SPECTRAL RESOLUTION IN A RICKART COMGROUP

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A comgroup is a compressible group with the general comparability property. A comgroup with the Rickart projection property is called a Rickart comgroup. We show that each element of a Rickart comgroup has a rational spectral resolution and a nonempty closed and bounded (real) spectrum. The rational spectral resolution and the spectrum are shown to have many of the properties of the spectral resolution and spectrum of a self-adjoint operator on a Hilbert space. Examples of Rickart comgroups include the additive group of self-adjoint elements in a von Neumann algebra and the Mundici group of a Heyting MV-algebra.

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1. Introduction

Our objective in this article is to construct and initiate a study of a "rational spectral resolution" $(p_\lambda)_{\lambda\in\mathbb{Q}}$ corresponding to an element g of a Rickart comgroup G. (For motivation, see Example 2.2 below.) Our work is based on material in [4] and [5] and, for the reader's convenience, we begin with a brief review of pertinent definitions and results from these references.

A *unital group* is a partially ordered abelian group G with positive cone $G^+ = \{g \in G \mid 0 \le g\}$ and with a distinguished element $u \in G^+$ such that (1) u is an *order unit*, i.e., for every $g \in G$, there is a positive integer N_g such that $g \le N_g u$, and (2) the interval $E := \{e \in G \mid 0 \le e \le u\}$ generates G^+ in the sense that, for every $g \in G^+$, there is a finite sequence $e_1, e_2, \ldots, e_n$ of (not necessarily distinct) elements of E such that $g = \sum_{i=1}^n e_i$. The distinguished element u in the positive cone G^+ of a unital group G is called the *unit* for (or of) G, the interval $E := \{e \in G \mid 0 \le e \le u\}$ is called the *unit interval* in G, and elements $e \in E$ are called *effects* in G. If G is a unital group, then G is *directed* [6, p. 4], i.e., $G = G^+ - G^+$.

If G and H are unital groups with units u and v, respectively, then an order-preserving group homomorphism $\phi : G \to H$ such that $\phi(u) = v$ is called a *unital* *morphism*. Regarded as a (totally) ordered abelian group under addition, the system $\mathbb{R}$ of real numbers forms a unital group with unit 1, and a unital morphism $\omega : G \to \mathbb{R}$ is called a *state* on the unital group G. If $G \neq \{0\}$ is a unital group, then the *state space* $\Omega(G)$ of G, i.e., the set of all states on G, is a non-empty [6, Corollary 4.4] σ-convex set [6, Proposition 6.5], and with the topology of pointwise convergence, it forms a compact Hausdorff space [6, Proposition 6.2]. If $G^+ = \{g \in G \mid 0 \leq \omega(g)$ for all $\omega \in \Omega(G)\}$, then the state space $\Omega(G)$ is said to be *order determining* (or *cone determining*). By [6, Theorem 4.14], $\Omega(G)$ is order determining iff G is *archimedean* in the sense that, if $g, h \in G$, then $ng \leq h$ for all positive integers n only if $g \leq 0$.

Let G be a unital group with unit u and unit interval E. A mapping $J : G \to G$ is called a *retraction* iff J is an order-preserving group endomorphism on G such that $J(u) \leq u$, and for every effect $e \in E$, $e \leq J(u) \Longrightarrow J(e) = e$. A retraction J on G is necessarily idempotent, i.e., $J = J \circ J$. A retraction J on G is called a *compression* iff, for every effect $e \in E$, $J(e) = 0 \Longrightarrow e + J(u) \leq u$.

Two retractions J and J' on the unital group G are said to be *quasicomplements* iff, for all $g \in G^+$, $J(g) = 0 \Longleftrightarrow J'(g) = g$ and $J(g) = g \Longleftrightarrow J'(g) = 0$. If a retraction J has a quasicomplement J', then both J and J' are compressions. By definition, a *compressible group* is a unital group G with unit u such that (1) if I and J are retractions on G, then $I(u) = J(u) \Longrightarrow I = J$, and (2) every retraction J on G has a quasicomplementary retraction J' on G. If G is a compressible group with unit u, then an element $p \in G$ is called a *projection* iff there is a retraction (hence a compression) J on G with $p = J(u)$.

STANDING ASSUMPTION. *In the sequel, we assume that $G \neq 0$ is a compressible group with unit u, $E = \{e \in G \mid 0 \leq e \leq u\}$ is the set of all effects in G, P is the set of all projections in G, and Ω is the state space of G. For each $p \in P$, we denote by J_p the unique compression on G such that $p = J_p(u)$.*

If $p \in P$, then $u - p \in P$ and J_{u-p} is the unique compression on G that is quasicomplementary to J_p. Partially ordered by the restriction to P of the partial order on G, and with $p \mapsto u - p$ as orthocomplementation, P forms an orthomodular poset (OMP) [13] with smallest element 0 and largest element u. Furthermore, P is *regular* [9] in the sense that a finite set $M \subseteq P$ of pairwise compatible elements in P is jointly compatible, i.e., M is contained in a Boolean subalgebra of P [5, Theorem 2.5]. If $p, q \in P$ and p and q have an infimum (respectively, a supremum) in P, then this infimum (respectively, supremum) is written as $p \wedge q$ (respectively, as $p \vee q$). More generally, the infimum (respectively, the supremum) of a family $(p_\iota)_{\iota \in I} \subseteq P$, if it exists, is written as $\bigwedge_{\iota \in I} p_\iota$ (respectively, as $\bigvee_{\iota \in I} p_\iota$). Existing infima and suprema as calculated in other subsets of G, including G itself, will be written with appropriate subscripts. For instance, if $g, h \in G$, the infimum of g and h in G, if it exists, will be written as $g \wedge_G h$.

If $p \in P$, we define $C(p) := \{g \in G \mid g = J_p(g) + J_{u-p}(g)\}$ and we say that g is *compatible with* p iff $g \in C(p)$. Evidently, $C(p)$ is a subgroup of G. For projections $p, q \in P$, we often write the condition $p \in C(q)$ in the form pCq.

By [4, Theorem 5.4 and Corollary 5.6], if $p, q \in P$, then pCq iff p and q are (Mackey) compatible in the OMP P, whence $pCq \Longleftrightarrow qCp \Longrightarrow p \wedge q$ and $p \vee q$ exist in P. Furthermore, $pCq \Longleftrightarrow J_p \circ J_q = J_q \circ J_p \Longrightarrow J_p \circ J_q = J_{p \wedge q}$.

If $g \in G$, we define the subgroup $CPC(g)$ of G by $CPC(g) := \bigcap\{C(p) \mid p \in P \text{ and } g \in C(p)\}$. Thus, $h \in CPC(g)$ iff h is compatible with every projection p with which g is compatible. Also, by definition, $C(P) := \bigcap\{C(p) \mid p \in P\}$, i.e., the elements of $C(P)$ are those that are compatible with every projection in P. If $G = C(P)$, we say that G is a *compatible* group.

By definition, if $g \in G$, then

$$P^{\pm}(g) := \{p \in P \cap CPC(g) \mid g \in C(p) \text{ and } J_{u-p}(g) \leq 0 \leq J_p(g)\}.$$

Each projection $p \in P^{\pm}(g)$ splits $g = J_p(g) + J_{u-p}(g)$ into a "positive part" $J_p(g) \geq 0$ and a "negative part" $J_{u-p}(g) \leq 0$. If $P^{\pm}(g)$ is nonempty for all $g \in G$, we say that the compressible group G has the *general comparability* property.

We say that G has the *Rickart projection property* iff there is a mapping $' : G \to P$, called the *Rickart mapping*, such that, for all $g \in G$, and all $p \in P$, $p \leq g' \Longleftrightarrow g \in C(p)$ and $J_p(g) = 0$ (cf. Example 2.3 below). If G has the Rickart projection property, then P is an *orthomodular lattice* (OML) [10]; in fact, for $p, q \in P$, $p' = u - p$ and the infimum of p and q in P is $p \wedge q = J_p((J_p(q'))')$ [5, Theorem 6.4]. If G has the Rickart projection property, then the mapping $g \mapsto g'' := (g')'$ is called the *iterated Rickart mapping* on G. Evidently, $g = 0$ iff $g'' = 0$ iff $g' = u$.

DEFINITION 1.1. For short, we refer to a compressible group with the general comparability property as a *comgroup*. A *Rickart comgroup* is a comgroup with the Rickart projection property.

Suppose that G is a comgroup. Then G is necessarily *unperforated*, i.e., if n is a positive integer, $g \in G$, and $ng \in G^+$, then $g \in G^+$, and, as an abelian group, G is torsion free [4, Lemma 4.8]. The comgroup G is lattice ordered iff it is a compatible group [5, Theorem 5.5].

Let $g \in G$, and choose $p \in P^{\pm}(g)$. Then by [5, Theorem 3.2], $g^+ := J_p(g)$ and $g^- := -J_{u-p}(g)$ are independent of the choice of $p \in P^{\pm}(g)$, and we have $g = g^+ - g^-$ with $0 \leq g^+, g^-$. In fact, by [5, Lemma 4.2], if $q \in P$, $g \in C(q)$, and $J_{u-q}(g) \leq 0 \leq J_q(g)$, then $g^+ = J_q(g)$ and $g^- = -J_{u-q}(g)$. We define $|g| := g^+ + g^-$. The comgroup G is lattice ordered iff the triangle inequality $|g + h| \leq |g| + |h|$ holds for all $g, h \in G$, in which case g^+ is the supremum $g \vee_G 0$ [5, Theorem 5.5]. If G is an archimedean comgroup and P satisfies the chain conditions, then G is a Rickart comgroup [5, Theorem 6.6].

THEOREM 1.1. *If G is a comgroup, then the following conditions are mutually equivalent:*

(i) *G is archimedean.*
(ii) *If $a, b \in G^+$ and $na \leq b$ for all positive integers n, then $a = 0$.*
(iii) *Ω is order determining.*

(iv) Ω *separates points in* G, *i.e., if* $g, h \in G$ *and* $\omega(g) = \omega(h)$ *for all* $\omega \in \Omega$, *then* $g = h$.
(v) *If* $g \in G$ *and* $\omega(g) = 0$ *for all* $\omega \in \Omega$, *then* $g = 0$.

Proof: That (i) $\Longleftrightarrow$ (ii) follows from [5, Lemma 3.5]. That (i) $\Longleftrightarrow$ (iii) follows from [6, Theorem 4.14]. Clearly, (iii) $\Longrightarrow$ (iv) and (iv) $\Longleftrightarrow$ (v). Assume (v) and the hypotheses of (ii). Then, if $\omega \in \Omega$, we have $n\omega(a) \le \omega(b)$ for every positive integer n and, since $0 \le \omega(a) \in \mathbb{R}$, it follows that $\omega(a) = 0$, whence $a = 0$. Therefore (v) $\Longrightarrow$ (ii). □

2. Examples

To fix ideas and provide motivation for our subsequent development, we now offer several examples of comgroups. Many of these examples arise from a certain class of rings as per the following definition.

DEFINITION 2.1. An *effect-ordered ring* is a ring A with unit 1 such that (1) under addition, A forms a partially ordered abelian group with positive cone A^+, (2) $1 \in A^+$, (3) the additive subgroup $G(A) := A^+ - A^+$ of A is a unital group with positive cone $G(A)^+ = A^+$ and unit 1, and (4) for all $a, b \in A^+$,

(i) $ab = ba \Rightarrow ab \in A^+$, (ii) $aba \in A^+$,
(iii) $aba = 0 \Rightarrow ab = ba = 0$, and (iv) $(a - b)^2 \in A^+$.

Let A be an effect-ordered ring. If the unital group $G(A)$ is archimedean, we say that A is an *archimedean* effect-ordered ring. If $G(A)$ is a compressible group, we call A a *compressible ring*.

Suppose that A is an effect-ordered ring and define $P(A) := \{p \in G(A) \mid p = p^2\}$. Then if $p \in P(A)$, the mapping $g \mapsto pgp$ for $g \in G(A)$ is a compression on $G(A)$ with $1 \mapsto p1p = p$ [3, Lemma 4.4]. An archimedean effect-ordered ring is a compressible ring [3, Corollary 4.6]. Let A be a compressible ring. Then $P(A)$ is the OMP of projections in $G(A)$, and for $p \in P(A)$ and $g \in G(A)$, we have $J_p(g) = pgp$. Furthermore, if $g \in G(A)$ and $p \in P(A)$, then $g \in C(p) \Longleftrightarrow gp = pg$.

EXAMPLE 2.1. If A is a unital C*-algebra, then with $A^+ := \{aa^* \mid a \in A\}$, A is an archimedean effect-ordered ring, hence it is a compressible ring. The archimedean compressible group $G(A)$ is the additive group of self-adjoint elements in A, $P(A)$ is the OMP of self-adjoint idempotents (i.e., projections) in A, and for $p \in P(A)$ and $g \in G(A)$, $J_p(g) = pgp$. If A is a Rickart C*-algebra, i.e., the right annihilating ideal of each element in A is a principal right ideal generated by a projection [7], then $G(A)$ has the Rickart projection property.

EXAMPLE 2.2. In Example 2.1, suppose that A is a von Neumann algebra. Then $P(A)$ is a complete OML. If $g, h \in G(A)$, then $g \in CPC(h)$ iff g "double commutes" with h, i.e., g commutes with every element in A that commutes with h. Furthermore, A is commutative iff $G(A)$ is a compatible group, and A is a von

Neumann factor iff $C(P(A)) = \{\lambda 1 \mid \lambda \in \mathbb{R}\}$. The von Neumann algebra A is a Rickart C*-algebra, so $G(A)$ has the Rickart projection property.

Owing to the spectral theorem, there corresponds to each element $g \in G(A)$ a (right continuous) *spectral resolution*, i.e., a one-parameter family of projections $(p_\lambda)_{\lambda\in\mathbb{R}}$ that is uniquely determined by the following conditions. For all $\lambda, \mu \in \mathbb{R}$: (i) $\lambda \leq \mu \Longrightarrow p_\lambda \leq p_\mu$. (ii) $\bigwedge_{\lambda<\mu} p_\mu = p_\lambda$. (iii) $\exists \alpha, \beta \in \mathbb{R}$, $p_\alpha = 0$ and $p_\beta = 1$. (iv) $gp_\lambda = p_\lambda g$. (v) $p_\lambda g \leq \lambda p_\lambda$. (vi) $\lambda(1-p_\lambda) \leq (1-p_\lambda)g$. If $L := \sup\{\lambda \mid p_\lambda = 0\}$ and $U := \inf\{\mu \mid p_\mu = 1\}$, then (i)–(vi) imply that $g = \int_{L-0}^{U} \lambda dp_\lambda$, where the integral is the limit in the sense of convergence in the norm $\|\cdot\|$ of sums of Stieltjes type. Furthermore, the *spectral bounds* L and U satisfy $L = \sup\{\lambda \mid \lambda 1 \leq g\}$, $U = \inf\{\mu \mid g \leq \mu 1\}$, and $\|g\| = \max\{|L|, |U|\}$.

If (p_λ) is the spectral resolution corresponding to $g \in G(A)$, then $1 - p_0 \in P^{\pm}(g)$. In fact, $1 - p_0$ is the smallest projection in $P^{\pm}(g)$. Thus, $P^{\pm}(g)$ is nonempty, and it follows that $G(A)$ is an archimedean Rickart comgroup. If $' : G(A) \to P(A)$ is the Rickart mapping on $G(A)$, then $g' = p_0 - \bigvee_{\lambda<0} p_\lambda$.

EXAMPLE 2.3. Let X be a compact Hausdorff space and let $A := C(X, \mathbb{R})$ be the commutative ring under pointwise operations of all continuous functions $f : X \to \mathbb{R}$. Then, with the constant function $1(x) = 1$ for all $x \in X$ as the unit, and with $A^+ := \{f \in A \mid 0 \leq f(x)$ for all $x \in X\}$, A is an archimedean compressible ring, $G(A) = A$ is the lattice-ordered additive group of A, $G(A)$ is a compatible group, and $P(A)$ is the Boolean algebra of characteristic set functions of all compact open subsets of X. The compressible group $G(A)$ is a Rickart comgroup iff X is basically disconnected, i.e., the closure of every open F_σ subset of X is open, in which case the Boolean algebra $P(A)$ is σ-complete.

EXAMPLE 2.4. Let $\mathbb{Z}$ be the additive group of integers, ordered as usual, let X be a nonempty set, and let $\mathcal{B}$ be a field of subsets of X. Define $A := \mathcal{F}(X, \mathcal{B}, \mathbb{Z})$ to be the set of all bounded functions $f : X \to \mathbb{Z}$ such that $f^{-1}(z) \in \mathcal{B}$ for every $z \in \mathbb{Z}$, and organize A into a ring with pointwise operations. The constant function $1(x) = 1$ for all $x \in X$ is a unit for the ring A, and with the positive cone $A^+ := \{f \in A \mid 0 \leq f(x)$ for all $x \in X\}$, A is an archimedean compressible ring. Furthermore, $G(A) = A$ is a lattice ordered Rickart comgroup and $P(A)$ is the Boolean algebra of all characteristic set functions χ_B of sets $B \in \mathcal{B}$.

By the Stone representation theorem, every Boolean algebra can be represented as the field $\mathcal{B}$ of compact open subsets of a totally-disconnected compact Hausdorff space X, hence, as per Example 2.4, every Boolean algebra can be represented as $P(A)$ for an archimedean compressible ring A.

A multitude of variations of the examples above suggest themselves. For instance, if F is a subfield of the field $\mathbb{C}$ of complex numbers, A is the ring of all n-by-n matrices over F, and A^+ is the cone in A consisting of all Hermitian positive semidefinite matrices, then A is an archimedean compressible ring. One can also construct numerous examples based on the nonclassical Hilbert spaces of H. Gross [8] and H. Keller [11].

Our next examples provide important classes of compressible groups that do not necessarily arise from compressible rings.

EXAMPLE 2.5. Let G be an *interpolation group with order unit* u [6]. Then G is a compatible compressible group [4, Theorem 3.5], the projections in G are the so called *characteristic* elements of G [6, p. 127], and the OML P of projections in G is a Boolean algebra. If $p \in P$ and $e \in E$, then $J_p(e) = p \wedge_E e$, the infimum of p and e as calculated in E. The interpolation group G may or may not be archimedean and, although G need not be a comgroup, the case in which it is has been studied in some detail as it is of considerable importance in the theory of interpolation groups [6, Chapter 8]. If G is a comgroup, then G is not only an interpolation group, but it is lattice ordered, and (as it is unperforated) it forms a so called *dimension group* [6, Chapter 3].

EXAMPLE 2.6. Let G be a lattice-ordered abelian group with order unit. Then G is automatically an interpolation group, whence, as in Example 2.5, it is a compatible compressible group, but again it need not be a comgroup. The unit interval E in G is a sublattice of G, and it forms a so called *MV-algebra* [1]. Conversely, by a theorem of D. Mundici [12], every MV-algebra can be realized as the unit interval E in a lattice-ordered abelian group G with order unit. The Mundici group G is uniquely determined up to a unital isomorphism by the MV-algebra E, and G need not be a comgroup. However, if G is Dedekind σ-complete, then by [6, Theorem 9.9], it is a comgroup. Furthermore, if G is a Rickart comgroup, then E is both a Heyting algebra and an MV-algebra [2].

3. Preliminary results

In this section we assemble some results that will be needed in Sections 4 and 5 below. We maintain our standing convention that $G \neq \{0\}$ *is a compressible group with unit* u*, unit interval* E*, OMP of projections* P*, and state space* Ω.

THEOREM 3.1. *Suppose* G *is a Rickart comgroup and* $g \in G$*. Then* $(g^+)'' \leq g'' \wedge (g^-)'$ *and* $(g^+)''$ *is the smallest element in* $P^{\pm}(g)$*. Conversely, if* $q \in P^{\pm}(g)$ *and* $q \leq g''$*, then* $q = (g^+)''$.

Proof: Let $p := (g^+)''$. Then $J_p(g^+) = g^+$ and $J_{p'}(g^+) = 0$. By [5, Theorem 6.5 (ii)], $p + (g^-)'' = p \vee (g^-)'' = g''$, and it follows that $p = g'' - (g^-)'' = g'' \wedge (g^-)'$. Since $p \leq (g^-)'$, we have $J_p(g^-) = 0$ and $J_{p'}(g^-) = g^-$. Therefore, $J_p(g) = J_p(g^+) - J_p(g^-) = g^+$ and $J_{p'}(g) = J_{p'}(g^+) - J_{p'}(g^-) = -g^-$, whence $g = g^+ - g^- = J_p(g) + J_{p'}(g)$, i.e., $g \in C(p)$. By [5, Theorem 6.5 (i)], $p = (g^+)'' \in CPC(g^+)$ and by [5, Lemma 4.3 (vii)], $g^+ \in CPC(g)$, and it follows that $p \in CPC(g)$. Thus, in view of the fact that $J_{p'}(g) = -g^- \leq 0 \leq g^+ = J_p(g)$, we have $p \in P^{\pm}(g)$. If $q \in P^{\pm}(g)$, then $J_q(g) = g^+$, so $J_q(g^+) = g^+$, and it follows that $p = (g^+)'' \leq q$. Therefore, p is the smallest element in $P^{\pm}(g)$.

Now suppose that $q \in P^{\pm}(g)$ and $q \leq g''$. We have shown above that $p \leq q$. Since $g^- = -J_{q'}(g)$, we have $J_{q'}(g^-) = g^-$, so $(g^-)'' \leq q'$, i.e., $q \leq (g^-)'$, whence $q \leq g'' \wedge (g^-)' = p$, and therefore $q = p$. $\square$

LEMMA 3.1. *Let G be a Rickart comgroup, let $g, h \in G$, let $p \in P$, and let $m, n \in \mathbb{Z}$. Then:*

(i) *If $n \neq 0$, then $g \in C(p) \Leftrightarrow ng - mu \in C(p)$.*
(ii) *If $0 < n$, then $(ng)^+ = ng^+$ and $(ng)' = g' = (-g)'$.*
(iii) *If $h \in CPC(g)$ and $g \leq h$, then $g^+ \leq h^+$.*
(iv) *If $0 \leq g \leq h$, then $h' \leq g'$.*
(v) $(g^-)' = (g^+)'' + g' = (g^+)'' \vee g'$.
(vi) $(g^+)' = 0 \Rightarrow 0 \leq g$.
(vii) *$g', g'' \in CPC(g)$ and $g \in C(g') = C(g'')$.*
(viii) $g \in C(p) \Rightarrow (J_p(g))'' = p \wedge g''$.
(ix) *If $d_1, d_2, \ldots, d_k \in P$, $d := \sum_{i=1}^k d_i \leq u$, and $g \in \bigcap_{i=1}^k C(d_i)$, then $d \in P$ and $J_d(g) = \sum_{i=1}^k J_{d_i}(g)$.*
(x) *If $g_1, g_2, \ldots, g_n \in G^+$, then $(\sum_{i=1}^n g_i)'' = \bigvee_{i=1}^n (g_i)''$.*

Proof: (i) Follows from [5, Lemma 2.2].

(ii) As $n > 0$, [5, Lemma 4.3 (x)] implies that $(ng)^+ = ng^+$. Let $p \in P$. As G is torsion free, $J_p(g) = 0 \Longleftrightarrow J_p(ng) = nJ_p(g) = 0$, and (i) implies that $g \in C(p) \Longleftrightarrow ng \in C(p)$, whence $(ng)' = g'$. Clearly, $(-g)' = g'$.

(iii) See [5, Lemma 4.4 (i)]. (iv) See [5, Lemma 6.2 (vi)]. By [5, Theorem 6.5 (ii)], $(g^+)'' + (g^-)'' = (g^+)'' \vee (g^-)''$, from which (v) follows. (vi) See [5, Theorem 6.5 (v)].

(vii) That $g', g'' \in CPC(g)$ follows from [5, Theorem 6.5 (i)]. That $g \in C(g') = C(g'')$ follows directly from the definition of the Rickart mapping.

(viii) By [5, Theorem 6.5 (ii)], $(J_p(g))'' = ((J_p(g))^+)'' \vee ((J_p(g))^-)''$. Also, by [5, Theorem 6.4 (vii)], $(J_p(g^+))'' = p \wedge (p' \vee (g^+)'')$ and $(J_p(g^-))'' = p \wedge (p' \vee (g^-)'')$, whence $(J_p(g))'' = (p \wedge (p' \vee (g^+)'')) \vee (p \wedge (p' \vee (g^-)''))$. By [5, Lemma 4.3 (vii)], $g^+, g^- \in C(p)$, whence by [5, Theorem 6.5 (i)], $(g^+)''Cp$ and $(g^-)''Cp$. Therefore, $p \wedge (p' \vee (g^+)'') = p \wedge (g^+)''$, $p \wedge (p' \vee (g^-)'') = p \wedge (g^-)''$, and $(J_p(g))'' = (p \wedge (g^+)'') \vee (p \wedge (g^-)'') = p \wedge ((g^+)'' \vee (g^-)'') = p \wedge g''$ by [5, Theorem 6.5 (ii)] again.

(ix) See [4, Corollary 5.2 (i)] and [5, Theorem 2.3 (ii)]. (x) See [5, Theorem 6.4 (iv)]. □

In the study of operators on a Hilbert space $\mathfrak{H}$, the notion of *reduction* of a bounded self-adjoint operator $T = T^*$ by a closed linear subspace $\mathfrak{M} \subseteq \mathfrak{H}$ plays an important role. If T commutes with the projection onto $\mathfrak{M}$, then $T(\mathfrak{M}) \subseteq \mathfrak{M}$, $T(\mathfrak{M}^\perp) \subseteq \mathfrak{M}^\perp$ and T is a "direct sum" of its own restrictions to $\mathfrak{M}$ and $\mathfrak{M}^\perp$. Likewise, if $v \in P$ and $g \in C(v)$, then $g = J_v(g) + J_{u-v}(g)$, the elements $J_v(g)$ and $J_{u-v}(g)$ are the analogues for g of the restrictions of T to $\mathfrak{M}$ and $\mathfrak{M}^\perp$, and the subgroups $H := J_v(G)$ and $K := J_{u-v}(G)$ of G are the analogues of the spaces of self-adjoint operators on $\mathfrak{M}$ and $\mathfrak{M}^\perp$, respectively.

Suppose that H is a subgroup of G. We understand that H is equipped with the *induced partial order*, i.e., the restriction to H of the partial order on G, whence the positive cone of H is $H^+ = H \cap G^+$. Obviously, if G is archimedean, then so is H. If f is a function defined on G, then $f|_H$ will denote the restriction of f to H. If, with the induced partial order, H is a unital group with its own unit v, then we denote the unit interval in H by $E_H := \{e \in H \mid 0 \leq e \leq v\}$ and we denote the state space of H by Ω_H. Similarly, if H is not only a unital group, but a compressible group as well, we distinguish between notions pertinent to G and the corresponding notions for H by using H as a subscript. For instance, P_H is the OMP of projections in H, $h \in C_H(p)$ means that $h \in H$, $p \in P_H$, and h is compatible with p in the compressible group H, and so on.

THEOREM 3.2. *Suppose that G is a compressible group, $0 \neq v \in P$, and $H := J_v(G) = \{J_v(g) \mid g \in G\} = \{h \in G \mid h = J_v(h)\}$. Then:*

(i) *With the induced partial order, H is a compressible group with unit v, $E_H = \{e \in E \mid e \leq v\}$, $P_H = \{p \in P \mid p \leq v\}$, and $H \subseteq C(v)$. For $p \in P_H$, the unique compression I on H for which $I(v) = p$ is the restriction $J_p|_H$ of J_p to H. If $p \in P_H$, then $C_H(p) = H \cap C(p)$ is the set of elements of H that are compatible with p in H. The state space of H is $\Omega_H = \{\omega|_H \mid \omega \in \Omega$ and $\omega(v) = 1\}$. If $\omega \in \Omega$, then $\omega(v) = 0 \Leftrightarrow \omega(h) = 0$ for all $h \in H$.*

(ii) *If G has the Rickart projection property, then so does H, the Rickart mapping on H is $h \mapsto h' \wedge v$, and the iterated Rickart mapping $h \mapsto (h' \wedge v)' \wedge v = h'' \in P_H$ is the restriction to H of the iterated Rickart mapping on G.*

(iii) *Suppose that G is a comgroup. If $h \in H$ and $q \in P^{\pm}(h)$, then qCv and $q \wedge v \in P_H^{\pm}(h) := (P_H)^{\pm}(h)$, hence H is a comgroup. Furthermore, $h^+, h^- \in H$, and h^+, $-h^-$ are the positive and negative parts of h as calculated either in G or in H.*

Proof: (i) By [4, Theorem 5.9], H is a compressible group with unit v and $P_H = \{p \in P \mid p \leq v\}$. Obviously, $E_H = \{e \in E \mid e \leq v\}$. Let $h \in H$ and $p \in P_H$. Then $h = J_v(h)$, so $J_{u-v}(h) = J_{u-v}(J_v(h)) = 0$, and it follows that $h = J_v(h) + J_{u-v}(h)$, i.e., $h \in C(v)$. Also, by [4, Theorem 5.9], the restriction $J_p|_H$ is the unique compression on H that maps v into p. Moreover, as $p \leq v$, we have $(u-p)Cv$, and $J_{u-p}(h) = J_{u-p}(J_v(h)) = J_{(u-p)\wedge v}(h) = J_{v-p}(h)$, whence $J_p|_H(h) + J_{v-p}|_H(h) = J_p(h) + J_{u-p}(h)$, and it follows that $h \in C_H(p)$ iff $h \in C(p)$.

If $\omega \in \Omega$ and $\omega(v) = 1$, it is clear that $\omega|_H \in \Omega_H$. Conversely, suppose that $\eta \in \Omega_H$ and define $\omega : G \to \mathbb{R}$ by $\omega := \eta \circ J_v$. As both $J_v : G \to H$ and $\eta : H \to \mathbb{R}$ are order-preserving group homomorphisms, so is ω. Also, as $J_v(u) = J_v(v) = v$, we have $\omega(u) = \omega(v) = \eta(v) = 1$, so $\omega \in \Omega$ with $\omega(v) = 1$. Furthermore, $\omega|_H = \eta$, and it follows that $\Omega_H = \{\omega|_H \mid \omega \in \Omega$ and $\omega(v) = 1\}$.

Suppose $\omega \in \Omega$ and $\omega(v) = 0$. Then, $\omega(e) = 0$ for all $e \in E_H$ and, since every element in H^+ is a finite sum of elements in E_H, it follows that $\omega(h) = 0$ for all $h \in H^+$. But $H = H^+ - H^+$, so $\omega(h) = 0$ for all $h \in H$.

(ii) Let $h \in H$. As G has the Rickart projection property, P is an OML, so the infimum $h' \wedge v$ exists in P. Since $h' \wedge v \leq v$, we have $h' \wedge v \in P_H$.

Also, as $p \leq v$, we have $p \leq h' \wedge v \Longleftrightarrow p \leq h' \Longleftrightarrow h \in C(p)$ with $J_p(h) = 0 \Longleftrightarrow h \in C_H(p)$ with $J_p|_H(h) = 0$, so H has the Rickart projection property and $h \mapsto h' \wedge v$ is the Rickart mapping. Also, as $h \in C(v)$, we have $h \in C(v')$ with $J_{v'}(h) = 0$, i.e., $v' \leq h'$, i.e., $h'' \leq v$, whence $h'' \in P_H$. Consequently, $(h' \wedge v)' \wedge v = (h'' \vee v') \wedge v = h'' \wedge v = v$.

(iii) Let $h \in H$ and let $q \in P^{\pm}(h)$, i.e., $h = J_v(h)$, $q \in P \cap CPC(h)$, $h \in C(q)$, and $J_{u-q}(h) \leq 0 \leq J_q(h)$. As $h \in C(v)$ and $q \in CPC(h)$ we have qCv, so $p := q \wedge v$ exists in P and, as $p \leq v$, it follows that $p \in P_H$. Suppose $r \in P_H$ with $h \in C_H(r)$. Then $r \in P$ with $h \in C(r)$, whence qCr. Also, $r \leq v$, so vCr, and therefore pCr, i.e., $p \in C_H(r)$. Consequently, $p \in C_H P_H C_H(h)$. As $h \in C(v)$ and $h \in C(q)$, it follows from [5, Corollary 2.4] that $h \in C(p)$, whence $h \in C_H(p)$. Since $h \in C(v)$ and $q \in CPC(h)$, we also have qCv, so $(u-q)Cv$. Therefore, $J_{v-p}|_H(h) = J_{v-(q\wedge v)}(h) = J_{(u-q)\wedge v}(h) = J_{u-q}(J_v(h)) = J_{u-q}(h) = -h^- \leq 0 \leq h^+ = J_q(h) = J_{q\wedge v}(h) = J_p|_H(h)$, and it follows that $p \in P_H^{\pm}(h)$. In particular, $P_H^{\pm}(h)$ is not empty, so H is a comgroup, and h^+, $-h^-$ are the positive and negative parts of h as calculated in H. □

COROLLARY 3.1. *If $0 \neq v \in P$, there exists a state $\omega \in \Omega$ with $\omega(v) = 1$.*

Suppose that G is a Rickart comgroup, $v \in P$ with $0 < v < u$, $H := J_v(G)$, and $K := J_{v'}(G)$. Then by Theorem 3.2, both H and K (with the partial orders induced from G) are Rickart comgroups with units v and v', respectively. Evidently, $H \cap K = \{0\}$, and $H + K = C(v)$, i.e., the subgroup $C(v)$ is a direct sum of H and K. It is not difficult to show that $C(v)$, with the partial order induced from G, is a Rickart comgroup with unit u, and that, as such, $C(g)$ is the direct sum *in the category of Rickart comgroups* of H and K. If $g \in C(v)$, then by analogy with the reduction of self-adjoint operators, we refer to decomposition of g into its components $J_v(g) \in H$ and $J_{v'}(g) \in K$ as the *reduction* of g by v (or by v').

LEMMA 3.2. *Suppose G is a Rickart comgroup and let $v \in P$, with $g \in C(v)$. Then:*

(i) $(J_v(g))' \wedge v = g' \wedge v$.
(ii) $J_v(g^+) = (J_v(g))^+$.
(iii) $((J_v(g))^+)' \wedge v = (g^+)' \wedge v$.

Proof: (i) As $g \in C(v)$, Lemma 3.1 (vii) implies that $g' \in C(v)$. Therefore, by Lemma 3.1 (viii), $(J_v(g))' \wedge v = (v' \vee g') \wedge v = g' \wedge v$.

(ii) Choose $q \in P^{\pm}(g)$. Then $q \in PCP(g)$, and as $g \in C(v)$, it follows that qCv. Thus, $J_q(J_v(g)) = J_v(J_q(g)) = J_v(g^+)$ and $J_{q'}(J_v(g)) = J_v(J_{q'}(g)) = J_v(-g^-)$, so $J_q(J_v(g)) + J_{q'}(J_v(g)) = J_v(g^+ - g^-) = J_v(g)$, i.e., $J_v(g) \in C(q)$. Therefore, as $J_{q'}(J_v(g)) = J_v(-g^-) \leq 0 \leq J_v(g^+) = J_q(J_v(g))$, [5, Lemma 4.3] implies that $(J_v(g))^+ = J_q(J_v(g)) = J_v(g^+)$.

(iii) Follows immediately from (i) and (ii). □

DEFINITION 3.1. For $g \in G$, define the *u-pseudonorm* $\|g\|$ of g by $\|g\| := \inf\{n/k \mid 0 < k, n \in \mathbb{Z} \text{ and } -nu \leq kg \leq nu\}$. If $\|\cdot\|$ is a norm on G, it is called the *u-norm* (cf. [6, p. 120]).

THEOREM 3.3. *Let $g, h \in G$, $m \in \mathbb{Z}$, and $p \in P$. Then:*

(i) $\|mg\| = |m| \cdot \|g\|$.
(ii) $\|g+h\| \leq \|g\| + \|h\|$.
(iii) $-h \leq g \leq h \Rightarrow \|g\| \leq \|h\|$.
(iv) $\|u\| = 1$.
(v) $\|g\| = \max\{|\omega(g)| \mid \omega \in \Omega\}$.
(vi) *If G is archimedean, then $\|\cdot\|$ is a norm on G.*
(vii) *If G is archimedean and $0 < k, n \in \mathbb{Z}$, then $\|g\| \leq k/n \Leftrightarrow -ku \leq ng \leq ku$.*
(viii) *If $p \neq 0$, then $\|p\| = 1$.*
(ix) $\|J_p(g)\| \leq \|g\|$. (x) *If $g_i \in G^+$ and $0 < m_i \in \mathbb{Z}$ with $m_i \leq \beta \in \mathbb{R}$ for $i = 1, 2, \ldots, N$, then $\|\sum_{i=1}^N m_i g_i\| \leq \beta \|\sum_{i=1}^N g_i\|$.*

Proof: Parts (i)–(v) follow from [6, Proposition 7.12], (vi) follows from (v) and the fact that, if G is archimedean, then Ω separates points in G. Part (vii) follows from Definition 3.1 and [6, Lemma 7.13 (iii)]. Part (viii) follows from (v) and Corollary 3.4. To prove (ix), we observe that, if $0 < k, n \in \mathbb{Z}$, then $kp \leq ku$, whence $-ku \leq ng \leq ku \Rightarrow -ku \leq -kp \leq nJ_p(g) \leq kp \leq ku$, so $\|J_p(g)\| \leq k/n$, and (ix) follows. Assume the hypotheses of (x). Then, by (v), $\|\sum_{i=1}^N m_i g_i\| = \max_{\omega\in\Omega} \omega(\sum_{i=1}^N m_i g_i) = \max_{\omega\in\Omega} \sum_{i=1}^N m_i \omega(g_i) \leq \max_{\omega\in\Omega} \sum_{i=1}^N \beta\omega(g_i) = \beta \max_{\omega\in\Omega} \omega(\sum_{i=1}^N g_i) = \beta \|\sum_{i=1}^N g_i\|$. □

In Example 2.1, the 1-norm as in Definition 3.1 is the restriction to $G(A)$ of the norm on the C*-algebra A.

THEOREM 3.4. *Let G be archimedean, and suppose that, for every positive integer k, there exists a positive integer n_k and an element $g_k \in G$ such that $\|n_k g - g_k\| \leq n_k/k$. Then, if $p \in P$ and $g_k \in C(p)$ for $k = 1, 2, \ldots$, it follows that $g \in C(p)$.*

Proof: For every positive integer k, we have $n_k\|g - J_p(g) - J_{u-p}(g)\| = \|n_k g - g_k + J_p(g_k) + J_{u-p}(g_k) - J_p(n_k g) - J_{u-p}(n_k g)\| \leq \|n_k g - g_k\| + \|J_p(g_k - n_k g)\| + \|J_{u-p}(g_k - n_k g)\| \leq 3\|n_k g - g_k\|$ by Theorem 3.3 (ix). Therefore, $n_k\|g - J_p(g) - J_{u-p}(g)\| \leq 3n_k/k$, i.e., $\|g - J_p(g) - J_{u-p}(g)\| \leq 3/k$. Letting $k \to +\infty$, we conclude that $g - J_p(g) - J_{u-p}(g) = 0$, so $g \in C(p)$. □

COROLLARY 3.2. *If G is archimedean and $p \in P$, then $C(p)$ is closed in the u-norm topology.*

Proof: If g is an accumulation point of $C(p)$ then for each positive integer k there is an element $g_k \in C(p)$ such that $\|g - g_k\| < 1/k$. Hence, by Theorem 3.4 with $n_k := 1$ for $k = 1, 2, \ldots$, we have $g \in C(p)$. □

DEFINITION 3.2. If $g \in G$, then the *spectral lower and upper bounds*, L_g and U_g, respectively, for g are defined by $L_g := \sup\{m/n \mid m, n \in \mathbb{Z}, 0 < n, mu \leq ng\}$ and $U_g := \inf\{m/n \mid m, n \in \mathbb{Z}, 0 < n, ng \leq mu\}$ (cf. [6, Proposition 4.7]).

THEOREM 3.5. *If $g \in G$, then:*

(i) $-\infty < L_g \leq U_g < \infty$.
(ii) $\omega \in \Omega \Rightarrow L_g \leq \omega(g) \leq U_g$.
(iii) *If $\rho \in \mathbb{R}$ and $L_g \leq \rho \leq U_g$, then there exists $\omega \in \Omega$ such that $\omega(g) = \rho$.*
(iv) $\|g\| = \max\{|L_g|, |U_g|\}$.

Proof: For (i)–(iii), see [6, Proposition 4.7], and for (iv), see the proof of [6, Proposition 4.7]. □

4. A rational spectral resolution

In what follows, *we assume that $G \neq \{0\}$ is a Rickart comgroup with unit u, P is the OML of projections in G, $' : G \to P$ is the Rickart mapping, $\Omega := \Omega(G)$ is the state space of G, and $g \in G$. We denote the lower and upper spectral bounds of g by $L := L_g$ and $U := U_g$. Also, as usual, $\mathbb{Q}$ is the ordered field of rational numbers.* Our purpose here is to define and study a "rational spectral resolution" $(p_\lambda)_{\lambda\in\mathbb{Q}}$ corresponding to g.

To motivate our subsequent development, we observe that, in Example 2.2, if $(p_\lambda)_{\lambda\in\mathbb{R}}$ is the spectral resolution corresponding to g, then for $\lambda \in \mathbb{R}$, $p_\lambda = ((g - \lambda\mathbf{1})^+)'$. Suppose $\lambda \in \mathbb{Q}$, say $\lambda = m/n$ with $n, m \in \mathbb{Z}$ and $n > 0$. Then $p_\lambda = ((g-(m/n)\mathbf{1})^+)' = (((1/n)(ng-m\mathbf{1}))^+)' = ((1/n)(ng-m\mathbf{1})^+)' = ((ng-m\mathbf{1})^+)'$. We note that the corresponding expression $((ng-mu)^+)'$ makes sense in any Rickart comgroup with unit u.

LEMMA 4.1. *Let $m, n, x, y \in \mathbb{Z}$ with $n, y > 0$. Then:*

(i) $m/n \leq x/y \Rightarrow ((ng - mu)^+)' \leq ((yg - xu)^+)'$.
(ii) $m/n = x/y \Rightarrow ((ng - mu)^+)' = ((yg - xu)^+)'$
(iii) $m/n = x/y \Rightarrow (ng - mu)' = (yg - xu)'$.

Proof: (i) Assume that $m/n \leq x/y$, i.e., $my \leq nx$. Then $myu \leq nxu$, and it follows that $n(yg - xu) = nyg - nxu \leq nyg - myu = y(ng - mu)$. Lemma 3.1 (i) implies that $y(ng - mu) \in CPC(n(yg - xu))$, hence by Lemma 3.1 (ii), (iii), $n(yg-xu)^+ = (n(yg-xu))^+ \leq (y(ng-mu))^+ = y(ng-mu)^+$. As $0 \leq n(yg-xu)^+ \leq y(ng-mu)^+$, Lemma 3.1 (iv) implies that $((y(ng-mu))^+)' \leq ((n(yg-xu))^+)'$, i.e., $((ng - mu)^+)' \leq ((yg - xu)^+)'$ by Lemma 3.1 (i) again. Part (ii) Follows directly from (i). (iii) Assume that $m/n = x/y$, i.e., $my = nx$. By Lemma 3.1 (ii), we have $(ng - mu)' = (nyg - myu)' = (nyg - nxu)' = (yg - xu)'$. □

DEFINITION 4.1. If $\lambda \in \mathbb{Q}$, choose $m, n \in \mathbb{Z}$ with $n \geq 0$ and $\lambda = m/n$. By parts (ii) and (iii) of Lemma 4.1, $p_{g,\lambda} := ((ng - mu)^+)' \in P$ and $d_{g,\lambda} := (ng - mu)' \in P$ are well defined. The family of projections $(p_{g,\lambda})_{\lambda\in\mathbb{Q}}$ is called the *rational spectral resolution* for (or corresponding to) $g \in G$. The projection $d_{g,\lambda}$ is called the *λ-eigenprojection* for g, and the family $(d_{g,\lambda})_{\lambda\in\mathbb{Q}}$ is called the *family of rational eigenprojections* for (or corresponding to) g. If $\lambda \in \mathbb{Q}$ and $d_{g,\lambda} \neq 0$, we say that λ is a *rational eigenvalue* of g. If g is understood, we write p_λ and d_λ rather than $p_{g,\lambda}$ and $d_{g,\lambda}$.

LEMMA 4.2. *If* $\lambda \in \mathbb{Q}$, *then* $p_{-g,\lambda} = d_{g,-\lambda} + (p_{g,-\lambda})' = d_{g,-\lambda} \vee (p_{g,-\lambda})'$ *and* $d_{-g,\lambda} = d_{g,-\lambda}$.

Proof: Let $\lambda = m/n$ with $m, n \in \mathbb{Z}$ and $0 < n$. Then $p_{-g,\lambda} = ((n(-g)-mu)^+)' = ((-(ng+mu))^+)' = ((ng+mu)^-)'$. Thus, by Lemma 3.1 (v), $d_{-g,\lambda} = ((ng+mu)^+)'' + (ng+mu)' = ((ng+mu)^+)'' \vee (ng+mu)'$. But, $((ng+mu)^+)'' = ((ng-(-m)u)^+)'' = (p_{g,-\lambda})'$ and $(ng+mu)' = (ng-(-m)u))' = d_{g,-\lambda}$. Also, by Lemma 3.1 (ii), $d_{-g,\lambda} = (n(-g)-mu)' = (-(ng-(-m)u))' = (ng-(-m)u)' = d_{g,-\lambda}$. □

For the remainder of this article, *we assume that* $(p_\lambda)_{\lambda\in\mathbb{Q}}$ *is the rational spectral resolution for* $g \in G$ *and that* $(d_\lambda)_{\lambda\in\mathbb{Q}}$ *is the family of rational eigenprojections for* g.

LEMMA 4.3. *If* $m, n \in \mathbb{Z}$ *with* $n > 0$ *and* $\lambda = m/n$, *then* p_λ *is uniquely determined by the following properties:*

(i) $p_\lambda \in P \cap CPC(g)$.
(ii) $g \in C(p_\lambda)$.
(iii) $nJ_{p_\lambda}(g) - mp_\lambda \le 0 \le nJ_{u-p_\lambda}(g) - m(u-p_\lambda)$.
(iv) $(ng-mu)' \le p_\lambda$.

Proof: Let $q := (p_\lambda)' = ((ng-mu)^+)''$. By Theorem 3.1, q is uniquely determined by the properties $q \in P^{\pm}(ng-mu)$ and $q \le (ng-mu)''$, i.e., $q \in P\cap CPC(ng-mu)$, $ng-mu \in C(q)$, $J_{q'}(ng-mu) \le 0 \le J_q(ng-mu)$, and $q \le (ng-mu)''$. By Lemma 3.1 (i), $q \in P\cap CPC(ng-mu) \Longleftrightarrow q \in P \cap CPC(g)$, and since $p_\lambda = u-q$, we have $q \in P\cap CPC(g) \Longleftrightarrow p_\lambda \in P \cap CPC(g)$. Likewise, by Lemma 3.1 (i) and the fact that $p_\lambda = u-q$, we have $ng-mu \in C(q) \Longleftrightarrow g \in C(p_\lambda)$. The condition $J_{q'}(ng-mu) \le 0 \le J_q(ng-mu)$ is equivalent to $nJ_{q'}(g) - mq' \le 0 \le nJ_q(g) - mq$, i.e., $nJ_{p_\lambda}(g) - mp_\lambda \le 0 \le nJ_{u-p_\lambda}(g) - m(u-p_\lambda)$. Finally, $q \le (ng-mu)'' \Longleftrightarrow d_\lambda = (ng-mu)' \le q' = p_\lambda$. □

THEOREM 4.1. *For all* $\lambda, \mu \in \mathbb{Q}$ *and all* $m, k, n \in \mathbb{Z}$ *with* $n > 0$:

(i) $p_\lambda, d_\lambda \in P \cap CPC(g)$, $g \in C(p_\lambda) \cap C(d_\lambda)$, *and* $p_\lambda C d_\mu$.
(ii) $\lambda = m/n \Rightarrow nJ_{p_\lambda}(g) - mp_\lambda \le 0 \le nJ_{(p_\lambda)'}(g) - m(p_\lambda)'$.
(iii) $\lambda \le \mu \Rightarrow p_\lambda \le p_\mu$ *and* $p_\mu - p_\lambda = p_\mu \wedge (p_\lambda)' \in P$.
(iv) $\lambda < \mu \Rightarrow d_\lambda \le p_\lambda \le (d_\mu)'$.
(v) $\lambda > U \Rightarrow p_\lambda = u$ *and* $\lambda < U \Rightarrow p_\lambda < u$.
(vi) $\lambda < L \Rightarrow p_\lambda = 0$ *and* $L < \lambda \Rightarrow 0 < p_\lambda$.
(vii) $L = \sup\{\lambda \in \mathbb{Q} \mid p_\lambda = 0\}$ *and* $U = \inf\{\lambda \in \mathbb{Q} \mid p_\lambda = u\}$.
(viii) *Suppose* $\lambda \le \mu$, *choose integers* m, k, n *with* $0 < n$, $\lambda = m/n$, *and* $\mu = k/n$, *and suppose that* $q \in P$ *with* $q \le p_\mu - p_\lambda = p_\mu \wedge (p_\lambda)'$. *Then* $mq \le nJ_q(g) \le kq$.
(ix) *If* $\lambda < \mu$, $q \in P$ *with* $q \le p_\mu \wedge (p_\lambda)'$, *and* $\omega \in \Omega$, *then* $\lambda\omega(q) \le \omega(J_q(g)) \le \mu\omega(q)$.

Proof: (i) By Lemma 4.3 (i) and (ii), we have $p_\lambda \in CPC(g)$, $g \in C(p_\lambda)$, and $g \in C(p_\mu)$. Let $\lambda = m/n$ with $m, n \in \mathbb{Z}$ and $0 < n$. Then $ng - mu \in CPC(g)$

by Lemma 3.1 (i) and $d_\lambda = (ng - mu)' \in CPC(ng - mu)$ by Lemma 3.1 (vii), from which it follows that $d_\lambda \in CPC(g)$. Thus, $g \in C(p_\mu)$ implies that $d_\lambda C p_\mu$. As $ng - mu \in C((mg - mu)') = C(d_\lambda)$, Lemma 3.1 (i) implies that $g \in C(d_\lambda)$.

Properties (ii) and (iii) follow directly from Lemma 4.3 (iii) and Lemma 4.1 (i), respectively.

(iv) That $d_\lambda \le p_\lambda$ is part (iv) of Lemma 4.3. Suppose that $\lambda < \mu$ and choose integers m, k, n with $n > 0$ such that $\lambda = m/n$ and $\mu = k/n$. Then $m < k$ and $nJ_{p_\lambda}(g) \le mp_\lambda$ by (ii). By (i), $J_{p_\lambda} \circ J_{d_\mu} = J_{d_\mu} \circ J_{p_\lambda}$ and $J_{p_\lambda}(d_\mu) = J_{d_\mu}(p_\lambda) = p_\lambda \wedge d_\mu$. Also, as $d_\mu = (ng - ku)'$, we have $J_{d_\mu}(ng - ku) = 0$, i.e., $kd_\mu = nJ_{d_\mu}(g)$. Therefore,

$$\begin{aligned} kJ_{p_\lambda}(d_\mu) &= J_{p_\lambda}(kd_\mu) = J_{p_\lambda}(nJ_{d_\mu}(g)) = J_{d_\mu}(nJ_{p_\lambda}(g)) \\ &\le J_{d_\mu}(mp_\lambda) = mJ_{d_\mu}(p_\lambda) = mJ_{p_\lambda}(d_\mu) \le kJ_{p_\lambda}(d_\mu), \end{aligned}$$

whence $kJ_{p_\lambda}(d_\mu) = mJ_{p_\lambda}(d_\mu)$, i.e., $(k - m)J_{p_\lambda}(d_\mu) = 0$. As $k - m > 0$, it follows that $J_{p_\lambda}(d_\mu) = 0$, whence $p_\lambda \le (d_\mu)'$.

(v) Let $\lambda \in \mathbb{Q}$, write $\lambda = x/y$ with $x, y \in \mathbb{Z}$ and $y > 0$, and suppose $\lambda > U = \inf\{m/n \mid m, n \in \mathbb{Z}, 0 < n, ng \le mu\}$. Then there exist $m, n \in \mathbb{Z}$ with $n > 0$, $m/n < x/y$, and $ng - mu \le 0$. Thus, $my < nx$, so $n(yg - xu) = nyg - nxu \le nyg - myu = y(ng - mu) \le 0$, and, since G is unperforated, it follows that $yg - xu \le 0$, i.e., $(yg - xu)^+ = 0$. Therefore, $p_\lambda = ((yg - xu)^+)' = 0' = u$. On the other hand, $p_\lambda = u \Longrightarrow ((yg - xu)^+)' = u \Longrightarrow (yg - xu)^+ = 0 \Longrightarrow yg - xu \le 0 \Longrightarrow U \le y/x = \lambda$, whence $\lambda < U \Longrightarrow p_\lambda < u$.

(vi) Let $\lambda \in \mathbb{Q}$, write $\lambda = x/y$ with $x, y \in \mathbb{Z}$ and $y > 0$, and suppose $\lambda < L = \sup\{m/n \mid m, n \in \mathbb{Z}, 0 < n, mu \le ng\}$. Then there exist $m, n \in \mathbb{Z}$ with $n > 0$, $x/y < m/n$, and $0 \le ng - mu$. Thus, $my - nx > 0$ and $nyg \ge myu$. Consequently, $n(yg - xu) = nyg - nxu \ge nyg - myu = y(ng - mu) \ge 0$, and it follows from Lemma 3.1 (ii) that $n(yg - xu)^+ = (n(yg - xu))^+ = n(yg - xu) = nyg - nxu \ge myu - nxu = (my - nx)u$ with $my - nx > 0$. Therefore, by Lemma 3.1 (iv), $((my - nx)u)' \ge (n(yg - xu)^+)'$, so by Lemma 3.1 (ii) again, $0 = u' \ge ((yg - xu)^+)' = p_\lambda$, i.e., $p_\lambda = 0$. On the other hand, by Lemma 3.1 (vi), $p_\lambda = 0 \Longrightarrow ((yg - xu)^+)' = 0 \Longrightarrow 0 \le yg - xu \Longrightarrow \lambda = y/x \le L$, whence $L < \lambda \Longrightarrow 0 < p_\lambda$.

(vii) Follows directly from (v) and (vi).

(viii) Assume the hypotheses. By (ii), $0 \le J_{(p_\lambda)'}(ng - mu)$ and, since $q \le (p_\lambda)'$, we have $0 \le J_q(J_{(p_\lambda)'}(ng - mu)) = J_q(ng - mu)$, i.e., $mq \le nJ_q(g)$. Also, by (ii), $J_{p_\mu}(ng - ku) \le 0$ and, since $q \le p_\mu$, we have $J_q(ng - ku) = J_q(J_{p_\mu}(ng - ku) \le 0$, i.e., $nJ_q(g) \le kq$, and (viii) follows.

(ix) Assume the hypotheses and choose $m, k, n \in \mathbb{Z}$ with $n > 0$, $\lambda = m/n$, $\mu = k/n$. By (viii), we have $mq \le nJ_q(g) \le kq$, whence $m\omega(q) \le n\omega(J_q(g)) \le k\omega(q)$, i.e., $\lambda\omega(q) \le \omega(J_q(g)) \le \mu\omega(q)$. □

As a consequence of Theorem 4.1 (iv), if $\alpha, \beta \in \mathbb{Q}$ and $\alpha \ne \beta$, then the eigenprojections d_α and d_β are orthogonal in the OML P, i.e., $d_\alpha + d_\beta \le u$. Additional useful properties of d_α are given in the following lemma.

LEMMA 4.4. *Let $\alpha = a/b$ with $a, b \in \mathbb{Z}$, $0 < b$, let $\lambda \in \mathbb{Q}$, and define $v := (d_\alpha)'$. Then:*

(i) $v = 0 \Leftrightarrow bg = au$.
(ii) $g, p_\lambda \in C(v) = C(v')$, $v' \leq p_\alpha$, *and* $p_\alpha - v' = p_\alpha \wedge v \in P$.
(iii) $\lambda < \alpha \Rightarrow p_\lambda \leq v$ *and* $\alpha \leq \lambda \Rightarrow v' \leq p_\lambda$.

Proof: (i) $0 = (d_\alpha)' = (bg - au)'' \Leftrightarrow bg - au = 0$. (ii) Follows directly from Theorem 4.1 (i) and (iv). (iii) If $\lambda < \alpha$, then $p_\lambda \leq (d_\alpha)' = v$ by Theorem 4.1 (iv). If $\alpha \leq \lambda$, then by Theorem 4.1 (iii), $p_\alpha \leq p_\lambda$, hence $v' \leq p_\lambda$ by (ii). □

THEOREM 4.2. *Suppose that $\lambda_0, \lambda_1, \ldots, \lambda_N \in \mathbb{Q}$ with $\lambda_0 < L < \lambda_1 < \cdots < \lambda_{N-1} < U < \lambda_N$ and that $\gamma_i \in \mathbb{Q}$ with $\lambda_{i-1} \leq \gamma_i \leq \lambda_i$ for $i = 1, 2, \ldots, N$. Choose integers $n > 0$, m_j, and k_i such that $\lambda_j = m_j/n$ for $j = 0, 1, \ldots, N$ and $\gamma_i = k_i/n$ for $i = 1, 2, \ldots, N$. Let $u_i := p_{\lambda_i} - p_{\lambda_{i-1}} \in P$ for $i = 1, 2, \ldots, N$ and let $\epsilon = \max\{\lambda_i - \lambda_{i-1} \mid i = 1, 2, \ldots, N\}$. Then:*

(i) $\sum_{i=1}^N u_i = u$.
(ii) $\sum_{i=1}^N m_{i-1}u_i \leq ng \leq \sum_{i=1}^N m_i u_i$.
(iii) $-\sum_{i=1}^N (m_i - m_{i-1})u_i \leq ng - \sum_{i=1}^N k_i u_i \leq \sum_{i=1}^N (m_i - m_{i-1})u_i$.
(iv) $\|ng - \sum_{i=1}^N k_i u_i\| \leq n\epsilon$.
(v) $\omega \in \Omega \Rightarrow |\omega(g) - \sum_{i=1}^N \gamma_i \omega(u_i)| \leq \epsilon$.

Proof: (i) By parts (v) and (vi) of Theorem 4.1, $\sum_{i=1}^N u_i = p_N - p_0 = u - 0 = u$.
(ii) By Theorem 4.1 (viii), $m_{i-1}u_i \leq nJ_{u_i}(g) \leq m_i u_i$ for $i = 1, 2, \ldots, N$, whence $\sum_{i=1}^N m_{i-1}u_i \leq n\sum_{i=1}^N J_{u_i}(g) \leq \sum_{i=1}^N m_i u_i$. Therefore, by (i), Theorem 4.1 (i), and Lemma 3.1 (ix), $\sum_{i=1}^N J_{u_i}(g) = g$, and (ii) follows.

(iii) As $m_{i-1} \leq k_i \leq m_i$ and $0 \leq u_i$, (ii) implies that $-\sum_{i=1}^N (m_i - m_{i-1})u_i \leq -\sum_{i=1}^N (k_i - m_{i-1})u_i \leq ng - \sum_{i=1}^N k_i u_i \leq \sum_{i=1}^N (m_i - k_i)u_i \leq \sum_{i=1}^N (m_i - m_{i-1})u_i$.

(iv) By (iii), Theorem 3.3 (iii), (i), and Theorem 3.3 (x), we have $\|ng - \sum_{i=1}^N k_i u_i\| \leq \|\sum_{i=1}^N (m_i - m_{i-1})u_i\| \leq n\epsilon$.

(v) By (iv) and Theorem 3.3 (v), $|n\omega(g) - \sum_{i=1}^N k_i \omega(u_i)| \leq n\epsilon$, and dividing by n, we obtain (v). □

Part (v) of Theorem 4.2 has the following immediate corollary.

COROLLARY 4.1. *If G is archimedean, then each element $g \in G$ is uniquely determined by its rational spectral resolution $(p_{g,\lambda})_{\lambda \in \mathbb{Q}}$.*

THEOREM 4.3. *If G is archimedean and $p \in P$, then $g \in C(p)$ if and only if $p_\lambda C p$ for all $\lambda \in \mathbb{Q}$.*

Proof: If $g \in C(p)$ and $\lambda \in \mathbb{Q}$, then $p_\lambda C p$ by Theorem 4.1 (i). Conversely, suppose $p_\lambda C p$ for all $\lambda \in \mathbb{Q}$. Let k be a positive integer and choose $\lambda_0, \lambda_1, \ldots, \lambda_N \in \mathbb{Q}$ with $\lambda_0 < L < \lambda_1 < \cdots < \lambda_{N-1} < U < \lambda_N$ and $\max\{\lambda_i - \lambda_{i-1} \mid i = 1, 2, \ldots, N\} < 1/k$. Choose a positive integer n_k and integers $m_1, m_2, \ldots, m_N$ such that $\lambda_j = m_j/n_k$

for $j = 0, 1, \ldots, N$, let $u_i := p_{\lambda_i} - p_{\lambda_{i-1}} \in P$ for $i = 1, 2, \ldots, N$, and define $g_k := \sum_{i=1}^{N} m_i u_i$. Then, by Theorem 4.2 (iv) with $n := n_k$ and $\gamma_i = \lambda_i$ for $i = 1, 2, \ldots, N$, we have $\|n_k g - g_k\| < n_k/k$. As $p_{\lambda_j} C p$ for $j = 0, 1, 2, \ldots, N$, it follows that $g_k \in C(p)$. Since k was an arbitrary positive integer, Theorem 3.4 implies that $g \in C(p)$. □

LEMMA 4.5. *Suppose G is archimedean, let $q \in P$ with $g \in C(q)$, and let $\alpha \in \mathbb{Q}$. Then:*

(i) *If $\alpha < \mu \in \mathbb{Q} \Rightarrow q \le p_\mu \wedge (p_\alpha)'$, then $q = 0$.*
(ii) *If $\alpha > \lambda \in \mathbb{Q} \Rightarrow q \le p_\alpha \wedge (p_\lambda)'$, then $q \le d_\alpha$.*

Proof: Choose integers a and b with $0 < b$ and $\alpha = b/a$.

(i) Assume that $\alpha < \mu \in \mathbb{Q} \Rightarrow q \le p_\mu \wedge (p_\alpha)'$. Then, by Theorem 4.1 (ix), $\alpha < \mu \in \mathbb{Q} \Rightarrow \alpha\omega(q) \le \omega(J_q(g)) \le \mu\omega(q)$ for every $\omega \in \Omega$. As μ is an arbitrary rational number greater than α, we have $\alpha\omega(q) = \omega(J_q(g))$, whence $\omega(aq) = \omega(J_q(bg))$, i.e., $\omega(J_q(bg - au)) = 0$ for every $\omega \in \Omega$. Since G is archimedean, it follows that $J_q(bg-au)) = 0$. Also, $g \in C(q)$ implies $bg-au \in C(q)$, whence $q \le (bg-au)' = d_\alpha$. But, $d_\alpha \le p_\alpha$ by Theorem 4.1 (iv), whence $q \le p_\alpha, (p_\alpha)'$, and it follows that $q = 0$.

(ii) Assume that $\alpha > \lambda \in \mathbb{Q} \Rightarrow q \le p_\alpha \wedge (p_\lambda)'$. Then, by Theorem 4.1 (ix), $\alpha > \lambda \in \mathbb{Q} \Rightarrow \lambda\omega(q) \le \omega(J_q(g)) \le \alpha\omega(q)$. As λ is an arbitrary rational number less than α, we have $\omega(J_q(g)) = \alpha\omega(q)$, whence $\omega(J_q(bg)) = \omega(aq)$ for every $\omega \in \Omega$. As in (i), it follows that $q \le d_\alpha$. □

Missing from the list of properties of the spectral resolution $(p_\lambda)_{\lambda\in\mathbb{Q}}$ in Theorem 4.1 is the property of "right continuity," i.e., for each $\lambda \in \mathbb{Q}$, $p_\lambda = \bigwedge_{\lambda<\mu\in\mathbb{Q}} p_\mu$. In fact, without further assumptions on the Rickart comgroup G, (e.g., that G is archimedean), this property may fail.

EXAMPLE 4.1. Let $G := \mathbb{Z} \times \mathbb{Z}$ with coordinatewise addition and define $\le$ on G (lexicographically) by $(a, b) \le (c, d) \iff a < c$ or ($a = c$ and $b \le d$). Define $u, j \in G$ by $u := (1, 0)$ and $j := (0, 1)$, noting that $u, j \in G^+$ and u, j is a free basis for the abelian group G. Evidently G is a totally ordered compatible compressible group with unit u. The countable set $\{nj \mid 0 < n \in \mathbb{Z}\}$ is bounded above by u, but fails to have a supremum in G, hence G is neither archimedean nor monotone σ-complete. The OMP of projections in G is $P = \{0, u\}$, and it forms a two-element Boolean algebra. There is only one state on G, namely $\omega : G \to \mathbb{R}$ defined by $\omega(a, b) := a$ for all $(a, b) \in G$. Thus $\Omega = \{\omega\}$ fails to separate points in G. The compressible group G is a Rickart comgroup; in fact, for $g \in G$, $g^+ = g \vee_G 0$ and $g' = 0$ unless $g = 0$. Let $(p_\lambda)_{\lambda\in\mathbb{Q}}$ be the rational spectral resolution corresponding to j. Then $p_\lambda = 0$ for $\lambda \le 0$ and $p_\lambda = u$ for $\lambda > 0$, so $\bigwedge_{0<\mu\in\mathbb{Q}} p_\mu = u \ne 0 = p_0$, and (p_λ) fails to be "continuous from the right." □

THEOREM 4.4. *Suppose G is archimedean and $\alpha \in \mathbb{Q}$. Then:*

(i) *$\bigwedge_{\alpha<\mu\in\mathbb{Q}} p_\mu$ exists in P and $\bigwedge_{\alpha<\mu\in\mathbb{Q}} p_\mu = p_\alpha$.*

(ii) $\bigvee_{\alpha>\lambda\in\mathbb{Q}} p_\lambda$ *exists in* P, *and* $\bigvee_{\alpha>\lambda\in\mathbb{Q}} p_\lambda = p_\alpha - d_\alpha$.

Proof: Assume the hypotheses.

(i) By Theorem 4.1 (iii), p_α is a lower bound in P for $\{p_\mu \mid \alpha < \mu \in \mathbb{Q}\}$. Suppose that $r \in P$ is another such lower bound for $\{p_\mu \mid \alpha < \mu \in \mathbb{Q}\}$. Then $p_\alpha \vee r$ is a lower bound in P for $\{p_\mu \mid \alpha < \mu \in \mathbb{Q}\}$. We have to prove that $r \le p_\alpha$. Let $q := (p_\alpha \vee r) - p_\alpha = (p_\alpha \vee r) \wedge (p_\alpha)' \in P$. It will suffice to prove that $q = 0$.

Let $\lambda \in \mathbb{Q}$. If $\lambda \le \alpha$, then $p_\lambda \le p_\alpha \le p_\alpha \vee r$ so $p_\lambda C((p_\alpha \vee r) \wedge (p_\alpha)')$, i.e., $p_\lambda Cq$. On the other hand, if $\alpha < \lambda$, then $q \le p_\alpha \vee r \le p_\lambda$, and again $p_\lambda Cq$. Therefore, $p_\lambda Cq$ for all $\lambda \in \mathbb{Q}$, and it follows from Theorem 4.3 that $g \in C(q)$. Also, if $\mu \in \mathbb{Q}$ with $\alpha < \mu$, then $q \le p_\alpha \vee r \le p_\mu$ and $q \le (p_\alpha)'$, so $q \le p_\mu \wedge (p_\alpha)'$, and it follows from Lemma 4.5 (i) that $q = 0$.

(ii) Let $v := (d_\alpha)'$. By Lemma 4.4 (ii), $d_\alpha = v' \le p_\alpha$ and $p_\alpha - d_\alpha = p_\alpha - v' = p_\alpha \wedge v \in P$. Thus, it will be sufficient to prove that $p_\alpha \wedge v$ is the least upper bound in P of $\{p_\lambda \mid \lambda \in \mathbb{Q}, \lambda < \alpha\}$.

Let $\lambda \in \mathbb{Q}$ with $\lambda < \alpha$. By Theorem 4.1 (iii), $p_\lambda \le p_\alpha$, and by Lemma 4.4 (iii), $p_\lambda \le v$, whence $p_\lambda \le p_\alpha \wedge v$, i.e., $p_\alpha \wedge v$ is an upper bound in P for $\{p_\lambda \mid \lambda \in \mathbb{Q}, \lambda < \alpha\}$. Suppose $r \in P$ is another such upper bound. We have to show that $p_\alpha \wedge v \le r$. As both $p_\alpha \wedge v$ and r are upper bounds in P for $\{p_\lambda \mid \lambda \in \mathbb{Q}, \lambda < \alpha\}$, it follows that $p_\alpha \wedge v \wedge r$ is also an upper bound in P for $\{p_\lambda \mid \lambda \in \mathbb{Q}, \lambda < \alpha\}$. Let $q := (p_\alpha \wedge v) - (p_\alpha \wedge v \wedge r) = p_\alpha \wedge v \wedge (p_\alpha \wedge v \wedge r)'$. It will be sufficient to prove that $q = 0$.

Let $\mu \in \mathbb{Q}$. If $\mu < \alpha$, then $p_\mu \le p_\alpha \wedge v \wedge r \le p_\alpha \wedge v$, so $p_\mu C(p_\alpha \wedge v \wedge r)$ and $p_\mu C(p_\alpha \wedge v)$, whence $p_\mu Cq$. On the other hand, if $\alpha \le \mu$, then $p_\alpha \wedge v \wedge r \le p_\alpha \wedge v \le p_\alpha \le p_\mu$, and again $p_\mu Cq$. Therefore, $p_\mu Cq$ for all $\mu \in \mathbb{Q}$, and it follows from Theorem 4.3 that $g \in C(q)$. Again, let $\lambda \in \mathbb{Q}$ with $\lambda < \alpha$. As $q \le (p_\alpha \wedge v \wedge r)'$, we have $p_\alpha \wedge v \wedge r \le q'$. Thus $p_\lambda \le p_\alpha \wedge v \wedge r \le q'$, so $q \le (p_\lambda)'$. Also, $q \le p_\alpha$, so Lemma 4.5 (ii) implies that $q \le d_\alpha = v'$. But, $q \le v$, so $q \le v, v'$, and it follows that $q = 0$. □

By Theorem 4.4 (ii), if G is archimedean, $\alpha \in \mathbb{Q}$, and $(p_\lambda)_{\lambda\in\mathbb{Q}}$ fails to be "continuous from the left" at α, then the eigenprojection d_α is the "jump" $p_\alpha - \bigvee_{\alpha>\lambda\in\mathbb{Q}} p_\lambda$ that occurs as $\lambda \uparrow \alpha$ in $\mathbb{Q}$.

The following theorem shows that the rational spectral resolution and the family of rational eigenprojections behave as expected under reduction.

THEOREM 4.5. *Let* $0 \ne v \in P$, *let* $H := J_v(G)$ *be organized into a Rickart comgroup with unit* v *and the induced partial order, and suppose that* $g \in C(v)$. *Then the rational spectral resolution of* $J_v(g)$ *in* H *is* $(p_\lambda \wedge v)_{\lambda\in\mathbb{Q}}$ *and the family of rational eigenprojections of* $J_v(g)$ *in* H *is* $(d_\lambda \wedge v)_{\lambda\in\mathbb{Q}}$.

Proof: Let $m, n \in \mathbb{Z}$ and $n > 0$. Then, $ng - mu \in C(v)$, and $nJ_v(g) - mv = J_v(ng - mu)$ and the desired results follow directly from Theorem 3.2 and Lemma 3.2. □

5. The spectrum in the archimedean case

We maintain the standing assumptions and notation of Section 4. Furthermore, in this section, *we assume that* G *is archimedean.*

DEFINITION 5.1. If $\rho \in \mathbb{R}$, we say that ρ belongs to the *resolvent set* of g iff there is an open interval $I \subseteq \mathbb{R}$ with $\rho \in I$ such that p_λ is constant on I, i.e., $p_\lambda = p_\mu$ for all $\lambda, \mu \in \mathbb{Q} \cap I$. If there is a positive real number ϵ such that p_λ is constant on each of the open intervals $(\rho - \epsilon, \rho)$ and $(\rho, \rho + \epsilon)$, we say that ρ belongs to the *relative resolvent set* of g. The *spectrum* of g, in symbols spec(g), is defined to be the complement in $\mathbb{R}$ of the resolvent set of g.

Clearly, the resolvent set of g is contained in the relative resolvent set of g, and a real number ρ belongs to the relative resolvent set of g, but not to the resolvent set of g, iff it is an isolated point of spec(g), i.e., $\rho \in \mathrm{spec}(g)$, but ρ is not an accumulation point of spec(g).

THEOREM 5.1. *The spectrum* spec(g) *of* g *is a closed nonempty subset of the closed interval* $[L, U] \subseteq \mathbb{R}$, *with* $L = \inf(\mathrm{spec}(g))$, *and* $U = \sup(\mathrm{spec}(g))$.

Proof: Follows directly from parts (v) and (vi) of Theorem 4.1. □

LEMMA 5.1. *Let* $\alpha \in \mathbb{Q}$, *suppose that* α *belongs to the relative resolvent set of* g, *and let* $v := (d_\alpha)'$. *Then, for all* $\lambda \in \mathbb{Q}$:

(i) *There is a positive real number* ϵ *such that* $\lambda \in (\alpha - \epsilon, \alpha) \Rightarrow p_\lambda = p_\alpha - v' = p_\alpha \wedge v$ *and* $\lambda \in [\alpha, \alpha + \epsilon) \Rightarrow p_\lambda = p_\alpha$.
(ii) $\alpha \in \mathrm{spec}(g) \Leftrightarrow \alpha$ *is a rational eigenvalue of* g.
(iii) *If* $v \neq 0$, $H := J_v(G)$ *is organized into a Rickart comgroup with unit* v, *and* $\mathrm{spec}_H(J_v(g))$ *is the spectrum of* $J_v(g)$ *as calculated in* H, *then* $\alpha \notin \mathrm{spec}_H(J_v(g))$ *and, for all* $\rho \in \mathbb{R}$ *with* $\rho \neq \alpha$, $\rho \in \mathrm{spec}_H(J_v(g)) \Leftrightarrow \rho \in \mathrm{spec}(g)$.

Proof: (i) There is a positive real number ϵ such that p_λ is constant on $(\alpha - \epsilon, \alpha)$ and on $(\alpha, \alpha + \epsilon)$. By Theorem 4.1 (iii) $\bigvee_{\alpha > \mu \in \mathbb{Q}} p_\mu$ exists and equals p_λ for every rational $\lambda \in (\alpha - \epsilon, \alpha)$; hence by Theorem 4.12 (ii), $\lambda \in (\alpha - \epsilon, \alpha) \Rightarrow p_\lambda = p_\alpha - d_\alpha = p_\alpha - v' = p_\alpha \wedge v$. Similarly, Theorem 4.1 (iii) implies that $\bigwedge_{\alpha < \mu \in \mathbb{Q}} p_\mu$ exists and equals p_λ for every rational $\lambda \in (\alpha, \alpha + \epsilon)$; hence by Theorem 4.4 (i), $\lambda \in [\alpha, \alpha + \epsilon) \Rightarrow p_\lambda = p_\alpha$.

(ii) If $\alpha \notin \mathrm{spec}(g)$, there exists a positive real number δ such that p_λ is constant on $(\alpha - \delta, \alpha + \delta)$. Choose rational numbers λ_1 and λ_2 with $\max(\alpha - \epsilon, \alpha - \delta) < \lambda_1 < \alpha < \lambda_2 < \min(\alpha + \epsilon, \alpha + \delta)$. Then $0 = p_{\lambda_2} - p_{\lambda_1} = d_\alpha$ by (i). Conversely, if $d_\alpha = 0$, then p_λ is constant on $(\alpha - \epsilon, \alpha + \epsilon)$ by (ii), whence $\alpha \notin \mathrm{spec}(g)$. Therefore, $\alpha \in \mathrm{spec}(g) \Longleftrightarrow d_\alpha \neq 0$. By Definition 4.1, $d_\alpha \neq 0$ iff α is a rational eigenvalue of g.

(iii) Since $v \neq 0$, Theorem 4.5 implies that the rational spectral resolution of $J_v(g)$ in H is $(p_\lambda \wedge v)_{\lambda \in \mathbb{Q}}$. By (i), $\lambda \in (\alpha - \epsilon, \alpha + \epsilon) \Rightarrow p_\lambda \wedge v = p_\alpha \wedge v$, so $p_\lambda \wedge v$ is constant on $(\alpha - \epsilon, \alpha + \epsilon)$, whence $\alpha \notin \mathrm{spec}_H(J_v(g))$.

By Lemma 4.4 (iii), if $\lambda \in \mathbb{Q}$, then $\lambda < \alpha \Rightarrow p_\lambda = p_\lambda \wedge v$ and $\alpha \leq \lambda \Rightarrow p_\lambda = (p_\lambda \wedge v) \vee v'$. Therefore, if $\alpha \neq \rho \in \mathbb{R}$, $I \subseteq \mathbb{R}$ is an open interval with $\rho \in I$ and $\alpha \notin I$, then p_λ is constant on I iff $p_\lambda \wedge v$ is constant on I. □

THEOREM 5.2. *Let $\alpha \in \mathbb{Q}$. Then:*

(i) *If α is an eigenvalue of g, then $\alpha \in$* spec(g).
(ii) *If α is an isolated point of* spec(g), *then α is a rational eigenvalue of g.*

Proof: (i) If $\alpha \in \mathbb{Q}$ and $d_\alpha \neq 0$, then α cannot belong to the resolvent set of g, else it would belong to the relative resolvent set of g, contradicting Lemma 5.1 (ii). Part (ii) also follows from Lemma 5.1 (ii). □

LEMMA 5.2. *Let $a, b \in \mathbb{Z}$ with $b > 0$ and let $\alpha := a/b$. Then the following conditions are mutually equivalent:*

(i) $d_\alpha = u$.
(ii) $bg = au$.
(iii) $\mathrm{spec}(g) = \{\alpha\}$.

Proof: That (i) $\Longleftrightarrow$ (ii) follows from Lemma 4.4 (i).

(ii) $\Longrightarrow$ (iii). Assume that $bg = au$ and let $m, n \in \mathbb{Z}$ with $n > 0$. Suppose that $m/n \leq \alpha$, i.e., $mb < na$. Then $mbu \leq nau = nbg$, and since G is unperforated, $mu \leq ng$. Therefore, by Definition 3.2, $m/n \leq \alpha \Longrightarrow m/n \leq L$, i.e., $\alpha \leq L$. Similarly, $U \leq \alpha$, whence $L = U = \alpha$, and it follows from Theorem 5.1 that $\mathrm{spec}(g) = \{\alpha\}$.

(iii) $\Longrightarrow$ (ii). Suppose that $\mathrm{spec}(g) = \{\alpha\}$ and let $v := (d_\alpha)' = (bg - au)''$. If $bg \neq au$, then $v \neq 0$, so by Lemma 5.1 (iii), $\mathrm{spec}_H(J_v(g))$ is empty, contradicting Theorem 5.1. □

LEMMA 5.3. *Let $a, b \in \mathbb{Z}$ with $b > 0$, let $\alpha := a/b$, suppose that α is an isolated point of* spec(g), *and let $v := (d_\alpha)'$. Then:*

(i) *$v' \neq 0$, so we can and do organize $K := J_{v'}(G)$ into a Rickart comgroup with unit v'.*
(ii) *The spectrum of $J_{v'}(g)$ as calculated in K is* $\mathrm{spec}_K(J_{v'}(g)) = \{\alpha\}$.
(iii) $bJ_{v'}(g) = av'$.

Proof: (i) That $v' = d_\alpha \neq 0$ follows from Theorem 5.2 (ii).

(ii) By Theorem 4.5, the rational spectral resolution of $J_{v'}(g)$ in K is $(p_\lambda \wedge v')_{\lambda\in\mathbb{Q}}$. By Lemma 4.4 (iii), $p_\lambda \leq v$ for $\lambda < \alpha$ and $v' \leq p_\lambda$ for $\alpha \leq \lambda$, whence $p_\lambda \wedge v' = 0$ for $\lambda < \alpha$ and $p_\lambda \wedge v' = v' \neq 0$ for $\alpha \leq \lambda$. Therefore, $\mathrm{spec}_K(J_{v'}(g)) = \{\alpha\}$.

(iii) That $bJ_{d_\alpha}(g) = ad_\alpha$ follows from (ii) and Lemma 5.2 applied to the Rickart comgroup K. □

THEOREM 5.3. *Suppose that* $\mathrm{spec}(g) = \{\alpha_1, \alpha_2, \ldots, \alpha_N\}$ *with $\alpha_i \in \mathbb{Q}$ for $i = 1, 2, \ldots, N$. Then $d_{\alpha_i} \neq 0$ for $i = 1, 2, \ldots, N$ and $\sum_{i=1}^N d_{\alpha_i} = u$. Moreover, if $b > 0$ and a_i are integers such that $\alpha_i = a_i/b$ for $i = 1, 2, \ldots, N$, then $bg = \sum_{i=1}^N a_i d_{\alpha_i}$.*

Proof: The proof is by induction on N. Lemma 5.2 takes care of the case $N = 1$, so we can and do assume that $N > 1$. Clearly, the rational numbers $\alpha_1, \alpha_2, \dots, \alpha_N$ are isolated points of the spectrum of g. Let $\alpha := \alpha_N = a_N/b$ and let $v := (d_\alpha)' = (bg - a_N u)''$. Then $v \neq 0$, else $bg = a_N u$, and $\operatorname{spec}(g) = \{\alpha_N\}$ by Lemma 5.2, contradicting $N > 1$. Thus, by Lemma 5.2 (iii), $H := J_v(G)$ is an archimedean Rickart comgroup and $\operatorname{spec}_H(J_v(g)) = \{\alpha_1, \alpha_2, \dots, \alpha_{N-1}\}$. Also, by Theorem 4.1 (iv), $d_{\alpha_i} \leq (d_\alpha)' = v$, i.e., $d_{\alpha_i} \wedge v = d_{\alpha_i}$, for $i = 1, 2, \dots, N-1$. Therefore, by the induction hypothesis,

$$d_{\alpha_i} \neq 0 \text{ for } i = 1, 2, \dots, N-1, \ \sum_{i=1}^{N-1} d_{\alpha_i} = v, \text{ and } bJ_v(g) = \sum_{i=1}^{N-1} a_i d_{\alpha_i}.$$

Also, by Lemma 5.3, $d_{\alpha_N} = d_\alpha \neq 0$ and $bJ_{d_{\alpha_N}}(g) = a_N d_{\alpha_N}$. Consequently, $\sum_{i=1}^N d_{\alpha_i} = v + v' = u$ and $bg = b(J_v(g) + J_{v'}(g)) = \sum_{i=1}^{N-1} a_i d_{\alpha_i} + a_N d_{\alpha_N} = \sum_{i=1}^N a_i d_{\alpha_i}$. □

Theorem 5.3 has the following converse.

THEOREM 5.4. *Let $0 < b \in \mathbb{Z}$, suppose that $a_i \in \mathbb{Z}$ are distinct integers, and define $\alpha_i := a_i/b$ for $i = 1, 2, \dots, N$. Let $0 \neq u_i \in P$ for $i = 1, 2, \dots N$ with $\sum_{i=1}^N u_i = u$, and suppose that $bg = \sum_{i=1}^N a_i u_i$. Then $\operatorname{spec}(g) = \{\alpha_1, \alpha_2, \dots, \alpha_N\}$ and $u_i = d_{\alpha_i}$ for $i = 1, 2, \dots, N$.*

Proof: The proof is by induction on N. Lemma 5.2 takes care of the case $N = 1$, so we can and do assume that $N > 1$. Also, we can and do assume that $a_N < a_{N-1} < \dots < a_1$. Let $v := \sum_{i=1}^{N-1} u_i$. As $\sum_{i=1}^N u_i = u$, we have $v = (u_N)'$, so $v' = u_N \neq 0$. Also, $0 < u_1 \leq v$ implies that $v \neq 0$.

Suppose $\lambda \in \mathbb{Q}$ with $\alpha_N \leq \lambda < \alpha_{N-1}$, and select $m, n \in \mathbb{Z}$ with $0 < n$ and $\lambda = m/n$. Then $na_N \leq mb < na_i$ for $i = 1, 2, .., N-1$. Also, for $i = 1, 2, \dots, N-1$, $J_v(u_i) = u_i$ and $J_{v'}(u_i) = 0$, whereas $J_v(u_N) = 0$ and $J_{v'}(u_N) = u_N$. Consequently, $bJ_v(g) = \sum_{i=1}^{N-1} a_i u_i$ and $bJ_{v'}(g) = a_N u_N$, so $b(J_v(g) + J_{v'}(g)) = bg$. Since $b \neq 0$ and G is torsion free, it follows that $g \in C(v)$. Furthermore, $J_v(nbg - mbu) = \sum_{i=1}^{N-1}(na_i - mb)u_i \geq 0$ and $J_{v'}(nbg - mbu) = (na_N - mb)u_N \leq 0$, so $(nbg - mbu)^+ = \sum_{i=1}^{N-1}(na_i - mb)u_i$. By parts (x) and (ii) of Lemma 3.1 and the fact that $\sum_{i=1}^N u_i = u$, it follows that $((nbg - mbu)^+)'' = \bigvee_{i=1}^{N-1}((na_i - mb)u_i)'' = \bigvee_{i=1}^{N-1} u_i = \sum_{i=1}^{N-1} u_i = v$. Hence, $p_\lambda = ((nbg - mbu)^+)' = v'$, so p_λ is constant on $[\alpha_N, \alpha_{N-1})$.

We have $bg - a_N u = \sum_{i=1}^N a_i u_i - \sum_{i=1}^N a_N u_i = \sum_{i=1}^{N-1}(a_i - a_N)u_i$ with $0 < a_i - a_N$ for $i = 1, 2, \dots, N-1$, whence $0 \leq bg - a_N u$. Also, by parts (x) and (ii) of Lemma 3.1 again, $(bg - a_N u)'' = \bigvee_{i=1}^{N-1}((a_i - a_N)u_i)'' = \bigvee_{i=1}^{N-1} u_i = \sum_{i=1}^{N-1} u_i = v$. Therefore, $d_{\alpha_N} = (bg - a_N u)' = v' = u_N \neq 0$. Consequently, α_N is a rational eigenvalue of g, hence $\alpha_N \in \operatorname{spec}(g)$ by Theorem 5.2 (i). As $0 \leq bg - a_N u$, it follows from Definition 3.2 that $\alpha_N \leq L$, hence that $(-\infty, \alpha_N)$ is contained in the resolvent set of g by Theorem 5.1. Therefore, α_N is an isolated point of $\operatorname{spec}(g)$.

We organize $H := J_v(G)$ into an archimedean Rickart comgroup with unit v under the induced partial order. If $i = 1, 2, \dots, N-1$, we have $u_i \leq v$, whence $0 \neq u_i \in P_H$ with $\sum_{i=1}^{N-1} u_i = v$. Moreover, $bJ_v(g) = \sum_{i=1}^{N-1} a_i u_i$, so by the induction hypothesis and Theorem 4.5, $\operatorname{spec}_H(J_v(g)) = \{\alpha_1, \alpha_2, \ldots, \alpha_{N-1}\}$ and $u_i = d_{\alpha_i} \wedge v$ for $i = 1, 2, \ldots, N-1$. But, if $i = 1, 2, \ldots, N-1$, then $\alpha_N < \alpha_i$, so by Theorem 4.1 (iv), $v' = d_{\alpha_N} \leq (d_{\alpha_i})'$, i.e., $d_{\alpha_i} \leq v$, whence $u_i = d_{\alpha_i} \wedge v = d_{\alpha_i}$. As $u_N = d_{\alpha_N}$, we have $u_i = d_{\alpha_i}$ for $i = 1, 2, \ldots, N$. Finally, by Lemma 5.2 (iii), $\operatorname{spec}(g) = \{\alpha_1, \alpha_2, \ldots, \alpha_{N-1}, \alpha_N\}$. □

THEOREM 5.5.
(i) $g \in G^+ \Leftrightarrow \operatorname{spec}(g) \subseteq [0, \infty)$.
(ii) $g \in E \Leftrightarrow \operatorname{spec}(g) \subseteq [0, 1]$.
(iii) $g \in P \Leftrightarrow \operatorname{spec}(g) \subseteq \{0, 1\}$.

Proof: (i) If $g \in G^+$, then $mu \leq ng$ holds for all integers $m, n \in \mathbb{Z}$ with $m < 0 < n$, whence $0 \leq L$ by Definition 3.2. Conversely, if $0 \leq L$, then by Theorem 3.5 (ii), $0 \leq \omega(g)$ for all $\omega \in \Omega$, and since G is archimedean, it follows that $g \in G^+$. Hence, $g \in G^+ \iff 0 \leq L$, and (i) follows from Theorem 5.1.

(ii) If $g \leq u$ then $U \leq 1$ by Definition 3.2. Conversely, if $U \leq 1$, then by Theorem 3.5 (ii), $\omega(g) \leq 1 = \omega(u)$ for all $\omega \in \Omega$, whence $g \leq u$. Therefore, by Theorem 5.1, $g \leq u \iff \operatorname{spec} g \subseteq (-\infty, 1]$, and (ii) now follows from (i).

(iii) By Lemma 5.2, $g = 0 \iff \operatorname{spec}(g) = \{0\}$ and $g = u \iff \operatorname{spec}(g) = \{1\}$, so we can and do assume that $g \neq 0, 1$. If g is a projection, put $u_1 := g'$, $u_2 := g$, $b := 1$, $a_1 := 0$ and $a_2 := 1$ in Theorem 5.4 to conclude that $\operatorname{spec}(g) = \{0, 1\}$. Conversely, if $\operatorname{spec} g = \{0, 1\}$, then Theorem 5.3 implies that $g = d_1 \in P$. □

DEFINITION 5.2. g is *singular* iff $0 \in \operatorname{spec}(g)$, otherwise g is *nonsingular.*

THEOREM 5.6. *For $g \in G^+$, the following conditions are mutually equivalent:*

(i) *g is nonsingular.*
(ii) $0 < L$.
(iii) *$0 < \omega(g)$ for all $\omega \in \Omega$.*
(iv) *g is an order unit in G.* (v) *There is a positive integer N such that $u \leq Ng$.*

Proof: (i) $\iff$ (ii). Follows from Theorems 5.1 and 5.2 (i). (ii) $\iff$ (iii). Follows from parts (ii) and (iii) of Theorem 3.5. (iii) $\iff$ (iv). Follows from [6, Corollary 4.13]. (iv) $\iff$ (v). As u is an order unit, it follows that g is an order unit iff (v) holds. □

To motivate the next definition, suppose that T is a bounded self-adjoint operator on a Hilbert space $\mathfrak{H}$. Then T is nonsingular in the sense of Definition 5.2 iff it is invertible. Also, T'' is the projection onto the closure $\mathfrak{M} := \overline{T(\mathfrak{H})}$ of the range $T(\mathfrak{H})$ of T, and T' is the projection onto the null space $T^{-1}(0)$ of T. Thus, $T(\mathfrak{H})$ is dense in $\mathfrak{H}$ iff $T' = \mathbf{0}$, and T is invertible iff its range is both dense and closed in $\mathfrak{H}$. The operator T is reduced by $\mathfrak{M}$, the restriction $T_{\mathfrak{M}}$ of T to the Hilbert space $\mathfrak{M}$ is bounded and self-adjoint, and the range of T is the same as the range of $T_{\mathfrak{M}}$, whence the range of $T_{\mathfrak{M}}$ is automatically dense in $\mathfrak{M}$. Consequently, $T_{\mathfrak{M}}$ is invertible (as an operator on $\mathfrak{M}$) iff the range of T is closed in $\mathfrak{H}$.

In generalizing the operator-theoretic notions in the last paragraph to the Rickart comgroup G, as we do in the next definition, we note that $g = J_{g''}(g)$, hence if $g \neq 0$, then $g'' \neq 0$ and g belongs to the Rickart comgroup $H := J_{g''}(G)$.

DEFINITION 5.3. We say that g is *range dense* iff $g'' = u$ (i.e., iff $g' = 0$). We say that g is *range closed* iff either $g = 0$ (i.e., $g'' = 0$) or $g \neq 0$ and g is a nonsingular element of the Rickart comgroup $H := J_{g''}(G)$.

THEOREM 5.7.
(i) *If $g \neq 0$, then g is range dense in the Rickart comgroup $J_{g''}(G)$.*
(ii) *g is range closed iff 0 belongs to the relative resolvent set of g.*
(iii) *g is nonsingular iff it is both range dense and range closed.*
(iv) *If $g \in G^+$, then g is range closed iff there is a positive integer N such that $g'' \leq Ng$.*

Proof: (i) Since $g \neq 0$, we have $g'' \neq 0$, so $J_{g''}(G)$ is a Rickart comgroup with unit g''. Thus, (i) follows from the facts that $H = J_{g''}(G)$ is closed under the iterated Rickart mapping and g'' is the unit in H.

(ii) If $g = 0$, then $\mathrm{spec}(g) = \{0\}$, so 0 is an isolated point in $\mathrm{spec}(g)$, i.e., 0 belongs to the relative resolvent set of g. Suppose $g \neq 0$. Then g is range closed iff g is nonsingular as an element of $H = J_{g''}(G)$, i.e., iff $0 \notin \mathrm{spec}_H(g)$. If 0 belongs to the relative resolvent set of g, then by Lemma 5.1 (iii) with $\alpha = 0$ and $v = (d_0)' = g''$, it follows that $0 \notin \mathrm{spec}_H(g)$. Conversely, suppose $0 \notin \mathrm{spec}_H(g)$, i.e., by Theorem 4.5, there is a positive real number ϵ such that $p_\lambda \wedge g''$ is constant on $(-\epsilon, \epsilon)$. By Lemma 4.4, $\lambda \in (-\epsilon, 0) \cap \mathbb{Q} \Longrightarrow p_\lambda = p_\lambda \wedge g''$, so p_λ is constant on $(-\epsilon, 0)$. Also, by Lemma 4.4, $\lambda \in (0, \epsilon) \cap \mathbb{Q} \Longrightarrow p_\lambda = (p_\lambda \wedge g'') \vee g'$, so p_λ is constant on $(0, \epsilon)$, and it follows that 0 belongs to the relative resolvent set of g.

(iii) Suppose g is nonsingular, i.e., there exists a positive real number ϵ such that p_λ is constant on $(-\epsilon, \epsilon)$. By (ii), g is range closed. Choose a positive rational number $\lambda < \epsilon$. Then, by Lemma 4.4, $g' \leq p_\lambda = p_{-\lambda} \leq g''$, whence $g' = g' \wedge g'' = 0$, i.e., g is range dense.

Conversely, suppose g is both range dense and range closed. As g is range dense, we have $g' = 0$ and $g'' = u$. As g is range closed, (ii) implies that 0 belongs to the relative resolvent set of g. Therefore, by Lemma 5.1 with $\alpha = 0$, there exists a real number $\epsilon > 0$ such that $\lambda \in (-\epsilon, 0) \cap \mathbb{Q} \Longrightarrow p_\lambda = p_0 - g' = p_0$ and $\lambda \in (0, \epsilon) \cap \mathbb{Q} \Longrightarrow p_\lambda = p_0$, i.e., p_λ is constant on $(-\epsilon, \epsilon)$, so g is nonsingular.

(iv) We can assume that $g \neq 0$. By Theorem 5.6 (iv), g is nonsingular in $H := J_{g''}(G)$ iff there is a positive integer N such that $g'' \leq Ng$. □

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