1 The JDG splitting

Let $\mathcal{M}$ be a von Neumann algebra, $\Phi : \mathcal{M} \to \mathcal{M}$ a normal unital cp map. Assume that ω is a faithful normal state invariant for Φ: $\omega \circ \Phi = \omega$. Let us denote $\mathbf{S}_0 = \{\Phi^k,\ k \in \mathbb{N}\}$ and let $\mathbf{S}$ be the closure of $\mathbf{S}_0$ in the point-weak*-topology. According to [**?**, **?**], $\mathbf{S}$ is a compact semitopological semigroup in this topology.

Let $x \in \mathcal{M}$ and let $O_0(x) := \{\Phi^k(x), k \in \mathbb{N}\}$ be the orbit of x under $\mathbf{S}_0$. Then the weak*-closure $\bar{O}_0(x)$ is the orbit of x under $\mathbf{S}$,

$$\bar{O}_0(x) = O(x) := \{T(x), T \in \mathbf{S}\}.$$

Indeed, we have by definition that $O(x) \subseteq \bar{O}_0(x)$ and the other inclusion follows by compactness: Let x_0 be a weak*-limit point of $O_0(x)$. For $\psi \in \mathcal{M}_*$ and $\epsilon > 0$, let $U_{\psi,\epsilon}$ be the set of $T \in \mathbf{S}$ such that $|\psi(T(x)) - \psi(x_0)| \leq \epsilon$. Then $U_{\psi,\epsilon}$ is a collection of closed subsets with a finite intersection property.

Further, $\mathbf{S}$ contains a minimal ideal $M(\mathbf{S})$ which is a compact topological group. Let E be the unit of this group, then $M(\mathbf{S}) = E \circ \mathbf{S}$. Since $E \in \mathbf{S}$ and $E^2 = E$, E is a normal conditional expectation preserving ω and we have $TE = ET$ for all $T \in \mathbf{S}$.

Let us put

$$\begin{aligned} \mathcal{M}_r &:= \{x \in \mathcal{M},\ T(x)^*T(x) = T(x^*x), \forall T \in \mathbf{S}\} \\ \mathcal{M}_s &:= \{x \in \mathcal{M},\ 0 \in O(x)\}. \end{aligned}$$

Lemma 1. $\mathcal{M}_r = E(\mathcal{M}) = \overline{\mathrm{span}\{x \in \mathcal{M},\ \Phi(x) = \lambda x,\ |\lambda| = 1\}}^{w*}$.

Proof. It is well known that for an ucp map T and $x \in \mathcal{M}$, we have $T(x^*x) = T(x)^*T(x)$ if and only if $T(x^*y) = T(x)^*T(y)$ for all $y \in \mathcal{M}$. This implies that $\mathcal{M}_r$ is a weak*-closed subspace in $\mathcal{M}$. Let us denote the last set on the RHS by $\mathcal{M}_0$. We will prove the chain of inclusions

$$\mathcal{M}_r \subseteq E(\mathcal{M}) \subseteq \mathcal{M}_0 \subseteq \mathcal{M}_r.$$

If $x \in \mathcal{M}_r$, then since $E \in \mathbf{S}$, we have $E(x^*x) = E(x)^*E(x)$. This implies $E((x - E(x))^*(x - E(x))) = 0$ and since E is faithful, we have $x = E(x) \in E(\mathcal{M})$.

To prove the second inclusion, let $x = E(x)$. Let $\widehat{M(\mathbf{S})}$ be the dual group and let $\chi \in \widehat{M(\mathbf{S})}$. Let us define

$$x_\chi := \int_{M(\mathbf{S})} \overline{\chi(T)} T(x) d\mu(T),$$

where μ is the normalized Haar measure over $M(\mathbf{S})$. The integral is defined in the weak*-topology, so we have $x_\chi \in \mathcal{M}$ and since for $T \in M(\mathbf{S})$, we have

$T = TE = ET$, we obtain

$$\Phi(x_\chi) = \int_{M(\mathbf{S})} \overline{\chi(T)} \Phi T(x) d\mu(T) = \int_{M(\mathbf{S})} \overline{\chi(T)} E\Phi T(x) d\mu(T)$$
$$= \chi(E\Phi) \int_{M(\mathbf{S})} \overline{\chi(T)} T(x) d\mu(T) = \chi(E\Phi) x_\chi$$

(since $E\Phi = \Phi E \in M(\mathbf{S})$), so that $x_\chi \in \mathcal{M}_0$. Let now $\psi \in \mathcal{M}_*$ be such that $\psi(y) = 0$ for any peripheral eigenvector y of Φ. Then

$$0 = \psi(x_\chi) = \int_{M(\mathbf{S})} \overline{\chi(T)} \psi(T(x)) d\mu(T), \qquad \forall \chi \in \widehat{M(\mathbf{S})}.$$

Since the characters span the space of square integrable functions on $M(\mathbf{S})$ and the function $T \mapsto \psi(T(x))$ is continuous, it follows that $\psi(T(x)) = 0$ for all $T \in M(\mathbf{S})$, in particular, $\psi(x) = \psi(E(x)) = 0$. This implies that $\{x,\ \Phi(x) = \lambda x,\ |\lambda| = 1\}^\perp \subseteq E(\mathcal{M})^\perp$, so that

$$E(\mathcal{M}) = E(\mathcal{M})^{\perp\perp} \subseteq \{x,\ \Phi(x) = \lambda x,\ |\lambda| = 1\}^{\perp\perp} = \mathcal{M}_0.$$

Finally, let $\Phi(x) = \lambda x$ for some $|\lambda| = 1$, then $\Phi^k(x) = \lambda^k x$ for all $k \in \mathbb{N}$. Let $S \in \mathbf{S}$, then there is a net $\Phi^{n_\alpha} \to S$, so that $S(x) = \lim \Phi^{n_\alpha}(x) = \lim \lambda^{n_\alpha} x$, so that $S(x) = \mu x$ with $\mu = \lim \lambda^{n_\alpha}$. By Schwartz inequality, $S(x^*x) \geq S(x)^* S(x) = x^*x$ and applying the faithful normal invariant state ω we obtain $S(x^*x) = x^*x$, so that $x \in \mathcal{M}_r$. This proves the last of the above chain of inclusions.

□

Lemma 2. *$\mathcal{M}_s = Ker(E)$.*

Proof. Since $E \in M(\mathbf{S})$, we clearly have $KerE \subseteq \mathcal{M}_s$. Conversely, let $x \in\in \mathcal{M}_s$ and let $S \in \mathbf{S}$ be such that $S(x) = 0$. Since $ES \in M(\mathbf{S})$, there is some $T \in M(\mathbf{S})$ such that $TES = E$, so that we have

$$E(x) = TES(x) = 0.$$

□