**Problem 6.18:** Let $M$ be a von Neumann algebra and $M_0 \subseteq M$ a subalgebra. Let $\varphi_1, \varphi_2$ be normal semifinite weights, we assume that $\varphi_2$ is faithful. Let $E : M \to M_0$ a faithful normal conditional expectation onto $M_0$. We show that the equality

$$[D(\varphi_1 \circ E) : D(\varphi_2 \circ E)]_t = [D\varphi_1 : D\varphi_2]_t, \qquad \forall t \in \mathbb{R} \tag{6.36}$$

holds.

It is known that (6.36) always holds if $\varphi_1$ is faithful. In general, let $e = s(\varphi_1)(\in M_0)$ and let $\varphi_0$ be a semifinite weight on $M_0$ such that $s(\varphi_0) = 1 - e$. Then $\varphi = \varphi_0 + \varphi_1$ is a faithful normal semifinite weight on $M_0$, such that $\varphi_1 = \varphi(e\cdot) = \varphi(\cdot e)$ and we have

$$[D\varphi_1 : D\varphi_2]_t = e[D\varphi : D\varphi_2]_t \qquad t \in \mathbb{R}.$$

We next show that we also have $s(\varphi_1 \circ E) = e$. Indeed, let $q = 1 - s(\varphi_1 \circ E)$, then $q$ is the largest projection in $M$ such that $\varphi_1 \circ E(q) = 0$. Since $\varphi_1 \circ E(1-e) = \varphi_1(1-e) = 0$, we obtain $1 - e \leq q$. On the other hand, since $e$ is the support projection of $\varphi_1$ and $\varphi_1(E(q)) = 0$, we obtain $E(q) \leq 1 - e \leq q$. Since $E$ is faithful, this implies $q = E(q) = 1 - e$, so that also $s(\varphi_1 \circ E) = e$.

The last equality implies that $\varphi_1 \circ E = \varphi \circ E(e\cdot) = \varphi \circ E(\cdot e)$, so that we have for all $t \in \mathbb{R}$:

$$[D(\varphi_1 \circ E) : D(\varphi_2 \circ E)]_t = e[D(\varphi \circ E) : D(\varphi_2 \circ E)]_t = e[D\varphi : D\varphi_2]_t = [D\varphi_1 : D\varphi_2]_t$$