Supplemental Material for:
Limitations of Classically-Simulable Measurements for Quantum State Discrimination

In this Supplemental Material, we provide detailed proofs of the theorems and propositions in the manuscript "Limitations of Classically-Simulable Measurements for Quantum State Discrimination". In Appendix I, we cover the basics of the discrete Wigner function. In Appendix II, we first present the detailed proofs for Lemma 2 and Proposition 3, which characterize the asymptotic limits of PWF POVMs for distinguishing a pure magic state and its orthogonal complement and a mixed magic state and its orthogonal complement, respectively. Then, we provide the proof of Theorem 4 which serves as an easy-to-compute criterion for when PWF POVMs cannot unambiguously distinguish two quantum states in the many-copy scenario. Appendix III introduces the primal and dual SDP for calculating the optimal error probability of distinguishing two quantum states via PWF POVMs. We further provide detailed proof of Proposition 5. Then in Appendix IV, we furnish detailed proofs for Theorem 6 and Proposition 7, both of which characterize the distinctions between the QRT of magic states and entanglement in QSD tasks.

I. THE DISCRETE WIGNER FUNCTION

We denote $\mathcal{H}_d$ as a Hilbert space of dimension d, and $\{|j\rangle\}_{j=0,\cdots,d-1}$ as the standard computational basis. Let $\mathcal{L}(\mathcal{H}_d)$ be the space of operators mapping $\mathcal{H}_d$ to itself. For odd prime dimension d, the unitary boost and shift operators $X, Z \in \mathcal{L}(\mathcal{H}_d)$ are defined as [1]:

$$X|j\rangle = |j \oplus 1\rangle, \quad Z|j\rangle = w^j |j\rangle, \tag{S1}$$

where $w = e^{2\pi i/d}$ and $\oplus$ denotes addition modulo d. The *discrete phase space* of a single d-level system is $\mathbb{Z}_d \times \mathbb{Z}_d$, which can be associated with a $d \times d$ cubic lattice. For a given point in the discrete phase space $\mathbf{u} = (a_1, a_2) \in \mathbb{Z}_d \times \mathbb{Z}_d$, the Heisenberg-Weyl operators are given by

$$T_{\mathbf{u}} = \tau^{-a_1 a_2} Z^{a_1} X^{a_2}, \tag{S2}$$

where $\tau = e^{(d+1)\pi i/d}$. These operators form a group, the Heisenberg-Weyl group, and are the main ingredient for representing quantum systems in finite phase space. The case of non-prime dimension can be understood to be a tensor product of $T_{\mathbf{u}}$ with odd prime dimension. For each point $\mathbf{u} \in \mathbb{Z}_d \times \mathbb{Z}_d$ in the discrete phase space, there is a phase-space point operator $A_{\mathbf{u}}$ defined as

$$A_{\mathbf{0}} := \frac{1}{d} \sum_{\mathbf{w}} T_{\mathbf{w}}, \quad A_{\mathbf{u}} := T_{\mathbf{u}} A_{\mathbf{0}} T_{\mathbf{u}}^{\dagger}. \tag{S3}$$

The discrete Wigner function of a state ρ at the point $\mathbf{u}$ is then defined as

$$W_\rho(\mathbf{u}) := \frac{1}{d} \operatorname{Tr}\left[A_{\mathbf{u}} \rho\right]. \tag{S4}$$

More generally, we can replace ρ with H for the discrete Wigner function of a Hermitian operator H. For the case of H being an effect E of some Positive Operator-Valued Measure (POVM), its discrete Wigner function is given by

$$W(E|\mathbf{u}) := \operatorname{Tr}[E A_{\mathbf{u}}]. \tag{S5}$$

There are several useful properties of the set $\{A_{\mathbf{u}}\}_{\mathbf{u}}$ as follows:

1. $A_{\mathbf{u}}$ is Hermitian;
2. $\sum_{\mathbf{u}} A_{\mathbf{u}}/d = \mathbb{1}$;
3. $\operatorname{Tr}[A_{\mathbf{u}} A_{\mathbf{u}'}] = d\delta(\mathbf{u}, \mathbf{u}')$;
4. $\operatorname{Tr}[A_{\mathbf{u}}] = 1$;
5. $H = \sum_{\mathbf{u}} W_H(\mathbf{u}) A_{\mathbf{u}}$;
6. $\{A_{\mathbf{u}}\}_{\mathbf{u}} = \{{A_{\mathbf{u}}}^T\}_{\mathbf{u}}$.

We say a Hermitian operator H has positive discrete Wigner functions (PWFs) if $W_H(\mathbf{u}) \geq 0, \forall \mathbf{u} \in \mathbb{Z}_d \times \mathbb{Z}_d$. According to the discrete Hudsons theorem [2], a pure state ρ is a stabilizer state, if and only if it has PWFs. Similarly, an n-valued POVM $\mathbf{E} = \{E_j\}_{j=0}^{n-1}$ is said to be a PWF POVM if each E_j has PWFs. The discrete Wigner function of each measurement outcome of a POVM $\{E_j\}_{j=0}^{n-1}$ has a conditional quasi-probability interpretation over the phase space

$$\sum_j W(E_j|\mathbf{u}) = 1, \tag{S6}$$

where $E_j \geq \mathbf{0}$ and $\sum_j E_j = \mathbb{1}$. In the case of E_j having PWFs, $W(E_j|\mathbf{u})$ can be interpreted as the probability of obtaining outcome j given that the system is at the phase space point $\mathbf{u}$. This property is crucial for efficiently simulating quantum computation classically. The total probability of obtaining outcome j from a measurement on state ρ is then given by

$$P(j|\rho) = \sum_{\mathbf{u}} W_\rho(\mathbf{u}) W(E_j|\mathbf{u}), \tag{S7}$$

where $P(j|\rho)$ can be effectively estimated [3, 4] when both ρ and E_j have PWFs, implying that both $W_\rho(\mathbf{u})$ and $W(E_j|\mathbf{u})$ possess classical probability interpretations. Therefore, negative quasi-probability is a vital resource for quantum speedup in stabilizer computation and has deep connections with contextuality in stabilizer measurements [5, 6]. In this sense, PWF POVMs are regarded as classically-simulable measurements [6], which strictly include all stabilizer measurements.

Lemma S1 *For any phase-space point operator $A_{\mathbf{u}} \in \mathcal{L}(\mathcal{H}_d)$, $\mathbf{u} \in \mathbb{Z}_d \times \mathbb{Z}_d$, $A_{\mathbf{u}}$ is a unitary operator with eigenvalues of $+1$ or -1, where eigenvalue +1 has a degeneracy of $\frac{d+1}{2}$ and eigenvalue -1 has a degeneracy of $\frac{d-1}{2}$.*

Proof Suppose $A_{\mathbf{0}}$ has a spectral decomposition

$$A_{\mathbf{0}} = \sum_i a_i |a_i\rangle\langle a_i|. \tag{S8}$$

Since $A_{\mathbf{u}} = T_{\mathbf{u}} A_{\mathbf{0}} T_{\mathbf{u}}^\dagger$, we have $A_{\mathbf{u}} = \sum_i a_i T_{\mathbf{u}} |a_i\rangle\langle a_i| T_{\mathbf{u}}^\dagger$ which means all possible $A_{\mathbf{u}}$ have same eigenvalues $\{a_i\}$ with corresponding eigenvectors $\{T_{\mathbf{u}}|a_i\rangle\}$. Note that $A_{\mathbf{0}} = \sum_{k\in\mathbb{Z}_d} |k\rangle\langle -k|$ [7], we conclude that the matrix representation of $A_{\mathbf{0}}$ is $A_{\mathbf{0}} = \begin{bmatrix} 1 & 0 \\ 0 & {\sigma_x}^{\otimes\lfloor\frac{d}{2}\rfloor} \end{bmatrix}$, where $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Now we consider the eigenvalues of $A_{\mathbf{0}}$. Notice that

$$\det(|aI_d - A_{\mathbf{0}}|) = \det\left(\left|aI_d - \begin{bmatrix} 1 & 0 \\ 0 & {\sigma_x}^{\otimes\lfloor\frac{d}{2}\rfloor} \end{bmatrix}\right|\right) = (1-a)\det\left|aI_{d-1} - {\sigma_x}^{\otimes\lfloor\frac{d}{2}\rfloor}\right| = (1-a)(1-a^2)^{\lfloor\frac{d}{2}\rfloor} = 0, \tag{S9}$$

where a denotes the eigenvalue of $A_{\mathbf{0}}$. Thus, eigenvalues of $A_{\mathbf{u}}$ are $+1$ or -1, and eigenvalue $+1$ has a degeneracy of $\frac{d+1}{2}$ and eigenvalue -1 has a degeneracy of $\frac{d-1}{2}$ due to $\mathrm{Tr}(A_{\mathbf{u}}) = 1$. We can further conclude that $A_{\mathbf{u}}$ is unitary according to the possible eigenvalues of $A_{\mathbf{u}}$. ■

II. ASYMPTOTIC LIMITS OF PWF POVMS

Lemma 2 *For a PWF unextendible subspace $\mathcal{S} \subseteq \mathcal{H}_d$, if there is a PWF state $\rho \in \mathcal{D}(\mathcal{S})$ such that $\mathrm{supp}(\rho) = \mathcal{S}$, then $\mathcal{S}$ is strongly PWF unextendible.*

Proof First, we will demonstrate that $\mathcal{S}^{\otimes 2}$ is PWF unextendible through a proof by contradiction. Suppose $\rho_s \in \mathcal{S}$ is a PWF state such that $\mathrm{supp}(\rho_s) = \mathrm{supp}(\mathcal{S})$. If there is a PWF state σ supporting on $(\mathcal{S}^{\otimes 2})^\perp$, then we have $\mathrm{Tr}[\sigma(\rho_s \otimes \rho_s)] = 0$ which leads to

$$\mathrm{Tr}\left[\rho_s \mathrm{Tr}_2[\sigma(\mathbb{1} \otimes \rho_s)]\right] = 0. \tag{S10}$$

Now we construct an operator $\sigma' \in \mathcal{L}(\mathcal{H}_d)$ by

$$\sigma' = \mathrm{Tr}_2[\sigma(\mathbb{1} \otimes \rho_s)]. \tag{S11}$$

It is easy to check that σ' is hermitian, $\sigma' \geq \mathbf{0}$ and $\mathrm{Tr}(\sigma' \rho_s) = 0$.

If $\sigma' = \mathbf{0}$, we know that $\mathrm{Tr}\,\sigma' = 0$ which indicates that $\mathrm{Tr}[\sigma(\mathbb{1} \otimes \rho_s)] = \mathrm{Tr}[\rho_s \mathrm{Tr}_1 \sigma] = 0$. We note that $\mathrm{Tr}_1 \sigma \neq \mathbf{0}$ otherwise $\sigma = \mathbf{0}$. Also, $\mathrm{Tr}_1 \sigma$ is PWF because partial trace preserves the positivity of the discrete Wigner functions which can be observed by expressing the state as $\sigma = \sum_{\mathbf{u}} W_\sigma(\mathbf{u}) A_{\mathbf{u}}$. Thus, we will get a PWF state supporting on $\mathcal{S}^\perp$ after normalizing $\operatorname{Tr}_1 \sigma$, a contradiction to the PWF unextendibility of $\mathcal{S}$.

If $\sigma' \neq \mathbf{0}$, we can calculate the Wigner functions of σ' and demonstrate their non-negativity as follows.

$$W_{\sigma'}(\mathbf{u}_1) = \frac{1}{d}\operatorname{Tr}(\sigma' A_{\mathbf{u}_1}) = \frac{1}{d^2}\sum_{\mathbf{u}_2} \operatorname{Tr}[\sigma(A_{\mathbf{u}_1} \otimes A_{\mathbf{u}_2})] \operatorname{Tr}(\rho_s A_{\mathbf{u}_2}). \tag{S12}$$

Since σ and ρ_s are PWF, i.e., $\operatorname{Tr}[\sigma(A_{\mathbf{u}_1} \otimes A_{\mathbf{u}_2})] \geq 0, \operatorname{Tr}(\rho_s A_{\mathbf{u}_2}) \geq 0, \forall \mathbf{u}_1, \mathbf{u}_2$, we have $W_{\sigma'}(\mathbf{u}_1) \geq 0$. Thus σ' is PWF. Consequently, we have obtained a PWF state supporting on $\mathcal{S}^\perp$ after normalizing σ', a contradiction to the PWF unextendibility of $\mathcal{S}$.

Hence, we conclude that $\mathcal{S}^{\otimes 2}$ is PWF unextendible. Using a similar technique, we can prove that $\mathcal{S}^{\otimes 3}$ is PWF unextendible by making a contradiction to the PWF unextendibility of $\mathcal{S}^{\otimes 2}$. In turn, we can conclusively demonstrate that $\mathcal{S}^{\otimes k}$ is PWF unextendible for any positive integer k, which completes the proof. ■

Proposition 3 *There exists a strongly PWF unextendible subspace $\mathcal{S} \subseteq \mathcal{H}_d$ of dimension $(d+1)/2$.*

Proof First, we construct a $(d-1)/2$ dimensional subspace $\mathcal{S}_m \subseteq \mathcal{H}_d$ that supports only magic states. Then we will show that $\mathcal{S}_m^\perp \subseteq \mathcal{H}_d$ is a strongly PWF unextendible subspace of dimension $(d+1)/2$. We consider the eigenspace of the phase-space point operator $A_{\mathbf{0}}$. Denote the set of all eigenvectors of $A_{\mathbf{0}}$ corresponding to eigenvalue of -1 as $S^- := \{|a_i^-\rangle\}_{i=1}^{\frac{d-1}{2}}$. We will show these states in S^- span a subspace $\mathcal{S}_m \in \mathcal{H}_d$ that contains no PWF states.

Obviously, any $|a_i^-\rangle \in S^-$ is a magic state due to $\operatorname{Tr}(|a_i^-\rangle\langle a_i^-|A_{\mathbf{0}}) = -1$. Suppose $|\psi\rangle$ is an arbitrary pure state in S^-. It can be written as $|\psi\rangle = \sum_i \alpha_i |a_i^-\rangle$. The Wigner function of $|\psi\rangle$ at the phase-space point $\mathbf{0}$ is

$$W_\psi(\mathbf{0}) = \frac{1}{d}\langle\psi|A_{\mathbf{0}}|\psi\rangle = \frac{1}{d}\sum_{i,j}\alpha_i^*\alpha_j\langle a_i^-|A_{\mathbf{0}}|a_j^-\rangle = -\frac{1}{d}\sum_{i,j}\alpha_i^*\alpha_j\langle a_i^-|a_j^-\rangle = -\frac{1}{d}\sum_i \alpha_i\alpha_i^* = -\frac{1}{d}, \tag{S13}$$

which tells $|\psi\rangle$ is a magic state. For any mixed state $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ on $\mathcal{S}_m$, we have

$$W_\rho(\mathbf{0}) = \sum_i p_i W_{\psi_i}(\mathbf{0}) = -\frac{1}{d}\sum_i p_i = -\frac{1}{d}. \tag{S14}$$

Thus, we construct a $(d-1)/2$ dimensional subspace $\mathcal{S}_m$ that contains no PWF states. Obviously, $\mathcal{S}_m^\perp$ is a PWF unextendible subspace of dimension $(d+1)/2$. $\mathcal{S}_m^\perp$ is spanned by the set of all eigenvectors of $A_{\mathbf{0}}$ corresponding to eigenvalue of $+1$, denoted as $S^+ := \{|a_i^+\rangle\}_{i=1}^{\frac{d+1}{2}}$. We show that $\rho_n = \frac{2}{d+1}\sum_j |a_j^+\rangle\langle a_j^+|$ is a PWF state on $\mathcal{S}_m^\perp$ as follows:

$$\begin{aligned} W_{\rho_n}(\mathbf{u}) &= \frac{1}{d}\operatorname{Tr}(A_{\mathbf{u}}\rho_n) && \text{(S15a)}\\ &= \frac{2}{(d+1)d}\operatorname{Tr}(A_{\mathbf{u}}\sum_j |a_j^+\rangle\langle a_j^+|) && \text{(S15b)}\\ &= \frac{2}{(d+1)d}\operatorname{Tr}[A_{\mathbf{u}}(I + A_{\mathbf{0}})/2] && \text{(S15c)}\\ &= \frac{1+\delta_{\mathbf{u},\mathbf{0}}}{(d+1)} > 0. && \text{(S15d)} \end{aligned}$$

From Eq. (S15b) to Eq. (S15c), we use the properties that $A_{\mathbf{0}} = \sum_j |a_j^+\rangle\langle a_j^+| - \sum_i |a_i^-\rangle\langle a_i^-|$ with spectral decomposition, and $I_d = \sum_j |a_j^+\rangle\langle a_j^+| + \sum_i |a_i^-\rangle\langle a_i^-|$. Note that $\operatorname{supp}(\rho_n) = \mathcal{S}_m^\perp$, combined with Lemma 2, we can conclude that $\mathcal{S}_m^\perp \subseteq \mathcal{H}_d$ is a strongly PWF unextendible subspace of dimension $(d+1)/2$. ■

Theorem 4 *Given $\rho_0, \rho_1 \in \mathcal{D}(\mathcal{H}_d)$, if any of them has strictly positive discrete Wigner functions, i.e., $W_{\rho_i}(\mathbf{u}) > 0, \forall \mathbf{u}$, then for any integer $n \in \mathbb{Z}^+$, $\rho_0^{\otimes n}$ and $\rho_1^{\otimes n}$ cannot be unambiguously distinguished by PWF POVMs.*

Proof Suppose the state ρ_0 and ρ_1 can be unambiguously distinguished by a PWF POVM $\{E_0, E_1\}$. By definition, we have

$$\operatorname{Tr}(E_0\rho_1) = 0 \text{ and } \operatorname{Tr}(E_1\rho_0) = 0. \tag{S16}$$

Then we are going to establish the theorem using a proof by contradiction. Without loss of generality, we suppose ρ_1 has strictly positive Wigner functions. Notice that

$$\operatorname{Tr}(E_0\rho_1) = \sum_{\mathbf{u}} W(E_0|\mathbf{u}) W_{\rho_1}(\mathbf{u}) = 0. \tag{S17}$$

By the strictly positivity of the Wigner functions of ρ_1, i.e., $W_{\rho_1}(\mathbf{u}) > 0, \forall \mathbf{u}$, we have that Eq. (S17) holds if and only if $W(E_0|\mathbf{u}) = 0, \forall \mathbf{u}$. Combining the fact that $\sum_{\mathbf{u}} W(E_0|\mathbf{u}) = d\,\mathrm{Tr}(E_0)$, we have $\mathrm{Tr}(E_0) = 0$. It follows that all eigenvalues of E_0 are equal to zero since $E_0 \geq 0$. Then we have $E_0 = \mathbf{0}$ which gives $\mathrm{Tr}(E_1\rho_0) = \mathrm{Tr}(\mathbb{1}\rho_0) = 1$, a contradiction.

Hence, there is no effect E_0 having PWFs such that $\mathrm{Tr}(E_0\rho_0) > 0$ and $\mathrm{Tr}(E_0\rho_1) = 0$ if ρ_1 has strictly positive Wigner functions. Similarly, we can show there is no effect E_1 having PWFs such that $\mathrm{Tr}(E_1\rho_0) = 0$ and $\mathrm{Tr}(E_1\rho_1) > 0$ if ρ_0 has strictly positive Wigner functions. Using the fact that

$$W_{\rho^{\otimes 2}}(\mathbf{u}_i \oplus \mathbf{u}_j) = W_\rho(\mathbf{u}_i)W_\rho(\mathbf{u}_j), \quad \forall \mathbf{u}_i, \mathbf{u}_j \in \mathbb{Z}_d \times \mathbb{Z}_d, \tag{S18}$$

we complete the proof. ■

III. MINIMUM ERROR DISCRIMINATION BY PWF POVMS

Note that given a two-valued PWF POVM $\{E, \mathbb{1} - E\}$, the discrete Wigner function of an effect E is $W(E|\mathbf{u}) = \mathrm{Tr}(EA_{\mathbf{u}})$. The SDP of discriminating an equiprobable pair of states $\{\rho_0, \rho_1\}$ via PWF POVMs can be written as

$$\begin{aligned} P_{\mathrm{e}}^{\text{PWF}}(\rho_0, \rho_1, \frac{1}{2}) = \min_{E}\ & \frac{1}{2} + \frac{1}{2}\mathrm{Tr}[E(\rho_1 - \rho_0)], \\ \text{s.t.}\ \ & 0 \leq E \leq \mathbb{1}, \\ & 0 \leq \mathrm{Tr}[EA_{\mathbf{u}}] \leq 1, \forall\, \mathbf{u}, \end{aligned} \tag{S19}$$

where $E \leq \mathbb{1}$ implies $\mathbb{1} - E$ is positive semidefinite. For different linear inequality constraints, we introduce corresponding dual variables V, U, $a_{\mathbf{u}}, b_{\mathbf{u}} \geq 0$. Then the Lagrange function of the primal problem can be written as

$$\begin{aligned} L(E, V, U, a_{\mathbf{u}}, b_{\mathbf{u}}) &= \frac{1}{2} + \frac{1}{2}\mathrm{Tr}[E(\rho_1 - \rho_0)] + \mathrm{Tr}[V(E - \mathbb{1})] - \mathrm{Tr}(UE) \\ &\quad - \sum_{\mathbf{u}} a_{\mathbf{u}}\mathrm{Tr}(EA_{\mathbf{u}}) + \sum_{\mathbf{u}} b_{\mathbf{u}}[\mathrm{Tr}(EA_{\mathbf{u}}) - 1] \\ &= \frac{1}{2} + \mathrm{Tr}\left[E\Big(V - U + \frac{1}{2}(\rho_1 - \rho_0) - \sum_{\mathbf{u}} a_{\mathbf{u}}A_{\mathbf{u}} + \sum_{\mathbf{u}} b_{\mathbf{u}}A_{\mathbf{u}}\Big)\right] - \mathrm{Tr}(V) - \sum_{\mathbf{u}} b_{\mathbf{u}} \end{aligned} \tag{S20}$$

The corresponding Lagrange dual function is

$$g(V, U, a_{\mathbf{u}}, b_{\mathbf{u}}) = \inf_{E} L(E, V, U, a_{\mathbf{u}}, b_{\mathbf{u}}). \tag{S21}$$

We can see that $V - U + \frac{1}{2}(\rho_1 - \rho_0) - \sum_{\mathbf{u}} a_{\mathbf{u}}A_{\mathbf{u}} + \sum_{\mathbf{u}} b_{\mathbf{u}}A_{\mathbf{u}} \geq 0$, otherwise $g(V, U, a_{\mathbf{u}}, b_{\mathbf{u}})$ is unbounded. Thus the dual SDP is

$$\begin{aligned} \max_{V, U, a_{\mathbf{u}}, b_{\mathbf{u}}}\ & \frac{1}{2} - \mathrm{Tr}(V) - \sum_{\mathbf{u}} b_{\mathbf{u}}, \\ \text{s.t.}\ \ & U \geq 0, V \geq 0, \\ & V - U + \frac{1}{2}(\rho_1 - \rho_0) \geq \sum_{\mathbf{u}} (a_{\mathbf{u}} - b_{\mathbf{u}})A_{\mathbf{u}}, \\ & a_{\mathbf{u}} \geq 0, b_{\mathbf{u}} \geq 0, \quad \forall\, \mathbf{u}. \end{aligned} \tag{S22}$$

Proposition 5 *Let ρ_0 be the Strange state $|\mathbb{S}\rangle\langle\mathbb{S}|$ and $\rho_1 = (\mathbb{1} - |\mathbb{S}\rangle\langle\mathbb{S}|)/2$ be its orthogonal complement. For $n \in \mathbb{Z}^+$, we have*

$$P_{\mathrm{e}}^{\text{PWF}}(\rho_0^{\otimes n}, \rho_1^{\otimes n}, \frac{1}{2}) = \frac{1}{2^{n+1}}. \tag{S23}$$

The optimal PWF POVM is $\{E, \mathbb{1} - E\}$, where $E = (|\mathbb{K}\rangle\langle\mathbb{K}| + |\mathbb{S}\rangle\langle\mathbb{S}|)^{\otimes n}$ and $|\mathbb{K}\rangle = (|1\rangle + |2\rangle)/\sqrt{2}$.

Proof First, we are going to prove $P_{\mathrm{e}}^{\text{PWF}}(\rho_0^{\otimes n}, \rho_1^{\otimes n}, \frac{1}{2}) \leq \frac{1}{2^{n+1}}$ using SDP (S19). We will show that $E = (|\mathbb{K}\rangle\langle\mathbb{K}| + |\mathbb{S}\rangle\langle\mathbb{S}|)^{\otimes n}$ is a feasible solution with a discrimination error $\frac{1}{2^{n+1}}$. In specific, it is easy to check $0 \leq E \leq \mathbb{1}$. Furthermore, we can check that $|0\rangle$, $|\mathbb{K}\rangle$ and $|\mathbb{S}\rangle$ are eigenvectors of $A_{\mathbf{0}}$ with eigenvalue $+1, +1$ and -1, respectively. It follows

$$\mathrm{Tr}[A_{\mathbf{u}}(|\mathbb{K}\rangle\langle\mathbb{K}| + |\mathbb{S}\rangle\langle\mathbb{S}|)] = \mathrm{Tr}[A_{\mathbf{u}}(\mathbb{1} - |0\rangle\langle 0|)] = 1 - \mathrm{Tr}(A_{\mathbf{u}}|0\rangle\langle 0|) \geq 0, \tag{S24}$$

where the inequality is due to the fact that $A_{\mathbf{u}}$ has eigenvalues no larger than 1. Also, we have $\mathrm{Tr}[A_{\mathbf{u}}(|\mathbb{K}\rangle\langle\mathbb{K}| + |\mathbb{S}\rangle\langle\mathbb{S}|)] = 1 - \mathrm{Tr}(A_{\mathbf{u}}|0\rangle\langle 0|) \leq 1$ as $|0\rangle\langle 0|$ is a stabilizer state with $\mathrm{Tr}(A_{\mathbf{u}}|0\rangle\langle 0|) \geq 0$. Thus, for the n-copy case, we have

$$0 \leq \prod_{i=1}^{n} \Big(\langle\mathbb{K}|A_{\mathbf{u}_i}|\mathbb{K}\rangle + \langle\mathbb{S}|A_{\mathbf{u}_i}|\mathbb{S}\rangle \Big) \leq 1, \tag{S25}$$

which makes E satisfies $0 \leq \mathrm{Tr}[EA_{\mathbf{u}}] \leq 1$. Hence, E is a feasible solution to the primal SDP (S19). Note that

$$\mathrm{Tr}[(|\mathbb{K}\rangle\langle\mathbb{K}| + |\mathbb{S}\rangle\langle\mathbb{S}|)\rho_0] = \langle\mathbb{K}|\mathbb{S}\rangle\langle\mathbb{S}|\mathbb{K}\rangle + \langle\mathbb{S}|\mathbb{S}\rangle\langle\mathbb{S}|\mathbb{S}\rangle = 1, \tag{S26a}$$

$$\mathrm{Tr}[(|\mathbb{K}\rangle\langle\mathbb{K}| + |\mathbb{S}\rangle\langle\mathbb{S}|)\rho_1] = \frac{1}{2}\langle\mathbb{K}|(\mathbb{1} - |\mathbb{S}\rangle\langle\mathbb{S}|)|\mathbb{K}\rangle + \frac{1}{2}\langle\mathbb{S}|(\mathbb{1} - |\mathbb{S}\rangle\langle\mathbb{S}|)|\mathbb{S}\rangle = \frac{1}{2}. \tag{S26b}$$

The corresponding discrimination error is

$$P^*_{pr} = \frac{1}{2} + \frac{1}{2}\,\mathrm{Tr}\left[(|\mathbb{K}\rangle\langle\mathbb{K}| + |\mathbb{S}\rangle\langle\mathbb{S}|)^{\otimes n}(\rho_1^{\otimes n} - \rho_0^{\otimes n})\right] \tag{S27a}$$

$$= \frac{1}{2} + \frac{1}{2}\,\mathrm{Tr}\left[(|\mathbb{K}\rangle\langle\mathbb{K}|\rho_1 + |\mathbb{S}\rangle\langle\mathbb{S}|\rho_1)^{\otimes n} - (|\mathbb{K}\rangle\langle\mathbb{K}|\rho_0 + |\mathbb{S}\rangle\langle\mathbb{S}|\rho_0)^{\otimes n}\right] \tag{S27b}$$

$$= \frac{1}{2} + \frac{1}{2}\left[\frac{1}{2^n} - 1\right] = \frac{1}{2^{n+1}}. \tag{S27c}$$

Second, we use the dual SDP (S22) to show $P_{\mathrm{e}}^{\mathrm{PWF}}(\rho_0^{\otimes n}, \rho_1^{\otimes n}, \frac{1}{2}) \geq \frac{1}{2^{n+1}}$. We will construct a valid $a_{\mathbf{u}}$ combined with $\{V = (2^n - 1)\rho_0/2^{n+1}, U = 0, b_{\mathbf{u}} = 0\}$ as a feasible solution to the dual problem. We note that $\rho_0 = (\mathbb{1} - A_{\mathbf{0}})/2$ and $\rho_1 = (\mathbb{1} + A_{\mathbf{0}})/4$ and introduce the following notation. Let $\mathbf{k} = (k_1, k_2, ..., k_n) \in \{0,1\}^n$ be a n-bit binary string and $|\mathbf{k}|$ be the Hamming weight of it. We then denote $A_{\mathbf{k}} = A_{k_1} \otimes A_{k_2} \otimes ... \otimes A_{k_n}$ where $A_{k_i} = A_{\mathbf{0}}$ if $k_i = 1$ and $A_{k_i} = \mathbb{1}$ if $k_i = 0$. Then we have

$$V - U + \frac{1}{2}(\rho_1^{\otimes n} - \rho_0^{\otimes n}) = \frac{2^n - 1}{2^{n+1}}\left(\frac{\mathbb{1} - A_{\mathbf{0}}}{2}\right)^{\otimes n} + \frac{1}{2}\left[\left(\frac{\mathbb{1} + A_{\mathbf{0}}}{4}\right)^{\otimes n} - \left(\frac{\mathbb{1} - A_{\mathbf{0}}}{2}\right)^{\otimes n}\right] \tag{S28a}$$

$$= \frac{1}{2^{2n+1}}(\mathbb{1} + A_{\mathbf{0}})^{\otimes n} - \frac{1}{2^{2n+1}}(\mathbb{1} - A_{\mathbf{0}})^{\otimes n} \tag{S28b}$$

$$= \frac{1}{2^{2n+1}} \sum_{\mathbf{k} \in \{0,1\}^n} \Big(1 - (-1)^{|\mathbf{k}|}\Big) A_{\mathbf{k}} \tag{S28c}$$

$$= \frac{1}{2^{2n+1}} \sum_{\mathbf{k} \in \{0,1\}^n} \Big(\frac{1 - (-1)^{|\mathbf{k}|}}{3^{n-|\mathbf{k}|}} \sum_{\mathbf{u}_{\mathbf{k}}} A_{\mathbf{u}_{\mathbf{k}}}\Big), \tag{S28d}$$

where $A_{\mathbf{u}_{\mathbf{k}}} = A_{\mathbf{u}_1} \otimes A_{\mathbf{u}_2} \otimes \cdots \otimes A_{\mathbf{u}_n}$ with $\mathbf{u}_j = \mathbf{0}$ if $k_j = 1$ for $j = 1, 2..., n$. To derive Eq. (S28d) from Eq. (S28c), we express each $A_{k_i} = \mathbb{1}$ with $k_i = 0$ in $A_{\mathbf{k}}$ as $\mathbb{1} = \frac{1}{3}\sum_{\mathbf{u}} A_{\mathbf{u}}$, where each $A_{\mathbf{k}}$ contains $(n - |\mathbf{k}|)$ occurrences of $\mathbb{1}$. Thus, we can find a set of $\hat{a}_{\mathbf{u}}$ such that

$$V - U + \frac{1}{2}(\rho_1^{\otimes n} - \rho_0^{\otimes n}) = \sum_{\mathbf{u}} \hat{a}_{\mathbf{u}} A_{\mathbf{u}}, \tag{S29}$$

by the following argument. For each $A_{\mathbf{u}'}$ in the n-copy system, we may find it as the sum of some terms in Eq. (S28d) with all coefficient positive since $\frac{1-(-1)^{|\mathbf{k}|}}{3^{n-|\mathbf{k}|}} \geq 0$. We can then let $\hat{a}_{\mathbf{u}}$ be the sum of those coefficients, which makes $\{V = (2^n - 1)\rho_0/2^{n+1}, \hat{a}_{\mathbf{u}}, U = 0, b_{\mathbf{u}} = 0\}$ a feasible solution of the dual SDP. Thus we have

$$P^*_{du} = \frac{1}{2} - \mathrm{Tr}(V) = \frac{1}{2^{n+1}}. \tag{S30}$$

Combining it with the primal part and utilizing Slater's condition for strong duality [8], we have that $P_{\mathrm{e}}^{\mathrm{PWF}}(\rho_0^{\otimes n}, \rho_1^{\otimes n}, \frac{1}{2}) = \frac{1}{2^{n+1}}$. ■

Notice that for a given measurement $\mathbf{M} := \{M_j\}_j$, we define the PWF robustness of measurement as

$$\mathbf{R}_{\mathcal{E}_{\mathrm{PWF}}}(\mathbf{M}) = \min\Big\{r \in \mathbb{R}_+ \Big| M_j + rN_j \in \mathcal{E}_{\mathrm{PWF}}\, \forall j, \{N_j\}_j \in \mathcal{M}\Big\}, \tag{S31}$$

where we denote by $\mathcal{M}$ the set of all possible POVMs, and denote by $\mathcal{E}_{\text{PWF}}$ the set of all PWF effects. An effect E belongs to $\mathcal{E}_{\text{PWF}}$ if it has PWFs. The *data-hiding ratio* [9] associated with PWF POVMs is defined in our manuscript as

$$R(\text{PWF}) = \max \frac{\|p\rho - (1-p)\sigma\|_{\text{All}}}{\|p\rho - (1-p)\sigma\|_{\text{PWF}}}, \tag{S32}$$

where the maximization ranges over all pairs of states ρ, σ and a priori probabilities p (here we also define $\|\cdot\|_{\text{PWF}}$ as the *distinguishability norm* associated with PWF POVMs). In an intuitive sense, we could imagine that a higher data-hiding ratio in Eq. (S32) will be obtained if the optimal POVMs for $\|\cdot\|_{\textbf{All}}$ exhibit 'less PWF'. This would suggest a more pronounced disparity allowing the agent to access the optimal discrimination strategy without a 'magic factory' in the given physical setting. Therefore, given an equiprobable pair of states $\{\rho, \sigma\}$, we define $\mathbf{R}^*_{\mathcal{E}_{\text{PWF}}}(\mathbf{M}_{\rho,\sigma})$ as the minimum PWF robustness of measurement that an optimal POVM must have to discriminate $\{\rho, \sigma\}$. It can be computed via the following SDP

$$\mathbf{R}^*_{\mathcal{E}_{\text{PWF}}}(\mathbf{M}_{\rho,\sigma}) = \min \ r \tag{S33a}$$

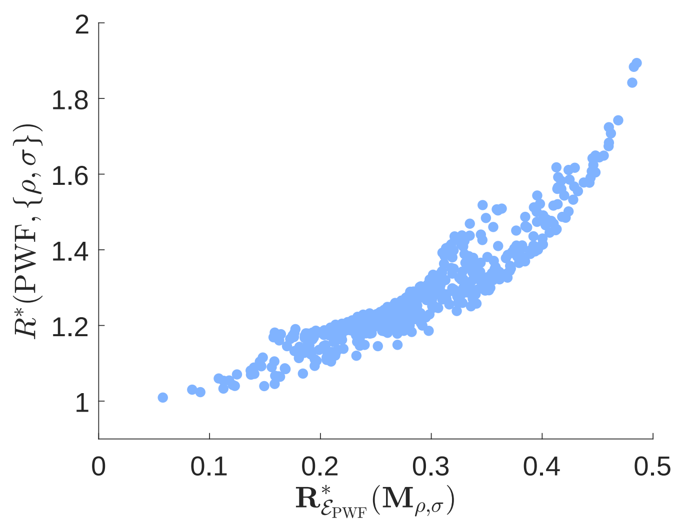
$$\text{s.t. } E_0, E_1, N_0, N_1 \geq 0, \tag{S33b}$$

$$E_0 + E_1 = \mathbb{1}, N_0 + N_1 = r \cdot \mathbb{1}, \tag{S33c}$$

$$\text{Tr}\left[(\rho - \sigma)E_0\right] = \frac{1}{2}\|\rho - \sigma\|_1, \tag{S33d}$$

$$W(E_0 + N_0|\mathbf{u}) \geq 0, W(E_1 + N_1|\mathbf{u}) \geq 0, \forall \mathbf{u}, \tag{S33e}$$

where the constraint in Eq. (S33d) ensures that optimal discrimination is achieved, and the constraints in Eq. (S33e) ensure that $E_j + rN_j \in$ PWF. We generate 500 equiprobable pair of states $\{\rho_j, \sigma_j\}_{j=1}^{500}$ where ρ_j is a random pure qutrit state according to the Haar measure and σ_j is its orthogonal complement. Then we compute the ratio $R^*(\text{PWF}, \{\rho, \sigma\}) = \|\frac{1}{2}\rho_j - \frac{1}{2}\sigma_j\|_{\text{All}} / \|\frac{1}{2}\rho - \frac{1}{2}\sigma\|_{\text{PWF}}$ and $\mathbf{R}^*_{\mathcal{E}_{\text{PWF}}}(\mathbf{M}_{\rho_j,\sigma_j})$. The numerical calculations are implemented in MATLAB [10] with the interpreters CVX [11] and QETLAB [12]. The results are depicted as follows.



We observe that there is a possible correlation between the PWF robustness of measurement and the data-hiding ratio associated with a state pair: as the optimal POVM for a state pair exhibits a higher PWF robustness, the corresponding data-hiding ratio also increases. However, their specific relationship remains unclear so far. This experiment also indicates that the data-hiding ratio associated with the Strange state and its orthogonal complement is already relatively high, which equals 2 when $n = 1$ as stated in Eq. (4) in our manuscript. A deeper relationship between the data-hiding ratio and the PWF robustness of measurement in the case of PWF POVMs merits further investigation.

IV. DISTINCTIONS BETWEEN THE QRT OF MAGIC STATES AND ENTANGLEMENT

Theorem 6 *For a subspace $\mathcal{S} \in \mathcal{H}_d$, if $\mathcal{S}$ has a set of basis $\{|\psi_i\rangle\}_{i=1}^n$ where every $|\psi_i\rangle$ is a stabilizer state, then $\mathcal{S}$ is PWF extendible.*

Proof Since $\{|\psi_i\rangle\}_{i=1}^n$ is a basis for $\mathcal{S}$, we have $|\psi_i\rangle$ and $|\psi_j\rangle$ are orthogonal which yields

$$\langle\psi_i|\psi_j\rangle = \sum_{\mathbf{u}} W_{\psi_i}(\mathbf{u}) W_{\psi_j}(\mathbf{u}) = 0. \tag{S34}$$

for any $i \neq j$. Note that every pure stabilizer state has Wigner functions 0 or $1/d$ [2]. Then we know that for a fixed point $\mathbf{u}'$, there is at most one state $|\psi_{j'}\rangle$ that has $W_{\psi_{j'}}(\mathbf{u}') = 1/d$. For any other states $|\psi_i\rangle, i \neq j'$, we have $W_{\psi_i}(\mathbf{u}') = 0$ otherwise $\langle\psi_{j'}|\psi_i\rangle \geq 1/d^2 > 0$, a contradiction to Eq. (S34). Thus, we have $\sum_{i=1}^n W_{\psi_i}(\mathbf{u}) = 0$ or $\sum_{i=1}^n W_{\psi_i}(\mathbf{u}) = 1/d$. Then we denote $P_{\mathcal{S}} = \sum_{i=1}^n |\psi_i\rangle\langle\psi_i|$ as the projection of $\mathcal{S}$ and consider its orthogonal complement $P_{\mathcal{S}}^\perp = \mathbb{1} - P_{\mathcal{S}}$. Considering $P_{\mathcal{S}}^\perp$ as an effect of the POVM $\{P_{\mathcal{S}}^\perp, P_{\mathcal{S}}\}$, we have

$$W(P_{\mathcal{S}}^\perp|\mathbf{u}) = 1 - d\sum_{i=1}^n W_{\psi_i}(\mathbf{u}) = 1 \text{ or } 0, \tag{S35}$$

which shows that $P_{\mathcal{S}}^\perp$ has PWFs. After normalization, we can obtain a PWF state supported on $\mathcal{S}^\perp$, which indicates that $\mathcal{S}$ is PWF extendible. ■

Proposition 7 *Let ρ_0 be the Strange state $|\mathbb{S}\rangle\langle\mathbb{S}|$ and $\rho_1 = (\mathbb{1} - |\mathbb{S}\rangle\langle\mathbb{S}|)/2$ be its orthogonal complement. $\rho_0 \otimes \tau^{\otimes k}$ and $\rho_1 \otimes \tau^{\otimes k}$ cannot be perfectly distinguished for any qutrit magic state τ and $k = 1$ or 2.*

Proof First, suppose there is a PWF POVM $\{E, \mathbb{1} - E\}$ that can perfectly distinguish $\rho_0 \otimes \tau^{\otimes k}$ and $\rho_1 \otimes \tau^{\otimes k}$. Then we have

$$\mathrm{Tr}[(\rho_0 \otimes \tau^{\otimes k})E] = 1, \mathrm{Tr}[(\rho_1 \otimes \tau^{\otimes k})E] = 0. \tag{S36}$$

We can write $\mathrm{Tr}[(\rho_0 \otimes \tau^{\otimes k})E] = \mathrm{Tr}[\rho_0 \, \mathrm{Tr}_2[(\mathbb{1} \otimes \tau^{\otimes k})E]] = 1$. Notice the fact that when ρ_0 is a pure state, $\mathrm{Tr}(\rho_0 X) = 1, \mathrm{Tr}(\rho_1 X) = 0$ if and only if $X = \rho_0$. Then for any $\mathbf{u}_1$, we have

$$\mathrm{Tr}(\rho_0 A_{\mathbf{u_1}}) = \frac{1}{d^k} \sum_{\mathbf{u_2}, \cdots, \mathbf{u_{k+1}}} \mathrm{Tr}(E A_{\mathbf{u_1}, \cdots, \mathbf{u_{k+1}}}) \mathrm{Tr}(\tau A_{\mathbf{u_2}}) \cdots \mathrm{Tr}(\tau A_{\mathbf{u_{k+1}}}). \tag{S37}$$

Suppose the value of $\mathrm{maxneg}(\rho_0)$ is obtained at phase point $\mathbf{u_1} = (a, b)$ for ρ_0, where $\mathrm{maxneg}(\rho) := -\min_{\mathbf{u}} W_\rho(\mathbf{u})$ denotes the maximal negativity of ρ. We consider the right hand of Eq. (S37) by choosing $\mathbf{u_1} = (a, b)$:

$$-d \cdot \mathrm{maxneg}(\rho_0) = \frac{1}{d^k} \sum_{\mathbf{u_2}, \cdots, \mathbf{u_{k+1}}} \mathrm{Tr}(E A_{(a,b), \cdots, \mathbf{u_{k+1}}}) \mathrm{Tr}(\tau A_{\mathbf{u_2}}) \cdots \mathrm{Tr}(\tau A_{\mathbf{u_{k+1}}}) \tag{S38}$$

$$\geq \max(\mathrm{Tr}(E A_{(a,b), \cdots, \mathbf{u_{k+1}}})) \frac{1}{d^k} \sum_{\mathbf{u_2}, \cdots, \mathbf{u_{k+1}}}^{<0} \mathrm{Tr}(\tau A_{\mathbf{u_2}}) \cdots \mathrm{Tr}(\tau A_{\mathbf{u_{k+1}}}) \tag{S39}$$

$$+ \min(\mathrm{Tr}(E A_{(a,b), \cdots, \mathbf{u_{k+1}}})) \frac{1}{d^k} \sum_{\mathbf{u_2}, \cdots, \mathbf{u_{k+1}}}^{\geq 0} \mathrm{Tr}(\tau A_{\mathbf{u_2}}) \cdots \mathrm{Tr}(\tau A_{\mathbf{u_{k+1}}}) \tag{S40}$$

$$\geq \frac{1}{d^k} \sum_{\mathbf{u_2}, \cdots, \mathbf{u_{k+1}}}^{<0} \mathrm{Tr}(\tau A_{\mathbf{u_2}}) \cdots \mathrm{Tr}(\tau A_{\mathbf{u_{k+1}}}) \tag{S41}$$

$$= -\,\mathrm{sn}(\tau^{\otimes k}), \tag{S42}$$

where the inequality in Eq. (S41) is due to the fact that $0 \leq W(E|\mathbf{u}) \leq 1$ for the PWF POVM $\{E, \mathbb{1} - E\}$ and $\mathrm{sn}(\rho) := \sum_{\mathbf{u}: W_\rho(\mathbf{u}) < 0} |W_\rho(\mathbf{u})|$ denotes the sum negativity of a magic state ρ. Thus, we have

$$d \cdot \mathrm{maxneg}(\rho_0) \leq \mathrm{sn}(\tau^{\otimes k}). \tag{S43}$$

Note that the Strange state $\rho_0 = |\mathbb{S}\rangle\langle\mathbb{S}|$ satisfies $d \cdot \mathrm{maxneg}(\rho_0) = 1$, which implies $\mathrm{sn}(\tau^{\otimes k}) \geq 1$. Since it has been shown that the maximal sum negativity of a qutrit state is $1/3$ [13], we conclude that

$$\mathrm{sn}(\tau^{\otimes k}) = [(2\mathrm{sn}(\tau) + 1)^k - 1]/2 \leq [(5/3)^k - 1]/2 < 1, \tag{S44}$$

for any qutrit magic state τ and $k = 1$ or 2, where we use the composition law of $\mathrm{sn}(\cdot)$ derived by Ref. [13]. Eq. (S44) is in contradiction with the inequality $\mathrm{sn}(\tau^{\otimes k}) \geq 1$. Thus, we complete the proof. ■

Similarly, we can conclude that for the case of Norell state $\rho_0 = |\mathbb{N}\rangle\langle\mathbb{N}|$, where $|\mathbb{N}\rangle = (-|0\rangle + 2|1\rangle - |2\rangle)/\sqrt{6}$ [13], $\rho_0 \otimes \tau$ and $\rho_1 \otimes \tau$ cannot be perfectly distinguished by PWF POVMs for any qutrit state τ.

[1] X. Wang, M. M. Wilde, and Y. Su, Quantifying the magic of quantum channels, New Journal of Physics **21**, 103002 (2019).
[2] D. Gross, Hudsons theorem for finite-dimensional quantum systems, Journal of mathematical physics **47** (2006).
[3] H. Pashayan, J. J. Wallman, and S. D. Bartlett, Estimating outcome probabilities of quantum circuits using quasiprobabilities, Physical Review Letters **115** (2015).
[4] A. Mari and J. Eisert, Positive wigner functions render classical simulation of quantum computation efficient, Physical Review Letters **109**, 1 (2012).
[5] V. Veitch, C. Ferrie, D. Gross, and J. Emerson, Negative quasi-probability as a resource for quantum computation, New Journal of Physics **14**, 1 (2012).
[6] M. Howard, J. Wallman, V. Veitch, and J. Emerson, Contextuality supplies the 'magic' for quantum computation, Nature **510**, 351 (2014).
[7] P.-E. Emeriau, M. Howard, and S. Mansfield, Quantum advantage in information retrieval, PRX Quantum **3**, 020307 (2022).
[8] S. P. Boyd and L. Vandenberghe, *Convex optimization* (Cambridge university press, 2004).
[9] L. Lami, C. Palazuelos, and A. Winter, Ultimate data hiding in quantum mechanics and beyond, Communications in Mathematical Physics **361**, 661 (2018).
[10] T. M. Inc., Matlab version: 9.13.0 (r2022b) (2022).
[11] M. Grant and S. Boyd, CVX: Matlab software for disciplined convex programming, version 2.1, `http://cvxr.com/cvx` (2014).
[12] N. Johnston, QETLAB: A MATLAB toolbox for quantum entanglement, version 0.9, `https://qetlab.com` (2016).
[13] V. Veitch, S. A. Hamed Mousavian, D. Gottesman, and J. Emerson, The resource theory of stabilizer quantum computation, New Journal of Physics **16**, 1 (2014).