Incompatibility in GPT and tensor norms

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General probabilistic theories: definition

(L. Lami, arXiv:1803.02902)

A GPT: $(V, V^+, \mathbb{1})$:

- V a finite dimensional real vector space
- $V^+ \subset V$ positive cone: proper convex cone

  (closed, pointed: $V^+ \cap -V^+ = \{0\}$ and generating: $V = V^+ - V^+$)
- $\mathbb{1}$ unit effect: a strictly positive functional on (V, V^+).

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$$K = \{\rho \in V^+,\ \langle \mathbb{1}, \rho \rangle = 1\}.$$

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$$K = \{\rho \in V^+,\ \langle \mathbb{1}, \rho \rangle = 1\}.$$

- a compact convex set
- a base of the cone V^+.

General probabilistic theories: duality and norms

The dual ordered space: (A, A^+)

- $A = V^*$ dual vector space
- $A^+ = (V^+)^*$ the dual cone: positive functionals

$$(V^+)^* := \{f \in A,\ \langle f, v \rangle \geq 0,\ \forall v \in V^+\}$$

- $\mathbb{1} \in \mathrm{int}(A^+)$ - order unit in A

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Base norm in V:

$$\|v\|_V := \inf\{\langle \mathbb{1}, v_+ + v_-\rangle :\ v = v^+ - v^-,\ v_\pm \in V^+\} = \|v\|_A^*.$$

General probabilistic theories: basic examples

Classical systems

- $V = \mathbb{R}^d$
- $V^+ = \mathbb{R}^d_+ = \{(x_1, \dots, x_d),\ x_i \geq 0\}$ simplicial cone
- $\mathbb{1} = (1, 1, \dots, 1)$

Classical state space: probability simplex

$$\Delta_d = \{(p_1, \dots, p_d),\ p_i \geq 0,\ \sum_i p_1 = 1\}$$

General probabilistic theories: basic examples

Quantum systems

- $V = B^{sa}(\mathcal{H})$ self-adjoint operators on a Hilbert space, $\dim(\mathcal{H}) < \infty$
- $V^+ = B^+(\mathcal{H})$ positive operators
- $\mathbb{1} = I$ identity operator

Quantum state space: set of density operators

$$D(\mathcal{H}) = \{\rho \in B^+(\mathcal{H}),\ \mathrm{Tr}\,\rho = 1\}$$

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Centrally symmetric GPT: $K \cong B_{\|\cdot\|}$ the unit ball in $(\mathbb{R}^{d-1}, \|\cdot\|)$

$$V \cong \mathbb{R}^d, \quad V^+ \cong \{(\alpha, x),\ \|x\| \leq \alpha\}, \quad \mathbb{1} \cong (1,0)$$
$$A \cong \mathbb{R}^d, \quad A^+ \cong \{(\beta, y),\ \|y\|^* \leq \beta\} \ - \ \text{the dual norm}$$

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- Cubic models: $K \cong B_{\|\cdot\|_\infty} = [-1, 1]^{d-1}$ - hypercube
- Spherical models: $K \cong B_{\|\cdot\|_2}$ - standard Euclidean ball

General probabilistic theories: effects and measurements

Effects: $f \in A^+$, $0 \leq f \leq \mathbb{1}$.

- map states to probabilities:

$$K \ni \rho \mapsto \langle f, \rho \rangle \in [0,1]$$

- binary measurements

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Measurements with outcomes in $[k] = \{1, \dots, k\}$:

- $f_1, \dots, f_k$ effects,

$$\langle f_i, \rho \rangle - \text{ probability of outcome } i \text{ in the state } \rho \in K$$

- $\sum_i f_i = \mathbb{1}$

Compatible effects in a GPT

A tuple $f = (f_1, \ldots, f_g)$ of effects (binary measurements) is compatible if all f_i are marginals of a joint measurement:

a measurement h with outcomes in $[2^g] \equiv [2]^g$ such that

$$f_i = \sum_{j_1,\ldots,j_{i-1},j_{i+1},\ldots,j_g} h_{j_1,\ldots,j_{i-1},1,j_{i+1},\ldots,j_g}$$

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Previous works:

- (A. Bluhm, I. Nechita, JMP 2018): quantum effects
- (AJ, PRA 2018): GPT setting

Tensor cross norms in Banach spaces

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. A (reasonable) cross norm is a norm $\|\cdot\|_\alpha$ on $X \otimes Y$ such that

- $\|x \otimes y\|_\alpha = \|x\|_X \|y\|_Y$, for all $x \in X$, $y \in Y$
- $\|x^* \otimes y^*\|_{\alpha^*} = \|x^*\|_{X^*} \|y^*\|_{Y^*}$, for all $x^* \in X^*$, $y^* \in Y^*$.

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Equivalently,

$$\|\cdot\|_\epsilon \leq \|\cdot\|_\alpha \leq \|\cdot\|_\pi,$$

- $\|\cdot\|_\epsilon$ - the injective cross norm
- $\|\cdot\|_\pi$ - the projective cross norm.

Effects and tensor cross norms

Fix a GPT $(V, V^+, \mathbb{1})$, $f = (f_1, \dots, f_g)$, $f_i \in A$. Put

$$\varphi_f := \sum_i e_i \otimes (2f_i - \mathbb{1}) \in \mathbb{R}^g \otimes A.$$

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An easy observation:

All f_i are effects if and only if

$$1 \geq \max_i \|2f_i - \mathbb{1}\|_A = \|\varphi_f\|_\epsilon$$

$\|\cdot\|_\epsilon$ - injective cross norm of $\ell_\infty^g = (\mathbb{R}^g, \|\cdot\|_\infty)$ and $(A, \|\cdot\|_A)$.

Compatibility and tensor cross norms

We introduce another norm in $\mathbb{R}^g \otimes A$:

$$\|\varphi\|_\alpha = \inf\{\|\sum_j h_j\|_A,\ \varphi = \sum_j z_j \otimes h_j,\ \|z_j\|_\infty = 1,\ h_j \in A^+\}.$$

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Characterization of compatibility

f is a compatible g-tuple of effects if and only if $\|\varphi_f\|_\alpha \leq 1$.

Compatibility region and degree

(P. Busch et al., EPL 2013; T. Heinosaari et al., J. Phys. A 2016)

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- different definitions by the choice of noise
- we choose trivial effects $\mathbb{1}/2$.

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$$f^s = (s_1 f_1 + (1 - s_1)\mathbb{1}/2, \ldots, s_g f_g + (1 - s_g)\mathbb{1}/2).$$

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Compatibility region for g effects in $(V, V^+, \mathbb{1})$:

$$\Gamma(g; V, V^+) = \cap_f \Gamma(f) = \{s \in [0,1]^g : \ f_s \text{ is compatible for all } f\}$$

Compatibility region and degree

Compatibility degree: diagonal elements in $\Gamma(f)$

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Compatibility degree for g effects:

$$\begin{aligned} \gamma(g; V, V^+) &= \min_f \gamma(f) \\ &= \max\{s \in [0,1] : \ (s, \ldots, s) \in \Gamma(g; V, V^+)\}. \end{aligned}$$

Compatibility region and degree

Each $\Gamma(f)$ is a convex set between Δ_{g+1} and the hypercube $[0,1]^g$:

- $\Gamma = [0,1]^g$ iff K is a simplex
- $\Gamma = \Delta_{g+1}$ iff $\exists$ a retraction $K \to [0,1]^g$

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We will give a dimension dependent lower bound on γ.

Compatibility region and inclusion constants

$f = (f_1, \ldots, f_g)$ effects, f_s the noisy effects, $s \in [0,1]^g$:

$$\varphi_{f_s} = s.\varphi_f = \sum_i e_i \otimes s_i(2f_i - \mathbb{1}).$$

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$$\Gamma(g; V, V^+) = \text{the set of inclusion constants for} \| \cdot \|_\epsilon \text{ and } \| \cdot \|_\alpha$$

$$= \{s \in [0,1]^g : \frac{\|s.\varphi\|_\alpha}{\|\varphi\|_\epsilon} \leq 1, \forall \varphi \in \mathbb{R}^g \otimes A\}$$

Compatibility degree and cross norms

Characterization of compatibility degree

1. For any g-tuple of effects, we have
$$\gamma(f) = 1/\|\varphi_f\|_\alpha$$
2. In any GPT $(V, V^+, \mathbb{1})$ and $g \in \mathbb{N}$, we have
$$\gamma(g; V, V^+) = \min_{\varphi \in \mathbb{R}^g \otimes A} \frac{\|\varphi\|_\epsilon}{\|\varphi\|_\alpha}$$
3. We have the following lower bound
$$\gamma(g; V, V^+) \geq \min_{\varphi \in \mathbb{R}^g \otimes A} \frac{\|\varphi\|_\epsilon}{\|\varphi\|_\pi} \geq \max\{1/g, 1/\dim(V)\}$$

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In the quantum case:

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There is some literature around the inclusion constants for these sets (related to operator systems, cp maps, free spectrahedra, etc.)

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A. Bluhm, I. Nechita, JMP 2018, arxiv:1807.01508)

In the quantum case:

Some results: for compatibility region and degree

1. $QC_q := \{s \in [0,1]^g,\ \sum_i s_i^2 \leq 1\} \subseteq \Gamma(g, \mathcal{H})$

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2. $\gamma(g, \mathcal{H}) \geq \max\{g^{-1/2}, 1/2n\}$
3. equality holds for large enough $n = \dim(\mathcal{H})$:

$$n \geq 2^{(g-1)/2}$$

Centrally symmetric GPTs

Let $(V, V^+, \mathbb{1})$ be centrally symmetric: for some $(\mathbb{R}^{d-1}, \|\cdot\|)$:

$$V = \mathbb{R} \oplus \mathbb{R}^{d-1}, \quad V^+ = \{(a, x),\ \|x\| \le a\}, \quad \mathbb{1} = (1, 0)$$

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Unbiased effects: $(1/2, \bar{f})$, $\|f\|^* \leq 1/2$.

Centrally symmetric GPTs: unbiased effects

Let $f = ((1/2, \bar{f}_1), \ldots, (1/2, \bar{f}_g))$ - a collection of unbiased effects:

$$\varphi_f = \sum_i e_i \otimes (0, 2\bar{f}_i) = 0 \oplus \sum_i e_i \otimes \bar{2} f_i \in \mathbb{R}^g \otimes \mathbb{R}^{d-1}$$

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Then

$$\|\varphi_f\|_\alpha = \|\varphi_f\|_\pi = \|\varphi_f\|_{\bar{\pi}}$$

$\|\cdot\|_{\bar{\pi}}$ is the projective cross norm of ℓ_∞^g and $(\mathbb{R}^{d-1}, \|\cdot\|^*)$.

Maximal incompatibility for tuples of effects in a centrally symmetric GPT is always attained on a tuple of unbiased effects.

Centrally symmetric GPT: compatibility region

Let $\|\cdot\|_{\bar{\epsilon}}$ denote the injective cross norm of ℓ_∞^g and $(\mathbb{R}^{d-1}, \|\cdot\|^*)$.

1. **Compatibility region:**

$$\Gamma(g; \|\cdot\|) = \{s \in [0,1]^g : \max_{\varphi \in \mathbb{R}^g \otimes \mathbb{R}^{d-1}} \frac{\|s.\varphi\|_{\bar{\pi}}}{\|\varphi\|_{\bar{\epsilon}}} \leq 1\}$$

2. **Compatibility degree:**

$$\gamma(g; \|\cdot\|) = \min_{\varphi \in \mathbb{R}^g \otimes \mathbb{R}^{d-1}} \frac{\|\varphi\|_{\bar{\epsilon}}}{\|\varphi\|_{\bar{\pi}}} \geq \max\{1/g, 1/(d-1)\}$$

3. **Tight** lower bound

Centrally symmetric GPTs: Examples

1. For the cubic model:

$$\Gamma(g; \|\cdot\|_\infty) = \{s \in [0,1]^g, \ \forall I \subseteq [g], \ |I| \le d-1, \ \sum_{i \in I} s_i \le 1\}$$

$$\gamma(g; \|\cdot\|_\infty) = \max\{1/g, 1/(d-1)\}.$$

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2. For the spherical model: (qubits)

$$\Gamma(g; \|\cdot\|_2) \supseteq \{s \in [0,1]^g,\ \sum_i s_i^2 \leq 1\}$$

$$\gamma(g; \|\cdot\|_2) \geq \max\{1/\sqrt{g}, 1/(d-1)\}$$

with equalities if $g \leq d-1$.

Incompatibility witnesses

A (strict) incompatibility witness: a linear functional

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- $\langle z, \varphi_f \rangle \leq 1$ for all compatible f;
- $\langle z, \varphi_f \rangle > 1$ for some f (which then must be incompatible).

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Compatibility degree:

$$\gamma(g; V, V^+) = 1 / \max_{\|z\|_\alpha^* \leq 1} \sum_i \|z_i\|_V$$

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Small compatibility degree $\equiv$ large hypercubes contained in V^+.

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- $$\max_\rho \sum_i \|\rho^{1/2} X_i \rho^{1/2}\|_1 > 1.$$

Incompatibility witnesses: centrally symmetric GPT

Let $(V, V^+, \mathbb{1})$ be centrally symmetric: for some $(\mathbb{R}^{d-1}, \|\cdot\|)$:

$$V = \mathbb{R} \oplus \mathbb{R}^{d-1}, \quad V^+ = \{(a, x),\ \|x\| \leq a\}, \quad \mathbb{1} = (1, 0)$$

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- we may restrict to witnesses with $z_i = (0, \bar{z}_i)$ and $z_0 = (1, 0)$
- hypercubes in the unit ball $B_{\|\cdot\|}$, with barycenter at 0.