On the properties of spectral effect algebras

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The aim of this paper is to show that there can be either only one or uncountably many contexts in any spectral effect algebra, answering a question posed in [S. Gudder, Convex and Sequential Effect Algebras, (2018), arXiv:1802.01265]. We also provide some results on the structure of spectral effect algebras and their state spaces and investigate the direct products and direct convex sums of spectral effect algebras. In the case of spectral effect algebras with sharply determining state space, stronger properties can be proved: the spectral decompositions are essentially unique, the algebra is sharply dominating and the set of its sharp elements is an orthomodular lattice. The article also contains a list of open questions that might provide interesting future research directions.

I. INTRODUCTION

Effect algebras were defined in [1] as a generalization of the set of projectors on a Hilbert space. Since then a variety of results has been developed in the field, it has been used in the so called general probabilistic theories and it has attracted the interest of both mathematicians and physicist alike.

We present an answer to the open question presented in [2] about the possible number of contexts contained in a spectral effect algebra. The article has slightly overgrown from a simple answer to a question into a broader look on some of the properties of the spectral effect algebras with its own set of open questions.

The article is organized as follows: in Sec. II we present the basic definitions and results. In Sec. III we introduce spectral effect algebras and prove some basic results on properties of spectral decompositions and the structure of the state space. In Sec. IV we prove the main result that a spectral effect algebra may contain only either one or uncountably many contexts. In Sec. V we inspect two standard constructions with convex effect algebras: the direct product and the direct convex sum and we show that while the direct product of spectral effect algebras is again a spectral effect algebra, the direct convex sum of spectral effect algebras is not spectral. In Sec. VI we look at a special case of spectral effect algebras that have sharply determining state space and we show that in such setting a result analogical to [3, Proposition 18] holds, moreover, the algebra is sharply dominating and the set of its sharp elements is an orthomodular lattice. The Sec. VII contains the conclusions and open questions.

II. PROPERTIES OF CONVEX EFFECT ALGEBRAS

In this section we are going to present definition and properties of convex effect algebras. We closely follow the definitions used in [2] with a slightly different notation, more natural for linear effect algebras that are closely related to general probabilistic theories.

Definition 1. An effect algebra is a system $(E, 0, 1, +)$, where E is a set containing at least one element, $0, 1 \in E$ and $+$ is a partial binary operation on E. Let $a, b \in E$, then we write $a + b \in E$ whenever $a + b$ is defined (and hence yields an element of E). Moreover we require that $(E, 0, 1, +)$ satisfies the following conditions:

(E1) if $a + b \in E$ then $b + a \in E$ and $a + b = b + a$,

(E2) if $a + b \in E$ and $(a + b) + c \in E$, then $a + (b + c) \in E$, $a + (b + c) = (a + b) + c$,

(E3) for every $a \in E$ there is unique $a' \in E$ such that $a + a' = 1$, we usually denote $a' = 1 - a$,

(E4) if $a + 1 \in E$, then $a = 0$.

Definition 2. An effect algebra E is convex if for every $a \in E$ and $\lambda \in [0, 1] \subset \mathbb{R}$ there is an element $\lambda a \in E$ such that for all $\lambda, \mu \in [0, 1]$ and $a, b \in E$ we have

(C1) $\mu(\lambda a) = (\lambda\mu)a$,

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(C2) if $\lambda + \mu \leq 1$, then $\lambda a + \mu a \in E$ and $(\lambda + \mu)a = \lambda a + \mu a$,

(C3) if $a + b \in E$, then $\lambda a + \lambda b \in E$ and $\lambda(a + b) = \lambda a + \lambda b$,

(C4) $1a = a$.

Definition 3. Let E, F be effect algebras. A map $\phi : E \mapsto F$ is called additive if for $a, b \in E$, $a + b \in E$ we have: $\phi(a) + \phi(b) \in F$ and $\phi(a + b) = \phi(a) + \phi(b)$. An additive map such that $\Phi(1) = 1$ is called a morphism. A morphism is an isomorphism if it is surjective and for $a, b \in E$, $\phi(a) + \phi(b) \in F$ implies $a + b \in E$. If E and F are convex effect algebras, then a morphism Φ is affine if

$$\Phi(\lambda a) = \lambda \Phi(a), \qquad a \in E, \lambda \in [0, 1].$$

Definition 4. A state on an effect algebra E is a morphism $s : E \mapsto [0, 1] \subset \mathbb{R}$. The set of states on an effect algebra will be denoted $\mathfrak{S}(E)$.

It was proved in [4] that any state on a convex effect algebra is affine. The set of states is also referred to as state space of the effect algebra and it will play an important role in later constructions.

Let V be a real vector space with a pointed convex cone P, that is $P \cap (-P) = \{0\}$ where 0 denotes the zero vector. For $v, w \in V$ we define $v \geq w$ if and only if $v - w \in P$. Then $\leq$ is a partial order in V and (V, P) is an ordered vector space. Let $u \in P$, then the set

$$[0, u] = \{v \in V : 0 \leq v \leq u\}$$

is an effect algebra with the operation $+$ defined as the sum of the vectors and for $a, b \in [0, u]$ we have $a + b \in [0, u]$ if and only if $a + b \leq u$ which is exactly why we have chosen such unusual notation in Def. 1. Also note that in this case u is the unit of the effect algebra $[0, u]$, i.e. we have $1 = u$.

Definition 5. A linear effect algebra is an effect algebra of the form $[0, u]$ for some ordered vector space (V, P) and $u \in P$.

The following is an important result.

Proposition 1. *Every convex effect algebra is affinely isomorphic to a linear effect algebra.*

Proof. See [5]. $\square$

From now on we are going to assume that all of the effect algebras we will work with are convex. Below, we omit the isomorphism and identify convex effect algebras with the linear effect algebras they are isomorphic to. Moreover, we may and will assume that the interval $[0, u]$ generates the ordered vector space (V, P), so that u is an order unit in (V, P), [6, Lemma 3.1].

Definition 6. Let $f \in E$. We say that

- f is one-dimensional if $f \neq 0$ and for $g \in E$ we have that $f \geq g$ implies $g = \lambda f$ for some $\lambda \in [0, 1]$;
- f is sharp if $f \geq g$ and $1 - f \geq g$ implies $g = 0$;
- f is extremal if $f = \lambda g_1 + (1 - \lambda) g_2$ for some $\lambda \in (0, 1)$ implies that $f = g_1 = g_2$.

The set of sharp elements will be denoted by $S(E)$. The set of sharp one-dimensional elements will be denoted by $S_1(E)$.

It was shown in [6, Lemma 4.4] that any extremal element is sharp, but the converse is not necessarily true. Moreover, for any sharp effect f there are states $s_0, s_1 \in \mathfrak{S}(E)$ such that $s_0(f) = 0$, $s_1(f) = 1$. Note that in general the states s_0 and s_1 do not have to be unique. In general probabilistic theories [7] one-dimensional effects are called indecomposable. Some authors also refer to sharp one-dimensional effects as atomic [8]. In what follows we will be interested in the properties of sharp one-dimensional effects.

Proposition 2. *Let E be a convex effect algebra, then a one-dimensional effect is sharp if and only if it is extremal.*

Proof. As noted above, every extremal element is sharp. Conversely, let $f \in S_1(E)$ and assume that we have $f = \lambda g_1 + (1-\lambda)g_2$ for some $g_1, g_2 \in E$ and $\lambda \in (0,1)$. It follows that we have $f \geq \lambda g_1$ and $f \geq (1-\lambda)g_2$ and since f is one-dimensional we must have $g_1 = \mu_1 f$ and $g_2 = \mu_2 f$, i.e. f must be a convex combination of its multiples. Let now $\mu \geq 1$ be such that $\mu f \in E$. Let $\eta = \min\{\frac{1}{\mu}, 1 - \frac{1}{\mu}\}$ then we have

$$f \geq \eta f$$

$$1 - f = \frac{1}{\mu}(1 - \mu f) + (1 - \frac{1}{\mu})1 \geq \eta 1 \geq \eta f.$$

Since f is sharp, this implies that $\eta = 0$ and hence $\mu = 1$. It follows that we must have $g_1 = g_2 = f$ and f is extremal. □

III. CONTEXTS AND SPECTRAL EFFECT ALGEBRAS

In this section we are going to introduce contexts and spectral effect algebras and provide some results on their structure.

Definition 7. A context in a convex effect algebra E is a finite collection $A = \{a_1, \ldots, a_n\} \subset S_1(E)$ such that $\sum_{i=1}^n a_i = 1$.

Let $A = \{a_1, \ldots, a_n\}$ be a context in a convex effect algebra E and let $\hat{a}_i = \{s_i \in \mathfrak{S}(E) : s_i(a_i) = 1\}$. By the remarks below Definition 6, the sets $\hat{a}_i$ are nonempty and it is easy to see that

$$s_i(a_j) = \delta_{ij}, \qquad \forall s_i \in \hat{a}_i, \; i, j = 1, \ldots, n,$$

where δ_{ij} is the Kronecker delta.

Let us denote by $\bar{A}$ the convex effect subalgebra generated by A, that is

$$\bar{A} = \left\{ \sum_i \mu_i a_i, \; \mu_i \in [0,1] \right\}.$$

Then $\bar{A}$ is the interval $[0, u_A]$ in the ordered vector space $V(A) := \{\sum_i t_i a_i, \; t_i \in \mathbb{R}\}$, with an obvious positive cone and order unit $u_A = \sum_i a_i$.

Definition 8. We say that a convex effect algebra E is spectral if for every $f \in E$ there is a context $A \subset E$ such that $f \in \bar{A}$.

Specifically, any $f \in E$ has the form

$$f = \sum_i \mu_i a_i \tag{1}$$

for some $\mu_i \in [0,1]$ and some context $A = \{a_1, \ldots, a_n\}$. Any expression of the form (1) will be called a spectral decomposition of f.

The most important examples are the algebras of finite dimensional classical and quantum effects, that were characterized in [2]. It is also easy to see that the algebra of effects over a finite dimensional real Hilbert space is spectral. Note that it is not assumed that the number of elements is the same in each context or that the space V generated by E is finite dimensional, although all the above examples have these properties.

We next discuss some properties of spectral effect algebras and their state spaces. The following will be a useful tool.

Lemma 1. *Let $f = \sum_{i=1}^n \mu_i a_i$ be a spectral decomposition of f. Then*

$$\max_{s \in \mathfrak{S}(E)} s(f) = \max\{\mu_1, \ldots, \mu_n\},$$
$$\min_{s \in \mathfrak{S}(E)} s(f) = \min\{\mu_1, \ldots, \mu_n\}.$$

Proof. Let $s \in \mathfrak{S}(E)$, then we have

$$s(f) = \sum_{i=1}^{n} \mu_i s(a_i) \leq \max\{\mu_1, \dots, \mu_n\} \sum_{i=1}^{n} s(a_i) = \max\{\mu_1, \dots, \mu_n\}.$$

To show that the bound is tight let $\max\{\mu_1, \dots, \mu_n\} = \mu_{i'}$ for some $i' \in \{1, \dots, n\}$ and let $s \in \hat{a}_{i'}$. Then we have

$$s(f) = \mu_{i'} = \max\{\mu_1, \dots, \mu_n\}.$$

The proof for the minimum is analogical. □

We next show that the elements of the embedding ordered vector space V have spectral decompositions as well.

Lemma 2. *Let E be spectral and let (V, P) be the ordered vector space with an order unit $u \in P$ such that $E = [0, u]$. Then for any $v \in V$ there is a context A such that $v \in V(A)$.*

Proof. Since u is a order unit, we have $-\lambda u \leq v \leq \lambda u$ for some $\lambda > 0$. Then $0 \leq v + \lambda u \leq 2\lambda u$, so that $a := \frac{1}{2\lambda} v + \frac{1}{2} u \in E$. Let $a = \sum_i \nu_i a_i$ for some context A, then $v = \lambda(2a - u) = \sum_i \lambda(2\nu_i - 1)a_i$. □

Proposition 3. *Any spectral effect algebra E has an order determining set of states, that is, $s(a) \leq s(b)$ for all $s \in \mathfrak{S}(E)$ implies that $a \leq b$.*

Proof. Let $a, b \in E$ be such that $s(a) \leq s(b)$ for all states s. By Lemma 2, there is some context A and real numbers μ_i such that $v = b - a = \sum_i \mu_i a_i$. By the assumption, $\mu_i = s_i(v) \geq 0$ for $s_i \in \hat{a}_i$, so that $b - a \geq 0$ and $a \leq b$. □

Let us remark that it was proved in [6, Theorem 3.6] that a convex effect algebra E has an order determining set of states if and only if it is Archimedean. In this case, the order unit seminorm

$$\|v\| := \inf\{\lambda > 0,\ -\lambda u \leq v \leq \lambda u\} = \sup_{s \in \mathfrak{S}(E)} |s(v)|$$

is a norm. Using Lemma 1, we obtain for $f \in E$ with spectral decomposition $f = \sum_i \mu_i a_i$ that

$$\|f\| = \max\{\mu_1, \dots, \mu_n\}, \qquad \|1 - f\| = 1 - \min\{\mu_1, \dots, \mu_n\}. \tag{2}$$

More generally, if $v = \sum_i \alpha_i a_i$ is a spectral decomposition, then we have for $s_i \in \hat{a}_i$

$$\|v\| = \sup_{s \in \mathfrak{S}(E)} |s(v)| = \max_i |\alpha_i| =: |\alpha_{i_{max}}| = |s_{i_{max}}(v)|.$$

An element $f \in E$ does not have to have a unique spectral decomposition. The following result is inspired by [3] and is immediate from (2).

Proposition 4. *Let $f \in E$ have two spectral decompositions*

$$\sum_{i=1}^{n} \mu_i a_i = f = \sum_{j=1}^{m} \nu_j b_j$$

Then $\max\{\mu_1, \dots, \mu_n\} = \max\{\nu_1, \dots, \nu_m\}$ *and* $\min\{\mu_1, \dots, \mu_n\} = \min\{\nu_1, \dots, \nu_m\}$.

Up to now it is not clear whether the sets $\hat{a}_i$ consist of a single state. Clearly, these sets are faces of $\mathfrak{S}(E)$. More generally, for any element $f \in E$, the set $\hat{f} := \{s \in \mathfrak{S}(E), s(f) = 1\}$ is a face of $\mathfrak{S}(E)$ (note that this also may be empty). A face of this form will be called E-exposed. If $\hat{f} = \{s\}$, we say that s is and E-exposed point of $\mathfrak{S}(E)$, in this case, we write $\hat{f} = s$. If $a \in S_1(E)$, $\hat{a}$ is a nonempty E-exposed proper face of $\mathfrak{S}(E)$. Note that if $A = \{a_1, \dots, a_n\}$ is a context, the faces $\hat{a}_i$ are affinely independent. We next show a property of the E-exposed points of $\mathfrak{S}(E)$.

Proposition 5. *Every E-exposed point of $\mathfrak{S}(E)$ has the form $\hat{a}$ for some $a \in S_1(E)$.*

Proof. Let $s \in \mathfrak{S}(E)$ be an E-exposed point, so that $s = \hat{f}$ for some $f \in E$. Then $1 = s(f) = \max_{s' \in \mathfrak{S}(E)} s'(f)$ and s is the unique point where this maximum is attained. We have already showed in the proof of Lemma 1 that every effect in E attains its maximum at some $s \in \hat{a}$ with $a \in S_1(E)$. The result follows. □

IV. NUMBER OF CONTEXTS IN A SPECTRAL EFFECT ALGEBRA

In this section we are going to prove the main result that answers the open question from [2] about the possible number of contexts in a spectral effect algebra.

Proposition 6. *Assume that every pair $a, a' \in S_1(E)$ is summable. Then there is only one context in E.*

Proof. Assume that there are two contexts $A \neq B$ in E. Then there is at least one effect a such that $a \in A$ but $a \notin B$. Let $B = \{b_1, \dots b_n\}$ and let $s \in \hat{a}$. For every b_i, $i \in \{1, \dots, n\}$, we must have $a + b_i \in E$. This implies

$$s(a + b_i) \leq 1$$

which gives $s(b_i) = 0$ for all $i \in \{1, \dots, n\}$ which is a contradiction with $\sum_{i=1}^{n} b_i = 1$. □

Proposition 7. *Every spectral effect algebra E contains either one context or uncountably many contexts.*

Proof. Assume that E contains at least two contexts. As a result of Prop. 6, there are $a, b \in S_1(E)$, $a \neq b$, that are not summable. Let $\lambda \in [0, 1]$ and denote

$$c_\lambda = \lambda a + (1 - \lambda) b.$$

We will show that every c_λ must belong to a different context, hence E we must contain uncountably infinite number of contexts.

Since E is spectral we have that for every $\lambda \in [0, 1]$ there is a context C_λ such that $c_\lambda \in \bar{C}_\lambda$. Assume that for some $\lambda \neq \mu$, $\mu \in [0, 1]$ we have $C_\lambda = C_\mu$. Then we have $a, b \in V(C_\lambda)$ as $c_\lambda, c_\mu \in V(C_\lambda)$ and we can express a and b as linear combinations of c_λ and c_μ.

Let $C_\lambda = \{c_1, \dots, c_n\}$, then we must have

$$a = \sum_{i=1}^{n} \alpha_i c_i$$

for some $\alpha_i \in \mathbb{R}$. Moreover, let $s_i \in \hat{c}_i$, then $\alpha_i = s_i(a) \in [0, 1]$ for all i implies that $a \in \bar{C}_\lambda$, which yields $a \in C_\lambda$ as a is sharp and one-dimensional. In a similar fashion we get $b \in C_\lambda$, which is a contradiction with the assumption that a and b are not summable. □

V. COMPOSITION OF SPECTRAL EFFECT ALGEBRAS

In the study of spectral effect algebras, it is a natural question whether spectrality is preserved by some constructions over convex effect algebras. In this section, we study the direct products and direct convex sums, note that these are the product and coproduct in the category of convex effect algebras.

Definition 9. Let E_1, E_2 be effect algebras, then their direct product is an effect algebra $E_1 \times E_2$ given as $E_1 \times E_2 = \{(f_1, f_2) : f_1 \in E_1, f_2 \in E_2\}$. The partial binary operation $+$ is given for $f_i, f_i' \in E_i$, such that $f_i + f_i' \in E_i$, $i \in \{1, 2\}$ as $(f_1, f_2) + (f_1', f_2') = (f_1 + f_1', f_2 + f_2')$ and the unit of $E_1 \times E_2$ is $(1, 1)$. If E_1, E_2 are convex effect algebras, then we can define a convex structure on $E_1 \times E_2$ by $\lambda(f_1, f_2) = (\lambda f_1, \lambda f_2)$, for $\lambda \in [0, 1]$. In this way, the direct product of convex effect algebras is a convex effect algebra.

Proposition 8. *The direct product of spectral effect algebras is a spectral effect algebra.*

Proof. It is easy to see that an element $(f_1, f_2) \in E_1 \times E_2$ is sharp if and only if both f_1 and f_2 are sharp. Moreover, since $(f_1, f_2) = (f_1, 0) + (0, f_2)$, such an element is one-dimensional if and only if one of the elements is one-dimensional and the other is 0. Let now $A = \{a_1, \dots, a_n\} \subset E_1$ and $B = \{b_1, \dots, b_m\} \subset E_2$ be contexts. It is straightforward to see that $(a_1, 0) + \ldots + (a_n, 0) + (0, b_1) + \ldots + (0, b_m) = (1, 1)$, hence

$$\{(a_1, 0), \dots, (a_n, 0), (0, b_1), \dots, (0, b_m)\} \subset E_1 \times E_2 \tag{3}$$

is a context. Moreover, any context in $E_1 \times E_2$ is of this form.

Finally let $(f_1, f_2) \in E_1 \times E_2$ be any element. Since E_1 and E_2 are spectral there are contexts $A \subset E_1$ and $B \subset E_2$ such that $f_1 \in \bar{A}$ and $f_2 \in \bar{B}$. It follows that we have

$$(f_1, f_2) = \sum_{i=1}^{n} \mu_i (a_i, 0) + \sum_{j=1}^{m} \nu_j (0, b_j)$$

which shows that $E_1 \times E_2$ is spectral effect algebra. □

The definition of the convex direct sum is a bit more involved.

Definition 10. Let E_1, E_2 be convex effect algebras, then they are affinely isomorphic to the intervals $[0, u_1] \subset C_1 \subset V_1$ and $[0, u_2] \subset C_2 \subset V_2$ respectively, where for $i \in \{1, 2\}$ we have that V_i are vectors spaces, C_i are pointed cones and $u_i \in C_i$. Take the vector space $V_1 \times V_2$ that corresponds to the coproduct of the respective vector spaces and define a relation of equivalence $\approx$ by $(u_1, 0) = (0, u_2)$, i.e. $(x_1, y_2) \approx (x_2, y_2)$ if and only if we have

$$(x_1, y_1) = (x_2, y_2) + \alpha(1, -1)$$

for some $\alpha \in \mathbb{R}$. It is clear that $\approx$ is reflexive, symmetric and transitive. Let V be the quotient space $V = (V_1 \times V_2)/_\approx$ and define the cone $C = (C_1 \times C_2)/_\approx \subset V$ as the set of equivalence classes containing an element of $C_1 \times C_2$. Clearly, C is a convex cone containing $u := [(u_1, 0)]_\approx = [(0, u_2)]_\approx$. To see that C is pointed, simply note that $[(x, y)]_\approx \in C \cap (-C)$ if and only if we can choose $x \in C_1$, $y \in C_2$ and for some $\alpha \in \mathbb{R}$ we have $x + \alpha u_1 \in (-C_1)$ and $y - \alpha u_2 \in (-C_2)$. This implies $0 \leq x \leq -\alpha u_1$ in (V_1, C_1) so that $\alpha \leq 0$. At the same time, $0 \leq y \leq \alpha u_2$ in (V_2, C_2), so that $\alpha \geq 0$. This implies $\alpha = 0$ and consequently $(x, y) = (0, 0)$.

Now we can define the direct convex sum of effect algebras E_1 and E_2 as

$$E_1 \oplus E_2 := [0, u] \subset C.$$

We can see that $E_1 \oplus E_2$ is a set of equivalence classes of the form $[(\lambda f_1, (1-\lambda) f_2)]_\approx$, where $f_1 \in E_1$, $f_2 \in E_2$ and $\lambda \in [0, 1]$. Note that $E_1 \oplus E_2$ is a convex effect algebra by construction as it is an interval in an ordered vector space.

In the less general circumstances of the general probabilistic theories the direct convex sum of the effect algebras can be introduced as the effect algebra corresponding to the direct product of state spaces [9, Definition 6].

Proposition 9. *The direct convex sum of spectral effect algebras is not a spectral effect algebra.*

Proof. In a sense we are going to mimic the proof of Prop. 8 with the only difference that now we will show that there are two types of contexts on $E_1 \oplus E_2$: they are either of the form $\{[(a_1, 0)]_\approx, \ldots, [(a_n, 0)]_\approx\}$ where $A = \{a_1, \ldots a_n\} \subset E_1$ is a context, or $\{[(0, b_1)]_\approx, \ldots, [(0, b_m)]_\approx\}$ where $B = \{b_1, \ldots b_m\} \subset E_2$ is a context.

Let $f = [(\lambda f_1, (1-\lambda) f_2)]_\approx \in S_1(E_1 \oplus E_2)$, then from

$$[(\lambda f_1, (1-\lambda) f_2)]_\approx = \lambda [(f_1, 0)]_\approx + (1-\lambda)[(0, f_2)]_\approx$$

we see that both $\lambda[(f_1, 0)]_\approx$ and $(1-\lambda)[(0, f_2)]_\approx$ must be multiples of f. This implies that there are some $t, \alpha \in [0, 1]$ such that

$$t\lambda f_1 = \alpha u_1, \qquad (1-t)(1-\lambda) f_2 = \alpha u_2.$$

Assuming that both f_1 and f_2 are nonzero, this implies that either both are multiples of identity or $\lambda \in \{0, 1\}$. In both cases, f is of the form $f = [g_1, 0]_\approx$ for some $g_1 \in E_1$ or $f = [0, g_2]_\approx$ for some $g_2 \in E_2$. It is clear that $[(g_1, 0)] \in S_1(E_1 \oplus E_2)$ if and only if $g_1 \in S_1(E_1)$ and similarly for elements of the form $[(0, g_2)]_\approx$. From the definition, we can see that in this case, $[(g_1, 0)]_\approx$ and $[(0, g_2)]_\approx$ are not summable in $E_1 \oplus E_2$, hence these cannot belong to the same context.

It follows that all contexts on $E_1 \oplus E_2$ are of the above two types. It is straightforward to see that if neither of f_1 or f_2 is a multiple of identity and $\lambda \in (0, 1)$, the element $[(\lambda f_1, (1-\lambda) f_2)]_\approx$ cannot be given as $\sum_{i=1}^n \mu_i [(a_i, 0)]_\approx$ or $\sum_{j=1}^m \nu_j [(0, b_j)]_\approx$. Hence $E_1 \oplus E_2$ is not spectral. $\square$

VI. SHARPLY DETERMINING STATE SPACES

Here we consider a special case of spectral effect algebras, for which stronger properties can be proved.

Definition 11. We say that the state space $\mathfrak{S}(E)$ is sharply determining if for any sharp $f \in E$ and any $g \in E$ such that $g \not\geq f$ there is a state $s \in \mathfrak{S}(E)$ such that $s(f) = 1 > s(g)$.

We first obtain a way stronger version of Prop. 4 that is similar to [3, Proposition 18].

Proposition 10. *Let E be a spectral effect algebra such that $\mathfrak{S}(E)$ is sharply determining. Let $A, B \subset E$ be contexts, $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots b_m\}$ such that for some $f \in E$ we have $f \in \bar{A} \cap \bar{B}$, specifically*

$$\sum_{i=1}^n \mu_i \left(\sum_{j_i=1}^{n_i} a_{i_j} \right) = f = \sum_{k=1}^m \nu_k \left(\sum_{l_k=1}^{m_k} b_{l_k} \right)$$

where $\mu_1 > \ldots > \mu_n > 0$ and $\nu_1 > \ldots > \nu_m > 0$. Then we have $n = m$, $\mu_i = \nu_i$ and $\sum_{j_i=1}^{n_i} a_{i_j} = \sum_{l_i=1}^{m_i} b_{l_i}$.

Proof. Denote $a_i' = \sum_{j_i=1}^{n_i} a_{i_j}$ and $b_j' = \sum_{l_k=1}^{m_k} b_{l_k}$. As a result of Prop. 4 we already know that $\mu_1 = \nu_1$. We will only show that $a_1' = b_1'$, the result will follow by repeating the same procedure for $f - \mu_1 a_1' = f - \nu_1 b_1'$.

As first note that a_i' and b_i' are sharp [6, Theorem 4.8]. Assume that $a_1' \ngeq b_1'$ then there is $s \in \mathfrak{S}(E)$ such that $s(b_1') = 1 > s(a_1')$. We have $s(f) = \sum_{k=1}^m \nu_k s(b_k') = \nu_1 = \mu_1$ as well as $s(f) = \sum_{i=1}^n \mu_i s(a_i') < \mu_1$ which is a contradiction. Hence we must have $a_1' \geq b_1'$ and by the same logic we must have also $b_1' \geq a_1'$ which together yields $a_1' = b_1'$. □

It was proved in [6, Theorem 4.8] that if $\mathfrak{S}(E)$ is sharply determining, then all sharp elements are extremal, moreover, the sum of sharp elements, if it exists, is sharp. If E is also spectral, then it is clear that the sharp elements are precisely the finite sums of one-dimensional sharp elements. The next results show that in this case E is sharply dominating, see [10] for a definition, and the set $S(E)$ of all sharp elements with ordering induced from E is an orthomodular lattice. We will need the following reformulation of the condition in Definition 11. For the proof, it is enough to realize that f is sharp if and only if $1 - f$ is sharp.

Lemma 3. *Let E be a convex effect algebra. Then $\mathfrak{S}(E)$ is sharply determining if and only if for any sharp $f \in E$ and any $g \in E$ such that $g \nleq f$ there is a state $s \in \mathfrak{S}(E)$ such that $s(f) = 0 < s(g)$.*

Proposition 11. *Let E be a spectral effect algebra with a sharply determining state space and let $a = \sum_i \mu_i a_i$ be a spectral decomposition. Then $a^0 := \sum_{i, \mu_i > 0} a_i$ is the smallest sharp element larger than a.*

Proof. It is clear that a^0 is a sharp element and $a \leq a^0$. Assume next that b is a sharp element such that $a \leq b$. Then for any $s \in \mathfrak{S}(E)$ such that $s(b) = 0$ we have $s(a) = \sum_i \mu_i s(a_i) = 0$, so that $s(a_i) = 0$ whenever $\mu_i > 0$. It follows that $s(a^0) = 0$. By Lemma 3, $a^0 \leq b$. □

Proposition 12. *Let E be a spectral effect algebra with a sharply determining state space. Let $f, g \in E$ be sharp. Then $(\lambda f + (1-\lambda) g)^0$ does not depend on $\lambda \in (1, 0)$ and we have $(\lambda f + (1-\lambda) g)^0 = f \vee g$ in $S(E)$. Moreover, we have $f \wedge g = 1 - ((1-f) \vee (1-g))$.*

Proof. Let us first observe that for any $f \in E$ and $s \in \mathfrak{S}(E)$, $s(f) = 0$ iff $s(f^0) = 0$, this is easy to see from the definition of f^0. Let f, g be sharp and for $\lambda \in (0,1)$ let $p_\lambda := (\lambda f + (1-\lambda) g)^0$. Then for $s \in \mathfrak{S}(E)$, $s(p_{1/2}) = 0$ iff $s(\frac{1}{2}(f+g)) = 0$ iff $s(f) = s(g) = 0$ iff $s(\lambda f + (1-\lambda) g) = 0$ iff $s(p_\lambda) = 0$. By Lemma 3 this implies that $p_\lambda = p_{1/2}$ for all $\lambda \in (0,1)$. If $h \in E$ is any sharp element such that $f, g \leq h$, then $s(h) = 0$ implies $s(f) = s(g) = 0$ so that $s(p_{1/2}) = 0$, hence $p_{1/2} \leq h$. By [6, Corollary 4.10], $h - p_{1/2} \in S(E)$, this shows that $p_\lambda = p_{1/2} = f \vee g$ in $S(E)$. The last assertion follows by de Morgan laws. □

Corollary 1. *Let E be a spectral effect algebra with a sharply determining state space. Then $S(E)$ is an orthomodular lattice.*

Proof. By [6, Corollary 4.9], $S(E)$ is a sub-effect algebra in E that is an orthoalgebra and by Proposition 12, $S(E)$ is a lattice. By [11, Prop. 1.5.8], any lattice ordered orthoalgebra is an orthomodular lattice. □

VII. CONCLUSIONS AND OPEN QUESTIONS

In this article we have proved that there can be either only one or uncountably many contexts contained in a spectral effect algebra as well as few other results concerning spectral effect algebras. There are still quite a few open questions that will be left for future research.

1. What are the extreme points of a state space of a spectral effect algebra? In Prop. 5 we have shown that all exposed points of the effect space of a spectral effect algebra have the form $s = \hat{a}$ for $a \in S_1(E)$. It is an open question whether one can show a similar result for the extreme points of the state space.

2. Let $A = \{a_1, \ldots, a_n\}$ be a context and let $\hat{A} = \mathrm{conv}(\cup_i \hat{a}_i)$. Since $\hat{a}_i$ are affinely independent, $\hat{A} = \oplus_c \hat{a}_i$ is their convex direct sum. Let I be a set that indexes all of the contexts A_α of an effect algebra E; does it hold that $\mathfrak{S}(E) = \cup_{\alpha \in I} \hat{A}_\alpha$? If $\hat{a}$ are E-exposed points for all $a \in S_1(E)$, this would be even a stronger result than the one proposed in the question above.

3. Is the cardinality of contexts always the same? It is straightforward to see that all contexts in the effect algebras used in quantum theory have the same number of elements. It would be very interesting to know whether this is true in general or whether there is a counter-example. As we have seen in Prop. 9 there are effect algebras that have contexts with different numbers of elements, but the given example is not a spectral effect algebra.

4. Is it possible to extend the results of Prop. 4 to a result similar to Prop. 10? The property in question is to show that $\sum_{i=1}^{n} \mu_i a_i = f = \sum_{j=1}^{m} \nu_j b_j$ where $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_m\}$ are contexts would imply that for every μ_i there is ν_j such that $\mu_i = \nu_j$. A stronger version of said result would be to show that Prop. 10 holds for all spectral effect algebras; this can be either done by showing that the state space of every spectral algebra is sharply determining or by other means.

5. For a given context $A = \{a_1, \dots, a_n\}$ is the set of states $\{\hat{a}_1, \dots, \hat{a}_n\}$ unique? The existence of the states $\hat{a}_i$ is simply implied by the fact that the effects a_i are sharp one-dimensional but it is rather easy to find convex effect algebras where the set $\{\hat{a}_1, \dots, \hat{a}_n\}$ is not unique.

6. Does spectrality of the effect algebra implies any kind of weak duality between the effect algebra and the state space? It is tempting to define a map $T : a_i \mapsto \hat{a}_i$ but it is of question whether the map would be well defined and whether the map would be affine. This would be a very strong result to prove or to at least find some conditions for when it holds. Still all of the examples of spectral effect algebras that we know of have said property.

7. Are there examples of spectral effect algebras that are not operator algebras nor classical? It is straightforward to see that the effect algebras for finite-dimensional real or complex quantum theory are spectral. It would be very interesting and possibly helpful to have other examples of spectral effect algebras.

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