On the characterization of the set of unital quantum channels by convex mixtures of extremal maps

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In this paper, we consider the set of unital quantum channels and their description in terms of extremal channels. In particular, we consider the extension of the well-known of set of mixed-unitary channels which is used to construct approximations of unital quantum channels. We do this by introducing a family of d-dimensional extremal channels which includes both unitary channels and other channels of higher rank. We consider the particular case of qutrit channels which is the smallest set of maps for which the existence of non-unitary extremal maps is known. In this setting, we see that our family of maps corresponds to maps with Kraus rank three or less. We also construct explicit examples of maps that belong to this family and we find the relation between one of the examples presented and the antisymmetric Werner-Holevo channel. Finally, we consider convex mixtures of our family of channels and we show that such channels have a special form with respect to their Choi representation.

1 Introduction

Quantum channels, also denoted as quantum maps, provide the most general characterisation of the arbitrary evolution of quantum systems and represent a vital ingredient in establishing quantum computing and communication. For instance, in quantum key distribution protocols, the amount of overall noise in the quantum channel determines the rate at which secret bits are distributed between authorised parties. Mathematically, quantum channels are characterized as completely positive trace-preserving (CPT) linear mappings between density matrices. An interesting class of these maps are the unital completely positive trace-preserving (UCPT) maps which are those quantum channels sending the noisiest state of the system, the maximally-mixed state, to itself. There are several good reasons to consider unital quantum maps instead of general quantum channels. For low dimensional systems, the additional constraint required by unitality often simplifies problems and allows for a geometrical intuition of the state space. Some crucial advances in QIT like the parametrization of qubit channels are connected to the characterization of unital maps [1, 2].

The most striking feature of qubit unital channels is that they always admit a decomposition in terms of convex combinations of unitary channels [3]. This property allows to associate the set of UCPT maps with the geometry of a 3-simplex in which the vertices correspond to the set of Pauli channels [4]. However, it was soon acknowledged that this property is not longer true for unital channels of higher dimension. Various investigations introduced examples of maps which were neither unitary channels nor could be decomposed in terms of convex combinations of unitary channels [5, 6]. The existence of unital maps which are not mixed-unitary establishes a crucial difference between qubit channels and higher dimensional channels. From the point of view of convex decomposition, the description of the set of unital qubit maps is analogous to the description doubly-stochastic matrices given by Birkhoff's theorem. This theorem establishes that doubly-stochastic matrices can be decomposed in terms of a convex combinations of permutation matrices. Interestingly, the existence of non-unitary extremal maps within the set of UCPT maps implies that Birkhoff's theorem cannot be extended to higher dimensional channels. This raises

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an interesting property for the the set of unital channels in that only those maps that are mixed-unitaries allow for classical error correction [7].

Research has considered the relation between the set of unital maps and the set of mixed-unitary channels. For example, Audenaert et al. considered the distance between both sets [8] and Mendl and Wolf provided computable criteria for the separation of the unital channels from the mixed-unitary set [9]. Here, we follow a different approach by considering a larger class of maps than the mixed-unitary set. In particular, just as mixed-unitary channels correspond to all possible convex combinations of rank one maps (unitary channels), we consider a class of unital maps arising from mixing UCPT-extremal maps apart from those that are rank one. Our reason to explore this set is that by using our family of mixed UCPT-extremal maps will allow for better approximations to the set of unital maps than working mixed-unitaries alone.

The work is organized as follows. In section 2, we outline the necessary tools required for the study of the UCPT maps and in particular those maps which are extremal within the set. In section 3, we propose a family of maps $\mathcal{E} : \mathcal{D}_d \to \mathcal{D}_d$ of rank $k \leq d$ as a candidate for the generalization of UCPT-extremal maps. We note that a similar family of maps was also introduced in [10] in the context of quantum circuit decomposition. We derive the constraints that guarantee that such maps are unital and trace-preserving and we also derive the conditions which determine if a given map of the family is an extreme point of the set of UCTP maps. The family of maps presented is given in terms of $2d^2 - 3d + 1$ real parameters. We note however that the general description of UCTP-extremal maps requires $2d^3 - 3d^2$ real parameters [11, 12]. In section 4, we consider the particular case of qutrit maps. We show that our parametrization of UCTP-extremal maps includes both unitary maps and the antisymmetric Werner-Holevo channel [13]. Finally, we consider convex mixtures of UCTP-extremal maps and their relation to generic unital maps.

2 Preliminaries

2.1 Notation

Let $n \in \mathbb{N}$ and denote by $\mathcal{H}_n$ the n-dimensional Hilbert space and by $\mathcal{M}_n$ the set of $n \times n$ complex matrices. The set of density operators ρ consisting of all hermitian, positive operators with unit trace acting on $\mathcal{H}_n$ is denoted by $\mathcal{D}_n \subset \mathcal{M}_n$. We write $\mathbb{1}_n$ to denote the identity operator acting on $\mathcal{H}_n$. A linear mapping $\mathcal{E} : \mathcal{D}_n \mapsto \mathcal{D}_m$ is said to be positive if it sends positive semi-definite matrices to positive semi-definite matrices, and completely positive if $\mathcal{E} \otimes \mathbb{1}_n$ is positive for all n. A completely positive mapping $\mathcal{E}$ is trace-preserving if and only if $\operatorname{Tr} \mathcal{E}(\rho) = \operatorname{Tr} \rho$ for all $\rho \in \mathcal{D}_n$. We denote the set of all CPT maps by $\Xi^T_{n,m}$. Simiraliry, a complete positive mapping $\mathcal{E}$ is unital if and only if it leaves the maximally mixed state invariant $\mathcal{E}(\mathbb{1}_n/n) = \mathbb{1}_m/m$. We denote the set of UCP maps by $\Xi^U_{n,m}$. The set of UCPT maps corresponds to all complete positive maps $\mathcal{E} : \mathcal{D}_n \mapsto \mathcal{D}_m$ which are unital and trace-preserving and is given by the intersection $\Xi^{UT}_{n,m} := \Xi^T_{n,m} \bigcap \Xi^U_{n,m}$.

2.2 Operator-sum representation

The mapping $\mathcal{E} : \mathcal{D}_n \mapsto \mathcal{D}_m$ is completely positive if and only if it admits a representation of the form

$$\mathcal{E}(\rho) = \sum_{i=1}^{r} K_i \rho K_i^\dagger \tag{1}$$

for all $\rho \in \mathcal{D}_n$, where the matrices $K_i \in \mathcal{M}_{m \times n}$ are referred to as the Kraus operators [14]. This form of expressing a map is known as the operator-sum representation of a map. A complete positive map $\mathcal{E} : \mathcal{D}_n \mapsto \mathcal{D}_m$ given by (1) is trace-preserving if

$$\sum_{i=1}^{r} K_i^\dagger K_i = \mathbb{1}_n \tag{2}$$

and $\mathcal{E}$ is unital if

$$\sum_{i=1}^{r} K_i K_i^\dagger = \mathbb{1}_m. \tag{3}$$

The operator-sum representation of a map is not unique. The following theorem establishes when two sets of operators represent the same map.

Theorem 1 (Unitary freedom in the operator-sum representation [15]). *Suppose $\{K_i\}_{i=0,\dots,n}$ and $\{G_j\}_{j=0,\dots,n}$ are the sets of Kraus operators defining the CP maps $\mathcal{E}$ and $\mathcal{F}$, respectively. Then $\mathcal{E}=\mathcal{F}$ if and only if there exist complex numbers u_{ij} such that $K_i=\sum_j u_{ij}G_j$ and $U=(u_{ij})_{i,j\in\mathbb{Z}_n}$ is an n by n unitary matrix.*

Two sets of Kraus operators with different cardinality represent the same map if by appending zero operators to the set with fewer elements, the unitary freedom condition is satisfied. The minimum number of Kraus operators such that the operator-sum representation of a map exists, is called the Kraus rank of the map r. Choi showed that the operator-sum representation of a map is minimal if and only if the set of Kraus operators $\{K_i\}_{i\in\mathbb{Z}_r}$ is linearly independent [14].

2.3 On the convexity of the set of UCPT maps

The set of trace-preserving maps, $\Xi^T_{n,m}$, and its adjoint, the set of unital maps, $\Xi^U_{n,m}$, are convex. For $0<p<1$ and $\mathcal{E}_1,\mathcal{E}_2\in\Xi^T_{n,m}(\Xi^U_{n,m})$, we have it that

$$\mathcal{E}_{12}(p)=\mathcal{E}_1+(1-p)\mathcal{E}_2\in\Xi^T_{n,m}(\Xi^U_{n,m}). \tag{4}$$

The elements of a set which do not admit a convex decomposition in terms of other elements within the set are called extreme points. The concise characterisation of extreme points of $\Xi^U_{n,m}$ was provided by Choi in the following theorem [14].

Theorem 2. *Consider the set of UCP maps $\mathcal{E}:\mathcal{D}_n\mapsto\mathcal{D}_m$ with minimal operator-sum representation $\mathcal{E}(\rho)=\sum_{i=1}^r K_i\rho K_i^\dagger$. Then, $\mathcal{E}$ is an extreme point of $\Xi^U_{n,m}$ if and only if the set $\{K_iK_j^\dagger\}_{i,j\in\mathbb{Z}_r}$ is linearly independent.*

Choi's theorem has a natural extension provided that the set of CPT maps is the dual of the set of UCP maps with respect to the complex conjugation. The following theorem establishes when a CPT map is an extreme point of the set $\Xi^T_{n,m}$.

Theorem 3. *Consider the set of CPT maps $\mathcal{E}:\mathcal{D}_n\mapsto\mathcal{D}_m$ with minimal operator-sum representation $\mathcal{E}(\rho)=\sum_{i=1}^r K_i\rho K_i^\dagger$. Then, $\mathcal{E}$ is an extreme point of $\Xi^T_{n,m}$ if and only if the set $\{K_i^\dagger K_j\}_{i,j\in\mathbb{Z}_r}$ is linearly independent.*

Theorem 2 and theorem 3 establish bounds to the Kraus rank of the extreme points of the set of unital maps and the set of trace-preserving maps, respectively. The Kraus rank of an extreme point of $\Xi^U_{n,m}$ is upper bounded by m. This follows from the fact that at most m^2 matrices $K_iK_j^\dagger$ can be linearly independent as its size is $m\times m$. For the CPT case, we have it that the Kraus rank of an extreme point of $\Xi^T_{n,m}$ is upper bounded by n. This follows from the fact that at most n^2 matrices $K_i^\dagger K_j\in\mathcal{M}_{n\times n}$ can be linearly independent. In this work, we focus on the set of unital and trace preserving maps in which $n=m=d$. The set $\Xi^{UT}_{d,d}$ is also convex and the following theorem originally published in [6] characterizes its extreme points.

Theorem 4. *Consider the set of UCPT maps $\mathcal{E}:\mathcal{D}_d\to\mathcal{D}_d$ where $\mathcal{E}(\rho):=\sum_{i=1}^r K_i\rho K_i^\dagger$ and $\sum_{i=1}^r K_iK_i^\dagger=\sum_{i=1}^r K_i^\dagger K_i=\mathbb{1}_d$. Then, $\mathcal{E}$ is an extreme point of $\Xi^{UT}_{n,m}$ if and only if the set of $2d\times 2d$ matrices*

$$\{K_i^\dagger K_j\oplus K_iK_j^\dagger\}_{i,j\in\mathbb{Z}_r} \tag{5}$$

is linearly independent.

From this theorem it follows that the Kraus rank of an extreme point of $\Xi^{UT}_{d,d}$ is upper bounded by $\lfloor\sqrt{2d^2}\rfloor$ since at most $2d^2$ matrices $K_i^\dagger K_j\oplus K_iK_j^\dagger$ can be linearly independent provided that the maximum number of non-zero values of these matrices is $2d^2$. For example, if we fix the dimension $d=3$ we get that the rank of the extreme points of unital and trace-preserving maps is upper bounded by $\lfloor\sqrt{18}\rfloor=4$. Extreme points of the set $\Xi^{UT}_{3,3}$ with Kraus rank one correspond to the case of unitary maps which can be expressed as $\mathcal{E}(\rho)=U\rho U$ where U is a unitary matrix. Landau and Streater established that there are no extreme points of $\Xi^{UT}_{3,3}$ of rank two as all of them can be decomposed as convex sums of rank one points [6]. However, extreme points of $\Xi^{UT}_{3,3}$ with rank three and four exist and several examples of such maps exist throughout the literature.

3 A family of UCPT-extremal maps

The objective of this section is to find a family of maps including both unitary maps and UCPT-extremal maps of higher rank. To do this, we look at a particular family of CP maps in which its Kraus operators have a special form with respect to the Heisenberg-Weyl basis. For this family of maps, we select a canonical parametrisation and we discuss in terms of those parameters which elements of the family correspond to UCTP maps. Finally, we discuss which of those maps correspond to extreme points of the UCTP set.

Let us consider the family of CP maps over dimension d, $\mathcal{E} : \mathcal{D}_d \mapsto \mathcal{D}_d$, whose operator-sum representation is given by

$$\mathcal{E}(\rho) = \sum_{i=0}^{d-1} K_i \rho K_i^\dagger \tag{6}$$

with the following Kraus operators

$$\begin{aligned} K_i &= \sum_{j}^{d-1} \alpha_{ij} X_i Z_j \\ &= \sum_{j,k=0}^{d-1} \alpha_{ij} \omega^{kj} \left|k+i\right\rangle \left\langle k\right| \end{aligned} \tag{7}$$

where $(\alpha_{ij})_{i,j\in\mathbb{Z}_d} \in \mathbb{C}^{d\times d}$ is a matrix of complex coefficients and $\omega = e^{\frac{2\pi}{d}i}$ is the dth primitive root of unity. We note that $\{X_i Z_j = \sum_{k=0}^{d-1} \omega^{kj} \left|k+i\right\rangle \left\langle k\right|\}_{i,j\in\mathbb{Z}_d}$ corresponds to the Heisenberg-Weyl basis over dimension d, a set of orthonormal matrices that generalize the set of Pauli matrices to arbitrary dimension. Theorem 1 establishes the relation between two sets of Kraus operators $\{K_i\}_{i=0,\dots,d-1}$ and $\{G_i\}_{i=0,\dots,d-1}$ representing the same map. Now suppose that $\{K_i\}_{i=0,\dots,d-1}$ and $\{G_i\}_{i=0,\dots,d-1}$ are given respectively by the Kraus operators in (7). Because of the orthogonality of the sets, $\mathrm{Tr}\,(K_i^\dagger G_j) \propto \delta_{i,j}$ for $i,j = 0,\dots,d-1$, we have it that the unitary matrix in theorem 1 is given by

$$U = \begin{pmatrix} e^{i\phi_1} & & \\ & \ddots & \\ & & e^{i\phi_d} \end{pmatrix} \tag{8}$$

and this means that the only possible freedom corresponds to multiplying each one of the d Kraus operators by an arbitrary phase. In terms of group theory, this freedom is determined by the action of the group generated by taking the direct product of d copies of the unitary group $U(1)$, $\bigotimes_{i\in\mathbb{Z}_d} U(1)$. The different maps given by the Kraus operators in (7) can be divided in equivalence classes determined by the action of $\bigotimes_{i\in\mathbb{Z}_d} U(1)$. We can represent each one of these classes of maps by fixing the phase of one of the columns of the matrix of coefficients $(\alpha_{ij})_{i,j\in\mathbb{Z}_d}$. To represent each class, we fix the phase of the first column of this matrix of coefficients to zero. We can discuss the properties of a given map of this family in terms of the coefficients α_{ij}. The following theorem establishes which maps are unital and trace preserving.

Theorem 5. *The map $\mathcal{E} : \mathcal{D}_d \mapsto \mathcal{D}_d$ given by the operator sum representation $\mathcal{E}(\rho) = \sum_{i=0}^{d-1} K_i \rho K_i^\dagger$ where $K_i = \sum_{j,k=0}^{d-1} \alpha_{ij} \omega^{jk} \left|k+i\right\rangle \left\langle k\right|$ is trace-preserving if*

$$\sum_{i,j=0}^{d-1} \alpha_{ij} \alpha_{ij}^* = 1 \tag{9}$$

and

$$\sum_{i,j=0}^{d-1} \alpha_{ij+l} \alpha_{ij}^* = 0, \quad l = 1,\dots,d-1. \tag{10}$$

The map $\mathcal{E}$ is unital if in addition to condition (9), we have it that

$$\sum_{i,j=0}^{d-1} \alpha_{ij+l}\alpha_{ij}^* \omega^{-il} = 0 \quad l = 1, \ldots, d-1. \tag{11}$$

Proof. The proof follows from the evaluation of the trace-preserving and unital conditions (2) and (3) for the Kraus set given by (7). Let us consider the set $\{K_i^\dagger K_i\}_{i\in\mathbb{Z}_d}$ in the $\{|a\rangle\langle b|\,,\ a,b\in\mathbb{Z}_d\}$ basis as

$$\begin{aligned}
K_i^\dagger K_i &= \left(\sum_{k,j=0}^{d-1} \alpha_{ij}^* \omega^{-kj} |k\rangle\langle k+i|\right)\left(\sum_{m,n=0}^{d-1} \alpha_{im}\omega^{mn} |n+i\rangle\langle n|\right) \\
&= \sum_{k,j,m,n=0}^{d-1} \alpha_{im}\alpha_{ij}^* \omega^{mn-kj} |k\rangle\langle k+i|n+i\rangle\langle n| \\
&= \sum_{k,j,m=0}^{d-1} \alpha_{im}\alpha_{ij}^* \omega^{k(m-j)} |k\rangle\langle k|\,.
\end{aligned} \tag{12}$$

To satisfy the trace-preserving condition $\sum_{i=0}^{d-1} K_i^\dagger K_i = \mathbb{1}_d$, it necessarily follows that

$$\sum_{i,j,m=0}^{d-1} \alpha_{im}\alpha_{ij}^* \omega^{k(m-j)} = 1 \text{ for } k = 0, \ldots, d-1. \tag{13}$$

By the change of index, $m - j = l$, (13) can be expressed as

$$\sum_{i,l=0}^{d-1}\sum_{j=0}^{d-1} \alpha_{ij+l}\alpha_{ij}^* \omega^{kl} = 1 \text{ for } k = 0, \ldots, d-1 \tag{14}$$

and using the change of variable $\beta_{il} = \sum_{j=0}^{d-1} \alpha_{ij+l}\alpha_{ij}^*$, we get that

$$\sum_{i,l=0}^{d-1} \beta_{il}\omega^{kl} = 1 \text{ for } k = 0, \ldots, d-1. \tag{15}$$

The unique solution to this system of d linearly independent equations in terms of the set of variables $\{\beta_{il}\}_{i,l\in\mathbb{Z}_d}$ corresponds to $\sum_{i=0}\beta_{i0} = 1$ and $\sum_{i=0}\beta_{il} = 0$ for $l = 1, \ldots d-1$. By expressing the solution of the system in terms of the original variables $\{\alpha_{ij}\}_{i,k\in\mathbb{Z}_d}$ we get precisely the equations (9) and (10).

Similarly, we can obtain the conditions required by a map to be unital. Let us consider the set $\{K_i K_i^\dagger\}_{i\in\mathbb{Z}_d}$ as

$$\begin{aligned}
K_i K_i^\dagger &= \left(\sum_{m,n=0}^{d-1} \alpha_{im}\omega^{mn} |n+i\rangle\langle n|\right)\left(\sum_{k,j=0}^{d-1} \alpha_{ij}^* \omega^{-kj} |k\rangle\langle k+i|\right) \\
&= \sum_{k,j,m,n=0}^{d-1} \alpha_{im}\alpha_{ij}^* \omega^{mn-kj} |k+i\rangle\langle k|n\rangle\langle n+i| \\
&= \sum_{k,j,m,n=0}^{d-1} \alpha_{im}\alpha_{ij}^* \omega^{mn-kj} \delta_{k,n} |k+i\rangle\langle n+i| \\
&= \sum_{k,j,m=0}^{d-1} \alpha_{im}\alpha_{ij}^* \omega^{(k-i)(m-j)} |k\rangle\langle k|\,.
\end{aligned} \tag{16}$$

To satisfy the unital condition $\sum_{i=0}^{d-1} K_i K_i^\dagger = \mathbb{1}_d$, it follows that

$$\sum_{i,j,m=0}^{d-1} \alpha_{im}\alpha_{ij}^* \omega^{(k-i)(m-j)} = 1 \text{ for } k = 0, \ldots, d-1. \tag{17}$$

By the change of index, $m - j = l$, (17) can be written as

$$\sum_{i,l=0}^{d-1} \sum_{j=0}^{d-1} \alpha_{ij+l}\alpha_{ij}^* \omega^{(k-i)l} = 1 \text{ for } k = 0, \ldots, d-1 \tag{18}$$

and using now the change of variable $\beta_{il} = \sum_{j=0}^{d-1} \alpha_{ij+l}\alpha_{ij}^*$ we get that

$$\sum_{i,l=0}^{d-1} \beta_{ik}\omega^{(k-i)l} = 1 \text{ for } k = 0, \ldots, d-1. \tag{19}$$

We get again a system of d equations in terms of $\{\beta_{ik}\}_{i,k\in\mathbb{Z}_d}$. The solution of this system is given by $\sum_{i,l=0} \beta_{i0} = 1$ and $\sum_i \beta_{il}\omega^{-l} = 0$ for $l = 1, \ldots d-1$. If we express the solution of the system in terms of the elements of the set $\{\alpha_{ij}\}_{i,j\in\mathbb{Z}_d}$ we get precisely the equations (9) and (11) which completes the proof. □

Equations (9), (10) and (11) represent $2(d-1)+1$ real constraints. To see this, we notice that (9) corresponds to one real constraint while (10) and (11) correspond to $(d-1)$ real constraints each one. The following theorem establishes whether a UCPT map given by the Kraus set in (7) corresponds to an extreme point of $\Xi_{d,d}^{UT}$.

Theorem 6. *A unital and trace-preserving map given by $\mathcal{E} : \mathcal{D}_d \mapsto \mathcal{D}_d$ with operator sum representation $\mathcal{E}(\rho) = \sum_{i=0}^{d-1} K_i \rho K_i^\dagger$ where $K_i = \sum_{j,k=0}^{d-1} \alpha_{ij}\omega^{jk} |k+i\rangle \langle k|$ corresponds to an extreme point of the set unital and trace-preserving maps iff the matrices $(M_l|N_l)$ are full-rank for $l = 0, ..., d-1$ where*

$$M_l = \left(\left(\sum_{j=0}^{d-1} \alpha_{i+lj}\omega^{j(k-l)} \right) \left(\sum_{j=0}^{d-1} \alpha_{ij}^* \omega^{-jk} \right) \right)_{i,k\in\mathbb{Z}_d} \tag{20}$$

and

$$N_l = \left(\left(\sum_{j=0}^{d-1} \alpha_{i+lj}\omega^{(k-i)j} \right) \left(\sum_{j=0}^{d-1} \alpha_{ij}^* \omega^{-(k-i)j} \right) \right)_{i,k\in\mathbb{Z}_d} . \tag{21}$$

Proof. By theorem 4, we have it that a map is an extreme point of $\Xi_{d,d}^{UT}$ if the set $\{K_i^\dagger K_j \oplus K_i K_j^\dagger\}_{i,j\in\mathbb{Z}_d}$ is linear independent. First, let us consider $K_i^\dagger K_{i+l}$ as

$$\begin{aligned}
K_i^\dagger K_{i+l} &= \left(\sum_{k,j=0}^{d-1} \alpha_{ij}^* \omega^{-kj} |k\rangle \langle k+i| \right) \left(\sum_{j,n=0}^{d-1} \alpha_{i+lj}\omega^{jn} |n+i+l\rangle \langle n| \right) \\
&= \sum_{k,n=0}^{d-1} \left(\sum_{j=0}^{d-1} \alpha_{i+lj}\omega^{jn} \right) \left(\sum_{j=0}^{d-1} \alpha_{ij}^* \omega^{-kj} \right) |k\rangle \langle k+i|n+i+l\rangle \langle n| \\
&= \sum_{k,n=0}^{d-1} \left(\sum_{j=0}^{d-1} \alpha_{i+lj}\omega^{j(n-l)} \right) \left(\sum_{j=0}^{d-1} \alpha_{ij}^* \omega^{-kj} \right) |k\rangle \langle k+i|n+i\rangle \langle n-l| \\
&= \sum_{k} \left(\sum_{j} \alpha_{i+lj}\omega^{j(k-l)} \right) \left(\sum_{j} \alpha_{ij}^* \omega^{-jk} \right) |k\rangle \langle k-l| .
\end{aligned} \tag{22}$$

The matrices in (22) can be expressed in vector form if we use $|k\rangle\langle k-l| \sim |k,k-l\rangle$ so that

$$K_i^\dagger K_{i+l} \cong \sum_{k=0}^{d-1} \gamma_{ikl} |k,k-l\rangle . \tag{23}$$

We take the inner product of two arbitrary vectors as

$$\langle K_j^\dagger K_{j+n} | K_i^\dagger K_{i+l}\rangle = \sum_{km}^{d-1} \gamma_{ikl}\gamma_{jmn}^* \langle m,m-n|k,k-l\rangle . \tag{24}$$

We see that $\langle K_j^\dagger K_{j+n} | K_i^\dagger K_{i+l}\rangle = 0$ if $l \neq n$ for $k,l,m,n \in \mathbb{Z}_d$. The non-zero coefficients of $\{K_i^\dagger K_{i+l}\}_{i\in\mathbb{Z}_d}$ can be expressed in matrix form as M_l for $l = 0,...,d-1$. Second, we consider $K_{i+l}K_i^\dagger$ as

$$\begin{aligned}
K_{i+l}K_i^\dagger &= \left(\sum_{j,n=0}^{d-1} \alpha_{i+lj}\omega^{nj} |n+i+l\rangle\langle n|\right)\left(\sum_{j,k=0}^{d-1} \alpha_{ij}^*\omega^{-jk} |k\rangle\langle k+i|\right) \\
&= \sum_{n,k=0}^{d-1}\left(\sum_{j=0}^{d-1} \alpha_{i+lj}\omega^{jn}\right)\left(\sum_{j=0}^{d-1} \alpha_{ij}^*\omega^{-jk}\right) |n+i+l\rangle\langle n|k\rangle\langle k+i| \\
&= \sum_{k=0}^{d-1}\left(\sum_{j=0}^{d-1} \alpha_{i+lj}\omega^{jk}\right)\left(\sum_{j=0}^{d-1} \alpha_{ij}^*\omega^{-jk}\right) |k+i+l\rangle\langle k+i| \\
&= \sum_{k=0}^{d-1}\left(\sum_{j=0}^{d-1} \alpha_{i+lj}\omega^{(k-i)j}\right)\left(\sum_{j=0}^{d-1} \alpha_{ij}^*\omega^{-(k-i)j}\right) |k+l\rangle\langle k| .
\end{aligned} \tag{25}$$

As we did before, we may vectorize the these matrices by using $|k+l\rangle\langle k| \sim |k+l,k\rangle$ so that

$$K_{i+l}K_i^\dagger \cong \sum_{k=0}^{d-1} \gamma_{ikl} |k+l,k\rangle \tag{26}$$

The inner product of two arbitrary vectors is expressed as

$$\langle K_{j+n}K_j^\dagger | K_{i+l}K_i^\dagger\rangle = \sum_{km}^{d-1} \gamma_{ikl}\gamma_{jmn}^* \langle m+n,m|k+l,k\rangle \tag{27}$$

and we note that $\langle K_{j+n}K_j^\dagger | K_{i+l}K_i^\dagger\rangle = 0$ in the case that $n \neq l$. In this case, the non-zero coefficients of the sets $\{K_{i+l}K_i^\dagger\}_{i\in\mathbb{Z}_d}$ are given by the matrices N_l with $l = 0,...,d-1$. As we saw, two elements of $\{K_{i+l}^\dagger K_i \oplus K_i K_{i+l}^\dagger\}_{i,l\in\mathbb{Z}_d}$ with different l are linear independent so we just require that the all the matrices $(M_l|N_l)$ with $l = 0,...,d-1$ to be full-rank as stated by the theorem. $\square$

The following theorem characterizes the set of maps given in terms of the operator-sum representation in (7) with are extremal with respect to the set of UCPT maps.

Theorem 7. *Consider the set of matrices given by*

$$\mathcal{A}(d) = \left\{ (\alpha_{ij})_{i,j\in\mathbb{Z}_d} \in \mathbb{C}^{d\times d} : \begin{array}{c} \sum_{i,j=0}^{d-1} \alpha_{ij}\alpha_{ij}^* = 1 \\ \sum_{i,j=0}^{d-1} \alpha_{ij+l}\alpha_{ij}^* = 0 \text{ for } l = 1,...,d-1 \\ \sum_{i,j=0}^{d-1} \alpha_{ij+l}\alpha_{ij}^*\omega^{-il} = 0 \text{ for } l = 1,...,d-1 \end{array} \right\} \tag{28}$$

where $\omega = e^{\frac{2\pi}{d} i}$ and further consider the set

$$\mathcal{B}_l(d) = \{(\alpha_{ij})_{i,j\in\mathbb{Z}_d} \in \mathbb{C}^{d\times d} : \det\left((M_l|N_l)(M_l|N_l)^\dagger\right) = 0\} \text{ for } l = 0, ..., d-1 \tag{29}$$

where the matrices $\{M_l\}_{l=0,\dots,d-1}$ and $\{N_l\}_{l=0,\dots,d-1}$ are given by (20) and (21), respectively. Then we have it that the set

$$\mathcal{X}(d) = \left(\mathcal{A}(d) - \left(\mathcal{A}(d) \cap \left(\bigcup_{l\in\mathbb{Z}_d} \mathcal{B}_l(d)\right)\right)\right) / \bigotimes_{i\in\mathbb{Z}_d} U(1) \tag{30}$$

corresponds to the set of all quantum maps defined by the Kraus operators in (7) which are extreme points of the set of unital and trace-preserving maps.

Proof. This theorem follows directly from the consideration of theorems 5 and 6. Let π denote the map that sends complex matrices to quantum maps as

$$\pi(\alpha_{ij}) = \mathcal{E} \tag{31}$$

where the action of the quantum map $\mathcal{E}$ on a density operator is given by

$$\mathcal{E}(\rho) = \sum_{i=0}^{d-1} \left(\sum_{j,k=0}^{d-1} \alpha_{ij}\omega^{kj} |k+i\rangle \langle k|\right) \rho \left(\sum_{j,k=0}^{d-1} \alpha^*_{ij}\omega^{-kj} |k\rangle \langle k+i|\right). \tag{32}$$

On the one hand, by theorem 5, we have it that if we apply the map π to $\mathcal{A}(d)$ we get as a result the set of maps which are trace-preserving and unital. On the other hand, if we apply the map π to the elements of the sets $\mathcal{B}_l(d)$ for $l = 0, \dots, d-1$ we get maps where the matrices $(M_l|N_l)$ are not full-rank. Therefore, by theorem 6, the map π applied over the set $\mathcal{A}(d) \cap \left(\bigcup_{l\in\mathbb{Z}_d} \mathcal{B}_l(d)\right)$ corresponds to the maps which are not UCPT-extremal. Note that the theorem is enunciated in terms of its complementary, the maps that are extreme within the set. Finally, it remains to consider the freedom in the choice of Kraus operators in order to obtain an injection between the set $\mathcal{A}(d)$ and the set of unital and trace-preserving maps. In this case, the the freedom in the choice of Kraus operators is given by the the action of the group $\prod_{i\in\mathbb{Z}_d} U(1)$ and therefore in order to get a one-to-one correspondence we take the quotient space of the set $\mathcal{A}(d)$ with respect to the action of this group. □

Corollary 8. *The set of unital and trace-preserving maps given by the Kraus operators defined by (7) has dimension $2d^2 - 3d + 1$.*

Proof. This lemma follows from the evaluation of the dimension of the set given by (30). We have it that

$$\dim(\mathcal{X}(d)) = \dim(\mathcal{A}(d)) - \dim(\bigotimes_{i\in\mathbb{Z}_d} U(1)) \tag{33}$$

Since $\mathcal{A}(d)$ is a $2d^2$ dimensional set with $2(d-1)+1$ real constraints and $\bigotimes_{i\in\mathbb{Z}_d} U(1)$ is a d dimensional group. We conclude that

$$\dim(\mathcal{X}(d)) = 2d^2 - (2(d-1)+1) - d = 2d^2 - 3d + 1 \tag{34}$$

□

4 Qutrit UCPT maps

In the previous section, we introduced a family of UCPT-extremal maps with the objective to extend the well-known set of mixed-unitary channels. The simplest type of UCPT maps for which the existence of non-unitary extremal maps is known are qutrit maps. For qutrit maps, by theorem 4, the rank of UCPT-extremal maps is bounded by $\lfloor\sqrt{18}\rfloor \approx \lfloor 4.24 \rfloor$. While all rank one UCTP maps are UCPT-extremal (unitary maps), it is the case that all rank two maps admit a decomposition in terms of other UCTP qutrit maps [6]. Rank three and rank four UCTP-extremal maps are key to characterize the convex structure of the set of UCPT qutrit maps. However, the rank of the maps of our family is bounded by d and consequently, for qutrit maps, this family only includes maps with rank three or less. In this section, we will see some examples of UCTP-extremal qutrit maps of this family. In particular, we will present examples of unitary maps but also one example of a rank three UCTP-extremal map.

Let $\mathcal{E} : \mathcal{D}_3 \mapsto \mathcal{D}_3$ be the map $\mathcal{E}(\rho) = \sum_{i=0}^{2} K_i \rho K_i^\dagger$ with Kraus operators given by (7). Theorem 5 maintains that a map given by this Kraus set is unital and trace-preserving if it satisfies the equations

$$\sum_{i,j=0}^{2} \alpha_{ij}\alpha_{ij}^* = 1, \tag{35}$$

$$\sum_{i,j=0}^{2} \alpha_{ij+1}\alpha_{ij}^* = 0 \tag{36}$$

and

$$\sum_{i,j=0}^{2} \alpha_{ij+1}\alpha_{ij}^*\omega^{-i} = 0. \tag{37}$$

We note that, for the case of qutrit maps, equations (36) and (37) can be expressed as the simple condition

$$\omega^i \sum_{j=0}^{d-1} \alpha_{i,j+1}\alpha_{ij}^* = \beta \quad \text{for} \quad i = 0, 1, 2 \tag{38}$$

where $\beta \in \mathbb{C}$. The family of maps given by the Kraus set in (7) includes the nine unitary Heisenberg-Weyl channels,

$$\mathcal{E}_{mn}^{HW}(\rho) = X_m Z_n \rho (X_m Z_n)^\dagger \quad \text{for} \quad m, n \in \mathbb{Z}_3. \tag{39}$$

These maps can be obtained from the Kraus operators in (7) by fixing the coefficients

$$\alpha_{ij} = \begin{cases} 1 & \text{if} \quad i = m, j = n \\ 0 & \text{if} \quad i \neq m, j \neq n \end{cases} \tag{40}$$

for $m, n \in \mathbb{Z}_3$. Our family of maps includes also UCPT-extremal maps of rank three and to see that, we present an example of such map. We also show that this example corresponds to a unitary transformation of the well-known antisymmetric qutrit Werner-Holevo channel $\mathcal{E}_{WH}$ [13].

Example 9. *We define the map $\mathcal{E}_a$ by the Kraus set in (7) where, in this case, the defining coefficients are given by the matrix*

$$(\alpha_{ij}) = \frac{\sqrt{2}}{6} \begin{pmatrix} 2 & e^{\frac{5\pi}{3}i} & e^{\frac{\pi}{3}i} \\ 2 & e^{\pi i} & e^{\pi i} \\ 2 & e^{\frac{\pi}{3}i} & e^{\frac{5\pi}{3}i} \end{pmatrix}. \tag{41}$$

To see the relation between $\mathcal{E}_a$ and $\mathcal{E}_{WH}$, we consider $\mathcal{E}_a(\rho) = \sum_{i=0}^{2} K_i \rho K_i^\dagger$ where

$$K_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, K_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } K_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \tag{42}$$

and the unitary matrix

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{43}$$

The conjugation of $\mathcal{E}_a$ with U results in the map

$$\begin{aligned} U\mathcal{E}_a(\rho)U^\dagger &= \sum_{i=0}^{2} UK_i\rho K_i^\dagger U^\dagger \\ &= \sum_{i=0}^{2} UK_i\rho(UK_i)^\dagger \end{aligned} \tag{44}$$

where the Kraus operators are

$$UK_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, UK_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \text{ and } UK_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{45}$$

This Kraus set corresponds precisely to the anti-symmetric Werner–Holevo channel over dimension three. One peculiarity of this map is that it maximizes the distance from the set of mixed-unitaries [9]. We also note that the Landau-Streater channel over dimension three can be obtained as a unitary conjugation of the Werner–Holevo channel [16] and consequently it can also be recovered as a unitary conjugation of the map $\mathcal{E}_a$.

Theorem 6 states that a map with the given Kraus form in 7 is an extreme point of the set of UCTP maps if the matrices $(M_l|N_l)$ are full-rank for $l = 0, ..., d-1$. The matrix M_1 corresponds to the conjugate of M_2 and the matrix N_1 corresponds to the conjugate of N_2. Therefore, it suffices to establish the rank of $(M_0|N_0)$ and $(M_1|N_1)$ in order to determine whether the given map corresponds to an extreme point of the set of UCPT maps. Noting that the rank of a matrix is given by the number of non-zero singular values, we shall consider the specific case of the map $\mathcal{E}_a$. For this map we get that

$$(M_0|N_0) = \frac{1}{2}\left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \end{array}\right) \tag{46}$$

and

$$(M_1|N_1) = \frac{1}{2}\left(\begin{array}{ccc|ccc} 0 & -1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \end{array}\right). \tag{47}$$

The singular values of the matrix $(M_0|N_0)$ are given by $\sigma(M_0|N_0) = \{\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\}$ and the singular values of the matrix $(M_1|N_1)$ are given by $\sigma(M_1|N_1) = \{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\}$.

5 Convex mixtures UCPT-extreme maps

Finally, we consider the set of maps given by all the convex combinations of UCPT-extreme qutrit maps introduced in the previous section. In this case, we will use the Choi representation of such maps to consider their convex mixtures. We will show that, by applying a particular unitary rotation to our family of UCPT-extreme maps, its Choi representation can be brought into a block-diagonal form.

The Choi matrix of the map $\mathcal{E}$ is given by

$$C_{\mathcal{E}} = (\mathcal{E} \otimes \mathbb{1}_d) |\psi\rangle \langle\psi| \tag{48}$$

where $|\psi\rangle$ represents a maximally entangled pure state i.e. $|\psi\rangle = \sum_{m=0}^{d-1} |m\rangle |m\rangle$. Now suppose that the Kraus representation of $\mathcal{E}$ is given by (7) and we fix $d = 3$. We have that the Choi representation of these maps is expressed as

$$\begin{aligned} C_{\mathcal{E}} &= \sum_{i=0}^{2} (K_i \otimes \mathbb{1}_3) |\psi\rangle \langle\psi| (K_i^\dagger \otimes \mathbb{1}_3) \\ &= \sum_{i=0}^{2} \left(\sum_{k,j=0}^{d-1} \alpha_{ij} \omega^{jk} |k+i\rangle |k\rangle \right) \left(\sum_{k,j=0}^{d-1} \alpha_{ij}^* \omega^{-jk} \langle k+i| \langle k| \right) \\ &= \sum_{i,k,l=0}^{2} \left(\sum_{j=0}^{d-1} \alpha_{ij} \omega^{jk} \right) \left(\sum_{j=0}^{2} \alpha_{ij}^* \omega^{-jl} \right) |k+i\rangle |k\rangle \langle l+i| \langle l| . \end{aligned} \tag{49}$$

Consider the convex hull of the set of UCPT-extremal maps $\mathcal{X}(3)$ in (30). Any map $\phi \in Conv(\mathcal{X}(3))$ admits a Choi representation given by

$$C_\phi = \sum_{i=0}^{8} a_i C_i \tag{50}$$

where $C_0, \ldots, C_8$ are different maps given by (49) and $\sum_{i=0}^{8} a_i = 1$. We seek to compare maps $\phi \in Conv(\mathcal{X}(3))$ with the set of all UCPT maps. To do this, we will use the fact that any hermitian matrix can be brought into a block-diagonal form [17]. For any map ϕ, we show that the same unitary transformation can be used to express C_ϕ as a block-diagonal matrix. To see this, consider the unitary operator

$$U = \sum_{q,j=0}^{2} |j-q\rangle |q\rangle \langle j| \langle q| \tag{51}$$

and the unitary conjugation over $C_\mathcal{E}$ given by

$$\begin{aligned}\tau(C_\mathcal{E}) &= U^\dagger C_\mathcal{E} U \\ &= \sum_{i=0}^{2} \left(\sum_{j=0}^{2} \alpha_{ij} \omega^{jk} |i\rangle |k\rangle \right) \left(\sum_{j,k=0}^{2} \alpha_{ij}^* \omega^{-jk} \langle i| \langle k| \right)\end{aligned} \tag{52}$$

The matrix $\tau(C_\mathcal{E})$ is block diagonal

$$\tau(C_\mathcal{E}) = \begin{pmatrix} P_0 & & \\ & P_1 & \\ & & P_2 \end{pmatrix} \tag{53}$$

and the blocks are given by

$$P_i = \left(\sum_{k=0}^{2} \sum_{j=0}^{2} \alpha_{ij} \omega^{jk} |k\rangle \right) \left(\sum_{k=0}^{2} \sum_{j=0}^{2} \alpha_{ij}^* \omega^{-jk} \langle k| \right). \tag{54}$$

6 Conclusions

In this paper, we considered the characterization of the convex structure of the set of unital quantum channels. In particular, we considered a family of unital and trace-preserving in terms of its operator sum-representation for which the Kraus operators have a special form with respect to the Heisenberg-Weyl basis of matrices. We derived firstly the equations needed to establish that the maps of the family are unital and trace-preserving. Secondly, we established the conditions which determine whether a given map corresponds to an extreme point of the set of unital and trace-preserving maps. As an application, we considered the case of maps of dimension three. For the qutrit case, we showed that the maps of our family correspond to both rank one and rank three maps and we presented examples of such maps. We also illustrated the relation between one of the examples presented and the Werner-Holevo channel of dimension three.

We can mention some ideas relating to this paper which would be interesting to explore in future works. As mentioned, rank three maps are not sufficient to describe all the extremal qutrit maps within the UCPT set as rank four qutrit UCTP-extremal maps also exist. We leave for future work the convex characterization of UCPT maps by further expanding our family by including also rank four maps. Since the set of UCPT maps is isomorphic to the set of maximally mixed bipartite states, it is a natural question to consider also the characterization of the set of the bipartite states associated to the family of maps introduced. We also aim to consider this problem at some point.

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