

# On the properties $\alpha - z$ Rényi divergences on general von Neumann algebras

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## 1 Introduction

## 2 Preliminaries

### 2.1 Basic definitions

Let  $\mathcal{M}$  be a von Neumann algebra and let  $\mathcal{M}^+$  be the cone of positive elements in  $\mathcal{M}$ . We denote the predual by  $\mathcal{M}_*$ , its positive part by  $\mathcal{M}_*^+$  and the set of normal states by  $\mathfrak{S}_*(\mathcal{M})$ . For  $\psi \in \mathcal{M}_*^+$ , we will denote by  $s(\psi)$  the support projection of  $\psi$ .

For  $0 < p \leq \infty$ , let  $L_p(\mathcal{M})$  be the Haagerup  $L_p$ -space over  $\mathcal{M}$  and let  $L_p(\mathcal{M})$  its positive cone, [5]. We will use the identifications  $\mathcal{M} \simeq L_\infty(\mathcal{M})$ ,  $\mathcal{M}_* \ni \psi \leftrightarrow h_\psi \in L_1(\mathcal{M})$  and the notation  $\text{Tr } h_\psi = \psi(1)$  for the trace in  $L_1(\mathcal{M})$ . In this way,  $\mathcal{M}_*^+$  is identified with the positive cone  $L_1(\mathcal{M})^+$  and  $\mathfrak{S}_*(\mathcal{M})$  with subset of elements in  $L_1(\mathcal{M})^+$  with unit trace. Precise definitions and further details on the spaces  $L_p(\mathcal{M})$  can be found in [7, Chap. 9], or in the notes [24]. Some technical results that will be used below can be found in Appendix A.

### 2.2 The $\alpha - z$ -Rényi divergences

In [14, 15], the  $\alpha - z$ -Rényi divergence for  $\psi, \varphi \in \mathcal{M}_*^+$  was defined as follows:

**Definition 2.1.** Let  $\psi, \varphi \in \mathcal{M}_*^+$ ,  $\psi \neq 0$  and let  $\alpha, z > 0$ ,  $\alpha \neq 1$ . The  $\alpha - z$ -Rényi divergence is defined as

$$D_{\alpha,z}(\psi\|\varphi) := \frac{1}{\alpha - 1} \log \frac{Q_{\alpha,z}(\psi\|\varphi)}{\psi(1)},$$

where

$$Q_{\alpha,z} = \begin{cases} \text{Tr} \left( h_\varphi^{\frac{1-\alpha}{2z}} h_\psi^\alpha h_\varphi^{\frac{1-\alpha}{2z}} \right)^z, & \text{if } 0 < \alpha < 1 \\ \|x\|_z^z, & \text{if } \alpha > 1 \text{ and} \\ h_\psi^\alpha = h_\varphi^{\frac{\alpha-1}{2z}} x h_\varphi^{\frac{\alpha-1}{2z}}, \text{ with } x \in s(\varphi)L_z(\mathcal{M})s(\varphi) & \\ \infty & \text{otherwise.} \end{cases}$$

In the case  $\alpha > 1$ , the following alternative form will be useful.

**Lemma 2.2.** [14, Lemma 7] Let  $\alpha > 1$  and  $\psi, \varphi \in \mathcal{M}_*^+$ . Then  $Q_{\alpha,z}(\psi\|\varphi) < \infty$  if and only if there is some  $y \in L_{2z}(\mathcal{M})s(\varphi)$  such that

$$h_{\psi}^{\frac{\alpha}{2z}} = y h_{\varphi}^{\frac{\alpha-1}{2z}}.$$

Moreover, in this case, such  $y$  is unique and we have  $Q_{\alpha,z}(\psi\|\varphi) = \|y\|_{2z}^{2z}$ .

The standard or Petz-type Rényi divergence [6, 8, 21] is contained in this range as  $D_{\alpha}(\psi\|\varphi) = D_{\alpha,1}(\psi\|\varphi)$ . The sandwiched Rényi divergence is obtained as  $\tilde{D}_{\alpha}(\psi\|\varphi) = D_{\alpha,\alpha}(\psi\|\varphi)$ , see [2, 8, 11, 12] for some alternative definitions and properties of  $\tilde{D}_{\alpha}$ . The definition in [11] and [12] is based on the Kosaki interpolation spaces  $L_p(\mathcal{M}, \varphi)$  with respect to a state [16]. These spaces and the complex interpolation method will be used frequently in the present work.

Many of the properties of  $D_{\alpha,z}(\psi\|\varphi)$  were extended from the finite dimensional case in [14]. In particular, the following variational expressions will be an important tool for our work.

**Theorem 2.3** (Variational expressions). Let  $\psi, \varphi \in \mathcal{M}_*^+$ ,  $\psi \neq 0$ .

(i) Let  $0 < \alpha < 1$  and  $\max\{\alpha, 1 - \alpha\} \leq z$ . Then

$$Q_{\alpha,z}(\psi\|\varphi) = \inf_{a \in \mathcal{M}^{++}} \left\{ \alpha \operatorname{Tr} \left( (a^{\frac{1}{2}} h_{\psi}^{\frac{\alpha}{z}} a^{\frac{1}{2}})^{\frac{z}{\alpha}} \right) + (1 - \alpha) \operatorname{Tr} \left( (a^{\frac{1}{2}} h_{\varphi}^{\frac{1-\alpha}{z}} a^{\frac{1}{2}})^{\frac{z}{1-\alpha}} \right) \right\}.$$

(ii) Let  $1 < \alpha$ ,  $\max\{\frac{\alpha}{2}, \alpha - 1\} \leq z$ . Then

$$Q_{\alpha,z}(\psi\|\varphi) = \sup_{a \in \mathcal{M}_+} \left\{ \alpha \operatorname{Tr} \left( (a^{\frac{1}{2}} h_{\psi}^{\frac{\alpha}{z}} a^{\frac{1}{2}})^{\frac{z}{\alpha}} \right) - (\alpha - 1) \operatorname{Tr} \left( (a^{\frac{1}{2}} h_{\varphi}^{\frac{\alpha-1}{2z}} a^{\frac{1}{2}})^{\frac{z}{\alpha-1}} \right) \right\}.$$

*Proof.* For part (i) see [14, Theorem 1 (vi)]. The inequality  $\geq$  in part (ii) holds for all  $\alpha$  and  $z$  and was proved in [14, Theorem 2 (vi)]. We now prove the opposite inequality.

Assume first that  $Q_{\alpha,z}(\psi\|\varphi) < \infty$ , so that there is some  $x \in s(\varphi)L_z(\mathcal{M})^+s(\varphi)$  such that  $h_{\psi}^{\frac{\alpha}{z}} = h_{\varphi}^{\frac{\alpha-1}{2z}} x h_{\varphi}^{\frac{\alpha-1}{2z}}$ . Plugging this into the right hand side, we obtain

$$\begin{aligned} & \sup_{a \in \mathcal{M}_+} \left\{ \alpha \operatorname{Tr} \left( (a^{\frac{1}{2}} h_{\psi}^{\frac{\alpha}{z}} a^{\frac{1}{2}})^{\frac{z}{\alpha}} \right) - (\alpha - 1) \operatorname{Tr} \left( (a^{\frac{1}{2}} h_{\varphi}^{\frac{\alpha-1}{2z}} a^{\frac{1}{2}})^{\frac{z}{\alpha-1}} \right) \right\} \\ &= \sup_{a \in \mathcal{M}_+} \left\{ \alpha \operatorname{Tr} \left( (a^{\frac{1}{2}} h_{\varphi}^{\frac{\alpha-1}{2z}} x h_{\varphi}^{\frac{\alpha-1}{2z}} a^{\frac{1}{2}})^{\frac{z}{\alpha}} \right) - (\alpha - 1) \operatorname{Tr} \left( (a^{\frac{1}{2}} h_{\varphi}^{\frac{\alpha-1}{2z}} a^{\frac{1}{2}})^{\frac{z}{\alpha-1}} \right) \right\} \\ &= \sup_{a \in \mathcal{M}_+} \left\{ \alpha \operatorname{Tr} \left( (x^{\frac{1}{2}} h_{\varphi}^{\frac{\alpha-1}{2z}} a h_{\varphi}^{\frac{\alpha-1}{2z}} x^{\frac{1}{2}})^{\frac{z}{\alpha}} \right) - (\alpha - 1) \operatorname{Tr} \left( (h_{\varphi}^{\frac{\alpha-1}{2z}} a h_{\varphi}^{\frac{\alpha-1}{2z}})^{\frac{z}{\alpha-1}} \right) \right\} \\ &= \sup_{w \in L_{\frac{z}{\alpha-1}}(\mathcal{M})^+} \left\{ \alpha \operatorname{Tr} \left( (x^{\frac{1}{2}} w x^{\frac{1}{2}})^{\frac{z}{\alpha}} \right) - (\alpha - 1) \operatorname{Tr} \left( w^{\frac{z}{\alpha-1}} \right) \right\}, \end{aligned}$$

where we used the fact that  $\operatorname{Tr}((h^*h)^p) = \operatorname{Tr}((hh^*)^p)$  for  $p > 0$ ,  $h \in L_{\frac{p}{2}}(\mathcal{M})$ , and Lemma A.1. Putting  $w = x^{\alpha-1}$  we get

$$\sup_{w \in L_{\frac{z}{\alpha-1}}(\mathcal{M})^+} \left\{ \alpha \operatorname{Tr} \left( (x^{\frac{1}{2}} w x^{\frac{1}{2}})^{\frac{z}{\alpha}} \right) - (\alpha - 1) \operatorname{Tr} \left( w^{\frac{z}{\alpha-1}} \right) \right\} \geq \operatorname{Tr}(x^z) = \|x\|_z^z = Q_{\alpha,z}(\psi\|\varphi).$$

This finishes the proof in the case that  $Q_{\alpha,z}(\psi\|\varphi) < \infty$ . Note that this holds if  $\psi \leq \lambda \varphi$  for some  $\lambda > 0$ . Indeed, since  $\frac{\alpha}{2z} \in (0, 1]$  by the assumption, we then have

$$h_{\psi}^{\frac{\alpha}{2z}} \leq \lambda^{\frac{\alpha}{2z}} h_{\varphi}^{\frac{\alpha}{2z}},$$

hence by [8, Lemma A.58] there is some  $b \in \mathcal{M}$  such that

$$h_{\psi}^{\frac{\alpha}{2z}} = bh_{\varphi}^{\frac{\alpha}{2z}} = yh_{\varphi}^{\frac{\alpha-1}{2z}},$$

where  $y = bh_{\varphi}^{\frac{1}{2z}} \in L_{2z}(\mathcal{M})$ . By Lemma 2.2 we get  $Q_{\alpha,z}(\psi\|\varphi) = \|y\|_{2z}^{2z} < \infty$ .

In the general case, the variational expression holds for  $Q_{\alpha,z}(\psi\|\varphi + \epsilon\psi)$  for all  $\epsilon > 0$ , so that we have

$$\begin{aligned} Q_{\alpha,z}(\psi\|\varphi + \epsilon\psi) &= \sup_{a \in \mathcal{M}_+} \left\{ \alpha \text{Tr} \left( (a^{\frac{1}{2}} h_{\psi}^{\frac{\alpha}{2z}} a^{\frac{1}{2}})^{\frac{z}{\alpha}} \right) - (\alpha - 1) \text{Tr} \left( (a^{\frac{1}{2}} h_{\varphi + \epsilon\psi}^{\frac{\alpha-1}{2z}} a^{\frac{1}{2}})^{\frac{z}{\alpha-1}} \right) \right\} \\ &\leq \sup_{a \in \mathcal{M}_+} \left\{ \alpha \text{Tr} \left( (a^{\frac{1}{2}} h_{\psi}^{\frac{\alpha}{2z}} a^{\frac{1}{2}})^{\frac{z}{\alpha}} \right) - (\alpha - 1) \text{Tr} \left( (a^{\frac{1}{2}} h_{\varphi}^{\frac{\alpha-1}{2z}} a^{\frac{1}{2}})^{\frac{z}{\alpha-1}} \right) \right\}, \end{aligned}$$

where the inequality above follows by Lemma A.3. Therefore, since lower semicontinuity [14, Theorem 2 (iv)] gives

$$Q_{\alpha,z}(\psi\|\varphi) \leq \liminf_{\epsilon \searrow 0} Q_{\alpha,z}(\psi\|\varphi + \epsilon\psi)$$

the desired inequality follows.  $\square$

We finish this section by investigation of the properties of the variational expression for  $0 < \alpha < 1$ , in the case when  $\lambda^{-1}\varphi \leq \psi \leq \lambda\varphi$  for some  $\lambda > 0$ . This will be denoted as  $\psi \sim \varphi$ .

**Lemma 2.4.** *Assume that  $\psi \sim \varphi$ . Then the infimum in the variational expression in Theorem 2.3 (i) is attained at a unique element  $\bar{a} \in \mathcal{M}^{++}$ . This element satisfies*

$$h_{\psi}^{\frac{\alpha}{2z}} \bar{a} h_{\psi}^{\frac{\alpha}{2z}} = (h_{\psi}^{\frac{\alpha}{2z}} h_{\varphi}^{\frac{1-\alpha}{z}} h_{\psi}^{\frac{\alpha}{2z}})^{\alpha} \quad (2.1)$$

$$h_{\varphi}^{\frac{1-\alpha}{2z}} \bar{a}^{-1} h_{\varphi}^{\frac{1-\alpha}{2z}} = (h_{\varphi}^{\frac{1-\alpha}{2z}} h_{\psi}^{\frac{\alpha}{z}} h_{\varphi}^{\frac{1-\alpha}{2z}})^{1-\alpha}. \quad (2.2)$$

*Proof.* We may assume that  $\varphi$  and hence also  $\psi$  is faithful. Following the proof of [14, Theorem 1 (vi)], we may use the assumptions and [8, Lemma A.58] to show that there are  $b, c \in \mathcal{M}$  such that

$$h_{\varphi}^{\frac{1-\alpha}{2z}} = b(h_{\varphi}^{\frac{1-\alpha}{2z}} h_{\psi}^{\frac{\alpha}{z}} h_{\varphi}^{\frac{1-\alpha}{2z}})^{\frac{1-\alpha}{2}}, \quad (h_{\varphi}^{\frac{1-\alpha}{2z}} h_{\psi}^{\frac{\alpha}{z}} h_{\varphi}^{\frac{1-\alpha}{2z}})^{\frac{1-\alpha}{2}} = ch_{\varphi}^{\frac{1-\alpha}{2z}}. \quad (2.3)$$

With  $\bar{a} := bb^* \in \mathcal{M}^{++}$  we have  $\bar{a}^{-1} = c^*c$  and  $\bar{a}$  is indeed a minimizer of

$$Q_{\alpha,z}(\psi\|\varphi) = \inf_{a \in \mathcal{M}^{++}} \left\{ \alpha \left\| h_{\psi}^{\frac{\alpha}{2z}} a h_{\psi}^{\frac{\alpha}{2z}} \right\|_{\frac{z}{\alpha}}^{\frac{z}{\alpha}} + (1 - \alpha) \left\| h_{\varphi}^{\frac{1-\alpha}{2z}} a^{-1} h_{\varphi}^{\frac{1-\alpha}{2z}} \right\|_{\frac{z}{1-\alpha}}^{\frac{z}{1-\alpha}} \right\}. \quad (2.4)$$

We next observe that such minimizer is unique. Indeed, suppose that the infimum is attained at some  $a_1, a_2 \in \mathcal{M}^{++}$ . Let  $a_0 := (a_1 + a_2)/2$ . Since the map  $L^p(\mathcal{M}) \ni k \mapsto \|k\|_p^p$  is convex for any  $p \geq 1$  and  $a_0^{-1} \leq (a_1^{-1} + a_2^{-1})/2$ , we have using Lemma A.2 in the second inequality that

$$\begin{aligned} \left\| h_{\psi}^{\frac{\alpha}{2z}} a_0 h_{\psi}^{\frac{\alpha}{2z}} \right\|_{\frac{z}{\alpha}}^{\frac{z}{\alpha}} &\leq \frac{1}{2} \left\{ \left\| h_{\psi}^{\frac{\alpha}{2z}} a_1 h_{\psi}^{\frac{\alpha}{2z}} \right\|_{\frac{z}{\alpha}}^{\frac{z}{\alpha}} + \left\| h_{\psi}^{\frac{\alpha}{2z}} a_2 h_{\psi}^{\frac{\alpha}{2z}} \right\|_{\frac{z}{\alpha}}^{\frac{z}{\alpha}} \right\}, \\ \left\| h_{\varphi}^{\frac{1-\alpha}{2z}} a_0^{-1} h_{\varphi}^{\frac{1-\alpha}{2z}} \right\|_{\frac{z}{1-\alpha}}^{\frac{z}{1-\alpha}} &\leq \left\| h_{\varphi}^{\frac{1-\alpha}{2z}} \left( \frac{a_1^{-1} + a_2^{-1}}{2} \right) h_{\varphi}^{\frac{1-\alpha}{2z}} \right\|_{\frac{z}{1-\alpha}}^{\frac{z}{1-\alpha}} \\ &\leq \frac{1}{2} \left\{ \left\| h_{\varphi}^{\frac{1-\alpha}{2z}} a_1^{-1} h_{\varphi}^{\frac{1-\alpha}{2z}} \right\|_{\frac{z}{1-\alpha}}^{\frac{z}{1-\alpha}} + \left\| h_{\varphi}^{\frac{1-\alpha}{2z}} a_2^{-1} h_{\varphi}^{\frac{1-\alpha}{2z}} \right\|_{\frac{z}{1-\alpha}}^{\frac{z}{1-\alpha}} \right\}. \end{aligned}$$

Hence we have

$$\left\| h_\varphi^{\frac{1-\alpha}{2z}} a_0^{-1} h_\varphi^{\frac{1-\alpha}{2z}} \right\|_{\frac{z}{1-\alpha}} = \left\| h_\varphi^{\frac{1-\alpha}{2z}} \left( \frac{a_1^{-1} + a_2^{-1}}{2} \right) h_\varphi^{\frac{1-\alpha}{2z}} \right\|_{\frac{z}{1-\alpha}},$$

which implies that  $a_0^{-1} = \frac{a_1^{-1} + a_2^{-1}}{2}$ , as easily verified by Lemma A.2. From this we easily have  $a_1 = a_2$ .

The equality (2.2) is obvious from the second equality in (2.3) and  $\bar{a}^{-1} = c^*c$ . Since  $Q_{\alpha,z}(\psi\|\varphi) = Q_{1-\alpha,z}(\varphi\|\psi)$ , we see by uniqueness that the minimizer of the infimum expression for  $Q_{1-\alpha,z}(\varphi\|\psi)$  (instead of (2.4)) is  $\bar{a}^{-1}$  (instead of  $\bar{a}$ ). This says that (2.1) is the equality corresponding to (2.2) when  $\psi, \varphi, \alpha$  are replaced with  $\varphi, \psi, 1 - \alpha$ , respectively.  $\square$

To make the next lemma more readable, we will use the following notations:

$$p := \frac{z}{\alpha}, \quad r := \frac{z}{1-\alpha}, \quad \xi_p(a) := h_\psi^{\frac{1}{2p}} a h_\psi^{\frac{1}{2p}}, \quad \eta_r(a) = h_\varphi^{\frac{1}{2r}} a^{-1} h_\varphi^{\frac{1}{2r}}.$$

We will also denote the function under the infimum in the variational expression in Theorem 2.3 (i) by  $f$ , that is,

$$f(a) = \alpha \|\xi_p(a)\|_p^p + (1-\alpha) \|\eta_r(a)\|_r^r, \quad a \in \mathcal{M}^{++}.$$

**Lemma 2.5.** *Assume that  $\psi \sim \varphi$  and let  $0 < \alpha < 1$ ,  $\max\{\alpha, 1-\alpha\} \leq z$ . If  $p > 1$ , then for every  $C \geq Q_{\alpha,z}(\psi\|\varphi)$  and  $\varepsilon > 0$  there is some  $\delta > 0$  such that whenever  $\|\xi_p(b)\|_p^p \leq C$  and  $\|\xi_p(b) - \xi_p(\bar{a})\|_p \geq \varepsilon$ , we have*

$$f(b) - Q_{\alpha,z}(\psi\|\varphi) \geq \delta.$$

A similar statement holds if  $r > 1$ .

*Proof.* By assumptions,  $p, r \geq 1$ . For  $a, b \in \mathcal{M}^{++}$  and  $s \in (0, 1/2)$ , we have

$$\begin{aligned} \|\xi_p(sb + (1-s)a)\|_p^p &= \|s\xi_p(b) + (1-s)\xi_p(a)\|_p^p = \|(1-2s)\xi_p(a) + 2s\frac{1}{2}(\xi_p(a) + \xi_p(b))\|_p^p \\ &\leq (1-2s)\|\xi_p(a)\|_p^p + 2s\left\|\frac{1}{2}(\xi_p(a) + \xi_p(b))\right\|_p^p. \end{aligned}$$

Similarly,

$$\|\eta_r(sb + (1-s)a)\|_r^r \leq (1-2s)\|\eta_r(a)\|_r^r + 2s\left\|\frac{1}{2}(\eta_r(a) + \eta_r(b))\right\|_r^r,$$

here we also used the fact that  $(ta + (1-t)b)^{-1} \leq ta^{-1} + (1-t)b^{-1}$  for  $t \in (0, 1)$  and Lemma A.2. It follows that

$$\begin{aligned} \langle \nabla f(a), b - a \rangle &= \lim_{s \rightarrow 0^+} s^{-1} [f(sb + (1-s)a) - f(a)] \\ &\leq 2\alpha \left( \left\| \frac{1}{2}(\xi_p(a) + \xi_p(b)) \right\|_p^p - \|\xi_p(a)\|_p^p \right) + 2(1-\alpha) \left( \left\| \frac{1}{2}(\eta_r(a) + \eta_r(b)) \right\|_r^r - \|\eta_r(a)\|_r^r \right) \\ &= f(b) - f(a) - 2 \left( \alpha \left( \frac{1}{2} \|\xi_p(a)\|_p^p + \frac{1}{2} \|\xi_p(b)\|_p^p - \left\| \frac{1}{2}(\xi_p(a) + \xi_p(b)) \right\|_p^p \right) \right. \\ &\quad \left. + (1-\alpha) \left( \frac{1}{2} \|\eta_r(a)\|_r^r + \frac{1}{2} \|\eta_r(b)\|_r^r - \left\| \frac{1}{2}(\eta_r(a) + \eta_r(b)) \right\|_r^r \right) \right). \end{aligned}$$

Since  $p, r \geq 1$ , both terms in brackets in the last expression above are nonnegative. Assume now that  $p > 1$ . Let  $\bar{a} \in \mathcal{M}^{++}$  be the minimizer as in Lemma 2.4, then  $f(\bar{a}) = Q_{\alpha, z}(\psi \|\varphi)$  and  $\nabla f(\bar{a}) = 0$ , so that we get

$$f(b) - Q_{\alpha, z}(\psi \|\varphi) \geq 2\alpha \left( \frac{1}{2} \|\xi_p(a)\|_p^p + \frac{1}{2} \|\xi_p(b)\|_p^p - \left\| \frac{1}{2} (\xi_p(a) + \xi_p(b)) \right\|_p^p \right).$$

The space  $L_p(\mathcal{M})$  is uniformly convex, so that the function  $h \mapsto \|h\|_p^p$  is uniformly convex on each set where it is bounded ([25, Thm. 3.7.7. and p. 288]). Hence for each  $C > 0$  and  $\epsilon > 0$  there is some  $\delta > 0$  such that for every  $h, k$  with  $\|h\|_p^p, \|k\|_p^p \leq C$  and  $\|h - k\|_p \geq \epsilon$ , we have

$$\frac{1}{2} \|h\|_p^p + \frac{1}{2} \|k\|_p^p - \left\| \frac{1}{2} (h + k) \right\|_p^p \geq \delta,$$

([25, Exercise 3.3]). The proof in the case  $r > 1$  is similar. □

### 3 Data processing inequality and reversibility of channels

Let  $\gamma : \mathcal{N} \rightarrow \mathcal{M}$  be a normal positive unital map. Then the predual of  $\gamma$  defines a positive linear map  $\gamma_* : L_1(\mathcal{M}) \rightarrow L_1(\mathcal{N})$  that preserves the trace, acting as

$$L_1(\mathcal{M}) \ni h_\rho \mapsto h_{\rho \circ \gamma} \in L_1(\mathcal{N}).$$

The support of  $\gamma$  will be denoted by  $s(\gamma)$ , recall that this is defined as the smallest projection  $e \in \mathcal{N}$  such that  $\gamma(e) = 1$  and in this case,  $\gamma(a) = \gamma(eae)$  for any  $a \in \mathcal{N}$ . For any  $\rho \in \mathcal{M}_*^+$  we clearly have  $s(\rho \circ \gamma) \leq s(\gamma)$ , with equality if  $\rho$  is faithful. It follows that  $\gamma_*$  maps  $L_1(\mathcal{M})$  to  $s(\gamma)L_1(\mathcal{N})s(\gamma) \equiv L_1(s(\gamma)\mathcal{N}s(\gamma))$ . For any  $\rho \in \mathcal{M}_*^+, \rho \neq 0$ , the map

$$\gamma_0 : s(\gamma)\mathcal{N}s(\gamma) \rightarrow s(\rho)\mathcal{M}s(\rho), \quad a \mapsto s(\rho)\gamma(a)s(\rho)$$

is a faithful normal positive unital map. Moreover, for any  $\sigma \in \mathcal{M}_*^+$  such that  $s(\sigma) \leq s(\rho)$ , we have for any  $a \in \mathcal{N}$ ,

$$\sigma(\gamma_0(s(\gamma)as(\gamma))) = \sigma(s(\rho)\gamma(a)s(\rho)) = \sigma(a).$$

Replacing  $\gamma$  by  $\gamma_0$  and  $\rho$  by the restriction  $\rho|_{s(\rho)\mathcal{M}s(\rho)}$ , we may assume that both  $\rho$  and  $\rho \circ \gamma$  are faithful.

The Petz dual of  $\gamma$  with respect to a faithful  $\rho \in \mathcal{M}_*^+$  is a map  $\gamma_\rho^* : \mathcal{M} \rightarrow \mathcal{N}$ , introduced in [23]. It was proved that it is again normal, positive and unital, in addition, it is  $n$ -positive whenever  $\gamma$  is. More generally, if  $e = s(\rho)$  and  $e_0 = s(\rho \circ \gamma)$ , we may use restrictions as above to define  $\gamma_\rho^* : e\mathcal{M}e \rightarrow e_0\mathcal{N}e_0$ . As explained in [11]  $\gamma_\rho^*$  is determined by the equality

$$(\gamma_\rho^*)_*(h_{\rho \circ \gamma}^{\frac{1}{2}} b h_{\rho \circ \gamma}^{\frac{1}{2}}) = h_\rho^{\frac{1}{2}} \gamma(b) h_\rho^{\frac{1}{2}}, \quad (3.1)$$

for all  $b \in \mathcal{N}^+$ , here  $(\gamma_\rho^*)_*$  is the predual map of  $\gamma_\rho^*$ . We also have

$$\rho \circ \gamma \circ \gamma_\rho^* = \rho, \quad (\gamma_\rho^*)_{\rho \circ \gamma}^* = \gamma.$$

In the special case that  $\gamma$  is the inclusion map  $\gamma : \mathcal{N} \hookrightarrow \mathcal{M}$  for a subalgebra  $\mathcal{N} \subseteq \mathcal{M}$ , the Petz dual is the generalized conditional expectation  $\mathcal{E}_{\mathcal{N}, \varphi} : \mathcal{M} \rightarrow \mathcal{N}$ , as introduced in [1]; see e.g. [8, Proposition 6.5]. Hence  $\mathcal{E}_{\mathcal{N}, \varphi}$  is a normal completely positive unital with range in  $\mathcal{N}$  and such that

$$\varphi \circ \mathcal{E}_{\mathcal{N}, \varphi} = \varphi.$$

### 3.1 Data processing inequality

In this paragraph we prove the data processing inequality (DPI) for  $D_{\alpha,z}$  with respect to normal positive unital maps. For standard Rényi divergence, that is for  $z = 1$ , DPI is known to hold for  $\alpha \in (0, 1) \cup (1, 2]$  under stronger positivity assumptions [6]. In the case of the sandwiched divergences  $\tilde{D}_\alpha$  with  $1/2 \leq \alpha \neq 1$ , DPI was proved in [11, 12], see also [2] for an alternative proof in the case when the maps are assumed completely positive. In the finite dimensional case, DPI with respect to completely positive maps was proved in [27].

**Lemma 3.1.** *Let  $\gamma : \mathcal{N} \rightarrow \mathcal{M}$  be a normal positive unital map and let  $\rho \in \mathcal{M}_*^+$ ,  $s(\rho) = e$ ,  $s(\rho \circ \gamma) = e_0$ . For any  $p \geq 1$ , the map  $\gamma_{\rho,p}^* : L_p(e_0 \mathcal{N} e_0) \rightarrow L_p(e \mathcal{M} e)$ , determined by*

$$\gamma_{\rho,p}^*(h_{\rho \circ \gamma}^{\frac{1}{2p}} b h_{\rho \circ \gamma}^{\frac{1}{2p}}) = h_\rho^{\frac{1}{2p}} \gamma(b) h_\rho^{\frac{1}{2p}}, \quad b \in \mathcal{N}$$

*is a contraction such that*

$$(\gamma_\rho^*)_*(h_{\rho \circ \gamma}^{\frac{p-1}{2p}} k h_{\rho \circ \gamma}^{\frac{p-1}{2p}}) = h_\rho^{\frac{p-1}{2p}} \gamma_{\rho,p}^*(k) h_\rho^{\frac{p-1}{2p}}, \quad k \in L_p(e_0 \mathcal{N} e_0).$$

*Moreover, if  $\rho_n \in \mathcal{M}_*^+$  are such that  $s(\rho) \leq s(\rho_n)$  and  $\|\rho_n - \rho\|_1 \rightarrow 0$ , then for any  $k \in L_p(e_0 \mathcal{N} e_0)$  we have  $\gamma_{\rho_n,p}^*(k) \rightarrow \gamma_{\rho,p}^*(k)$  in  $L_p(\mathcal{M})$ .*

*Proof.* For  $b \in \mathcal{N}$ , let  $\sigma \in e_0(\mathcal{N}_*)e_0$  be such that  $h_\sigma = h_{\rho \circ \gamma}^{\frac{1}{2}} b h_{\rho \circ \gamma}^{\frac{1}{2}}$ . Then

$$(\gamma_\rho^*)_*(h_{\rho \circ \gamma}^{\frac{1}{2}} b h_{\rho \circ \gamma}^{\frac{1}{2}}) = h_\rho^{\frac{1}{2}} \gamma(b) h_\rho^{\frac{1}{2}} = h_\rho^{\frac{p-1}{2p}} \gamma_{\rho,p}^*(h_{\rho \circ \gamma}^{\frac{1}{2p}} b h_{\rho \circ \gamma}^{\frac{1}{2p}}) h_\rho^{\frac{p-1}{2p}}.$$

Since  $\gamma_\rho^*$  is a normal positive unital map, its predual  $(\gamma_\rho^*)_*$  defines a contraction on the Kosaki  $L_p$ -spaces  $L_p(\mathcal{N}, \rho \circ \gamma) \rightarrow L_p(\mathcal{M}, \rho)$ , so that

$$\|h_\rho^{\frac{1}{2p}} \gamma(b) h_\rho^{\frac{1}{2p}}\|_p = \|(\gamma_\rho^*)_*(h_{\rho \circ \gamma}^{\frac{1}{2}} b h_{\rho \circ \gamma}^{\frac{1}{2}})\|_{p,\rho} \leq \|h_{\rho \circ \gamma}^{\frac{1}{2}} b h_{\rho \circ \gamma}^{\frac{1}{2}}\|_{p,\rho \circ \gamma} = \|h_{\rho \circ \gamma}^{\frac{1}{2p}} b h_{\rho \circ \gamma}^{\frac{1}{2p}}\|_p$$

Since  $h_{\rho \circ \gamma}^{\frac{1}{2p}} \mathcal{N} h_{\rho \circ \gamma}^{\frac{1}{2p}}$  is dense in  $L_p(e_0 \mathcal{N} e_0)$ , this proves the first part of the statement. Let  $\rho_n$  be a sequence as required and let  $k \in L_p(e_0 \mathcal{N} e_0)$ . By the assumptions on the supports,  $\gamma_{\rho_n,p}^*$  is well defined on  $k$  for all  $n$ . Further, we may assume that  $k = h_{\rho \circ \gamma}^{\frac{1}{2p}} b h_{\rho \circ \gamma}^{\frac{1}{2p}}$  for some  $b \in \mathcal{N}$ , since the set of such elements is dense in  $L_p(e_0 \mathcal{N} e_0)$  and all the maps are contractions. Put  $k_n := h_{\rho_n \circ \gamma}^{\frac{1}{2p}} b h_{\rho_n \circ \gamma}^{\frac{1}{2p}}$ , then we have

$$\gamma_{\rho,p}^*(k) = h_\rho^{\frac{1}{p}} \gamma(b) h_\rho^{\frac{1}{p}}, \quad \gamma_{\rho_n,p}^*(k_n) = h_{\rho_n}^{\frac{1}{p}} \gamma(b) h_{\rho_n}^{\frac{1}{p}}$$

and we have  $k_n \rightarrow k$  in  $L_p(\mathcal{N})$  and  $\gamma_{\rho_n,p}^*(k_n) \rightarrow \gamma_{\rho,p}^*(k)$  in  $L_p(\mathcal{M})$ , this follows by the Hölder inequality and continuity of the map  $L_1(\mathcal{M})^+ \ni h \mapsto h^{\frac{1}{p}} \in L_p(\mathcal{M})^+$ , [17]. Therefore

$$\|\gamma_{\rho_n,p}^*(k) - \gamma_{\rho,p}^*(k)\|_p \leq \|\gamma_{\rho_n,p}^*(k - k_n)\|_p + \|\gamma_{\rho_n,p}^*(k_n) - \gamma_{\rho,p}^*(k)\|_p \rightarrow 0.$$

□

**Lemma 3.2.** *Let  $\gamma : \mathcal{N} \rightarrow \mathcal{M}$  be a normal positive unital map and let  $\rho \in \mathcal{M}_*^+$ ,  $b \in \mathcal{N}^+$ .*

*(i) If  $p \in [1/2, 1)$ , then*

$$\|h_{\rho \circ \gamma}^{\frac{1}{2p}} b h_{\rho \circ \gamma}^{\frac{1}{2p}}\|_p \leq \|h_\rho^{\frac{1}{2p}} \gamma(b) h_\rho^{\frac{1}{2p}}\|_p.$$

(ii) If  $p \in [1, \infty]$ , the inequality reverses.

*Proof.* Let us denote  $\beta := \gamma_\rho^*$  and let  $\omega \in \mathcal{M}_*^+$  be such that  $h_\omega := h_{\rho \circ \gamma}^{\frac{1}{2}} b h_{\rho \circ \gamma}^{\frac{1}{2}} \in L_1(\mathcal{N})^+$ . Then  $\beta$  is a normal positive unital map and we have

$$\beta_*(h_\omega) = h_\rho^{\frac{1}{2}} \gamma(b) h_\rho^{\frac{1}{2}}, \quad \beta_*(h_{\rho \circ \gamma}) = h_\rho.$$

Let  $p \in [1/2, 1)$ , then

$$\begin{aligned} \|h_\rho^{\frac{1}{2p}} \gamma(b) h_\rho^{\frac{1}{2p}}\|_p^p &= \|h_\rho^{\frac{1-p}{2p}} \beta_*(h_\omega) h_\rho^{\frac{1-p}{2p}}\|_p^p = Q_{p,p}(\beta_*(h_\omega) \|h_\rho\|) = Q_{p,p}(\beta_*(h_\omega) \|\beta_*(h_{\rho \circ \gamma})\|) \\ &\geq Q_{p,p}(h_\omega \|h_{\rho \circ \gamma}\|) = \|h_{\rho \circ \gamma}^{\frac{1-p}{2p}} h_\omega h_{\rho \circ \gamma}^{\frac{1-p}{2p}}\|_p^p = \|h_{\rho \circ \gamma}^{\frac{1}{2p}} b h_{\rho \circ \gamma}^{\frac{1}{2p}}\|_p^p. \end{aligned}$$

Here we have used the DPI for the sandwiched Rényi divergence  $D_{\alpha,\alpha}$  for  $\alpha \in [1/2, 1)$ , [12, Theorem 4.1]. This proves (i). The case (ii) is immediate from Lemma 3.1. This was proved also in [14] (see Eq. (22) therein), using the same argument.  $\square$

**Theorem 3.3 (DPI).** *Let  $\psi, \varphi \in \mathcal{M}_*^+$ ,  $\psi \neq 0$  and let  $\gamma : \mathcal{N} \rightarrow \mathcal{M}$  be a normal positive unital map. Assume either of the following conditions:*

- (i)  $0 < \alpha < 1$ ,  $\max\{\alpha, 1 - \alpha\} \leq z$
- (ii)  $\alpha > 1$ ,  $\max\{\alpha/2, \alpha - 1\} \leq z \leq \alpha$ .

Then we have

$$D_{\alpha,z}(\psi \circ \gamma \| \varphi \circ \gamma) \leq D_{\alpha,z}(\psi \| \varphi).$$

*Proof.* Under the conditions (i), the DPI was proved in [14, Theorem 1 (viii)]. Since parts of this proof will be used below, we repeat it here.

Assume the conditions in (i) and put  $p := \frac{z}{\alpha}$ ,  $r := \frac{z}{1-\alpha}$ , so that  $p, r \geq 1$ . For any  $b \in \mathcal{N}^{++}$ , we have by the Choi inequality [3] that  $\gamma(b)^{-1} \leq \gamma(b^{-1})$ , so that by Lemma A.2 and 3.2 (ii), we have

$$\|h_{\varphi}^{\frac{1}{2r}} \gamma(b)^{-1} \varphi^{\frac{1}{2r}}\|_r \leq \|h_{\varphi}^{\frac{1}{2r}} \gamma(b^{-1}) \varphi^{\frac{1}{2r}}\|_r \leq \|h_{\varphi \circ \gamma}^{\frac{1}{2r}} b^{-1} h_{\varphi \circ \gamma}^{\frac{1}{2r}}\|_r^r. \quad (3.2)$$

Using the variational expression in Theorem 2.3 (i), we have

$$Q_{\alpha,z}(\psi \| \varphi) \leq \alpha \|h_{\psi}^{\frac{1}{2p}} \gamma(b) h_{\psi}^{\frac{1}{2p}}\|_p^p + (1 - \alpha) \|h_{\varphi}^{\frac{1}{2r}} \gamma(b)^{-1} h_{\varphi}^{\frac{1}{2r}}\|_r^r \quad (3.3)$$

$$\leq \alpha \|h_{\psi \circ \gamma}^{\frac{1}{2p}} b h_{\psi \circ \gamma}^{\frac{1}{2p}}\|_p^p + (1 - \alpha) \|h_{\varphi \circ \gamma}^{\frac{1}{2r}} b^{-1} h_{\varphi \circ \gamma}^{\frac{1}{2r}}\|_r^r. \quad (3.4)$$

Since this holds for all  $b \in \mathcal{N}^{++}$ , it follows that  $Q_{\alpha,z}(\psi \| \varphi) \leq Q_{\alpha}(\psi \circ \gamma \| \varphi \circ \gamma)$ , which proves the DPI in this case.

Assume next the condition (ii), and put  $p := \frac{z}{\alpha}$ ,  $q := \frac{z}{\alpha-1}$ , so that  $p \in [1/2, 1)$  and  $q \geq 1$ . Using Theorem 2.3 (ii), we get for any  $b \in \mathcal{N}^+$ ,

$$\begin{aligned} Q_{\alpha,z}(\psi \| \varphi) &\geq \alpha \|h_{\psi}^{\frac{1}{2p}} \gamma(b) h_{\psi}^{\frac{1}{2p}}\|_p^p - (\alpha - 1) \|h_{\varphi}^{\frac{1}{2q}} \gamma(b) h_{\varphi}^{\frac{1}{2q}}\|_q^q \\ &\geq \alpha \|h_{\psi \circ \gamma}^{\frac{1}{2p}} b h_{\psi \circ \gamma}^{\frac{1}{2p}}\|_p^p - (\alpha - 1) \|h_{\varphi \circ \gamma}^{\frac{1}{2q}} b h_{\varphi \circ \gamma}^{\frac{1}{2q}}\|_q^q, \end{aligned}$$

here we used both (i) and (ii) in Lemma 3.2. Again, since this holds for all  $b \in \mathcal{N}^+$ , we get the desired inequality.  $\square$

### 3.2 Martingale convergence

An important consequence of DPI is the martingale convergence property that will be proved in this paragraph.

Let  $\mathcal{M}$  be a  $\sigma$ -finite von Neumann algebra. Let  $\{\mathcal{M}_i\}$  be an increasing net of von Neumann subalgebras of  $\mathcal{M}$  containing the unit of  $\mathcal{M}$  such that  $\mathcal{M} = (\bigcup_i \mathcal{M}_i)''$ .

**Theorem 3.4.** *Assume that  $\alpha$  and  $z$  satisfy the DPI bounds (that is, conditions (i) or (ii) in Theorem 3.3). Then we have*

$$D_{\alpha,z}(\psi\|\varphi) = \lim_i D_{\alpha,z}(\psi|_{\mathcal{M}_i}\|\varphi|_{\mathcal{M}_i}) \quad \text{increasingly.} \quad (3.5)$$

*Proof.* Let  $\varphi_i := \varphi|_{\mathcal{M}_i}$  and  $\psi_i := \psi|_{\mathcal{M}_i}$ . From Theorem 3.3, it follows that  $D_{\alpha,z}(\psi\|\varphi) \geq D_{\alpha,z}(\psi_i\|\varphi_i)$  for all  $i$  and  $i \mapsto D_{\alpha,z}(\psi_i\|\varphi_i)$  is increasing. Hence, to show (3.5), it suffices to prove that

$$D_{\alpha,z}(\psi\|\varphi) \leq \sup_i D_{\alpha,z}(\psi_i\|\varphi_i). \quad (3.6)$$

To do this, we may assume that  $\varphi$  is faithful. Indeed, assume that (3.6) has been shown when  $\varphi$  is faithful. For general  $\varphi \in \mathcal{M}_*^+$ , from the assumption of  $\mathcal{M}$  being  $\sigma$ -finite, there exists a  $\varphi_0 \in \mathcal{M}_*^+$  with  $s(\varphi_0) = \mathbf{1} - s(\varphi)$ . Let  $\varphi^{(n)} := \varphi + n^{-1}\varphi_0$  and  $\varphi_i^{(n)} := \varphi^{(n)}|_{\mathcal{M}_i}$  for each  $n \in \mathbb{N}$ . Thanks to the lower semi-continuity [14, Theorem 1(iv) and Theorem 2(iv)] and the order relation [14, Theorem 1(iii) and Theorem 2(iii)] we have

$$\begin{aligned} D_{\alpha,z}(\psi\|\varphi) &\leq \liminf_{n \rightarrow \infty} D_{\alpha,z}(\psi\|\varphi^{(n)}) \\ &\leq \liminf_{n \rightarrow \infty} \sup_i D_{\alpha,z}(\psi_i\|\varphi_i^{(n)}) \\ &\leq \sup_i D_{\alpha,z}(\psi_i\|\varphi_i), \end{aligned}$$

proving (3.6) for general  $\varphi$ . Below we assume the faithfulness of  $\varphi$  and write  $\mathcal{E}_{\mathcal{M}_i,\varphi}$  for the generalized conditional expectation from  $\mathcal{M}$  to  $\mathcal{M}_i$  with respect to  $\varphi$ . Then we note that we have by [10, Theorem 3],

$$\psi_i \circ \mathcal{E}_{\mathcal{M}_i,\varphi} = \psi \circ \mathcal{E}_{\mathcal{M}_i,\varphi} \rightarrow \psi \quad \text{in the norm,} \quad (3.7)$$

as well as

$$\varphi_i \circ \mathcal{E}_{\mathcal{M}_i,\varphi} = \varphi \circ \mathcal{E}_{\mathcal{M}_i,\varphi} = \varphi. \quad (3.8)$$

Using lower semicontinuity and DPI, we obtain

$$D_{\alpha,z}(\psi\|\varphi) \leq \liminf_i D_{\alpha,z}(\psi_i \circ \mathcal{E}_{\mathcal{M}_i,\varphi}\|\varphi) \leq \liminf_i D_{\alpha,z}(\psi_i\|\varphi) \leq \sup_i D_{\alpha,z}(\psi_i\|\varphi).$$

□



### 3.3 Equality in DPI and reversibility of channels

In what follows, a channel is a normal 2-positive unital map  $\gamma : \mathcal{N} \rightarrow \mathcal{M}$ .

**Definition 3.5.** Let  $\gamma : \mathcal{N} \rightarrow \mathcal{M}$  be a channel and let  $\mathcal{S} \subset \mathcal{M}_*^+$ . We say that  $\gamma$  is reversible (or sufficient) with respect to  $\mathcal{S}$  if there exists a channel  $\beta : \mathcal{M} \rightarrow \mathcal{N}$  such that

$$\rho \circ \gamma \circ \beta = \rho, \quad \forall \rho \in \mathcal{S}.$$

The notion of sufficient channels was introduced by Petz [22, 23], who also obtained a number of conditions characterizing this situation. In particular, it was proved in [23] that sufficient channels can be characterized by equality in DPI for the relative entropy  $D(\psi\|\varphi)$ : if  $D(\psi\|\varphi) < \infty$ , then a channel  $\gamma$  is sufficient with respect to  $\{\psi, \varphi\}$  if and only if

$$D(\psi \circ \gamma \| \varphi \circ \gamma) = D(\psi \| \varphi).$$

This characterization has been proved for a number of other divergence measures, including the standard Rényi divergences  $D_{\alpha,1}$  with  $0 < \alpha < 2$  and the sandwiched Rényi divergences  $D_{\alpha,\alpha}$  for  $\alpha > 1/2$  ([8, 11, 12]). Another important result of [23] shows that the Petz dual  $\gamma_\varphi^*$  is a universal recovery map, in the sense given in the proposition below.

**Proposition 3.6.** *Let  $\gamma : \mathcal{N} \rightarrow \mathcal{M}$  be a channel and let  $\varphi \in \mathcal{M}_*^+$  be such that both  $\varphi$  and  $\varphi \circ \gamma$  are faithful. Then for any  $\psi \in \mathcal{M}_*^+$ ,  $\gamma$  is sufficient with respect to  $\{\psi, \varphi\}$  if and only if  $\psi \circ \gamma \circ \gamma_\varphi^* = \psi$ .*

*Consequently, there is a faithful normal conditional expectation  $\mathcal{E}$  on  $\mathcal{M}$  such that  $\varphi \circ \mathcal{E} = \varphi$  and  $\gamma$  is sufficient with respect to  $\{\psi, \varphi\}$  if and only if also  $\psi \circ \mathcal{E} = \psi$ .*

Note that the range of the conditional expectation  $\mathcal{E}$  in the above proposition is the set of fixed points of the channel  $\gamma \circ \gamma_\varphi^*$ .

Our aim in this section is to prove that equality in DPI for  $D_{\alpha,z}$  with values of the parameters (strictly) contained in the DPI bounds of Theorem 3.3 characterizes sufficiency of channels. Throughout this section, we use the notations  $\psi_0 = \psi \circ \gamma$  and  $\varphi_0 = \varphi \circ \gamma$ . We also denote

$$p := \frac{z}{\alpha}, \quad r := \frac{z}{1-\alpha}, \quad q := -r = \frac{z}{\alpha-1}.$$

#### 3.3.1 The case $\alpha \in (0, 1)$

Here we study equality in DPI for  $D_{\alpha,z}$  with  $\alpha \in (0, 1)$ , for a pair of positive normal functionals  $\psi, \varphi \in \mathcal{M}_*^+$  and a normal positive unital map  $\gamma : \mathcal{N} \rightarrow \mathcal{M}$ . We first prove some equality conditions in the case  $\psi \sim \varphi$ .

**Proposition 3.7.** *Let  $0 < \alpha < 1$ ,  $\max\{\alpha, 1-\alpha\} \leq z$ . Assume that  $\psi \sim \varphi$  and let  $\gamma : \mathcal{N} \rightarrow \mathcal{M}$  be a normal positive unital map. Let  $\bar{a} \in \mathcal{M}^{++}$  be the unique minimizer as in Lemma 2.4 for  $Q_{\alpha,z}(\psi\|\varphi)$  and let  $\bar{a}_0 \in \mathcal{N}^{++}$  be the minimizer for  $Q_{\alpha,z}(\psi_0\|\varphi_0)$ . The following conditions are equivalent:*

- (i)  $D_{\alpha,z}(\psi_0\|\varphi_0) = D_{\alpha,z}(\psi\|\varphi)$ , i.e.,  $Q_{\alpha,z}(\psi_0\|\varphi_0) = Q_{\alpha,z}(\psi\|\varphi)$ .
- (ii)  $\gamma(\bar{a}_0) = \bar{a}$  and  $\|h_{\psi}^{\frac{1}{2p}} \gamma(\bar{a}_0) h_{\psi}^{\frac{1}{2p}}\|_p = \|h_{\psi_0}^{\frac{1}{2p}} \bar{a}_0 h_{\psi_0}^{\frac{1}{2p}}\|_p$ .
- (iii)  $\|h_{\psi}^{\frac{1}{2p}} \bar{a} h_{\psi}^{\frac{1}{2p}}\|_p = \|h_{\psi_0}^{\frac{1}{2p}} \bar{a}_0 h_{\psi_0}^{\frac{1}{2p}}\|_p$ .

$$(iv) \quad \gamma(\bar{a}_0^{-1}) = \bar{a}^{-1} \text{ and } \|h_{\varphi}^{\frac{1}{2r}} \gamma(\bar{a}_0^{-1}) h_{\varphi}^{\frac{1}{2r}}\|_r = \|h_{\varphi_0}^{\frac{1}{2r}} \bar{a}_0^{-1} h_{\varphi_0}^{\frac{1}{2r}}\|_r.$$

$$(v) \quad \|h_{\varphi}^{\frac{1}{2r}} \bar{a}^{-1} h_{\varphi}^{\frac{1}{2r}}\|_r = \|h_{\varphi_0}^{\frac{1}{2r}} \bar{a}_0^{-1} h_{\varphi_0}^{\frac{1}{2r}}\|_r.$$

*Proof.* By the assumptions,  $s(\psi) = s(\varphi)$  and since also  $\psi_0 \sim \varphi_0$ , we have  $s(\psi_0) = s(\varphi_0)$ . Using restrictions, we may assume that all  $\psi, \varphi, \psi_0, \varphi_0$  are faithful.

(i)  $\implies$  (ii) & (iv). By Lemma 3.2 (ii)

$$\left\| h_{\psi}^{\frac{1}{2p}} \gamma(\bar{a}_0) h_{\psi}^{\frac{1}{2p}} \right\|_p \leq \left\| h_{\psi_0}^{\frac{1}{2p}} \bar{a}_0 h_{\psi_0}^{\frac{1}{2p}} \right\|_p, \quad (3.9)$$

and by (3.2) one has

$$\left\| h_{\varphi}^{\frac{1}{2r}} \gamma(\bar{a}_0)^{-1} h_{\varphi}^{\frac{1}{2r}} \right\|_r \leq \left\| h_{\varphi}^{\frac{1}{2r}} \gamma(\bar{a}_0^{-1}) h_{\varphi}^{\frac{1}{2r}} \right\|_r \leq \left\| h_{\varphi_0}^{\frac{1}{2r}} \bar{a}_0^{-1} h_{\varphi_0}^{\frac{1}{2r}} \right\|_r. \quad (3.10)$$

From (3.9) and (3.10) it follows that

$$\begin{aligned} Q_{\alpha,z}(\psi\|\varphi) &\leq \alpha \left\| h_{\psi}^{\frac{1}{2p}} \gamma(\bar{a}_0) h_{\psi}^{\frac{1}{2p}} \right\|_p^p + (1-\alpha) \left\| h_{\varphi}^{\frac{1}{2r}} \gamma(\bar{a}_0)^{-1} h_{\varphi}^{\frac{1}{2r}} \right\|_r^r \\ &\leq \alpha \left\| h_{\psi_0}^{\frac{1}{2p}} \bar{a}_0 h_{\psi_0}^{\frac{1}{2p}} \right\|_p^p + (1-\alpha) \left\| h_{\varphi_0}^{\frac{1}{2r}} \bar{a}_0^{-1} h_{\varphi_0}^{\frac{1}{2r}} \right\|_r^r = Q_{\alpha,z}(\psi_0\|\varphi_0) = Q_{\alpha,z}(\psi\|\varphi). \end{aligned}$$

By uniqueness in Lemma 2.4 we find that  $\gamma(\bar{a}_0) = \bar{a}$  and all the inequalities in (3.9) and (3.10) must become equalities. Since  $\gamma(\bar{a}_0^{-1}) \geq \gamma(\bar{a}_0)^{-1}$ , we verify by Lemma A.2 that the equality in (3.10) yields  $\gamma(\bar{a}_0^{-1}) = \gamma(\bar{a}_0)^{-1} = \bar{a}^{-1}$ . Therefore, (ii) and (iv) hold.

The implications (ii)  $\implies$  (iii) and (iv)  $\implies$  (v) are obvious.

(iii)  $\implies$  (i). By (iii) with the equality (2.1) in Lemma 2.4 we have

$$\begin{aligned} Q_{\alpha,z}(\psi\|\varphi) &= \text{Tr} \left( h_{\psi}^{\frac{1}{2p}} h_{\varphi}^{\frac{1}{p}} h_{\psi}^{\frac{1}{2p}} \right)^z = \text{Tr} \left( h_{\psi}^{\frac{1}{2p}} \bar{a} h_{\psi}^{\frac{1}{2p}} \right)^p \\ &= \text{Tr} \left( h_{\psi_0}^{\frac{1}{2p}} \bar{a}_0 h_{\psi_0}^{\frac{1}{2p}} \right)^p = \text{Tr} \left( h_{\psi_0}^{\frac{1}{2p}} h_{\varphi_0}^{\frac{1}{p}} h_{\psi_0}^{\frac{1}{2p}} \right)^z \\ &= Q_{\alpha,z}(\psi_0\|\varphi_0). \end{aligned}$$

(v)  $\implies$  (i). Similarly, by (v) with the equality (2.2) in Lemma 2.4 we have

$$\begin{aligned} Q_{\alpha,z}(\psi\|\varphi) &= \text{Tr} \left( h_{\varphi}^{\frac{1}{2r}} h_{\psi}^{\frac{1}{p}} h_{\varphi}^{\frac{1}{2r}} \right)^z = \text{Tr} \left( h_{\varphi}^{\frac{1}{2r}} \bar{a}^{-1} h_{\varphi}^{\frac{1}{2r}} \right)^r \\ &= \text{Tr} \left( h_{\varphi_0}^{\frac{1}{2r}} \bar{a}_0^{-1} h_{\varphi_0}^{\frac{1}{2r}} \right)^r = \text{Tr} \left( h_{\varphi_0}^{\frac{1}{2r}} h_{\psi_0}^{\frac{1}{p}} h_{\varphi_0}^{\frac{1}{2r}} \right)^z \\ &= Q_{\alpha,z}(\psi_0\|\varphi_0). \end{aligned}$$

□

*Remark 3.8.* Note that the above conditions extend the results obtained in [19] and [26] in the finite dimensional case. Indeed, the condition (ii) with  $\alpha = z$  is equivalent to the condition in [19, Thm. 1], note here that in this case the second condition in (ii) is automatic. Moreover, (ii) extends the necessary condition in [26, Thm. 1.2 (2)] to a necessary and sufficient one. While in both these works  $\gamma$  was required to be completely positive, only positivity is enough for our result.

**Theorem 3.9.** *Let  $0 < \alpha < 1$  and  $\alpha, 1 - \alpha \leq z$ . Let  $\psi, \varphi \in \mathcal{M}_*^+$  and assume that  $\alpha < z$  and  $s(\varphi) \leq s(\psi)$  or  $1 - \alpha < z$  and  $s(\psi) \leq s(\varphi)$ . A channel  $\gamma : \mathcal{N} \rightarrow \mathcal{M}$  is reversible with respect to  $\{\psi, \varphi\}$  if and only if*

$$D_{\alpha,z}(\psi\|\varphi) = D_{\alpha,z}(\psi \circ \gamma\|\varphi \circ \gamma).$$

*Proof.* This proof is a modification of the proof of [12, Thm. 5.1]. We will assume that  $s(\varphi) \leq s(\psi)$  and  $\alpha < z$ , that is  $p > 1$ . In the other case we may exchange the role of  $p, r$  and  $\psi, \varphi$  by the equality  $Q_{\alpha,z}(\psi\|\varphi) = Q_{1-\alpha,z}(\varphi\|\psi)$ . As before, we may assume that both  $\psi$  and  $\psi_0$  are faithful.

The strategy of the proof is to use known results for the sandwiched Rényi divergence  $D_{p,p}$  with  $p > 1$ , [11]. For this, notice that

$$Q_{z,\alpha}(\psi\|\varphi) = Q_{p,p}(\omega\|\psi),$$

where  $\omega \in \mathcal{M}_*^+$  is such that

$$h_\omega = h_\psi^{\frac{p-1}{2p}} h_\mu^{\frac{1}{p}} h_\psi^{\frac{p-1}{2p}}, \quad h_\mu = |h_\varphi^{\frac{1}{2r}} h_\psi^{\frac{1}{2p}}|^{2z}. \quad (3.11)$$

Let  $\omega_0, \mu_0 \in \mathcal{N}_*^+$  be similar functionals obtained from  $\psi_0, \varphi_0$ . Then we have the equality

$$Q_{p,p}(\omega_0\|\psi_0) = Q_{\alpha,z}(\psi_0\|\varphi_0) = Q_{\alpha,z}(\psi\|\varphi) = Q_{p,p}(\omega\|\psi). \quad (3.12)$$

Our first goal is to show that  $\omega_0 = \omega \circ \gamma$ , which implies by [11] that  $\gamma$  is sufficient with respect to  $\{\omega, \psi\}$ . We then apply Proposition 3.6 and the properties of the extensions of the conditional expectation  $\mathcal{E}$  to Haagerup  $L_p$ -spaces proved in [13].

Let us remark here that if  $\psi \sim \varphi$ , we have  $h_\omega = h_\psi^{\frac{1}{2}} \bar{a} h_\psi^{\frac{1}{2}}$  and  $h_{\omega_0} = h_{\psi_0}^{\frac{1}{2}} \bar{a}_0 h_{\psi_0}^{\frac{1}{2}}$ , so that from the condition (iii) in Proposition 3.7, we immediately get

$$(\gamma_\psi^*)_*(h_{\omega_0}) = h_\psi^{\frac{1}{2}} \gamma(\bar{a}_0) h_\psi^{\frac{1}{2}} = h_\omega.$$

This and (3.12) shows that  $\gamma_\psi^*$  is sufficient with respect to  $\{\omega_0, \psi_0\}$ . By Proposition 3.6 and the fact that the Petz dual  $(\gamma_\psi^*)_{\psi_0}^*$  is  $\gamma$  itself, this implies the desired equality

$$\omega \circ \gamma = \omega_0 \circ \gamma_\psi^* \circ \gamma = \omega_0.$$

In the case  $\psi \not\sim \varphi$  some more work is required. Let  $\psi_n := \psi + \frac{1}{n}\varphi$  and  $\varphi_n := \varphi + \frac{1}{n}\psi$ . Then all  $\psi_n, \varphi_n$  are faithful,  $\psi_n \rightarrow \psi, \varphi_n \rightarrow \varphi$  in  $\mathcal{M}_*^+$ , moreover,  $\psi_n \sim \varphi_n$  for all  $n$ . Then  $\psi_n \circ \gamma \sim \varphi_n \circ \gamma$ ,  $\psi_n \circ \gamma \rightarrow \psi_0, \varphi_n \circ \gamma \rightarrow \varphi_0$  and by joint continuity of  $Q_{\alpha,z}$  ([14, Thm. 1 (iv)]), we have

$$\lim_n Q_{\alpha,z}(\psi_n \circ \gamma\|\varphi_n \circ \gamma) = Q_{\alpha,z}(\psi_0\|\varphi_0) = Q_{\alpha,z}(\psi\|\varphi) = \lim_n Q_{\alpha,z}(\psi_n\|\varphi_n).$$

Let  $\bar{b}_n \in \mathcal{N}^{++}$  be the minimizer for the variational expression for  $Q_{\alpha,z}(\psi_n \circ \gamma\|\varphi_n \circ \gamma)$ . Let also  $\bar{a}_n$  be the minimizer for  $Q_{\alpha,z}(\psi_n\|\varphi_n)$  and let  $f_n : \mathcal{M}^{++} \rightarrow \mathbb{R}^+$  be the function minimized in the expression for  $Q_{\alpha,z}(\psi_n\|\varphi_n)$ . We then have

$$\begin{aligned} Q_{\alpha,z}(\psi_n \circ \gamma\|\varphi_n \circ \gamma) - f_n(\gamma(\bar{b}_n)) &= \alpha \left( \|h_{\psi_n \circ \gamma}^{\frac{1}{2p}} \bar{b}_n h_{\psi_n \circ \gamma}^{\frac{1}{2p}}\|_p^p - \|h_{\psi_n}^{\frac{1}{2p}} \gamma(\bar{b}_n) h_{\psi_n}^{\frac{1}{2p}}\|_p^p \right) \\ &\quad + (1 - \alpha) \left( \|h_{\varphi_n \circ \gamma}^{\frac{1}{2r}} \bar{b}_n^{-1} h_{\varphi_n \circ \gamma}^{\frac{1}{2r}}\|_r^r - \|h_{\varphi_n}^{\frac{1}{2r}} \gamma(\bar{b}_n)^{-1} h_{\varphi_n}^{\frac{1}{2r}}\|_r^r \right) \geq 0, \end{aligned}$$

where the inequality follows from Lemma 3.2 (ii) and (3.2). We obtain

$$Q_{\alpha,z}(\psi_n \circ \gamma\|\varphi_n \circ \gamma) - Q_{\alpha,z}(\psi_n\|\varphi_n) \geq f_n(\gamma(\bar{b}_n)) - Q_{\alpha,z}(\psi_n\|\varphi_n) \geq 0. \quad (3.13)$$

Now let  $\mu_{n,0} \in \mathcal{N}_*^+$  and  $\mu_n \in \mathcal{M}_*^+$  be such that (using (2.1) in Lemma 2.4)

$$h_{\mu_{n,0}}^{\frac{1}{p}} = |h_{\varphi_n \circ \gamma}^{\frac{1}{2r}} h_{\psi_n \circ \gamma}^{\frac{1}{2p}}|^{2\alpha} = h_{\psi_n \circ \gamma}^{\frac{1}{2p}} \bar{b}_n h_{\psi_n \circ \gamma}^{\frac{1}{2p}}, \quad h_{\mu_n}^{\frac{1}{p}} = |h_{\varphi_n}^{\frac{1}{2r}} h_{\psi_n}^{\frac{1}{2p}}|^{2\alpha} = h_{\psi_n}^{\frac{1}{2p}} \bar{a}_n h_{\psi_n}^{\frac{1}{2p}}.$$

Then  $h_{\mu_{n,0}}^{\frac{1}{p}} \rightarrow h_{\mu_0}^{\frac{1}{p}}$  in  $L_p(\mathcal{N})$ , this follows by the Hölder inequality and the fact that the map  $L_{2z}(\mathcal{N}) \rightarrow L_p(\mathcal{N})$ , given as  $h \mapsto |h|^{2\alpha}$  is norm to norm continuous, [17]. Similarly,  $h_{\mu_n}^{\frac{1}{p}} \rightarrow h_{\mu}^{\frac{1}{p}}$  in  $L_p(\mathcal{M})$ . Next, note that since  $Q_{\alpha,z}(\psi_n \circ \gamma \|\varphi_n \circ \gamma)$  and  $Q_{\alpha,z}(\psi_n \|\varphi_n)$  have the same limit, we see from (3.13) and Lemma 2.5 that  $h_{\psi_n}^{\frac{1}{2p}} \gamma(\bar{b}_n) h_{\psi_n}^{\frac{1}{2p}} - h_{\mu_n}^{\frac{1}{p}} \rightarrow 0$  in  $L_p(\mathcal{M})$ . On the other hand, let  $\gamma_{\psi_n,p}^*, \gamma_{\psi,p}^*$  be the contractions defined in Lemma 3.1. We then have

$$h_{\psi_n}^{\frac{1}{2p}} \gamma(\bar{b}_n) h_{\psi_n}^{\frac{1}{2p}} = \gamma_{\psi_n,p}^*(h_{\mu_{n,0}}^{\frac{1}{p}})$$

and since  $\gamma_{\psi_n,p}^*(k) \rightarrow \gamma_{\psi,p}^*(k)$  in  $L_p(\mathcal{M})$  for any  $k \in L_p(s(\psi \circ \gamma)\mathcal{N}s(\psi \circ \gamma))$  by Lemma 3.1, we have

$$\|\gamma_{\psi,p}^*(h_{\mu_0}^{\frac{1}{p}}) - \gamma_{\psi_n,p}^*(h_{\mu_{n,0}}^{\frac{1}{p}})\|_p \leq \|(\gamma_{\psi,p}^* - \gamma_{\psi_n,p}^*)(h_{\mu_0}^{\frac{1}{p}})\|_p + \|\gamma_{\psi_n,p}^*(h_{\mu_0}^{\frac{1}{p}} - h_{\mu_{n,0}}^{\frac{1}{p}})\|_p \rightarrow 0.$$

Putting all together, we obtain that

$$h_{\mu}^{\frac{1}{p}} = \lim_n h_{\mu_n}^{\frac{1}{p}} = \lim_n \gamma_{\psi_n,p}^*(h_{\mu_{n,0}}^{\frac{1}{p}}) = \gamma_{\psi,p}^*(h_{\mu_0}^{\frac{1}{p}}).$$

It follows that

$$(\gamma_{\psi}^*)_*(h_{\omega_0}) = h_{\psi}^{\frac{p-1}{2p}} \gamma_{\psi,p}^*(h_{\mu_0}^{\frac{1}{p}}) h_{\psi}^{\frac{p-1}{2p}} = h_{\psi}^{\frac{p-1}{2p}} h_{\mu}^{\frac{1}{p}} h_{\psi}^{\frac{p-1}{2p}} = h_{\omega}.$$

As we have seen above, this and (3.12) implies

$$\omega \circ \gamma = \omega_0 \circ \gamma_{\psi}^* \circ \gamma = \omega_0.$$

Next, let  $\mathcal{E}$  be the faithful normal conditional expectation onto the set of fixed points of  $\gamma \circ \gamma_{\psi}^*$ , as in Proposition 3.6. Then  $\mathcal{E}$  preserves both  $\psi$  and  $\omega$ , which by [13] implies that

$$h_{\psi}^{\frac{p-1}{2p}} h_{\mu}^{\frac{1}{p}} h_{\psi}^{\frac{p-1}{2p}} = h_{\omega} = \mathcal{E}_*(h_{\omega}) = h_{\psi}^{\frac{p-1}{2p}} \mathcal{E}_p(h_{\mu}^{\frac{1}{p}}) h_{\psi}^{\frac{p-1}{2p}},$$

so that  $|h_{\varphi}^{\frac{1}{2r}} h_{\psi}^{\frac{1}{2p}}|^{2\alpha} = h_{\mu}^{\frac{1}{p}} \in L_p(\mathcal{E}(\mathcal{M}))$  and consequently  $|h_{\varphi}^{\frac{1}{2r}} h_{\psi}^{\frac{1}{2p}}| = h_{\mu}^{\frac{1}{2z}} \in L_{2z}(\mathcal{E}(\mathcal{M}))$ . Note that by the assumptions  $2z > 1$ , so that we may use the multiplicativity properties of the extension of  $\mathcal{E}$ . Let

$$h_{\varphi}^{\frac{1}{2r}} h_{\psi}^{\frac{1}{2p}} = u |h_{\varphi}^{\frac{1}{2r}} h_{\psi}^{\frac{1}{2p}}|$$

be the polar decomposition in  $L_{2z}(\mathcal{M})$ , then we have

$$u^* h_{\varphi}^{\frac{1}{2r}} h_{\psi}^{\frac{1}{2p}} = \mathcal{E}_{2z}(u^* h_{\varphi}^{\frac{1}{2r}} h_{\psi}^{\frac{1}{2p}}) = \mathcal{E}_{2r}(u^* h_{\varphi}^{\frac{1}{2r}}) h_{\psi}^{\frac{1}{2p}},$$

which implies that  $u^* h_{\varphi}^{\frac{1}{2r}} \in L_{2r}(\mathcal{E}(\mathcal{M}))$ . Since  $\psi$  is faithful, we have  $uu^* = \text{ran}(h_{\varphi}^{\frac{1}{2r}} h_{\psi}^{\frac{1}{2p}}) = s(\varphi)$ , so that by uniqueness of the polar decomposition in  $L_{2r}(\mathcal{M})$  and  $L_{2r}(\mathcal{E}(\mathcal{M}))$ , we must have  $h_{\varphi}^{\frac{1}{2r}} \in L_{2r}(\mathcal{E}(\mathcal{M}))$  and  $u \in \mathcal{E}(\mathcal{M})$ . Hence  $\varphi \circ \mathcal{E} = \varphi$  and  $\gamma$  is sufficient with respect to  $\{\psi, \varphi\}$ .  $\square$

### 3.3.2 The case $\alpha > 1$

We now turn to the case  $\alpha > 1$ . We will put  $p := \frac{z}{\alpha}$  and  $q := \frac{z}{\alpha-1}$ , then within the DPI bounds, we have  $p \in [1/2, 1]$  and  $q \geq 1$ , note that we always have  $p < q$ . Here we need to assume that  $D_{\alpha,z}(\psi\|\varphi) < \infty$ , so that by Lemma 2.2 there is some (unique)  $y \in L_{2z}(\mathcal{M})$  such that

$$h_{\psi}^{\frac{1}{2p}} = y h_{\varphi}^{\frac{1}{2q}}.$$

By the proof of Theorem 2.3, we have the following variational expression

$$Q_{\alpha,z}(\psi\|\varphi) = \sup_{w \in L_q(\mathcal{M})^+} \alpha \text{Tr}((ywy^*)^p) - (\alpha - 1) \text{Tr}(w^q). \quad (3.14)$$

The supremum is attained at a unique point  $\bar{w} = (y^*y)^{\alpha-1} \in L_q(\mathcal{M})^+$ , uniqueness follows from strict concavity of the function  $w \mapsto \alpha \text{Tr}((ywy^*)^p) - (\alpha - 1) \text{Tr}(w^q)$ .

By DPI, we have  $D_{\alpha,z}(\psi_0\|\varphi_0) \leq D_{\alpha,z}(\psi\|\varphi) < \infty$ , so that there is some (unique)  $y_0 \in L_{2z}(\mathcal{N})$  such that

$$h_{\psi_0}^{\frac{1}{2p}} = y_0 h_{\varphi_0}^{\frac{1}{2q}}.$$

Since  $D_{\alpha,z}(\psi\|\varphi) < \infty$  implies that  $s(\psi) \leq s(\varphi)$ , we may assume that both  $\varphi$  and  $\varphi_0$  are faithful.

**Lemma 3.10.** *Let  $\gamma : \mathcal{N} \rightarrow \mathcal{M}$  be a normal positive unital map. Let  $\gamma_{\varphi,q}^* : L_q(\mathcal{N}) \rightarrow L_q(\mathcal{M})$  be the contraction as in lemma 3.1. Let  $\bar{w} : (y^*y)^{\alpha-1} \in L_q(\mathcal{M})$  and  $\bar{w}_0 := (y_0^*y_0)^{\alpha-1} \in L_q(\mathcal{N})$ . Then equality in DPI holds if and only if*

$$\bar{w} = \gamma_{\varphi,q}^*(\bar{w}_0) \quad \text{and} \quad \text{Tr}(\bar{w}^q) = \text{Tr}(\gamma_{\varphi,q}^*(\bar{w}_0)^q). \quad (3.15)$$

*Proof.* We first show that for any  $w_0 \in L_q(\mathcal{N})^+$ ,

$$\text{Tr}((y\gamma_{\varphi,q}^*(w_0)y^*)^p) \geq \text{Tr}((y_0w_0y_0^*)^p). \quad (3.16)$$

Let us first assume that  $w_0 = h_{\varphi_0}^{\frac{1}{2q}} b h_{\varphi_0}^{\frac{1}{2q}}$  for some  $b \in \mathcal{N}_+$ . Then  $\gamma_{\varphi,q}^*(w_0) = h_{\varphi}^{\frac{1}{2q}} \gamma(b) h_{\varphi}^{\frac{1}{2q}}$ . Therefore

$$\begin{aligned} \text{Tr}((y\gamma_{\varphi,q}^*(w_0)y^*)^p) &= \text{Tr}((y h_{\varphi}^{\frac{1}{2q}} \gamma(b) h_{\varphi}^{\frac{1}{2q}} y^*)^p) = \text{Tr}((h_{\psi}^{\frac{1}{2p}} \gamma(b) h_{\psi}^{\frac{1}{2p}})^p) \geq \text{Tr}((h_{\psi_0}^{\frac{1}{2p}} b h_{\psi_0}^{\frac{1}{2p}})^p) \\ &= \text{Tr}((y_0 h_{\varphi_0}^{\frac{1}{2q}} b h_{\varphi_0}^{\frac{1}{2q}} y_0^*)^p) = \text{Tr}((y_0 w_0 y_0^*)^p), \end{aligned}$$

here the inequality is from Lemma 3.2 (i). The proof of the inequality (3.16) is finished by Lemma A.1. Using this and the fact that  $\gamma_{\varphi,q}^*$  is a contraction, we get from the variational expression (3.14)

$$\begin{aligned} Q_{\alpha,z}(\psi\|\varphi) &\geq \alpha \text{Tr}((y\gamma_{\varphi,q}^*(\bar{w}_0)y^*)^p) - (\alpha - 1) \text{Tr}(\gamma_{\varphi,q}^*(\bar{w}_0)^q) \\ &\geq \alpha \text{Tr}((y_0 \bar{w}_0 y_0^*)^p) - (\alpha - 1) \text{Tr}(\bar{w}_0^q) = Q_{\alpha,z}(\psi_0\|\varphi_0). \end{aligned}$$

Suppose that  $Q_{\alpha,z}(\psi_0\|\varphi_0) = Q_{\alpha,z}(\psi\|\varphi)$ , then both the inequalities must be equalities. Since  $\bar{w} \in L_q(\mathcal{M})^+$  and  $\bar{w}_0 \in L_q(\mathcal{N})^+$  are the unique elements such that the suprema in the variational expression (3.14) for  $Q_{\alpha,z}(\psi\|\varphi)$  resp.  $Q_{\alpha,z}(\psi_0\|\varphi_0)$  are attained, this proves (3.15). Conversely, if the equalities in (3.15) hold, then

$$Q_{\alpha,z}(\psi\|\varphi) = \text{Tr}((y^*y)^z) = \text{Tr}(\bar{w}^q) = \text{Tr}(\bar{w}_0^q) = \text{Tr}((y_0^*y_0)^z) = Q_{\alpha,z}(\psi_0\|\varphi_0).$$

□

**Theorem 3.11.** *Let  $\gamma : \mathcal{N} \rightarrow \mathcal{M}$  be a channel and let  $\psi, \varphi \in \mathcal{M}_*^+$  be such that  $s(\psi) \leq s(\varphi)$  and  $D_{\alpha,z}(\psi\|\varphi) < \infty$ . Then  $D_{\alpha,z}(\psi \circ \gamma\|\varphi \circ \gamma) = D_{\alpha,z}(\psi\|\varphi)$  if and only if  $\gamma$  is sufficient with respect to  $\{\psi, \varphi\}$ .*

*Proof.* Let  $\bar{w}$  and  $\bar{w}_0$  be as in Lemma 3.10. Let  $\omega \in \mathcal{M}_*^+$  and  $\omega_0 \in \mathcal{N}_*^+$  be such that

$$h_\omega = h_\varphi^{\frac{1}{2\alpha}} \bar{w} h_\varphi^{\frac{1}{2\alpha}}, \quad h_{\omega_0} = h_{\varphi_0}^{\frac{1}{2\alpha}} \bar{w}_0 h_{\varphi_0}^{\frac{1}{2\alpha}}.$$

Assume the equality in DPI holds, then by Lemma 3.10 we have

$$Q_{\alpha,z}(\omega_0\|\varphi_0) = \text{Tr}(\bar{w}_0^q) = \text{Tr}(\bar{w}^q) = Q_{\alpha,z}(\omega\|\varphi).$$

and using also Lemma 3.1, we have

$$(\gamma_\varphi^*)_*(h_{\omega_0}) = h_\varphi^{\frac{1}{2\alpha}} \gamma_{\varphi,q}^*(\bar{w}_0) h_\varphi^{\frac{1}{2\alpha}} = h_\omega.$$

Similarly as in the proof of Theorem 3.11, this shows that  $\gamma$  is sufficient with respect to  $\{\omega, \varphi\}$ . Hence  $\omega \circ \mathcal{E} = \omega$ , where  $\mathcal{E}$  is the conditional expectation onto the fixed points of  $\gamma \circ \gamma_\varphi^*$ . Using the extensions of  $\mathcal{E}$  and their properties, we get

$$h_\varphi^{\frac{q-1}{2q}} \bar{w} h_\varphi^{\frac{q-1}{2q}} = h_\omega = \mathcal{E}(h_\omega) = h_\varphi^{\frac{q-1}{2q}} \mathcal{E}(\bar{w}) h_\varphi^{\frac{q-1}{2q}},$$

which implies that  $\bar{w} = \mathcal{E}(\bar{w}) \in L_q(\mathcal{E}(\mathcal{M}))^+$ . But then also

$$|y| = \bar{w}^{\frac{1}{2(\alpha-1)}} = \bar{w}^{\frac{q}{2z}} \in L_{2z}(\mathcal{E}(\mathcal{M}))^+$$

Let  $y = u|y|$  be the polar decomposition of  $y$ , then we obtain from the definition of  $y$  that  $uu^* = s(yy^*) = s(\psi)$ . Further,

$$u^* h_\psi^{\frac{1}{2p}} = |y| h_\varphi^{\frac{1}{2q}} \in L_{2p}(\mathcal{E}(\mathcal{M}))$$

and by uniqueness of the polar decomposition in  $L_{2p}(\mathcal{M})$  and  $L_{2p}(\mathcal{E}(\mathcal{M}))$ , we obtain that  $h_\psi^{\frac{1}{2p}} \in L_{2p}(\mathcal{E}(\mathcal{M}))^+$ ,  $u \in \mathcal{E}(\mathcal{M})$ . Hence we must have  $h_\psi \in L_1(\mathcal{E}(\mathcal{M}))$  so that  $\psi \circ \mathcal{E} = \psi$  and  $\gamma$  is sufficient with respect to  $\{\psi, \varphi\}$  by Proposition 3.6. The converse is clear from DPI.  $\square$

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## 4 Monotonicity in the parameter $z$

It is well known [2, 6, 11] that the standard Rényi divergence  $D_{\alpha,1}(\psi\|\varphi)$  is monotone increasing in  $\alpha \in (0, 1) \cup (1, \infty)$  and the sandwiched Rényi divergence  $D_{\alpha,\alpha}(\psi\|\varphi)$  is monotone increasing in  $\alpha \in [1/2, 1) \cup (1, \infty)$ . It is also known [2, 6, 11] that

$$\lim_{\alpha \nearrow 1} D_{\alpha,1}(\psi\|\varphi) = \lim_{\alpha \nearrow 1} D_{\alpha,\alpha}(\psi\|\varphi) = D_1(\psi\|\varphi),$$

and if  $D_{\alpha,1}(\psi\|\varphi) < \infty$  (resp.,  $D_{\alpha,\alpha}(\psi\|\varphi) < \infty$ ) for some  $\alpha > 1$ , then

$$\lim_{\alpha \searrow 1} D_{\alpha,1}(\psi\|\varphi) = D_1(\psi\|\varphi) \quad \left( \text{resp., } \lim_{\alpha \searrow 1} D_{\alpha,\alpha}(\psi\|\varphi) = D_1(\psi\|\varphi) \right).$$

In the rest of the paper we will discuss similar monotonicity properties and limits for  $D_{\alpha,z}(\psi\|\varphi)$ . We consider monotonicity in the parameter  $z$  in Sec. 4 and monotonicity in the parameter  $\alpha$  in Sec. 5.

## 4.1 The finite von Neumann algebra case

Assume that  $(\mathcal{M}, \tau)$  is a semi-finite von Neumann algebra  $\mathcal{M}$  with a faithful normal semi-finite trace  $\tau$ . Then the Haagerup  $L_p$ -space  $L_p(\mathcal{M})$  is identified with the  $L_p$ -space  $L_p(\mathcal{M}, \tau)$  with respect to  $\tau$  [7, Example 9.11]. Hence one can define  $Q_{\alpha, z}(\psi \parallel \varphi)$  for  $\psi, \varphi \in \mathcal{M}_*^+$  by replacing, in Definition 2.1,  $L_p(\mathcal{M})$  with  $L_p(\mathcal{M}, \tau)$  and  $h_\psi \in L_1(\mathcal{M})_+$  with the Radon–Nikodym derivative  $d\psi/d\tau \in L_1(\mathcal{M}, \tau)^+$ . Below we use the symbol  $h_\psi$  to denote  $d\psi/d\tau$  as well. Note that  $\tau$  on  $\mathcal{M}_+$  is naturally extended to the positive part  $\widetilde{\mathcal{M}}^+$  of the space  $\widetilde{\mathcal{M}}$  of  $\tau$ -measurable operators. We then have [7, Proposition 4.20]

$$\tau(a) = \int_0^\infty \mu_s(a) ds, \quad a \in \widetilde{\mathcal{M}}^+, \quad (4.1)$$

where  $\mu_s(a)$  is the generalized  $s$ -number of  $a$  [4].

Throughout this subsection we assume that  $\mathcal{M}$  is a finite von Neumann algebra with a faithful normal finite trace  $\tau$ ; then  $\widetilde{\mathcal{M}}^+$  consists of all positive self-adjoint operators affiliated with  $\mathcal{M}$ .

**Lemma 4.1.** *For every  $\psi, \varphi \in \mathcal{M}_*^+$  with  $\psi \neq 0$  and for any  $\alpha, z > 0$  with  $\alpha \neq 1$ ,*

$$D_{\alpha, z}(\psi \parallel \varphi) = \lim_{\varepsilon \searrow 0} D_{\alpha, z}(\psi \parallel \varphi + \varepsilon \tau) \quad \text{increasingly}, \quad (4.2)$$

and hence  $D_{\alpha, z}(\psi \parallel \varphi) = \sup_{\varepsilon > 0} D_{\alpha, z}(\psi \parallel \varphi + \varepsilon \tau)$ .

*Proof.* Case  $0 < \alpha < 1$ . We need to prove that

$$Q_{\alpha, z}(\psi \parallel \varphi) = \lim_{\varepsilon \searrow 0} Q_{\alpha, z}(\psi \parallel \varphi + \varepsilon \tau) \quad \text{decreasingly}. \quad (4.3)$$

In the present setting we have by (4.1)

$$Q_{\alpha, z}(\psi \parallel \varphi) = \tau \left( \left( h_\psi^{\alpha/2z} h_\varphi^{\frac{1-\alpha}{z}} h_\psi^{\alpha/2z} \right)^z \right) = \int_0^\infty \mu_s \left( h_\psi^{\alpha/2z} h_\varphi^{\frac{1-\alpha}{z}} h_\psi^{\alpha/2z} \right)^z ds, \quad (4.4)$$

and similarly

$$Q_{\alpha, z}(\psi \parallel \varphi + \varepsilon \tau) = \int_0^\infty \mu_s \left( h_\psi^{\alpha/2z} h_{\varphi + \varepsilon \tau}^{\frac{1-\alpha}{z}} h_\psi^{\alpha/2z} \right)^z ds.$$

Since  $h_{\varphi + \varepsilon \tau}^{\frac{1-\alpha}{z}} = (h_\varphi + \varepsilon \mathbf{1})^{\frac{1-\alpha}{z}}$  decreases to  $h_\varphi^{\frac{1-\alpha}{z}}$  in the measure topology as  $\varepsilon \searrow 0$ , it follows that  $h_\psi^{\alpha/2z} h_{\varphi + \varepsilon \tau}^{\frac{1-\alpha}{z}} h_\psi^{\alpha/2z}$  decreases to  $h_\psi^{\alpha/2z} h_\varphi^{\frac{1-\alpha}{z}} h_\psi^{\alpha/2z}$  in the measure topology. Hence by [4, Lemma 3.4] we have  $\mu_s \left( h_\psi^{\alpha/2z} h_{\varphi + \varepsilon \tau}^{\frac{1-\alpha}{z}} h_\psi^{\alpha/2z} \right) \searrow \mu_s \left( h_\psi^{\alpha/2z} h_\varphi^{\frac{1-\alpha}{z}} h_\psi^{\alpha/2z} \right)$  as  $\varepsilon \searrow 0$  for almost every  $s > 0$ . Since  $s \mapsto \mu_s \left( h_\psi^{\alpha/2z} h_{\varphi + \varepsilon \tau}^{\frac{1-\alpha}{z}} h_\psi^{\alpha/2z} \right)$  is integrable on  $(0, \infty)$ , the Lebesgue convergence theorem gives (4.3).

Case  $\alpha > 1$ . We need to prove that

$$Q_{\alpha, z}(\psi \parallel \varphi) = \lim_{\varepsilon \searrow 0} Q_{\alpha, z}(\psi \parallel \varphi + \varepsilon \tau) \quad \text{increasingly}. \quad (4.5)$$

For any  $\varepsilon > 0$ , since  $h_{\varphi + \varepsilon \tau} = h_\varphi + \varepsilon \mathbf{1}$  has the bounded inverse  $h_{\varphi + \varepsilon \tau}^{-1} = (h_\varphi + \varepsilon \mathbf{1})^{-1} \in \mathcal{M}^+$ , one can define  $x_\varepsilon := (h_\varphi + \varepsilon \mathbf{1})^{-\frac{\alpha-1}{2z}} h_\psi^{\alpha/z} (h_\varphi + \varepsilon \mathbf{1})^{-\frac{\alpha-1}{2z}} \in \widetilde{\mathcal{M}}^+$  so that

$$h_\psi^{\alpha/z} = (h_\varphi + \varepsilon \mathbf{1})^{\frac{\alpha-1}{2z}} x_\varepsilon (h_\varphi + \varepsilon \mathbf{1})^{\frac{\alpha-1}{2z}}.$$

In the present setting one can write by (4.1)

$$Q_{\alpha,z}(\psi\|\varphi + \varepsilon\tau) = \tau(x_\varepsilon^z) = \int_0^\infty \mu_s(x_\varepsilon)^z ds \quad (\in [0, \infty]). \quad (4.6)$$

Let  $0 < \varepsilon \leq \varepsilon'$ . Since  $(h_\varphi + \varepsilon\mathbf{1})^{-\frac{\alpha-1}{z}} \geq (h_\varphi + \varepsilon'\mathbf{1})^{-\frac{\alpha-1}{z}}$ , one has  $\mu_s(x_\varepsilon) \geq \mu_s(x_{\varepsilon'})$  for all  $s > 0$ , so that

$$Q_{\alpha,z}(\psi\|\varphi + \varepsilon\tau) \geq Q_{\alpha,z}(\psi\|\varphi + \varepsilon'\tau).$$

Hence  $\varepsilon > 0 \mapsto D_{\alpha,z}(\psi\|\varphi + \varepsilon\tau)$  is decreasing.

First, assume that  $s(\psi) \not\leq s(\varphi)$ . Then  $\mu_{s_0}(h_\psi^{\alpha/2z} s(\varphi)^\perp h_\psi^{\alpha/2z}) > 0$  for some  $s_0 > 0$ ; indeed, otherwise,  $h_\psi^{\alpha/2z} s(\varphi)^\perp h_\psi^{\alpha/2z} = 0$  so that  $s(\psi) \leq s(\varphi)$ . Hence we have

$$\mu_s(x_\varepsilon) = \mu_s(h_\psi^{\alpha/2z} (h_\varphi + \varepsilon\mathbf{1})^{-\frac{\alpha-1}{z}} h_\psi^{\alpha/2z}) \geq \varepsilon^{-\frac{\alpha-1}{z}} \mu_s(h_\psi^{\alpha/2z} s(\varphi)^\perp h_\psi^{\alpha/2z}) \nearrow \infty \quad \text{as } \varepsilon \searrow 0$$

for all  $s \in (0, s_0]$ . Therefore, it follows from (4.6) that  $Q_{\alpha,z}(\psi\|\varphi + \varepsilon\tau) \nearrow \infty = Q_{\alpha,z}(\psi\|\varphi)$ .

Next, assume that  $s(\psi) \leq s(\varphi)$ . Take the spectral decomposition  $h_\varphi = \int_0^\infty t de_t$  and define  $y, x \in \widetilde{\mathcal{M}}_+$  by

$$y := h_\varphi^{-\frac{\alpha-1}{z}} s(\varphi) = \int_{(0,\infty)} t^{-\frac{\alpha-1}{z}} de_t, \quad x := y^{1/2} h_\psi^{\alpha/z} y^{1/2}.$$

Since

$$h_\psi^{\alpha/z} = s(\varphi) h_\psi^{\alpha/z} s(\varphi) = h_\varphi^{\frac{\alpha-1}{2z}} y^{1/2} h_\psi^{\alpha/z} y^{1/2} h_\varphi^{\frac{\alpha-1}{2z}} = h_\varphi^{\frac{\alpha-1}{2z}} x h_\varphi^{\frac{\alpha-1}{2z}},$$

one has, similarly to 4.6,

$$Q_{\alpha,z}(\psi\|\varphi) = \tau(x^z) = \int_0^\infty \mu_s(x)^z ds. \quad (4.7)$$

We write  $(h_\varphi + \varepsilon\mathbf{1})^{-\frac{\alpha-1}{z}} s(\varphi) = \int_{(0,\infty)} (t + \varepsilon)^{-\frac{\alpha-1}{z}} de_t$ , and for any  $\delta > 0$  choose a  $t_0 > 0$  such that  $\tau(e_{(0,t_0)}) < \delta$ . Then, since  $\int_{[t_0,\infty)} (t + \varepsilon)^{-\frac{\alpha-1}{z}} de_t \rightarrow \int_{[t_0,\infty)} t^{-\frac{\alpha-1}{z}} de_t$  in the operator norm as  $\varepsilon \searrow 0$ , we obtain  $(h_\varphi + \varepsilon\mathbf{1})^{-\frac{\alpha-1}{z}} s(\varphi) \nearrow y$  in the measure topology (see [4, 1.5]), so that  $h_\psi^{\alpha/2z} (h_\varphi + \varepsilon\mathbf{1})^{-\frac{\alpha-1}{z}} h_\psi^{\alpha/2z} \nearrow h_\psi^{\alpha/2z} y h_\psi^{\alpha/2z}$  in the measure topology as  $\varepsilon \searrow 0$ . Hence we have by [4, Lemma 3.4]

$$\mu_s(x_\varepsilon) = \mu_s(h_\psi^{\alpha/2z} (h_\varphi + \varepsilon\mathbf{1})^{-\frac{\alpha-1}{z}} h_\psi^{\alpha/2z}) \nearrow \mu_s(h_\psi^{\alpha/2z} y h_\psi^{\alpha/2z}) = \mu_s(x) \quad (4.8)$$

for all  $s > 0$ . Therefore, by (4.6) and (4.7) the monotone convergence theorem gives (4.5).  $\square$

**Lemma 4.2.** *Let  $(\mathcal{M}, \tau)$  and  $\psi, \varphi$  be as above, and let  $0 < z \leq z'$ . Then*

$$\begin{cases} D_{\alpha,z}(\psi\|\varphi) \leq D_{\alpha,z'}(\psi\|\varphi), & 0 < \alpha < 1, \\ D_{\alpha,z}(\psi\|\varphi) \geq D_{\alpha,z'}(\psi\|\varphi), & \alpha > 1. \end{cases}$$

*Proof.* The case  $0 < \alpha < 1$  was shown in [14, Theorem 1(x)] for general von Neumann algebras. For the case  $\alpha > 1$ , by Lemma 4.1 it suffices to show that, for every  $\varepsilon > 0$ ,

$$\tau\left(\left(y_\varepsilon^{\frac{\alpha-1}{2z}} h_\psi^{\alpha/z} y_\varepsilon^{\frac{\alpha-1}{2z}}\right)^z\right) \geq \tau\left(\left(y_\varepsilon^{\frac{\alpha-1}{2z'}} h_\psi^{\alpha/z'} y_\varepsilon^{\frac{\alpha-1}{2z'}}\right)^{z'}\right),$$



where  $y_\varepsilon := (h_\varphi + \varepsilon \mathbf{1})^{-1} \in \mathcal{M}_+$ . The above is equivalently written as

$$\tau\left(\left|h_\psi^{\alpha/2z'}\right)^r(y_\varepsilon^{(\alpha-1)/2z'})^r\right|^{2z} \geq \tau\left(\left|h_\psi^{\alpha/2z'}y_\varepsilon^{(\alpha-1)/2z'}\right|^{2zr}\right),$$

where  $r := z'/z \geq 1$ . Hence the desired inequality follows from Kosaki's ALT inequality [18, Corollary 3].  $\square$

When  $(\mathcal{M}, \tau)$  and  $\psi, \varphi$  are as in Lemma 4.1, one can define, thanks to Lemma 4.2, for any  $\alpha \in (0, \infty) \setminus \{1\}$ ,

$$\begin{aligned} Q_{\alpha, \infty}(\psi \| \varphi) &:= \lim_{z \rightarrow \infty} Q_{\alpha, z}(\psi \| \varphi) = \inf_{z > 0} Q_{\alpha, z}(\psi \| \varphi), \\ D_{\alpha, \infty}(\psi \| \varphi) &:= \frac{1}{\alpha - 1} \log \frac{Q_{\alpha, \infty}(\psi \| \varphi)}{\psi(\mathbf{1})} \\ &= \lim_{z \rightarrow \infty} D_{\alpha, z}(\psi \| \varphi) = \begin{cases} \sup_{z > 0} D_{\alpha, z}(\psi \| \varphi), & 0 < \alpha < 1, \\ \inf_{z > 0} D_{\alpha, z}(\psi \| \varphi), & \alpha > 1. \end{cases} \end{aligned} \quad (4.9)$$

If  $h_\psi, h_\varphi \in \mathcal{M}^{++}$  (i.e.,  $\delta\tau \leq \psi, \varphi \leq \delta^{-1}\tau$  for some  $\delta \in (0, 1)$ ), then the Lie–Trotter formula gives

$$Q_{\alpha, \infty}(\psi \| \varphi) = \tau(\exp(\alpha \log h_\psi + (1 - \alpha) \log h_\varphi)). \quad (4.10)$$

**Lemma 4.3.** *Let  $(\mathcal{M}, \tau)$  and  $\psi, \varphi$  be as above. Then for any  $z > 0$ ,*

$$\begin{cases} D_{\alpha, z}(\psi \| \varphi) \leq D_1(\psi \| \varphi), & 0 < \alpha < 1, \\ D_{\alpha, z}(\psi \| \varphi) \geq D_1(\psi \| \varphi), & \alpha > 1. \end{cases}$$

*Proof.* First, assume that  $h_\psi, h_\varphi \in \mathcal{M}^{++}$ . Set self-adjoint  $H := \log h_\psi$  and  $K := \log h_\varphi$  in  $\mathcal{M}$  and define  $F(\alpha) := \log \tau(e^{\alpha H + (1-\alpha)K})$  for  $\alpha > 0$ . Then by (4.10),  $F(\alpha) = \log Q_{\alpha, \infty}(\psi \| \varphi)$  for all  $\alpha \in (0, \infty) \setminus \{1\}$ , and we compute

$$\begin{aligned} F'(\alpha) &= \frac{\tau(e^{\alpha H + (1-\alpha)K}(H - K))}{\tau(e^{\alpha H + (1-\alpha)K})}, \\ F''(\alpha) &= \frac{\{\tau(e^{\alpha H + (1-\alpha)K}(H - K))\}^2 - \tau(e^{\alpha H + (1-\alpha)K}(H - K)^2)}{\{\tau(e^{\alpha H + (1-\alpha)K})\}^2}. \end{aligned}$$

I think the second derivative should be

$$F''(\alpha) = \frac{\tau(e^{\alpha H + (1-\alpha)K}(H - K)^2)\tau(e^{\alpha H + (1-\alpha)K}) - \{\tau(e^{\alpha H + (1-\alpha)K}(H - K))\}^2}{\{\tau(e^{\alpha H + (1-\alpha)K})\}^2}.$$

The rest of the argument below is fine.

Since  $F''(\alpha) \geq 0$  on  $(0, \infty)$  thanks to the Schwarz inequality, we see that  $F(\alpha)$  is convex on  $(0, \infty)$  and hence

$$D_{\alpha, \infty}(\psi \| \varphi) = \frac{F(\alpha) - F(1)}{\alpha - 1}$$

is increasing in  $\alpha \in (0, \infty)$ , where for  $\alpha = 1$  the above RHS is understood as

$$F'(1) = \frac{\tau(e^H(H - K))}{\tau(e^H)} = \frac{\tau(h_\psi(\log h_\psi - \log h_\varphi))}{\tau(h_\psi)} = D_1(\psi \| \varphi).$$

Hence by (4.9) the assertion holds when  $h_\psi, h_\varphi \in \mathcal{M}^{++}$ . Below we extend it to general  $\psi, \varphi \in \mathcal{M}_*^+$ .

*Case  $0 < \alpha < 1$ .* Let  $\psi, \varphi \in \mathcal{M}_*^+$  and  $z > 0$ . From [14, Theorem 1(iv)] and [8, Corollary 2.8(3)] we have

$$\begin{aligned} D_{\alpha,z}(\psi\|\varphi) &= \lim_{\varepsilon \searrow 0} D_{\alpha,z}(\psi + \varepsilon\tau\|\varphi + \varepsilon\tau), \\ D_1(\psi\|\varphi) &= \lim_{\varepsilon \searrow 0} D_1(\psi + \varepsilon\tau\|\varphi + \varepsilon\tau), \end{aligned}$$

so that we may assume that  $\psi, \varphi \geq \varepsilon\tau$  for some  $\varepsilon > 0$ . Take the spectral decompositions  $h_\psi = \int_0^\infty t de_t^\psi$  and  $h_\varphi = \int_0^\infty t de_t^\varphi$ , and define  $e_n := e_n^\psi \wedge e_n^\varphi$  for each  $n \in \mathbb{N}$ . Then  $\tau(e_n^\perp) \leq \tau((e_n^\psi)^\perp) + \tau((e_n^\varphi)^\perp) \rightarrow 0$  as  $n \rightarrow \infty$ , so that  $e_n \nearrow \mathbf{1}$ . We set  $\psi_n := \psi(e_n \cdot e_n)$  and  $\varphi_n := \varphi(e_n \cdot e_n)$ ; then  $h_{\psi_n} = e_n h_\psi e_n$  and  $h_{\varphi_n} = e_n h_\varphi e_n$  are in  $(e_n \mathcal{M} e_n)^{++}$ . Note that

$$\begin{aligned} \|h_\psi - e_n h_\psi e_n\|_1 &\leq \|(\mathbf{1} - e_n)h_\psi\|_1 + \|e_n h_\psi (\mathbf{1} - e_n)\|_1 \\ &\leq \|(\mathbf{1} - e_n)h_\psi^{1/2}\|_2 \|h_\psi^{1/2}\|_2 + \|e_n h_\psi^{1/2}\|_2 \|h_\psi^{1/2}(\mathbf{1} - e_n)\|_2 \\ &= \psi(\mathbf{1} - e_n)^{1/2} \psi(\mathbf{1})^{1/2} + \psi(e_n)^{1/2} \psi(\mathbf{1} - e_n)^{1/2} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

and similarly  $\|h_\varphi - e_n h_\varphi e_n\|_1 \rightarrow 0$ . Hence by [14, Theorem 1(iv)] one has  $D_{\alpha,z}(e_n \psi e_n \| e_n \varphi e_n) \rightarrow D_{\alpha,z}(\psi\|\varphi)$ . On the other hand, one has  $D_1(e_n \psi e_n \| e_n \varphi e_n) \rightarrow D_1(\psi\|\varphi)$  by [8, Proposition 2.10]. Since  $D_{\alpha,z}(e_n \psi e_n \| e_n \varphi e_n) \leq D_1(e_n \psi e_n \| e_n \varphi e_n)$  holds by regarding  $e_n \psi e_n, e_n \varphi e_n$  as functionals on the reduced von Neumann algebra  $e_n \mathcal{M} e_n$ , we obtain the desired inequality for general  $\psi, \varphi \in \mathcal{M}_*^+$ .

*Case  $\alpha > 1$ .* We show the extension to general  $\psi, \varphi \in \mathcal{M}_*^+$  by dividing four steps as follows, where  $h_\psi = \int_0^\infty t e_t^\psi$  and  $h_\varphi = \int_0^\infty t de_t^\varphi$  are the spectral decompositions.

(1) Assume that  $h_\psi \in \mathcal{M}^+$  and  $h_\varphi \in \mathcal{M}^{++}$ . Set  $\psi_n \in \mathcal{M}_*^+$  by  $h_{\psi_n} = (1/n)e_{[0,1/n]}^\psi + \int_{(1/n,\infty)} t de_t^\psi$  ( $\in \mathcal{M}^{++}$ ). Since  $h_{\psi_n}^{\alpha/z} \searrow h_\psi^{\alpha/z}$  in the operator norm, we have by (4.4) and [4, Lemma 3.4]

$$\begin{aligned} Q_{\alpha,z}(\psi\|\varphi) &= \int_0^\infty \mu_s((h_\varphi^{-1})^{\frac{\alpha-1}{2z}} h_\psi^{\alpha/z} (h_\varphi^{-1})^{\frac{\alpha-1}{2z}})^z ds \\ &= \lim_{n \rightarrow \infty} \int_0^\infty \mu_s((h_\varphi^{-1})^{\frac{\alpha-1}{2z}} h_{\psi_n}^{\alpha/z} (h_\varphi^{-1})^{\frac{\alpha-1}{2z}})^z ds = \lim_{n \rightarrow \infty} Q_{\alpha,z}(\psi_n\|\varphi). \end{aligned} \tag{4.11}$$

From this and the lower semicontinuity of  $D_1$  the extension holds in this case.

(2) Assume that  $h_\psi \in \mathcal{M}^+$  and  $h_\varphi \geq \delta \mathbf{1}$  for some  $\delta > 0$ . Set  $\varphi_n \in \mathcal{M}_*^+$  by  $h_{\varphi_n} = \int_{[\delta,n]} t de_t^\varphi + ne_{(n,\infty)}^\varphi$  ( $\in \mathcal{M}^{++}$ ). Since  $h_{\varphi_n}^{-\frac{\alpha-1}{z}} \searrow h_\varphi^{-\frac{\alpha-1}{z}}$  in the operator norm, we have by (4.4) and [4, Lemma 3.4] again

$$\begin{aligned} Q_{\alpha,z}(\psi\|\varphi) &= \int_0^\infty \mu_s(h_\psi^{\alpha/2z} h_\varphi^{-\frac{\alpha-1}{z}} h_\psi^{\alpha/2z})^z ds \\ &= \lim_{n \rightarrow \infty} \int_0^\infty \mu_s(h_\psi^{\alpha/2z} h_{\varphi_n}^{-\frac{\alpha-1}{z}} h_\psi^{\alpha/2z})^z ds = \lim_{n \rightarrow \infty} Q_{\alpha,z}(\psi\|\varphi_n). \end{aligned}$$

From this and (1) above the extension holds in this case too.

(3) Assume that  $\psi$  is general and  $\varphi \geq \delta \tau$  for some  $\delta > 0$ . Set  $\psi_n \in \mathcal{M}_*^+$  by  $h_{\psi_n} = \int_{[0,n]} t de_t^\psi + ne_{(n,\infty)}^\psi$  ( $\in \mathcal{M}_+$ ). Since  $h_{\psi_n}^{\alpha/z} \nearrow h_\psi^{\alpha/z}$  in the measure topology, one can argue as in (4.11) with use of the monotone convergence theorem to see from (2) that the extension holds in this case too.

(4) Finally, from (3) with Lemma 4.1 and [8, Corollary 2.8(3)] it follows that the desired extension holds for general  $\psi, \varphi \in \mathcal{M}_*^+$ .  $\square$

In the next proposition, we summarize inequalities for  $D_{\alpha,z}$  obtained so far in Lemmas 4.2 and 4.3.

**Proposition 4.4.** *Assume that  $\mathcal{M}$  is a finite von Neumann algebra with a faithful normal finite trace  $\tau$ . Let  $\psi, \varphi \in \mathcal{M}_*^+$ ,  $\psi \neq 0$ . If  $0 < \alpha < 1 < \alpha'$  and  $0 < z \leq z' \leq \infty$ , then*

$$D_{\alpha,z}(\psi\|\varphi) \leq D_{\alpha,z'}(\psi\|\varphi) \leq D_1(\psi\|\varphi) \leq D_{\alpha',z'}(\psi\|\varphi) \leq D_{\alpha',z}(\psi\|\varphi).$$

**Corollary 4.5.** *Let  $(\mathcal{M}, \tau)$  and  $\psi, \varphi$  be as in Proposition 4.4. Then for any  $z \in [1, \infty]$ ,*

$$\lim_{\alpha \nearrow 1} D_{\alpha,z}(\psi\|\varphi) = D_1(\psi\|\varphi). \quad (4.12)$$

Moreover, if  $D_{\alpha,\alpha}(\psi\|\varphi) < \infty$  for some  $\alpha > 1$  then for any  $z \in (1, \infty]$ ,

$$\lim_{\alpha \searrow 1} D_{\alpha,z}(\psi\|\varphi) = D_1(\psi\|\varphi). \quad (4.13)$$

*Proof.* Let  $z \geq 1$ . For every  $\alpha \in (0, 1)$ , Proposition 4.4 gives

$$D_{\alpha,1}(\psi\|\varphi) \leq D_{\alpha,z}(\psi\|\varphi) \leq D_1(\psi\|\varphi).$$

Hence (4.12) follows since it holds for  $D_{\alpha,1}$  [6, Proposition 5.3(3)].

Next, assume that  $D_{\alpha,\alpha}(\psi\|\varphi) < \infty$  for some  $\alpha > 1$ . Let  $z > 1$ . For every  $\alpha \in (1, z]$ , Proposition 4.4 gives

$$D_1(\psi\|\varphi) \leq D_{\alpha,z}(\psi\|\varphi) \leq D_{\alpha,\alpha}(\psi\|\varphi).$$

Hence (4.13) follows since it holds for  $D_{\alpha,\alpha}$  [11, Proposition 3.8(ii)].  $\square$

In this subsection, in the specialized setting of finite von Neumann algebras, we have given monotonicity of  $D_{\alpha,z}$  in the parameter  $z$  in an essentially similar way to the finite-dimensional case [20]. In the next subsection we will extend it to general von Neumann algebras under certain restrictions of  $\alpha, z$ .

## 4.2 The general von Neumann algebra case

**Theorem 4.6.** *For every  $\psi, \varphi \in \mathcal{M}_*^+$ ,  $\psi \neq 0$ , and  $0 < \alpha < 1$ , we have:*

(1) *If  $0 < \alpha < 1$  and  $\max\{\alpha, 1 - \alpha\} \leq z \leq z'$ , then*

$$D_{\alpha,z}(\psi\|\varphi) \leq D_{\alpha,z'}(\psi\|\varphi) \leq D_1(\psi\|\varphi).$$

(2) *If  $\alpha > 1$  and  $\max\{\alpha/2, \alpha - 1\} \leq z \leq z' \leq \alpha$ , then*

$$D_1(\psi\|\varphi) \leq D_{\alpha,z'}(\psi\|\varphi) \leq D_{\alpha,z}(\psi\|\varphi).$$

**Hiai (12/8/2023)** In fact, (2) is improved in Theorem 6.

**Theorem 4.7.** *For every  $\psi, \varphi \in \mathcal{M}_*^+$ ,  $\psi \neq 0$ , and  $\alpha > 1$ , the function  $z \mapsto D_{\alpha,z}(\psi\|\varphi)$  is monotone decreasing on  $[\alpha/2, \infty)$ .*

**Anna (Jan. 23, 2024)**

## 5 Monotonicity in the parameter $\alpha$

### 5.1 The case $\alpha < 1$ and all $z > 0$

**Theorem 5.1.** *Let  $\psi, \varphi \in \mathcal{M}_*^+$  and  $z > 0$ . Then we have*

- (1)  $\alpha \mapsto \log Q_{\alpha,z}(\psi\|\varphi)$  is convex on  $(0, 1)$ ,
- (2)  $\alpha \mapsto D_{\alpha,z}(\psi\|\varphi)$  is monotone increasing on  $(0, 1)$ .

Anna (Jan. 10, 2024), Hiai (1/16/2024)

### 5.2 The case $1 < \alpha \leq 2z$

**Theorem 5.2.** *Let  $\psi, \varphi \in \mathcal{M}_*^+$  and  $z > 1/2$ . Then we have*

- (1)  $\alpha \mapsto \log Q_{\alpha,z}(\psi\|\varphi)$  is convex on  $(1, 2z]$ ,
- (2)  $\alpha \mapsto D_{\alpha,z}(\psi\|\varphi)$  is monotone increasing on  $(1, 2z]$ .

Anna (Jan. 23, 2024), Hiai (12/31/2023)

### 5.3 Limits as $\alpha \nearrow 1$ and $\alpha \searrow 1$

**Theorem 5.3.** *Let  $\psi, \varphi \in \mathcal{M}_*^+$ ,  $\psi \neq 0$ . For every  $z \in (0, 1]$  we have*

$$\lim_{\alpha \nearrow 1} D_{\alpha,z}(\psi\|\varphi) = D_1(\psi\|\varphi).$$

Anna (Dec. 7, 2023)

**Theorem 5.4.** *Let  $\psi, \varphi \in \mathcal{M}_*^+$ ,  $\psi \neq 0$ , and  $z > 1/2$ . Assume that  $D_{\alpha,z}(\psi\|\varphi) < \infty$  for some  $\alpha \in (1, 2z]$ . Then we have*

$$\lim_{\alpha \searrow 1} D_{\alpha,z}(\psi\|\varphi) = D_1(\psi\|\varphi).$$

Anna (Jan. 23, 2024)

## A Haagerup $L_p$ -spaces

The following lemmas are well known, proofs are given for completeness.

**Lemma A.1.** *For any  $0 < p < \infty$  and  $\varphi \in \mathcal{M}_*^+$ ,  $h_\varphi^{\frac{1}{2p}} \mathcal{M}^+ h_\varphi^{\frac{1}{2p}}$  is dense in  $L_p(\mathcal{M})^+$  with respect to the (quasi)-norm  $\|\cdot\|_p$ .*

*Proof.* We may assume that  $\varphi$  is faithful. By [13, Lemma 1.1],  $\mathcal{M} h_\varphi^{\frac{1}{2p}}$  is dense in  $L_{2p}(\mathcal{M})$  for any  $0 < p < \infty$ . Let  $y \in L_p(\mathcal{M})^+$ , then  $y^{\frac{1}{2}} \in L_{2p}(\mathcal{M})$ , hence there is a sequence  $a_n \in \mathcal{M}$  such that  $\|a_n h_\varphi^{\frac{1}{2p}} - y^{\frac{1}{2}}\|_{2p} \rightarrow 0$ . Then also

$$\|h_\varphi^{\frac{1}{2p}} a_n^* - y^{\frac{1}{2}}\|_p = \|(a_n h_\varphi^{\frac{1}{2p}} - y^{\frac{1}{2}})^*\|_p = \|a_n h_\varphi^{\frac{1}{2p}} - y^{\frac{1}{2}}\|_p \rightarrow 0$$

and

$$\|h_\varphi^{\frac{1}{2p}} a_n^* a_n h_\varphi^{\frac{1}{2p}} - y\|_p = \|(h_\varphi^{\frac{1}{2p}} a_n^* - y^{\frac{1}{2}}) a_n h_\varphi^{\frac{1}{2p}} + y^{\frac{1}{2}} (a_n h_\varphi^{\frac{1}{2p}} - y^{\frac{1}{2}})\|_p$$

Since  $\|\cdot\|_p$  is a (quasi)-norm, the above expression goes to 0 by the Hölder inequality.  $\square$

**Lemma A.2.** *Let  $0 < p \leq \infty$  and let  $h, k \in L_p(\mathcal{M})^+$  be such that  $h \leq k$ . Then  $\|h\|_p \leq \|k\|_p$ . Moreover, if  $1 \leq p < \infty$ , then*

$$\|k - h\|_p^p \leq \|k\|_p^p - \|h\|_p^p.$$

*Proof.* The first statement follows from [4, Lemma 2.5 (iii) and Lemma 4.8]. The second statement is from [4, Lemma 5.1].  $\square$

**Lemma A.3.** *Let  $\psi, \varphi \in \mathcal{M}_*^+$  with  $\psi \leq \varphi$ . Then for any  $a \in \mathcal{M}$  and  $p \in [1, \infty)$ ,*

$$\mathrm{Tr} \left( (a^* h_\psi^{\frac{1}{p}} a)^p \right) \leq \mathrm{Tr} \left( (a^* h_\varphi^{\frac{1}{p}} a)^p \right)$$

*Proof.* Since  $1/p \in (p, 1]$ , it follows (see [8, Lemma B.7] and [9, Lemma 3.2]) that  $h_\psi^{1/p} \leq h_\varphi^{1/p}$ . Hence  $a^* h_\psi^{1/p} a \leq a^* h_\varphi^{1/p} a$ . Therefore, by Lemma A.2, we have the statement.  $\square$

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