

On $\alpha - z$ -Rényi divergences in von Neumann algebras

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The α - z -Rényi divergences

For density operators ρ, σ on a finite dimensional Hilbert space:

$$D_{\alpha,z}(\rho\|\sigma) = \frac{1}{\alpha-1} \log \frac{\text{Tr}(\rho^{\frac{\alpha}{2z}} \sigma^{\frac{1-\alpha}{z}} \rho^{\frac{\alpha}{2z}})^z}{\text{Tr} \rho},$$

where $0 < \alpha \neq 1$ and $z > 0$.

For each $z > 0$: a quantum extension of classical Rényi α -divergences for probability vectors p, q :

$$D_{\alpha}(p\|q) = \frac{1}{\alpha-1} \log(\sum_i p_i^{\alpha} q_i^{1-\alpha}).$$

The α - z -Rényi divergences

Important special cases:

- Relative entropy:

$$\lim_{\alpha \rightarrow 1} D_{\alpha,z}(\rho \parallel \sigma) = D_1(\rho \parallel \sigma) = \frac{\text{Tr}(\rho(\log \rho - \log \sigma))}{\text{Tr} \rho}$$

- Petz-type (standard) Rényi divergence: $z = 1$, $0 < \alpha \neq 1$

$$D_{\alpha}(\rho \parallel \sigma) = \frac{1}{\alpha - 1} \log \frac{\text{Tr}(\rho^{\alpha} \sigma^{1-\alpha})}{\text{Tr} \rho}$$

- Sandwiched Rényi divergence: $0 < z = \alpha \neq 1$

$$\tilde{D}_{\alpha}(\rho \parallel \sigma) = \frac{1}{\alpha - 1} \log \frac{\text{Tr}(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}})}{\text{Tr} \rho}$$

Data processing inequality (DPI)

For a quantum channel (CPTP map) Φ and any ρ, σ :

$$D_{\alpha,z}(\Phi(\rho)\|\Phi(\sigma)) \leq D_{\alpha,z}(\rho\|\sigma)$$

- not true for all values of α, z :

- Petz-type:¹ $\alpha \in (0, 1) \cup (1, 2]$;
- sandwiched:² $\alpha \in [1/2, 1) \cup (1, \infty]$;
- general case:³

$$0 < \alpha < 1, \quad \max\{\alpha, 1 - \alpha\} \leq z$$

or

$$\alpha > 1, \quad \max\{\alpha/2, \alpha - 1\} \leq z \leq \alpha.$$

¹Ando's convexity theorem, 1979

²S. Beigi, 2013; Frank and Lieb, 2013

³Carlen, Frank and Lieb, 2018; Zhang, 2020

Outline of this talk

- extension of $D_{\alpha,z}$ to the setting of von Neumann algebras
- DPI with respect to positive trace preserving maps (within the same bounds on parameters as in finite dimensions)
- equality in DPI implies sufficiency (reversibility) for 2-positive trace preserving maps

Our tools

- variational formula for $D_{\alpha,z}$
- known results in the sandwiched case
- properties of conditional expectations

von Neumann algebra extensions

The Rényi divergences were defined for normal positive functionals ψ, φ on a von Neumann algebra, using some technical tools:

- Araki relative entropy⁴: relative modular operator $\Delta_{\psi, \varphi}$
- Petz-type (Petz quasi divergence)⁵: $\Delta_{\psi, \varphi}$
- sandwiched Rényi divergence:⁶ Araki-Masuda or Kosaki L^p -spaces
- general α - z Rényi divergences:⁷ Haagerup L^p -spaces

⁴Araki, 1976

⁵Petz, 1985

⁶Berta, Scholtz and Tomamichel, 2018; AJ, 2018; 2021

⁷Kato and Ueda, 2023; Kato, 2024

von Neumann algebras and Haagerup L^p -spaces

Let \mathcal{M} be a von Neumann algebra \mathcal{M} , with predual \mathcal{M}_* .

- Haagerup L^p -space $L^p(\mathcal{M})$, $0 < p \leq \infty$
- $\mathcal{M} = L^\infty(\mathcal{M})$, $\mathcal{M}_* \simeq L^1(\mathcal{M})$, $\varphi \mapsto h_\varphi$, $\text{tr}(h_\varphi) = \varphi(1)$
- order isomorphism: $\mathcal{M}_*^+ \ni \varphi \mapsto h_\varphi \in L^1(\mathcal{M})^+$
- polar decomposition: for $0 < p < \infty$, $k \in L^p(\mathcal{M})$, $k = u|k|$:

$u \in \mathcal{M}$ partial isometry, $|k| = h_\varphi^{1/p} \in L^p(\mathcal{M})^+$, $\varphi \in \mathcal{M}_*^+$

von Neumann algebras and Haagerup L^p -spaces

For $0 < p < \infty$, $k \in L^p(\mathcal{M})$, put $\|k\|_p = (\operatorname{tr} |k|^p)^{1/p}$.

- For $1 < p < \infty$, $\|k\|_p$ is a norm in $L^p(\mathcal{M})$, which is a reflexive Banach space, with dual $L^p(\mathcal{M})^* \simeq L^q(\mathcal{M})$, $1/p + 1/q = 1$
- $\|k\|_p$ is a quasi norm for $0 < p < 1$
- Hölder inequality: for $1/p + 1/q = 1/r$, $0 < p, q, r \leq \infty$, $h \in L^p(\mathcal{M})$, $k \in L^q(\mathcal{M})$:

$$hk \in L^r(\mathcal{M}) \quad \text{and} \quad \|hk\|_r \leq \|h\|_p \|k\|_q$$

$D_{\alpha,z}$ for von Neumann algebras

Let $0 < \alpha \neq 1$, $0 < z$. For $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$, we define⁸

$$D_{\alpha,z}(\psi\|\varphi) = \frac{1}{\alpha-1} \log \frac{Q_{\alpha,z}(\psi\|\varphi)}{\psi(1)}$$

where

$$Q_{\alpha,z}(\psi\|\varphi) := \begin{cases} \operatorname{tr} \left(h_{\varphi}^{\frac{1-\alpha}{2z}} h_{\psi}^{\frac{\alpha}{z}} h_{\varphi}^{\frac{1-\alpha}{2z}} \right)^z, & \text{if } 0 < \alpha < 1, \\ \|x\|_z^z, & \text{if } \alpha > 1 \text{ and } h_{\psi}^{\frac{\alpha}{z}} = h_{\varphi}^{\frac{\alpha-1}{2z}} x h_{\varphi}^{\frac{\alpha-1}{2z}} \\ & \text{with } x \in s(\varphi)L^z(\mathcal{M})s(\varphi), \\ \infty, & \text{otherwise.} \end{cases}$$

⁸Kato and Ueda, 2023; Kato 2024

Positive maps and the Petz dual

Let \mathcal{M}, \mathcal{N} be von Neumann algebras, $\gamma : \mathcal{N} \rightarrow \mathcal{M}$ positive unital normal map.

- The **predual map**: $\gamma_* : L^1(\mathcal{M}) \rightarrow L^1(\mathcal{N})$,

$$\gamma_*(h_\omega) := h_{\omega \circ \gamma}, \quad \text{positive, trace preserving}$$

- Let $\rho \in \mathcal{M}_*^+$, $e := s(\rho)$, $e_0 := s(\rho \circ \gamma)$. The **Petz dual** $\gamma_\rho^* : e\mathcal{M}e \rightarrow e_0\mathcal{N}e_0$ is determined by

$$(\gamma_\rho^*)_*(h_{\rho \circ \gamma}^{1/2} b h_{\rho \circ \gamma}^{1/2}) = h_\rho^{1/2} \gamma(b) h_\rho^{1/2}, \quad b \in \mathcal{N}^+.$$

- positive, unital and normal,
- n -positive whenever γ is.

DPI in von Neumann algebra setting

For any $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$, and a positive unital normal map $\gamma : \mathcal{N} \rightarrow \mathcal{M}$:

$$D_{\alpha,z}(\psi \circ \gamma \| \varphi \circ \gamma) \leq D_{\alpha,z}(\psi \| \varphi).$$

This was already proved for:

- Petz type: $\alpha \in (0, 1) \cup (1, 2]$, γ a Schwarz map⁹,
- sandwiched: $\alpha \in [1/2, 1) \cup (1, \infty]$, γ completely positive¹⁰, γ positive¹¹
- $D_{\alpha,z}$ with $0 < \alpha < 1$, $\max\{\alpha, 1 - \alpha\} \leq z$, γ positive¹²

⁹Petz, 1985

¹⁰Berta, Scholz and Tomamichel, 2018

¹¹AJ, 2018, 2021

¹²Kato, 2024

Variational expressions

Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$.

(i) Let $0 < \alpha < 1$, $\max\{\alpha, 1 - \alpha\} \leq z$. Then

$$Q_{\alpha,z}(\psi\|\varphi) = \inf_{a \in \mathcal{M}^{++}} \left\{ \alpha \operatorname{tr} \left(\left(h_{\psi}^{\frac{\alpha}{2z}} a h_{\psi}^{\frac{\alpha}{2z}} \right)^{\frac{z}{\alpha}} \right) \right. \\ \left. + (1 - \alpha) \operatorname{tr} \left(\left(h_{\varphi}^{\frac{1-\alpha}{2z}} a^{-1} h_{\varphi}^{\frac{1-\alpha}{2z}} \right)^{\frac{z}{1-\alpha}} \right) \right\}.$$

(ii) Let $\alpha > 1$, $\max\{\alpha/2, \alpha - 1\} \leq z$. Then

$$Q_{\alpha,z}(\psi\|\varphi) = \sup_{a \in \mathcal{M}^+} \left\{ \alpha \operatorname{tr} \left(\left(h_{\psi}^{\frac{\alpha}{2z}} a h_{\psi}^{\frac{\alpha}{2z}} \right)^{\frac{z}{\alpha}} \right) \right. \\ \left. - (\alpha - 1) \operatorname{tr} \left(\left(h_{\varphi}^{\frac{\alpha-1}{2z}} a h_{\varphi}^{\frac{\alpha-1}{2z}} \right)^{\frac{z}{\alpha-1}} \right) \right\}.$$

Useful inequalities

$\gamma : \mathcal{N} \rightarrow \mathcal{M}$ a normal positive unital map, $\rho \in \mathcal{M}_*^+$, $b \in \mathcal{N}^+$.

(1) If $p \in [1/2, 1)$, then

$$\left\| h_{\rho \circ \gamma}^{\frac{1}{2p}} b h_{\rho \circ \gamma}^{\frac{1}{2p}} \right\|_p \leq \left\| h_{\rho}^{\frac{1}{2p}} \gamma(b) h_{\rho}^{\frac{1}{2p}} \right\|_p.$$

Proof.

Let $\omega \in \mathcal{N}_*^+$, $h_{\omega} = h_{\rho \circ \gamma}^{\frac{1}{2}} b h_{\rho \circ \gamma}^{\frac{1}{2}}$.

$$\begin{aligned} \left\| h_{\rho}^{\frac{1}{2p}} \gamma(b) h_{\rho}^{\frac{1}{2p}} \right\|_p^p &= Q_{p,p}(\omega \circ \gamma_{\rho}^* \| \rho) = Q_{p,p}(\omega \circ \gamma_{\rho}^* \| \rho \circ \gamma \circ \gamma_{\rho}^*) \\ &\geq {}^{13} Q_{p,p}(\omega \| \rho \circ \gamma) = \left\| h_{\rho \circ \gamma}^{\frac{1}{2p}} b h_{\rho \circ \gamma}^{\frac{1}{2p}} \right\|_p^p \end{aligned}$$

□

Useful inequalities

$\gamma : \mathcal{N} \rightarrow \mathcal{M}$ a normal positive unital map, $\rho \in \mathcal{M}_*^+$, $b \in \mathcal{N}^+$.

(2) If $p \in [1, \infty]$, then

$$\left\| h_{\rho \circ \gamma}^{\frac{1}{2p}} b h_{\rho \circ \gamma}^{\frac{1}{2p}} \right\|_p \geq \left\| h_{\rho}^{\frac{1}{2p}} \gamma(b) h_{\rho}^{\frac{1}{2p}} \right\|_p.$$

Proof.

Let $\omega \in \mathcal{N}_*^+$, $h_{\omega} = h_{\rho \circ \gamma}^{\frac{1}{2}} b h_{\rho \circ \gamma}^{\frac{1}{2}}$.

$$\begin{aligned} \left\| h_{\rho}^{\frac{1}{2p}} \gamma(b) h_{\rho}^{\frac{1}{2p}} \right\|_p^p &= Q_{p,p}(\omega \circ \gamma_{\rho}^* \| \rho) = Q_{p,p}(\omega \circ \gamma_{\rho}^* \| \rho \circ \gamma \circ \gamma_{\rho}^*) \\ &\leq {}^{14} Q_{p,p}(\omega \| \rho \circ \gamma) = \left\| h_{\rho \circ \gamma}^{\frac{1}{2p}} b h_{\rho \circ \gamma}^{\frac{1}{2p}} \right\|_p^p \end{aligned}$$

□

DPI in the von Neumann algebra setting

Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$ and let $\gamma : \mathcal{N} \rightarrow \mathcal{M}$ be a normal positive unital map.

Assume either of the following conditions:

- (i) $0 < \alpha < 1$, $\max\{\alpha, 1 - \alpha\} \leq z$,
- (ii) $\alpha > 1$, $\max\{\alpha/2, \alpha - 1\} \leq z \leq \alpha$.

Then we have

$$D_{\alpha,z}(\psi \circ \gamma \| \varphi \circ \gamma) \leq D_{\alpha,z}(\psi \| \varphi).$$

DPI in the von Neumann algebra setting

Let $0 < \alpha < 1$, $\max\{\alpha, 1 - \alpha\} \leq z$. We have

$$Q_{\alpha,z}(\psi\|\varphi) = \inf_{a \in \mathcal{M}^{++}} \left\{ \alpha \left\| h_{\psi}^{\frac{1}{2p}} a h_{\psi}^{\frac{1}{2p}} \right\|_p^p + (1 - \alpha) \left\| h_{\varphi}^{\frac{1}{2r}} a^{-1} h_{\varphi}^{\frac{1}{2r}} \right\|_r^r \right\},$$

with $p := \frac{z}{\alpha}$, $r := \frac{z}{1-\alpha}$. In the above bounds, $p, r \geq 1$.

By the inequality (2) and the Choi inequality:

$$\gamma(b)^{-1} \leq \gamma(b^{-1}),$$

we get

$$Q_{\alpha,z}(\psi \circ \gamma \| \varphi \circ \gamma) \geq Q_{\alpha,z}(\psi \| \varphi).$$

DPI in the von Neumann algebra setting

Let $\alpha > 1$, $\max\{\alpha/2, \alpha - 1\} \leq z \leq \alpha$. We have

$$Q_{\alpha,z}(\psi\|\varphi) = \sup_{a \in \mathcal{M}^+} \left\{ \alpha \left\| h_{\psi}^{\frac{1}{2p}} a h_{\psi}^{\frac{1}{2p}} \right\|_p^p - (\alpha - 1) \left\| h_{\varphi}^{\frac{1}{2q}} a h_{\varphi}^{\frac{1}{2q}} \right\|_q^q \right\},$$

with $p := \frac{z}{\alpha}$, $q := \frac{z}{\alpha-1}$. In the above bounds, $p \in [1/2, 1)$, $q \geq 1$.

By the inequalities (1) and (2) we get

$$Q_{\alpha,z}(\psi \circ \gamma \| \varphi \circ \gamma) \leq Q_{\alpha,z}(\psi \| \varphi).$$

Sufficient channels and equality in DPI

A **channel** is a 2-positive unital normal map $\gamma : \mathcal{N} \rightarrow \mathcal{M}$.

Let $\psi, \varphi \in \mathcal{M}_*^+$. We say that γ is **sufficient** with respect to $\{\psi, \varphi\}$ if there exists a **recovery channel** $\beta : \mathcal{M} \rightarrow \mathcal{N}$ such that

$$\psi \circ \gamma \circ \beta = \psi, \quad \varphi \circ \gamma \circ \beta = \varphi.$$

Petz theorem: Assume that $D_1(\psi \| \varphi) < \infty$. Then γ is sufficient with respect to $\{\psi, \varphi\}$ if and only if

$$D_1(\psi \circ \gamma \| \varphi \circ \gamma) = D_1(\psi \| \varphi).$$

A similar result holds for the transition probability $(D_{\frac{1}{2},1})$.

Known results on equality in DPI

Characterization of sufficient channels:

- Petz-type: $D_{\alpha,1}$, $\alpha \in (0,1) \cup (1,2)$ ¹⁵
- sandwiched: $D_{\alpha,\alpha}$, $\alpha \in (1/2,1) \cup (1,\infty)$ ¹⁶

Other equality conditions for $D_{\alpha,z}$ were found in finite dimensions¹⁷

- no clear relation to sufficiency of channels (apart from some special cases)¹⁸

¹⁵AJ and Petz, 2006; Hiai et al, 2011; Hiai and Mosonyi 2017; Hiai, 2018

¹⁶AJ, 2018, 2021

¹⁷Leditzky, Rouzé and Datta, 2017; Zhang 2020

¹⁸Hiai and Mosonyi, 2017

Universal recovery channel

The Petz dual γ_φ^* is a **universal recovery channel**:

- $\varphi \circ \gamma \circ \gamma_\varphi^* = \varphi$
- Let $\psi \in \mathcal{M}_*^+$ be such that $s(\psi) \leq s(\varphi)$. Then γ is sufficient with respect to $\{\psi, \varphi\}$ if and only if

$$\psi \circ \gamma \circ \gamma_\varphi^* = \psi.$$

- Equivalently, $\psi \circ \mathcal{E} = \psi$ for the **conditional expectation** $\mathcal{E} : \mathcal{M} \rightarrow \mathcal{M}$ onto the fixed points of $\gamma \circ \gamma_\varphi^*$.

Sufficient channels and equality in DPI for $D_{\alpha,z}$

Let $0 < \alpha < 1$, $\max\{\alpha, 1 - \alpha\} \leq z$.

Let $\psi, \varphi \in \mathcal{M}_*^+$ and let $\gamma : \mathcal{N} \rightarrow \mathcal{M}$ be a channel. Assume that

$$\alpha < z \text{ and } s(\varphi) \leq s(\psi) \quad \text{or} \quad 1 - \alpha < z \text{ and } s(\psi) \leq s(\varphi).$$

Then γ is sufficient with respect to $\{\psi, \varphi\}$ if and only if

$$D_{\alpha,z}(\psi \circ \gamma \| \varphi \circ \gamma) = D_{\alpha,z}(\psi \| \varphi).$$

Sufficient channels and equality in DPI for $D_{\alpha,z}$

Let $\alpha > 1$, $\max\{\alpha/2, \alpha - 1\} \leq z \leq \alpha < z + 1$.

Let $\psi, \varphi \in \mathcal{M}_*^+$ be such that $D_{\alpha,z}(\psi\|\varphi) < \infty$. A channel $\gamma : \mathcal{N} \rightarrow \mathcal{M}$ is sufficient with respect to $\{\psi, \varphi\}$ if and only if

$$D_{\alpha,z}(\psi \circ \gamma \|\varphi \circ \gamma) = D_{\alpha,z}(\psi \|\varphi).$$

A sketch of proof for $\alpha > 1$

Put $\psi_0 := \psi \circ \gamma$, $\varphi_0 := \varphi \circ \gamma$, $p = \frac{z}{\alpha}$, $q = \frac{z}{\alpha-1} > 1$.

- By the assumptions, $D_{\alpha,z}(\psi_0 \| \varphi_0) \leq D_{\alpha,z}(\psi \| \varphi) < \infty$.
- In this case, for some $y \in L^{2z}(\mathcal{M})$, $y_0 \in L^{2z}(\mathcal{N})$:

$$h_{\psi}^{\frac{1}{2p}} = y h_{\varphi}^{\frac{1}{2q}}, \quad h_{\psi_0}^{\frac{1}{2p}} = y_0 h_{\varphi_0}^{\frac{1}{2q}}$$

- Variational expression:

$$Q_{\alpha,z}(\psi \| \varphi) = \sup_{L^q(\mathcal{M})^+} \{ \alpha \operatorname{tr}((y w y^*)^p) - (\alpha - 1) \operatorname{tr}(w^q) \},$$

uniquely attained at $\bar{w} := (y^* y)^{\alpha-1} \in L^q(\mathcal{M})^+$.

- Similarly for ψ_0 , φ_0 , $\bar{w}_0 := (y_0^* y_0)^{\alpha-1} \in L^q(\mathcal{N})^+$.

A sketch of proof for $\alpha > 1$

- Let $\omega \in \mathcal{M}_*^+$, $\omega_0 \in \mathcal{N}_*^+$ be such that

$$h_\omega = h_\varphi^{\frac{q-1}{2q}} \bar{w} h_\varphi^{\frac{q-1}{2q}}, \quad h_{\omega_0} = h_{\varphi_0}^{\frac{q-1}{2q}} \bar{w}_0 h_{\varphi_0}^{\frac{q-1}{2q}}.$$

- Then

$$D_{q,q}(\omega_0 \|\varphi_0) = D_{\alpha,z}(\psi_0 \|\varphi_0) = D_{\alpha,z}(\psi \|\varphi) = D_{q,q}(\omega \|\varphi).$$

- By the variational formula and uniqueness, we get $\omega_0 = \omega \circ \gamma$.
- By known properties of $D_{q,q}$, $q > 1$, γ is sufficient with respect to $\{\omega, \varphi\}$, so that

$$\omega \circ \mathcal{E} = \omega.$$

- Extensions of \mathcal{E} to the L^p -spaces and their properties¹⁹ give us $\psi \circ \mathcal{E} = \psi$.

Further results on $D_{\alpha,z}$

- Monotonicity in z : $z \mapsto D_{\alpha,z}(\psi\|\varphi)$ is
 - increasing on $(0, \infty)$ for $0 < \alpha < 1$,
 - decreasing on $[\alpha/2, \infty)$ for $\alpha > 1$.
- Monotonicity in α : $\alpha \mapsto D_{\alpha,z}(\psi\|\varphi)$ is
 - increasing on $(0, 1)$ for all $z > 0$,
 - increasing on $(1, 2z]$ for $z > 1/2$.
- The limits $\alpha \rightarrow 1$:
 - $\lim_{\alpha \nearrow 1} D_{\alpha,z}(\psi\|\varphi) = D_1(\psi\|\varphi)$ for $z > 0$,
 - $\lim_{\alpha \searrow 1} D_{\alpha,z}(\psi\|\varphi) = D_1(\psi\|\varphi)$ for $z > 1/2$ if $D_{\alpha,z}(\psi\|\varphi) < \infty$ for some $\alpha \in (1, 2z]$.

Thank you for your attention.