JORDAN ALGEBRAS OF SELF-ADJOINT OPERATORS

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1. **Introduction.** A Jordan algebra of self-adjoint operators on a Hilbert space, or simply, a *J-algebra*, is a real linear space of such operators closed under the product $A \circ B = \frac{1}{2}(AB+BA)$. A *JC-algebra*, respectively, a *JW-algebra*, is a uniformly closed, respectively, weakly closed J-algebra (we show in §3 that σ-weakly closed J-algebras are weakly closed). In a recent paper [4], D. Topping has shown that many of the techniques used in the study of self-adjoint algebras of operators are applicable to J-algebras. We continue in this direction, proving that various problems are simplified by passing to the second dual.

We begin by showing that the second dual $\mathfrak{A}^{**}$ of a J-algebra $\mathfrak{A}$ is isometric to a JW-algebra. We then use $\mathfrak{A}^{**}$, together with the second dual of the C^*-algebra $[\mathfrak{A}]$ generated by $\mathfrak{A}$ to investigate the uniformly closed Jordan ideals in $\mathfrak{A}$. If $\mathfrak{J}$ is such an ideal we prove that $\mathfrak{A}/\mathfrak{J}$ is isometrically isomorphic to a J-algebra. We also show that if $[\mathfrak{J}]$ is the C^*-algebra generated by $\mathfrak{J}$, then $\mathfrak{J} = [\mathfrak{J}] \cap \mathfrak{A}$, and $[\mathfrak{J}]$ is an ideal in $[\mathfrak{A}]$. We conclude with a characterization of the uniformly closed Jordan ideals in $\mathfrak{A}$. The simplified proof of this result was suggested to us by J. Ringrose.

We are indebted to D. Topping for invaluable correspondence on the subject.

2. **Topological preliminaries.** We recall that if $\mathfrak{X}$ is a real or complex normed linear space, there is a canonical isometry of $\mathfrak{X}$ into the second dual $\mathfrak{X}^{**}$, sending the weak (i.e., $\sigma(\mathfrak{X}, \mathfrak{X}^*)$) topology into the relative weak* (i.e., $\sigma(\mathfrak{X}^{**}, \mathfrak{X}^*)$) topology. We shall identify $\mathfrak{X}$ with its image. If $\mathfrak{K}$ is a uniformly closed convex subset of $\mathfrak{X}$, then it is weakly closed (see [2, p. 67]). It follows that if $\bar{\mathfrak{K}}$ is the weak* closure of $\mathfrak{K}$ in $\mathfrak{X}^{**}$, $\mathfrak{K} = \bar{\mathfrak{K}} \cap \mathfrak{X}$. If $\mathfrak{L}$ is a convex set in $\mathfrak{X}^{**}$ containing 0, the double polar $\mathfrak{L}^{00}$ coincides with the weak* closure $\bar{\mathfrak{L}}$.

We shall use a subscript 1 (resp., 0) to indicate the closed (resp., open) unit ball of a given normed linear space.

If $\mathfrak{X}$ and $\mathfrak{Y}$ are normed linear spaces, and S is a linear isometry of $\mathfrak{X}$ into $\mathfrak{Y}$, the adjoint S^* maps $\mathfrak{Y}_1^*$ onto $\mathfrak{X}_1^*$, and kernel $S^* = S(\mathfrak{X})^0$. On the other hand, if $T: \mathfrak{X} \to \mathfrak{Y}$ is a linear map with $T(\mathfrak{X}_0) = \mathfrak{Y}_0$, T^* is an isometry and a weak* homeomorphism (see [2, p. 101]) of $\mathfrak{Y}^*$ into $\mathfrak{X}^*$, and $T^*(\mathfrak{Y}^*) = (\text{kernel } T)^0$. Turning to the second adjoints, S^{**} is an isometry and weak* homeomorphism of $\mathfrak{X}^{**}$ into $\mathfrak{Y}^{**}$, extending S, and $S^{**}(\mathfrak{X}^{**}) = S(\mathfrak{X})^{00} = S(\mathfrak{X})^-$. Similarly, $T^{**}(\mathfrak{X}_1^{**}) = \mathfrak{Y}_1^{**}$, and if $\mathfrak{K} = \text{kernel } T$, kernel $T^{**} = \mathfrak{K}^{00} = \bar{\mathfrak{K}}$. T^{**} induces an isometry of $\mathfrak{X}^{**}/\bar{\mathfrak{K}}$ onto $\mathfrak{Y}^{**}$.

Received by the editors October 31, 1965.

3. The second dual.

THEOREM 1. *Let $\mathfrak{A}$ be a J-algebra. There is an isometry of $\mathfrak{A}^{**}$ onto a JW-algebra, carrying the weak* topology onto the weak operator topology, and providing $\mathfrak{A}^{**}$ with a multiplication extending that of $\mathfrak{A}$.*

Proof. Let $\mathfrak{C}$ be the C^*-algebra $[\mathfrak{A}]$. Due to the Sherman, Takeda, Grothendieck theory (see [3]), we may identify $\mathfrak{C}^{**}$ with a von Neumann algebra, for which the weak and σ-weak operator topologies coincide with the weak* topology, and the multiplication extends that of $\mathfrak{C}$. From §2, the second adjoint of the identity map of $\mathfrak{A}$ into $\mathfrak{C}$ is an isometry and a weak* homeomorphism of $\mathfrak{A}^{**}$ onto $\bar{\mathfrak{A}}$, the weak operator closure of the J-algebra $\mathfrak{A}$. $\bar{\mathfrak{A}}$ is again a J-algebra (see [4]), hence a JW-algebra. As the isometry reduces to the identity on $\mathfrak{A}$, the resulting Jordan product on $\mathfrak{A}^{**}$ extends that on $\mathfrak{A}$.

It is readily verified that the product on $\mathfrak{A}^{**}$ coincides with both extensions of that on $\mathfrak{A}$ defined by Arens [1].

With the exception of the following paragraph, it will not be necessary to discuss the σ-weak operator topology. As it coincides with the weak operator topology on $\mathfrak{C}^{**}$, the same is true on $\mathfrak{A}^{**}$. In addition, any σ-weakly closed J-algebra $\mathfrak{A}$ is weakly closed. To see this we note that the unit ball $\mathfrak{A}_1$ is strongly, hence weakly dense in $(\bar{\mathfrak{A}})_1$. For, as has been indicated to us by Topping, the Kaplansky Density Theorem is valid for J-algebras (see, e.g., the proof in [5]). But $\mathfrak{A}_1$ is weakly compact, hence $\mathfrak{A}=\bar{\mathfrak{A}}$.

A *representation* ϕ of a J-algebra $\mathfrak{A}$ is a bounded linear map of $\mathfrak{A}$ into $\mathfrak{L}_{SA}$, the self-adjoint operators on some Hilbert space, satisfying $\phi(A \circ B)=\phi(A) \circ \phi(B)$. As $\phi: \mathfrak{A} \to \mathfrak{L}$ is continuous in the $\sigma(\mathfrak{A}, \mathfrak{A}^*)$ and $\sigma(\mathfrak{L}, \mathfrak{L}_*)$ topologies, the restriction map $\phi^*: \mathfrak{L}_* \to \mathfrak{A}^*$ is defined, and we obtain a σ-weakly continuous representation $\phi^{**}: \mathfrak{A}^{**} \to \mathfrak{L}$. Any σ-weakly continuous homomorphism of JW-algebras splits into an isometric isomorphism, and a zero map. As we shall not use this fact, and the proof is essentially the same as that for von Neumann algebras, we omit the details.

4. Ideal theory.

A *Jordan ideal* $\mathfrak{J}$ in a J-algebra $\mathfrak{A}$ is a linear subspace of $\mathfrak{A}$ such that if $A \in \mathfrak{A}$ and $D \in \mathfrak{J}$, then $A \circ D \in \mathfrak{J}$. If $\mathfrak{J}$ is uniformly closed, the quotient space $\mathfrak{A}/\mathfrak{J}$ forms a (nonassociative) normed algebra in the usual way. The weak operator closure $\bar{\mathfrak{J}}$ is a Jordan ideal in $\bar{\mathfrak{A}}$. For if $A \in \mathfrak{A}$ and $D \in \bar{\mathfrak{J}}$, choose a net $D_\alpha \in \mathfrak{J}$ with $D_\alpha \to D$ weakly. Then $A \circ D_\alpha \in \mathfrak{J}$ and $A \circ D_\alpha \to A \circ D$, hence the latter is in $\bar{\mathfrak{J}}$. If $A \in \bar{\mathfrak{A}}$ and $D \in \bar{\mathfrak{J}}$, choose a net $A_\alpha \to A$. Then $A_\alpha \circ D \in \bar{\mathfrak{J}}$, $A_\alpha \circ D \to A \circ D$, and $A \circ D \in \bar{\mathfrak{J}}$.

Topping proved [4] that if $\mathfrak{A}$ is a JW-algebra, the lattice of projections in $\mathfrak{A}$ must be complete. Letting E be the maximal projection in $\mathfrak{A}$, $A=AE=EA$ for all $A \in \mathfrak{A}$. He then pointed out that if $\mathfrak{J}$ is a weakly closed ideal in $\mathfrak{A}$, and F is the maximal projection in $\mathfrak{J}$, then $\mathfrak{J}=\mathfrak{A}F$, and F commutes with all the elements of $\mathfrak{A}$.

THEOREM 2. *Let $\mathfrak{J}$ be a uniformly closed Jordan ideal in a J-algebra $\mathfrak{A}$. Then $\mathfrak{A}/\mathfrak{J}$ is isometrically isomorphic with a J-algebra, $\mathfrak{J}=[\mathfrak{J}] \cap \mathfrak{A}$, and $[\mathfrak{J}]$ is an ideal in $[\mathfrak{A}]$.*

Proof. As we showed in the proof of Theorem 1, $\mathfrak{A}^{**}$ may be identified with the weak closure of $\mathfrak{A}$ in $\mathfrak{C}^{**}$, where $\mathfrak{C}=[\mathfrak{A}]$. From above, the weak closure $\mathfrak{J}^{00}=\bar{\mathfrak{J}}$ is an ideal in $\bar{\mathfrak{A}}$. The maximal projection F of $\bar{\mathfrak{J}}$ must lie in the center of $\mathfrak{C}^{**}=[\mathfrak{A}]^{-}$.

Let T be the canonical homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/\mathfrak{J}$. Then $T(\mathfrak{A}_0)=(\mathfrak{A}/\mathfrak{J})_0$ and T^{**} induces an isometry of $\mathfrak{A}^{**}/\bar{\mathfrak{J}}$ onto $(\mathfrak{A}/\mathfrak{J})^{**}$ (see §2). Composing the inverse of this map with the canonical isometry of $\mathfrak{A}/\mathfrak{J}$ into $(\mathfrak{A}/\mathfrak{J})^{**}$, we obtain the map $A+\mathfrak{J} \to A+\bar{\mathfrak{J}}$, and conclude the latter is an isometric isomorphism of $\mathfrak{A}/\mathfrak{J}$ into $\bar{\mathfrak{A}}/\bar{\mathfrak{J}}$. On the other hand, the homomorphism $B+\bar{\mathfrak{J}} \to B(I-F)$ of $\bar{\mathfrak{A}}/\bar{\mathfrak{J}}$ onto $\bar{\mathfrak{A}}(I-F)$ is an isometry, as if $D \in \bar{\mathfrak{J}}$,

$$\|B+D\| = \|B(I-F)+(B+D)F\| = \max\,[\|B(I-F)\|, \|(B+D)F\|],$$

the von Neumann algebra $\mathfrak{C}^{**}$ being isomorphic, hence isometric to $\mathfrak{C}^{**}F \oplus \mathfrak{C}^{**}(I-F)$. Thus $A+\mathfrak{J} \to A(I-F)$ is an isometric isomorphism of $\mathfrak{A}/\mathfrak{J}$ into the JW-algebra $\bar{\mathfrak{A}}(I-F)$.

If $\mathfrak{J}'$ is the weak* closure of $\mathfrak{J}$ in $\mathfrak{A}^{**}$, we have from §2 that $\mathfrak{J}=\mathfrak{J}' \cap \mathfrak{A}$. As the injection of $\mathfrak{A}^{**}$ into $\mathfrak{C}^{**}$ is a weak* homeomorphism, $\mathfrak{J}=\bar{\mathfrak{J}} \cap \mathfrak{A}$. As $AF=A$ for all $A \in \mathfrak{J}$, $CF=C$ for all $C \in [\mathfrak{J}]$, i.e., $[\mathfrak{J}] \subseteq \mathfrak{C}^{**}F$. Thus

$$[\mathfrak{J}] \cap \mathfrak{A} \subseteq \mathfrak{C}^{**}F \cap \bar{\mathfrak{A}} = \bar{\mathfrak{A}}F = \bar{\mathfrak{J}},$$

and $[\mathfrak{J}] \cap \mathfrak{A} \subseteq \mathfrak{J}$. The converse inclusion is trivial.

We next prove that $[\mathfrak{J}]^{-}=\mathfrak{C}^{**}F$. It suffices to show that if $B \in \mathfrak{C}^{**}=[\mathfrak{A}]^{-}$ is such that $B=BF$, then $B \in [\mathfrak{J}]^{-}$. B is a weak limit of finite linear combinations of terms of the form $A_1 \cdots A_n$ with $A_i \in \mathfrak{A}$. Multiplying these sums by F, we may instead assume the A_i lie in $\bar{\mathfrak{J}}=\bar{\mathfrak{A}}F$. Using the Kaplansky Density Theorem, we may select $A_i^{\alpha} \in \mathfrak{J}$ with $\|A_i^{\alpha}\| \leq \|A_i\|$, and $A_i^{\alpha} \to A_i$ strongly. Thus $A_1^{\alpha} \cdots A_n^{\alpha} \to A_1 \cdots A_n$ strongly, $A_1 \cdots A_n \in [\mathfrak{J}]^{-}$, and $B \in [\mathfrak{J}]^{-}$.

It follows that $[\mathfrak{J}]^{-}$ is an ideal in $\mathfrak{C}^{**}$, and $[\mathfrak{J}]=[\mathfrak{J}]^{-} \cap \mathfrak{C}$ is an ideal in $\mathfrak{C}$.

5. A characterization of Jordan ideals.

THEOREM 3. *Let $\mathfrak{J}$ be a subspace of a J-algebra $\mathfrak{A}$ with a multiplicative identity, or a uniformly closed subspace of an arbitrary J-algebra $\mathfrak{A}$. Then the following are equivalent:*

(1) *$\mathfrak{J}$ is a Jordan ideal in $\mathfrak{A}$.*

(2) *If $A \in \mathfrak{A}$ and $D \in \mathfrak{J}$, then $ADA \in \mathfrak{J}$.*

Proof. That (1) always implies (2) follows from the equation

$$ADA = \tfrac{1}{2}[(A(DA+AD)+(DA+AD)A)-(A^2D+DA^2)].$$

Conversely, say that (2) is satisfied and E is a multiplicative identity for $\mathfrak{A}$. If $A \in \mathfrak{A}$ and $D \in \mathfrak{J}$, the $A \circ D \in \mathfrak{J}$ as

$$AD+DA = (A+E)D(A+E)-ADA-D.$$

If $\mathfrak{A}$ does not have an identity, let E be the maximal projection in the JW-algebra $\mathfrak{A}^{**}$. From the Kaplansky Density Theorem, there is a net $B_\alpha \in \mathfrak{A}$ with $\|B_\alpha\| \leqq 1$, converging strongly to E. It follows that

$$(A+B_\alpha)D(A+B_\alpha)-ADA-D \to AD+DA$$

in the strong, hence weak topologies, and $AD+DA \in \bar{\mathfrak{J}} \cap \mathfrak{A}$. As $\mathfrak{J}$ is convex and uniformly closed, we have from §2, $\bar{\mathfrak{J}} \cap \mathfrak{A} = \mathfrak{J}$.

Bibliography

1. R. Arens, *The adjoint of a bilinear operation*, Proc. Amer. Math. Soc. **2** (1951), 839–848.

2. N. Bourbaki, *Espaces vectoriels topologique*, Actualités Sci. Indust. No. 1229, Hermann, Paris, 1955.

3. Z. Takeda, *Conjugate spaces of operator algebras*, Proc. Japan Acad. **30** (1954), 90–95.

4. D. Topping, *Jordan algebras of self-adjoint operators*, Mem. Amer. Math. Soc. No. 53 (1965), 48 pp.

5. J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien*, Gauthier-Villars, Paris, 1957.

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