S. Sreekumar and M. Berta: Limit distribution theory for quantum divergences

Referee report

The paper investigates asymptotic properties of estimators of quantum divergences, namely the quantum relative entropy and two quantum versions of Rényi divergences. The estimators are obtained as the divergence $D(\hat{\rho}_n, \hat{\sigma}_n)$ for estimators $\hat{\rho}_n$ and $\hat{\sigma}_n$ of states ρ resp. σ. Under some assumption on the asymptotic behaviour of the error of the two sequences of estimators, the limit distributions for the divergences are obtained using their Taylor expansions. The results are applied to problems of multiple hypothesis testing.

Overall evaluation

The topic of the paper is very interesting and may have important applications in quantum information theory, estimation and hypothesis testing, etc. However, the present version is not well written, the arguments are unclear and there are some mistakes that do not seem crucial, but create some doubts about the other content of the paper.

A large part of the paper is devoted to the derivatives and Taylor expansions of the divergences and the involved functions. The derivatives are obtained via integral representations and checking of the regularity conditions for exchanging of differentiation and integral, or traces and integral, take up a large part of the paper. I think such derivatives were already considered before and I am sure the authors can find the necessary computations in the literature. On the other hand, very small space is devoted to the random density operators which are the object of the study, the mode of their convergence and its properties and the techniques which are used in the proofs. Such techniques may be not so widespread in the quantum information community for which this paper surely would be interesting.

Although the authors briefly describe the main ideas of the proofs in the main body of the paper, which is a good thing, the proof themselves are quite unclear. The main techniques are only sparsely mentioned and it seems that many of the steps are just skipped. For example, I do not understand how the Skorohod representation theorem is applied in the subsequence argument in the proof of the theorems, the authors just write that "This is possible by Skorohods representation theorem (see e.g. [95])", as if a rabbit was just taken out of the hat. This seems to be the crucial argument in most of the proofs and should be better explained. There are also other points, for example the use of the portmanteau theorem at the beginning of p. 19 (and similarly also in other proofs).

There are also mistakes, most notably, Lemma 1, part (i). This is quite obviously wrong, and its proof does not make any sense. This part is used also in the proof of part (ii), which, fortunately, is easy to see to be true in the case when $A \geq 0$, which is the only case when it was used in the paper. There are also further mistakes (some of them listed below), which make it difficult to trust the authors that reading through the unclear parts is worth the effort.

In conclusion, the paper should be rewritten, the techniques explained and the arguments clarified before the paper can be reviewed.

Some further comments

1. page 10, last line: what is the meaning of Λ_j^+, Λ_j^-? I would say that measuring γ_j has outcomes just ± 1

2. Eq. (19a): this is not a density operator in the case that $\|\hat{s}^{(n)}(\rho)\|_2 > 1$. The definition of $\hat{\rho}_n$ should be modified in an obvious way, which seems to be also used in the proof of Proposition 1.

3. in Eq. (19a): also the notation $\mathbb{1}_{\|s^{(n)}\|_2 \leq 1}$ etc, should better be explained.

4. Eqs. (30) (and elsewhere) it would really be better to use a notation that shows that these are also functions of t

5. page 20, last set of displayed equations: in the last line, all the "tilded" terms are equal to their "untilded" versions, except for the last one, where it is not so automatic. This is because we have $\rho_n \ll \sigma_n \ll \rho \ll P$, but $P \ll \sigma$, so I do not see that $U \log \sigma U^\dagger = \log \tilde{\sigma}$.

6. p. 33, first equation: which norm is this? (maybe it should be $\| \cdot \|_1$?)

7. p. 33, line 13 (displayed equation): the first term is not correct

8. Since the statements and their proofs are at different places in the paper, I would suggest to use one counter for all Lemmas, Propositions, Theorems, etc. Separate numbering makes them harder to find in the text.