Notes on incompatibility witnesses and free spectrahedra

Anna Jenčová

October 12, 2018

Here are some preliminary remarks on the relation of the papers [1, 2] and [3]. Only only two-outcome measurements (effects) ared considered, the results for more outcomes are obtained similarly.

1 Compatibility of effects in GPT

We first recall briefly the notation of [3]. Here a state space K is a compact convex subset of a finite dimensional vector space, $A(K)$ is the real vector space of affine functions $K \to \mathbb{R}$, with the cone $A(K)^+$ of positive functions. Then $(A(K), A(K)^+)$ is an ordered vector space, with an order unit 1_K - the constant 1. The elements $f \in A(K)$, $0 \leq f \leq 1_K$, are called effects. The set of effects is denoted by $E(K)$.

Let $V(K) = A(K)^*$ and $V(K)^+ = (A(K)^+)^*$ be the dual space and cone, then K can be identified with a base of $V(K)^+$, consisting of positive linear functionals satisfying $\langle \varphi, 1_K \rangle = 1$ and with this identification, $V(K) = \mathrm{span}(K)$.

For $g \in \mathbb{N}$, let Δ_g be the g-dimensional simplex, with vertices δ_i^g. Let $\Box_g = \Delta_1^g$ be the g-dimensional hypercube, with vertices $s_{n_1,\dots,n_g} = (\delta^1_{n_1}, \dots, \delta^1_{n_g})$, $n_i \in \{0, 1\}$. The projections $\pi_j : \Box_g \to \Delta_1$ onto the j-th component define effects $m_j \in E(\Box_g)$, given as

$$m_j(s_{n_1,\dots,n_g}) = \begin{cases} 1 & \text{if } n_j = 0 \\ 0 & \text{otherwise} \end{cases}$$

For ordered vector spaces (L_1, L_1^+) and (L_2, L_2^+), $L_1^+ \otimes_{min} L_2^+$ and $L_1^+ \otimes_{max} L_2^+$ denote the minimal and maximal tensor product, respectively.

Let $x_1, \dots, x_N \in V(K)$ and $e_1, \dots, e_N \in A(K)$ be a pair of dual bases of the vector spaces $V(K)$ and $A(K)$. Put

$$\chi_K = \sum_i x_i \otimes e_i,$$

then $\chi_K \in V(K)^+ \otimes_{max} A(K)^+$ and this element does not depend on the choice of the bases. For any (L, L^+), positive linear maps $(A(K), A(K)^+) \to (L, L^+)$ can be identified with elements in $V(K)^+ \otimes_{max} L^+$ by $Y \mapsto (id \otimes Y)(\chi_K)$.

1.1 Compatibility and ETB maps

Let $g \in \mathbb{N}$ and let $f_1, \dots f_g \in E(K)$ be a collection of effects. Let us denote

$$F := (f_1, \dots, f_g) : K \to \Box_g, \qquad F(x) = (f_1(x), \dots, f_g(x)).$$

By [3, Thm. 1], $f_1, \dots f_g$ are compatible iff F is entanglement breaking (ETB). The following lemma lists various expressions of this condition.

Lemma 1. *Let $F^* : A(\square_g) \to A(K)$ be the adjoint map. The following conditions are equivalent.*

(i) F is ETB.

(ii) F^* is ETB.

(iii) For any ordered vector space (L, L^+) and any $z \in A(\square_g)^+ \otimes_{max} L^+$, $(F^* \otimes id)(z)$ is separable.

(iv) For any $z \in A(\square_g)^+ \otimes_{max} V(K)^+$, $(F^* \otimes id)(z)$ is separable.

(v) For any $z \in A(\square_g)^+ \otimes_{max} V(K)^+$, $\langle \chi_K, (F^* \otimes id)(z) \rangle \geq 0$.

Let now (L, L^+) be an ordered vector space. We want to characterize the cone $A(\square_g)^+ \otimes_{max} L^+$. According to [3, Lemma 1], the elements

$$\{1, 2m_1 - 1, \dots, 2m_g - 1\}$$

form a basis of of $A(\square_g)$, here $1 = 1_{\square_g}$, so that any $z \in A(\square_g) \otimes L$ has the form

$$z = 1 \otimes z_0 + \sum_{j=1}^{g} (2m_j - 1) \otimes z_j, \qquad z_0, \dots, z_g \in L. \tag{1}$$

The element z defines a map $W : V(\square_g) \to L$ by

$$\langle W(s), x^* \rangle = \langle z, s \otimes x^* \rangle, \qquad \forall s \in \square_g,\ x^* \in L^*.$$

It is well known that $z \in A(\square_g)^+ \otimes_{max} L^+$ iff this map is positive iff the images of all vertices $w_{n_1,\dots,n_g} := W(s_{n_1,\dots,n_g}) \in L^+$ for all $(n_1, \dots, n_g) \in \{0,1\}^g$. It is easily seen that

$$w_{n_1,\dots,n_g} = z_0 + \sum_{j=1}^{g} (-1)^{n_j} z_j. \tag{2}$$

Equivalently, W can be seen as an affine map of the hypercube $\square_g$ into L^+, such a map has a unique linear extension to $V(\square_g)$. Conversely, let W be such a map. Let z_0 be the image of the barycenter of $\square_g$ under W and let $2z_i$ be the images of the edges adjacent to the vertex $s_{1,\dots,1}$, so that

$$z_i = \frac{1}{2}(w_{1,1,\dots,0,1,\dots,1} - w_{1,\dots,1}), \quad 0 \text{ at the } i\text{-th place}.$$

Then (2) holds.

We have established a 1-1 correspondence between the following three sets:

(1) The set

$$\mathcal{D}_{g,L} := \{z_0, z_1 \dots, z_g,\ z_0 + \sum_{j=1}^{g} (-1)^{n_j} z_j \in L^+,\ \forall (n_1, \dots, n_g) \in \{0,1\}^g\};$$

(2) the convex cone $A(\Box_g)^+ \otimes_{max} L^+$;

(3) the set of affine maps $\Box_g \to L^+$.

Using these identifications and Lemma 1, we obtain the following characterizations of compatibility.

Proposition 1. *Let $F = (f_1, \dots, f_g)$. The following conditions are equivalent.*

(i) The collection of effects $(f_1, \dots, f_g)$ is compatible;

(ii) For any (L, L^+) and any tuple $(z_0, \dots, z_g) \in \mathcal{D}_{g,L}$, the element

$$1_K \otimes z_0 + \sum_{j=1}^{g} (2f_j - 1) \otimes z_j$$

is separable;

(iii) For any tuple $(z_0, \dots, z_g) \in \mathcal{D}_{g,V(K)}$, the element

$$1_K \otimes z_0 + \sum_{j=1}^{g} (2f_j - 1) \otimes z_j$$

is separable;

(iv) For any affine map $W : \Box_g \to V(K)^+$, $\mathrm{Tr}\, FW \geq 0$.

Proof. It is not difficult to see that $F^*(m_j) = f_j$, for $j = 1, \dots, g$, so that

$$(F^* \otimes id)(z) = 1_K \otimes z_0 + \sum_{j=1}^{g} (2f_j - 1) \otimes z_j.$$

The equivalence of (i)–(iii) follows by [3, Thm. 1] and Lemma 1. The condition (iv) was proved in [3]. $\Box$

1.2 Incompatibility witnesses

In [3], an incompatibility witness is defined as an affine map $W : \Box_g \to V(K)^+$, such that $\mathrm{Tr}\, FW < 0$. It would be also possible to define a witness as a tuple in $\mathcal{D}_{g,L}$, such that the corresponding element is not separable. These two concepts are not exactly the same, but closely related: Let a map W be a witness and let $z \in A(\Box_g)^+ \otimes_{max} V(K)^+$ be the corresponding element. Then

$$\langle \chi_K, (F^* \otimes id)(z) \rangle = \mathrm{Tr}\, FW < 0,$$

so that $(F^* \otimes id)(z)$ cannot be separable, hence the corresponding tuple $(z_0, \dots, z_g) \in \mathcal{D}_{g,V(K)}$ is a witness as well. Conversely, let $(z_0, \dots, z_g) \in \mathcal{D}_{g,L}$ be a witness, so that $(F^* \otimes id)(z)$ is not separable, then there is some element $y \in V(K)^+ \otimes_{max} (L^+)^*$ such that $\langle y, (F^* \otimes id)(z) \rangle < 0$. There is a unique positive map $Y : A(K) \to L^*$, such that $y = (id \otimes Y)(\chi_K)$ and we have

$$0 > \langle y, (F^* \otimes id)(z) \rangle = \langle (id \otimes Y)(\chi_K), (F^* \otimes id)(z) \rangle = \langle \chi_K, (F^* \otimes Y^*)(z) \rangle = \mathrm{Tr}\, FY^*W,$$

where $W : \Box_g \to L^+$ is the affine map given by the tuple. Hence Y^*W is a witness in the sense of [3].

2 Compatibility of quantum effects

In this paragraph, we assume that K is a quantum state space, that is, the set of $d \times d$ density matrices. The quantum effects $f \in E(K)$ can be identified with operators $E \in M_d^{sa}$, $0 \le E \le I_d$.

We first note the relation between the set $\mathcal{D}_{g,L}$ for $(L, L^+) = (M_n^{sa}, M_n^+)$ with the matrix diamond $\mathcal{D}_{\diamond,g}(n)$, see [1]:

Lemma 2. *Let $(z_0, \dots, z_g) \in (M_n^{sa})^g$. Then $(z_0, \dots, z_g) \in \mathcal{D}_{g,M_n^{sa}}$ iff $z_0 \ge 0$ and $z_i = z_0^{1/2} x_i z_0^{1/2}$, $i = 1, \dots, g$, for some tuple $(x_1, \dots, x_g) \in \mathcal{D}_{\diamond,g}(n)$.*

Proof. Let $(z_0, \dots, z_g) \in \mathcal{D}_{g,M_n^{sa}}$. Note that

$$z_0 = \frac{1}{2^g} \sum_{(n_1,\dots,n_g)\in\{0,1\}^g} w_{n_1,\dots,n_g},$$

hence $z_0 \ge 0$, moreover, all $w_{n_1,\dots,n_g}$ have supports contained in $\mathrm{supp}(z_0) =: p$. It follows that $z_i = p z_i p$ for all $i = 0, \dots, g$. Put

$$x_i = z_0^{-1/2} z_i z_0^{-1/2}, \qquad i = 1, \dots, g,$$

where the inverse is restricted to p. Then by the definition of $\mathcal{D}_{g,L}$,

$$\sum_{i=1}^g \epsilon_i x_i \le p \le I_n, \qquad \forall (\epsilon_1, \dots, \epsilon_g) \in \{-1, 1\}^g$$

so that $(x_1, \dots, x_g) \in \mathcal{D}_{\diamond,g}(n)$, moreover, $z_0^{1/2} x_i z_0^{1/2} = p z_i p = z_i$. The converse is clear.

□

The following statement was proved in [1]. We will show how it is proved from the above results for GPT.

Proposition 2. *A collection $(E_1, \dots, E_g)$ of quantum effects is compatible iff for any n and any tuple $(x_1, \dots, x_g) \in \mathcal{D}_{\diamond,g}(n)$, we have*

$$\sum_{i=1}^g (2E_i - I_d) \otimes x_i \le I_{dn}.$$

Proof. By symmetry of the matrix diamond, the condition of the theorem is equivalent to

$$I_d \otimes I_n + \sum_{i=1}^g (2E_i - I_d) \otimes x_i \ge 0$$

for all tuples in $\mathcal{D}_{\diamond,g}(n)$. Assume that this is satisfied for all n. We will use the condition (v) of Lemma 1. Note that in this case, $V(K)^+ \simeq A(K)^+ \simeq M_d^+$ and with this identification, χ_K corresponds to the maximally entangled state Ψ.

Take $z \in A(\square_g)^+ \otimes_{max} M_d^+$, with the corresponding tuple $(z_0, \dots, z_g) \in \mathcal{D}_{g,M_d^{sa}}$. Let $(x_1, \dots, x_g) \in \mathcal{D}_{\diamond,g}(d)$ be as in Lemma 2. Then

$$\begin{aligned}\langle \chi_K, (F^* \otimes id)(z) \rangle &= \mathrm{Tr}\,[\Psi(I_d \otimes z_0 + \sum_j (2E_i - I_d) \otimes z_i)] \\ &= \mathrm{Tr}\,[(I \otimes z_0^{1/2})\Psi(I \otimes z_0^{1/2})(I_d \otimes I + \sum_j (2E_i - I_d) \otimes x_i)] \ge 0\end{aligned}$$

by the assumption. By Lemma 1 (v), this implies that F is ETB and the collection is compatible.

For the converse, note that if $(x_1, \ldots, x_g) \in \mathcal{D}_{\diamond,g}(n)$, then $(I, x_1, \ldots, x_g) \in \mathcal{D}_{g,M_n^{sa}}$. By Proposition 1,

$$I_d \otimes I_n + \sum_{i=1}^{g} (2E_i - I_d) \otimes x_i$$

is separable, hence also positive.

□

2.1 Incompatibility witnesses

In [2], an incompatibility witness is defined as a tuple in the matrix diamond $\mathcal{D}_{\diamond,g}$. Assume that for such a tuple, $I_d \otimes I_n + \sum_{i=1}^{g}(2E_i - I_d) \otimes x_i$ is not positive. Hence there must be a pure state $\rho \in M_{dn}^+$ such that

$$\operatorname{Tr} \rho[I_d \otimes I_n + \sum_{i=1}^{g} (2E_i - I_d) \otimes x_i] < 0$$

Let $R : \mathbb{C}^n \to \mathbb{C}^d$ be a linear map such that $\rho = (I \otimes R^*)\Psi(I \otimes R)$. Then $(RR^*, Rx_1R^*, \ldots, Rx_gR^*) \in \mathcal{D}_{g,M_d^{sa}}$. Let $z \in A(\square_g)^+ \otimes_{max} M_d^+$ and $W : \square_g \to M_d^+$ be the corresponding element and affine map, then

$$\operatorname{Tr} \rho[I_d \otimes I_n + \sum_{i=1}^{g} (2E_i - I_d) \otimes x_i] = \operatorname{Tr} \Psi(F^* \otimes id)(z) = \operatorname{Tr} FW,$$

hence W is an incompatibility witness in the sense of [3]. Conversely, to any witness W there is a corresponding tuple $(x_1, \ldots, x_g) \in \mathcal{D}_{\diamond,g}(d)$ which is a witness in the sense of [2].

References

[1] A. Bluhm and I. Nechita, Joint measurability of quantum effects and the matrix diamond, arXiv:1807.01508

[2] A. Bluhm and I. Nechita, Compatibility of quantum measurements and inclusion constants for the matrix jewel, arXiv:1809.04514

[3] A. Jenčová, Incompatible measurements in a class of general probabilistic theories, Phys. Rev. A 98, 012133 (2018), arXiv:1705.08008