E.A. Carlen, A. Vershynina: Recovery map stability for the Data Processing Inequality, revised version

Referee report

The authors have removed the part of the paper containing already known results and added some more explanations. The content of the paper is now satisfactory and worth publication. After dealing with some minor points listed below, I will be happy to recommend the paper for publication.

(a) I agree with the authors that the part 3 on conditional expectations has its merits and is valuable not only for giving a clear and accessible proof in finite dimensions, but also for the more detailed study of the conditions in this case. My remark is that the results could be stated (and proved) in a clearer and a bit less redundant form. Specifically, Theorem 3.3 is but a Corollary of Thm 3.1 and the proof repeats a lot of the arguments present in the proof of 3.1. In fact, I think it would be much clearer and elegant to state Thm. 3.1. as a list of equivalent conditions, starting with "$\mathcal{P}_\rho$ is real" and including the fact that $\mathcal{P}_\rho$ is a conditional expectation.

Also the present proof could be shortened, for example, to prove the converse part of (1), it is enough to realize that $\mathcal{K}$ is both self-adjoint and invariant under $\Delta_\rho^{1/2}$ and use the fact that $\Delta_\rho^{1/2}$ is self-adjoint with respect to the HS inner product, so that $\Delta_\rho^{-1/2}((H\rho^{1/2} - A\rho^{1/2})^*) = H\rho^{1/2} - A^*\rho^{1/2}$ must be orthogonal to $\mathcal{K}$, and then argue by uniqueness.

At the very least, the last two displayed equations on p. 16 are exactly the same, so something should be done here.

Similarly, there is some mess on p. 17, the lines below "Finally, we show that (3.1) is valid...). Again some redundancy here.

(b) There is quite a number of typos. A list of some of them is below.

p.1 - will exists
- realtively simple
- "...and let $\mathcal{N}$ the functions..." perhaps missing "denote"?
- "...if and only ω..." missing "if"

p.5 - Eq.(1.1): ρ_n should be $\rho_{\mathcal{N}}$ and some mixed up parentheses
- exactly the same in the displayed eq. above (1.13)
- "Because of (??)..." missing equation label

p.6 - "...condition expectation of Y given the sigma-algebra $\mathcal{M}$." conditional exp. and $\mathcal{M}$ is not a σ-algebra
- last displayed eq: γ is missing

p.7 - Eq. (1.19) an inequality should appear somewhere

- will yields

p.8 - first paragraph ends with a comma ","

- line below Eq. (1.27), sentence beginning with "Then by DPI...": something is quite wrong here

p.9 - As notes above

- Corollary 1.6: 1.6. is a Lemma

p.10 last sentence: perhaps equalities should be minus signs here? Also the statement that the two norms are "comparable in size" seems rather confusing to me, but perhaps I misunderstand something

p.13 - $U(\mathcal{N}\rho)$ should be $U(\rho_{\mathcal{N}})$

- displayed equation just below this: perhaps there should be $-w_t$ (but this noes not influence the arguments below)

p.17 - line 2: "$Z\ in\mathcal{N}$"

- $\delta_\rho(B)$

- "$\rho(\mathcal{A}_\rho(X))X$ for all $X \in \mathcal{M}$" ??