

# Various definitions

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## 1 Symmetric monoidal categories (SMC)

**Monoidal category:** A category  $C$  equipped with

- A functor  $\otimes : C \times C \rightarrow C$ ;
- unit object  $I \in C$ ;
- associator: natural iso  $(a \otimes b) \otimes c \xrightarrow{\alpha_{a,b,c}} a \otimes (b \otimes c)$ ;
- left unitor: natural iso  $I \otimes a \xrightarrow{\lambda_a} a$ ;
- right unitor: natural iso  $a \otimes I \xrightarrow{\rho_a} a$ ;
- **symmetric** if there is a symmetry: natural iso  $a \otimes b \xrightarrow{\sigma_{a,b}} b \otimes a$  such that  $\sigma_{b,a} = \sigma_{a,b}^{-1}$ ,

satisfying triangle, pentagon (+ symmetry) diagrams.

We will always assume that  $C$  is a SMC.

### 1.1 Closed SMC

A SMC  $C$  is **closed** if:

for every  $b \in C$ , the endofunctor  $- \otimes b$  has a right adjoint  $[b, -]$  (internal hom).

What does this mean?

- (1) For all  $a, c \in C$ ,  $C(a \otimes b, c) \simeq C(a, [b, c])$ , naturally in  $a, c$ .
- (2) unit  $\eta_a^b : a \rightarrow [b, a \otimes b]$ , counit:  $\epsilon_a^b : [b, a] \otimes b \rightarrow a$ , natural transformations, triangle identities

‘ Relation of the two:

- Let  $i$  be the iso of (1):

$$\begin{aligned} \eta_a^b &\in C(a, [b, a \otimes b]) \simeq C(a \otimes b, a \otimes b), & \eta_a^b &= i(id_{a \otimes b}) \\ \epsilon_a^b &\in C([b, a] \otimes b, a) \simeq C([b, a], [b, a]), & \epsilon_a^b &= i^{-1}(id_{[b, a]}). \end{aligned}$$

- Conversely, from  $\eta^b, \epsilon^b$  of (2), we define  $i$  as

$$g \in C(a \otimes b, c), \quad i(g) = [b, g] \circ \eta_a^b$$

with inverse

$$h \in C(a, [b, c]), \quad i^{-1}(h) = \epsilon_c^b \circ (h \otimes b)$$

Equivalently: a SMC  $C$  is closed if and only if for all  $b, c \in C$ , there is an object  $[b, c]$  and an **evaluation map**  $eval_{b,c} : [b, c] \otimes b \rightarrow c$  that has the following **universal property**: for all  $a \in C$  and  $f : a \otimes b \rightarrow c$  there is a unique  $h : a \rightarrow [b, c]$  such that

$$f = eval_{b,c} \circ (h \otimes b).$$

The evaluation map is the counit  $eval_{b,c} = \epsilon_c^b$  above.

Internal hom is a functor  $[-, -] : C^{op} \times C \rightarrow C$  and the isomorphism in (1) is natural in all 3 variables  $a, b, c$ . This follows by Yoneda (nLab).

## 1.2 Compact SMC

A SMC is **compact** if each object  $a \in C$  has a dual  $a^* \in C$  such that there are maps  $\cup_a : I \rightarrow a^* \otimes a$  and  $\cap_a : a \otimes a^* \rightarrow I$  satisfying the snake identities

$$(\cap_a \otimes id_a) \circ (id_a \otimes \cup_a) = id_a, \quad (id_{a^*} \otimes \cap_a) \circ (\cup_a \otimes id_{a^*}) = id_{a^*}.$$

The following are easily seen by the snake identities (pictures):

(1)  $a^*$  is determined up to iso;

(1)  $I^* \simeq I$ , by the isomorphisms

$$\rho_{I^*} \circ \cup_I : I \rightarrow I^*, \quad \cap_I \circ \lambda_{I^*}^{-1} : I^* \rightarrow I;$$

(2)  $a^{**} \simeq a$ , indeed, we may define  $\cup_{a^*} : I \rightarrow a \otimes a^*$  and  $\cap_{a^*} : a^* \otimes a \rightarrow I$  as

$$\cup_{a^*} = \sigma_{a^*,a} \circ \cup_a, \quad \cap_{a^*} = \cap_a \circ \sigma_{a^*,a},$$

so that  $a$  is dual to  $a^*$ , and use (1);

(3) if we fix  $a^*$  and  $\cup_a$  ( $\cap_a$ ), then  $\cap_a$  ( $\cup_a$ ) is uniquely determined;

(4) any assignment  $a \mapsto a^*$  defines a functor  $C \rightarrow C^{op}$  (if  $f : a \rightarrow b$ , we can use  $\cup_a$  and  $\cap_b$  to "bend the wires" to obtain a map  $b^* \rightarrow a^*$ , this is obviously functorial);

(5)  $(a \otimes b)^* \simeq a^* \otimes b^*$ , we can clearly put (using symmetry)

$$\cup_{a \otimes b} = \cup_a \otimes \cup_b, \quad \cap_{a \otimes b} = \cap_a \otimes \cap_b$$

(5)  $C$  is closed, with  $[b, c] = b^* \otimes c$ : the iso  $i : C(a \otimes b, c) \simeq C(a, b^* \otimes c)$  can be obtain (roughly) as

$$i(g) = g \circ \cup_b, \quad i^{-1}(h) = \cap_b \circ h,$$

the maps are inverses by the snake identities, naturality is obvious, since  $i$  does nothing on  $a$  or  $c$ . The unit and counit of the adjunction are given as

$$\eta_a^b = a \otimes \cup_b : a \rightarrow b^* \otimes a \otimes b, \quad \epsilon_a^b = \cap_b \otimes a : b^* \otimes a \otimes b \rightarrow a.$$

- (6) Can we state a theorem like:  $C$  is compact if and only if for each  $b \in C$  there is some  $b^* \in C$  such that  $b^* \otimes -$  is the right adjoint of  $- \otimes b$  and ...? What should be the additional conditions?

## 2 Kleisli categories and monoidal monads

A **monad** on  $C$  is a triple  $(P, \eta, \mu)$ , where:

- $P : C \rightarrow C$  is an endofunctor;
- $\eta : Id_C \rightarrow P$ ,  $\mu : P^2 \rightarrow P$  are natural transformations satisfying some triangles and squares.

### 2.1 Kleisli categories

The **Kleisli category**  $C_P$  has the same objects as  $C$ , with morphisms:

$$C_p(a, b) = C(a, P(b)),$$

the identity  $id_a = \eta_a$  and for  $f \in C_p(a, b)$ ,  $g \in C_p(b, c)$ , the composition is defined as

$$g \circ f := \mu_c \circ P(g) \circ f.$$

We have the following adjunction:

- the **left adjoint functor**  $F_P : C \rightarrow C_P$  is defined as  $a \mapsto a$  and for  $f : a \rightarrow b$ , we put  $F_P(f) \in C_P(a, b) = C(a, P(b))$  as  $\eta_b \circ f$ ;
- the **right adjoint functor**  $G_P : C_P \rightarrow C$  is given as  $a \mapsto P(a)$  and for  $f \in C_P(a, b) = C(a, P(b))$  we put  $G_P(f) \in C(P(a), P(b))$  as  $G_P(f) = \mu_b \circ P(f)$ .

This is indeed an adjunction, where the unit is given by  $\eta$  and the counit is determined as  $\epsilon_a = id_{P(a)} \in C_P(P(a), a)$ .

### 2.2 Monoidal monads

A monad is **monoidal** [?] if there are maps

$$\kappa_{a,b} : Pa \otimes Pb \rightarrow P(a \otimes b), \quad a, b \in C,$$

natural in  $a, b$  and such that

- $(P, \eta, \kappa)$  is a **monoidal functor**, that is, some diagrams involving  $P$ ,  $\alpha$ ,  $\lambda$ ,  $\rho$ ,  $\kappa$  and  $\eta$  commute;
- additional diagrams containing  $\mu$  commutes;
- **symmetric**: additionally a diagram with  $\sigma$  commutes.

A monoidal functor is **strict** if  $\kappa$  is iso.

**Proposition 1.** [?, Prop. 1.2.2] *There is a bijective correspondence between:*

- (i) families of morphisms  $\{\kappa_{a,b}\}$  such that  $(P, \eta, \mu, \kappa)$  is a (symmetric) monoidal monad;
- (ii) (symmetric) monoidal structures on  $C_P$  such that the left adjoint functor  $F_P : C \rightarrow C_P$  is strict monoidal.

If  $(P, \eta, \mu, \kappa)$  is a symmetric monoidal monad, we define the monoidal structure on  $C_P$  as follows. The functor

$$\otimes_P : C_P \times C_P \rightarrow C_P$$

is given as  $a \otimes_P b = a \otimes b$  on objects, and for  $f \in C_P(a, c) = C(a, P(c))$  and  $g \in C_P(b, d) = C(b, P(d))$ , we define  $f \otimes_P g \in C_P(a \otimes_P b, c \otimes_P d) = C(a \otimes b, P(c \otimes d))$  as

$$f \otimes_P g := \kappa_{b,d} \circ (f \otimes g).$$

The associator and unitors and symmetry in  $C_P$  can be defined from those in  $C$  by composition with  $\eta$ .

### 3 Compact Kleisli categories

Assume that  $C$  is a symmetric monoidal closed category. Assume that  $(P, \eta, \mu, \kappa)$  is a monoidal monad, such that the category  $C_P$  with corresponding monoidal structure is compact. We will study some consequences of this.

#### 3.1 First consequences

It follows that each object  $a \in C_P$  has a dual object  $a^*$ , such that there is an isomorphism

$$i : C_P(a \otimes_P b^*, c) \simeq C_P(a, b \otimes_P c),$$

which is natural (in  $C_P$ ) in  $a$  and  $c$ . This means that for any arrows  $a \xrightarrow{f} a'$  and  $a' \otimes b^* \xrightarrow{h'} c$  in  $C_P$ , we have

$$i(h' \circ_P (f \otimes_P id_{b^*}^P)) = i(h') \circ_P f,$$

and similarly, for  $c \xrightarrow{g} c'$  and  $a \otimes b^* \xrightarrow{h} c$  in  $C_P$ , we get

$$i(g \circ_P h) = (id_b^P \otimes_P g) \circ_P i(h).$$

By definition of the Kleisli category,  $i$  is an isomorphism (that is, a bijection of sets)

$$i : C(a \otimes b^*, P(c)) \simeq C(a, P(b \otimes c)).$$

We would like to show that  $i$  is natural in  $a$  and  $c$  also in the category  $C$ .

So let  $f \in C(a, a')$  and let  $\tilde{f} := \eta_{a'} \circ f \in C(a, P(a')) = C_P(a, a')$ . Let  $h' \in C(a' \otimes b^*, P(c))$ , then

$$\begin{aligned} h' \circ_P (\tilde{f} \otimes_P b^*) &= \mu_c \circ P(h') \circ (\tilde{f} \otimes_P \eta_{b^*}) = \mu_c \circ P(h') \circ \kappa_{a', b^*} \circ (\eta_{a'} \otimes \eta_{b^*}) \circ (f \otimes b^*) \\ &= \mu_c \circ P(h') \circ \eta_{a' \otimes b^*} \circ (f \otimes b^*) = \mu_c \circ \eta_{P(c)} \circ h' \circ (f \otimes b^*) \\ &= h' \circ (f \otimes b^*), \end{aligned}$$

where we used that  $\kappa_{a',b^*} \circ (\eta_{a'} \otimes \eta_{b^*}) = \eta_{a' \otimes b^*}$ , naturality of  $\eta$  and the triangle identity. Similarly, we get for any  $\bar{h}' \in C(a, P(b \otimes c))$ ,

$$\bar{h}' \circ_P f = \mu_{b \otimes c} \circ P(\bar{h}') \circ \eta_{a'} \circ f = \bar{h}' \circ f,$$

in particular, putting these together, this implies

$$i(h' \circ (f \otimes b^*)) = i(h' \circ_P (\tilde{f} \otimes_P id_{b^*})) = i(h') \circ_P \tilde{f} = i(h') \circ f.$$

Naturality in  $c$  is proved similarly.

It follows that there is an isomorphism

$$C(a, P(b \otimes c)) \simeq C(a \otimes b^*, P(c)) \simeq C(a, [b^*, P(c)]),$$

natural in  $a$  and  $c$ . By Yoneda, we get the isomorphism

$$P(b \otimes c) \simeq [b^*, P(c)],$$

natural in  $c$ . Putting  $c = I$ , we obtain

$$P(b) \simeq P(b \otimes I) \simeq [b^*, P(I)].$$

### 3.2 A construction of a monoidal monad

Fix an object  $p \in C$  and assume that

- there is a bijective map  $a \mapsto a^*$  on objects;
- for each  $a \in C$ , there is a morphism  $\theta_a \in C(a, [a^*, p])$ ;
- for each morphism  $f \in C(a, [b^*, p])$  there is some  $\hat{f} \in C([a^*, p], [b^*, p])$

such that

- (i)  $\hat{\theta}_a = id_{[a^*, p]}$ ;
  - (ii) for  $f \in C(a, [b^*, p])$ ,  $\hat{f} \circ \theta_a = f$ ;
  - (iii) for  $f \in C(a, [b^*, p])$  and  $g \in C(b, [c^*, p])$ ,
- $$(\hat{g} \circ f)^\wedge = \hat{g} \circ \hat{f}. \tag{1}$$

From this data, we may define a monad  $(P_p, \theta, \nu)$ . Here the functor  $P_p$  acts as  $a \mapsto [a^*, p]$  on objects and for  $f \in C(a, b)$ , we define  $P_p(f) \in C([a^*, p], [b^*, p])$  by

$$P_p(f) := (\theta_b \circ f)^\wedge.$$

Moreover,  $\nu$  is defined as  $\nu_a = \hat{id}_{[a^*, p]}$ . The fact that this is a monad follows easily from the properties (i)-(iii).

To make it monoidal, we add family of maps

$$\kappa_{a,b} : [a^*, p] \otimes [b^*, p] \rightarrow [(a \otimes b)^*, p],$$

such that

(iv) for all  $a, b \in C$ ,

$$\theta_{a \otimes b} = \kappa \circ (\theta_a \otimes \theta_b);$$

(v) for  $f \in C(a, [b^*, p])$  and  $g \in C(c, [d^*, p])$ ,

$$\kappa_{b,d} \circ (\hat{f} \otimes \hat{g}) = (\kappa_{b,d} \circ (f \otimes g))^\wedge \circ \kappa_{a,c}.$$

Then one can check that  $\kappa_{a,b}$  are natural in  $a, b$  and that

$$\nu_{a \otimes b} \circ P_p(\kappa_{a,b}) \circ \kappa_{[a^*, p], [b^*, p]} = \kappa_{a,b} \circ (\nu_a \otimes \nu_b).$$

We also need some properties with respect to  $\alpha, \lambda, \rho$  and  $\sigma$ :

(vi) for all  $a, b, c$ ,

$$\kappa_{a, b \otimes c} \circ (id_{[a^*, p]} \otimes \kappa_{b,c}) \circ \alpha_{[a^*, p], [b^*, p], [c^*, p]} = P_p(\alpha_{a,b,c}) \circ \kappa_{a \otimes b, c} \circ (\kappa_{a,b} \otimes id_{[c^*, p]})$$

(vii) for all  $a$ ,

$$\begin{aligned} (\theta_a \circ \lambda_a)^\wedge \circ \kappa_{I,a} \circ (\theta_I \otimes id_{[a^*, p]}) &= \lambda_{[a^*, p]} \\ (\theta_a \circ \rho_a)^\wedge \circ \kappa_{I,a} \circ (id_{[a^*, p]} \otimes \theta_I) &= \rho_{[a^*, p]}; \end{aligned}$$

(viii) for all  $a, b$ ,

$$(\theta_{b \otimes a} \circ \sigma_{a,b})^\wedge \circ \kappa_{a,b} = \kappa_{b,a} \circ \sigma_{[a^*, p], [b^*, p]}.$$

Then  $(P_p, \theta, \nu, \kappa)$  is a monoidal monad, [? ].

### 3.3 The Kleisli category $C_p$

The Kleisli category  $C_p := C_{P_p}$  has the same objects as  $C$ , with morphisms  $C_p(a, b) = C(a, [b^*, p])$ , the identity is  $id_a^p = \theta_a$  and for  $f \in C_p(a, b)$ ,  $g \in C_p(b, c)$ , the composition is given as

$$g \circ_p f = \hat{g} \circ f.$$

*Remark 1.* Let  $j$  be the natural iso (in  $C$ ):

$$j : C(a \otimes b, c) \simeq C(a, [b, c])$$

Note that  $C_p(a, b)$  can be identified with  $C(a \otimes b^*, p)$ , with composition given by

$$j^{-1}(j(\psi)^\wedge \circ j(\varphi)), \quad \varphi \in C(a \otimes b^*, p), \quad \psi \in C(b \otimes c^*, p).$$

We equip  $C_p$  with the tensor product  $\otimes_p$  defined by  $a \otimes_p b = a \otimes b$  on objects and  $f \otimes_p g = \kappa \circ (f \otimes g)$  on morphisms. Then  $(C_p, \otimes_p, I)$  is a symmetric monoidal category, with the natural isomorphisms  $\alpha, \lambda, \rho, \sigma$  extended by  $\theta$ , that is,  $\alpha^p := \theta \circ \alpha$ ,  $\lambda^p := \theta \circ \lambda$ ,  $\rho^p := \theta \circ \rho$ ,  $\sigma^p := \theta \circ \sigma$ .

### 3.4 When is $C_p$ closed?

We need to define the internal hom  $b \xrightarrow{p} c$ , such that  $b \xrightarrow{p} -$  is the right adjoint of  $b \otimes_p -$  in  $C_p$ . In fact, it is enough to specify  $b \xrightarrow{p} c$  on objects and to find an iso

$$C_p(a \otimes_p b, c) \simeq C_p(a, b \xrightarrow{p} c)$$

natural in  $a$ . As for the isomorphism, we must have

$$C(a \otimes b, [c^*, p]) \simeq C(a, [(b \xrightarrow{p} c)^*, p])$$

Since  $C$  is SMC, we have

$$C(a \otimes b, [c^*, p]) \simeq C((a \otimes b) \otimes c^*, p) \simeq C(a \otimes (b \otimes c^*), p) \simeq C(a, [b \otimes c^*, p])$$

and the isomorphisms are natural (in  $C$ ) in all variables. This suggests to define  $b \xrightarrow{p} c$  as the object such that  $(b \xrightarrow{p} c)^* = b \otimes c^*$ . Since  $(-)^*$  is bijective, such an object exists and is unique.

As for naturality of the isomorphism, let us denote by  $i$  the resulting isomorphism

$$i : C(a \otimes b, [c^*, p]) \simeq C(a, [b \otimes c^*, p])$$

Let  $f \in C_p(a', a) = C(a', [a^*, p])$ . Then naturality means that we require

$$i(h \circ_p (f \otimes_p id_b^p)) = i(h) \circ_p f = i(h)^\wedge \circ f.$$

on the left hand side we obtain

$$h \circ_p (f \otimes_p id_b^p) = \hat{h} \circ \kappa_{a,b} \circ (f \otimes \theta_b) = \hat{h} \circ s_{a,b} \circ (f \otimes b),$$

where  $\hat{h} \circ s_{a,b} \in C([a^*, p] \otimes b, [c^*, p])$ . By naturality in  $C$ , we see that

$$i(\hat{h} \circ s_{a,b} \circ (f \otimes b)) = i(\hat{h} \circ s_{a,b}) \circ f,$$

where  $i(\hat{h} \circ s_{a,b}) \in C([a^*, p], [b \otimes c^*, p])$ . It follows that we need to have

$$i(\hat{h} \circ s_{a,b}) = i(h)^\wedge. \tag{2}$$