Equality conditions of DPI

Here we consider equality conditions of the DPI of $D_{\alpha,z}$ for $\alpha<1$. Assume that $0<\alpha<1$ and $z\ge\max\{\alpha,1-\alpha\}$. For simplicity, assume that $\lambda^{-1}\varphi\le\psi\le\lambda\varphi$ for some $\lambda>0$. Let $\gamma:\mathcal{N}\to\mathcal{M}$ be a unital normal positive map, and set $\psi_0:=\psi\circ\gamma$ and $\varphi_0:=\varphi\circ\gamma$. Moreover, set $e:=s(\psi)=s(\varphi)$ and $e_0:=s(\psi_0)=s(\varphi_0)$ (since $\lambda\varphi_0\le\psi_0\le\lambda\varphi_0$). Consider $\gamma_{e_0,e}:e_0\mathcal{N}e_0\to e\mathcal{M}e$ defined by $\gamma_{e_0,e}(y):=e\gamma(e_0ye_0)e$ for $y\in e_0\mathcal{N}e_0$. Then for every $y\in e_0\mathcal{N}e_0$,

$$\psi\circ\gamma_{e_0,e}(y)=\psi(e\gamma(e_0ye_0)e)=\psi(\gamma(e_0ye_0)e)=\psi_0(e_0ye_0)=\psi_0(y),$$

so that we have $\psi\circ\gamma_{e_0,e}=\psi_0|_{e_0\mathcal{N}e_0}$ and similarly $\varphi\circ\gamma_{e_0,e}=\varphi_0|_{e_0\mathcal{N}e_0}$. Hence, by replacing γ with $\gamma_{e_0,e}$ we may assume that $\psi,\varphi,\psi_0,\varphi_0$ are all faithful.

One can define $b,c\in\mathcal{M}$ in such a way that

$$h_\varphi^{\frac{1-\alpha}{2z}}=b\bigl(h_\varphi^{\frac{1-\alpha}{2z}}h_\psi^{\frac{\alpha}{z}}h_\varphi^{\frac{1-\alpha}{2z}}\bigr)^{\frac{1-\alpha}{2}},\qquad\bigl(h_\varphi^{\frac{1-\alpha}{2z}}h_\psi^{\frac{\alpha}{z}}h_\varphi^{\frac{1-\alpha}{2z}}\bigr)^{\frac{1-\alpha}{2}}=ch_\varphi^{\frac{1-\alpha}{2z}}.\tag{0.1}$$

Then $a:=bb^*\in\mathcal{M}^{++}$ and $a^{-1}=c^*c$. By [2, Theorem 1 (vi)] we have

$$Q_{\alpha,z}(\psi\|\varphi)=\inf_{x\in\mathcal{M}^{++}}\Bigl\{\alpha\Bigl\|h_\psi^{\frac{\alpha}{2z}}xh_\psi^{\frac{\alpha}{2z}}\Bigr\|_{z/\alpha}^{z/\alpha}+(1-\alpha)\Bigl\|h_\varphi^{\frac{1-\alpha}{2z}}x^{-1}h_\varphi^{\frac{1-\alpha}{2z}}\Bigr\|_{z/(1-\alpha)}^{z/(1-\alpha)}\Bigr\},\tag{0.2}$$

and a is a minimizer of the above infimum expression, so that

$$Q_{\alpha,z}(\psi\|\varphi)=\alpha\Bigl\|h_\psi^{\frac{\alpha}{2z}}ah_\psi^{\frac{\alpha}{2z}}\Bigr\|_{z/\alpha}^{z/\alpha}+(1-\alpha)\Bigl\|h_\varphi^{\frac{1-\alpha}{2z}}a^{-1}h_\varphi^{\frac{1-\alpha}{2z}}\Bigr\|_{z/(1-\alpha)}^{z/(1-\alpha)}.\tag{0.3}$$

One can also define $b_0,c_0\in\mathcal{N}$ in such a way that

$$h_{\varphi_0}^{\frac{1-\alpha}{2z}}=b_0\bigl(h_{\varphi_0}^{\frac{1-\alpha}{2z}}h_{\psi_0}^{\frac{\alpha}{z}}h_{\varphi_0}^{\frac{1-\alpha}{2z}}\bigr)^{\frac{1-\alpha}{2}},\qquad\bigl(h_{\varphi_0}^{\frac{1-\alpha}{2z}}h_{\psi_0}^{\frac{\alpha}{z}}h_{\varphi_0}^{\frac{1-\alpha}{2z}}\bigr)^{\frac{1-\alpha}{2}}=c_0h_{\varphi_0}^{\frac{1-\alpha}{2z}}.\tag{0.4}$$

Then $a_0:=b_0b_0^*\in\mathcal{N}^{++}$ and $a_0^{-1}=c_0^*c_0$, and we have

$$Q_{\alpha,z}(\psi_0\|\varphi_0)=\alpha\Bigl\|h_{\psi_0}^{\frac{\alpha}{2z}}a_0h_{\psi_0}^{\frac{\alpha}{2z}}\Bigr\|_{z/\alpha}^{z/\alpha}+(1-\alpha)\Bigl\|h_{\varphi_0}^{\frac{1-\alpha}{2z}}a_0^{-1}h_{\varphi}^{\frac{1-\alpha}{2z}}\Bigr\|_{z/(1-\alpha)}^{z/(1-\alpha)}.\tag{0.5}$$

Lemma 0.1. *The operator $a\in\mathcal{M}^{++}$ is uniquely determined by equality* (0.3). *Similarly, $a_0\in\mathcal{N}^{++}$ is uniquely determined by equality* (0.5).

Proof. Suppose that $a_1,a_2\in\mathcal{M}^{++}$ satisfy equality (0.3). Let $a_0:=(a_1+a_2)/2$. Since $k\in L^{z/\alpha}(\mathcal{M})\mapsto\|k\|_{z/\alpha}^{z/\alpha}$ and $k\in L^{z/(1-\alpha)}(\mathcal{M})\mapsto\|k\|_{z/(1-\alpha)}^{z/(1-\alpha)}$ are convex and $a_0^{-1}\le(a_1^{-1}+a_2^{-1})/2$, we have

$$\Bigl\|h_\psi^{\frac{\alpha}{2z}}a_0h_\psi^{\frac{\alpha}{2z}}\Bigr\|_{z/\alpha}^{z/\alpha}\le\frac12\Bigl\{\Bigl\|h_\psi^{\frac{\alpha}{2z}}a_1h_\psi^{\frac{\alpha}{2z}}\Bigr\|_{z/\alpha}^{z/\alpha}+\Bigl\|h_\psi^{\frac{\alpha}{2z}}a_2h_\psi^{\frac{\alpha}{2z}}\Bigr\|_{z/\alpha}^{z/\alpha}\Bigr\},$$

$$\begin{aligned}\Bigl\|h_\varphi^{\frac{1-\alpha}{2z}}a_0^{-1}h_\varphi^{\frac{1-\alpha}{2z}}\Bigr\|_{z/(1-\alpha)}^{z/(1-\alpha)}&\le\Bigl\|h_\varphi^{\frac{1-\alpha}{2z}}\Bigl(\frac{a_1^{-1}+a_2^{-1}}{2}\Bigr)h_\varphi^{\frac{1-\alpha}{2z}}\Bigr\|_{z/(1-\alpha)}^{z/(1-\alpha)}\\&\le\frac12\Bigl\{\Bigl\|h_\varphi^{\frac{1-\alpha}{2z}}a_1^{-1}h_\varphi^{\frac{1-\alpha}{2z}}\Bigr\|_{z/(1-\alpha)}^{z/(1-\alpha)}+\Bigl\|h_\varphi^{\frac{1-\alpha}{2z}}a_2^{-1}h_\varphi^{\frac{1-\alpha}{2z}}\Bigr\|_{z/(1-\alpha)}^{z/(1-\alpha)}\Bigr\}.\end{aligned}$$

Hence we have

$$\Big\|h_\varphi^{\frac{1-\alpha}{2z}}a_0^{-1}h_\varphi^{\frac{1-\alpha}{2z}}\Big\|_{z/(1-\alpha)}=\Big\|h_\varphi^{\frac{1-\alpha}{2z}}\biggl(\frac{a_1^{-1}+a_2^{-1}}{2}\biggr)h_\varphi^{\frac{1-\alpha}{2z}}\Big\|_{z/(1-\alpha)},$$

which implies that $a_0^{-1}=\frac{a_1^{-1}+a_2^{-1}}{2}$, as easily verified. From this we easily have $a_1=a_2$. $\square$

Furthermore, one has

$$\begin{aligned}
h_\psi^{\frac{\alpha}{2z}}ah_\psi^{\frac{\alpha}{2z}}&=\bigl(h_\psi^{\frac{\alpha}{2z}}h_\varphi^{\frac{1-\alpha}{z}}h_\psi^{\frac{\alpha}{2z}}\bigr)^\alpha, &(0.6)\\
h_\varphi^{\frac{1-\alpha}{2z}}a^{-1}h_\varphi^{\frac{1-\alpha}{2z}}&=\bigl(h_\varphi^{\frac{1-\alpha}{2z}}h_\psi^{\frac{\alpha}{z}}h_\varphi^{\frac{1-\alpha}{2z}}\bigr)^{1-\alpha}, &(0.7)\\
h_{\psi_0}^{\frac{\alpha}{2z}}a_0h_{\psi_0}^{\frac{\alpha}{2z}}&=\bigl(h_{\psi_0}^{\frac{\alpha}{2z}}h_{\varphi_0}^{\frac{1-\alpha}{z}}h_{\psi_0}^{\frac{\alpha}{2z}}\bigr)^\alpha, &(0.8)\\
h_{\varphi_0}^{\frac{1-\alpha}{2z}}a_0^{-1}h_{\varphi_0}^{\frac{1-\alpha}{2z}}&=\bigl(h_{\varphi_0}^{\frac{1-\alpha}{2z}}h_{\psi_0}^{\frac{\alpha}{z}}h_{\varphi_0}^{\frac{1-\alpha}{2z}}\bigr)^{1-\alpha}. &(0.9)
\end{aligned}$$

Indeed, (0.7) is obvious from the second equality in (0.1) and $a^{-1}=c^*c$. Since $Q_{\alpha,z}(\psi\|\varphi)=Q_{1-\alpha,z}(\varphi\|\psi)$, we see in view of Lemma 0.1 that the minimizer of the infimum expression for $Q_{1-\alpha,z}(\varphi\|\psi)$ (instead of (0.2)) is a^{-1} (instead of a). This says that (0.6) is the equality corresponding to (0.7) when ψ,φ,α are replaced with $\varphi,\psi,1-\alpha$, respectively. (0.8) and (0.9) are similar.

Proposition 0.2. *In the above stated situation the following conditions are equivalent:*

(i) $D_{\alpha,z}(\psi_0\|\varphi_0)=D_{\alpha,z}(\psi\|\varphi)$, *i.e.,* $Q_{\alpha,z}(\psi_0\|\varphi_0)=Q_{\alpha,z}(\psi\|\varphi)$.

(ii) $\gamma(a_0)=a$ *and* $\|h_\psi^{\frac{\alpha}{2z}}\gamma(a_0)h_\psi^{\frac{\alpha}{2z}}\|_{z/\alpha}=\|h_{\psi_0}^{\frac{\alpha}{2z}}a_0h_{\psi_0}^{\frac{\alpha}{2z}}\|_{z/\alpha}$.

(iii) $\|h_\psi^{\frac{\alpha}{2z}}ah_\psi^{\frac{\alpha}{2z}}\|_{z/\alpha}=\|h_{\psi_0}^{\frac{\alpha}{2z}}a_0h_{\psi_0}^{\frac{\alpha}{2z}}\|_{z/\alpha}$.

(iv) $\gamma(a_0^{-1})=a^{-1}$ *and* $\|h_\varphi^{\frac{1-\alpha}{2z}}\gamma(a_0^{-1})h_\varphi^{\frac{1-\alpha}{2z}}\|_{z/(1-\alpha)}=\|h_{\varphi_0}^{\frac{1-\alpha}{2z}}a_0^{-1}h_{\varphi_0}^{\frac{1-\alpha}{2z}}\|_{z/(1-\alpha)}$.

(v) $\|h_\varphi^{\frac{1-\alpha}{2z}}a^{-1}h_\varphi^{\frac{1-\alpha}{2z}}\|_{z/(1-\alpha)}=\|h_{\varphi_0}^{\frac{1-\alpha}{2z}}a_0^{-1}h_{\varphi_0}^{\frac{1-\alpha}{2z}}\|_{z/(1-\alpha)}$.

Proof. (i) $\Longrightarrow$ (ii) & (iv). By [2, (22)] one has

$$\Big\|h_\psi^{\frac{\alpha}{2z}}\gamma(a_0)h_\psi^{\frac{\alpha}{2z}}\Big\|_{z/\alpha}\le\Big\|h_{\psi_0}^{\frac{\alpha}{2z}}a_0h_{\psi_0}^{\frac{\alpha}{2z}}\Big\|_{z/\alpha}, \tag{0.10}$$

$$\Big\|h_\varphi^{\frac{1-\alpha}{2z}}\gamma(a_0^{-1})h_\varphi^{\frac{1-\alpha}{2z}}\Big\|_{z/(1-\alpha)}\le\Big\|h_{\varphi_0}^{\frac{1-\alpha}{2z}}a_0^{-1}h_{\varphi_0}^{\frac{1-\alpha}{2z}}\Big\|_{z/(1-\alpha)}, \tag{0.11}$$

Moreover, since $\gamma(a_0^{-1})\ge\gamma(a_0)^{-1}$ due to Choi's inequality [1, Corollary 2.3], one has

$$\Big\|h_\varphi^{\frac{1-\alpha}{2z}}\gamma(a_0)^{-1}h_\varphi^{\frac{1-\alpha}{2z}}\Big\|_{z/(1-\alpha)}\le\Big\|h_\varphi^{\frac{1-\alpha}{2z}}\gamma(a_0^{-1})h_\varphi^{\frac{1-\alpha}{2z}}\Big\|_{z/(1-\alpha)} \tag{0.12}$$

From (0.10)–(0.12) it follows that

$$\begin{aligned}
&\alpha\|h_\psi^{\frac{\alpha}{2z}}\gamma(a_0)h_\psi^{\frac{\alpha}{2z}}\|_{z/\alpha}+(1-\alpha)\|h_\varphi^{\frac{1-\alpha}{2z}}\gamma(a_0)^{-1}h_\varphi^{\frac{1-\alpha}{2z}}\|_{z/(1-\alpha)}\\
&\quad\le\alpha\|h_{\psi_0}^{\frac{\alpha}{2z}}a_0h_{\psi_0}^{\frac{\alpha}{2z}}\|_{z/\alpha}+(1-\alpha)\|h_{\varphi_0}^{\frac{1-\alpha}{2z}}a_0^{-1}h_{\varphi_0}^{\frac{1-\alpha}{2z}}\|_{z/(1-\alpha)}
\end{aligned}$$

$$= Q_{\alpha,z}(\psi_0\|\varphi_0) = Q_{\alpha,z}(\psi\|\varphi).$$

By Lemma 0.1 we find that $\gamma(a_0) = a$ and all the inequalities in (0.10)–(0.12) must become equalities. Since $\gamma(a_0^{-1}) \ge \gamma(a_0)^{-1}$, we easily verify that the equality in (0.12) yields $\gamma(a_0^{-1}) = \gamma(a_0)^{-1}$ and hence $\gamma(a_0^{-1}) = a^{-1}$. Therefore, (ii) and (iv) hold.

(ii) $\Longrightarrow$ (iii) and (iv) $\Longrightarrow$ (v) are obvious.

(iii) $\Longrightarrow$ (i). By (iii) with (0.6) and (0.8) we have

$$\begin{aligned}
Q_{\alpha,z}(\psi\|\varphi) &= \mathrm{tr}\,\big(h_\psi^{\frac{\alpha}{2z}} h_\varphi^{\frac{1-\alpha}{z}} h_\psi^{\frac{\alpha}{2z}}\big)^z = \mathrm{tr}\,\big(h_\psi^{\frac{\alpha}{2z}} a h_\psi^{\frac{\alpha}{2z}}\big)^{z/\alpha} \\
&= \mathrm{tr}\,\big(h_{\psi_0}^{\frac{\alpha}{2z}} a_0 h_{\psi_0}^{\frac{\alpha}{2z}}\big)^{z/\alpha} = \mathrm{tr}\,\big(h_{\psi_0}^{\frac{\alpha}{2z}} h_{\varphi_0}^{\frac{1-\alpha}{z}} h_{\psi_0}^{\frac{\alpha}{2z}}\big)^z \\
&= Q_{\alpha,z}(\psi_0\|\varphi_0).
\end{aligned}$$

(v) $\Longrightarrow$ (i). By (iii) with (0.7) and (0.9) we have

$$\begin{aligned}
Q_{\alpha,z}(\psi\|\varphi) &= \mathrm{tr}\,\big(h_\varphi^{\frac{1-\alpha}{2z}} h_\psi^{\frac{\alpha}{z}} h_\varphi^{\frac{1-\alpha}{2z}}\big)^z = \mathrm{tr}\,\big(h_\varphi^{\frac{1-\alpha}{2z}} a^{-1} h_\varphi^{\frac{1-\alpha}{2z}}\big)^{z/(1-\alpha)} \\
&= \mathrm{tr}\,\big(h_{\varphi_0}^{\frac{1-\alpha}{2z}} a_0^{-1} h_{\varphi_0}^{\frac{1-\alpha}{2z}}\big)^{z/(1-\alpha)} = \mathrm{tr}\,\big(h_{\varphi_0}^{\frac{1-\alpha}{2z}} h_{\psi_0}^{\frac{\alpha}{z}} h_{\varphi_0}^{\frac{1-\alpha}{2z}}\big)^z \\
&= Q_{\alpha,z}(\psi_0\|\varphi_0).
\end{aligned}$$

□

Remark 0.3. Assume that $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\mathcal{N} = \mathcal{B}(\mathcal{K})$ with finite-dimensional Hilbert spaces $\mathcal{H}, \mathcal{K}$ and $\gamma = \Phi^*$ with a trace-preserving positive map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$. For $\psi = \rho$, $\varphi = \sigma$ we write

$$a = \sigma^{\frac{1-\alpha}{2z}} \big(\sigma^{\frac{1-\alpha}{2z}} \rho^{\frac{\alpha}{z}} \sigma^{\frac{1-\alpha}{2z}}\big)^{\alpha-1} \sigma^{\frac{1-\alpha}{2z}} = \rho^{-\frac{\alpha}{2z}} \big(\rho^{\frac{\alpha}{2z}} \sigma^{\frac{1-\alpha}{z}} \rho^{\frac{\alpha}{2z}}\big)^{\alpha} \rho^{-\frac{\alpha}{2z}}$$

and similarly for $\rho_0 := \Phi(\rho)$, $\sigma_0 := \Phi(\sigma)$,

$$a_0 = \rho_0^{-\frac{\alpha}{2z}} \big(\rho_0^{\frac{\alpha}{2z}} \sigma_0^{\frac{1-\alpha}{z}} \rho_0^{\frac{\alpha}{2z}}\big)^{\alpha} \rho_0^{-\frac{\alpha}{2z}}.$$

Consequently, the equality $\Phi^*(a_0) = a$ in (ii) coincides with [3, Theorem I.2 (2)]. The conditions given in [3, Theorem I.2 (3) and (4)] are

$$\begin{aligned}
\Phi\big((a^{1/2} \rho^{\frac{\alpha}{z}} a^{1/2})^{z/\alpha}\big) &= \big(a_0^{1/2} \rho_0^{\frac{\alpha}{z}} a_0^{1/2}\big)^{z/\alpha}, \\
\Phi\big((a^{-1/2} \sigma^{\frac{1-\alpha}{z}} a^{-1/2})^{z/(1-\alpha)}\big) &= \big(a_0^{-1/2} \sigma_0^{\frac{1-\alpha}{z}} a_0^{-1/2}\big)^{z/(1-\alpha)}.
\end{aligned}$$

In the setting of Proposition 0.2 these correspond to

$$\gamma_*\big((a^{1/2} h_\psi^{\frac{\alpha}{z}} a^{1/2})^{z/\alpha}\big) = \big(a_0 h_{\psi_0}^{\frac{\alpha}{z}} a_0^{1/2}\big)^{z/\alpha}, \tag{0.13}$$

$$\gamma_*\big((a^{-1/2} h_\varphi^{\frac{1-\alpha}{z}} a^{-1/2})^{z/(1-\alpha)}\big) = \big(a_0^{-1/2} h_{\varphi_0}^{\frac{1-\alpha}{z}} a_0^{-1/2}\big)^{z/(1-\alpha)}. \tag{0.14}$$

Since

$$\mathrm{tr}\,\gamma_*\big((a^{1/2} h_\psi^{\frac{\alpha}{z}} a^{1/2})^{z/\alpha}\big) = \mathrm{tr}\,\big(a^{1/2} h_\psi^{\frac{\alpha}{z}} a^{1/2}\big)^{z/\alpha} = \mathrm{tr}\,\big(h_\psi^{\frac{\alpha}{2z}} a h_\psi^{\frac{\alpha}{2z}}\big)^{z/\alpha} = \big\|h_\psi^{\frac{\alpha}{2z}} a h_\psi^{\frac{\alpha}{2z}}\big\|_{z/\alpha}^{z/\alpha},$$

we note that (0.13) is a stronger version of Proposition 0.2 (iii). Similarly, (0.14) is a stronger version of Proposition 0.2 (v). In view of [3, Theorem I.2 (iii) and (iv)] we may conjecture that Proposition 0.2 (i) $\Longrightarrow$ (0.13) whenever $z \ne \alpha$, and that Proposition 0.2 (i) $\Longrightarrow$ (0.14) whenever $z \ne 1-\alpha$, if γ is a unital normal CP map.

References

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