Classical XOR games: of size n: $R = (R_{s,t})_{s,t \in [n]}$, $R_{s,t} \in \mathbb{R}$,

$\sum_{s,t=1}^{n} |R_{s,t}| = 1 \Rightarrow \pi$ probab. distribution on $[n] \times [n]$.

Referee $\rightarrow$ $(s,t) \in [n] \times [n] \sim \pi$, $s \rightarrow$ Alice, $t \rightarrow$ Bob

Responses $\rightarrow$ $A^s \rightarrow a$, $B^t \rightarrow b$ $a, b \in \{0,1\}$

result $\rightarrow$ win if $(-1)^{a \oplus b} = \operatorname{sign}(R_{s,t})$

- Strategy: A: $s \mapsto p(s)$ probab. of $a(s) = 0$ $p(s), q(s) \in [0,1]$
  B: $t \rightarrow q(t)$ — " — $b(t) = 0$

- success probability:

$P_{succ}(R, p, q) = \sum_{s,t} |R_{st}| P_{succ}(s,t, p, q)$, where

$$P_{succ}(s,t, p,q) = \begin{cases} p(s) q(t) + (1-p(s))(1-q(t)) & \text{if } \operatorname{sgn} R_{s,t} = 1 \\ p(s)(1-q(t)) + (1-p(s)) q(t) & \text{if } \operatorname{sgn} R_{s,t} = -1 \end{cases}$$

- random strategy: $p(s) = q(t) = \frac{1}{2}$ $\forall s,t$

$\Rightarrow$ $P_{succ}(s,t, p,q) = \frac{1}{2}$ $\forall s,t$

$\Rightarrow P_{succ}(R, p,q) = \frac{1}{2} \sum_{s,t} |R_{s,t}| = \frac{1}{2}$

- $P_{succ}(R, p,q) - \frac{1}{2}$ bias: $\omega(R)$

$\operatorname{sgn} R_{s,t} = 1 \Rightarrow P_{succ}(s,t;p,q) - \frac{1}{2} = 2p(s)q(t) + \frac{1}{2} - q(t) - p(s)$

$\operatorname{sgn} R_{s,t} = -1 \Rightarrow P_{succ}(s,t;p,q) - \frac{1}{2} = p(s) - 2p(s)q(t) + q(t) - \frac{1}{2}$

$\Rightarrow P_{succ}(s,t;p,q) - \frac{1}{2} = \operatorname{Sgn} R_{s,t} \left(2p(s)q(t) - q(t) - p(s) + \frac{1}{2} \right)$

Classical XOR games: of size n: $R = (R_{s,t})_{s,t \in [n]}$, $R_{s,t} \in \mathbb{R}$,

$$\sum_{s,t=1}^{n} |R_{s,t}| = 1 \equiv \pi \text{ -probab. distribution on } [n] \times [n].$$

Referee $\to$ $(s,t) \in [n] \times [n] \sim \pi$, $\quad s \to$ Alice, $\quad t \to$ Bob

Response $\to$ $A^s \to a$, $\quad B^t \mapsto b$ $\quad a, b \in \{0,1\}$

result $\to$ win if $(-1)^{a \oplus b} = \operatorname{sign}(R_{s,t})$

• strategy: $A: s \mapsto p(s)$ probab. of $a(s) = 0$ $\quad p(s), q(t) \in [0,1]$

$B: t \to q(t)$ — " — $b(t) = 0$

• success probability:

$$P_{succ}(R, p, q) = \sum_{s,t} |R_{s,t}| \, P_{succ}(s,t, p, q), \text{ where}$$

$$P_{succ}(s,t,p,q) = \begin{cases} p(s) q(t) + (1-p(s))(1-q(t)) & \text{if } \operatorname{sgn} R_{s,t} = 1 \\ p(s)(1-q(t)) + (1-p(s)) q(t) & \text{if } \operatorname{sgn} R_{s,t} = -1 \end{cases}$$

• random strategy: $p(s) = q(t) = \frac{1}{2}$ $\forall s,t$

$\Rightarrow$ $P_{succ}(s,t,p,q) = \frac{1}{2}$ $\forall s,t$

$$\Rightarrow P_{succ}(R, p, q) = \frac{1}{2} \sum_{s,t} |R_{s,t}| = \frac{1}{2}$$

• $P_{succ}(R, p, q) - \frac{1}{2}$ bias: $\omega(R)$

$\operatorname{sgn} R_{s,t} = 1 \Rightarrow P_{succ}(s,t,p,q) - \frac{1}{2} = 2p(s)q(t) + \frac{1}{2} - q(t) - p(s)$

$\operatorname{sgn} R_{s,t} = -1 \Rightarrow P_{succ}(s,t,p,q) - \frac{1}{2} = p(s) - 2p(s)q(t) + q(t) - \frac{1}{2}$

$$\Rightarrow P_{succ}(s,t; p,q) - \frac{1}{2} = \operatorname{Sgn} R_{s,t} \left(2p(s)q(t) - q(t) - p(s) + \frac{1}{2} \right)$$

$$P_{succ}(s,t;p,q) - \tfrac{1}{2} = \mathrm{sgn}\, R_{s,t}\left((2p(s)\cancel{\;} - 1)\,q(t) - p(s) + \tfrac{1}{2}\right) =$$

$$= \mathrm{sgn}\, R_{s,t}\left((2p(s)-1)(q(t)-\tfrac{1}{2})\right) =$$

$$= \tfrac{1}{2}\,\mathrm{sgn}(R_{s,t})\left((2p(s)-1)(2q(t)-1)\right)$$

Bias (R,p,q)

$$\to \cancel{P_{succ}(\ldots)} = \sum_{s,t} |R_{s,t}|\left(P_{succ}(s,t;p,q) - \tfrac{1}{2}\right) =$$

$$= \tfrac{1}{2}\sum_{s,t} R_{s,t}\,(2p(s)-1)(2q(t)-1)$$

classical bias

$$\boxed{\to \max_{p,q} = \tfrac{1}{2}\max_{\substack{x,y\\ \|x\|_\infty \le 1\\ \|y\|_\infty \le 1}} \left|\sum_{s,t} R_{s,t}\, x_s y_t\right| = \tfrac{1}{2}\max_{x_s, y_t \in \{-1,1\}} \left|\sum_{s,t} R_{s,t}\, x_s y_t\right|}$$

$\omega(R)$

CLASSICAL BIAS

replace real x_s, y_t by complex

COMPLEX BIAS

$$\boxed{\omega^{\mathbb{C}}(R) = \max_{\substack{x,y \in \mathbb{C}^n\\ \|x\|_\infty, \|y\|_\infty \le 1}} \left|\sum_{s,t} R_{s,t}\, x_s y_t\right|}$$

$x, y \in \mathbb{C}^n$ $\quad x = \sum x_s |e_s\rangle$ $\quad y = \sum y_t |e_t\rangle$ ONB

$$\omega^{\mathbb{C}}(R)_{(\bar{x},y)} = \sum_{s,t} R_{s,t} \langle e_s|\bar{x}\rangle\langle e_t|y\rangle = \sum_{s,t} R_{s,t} |e_s\rangle\langle e_t|$$

$$= \langle x|R\, y\rangle$$

$$\omega^{\mathbb{C}}(R) = \max_{\max_{s,t} |x_s|, |y_t| \le 1} \langle x|R y\rangle$$

- ~~SDP relaxation~~:

$$\omega^{sdp}(R) = \sup_{\substack{d, x_s, y_t \in \mathbb{C}^d \\ \|x_s\|, \|y_t\| \leq 1}} \left| \sum_{s,t} R_{s,t} \langle \bar{x}_s, y_t \rangle \right|$$

(semidefinite program) (replace x_s, y_t by d-dim. vectors $\bar{x}_s, \bar{y}_t$
(hm.)

BOUNDS FROM GROTHENDIECK inequality

$\omega(R) \leq \omega^{sdp}(R) \leq K_G^{\mathbb{R}}(R)$ for real R

$\omega^{\mathbb{C}}(R) \leq \omega^{sdp}(R) \leq K_G^{\mathbb{C}}(R)$ complex R

$K_G^{\mathbb{R}}, K_G^{\mathbb{C}}$ Grothendieck constants

ENTANGLED BIAS ~~A and B share a state $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$~~

A and B have local quantum systems $\mathcal{H}_A, \mathcal{H}_B \simeq \mathbb{C}^d$ and share a state $|\psi\rangle \in \mathcal{H}_{AB}$.

- strategy: with some $0 \leq A_s^0, B_t^0 \leq I$, ~~then~~

$\to s \mapsto$ A measures $A_s^0, I - A_s^0$ on her part $\dashrightarrow$ outcomes

$t \mapsto$ B — " — $B_t^0, I - B_t^0$ - - - - - - $\dashrightarrow$

- bias $\sum_{s,t} R_{s,t} \langle \psi | A_s \otimes B_t | \psi \rangle$

$$\omega^*(R) = \sup_{d, A_s, B_t, |\psi\rangle} \left| \sum_{s,t} R_{s,t} \langle \psi | A_s \otimes B_t | \psi \rangle \right|$$

Entangled bias

$\omega^c(R) \le \omega^*(R)$: choose $|\psi\rangle, A_s, B_t$ such that

$$\langle\psi| A_s \otimes B_t |\psi\rangle = x_s y_t \quad \text{for any given } x_s, y_t \in \mathbb{C}$$

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \Rightarrow \overset{\shortparallel}{\langle\psi_1 A_s \psi_1\rangle\langle\psi_2 B_t \psi_2\rangle}$$

$|\psi_1\rangle = |\psi_2\rangle = |\xi\rangle$ unit vector

$$A_s = x_s |\xi\rangle\langle\xi| \,, \quad B_t = y_t |\xi\rangle\langle\xi|$$

$\boxed{\textbf{Tsirelson:} \text{ For any real } R, \quad \omega^*(R) \le \omega^{sdp}(R).}$

Proof: Let U_s, V_t be <u>symmetries</u>, $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ unit vector $\to$ strategy

(the supremum can be taken over such strategies $\to$ linearity!)

- we may write $|\psi\rangle = \sum_i \lambda_i |i\rangle|i\rangle$ (up to local unitaries)

$$\Rightarrow \langle\psi| U_s \otimes V_t |\psi\rangle = \sum_{i,j} \lambda_i \lambda_j \langle i|U_s|j\rangle \langle i|V_t|j\rangle$$

put $x_{s,ij} = \{\lambda_i \langle i|U_s|j\rangle\}$

$y_{t,ij} = \{\lambda_j \langle i|V_t|j\rangle\}$

then $\langle \overline{x_s}, y_t\rangle = \sum_{i,j} \lambda_i \lambda_j \langle i|U_s|j\rangle\langle i|V_t|j\rangle$
$= \langle\psi| U_s \otimes V_t |\psi\rangle$

and $\|x_s\|^2 = \sum_{i,j} \lambda_i^2 \overline{\langle i|U_s|j\rangle}\langle i|U_s|j\rangle =$

$= \sum_{i,j} \lambda_i^2 \langle j|U_s|i\rangle\langle i|U_s|j\rangle =$

$= \mathrm{Tr}\, U_s(\mathrm{diag}\,\lambda_j^2) U_s = \sum_j \lambda_j^2 = 1$

similarly $\|y_t\|^2 = 1$ $\square$

In fact, Tsirelson proved that $\omega^*(R) = \omega^{sdp}(R)$, with the <u>max. entangled state</u>

$R_{11} = R_{12} = R_{21} = \frac{1}{4}, \quad R_{22} = -\frac{1}{4}$

- $\omega(R) = \max_{x_s, y_t \in \{\pm 1\}} \frac{1}{4}\left(x_1 y_1 + x_2 y_1 + x_1 y_2 - x_2 y_2\right)$

$= \max_{x_s, y_t \in \{\pm 1\}} \frac{1}{4}\left(x_1(y_1+y_2) + x_2(y_1 - y_2)\right) = \frac{1}{2}$

- $\omega^{\mathbb{C}}(R) = \max_{\substack{x_s, y_t \in \mathbb{C} \\ \max |x_s|, |y_t| \le 1}} \frac{1}{4}\left| x_1(y_1+y_2) + x_2(y_1-y_2)\right| = \frac{\sqrt{2}}{2}$

$\cdot 1,\ \omega^{\mathbb{C}}(R) \le \frac{1}{4} \max_{|y_{1/2}| \le 1} \left(\|y_1+y_2\| + \|y_1 - y_2\|\right)$

$\le \frac{\sqrt{2}}{4}\left(|y_1+y_2|^2 + |y_1-y_2|^2\right)^{1/2} \le \frac{\sqrt{2}}{4}\left(2\left(\|y_1\|^2 + \|y_2\|^2\right)\right)^{1/2}$

$\le \frac{\sqrt{2}}{2}$

$y_1 + y_2 = a_1 + a_2 + i(b_1 + b_2)$

$y_j = a_j + i b_j$

$|a_1 + a_2|^2 + |b_1 + b_2|^2 + |a_1 - a_2|^2 + |b_1 - b_2|^2$

$y_1^2 + y_2^2 + \ldots$

$|y_1+y_2|^2 + |y_1 - y_2|^2 \le 4$

$\langle y_1 + y_2, y_1 + y_2 \rangle + \langle y_1 - y_2, y_1 - y_2 \rangle$

$= \langle y_1, y_1 \rangle + \langle y_1, y_2 \rangle + \langle y_2, y_1 \rangle + \langle y_2, y_2 \rangle$

$+ \langle y_1, y_1 \rangle + \langle y_2, y_2 \rangle - \ldots$

$= 2\left(\|y_1\|^2 + \|y_2\|^2\right)$

- conversely, put $|x_i| = |y_j| = 1$, $x_1 = 1$, $x_2 = -i$, $y_1 = \frac{(1+i)}{\sqrt{2}}$, $y_2 = \frac{(1-i)}{\sqrt{2}}$

$\omega^{sdp}(R) = \frac{\sqrt{2}}{2} \to$ the same argument.

$$|x_1(y_1+y_2)+x_2(y_1-y_2)| = \frac{1}{\sqrt{2}}\left(1.(1-i+1-i)+i(1+i-1+i)\right) =$$

$$= \frac{1}{\sqrt{2}}\left(2-i(2i)\right) = \frac{1}{\sqrt{2}}4 \quad \cdot \frac{1}{4} = \frac{\sqrt{2}}{2}. \quad \square$$

$\boxed{\text{Classical XOR games:}}$ biases: <u>For real matrix R:</u>

$$\underset{\substack{\uparrow\\ \text{complex bias}}}{\omega(R)} \le \underset{\substack{\llcorner\\ \text{bounded entanglement}}}{\omega^{\mathbb{C}}(R)} \le \omega^{*}(R) = \underset{\substack{\llcorner\\ \text{the SDP relaxation}}}{\omega^{sdp}(R)} \le \underset{\substack{\searrow\\ \text{Grothendieck constant}}}{K_G^{\mathbb{R}}}\,\omega(R)$$

$\boxed{\text{QUANTUM XOR GAMES}}$ of size n: A Hermitian matrix $M \in M(\mathbb{C}^{n^2})$,

$\|M\|_1 \le 1$. Strategy: $A \in Obs(\mathbb{C}^n \otimes \mathcal{H}_A)$, $B \in Obs(\mathbb{C}^n \otimes \mathcal{H}_B)$

$|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. (symmetries) (Hermitian unitaries)

The bias: $\omega(A, B, |\psi\rangle, M) = Tr\,(A\otimes B)(M \otimes |\psi\rangle\langle\psi|)$

<u>Classical</u> ? ~~[illegible]~~ R, $A_s \in Obs(\mathcal{H}_A)$, $B_s \in Obs(\mathcal{H}_B)$

$|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$

$$Tr\,(A\otimes B)(M\otimes|\psi\rangle\langle\psi|) = \sum_{t,s} R_{s,t}\, Tr(A_s \otimes B_t)|\psi\rangle\langle\psi|$$

$$A = \sum_s |e_s\rangle\langle e_s| \otimes A_s$$

$$B = \sum_t |e_t\rangle\langle e_t| \otimes B_t$$

$$\Rightarrow Tr(A\otimes B)(M\otimes|\psi\rangle\langle\psi|) = \sum_{s,t} Tr\big(|e_s\rangle\langle e_s|\otimes A_s \otimes |e_t\rangle\langle e_t| \otimes B_t\big)(M\otimes|\psi\rangle\langle\psi|)$$

$$= \sum_{s,t} \underbrace{\langle e_s\otimes e_t| M | e_s \otimes e_t\rangle}_{R_{s,t}\,?} Tr(A_s\otimes B_t)|\psi\rangle\langle\psi|$$

$$M = \sum R_{s,t}\, |e_s\otimes e_t\rangle\langle e_s \otimes e_t|$$

diagonal matrix

$$\|M\|_1 = \sum_{s,t} |R_{s,t}| = 1.$$