R. Beneduci, Fuzzy observables: from weak Markov kernels to Markov kernels

Referee report

It is already well understood that a fuzzy observable (which is a commutative POVM) is always obtained by a fuzzification of a sharp observable (a PVM) by a weak Markov kernel. Moreover, if the observable is defined on a second countable completely metrizable space, then the weak Markov kernel can be replaced by an equivalent Markov kernel. It was also proved by the present author that under some additional conditions, we may assume that the Markov kernel is a Feller Markov kernel, which means that it preserves continuity of functions.

As far as I understand, the present work provides a generalization of these results to arbitrary fuzzy observables defined on a second countable metrizable space. This is achieved by studying weak Markov kernels defined on a subset of the unit interval and the Borel sets of a second countable metrizable topological space. These results are quite technical and their significance is not easily seen. In my opinion, this is largely due to the presentation, which should be improved before acceptance.

Some specific comments that might help to increase the value of the paper:

1. The Introduction could be somewhat improved, so that also a non-expert reader would understand the problem (e.g. the difference between Markov kernels and weak Markov kernels) and the obtained new results.

2. I particular, weak Markov kernels first appear on p. 3 without any explanation, it would be better to at least add something like ...see Def. 2 below...

3. The main results are based on Sec. 2, but it is not clear what is the role of the metrizability condition? It does not appear anywhere in the assumptions in Sec. 2, in particular, there is no such assumption in Thm. 3.

4. It would be good, apart from the technical proofs, to provide a paragraph in Sec. 2, explaining the methods and perhaps comparing them to those used previously. Otherwise, the reader is quickly lost in technical details, without seeing the whole picture.

5. In Sec. 3, the word "smearing" appears. It had been used in the literature before, but in this paper the term "fuzzification" is used in the previous sections, so it would be better to stick to that.

6. On p. 9, line 7, it is written that ...avoiding to require that F is regular... But the assumption of regularity is present in Thm. 5.

7. p. 9, line 7: It is not clear what O_2 is.

8. Definition 5: As far as I can see, $\mathscr{E}(\mathscr{H})$ was not defined before. Also PVM appears for the first time (better use the term sharp observable).

9. Thm. 5: It is not clear what $\mathscr{F}(\mathscr{H})$ is.