Haagerup reduction, monotonicity in z, limit $\alpha \neq 1$

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December 12, 2023

1 Haagerup reduction and L_p-spaces

Haagerup reduction theorem [**?** , Thm. 2.1] says that there is a von Neumann algebra $\mathcal{R}$ with a faithful normal state ϕ and a sequence of von Neumann algebras $(\mathcal{R}_n)_{n\geq 1}$ such that

(i) $\mathcal{M} \subseteq \mathcal{R}$ and there is a conditional expectation $\mathcal{E}$ on $\mathcal{R}$ onto $\mathcal{M}$ such that $\phi \circ \mathcal{E} = \phi$,

(ii) $\mathcal{R}_n \subseteq \mathcal{R}_{n+1}$ and each $\mathcal{R}_n$ is finite,

(iii) $\bigcup_n \mathcal{R}_n$ is w*-dense in $\mathcal{R}$,

(iv) for each n there is a conditional expectation on $\mathcal{R}$ onto $\mathcal{R}_n$ such that $\phi \circ \mathcal{E}_n = \phi$.

This can be applied to Haagerup L_p-spaces $L_p(\mathcal{M})$ as follows, [**?** , Thm. 3.1], we will closely follow [**?** , Sec. 2] Let $\sigma = \sigma^\phi$, then σ_t preserves $\mathcal{M}$ by the Takesaki theorem. We then have $\sigma^{\phi|_\mathcal{M}} = \sigma|_\mathcal{M}$ and the crossed product $\mathcal{N}_\mathcal{M} := \mathcal{M} \rtimes_{\sigma^{\phi|_\mathcal{M}}} \mathbb{R}$ can be identified with a subalgebra in $\mathcal{N}_\mathcal{R} := \mathcal{R} \rtimes_\sigma \mathbb{R}$. The dual action $\hat\sigma_\mathcal{M}$ on $\mathcal{N}_\mathcal{M}$ is the restriction of the dual action $\hat\sigma$ on $\mathcal{N}_\mathfrak{R}$. It follows that the dual weight $\hat\phi_\mathcal{M}$ is the restriction of $\hat\phi$ and that the canonical trace $\tau_\mathcal{M}$ is the restriction of the canonical trace τ. It follows that the space of $\tau_\mathcal{M}$-measurable elements $L_0(\mathcal{M})$ can be identified with a subspace in $L_0(\mathcal{R})$ and similarly $L_p(\mathcal{M}) \subseteq L_p(\mathcal{R})$, $0 < p \leq \infty$.

By [**?** , Prop. 2.3], for $1 \leq p \leq \infty$, $\mathcal{E}$ can be extended to a contractive projection of $L_p(\mathcal{R})$ onto $L_p(\mathcal{M})$. In particular, for $p = 1$ and $h_\omega \in L_1(\mathcal{R})$ we obtain

$$\mathcal{E}_1(h_\omega) = h_{\omega\circ\mathcal{E}}.$$

Moreover, for $1 \leq p, q, r, s \leq \infty$ such that $1/p + 1/q + 1/r = 1/s$, we have

$$\mathcal{E}_s(hxk) = h\mathcal{E}_r(x)k, \qquad h \in L_p(\mathcal{M}),\ k \in L_q(\mathcal{M}),\ x \in L_r(\mathcal{R}).$$

Now let $\psi \in \mathcal{M}_*^+$, then in the above identification $L_1(\mathcal{M}) \subseteq L_1(\mathcal{R})$, we get $h_\psi \equiv h_{\psi\circ\mathcal{E}} = h_{\hat\psi}$. Assume that $\frac{1}{2} < \frac{\alpha}{2} \leq z$ and $\psi, \varphi\mathcal{M}_*^+$, we will also assume for simplicity that φ is faithful. Suppose that $Q_{\alpha,z}(\psi\|\varphi) < \infty$, then there is some $y \in L_{2z}(\mathcal{M})$ such that

$$h_\psi^{\frac{\alpha}{2z}} = y h_\varphi^{\frac{\alpha-1}{2}z}.$$

Since $h_\psi = h_{\hat\psi}$, $h_\varphi = h_{\hat\varphi}$ and $y \in L_{2z}(\mathcal{M}) \subseteq L_{2z}(\mathcal{R})$, we obtain that

$$Q_{\alpha,z}(\hat\psi\|\hat\varphi) = \|y\|_{2z}^{2z} = Q_{\alpha,z}(\psi\|\varphi).$$

Conversely, assume that $Q_{\alpha,z}(\hat{\psi}\|\hat{\varphi}) < \infty$, so there is some $\hat{y} \in L_{2z}(\mathcal{R})$ such that

$$h_{\hat{\psi}}^{\frac{\alpha}{2z}} = y h_{\hat{\varphi}}^{\frac{\alpha-1}{2z}}.$$

Then we have

$$h_{\psi}^{\frac{\alpha}{2z}} = h_{\hat{\psi}}^{\frac{\alpha}{2z}} = \mathcal{E}_{\frac{2z}{\alpha}}(h_{\hat{\psi}}^{\frac{\alpha}{2z}}) = \mathcal{E}_{2z}(\hat{y}) h_{\hat{\varphi}}^{\frac{\alpha-1}{2z}} = \mathcal{E}_{2z}(\hat{y}) h_{\varphi}^{\frac{\alpha-1}{2z}}$$

1.1 DPI

This is proved for

$$\alpha \in (0,1),\ \max\{\alpha, 1-\alpha\} \le z \qquad \text{and} \qquad \alpha > 1,\ \max\{\alpha/2, \alpha-1\} \le z \le \alpha$$

See [5, Thm. 1(viii)], [4]. It is no possible to go beyond these bounds, [9].

1.2 Lower semicontinuity (LS)

Holds for $\alpha \in (0,1)$ and for $\alpha > 1$, $z \ge \alpha/2$, [5].

1.3 Variational expressions

For $\alpha \in (0,1)$, $z \ge \max\{\alpha, 1-\alpha\}$, we have [5, Theorem 1(vi)]

$$Q_{\alpha,z}(\psi\|\varphi) = \inf_{a \in \mathcal{M}_{++}} \left\{ \alpha \mathrm{Tr}\left((a^{1/2} h_{\psi}^{\alpha/z} a^{1/2})^{z/\alpha} \right) + (1-\alpha)\mathrm{Tr}\left(a^{-1/2} h_{\varphi}^{(1-\alpha)/z} a^{-1/2} \right) \right\}.$$

For $\alpha > 1$, $z \ge \alpha/2$, we have [5, Theorem 2(vi)] and [6]

$$Q_{\alpha,z}(\psi\|\varphi) = \sup_{a \in \mathcal{M}_{+}} \left\{ \alpha \mathrm{Tr}\left((a^{1/2} h_{\psi}^{\alpha/z} a^{1/2})^{z/\alpha} \right) - (\alpha-1)\mathrm{Tr}\left(a^{1/2} h_{\varphi}^{(\alpha-1)/z} a^{1/2} \right) \right\}.$$

The lower bound comes from the lower bound in LS. Proved for all $z > 0$ in type I case [8].

1.4 Martingale convergence

Holds in the bounds for DPI, [3]. A remark to this proof: I think that the proof of [3, Eq. (0.4)] can be simplified. The key ingredient here is the martingale convergence of the generalized conditional expectations, which gives [3, Eqs. (0.5)]

$$\psi_i \circ \mathcal{E}_{\mathcal{M}_i,\varphi} \to \psi \qquad \text{in norm.}$$

(We denote $\varphi_i := \varphi|_{\mathcal{M}_i}$ and $\psi_i = \psi|_{\mathcal{M}_i}$.) In the bounds for DPI, we also have LS, so that

$$D_{\alpha,z}(\psi_i\|\varphi_i) \ge D_{\alpha,z}(\psi_i \circ \mathcal{E}_{\mathcal{M}_i,\varphi}\|\varphi_i \circ \mathcal{E}_{\mathcal{M}_i,\varphi})$$

and (using also [3, Eq.(0.5)])

$$\sup_i D_{\alpha,z}(\psi_i\|\varphi_i) \ge \liminf D_{\alpha,z}(\psi_i \circ \mathcal{E}_{\mathcal{M}_i,\varphi}\|\varphi_i \circ \mathcal{E}_{\mathcal{M}_i,\varphi}) \ge D_{\alpha,z}(\psi\|\varphi)$$

1.5 Monotonicity in z

For $\alpha > 1$, monotonicity is as

$$0 < z \leq z' \implies D_{\alpha,z}(\psi\|\varphi) \geq D_{\alpha,z'}(\psi\|\varphi).$$

This was proved in [2, Sec. 3] in type II_1 algebras, under the condition of lower semicontinuity of the map $\varphi \mapsto D_{\alpha,z'}(\psi\|\varphi)$, in particular, it holds for $\alpha/2 \leq z'$. Indeed, by the proof in [2, Sec. 3], we have for $z \leq z'$ and all $\varepsilon > 0$

$$Q_{\alpha,z}(\psi\|\varphi) \geq Q_{\alpha,z}(\psi\|\varphi + \varepsilon\tau) \geq Q_{\alpha,z'}(\psi\|\varphi + \varepsilon\tau),$$

so that the inequality follows by LS for z'.

Extension to the general case can be done using Haagerup reduction. For this, we so far need to assume that

$$\alpha - 1, \alpha/2 \leq z \leq z' \leq \alpha.$$

Let $\hat{\mathcal{M}}$, $\hat{\mathcal{M}}_n$, $\hat{\psi}$, $\hat{\varphi}$ be as in the Haagerup reduction and put $\psi_n = \hat{\psi}|_{\mathcal{M}_n}$, $\varphi_n = \hat{\varphi}|_{\mathcal{M}_n}$. Using [2, Sec. 3], we have for $z \leq z'$ and all n

$$D_{\alpha,z'}(\psi_n\|\varphi_n) \leq D_{\alpha,z}(\psi_n\|\varphi_n).$$

Using DPI and LS, we obtain

$$D_{\alpha,z'}(\psi\|\varphi) = D_{\alpha,z'}(\hat{\psi}\|\hat{\varphi}) \leq \liminf_n D_{\alpha,z'}(\psi_n\|\varphi_n) \leq \liminf_n D_{\alpha,z}(\psi_n\|\varphi_n) \leq D_{\alpha,z}(\hat{\psi}\|\hat{\varphi}) = D_{\alpha,z}(\psi\|\varphi).$$

It would be very useful if we could extend the use of Haagerup reduction beyond these bounds, see for example Section 2.2 below.

2 Some further results and remarks

2.1 Convergence of $D_{\alpha,z}$ as $\alpha \nearrow 1$

Using monotonicity in z, we can prove this for $z \leq 1$.

Lemma 1. *Assume that $0 \leq 1 - z < \alpha < 1$. Then for any normal state ψ,*

$$D_{\beta,1}(\psi\|\varphi) \leq D_{\alpha,z}(\psi\|\varphi) \leq D_{\alpha,1}(\psi\|\varphi),$$

where $\beta = \frac{\alpha-1+z}{z}$.

Proof. The statement is trivial for $z = 1$, so we may assume $0 < 1 - z$. The second inequality follows by monotonicity of $z \mapsto D_{\alpha,z}(\psi\|\varphi)$ for $\alpha \in (0,1)$. For the first inequality, note first that by the assumption, $\beta \in (0,1)$ and by Hölder

$$Q_{\alpha,z}(\psi\|\varphi)^{\frac{1}{2z}} = \|h_\psi^{\frac{\alpha}{2z}} h_\varphi^{\frac{1-\alpha}{2z}}\|_{2z} = \|h_\psi^{\frac{1-z}{2z}} h_\psi^{\frac{\beta}{2}} h_\varphi^{\frac{1-\beta}{2}}\|_{2z} \leq \|h_\psi^{\frac{\beta}{2}} h_\varphi^{\frac{1-\beta}{2}}\|_2 = Q_{\beta,1}(\psi\|\varphi)^{\frac{1}{2}}.$$

This proves the statement. □

Using the lemma for $1 - \alpha$ small enough and properties of $D_{\alpha,1}$, we get for any $z \leq 1$:

$$\lim_{\alpha \nearrow 1} D_{\alpha,z}(\psi\|\varphi) = D(\psi\|\varphi).$$

2.2 Convergence of $D_{\alpha,z}$ as $\alpha \searrow 1$

The strategy of the previous section is limited by the bounds for which we currently have monotonicity in z for $\alpha > 1$. Namely, the inequality $D_{\alpha,z}(\psi\|\varphi) \le D_{\alpha,1}(\psi\|\varphi)$ is currently only proved for

$$1 \le z \le \alpha \le 2,$$

which is violated as $\alpha \searrow 1$ (unless $z = 1$ or $z = \alpha$). We have the following lower bound.

Lemma 2. *Let $1 < \alpha \le z$. Then for any normal state ψ, we have*

$$D_{\beta,1}(\psi\|\varphi) \le D_{\alpha,z}(\psi\|\varphi),$$

where $\beta = \frac{\alpha-1+z}{z}$.

Proof. Assume that $D_{\alpha,z}(\psi\|\varphi) < \infty$, otherwise there is nothing to prove. Then there is some $y \in L_{2z}(\mathcal{M})$ such that

$$h_\psi^{\frac{\alpha}{2z}} = y h_\varphi^{\frac{\alpha-1}{2z}}, \qquad Q_{\alpha,z}(\psi\|\varphi) = \|y\|_{2z}^{2z}.$$

Since $\alpha \le z$, we have $\frac{1}{2} - \frac{\alpha}{2z} \ge 0$ and

$$h_\psi^{1/2} = h_\psi^{\frac{1}{2}-\frac{\alpha}{2z}} y h_\varphi^{\frac{\alpha-1}{2z}}.$$

It follows that $h_\psi^{1/2} \in \mathcal{D}(\Delta_{\psi,\varphi}^{\frac{\alpha-1}{2z}})$ and

$$\Delta_{\psi,\varphi}^{\frac{\alpha-1+z}{2z}} h_\varphi^{1/2} = \Delta_{\psi,\varphi}^{\frac{\alpha-1}{2z}} \Delta_{\psi,\varphi}^{1/2} h_\varphi^{1/2} = \Delta_{\psi,\varphi}^{\frac{\alpha-1}{2z}} h_\psi^{1/2} = h_\psi^{\frac{z-1}{2z}} y.$$

It follows that

$$\|\Delta_{\psi,\varphi}^{\frac{\alpha-1+z}{2z}} h_\varphi^{1/2}\|_2 \le \psi(1)^{\frac{z-1}{2z}} \|y\|_{2z}.$$

We therefore have for $\psi(1) = 1$,

$$D_{\alpha,z}(\psi\|\varphi) = \frac{1}{\alpha-1}\log\|y\|_{2z}^{2z} \ge \frac{z}{\alpha-1}\log\|\Delta_{\psi,\varphi}^{\frac{\alpha-1+z}{2z}} h_\varphi^{1/2}\|_2^2 = D_{\frac{\alpha-1+z}{z},1}(\psi\|\varphi).$$

□

2.3 Complex interpolation

This is to remark that for $\alpha > 1$ and $z \ge \alpha/2$, $Q_{\alpha,z}(\psi\|\phi)$ can be written using the Kosaki interpolation norms. Assume that both ψ and φ are faithful and $Q_{\alpha,z}(\psi\|\phi) < \infty$. We then have

$$h_\psi = h_\psi^{1-\frac{\alpha}{2z}} y h_\varphi^{\frac{\alpha-1}{2z}} = h_\psi^{\eta/q} y h_\varphi^{(1-\eta)/q}$$

for some $y \in L_{2z}(\mathcal{M})$, $q = \frac{2z}{2z-1}$ (the dual parameter to $2z$) and $\eta = \frac{2z-\alpha}{2z-1} \in [0,1)$. Hence h_ψ belongs to the space $L_{2z}^\eta(\mathcal{M},\psi,\varphi)$, where

$$L_p^\eta(\mathcal{M},\psi,\varphi) := C_{1/p}(h_\psi^{\frac{\eta}{2}} \mathcal{M} h_\varphi^{\frac{1-\eta}{2}}, \mathcal{M}_*) = C_\eta(L_p(\mathcal{M},\varphi)_L, L_p(\mathcal{M},\psi)_R),$$

[7, Thm. 11.1]. Let $\|\cdot\|_{2z,\psi,\varphi,\eta}$ denote the norm in this space, then it is easily seen that

$$Q_{\alpha,z}(\psi\|\phi) = \|y\|_{2z}^{2z} = \|h_\psi\|_{2z,\psi,\varphi,\eta}^{2z}.$$

We may be able to use the fact that $L_p^\eta(\mathcal{M},\psi,\varphi)$ form an interpolating family with respect to both p and η to prove some results, e.g. monotonicity in z or in α. It may be possible to extend this for other values of α and z using [1].

References

[1] J. Gu, Z. Yin, and H. Zhang. Interpolation of quasi noncommutative L_p-spaces. *arXiv:1905.08491*, 2019.

[2] F. Hiai, Questions, note.

[3] F. Hiai, Martingale convergence for $D_{\alpha,z}$, note.

[4] A. Jenčová, DPI for $\alpha - z$-Rényi divergence, note.

[5] S. Kato. On α-z-Rényi divergence in the von Neumann algebra setting. *arXiv preprint arXiv:2311.01748*, 2023.

[6] S. Kato, Variational expression for $\alpha > 1$, note.

[7] H. Kosaki. Applications of the complex interpolation method to a von Neumann algebra: Non-commutative L_p-spaces. *J. Funct. Anal.*, 56:26–78, 1984.

[8] M. Mosonyi. The strong converse exponent of discriminating infinite-dimensional quantum states. *Communications in Mathematical Physics*, 400(1):83–132, 2023.

[9] H. Zhang. From Wigner-Yanase-Dyson conjecture to Carlen-Frank-Lieb conjecture. *Advances in Mathematics*, 365:107053, 2020.