Physics Letters A

Tight conic approximation of testing regions for quantum statistical models and measurements

--Manuscript Draft--

Manuscript Number:	PHYLA-D-24-00643
Article Type:	Letter
Section/Category:	Quantum physics
Keywords:	Testing region; Lorenz curce; quantum statistical model; quantum measurement; conical approximation

Abstract:	Quantum statistical models and quantum measurements can be regarded as linear maps: the former, mapping the space of effects to the space of probabilities; the latter, mapping the space of states to the space of probability distributions. The images of such linear maps are called testing regions. Our first result is to provide an implicit outer approximation of the testing region of any given quantum statistical model or measurement: namely, a region in probability space that contains the desired image, but is defined using a formula that depends only on the model or measurement. The outer approximation that we construct is minimal among all such outer approximations, and close, in the sense that it becomes the maximal inner approximation up to a constant scaling factor. Finally, we apply our approximation formulas to characterize, in a semi-device independent way, the ability to transform one quantum statistical model or measurement into another.

* We derive conical approximations of the testing regions of any given quantum statistical model and measurement.

* Such approximations are tight, namely, minimal in volume among all conical approximations.

* Such approximations are close, namely, there is just a constant rescaling between the outer and inner approximations.

* We apply our results to the semi-device independent testing of the simulability of quantum statistical models and measurements.

Tight conic approximation of testing regions for quantum statistical models and measurements

Michele Dall'Arno[1,2] and Francesco Buscemi[3]

[1]Department of Computer Science and Engineering, Toyohashi University of Technology, Japan
[2]Yukawa Institute for Theoretical Physics, Kyoto University, Japan
[3]Graduate School of Informatics, Nagoya University, Chikusa-ku, 464-8601 Nagoya, Japan

Quantum statistical models (i.e., families of normalized density matrices) and quantum measurements (i.e., positive operator-valued measures) can be regarded as linear maps: the former, mapping the space of effects to the space of probability distributions; the latter, mapping the space of states to the space of probability distributions. The images of such linear maps are called the testing regions of the corresponding model or measurement. Testing regions are notoriously impractical to treat analytically in the quantum case. Our first result is to provide an implicit outer approximation of the testing region of any given quantum statistical model or measurement in any finite dimension: namely, a region in probability space that contains the desired image, but is defined implicitly, using a formula that depends only on the given model or measurement. The outer approximation that we construct is *minimal* among all such outer approximations, and *close*, in the sense that it becomes the *maximal inner* approximation up to a constant scaling factor. Finally, we apply our approximation formulas to characterize, in a semi-device independent way, the ability to transform one quantum statistical model or measurement into another.

1 Introduction

In statistics, information theory, and mathematical economics one is often faced with the problem of comparing two setups in terms of their expected performances on a particular task of interest. For example, one might compare two statistical models by comparing their informativeness in a given parameter estimation problem, or two noisy channels with respect to a given communication figure of merit, or again two portfolios with respect to their expected utility in a given betting scenario. The comparison could also be extended, so to ask when a given setup is *always* better than another one, i.e., independent of any particular task at hand. Such "global" comparisons, generally described by a preorder relation, play a crucial role in the formulation of mathematical statistics.

The simplest example of one such preorder in statistics is given by the *majorization preorder* of probability distributions [1, 2, 3, 4]. Generalizing this, we find the comparison of families comprising two or more probability distributions. The case of pairs of probability distributions (i.e., *dichotomies*) is also known as *relative* majorization [5, 6, 7, 8], whereas the case of multiple elements is usually referred to as comparison of statistical *experiments* or *models* [5, 6, 7, 9].

The relevance of such preorder relations is epitomized by Blackwell's theorem [5, 6], which establishes the equivalence between the above mentioned statistical comparisons, and the existence of a suitable stochastic map that transforms one setup (the "always better" one) into the other (the "always worse" one). For this reason, Blackwell's theorem and its variants provide a powerful framework for general resource theories [10], and indeed recent quantum extensions of Blackwell's theorem [11, 12, 13] have found fruitful application in the study of quantum entanglement [14], quantum thermodynamics [15, 16], and quantum measurement theory [17, 18, 19], for example.

Michele Dall'Arno: michele.dallarno.mv@tut.jp
Francesco Buscemi: buscemi@i.nagoya-u.ac.jp

Mathematically, equivalence theorems *à la* Blackwell start from the characterization of suitably defined *testing regions*, corresponding to the statistical models at hand. In the simplest scenario, the testing region of a statistical model $\{\rho_i : 1 \leq i \leq n\}$ is constructed as follows: for any effect $0 \leq \pi \leq \mathbb{1}$, one computes the n-dimensional real vector whose i-th component is $\mathrm{Tr}[\pi\ \rho_i]$; the collection of all such vectors, for varying effect π, constitute the testing region of $\{\rho_i\}_i$[1]. In other words, the testing region of a statistical model is the *image* of the set of effect through the linear map induced by the former. For this reason, in what follows we will use the terms "testing region" and "image" interchangeably. Two statistical models with the same number of elements can then be compared by looking at their testing regions. A particularly relevant condition occurs when the testing region of one statistical model contains that of the other one. In the case of dichotomies, the inclusion relation for testing regions corresponds exactly with the preorder of relative majorization [8, 13].

Unfortunately, due to the non-commutativity of the underlying algebra, the quantum version of Blackwell's equivalence [11] turns out to be more convoluted than its original classical variant. One reason for this is that testing regions quickly become impractical to treat analytically[2]. This is particularly evident already in the case of relative majorization: while classical relative majorization can be summarized in a finite collection of easily computable inequalities [5, 8], in the quantum case (with the notable exceptions of qubits [20, 21]) an infinite number of scalar inequalities must be evaluated [13]. The situation becomes even more cumbersome in the case of quantum statistical models [11].

In this paper, in order to shed more light on the structure of quantum testing regions, we provide techniques to construct implicit approximations of the testing region of arbitrary quantum statistical models and measurements, in any finite dimension. More precisely, we construct conic regions in probability space that contain (outer approximations), or are contained (inner approximations) by, the desired testing region. Such approximations, unlike the testing region, can be defined implicitly, using a formula that depends only on the given setup (i.e., quantum statistical model or measurement). The approximations that we construct are *optimal* among all such approximations, that is, we prove that they are the minimal outer and the maximal inner conic approximations. They are moreover *close*, in the sense that the minimal outer approximation becomes the maximal inner approximation up to a constant scaling factor. Our approximation techniques thus generalize the bounding recently provided in Ref. [22] by Xu, Schwonnek, and Winter: first, the extension is from Pauli strings to arbitrary measurements; second, the optimization is not restricted to the radius of fixed-axis ellipsoids, but it is a *global* optimization over all the parameters of the ellispoid. As an application, we utilize our approximation formulas to characterize, in a semi-device independent way, the ability to transform one quantum measurement into another, or one quantum statistical model into another.

2 Main Results

2.1 Quantum measurements

Given a d-dimensional quantum measurement $\boldsymbol{\pi} = \{\pi_i : 1 \leq i \leq n\}$, $\pi_i \geq 0$, $\sum_i \pi_i = \mathbb{1}$, its testing region is defined as the image $\boldsymbol{\pi}(\mathbb{S}_d)$ of the set $\mathbb{S}_d$ of d-dimensional states through $\boldsymbol{\pi}$. By definition, this is given in parametric form, that is, it is a body in the probability space parameterized by states in the state space. Ideally, one would aim at implicitizing it, that is, writing it in the form $f(p) \leq 1$, for probability distributions p. However, due to intractability of the strucuture of the state space, we resort here to providing inclusion conditions in terms of implicit bodies.

Definition 1. *For any d-dimensional, n-outcome measurement $\boldsymbol{\pi} = \{\pi_i\}_{i=1}^n$, we define the family*

[1] The definition of testing region can be straightforwardly extended also to families of effects $\boldsymbol{\pi} = \{\pi_i : 1 \leq i \leq n\}$. In this case the region in $\mathbb{R}^n$ to consider is the collection of vectors whose components are given by $\mathrm{Tr}[\pi_i\ \rho]$, for varying ρ in the set of all states.

[2] Another reason is that the requirement of *complete positivity* demands an extended comparison [11].

$\{\mathcal{E}_r(\boldsymbol{\pi})\}_{r\in\mathbb{R}}$ *of hyper-ellipsoids given by:*

$$\mathcal{E}_r(\boldsymbol{\pi}) := \left\{ \mathbf{p} \in \boldsymbol{\pi}\left(\mathbb{C}^d\right) \,\middle|\, \left|\sqrt{Q^+}\left(\mathbf{p}-\mathbf{t}\right)\right|_2^2 \leq \frac{1}{r^2} \right\},$$

where $Q \in \mathbb{R}^{n\times n}$ *is the symmetric positive semi-definite covariance matrix given by*

$$Q_{ij} = \frac{d-1}{d}\left(\operatorname{Tr}\left[\pi_i\pi_j\right] - \frac{\operatorname{Tr}\left[\pi_i\right]\operatorname{Tr}[\pi_j]}{d}\right),$$

for any $0 \leq i,j \leq n$*, and* $\mathbf{t} \in \mathbb{R}^n$ *is the vector*

$$t_i = \frac{1}{d}\operatorname{Tr}\left[\pi_i\right], \qquad 1 \leq i \leq n.$$

Theorem 1. *For any* d*-dimensional,* n*-outcome informationally complete measurement* $\boldsymbol{\pi}$*, one has that* $\mathcal{E}_{d-1}(\boldsymbol{\pi})$ *is the maximum volume ellipsoid enclosed in* $\boldsymbol{\pi}(\mathbb{S}_d)$ *and* $\mathcal{E}_1(\boldsymbol{\pi})$ *is the minimum volume ellipsoid enclosing* $\boldsymbol{\pi}(\mathbb{S}_d)$*.*

If measurement $\boldsymbol{\pi}$ is not informationally complete, ellipsoids $\mathcal{E}_{d-1}(\boldsymbol{\pi})$ and $\mathcal{E}_1(\boldsymbol{\pi})$ still are inner and outer approximations of $\boldsymbol{\pi}(\mathbb{S}_d)$, although not necessarily maximal and minimal in volume, respectively.

We postpone the proof of Theorem 1 to Section 3.2.

As examples, let us consider symmetric, informationally complete (SIC) and mutually unbiased basis (MUB) measurements.

A d-dimensional measurement $\boldsymbol{\pi}$ is SIC if and only if it has $n = d^2$ effects satisfying the condition $\operatorname{Tr}\pi_i\pi_j = (d\delta_{i,j}+1)/(d^2(d+1))$. By explicit computation one has

$$Q = \frac{d-1}{d^2\left(d+1\right)}\left(\mathbb{1}_{d^2} - \hat{\mathbf{u}}\hat{\mathbf{u}}^T\right).$$

As expected, Q is a $d^2\times d^2$ matrix of rank d^2-1, and it is proportional to a projector [23]. Its pseudo-inverse is then given by

$$Q^+ = \frac{d^2\left(d+1\right)}{d-1}\left(\mathbb{1}_{d^2} - \hat{\mathbf{u}}\hat{\mathbf{u}}^T\right).$$

A d-dimensional measurement $\boldsymbol{\pi}$ is a complete MUB if and only if it has $n = d(d+1)$ effects satisfying the condition $\operatorname{Tr}[\pi_{i,j}\pi_{k,l}] = (\delta_{i,k}\delta j, l + (1-\delta_{i,k})/d)/(d+1)^2$, where indices i,k denote the basis and indices j,l denote the effect within the basis. By explicit copmputation one has

$$Q = \frac{d-1}{d\left(d+1\right)^2}\left(\mathbb{1}_{d(d+1)} - \oplus_{i=1}^{d+1}\hat{\mathbf{u}}_d^i\hat{\mathbf{u}}_d^{iT}\right),$$

where $\mathbf{u}_d^i$ is the vector with ones for the entries corresponding to basis i and zero otherwise. As expected, Q is a $d(d+1)\times d(d+1)$ matrix of rank d^2-1, and it is proportional to a projector [23]. Its pseudo-inverse is then given by

$$Q^+ = \frac{d\left(d+1\right)^2}{d-1}\left(\mathbb{1}_{d(d+1)} - \oplus_{i=1}^{d+1}\hat{\mathbf{u}}_d^i\hat{\mathbf{u}}_d^{iT}\right).$$

Now that we have a close approximation of the image of the set of states through any given measurement, we turn our attention to applying it to semi-device independent tests of simulability. A test is semi-device independent if it only assumes the dimension of the devices involved, but does not otherwise assume their mathematical description. We say that a d_1-dimensional, n-outcome measurement $\boldsymbol{\pi}_1$ simulates a d_0-dimensional, n-outcome measurement $\boldsymbol{\pi}_0$ if and only if there exists a completely positive map $\mathcal{C} : \mathcal{L}(\mathbb{C}^{d_0}) \to \mathcal{L}(\mathbb{C}^{d_1})$ such that

$$\boldsymbol{\pi}_1 \circ \mathcal{C} = \boldsymbol{\pi}_0. \tag{1}$$

The following corollary generalizes Corollary 2 of Ref. [21] to the arbitrary dimensional case, providing a semi-device independent test of Eq. (1).

Corollary 1 (Semi-device independent simulability test)**.** *Given a set $\mathcal{P}$ of n-element probability distributions generated by a d_1-dimensional (otherwise unspecified) measurement $\boldsymbol{\pi}_1$, for any d_0 and for any d_0-dimensional n-outcome measurement $\boldsymbol{\pi}_0$ such that*

$$\mathcal{E}_1\left(\boldsymbol{\pi}_0\right) \subseteq \operatorname{conv}\mathcal{P},$$

there exists a trace preserving map $\mathcal{C}$ that is positive on the support of $\boldsymbol{\pi}_0$ such that Eq. (1) holds. Moreover, if $d_1 = 2$, $n \leq 3$, and $d_0 \leq 3$, map $\mathcal{C}$ in Eq. (1) is completely positive, that is, measurement $\boldsymbol{\pi}_1$ simulates measurement $\boldsymbol{\pi}_0$.

Proof. The first part of the statement follows from Theorem 1 and from Proposition 7.1 of Ref. [24]. The second part of the statement follows from Theorem 1 and from Theorem 2 of Ref. [21]. □

2.2 Quantum statistical models

Given a d-dimensional quantum statistical model $\boldsymbol{\rho} = \{\rho_i : 1 \leq i \leq n\}$, $\rho_i \geq 0$, $\mathrm{Tr}[\rho_i] = 1$, its testing region is defined as the image $\boldsymbol{\rho}(\mathbb{E})$ of the cone $\mathbb{E}$ of effects through $\boldsymbol{\rho}$, seen as a classical-quantum (c-q for short) channel. By definition, the testing region $\boldsymbol{\rho}(\mathbb{E})$ is given in parametric form, that is, it is a body in the probability space parameterized by effects in the effect space. Ideally, one would aim at implicitizing it, that is, write it in the form $f(q) \leq 1$, for vectors of probabilities q. However, due to the intractability of the structure of the effect space, we resort here to providing inclusion conditions in terms of implicit bodies.

Definition 2. *For any d-dimensional family $\boldsymbol{\rho} = \{\rho_i\}_{i=1}^n$ of n states, let $\{\mathcal{E}_r^k(\boldsymbol{\rho})\}_{r\in\mathbb{R}}^{k=0,\dots d}$ be the following family of hyper-ellipsoids:*

$$\mathcal{E}_r^k\left(\boldsymbol{\rho}\right) = \left\{ \mathbf{q} \in \boldsymbol{\rho}\left(\mathbb{C}^d\right) \middle| \left|\sqrt{Q_k^+}\left(\mathbf{q} - \frac{k}{d}\mathbf{u}\right)\right|_2^2 \leq \frac{1}{r^2} \right\},$$

where $Q_k \in \mathbb{R}^{n\times n}$ is the symmetric positive semi-definite covariance matrix given by

$$\left(Q_k\right)_{ij} = \left(k - \frac{k^2}{d}\right)\left(\mathrm{Tr}\left[\rho_i\rho_j\right] - \frac{1}{d}\right),$$

for any $0 \leq i, j \leq n$, and $\mathbf{u} \in \mathbb{R}^n$ is the vector with all unit entries.

We introduce a d-cone as a generalization of the bicone. A d-cone in $\mathbb{R}^n$ is the convex hull of the origin and d arbitrary $(n-1)$-balls with aligned and equidistant centers lying on hyperplanes orthogonal to the line of the centers. Let $r(x)$ be the radius of the ball at distance x from the origin and L be the distance of the furthest ball. If $r(x)$ is symmetric, that is $r(x) = r(L-x)$ for any $0 \leq x \leq L$, then we say that the d-cone is symmetric. The usual bicone is recovered as the symmetric 2-cone. A pictorial representation of d-cones is given in Fig. 1. An elliptical d-cone is the image of a d-cone through a linear transformation that preserves the line joining the centers of the balls.

Theorem 2. *For any d-dimensional, n-outcome informationally complete family $\boldsymbol{\rho}$ of states, one has that $\operatorname{conv}\cup_{k=0}^d \mathcal{E}_{\eta(k,d)}^k(\boldsymbol{\rho})$ is the maximum volume elliptical d-cone enclosed in $\boldsymbol{\rho}(\mathbb{E})$ and $\operatorname{conv}\cup_{k=0}^d \mathcal{E}_1^k(\boldsymbol{\rho})$ is the minimum volume elliptical d-cone enclosing $\boldsymbol{\rho}(\mathbb{E})$, where*

$$\eta\left(k,d\right)^2 = \left(d-1\right)\frac{k\left(d-k\right)}{\min\left(k^2, \left(d-k\right)^2\right)}.$$

If family $\boldsymbol{\rho}$ of states is not informationally complete, elliptical d-cones $\operatorname{conv}\cup_{k=0}^d \mathcal{E}_{d-1}^k(\boldsymbol{\rho})$ and $\operatorname{conv}\cup_{k=0}^d \mathcal{E}_1^k(\boldsymbol{\rho})$ still are inner and outer approximations of $\boldsymbol{\rho}(\mathbb{E})$, although not necessarily maximal and minimal in volume, respectively.

We postpone the proof of Theorem 2 to Section 3.3.

As examples, let us consider symmetric, informationally complete (SIC) and mutually unbiased basis (MUB) families of states.

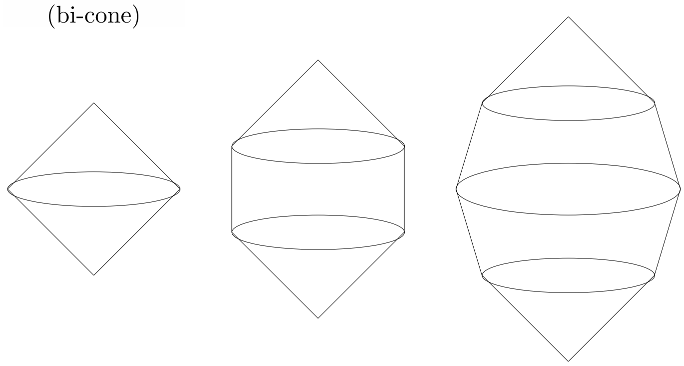


Figure 1: A pictorial representation of symmetric d-cones in $\mathbb{R}^3$, for $d = 2, 3, 4$.

A d-dimensional family $\boldsymbol{\rho}$ of states is SIC if and only if it has $n = d^2$ states satisfying the condition $\operatorname{Tr} \rho_i \rho_j = (d\delta_{i,j} + 1)/(d + 1)$. By explicit computation one has

$$Q_k = \frac{kd - k^2}{d^2 (d+1)} \left(\mathbb{1}_{d^2} - \hat{\mathbf{u}}\hat{\mathbf{u}}^T \right),$$

where $\hat{\mathbf{u}}$ denotes the unit vector with all equal entries. As expected, Q_k are $d^2 \times d^2$ matrix of rank $d^2 - 1$, and they are proportional to a projector. Their pseudo-inverses are then given by

$$Q_k^+ = \frac{d^2 (d+1)}{kd - k^2} \left(\mathbb{1}_{d^2} - \hat{\mathbf{u}}\hat{\mathbf{u}}^T \right).$$

A d-dimensional family $\boldsymbol{\rho}$ of states is a complete MUB if and only if it has $n = d(d+1)$ states satisfying the condition $\operatorname{Tr}[\rho_{i,j}\rho_{k,l}] = (\delta_{i,k}\delta j, l + (1 - \delta_{i,k})/d)$, where indices i, k denote the basis and indices j, l denote the effect within the basis. By explicit copmputation one has

$$Q_k = \frac{kd - k^2}{d\,(d+1)^2} \left(\mathbb{1}_{d(d+1)} - \oplus_{i=1}^{d+1} \hat{\mathbf{u}}_d^i \hat{\mathbf{u}}_d^{iT} \right),$$

where $\mathbf{u}_d^i$ is the vector with ones for the entries corresponding to basis i and zero otherwise. As expected, Q_k are $d(d+1) \times d(d+1)$ matrces of rank $d^2 - 1$, and they are proportional to a projector. Their pseudo-inverses are then given by

$$Q_k^+ = \frac{d\,(d+1)^2}{kd - k^2} \left(\mathbb{1}_{d(d+1)} - \oplus_{i=1}^{d+1} \hat{\mathbf{u}}_d^i \hat{\mathbf{u}}_d^{iT} \right).$$

Now that we have a close approximation of the image of the set of effects through any given family of states, we turn our attention to applying it to semi-device independent tests of simulability. We say that a d_1-dimensional, n-outcome family of states $\boldsymbol{\rho}$ simulates a d_0-dimensional, n-outcome measurement $\boldsymbol{\rho}_0$ if and only if there exists a completely positive trace preserving map (a quantum channel) $\mathcal{C} : \mathcal{L}(\mathbb{C}^{d_1}) \to \mathcal{L}(\mathbb{C}^{d_0})$ such that

$$\mathcal{C} \circ \boldsymbol{\rho}_1 = \boldsymbol{\rho}_0. \tag{2}$$

The following corollary generalizes Corollary 1 of Ref. [21] to the arbitrary dimensional case, providing a semi-device independent test of Eq. (2).

Corollary 2 (Semi-device independent simulability test)**.** *Given a set $\mathcal{Q}$ of n-element vectors of probabilities generated by a d_1-dimensional (otherwise unspecified) family of n states $\boldsymbol{\rho}_1$, for any d_0 and for any d_0-dimensional family of n states $\boldsymbol{\rho}_0$ such that*

$$\operatorname{conv} \cup_{k=0}^{d} \mathcal{E}_1^k (\boldsymbol{\rho}_0) \subseteq \operatorname{conv} \mathcal{Q},$$

there exists a (not necessarily trace preserving) map $\mathcal{C}$ that is positive on the support of $\boldsymbol{\rho}_0$ such that Eq. (1) holds. Moreover, if $d_1 = 2$, $n = 2$, and $d_0 = 2$, map $\mathcal{C}$ in Eq. (1) is completely positive trace preserving, that is, family $\boldsymbol{\rho}_1$ of states simulates family $\boldsymbol{\rho}_0$ of states.

Proof. The first part of the statement follows from Theorem 2. The second part of the statement follows from Theorem 2 and from Theorem 1 of Ref. [21]. □

3 Proofs

3.1 Formalization

For any positive integer d, let $\mathcal{L}(\mathbb{C}^d)$ denote the space of Hermitian operators on $\mathbb{C}^d$ equipped with the Hilbert-Schmidt product, that is, for any $\rho, \pi \in \mathcal{L}(\mathbb{C}^d)$ we have $\rho \cdot \pi = \mathrm{Tr}[\rho\pi]$. For any positive integer n, let $\mathbb{R}^n$ denote the space of n-dimensional real vectors equipped with the usual inner product, that is, for any $p, q \in \mathbb{R}^n$ we have $p \cdot q = \sum_{i=1}^n p_i^\dagger q_i$.

A d-dimensional, n-outcome measurement is a map

$$\boldsymbol{\pi} : \mathcal{L}\left(\mathbb{C}^d\right) \to \mathbb{R}^n.$$

Any measurement $\boldsymbol{\pi}$ can be represented as an indexed family $\{\pi_i \in \mathcal{L}(\mathbb{C}^d)\}_{i=1}^n$ of operators as follows. Recalling that the space $\mathcal{L}(\mathbb{C}^d)$ is equipped with the Hilbert-Schmidt product, the action of $\boldsymbol{\pi}$ on an operator $\rho \in \mathcal{L}(\mathbb{C}^d)$ is naturally given by

$$\boldsymbol{\pi}(\rho) := \begin{bmatrix} \langle\langle \pi_1 | \\ \vdots \\ \langle\langle \pi_n | \end{bmatrix} |\rho\rangle\rangle = \begin{bmatrix} \mathrm{Tr}[\pi_1 \rho] \\ \vdots \\ \mathrm{Tr}[\pi_n \rho] \end{bmatrix} \in \mathbb{R}^n,$$

where $\langle\langle \pi | : \mathcal{L}(\mathbb{C}^d) \to \mathbb{R}$ is given by $\langle\langle \pi | \rho \rangle\rangle = \mathrm{Tr}[\pi\rho]$.

Recalling that the space $\mathbb{R}^n$ is instead equipped with the usual inner product, the action of the Hermitian conjugate $\boldsymbol{\pi}^\dagger$ on a vector $\mathbf{p} \in \mathbb{R}^n$ is naturally given by

$$\begin{aligned} \boldsymbol{\pi}^\dagger \mathbf{p} &= \begin{bmatrix} |\pi_1\rangle\rangle & \dots & |\pi_n\rangle\rangle \end{bmatrix} \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} \\ &= \sum_{i=1}^n p_i |\pi_i\rangle\rangle \in \mathcal{L}\left(\mathbb{C}^d\right). \end{aligned}$$

Finally, for any measurements $\boldsymbol{\pi}$ and $\boldsymbol{\tau}$, the compositions $\boldsymbol{\tau}^\dagger \boldsymbol{\pi}$ and $\boldsymbol{\pi}\boldsymbol{\tau}^\dagger$ are given by

$$\begin{aligned} \boldsymbol{\tau}^\dagger \boldsymbol{\pi} &= \begin{bmatrix} |\tau_1\rangle\rangle & \dots & |\tau_n\rangle\rangle \end{bmatrix} \begin{bmatrix} \langle\langle \pi_1 | \\ \vdots \\ \langle\langle \pi_n | \end{bmatrix} \\ &= \sum_{i=1}^n |\tau_i\rangle\rangle\langle\langle \pi_i | \in \mathcal{L}\left(\mathbb{C}^d\right) \to \mathcal{L}\left(\mathbb{C}^d\right), \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{\pi}\boldsymbol{\tau}^\dagger &= \begin{bmatrix} \langle\langle \pi_1 | \\ \vdots \\ \langle\langle \pi_n | \end{bmatrix} \begin{bmatrix} |\tau_1\rangle\rangle & \dots & |\tau_n\rangle\rangle \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{Tr}[\pi_1 \tau_1] & \dots & \mathrm{Tr}[\pi_1 \tau_n] \\ \vdots & & \vdots \\ \mathrm{Tr}[\pi_n \tau_1] & \dots & \mathrm{Tr}[\pi_n \tau_n] \end{bmatrix} \in \mathbb{R}^n \to \mathbb{R}^n. \end{aligned}$$

For any d-dimensional, n-outcome measurement $\boldsymbol{\pi}$, its pseudo-inverse $\boldsymbol{\pi}^+$ is the unique n-elements row vector of operators in $\mathcal{L}(\mathbb{C}^d)$ such that

$$
\begin{aligned}
\boldsymbol{\pi}\boldsymbol{\pi}^+\boldsymbol{\pi} &= \boldsymbol{\pi}, \\
\boldsymbol{\pi}^+\boldsymbol{\pi}\boldsymbol{\pi}^+ &= \boldsymbol{\pi}^+, \\
\boldsymbol{\pi}^+\boldsymbol{\pi} &= \left(\boldsymbol{\pi}^+\boldsymbol{\pi}\right)^\dagger, \\
\boldsymbol{\pi}\boldsymbol{\pi}^+ &= \left(\boldsymbol{\pi}\boldsymbol{\pi}^+\right)^\dagger.
\end{aligned}
$$

3.2 Quantum measurements

Leveraging on the formalism introduced in Section 3.1, for any d-dimensional, n-outcome measurement $\boldsymbol{\pi}$ we can provide the following definitions of covariance matrix Q and probability distribution $\mathbf{t}$:

$$
Q := \frac{d-1}{d} \left(\boldsymbol{\pi} - \boldsymbol{\tau}\right)\left(\boldsymbol{\pi} - \boldsymbol{\tau}\right)^\dagger,
$$

and

$$
\mathbf{t} := \boldsymbol{\tau} \frac{|\mathbb{1}\rangle\rangle}{d},
$$

where $\boldsymbol{\tau}$ is the d-dimensional, n-outcome measurement given by

$$
\boldsymbol{\tau} := \frac{1}{d} \begin{bmatrix} \operatorname{Tr}\left[\pi_1\right] \langle\langle \mathbb{1}| \\ \vdots \\ \operatorname{Tr}\left[\pi_n\right] \langle\langle \mathbb{1}| \end{bmatrix}.
$$

Notice that these definitions are consistent with those in Def. 1.

For any dimension d we denote with $\mathbb{B}_d$ the ball whose extremal points include all pure states, that is

$$
\mathbb{B}_d := \left\{ \rho \in \mathcal{L}\left(\mathbb{C}^d\right) \,\middle|\, \operatorname{Tr}\left[\rho\right] = 1,\ \operatorname{Tr}\left[\rho^2\right] \leq 1 \right\}.
$$

Consider the image $\boldsymbol{\pi}(\mathbb{B}_d)$ of the ball $\mathbb{B}_d$ through a measurement $\boldsymbol{\pi}$. Again, this expression describes a body in the probability space parameterized by a body in the state space. The following lemma makes implicit this parametric equation by removing the dependence on the states and expressing the image of $\mathbb{B}_d$ in the form $f(\mathbf{p}) \leq 0$. The lemma generalizes Theorem 1 of Ref. [25] from the qubit case to the arbitrary dimensional case.

Lemma 1 (Implicitization of $\boldsymbol{\pi}(\mathbb{B}_d)$)**.** *For any d-dimensional, n-outcome measurement $\boldsymbol{\pi}$, the image $\boldsymbol{\pi}(\mathbb{B}_d)$ is given by the following hyper-ellipsoid:*

$$
\boldsymbol{\pi}\left(\mathbb{B}_d\right) := \mathcal{E}_1\left(\boldsymbol{\pi}\right).
$$

Proof. One has

$$
\begin{aligned}
\mathbf{p} &= \boldsymbol{\pi} \, |\rho\rangle\rangle \\
&= \left(\boldsymbol{\pi} - \boldsymbol{\tau} + \boldsymbol{\tau}\right) |\rho\rangle\rangle \\
&= \left(\boldsymbol{\pi} - \boldsymbol{\tau}\right) |\rho\rangle\rangle + \boldsymbol{\tau} \, |\rho\rangle\rangle \\
&= \left(\boldsymbol{\pi} - \boldsymbol{\tau}\right) |\rho\rangle\rangle + \mathbf{t}.
\end{aligned}
$$

Hence

$$
\boldsymbol{\pi}\left(\mathbb{B}_d\right) = \left\{ \mathbf{p} = \left(\boldsymbol{\pi} - \boldsymbol{\tau}\right) |\rho\rangle\rangle + \mathbf{t} \,\middle|\, \operatorname{Tr}\rho = 1, \operatorname{Tr}\rho^2 \leq 1 \right\}.
$$

Solutions of $\left(\boldsymbol{\pi} - \boldsymbol{\tau}\right) |\rho\rangle\rangle = \mathbf{p} - \mathbf{t}$ in ρ exist if and only if $\mathbf{p} - \mathbf{t}$ belongs to the range of $\boldsymbol{\pi} - \boldsymbol{\tau}$. Solutions are given by

$$
|\rho\rangle\rangle = \left(\boldsymbol{\pi} - \boldsymbol{\tau}\right)^+ \left(\mathbf{p} - \mathbf{t}\right) + \left(\mathbb{1} - \Pi\right) |\sigma\rangle\rangle, \tag{3}
$$

where $\Pi := (\boldsymbol{\pi} - \boldsymbol{\tau})^{+}(\boldsymbol{\pi} - \boldsymbol{\tau})$, for any $\sigma \in \mathcal{L}(\mathbb{C}^d)$. Notice that $\Pi |\mathbb{1}\rangle\rangle = 0$ since $(\boldsymbol{\pi} - \boldsymbol{\tau}) |\mathbb{1}\rangle\rangle = \mathbf{t} - \mathbf{t}$. Hence Eq. (3) is equivalent to

$$\begin{aligned} |\rho\rangle\rangle =& (\boldsymbol{\pi} - \boldsymbol{\tau})^{+} (\mathbf{p} - \mathbf{t}) + \lambda \frac{|\mathbb{1}\rangle\rangle}{d} \\ &+ \left(\mathbb{1} - \frac{1}{d} |\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}| - \Pi\right) |\sigma\rangle\rangle , \end{aligned}$$

again for any $\sigma \in \mathcal{L}(\mathbb{C}^d)$.

The condition $\operatorname{Tr} \rho = 1$ immediately implies $\lambda = 1$. Moreover, due to the Hilbert-Schmidt orthogonality of $(\boldsymbol{\pi} - \boldsymbol{\tau})^{+}(\mathbf{p} - \mathbf{t})$ and $(\mathbb{1} - |\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}| /d - \Pi) |\sigma\rangle\rangle$, one has that for any σ such that $\operatorname{Tr} \rho^2 \leq 1$, the same condition is also verified for $\sigma = 0$. Hence, without loss of generality we take $\sigma = 0$. Thus we have

$$|\rho\rangle\rangle = (\boldsymbol{\pi} - \boldsymbol{\tau})^{+} (\mathbf{p} - \mathbf{t}) + \frac{|\mathbb{1}\rangle\rangle}{d}.$$

Hence,

$$\operatorname{Tr} \rho^2 = (\mathbf{p} - \mathbf{t})^T (\boldsymbol{\pi} - \boldsymbol{\tau})^{+\dagger} (\boldsymbol{\pi} - \boldsymbol{\tau})^{+} (\mathbf{p} - \mathbf{t}) + \frac{1}{d}.$$

Thus, condition $\operatorname{Tr} \rho^2 \leq 1$ becomes

$$(\mathbf{p} - \mathbf{t})^T (\boldsymbol{\pi} - \boldsymbol{\tau})^{+\dagger} (\boldsymbol{\pi} - \boldsymbol{\tau})^{+} (\mathbf{p} - \mathbf{t}) \leq 1 - \frac{1}{d}.$$

Hence the statement follows. □

We are now in a position to prove Theorem 1, that we rewrite here for convenience.

Theorem 1. *For any d-dimensional, n-outcome informationally complete measurement $\boldsymbol{\pi}$, one has that $\mathcal{E}_{d-1}(\boldsymbol{\pi})$ is the maximum volume ellipsoid enclosed in $\boldsymbol{\pi}(\mathbb{S}_d)$ and $\mathcal{E}_1(\boldsymbol{\pi})$ is the minimum volume ellipsoid enclosing $\boldsymbol{\pi}(\mathbb{S}_d)$.*

Proof. First, we prove that the image $\boldsymbol{\pi}(\mathbb{B}_d)$ coincides with the minimum volume ellipsoid $\mathcal{E}(\boldsymbol{\pi}(\mathbb{S}_d))$ enclosing the image of $\mathbb{S}_d$. This can be shown as follows. First, we show that any 2-design $\{\lambda_k, \rho_k\}_k$ is a scalable frame, that is, a family of weights over states such that

$$\sum_k \lambda_k |\rho_k - \frac{\mathbb{1}}{d}\rangle\rangle\langle\langle\rho_k - \frac{\mathbb{1}}{d}| = \left(\mathbb{1} - \frac{1}{d} |\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|\right).$$

Indeed, for any state ρ we have

$$\begin{aligned} &\sum_k \lambda_k \left(\rho_k - \frac{\mathbb{1}}{d}\right) \operatorname{Tr}\left[\left(\rho_k - \frac{\mathbb{1}}{d}\right)\left(\rho - \frac{\mathbb{1}}{d}\right)\right] \\ =& \sum_k \lambda_k \rho_k \operatorname{Tr}\left[\rho_k \left(\rho - \frac{\mathbb{1}}{d}\right)\right] \\ =& \operatorname{Tr}_2\left[\sum_k \lambda_k \rho_k^{\otimes 2} \left(\mathbb{1} \otimes \left(\rho - \frac{\mathbb{1}}{d}\right)\right)\right] \\ =& \operatorname{Tr}_2\left[(\mathbb{1} + S)\left(\mathbb{1} \otimes \left(\rho - \frac{\mathbb{1}}{d}\right)\right)\right] \\ =& \operatorname{Tr}_2\left[S\left(\mathbb{1} \otimes \left(\rho - \frac{\mathbb{1}}{d}\right)\right)\right] \\ =& \left(\rho - \frac{\mathbb{1}}{d}\right), \end{aligned}$$

where S denotes the swap operator. Notice that, from Sections 6.9 and 6.11 of Ref. [27] it immediately follows that finite 2-designs exist in any dimension d, hence the existence of scalable frames in any dimension d. Then, the statement immediately follows from Theorem 2.11 of Ref. [26].

Notice that, if rescaled by constant factor $d^2 - 1$, minimum volume enclosing ellipsoids are enclosed in the convex body (see e.g. Section 8.4.1 of Ref. [28]). However, the lower bound in Theorem 1 is tighter than this, hence the need for the following independent proof.

The inner ellipsoid must include boundary states, otherwise it would not maximize the volume. Among all boundary states, the ones that minimize the 2-norm are the projectors of rank $d-1$. Since $\mathrm{Tr}[|\phi\rangle\langle\phi|]^{1/2} = 1$ and $\mathrm{Tr}[\mathbb{1}/d]^{1/2} = 1/\sqrt{d}$, one has that the radius of the outer ellipsoid is given by $\sqrt{1-1/d} = \sqrt{(d-1)/d}$. Since $\mathrm{Tr}[(\mathbb{1} - |\phi\rangle\langle\phi|)/(d-1)^2]^{1/2} = 1/\sqrt{d-1}$, one has that the radius of the inner ellipsoid is given by $\sqrt{1/(d-1) - 1/d} = \sqrt{1/(d(d-1))}$. Hence, the ratio of the two radii is $\sqrt{(d-1)/d}\sqrt{(d(d-1))} = d-1$.

Using Theorem [J] of Ref. [29], we have that the lower bound in Theorem 1 holds again in any dimension in which there exists a finite scalable frame $\{\lambda_k, \rho_k\}$ of states proportional to rank-$(d-1)$ projectors. Since for any pure state ϕ one has

$$\frac{\mathbb{1} - |\phi\rangle\langle\phi|}{d-1} - \frac{\mathbb{1}}{d} = -\frac{d}{d-1}\left(|\phi\rangle\langle\phi| - \frac{\mathbb{1}}{d}\right),$$

one has that such a scalable frame exists if and only if a scalable frame of pure states exists, hence the proof of the lower bound goes along that of the upper bound. □

3.3 Quantum states

Leveraging on the formalism introduced in Section 3.1, for any d-dimensional, n-outcome family $\boldsymbol{\rho}$ of states we can provide the following definition of covariance matrix Q:

$$Q_k := \left(k - \frac{k^2}{d}\right)(\boldsymbol{\rho} - \boldsymbol{\sigma})(\boldsymbol{\rho} - \boldsymbol{\sigma})^\dagger,$$

where $\boldsymbol{\sigma}$ is the d-dimensional, n-outcome c-q channel given by

$$\boldsymbol{\sigma} := \frac{1}{d}\begin{bmatrix} \langle\langle \mathbb{1}| \\ \vdots \\ \langle\langle \mathbb{1}| \end{bmatrix}.$$

Notice that this definition is consistent with that in Def. 2.

For any dimension d and any $0 \le k \le d$ we denote with $\mathbb{B}_d^k$ the ball whose extremal points include all extremal effects with trace k, that is

$$\mathbb{B}_d^k := \left\{\pi \in \mathcal{L}\left(\mathbb{C}^d\right) \,\middle|\, \mathrm{Tr}\left[\pi\right] = k,\ \mathrm{Tr}\left[\pi^2\right] \le k\right\}.$$

We denote with $\mathbb{D}_d$ the symmetric d-cone whose extremal points include all extremal effects, that is

$$\mathbb{D}_d := \mathrm{conv} \cup_{k=0}^d \mathbb{B}_d^k.$$

Consider the image $\boldsymbol{\rho}(\mathbb{D}_d)$ of the d-cone $\mathbb{D}_d$ through a c-q channel $\boldsymbol{\rho}$. Again, this expression describes a body in the probability space parameterized by a body in the effect space. The following lemma makes implicit this parametric equation by removing the dependence on the effects and expressing the image of $\mathbb{D}_d$ in the form $f(\mathbf{q}) \le 0$. The lemma generalizes Proposition 2 of Ref. [30] from the qubit case to the arbitrary dimensional case.

Lemma 2 (Implicitization of $\boldsymbol{\rho}(\mathbb{D}_d)$)**.** *For any d-dimensional, n-outcome c-q channel $\boldsymbol{\rho}$, the image $\boldsymbol{\rho}(\mathbb{D}_d)$ is given by the following convex hull of hyper-ellipsoids:*

$$\boldsymbol{\rho}\left(\mathbb{D}_d\right) := \mathrm{conv} \cup_{k=0}^d \mathcal{E}_1^k\left(\boldsymbol{\rho}\right).$$

Proof. One has

$$\begin{aligned}\mathbf{q} &= \boldsymbol{\rho} \,|\pi\rangle\!\rangle \\ &= (\boldsymbol{\rho} - \boldsymbol{\sigma} + \boldsymbol{\sigma}) \,|\pi\rangle\!\rangle \\ &= (\boldsymbol{\rho} - \boldsymbol{\sigma}) \,|\pi\rangle\!\rangle + \boldsymbol{\sigma} \,|\pi\rangle\!\rangle \\ &= (\boldsymbol{\rho} - \boldsymbol{\sigma}) \,|\pi\rangle\!\rangle + \frac{k}{d}\mathbf{u}.\end{aligned}$$

Hence

$$\begin{aligned}&\boldsymbol{\rho}\left(\mathbb{B}_d^k\right) \\ =&\left\{\mathbf{q} = (\boldsymbol{\rho} - \boldsymbol{\sigma}) \,|\pi\rangle\!\rangle + \frac{k}{d}\mathbf{u} \,\middle|\, \operatorname{Tr}\pi = k, \operatorname{Tr}\pi^2 \le k\right\}.\end{aligned}$$

Solutions of $(\boldsymbol{\rho} - \boldsymbol{\sigma}) \,|\pi\rangle\!\rangle = \mathbf{q} - k\mathbf{u}/d$ in π exist if and only if $\mathbf{q} - k\mathbf{u}/d$ belongs to the range of $\boldsymbol{\rho} - \boldsymbol{\sigma}$. Solutions are given by

$$|\pi\rangle\!\rangle = (\boldsymbol{\rho} - \boldsymbol{\sigma})^{+} \left(\mathbf{q} - \frac{k}{d}\mathbf{u}\right) + (\mathbb{1} - \Pi) \,|\tau\rangle\!\rangle\,, \tag{4}$$

where $\Pi := (\boldsymbol{\rho} - \boldsymbol{\sigma})^{+}(\boldsymbol{\rho} - \boldsymbol{\sigma})$, for any $\tau \in \mathcal{L}(\mathbb{C}^d)$. Notice that $\Pi \,|\mathbb{1}\rangle\!\rangle = 0$ since $(\boldsymbol{\rho} - \boldsymbol{\sigma}) \,|\mathbb{1}\rangle\!\rangle = k\mathbf{u}/d - k\mathbf{u}/d$. Hence Eq. (4) is equivalent to

$$\begin{aligned}|\pi\rangle\!\rangle =&(\boldsymbol{\rho} - \boldsymbol{\sigma})^{+} \left(\mathbf{q} - \frac{k}{d}\mathbf{u}\right) + \lambda\frac{k}{d} \,|\mathbb{1}\rangle\!\rangle \\ &+ \left(\mathbb{1} - \frac{1}{d} \,|\mathbb{1}\rangle\!\rangle\!\langle\!\langle\mathbb{1}| - \Pi\right) |\tau\rangle\!\rangle\,,\end{aligned}$$

again for any $\tau \in \mathcal{L}(\mathbb{C}^d)$.

The condition $\operatorname{Tr}\pi = k$ immediately implies $\lambda = 1$. Moreover, due to the Hilbert-Schmidt orthogonality of $(\boldsymbol{\rho} - \boldsymbol{\sigma})^{+}(\mathbf{q} - k\mathbf{u}/d)$ and $(\mathbb{1} - |\mathbb{1}\rangle\!\rangle\!\langle\!\langle\mathbb{1}| /d - \Pi) \,|\sigma\rangle\!\rangle$, one has that for any τ such that $\operatorname{Tr}\pi^2 \le k$, the same condition is also verified for $\tau = 0$. Hence, without loss of generality we take $\tau = 0$. Thus we have

$$|\pi\rangle\!\rangle = (\boldsymbol{\rho} - \boldsymbol{\sigma})^{+} \left(\mathbf{q} - \frac{k}{d}\mathbf{u}\right) + \frac{k}{d} \,|\mathbb{1}\rangle\!\rangle\,.$$

Hence,

$$\begin{aligned}&\operatorname{Tr}\pi^2 \\ =&\left(\mathbf{q} - \frac{k}{d}\mathbf{u}\right)^T (\boldsymbol{\rho} - \boldsymbol{\sigma})^{+\dagger} (\boldsymbol{\rho} - \boldsymbol{\sigma})^{+} \left(\mathbf{q} - \frac{k}{d}\mathbf{u}\right) + \frac{k^2}{d}.\end{aligned}$$

Thus, condition $\operatorname{Tr}\pi^2 \le k$ becomes

$$\begin{aligned}&\left(\mathbf{q} - \frac{k}{d}\mathbf{u}\right)^T (\boldsymbol{\rho} - \boldsymbol{\sigma})^{+\dagger} (\boldsymbol{\rho} - \boldsymbol{\sigma})^{+} \left(\mathbf{q} - \frac{k}{d}\mathbf{u}\right) \\ \le&k - \frac{k^2}{d}.\end{aligned}$$

Hence the statement follows. □

We are now in a position to prove Theorem 2, that we rewrite here for convenience.

Theorem 2. *For any d-dimensional, n-outcome informationally complete family $\boldsymbol{\rho}$ of states, one has that $\operatorname{conv}\cup_{k=0}^{d}\mathcal{E}_{d-1}^{k}(\boldsymbol{\rho})$ is the maximum volume elliptical d-cone enclosed in $\boldsymbol{\rho}(\mathbb{E})$ and $\operatorname{conv}\cup_{k=0}^{d}\mathcal{E}_{1}^{k}(\boldsymbol{\rho})$ is the minimum volume elliptical d-cone enclosing $\boldsymbol{\rho}(\mathbb{E})$.*

Proof. An effect $0 \leq \pi \leq \mathbb{1}$ is extremal if and only if it is a projector. Hence, the set $\mathbb{E}_d$ of effects is the convex hull of projectors, that is

$$\mathbb{E}_d = \operatorname{conv} \cup_{k=0}^{d} \left\{ \pi \in \mathcal{L}\left(\mathbb{C}^d\right) \middle| \operatorname{Tr}[\pi] = k, \pi^2 = \pi \right\}.$$

The proof proceeds along the lines of the proof of Theorem 1. First, due to Sections 6.9 and 6.11 of Ref. [27], for any dimension there exists a finite scalable frame of k-trace projectors. Then, due to Theorem 2.11 of Ref. [26], the minimum volume ellipsoid enclosing k-trace projectors is the ball $\mathbb{B}_k^d$. □

4 Conclusion and outlook

In this paper we provided an implicit outer approximation of the image of any given quantum measurement in any finite dimension, thus generalizing a recent result [22] by Xu, Schwonnek, and Winter on the image of Pauli strings. The outer approximation that we constructed is *minimal* among all such outer approximations, and *close*, in the sense that it becomes the *maximal inner* approximation up to a constant scaling factor. We also obtained a similar result for the dual problem of implicitizing the image of the set of effects through a family of quantum states. Finally, we applied our approximation formulas to characterize, in a semi-device independent way, the ability to transform one quantum measurement into another, or one quantum statistical model into another.

5 Acknowledgments

M. D. is grateful to Anna Szymusiak for insightful comments and suggestions. M. D. acknowledges support from the Department of Computer Science and Engineering, Toyohashi University of Technology, from the International Research Unit of Quantum Information, Kyoto University, and from the JSPS KAKENHI grant number JP20K03774. F. B. acknowledges support from MEXT Quantum Leap Flagship Program (MEXT QLEAP) Grant No. JPMXS0120319794; from MEXT-JSPS Grant-in-Aid for Transformative Research Areas (A) "Extreme Universe", No. 21H05183; from JSPS KAKENHI Grants No. 20K03746 and No. 23K03230.

References

[1] Max O. Lorenz. Methods of measuring the concentration of wealth. *Publ. Am. Stat. Assoc.*, 9(70):209–219, 1905.

[2] Godfrey Harold Hardy, John Edensor Littlewood, and George Polya. *Inequalities*. Cambridge university press, 1952.

[3] Barry C. Arnold. *Majorization and the Lorenz Order: A Brief Introduction*. Springer New York, 1987. doi:10.1007/978-1-4615-7379-1.

[4] Albert W. Marshall, Ingram Olkin, and Barry C. Arnold. *Inequalities: theory of majorization and its applications*. Springer, 2010.

[5] David Blackwell. Equivalent Comparisons of Experiments. *The Annals of Mathematical Statistics*, 24(2):265–272, 1953. URL: http://www.jstor.org/stable/2236332, doi:10.1214/aoms/1177729032.

[6] D.A. Blackwell and M.A. Girshick. *Theory of Games and Statistical Decisions*. Dover Books on Mathematics. Dover Publications, 1979. URL: https://books.google.co.jp/books?id=fwTLAgAAQBAJ.

[7] Erik Torgersen. *Comparison of Statistical Experiments*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1991. doi:10.1017/CBO9780511666353.

[8] Joseph M. Renes. Relative submajorization and its use in quantum resource theories. *Journal of Mathematical Physics*, 57(12):122202, 12 2016. arXiv:https://pubs.aip.org/aip/jmp/article-pdf/doi/10.1063/1.4972295/16083406/122202_1_online.pdf, doi:10.1063/1.4972295.

[9] F. Liese and K.-J. Miescke. *Statistical Decision Theory*. Springer, 2008.

[10] Eric Chitambar and Gilad Gour. Quantum resource theories. *Rev. Mod. Phys.*, 91:025001, Apr 2019. URL: https://link.aps.org/doi/10.1103/RevModPhys.91.025001, doi:10.1103/RevModPhys.91.025001.

[11] Francesco Buscemi. Comparison of quantum statistical models: Equivalent conditions for sufficiency. *Communications in Mathematical Physics*, 310(3):625–647, 2012. doi:10.1007/s00220-012-1421-3.

[12] Anna Jencova. Comparison of quantum channels and statistical experiments, 2015. URL: https://arxiv.org/abs/1512.07016, doi:10.48550/ARXIV.1512.07016.

[13] Francesco Buscemi and Gilad Gour. Quantum relative lorenz curves. *Phys. Rev. A*, 95:012110, Jan 2017. URL: https://link.aps.org/doi/10.1103/PhysRevA.95.012110, doi:10.1103/PhysRevA.95.012110.

[14] Francesco Buscemi. All entangled quantum states are nonlocal. *Phys. Rev. Lett.*, 108:200401, May 2012. URL: https://link.aps.org/doi/10.1103/PhysRevLett.108.200401, doi:10.1103/PhysRevLett.108.200401.

[15] Francesco Buscemi. Fully quantum second-law–like statements from the theory of statistical comparisons, 2015. URL: https://arxiv.org/abs/1505.00535, doi:10.48550/ARXIV.1505.00535.

[16] Gilad Gour, David Jennings, Francesco Buscemi, Runyao Duan, and Iman Marvian. Quantum majorization and a complete set of entropic conditions for quantum thermodynamics. *Nature Communications*, 9(1):5352, 2018. doi:10.1038/s41467-018-06261-7.

[17] Francesco Buscemi, Eric Chitambar, and Wenbin Zhou. Complete resource theory of quantum incompatibility as quantum programmability. *Phys. Rev. Lett.*, 124:120401, Mar 2020. URL: https://link.aps.org/doi/10.1103/PhysRevLett.124.120401, doi:10.1103/PhysRevLett.124.120401.

[18] Francesco Buscemi, Kodai Kobayashi, Shintaro Minagawa, Paolo Perinotti, and Alessandro Tosini. Unifying different notions of quantum incompatibility into a strict hierarchy of resource theories of communication, 2022. URL: https://arxiv.org/abs/2211.09226, doi:10.48550/ARXIV.2211.09226.

[19] Francesco Buscemi, Kodai Kobayashi, and Shintaro Minagawa. A complete and operational resource theory of measurement sharpness, 2023. arXiv:2303.07737.

[20] Peter M. Alberti and Armin Uhlmann. *Stochasticity and Partial Order Doubly Stochastic Maps and Unitary Mixing*. Springer Netherlands, 1982.

[21] Michele Dall'Arno, Francesco Buscemi, and Valerio Scarani. Extension of the Alberti-Ulhmann criterion beyond qubit dichotomies. *Quantum*, 4:233, February 2020. doi:10.22331/q-2020-02-20-233.

[22] Zhen-Peng Xu, René Schwonnek, and Andreas Winter. Bounding the joint numerical range of pauli strings by graph parameters, 2023. arXiv:2308.00753.

[23] Wojciech Słomczyński and Anna Szymusiak. Morphophoric POVMs, generalised qplexes, and 2-designs. *Quantum*, 4:338, September 2020. doi:10.22331/q-2020-09-30-338.

[24] Francesco Buscemi, Michael Keyl, Giacomo Mauro D'Ariano, Paolo Perinotti, and Reinhard F. Werner. Clean positive operator valued measures. *Journal of Mathematical Physics*, 46(8):082109, 2005. arXiv:https://doi.org/10.1063/1.2008996, doi:10.1063/1.2008996.

[25] Michele Dall'Arno, Sarah Brandsen, Francesco Buscemi, and Vlatko Vedral. Device-independent tests of quantum measurements. *Phys. Rev. Lett.*, 118:250501, Jun 2017. URL: https://link.aps.org/doi/10.1103/PhysRevLett.118.250501, doi:10.1103/PhysRevLett.118.250501.

[26] Xuemei Chen, Gitta Kutyniok, Kasso A. Okoudjou, Friedrich Philipp, and Rongrong Wang. Measures of scalability. *IEEE Transactions on Information Theory*, 61(8):4410–4423, 2015. doi:10.1109/TIT.2015.2441071.

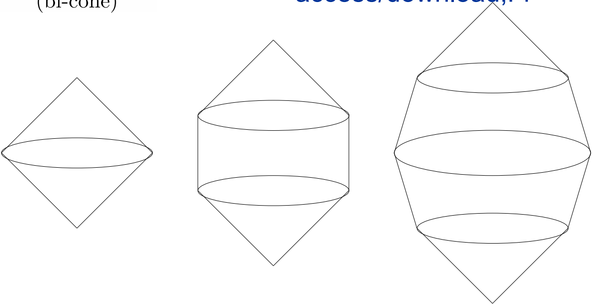
[27] Shayne FD Waldron. *An introduction to finite tight frames*. Springer, 2018.

[28] S.P. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004. URL: https://books.google.co.jp/books?id=mYm0bLd3fcoC.

[29] Keith Ball. Ellipsoids of maximal volume in convex bodies. *Geometriae Dedicata*, 41(2):241–250, 1992. doi:10.1007/BF00182424.

[30] Michele Dall'Arno. Device-independent tests of quantum states. *Phys. Rev. A*, 99:052353, May 2019. URL: https://link.aps.org/doi/10.1103/PhysRevA.99.052353, doi:10.1103/PhysRevA.99.052353.

$d = 2$ symmetric 2-cone (bi-cone) $d = 3$ symmetric 3-cone $d = 4$ symmetric 4-cone



Click here to access/download
LaTeX Souce Files
range.bbl

Click here to access/download

LaTeX Souce Files

range.tex