On the structure of higher order quantum maps

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September 3, 2024

1 Preliminaries

1.1 The category FinVect

sec:fv

Let FinVect be the category of finite dimensional real vector spaces with linear maps. We will denote the usual tensor product by $\otimes$, then $(\mathrm{FinVect}, \otimes, I = \mathbb{R})$ is a symmetric monoidal category, with the associators, unitors and symmetries given by the obvious isomorphisms

$$\begin{aligned} \alpha_{U,V,W} &: (U \otimes V) \otimes W \simeq U \otimes (V \otimes W), \\ \lambda_V &: I \otimes V \simeq V, \qquad \rho_V : V \otimes I \simeq V, \\ \sigma_{U,V} &: U \otimes V \simeq V \otimes U. \end{aligned}$$

Let $(-)^* : V \mapsto V^*$ be the usual vector space dual, with duality denoted by $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}$. We will use the canonical identification $V^{**} = V$ and $(V_1 \otimes V_2)^* = V_1^* \otimes V_2^*$. With this duality, FinVect is compact closed. This means that for each object V, there are maps $\eta_V : I \to V^* \otimes V$ (the "cup") and $\epsilon_V : V \otimes V^* \to I$ (the "cap") such that the following snake identities hold:

$$(\epsilon_V \otimes V) \circ (V \otimes \eta_V) = V, \qquad (V^* \otimes \epsilon_V) \circ (\eta_V \otimes V^*) = V^*, \tag{1}$$

eq:snake

here we denote the identity map on the object V by V. Indeed, η_V can be identified with an element $\eta_V(1) \in V^* \otimes V$ and $\epsilon_V \in (V \otimes V^*)^* = V^* \otimes V$ is again an element of the same space. Choose a basis $\{e_i\}$ of V, let $\{e_i^*\}$ be the dual basis of V^*, that is, $\langle e_i^*, e_j \rangle = \delta_{i,j}$. Let us then define

$$\eta_V(1) = \epsilon_V := \sum_i e_i^* \otimes e_i.$$

It is easy to see that this definition does not depend on the choice of the basis, indeed, ϵ_V is the linear functional on $V \otimes V^*$ defined by

$$\langle \epsilon_V, x \otimes x^* \rangle = \langle x^*, x \rangle, \qquad x \in V, \; x^* \in V^*.$$

It is also easily checked that the snake identities (eq:snake 1) hold.

For two objects V and W in FinVect, we will denote the set of all morphisms (i.e. linear maps) $V \to W$ $L(V, W)$ by $\mathrm{FinVect}(V, W)$. Then $\mathrm{FinVect}(V, W)$ is itself a real linear space and we have the well-known identification $\mathrm{FinVect}(V, W) \simeq V^* \otimes W$. This can be given as follows: for each $f \in FV(V, W)$, we have $C_f := (V^* \otimes f)(\epsilon_V) = \sum_i e_i^* \otimes f(e_i) \in V^* \otimes W$. Conversely, since $\{e_i^*\}$ is a basis of V^*, any element $w \in V^* \otimes W$ can be uniquely written as $w = \sum_i e_i^* \otimes w_i$ for $w_i \in W$, and since $\{e_i\}$ is a basis of V, the assignment $f(e_i) := w_i$ determines a unique map $f : V \to W$. The relations between $f \in \mathrm{FinVect}(V, W)$ and $C_f \in V^* \otimes W$ can be also written as

$$\langle\, C_f, x \otimes y^* \,\rangle = \langle\, \epsilon_V, x \otimes f^*(y^*) \,\rangle = \langle\, f^*(y^*), x \,\rangle = \langle\, y^*, f(x) \,\rangle, \qquad x \in V,\ y^* \in W^*,$$

here $f^* : W^* \to V^*$ is the adjoint of f. Note that by compactness, the internal hom in FinVect satisfies $[V, W] \simeq V^* \otimes W$, so that in the case of FinVect, the object $[V, W]$ can be identified with the space of linear maps $\mathrm{FinVect}(V, W)$.

lassical *Example* 1. Let $V = \mathbb{R}^N$. In this case, we fix the canonical basis $\{|i\,\rangle,\ i = 1, \dots, N\}$. We will identify $(\mathbb{R}^N)^* = \mathbb{R}^N$, with duality $\langle\, x, y \,\rangle = \sum_i x_i y_i$, in particular, we identify $I = I^*$. We then have $\epsilon_V = \sum_i |i\,\rangle \otimes |i\,\rangle$ and if $f : \mathbb{R}^N \to \mathbb{R}^M$ is given by the matrix A in the two canonical bases, then $C_f = \sum_i |i\,\rangle \otimes A|i\,\rangle$ is the vectorization of A.

:quantum *Example* 2. Let $V = M_n^h$ be the space of $n \times n$ complex hermitian matrices. We again identify $(M_n^h)^* = M_n^h$, with duality $\langle\, A, B \,\rangle = \mathrm{Tr}\, A^T B$, where A^T is the usual transpose of the matrix A. Let us choose the basis in M_n^h, given as

$$\Big\{ |j\,\rangle\langle\, k| + |k\,\rangle\langle\, j|,\ j \le k,\ i\Big(|j\,\rangle\langle\, k| - |k\,\rangle\langle\, j|\Big),\ j < k \Big\}.$$

Then one can check that

$$\Big\{ \frac{1}{2}\Big(|j\,\rangle\langle\, k| + |k\,\rangle\langle\, j|\Big),\ j \le k,\ \frac{i}{2}\Big(|k\,\rangle\langle\, j| - |j\,\rangle\langle\, k|\Big),\ j < k \Big\}$$

is the dual basis and we have

$$\epsilon_V = \sum_{j,k} |j\,\rangle\langle\, k| \otimes |j\,\rangle\langle\, k|.$$

For any $f : M_n^h \to M_m^h$,

$$C_f = \sum_{j,k} |j\,\rangle\langle\, k| \otimes f(|j\,\rangle\langle\, k|)$$

is the Choi matrix of f.

1.2 Boolean functions

We will work with certain boolean functions, which are defined as functions from binary strings $\{0, 1\}^*$ to $\{0, 1\}$. We now list some basic notations used below.

For $s \in \{0, 1\}$, we denote $\bar{s} := 1 - s$. For binary strings of fixed length n, that is, elements of $\{0, 1\}^n$, we will denote by 0_n or just 0 the string $00 \dots 0$ and

$$e^i = \delta_{i,1} \dots \delta_{i,n}.$$

For $m, n \in \mathbb{N}$, the concatenation of strings $s \in \{0, 1\}^m$ and $t \in \{0, 1\}^n$ will be denoted by st, that is,

$$st = s_1 \dots s_m t_1 \dots t_n \in \{0, 1\}^{m+n}.$$

For a string $x \in \{0, 1\}^n$ and any set of indices $\{i_1, \dots, i_k\} \subseteq [n]$, we will denote by $x^{i_1 \dots i_k}$ the string in $\{0, 1\}^{n-k}$ obtained from x by removing $x_{i_1}, \dots, x_{i_k}$. For any permutation $\sigma \in S_n$, we will denote by the same symbol the obvious action on $\{0, 1\}^n$, that is

$$\sigma(s_1 \dots s_n) = s_{\sigma(1)} \dots s_{\sigma(n)}.$$

Let us introduce the subset

$$\mathcal{F}_n := \{f : \{0,1\}^n \to \{0,1\},\ f(0) = 1\}.$$

With the poitwise ordering, $\mathcal{F}_n$ is a (finite) distributive lattice, with top element the constant 1 function and the bottom element $p_n := \chi_0$, the characteristic function of the zero string. We may also define complementation in $\mathcal{F}_n$ as

$$f^* := 1 - f + p_n.$$

It can be easily checked that with these structures $\mathcal{F}_n$ is a boolean algebra.

We now introduce some more operations in $\mathcal{F}_n$. For $f \in \mathcal{F}_n$ and any permutation $\sigma \in S_n$, we see that $f \circ \sigma \in \mathcal{F}_n$. For $f \in \mathcal{F}_m$ and $g \in \mathcal{F}_n$, we define the function $f \otimes g \in \mathcal{F}_{m+n}$ as

$$(f \otimes g)(st) = f(s)g(t).$$

As it is, this tensor product is not symmetric, but there is a permutation $\sigma \in S_{m+n}$ such that $(g \otimes f) = (f \otimes g) \circ \sigma$ for any $f \in \mathcal{F}_m$ and $g \in \mathcal{F}_n$.

fproduct **Lemma 1.** *For $f \in \mathcal{F}_m$, $g \in \mathcal{F}_n$, we have*

$$f \otimes g \leq (f^* \otimes g^*)^*.$$

Equality holds if and only if f and g are either both the top or both the bottom elements in $\mathcal{F}_m$ resp. $\mathcal{F}_n$.

Proof. The inequality is easily checked, since $(f \otimes g)(st)$ can be 1 only if $f(s) = g(t) = 1$. If both s and t are the zero strings, then $st = 0_{m+n}$ and both sides are equal to 1. Otherwise, the condition $f(s) = g(t) = 1$ implies that $(f^* \otimes g^*)(st) = 0$, which implies that the right hand side must be 1. If f and g are both constant 1, then $(1 \otimes 1)^* = 1^* = p_{n+m} = 1^* \otimes 1^*$, in the case when both f and g are the minimal elements equality follows by duality. Finally, asume the equality holds and that $f \neq 1$, so that there is some s such that $f(s) = 0$. But then $s \neq 0$ and for any t,

$$0 = (f \otimes g)(st) = (f^* \otimes g^*)^*(st) = 1 - f^*(s)g^*(t) + p_{m+n}(st) = 1 - g^*(t),$$

which implies that $g(t) = 0$ for all $t \neq 0$, that is, $g = p_n$. By the same argument, $f = p_m$ if $g \neq 1$, which implies that either $f = 1$ and $g = 1$, or $f = p_m$ and $g = p_n$.

□

We now show an important example.

ex:ps *Example* 3. Let $S \subseteq [n]$ be any subset. We will denote

$$p_S(s) := \prod_{i \in S} \bar{s}_i.$$

It is clear that $p_S \in \mathcal{F}_n$, $p_\emptyset = 1$, $p_{[n]} = \chi_0 = p_n$. The following properies are also easy to see for $S, T \subseteq [n]$:

(i) if $S \subseteq T$, then $p_T \leq p_S$,

(ii) $p_S \wedge p_T = p_S p_T = p_{S \cup T}$,

(iii) $p_S \vee p_T = p_S + p_T - p_{S\cup T}$.

(iv) let $S \subseteq [m]$ and $T \subseteq [n]$, then

$$p_S \otimes p_T = p_{S\cup(m+T)}.$$

We will use the above functions to introduce a convenient parametrization to $\mathcal{F}_n$. For this, we first include $\mathcal{F}_n$ into a larger set

$$\mathcal{F}_n \subseteq \{f : \{0,1\}^n \to \mathbb{R}\} =: \mathcal{V},$$

which is a 2^n-dimensional real vector space. It becomes a Hilbert space with respect to the inner product

$$\langle\, f, g \,\rangle = \sum_{s\in\{0,1\}^n} f(s)g(s).$$

a:Vbasis **Lemma 2.** *The set* $\{p_S,\ S \subseteq [n]\}$ *is a basis of* $\mathcal{V}$*. Any* $f \in \mathcal{V}$ *can be uniquely written as*

$$f = \sum_{S\subseteq[n]} \hat{f}_S p_S,$$

where the coefficients $\hat{f}_S \in \mathbb{R}$ *are obtained as*

$$\hat{f}_S = \sum_{\substack{x\in\{0,1\}^n \\ x_i=1,\forall i\in S^c}} (-1)^{\sum_{i\in S} x_i} f(x).$$

Proof. For $T \subseteq [n]$, let us define the function $p_T^\perp$ as

$$p_T^\perp(x) := (-1)^{\sum_{i\in T} x_i} \prod_{i\in T^c} x_i.$$

We prove that for $S, T \subseteq [n]$,

$$\langle\, p_S, p_T^\perp \,\rangle = \delta_{S,T},$$

which shows that $\{p_S,\ S \subseteq [n]\}$ is a basis and $\{p_T^\perp,\ T \subseteq [n]\}$ is the dual basis. We compute

$$\langle\, p_S, p_T^\perp \,\rangle = \sum_x p_S(x) p_T^\perp(x) = \sum_x (-1)^{\sum_{i\in T} x_i} \prod_{i\in S} \bar{x}_i \prod_{j\in T^c} x_j.$$

This expression can be nonzero only if $S \cap T^c = \emptyset$, that is, $S \subseteq T$. In this case, the last sum is equal to

$$\sum_{\substack{x\in\{0,1\}^n \\ x_i=0,\forall i\in S \\ x_i=1,\forall i\in T^c}} (-1)^{\sum_{j\in T\setminus S} x_j} = \begin{cases} 0 & \text{if } S \subsetneq T \\ 1 & \text{if } S = T \end{cases}$$

It is now clear that the coefficients

$$\hat{f}_S = \langle\, f, p_S^\perp \,\rangle$$

have the given form.

□

Remark 1. This can be also obtained using Möbius inversion formula, see [Stanley, Sec. 3.7].

Let $\mathcal{L}_n$ be the lattice of subsets in $[n]$. We can visualise $\mathcal{L}_n$ as a hypercube, and the coefficients $\hat{f}_S$ of f as labels for its vertices. Let $s \in \{0,1\}^n$ be a string such that $s_j = 1$ if and only if $j \in T$. Then $p_S(s) = 1$ if and only if $T \cap S = \emptyset$. The fact that the function f takes values in $\{0,1\}$ means that we must have

$$f(s) = \sum_{\substack{S \subseteq [n] \\ S \cap T = \emptyset}} \hat{f}_S \in \{0,1\}.$$

It follows that the sum of labels $\hat{f}_S$ over any face of the hypercube $\mathcal{L}_n$ containing the vertex $\emptyset$ must be 0 or 1. In particular, $\hat{f}_\emptyset = f(11\ldots1) \in \{0,1\}$. The fact that $f \in \mathcal{F}_n$ means that in addition,

$$f(0) = \sum_{S \subseteq [n]} \hat{f}_S = 1.$$

2 The category of affine subspaces

2.1 The category Af

We now introduce the category Af, whose objects are of the form $X = (V_X, A_X)$, where V_X is an object in FinVect and $A_X \subseteq V_X$ a proper affine subspace, which means that $0 \notin A_X \neq \emptyset$. Morphisms $X \xrightarrow{f} Y$ in Af are linear maps $f : V_X \to V_Y$ such that $f(A_X) \subseteq A_Y$. For any object X, we put

$$L_X = \mathrm{Lin}(A_X) := \{a_1 - a_2,\ a_1, a_2 \in A_X\} = \{a - a_X,\ a \in A_X\}, \qquad S_X := \mathrm{Span}(A_X).$$

Here a_X is any element in A_X and L_X does not depend on this choice. Then L_X and S_X are linear subspaces such that $d_X := \dim(L_X) = \dim(S_X) - 1$. We will also denote $D_X = \dim(V_X)$. For any element $a_X \in A_X$, the affine subspace is determined as

$$A_X = a_X + L_X.$$

Let us now define the duality of affine subspaces as follows. Let V be an object in FinVect and let $C \subseteq V$ be any subset. Let

$$\tilde{C} := \{v^* \in V^*,\ \langle v^*, c\rangle = 1\}.$$

The following lemma collects some properties that are easily proven.

mma:dual **Lemma 3.** *(i) $\tilde{C}$ is an affine subspace.*

(ii) $0 \in \tilde{C}$ if and only if $C = \emptyset$ and $\tilde{C} = \emptyset$ if and only if $0 \in \mathrm{Aff}(C)$.

(iii) Let $0 \notin \mathrm{Aff}(C)$, then $\mathrm{Aff}(C) = \tilde{\tilde{C}}$ and we have

$$\begin{aligned} \mathrm{Lin}(C) &= \mathrm{Lin}(\tilde{\tilde{C}}) = \tilde{C}^\perp = \mathrm{Span}(\tilde{C})^\perp, & \qquad \mathrm{Lin}(\tilde{C}) &= C^\perp = \mathrm{Span}(C)^\perp \\ \mathrm{Span}(C) &= C^{\perp\perp} = \mathrm{Lin}(\tilde{C})^\perp, & \mathrm{Span}(\tilde{C}) &= \mathrm{Lin}(C)^\perp. \end{aligned}$$

For any $\tilde{a}_X \in \tilde{A}_X$, the subspace A_X is determined as

$$A_X = S_X \cap \{\tilde{a}_X\}^\sim.$$

The relation between the subspaces L_X and S_X is given as

$$S_X = L_X \oplus \mathbb{R}a_X, \qquad L_X = S_X \cap \{\tilde{a}_X\}^\perp.$$

By Lemma 3 above, $\tilde{A}_X$ is a proper affine subspace in V_X^*, so that $X^* := (V_X^*, \tilde{A}_X)$ is an object in Af. We have $X^{**} = X$ and the corresponding subspaces are related as

$$L_{X^*} = S_X^\perp, \qquad S_{X^*} = L_X^\perp. \tag{2}$$

Note also that for $X \xrightarrow{f} Y$, the adjoint map satisfies $f^*(\tilde{A}_Y) \subseteq \tilde{A}_X$, so that $Y^* \xrightarrow{f^*} X^*$ and the duality $(-)^*$ is a full and faithful functor $\mathrm{Af}^{op} \to \mathrm{Af}$.

We will next introduce a monoidal structure $\otimes$ as follows. For two objects X and Y, we put $V_{X\otimes Y} = V_X \otimes V_Y$ and construct the affine subspace $A_{X\otimes Y}$ as the affine span of

$$A_X \otimes A_Y = \{a \otimes b,\ a \in A_X,\ b \in A_Y\}.$$

Fix any $\tilde{a}_X \in \tilde{A}_X$ and $\tilde{a}_Y \in \tilde{A}_Y$. Since $A_X \otimes A_Y \subset \{\tilde{a}_X \otimes \tilde{a}_Y\}^\sim$, the affine span of $A_X \otimes A_Y$ is a proper affine subspace and we have by Lemma 3

$$A_{X\otimes Y} := \mathrm{Aff}(A_X \otimes A_Y) = \{A_X \otimes A_Y\}^{\approx}.$$

Lemma 4. *For any $a_X \in A_X$, $a_Y \in A_Y$, we have*

$$\begin{aligned} L_{X\otimes Y} &= \mathrm{Lin}(A_X \otimes A_Y) = \mathrm{Span}(\{x \otimes y - a_X \otimes a_Y,\ x \in A_X,\ y \in A_Y\}) && (3)\\ &= (a_X \otimes L_Y) + (L_X \otimes a_Y) + (L_X \otimes L_Y) && (4) \end{aligned}$$

(here + denotes the direct sum of subspaces). We also have

$$S_{X\otimes Y} = S_X \otimes S_Y.$$

Proof. The equality (3) follows from Lemma 3. For any $x \in A_X$, $y \in A_Y$ we have

$$x \otimes y - a_X \otimes a_Y = a_X \otimes (y - a_Y) + (x - a_X) \otimes a_Y + (x - a_X) \otimes (y - a_Y),$$

so that $L_{X\otimes Y} = \mathrm{Lin}(A_X \otimes A_Y)$ is contained in the subspace on the RHS of (4). Let d be the dimension of this subspace, then clearly

$$d_{X\otimes Y} \le d \le d_X + d_Y + d_X d_Y.$$

On the other hand, any element of S_X has the form tx for some $t \in \mathbb{R}$ and $x \in A_X$, so that it is easily seen that $S_X \otimes S_Y = S_{X\otimes Y}$. Hence

$$\begin{aligned} d_{X\otimes Y} &= \dim(L_{X\otimes Y}) = \dim(S_{X\otimes Y}) - 1 = \dim(S_X)\dim(S_Y) - 1 = (d_X + 1)(d_Y + 1) - 1\\ &= d_X + d_Y + d_X d_Y. \end{aligned}$$

This completes the proof. $\square$

Lemma 5. *Let $I = (\mathbb{R}, \{1\})$. Then $(\mathrm{Af}, \otimes, I)$ is a symmetric monoidal category.*

Proof. Note that this structure is inherited from the symmetric monoidal structure in FinVect. To show that $\otimes$ is a functor, we have to check that for $X_1 \xrightarrow{f} Y_1$ and $X_2 \xrightarrow{g} Y_2$ in Af, we have $X_1 \otimes Y_1 \xrightarrow{f\otimes g} X_2 \otimes Y_2$ which amounts to showing that

$$(f \otimes g)(A_{X_1 \otimes Y_1}) \subseteq A_{X_2 \otimes Y_2}.$$

Let $x \in A_{X_1}$, $y \in A_{Y_1}$, then $f(x) \otimes g(y) \in A_{X_2} \otimes A_{Y_2} \subseteq A_{X_2 \otimes Y_2}$. Since $A_{X_1 \otimes Y_1}$ is the affine subspace generated by $A(X_1) \otimes A(Y_1)$, the above inclusion follows by linearity of $f \otimes g$.

It only remains to prove that the associators, unitors and symmetries from FinVect are morphisms in Af. We will prove this for the associators $\alpha_{X,Y,Z} : V_X \otimes (V_Y \otimes V_Z) \to (V_X \otimes V_Y) \otimes V_Z$, the other proofs are similar. We need to check that $\alpha_{X,Y,Z}(A_{X\otimes(Y\otimes Z)}) \subseteq A_{(X\otimes Y)\otimes Z}$. It is easily checked that $A_{X\otimes(Y\otimes Z)}$ is the affine span of elements of the form $x \otimes (y \otimes z)$, $x \in A_X$, $y \in A_Y$ and $z \in A_Z$, and we have

$$\alpha_{X,Y,Z}(x \otimes (y \otimes z)) = (x \otimes y) \otimes z \in A_{(X\otimes Y)\otimes Z}$$

for all such elements. The desired inclusion follows by linearity.

□

Theorem 1. *$(\mathrm{Af}, \otimes, I)$ is a *-autonomous category, with duality $(-)^*$, such that $I^* = I$.*

lemma:monoidal

Proof. By Lemma 5, we have that $(\mathrm{Af}, \otimes I)$ is a symmetric monoidal category. We have also seen that the duality $(-)^*$ is a full and faithful contravariant functor. We only need to check the natural isomorphisms

$$\mathrm{Af}(X \otimes Y, Z^*) \simeq \mathrm{Af}(X, (Y \otimes Z)^*).$$

Since FinVect is compact, we have the natural isomorphisms

$$\mathrm{FinVect}(V_X \otimes V_Y, V_Z^*) \simeq \mathrm{FinVect}(V_X, V_Y^* \otimes V_Z^*),$$

determined by the equalities

$$\langle\, f(x \otimes y), z \,\rangle = \langle\, h(x), y \otimes z \,\rangle, \qquad x \in V_X,\ y \in V_Y,\ z \in V_Z,$$

for $f \in \mathrm{FinVect}(V_X \otimes V_Y, V_Z^*)$ and $h \in \mathrm{FinVect}(V_Z, V_Y^* \otimes V_Z^*)$. Since $A_{X\otimes Y}$ is an affine span of $A_X \otimes A_Y$, we see that f is in $\mathrm{Af}(X \otimes Y, Z^*)$ if and only if $f(x \otimes y) \in \tilde{A}_Z$ for all $x \in A_X$, $y \in A_Y$, that is,

$$1 = \langle\, f(x \otimes y), z \,\rangle = \langle\, h(x), y \otimes z \,\rangle \qquad \forall x \in A_X,\ \forall y \in A_Y,\ \forall z \in A_Z.$$

But this is equivalent to

$$h(x) \in (A_Y \otimes A_Z)^{\sim} = \tilde{A}_{Y\otimes Z}, \qquad \forall x \in A_X,$$

which means that $h \in \mathrm{Af}(X, (Y \otimes Z)^*)$.

□

A *-autonomous category is compact closed if it satisfies $(X \otimes Y)^* = X^* \otimes Y^*$. In general, $X \odot Y = (X^* \otimes Y^*)^*$ defines a dual symmetric monoidal structure that is different from $\otimes$. We next show that Af is not compact.

ncompact

Proposition 1. *For objects in* Af*, we have $(X \otimes Y)^* = X^* \otimes Y^*$ exactly in one of the following situations:*

(i) $X \simeq I$ *or* $Y \simeq I$*,*

(ii) $d_X = d_Y = 0$*,*

(iii) $d_{X^*} = d_{Y^*} = 0$*.*

Proof. It is easily seen by definition that $A_{X^*} \otimes A_{Y^*} = \tilde{A}_X \otimes \tilde{A}_Y \subseteq \tilde{A}_{X \otimes Y} = A_{(X\otimes Y)^*}$. Hence the equality holds if and only if $d_{X^*\otimes Y^*} = d_{(X\otimes Y)^*}$. From Lemma 4, we see that

$$d_{X^*\otimes Y^*} = d_{X^*} + d_{Y^*} + d_{X^*}d_{Y^*}.$$

On the other hand, we have using (2) that $L_{(X\otimes Y)^*} = S^{\perp}_{X\otimes Y} = (S_X \otimes S_Y)^{\perp}$, so that

$$d_{(X\otimes Y)^*} = D_X D_Y - \dim(S_X)\dim(S_Y) = D_X D_Y - (d_X + 1)(d_Y + 1).$$

Taking into account that by (2) we have $d_{X^*} = D_X - d_X - 1$, similarly for d_{Y^*}, we obtain

$$d_{(X\otimes Y)^*} - d_{X^*\otimes Y^*} = d_X d_{Y^*} + d_{X^*} d_Y.$$

This is equal to 0 iff $d_X d_{Y^*} = d_Y d_{X^*} = 0$, which amounts to the conditions in the lemma.

□

In a *-autonomous category, the internal hom can be identified as $[X, Y] = (X \otimes Y^*)^*$. The underlying vector space is $V_{[X,Y]} = (V_X \otimes V_Y^*)^* = V_X^* \otimes V_Y$ and we have seen in Section 1.1 that we may identify this space with $\mathrm{FinVect}(V_X, V_Y)$, by $f \leftrightarrow C_f$. This property is extended to Af, in the following sense.

Proposition 2. *For any objects* X, Y *in* Af*, the map* $f \mapsto C_f$ *is a bijection of* $\mathrm{Af}(X, Y)$ *onto* $A_{[X,Y]}$*.*

Proof. Let $f \in \mathrm{FinVect}(V_X, V_Y)$. Since by definition $A_{[X,Y]} = \tilde{A}_{X\otimes Y^*}$ and $A_{X\otimes Y^*}$ is an affine span of $A_X \otimes A_Y^*$, we see that $C_f \in A_{[X,Y]}$ if and only if for all $x \in A_X$ and $y^* \in \tilde{A}_Y$, we have

$$1 = \langle C_f, x \otimes y^* \rangle = \langle y^*, f(x) \rangle.$$

This latter statement is clearly equivalent to $f(A_X) \subseteq A_Y$, so that $f \in \mathrm{Af}(X, Y)$. □

In the next result, we restrict the objects to spaces of hermitian matrices, as in Example 2 and morphisms to completely positive maps. We show that this restriction amounts to taking an intersection of $A_{[X,Y]}$ with the cone of positive semidefinite matrices. This, and subsequent examples, shows that for characterization of sets of quantum objects such as states, channels, combs and transformations between them, it is enough to work with the category Af.

An object X of Af will be called quantum if $V_X = M_n^h$ for some n and A_X is an affine subspace such that both A_X and $\tilde{A}_X$ have nonempty intersection with the interior of the positive cone $int(M_n^+)$ (recall that we identify $(M_n^h)^* = M_n^h$).

Proposition 3. *Let* X*,* Y *be quantum objects in* Af*. Then*

(i) X^* *and* $X \otimes Y$ *are quantum objects as well*

(ii) Let $V_X = M_n^h$*,* $V_Y = M_m^h$*. Then for any* $f \in \mathrm{FinVect}(M_n^h, M_m^h)$*, we have* $C_f \in A_{[X,Y]} \cap M_{mn}^+$ *if and only if* f *is completely positive and*

$$f(A_X \cap M_n^+) \subseteq A_Y \cap M_m^+.$$

Proof. The statement (i) is easily seen from $A_X \otimes A_Y \subseteq A_{X\otimes Y}$ and $\tilde{A}_X \otimes \tilde{A}_Y \subseteq \tilde{A}_{X\otimes Y}$, together with the fact that $int(M_n^+) \otimes int(M_m^+) \subseteq int(M_{mn}^+)$. To show (ii), let $C_f \in A_{[X,Y]} \cap M_{mn}^+$. By the properties of the Choi isomorphism f is completely positive and by Proposition 2, $f(A_X) \subseteq A_Y$, this proves one implication. For the converse, note that we only need to prove that under the given assumptions, $f(A_X) \subseteq A_Y$, for which it is enough to show that $A_X \subseteq \mathrm{Aff}(A_X \cap M_n^+)$. To see this, pick some $a_X \in A_X \cap int(M_n^+)$. Any element in A_X can be written in the form $a_X + v$ for some $v \in L_X$. Since $a_X \in int(M_n^+)$, there is some $s > 0$ such that $a_\pm := a_X \pm sv \in M_n^+$, and since $\pm sv \in L_X$, we see that $a_\pm \in A_X \cap M_n^+$. It is now easily checked that

$$a_X + v = \frac{1+s}{2s} a_+ + \frac{s-1}{2s} a_- \in \mathrm{Aff}(A_X \cap M_n^+).$$

□

We can define classical obejcts in Af in a similar way, replacing M_n^h by $\mathbb{R}^N$ and the positive cone by $\mathbb{R}_+^N$. A similar statement holds in this case, with complete positivity replaced by positivity. We can similarly treat classical-to-quantum and quantum-to-classical maps as morphisms between these types of objects, satisfying appropriate positivity assumptions.

Example 4. States, channels, combs, nonsignaling, etb, dual, process matrices

Example 5. POVMs, instruments, multimeters.

2.2 First order and higher order objects

We say that an object X in Af is first order if $d_X = D_X - 1$, equivalently, $S_X = V_X$. Another equivalent condition is $d_{X^*} = 0$, which means that A_X is determined by a single element $\tilde{a}_X \in V_X^*$ as

$$A_X = \{\tilde{a}_X\}^\sim, \qquad \tilde{A}_X = \{\tilde{a}_X\}.$$

In the case of first order quantum objects we additionally require that $\tilde{a}_X \in int(M_n^+)$, similarly for classical first order objects. Note that first order objects, resp. their duals, are exactly those satisfying condition (iii), resp. condition (ii), in Proposition 1, in particular, $(X \otimes Y)^* = X^* \otimes Y^*$ for first order objects X and Y.

For a first order object $X = (V_X, \{\tilde{a}_X\}^\sim)$, let us pick an element $a_X \in A_X$, then we have a direct sum decomposition

$$V_X = L_{X,0} \oplus L_{X,1},$$

where $L_{X,0} := \mathbb{R}\{a_X\}$, $L_{X,1} := \{\tilde{a}_X\}^\perp = L_X$. We also define the *conjugate object* $\tilde{X} = (V_X^*, \{a_X\}^\sim)$, note that we always have $\tilde{a}_X \in A_{\tilde{X}}$ and with the choice $a_{\tilde{X}} = \tilde{a}_X$, we have $\tilde{\tilde{X}} = X$ and

$$L_{\tilde{X},u} = L_{X,\bar{u}}^\perp, \qquad u \in \{0,1\}. \tag{5}$$

These definitions depend on the choice of a_X, but we will assume below that this choice is fixed and that we choose $a_{\tilde{X}} = \tilde{a}_X$. For quantum objects we will assume that $a_X \in int(M_n^+)$.

Example 6. Example: quantum states, multiple of identity...

Higher order objects are those obtained from a finite set $\{X_1, \ldots, X_n\}$ of first order objects by taking tensor products and duals, and applying any permuations of the spaces. The above is indeed a set, so that all the objects are different (though they may be isomorphic) and the ordering is not essential. We will also assume that the tensor unit is not contained in this set. Of course, any first order object is also higher order with $n = 1$. Note that we cannot say that such an object is automatically "of order n", as the following lemma shows.

ertensor **Lemma 6.** *Let X, Y be first order, then $X \otimes Y$ is first order as well.*

Proof. We have

$$S_{X\otimes Y} = S_X \otimes S_Y = V_X \otimes V_Y = V_{X\otimes Y}.$$

□

Example 7. (states quantum first order, channels, supermaps - quantum higher order)

Example 8. replacement $X^* \otimes Y$, quantum

2.3 Description of higher order objects

We start by noticing that there are certain objects in Af that can be constructed from a set of first order objects and functions in $\mathcal{F}_n$.

Let $X_1, \dots, X_n$ be first order objects in Af. Let $a_{X_i} \in A_{X_i}$ be fixed and let $\tilde{X}_i$ be the conjugate first order objects. Let us denote $V_i = V_{X_i}$ and

$$L_{i,u} := L_{X_i,u}, \qquad \tilde{L}_{i,u} := L_{\tilde{X}_i,u} \qquad u \in \{0,1\},\ i \in [n].$$

For $s \in \{0,1\}^n$, we define

$$L_s := L_{1,s_1} \otimes \dots \otimes L_{n,s_n}, \qquad \tilde{L}_s := \tilde{L}_{1,s_1} \otimes \dots \otimes \tilde{L}_{n,s_n},$$

then we have the direct sum decompositions

$$V := V_1 \otimes \dots \otimes V_n = \bigoplus_{s\in\{0,1\}^n} L_s, \qquad V^* = V_1^* \otimes \dots \otimes V_n^* = \bigoplus_{s\in\{0,1\}^n} \tilde{L}_s.$$

ma:Lperp **Lemma 7.** *For any $s \in \{0,1\}^n$, we have*

$$L_s^\perp = \bigoplus_{t\in\{0,1\}^n} \bar{\chi}_s(t)\tilde{L}_t, \qquad \tilde{L}_s^\perp = \bigoplus_{t\in\{0,1\}^n} \bar{\chi}_s(t) L_t.$$

Here $\chi_s : \{0,1\}^n \to \{0,1\}$ is the characteristic function of s, $\bar{\chi}_s = 1 - \chi_s$.

Proof. Using (eq:complement 5) and the direct sum decomposition of V_i^*, we get

$$\begin{aligned}
(L_{1,s_1} \otimes \dots \otimes L_{n,s_n})^\perp &= \bigvee_j \left(V_1^* \otimes \dots \otimes V_{j-1}^* \otimes \tilde{L}_{j,\bar{s}_j} \otimes V_{j+1}^* \otimes \dots \otimes V_n^* \right) \\
&= \bigvee_j \left(\bigoplus_{\substack{t\in\{0,1\}^n \\ t_j \neq s_j}} \tilde{L}_{1,t_1} \otimes \dots \otimes \tilde{L}_{n,t_n} \right) \\
&= \bigoplus_{\substack{t\in\{0,1\}^n \\ t \neq s}} \left(\tilde{L}_{1,t_1} \otimes \dots \otimes \tilde{L}_{n,t_n} \right).
\end{aligned}$$

The proof of the other equality is the same.

□

lemma:Xf **Lemma 8.** *Put* $a := a_1 \otimes \cdots \otimes a_n$, $\tilde a := \tilde a_1 \otimes \cdots \otimes \tilde a_n$. *For* $f \in \mathcal{F}_n$ *define*

$$S_f := \bigoplus_{s\in\{0,1\}^n} f(s) L_s, \qquad A_f = A_f(X_1,\dots,X_n) := S_f \cap \{\tilde a\}^\sim.$$

Then A_f *is a proper affine subspace in* V *containing* a. *Moreover,*

$$L_{A_f} = \bigoplus_{s\in\{0,1\}^n\setminus\{0\}} f(s) L_s, \qquad S_{A_f} = S_f$$

and the dual affine subspace satisfies

$$\tilde A_f(X_1,\dots,X_n) = A_{f^*}(\tilde X_1,\dots,\tilde X_n) = \bigoplus_{s\in\{0,1\}^n} f^*(s)\tilde L_s \cap \{a\}^\sim.$$

Proof. It is clear from definition that A_f is an affine subspace. Since $f(0) = 1$, the space S_f always contains the subspace $L_0 = L_{1,0} \otimes \cdots \otimes l_{n,0} = \mathbb{R}\{a\}$ and it is clear that $L_s \subseteq \{\tilde a\}^\perp$ for any $s \neq 0$. It follows that $a \in A_f$, so that $A_f \neq \emptyset$, and since $A_f \subseteq \{\tilde a\}^\sim$, we see that A_f is proper and $\tilde a \in \tilde A_f$. The expressions for L_{A_f} and S_{A_f} are immediate from the definition and $L_{A_f} = S_{A_f} \cap \{\tilde a\}^\sim$. To obtain the dual affine subspace, we compute using Lemma 7 and the fact that the subspaces form an independent decomposition,

$$\begin{aligned} S_{\tilde A_f} = L_{A_f}^\perp &= \Bigg(\bigoplus_{s\in\{0,1\}^n\setminus\{0\}} f(s)L_s\Bigg)^\perp = \bigwedge_{\substack{s\in\{0,1\}^n\\ s\neq 0, f(s)=1}} L_s^\perp = \bigwedge_{\substack{s\in\{0,1\}^n\\ s\neq 0, f(s)=1}} \Bigg(\bigoplus_{t\in\{0,1\}^n} \bar\chi_s(t)\tilde L_t\Bigg) \\ &= \bigoplus_{t\in\{0,1\}^n} \Bigg(\bigwedge_{\substack{s\in\{0,1\}^n\\ s\neq 0, f(s)=1}} \bar\chi_s(t)\tilde L_t\Bigg) = \bigoplus_{t\in\{0,1\}^n} f^*(t)\tilde L_t. \end{aligned}$$

To see the last equality, note that

$$\bigwedge_{\substack{s\in\{0,1\}^n\\ s\neq 0, f(s)=1}} \bar\chi_s(t) = \begin{cases} 1 & \text{if } t = 0 \\ 1 - f(t) & \text{if } t \neq 0 \end{cases} = f^*(t).$$

□

Since L_s, $s \in \{0,1\}$ is an independent decomposition, the map $f \mapsto S_f$, and hence also $f \mapsto A_f$, is injective. This map has the following further properties, which are easily checked:

(i) For the bottom and top elements in $\mathcal{F}_n$ we have

$$A_{p_n} = \{a\}, \qquad A_{1_n} = \{\tilde a\}^\sim,$$

(ii) We have $f \leq g$ if and only if $A_f \subseteq A_g$,

(iii) $A_{f\wedge g} = A_f \cap A_g$,

(iv) $A_{f\vee g} = A_f \vee A_g := \mathrm{Aff}(A_f \cup A_g)$.

It follows that the set $\{A_f,\ f \in \mathcal{F}_n\}$ is a distributive lattice, with respect to the lattice operations $\cap$ and $\vee$.

Since all the affince subspaces $A_f \subseteq V$ are proper, there are objects $X_f := (V, A_f)$ in Af. The above relations can be rephrased as follows:

(i) $X_{1_n} = (V, \{\tilde{a}\}^\sim)$ is a first order object, $X_{p_n} = (V^*, \{a\}^\sim)^\sim$ is a dual first order object.

(ii) We have $f \leq g$ if and only if id_V is a morphism $X_f \to X_g$ in Af,

(iii) Let $f, g \leq h$. The following is a pullback diagram:

$$\begin{tikzcd} X_{f\wedge g} \arrow[r, "id_V"] \arrow[d, "id_V"'] & X_f \arrow[d, "id_V"] \\ X_g \arrow[r, "id_V"'] & X_h \end{tikzcd}$$

(iv) Let $h \leq f, g$. The following is a pushout diagram:

$$\begin{tikzcd} X_h \arrow[r, "id_V"] \arrow[d, "id_V"'] & X_f \arrow[d, "id_V"] \\ X_g \arrow[r, "id_V"'] & X_{f\vee g} \end{tikzcd}$$

In particular, it follows that $\{X_f,\ f \in \mathcal{F}_n\}$, is a distributive lattice, with pullbacks and pushouts as lattice operations. Furthermore, using the conjugate objects, we may construct

$$\tilde{X}_f := (V^*, A_f(\tilde{X}_1, \ldots, \tilde{X}_n))$$

and we see from Lemma lemma:Xf 8 that

$$X_f^* = \tilde{X}_{f^*}, \qquad f \in \mathcal{F}_n. \tag{6}$$ eq:duali

We next observe that the higher order objects are of the form X_f, for some choice of the first order objects $X_1, \ldots, X_n$ and a function f that belongs to a special subclass of $\mathcal{F}_n$. So assume that Y is a higher order object constructed from a set of distinct first order objects $Y_1, \ldots, Y_n$, $Y_i = (V_{Y_i}, \{\tilde{a}_{Y_i}\}^\sim)$, we will write $Y \sim \{Y_1, \ldots, Y_n\}$ in this case. Let us fix elements $a_{Y_i} \in A_{Y_i}$ and construct the objects $\tilde{Y}_i$.

By compactness of FinVect, we may assume (relabeling the objects if necessary) that the vector space of Y has the form

$$V_Y = V := V_1 \otimes \cdots \otimes V_n,$$

where V_i is either V_{Y_i} or $V_{Y_i}^*$, according to whether Y_i was subjected to taking duals an even or odd number of times. The indices such that the first case is true will be called the outputs and the subset of outputs in $[n]$ will be denoted by O, or O_Y, when we need to specify the object. The set $I = I_Y := [n] \setminus O_Y$ is the set of inputs. The reason for this terminology will become clear later.

:boolean **Proposition 4.** *For $i \in [n]$, let $X_i = Y_i$ if $i \in O_Y$ and $X_i = \tilde{Y}_i$ for $i \in I_Y$. There is a unique function $f \in \mathcal{F}_n$ such that*

$$Y = X_f = (V, A_f(X_1, \ldots, X_n)).$$

Proof. Since the map $f \mapsto X_f$ is injective, uniqueness is clear. To show existence of this function, we will proceed by induction on n. For $n = 1$, the assertion is easily seen to be true, since in this case, we we have either $Y = Y_1$ or $Y = Y_1^*$. In the first case, $O = [1]$, $X_1 = Y_1$ and

$$S_Y = V_Y = V_1 = L_{1,0} \oplus L_{1,1} = 1(0)L_{1,0} \oplus 1(1)L_{1,1},$$

so in this case f is the constant 1. If $Y = Y_1^*$, we have $O = \emptyset$, $X_1 = \tilde{Y}_1$, and then

$$S_Y = \mathbb{R}\{\tilde{a}_{Y_1}\} = L_{1,0} = 1^*(0)L_{1,0} \oplus 1^*(1)L_{1,1},$$

so that $f = 1^*$.

Assume now that the assertion is true for all $m < n$. By construction, Y is either the tensor product $Y = Z_1 \otimes Z_2$, with

$$Z_1 \sim \{Y_1, \dots, Y_m\}, \qquad Z_2 \sim \{Y_{m+1}, \dots, Y_n\},$$

or Y is the dual of such a product. Let us assume the first case. It is clear that $O_{Z_1} \cup O_{Z_2} = O_Y$, and similarly for I, so that the corresponding objects $X_1, \dots, X_m$ and $X_{m+1}, \dots, X_n$ remain the same. By the induction assumption, there are functions $f_1 \in \mathcal{F}_m$ and $f_2 \in \mathcal{F}_{n-m}$ such that

$$S_Y = S_{Z_1} \otimes S_{Z_2} = \bigoplus_{\substack{s \in \{0,1\}^m \\ t \in \{0,1\}^{n-m}}} f_1(s)f_2(t) L_{1,s_1} \otimes \cdots \otimes L_{m,s_m} \otimes L_{m+1,t_1} \otimes \cdots \otimes L_{n,t_{n-m}}.$$

This implies the assertion, with $f = f_1 \otimes f_2$.

To finish the proof, it is now enough to observe that if the assertion holds for Y then it also holds for Y^*. So assume that $Y = X_f = (V, A_f(X_1, \dots, X_n))$ for some $f \in \mathcal{F}_n$, then $Y^* = X_f^* = \tilde{X}_{f^*} = (V^*, A_{f^*}(\tilde{X}_1, \dots, \tilde{X}_n))$. In is now enough to notice that $\tilde{X}_i = \tilde{\tilde{Y}}_i = Y_i$ if $i \in I_Y$ and $\tilde{X}_i = \tilde{Y}_i$ if $i \in O_Y$. Since by definition $O_{Y^*} = I_Y$, this proves the statement.

□

Let us stress that in general, the objects X_f depend on the choice of the elements a_{X_i}. From the above proof, it is clear that the description in Proposition 4 does not depend on the choice of the elements $a_{Y_i} \in A_{Y_i}$.

2.4 Type funtions and higher order objects

Let $\mathcal{T}_n \subseteq \mathcal{F}_n$ be defined as the subset generated from the constant function 1 on $\{0, 1\}$ by taking duals and tensor products. For example, we have

$$\mathcal{T}_1 = \mathcal{F}_1 = \{1, 1^*\}, \quad \mathcal{T}_2 = \{1 \otimes 1, (1 \otimes 1)^*, 1 \otimes 1^*, 1^* \otimes 1, (1^* \otimes 1)^*, (1 \otimes 1^*)^*\},$$

etc. Elements of $\mathcal{T}_n$ will be called *type functions.* Similarly as for the higher order objects, the indexes in $[n]$ such that the corresponding component was subjected to taking the dual an even number of times will be called the outputs (of f) and denoted by $O = O_f$, indexes in $I = I_f := [n] \setminus O_f$ will be called inputs. From the proof of Proposition 4, it is easily seen that a higher order object is of the form $Y = X_f$ for a function $f \in \mathcal{T}_n$ with the same outputs (and of course also inputs) as Y. We next show that the converse is true.

Proposition 5. *Let $\{X_1, \dots, X_n\}$ be first order objects and let $f \in \mathcal{T}_n$. Then $Y = X_f$ is a higher order object with $O_Y = O_f$ and $Y \sim \{Y_1, \dots, Y_n\}$, where $Y_i = X_i$ for $i \in O_f$ and $Y_i = \tilde{X}_i$ for $i \in I_f$.*

Proof. As before, we will proceed by induction on n. For $n = 1$, we only have the possibilities $f = 1$ or $f = 1^*$. In the first case, $O = [1]$ and we get

$$S_f = 1L_{1,0} \oplus 1L_{1,1} = V_1,$$

so that $X_f = (V_1, \{\tilde{a}_1\}^{\sim}) = X_1$. In the second case, $O = \emptyset$ and

$$S_f = 1L_{1,0} = \mathbb{R}\{a_1\},$$

so that $X_f = (V_1, \{a_1\}) = \tilde{X}_1^*$. Assume next that the statement is true for all $m < n$ and assume that $f = f_1 \otimes f_2$ for some $f_1 \in \mathcal{F}_m$, $f_2 \in \mathcal{F}_{n-m}$, then it is easily seen that $Y = Z_1 \otimes Z_2$ for $Z_1 = X_{f_1}$ and $Z_2 = X_{f_2}$, constructed from $\{X_1, \dots, X_m\}$ resp. $\{X_{m+1}, \dots, X_n\}$. By the induction assumption, Z_1 and Z_2 are higher order objects, with $O_{Z_i} = O_{f_i}$, it follows that Y is a higher order object with $O_Y = O_{Z_1} \cup O_{Z_2} = O_{f_1} \cup O_{f_2} = O_f$.

Finally, assume that the statement is true for $f \in \mathcal{F}_n$, we will show that it holds for f^*. From (6), we see that $X_{f^*} = \tilde{X}_f$, which shows that $X_{f^*} \sim \{\tilde{Y}_1, \dots, \tilde{Y}_n\}$. Since taking duals will switch inputs and outputs, this finishes the proof. □

Let $\{Y_1, \dots, Y_n\}$ be first order objects. The above results show that any of higher order object $Y \sim \{Y_1, \dots, Y_n\}$ with fixed set of outputs $O_Y = O$ satisfies $Y \simeq X_f$ for a unique type function $f \in \mathcal{T}_n$, $O_f = O$, and a fixed set of objects $\{X_1, \dots, X_n\}$, where the isomorphism is given by the action of some permutation in S_n on the space $V_1 \otimes \dots \otimes V_n$. Conversely, any object of this form has the above properties. A basic example of such a function is (see Section ??)

$$p_I(s) = \Pi_{i \in I} \bar{s}_i = \otimes_{i \in I} 1^*(s_i), \qquad s \in \{0,1\}^n,$$

where $I = [n] \setminus O$. Clearly, $p_I \in \mathcal{T}_n$ and the set of outputs of p_I is O. It is easy to see that $X_{p_I} \simeq Y_I^* \otimes Y_O$, where $Y_I = \otimes_{i \in I} Y_i$ and $Y_O = \otimes_{i \in O} Y_i$ are first order objects, so that the corresponding higher order object can be identified with the set of replacement channels $Y_I \to Y_O$. Similarly, the function $p_O^* \in \mathcal{T}_n$ has output set O and $X_{p_O^*} \simeq (Y_I \otimes Y_O^*)^* = [Y_I, Y_O]$, the set of all channels $Y_I \to Y_O$.

Lemma 9. *Let $f \in \mathcal{T}_n$ and let $O_f = O$, $I = I_f$. Then*

$$p_I \leq f \leq p_O^*.$$

Proof. This is obviously true for $n = 1$. Indeed, in this case, $\mathcal{T}_1 = \mathcal{F}_1 = \{1 = p_\emptyset, 1^* = p_{[1]}\}$. If $f = 1$, then $O = [1]$, $I = \emptyset$ and

$$p_I = p_\emptyset = 1 = p_O^*,$$

the case $f = 1^*$ is obtained by taking complements. Assume that the assertion holds for $m < n$. Let $f \in \mathcal{T}_n$ and assume that $f = g \otimes h$ for some $g \in \mathcal{T}_m$, $h \in \mathcal{T}_{n-m}$. By the assumption,

$$p_{I_g} \otimes p_{I_h} \leq g \otimes h \leq p_{O_g}^* \otimes p_{O_h}^* \leq (p_{O_g} \otimes p_{O_f})^*,$$

the last inequality follows from Lemma 1. We have $O_f = O_g \cup (m + O_h)$, $I_f = I_g \cup (m + I_h)$, so that $p_{O_f} = p_{O_g} \otimes p_{O_h}$ and similarly for p_{I_f}. Now notice that any $f \in \mathcal{T}_n$ is either of the form

$(f \otimes g) \circ \sigma$ or of the form $(f \otimes g)^* \circ \sigma$, for some permutation σ. Since the inequality is easily seen to be preserved by permutations, and reversed by duality which also swiches the input and output sets, the assertion is proved.

□

Combining this with the remarks below Lemma lemma:Xf 8, we get the following result (cf. cite). Recall that a bimorphism in a category is a morphism $X \xrightarrow{\varphi} Y$ which is both mono and epi, that is, such that for any pairs of arrows (with appropriate sources and targets) we have $\psi \circ f = \psi \circ g \iff f = g$ and $k \circ \psi = l \circ \psi \iff k = l$. It can be shown that the bimorphisms in Af are precisely those morphisms that are given by isomorphisms in FinVect.

:setting **Theorem 2.** *Let $Y \sim \{Y_1, \dots, Y_n\}$ be such that $O_Y = O$, $I_Y = I$. Then there exist bimorphisms*

$$Y_I^* \otimes Y_O \xrightarrow{\varphi} Y \xrightarrow{\psi} [Y_I, Y_O].$$

The bimorphisms are given by permutations.

We will see below that $\mathcal{T}_n$ is not a lattice for $n \geq 2$, so that for $f_1, f_2 \in \mathcal{T}_n$, neither of $f_1 \wedge f_2$ or $f_1 \vee f_2$ has to be a type function. Nevertheless, we have by the above results that if $O_{f_1} = O_{f_2}$,

$$Y_I^* \otimes Y_0 \xrightarrow{\varphi} X_{f_1 \wedge f_2} \xrightarrow{id_V} X_{f_1 \vee f_2} \xrightarrow{\psi} [Y_I, Y_O]$$

for some suitable bimorphisms φ, ψ, moreover, the objects $X_{f_1 \wedge f_2}$ and $X_{f_1 \vee f_2}$ are obtained as a pullback resp. pushout. It follows that although these objects may not be higher order objects themselves, they are included in some higher order object (e.g. $[Y_I, Y_O]$) with the same sets of inputs and outputs.

We finish this section by showing a simple way to obtain the output set of a type function.

_outputs **Proposition 6.** *For $f \in \mathcal{T}_n$, $i \in O_f$ if and only if $f(e^i) = 1$.*

Proof. Let $i \in O_f$, then by Lemma lemma:fh_setting 9, $p_{I_f}(e^i) = 1 \leq f(e^i)$, so that $f(e^i) = 1$. Conversely, if $f(e^i) = 1$, then by the other inequality in lemma lemma:fh_setting 9, $p_{O_f}(e^i) = 0$, whence $i \in O_f$.

□

3 Characterizations of type functions

We have the following description of the sets of type functions.

type_min **Proposition 7.** *The set $\mathcal{T}_n$ is the smallest subset in $\mathcal{F}_n$ such that:*

1. *$\mathcal{T}_n$ is invariant under permutations: if $f \in \mathcal{T}_n$, then $f \circ \sigma \in \mathcal{T}_n$ for any permutation $\sigma \in S_n$,*
2. *$\mathcal{T}_n$ is invariant under taking duals: if $f \in \mathcal{T}_n$ then $f^* \in \mathcal{T}_n$,*
3. *$\mathcal{T}_m \otimes \mathcal{T}_n \subseteq \mathcal{T}_{m+n}$,*
4. *$\mathcal{T}_1 = \{1, p_1\} = \mathcal{F}_1$.*

Proof. It is clear by construction that any system of subsets $\{\mathcal{S}_n\}_n$ with these properties must contain the type functions and that $\{\mathcal{T}_n\}_n$ itself has these properties.

□

Our goal is to find some characterization of the type functions. We start by looking at some examples and non-examples.

exm:T2 *Example* 9. The type functions for $n = 2$ are given as

$$s \quad \mapsto \quad 1, \quad \bar{s}_1\bar{s}_2, \quad \bar{s}_1, \quad 1 - \bar{s}_1 + \bar{s}_1\bar{s}_2,$$

and functions obtained from these by exchanging $s_1 \leftrightarrow s_2$, which gives 6 elements. It can be seen that $\mathcal{F}_n$ has 2^{2^n-1} elements, so that $\mathcal{F}_2$ has 8 elements in total. The two of them that are not type functions are

$$g(s) = 1 - \bar{s}_1 - \bar{s}_2 + 2\bar{s}_1 s_2, \qquad g^*(s) = \bar{s}_1 + \bar{s}_2 - \bar{s}_1\bar{s}_2.$$

This can be checked directly from Propositions 9 and 6. Indeed, if $g \in \mathcal{T}_2$, we would have $O_g = \emptyset$, so that $p_2 \leq g \leq p^*_\emptyset = p_2$, which is obviously not the case. Clearly, also the complement $g^* \notin \mathcal{T}_2$. Notice also that $g^* = p_{\{1\}} \vee p_{\{2\}}$, so that $\mathcal{T}_2$ is not a lattice. Since $\mathcal{F}_2$ can be identified as a sublattice in $\mathcal{F}_n$ for all $n \geq 2$ as $f \mapsto (s \mapsto f(s_1 s_2))$, we see that $\mathcal{T}_n$, $n \geq 2$ is a subposet in the distributive lattice $\mathcal{F}_n$ but itself not a lattice.

3.1 The poset $\mathcal{P}_f$

It will be convenient to use the representation

$$f = \sum_{S \subseteq [n]} \hat{f}_S p_S,$$

obtained in Lemma 2. For $f \in \mathcal{T}_n$, let $\mathcal{P}_f$ be the subposet in $\mathcal{L}_n$ of elements such that $\hat{f}_S \neq 0$. We will show that f is fully determined by $\mathcal{P}_f$. We say that a poset $\mathcal{P}$ is graded of rank k if every maximal chain in $\mathcal{P}$ has the same length equal to k (recall that the length of a chain is defined as number of its elements -1). Equivalently, there is a unique rank function $\rho : \mathcal{P} \to \{0, 1, \ldots, k\}$ such that $\rho(S) = 0$ if S is a minimal element of $\mathcal{P}$ and $\rho(T) = \rho(S) + 1$ if T covers S, that is, $S \leq T$ and for any R such that $S \leq R \leq T$ we have $R = T$ or $R = S$.

p:graded **Proposition 8.** *Let $f \in \mathcal{T}_n$, then $\mathcal{P}_f$ is a graded poset with even rank $k \leq n$. If ρ is the rank function, then we have*

$$f = \sum_{S \in \mathcal{P}_f} (-1)^{\rho(S)} p_S.$$

Then rank of $\mathcal{P}_f$ will be denoted by $r(f)$ and called the rank of f. Note that the assertion means that for $f \in \mathcal{T}_n$,

$$\hat{f}_S = \begin{cases} (-1)^{\rho(S)}, & \text{if } S \in \mathcal{P}_f \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We first note that the property in the statement is invariant under permutations and complements. Assume the statement holds for f and let us take any $\sigma \in S_n$. It is easily seen from Lemma 2 that $\widehat{f \circ \sigma}_S = \hat{f}_{\sigma(S)}$ so that $S \mapsto \sigma(S)$ is an isomprphism of $\mathcal{P}_{f\circ\sigma}$ onto $\mathcal{P}_f$. Hence if $\mathcal{P}_f$ is graded with rank function ρ, then $\mathcal{P}_{f\circ\sigma}$ is graded with the same rank, and rank function $\rho \circ \sigma$. By the assumption we have

$$f \circ \sigma = \sum_{S \in \mathcal{P}_f} (-1)^{\rho(S)} p_S \circ \sigma = \sum_{S \in \mathcal{P}_f} (-1)^{\rho(S)} p_{\sigma^{-1}(S)} = \sum_{S \in \mathcal{P}_{f\circ\sigma}} (-1)^{\rho\circ\sigma(S)} p_S.$$

For the complement we write

$$f^* = 1 - f + p_n = (1 - \hat{f}_\emptyset)1 - \sum_{\substack{S \in \mathcal{P}_f \\ \emptyset \neq S, [n] \neq S}} (-1)^{\rho(S)} p_S + (1 - \hat{f}_{[n]}) p_n. \tag{7}$$ eq:dual_

If $\emptyset \in \mathcal{P}_f$, then $\emptyset$ is the least element of $\mathcal{P}_f$, so that $\rho(\emptyset) = 0$ and therefore $\hat{f}_\emptyset = (-1)^0 = 1$. Similarly, if $[n] \in \mathcal{P}_f$, then $[n]$ is the largest element in $\mathcal{P}_f$, hence it is the last element in any maximal chain. It follows that $\rho([n]) = k$ and hence $\hat{f}_{[n]} = (-1)^k = 1$ (since k is even). Therefore the equality eq:dual_rank (7) implies that $\mathcal{P}_{f^*}$ differs from $\mathcal{P}_f$ only in the bottom and top elements: $\emptyset \in \mathcal{P}_f$ iff $\emptyset \notin \mathcal{P}_{f^*}$ and $p_n \in \mathcal{P}_f$ iff $p_n \notin \mathcal{P}_{f^*}$. It follows that $\mathcal{P}_{f^*}$ is graded as well, with rank equal to $k-2$, k or $k+2$, which in any case is even. Furthermore, let ρ^* be the rank function of f^*, then this also implies that for all $S \in \mathcal{P}_f$, $S \notin \{\emptyset, [n]\}$, we have $\rho^*(S) = \rho(S) \pm 1$, according to whether $\emptyset$ was added or removed. The statement now follows from eq:dual_rank (7).

We now proceed by induction on n. For $n = 1$, we have $\mathcal{L}_1 = \{\emptyset, [1]\}$ and $\mathcal{T}_1 = \{1, 1^*\}$. For $f = 1$, $\mathcal{P}_f = \{\emptyset\}$ is a singleton, which is clearly a graded poset, with rank $k = 0$ and trivial rank function ρ. We have

$$f = 1 = p_\emptyset = (-1)^{\rho(\emptyset)} p_\emptyset.$$

The proof for $f = 1^*$ is similar, replacing $\emptyset$ by $[1]$.

To finish the proof, assume that the statement is true for $m < n$ and let $f \in \mathcal{T}_n$. Then f is either a permutation of a product of some $f_1 \in \mathcal{F}_m$ and $f_2 \in \mathcal{T}_{n-m}$, or a dual of such an element. By the first part of the proof, we only need to prove that the statement holds for $f = f_1 \otimes f_2$. But in this case, by the induction assumption, $\mathcal{P}_{f_i}$ is graded with even rank k_i and rank function ρ_i. We also have

$$f = f_1 \otimes f_2 = \sum_{S \subseteq [m], T \subseteq [m-n]} (\widehat{f_1})_S (\widehat{f_2})_T p_S p_T = \sum_{S \subseteq [m], T \subseteq [m-n]} (-1)^{\rho_1(S) + \rho_2(T)} p_{S \cup (m+T)}.$$

It follows that $\mathcal{P}_f = \mathcal{P}_{f_1} \times \mathcal{P}_{f_2}$ is the product of the two posets, which is a graded poset with rank $k = k_1 + k_2$ and rank function $\rho = \rho_1 + \rho_2$. This proves the statement.

□

We next show that the input and output sets of $f \in \mathcal{T}_n$ can be obtained from $\mathcal{P}_f$. For an index $i \in [n]$, let $\mathcal{P}_{f,i} := \{S \in \mathcal{P}_f,\ i \in S\}$. Note that $\mathcal{P}_{f,i}$ may be empty.

:pfinput **Proposition 9.** *Let $f \in \mathcal{T}_n$ and $i \in [n]$. Then*

1. *If $\mathcal{P}_{f,i} \neq \emptyset$, then all minimal elements in $\mathcal{P}_{f,i}$ have the same rank, which will be denoted by $r_f(i)$. If $\mathcal{P}_{f,i} = \emptyset$, we put $r_f(i) := r(f) + 1$.*

2. *$i \in O_f$ if and only if $\rho_f(i)$ is odd.*

Proof. Since $\mathcal{P}_f \simeq \mathcal{P}_{f \circ \sigma}$, it is quite clear that the two properties are preserved by permutations. We will show that they are preserved by complementation. Observe first that $\mathcal{P}_{f,i} = \emptyset$ if and only if $\mathcal{P}_{f^*,i} = \{[n]\}$, since $\mathcal{P}_{f^*}$ differs from $\mathcal{P}_f$ only up to adding/removing the least and greatest elements $\emptyset$ and $[n]$. If $\mathcal{P}_{f,i}$ is empty, then $p_S(e^i) = 1$ for all $S \in \mathcal{P}_f$, so that $f(e^i) = f(0) = 1$ and $i \in O_f$, we also see that $r_f(i) = r(f) + 1$ is odd. If $\mathcal{P}_{f,i} = [n]$, then $r_f(i) = \rho_f([n]) = r(f)$ by definition of the rank, hence $r_f(i)$ is even. As we have seen, $i \in O_{f^*} = I_f$.

Let us assume that $\mathcal{P}_{f,i}$ is not equal to $\emptyset$ or $\{[n]\}$. Then the same is true for $\mathcal{P}_{f^*,i}$, in fact, it differs from $\mathcal{P}_{f,i}$ only up to adding/removing the maximal element $[n]$. Therefore both posets prop:graded

have the same minimal elements, and by the proof of Proposition 8, we have for any such minimal element S that $\rho_{f^*}(S) = \rho_f(S) \pm 1$, depending only on the fact whether $\emptyset \in \mathcal{P}_f$. This implies that the properties are preserved by complementation.

We will now proceed by induction on n as before. Both assertions are quite trivial for $n = 1$, so assume the statements hold for $m < n$. It is enough to assume that $f = g \otimes h$ for some $g \in \mathcal{T}_m$ and $h \in \mathcal{T}_{n-m}$. Suppose without loss of generality that $i \in [m]$, then clearly $\mathcal{P}_{g,i} \neq \emptyset$ and the minimal elements in $\mathcal{P}_{f,i}$ are of the form $S \cup T$, with S minimal in $\mathcal{P}_{g,i}$ and T a minimal in $\mathcal{P}_h$. Since $\rho_h(T) = 0$ for any minimal element T, we have by the induction assumption

$$\rho_f(S \cup T) = \rho_g(S) + \rho_h(T) = r_g(i).$$

The statement (ii) follows from the fact that $i \in O_f$ if and only if $i \in O_g$.

□

Napisat toto dolu lepsie, definicie do prelim, ako vyzera f?

rop:isom **Proposition 10.** *Let $\ell : [m] \to [n]$ be a map. Let $f \in \mathcal{T}_n$ and let*

$$g = \sum_{S \subseteq [n]} \hat{f}_S p_{\ell^{-1}(S)}.$$

Then $g \in \mathcal{T}_m$.

Note that $S \mapsto \ell^{-1}(S)$ defines a homomorphism of the Boolean algebras $\mathcal{L}_n$ and $\mathcal{L}_m$, in particular, it preseves unions and intersections, and $\ell^{-1}(\emptyset) = \emptyset$, $\ell^{-1}([m]) = [n]$.

Proof. Let us denote $g = \ell(f)$. For any permutation $\sigma \in S_m$, we have

$$g \circ \sigma = \sum_{S \subseteq [n]} \hat{f}_S p_{\ell^{-1}(S)} \circ \sigma = \sum_{S \subseteq [n]} \hat{f}_S p_{\sigma^{-1}(\ell^{-1}(S))} = (\ell \circ \sigma)(f),$$

so that $\ell(f) \circ \sigma = (\ell \circ \sigma)(f)$. We can similarly show that for any $\tau \in S_n$, $\ell(f \circ \tau) = (\tau \circ \ell)(f)$. We may therefore assume that either $f = f_1 \otimes f_2$ for some $f_1 \in \mathcal{T}_k$, $f_2 \in \mathcal{T}_{n-k}$ or f is the dual of such a product. We then have

$$\ell(f) = \sum_{\substack{S \subseteq [k] \\ T \subseteq [n-k]}} (\hat{f}_1)_S (\hat{f}_2)_T p_{\ell^{-1}(S \cup (k+T))} = \sum_{\substack{S \subseteq [k] \\ T \subseteq [n-k]}} (\hat{f}_1)_S (\hat{f}_2)_T p_{\ell^{-1}(S) \cup \ell^{-1}(k+T)}$$

Composing by a suitable permutation if necessary, we may assume that there are some maps $\ell_1 : [k_1] \to [k]$ and $\ell_2 : [k_2] \to [n-k]$ such that $\ell^{-1}(S) = \ell_1^{-1}(S)$ for $S \subseteq [k]$ and $\ell^{-1}(k+T) = k_1 + \ell_2^{-1}(T)$, $T \subseteq [n-k]$. It follows that

$$\ell(f) = \ell_1(f_1) \otimes \ell_2(f_2).$$

eq:dual_rank

Since $\ell^{-1}(\emptyset) = \emptyset$ and $\ell^{-1}([n]) = [m]$, we can infer from (7) that $\ell(f^*) = \ell(f)^*$. Since clearly $\ell(1) = 1$, we can prove the statement by induction on n as before.

□

3.2 Chains and combs

Let $\mathcal{P} = \{S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_N\}$ be a chain in $\mathcal{L}_n$. Then $\mathcal{P}$ is graded with rank $N-1$ and rank function $\rho(S_i) = i-1$.

p:chains **Proposition 11.** *For a chain $\mathcal{P} = \{S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_N\}$, the function*

$$f = f_{\mathcal{P}} := \sum_i (-1)^{i-1} p_{S_i}$$

is a type function if and only if N odd.

Proof. By Proposition prop:graded 8, if $f \in \mathcal{T}_n$, then the rank of f must be even, so that N must be odd. We will show that the converse is true. We proceed by induction on N. For $N = 1$, we have $f = p_{S_1} \in \mathcal{T}_n$. Assume that the statement holds for all odd numbers $M < N$ and let $\mathcal{P}$ be a chain as above. Then we have

$$f = p_{S_1} \otimes g \otimes 1_{[n]\setminus S_N}$$

where g is the function for the chain $\emptyset = S'_1 \subsetneq S'_2 \subsetneq \cdots \subsetneq S'_N$, with $S'_i := S_i \setminus S_1$. Since f is a type function if g is, this shows that we may assume that the chain contains $\emptyset$ and $[n]$. But then

$$f = 1 + \sum_{j=2}^{N-1} (-1)^{j-1} p_{S_j} + p_n$$

and

$$f^* = 1 - f + p_n = \sum_{j=1}^{N-2} (-1)^{j-1} p_{T_j},$$

where $T_j := S_{j+1}$. By the induction assumption $f^* \in \mathcal{T}_n$, hence also $f = f^{**} \in \mathcal{T}_n$. $\square$

As we can see from Example exm:T2 9, all elements in $\mathcal{T}_2$ are chains. This is also true for $n = 3$. Indeed, up to a permutation that does not change the chain structure, any $f \in \mathcal{T}_3$ is either a product of two elements $g \in \mathcal{T}_2$ and $h \in \mathcal{T}_1$, or the dual of such a product. Since g must be a chain and $|\mathcal{P}_h| = 1$, their product must be a chain as well. Taking the dual of a chain only adds/removes the least/largest elements, so the dual of a chain must be a chain as well.

We now show that chains correspond to important higher order objects. Let k be odd and let $\mathcal{P} = \{\emptyset \subsetneq S_1 \subsetneq \cdots \subsetneq S_k \subsetneq [n]\}$. Let $f = f_{\mathcal{P}}$, then f is a type function. By Proposition prop:type_hom 5, for any first order objects $X_1, \dots, X_n$, $Y = X_f$ is a higher order object such that $Y \sim \{Y_1, \dots, Y_n\}$.

ns_combs **Proposition 12.** *Let $T_1 = S_1$, $T_i = S_i \setminus S_{i-1}$ for $i = 1, \dots, k$ and $T_{k+1} = S_k^c$. For $S \subseteq [n]$, we denote $Y_S := \otimes_{i\in S} Y_i$. Then for $k = 1$, $Y \simeq [Y_{T_2}, Y_{T_1}]$ and for any odd $k > 1$,*

$$Y \simeq [Y_{T_{k+1}}, [[Y_{T_k}, [[....], Y_{T_2}]], Y_{T_1}]].$$

Remark (quantum) comb.

Proof. It is easily checked by Proposition prop:fh_outputs 6 that $T_i \subseteq O_f$ if i is odd and $T_i \subseteq I_f$ otherwise. We therefore have

$$Y_{T_i} = \begin{cases} \otimes_{i\in T_i} X_i, & \text{if } i \text{ is odd,} \\ \otimes_{i\in T_i} \tilde{X}_i, & \text{if } i \text{ is even.} \end{cases}$$

Let $k=1$, then $f=1-p_{T_1}+p_n$, and $f^*=p_{T_1}=p_{T_1}\otimes 1_{T_2}$. Then $X_f=\tilde{X}^*_{f^*}$, and we see that $\tilde{X}_{f^*}=X^*_{T_1}\otimes\tilde{X}_{T_2}=Y^*_{T_1}\otimes Y_{T_2}$. It follows that $X_f=(Y^*_{T_1}\otimes Y_{T_2})^*\simeq[Y_{T_2},Y_{T_1}]$, where the isomorphism is given by swapping the spaces $V^*_{T_1}$ and V_{T_2}. Assume the assertion is true for $k-2$. As in the proof of Proposition prop:chains 11, we see that

$$f^*=\sum_{i=1}^{k}(-1)^{i-1}p_{S_i}=p_{T_1}\otimes g\otimes 1_{T_{k+1}}$$

where $g=1-\sum_{i=2}^{k-1}(-1)^{i-1}p_{S'_i}+p_{S'_k}$ is the function for the chain $\mathcal{P}'=\{\emptyset\subsetneq S'_2\subsetneq\cdots\subsetneq S'_k\}$ in $S'_k\simeq[n']$ for $n'=|S'_k|$, $S'_i=S_i\setminus S_1=\cup_{j=2}^{i}T_j$. We have

$$X_f=\tilde{X}^*_{f^*}=(X^*_{Y_1}\otimes\tilde{X}_g\otimes\tilde{X}_{T_{k+1}})^*\simeq(\tilde{X}_{T_{k+1}}\otimes[\tilde{X}_g,X_{T_1}]^*)^*=[\tilde{X}_{T_{k+1}},[\tilde{X}_g,X_{T_1}]].$$

Here we have used the fact that $\tilde{X}_{f^*}$ is constructed from $\tilde{X}_1,\ldots,\tilde{X}_n$. By induction assumption, we get

$$[\tilde{X}_{T_{k+1}},[\tilde{X}_g,X_{T_1}]]=[\tilde{X}_{T_{k+1}},[[X_{T_k},[[....],\tilde{X}_{T_2}]],X_{T_1}],$$

which is as required.

□

3.3 Separating type functions

We say that a function $f\in\mathcal{F}_n$ separates $i,j\in[n]$ if $f\neq f^{ij}$, where $f^{ij}\in\mathcal{F}_n$ is such that for any $s\in\{0,1\}^n$, $f^{ij}(s)=f(\hat{s}^{ij})$, here $\hat{s}^{ij}$ is the string obtained from s by replacing both s_i and s_j by $s_i\vee s_j$. Equivalently, there exists some $S\in\mathcal{P}_f$ such that $S\cap\{i,j\}$ is a singleton.

To see this equivalence, note that $p_S(s)=p_S(\hat{s}^{ij})$ whenever $\{i,j\}\subseteq S$ or $\{i,j\}\cap S=\emptyset$. Hence, if f separates i and j, there must be at least one $S\in\mathcal{P}_f$ that contains one of the indices but not the other. Conversely, let S be minimal with the property that, say, $i\in S$ and $j\notin S$. Let s be the string such that $s_k=0$ if and only if $k\in S$, in particular, $s_i=0$, $s_j=1$ and $s_i\vee s_j=1$. Then

$$f(s)=\sum_{T\subseteq S}\hat{f}_T p_T(s)=\sum_{T\subseteq S\setminus\{i\}}\hat{f}_T p_T(s)+\hat{f}_S p_S(s),$$

the second equality follows from the fact that S is minimal set with $\hat{f}_S\neq 0$ containing i but not j. On the other hand,

$$f(\hat{s}^{ij})=\sum_{T\subseteq S\setminus\{i\}}\hat{f}_T(s),$$

so that $f(s)-f^{ij}(s)=\hat{f}_S\neq 0$.

parateij **Lemma 10.** *Let $f\in\mathcal{F}_n$ and assume that f does not separate i and j. Then there is some $g\in\mathcal{F}_{n-1}$ such that $f(s)=g(s^{ij}(s_i\vee s_j))$. We have $\mathcal{P}_f\simeq\mathcal{P}_g$ (as posets) and $f\in\mathcal{T}_n$ if and only if $g\in\mathcal{T}_{n-1}$.*

Proof. We may assume $i=n-1$, $j=n$. Put $g(s_1\ldots s_{n-1}):=f(s_1\ldots s_{n-1}s_{n-1})$. Then clearly $g\in\mathcal{F}_{n-1}$ and for $s\in\{0,1\}^n$,

$$g(s_1\ldots s_{n-2}(s_{n-1}\vee s_n))=f(\hat{s}^{n(n-1)})=f(s).$$

Since any $S \in \mathcal{P}_f$ either contains none of $n-1, n$ or both of them, we have

$$\begin{aligned} g(s_1 \dots s_{n-1}) = f(s_1 \dots s_{n-1} s_{n-1}) &= \sum_{S \subseteq [n]} \hat{f}_S p_S(s_1 \dots s_{n-1} s_{n-1}) \\ &= \sum_{S \subseteq [n-2]} \hat{f}_S p_S(s_1 \dots s_{n-1}) + \sum_{\substack{S \subseteq [n] \\ \{n-1,n\} \subseteq S}} \hat{f}_S p_S(s_1 \dots s_{n-1} s_{n-1}) \\ &= \sum_{S \subseteq [n]} \hat{f}_S p_{S \setminus \{n\}}(s_1 \dots s_{n-1}). \end{aligned}$$

Note that for $S \subseteq [n]$, we have $S \setminus \{n\} = \ell^{-1}(S)$, where ℓ is the inclusion map $[n-1] \hookrightarrow [n]$. It follows by Proposition 10 that if $f \in \mathcal{T}_n$ then $g \in \mathcal{T}_{n-1}$. Moreover, ℓ^{-1} is not injective on $\mathcal{L}_n$, since ℓ is not surjective, but its restriction to $\mathcal{P}_f$ is, so that $\mathcal{P}_f \simeq \mathcal{P}_g$.

Assume $g \in \mathcal{T}_{n-1}$, then since $\overline{s_{n-1} \vee s_n} = \bar{s}_{n-1} \bar{s}_n$, we get

$$\begin{aligned} f(s) &= \sum_{T \subseteq [n-1]} \hat{g}_T p_T(s_1 \dots s_{n-2}(s_{n-1} \vee s_n)) \\ &= \sum_{T \subseteq [n-2]} \hat{g}_T p_T(s) + \sum_{T \subseteq [n-1], n-1 \in T} \hat{g}_T p_{T \cup \{n\}}(s) = \sum_{T \subseteq [n-1]} \hat{g}_T p_{\ell_1^{-1}(T)}(s), \end{aligned}$$

were $\ell_1 : [n] \to [n-1]$ is the map such that $\ell_1(i) = i$ for $i \neq n$ and $\ell_1(n) = n-1$. Again, using Proposition 10, this implies that $f \in \mathcal{T}_n$.

□

We say that f is separating, or separates the poits of $[n]$, if it separates all pairs $i, j \in [n]$. Notice that in this case both $\cap \mathcal{P}_f$ and $(\cup \mathcal{P}_f)^c$ are at most singletons.

Theorem 3. *For any $f \in \mathcal{T}_n$ there is some $g \in \mathcal{T}_k$ such that g is separating, $\mathcal{P}_f \simeq \mathcal{P}_g$ and there are subsets $S_1, \dots, S_l$ in $[n]$ such that $f(s) = g(s^{\cup_j S_j}(\vee_{j \in I_1} s_j) \dots (\vee_{j \in I_l} s_j))$.*

Proof. We can construct the function g by repeated application of Lemma 10, glueing together nonseparable points until, after a finite number of steps, none are left, so that the resulting function g separates the points of m.

□

Let us remark that the higher order object constructed from f and a set of first order objects $X_1, \dots, X_n$ is isomorphic to the object constructed from g and a sequence $Z_1, \dots, Z_k$, which is obtained by removing the first order objects X_i, $i \in S_j$ and replacing them by their tensor product $\otimes_{i \in S_j} X_i$, for any set of nonseparable indices S_j.

3.3.1 Separating chains

We will describe all separating chains, up to permutations. Clearly, a chain $\mathcal{P} = \{S_1 \subsetneq \dots \subsetneq S_N\}$ in $\mathcal{L}_n$ is separating if and only if $S_i \setminus S_{i-1}$ is a singleton for $i = 2, \dots, N$ and both S_1 and S_N^c are at most singletons (that is, these might be also empty). We have seen that the correspnding function is in $\mathcal{T}_n$ if and only if $N = r(\mathcal{P}) + 1$ is odd. By the separating property, we must have

$n-1 \leq N \leq n+1$, depending on whether S_1 or S_N^C are empty or not. Since N must be odd, we see that the only possibilities are

$$\begin{aligned} &N = n-1 \text{ or } N = n+1 && \text{if } n \text{ is even} \\ &N = n && \text{if } n \text{ is odd.} \end{aligned}$$

For even n, put

$$\gamma_n(s) := \sum_{l=0}^{n} (-1)^l p_{[l]}(s) = 1 - \bar{s}_1 + \bar{s}_1\bar{s}_2 - \cdots + \bar{s}_1 \ldots \bar{s}_n.$$

Then γ_n is a separating chain in $\mathcal{T}_n$ and the corresponding sets of input and output indices are

$$I = \{2j, j = 1, \ldots n/2\}, \qquad O = \{2j-1,\ j = 1, \ldots n/2\}.$$

Up to a permutation, any separating chain in $\mathcal{T}_n$ is of the form

$$f = \gamma_n \text{ or } \ f = p_1 \otimes \gamma_{n-2} \otimes 1_1.$$

These two chains are easily seen to be each others complement. Similarly, if n is odd, then (up to permutation) any separating chain in $\mathcal{T}_n$ must either of the two comlementary forms

$$f = p_1 \otimes \gamma_{n-1} \text{ or } \ f = \gamma_{n-1} \otimes 1_1.$$

From this and Example ex:chains ??, we see that there are basically two separating elements in $\mathcal{T}_2$:

$$f = \gamma_2, \qquad f^* = p_1 \otimes 1_1.$$

Similarly, there are basically two complementary forms of separating elements in $\mathcal{T}_3$:

$$f = p_1 \otimes \gamma_2, \qquad f^* = \gamma_2 \otimes 1_1.$$