The testing range of quantum statistical models and measurements

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Abstract

A positive operator-valued measure (i.e., a quantum measurement) can be regarded as a linear map from the space of states to the space of probability distributions. The image of the set of states through such a map is naturally defined in parametric form, that is, as a body in the space of probability distributions parameterized by the set of states itself. In such a form, the image of a quantum measurement is impractical to treat analytically. Our first result is to provide an implicit outer approximation of the image of any given quantum measurement in any finite dimension: namely, a region in probability space that contains the desired image, but is defined implicitly, using a formula that depends only on the given quantum measurement. This generalizes the bounding recently provided in Ref. [22] by Xu, Schwonnek, and Winter: first, the extension is from Pauli strings to arbitrary measurements; second, the optimization is not restricted to the radius of fixed-axis ellipsoids, but it is a *global* optimization over all the parameters of the ellispoid. The outer approximation that we construct is *minimal* among all such outer approximations, and *close*, in the sense that it becomes the *maximal inner* approximation up to a constant scaling factor. We also obtain a similar result for the dual problem of implicitizing the image of the set of effects through a family of quantum states (i.e., a quantum statistical model). Finally, we apply our approximation formulas to characterize, in a semi-device independent way, the ability to transform one quantum measurement into another, or one quantum statistical model into another.

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In statistics, information theory, and mathematical economics one is often faced with the problem of comparing two setups in terms of their expected performances on a particular task of interest. For example, one might compare two statistical models by comparing their informativeness in a given parameter estimation problem, or two noisy channels with respect to a given communication figure of merit, or again two portfolios with respect to their expected utility in a given betting scenario. The comparison could also be extended, so to ask when a given setup is *always* better than another one, i.e., independent of any particular task at hand. Such "global" comparisons, generally described by a preorder relation, play a crucial role in the formulation of mathematical statistics.

The simplest example of one such preorder in statistics is given by the *majorization preorder* of probability distributions [1, 2, 3, 4]. Generalizing this, we find the comparison of families comprising two or more probability distributions. The case of pairs of probability distributions (i.e., *dichotomies*) is also known as *relative* majorization [5, 6, 7, 8], whereas the case of multiple elements is usually referred to as comparison of statistical *experiments* or *models* [5, 6, 7, 9].

The relevance of such preorder relations is epitomized by Blackwell's theorem [5, 6], which establishes the equivalence between the above mentioned statistical comparisons, and the existence of a suitable stochastic map that transforms one setup (the "always better" one) into the other (the "always worse" one). For this reason, Blackwell's theorem and its variants provide a powerful framework for general resource theories [10], and indeed recent quantum extensions of Blackwell's theorem [11, 12, 13] have found fruitful application in the study of quantum entanglement [14], quantum thermodynamics [15, 16], and quantum measurement theory [17, 18, 19], for example.

Unfortunately, due to the non-commutativity of the underlying algebra, the quantum version of Blackwell's theorem [11] turns out to be necessarily more convoluted than its original classical variant. This is particularly evident in the case of relative majorization: while classical relative majorization can be summarized in a finite collection of easily computable inequalities [5, 8], in the quantum case (with the notable exceptions of qubits [20, 21]) an infinite number of scalar inequalities must be evaluated [13]. The situation becomes even more cumbersome in the case of quantum statistical models [11].

The image $\boldsymbol{\pi}(\mathbb{S}_d)$ of the set $\mathbb{S}_d$ of states through a measurement $\boldsymbol{\pi}$ is by definition given in parametric form, that is, it is a body in the probability space parameterized by states in the state space. Ideally, one would aim at implicitizing it, that is, writing it in the form $f(p) \leq 1$, for probability distributions p. However, due to intractability of the strucuture of the state space, we provide here inclusion conditions in terms of implicit bodies.

Definition 1. *For any d-dimensional, n-outcome measurement $\boldsymbol{\pi} = \{\pi_i\}_{i=1}^n$, let $\{\mathcal{E}_r(\boldsymbol{\pi})\}_{r\in\mathbb{R}}$ be the following family of hyper-ellipsoids parameterized by r:*

$$\mathcal{E}_r(\boldsymbol{\pi}) := \left\{ \mathbf{p} \in \boldsymbol{\pi}\left(\mathbb{C}^d\right) \ \ s.t.\ \left| \sqrt{Q^+}\left(\mathbf{p} - \mathbf{t}\right)\right|_2^2 \leq \frac{1}{r^2} \right\},$$

where $Q \in \mathbb{R}^{n\times n}$ is the positive semi-definite covariance matrix given by

$$Q_{ij} = \frac{d-1}{d}\left(\mathrm{Tr}\left[\pi_i\pi_j\right] - \frac{\mathrm{Tr}\left[\pi_i\right]\mathrm{Tr}[\pi_j]}{d}\right), \quad 0 \leq i, j \leq n,$$

and $\mathbf{t} \in \mathbb{R}^n$ is the vector with entries $t_i = \mathrm{Tr}\left[\pi_i\right]/d$.

Theorem 1. *For any d-dimensional, n-outcome informationally complete measurement $\boldsymbol{\pi}$, one has that $\mathcal{E}_{\sqrt{d^2-1}}(\boldsymbol{\pi})$ is the maximum volume ellipsoid enclosed in $\boldsymbol{\pi}(\mathbb{S}_d)$ and $\mathcal{E}_1(\boldsymbol{\pi})$ is the minimum volume ellipsoid enclosing $\boldsymbol{\pi}(\mathbb{S}_d)$.*

This generalizes the bounding recently provided in Ref. [22] by Xu, Schwonnek, and Winter: first, the extension is from Pauli strings to arbitrary measurements; second, the optimization is not restricted to the radius of fixed-axis ellipsoids, but it is a *global* optimization over all the parameters of the ellispoid.

Now that we have a close approximation of the image of the set of states through any given measurement, we turn our attention to applying it to semi-device independent tests of simulability. We say that a d_1-dimensional, n-outcome measurement $\boldsymbol{\pi}_1$ simulates a d_0-dimensional, n-outcome measurement $\boldsymbol{\pi}_0$ if and only if there exists a completely positive map $\mathcal{C} : \mathcal{L}(\mathbb{C}^{d_0}) \to \mathcal{L}(\mathbb{C}^{d_1})$ such that

$$\boldsymbol{\pi}_1 \circ \mathcal{C} = \boldsymbol{\pi}_0. \tag{1}$$

The following corollary generalizes Corollary 2 of Ref. [21] to the arbitrary dimensional case, providing a semi-device independent test of Eq. (1).

Corollary 1 (Semi-device independent simulability test)**.** *Given a set $\mathcal{P}$ of n-element probability distributions generated by a d_1-dimensional (otherwise unspecified) measurement $\boldsymbol{\pi}_1$, for any d_0 and for any d_0-dimensional n-outcome measurement $\boldsymbol{\pi}_0$ such that*

$$\mathcal{E}_1\left(\boldsymbol{\pi}_0\right) \subseteq \mathcal{P},$$

there exists a trace preserving map $\mathcal{C}$ that is positive on the support of $\boldsymbol{\pi}_0$ such that Eq. (1) *holds. Moreover, if $D = 2$, $n \leq 3$, and $d \leq 3$, map $\mathcal{C}$ in Eq.* (1) *is completely positive, that is, measurement $\boldsymbol{\pi}_1$ simulates measurement $\boldsymbol{\pi}_0$.*

We also obtain a similar result for the dual problem of implicitizing the image of the set of effects through a family of quantum states (i.e., a quantum statistical model).

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