On the convex characterisation of the set of unital quantum channels

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In this paper, we consider the convex set of d dimensional unital quantum channels. In particular, we parametrise a family of maps and through this parametrisation we provide a partial characterisation of the set of unital quantum maps with respect to this family of channels. For the case of qutrit channels, we consider the extreme points of the set and their classification with respect to the Kraus rank. In this setting, we see that the parametrised family of maps corresponds to maps with Kraus rank three. Furthermore, we introduce a novel family of qutrit unital quantum channels with Kraus rank four to consider the extreme points of the set over all possible Kraus ranks. We construct explicit examples of these two families of channels and we consider the question of whether these channels correspond to extreme points of the set of quantum unital channels. Finally, we demonstrate how well-known channels relate to the examples presented.

1 Introduction

Quantum channels provide the most general characterisation of an arbitrary evolution of a quantum system and represent a vital ingredient in establishing quantum computing and communication. For instance, in quantum key distribution protocols, the amount of overall noise in the quantum channel determines the rate at which secret bits are distributed between authorised parties. Quantum channels serve as a departure point for areas of intensive study, including quantum tomographic protocols, quantum simulation, and quantum channel capacity. In the particular case concerning channel capacity, whereas classical channels centre on the transmission of classical information, quantum channels offer a variety of transmission possibilities including classical information, entanglement-assisted classical information, private classical information and quantum information.

Quantum channels, also known as quantum maps, represent the set of physically allowed transformations that can be applied to quantum states. In particular, a quantum map from the Hilbert space $\mathcal{H}$ of dimension $d_{\mathcal{H}}$ to the Hilbert space $\mathcal{K}$ of dimension $d_{\mathcal{K}}$ corresponds to a completely positive trace-preserving linear transformation from the set of $d_{\mathcal{H}} \times d_{\mathcal{H}}$ complex matrices to the set of $d_{\mathcal{K}} \times d_{\mathcal{K}}$ complex matrices. In this work, we investigate the set of unital quantum channels which are those that map the maximally mixed state to itself. Unital quantum channels are given by the set of unital completely positive trace-preserving (UCPT) linear maps. The set of UCPT maps is convex and corresponds to the intersection of the set of completely positive trace-preserving (CPT) maps with its dual, the set of unital completely positive (UCP) maps. For the case $d_{\mathcal{H}} = d_{\mathcal{K}} = 2$, the convex structure of the set of UCPT maps is known. Krummerer proved that the extreme points of the set of qubit UCPT maps correspond to unitary conjugations, that is, maps with Kraus rank $k = 1$ [1]. For $d \geq 3$, the convex structure of the set of qudit UCPT maps is more complex and many questions about it remain unresolved [2].

Some works improved the understanding of the convex structure of the set of UCPT maps by introducing examples of extreme points with Kraus rank different from one. Tregub [3] presented an example of a UCPT map with dimension $d = 4$ and rank $k = 4$. Landau and Streater introduced

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an extremal channel which can be constructed for arbitrary dimension [4]. In the case of the Landau and Streater channel, the rank of the map corresponds to its dimension $k = d$. Werner and and Holevo introduced a non-unitary UCPT map which corresponds to a counterexample to the additivity conjecture of the purity of quantum channels [5]. For $d = 3$, this map corresponds to a rank three map, $k = 3$, which is extremal within the set of UCPT maps. Ohno in [6] presented two examples of UCPT maps with dimension $d = 3$ and Kraus rank $k = 4$ these channels have the particularity of being extreme for the set of UCPT maps in spite of not being extreme for the sets of CPT maps or the set of UCP maps. More recently, Haagerup et al. presented a family of maps that generalize the Werner-Holevo channel to arbitrary dimension and also presented a $2-$parameter family of UCPT maps with $d = 3$ and $k = 4$ [7].

In this work, we present a family of UCPT maps of dimension d and rank $k = d$ in terms of $2d^2 - 3d + 1$ real parameters. We also present the conditions which determine whether a given map corresponds to an extreme point of the set of UCPT maps. This family of UCTP maps can be seen as a partial parametrisation of the set of UCPT maps with fixed rank as a complete description of the set would require $2d^3 - 3d^2$ real parameters. We refer to [8, 9] for a detailed dimensional analysis of the set of CPT maps over dimension d and rank k.

The work is organized as follows. In section 2, we outline the necessary tools required for the study of UCPT maps. In section 3, we discuss the family of maps where its Kraus operators are given by convex sums of the elements of the Heisenberg-Weyl basis. We note that Kraus sets formed with elements of the Heisenberg-Weyl were studied as a tool to study efficient quantum circuit decomposition [10]. We derive the set of conditions that the parameters should satisfy in order to represent unital and trace-preserving maps. We also present the set of conditions which determines if a given map of the family is a extreme point of the set of UCTP maps. In section 4, we offer explicit examples of UCPT maps over dimension $d = 3$. We provide one example of a map of the family that corresponds to a unitary conjugation of the Werner-Holevo channel and we provide a different example of a map of the family that corresponds to a mixed unitary channel. We also introduce a novel class of UCTP maps of dimension $d = 3$ and rank $k = 4$ and we provide further examples of these class of maps. Finally, we determine whether the examples of maps presented correspond to extreme points of the set of UCTP maps.

2 Preliminaries

2.1 Notation

Let $n \in \mathbb{N}$ and denote by $\mathcal{H}_n$ the n-dimensional Hilbert space and by $\mathcal{M}_n$ the set of $n \times n$ complex matrices. The set of density operators ρ consisting of all hermitian, positive operators with unit trace acting on $\mathcal{H}_n$ is denoted by $\mathcal{D}_n \subset \mathcal{M}_n$. We write $\mathbb{1}_n$ to denote the identity matrix on $\mathcal{H}_n$. A linear mapping $\mathcal{E} : \mathcal{D}_n \mapsto \mathcal{D}_m$ is said to be positive if it sends positive semi-definite matrices to positive semi-definite matrices, and completely positive (CP) if $\mathcal{E} \otimes \mathbb{1}_n$ is positive for all n. A CP mapping $\mathcal{E}$ is trace-preserving if and only if $\operatorname{Tr} \mathcal{E}(\rho) = \operatorname{Tr} \rho$ for all $\rho \in \mathcal{D}_n$. We denote the set of all CPT maps by $\Xi^T_{n,m}$. A CP mapping $\mathcal{E}$ is unital if and only if it leaves the maximally mixed state invariant $\mathcal{E}(\mathbb{1}_n) = \mathbb{1}_m$. We denote the set of all UCP maps by $\Xi^U_{n,m}$. The set of all CP maps $\mathcal{E} : \mathcal{D}_n \mapsto \mathcal{D}_m$ which are unital and trace-preserving is given by the intersection $\Xi^{UT}_{n,m} := \Xi^T_{n,m} \bigcap \Xi^U_{n,m}$.

2.2 Operator-sum representation

The mapping $\mathcal{E} : \mathcal{D}_n \mapsto \mathcal{D}_m$ is completely positive if and only if it admits a representation of the form

$$\mathcal{E}(\rho) = \sum_{i=1}^{r} K_i \rho K_i^\dagger \tag{1}$$

for all $\rho \in \mathcal{D}_n$, where the matrices $K_i \in \mathbb{C}^{m \times n}$ are referred to as the Kraus operators [11]. This form of expressing a map is known as the operator-sum representation of a map. A CP map

$\mathcal{E} : \mathcal{D}_n \mapsto \mathcal{D}_m$ given by (1) is trace-preserving if

$$\sum_{i=1}^{r} K_i^\dagger K_i = \mathbb{1}_n \tag{2}$$

and $\mathcal{E}$ is unital if

$$\sum_{i=1}^{r} K_i K_i^\dagger = \mathbb{1}_m. \tag{3}$$

The operator-sum representation of a map is not unique. The following theorem establishes when two sets of operators represent the same map.

Theorem 1 (Unitary freedom in the operator-sum representation [12]). *Suppose $\{K_i\}_{i=0,\ldots,n}$ and $\{G_j\}_{j=0,\ldots,n}$ are the sets of Kraus operators defining the CP maps $\mathcal{E}$ and $\mathcal{F}$, respectively. Then $\mathcal{E} = \mathcal{F}$ if and only if there exist complex numbers u_{ij} such that $K_i = \sum_j u_{ij} G_j$ and $U = (u_{ij})_{i,j\in\mathbb{Z}_n}$ is an n by n unitary matrix.*

Two sets of Kraus operators with different cardinality represent the same map if by appending zero operators to the set with fewer elements, the unitary freedom condition is satisfied. The minimum number of Kraus operators such that the operator-sum representation of a map exists, is called the Kraus rank of the map r. Choi showed that the operator-sum representation of a map is minimal if and only if the set of Kraus operators $\{K_i\}_{i\in\mathbb{Z}_r}$ is linearly independent [11].

2.3 Properties of set of UCPT maps

The set $\Xi^T_{n,m}$ and its dual the set $\Xi^U_{n,m}$ are convex. This means that for $0 < p < 1$ and $\mathcal{E}_1, \mathcal{E}_2 \in \Xi^T_{n,m}(\Xi^U_{n,m})$, we have it that

$$\mathcal{E}_{12}(p) = \mathcal{E}_1 + (1-p)\mathcal{E}_2 \in \Xi^T_{n,m}(\Xi^U_{n,m}). \tag{4}$$

The elements of a set which do not admit a convex decomposition in terms of other elements within the set are called extreme points. The concise characterisation of extreme points of the set $\Xi^U_{n,m}$ was provided by Choi [11] with the following theorem.

Theorem 2. *Consider the set of UCP maps $\mathcal{E} : \mathcal{D}_n \mapsto \mathcal{D}_m$ with minimal operator-sum representation $\mathcal{E}(\rho) = \sum_{i=1}^r K_i \rho K_i^\dagger$. Then, $\mathcal{E}$ is an extreme point of $\Xi^U_{n,m}$ if and only if the set $\{K_i K_j^\dagger\}_{i,j\in\mathbb{Z}_r}$ is linearly independent.*

Choi's theorem has a natural extension provided that the set of CPT maps is the dual of the set of UCP maps. The following theorem establishes when a CPT map is an extreme point of the set $\Xi^T_{n,m}$.

Theorem 3. *Consider the set of CPT maps $\mathcal{E} : \mathcal{D}_n \mapsto \mathcal{D}_m$ with minimal operator-sum representation $\mathcal{E}(\rho) = \sum_{i=1}^r K_i \rho K_i^\dagger$. Then, $\mathcal{E}$ is an extreme point of $\Xi^T_{n,m}$ if and only if the set $\{K_i^\dagger K_j\}_{i,j\in\mathbb{Z}_r}$ is linearly independent.*

Theorem 2 and Theorem 3 establish bounds to the Kraus rank of the extreme points of the set of unital maps and the set of trace-preserving maps, respectively. The Kraus rank of an extreme point of $\Xi^U_{n,m}$ is upper bounded by m. This follows from the fact that at most m^2 matrices $K_i K_j^\dagger$ can be linearly independent as its size is $m \times m$. For the CPT case, we have it that the Kraus rank of an extreme point of $\Xi^T_{n,m}$ is upper bounded by n. This follows from the fact that at most n^2 matrices $K_i^\dagger K_j$ can be linearly independent since its size is $n \times n$. In this work, we focus on the set of unital and trace preserving maps in which $n = m = d$. The set $\Xi^{UT}_{d,d}$ is also convex and the following theorem originally published in [4] characterizes its extreme points.

Theorem 4. *Consider the set of UCPT maps $\mathcal{E} : \mathcal{D}_d \to \mathcal{D}_d$ where $\mathcal{E}(\rho) := \sum_{i=1}^r K_i \rho K_i^\dagger$ and $\sum_{i=1}^r K_i K_i^\dagger = \sum_{i=1}^r K_i^\dagger K_i = \mathbb{1}_d$. Then, $\mathcal{E}$ is an extreme point of $\Xi^{UT}_{n,m}$ if and only if the set of $2d \times 2d$ matrices*

$$\{K_i^\dagger K_j \oplus K_i K_j^\dagger\}_{i,j\in\mathbb{Z}_r} \tag{5}$$

is linearly independent.

From this theorem it follows that the Kraus rank of an extreme point of $\Xi_{d,d}^{UT}$ is upper bounded by $\lfloor\sqrt{2d^2}\rfloor$ since at most $2d^2$ matrices $K_i^\dagger K_j \oplus K_i K_j^\dagger$ can be linearly independent provided that the maximum number of non-zero values of these matrices is $2d^2$. For example, if we fix the dimension $d = 3$ we get that the rank of the extreme points of unital and trace-preserving maps is upper bounded by $\lfloor\sqrt{18}\rfloor = 4$. Extreme points of the set $\Xi_{3,3}^{UT}$ with Kraus rank one correspond to the case of unitary maps which can be expressed as $\mathcal{E}(\rho) = U\rho U$ where U is a unitary matrix. Landau and Streater established that there are no extreme points of $\Xi_{3,3}^{UT}$ of rank two as all of them can be decomposed as convex sums of rank one points [4]. However, extreme points of $\Xi_{3,3}^{UT}$ with rank three and four exist and several examples of such maps exist throughout the literature.

3 A canonical family of UCPT maps.

In this section we look at a particular family of CP maps in which its Kraus operators are given by linear combinations of elements of the Heisenberg-Weyl basis. We introduce a canonical parametrisation of the elements of this family and we discuss in terms of the parameters which elements of the family correspond to UCTP maps. We also discuss which of those UCTP maps correspond to extreme points of the set.

Let us consider the family of CP maps over dimension d, $\mathcal{E} : \mathcal{D}_d \mapsto \mathcal{D}_d$, whose operator-sum representation is given by

$$\mathcal{E}(\rho) = \sum_{i=0}^{d-1} K_i \rho K_i^\dagger \tag{6}$$

where

$$\begin{aligned} K_i &= \sum_{j}^{d-1} \alpha_{ij} X_i Z_j \\ &= \sum_{j,k=0}^{d-1} \alpha_{ij}\omega^{kj} \left|k+i\right\rangle \left\langle k\right|, \end{aligned} \tag{7}$$

$(\alpha_{ij})_{i,j\in\mathbb{Z}_d} \in \mathbb{C}^{d\times d}$ is the matrix of coefficients that parametrises the family and $\omega = e^{\frac{2\pi}{d}i}$ is a dth primitive root of unity. Note that the elements of the set $\{X_i Z_j = \sum_{k=0}^{d-1}\omega^{kj}\left|k+i\right\rangle\left\langle k\right|\}_{i,j\in\mathbb{Z}_d}$ are also elements of the Heisenberg-Weyl basis over dimension d. We shall recall that the Heisenberg-Weyl basis is a set of orthonormal matrices that generalize the set of Pauli matrices to arbitrary dimension. To establish a canonical family of maps, we require a one-to-one correspondence between a map and its representation. Theorem 1 gives us the relation between two sets of Kraus operators $\{K_i\}_{i=0,\ldots,d-1}$ and $\{G_i\}_{i=0,\ldots,d-1}$ representing the same map. Now suppose that $\{K_i\}_{i=0,\ldots,d-1}$ and $\{G_i\}_{i=0,\ldots,d-1}$ are given respectively by the Kraus operators in (7). Because of the orthogonality of the sets, $\mathrm{Tr}\,(K_i^\dagger G_j) \propto \delta_{i,j}$ for $i,j = 0,\ldots,d-1$, we have it that the unitary matrix in Theorem 1 is given by

$$u_{ij} = \begin{pmatrix} e^{i\phi_1} & & \\ & \ddots & \\ & & e^{i\phi_d} \end{pmatrix} \tag{8}$$

and this means that the only possible freedom corresponds to multiplying each one of the d Kraus operators by an arbitrary phase. In terms of group theory, this freedom is determined by the action of the group generated by taking the direct product of d copies of the unitary group $U(1)$, this is $\bigotimes_{i\in\mathbb{Z}_d} U(1)$. The different maps given by the Kraus operators in (7) can be divided in equivalence classes determined by the action of the group $\bigotimes_{i\in\mathbb{Z}_d} U(1)$. We can represent each one of these classes of maps by fixing the phase of one of the columns of the matrix of coefficients $(\alpha_{ij})_{i,j\in\mathbb{Z}_d}$. For example, we may fix the phase of the first column of the matrix $(\alpha_{ij})_{i,j\in\mathbb{Z}_d}$ to zero. We now discuss the properties of this family of maps in terms of $(\alpha_{ij})_{i,j\in\mathbb{Z}_d}$. The following theorem establishes which maps are unital and trace preserving.

Theorem 5. *The map $\mathcal{E}:\mathcal{D}_d \mapsto \mathcal{D}_d$ given by the operator sum representation $\mathcal{E}(\rho)=\sum_{i=0}^{d-1} K_i\rho K_i^\dagger$ where $K_i=\sum_{j,k=0}^{d-1}\alpha_{ij}\omega^{jk}\,|k+i\rangle\langle k|$ is trace-preserving if*

$$\sum_{i,j=0}^{d-1}\alpha_{ij}\alpha_{ij}^* = 1 \tag{9}$$

and

$$\sum_{i,j=0}^{d-1}\alpha_{ij+l}\alpha_{ij}^* = 0, \quad l=1,\ldots,d-1. \tag{10}$$

The map $\mathcal{E}$ is unital if in addition to condition (9), we have it that

$$\sum_{i,j=0}^{d-1}\alpha_{ij+l}\alpha_{ij}^*\omega^{-il} = 0 \quad l=1,\ldots,d-1. \tag{11}$$

Proof. The proof follows from the evaluation of the trace-preserving and unital conditions (2) and (3) for the Kraus set given by (7). Let us consider the set $\{K_i^\dagger K_i\}_{i\in\mathbb{Z}_d}$ in the $\{|a\rangle\langle b|\,,\ a,b\in\mathbb{Z}_d\}$ basis as

$$\begin{aligned}
K_i^\dagger K_i &= \left(\sum_{k,j=0}^{d-1}\alpha_{ij}^*\omega^{-kj}\,|k\rangle\langle k+i|\right)\left(\sum_{m,n=0}^{d-1}\alpha_{im}\omega^{mn}\,|n+i\rangle\langle n|\right)\\
&= \sum_{k,j,m,n=0}^{d-1}\alpha_{im}\alpha_{ij}^*\omega^{mn-kj}\,|k\rangle\langle k+i|n+i\rangle\langle n|\\
&= \sum_{k,j,m,n=0}^{d-1}\alpha_{im}\alpha_{ij}^*\omega^{mn-kj}\delta_{k,n}\,|k\rangle\langle n|\\
&= \sum_{k,j,m=0}^{d-1}\alpha_{im}\alpha_{ij}^*\omega^{k(m-j)}\,|k\rangle\langle k|
\end{aligned} \tag{12}$$

where $\delta_{k,n}$ is the Kronecker delta;

$$\delta_{k,n} = \begin{cases} 1, & \text{if} \quad k=n \\ 0, & \text{if} \quad k\neq n. \end{cases} \tag{13}$$

To satisfy the trace-preserving condition $\sum_{i=0}^{d-1}K_i^\dagger K_i = \mathbb{1}_d$, it necessarily follows that

$$\sum_{i,j,m=0}^{d-1}\alpha_{im}\alpha_{ij}^*\omega^{k(m-j)} = 1 \text{ for } k=0,\ldots,d-1. \tag{14}$$

By the change of index, $m-j=l$, (14) can be expressed as

$$\sum_{i,l=0}^{d-1}\sum_{j=0}^{d-1}\alpha_{ij+l}\alpha_{ij}^*\omega^{kl} = 1 \text{ for } k=0,\ldots,d-1 \tag{15}$$

and using the change of variable $\beta_{il}=\sum_{j=0}^{d-1}\alpha_{ij+l}\alpha_{ij}^*$, we get that

$$\sum_{i,l=0}^{d-1}\beta_{il}\omega^{kl} = 1 \text{ for } k=0,\ldots,d-1. \tag{16}$$

The unique solution to this system of d linearly independent equations in terms of the set of variables $\{\beta_{il}\}_{i,l\in\mathbb{Z}_d}$ corresponds to $\sum_{i=0}\beta_{i0}=1$ and $\sum_{i=0}\beta_{il}=0$ for $l=1,\ldots d-1$. By expressing the solution of the system in terms of the original variables $\{\alpha_{ij}\}_{i,k\in\mathbb{Z}_d}$ we get precisely the equations (9) and (10).

Similarly, we can obtain the conditions required by a map to be unital. Let us consider the set $\{K_iK_i^\dagger\}_{i\in\mathbb{Z}_d}$ as

$$\begin{aligned} K_iK_i^\dagger &= \left(\sum_{m,n=0}^{d-1} \alpha_{im}\omega^{mn} \left|n+i\right\rangle \left\langle n\right|\right)\left(\sum_{k,j=0}^{d-1} \alpha_{ij}^*\omega^{-kj} \left|k\right\rangle \left\langle k+i\right|\right) \\ &= \sum_{k,j,m,n=0}^{d-1} \alpha_{im}\alpha_{ij}^*\omega^{mn-kj} \left|k+i\right\rangle \left\langle k|n\right\rangle \left\langle n+i\right| \\ &= \sum_{k,j,m,n=0}^{d-1} \alpha_{im}\alpha_{ij}^*\omega^{mn-kj}\delta_{k,n} \left|k+i\right\rangle \left\langle n+i\right| \\ &= \sum_{k,j,m=0}^{d-1} \alpha_{im}\alpha_{ij}^*\omega^{(k-i)(m-j)} \left|k\right\rangle \left\langle k\right|. \end{aligned} \tag{17}$$

To satisfy the unital condition $\sum_{i=0}^{d-1} K_iK_i^\dagger = \mathbb{1}_d$, it follows that

$$\sum_{i,j,m=0}^{d-1} \alpha_{im}\alpha_{ij}^*\omega^{(k-i)(m-j)} = 1 \text{ for } k = 0,\dots,d-1. \tag{18}$$

By the change of index, $m-j=l$, (18) can be written as

$$\sum_{i,l=0}^{d-1}\sum_{j=0}^{d-1} \alpha_{ij+l}\alpha_{ij}^*\omega^{(k-i)l} = 1 \text{ for } k = 0,\dots,d-1 \tag{19}$$

and using now the change of variable $\beta_{il} = \sum_{j=0}^{d-1} \alpha_{ij+l}\alpha_{ij}^*$ we get that

$$\sum_{i,l=0}^{d-1} \beta_{ik}\omega^{(k-i)l} = 1 \text{ for } k = 0,\dots,d-1. \tag{20}$$

We get again a system of d equations in terms of $\{\beta_{ik}\}_{i,k\in\mathbb{Z}_d}$. The solution of this system is given by $\sum_{i,l=0}\beta_{i0} = 1$ and $\sum_i \beta_{il}\omega^{-l} = 0$ for $l = 1,\dots d-1$. If we express the solution of the system in terms of the elements of the set $\{\alpha_{ij}\}_{i,j\in\mathbb{Z}_d}$ we get precisely the equations (9) and (11) which completes the proof. □

Equations (9), (10) and (11) represent $2(d-1)+1$ real constraints. First, we notice that (9) corresponds to one real constraint. Second, we notice that equations (10) and (11) correspond to $(d-1)$ real constraints each one. This is because for every equation in the set of $(d-1)$ complex equations in (10) and (11) we find that its conjugate is also part of the set. An equation and its conjugate correspond to two real constraints and if we consider all the equations given by (10) and (11) we see that each one corresponds to $(d-1)$ real constraints.

The following theorem establishes whether a UCPT map given by the Kraus set in (7) corresponds to an extreme point of $\Xi_{d,d}^{UT}$.

Theorem 6. *A unital and trace-preserving map given by $\mathcal{E}:\mathcal{D}_d\mapsto\mathcal{D}_d$ with operator sum representation $\mathcal{E}(\rho) = \sum_{i=0}^{d-1} K_i\rho K_i^\dagger$ where $K_i = \sum_{j,k=0}^{d-1}\alpha_{ij}\omega^{jk}\left|k+i\right\rangle\left\langle k\right|$ corresponds to an extreme point of the set unital and trace-preserving maps if the matrices $(M_l|N_l)$ are full-rank for $l = 0,...,d-1$ where*

$$M_l = \left(\left(\sum_j^{d-1} \alpha_{i+lj}\omega^{j(k-l)}\right)\left(\sum_j^{d-1} \alpha_{ib}^*\omega^{-jk}\right)\right)_{i,k\in\mathbb{Z}_d} \tag{21}$$

and

$$N_l = \left(\left(\sum_{j=0}^{d-1} \alpha_{i+lj} \omega^{(k-i)j} \right) \left(\sum_{j=0}^{d-1} \alpha^*_{ij} \omega^{-(k-i)j} \right) \right)_{i,k\in\mathbb{Z}_d} . \tag{22}$$

Proof. The proof consists on showing that the set $\{K^\dagger_{i+l}K_i\}_{i,l\in\mathbb{Z}_d}$ is linearly independent provided that the matrices M_l are full-rank for $l = 0, ..., d-1$ and similarly showing that the set $\{K_{i+l}K^\dagger_i\}_{i,l\in\mathbb{Z}_d}$ is linearly independent if N_l are full-rank for $l = 0, ..., d-1$. By Theorem 4 we have it that a map is an extreme point of $\Xi^{UT}_{d,d}$ if the set $\{K^\dagger_i K_j \oplus K_i K^\dagger_j\}_{i,j\in\mathbb{Z}_r}$ is linear independent. This condition is satisfied if the matrices given by $(M_l|N_l)$ are full-rank for $l = 0, ..., d-1$. Let us consider $K^\dagger_i K_{i+l}$ as

$$\begin{aligned}
K^\dagger_i K_{i+l} &= \left(\sum_{k,j=0}^{d-1} \alpha^*_{ij} \omega^{-kj} |k\rangle \langle k+i| \right) \left(\sum_{j,n=0}^{d-1} \alpha_{i+lj} \omega^{jn} |n+i+l\rangle \langle n| \right) \\
&= \sum_{k,n=0}^{d-1} \left(\sum_{j=0}^{d-1} \alpha_{i+lj} \omega^{jn} \right) \left(\sum_{j=0}^{d-1} \alpha^*_{ij} \omega^{-kj} \right) |k\rangle \langle k+i|n+i+l\rangle \langle n| \\
&= \sum_{k,n=0}^{d-1} \left(\sum_{j=0}^{d-1} \alpha_{i+lj} \omega^{j(n-l)} \right) \left(\sum_{j=0}^{d-1} \alpha^*_{ij} \omega^{-kj} \right) |k\rangle \langle k+i|n+i\rangle \langle n-l| \\
&= \sum_{k} \left(\sum_{j} \alpha_{i+lj} \omega^{j(k-l)} \right) \left(\sum_{j} \alpha^*_{ij} \omega^{-jk} \right) |k\rangle \langle k-l| .
\end{aligned} \tag{23}$$

By the Choi-Jamilkowsky [13] isomorphism we may see operators of dimension d as states of dimension d^2. The elements of the basis in (23) correspond to states $|k\rangle \langle k-l| \sim |k, k-l\rangle$ and

$$K^\dagger_i K_{i+l} \cong \sum_{k=0}^{d-1} \gamma_{ikl} |k, k-l\rangle . \tag{24}$$

If we take the inner product of two arbitrary states

$$\langle (K^\dagger_j K_{j+n})^\dagger | K^\dagger_i K_{i+l} \rangle = \sum_{km}^{d-1} \gamma_{ikl} \gamma^*_{jmn} \langle m, m-n | k, k-l \rangle \tag{25}$$

we see that two elements of the set $\{K^\dagger_i K_{i+l}\}_{i,l\in\mathbb{Z}_d}$ with different l are orthogonal since we get that $\langle m, m-n|k, k-l\rangle = 0$ if $l \neq n$ with $k, l, m, n \in \mathbb{Z}_d$. The linear independence of the whole set follows from the linear independence of the individual sets $\{K^\dagger_i K_{i+l}\}_{i\in\mathbb{Z}_d}$ for $l = 0, ..., d-1$. This is the case if all the matrices M_l for $l = 0, ..., d-1$ are full-rank. Now let us consider $K_{i+l}K^\dagger_i$ as

$$\begin{aligned}
K_{i+l} K^\dagger_i &= \left(\sum_{j,n=0}^{d-1} \alpha_{i+lj} \omega^{nj} |n+i+l\rangle \langle n| \right) \left(\sum_{j,k=0}^{d-1} \alpha^*_{ij} \omega^{-jk} |k\rangle \langle k+i| \right) \\
&= \sum_{n,k=0}^{d-1} \left(\sum_{j=0}^{d-1} \alpha_{i+lj} \omega^{jn} \right) \left(\sum_{j=0}^{d-1} \alpha^*_{ij} \omega^{-jk} \right) |n+i+l\rangle \langle n|k\rangle \langle k+i| \\
&= \sum_{k=0}^{d-1} \left(\sum_{j=0}^{d-1} \alpha_{i+lj} \omega^{jk} \right) \left(\sum_{j=0}^{d-1} \alpha^*_{ij} \omega^{-jk} \right) |k+i+l\rangle \langle k+i| \\
&= \sum_{k=0}^{d-1} \left(\sum_{j=0}^{d-1} \alpha_{i+lj} \omega^{(k-i)j} \right) \left(\sum_{j=0}^{d-1} \alpha^*_{ij} \omega^{-(k-i)j} \right) |k+l\rangle \langle k| .
\end{aligned} \tag{26}$$

By the Choi-Jamilkowsky isomorphism we have it that an operator of dimension d can be seen as a state of dimension d^2. The elements of the basis in (26) can be expressed as the states $|k+l\rangle\langle k| \sim |k+l,k\rangle$ and

$$K_{i+l}K_i^\dagger \cong \sum_{k=0}^{d-1} \gamma_{ikl} |k+l,k\rangle \tag{27}$$

if we take the inner product of two arbitrary states

$$\langle (K_{j+n}K_j^\dagger)^\dagger | K_{i+l}K_i^\dagger\rangle = \sum_{km}^{d-1} \gamma_{ikl}\gamma_{jmn}^* \langle m+n,m|k+l,k\rangle$$

we get that two elements in $\{K_{i+l}K_i^\dagger\}_{i,l\in\mathbb{Z}_d}$ with different l are orthogonal, $\langle m+n,m|k+l,k\rangle = 0$ if $n \neq l$ for $k,l,m,n \in \mathbb{Z}_d$. In order to establish the linear independence of the whole set it only remains to impose the linear independence of the sets $\{K_{i+l}K_i^\dagger\}_{i\in\mathbb{Z}_d}$ separately for $l = 0,\ldots,d-1$ which follows from the linear independence of the matrices N_l for $l = 0,\ldots,d-1$. $\square$

The following theorem characterizes the convex structure of the set of maps defined by the Kraus operators in (7) with respect to the set of trace preserving and unital maps.

Theorem 7. *Consider the set of matrices given by*

$$\mathcal{A}(d) = \left\{ (\alpha_{ij})_{i,j\in\mathbb{Z}_d} \in \mathbb{C}^{d\times d} : \begin{array}{r} \sum_{i,j=0}^{d-1} \alpha_{ij}\alpha_{ij}^* = 1 \\ \sum_{i,j=0}^{d-1} \alpha_{ij+l}\alpha_{ij}^* = 0 \text{ for } l = 1,\ldots,d-1 \\ \sum_{i,j=0}^{d-1} \alpha_{ij+l}\alpha_{ij}^*\omega^{-il} = 0 \text{ for } l = 1,\ldots,d-1 \end{array} \right\} \tag{28}$$

where $\omega = e^{\frac{2\pi}{d}i}$ and further consider the set

$$\mathcal{B}_l(d) = \{(\alpha_{ij})_{i,j\in\mathbb{Z}_d} \in \mathbb{C}^{d\times d} : \det\left((M_l|N_l)(M_l|N_l)^\dagger\right) = 0\} \text{ for } l = 0,\ldots,d-1 \tag{29}$$

where the matrices $\{M_l\}_{l=0,\ldots,d-1}$ and $\{N_l\}_{l=0,\ldots,d-1}$ are given by (21) and (22), respectively. Then we have it that the set

$$\mathcal{X}(d) = \left(\mathcal{A}(d) - \left(\mathcal{A}(d) \cap \left(\bigcup_{l\in\mathbb{Z}_d} \mathcal{B}_l(d)\right)\right)\right) / \bigotimes_{i\in\mathbb{Z}_d} U(1) \tag{30}$$

corresponds to the set of all quantum maps defined by the Kraus operators in (7) which are extreme points of the set of unital and trace-preserving maps.

Proof. The proof follows from considering Theorem 5 and Theorem 6 together. Let π denote the map that sends complex matrices to quantum maps as

$$\pi(\alpha_{ij}) = \mathcal{E} \tag{31}$$

where the action of the quantum map $\mathcal{E}$ on a density operator is given by

$$\mathcal{E}(\rho) = \sum_{i=0}^{d-1} \left(\sum_{j,k=0}^{d-1} \alpha_{ij}\omega^{kj} |k+i\rangle\langle k| \right) \rho \left(\sum_{j,k=0}^{d-1} \alpha_{ij}^*\omega^{-kj} |k\rangle\langle k+i| \right). \tag{32}$$

On the one hand, by Theorem 5, we have it that if we apply the map π to the elements of the set $\mathcal{A}(d)$ we get as a result the set of maps which are trace-preserving and unital. On the other hand, if we apply the map π to the elements of the sets $\mathcal{B}_l(d)$ for $l = 0,\ldots,d-1$ we get maps where the matrices $(M_l|N_l)$ are not full-rank. Therefore, by Theorem 6, the map π applied over the set $\mathcal{A}(d) \cap \left(\bigcup_{l\in\mathbb{Z}_d} \mathcal{B}_l(d)\right)$ corresponds to the maps which are not extreme within the set of unital and trace-preserving maps. Note that the theorem is enunciated in terms of its complementary, the maps that are extreme within the set. Finally, it remains to consider the freedom in the choice of Kraus operators in order to obtain an injection between the set $\mathcal{A}(d)$ and the set of unital and trace-preserving maps. In this case, the the freedom in the choice of Kraus operators is given by the the action of the group $\prod_{i\in\mathbb{Z}_d} U(1)$ and therefore in order to get a one-to-one correspondence we take the quotient space of the set $\mathcal{A}(d)$ with respect to the action of this group. □

Corollary 8. *The set of unital and trace-preserving maps given by the Kraus operators defined by (7) has dimension* $2d^2 - 3d + 1$.

Proof. This lemma follows from the evaluation of the dimension of the set given by (30). We have it that

$$\dim(\mathcal{X}(d)) = \dim(\mathcal{A}(d)) - \dim(\bigotimes_{i\in\mathbb{Z}_d} U(1)) \tag{33}$$

Since $\mathcal{A}(d)$ is a $2d^2$ dimensional set with $2(d-1)+1$ real constraints and $\bigotimes_{i\in\mathbb{Z}_d} U(1)$ is a d dimensional group. We conclude that

$$\dim(\mathcal{X}(d)) = 2d^2 - (2(d-1)+1) - d = 2d^2 - 3d + 1 \tag{34}$$

□

4 Qutrit UCPT maps

In the previous section, we discussed a canonical family of CP maps in terms of a set of parameters and we derived the equations that determine if a map of the family is trace-preserving and unital. We also established the conditions that determine whether these maps correspond to extreme points of the set of UCPT maps. In this section, we use the theorems derived in the previous section to study the convex structure of the set of qutrit UCPT maps. By Theorem 4, we have it that the rank of the extreme points of the set of unital and trace-preserving qutrit maps is bounded by $\lfloor\sqrt{18}\rfloor \approx \lfloor 4.24 \rfloor$. Consequently, extremal points within the set of UCPT qutrit maps have rank ranging from one to four. While all rank one UCTP maps are extremal points, it is the case that all rank two maps admit a decomposition in terms of other UCTP qutrit maps [4]. The question of determining whether rank three and rank four maps correspond to extremal points is key to fully understand the convex structure of the set of UCPT qutrit maps. In section 4.1, we construct examples of UCPT qutrit maps with rank three which correspond to the maps of the family given by the Kraus set in (7) over dimension three. In section 4.2, we introduce a novel family of UCPT qutrit maps with rank four and we construct explicit examples of this family of maps. In section 4.3, we evaluate whether all the constructed examples correspond to extreme points of the set of qutrit UCPT maps.

4.1 Rank three qutrit UCPT maps

Let $\mathcal{E} : \mathcal{D}_3 \mapsto \mathcal{D}_3$ be the map $\mathcal{E}(\rho) = \sum_{i=0}^{2} K_i \rho K_i^\dagger$ where the Kraus operators are given by

$$K_0 = \begin{pmatrix} \alpha_{00} & 0 & 0 \\ 0 & \alpha_{00} & 0 \\ 0 & 0 & \alpha_{00} \end{pmatrix} + \begin{pmatrix} \alpha_{01} & 0 & 0 \\ 0 & \alpha_{01}\omega & 0 \\ 0 & 0 & \alpha_{01}\omega^2 \end{pmatrix} + \begin{pmatrix} \alpha_{02} & 0 & 0 \\ 0 & \alpha_{02}\omega^2 & 0 \\ 0 & 0 & \alpha_{02}\omega \end{pmatrix}, \tag{35}$$

$$K_1 = \begin{pmatrix} 0 & 0 & \alpha_{10} \\ \alpha_{10} & 0 & 0 \\ 0 & \alpha_{10} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \alpha_{11}\omega^2 \\ \alpha_{11} & 0 & 0 \\ 0 & \alpha_{11}\omega & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \alpha_{12}\omega \\ \alpha_{12} & 0 & 0 \\ 0 & \alpha_{12}\omega^2 & 0 \end{pmatrix} \tag{36}$$

and

$$K_2 = \begin{pmatrix} 0 & \alpha_{20} & 0 \\ 0 & 0 & \alpha_{20} \\ \alpha_{20} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \alpha_{21}\omega & 0 \\ 0 & 0 & \alpha_{21}\omega^2 \\ \alpha_{21} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \alpha_{22}\omega^2 & 0 \\ 0 & 0 & \alpha_{22}\omega \\ \alpha_{22} & 0 & 0 \end{pmatrix} \tag{37}$$

where $\omega = e^{\frac{2\pi i}{3}}$ denotes the third root of unity. Theorem 5 maintains that a map given by this Kraus set is unital and trace-preserving if it satisfies the equations

$$\sum_{i,j=0}^{2} \alpha_{ij}\alpha_{ij}^* = 1, \tag{38}$$

$$\sum_{i,j=0}^{2} \alpha_{ij+1}\alpha_{ij}^* = 0 \tag{39}$$

and

$$\sum_{i,j=0}^{2} \alpha_{ij+1}\alpha_{ij}^*\omega^{-i} = 0. \tag{40}$$

We note that, for the case of qutrit maps, equations (39) and (40) can be expressed as the simple condition

$$\begin{aligned} \sum_{j=0}^{d-1} \alpha_{0,j+1}\alpha_{0j}^* &= \omega \sum_{j=0}^{d-1} \alpha_{1,j+1}\alpha_{1,j}^* \\ &= \omega^2 \sum_{j=0}^{d-1} \alpha_{2,j+1}\alpha_{2,j}^*. \end{aligned} \tag{41}$$

We now construct examples of unital and trace-preserving maps based the Kraus operators (35), (36) and (37) where its defining parameters are explicitly given by a matrix of coefficients. In particular, we present three different maps of this class given by of the matrices $(a_{ij})_{i,j\in\mathbb{Z}_d}$, $(b_{ij})_{i,j\in\mathbb{Z}_d}$ and $(c_{ij})_{i,j\in\mathbb{Z}_d}$.

Example 9. *Let $\mathcal{E}_a$ denote the map defined in terms of the Kraus operators given by the matrix of coefficients*

$$(a_{ij}) = \frac{\sqrt{5}}{30} \begin{pmatrix} 3\sqrt{3} & 6\,e^{\frac{\pi}{4}i} & 3\sqrt{3} \\ \sqrt{3} & \sqrt{3}\,e^{\frac{4\pi}{3}i} & 4\sqrt{3}\,e^{\frac{2\pi}{3}i} \\ 3\sqrt{2} & 3\,e^{\frac{5\pi}{12}i} & 3\,e^{\frac{13\pi}{12}i} \end{pmatrix}. \tag{42}$$

A second example is the map $\mathcal{E}_b$ in which the Kraus operators are given now by the matrix

$$(b_{ij}) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{43}$$

Finally, a third example of unital and trace-preserving map is $\mathcal{E}_c$ where the coefficients defining the Kraus set are given by the matrix

$$(c_{ij}) = \frac{\sqrt{2}}{6} \begin{pmatrix} 2 & e^{\frac{5\pi}{3}i} & e^{\frac{\pi}{3}i} \\ 2 & e^{\pi i} & e^{\pi i} \\ 2 & e^{\frac{\pi}{3}i} & e^{\frac{5\pi}{3}i} \end{pmatrix}. \tag{44}$$

The sets of coefficients outlined by the matrices in (42), (43) and (44) satisfy the conditions given by Theorem 5. Consequently, the maps $\mathcal{E}_a$, $\mathcal{E}_b$ and $\mathcal{E}_c$ correspond to trace-preserving and unital maps. We have that $\mathcal{E}_b$ and $\mathcal{E}_c$ can be related to well-known maps. In particular, $\mathcal{E}_b$ corresponds to a convex sum of unitary maps $\mathcal{E}_b(\rho) = \sum_{i=0}^{2} \frac{1}{3} U_i \rho U_i^\dagger$ with

$$U_0 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, U_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } U_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{45}$$

Furthermore, the map $\mathcal{E}_c$ corresponds to a unitary conjugation of the antisymmetric version of the Werner-Holevo map over dimension three [5]. To see this, consider the map $\mathcal{E}_c(\rho) = \sum_{i=0}^{2} K_i \rho K_i^\dagger$ where

$$K_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, K_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } K_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \tag{46}$$

and choose the unitary matrix

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{47}$$

The conjugation of $\mathcal{E}_c$ with U is given by the map

$$\begin{aligned} U\mathcal{E}_c(\rho)U^\dagger &= \sum_{i=0}^{2} UK_i \rho K_i^\dagger U^\dagger \\ &= \sum_{i=0}^{2} UK_i \rho (UK_i)^\dagger \end{aligned} \tag{48}$$

where the Kraus operators are

$$UK_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, UK_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \text{ and } UK_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \tag{49}$$

which corresponds to the anti-symmetric Werner–Holevo channel over dimension three. We note that the Landau-Streater channel over dimension three can be obtained as a unitary conjugation of the Werner–Holevo channel [14] and consequently it can also be recovered as a unitary conjugation of the map $\mathcal{E}_c$.

4.2 Rank four qutrit UCPT maps

Extremal qutrit maps with rank three are not enough to describe the convex set of UCPT maps. This is because Theorem 4 establishes that the rank of qutrit maps which are extremal within the set of UCPT maps is bounded by four. Besides rank one and rank three there are also rank four extremal qutrit maps. We shall introduce a novel family of rank four qutrit maps. Let $\mathcal{F} : \mathcal{D}_3 \mapsto \mathcal{D}_3$ be the map $\mathcal{F}(\rho) = \sum_{i=0}^{3} K_i \rho K_i^\dagger$ where the Kraus operators are given by

$$K_0 = \begin{pmatrix} \alpha_{00} & 0 & 0 \\ 0 & \alpha_{00} & 0 \\ 0 & 0 & \alpha_{00} \end{pmatrix} + \begin{pmatrix} \alpha_{01} & 0 & 0 \\ 0 & \alpha_{01}\omega & 0 \\ 0 & 0 & \alpha_{01}\omega^2 \end{pmatrix} + \begin{pmatrix} \alpha_{02} & 0 & 0 \\ 0 & \alpha_{02}\omega^2 & 0 \\ 0 & 0 & \alpha_{02}\omega \end{pmatrix}, \tag{50}$$

$$K_1 = \begin{pmatrix} 0 & 0 & 0 \\ \alpha_{10} & 0 & 0 \\ 0 & \alpha_{10} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ \alpha_{11} & 0 & 0 \\ 0 & \alpha_{11}\omega & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ \alpha_{12} & 0 & 0 \\ 0 & \alpha_{12}\omega^2 & 0 \end{pmatrix}, \tag{51}$$

$$K_2 = \begin{pmatrix} 0 & 0 & \alpha_{10} \\ 0 & 0 & 0 \\ \alpha_{20} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \alpha_{11}\omega^2 \\ 0 & 0 & 0 \\ \alpha_{21} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \alpha_{12}\omega \\ 0 & 0 & 0 \\ \alpha_{22} & 0 & 0 \end{pmatrix} \tag{52}$$

and

$$K_3 = \begin{pmatrix} 0 & \alpha_{20} & 0 \\ 0 & 0 & \alpha_{20} \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \alpha_{21}\omega & 0 \\ 0 & 0 & \alpha_{21}\omega^2 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \alpha_{22}\omega^2 & 0 \\ 0 & 0 & \alpha_{22}\omega \\ 0 & 0 & 0 \end{pmatrix} \tag{53}$$

where $\omega = e^{\frac{2\pi i}{3}}$ denotes the primitive cube root of unity. We note that the Kraus set given by (50), (51), (52) and (53) corresponds to an augmentation of the Kraus set given by (35), (36) and (37). In particular, the first Kraus set is obtained by inserting rows of zeros in the operators of the second Kraus set with periodicity four. Despite the differences, the families of maps defined by these two Kraus sets share some properties. For example, the conditions given by Theorem 5, which determine whether a map is trace-preserving and unital, are equivalent for the two families. This follows from the fact that the diagonal of the Choi matrices associated to both families of maps are identical. We will now construct examples of unital and trace-preserving rank four qutrit maps given by the Kraus operators in (50), (51), (52) and (53). In particular, we construct three examples of rank four UCTP maps given by the matrices of coefficients $(a_{ij})_{i,j\in\mathbb{Z}_d}$, $(b_{ij})_{i,j\in\mathbb{Z}_d}$ and $(c_{ij})_{i,j\in\mathbb{Z}_d}$ which where also used to construct three rank UCPT maps.

Example 10. *Let $\mathcal{F}_a$ denote the rank four map defined by the Kraus operators given by the matrix of coefficients*

$$(a_{ij}) = \frac{\sqrt{5}}{30}\begin{pmatrix} 3\sqrt{3} & 6\,e^{\frac{\pi}{4}i} & 3\sqrt{3} \\ \sqrt{3} & \sqrt{3}\,e^{\frac{4\pi}{3}i} & 4\sqrt{3}\,e^{\frac{2\pi}{3}i} \\ 3\sqrt{2} & 3\,e^{\frac{5\pi}{12}i} & 3\,e^{\frac{13\pi}{12}i} \end{pmatrix}. \tag{54}$$

A second example is the rank four map $\mathcal{F}_b$ given by the matrix

$$(b_{ij}) = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{55}$$

Finally, as a third example of a rank four map, let $\mathcal{F}_c$ be the map defined by the Kraus set given by the matrix of coefficients

$$(c_{ij}) = \frac{\sqrt{2}}{6}\begin{pmatrix} 2 & e^{\frac{5\pi}{3}i} & e^{\frac{\pi}{3}i} \\ 2 & e^{\pi i} & e^{\pi i} \\ 2 & e^{\frac{\pi}{3}i} & e^{\frac{5\pi}{3}i} \end{pmatrix}. \tag{56}$$

4.3 Parametrising extremal UCTP qutrit maps

We introduced two families of unital and trace-preserving qutrit maps. Now we present the conditions which determine whether these maps correspond to extreme points of the set of UCTP maps. For the case of the family of rank three qutrit maps, these particular conditions are established by Theorem 6. However, for the case of the family of rank four qutrit maps the same result does not hold. To overcome this issue, we construct a set of matrices which determine whether the maps of the family correspond to extreme points of the set of UCTP maps. We also evaluate whether the maps given in Examples 9 and 10 can be classified as extreme points of the UCPT set.

4.3.1 Rank three qutrit maps

Theorem 6 states that a map with the given Kraus form is an extreme point of the set of UCTP maps if the matrices $(M_l|N_l)$ are full-rank for $l = 0, ..., d-1$. In the case of dimension $d = 3$, we write the explicit form of these matrices. Letting $\omega = e^{\frac{2\pi i}{3}}$ we have it that for $l = 0$

$$M_0 = \begin{pmatrix} \left(\sum_j^2 \alpha_{0j}\right)\left(\sum_j^2 \alpha_{0j}^*\right) & \left(\sum_j^2 \alpha_{0j}\omega^j\right)\left(\sum_j^2 \alpha_{0j}^*\omega^{-j}\right) & \left(\sum_j^2 \alpha_{0j}\omega^{2j}\right)\left(\sum_j^2 \alpha_{0j}^*\omega^{-2j}\right) \\ \left(\sum_j^2 \alpha_{2j}\right)\left(\sum_j^2 \alpha_{2j}^*\right) & \left(\sum_j^2 \alpha_{2j}\omega^j\right)\left(\sum_j^2 \alpha_{2j}^*\omega^{-j}\right) & \left(\sum_j^2 \alpha_{2j}\omega^{2j}\right)\left(\sum_j^2 \alpha_{2j}^*\omega^{-2j}\right) \\ \left(\sum_j^2 \alpha_{1j}\right)\left(\sum_j^2 \alpha_{1j}^*\right) & \left(\sum_j^2 \alpha_{1j}\omega^j\right)\left(\sum_j^2 \alpha_{1j}^*\omega^{-j}\right) & \left(\sum_j^2 \alpha_{1j}\omega^{2j}\right)\left(\sum_j^2 \alpha_{1j}^*\omega^{-2j}\right) \end{pmatrix} \tag{57}$$

and

$$N_0 = \begin{pmatrix} \left(\sum_j^2 \alpha_{0j}\right)\left(\sum_j^2 \alpha_{0j}^*\right) & \left(\sum_j^2 \alpha_{0j}\omega^j\right)\left(\sum_j^2 \alpha_{0j}^*\omega^{-j}\right) & \left(\sum_j^2 \alpha_{0j}\omega^{-j}\right)\left(\sum_j^2 \alpha_{0j}^*\omega^{j}\right) \\ \left(\sum_j^2 \alpha_{2j}\omega^j\right)\left(\sum_j^2 \alpha_{2j}^*\omega^{-j}\right) & \left(\sum_j^2 \alpha_{2j}\omega^{-j}\right)\left(\sum_j^2 \alpha_{2j}^*\omega^{j}\right) & \left(\sum_j^2 \alpha_{2j}\right)\left(\sum_j^2 \alpha_{2j}^*\right) \\ \left(\sum_j^2 \alpha_{1j}\omega^{-j}\right)\left(\sum_j^2 \alpha_{1j}^*\omega^{j}\right) & \left(\sum_j^2 \alpha_{1j}\right)\left(\sum_j^2 \alpha_{1j}^*\right) & \left(\sum_j^2 \alpha_{1j}\omega^{j}\right)\left(\sum_j^2 \alpha_{1j}^*\omega^{-j}\right) \end{pmatrix}. \tag{58}$$

Simiraly, for $l=1$, we have it that

$$M_1 = \begin{pmatrix} \left(\sum_j^2 \alpha_{2j}\omega^j\right)\left(\sum_j^2 \alpha_{1j}^*\omega^j\right) & \left(\sum_j^2 \alpha_{2j}\omega^{-j}\right)\left(\sum_j^2 \alpha_{1j}^*\right) & \left(\sum_j^2 \alpha_{2j}\right)\left(\sum_j^2 \alpha_{1j}^*\omega^{-j}\right) \\ \left(\sum_j^2 \alpha_{1j}\omega^j\right)\left(\sum_j^2 \alpha_{0j}^*\omega^j\right) & \left(\sum_j^2 \alpha_{1j}\omega^{-j}\right)\left(\sum_j^2 \alpha_{0j}^*\right) & \left(\sum_j^2 \alpha_{1j}\right)\left(\sum_j^2 \alpha_{0j}^*\omega^{-j}\right) \\ \left(\sum_j^2 \alpha_{0j}\omega^j\right)\left(\sum_j^2 \alpha_{2j}^*\omega^j\right) & \left(\sum_j^2 \alpha_{0j}\omega^{-j}\right)\left(\sum_j^2 \alpha_{2j}^*\right) & \left(\sum_j^2 \alpha_{0j}\right)\left(\sum_j^2 \alpha_{2j}^*\omega^{-j}\right) \end{pmatrix} \tag{59}$$

and

$$N_1 = \begin{pmatrix} \left(\sum_j^2 \alpha_{2j}\omega^{-j}\right)\left(\sum_j^2 \alpha_{1j}^*\omega^j\right) & \left(\sum_j^2 \alpha_{2j}\right)\left(\sum_j^2 \alpha_{1j}^*\right) & \left(\sum_j^2 \alpha_{2j}\omega^j\right)\left(\sum_j^2 \alpha_{1j}^*\omega^{-j}\right) \\ \left(\sum_j^2 \alpha_{1j}\right)\left(\sum_j^2 \alpha_{0j}^*\right) & \left(\sum_j^2 \alpha_{1j}\omega^j\right)\left(\sum_j^2 \alpha_{0j}^*\omega^{-j}\right) & \left(\sum_{2j}^2 \alpha_{1j}\omega^{-j}\right)\left(\sum_j^2 \alpha_{0j}^*\omega^j\right) \\ \left(\sum_j^2 \alpha_{0j}\omega^j\right)\left(\sum_j^2 \alpha_{2j}^*\omega^{-j}\right) & \left(\sum_j^2 \alpha_{0j}\omega^{-j}\right)\left(\sum_j^2 \alpha_{2j}^*\omega^j\right) & \left(\sum_j^2 \alpha_{0j}\right)\left(\sum_j^2 \alpha_{2j}^*\right) \end{pmatrix}. \tag{60}$$

Finally, for $l=2$, we have

$$M_2 = \begin{pmatrix} \left(\sum_j^2 \alpha_{1j}\omega^{-j}\right)\left(\sum_j^2 \alpha_{2j}^*\omega^{-j}\right) & \left(\sum_j^2 \alpha_{1j}\right)\left(\sum_j^2 \alpha_{2j}^*\omega^j\right) & \left(\sum_j^2 \alpha_{1j}\omega^j\right)\left(\sum_j^2 \alpha_{2j}^*\right) \\ \left(\sum_j^2 \alpha_{0j}\omega^{-j}\right)\left(\sum_j^2 \alpha_{1j}^*\omega^{-j}\right) & \left(\sum_j^2 \alpha_{0j}\right)\left(\sum_j^2 \alpha_{1j}^*\omega^j\right) & \left(\sum_j^2 \alpha_{0j}\omega^j\right)\left(\sum_j^2 \alpha_{1j}^*\right) \\ \left(\sum_j^2 \alpha_{2j}\omega^{-j}\right)\left(\sum_j^2 \alpha_{0j}^*\omega^{-j}\right) & \left(\sum_j^2 \alpha_{2j}\right)\left(\sum_j^2 \alpha_{0j}^*\omega^j\right) & \left(\sum_j^2 \alpha_{2j}\omega^j\right)\left(\sum_j^2 \alpha_{0j}^*\right) \end{pmatrix} \tag{61}$$

and

$$N_2 = \begin{pmatrix} \left(\sum_j^2 \alpha_{1j}\omega^{-j}\right)\left(\sum_j^2 \alpha_{2j}^*\omega^j\right) & \left(\sum_j^2 \alpha_{1j}\right)\left(\sum_j^2 \alpha_{2j}^*\right) & \left(\sum_j^2 \alpha_{1j}\omega^j\right)\left(\sum_j^2 \alpha_{2j}^*\omega^{-j}\right) \\ \left(\sum_j^2 \alpha_{0j}\right)\left(\sum_j^2 \alpha_{1j}^*\right) & \left(\sum_j^2 \alpha_{0j}\omega^j\right)\left(\sum_j^2 \alpha_{1j}^*\omega^{-j}\right) & \left(\sum_j^2 \alpha_{0j}\omega^{-j}\right)\left(\sum_j^2 \alpha_{1j}^*\omega^j\right) \\ \left(\sum_j^2 \alpha_{2j}\omega^j\right)\left(\sum_j^2 \alpha_{0j}^*\omega^{-j}\right) & \left(\sum_j^2 \alpha_{2j}\omega^{-j}\right)\left(\sum_j^2 \alpha_{0j}^*\omega^j\right) & \left(\sum_j^2 \alpha_{2j}\right)\left(\sum_j^2 \alpha_{0j}^*\right) \end{pmatrix}. \tag{62}$$

The matrix M_1 corresponds to the conjugate of M_2 and the matrix N_1 corresponds to the conjugate of N_2. Therefore, it suffices to establish the rank of $(M_0|N_0)$ and $(M_1|N_1)$ in order to determine whether the given map corresponds to an extreme point of the set of UCPT maps. Noting that the rank of a matrix is given by the number of non-zero singular values, we shall consider the specific case of the maps introduced in Example 9. For the map $\mathcal{E}_a$, defined by the matrix of coefficients given in (42), we have it that

$$(M_0|N_0) = \frac{1}{20}\left(\begin{array}{ccc|ccc} 16 & 7 & 7 & 16 & 7 & 7 \\ 3 & 12 & 3 & 3 & 3 & 12 \\ 1 & 1 & 10 & 1 & 10 & 1 \end{array}\right) \tag{63}$$

and

$$(M_1|N_1) = \frac{1}{20}\left(\begin{array}{ccc|ccc} \sqrt{3}\,e^{\frac{-13\pi}{12}i} & \sqrt{30}\,e^{\frac{-2\pi}{3}-\phi_2 i} & 2\sqrt{3}\,e^{\frac{\pi}{4}i} & \sqrt{30}\,e^{\frac{4\pi}{3}-\phi_2 i} & \sqrt{3}\,e^{\frac{5\pi}{12}i} & 2\sqrt{3}\,e^{\frac{-\pi}{4}i} \\ 2\sqrt{21}\,e^{\frac{-\pi}{3}+\phi_1 i} & 4\sqrt{21}\,e^{\frac{7\pi}{6}i} & \sqrt{21}\,e^{\pi+\phi_1 i} & \sqrt{21}\,e^{-\pi-\phi_1 i} & 4\sqrt{3}\,e^{\frac{-\pi}{2}i} & 2\sqrt{21}\,e^{\frac{-\pi}{3}-\phi_1 i} \\ \sqrt{70}\,e^{\frac{-\pi}{3}+\phi_2-\phi_1 i} & \sqrt{7}\,e^{\frac{\pi}{12}-\phi_1 i} & 4\,e^{\frac{-\pi}{12}i} & \sqrt{70}\,e^{\frac{-\pi}{3}+\phi_1-\phi_2 i} & 4\,e^{\frac{\pi}{12}i} & \sqrt{7}\,e^{\frac{7\pi}{12}-\phi_1 i} \end{array}\right) \tag{64}$$

where $\phi_1 = \arctan\left(\frac{2\sqrt{3}}{3}\right)$ and $\phi_2 = \arctan\left(\frac{1}{2}\right)$. Let $\sigma(A)$ denote the set of singular values of the matrix A. The singular values of the matrix $(M_0|N_0)$ are given by $\sigma\left(M_0|N_0\right) = \{1.529, 0.6364, 0.5885\}$ and the singular values of $(M_1|N_1)$ are given by $\sigma\left(M_1|N_1\right) = \{0.9318, 0.6332, 0.4308\}$. This yields that $\mathcal{E}_a$ is an extreme point of the set of UCTP maps. In the same way, we can study the map $\mathcal{E}_b$ which is defined in terms of the set of parameters given in (43). In this case we have it that

$$(M_0|N_0) = \frac{1}{3}\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right) \tag{65}$$

and

$$(M_1|N_1)=\frac{1}{3}\left(\begin{array}{ccc|ccc}1&1&1&1&1&1\\1&1&1&1&1&1\\1&1&1&1&1&1\end{array}\right). \tag{66}$$

The singular values of the matrix $(M_0|N_0)$ are given by $\sigma\left(M_0|N_0\right)=\{\sqrt{2},0,0\}$ and the singular values of the matrix $(M_1|N_1)$ are given by $\sigma\left(M_1|N_1\right)=\{\sqrt{2},0,0\}$. Since these matrices are not full-rank, we conclude that $\mathcal{E}_b$ is not an extreme point of the set of UCTP maps. Finally, we can also study the case of the map $\mathcal{E}_c$ which is defined in terms of the set of parameters given by the matrix in (44). For $\mathcal{E}_c$ we have it that

$$(M_0|N_0)=\frac{1}{2}\left(\begin{array}{ccc|ccc}0&1&1&1&0&1\\1&1&0&1&1&0\\1&0&1&0&1&1\end{array}\right) \tag{67}$$

and

$$(M_1|N_1)=\frac{1}{2}\left(\begin{array}{ccc|ccc}0&-1&0&-1&0&0\\-1&0&0&0&-1&0\\0&0&-1&0&0&-1\end{array}\right). \tag{68}$$

The singular values of the matrix $(M_0|N_0)$ are given by $\sigma\left(M_0|N_0\right)=\{\sqrt{2},\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\}$ and the singular values of the matrix $(M_1|N_1)$ are given by $\sigma\left(M_1|N_1\right)=\{\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\}$. This yields that the map $\mathcal{E}_c$ is also an extreme point of the set of unital and trace-preserving maps.

4.3.2 Rank four qutrit maps

For the case of rank four qutrit maps, Theorem 6 cannot be applied. However, a similar result can be derived for this particular family of maps. We have that a map given by the Kraus operators (50), (51), (52) and (53) is an extreme point of the set of UCTP maps if the matrices $(M_l'|N_l')$ are full-rank for $l=0,...,3$. The specific form of these matrices is given as follows. Letting $\omega=e^{\frac{2\pi i}{3}}$ we have it that for $l=0$

$$M_0'=\begin{pmatrix}\left(\sum_j^2\alpha_{0j}\right)\left(\sum_j^2\alpha_{0j}^*\right) & \left(\sum_j^2\alpha_{0j}\omega^j\right)\left(\sum_j^2\alpha_{0j}^*\omega^{-j}\right) & \left(\sum_j^2\alpha_{0j}\omega^{2j}\right)\left(\sum_j^2\alpha_{0j}^*\omega^{-2j}\right)\\ \left(\sum_j^2\alpha_{1j}\right)\left(\sum_j^2\alpha_{1j}^*\right) & \left(\sum_j^2\alpha_{1j}\omega^j\right)\left(\sum_j^2\alpha_{1j}^*\omega^{-j}\right) & 0\\ \left(\sum_j^2\alpha_{2j}\right)\left(\sum_j^2\alpha_{2j}^*\right) & 0 & \left(\sum_j^2\alpha_{1j}\omega^{2j}\right)\left(\sum_j^2\alpha_{1j}^*\omega^{-2j}\right)\\ 0 & \left(\sum_j^2\alpha_{2j}\omega^j\right)\left(\sum_j^2\alpha_{2j}^*\omega^{-j}\right) & \left(\sum_j^2\alpha_{2j}\omega^{2j}\right)\left(\sum_j^2\alpha_{2j}^*\omega^{-2}\right)\end{pmatrix} \tag{69}$$

and

$$N_0'=\begin{pmatrix}\left(\sum_j^2\alpha_{0j}\right)\left(\sum_j^2\alpha_{0j}^*\right) & \left(\sum_j^2\alpha_{0j}\omega^j\right)\left(\sum_j^2\alpha_{0j}^*\omega^{-j}\right) & \left(\sum_j^2\alpha_{0j}\omega^{-j}\right)\left(\sum_j^2\alpha_{0j}^*\omega^{j}\right)\\ 0 & \left(\sum_j^2\alpha_{2j}\omega^{-j}\right)\left(\sum_j^2\alpha_{2j}^*\omega^{j}\right) & \left(\sum_j^2\alpha_{2j}\right)\left(\sum_j^2\alpha_{2j}^*\right)\\ \left(\sum_j^2\alpha_{2j}\omega^j\right)\left(\sum_j^2\alpha_{2j}^*\omega^{-j}\right) & 0 & \left(\sum_j^2\alpha_{1j}\omega^{j}\right)\left(\sum_j^2\alpha_{1j}^*\omega^{-j}\right)\\ \left(\sum_j^2\alpha_{1j}\omega^{-j}\right)\left(\sum_j^2\alpha_{1j}^*\omega^{j}\right) & \left(\sum_j^2\alpha_{1j}\right)\left(\sum_j^2\alpha_{1j}^*\right) & 0\end{pmatrix}. \tag{70}$$

Similary, for the case in which $l=1$, we have that

$$M_1'=\begin{pmatrix}\left(\sum_j^2\alpha_{1j}\right)\left(\sum_j^2\alpha_{0j}^*\omega^{-j}\right) & \left(\sum_j^2\alpha_{1j}\omega^j\right)\left(\sum_j^2\alpha_{0j}^*\omega^{-2j}\right)\\ \left(\sum_j^2\alpha_{2j}\right)\left(\sum_j^2\alpha_{1j}^*\omega^{-j}\right) & 0\\ 0 & \left(\sum_j^2\alpha_{2j}\omega^j\right)\left(\sum_j^2\alpha_{1j}^*\omega^{-2j}\right)\\ \left(\sum_j^2\alpha_{0j}\right)\left(\sum_j^2\alpha_{2j}^*\omega^{-j}\right) & \left(\sum_j^2\alpha_{0j}\omega^j\right)\left(\sum_j^2\alpha_{2j}^*\omega^{-2j}\right)\end{pmatrix} \tag{71}$$

and

$$N_1' = \begin{pmatrix} \left(\sum_j^2 \alpha_{1j}\right)\left(\sum_j^2 \alpha_{0j}^*\right) & \left(\sum_j^2 \alpha_{1j}\omega^j\right)\left(\sum_j^2 \alpha_{0j}^*\omega^{-j}\right) \\ 0 & \left(\sum_j^2 \alpha_{2j}\right)\left(\sum_j^2 \alpha_{1j}^*\right) \\ \left(\sum_j^2 \alpha_{2j}\omega^{2j}\right)\left(\sum_j^2 \alpha_{1j}^*\omega^{-2j}\right) & 0 \\ \left(\sum_j^2 \alpha_{0j}\omega^j\right)\left(\sum_j^2 \alpha_j^*\omega^{-j}\right) & \left(\sum_j^2 \alpha_{0j}\omega^{2j}\right)\left(\sum_j^2 \alpha_{2j}^*\omega^{-2j}\right) \end{pmatrix}. \tag{72}$$

For $l = 2$, we have it that

$$M_2' = \begin{pmatrix} \left(\sum_j^2 \alpha_{1j}\omega^{2j}\right)\left(\sum_j^2 \alpha_{0j}^*\right) & \left(\sum_j^2 \alpha_{2j}\right)\left(\sum_j^2 \alpha_{0j}^*\omega^{-2j}\right) \\ \left(\sum_j^2 \alpha_{2j}\omega^{2j}\right)\left(\sum_j^2 \alpha_{1j}^*\right) & 0 \\ \left(\sum_j^2 \alpha_{0j}\omega^{2j}\right)\left(\sum_j^2 \alpha_{2j}^*\right) & \left(\sum_j^2 \alpha_{0j}\right)\left(\sum_j^2 \alpha_{1j}^*\omega^{-2j}\right) \\ 0 & \left(\sum_j^2 \alpha_{1j}\right)\left(\sum_j^2 \alpha_{2j}^*\omega^{-2j}\right) \end{pmatrix} \tag{73}$$

and

$$N_2' = \begin{pmatrix} \left(\sum_j^2 \alpha_{1j}\omega^{2j}\right)\left(\sum_j^2 \alpha_{0j}^*\omega^{-2j}\right) & \left(\sum_j^2 \alpha_{2j}\right)\left(\sum_j^2 \alpha_{0j}^*\right) \\ \left(\sum_j^2 \alpha_{2j}\omega^j\right)\left(\sum_j^2 \alpha_{1j}^*\omega^{-j}\right) & 0 \\ \left(\sum_j^2 \alpha_{0j}\right)\left(\sum_j^2 \alpha_{2j}^*\right) & \left(\sum_j^2 \alpha_{0j}\omega^{2j}\right)\left(\sum_j^2 \alpha_{1j}^*\omega^{-2j}\right) \\ 0 & \left(\sum_j^2 \alpha_{1j}\right)\omega^j\left(\sum_j^2 \alpha_{2j}^*\omega^{-j}\right) \end{pmatrix}. \tag{74}$$

Finally, for $l = 3$, we have it that

$$M_3' = \begin{pmatrix} \left(\sum_j^2 \alpha_{0j}\omega^j\right)\left(\sum_j^2 \alpha_{1j}^*\right) & \left(\sum_j^2 \alpha_{0j}\omega^{2j}\right)\left(\sum_j^2 \alpha_{1j}^*\omega^{-j}\right) \\ \left(\sum_j^2 \alpha_{1j}\omega^j\right)\left(\sum_j^2 \alpha_{2j}^*\right) & 0 \\ 0 & \left(\sum_j^2 \alpha_{1j}\omega^{2j}\right)\left(\sum_j^2 \alpha_{2j}^*\omega^{-j}\right) \\ \left(\sum_j^2 \alpha_{2j}\omega^j\right)\left(\sum_j^2 \alpha_{0j}^*\right) & \left(\sum_j^2 \alpha_{2j}\omega^{2j}\right)\left(\sum_j^2 \alpha_{0j}^*\omega^{-2j}\right) \end{pmatrix} \tag{75}$$

and

$$N_3' = \begin{pmatrix} \left(\sum_j^2 \alpha_{0j}\right)\left(\sum_j^2 \alpha_{1j}^*\right) & \left(\sum_j^2 \alpha_{0j}\omega^j\right)\left(\sum_j^2 \alpha_{1j}^*\omega^{-j}\right) \\ 0 & \left(\sum_j^2 \alpha_{1j}\right)\left(\sum_j^2 \alpha_{2j}^*\right) \\ \left(\sum_j^2 \alpha_{1j}\omega^{2j}\right)\left(\sum_j^2 \alpha_{2j}^*\omega^{-2j}\right) & 0 \\ \left(\sum_j^2 \alpha_j\omega^j\right)\left(\sum_j^2 \alpha_{0j}^*\omega^{-j}\right) & \left(\sum_j^2 \alpha_{2j}\omega^{2j}\right)\left(\sum_j^2 \alpha_{0j}^*\omega^{-2j}\right) \end{pmatrix}. \tag{76}$$

The specific form of the matrices $\{(M_l'|N_l')\}_{l\in\mathbb{Z}_4}$ is obtained by using the following process. Firstly, we see that the elements of the sets $\{K_{i+l}^\dagger K_i\}_{i,l\in\mathbb{Z}_4}$ and $\{K_{i+k}^\dagger K_i\}_{i,k\in\mathbb{Z}_4}$ are orthogonal for $k \neq l$. A similar case applies to the sets $\{K_{i+l}K_i^\dagger\}_{i,l\in\mathbb{Z}_4}$ and $\{K_{i+k}K_i^\dagger\}_{i,k\in\mathbb{Z}_4}$. Secondly, for each one of the four sets of four matrices $\{K_{i+l}^\dagger K_i\}_{i\in\mathbb{Z}_4}$ with $l = 0, 1, 2$ and 3, we arrange its elements as vectors, we take all zero terms in each vector and we arrange the four vectors of the same set as a matrix (where i is the row index). We get as a result the matrices M_0', M_1', M_2' and M_3'. Similarly, the repeat this process for each one of the four sets of four matrices $\{K_{i+l}K_i^\dagger\}_{i\in\mathbb{Z}_3}$ with $l = 0, 1, 2$ and 3. In this case, we get as a result the matrices N_0', N_1', N_2' and N_3'. Finally, by Theorem 4 we conclude that a map given by the Kraus operators (50), (51), (52) and (53) is extremal within the set UCTP maps if the matrices $(M_0'|N_0')$, $(M_1'|N_1')$, $(M_2'|N_2')$ and $(M_3'|N_3')$ are full-rank. However, if we look at the specific form of these matrices, we note that the matrix M_1' corresponds to the conjugate of M_3' and the matrix N_1' corresponds to the conjugate of N_3'. Therefore, it suffices to evaluate the rank of the matrices $(M_0'|N_0')$, $(M_1'|N_1')$ and $(M_2'|N_2')$ in order to establish whether a given map corresponds to an extreme point of the set of UCTP maps.

Now we consider the specific case of the maps introduced in Example 10. For the map $\mathcal{F}_a$, defined by the set of parameters given in (54), we have it that

$$(M_0'|N_0') = \frac{1}{20}\left(\begin{array}{ccc|ccc} 16 & 7 & 7 & 16 & 7 & 7 \\ 3 & 12 & 0 & 0 & 3 & 12 \\ 1 & 0 & 3 & 3 & 0 & 1 \\ 0 & 1 & 10 & 1 & 10 & 0 \end{array}\right), \tag{77}$$

$$(M_1'|N_1') = \frac{1}{20}\left(\begin{array}{cc|cc} 4\sqrt{21}\,e^{\frac{7\pi}{6}i} & \sqrt{21}\,e^{\pi+\phi_1 i} & \sqrt{21}\,e^{-\pi-\phi_1 i} & 4\sqrt{3}\,e^{\frac{-\pi}{2}i} \\ \sqrt{7}\,e^{\frac{\pi}{12}-\phi_1 i} & 0 & 0 & 4\,e^{\frac{\pi}{12}i} \\ 0 & 4\,e^{\frac{-\pi}{12}i} & \sqrt{70}\,e^{\frac{-\pi}{3}+\phi_1-\phi_2 i} & 0 \\ \sqrt{30}\,e^{\frac{-2\pi}{3}-\phi_2 i} & 2\sqrt{3}\,e^{\frac{\pi}{4}i} & \sqrt{30}\,e^{\frac{4\pi}{3}-\phi_2 i} & \sqrt{3}\,e^{\frac{5\pi}{12}i} \end{array}\right) \tag{78}$$

and

$$(M_2'|N_2') = \frac{1}{20}\left(\begin{array}{cc|cc} 2\sqrt{21}\,e^{\frac{-\pi}{3}+\phi_1 i} & \sqrt{70}\,e^{\frac{\pi}{3}-\phi_2+\phi_1} & 2\sqrt{21}\,e^{\frac{-\pi}{3}-\phi_1 i} & \sqrt{7}\,e^{\frac{-7\pi}{12}+\phi_1 i} \\ \sqrt{3}\,e^{\frac{-13\pi}{12}i} & 0 & 2\sqrt{3}\,e^{\frac{-\pi}{4}i} & 0 \\ \sqrt{70}\,e^{\frac{-\pi}{3}+\phi_2-\phi_1} & 2\sqrt{21}\,e^{\frac{\pi}{3}-\phi_1 i} & \sqrt{7}\,e^{\frac{7\pi}{12}-\phi_1 i} & 2\sqrt{21}\,e^{\frac{\pi}{3}+\phi_1 i} \\ 0 & \sqrt{3}\,e^{\frac{13\pi}{12}i} & 0 & 2\sqrt{3}\,e^{\frac{\pi}{4}i} \end{array}\right) \tag{79}$$

where $\phi_1 = \arctan\left(\frac{2\sqrt{3}}{3}\right)$ and $\phi_2 = \arctan\left(\frac{1}{2}\right)$. The singular values of the matrix $(M_0'|N_0')$ are given by $\sigma\left(M_0'|N_0'\right) = \{1.475, 0.7347, 0.5996, 0.1229\}$, the singular values of $(M_1'|N_1')$ are given by $\sigma\left(M_1'|N_1'\right) = \{0.9015, 0.5009, 0.2245, 0.0768\}$ and the singular values of $(M_2'|N_2')$ are given by $\sigma\left(M_2'|N_2'\right) = \{0.6511, 0.3698, 0.3075, 0.1214\}$. This yields that $\mathcal{F}_a$ corresponds to an extreme point of the set UCTP maps. We can study the map $\mathcal{F}_b$ which is defined in terms of the set of parameters given in (55). In this case we have it that

$$(M_0'|N_0') = \frac{1}{3}\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{array}\right), \tag{80}$$

$$(M_1'|N_1') = \frac{1}{3}\left(\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{array}\right) \tag{81}$$

and

$$(M_2'|N_2') = \frac{1}{3}\left(\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right). \tag{82}$$

The singular values of $(M_0'|N_0')$ are given by $\sigma\left(M_0'|N_0'\right) = \{1.247, 0.4714, 0, 4714, 0\}$, the singular values of the matrix $(M_1'|N_1')$ are given by $\sigma\left(M_1'|N_1'\right) = \{1.054, 0.4714, 0, 0\}$ and the singular values of the matrix $(M_2|N_2)$ are given by $\sigma\left(M_2'|N_2'\right) = \{1.054, 0.4714, 0, 0\}$. Since these matrices are not full-rank, we conclude that $\mathcal{F}_b$ is not an extreme point of the set of UCTP maps. Finally, we shall study also the map $\mathcal{F}_c$ which is determined by the set of parameters (56). In this case we have it that

$$(M_0'|N_0') = \frac{1}{2}\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right), \tag{83}$$

$$(M_1'|N_1') = \frac{1}{2}\left(\begin{array}{cc|cc} 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right) \tag{84}$$

and

$$(M_2'|N_2') = \frac{1}{2}\left(\begin{array}{cc|cc} -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right). \tag{85}$$

The singular values of $(M_0'|N_0')$ are given by $\sigma\left(M_0'|N_0'\right) = \{1.3066, 0.7071, 0.7071, 0.5412\}$, the singular values of the matrix $(M_1'|N_1')$ are given by $\sigma\left(M_1'|N_1'\right) = \{0.5, 0.5, 0.5, 0.5\}$ and the singular values of the matrix $(M_2'|N_2')$ are given by $\sigma\left(M_2'|N_2'\right) = \{0.7071, 0.7071, 0, 0\}$. We conclude that $\mathcal{F}_c$ is not an extreme point of the set of UCTP maps, since the matrix $(M_2'|N_2')$ is not full-rank for this map.

5 Conclusions

In this paper, we considered the characterisation of the convex structure of the set of unital quantum channels. We considered a particular family of unital and trace-preserving maps defined in terms of the operator sum-representation. In particular, we considered maps where the Kraus operators are given by linear combinations of the Heisenberg-Weyl basis. We parametrized our family of maps and through this parametrisation, we were able to consider the convex structure of these maps. We derived firstly the equations needed to establish that the maps of the family are unital and trace-preserving. Secondly, we established the conditions which determine whether a given map corresponds to an extreme point of the set of unital and trace-preserving maps.

As an application, we considered the convex structure of the set of UCPT maps over dimension three. For the qutrit case, the maps of our family correspond to rank three maps. Interestingly, however, rank three maps are not sufficient to describe all the extreme points of the set UCTP maps, which is because rank four maps extremal within the set UCTP maps also exist. In considering this additional class of UCTP qutrit maps we introduced a novel family of rank four qutrit maps. Furthermore, we presented explicit examples of rank three and rank four unital and trace-preserving qutrit maps and we determined whether the examples presented correspond extreme points of the set of UCTP maps. We also illustrated the relations between the examples presented and well-known examples of extremal and non-extremal maps of the set of unital and trace-preserving qutrit maps.

Finally, after considering this partial parametrisation of the set of unital and trace-preserving maps, we believe that a complete parametrisation of UCTP qutrit maps is attainable. For future work, we will consider the features of the bipartite states associated to the families of maps presented. In particular, we will consider the application the Choi-Jamilkowski isomorphism between such bipartite states and our maps.

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