R. Carbone & F. Girotti:

ABSORPTION OPERATORS - notes

1) p-subharmonic: $\Phi(p) \geq p$

$$p\,\Phi(pxp)\,p = p\,\Phi(x)\,p \quad ?$$

$$\Phi(1-p) = 1-\Phi(p) \leq 1-p$$

$$\Phi((1-p)x(1-p)) \leq \Phi(1-p) \leq 1-p = p'$$

$0 \leq x \leq 1$

$$\Phi(xp')$$

$$\Phi(p'x^*)\,\Phi(xp') \leq \Phi(p'x^*xp') \leq p'$$

$$\Rightarrow p\,\Phi(p'x^*)\,\Phi(xp')\,p = 0$$

$$\Rightarrow \Phi(xp')\,p = 0$$

$$p\,\Phi(x)\,p = p\,\Phi(x(p+p'))\,p = p\,\Phi(xp)\,p + p\,\Phi(xp')\,p = p\,\Phi(xp)\,p$$

$$= p\,\Phi\big(((p'+p)x_p)p\big) =$$

$$= \underbrace{p\,\Phi(p'x_p)p}_{=0} + p\,\Phi(px_p)p$$

$$= p\,\Phi(px_p)p \checkmark$$

2) Classical Markov chains:

V - a set (finite for simplicity)

Markov chain : a Markov kernel μ on V

$X = (X_n)_{n\in\mathbb{N}}$, values in V

$X_0 = i_0$

$$\mu_{ij} = \mathrm{Prob}(X_n = j \mid X_{n-1} = i)$$

• "Non-commutative" picture:

• $\mathcal{H} = \mathbb{C}^{|V|}$

• a commutative subalgebra in $B(\mathcal{H})$ $\rightarrow \mathcal{D} = C(V)$

$=$ bounded functions $f : V \to \mathbb{C}$

diagonal operators

$$O_f = \sum_i f(i)|i\rangle\langle i|$$

- states $\equiv$ densities (diagonal in the fixed basis)

$$p \in \mathcal{P}(V) \equiv \rho_p = \sum p_i |i\rangle\langle i|$$

- channels : (completely) positive maps

$$\mathcal{D} \to \mathcal{D} \quad : \quad \mathcal{O}_f \to \mathcal{O}_g$$

$\equiv$ positive $\overset{\text{linear}}{\text{maps}}$ on $C(V)$ $\underline{\underline{\mu}}$

$\shortparallel$

Markov kernels

$\to$ the function $\delta_j(i) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

is mapped onto some

$$\mu_j \in C(V)^+ \ (\text{positive}).$$

- unital :

$$\mu(1) = \mu\left(\sum \delta_j\right) = \sum \mu_j = 1$$

We may identify $\mu \equiv \mu_{ij}$, $\sum_j \mu_{ij} = 1$

- action on $C(V)$:

$$f = \sum_j f(j)\, \delta_j$$

$$\mu(f)(i) = \sum_j f(j)\, \mu_{ij}$$

- dual action : on $\mathcal{P}(V)$:

$$\mu_*(p)_j = \sum_i p_i \mu_{ij}$$

- <u>Enclosure</u>: a subset $C \subseteq V$ such that $\mu_{ij} = \mu(j|i) = 0$ if $i \in C$, $j \notin C$

a <u>matrix</u>: (μ_{ij}) with probabs in <u>rows</u>

$\{1,2\}$ an enclosure:

$$\begin{pmatrix} \mu_{11} & \mu_{12} & 0 & \cdots & 0 \\ \mu_{21} & \mu_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & & & \end{pmatrix}$$ <u>rows</u>

- the <u>projection</u>: χ_C

$$\boxed{\begin{array}{l} \bullet \text{ absorption operator:} \\ \lim_{n \to \infty} \mu^n(\chi_C) \end{array}}$$

$$\mu(\chi_C)(i) = \sum_j \chi_C(j) \mu_{ij} = \sum_{j \in C} \mu_{ij} = \mathbb{P}(X_1 \in C | X_0 = i)$$

$$\mu^2(\chi_C)(k) = \sum_\ell \mu(\chi_C)(\ell) \mu_{k\ell}$$

$$= \sum_{\ell} \sum_{j \in C} \mu_{\ell j} \mu_{k \ell} =$$

$$= \sum_{\ell} \mathbb{P}(X_1 \in C \mid X_0 = \ell)\, \mathbb{P}(X_1 = \ell \mid X_0 = k)$$

$$= \mathbb{P}(X_2 \in C \mid X_0 = k)$$

$$= \mathbb{P}\big((X_2 \in C) \cup (X_1 \in C) \mid X_0 = k\big)$$

$$\boxed{\begin{aligned} A(C)^{(i)} &= \lim_{n \to \infty} \mathbb{P}(X_n \in C \mid X_0 = i) \\ &= \mathbb{P}\Big(\underbrace{\bigcup_n \{X_n \in C\}} \;\Big|\; X_0 = i\Big) \quad \cancel{\lim} \end{aligned}}$$

lebo: enclosure uzavrené, tj.:

$$\{X_n \in C\} \subseteq \{X_{n+1} \in C\} \quad \checkmark$$

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3) Smallest enclosure containing $|x\rangle$:

- Enclosure: a subspace $\mathcal{V} \subseteq \mathcal{H}$

  $$\Phi(P_{\mathcal{V}}) \geq P_{\mathcal{V}}$$

- if $\underset{\|x\|=1}{x} \in \mathcal{V}$ an enclosure, then

$$1 \geq \mathrm{Tr}\, \Phi(P_{\mathcal{V}}) |x\rangle\langle x| \geq \mathrm{Tr}\, P_{\mathcal{V}} |x\rangle\langle x|$$

$$= \mathrm{Tr}\, |x\rangle\langle x| = 1$$

$$\Rightarrow 1 = \mathrm{Tr}\left(\Phi(P_V)\,|x\rangle\langle x|\right) = \mathrm{Tr}\left(P_V\,\Phi_*(|x\rangle\langle x|)\right)$$

$$\Rightarrow \operatorname{supp} \Phi_*(|x\rangle\langle x|) \le P_V$$

- If V is an enclosure for a semigroup $\mathcal{P} = \{\Phi_t\}$

$$\Phi_t(P_V) \ge P_V \quad \forall t$$

$$\Rightarrow \underbrace{\operatorname{supp} \Phi_{t,*}(|x\rangle\langle x|)}_{\text{some projections on } \mathcal{H}} \le P_V \quad \forall t$$

$$\Rightarrow Q = \underbrace{\bigvee_t \operatorname{supp} \Phi_{t,*}(|x\rangle\langle x|)}_{\text{we also have to prove that}} \le P_V$$

<u>this is an enclosure:</u>

- $\Phi_t(Q) \ge Q$ or

hm: $\operatorname{supp} \sigma \le \overbrace{\operatorname{supp} \Phi_{t,*}(|x\rangle\langle x|)}^{\text{or } \Phi_t}$ for some σ

?. $\operatorname{supp} \sigma \le \operatorname{supp} \rho \Rightarrow \operatorname{supp} \Phi_*(\sigma) \le \operatorname{supp} \Phi_*(\rho)$?

naozaj?.

$\mathrm{Tr}\, \Phi_*(\rho)\, a \Rightarrow 0$
$\Rightarrow \mathrm{Tr}\, \rho\, \Phi(a) = 0$

$\rightarrow \quad \operatorname{supp} \phi(a) \leq 1 - \operatorname{supp} \rho \leq 1 - \operatorname{supp} \sigma$

$\Rightarrow 0 = \operatorname{Tr} \sigma \phi(a) = \operatorname{Tr} \phi_*(\sigma) a \quad \ldots$

$\Rightarrow \operatorname{supp} \Phi_{s,*}(\sigma) \leq \operatorname{supp} \phi_{s+t,*}(|x\rangle\langle x|) \leq Q$

$$\boxed{\Rightarrow \operatorname{supp} \sigma \leq Q_t \Rightarrow \operatorname{supp} \phi_{s,*}(\sigma) \leq Q \quad \forall_{s,t}}$$

Let $\operatorname{supp} \sigma = Q_t$, then this implies

$$0 = \operatorname{Tr}\left(\Phi_{s,*}(\sigma)(1-Q)\right) = \operatorname{Tr} \sigma \phi_s(1-Q)$$

$$\Rightarrow \operatorname{Tr} Q_t \phi_s(1-Q) = 0$$

$$\Rightarrow \quad Q_t \phi_s(1-Q) = 0$$

$$Q_t = Q_t \phi_s(Q)$$

$$\Rightarrow Q_t \leq \Phi_s(Q) \quad \forall_{s,t}$$

$$\Rightarrow Q = \bigvee_t Q_t \leq \Phi_s(Q)$$

$\square$ ✓

4) $A(v)$ is a fixed point of (Φ_t):

$$0 \leq P_v \leq 1 \Rightarrow 0 \leq \Phi_t(P_v) \leq 1$$

$P_r \to$ subharmonic $\Rightarrow$

$$\Phi_t(P_r) \geq \Phi_s(P_r), \quad t \geq s$$

nondecreasing : $\Phi_t(P_r) \nearrow A(r)$

in the w*- topology

Since all Φ_s are normal :

$$\Phi_s(\Phi_t(P_r)) \longrightarrow \Phi_s(A(r))$$

but : $\lim_{t\to\infty} \Phi_{s+t}(P_r) = \lim_{t\to\infty} \Phi_t(P_r) = A(r)$

$\to \Phi_s(A(r)) = A(r) \quad \forall s.$

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5) Kraus operators :

$$\Phi(x) = \sum_i V_i^* x V_i$$

$P_K \to$ kernel proj of $A(r)$ $\to$ this is inv "r"

$q \to$ range proj of $A(r) - r$

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then:

- $P_r V_i P_K = P_K V_i P_r = 0 \quad \forall i$:

P_K - (max.) enclosure orthogonal to r

$\Rightarrow$ we have

$$P_K \Phi(P_\gamma) P_K = 0$$

$$= \sum_i P_K V_i^* P_\gamma V_i P_K \Rightarrow P_K V_i^* P_\gamma V_i P_K = 0$$

$$\Rightarrow P_\gamma P_i P_K = 0.$$

Since P_K is an enclosure and P_γ is orthogonal to γ_K, we similarly have

$$P_K V_i P_\gamma = 0$$

- if $q \neq 0 \Rightarrow P_\gamma V_i q \neq 0$ for some i

$$A(v) - v \neq 0$$

$$\Phi(P_v) = P_v + P_v^\perp \phi(P_v) P_v^\perp$$

$$A(v) = P_v + P_v^\perp A(v) P_v^\perp$$

$$A(v) - v = P_v^\perp A(v) P_v^\perp \geq P_v^\perp \phi(P_v) P_v^\perp$$

if $P_\gamma V_i q = 0 \quad \forall i \Rightarrow$

$$\Phi(P_v) q = \sum_i V_i^* P_v V_i q = 0$$

but then also $\Phi(P_v)(A(v) - v) = 0$

$$\hookrightarrow = \left(P_r + P_r^{\perp}\Phi(P_r)P_r^{\perp}\right)\left(P_r^{\perp} A(r) P_r^{\perp}\right)$$

$$= \underbrace{P_r^{\perp}\,\Phi(P_r)P_r^{\perp}}_{B}\;\underbrace{P_r^{\perp}\,A(r)\,P_r^{\perp}}_{A}$$

but:

$$B \leq A \;,\quad BA = 0 \Rightarrow$$

$$\Rightarrow B^{1/2} A B^{1/2} = 0 \Rightarrow$$

$$0 = B^{1/2}(B + A - B)B^{1/2} = B^2 + B^{1/2}(A-B)B^{1/2} \geq B^2 \geq 0$$

$$\Rightarrow B = 0$$

But then $\Phi(P_r) = P_r$ and $\Omega = 0$!

6) Recurrent, positive recurrent, transient

$$J = \bigvee \{\operatorname{supp} \mathcal{U}(x),\ x \in \mathcal{B}_{int}\}$$

here $x \in \mathcal{B}_{int} \subset \mathcal{B}(\mathcal{X})^{+}$ means that the

quadratic form $v \longmapsto \int_0^\infty \langle v, \Phi_t(x) v\rangle \, d\mu(t)$ is

<u>bounded</u>

$$\hookrightarrow = \int_0^\infty \operatorname{Tr} \Phi_{t,*}(|v\rangle\langle v|)\, x \, dm(t)$$

- then $\mathcal{U}(x)$ is the corresp. bounded positive operator:

$$\mathcal{U}(x) = \int_0^\infty \Phi_t(x)\, dm(t)$$

$\tilde{J} \equiv$ <u>transient subspace</u>

$R = J^\perp$ <u>- recurrent subspace</u>

$$R_+ = \bigvee \{ \operatorname{supp}(\rho),\ \Phi_{t,*}(\rho) = \rho,\ \forall t \in I \}$$

¿ $R_+ \subseteq R = J^\perp$?

assume that ρ is an invariant state

$x \in \mathcal{B}_{int}$, then

$$\operatorname{Tr} \rho\, \mathcal{U}(x) = \operatorname{Tr} \rho \int_0^\infty \Phi_t(x)\, dm\, t =$$

$$= \int_0^\infty \operatorname{Tr} \rho x \, dm\, t < \infty \Rightarrow \underline{\underline{\operatorname{Tr} \rho x = 0}}$$

$\Rightarrow \operatorname{Tr} \rho\, \mathcal{U}(x) = 0 \Rightarrow$

$\operatorname{supp}(\rho) \perp \operatorname{supp} \mathcal{U}(x)$ $\quad \forall \rho$ - invariant state

$\forall x \in \mathcal{B}(\text{int})$

$\Rightarrow \operatorname{supp}(\rho) \leq 1 - \operatorname{supp} \mathcal{U}(x) \quad \forall x \in \text{dual}$

$\Rightarrow \operatorname{supp}(\rho) \leq \bigwedge \left(1 - \operatorname{supp} \mathcal{U}(x)\right) =$

$= 1 - \tau$

$\Rightarrow \mathcal{R}_+ \subseteq \mathcal{T}^\perp$

$\square$

- what is then $\mathcal{R} - \mathcal{R}_+$?

$v \in \mathcal{R}_0 \quad |v\rangle \in \mathcal{T}^\perp$

$|v\rangle\langle v| \perp \operatorname{supp}(\mathcal{U}(x)) \quad \forall x$

$0 = \langle v | \mathcal{U}(x) v\rangle \quad \int_0^\infty \langle v, \Phi_t(x) v\rangle \, dm(t)$

$= \int_0^\infty \operatorname{Tr} \Phi_{t,*}(|v\rangle\langle v|)\, x \, dm(t)$

$\Rightarrow \operatorname{Tr}\left(\Phi_{t,*}(|v\rangle\langle v|)\, x\right) = 0$ for almost all t
$\underset{\text{hm.}}{}$

$\square$

7) Lemma 10 :

$$\boxed{\begin{array}{l} \mathcal{V}\text{- enclosure, } w \in \mathcal{V}^\perp, \\ \Phi_t(\mathcal{V}) \geq \varepsilon |w\rangle\langle w|, \text{ for some } \\ \qquad \varepsilon > 0,\ t \in \mathcal{I}. \end{array}}$$

- $w \in \mathrm{Ker}(A(v))^{\perp}$?

$$A(v) \geq \Phi_t(v) \geq \varepsilon |w\rangle\langle w| \ , \ \varepsilon > 0$$

$$\Rightarrow$$

- $v^{\perp} \cap \mathrm{Ker}(A(W))^{\perp}$ superharmonic?

P_v - subharmonic

$Q = P_{\mathrm{Ker}(A(v))}$ - subharmonic

$$\boxed{W := P_v^{\perp} \wedge Q^{\perp}} \leq Q^{\perp} = 1 - Q$$

$$\Phi(P^{\perp} \wedge Q^{\perp}) \leq \Phi(1-Q) \leq 1 - Q = Q^{\perp}$$

$$\leq \Phi(1-P) \leq 1 - P = P^{\perp}$$

$$\Rightarrow \leq P^{\perp} \wedge Q^{\perp}$$

$\square$

- $t \geq \bar{t}$: $|v\rangle$

$$\Phi_{\bar{t}}(W) \leq W$$

$$\mathrm{Tr}\, \Phi_{t*}(|v\rangle\langle v|) W =$$

$$= \mathrm{Tr}\, \Phi_{t-\bar{t},*}(|v\rangle\langle v|) \Phi_{\bar{t}}(W) =$$

$$= \mathrm{Tr}\left[W \Phi_{t-\bar{t},*}(|v\rangle\langle v|) W \Phi_{\bar{t}}(W)\right]$$

$$\overset{?}{\leq} \operatorname{Tr}\left[\Phi_{t-\bar{t},*}(|v\rangle\langle v|)\,(w - w\Phi_{\bar{t}}(v)w)\right]$$

$\because$ $w \leq v^{\perp} = 1 - v$

$$\Phi_{\bar{t}}(w) \leq 1 - \Phi_{\bar{t}}(v)$$

$$w\,\Phi_{\bar{t}}(w)\,w \leq w\,(1 - \Phi_{\bar{t}}(v))\,w$$

$$= w - w\,\Phi_{\bar{t}}(v)\,w \checkmark$$

$\square$

- $\alpha_t^{(v)} = \operatorname{Tr}\left(\Phi_{t,*}(|v\rangle\langle v|)\,w\right)$

$$\alpha_{t+s}(v) = \operatorname{Tr}\,\Phi_{t,*}\,|v\rangle\langle v|\,\Phi_s(w))$$

superharmonic! $\rightarrow$ $\leq \operatorname{Tr}\,\Phi_{t,*}\,|v\rangle\langle v|\,w)$

$$= \alpha_t(v)$$

my potrzebujeme:

$$\int_0^s \left(\alpha_t(v) - \alpha_{t+\bar{t}}(v)\right) dm(t) \overset{s \geq \bar{t}}{\leq} \int_0^{\bar{t}} \alpha_t(v)\, dm(t)$$

??

$$\int_0^s \left(\alpha_t(v) - \alpha_{t+\bar t}(v)\right) dm(t) =$$

$$= \int_0^{\bar t} \alpha_t(v) + \boxed{\int_{\bar t}^{s} \left(\alpha_t(v) - \alpha_{t+\bar t}(v)\right) - \int_0^{\bar t} \alpha_{t+\bar t}(v)}$$

≤ 0 ?

$$\int_{\bar t}^{s} \alpha_t(v)\, dm(t) - \int_0^{s} \alpha_{t+\bar t}(v)\, dm(t)$$

$u = t + \bar t$

$u \dots \bar t$

$$= \int_{\bar t}^{s} \alpha_t(v) - \int_{\bar t}^{s+\bar t} \alpha_t(v)$$

$$= \int_{\bar t}^{s} \alpha_t - \int_{\bar t}^{\cancel{s}} \alpha_t(v) - \int_{s}^{s+\bar t} \alpha_t(v) \leq 0 \;!!$$

$\smile$!

- $w \in \operatorname{supp} \mathcal{U}(|w\rangle\langle w|) \overset{s}{=}$?

$$\int_0^\infty \langle w, \Phi_t(|w\rangle\langle w|)|w\rangle$$

$w \in \operatorname{supp} \mathcal{U} \Leftrightarrow |w\rangle\langle w| \leq \operatorname{supp} \mathcal{U}$

$\Leftrightarrow |w\rangle\langle w| \operatorname{supp} \mathcal{U} = |w\rangle\langle w|$ hm

$|w\rangle\langle w| (1 - \operatorname{supp} \mathcal{U}) = 0$

$\Leftrightarrow \mathcal{U}_a = 0 \Rightarrow |w\rangle\langle w| a = 0$

$$\operatorname{Tr} \mathcal{U}(a) = \int_0^\infty \operatorname{Tr} \Phi_t(|w\rangle\langle w|) a =$$

$$= \int_0^\infty \operatorname{Tr}\left(|w\rangle\langle w| \Phi_t^*(a)\right) = 0$$

discrete : $\sum_{n=0}^\infty t_n \langle w| \Phi_*^n(a) |w\rangle = 0$

$\Rightarrow \langle w | a | w \rangle = 0$ ✓

clms : since $t \mapsto \langle w | \Phi_{t*} a |w\rangle$ is cts

$\langle w | a w \rangle > 0 \Rightarrow \int_0^\infty \langle w, \Phi_{t*}(\cdot) w \rangle > 0$

$\Rightarrow \langle w | a w \rangle = 0$! $\square$

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8) Proof of Thm. 9 :

• $V \subseteq Z \subseteq R$, $\quad W = Z \cap V^{\perp}$ is an enclosure $\leq V^{\perp}$

Assume $PAP \neq 0$, $A \geq 0$, then

$\exists\, x \in P$ such that $PAP \geq \varepsilon |x\rangle\langle x|$: $\|x\| = 1$
$\varepsilon > 0$

Indeed, let $y \in \mathcal{H}$, $\|y\| = 1$ be such that $\langle y, PAPy\rangle = \varepsilon > 0$.

Put $x := \varepsilon^{-1/2}(PAP)^{1/2} y$, then

$$\|x\|^2 = \varepsilon^{-1}\langle y, PAPy\rangle = 1$$

and $P^{\perp}x = 0 \Rightarrow x \in P$

moreover, we have:

$$PAP - \varepsilon|x\rangle\langle x| = PAP - (PAP)^{1/2}|y\rangle\langle y|(PAP)^{1/2}$$
$$= (PAP)^{1/2}(I - |y\rangle\langle y|)(PAP)^{1/2} \geq 0 \;\square$$

So, if $W \Phi_t(v) W \neq 0$:

$\exists \boxed{\begin{matrix}\|y\| = 1 \\ y \in W\end{matrix}}$ s.t. $\langle w, \Phi_t(v) w\rangle = \varepsilon > 0$

But since $y \in W \leq V^{\perp}$, this implies that

$\langle y, V^{\perp}\Phi_t(v) V^{\perp} y\rangle = \varepsilon$ hm.... ε but to vanish'm!

9) Corollary 13: $v \le R$ is harmonic

for $\Phi_t^R = R(\Phi_t(R \cdot R))R$

$\Rightarrow v$ is an enclosure:

$\Phi_t(v) \geq v$:

$$\Phi_t^R(v) = R\,\Phi_t(v)R = v$$

$$\neq v$$

$$\Phi_t(v) = \overbrace{R\,\Phi_t(v)R} + J\Phi_t(v)R + Q\,\Phi_t(v)\tilde{J} + J(\Phi_t(v))\tilde{J}$$

$$R\,\Phi_t(v)\tilde{J} = \underbrace{(R-v)\Phi_t(v)\tilde{J}}_{=0} + v\,\Phi_t(v)\tilde{J} = 0 \quad \checkmark$$

$\square$

$0 \le \Phi_t(v) \le 1 \qquad v\,\Phi_t(v)v = v$

$0 \le A \le 1, \qquad PAP = P$

$\Rightarrow \quad 1 - A = 1 - P + P - A =$

$= 1 - P + \cancel{P} - \cancel{PAP} - P^\perp AP - PAP^\perp - P^\perp AP^\perp$

$= P^\perp(I - A)P^\perp - P^\perp AP - PAP^\perp \not\geq 0$

unless $\to \leq 0$

Thm. 9 este raz!

odbočka pre znalcov:
SUPPORT OF an OPERATOR $A \in \mathcal{B}(\mathcal{H})^+$

$$\operatorname{supp}(A) = R(A) = \operatorname{Ker}(A)^{\perp}$$

Also
$$\boxed{\operatorname{supp}(A) = \operatorname{cl}\{x \in \mathcal{H},\ \exists \varepsilon > 0 : A \geq \varepsilon |x\rangle\langle x|\}}:$$

Lebo: $\operatorname{supp}(A) = R(A)$

$$= R(A^{1/2})$$

$$= \operatorname{cl}\{A^{1/2}y,\ y \in \mathcal{H}\}.$$

Let $x = A^{1/2}y$ for some $y \in \mathcal{H}$.

$$A - \|y\|^{-2}|x\rangle\langle x| = A - \|y\|^{-2}A^{1/2}|y\rangle\langle y|A^{1/2}$$

$$= A^{1/2}(I - P_y)A^{1/2} \geq 0$$

So we have $A \geq \varepsilon|x\rangle\langle x|$ with $\varepsilon = \|y\|^{-2}$

$\square$

Koniec odbočky pre znalcov

Let v be an enclosure, then

$$(*) \quad \boxed{\Phi_t(v) = v + v^{\perp}\Phi_t(v)v^{\perp}, \ \forall t \in I}$$

Lemma 10 says that:

$$\boxed{\operatorname{supp}\left(v^{\perp}\Phi_t(v)v^{\perp}\right) \subseteq \mathcal{J} \quad \forall t \in I}$$

$\mathcal{J}$ – harmonic subspace

Indeed, let x and $\varepsilon > 0$, $t \in I$ be such that

$$v^{\perp}\Phi_t(v)v^{\perp} \geq \varepsilon |x\rangle\langle x|,$$

then by $(*)$ $\boxed{\Phi_t(v) \geq \varepsilon |x\rangle\langle x|}$

$\to$ by Lemma 10 : $x \in \mathcal{J}$

$\Rightarrow$

$$\operatorname{supp}\left(v^{\perp}\Phi_t(v)v^{\perp}\right) = \operatorname{cl}\left\{x, \ v^{\perp}\Phi_t(v)v^{\perp} \geq \varepsilon |x\rangle\langle x| \text{ for some } \varepsilon > 0\right\}$$

$$\subseteq \mathcal{J}$$

$\square$

Theorem 10 follows:

Let $v \leq z \leq R$

$$w = z \wedge v^{\perp} = z - v \quad \text{(as projections)}$$

Then

$$W \bar{\Phi}_t(V) W = W \left(V + V^\perp \bar{\Phi}_t(V) V^\perp \right) W$$

since $W \subseteq V^\perp$ $\hookrightarrow = W V^\perp \bar{\Phi}_t(V) V^\perp W$

since $W \subseteq R = J^\perp$ and supp $\to = 0$

$\subseteq J$.

Look again at the relations?

Then

$$\bar{\Phi}_t(W) = \bar{\Phi}_t(Z - V) = \bar{\Phi}_t(Z) - \bar{\Phi}_t(V)$$

But Z, V are enclosures $\subseteq R$:

$$J = R^\perp \subseteq Z^\perp \subseteq V^\perp$$

$$\bar{\Phi}_t(Z) = Z + Z^\perp \bar{\Phi}_t(Z) Z^\perp \quad \leftarrow \text{by Lemma 10}$$

$$= Z + J \bar{\Phi}_t(Z) J$$

$$\bar{\Phi}_t(V) = V + J \bar{\Phi}_t(V) J$$

$$\bar{\Phi}_t(W) = Z - V + J \bar{\Phi}_t(Z) J - J \bar{\Phi}_t(V) J$$

$$= W + J \bar{\Phi}_t(W) J \quad \checkmark$$

10) $\boxed{\text{Corollary 17}}$: any enclosure $V \subseteq R$ commutes with R_+:

We need to show that if ρ is invariant for Φ_t then $V_\rho V$ is invariant as well

- one way to prove this, if Φ_t is cp (or at least a Schwartz map):

by Cor. 13, V is harmonic for the reduced semigroup Φ_t^R ← recurrent subspace

$$\Phi_t^R(V) = V$$

here $\Phi_t^R(A) = R\,\Phi_t(A)\big|_{R\cdot R} R$

Schwartz map as well

$$\Phi_t^R(A^*A) = R\,\Phi_t(A^*A)R \geq \ldots$$

hence $\Phi_t^R(AV) = \Phi_t^R(A)V$

- ρ is an invariant state for Φ_t^R:

$$\Phi_{t*}^R(\rho) = \rho$$

- Hence: $\operatorname{Tr} \Phi_{t,*}(V_\rho V) A = \quad \forall A$

$$= \mathrm{Tr}\ \Phi_{t*}(V\rho V) RAR$$

$$= \mathrm{Tr}\ V\rho V\ \Phi_t(RAR) =$$

$$= \mathrm{Tr}\ V\rho V\ \Phi_t^R(RAR) =$$

$$= \mathrm{Tr}\ \rho\ \Phi_t^R(VRARV) =$$

$$= \mathrm{Tr}\ \rho R\ \Phi_t(VAV) R =$$

$$= \mathrm{Tr}\ \rho\ \Phi_t(VAV) = \mathrm{Tr}\ V\rho V A \checkmark$$

$$\square$$

11) Structure of fixed points if $A(R_+) = 1$ (Thm. 22).

- $\mathcal{F}(P) = \mathcal{F}(P^R) \oplus \mathcal{J}\mathcal{F}(P)\mathcal{J}$

$F(P) = \mathrm{span}\,\{A(V),\ V \subseteq R\}$ ✓

$$A(V) = V \oplus \mathcal{J} A(V) \mathcal{J}$$

$1 \in \mathcal{F}(P^R)$ / this only says that any $X \in \mathcal{F}(P)$ commutes with $\mathcal{J}$ $(\mathcal{J} \in \mathcal{F}(P)')$

and $R\,F(P)\,R = F(P^R)$

- $y \in \mathcal{F}(P)$

$$y = y_0 + JyJ$$

here : $y_0 = RyR \in \mathcal{F}(P^R)$

and

$$\underline{y} = \mathcal{E}(y) = \mathcal{E}(y_0) + \underbrace{\mathcal{E}(JyJ)}_{=0} = \underline{\mathcal{E}(y_0)}$$

- so $y_0 = RyR$ is the <u>unique</u> elements in $\mathcal{F}(P^R)$ and that $\mathcal{E}(y_0) = y$!

$$\mathcal{F}(P) = \mathcal{F}(P^R) \oplus J\mathcal{F}(P)J$$

suggests that for <u>any</u> $x \in \mathcal{F}(P^R)$ and $y \in \mathcal{F}(P)$, we have

$$x + JyJ \in \mathcal{F}(P)$$

but this means that

$$x + JyJ = \mathcal{E}(x + JyJ) = \mathcal{E}(x)$$

so that $x + JyJ \in \mathcal{F}(P)$ iff

$$JyJ = \mathcal{E}(x) - x$$ $\leftrightarrow$ determined by x

this is in Remark 24

12) Remark 24:

$$\Psi : \mathcal{F}(P^R) \to J\,\mathcal{F}(P)J :$$

any $x \in \mathcal{F}(P)$ can be written as

$$x = R \times R + J \times J$$
$$= R \times R + \Psi(R \times R)$$

(discrete time)

$$\Phi(x) = \Phi(R \times R) + \Phi(J \times J) = x$$

$$0 = \Phi(x) - x = \Phi(R \times R) - R \times R +$$
$$+ \Phi(J \times J) - J \times J$$

$\mathcal{L} = \Phi - id$

$$0 = \mathcal{L}(R \times R) + \mathcal{L}(J \times J)$$

$$0 = \mathcal{L}(R \times R) + \mathcal{L}(\Psi(R \times R))$$

$$\mathcal{L}(R \times R) = -\mathcal{L}(\Upsilon(R \times R))$$
$$= -\mathcal{L}(T \,y\, T) \quad \text{2. } \underline{\underline{\text{[illegible]}}}$$

$$- \quad R\,\bar{\Phi}(R \times R) R = R \times R = R\,\bar{\Phi}(x)\,R$$
$$= R\,\bar{\Phi}(R \times R) R + \underbrace{R\,\bar{\Phi}(T \times T) R}_{=0}$$
$$\Rightarrow \operatorname{supp} \bar{\Phi}(T \times T) \leq T = R^{\perp}$$

$$\mathcal{L}(R \times R) = \bar{\Phi}(R \times R) - R \times R = \underbrace{x - R \times R}_{= T \times T}$$

$$T\,\mathcal{L}(R \times R)\,T = -\mathcal{L}(T \,y\, T)$$
$$\dots \qquad \square$$

12) $A(R_+) = 1$, $\mathcal{E}(x) = ?$ (using decomp on $R_+ = R$)
(Ergodic theory)

$$\mathcal{F}(\mathcal{P}^{R}) = \bigoplus_i U_i \left(B(\mathcal{K}^{(1)}_i) \otimes 1_{\mathcal{K}^2_i} \right) U_i$$

....

$$\mathcal{E}(x) \in \mathcal{F}(\mathcal{P})$$
$$\mathcal{E}(x) = R\,\mathcal{E}(x)\,R + T\,\mathcal{E}(x)\,T$$

we must have $\quad T\,\mathcal{E}(x)\,T = \mathcal{E}(R\,\mathcal{E}(x)\,R) - \mathcal{E}(x)$

Fixed point:

$$y = RyR + JyJ$$

$$y = \mathcal{E}(y) = \mathcal{E}(RyR) + \mathcal{E}(JyJ) = \mathcal{E}(RyR)$$

$$\mathcal{E}(x) = \mathcal{E}(R \underbrace{\mathcal{E}(x)}_{\text{Fixed point of } \rho^R} R)$$

$$R\,\mathcal{E}(x)\,R = \bigoplus_i V_i (x_i \otimes 1_{\mathcal{H}_i^2}) V_i^*$$

$$= \bigoplus_i V_i \left(\sum x_i^{kl} |k\rangle\langle l| \otimes 1_{\mathcal{H}_i^2} \right) V_i^*$$

$$x_i^{kl} = \mathrm{Tr}\; R\,\mathcal{E}(x) R\, V_i (|l\rangle\langle k| \otimes \rho_i) V_i^*$$

$$= \mathrm{Tr}\; \mathcal{E}(x)\, V_i (|l\rangle\langle k| \otimes \rho_i) V_i^*$$

$$= \mathrm{Tr}\; x\, \mathcal{E}^*(V_i |l\rangle\langle k| \otimes \rho_i) V_i^*)$$

$$\Rightarrow R\,\mathcal{E}(x)\,R = \bigoplus_i V_i \left(\sum_{kl} \mathrm{Tr}\, x\, \mathcal{E}^*(V_i(|l\rangle\langle k| \otimes \rho_i) V_i^*)\, |k\rangle\langle l| \otimes 1_{\mathcal{H}_i^2} \right)$$

$$= \bigoplus_i V_i \sum_{kl} \mathrm{Tr}(RxR)\, V_i (|l\rangle\langle k| \otimes \rho_i) V_i^*\, |k\rangle\langle l| \otimes 1_{\mathcal{H}_i^2}$$

$$\mathcal{E}(R\,\mathcal{E}(x)\,R) =$$

$$= \sum_i \left[\mathrm{Tr}\,(RxR) V_i (|l\rangle\langle k| \otimes \rho_i) V_i^* \right] \underbrace{\mathcal{E}(|k\rangle\langle l| \otimes 1_{\mathcal{H}_i^2})}_{Q_{i,k,l}}$$

We see that actually

$$\mathcal{E}(X) = \mathcal{E}(R\,\mathcal{E}(X)\,R) = \boxed{\mathcal{E}(\mathcal{E}^R(RXR))}$$

$$\bigoplus_i \sum_{kl} \left[\mathrm{Tr}\; y\, U_i(|e\rangle\langle k| \otimes \rho_i) U_i^* \right] (|k\rangle\langle l| \otimes 1_{\mathcal{H}_i})$$

$$= \bigoplus_i \sum_{kl} \mathrm{Tr}\,\mathrm{Tr}_2 \left[U_i^* y U_i (|l\rangle\langle k| \otimes \rho_i) \right] |k\rangle\langle l| \otimes 1_{\mathcal{H}_i}$$

$$= \bigoplus_i \sum_{kl} \langle k| \mathrm{Tr}_2\, U_i^* y U_i ((1_{\mathcal{H}_i} \otimes \rho_i)|l\rangle |k\rangle\langle l| \otimes 1_{\mathcal{H}_i}$$

$$= \bigoplus_i \left(\mathrm{Tr}_2\, U_i^* y U_i (1_{\mathcal{H}_i} \otimes \rho_i) \right) \otimes 1_{\mathcal{H}_i} = \mathcal{E}^R(y)$$

13) Pathwise ergodic theorem ($I = \mathbb{N}$)

- a quantum channel $\Phi = \sum_i V_i^* \cdot V_i$

  $P = \{\Phi^n\}_{n \in \mathbb{N}}$

- choose a state ϕ: uniquely determines a probability measure on $\Omega = \mathbb{N}^\infty$:

$$\forall n \quad \boxed{\mathbb{P}^\phi(\omega_1 = i_1, \dots, \omega_n = i_n) = \mathrm{Tr}\, V_{i_n} V_{i_{n-1}} \cdots V_{i_1} \phi V_{i_1}^* \cdots V_{i_n}^*}$$

$\theta_n : \Omega \to L_1(\mathcal{H})$ random variable $L_1(\mathcal{H})$-valued

$$\theta_n(\omega) = V_{\omega_n} \cdots V_{\omega_1} \phi V_{\omega_1}^* \cdots V_{\omega_n}^* \Big/ \mathrm{Tr}(\quad)$$

$\theta_n = V_{\omega_n} \cdots V_{\omega_1} \phi V_{\omega_1}^* \cdots V_{\omega_n}^*$ with probability $P^\phi(\omega_1, \dots, \omega_n)$

a Markov chain with values in quantum states.

v - endomorphism: ϕ a state

$$\operatorname{Tr}\phi\, \mathcal{E}(v) = \lim_{N\to\infty} \operatorname{Tr}\phi \frac{1}{N}\sum_{n=0}^{N-1}\Phi^n(v) =$$

$= \operatorname{Tr}\phi\,\mathcal{E}(v)$

$$= \lim_{N\to\infty} \operatorname{Tr} v \frac{1}{N}\sum_{n=0}^{N-1}\sum_{i_1,\dots,i_n} V_{i_n}\cdots V_{i_1}\phi V_{i_1}^*\cdots V_{i_n}^* =$$

$$= \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \operatorname{Tr} v \int \Theta_n(\omega)\, dP^\phi(\omega) =$$

$$= \lim_{N\to\infty} \mathbb{E}_\phi\left[\frac{1}{N}\sum_{n=0}^{N-1}\operatorname{Tr} v\,\Theta_n\right] = \mathbb{E}_\phi\left[\operatorname{Tr} v\,\Theta_\infty\right]$$

$$P_\phi \text{ a.s} \longrightarrow \Theta_\infty = \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\Theta_n$$

Θ_∞ an invariant state for Φ

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15) If $\mathcal{F}(\mathcal{P})$ is a subalgebra, $A(R_+)=1$

$\tilde{\mathcal{E}} \stackrel{\perp}{=} \mathcal{E}\big|_{\mathcal{F}(\mathcal{P}^R)}$ and we have

$\tilde{\mathcal{E}} : \mathcal{F}(\mathcal{P}^R) \to \mathcal{F}(\mathcal{P})$ bijection (homeomorph)

$\tilde{\mathcal{E}}^{-1} = R\cdot R : \mathcal{F}(\mathcal{P}) \to \mathcal{F}(\mathcal{P}^R)$

$x \in \mathcal{F}(P) \Rightarrow x^* x \in \mathcal{F}(P)$

$$\tilde{\mathcal{E}}(R x^* x R) = \mathcal{E}(R x^* x R) = x^* x = \tilde{\mathcal{E}}(R x^* R)\, \tilde{\mathcal{E}}(R x R)$$

$$\leq \mathcal{E}(R x^* R x R) \leq \mathcal{E}(R x^* x R) = \tilde{\mathcal{E}}(R x^* x R)$$

$$\Rightarrow \quad \tilde{\mathcal{E}}(R x^* R)\, \tilde{\mathcal{E}}(R x R) = \mathcal{E}(R x^* R R x R)$$

$\square$