On $\alpha - z$-Rényi divergences in von Neumann algebras

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The α-z-Rényi divergences

For density operators ρ, σ on a finite dimensional Hilbert space:

$$D_{\alpha,z}(\rho\|\sigma) = \frac{1}{\alpha-1}\log\frac{\operatorname{Tr}\left(\rho^{\frac{\alpha}{2z}}\sigma^{\frac{1-\alpha}{z}}\rho^{\frac{\alpha}{2z}}\right)^z}{\operatorname{Tr}\rho},$$

where $0 < \alpha \neq 1$ and $z > 0$.

For each $z > 0$, $D_{\alpha,z}$ is a quantum extension of classical Rényi α-divergences for probability vectors p, q:

$$D_\alpha(p\|q) = \frac{1}{\alpha-1}\log(\sum_i p_i^\alpha q_i^{1-\alpha}).$$

Jakšić, Ogata, Pautrat and Pillett, 2011
Audenaert and Datta, 2015

The α-z-Rényi divergences

Important special cases:

- Relative entropy:

$$\lim_{\alpha\to 1} D_{\alpha,z}(\rho\|\sigma) = D_1(\rho\|\sigma) = \frac{\operatorname{Tr}\big(\rho(\log\rho - \log\sigma)\big)}{\operatorname{Tr}\rho}$$

- Petz-type (standard) Rényi divergence: $z = 1$, $0 < \alpha \neq 1$

$$D_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1}\log\frac{\operatorname{Tr}\big(\rho^\alpha\sigma^{1-\alpha}\big)}{\operatorname{Tr}\rho}$$

- Sandwiched Rényi divergence: $0 < z = \alpha \neq 1$

$$\tilde{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1}\log\frac{\operatorname{Tr}\big(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}}\big)}{\operatorname{Tr}\rho}$$

Data processing inequality (DPI)

For a quantum channel (CPTP map) Φ and any ρ, σ:

$$D_{\alpha,z}(\Phi(\rho)\|\Phi(\sigma)) \leq D_{\alpha,z}(\rho\|\sigma)$$

- not true for all values of α, z:

- Petz-type:[1] $\alpha \in (0,1) \cup (1,2]$;
- sandwiched:[2] $\alpha \in [1/2,1) \cup (1,\infty]$;
- general case:[3]

$$0 < \alpha < 1, \quad \max\{\alpha, 1-\alpha\} \leq z$$

or

$$\alpha > 1, \quad \max\{\alpha/2, \alpha-1\} \leq z \leq \alpha.$$

[1] Ando's convexity theorem, 1979

[2] S. Beigi, 2013; Frank and Lieb, 2013

[3] Carlen, Frank and Lieb, 2018; Zhang, 2020

Outline of this talk

- extension of $D_{\alpha,z}$ to the setting of von Neumann algebras
- DPI with respect to positive trace preserving maps (within the same bounds on parameters as in finite dimensions)
- equality in DPI implies sufficiency (reversibility) for 2-positive trace preserving maps

Our tools

- variational formula for $D_{\alpha,z}$
- known results in the sandwiched case
- properties of conditional expectations

von Neumann algebra extensions

The Rényi divergences were defined for normal positive functionals ψ, φ on a von Neumann algebra, using some technical tools:

- Araki relative entropy[4]: relative modular operator $\Delta_{\psi,\varphi}$
- Petz-type (Petz quasi divergence)[5]: $\Delta_{\psi,\varphi}$
- sandwiched Rényi divergence:[6] Araki-Masuda or Kosaki L^p-spaces
- general α-z Rényi divergences:[7] Haagerup L^p-spaces

[4] Araki, 1976
[5] Petz, 1985
[6] Berta, Scholtz and Tomamichel, 2018; AJ, 2018; 2021
[7] Kato and Ueda, 2023; Kato, 2024

von Neumann algebras and Haagerup L^p-spaces

Let $\mathcal{M}$ be a von Neumann algebra $\mathcal{M}$, with predual $\mathcal{M}_*$.

- Haagerup L^p-space $L^p(\mathcal{M})$, $0 < p \leq \infty$
- $\mathcal{M} = L^\infty(\mathcal{M})$, $\mathcal{M}_* \simeq L^1(\mathcal{M})$, $\varphi \mapsto h_\varphi$, $\mathrm{tr}\,(h_\varphi) = \varphi(1)$
- order isomorphism: $\mathcal{M}_*^+ \ni \varphi \mapsto h_\varphi \in L^1(\mathcal{M})^+$
- polar decomposition: for $0 < p < \infty$, $k \in L^p(\mathcal{M})$, $k = u|k|$:

$$u \in \mathcal{M} \text{ partial isometry, } \ |k| = h_\varphi^{1/p} \in L^p(\mathcal{M})^+, \ \varphi \in \mathcal{M}_*^+$$

von Neumann algebras and Haagerup L^p-spaces

For $0 < p < \infty$, $k \in L^p(\mathcal{M})$, put $\|k\|_p = (\mathrm{tr}\, |k|^p)^{1/p}$.

- For $1 < p < \infty$, $\|k\|_p$ is a norm in $L^p(\mathcal{M})$, which is a reflexive Banach space, with dual $L^p(\mathcal{M})^* \simeq L^q(\mathcal{M})$, $1/p + 1/q = 1$
- $\|k\|_p$ is a quasi norm for $0 < p < 1$
- Hölder inequality: for $1/p + 1/q = 1/r$, $0 < p, q, r \leq \infty$, $h \in L^p(\mathcal{M})$, $k \in L^q(\mathcal{M})$:

$$hk \in L^r(\mathcal{M}) \quad \text{and} \quad \|hk\|_r \leq \|h\|_p \|k\|_q$$

$D_{\alpha,z}$ for von Neumann algebras

Let $0 < \alpha \neq 1$, $0 < z$. For $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$, we define

$$D_{\alpha,z}(\psi\|\varphi) = \frac{1}{\alpha-1}\log\frac{Q_{\alpha,z}(\psi\|\varphi)}{\psi(1)}$$

where

$$Q_{\alpha,z}(\psi\|\varphi) := \begin{cases} \mathrm{tr}\left(h_\varphi^{\frac{1-\alpha}{2z}} h_\psi^{\frac{\alpha}{z}} h_\varphi^{\frac{1-\alpha}{2z}}\right)^z, & \text{if } 0<\alpha<1, \\ \|x\|_z^z, & \text{if } \alpha>1 \text{ and } h_\psi^{\frac{\alpha}{z}} = h_\varphi^{\frac{\alpha-1}{2z}} x h_\varphi^{\frac{\alpha-1}{2z}} \\ & \text{with } x \in s(\varphi)L^z(\mathcal{M})s(\varphi), \\ \infty, & \text{otherwise.} \end{cases}$$

Kato and Ueda, 2023; Kato 2024

Positive maps and the Petz dual

Let $\mathcal{M}, \mathcal{N}$ be von Neumann algebras, $\gamma : \mathcal{N} \to \mathcal{M}$ positive unital normal map.

- The predual map: $\gamma_* : L^1(\mathcal{M}) \to L^1(\mathcal{N})$,

$$\gamma_*(h_\omega) := h_{\omega\circ\gamma}, \quad \text{positive, trace preserving}$$

- Let $\rho \in \mathcal{M}_*^+$, $e := s(\rho)$, $e_0 := s(\rho\circ\gamma)$. The Petz dual $\gamma_\rho^* : e\mathcal{M}e \to e_0\mathcal{N}e_0$ is determined by

$$(\gamma_\rho^*)_*\big(h_{\rho\circ\gamma}^{1/2} b h_{\rho\circ\gamma}^{1/2}\big) = h_\rho^{1/2}\gamma(b)h_\rho^{1/2}, \qquad b \in \mathcal{N}^+.$$

  - positive, unital and normal,
  - n-positive whenever γ is.

DPI in von Neumann algebra setting

For any $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$, and a positive unital normal map $\gamma : \mathcal{N} \to \mathcal{M}$:

$$D_{\alpha,z}(\psi \circ \gamma \| \varphi \circ \gamma) \leq D_{\alpha,z}(\psi \| \varphi).$$

This was already proved for:

- Petz type: $\alpha \in (0,1) \cup (1,2]$, γ a Schwarz map[8],
- sandwiched: $\alpha \in [1/2, 1) \cup (1, \infty]$, γ completely positive[9], γ positive[10]
- $D_{\alpha,z}$ with $0 < \alpha < 1$, $\max\{\alpha, 1-\alpha\} \leq z$, γ positive[11]

[8] Petz, 1985
[9] Berta, Scholz and Tomamichel, 2018
[10] AJ, 2018, 2021
[11] Kato, 2024

DPI for sandwiched Rényi divergences, $\alpha > 1$

Let $\tilde{D}_\alpha(\psi\|\varphi) = D_{\alpha,\alpha}(\psi\|\varphi) < \infty$. Then $h_\psi = h_\varphi^{\frac{\alpha-1}{2\alpha}} x h_\varphi^{\frac{\alpha-1}{2\alpha}}$ for some $x \in L^\alpha(\mathcal{M})^+$ and

$$\tilde{Q}_\alpha(\psi\|\varphi) = Q_{\alpha,\alpha}(\psi\|\varphi) = \|x\|_\alpha^\alpha = \|h_\psi\|_{\alpha,\varphi}^\alpha$$

Kosaki L^p-norm: complex interpolation between

$$\|h\|_1 \text{ and } \|h_\varphi^{\frac{1}{2}} a h_\varphi^{\frac{1}{2}}\|_{\infty,\varphi} = \|a\|.$$

Since $\|\gamma_*(h)\|_1 \leq \|h\|_1$ and $\|\gamma_*(h_\varphi^{\frac{1}{2}} a h_\varphi^{\frac{1}{2}})\|_{\infty,\varphi\circ\gamma} = \|\gamma_\varphi^*(a)\| \leq \|a\|$,

$$\tilde{Q}_\alpha(\psi\circ\gamma\|\varphi\circ\gamma) = \|\gamma_*(h_\psi)\|_{\alpha,\varphi\circ\gamma}^\alpha \leq \|h_\psi\|_{\alpha,\varphi} = \tilde{Q}_\alpha(\psi\|\varphi), \quad \alpha > 1$$

by interpolation.

Beigi, 2013; AJ, 2018

Variational expressions

Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$.

(i) Let $0 < \alpha < 1$, $\max\{\alpha, 1-\alpha\} \le z$. Then

$$Q_{\alpha,z}(\psi\|\varphi) = \inf_{a \in \mathcal{M}^{++}} \left\{ \alpha \operatorname{tr}\left(\left(h_\psi^{\frac{\alpha}{2z}} a h_\psi^{\frac{\alpha}{2z}} \right)^{\frac{z}{\alpha}} \right) + (1-\alpha) \operatorname{tr}\left(\left(h_\varphi^{\frac{1-\alpha}{2z}} a^{-1} h_\varphi^{\frac{1-\alpha}{2z}} \right)^{\frac{z}{1-\alpha}} \right) \right\}.$$

(ii) Let $\alpha > 1$, $\max\{\alpha/2, \alpha - 1\} \le z$. Then

$$Q_{\alpha,z}(\psi\|\varphi) = \sup_{a \in \mathcal{M}^{+}} \left\{ \alpha \operatorname{tr}\left(\left(h_\psi^{\frac{\alpha}{2z}} a h_\psi^{\frac{\alpha}{2z}} \right)^{\frac{z}{\alpha}} \right) - (\alpha-1) \operatorname{tr}\left(\left(h_\varphi^{\frac{\alpha-1}{2z}} a h_\varphi^{\frac{\alpha-1}{2z}} \right)^{\frac{z}{\alpha-1}} \right) \right\}.$$

Finite dimensions: Zhang, 2020

A useful inequality

$\gamma : \mathcal{N} \to \mathcal{M}$ a normal positive unital map, $\rho \in \mathcal{M}_*^+$, $b \in \mathcal{N}^+$.

(1) If $p \in [1, \infty]$, then

$$\left\| h_{\rho\circ\gamma}^{\frac{1}{2p}} b h_{\rho\circ\gamma}^{\frac{1}{2p}} \right\|_p^p \geq \left\| h_{\rho}^{\frac{1}{2p}} \gamma(b) h_{\rho}^{\frac{1}{2p}} \right\|_p^p.$$

Proof.

Let $\omega \in \mathcal{N}_*^+$, $h_\omega = h_{\rho\circ\gamma}^{\frac{1}{2}} b h_{\rho\circ\gamma}^{\frac{1}{2}}$.

$$\begin{aligned} \left\| h_{\rho}^{\frac{1}{2p}} \gamma(b) h_{\rho}^{\frac{1}{2p}} \right\|_p^p &= Q_{p,p}(\omega \circ \gamma_\rho^* \| \rho) = Q_{p,p}(\omega \circ \gamma_\rho^* \| \rho \circ \gamma \circ \gamma_\rho^*) \\ &\leq [12] Q_{p,p}(\omega \| \rho \circ \gamma) = \left\| h_{\rho\circ\gamma}^{\frac{1}{2p}} b h_{\rho\circ\gamma}^{\frac{1}{2p}} \right\|_p^p \end{aligned}$$

□

[12] Sandwiched case: AJ, 2018

DPI in the von Neumann algebra setting

Let $0 < \alpha < 1$, $\max\{\alpha, 1-\alpha\} \leq z$. We have

$$Q_{\alpha,z}(\psi\|\varphi) = \inf_{a \in \mathcal{M}^{++}} \left\{ \alpha \left\| h_\psi^{\frac{1}{2p}} a h_\psi^{\frac{1}{2p}} \right\|_p^p + (1-\alpha) \left\| h_\varphi^{\frac{1}{2r}} a^{-1} h_\varphi^{\frac{1}{2r}} \right\|_r^r \right\},$$

with $p := \frac{z}{\alpha}$, $r := \frac{z}{1-\alpha}$. In the above bounds, $p, r \geq 1$.

By the inequality (1) and the Choi inequality:

$$\gamma(b)^{-1} \leq \gamma(b^{-1}),$$

we get

$$Q_{\alpha,z}(\psi \circ \gamma \| \varphi \circ \gamma) \geq Q_{\alpha,z}(\psi\|\varphi).$$

A useful inequality

$\gamma : \mathcal{N} \to \mathcal{M}$ a normal positive unital map, $\rho \in \mathcal{M}_*^+$, $b \in \mathcal{N}^+$.

(2) If $p \in [1/2, 1)$, then

$$\left\| h_{\rho\circ\gamma}^{\frac{1}{2p}} b h_{\rho\circ\gamma}^{\frac{1}{2p}} \right\|_p^p \leq \left\| h_\rho^{\frac{1}{2p}} \gamma(b) h_\rho^{\frac{1}{2p}} \right\|_p^p.$$

Proof.

Let $\omega \in \mathcal{N}_*^+$, $h_\omega = h_{\rho\circ\gamma}^{\frac{1}{2}} b h_{\rho\circ\gamma}^{\frac{1}{2}}$.

$$\begin{aligned} \left\| h_\rho^{\frac{1}{2p}} \gamma(b) h_\rho^{\frac{1}{2p}} \right\|_p^p &= Q_{p,p}(\omega \circ \gamma_\rho^* \| \rho) = Q_{p,p}(\omega \circ \gamma_\rho^* \| \rho \circ \gamma \circ \gamma_\rho^*) \\ &\geq {}^{13} Q_{p,p}(\omega \| \rho \circ \gamma) = \left\| h_{\rho\circ\gamma}^{\frac{1}{2p}} b h_{\rho\circ\gamma}^{\frac{1}{2p}} \right\|_p^p \end{aligned}$$

□

[13] Sandwiched case: AJ, 2021

DPI in the von Neumann algebra setting

Let $\alpha > 1$, $\max\{\alpha/2, \alpha - 1\} \leq z \leq \alpha$. We have

$$Q_{\alpha,z}(\psi\|\varphi) = \sup_{a\in\mathcal{M}^+} \left\{ \alpha \left\| h_\psi^{\frac{1}{2p}} a h_\psi^{\frac{1}{2p}} \right\|_p^p - (\alpha - 1) \left\| h_\varphi^{\frac{1}{2q}} a h_\varphi^{\frac{1}{2q}} \right\|_q^q \right\},$$

with $p := \frac{z}{\alpha}$, $q := \frac{z}{\alpha - 1}$. In the above bounds, $p \in [1/2, 1)$, $q \geq 1$.

By the inequalities (1) and (2) we get

$$Q_{\alpha,z}(\psi \circ \gamma\|\varphi \circ \gamma) \leq Q_{\alpha,z}(\psi\|\varphi).$$

DPI in the von Neumann algebra setting

Theorem

Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$ and let $\gamma : \mathcal{N} \to \mathcal{M}$ be a normal positive unital map.

Assume either of the following conditions:

(i) $0 < \alpha < 1$, $\max\{\alpha, 1-\alpha\} \leq z$,

(ii) $\alpha > 1$, $\max\{\alpha/2, \alpha - 1\} \leq z \leq \alpha$.

Then we have

$$D_{\alpha,z}(\psi \circ \gamma \| \varphi \circ \gamma) \leq D_{\alpha,z}(\psi \| \varphi).$$

Sufficient channels and equality in DPI

A channel is a 2-positive unital normal map $\gamma : \mathcal{N} \to \mathcal{M}$.

Let $\psi, \varphi \in \mathcal{M}_*^+$. We say that γ is sufficient with respect to $\{\psi, \varphi\}$ if there exists a recovery channel $\beta : \mathcal{M} \to \mathcal{N}$ such that

$$\psi \circ \gamma \circ \beta = \psi, \qquad \varphi \circ \gamma \circ \beta = \varphi.$$

Petz theorem: Assume that $D_1(\psi\|\varphi) < \infty$. Then γ is sufficient with respect to $\{\psi, \varphi\}$ if and only if

$$D_1(\psi \circ \gamma \| \varphi \circ \gamma) = D_1(\psi\|\varphi).$$

A similar result holds for the transition probability ($D_{\frac{1}{2},1}$).

Petz, 1986, 1988

Known results on equality in DPI

Characterization of sufficient channels:

- Petz-type: $D_{\alpha,1}$, $\alpha \in (0,1) \cup (1,2)$[14]
- sandwiched: $D_{\alpha,\alpha}$, $\alpha \in (1/2,1) \cup (1,\infty)$[15]

Other equality conditions for $D_{\alpha,z}$ were found in finite dimensions[16]

- no clear relation to sufficiency of channels (apart from some special cases)[17]

[14] AJ and Petz, 2006; Hiai et al, 2011; Hiai and Mosonyi 2017; Hiai, 2018
[15] AJ, 2018, 2021
[16] Leditzky, Rouzé and Datta, 2017; Zhang 2020
[17] Hiai and Mosonyi, 2017

Universal recovery channel

The Petz dual γ_φ^* is a universal recovery channel:

- $\varphi \circ \gamma \circ \gamma_\varphi^* = \varphi$
- Let $\psi \in \mathcal{M}_*^+$ be such that $s(\psi) \leq s(\varphi)$. Then γ is sufficient with respect to $\{\psi, \varphi\}$ if and only if

$$\psi \circ \gamma \circ \gamma_\varphi^* = \psi.$$

- Equivalently, $\psi \circ \mathcal{E} = \psi$ for the conditional expectation $\mathcal{E} : \mathcal{M} \to \mathcal{M}$ onto the fixed points of $\gamma \circ \gamma_\varphi^*$.

Petz, 1986, 1988

Sufficient channels and equality in DPI for $D_{\alpha,z}$

Theorem

Let $0 < \alpha < 1$, $\max\{\alpha, 1-\alpha\} \leq z$.

Let $\psi, \varphi \in \mathcal{M}_*^+$ and let $\gamma : \mathcal{N} \to \mathcal{M}$ be a channel. Assume that

$\alpha < z$ and $s(\varphi) \leq s(\psi)$ or $1-\alpha < z$ and $s(\psi) \leq s(\varphi)$.

Then γ is sufficient with respect to $\{\psi, \varphi\}$ if and only if

$$D_{\alpha,z}(\psi \circ \gamma \| \varphi \circ \gamma) = D_{\alpha,z}(\psi \| \varphi).$$

Sufficient channels and equality in DPI for $D_{\alpha,z}$

Theorem

Let $\alpha > 1$, $\max\{\alpha/2, \alpha - 1\} \le z \le \alpha < z + 1$.

Let $\psi, \varphi \in \mathcal{M}_*^+$ be such that $D_{\alpha,z}(\psi\|\varphi) < \infty$. A channel $\gamma : \mathcal{N} \to \mathcal{M}$ is sufficient with respect to $\{\psi, \varphi\}$ if and only if

$$D_{\alpha,z}(\psi \circ \gamma \| \varphi \circ \gamma) = D_{\alpha,z}(\psi\|\varphi).$$

A sketch of proof for $\alpha > 1$

Put $\psi_0 := \psi \circ \gamma$, $\varphi_0 := \varphi \circ \gamma$, $p = \frac{z}{\alpha}$, $q = \frac{z}{\alpha-1} > 1$.

- By the assumptions, $D_{\alpha,z}(\psi_0 \| \varphi_0) \le D_{\alpha,z}(\psi \| \varphi) < \infty$.
- In this case, for some $x \in L^z(\mathcal{M})^+$, $x_0 \in L^z(\mathcal{N})^+$:

$$h_\psi^{\frac{1}{p}} = h_\varphi^{\frac{1}{2q}} x h_\varphi^{\frac{1}{2q}}, \qquad h_{\psi_0}^{\frac{1}{p}} = h_{\varphi_0}^{\frac{1}{2q}} x_0 h_{\varphi_0}^{\frac{1}{2q}}$$

- Variational expression:

$$Q_{\alpha,z}(\psi \| \varphi) = \sup_{w \in L^q(\mathcal{M})^+} \left\{ \alpha \operatorname{tr}\left((x^{\frac{1}{2}} w x^{\frac{1}{2}})^p \right) - (\alpha - 1) \operatorname{tr}\left(w^q \right) \right\},$$

  uniquely attained at $\bar{w} := x^{\alpha-1} \in L^q(\mathcal{M})^+$.

- Similarly for ψ_0, φ_0, $\bar{w}_0 := x_0^{\alpha-1} \in L^q(\mathcal{N})^+$.

A sketch of proof for $\alpha > 1$

- Let $\omega \in \mathcal{M}_*^+$, $\omega_0 \in \mathcal{N}_*^+$ be such that

$$h_\omega = h_\varphi^{\frac{q-1}{2q}} \bar{w} h_\varphi^{\frac{q-1}{2q}}, \qquad h_{\omega_0} = h_{\varphi_0}^{\frac{q-1}{2q}} \bar{w}_0 h_{\varphi_0}^{\frac{q-1}{2q}}.$$

- Then

$$D_{q,q}(\omega_0\|\varphi_0) = D_{\alpha,z}(\psi_0\|\varphi_0) = D_{\alpha,z}(\psi\|\varphi) = D_{q,q}(\omega\|\varphi).$$

- By the variational formula and uniqueness, we get $\omega_0 = \omega \circ \gamma$.
- By known properties of $D_{q,q}$, $q > 1$, γ is sufficient with respect to $\{\omega, \varphi\}$, so that

$$\omega \circ \mathcal{E} = \omega.$$

- Extensions of $\mathcal{E}$ to the L^p-spaces and their properties[18] give us $\psi \circ \mathcal{E} = \psi$.

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[18] Junge and Xu, 2003

Further results on $D_{\alpha,z}$

- Monotonicity in z: $z \mapsto D_{\alpha,z}(\psi\|\varphi)$ is

  - increasing on $(0,\infty)$ for $0<\alpha<1$,
  - decreasing on $[\alpha/2,\infty)$ for $\alpha>1$.

- Monotonicity in α: $\alpha \mapsto D_{\alpha,z}(\psi\|\varphi)$ is

  - increasing on $(0,1)$ for all $z>0$,
  - increasing on $(1,2z]$ for $z>1/2$.

- The limits $\alpha\to 1$:

  - $\lim_{\alpha\nearrow 1} D_{\alpha,z}(\psi\|\varphi) = D_1(\psi\|\varphi)$ for $z>0$,
  - $\lim_{\alpha\searrow 1} D_{\alpha,z}(\psi\|\varphi) = D_1(\psi\|\varphi)$ for $z>1/2$ if

  $D_{\alpha,z}(\psi\|\varphi)<\infty$ for some $\alpha\in(1,2z]$.

Thank you for your attention.