Generalized channels: Channels for convex subsets of the state space

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Let K be a convex subset of the state space of a finite-dimensional C^*-algebra. We study the properties of channels on K, which are defined as affine maps from K into the state space of another algebra, extending to completely positive maps on the subspace generated by K. We show that each such map is the restriction of a completely positive map on the whole algebra, called a generalized channel. We characterize the set of generalized channels and also the equivalence classes of generalized channels having the same value on K. Moreover, if K contains the tracial state, the set of generalized channels forms again a convex subset of a multipartite state space, this leads to a definition of a generalized supermap, which is a generalized channel with respect to this subset. We prove a decomposition theorem for generalized supermaps and describe the equivalence classes. The set of generalized supermaps having the same value on equivalent generalized channels is also characterized. Special cases include quantum combs and process positive operator valued measures (POVMs). © *2012 American Institute of Physics.* [doi:10.1063/1.3676294]

I. INTRODUCTION

The first motivation for this paper comes from the problem of measurement of a quantum channel. A mathematical framework for such measurements, or more generally, for measurements on quantum networks, was introduced in Ref. 4, in terms of testers.[5] For quantum channels, these were called process POVMs or PPOVMs in Ref. 16. Similar to POVMs, a PPOVM is a collection of positive operators $(F_1, \ldots, F_m)$ in the tensor product of the input and output spaces, but summing up to an operator $I_{\mathcal{H}_1} \otimes \omega$ for some state ω on the input space. The output probabilities of the corresponding channel measurement with values in $\{1, \ldots, m\}$ are then given by

$$p_i(\mathcal{E}) = \mathrm{Tr}\,(M_i X_{\mathcal{E}}), \qquad i = 1, \ldots, m,$$

where $X_{\mathcal{E}}$ is the Choi matrix of the channel $\mathcal{E}$. Via the Choi isomorphism, the set of channels $\mathcal{C}(\mathcal{H}_0, \mathcal{H}_1)$ can be viewed as (a multiple of) an intersection of the set of states in $B(\mathcal{H}_1 \otimes \mathcal{H}_0)$ with a self-adjoint vector subspace J. This is a convex set, and a measurement on channels can be naturally defined as an affine map from this set to the set of probability measures on the set of outcomes.

A natural question arising in this context is the following: Are all such affine maps given by PPOVMs? And if so, is this correspondence one-to-one?

Further, the concept of a quantum supermap was introduced in Ref. 6, which is a map $B(\mathcal{H}_1 \otimes \mathcal{H}_0) \to B(\mathcal{H}_3 \otimes \mathcal{H}_2)$ sending channels to channels. It was argued that such a map should be linear and completely positive. But it is clear that it is enough to consider completely positive maps $J \to B(\mathcal{H}_3 \otimes \mathcal{H}_2)$ sending channels to channels. We may then ask whether all such maps extend to a completely positive map on $B(\mathcal{H}_1 \otimes \mathcal{H}_0)$, and if this extension is unique.

Supermaps on supermaps were defined similarly, these are the so-called quantum combs, which are used in description of quantum networks.[4,8] It was proved that all quantum combs can be

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represented by memory channels, which are given by a sequence of channels connected by an ancilla, these form the "teeth" of the comb. The theory of quantum combs was subsequently used for optimal cloning[7] and learning[2] of unitary transformations and measurements.[3] As it turns out, the set of all N-combs forms again (a multiple of) an intersection of the set of multipartite states by a vector subspace.

To deal with these questions in full generality, we introduce the notion of a channel on a convex subset K of the state space, which is an affine map from K into another state space, extending to a completely positive map on the vector subspace generated by K. In order to include all channels, POVMs and instruments, and other similar objects, we work with finite-dimensional C^*-algebras rather than matrix algebras. We show that each such map can be extended to a completely positive map on the whole algebra, these maps are called generalized channels (with respect to K). Further, a measurement on K is defined as an affine map from K into the set of probability distributions and it is shown that each such measurement is given by a (completely) positive map on the whole algebra if and only if K is a section of the state space, that is, an intersection of the set of states by a linear subspace. This special kind of a generalized channel is called a generalized POVM.

We describe the equivalence class of generalized channels restricting to the same channel on K. Moreover, we show that if K contains the tracial state, the set of generalized channels, via Choi representation, is again (a multiple of) a section of some state space, so that we may apply our results on the set of generalized channels themselves and repeat the process infinitely. This leads to the definition of a generalized supermap. We show that the quantum combs and testers are particular cases of generalized supermaps, other examples treated here include channels and measurements on POVMs and PPOVMs, and supermaps on instruments. We also describe channels on the set of states having the same output probabilities for a POVM or a finite number of POVMs.

The outline of the paper is as follows: After Sec. II, we consider extensions of completely positive maps on subspaces of the algebra and of positive affine functions on K. If the subspace is self-adjoint and generated by its positive elements, then a consequence of Arveson's extension theorem shows that any completely positive map can be extended to the whole algebra. For positive functionals on K, we show that these extend to positive linear functionals on the whole algebra if and only if K is a section of the state space. These results are used in Sect. IV for extension theorems for channels and measurements on K. We characterize the generalized channels with respect to K and their equivalence classes. We show that a generalized channel can be decomposed to a so-called simple generalized channel and a channel.

In Sec. V, we prove that the set of generalized channels is again a section of a state space and introduce the generalized supermaps. We give a characterization of generalized supermaps as sections of a multipartite state space and show that the quantum combs are a particular case. We prove a decomposition theorem for the generalized supermaps, similar to the realization of quantum combs by memory channels proved in Ref. 8. In particular, we show that a generalized comb can be decomposed as a simple generalized channel and a comb. Finally, we describe the equivalence classes for generalized supermaps and consider the set of supermaps having the same value on equivalence classes.

II. PRELIMINARIES

Let $\mathcal{A}$ be a finite-dimensional C^*-algebra. Then $\mathcal{A}$ is isomorphic to a direct sum of matrix algebras, that is, there are finite-dimensional Hilbert spaces $\mathcal{H}_1, \ldots \mathcal{H}_n$, such that

$$\mathcal{A} \equiv \bigoplus_j B(\mathcal{H}_j).$$

Below we always assume that $\mathcal{A}$ has this form, so that $\mathcal{A}$ is a subalgebra of block-diagonal elements in the matrix algebra $B(\mathcal{H})$, with $\mathcal{H} = \oplus_j \mathcal{H}_j$. The identity in $\mathcal{A}$ will be denoted by $I_{\mathcal{A}}$. We fix a trace $\mathrm{Tr}_{\mathcal{A}}$ on $\mathcal{A}$ to be the restriction of the trace in $B(\mathcal{H})$, we omit the subscript $\mathcal{A}$ if no confusion is possible. If $\mathcal{A} = B(\mathcal{H})$ is a matrix algebra, then we write $I_{\mathcal{H}}$ and $\mathrm{Tr}_{\mathcal{H}}$ instead of $I_{B(\mathcal{H})}$ and $\mathrm{Tr}_{B(\mathcal{H})}$. We will sometimes use the notation $\mathcal{H}_A$, $\mathcal{H}_B$, etc., for the Hilbert spaces, and $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, $\mathrm{Tr}_A = \mathrm{Tr}_{\mathcal{H}_A}$, $I_A = I_{\mathcal{H}_A}$.

If $\mathcal{B}$ is another C*-algebra, then $\mathrm{Tr}_{\mathcal{A}}^{\mathcal{A}\otimes\mathcal{B}}$ will denote the partial trace on the tensor product $\mathcal{A}\otimes\mathcal{B}$, $\mathrm{Tr}_{\mathcal{A}}(a\otimes b)=\mathrm{Tr}(a)b$. If the input space is clear, we will sometimes denote the partial trace just by $\mathrm{Tr}_{\mathcal{A}}$.

For $a\in\mathcal{A}$, we denote by a^T the transpose of a. Note that $\mathrm{Tr}_{\mathcal{A}}^{\mathcal{A}\otimes\mathcal{B}}(x^T)=(\mathrm{Tr}_{\mathcal{A}}^{\mathcal{A}\otimes\mathcal{B}}x)^T$ for $x\in\mathcal{A}\otimes\mathcal{B}$. If $A\subset\mathcal{A}$, then $A^T=\{a^T,a\in A\}$.

We denote by $\mathcal{A}^h$ the set of all self-adjoint elements in $\mathcal{A}$, $\mathcal{A}^+$ the convex cone of positive elements in $\mathcal{A}$ and $\mathfrak{S}(\mathcal{A})$ the set of states on $\mathcal{A}$, which will be identified with the set of density operators in $\mathcal{A}$, that is, elements $\rho\in\mathcal{A}^+$ with $\mathrm{Tr}\,\rho=1$. If $\rho\in\mathfrak{S}(\mathcal{A})$ is invertible, then we say that ρ is a faithful state. The projection onto the support of ρ will be denoted by $\mathrm{supp}\,(\rho)$. If $\mathcal{A}=B(\mathcal{H})$, then we denote the set of states by $\mathfrak{S}(\mathcal{H})$. Let $\tau_{\mathcal{A}}$ denote the tracial state $t_{\mathcal{A}}^{-1}I_{\mathcal{A}}$, here $t_{\mathcal{A}}=\mathrm{Tr}\,(I_{\mathcal{A}})$. Later on, we will also need the set $\mathfrak{S}_c(\mathcal{A})=\{a\in\mathcal{A}^+,\mathrm{Tr}\,(ca)=1\}$ for a positive invertible element $c\in\mathcal{A}$, note that $\mathfrak{S}_{I_{\mathcal{A}}}(\mathcal{A})=\mathfrak{S}(\mathcal{A})$.

The trace defines an inner product in $\mathcal{A}$ by $\langle a,b\rangle=\mathrm{Tr}\,(a^*b)$, with this $\mathcal{A}$ becomes a Hilbert space. If $A\subset\mathcal{A}$ is any subset, then $A^\perp$ will denote the orthogonal complement of A. Then $A^{\perp\perp}=:[A]$ is the linear subspace, spanned by A. The subspace spanned by a single element a will be denoted by $[a]$.

Let now $L\subseteq\mathcal{A}$ be a (complex) linear subspace. We denote by L^h the set of self-adjoint elements in L, then L^h is a real vector subspace in $\mathcal{A}^h$. The subspace L is self-adjoint if $a^*\in L$ whenever $a\in L$. In this case $L=L^h\oplus iL^h$. If also $I_{\mathcal{A}}\in L$, then L is called an operator system.[15] If L is generated by positive elements, then we say that L is positively generated. If L_1 and L_2 are subspaces in $\mathcal{A}$, then $L_1\vee L_2$ denotes the smallest subspace containing both L_1 and L_2, and $L_1\wedge L_2=L_1\cap L_2$.

A. Channels, instruments, and POVMs

Let $\mathcal{H},\mathcal{K}$ be finite-dimensional Hilbert spaces. For any linear map $T:B(\mathcal{H})\to B(\mathcal{K})$, there is an element $X_T\in B(\mathcal{K}\otimes\mathcal{H})$, given by

$$X_T:=(T\otimes id_{\mathcal{H}})(\Psi_{\mathcal{H}}),\qquad \Psi_{\mathcal{H}}=\sum_{i,j}|i\rangle\langle j|\otimes|i\rangle\langle j| \tag{1}$$

for $|i\rangle$ a canonical basis in $\mathcal{H}$. Conversely, each operator X in $B(\mathcal{K}\otimes\mathcal{H})$ defines a linear map $T_X:B(\mathcal{H})\to B(\mathcal{K})$ by

$$T_X(a)=\mathrm{Tr}_{\mathcal{H}}[(I_{\mathcal{K}}\otimes a^T)X],\qquad a\in B(\mathcal{H}). \tag{2}$$

It is easy to see that $T_{X_T}=T$ and $X_{T_X}=X$, so that the two maps are each other's inverses. The matrix X_T is called the Choi matrix of T. We have the following:

(i) T is completely positive (cp) if and only if $X_T\geq 0$.[11]
(ii) T is trace-preserving if and only if $\mathrm{Tr}_{\mathcal{K}}X_T=I_{\mathcal{H}}$.

Let now $\mathcal{A}=\oplus_i B(\mathcal{H}_i)$ and $\mathcal{B}=\oplus_j B(\mathcal{K}_j)$ be finite-dimensional C*-algebras. For any linear map $T:\mathcal{A}\to\mathcal{B}$ there are linear maps $T_{ij}:B(\mathcal{H}_i)\to B(\mathcal{K}_j)$ such that $T(a_i)=\oplus_j T_{ij}(a_i),a_i\in B(\mathcal{H}_i)$. It is clear that T is a cp map if and only if all T_{ij} are cp maps. Put

$$X_T:=\oplus_{i,j}X_{T_{ij}}\in\mathcal{B}\otimes\mathcal{A}. \tag{3}$$

Then it is easy to see that Eq. (2) and both (i) and (ii) hold with $\mathcal{H}=\oplus_i\mathcal{H}_i$ and $\mathcal{K}=\oplus_j\mathcal{K}_j$ (hence, we may replace $\mathrm{Tr}_{\mathcal{H}}$ and $\mathrm{Tr}_{\mathcal{K}}$ by $\mathrm{Tr}_{\mathcal{A}}$ and $\mathrm{Tr}_{\mathcal{B}}$, similarly for $I_{\mathcal{H}}$ and $I_{\mathcal{K}}$). The matrix X_T is again called the Choi matrix of T.

Next we describe instruments and POVMs as special kinds of channels. Let $\mathcal{K}_j\equiv\mathcal{K}$ for all $j=1,\ldots,m$, so that $\mathcal{B}=\mathbb{C}^m\otimes B(\mathcal{K})$. Then a channel $T:\mathcal{A}\to\mathcal{B}$ is called an instrument $\mathcal{A}\to B(\mathcal{K})$, with values in $\{1,\ldots,m\}$.[14] Note that T is a channel if and only if T_{ij} are cp maps, such that for each i, $T_i:=\sum_j T_{ij}$ is a channel $B(\mathcal{H}_i)\to B(\mathcal{K})$. The Choi matrix of an instrument has the form $X_T=\oplus_i\sum_{j=1}^m|j\rangle\langle j|\otimes X_{ij}$, with

$$\mathrm{Tr}_{\mathcal{B}}X_T=\oplus_i\sum_j\mathrm{Tr}_{\mathcal{K}}X_{ij}=\oplus_i I_{\mathcal{H}_i}=I_{\mathcal{H}}=I_{\mathcal{A}}.$$

Let us now suppose that $\mathcal{K} = \mathbb{C}$, then $\mathcal{B}$ is the commutative C*-algebra $\mathcal{B} = \mathbb{C}^m$. A channel $T : \mathcal{A} \to \mathcal{B}$ maps states onto probability distributions, hence it is given by a POVM $M_1, \ldots, M_m \in \mathcal{A}^+$, $\sum_k M_k = I_{\mathcal{A}}$ as

$$T(a) = (\operatorname{Tr} M_1 a, \ldots, \operatorname{Tr} M_m a). \tag{4}$$

The Choi matrix is $X_T = \sum_k |k\rangle\langle k| \otimes M_k^T$, with $\operatorname{Tr}_{\mathcal{B}} X_T = \sum_j M_j^T = I_{\mathcal{A}}$.

1. The link product

Let $\mathcal{H}_i$ be Hilbert spaces, for $i = 1, 2, \ldots$ and let $\mathcal{M} \subset \mathbb{N}$ be a finite set of indices. We denote $\mathcal{H}_{\mathcal{M}} := \bigotimes_{i \in \mathcal{M}} \mathcal{H}_i$. Let $\mathcal{N} \subseteq \mathbb{N}$ be another finite set and let $X \in \mathcal{H}_{\mathcal{M}}$, $Y \in \mathcal{H}_{\mathcal{N}}$ be any operators. The link product of X and Y was defined in Ref. 8 as the operator $X * Y \in B(\mathcal{H}_{\mathcal{M} \backslash \mathcal{N}} \otimes \mathcal{H}_{\mathcal{N} \backslash \mathcal{M}})$, given by

$$X * Y = \operatorname{Tr}_{\mathcal{M} \cap \mathcal{N}}[(I_{\mathcal{M} \backslash \mathcal{N}} \otimes Y^{T_{\mathcal{M} \cap \mathcal{N}}})(X \otimes I_{\mathcal{N} \backslash \mathcal{M}})], \tag{5}$$

where $T_{\mathcal{M} \cap \mathcal{N}}$ is the partial transpose on the space $\mathcal{H}_{\mathcal{M} \cap \mathcal{N}}$. In particular, $X * Y = X \otimes Y$, if $\mathcal{M} \cap \mathcal{N} = \emptyset$ and $X * Y = \operatorname{Tr}(Y^T X)$, if $\mathcal{M} = \mathcal{N}$.

Proposition 1 (Ref. 8): The link product has the following properties.

1. *(Associativity): Let $\mathcal{M}_i$, $i = 1, 2, 3$ be sets of indices, such that $\mathcal{M}_1 \cap \mathcal{M}_2 \cap \mathcal{M}_3 = \emptyset$. Then for $X_i \in \mathcal{H}_{\mathcal{M}_i}$,*

$$(X_1 * X_2) * X_3 = X_1 * (X_2 * X_3).$$

2. *(Commutativity): Let $X \in \mathcal{H}_{\mathcal{M}}$, $Y \in \mathcal{H}_{\mathcal{N}}$, then*

$$Y * X = E(X * Y)E,$$

   where E is the unitary swap on $\mathcal{H}_{\mathcal{M} \backslash \mathcal{N}} \otimes \mathcal{H}_{\mathcal{N} \backslash \mathcal{M}}$.
3. *(Positivity): If X and Y are positive, then $X * Y$ is positive.*

The interpretation of the link product is the following: If $X \in B(\mathcal{H}_1 \otimes \mathcal{H}_0)$ and $Y \in B(\mathcal{H}_2 \otimes \mathcal{H}_1)$ are the Choi matrices of maps $T_X : B(\mathcal{H}_0) \to B(\mathcal{H}_1)$ and $T_Y : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$, then $X * Y$ is the Choi matrix of their composition $T_Y \circ T_X$. For $X \in B(\mathcal{H}_1)$, we have

$$Y * X = T_Y(X). \tag{6}$$

Let now $X \in \mathcal{H}_{\mathcal{M}}$ be a multipartite operator and let $\mathcal{I} \cup \mathcal{O} = \mathcal{M}$ be a partition of $\mathcal{M}$, then X defines a linear map $\Phi_{X;\mathcal{I},\mathcal{O}} : \mathcal{H}_{\mathcal{I}} \to \mathcal{H}_{\mathcal{O}}$, by

$$\Phi_{X;\mathcal{I},\mathcal{O}}(a_{\mathcal{I}}) = \operatorname{Tr}_{\mathcal{H}_{\mathcal{I}}}(I_{\mathcal{H}_{\mathcal{O}}} \otimes a_{\mathcal{I}}^T)X, \qquad a_{\mathcal{I}} \in \mathcal{H}_{\mathcal{I}}. \tag{7}$$

As it was emphasized in Ref. 8, X is the Choi matrix of many different maps, depending on how we choose the input and output spaces $\mathcal{I}$ and $\mathcal{O}$. The flexibility of the link product is in that it accounts for these possibilities. For example, let $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_0$ and $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{M}_0$ be partitions of $\mathcal{M}$ and $\mathcal{N}$. Put $\Phi_X := \Phi_{X;\mathcal{M}_1,\mathcal{M}_0 \cup \mathcal{M}_2}$ and $\Phi_Y := \Phi_{Y;\mathcal{N}_1 \cup \mathcal{M}_0,\mathcal{N}_2}$. Then $X * Y$ is the Choi matrix of the map $B(\mathcal{H}_{\mathcal{M}_1 \cup \mathcal{N}_1}) \to B(\mathcal{H}_{\mathcal{M}_2 \cup \mathcal{N}_2})$, given by

$$\Phi_{Y*X;\mathcal{M}_1 \cup \mathcal{N}_1,\mathcal{M}_2 \cup \mathcal{N}_2} = (\Phi_Y \otimes id_{\mathcal{M}_2}) \circ (id_{\mathcal{N}_1} \otimes \Phi_X).$$

In the case when the input and output spaces are fixed, we will often treat a cp map and its Choi matrix as one and the same object, to shorten the discussion.

III. EXTENSIONS OF CP MAPS AND POSITIVE FUNCTIONALS

The main goal of this paper is to study cp maps and channels from a convex subset K of the state space into another C*-algebra. To characterize such maps, it is crucial to know whether or when these can be extended to cp maps on the whole algebra. This section contains an extension theorem

for cp maps on a vector subspace. We also prove that positive affine functionals on K have positive extensions if and only if K is a section, that is an intersection of the state space by a vector subspace.

A. An extension theorem for cp maps

Let $J \subseteq \mathcal{A}$ be a subspace and let $\mathcal{K}$ be a finite-dimensional Hilbert space. Let $\mathcal{B} \subseteq B(\mathcal{K})$ be a C^*-algebra.

A map $\Xi : J \to \mathcal{B}$ is positive, if it maps $J \cap \mathcal{A}^+$ into the positive cone $\mathcal{B}^+$ and Ξ is completely positive, if the map

$$id_{\mathcal{K}_0} \otimes \Xi : B(\mathcal{K}_0) \otimes J \to B(\mathcal{K}_0) \otimes \mathcal{B}$$

is positive, for every finite-dimensional Hilbert space $\mathcal{K}_0$. If J is an operator system, that is a self-adjoint subspace containing the unit, then Arveson's extension theorem[1,15] states that any completely positive map $\Xi : J \to B(\mathcal{K})$ can be extended to a cp map $\mathcal{A} \to B(\mathcal{K})$.

The following is a consequence of this theorem in finite dimensions.

Theorem 1: *Let $J \subseteq \mathcal{A}$ be a self-adjoint positively generated subspace. Then any cp map $J \to \mathcal{B}$ can be extended to a cp map $\mathcal{A} \to \mathcal{B}$.*

Proof: Let $J^+ = J \cap \mathcal{A}^+$, so that J is generated by J^+. There is some $\rho \in J^+$ such that the support of ρ contains the supports of all other elements in J^+. Let us denote $p := \text{supp}(\rho)$, then J is a subspace in the algebra $\mathcal{A}_p := p\mathcal{A}p$. Denote

$$\Delta : \mathcal{A}_p \to \mathcal{A}_p, \qquad \Delta(a) = \rho^{1/2} a \rho^{1/2}.$$

Then $J' := \Delta^{-1}(J)$ is an operator system in $\mathcal{A}_p$. Moreover, $\Xi : J \to \mathcal{B}$ is a cp map if and only if $\Xi' := \Xi \circ \Delta$ is a cp map $J' \to \mathcal{B} \subseteq B(\mathcal{K})$. By Arveson's extension theorem, Ξ' can be extended to a cp map $\Phi' : \mathcal{A}_p \to B(\mathcal{K})$. Let $E_\mathcal{B} : B(\mathcal{K}) \to \mathcal{B}$ be the trace preserving conditional expectation, then $\Phi := E_\mathcal{B} \circ \Phi' \circ \Delta^{-1}$ is a cp map $\mathcal{A}_p \to \mathcal{B}$ extending Ξ. This can be obviously extended to $\mathcal{A}$ □

B. Sections of the state space

Let f be an affine function $\mathfrak{S}(\mathcal{A}) \to \mathbb{R}^+$. Then, since $\mathfrak{S}(\mathcal{A})$ generates the positive cone $\mathcal{A}^+$, f can be extended to a positive linear functional on $\mathcal{A}$. Below we discuss the possibility of such extension if f is defined on some convex subset $K \subset \mathfrak{S}(\mathcal{A})$. Let us first describe a special type of such subset.

Let $K \subseteq \mathfrak{S}(\mathcal{A})$ be a convex subset and let Q be the convex cone generated by K, then $Q = \{\lambda K, \lambda \geq 0\} \subseteq \mathcal{A}^+$. The vector subspace $[K]$ generated by K is self-adjoint and $[K] = Q - Q + i(Q - Q)$.

We say that K is a section of $\mathfrak{S}(\mathcal{A})$, if

$$K = [K] \cap \mathfrak{S}(\mathcal{A}). \tag{8}$$

It is clear that a section of $\mathfrak{S}(\mathcal{A})$ is convex and compact. It is also clear that (8) is equivalent with

$$Q = [K] \cap \mathcal{A}^+. \tag{9}$$

Sections of the state space can be characterized as follows.

Proposition 2: Let $K \subset \mathfrak{S}(\mathcal{A})$ be a compact convex subset and let $Q = \{\lambda K, \lambda \geq 0\}$. Then K is the section of $\mathfrak{S}(\mathcal{A})$ if and only if $a, b \in Q$, and $b \leq a$ implies $a - b \in Q$.

Proof: Since we always have $Q \subseteq [K] \cap \mathcal{A}^+$, it is enough consider the inclusion $[K] \cap \mathcal{A}^+ \subseteq Q$. But $[K] \cap \mathcal{A}^+ = (Q - Q) \cap \mathcal{A}^+$ and hence any element $y \in [K] \cap \mathcal{A}^+$ has the form $y = a - b$ with $a, b \in Q$, and $b \leq a$.   □

Proposition 3: Let $K \subseteq \mathfrak{S}(\mathcal{A})$. Then K is a section of $\mathfrak{S}(\mathcal{A})$ if and only if there is a subspace $J \subseteq \mathcal{A}$, such that $K = J \cap \mathfrak{S}(\mathcal{A})$.

Proof: If K is a section of $\mathfrak{S}(\mathcal{A})$, then we can put $J = [K]$. Conversely, let $K = J \cap \mathfrak{S}(\mathcal{A})$ for some subspace $J \subseteq \mathcal{A}$. Then $Q = J \cap \mathcal{A}^+$ and if $a, b \in Q$ with $b \leq a$, then obviously $a - b \in J \cap \mathcal{A}^+ = Q$. By Proposition 2, K is a section of $\mathfrak{S}(\mathcal{A})$.   □

Note that if $K = J \cap \mathfrak{S}(\mathcal{A})$ for some subspace J, we do not necessarily have $J = [K]$, even if J is self-adjoint. The next proposition clarifies this situation.

Proposition 4: Let $J \subseteq \mathcal{A}$ be a self-adjoint subspace and let $K = J \cap \mathfrak{S}(\mathcal{A}) \neq \emptyset$. Then there is a projection $p \in \mathcal{A}$, such that $[K] = J \cap \mathcal{A}_p$. In particular, $J = [K]$ if J contains a positive invertible element.

Proof: Suppose first that J contains a positive invertible element ρ and let $K = J \cap \mathfrak{S}(\mathcal{A})$, equivalently, $Q = J \cap \mathcal{A}^+$. Since $\mathcal{A}$ is finite-dimensional, for any $a \in J^h$, there is some $M > 0$, such that $a \leq M\rho$, and then

$$a = M\rho - (M\rho - a) \in Q - Q.$$

This implies $J^h = Q - Q$ and since J is self-adjoint, $J = [K]$.

For the general case, choose some state $\rho \in K$ such that its support contains the supports of all $\sigma \in K$, so that $K \subseteq \mathcal{A}_p$, where $p := \mathrm{supp}(\rho)$. Then $J_p := J \cap \mathcal{A}_p$ is a subspace in $\mathcal{A}_p$, containing the positive invertible element ρ and $K = J_p \cap \mathfrak{S}(\mathcal{A}_p)$. Hence, by the first part of the proof, $[K] = J_p$.   □

C. Positive affine functions on *K*

Let $A(K)$ be the vector space of real affine functions and $A(K)^+$ the convex cone of positive affine functions over K. In this paragraph, we study elements in $A(K)^+$ that can be extended to a positive affine functional on $\mathfrak{S}(\mathcal{A})$, hence, are given by positive elements in $\mathcal{A}$.

Any element in $A(K)$ extends to a (unique) real linear functional on $[K]^h$ and conversely, any linear functional on $[K]^h$ defines an element in $A(K)$, so that

$$A(K) \equiv ([K]^h)^* \equiv \mathcal{A}^h|_{K^\perp} := \{a + K^\perp, \ a \in \mathcal{A}^h\}.$$

In other words, any element $\phi \in A(K)$ has the form $\phi(\sigma) = \mathrm{Tr}\, a\sigma$ for some $a \in \mathcal{A}^h$ and two elements $a_1, a_2 \in \mathcal{A}^h$ define the same $\phi \in A(K)$ if and only if $a_1 = a_2 + x$ for some $x \in K^\perp$.

Let $\pi_{K^\perp} : a \mapsto a + K^\perp$ be the quotient map. Then it is clear that $\pi_{K^\perp}(\mathcal{A}^+) \subseteq A(K)^+$. We are interested in the converse. Note that if $\bar{K}$ is the closure of K, then $\bar{K}$ is convex and $K^\perp = \bar{K}^\perp$, $[K] = [\bar{K}]$ and $A(K) = A(\bar{K})$, $A(K)^+ = A(\bar{K})^+$.

Theorem 2: *Let $K \subseteq \mathfrak{S}(\mathcal{A})$ be a nonempty convex subset. Then $A(K)^+ = \pi_{K^\perp}(\mathcal{A}^+)$ if and only if $\bar{K}$ is a section of $\mathfrak{S}(\mathcal{A})$.*

Proof: It is clear by the remark preceding the theorem that we may suppose that K is closed.

Let K be a section of $\mathfrak{S}(\mathcal{A})$, then any positive affine function on K extends to a positive linear functional on $[K]$. Since positive functionals are completely positive and $[K]$ is positively generated, the assertion follows by Theorem 1.

Conversely, suppose that K is not a section of $\mathfrak{S}(\mathcal{A})$. Then there is some $x \in [K] \cap \mathcal{A}^+$, such that $x \notin Q$. Since Q is closed and convex, by Hahn-Banach separation theorem there is a linear functional f on $\mathcal{A}^h$, such that $f(x) < s < \inf\{f(a), a \in Q\}$, for some $s \in \mathbb{R}$. This implies that $s < f(0) = 0$ and, moreover, $\lambda f(\sigma) > s$ for all $\lambda \geq 0$, $\sigma \in K$, hence $f(\sigma) \geq 0$ and f defines an element $\phi \in A(K)^+$. But ϕ has a unique extension to $[K]$, namely, f and $f(x) < s < 0$, so that ϕ cannot be given by an element in $\mathcal{A}^+$.   □

IV. GENERALIZED CHANNELS

Let $K \subseteq \mathfrak{S}(\mathcal{A})$ be a convex set and let $\Xi : K \to \mathcal{B}^+$ be an affine map. Then Ξ extends to a linear map $[K] \to \mathcal{B}$. (Note that in general, this extension does not need to be positive.) We will say that Ξ is a cp map on K, if this extension of Ξ is completely positive. If Ξ also preserves trace (equivalently, $\Xi(K) \subseteq \mathfrak{S}(\mathcal{B})$), then Ξ will be called a channel on K.

Remark 1: Note that by this definition, Ξ is a cp map (resp. channel) on K if and only if (the extension of) Ξ is a cp map (resp. channel) on $\tilde{K} := [K] \cap \mathfrak{S}(\mathcal{A})$, the smallest section of $\mathfrak{S}(\mathcal{A})$ containing K. Therefore, without any loss of generality we may suppose that K is a section of $\mathfrak{S}(\mathcal{A})$.

Theorem 3: *Let $K \subseteq \mathfrak{S}(\mathcal{A})$ be a convex subset. Then any cp map on K has a cp extension to $\mathcal{A}$. If $\Phi : \mathcal{A} \to \mathcal{B}$ is a cp map, then Φ defines a channel on K if and only if its Choi matrix satisfies*

$$\mathrm{Tr}_\mathcal{B} X_\Phi \in I_\mathcal{A} + (K^T)^\perp. \tag{10}$$

Two cp maps $\Phi_1, \Phi_2 : \mathcal{A} \to \mathcal{B}$ define the same cp map on K if and only if

$$X_{\Phi_1} - X_{\Phi_2} \in \mathcal{B} \otimes (K^T)^\perp. \tag{11}$$

Proof: Since $[K]$ is positively generated, the first statement follows from Theorem 1. The map Φ defines a channel on K if and only if $\mathrm{Tr}\,(\Phi(a)) = 1$ for all $a \in K$, that is,

$$\mathrm{Tr}\,(a^T) = 1 = \mathrm{Tr}\,(\Phi(a)) = \mathrm{Tr}\,((I_\mathcal{B} \otimes a^T)X_\Phi) = \mathrm{Tr}\,(a^T \mathrm{Tr}_\mathcal{B} X_\Phi), \qquad a \in K,$$

equivalently, $\mathrm{Tr}_\mathcal{B} X_\Phi \in I_\mathcal{A} + (K^T)^\perp$. Furthermore, Φ_1 and Φ_2 have the same value on K if and only if

$$\mathrm{Tr}\,(b(\Phi_1(a) - \Phi_2(a))) = \mathrm{Tr}\,(b \otimes a^T)(X_{\Phi_1} - X_{\Phi_2}) = 0, \qquad \forall a \in K, \ b \in \mathcal{B},$$

that is, $X_{\Phi_1} - X_{\Phi_2} \in (\mathcal{B} \otimes K^T)^\perp = \mathcal{B} \otimes (K^T)^\perp$. $\qquad\square$

Any cp map $\Phi : \mathcal{A} \to \mathcal{B}$, satisfying (10) will be called a generalized channel. Two generalized channels having the same value on K will be called equivalent. If we want to stress the set K (or the subspace $[K]$), we will say that Φ is a generalized channel with respect to K (or $[K]$).

We will next introduce an example that will be used repeatedly throughout the paper. Let $\mathcal{A}_0$ be a finite-dimensional C*-algebra and let $S : \mathcal{A} \to \mathcal{A}_0$, $T : \mathcal{A}_0 \to \mathcal{A}$ be completely positive maps. Let $J_0 \subseteq \mathcal{A}_0$ be a self-adjoint vector subspace. Then $S^{-1}(J_0) = \{a \in \mathcal{A}, \ S(a) \in J_0\}$, and $T(J_0)$ are self-adjoint subspaces in $\mathcal{A}$. In particular, if $J_0 = [S(\rho)]$ is the one-dimensional subspace generated by $S(\rho)$ for some $\rho \in \mathfrak{S}(\mathcal{A})$, then $S^{-1}(J_0) \cap \mathfrak{S}(\mathcal{A})$ is the equivalence class containing ρ for the equivalence relation on $\mathfrak{S}(\mathcal{A})$ induced by S.

Lemma 1: Let $S : \mathcal{A} \to \mathcal{A}_0$ be a cp map and let J_0 be a subspace in $\mathcal{A}_0$. Then $S^{-1}(J_0)^\perp = S^*(J_0^\perp)$, where $S^* : \mathcal{A}_0 \to \mathcal{A}$ is the adjoint of S with respect to $\langle a, b \rangle = \mathrm{Tr}\,(a^*b)$.

Proof: Let $a \in \mathcal{A}$, then $\mathrm{Tr}\,(a^*S^*(b)) = \mathrm{Tr}\,(S(a^*)b) = \mathrm{Tr}\,(S(a)^*b) = 0$ for all $b \in J_0^\perp$ if and only if $S(a) \in J_0$, this implies that $S^*(J_0^\perp)^\perp = S^{-1}(J_0)$, so that $S^{-1}(J_0)^\perp = S^*(J_0^\perp)$. $\qquad\square$

We denote by S^T the linear map $\mathcal{A} \to \mathcal{A}_0$, defined by $S^T(a) = [S(a^T)]^T$. Note that the Choi matrix of S^T satisfies $X_{S^T} = X_S^T$, so that S is a channel if and only if S^T is a channel.

Lemma 2: Let $S : \mathcal{A} \to \mathcal{A}_0$ be a channel and let $J_0 \subseteq \mathcal{A}_0$ be a subspace. Let $J = S^{-1}(J_0)$. Then

(i) $(J^T)^\perp = (S^T)^*((J_0^T)^\perp)$,
(ii) $I_\mathcal{A} + (J^T)^\perp = (S^T)^*(I_{\mathcal{A}_0} + (J_0^T)^\perp)$.

Proof: We have

$$S^{-1}(J_0)^T = \{a, \ S(a^T) \in J_0\} = \{a, \ S^T(a) \in J_0^T\} = (S^T)^{-1}(J_0^T),$$

(i) now follows by Lemma 1 and (ii) follows from the fact that S^T is a channel, so that $(S^T)^*$ is unital. $\qquad\square$

Example 1 (Channels on channels): Let $\mathcal{A} = \mathcal{B}_1 \otimes \mathcal{B}_0$, $\mathcal{A}_0 = \mathcal{B}_0$ and let $S : \mathcal{B}_1 \otimes \mathcal{B}_0 \to \mathcal{B}_0$ be the partial trace $\mathrm{Tr}_{\mathcal{B}_1}$. Let $J_0 = [I_{\mathcal{B}_0}] = \mathbb{C}I_{\mathcal{B}_0}$. The set

$$\mathcal{C}(\mathcal{B}_0, \mathcal{B}_1) := \mathrm{Tr}_{\mathcal{B}_1}^{-1}([I_{\mathcal{B}_0}]) \cap t_{\mathcal{B}_0}\mathfrak{S}(\mathcal{B}_1 \otimes \mathcal{B}_0) \tag{12}$$

is the set of all Choi matrices of channels $\mathcal{B}_0 \to \mathcal{B}_1$. Denote $J := \mathrm{Tr}_{\mathcal{B}_1}^{-1}([I_{\mathcal{B}_0}])$ and $K = J \cap \mathfrak{S}(\mathcal{B}_1 \otimes \mathcal{B}_0)$, then K is a section of the state space and $\mathcal{C}(\mathcal{B}_0, \mathcal{B}_1) = t_{\mathcal{B}_0}K$. It follows that Ξ is a channel on K if and only if $t_{\mathcal{B}_0}^{-1}\Xi$ is a channel on $\mathcal{C}(\mathcal{B}_0, \mathcal{B}_1)$. Hence, any channel $\mathcal{C}(\mathcal{B}_0, \mathcal{B}_1) \to \mathfrak{S}(\mathcal{B})$ is given by a cp map $\Phi : \mathcal{B}_1 \otimes \mathcal{B}_0 \to \mathcal{B}$, such that $\mathrm{Tr}_\mathcal{B}X_\Phi \in t_{\mathcal{B}_0}^{-1}I_{\mathcal{B}_1 \otimes \mathcal{B}_2} + (K^T)^\perp$.

Since $I_{\mathcal{B}_1 \otimes \mathcal{B}_0} \in J$, we have $J = [K]$ by Proposition 4, so that $(K^T)^\perp = (J^T)^\perp$. Note also that $S^T = S$ and $S^*(a) = I_{\mathcal{B}_1} \otimes a$ for $a \in \mathcal{B}_0$. By Lemma 2,

$$(K^T)^\perp = I_{\mathcal{B}_1} \otimes [I_{\mathcal{B}_0}]^\perp$$

and taking into account that $X_\Phi \geq 0$, we get

$$\mathrm{Tr}_\mathcal{B}X_\Phi \in [I_{\mathcal{B}_1} \otimes (\tau_{\mathcal{B}_0} + [I_{\mathcal{B}_0}]^\perp)] \cap (\mathcal{B}_1 \otimes \mathcal{B}_0)^+ = I_{\mathcal{B}_1} \otimes \mathfrak{S}(\mathcal{B}_0). \tag{13}$$

Moreover, Φ_1 and Φ_2 are equivalent if and only if

$$X_{\Phi_1} - X_{\Phi_2} = I_{\mathcal{B}_1} \otimes Y, \quad Y \in \mathcal{B} \otimes \mathcal{B}_0, \ \mathrm{Tr}_{\mathcal{B}_0}Y = 0. \qquad\square$$

Example 2 (Channels on POVMs): Put $\mathcal{B}_1 = \mathbb{C}^m$ in example 1, then $\mathcal{C}(\mathcal{B}_0, \mathbb{C}^m)$ is the set of all POVMs on $\mathcal{B}_0$, with values in $\{1, \ldots, m\}$. If $\Phi : \mathcal{B}_1 \otimes \mathcal{B}_0 \to \mathcal{B}$ is a cp map, then the Choi matrix has the form $X_\Phi = \sum_{j=1}^m |j\rangle\langle j| \otimes X_j$, $X_j \in (\mathcal{B} \otimes \mathcal{B}_0)^+$. The condition (13) becomes

$$X_\Phi = \sum_{j=1}^m |j\rangle\langle j| \otimes X_j, \quad \mathrm{Tr}_\mathcal{B}X_j = \omega \ \forall j, \quad \omega \in \mathfrak{S}(\mathcal{B}_0), \tag{14}$$

and Φ_1 and Φ_2 are equivalent if and only if $X_{\Phi_i} = \sum_j |j\rangle\langle j| \otimes X_{ij}$, $i = 1, 2$, with

$$X_{1j} - X_{2j} = Y \ \forall j, \quad Y \in \mathcal{B} \otimes \mathcal{B}_0, \quad \mathrm{Tr}_{\mathcal{B}_0}Y = 0. \qquad\square$$

Example 3: Let $\mathcal{A} = B(\mathcal{H})$ and let $E = (E_1, \ldots, E_k)$ be a POVM on $B(\mathcal{H})$. Then E defines a channel $S_E : B(\mathcal{H}) \to \mathbb{C}^k$ by $a \mapsto (\mathrm{Tr}(E_1a), \ldots, \mathrm{Tr}(E_ka))$. Let ρ be a faithful state and let $S_E(\rho) = \lambda = (\lambda_1, \ldots, \lambda_k)$. Let $J = S_E^{-1}([\lambda])$ and let

$$K = J \cap \mathfrak{S}(\mathcal{A}) = \{\sigma \in \mathfrak{S}(\mathcal{H}), \ \mathrm{Tr}(\sigma E_i) = \lambda_i, i = 1, \ldots, k\}.$$

We have $S_E^T = S_{E^T}$ and $(S_E^T)^*(x) = \sum_i x_i E_i^T$ for $x \in \mathbb{C}^k$, and since $\rho \in J$ is invertible, $(K^T)^\perp = (J^T)^\perp = S_{E^T}^*([\lambda]^\perp)$, by Lemma 2. It follows that channels $K \to \mathfrak{S}(\mathcal{B})$ are given by cp maps $\Phi : B(\mathcal{H}) \to \mathcal{B}$, such that

$$\mathrm{Tr}_\mathcal{B}X_\Phi = \sum_i c_i E_i^T, \qquad \sum_i c_i \lambda_i = 1.$$

Note that if E is a projection valued measure, then E^T is a PVM as well and positivity of X_Φ implies that we must have $c_i \geq 0$ for all i. Moreover, Φ_1 and Φ_2 are equivalent if and only if

$$X_{\Phi_1} - X_{\Phi_2} = \sum_j y_j \otimes E_j^T, \quad y_j \in \mathcal{B}, \ \sum_j \lambda_j y_j = 0.$$

More generally, let $E^i = (E_1^i, \ldots, E_{k_i}^i), i = 1, \ldots, n$ be POVMs. Put $J = \cap_i J_i$, for $J_i = S_{E^i}^{-1}([\lambda^i])$, with $\lambda_j^i = \mathrm{Tr}(E_j^i\rho), j = 1, \ldots, k_i, i = 1, \ldots, n$, and

$$K = J \cap \mathfrak{S}(\mathcal{A}) = \{\sigma \in \mathfrak{S}(\mathcal{A}), \ \mathrm{Tr}(\sigma E_j^i) = \lambda_j^i, \ j = 1, \ldots, k_i, \ i = 1, \ldots, n\}.$$

Again, $\rho \in J$, so that

$$(K^T)^\perp = (J^T)^\perp = (\cap_i J_i^T)^\perp = \vee_i(J_i^T)^\perp = \vee_i S_{E_i^T}^*([\lambda^i]^\perp).$$

It follows that channels $K \to \mathfrak{S}(\mathcal{B})$ are given by cp maps $\Phi : B(\mathcal{H}) \to \mathcal{B}$, satisfying

$$\mathrm{Tr}_\mathcal{B} X_\Phi = \sum_{i=1}^n \sum_{j=1}^{k_i} d_j^i (E_j^i)^T, \quad \sum_{i,j} d_j^i \lambda_j^i = 1,$$

and Φ_1, Φ_2 are equivalent if and only if

$$X_{\Phi_1} - X_{\Phi_2} = \sum_{ij} y_{ij} \otimes (E_j^i)^T, \quad y_{ij} \in \mathcal{B}, \quad \sum_j y_{ij} \lambda_j^i = 0, \ \forall i. \qquad \square$$

A. Measurements and instruments on *K*

Let $\mathcal{B} = \mathbb{C}^m \otimes B(\mathcal{K}_1)$ and let $\Phi : \mathcal{A} \to \mathcal{B}$ be a generalized channel with respect to K. Then there are cp maps $\Phi_j : \mathcal{A} \to B(\mathcal{K}_1), j = 1, \ldots, m$, such that $\Phi(a) = \sum_j |j\rangle\langle j| \otimes \Phi_j(a)$. Since

$$1 = \mathrm{Tr}\,(\Phi(a)) = \sum_j \mathrm{Tr}\,(\Phi_j(a)), \qquad a \in K,$$

$\sum_j \Phi_j$ is a generalized channel with respect to K. In this case, we will say that Φ is a generalized instrument with respect to K.

In particular, let $\mathcal{B} = \mathbb{C}^m$, then any cp map $\Phi : \mathcal{A} \to \mathcal{B}$ has the form (4) with some positive elements $M_j \in \mathcal{A}$ and the Choi matrix is $X_\Phi = \sum_j |j\rangle\langle j| \otimes M_j^T$. Then Φ is a generalized channel with respect to K if and only if

$$\sum_j M_j = \mathrm{Tr}_\mathcal{B} X_\Phi^T \in I_\mathcal{A} + K^\perp. \tag{15}$$

Any such collection of positive operators will be called a generalized POVM (with respect to K). If M and N are generalized POVMs, then they are equivalent if and only if

$$M_j - N_j \in K^\perp, \quad \forall j. \tag{16}$$

Now let K be any convex subset of $\mathfrak{S}(\mathcal{A})$. A measurement on K with values in a finite set X is naturally defined as an affine map from K to the set of probability measures on X. It is clear that any generalized POVM with respect to K defines a measurement on K by

$$p_j(a) = \mathrm{Tr}\,(M_j a), \quad j \in X, \qquad a \in K.$$

Conversely, any measurement on K is given by a collection of functions $\lambda_i \in A(K)^+$, $i \in X$, such that $\sum_i \lambda_i = 1$ (here 1 is the function identically 1 on K). Each λ_i is given by some element $M_i \in \mathcal{A}^h$, such that $\sum_i M_i \in I_\mathcal{A} + K^\perp$. By Theorem 2, all M_i can be chosen positive, and hence form a generalized POVM, if and only if λ extends to a measurement on the section $\tilde{K}$, see Remark 1. If K is a section of $\mathfrak{S}(\mathcal{A})$, then measurements on K are precisely the equivalence classes of generalized POVMs. If K is not a section, then Theorem 2 implies that there are measurements on K that cannot be obtained by a generalized POVM.

Example 4 (PPOVMs): Let $\mathcal{B}_0 = B(\mathcal{H}_0)$, $\mathcal{B}_1 = B(\mathcal{H}_1)$, and $\mathcal{B} = \mathbb{C}^k$ in example 1. Let us denote $\mathcal{C}(\mathcal{H}_0, \mathcal{H}_1) := \mathcal{C}(\mathcal{B}_0, \mathcal{B}_1)$ in this case. Since this is (a multiple of) a section of $\mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_0)$, measurements on $\mathcal{C}(\mathcal{H}_0, \mathcal{H}_1)$ are given by generalized POVMs. A collection $(M_1, \ldots, M_m)$ of operators $M_i \in B(\mathcal{H}_1 \otimes \mathcal{H}_0)^+$ is a generalized POVM with respect to $\mathcal{C}(\mathcal{H}_0, \mathcal{H}_1)$ if and only if

$$\sum_j M_j = I_{\mathcal{H}_1} \otimes \omega, \qquad \omega \in \mathfrak{S}(\mathcal{H}_0).$$

Note that these are exactly the quantum 1-testers,[5] also called process POVMs, or PPOVMs, in Ref. 16. Moreover, two PPOVMs, M and N, are equivalent if and only if

$$M_j - N_j = I_{\mathcal{H}_1} \otimes y_j, \quad \mathrm{Tr}\,(y_j) = 0, \ \forall j.$$

Similarly, if we put $\mathcal{B}_0 = B(\mathcal{H}_0)$, $\mathcal{B}_1 = \mathbb{C}^m$, and $\mathcal{B} = \mathbb{C}^k$, we get that any measurement on the set $\mathcal{C}(B(\mathcal{H}_0), \mathbb{C}^m)$ has the form $(M_1, \ldots, M_k)$, with

$$M_j = \sum_{i=1}^m |i\rangle\langle i| \otimes M_{ij}, \ M_{ij} \in B(\mathcal{H}_0)^+, \quad \sum_j M_{ij} = \omega \in \mathfrak{S}(\mathcal{H}_0), \ \forall i$$

and M and N define the same measurement if and only if

$$M_{ij} - N_{ij} = y_j, \ \forall i, \quad \mathrm{Tr}\,(y_j) = 0, \ \forall j. \qquad \qquad \square$$

B. Decomposition of generalized channels

Let $c \in \mathcal{A}^+$. We denote $\chi_c: a \mapsto c^{1/2}ac^{1/2}$. Then χ_c is a completely positive map $\mathcal{A} \to \mathcal{A}$ and χ_c defines a channel on K if and only if $\mathrm{Tr}\,(\chi_c(a)) = \mathrm{Tr}\,(ac) = 1$, that is, $\mathrm{Tr}\,((I_\mathcal{A} - c)a) = 0$ for all $a \in K$. This shows that χ_c is a generalized channel if and only if

$$c \in \bigcap_{\sigma \in K} \mathfrak{S}_\sigma(\mathcal{A}) = (I_\mathcal{A} + K^\perp) \cap \mathcal{A}^+.$$

Such generalized channels with respect to K will be called simple.

Proposition 5: Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a generalized channel with respect to K. Then there is a pair (χ, Λ), with $\chi = \chi_c$ a simple generalized channel with respect to K and $\Lambda : \mathcal{A} \to \mathcal{B}$ a channel, such that

$$\Phi = \Lambda \circ \chi.$$

Conversely, each such pair defines a generalized channel. If in each pair (χ, Λ) we take the restriction $\Lambda|_{\mathcal{A}_p}$ with $p = supp\,(c)$, then the correspondence is one-to-one.

Proof: Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a generalized channel. Then $\mathrm{Tr}_\mathcal{B} X_\Phi \in (I_\mathcal{A} + (K^T)^\perp) \cap \mathcal{A}^+$ or equivalently,

$$\Phi^*(I_\mathcal{B}) \in (I_\mathcal{A} + K^\perp) \cap \mathcal{A}^+.$$

Put $c = \Phi^*(I_\mathcal{B})$ and let $p = \mathrm{supp}\,(c)$. Then since $b \leq \|b\| I_\mathcal{B}$ for $b \in \mathcal{B}^+$, we have $\Phi^*(b) \leq \|b\| c \leq \|b\| \|c\| p$. This implies that $p\Phi^*(b)p = \Phi^*(b)p = \Phi^*(b)$ for all $b \in \mathcal{B}^+$, and hence for all $b \in \mathcal{B}$, so that Φ^* maps $\mathcal{B}$ into $\mathcal{A}_p$. It follows that $\chi_{c^{-1}} \circ \Phi^*$ is well defined and unital map $\mathcal{B} \to \mathcal{A}_p$. Let Λ_p be the adjoint map, $\Lambda_p = \Phi \circ \chi_{c^{-1}}$, then Λ_p is a channel $\mathcal{A}_p \to \mathcal{B}$ and $\Phi = \Lambda_p \circ \chi_c$.

The channel Λ_p can be extended to a channel $\Lambda : \mathcal{A} \to \mathcal{B}$ as

$$\Lambda(a) = \Lambda_p(a) + \omega \mathrm{Tr}\, a(1 - p), \qquad a \in \mathcal{A},$$

where $\omega \in \mathcal{B}$ is any state, and $\Phi = \Lambda \circ \chi_c$. The converse is quite obvious.

Suppose now that there are (χ_i, Λ_i), $i = 1, 2$, such that $\Phi_1 := \Lambda_1 \circ \chi_1 = \Lambda_2 \circ \chi_2 =: \Phi_2$. Let $\chi_i = \chi_{c_i}$. Then since $\Phi_i^*(I_\mathcal{B}) = c_i$, we have $c_1 = c_2 =: c$ and $\chi_1 = \chi_2 =: \chi$. Let $p := \mathrm{supp}\, c$. But then it is clear that if Λ_i are defined on $\mathcal{A}_p$, then we must have $\Lambda_i = \Phi \circ \chi_c^{-1}$. $\qquad \square$

We apply this result to the set of channels on $\mathcal{C}(\mathcal{H}_0, \mathcal{H}_1)$, see example 1.

Theorem 4: *For any channel $\Xi : \mathcal{C}(\mathcal{H}_0, \mathcal{H}_1) \to \mathfrak{S}(\mathcal{B})$, there exists an ancillary Hilbert space $\mathcal{H}_A$, a pure state $\rho \in B(\mathcal{H}_0 \otimes \mathcal{H}_A)$ and a channel $\Lambda : B(\mathcal{H}_1 \otimes \mathcal{H}_A) \to \mathcal{B}$, such that*

$$\Xi(X_\mathcal{E}) = \Lambda \circ (\mathcal{E} \otimes id_{\mathcal{H}_A})(\rho), \qquad \mathcal{E} \in \mathcal{C}(\mathcal{H}_0, \mathcal{H}_1). \qquad (17)$$

Conversely, let $\mathcal{H}_A$ be an ancillary Hilbert space and let $\rho \in B(\mathcal{H}_0 \otimes \mathcal{H}_A)$ be a state. Let $\Lambda : B(\mathcal{H}_1 \otimes \mathcal{H}_A) \to \mathcal{B}$ be a channel. Then (17) defines a channel $\mathcal{C}(\mathcal{H}_0, \mathcal{H}_1) \to \mathfrak{S}(\mathcal{B})$.

Proof: By example 1 and Proposition 5,

$$\Phi = \Lambda \circ \chi_{I_{\mathcal{H}_1} \otimes \omega}$$

with $\omega \in \mathfrak{S}(\mathcal{H}_0)$ and $\Lambda : B(\mathcal{H}_1 \otimes p\mathcal{H}_0) \to B(\mathcal{K})$ a channel, $p = \text{supp}\,\omega$. Let now $\mathcal{E} : B(\mathcal{H}_0) \to B(\mathcal{H}_1)$ be a channel. Then we have

$$\chi_{I\otimes\omega}(X_\mathcal{E}) = (I_{\mathcal{H}_1} \otimes \omega^{1/2})(\mathcal{E} \otimes id_{\mathcal{H}_0})(\Psi_{\mathcal{H}_0})(I_{\mathcal{H}_1} \otimes \omega^{1/2}) = (\mathcal{E} \otimes id_{p\mathcal{H}_0})(\rho),$$

where $\rho = (I_{\mathcal{H}_0} \otimes \omega^{1/2})\Psi_{\mathcal{H}_0}(I_{\mathcal{H}_0} \otimes \omega^{1/2})$ is a pure state in $B(\mathcal{H}_0 \otimes p\mathcal{H}_0)$. Then (17) holds with $\mathcal{H}_A = p\mathcal{H}_0$.

To prove the converse, let $\mathcal{R} : B(\mathcal{H}_0) \to B(\mathcal{H}_A)$ be the cp map with Choi matrix ρ, then $\rho = (id_{\mathcal{H}_0} \otimes \mathcal{R})(\Psi_{\mathcal{H}_0})$. We have

$$(\mathcal{E} \otimes id_{\mathcal{H}_A})(\rho) = (\mathcal{E} \otimes id_{\mathcal{H}_A})(id_{\mathcal{H}_0} \otimes \mathcal{R})(\Psi_{\mathcal{H}_0}) = (id_{\mathcal{H}_1} \otimes \mathcal{R})(\mathcal{E} \otimes id_{\mathcal{H}_0})(\Psi_{\mathcal{H}_0}).$$

Put $\Phi = \Lambda \circ (id_{\mathcal{H}_1} \otimes \mathcal{R})$, then Φ is a cp map $B(\mathcal{H}_1 \otimes \mathcal{H}_0) \to \mathcal{B}$ and

$$\Phi^*(I_\mathcal{B}) = (id_{\mathcal{H}_1} \otimes \mathcal{R}^*)(I_{\mathcal{H}_1 \otimes \mathcal{H}_A}) = I_{\mathcal{H}_1} \otimes \omega,$$

where $\omega = \mathcal{R}^*(I_{\mathcal{H}_A}) = \text{Tr}_{\mathcal{H}_A}\rho^T$ is a state in $B(\mathcal{H}_0)$. $\qquad\square$

Note that the analog to the above theorem for PPOVMs was proved in Ref. 16.

V. GENERALIZED SUPERMAPS

Quantum supermaps were defined in Ref. 6 as completely positive map transforming a quantum operation to another quantum operation. More generally, supermaps on supermaps, or quantum combs, were introduced in Ref. 4. In this section, we define generalized supermaps as channels on generalized channels and show the relation to quantum combs.

Let $J \subseteq \mathcal{A}$ be a self-adjoint subspace. Denote by $\tilde{J}$ the vector subspace generated by $I_\mathcal{A} + (J^T)^\perp$. Then it is easy to see that $\tilde{J}$ is self-adjoint and

$$\tilde{J} = [I_\mathcal{A}] \vee (J^T)^\perp.$$

Lemma 3:

(i)   If $\rho \in J$ is any state, then

$$(I_\mathcal{A} + (J^T)^\perp) \cap \mathcal{A}^+ = \tilde{J} \cap \mathfrak{S}_{\rho^T}(\mathcal{A}).$$

(ii)   If $I_\mathcal{A} \in J$, then $\tilde{\tilde{J}} = J$.
(iii)   If $J = S^{-1}(J_0)$ for a channel $S : \mathcal{A} \to \mathcal{A}_0$ and a self-adjoint subspace $J_0 \subseteq \mathcal{A}_0$, then $\tilde{J} = (S^T)^*(\tilde{J}_0)$.

Proof: (i) An element $x \in \tilde{J}$ has the form $x = cI_\mathcal{A} + x_0$, where $x_0 \in (J^T)^\perp$ and $c = \text{Tr}\,\rho^T x$ for any state $\rho \in J$, (ii) follows from the fact that if $I_\mathcal{A} \in J$, then

$$\tilde{\tilde{J}} = [I_\mathcal{A}] \vee (\tilde{J}^T)^\perp = [I_\mathcal{A}] \vee ([I_\mathcal{A}]^\perp \wedge J) = J,$$

(iii) follows from Lemma 2. $\qquad\square$

Let K be a section of $\mathfrak{S}(\mathcal{A})$ and let $J = [K]$. We denote by $\mathcal{C}_K(\mathcal{A}, \mathcal{B})$ or $\mathcal{C}_J(\mathcal{A}, \mathcal{B})$ the set of all generalized channels $\mathcal{A} \to \mathcal{B}$ with respect to J. In particular, if $K = \mathfrak{S}(\mathcal{A})$, we get the set of all channels $\mathcal{C}(\mathcal{A}, \mathcal{B})$. An element $\Phi \in \mathcal{C}_J(\mathcal{A}, \mathcal{B})$ will be identified with its Choi matrix $X_\Phi \in \mathcal{B} \otimes \mathcal{A}$. In the next proposition, we characterize the set $\mathcal{C}_J(\mathcal{A}, \mathcal{B})$.

Proposition 6: Let K be a section of $\mathfrak{S}(\mathcal{A})$ and let $J = [K]$. Then

$$\mathcal{C}_J(\mathcal{A}, \mathcal{B}) = \text{Tr}_\mathcal{B}^{-1}(\tilde{J}) \cap \mathfrak{S}_{I_\mathcal{B}\otimes\rho^T}(\mathcal{B} \otimes \mathcal{A}),$$

where ρ is any element in K. In particular, if K contains the tracial state $\tau_\mathcal{A}$, then $\mathcal{C}_J(\mathcal{A}, \mathcal{B}) = \text{Tr}_\mathcal{B}^{-1}(\tilde{J}) \cap t_\mathcal{A}\mathfrak{S}(\mathcal{B} \otimes \mathcal{A})$.

Proof: An element $X \in \mathcal{B} \otimes \mathcal{A}$ is the Choi matrix of a generalized channel with respect to J if and only if X is positive and

$$\mathrm{Tr}_{\mathcal{B}} X \in (I_{\mathcal{A}} + (J^T)^{\perp}) \cap \mathcal{A}^+ = \tilde{J} \cap \mathfrak{S}_{\rho^T}(\mathcal{A}),$$

by Lemma 3 (i), which is equivalent with $\mathrm{Tr}_{\mathcal{B}} X \in \tilde{J}$ and $1 = \mathrm{Tr}\, \rho^T \mathrm{Tr}_{\mathcal{B}} X = \mathrm{Tr}\,(I_{\mathcal{B}} \otimes \rho^T) X$.

If $\tau_{\mathcal{A}} \in K$, then $\mathfrak{S}_{I_{\mathcal{B}} \otimes \tau_{\mathcal{A}}^T}(\mathcal{B} \otimes \mathcal{A}) = t_{\mathcal{A}} \mathfrak{S}(\mathcal{B} \otimes \mathcal{A})$. □

This implies that if K contains the tracial state, then the set of generalized channels forms a constant multiple of a section of the state space $\mathfrak{S}(\mathcal{B} \otimes \mathcal{A})$. Then any cp map that maps $C_J(\mathcal{A}, \mathcal{B})$ to another state space is a constant multiple of a generalized channel. Since the set $\mathrm{Tr}_{\mathcal{B}}^{-1}(\tilde{J})$ always contains the unit, we can repeat the process infinitely. The generalized channels obtained in this way will be called generalized supermaps.

Let $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \ldots$ be finite-dimensional C*-algebras and let K be a section of the state space $\mathfrak{S}(\mathcal{B}_0)$, such that $\tau_{\mathcal{B}_0} \in K$. Let $J = [K]$. We denote by $\mathcal{C}_J(\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_n)$ the set of all cp maps that map $\mathcal{C}_J(\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_{n-1})$ into $\mathfrak{S}(\mathcal{B}_n)$. We further introduce the following notations. Let $\mathcal{A}_n := \mathcal{B}_n \otimes \mathcal{B}_{n-1} \otimes \cdots \otimes \mathcal{B}_0$, $n = 0, 2, \ldots$. Let $S_n : \mathcal{A}_n \to \mathcal{A}_{n-1}$ denote the partial trace $\mathrm{Tr}_{\mathcal{B}_n}^{\mathcal{A}_n}$, $n = 1, 2, \ldots$.

Theorem 5: *We have for $n = 1, 2, \ldots$,*

$$\mathcal{C}_J(\mathcal{B}_0, \ldots, \mathcal{B}_n) = J_n \cap c_n \mathfrak{S}(\mathcal{A}_n),$$

where

$$J_{2k-1} = J_{2k-1}(J, \mathcal{B}_1, \ldots, \mathcal{B}_{2k-1}) := S_{2k-1}^{-1}(S_{2k-2}^*(S_{2k-3}^{-1}(\ldots S_1^{-1}(\tilde{J})\ldots))),$$

$$J_{2k} = J_{2k}(J, \mathcal{B}_1, \ldots, \mathcal{B}_{2k}) := S_{2k}^{-1}(S_{2k-1}^*(S_{2k-2}^{-1}(\ldots S_1^*(J)\ldots))),$$

and $c_n = c_n(J, \mathcal{B}_1, \ldots, \mathcal{B}_{2k-1}) := \Pi_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} t_{\mathcal{B}_{n-1-2l}}$.

Proof: We will prove the statement by induction on n, together with the fact that $J_n = S_n^{-1}(\tilde{J}_{n-1})$ for $n = 1, 2, \ldots$, where we put $J_0 := J$.

For $n = 1$, the statement is proved in Proposition 6 and $J_1 = S_1^{-1}(\tilde{J})$ by definition. Suppose now that this holds for some n. Note that since $\tilde{J}_{n-1}$ contains the unit $I_{\mathcal{A}_{n-1}}$, $J_n = S_n^{-1}(\tilde{J}_{n-1})$ contains the unit as well. Then

$$\mathcal{C}_J(\mathcal{B}_0, \ldots, \mathcal{B}_{n+1}) = \frac{1}{c_n} \mathcal{C}_{J_n}(\mathcal{A}_n, \mathcal{B}_{n+1}) \tag{18}$$

and by Proposition 6,

$$\mathcal{C}_{J_n}(\mathcal{A}_n, \mathcal{B}_{n+1}) = S_{n+1}^{-1}(\tilde{J}_n) \cap t_{\mathcal{A}_n} \mathfrak{S}(\mathcal{A}_{n+1}).$$

Since $S_n^T = S_n$, we have by Lemma 3 (ii) and (iii) that

$$\tilde{J}_n = S_n^*(\tilde{\tilde{J}}_{n-1}) = S_n^*(J_{n-1}), \tag{19}$$

so that $S_{n+1}^{-1}(\tilde{J}_n) = J_{n+1}$. Finally, the proof follows from

$$\frac{t_{\mathcal{A}_n}}{c_n} = \frac{\Pi_{l=0}^n t_{\mathcal{B}_l}}{\Pi_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} t_{\mathcal{B}_{n-1-2l}}} = \Pi_{l=0}^{\lfloor \frac{n}{2} \rfloor} t_{\mathcal{B}_{n-2l}} = c_{n+1}.$$

□

The above theorem can be written in the following form.

Theorem 6: *Let $k := \lfloor \frac{n}{2} \rfloor$. Then $X \in \mathcal{C}_J(\mathcal{B}_0, \ldots, \mathcal{B}_n)$ if and only if there are positive elements $Y^{(m)} \in \mathcal{A}_{n-2m}$ for $m = 0, \ldots, k$, such that*

$$\mathrm{Tr}_{\mathcal{B}_{n-2m}} Y^{(m)} = I_{\mathcal{B}_{n-2m-1}} \otimes Y^{(m+1)}, \ m = 0, \ldots, k-1, \tag{20}$$

$Y^{(0)} := X$, $Y^{(k)} \in \mathcal{C}_J(\mathcal{B}_0, \mathcal{B}_1)$ if $n = 2k + 1$ and $Y^{(k)} \in K$ if $n = 2k$.

Example 5 (Channels on generalized POVMs): Let $X \in \mathcal{C}_J(\mathcal{A}, \mathbb{C}^m, \mathcal{B})$, then X defines a channel on the set $\mathcal{C}_J(\mathcal{A}, \mathbb{C}^m)$ of generalized POVMs. Since $X \in \mathcal{B} \otimes \mathbb{C}^m \otimes \mathcal{A}$, we must have $X = \sum_{j=1}^m |j\rangle\langle j| \otimes X_j$, $X_j \in \mathcal{B} \otimes \mathcal{A}$. By Theorem 6, $\mathrm{Tr}_\mathcal{B} X = I_{\mathbb{C}^m} \otimes X_0$ for some $X_0 \in K$. It follows that if X is positive,

$$X \in \mathcal{C}_J(\mathcal{A}, \mathbb{C}^m, \mathcal{B}) \iff X = \sum_{j=1}^m |j\rangle\langle j| \otimes X_j, \ \mathrm{Tr}_\mathcal{B} X_j = X_0 \in K, \ \forall j. \qquad (21)$$

$\square$

Note that example 2 is a special case of the above example. Another special case is the following:

Example 6 (Channels and measurements on PPOVMs): Let $\mathcal{H}_0, \mathcal{H}_1$ be finite-dimensional Hilbert spaces. Then $\mathcal{C}(B(\mathcal{H}_0), B(\mathcal{H}_1), \mathbb{C}^m)$ is the set of all measurements on $\mathcal{C}(\mathcal{H}_0, \mathcal{H}_1)$ with values in $\{1, \ldots, m\}$, that is, the set of all PPOVMs. By (18),

$$\mathcal{C}(B(\mathcal{H}_0), B(\mathcal{H}_1), \mathbb{C}^m) = \frac{1}{\dim \mathcal{H}_0} \mathcal{C}_{J_1}(B(\mathcal{H}_1 \otimes \mathcal{H}_0), \mathbb{C}^m),$$

so that

$$\mathcal{C}(B(\mathcal{H}_0), B(\mathcal{H}_1), \mathbb{C}^m, \mathcal{B}) = (\dim \mathcal{H}_0)\mathcal{C}_{J_1}(B(\mathcal{H}_1 \otimes \mathcal{H}_0), \mathbb{C}^m, \mathcal{B}),$$

here $J_1 = \mathrm{Tr}_{\mathcal{H}_1}^{-1}([I_{\mathcal{H}_0}])$. By (21), $X \in \mathcal{C}(B(\mathcal{H}_0), B(\mathcal{H}_1), \mathbb{C}^m, \mathcal{B})$ if and only if

$$X = \sum_{j=1}^m |j\rangle\langle j| \otimes X_j, \quad \mathrm{Tr}_\mathcal{B} X_j = X_0 \in \mathcal{C}(B(\mathcal{H}_0), B(\mathcal{H}_1)), \ \forall j.$$

Note that by Theorem 7 below, this also describes all cp maps sending POVMs with values in $\{1, \ldots, m\}$ to channels $B(\mathcal{H}_0) \to \mathcal{B}$.

In particular, by putting $\mathcal{B} = \mathbb{C}^k$, we get that measurements on PPOVMs are given by collections of instruments $\Lambda_j : B(\mathcal{H}_0) \to B(\mathcal{H}_1)$ with values in $\{1, \ldots, k\}$, such that their components $\Lambda_{1j}, \ldots, \Lambda_{kj}$ sum to the same channel, for all $j \in \{1, \ldots, m\}$.

Let now $K = \mathfrak{S}(\mathcal{B}_0)$. Then $J = \mathcal{B}_0$ and $\tilde{J} = [I_{\mathcal{B}_0}]$, so that Proposition 6 gives the usual characterization of the set $\mathcal{C}(\mathcal{B}_0, \mathcal{B}_1)$ of all Choi matrices of channels $\mathcal{B}_0 \to \mathcal{B}_1$. For $n > 1$, we have the characterization in Theorem 6 with $Y^{(k)} \in \mathfrak{S}(\mathcal{B}_0)$ if $n = 2k$ and $\mathrm{Tr}_{\mathcal{B}_1} Y^{(k)} = I_{\mathcal{B}_0}$ for $n = 2k + 1$. Suppose that all $\mathcal{B}_j, j = 0, 1, \ldots$, are matrix algebras, $\mathcal{B}_j = B(\mathcal{H}_j)$. Then, comparing Theorem 6 with the results in Ref. 8, we see that for $n = 2k - 1$, the set $\mathcal{C}(B(\mathcal{H}_0), \ldots, B(\mathcal{H}_n))$ is precisely the set of k-combs on $(\mathcal{H}_0, \ldots, \mathcal{H}_{2k-1})$. We give the definition below and also give an alternative proof of the characterization of quantum combs. Note that a similar characterization was obtained in Ref. 13 for Choi matrices of strategies and co-strategies of quantum games.

A. Quantum combs

Quantum N-combs were defined in Ref. 8 as a tool for description of quantum networks. A quantum 1-comb on $(\mathcal{H}_0, \mathcal{H}_1)$ is the Choi matrix of a channel $B(\mathcal{H}_0) \to B(\mathcal{H}_1)$. A quantum N-comb on $(\mathcal{H}_0, \mathcal{H}_1, \ldots, \mathcal{H}_{2N-1})$ is the Choi matrix of a cp map, transforming $(N-1)$-combs on $(\mathcal{H}_1, \ldots, \mathcal{H}_{2N-2})$ to 1-combs on $(\mathcal{H}_0, \mathcal{H}_{2N-1})$. We use the definition of N-combs with the matrix algebras $B(\mathcal{H}_j)$ replaced by finite-dimensional C*-algebras $\mathcal{B}_j, j = 0, \ldots, 2N - 1$. This corresponds to conditional combs introduced in Ref. 9, which describe quantum networks with classical inputs and outputs. We show below that the N-combs are precisely the generalized supermaps $\mathcal{C}(\mathcal{B}_0, \ldots, \mathcal{B}_{2N-1})$.

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be finite-dimensional C*-algebras and let K be a section of $\mathfrak{S}(\mathcal{A})$, let $J = [K]$. We will describe the set of all cp maps $\mathcal{A} \to \mathcal{C} \otimes \mathcal{B}$ that transform K into the set of all channels $\mathcal{B} \to \mathcal{C}$, this will be denoted by $\mathrm{Comb}_J(\mathcal{A}, \mathcal{B}, \mathcal{C})$. It will be convenient to consider this set as a subset in $\mathcal{C} \otimes \mathcal{A} \otimes \mathcal{B}$.

It is quite clear that if $X \in (\mathcal{C} \otimes \mathcal{A} \otimes \mathcal{B})^+$, then $X \in \mathrm{Comb}_J(\mathcal{A}, \mathcal{B}, \mathcal{C})$ if and only if $X * \rho \in C_J(\mathcal{A}, \mathcal{C})$ for all $\rho \in \mathfrak{S}(\mathcal{B})$, this follows from (6) and from

$$(X * \rho) * a = X * (a \otimes \rho) = (X * a) * \rho$$

for all $\rho \in \mathfrak{S}(\mathcal{B})$ and $a \in K$.

Proposition 7: Suppose $\tau_{\mathcal{A}} \in K$. Then

$$\mathrm{Comb}_J(\mathcal{A}, \mathcal{B}, \mathcal{C}) = \mathcal{C}_{J \otimes \mathcal{B}}(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}).$$

Proof: Let X be a positive element in $\mathcal{C} \otimes \mathcal{A} \otimes \mathcal{B}$. As we already argued above, $X \in \mathrm{Comb}_J(\mathcal{A}, \mathcal{B}, \mathcal{C})$ if and only in $X * \rho \in C_J(\mathcal{A}, \mathcal{C})$ for all $\rho \in \mathfrak{S}(\mathcal{B})$, in other words,

$$\mathrm{Tr}_{\mathcal{C}}(X * \rho) = (\mathrm{Tr}_{\mathcal{C}} X) * \rho \in \tilde{J}, \qquad \rho \in \mathfrak{S}(\mathcal{B}) \tag{22}$$

and, simultaneously,

$$\mathrm{Tr}\,(X * \rho) = \mathrm{Tr}\,(\rho^T [\mathrm{Tr}_{\mathcal{C} \otimes \mathcal{A}} X]) = t_{\mathcal{A}}, \qquad \rho \in \mathfrak{S}(\mathcal{B}), \tag{23}$$

which means that $\mathrm{Tr}_{\mathcal{C} \otimes \mathcal{A}} X = t_{\mathcal{A}} I_{\mathcal{B}}$. Moreover, we can write (22) as

$$0 = \mathrm{Tr}\,[((\mathrm{Tr}_{\mathcal{C}} X) * \rho) a] = \mathrm{Tr}\,[(\mathrm{Tr}_{\mathcal{C}} X)(a \otimes \rho^T)]$$

for all $\rho \in \mathcal{B}$ and $a \in \tilde{J}^{\perp}$, which is the same as $\mathrm{Tr}_{\mathcal{C}} X \in (\tilde{J}^{\perp} \otimes \mathcal{B})^{\perp} = \tilde{J} \otimes \mathcal{B}$. Putting this together, we get $X \in \mathrm{Comb}_J(\mathcal{A}, \mathcal{B}, \mathcal{C})$ if and only if

$$\mathrm{Tr}_{\mathcal{C}} X \in [\tilde{J} \otimes \mathcal{B}] \wedge S_{\mathcal{A}}^{-1}([I_{\mathcal{B}}]), \qquad \mathrm{Tr}\, X = t_{\mathcal{A} \otimes \mathcal{B}},$$

where $S_{\mathcal{A}} := \mathrm{Tr}_{\mathcal{A}}^{\mathcal{A} \otimes \mathcal{B}}$.

Let $Y \in \tilde{J} \otimes \mathcal{B}$, then $Y = \sum_i (t_i I_{\mathcal{A}} + x_i) \otimes b_i$, with $b_i \in \mathcal{B}$ and $x_i \in (J^T)^{\perp}$. Since $\tau_{\mathcal{A}} \in K$, we have $\mathrm{Tr}_{\mathcal{A}} Y = t_{\mathcal{A}} \sum_i t_i b_i$, so that $\mathrm{Tr}_{\mathcal{A}} Y \in [I_{\mathcal{B}}]$ if and only if $Y = c I_{\mathcal{A} \otimes \mathcal{B}} + \sum_i x_i \otimes b_i$ for some $c \in \mathbb{C}$, this implies that

$$Y \in [I_{\mathcal{A} \otimes \mathcal{B}}] \vee ((J^T)^{\perp} \otimes \mathcal{B}) = (J \otimes \mathcal{B})\tilde{\ }.$$

Conversely, let $Y \in (J \otimes \mathcal{B})\tilde{\ }$ and let $\{b_k\}_k$ be a basis in $\mathcal{B}$, such that $b_1 = I_{\mathcal{B}}$. Then there are $x_k \in (J^T)^{\perp}$, such that $Y = c I_{\mathcal{A} \otimes \mathcal{B}} + \sum_k x_k \otimes b_k = \sum_k (t_k I_{\mathcal{A}} + x_k) \otimes b_k$, with $t_1 = c$ and $t_k = 0$ for $k \neq 1$. Hence $Y \in \tilde{J} \otimes \mathcal{B}$ and clearly, $\mathrm{Tr}_{\mathcal{A}} Y \in [I_{\mathcal{B}}]$. This proves that $[\tilde{J} \otimes \mathcal{B}] \wedge S_{\mathcal{A}}^{-1}([I_{\mathcal{B}}]) = (J \otimes \mathcal{B})\tilde{\ }$, so that by Proposition 6,

$$\mathrm{Comb}_J(\mathcal{A}, \mathcal{B}, \mathcal{C}) = \mathrm{Tr}_{\mathcal{C}}^{-1}((J \otimes \mathcal{B})\tilde{\ }) \cap t_{\mathcal{A} \otimes \mathcal{B}} \mathfrak{S}(\mathcal{C} \otimes \mathcal{A} \otimes \mathcal{B}) = \mathcal{C}_{J \otimes \mathcal{B}}(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}).$$

$\square$

Let us now denote by $\mathrm{Comb}(\mathcal{B}_0, \ldots, \mathcal{B}_{2N-1})$ the set of N-combs.

Theorem 7: $\mathrm{Comb}(\mathcal{B}_0, \ldots, \mathcal{B}_{2N-1}) = \mathcal{C}(\mathcal{B}_0, \ldots, \mathcal{B}_{2N-1})$.

Proof: For $N = 1$, the statement is trivial. Suppose that it is true for some N. Let $\hat{\mathcal{A}}_{2N-1} := \mathcal{B}_{2N} \otimes \cdots \otimes \mathcal{B}_1$ and let $\hat{J}_{2N-1} := J_{2N-1}(\mathcal{B}_1, \ldots, \mathcal{B}_{2N})$ and $\hat{c}_{2N-1} = c_{2N-1}(\mathcal{B}_1, \ldots, \mathcal{B}_{2N})$, with the notations from Theorem 5. Then

$$\mathrm{Comb}(\mathcal{B}_1, \ldots, \mathcal{B}_{2N}) = \mathcal{C}(\mathcal{B}_1, \ldots, \mathcal{B}_{2N}) = \hat{J}_{2N-1} \cap \hat{c}_{2N-1} \mathfrak{S}(\hat{\mathcal{A}}_{2N-1}). \tag{24}$$

Next, let $\mathcal{A}_{2N} = \hat{\mathcal{A}}_{2N-1} \otimes \mathcal{B}_0$, $J_{2N} = J_{2N}(\mathcal{B}_0, \ldots, \mathcal{B}_{2N})$, and $c_{2N} = c_{2N}(\mathcal{B}_0, \ldots, \mathcal{B}_{2N})$. Then it is not difficult to see that $J_{2N} = \hat{J}_{2N-1} \otimes \mathcal{B}_0$ and $c_{2N} = \hat{c}_{2N-1}$. By (24) and Proposition 7,

$$\mathrm{Comb}(\mathcal{B}_0, \ldots, \mathcal{B}_{2N+1}) = \frac{1}{\hat{c}_{2N-1}} \mathrm{Comb}_{\hat{J}_{2N-1}}(\hat{\mathcal{A}}_{2N-1}, \mathcal{B}_0, \mathcal{B}_{2N+1}),$$

$$= \frac{1}{c_{2N}} \mathcal{C}_{J_{2N}}(\mathcal{A}_{2N}, \mathcal{B}_{2N+1}),$$

$$= \mathcal{C}(\mathcal{B}_0, \ldots, \mathcal{B}_{2N+1}),$$

the last equality follows from (18).   □

In accordance with this result, the elements in $\mathcal{C}_J(\mathcal{B}_0, \ldots, \mathcal{B}_{2N-1})$ will be called generalized N-combs.

Note that an element $X \in \mathcal{C}(\mathcal{B}_0, \ldots, \mathcal{B}_{2N-1})$ is the Choi matrix of a generalized supermap $\mathcal{B}_{2N-2} \otimes \cdots \otimes \mathcal{B}_0 \to \mathcal{B}_{2N-1}$, whereas the same operator as an element in $\text{Comb}(\mathcal{B}_0, \ldots, \mathcal{B}_{2N-1})$ is viewed as the Choi matrix of a cp map $\mathcal{B}_{2N-2} \otimes \cdots \otimes \mathcal{B}_1 \to \mathcal{B}_{2N-1} \otimes \mathcal{B}_0$. Note also that the set $\mathcal{C}(\mathcal{B}_0, \ldots, \mathcal{B}_{2N-1}, \mathbb{C}^k)$ is precisely the set of N-testers with k values,[5] so that quantum testers are a special class of generalized POVMs.

B. Decomposition of generalized supermaps

Let $k = \lfloor \frac{n}{2} \rfloor$. Let us write the algebra $\mathcal{A}_n$ as

$$\mathcal{A}_n = \mathcal{B}'_{2k} \otimes \mathcal{B}'_{2k-1} \otimes \cdots \otimes \mathcal{B}'_0, \tag{25}$$

where $\mathcal{B}'_j = \mathcal{B}_j$ for $j = 0, \ldots, n$ if $n = 2k$, and $\mathcal{B}'_j = \mathcal{B}_{j+1}$ for $j = 1, \ldots, 2k$ and $\mathcal{B}'_0 = \mathcal{B}_1 \otimes \mathcal{B}_0$ if $n = 2k + 1$. Further, let us suppose that $\mathcal{B}'_j = \oplus_{l=1}^{n_j} B(\mathcal{H}_{B_l^j})$, with minimal central projections $\{q_{k_j}^j\}$, $j = 0, 1, \ldots, 2k$. Let us denote

$$\mathcal{I}_k := \{I = (I_{2k}, \ldots, I_0) \in \mathbb{N}^{2k+1}, \ I_j \in \{1, \ldots, n_j\}, \ j = 0, \ldots, 2k\}$$

be the set of multi-indices. For $I \in \mathcal{I}_k$ and $l \leq k$, we denote $I^l = (I_{2l}, \ldots, I_0) \in \mathcal{I}_l$. Let $q(I) := \otimes_{l=0}^{2k} q_{I_{2k-l}}^{2k-l}$ and $\mathcal{H}_{B(I)} := \mathcal{H}_{B_{I_{2k}}^{2k} \ldots B_{I_0}^0}$, then

$$\mathcal{A}_n = \bigoplus_{I \in \mathcal{I}_k} \mathcal{H}_{B(I)}$$

and $q(I)$ are the minimal central projections in $\mathcal{A}_n$.

Theorem 8: *Let $X \in \mathcal{C}_J(\mathcal{B}_0, \ldots, \mathcal{B}_n)$. Let $k = \lfloor \frac{n}{2} \rfloor$. Then there are the following:*

1. *an ancillary Hilbert space $\mathcal{H}_D = \mathcal{H}_{D_0} = \mathcal{H}_{D_1} = \cdots = \mathcal{H}_{D_k}$,*
2. *elements $X_m(I^{m-1}) \in \mathcal{C}(\mathcal{B}'_{2m-1} \otimes B(\mathcal{H}_{D_{m-1}}), B(\mathcal{H}_{D_m}) \otimes \mathcal{B}'_{2m})$ for m = 1, ..., k and for every multi-index $I \in \mathcal{I}_k$,*
3. *a state $X_0 \in B(\mathcal{H}_{D_0}) \otimes J$, if n = 2k or a generalized channel $X_0 \in \mathcal{C}_J(\mathcal{B}_0, B(\mathcal{H}_{D_0}) \otimes \mathcal{B}_1)$, if n = 2k + 1*

such that, for all $I \in \mathcal{I}_k$,

$$q(I)X = I_{D_k} * X_k(I^k) * \cdots * X_1(I^1) * X_0(I_0), \tag{26}$$

where

$$X_m(I^m) := (I_{D_m} \otimes q_{I_{2m}}^{2m} \otimes q_{I_{2m-1}}^{2m-1} \otimes I_{D_{m-1}})X_m(I^{m-1}), m = 1, \ldots, k \tag{27}$$

and $X_0(I_0) = (I_{\mathcal{H}_{D_0}} \otimes q_{I_0}^0)X_0$.

Proof: We proceed by induction on k. If $k = 0$, then we must have $n = 1$ and the statement is trivial. Suppose now that the theorem holds for some k.

Let n be such that $\lfloor \frac{n}{2} \rfloor = k + 1$. Then $\mathcal{A}_n = \mathcal{B}'_{2k+2} \otimes \mathcal{B}'_{2k+1} \otimes \mathcal{A}_{n-2}$ and by Theorem 6, $X \in \mathcal{C}_J(\mathcal{B}_0, \ldots, \mathcal{B}_n)$ if and only if X is positive and there is some $Y^{(1)} \in \mathcal{C}_J(\mathcal{B}_0, \ldots, \mathcal{B}_{n-2})$ such that

$$\text{Tr}_{\mathcal{B}'_{2k+2}} X = I_{\mathcal{B}'_{2k+1}} \otimes Y^{(1)}.$$

Now by Theorem 11 from the Appendix, the last equation holds if and only if there is an ancillary Hilbert space $\mathcal{H}_D = \mathcal{H}_{D_k} = \mathcal{H}_{D_{k+1}}$ and

$$X_1(I_{2k+1}, \Pi_j I_j^k) \in \mathcal{C}(B(\mathcal{H}_{B_{I_{2k+1}}^{2k+1} D_k}), \mathcal{B}'_{2k+2}), \quad X_0(\Pi_j I_j^k) \in B(\mathcal{H}_{D_k B(I^k)})$$

with

$$\mathrm{Tr}_{D_k} X_0(\Pi_j I_j^k) = q(I^k)Y^{(1)} \tag{28}$$

such that

$$(I_{\mathcal{B}'_{2k+2}} \otimes q_{I_{2k+1}}^{2k+1} \otimes q(I^k))X = X_1(I_{2k+1}, \Pi_j I_j^k) * X_0(\Pi_j I_j^k)$$

for any multi-index $I \in \mathcal{I}_{k+1}$. Put

$$X_{k+1}(I^k) := \omega_{D_{k+1}} \otimes \left(\bigoplus_{i=1}^{n_{2k+1}} X_1(i, \Pi_j I_j^k) \right)$$

with an arbitrary state $\omega_{D_{k+1}} \in B(\mathcal{H}_{D_{k+1}})$. Then $X_{k+1}(I^k) \in \mathcal{C}(\mathcal{B}'_{2k+1} \otimes B(\mathcal{H}_{D_k}), B(\mathcal{H}_{D_{k+1}}) \otimes \mathcal{B}'_{2k+2})$, and

$$q(I)X = I_{D_{k+1}} * X_{k+1}(I^{k+1}) * X_0(\Pi_j I_j^k),$$

where $X_{k+1}(I^{k+1})$ is given by (27). Let now $X'_k := \bigoplus_{J \in \mathcal{I}_k} X_0(\Pi_j J_j) \in B(\mathcal{H}_{D_k}) \otimes \mathcal{A}_{n-2}$. Then by (28) and $Y^{(1)} \in \mathcal{C}_J(\mathcal{B}_0, \ldots, \mathcal{B}_{n-2})$, we get

$$\mathrm{Tr}_{B(\mathcal{H}_{D_k}) \otimes \mathcal{B}_{n-2}} X'_k = \mathrm{Tr}_{\mathcal{B}_{n-2}} Y^{(1)} = I_{\mathcal{B}_{n-3}} \otimes Y^{(2)}, \quad Y^{(2)} \in \mathcal{C}_J(\mathcal{B}_0, \ldots, \mathcal{B}_{n-4}),$$

which is equivalent with $X'_k \in \mathcal{C}_J(\mathcal{B}_0, \ldots, B(\mathcal{H}_{D_k}) \otimes \mathcal{B}_{n-2})$. Since $\lfloor \frac{n-2}{2} \rfloor = k$, we may apply the induction hypothesis to X'_k. Hence there is some ancilla $\mathcal{H}_E = \mathcal{H}_{E_0} = \cdots = \mathcal{H}_{E_k}$, elements $X_m(J^{m-1}) \in \mathcal{C}(\mathcal{B}'_{2m-1} \otimes B(\mathcal{H}_{E_{m-1}}), B(\mathcal{H}_{E_m}) \otimes \mathcal{B}'_{2m})$ for $m = 1, \ldots, k-1$, an element $X''_k(I^{k-1}) \in \mathcal{C}(\mathcal{B}'_{2k-1} \otimes B(\mathcal{H}_{E_{k-1}}), B(\mathcal{H}_{E_k D_k}) \otimes \mathcal{B}'_{2k})$, and $X_0 \in \mathcal{B}_0$ satisfying 3, such that for every $J \in \mathcal{I}_k$,

$$X_0(\Pi_j J_j) = q(J)X'_k = I_{E_k} * X''_k(J) * \cdots * X_0(I_0).$$

Note also that we may suppose $\mathcal{H}_E = \mathcal{H}_D$, exactly as in the proof of Theorem 11. By putting $X_k(J) = I_{E_k} * X''_k(J)$, we obtain the result. $\qquad \square$

Theorem 8, together with Proposition 5, gives the following Corollary.

Corollary 1: For $k \geq 1$ and for any generalized k-comb $X \in \mathcal{C}_J(\mathcal{B}_0, \ldots, \mathcal{B}_{2k-1})$, there exists a pair (χ, Λ), where $\chi : \mathcal{B}_0 \to \mathcal{B}_0$ is a simple generalized channel with respect to J and $X_\Lambda \in$ Comb$(\mathcal{B}_0, \ldots, \mathcal{B}_{2k-1})$, such that

$$\Phi_X = \Lambda \circ (id_{\mathcal{B}_{2k-1} \otimes \cdots \otimes \mathcal{B}_1} \otimes \chi).$$

Conversely, each such pair defines an element in $\mathcal{C}_J(\mathcal{B}_0, \ldots, \mathcal{B}_{2k+1})$. In particular, $\mathcal{C}_J(\mathcal{B}_0, \ldots, \mathcal{B}_{2k+1})$ is the set of cp maps sending $\mathcal{C}(\mathcal{B}_1, \ldots, \mathcal{B}_{2k})$ to the set of generalized channels $\mathcal{C}_J(\mathcal{B}_0, \mathcal{B}_{2k+1})$.

We will now describe how an element $Y \in \mathcal{C}_J(\mathcal{B}_0, \ldots, \mathcal{B}_{n+1})$ acts on $X \in \mathcal{C}_J(\mathcal{B}_0, \ldots, \mathcal{B}_n)$. Let $\Phi_Y : \mathcal{C}_J(\mathcal{B}_0, \ldots, \mathcal{B}_n) \to \mathcal{B}_{n+1}$ be the cp map with Choi matrix Y. By (6),

$$\Phi_Y(X) = Y * X = \mathrm{Tr}_{\mathcal{A}_n}[(I_{\mathcal{B}_{n+1}} \otimes X^T)Y],$$

$$= \mathrm{Tr}_{\mathcal{A}_n}[(I_{\mathcal{B}_{n+1}} \otimes \bigoplus_I q(I)X^T) \bigoplus_{i,J} (q_i^{n+1} \otimes q(J))Y],$$

$$= \mathrm{Tr}_{\mathcal{A}_n}[\bigoplus_{i,I}[(I_{\mathcal{B}_{n+1}} \otimes q(I)X^T)(q_i^{n+1} \otimes q(I))Y],$$

$$= \bigoplus_i \sum_I \mathrm{Tr}_{B(I)}[(I_{\mathcal{B}_{n+1}} \otimes (q(I)X)^T)(q_i^{n+1} \otimes q(I))Y],$$

$$= \bigoplus_i \sum_I ((q_i^{n+1} \otimes q(I))Y) * (q(I)X).$$

Let now $n = 2k$, so that $\lfloor \frac{n}{2} \rfloor = \lfloor \frac{n+1}{2} \rfloor = k$. Then

$$q(I)X = I_{D_k} * X_k(I^k) * \cdots * X_1(I^1) * X_0(I_0),$$

$$(q_i^{n+1} \otimes q(I))Y = I_{E_k} * Y_k(\bar{I}^k) * \cdots * Y_1(\bar{I}^1) * Y_0(\bar{I}_0).$$

Here, $\bar{I}$ is he multi-index in $\mathcal{I}_k$, such that $\bar{I}_{2k} = i$, $\bar{I}_j = I_{j+1}, j = 1, \ldots, 2k - 1$, and $\bar{I}_0 = I_0 I_1$. Then

$$((q_i^{n+1} \otimes q(I))Y) * (q(I)X) = I_{D_k E_k} * Y_k(\bar{I}^k) * X_k(I^k) * \cdots * Y_0(\bar{I}_0) * X_0(I_0),$$

this follows from Proposition 1 and 2. More explicitly, we first apply the components of the channel $Y_0(\bar{I}_0)$ to the part of $X_0(I_0)$ in $\mathcal{B}_0$, then on the part of the result in $\mathcal{B}_1$, we apply the components of the channel $X_1(I_1)$, etc., both ancillas are traced out at the end.

Similarly, if $n = 2k + 1$, then $\lfloor \frac{n+1}{2} \rfloor = k + 1$ and

$$(q_i^{n+1} \otimes q(I))Y = I_{E_{k+1}} * Y_{k+1}(\hat{I}^{k+1}) * \cdots * Y_1(\hat{I}^1) * Y_0(\hat{I}_0),$$

where $\hat{I} \in \mathcal{I}_{k+1}$ is such that $\hat{I}_{2k+2} = i$, $\hat{I}_j = I_{j-1}$ for $j = 2, \ldots 2k + 1$ and $I_0 = \hat{I}_1 \hat{I}_0$. Then

$$((q_i^{n+1} \otimes q(I))Y) * (q(I)X) = I_{D_k E_{k+1}} * Y_{k+1}(\hat{I}^{k+1}) * X_k(I^k) * \cdots * Y_1(\hat{I}_1) * X_0(I_0) * Y_0(\hat{I}_0).$$

Note that here $X_0(I_0)$ is a channel, which we apply to $Y_0(\hat{I}_0)$, etc.

Example 7 (PPOVMs): Let $Y \in \mathcal{C}(B(\mathcal{H}_0), B(\mathcal{H}_1), \mathbb{C}^m)$. By Theorem 8, there is some ancilla $\mathcal{H}_D$, a POVM $M(= I_{D_1} * Y_1) \in \mathcal{C}(B(\mathcal{H}_1 \otimes \mathcal{H}_D), \mathbb{C}^m)$ and a state $\rho(= Y_0) \in B(\mathcal{H}_D \otimes \mathcal{H}_0)$, such that $Y = M * \rho$. For any $X \in \mathcal{C}(\mathcal{H}_0, \mathcal{H}_1)$, we have

$$Y * X = M * X * \rho = \bigoplus_{i=1}^{m} \operatorname{Tr} M_i (id_D \otimes \Phi_X)(\rho),$$

where $M = (M_1, \ldots, M_m)$, compare this to Theorem 4. We will write such decomposition as $Y = (\mathcal{H}_D, (M_1, \ldots, M_m), \rho)$.

Next, let $Z \in \mathcal{C}(B(\mathcal{H}_0), B(\mathcal{H}_1), \mathbb{C}^m, B(\mathcal{H}_3) \otimes \mathbb{C}^l)$, which is the set of all instruments from PPOVMs to $B(\mathcal{H}_3)$, with values in $\{1, \ldots, l\}$. Then there is an ancilla $\mathcal{H}_E$ a channel $\xi \in \mathcal{C}(B(\mathcal{H}_0), B(\mathcal{H}_E \otimes \mathcal{H}_1))$ and an instrument $\Lambda \in \mathcal{C}(\mathbb{C}^m \otimes B(\mathcal{H}_E), B(\mathcal{H}_3) \otimes \mathbb{C}^l)$, such that

$$Z = \Lambda * \xi.$$

Here $\Lambda = \oplus_{j=1}^m \Lambda_j$, where each $\Lambda_j : B(\mathcal{H}_E) \to B(\mathcal{H}_3) \otimes \mathbb{C}^l$ is an instrument, with components $(\Lambda_{1j}, \ldots, \Lambda_{lj})$. We write $Z = (\mathcal{H}_E, (\Lambda_1, \ldots, \Lambda_m), \xi)$. Let now $Y = (\mathcal{H}_D, (M_1, \ldots, M_m), \rho)$ be a PPOVM. We have

$$Z * Y = \bigoplus_i \sum_j \Lambda_{ij} (\operatorname{Tr}_{\mathcal{H}_D \otimes \mathcal{H}_1}(I_E \otimes M_j)(id_D \otimes \xi)(\rho)),$$

$$= \bigoplus_i \sum_j \operatorname{Tr}_{\mathcal{H}_D \otimes \mathcal{H}_1}(M_j \otimes I_{\mathcal{H}_3})(id_D \otimes [(\Lambda_{ij} \otimes id_{\mathcal{H}_1}) \circ \xi])(\rho),$$

$$= \bigoplus_i \sum_j \operatorname{Tr}_{\mathcal{H}_D \otimes \mathcal{H}_1}(M_j \otimes I_{\mathcal{H}_3})(id_D \otimes \hat{\Lambda}_{ij})(\rho),$$

where $\hat{\Lambda}_j := (\Lambda_j \otimes id_{\mathcal{H}_1}) \circ \xi$ is an instrument $B(\mathcal{H}_0) \to B(\mathcal{H}_3 \otimes \mathcal{H}_1)$, with values in $\{1, \ldots, l\}$, such that $\sum_i \operatorname{Tr}_{\mathcal{H}_3} \circ \hat{\Lambda}_{ij} = \operatorname{Tr}_E \circ \xi$ for all j, compare this with example 6.

Example 8 (Supermaps on instruments): We next describe the set $\operatorname{Comb}(B(\mathcal{H}_0), \mathcal{B}(\mathcal{H}_1), \mathbb{C}^m \otimes B(\mathcal{H}_2), B(\mathcal{H}_3))$, that is, the set of cp maps from instruments $B(\mathcal{H}_1) \to B(\mathcal{H}_2)$ to channels $B(\mathcal{H}_0) \to B(\mathcal{H}_3)$. By Theorems 7 and 8, for any such map, there is an ancillary Hilbert space $\mathcal{H}_D$, channels $\xi_j : B(\mathcal{H}_D \otimes \mathcal{H}_2) \to B(\mathcal{H}_3), j = 1, \ldots, m$ and a channel $\xi : B(\mathcal{H}_0) \to B(\mathcal{H}_D \otimes \mathcal{H}_1)$ such that the map has the form

$$(\Lambda_1, \ldots, \Lambda_m) \mapsto \sum_j \xi_j \circ (id_D \otimes \Lambda_j) \circ \xi.$$

This seems to be more general than the supermaps considered in Ref. 6, more precisely, this map consists of m supermaps in the sense of Ref. 6, which have the first channel equal to the same ξ. $\square$

The decomposition given in this section can be understood as a physical realization of generalized supermaps in $\mathcal{C}_J(\mathcal{B}_0, \ldots, \mathcal{B}_n)$. It is not unique, indeed, for example, by Theorem 4, any state ρ on $\mathcal{H}_0 \otimes \mathcal{H}_A$ and a POVM on $B(\mathcal{H}_1 \otimes \mathcal{H}_A)$ define a PPOVM, but (by the first part of this Theorem), we can always have a decomposition where the state is pure. The elements in $\mathcal{C}_J(\mathcal{B}_0, \ldots, \mathcal{B}_{n+1})$ do not distinguish between these different realizations, but only the generalized channels they define. We may go a step further and consider maps which recognize only the channels on K, defined by the generalized channels, that is, maps which give the same result on equivalent channels. This is the content of the next paragraph.

C. Equivalence of generalized supermaps

By Theorems 3 and 5, two elements $X_1, X_2 \in \mathcal{C}_J(\mathcal{B}_0, \ldots, \mathcal{B}_n)$ are equivalent if and only if

$$X_1 - X_2 \in \mathcal{B}_n \otimes (J_{n-1}^T)^\perp. \tag{29}$$

Using Lemma 1, we get

$$(J_{n-1}^T)^\perp = S_{n-1}^*(S_{n-2}^{-1}(S_{n-3}^*(\ldots(L^T)^\perp \ldots))),$$

where $(L^T)^\perp = S_1^*([I_A]^\perp \cap J)$ if n is even and $(L^T)^\perp = S_1^{-1}((J^T)^\perp)$ if n is odd. From this, we get the following proposition.

Proposition 8: Let $k = \lfloor \frac{n}{2} \rfloor$. Two elements $X_1, X_2 \in \mathcal{C}_J(\mathcal{B}_0, \ldots, \mathcal{B}_n)$ are equivalent if and only if there are elements $W^{(m)} \in \mathcal{B}_n \otimes \mathcal{A}_{n-2m}$, $m = 1, \ldots k$, such that

$$X_1 - X_2 = I_{\mathcal{B}_{n-1}} \otimes W^{(1)},$$

$$\mathrm{Tr}_{\mathcal{B}_{n-2m}} W^{(m)} = I_{\mathcal{B}_{n-2m-1}} \otimes W^{(m+1)}, \ m = 1, \ldots, k-1,$$

$$W^{(k)} \in \mathcal{B}_n \otimes J, \quad \mathrm{Tr}_{\mathcal{B}_0} W^{(k)} = 0 \qquad if \ n = 2k,$$

$$\mathrm{Tr}_{\mathcal{B}_1} W^{(k)} \in \mathcal{B}_n \otimes (J^T)^\perp \qquad if \ n = 2k + 1.$$

It is not clear in the present how to interpret this equivalence, in terms of the physical realizations of the channels. The next theorem gives a characterization of elements in $\mathcal{C}_J(\mathcal{B}_0, \ldots, \mathcal{B}_{n+1})$ which respect this equivalence.

Theorem 9: *The set of all elements in $\mathcal{C}_J(\mathcal{B}_0, \ldots, \mathcal{B}_{n+1})$ having the same value on each equivalence class of elements in $\mathcal{C}_J(\mathcal{B}_0, \ldots, \mathcal{B}_n)$ is*

$$J_{n+1} \cap (\mathcal{B}_{n+1} \otimes \mathcal{B}_n \otimes J_{n-1}) \cap c_{n+1} \mathfrak{S}(\mathcal{A}_{n+1}).$$

In particular, if $K = \mathfrak{S}(\mathcal{B}_0)$, then this set has the form

$$\mathcal{C}(\mathcal{B}_0, \ldots, \mathcal{B}_{n+1}) \cap \mathcal{C}(\mathcal{B}_0, \mathcal{B}_n, \mathcal{B}_{n+1}, \mathcal{B}_1, \ldots, \mathcal{B}_{n-1}), \ \text{if } n \text{ is odd,}$$

$$\mathcal{C}(\mathcal{B}_0, \ldots, \mathcal{B}_{n+1}) \cap \mathcal{C}(\mathcal{B}_n, \mathcal{B}_{n+1}, \mathcal{B}_0, \ldots, \mathcal{B}_{n-1}), \ \text{if } n \text{ is even.}$$

Proof: Let $X \in \mathcal{C}_J(\mathcal{B}_0, \ldots, \mathcal{B}_{n+1})$, then it is clear from (29) that the corresponding map has the same value on equivalent elements if and only if it is equal to 0 on $\mathcal{B}_n \otimes (J_{n-1}^T)^\perp$. Equivalently,

$$0 = \mathrm{Tr}\left(b\mathrm{Tr}_{\mathcal{A}_n}[(I_{\mathcal{B}_{n+1}} \otimes Y^T)X]\right) = \mathrm{Tr}((b \otimes Y^T)X)$$

for all $b \in \mathcal{B}_{n+1}$ and $Y \in \mathcal{B}_n \otimes (J_{n-1}^T)^\perp$, that is, $X \in (\mathcal{B}_{n+1} \otimes \mathcal{B}_n \otimes J_{n-1}^\perp)^\perp = \mathcal{B}_{n+1} \otimes \mathcal{B}_n \otimes J_{n-1}$. Since $X \in \mathcal{C}_J(\mathcal{B}_0, \ldots, \mathcal{B}_{n+1})$, we get the result.

Suppose $K = \mathfrak{S}(\mathcal{B}_0)$ and let $k = \lfloor \frac{n+1}{2} \rfloor$. Since $X \in \mathcal{B}_{n+1} \otimes \mathcal{B}_n \otimes J_{n-1}$, there are positive elements $Z^{(m)} \in \mathcal{B}_{n+1} \otimes \mathcal{B}_n \otimes \mathcal{A}_{n-1-2m}$, such that

$$\mathrm{Tr}_{\mathcal{B}_{n-1-2m}} Z^{(m)} = I_{\mathcal{B}_{n-2-2m}} \otimes Z^{(m+1)}, \qquad m = 0, \ldots, k-2, \tag{30}$$

$$Z^{(k-1)} \in \mathcal{B}_{n+1} \otimes \mathcal{B}_n \otimes J \ \text{ if } n \text{ is odd} , \tag{31}$$

$$Z^{(k-1)} \in \mathcal{B}_{n+1} \otimes \mathcal{B}_n \otimes S_1^{-1}(\tilde{J}) \ \text{ if } n \text{ is even} \tag{32}$$

and $Z^{(0)} = X$. Suppose n is odd, then by Theorem 6, we get

$$\mathrm{Tr}_{\mathcal{B}_{n+1}} \mathrm{Tr}_{\mathcal{B}_{n-1}} \dots \mathrm{Tr}_{\mathcal{B}_2} X = I_{\mathcal{B}_n \otimes \mathcal{B}_{n-2} \otimes \cdots \otimes \mathcal{B}_1} \otimes Y^{(k)}$$

with $Y^{(k)} \in \mathfrak{S}(\mathcal{B}_0)$, and from (30), we have

$$\mathrm{Tr}_{\mathcal{B}_{n-1}} \mathrm{Tr}_{\mathcal{B}_{n-3}} \dots \mathrm{Tr}_{\mathcal{B}_2} X = I_{\mathcal{B}_{n-2} \otimes \mathcal{B}_{n-4} \otimes \cdots \otimes \mathcal{B}_1} \otimes Z^{(k-1)}.$$

This implies $\mathrm{Tr}_{\mathcal{B}_{n+1}} Z^{(k-1)} = I_{\mathcal{B}_n} \otimes Y^{(k)}$. If $J = \mathcal{B}_0$, this together with (30) and (31) is equivalent with $X \in \mathcal{C}(\mathcal{B}_0, \mathcal{B}_n, \mathcal{B}_{n+1}, \mathcal{B}_1, \dots, \mathcal{B}_{n-1})$. Similarly, if $J = \mathcal{B}_0$ and n is even, we have

$$\mathrm{Tr}_{\mathcal{B}_{n+1}} \mathrm{Tr}_{\mathcal{B}_{n-1}} \dots \mathrm{Tr}_{\mathcal{B}_1} X = I_{\mathcal{B}_n \otimes \mathcal{B}_{n-2} \otimes \cdots \otimes \mathcal{B}_0}$$

and by (32), there is some positive element $Z^{(k)} \in \mathcal{B}_{n+1} \otimes \mathcal{B}_n$, such that

$$\mathrm{Tr}_{\mathcal{B}_1} Z^{(k-1)} = I_{\mathcal{B}_0} \otimes Z^{(k)}. \tag{33}$$

Then

$$\mathrm{Tr}_{\mathcal{B}_{n-1}} \mathrm{Tr}_{\mathcal{B}_{n-3}} \dots \mathrm{Tr}_{\mathcal{B}_1} X = I_{\mathcal{B}_{n-2} \otimes \mathcal{B}_{n-4} \otimes \cdots \otimes \mathcal{B}_0} \otimes Z^{(k)},$$

so that we must have $\mathrm{Tr}_{\mathcal{B}_{n+1}} Z^{(k)} = I_{\mathcal{B}_n}$. This, together with (30) and (33), is equivalent with $X \in \mathcal{C}(\mathcal{B}_n, \mathcal{B}_{n+1}, \mathcal{B}_0, \dots, \mathcal{B}_{n-1})$. $\qquad\square$

Example 9 (Equivalence on PPOVMs): Suppose that Z is a generalized POVM on the set of PPOVMs, that is, $Z \in \mathcal{C}(B(\mathcal{H}_0), B(\mathcal{H}_1), \mathbb{C}^m, \mathbb{C}^k)$. Then by example 6, $Z = \sum_{i=1}^{k} \sum_{j=1}^{m} |i\rangle\langle i| \otimes |j\rangle\langle j| \otimes Z_{ij}$ and each Z_{ij} is the Choi matrix of a cp map $\Lambda_{ij} : B(\mathcal{H}_0) \to B(\mathcal{H}_1)$, such that there is a channel ξ with $\sum_j \Lambda_{ij} = \xi$ for all i. If Z attains the same value on equivalent elements, then it defines a measurement on the set of equivalence classes of PPOVMs, that is, on the set of measurements on channels $B(\mathcal{H}_0) \to B(\mathcal{H}_1)$. By Theorem 9, this happens if and only if Z is also in $\mathcal{C}(\mathbb{C}^m, \mathbb{C}^k, B(\mathcal{H}_0), B(\mathcal{H}_1))$. Using Theorem 6, we get that there are some numbers $\mu_{ij} \geq 0$, with $\sum_j \mu_{ij} = 1$ for all i, such that $\mathrm{Tr}_{\mathcal{H}_1} Z_{ij} = \mu_{ij} I_{\mathcal{H}_0}$. It follows that there are channels ξ_{ij}, such that $\Lambda_{ij} = \mu_{ij} \xi_{ij}$. We have proved the following.

For any measurement on measurements on $\mathcal{C}(B(\mathcal{H}_0), B(\mathcal{H}_1))$ with values in $\{1, \dots, m\}$, there are $\xi_{ij} \in \mathcal{C}(B(\mathcal{H}_0), B(\mathcal{H}_1))$ and numbers $\mu_{ij} \geq 0$, $\sum_j \mu_{ij} = 1$, satisfying $\sum_j \mu_{ij} \xi_{ij} = \xi$ for all i, such that, if a measurement on $\mathcal{C}(B(\mathcal{H}_0), B(\mathcal{H}_1))$ has an implementation $(\mathcal{H}_D, (M_1, \dots, M_m), \rho)$, then the corresponding probabilities are given by

$$p_i(\mathcal{H}_D, (M_1, \dots, M_m), \rho) = \sum_j \mu_{ij} \mathrm{Tr}\,(M_j(\xi_{ij} \otimes id_D)(\rho)).$$

Conversely, any such ξ_{ij}, μ_{ij} define a measurement on measurements on $\mathcal{C}(B(\mathcal{H}_0), B(\mathcal{H}_1))$. Note that if $(\mathcal{H}_D, M, \rho)$ and $(\mathcal{H}_E, N, \sigma)$ are implementations of PPOVMs, then these are equivalent if and only if $\mathrm{Tr}\,(M_j(\xi \otimes id_D)(\rho)) = \mathrm{Tr}\,(N_j(\xi \otimes id_E)(\sigma))$ for any channel ξ. $\qquad\square$

D. Equivalence of combs

Any N-comb $X \in \mathrm{Comb}(\mathcal{B}_0, \dots, \mathcal{B}_{2N+1})$ is a cp map $\mathrm{Comb}(\mathcal{B}_1, \dots, \mathcal{B}_{2N}) \to \mathcal{B}_{2N+1} \otimes \mathcal{B}_0$. By (24) and Theorem 3, two N-combs X_1 and X_2 are equivalent if and only if

$$X_1 - X_2 \in \mathcal{B}_{2N+1} \otimes (\hat{J}_{2N-1}^T)^\perp \otimes \mathcal{B}_0,$$

where $\hat{J}_{2N-1} := J_{2N-1}(\mathcal{B}_1, \dots, \mathcal{B}_{2N})$.

Proposition 9: Two elements X_1, $X_2 \in \mathrm{Comb}(\mathcal{B}_0, \ldots, \mathcal{B}_{2N-1})$ are equivalent if and only if there are elements $V^{(m)} \in \mathcal{B}_{2N+1} \otimes \mathcal{A}_{2m-1}$, $m = 1, \ldots, N$, such that

$$X_1 - X_2 = I_{\mathcal{B}_{2N}} \otimes V^{(N)},$$

$$\mathrm{Tr}_{\mathcal{B}_{2m-1}} V^{(m)} = I_{\mathcal{B}_{2m-2}} \otimes V^{(m-1)}, \quad m = 2, \ldots, N,$$

$$\mathrm{Tr}_{\mathcal{B}_1} V^{(1)} = 0.$$

The proof of the next theorem is the same as of Theorem 9.

Theorem 10: *The set elements in $\mathrm{Comb}(\mathcal{B}_0, \ldots, \mathcal{B}_{2N+1})$ having the same value on equivalent elements in $\mathrm{Comb}(\mathcal{B}_1, \ldots, \mathcal{B}_{2N})$ is equal to*

$$\mathrm{Comb}(\mathcal{B}_0, \ldots, \mathcal{B}_{2N+1}) \cap \mathrm{Comb}(\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_{2N}, \mathcal{B}_{2N+1}, \mathcal{B}_2, \ldots, \mathcal{B}_{2N-1}).$$

VI. FINAL REMARKS

We have introduced the concept of a channel on a section of the state space of a finite-dimensional C*-algebra. We proved that such channels are restrictions of completely positive maps, called generalized channels. If the section K contains the tracial state, the Choi matrices of generalized channels with respect to K form again a section of the state space of some C*-algebra. This allows us to define generalized supermaps as completely positive maps sending generalized channels (or generalized supermaps) to states. The set of generalized supermaps is characterized as an intersection of the state space by a subspace. This might be useful, for example, in optimization problems with respect to supermaps.

Although the condition $\tau_A \in K$ includes the most important examples of channels and combs, it might be interesting to consider supermaps for arbitrary generalized channels. By Proposition 6, this should be possible by extending our theory using the set $\mathfrak{S}_\rho(\mathcal{A})$ instead of $\mathfrak{S}(\mathcal{A})$, with an invertible element $\rho \in \mathcal{A}^+$. This can be done along similar lines.

Another possible extension of the theory is to look at the generalized channels sending a section K_1 to a given convex subset K_2 of the target state space. The set $\mathrm{Comb}_J(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is a particular example of this, but arbitrary convex subset can be considered, using similar tools as were used in the present paper.

A natural question is an extension of these results to infinite dimension. For example, in the setting of the algebras of bounded operators $B(\mathcal{H})$ for infinite-dimensional Hilbert space $\mathcal{H}$, quantum supermaps were studied in Ref. 10. Channels and measurements on sections of the state space can be studied also in this case and similar results can be expected. But the identification of the set of channels with a section of a state space fails.

ACKNOWLEDGMENTS

APPENDIX: DECOMPOSITION OF SEMICAUSAL MAPS

Let $\mathcal{A} = \oplus_n B(\mathcal{H}_{A_n})$ be a finite-dimensional C*-algebra and let $\mathcal{H}_A$, $\mathcal{H}_B$, $\mathcal{H}'_B$ be finite-dimensional Hilbert spaces. Let $T : \mathcal{A} \otimes B(\mathcal{H}_B) \to B(\mathcal{H}_{AB'})$ be a cp map. Then, we say that T is semicausal if

$$T(I_A \otimes b) = I_A \otimes S(b) \tag{A1}$$

for some cp map $S : B(\mathcal{H}_B) \to B(\mathcal{H}_{B'})$, and T is semilocalizable, if

$$T = (id_A \otimes G) \circ (F \otimes id_B) \tag{A2}$$

for some unital cp map $F : \mathcal{A} \to B(\mathcal{H}_{AD})$ and a cp map $G : B(\mathcal{H}_{DB}) \to B(\mathcal{H}_{B'})$, where $\mathcal{H}_D$ is some (finite-dimensional) Hilbert space. The following statement was proved in Ref. 12, in the case that $\mathcal{A}$ is a matrix algebra. For the convenience of the reader, we give the modification of the proof in Ref. 12 for our slightly more general case.

Lemma 4: Let $T : \mathcal{A} \otimes B(\mathcal{H}_B) \to \mathcal{H}_{AB'}$ be a cp map. Then T is semicausal if and only if T is semilocalizable.

Proof: Any representation of $\mathcal{A} \otimes B(\mathcal{H}_B)$ has the form

$$\Pi(a \otimes b) = \oplus_n I_{E_n} \otimes a_n \otimes b = (\oplus_n I_{E_n} \otimes a_n) \otimes b$$

for some Hilbert spaces $\mathcal{H}_{E_n}$, where $a = \oplus_n a_n \in \mathcal{A}$ and $b \in B(\mathcal{H}_B)$. Hence by Stinespring representation, T has the form

$$T(a \otimes b) = V^*((\oplus_n I_{E_n} \otimes a_n) \otimes b)V$$

for some linear map $V : \mathcal{H}_{AB'} \to \oplus_n \mathcal{H}_{E_n A_n B}$. Let now

$$S(b) = W^*(1_D \otimes b)W$$

be a minimal Stinespring representation of S. Then (A1) implies that

$$V^*(I_{\oplus_n \mathcal{H}_{E_n A_n}} \otimes b)V = (I_A \otimes W^*)(I_{AD} \otimes b)(W \otimes I_B).$$

Exactly as in Ref. 12, we get by minimality of the Stinespring representation that there is some isometry $U : \mathcal{H}_{AD} \to \oplus_n \mathcal{H}_{E_n A_n}$, such that

$$V = (U \otimes I_B)(I_A \otimes W).$$

Hence,

$$\Phi(a \otimes b) = (I_A \otimes W^*)(U^*(\oplus_n I_{E_n} \otimes a_n)U \otimes b)(I_A \otimes W),$$

so that

$$\Phi = (id_A \otimes G) \circ (F \otimes id_B) \tag{A3}$$

for the unital cp map $F : \mathcal{A} \to B(\mathcal{H}_{AD})$, given by $F(a) = U^*(\oplus_n I_{E_n} \otimes a_n)U$ and the cp map $G : B(\mathcal{H}_{DB}) \to B(\mathcal{H}'_B)$, defined as $G(d \otimes b) = W^*(d \otimes b)W$.

Conversely, if T is of the form (A3), then it is clear that T satisfies (A1), with

$$S(b) = G(1_D \otimes b). \tag{A4}$$

$\square$

Theorem 11: *Let $\mathcal{A} = \oplus B(\mathcal{H}_{A_k})$, $\mathcal{B} = \oplus B(\mathcal{H}_{B_m})$, $\mathcal{C} = \oplus B(\mathcal{H}_{C_n})$ be finite-dimensional C*-algebras, with minimal central projections $\{p_k\}_k$, $\{q_m\}_m$, and $\{r_n\}_n$, respectively. Let $X \in \mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C}$ be positive. Then the following are equivalent:*

(i)   *There is some positive element $Y \in \mathcal{C}$ such that*

$$\mathrm{Tr}_A X = I_{\mathcal{B}} \otimes Y.$$

(ii)   *There is an auxiliary Hilbert space $\mathcal{H}_D$, positive elements $X_0(n) \in B(\mathcal{H}_{DC_n})$ and $X_1(m, n) \in \mathcal{C}(B(\mathcal{H}_{B_m D}), \mathcal{A})$ such that*

$$X_{m,n} := (I_A \otimes q_m \otimes r_n)X = X_1(m, n) * X_0(n).$$

Moreover, we have

$$\mathrm{Tr}_D X_0(n) = Y_n := r_n Y.$$

Proof: Suppose first that $\mathcal{B} = B(\mathcal{H}_B)$ and $\mathcal{C} = B(\mathcal{H}_C)$ are matrix algebras. We can always write $\mathcal{H}_C = \mathcal{H}_{C_1} \otimes \mathcal{H}_{C_2}$. Let us define the map $\Phi : B(\mathcal{H}_{BC_1}) \to \mathcal{A} \otimes B(\mathcal{H}_{C_2})$ by

$$\Phi(a) = X * a, \qquad a \in B(\mathcal{H}_{BC_1}).$$

Then Φ is a cp map and

$$\mathrm{Tr}_{\mathcal{A}}\Phi(a) = [\mathrm{Tr}_{\mathcal{A}}X] * a, \qquad a \in B(\mathcal{H}_{BC_1}),$$

so that $\mathrm{Tr}_{\mathcal{A}}X$ is the Choi matrix of $\mathrm{Tr}_{\mathcal{A}} \circ \Phi$. Similarly, if $\xi : B(\mathcal{H}_{C_1}) \to B(\mathcal{H}_{C_2})$ is the cp map with C-J matrix Y, then $I_{\mathcal{A}} \otimes Y$ is the C-J matrix of $\xi \circ \mathrm{Tr}_{\mathcal{A}}$. It follows that the maps Φ and ξ satisfy

$$\mathrm{Tr}_{\mathcal{A}} \circ \Phi = \xi \circ \mathrm{Tr}_{\mathcal{A}}.$$

For the adjoints, this condition has he form $\Phi^*(I_{\mathcal{A}} \otimes c) = I_B \otimes \xi^*(c)$, for all $c \in B(\mathcal{H}_{C_2})$ which means that the map Φ^* is semicausal. By Lemma 4, (i) is equivalent with

$$\Phi = (F^* \otimes id_{C_2}) \circ (id_B \otimes G^*)$$

for a cp map $G^* : B(\mathcal{H}_{C_1}) \to B(\mathcal{H}_{DC_2})$ and a channel $F^* : B(\mathcal{H}_{BD}) \to \mathcal{A}$, with some Hilbert space $\mathcal{H}_D$. By putting X_1 and X_0 the Choi matrices of F and G, respectively, we get (ii). Finally, (A4) implies $\mathrm{Tr}_D X_0 = Y$.

For the general case, note that $X_{m,n} \in \mathcal{A} \otimes B(\mathcal{H}_{B_m C_n})$ and

$$\mathrm{Tr}_{\mathcal{A}}X_{m,n} = (q_m \otimes r_n)\mathrm{Tr}_{\mathcal{A}}X,$$

so that (i) is equivalent with

$$\mathrm{Tr}_{\mathcal{A}}X_{m,n} = I_{B_m} \otimes Y_n, \qquad \forall m, n,$$

where $Y_n = r_n Y \in B(\mathcal{H}_{C_n})^+$. By the first part of the proof, we get that (i) holds if and only if

$$X_{m,n} = X_1'(m,n) * X_0'(m,n)$$

with positive elements $X_0'(m,n) \in B(\mathcal{H}_{D_{m,n}C_n})$, $X_1'(m,n) \in \mathcal{C}(B(\mathcal{H}_{B_m D_{m,n}}), \mathcal{A})$ for some ancillary Hilbert spaces $\mathcal{H}_{D_{m,n}}$, and such that $\mathrm{Tr}_{D_{m,n}}X_0'(m,n) = Y_n$. Note further that in the proof of Lemma 4, the cp map G and the ancilla $\mathcal{H}_D$ are given by a minimal Stinespring representation of S. Hence $X_0'(m,n)$ and the ancilla are determined by Y_n, so that these depend only on n. Moreover, there are some $\mathcal{H}_{D_n'}$ and $\mathcal{H}_D$, such that $\mathcal{H}_D = \mathcal{H}_{D_n D_n'}$ for all n. Choose some state $\omega_n \in B(\mathcal{H}_{D_n'})$ for all n and put

$$X_0(n) := \omega_n \otimes X_0'(n), \quad X_1(m,n) := X_1'(m,n) \otimes I_{D_n'}.$$

Then $X_0(n) \in B(\mathcal{H}_{DC_n})$, $X_1(m,n) \in \mathcal{C}(B(\mathcal{H}_{B_m D}), \mathcal{A})$, and

$$X_1(m,n) * X_0(n) = X_1(m,n) * I_{D_n'} * \omega_n * X_0'(n) = X_1'(m,n) * X_0'(n) = X_{m,n}.$$

Clearly, also

$$\mathrm{Tr}_D X_0(n) = \mathrm{Tr}_{D_n} X_0'(n) = Y_n.$$

$\square$

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