variational expression for $\alpha > 1$ (short proof)

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We give the element which attains the supremum of inequality (14) in [1] when $Q_{\alpha,z}(\psi||\varphi)$ is finite.

If $Q_{\alpha,z}(\psi||\varphi) < \infty$, that is, there exists an $x \in (s(\varphi)L^z(\mathcal{M})s(\varphi))_+$ such that $h_\psi^{\alpha/z} = h_\varphi^{(\alpha-1)/2z} x h_\varphi^{(\alpha-1)/2z}$ holds, then the element $w_0 = x^{\alpha-1} \in (s(\varphi)L^{z/(\alpha-1)}s(\varphi))_+$ attains the supremum of the right-hand side of inequality (14) in [1] since $\{h_\varphi^{(\alpha-1)/2z} a h_\varphi^{(\alpha-1)/2z} \,;\, a \in \mathcal{M}_+\}$ is dense in $(s(\varphi)L^{z/(\alpha-1)}s(\varphi))_+$.

Proof. We show the element $w_0 = x^{\alpha-1} \in (s(\varphi)L^{z/(\alpha-1)}s(\varphi))_+$ attains the supremum of the right-hand side of inequality (14) when $Q_{\alpha,z}(\psi||\varphi) < \infty$. For any $a \in \mathcal{M}_+$, we observe that

$$\begin{aligned}
&\alpha \operatorname{tr}\left((a^{1/2}h_\psi^{\alpha/z}a^{1/2})^{z/\alpha}\right) - (\alpha-1)\operatorname{tr}\left((a^{1/2}h_\varphi^{(\alpha-1)/z}a^{1/2})^{z/(\alpha-1)}\right)\\
&= \alpha \operatorname{tr}\left((a^{1/2}h_\varphi^{(\alpha-1)/2z}xh_\varphi^{(\alpha-1)/2z}a^{1/2})^{z/\alpha}\right) - (\alpha-1)\operatorname{tr}\left((a^{1/2}h_\varphi^{(\alpha-1)/z}a^{1/2})^{z/(\alpha-1)}\right)\\
&= \alpha \operatorname{tr}\left((x^{1/2}h_\varphi^{(\alpha-1)/2z}ah_\varphi^{(\alpha-1)/2z}x^{1/2})^{z/\alpha}\right) - (\alpha-1)\operatorname{tr}\left((h_\varphi^{(\alpha-1)/2z}ah_\varphi^{(\alpha-1)/2z})^{z/(\alpha-1)}\right)
\end{aligned}$$

by identity (♠) and Lemma 3 in [1]. Here we use the fact that $\{h_\varphi^{(\alpha-1)/2z} a h_\varphi^{(\alpha-1)/2z} \,;\, a \in \mathcal{M}_+\}$ is dense in $(s(\varphi)L^{z/(\alpha-1)}s(\varphi))_+$. Hence, we have

$$\begin{aligned}
&\sup_{a\in\mathcal{M}_+}\left\{\alpha \operatorname{tr}\left((a^{1/2}h_\psi^{\alpha/z}a^{1/2})^{z/\alpha}\right) - (\alpha-1)\operatorname{tr}\left((a^{1/2}h_\varphi^{(\alpha-1)/z}a^{1/2})^{z/(\alpha-1)}\right)\right\}\\
&= \sup_{w\in(s(\varphi)L^{z/(\alpha-1)}s(\varphi))_+}\left\{\alpha\operatorname{tr}((x^{1/2}wx^{1/2})^{z/\alpha}) - (\alpha-1)\operatorname{tr}(w^{z/(\alpha-1)})\right\}.
\end{aligned}$$

Taking $w_0 := x^{\alpha-1} \in (s(\varphi)L^{z/(\alpha-1)}s(\varphi))_+$, we have

$$\alpha\operatorname{tr}((x^{1/2}w_0x^{1/2})^{z/\alpha}) - (\alpha-1)\operatorname{tr}(w_0^{z/(\alpha-1)}) = \alpha\operatorname{tr}(x^z) - (\alpha-1)\operatorname{tr}(x^z) = \operatorname{tr}(x^z) = Q_{\alpha,z}(\psi||\varphi),$$

which implies that inequality (14) becomes equality. □

Thus, we can show that inequality (14) becomes equality when $z \geq \alpha/2$ by using the joint lower semi-continuity (Theorem 2(iv)) and Lemma 6 in [1].

References

[1] Shinya Kato, On α-z-Rényi divergence in the von Neumann algebra setting 2023, arXiv:2311.01748