$$f(w) = f\Bigl(\frac{1}{\alpha}\Bigr) + \Bigl(w - \frac{1}{\alpha}\Bigr)h + o\Bigl(w - \frac{1}{\alpha}\Bigr),$$

where $h \in L_{2z,L}$ is the derivative of f at $w = 1/\alpha$ and $\|o(\zeta)\|_{L_{2z,L}}/|\zeta| \to 0$ as $|\zeta| \to 0$. For each $\alpha' \in (1, \alpha)$ it follows that

$$f\Bigl(\frac{\alpha'}{\alpha}\Bigr) = f\Bigl(\frac{1}{\alpha}\Bigr) + \frac{\alpha' - 1}{\alpha} h + o\Bigl(\frac{\alpha' - 1}{\alpha}\Bigr) \quad \text{as } \alpha' \searrow 1.$$

Furthermore, as in the proof of Theorem 4.8, we have $f(\alpha'/\alpha) = h_\psi^{\frac{\alpha'}{2z}} h_\varphi^{1-\frac{\alpha'}{2z}} = y' h_\varphi^{\frac{2z-1}{2z}}$ for some $y' \in L_{2z}(\mathcal{M})$, so that $Q_{\alpha' z}(\psi\|\varphi) = \|y\|_{2z}^{2z} = \|f(\alpha'/\alpha)\|_{L_{2z,L}}$.

Now let us recall that $L_{2z,L}$ is uniformly convex thanks to $2z > 1$ (see [10], [23, Theorem 4.2]), so that the norm $\|\cdot\|_{2z,L}$ is uniformly Fréchet differentiable (see, e.g., [2, Part 3, Chap. II]). We set $a_0 \in L_{\frac{2z}{2z-1},L}$ with the unit norm by

$$a_0 := \left(\frac{h_\psi}{\psi(1)}\right)^{\frac{2z-1}{2z}} h_\varphi^{\frac{1}{2z}},$$

so that $\langle a_0, f(1/\alpha)\rangle = \|f(1/\alpha)\|_{L_{2z,L}}$, where the dual pairing of $L_{\frac{2z}{2z-1},L}$ and $L_{2z,L}$ is given in (C.5) in Appendix C with $p = \frac{2z}{2z-1}$. Then the uniform Fréchet differentiability of $\|\cdot\|_{2z,L}$ at $1/\alpha$ implies that

$$\langle a_0, f(1/\alpha)\rangle = \lim_{\alpha'\searrow \mathbf{1}} \frac{\|f(\alpha'/\alpha)\|_{L_{2z,L}} - \|f(1/\alpha)\|_{L_{2z,L}}}{\frac{\alpha'-1}{\alpha}} \tag{5.13}$$

and also

$$\begin{aligned}
\langle a_0, f(1/\alpha)\rangle &= \lim_{t\to 0} \frac{1}{it}\langle a_0, f((1/\alpha)+it) - f(1/\alpha)\rangle \\
&= \psi(\mathbf{1})^{-\frac{2z-1}{2z}} \lim_{t\to 0}\frac{1}{it}\Big\langle h_\psi^{\frac{2z-1}{2z}} h_\varphi^{\frac{1}{2z}}, h_\psi^{\frac{1}{2z}}\Big(h_{\psi_0}^{\frac{\alpha}{2z}it} h_\varphi^{-\frac{\alpha}{2z}it} - \mathbf{1}\Big) h_\varphi^{\frac{2z-1}{2z}}\Big\rangle \\
&= \psi(\mathbf{1})^{-\frac{2z-1}{2z}} \lim_{t\to 0}\frac{1}{it}\mathrm{tr}\Big[h_\psi^{\frac{2z-1}{2z}} h_\psi^{\frac{1}{2z}}\big(h_{\psi_0}^{\frac{\alpha}{2z}it} h_\varphi^{-\frac{\alpha}{2z}it} - 1\big)\Big] \\
&= \psi(\mathbf{1})^{-\frac{2z-1}{2z}} \frac{\alpha}{2z}\lim_{t\to 0}\mathrm{tr}\big[h_\psi\big(h_{\psi_0}^{it} h_\varphi^{-it} - \mathbf{1}\big)\big] \\
&= \psi(\mathbf{1})^{-\frac{2z-1}{2z}} \frac{\alpha}{2z} D(\psi\|\varphi),
\end{aligned} \tag{5.14}$$

where we have used (C.5) for the third equality and [30, Theorem 5.7] for the last equality. Since $\psi(\mathbf{1}) = \|f(1/\alpha)\|_{L_{2z,L}}^{2z}$, it follows from (5.13) and (5.14) that

$$\begin{aligned}
D_{\alpha',z}(\psi\|\varphi) &= \frac{\log Q_{\alpha',z}(\psi\|\varphi) - \log\psi(\mathbf{1})}{\alpha'-1} = \frac{2z\log\|f(\alpha'/\alpha)\|_{L_{2z,L}} - 2z\log\|f(1/\alpha)\|_{L_{2z,L}}}{\alpha'-1} \\
&= \left(\frac{\log\|f(\alpha'/\alpha)\|_{L_{2z,L}} - \log\|f(1/\alpha)\|_{L_{2z,L}}}{\|f(\alpha'/\alpha)\|_{L_{2z,L}} - \|f(1/\alpha)\|_{L_{2z,L}}}\right)\frac{2z}{\alpha}\left(\frac{\|f(\alpha'/\alpha)\|_{L_{2z,L}} - \|f(1/\alpha)\|_{L_{2z,L}}}{\frac{\alpha'-1}{\alpha}}\right) \\
&\to \frac{1}{\psi(\mathbf{1})^{\frac{1}{2z}}}\frac{2z}{\alpha}\psi(\mathbf{1})^{-\frac{2z-1}{2z}}\frac{\alpha}{2z}D(\psi\|\varphi) = \frac{D(\psi\|\varphi)}{\psi(\mathbf{1})} = D_1(\psi\|\varphi)
\end{aligned}$$

as $\alpha \searrow 1$, as desired. ☐

6 Concluding remarks

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Acknowledgments

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A Haagerup L_p-spaces

Let $\mathcal{R} := \mathcal{M} \rtimes_{\sigma^\omega} \mathbb{R}$ be the crossed product of $\mathcal{M}$ by the *modular automorphism group* σ_t^ω, $t \in \mathbb{R}$, for a faithful normal semi-finite weight ω on $\mathcal{M}$. Then $\mathcal{R}$ is a semi-finite von Neumann algebra with the canonical trace τ. Let θ_s, $s \in \mathbb{R}$, be the *dual action* on $\mathcal{R}$ having the τ-scaling property $\tau \circ \theta_s = e^{-s}\tau$, $s \in \mathbb{R}$; see [34, Chap. X] (also [13, Chap. 8]). Let $\widetilde{\mathcal{R}}$ denote the space of τ-measurable operators affiliated with $\mathcal{R}$; see [8] (also [13, Chap. 4]). For $0 < p \leq \infty$, the *Haagerup L_p-space* $L_p(\mathcal{M})$ [10, 35] (also [13, Chap. 9]) is defined by

$$L_p(\mathcal{M}) := \{a \in \widetilde{\mathcal{R}} : \theta_s(a) = e^{-s/p}a,\ s \in \mathbb{R}\}.$$

In particular, $\mathcal{M} = L_\infty(\mathcal{M})$ and we have an order isomorphism $\mathcal{M}_* \cong L_1(\mathcal{M})$ given as $\psi \in \mathcal{M}_* \leftrightarrow h_\psi \in L_1(\mathcal{M})$, so that $\operatorname{tr} h_\psi = \psi(1)$, $\psi \in L_1(\mathcal{M})$, defines a positive linear functional tr on $L_1(\mathcal{M})$. For $0 < p < \infty$ the L_p-norm (quasi-norm for $0 < p < 1$) of $a \in L_p(\mathcal{M})$ is defined by $\|a\|_p := (\operatorname{tr}|a|^p)^{1/p}$, and the L_∞-norm $\|\cdot\|_\infty$ is the operator norm $\|\cdot\|$ on $\mathcal{M}$. For $1 \leq p < \infty$, $L_p(\mathcal{M})$ is a Banach space whose dual Banach space is $L^q(\mathcal{M})$, where $1/p + 1/q = 1$, by the duality pairing

$$(a, b) \in L_p(\mathcal{M}) \times L_q(\mathcal{M}) \mapsto \operatorname{tr}(ab) = \operatorname{tr}(ba). \tag{A.1}$$

The following lemmas are well known, proofs are given for completeness.

Lemma A.1. *For any $0 < p < \infty$ and $\varphi \in \mathcal{M}_*^+$, $h_\varphi^{\frac{1}{2p}} \mathcal{M}^+ h_\varphi^{\frac{1}{2p}}$ is dense in $L_p(\mathcal{M})^+$ with respect to the (quasi)-norm $\|\cdot\|_p$.*

Proof. We may assume that φ is faithful. By [20, Lemma 1.1], $\mathcal{M}h_\varphi^{\frac{1}{2p}}$ is dense in $L_{2p}(\mathcal{M})$ for any $0 < p < \infty$. Let $y \in L_p(\mathcal{M})^+$, then $y^{\frac{1}{2}} \in L_{2p}(\mathcal{M})$, hence there is a sequence $a_n \in \mathcal{M}$ such that $\|a_n h_\varphi^{\frac{1}{2p}} - y^{\frac{1}{2}}\|_{2p} \to 0$. Then also

$$\left\|h_\varphi^{\frac{1}{2p}} a_n^* - y^{\frac{1}{2}}\right\|_p = \left\|(a_n h_\varphi^{\frac{1}{2p}} - y^{\frac{1}{2}})^*\right\|_p = \left\|a_n h_\varphi^{\frac{1}{2p}} - y^{\frac{1}{2}}\right\|_p \to 0$$

and

$$\left\|h_\varphi^{\frac{1}{2p}} a_n^* a_n h_\varphi^{\frac{1}{2p}} - y\right\|_p = \left\|(h_\varphi^{\frac{1}{2p}} a_n^* - y^{\frac{1}{2}}) a_n h_\varphi^{\frac{1}{2p}} + y^{\frac{1}{2}}(a_n h_\varphi^{\frac{1}{2p}} - y^{\frac{1}{2}})\right\|_p.$$

Since $\|\cdot\|_p$ is a (quasi)-norm, the above expression goes to 0 by the Hölder inequality. ☐

Lemma A.2. *Let $0 < p \leq \infty$ and let $h, k \in L_p(\mathcal{M})^+$ be such that $h \leq k$. Then $\|h\|_p \leq \|k\|_p$. Moreover, if $1 \leq p < \infty$, then*

$$\|k - h\|_p^p \leq \|k\|_p^p - \|h\|_p^p.$$

Proof. The first statement follows from [8, Lemmas 2.5(iii) and 4.8]. The second statement is from [8, Lemma 5.1]. ☐

Lemma A.3. *Let $\psi, \varphi \in \mathcal{M}_*^+$ with $\psi \leq \varphi$. Then for any $a \in \mathcal{M}$ and $p \in [1, \infty)$,*

$$\operatorname{tr}\Big(\big(a^* h_\psi^{1/p} a\big)^p\Big) \leq \operatorname{tr}\Big(\big(a^* h_\varphi^{1/p} a\big)^p\Big).$$

Proof. Since $1/p \in (0, 1]$, it follows (see [14, Lemma B.7] and [16, Lemma 3.2]) that $h_\psi^{1/p} \leq h_\varphi^{1/p}$. Hence $a^* h_\psi^{1/p} a \leq a^* h_\varphi^{1/p} a$. Therefore, by Lemma A.2, we have the statement. ☐

B Haagerup's reduction theorem

In this appendix let us recall Haagerup's reduction theorem, which was presented in [11, Sec. 2] (a compact survey is also found in [9, Sec. 2.5]). Let $\mathcal{M}$ be a general σ-finite von Neumann algebra. Let ω be a faithful normal state of $\mathcal{M}$ and σ_t^ω ($t \in \mathbb{R}$) be the associated modular automorphism group. Consider the discrete additive group $G := \bigcup_{n\in\mathbb{N}} 2^{-n}\mathbb{Z}$ and define $\hat{\mathcal{M}} := \mathcal{M} \rtimes_{\sigma^\omega} G$, the crossed product of $\mathcal{M}$ by the action $\sigma^\omega|_G$. Then the dual weight $\hat{\omega}$ is a faithful normal state of $\hat{\mathcal{M}}$, and we have $\hat{\omega} = \omega \circ E_{\mathcal{M}}$, where $E_{\mathcal{M}} : \hat{\mathcal{M}} \to \mathcal{M}$ is the canonical conditional expectation (see, e.g., [13, Sec. 8.1], also [9, Sec. 2.5]).

Haagerup's reduction theorem is summarized as follows:

Theorem B.1 ([11])**.** *In the above setting, there exists an increasing sequence $\{\mathcal{M}_n\}_{n\ge1}$ of von Neumann subalgebras of $\hat{\mathcal{M}}$, containing the unit of $\hat{\mathcal{M}}$, such that the following hold:*

(i) *Each $\mathcal{M}_n$ is finite with a faithful normal tracial state τ_n.*

(ii) $\big(\bigcup_{n\ge1} \mathcal{M}_n\big)'' = \hat{\mathcal{M}}$.

(iii) *For every n there exist a (unique) faithful normal conditional expectation $E_{\mathcal{M}_n} : \hat{\mathcal{M}} \to \mathcal{M}_n$ satisfying*

$$\hat{\omega} \circ E_{\mathcal{M}_n} = \hat{\omega}, \qquad \sigma_t^{\hat{\omega}} \circ E_{\mathcal{M}_n} = E_{\mathcal{M}_n} \circ \sigma_t^{\hat{\omega}}, \quad t \in \mathbb{R}.$$

Moreover, for any $x \in \hat{\mathcal{M}}$, $E_{\mathcal{M}_n}(x) \to x$ in the σ-strong topology.

Furthermore, for any $\psi \in \mathcal{M}_*^+$, if we define $\hat{\psi} := \psi \circ E_{\mathcal{M}}$ then $\hat{\psi} \circ E_{\mathcal{M}_n} \to \hat{\psi}$ in the norm, as seen from [17, Theorem 4]. [It is written in [9, Sec. 2.5] that this was proved in [11, Theorem 3.1]. However it is not clear to me.]

C Kosaki's interpolation L_p-spaces

Assume that $\mathcal{M}$ is a σ-finite von Neumann algebra and let faithful $\psi_0, \varphi_0 \in \mathcal{M}_*^+$ be given. For each $\eta \in [0,1]$ consider an embedding $\mathcal{M} \hookrightarrow L^1(\mathcal{M})$ by $x \mapsto h_{\psi_0}^\eta x h_{\varphi_0}^{1-\eta}$. Defining $\|h_{\psi_0}^\eta x h_{\varphi_0}^{1-\eta}\|_\infty := \|x\|$ (the operator norm of x) on $h_{\psi_0}^\eta \mathcal{M} h_{\varphi_0}^{1-\eta}$ we have a pair $\big(h_{\psi_0}^\eta \mathcal{M} h_{\varphi_0}^{1-\eta}, L^1(\mathcal{M})\big)$ of compatible Banach spaces (see, e.g., [3]). For $1 < p < \infty$, *Kosaki's interpolation L_p-space* with respect to ψ_0, φ_0 and η [23] (also see [13, Sec. 9.3] for a compact survey) is defined as the complex interpolation Banach space:

$$L_p(\mathcal{M}, \psi_0, \varphi_0)_\eta := C_{1/p}\big(h_{\psi_0}^\eta \mathcal{M} h_{\varphi_0}^{1-\eta}, L^1(\mathcal{M})\big) \tag{C.1}$$

equipped with the interpolation norm $\|\cdot\|_{p,\psi_0,\varphi_0,\eta} := \|\cdot\|_{C_{1/p}}$. Then, Kosaki's theorem [23, Theorem 9.1] says that for every $\eta \in [0,1]$ and $p \in (1,\infty)$ with $1/p + 1/q = 1$,

$$L_p(\mathcal{M}, \psi_0, \varphi_0)_\eta = h_{\psi_0}^{\eta/q} L_p(\mathcal{M}) h_{\varphi_0}^{(1-\eta)/q} \ (\subset L_1(\mathcal{M})),$$
$$\|h_{\psi_0}^{\eta/q} a h_{\varphi_0}^{(1-\eta)/q}\|_{p,\psi_0,\varphi_0,\eta} = \|a\|_p, \qquad a \in L^p(\mathcal{M}),$$

that is, $L_p(\mathcal{M}) \cong L_p(\mathcal{M}, \psi_0, \varphi_0)_\eta$ by the isometry $a \mapsto h_{\psi_0}^{\eta/q} a h_{\varphi_0}^{(1-\eta)/q}$. In the main body of this paper we use the special cases where $\eta = 0, 1$, that is,

$$L_p(\mathcal{M}, \varphi_0)_L := C_{1/p}\big(\mathcal{M} h_{\varphi_0}, L_1(\mathcal{M})\big) = L_p(\mathcal{M}) h_{\varphi_0}^{1/q}, \tag{C.2}$$

$$L_p(\mathcal{M}, \psi_0)_R := C_{1/p}\big(h_{\psi_0} \mathcal{M}, L_1(\mathcal{M})\big) = h_{\psi_0}^{1/q} L_p(\mathcal{M}), \tag{C.3}$$

which are called Kosaki's *left and right L_p-spaces*, respectively. Another special case we use is the *symmetric L_p-space $L_p(\mathcal{M}, \varphi_0)$* where $\eta = 1/2$ and $\psi_0 = \varphi_0$, i.e.,

$$L_p(\mathcal{M}, \varphi_0) = C_{1/p}\big(h_{\varphi_0}^{1/2} \mathcal{M} h_{\varphi_0}^{1/2}, L_1(\mathcal{M})\big) = h_{\varphi_0}^{1/2q} L_p(\mathcal{M}) h_{\varphi_0}^{1/2q}, \tag{C.4}$$

whose interpolation norm is denoted by $\|\cdot\|_{p,\varphi_0}$. The L_p-L_q duality of Kosaki's L_p-spaces can be given by transforming the duality paring in (A.1); in particular, the duality pairing between $L_p(\mathcal{M}, \varphi_0)_L$ and $L_q(\mathcal{M}, \varphi_0)_L$ for $1 \le p < \infty$ and $1/p + 1/q = 1$ is written as

$$\langle a h_{\varphi_0}^{1,q}, b h_{\varphi_0}^{1/p} \rangle = \mathrm{tr}(ab), \qquad a \in L_p(\mathcal{M}),\ b \in L_q(\mathcal{M}). \tag{C.5}$$

Kosaki's non-commutative Stein–Weiss interpolation theorem [23, Theorem 11.1] says that for each $\eta \in (0,1)$ and $p \in (1,\infty)$, Kosaki's L_p-space $L_p(\mathcal{M}, \psi_0, \varphi_0)_\eta$ given in (C.1) is the complex interpolation space of the left and right L_p-spaces in (C.2) and (C.3) with equal norms, that is,

$$L_p(\mathcal{M}, \psi_0, \varphi_0)_\eta = C_{1/p}\big(h_{\psi_0}^{\eta} \mathcal{M} h_{\varphi_0}^{1-\eta}, L^1(\mathcal{M})\big) = C_\eta(L_p(\mathcal{M}, \varphi_0)_L, L_p(\mathcal{M}, \psi_0)_R). \tag{C.6}$$

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