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Lyapunov exponents for Quantum Channels: an entropy formula and generic properties

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On a previous work the authors show that for a fixed measure μ it is generic on the function L the Φ-Erg property (also irreducibility). Here we will show that the purification property is also generic on L for a fixed μ.
Given L and μ there are two related stochastic process: one takes values on the projective space $ P(\C^k)$ and the other on matrices in M_k.
The Φ-Erg property and the purification condition are good hypothesis for the discrete time evolution given by the natural transition probability. In this way it will follow that generically on L,
if $\int |L(v)|^2 \log |L(v)|\, \ d\mu(v)<\infty$, then the Lyapunov exponents $\infty > \gamma_1\geq \gamma_2\geq ...\geq \gamma_k\geq -\infty$ are well defined.
On the previous work it was presented the concepts of entropy of a channel and of Gibbs channel; and also an example (associated to a stationary Markov chain) where this definition of entropy (for a quantum channel) matches the Kolmogorov-Shanon definition of entropy. We estimate here the larger Lyapunov exponent for the above mentioned example and we show that it is equal to $-\frac{1}{2} \,h$, where h is the entropy of the associated Markov probability.

Lyapunov exponents for Quantum Channels: an entropy formula and generic properties

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Abstract

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On a previous work the authors show that for a fixed measure μ it is generic on the function L the Φ-Erg property (also irreducibility). Here we will show that the purification property is also generic on L for a fixed μ.

Given L and μ there are two related stochastic process: one takes values on the projective space $P(\mathbb{C}^k)$ and the other on matrices in M_k. The Φ-Erg property and the purification condition are good hypothesis for the discrete time evolution given by the natural transition probability. In this way it will follow that generically on L, if $\int |L(v)|^2 \log |L(v)| \; d\mu(v) < \infty$, then the Lyapunov exponents $\infty > \gamma_1 \geq \gamma_2 \geq ... \geq \gamma_k \geq -\infty$ are well defined.

On the previous work it was presented the concepts of entropy of a channel and of Gibbs channel; and also an example (associated to a stationary Markov chain) where this definition of entropy (for a quantum channel) matches the Kolmogorov-Shanon definition of entropy. We estimate here the larger Lyapunov exponent for the above mentioned example and we show that it is equal to $-\frac{1}{2}\, h$, where h is the entropy of the associated Markov probability.

1 Introduction

We consider quantum channels of the form $\phi_L : M_k \to M_k$, where M_k is the set of complex k by k matrices, $\rho \to \phi_L(\rho) = \int_{M_k} L(v)\rho L(v)^\dagger \, d\mu(v)$, $L : M_k \to M_k$ is a measurable function and μ is a measure on M_k.

In the recent paper [10] the authors considered Lyapunov exponents for this class of channels ϕ_L when L was constant and equal to the identity matrix. The Φ-Erg property and the purification condition (see definitions on section 6) were also considered on this mentioned paper.

On a previous paper [12] we show that for a fixed measure μ it is generic on the function L the Φ-Erg property (in fact we show that the irreducible condition is generic). The novelty here is that we will show that the purification condition is also generic on L for a fixed measure μ (see section 9).

The introduction of this variable L allows us to consider questions of generic nature in this type of problem. We use the C^0 topology in the set of complex matrices.

Following [10] one can consider associated to L and μ two related process: one denoted by X_n, $n \in \mathbb{N}$, takes values on the projective space $P(\mathbb{C}^k)$, and the other, denoted by ρ_n, $n \in \mathbb{N}$, takes values on D_k (where D_k is the set of density operators). The natural transition probability is defined in [10].

The Φ-Erg property and the purification property play an important role when analyzing the ergodic properties of these two processes.

For a fixed μ and a general L it was presented in [12] a natural concept of entropy (see future section 3) for a channel in order to develop a version of Gibbs formalism. It was also presented in example 8.5 in [12] a certain channel (related to stationary Markov Chains) where the value obtained with this definition coincides with the classical value of entropy. This shows that the concept that was introduced is natural. This definition of entropy is a generalization of the concept described on the papers [6], [8] and [7]. This particular way of defining entropy is in some sense inspired by results of [26] which considers iterated function systems.

The main contribution to the topic of Lyapunov exponents of quantum channels is the paper [10]. We adapt here the formalism of [10] to the case of a general L in order to estimate the Lyapunov exponents of the associated dynamical time evolution. We will just outline the proof (see section 7) because is basically the one in [10]). We describe the sufficient conditions for the Lyapunov exponents to be finite. Irreducibility and the purification condition are the good assumptions. Proposition 9.1 and [12] show that these conditions are true for a fixed μ and a generic L.

Results relating entropy and Lyapunov exponents (both in the classical sense) are quite important in Ergodic Theory (see for instance [9], [20] and [18]).

Another important issue here is the entropy formula. We compute the first Lyapunov exponent (which is negative) for the above mentioned example (see Section 8) and we show that it is equal to $-\frac{1}{2}\,h$, where h is the entropy of the associated Markov probability. We also show that the second Lyapunov exponent in this case will be $-\infty$. Of course, a general result for the class of all quantum channels is not reachable due to its inherent generality.

We point out that the definition of entropy for a (normalized) channel presented in [12] explore the use of a "kind" of Ruelle operator. This procedure uses a natural *a priori* probability and this makes sense due to the fact that the "set of preimages" can be an uncountable set (see [19]). The main issue on the reasoning in [12] is invariance (in time one), however, there the concept of entropy is not directly associated to time evolution. We do not use in this way ergodicity (or, the limit of measures of an increasing family of partitions, etc..) in the definition of entropy. On the other hand, we point out that the values of the Lyapunov exponents are of *dynamical nature.* This dynamical discrete time evolution is described by a stochastic process taking values on the set of matrices in M_k (see sections 6 and 7). The example we consider here in section 8 shows that the concept of entropy of a channel (at least in this case) presented in [12] can be linked to the natural dynamical time evolution via the main Lyapunov exponent.

Nice references for Quantum Channels are [16], [23], [1] and [29]. The book [28] presents several important results for the general theory of Lyapunov exponents (see also [14], [2], [3], [4], [5], [17] and [13]). [27] and [22] describe basic result in Ergodic Theory.

We thanks S. Klein for supplying us with references.

2 Basic results

We denote by M_k the set of complex k by k matrices. We denote by Id the identity matrix on M_k.

We consider the standard Borel sigma-algebra over M_k and the canonical Euclidean inner product on $\mathbb{C}^k$

According to our notation $\dagger$ denotes the operation of taking the dual of a matrix with respect to the canonical inner product on $\mathbb{C}^k$.

Here tr denotes the trace of a matrix.

Given two matrices A and B we define the Hilbert-Schmidt product

$$\langle\, A\,,\, B\,\rangle \;=\; \text{tr}\,(A\,B^\dagger).$$

This induces a norm $\|A\| = \sqrt{\langle\, A\,,\, A\,\rangle}$ on the Hilbert space M_k which will be called the Hilbert-Schmidt norm.

Definition 2.1 *Given a linear operator Φ on M_k we denote by $\Phi^* : M_k \to M_k$ the dual linear operator in the sense of Hilbert-Schmidt, that is, if for all X, Y we get*

$$\langle\, \Phi(X)\,,\, Y\,\rangle = \langle\, X\,,\, \Phi^*(Y)\,\rangle.$$

Consider a measure μ on the Borel sigma-algebra over M_k. For an integrable transformation $F : M_k \to M_k$:

$$\int_{M_k} F(v)\, d\mu(v) = \left(\int_{M_k} F(v)_{i,j}\, d\mu(v) \right)_{i,j},$$

where $F(v)_{i,j}$ is the entry (i, j) of the matrix $F(v)$.

Definition 2.2 *Given a measure μ on M_k and a measurable funtion $L : M_k \to M_k$, we say that μ is L-square integrable, if*

$$\int_{M_k} \|L(v)\|^2 \, d\mu(v) < \infty.$$

For a fixed L we denote by $\mathcal{M}(L)$ the set of L-square integrable measures. We also denote $\mathcal{P}(L)$ the set of L-square integrable probabilities.

ϕ_L is well defined for $L \in \mathcal{M}(L)$.

Proposition 2.3 *Given a measurable function $L : M_k \to M_k$ and a square integrable measure μ, then, the dual transformation ϕ_L^* is given by*

$$\phi_L^*(\rho) = \int_{M_k} L(v)^\dagger \rho L(v)\, d\mu(v).$$

Definition 2.4 *Given a measure μ over M_k and a square integrable transformation $L : M_k \to M_k$ we say that L is a* **stochastic square integrable transformation** *if*

$$\phi_L^*(Id) = \int_{M_k} L(v)^\dagger L(v)\, d\mu(v) = Id.$$

Definition 2.5 *A linear map $\phi : M_k \to M_k$ is called* **positive** *if takes positive matrices to positive matrices.*

Definition 2.6 *A positive linear map $\phi : M_k \to M_k$ is called* **completely positive**, *if for any m, the linear map $\phi_m = \phi \otimes I_m : M_k \otimes M_m \to M_k \otimes M_m$ is positive, where I_m is the identity operator acting on the matrices in M_m.*

Definition 2.7 *If $\phi : M_k \to M_k$ is square integrable and satisfies*

1. *ϕ is completely positive;*

2. *ϕ preserves trace,*

then, we say that ϕ is a quantum channel.

Theorem 2.8 *Given μ and L square integrable then the associated transformation ϕ_L is completely positive. Moreover, if ϕ_L is stochastic, then it preserves trace.*

For the proof see [12].

Remark 2.9 *ϕ_L^* is also completely positive. We say that ϕ_L preserves unity if $\phi_L(\mathrm{Id}_k) = \mathrm{Id}_k$. In this case, ϕ_L^* preserves trace. If ϕ_L^* preserves the identity then ϕ_L preserves trace.*

Definition 2.10 (Irreducibility) *We say that $\phi : M_k \to M_k$ is an* **irreducible channel** *if one of the equivalent properties is true*

- *Does not exists $\lambda > 0$ and a projection p in a proper subspace of $\mathbb{C}^k$, such that, $\phi(p) \leq \lambda p$;*
- *For all non null $A \geq 0$, $(\boldsymbol{1} + \phi)^{k-1}(A) > 0$;*
- *For all non null $A \geq 0$ there exists $t_A > 0$, such that, $(e^{t_A \phi})(A) > 0$;*
- *For all pair of non null positive matrices $A, B \in M_k$ there exists a natural number $n \in \{1, ..., k-1\}$, such that, $\mathrm{tr}\,[B\phi^n(A)] > 0$.*

For the proof of the equivalences we refer the reader to [15], [25] and [29].

Definition 2.11 (Irreducibility) *Given μ we will say (by abuse of language) that L is irreducible if the associated ϕ_L is an irreducible channel.*

Theorem 2.12 (Spectral radius of ϕ_L and ϕ_L^*) *Given a square integrable $L : M_k \to M_k$ assume that the associated ϕ_L is irreducible. Then, the spectral radius $\lambda_L > 0$ of ϕ_L and ϕ_L^* is the same and the eigenvalue is simple. We denote, respectively, by $\rho_L > 0$ and $\sigma_L > 0$, the eigenmatrices, such that, $\phi_L(\rho_L) = \lambda_L \rho_L$ and $\phi_L^*(\sigma_L) = \lambda_L \sigma_L$, where ρ_L and σ_L are the unique non null eigenmatrices (up to multiplication by scalar).*

The above theorem is the natural version of the Perron-Frobenius Theorem for the present setting.

It is natural to think that ϕ_L acts on density states and ϕ_L^* acts in self-adjoint matrices.

Definition 2.13 *Given the measure μ over M_k we denote by $\mathfrak{L}(\mu)$ the set of all integrable L such that the associated ϕ_L is irreducible.*

Definition 2.14 *Suppose L is in $\mathfrak{L}(\mu)$. We say that L is* **normalized** *if ϕ_L has spectral radius 1 and preserves trace. We denote by $\mathfrak{N}(\mu)$ the set of all normalized L.*

If $L \in \mathfrak{N}(\mu)$, then, we get from Theorem 2.12 and the fact that $\phi_L^*(\mathrm{Id}_k) = \mathrm{Id}_k$, that $\lambda_L = 1$. That is, there exists ρ_L such that $\phi_L(\rho_L) = \rho_L$ and ρ_L is the only fixed point. Moreover, the spectral radius is equal to 1.

Theorem 2.15 (Ergodicity and temporal means) *Suppose $L \in \mathfrak{N}(\mu)$. Then, for all density matrix $\rho \in M_k$ it is true that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \phi_L^n(\rho) = \rho_L,$$

where ρ_L is the density matrix associated to L.

Proof: The proof follows from Theorem 2.12 and corollary 6.3 in [29].

$\square$

The above result connects irreducibility and ergodicity (the temporal means have a unique limit).

3 Entropy

A measure μ over M_k (which plays the role of the *a priori* probability) is fixed. In this way given $L \in \mathfrak{L}(\mu)$ we will associate in a natural way the transformation $\phi_L : M_k \to M_k$.

Definition 3.1 *We denote by $\Phi = \Phi_\mu$ the set of all L such that the associated $\phi_L : M_k \to M_k$ is irreducible and stochastic.*

Suppose L is irreducible and stochastic.

Given L consider the density matrix ρ_L which is invariant for ϕ_L (see Theorem 2.12).

Definition 3.2 *We define entropy for L (or, for ϕ_L) by the expression (when finite) :*

$$h(L) = h_\mu(L) := -\int_{M_k \times M_k} \mathrm{tr}\,(L(v)\rho_L L(v)^\dagger) P_L(v,w) \log P_L(v,w)\; d\mu(v) d\mu(w),$$

where

$$P_L(v,w) := \frac{\mathrm{tr}\,(L(w)L(v)\rho_L L(v)^\dagger L(w)^\dagger)}{\mathrm{tr}\,(L(v)\rho_L L(v)^\dagger)}.$$

This definition is a generalization of the analogous concept presented on the papers [6], [8] and [7].

An example in [12] shows that the above definition of entropy is indeed a natural generalization of the classical one in Ergodic Theory. Later we will consider again this example when analyzing Lyapunov exponents (see section 8).

4 Process X_n, $n \in \mathbb{N}$, taking values on $P(\mathbb{C}^k)$

Consider a fixed measure μ on M_k and a fixed $L : M_k \to M_k$, such that, $\int_{M_k} \|L(v)\|^2\, d\mu(v) < \infty$, and, also assume that ϕ_L is irreducible and stochastic.

Note that if, for example, μ is a probability and the the function $v \to \|L(v)\|$ is bounded we get that $\int_{M_k} \|L(v)\|^2\, d\mu(v) < \infty$.

We follow the notation of [10] (and, also [12])

Denote by $P(\mathbb{C}^k)$ the projective space on $\mathbb{C}^k$ with the metric $d(\hat{x},\hat{y}) = (1 - |\langle\, x\,,\, y\,\rangle|^2)^{1/2}$, where x, y are representatives with norm 1 and $\langle\,\cdot\,,\,\cdot\,\rangle$ is the canonical inner product.

Take $\hat{x} \in P(\mathbb{C}^k)$ and $S \subset P(\mathbb{C}^k)$. For a stochastic ϕ_L we consider the kernel

$$\Pi_L(\hat{x}, S) = \int_{M_k} \mathbf{1}_S(L(v)\cdot\hat{x})\, \|L(v)x\|^2\; d\mu(v), \tag{1}$$

where the norm above is the Hilbert-Schmidt one.

This discrete time process (described by the kernel) taking values on $P(\mathbb{C}^k)$ is determined by such μ and L. If ν is a probability on the Borel σ-algebra $\mathcal{B}$ of $P(\mathbb{C}^k)$ define

$$\begin{aligned}\nu\Pi_L(S) &= \int_{P(\mathbb{C}^k)} \Pi_L(\hat{x}, S)\; d\nu(\hat{x}) \\ &= \int_{P(\mathbb{C}^k)\times M_k} \mathbf{1}_S(L(v)\cdot\hat{x})\, \|L(v)x\|^2\;\; d\nu(\hat{x}) d\mu(v).\end{aligned}$$

$\nu\Pi_L$ is a new probability on $P(\mathbb{C}^k)$ and Π_L is a Markov operator. The above definition of $\nu \to \nu\Pi_L$ is a simple generalization of the one in [10], where the authors take the L consider here as the identity transformation.

The map $\nu \to \nu\,\Pi_L$ (acting on probabilities ν) is called the Markov operator obtained from ϕ_L in the paper [21] . There the *a priori* measure μ is a sum of Dirac probabilities. Here we consider a more general setting.

Definition 4.1 *We say that the probability ν over $P(\mathbb{C}^k)$ is invariant for Π_L, if $\nu\Pi_L = \nu$.*

See [12] (and also future Proposition 6.7) for the existence of invariant probabilities for Π_L

5 The process ρ_n, $n \in \mathbb{N}$, taking values on D_k

For a fixed μ over M_k and L such ϕ_L is irreducible and stochastic, one can naturally define a process (ρ_n) on $D_k = \{\rho \in M_k : \operatorname{tr}\rho = 1 \text{ and } \rho \geq 0\}$, which is called *quantum trajectory.*

We follow the notation of [10] (and, also [12])

Given a ρ_0 initial state, we get

$$\rho_n = \frac{L(v)\rho_{n-1}L(v)^*}{\operatorname{tr}(L(v)\rho_{n-1}L(v)^*)}$$

with probability

$$\operatorname{tr}(L(v)\rho_{n-1}L(v)^*)d\mu(v), \quad n \in \mathbb{N}.$$

We want to relate the invariant probabilities of last section with the fixed point $\rho_{inv} = \rho_{inv}^L$ of ϕ_L.

First, denote $\Omega := M_k^{\mathbb{N}}$, and for $\omega = (\omega_i)_{i\in\mathbb{N}}$, take $\phi_n(\omega) = (\omega_1, ..., \omega_n)$.

We denote π_n the projection of ω in its first n coordinates.

We also denote by $\mathcal{M}$ the Borel sigma algebra M_k. For all, $n \in \mathbb{N}$, consider $\mathcal{O}_n$ the sigma algebra on Ω generated by the cylinder sets of size n, that is, $\mathcal{O}_n := \pi_n^{-1}(\mathcal{M}^n)$. We equip Ω with the smaller sigma algebra $\mathcal{O}$ which contains all $\mathcal{O}_n$, $n \in \mathbb{N}$.

Denote $\mathcal{J}_n := \mathcal{B} \otimes \mathcal{O}_n$ and $\mathcal{J} := \mathcal{B} \otimes \mathcal{O}$. In this way, $(P(C^k) \times \Omega, \mathcal{J})$ is an integrable space. By abuse of language we consider $V_i : \Omega \to M_k$ as a random variable $V_i(\omega) = \omega_i$. We also introduce another random variable

$$W_n := L(V_n)...L(V_1), \text{ where } W_n(\omega) = L(\omega_n)...L(\omega_1).$$

We point out that the symbol $\otimes$ does not represents tensor product.

For a given a probability ν on $P(\mathbb{C}^k)$, we define for $S \in \mathcal{B}$ and $O_n \in \mathcal{O}_n$, another probability

$$\mathbb{P}_\nu(S \times O_n) := \int_{S\times O_n} \|W_n(\omega)x\|^2 \; d\nu(\hat{x})d\mu^{\otimes n}(\omega).$$

Denote $\mathbb{E}_\nu$ the expected value with respect to $\mathbb{P}_\nu$. Now observe that for a ν probability on $P(\mathbb{C}^k)$, if $\pi_{X_0} : \mathbb{C}^k - \{0\} \to P(\mathbb{C}^k)$ is the orthogonal projection on subspace generated by X_0 on $\mathbb{C}^k$, we have

$$\rho_\nu := \mathbb{E}_\nu(\pi_{X_0}) = \int_{P(\mathbb{C}^k)} \pi_{x_0} \; d\nu(x_0).$$

We call ρ_ν barycenter of ν, and it's easy to see that $\rho_\nu \in D_k$.

Proposition 5.1 *If ν is invariant for Π_L, then*

$$\rho_\nu = \mathbb{E}_\nu(\pi_{\hat{X}_0}) = \mathbb{E}_\nu(\pi_{\hat{X}_1}) = \phi_L(\rho_\nu).$$

Therefore, for an irreducible L, every invariant measure ν for Π_L has the same barycenter.

We point out that in this way we can recover ρ_{inv}, the fixed point of ϕ_L, by taking the barycenter of any invariant probability (the quantum channel ϕ_L admits only one fixed point). That is, for any invariant probability ν for Π_L, we get that $\rho_\nu = \rho_{inv}$.

Note that the previous process can be seen as $\rho_n : \Omega \to D_k$, such that, $\rho_0(\hat{x}, \omega) = \rho_\nu$ and

$$\rho_n(\omega) = \frac{W_n(\omega)\rho_0 W_n(\omega)^*}{\mathrm{tr}\,(W_n(\omega)\rho_0 W_n(\omega)^*)}.$$

Using an invariant ρ we can define an Stationary Stochastic Process taking values on M_k. That is, we will define a probability $\mathbb{P}$ over $\Omega = (M_k)^{\mathbb{N}}$.

Take $O_n \in \mathcal{O}_n$ and define

$$\mathbb{P}^\rho(O_n) = \int_{O_n} \mathrm{tr}\,(W_n(\omega)\rho W_n(\omega)^*) \; d\mu^{\otimes n}(\omega).$$

The probability $\mathbb{P}$ on Ω defines a Stationary Stochastic Process.

6 Irreducibility, the Φ-Erg property and the purification condition

We will use in this section the notation of [10].

Definition 6.1 *Given $L : M_k \to M_k$, μ on M_k and E subspace of $\mathbb{C}^k$, we say that E is (L, μ)-invariant, if $L(v)E \subset E$, for all $v \in supp\, \mu$.*

Definition 6.2 *Given $L : M_k \to M_k$, μ on M_k, we say that L* **is Φ-Erg for** *μ, if there exists an unique minimal non-trivial space E, such that, E is (L, μ)-invariant.*

In [25] it is shown that if the above space E is equal to $\mathbb{C}^k$, then L is **irreducible** for μ (or, μ-irreducible) in the sense of Definition 2.11.

The relation of $\mathbb{P}^\rho$ and $\mathbb{P}_\nu$ (described on last sections) is described in the next result.

Proposition 6.3 *The marginal of $\mathbb{P}_\nu$ on $\mathcal{O}$ is $\mathbb{P}^{\rho_\nu}$. In the case the Φ-Erg is true, then for any two Π-invariant probabilities ν_a and ν_b, we get $\mathbb{P}^{\rho_{\nu_a}} = \mathbb{P}^{\rho_{\nu_b}}$.*

The proof of the above result when L is the identity was done in Proposition 2.1 in [10]. The proof for the case of a general L is analogous.

Given two operators A and B we say that $A \propto B$, if there exists $\beta \in \mathbb{C}$, such that, $A = \beta\, B$.

Definition 6.4 *Given $L : M_k \to M_k$, μ on M_k, we say that the pair (L, μ) satisfies the* **purification condition***, if an orthogonal projector π, such that, for any $n \in \mathbb{N}$*

$$\pi L(V_1)^* ... L(V_n)^*\, L(V_n) ... L(V_1)\, \pi \;\propto\; \pi,$$

for $\mu^{\otimes n}$-almost all $(v_1, v_2, .., v_n)$, it is necessarily of rank one.

Following [10] we denote $\mathbb{P}^{ch} = \mathbb{P}^{\frac{1}{k}\, Id}$.

We denote by Y_n, $n \in \mathbb{N}$, the matrix-valued random variable

$$Y_n \;=\; \frac{W_n^*\, W_n}{\text{tr } (W_n^*\, W_n)}, \text{ if tr } (W_n^*\, W_n) \neq 0,$$

where we extend the definition in arbitrary way when tr $(W_n^*\, W_n)$=0 .

The next two propositions are of fundamental importance in the theory and they were proved in Proposition 2.2 in [10] (the same proofs works in our setting).

Proposition 6.5 *For any probability ν over $P(\mathbb{C}^k)$ the stochastic process Y_n, $n \in \mathbb{N}$, is a martingale with respect to the sequence of sigma-algebras $\mathcal{O}_n$, $n \in \mathbb{N}$. Therefore, there exists a random variable Y_∞ which is the almost sure limit of Y_n for the probability $\mathcal{P}_\nu$ and also in the L^1 norm.*

Proposition 6.6 *For any probability ν over $P(\mathbb{C}^k)$ and $\rho \in \mathcal{D}_k$*

$$\frac{d\,\mathbb{P}^\rho}{d\,\mathbb{P}^{ch}} = k\ tr\,(\rho\, Y_\infty).$$

Moreover, μ and L satisfy the purification condition, if and only if, Y_∞ is $\mathbb{P}_\nu$-a.s a rank one projection for any probability ν over $P(\mathbb{C}^k)$.

Proposition 6.7 *If the pair (L, μ) satisfies the ϕ-Erg and the purification condition, then, the Markov kernel Π admits a unique invariant probability.*

$x_1 \wedge x_2 \wedge ... \wedge x_n$, with $x_j \in \mathbb{C}^k$, denotes the classical wedge product (an alternate form on $\mathbb{C}^k$).

One can consider an inner product

$$\langle\, r_1 \wedge r_2 \wedge ... \wedge r_n\ ,\ s_1 \wedge s_2 \wedge ... \wedge s_n\ \rangle = \ \det\,(r_i s_j))_{i,j=1,2,...,n},$$

and, the associated norm $|x_1 \wedge x_2 \wedge ... \wedge x_n\,|$.

Given an operator $X : \mathbb{C}^k \to \mathbb{C}^k$ we define $\bigwedge^n X\ : \wedge^n\mathbb{C}^k \to \wedge^n\mathbb{C}^k$ by $\bigwedge^n X\,(x_1 \wedge x_2 \wedge ... \wedge x_n)\ = X(x_1) \wedge X(x_2) \wedge ... \wedge X(x_n)$.

Proposition 6.8 *Assume the pair (L, μ) satisfies the purification condition, then, there are two constants $C > 0$ and $\beta < 1$, such that, for each n*

$$\int_{M_k^n} |\wedge^2\,(\,L(v_n)...L(v_1)\,)\,|\,d\,\mu^{\otimes\, n}(v_1, v_2, ..., v_n)\ = \mathbb{E}^{ch}\,(k\,\frac{|\bigwedge^2 W_n\,|}{tr\,(\,W_n^* W_n\,)}\,) \leq C\,\beta^n.$$

The proofs of the two propositions above are similar to the corresponding ones in [10].

Consider $\mathcal{B}(M_k) = \{L : M_k \to M_k\,|\,\text{L is continuous and bounded}\}$ where $\|L\| = \sup_{v \in M_k} \|L(v)\|$.

Definition 6.9 *For a fixed a measure μ over M_k, define*

$$\mathcal{B}_\mu(M_k) = \{L \in \mathcal{B}\,|\,L\ \textit{is}\ \mu\textit{-irreducible}\},$$

and

$$\mathcal{B}_\mu^\Phi(M_k) = \{L \in \mathcal{B}\,|\,L\ \textit{is}\ \Phi\textit{-Erg for}\ \mu\}.$$

Proposition 6.10 *Given μ over M_k with $\# supp\, \mu > 1$, $\mathcal{B}_\mu^\Phi(M_k)$ is open and dense on $\mathcal{B}(M_k)$.*

Proof: See [12].

□

In section 9 we will prove:

Proposition 6.11 *Given μ over M_k with $\# supp\, \mu > 1$, the set of L satisfying the purification condition is generic in $\mathcal{B}(M_k)$.*

7 Lyapunov exponents for quantum channels

In this section we will consider a discrete time process taking values on M_k.

Take μ over M_k and $L : M_k \to M_k$ in such way that the associated channel Φ defines a Φ-Erg stochastic map. We assume in this section that $\rho \in D_k$ is such that $\Phi(\rho) = \rho$. Such ρ plays the role of the initial vector of probability (in the analogy with the theory of Markov Chains).

We follow the notation of [10].

Take $\Omega = M_k^{\mathbb{N}}$, and for $n \in \mathbb{N}$ let $\mathcal{O}_n$ be the σ-algebra on Ω generated by the n-cylinder sets (as in Section 5).

An element on Ω is denoted by $(\omega_1, \omega_2, ..., \omega_n, ...)$. Following section 5 we denote $W_n(\omega) = L(\omega_n)...L(\omega_1)$.

Taking $O_n \in \mathcal{O}_n$ we define

$$\mathbb{P}(O_n) = \int_{O_n} \operatorname{tr}\left(W_n(\omega)\rho W_n(\omega)^*\right)\, d\mu^{\otimes n}(\omega).$$

If $\mathcal{O}$ is the smallest σ-algebra of Ω that contains all $\mathcal{O}_n$ we can extend the action of $\mathbb{P}$ to this σ-algebra.

The probability $\mathbb{P}$ on Ω defines a Stationary Stochastic Process.

Theorem 7.1 *$(\Omega, \mathbb{P}, \theta)$ is ergodic where θ is the shift map.*

The above theorem has been proved in Lemma 4.2 in [10].

Theorem 7.2 *Suppose the pair (L, μ) satisfies irreducibility, the ϕ-Erg and the purification condition. Assume also that $\int |L(v)|^2 \log |L(v)|\ \, d\mu(v) < \infty$, then, there exists numbers*

$$\infty > \gamma_1 \geq \gamma_2 \geq ... \geq \gamma_k \geq -\infty,$$

such that, for any probability ν over $P(\mathbb{C}^k)$ and any $p \in \{1, 2, .., k\}$

$$\lim_{n\to\infty} \frac{1}{n} \log | \bigwedge^p W_n | = \sum_{j=1}^{p} \gamma_j,$$

$\mathbb{P}_\nu$-a.s.

The above theorem was proved in [10] and the same proof works here in our setting. We point out that a key ingredient in this proof (see (35) in [10]) is the fact that if (L, μ) is ϕ-Erg and irreducible, then, $\rho_{inv} > 0$, and for any $\rho \in \mathcal{D}_k$

$$\mathbb{P}^\rho << \mathbb{P}^{\rho_{inv}}.$$

Proposition 6.3 is also used in the proof (corresponds to proposition 2.1 in [10]).

The numbers

$$\gamma_1 \geq \gamma_2 \geq ... \geq \gamma_k,$$

are called the **Lyapunov exponents**.

Theorem 7.3 *Suppose that L is generic for μ and $\int |L(v)|^2 \log |L(v)| \; d\mu(v) < \infty$, then, the Lyapunov exponents*

$$\infty > \gamma_1 \geq \gamma_2 \geq ... \geq \gamma_k \geq -\infty$$

are well defined.

(a) $\gamma_2 - \gamma_1 < 0$, where $\gamma_2 - \gamma_1$ is the limit

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{| \bigwedge^2 W_n |}{| W_n |^2},$$

whenever $\gamma_1 = -\infty$

(b) For $\mathbb{P}_\nu$-almost sure x we have that

$$\lim_{n\to\infty} \frac{1}{n} (\log | W_n (x) | - \log | W_n |) = 0.$$

proof:

The proof of (a) is similar to the one in [10].

Proof of (b): We start with

$$\frac{\|W_n x\|}{\|W_n\|} = \frac{[\mathrm{tr}\,(W_n \pi_x W_n^\dagger)\;]^{1/2}}{[\mathrm{tr}\,(W_n W_n^\dagger)\;]^{1/2}} = \left[\frac{\mathrm{tr}\,(W_n^\dagger W_n \pi_x)}{\mathrm{tr}\,(W_n W_n^\dagger)}\right]^{1/2}$$

$$= \left[\mathrm{tr}\left(\frac{W_n^\dagger W_n}{\mathrm{tr}\,(W_n W_n^\dagger)}\ \pi_x \right) \right]^{1/2} = [\mathrm{tr}\ (M_n \pi_x)]^{1/2}\,.$$

These calculations are valid for all ω such that $W_n W_n^\dagger(\omega) \neq 0$. Since $\mathbb{P}_\nu(W_n = 0) = 0$, no extra work is required. Now we use proposition 2.2 in [10] which says that M_n converges $\mathbb{P}_\nu - a.s.$ and in L^1 norm to a $\mathcal{O}$-measurable random variable M_∞. By continuity of the trace and square root, we have

$$\lim_{n\to\infty} [\mathrm{tr}\ (M_n \pi_x)]^{1/2} = [\mathrm{tr}\ (M_\infty \pi_x)]^{1/2}\,,\ \text{for } \mathbb{P}_\nu - \text{a.s. } x \in P(\mathbb{C}^k).$$

The proof is similar to the one in [10].

$\square$

8 The main example - Lyapunov exponents and entropy

Now we will present an example where we can estimate the Lyapunov exponents and show a relation with entropy.

Let $V_{ij} = \sqrt{p_{ij}}|\, i\,\rangle\langle\, j\,|$ where $P = (p_{ij})$ is a irreducible (in the classical sense for a Markov chain) k by k column stochastic matrix, $\mu = \sum_{ij} \delta_{V_{ij}}$ and $L = I$. In this case, we get that

$$V_{ij}^* V_{ij} = p_{ij}|\, j\,\rangle\langle\, j\,|,$$

is a diagonal matrix and, if $A = (a_{ij})$,

$$V_{ij}^* A V_{ij} = p_{ij} a_{ii}|\, j\,\rangle\langle\, j\,|.$$

Therefore, when $\omega = (V_{i_n j_n})$, we have

$$W_2(\omega)^* W_2(\omega) = p_{i_1 j_1} p_{i_2 j_2}|\, j_2\,\rangle\langle\, j_2\,|\delta_{i_2 j_1},$$

where $\delta_{ij} = 0$ if $i \neq j$ and 1 if $i = j$. By induction,

$$W_n(\omega)^* W_n(\omega) = \left(\prod_{k=1}^{n} p_{i_k j_k} \right) \left(\prod_{k=1}^{n-1} \delta_{i_{k+1} j_k} \right) |\, j_n\,\rangle\langle\, j_n\,|.$$

Thus, $W_n(\omega)^* W_n(\omega)$ is 0 or a diagonal matrix with a unique entry different from 0. This entry is exactly $(\prod_{k=1}^{n} p_{i_k j_k})$ which implies that

$$\|W_n(\omega)^* W_n(\omega)\| = \left(\prod_{k=1}^{n} p_{i_k j_k} \right) = p_{11}^{X_{11,n}(\omega)} p_{12}^{X_{12,n}(\omega)} p_{21}^{X_{21,n}(\omega)} p_{22}^{X_{22,n}(\omega)},$$

where

$$X_{ij,n}(\omega) = \sum_{k=0}^{n-1} 1_{[V_{ij}]} \circ \theta^k(\omega),$$

with $1_{[V_{ij}]}$ being the characteristic function of cylinder $[V_{ij}]$.

Note that under the ergodicity hypothesis we would have the property: for any i, j and $\mathbb{P}$-almost sure ω, we have that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_{[V_{ij}]} \circ \theta^k(\omega) = \mathbb{P}([V_{ij}]).$$

It follows from the arguments in example 8.5 in [12] that the pair (L, μ) satisfies the purification condition, the ϕ-Erg conditions and also irreduciblil-ity.

Remember that for given matrix $A \in M_k(\mathbb{C})$, $a_1(A) \geq a_2(A) \geq \ldots \geq a_k(A)$ are the singular values of A, i.e., the square roots of eigenvalues of A^*A, labeled in decreasing order. From Lemma III.5.3 in [11] we have

$$\left\| \bigwedge^p W_n(\omega) \right\| = a_1(W_n)\ldots a_p(W_n).$$

Therefore, $\| \bigwedge^1 W_n(\omega) \| = a_1(W_n) = \|W_n^* W_n\|^{\frac{1}{2}}$.

Following Proposition 7.2 (which corresponds to Proposition 4.3 in [10]) we can obtain the greater Lyapunov exponent γ_1 taking the limit

$$\begin{aligned} \gamma_1 : &= \lim_{n\to\infty} \frac{1}{n} \left\| \bigwedge^1 W_n(\omega) \right\| \\ &= \lim_{n\to\infty} \frac{1}{n} \log(\|W_n(\omega)^* W_n(\omega)\|^{\frac{1}{2}}) \\ &= \frac{1}{2} \sum_{ij} \lim_{n\to\infty} \frac{X_{ij,n}(\omega)}{n} \log(p_{ij}) \\ &= \sum_{ij} \frac{\mathbb{P}([V_{ij}])}{2} \log(p_{ij}). \end{aligned}$$

We know from [12] that if Φ is the channel defined for such μ and L, then, Φ is Φ-Erg. Moreover, the unique ρ, such that, $\Phi(\rho) = \rho$, is exactly the diagonal matrix ρ with entries $\pi_1, \pi_2, \ldots, \pi_k$, where $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ is the invariant probability vector for the stochastic matrix P. We also know in this case that the entropy (see example 8.5 in [12]) for a channel defined in [12] is equal to the classical Shannon-Kolmogorov entropy of the stationary Markov Process associated to the column stochastic matrix $P = (p_{ij})$ (see formula in [24]).

Now, we can estimate

$$\begin{aligned}\mathbb{P}([V_{ij}]) &= \int_{[V_{ij}]} \operatorname{tr}(v\rho v^*)\, d\mu(v) = \operatorname{tr}(V_{ij}^* V_{ij}\rho) = \operatorname{tr}(p_{ij}|\, j\,\rangle\langle\, j\,|\rho) = p_{ij}\langle\, j\,|\rho|\, j\,\rangle \\ &= p_{ij}\pi_j.\end{aligned}$$

Therefore,

$$\gamma_1 = \frac{1}{2}\sum_{i,j\in\{0,1\}} \pi_j p_{ij}\log(p_{ij}) = -\frac{1}{2}h,$$

where h is the entropy of the Markov invariant measure associated to the matrix P.

The value $\frac{1}{2}$ which multiplies the entropy on the above expression is due to the fact that we considered the norm $\|A\| = \langle\, A\,,\, A\,\rangle^{1/2}$.

Now we estimate the second Lyapunov exponent γ_2.

We showed that $W_n(\omega)^* W_n(\omega) = \left(\prod_{k=1}^{n} p_{i_k j_k}\right)\left(\prod_{k=1}^{n-1}\delta_{i_{k+1}j_k}\right)|\, j_n\,\rangle\langle\, j_n\,|$, which implies that the second eigenvalue is 0 and therefore $a_2(W_n(\omega)) = 0$.

Now, we can get γ_2, indeed,

$$\gamma_1 + \gamma_2 = \lim_n \frac{1}{n}\log(a_1(W_n(\omega))a_2(W_n(\omega))) = \lim_n \frac{1}{n}\log(0),$$

which implies that $\gamma_2 = -\infty$.

9 The purification condition is generic

The measure μ is fixed from now on.

Our main goal in this section is to show:

Proposition 9.1 *Given μ over M_k with $\#\mathrm{supp}\,\mu > 1$, the set of L satisfying the purification condition is generic in $\mathcal{B}(M_k)$.*

This will follow from Lemma 9.14.

Definition 9.2 *We say that the projection π n-purifies $L : M_k \to M_k$, where rank $\pi \geq 2$, if there exists $E \in \mathcal{O}_n$, with $\mu^{\otimes n}(E) > 0$, such that,*

$$\pi W_n(\omega) W_n(\omega)\pi \not\propto \pi,$$

for all $\omega \in E$.

In order show that a certain L satisfies the purification condition we have to consider all possible projections π (see definition 6.4).

Observe that if Q is a unitary matrix, π has rank great or equal to 2 and n-purifies L for $E \in \mathcal{O}_n$, then

$$Q\pi Q^* Q W_n(\omega)^* W_n(\omega) Q^* Q \pi Q^* \not\propto Q\pi Q^*.$$

Besides that, $W_n(\omega) = L(\omega_n)...L(\omega_1)$, so, as $Q^*Q = \mathrm{Id}_k$, we have

$$QW_n(\omega)^* W_n(\omega) Q^* =$$
$$QL(\omega_1)^* Q^* Q ... Q^* Q L(\omega_n)^* Q^* Q L(\omega_n) Q^* Q ... Q^* Q L(\omega_1) Q^* Q.$$

From this follows:

Proposition 9.3 *If $L_Q(v) := QL(v)Q^*$, then for a projection π, such that, rank $\pi \geq 2$ and an unitary matrix Q, it's true that*

$$\pi \text{ } n\text{-purifies } L \iff Q\pi Q^* \text{ } n\text{-purifies } L_Q.$$

Definition 9.4 *For an orthogonal projection π and $n \in \mathbb{N}$ we define*

$$Pur_\pi^n = \{L \in \mathcal{B}(M_k) \,|\, \pi \text{ } n\text{-purifies } L\}.$$

Note that if

$$Pur = \{L \in \mathcal{B}(M_k) \,|\, \Phi_L \text{ satisfies (Pur) condition } \},$$

and we denote

$$P_2 = \{\pi \text{ orthogonal projection } \,|\, \operatorname{rank} \pi \geq 2\},$$

it follows that

$$Pur = \bigcap_{\pi \in P_2} \bigcup_{n \in \mathbb{N}} Pur_\pi^n.$$

Proposition 9.5 *For any $\pi \in P_2$ and $n \in \mathbb{N}$, Pur_π^n is open.*

proof: Take $\pi \in P_2$ with $\operatorname{rank} \pi = l$, Q an unitary matrix that diagonalizes π. Suppose that

$$\tilde{\pi} := Q\pi Q^* = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & 0 & \dots & 0 \end{bmatrix}.$$

So, if $L \in Pur_{\pi}^{n}$, we know that $\tilde{\pi}$ n-purifies L_Q.

If

$$(s_{ij}^{L}(\omega)) := QW_n(\omega)^* W_n(\omega) Q^*$$

we know that s_{ij}^{L} are continuous functions. Moreover, there exists $\omega_0 \in \operatorname{supp} \mu^{\otimes n}$ such that at least one of following conditions occurs:

1. There exists $i \neq j$, such that, $s_{ij}^{L}(\omega_0) \neq 0$, or there exists $j = i >$ rank $\tilde{\pi}$, such that, $s_{ii}^{L}(\omega_0) \neq 0$.
In this case, we define the matrix $s^{L}(\omega) := s_{ij}^{L}(\omega)$;

2. There exists $i \neq j$, such that, $s_{ii}^{L}(\omega_0) - s_{jj}^{L}(\omega_0) \neq 0$.
In this case, we define $s^{L}(\omega) := s_{ii}^{L}(\omega) - s_{jj}^{L}(\omega)$.

Of course that for $F \in \mathcal{B}(M_k)$ with $\|L - F\| \leq \varepsilon$, we have, for ε small enough, that $s^{F}(\omega_0) \neq 0$, because $s^{F}(\omega_0)$ has a continuous dependence on F. Furthermore, s^{F} is continuous, then exists a open set B with $\omega_0 \in B$, such that, $s^{F}(\omega) \neq 0$, for $\omega \in B$. Moreover, $\omega_0 \in \operatorname{supp} \mu^{\otimes n}$ which implies that $\mu^{\otimes n}(B) > 0$, and therefore $F \in Pur_{\pi}^{n}$.

$\square$

We point out that $s^{L}(w_0) \neq 0$ is the good condition for purification.

Proposition 9.6 *For any $\pi \in P_2$, Pur_{π}^{1} is dense.*

proof: Take $L \notin Pur_{\pi}^{1}$ and Q unitary matrix that diagonalizes π as above. If $L \notin Pur_{\pi}^{1}$, defining (s_{ij}^{L}) as in the previous proposition, we know that $s_{11}(v) = s_{22}(v)$, for almost every v. If $D = |\,1\,\rangle\langle\,1\,|$, and $\varepsilon > 0$, we consider $L_Q^{\varepsilon}(v) = L_Q(v) + \varepsilon D$. So, we have

$$\begin{aligned} L_Q^{\varepsilon}(v)^* L_Q^{\varepsilon}(v) &= (L_Q(v) + \varepsilon D)^* (L_Q(v) + \varepsilon D) \\ &= L_Q(v)^* L_Q(v) + \varepsilon L_Q^{*}(v) D + \varepsilon D L_Q(v) + \varepsilon^2 D. \end{aligned}$$

This perturbation change $s^L_{11}(v)$ but not $s^L_{22}(v)$, which implies that $Q\pi Q^*$ n-purifies L^ε_Q. Now, we need just to take $L^\varepsilon(v) = L(v) + \varepsilon Q^* D Q$ and, as $\|L - L^\varepsilon\|$ is small, it follows the density property at once.

□

Definition 9.7 *For any $\pi \in P_2$, we define $Pur_\pi = \bigcup_{n\in\mathbb{N}} Pur^n_\pi$.*

Note that by the two propositions above given a fixed π the set Pur_π is an open and dense on $\mathcal{B}(M_k)$. The purification condition requires to consider all possible projections π (see definition 6.4).

Lemma 9.8 *If $\pi_1, \pi_2 \in P_2$ has the same rank, $E_i := Im\ \pi_i$, and $\{x_i\}$ is an orthonormal basis for E_i, then, $\{\pi_2 x_i\}$ is a basis for E_2 if π_1 and π_2 are close enough.*

proof: The proof will be by contradiction. Suppose $\operatorname{rank} \pi_1 = l$ and $y_i := \pi_2 x_i$ are linearly dependent. So, the dimension generated by $\{y_i\}$ is at most $l-1$. Then, there exists a vector $\hat{y} \in E_2$ which has norm 1 and it is orthogonal to the subspace generated by $\{y_i\}$. Therefore, $\langle\, \hat{y}\,,\, y_i\,\rangle = 0$, for all i. This implies that $\langle\, \hat{y}\,,\, \pi_2 x_i\,\rangle = \langle\, \pi_2\hat{y}\,,\, x_i\,\rangle = \langle\, \hat{y}\,,\, x_i\,\rangle = 0$ and moreover $\pi_1\hat{y} = 0$. Finally, we get $\|\pi_1 - \pi_2\| \geq \|\pi_1\hat{y} - \pi_2\hat{y}\| = \|\hat{y}\| = 1$.

If we assume that $\|\pi_1 - \pi_2\| < 1$ we are done.

□

Observe that, for $i \neq j$ and $\|\pi_2 - \pi_1\| < \varepsilon$,

$$
\begin{aligned}
|\langle\, y_i\,,\, y_j\,\rangle| = |\langle\, y_i\,,\, x_j\,\rangle| = |\langle\, y_i - x_i\,,\, x_j\,\rangle| &\leq \|y_i - x_i\|\,\|x_j\| = \|y_i - x_i\| \\
&\leq \|\pi_2 - \pi_1\| < \varepsilon
\end{aligned}
\tag{2}
$$

The set of y_i is not an orthonormal basis.

We would like to get an orthonormal basis close to the orthogonal basis $x_1, .., x_n$. Our aim is to prove corollary 9.12 which claims that, given ε there exists an orthonormal basis (u_i) for E_2 with $\|u_i - x_i\| < C\varepsilon$, for some constant $C > 0$. In this direction we will perform a Gram-Schmidt normalization procedure.

Denote $u_1 := \dfrac{y_1}{\|y_1\|}$, $N_i := y_i - \sum_{j=1}^{i-1} \langle\, y_i\,,\, u_j\,\rangle u_j$ and $u_i := \dfrac{N_i}{\|N_i\|}$, for $i > 1$. Then, we have

$$
\begin{aligned}
\|u_1 - x_1\| &= \left\| \, \|\pi_2 x_1\|^{-1} \pi_2 x_1 - \|\pi_2 x_1\|^{-1} x_1 + \|\pi_2 x_1\|^{-1} x_1 - x_1 \right\| \\
&\leq \|\pi_2 x_1\|^{-1} \|\pi_2 x_1 - \pi_1 x_1\| + \|x_1\| \left| \, \|\pi_2 x_1\|^{-1} - 1 \, \right| \\
&\leq \|\pi_2 x_1\|^{-1} \varepsilon + \left| \, \|\pi_2 x_1\|^{-1} - 1 \, \right| \\
&< \varepsilon(1-\varepsilon)^{-1} + (1-\varepsilon)^{-1}\varepsilon \\
&< 4\varepsilon,
\end{aligned}
$$

and

$$
|\|\pi_2 x_i\| - \|x_i\|| \leq \|\pi_2 x_i - \pi_1 x_i\| < \varepsilon \implies 1 - \varepsilon < \|\pi_2 x_i\| < 1 + \varepsilon.
$$

Furthermore,

$$
\begin{aligned}
\|y_2 - \langle \, y_2 \, , \, u_1 \, \rangle u_1\| = &\left\| y_2 - \|\pi_2 x_i\|^{-2} \langle \, y_2 \, , \, y_1 \, \rangle y_1 \right\| \\
&\leq \|x_2\| + (1-\varepsilon)^{-2}\varepsilon \, \|x_1\| \\
&< 1 + 4\varepsilon.
\end{aligned}
$$

Proposition 9.9 *For any $i \in \{1, ..., n\}$, $j < i$, there is $C_{ij} > 0$, such that, $|\langle \, y_i \, , \, u_j \, \rangle| < C_{ij}\varepsilon$.*

proof:

Take $N := \min \|N_i\| > 0$.

Observe that $|\langle \, y_2 \, , \, u_1 \, \rangle| \leq N^{-1}\varepsilon$ (this follows from a similar procedure as in (2) and the Cauchy-Schwartz inequality).

Then suppose, for all $l < i$, $|\langle \, y_l \, , \, u_j \, \rangle| \leq C_{lj}\varepsilon$, for all $j < l$, with $C_{lj} > 0$. If $j < i$, we have

$$
\begin{aligned}
|\langle \, y_i \, , \, u_j \, \rangle| &\leq N^{-1} |\langle \, y_i \, , \, y_j \, \rangle| + N^{-1} \sum_{k=1}^{j-1} |\langle \, y_j \, , \, u_k \, \rangle| \\
&\leq N^{-1}\varepsilon + N^{-1} \sum_{k=1}^{j-1} C_{jk}\varepsilon \\
&= \left(1 + \sum_{k=1}^{j-1} C_{jk} \right) N^{-1}\varepsilon.
\end{aligned}
$$

Taking $C_{ij} = \left(1 + \sum_{k=1}^{j-1} C_{jk} \right) N^{-1}$ the claim follows by induction.

$\square$

Proposition 9.10 *For all i, $|\|N_i\| - 1| < K\varepsilon$, for some $K > 0$.*

proof:

$$
\begin{aligned}
\|N_i\| &\leq 1 + \|y_i - x_i\| + \sum_{j=1}^{i-1} |\langle\, y_i \,,\, u_j \,\rangle| \\
&\leq 1 + \varepsilon + \left(\sum_{j=1}^{i-1} C_{ij} \sum \right)\varepsilon \\
&= 1 + \left(1 + \left(\sum_{j=1}^{i-1} C_{ij} \right)\right)\varepsilon.
\end{aligned}
$$

Taking $K = \left| 1 + \sum_{j=1}^{i-1} C_{ij} \right|$ the proof is done.

□

Proposition 9.11 *For all i, we have $\|u_i - x_i\| < C_i\varepsilon$.*

proof:

$$
\begin{aligned}
\|u_i - x_i\| &\leq N^{-1}\, \|N_i - x_i\| + \|\|N_i\|\, x_i - x_i\| \\
&\leq N^{-1}\, \|y_i - x_i\| + N^{-1} \sum_{j=0}^{i-1} |\langle\, y_i \,,\, u_j \,\rangle| + |\|N_i\| - 1| \\
&\leq N^{-1}\varepsilon + N^{-1} \sum_{j=1}^{i-1} C_{ij} + K\varepsilon \\
&= \left(N^{-1} + N^{-1} \sum_{j=1}^{i-1} C_{ij} + K \right)\varepsilon.
\end{aligned}
$$

Define $C_i := N^{-1} + N^{-1} \sum_{j=1}^{i-1} C_{ij} + K$ and the statement has been proved.

□

Corollary 9.12 *There exists $C > 0$, such that, for all $\varepsilon > 0$, there exists an orthonormal basis (u_i) for E_2 with $\|u_i - x_i\| < C\varepsilon$.*

If we repeat the process, but now for $E_1^{\perp}$ and $E_2^{\perp}$, we get another constant C_2, and in a similar way we obtain new vectors (u_i) from the (x_i). These u_i define an orthonormal basis for $\mathbb{C}^k$ with $\|u_i - x_i\| < C_2\varepsilon$.

Now we define Q_1, Q_2 , such that $Q_1 x_i = e_i$ and $Q_2 u_i = e_i$, where (e_i) is canonical basis for $\mathbb{C}^k$. Then, $Q_i \pi_i Q_i^*$ is a diagonalization for π_i. Observe that

$$\|Q_2 Q_1^* - I\| = \|Q_2 Q_1^* - Q_2\, Q_2^*\| < C_3 \varepsilon,$$

the map $A \to Q_2 Q_1^* A (Q_2 Q_1^*)^*$ is continuous and moreover this map is close to the identity map.

We know if $L \in Pur^n_{\pi_1}$, then, we take s_1^L from the π_1, Q_1 and L, as in proposition 9.5. In the same way there exists $\omega_0 \in \operatorname{supp} \mu$ with $s_1^L(\omega_0) \neq 0$. Observe that $s_1^L(\omega_0)$ depends only on the cordinates of $Q_1 W_n^L(\omega_0)^* W_n^L(\omega_0) Q_1^*$. Now, applying $Q_2 Q_1^* (\cdot) (Q_2 Q_1^*)^*$ we get $Q_2 W_n^L(\omega_0)^* W_n^L(\omega_0) Q_2^*$. Note that this is the same as to consider π_2, Q_2, L and the associated $s_2^L(\omega_0)$. If ε is small enough, $s_2^L(\omega_0) \neq 0$ and we can repeat the argument used in proposition 9.5 in order to obtain an open set B, such that, $\omega_0 \in B$ and, moreover, if $\omega \in B$ then $s_2^L(\omega) \neq 0$. Therefore, $L \in Pur^n_{\pi_2}$.

The previous arguments prove the following lemma.

Lemma 9.13 *If $L \in Pur_{\pi_1}$ and π_1, π_2 are close enough, then $L \in Pur_{\pi_2}$.*

Lemma 9.14 *Take K_2* **a countable dense subset of** *P_2. Then,*

$$Pur = \bigcap_{\pi \in K_2} Pur_\pi.$$

proof: We will use the classical Baire Theorem. Suppose that $L \in \bigcap_{\pi \in K_2} Pur_\pi$, then for lemma 9.13, for each $\pi \in K_2$, as $L \in Pur_\pi$, there exists $\varepsilon(\pi)$, such that, if $\hat{\pi} \in P_2$ and $\|\pi - \hat{\pi}\| < \varepsilon(\pi)$, then $L \in Pur_{\hat{\pi}}$. Furthermore, if we define $B(\pi) = \{\hat{\pi} \in P_2 | \, \|\hat{\pi} - \pi\| < \varepsilon(\pi)\}$, then $\bigcup_{\pi \in K_2} B(\pi)$ covers P_2. Therefore, for any $\hat{\pi} \in P_2$, there is $\pi \in K_2$, such that, $\hat{\pi} \in B(\pi)$ and thus $L \in Pur_{\hat{\pi}}$. This implies that $L \in \bigcap_{\pi \in P_2} Pur_\pi = Pur$.

$\square$

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