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On Period, Cycles and Fixed Points of a Quantum Channel

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Abstract. We consider a quantum channel acting on an infinite-dimensional von Neumann algebra of operators on a separable Hilbert space. When there exists an invariant normal faithful state, the cyclic properties of such channels are investigated passing through the decoherence-free algebra and the fixed points domain. Both these spaces are proved to be images of a normal conditional expectation so that their consequent atomic structures are analyzed in order to give a better description of the action of the channel and, for instance, of its Kraus form and invariant densities.

1. Introduction

Quantum channels are basic tools in quantum theory. As a representation of a communication channel, they play a central role in quantum information theory and quantum information processing. They are seen as the counterpart of Markov operators in the noncommutative models, and they are generally used to represent the evolution of an open quantum system in discrete-time models.

Some classical results related to Markov chains still need to be clarified in their noncommutative version, and the quantum theory reveals to be richer and more complicated due to the different framework and techniques we are dealing with. In particular, while the fixed points are a quite natural topic, already extensively studied also in the quantum case, the cyclic behavior (related to what is classically called period for a Markov chain) has still many mysterious aspects, starting from the fact that a good definition of a period for any irreducible quantum channel is not recognized by now; moreover, these cycles have showed some typically noncommutative features. Both objects (cycles and fixed points), however, display a kind of rigidity in the structure of the channel which can link different irreducible components of the evolution;

this is a strongly quantum feature in the sense that it is something that cannot be observed in a purely classical context.

Some aspects of this rigidity were already known and were object of interest in many papers in the last years, related to different problems, e.g., the structure of the invariant states and irreducible decompositions [8,12], decoherence-free algebra and environmental decoherence [13,16], the notion of sufficiency in quantum statistics [28,32,34], periodicity and ergodic properties [11]. In finite dimension, the structure of the channel and its spectrum, cycles and multiplicative properties were investigated in [42,43]. In particular, multiplicative properties were studied in view of applications to quantum information theory, such as quantum error correction and private subspaces (e.g., [15,29,35]) or entanglement breaking channels [36].

In the present paper, a quantum channel is a unital normal completely positive operator Φ on the algebra $B(\mathcal{H})$ of the bounded operators on a separable (infinite-dimensional) Hilbert space $\mathcal{H}$. In this setting, we study the subspace of fixed points and the so-called decoherence-free algebra (DFA) of the channel. The aim is to obtain a unified description of these spaces and their relations, together with the restricted action of the channel, in the presence of a faithful normal invariant state (i.e., when the channel is positive recurrent).

Under the last assumption, the fixed point subspace is easily seen to be a subalgebra; moreover, it follows by the mean ergodic theorem for quantum dynamical systems [22,30] that it is the range of a (faithful normal) conditional expectation, contained in the w^*-closed convex hull of the semigroup generated by the channel. Let us remark that the situation is more complicated in the general case, where the fixed point subspace may be not a subalgebra, see, e.g., [1,10] for some descriptions, constructions and examples.

The second main object of interest, the decoherence-free algebra of the channel, can be informally defined as the largest subalgebra on which Φ acts as a *-endomorphism. For dynamical semigroups of channels, the DFA was a very popular object already in the 1970s and 1980s, extensively used in order to study asymptotic properties of the semigroup (see, e.g., [22,30,37]), in particular, together with the fixed points, in order to distinguish between ergodicity and mean ergodicity.

More recently, the DFA appeared again in the literature and was reconsidered because of the interest in reversible subsystems arising in quantum information and in relation to environmental decoherence, as defined in [9]. Most of these previous studies are generally concentrated in the case of a continuous-time Markov semigroup. For instance, in [17] a characterization of fixed points and of the DFA was found in terms of the Lindblad form of the generator of the quantum dynamical semigroup. The DFA also appears in [13] or [27], linked with environmental decoherence and other forms of decompositions of the algebra.

Assuming the existence of a faithful invariant state, the analysis of the peripheral eigenvectors and a structural approach to the Perron–Frobenius spectral theory in [25], and more recently and in more generality in [7], produce the opportunity to split the algebra into a "stable" and "reversible" part with respect to the semigroup (a Jacobs–DeLeeuw–Glicksberg-type decomposition). The reversible part is a subalgebra spanned by the peripheral eigenvectors, and it is the range of a (faithful normal) conditional expectation commuting with the channel. This subalgebra is contained in the DFA, and it is easily seen that in finite dimensions, these two subalgebras coincide, [42].

As one of our main results, we prove that, for a channel acting on an atomic von Neumann algebra and with a faithful invariant state, the reversible subalgebra coincides with DFA also in infinite dimensions. Note that this implies that the DFA is the range of a conditional expectation, in particular it is atomic. This allows us to deduce the structural properties of the DFA and the action of the channel; in particular, we obtain a decomposition of the channel into blocks with a finite cyclic structure. On the other hand, existence of a conditional expectation or more generally the atomicity of DFA was commonly assumed as a hypothesis (see [9,13,16]) that allowed to obtain more precise results on environmental decoherence and the structure of the semigroup. Notice that our proof can be applied also to the continuous-time case, so we can conclude that these results hold more generally. Furthermore, one could use the conditional expectation for the study of the decoherence time and spectral gap inequalities as in [4,6] in finite dimensions. These possibilities are remarked on but not pursued further in the paper and left for future work.

Also, it would be interesting to investigate different generalizations of our result. For instance, a natural question would be wondering what happens when we have only an invariant faithful weight instead of a state. Here the question can become much more difficult but an important step in this direction can be found in [33]: under some additional suitable conditions and the presence of an invariant faithful weight, the authors can prove the existence of a compatible conditional expectation on the algebra of detectable observables (the space $\mathcal{M}_1$ in the definition of decoherence in Remark 3) for a continuous-time evolution. This result is someway parallel to our Theorem 1 and Corollary 1 and can potentially be the starting point to develop other structure results we could deduce here.

The paper is structured as follows: Section 2 contains a characterization of the fixed points and of the DFA as commutants of suitable algebras defined in terms of the Kraus operators of the channel. This can be seen as a discrete-time counterpart of [17] for quantum dynamical semigroups, where the two spaces are characterized using the Lindblad form of the generator. The relation between the DFA and the reversible subalgebra is also proved here (Theorem 1).

Afterward, in Sect. 3, we introduce the study of cycles, using also the notion of period, as introduced in the quantum context in [21], and generalizing it to the infinite-dimensional case. We start from the irreducible case, where the relation between the DFA, the fixed points of powers of the channel, and the cyclic decomposition is evident and can be clearly described (Corollary 2 and Proposition 6). Then, we turn to the reducible case, where we exploit the fact that the two algebras are atomic, to deduce ad hoc decompositions of the invariant states, relations with the Kraus operators, a better description of the conditional expectations and of the cyclic behavior of the channel (Proposition 8 and Theorem 2). These results are strictly related to the studies in [8,16] and the decomposition appearing in the last part of [43]. Finally, in Sect. 4, we apply our results to analyze a remarkable family of quantum channels, i.e., the so-called open quantum random walks. This will give us the opportunity to show in detail some explicit examples: in the last one, we try to throw a glance to a channel without invariant state.

2. Multiplicative Domain, Decoherence-Free Algebra and Fixed Points

Let $\mathcal{H}$ be a separable Hilbert space. We will denote the algebra of bounded operators over $\mathcal{H}$ by $B(\mathcal{H})$, the predual of $B(\mathcal{H})$ by $B(\mathcal{H})_*$ and the set of normal states on $B(\mathcal{H})$ by $\mathfrak{S}(\mathcal{H})$. The predual $B(\mathcal{H})_*$ will be identified with the set of trace-class operators in $B(\mathcal{H})$, and then, $\mathfrak{S}(\mathcal{H})$ is the set of positive operators with unit trace. The identity operator on $\mathcal{H}$ will be sometimes denoted by $I_{\mathcal{H}}$ if the space $\mathcal{H}$ has to be emphasized.

The main object studied in this paper is a unital normal completely positive map $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$; such maps are called (quantum) channels. The preadjoint of Φ is the map $\Phi_* : B(\mathcal{H})_* \to B(\mathcal{H})_*$, defined by

$$\mathrm{Tr}\,[\Phi_*(\rho)A] = \mathrm{Tr}\,[\rho\Phi(A)], \qquad \rho \in B(\mathcal{H})_*, \qquad A \in B(\mathcal{H}).$$

The preadjoint of a channel is completely positive and preserves trace.

We consider here only normal states, so a state is for us a nonnegative operator in $B(\mathcal{H})_*$ with unit trace; a strictly positive state is called faithful. Moreover, a state is invariant for the channel when it is a fixed point for the preadjoint Φ_*. In analogy with the classical probability environment, we say that a channel is positive recurrent when it has an invariant faithful state (not necessarily unique).

It is well known that any channel Φ has a representation of the form

$$\Phi(A) = \sum_{k=1}^{\infty} V_k^* A V_k, \qquad A \in B(\mathcal{H}), \tag{1}$$

where the Kraus operators $V_k \in B(\mathcal{H})$ are such that $\sum_k V_k^* V_k = I$. Let $\mathcal{K}$ be a separable Hilbert space, and let $\{e_k\}$ be an orthonormal basis of $\mathcal{K}$. Let us define

$$V = \sum_k V_k \otimes |e_k\rangle, \tag{2}$$

then $V : \mathcal{H} \to \mathcal{H} \otimes \mathcal{K}$ is an isometry, and we obtain the Stinespring representation

$$\Phi(A) = V^*(A \otimes I_{\mathcal{K}})V, \qquad A \in B(\mathcal{H}). \tag{3}$$

We will consider the following sets of operators:

- the fixed points' domain

$$\mathcal{F}(\Phi) := \{A \in B(\mathcal{H}), \Phi(A) = A\};$$

- the multiplicative domain

$$\mathcal{M}(\Phi) := \{A \in B(\mathcal{H}), \Phi(A^*A) = \Phi(A)^*\Phi(A), \Phi(AA^*) = \Phi(A)\Phi(A)^*\};$$

- the decoherence-free algebra (DFA)

$$\mathcal{N}(\Phi) := \cap_n \mathcal{M}(\Phi^n).$$

Since the map Φ will be fixed throughout, we will mostly use the notations $\mathcal{M} = \mathcal{M}(\Phi)$, $\mathcal{N} = \mathcal{N}(\Phi)$ and $\mathcal{F} = \mathcal{F}(\Phi)$.

We now collect some basic facts about these sets. The proofs are included for the convenience of the reader.

First, notice that the set of fixed points is in general not a subalgebra (in contrast, as we will see, to $\mathcal{M}$ and $\mathcal{N}$). An example can easily be constructed simply using a classical Markov chain with a transient class which can have access to two different positive recurrent classes. For quantum examples and discussion around the characterization of $\mathcal{F}$ and the following proposition, see, for example, [1] or [10, Section 3].

Proposition 1. *$\mathcal{F}$ is a von Neumann algebra if and only if it is contained in $\mathcal{N}$. In this case, we have*

$$\mathcal{F} = \{V_j, V_j^*,\ j = 1, 2 \ldots\}',$$

where $\{\ \}'$ denotes the commutant.

Proof. The first statement is quite obvious. Assume now that $\mathcal{F}$ is a von Neumann algebra and let $A \in \mathcal{F}$. Then

$$0 = \Phi(A^*A) - A^*A = \sum_j (V_jA - AV_j)^*(V_jA - AV_j),$$

this implies $AV_j = V_jA$. Similarly, we obtain $AV_j^* = V_j^*A$. It follows that $\mathcal{F} \subseteq \{V_j, V_j^*,\ j = 1, 2 \ldots\}'$. The converse inclusion is clear. □

We point out that when there is a faithful normal invariant state, $\mathcal{F}$ is included in $\mathcal{N}$ and so the previous characterization holds.

We now turn to the multiplicative domain $\mathcal{M}$. It was proved by Choi [14] that $\mathcal{M}$ satisfies the following multiplicative property:

$$\Phi(AB) = \Phi(A)\Phi(B),\ \Phi(BA) = \Phi(B)\Phi(A),\ \text{for all } A \in B(\mathcal{H}) \text{ and } B \in \mathcal{M}. \tag{4}$$

Consequently, $\mathcal{M}$ is a von Neumann subalgebra in $B(\mathcal{H})$ and the restriction of Φ to $\mathcal{M}$ is a *-homomorphism. We have the following characterization of $\mathcal{M}$.

Proposition 2. *Let $V_1, V_2, \ldots$ be Kraus operators as in (1). Then,*

$$\mathcal{M} = \{V_jV_k^*,\ j, k = 1, 2, \ldots\}'.$$

Proof. It will be convenient to use the Stinespring representation (3). Let V be as in (2) and let $P = VV^*$, then $P \in B(\mathcal{H} \otimes \mathcal{K})$ is a projection, and we have $A \in \mathcal{M}$ if and only if $A \otimes I$ commutes with P. Indeed, suppose $A \in \mathcal{M}$, then

$$V^*(A^*A \otimes I)V = V^*(A^* \otimes I)P(A \otimes I)V.$$

It follows that $P(A^* \otimes I)(1-P)(A \otimes I)P = 0$, hence $(1-P)(A \otimes I)P = 0$, so that

$$(A \otimes I)P = P(A \otimes I)P.$$

Similarly, we get the same for A^*, and this implies that

$$P(A \otimes I) = P(A \otimes I)P = (A \otimes I)P.$$

The converse is easy. Now notice that $P = \sum_{j,k} V_j V_k^* \otimes |e_j\rangle\langle e_k|$, and this implies the statement. □

It is clear from the definition that the DFA $\mathcal{N}$ is a von Neumann subalgebra as well, and it is also easy to see that $\mathcal{N}$ is the smallest subalgebra such that the restriction $\Phi|_{\mathcal{N}}$ is a *-endomorphism.

Remark 1. Notice that $\Phi|_{\mathcal{N}}$ is not always a *-automorphism. Indeed, $\mathcal{N}$ can have, for instance, a nontrivial intersection with the kernel of Φ. Since this intersection is a subalgebra, it then contains a nonzero projection $0 \neq P \in \mathrm{Ker}(\Phi) \cap \mathcal{N}$. On the other hand, any projection in $\mathrm{Ker}(\Phi)$ is necessarily in $\mathcal{N}$, so that this happens if and only if Φ is not faithful.

Proposition 3. *We have the following characterizations of $\mathcal{N}$:*

(i) $\mathcal{N} = \{V_{i_1} \dots V_{i_n} V_{j_1}^* \dots V_{j_n}^*,\ i_k, j_k = 1, 2, \dots;\ n \in \mathbb{N}\}'$.

(ii) *$\mathcal{N}$ is the von Neumann algebra generated by the preserved projections, i.e., by the set*

$$\{Q \in B(\mathcal{H}) \,:\, \Phi^n(Q) \text{ is a projection } \forall\, n \geq 0\}.$$

Proof. (i) is immediate from Proposition 2. (ii) holds since a projection Q is in $\mathcal{M}$ if and only if $\Phi(Q)$ is again a projection. □

Point (ii) already appeared in [13] (see also references therein) and was the original representation of the decoherence-free algebra used in [9] when introducing environmental decoherence.

The following results are well known.

Proposition 4. *Assume that there is a faithful normal invariant state for Φ. Then*

(i) *$\mathcal{F}$ is a von Neumann subalgebra;*

(ii) *the restriction $\Phi|_{\mathcal{N}}$ is a *-automorphism.*

Proof. See, for example, [37] and references therein. □

2.1. Maps with a Faithful Invariant State

In this section, we assume that there is a faithful normal state $\rho \in \mathfrak{S}(\mathcal{H})$ for Φ. In this case, there is another special subalgebra investigated in the literature, e.g., [7,25], appearing in some asymptotic splitting, usually called the *reversible subalgebra* and denoted by $\mathcal{M}_r$. We describe the reversible subalgebra, following [7,25] or [27]. Let $\mathbf{S}$ be the closure of the semigroup of channels $\{\Phi^n,\ n \in \mathbb{N}\}$ in the point-ultraweak topology and define

$$\mathcal{M}_r := \{X \in B(\mathcal{H}),\ T(X)^* T(X) = T(X^* X), \forall T \in \mathbf{S}\}.$$

We will show below, in Theorem 1, that the equality $\mathcal{M}_r = \mathcal{N}$ holds for channels on $B(\mathcal{H})$ (or more generally on atomic von Neumann algebras).

Due to the presence of a faithful normal invariant state, for any $\varphi \in B(\mathcal{H})_*$, the set

$$\{\Phi_*^n \varphi,\ n \in \mathbb{N}\}$$

is weakly relatively compact, and equivalently, the set $\mathbf{S}$ consists of normal operators and is a compact semitopological semigroup ([25, Proposition 2.1]). Further, $\mathbf{S}$ contains a minimal ideal $M(\mathbf{S})$, which is a compact topological group. Let F be the unit of this group; then, $M(\mathbf{S}) = F \circ \mathbf{S}$ and F is a normal conditional expectation preserving the invariant state ρ such that $TF = FT$ for all $T \in \mathbf{S}$. Finally, $\mathcal{M}_r$ is a von Neumann algebra, and the minimal ideal $M(\mathbf{S})$ acts as a compact group of *-automorphisms on $\mathcal{M}_r$ ([7, Theorem 1.2 and Corollary 1.3]).

This last fact trivially implies, in particular, that $\mathcal{M}_r \subseteq \mathcal{N}$ and that equality holds in finite dimension, but the infinite-dimensional case is quite more delicate and tricky.

Now let $X \in B(\mathcal{H})$ and let $O_0(X) := \{\Phi^k(X), k \in \mathbb{N}\}$ be the orbit of X under $\{\Phi^k\}_{k\geq 0}$. Then the weak*-closure $\bar{O}_0(X)$ is the orbit of X under $\mathbf{S}$,

$$\bar{O}_0(X) = O(X) := \{T(X), T \in \mathbf{S}\}$$

and we can define the *stable subspace* as

$$\mathcal{M}_s := \{X \in B(\mathcal{H}),\ 0 \in O(X)\}.$$

The following lemma can be deduced from [27, Theorem 2.1], [7, Theorem 1.2] and [25, Proposition 2.2]. Since we did not find an explicit and comprehensive statement in the literature, we reconstruct here the detailed result that we need and the proof for the convenience of the reader.

Lemma 1. *Let F be the normal conditional expectation introduced before.*

1. $\mathcal{M}_r = F(B(\mathcal{H})) = \overline{\mathrm{span}\{X \in B(\mathcal{H}),\ \Phi(X) = \lambda X,\ |\lambda| = 1\}}^{w^*}$;
2. $\mathcal{M}_s = Ker(F)$.

Proof. 1. Let us denote the last set on the RHS by $\mathcal{M}_0$. We will prove the chain of inclusions

$$\mathcal{M}_r \subseteq F(B(\mathcal{H})) \subseteq \mathcal{M}_0 \subseteq \mathcal{M}_r.$$

First, if $X \in \mathcal{M}_r$, then since $F \in \mathbf{S}$, we have $F(X^*X) = F(X)^*F(X)$. This implies

$$F((X - F(X))^*(X - F(X))) = 0$$

and since F is faithful, we have $X = F(X) \in F(B(\mathcal{H}))$.

To prove the second inclusion, let $X = F(X)$. Let $\widehat{M(\mathbf{S})}$ be the dual group and let $\chi \in \widehat{M(\mathbf{S})}$. Let us define

$$X_\chi := \int_{M(\mathbf{S})} \overline{\chi(T)}T(X)\mathrm{d}\mu(T),$$

where μ is the normalized Haar measure over $M(\mathbf{S})$. The integral is defined in the weak*-topology, so we have $X_\chi \in B(\mathcal{H})$, and since for $T \in M(\mathbf{S})$, we have $T = TF = FT$, we obtain

$$\Phi(X_\chi) = \int_{M(\mathbf{S})} \overline{\chi(T)} \Phi T(X) \mathrm{d}\mu(T) = \int_{M(\mathbf{S})} \overline{\chi(T)} F \Phi T(X) \mathrm{d}\mu(T)$$
$$= \chi(F\Phi) \int_{M(\mathbf{S})} \overline{\chi(T)} T(X) \mathrm{d}\mu(T) = \chi(F\Phi) X_\chi$$

(since $F\Phi = \Phi F \in M(\mathbf{S})$), so that $X_\chi \in \mathcal{M}_0$. Let now $\psi \in B(\mathcal{H})_*$ be such that $\mathrm{Tr}\,[\psi Y] = 0$ for any peripheral eigenvector Y of Φ. Then

$$0 = \mathrm{Tr}\,[\psi X_\chi] = \int_{M(\mathbf{S})} \overline{\chi(T)} \mathrm{Tr}\,[\psi T(X)] \mathrm{d}\mu(T), \qquad \forall \chi \in \widehat{M(\mathbf{S})}.$$

Since the characters span the space of square integrable functions on $M(\mathbf{S})$ and the function $T \mapsto \mathrm{Tr}\,[\psi T(X)]$ is continuous, it follows that $\mathrm{Tr}\,[\psi T(X)] = 0$ for all $T \in M(\mathbf{S})$; in particular, $\mathrm{Tr}\,[\psi X] = \mathrm{Tr}\,[\psi F(X)] = 0$. This implies that $\{Y,\ \Phi(Y) = \lambda Y,\ |\lambda| = 1\}^\perp \subseteq F(B(\mathcal{H}))^\perp$, so that

$$F(B(\mathcal{H})) = F(B(\mathcal{H}))^{\perp\perp} \subseteq \{Y,\ \Phi(Y) = \lambda Y,\ |\lambda| = 1\}^{\perp\perp} = \mathcal{M}_0.$$

Finally, let $\Phi(X) = \lambda X$ for some $|\lambda| = 1$; then $\Phi^k(X) = \lambda^k X$ for all $k \in \mathbb{N}$. Let $S \in \mathbf{S}$; then, there is a net $\Phi^{n_\alpha} \to S$, so that $S(X) = \lim \Phi^{n_\alpha}(X) = \lim \lambda^{n_\alpha} X$, hence $S(X) = \mu X$ with $\mu = \lim \lambda^{n_\alpha}$. By Schwartz inequality, $S(X^*X) \geq S(X)^* S(X) = X^*X$ and applying the faithful normal invariant state ρ we obtain $S(X^*X) = X^*X$, so that $X \in \mathcal{M}_r$. This proves the last of the above chain of inclusions.

2. Since $F \in M(\mathbf{S})$, we clearly have $\mathrm{Ker} F \subseteq \mathcal{M}_s$. Conversely, let $X \in \mathcal{M}_s$ and let $S \in \mathbf{S}$ be such that $S(X) = 0$. Since $FS \in M(\mathbf{S})$, there is some $T \in M(\mathbf{S})$ such that $TFS = F$, so that we have

$$F(X) = TFS(X) = 0.$$

This concludes the proof. □

We will now prove the main result of this section.

Theorem 1. *Assume that a quantum channel $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$ admits a faithful normal invariant state ρ. Then $\mathcal{N} = \mathcal{M}_r$.*

Proof. (This proof is inspired by [2]) Let $\mathcal{B}_1$, $\mathcal{N}_1$ and $\mathcal{R}_1$ be the unit balls of $B(\mathcal{H})$, $\mathcal{N}$ and $\mathcal{M}_r$, respectively. Then

$$\mathcal{R}_1 \subseteq \mathcal{N}_1 \subseteq \bigcap_n \Phi^n(\mathcal{B}_1).$$

Indeed, the first inclusion follows from $\mathcal{M}_r \subseteq \mathcal{N}$ and the second from the fact that the restriction $\Phi|_\mathcal{N}$ is an automorphism. We will show that $\mathcal{R}_1 = \bigcap_n \Phi^n(\mathcal{B}_1)$, which implies the statement.

We will use a Hahn–Banach separation argument. So let $X \in \bigcap_n \Phi^n(\mathcal{B}_1) \setminus \mathcal{R}_1$. Since $\mathcal{R}_1 \subset B(\mathcal{H})$ is convex and compact in the weak*-topology, there exists some $\psi \in B(\mathcal{H})_*$ such that

$$\mathrm{Tr}\,[\psi X] > \sup_{Y\in\mathcal{R}_1} \mathrm{Tr}\,[\psi Y] = \|\psi|_{\mathcal{M}_r}\|_1 = \|F_*\psi\|_1.$$

For each $n \in \mathbb{N}$, there is some $Y_n \in \mathcal{B}_1$ such that $X = \Phi^n(Y_n)$, and we have

$$\|\Phi_*^n\psi\|_1 \geq \mathrm{Tr}\,[\Phi_*^n(\psi)Y_n] = \mathrm{Tr}\,[\psi X].$$

Note that since Φ_* is a contraction, $\{\|\Phi_*^n\psi\|_1\}_n$ is a bounded nonincreasing sequence and we have

$$\lim_n \|\Phi_*^n\psi\|_1 \geq \mathrm{Tr}\,[\psi X] > \|F_*\psi\|_1.$$

On the other hand, for any $\varphi \in B(\mathcal{H})_*$, the orbit

$$\mathbf{S}_*\varphi := \{S_*\varphi,\ S \in \mathbf{S}\} = \{\Phi_*^n\varphi,\ n \in \mathbb{N}\}^{-w}$$

is weakly compact. Since $F \in \mathbf{S}$, $\mathbf{S}_*\varphi$ contains $F_*\varphi$, and since $B(\mathcal{H})_*$ is a separable Banach space, the weak topology on the orbit is a metric topology ([19, Theorem V.6.3]). Hence, there is a subsequence of $\Phi_*^n\varphi$ converging weakly to $F_*\varphi$.

Let Φ^{n_k} be such that $\Phi_*^{n_k}\psi \to F_*\psi$ and let $\varphi_1,\ldots,\varphi_4 \in B(\mathcal{H})_*^+$ be such that $\psi = \sum_i c_i\varphi_i$. Then, we may assume that $\Phi_*^{n_k}\varphi_i$ are all weakly convergent, restricting to subsequences if necessary ([19, Theorem V.6.1]). By [40, Corollary III.5.11], $\Phi_*^{n_k}\varphi_i$ are all norm convergent. It follows that $\Phi_*^{n_k}\psi \to F_*\psi$ in norm, so that

$$\lim \|\Phi_*^n\psi\|_1 = \lim \|\Phi_*^{n_k}\psi\|_1 = \|F_*\psi\|_1,$$

a contradiction. □

Corollary 1. *Let $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$ be a channel admitting a faithful normal invariant state ρ. Then, $\mathcal{N}$ is the range of a normal conditional expectation $E_{\mathcal{N}}$ preserving ρ and commuting with Φ. Consequently, $\mathcal{N}$ is an atomic von Neumann algebra.*

Proof. Put $E_{\mathcal{N}} = F$ and the fact that $\mathcal{N}$ must be atomic follows by [41]. □

Remark 2. Note that Theorem 1 and Corollary 1 hold for quantum channels on any atomic von Neumann algebra $\mathcal{M}$. The same proof can be used also in continuous-time case.

Remark 3. As we mentioned in introduction, the DFA is a basic object in the study of the problem of environmental decoherence. According to the theory introduced by Blanchard and Olkiewicz ([9]), a system undergoing the evolution $(\Phi^n)_n$ displays environmental decoherence if there exist two subspaces $\mathcal{M}_1$ and $\mathcal{M}_2$, both preserved by the channel, and such that

- $B(\mathcal{H}) = \mathcal{M}_1 \oplus \mathcal{M}_2$,
- $\mathcal{M}_1$ is a von Neumann algebra and Φ is a $*$-automorphism when restricted to $\mathcal{M}_1$,
- $w^* - \lim_n \Phi^n(x) = 0$ for all x in $\mathcal{M}_2$.

The fact that $\mathcal{M}_1$ coincides with $\mathcal{N}$ and can be the image of a normal conditional expectation is in general an interesting but not clear point as far as we know ([13]). The previous theorem allows us to prove that this is true whenever the channel has an invariant faithful density; moreover, in the same case, we can deduce there is environmental decoherence choosing the decomposition $\mathcal{M}_1 = \mathcal{N}$ and $\mathcal{M}_2 = \mathcal{M}_s$. This last consideration is an almost direct consequence stated, for instance, in [13, Proposition 31].

Due to the previous remark, these conclusions hold also for the continuous-time case, so, for instance, it can generalize many of the results concerning EID for quantum dynamical semigroups as treated in [13] (see in particular Section IV).

Finally, we emphasize that the existence of a conditional expectation with range $\mathcal{N}$, commuting with the channel, can be a useful tool to study the velocity of decoherence, but we shall come back to this point later, in Remark 4.

3. Cyclic Decompositions

In this section, we shall investigate the cyclic behavior of a quantum channel. We shall start with irreducible maps: Here the cycles can be analyzed using the period and we can prove that the DFA is commutative. Then, we shall go to the general case, where the study of cycles is more demanding.

Following conventional terminology already introduced in the 1970s (see [20] and references therein), we say that the map Φ is irreducible if there are no proper hereditary subalgebras preserved by the channel; equivalently, if there exist no nontrivial subharmonic projections, that is, if $P \in B(\mathcal{H})$ is a projection such that $\Phi(P) \geq P$, then $P = 0$ or $P = I$. If there is a faithful normal invariant state, this clearly happens if and only if $\mathcal{F} = \mathbb{C}I$. Moreover, it follows by the Perron–Frobenius theory for positive maps on trace-class operators [38] that there is at most one invariant faithful state for irreducible Φ.

3.1. Irreducible Quantum Channels

We shall concentrate here on irreducible quantum channels with an invariant faithful state. In this case, the cycles of the channel are clearly related to the decoherence-free algebra, we can use the notion of period (which consists in a precise structure of the peripheral spectrum of the channel), and this will give a precise link with the fixed points domain of the powers of the channel.

First, we introduce the definition of period as was made for the finite-dimensional case in [21] (but see also [20,38]).

Definition 1. Period of Φ. Let Φ be an irreducible quantum channel. Then, the period d is the maximal integer d such that there exists a resolution of the identity $Q_0, \ldots, Q_{d-1}$ verifying $\Phi(Q_j) = Q_{j-1}$ for all j (subtractions on indices are modulo d).

Each Q_j is called a cyclic projection, and the set $Q_0, ...Q_{d-1}$ will be called cyclic decomposition (or cyclic resolution) of Φ.

This is a good definition in the context of finite-dimensional Hilbert spaces. When we work on infinite-dimensional spaces, we need to prove that (or when) the period d is finite. For this, we need to use some spectral properties of the channel.

Proposition 5 ([26], Groh [25] and Batkai et al [7, Propositions 6.1 and 6.2]). *Let Φ be an irreducible quantum channel on $B(\mathcal{H})$ with an invariant faithful state. Then the peripheral point spectrum of Φ is the group of all the d-th roots of unity for some $d \geq 1$, and all the eigenvalues in the peripheral point spectrum are simple. Moreover, there exists a unitary operator U such that $U^d = I$ and $\Phi(U^n) = \exp(i2\pi n/d)U^n$ for all integer n.*

In the finite-dimensional case, this result was proved in [20]. Here the existence of a faithful invariant state is implied by irreducibility, and it is enough to assume that Φ is a Schwarz map. On the other hand, [23, Example 1.3] shows that if the map is only positive, the peripheral spectrum may not be a subgroup of the unit circle.

Corollary 2. *Let Φ be an irreducible quantum channel on $B(\mathcal{H})$ with an invariant faithful state. Then Φ has finite period, the cyclic resolution of Φ is unique and $\mathcal{N}$ is an abelian algebra spanned by the cyclic projections of Φ.*

Proof. Let ω be the primary d-th root of unity and U the unitary operator satisfying $U^d = I$ and $\Phi(U^n) = \omega^n U^n$ of Proposition 5. It follows that U^n is the unique (up to multiplicative constants) eigenvector associated with the eigenvalue ω^n. By Theorem 1,

$$\mathcal{N} = \mathcal{M}_r = \operatorname{span}\{I, U, \dots U^{d-1}\} = \{U\}''.$$

In particular, it follows that the abelian subalgebra generated by U is finite dimensional and U admits a spectral representation

$$U = \sum_{j=0}^{d-1} \omega^j Q_j$$

for some orthogonal projections Q_j summing up to I. We immediately deduce that, since $\Phi(U) = \omega U$, then $\Phi(Q_j) = Q_{j-1}$ for all j, so that $Q_0, ..Q_{d-1}$ is a cyclic decomposition of Φ, and we have

$$\mathcal{N} = \{U\}'' = \operatorname{span}\{Q_0, \dots, Q_{d-1}\}.$$

To prove uniqueness, assume that $P_0, \dots, P_{d-1}$ is another cyclic resolution of Φ. Then we can construct the unitary operator

$$V = \sum_{j=0}^{d-1} \omega^j P_j,$$

which is an eigenvector for Φ corresponding to ω. Since the eigenvalues are simple, we must have $V = zU$ for some $z \in \mathbb{C}$, $|z| = 1$, and it is easy to see that $z = \omega^k$ for some $k \in \{0, \dots, d-1\}$, and then for each j, we must have $P_j = Q_{j-k}$ (subtraction modulo d).

$\square$

Proposition 6. *Suppose Φ is an irreducible quantum channel with an invariant faithful state, and let $Q_0,\ldots,Q_{d-1}$ be the cyclic resolution for Φ. Then*

1. *$\mathcal{F}(\Phi^m)$ is a subalgebra of $\mathcal{N}$ for any m;*
2. *$\mathcal{F}(\Phi^d)=\mathcal{N}$ and d is the smallest integer with this property;*
3. *$\mathcal{F}(\Phi^m)=\mathbb{C}1$ if and only if $GCD(m,d)=1$.*

Moreover, denote by $\Phi^d_{|k}$ the restriction of Φ^d to the subalgebra $Q_kB(\mathcal{H})Q_k$, then $\Phi^d_{|k}$ is irreducible, positive recurrent and aperiodic, and consequently ergodic.

Proof. Let $\rho\in\mathfrak{S}(\mathcal{H})$ be the (unique) faithful invariant state of Φ, then ρ is also invariant for Φ^m, so that by Propositions 1 and 4, $\mathcal{F}(\Phi^m)$ is a subalgebra in $\mathcal{N}(\Phi^m)$. Note that for any $n\in\mathbb{N}$ and $X\in\mathcal{M}(\Phi^n)$, we have by Schwarz inequality that

$$\begin{aligned}\Phi^n(X^*X)&=\Phi(\Phi^{n-1}(X^*X))\geq\Phi(\Phi^{n-1}(X)^*\Phi^{n-1}(X))\\&\geq\Phi^n(X)^*\Phi^n(X)=\Phi^n(X^*X).\end{aligned}$$

Using the fact that $\Phi^{n-1}(X^*X)-\Phi^{n-1}(X)^*\Phi^{n-1}(X)\geq 0$ and that ρ is a faithful invariant state, we obtain that $\Phi^{n-1}(X^*X)=\Phi^{n-1}(X)^*\Phi^{n-1}(X)$. This implies that $\mathcal{M}(\Phi^n)\subseteq\mathcal{M}(\Phi^{n-1})$ for all n and hence

$$\mathcal{N}(\Phi^m)=\cap_n\mathcal{M}(\Phi^{mn})=\cap_n\mathcal{M}(\Phi^n)=\mathcal{N}.$$

This proves 1.

By definition of cyclic decomposition, we have $Q_j\in\mathcal{F}(\Phi^d)$ for all j, this implies $\mathcal{N}\subseteq\mathcal{F}(\Phi^d)$. The converse inclusion holds by part 1. If $n<d$, then $\Phi^n(Q_{d-1})=Q_{d-n-1}\neq Q_{d-1}$, so that $Q_{d-1}\notin\mathcal{F}(\Phi^n)$ and hence $\mathcal{F}(\Phi^n)\neq\mathcal{N}$, and this proves 2.

Assume now that $\mathcal{F}(\Phi^m)\neq\mathbb{C}1$, then there is some nontrivial minimal projection $P\in\mathcal{F}(\Phi^m)$, which by part 1 must be of the form $P=Q_{j_1}+\cdots+Q_{j_k}$ for some (distinct) indices $0\leq j_i\leq d-1$ and $k<d$. Let $P_i=\Phi^i(P)$, $i=0,\ldots,m-1$, and then, all P_i are minimal projections in $\mathcal{F}(\Phi^m)$, so that for $i\neq j$, either $P_iP_j=0$ or $P_i=P_j$. By rearranging the indices if necessary, we may assume that $P_0,\ldots,P_{l-1}$ are mutually orthogonal and all other P_i are contained in $\{P_0,\ldots,P_{l-1}\}$. Then $\sum_{i=0}^{m-1}P_i=\sum_{j=0}^{l-1}n_jP_j$ for some integers n_j. On the other hand, we have $\sum_{i=0}^{m-1}P_i\in\mathcal{F}=\mathbb{C}1$ since Φ is irreducible. It follows that $n_1=\cdots=n_{l-1}=:n$ and

$$\sum_{i=0}^{m-1}P_i=nI=n\sum_{j=0}^{l-1}P_j.$$

This implies $m=nl$. Further, $\sum_{j=0}^{l-1}P_j=I$ implies that $d=kl$ by the definition of P_j. Note also that $l>1$ since otherwise we would have $\Phi(P)=P$, which is not possible. Conversely, assume that $GCD(m,d)=l>1$ and let $d=kl$. Put $P=Q_0+Q_l+\cdots+Q_{(k-1)l}$, then clearly P is a projection, $P\neq 0,1$ and $\Phi^l(P)=P$ and also $\Phi^m(P)=P$, since m is a multiple of l, so that $P\in\mathcal{F}(\Phi^m)$ and $\mathcal{F}(\Phi^m)\neq\mathbb{C}1$.

To prove the last statement, observe that the restriction of the Φ-invariant state will give a faithful $\Phi^d_{|k}$-invariant state. Let Q, $0 \le Q \le Q_k$ be a $\Phi^d_{|k}$-harmonic projection, i.e., such that $\Phi^d(Q) = Q$. Then by point 2, Q is in $\mathcal{N}$ and since

$$\mathcal{N} = \operatorname{span}\{Q_0, \dots, Q_{d-1}\}$$

by Corollary 2, we must have either $Q = 0$ or $Q = Q_k$. Hence, $\Phi^d_{|k}$ is irreducible and positive recurrent. Similarly for the period, since $\mathcal{N}(\Phi^d_{|k}) \subseteq Q_k\mathcal{N} = \mathbb{C}Q_k$, any cyclic projection of $\Phi^d_{|k}$ must be trivial, so $\Phi^d_{|k}$ is aperiodic. $\square$

Remark 4. On the line of Remark 3, we can now give some more details on how Theorem 1 can help in evaluating the "time for decoherence" for an irreducible channel with an invariant faithful density. In particular, it is in general interesting to understand when the evolution of the channel tends to become reversible exponentially fast, or equivalently when the elements of the stable space $\mathcal{M}_s$ converge to 0 exponentially fast and with uniform rate; this can be characterized using a kind of spectral gap parameter.

In the standard literature on this topic, the convergence is considered with respect to the L^2 structure induced by the invariant faithful density, say ρ, i.e., one usually defines a scalar product $\langle x, y\rangle_\rho = \operatorname{Tr}(\rho x^* y)$ and consequently a norm $\|x\|^2_{2,\rho} = \operatorname{Tr}(\rho x^* x)$, for x and y in $B(\mathcal{H})$. Then the suitable $L^2(\mathcal{H}, \rho)$ space will be the completion of $B(\mathcal{H})$ with respect to this norm.

- The first fact worth to be noticed is that the good behavior of the conditional expectation F given by Theorem 1 implies that it gives an orthogonal decomposition in $L^2(\mathcal{H}, \rho)$, and this orthogonality is preserved by Φ, in the sense that

  $$\langle Fx, (1-F)y\rangle = \langle \Phi(Fx), \Phi((1-F)y)\rangle = 0 \qquad \forall\, x, y.$$

  Indeed, for any n,

  $$\begin{aligned}\langle \Phi^n(Fx), \Phi^n((1-F)y)\rangle &= \operatorname{Tr}(\rho\Phi^n(Fx^*)\Phi^n((1-F)y)) \\ &= \operatorname{Tr}(\rho\Phi^n(Fx^*(1-F)y)) \\ &= \operatorname{Tr}(\rho F x^*(1-F)y) \\ &= \langle Fx, (1-F)y\rangle.\end{aligned}$$

  The previous, for $n = 1$, gives the first equality and we can deduce the second taking $n = md + k$, where d is the period of the channel and $k = 0, \dots d-1$, since we can also write

  $$\langle \Phi^n(Fx), \Phi^n((1-F)y)\rangle = \operatorname{Tr}(\rho\Phi^k(Fx^*)\Phi^{md+k}((1-F)y)) \to_{m\to\infty} 0$$

  and then repeat for all possible k.
- Moreover, Φ is contractive also with respect to this new norm due to the Schwarz property

  $$\|\Phi(x)\|^2_{2,\rho} = \operatorname{Tr}(\rho\Phi(x^*)\Phi(x)) \le \operatorname{Tr}(\rho\Phi(x^*x)) = \operatorname{Tr}(\rho x^* x)) = \|x\|^2_{2,\rho};$$

and it is isometric on the range $\mathcal{N}$ of F since, for x in $\mathcal{N}$, the multiplicativity property will transform the inequality in the previous line into an equality.

In this context, we could define the decoherence spectral gap as the maximum ϵ such that

$$\|\Phi^n(x) - \Phi^n(Fx)\|_{2,\rho} \le e^{-\epsilon n}\|x - Fx\|_{2,\rho} \qquad \text{for all } x \text{ and } n.$$

The existence of a strictly positive ϵ, uniform in n and x, would give the uniform exponential convergence of the evolution to the decoherence-free algebra. Obviously, such optimal ϵ can also be characterized as

$$\epsilon = \inf_{n>0, x\in B(\mathcal{H})} \frac{1}{n} \ln \frac{\|\Phi^n(x) - \Phi^n(Fx)\|_{2,\rho}}{\|x - Fx\|_{2,\rho}} = \inf_{n>0, x\perp\mathcal{N}} \frac{1}{n} \ln \frac{\|\Phi^n(x)\|_{2,\rho}}{\|x\|_{2,\rho}}$$

A common technique for estimating the usual spectral gap for finite classical Markov chains consists in using continuous-time generators. The same ideas can be applied to our case of interest, with the proper adaptation. Briefly:

- first, we consider the operator Φ^d, where d is the period of the channel, so that the algebra $\mathcal{N}$ is the fixed space of the new operator,
- second, we introduce the infinitesimal Lindblad generator $\mathcal{L} := (\Phi^d - 1)$, inheriting the invariant faithful state of Φ, and we compute the spectral gap of $\mathcal{L}$ with the usual standard techniques.

Some interesting similar problems in the continuous setting have been studied in [4,6].

3.2. Reducible Maps

By Corollary 1, if Φ admits a faithful normal invariant state ρ, the decoherence-free algebra $\mathcal{N}$ is the range of a faithful normal conditional expectation $E_{\mathcal{N}}$ and consequently must be atomic. On the other hand, it is known [22,30] that the limit

$$\lim_n \frac{1}{n} \sum_{k=0}^{n-1} \Phi^k \tag{5}$$

exists in the point-ultraweak topology and gives a faithful normal conditional expectation $E_{\mathcal{F}}$ onto $\mathcal{F}$, satisfying

$$E_{\mathcal{F}} \circ \Phi = \Phi \circ E_{\mathcal{F}} = E_{\mathcal{F}}. \tag{6}$$

Hence, $\mathcal{F}$ is an atomic von Neumann subalgebra of $\mathcal{N}$. In this section, we study the structure of the channel induced by the two algebras $\mathcal{F}$ and $\mathcal{N}$.

First of all, we explain, in Lemma 2, how the minimal central projections of either $\mathcal{F}$ or $\mathcal{N}$ are related to a better description of the corresponding algebra, the action of the associated conditional expectation and its invariant states. Then, in Proposition 8 and Theorem 2, we detail the study of the channel with respect to the structural properties of $\mathcal{F}$ and $\mathcal{N}$. This will lead to a simplified characterization of the channel, its Kraus operators and invariant states. The simplification essentially follows from the fact that Φ can be described by a collection of "lower-dimensional" operators.

We first describe a general form of a faithful normal conditional expectation on $B(\mathcal{H})$.

Lemma 2. *Let $E : B(\mathcal{H}) \to B(\mathcal{H})$ be a faithful normal conditional expectation, and let $\mathcal{R} = E(B(\mathcal{H}))$ be its range. Then*

(i) *$\mathcal{R}$ is atomic, so that there is a direct sum decomposition $\mathcal{H} = \bigoplus_j \mathcal{H}_j$, Hilbert spaces $\mathcal{H}_j^L$, $\mathcal{H}_j^R$ and unitaries $U_j : \mathcal{H}_j \to \mathcal{H}_j^L \otimes \mathcal{H}_j^R$ such that*
$$\mathcal{R} = \bigoplus_j U_j^*(B(\mathcal{H}_j^L) \otimes I_{\mathcal{H}_j^R})U_j;$$
(ii) *the orthogonal projections P_j onto $\mathcal{H}_j$ are minimal central projections in $\mathcal{R}$ and*
$$E(A) = \sum_j E(P_j A P_j);$$
(iii) *identifying $P_j B(\mathcal{H}) P_j$ with $B(\mathcal{H}_j)$, the restriction of E to $P_j B(\mathcal{H}) P_j$ is determined, for all $A_j \in B(\mathcal{H}_j^L)$ and $B_j \in B(\mathcal{H}_j^R)$, by*
$$E(U_j^*(A_j \otimes B_j)U_j) = U_j^*(A_j \otimes \mathrm{Tr}\,[\rho_j B_j] I_{\mathcal{H}_j^R})U_j,$$
where each $\rho_j \in \mathfrak{S}(\mathcal{H}_j^R)$ is a (fixed) faithful normal state;

(iv) *a normal state $\omega \in \mathfrak{S}(\mathcal{H})$ is invariant under E if and only if*
$$\omega = \oplus_j \lambda_j U_j^*(\omega_j^L \otimes \rho_j)U_j,$$
where ρ_j are as in (iii), *$\lambda_j \in [0,1]$, $\sum_j \lambda_j = 1$ and $\omega_j^L \in \mathfrak{S}(\mathcal{H}_j^L)$.*

Proof. The range $\mathcal{R}$ is atomic by [41]. Let $\{P_j\}$ be the minimal central projections in $\mathcal{R}$ and let $\mathcal{H}_j = P_j\mathcal{H}$. Since $\mathcal{R}P_j$ is a type I factor acting on $\mathcal{H}_j$, there are Hilbert spaces $\mathcal{H}_j^L$, $\mathcal{H}_j^R$ and a unitary $U_j : \mathcal{H}_j \to \mathcal{H}_j^L \otimes \mathcal{H}_j^R$ such that
$$\mathcal{R}P_j = U_j^*(B(\mathcal{H}_j^L) \otimes I_{\mathcal{H}_j^R})U_j,$$
this proves (i). By the properties of conditional expectations,
$$E(P_j A P_k) = P_j E(A) P_k = P_j P_k E(A)$$
for any $A \in B(\mathcal{H})$, this proves (ii). It also follows that under the identification in (iii), $E(B(\mathcal{H}_j)) \subseteq B(\mathcal{H}_j)$ for all j and the restriction of E is a faithful normal conditional expectation on $B(\mathcal{H}_j)$, with range $U_j^*(B(\mathcal{H}_j^L) \otimes I_{\mathcal{H}_j^R})U_j$. Let $B_j \in B(\mathcal{H}_j^R)$, and then the multiplicative property of E implies that $E(U_j^*(I \otimes B_j)U_j)$ must commute with all elements in $U_j^*(B(\mathcal{H}_j^L) \otimes I))U_j$. It follows that there is some $c_j(B_j) \in \mathbb{C}$ such that $E(U_j^*(I \otimes B_j)U_j) = c_j(B_j)P_j$. It is clear that $B_j \mapsto c_j(B_j)$ defines a normal state on $B(\mathcal{H}_j^R)$ with corresponding density $\rho_j \in \mathfrak{S}(\mathcal{H}_j^R)$, which must be faithful since E is. This proves (iii).

For (iv), let $\omega \in \mathfrak{S}(\mathcal{H})$. It is immediate that if $E_*(\omega) = \omega$, then we must have $\omega = \sum_j \lambda_j \omega_j$ for some $\omega_j \in \mathfrak{S}(\mathcal{H}_j)$ and $\lambda_j = \mathrm{Tr}\,[P_j\omega]$. Let $\omega_j^L \in \mathfrak{S}(\mathcal{H}_j^L)$ be the partial trace $\mathrm{Tr}_{\mathcal{H}_j^R}[U_j\omega_j U_j^*]$. Then ω_j, and consequently also ω, is invariant under E if
$$\omega_j = E_*(\omega_j) = U_j^*(\omega_j^L \otimes \rho_j)U_j$$

and this concludes the proof. $\square$

The previous lemma, applied with $\mathcal{R}$ equal to either $\mathcal{F}$ or $\mathcal{N}$, will give us two different decompositions of the Hilbert space $\mathcal{H}$, into ranges of minimal central projections. We can better detail these two decompositions separately, exploiting their peculiar features, but we mainly want to fit the two together, in order to optimize our knowledge. Therefore, searching for the finest decomposition of $\mathcal{H}$ which contains both the previous decompositions takes us to consider the algebra

$$\mathcal{Z} = \mathcal{Z}(\mathcal{F}) \cap \mathcal{Z}(\mathcal{N}),$$

where $\mathcal{Z}(\mathcal{F})$ and $\mathcal{Z}(\mathcal{N})$ denote the centers of $\mathcal{F}$ and $\mathcal{N}$, respectively. Clearly, $\mathcal{Z}$ is a discrete abelian von Neumann algebra, and the minimal projections in $\mathcal{Z}$, say $\{Z_1, Z_2, \ldots\}$, will be denumerable and give a resolution of the identity. We shall call any Z_i a MFNC (minimal and $\mathcal{F}/\mathcal{N}$-central) projection. Identifying $Z_i B(\mathcal{H}) Z_i$ with $B(Z_i\mathcal{H})$, we have $\Phi(B(Z_i\mathcal{H})) \subseteq B(Z_i\mathcal{H})$, so that $\Phi_i := \Phi|_{B(Z_i\mathcal{H})}$ is a quantum channel on $B(Z_i\mathcal{H})$, with $\mathcal{N}(\Phi_i) = \mathcal{N}Z_i =: \mathcal{N}_i$; Φ_i will be denominated a MFNC component of the channel.

Similarly as for the irreducible case, the first point will be to show that there is a well-defined notion of a period and cyclic resolution for a MFNC component. This is the content of Proposition 7.

Proposition 7. *Let Φ be a quantum channel with an invariant faithful density, and let $\{Z_i\}_i$ and $\{\Phi_i\}_i$ be its MFNC projections and components as previously introduced. For each MFNC component Φ_i*

- *the dimension d_i of $\mathcal{Z}(\mathcal{N}_i)$ is finite,*
- *the minimal projections $Q_0^i, \ldots, Q_{d_i-1}^i$ of the abelian algebra $\mathcal{Z}(\mathcal{N}_i)$ can be numbered in such a way that*

$$Z_i = \sum_{m=0}^{d_i-1} Q_m^i \qquad \text{and} \qquad \Phi(Q_m^i) = Q_{m-1}^i$$

(where the subtraction on indices is modulo d_i).

Proof. Let $Q_0^i, Q_1^i, \ldots$ be minimal central projections in $\mathcal{N}_i$; then, clearly all Q_m^i are minimal central projections in $\mathcal{N}$ and we have $\sum_m Q_m^i = Z_i$. Since the restriction of Φ_i to $\mathcal{N}_i$ is a *-automorphism, $\Phi(Q_m^i) = \Phi_i(Q_m^i)$ is a minimal central projection as well. Put

$$d_i := \inf\{m, \Phi^m(Q_0^i) = Q_0^i\},$$

we will show that $d_i < \infty$. Assume that $0 \leq m < n < d_i$ are such that $\Phi^m(Q_0^i) = \Phi^n(Q_0^i)$, then the element $R = \sum_{k=m}^{n-1} \Phi^k(Q_0^i)$ satisfies $\Phi(R) = R$, so that $R \in \mathcal{Z}$. Since $Z_i R = R$ and Z_i is minimal, R must be a multiple of Z_i. But then $Q_0^i \leq R$, so that, since all $\Phi^k(Q_0^i)$ are minimal central projections, we must have $Q_0^i = \Phi^k(Q_0^i)$ for some $m \leq k < n$, a contradiction with the definition of d_i. It follows that $\sum_{0 \leq k < d_i} \Phi^k(Q_0^i) \leq Z_i$. Since Φ preserves the faithful state ρ, this implies that $\rho(Z_i) \geq \rho(\sum_{0 \leq k \leq D} \Phi^k(Q_0^i)) = D\rho(Q_0^i)$ for all finite $D < d_i$ and so $d_i < \infty$.

Assume that the projections are numbered so that

$$Q^i_{d_i-m} = \Phi^m(Q^i_0), \qquad m = 0, \ldots, d_i - 1.$$

Put $Q^i := \sum_{m=0}^{d_i-1} Q^i_m$, then by a similar reasoning as above $Q^i \in \mathcal{Z}(\mathcal{N})$ and $\Phi(Q^i) = Q^i$, so that $Q^i \in \mathcal{Z}$. Since also $Q^i \leq Z_i$ and Z_i is minimal in $\mathcal{Z}$, we must have $Q^i = Z_i$. $\square$

These results lead to the following definition.

Definition 2. Period of a MFNC component.
Let Φ be a MFNC component (or equivalently a quantum channel for which the algebra $\mathcal{Z} = \mathcal{Z}(\mathcal{F}) \cap \mathcal{Z}(\mathcal{N})$ is trivial) with an invariant faithful density. Then, the period of Φ is the dimension d of the algebra $\mathcal{Z}(\mathcal{N})$.
Further, the collection of minimal projections $\{Q_m\}_{m\in\mathbb{Z}_d} \in \mathcal{Z}(\mathcal{N})$ summing up to the identity and such that $\Phi(Q_m) = Q_{m-1}$ is called cyclic resolution (or decomposition) for Φ.

Remark 5. 1. If Φ is irreducible, $\mathcal{Z}$ is trivial, so that Φ itself is the unique MFNC component. Then Definitions 1 and 2 coincide and will give the same period and cyclic resolution since $\mathcal{Z}(\mathcal{N}) = \mathcal{N}$.

2. For reducible MFNC components Φ, Proposition 7 shows that the period and cyclic resolution are well defined and unique. We will see later in Theorem 2 that d is the common period of all irreducible restrictions of the component Φ.

3. Obviously, a generic channel does not need to have finite period. An easy example can be constructed choosing the system Hilbert space as the space of the square summable sequences indexed in $\mathbb{Z}$, $\mathcal{H} = l^2(\mathbb{Z})$, with the canonical basis $(e_k)_{k\in\mathbb{Z}}$, and considering the shift operator $S = \sum_k |e_{k+1}\rangle\langle e_k|$ to define the channel $\Phi(x) = S^* x S$, which is unitary, transient and with irreducible components with infinite period. Indeed, one can consider the action of the channel on the projections on the single vectors e_k and remark that $\Phi^n(|e_k\rangle\langle e_k|) = |e_{k-n}\rangle\langle e_{k-n}|$ for any $n \geq 0$. However, this channel would need a completely different study with respect to the approach we have described here, because it loses any good property we could deduce from positive recurrence.

We now describe the action of Φ_i on one component $\mathcal{N}_i$. For simplicity, we drop the index i, and this corresponds to assuming that there is only one such component, so that $\mathcal{Z}$ is trivial, d is the period of Φ and $Q_0, \ldots, Q_{d-1}$ is the cyclic resolution of Φ (as in Definition 2).

Since $\mathcal{N}$ is the range of $E_\mathcal{N}$, we may use Lemma 2 to describe its structure. Let us denote $\mathcal{K}_m := Q_m\mathcal{H}$, and then, there are Hilbert spaces $\mathcal{K}^L_m, \mathcal{K}^R_m$, $m = 0, \ldots, d-1$ and unitaries $S_m : \mathcal{K}_m \to \mathcal{K}^L_m \otimes \mathcal{K}^R_m$ such that

$$\mathcal{N} = \bigoplus_{m=0}^{d-1} S^*_m(B(\mathcal{K}^L_m) \otimes I^R_m)S_m. \tag{7}$$

Here we put $I^R_m = I_{\mathcal{K}^R_m}$ to simplify notations, and we will use a similar notation for $I_{\mathcal{K}^L_m}$. Let also $\rho_m \in \mathfrak{S}(\mathcal{K}^R_m)$ denote the states determining $E_\mathcal{N}$, as in

Lemma 2 (iii). The following proposition clarifies some aspects in the structure of the channel Φ and its action on the DFA.

Proposition 8. *Assume that $\mathcal{Z}$ is trivial and let the period of Φ be d. Then there are*

(a) *unitaries $T_m : \mathcal{K}_m^L \to \mathcal{K}_{m-1}^L$, $m \in \mathbb{Z}_d$,*

(b) *quantum channels $\Xi_m : B(\mathcal{K}_m^R) \to B(\mathcal{K}_{m-1}^R)$, $m \in \mathbb{Z}_d$,*

such that, for all m,

(i) *$(\Xi_m)_*(\rho_{m-1}) = \rho_m$;*

(ii) *$\Xi_{m-(d-1)} \circ \cdots \circ \Xi_{m-1} \circ \Xi_m$ is irreducible and aperiodic;*

(iii) *the restriction $\Phi|_{B(\mathcal{K}_m)}$ is a quantum channel $B(\mathcal{K}_m) \to B(\mathcal{K}_{m-1})$, determined as*

$$\Phi(S_m^*(A_m \otimes B_m)S_m) = S_{m-1}^*(T_m A_m T_m^* \otimes \Xi_m(B_m))S_{m-1};$$

(iv) *Φ has a Kraus representation $\Phi(A) = \sum_k V_k^* A V_k$, such that*

$$V_k = \sum_m S_m^*(T_m^* \otimes L_{m,k})S_{m-1},$$

where $\Xi_m = \sum_k L_{m,k}^* \cdot L_{m,k}$ is a Kraus representation of Ξ_m.

Remark 6. The results in this proposition are in some points parallel to what discussed in [16] for continuous-time Markov semigroups: what is intrinsically different here in our paper is the presence of a supplementary decomposition due to the period, which cannot appear in continuous time.

Proof. Let $A_m \in B(\mathcal{K}_m^L)$. Since $\Phi(Q_m\mathcal{N}) = Q_{m-1}\mathcal{N}$, we have

$$\Phi(S_m^*(A_m \otimes I_m^R)S_m) = S_{m-1}^*(A_m' \otimes I_{m-1}^R)S_{m-1}$$

for some $A_m' \in B(\mathcal{K}_{m-1}^L)$ and the map $A_m \mapsto A_m'$ defines a *-isomorphism of $B(\mathcal{K}_m^L)$ onto $B(\mathcal{K}_{m-1}^L)$. Hence, there is a unitary operator $T_m : \mathcal{K}_m^L \to \mathcal{K}_{m-1}^L$, such that $A_m' = T_m A_m T_m^*$. Moreover, by the multiplicativity properties of Φ on $\mathcal{N}$ [see Eq. (4)], we have, for all $B_m \in B(\mathcal{K}_m^R)$,

$$\begin{aligned}\Phi(S_m^*(A_m \otimes B_m)S_m) &= \Phi(S_m^*(A_m \otimes I_m^R)S_m)\Phi(S_m^*(I_m^L \otimes B_m)S_m)\\ &= \Phi(S_m^*(I_m^L \otimes B_m)S_m)\Phi(S_m^*(A_m \otimes I_m^R)S_m).\end{aligned}$$

It follows that $\Phi(S_m^*(I_m^L \otimes B_m)S_m)$ is an element in $B(\mathcal{K}_{m-1})$, commuting with all elements in $S_{m-1}^*(B(\mathcal{K}_{m-1}^L) \otimes I_{m-1}^R)S_{m-1}$, so that

$$\Phi(S_m^*(I_m^L \otimes B_m)S_m) = S_{m-1}^*(I_{m-1}^L \otimes B_m')S_{m-1}$$

for some $B_m' \in B(\mathcal{K}_{m-1}^R)$. It is clear that $B_m \mapsto B_m'$ defines a quantum channel $\Xi_m : B(\mathcal{K}_m^R) \to B(\mathcal{K}_{m-1}^R)$. Putting all together proves (iii).

To see (ii), let $\tilde{\Xi}_m = \Xi_{m-(d-1)} \circ \cdots \circ \Xi_{m-1} \circ \Xi_m$ be the given composition, and let $R_m \in B(\mathcal{K}_m^R)$ be a projection that is either fixed or decoherence-free for $\tilde{\Xi}_m$. Then, $S_m^*(I_m^L \otimes R_m)S_m$ is in $\mathcal{N}$, so that R_m must be trivial.

Further, for (i) note that by Corollary 1, $E_\mathcal{N}$ commutes with Φ. For $B_m \in B(\mathcal{K}_m^R)$, we have by Lemma 2

$$\Phi \circ E_\mathcal{N}(S_m^*(I_m^L \otimes B_m)S_m) = \mathrm{Tr}\,[\rho_m B_m]\Phi(Q_m) = \mathrm{Tr}\,[\rho_m B_m]Q_{m-1}$$

and

$$E_{\mathcal{N}} \circ \Phi(S_m^*(I_m^L \otimes B_m)S_m) = E_{\mathcal{N}}(S_{m-1}^*(I_{m-1}^L \otimes \Xi_m(B_m))S_{m-1})$$
$$= \mathrm{Tr}\,[\rho_{m-1}\Xi_m(B_m)]Q_{m-1},$$

so that (i) holds.

Finally, let $\Phi = \sum_k V_k^* \cdot V_k$ be any Kraus representation of Φ. Then, we have

$$\Phi(A) = \sum_{m,n=0}^{d-1} \Phi(Q_m A Q_n) = \sum_{m,n=0}^{d-1} Q_{m-1}\Phi(Q_m A Q_n)Q_{n-1},$$

so that we may assume that $V_k = \sum_m V_{k,m}$, with $V_{k,m} = Q_m V_k Q_{m-1}$, for all k and m. Moreover, for each m, $\sum_k V_{k,m}^* \cdot V_{k,m}$ is a Kraus representation of the restriction $\Phi|_{B(\mathcal{K}_m)}$.

Let $\Xi_m = \sum_l K_{m,l}^* \cdot K_{m,l}$ be a minimal Kraus representation. It follows from (iii) that

$$\Phi|_{B(\mathcal{K}_m)} = \sum_l S_{m-1}^*(T_m \otimes K_{m,l}^*)S_m \cdot S_m^*(T_m^* \otimes K_{m,l})S_{m-1}$$

is another Kraus representation of $\Phi|_{B(\mathcal{K}_m)}$, and hence, there are some $\{\eta_{k,l}^j\}$ such that $\sum_i \eta_{i,k}^j \bar{\eta}_{i,l}^j = \delta_{k,l}$ and

$$V_{k,m} = \sum_l \eta_{k,l}^j S_m^*(T_m^* \otimes K_{m,l})S_{m-1} = S_m^*(T_m^* \otimes L_{m,k})S_{m-1},$$

where $L_{m,k} := \sum_l \eta_{k,l}^m K_{m,l}$, this proves (iv). $\square$

Note that by identifying

$$\mathcal{H} = \bigoplus_m \mathcal{K}_m \simeq \sum_m \mathcal{K}_m \otimes \mathbb{C}|m\rangle$$

and

$$\mathcal{K} := \bigoplus_m \mathcal{K}_m^L \otimes \mathcal{K}_m^R \simeq \sum_m \mathcal{K}_m^L \otimes \mathcal{K}_m^R \otimes \mathbb{C}|m\rangle,$$

(7) can be written as

$$\mathcal{N} = S^* \left(\sum_{m=0}^{d-1} B(\mathcal{K}_m^L) \otimes I_m^R \otimes |m\rangle\langle m| \right) S,$$

where $S : \mathcal{H} \to \mathcal{K}$ is a unitary given as $S = \sum_m S_m \otimes |m\rangle\langle m|$. We will also use the notation

$$\mathcal{K}^R := \bigoplus_m \mathcal{K}_m^R \simeq \sum_m \mathcal{K}_m^R \otimes \mathbb{C}|m\rangle$$

and put $I^R := I_{\mathcal{K}^R}$. We are now ready to describe the subalgebra $\mathcal{F}$. In the following proposition, we keep the notations of Proposition 8. We can see the next step as an improvement of Lemma 2 applied to the fixed points domain $\mathcal{F}$: we can give a more detailed description of $\mathcal{F}$ and construct some of the mathematical objects appearing in the lemma using the items introduced in

Proposition 8. Going to the predual vision, we can then consider the structure of the invariant states, and finally, we can present the action of the channel on the subsystems associated with the central projections of $\mathcal{F}$.

Theorem 2. *Assume that $\mathcal{Z}$ is trivial and let the period of Φ be d. Let us denote*

$$\tilde{T}_m : \mathcal{K}_0^L \to \mathcal{K}_m^L, \quad \tilde{T}_m := T_{m+1} \dots T_{d-1} T_0, \ m = 0, \dots, d-2; \quad \tilde{T}_{d-1} := T_0$$

and let $T : \mathcal{K}_0^L \otimes \mathcal{K}^R \to \mathcal{K}$ be the unitary defined as

$$T = \sum_{m=0}^{d-1} \tilde{T}_m \otimes I_m^R \otimes |m\rangle\langle m|.$$

(i) *The operator $\tilde{T}_0 \in \mathcal{U}(\mathcal{K}_0^L)$ has a discrete spectrum. Let R_j be its minimal spectral projections and let $\mathcal{L}_j := R_j \mathcal{K}_0^L$, then*

$$\mathcal{F} = S^* T \left(\bigoplus_j B(\mathcal{L}_j) \otimes I^R \right) T^* S.$$

(ii) *Let $\sigma_j \in \mathfrak{S}(\mathcal{K}^R)$ be the faithful normal states corresponding to $E_{\mathcal{F}}$ as in Lemma 2 (iii) and (iv). Then,*

$$\sigma_j \equiv \sigma := \frac{1}{d} \sum_{m=0}^{d-1} \rho_m \otimes |m\rangle\langle m|, \qquad \forall j.$$

(iii) Invariant states $\xi \in \mathfrak{S}(\mathcal{H})$ for Φ are precisely those of the form

$$\xi = S^* T \, (\omega \otimes \sigma) \, T^* S,$$

where $\omega = \sum_j \lambda_j \omega_j \otimes |j\rangle\langle j|$ for some probabilities $\{\lambda_j\}$ and states $\omega_j \in \mathfrak{S}(\mathcal{L}_j)$.

(iv) Let $P_j := S^* T (R_j \otimes I^R) T^* S$ be the minimal central projections in $\mathcal{F}$. The restrictions $\Phi|_{B(P_j \mathcal{H})}$ have the form

$$\Phi|_{B(P_j \mathcal{H})}(S^* T (A_j \otimes B) T^* S) = S^* T (A_j \otimes \Psi_j(B)) T^* S,$$
$$A_j \in B(\mathcal{L}_j), B \in B(\mathcal{K}^R),$$

where Ψ_j are irreducible quantum channels on $B(\mathcal{K}^R)$. Moreover, all Ψ_j coincide on block-diagonal elements of the form $\sum_m B_{mm} \otimes |m\rangle\langle m|$, and we have

$$\Psi_j(\sum_m B_{mm} \otimes |m\rangle\langle m|) = \sum_m \Xi_m(B_{mm}) \otimes |m-1\rangle\langle m-1|.$$

In particular, for all j, Ψ_j has period d, $\mathcal{N}(\Psi_j) = span\{I_m^R \otimes |m\rangle\langle m|, m = 0, \dots, d-1\}$ and σ of (ii) is the unique invariant state.

Proof. Since $\mathcal{F} \subseteq \mathcal{N}$, we may apply Proposition 8. It can be easily checked that an element of $\mathcal{N}$ is in $\mathcal{F}$ if and only if it is of the form

$$S^* T (A \otimes I^R) T^* S$$

with $A \in \mathcal{A} := \{\tilde{T}_0\}' \cap B(\mathcal{H}_0^L)$. Note that the commutant $\mathcal{A}' := \{\tilde{T}_0\}'' \cap B(\mathcal{H}_0^L) = \mathcal{Z}(\mathcal{A})$ is abelian. Further, we have $\mathcal{F} \simeq \mathcal{A}$ and since $\mathcal{F}$ is atomic, $\mathcal{A}$ must be such as well, so that $\{\tilde{T}_0\}'' \cap B(\mathcal{H}_0^L)$ must be discrete. This proves (i).

By Lemma 2, there are some states $\sigma_j \in \mathfrak{S}(\mathcal{K}^R)$ such that

$$E_{\mathcal{F}}(S^*T(R_j \otimes B)T^*S) = \mathrm{Tr}\,[\sigma_j B]P_j, \tag{8}$$

where $B \in B(\mathcal{K}^R)$ and $P_j := S^*T(R_j \otimes I^R)T^*S$ are the minimal central projections in $\mathcal{F}$. Moreover, since $E_{\mathcal{F}}$ is given by (5) and satisfies (6), we see that a state ξ is invariant for Φ if and only if it is invariant for $E_{\mathcal{F}}$. Consequently, by Lemma 2 (iv), any state of the form $\psi = S^*T(\omega_j \otimes \sigma_j)T^*S$ with $\omega_j \in \mathfrak{S}(\mathcal{L}_j)$ is an invariant state for Φ. It follows that for any $m = 0, \ldots d-1$,

$$\begin{aligned}
\mathrm{Tr}\,[\sigma_j(I_m \otimes |m\rangle\langle m|)] &= \mathrm{Tr}\,[\psi S^*T(R_j \otimes I_m \otimes |m\rangle\langle m|)T^*S] \\
&= \mathrm{Tr}\,[\Phi_*(\psi) S^*T(R_j \otimes I_m \otimes |m\rangle\langle m|)T^*S] \\
&= \mathrm{Tr}\,[\psi \Phi(S^*T(R_j \otimes I_m \otimes |m\rangle\langle m|)T^*S)] \\
&= \mathrm{Tr}\,[\psi S^*T(R_j \otimes I_{m-1} \otimes |m-1\rangle\langle m-1|)T^*S] \\
&= \mathrm{Tr}\,[\sigma_j(I_{m-1} \otimes |m-1\rangle\langle m-1|)]
\end{aligned}$$

so that $\mathrm{Tr}\,[\sigma_j(I_m \otimes |m\rangle\langle m|)] = 1/d$. Let now $B = \sum_{m,n} B_{mn} \otimes |m\rangle\langle n| \in B(\mathcal{K}^R)$. Since $E_{\mathcal{N}} \in \mathbf{S}$, we obtain from (6) that also $E_{\mathcal{F}} \circ E_{\mathcal{N}} = E_{\mathcal{N}} \circ E_{\mathcal{F}} = E_{\mathcal{F}}$. Using Lemma 2 (ii) for $E_{\mathcal{N}}$, we get

$$\begin{aligned}
E_{\mathcal{F}}(S^*T(R_j \otimes B)T^*S) &= \sum_m E_{\mathcal{F}} \circ E_{\mathcal{N}}(Q_m S^*T(R_j \otimes B)T^*SQ_m) \\
&= \sum_m E_{\mathcal{F}} \circ E_{\mathcal{N}}(S_m^*(\tilde{T}_m R_j \tilde{T}_m^* \otimes B_{mm})S_m) \\
\text{[by Lemma 2 (iii)]} &= \sum_m \mathrm{Tr}\,[\rho_m B_{mm}] E_{\mathcal{F}}(S_m^*(\tilde{T}_m R_j \tilde{T}_m^* \otimes I_m^R)S_m) \\
&= \sum_m \mathrm{Tr}\,[\rho_m B_{mm}] E_{\mathcal{F}}(S^*T(R_j \otimes I_m^R \otimes |m\rangle\langle m|)T^*S) \\
&= \frac{1}{d}\sum_m \mathrm{Tr}\,[\rho_m B_{mm}]P_j,
\end{aligned}$$

where the last equality follows from (8) and the previously obtained equality $\mathrm{Tr}\,[\sigma_j I_m \otimes |m\rangle\langle m|] = 1/d$. This and (8) prove (ii).

Point (iii) now directly follows from Lemma 2 (iv).

Finally, we prove (iv). We see by the multiplicativity properties of Φ on $\mathcal{N}$ that $\Phi(B(P_j\mathcal{H})) \subseteq B(P_j\mathcal{H})$ and that the restrictions have the given form with some quantum channel Ψ_j on $B(\mathcal{K}^R)$. Since any fixed point of Ψ_j is related to a fixed point of Φ, we can see that it must be trivial, so that Ψ_j are irreducible. For any $B_m \in B(\mathcal{K}_m^R)$, we have by Proposition 8,

$$\begin{aligned}
\Phi(S^*T(R_j \otimes B_m \otimes |m\rangle\langle m|)T^*S &= \Phi(S_m^*(\tilde{T}_{m-1} R_j \tilde{T}_{m-1}^* \otimes B_m)S_m) \\
&= S_{m-1}^*(\tilde{T}_m R_j \tilde{T}_m^* \otimes \Xi_m(B_m))S_{m-1}) \\
&= S^*T(R_j \otimes \Xi_m(B_m) \otimes |m-1\rangle\langle m-1|)T^*S.
\end{aligned}$$

It follows that $\Psi_j(B_m \otimes |m\rangle\langle m|) = \Xi_m(B_m) \otimes |m-1\rangle\langle m-1|$ and $I_m^R \otimes |m\rangle\langle m| \in \mathcal{N}(\Psi_j)$ for all m and j. Hence, any minimal projection in $\mathcal{N}(\Psi_j)$ must be of the form $Q \otimes |m\rangle\langle m|$ for some $m = 0, \ldots, d-1$ and some projection $Q \in B(\mathcal{K}_m^R)$. But then it easily follows that $I_m \otimes Q$ is in $\mathcal{N}$, so that we must have $Q = I_m^R$. Finally, the fact that σ is an invariant state for Ψ_j follows easily from (iii). □

Conclusions. We are aware that the contents of this subsection are technical and the relations between different representations and decompositions are intricate, in particular for a reader who is not involved in similar research topics. For a full comprehension, it can be useful to insert it in the surrounding literature. As we already mentioned in Introduction, the results of this section include sometimes a revision and improvements or generalizations of different previous studies. The structure of the fixed points domain has already been investigated, and one can find various papers in last two decades, see, for instance, [1,8,10,11,28] and references therein. For the structure of the DFA, there is some interest growing from different fields, and we could improve its description in Theorem 2. We can underline that here we study a dual version in infinite dimension of the decomposition appearing in [43, Theorem 8] and [42]; further, this section includes a generalization, in discrete-time version (which has a richer structure, due to period) of [16], without the need of imposing atomicity condition.

4. Application to Open Quantum Walks

In this section, we discuss an important type of quantum channels.

Let $\mathcal{H} = \oplus_{i\in V}\mathfrak{h}_i$, where V is a countable set of vertices and $\mathfrak{h}_i$ are separable Hilbert spaces. Note that we may express $\mathcal{H}$ as $\mathcal{H} = \sum_{i\in V}\mathfrak{h}_i \otimes |i\rangle$. An open quantum random walk (OQRW) ([3]) is a completely positive trace-preserving map $\mathfrak{M}$ on the space $B(\mathcal{H})_*$ of trace-class operators, of the form

$$\mathfrak{M} : \rho \mapsto \sum_{i,j} V_{i,j}\rho V_{i,j}^*,$$

where $V_{i,j} = L_{i,j} \otimes |i\rangle\langle j|$ and $L_{i,j}$ are bounded operators $\mathfrak{h}_j \to \mathfrak{h}_i$ satisfying

$$\sum_{i\in V} L_{i,j}^* L_{i,j} = I_j, \qquad \forall j \in V. \tag{9}$$

Put $\Phi = \mathfrak{M}^*$, and then, Φ is a quantum channel. Note that any operator $A \in B(\mathcal{H})$ can be written as

$$A = \sum_{i,j\in V} A_{i,j} \otimes |i\rangle\langle j|,$$

where $A_{i,j}$ is a bounded operator $\mathfrak{h}_j \to \mathfrak{h}_i$, and the action of Φ has the form

$$\Phi(A) = \sum_j \sum_i L_{i,j}^* A_{i,i} L_{i,j} \otimes |j\rangle\langle j|.$$

This family of quantum channels has recently become quite popular and have been extensively studied (see [5,11,18,26,31,39]). Here we want to investigate the structure of the DFA associated with an OQRW: We obtain some results in the general case and then expound some particular remarkable classes. Finally, we go exploring a nonpositive recurrent family of models considering homogeneous OQRWs on the group $V = \mathbb{Z}$.

We next characterize the multiplicative domain $\mathcal{M}$ and the decoherence-free subalgebra $\mathcal{N}$ of Φ by the transition operators $L_{i,j}$.

To obtain $\mathcal{N}$, we invoke the notation of [11]. For $i, j \in V$, let $\mathcal{P}_n(i,j)$ be the set of all paths

$$\pi = (i_0 = i, i_1, \ldots, i_n = j)$$

from i to j of length n. For each $\pi \in \mathcal{P}_n(i,j)$, we define the operator $L_\pi : \mathfrak{h}_i \to \mathfrak{h}_j$ as

$$L_\pi = L_{j,i_{n-1}} L_{i_{n-1},i_{n-2}} \ldots L_{i_1,i}.$$

Proposition 9. *Let Φ be an OQRW.*
1. *$A \in \mathcal{M}$ if and only if*

$$A_{i,i} L_{i,j} L_{k,j}^* = L_{i,j} L_{k,j}^* A_{k,k}, \quad \forall i, j, k \tag{10}$$

$$\textit{and} \qquad A_{l,i} L_{i,j} = 0 = L_{l,j}^* A_{l,i}, \quad \forall i, j, l \in V, \ i \neq l \tag{11}$$

2. *$A \in \mathcal{N}$ if and only if for all $i, j, k, l \in V$, $l \neq i$ and $n \in \mathbb{N}$,*

$$A_{i,i} L_\pi L_{\pi'}^* = L_\pi L_{\pi'}^* A_{k,k}, \quad \forall \pi \in \mathcal{P}_n(j,i), \ \pi' \in \mathcal{P}_n(j,k) \tag{12}$$

$$\textit{and} \qquad A_{l,i} L_{i,j} = 0 = L_{l,j}^* A_{l,i}. \tag{13}$$

Proof. 1. It is easy to see from Proposition 2 that $A \in \mathcal{M}$ if and only if A commutes with all operators of the form $L_{i,j} L_{k,j}^* \otimes |i\rangle\langle k|$, $i, j, k \in V$. This is equivalent to (10), together with

$$A_{l,i} L_{i,j} L_{k,j}^* = 0 = L_{k,j} L_{l,j}^* A_{l,i}, \quad \forall i, j, k, l \in V, \ l \neq i \tag{14}$$

It is clear that (11) implies (14). For the converse, multiply the first equality of (14) by $L_{k,j}$ from the right and sum over $k \in V$, then (9) implies the first equality of (11). The second equality is proved similarly.

2. Since the Kraus operators of Φ^n are operators of the form $L_\pi \otimes |i\rangle\langle j|$ for $\pi \in \mathcal{P}_n(j,i)$, the second statement can be proved exactly as the previous one. $\square$

Due to the characterization in the previous proposition, we can deduce a decomposition of the decoherence-free algebra in block-diagonal and block off-diagonal operators.

Corollary 3. *$\mathcal{N} = \mathcal{N}_D \oplus \mathcal{N}_{OD}$ where:*

$$\mathcal{N}_D := \left\{ A = \sum_{i \in V} A_i \otimes |i\rangle\langle i|, A \in \mathcal{N} \right\}$$

$$= \left\{ \sum_{i \in V} A_i \otimes |i\rangle\langle i| \ : \ A_i L_\pi L_{\pi'}^* = L_\pi L_{\pi'}^* A_k, \forall i, k \in V, \forall (\pi, \pi') \right.$$

$$\left.\in \cup_{j,n}(\mathcal{P}_n(j,i) \times \mathcal{P}_n(j,k))\right\},$$

$$\mathcal{N}_{OD} := \left\{ A = \sum_{i \neq j \in V} A_{i,j} \otimes |i\rangle\langle j|, A \in \mathcal{N} \right\}$$
$$= \left\{ \sum_{i \neq j \in V} A_{i,j} \otimes |i\rangle\langle j| \,:\, A_{i,j}L_{j,l} = 0 = L_{i,l}^* A_{i,j},\ \forall i \neq j, l \in V \right\}.$$

Assume that $\mathcal{N}_{OD}$ is nontrivial, so that there is some $0 \neq A \in \mathcal{N}_{OD}$. Since $\Phi(A) = 0$, $A \in \mathcal{N} \cap \ker \Phi$ and clearly also $A^*A \in \mathcal{N} \cap \ker \Phi$. The block-diagonal part $(A^*A)_D := \sum_{i \in V} (A^*A)_{i,i} \otimes |i\rangle\langle i|$ is a nonzero positive operator in

$$\mathcal{N} \cap \ker \Phi \cap \{\text{block diagonal operators}\} = \mathcal{N}_D \cap \ker \Phi.$$

Summing up, we deduce

$$\mathcal{N}_{OD} \neq \{0\} \quad \Rightarrow \quad \mathcal{N} \cap \ker \Phi \neq \{0\} \quad \Leftrightarrow \quad \mathcal{N}_D \cap \ker \Phi \neq \{0\}.$$

Corollary 4. *If Φ admits a faithful normal invariant state, then $\mathcal{N} = \mathcal{N}_D$. In the general case, put $\mathcal{W}_i := \cap_{j \in V} \mathrm{Range}(L_{i,j})^\perp$, and then, $\mathcal{N} = \mathcal{N}_D$ if and only if there is at most one index $i \in V$, such that $\mathcal{W}_i \neq \{0\}$.*

Proof. As shown above, if $\mathcal{N}_{OD} \neq \{0\}$, then also $\mathcal{N}_D \cap \ker \Phi \neq \{0\}$, and since this is a von Neumann algebra, $\mathcal{N}_D \cap \ker \Phi$ must contain a nontrivial projection. This is clearly not possible if Φ admits a faithful normal invariant state, since then Φ is faithful and there can be no projections in $\ker \Phi$. The general case is clear from Corollary 3. □

4.1. Homogeneous OQRWs

An OQRW is called homogeneous if V is an abelian group, $\mathfrak{h}_i = \mathfrak{h}$ does not depend on i and transition operators are translation invariant, i.e., $L_{i,j} = L_{i+n,j+n} =: L_{j-i}$ for any $i, j, n \in V$. We can define the local operator $\mathcal{L}$, acting on $B(\mathfrak{h})_*$ as

$$\mathcal{L}(\rho) = \sum_{k \in V} L_k \rho L_k^*.$$

Let $\rho^{\text{inv}} \in \mathfrak{S}(\mathfrak{h})$ be an invariant state for $\mathcal{L}$. If V is finite, then

$$\frac{1}{|V|} \sum_i \rho^{\text{inv}} \otimes |i\rangle\langle i|$$

is a normal invariant state for Φ, which is faithful iff ρ^{inv} is. If V is infinite, we can only obtain an invariant weight in this way, given as

$$\omega\left(\sum x_{ij} \otimes |i\rangle\langle j|\right) := \sum_j \mathrm{Tr}\,[\rho^{\text{inv}} x_{jj}],$$

for all positive $x = \sum x_{ij} \otimes |i\rangle\langle j|$ in $B(\mathcal{H})$. In particular, if Φ is irreducible, the invariant states must be translation invariant, and hence, there are no invariant states if V is infinite, [11, Prop. 9.3.].

We will consider the nearest-neighbor case with $V = \mathbb{Z}$ (or $V = \mathbb{Z}_d$) and $L_{i-1,i} = L_-$, $L_{i+1,i} = L_+$, all the other $L_{i,j} = 0$. An immediate application of Proposition 9 will give us the following.

Corollary 5. *Let Φ be a homogeneous nearest-neighbor OQRW on $\mathbb{Z}$ (or $\mathbb{Z}_d$). Then, $A \in \mathcal{M}$ if and only if*

$$A_{i,i}L_+L_-^* = L_+L_-^*A_{i-2,i-2}, \quad A_{i-2,i-2}L_-L_+^* = L_-L_+^*A_{i,i}, \quad A_{i,i} \in \{|L_+^*|, |L_-^*|\}' \quad \forall i,$$

and

$$A_{ik}L_- = A_{ik}L_+ = L_-^*A_{ik} = L_+^*A_{ik} = 0, \quad \forall i,k \in V,\ i \neq k. \tag{15}$$

In particular, when at least one transition operator is invertible, $\mathcal{M}$ contains only block-diagonal operators.

4.2. Examples

We will consider three examples of open quantum random walks and describe their decoherence-free algebras. As we will see in the first example, the action of any quantum channel on a cyclic component of $\mathcal{N}$ is described by an OQRW of a certain type. The second example is a homogeneous OQRW with two vertices and finite-dimensional local spaces. In the last example, we describe the decoherence-free algebra for a homogeneous OQRW without a faithful normal invariant state.

4.2.1. A Cyclic Shift OQRW. We consider an OQRW with $V = \mathbb{Z}_d$ and $\mathfrak{h}_0 \simeq \mathfrak{h}_1 \simeq \cdots \simeq \mathfrak{h}_{d-1}$. Let $L_{i,i-1} = U_i$ be a unitary $\mathfrak{h}_{i-1} \to \mathfrak{h}_i$, $i = 0, \dots, d-1$, $L_{i,j} = 0$ for $i - j \neq 1$ (addition and subtraction on indices are in $\mathbb{Z}_d$). We can explicitly write the action of Φ and its preadjoint as

$$\Phi(A) = \sum_{i=0}^{d-1} U_i^* A_{i,i} U_i \otimes |i-1\rangle\langle i-1|, \qquad \Phi_*(\rho) = \sum_{i=0}^{d-1} U_i \rho_{i-1,i-1} U_i^* \otimes |i\rangle\langle i|,$$

where $A = \sum_{i,j} A_{i,j} \otimes |i\rangle\langle j|$ and $\rho = \sum_{k,l} \rho_{k,l} \otimes |k\rangle\langle l|$. It is clear from this expression that the fixed points of Φ are precisely the block-diagonal operators such that

$$U_i^* A_{i,i} U_i = A_{i-1,i-1}, \qquad i = 0, \dots, d-1.$$

Putting $\tilde{U}_i := U_i \dots U_1 : \mathfrak{h}_0 \to \mathfrak{h}_i$, $i = 1, \dots, d-1$, $\tilde{U}_0 := I_{\mathfrak{h}_0}$ and $\tilde{U} := U_0 \tilde{U}_{d-1} \in B(\mathfrak{h}_0)$, we obtain that

$$\mathcal{F} = \left\{ \sum_{i=0}^{d-1} \tilde{U}_i A_0 \tilde{U}_i^* \otimes |i\rangle\langle i|,\ A_0 \in B(\mathfrak{h}_0)\, s.t.\, [A_0, \tilde{U}] = 0 \right\}.$$

It follows that $\mathcal{F}$ is a von Neumann algebra isomorphic to $\{\tilde{U}\}' \cap B(\mathfrak{h}_0)$, and hence, we always have $\mathcal{F} \subseteq \mathcal{N}$ (Proposition 1). Similarly, the invariant normal states have the form

$$\rho = \frac{1}{d}\sum_{i=0}^{d-1} \tilde{U}_i \rho_0 \tilde{U}_i^* \otimes |i\rangle\langle i|, \qquad \rho_0 \in \mathfrak{S}(\mathfrak{h}_0), \quad \tilde{U}\rho_0\tilde{U}^* = \rho_0.$$

It follows that normal invariant states for Φ are obtained from normal invariant states for the unitary conjugation $\tilde{U}\cdot\tilde{U}^*$.

Due to Corollary 4, $\mathcal{N} = \mathcal{N}_D$ is block diagonal; moreover, since for any path π of length n, the operator L_π is nonzero if and only if $\pi = (i, i+1, \ldots, i+n) \in \mathcal{P}_n(i, i+n)$, we can see from Corollary 3 that $\mathcal{N}$ consists of all block-diagonal operators, i.e.,

$$\mathcal{N} = \left\{ \sum_{i=0}^{d-1} A_i \otimes |i\rangle\langle i|,\ A_i \in B(\mathfrak{h}_i),\ i = 0, \ldots, d-1 \right\}$$

with minimal central projections $I_{\mathfrak{h}_i} \otimes |i\rangle\langle i|$, $i = 0, \ldots, d-1$.

It is then clear that Φ has a unique MFNC component, i.e., $\mathcal{Z} = \mathcal{Z}(\mathcal{F}) \cap \mathcal{Z}(\mathcal{N})$ is trivial, with period d. It can be instructive to compare the above example to the results of Sect. 3.2, when a faithful normal invariant state exists. In the notations of Proposition 8, we have $\mathcal{K}_m^L = \mathfrak{h}_m$, $\dim(\mathcal{K}_m^R) = 1$ and $T_m = U_m^*$, $m = 0, \ldots, d-1$. Moreover, we see that the obtained form of $\mathcal{F}$ and of invariant normal states corresponds to the results of Theorem 2, where here $\mathcal{K}^R \simeq \mathbb{C}^d$ and σ is the tracial state on $B(\mathbb{C}^d)$. In fact, we can observe the following result.

Proposition 10. *Let Φ be a quantum channel admitting a faithful normal invariant state, with a unique MFNC component. Then there exists an OQRW Ψ of the above form, such that $\mathcal{N}(\Phi) \simeq \mathcal{N}(\Psi)$ and $\Phi|_{\mathcal{N}(\Phi)} \simeq \Psi|_{\mathcal{N}(\Psi)}$.*

4.2.2. A Homogeneous OQRW with Generalized Pauli Operators. Let $V = \mathbb{Z}_2$ and $\mathfrak{h}_0 = \mathfrak{h}_1 = \mathfrak{h}$. Let $L_{00} = L_{11} = \sqrt{\alpha}U_0$, $L_{01} = L_{10} = \sqrt{1-\alpha}U_1$, with $\alpha \in (0,1)$ and U_0, U_1 unitaries on $\mathfrak{h}$. Explicitly, Φ acts as

$$\begin{aligned}\Phi(A) = {} & [\alpha(U_0^* A_{00} U_0) + (1-\alpha)U_1^* A_{11} U_1] \otimes |0\rangle\langle 0| \\ & + [(1-\alpha)U_1^* A_{00} U_1 + \alpha U_0^* A_{11} U_0] \otimes |1\rangle\langle 1|.\end{aligned}$$

Assume $d := \dim(\mathfrak{h}) < \infty$, then $(2d)^{-1} I_{\mathfrak{h}} \otimes I_2$ is a faithful invariant state for Φ. We next investigate the fixed points and decoherence-free subalgebra in the case when U_0 and U_1 are generalized Pauli operators described below.

Let $\{|j\rangle,\ j = 0, \ldots, d-1\}$ denote a fixed ONB in $\mathfrak{h}$, and let $\oplus$ be addition modulo d. Put $\omega = e^{i2\pi/d}$ and define the operators Z and X, for all j, as

$$\begin{aligned} Z|j\rangle &= \omega^j |j\rangle, \\ X|j\rangle &= |j \oplus 1\rangle. \end{aligned}$$

Then Z and X are unitaries satisfying the commutation relation

$$ZX = \omega XZ.$$

Let us also denote

$$W(p) = Z^p X^{-p}, \qquad p \in \mathbb{Z},$$

then $W(p)$ satisfy the relations

$$W(p)W(q) = W(q)W(p) = \omega^{pq}W(p+q). \tag{16}$$

Let Φ be an OQRW as above, with

$$U_0 = Z, \qquad U_1 = X.$$

We first find the fixed point subalgebra of Φ, and this can be done using Proposition 1. We see that

$$\mathcal{F} = \{Z \otimes |0\rangle\langle 0|,\ Z \otimes |1\rangle\langle 1|,\ X \otimes |0\rangle\langle 1|, X \otimes |1\rangle\langle 0|\}'$$

and from this, we get

$$\mathcal{F} = \left\{ \begin{pmatrix} A & 0 \\ 0 & XAX^*, \end{pmatrix},\ A \in \{Z, X^2\}' \right\}. \tag{17}$$

The condition $A \in \{Z, X^2\}'$ implies that A is diagonal in the basis $\{|j\rangle\}$ and

$$A = X^2 A (X^*)^2 \implies \sum_j a_j |j\rangle\langle j| = \sum_j a_j |j \oplus 2\rangle\langle j \oplus 2|,$$

so that $a_j = a_{j\oplus 2}$ for $j = 0, \ldots, d-1$.

Assume now that d is odd. Then it follows that $a_j = a_0$ for all j, so that $\mathcal{F}$ is trivial. Hence, in this case, Φ is irreducible. Put $W = W(1) = ZX^*$, then

$$Z^*WZ = X^*WX = \omega W.$$

It follows that $\Phi(W \otimes I_2) = \omega(W \otimes I_2)$, so $\tilde{W} := W \otimes I_2$ is an eigenvector related to the peripheral eigenvalue ω. The eigenvalues of W are ω^k, $k = 0, \ldots, d-1$, each with an eigenvector $|x_k\rangle$. Hence, the period of Φ is d, and we have the cyclic decomposition

$$\{Q_m = |x_m\rangle\langle x_m| \otimes I_2,\ m = 0, \ldots, d-1\}.$$

By the results of Sect. 3.1, $\mathcal{N}$ is spanned by $\{Q_0, \ldots, Q_{d-1}\}$.

We next turn to the more interesting case when d is even. Put $q = d/2$. Then (17) holds, with $A = a_+P_+ + a_-P_-$, where $a_+, a_- \in \mathbb{C}$ and

$$P_+ = \sum_{k=0}^{q-1} |2k\rangle\langle 2k|, \qquad P_- = \sum_{k=0}^{q-1} |2k+1\rangle\langle 2k+1|.$$

So $\mathcal{F}$ is isomorphic to the abelian algebra spanned by these two projections. Note that we have $XP_+X^* = P_-$, $XP_-X^* = P_+$, so that we may write

$$\mathcal{F} = \operatorname{span}\left\{ \tilde{P}_+ := \begin{pmatrix} P_+ & 0 \\ 0 & P_- \end{pmatrix}, \tilde{P}_- := \begin{pmatrix} P_- & 0 \\ 0 & P_+ \end{pmatrix} \right\}.$$

Let us compute $\mathcal{N}$ using Proposition 9. Note first that by the commutation relations, we have for $\pi \in \mathcal{P}_n(i,j)$,

$$L_\pi = xZ^{n-l}X^l,$$

where $x \neq 0$ is some constant and $l \in \mathbb{N}$ is even if and only if $i = j$. It follows that if $\pi \in \mathcal{P}_n(j,i)$, $\pi' \in \mathcal{P}_n(j,k)$, we have

$$L_\pi L_{\pi'}^* = yZ^pX^{-p} = yW(p),$$

where $0 \neq y \in \mathbb{C}$ and $|p|$ is even iff $k = i$. Since all $L_{i,j}$ are (nonzero) multiples of unitary operators, we must have $\mathcal{N}_{OD} = \{0\}$. From the conditions on the diagonal blocks, we obtain that $A_{i,i}$ must commute with $W(p)$ for all even $|p|$ and $A_{i,i} = W(p)^* A_{j,j} W(p)$ for all $|p|$ odd if $i \neq j$. Using (16), we obtain that

$$\mathcal{N} = \left\{ \begin{pmatrix} A & 0 \\ 0 & WAW^* \end{pmatrix},\ A \in \{W(2)\}' \right\}.$$

It follows that $\mathcal{N}$ is isomorphic to the algebra $\{W(2)\}'$. One can see by (16) and $d = 2q$ that $W(2)^q = I$, so that the eigenvalues of $W(2)$ are the q-th roots of unity, that is, $\mu_m = \omega^{2m}$, $m = 0, \dots, q-1$. Let us denote

$$|m,+\rangle := \frac{1}{\sqrt{q}} \sum_{l=0}^{q-1} \omega^{2l(m-l+1)} |2l\rangle, \qquad |m,-\rangle := \frac{1}{\sqrt{q}} \sum_{l=0}^{q-1} \omega^{2l(m-l)} |2l \oplus 1\rangle$$

then one can check that

$$Q_m := |m,+\rangle\langle m,+| + |m,-\rangle\langle m,-|$$

is the eigenprojection corresponding to the eigenvalue μ_m. Since W commutes with $W(2)$ by (16), we have $WQ_mW^* = Q_m$, so that the center of $\mathcal{N}$ is spanned by the projections

$$\tilde{Q}_m := Q_m \otimes I_2, \qquad m = 0, \dots, q-1.$$

Further, it is easily checked that for $m = 0, \dots, q-1$, we have

$$Z|m,+\rangle = |m \oplus_q 1, +\rangle, \qquad Z|m,-\rangle = \omega |m \oplus_q 1, -\rangle$$

and

$$X|m,+\rangle = |m \oplus_q 1, -\rangle, \qquad X|m,-\rangle = \bar{\omega}^{2(m+1)} |m \oplus_q 1, +\rangle.$$

Since the action of Φ on elements of $\mathcal{N}$ has the form

$$\Phi \begin{pmatrix} A & 0 \\ 0 & WAW^* \end{pmatrix} = \begin{pmatrix} Z^*AZ & 0 \\ 0 & X^*AX \end{pmatrix},$$

we obtain $\Phi(\tilde{Q}_m) = \tilde{Q}_{m \ominus_q 1}$. It follows that there is a unique cycle of length q and consequently only one MFNC component $\mathcal{N}_1 = \mathcal{N}$, with period q.

We will identify the objects described in Sect. 3.2 for this special case. We have $\mathcal{K}_m^L = Q_m \mathfrak{h} \simeq \mathbb{C}^2$, $\mathcal{K}_m^R = \mathbb{C}^2$ and $\mathcal{K}^R = \sum_m \mathbb{C}^2 \otimes |m\rangle\langle m| \simeq \mathfrak{h}$. Put $S_m = I_m^L \otimes |0\rangle\langle 0| + W^*|_{\mathcal{K}_m^L} \otimes |1\rangle\langle 1|$, $m = 0, \dots, q-1$, and then, we have

$$\mathcal{N} = \oplus_m S_m^* (B(\mathcal{K}_m^L) \otimes I_2) S_m.$$

The unitaries $T_m : \mathcal{K}_m^L \to \mathcal{K}_{m \ominus_q 1}^L$ are given by the restrictions $T_m = Z^*|_{\mathcal{K}_m^L}$, so the action of Φ on $\mathcal{N}$ is described by the homogeneous cyclic shift OQRW on $\mathbb{Z}_q$, with local spaces $\mathbb{C}^2$ and unitaries $U_i \equiv U = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}$, $i = 0, \dots, q-1$ (cf. Proposition 10).

Let us compute the states ρ_m and maps Ξ_m defined in Proposition 8. Let $\Delta, \bar{\Delta} : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ be given by

$$\Delta \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} = \begin{pmatrix} b_{00} & 0 \\ 0 & b_{11} \end{pmatrix}, \quad \bar{\Delta} \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} = \begin{pmatrix} b_{11} & 0 \\ 0 & b_{00} \end{pmatrix}.$$

It is easily checked that for each m, the map $\Xi_m : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ is defined as

$$\Xi_m = \Xi := \alpha\Delta + (1-\alpha)\bar{\Delta}.$$

It follows by Proposition 8 (i) that the states ρ_m must all be equal to the unique invariant state $\rho = \frac{1}{2}I_2$ of Ξ.

Let us now turn to Theorem 2. We have $\tilde{T}_m = Z^{-m}|_{\mathcal{K}_0^L}$, $m = 0, \ldots, q-1$, in particular, the unitary $\tilde{T}_0$ has two eigenvalues ± 1, with eigenvectors $|0, \pm\rangle$, so that

$$\tilde{T}_0 = |0,+\rangle\langle 0,+| - |0,-\rangle\langle 0,-|.$$

The subalgebra $\{\tilde{T}_0\}' \cap B(\mathcal{K}_0^L)$ of Theorem 2 is the abelian subalgebra spanned by the projections $|0,\pm\rangle\langle 0,\pm|$. Note that we have

$$\sum_{m=0}^{q-1} \tilde{T}_m |0,\pm\rangle\langle 0,\pm| \tilde{T}_m^* = \sum_{m=0}^{q-1} Z^{-m}|0,\pm\rangle\langle 0,\pm|Z^m = \sum_{m=0}^{q-1} |m,\pm\rangle\langle m,\pm| = P_\pm,$$

so that

$$S^*T(|0,\pm\rangle\langle 0,\pm| \otimes I^R)T^*S = \tilde{P}_\pm$$

are the central projections of $\mathcal{F}$, which corresponds to Theorem 2 (i). For $s \in [0,1]$, put $\omega_s := s|0,+\rangle\langle 0,+| + (1-s)|0,-\rangle\langle 0,-|$, then we can see from Theorem 2 (ii) and (iii) that the invariant states of Φ are precisely those of the form

$$\xi_s := S^*T\left(\omega_s \otimes \frac{1}{d}I_d\right)T^*S = \frac{1}{d}\left(s\tilde{P}_+ + (1-s)\tilde{P}_-\right).$$

Finally, let $\Psi_\pm$ be the irreducible channels on $B(\mathcal{K}^R)$ corresponding to the restrictions of Φ by the projections $\tilde{P}_\pm$ as in Theorem 2 (iv). Let X_q, Z_q be the generalized Pauli operators on the q-dimensional Hilbert space with standard basis $\{|m\rangle\}$. One can check that we have

$$\Psi_+ = (\alpha\Delta + (1-\alpha)\bar{\Delta}) \otimes (X_q \cdot X_q^*) = \Xi \otimes (X_q \cdot X_q^*)$$

and

$$\Psi_- = \alpha\Delta \otimes (X_q \cdot X_q^*) + (1-\alpha)\bar{\Delta} \otimes (X_q Z_q^* \cdot Z_q X_q^*).$$

4.2.3. A Homogeneous Nearest-Neighbor OQRW on $\mathbb{Z}$. Let us consider a homogeneous nearest-neighbor OQRW Φ on $V = \mathbb{Z}$, with local space $\mathbb{C}^2$. We will assume that the transition operators L_+, L_- are invertible and that Φ is irreducible. The last condition implies that no invariant state exists, so the results of previous sections cannot be applied. Nevertheless, we show that also in this case the decoherence-free algebra is generated by the cyclic resolution of Φ, cf. Corollary 2.

Proposition 11. *Let us denote* $P_{\text{even}} := \sum_i I_2 \otimes |2i\rangle\langle 2i|$, $P_{\text{odd}} := \sum_i I_2 \otimes |2i+1\rangle\langle 2i+1|$. *Then*

$$\mathcal{N} = \operatorname{span}\{P_{\text{odd}}, P_{\text{even}}\}$$

unless there exists an orthonormal basis $\{f_0, f_1\}$ *such that* L_- *and* L_+ *are one diagonal and one off-diagonal in this basis. In the last case,* $\mathcal{N}$ *is generated by the cyclic projections*

$$P_{\epsilon,\delta} = \sum_j (|f_\epsilon\rangle\langle f_\epsilon| \otimes |4j+\delta\rangle\langle 4j+\delta| + |f_{1-\epsilon}\rangle\langle f_{1-\epsilon}| \otimes |4j+2+\delta\rangle\langle 4j+2+\delta|),$$

with $\epsilon, \delta = 0, 1$ *and the period is 4. Otherwise, the period is 2 with cyclic projections* P_{odd}, P_{even}.

The period was already computed in [11].

Proof. By Corollary 4, we know that the decoherence-free algebra $\mathcal{N}$ consists only of block-diagonal operators. Then, a projection P in $\mathcal{N}$ will have the form

$$P = \sum_j P_j \otimes |j\rangle\langle j|,$$

where, by Corollary 5, P_j are projections satisfying at least the conditions

$$P_j \in \{|L_+^*|, |L_-^*|\}', \qquad P_{j-1} L_- L_+^* = L_- L_+^* P_{j+1} \qquad \forall j. \tag{18}$$

We can write the action of Φ explicitly, in particular

$$\begin{aligned}\Phi(P) &= \sum_j (L_+^* P_{j+1} L_+ + L_-^* P_{j-1} L_-) \otimes |j\rangle\langle j|,\\ \Phi^2(P) &= \sum_j (L_+^{*2} P_{j+2} L_+^2 + L_-^{*2} P_{j-2} L_-^2 + L_-^* L_+^* P_j L_+ L_- \\ &\quad + L_+^* L_-^* P_j L_- L_+) \otimes |j\rangle\langle j|.\end{aligned} \tag{19}$$

By these relations, it is easily deduced that $\Phi^n(P_{\text{odd}})$ is equal to P_{odd} for even n and to P_{even} for odd n (and similarly for $\Phi^n(P_{\text{even}})$). In particular, $\Phi^n(P_{\text{odd}}), \Phi^n(P_{\text{even}})$ are always projections, and this allows us to conclude that P_{odd} and P_{even} belong to $\mathcal{N}$, Proposition 3 (ii). Moreover, they are trivially central, i.e., for any other projection P in $\mathcal{N}$, $PP_{\text{odd}} = P_{\text{odd}}P$ and $PP_{\text{even}} = P_{\text{even}}P$.

When there exists an orthonormal basis $\{f_0, f_1\}$ such that L_- and L_+ are one diagonal and one off-diagonal in this basis, it is easy to see that the projections $P_{\epsilon,\delta}$ in the statement are cyclic. It is a little more complicated to see that these cyclic projections can exist only in that case and anyway no other minimal projection can then appear.

So now we want to consider, for a homogeneous irreducible OQRW, whether there exists a projection P in $\mathcal{N}\backslash\text{span}\{P_{\text{odd}}, P_{\text{even}}\}$. We shall see that this is not possible, unless we are in the special case described in the statement.

If such a P exists, then $P = PP_{\text{odd}} + PP_{\text{even}}$ and the two addends are both in $\mathcal{N}$, so, by homogeneity, it will be sufficient to search for a projection P in $\mathcal{N}$ such that $P = PP_{\text{even}}$ and $0 < P < P_{\text{even}}$. Then, we consider $P = \sum_j P_{2j} \otimes |2j\rangle\langle 2j|$.

Relations (18) imply that all the P_{2j}'s have the same rank (since the transition operators are invertible). Then, if P is different from 0 and from P_{even}, the only possibility is that P_{2j} is a rank one projection for any j. Calling

u a norm one vector such that $P_0 = |u\rangle\langle u|$, and denoting $V := L_- L_+^*$, we deduce

$$P = \sum_j |V^{-j}u\rangle\langle V^{*j}u| \otimes |2j\rangle\langle 2j|,$$

where $V^{-j}u$ is proportional to $V^{*j}u$ because any P_{2j} is a projection and, due to the first condition in (18), $V^{*j}u$ is a common eigenvector of $|L_+^*|$ and $|L_-^*|$ for any j.

Similar considerations will hold for $\Phi^n(P)$, but considering only odd vertices instead of even vertices when n is odd. Indeed, starting with $n = 1$ (for $\Phi^n(P)$ we simply proceed inductively),

– $\Phi(P)$ is a projection in $\mathcal{N}$, $\Phi(P) \leq P_{\mathrm{odd}}$ due to the fact that $0 \leq P \leq P_{\mathrm{even}}$ and Φ is positive,
– moreover, when $P \neq P_{\mathrm{even}}$ then $\Phi(P) \neq P_{\mathrm{odd}}$ by irreducibility; indeed, if we had, for instance, $P \neq P_{\mathrm{even}}$ and $\Phi(P) = P_{\mathrm{odd}}$, then $P_{\mathrm{even}} - P$ would be a nonzero projection in the kernel of Φ, and this contradicts irreducibility.

Then, using (19), we need that

$$\Phi^2(P)(I_2 \otimes |0\rangle\langle 0|) = (L_+^{*2} P_2 L_+^2 + L_-^{*2} P_{-2} L_-^2 + L_-^* L_+^* P_0 L_+ L_- + L_+^* L_-^* P_0 L_- L_+)$$

is a one-dimensional projection. This implies in particular that $L_-^* L_+^* u \parallel L_+^* L_-^* u$, so that u is an eigenvector for $(L_+^* L_-^*)^{-1} L_-^* L_+^*$.

Also, calling $u^\perp$ a norm one vector orthogonal to u, $P' := P_{\mathrm{even}} - P = \sum_j |V^{-j}u^\perp\rangle\langle V^{*j}u^\perp| \otimes |2j\rangle\langle 2j|$, will be a projection in $\mathcal{N}$ and so $u^\perp$ will satisfy the same conditions as u.

Summing up, we have that u and $u^\perp$ should be two distinct eigenvectors for the operators

$$|L_+^*|, \qquad |L_-^*|, \qquad W := (L_+^* L_-^*)^{-1} L_-^* L_+^*. \tag{20}$$

Now, we claim that, due to irreducibility, the previous operators cannot be all proportional to the identity, and we postpone of some lines the proof of this claim.

This fact implies that, either such vectors u and $u^\perp$ do not exist, and so $\mathcal{N} = \mathrm{span}\{P_{\mathrm{odd}}, P_{\mathrm{even}}\}$, or they can be chosen in a unique way, up to multiplicative constants, as the orthonormal basis which diagonalize all the three operators above. In the latter case, we now look at the form of $\Phi(P)$ given in (19) and we see that

$$\Phi(P)(I_2 \otimes |j\rangle\langle j|) = L_+^* P_{j+1} L_+ + L_-^* P_{j-1} L_-$$

should be a one-dimensional projection on a vector v which should be an eigenvector of the same three operators. This implies that

$$L_\epsilon^* u, L_\epsilon^* u^\perp \in \mathrm{span}\{u\} \cup \mathrm{span}\{u^\perp\}, \qquad \epsilon = +, -$$

and consequently that the operators L_+ and L_- should be either diagonal or off-diagonal in the basis $\{u, u^\perp\}$, but they cannot be both diagonal or both off-diagonal, because this would contradict irreducibility. So the conclusion follows choosing $\{f_0, f_1\} = \{u, u^\perp\}$.

Finally, we go back to prove the claim. By contradiction, we suppose that all the operators in (20) are proportional to the identity, so that

$$L_+ = c_+ U_+, \qquad L_- = c_- U_-, \qquad W = c1,$$

for some complex numbers c_+, c_-, c and unitary operators U_+, U_-. Then we can rewrite

$$W = c1 = U_- U_+ U_-^* U_+^* \qquad \Rightarrow \qquad U_- = cU_+ U_- U_+^*.$$

But now write the diagonal form of the unitary U_+, $U_+^* = \sum_{k=0,1} \lambda_k |v_k\rangle\langle v_k|$, with λ_0, λ_1 in the unit circle and $\{v_0, v_1\}$ orthonormal basis, and consider

$$\langle v_k, U_- v_j\rangle = c\langle v_k, U_+ U_- U_+^* v_j\rangle = c\overline{\lambda}_k \lambda_j \langle v_k, U_- v_j\rangle \qquad \text{for } j,k = 0,1.$$

This implies $c = 1$ and $\lambda_0 = \lambda_1$ which requires that U_+ and so L_+ are proportional to the identity. But this contradicts irreducibility. □

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