# Rényi relative entropies and noncommutative $L_p$-spaces II

Anna Jenčová *

*Mathematical Institute, Slovak Academy of Sciences*

*Štefánikova 49, 814 73 Bratislava, Slovakia*

**Abstract**

We show the relation between two versions of sandwiched Rényi relative entropies for von Neumann algebras, introduced recently in [M. Berta et al, arXiv:1608.05317] and [A. Jenčová, arXiv:1609.08462]. It is also proved that equality in data processing inequality for a quantum channel and $\alpha \in (1/2, 1)$ is equivalent to sufficiency (reversibility) of the channel.

# 1 Introduction

In [4], we introduced a version of sandwiched Rényi relative $\alpha$-entropy $\tilde{D}_\alpha$ with $\alpha > 1$ for normal positive linear functionals on a von Neumann algebra. Our definition is based on non-commutative $L_p$ spaces with respect to a state, defined by Kosaki [6]. Another version, called the Araki-Masuda divergences which we will denote by $D_\alpha^{BST}$, was introduced in [2], based on the weighted $L_p$-norms of Araki and Masuda [1], this definition works for all $\alpha \in [1/2, 1) \cup (1, \infty]$. We show that for $\alpha > 1$ these two versions are equal and we give an expression for $D_\alpha^{BST}$, $\alpha \in [1/2, 1)$, in the framework of [4]. For this, we use the polar decomposition in the Araki-Masuda $L_p$-spaces. Similar results, by different methods, were independently obtained by Hiai, [3]. We also prove that for a quantum channel $\Phi$, two normal states $\psi, \varphi$ such that the support projections satisfy $s(\psi) \leq s(\varphi)$ and $\alpha \in (1/2, 1)$, the equality

$$D_\alpha^{BST}(\psi\|\varphi) = D_\alpha^{BST}(\Phi(\psi)\|\Phi(\varphi))$$

implies that the channel $\Phi$ is sufficient for $\{\psi, \varphi\}$.

The present paper is intended as a continuation of [4] and all the basic definitions and notations introduced therein will be used freely, without a separate introduction. We will also refer to the definitions and properties of Haagerup $L_p$-spaces, relative modular operators and conditional expectations, listed in [4, Appendix].

*jenca@mat.savba.sk

## 2 The Araki-Masuda divergences

In this section, we recall the definition of the Araki-Masuda divergences of [2] and prove that they are equal to $\tilde{D}_\alpha$ for $\alpha > 1$. We first introduce the Araki-Masuda $L_p$-spaces and their properties, in particular the norm duality and polar decompositions that are crucial for our results, and prove their relation to the norms $\|\cdot\|_{p,\varphi}$. Then we discuss the Araki-Masuda divergences and $\tilde{D}_\alpha$. If not stated otherwise, we will work in the standard form $(l(\mathcal{M}), L_2(\mathcal{M}), L_2(\mathcal{M})^+, J = *)$, [4, Appendix A.1].

### 2.1 The Araki-Masuda weighted $L_p$-spaces

Let us assume that $\varphi \in \mathcal{M}_*^+$ is faithful. The Araki-Masuda noncommutative $L_p$-spaces with respect to $\varphi$ are defined as follows [1]:

1. for $2 \le p \le \infty$, $L_p^{AM}(\mathcal{M}, \varphi)$ is a subspace in $L_2(\mathcal{M})$ of elements
$$k \in \cap_{\sigma \in \mathfrak{S}_*(\mathcal{M})} \mathcal{D}(\Delta_{\sigma,\varphi}^{1/2-1/p}), \quad \|k\|_{p,\varphi}^{AM} := \sup_{\sigma \in \mathfrak{S}_*(\mathcal{M})} \|\Delta_{\sigma,\varphi}^{1/2-1/p} k\|_2 < \infty$$
2. for $1 \le p < 2$, $L_p^{AM}(\mathcal{M}, \varphi)$ is the completion of $L_2(\mathcal{M})$ under the norm
$$\|k\|_{p,\varphi}^{AM} := \inf_{\sigma \in \mathfrak{S}_*(\mathcal{M}), s(\omega_k) \le s(\sigma)} \|\Delta_{\sigma,\varphi}^{1/2-1/p} k\|_2.$$

Here $\Delta_{\sigma,\psi}$ for $\sigma, \psi \in \mathcal{M}_*^+$ is the relative modular operator ([1, Appendix C], see [4, Appendix A.1] for its properties in the present standard form).

With these norms, $L_p^{AM}(\mathcal{M}, \varphi)$ are Banach spaces for $1 \le p \le \infty$. Let $1/p + 1/q = 1$. By [1, Theorem 1], the inner product $(\cdot, \cdot)$ restricted to $[L_p^{AM}(\mathcal{M}, \varphi) \cap L_2(\mathcal{M})] \times [L_q^{AM}(\mathcal{M}, \varphi) \cap L_2(\mathcal{M})]$ extends uniquely to a continuous sesquilinear form $\langle \cdot, \cdot \rangle_{p,\varphi}^{AM}$ on $L_p^{AM}(\mathcal{M}, \varphi) \times L_q^{AM}(\mathcal{M}, \varphi)$, through which $L_q^{AM}(\mathcal{M}, \varphi)$ is the dual of $L_p^{AM}(\mathcal{M}, \varphi)$ for $1 \le p < \infty$. In particular, we have

$$\|k\|_{p,\varphi}^{AM} = \sup\{|(k, k')|, \ k' \in L_2(\mathcal{M}), \ \|k'\|_{q,\varphi}^{AM} \le 1\} \tag{1}$$

for $k \in L_p^{AM}(\mathcal{M}, \varphi)$ and $1 \le p \le \infty$.

By [1, Theorem 3], we have the following polar decomposition for $k \in L_p^{AM}(\mathcal{M}, \varphi)$: there is a (unique) partial isometry $u \in \mathcal{M}$ and $\rho \in \mathcal{M}_*^+$, such that $uu^* = s(\omega_k)$, $u^*u = s(\rho)$ and

$$k = u\Delta_{\rho,\varphi}^{1/p} h_\varphi^{1/2} = u h_\rho^{1/p} h_\varphi^{1/2-1/p}$$

if $2 \le p < \infty$ and

$$\langle k, k' \rangle_{p,\varphi}^{AM} = (\Delta_{\rho,\varphi}^{1/2} h_\varphi^{1/2}, \Delta_{\rho,\varphi}^{1/p-1/2} u^* k') = (h_\rho^{1/2}, \Delta_{\rho,\varphi}^{1/p-1/2} u^* k')$$

for all $k' \in L_q^{AM}(\mathcal{M}, \varphi)$ if $1 \le p \le 2$. Conversely, any element of this form is in $L_p^{AM}(\mathcal{M}, \varphi)$ and $\|k\|_{p,\varphi}^{AM} = \rho(1)^{1/p}$. In this case, we will symbolically write

$$k = u\rho^{1/p}.$$

Moreover, for $1 < p < \infty$ and $k = u\rho^{1/p}$, $k' = \rho(1)^{-1/q} u \rho^{1/q}$ is the unique element in the unit ball of $L_q^{AM}(\mathcal{M}, \varphi)$ such that $\langle k, k' \rangle_{p,\varphi}^{AM} = \|k\|_{p,\varphi}^{AM}$.

We next find the relation to the Kosaki $L_p$-norm $\|\cdot\|_{p,\varphi}$.

**Proposition 1.** *Let $k \in L_2(\mathcal{M})$, $1 < p < \infty$. Then $k \in L_{2p}^{AM}(\mathcal{M}, \varphi)$ if and only if $k^*k \in L_p(\mathcal{M}, \varphi)$ and $\|k\|_{2p,\varphi}^{AM} = \|k^*k\|_{p,\varphi}^{1/2}$.*

*Proof.* Let $k \in L_{2p}^{AM}(\mathcal{M}, \varphi)$ and let $k = u\rho^{1/2p}$ be the polar decomposition, so that $k = uh_\rho^{1/2p}h_\varphi^{1/2-1/2p}$. Then $k^*k = h_\varphi^{1/2q}h_\rho^{1/p}h_\varphi^{1/2q} \in L_p(\mathcal{M}, \varphi)$, moreover, $\|k\|_{2p,\varphi}^{AM} = \rho(1)^{1/2p} = \|k^*k\|_{p,\varphi}^{1/2}$.

For the converse, let $k = vh_\psi^{1/2}$ be the (unique) polar decomposition of $k$ as an element in $L_2(\mathcal{M})$. Then $v^*v = s(\psi)$, $vv^* = s(\omega_k)$ and $h_\psi = k^*k \in L_p(\mathcal{M}, \varphi)^+$. Hence there is some $\rho \in \mathcal{M}_*^+$ such that $h_\psi = h_\varphi^{1/2q}h_\rho^{1/p}h_\varphi^{1/2q}$. Let $k' := h_\rho^{1/2p}h_\varphi^{1/2q}$, then $k' \in L_2(\mathcal{M})$ has the polar decomposition $k' = wh_\psi^{1/2}$, with $w^*w = v^*v = s(\psi)$. It follows that

$$k = vh_\psi^{1/2} = vw^*wh_\psi^{1/2} = vw^*k' = vw^*h_\rho^{1/2p}h_\varphi^{1/2q},$$

and since $vw^*wv^* = vv^*vv^* = vv^* = s(\omega_k)$, we obtain $k \in L_{2p}^{AM}(\mathcal{M}, \varphi)$, the equality for the norms holds as before.

□

*Remark* 2. Let us note that the Araki-Masuda $L_p$-spaces can be obtained by complex interpolation as in [6, Section 3], using the embeddings $\mathcal{M} \hookrightarrow L_2(\mathcal{M}) \hookrightarrow L_1(\mathcal{M}) \simeq \mathcal{M}_*$, given by

$$\mathcal{M} \ni x \mapsto xh_\varphi^{1/2} \in L_2(\mathcal{M}), \quad L_2(\mathcal{M}) \ni k \mapsto (h_\varphi^{1/2}, \cdot k) \in \mathcal{M}_*.$$

We then have the isometric isomorphisms

$$\begin{aligned} L_{p,\varphi}^{AM} &\simeq C_{1/p}(\mathcal{M}, L_1(\mathcal{M})) \simeq C_{2/p}(\mathcal{M}, L_2(\mathcal{M})), \quad 2 \le p \le \infty \\ L_{p,\varphi}^{AM} &\simeq C_{1/p}(\mathcal{M}, L_1(\mathcal{M})) \simeq C_{2/p-1}(L_2(\mathcal{M}), L_1(\mathcal{M})), \quad 1 \le p \le 2. \end{aligned}$$

This can be seen from [1, Thm 4], the polar decompositions and [6, Thm 9.1].

## 2.2 The Araki-Masuda divergences

In this paragraph, $\varphi \in \mathcal{M}_*^+$ is not assumed faithful and $\pi : \mathcal{M} \to B(\mathcal{H})$ is any $*$-representation. For $\xi \in \mathcal{H}$, let $\omega_\xi$ be the vector state given by $\xi$, that is $\omega_\xi(a) = (\xi, \pi(a)\xi)$. We also denote by $\omega'_\xi$ the corresponding state on the commutant: $\omega'_\xi(a') = (\xi, a'\xi)$, $a' \in \pi(\mathcal{M})'$. Let $\Delta(\xi/\varphi)$ denote the spatial derivative as defined in [2, Sec. 2.2] (we give this definition in the Appendix). The $\varphi$-weighted $p$-norm of $\xi \in \mathcal{H}$ is defined as:[1]

1. for $2 \le p \le \infty$,

$$\|\xi\|_{p,\varphi}^{BST} := \sup_{\zeta \in \mathcal{H}, \|\zeta\|=1} \|\Delta(\zeta/\varphi)^{1/2-1/p}\xi\|$$

   if $s(\omega_\xi) \le s(\varphi)$ and $+\infty$ otherwise. Note that the supremum can be infinite also if the condition on the supports holds.

[1]The expression in 2. is slightly different from [2] but it seems it does not work otherwise

2. for $1 \leq p < 2$, we define

$$\|\xi\|_{p,\varphi}^{BST} := \inf_{\zeta\in\mathcal{H},\|\zeta\|=1,s(\omega'_\zeta)\geq s(\omega'_\xi)} \|\Delta(\zeta/\varphi)^{1/2-1/p}\xi\|.$$

The following relation to the Araki-Masuda $L_p$-norms is immediate from the results in the Appendix and properties of the standard representation on $L_2(\mathcal{M})$.

**Proposition 3.** *Let $\varphi \in \mathcal{M}_*^+$ be faithful and let $k \in L_2(\mathcal{M})$, $1 \leq p \leq \infty$. Then $\|k\|_{p,\varphi}^{BST} = \|k^*\|_{p,\varphi}^{AM}$.*

The use of the BST-norms has the advantage that this definition works for non-faithful $\varphi$ and does not depend on the representation $\pi$ nor the particular vector representing the functional $\omega_\xi$. We now recall the definition of Araki-Masuda divergences.

**Definition 1.** [2] Let $\varphi \in \mathcal{M}_*^+, \psi \in \mathfrak{S}_*(\mathcal{M})$ and $\alpha \in [1/2, 1) \cup (1, \infty)$. Let $\xi_\psi$ be any vector representative of $\psi$ for a $*$-representation $\pi : \mathcal{M} \to B(\mathcal{H})$. Then

$$D_\alpha^{BST}(\psi\|\varphi) = \frac{2\alpha}{\alpha-1}\log\|\xi_\psi\|_{2\alpha,\varphi}^{BST} \tag{2}$$

## 2.3 The relation of $D_\alpha^{BST}$ and $\tilde{D}_\alpha$

We now prove equality of the two versions of Rényi relative entropies for $\alpha > 1$ and find a suitable expression for $D_\alpha^{BST}$, $\alpha \in [1/2, 1)$, in terms of the operators $h_\psi, h_\varphi \in L_1(\mathcal{M})$. We will need the following result.

**Lemma 4.** *Let $1 < p \leq 2$ and let $\varphi \in \mathcal{M}_*^+$, $k \in L_2(\mathcal{M})$. Then*

$$\|k\|_{p,\varphi}^{BST} = \rho(1)^{1/p},$$

*where $\rho \in \mathcal{M}_*^+$ is obtained from the polar decomposition $k^*h_\varphi^{1/p-1/2} = uh_\rho^{1/p}$ in $L_p(\mathcal{M})$. Moreover, if $\varphi$ is faithful, then $k^* = u\rho^{1/p}$ is the polar decomposition of $k^*$ in $L_p^{AM}(\mathcal{M}, \varphi)$.*

*Proof.* Similarly as before, using Appendix and the properties of a standard representation we obtain

$$\|k\|_{p,\varphi}^{BST} = \inf_{\sigma\in\mathfrak{S}_*(\mathcal{M}),s(\sigma)\geq s(\omega_{k^*})} \|\Delta_{\sigma,\varphi}^{1/2-1/p}k^*\|_2.$$

Since $k^* \in L_2(\mathcal{M})$, we have $k^*h_\varphi^{1/p-1/2} \in L_p(\mathcal{M})$, so that $k^*h_\varphi^{1/p-1/2} = uh_\rho^{1/p}$ for some $\rho \in \mathcal{M}_*^+$. Assume that $\sigma \in \mathfrak{S}_*(\mathcal{M})$ is such that $s(\omega_{k^*}) \leq s(\sigma)$ and $k^* \in \mathcal{D}(\Delta_{\sigma,\varphi}^{1/2-1/p})$. Then (see [4, Appendix A]) $\Delta_{\sigma,\varphi}^{1/2-1/p}k^* =: k' \in L_2(\mathcal{M})$ satisfies

$$uh_\rho^{1/p} = s(\sigma)k^*h_\varphi^{1/p-1/2} = h_\sigma^{1/p-1/2}k'.$$

By Hölder's inequality, we obtain

$$\rho(1)^{1/p} = \|uh_\rho^{1/p}\|_p \leq \|h_\sigma^{1/p-1/2}\|_{2p/(2-p)}\|k'\|_2 = \|k'\|_2. \tag{3}$$

On the other hand, put $\rho_u(a) = \rho(u^*au)$, then $s(\rho_u) = uu^* \leq s(\omega_{k^*})$. Let $\sigma_0 \in \mathfrak{S}_*(\mathcal{M})$ be any state such that $s(\sigma_0) = s(\omega_{k^*}) - s(\rho_u)$ and put $\sigma_\epsilon = \epsilon\rho(1)^{-1}\rho_u + (1-\epsilon)\sigma_0$. Then $\sigma_\epsilon \in \mathfrak{S}_*(\mathcal{M})$, $s(\sigma_\epsilon) = s(\omega_{k^*})$ and we have

$$k^* h_\varphi^{1/p-1/2} = uh_\rho^{1/p} = h_{\sigma_\epsilon}^{1/p-1/2}k'$$

where $k' = \epsilon^{1/2-1/p}\rho(1)^{1/p}h^{1/2}_{\rho_u(1)^{-1}\rho_u}u$. From this and (3), it follows that

$$\rho(1)^{1/p} \leq \|k\|_{p,\varphi}^{BST} \leq \|\Delta_{\sigma_\epsilon,\varphi}^{1/2-1/p}k^*\|_2 = \|k'\|_2 = \epsilon^{1/2-1/p}\rho(1)^{1/p}$$

for all $\epsilon \in (0,1)$. Letting $\epsilon \to 1$, we get $\rho(1)^{1/p} = \|k\|_{p,\varphi}^{BST}$.

Assume next that $\varphi$ is faithful and let $k' \in L_q^{AM}(\mathcal{M},\varphi) \subseteq L_2(\mathcal{M})$, with polar decomposition $k' = v\sigma^{1/q}$. Then

$$\begin{aligned}\langle k^*, k'\rangle_{p,\varphi}^{AM} &= (k^*, k') = (k^*, vh_\sigma^{1/q}h_\varphi^{1/p-1/2}) = \operatorname{Tr} h_\sigma^{1/q}v^*k^*h_\varphi^{1/p-1/2} \\ &= \operatorname{Tr} h_\sigma^{1/q}v^*uh_\rho^{1/p} = (h_\rho^{1/2}, \Delta_{\rho,\varphi}^{1/p-1/2}u^*k')\end{aligned}$$

so that $k^* = u\rho^{1/p}$ is the polar decomposition of $k^*$ in $L_p^{AM}(\mathcal{M},\varphi)$.

□

**Theorem 5.** *Let $\psi, \varphi \in \mathcal{M}_*^+$. Then*

*(i) for $\alpha \in (1,\infty)$, $D_\alpha^{BST}(\psi\|\varphi) = \tilde{D}_\alpha(\psi\|\varphi)$.*

*(ii) for $\alpha \in [1/2, 1)$, we have*

$$D_\alpha^{BST}(\psi\|\varphi) = \frac{1}{\alpha-1}\log \operatorname{Tr}(h_\varphi^{\frac{1-\alpha}{2\alpha}}h_\psi h_\varphi^{\frac{1-\alpha}{2\alpha}})^\alpha$$

*Proof.* For (i), we may assume that $s(\psi) \leq s(\varphi)$, otherwise both expressions are infinite. By restriction to the compressed algebra $s(\varphi)\mathcal{M}s(\varphi)$, we may suppose that $\varphi$ is faithful. The statement then follows by Prop. 1.

For (ii), let $\alpha \in [1/2, 1)$. Then $h_\psi^{1/2} \in L_2(\mathcal{M}) \cap L_{2\alpha}^{AM}(\mathcal{M},\varphi)$ and by Lemma 4, we have that

$$(\|h_\psi^{1/2}\|_{2\alpha,\varphi}^{BST})^{2\alpha} = \|h_\psi^{1/2}h_\varphi^{1/2\alpha-1/2}\|_{2\alpha}^{2\alpha} = \operatorname{Tr}(h_\varphi^{\frac{1-\alpha}{2\alpha}}h_\psi h_\varphi^{\frac{1-\alpha}{2\alpha}})^\alpha.$$

□

# 3 Monotonicity, equality and sufficiency

Let $\Phi : L_1(\mathcal{M}) \to L_1(\mathcal{N})$ be a quantum channel (that is, a completely positive trace preserving map). Then the dual map $\Phi^* : \mathcal{N} \to \mathcal{M}$ is a completely positive unital normal map. Using Stinespring representation, there exists a Hilbert space $\mathcal{K}$, a normal *-representation $\pi : \mathcal{N} \to B(\mathcal{K})$ and an isometry $T : L_2(\mathcal{M}) \to \mathcal{K}$ such that

$$\Phi^*(a) = T^*\pi(a)T, \qquad a \in \mathcal{N}.$$

Let $k \in L_2(\mathcal{M})$ be a representing vector for $\psi \in \mathcal{M}_*^+$, then $Tk \in \mathcal{K}$ is a representing vector for $\Phi(\psi)$, hence we have

$$D_\alpha^{BST}(\Phi(\psi), \Psi(\varphi)) = \frac{2\alpha}{\alpha - 1} \log \|Tk\|_{2\alpha,\Phi(\varphi)}^{BST}.$$

The following data processing inequality (DPI) for $D_\alpha^{BST}$ was proved in [2]:

$$D_\alpha^{BST}(\psi\|\varphi) \geq D_\alpha^{BST}(\Phi(\psi)\|\Phi(\varphi)), \qquad \alpha \in [1/2, 1) \cup (1, \infty].$$

This is equivalent to

$$\|Tk\|_{p,\Phi(\varphi)}^{BST} \leq \|k\|_{p,\varphi}^{BST},\ 2 < p \leq \infty; \qquad \|Tk\|_{p,\Phi(\varphi)}^{BST} \geq \|k\|_{p,\varphi}^{BST},\ 1 \leq p < 2 \quad (4)$$

for any Stinespring dilation $(\mathcal{K}, \pi, T)$. We next show that equality in DPI implies that the channel $\Phi$ is sufficient with respect to $\{\psi, \varphi\}$.

**Theorem 6.** *Assume that $s(\psi) \leq s(\varphi)$ and let $\alpha \in (1/2, 1)$. Then $D_\alpha^{BST}(\psi\|\varphi) = D_\alpha^{BST}(\Phi(\psi)\|\Phi(\varphi))$ if and only if $\Phi$ is sufficient for $\{\psi, \varphi\}$.*

*Proof.* Because of the assumption on the supports, we may suppose that both $\varphi$ and $\Phi(\varphi)$ are faithful. Assume that the equality holds, so that $\|h_\psi^{1/2}\|_{p,\varphi}^{BST} = \|Th_\psi^{1/2}\|_{p,\Phi(\varphi)}^{BST}$, here $p = 2\alpha \in (1, 2)$. Let $h_\psi^{1/2} = u\rho^{1/p}$ be the polar decomposition in $L_p^{AM}(\mathcal{M}, \varphi)$, then

$$\|h_\psi^{1/2}\|_{p,\varphi}^{BST} = \|h_\psi^{1/2}\|_{p,\varphi}^{AM} = (\|k\|_{q,\varphi}^{AM})^{-1}(k, h_\psi^{1/2})_{L_2(\mathcal{M})},$$

where $1/p + 1/q = 1$ and $k \in L_q^{AM}(\mathcal{M}, \varphi)$ has polar decomposition $k = u\rho^{1/q}$. By Lemma 4, $h_\psi^{1/2} h_\varphi^{1/p - 1/2} = uh_\rho^{1/p}$ and we have $k = uh_\rho^{1/q} h_\varphi^{1/2 - 1/q}$. Since $T$ is an isometry, we get using the norm duality in [2, Sec. 3.2]

$$\begin{aligned} (k, h_\psi^{1/2})_{L_2(\mathcal{M})} = (h_\psi^{1/2}, k^*)_{L_2(\mathcal{M})} &= (Th_\psi^{1/2}, Tk^*)_{\mathcal{K}} \\ &\leq \|Th_\psi^{1/2}\|_{p,\Phi(\varphi)}^{BST} \|Tk^*\|_{q,\Phi(\varphi)}^{BST} \end{aligned}$$

By the assumption and Proposition 3,

$$\|Th_\psi^{1/2}\|_{p,\Phi(\varphi)}^{BST} = \|h_\psi^{1/2}\|_{p,\varphi}^{BST} \leq (\|k^*\|_{q,\varphi}^{BST})^{-1} \|Tk^*\|_{q,\Phi(\varphi)}^{BST} \|Th_\psi^{1/2}\|_{p,\Phi(\varphi)}^{BST},$$

which implies that $\|Tk^*\|_{q,\Phi(\varphi)}^{BST} \geq \|k^*\|_{q,\varphi}^{BST}$. By (4) for $q > 2$, we get the equality $\|Tk^*\|_{q,\Phi(\varphi)}^{BST} = \|k^*\|_{q,\varphi}^{BST}$ which by Theorem 5 yields

$$\tilde{D}_\beta(\omega\|\varphi) = D_\beta^{BST}(\omega\|\varphi) = D_\beta^{BST}(\Phi(\omega)\|\Phi(\varphi)) = \tilde{D}_\beta(\Phi(\omega)\|\Phi(\varphi)),$$

where $\beta := q/2$ and $h_\omega = \|k\|_2^{-2} k^* k$ is the state given by the (normalized) vector $k^*$. By [4, Thm. 7], this equality implies that $\Phi$ is sufficient with respect to $\{\omega, \varphi\}$. Since $h_\omega = \|k\|_2^{-2} h_\varphi^{1/2\alpha} h_\rho^{1/\beta} h_\varphi^{1/2\alpha}$, [4, Lemma 8] implies that $\Phi$ is sufficient with respect to $\{\rho(1)^{-1}\rho, \varphi\}$.

Let $E : \mathcal{M} \to \mathcal{M}$ be a faithful normal conditional expectation as in [4, Lemma 7], so that $\varphi \circ E = \varphi$ and $\Phi$ is sufficient for $\{\psi, \varphi\}$ if and only if

$\psi \circ E = \psi$. Let $E_p$ be the extension of $E$ to $L_p(\mathcal{M})$ ([5], [4, Appendix A.3]). We have by [4, Eq. (A.5)],

$$u^* h_\psi^{1/2} h_\varphi^{1/p-1/2} = h_\rho^{1/p} = E_p(h_\rho^{1/p}) = E_2(u^* h_\psi^{1/2}) h_\varphi^{1/p-1/2}.$$

Since $\varphi$ is faithful, we have $uu^* = s(\psi)$ by the properties of polar decomposition, and the above equalities imply that $u^* h_\psi^{1/2} = E_2(u^* h_\psi^{1/2})$, hence

$$h_{\psi\circ E} = E_1(h_\psi) = h_\psi^{1/2} uu^* h_\psi^{1/2} = h_\psi$$

so that $\Phi$ is sufficient for $\{\psi, \varphi\}$. The converse is obvious from DPI.

$\square$

# Appendix: The spatial derivative

We recall the definition of the spatial derivative $\Delta(\eta/\varphi)$ of [2], using the standard representation $(l(\mathcal{M}), L_2(\mathcal{M}), L_2(\mathcal{M})^+, \cdot^*)$. Let $\mathcal{H}_\varphi := [\mathcal{M} h_\varphi^{1/2}] = L_2(\mathcal{M}) s(\varphi)$ and let $k \in L_2(\mathcal{M})$ be such that the corresponding functional is majorized by $\varphi$:

$$\omega_k(a^* a) = \|ak\|^2 \le C_k \varphi(a^* a), \qquad \forall a \in \mathcal{M},$$

for some positive constant $C_k$. Then

$$R^\varphi(k) : a h_\varphi^{1/2} \mapsto ak, \qquad a \in \mathcal{M}$$

extends to a bounded linear operator $\mathcal{H}_\varphi \to L_2(\mathcal{M})$. Obviously, $R^\varphi(k)$ extends to a bounded linear operator on $L_2(\mathcal{M})$ by putting it equal to 0 on $L_2(\mathcal{M})(1-s(\varphi))$. Moreover, this operator commutes with the left action of $\mathcal{M}$, so that it belongs to $l(\mathcal{M})' = r(\mathcal{M})$, where $r$ is the right action $r(a) : h \mapsto ha$, $h \in L_2(\mathcal{M})$. In fact, $\omega_k$ is majorized by $\varphi$ if and only if $k \in h_\varphi^{1/2}\mathcal{M}$, so that there is some $y_k \in \mathcal{M}$ such that $k = h_\varphi^{1/2} y_k$, $s(\varphi) y_k = y_k$ and we have $R^\varphi(k) = r(y_k)$.

Let now $h \in L_2(\mathcal{M})$, $\omega := \omega_h$. The spatial derivative $\Delta(h/\varphi)$ is a positive self-adjoint operator associated with the quadratic form $k \mapsto (h, R^\varphi(k) R^\varphi(k)^* h)$ as

$$\begin{aligned}(k, \Delta(h/\varphi)k) &= (\Delta(h/\varphi)^{1/2} k, \Delta(h/\varphi)^{1/2} k) = (h, R^\varphi(k) R^\varphi(k)^* h) \\ &= (R^\varphi(k)^* h, R^\varphi(k)^* h) = (h y_k^* s(\varphi), h y_k^* s(\varphi)) = (F_{h, h_\varphi^{1/2}} k, F_{h, h_\varphi^{1/2}} k),\end{aligned}$$

(see [4, Appendix A], for the definition of $F_{\eta,\xi}$). Since $h_\varphi^{1/2}\mathcal{M} + (1-s(\varphi))L_2(\mathcal{M})$ is a core for both $\Delta(h/\varphi)$ and $F_{h, h_\varphi^{1/2}}$, it follows that

$$\Delta(h/\varphi) = F^*_{h, h_\varphi^{1/2}} F_{h, h_\varphi^{1/2}} = J \Delta_{\omega,\varphi} J.$$

This implies that for any $k \in L_2(\mathcal{M})$ and $\gamma \in \mathbb{C}$, we have

$$\|\Delta(h/\varphi)^\gamma k\|_2 = \|\Delta^\gamma_{\omega,\varphi} J k\|_2 = \|\Delta^\gamma_{\omega,\varphi} k^*\|_2.$$

# References


[1] H. Araki and T. Masuda. Positive cones and $L_p$-spaces for von Neumann algebras. *Publ. RIMS, Kyoto Univ.*, 18:339411, 1982.

[2] M. Berta, V. B. Scholz, and M. Tomamichel. Rnyi divergences as weighted non-commutative vector valued $L_p$-spaces. arXiv:1608.05317, 2016.

[3] F. Hiai. Unpublished notes, 2017.

[4] A. Jenčová. Rényi relative entropies and noncommutative $L_p$-spaces. arXiv:1604.08462, 2016.

[5] M. Junge and Q. Xu. Noncommutative Burkholder/Rosenthal inequalities. *Ann. Probab.*, 31:948–995, 2003.

[6] H. Kosaki. Applications of the complex interpolation method to a von Neumann algebra: Non-commutative $L_p$-spaces. *J. Funct. Anal.*, 56:26–78, 1984.