Perfect Discrimination of Non-Orthogonal Separable Pure States on Bipartite System in General Probabilistic Theory

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Abstract. This paper addresses perfect discrimination of two separable states. When available states are restricted to separable states, we can theoretically consider a larger class of measurements than the class of measurements allowed in quantum theory. The pair of the class of separable states and the above extended class of measurements is a typical example of general probabilistic theories. In this framework, we give a necessary and sufficient condition to discriminate two separable pure states perfectly. In particular, we derive measurements explicitly to discriminate two separable pure states perfectly. Finally, we mention that the maximum number of states that are distinguishable simultaneously and perfectly in the above framework is equal to that in quantum theory.

Keywords: perfect discrimination, separable states, general probabilistic theories

1 Introduction

Entanglement is a resource for miracle performance of quantum information processing [1, 2]. Even when a quantum state has no entanglement, entanglement in a measuring process brings us performance that measuring processes without quantum correlation cannot realize. In fact, when we discriminate the n-fold tensor products of two quantum states, the performance of measurements with quantum correlation is beyond that of any measurement without quantum correlation, e.g., local operation and classical communication (LOCC) and separable measurement [3–5]. The difference between the first and second performance can be derived from the following two classes of measurements. One is the class of measurements allowed in quantum theory and the other is the class of measurements with only separable form. The first class achieves strictly better performance than the second class in the above discrimination.

However, if available states are restricted to separable states, it is possible to consider a larger class of measurements on a composite system than the class of measurements of quantum theory. Such a framework is discussed in general probabilistic theories (GPTs) [6–20], which are generalization of quantum theory and classical probability theory. GPTs are the most general framework to characterize states, measurements, and time evolutions. Some preceding studies compared composite systems in GPTs with that in quantum theory [7, 10, 15]. Although the paper [18] discussed cloning and broadcasting in the GPTs, few study clarified the difference between quantum theory and other GPTs in the viewpoint of state discrimination. Hence, to clarify the difference, this paper focuses on the following typical GPT on a bipartite system: we restrict available states to separable states on the composite system and this restriction allows us to consider theoretically measurements that are not allowed in quantum theory. The pair of the class of separable states and the class of such measurements is a typical example of GPTs and is denoted by SEP.

The difference between quantum theory and SEP can be characterized by the relation between the *positive and dual cones* appeared in quantum theory and SEP. A *positive cone* defines the set of all states in a GPT so that a state is given as an element of a positive cone whose trace is one. For example, the positive cone of quantum theory is the set of all positive semi-definite matrices and the positive cone of SEP is the set of all matrices with separable form. Thus, states in SEP are restricted to separable states, and the positive cone of SEP is smaller than that of quantum theory. This restriction makes bit commitment possible under SEP [19]. Furthermore, the *dual cone* of a positive cone defines measurements of a GPT so that a measurement is given as a decomposition $\{M_i\}_i$ of the identity matrix I. More precisely, all elements M_i lie in the dual cone and satisfy $\sum_i M_i = I$. For example, the dual cone of quantum theory is also the set of all positive semi-definite matrices and the dual cone of SEP is the set of all matrices Y that satisfy $\mathrm{Tr}\, XY \geq 0$ for all matrices X with separable form. Thus the dual cone of SEP is larger than that of quantum theory. Therefore, measurements of SEP contain not only ones of quantum theory but also ones that quantum theory cannot realize.

This paper addresses perfect discrimination of two pure states in SEP because state discrimination is a fundamental task in information theory and physics. A main goal of this paper is to reveal how much better the performance of perfect discrimination in SEP is than that in quantum theory. In quantum theory, it is well-known that the orthogonality of two states is necessary and sufficient to discriminate two states perfectly [21]. However, as shown in this paper, there exists a non-orthogonal pair of two separable pure states that can be discriminated in SEP. In this sense, SEP is completely different from

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quantum theory. Moreover, we derive a necessary and sufficient condition for state discrimination in SEP.

Since our necessary and sufficient condition reveals that some non-orthogonal states in SEP can be discriminated perfectly, one might think that the maximum number of simultaneously and perfectly distinguishable states in SEP is greater than that of quantum theory. The maximum number is called the *capacity*, which expresses the limit of communication quantity per single use of the quantum communication. The capacity in quantum theory is equal to the dimension of a quantum system, and an interesting relation for the capacities in GPTs has been derived [9]. Using the relation [9, Lemma 24], we find that the capacity in SEP is equal to that in quantum theory.

The remaining of this paper is organized as follows. The beginning of Section 2 formulates our extended class of measurements and gives a perfectly distinguishable pair of two separable pure states that are not orthogonal. The latter of Section 2 gives a necessary and sufficient condition to discriminate two separable pure states in SEP perfectly (Theorem 2). Also, the latter of Section 2 discusses the capacity in SEP (Theorem 3). The full version of this paper is available at [22].

2 Perfectly distinguishable pairs of two pure states in SEP

This section gives a necessary and sufficient condition to discriminate two pure states in SEP and mentions that the capacity in SEP is equal to the dimension of a quantum system. Before giving our necessary and sufficient condition, let us describe our framework SEP and notational conventions. First, we describe states in SEP. Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be two finite-dimensional complex Hilbert spaces. Then $\mathcal{T}(AB)$ and $\mathcal{T}_+(AB)$ denote the set of all Hermitian matrices on $\mathcal{H}_A \otimes \mathcal{H}_B$ and the set of all positive semi-definite matrices on $\mathcal{H}_A \otimes \mathcal{H}_B$, respectively. Similarly, $\mathcal{T}(A)$, $\mathcal{T}(B)$, $\mathcal{T}_+(A)$, and $\mathcal{T}_+(B)$ are defined. In quantum theory, available states are elements of $\mathcal{T}_+(AB)$ whose traces are one. However, this paper restricts available states to separable states: available states are elements of

$$\mathrm{SEP}(A;B) := \left\{ \sum_i X_i^A \otimes X_i^B \;\middle|\; X_i^A \in \mathcal{T}_+(A),\ X_i^B \in \mathcal{T}_+(B)\ (\forall i) \right\}$$

whose traces are one.

Next, we describe measurements of SEP. In quantum theory, measurements are given as positive-operator valued measures (POVMs). That is, a measurement $\{M_i\}_i$ satisfies $M_i \in \mathcal{T}_+(AB)$ and $\sum_i M_i = I$ for any outcome i. However, since this paper restricts available states to separable states, measurements of SEP are a larger class than ones of quantum theory. A measurement $\{M_i\}_i$ of SEP is defined by the conditions

$$M_i \in \mathrm{SEP}^*(A;B)\ (\forall i), \quad \sum_i M_i = I,$$

where $\mathrm{SEP}^*(A;B)$ denotes the dual cone of $\mathrm{SEP}(A;B)$ and is defined as

$$\begin{aligned}&\mathrm{SEP}^*(A;B)\\ &= \{\, Y \in \mathcal{T}(AB) \mid \operatorname{Tr} XY \geq 0\ (\forall X \in \mathcal{T}_+(AB)) \,\}.\end{aligned}$$

Since the inclusion relation $\mathrm{SEP}^*(A;B) \subset \mathcal{T}_+(AB)$ holds, measurements of SEP are a larger class than ones of quantum theory.

Now, let us consider state discrimination in SEP. Let $\{\rho_i\}_{i=1}^n$ be a family of n states. Then we say that $\{\rho_i\}_{i=1}^n$ is *perfectly distinguishable* in SEP (resp. quantum theory) if there exists a measurement $\{M_j\}_{j=1}^n$ of SEP (resp. quantum theory) such that $\operatorname{Tr} M_j \rho_i = \delta_{ij}$, where δ_{ij} denotes the Kronecker delta. It is well-known that $\{\rho_i\}_{i=1}^n$ is perfectly distinguishable in quantum theory if and only if any two distinct states of $\{\rho_i\}_{i=1}^n$ are orthogonal, i.e., $\operatorname{Tr} \rho_i \rho_j = \delta_{ij}$ for all $i \neq j$. This paper addresses the case $n = 2$ mainly.

Example 1 gives an example that two states are perfectly distinguishable and not orthogonal. For this purpose, we consider the case where $\mathcal{H}_A$ and $\mathcal{H}_B$ are two-dimensional (hereinafter, it is called the $(2,2)$-dimensional case). In this case, the dual cone $\mathrm{SEP}^*(A;B)$ can be expressed explicitly by using the partial transpose operation. Thus, let us define the partial transpose operation Γ. This paper defines it as the transpose operation on the subsystem $\mathcal{H}_B$. Since for 2×2 matrices $C = (c_{ij})_{i,j}$ and $D = (d_{ij})_{i,j}$ the tensor product matrix $C \otimes D$ is expressed as

$$C \otimes D = \left[\begin{array}{cc|cc} c_{11}d_{11} & c_{11}d_{12} & c_{12}d_{11} & c_{12}d_{12} \\ c_{11}d_{21} & c_{11}d_{22} & c_{12}d_{21} & c_{12}d_{22} \\ \hline c_{21}d_{11} & c_{21}d_{12} & c_{22}d_{11} & c_{22}d_{12} \\ c_{21}d_{11} & c_{21}d_{12} & c_{22}d_{11} & c_{22}d_{12} \end{array}\right],$$

the partial transpose $\Gamma(X)$ of a matrix $X = (x_{ij})_{i,j}$ is

$$\Gamma(X) = \left[\begin{array}{cc|cc} x_{11} & x_{21} & x_{13} & x_{23} \\ x_{12} & x_{22} & x_{14} & x_{24} \\ \hline x_{31} & x_{41} & x_{33} & x_{43} \\ x_{32} & x_{42} & x_{34} & x_{44} \end{array}\right].$$

As stated above, we can express the dual cone $\mathrm{SEP}^*(A;B)$ explicitly. Indeed, the combination of [23] and [24] implies the following proposition.

Proposition 1. *If* $(\dim \mathcal{H}_A, \dim \mathcal{H}_B) = (2,2)$*, then*

$$\mathrm{SEP}^*(A;B) = \{\, T + \Gamma(T') \mid T, T' \in \mathcal{T}_+(AB) \,\}.$$

Next, we give a perfectly distinguishable pair of two pure states in SEP even when the two states are not orthogonal. The pair given below is such an example and is also a special case of our main result.

Example 1. Suppose that two pure states $\rho_1, \rho_2 \in \mathrm{SEP}(A;B)$ are given as

$$\begin{aligned}\rho_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \\ \rho_2 &= \begin{bmatrix} 1-\alpha_1 & \beta_1 \\ \beta_1 & \alpha_1 \end{bmatrix} \otimes \begin{bmatrix} 1-\alpha_2 & \beta_2 \\ \beta_2 & \alpha_2 \end{bmatrix},\end{aligned} \tag{1}$$

Table 1: Necessary and sufficient (NS) conditions of perfect discrimination of two pure states, and capacities.

GPTs	SEP	Quantum theory
NS condition	$\mathrm{Tr}\,\rho_1^A\rho_2^A + \mathrm{Tr}\,\rho_1^B\rho_2^B \le 1$	$(\mathrm{Tr}\,\rho_1^A\rho_2^A)(\mathrm{Tr}\,\rho_1^B\rho_2^B) = 0$
Capacity	$\dim(\mathcal{H}_A \otimes \mathcal{H}_B)$	$\dim(\mathcal{H}_A \otimes \mathcal{H}_B)$

where $\alpha_i \in [0,1]$, $\beta_i \ge 0$, and $\beta_i^2 = \alpha_i(1-\alpha_i)$ for all $i = 1,2$. The perfect discrimination of ρ_1 and ρ_2 is possible in quantum theory if and only if ρ_1 and ρ_2 are orthogonal. However, a measurement of SEP can discriminate ρ_1 and ρ_2 perfectly even when ρ_1 and ρ_2 are not orthogonal. To see this fact, we give a measurement $\{T_1 + \Gamma(T_1), T_2 + \Gamma(T_2)\}$ with positive semi-definite matrices T_1 and T_2. Since T_1 and T_2 are positive semi-definite, proposition 1 implies that $T_i + \Gamma(T_i) \in \mathrm{SEP}^*(A;B)$ for all $i = 1,2$. Now, we set the positive semi-definite matrices T_1 and T_2 as

$$T_1 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \quad T_2 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $\{T_1 + \Gamma(T_1), T_2 + \Gamma(T_2)\}$ is a measurement of SEP because $(T_1 + \Gamma(T_1)) + (T_2 + \Gamma(T_2)) = I$. The measurement $\{T_1 + \Gamma(T_1), T_2 + \Gamma(T_2)\}$ discriminates ρ_1 and ρ_2 perfectly if $\alpha_1 + \alpha_2 = 1$. Indeed, it follows that $\mathrm{Tr}\,\rho_1(T_2 + \Gamma(T_2)) = \mathrm{Tr}\,\rho_2(T_1 + \Gamma(T_1)) = 0$ from $\alpha_1 + \alpha_2 = 1$. Let us verify it. First, the equation $\mathrm{Tr}\,\rho_1(T_2 + \Gamma(T_2)) = 0$ follows from the definitions. Next, note that the assumption $\alpha_1 + \alpha_2 = 1$ implies $\beta_1 = \beta_2 = \sqrt{\alpha_1\alpha_2}$. Since (i) $\Gamma(\rho_2) = \rho_2$ and (ii) $\alpha_1 + \alpha_2 = 1$ ($\beta_1 = \beta_2 = \sqrt{\alpha_1\alpha_2}$), we have

$$\begin{aligned} &\mathrm{Tr}\,\rho_2(T_1 + \Gamma(T_1)) \overset{\text{(i)}}{=} 2\,\mathrm{Tr}\,\rho_2 T_1 \\ &= (1-\alpha_1)(1-\alpha_2) + \alpha_1\alpha_2 - 2\beta_1\beta_2 \overset{\text{(ii)}}{=} 0. \end{aligned}$$

Thus the equation $\mathrm{Tr}\,\rho_2(T_1 + \Gamma(T_1)) = 0$ also follows. Therefore, the measurement $\{T_1 + \Gamma(T_1), T_2 + \Gamma(T_2)\}$ discriminates ρ_1 and ρ_2 perfectly. Here, note that ρ_1 and ρ_2 are not orthogonal if $\alpha_1, \alpha_2 \neq 1$. Thus perfect discrimination of two pure states in SEP is possible even when the two states are not orthogonal.

Example 1 gives a sufficient condition of perfect discrimination, but it does not give a necessary condition. Thus we give the following theorem as a necessary and sufficient condition for two pure states to be discriminated perfectly.

Theorem 2. *Two pure states $\rho_1 = \rho_1^A \otimes \rho_1^B$ and $\rho_2 = \rho_2^A \otimes \rho_2^B$ are perfectly distinguishable in SEP if and only if*

$$\mathrm{Tr}\,\rho_1^A\rho_2^A + \mathrm{Tr}\,\rho_1^B\rho_2^B \le 1.$$

Here, let us compare the necessary and sufficient condition in SEP with that in quantum theory. In quantum theory, the condition $(\mathrm{Tr}\,\rho_1^A\rho_2^A)(\mathrm{Tr}\,\rho_1^B\rho_2^B) = 0$ is necessary and sufficient to discriminate the two state in Theorem 2 perfectly. Thus we can find that measurements of SEP improve the performance of state discrimination.

Surprisingly, measurements of SEP improve the performance of multiple-copy state discrimination more dramatically. To see this fact, let us consider perfect discrimination of two $2n$-copies $\rho_1^{\otimes 2n}$ and $\rho_2^{\otimes 2n}$ of two pure states. Then $\rho_i^{\otimes 2n} = \rho_i^{\otimes n} \otimes \rho_i^{\otimes n}$ is a separable pure state on a bipartite system for $i = 1,2$. Thus $\rho_1^{\otimes 2n}$ and $\rho_2^{\otimes 2n}$ are perfectly distinguishable in SEP if $2(\mathrm{Tr}\,\rho_1\rho_2)^n = \mathrm{Tr}\,\rho_1^{\otimes n}\rho_2^{\otimes n} + \mathrm{Tr}\,\rho_1^{\otimes n}\rho_2^{\otimes n} \le 1$. This inequality always holds for a sufficiency large n if $\rho_1 \neq \rho_2$. Therefore, $\rho_1^{\otimes 2n}$ and $\rho_2^{\otimes 2n}$ are perfectly distinguishable in SEP. Of course, measurements in quantum theory are impossible to do the above perfect discrimination.

Next, we discuss how many states are perfectly distinguishable simultaneously in SEP. That is, our interest is the *capacity* N_{SEP} in SEP which is defined as the maximum number of simultaneously and perfectly distinguishable states in SEP:

$$\begin{aligned} &N_{\mathrm{SEP}} := \\ &\max\{\, n \in \mathbb{N} \mid \exists\{\rho_i\}_{i=1}^n,\ \exists\{M_j\}_{j=1}^n \text{ s.t. } \mathrm{Tr}\,\rho_i M_j = \delta_{ij} \,\}, \end{aligned}$$

where $\{\rho_i\}_{i=1}^n$ and $\{M_j\}_{j=1}^n$ are a family of states in SEP and a measurement of SEP, respectively. As stated in the previous paragraph, the performance of state discrimination in SEP is higher than that in quantum theory. Hence one might guess that the capacity in SEP is greater than that in quantum theory. However, the following proposition denies this guess.

Proposition 3. *The capacity N_{SEP} is $\dim(\mathcal{H}_A \otimes \mathcal{H}_B)$.*

Since the capacity in quantum theory is equal to the dimension of a quantum system, Proposition 3 means that SEP has the same capacity as quantum theory. Actually, Proposition 3 follows from [9, Lemma 24 (iii)] which is a more general statement on capacities. However, using [9, Lemma 24 (iii)] needs to be careful. For details, see the full version [22].

Table 1 summarizes the necessary and sufficient conditions of perfect discrimination and the capacities in quantum theory and SEP. The performance of perfect discrimination in SEP is better than that in quantum theory but the capacity in SEP is equal to that in quantum theory.

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