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SANDWICHED RÉNYI RELATIVE ENTROPY ON DENSITY OPERATORS

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ABSTRACT. Relative entropies play important roles in classical and quantum information theory. In this paper, we first give the definition of sandwiched Rényi relative entropy for $\alpha \in (0,1)$ on $\mathcal{T}(H)^+$ (the cone of positive trace-class operators acting on an infinite-dimensional complex Hilbert space H); and then characterize all surjective maps preserving the sandwiched Rényi relative entropy on $\mathcal{T}(H)^+$. Such transformations have the form $T \mapsto cUTU^*$ for each $T \in \mathcal{T}(H)^+$, where $c > 0$ and U is either a unitary or an anti-unitary operator on H. Particularly, the definition of sandwiched Rényi relative entropy on the set of all density operators $\mathcal{S}(H)$ is given and all surjective maps preserving sandwiched Rényi relative entropy on $\mathcal{S}(H)$ are necessarily implemented by either a unitary or an anti-unitary operator.

1. INTRODUCTION

Relative entropies play important roles in classical information theory. For any two probability distributions $p = \{p(x)\}$ and $q = \{q(x)\}$ on the same index set $\{x\}$, their α-dependent Rényi relative entropy ($\alpha \in (0, +\infty)$ with $\alpha \neq 1$) is the quantity

$$D_\alpha(p\|q) = \frac{1}{\alpha - 1}\ln(\sum_x p(x)^\alpha q(x)^{1-\alpha}).$$

An important special case is the relative entropy $D(p\|q)$, where

$$D(p\|q) = \lim_{\alpha \to 1} D_\alpha(p\|q) = \sum_x p(x)\ln\frac{p(x)}{q(x)}.$$

These Rényi relative entropies have been found operational interpretations in these contexts in terms of error exponents or strong converse exponents, which, respectively, characterize the exponential rate at which error probabilities decay to zero or increase to one for a given information-processing task (for example, see [3, 4, 13]).

The above classical relative entropies can be generalized to quantum information theory setting. It is well known that a quantum mechanical system can be represented by a complex separable Hilbert space. As the non-commutative generalization of α-Rényi relative entropies,

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quantum Rényi divergence of positive operators in finite-dimensional systems is introduced in [7] and [16]. Assume that H is a finite-dimensional Hilbert space. Let $\mathcal{B}(H)$ be the algebra of all bounded linear operators on H and $\mathcal{B}(H)^+$ the cone of all positive semi-definite operators in $\mathcal{B}(H)$. An operator $\rho \in \mathcal{B}(H)^+$ is called a quantum state if its trace is 1, that is, $\mathrm{tr}\rho = 1$. Denote by $\mathcal{S}(H)$ the set of all quantum states in $\mathcal{B}(H)$. For any $T, S \in \mathcal{B}(H)^+$ with $T \neq 0$ and for $\alpha > 0$ with $\alpha \neq 1$, the quantum Rényi divergence of T and S is defined as follows:

$$D_\alpha(T\|S) = \begin{cases} \frac{1}{\alpha-1}\log((\mathrm{tr}T)^{-1}\mathrm{tr}(S^{\frac{1-\alpha}{2\alpha}}TS^{\frac{1-\alpha}{2\alpha}})^\alpha) & \text{if } \ T \not\perp S \wedge (\mathrm{supp}T \subseteq \mathrm{supp}S \vee \alpha < 1), \\ +\infty & \text{otherwise}, \end{cases}$$

where $\mathrm{supp}(T)$ is the support of T (that is, the span of eigenvectors of corresponding to nonzero eigenvalues) and the base of the logarithm is taken to 2. Particularly, in the case $T, S \in \mathcal{S}(H)$, the definition of quantum Rényi divergence is given independently in [1]. Some properties about sandwiched Rényi relative entropy are also studied in [1, 7, 16]. We point out here that this quantity is also called quantum sandwiched Rényi relative entropy ([1, 2, 8, 11, 16]), α-sandwiched Rényi divergence ([6]) and Rényi α-entropies ([10]). In this paper, we will use the terminology "sandwiched Rényi relative entropy".

On the other hand, preservers of sandwiched Rényi relative entropy attracted many authors's attention. Assume that $\mathcal{A}$ is a subalgebra of $\mathcal{B}(H)$ with $\dim H < \infty$. Recall that a map $\phi : \mathcal{A} \to \mathcal{A}$ is sandwiched Rényi relative entropy preserving if $D_\alpha(T\|S) = D_\alpha(\phi(T)\|\phi(S))$ for all $T, S \in \mathcal{A}$. In [5], Gaál proved that a map $\phi : \mathcal{S}(H) \to \mathcal{S}(H)$ preserves the sandwiched Rényi relative entropy if and only if $\phi(\rho) = U\rho U^*$ holds for all $\rho \in \mathcal{S}(H)$, where U is either a unitary or an antiunitary operator on H; and a bijective map ϕ on $\mathcal{B}(H)^{++}$ (the set of all positive invertible elements in $\mathcal{B}(H)$) preserves sandwiched Rényi relative entropy if and only if $\phi(T) = cUTU^*$ for every $T \in \mathcal{B}(H)^{++}$, where $c > 0$ and U is either a unitary or an antiunitary operator on H. Recently, Molnár in [9] generalized the above results to general finite C^*-algebras and showed that a surjective map Φ on positive invertible cone $\mathcal{A}_+^{-1}$ of C^*-algebra $\mathcal{A}$ is the sandwiched Rényi relative entropy preserving if and only if there are a central element $C \in \mathcal{A}_+^{-1}$ and a Jordan $*$-isomorphism $J : \mathcal{A}_+^{-1} \to \mathcal{A}_+^{-1}$ such that $\Phi(A) = CJ(A)$ for all $A \in \mathcal{A}_+^{-1}$. Here, a bijective linear map J from a $*$-algebra $\mathcal{A}$ into another $*$-algebra is called a Jordan $*$-isomorphism if it satisfies $J(AB + BA) = J(A)J(B) + J(B)J(A)$ and $J(A)^* = J(A^*)$ for all $A, B \in \mathcal{A}$.

In the present paper, we will first generalize the definition of the above sandwiched Rényi relative entropy to infinite-dimensional Hilbert spaces (see Section 2), and then give a characterization of maps preserving sandwiched Rényi relative entropy on density operators (see Section 3).

Finally, we fix some notions and notations. Assume that H is an infinite-dimensional Hilbert space. For any self-adjoint operators $A, B \in \mathcal{B}(H)$, $B \geq A$ if $B - A \geq 0$; $A \perp B$ if the range of A and the range of B are orthogonal. For nonzero vectors $x, y \in H$, the symbol $x \otimes y$ stands for the rank one operator in $\mathcal{B}(H)$ defined by $(x \otimes y)z = \langle z, y\rangle x$ for all $z \in H$. Let $\mathcal{K}(H)$ stand for the set of all compact operators in $\mathcal{B}(H)$ and $\mathcal{K}(H)^+$ the set of all positive operators in $\mathcal{K}(H)$. For $A \in \mathcal{K}(H)$, let $x_1, x_2, \cdots$ be the eigenvalues of $|A| = (A^\dagger A)^{\frac{1}{2}}$ in decreasing order and repeated according to multiplicity. A is said to be a trace-class operator if $\sum_i x_i < +\infty$, and in this case, the trace norm of A is defined as $\|A\|_1 = \sum_i x_i$. Denote by $\mathcal{T}(H)$ and $\mathcal{T}(H)^+$ the set of all trace-class operators and positive trace-class operators in $\mathcal{B}(H)$, respectively. Let $\mathcal{S}(H)$ be the set of all quantum states in $\mathcal{B}(H)$. A state $\rho \in \mathcal{S}(H)$ is called pure if $\mathrm{tr}(\rho^2) = 1$. It is clear that a pure state is a rank one projection.

2. Sandwiched Rényi relative entropy for infinite-dimensional quantum systems

Assume that H is an infinite-dimensional Hilbert space. For any $T, S \in \mathcal{T}(H)^+$ with $T \neq 0$ and any $\alpha \in (0,1)$, we define the sandwiched Rényi relative entropy of T and S by

$$D_\alpha(T\|S) = \frac{1}{\alpha - 1}\log((\mathrm{tr}T)^{-1}\mathrm{tr}(S^{\frac{1-\alpha}{2\alpha}}TS^{\frac{1-\alpha}{2\alpha}})^\alpha). \tag{2.1}$$

Note that $D_\alpha(T\|S)$ may be divergent. In fact, we have

$$D_\alpha(T\|S) = +\infty \Leftrightarrow T \perp S \tag{2.2}$$

and

$$D_\alpha(T\|S) = -\infty \Leftrightarrow (S^{\frac{1-\alpha}{2\alpha}}TS^{\frac{1-\alpha}{2\alpha}})^\alpha \notin \mathcal{T}(H)^+; \tag{2.3}$$

and for other cases, $D_\alpha(T\|S) < \infty$.

To see this, assume that $T, S \in \mathcal{T}(H)^+$ with $T \neq 0$. If $T \perp S$, then $TS = 0$. We claim that $TS^{\frac{1-\alpha}{2\alpha}} = 0$. In fact, if $\frac{1-\alpha}{2\alpha} \geq 1$, then it is obvious that $TS^{\frac{1-\alpha}{2\alpha}} = 0$. If $0 < \frac{1-\alpha}{2\alpha} < 1$, then $TST = (TS^{\frac{1}{2}})(TS^{\frac{1}{2}})^* = 0$, which implies $TS^{\frac{1}{2}} = 0$. Repeating this processing, one can get $TS^{\frac{1}{2^n}} = 0$ for any positive integer n. Now, by taking some suitable n such that $\frac{1}{2^n} < \frac{1-\alpha}{2\alpha}$, we have $TS^{\frac{1-\alpha}{2\alpha}} = 0$. So $\log\mathrm{tr}(S^{\frac{1-\alpha}{2\alpha}}TS^{\frac{1-\alpha}{2\alpha}})^\alpha = -\infty$, and hence $D_\alpha(T\|S) = +\infty$.

Conversely, if $D_\alpha(T\|S) = +\infty$, then $\mathrm{tr}(S^{\frac{1-\alpha}{2\alpha}}TS^{\frac{1-\alpha}{2\alpha}})^\alpha = 0$. If follows from the faithfulness of trace that $(S^{\frac{1-\alpha}{2\alpha}}TS^{\frac{1-\alpha}{2\alpha}})^\alpha = 0$, and so $S^{\frac{1-\alpha}{2\alpha}}TS^{\frac{1-\alpha}{2\alpha}} = 0$. As $(T^{\frac{1}{2}}S^{\frac{1-\alpha}{2\alpha}})^*(T^{\frac{1}{2}}S^{\frac{1-\alpha}{2\alpha}}) = S^{\frac{1-\alpha}{2\alpha}}TS^{\frac{1-\alpha}{2\alpha}} = 0$, one has $T^{\frac{1}{2}}S^{\frac{1-\alpha}{2\alpha}} = 0$. This means $TS^{\frac{1-\alpha}{2\alpha}} = 0$. Now, by a similar argument to that of the above, one can obtain $TS = 0$. Hence Eq.(2.2) holds.

Next, if $(S^{\frac{1-\alpha}{2\alpha}}TS^{\frac{1-\alpha}{2\alpha}})^\alpha \notin \mathcal{T}(H)^+$, then $\mathrm{tr}(S^{\frac{1-\alpha}{2\alpha}}TS^{\frac{1-\alpha}{2\alpha}})^\alpha = +\infty$ as $(S^{\frac{1-\alpha}{2\alpha}}TS^{\frac{1-\alpha}{2\alpha}})^\alpha$ is a positive operator, which, together with $\alpha \in (0,1)$, gives $D_\alpha(T\|S) = -\infty$. Conversely, if

$D_\alpha(T\|S) = -\infty$, then $\mathrm{tr}(S^{\frac{1-\alpha}{2\alpha}} T S^{\frac{1-\alpha}{2\alpha}})^\alpha = +\infty$. This means $(S^{\frac{1-\alpha}{2\alpha}} T S^{\frac{1-\alpha}{2\alpha}})^\alpha \notin \mathcal{T}(H)^+$. Hence Eq.(2.3) is true.

In addition, by Def.(2.1), the following properties are also apparent.

Proposition 2.1. $D_\alpha(\cdot\|\cdot)$ is unitary invariant, that is, for $T, S \in \mathcal{T}(H)^+$ with $T \neq 0$, we have $D_\alpha(T\|S) = D_\alpha(UTU^*\|USU^*)$ for all unitary operators $U \in \mathcal{B}(H)$.

Proposition 2.2. Assume that $T, S \in \mathcal{T}(H)^+$ with at least one finite rank operator. Then, for $\alpha \in (\frac{1}{2}, 1)$, if $T \geq S$, then $D_\alpha(T\|S) \geq 0$; if $T \leq S$, then $D_\alpha(T\|S) \leq 0$.

Proposition 2.3. Assume that $T, S_1, S_2 \in \mathcal{T}(H)^+$ with T finite rank operator. Then, for $\alpha \in (\frac{1}{2}, 1)$, $S_1 \geq S_2$ implies $D_\alpha(T\|S_1) \leq D_\alpha(T\|S_2)$.

3. Maps preserving sandwiched Rényi relative entropy on $\mathcal{S}(H)$

In this section, we will give a characterization of all maps preserving sandwiched Rényi relative entropy on $\mathcal{S}(H)$.

The following is our main result in this section.

Theorem 3.1. *Let H be a complex infinite-dimensional Hilbert space. Assume that $\phi : \mathcal{S}(H) \to \mathcal{S}(H)$ is a surjective map and $\alpha \in (0, 1)$ is any real number. Then ϕ preserves sandwiched Rényi relative entropy, that is, ϕ satisfies*

$$D_\alpha(\rho\|\sigma) = D_\alpha(\phi(\rho)\|\phi(\sigma)) \quad \text{for all} \quad \rho, \sigma \in \mathcal{S}(H)$$

if and only if there exists either a unitary or an anti-unitary operator $U : H \to H$ such that $\phi(\rho) = U\rho U^*$ for all $\rho \in \mathcal{S}(H)$.

In fact, we can obtain a more general result, that is, we can give a characterization of all maps preserving sandwiched Rényi relative entropy on positive trace-class operators $\mathcal{T}(H)^+$.

Theorem 3.2. *Let H be a complex infinite-dimensional Hilbert space. Assume that $\phi : \mathcal{T}(H)^+ \to \mathcal{T}(H)^+$ is a surjective map and $\alpha \in (0, 1)$ is any real number. Then ϕ satisfies*

$$D_\alpha(T\|S) = D_\alpha(\phi(T)\|\phi(S)) \quad \text{for all} \quad T, S \in \mathcal{T}(H)^+,$$

if and only if there exists a scalar $c > 0$ and either a unitary or an anti-unitary operator $U : H \to H$ such that $\phi(T) = cUTU^*$ for each $T \in \mathcal{T}(H)^+$.

It is enough to give a proof of Theorem 3.2. To do this, we first need several lemmas and propositions.

Denote by $\mathcal{P}_1(H)$ the set of all rank one projections (that is, pure states) in $\mathcal{B}(H)$. The first lemma is the celebrated Wigner's symmetry representation theorem.

Lemma 3.3. (see [12] or [14]) *Let H be a complex Hilbert space with* $\dim H > 1$ *and* $\varphi : \mathcal{P}_1(H) \to \mathcal{P}_1(H)$ *be a surjective map. If* φ *preserves the transition probability, i.e.,*

$$\mathrm{tr(PQ)} = \mathrm{tr}(\varphi(\mathrm{P})\varphi(\mathrm{Q})) \quad \text{for all } P, Q \in \mathcal{P}_1(H),$$

then there is an either unitary or anti-unitary operator U on H such that $\varphi(P) = UPU^*$ for all $P \in \mathcal{P}_1(H)$.

Lemma 3.4. ([15]) *Assume that H is a Hilbert space and $A, B \in \mathcal{K}(H)^+$ with $A \leq B$. Then, for any $r > 0$, we have* $\mathrm{tr}(A^r) \leq \mathrm{tr}(B^r)$.

The following proposition gives a necessary and sufficient condition for two positive trace-class operators preserving order relation.

Proposition 3.5. *Let H be a complex Hilbert space with* $\dim H > 1$. *Assume that $T, S \in \mathcal{T}(H)^+$ and $\alpha \in (0, 1)$. Then $T \leq S$ if and only if* $\mathrm{tr}((X^{\frac{1-\alpha}{2\alpha}} T X^{\frac{1-\alpha}{2\alpha}})^\alpha) \leq \mathrm{tr}((X^{\frac{1-\alpha}{2\alpha}} S X^{\frac{1-\alpha}{2\alpha}})^\alpha)$ *holds for all $X \in \mathcal{T}(H)^+$.*

Proof. Take any $T, S \in \mathcal{T}(H)^+$ and $\alpha \in (0, 1)$. If $T \leq S$, then

$$X^{\frac{1-\alpha}{2\alpha}} T X^{\frac{1-\alpha}{2\alpha}} \leq X^{\frac{1-\alpha}{2\alpha}} S X^{\frac{1-\alpha}{2\alpha}} \quad \text{for all} \quad X \in \mathcal{T}(H)^+.$$

Note that $X^{\frac{1-\alpha}{2\alpha}} T X^{\frac{1-\alpha}{2\alpha}}, X^{\frac{1-\alpha}{2\alpha}} S X^{\frac{1-\alpha}{2\alpha}} \in \mathcal{T}(H)^+$. By Lemma 3.4, it is obvious that

$$\mathrm{tr}((X^{\frac{1-\alpha}{2\alpha}} T X^{\frac{1-\alpha}{2\alpha}})^\alpha) \leq \mathrm{tr}((X^{\frac{1-\alpha}{2\alpha}} S X^{\frac{1-\alpha}{2\alpha}})^\alpha).$$

Now, assume that $\mathrm{tr}((X^{\frac{1-\alpha}{2\alpha}} T X^{\frac{1-\alpha}{2\alpha}})^\alpha) \leq \mathrm{tr}((X^{\frac{1-\alpha}{2\alpha}} S X^{\frac{1-\alpha}{2\alpha}})^\alpha)$ holds for all $X \in \mathcal{T}(H)^+$. Particularly, take $X = x \otimes x$, where $x \in H$ is any vector with $\|x\| = 1$. Then we have

$$\mathrm{tr}(((x \otimes x)T(x \otimes x))^\alpha) \leq \mathrm{tr}(((x \otimes x)S(x \otimes x))^\alpha).$$

A simple calculation gives $\langle Tx, x\rangle^\alpha \leq \langle Sx, x\rangle^\alpha$, which implies $\langle Tx, x\rangle \leq \langle Sx, x\rangle$ for each unit vector $x \in H$. So $T \leq S$.

The proof is completed. □

The following proposition gives a new characterization of rank one operators.

Proposition 3.6. *Let H be a complex Hilbert space with* $\dim H > 1$. *Assume that $T \in \mathcal{T}(H)^+$ and $\alpha \in (0, 1)$. Then the rank of T is 1 if and only if the following statements hold.*

(1) *The set $\mathcal{D} = \{A \in \mathcal{T}(H)^+ : (\frac{\mathrm{tr}T}{\mathrm{tr}A})^{\frac{1}{\alpha}} A \leq T\}$ is infinite.*

(2) *For any $A_1, A_2 \in \mathcal{D}$, $(\frac{\mathrm{tr}A_1}{\mathrm{tr}A_2})^{\frac{1}{\alpha}} A_2$ and A_1 are comparable with respect to the order $\leq$.*

Proof. Assume $T \in \mathcal{T}(H)^+$ and $\alpha \in (0, 1)$. If the rank of T is 1, T can be written as $T = t(x \otimes x)$ for some vector $x \in H$ with $\|x\| = 1$ and some $t > 0$. It is easily seen that the set $\mathcal{D} = \{A \in \mathcal{T}(H)^+ : (\frac{\mathrm{tr}T}{\mathrm{tr}A})^{\frac{1}{\alpha}} A \leq T\}$ is infinite and every element in $\mathcal{D}$ is rank one, that is, (1) holds. In addition, for any $A_1, A_2 \in \mathcal{D}$, write $A_1 = a_1(x \otimes x)$, $A_2 = a_2(x \otimes x)$. Then $a_1, a_2 \geq t$.

As $(\frac{\mathrm{tr}A_1}{\mathrm{tr}A_2})^{\frac{1}{\alpha}}A_2=a_1^{\frac{1}{\alpha}}a_2^{1-\frac{1}{\alpha}}(x\otimes x)$, it is clear that $a_1^{\frac{1}{\alpha}}a_2^{1-\frac{1}{\alpha}}(x\otimes x)$ and A_1 are comparable. So (2) also holds.

Now, suppose that (1) and (2) hold. On the contrary, if the rank of T is greater than 1, then under the space decomposition $H=N\oplus N^{\perp}$ (here, N is some invariant subspace of T), T has a matrix representation $T=\begin{pmatrix} T_1 & 0\\ 0 & T_2\end{pmatrix}$ with $T_1\in\mathcal{T}(N)^+$ and $T_2\in\mathcal{T}(N^{\perp})^+$. Take

$$A_1=2^{\frac{\alpha}{1-\alpha}}\begin{pmatrix} T_1 & T_1^{\frac{1}{2}}D_1T_2^{\frac{1}{2}}\\ T_2^{\frac{1}{2}}D_1^*T_1^{\frac{1}{2}} & T_2\end{pmatrix},$$

where $D_1\in\mathcal{B}(H)$ is a contraction operator, that is, $\|D_1\|\leq 1$. Then $A_1\in\mathcal{T}(H)^+$ and

$$T-(\frac{\mathrm{tr}T}{\mathrm{tr}A_1})^{\frac{1}{\alpha}}A_1=\frac{1}{2}\begin{pmatrix} T_1 & -T_1^{\frac{1}{2}}D_1T_2^{\frac{1}{2}}\\ -T_2^{\frac{1}{2}}D_1^*T_1^{\frac{1}{2}} & T_2\end{pmatrix}\geq 0.$$

Similarly, by taking

$$A_2=2^{\frac{\alpha}{1-\alpha}}\begin{pmatrix} T_1 & T_1^{\frac{1}{2}}D_2T_2^{\frac{1}{2}}\\ T_2^{\frac{1}{2}}D_2^*T_1^{\frac{1}{2}} & T_2\end{pmatrix}$$

with $D_2\in\mathcal{B}(H)$ another contraction operator, we have $A_2\in\mathcal{T}(H)^+$ and $T-(\frac{\mathrm{tr}T}{\mathrm{tr}A_2})^{\frac{1}{\alpha}}A_2\geq 0$. However, $(\frac{\mathrm{tr}A_i}{\mathrm{tr}A_j})^{\frac{1}{\alpha}}A_j$ and A_i are not comparable since

$$A_i-(\frac{\mathrm{tr}A_i}{\mathrm{tr}A_j})^{\frac{1}{\alpha}}A_j=2^{\frac{\alpha}{1-\alpha}}\begin{pmatrix} 0 & T_1^{\frac{1}{2}}(D_i-D_j)T_2^{\frac{1}{2}}\\ T_2^{\frac{1}{2}}(D_i^*-D_j^*)T_1^{\frac{1}{2}} & 0\end{pmatrix}$$

is not a positive operator ($i\neq j\in\{1,2\}$). So the rank of T is 1.

The proof is finished. □

Now, we are in a position to prove our main result.

Proof of Theorem 3.2. By Proposition 2.1, the "if" part is clear. For the "only if" part, we will show it by several claims.

In the sequel, we always assume that $\alpha\in(0,1)$ and $\phi:\mathcal{T}(H)^+\to\mathcal{T}(H)^+$ is a surjective map satisfying

$$D_\alpha(T\|S)=D_\alpha(\phi(T)\|\phi(S))\ \text{ for all }\ T,S\in\mathcal{T}(H)^+. \tag{3.1}$$

Claim 1. For any $T,S\in\mathcal{T}(H)^+$ with $T,S\neq 0$, we have

$$(\frac{\mathrm{tr}S}{\mathrm{tr}T})^{\frac{1}{\alpha}}T\leq S\ \text{ if and only if }\ (\frac{\mathrm{tr}\phi(S)}{\mathrm{tr}\phi(T)})^{\frac{1}{\alpha}}\phi(T)\leq\phi(S).$$

In fact, if $(\frac{\mathrm{tr}S}{\mathrm{tr}T})^{\frac{1}{\alpha}}T\leq S$, then $(\mathrm{tr}T)^{-\frac{1}{\alpha}}T\leq(\mathrm{tr}S)^{-\frac{1}{\alpha}}S$. By Proposition 3.5, one gets

$$\mathrm{tr}((X^{\frac{1-\alpha}{2\alpha}}(\mathrm{tr}T)^{-\frac{1}{\alpha}}TX^{\frac{1-\alpha}{2\alpha}})^{\alpha})\leq\mathrm{tr}((X^{\frac{1-\alpha}{2\alpha}}(\mathrm{tr}S)^{-\frac{1}{\alpha}}SX^{\frac{1-\alpha}{2\alpha}})^{\alpha}),$$

that is,

$$\frac{\mathrm{tr}((X^{\frac{1-\alpha}{2\alpha}}TX^{\frac{1-\alpha}{2\alpha}})^{\alpha})}{\mathrm{tr}T} \leq \frac{\mathrm{tr}((X^{\frac{1-\alpha}{2\alpha}}SX^{\frac{1-\alpha}{2\alpha}})^{\alpha})}{\mathrm{tr}S} \tag{3.2}$$

for all $X \in \mathcal{T}(H)^+$.

Next, we will prove that

$$D_\alpha(T\|X) \geq D_\alpha(S\|X) \text{ holds for all } X \in \mathcal{T}(H)^+. \tag{3.3}$$

Case 1.1. $\mathrm{tr}((X^{\frac{1-\alpha}{2\alpha}}SX^{\frac{1-\alpha}{2\alpha}})^{\alpha}) = +\infty$.

In this case, $D_\alpha(S\|X) = -\infty$. So $D_\alpha(T\|X) \geq D_\alpha(S\|X)$ is obvious.

Case 1.2. $0 < \mathrm{tr}((X^{\frac{1-\alpha}{2\alpha}}SX^{\frac{1-\alpha}{2\alpha}})^{\alpha}) < +\infty$.

By Ineq.(3.2), we have $0 \leq \mathrm{tr}((X^{\frac{1-\alpha}{2\alpha}}TX^{\frac{1-\alpha}{2\alpha}})^{\alpha}) < +\infty$. If $\mathrm{tr}((X^{\frac{1-\alpha}{2\alpha}}TX^{\frac{1-\alpha}{2\alpha}})^{\alpha}) = 0$, then $D_\alpha(T\|X) = +\infty$, and so $D_\alpha(T\|X) \geq D_\alpha(S\|X)$. If $0 < \mathrm{tr}((X^{\frac{1-\alpha}{2\alpha}}TX^{\frac{1-\alpha}{2\alpha}})^{\alpha}) < +\infty$, then both $D_\alpha(T\|X)$ and $D_\alpha(S\|X)$ are finite. Thus, by using the monotonicity of the logarithm function and the fact $\alpha \in (0,1)$, Ineq.(3.2) implies $D_\alpha(T\|X) \geq D_\alpha(S\|X)$.

Case 1.3. $\mathrm{tr}((X^{\frac{1-\alpha}{2\alpha}}SX^{\frac{1-\alpha}{2\alpha}})^{\alpha}) = 0$.

In this case, by Ineq.(3.2) and $\frac{\mathrm{tr}((X^{\frac{1-\alpha}{2\alpha}}TX^{\frac{1-\alpha}{2\alpha}})^{\alpha})}{\mathrm{tr}T} \geq 0$, we have $\mathrm{tr}((X^{\frac{1-\alpha}{2\alpha}}TX^{\frac{1-\alpha}{2\alpha}})^{\alpha}) = 0$. So $D_\alpha(S\|X) = D_\alpha(T\|X) = +\infty$.

Now, combining Cases 1.1-1.3, the relation Eq.(3.3) holds. Thus, Eq.(3.1) and Ineq.(3.3) yield

$$D_\alpha(\phi(T)\|\phi(X)) \geq D_\alpha(\phi(S)\|\phi(X)) \text{ for all } X \in \mathcal{T}(H)^+.$$

Since ϕ is surjective, the above inequality implies that

$$D_\alpha(\phi(T)\|X) \geq D_\alpha(\phi(S)\|X) \text{ holds for all } X \in \mathcal{T}(H)^+,$$

that is,

$$\mathrm{tr}((X^{\frac{1-\alpha}{2\alpha}}\mathrm{tr}(\phi(T))^{-\frac{1}{\alpha}}\phi(T)X^{\frac{1-\alpha}{2\alpha}})^{\alpha}) \leq \mathrm{tr}((X^{\frac{1-\alpha}{2\alpha}}\mathrm{tr}(\phi(S))^{-\frac{1}{\alpha}}\phi(S)X^{\frac{1-\alpha}{2\alpha}})^{\alpha})$$

holds for each $X \in \mathcal{T}(H)^+$. It follows from Proposition 3.5 that

$$\mathrm{tr}(\phi(T))^{-\frac{1}{\alpha}}\phi(T) \leq \mathrm{tr}(\phi(S))^{-\frac{1}{\alpha}}\phi(S).$$

So $(\frac{\mathrm{tr}\phi(S)}{\mathrm{tr}\phi(T)})^{\frac{1}{\alpha}}\phi(T) \leq \phi(S)$.

Conversely, if $(\frac{\mathrm{tr}\phi(S)}{\mathrm{tr}\phi(T)})^{\frac{1}{\alpha}}\phi(T) \leq \phi(S)$, then $\mathrm{tr}(\phi(T))^{-\frac{1}{\alpha}}\phi(T) \leq \mathrm{tr}(\phi(S))^{-\frac{1}{\alpha}}\phi(S)$. Again, by Proposition 3.5, one gets

$$\mathrm{tr}((X^{\frac{1-\alpha}{2\alpha}}\mathrm{tr}(\phi(T))^{-\frac{1}{\alpha}}\phi(T)X^{\frac{1-\alpha}{2\alpha}})^{\alpha}) \leq \mathrm{tr}((X^{\frac{1-\alpha}{2\alpha}}\mathrm{tr}(\phi(S))^{-\frac{1}{\alpha}}\phi(S)X^{\frac{1-\alpha}{2\alpha}})^{\alpha})$$

for all $X \in \mathcal{T}(H)^+$. Similar to the proof of Ineq.(3.3), one can show that

$$D_\alpha(\phi(T)\|X) \geq D_\alpha(\phi(S)\|X) \text{ holds for all } X \in \mathcal{T}(H)^+.$$

Note that $\phi(\mathcal{T}(H)^+) \subseteq \mathcal{T}(H)^+$. The above relation implies

$$D_\alpha(\phi(T)\|\phi(Y)) \geq D_\alpha(\phi(S)\|\phi(Y)) \text{ for all } Y \in \mathcal{T}(H)^+,$$

which and Eq.(3.1) yield

$$D_\alpha(T\|Y) \geq D_\alpha(S\|Y) \text{ for all } Y \in \mathcal{T}(H)^+.$$

By the definition, one gets $\mathrm{tr}((Y^{\frac{1-\alpha}{2\alpha}}(\mathrm{tr}T)^{-\frac{1}{\alpha}}TY^{\frac{1-\alpha}{2\alpha}})^\alpha) \leq \mathrm{tr}((Y^{\frac{1-\alpha}{2\alpha}}(\mathrm{tr}S)^{-\frac{1}{\alpha}}SY^{\frac{1-\alpha}{2\alpha}})^\alpha)$ for all $Y \in \mathcal{T}(H)^+$. By Proposition 3.5 again, we obtain $(\frac{\mathrm{tr}S}{\mathrm{tr}T})^{\frac{1}{\alpha}}T \leq S$.

Claim 2. ϕ is an injective map. Therefore, ϕ is a bijective map.

Assume that $\phi(T) = \phi(S)$ for some $T, S \in \mathcal{T}(H)^+$. It is clear that $(\frac{\mathrm{tr}\phi(S)}{\mathrm{tr}\phi(T)})^{\frac{1}{\alpha}}\phi(T) = \phi(S)$. Then, by using the relation $(\frac{\mathrm{tr}\phi(S)}{\mathrm{tr}\phi(T)})^{\frac{1}{\alpha}}\phi(T) \leq \phi(S)$ and Claim 1, one has

$$(\frac{\mathrm{tr}S}{\mathrm{tr}T})^{\frac{1}{\alpha}}T \leq S;$$

by the relation $(\frac{\mathrm{tr}\phi(S)}{\mathrm{tr}\phi(T)})^{\frac{1}{\alpha}}\phi(T) \geq \phi(S)$ (that is, $(\frac{\mathrm{tr}\phi(T)}{\mathrm{tr}\phi(S)})^{\frac{1}{\alpha}}\phi(S) \leq \phi(T)$) and Claim 1, one gets $(\frac{\mathrm{tr}T}{\mathrm{tr}S})^{\frac{1}{\alpha}}S \leq T$, that is,

$$(\frac{\mathrm{tr}S}{\mathrm{tr}T})^{\frac{1}{\alpha}}T \geq S.$$

The above two relations imply $(\frac{\mathrm{tr}T}{\mathrm{tr}S})^{\frac{1}{\alpha}}S = T$. By taking the trace on this equation, we obtain $\mathrm{tr}T = \mathrm{tr}S$. So $T = S$. Hence ϕ is injective.

Claim 3. ϕ preserves rank-one operators in both directions.

By the bijectivity of ϕ, we only need to prove that ϕ preserves rank-one operators. To do this, take any rank-one operator $T \in \mathcal{T}(H)^+$. By Proposition 3.6, $\{A \in \mathcal{T}(H)^+ : (\frac{\mathrm{tr}T}{\mathrm{tr}A})^{\frac{1}{\alpha}}A \leq T\}$ is an infinite set and the operators $\frac{\mathrm{tr}A_1}{\mathrm{tr}A_2}A_2$, A_1 are comparable for any A_1, A_2. It follows from Claims 1-2 that $\{\phi(A) \in \mathcal{T}(H)^+ : (\frac{\mathrm{tr}\phi(T)}{\mathrm{tr}\phi(A)})^{\frac{1}{\alpha}}\phi(A) \leq \phi(T)\}$ is also an infinite set and that $\frac{\mathrm{tr}\phi(A_1)}{\mathrm{tr}\phi(A_2)}\phi(A_2)$ and $\phi(A_1)$ are comparable for any $\phi(A_1), \phi(A_2)$. By Proposition 3.6 again, $\phi(T)$ is a rank-one operator.

Claim 4. There exists a real number $c > 0$ such that, for any unit vector $x \in H$, we have $\phi(x \otimes x) = cy \otimes y$ for some unit vector $y \in H$.

For any vector $x \in H$ with $\|x\| = 1$, by Claim 3, there is a scalar $c_x > 0$ and a unit vector $y \in H$ such that

$$\phi(x \otimes x) = c_x y \otimes y.$$

To complete the proof of the claim, one only needs to check that c_x is independent with x. Pick any unit vectors $x_1, x_2 \in H$ and write

$$\phi(x_1 \otimes x_1) = c_{x_1}(y_1 \otimes y_1) \text{ and } \phi(x_2 \otimes x_2) = c_{x_2}(y_2 \otimes y_2)$$

for some $c_{x_1}, c_{x_2} > 0$ and some vectors y_1, $y_2 \in H$ with $\|y_1\| = \|y_2\| = 1$.

If $\langle x_1, x_2\rangle \neq 0$, by Eq.(2.2), one gets $\langle y_1, y_2\rangle \neq 0$. Note that

$$D_\alpha(x_1\otimes x_1\|x_2\otimes x_2) = \frac{1}{\alpha-1}\log|\langle x_2, x_1\rangle|^{2\alpha}, \tag{3.4}$$

$$D_\alpha(x_2\otimes x_2\|x_1\otimes x_1) = \frac{1}{\alpha-1}\log|\langle x_1, x_2\rangle|^{2\alpha}, \tag{3.5}$$

$$D_\alpha(\phi(x_1\otimes x_1)\|\phi(x_2\otimes x_2)) = \frac{1}{\alpha-1}\log c_{x_1}^{\alpha-1}c_{x_2}^{1-\alpha}|\langle y_2, y_1\rangle|^{2\alpha} \tag{3.6}$$

and

$$D_\alpha(\phi(x_2\otimes x_2)\|\phi(x_1\otimes x_1)) = \frac{1}{\alpha-1}\log c_{x_2}^{\alpha-1}c_{x_1}^{1-\alpha}|\langle y_1, y_2\rangle|^{2\alpha}. \tag{3.7}$$

Combining Eq.(3.1) and Eqs.(3.4)-(3.7), we obtain

$$|\langle x_2, x_1\rangle|^{2\alpha} = c_{x_1}^{\alpha-1}c_{x_2}^{1-\alpha}|\langle y_2, y_1\rangle|^{2\alpha} \text{ and } |\langle x_1, x_2\rangle|^{2\alpha} = c_{x_2}^{\alpha-1}c_{x_1}^{1-\alpha}|\langle y_1, y_2\rangle|^{2\alpha}.$$

These imply $c_{x_1}^{\alpha-1}c_{x_2}^{1-\alpha} = c_{x_2}^{\alpha-1}c_{x_1}^{1-\alpha}$, that is, $(\frac{c_{x_1}}{c_{x_2}})^{1-\alpha} = (\frac{c_{x_2}}{c_{x_1}})^{1-\alpha}$. So $c_{x_1} = c_{x_2}$.

If $\langle x_1, x_2\rangle = 0$, then there is a unit vector $x_3 \in H$ such that $\langle x_3, x_1\rangle \neq 0$ and $\langle x_3, x_2\rangle \neq 0$. Let $\phi(x_3\otimes x_3) = c_{x_3}(y_3\otimes y_3)$ for some $c_{x_3} > 0$ and some unit vector $y_3 \in H$. By the above discussion, one gets $c_{x_1} = c_{x_3} = c_{x_2}$.

The proof of the claim is completed.

Now, define a map $\Phi : \mathcal{T}(H)^+ \to \mathcal{T}(H)^+$ by

$$\Phi(T) = c^{-1}\phi(T) \text{ for each } T \in \mathcal{T}(H)^+.$$

Clearly, Φ is a bijective map and preserves sandwiched Rényi relative entropy. In addition, by Claim 4, Φ preserves rank-one projections in both directions.

Claim 5. There exists either a unitary or an anti-unitary operator $U : H \to H$ such that $\Phi(P) = UPU^*$ for all rank one projections $P \in \mathcal{P}_1(H)$.

Take any rank one projections $P_1, P_2 \in \mathcal{P}_1(H)$, and write $P_1 = x_1\otimes x_1$ and $P_2 = x_2\otimes x_2$ with unit vectors $x_1, x_2 \in H$. We have

$$\begin{aligned} D_\alpha(P_1\|P_2) = & \tfrac{1}{\alpha-1}\log\operatorname{tr}(((x_2\otimes x_2)^{\frac{1-\alpha}{2\alpha}}(x_1\otimes x_1)(x_2\otimes x_2)^{\frac{1-\alpha}{2\alpha}})^\alpha) \\ = & \tfrac{1}{\alpha-1}\log\operatorname{tr}(((x_2\otimes x_2)(x_1\otimes x_1)(x_2\otimes x_2))^\alpha) \\ = & \tfrac{1}{\alpha-1}\log(\langle x_1, x_2\rangle^\alpha\langle x_2, x_1\rangle^\alpha) = \tfrac{1}{\alpha-1}\log(\operatorname{tr}(P_1P_2))^\alpha. \end{aligned} \tag{3.8}$$

Since $\Phi(P_1)$ and $\Phi(P_2)$ are also rank-one projections, a similar argument to that of Eq.(3.8) gives

$$D_\alpha(\Phi(P_1)\|\Phi(P_2)) = \frac{1}{\alpha-1}\log(\operatorname{tr}(\Phi(P_1)\Phi(P_2)))^\alpha. \tag{3.9}$$

Comparing Eqs.(3.8)-(3.9) and by Eq.(3.1), we obtain

$$\operatorname{tr}(P_1P_2) = \operatorname{tr}(\Phi(P_1)\Phi(P_2)) \text{ for all } P_1, P_2 \in \mathcal{P}_1(H).$$

Thus, by Lemma 3.3, there exists either a unitary or anti-unitary operator U on H such that $\Phi(P) = UPU^*$ for all $P \in \mathcal{P}_1(H)$.

Claim 6. There exists either a unitary or an anti-unitary operator $U : H \to H$ such that $\Phi(T) = UTU^*$ for all $T \in \mathcal{T}(H)^+$.

For any $T \in \mathcal{T}(H)^+$ and any rank-one projection $x \otimes x \in \mathcal{P}_1(H)$, by Claim 5, we have

$$D_\alpha(x \otimes x \| T) = \frac{1}{\alpha - 1} \log \operatorname{tr}((T^{\frac{1-\alpha}{2\alpha}} x \otimes T^{\frac{1-\alpha}{2\alpha}} x)^\alpha)$$

and

$$D_\alpha(\Phi(x \otimes x) \| \Phi(T)) = \frac{1}{\alpha - 1} \log \operatorname{tr}((\Phi(T)^{\frac{1-\alpha}{2\alpha}} Ux \otimes \Phi(T)^{\frac{1-\alpha}{2\alpha}} Ux)^\alpha).$$

These and Eq.(3.1) yield

$$\operatorname{tr}((T^{\frac{1-\alpha}{2\alpha}} x \otimes T^{\frac{1-\alpha}{2\alpha}} x)^\alpha) = \operatorname{tr}((\Phi(T)^{\frac{1-\alpha}{2\alpha}} Ux \otimes \Phi(T)^{\frac{1-\alpha}{2\alpha}} Ux)^\alpha),$$

which implies $\langle T^{\frac{1-\alpha}{2\alpha}} x, T^{\frac{1-\alpha}{2\alpha}} x \rangle^\alpha = \langle \Phi(T)^{\frac{1-\alpha}{2\alpha}} Ux, \Phi(T)^{\frac{1-\alpha}{2\alpha}} Ux \rangle^\alpha$. So

$$\langle T^{\frac{1-\alpha}{2\alpha}} x, T^{\frac{1-\alpha}{2\alpha}} x \rangle = \langle \Phi(T)^{\frac{1-\alpha}{2\alpha}} Ux, \Phi(T)^{\frac{1-\alpha}{2\alpha}} Ux \rangle \ \text{ holds for all unit vectors } \ x \in H,$$

that is,

$$\langle (T^{\frac{1-\alpha}{\alpha}} - U^*\Phi(T)^{\frac{1-\alpha}{\alpha}} U)x, x \rangle = 0 \ \text{ holds for all unit vectors } \ x \in H. \tag{3.10}$$

Note that $T^{\frac{1-\alpha}{\alpha}} - U^*\Phi(T)^{\frac{1-\alpha}{\alpha}} U$ is a self-adjoint operator. It follows from Eq.(3.10) that $U^*\Phi(T)^{\frac{1-\alpha}{\alpha}} U - T^{\frac{1-\alpha}{\alpha}} = 0$, and so $\Phi(T) = UTU^*$ for all $T \in \mathcal{T}(H)^+$.

Now, by the definition of Φ, one achieves that $\phi(T) = cUTU^*$ holds for all $T \in \mathcal{T}(H)^+$. The proof of the theorem is finished. □

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