Andrzej Łuczak, Hanna Podsedkowska, and Rafał Wieczorek: Mappings preserving quantum Renyi's entropies in von Neumann algebras

Referee report

The aim of this paper is to characterize normal positive unital maps that preserve the Rényi entropy of a state in a semifinite von Neumann algebra. The main result, stated as the Main Theorem, shows that if such a map is also trace preserving, then it preserves the Rényi entropy of a state ρ if and only if its restriction to the subalgebra generated by the density of ρ is a *-isomorphism.

This result is proved for any value of the parameter α in $(0,1)\cup(1,+\infty)$, the proof is divided into the cases $\alpha \in (0,1)$, $\alpha \in (1,2]$, and $\alpha > 2$ where an additional assumption is required. The proof in the first case is based on integral repsresentation of the function $t \mapsto t^\alpha$ and the previous result of some of the authors [2], where a similar statement is proved for the Segal entropy. The other cases are reduced to this case by extensions of the known Jensen operator or trace inequalities for bounded operators to the case of measurable operators, this is done by standard approximation techniques.

Overall evaluation

The results of the paper are not particlularly surprising. It is basically an extension of the known results on the equality in Jensen's inequality from the case of bounded operators [4]. The proofs are not very sofisticated and it seems that they could be made more effective by using known properties of the measurable operators. For example, Proposition 1 follows easily from operator monotonicity of the function $t \mapsto t^\alpha$ for $\alpha \in (0,1)$, the properties of the singular numbers of measurable operators (Def. 2.1 in [T. Fack and H. Kosaki, Generalized s-numbers of τ-measurable operators, Pacific J. Math. 123(2): 269-300 (1986)]) and some convergence theorems (see Lemmas 2.5, 3.4 and Corollary 2.8 of the Fack-Kosaki paper above).

The paper is also not very well written. I would be more inclined towards publication if it was more self-contained, giving a more precise exposition about the main ingredients of the paper. For example, a more precise definition of the space $L_1(M,\tau)$ and the extension of the map Φ to it could be given without taking up too much space. Also, the final argument of the proof in case 1 could be included at least in a concise form, instead of just referring to [2]. The authors also should give precise formulations and references to the (bounded operator case) Jensen operator and trace inequalities they are using throughout the paper.

The last paragraph contains the following statement:

"The above reasoning shows that some results which state e.g. that a map leaving the relative entropy, or the H as in (18), invariant must be of the form $\Phi(x) = uxu^*$ where u is a unitary or antiunitary or an isometry are incorrect (a good example of such a Jordan *-isomorphism which does not change the entropy but is not of the form as above, is transposition in $B(H)$ with respect to a given orthonormal basis)."

This is quite puzzling, since by the characterization by Kadison, any Jordan *-isomorphism of a factor is precisely of the stated form, for a unitary or antiunitary operator u. I particular, transposition coincodes with complex conjugation on positive operators and is therefore given by an antiunitary transformation.

Some specific comments

The proof of Proposition 1 can be simplified using known results on measurable operators [T. Fack and H. Kosaki, Generalized s-numbers of τ-measurable operators, Pacific J. Math.

123(2): 269-300 (1986)] (all the references in this paragraph are to this paper). Indeed, assume that $0 \le x_n \nearrow x$ in $L^1(M, \tau)$ and let $0 < \alpha < 1$. Then by operator monotonicity of $t \mapsto t^\alpha$ we have $0 \le x_n^\alpha \le x^\alpha$. Further, by Theorem 3.7, $x_n \to x$ in measure topology. Using Lemmas 2.5 (iii) and 3.4 (ii), we get $\mu_t(x_n) \nearrow \mu_t(x)$ for all $t > 0$, where $\mu_t(T)$ is the t-th singular number of the measurable operator T (Def. 2.1). Hence $\mu_t(x_n)^\alpha \nearrow \mu_t(x)^\alpha$ for all $0 < \alpha < 1$, so by the Lebesgue monotone convergence theorem and Corollary 2.8,

$$\tau(x_n^\alpha) = \int \mu_t(x_n)^\alpha dt \nearrow \int \mu_t(x)^\alpha dt = \tau(x^\alpha).$$