$\lim_{\alpha\nearrow 1}$ and $\lim_{\alpha\searrow 1}$ of $D_{\alpha,z}$

The following are slight updates of [1, Corollary 2.5] and the $z \ge 1$ case of [3, Corollary 2]. (The $1/2 < z < 1$ case of [3, Corollary 2] is still unclear to me. I probably overlooked something.) In this way we can include the $\alpha \to 1$ limits of $D_{\alpha,\alpha}(\psi\|\varphi)$ and $D_{\alpha,1}(\psi\|\varphi)$ as special cases. Note that a very general result was known in [2, Proposition III.36] in the finite-dimensional case.

Proposition 0.1. *Let $\delta \in (0,1)$ and $\alpha \mapsto z(\alpha) \in (0,\infty)$ be a function on $(1-\delta, 1+\delta)$. Let $\psi, \varphi \in \mathcal{M}_*^+$.*

(i) If $z(\alpha) \ge \alpha$ for any $\alpha \in (1-\delta, 1)$, then

$$\lim_{\alpha\nearrow 1} D_{\alpha,z(\alpha)}(\psi\|\varphi) = D_1(\psi\|\varphi).$$

In particular, $\lim_{\alpha\nearrow 1} D_{\alpha,z}(\psi\|\varphi) = D_1(\psi\|\varphi)$ for any $z \ge 1$.

(ii) Assume that $D_{\alpha_0,z_0}(\psi\|\varphi) < \infty$ for some $\alpha_0 > 1$ and $z_0 \ge \min\{1, \alpha_0/2\}$. If $z(\alpha) \ge 1$ for any $\alpha \in (1, 1+\delta)$, then

$$\lim_{\alpha\searrow 1} D_{\alpha,z(\alpha)}(\psi\|\varphi) = D_1(\psi\|\varphi).$$

In particular, $\lim_{\alpha\searrow 1} D_{\alpha,z}(\psi\|\varphi) = D_1(\psi\|\varphi)$ for any $z \ge 1$.

Proof. (i) By [1, Proposition 2.3 (1)],

$$D_{\alpha,\alpha}(\psi\|\varphi) \le D_{\alpha,z(\alpha)}(\psi\|\varphi) \le D_1(\psi\|\varphi), \qquad \alpha \in (1-\delta, 1).$$

Since $D_{\alpha,\alpha}(\psi\|\varphi) \to D_1(\psi\|\varphi)$ as $\alpha \nearrow 1$, the result follows.

(ii) Under the assumption of (ii), we have $D_{\beta,1}(\psi\|\varphi) < \infty$ for some $\beta > 1$ as noted in the proof of [3, Corollary 2]. For each $\alpha \in (1, 1+\delta)$, assume that $z(\alpha) \ge 1$. Then letting $\beta(\alpha) := \frac{\alpha+z(\alpha)-1}{z(\alpha)} > 1$ we have

$$D_1(\psi\|\varphi) \le D_{\beta(\alpha),1}(\psi\|\varphi) \le D_{\alpha,z(\alpha)}(\psi\|\varphi) \le D_{\alpha,1}(\psi\|\varphi),$$

where the second inequality is due to [3, Lemma 2] and the last inequality is due to [3, Corollary 1]. Since $D_{\alpha,1}(\psi\|\varphi) \to D_1(\psi\|\varphi)$ as $\alpha \searrow 1$, the result follows. $\square$

Remark 0.2. In the situation of (ii) above, if $\alpha/2 \le z(\alpha) < 1$ ($< \alpha$), then we have

$$D_1(\psi\|\varphi) \le D_{\alpha,1}(\psi\|\varphi) \le D_{\alpha,z(\alpha)}(\psi\|\varphi) \le D_{\alpha,\alpha/2}(\psi\|\varphi).$$

But it is not clear to me how we can show that $D_{\alpha,z(\alpha)}(\psi\|\varphi) \to D_1(\psi\|\varphi)$ as $\alpha \searrow 1$.

References

[1] F. Hiai, Monotonicity of $z \mapsto D_{\alpha,z}(\psi\|\varphi)$, Notes (12/3/2023, 12/8/2023).

[2] M. Mosonyi and F. Hiai, Some continuity properties of quantum Rényi divergences, *IEEE Trans. Inform. Theory*, to appear, DOI 10.1109/TIT.2023.3324758.

[3] A. Jenčová, Note on the limit $\alpha \searrow 1$, Notes, December 19, 2023.