Some notes on sufficient channels 2

October 18, 2023

Let $\mathcal{M}, \mathcal{N}$ be von Neumann algebras and let $\mathcal{S}$ be a set of normal states. We will assume that $\mathcal{S}$ is convex and contains a faithful state $\omega \in \mathcal{S}$.

Let $\Phi : \mathcal{N} \to \mathcal{M}$ be a unital normal completely positive (or 2-positive) map. Such a map, or its predual $\Phi_* : \mathcal{M}_* \to \mathcal{N}_*$, will be called a quantum channel. We will also assume that $\tilde{\omega} := \Phi_*(\omega)$ is faithful as well.

We say that Φ is sufficient with respect to $\mathcal{S}$ if there exists a recovery channel $\Psi : \mathcal{M} \to \mathcal{N}$ such that

$$\Psi_* \circ \Phi_*(\rho) = \rho, \qquad \forall \rho \in \mathcal{S}.$$

If $\mathcal{N} \subseteq \mathcal{M}$ is a subalgebra and Φ is the inclusion $\mathcal{N} \hookrightarrow \mathcal{M}$, then we say that $\mathcal{N}$ is a sufficient subalgebra with respect to $\mathcal{S}$.

1 Minimal sufficient subalgebra

Let us denote a set of quantum channels

$$\mathcal{L} := \{\Theta : \mathcal{M} \to \mathcal{M},\ \Psi_*(\rho) = \rho,\ \forall \rho \in \mathcal{S}\}.$$

Then $\mathcal{L}$ is a semigroup (closed under composition), which is convex and closed in the point - σ-weak topology. By the mean ergodic theorem, there exists an element $E \in \mathcal{L}$ such that

$$E \circ \Theta = \Theta \circ E = E, \qquad \forall \Theta \in \mathcal{L}.$$

Then

1. E is clearly idempotent ($E^2 = E$) and preserves ω. Hence E is a conditional expectation and its range is a subalgebra in $\mathcal{M}$, invariant under the modular group σ_t^ω.

2. A channel $\Theta : \mathcal{M} \to \mathcal{M}$ is in $\mathcal{L}$ if and only if $\Theta \circ E = E$.

3. The range of E is
$$\mathcal{R}(E) = \{x \in \mathcal{M},\ \Theta(x) = x,\ \forall \Theta \in \mathcal{L}\}.$$

We will use the notation $E_{\mathcal{S}} := E$ and $\mathcal{M}_{\mathcal{S}} := \mathcal{R}(E)$.

Proposition 1. *$\mathcal{M}_{\mathcal{S}}$ is a minimal sufficient subalgebra with respect to $\mathcal{S}$.*

Proof. It is easily seen that $E_{\mathcal{S}} : \mathcal{M} \to \mathcal{M}_{\mathcal{S}}$ is a recovery channel. Let $\mathcal{N} \subseteq \mathcal{M}$ be any sufficient subalgebra with respect to $\mathcal{S}$ and let $\Psi : \mathcal{M} \to \mathcal{N} \subseteq \mathcal{M}$ be the recovery channel, then $\Psi \in \mathcal{L}$ and we have for any $a \in \mathcal{M}_{\mathcal{S}}$:

$$a = E(a) = \Psi \circ E(a) = \Psi(a) \in \mathcal{N},$$

so that $\mathcal{M}_{\mathcal{S}} \subseteq \mathcal{N}$.

□

Proposition 2. *1. $\mathcal{M}_{\mathcal{S}}$ is generated by the set of Connes cocycles*

$$\{[D\rho : D\omega]_t,\ \rho \in \mathcal{S},\ t \in \mathbb{R}\}$$

2. *If the commutant Radon-Nikodym derivatives $d(\rho, \omega)$ exist for $\rho \in \mathcal{S}$, then $\mathcal{M}_{\mathcal{S}}$ is generated by $\{\sigma_t^{\omega}(d(\rho, \omega)),\ \rho \in \mathcal{S},\ t \in \mathbb{R}\}$.*

Proof. □

Proposition 3. *If $\mathcal{M} = B(\mathcal{H})$, then there is a decomposition...*
Any channel preserving $\mathcal{S}$ must satisfy....

Example 1.

2 Sufficient channels

A channel $\Phi : \mathcal{N} \to \mathcal{M}$ is sufficient with respect to $\mathcal{S}$ if and only if there is some channel $\Psi : \mathcal{M} \to \mathcal{N}$ such that $\Phi \circ \Psi = E_{\mathcal{S}}$. In this case,

1. $\Psi|_{\mathcal{M}_{\mathcal{S}}}$ is an isomorphism onto the subalgebra $\tilde{\mathcal{M}} = \Psi(\mathcal{M}_{\mathcal{S}})$.

2. $\Phi|_{\tilde{\mathcal{M}}}$ is an isomorphism onto $\mathcal{M}_{\mathcal{S}}$ and $\Psi|_{\mathcal{M}_{\mathcal{S}}}$ is its inverse.

3. $\tilde{E} := \Psi \circ E_{\mathcal{S}} \circ \Phi$ is a faithful conditional expectation onto $\tilde{\mathcal{M}}$ that preserves $\tilde{\omega}$.

4. Let $\tilde{\mathcal{S}} = \{\Phi_*(\rho),\ \rho \in \mathcal{S}\}$, then $\tilde{E} = E_{\tilde{\mathcal{S}}}$.

Proof. 1. For $a \in \mathcal{M}_{\mathcal{S}}$, we have, since $\tilde{\omega} \circ \Psi = \omega \circ E_{\mathcal{S}}$,

$$\omega(a^*a) = \tilde{\omega}(\Psi(a^*a)) \geq \tilde{\omega}(\Psi(a)^*\Psi(a)) \geq \omega(\Phi(\Psi(a))^*\Phi(\Psi(a))) = \omega(a^*a).$$

It follows in particular that the first inequality must be an equality. Since $\Psi(a^*a) \geq \Psi(a)^*\Psi(a)$ and ω is faithful, it follows that $\Psi(a^*a) = \Psi(a)^*\Psi(a)$. Hence $\Psi|_{\mathcal{M}_{\mathcal{S}}}$ is a homomorphism. Since $\tilde{\omega} \circ \Psi = \omega \circ E_{\mathcal{S}} = \omega$ is faithful, Ψ must be an isomorphism.

2. For any $a \in \mathcal{M}_{\mathcal{S}}$, we have $\Phi(\Psi(a)) = E_{\mathcal{S}}(a) = a$, so that $\Phi|_{\tilde{\mathcal{M}}}$ is the inverse of $\Psi|_{\mathcal{M}_{\mathcal{S}}}$, this clearly proves the point 2.

3. It is easily seen that $\tilde{E}^2 = \tilde{E}$ and that $\tilde{\omega} \circ \tilde{E} = \tilde{\omega}$, so that $\tilde{E}$ is a faithful conditional expectation with range $\tilde{\mathcal{M}}$.

4. It is easily seen that $\tilde{E}$ preserves $\rho \circ \Phi$ for all $\rho \in \mathcal{S}$, so that we have $\tilde{E} \circ E_{\tilde{\mathcal{S}}} = E_{\tilde{\mathcal{S}}} \circ \tilde{E} = E_{\tilde{\mathcal{S}}}$. On the other hand, we have

$$\rho \circ (\Phi \circ E_{\tilde{\mathcal{S}}} \circ \Psi) = (\rho \circ \Phi) \circ E_{\tilde{\mathcal{S}}} \circ \Psi = \rho \circ \Phi \circ \Psi = \rho,$$

so that $(\Phi \circ E_{\tilde{\mathcal{S}}} \circ \Psi) \circ E_{\mathcal{S}} = E_{\mathcal{S}}$. It follows that

$$\begin{aligned} E_{\tilde{\mathcal{S}}} &= \tilde{E} \circ E_{\tilde{\mathcal{S}}} \circ \tilde{E} = (\Psi \circ E_{\mathcal{S}} \circ \Phi) \circ E_{\tilde{\mathcal{S}}} \circ (\Psi \circ E_{\mathcal{S}} \circ \Phi) = \Psi \circ E_{\mathcal{S}} \circ (\Phi \circ E_{\tilde{\mathcal{S}}} \circ \Psi \circ E_{\mathcal{S}}) \circ \Phi \\ &= \Psi \circ E_{\mathcal{S}} \circ \Phi = \tilde{E}. \end{aligned}$$

□

3 Rényi relative entropies and sufficiency

For $\alpha \in (0, \infty) \setminus \{1\}$, the sandwiched (minimal) Rényi relative entropy is defined as

$$\tilde{D}_\alpha(\rho\|\sigma) := \frac{1}{\alpha - 1} \log \frac{\tilde{Q}_\alpha(\rho\|\sigma)}{\operatorname{Tr} \rho},$$

$$\tilde{Q}_\alpha(\rho\|\sigma) := \begin{cases} \operatorname{Tr} \left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha & \text{if } \alpha \in (0,1) \text{ or } \operatorname{supp}(\rho) \leq \operatorname{supp}(\sigma) \\ \infty & \text{otherwise.} \end{cases}$$

Let us assume $\alpha \in [1/2, 1)$. We have the following variational expression [**? ?**]

$$\tilde{Q}_\alpha(\rho\|\sigma) = \inf_{x \in \mathcal{M}^{++}} \alpha \operatorname{Tr} \rho x + (1-\alpha) \operatorname{Tr} \left(\sigma^{\frac{1-\alpha}{2\alpha}} x^{-1} \sigma^{\frac{1-\alpha}{2\alpha}}\right)^{\frac{\alpha}{1-\alpha}} \tag{1}$$

Proposition 4. *Assume that $\alpha > 1/2$ and put $\gamma := \frac{\alpha}{1-\alpha}$, then $\gamma > 1$ and we have*

$$\tilde{Q}_\alpha(\rho\|\sigma) = \inf_{x \in \mathcal{M}^{++}} \alpha \operatorname{Tr} \rho x + (1-\alpha) \tilde{Q}_\gamma(\sigma^{1/2} x^{-1} \sigma^{1/2} \|\sigma).$$

If $\operatorname{supp}(\rho) = \operatorname{supp}(\sigma)$, *then the infimum is attained at a unique $\bar{x} \in \mathcal{M}^{++}$ such that*

$$\sigma^{1/2} \bar{x}^{-1} \sigma^{1/2} = \sigma^{\frac{\gamma-1}{2\gamma}} \left(\sigma^{\frac{1}{2\gamma}} \rho \sigma^{\frac{1}{2\gamma}}\right)^{1-\alpha} \sigma^{\frac{\gamma-1}{2\gamma}}$$

Proof. By the properties of the L_p-norms, the infimum can be attained at a unique element in $\mathcal{M}^{++}$. Let $\bar{x}$ be such that the above equality holds, then

$$\bar{x} = \sigma^{\frac{1}{2\gamma}} (\sigma^{\frac{1}{2\gamma}} \rho \sigma^{\frac{1}{2\gamma}})^{\alpha - 1} \sigma^{\frac{1}{2\gamma}}.$$

It is easily checked that

$$\tilde{Q}_\gamma(\sigma^{1/2} \bar{x}^{-1} \sigma^{1/2} \|\sigma) = \tilde{Q}_\alpha(\rho\|\sigma) = \operatorname{Tr} \bar{x} \rho,$$

so that the infimum is attained.

□

Theorem 1. *Let Φ be a 2-positive trace preseving map. Then for $\alpha \in [1/2, 1)$, $\tilde{D}_\alpha(\Phi(\rho)\|\Phi(\sigma)) \leq \tilde{D}_\alpha(\rho\|\sigma)$.*

Proof. Assume first that $\alpha = 1/2$. Then we obtain from (1) that for any $y \in \mathcal{N}^{++}$,

$$\begin{aligned}\tilde{Q}_{1/2}(\rho\|\sigma) &= \frac{1}{2}\inf_{x\in\mathcal{M}^{++}} \operatorname{Tr}\rho x + \operatorname{Tr}\sigma x^{-1} \leq \frac{1}{2}(\operatorname{Tr}\rho\Phi^*(y) + \operatorname{Tr}\sigma\Phi^*(y)^{-1})\\ &\leq \frac{1}{2}(\operatorname{Tr}\Phi(\rho)x + \operatorname{Tr}\Phi(\sigma)y^{-1}),\end{aligned}$$

the second inequality follows by the Choi inequality $\Phi^*(y)^{-1} \leq \Phi^*(y^{-1})$ for a unital positive map. Taking the infimum over all $y \in \mathcal{N}^{++}$ implies the result.

Let us now assume that $\alpha \in (1/2, 1)$. Then, similarly as above, for any $y \in \mathcal{N}^{++}$ we have

$$\begin{aligned}\tilde{Q}_\alpha(\rho\|\sigma) &\leq \alpha\operatorname{Tr}\rho\Phi^*(y) + (1-\alpha)\tilde{Q}_\gamma(\sigma^{1/2}\Phi^*(y)^{-1}\sigma^{1/2}\|\sigma)\\ &\leq \alpha\operatorname{Tr}\Phi(\rho)y + (1-\alpha)\tilde{Q}_\gamma(\sigma^{1/2}\Phi^*(y^{-1})\sigma^{1/2}\|\sigma)\\ &= \alpha\operatorname{Tr}\Phi(\rho)y + (1-\alpha)\tilde{Q}_\gamma(\Phi_\sigma(\Phi(\sigma)^{1/2}y^{-1}\Phi(\sigma)^{1/2})\|\sigma)\\ &\leq \alpha\operatorname{Tr}\Phi(\rho)y + (1-\alpha)\tilde{Q}_\gamma(\Phi(\sigma)^{1/2}y^{-1}\Phi(\sigma)^{1/2}\|\Phi(\sigma))\end{aligned}$$

Here the second inequality follows by the Choi inequality and the fact that $\tilde{Q}_\gamma$ is nondecreasing in the first variable, the next equality is by definition of the Petz recovery map. The third inequality is by the fact that $\Phi_\sigma(\Phi(\sigma)) = \sigma$ and monotonicity of $\tilde{Q}_\gamma$ under positive maps. Taking infimum over $y \in \mathcal{N}^{++}$ we get the result. □

Theorem 2. *Let* $\operatorname{supp}(\rho) \leq \operatorname{supp}(\sigma)$ *and let* $\alpha \in (1/2, 1)$. *Then*

$$\tilde{Q}_\alpha(\rho\|\sigma) = \tilde{Q}_\alpha(\Phi(\rho)\|\Phi(\sigma)) \iff \Phi \textit{ is sufficient with respect to } \{\rho,\sigma\}.$$

Proof. We will prove this for the case when $\operatorname{supp}(\rho) = \operatorname{supp}(\sigma)$. Then also $\operatorname{supp}(\Phi(\rho)) = \operatorname{supp}(\Phi(\sigma))$, so we may assume that all the states are faithful. The infima in the variational expressions are attained at some (unique) $\bar{x} \in \mathcal{M}^{++}$ and $\bar{y} \in \mathcal{N}^{++}$.

$$\tilde{Q}_\alpha(\Phi(\rho)\|\Phi(\sigma)) = \alpha\operatorname{Tr}\rho\Phi^*(\bar{y}) + (1-\alpha)\tilde{Q}_\gamma(\Phi(\sigma)^{1/2}\bar{y}^{-1}\Phi(\sigma)^{1/2}\|\Phi(\sigma)).$$

Inserting $\bar{y}$ into the chain of (in)equalities in the proof of Theorem 1, we obtain that all the inequalities must be equalities. This implies that the infimum for in the variational expression for $\tilde{Q}(\rho\|\sigma)$ is attained at $\Phi^*(\bar{y})$, so that we must have $\bar{x} = \Phi^*(\bar{y})$. We also hav the following chain of equalities:

$$\begin{aligned}\tilde{Q}_\gamma(\sigma^{1/2}\Phi^*(\bar{y})^{-1}\sigma^{1/2}\|\sigma) &= \tilde{Q}_\gamma(\sigma^{1/2}\Phi^*(\bar{y}^{-1})\sigma^{1/2}\|\sigma) = \tilde{Q}_\gamma(\Phi_\sigma(\Phi(\sigma)^{1/2}\bar{y}^{-1}\Phi(\sigma)^{1/2})\|\sigma)\\ &= \tilde{Q}_\gamma(\Phi(\sigma)^{1/2}\bar{y}^{-1}\Phi(\sigma)^{1/2}\|\Phi(\sigma)) \qquad (2)\end{aligned}$$

From $\bar{x} = \Phi^*(\bar{y})$ and the first equality, we get (by the properties of the L_p-norms (Fack-Kosaki inequality), that

$$\mu := \sigma^{1/2}\bar{x}^{-1}\sigma^{1/2} = \sigma^{1/2}\Phi^*(\bar{y}^{-1})\sigma^{1/2} = \Phi_\sigma(\nu),$$

where $\nu := \Phi(\sigma)^{1/2}\bar{y}^{-1}\Phi(\sigma)^{1/2}$. But we also have

$$\tilde{Q}_\gamma(\nu\|\Phi(\sigma)) = \tilde{Q}_\alpha(\Phi(\rho)\|\Phi(\sigma)) = \tilde{Q}_\alpha(\rho\|\sigma) = \tilde{Q}_\gamma(\mu\|\sigma) = \tilde{Q}_\gamma(\Phi_\sigma(\nu)\|\Phi_\sigma(\Phi(\sigma))).$$

Since $\gamma > 1$, this implies that Φ_σ is sufficient with respect to the pair $\{\nu, \Phi(\sigma)\}$. Now note that the Petz dual of Φ_σ with respect to $\Phi(\sigma)$ is Φ itself, we obtain that $\Phi(\Phi_\sigma(\nu)) = \nu$. Hence

$$\Phi_\sigma \circ \Phi(\mu)) = \Phi_\sigma \circ \Phi \circ \Phi_\sigma(\nu) = \Phi_\sigma(\nu) = \mu,$$

so that Φ is sufficient with respect to $\{\mu, \sigma\}$. □