SPECTRAL RESOLUTION IN AN ORDER-UNIT SPACE

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The operational approach to quantum physics employs an order-unit space in duality with a base-normed space, and in this context, a suitable spectral theory is a prerequisite for the representation of quantum-mechanical observables. An order-unit space is called spectral if it is enriched by a compression base with the comparability and projection cover properties. These notions are explicated in the article. We show that each element in a spectral order-unit space determines and is determined by a spectral resolution and it has a spectrum which is a nonempty closed bounded subset of the real numbers. Our theory is a generalization and a more algebraic version of the well-known non-commutative spectral theory of Alfsen and Shultz.

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1. Introduction and basic definitions

An *order-unit space* [1, p. 69] is an archimedean partially-ordered real vector space A with a distinguished (strong) *order unit* u. The *operational approach* to quantum physics [5] features an order-unit space A in duality with a base-normed space V, the cone base in V being the convex state space of a quantum-mechanical system $\mathcal{S}$. According to the modern quantum theory of measurement [4], (possibly "unsharp") observables for $\mathcal{S}$ are represented by measures defined on Borel spaces and taking on values in the unit interval E of A. The "sharp" observables for $\mathcal{S}$ correspond to

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spectral (projection-valued) measures, and their mathematical representation requires an appropriate spectral theory for the order-unit space A.

Every order-unit space is a partially ordered additive abelian group with order unit, so the results in [11] pertaining to such groups are applicable to order-unit spaces. In what follows, $\mathbb{R}$ is the ordered field of real numbers and $\mathbb{N}$ is the set of positive integers.

STANDING ASSUMPTION 1.1. *In the sequel, A is an order-unit space with order unit u and positive cone $A^+ := \{a \in A : 0 \leq a\}$.*[1] *The "unit interval" in A is denoted by $E := \{e \in A : 0 \leq e \leq u\}$. The order-unit norm $\|\cdot\|$ on A is defined by $\|a\| = \inf\{\lambda \in \mathbb{R} : 0 < \lambda$ and $-\lambda u \leq a \leq \lambda u\}$ for every $a \in A$* [1, pp. 68–69]. *To avoid trivialities, we also assume that $A \neq \{0\}$, i.e. $0 < u$.*

The norm $\|\cdot\|$ satisfies the condition $-\|a\|u \leq a \leq \|a\|u$ for all $a \in A$ [1, Proposition II.1.2]. Moreover, by [11, Proposition 7.12], if $a, b \in A$, then $-a \leq b \leq a \Rightarrow \|b\| \leq \|a\|$. Obviously, $\|u\| = 1$.

If $a \in A$, then since u is an order unit, there exists $n \in \mathbb{N}$ such that $a \leq nu$; hence, if $a \in A^+$, then $e := (1/n)a \in E$. From this, it follows easily that $A = A^+ - A^+$, i.e. A is directed, and $A^+ = \{ne : n \in \mathbb{N}$ and $e \in E\}$.

Our main purpose in this article is to show that, if the order-unit space A is enriched by a so-called *spectral compression base* (Definition 1.7 below), then every element $a \in A$ has a spectral resolution and a corresponding spectrum. In Examples 1.1, 1.2, 1.4, and 1.5 below, we illustrate, compare, and contrast our work with an important special case, namely the well-known non-commutative spectral theory of E.M. Alfsen and F.W. Shultz [2, Sections 1–7]. The simpler Examples 1.3, 1.6, and 1.7 will also help to fix ideas.

EXAMPLE 1.1. In [2], Alfsen and Shultz study an order-unit space[2] (A, u) and a base-normed space (V, K) in separating norm and order duality under a bilinear form $\langle\cdot,\cdot\rangle$ (see Conditions (1.1) and (2.1) in [2]). In [2, Section 4], two additional conditions are imposed: A is pointwise monotone σ-complete (Condition (4.1)), and every exposed face of K is projective (Condition (4.2)). Condition (4.1) implies that A is monotone σ-complete, which in turn implies that A is a Banach space.

Our approach to spectral theory for A is more algebraic, we make no overt use of a base-normed space V in duality with A, and we do not necessarily assume that A is a Banach space.

DEFINITION 1.1. A *compression with focus p* on A [7, Definition 2.4] is a mapping J: $A \to A$ such that, for all $a, b \in A$ and all $e \in E$, (i) $p = J(u) \in E$, (ii) $J(a+b) = J(a) + J(b)$, (iii) $a \leq b \Rightarrow J(a) \leq J(b)$, (iii) $e \leq p \Rightarrow J(e) = e$, and (iv) $J(e) = 0 \Rightarrow e \leq u - p$.

[1]The notation $:=$ means "equals by definition."

[2]In [2], the order unit is denoted by e.

LEMMA 1.1. *If J is a compression on A, then J is an idempotent linear mapping on A and $\|J(a)\| \leq \|a\|$ holds for all $a \in A$.*

Proof: By [7, Lemma 2.2], J is idempotent. Let $b \in A^+$ and $\lambda \in \mathbb{R}$. As J is additive, it follows that $J(\alpha b) = \alpha J(b)$ for all rational numbers α. Let $n \in \mathbb{N}$, and choose rational numbers α, β with $0 < \lambda - \alpha < 1/n$ and $\lambda + (\lambda - \alpha) < \beta < \lambda + 1/n$. Then $-(\beta - \lambda) < \alpha - \lambda$ and $0 < \beta - \lambda < 1/n$. As $\alpha < \lambda < \beta$ and $0 \leq b$, we have $\alpha b \leq \lambda b \leq \beta b$, whence $\alpha J(b) \leq J(\lambda b) \leq \beta J(b)$, and it follows that

$$-(\beta - \lambda)J(b) \leq (\alpha - \lambda)J(b) \leq J(\lambda b) - \lambda J(b) \leq (\beta - \lambda)J(b).$$

Consequently, $\|J(\lambda b) - \lambda J(b)\| \leq (\beta - \lambda)\|J(b)\| \leq (1/n)\|J(b)\|$, and letting $n \to \infty$, we conclude that $J(\lambda b) = \lambda J(b)$. Since $A = A^+ - A^+$, it follows that $J(\lambda a) = \lambda J(a)$ for all $a \in A$, and therefore J is linear.

Let $a \in A$ and let $p = J(u)$. Then $0 \leq p \leq u$, so $\|p\| \leq \|u\| = 1$. Also, $-\|a\|u \leq a \leq \|a\|u$, so $-\|a\|p \leq J(a) \leq \|a\|p$, whence $\|J(a)\| \leq \|a\|\|p\| \leq \|a\|$. □

The unit interval E in A is a convex subset of A^+ and it forms a so-called (*interval*) *effect algebra* [3]; hence we refer to elements $e \in E$ as *effects*.

DEFINITION 1.2. A subset $S \subseteq E$ is called a *sub-effect algebra* of E iff[3] (i) $0, u \in S$, (ii) $s \in S \Rightarrow u - s \in S$, and (iii) if $s, t \in S$, then $s + t \leq u \Rightarrow s + t \in S$. If S is a sub-effect algebra of E and $s, t \in S$, then s and t are *Mackey compatible in* S, in symbols sC_St, iff there are elements $r, s_1, t_1 \in S$ such that $r + s_1 + t_1 \in S$, $s = r + s_1$, and $t = r + t_1$. A sub-effect algebra S of E is said to be *normal* iff, for all $d, e, f \in E$ such that $d+e+f \leq u$, we have $d+e, d+f \in S \Rightarrow d \in S$ [9, Definition 1].

If S is a sub-effect algebra of E and $s, t \in S$, then $s \leq t \Leftrightarrow t - s \in S$; moreover, if S is normal, then $sC_St \Leftrightarrow sC_Et$.

DEFINITION 1.3. A *compression base* for A [9, Definition 2] is a family $(J_p)_{p \in P}$ of compressions on A, indexed by their own foci, such that P is a normal sub-effect algebra of E, and whenever $p, q, r \in P$ and $p + q + r \leq u$, then $J_{p+r} \circ J_{q+r} = J_r$.

EXAMPLE 1.2. In Example 1.1, every P-projection J on A [2, p. 8] is a compression on A, and by [2, Lemma 2.16], J is uniquely determined by its focus $p = J(u)$. If Conditions (4.1) and (4.2) in [2] are satisfied, then the family $(J_p)_{p \in P}$ of all P-projections on A, indexed by their own foci, is a compression base for A and P is the set of all so-called projective units in A [2, p. 12].[4]

STANDING ASSUMPTION 1.2. *In the sequel, we assume that $(J_p)_{p \in P}$ is a compression base for A, and we refer to elements $p \in P$ as "projections."*

EXAMPLE 1.3. Let X be a compact Hausdorff space and let $A := C(X, \mathbb{R})$ be the set of all continuous functions $f\colon X \to \mathbb{R}$. Define P to be the subset of A consisting of all characteristic set functions (indicator functions) of compact open subsets of

[3] We use "iff" as an abbreviation for "if and only if."

[4] In [2], the set of projective units is denoted by $\mathcal{U}$ and a generic P-projection is denoted by P.

X, and denote by $1 \in P$ the constant function $1(x) := 1$ for all $x \in X$. Then, with pointwise addition and partial order, A is an order-unit space with order unit 1. If $p \in P$, then J_p: $A \to A$ defined by $J_p(f) := pf$ (pointwise product) for all $f \in A$ is a compression on A with focus p, and every compression on A has the form J_p for a uniquely determined $p \in P$. The family $(J_p)_{p \in P}$ is a compression base for A, and P is a Boolean algebra isomorphic to the field of compact open subsets of X.

By parts (iii) and (iv) of the following lemma, if $p \in P$, then J_p and J_{u-p} are *quasicomplements* in the sense of [2, (1.22)].

LEMMA 1.2. *Let $a \in A$, and let $p \in P$. Then:* (i) $J_0(a) = 0$ *and* $J_u(a) = a$. (ii) $J_{u-p} \circ J_p = J_0$. (iii) $J_p(a) = a \Rightarrow J_{u-p}(a) = 0$. (iv) *If* $0 \leq a$, *then* $J_{u-p}(a) = 0 \Rightarrow J_p(a) = a$.

Proof: Part (i) is obvious. As $(u - p) + p + 0 = u$, we have $J_{u-p} \circ J_p = J_{(u-p)+0} \circ J_{p+0} = J_0$, proving (ii), and (iii) follows from (ii). To prove (iv), suppose $0 \leq a$ and $J_{u-p}(a) = 0$. As $0 \leq a$, there exists $e \in E$ and $n \in \mathbb{N}$ such that $a = ne$. Therefore, $J_{u-p}(e) = 0$, so $e \leq u - (u - p) = p$, and it follows that $J_p(e) = e$, whence $J_p(a) = a$. □

DEFINITION 1.4. If $e \in E$ and $p \in P$, then p is a *projection cover* for e [6, Definition 6.1] iff, for all $q \in P$, $e \leq q \Leftrightarrow p \leq q$. The compression base $(J_p)_{p \in P}$ has the *projection cover property* iff every effect $e \in E$ has a (necessarily unique) projection cover.

If A is regarded as an additive partially ordered abelian group with order unit, then A forms a so-called *unital group* [6, p. 436]; hence by [6, Theorem 6.3], we have the following result.[5]

THEOREM 1.1. *Suppose that $(J_p)_{p \in P}$ has the projection cover property. Then, with the partial order inherited from A, and with $p \mapsto u - p$ as orthocomplementation, P is an orthomodular lattice* (OML) [12].

EXAMPLE 1.4. In Examples 1.1 and 1.2, suppose A satisfies Conditions (4.1) and (4.2). Then, if $a \in A^+$, there is a smallest element $p \in P$ such that a belongs to the face of A^+ generated by p [2, Proposition 4.7], and by definition $\mathrm{rp}(a) := p$ [2, p. 31]. If $e \in E$, then $\mathrm{rp}(e)$ is the smallest element $p \in P$ such that $e \leq p$; hence A has the projection cover property. By Theorem 1.1 above, P is an OML, but an independent proof is given in [2, Theorem 4.5]. Furthermore, Condition (4.1) implies that P is σ-complete [2, Proposition 4.2].

DEFINITION 1.5. Let $(J_p)_{p \in P}$ be a compression base for A and let $a \in A$.
(i) If $p \in P$, then $C(p) := \{a \in A : a = J_{u-p}(a) + J_p(a)\}$, and we say that a is *compatible with* p iff $a \in C(p)$ (cf. [2, p. 32]).

[5] Theorem 6.3 in [6] was proved under hypotheses stronger than in Theorem 1.1, but the proof given in [6] goes through under our current hypotheses. A similar observation applies to the proofs in [7] and [8] that are cited below.

(ii) If $Q \subseteq P$, then $C(Q) := \bigcap_{p \in Q} C(p)$.
(iii) $CPC(a) := C(\{p \in P : a \in C(p)\})$.

THEOREM 1.2. *Let $v \in P$, $H := J_v(A)$, $C := C(v)$, and $P_H := \{p \in P : p \leq v\}$. If $p \in P$ and $M \subseteq A$, let J_p^M be the restriction of J_p to M. Then:*

(i) *H is a norm-closed linear subspace of A, H is an order-unit space with order unit v, and $(J_p^H)_{p \in P_H}$ is a compression base for H.*
(ii) *C is a norm-closed linear subspace of A, C is an order-unit space with order unit u, and $(J_p^C)_{p \in C \cap P}$ is a compression base for C.*

Proof: By Lemma 1.1, $J_u - J_v$ and $J_u - J_v - J_{u-v}$ are norm-continuous linear transformations on A; hence their respective null spaces H and C are norm-closed linear subspaces of A. The remainder of the proof can be found in [9, Theorems 4 and 5]. □

If $a, b \in A$, then the condition $b \in CPC(a)$ requires that b is compatible with every projection $p \in P$ such that a is compatible with p (cf. [2, p. 35]). For the following result, see [9, Lemma 3 and Theorem 3].

LEMMA 1.3. *If $p, q \in P$ and $a \in A$, then:* (i) $J_p(a) \leq a \Rightarrow a \in C(p)$. (ii) *If $a \in A^+$, then $a \in C(p) \Leftrightarrow J_p(a) \leq a$.* (iii) $pC_Pq \Leftrightarrow p \in C(q) \Leftrightarrow q \in C(p)$. (iv) $p \in C(q) \Leftrightarrow J_p \circ J_q = J_q \circ J_p$.

If $p, q \in P$, then in view of Lemma 1.3 (iii), *we shall write the condition $p \in C(q)$, or equivalently, pC_Pq, simply as pCq.*

DEFINITION 1.6.

(i) If $a \in A$, then $P^{\pm}(a) := \{p \in P \cap CPC(a) : a \in C(p)$ and $J_{u-p}(a) \leq 0 \leq J_p(a)\}$.
(ii) The compression base $(J_p)_{p \in P}$ has the *comparability property* iff, for every $a \in A$, $P^{\pm}(a) \neq \emptyset$.

Suppose that $(J_p)_{p \in P}$ satisfies the comparability property, let $a \in A$, let $\lambda \in \mathbb{R}$, and choose $p \in P^{\pm}(a - \lambda u)$. Then $a = J_{u-p}(a) + J_p(a)$ with $J_{u-p}(a) \leq \lambda(u - p)$ and $\lambda p \leq J_p(a)$ (cf. [2, (6.3)]).

DEFINITION 1.7. The compression base $(J_p)_{p \in P}$ is *spectral* iff it has both the projection cover and the comparability properties. A *spectral order-unit space* is an order unit space equipped with a spectral compression base.

EXAMPLE 1.5. According to [2, p. 55], the order-unit space (A, u) and the base norm space (V, K) in separating order and norm duality (Example 1.1) are said to be in spectral duality iff A is pointwise monotone σ-complete (Condition (4.1)) and Condition (7.1) [2, p. 55] holds. It is easily seen that Condition (7.1) implies that the compression base $(J_p)_{p \in P}$ (Example 1.2) has the comparability property. Moreover, Condition (7.1) obviously implies Condition (6.3), and by [2, Proposition 6.2], it also implies Condition (4.2), so $(J_p)_{p \in P}$ has the projection cover property (Example 1.4). Therefore if A and V are in spectral duality, then A is a spectral order-unit space.

In view of Example 1.5, all of the examples in [2] of pairs A and V in spectral duality yield examples of spectral order-unit spaces A. For instance, if K is a Choquet simplex, then the space $A^b(K)$ of all bounded affine functionals on K is a spectral order-unit space [2, Theorem 10.4]. Also, if A is the self-adjoint part of a unital von Neumann algebra, then A is a spectral order-unit space [2, Section 11]. Of course, all of the spectral order-unit spaces A arising from pairs A, V in spectral duality are pointwise monotone σ-complete, they are Banach spaces, and their projections form a σ-OML.

EXAMPLE 1.6. Let $A := C(X, \mathbb{R})$ with compression base $(J_p)_{p\in P}$ as in Example 1.3. If the space X is connected and consists of more than one point, then $P = \{0, 1\}$ and $(J_p)_{p\in P}$ has the projection cover property (trivially), but $(J_p)_{p\in P}$ does not have the comparability property. If X is basically disconnected (i.e. the closure of every open F_σ subset of X remains open), then A is a spectral order unit space. If X is totally disconnected, but not basically disconnected, then $(J_p)_{p\in P}$ does not have the projection cover property.

The following example, a variation on Examples 1.3 and 1.6, shows that a spectral order-unit space need not be monotone σ-complete.

EXAMPLE 1.7. Let $\mathcal{B}$ be a field of subsets of the nonempty set X and define A to be the set of all functions f: $X \to \mathbb{R}$ such that $f(X) = \{f(x) : x \in X\}$ is finite and $f^{-1}(\lambda) \in \mathcal{B}$ for all $\lambda \in \mathbb{R}$. Then, with the constant function $1(x) = 1$ for all $x \in X$ as the order unit, and with pointwise addition and partial order, A is an order-unit space. Define $P := \{p \in A : p(X) \subseteq \{0, 1\}\}$, and for each $p \in P$, define $J_p(f) := pf$ (pointwise product) for all $f \in A$. Then, with $(J_p)_{p\in P}$ as compression base, A is a spectral order-unit space. In particular, if B is a Boolean algebra that is not σ-complete, X is the Stone space of B, and $\mathcal{B}$ is the field of compact open subsets of X, then A is not monotone σ-complete.

2. Basic properties of a spectral order-unit space

STANDING ASSUMPTION 2.1. *In the sequel, we assume that A is a spectral order-unit space with order unit $u > 0$, unit interval E, and compression base $(J_p)_{p\in P}$.*

The following theorem is a consequence of [8, Theorem 6.5].

THEOREM 2.1. *There is a uniquely determined mapping $'$: $A \to P$ such that for all $a \in A$ and all $p \in P$, $p \leq a' \Leftrightarrow a \in C(p)$ with $J_p(a) = 0$.*

In what follows, we make extensive use of the mapping $'$: $A \to P$, which we call the *Rickart mapping*. By part (ii) of the following lemma, if $p \in P$, then $p' = u - p$ is the orthocomplement of p in the OML P.

LEMMA 2.1. *For all* $a, b \in A$, *all* $p \in P$, *and all* $e \in E$: (i) $J_{a'}(a) = 0$ *and* $a \in C(a')$. (ii) $p' = u - p$. (iii) $a'' := (a')' = u - a'$ *and* $C(a'') = C(a')$, (iv) $0 \le a \le b \Rightarrow a'' \le b''$. (v) e'' *is the projection cover of* e. (vi) $a', a'' \in CPC(a)$. (vii) $J_p(a) = a \Leftrightarrow a'' \le p$.

Proof: Parts (i)–(v) follow from [8, Lemma 6.2], and (vi) follows from [8, Theorem 6.5]. If $J_p(a) = a$, then $J_{u-p}(a) = 0$ by Lemma 1.2 (iii) and $a \in C(p) = C(u-p)$ by Lemma 1.3 (i), whence $u - p \le a'$, so by (iii), $a'' = u - a' \le p$. Conversely, suppose $a'' \le p$. Then $u - p \le a'$, whence $J_{u-p}(a) = 0$ and $a \in C(u-p) = C(p)$, and it follows that $a = J_p(a) + J_{u-p}(a) = J_p(a)$, proving (vii). □

If $p, q \in P$, we denote the supremum and infimum of p and q in the OML P by $p \vee q$ and $p \wedge q$, respectively. More generally, an existing supremum or infimum in P of a set $Q \subseteq P$ is written as $\bigvee Q$ or $\bigwedge Q$, respectively. As usual, elements p and q in P are said to be *orthogonal*, in symbols $p \perp q$, iff $p \le u - q = q'$.

LEMMA 2.2. *Let* $p, q \in P$. *Then:* (i) $p \perp q \Leftrightarrow p+q \in P$. (ii) $p \le q \Leftrightarrow q-p \in P$. (iii) $p \perp q \Rightarrow p + q = p \vee q$. (iv) $p \le q \Rightarrow q - p = q \wedge (u - p) = q \wedge p'$. (v) $p \le q \Leftrightarrow J_p \circ J_q = J_q \circ J_p = J_p$. (vi) $pCq \Leftrightarrow J_p \circ J_q = J_q \circ J_p \Leftrightarrow J_p(q) = J_q(p) = p \wedge q$. (vii) *If* pCq, *then* $C(p) \cap C(q) \subseteq C(p \wedge q) \cap C(p \vee q)$.

Proof: As P is a sub-effect algebra of E, (i) and (ii) are obvious. Suppose that $p+q \in P$. Then clearly, $p+q$ is an upper bound in P for p and q. Also, if $r \in P$ and $p, q \le r$, then $p = J_r(p)$ and $q = J_r(q)$, whence $p+q = J_r(p+q) \le J_r(u) = r$, so $p + q = p \vee q$, proving (iii). If $p \le q$, then $q = p + (q - p) = p \vee (q - p)$ by (i) and (iii), and (iv) follows since P is an OML. Part (v) is [9, Lemma 2], part (vi) is [9, Theorem 3 and Corollary 1], and part (vii) is a consequence of [8, Corollary 2.4]. □

LEMMA 2.3. *Let* $p_1, p_2, \ldots, p_n \in P$ *with* $p := \sum_{i=1}^n p_i \le u$. *Then:* (i) $p = \bigvee_{i=1}^n p_i \in P$. (ii) $c \in C(\{p_1, p_2, \ldots, p_n\}) \Rightarrow J_p(c) = \sum_{i=1}^n J_{p_i}(c)$. (iii) $a_1, a_2, \ldots, a_n \in A^+ \Rightarrow (\sum_{i=1}^n a_i)'' = \bigvee_{i=1}^n (a_i)''$.

Proof: Part (i) follows from parts (i) and (iii) of Lemma 2.2 and induction on n, part (ii) is [8, Theorem 2.3 (ii)], and part (iii) is [8, Theorem 6.4 (ii)]. □

DEFINITION 2.1. Let $a \in A$, and let $p \in P^{\pm}(a)$. By [8, Theorem 3.2], $J_p(a)$ and $J_{u-p}(a)$ are independent of the choice of $p \in P^{\pm}(a)$, hence we can and do define $a^+ := J_p(a)$, $a^- := -J_{u-p}(a)$, and $|a| := a^+ + a^-$. We refer to a^+ as the *positive part* of a and to $|a|$ as the *absolute value* of a.

In the sequel, we also make extensive use of the mapping $^+$: $A \to A^+$.

LEMMA 2.4. *Let* $a, b \in A$. *Then:* (i) $0 \le a^+, a^-, |a|$. (ii) $a = a^+ - a^-$. (iii) $a^- = (-a)^+$. (iv) $a^+, a^-, |a| \in CPC(a)$. (v) $a \le b \in CPC(a) \Rightarrow a^+ \le b^+ \Rightarrow (a^+)'' \le (b^+)''$. (vi) $a' = |a|' \in CPC(a)$. (vii) $(a^+)'' \perp (a^-)''$ *with* $(a^+)'' + (a^-)'' = (a^+)'' \vee (a^-)'' = a''$. (viii) *If* $p \in P$, $a \in C(p)$, *and* $J_{p'}(a) \le 0 \le J_p(a)$, *then* $J_{p'}(a) = -a^-$ *and* $J_p(a) = a^+$.

Proof: Let $p \in P^{\pm}(a)$. Parts (i), (ii), and (iii) are obvious. To prove (iv), suppose $q \in P$ and $a \in C(q)$. Then pCq, so by Lemma 1.3 (iv), $a^+ = J_p(a) = J_p(J_q(a) + J_{q'}(a)) = J_p(J_q(a)) + J_p(J_{q'}(a)) = J_q(J_p(a)) + J_{q'}(J_p(a)) = J_q(a^+) + J_{q'}(a^+)$, whence $a^+ \in C(q)$. Likewise, $a^- \in C(q)$, and so $|a| = a^+ + a^- \in C(q)$.

The first implication in (v) follows from [8, Lemma 4.4 (i)], and the second implication is a consequence of Lemma 2.1 (iv). Parts (vi) and (vii) follow from parts (i) and (ii) of [8, Theorem 6.5] and part (viii) follows from [8, Lemma 4.2]. □

THEOREM 2.2. *Let $a \in A$. Then:*
(i) $(a^+)'' \leq a''$ *and* $(a^+)''$ *is the smallest projection in* $P^{\pm}(a)$.
(ii) *If* $q \in P^{\pm}(a)$ *and* $q \leq a''$, *then* $q = (a^+)''$.
(iii) *If* $(a^+)'' = u$, *then* $0 \leq a$.

Proof: Let $p := (a^+)''$. Then $J_{p'}(a^+) = 0$ and $J_p(a^+) = a^+$ by parts (i) and (vii) of Lemma 2.1. By Lemma 2.4 (vii), $p + (a^-)'' = a''$, so $p = a'' - (a^-)'' = a'' \wedge (a^-)'$ by Lemma 2.2 (iv). As $p \leq (a^-)'$, we have $J_p(a^-) = 0$, and since $0 \leq a^-$, Lemma 1.2 (iv) implies that $J_{p'}(a^-) = a^-$. Therefore, $J_p(a) = J_p(a^+ - a^-) = J_p(a^+) - J_p(a^-) = a^+$, and likewise, $J_{p'}(a) = -a^-$, whence $a = J_p(a) + J_{p'}(a)$, i.e. $a \in C(p)$. By Lemma 2.1 (vi), $p = (a^+)'' \in CPC(a^+)$, and by Lemma 2.4 (iv), $a^+ \in CPC(a)$, so $p \in CPC(a)$. Consequently, since $J_{p'}(a) = -a^- \leq 0 \leq a^+ = J_p(a)$, we have $p \in P^{\pm}(a)$.

Suppose that $q \in P^{\pm}(a)$. Then $J_q(a) = a^+$, so $J_q(a^+) = a^+$, and it follows from Lemma 2.1 (vii) that $p \leq q$, completing the proof of (i). Now suppose that $q \leq a''$. Since $a^- = -J_{q'}(a)$, we have $J_{q'}(a^-) = a^-$, so $(a^-)'' \leq q'$, whence $q \leq (a^-)'$, and it follows that $q \leq a'' \wedge (a^-)' = p$, whereupon $q = p$, and (ii) is proved. Part (iii) follows immediately from (i). □

We omit the straightforward proofs of the following two theorems.

THEOREM 2.3. *As in Theorem 1.2, let $v \in P$ and $H := J_v(A)$. Then, for all $h \in H$:* (i) *H is a spectral order-unit space with order unit v.* (ii) *$v' \leq h'$ and the Rickart mapping on H is given by $h \mapsto h' - v' = h' \wedge v$.* (iii) *$h^+ \in H$ and h^+ is the positive part of h as calculated in H.* (iv) *If $a \in C(v)$, then $(J_v(a))' \wedge v = a' \wedge v$ and $(J_v(a))^+ = J_v(a^+)$.*

THEOREM 2.4. *As in Theorem 1.2, let $v \in P$ and $C := C(v)$. Then, for all $c \in C$:* (i) *C is a spectral order-unit space with order unit u.* (ii) *$c' \in C$ and the Rickart mapping on C is the restriction to C of the Rickart mapping on A.* (iii) *$c^+ \in C$ and c^+ is the positive part of c as calculated in C.*

3. Spectral resolution

DEFINITION 3.1. If $a \in A$, then the *spectral lower and upper bounds* for a are defined by $L_a := \sup\{\lambda \in \mathbb{R} : \lambda u \leq a\}$ and $U_a := \inf\{\lambda \in \mathbb{R} : a \leq \lambda u\}$, respectively.

THEOREM 3.1. *If* $a \in A$, *then:* (i) $-\infty < L_a \leq U_a < \infty$. (ii) $\|a\| = \max\{|L_a|, |U_a|\}$. (iii) $L_{-a} = -U_a$ *and* $U_{-a} = -L_a$. (iv) $L_a u \leq a \leq U_a u$.

Proof: Parts (i) and (ii) follow as in the proof of [11, Proposition 4.7], and (iii) is obvious. For every $n \in \mathbb{N}$, it is clear that $(L_a - 1/n)u \leq a$, whence $n(L_a u - a) \leq u$, and as A is archimedean, it follows that $L_a u - a \leq 0$, i.e., $L_a u \leq a$. Therefore, by (iii), $-U_a u \leq -a$, so $a \leq U_a u$, completing the proof of (iv). □

DEFINITION 3.2. Let $a \in A$, $\lambda \in \mathbb{R}$, and define

$$p_\lambda := ((a-\lambda)^+)' \in P \text{ and } d_\lambda := (a-\lambda)' \in P.$$

The family of projections $(p_\lambda)_{\lambda\in\mathbb{R}}$ is called the *spectral resolution* for a. For $\lambda \in \mathbb{R}$, d_λ is called the *λ-eigenprojection* for a, and λ is called an *eigenvalue* of a iff $d_\lambda \neq 0$.

STANDING ASSUMPTION 3.1. *In what follows:* $a \in A$; $L := L_a$ *and* $U := U_a$ *are the spectral bounds for* a; $(p_\lambda)_{\lambda\in\mathbb{R}}$ *is the spectral resolution of* a; *and* $(d_\lambda)_{\lambda\in\mathbb{R}}$ *is the family of eigenprojections for* a.

For the spectral resolution $(p_\lambda)_{\lambda\in\mathbb{R}}$ of a, we have the following uniqueness theorem.[6]

THEOREM 3.2. *For each* $\lambda \in \mathbb{R}$, p_λ *is uniquely determined by the following properties:*
(i) $p_\lambda \in P \cap CPC(a)$ *and* $a \in C(p_\lambda)$.
(ii) $J_{p_\lambda}(a) - \lambda p_\lambda \leq 0 \leq J_{u-p_\lambda}(a) - \lambda(u - p_\lambda)$.
(iii) $(a - \lambda u)' \leq p_\lambda$.

Proof: Let $q := (p_\lambda)' = ((a-\lambda u)^+)''$. By Theorem 2.2, q, hence also $p_\lambda = q'$, is uniquely determined by the conditions $q \in P^\pm(a-\lambda u)$ and $q \leq (a-\lambda u)''$. Therefore, q is uniquely determined by the following properties:

$$q \in P \cap CPC(a - \lambda u) = P \cap CPC(a); \quad a - \lambda u \in C(q), \text{ i.e. } a \in C(q);$$

$$J_{q'}(a) - \lambda q' = J_{q'}(a - \lambda u) \leq 0 \leq J_q(a - \lambda u) = J_q(a) - \lambda q;$$

$$\text{and } q \leq (a - \lambda u)'', \text{ i.e. } (a - \lambda u)' \leq q'.$$

Thus, properties (i), (ii), and (iii) are obtained upon replacing q above by $(p_\lambda)' = u - p_\lambda$. □

LEMMA 3.1. *Let* $\alpha \in \mathbb{R}$. *Then, for all* $\lambda \in \mathbb{R}$: (i) *The spectral resolution* $(q_\lambda)_{\lambda\in\mathbb{R}}$ *of* $a - \gamma u$ *is given by* $q_\lambda = p_{\gamma+\lambda}$. (ii) *The* λ*-eigenprojection for* $a - \gamma u$ *is* $d_{\gamma+\lambda}$. (iii) *The spectral resolution* $(s_\lambda)_{\lambda\in\mathbb{R}}$ *of* $-a$ *is given by* $s_\lambda = (u - p_{-\lambda}) + d_{-\lambda} = (u - p_{-\lambda}) \vee d_{-\lambda}$. (iv) *The* λ*-eigenprojection for* $-a$ *is* $d_{-\lambda}$.

[6] Right continuity of the spectral resolution is a *consequence* of Definition 3.2 (Theorem 3.5 below) and is not required *a priori* for the uniqueness (cf. [2, Definition p. 49] and [2, Theorem 7.2]).

Proof: Parts (i) and (ii) follow directly from Definition 3.2. By Lemma 2.4 (iii), we have

$$u - s_\lambda = ((-a - \lambda u)^+)'' = ((-(a - (-\lambda)u))^+)'' = ((a - (-\lambda)u)^-)''.$$

Therefore, by Lemma 2.4 (vii),

$$\begin{aligned}(u - p_{-\lambda}) + (u - s_\lambda) &= ((a - (-\lambda)u)^+)'' + ((a - (-\lambda)u)^-)'' \\ &= (a - (-\lambda)u)'' = u - d_{-\lambda},\end{aligned}$$

whence, by parts (i) and (iii) of Lemma 2.2, $s_\lambda = (u - p_{-\lambda}) + d_{-\lambda} = (u - p_{-\lambda}) \vee d_{-\lambda}$, proving (iii). To prove (iv), we note that $(-h)' = h'$ for all $h \in A$, so

$$(-a - \lambda u)' = (-(a - (-\lambda)u))' = (a - (-\lambda)u)' = d_{-\lambda}. \qquad \square$$

THEOREM 3.3. *For all* $\lambda, \mu \in \mathbb{R}$:

(i) $p_\lambda, d_\lambda \in P \cap CPC(a)$, $a \in C(p_\lambda) \cap C(d_\lambda)$, *and* $d_\lambda C p_\lambda$.
(ii) $J_{p_\lambda}(a) - \lambda p_\lambda \leq 0 \leq J_{u-p_\lambda}(a) - \lambda(u - p_\lambda)$.
(iii) $\lambda \leq \mu \Rightarrow p_\lambda \leq p_\mu$ *and* $p_\mu - p_\lambda = p_\mu \wedge (u - p_\lambda)$.
(iv) $\lambda < \mu \Rightarrow d_\lambda \leq p_\lambda \leq u - d_\mu \Rightarrow d_\lambda \perp d_\mu$.
(v) $\lambda > U \Rightarrow p_\lambda = u$, *and* $\lambda < U \Rightarrow p_\lambda < u$.
(vi) $\lambda < L \Rightarrow p_\lambda = 0$, *and* $L < \lambda \Rightarrow 0 < p_\lambda$.
(vii) $L = \sup\{\lambda \in \mathbb{R} : p_\lambda = 0\}$, *and* $U = \inf\{\lambda \in \mathbb{R} : p_\lambda = u\}$.
(viii) *If* $\lambda \leq \mu$ *and* $q \in P$ *with* $q \leq p_\mu - p_\lambda$, *then* $\lambda q \leq J_q(a) \leq \mu q$.

Proof: (i) By Theorem 3.2, $p_\lambda \in P \cap CPC(a)$ and $a \in C(p_\lambda)$. By parts (vi) and (i) of Lemma 2.1, $d_\lambda \in CPC(a - \lambda u)$ and $a - \lambda u \in C(d_\lambda)$, whence $d_\lambda \in P \cap CPC(a)$ and $a \in C(d_\lambda)$. As $a \in C(p_\lambda)$ and $d_\lambda \in CPC(a)$, we also have $d_\lambda C p_\lambda$.

(ii) Part (ii) follows from Theorem 3.2.

(iii) Assume that $\lambda \leq \mu$. Then $a - \mu u \leq a - \lambda u$, and $a - \mu u \in CPC(a - \lambda u)$; hence $p_\lambda \leq p_\mu$ follows from Lemma 2.4 (v). Also, $p_\mu - p_\lambda = p_\mu \wedge (u - p_\lambda)$ by Lemma 2.2 (iv).

(iv) By Theorem 3.2 (iii), $d_\lambda \leq p_\lambda$. Assume that $\lambda < \mu$. By (i), $d_\mu \in CPC(a)$ and $a \in C(p_\lambda)$, so $d_\mu C p_\lambda$. Therefore, by Lemma 1.3 (iv), $J_{d_\mu} \circ J_{p_\lambda} = J_{p_\lambda} \circ J_{d_\mu}$, and by Lemma 2.2 (vi), $J_{p_\lambda}(d_\mu) = J_{d_\mu}(p_\lambda) = d_\mu \wedge p_\lambda$. As $d_\mu = (a - \mu u)'$, we have $J_{d_\mu}(a - \mu u) = 0$, i.e., $\mu d_\mu = J_{d_\mu}(a)$. Also, by (ii), $J_{p_\lambda}(a) \leq \lambda p_\lambda$, and it follows that

$$\begin{aligned}\mu(p_\lambda \wedge d_\mu) &= \mu J_{p_\lambda}(d_\mu) = J_{p_\lambda}(\mu d_\mu) = J_{p_\lambda}(J_{d_\mu}(a)) = J_{d_\mu}(J_{p_\lambda}(a)) \\ &\leq J_{d_\mu}(\lambda p_\lambda) = \lambda J_{d_\mu}(p_\lambda) = \lambda(p_\lambda \wedge d_\mu) \leq \mu(p_\lambda \wedge d_\mu),\end{aligned}$$

and therefore $\lambda(p_\lambda \wedge d_\mu) = \mu(p_\lambda \wedge d_\mu)$, i.e. $(\mu - \lambda)(p_\lambda \wedge d_\mu) = 0$. As $0 < \mu - \lambda$, it follows that $p_\lambda \wedge d_\mu = 0$. But $p_\lambda C d_\mu$, and therefore $p_\lambda \leq u - d_{mu}$.

(v) If $\lambda > U$, there exists $\mu \in \mathbb{R}$ such that $\mu < \lambda$ and $a \leq \mu u \leq \lambda u$, whereupon $a - \lambda u \leq 0$, i.e. $(a - \lambda u)^+ = 0$, so $((a - \lambda u)^+)'' = 0$, and it follows that $p_\lambda = u$. Conversely, if $p_\lambda = u$, then $((a - \lambda u)^+)'' = 0$, so $(a - \lambda u)^+ = 0$, whence $a - \lambda u \leq 0$, and it follows that $U \leq \lambda$; consequently, $\lambda < U \Rightarrow p_\lambda < u$.

(vi) Suppose $\lambda < L$. Then there exists $\mu \in \mathbb{R}$ such that $\lambda < \mu$ and $\mu u \leq a$. Therefore, $u \leq (\mu - \lambda)u = \mu u - \lambda u \leq a - \lambda u = (a - \lambda u)^+$, and it follows from Lemma 2.1 (iv) that $u = u'' \leq ((a - \lambda u)^+)'' = u - p_\lambda$, whence $p_\lambda = 0$. Conversely, if $p_\lambda = 0$, then $((a - \lambda u)^+)'' = u$, whence $0 \leq a - \lambda u$, i.e., $\lambda u \leq a$, by Theorem 2.2 (iii), whereupon $\lambda \leq L$; consequently, $L < \lambda \Rightarrow 0 < p_\lambda$.

(vii) Follows directly from (v) and (vi).

(viii) Assume the hypotheses. By (iii), $q \leq p_\mu$ and $q \leq u - p_\lambda$; hence $J_q \circ J_{p_\mu} = J_q \circ J_{u-p_\lambda} = J_q$ by Lemma 2.2 (v). As $q \leq p_\mu$, we have $q = J_q(q) \leq J_q(p_\mu) \leq J_q(u) = q$, whence $q = J_q(p_\mu)$, and likewise, $q = J_q(u - p_\lambda)$. Also, by (ii),

$$\lambda(u - p_\lambda) \leq J_{u-p_\lambda}(a) \quad \text{and} \quad J_{p_\mu}(a) \leq \mu p_\mu \text{ ; hence}$$

$$\begin{aligned} \lambda q &= \lambda J_q(u - p_\lambda) = J_q(\lambda(u - p_\lambda)) \leq J_q(J_{u-p_\lambda}(a)) \\ &= J_q(a) = J_q(J_{p_\mu}(a)) \leq J_q(\mu p_\mu) = \mu J_q(p_\mu) = \mu q. \end{aligned}$$

Consequently, $\lambda q \leq J_q(a) \leq \mu q$. □

THEOREM 3.4. *Suppose that* $\lambda_0, \lambda_1, \ldots, \lambda_n \in \mathbb{R}$ *with* $\lambda_0 < L < \lambda_1 < \cdots < \lambda_{n-1} < U < \lambda_n$, *and let* $\gamma_i \in \mathbb{R}$ *with* $\lambda_{i-1} \leq \gamma_i \leq \lambda_i$ *for* $i = 1, 2, \ldots, n$. *Define* $u_i := p_{\lambda_i} - p_{\lambda_{i-1}}$ *for* $i = 1, 2, \ldots, n$, *and let* $\epsilon := \max\{\lambda_i - \lambda_{i-1} : i = 1, 2, \ldots, n\}$. *Then:*

$$u_1, u_2, \ldots, u_n \in P \cap CPC(a), \qquad a \in C(\{u_1, u_2, \ldots, u_n\}),$$

$$\sum_{i=1}^{n} u_i = 1, \ \textit{and} \ \|a - \sum_{i=1}^{n} \gamma_i u_i\| \leq \epsilon.$$

Proof: In the proof, we understand that $i = 1, 2, \ldots, n$ and that all sums are from $i = 1$ to $i = n$. By parts (i) and (iii) of Theorem 3.3, we have $p_{\lambda_{i-1}} \leq p_{\lambda_i}$ with $p_{\lambda_{i-1}}, p_{\lambda_i} \in P \cap CPC(a)$, whence $u_i \in P \cap CPC(a)$. Also, $a \in C(p_{\lambda_i})$, so $a \in C(u_i)$ by Lemma 2.2 (vii). That $\sum u_i = u$ follows from parts (v) and (vi) of Theorem 3.3. Theorem 3.3 (viii) with $q := u_i$ implies that $\lambda_{i-1}u_i \leq J_{u_i}(a) \leq \lambda_i u_i$, whence $\sum \lambda_{i-1}u_i \leq \sum J_{u_i}(a) \leq \sum \lambda_i u_i$. By Lemma 2.3, $\sum J_{u_i}(a) = J_1(a) = a$, and we have

$$\sum \lambda_{i-1}u_i \leq a \leq \sum \lambda_i u_i.$$

The latter inequalities together with $\lambda_{i-1} \leq \gamma_i \leq \lambda_i$ and $0 \leq u_i$ imply that

$$-\sum(\lambda_i - \lambda_{i-1})u_i \leq a - \sum \gamma_i u_i \leq \sum(\lambda_i - \lambda_{i-1})u_i,$$

whence

$$\left\|a - \sum \gamma_i u_i\right\| \leq \left\|\sum(\lambda_i - \lambda_{i-1})u_i\right\| \leq \epsilon \left\|\sum u_i\right\| = \epsilon \cdot 1 = \epsilon. \qquad \square$$

COROLLARY 3.1. *There exists an ascending sequence* $a_1 \leq a_2 \leq \cdots$ *in* $CPC(a)$ *such that each* a_n *is a finite linear combination of projections in the family* $(p_\lambda)_{\lambda \in \mathbb{R}}$ *and* $\|a - a_n\| \to 0$.

Proof: Choose and fix $\alpha, \beta \in \mathbb{R}$ with $\alpha < L$ and $\beta > U$. As usual, a partition of the closed interval $[\alpha, \beta] \subseteq \mathbb{R}$ is understood to be a finite sequence $\Lambda = (\lambda_i)_{i=0,1,2,\ldots,n} \subseteq [\alpha, \beta]$ such that $\alpha = \lambda_0 < \lambda_1 < \cdots \lambda_{n-1} < \lambda_n = \beta$. The closed interval $[\lambda_{i-1}, \lambda_i]$ is called the ith subinterval of Λ for $i = 1, 2, \ldots, n$, and we define $\epsilon(\Lambda) := \max\{\lambda_i - \lambda_{i-1} : i = 1, 2, \ldots, n\}$. For the partition Λ, we also define $a(\Lambda) := \sum_{i=1}^{n} \lambda_{i-1}(p_{\lambda_i} - p_{\lambda_{i-1}}) \in CPC(a)$, and we have $\|a - a(\Lambda)\| \leq \epsilon(\Lambda)$ by Theorem 3.4 with $\gamma_i = \lambda_{i-1}$ for $i = 1, 2, \ldots, n$.

By recursion, we define a sequence $(\Lambda_n)_{n\in\mathbb{N}}$ of partitions of $[\alpha, \beta]$ as follows: Λ_1 is the partition $\alpha = \lambda_0 < \lambda_1 = \beta$ having only one subinterval, namely $[\alpha, \beta]$ itself. For each partition Λ_n, we form the refined partition Λ_{n+1}, with twice as many subintervals as Λ_n, by appending to the partition Λ_n the midpoints of all its subintervals. Define $a_n := a(\Lambda_n)$ for all $n \in \mathbb{N}$. Evidently, $a_1 \leq a_2 \leq \cdots$. Obviously, $\epsilon(\Lambda_n) = (\beta - \alpha)/2^{n-1}$, and we have $\|a - a_n\| \to 0$. □

COROLLARY 3.2. *If $p \in P$, then $a \in C(p) \Leftrightarrow p_\lambda \in C(p)$ for all $\lambda \in \mathbb{R}$.*

Proof: If $a \in C(p)$ and $\lambda \in \mathbb{R}$, then $p_\lambda C p$ by Theorem 3.3 (i). Conversely, suppose that $p_\lambda C p$ for all $\lambda \in \mathbb{R}$ and let $(a_n)_{n\in\mathbb{N}}$ be the sequence in Corollary 3.1. Then $a_n \in C(p)$ for all $n \in \mathbb{N}$, whence $a \in C(p)$ by Theorem 1.2 (ii). □

The following theorem indicates the sense in which the spectral resolution of a is "continuous from the right."

THEOREM 3.5. *If $\alpha \in \mathbb{R}$, then $p_\alpha = \bigwedge\{p_\mu : \alpha < \mu \in \mathbb{R}\}$.*

Proof: Let $D := \{p_\mu : \alpha < \mu \in \mathbb{R}\}$. By Theorem 3.3 (iii), p_α is a lower bound for D. Suppose that $r \in P$ is another lower bound for D. We have to prove that $r \leq p_\alpha$. Evidently, $p_\alpha \vee r$ is a lower bound for D. Define $q := (p_\alpha \vee r) - p_\alpha = (p_\alpha \vee r) \wedge (u - p_\alpha)$. It will be sufficient to prove that $q = 0$. Let $\lambda \in \mathbb{R}$. If $\lambda \leq \alpha$, then $p_\lambda \leq p_\alpha \leq p_\alpha \vee r$, so $p_\lambda C q$. If $\alpha < \lambda$, then $p_\lambda \in D$, so $q \leq p_\alpha \vee r \leq p_\lambda$, and again $p_\lambda C q$; hence $a \in C(q)$ by Corollary 3.2.

Now suppose that $\alpha < \mu \in \mathbb{R}$. Then $p_\mu \in D$, so $q \leq p_\alpha \vee r \leq p_\mu$ and $q \leq u - p_\alpha$, so $q \leq p_\mu \wedge (u - p_\alpha) = p_\mu - p_\alpha$, and it follows from Theorem 3.3 (viii) that $\alpha q \leq J_q(a) \leq \mu q$. Therefore, $0 \leq J_q(a) - \alpha q \leq (\mu - \alpha) q$, whence $\|J_q(a) - \alpha q\| \leq (\mu - \alpha)\|q\|$, and since $\mu - \alpha$ can be made arbitrarily small, we have $J_q(a - \alpha u) = J_q(a) - \alpha q = 0$ with $a - \alpha u \in C(q)$. Therefore, $q \leq (a - \alpha u)' = d_\alpha \leq p_\alpha$ by Theorem 3.3 (iv). But $q \leq u - p_\alpha$, so $q = 0$. □

REMARK 3.1. By Theorem 3.5, $p_U = u$; hence in the proof of Theorem 3.4, we can take $\lambda_n = U$. Consequently, a can be written as a norm-convergent integral $a = \int_{L-0}^{U} \lambda \, dp_\lambda$ of Riemann–Stieltjes type (cf. [2, Theorem 6.8]).

According to the next theorem, in the same sense as Theorem 3.5, the eigenprojection d_α may be interpreted as the "jump" that occurs as λ approaches α from the left.

THEOREM 3.6. *If $\alpha \in \mathbb{R}$, then $p_\alpha - d_\alpha = \bigvee\{p_\mu : \alpha > \mu \in \mathbb{R}\}$.*

Proof: Let $D := \{p_\mu : \alpha > \mu \in \mathbb{R}\}$. By Theorem 3.3 (iv), $d_\alpha \leq p_\alpha$, so $p_\alpha - d_\alpha = p_\alpha \wedge (u - d_\alpha) \in P$. Let $(s_\lambda)_{\lambda \in \mathbb{R}}$ be the spectral resolution of $-a$. By Lemma 3.1 (iii) and Theorem 3.5, $s_{-\alpha} = u - p_\alpha + d_\alpha$ is the infimum in P of

$$\{s_\lambda : -\alpha < \lambda \in \mathbb{R}\} = \{u - p_{-\lambda} + d_{-\lambda} : \alpha > -\lambda \in \mathbb{R}\} = \{u - p_\mu + d_\mu : \alpha > \mu \in \mathbb{R}\};$$

hence by the De Morgan law in P, $u - (u - p_\alpha + d_\alpha) = p_\alpha - d_\alpha$ is the supremum in P of

$$C := \{u - (u - p_\mu + d_\mu) : \alpha > \mu \in \mathbb{R}\} = \{p_\mu - d_\mu : \alpha > \mu \in \mathbb{R}\}.$$

We have to show that $p_\alpha - d_\alpha = p_\alpha \wedge (u - d_\alpha)$ is also the supremum in P of D. If $\mu < \alpha$, then by parts (iii) and (iv) of Theorem 3.3, $p_\mu \leq p_\alpha \wedge (u - d_\alpha)$, i.e. $p_\alpha \wedge (u - d_\alpha)$ is an upper bound for D. Suppose that $r \in P$ is another upper bound for D. Then, if $\mu < \alpha$, we have $p_\mu - d_\mu \leq p_\mu \leq r$, i.e. r is an upper bound for C; hence $p_\alpha - d_\alpha \leq r$, so $p_\alpha - d_\alpha$ is the supremum of D. □

LEMMA 3.2. *Suppose $v \in P$, $a \in C(v)$, and $H := J_v(A)$. Then:* (i) vCp_λ, vCd_λ *for all* $\lambda \in \mathbb{R}$. (ii) *The spectral resolution of $J_v(a)$ in the spectral order-unit space H (Theorem* 2.3) *is* $(v \wedge p_\lambda)_{\lambda \in \mathbb{R}}$. (iii) *The family of eigenprojections of $J_v(a)$ in H is* $(v \wedge d_\lambda)_{\lambda \in \mathbb{R}}$.

Proof: As $p_\lambda, d_\lambda \in CPC(a)$ and $a \in C(v)$, we have (i). Let $\lambda \in \mathbb{R}$. As $a - \lambda u \in C(v)$ and v is the order unit in H, Theorem 2.3 implies that the projection corresponding to λ in the spectral resolution of $J_v(a)$ in H is

$$\begin{aligned} ((J_v(a) - \lambda v)^+)' \wedge v &= ((J_v(a - \lambda u))^+)' \wedge v = (J_v((a - \lambda u)^+))' \wedge v \\ &= ((a - \lambda u)^+)' \wedge v = p_\lambda \wedge v, \end{aligned}$$

proving (ii), and a similar computation takes care (iii). □

4. The spectrum

In this section, Assumption 3.1 *remains in force*, i.e. $a \in A$; $L := L_a$, $U := U_a$, $(p_\lambda)_{\lambda \in \mathbb{R}}$ is the spectral resolution of a, and $(d_\lambda)_{\lambda \in \mathbb{R}}$ is the family of eigenprojections for a.

DEFINITION 4.1. Let $\alpha, \rho \in \mathbb{R}$. We say that ρ belongs to the *resolvent set* of a iff there exists $0 < \epsilon \in \mathbb{R}$ such that p_λ is constant for λ in the open interval $(\rho - \epsilon, \rho + \epsilon)$. The *spectrum* of a, in symbols, spec(a), is defined to be the complement in $\mathbb{R}$ of the resolvent set of a. If there exists $0 < \epsilon \in \mathbb{R}$ such that both open intervals $(\alpha - \epsilon, \alpha)$ and $(\alpha, \alpha + \epsilon)$ are contained in the resolvent set of a, we shall say that α belongs to the *relative resolvent set of a*.

As is easily seen, if I is an open interval contained in the resolvent set of a, then p_λ is constant for $\lambda \in I$.

LEMMA 4.1. *Let $0 < \epsilon \in \mathbb{R}$, let $\alpha \in \mathbb{R}$, and let $I \subseteq \mathbb{R}$ be an open interval. Then:* (i) *p_λ is constant for λ in the open interval $(\alpha - \epsilon, \alpha)$ iff $p_\lambda = p_\alpha - d_\alpha$*

for all $\lambda \in (\alpha - \epsilon, \alpha)$. (ii) p_μ *is constant for* μ *in the open interval* $(\alpha, \alpha + \epsilon)$ *iff* $p_\mu = p_\alpha$ *for all* $\mu \in (\alpha, \alpha + \epsilon)$. (iii) *If* p_λ *is constant for* $\lambda \in I$, *then* $d_\lambda = 0$ *for all* $\lambda \in I$. (iv) $p_\lambda - d_\lambda$ *is constant for* $\lambda \in I$ *iff* p_λ *is constant for* $\lambda \in I$.

Proof: By Theorems 3.5 and 3.6, $p_\alpha = \inf\{p_\mu : \mu \in (\alpha, \alpha + \epsilon)\}$ and $p_\alpha - d_\alpha = \sup\{p_\lambda : \lambda \in (\alpha - \epsilon, \alpha)\}$, from which (i) and (ii) follow immediately.

Assume the hypothesis of (iii), let $\alpha \in I$, and choose $0 < \epsilon \in \mathbb{R}$ such that $(\alpha - \epsilon, \alpha) \subseteq I$. Choose any $\lambda \in (\alpha - \epsilon, \alpha)$. By part (ii) above, $p_\lambda = p_\alpha - d_\alpha$, and since $\lambda, \alpha \in I$, $p_\lambda = p_\alpha$, whence $d_\alpha = 0$. As $\alpha \in I$ was arbitrary, (iii) follows.

To prove (iv), suppose first that $p_\lambda - d_\lambda$ is constant for $\lambda \in I$. Let $\lambda \in I$ and choose any $\mu \in I$ with $\lambda < \mu$, so that $p_\lambda \leq p_\mu$. As $\lambda, \mu \in I$, we also have $p_\lambda - d_\lambda = p_\mu - d_\mu$. Therefore, $0 \leq p_\mu - p_\lambda = d_\mu - d_\lambda$, so $d_\lambda \leq d_\mu$. But, as $\lambda < \mu$, it follows that $d_\lambda \leq u - d_\mu$, whence $d_\lambda = 0$. Therefore, if $p_\lambda - d_\lambda$ is constant for $\lambda \in I$, then $d_\lambda = 0$ for all $\lambda \in I$, and so p_λ is constant for $\lambda \in I$. The converse implication follows from (ii) above, and (iv) is proved. □

THEOREM 4.1. (i) *If* $\gamma \in \mathbb{R}$, *then* $\mathrm{spec}(a - \gamma u) = \{\alpha - \gamma : \alpha \in \mathrm{spec}(a)\}$. (ii) $\mathrm{spec}(-a) = \{-\alpha : \alpha \in \mathrm{spec}(a)\}$

Proof: Part (i) follows from Lemma 3.1 (i), and part (ii) follows from Lemma 3.1 (iii) and Lemma 4.1 (iv). □

THEOREM 4.2. *Let* $\alpha \in \mathbb{R}$. *Then:* (i) α *is an isolated point of* $\mathrm{spec}(a)$ *iff* α *is in the relative resolvent set of* a, *but not in the resolvent set of* a. (ii) *If* α *is an isolated point of* $\mathrm{spec}(a)$, *then* α *is an eigenvalue of* a. (iii) *If* α *is an eigenvalue of* a, *then* $\alpha \in \mathrm{spec}(a)$. (iv) α *is in the resolvent set of* a *iff* α *is in the relative resolvent set of* a, *but* α *is not an eigenvalue of* a.

Proof: Part (i) follows directly from the definitions involved. Suppose that α is an isolated point of $\mathrm{spec}(a)$. Then there exists $0 < \epsilon \in \mathbb{R}$ such that p_λ is constant on $(\alpha - \epsilon, \alpha)$ and on $(\alpha, \alpha + \epsilon)$; hence by parts (i) and (ii) of Lemma 4.1, $p_\lambda = p_\alpha - d_\alpha$ for $\alpha - \epsilon < \lambda < \alpha$ and $p_\lambda = p_\alpha$ for $\alpha \leq \lambda < \alpha + \epsilon$. Since α is not in the resolvent set of a, p_λ cannot be constant on $(\alpha - \epsilon, \alpha + \epsilon)$, and it follows that $d_\alpha \neq 0$, proving (ii). Part (iii) is a consequence of Lemma 4.1 (iii). As the resolvent set of a is contained in the relative resolvent set of a, (iv) follows from (i), (ii), and (iii). □

THEOREM 4.3. $\mathrm{spec}(a)$ *is a closed nonempty subset of the closed interval* $[L, U] \subseteq \mathbb{R}$, $L = \inf(\mathrm{spec}(a)) \in \mathrm{spec}(a)$, $U = \sup(\mathrm{spec}(a)) \in \mathrm{spec}(a)$, *and* $\|a\| = \sup\{|\alpha| : \alpha \in \mathrm{spec}(a)\}$.

Proof: Clearly, the resolvent set of a is open, so $\mathrm{spec}(a)$ is closed. By parts (v) and (vi) of Theorem 3.3, $(-\infty, L)$ and (U, ∞) are contained in the resolvent set of a, whereas $L, U \in \mathrm{spec}(a)$, and it follows that $L = \inf(\mathrm{spec}(a))$ and $U = \sup(\mathrm{spec}(a))$. Therefore, by Theorem 3.1 (ii), $\|a\| = \sup\{|\alpha| : \alpha \in \mathrm{spec}(a)\}$. □

COROLLARY 4.1. *If $\alpha \in \mathbb{R}$, then the following conditions are mutually equivalent:* (i) $d_\alpha = u$. (ii) $a = \alpha u$. (iii) $\operatorname{spec}(a) = \{\alpha\}$.

Proof: As $d_\alpha = (a - \alpha u)'$ and $(a - \alpha u)'' = 0 \Leftrightarrow a - \alpha u = 0$, it is clear that (i) $\Leftrightarrow$ (ii). If $a = \alpha u$, then $L = \sup\{\lambda \in \mathbb{R} : \lambda u \le a\} = \alpha$ and $U = \inf\{\mu \in \mathbb{R} : a \le \mu u\} = \alpha$, whence $\operatorname{spec}(a) = \{\alpha\}$ by Theorem 4.3, so (ii) $\Rightarrow$ (iii). Conversely, if $\operatorname{spec}(a) = \{\alpha\}$, then $L = \alpha = U$ by Theorem 4.3, and therefore $a = \alpha u$ by Theorem 3.1 (iv); hence (iii) $\Rightarrow$ (ii). □

THEOREM 4.4. *The following conditions are mutually equivalent:* (i) $0 \le a$. (ii) $0 \le L$. (iii) *If $\lambda \in \mathbb{R}$, then $\lambda < 0 \Rightarrow p_\lambda = 0$.* (iv) $\operatorname{spec}(a) \subseteq [0, \infty)$.

Proof: That (i) $\Rightarrow$ (ii) follows from the definition of L, and the converse implication is a consequence of Theorem 3.1 (iv); hence (i) $\Leftrightarrow$ (ii). That (ii) $\Leftrightarrow$ (iii) follows from Theorem 3.3 (vi). By Theorem 4.3, $L = \inf(\operatorname{spec}(a))$, from which (ii) $\Leftrightarrow$ (iv) follows. □

The terminology in the next definition is suggested by the fact that, if A is the self-adjoint part of a unital von Neumann algebra and $a \in A$, then a is von Neumann regular in A iff $|a|$ is an order unit in $a''Aa''$.

DEFINITION 4.2. The element a is *regular* iff $|a|$ is an order unit in the spectral order-unit space $J_{a''}(A)$ (see Theorem 2.3).[7] The element a is *nonsingular* iff $a'' = u$ and a is regular. If a fails to be nonsingular, we say that a is *singular*.

We omit the straightforward proof of the following.

LEMMA 4.2. *$|a|$ is an order unit in A iff there exists $n \in \mathbb{N}$ such that $u \le n|a|$, and if $|a|$ is an order unit in A, then $a'' = |a|'' = u$. Moreover, the following conditions are mutually equivalent:* (i) *a is regular.* (ii) *There exists $n \in \mathbb{N}$ such that $a'' \le n|a|$.* (iii) *There exists $0 < \epsilon \in \mathbb{R}$ such that $\epsilon a'' \le |a|$.* (iv) *a is nonsingular in $J_{a''}(A)$.*

The well-known Gelfand–Mazur theorem for Banach algebras has the following analogue for the spectral order-unit space A.

THEOREM 4.5. *The following conditions are mutually equivalent:* (i) $A = \mathbb{R}u$. (ii) *Every nonzero element in A^+ is nonsingular.* (iii) $P = \{0, u\}$.

Proof: Obviously, (i) $\Rightarrow$ (ii). Assume (ii) and suppose $p \in P$ with $p \ne 0$. Then there exists $n \in \mathbb{N}$ with $u \le np$; whence $p' = J_{p'}(u) \le nJ_{p'}(p) = 0$, so $p' = 0$, and therefore $p = u$. Assume (iii). Then, by Theorem 3.3 (vi), $p_\mu = u$ for $U \le \mu$ and $p_\lambda = 0$ for $\lambda < U$; hence $\operatorname{spec}(a) = \{U\}$. Therefore, $a = Uu$ by Corollary 4.1, and since a is an arbitrary element of A, (i) follows. □

THEOREM 4.6. *a is regular iff both a^+ and a^- are regular.*

[7] By Lemma 2.4 (vi), $a'' = |a|''$, whence $|a| = J_{a''}(|a|) \in J_{a''}(A)$.

Proof: Let $p := (a^+)''$ and $q := (a^-)''$. Then $a^+ = J_p(a)$, $a^- = J_q(a)$, $J_p(q) = J_q(p) = J_p(a^-) = J_q(a^+) = 0$, and $p + q = p \vee q = a''$ by Lemma 2.4 (vii).

Suppose that a is regular. Then there exists $0 < \epsilon \in \mathbb{R}$ with $\epsilon(p+q) = \epsilon a'' \leq |a| = a^+ + a^-$, so $\epsilon p = J_p(\epsilon(p+q)) \leq J_p(a^+) + J_p(a^-) = a^+$, whence a^+ is regular. Likewise, $\epsilon q \leq a^-$, so a^- is regular. Conversely, if both a^+ and a^- are regular, there exist $0 < \alpha, \beta$ such that $\alpha p \leq a^+$ and $\beta q \leq a^-$; hence with $\epsilon := \min\{\alpha, \beta\}$, we have $\epsilon a'' = \epsilon(p+q) \leq a^+ + a^- = |a|$, and it follows that a is regular. □

COROLLARY 4.2. *a is nonsingular iff $a'' = u$ and both a^+ and a^- are regular.*

LEMMA 4.3. *Suppose that $0 \leq a$. Then the following conditions are mutually equivalent:* (i) *a is nonsingular.* (ii) $0 < L$. (iii) $0 \notin \text{spec}(a)$.

Proof: As $0 \leq a$, a is nonsingular iff there exists $0 < \epsilon \in \mathbb{R}$. such that $\epsilon u \leq a$. As $L = \sup\{\lambda : \lambda u \leq a\}$, it follows that (i) $\Leftrightarrow$ (ii). By Theorem 3.3 (vi), $p_\lambda = 0$ for all $\lambda \in (-\infty, L)$, whence if (ii) holds, then 0 belongs to the resolvent set of a. Therefore, (ii) $\Rightarrow$ (iii). Conversely, suppose that (iii) holds. Then there exists $0 < \epsilon \in \mathbb{R}$ such that p_λ is constant on $(-\epsilon, \epsilon)$. By Lemma 4.4, $p_\lambda = 0$ for $-\epsilon < \lambda < 0$, whence $p_\lambda = 0$ for $-\epsilon < \lambda < \epsilon$, and it follows from Theorem 3.3 (vi) that $0 < \epsilon \leq L$. Consequently, (iii) $\Rightarrow$ (ii). □

LEMMA 4.4. *Suppose $0 \leq a$. Then the following conditions are mutually equivalent:* (i) *a is regular.* (ii) *There exists $0 < \epsilon \in \mathbb{R}$ such that, for all $\lambda \in \mathbb{R}$, $0 \leq \lambda < \epsilon \Rightarrow p_\lambda = p_0$.* (iii) *There exists $0 < \epsilon \in \mathbb{R}$ such that p_λ is constant for $\lambda \in (0, \epsilon)$.*

Proof: Let $v := a''$ and $H := J_v(A)$. Then $0 \leq a = J_v(a) \in H$, $p_0 = (a^+)' = a' = v'$, and $J_v(p_0) = 0$. Let ϵ be the lower spectral bound of a as calculated in H. By Theorem 3.3 (vii) and Lemma 3.2, $\epsilon = \sup\{\lambda \in \mathbb{R} : v \wedge p_\lambda = 0\}$, and as $v \wedge p_0 = 0$, we have $0 \leq \epsilon = \sup\{0 \leq \lambda \in \mathbb{R} : v \wedge p_\lambda = 0\}$. Let $0 \leq \lambda \in \mathbb{R}$. Then $p_0 \leq p_\lambda$, whence $v \wedge p_\lambda = 0 \Leftrightarrow p_\lambda \leq p_0 \Leftrightarrow p_\lambda = p_0$, and it follows that

$$\epsilon = \sup\{0 \leq \lambda \in \mathbb{R} : p_\lambda = p_0\}.$$

By Lemma 4.3, a is an order unit in H iff $0 < \epsilon$, hence (i) and (ii) are equivalent. That (ii) $\Leftrightarrow$ (iii) follows from Lemma 4.1 (ii). □

LEMMA 4.5. *Let $a \in A$ and let $0 \leq \lambda \in \mathbb{R}$. Then $(a - \lambda u)^+ = (a^+ - \lambda u)^+$.*

Proof: Obviously, $(a^+)^+ = a^+$, so we can and do assume that $0 < \lambda$. If $b \in A$, it is clear that $(\lambda b)^+ = \lambda b^+$; hence $(a^+ - \lambda u)^+ = \lambda((\lambda^{-1}a)^+ - u)^+$ and $(a - \lambda u)^+ = \lambda(\lambda^{-1}a - u)^+$. Thus, replacing $\lambda^{-1}a$ by a, we need only prove the lemma for the case $\lambda = 1$, i.e. we need only prove that $(a - u)^+ = (a^+ - u)^+$. Put

$$p := (a^+)'' = u - p_0 \quad \text{and} \quad q := ((a-u)^+)'' = u - p_1.$$

By Lemma 2.4 (iv), $a^+ \in CPC(a)$ and by Theorem 3.3 (i), $a \in C(p_1) = C(q)$, whence $a^+ \in C(q)$, and it follows that $a^+ - u \in C(q)$. Also, as $p_0 \leq p_1$, we have

$q \leq p$. As a consequence of Theorem 2.2, $p \in P^{\pm}(a)$ and $q \in P^{\pm}(a-u)$. Therefore, $J_{q'}(a) - q' = J_{q'}(a-u) \leq 0$, so

$$J_{q'}(a) \leq q' \text{ with } J_p(a) = a^+ \text{ and } J_q(a-u) = (a-u)^+. \tag{1}$$

As $q \leq p$, Lemma 2.2 (iv) implies that $J_q \circ J_p = J_q$, so by (1), $J_q(a^+) = J_q(J_p(a)) = J_q(a)$, and therefore

$$J_q(a^+ - u) = J_q(a^+) - q = J_q(a) - q = J_q(a-u) = (a-u)^+ \geq 0. \tag{2}$$

The condition $q \leq p$ also implies that $q'Cp$, so by Lemma 1.3 (iv), (1), and Lemma 2.2 (vi), $J_{q'}(J_p(a)) = J_p(J_{q'}(a)) \leq J_p(q') = p \wedge q'$, and we have

$$J_{q'}(a^+ - u) = J_{q'}(J_p(a)) - q' = p \wedge q' - q' \leq 0. \tag{3}$$

By (2) and (3), $J_{q'}(a^+ - u) \leq 0 \leq J_q(a^+ - u)$, and as $a^+ - u \in C(q)$, Lemma 2.4 (viii) implies that $J_q(a^+ - u) = (a^+ - u)^+$. Therefore, $(a-u)^+ = (a^+ - u)^+$ by (1). □

LEMMA 4.6. *Let $(q_\lambda)_{\lambda \in \mathbb{R}}$ be the spectral resolution of a^+, and let $(r_\lambda)_{\lambda \in \mathbb{R}}$ be the spectral resolution of a^-. Then:* (i) $0 \leq \lambda \in \mathbb{R} \Rightarrow q_\lambda = p_\lambda$ *and* $r_\lambda = u - (p_{-\lambda} - d_{-\lambda})$. (ii) $0 > \lambda \in \mathbb{R} \Rightarrow q_\lambda = r_\lambda = 0$.

Proof: Let $0 \leq \lambda \in \mathbb{R}$. That $q_\lambda = p_\lambda$ follows directly from Lemma 4.5. By Lemma 3.1 (iii), the spectral resolution $(s_\mu)_{\mu \in \mathbb{R}}$ of $-a$ is given by $s_\mu = u - (p_{-\mu} - d_{-\mu})$ for all $\mu \in \mathbb{R}$; hence, applying what has just been proved to $-a$, we find that $r_\lambda = s_\lambda = u - (p_{-\lambda} - d_{-\lambda})$, completing the proof of (i). As $0 \leq a^+, a^-$, (ii) follows from Lemma 4.4. □

THEOREM 4.7. *Let $\alpha \in \mathbb{R}$. Then:* (i) *a is regular iff 0 belongs to the relative resolvent set of a.* (ii) *$a - \alpha u$ is regular iff α belongs to the relative resolvent set of a.* (iii) *a is nonsingular iff 0 belongs to the resolvent set of a.* (iv) $\operatorname{spec}(a) = \{\alpha \in \mathbb{R} : a - \alpha u$ *is singular*$\}$.

Proof: (i) Let $(q_\lambda)_{\lambda \in \mathbb{R}}$ be the spectral resolution of a^+, and let $(r_\lambda)_{\lambda \in \mathbb{R}}$ be the spectral resolution of a^-. By Theorem 4.6 and Lemma 4.4, it will be sufficient to show that, if $0 < \epsilon \in \mathbb{R}$, then p_λ is constant on $(-\epsilon, 0)$ and also on $(0, \epsilon)$ iff both q_λ and r_λ are constant on $(0, \epsilon)$. By Lemma 4.6 (i), q_λ is constant on $(0, \epsilon)$ iff p_λ is constant on $(0, \epsilon)$. Also, by Lemma 4.6 (i) and Lemma 4.1 (iv), r_λ is constant on $(0, \epsilon)$ iff $p_\lambda - d_\lambda$ is constant on $(-\epsilon, 0)$ iff p_λ is constant on $(-\epsilon, 0)$.

(ii) Part (ii) follows from (i) and Lemma 3.1 (i).

(iii) By (i) above, a is nonsingular iff 0 belongs to the relative resolvent set of a and $a'' = u$. But $a'' = u - d_0$, so a is nonsingular iff 0 belongs to the relative resolvent set of a and 0 is not an eigenvalue of a. Therefore, (iii) follows from Theorem 4.2 (iv).

(iv) Part (iv) follows from (iii) and Lemma 3.1 (i). □

5. Simple elements of A

In this section, *we maintain Standing Assumption* 3.1.

DEFINITION 5.1. We shall say that an element in A is *simple* iff it can be written as a finite linear combination of pairwise compatible projections.

REMARK 5.1. By Corollary 3.1, each element $a \in A$ is a norm limit of an ascending sequence $a_1 \leq a_2 \leq \cdots$ of simple elements; hence, the simple elements are norm dense in A.

LEMMA 5.1. *$a \in A$ is simple iff there are real numbers $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ and projections $0 \neq u_i \in P$ for $i = 1, 2, \ldots, n$ such that $a = \sum_{i=1}^{n} \alpha_i u_i$, and $\sum_{i=1}^{n} u_i = u$.*

Proof: Suppose that a is simple. Then there exist real numbers γ_k and pairwise compatible projections $v_k \in P$ for $k = 1, 2, \ldots, M$ such that $a = \sum_{k=1}^{M} \gamma_k v_k$. The pairwise compatible projections v_k, $k = 1, 2, \ldots, M$, generate a finite Boolean sublattice B of the OML P. Let w_j, $j = 1, 2, \ldots, m$ be the (distinct) atoms in B. Then every v_k can be written as a sum of w_j's, and by collecting terms and inserting terms with zero coefficients if necessary, we can write $a = \sum_{j=1}^{m} \beta_j w_j$ with $\beta_j \in \mathbb{R}$ for $j = 1, 2, \ldots, m$. Let $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ be a listing of the distinct coefficients β_j, $j = 1, 2, \ldots, m$. Again by collecting terms, we can write $a = \sum_{i=1}^{n} \alpha_i u_i$, where each u_i is the sum of all the w_j's for which $\alpha_i = \beta_j$. Since $\sum_{j=1}^{m} w_j = u$, it follows that $\sum_{i=1}^{n} u_i = u$. The converse is obvious. □

DEFINITION 5.2. If a is a simple element of A, we shall refer to the representation $a = \sum_{i=1}^{n} \alpha_i u_i$ satisfying the conditions in Lemma 5.1 as the *reduced representation* of a.

LEMMA 5.2. *Suppose $\alpha \in \mathbb{R}$ and $a - \alpha u$ is regular. Define $v := (a - \alpha u)'' = u - d_\alpha$. Then:* (i) *If $a \neq \alpha u$, then $0 \neq v \in CPC(a)$, $J_v(a) \in J_v(A)$, and the spectrum of $J_v(a)$ as calculated in $J_v(A)$ is* $\mathrm{spec}(a) \setminus \{\alpha\}$. (ii) *If $\alpha \in$* $\mathrm{spec}(a)$, *then $0 \neq v' \in CPC(a)$ and $J_{v'}(a) = \alpha v'$.*

Proof: Since $(a - \alpha u)'' \in CPC(a)$, we have $v, v' \in CPC(a)$, whence $v, v' \in C(p_\lambda)$ for all $\lambda \in \mathbb{R}$. Also, since $a - \alpha u$ is regular, α belongs to the relative resolvent set of a by Theorem 4.7 (ii), so by parts (i) and (ii) of Lemma 4.1, there exists $0 < \epsilon \in \mathbb{R}$ such that, for all $\lambda \in \mathbb{R}$,

$$\alpha \leq \lambda < \alpha + \epsilon \Rightarrow p_\lambda = p_\alpha \quad \text{and} \quad \alpha - \epsilon < \lambda < \alpha \Rightarrow p_\lambda = p_\alpha - d_\alpha. \tag{1}$$

(i) By Lemma 3.2, the spectral resolution of $J_v(a)$ in $J_v(A)$ is $(v \wedge p_\lambda)_{\lambda \in \mathbb{R}}$. As $d_\alpha = v'$, we have $v \wedge d_\alpha = 0$, so by (1),

$$\alpha - \epsilon < \lambda < \alpha + \epsilon \Rightarrow v \wedge p_\lambda = v \wedge p_\alpha, \tag{2}$$

whence α does not belong to the spectrum of a as calculated in $J_v(A)$. By parts

(iii) and (iv) of Theorem 3.3, we have

$$\alpha \leq \lambda \Rightarrow v' = d_\alpha \leq p_\alpha \leq p_\lambda \Rightarrow v \wedge p_\lambda = p_\lambda - v'. \tag{3}$$

and

$$\lambda < \alpha \Rightarrow p_\lambda \leq u - d_\alpha = v \Rightarrow v \wedge p_\lambda = p_\lambda. \tag{4}$$

Suppose $\beta \in \mathbb{R}$ and $\beta \neq \alpha$. By (3) and (4), if I is an open interval in $\mathbb{R}$ with $\beta \in I$ and $\alpha \notin I$, then p_λ is constant on I iff $v \wedge p_\lambda$ is constant on I, i.e., β belongs to the spectrum of $J_v(a)$ as calculated in $J_v(A)$ iff β belongs to spec(a).

(ii) Assume that $\alpha \in$ spec(a). Then, by parts (i) and (ii) of Theorem 4.2, $v' = d_\alpha \neq 0$, and by Lemma 3.2, the spectral resolution of $J_{v'}(a)$ in $J_{v'}(A)$ is $(v' \wedge p_\lambda)_{\lambda \in \mathbb{R}}$. By parts (iii) and (iv) of Theorem 3.3, we have

$$p_\lambda \leq v \text{ for } \lambda < \alpha \text{ and } v' = d_\alpha \leq p_\alpha \leq p_\lambda \text{ for } \alpha \leq \lambda,$$

whence

$$v' \wedge p_\lambda = 0 \text{ for } \lambda < \alpha \text{ and } v' \wedge p_\lambda = v' \neq 0 \text{ for } \alpha \leq \lambda.$$

Thus, the spectrum of $J_{v'}(a)$ as calculated in $J_{v'}(A)$ is $\{\alpha\}$, and $J_{v'}(a) = \alpha v'$ follows from Corollary 4.1. □

THEOREM 5.1. *Suppose that* spec$(a) = \{\alpha_i : i = 1, 2, \ldots, n\}$. *Then* $d_{\alpha_i} \neq 0$ *for* $i = 1, 2, \ldots, n$, $\sum_{i=1}^n d_{\alpha_i} = u$ *and* $a = \sum_{i=1}^n \alpha_i d_{\alpha_i}$.

Proof: As each α_i is an isolated point of spec(a), Theorem 4.2(ii) implies that $d_{\alpha_i} \neq 0$ for $i = 1, 2, \ldots, n$. The remainder of the proof is by induction on n. Corollary 4.1 takes care of the case $n = 1$. Suppose $n > 1$, let $\alpha := \alpha_n$, and let $v := u - d_{\alpha_n}$. By Corollary 4.1 again, $a \neq \alpha u$. By Theorem 4.2 (i), and Theorem 4.7 (ii), $a - \alpha u$ is regular. Thus by Lemma 5.2 (i), $v \neq 0$ and the spectrum of $J_v(a)$ in $J_v(A)$ is $\{\alpha_i : i = 1, 2, \ldots, n-1\}$; moreover, by Lemma 3.2, $(v \wedge d_\lambda)_{\lambda \in \mathbb{R}}$ is the family of eigenprojections of $J_v(a)$ in $J_v(A)$. Also, by Theorem 3.3 (iv), $d_{\alpha_i} \leq u - d_{\alpha_n} = v$, whence $v \wedge d_{\alpha_i} = d_{\alpha_i}$ for $1 \leq i \leq n-1$. Thus, by the inductive hypothesis,

$$J_v(a) = \sum_{i=1}^{n-1} \alpha_i (v \wedge d_{\alpha_i}) = \sum_{i=1}^{n-1} \alpha_i d_{\alpha_i} \text{ and } \sum_{i=1}^{n-1} d_{\alpha_i} = v.$$

Also, by Lemma 5.2 (ii), $J_{v'}(a) = \alpha v' = \alpha d_\alpha = \alpha_n d_{\alpha_n}$, and it follows that

$$a = J_v(a) + J_{v'}(a) = \sum_{i=1}^{n-1} \alpha_i d_{\alpha_i} + \alpha_n d_{\alpha_n} = \sum_{i=1}^{n} \alpha_i d_{\alpha_i}$$

with

$$\sum_{i=1}^{n} d_{\alpha_i} = \sum_{i=1}^{n-1} d_{\alpha_i} + d_{\alpha_n} = v + v' = u.$$

□

LEMMA 5.3. *Let* $u_i \in P$ *with* $v := \sum_{i=1}^n u_i \leq u$, *suppose that* $0 \neq \alpha_i \in \mathbb{R}$ *for* $i = 1, 2, \ldots, n$, *and let* $a := \sum_{i=1}^n \alpha_i u_i$. *Define* $I^+ := \{i = 1, 2, \ldots, n : \alpha_i > 0\}$,

$I^- := \{i = 1, 2, \ldots, n : \alpha_i < 0\}$, $p := \sum_{i \in I^+} u_i$ *and* $q := \sum_{i \in I^-} u_i$. *Then* (i) $u_1, u_2, \ldots u_n$ *are pairwise orthogonal,* $v, p, q \in P$, *and* $v = p + q = \bigvee_{i=1}^n u_i$. (ii) $J_{u_i}(a) = \alpha_i u_i$ *and* $a \in C(u_i)$ *for* $i = 1, 2, \ldots, n$. (iii) $J_p(a) = J_{q'}(a) = \sum_{i \in I^+} \alpha_i u_i \geq 0$ *and* $J_q(a) = J_{p'}(a) = \sum_{i \in I^-} \alpha_i u_i \leq 0$. (iv) $J_p(a) = a^+$, $J_q(a) = -a^-$, *and* $|a| = \sum_{i=1}^n |\alpha_i| u_i$. (v) a *is regular and* $a'' = v$.

Proof: Part (i) follows from Lemma 2.3 (i). As a consequence of the pairwise orthogonality of the u_i's, we have $J_{u_i}(a) = \alpha_i u_i$ and $J_{u-u_i}(a) = a - \alpha_i u_i$, so $a \in C(u_i)$ for $i = 1, 2, \ldots, n$, proving (ii). As $u_i \leq p, q'$ for $i \in I^+$ and $u_i \leq q, p'$ for $i \in I^-$, (iii) is clear. Since $a \in C(u_i)$ for all $i \in I^+$, it follows that $a \in C(p)$, hence $J_{p'}(a) \leq 0 \leq J_p(a)$ implies that $J_p(a) = a^+$ and $J_{p'}(a) = -a^-$ by Lemma 2.4 (viii). Similarly $J_q(a) = -a^-$ and $J_{q'}(a) = a^+$, whence $|a| = a^- + a^+ = \sum_{i \in I^-}(-\alpha_i)u_i + \sum_{i \in I^+} \alpha_i u_i = \sum_{i=1}^n |\alpha_i| u_i$, proving (iv). As $0 \neq |\alpha_i|$, it is clear that $(|\alpha_i| u_i)'' = u_i$ for $i = 1, 2, \ldots n$, whence $a'' = |a|'' = (\sum_{i \in I^+} |\alpha_i| u_i)'' = \bigvee_{i=1}^n u_i = v$ by Lemma 2.4 (vi) and Lemma 2.3 (iii). Finally, with $\epsilon := \min\{|\alpha_i| : i = 1, 2, \ldots n\}$ we obviously have $\epsilon a'' = \epsilon v \leq |a|$, so a is regular by Lemma 4.2, proving (iv). □

THEOREM 5.2. *Suppose that* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *are distinct real numbers,* $u_1, u_2, \ldots, u_n \in P$ *are nonzero projections,* $a = \sum_{i=1}^n \alpha_i u_i$, *and* $\sum_{i=1}^n u_i = u$. *Then* $\operatorname{spec}(a) = \{\alpha_i : i = 1, 2, \ldots, n\}$ *and* $u_i = d_{\alpha_i}$ *for* $i = 1, 2, \ldots, n$.

Proof: By Corollary 4.1, we need only consider the case $n > 1$. If $\alpha \in \mathbb{R}$, then

$$a - \alpha u = \sum_{i=1}^n \alpha_i u_i - \alpha \sum_{i=1}^n u_i = \sum_{i=1}^n (\alpha_i - \alpha) u_i.$$

On the one hand, if $\alpha \neq \alpha_i$ for $i = 1, 2, \ldots, n$, then the coefficients $\alpha_i - \alpha$ are nonzero, Lemma 5.3 (iv) implies that $a - \alpha u$ is regular with $(a - \alpha u)'' = \sum_{i=1}^n u_i = u$, so $a - \alpha u$ is nonsingular, and it follows from Theorem 4.7 (iv) that $\alpha \notin \operatorname{spec}(a)$. On the other hand, putting $\alpha := \alpha_1$, we find that $a - \alpha_1 u = \sum_{i=2}^n (\alpha_i - \alpha_1) u_i$, where again the coefficients are nonzero; hence $(a - \alpha_1 u)'' = \sum_{i=2}^n u_i = u - u_1 \neq u$, so $a - \alpha_1 u$ is singular, and therefore $\alpha_1 \in \operatorname{spec}(a)$. Moreover, $d_{\alpha_1} = u - (a - \alpha_1 u)'' = u_1$. By symmetry, $\alpha_i \in \operatorname{spec}(a)$ with $u_i = d_{\alpha_i}$ for $i = 1, 2, \ldots, n$. □

COROLLARY 5.1. *a is a projection iff* $\operatorname{spec}(a) \subseteq \{0, 1\}$.

Proof: By Corollary 4.1, $a = 0 \Leftrightarrow \operatorname{spec}(a) = \{0\}$ and $a = u \Leftrightarrow \operatorname{spec}(a) = \{1\}$. If a is a projection, $a \neq 0, u$, then $a = 0(u - a) + 1a$ where the projections $u - a$ and a are nonzero and $(u - a) + a = u$; hence $\operatorname{spec}(a) = \{0, 1\}$ by Theorem 5.2. Conversely, if $\operatorname{spec}(a) = \{0, 1\}$, then $a = 0d_0 + 1d_1 = d_1 \in P$ by Theorem 5.1. □

As a consequence of Theorems 4.3, 5.1, 5.2, and Lemma 5.3 we have the following.

THEOREM 5.3. *The simple elements in A are precisely those with finite spectrum, each simple element* $a \in A$ *has a unique reduced representation* $a = \sum_{i=1}^n \alpha_i u_i$, *with* $\alpha_1 < \alpha_2 < \cdots < \alpha_n$, $0 \neq u_i \in P$, $\sum_{i=1}^n u_i = u$, $\operatorname{spec}(a) = \{\alpha_i : i = 1, 2, \ldots, n\}$,

$|a| = \sum_{i=1}^{n} |\alpha_i| u_i$, *and* $\|a\| = \max\{|\alpha_i| : i = 1, 2, \ldots, n\}$. *Moreover, every simple element in* A *is regular.*

THEOREM 5.4. *The following conditions are mutually equivalent:* (i) a *is simple.* (ii) $a - \lambda u$ *is regular for all* $\lambda \in \mathbb{R}$. (iii) $\operatorname{spec}(a)$ *consists entirely of isolated points.* (iv) $\operatorname{spec}(a)$ *is finite.* (v) $\{p_\lambda : \lambda \in \mathbb{R}\}$ *is a finite chain in the OML* P.

Proof: (i) $\Rightarrow$ (ii). Suppose a is simple and let $a = \sum_{i=1}^{n} \alpha_i u_i$ be the reduced representation of a. Then, for $\lambda \in \mathbb{R}$, $a - \lambda u = \sum_{i=1}^{n} (\alpha_i - \lambda) u_i$, whence $a - \lambda u$ is simple, and therefore regular by Theorem 5.3.

(ii) $\Rightarrow$ (iii). Assume (ii). Then by Theorem 4.7 (ii), every real number belongs to the relative resolvent set of a; hence, (iii) follows from Theorem 4.2 (i).

(iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (i). Since $\operatorname{spec} a$ is a compact subset of $\mathbb{R}$, it consists entirely of isolated points iff it is finite, and $\operatorname{spec}(a)$ is finite iff a is simple by Theorem 5.3.

(iv) $\Leftrightarrow$ (v). Assume that $\operatorname{spec} a$ is finite and list its elements as $L = \alpha_1 < \alpha_2 < \cdots < \alpha_n = U$. Then, p_λ is constant on each of the intervals $(-\infty, \alpha_1)$, $[\alpha_1, \alpha_2), \ldots, [\alpha_{n-1}, \alpha_n)$, $[\alpha_n, \infty)$, whence $\{p_\lambda : \lambda \in \mathbb{R}\}$ is finite.

Conversely, suppose that $\{p_\lambda : \lambda \in \mathbb{R}\}$ is finite and list its elements as $0 = q_0 < q_1 < q_2 < \cdots < q_n = u$. For each $i = 1, 2, \ldots, n$, let $\alpha_i := \inf\{\lambda \in \mathbb{R} : p_\lambda = q_i\}$. Then $q_i = p_{\alpha_i}$ for each $i = 1, 2, \ldots, n$,

$$L = \alpha_1 < \alpha_2 < \cdots < \alpha_{n-1} < \alpha_n = U,$$

and for all $\mu \in \mathbb{R}$ and all $i = 1, 2, \ldots, n$,

$$\alpha_{i-1} \leq \mu < \alpha_i \Rightarrow p_\mu = p_{\alpha_{i-1}} = q_{i-1};$$

hence each open interval (α_{i-1}, α_i) is contained in the resolvent set of a. Consequently, $\operatorname{spec}(a) = \{L, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}, U\}$. □

COROLLARY 5.2. (i) *If* A *has finite dimension, then* P *satisfies the chain conditions (i.e. there are no infinite properly ascending or descending sequences in* P*).* (ii) *If* P *satisfies the chain conditions, then every element in* A *is simple, hence every element in* A *is regular.*

Proof: Part (i) follows from the observation that, for each $p \in P$, $J_p(A)$ is a vector subspace of A, and if $p, q \in P$, then $p \leq q \Leftrightarrow J_p(A) \subseteq J_q(A)$. Part (ii) is a consequence of the fact that a totally ordered set such as $\{p_\lambda : \lambda \in \mathbb{R}\}$ satisfies the chain conditions, then it is finite. □

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