$\lim_{\alpha\searrow 1} D_{\alpha,z} = D_1$ when $1/2 < z < 1$

Assume that $\mathcal{M}$ is a general (σ-finite) von Neumann algebra and $\psi,\varphi\in\mathcal{M}_*^+$ with $\psi\ne 0$. Let $1/2<z<1$.

Proposition 0.1. *If $\lambda^{-1}\varphi\le\psi\le\lambda\varphi$ for some $\lambda\ge 1$, then*

$$\lim_{\alpha\searrow 1} D_{\alpha,z}(\psi\|\varphi)=D_1(\psi\|\varphi). \tag{0.1}$$

Proof. We may and do assume that φ and hence ψ are faithful. The proof is based on state perturbation theory due to Araki [2]. By [2, Theorem 6.3] (see also [5, Theorem B.1]) there exists an $h\in\mathcal{M}_{\mathrm{sa}}$ such that $\psi=\varphi^h$ and $-\log\lambda\le h\le\log\lambda$. Here the perturbed functional φ^h is given by $\varphi^h=\langle\Phi^h,\cdot\,\Phi^h\rangle$ with $\Phi=h_\varphi^{1/2}\in L^2(M)_+$, where the perturbed vector $\Phi^h\in L^2(M)_+$ is defined by

$$\Phi^h:=\sum_{n=0}^\infty\int_0^{1/2}dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}dt_n\Delta_\varphi^{t_n}h\Delta_\varphi^{t_{n-1}-t_n}h\cdots\Delta_\varphi^{t_1-t_2}h\Phi.$$

It is known [1, 2] that

$$[D\psi:D\varphi]_t=[D\varphi^h:D\varphi]_t=\sum_{n=0}^\infty i^n\int_0^t dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}dt_n\sigma_{t_n}^\varphi(h)\cdots\sigma_{t_1}^\varphi(h), \tag{0.2}$$

Moreover, we have by [3, Theorem 3.10]

$$D(\psi\|\varphi)=\psi(h). \tag{0.3}$$

It is also known (see, e.g., [5, Lemma A.2]) that $[D\psi:D\varphi]_t$ extends to a strongly continuous ($\mathcal{M}$-valued) function $[D\psi:D\varphi]_z$ on $-1/2\le\operatorname{Im}z\le 1/2$ that is analytic in the interior, where $[D\psi:D\varphi]_{\bar z}^*=[D\varphi:D\psi]_z$. Moreover, we have

$$h_\psi^p=[D\psi:D\varphi]_{-ip}h_\varphi^p,\qquad 0<p\le 1/2. \tag{0.4}$$

Let $1<\alpha<2z$ and set $p=p(\alpha):=\frac{\alpha-1}{2z}$; then $0<p<\frac{2z-1}{2z}<1/2$ (thanks to $1/2<z<1$). Since (0.4) gives

$$h_\psi^{\frac{\alpha}{2z}}=h_\psi^{\frac{1}{2z}}h_\psi^p=h_\psi^{\frac{1}{2z}}[D\psi:D\varphi]_{-ip}h_\varphi^p,$$

we have

$$Q_{\alpha,z}(\psi\|\varphi)=\big\|h_\psi^{\frac{1}{2z}}[D\psi:D\varphi]_{-ip}\big\|_{2z}^{2z}. \tag{0.5}$$

The expansion in (0.2) shows that

$$[D\psi:D\varphi]_t=\mathbf{1}+i\int_0^t\sigma_{t_1}^\varphi(h)\,dt_1+o(t)=\mathbf{1}+ith+o(t), \tag{0.6}$$

where $o(t)$ means that $o(t)/t\to 0$ strongly as $t\to 0$. On the other hand, the analyticity of $[D\psi:D\varphi]_z$ in a neighborhood of 0 shows that

$$[D\psi:D\varphi]_z=\mathbf{1}+za+o(z) \tag{0.7}$$

for some $a \in \mathcal{M}$, where $o(z)$ means that $\|o(z)\|/|z| \to 0$ as $z \to 0$. Comparing (0.6) and (0.7) gives $a = ih$ so that

$$[D\psi : D\varphi]_{-ip} = \mathbf{1} + ph + o(p) \quad \text{as } p \searrow 0, \tag{0.8}$$

where $o(p)$ means that $\|o(p)\|/p \to 0$ as $p \searrow 0$. Therefore, we have

$$h_\psi^{\frac{1}{2z}}[D\psi : D\varphi]_{-ip} = h_\psi^{\frac{1}{2z}} + ph_\psi^{\frac{1}{2z}}h + \varepsilon(p) \quad \text{as } p \searrow 0, \tag{0.9}$$

where $\varepsilon(p) \in L^{2z}(\mathcal{M})$ and $\|\varepsilon(p)\|_{2z}/p \to 0$ as $p \searrow 0$.

Now let us recall that $L^{2z}(M)$ is uniformly convex (thanks to $2z > 1$) so that the norm $\|\cdot\|_{2z}$ is uniformly Fréchet differentiable (see, e.g., [4, Part 3, Chap. II]). Since $a_0 := h_\psi^{\frac{2z-1}{2z}}/\psi(1)^{\frac{2z-1}{2z}}$ is an element of $L^{\frac{2z}{2z-1}}(\mathcal{M})$ (the dual space of $L^{2z}(\mathcal{M})$) such that $\|v\|_{\frac{2z}{2z-1}} = 1$ and

$$\operatorname{tr} a_0 h_\psi^{\frac{1}{2z}} = \left\|h_\psi^{\frac{1}{2z}}\right\|_{2z},$$

the uniform Fréchet differentiability at $h_\psi^{\frac{1}{2z}}$ says that

$$\left| \frac{\left\|h_\psi^{\frac{1}{2z}} + tb\right\|_{2z} - \left\|h_\psi^{\frac{1}{2z}}\right\|_{2z}}{t} - \operatorname{tr} a_0 b \right| = o(t) \quad \text{as } t \to 0 \tag{0.10}$$

uniformly for $b \in L^{2z}(\mathcal{M})$ with $\|b\|_{2b} \le k$ for any $k > 0$. Letting $f(p) := \|h_\psi^{\frac{1}{2z}}[D\psi : D\varphi]_{-ip}\|_{2z}$ for $0 \le p < 1/2$ with $f(0) := \left\|h_\psi^{\frac{1}{2z}}\right\|_{2z} = \psi(1)^{\frac{1}{2z}}$, by (0.9) we have

$$\begin{aligned}
\frac{f(p) - f(0)}{p} - \operatorname{tr} a_0 h_\psi^{\frac{1}{2z}} h &= \frac{\left\|h_\psi^{\frac{1}{2z}} + ph_\psi^{\frac{1}{2z}}h + \varepsilon(p)\right\|_{2z} - \left\|h_\psi^{\frac{1}{2z}}\right\|_{2z}}{p} - \operatorname{tr} a_0 h_\psi^{\frac{1}{2z}} h \\
&\begin{cases} \le \frac{\left\|h_\psi^{\frac{1}{2z}} + ph_\psi^{\frac{1}{2z}}h\right\|_{2z} - \left\|h_\psi^{\frac{1}{2z}}\right\|_{2z}}{p} - \operatorname{tr} a_0 h_\psi^{\frac{1}{2z}} h + \frac{\|\varepsilon(p)\|_{2z}}{p}, \\ \ge \frac{\left\|h_\psi^{\frac{1}{2z}} + ph_\psi^{\frac{1}{2z}}h\right\|_{2z} - \left\|h_\psi^{\frac{1}{2z}}\right\|_{2z}}{p} - \operatorname{tr} a_0 h_\psi^{\frac{1}{2z}} h - \frac{\|\varepsilon(p)\|_{2z}}{p}. \end{cases}
\end{aligned}$$

Therefore, it follows from (0.10) with $b = h_\psi^{\frac{1}{2z}} h$ and from (0.3) that

$$\lim_{p \searrow 0} \frac{f(p) - f(0)}{p} = \operatorname{tr} a_0 h_\psi^{\frac{1}{2z}} h = \frac{\operatorname{tr} h_\psi h}{\psi(1)^{\frac{2z-1}{2z}}} = \frac{D(\psi\|\varphi)}{\psi(1)^{\frac{2z-1}{2z}}}. \tag{0.11}$$

Since $p = \frac{\alpha-1}{2z}$, we arrive at (0.1) as follows:

$$\begin{aligned}
\lim_{\alpha \searrow 1} D_{\alpha,z}(\psi\|\varphi) &= \lim_{\alpha \searrow 1} \frac{\log Q_{\alpha,z}(\psi\|\varphi) - \log \psi(1)}{\alpha - 1} \\
&= \lim_{p \searrow 0} \frac{2z \log f(p) - 2z \log f(0)}{2zp} \qquad \text{(by (0.5))} \\
&= \lim_{p \searrow 0} \frac{\log f(p) - \log f(0)}{f(p) - f(0)} \cdot \frac{f(p) - f(0)}{p} \\
&= \frac{1}{f(0)} \cdot \frac{D(\psi\|\varphi)}{\psi(1)^{\frac{2z-1}{2z}}} \qquad \text{(by (0.11))} \\
&= \frac{D(\psi\|\varphi)}{\psi(1)} = D_1(\psi\|\varphi).
\end{aligned}$$

$\square$

Remark 0.2. The assumption $\lambda^{-1}\varphi \le \psi \le \lambda\varphi$ is too strong. The proposition and its proof are valid in a slightly weaker assumption (though still too strong) that $\lambda^{-1}h_\varphi^\delta \le h_\psi^\delta \le \lambda h_\varphi^\delta$ for some $\delta > 0$ and some $\lambda > 1$. In this case, $\psi = \varphi^h$ for some $h \in \mathcal{M}_{\rm sa}$ and $[D\psi : D\varphi]_t$ extends to a strongly continuous ($\mathcal{M}$-valued) function $[D\psi : D\varphi]_z$ on $-\delta/2 \le z \le \delta/2$ that is analytic in the interior. It is desirable to prove (0.1) under the one-side dominance assumption $\psi \le \lambda\varphi$, though seems difficult.

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