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TWO THEOREMS ABOUT $\mathscr{C}_p$ *

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We show (1) a theorem about the continuity of multiplication with operators in $\mathscr{C}_p$; (2) strong convergence of a sequence of operators in $\mathscr{C}_p$ and their adjoints together with convergence of their p-norms implies convergence in the topology of $\mathscr{C}_p$. Applications of these theorems are found especially in quantum statistical mechanics.

1. Introduction

We recall some familiar facts about norm ideals in $\mathscr{B}(\mathscr{H})$, the space of all bounded operators on a separable Hilbert space $\mathscr{H}$. They are all contained in the maximal ideal $\mathscr{C}(\mathscr{H})$ [4]—the space of compact operators—and have been characterized by Schatten [9]. Much research has gone into the special norm ideals $\mathscr{C}_p$; a collection of their properties can be found in [1]. We recall their definition [8]:

$$A\in\mathscr{C}_p \leftrightarrow \text{there exists } M \text{ such that } \sum |\langle\psi_i|A|\psi_i\rangle|^p < M \leqslant +\infty \tag{1}$$

($\{|\psi_i\rangle\}$ any orthonormal system).

$\mathscr{C}_p$ becomes a Banach space with the following norm:

$$\|A\|_p = (\sum_i |\lambda_i|^p)^{1/p} \tag{2}$$

(we call this the *p-norm* for simplicity).

λ_i are the eigenvalues of the compact and positive operator $(A^+A)^{1/2}$. We remark that we can imbed isometrically l_p (the space of sequences $\{a_i\}$ with $\sum_i |a_i|^p < \infty$) in $\mathscr{C}_p$ given any couple of infinite orthonormal systems $\{|\psi_i\rangle\}$ and $\{|\varphi_i\rangle\}$: let $\{a_i\}$ go into $\sum_i a_i|\psi_i\rangle\langle\varphi_i|$. The dual space of $\mathscr{C}_p$ as a Banach space is $\mathscr{C}_q$ with $1/p+1/q=1$ under the identification

$$A'\in\mathscr{C}_q \leftrightarrow l_{A'}\in(\mathscr{C}_p)^*:\ l_{A'}(A)=\operatorname{tr}(A'A),\qquad A\in\mathscr{C}_p; \tag{3}$$

this is well-defined and continuous due to the well-known inequalities ([8])

$$\|AB\|_r \leqslant \|A\|_p\|B\|_q \quad (r^{-1}=p^{-1}+q^{-1}) \quad \text{and} \quad |\operatorname{tr} A| \leqslant \|A\|_1 . \tag{4}$$

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Weak convergence in $\mathscr{C}_p$ as a Banach space is equivalent to weak convergence in $\mathscr{B}(\mathscr{H})$ in the usual sense together with boundedness of the p-norms: this is due to (3) and the fact that operators of finite rank are dense in all $\mathscr{C}_p$'s in their resp. topologies ([8]).

Some notations: $\rightharpoonup$ denotes weak, $\rightarrow$ strong, $\Rightarrow$ norm, and $\xrightarrow{p}$ convergence in the topology of $\mathscr{C}_p$.

B_1 is the unit sphere in $\mathscr{B}(\mathscr{H})$. If $p=\infty$, $\mathscr{C}_\infty=\mathscr{C}$ and $\xrightarrow{\infty}$ is the same as $\Rightarrow$. (To avoid redundancy in the statement of the theorems.) $\mathscr{M}^\perp$ is the orthogonal complement of the linear subspace $\mathscr{M}$.

2. Continuity of multiplication

THEOREM 1. *The maps $(A, B) \rightarrow AB$, $(A, B) \rightarrow BA$ are continuous on $B_1 \times \mathscr{C}_p \rightarrow \mathscr{C}_p$, B_1 carrying the strong operator topology and $\mathscr{C}_p$ its norm topology.* $1 \leqslant p \leqslant \infty$; *$B_1$ can of course be replaced by any bounded set in $\mathscr{B}(\mathscr{H})$.*

Proof: We prove the theorem only for one order of the factors; the other one is similar. The restriction of the strong topology to any bounded set in $\mathscr{B}(\mathscr{H})$ is metrizable ([5]). Therefore it suffices to show that $A_nB_n \xrightarrow{p} AB$ if $A_n \rightarrow A$, $B_n \xrightarrow{p} B$. Now

$$\|A_nB_n - AB\|_p \leqslant \|A_n(B_n-B)\|_p + \|(A_n-A)B\|_p = \sup_n \|A_n\| \, \|B_n-B\|_p + \|(A_n-A)B\|_p \tag{5}$$

where we used equation (3) in the first term; it goes to 0 since $B_n \xrightarrow{p} B$ and $\sup\limits_n \|A_n\| < \infty$.

(a) $p<\infty$. We write $B=B_1+B_2$ with B_1 of finite rank and $\|B_2\|_p < \frac{1}{4}\varepsilon\,(\sup\limits_n \|A_n\|)^{-1}$. Again

$$\|(A_n-A)B\|_p \leqslant 2\sup_n \|A_n\| \, \|B_2\|_p + \|(A_n-A)B_1\|_p \tag{6}$$

since $\|A \leqslant \|\sup\limits_n \|A_n\|$ ([5]). The first term is less than ε; in the second one we write $B_1 = |B_1|\, U$ with $|B_1| \geqslant 0$ and of finite rank, U unitary.[1] Now $\|(A_n-A)\, B_1\|_p = \|(A_n-A) \cdot |B_1|\, \|_p = \|\, |B_1|\, (A_n-A)^\dagger (A_n-A)\, |B_1|\, \|_{p/2}$.[2] Now $A_n \rightarrow A$ is equivalent to $(A_n-A)^\dagger(A_n-A) \rightharpoonup 0$. But on the finite-dimensional space range $(|B_1|)$ all operator topologies are equivalent. Therefore, for n large enough, $P(A_n-A)^\dagger(A_n-A)\,P < P\,\frac{1}{2}\varepsilon\,\|B\|_p$ [3] with P the projection on range $(|B_1|)$. So $\|(A_n-A)\,B_1\|_p < \frac{1}{2}\varepsilon$ which, together with (6), implies the theorem.

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[1] $\|A\|_p = \|AU\|_p$; this can easily be seen from the formula in [8]

$$\|A\|_p^p = \sup_{\{\psi_i\}\,\{\varphi_i\}} \sum_i |\langle \psi_i \,|A|\, \varphi_i\rangle|^p$$

with $\{\psi_i\}$, $\{\varphi_i\}$ any pair of orthonormal systems.

[2] If $p<2$ this is again defined by (2).

[3] All eigenvalues of $P(A_n^\dagger - A^\dagger)(A_n-A)\,P$ go to zero; take n large enough such that the greatest eigenvalue is less than

$$\tfrac{1}{2}\varepsilon(\|B\|_p)^{-1}.$$

(b) $p=\infty$. B throws the unit sphere of $\mathscr{H}$ into a compact set. Strong operator convergence implies uniform convergence on every compact set ([10]) and $A_n \Rightarrow A$ is equivalent to $A_n|\psi\rangle \to A\,|\psi\rangle$ uniformly in $|\psi\rangle$ ranging over the unit sphere.

APPLICATION. For every increasing sequence of finite-dimensional subspaces $\mathscr{M}_n \subset \mathscr{H}$ with $\bigcup_n \mathscr{M}_n$ dense in $\mathscr{H}$, the compressions to $\mathscr{M}_n$ of every operator $A \in \mathscr{C}_p$ converge to A in the $\mathscr{C}_p$-topology.[4] This follows from $P_n \to I$, P_n projecting on $\mathscr{M}_n$; therefore $P_n A P_n \xrightarrow{p} A$. This can be applied to approximations of operators in $\mathscr{C}_p$ by finite-dimensional ones. M. Breitenecker in [1] has proven along these lines the important inequalities $\operatorname{tr} e^{A+B} \leqslant \operatorname{tr} e^A e^B$ and $|\operatorname{tr} e^{A+iB}| \leqslant \operatorname{tr} e^A$ which are very useful in the study of Green's functions with different conditions as in [3].

3. S^*-convergence and p-convergence

For our second theorem, we recall a definition:

DEFINITION. $A_n \xrightarrow{S^*} A$ *(strong $*$-convergence)* $\Leftrightarrow$

$$A_n \to A \quad \text{and} \quad A_n^\dagger \to A^\dagger .$$

(For sequences of self-adjoint operators or sequences of normal operators converging to normal operators, this concept coincides with strong convergence[5].)

THEOREM 2. *Let $A_n, A \in \mathscr{C}_p$, $1 \leqslant p < \infty$ with $A_n \xrightarrow{S^*} A$ and $\||A_n\||_p \to \||A\||_p$. Then $A_n \xrightarrow{p} A$.*

For $p=\infty$ this theorem is trivially false.

Proof: Since the product of two strongly convergent sequences is again strongly convergent,

$$A_n^\dagger A_n \to A^\dagger A \quad \text{and} \quad A_n A_n^\dagger \to A A^\dagger . \tag{7}$$

We use the spectral theorem for compact operators to write according to [8]:

$$A_n = \sum_i \lambda_i^n |\psi_i^n\rangle \langle \varphi_i^n| \quad \text{and} \quad A = \sum_i \lambda_i |\psi_i\rangle \langle \varphi_i| . \tag{8}$$

λ_i resp. λ_i^n being the square roots of the positive eigenvalues of $A^\dagger A$ resp. $A_n^\dagger A_n$, $|\varphi_i\rangle$ resp. $|\varphi_n^i\rangle$ ($|\psi_i\rangle$ resp. $|\psi_i^n\rangle$) the corresponding eigenvalues of $A^\dagger A$ resp. $A_n^\dagger A_n$ ($AA^\dagger$ resp. $A_n A_n^\dagger$). For $A_n \to A$, all A_n and A self-adjoint, we have $E_\Delta(A_n) \to E_\Delta(A)$ for every interval Δ of the real axis whose endpoints do not belong to the point spectrum of A ([5], p. 923; $E_\Delta(A)$ denoting the spectral projection of A on Δ). The compactness of all occuring operators now implies that (this argument is treated in more detail in [2], p. 18).

$$\lambda_i^n \to \lambda_i , \quad |\varphi_i^n\rangle \Rightarrow |\varphi_i\rangle , \quad |\psi_i^n\rangle \Rightarrow |\psi_i\rangle . \tag{9}$$

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[4] The compression of an operator A to a closed subspace $\mathscr{M}$ is defined as PAP, P projecting on $\mathscr{M}$ [6].

[5] The strong limit of normal operators is usually *not* normal; see [1].

It is obvious that f.i., the $|\varphi^i\rangle$ span $(\ker A)^\perp$ and the $|\psi_i\rangle$ range A; similarly for the $|\varphi_i^n\rangle$ and $|\psi_i^n\rangle$.

We now want to transform the sequence $\{A_n\}$ into a new one $\{A_n'\}$ in such a way that $\{A_n'\}$ and A lie together in the image of a definite imbedding of ℓ_p in $\mathscr{C}_p$; well-known theorems about ℓ_p will then imply the desired results. To this end we introduce two sequences of partial isometries:

$$\begin{aligned} U_n|\psi_i^n\rangle &= |\psi_i\rangle, \qquad U_n|_{(\mathrm{range}\,A)^\perp} = 0. \\ V_n^*|\varphi_i^n\rangle &= |\varphi_i\rangle, \qquad V_n^*|_{\ker A_n} \equiv 0, \end{aligned} \tag{10}$$

Obviously, range U_n = range A, range $V_n^* = (\ker A)^\perp$ and $U_n \xrightarrow{S*} P_r = P\,(\overline{\mathrm{range}\,A})$, $V_n \xrightarrow{S*} P_k = P\,(\ker A^\perp)$ where $P(\mathscr{M})$ denotes the orthogonal projection onto the closed subspace $\mathscr{M}$.

We define

$$A_n' = U_n A_n V_n^* = \sum \lambda_i^n |\psi_i\rangle\langle\varphi_i|\,. \tag{11}$$

From the assumptions of the theorem we have

$$\sum_n |\lambda_i^n|^p \to \sum_i |\lambda_i|^p. \tag{12}$$

This, together with (9) implies in ℓ_p[6] $\sum_i |\lambda_i^n - \lambda_i|^p = \|A_n' - A\|_p^p \to 0$. We now transform back to the original A_n:

$$A_n = U_n^* A_n' V_n \xrightarrow{p} P_r A P_k = A \tag{13}$$

since $U_n^* U_n = P\,(\overline{\mathrm{range}\,A_n})$, $V_n V_n^* = P\,(\ker A_n^\perp)$. Q.E.D.

Application. Let A be a self-adjoint operator, semibounded from below, with $1/(A-z) \in \mathscr{C}_r$ for some r and z. Then $e^{-A+B} \in \mathscr{C}_1$ for every bounded B [3]. If $B_n \to B$, and all B_n and B commute with A, then

$$\mathrm{tr}\, e^{-A+iB_n} \to \mathrm{tr}\, e^{-A+iB} \tag{14}$$

since, obviously, $e^{-A+iB_n} \xrightarrow{S*} e^{-A+iB}$ and

$$\|e^{-A+iB_n}\|_1 = \mathrm{tr}\, e^{-A} = \|e^{-A+iB}\|_1\,. \tag{15}$$

This result can be extended to certain non-commuting cases and used to show analyticity of the partition function.

Remark. It is known ([11]) that for positive A_n and A, Theorem 2 is true if one assumes only weak convergence of the $A_n \rightharpoonup A$.

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[6] This calculation can be found in [1].

Appendix

We show the result about convergence in l_p required for the proof of Theorem 2.

Let the sequences $\{a_i^n\}$ and $\{a_i\}$ belong to l_p $(p<\infty)$. Furthermore suppose that

$$a_i^n \to a_i \; \forall i \quad \text{and} \quad \sum_{i=1}^{\infty} |a_i^n|^p \to \sum_{i=1}^{\infty} |a_i|^p. \tag{A1}$$

Then, obviously, $\sum_{i_0}^{\infty} |a_i^n|^p \to \sum_{i_0}^{\infty} |a_i|^p$ as well, for every i_0. There is an i_0 with $\sum_{i_0}^{\infty} |a_i|^p < \frac{1}{6}\varepsilon$ and an n_0 with

$$\left|\sum_{i_0}^{\infty} |a_i^n|^p - \sum_{i_0}^{\infty} |a_i|^p\right| < \tfrac{1}{6}\varepsilon \; \forall n \geqslant n_0. \tag{A2}$$

Now

$$\sum_i |a_i^n - a_i|^p \leqslant \sum_{i=1}^{i_0-1} |a_i^n - a_i|^p + \sum_i^{\infty} |a_i^n|^p + \sum_i^{\infty} |a_i|^p. \tag{A3}$$

The second and third terms together are smaller than $\frac{1}{2}\varepsilon$ $\forall n \geqslant n_0$; the first can be made as small as desired by the individual convergence of the a_i^n. QED.

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