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Quantum process discrimination with restricted strategies
by Kenji Nakahira

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Quantum process discrimination with restricted strategies

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(Dated: August 24, 2021)

The discrimination of quantum processes, including quantum states, channels, and superchannels, is a fundamental topic in quantum information theory. It is often of interest to analyze the optimal performance that can be achieved when discrimination strategies are restricted to a given subset of all strategies allowed by quantum mechanics. In this Letter, we present a general formulation of the task of finding the maximum success probability for discriminating quantum processes as a convex optimization problem whose Lagrange dual problem exhibits zero duality gap. The proposed formulation can be applied to any restricted strategy. We provide a simple example in which the dual problem given by our formulation can be much easier to solve than the original problem. We also show that the optimal performance of each restricted process discrimination problem can be written in terms of a certain robustness measure. This finding has the potential to provide a deeper insight into the discrimination performance of various restricted strategies.

Quantum processes are fundamental building blocks of quantum information theory. The tasks of discriminating between quantum processes are of crucial importance in quantum communication, quantum metrology, quantum cryptography, etc. In many situations, it is reasonable to assume that the available discrimination strategies (also known as quantum testers) are restricted to a certain subset of all possible testers in quantum mechanics. For example, in practical situations, we are usually concerned only with discrimination strategies that are readily implementable with current technology. Another example is a setting where discrimination is performed by two or more parties whose communication is limited. In such settings, one may naturally ask how the performance of an optimal restricted tester can be evaluated. To answer this question, different individual problems of distinguishing quantum states [1–5], measurements [6–9], and channels [10–19] have been investigated.

It is known that if all quantum testers are allowed, then the problem of finding the maximum success probability of guessing which process was applied can be formalized as a semidefinite programming problem, and its Lagrange dual problem has zero duality gap [20]. Many discrimination problems of quantum states, measurements, and channels have been addressed through the analysis of their dual problems [7, 16, 19–31]. However, in a general case where the allowed testers are restricted, the problem cannot be formalized as a semidefinite programming problem.

In this Letter, we provide a general method to analyze quantum process discrimination problems in which discrimination testers are restricted to given types of testers. We show that the task of finding the maximum success probability for discriminating any quantum processes can be formulated as a convex optimization problem even if the allowed testers are restricted to any subset of all testers and that its Lagrange dual problem has zero duality gap. It should be mentioned that, to our knowledge, a convex programming formulation applicable to any restricted strategy has not yet been reported even in quantum state discrimination problems. In some scenarios, the dual problem can be much easier to solve analytically or numerically than the original problem, as we will demonstrate through a simple example. Our approach can deal with process discrimination problems in both cases with and without the restriction of testers within a common framework, which makes it easy to compare their optimal values. Note that we use the quantum mechanical notation for convenience, but since our method essentially relies only on convex analysis, our techniques are applicable to a general operational probabilistic theory (including a theory that does not obey the no-restriction hypothesis [32]).

The robustness of a resource, which is a topic closely related to discrimination problems, has been recently extensively investigated. It is known that the robustness of a process can be seen as a measure of its advantage over all resource-free processes in some discrimination task [33–40]. Conversely, we show that the optimal performance of any restricted process discrimination problem is characterized by a certain robustness measure.

Quantum process discrimination — We first review quantum process discrimination problems where all possible testers are allowed. Let $\mathbb{C}$ and $\mathbb{R}_+$ denote, respectively, the sets of all complex and nonnegative real numbers. Also, let Her_V, Pos_V, and Den_V be, respectively, the sets of all Hermitian, positive semidefinite, and density matrices on a system V. We refer to a concatenation of one or more quantum operations (i.e., completely positive maps) as a quantum process. We here address the problem of discriminating M processes, $\hat{\mathcal{E}}_1, \dots, \hat{\mathcal{E}}_M$, where each $\hat{\mathcal{E}}_m$ is the concatenation of T channels $\hat{\Lambda}_m^{(1)}, \dots, \hat{\Lambda}_m^{(T)}$ with ancillary systems (see Fig. 1). $\hat{\mathcal{E}}_m$ is expressed by $\hat{\mathcal{E}}_m = \hat{\Lambda}_m^{(T)} \circledast \cdots \circledast \hat{\Lambda}_m^{(1)}$, where $\circledast$ denotes the concatenation. Such a process represented by a concatenation of T channels is referred to as a quantum comb with T time steps [41]. States, channels, and superchannels, which are processes that transform quantum channels to quantum channels, are special cases of quantum combs. In the particular case where, for each m, $\hat{\mathcal{E}}_m$ has no ancillary system and $\hat{\Lambda}_m^{(1)}, \dots, \hat{\Lambda}_m^{(T)}$ are the same channel, denoted by $\hat{\Lambda}_m$, the problem reduces to the problem of discriminating M channels $\hat{\Lambda}_1, \dots, \hat{\Lambda}_M$ with T uses. For simplicity, we restrict ourselves to the case where each $\hat{\mathcal{E}}_m$ is a quantum comb with $T = 2$ time steps in this Letter, but our approach can be readily extended to the case where each $\hat{\mathcal{E}}_m$ is a more general quantum process. As shown in Fig. 1, to

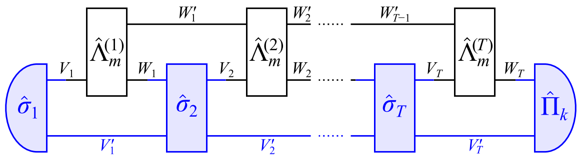

FIG. 1. General protocol for the discrimination of quantum combs $\{\hat{\mathcal{E}}_m = \hat{\Lambda}_m^{(T)} \circledast \cdots \circledast \hat{\Lambda}_m^{(1)}\}_{m=1}^M$ (plotted in black) with T time steps, where $W_1', \dots, W_{T-1}'$ are ancillary systems. Any physically allowed discrimination strategy can be represented by a tester (plotted in blue), which consists of a state $\hat{\sigma}_1$, channels $\hat{\sigma}_2, \dots, \hat{\sigma}_T$, and a measurement $\{\hat{\Pi}_k\}_{k=1}^M$.

discriminate between given combs, we first prepare a bipartite system $V_1 \otimes V_1'$ in an initial state $\hat{\sigma}_1$. One part V_1 is sent through the channel $\hat{\Lambda}_m^{(1)}$, followed by a channel $\hat{\sigma}_2$. After that, we send the system V_2 through the channel $\hat{\Lambda}_m^{(2)}$ and perform a measurement $\{\hat{\Pi}_k\}_{k=1}^M$ on the system $W_2 \otimes V_2'$. Such a collection of $\{\hat{\sigma}_1, \hat{\sigma}_2, \{\hat{\Pi}_k\}_{k=1}^M\}$, which can be expressed as $\{\hat{\Phi}_k := \hat{\Pi}_k \circledast \hat{\sigma}_2 \circledast \hat{\sigma}_1\}_{k=1}^M$, is called a tester. Any discrimination strategy, including an entanglement-assisted strategy and an adaptive strategy, can be represented by a tester. Let $\tilde{V} := W_2 \otimes V_2 \otimes W_1 \otimes V_1$. A comb $\hat{\mathcal{E}}_m$ is uniquely specified by its Choi-Jamiołkowski representation [41–43], which is defined as

$$\mathcal{E}_m := (\hat{\mathcal{E}}_m \otimes \mathbb{1}_{V_2 \otimes V_1})(|I_{V_2 \otimes V_1}\rangle\!\rangle\langle\!\langle I_{V_2 \otimes V_1}|) \in \mathsf{Pos}_{\tilde{V}},$$

where $|I_V\rangle\!\rangle := \sum_n |n\rangle |n\rangle \in V \otimes V$. For each tester element $\hat{\Phi}_k$, let

$$\Phi_k := (\hat{\Phi}_k \otimes \mathbb{1}_{V_2 \otimes V_1})^\dagger (|I_{V_2 \otimes V_1}\rangle\!\rangle\langle\!\langle I_{V_2 \otimes V_1}|) \in \mathsf{Pos}_{\tilde{V}},$$

where † is the adjoint operator. When there is no confusion, we simply refer to these representations $\mathcal{E}_m$ and $\{\Phi_k\}$ (which are denoted by the same letters without the hat) a comb and tester, respectively. Let $\mathcal{P}_{\mathrm{G}}$ be the set of all such testers $\Phi := \{\Phi_k\}_{k=1}^M$, which can be written as (see [41] for details)

$$\mathcal{P}_{\mathrm{G}} = \left\{ \{\Phi_m\}_{m=1}^M \subset \mathsf{Pos}_{\tilde{V}} : \sum_{m=1}^M \Phi_m \in \mathcal{S}_{\mathrm{G}} \right\}, \tag{1}$$

where

$$\mathcal{S}_{\mathrm{G}} := \{ I_{W_2} \otimes \tau_2 : \tau_2 \in \mathsf{Pos}_{V_2 \otimes W_1 \otimes V_1}, \\ \tau_1 \in \mathsf{Den}_{V_1}, \ \mathrm{Tr}_{V_2} \tau_2 = I_{W_1} \otimes \tau_1 \}.$$

The probability that a tester Φ gives the outcome k for the comb $\mathcal{E}_m$ is given by $\langle \Phi_k, \mathcal{E}_m \rangle := \mathrm{Tr}(\Phi_k \mathcal{E}_m)$. The task of finding the maximum success probability for discriminating the given quantum combs $\{\mathcal{E}_m\}_{m=1}^M$ with prior probabilities $\{p_m\}_{m=1}^M$ can be formulated as an optimization problem, namely [20]

$$\begin{aligned} &\text{maximize} \quad P(\Phi) := \sum_{m=1}^M p_m \langle \Phi_m, \mathcal{E}_m \rangle \\ &\text{subject to} \quad \Phi \in \mathcal{P}_{\mathrm{G}}. \end{aligned}$$

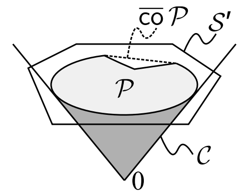

FIG. 2. Schematic diagram of the closed convex hull of $\mathcal{P}$, which is the intersection of a closed convex cone $\mathcal{C}$ and a convex set $\mathcal{S}' := \{\Phi : \sum_{m=1}^M \Phi_m \in \mathcal{S}\}$.

Restricted discrimination — We now consider the situation that the allowed testers are restricted to a nonempty subset $\mathcal{P}$ of $\mathcal{P}_{\mathrm{G}}$; in this case, the problem is formulated as

$$\begin{aligned} &\text{maximize} \quad P(\Phi) \\ &\text{subject to} \quad \Phi \in \mathcal{P}. \end{aligned} \tag{P}$$

Let us interpret each tester as a vector in the real vector space $\mathsf{Her}_{\tilde{V}}^M$. This means that one can work with linear combinations of testers $\Phi^{(1)}, \Phi^{(2)}, \dots$; a tester that applies $\Phi^{(i)} := \{\Phi_k^{(i)}\}_{k=1}^M$ with probability q_i is represented as $\sum_i q_i \Phi^{(i)} = \{\sum_i q_i \Phi_k^{(i)}\}_{k=1}^M$. Let us denote the closure and the closed convex hull of $\mathcal{P}$ by, respectively, $\overline{\mathcal{P}}$ and $\overline{\mathrm{co}}\,\mathcal{P}$. Then, one can easily see that the optimal value of Problem (P) remains the same if the feasible set $\mathcal{P}$ is replaced by $\overline{\mathrm{co}}\,\mathcal{P}$. Indeed, an optimal solution, denoted by $\Phi^\star \in \overline{\mathrm{co}}\,\mathcal{P}$, to Problem (P) with $\mathcal{P}$ relaxed to $\overline{\mathrm{co}}\,\mathcal{P}$ can be represented as a probabilistic mixture of $\Phi^{(1)}, \Phi^{(2)}, \dots \in \overline{\mathcal{P}}$, i.e., $\Phi^\star = \sum_i \nu_i \Phi^{(i)}$ for some probability distribution $\{\nu_i\}_i$[44]. Since $P(\Phi^\star) \le P[\Phi^{(i)}]$ holds for some i, $\Phi^{(i)} \in \overline{\mathcal{P}}$ must be an optimal solution to the relaxed problem. Thus, Problem (P), whose objective function is convex by construction, is transformed into a convex optimization problem by relaxing $\mathcal{P}$ to $\overline{\mathrm{co}}\,\mathcal{P}$. However, this relaxed problem is often very difficult to solve directly.

We find that, for any feasible set $\mathcal{P}$, each tester $\Phi \in \mathcal{P}$ can be interpreted as an element in some convex cone such that the sum $\sum_{m=1}^M \Phi_m$ is in some convex set. Specifically, we can choose a closed convex cone $\mathcal{C}$ and a closed convex set $\mathcal{S}$ such that (see Fig. 2)

$$\overline{\mathrm{co}}\,\mathcal{P} = \left\{ \Phi \in \mathcal{C} : \sum_{m=1}^M \Phi_m \in \mathcal{S} \right\}, \quad \mathcal{C} \subseteq \mathcal{C}_{\mathrm{G}}, \quad \mathcal{S} \subseteq \mathcal{S}_{\mathrm{G}}, \tag{2}$$

where $\mathcal{C}_{\mathrm{G}} := \mathsf{Pos}_{\tilde{V}}^M$. Such $\mathcal{C}$ and $\mathcal{S}$ always exist [45]. Equation (1) can be regarded as a special case of this equation with $\mathcal{P} = \mathcal{P}_{\mathrm{G}}$, $\mathcal{C} = \mathcal{C}_{\mathrm{G}}$, and $\mathcal{S} = \mathcal{S}_{\mathrm{G}}$ (note that $\overline{\mathrm{co}}\,\mathcal{P}_{\mathrm{G}} = \mathcal{P}_{\mathrm{G}}$ holds). Let

$$\mathcal{D}_{\mathcal{C}} := \left\{ \chi \in \mathsf{Her}_{\tilde{V}} : \sum_{m=1}^M \langle \Phi_m, \chi - p_m \mathcal{E}_m \rangle \ge 0 \ (\forall \Phi \in \mathcal{C}) \right\};$$

then, we can easily verify that

$$D_{\mathcal{S}}(\chi) := \max_{\varphi \in \mathcal{S}} \langle \varphi, \chi \rangle \ge \sum_{m=1}^M \langle \Phi_m^\star, \chi \rangle \ge P(\Phi^\star)$$

holds for any $\chi \in \mathcal{D}_C$. The first and second inequalities follow from $\sum_{m=1}^{M} \Phi_m^\star \in \mathcal{S}$ and $\Phi^\star \in C$, respectively. Thus, the optimal value of the following problem

$$\begin{array}{ll} \text{minimize} & D_{\mathcal{S}}(\chi) \\ \text{subject to} & \chi \in \mathcal{D}_C \end{array} \tag{D}$$

is not less than that of Problem (P). We can see that Problem (D), which is the so-called Lagrange dual problem of Problem (P), has zero duality gap, as shown in the following theorem [proved in Sec. II of the Supplemental Material (SM)]:

Theorem 1 Let us arbitrarily choose a closed convex cone C and a closed convex set $\mathcal{S}$ satisfying Eq. (2); then, the optimal values of Problems (P) and (D) are the same.

In several problems, Problem (D) provides an efficient way to find the optimal value of Problem (P). Note that the difficulty of solving Problem (D) depends on the choice of C and $\mathcal{S}$. Using Theorem 1, we can easily derive a necessary and sufficient condition for an optimal restricted strategy to be optimal within the set of all strategies (see Sec. V of the SM).

Example — We illustrate the use of Theorem 1 in the following example. Let us consider the problem of discriminating three qubit channels $\hat{\Lambda}_1, \hat{\Lambda}_2, \hat{\Lambda}_3$ with $T = 2$ uses, in which case V_1, W_1, V_2, and W_2 are all qubit systems and $\hat{\mathcal{E}}_m = \hat{\Lambda}_m \circledast \hat{\Lambda}_m$ (i.e., $\mathcal{E}_m = \Lambda_m \otimes \Lambda_m$) holds. Assume that the prior probabilities are equal and that each $\hat{\Lambda}_m$ is the unitary channel represented by $\hat{\Lambda}_m(\rho) = U^m \rho U^{-m}$, where $U := \mathrm{diag}(1, \omega)$ and $\omega := \exp(2\pi\sqrt{-1}/3)$. Then, we have

$$\Lambda_m = \begin{bmatrix} 1 & 0 & 0 & \omega^{-m} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \omega^m & 0 & 0 & 1 \end{bmatrix}.$$

We consider the case where a tester is restricted to a sequential one. The set of all sequential testers in $\mathcal{P}_\mathrm{G}$, denoted by $\mathcal{P}_\mathrm{seq}$, is (see Fig. 3 and Sec. III of the SM)

$$\mathcal{P}_\mathrm{seq} := \left\{ \left\{ \sum_j B_m^{(j)} \otimes A_j \right\}_{m=1}^{3} : \{A_j\} \in \mathsf{Test},\ \{B_m^{(j)}\}_m \in \mathsf{Test}_3 \right\},$$

where

$$\mathsf{Test}_M := \left\{ \{B_m\}_{m=1}^{M} \subset \mathsf{Pos}_4 : \sum_{m=1}^{M} B_m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \rho,\ \rho \in \mathsf{Den}_2 \right\},$$

$\mathsf{Test} := \bigcup_{M=1}^{\infty} \mathsf{Test}_M$, and Pos_n and Den_n are, respectively, the sets of all positive semidefinite and density matrices of order n. Problem (P) with $\mathcal{P} = \mathcal{P}_\mathrm{seq}$ can be written as the following non-convex programming problem:

$$\begin{array}{ll} \text{maximize} & \displaystyle\frac{1}{3} \sum_{m=1}^{3} \sum_j \langle B_m^{(j)}, \Lambda_m \rangle \langle A_j, \Lambda_m \rangle \\ \text{subject to} & \{A_j\} \in \mathsf{Test},\ \{B_m^{(j)}\}_m \in \mathsf{Test}_3. \end{array} \tag{3}$$

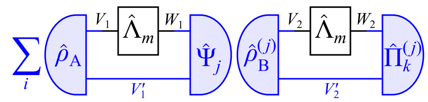

FIG. 3. Discrimination scheme for $\{\hat{\mathcal{E}}_m = \hat{\Lambda}_m \circledast \hat{\Lambda}_m\}_{m=1}^{3}$ with a sequential tester. Consider the task to be performed by two parties, Alice and Bob. Alice prepares a quantum state $\hat{\rho}_\mathrm{A} \in \mathsf{Den}_{V_1 \otimes V_1'}$, feeds the system V_1 into $\hat{\Lambda}_m$, and performs a measurement $\{\hat{\Psi}_j\}_{j \in \mathcal{J}}$ on $W_1 \otimes V_1'$, where $\mathcal{J}$ is a set, which may contain any number of elements. According to its outcome, j, Bob prepares a state $\hat{\rho}_\mathrm{B}^{(j)} \in \mathsf{Den}_{V_2 \otimes V_2'}$, feeds the system V_2 into $\hat{\Lambda}_m$, and performs a measurement $\{\hat{\Pi}_k^{(j)}\}_{k=1}^{3}$ on $W_2 \otimes V_2'$. $\{\hat{A}_j := \hat{\Psi}_j \circledast \hat{\rho}_\mathrm{A}\}_{j \in \mathcal{J}}$ and $\{\hat{B}_k^{(j)} := \hat{\Pi}_k^{(j)} \circledast \hat{\rho}_\mathrm{B}^{(j)}\}_{k=1}^{3}$ $(\forall j \in \mathcal{J})$ are testers.

This problem is very hard to solve due to two main reasons: i) both $\{A_j\}$ and $\{B_m^{(j)}\}_m$ $(\forall j)$ need to be optimized and ii) how many elements an optimal tester $\{A_j\}$ has is unknown. Here, we pay attention to the fact that Eq. (2) with

$$C := \left\{ \left\{ \sum_j B_m^{(j)} \otimes A_j \right\}_{m=1}^{3} : A_j \in \mathsf{Pos}_4,\ \{B_m^{(j)}\}_m \in \mathsf{Test}_3 \right\},$$
$$\mathcal{S} := \mathcal{S}_\mathrm{G}$$

holds (see Sec. III of the SM). In this situation, Problem (D) is expressed as

$$\begin{array}{ll} \text{minimize} & D_{\mathcal{S}_\mathrm{G}}(\chi) \\ \text{subject to} & \displaystyle\sum_{m=1}^{3} \left\langle \sum_j B_m^{(j)} \otimes A_j, \chi - \frac{1}{3} \Lambda_m \otimes \Lambda_m \right\rangle \geq 0 \\ & \left[\forall A_j \in \mathsf{Pos}_4,\ \{B_m^{(j)}\}_m \in \mathsf{Test}_3\right] \end{array}$$

with $\chi \in \mathsf{Her}_{\tilde{V}}$. After some algebra, this problem is reduced to (see Sec. III of the SM)

$$\begin{array}{ll} \text{minimize} & \lambda \\ \text{subject to} & \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \geq \displaystyle\frac{1}{3} \sum_{m=1}^{3} \langle B_m, \Lambda_m \rangle \begin{bmatrix} 1 & \omega^{-m} \\ \omega^m & 1 \end{bmatrix} \\ & (\forall \{B_m\} \in \mathsf{Test}_3) \end{array} \tag{4}$$

with $\lambda \in \mathbb{R}_+$. This problem is much easier to solve than Problem (3). Also, Theorem 1 guarantees that the optimal value of Problem (4), which is numerically found to be around 0.933, is equal to the maximum success probability. Note that in the case where any physically allowed discrimination strategy can be used, we can easily see that the three channels can be perfectly distinguished with two uses (see Sec. III of the SM). We restrict our discussion here to the discrimination problem for symmetric unitary qubit channels, but our method can be applied to more general combs. Other examples of different restricted strategies are shown in Sec. IV of the SM.

Relationship with robustnesses — In resource theory, robustness has been used as a measure of the resourcefulness of a quantum comb, such as a state, measurement, or channel. For a given closed set $\mathcal{F}$, called a free set, and a closed convex cone $\mathcal{K}$ of $\mathsf{Her}_{\tilde{V}}$, the robustness of a comb $\mathcal{E} \in \mathsf{Pos}_{\tilde{V}}$ against

$\mathcal{K}$ can be defined as [46, 47]

$$R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E}) := \inf\left\{\lambda \in \mathbb{R}_+ : \frac{\mathcal{E}+\lambda\mathcal{E}'}{1+\lambda} \in \mathcal{F},\ \mathcal{E}' \in \mathcal{K}\right\}. \quad (5)$$

$R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E})$ can be intuitively interpreted as the minimal amount, λ, of mixing with a process, $\mathcal{E}' \in \mathcal{K}$, necessary in order for the mixed and renormalized process, $(\mathcal{E}+\lambda\mathcal{E}')/(1+\lambda)$, to be in $\mathcal{F}$. As already mentioned in the introduction, it has been shown that the robustness of $\mathcal{E}$ is characterized as a measure of the advantage of $\mathcal{E}$ over all the processes in $\mathcal{F}$ in some discrimination problem [33–40] (see also Sec. VII of the SM). Note that this problem is somewhat different from a process discrimination problem that this Letter deals with. Conversely, we show that the optimal value of Problem (P) is characterized by a robustness measure.

For the problem of discriminating quantum combs $\{\hat{\mathcal{E}}_m\}_{m=1}^M$ with prior probabilities $\{p_m\}_{m=1}^M$, let us suppose that a party, Alice, chooses a state $|m\rangle\langle m|$ with the probability p_m, where $\{|m\rangle\}$ is the standard basis of a classical system W_A, and sends the corresponding comb $\hat{\mathcal{E}}_m$ to another party, Bob. The Choi-Jamiołkowski representation of the comb shared by Alice and Bob is expressed as

$$\mathcal{E}^{\mathrm{ex}} := \sum_{m=1}^M p_m |m\rangle\langle m| \otimes \mathcal{E}_m \in \mathsf{Pos}_{W_A\otimes\tilde{V}}.$$

Bob tries to infer which state Alice has. When he uses a tester $\{\Phi_m\}_m$, the success probability is written as $\sum_{m=1}^M \langle |m\rangle\langle m| \otimes \Phi_m, \mathcal{E}^{\mathrm{ex}}\rangle = P(\Phi)$. Using Theorem 1, we can see that the optimal value of Problem (P) is characterized by a robustness measure.

Corollary 2 Let

$$\mathcal{K} := \left\{Y \in \mathsf{Her}_{W_A\otimes\tilde{V}} : \sum_{m=1}^M \langle |m\rangle\langle m| \otimes \Phi_m, Y\rangle \geq 0\ (\forall\Phi \in C)\right\},$$
$$\mathcal{F} := \{I_{W_A} \otimes \chi' : \chi' \in \mathsf{Her}_{\tilde{V}},\ D_{\mathcal{S}}(\chi') = 1/M\};$$

then, the optimal value of Problem (P) is equal to $[1 + R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E}^{\mathrm{ex}})]/M$.

Proof From Eq. (5), we have

$$\frac{1+R_{\mathcal{K}}^{\mathcal{F}}(\mathcal{E}^{\mathrm{ex}})}{M} = \inf\left\{\frac{1+\lambda}{M} : \frac{\mathcal{E}^{\mathrm{ex}}+\delta}{1+\lambda} = I_{W_A}\otimes\chi',\ \chi' \in \mathsf{Her}_{\tilde{V}},\ D_{\mathcal{S}}(\chi') = \frac{1}{M},\ \delta \in \mathcal{K}\right\}.$$

Letting $\chi := (1+\lambda)\chi'$ and using some algebra, the right-hand side becomes

$$\begin{aligned}
&\inf\{D_{\mathcal{S}}(\chi) : \chi \in \mathsf{Her}_{\tilde{V}},\ I_{W_A}\otimes\chi - \mathcal{E}^{\mathrm{ex}} \in \mathcal{K}\}\\
&= \inf\left\{D_{\mathcal{S}}(\chi) : \chi \in \mathsf{Her}_{\tilde{V}},\ \sum_{m=1}^M |m\rangle\langle m| \otimes (\chi - p_m\mathcal{E}_m) \in \mathcal{K}\right\}\\
&= \inf\{D_{\mathcal{S}}(\chi) : \chi \in \mathcal{D}_C\} = D^\star,
\end{aligned}$$

where $D^\star$ is the optimal value of Problem (D). Thus, Theorem 1 completes the proof. ■

If $\mathcal{E}^{\mathrm{ex}}$ belongs to the free set $\mathcal{F}$, then $p_1 = \cdots = p_M = 1/M$ and $\mathcal{E}_1 = \cdots = \mathcal{E}_M$ must hold, which implies that $\mathcal{F}$ can intuitively be thought of as a set that includes all $\mathcal{E}^{\mathrm{ex}}$ corresponding to trivial process discrimination problems. This robustness measure indicates how far $\mathcal{E}^{\mathrm{ex}}$ is from $\mathcal{F}$. This interpretation has the potential to provide a deeper insight into optimal discrimination of quantum processes with restricted testers.

Conclusions — We have presented a general approach for solving quantum process discrimination problems with restricted testers based on convex programming. Our analysis indicates that a dual problem exhibiting zero duality gap is obtained regardless of the set of all restricted testers. We have shown that the optimal value of each process discrimination problem can be written in terms of a robustness measure. In comparison to previous theoretical works, our approach would allow a unified analysis for a large class of process discrimination problems in which the allowed testers are restricted. A meaningful direction for subsequent work would be to apply our approach to practical fields, such as quantum communication and quantum metrology.

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