

# On the category of affine subspaces

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We present some important categories.

## 0.1 The category FinVect

Let FinVect be the category of finite dimensional real vector spaces with linear maps. Then  $(\text{FinVect}, \otimes, \mathbb{R})$  is a symmetric monoidal category, with the usual tensor product of vector spaces. With the usual duality  $(-)^* : V \mapsto V^*$  of vector spaces, FinVect is compact closed. We will always use the identification  $V = V^{**}$  by the canonical isomorphism.

Put

$$e_U : U \otimes U^* \rightarrow \mathbb{R}, \quad e_U(u \otimes u^*) = \langle u^*, u \rangle,$$

then  $e_U$  is the cap (counit) for the duality of  $U$  and  $U^*$ . The unit (cup) of the duality is

$$\eta_U : \mathbb{R} \rightarrow U^* \otimes U, \quad \eta_U(1) = \sum_i e_i^* \otimes e_i,$$

where  $\{e_i\}$  is a basis of  $U$  and  $\{e_i^*\}$  the dual basis of  $U^*$ , determined by  $\langle e_i^*, e_j \rangle = \delta_{ij}$ . It is easily verified that  $\eta_U$  does not depend on the choice of the basis  $\{e_i\}$ . We will identify  $\eta_U := \eta_U(1)$ . Note that  $\eta_U$  is just  $e_U$  seen as an element in  $(U \otimes U^*)^* = U^* \otimes U$ .

By compactness, the internal hom is  $[U, V] = U^* \otimes V$ . Note that we have the usual identification

$$\langle f(u), v^* \rangle = \langle w, u \otimes v^* \rangle, \quad u \in U, \ v^* \in V^* \tag{1}$$

between maps  $f : U \rightarrow V$  and elements  $w \in U^* \otimes V = [U, V]$ , so the internal hom is identified with the set of all morphisms  $U \rightarrow V$ . The above relation of  $f$  and  $w$  can be also written as

$$f(u) = (e_U \otimes V)(u \otimes w), \quad w = (f^* \otimes V)(\eta_U),$$

(we write  $V$  for the identity map  $id_V$ ). Here  $f^*$  is the usual adjoint map  $f^* : V^* \rightarrow U^*$ .

The composition  $\circ$  defines a map

$$[U, V] \otimes [V, W] \rightarrow [U, W],$$

using the identification  $[U, V] = U^* \otimes V$ ,  $\circ$  is identified with

$$U^* \otimes V \otimes V^* \otimes W \xrightarrow{U^* \otimes e_V \otimes W} U^* \otimes W.$$

Using symmetry and  $e_V$ , we can also define partial compositions through  $V$  in an obvious way. This can be depicted graphically in a nice way.

## 0.2 Affine subspaces

A subset  $A \subseteq V$  of a finite dimensional vector space  $V$  is an affine subspace if  $\sum_i \alpha_i a_i \in A$  whenever all  $a_i \in A$  and  $\sum_i \alpha_i = 1$ . We say that  $A$  is proper if  $0 \neq A$  and  $A \neq \emptyset$ . We will always mean that an affine subspace is proper (if not explicitly stated otherwise).

### 0.2.1 Description

An affine subspace can be determined in two ways:

- (i) Let  $L \subseteq V$  be a linear subspace and  $a_0 \notin L$ . Then

$$A = a_0 + L$$

is a proper affine subspace. Note that  $a_0 \in A$  and  $A \cap L = \emptyset$ . Conversely, any proper affine subspace  $A$  can be given in this way, with  $a_0$  an arbitrary element in  $A$  and

$$L = \text{Lin}(A) := \{a_1 - a_2, a_1, a_2 \in A\} = \{a - a_0, a \in A\}.$$

- (ii) Let  $S \subseteq V$  be a linear subspace and  $a_0^* \in V^* \setminus S^\perp$ . Then

$$A = \{a \in S, \langle a_0^*, a \rangle = 1\}$$

is a proper affine subspace. Conversely, any proper affine subspace  $A$  is given in this way, with  $S = \text{Span}(A)$  and  $a_0^*$  an arbitrary element in the dual

$$\tilde{A} = \{a^* \in V^*, \langle a^*, a \rangle = 1, \forall a \in A\}.$$

For an affine subspace  $A$ , the relation of  $L = \text{Lin}(A)$  and  $S = \text{Span}(A)$  is as follows:

$$S = L + \mathbb{R}a, \quad L = S \cap \{\tilde{a}\}^\perp,$$

here  $a \in A$  and  $\tilde{a} \in \tilde{A}$  are arbitrary elements.

### 0.2.2 Duality

For an affine subspace  $A$ ,  $\tilde{A}$  is an affine subspace as well. If  $A$  is proper, then  $\tilde{A}$  is proper and we have  $\tilde{\tilde{A}} = A$ . More generally, if  $\emptyset \neq C \subseteq A$  is any subset of a proper affine subspace  $A$ , then  $\tilde{C}$  is a proper affine subspace and  $\tilde{\tilde{C}}$  is the affine hull of  $C$ , that is,

$$\tilde{\tilde{C}} = \text{Aff}(C) := \left\{ \sum_i \alpha_i c_i, c_i \in C, \sum_i \alpha_i = 1 \right\}.$$

In this case, we may write  $\tilde{\tilde{C}}$  as

$$\tilde{\tilde{C}} = c_0 + \text{Lin}(C) = c_0 + \text{Span}(\{c_1 - c_2, c_1, c_2 \in C\})$$

with an arbitrary element  $c_0 \in C$ , or as

$$\tilde{\tilde{C}} = \{c \in \text{Span}(C), \langle a_0^*, c \rangle = 1\}$$

for an arbitrary element  $a_0^* \in \tilde{A}$ .

**Lemma 1.** *Let  $A$  be a proper affine subspace and let  $C \subseteq A$  be any subset. Then*

$$\begin{aligned} \text{Lin}(C) &= \text{Lin}(\tilde{\tilde{C}}) = \tilde{C}^\perp = \text{Span}(\tilde{C})^\perp, & \text{Lin}(\tilde{C}) &= C^\perp = \text{Span}(C)^\perp \\ \text{Span}(C) &= C^{\perp\perp} = \text{Lin}(\tilde{C})^\perp, & \text{Span}(\tilde{C}) &= \text{Lin}(C)^\perp. \end{aligned}$$

### 0.2.3 The lattice of affine subspaces

Let  $\mathcal{A}(V)$  be the set of all affine subspaces in a finite dimensional vector space  $V$ . Then  $\mathcal{A}(V)$  can be ordered by inclusion and it is a complete lattice, with

$$\wedge \mathcal{A} = \cap \mathcal{A}, \quad \vee \mathcal{A} = \left\{ \sum_i \alpha_i a_i, \ a_i \in A_i \in \mathcal{A}, \sum_i \alpha_i = 1 \right\}$$

for any subset  $\mathcal{A} \subseteq \mathcal{A}(V)$ . Let us choose any nonzero elements  $a \in V$ ,  $\tilde{a} \in V^*$  and put

$$\mathcal{A}_{a,\tilde{a}}(V) = \{A \in \mathcal{A}(V), \ a \in A, \ \tilde{a} \in \tilde{A}\}.$$

Note that any subspace in  $\mathcal{A}_{a,\tilde{a}}$  is proper and it is a complete sublattice in  $\mathcal{A}(V)$ . Moreover, we have

$$\text{Lin}(\wedge \mathcal{A}) = \wedge \{\text{Lin}(A), \ A \in \mathcal{A}\}, \quad \text{Lin}(\vee \mathcal{A}) = \vee \{\text{Lin}(A), \ A \in \mathcal{A}\}$$

and similarly for Span.

We say that  $A, B \in \mathcal{A}_{a,\tilde{a}}(V)$  are independent if  $A \cap B = \{a\}$ , equivalently,  $\text{Lin}(A) \cap \text{Lin}(B) = \{0\}$ , that is,  $\text{Lin}(A)$  and  $\text{Lin}(B)$  are independent linear subspaces. A family  $\{A_1, \dots, A_n\} \subseteq \mathcal{A}_{a,\tilde{a}}(V)$  is independent if  $A_i$  and  $\vee_{j \in I} A_j$  are independent for any  $i \in \{1, \dots, n\}$  and  $i \neq I \subseteq \{1, \dots, n\}$ . Equivalently,  $\{\text{Lin}(A_1), \dots, \text{Lin}(A_n)\}$  is an independent family of subspaces in  $V$ .

**Lemma 2.** *Let  $\{A_1, \dots, A_n\} \subseteq \mathcal{A}_{a,\tilde{a}}(V)$  be an independent family. Then the sublattice generated by  $\{A_1, \dots, A_n\}$  is distributive.*

*Proof.* Clear from a similar property of linear subspaces. □

## 0.3 The category Af

The objects of Af are of the form  $X = (V_X, A_X)$ , where  $V_X$  is an object in FinVect and  $A_X \subseteq V_X$  a proper affine subspace. Morphisms  $X \rightarrow Y$  are morphisms  $f : V_X \rightarrow V_Y$  in FinVect such that  $f(A_X) \subseteq A_Y$ . We may also add two special objects: the initial object  $\emptyset := (\{0\}, \emptyset)$  and the terminal object  $0 := (\{0\}, \{0\})$ , here the affine subspaces are obviously not proper.

For any object  $X$ , we also put

$$L_X := \text{Lin}(A_X) \quad S_X := \text{Span}(A_X), \quad d_X := \dim(L_X), \quad D_X := \dim(V_X).$$

Note that  $X$  is uniquely determined also by the triple  $(V_X, L_X, a_X)$  with an element  $a_X \in A_X$ , or by  $(V_X, S_X, \tilde{a}_X)$  with an element  $\tilde{a}_X \in \tilde{A}_X$ .

### 0.3.1 Limits and colimits

Limits and colimits should be obtained from those in FinVect, we have to specify the other structures and check whether the corresponding arrows are in Af.

First, note that  $\{0\}$  is both initial and terminal in FinVect. In Af, it is easily seen that  $\emptyset$  is initial and  $0$  is terminal in Af.

Let  $X, Y$  be two objects in Af. Assume first that both are proper. We define their product as

$$X \times Y := (V_X \times V_Y, A_X \times A_Y, (a_X, a_Y), \frac{1}{2}(\tilde{a}_X, \tilde{a}_Y)),$$

where

$$A_X \times A_Y := \{(x, y) \in V_X \times V_Y, x \in A_X, y \in A_Y\}$$

is the direct product of  $A_X$  and  $A_Y$ . It is easily verified that this is indeed an affine subspace and the usual projections  $\pi_X : V_X \times V_Y \rightarrow V_X$  and  $\pi_Y : V_X \times V_Y \rightarrow V_Y$  are in Af. Moreover, for  $f : Z \rightarrow X$  and  $g : Z \rightarrow Y$ , the map  $f \times g(z) = (f(z), g(z))$  is also clearly a morphism  $Z \rightarrow X \times Y$  in Af. We have

$$L_{X \times Y} = L_X \times L_Y, \quad S_{X \times Y} = (L_X \times L_Y) \vee \mathbb{R}(a_X, a_Y) = (S_X \times S_Y) \wedge \{(\tilde{a}_X, -\tilde{a}_Y)\}^\perp$$

for an arbitrary choice  $a_X \in A_X$ ,  $a_Y \in A_Y$  and  $\tilde{a}_X \in \tilde{A}_X$ ,  $\tilde{a}_Y \in \tilde{A}_Y$ .

Next, we put  $X \times \emptyset = \emptyset$ , with the unique morphisms  $\pi_X : \emptyset \rightarrow X$  and  $\pi_\emptyset : \emptyset \rightarrow \emptyset$ . If  $Y \xrightarrow{f} X$  and  $Y \xrightarrow{g} \emptyset$ , then it is clear that  $Y = \emptyset$ , this shows that this is indeed the product. Further, put  $X \times 0 = X$ , with  $\pi_X = id_X$  and  $\pi_0 : X \xrightarrow{!} 0$ . It is also readily verified that this is the product.

The coproduct for proper objects  $X, Y$  is defined as

$$X \oplus Y = (V_X \times V_Y, A_X \oplus A_Y)$$

where

$$A_X \oplus A_Y := \{(tx, (1-t)y), x \in A_X, y \in A_Y, t \in \mathbb{R}\}$$

is the direct sum. To check that this is an affine subspace, let  $x_i \in A_X$ ,  $y_i \in A_Y$ ,  $s_i \in \mathbb{R}$  and let  $\sum_i \alpha_i = 1$ , then

$$\sum_i \alpha_i (s_i x_i, (1-s_i)y_i) = (\sum_i s_i \alpha_i x_i, \sum_i (1-s_i) \alpha_i y_i) = (sx, (1-s)y) \in A_X \oplus A_Y,$$

where  $s = \sum_i s_i \alpha_i$ ,  $x = s^{-1} \sum_i s_i \alpha_i x_i$  if  $s \neq 0$  and is arbitrary in  $A_X$  otherwise, similarly  $y = (1-s)^{-1} \sum_i (1-s_i) \alpha_i y_i$  if  $s \neq 1$  and is arbitrary otherwise. The usual embeddings  $p_X : V_X \rightarrow V_X \times V_Y$  and  $p_Y : V_Y \rightarrow V_X \times V_Y$  are easily seen to be morphisms in Af.

Let  $f : X \rightarrow Z$ ,  $g : Y \rightarrow Z$  be any morphisms in Af and consider the map  $V_X \times V_Y \rightarrow V_Z$  given as  $f \oplus g(u, v) = f(u) + g(v)$ . We need to show that it preserves the affine subspaces. So let  $x \in A_X$ ,  $y \in A_Y$ , then since  $f(x), g(y) \in A_Z$ , we have for any  $s \in \mathbb{R}$ ,

$$f \oplus g(sx, (1-s)y) = sf(x) + (1-s)g(y) \in A_Z.$$

We also have

$$L_{X \oplus Y} = (L_X \times L_Y) \vee \mathbb{R}\{(a_X, -a_Y)\}, \quad S_{X \oplus Y} = S_X \times S_Y$$

for some  $a_X \in A_X$ ,  $a_Y \in A_Y$ .

Similarly as in the case of products, it is verified that  $X \oplus \emptyset = X$  and  $X \oplus 0 = 0$ . (All the statements for coproducts can be obtained from duality defined below).

One can also discuss equalizers and coequalizers, here we only note that these may be trivial even for proper objects. We will consider pullbacks and pushouts for some special morphisms that will be needed below.

### 0.3.2 Mono-, epi- and bimorphisms

A morphism  $X \xrightarrow{f} Y$  is a monomorphism if  $f \circ h = f \circ g$  implies  $h = g$  for any morphisms  $g, h$ , and an epimorphism if  $h \circ f = g \circ f$  implies  $h = g$ . A morphism that is both mono and epi is called a bimorphism.

**Lemma 3.** *A morphism  $X \xrightarrow{f} Y$  is a monomorphism if and only if it is injective as a map  $f : V_X \rightarrow V_Y$ . Similarly,  $f$  is an epimorphism if and only if it is surjective.*

Consequently,  $f$  is a bimorphism if and only if it is an isomorphism of  $V_X$  and  $V_Y$ . Note that a bimorphism is not necessarily an isomorphism in Af.

*Proof.* Since injective maps are monomorphisms in  $FV$ , it is clear that any such map is a monomorphism also in Af. For the converse, let  $f$  be a monomorphism in Af and let  $K = \text{Ker}(f)$ . Let  $Z = (V_X \times K, A_X \times \{0\})$ , then  $Z$  is an object in Af. Let  $g, h : V_Z \rightarrow V_X$  be defined as  $g = \pi_{V_X}$ ,  $h = \pi_{V_X} + \pi_K$ , then  $g, h : Z \rightarrow X$  are morphisms in Af and we have

$$f \circ g(x, y) = f(x) = f(x) + f(y) = f \circ h(x, y), \quad \forall (x, y) \in V_Z.$$

Hence  $h = g$ , so that we must have  $K = \{0\}$ .

Similarly, let  $f$  be an epimorphism and let  $R = f(V_X) \subseteq V_Y$ . Let  $Z = (V_Y \times V_Y|_R, A_Y \times \{[0]\})$  and let  $g, h : V_Z \rightarrow V_Y$  be given by  $g(y) = (y, [0])$ ,  $h(y) = (y, q(y))$ , where  $q : V_Y \rightarrow V_Y|_R$  is the quotient map. Since  $A_Y \subseteq R$ , we have  $q(A_Y) = \{[0]\}$ , so that both  $g, h$  are morphisms in Af. Moreover,

$$g \circ f(x) = (f(x), [0]) = (f(x), q(f(x))) = h \circ f,$$

so that  $g = h$ , but this implies that  $R = V_Y$ . □

### 0.3.3 Pullbacks and pushouts

Let  $X, Y, Z$  be proper objects such that there are bimorphisms

$$Z \xrightarrow{f} X, \quad Z \xrightarrow{g} Y.$$

Note that in particular  $\psi := f \circ g^{-1}$  is an isomorphism of  $V_Y$  onto  $V_X$ .

Let us define  $X \sqcup_{f,g} Y := (V_X, A_{X \sqcup_{f,g} Y})$ , with

$$A_{X \sqcup_{f,g} Y} = \{sa + (1-s)\psi(b), \ a \in A_X, \ b \in A_Y, \ s \in \mathbb{R}\}.$$

Note first that this is a proper object in Af if and only if

$$\forall b \in A_Y, \quad t\psi(b) \in A_X \implies t = 1. \quad (2)$$

Indeed, we only have to check that  $0 \notin A_{X \sqcup_{f,g} Y}$  which is easily seen to be equivalent to (2).

Assume (2), then  $X \sqcup_{f,g} Y$  together with the morphisms given by the linear maps  $id : V_X \rightarrow V_X$  and  $\psi : V_Y \rightarrow V_X$ , is the **pushout** of the above diagram. Indeed, these are clearly bimorphisms  $X \rightarrow X \sqcup_{f,g} Y$  and  $Y \rightarrow X \sqcup_{f,g} Y$  in Af, and we have

$$id \circ f = f = \psi \circ g.$$

Also, if  $W$  is an object in  $\text{Af}$  and  $X \xrightarrow{i} W$  and  $Y \xrightarrow{j} W$  are such that  $i \circ f = j \circ g$ , then  $i = i \circ id$ ,  $j = i \circ \psi$ , so the map  $i$  defines a morphism  $X \sqcup_{f_0, g_0} Y \rightarrow W$ , obviously unique, with the required properties. We have

$$L_{X \sqcup_{f,g} Y} = L_X \vee \psi(L_Y), \quad S_{X \sqcup_{f,g} Y} = S_X \vee \psi(S_Y).$$

Let us also note that if (2) is not satisfied, there is some  $z \in V_Z$  such that for some  $t \neq 1$ ,

$$tf(z) \in A_X, \quad g(z) \in A_Y.$$

If there are some  $X \xrightarrow{i} W$  and  $Y \xrightarrow{j} W$  as above, then  $ti \circ f(z) \in A_W$ , but also  $i \circ f(z) = j \circ g(z) \in A_W$ , so that  $W$  is not a proper object, in this case the pushout is the terminal object  $0$ , with the unique arrows  $X \xrightarrow{!} 0$ ,  $Y \xrightarrow{!} 0$ .

Similarly, let

$$X \xrightarrow{f} Z, \quad Y \xrightarrow{g} Z$$

be bimorphisms and let  $\psi = f^{-1} \circ g$ . If

$$\phi(A_Y) \cap A_X \neq \emptyset, \tag{3}$$

the **pullback** of  $f, g$  is  $X \sqcap_{f,g} Y = (V_X, A_X \cap \phi(A_Y))$ , with the bimorphisms given by  $id_X$  and  $\phi^{-1}$ . In this case

$$L_{X \sqcap_{f,g} Y} = L_X \cap \phi(L_Y), \quad S_{X \sqcap_{f,g} Y} = S_X \cap \phi(S_Y).$$

Without condition (3), the above is not a proper object and in this case the pullback is the initial object  $\emptyset$ .

### 0.3.4 Monoidal structure

Let  $X, Y$  be objects in  $\text{Af}$ . Let us define

$$A_{X \otimes Y} := \{x \otimes y, x \in A_X, y \in A_Y\}^{\approx}.$$

In other words,  $A_{X \otimes Y}$  is the smallest affine subspace in  $V_X \otimes V_Y$  containing  $A_X \otimes A_Y$ .

**Lemma 4.** *For any  $a_X \in A_X$ ,  $a_Y \in A_Y$ , we have*

$$\begin{aligned} L_{X \otimes Y} &= \text{Lin}(A_X \otimes A_Y) = \text{Span}(\{x \otimes y - a_X \otimes a_Y, x \in A_X, y \in A_Y\}) \\ &= (a_X \otimes L_Y) + (L_X \otimes a_Y) + (L_X \otimes L_Y) \end{aligned} \tag{4}$$

$$= S_X \otimes L_Y + L_X \otimes a_Y = a_X \otimes L_Y + L_X \otimes S_Y \tag{5}$$

(here  $+$  denotes the direct sum of subspaces). We also have

$$S_{X \otimes Y} = S_X \otimes S_Y.$$

*Proof.* Let  $x \in A_X$ ,  $y \in A_Y$ , then

$$x \otimes y - a_X \otimes a_Y = a_X \otimes (y - a_Y) + (x - a_X) \otimes a_Y + (x - a_X) \otimes (y - a_Y),$$

so that  $L_{X \otimes Y} = \text{Lin}(A_X \otimes A_Y)$  is contained in the subspace on the RHS of (4). Let  $d$  be the dimension of this subspace, then clearly

$$d_{X \otimes Y} \leq d \leq d_X + d_Y + d_X d_Y.$$

On the other hand, any element of  $S_X$  has the form  $tx$  for some  $t \in \mathbb{R}$  and  $x \in A_X$ , so that it is easily seen that  $S_X \otimes S_Y = S_{X \otimes Y}$ . Hence

$$\begin{aligned} d_{X \otimes Y} &= \dim(L_{X \otimes Y}) = \dim(S_{X \otimes Y}) - 1 = \dim(S_X) \dim(S_Y) - 1 = (d_X + 1)(d_Y + 1) - 1 \\ &= d_X + d_Y + d_X d_Y. \end{aligned}$$

This completes the proof. □

For  $X, Y$  in  $\text{Af}$ , put  $X \otimes Y := (V_X \otimes V_Y, A_{X \otimes Y})$ . Note that by this definition,  $X \otimes \emptyset = \emptyset$  and  $X \otimes 0 = 0$  unless  $X = \emptyset$ , in which case  $\emptyset \otimes 0 = \emptyset$ . Also let  $I := (\mathbb{R}, \{1\})$ .

**Lemma 5.**  *$(\text{Af}, \otimes, I)$  is a symmetric monoidal category.*

*Proof.* Note that this structure is inherited from the symmetric monoidal structure in  $\text{FinVect}$ . To show that  $\otimes$  is a functor, we have to check that for  $X_1 \xrightarrow{f} Y_1$  and  $X_2 \xrightarrow{g} Y_2$  in  $\text{Af}$ , we have  $X_1 \otimes Y_1 \xrightarrow{f \otimes g} X_2 \otimes Y_2$  which amounts to showing that

$$(f \otimes g)(A_{X_1 \otimes Y_1}) \subseteq A_{X_2 \otimes Y_2}.$$

Let  $x \in A_{X_1}$ ,  $y \in A_{Y_1}$ , then  $f(x) \otimes g(y) \in A_{X_2} \otimes A_{Y_2} \subseteq A_{X_2 \otimes Y_2}$ . Since  $A_{X_1 \otimes Y_1}$  is the affine subspace generated by  $A(X_1) \otimes A(Y_1)$ , the above inclusion follows by linearity of  $f \otimes g$ .

It only remains to prove that the associators, unitors and symmetries from  $\text{FinVect}$  are morphisms in  $\text{Af}$ . Indeed, let  $\alpha_{X,Y,Z} : V_X \otimes (V_Y \otimes V_Z) \rightarrow (V_X \otimes V_Y) \otimes V_Z$  be the associator in  $\text{FinVect}$ . We need to check that  $\alpha_{X,Y,Z}(A_{X \otimes (Y \otimes Z)}) \subseteq A_{(X \otimes Y) \otimes Z}$ . It is easily checked that  $A_{X \otimes (Y \otimes Z)}$  is the affine span of elements of the form  $x \otimes (y \otimes z)$ ,  $x \in A_X$ ,  $y \in A_Y$  and  $z \in A_Z$ , and we have

$$\alpha_{X,Y,Z}(x \otimes (y \otimes z)) = (x \otimes y) \otimes z \in A_{(X \otimes Y) \otimes Z}$$

for all such elements. The desired inclusion follows by linearity. The proof of the other inclusions is similar. □

### 0.3.5 Duality

We define  $X^* := (V_X^*, \tilde{A}_X)$ . Note that we have

$$L_{X^*} = S_X^\perp, \quad S_{X^*} = L_X^\perp, \quad d_{X^*} = D_X - d_X - 1.$$

It is easily seen that  $(-)^*$  defines a full and faithful functor  $\text{Af}^{op} \rightarrow \text{Af}$ , moreover,  $X^{**} = X$  (we will use the canonical identification of any  $V$  in  $\text{FinVect}$  with its second dual).

**Theorem 1.**  *$(\text{Af}, \otimes, I)$  is a  $*$ -autonomous category, with duality  $(-)^*$ , such that  $I^* = I$ .*

*Proof.* We only need to check the natural isomorphisms

$$\text{Af}(X \otimes Y, Z^*) \simeq \text{Af}(X, (Y \otimes Z)^*).$$

Since  $\text{FinVect}$  is compact, we have the natural isomorphisms

$$\text{FinVect}(V_X \otimes V_Y, V_Z^*) \simeq \text{FinVect}(V_X, V_Y^* \otimes V_Z^*),$$

determined by the equalities

$$\langle f(x \otimes y), z \rangle = \langle h(x), y \otimes z \rangle, \quad x \in V_X, y \in V_Y, z \in V_Z,$$

for  $f \in \text{FinVect}(V_X \otimes V_Y, V_Z^*)$  and  $h \in \text{FinVect}(V_Z, V_Y^* \otimes V_X^*)$ . Since  $A_{X \otimes Y}$  is an affine span of  $A_X \otimes A_Y$ , we see that  $f$  is in  $\text{Af}$  if and only if  $f(x \otimes y) \in \tilde{A}_Z$  for all  $x \in A_X, y \in A_Y$ , that is,

$$1 = \langle f(x \otimes y), z \rangle = \langle h(x), y \otimes z \rangle \quad \forall x \in A_X, \forall y \in A_Y, \forall z \in A_Z.$$

But this is equivalent to

$$h(x) \in (A_Y \otimes A_Z)^\sim = \tilde{A}_{Y \otimes Z}, \quad \forall x \in A_X,$$

which means that  $h \in \text{Af}$ . □

### 0.3.6 The dual tensor product

Let us define the dual tensor product by  $\odot$ , that is

$$X \odot Y = (X^* \otimes Y^*)^*.$$

We then have

$$\begin{aligned} L_{X \odot Y} &= S_{X^* \otimes Y^*}^\perp = (S_{X^*} \otimes S_{Y^*})^\perp = (L_X^\perp \otimes L_Y^\perp)^\perp \\ S_{X \odot Y} &= L_{X^* \otimes Y^*}^\perp = (\tilde{a}_X \otimes S_Y^\perp)^\perp \wedge (S_X^\perp \otimes \tilde{a}_Y)^\perp \wedge (S_X^\perp \otimes S_Y^\perp)^\perp \end{aligned}$$

In particular,

$$d_{X \odot Y} = D_X d_Y + d_X D_Y - d_X d_Y.$$

**Lemma 6.** *Let  $X, Y$  be nontrivial. Then  $X \otimes Y = X \odot Y$  exactly in one of the following situations:*

1.  $X \simeq I$  or  $Y \simeq I$ ,
2.  $d_X = d_Y = 0$ ,
3.  $D_X = d_X + 1$  and  $D_Y = d_Y + 1$  (Objects with this property will be called first order).

*Proof.* It is easy to see that (when identifying  $X = X^{**}$ ), we have  $A_X \otimes A_Y \subseteq \tilde{A}_{X^* \otimes Y^*}$ , hence  $A_{X \otimes Y} \subseteq A_{X \odot Y}$ . We see from the above computations that

$$d_{X \odot Y} - d_{X \otimes Y} = d_X(D_Y - 1) + (D_X - 1)d_Y - 2d_X d_Y \geq 0,$$

with equality if and only if the conditions of the lemma hold. □



### 0.3.7 The no signaling product

**Lemma 7.** *The space  $A_{X \otimes Y}$  is precisely the affine subspace of elements  $w \in A_{X \odot Y}$ , such that  $\langle w, \cdot \otimes \tilde{a}_Y \rangle$  and  $\langle w, \tilde{a}_X \otimes \cdot \rangle$  do not depend on the choice of  $\tilde{a}_Y \in \tilde{A}_Y$  and  $\tilde{a}_X \in \tilde{A}_X$ .*

*Proof.* Any element  $w \in A_{X \otimes Y}$  has the form  $w = \sum_i \alpha_i x_i \otimes y_i$ , for  $x_i \in A_X$ ,  $y_i \in A_Y$  and  $\sum_i \alpha_i = 1$ . It follows that for any  $\tilde{a}_X \in \tilde{A}_X$  and  $\tilde{a}_Y \in \tilde{A}_Y$ ,

$$\langle w, \cdot \otimes \tilde{a}_Y \rangle = \sum_i \alpha_i x_i, \quad \langle w, \tilde{a}_X \otimes \cdot \rangle = \sum_i \alpha_i y_i.$$

Conversely, assume that  $w \in A_{X \odot Y}$  has this property, then for any  $\tilde{x} \in L_{X^*}$  and  $\tilde{y} \in L_{Y^*}$ , we have

$$\langle w, \cdot \otimes \tilde{y} \rangle = 0, \quad \langle w, \tilde{x} \otimes \cdot \rangle = 0.$$

It follows that

$$w \in (V_X^* \otimes L_{Y^*})^\perp \cap (L_{X^*} \otimes V_Y^*)^\perp = (V_X \otimes S_Y) \cap (S_X \otimes V_Y) = S_X \otimes S_Y$$

Since  $w \in A_{X \odot Y}$ , we have  $\langle w, \tilde{a}_X \otimes \tilde{a}_Y \rangle = 1$  for any choice of  $\tilde{a}_X \in \tilde{A}_X$ ,  $\tilde{a}_Y \in \tilde{A}_Y$ . Since  $\tilde{a}_X \otimes \tilde{a}_Y \in \tilde{A}_{X \otimes Y}$  and  $S_{X \otimes Y} = S_X \otimes S_Y$ , this implies  $w \in A_{X \otimes Y}$ .  $\square$

We will now define one-sided variants of this property. Namely, let

$$\begin{aligned} A_{X \prec Y} &:= \{w \in A_{X \odot Y}, \langle w, \cdot \otimes \tilde{a}_Y \rangle \text{ does not depend on } \tilde{a}_Y \in \tilde{A}_Y\} \\ A_{X \succ Y} &:= \{w \in A_{X \odot Y}, \langle w, \tilde{a}_X \otimes \cdot \rangle \text{ does not depend on } \tilde{a}_X \in \tilde{A}_X\}. \end{aligned}$$

We then put  $X \prec Y = (V_X \otimes V_Y, A_{X \prec Y})$  and  $X \succ Y = (V_X \otimes V_Y, A_{X \succ Y})$ .

**Lemma 8.** *For any choice of  $a_Y \in A_Y$ , we have*

$$L_{X \prec Y} = V_X \otimes L_Y + L_X \otimes a_Y, \quad S_{X \prec Y} = V_X \otimes L_Y + S_X \otimes a_Y.$$

*Similarly, for any  $a_X \in A_X$ , we have*

$$L_{X \succ Y} = L_X \otimes V_Y + a_X \otimes L_Y, \quad S_{X \succ Y} = L_X \otimes V_Y + a_X \otimes S_Y.$$

*Proof.* By the definition, we have

$$L_{X \prec Y} = L_{X \odot Y} \cap (V_X^* \otimes L_{Y^*})^\perp = (S_{X^*} \otimes S_{Y^*})^\perp \cap (V_X \otimes S_Y) = V_X \otimes L_Y + L_X \otimes a_Y,$$

for any element  $a_Y \in A_Y$ . The proof for  $\succ$  is similar.  $\square$

For the rest of this section, we fix some  $a_X \in A_X$  and  $\tilde{a}_X \in \tilde{A}_X$ . We will use the notation  $X_{\min} := (V_X, \{a_X\})$  and  $X_{\max} = (V_X, \{\tilde{a}_X\}^\sim)$ .

We also introduce a decomposition of  $V_X$  into an independent family of subspaces  $L_X^0, L_X^1, L_X^2$  as  $L_X^0 := \mathbb{R}a_X$ ,  $L_X^1 := L_X$  and  $L_X^2$  is any complement of  $L_X$  in the subspace  $\{\tilde{a}_X\}^\perp$ . We see that the  $L$  or  $S$  spaces of any of the objects discussed in this paragraph is a union of some of the subspaces  $L_X^i \otimes L_Y^j$ ,  $i, j = 0, 1, 2$ . We may therefore represent the subspaces in question by  $3 \times 3$

matrices such that the  $i, j$ -th element is 1 if the subspace contains  $L_X^i \otimes L_Y^j$  and 0 otherwise. For example, we have

$$L_{X \otimes Y} \equiv \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad L_{X \odot Y} \equiv \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad L_{X \prec Y} \equiv \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad L_{X \succ Y} \equiv \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \quad (6)$$

The  $S$ -spaces can be obtained from these by putting the top left element to 1. Notice also that all these objects belong to the set of objects  $Z = (V_X \otimes V_Y, A_Z)$  such that

$$X_{\min} \otimes Y_{\min} \hookrightarrow Z \hookrightarrow X_{\max} \otimes Y_{\max} (= X_{\max} \odot Y_{\max})$$

and this set forms a lattice under inclusion ordering with  $Z_0 \wedge Z_1 = Z_0 \sqcap Z_1$ ,  $Z_0 \vee Z_1 = Z_0 \sqcup Z_1$ , where  $\sqcap$  and  $\sqcup$  are the pulback and pushout of the inclusions  $X_{\min} \otimes Y_{\min} \hookrightarrow Z_i \hookrightarrow X_{\max} \otimes Y_{\max}$ . Furthermore, since  $\{L_X^i \otimes L_Y^j\}$  is an independent decomposition of  $V_X \otimes V_Y$ , all the objects in (6) are contained in a distributive sublattice of objects such that  $A_Z = a_X \otimes a_Y + L$ , where  $L$  is a subspace represented by a matrix  $M_Z$  with the top left element equal to 0. For such elements  $Z_1$  and  $Z_2$ , the representing matrices  $M_{Z_1 \sqcap Z_2}$  resp.  $M_{Z_1 \sqcup Z_2}$  are given by poinwise minimum resp. maximum of  $M_{Z_1}$  and  $M_{Z_2}$ .

Some further useful elements of this sublattice are represented as

$$L_{X_{\max} \otimes Y} \equiv \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad L_{X \otimes Y_{\max}} \equiv \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad L_{X_{\min} \odot Y} \equiv \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad L_{X \odot Y_{\min}} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

From these consideration, the following is immediate.

**Lemma 9.** *We have*

$$\begin{aligned} X \prec Y &= (X \otimes Y) \sqcup (X_{\min} \odot Y) = (X \odot Y) \sqcap (X_{\max} \otimes Y) \\ X \succ Y &= (X \otimes Y) \sqcup (X \odot Y_{\min}) = (X \odot Y) \sqcap (X \otimes Y_{\max}) \\ X \otimes Y &= (X \prec Y) \sqcap (X \succ Y) \\ X \odot Y &= (X \prec Y) \sqcup (X \succ Y). \end{aligned}$$

We have the inclusions

$$X_a \rightarrow X \rightarrow X^{\bar{a}}.$$

We also fix  $b \in A_Y$ ,  $\tilde{b} \in \tilde{A}_Y$ . By inclusions, we have the following diagrams

$$X_a \otimes Y_b \rightarrow X \otimes Y, \quad X_a \otimes Y_b \rightarrow X_a \odot Y$$

and

$$X \odot Y \rightarrow X^{\bar{a}} \odot Y^{\bar{b}} = X^{\bar{a}} \otimes Y^{\bar{b}}, \quad X \otimes Y^{\bar{b}} \rightarrow X^{\bar{a}} \otimes Y^{\bar{b}}.$$

**Lemma 10.**  $X \prec Y = X \otimes Y \sqcup X \odot Y_b = X \odot Y \sqcap X \otimes Y^{\bar{b}}.$

We see that the identity map  $id_{V_X \otimes V_Y}$  defines bimorphisms

$$X \otimes Y \rightarrow X \prec Y \rightarrow X \odot Y, \quad X \otimes Y \rightarrow X \succ Y \rightarrow X \odot Y.$$

**Lemma 11.** *The pushout and pullback of the above diagram are*

$$X \otimes Y = (X \prec Y) \sqcap (X \succ Y), \quad X \odot Y = (X \prec Y) \sqcup (X \succ Y).$$

*Proof.* We have by Lemma 7 that

$$A_{X \otimes Y} = A_{X \prec Y} \cap A_{X \succ Y} = A_{(X \prec Y) \sqcap (X \succ Y)},$$

we clearly have the last equality since the intersection  $A_{X \prec Y} \cap A_{X \succ Y}$  is nonempty. For the second part, □

To each object  $X = (V_X, A_X, a_X, \tilde{a}_X)$  we may define two object

$$X_{\min} := (V_X, \{a_X\}, a_X, \tilde{a}_X), \quad X_{\max} := (V_X, \{\tilde{a}_X\}^\sim, a_X, \tilde{a}_X).$$

It is easily seen that  $X_{\min} = (X_{\max}^*)^*$  and  $X_{\max} = (X_{\min}^*)^*$ , moreover,  $X_{\max}$  and  $(X_{\min})^*$  are first order objects. We have the inclusions

$$X_{\min} \xrightarrow{id} X \xrightarrow{id} X_{\max}.$$

We also have the inclusions

$$X \otimes Y \rightarrow X \odot Y \rightarrow X_{\max} \odot Y_{\max} = X_{\max} \otimes Y_{\max}$$

and

$$X \otimes Y \rightarrow X_{\max} \otimes Y \rightarrow X_{\max} \otimes Y_{\max}, \quad X \otimes Y \rightarrow X \otimes Y_{\max} \rightarrow X_{\max} \otimes Y_{\max}.$$

We can therefore define pullbacks and pushouts, which then becomes

$$(X_{\max} \otimes Y) \sqcap (X \otimes Y_{\max}) = X \otimes Y, \quad (X_{\max} \otimes Y) \sqcup (X \otimes Y_{\max}) = X_{\max} \otimes Y_{\max}.$$

Hence we may decompose  $X \odot Y$  into two parts

$$X \prec Y := (X \odot Y) \sqcap (X_{\max} \otimes Y), \quad X \succ Y := (X \odot Y) \sqcap (X \otimes Y_{\max}).$$

Note that these forms do not depend on the choice of the elements  $a_X, a_Y \dots$ !

**Lemma 12.** *We have*

$$X \otimes Y = (X \prec Y) \sqcap (X \succ Y), \quad X \odot Y = (X \prec Y) \sqcup (X \succ Y).$$

*Proof.* We have

$$\begin{aligned} (X \prec Y) \sqcap (X \succ Y) &= ((X \odot Y) \sqcap (X_{\max} \otimes Y)) \sqcap ((X \odot Y) \sqcap (X \otimes Y_{\max})) \\ &= (X \odot Y) \sqcap ((X_{\max} \otimes Y) \sqcap (X \otimes Y_{\max})) = (X \odot Y) \sqcap (X \otimes Y) = X \otimes Y. \end{aligned}$$

The second equality follows easily from Lemma 8 □

We next show that  $- \prec -$  and  $- \succ -$  define a functor  $\text{Af} \times \text{Af} \rightarrow \text{Af}$ . Let  $X_1 \xrightarrow{f} Y_1$  and  $X_2 \xrightarrow{g} Y_2$ , we will show that  $X_1 \prec Y_1 \xrightarrow{f \otimes g} X_2 \prec Y_2$ . For this, we need to prove that  $(f \otimes g)(A_{X_1 \prec Y_1}) \subseteq A_{X_2 \prec Y_2}$  and  $(f \otimes g)(A_{X_1 \succ Y_1}) \subseteq A_{X_2 \succ Y_2}$ . This is clear from Lemma 8.

**Lemma 13.**  $(\text{Af}, \prec, I)$  is a monoidal category.

*Proof.* It is easily checked from Lemma 8 that  $\alpha_{X,Y,X}(L_{(X \prec Y) \prec Z}) = L_{X \prec (Y \prec Z)}$  and clearly also  $\alpha_{X,Y,Z}(a_X \otimes a_Y \otimes a_Z) = a_X \otimes a_Y \otimes a_Z$ , so that  $\alpha$  is the associator. Since  $I \otimes X = I \odot X$  and  $X \odot I = X \otimes I$ , we have  $I \prec X = I \otimes X$  and  $X \prec I = X \otimes I$ , so  $\lambda$  and  $\rho$  are the unitors. But note that  $\sigma_{X,Y}(A_{X \prec Y}) = A_{Y \succ X}$ , so this structure is not symmetric.  $\square$

We have  $(X \prec Y)^* = X^* \prec Y^*$ . Indeed, by duality,

$$\begin{aligned} (X \prec Y)^* &= ((X \odot Y) \sqcap (X_{\max} \odot Y))^* = (X \odot Y)^* \sqcup (X_{\max} \otimes Y)^* \\ &= (X^* \otimes Y^*) \sqcup (X_{\max}^* \odot Y^*) \end{aligned}$$

### 0.3.8 Internal hom

The internal hom has the form

$$[X, Y] = (X \otimes Y^*)^* = X^* \odot Y. \quad (7)$$

We then have

$$\begin{aligned} L_{[X,Y]} &= (S_X \otimes L_Y^\perp)^\perp = (V_X^* \otimes L_Y) \vee (L_{X^*} \otimes V_Y), \\ S_{[X,Y]} &= (\tilde{a}_X \otimes S_Y^\perp)^\perp \wedge (L_X \otimes \tilde{a}_Y)^\perp \wedge (L_X \otimes S_Y^\perp)^\perp = (V_X^* \otimes L_Y) \vee (L_{X^*} \otimes V_Y) \vee (\tilde{a}_X \otimes a_Y). \end{aligned}$$

and

$$d_{[X,Y]} = D_X D_Y - (d_X + 1)(D_Y - d_Y).$$

As we have seen in  $\text{FinVect}$ , the space  $V_{[X,Y]} = V_X^* \otimes V_Y$  is identified with the space of all linear maps  $V_X \rightarrow V_Y$ , by (1). We will show that  $A_{[X,Y]}$  corresponds to the affine subspace of maps mapping  $A_X$  into  $A_Y$ , that is, morphisms in  $\text{Af}$ . Indeed, we see from (1) that  $f$  is in  $\text{Af}$  if and only if

$$\langle f(x), y^* \rangle = \langle w, x \otimes y^* \rangle = 1, \quad x \in A_X, \ y^* \in \tilde{A}_Y,$$

which is equivalent to  $w \in (A_X \otimes \tilde{A}_Y)^\sim = \tilde{A}_{X \otimes Y^*}$ .

### 0.3.9 The no signaling product

For two objects  $X, Y$  we define

$$X \prec Y := (V_X \otimes V_Y, A_{X \prec Y}, a_X \otimes a_Y, \tilde{a}_X \otimes \tilde{a}_Y),$$

where  $A_{X \prec Y}$  is determined by

$$S_{X \prec Y} = V_X \otimes S_Y \cap S_{X \odot Y} = (V_X \otimes S_Y) \cap (L_{X \odot Y} \vee \{a_X \otimes a_Y\}) = (V_X \otimes S_Y) \cap ((V_X \otimes L_Y) \vee (L_X \otimes V_Y) \vee \{a_X \otimes a_Y\})$$

**Lemma 14.**

We may similarly define  $X \succ Y$ . Setting  $f \prec g = f \otimes g$  for  $X_1 \xrightarrow{f} X_2, Y_1 \xrightarrow{g} Y_2$ , we see that  $(- \prec -)$  is functorial. Indeed, to show that  $X_1 \prec Y_1 \xrightarrow{f \otimes g} X_2 \prec Y_2$ , we need to show that  $f \otimes g(A_{X_1 \prec Y_1}) \subseteq A_{X_2 \prec Y_2}$ . Assume  $w \in A_{X_1 \prec Y_1}$ , that is,  $w \in V_{X_1} \otimes S_{Y_1}$  and  $\langle w, \tilde{a}_{X_1} \otimes \tilde{a}_{Y_1} \rangle = 1$ .

### 0.3.10 Dualizable (nuclear) objects

An object in  $\mathbf{Af}$  is nuclear if the natural map  $X^* \otimes X \rightarrow [X, X]$  is an isomorphism (santocanale). That is, the inclusion  $X^* \otimes X \subseteq X^* \odot X$  that comes from the embedding

$$\tilde{A}_X \otimes A_X \subseteq (A_X \otimes \tilde{A}_X)^\sim$$

becomes an equality. As we have seen in Lemma 6, for proper objects we have  $X^* \otimes X = X^* \odot X$  if and only if

$$d_X + 1 = D_X = D_{X^*} = d_{X^*} + 1 = D_X - d_X.$$

It follows that  $d_X = 0$  and  $D_X = 1$ , so that  $X \simeq I$ . Hence the tensor unit is the unique dualizable (or nuclear) object in  $\mathbf{Af}$ .

### 0.3.11 No signaling

We say that  $X \xrightarrow{f} Y$  is no signaling if

$$y^* \circ f = \tilde{a}_Y \circ f, \quad \forall y \in Y^* = [Y, I],$$

in other words

$$y \circ f = 0, \quad \forall y^* \in L_{Y^*} = S_Y^\perp.$$

Taking  $w \in A_{[X,Y]}$  be the corresponding elements, this means that

$$\langle w, x \otimes y^* \rangle = 0, \quad \forall x \in V_X, y^* \in S_Y^\perp,$$

in other words

$$w \in (V_X \otimes S_Y^\perp)^\perp = V_X^* \otimes S_Y,$$

so that

$$w \in A_{[X,Y]} \cap (V_X^* \otimes S_Y).$$

Since  $a_{[X,Y]} = \tilde{a}_X \otimes a_Y \in V_X^* \otimes S_Y$ , we have that

$$a_{[X,Y]} - w \in L_{[X,Y]} \cap V_X^* \otimes S_Y = (S_X \otimes L_Y^\perp)^\perp \cap V_X^* \otimes S_Y = (V_X^* \otimes L_Y) + (L_{X^*} \otimes a_Y).$$

We can also define no signaling in the oposite way, that is,

$$f(x) = f(a_X), \quad \forall x \in A_X.$$

This is of course the same as

$$f(x) = 0, \quad \forall x \in L_X,$$

or

$$\langle w, x \otimes y^* \rangle = 0, \quad \forall x \in L_X, y^* \in V_Y^*,$$

that is,

$$w \in (L_X \otimes V_Y^*)^\perp = L_X^\perp \otimes V_Y = S_{X^*} \otimes V_Y.$$

It follows that

$$\tilde{a}_X \otimes a_Y - w \in L_{[X,Y]} \cap S_{X^*} \otimes V_Y = L_{X^*} \otimes V_Y + \tilde{a}_X \otimes L_Y.$$

## 0.4 Once more on the monoidal structures

### 0.4.1 Tensor product

We have

$$L_{X \otimes Y} = (a_X \otimes L_Y) + (L_X \otimes a_Y) + (L_X \otimes L_Y) = (S_X \otimes L_Y) + (L_X \otimes a_Y) = (L_X \otimes S_Y) + (a_X \otimes L_Y)$$

A closed symmetric monoidal structure. We have

$$L_{X_{\max} \otimes Y} = V_X \otimes L_Y + \{\tilde{a}_X\}^\perp \otimes a_Y, \quad L_{X_{\min} \otimes Y} = a_X \otimes L_Y.$$

**Lemma 15.** *We have*

$$X \otimes Y = (X_{\max} \otimes Y) \sqcap (X \otimes Y_{\max}).$$

*Proof.* This is easy, since

$$S_{X_{\max} \otimes Y} \cap S_{X \otimes Y_{\max}} = (V_X \otimes S_Y) \cap (S_X \otimes V_Y) = S_X \otimes S_Y.$$

□

### 0.4.2 Dual product

By definition,  $X \odot Y = (X^* \otimes Y^*)^*$ . We have

$$L_{X \odot Y} = (L_X^\perp \otimes L_Y^\perp)^\perp = (V_X \otimes L_Y) \vee (L_X \otimes V_Y).$$

We have

$$L_{X_{\max} \odot Y} = (V_X \otimes L_Y) \vee (\{\tilde{a}_X\}^\perp \otimes V_Y), \quad L_{X_{\min} \odot Y} = V_X \otimes L_Y.$$

**Lemma 16.** *We have*

$$\begin{aligned} X_{\max} \otimes Y_{\max} &= (X_{\max} \odot Y) \sqcup (X \odot Y_{\max}) \\ X \odot Y &= (X_{\min} \odot Y) \sqcup (X \odot Y_{\min}) \\ X \otimes Y &= (X_{\min} \otimes Y) \sqcup (X \otimes Y_{\min}) \sqcup (X_{\min} \odot Y \sqcap X \odot Y_{\min}) \end{aligned}$$

*Proof.* The first is easy, the second follows from Lemma 15 by duality, the third is also easy.

□

## 0.5 The no signalling product

Let us define  $X \prec Y := (X \odot Y) \sqcap (X_{\max} \otimes Y)$ . We have

$$L_{X \prec Y} = V_X \otimes L_Y + L_X \otimes a_Y = L_{X_{\min} \odot Y} + L_{Y \otimes Y_{\min}}.$$

So that

$$X \prec Y := (X \odot Y) \sqcap (X_{\max} \otimes Y) = (X_{\min} \odot Y) + (X \otimes Y_{\min})$$

**Lemma 17.** *We have*

$$\begin{aligned} (X \otimes Y) \sqcup (X_{\min} \odot Y) &= (X \odot Y) \sqcap (X_{\max} \otimes Y) \\ &= (X_{\min} \odot Y) + (X \otimes Y_{\min}) = (X_{\max} \otimes Y) \sqcap (X \odot Y_{\max}). \end{aligned}$$

Let us denote the above object by  $X \prec Y$ . Then  $A_{X \prec Y}$  is the set of elements in  $V_X \otimes V_Y$  such that  $\langle w, \cdot \otimes y^* \rangle$  is a fixed element in  $A_X$ , independently of  $y^* \in \tilde{A}_Y$ .

Blbe uvedenie, definicia!

*Proof.* We see that  $A_{X \prec Y} \subseteq A_{X \odot Y}$ , moreover,

$$A_{X \prec Y} = \{w \in A_{X \odot Y}, \langle w, id_X \otimes y^* \rangle = 0, \forall y^* \in L_{Y^*}\}.$$

In other words, since clearly  $a_X \otimes a_Y \in A_{X \prec Y}$ ,

$$\begin{aligned} L_{X \prec Y} &= \{w - a_X \otimes a_Y, w \in A_{X \prec Y}\} = L_{X \odot Y} \cap (V_X^* \otimes L_Y^*)^\perp = (L_X^\perp \otimes S_{Y^*})^\perp \cap (V_X^* \otimes L_{Y^*})^\perp \\ &= ((L_X^\perp \otimes S_{Y^*}) \vee (V_X^* \otimes L_{Y^*}))^\perp = ((V_X^* \otimes L_{Y^*}) + (S_X^* \otimes \tilde{a}_Y))^\perp = S_{(X \prec Y)^*}^\perp \end{aligned}$$

But also

$$\begin{aligned} L_{X \odot Y} \cap (V_X^* \otimes L_Y^*)^\perp &= ((V_X \otimes L_Y) \vee L_X \otimes V_Y) \cap (V_X \otimes S_Y) \\ &= ((V_X \otimes L_Y) + (L_X \otimes a_Y) + (L_X \otimes S_Y')) \cap ((V_X \otimes L_Y) + (L_X \otimes a_Y) + (L_X' \otimes a_Y)) \\ &= ((V_X \otimes L_Y) + (L_X \otimes a_Y)). \end{aligned}$$

First note that we have

$$L_{X_{\min} \odot Y} = V_X \otimes L_Y, \quad L_{X \otimes Y_{\min}} = L_X \otimes a_Y$$

and therefore

$$L_{X \otimes Y_{\min}} \cap L_{X_{\min} \odot Y} = \{0\}.$$

Further,

$$L_{X \otimes Y} = (S_X \otimes L_Y) + (L_X \otimes a_Y) = (S_X \otimes L_Y) + L_{X \otimes Y_{\min}}$$

and

$$L_{X \odot Y} = (V_X \otimes L_Y) \vee (L_X \otimes V_Y), \quad L_{X_{\max} \otimes Y} = (V_X \otimes L_Y) + (\{\tilde{a}_X\}^\perp \otimes a_Y)$$

We have

$$S_{(X \otimes Y) \sqcup (X_{\min} \odot Y)} = S_{X \otimes Y} \vee S_{X_{\min} \odot Y} = S_{X \otimes Y} \vee L_{X_{\min} \odot Y}$$

□

## 0.6 The category AfH

The category AfH will be constructed as a subcategory in Af.

### 0.6.1 First order objects

It is easily seen that the following are equivalent:

1.  $D_X = d_X + 1$ ;
2.  $S_X = V_X$ ;
3.  $L_X = \{\tilde{a}_X\}^\perp$ ;
4.  $S_{X^*} = \mathbb{R}\tilde{a}_X$ ;
5.  $L_{X^*} = \{0\}$ .

We say that an object  $X$  is first order if any of these conditions is fulfilled. We have seen that for proper objects,  $X \otimes Y = X \odot Y$  if and only if both  $X$  and  $Y$  are first order. We also have

**Lemma 18.**  *$X$  is first order if and only if  $X \otimes X = X \odot X$ .*

**Lemma 19.** *Let  $X, Y$  be first order, then  $X \otimes Y$  is first order.*

*Proof.* We have

$$S_{X \otimes Y} = S_X \otimes S_Y = V_X \otimes V_Y = V_{X \otimes Y}.$$

□

### 0.6.2 Channels

A channel is an object  $[X, Y]$  where  $X$  and  $Y$  are first order. As we have seen,

$$X^* \otimes Y \subseteq X^* \odot Y = [X, Y].$$

If  $X$  is first order,  $\tilde{A}_X = \{\tilde{a}_X\}$  and the elements of  $A_{X^* \otimes Y} = \tilde{a}_X \otimes A_Y$  are identified with channels of the form

$$f(x) = \langle \tilde{a}_X, x \rangle y, \quad x \in V_X,$$

for some  $y \in A_Y$ . Such maps will be called replacement channels.

**Lemma 20.** *Let  $X, Y$  be first order and let  $w \in V_X^* \otimes V_Y$ . Then  $w \in A_{[X, Y]}$  if and only if*

$$\circ_Y : w_{X^* Y} \otimes \tilde{a}_Y \mapsto \tilde{a}_X.$$

*Proof.* Let  $f : V_X \rightarrow V_Y$  be the map corresponding to  $w$ , then

$$\circ_Y(w \otimes \tilde{a}_Y) = (V_X^* \otimes e_{V_Y})(w \otimes \tilde{a}_Y) = \tilde{a}_Y \circ f,$$

where  $\tilde{a}_Y \in V_Y^*$  is seen as a map  $V_Y \rightarrow \mathbb{R}$ . So  $\tilde{a}_Y \circ f : V_X \rightarrow \mathbb{R}$  is an element in  $V_X^*$ . We know that  $w \in A_{[X, Y]}$  iff  $f(A_X) \subseteq A_Y$ , which is equivalent to  $\tilde{a}_Y \circ f(x) = 1$  for all  $x \in A_X$ , so that  $\tilde{a}_Y \circ f \in \tilde{A}_X = \{\tilde{a}_X\}$ , since  $X$  is first order. □

**Lemma 21.** *Let  $Y$  be first order and  $w \in V_X^* \otimes V_Y$ . Then  $w \in A_{[X, Y]}$  if and only if*

$$\circ_Y(w_{X^* Y} \otimes \tilde{a}_Y) \in \tilde{A}_X.$$

Moreover,

$$\tilde{A}_{[X, Y]} = A_X \otimes \{\tilde{a}_Y\}.$$

*Proof.* Since  $Y$  is first order, we have  $A_{Y^*} = \tilde{A}_Y = \{\tilde{a}_Y\}$  and by (7)

$$\tilde{A}_{[X, Y]} = A_{X \otimes Y^*} = A_X \otimes \{\tilde{a}_Y\}.$$

As in the above proof, let  $f : V_X \rightarrow V_Y$  be the map corresponding to  $w$ . Then  $\tilde{a}_Y \circ f \in V_X^*$  and  $w \in A_{[X, Y]}$  iff  $f(A_X) \subseteq A_Y$ . This means that

$$\tilde{a}_Y \circ f(x) = 1, \quad \forall x \in A_X,$$

which means that  $\tilde{a}_Y \circ f \in \tilde{A}_X$ . □



### 0.6.3 AfH

The category AfH is the full subcategory in Af created from first order objects by taking tensor products and duals. We will add more later. We will use the notation  $V_{XY^*}$  for  $V_X \otimes V_Y^*$ , etc.

Any object  $X$  in AfH is created from first order objects  $X_1, \dots, X_k$ , so that  $V_X = \tilde{V}_{X_1} \otimes \dots \otimes \tilde{V}_{X_k}$ , where  $\tilde{V}_{X_i}$  is either  $V_{X_i}$  or  $V_{X_i}^*$ ,  $i = 1, \dots, k$ . We will next show that any object is a set of channels that contains all replacement channels.

**Proposition 1.** *Let  $X$  be an object in AfH. Then there are first order objects  $Y_I$  and  $Y_O$  and inclusions  $f, g$  such that*

$$Y_I^* \otimes Y_O \xrightarrow{f} X \xrightarrow{g} [Y_I, Y_O]. \quad (8)$$

*Proof.* Let  $X$  be first order, then since  $I$  is first order,

$$I^* \otimes X = I \otimes X \xrightarrow{\lambda_X} X \xrightarrow{\lambda_X^{-1}} I \otimes X = I \odot X = [I, X].$$

Clearly,  $f = \lambda_X$  and  $g = \lambda_X^{-1}$  are inclusions. Now assume that  $Z$  satisfies (8) and let  $X = Z^*$ . Taking duals and composing with symmetries, we get

$$Y_O^* \otimes Y_I \xrightarrow{\sigma_{Y_O^*, V_{Y_I}}} Y_I \otimes Y_O^* = [Y_I \otimes Y_O]^* \xrightarrow{g^*} X \xrightarrow{f^*} (Y_I^* \otimes Y_O)^* \xrightarrow{\sigma_{Y_I^*, V_{Y_O}^*}} (Y_O \otimes Y_I)^* = [Y_O, Y_I].$$

Since the compositions of  $f^*$  and  $g^*$  with symmetries are inclusions, we see that  $X$  satisfies (8).

Next, let  $X_1$  and  $X_2$  satisfy (8) with some first order objects  $Y_I^i, Y_O^i$  and inclusions  $f^i, g^i$ ,  $i = 1, 2$ , and let  $X = X_1 \otimes X_2$ . We then have, using the appropriate symmetries

$$Y_I^1 Y_I^2 \otimes (Y_O^1 Y_O^2)^* \xrightarrow{\sigma_{Y_I^2, Y_O^1}} Y_I^1 \otimes (Y_O^1)^* \otimes Y_I^2 \otimes (Y_O^2)^* \xrightarrow{f^1 \otimes f^2} X \xrightarrow{g^1 \otimes g^2} [Y_I^1, Y_O^1] \otimes [Y_I^2, Y_O^2] \xrightarrow{\sigma_{Y_O^1, Y_O^2}} [Y_I^1 Y_I^2, Y_O^1 Y_O^2].$$

Perhaps the last arrow needs some checking, so let us do it properly. We need to show that for  $w \in A_{[Y_I^1, Y_O^1] \otimes [Y_I^2, Y_O^2]}$ , we have  $\sigma_{Y_O^1, Y_O^2}(w) \in A_{[Y_I^1 Y_I^2, Y_O^1 Y_O^2]}$ , but this is clear using Lemma 20.  $\square$

The pair  $(Y_I, Y_O)$  for an object  $X$  will be called the setting of  $X$ . For objects of the same setting we may take pullbacks and pushouts of the corresponding inclusions.

Pullbacks are intersections, pushouts the affine mixture.

Channels into (from) products and coproducts

We define AfH as the full subcategory of Af containing all first order objects and closed under (finite products,) duals and tensor products .