Martingale convergence for $D_{\alpha,z}$

Let $\mathcal{M}$ be a σ-finite von Neumann algebra. Let $\{\mathcal{M}_i\}$ be an increasing net of von Neumann subalgebras of $\mathcal{M}$ containing the unit of $\mathcal{M}$ such that $\mathcal{M} = \big(\bigcup_i \mathcal{M}_i\big)''$.

Theorem 0.1. *Assume that either*

$$\alpha \in (0,1), \qquad z \ge \max\{\alpha, 1-\alpha\}, \tag{0.1}$$

or

$$\alpha > 1, \qquad \max\{\alpha/2, \alpha-1\} \le z \le \alpha. \tag{0.2}$$

Then we have

$$D_{\alpha,z}(\psi\|\varphi) = \lim_i D_{\alpha,z}(\psi|_{\mathcal{M}_i}\|\varphi|_{\mathcal{M}_i}) \quad \textit{increasingly.} \tag{0.3}$$

Proof. Let $\varphi_i := \varphi|_{\mathcal{M}_i}$ and $\psi_i := \psi|_{\mathcal{M}_i}$. From the DPI of $D_{\alpha,z}$ proved in [6, Theorem 1(viii)] and [5], it follows that $D_{\alpha,z} \ge D_{\alpha,z}(\psi_i\|\varphi_i)$ for all i and $i \mapsto D_{\alpha,z}(\psi_i\|\varphi_i)$ is increasing. Hence, to show (0.3), it suffices to prove that

$$D_{\alpha,z}(\psi\|\varphi) \le \sup_i D_{\alpha,z}(\psi_i\|\varphi_i). \tag{0.4}$$

To do this, we may assume that φ is faithful. Indeed, assume that (0.4) has been sbown when φ is faithful. For general $\varphi \in \mathcal{M}_*^+$, from the assumption of $\mathcal{M}$ being σ-finite, there exists a $\varphi_0 \in \mathcal{M}_*^+$ with $s(\varphi_0) = \mathbf{1} - s(\varphi)$. Let $\varphi^{(n)} := \varphi + n^{-1}\varphi_0$ and $\varphi_i^{(n)} := \varphi^{(n)}|_{\mathcal{M}_i}$ for each $n \in \mathbb{N}$. Thanks to the lower semi-continuity [6, Theorem 1(iv) and Theorem 2(iv)] and the order relation [6, Theorem 1(iii) and Theorem 2(iii)] we have

$$\begin{aligned} D_{\alpha,z}(\psi\|\varphi) &\le \liminf_{n\to\infty} D_{\alpha,z}(\psi\|\varphi_i^{(n)}) \\ &\le \liminf_{n\to\infty} \sup_i D_{\alpha,z}(\psi_i\|\varphi_i^{(n)}) \\ &\le \sup_i D_{\alpha,z}(\psi_i\|\varphi_i), \end{aligned}$$

proving (0.4) for general φ. Below we assume the faithfulness of φ and write $\mathcal{E}_{\mathcal{M}_i,\varphi}$ for the generalized conditional expectation from $\mathcal{M}$ to $\mathcal{M}_i$ with respect to φ, that is, Petz' recovery map of the inclusion map $\mathcal{M}_i \hookrightarrow \mathcal{M}$ with respect to φ; see, e.g., [3, Proposition 6.5]. Then we note by [4, Theorem 3] that

$$\psi_i \circ \mathcal{E}_{\mathcal{M}_i,\varphi} = \psi \circ \mathcal{E}_{\mathcal{M}_i,\varphi} \to \psi \quad \text{in the norm,} \tag{0.5}$$

as well as

$$\varphi_i \circ \mathcal{E}_{\mathcal{M}_i,\varphi} = \varphi \circ \mathcal{E}_{\mathcal{M}_i,\varphi} = \varphi. \tag{0.6}$$

Now we divide the proof into two cases (0.1) and (0.2).

Case (0.1). We need to prove that

$$Q_{\alpha,z}(\psi\|\varphi) \ge \inf_i Q_{\alpha,z}(\psi_i\|\varphi_i). \tag{0.7}$$

To do this, we may and do assume that ψ is also faithful. Indeed, assume that (0.7) has been proved when ψ is faithful. Then, for general ψ we have by [6, Theorem 1(iii), (iv)]

$$\begin{aligned}Q_{\alpha,z}(\psi\|\varphi)&=\inf_{\varepsilon>0}Q_{\alpha,z}(\psi+\varepsilon\varphi\|\varphi)=\inf_{\varepsilon>0}\inf_i Q_{\alpha,z}(\psi_i+\varepsilon\varphi_i\|\varphi_i)\\&=\inf_i\inf_{\varepsilon>0}Q_{\alpha,z}(\psi_i+\varepsilon\varphi_i\|\varphi_i)=\inf_i Q_{\alpha,z}(\psi_i\|\varphi_i).\end{aligned}$$

Define $p:=z/\alpha$ and $q:=z/(1-\alpha)$, which are in $[1,\infty)$ by assumption (0.1). For every $a\in\mathcal{M}_{++}$ we have

$$\begin{aligned}&\alpha\mathrm{tr}\big(h_\psi^{\alpha/2z}ah_\psi^{\alpha/2z}\big)^{z/\alpha}+(1-\alpha)\mathrm{tr}\big(h_\varphi^{(1-\alpha)/2z}a^{-1}h_\varphi^{(1-\alpha)/2z}\big)^{z/(1-\alpha)}\\&\quad=\alpha\big\|h_\psi^{1/2p}ah_\psi^{1/2p}\big\|_p^p+(1-\alpha)\big\|h_\varphi^{1/2q}a^{-1}h_\varphi^{1/2q}\big\|_q^q\\&\quad=\lim_i\Big[\alpha\big\|h_{\psi_i\circ\mathcal{E}_{\mathcal{M}_i,\varphi}}^{1/2p}ah_{\psi_i\circ\mathcal{E}_{\mathcal{M}_i,\varphi}}^{1/2p}\big\|_p^p+(1-\alpha)\big\|h_{\varphi_i\circ\mathcal{E}_{\mathcal{M},\varphi}}^{1/2q}a^{-1}h_{\varphi_i\circ\mathcal{E}_{\mathcal{M}_i,\varphi}}^{1/2q}\big\|_q^q\Big],\end{aligned}\tag{0.8}$$

where the last equality is seen from [6, Lemma 6] with (0.5) and (0.6). For each i, by [6, (22)] we obtain

$$\big\|h_{\psi_i\circ\mathcal{E}_{\mathcal{M}_i,\varphi}}^{1/2p}ah_{\psi_i\circ\mathcal{E}_{\mathcal{M}_i,\varphi}}^{1/2p}\big\|_p^p\ge\big\|h_{\psi_i}^{1/2p}\mathcal{E}_{\mathcal{M}_i,\varphi}(a)h_{\psi_i}\big\|_p^p=\mathrm{tr}\big(h_{\psi_i}^{1/2p}\mathcal{E}_{\mathcal{M}_i,\varphi}(a)h_{\psi_i}^{1/2p}\big)^p,\tag{0.9}$$

and

$$\begin{aligned}\big\|h_{\varphi_i\circ\mathcal{E}_{\mathcal{M},\varphi}}^{1/2q}a^{-1}h_{\varphi_i\circ\mathcal{E}_{\mathcal{M}_i,\varphi}}^{1/2q}\big\|_q^q&\ge\big\|h_{\varphi_i}^{1/2q}\mathcal{E}_{\mathcal{M}_i,\varphi}(a^{-1})h_{\varphi_i}\big\|_q^q\ge\big\|h_{\varphi_i}^{1/2q}\mathcal{E}_{\mathcal{M}_i,\varphi}(a)^{-1}h_{\varphi_i}\big\|_q^q\\&=\mathrm{tr}\big(h_{\varphi_i}^{1/2q}\mathcal{E}_{\mathcal{M}_i,\varphi}(a)^{-1}h_{\varphi_i}^{1/2q}\big)^q,\end{aligned}\tag{0.10}$$

where the second inequality in (0.10) follows from [1, Corollary 2.3]. Combining (0.8)–(0.10) gives

$$\begin{aligned}&\alpha\mathrm{tr}\big(h_\psi^{\alpha/2z}ah_\psi^{\alpha/2z}\big)^{z/\alpha}+(1-\alpha)\mathrm{tr}\big(h_\varphi^{(1-\alpha/2z}a^{-1}h_\varphi^{(1-\alpha)/2z}\big)^{z/(1-\alpha)}\\&\quad\ge\inf_i\Big[\mathrm{tr}\big(h_{\psi_i}^{\alpha/2z}\mathcal{E}_{\mathcal{M}_i,\varphi}(a)h_{\psi_i}^{\alpha/2z}\big)^{z/\alpha}+(1-\alpha)\mathrm{tr}\big(h_{\varphi_i}^{(1-\alpha)/2z}\mathcal{E}_{\mathcal{M}_i,\varphi}(a)^{-1}h_{\varphi_i}^{(1-\alpha)/2z}\big)^{z/(1-\alpha)}\Big]\\&\quad\ge\inf_i Q_{\alpha,z}(\psi_i\|\varphi_i)\end{aligned}$$

due to [6, Theorem 1(vi)]. This implies (0.7) by [6, Theorem 1(vi)] again.

Case (0.2). We need to prove that

$$Q_{\alpha,z}(\psi\|\varphi)\le\sup_i Q_{\alpha,z}(\psi_i\|\varphi_i).\tag{0.11}$$

Define $p:=z/\alpha$ and $q:=z/(\alpha-1)$ in this case. Then $p\in[1/2,1]$ and $q\in[1,\infty)$ by assumption (0.2). For every $a\in\mathcal{M}_+$ we have

$$\begin{aligned}&\alpha\mathrm{tr}\big(h_\psi^{\alpha/2z}ah_\psi^{\alpha/2z}\big)^{z/\alpha}-(\alpha-1)\mathrm{tr}\big(h_\varphi^{(\alpha-1)/2z}ah_\varphi^{(\alpha-1)/2z}\big)^{z/(\alpha-1)}\\&\quad=\alpha\big\|h_\psi^{1/2p}ah_\psi^{1/2p}\big\|_p^p-(\alpha-1)\big\|h_\varphi^{1/2q}ah_\varphi^{1/2q}\big\|_q^q\\&\quad=\lim_i\Big[\alpha\big\|h_{\psi_i\circ\mathcal{E}_{\mathcal{M}_i,\varphi}}^{1/2p}ah_{\psi_i\circ\mathcal{E}_{\mathcal{M}_i,\varphi}}^{1/2p}\big\|_p^p-(\alpha-1)\big\|h_{\varphi_i\circ\mathcal{E}_{\mathcal{M},\varphi}}^{1/2q}ah_{\varphi_i\circ\mathcal{E}_{\mathcal{M}_i,\varphi}}^{1/2q}\big\|_q^q\Big].\end{aligned}\tag{0.12}$$

where the last equality is seen as follows: By [2, Theorem 4.9(iii)] and the Hölder inequality we have

$$\Big|\,\big\|h_\psi^{1/2p}ah_\psi^{1/2p}\big\|_p^p-\big\|h_{\psi_i\circ\mathcal{E}_{\mathcal{M}_i,\varphi}}^{1/2p}ah_{\psi_i\circ\mathcal{E}_{\mathcal{M}_i,\varphi}}^{1/2p}\big\|_p^p\Big|$$

$$
\begin{aligned}
&\le \big\|h_\psi^{1/2p}ah_\psi^{1/2p} - h_{\psi_i\circ\mathcal{E}_{\mathcal{M}_i,\varphi}}^{1/2p}ah_{\psi_i\circ\mathcal{E}_{\mathcal{M}_i,\varphi}}^{1/2p}\big\|_p^p \\
&\le \big\|\big(h_\psi^{1/2p} - h_{\psi_i\circ\mathcal{E}_{\mathcal{M}_i,\varphi}}^{1/2p}\big)ah_\psi^{1/2p}\big\|_p^p + \big\|h_{\psi_i\circ\mathcal{E}_{\mathcal{M}_i,\varphi}}^{1/2p}a\big(h_\psi^{1/2p} - h_{\psi_i\circ\mathcal{E}_{\mathcal{M}_i,\varphi}}^{1/2p}\big)\big\|_p^p \\
&\le \|a\|^p\Big(\|h_\psi^{1/2p}\|_{2p}^p + \big\|h_{\psi_i\circ\mathcal{E}_{\mathcal{M}_i,\varphi}}^{1/2p}\big\|_{2p}^p\Big)\big\|h_\psi^{1/2p} - h_{\psi_i\circ\mathcal{E}_{\mathcal{M}_i,\varphi}}^{1/2p}\big\|_{2p}^p \\
&\le \|a\|^p\big(\|\psi\|^{1/2} + \big\|\psi_i\circ\mathcal{E}_{\mathcal{M}_i,\varphi}\big\|^{1/2}\big)\big\|h_\psi^{1/2p} - h_{\psi_i\circ\mathcal{E}_{\mathcal{M}_i,\varphi}}^{1/2p}\big\|_{2p}^p,
\end{aligned}
$$

which converges to 0 due to (0.5) and [6, Lemma 6]. For each i, by [6, (22)] we obtain

$$
\big\|h_{\varphi_i\circ\mathcal{E}_{\mathcal{M},\varphi}}^{1/2q}ah_{\varphi_i\circ\mathcal{E}_{\mathcal{M}_i,\varphi}}^{1/2q}\big\|_q^q \ge \big\|h_{\varphi_i}^{1/2q}\mathcal{E}_{\mathcal{M}_i,\varphi}(a)h_{\varphi_i}^{1/2q}\big\|_q^q \quad = \mathrm{tr}\big(h_{\varphi_i}^{1/2q}\mathcal{E}_{\mathcal{M}_i,\varphi}(a)h_{\varphi_i}^{1/2q}\big)^q. \tag{0.13}
$$

On the other hand, by [5, (1)] we obtain

$$
\big\|h_{\psi_i\circ\mathcal{E}_{\mathcal{M}_i,\varphi}}^{1/2p}ah_{\psi_i\circ\mathcal{E}_{\mathcal{M}_i,\varphi}}^{1/2p}\big\|_p^p \le \big\|h_{\psi_i}^{1/2p}\mathcal{E}_{\mathcal{M}_i,\varphi}(a)h_{\psi_i}^{1/2p}\big\|_p^p = \mathrm{tr}\big(h_{\psi_i}^{1/2p}\mathcal{E}_{\mathcal{M}_i,\varphi}(a)h_{\psi_i}^{1/2p}\big)^p. \tag{0.14}
$$

(Note that the assumption of ψ being faithful in [5] can easily been removed by a convergence argument.) Combining (0.12)–(0.14) yields

$$
\begin{aligned}
&\alpha\mathrm{tr}\big(h_\psi^{\alpha/2z}ah_\psi^{\alpha/2z}\big)^{z/\alpha} - (\alpha-1)\mathrm{tr}\big(h_\varphi^{(\alpha-1)/2z}ah_\varphi^{(\alpha-1)/2z}\big)^{z/(\alpha-1)} \\
&\quad \le \sup_i\Big[\mathrm{tr}\big(h_{\psi_i}^{\alpha/2z}\mathcal{E}_{\mathcal{M}_i,\varphi}(a)h_{\psi_i}^{\alpha/2z}\big)^{z/\alpha} - (\alpha-1)\mathrm{tr}\big(h_{\varphi_i}^{(\alpha-1)/2z}\mathcal{E}_{\mathcal{M}_i,\varphi}(a)h_{\varphi_i}^{(\alpha-1)/2z}\big)^{z/(1-\alpha)}\Big] \\
&\quad \le \sup_i Q_{\alpha,z}(\psi_i\|\varphi_i)
\end{aligned}
$$

due to the strengthened version of [6, Theorem 2(vi)]. By using this again, we obtain (0.11). □

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