Contents

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1 Excellence

1.1 PROJECT OBJECTIVES

1.1.1 Overview

Our goal is to make profound contributions in a number of directions related to the mathematical/statistical foundations of quantum information science, including problems related to state and channel discrimination, state convertibility, the study of quantum correlations in statistical physics and communication, and the mathematical study of various information measures and their applications. The problems we set out to study fall in three main groups that either open up completely new research directions, or address some major open questions in quantum information theory.

1. Explore the phenomenon of correlated state discrimination with super-exponentially vanishing error probabilities, discovered by the PI of this proposal and coauthors; determine the fastest possible speed of decrease, establish trade-off relations on a super-exponential scale and identify the relevant information measures, find quantitative relations to correlation/entanglement asymptotics, and link the effect to criticality of states.

2. Initiate the systematic mathematical study of multi-variate quantum Renyi divergences, and explore their applications in quantum information theory, in particular, in giving necessary and sufficient conditions for state convertibility problems, and establishing trade-off relations in tasks with more than two competing operational quantities, like state discrimination with more than two hypotheses, or quantum state redistribution.

3. Find explicit trade-off relations in problems where quantum effects prohibit single-letter expressions, like composite state or channel discrimination, and complete coding theorems in these problems by establishing their strong converse property.

The proposed project is truly interdisciplinary, at the interface of matrix analysis, functional analysis, information theory, and mathematical physics, addressing problems and using tools from, and promising results relevant for, each of these fields.

1.1.2 Background: Rényi divergences and trade-off relations

Most problems in (quantum) information theory can be characterized by a number of competing quantities (like the coding rate and the decoding error in noisy channel coding), and one's aim is to find their joint optima, constrained by their respective trade-off relations. In this sense, a complete solution of a problem is attained when the trade-off curves (or higher dimensional surfaces) bounding the region of the achievable values of the relevant quantities of the problem, are given in an explicit form. In the asymptotic setting, such explicit forms can invariably be expressed in terms of some information measures related to dissimilarity measures (divergences) of quantum states, typically Rényi divergences. Such results, in turn, provide operational justification for the use of the corresponding information measures.

Many of these problems are intimately connected to some form of state discrimination problem, where a sender encodes one of finitely many possible messages into the state of a quantum system, and a receiver has to read it out using some prior knowledge about the possible code states. Already this basic problem can be studied in a great variety of settings: considering only two possible messages (binary case) or more; encoding the messages into a single copy of the system (single-shot), or multiple copies (asymptotic); assuming the copies to be identical and independent (i.i.d.) for every message, or allowing correlations; allowing arbitrary collective measurements on multiple copies, or imposing some locality constraints; assuming the identity of the code states to be perfectly known by the receiver (simple case), or only certain sets into which they fall (composite case); using only finite-dimensional systems, or allowing systems described by infinite-dimensional Hilbert spaces, operator algebraic models, or generalized probabilistic theories (GPTs). In every setting, the aim of the receiver is to find measurements that minimize the erroneous identification of the messages according to some constraints. In the binary asymptotic i.i.d. setting, the optimal asymptotics achievable for the two possible error probabilities $\gamma_{\mathrm{I}}^{(n)}$ and $\gamma_{\mathrm{II}}^{(n)}$ (mistakenly dismissing the first or the second possible message, respectively), is an exponential decay in the number of copies of the system n, of the form $\gamma_{\mathrm{I}}^{(n)} \sim e^{-nr_1}$, $\gamma_{\mathrm{II}}^{(n)} \sim e^{-nr_2}$, with a trade-off between the two exponents, given by the *direct exponent* $\mathrm{d}_r(\rho\|\sigma) := \sup\{r_1 : r_2 \geq r\} = \sup_{\alpha\in(0,1)} \frac{\alpha-1}{\alpha}[r - D_\alpha(\rho\|\sigma)] =: H_r(\rho\|\sigma)$ [6, 31, 42, 84]), where $D_\alpha(\rho\|\sigma)$ is the *Petz-type Rényi α-divergence* [87, 88] of the states (density operators) ρ and σ encoding the first and the second message, respectively ($D_\alpha(\rho\|\sigma) = (\alpha - 1)^{-1}\log \operatorname{Tr}\rho^\alpha\sigma^{1-\alpha}$ in the finite-dimensional case). This gives that the optimal type II error exponent under the constraint of asymptotically vanishing type I error (the *direct Stein exponent* $\mathrm{s}(\rho\|\sigma)$) is lower bounded by the *Umegaki relative entropy* [99] $D(\rho\|\sigma) := \operatorname{Tr}\rho(\log\rho - \log\sigma) = \lim_{\alpha\to 1} D_\alpha(\rho\|\sigma)$. On the other hand, if the measurements are chosen such that the type II errors decay with an exponent $r > D(\rho\|\sigma)$ then the type I errors go to 1 as $1 - e^{-n\cdot\mathrm{sc}_r(\rho\|\sigma)}$ (strong converse of Stein's lemma [83, 86]), where the exact strong converse exponent has been determined by the PI and coauthors [37, 71, 75] as $\mathrm{sc}_r(\rho\|\sigma) = \sup_{\alpha>1}\frac{\alpha-1}{\alpha}[r - D^*_\alpha(\rho\|\sigma)]$, where $D^*_\alpha(\rho\|\sigma)$ is the *sandwiched Rényi α-divergence* of the states ρ and σ [11, 43, 44, 82, 100] ($D^*_\alpha(\rho\|\sigma) = (\alpha-1)^{-1}\log\operatorname{Tr}(\rho^{1/2}\sigma^{\frac{1-\alpha}{\alpha}}\rho^{1/2})^\alpha$ in the finite-dimensional case). These results on the one hand give single-letter expressions for the asymptotically defined error exponents, which are therefore computable (at least numerically), while on the other hand they identify the Petz-type and the sandwiched Rényi divergences as operationally relevant measures of dissimilarity of states; in other words, they give *operational interpretations* to these quantities.

Interestingly, quantum Rényi divergences and related information measures not only appear in the solution part of information theoretic problems, as quantifiers of trade-off relations, but they also turn out to be useful technical tools in proving many of the coding theorems. Nagaoka [83] used their monotonicity under measurements to give elegant proofs for the strong converse property of the quantum Stein's lemma and classical-quantum channel coding, and his proof method was successfully applied later to establish the strong converse property in a large number of problems; e.g., [52, 57, 93, 94, 97, 100]. The PI has found simple and robust proofs for the quantum Stein's lemma with composite null hypothesis and for compound classical-quantum channel coding, using a subadditivity property of the sandwiched Rényi divergences [70], and the PI with Hiai and Ogawa [40, 76] showed that the asymptotic properties of Rényi divergences are intimately related to the large deviation laws governing the decay of the error probabilities in binary state discrimination between correlated quantum states. In the same time, questions related to the properties of quantum Rényi divergences have been a driving force behind intensive research in pure mathematics in the subject of convexity and monotonicity properties of trace functions; see, e.g., [8, 11, 18, 20, 19, 27, 34, 43, 44, 53, 87, 88, 102] for a selection of results using various techniques from matrix analysis, complex function theory, complex interpolation theory, non-commutative L^p-spaces, and more.

In the proposed project, we plan to go beyond the state of the art in various directions, and explore qualitatively different phenomena. Below we give an overview of the three main directions of the planned research.

1.1.3 Objective 1: Super-exponential state discrimination

As explained above, the optimal asymptotics of the error probabilities in simple binary i.i.d. state discrimination is an exponential decay of the two error probabilities, and it is also easy to see that a faster (super-exponential) decrease of both error probabilities is only possible in the trivial case when the states ρ and σ are supported on orthogonal subspaces, and hence they can be perfectly distinguished using only a single copy of the system. For this reason, it is natural to study the trade-off between the type I and the type II error probabilities on the level of exponents, and more generally, studying trade-off relations on the exponential scale is a general paradigm all over (quantum) information theory, not least due to the central role of sate discrimination in many other information theoretic tasks. In fact, the very principle of measuring information in bits (i.e., the base 2 logarithm of the number of messages, or the size of the system carrying the information) reflects the fundamental role of the exponential scale in information theory. However, in a recent paper [17], the PI and coauthors have shown that there exist correlated state discrimination problems where the exponential paradigm does not apply anymore. Namely, a general construction of correlated translation-invariant state pairs on an infinite spin chain was given in [17] such that the error probabilities of discriminating their n-site restrictions decay super-exponentially in n, at least with the speed $\gamma_{\mathrm{I}}^{(n)} \le e^{-cn\log n}$, $\gamma_{\mathrm{II}}^{(n)} \le e^{-cn\log n}$, with some positive constant c, while all their local density operators are invertible. This striking phenomenon leads to a host of exciting open questions that we plan to investigate. For instance, it is not known what the exact asymptotics is in our examples, whether different/faster decrease of the error probabilities is possible with other constructions, and if this depends on the dimension of the single-site Hilbert spaces (in our example it is the smallest possible, 2), and maybe most importantly, what types of correlations are needed for a super-exponential asymptotics. We will aim at answering these questions in the project. In particular, we intend to find quantitative relations between the speed of the error decay and measures of entanglement (our constructions lead to highly entangled states). We will also aim at establishing tight trade-off relations between the error probabilities on the super-exponential scale; we expect that these may be obtained in terms of Rényi divergences regularized on a suitably chosen superlinear scale. These problems are also very strongly connected to quantum statistical physics: our construction in [17] uses gauge-invariant and translation-invariant quasifree states of fermionic lattice systems, and includes as a special case the family of thermodynamical limit ground states of the XX-model on a 1-dimensional spin chain corresponding to different transversal magnetic fields. We expect to prove similar results for other relevant physical models, including bosonic lattice states and ground states of critical spin models, and to explore the relation of super-exponential distinguishability and criticality, with a detailed analysis of the change of the error asymptotics in the vicinity of a phase transition point.

1.1.4 Objective 2: Multi-variate Rényi divergences and their applications

All the problems in quantum information theory where exact trade-off relations have been obtained so far can be characterized by two competing quantities, and the relevant information measures are derived from 2-variable Rényi divergences. There are, however, problems where more than two operational quantities have to be jointly optimized, e.g., state discrimination with more than 2 hypotheses, or state redistribution [22, 101], a common generalization of many particular information processing tasks, where the the encoder and the decoder both have access to quantum side information and shared entanglement to compress many copies of a quantum state, and the competing quantities are the "goodness" of the recovery, the size of the system carrying the compressed state, and the entanglement cost. It is reasonable to expect that in order to obtain exact trade-off relations in such problems, one would need a theory of multi-variate extensions of quantum Rényi divergences, which, however, does not seem to have been developed yet; in fact, even the very definition of such quantities seems highly non-trivial. To be more precise, we are looking for quantum extensions of the classical multi-variate Rényi divergences (also called Hellinger transforms), defined for a finite set of commuting density operators[1] $(W_x)_{x\in\mathcal{X}}$ and a signed probability distribution P on $\mathcal{X}$ as

$$Q_P(W) := \operatorname{Tr}\prod_{x\in\mathcal{X}} W_x^{P(x)}, \qquad D_P(W) := \frac{-\log Q_P(W)}{\prod_{x\in\mathcal{X}}(1-P(x))}, \tag{1.1}$$

where we assume that $P(x) \neq 1$ for any x. It is easy to verify [25, 81] that D_P is monotone non-increasing under the joint application of a stochastic map on every W (i.e., it satisfies the *data processing inequality*) if and only if either all $P(x) \ge 0$, (we will denote the set of such P by $\mathcal{P}(\mathcal{X})$) or there exists exactly one x such that $P(x) > 0$, and $P(y) \le 0$ for $y \in \mathcal{X}\setminus\{x\}$ (we will denote the set of such P by $\mathcal{P}^-(\mathcal{X})$). This monotonicity property is crucial for any divergence to possibly have an operational interpretation. It has been shown very recently in [66] that these quantities find operational significance in the determination of the optimal exponential decay of the sum of the error probabilities in the problem of state exclusion, where one's aim is to determine which one out of finitely many candidates is *not* the true state of the system. Note that in the binary case where $\mathcal{X} = \{0,1\}$, state exclusion is equivalent to state discrimination, and accordingly, $D_P(W) = D_\alpha(\rho\|\sigma)$ with $\alpha := P(0)$, $\rho := W_0$, $\sigma := W_1$, i.e., D_P is indeed the correct multi-variate extension of the 2-variable Rényi divergences, both formally and in their operational role.

Approaching from a different direction, one may consider the central problem of convertibility of a set of states $(W_x)_{x\in\mathcal{X}}$ into another set of states $(\widehat{W}_x)_{x\in\mathcal{X}}$ with a joint quantum operation. This problem can be studied in a large variety of settings; single-shot or asymptotic, exact or approximate, with or without catalysts, allowing arbitrary quantum operations or only those respecting some symmetry or being free operations of some resource theory, any combination of these, and more. Necessary conditions for convertibility can be obtained using multi-variate functions on quantum states with suitable mathematical properties; for instance, for exact single-shot convertibility, $F((W_x)_{x\in\mathcal{X}'}) \ge F((\widehat{W}_x)_{x\in\mathcal{X}'})$ has to hold for any function F whose variables are indexed by

[1] Here we follow a notational convention often used for classical-quantum channels, and denote states by W instead of ρ, σ, etc. Also, we assume that all W_x are invertible, to avoid going into technically important but conceptually irrelevant details regarding supports.

a subset $\mathcal{X}' \subseteq \mathcal{X}$ and which is monotone non-increasing under the joint application of an allowed quantum operation on its arguments; the same has to hold also for multi-copy or catalytic single-shot convertibility, if F is additionally additive on tensor products, and for asymptotic catalytic convertibility, if, moreover, F is lower semi-continuous in its variables. Many of the 2-variable quantum Rényi divergences provide such functions on pairs of states (i.e., $|\mathcal{X}'| = 2$), including the maximal Rényi α-divergences [65] with $\alpha \in [0, 2]$, and the Rényi (α, z)-divergences [7] for certain values of α and z [102]. In the converse direction, sufficient conditions in terms of Rényi divergences have been given for the convertibility of pairs of commuting states in [15, 49, 51, 81, 98], and these have been extended very recently in [25] to a complete characterization of asymptotic as well as approximate catalytic convertibility between finite sets of commuting states in terms of the monotonicity of the multi-variate Rényi divergence D_P defined above.

Our first goal is to define quantum extensions of the multi-variate Rényi divergences, i.e., quantities Q_P^q and D_P^q that reduce to the ones in (1.1) when the states W_x are commuting. Of course, these are only interesting when they satisfy a number of properties, most importantly *monotonicity*, or the *data-processing inequality*, i.e., for any states W_x on a finite-dimensional Hilbert space $\mathcal{H}$ and any completely positive trace-preserving map (quantum channel) Φ on $\mathcal{B}(\mathcal{H})$,

$$D_P^q((\Phi(W_x))_{x\in\mathcal{X}}) \le D_P^q((W_x)_{x\in\mathcal{X}}). \tag{1.2}$$

Other important properties include *additivity*, i.e., that for any $(W_x)_{x\in\mathcal{X}}$ and $(W_x')_{x\in\mathcal{X}}$,

$$D_P^q((W_x \otimes W_x')_{x\in\mathcal{X}}) = D_P^q((W_x)_{x\in\mathcal{X}}) + D_P^q((W_x')_{x\in\mathcal{X}}),$$

and lower semi-continuity in the variables. In the 2-variable case, all the operationally relevant Rényi divergences are also *reversibility witnesses*, i.e., equality in the data processing inequality holds if and only if there exists a quantum channel Ψ such that $\Psi(\Phi(W_x)) = W_x$, $x \in \mathcal{X}$. We outline a number of possible approaches in the Methodology part, Section 1.2.2, that we will consider to define monotone and additive multi-variate quantum Rényi divergences. Our second goal will be to apply these Rényi divergences in quantum information theory; in particular, in quantifying the trade-off between the different error exponents in quantum state exclusion, and its natural generalization, quantum channel exclusion. Explicit expressions for such trade-off relations will identify the operationally relevant quantum extensions of the classical Rényi divergences, while other extensions might also be useful in giving non-trivial bounds on the error exponents where explicit expressions are not yet available. Our third goal is identifying multi-variate quantum Rényi divergences that are reversibility witnesses; this can be seen as a special case of the multi-state conversion problem described above, where $W_x = \Phi(\rho_x)$ and $\widehat{W}_x = \rho_x$ for some quantum states ρ_x, $x \in \mathcal{X}$, and a quantum channel Φ.

1.1.5 Objective 3: Trade-off relations for composite hypothesis testing

Another departure from the fully solved case of simple binary i.i.d. state discrimination is the problem of composite state discrimination, where for n copies the receiver's knowledge of the code states is described by sets of the form $\{\rho_i^{\otimes n}\}_{i\in\mathcal{I}}$ and $\{\sigma_j^{\otimes n}\}_{j\in\mathcal{J}}$. It is easy to see that $\mathrm{d}_r(\{\rho_i\}_{i\in\mathcal{I}}\|\{\sigma_j\}_{j\in\mathcal{J}}) \le \inf_{i\in\mathcal{I},j\in\mathcal{J}} \mathrm{d}_r(\rho_i\|\sigma_j) = \inf_{i\in\mathcal{I},j\in\mathcal{J}} H_r(\rho_i\|\sigma_j)$, i.e., the direct exponent of distinguishing two sets of states cannot be better than the worst-case pairwise direct exponents of discriminating individual members of the two sets. (The error exponents are defined analogously to Section 1.1.2, with worst-case error probabilities over the respective sets.) The PI and coauthors have shown in [80] (based on [16]) that equality holds in the finite-dimensional classical case if both sets are finite, or both sets are convex and compact (in fact, even in the more general correlated settings of arbitrarily varying or adversarial hypothesis testing). On the other hand, it was also shown in [80] that, in sharp contrast with the classical case, strict inequality holds for quantum states in general, already in the simplest case when $\mathcal{I}$ has only one element, and $\mathcal{J}$ has only two. Very little is known about exact expressions for the error exponents in the general case; in fact, the only general result is the expression for the Stein exponent [10]

$$\mathrm{s}((\{\rho_i\}_{i\in\mathcal{I}}\|\{\sigma_j\}_{j\in\mathcal{J}}) = \lim_{n\to+\infty} \frac{1}{n} \inf\left\{ D\left(\rho_i^{\otimes n}\|\sigma_n\right) : i \in \mathcal{I},\, \sigma_n \in \overline{\mathrm{co}}\left(\{\sigma_j^{\otimes n}\}_{j\in\mathcal{J}}\right)\right\}, \tag{1.3}$$

where $\overline{\mathrm{co}}$ stands for the closed convex hull. This retains the desirable conceptual feature that it equates the operational quantity of the problem to an entropic quantity; however, the latter is a complicated regularized expression that cannot be explicitly evaluated. It is therefore highly desirable to find efficiently computable single-letter bounds for the various error exponents, preferably ones that are also sharp, i.e., are attainable by certain special configurations of states, and this is what we set out to obtain in this project point. We note that the expression in (1.3) for the composite Stein exponent was proved in [10] with a *weak converse*, i.e., if the type II exponent is above (1.3) then the type I errors cannot go to 0; we will also aim at strengthening this to a *strong converse*, i.e., showing that if the type II exponent is above (1.3) then the type I errors go to 1 with an exponential speed.

1.2 METHODOLOGY

In general, we follow the usual methodology of mathematical research: we formulate conjectures, test them on special cases and also numerically when it is feasible, after which we set out to prove or disprove them analytically. Below we outline in more detail some possible approaches and some technical tools that we expect to be useful/necessary to attain the project objectives.

1.2.1 Methodology for Objective 1

We will attack the questions outlined in Section 1.1.3 by studying error exponents in concrete physical spin chain and lattice models where correlations can be efficiently described. This should not only provide an abundance of examples on which the new phenomenon of super-exponential state discrimination can be understood, but it is also expected to lead to results with direct physical significance. In particular, we plan to study state discrimination of quasi-free fermionic and bosonic lattice systems, and of finitely correlated states.

A translation-invariant and gauge-invariant quasi-free state on a fermionic chain is given by a measurable function $q : [0, 2\pi) \to [0, 1]$, called its symbol, and any such state is mapped into a translation-invariant state on the infinite spin chain $\otimes_{k\in\mathbb{Z}}\mathcal{B}(\mathbb{C}^2)$ by a variant of the Jordan-Wigner isomorphism [3, 17]. This mapping preserves locality in the sense that the density operator of the restriction of the original state onto an interval $[k, l]$ of sites is mapped into the density operator of the restriction of the image state onto the same interval. Hence, the asymptotic discrimination problem of two such states ω_q and ω_r with symbols q and r is equivalent to the asymptotic discrimination problem of their images. It was shown by the PI and coauthors in [74] that if $\eta \le q, r \le 1 - \eta$ for some constant $\eta \in (0, 1/2)$ then the regularized Petz-type Rényi divergences of the two states can be explicitly expressed as

$$\overline{D}_\alpha(\omega_q\|\omega_r) := \lim_n \frac{1}{n} D_\alpha(\omega_q|_{[1,n]}\|\omega_r|_{[1,n]}) = \frac{1}{2\pi}\int_{[0,2\pi)} D_\alpha((q(x), 1 - q(x))\|(r(x), 1 - r(x)))\, dx, \quad (1.4)$$

with the classical Rényi divergences of the given probability distributions appearing in the right-hand side. Moreover, it has been shown by the PI and Ogawa in [76] that regularizing the sandwiched Rényi divergences yields the same expression. These were proved using a multivariate generalization of Szegő's theorem [30], also established in [74]. With this result in hand, it is easy to show that all the error exponents discussed in Section 1.1.2 can be expressed in the same form as in the i.i.d. case, with the regularized Rényi divergences in place of the single-copy Rényi divergences.

On the other hand, we have shown in a recent paper [17] that if q is constant 1 and r is constant 0 on some non-degenerate interval $[\mu, \nu]$ then there exists a sequence of tests along which both types of error probabilities decay at least with the speed $e^{-cn\log n}$ in the system size n, where c is a positive constant. Particular examples of such states include the thermodynamical limit ground states of the XX-model corresponding to different transversal magnetic fields [68], showing the physical relevance of this problem. It is an open question at the moment what the exact asymptotics of the error probabilities is in this case, and our first goal in this project point will be to settle this question. Clearly, the right-hand side of (1.4) is $+\infty$ in this case, and we expect that the Rényi divergences have to be regularized on a different (superlinear) scale to get a meaningful quantity. Moreover, we expect that this scale also determines the scale on which the asymptotics of the error probabilities should be studied, and on which meaningful trade-off relations between them can be established, which we furthermore expect to be quantifiable by the correctly regularized Rényi divergences. Solving these problems will require a new type of Szegő theorem (for symbols with singularities), which might be of independent mathematical interest. In the opposite direction, we will try to extend Szegő's theorem to pairs of functions for which the RHS of (1.4) is finite, and show that the equality in (1.4) still holds in this case, whence the error probabilities can only decrease exponentially fast.

The next question we aim to study is the kind of error asymptotics that might be achievable in the state discrimination problem of translation-invariant correlated states on general spin chains, of possibly higher local dimensions. We will start exploring this landscape using finitely correlated states [23, 24]. On the one hand, these are practical as they can be recursively generated from a single site by the help of an auxiliary algebra and a quantum channel, and therefore they can be specified by a number of parameters that does not grow with the system size. On the other hand, this class of states is highly relevant for physics, as they are widely used for approximations of ground states of Hamiltonian models. In particular, we intend to explore whether there is an upper bound on the achievable speed of decrease of the error probabilities, and whether this depends on the local dimension of the spin chain.

We will also study super-exponential distinguishability in other important physical models, e.g., non-gauge-invariant fermionic chains and translation-invariant Guassian states of bosonic chains, which are relevant for quantum optical information theory. While the local algebras at each site are infinite-dimensional in the latter case, the correlations are described by mathematical objects of the same kind as in the fermionic case, and it was shown by the PI in [69] that for suitably bounded symbols, an analogous expression to (1.4) holds, and the error probabilities decrease exponentially, with the exponent expressed in terms of the regularized Rényi divergences. Thus, we expect the existence of non-trivial super-exponential asymptotics also in this case, for suitably chosen pairs of symbols.

1.2.2 Methodology for Objective 2

The classical multi-variate Rényi divergences can be characterized in a number of different ways, and quantum generalizations can be obtained using any of these characterizations. Below we explain six different approaches that we plan to explore in our search for quantum multi-variate Rényi divergences with good mathematical properties.

First, if G_P^q is a non-commutative P-weighted operator mean, i.e., an operator-valued function G_P^q on collections of PSD operators that reduces to the usual P-weighted geometric mean for commuting operators, then $Q_P^q(W) := \operatorname{Tr} G_P^q(W)$ gives a quantum extension of Q_P. Geometric means with good mathematical properties (e.g., multiplicativity under tensor product and concavity in the arguments) are provided by iterated Kubo-Ando geometric means [54] or the Karcher mean (see, e.g., [64] for the latter).

Second, for any collection of quantum relative entropies $D^{\mathbf{q}} := (D^{q_x})_{x\in\mathcal{X}}$ (i.e., quantum extensions of the Kullback-Leibler divergence), and any probability distribution $P \in \mathcal{P}(\mathcal{X}) \setminus \{\mathbf{1}_{\{x\}} : x \in \mathcal{X}\}$, one may define

$$Q_P^{\mathrm{b},\mathbf{q}}(W) := \sup_{\tau}\Big\{ \operatorname{Tr}\tau - \sum_x P(x) D^{q_x}(\tau\|W_x)\Big\}, \tag{1.5}$$

with optimization over all positive semi-definite (PSD) operators τ on the given Hilbert space. These quantities were introduced by the PI and coauthors in [72], where it was shown that the corresponding quantum Rényi divergence can be expressed as

$$D_\alpha^{\mathrm{b},\mathbf{q}}(W) = \frac{1}{\prod_{x\in\mathcal{X}}(1-P(x))}\inf_{\omega}\sum_x P(x) D^{q_x}(\omega\|W_x),$$

where the infimum is over all states ω on the given Hilbert space; thus, it is called the *barycentric Rényi divergence* of W induced by $D^{\mathbf{q}}$. As it was shown in [72], the optimal τ in (1.5) gives a non-commutative P-weighted geometric mean $G_P^{\mathrm{b},\mathbf{q}}(W)$ of the W_x, for which $Q_P^{\mathrm{b},\mathbf{q}}(W) = \operatorname{Tr} G_P^{\mathrm{b},\mathbf{q}}(W)$ holds. It was also shown in [72] that $D_P^{\mathrm{b},\mathbf{q}}$ inherits lower semi-continuity and monotonicity from the parent quantities $D^{\mathbf{q}}$, while it is an open question whether additivity is inherited, too. (Note that subadditivity, on the other hand, follows trivially.) Of course, additivity may be easily established when an explicit expression for $D_P^{\mathrm{b},\mathbf{q}}$ is available; for instance, when all the generating relative entropies coincide with the Umegaki relative entropy D^{Um} [99], we get $Q_P^{\mathrm{b},\mathrm{Um}}(W) = \operatorname{Tr} G_P^{D^{\mathrm{Um}}}(W)$ with $G_P^{D^{\mathrm{Um}}}(W) := \exp(\sum_x P(x)\log W_x)$, whence $D_P^{\mathrm{b},\mathrm{Um}}$ is clearly additive. This leads to another open question: to find explicit expressions for $D_P^{\mathrm{b},\mathbf{q}}$ at least for the most important quantum relative entropies; e.g., when each D^{q_x} is equal to the maximal (or Belavkin-Staszewski) relative entropy [9, 65]. Of course, it might turn out that additivity does not hold in general; in this case the natural continuation would be the study of the regularized quantity

$$\overline{D}_P^{\mathrm{b},\mathbf{q}}(W) := \inf_{n\in\mathbb{N}}\frac{1}{n} D_P^{\mathrm{b},\mathbf{q}}(W^{\otimes n}) = \lim_{n\to+\infty}\frac{1}{n} D_P^{\mathrm{b},\mathbf{q}}(W^{\otimes n}),$$

where the equality follows from subadditivity. While this quantity is weakly additive by definition, establishing its additivity is still a non-trivial problem. We remark that there are a number of notable 2-variable quantum Rényi divergences that are not additive, but have very important regularizations; for instance, $D_\alpha^{\#}$ from [26], or the Rényi divergences considered very recently in [28, 41].

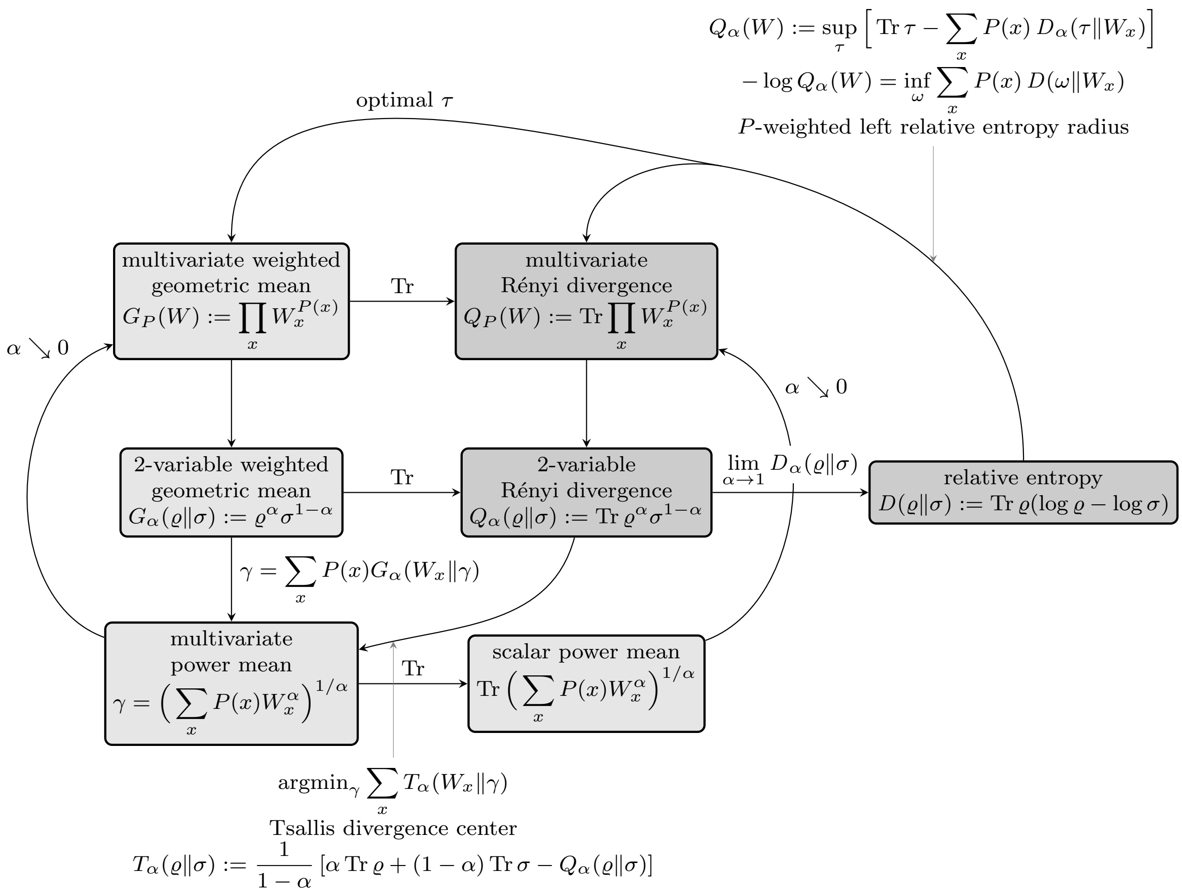
Third, for any collection $D_\alpha^{q_x}$, $x \in \mathcal{X}$, of (2-variable) quantum Rényi α-divergences, with corresponding $Q_\alpha^{q_x} = e^{(\alpha-1)D_\alpha^{q_x}}$, any optimizer $\tau_P^{\mathbf{q},\alpha}$ of $\inf_\tau \sum_x P(x)(1-\alpha)^{-1}[\alpha\operatorname{Tr} W_x + (1-\alpha)\operatorname{Tr}\tau - Q_\alpha^{q_x}(W_x\|\tau)]$

(called the *Tsallis divergence center*) gives a non-commutative generalization of the P-weighted α-power mean $(\sum_x P(x)W_x^\alpha)^{1/\alpha}$. The limit $Q_P^{\mathrm{T},\mathbf{q}}(W) := \lim_{\alpha\searrow 0} \operatorname{Tr} \tau_\alpha^{\mathbf{q},\alpha}$, if exists, defines a multi-variate quantum Rényi divergence. If, moreover, the limit exists without taking the trace, then the resulting operator gives a P-weighted geometric mean $G_P^{\mathrm{T},\mathbf{q}}(W)$ of the W_x (and obviously $Q_P^{\mathrm{T},\mathbf{q}}(W) = \operatorname{Tr} G_P^{\mathrm{T},\mathbf{q}}(W)$.)

Fourth, for any family $(G_\alpha^{q_x})_{x\in\mathcal{X}}$ of non-commutative extensions of the 2-variable α-weighted geometric mean G_α, any solution $\gamma_P^{\mathbf{q},\beta}$ of the fixed point equation $\gamma = \sum_x P(x) G_\alpha^{q_x}(W_x\|\gamma)$ gives again a non-commutative generalization of the P-weighted α-power mean, and we can repeat the same procedure as above. As it was pointed out by the PI and Ogawa in [78], for $G_\alpha^{q_x}(W_x\|\gamma) := G_{\alpha,z(\alpha)}(W_x\|\gamma) := (\gamma^{\frac{1-\alpha}{2z(\alpha)}} W_x^{\alpha/z(\alpha)} \gamma^{\frac{1-\alpha}{2z(\alpha)}})^{z(\alpha)}$ and $Q_\alpha^{q_x} := Q_{\alpha,z\alpha)} := \operatorname{Tr} G_{\alpha,z(\alpha)}$, $x\in\mathcal{X}$, we get that $\tau_P^{\mathbf{q},\alpha} = \gamma_P^{\mathbf{q},\alpha}$ for all $\alpha\in(0,1)$ and $z(\alpha) > 1-\alpha$, whence the third and the fourth approaches coincide. In particular, in the case of the Petz-type quantities corresponding to $z(\alpha)=1$, one can show [78] that $\tau_P^{\mathbf{q},\alpha} = \gamma_P^{\mathbf{q},\alpha} = (\sum_x P(x) W_x^\alpha)^{1/\alpha}$, and it is easy to see that its limit at $\alpha\to 1$ is $G_P^{D^{\mathrm{Um}}}(W) = \exp(\sum_x P(x)\log W_x)$. In contrast, if G_α is the α-weighted Kubo-Ando geometric mean [54] then $\tau_P^{\mathbf{q},\alpha} \neq \gamma_P^{\mathbf{q},\alpha}$ may happen, according to [90]. In this latter case, the $\gamma_\alpha^{\mathbf{q},\alpha}$ converge to the Karcher mean in the $\alpha\to 0$ limit, as was shown in [64].

Figure 1 below gives a graphical illustration of the above ideas, showing some relations between 2-variable and multi-variate weighted geometric means, Rényi divergences, weighted power means, divergence radii and centers, and fixpoint equations, an arrow indicating that one quantity can be obtained from the other. These work when all operators are commuting; non-commutative extensions of any of these quantities may by obtained by extending any quantity in the diagram to non-commutative variables, and trying to follow the arrows to obtain non-commutative extensions of other quantities. The examples in the above paragraph show that the diagram is not commutative in the quantum case.

Figure 1: Different approaches to multi-variate Rényi divergences



The fifth approach is somewhat different from the above, and is based on variational representations of the Rényi (α,z)-divergences [7] given by the PI in [71] based on [102]. (Note that the Rényi (α,z)-divergences give common generalizations of the operationally relevant families of the Petz-type and the sandwiched Rényi divergences, which we recover when $z=\alpha$ and $z=1$, respectvely.) Assume for simplicity that $\mathcal{X}=\{0,\dots,r\}$. When $P(0)>1$ and $P(k)\le 1$, $k=1,\dots,r$, we define

$$Q_{P,z}(W):=\sup_{H\in\mathcal{B}(\mathcal{H})_{\ge 0}}\left\{P(0)\operatorname{Tr}\left(H^{\frac{1}{2}}W_0^{\frac{P(0)}{z}}H^{\frac{1}{2}}\right)^{\frac{z}{P(0)}}+\sum_{k=1}^{r}P(k)\operatorname{Tr}\left(H^{\frac{1}{2}}W_0^{-\frac{P(k)}{z}}H^{\frac{1}{2}}\right)^{-\frac{z}{P(k)}}\right\}.$$

This coincides with $Q_{\alpha,z}(\rho\|\sigma)$ when $r=1$, $P(0)=\alpha>1$, and $W_0=\rho$, $W_1=\sigma$; see [27, 71, 102]. Note that for any $B\in\mathcal{B}(\mathcal{H})$, $\mathcal{B}(\mathcal{H})_{\ge 0}\ni A\mapsto\operatorname{Tr}(B^*A^pB)^{\frac{1}{p}}$ is concave if $0\le p\le 1$, and convex if $1\le p\le 2$ [19, 27]. Thus, $Q_{P,z}(W)$ is convex in W whenever $P(0)/2\le z\le P(0)$ and $-P(k)\le z$, $k=1,\dots,r$, and according well-known arguments, $D_{P,z}$ is monotone under CPTP maps if and only if $Q_{P,z}(W)$ is convex in W. In particular, the natural extension of the sandwiched Rényi divergences seems to be the choice when $z=P(0)$, in which case

$$1=P(0)+P(k)+\underbrace{\sum_{i\ne k}P(i)}_{\le 0}\implies 1\le P(0)+P(k)\implies -P(k)\le P(0)=z,\qquad k=1,\dots,r,$$

i.e., $Q_{P,P(0)}(W)$ is convex in W for any P with $P(0)>1$. The natural extension of the Petz-type Rényi divergences in this case is probably given by the choice $z=1$, in which case convexity holds whenever $P(0)\in(1,2]$ (since $P(k)\ge -1$, $k=1,\dots,r$, holds due to $P(0)+\sum_{k=1}^{r}P(k)=1$). One of the obvious questions here is whether $Q_{P,z}$ has a closed expression, at least in the multi-variate sandwiched and Petz-type versions. Another natural question is whether

$$\log Q_{P,z}(W)=\sup_{H\in\mathcal{B}(\mathcal{H})_{\ge 0}}\left\{P(0)\log\operatorname{Tr}\left(H^{\frac{1}{2}}W_0^{\frac{P(0)}{z}}H^{\frac{1}{2}}\right)^{\frac{z}{P(0)}}+\sum_{k=1}^{r}P(k)\log\operatorname{Tr}\left(H^{\frac{1}{2}}W_0^{-\frac{P(k)}{z}}H^{\frac{1}{2}}\right)^{-\frac{z}{P(k)}}\right\},$$

analogously to the 2-variable case given in [71, Lemma 3.23]. Note that this could be an alternative definition of the multi-variate Rényi (α,z)-divergences. In the 2-variable case, the Rényi (α,z)-divergences can be expressed by a variational formula similar to the above also when $\alpha\in(0,1)$; one of the main challenges here is to find the right multi-variate generalization of that formula for the case when $P\ge 0$. In particular, we expect that in the case $z=1$ this would give the operationally relevant multi-variate extensions of the Petz-type Rényi divergences for the purpose of quantifying the trade-off between the error exponents in the problem of state exclusion described in Section 1.1.4. We remark that another viable approach to define multi-variate extensions of the Petz-type Rényi divergences may go via finding a characterization of the Nussbaum-Szkoła distributions [85] that can be generalized to multiple states.

Finally, the sixth approach goes via optimizations of classical Rényi divergences over different classicalizations of the states. One is the *measured Rényi divergence* $D_P^{\mathrm{meas}}:=\sup D_P((\mathcal{M}(W_x))_{x\in\mathcal{X}})$, where the supremum goes over every quantum-to-classical channel $\mathcal{M}$ (naturally identified with a quantum measurement). This quantity is known not to be additive in the 2-variable case, whence it is natural to consider the *regularized measured Rényi divergence* $\overline{D}_P^{\mathrm{meas}}:=\lim_n(1/n)D_P^{\mathrm{meas}}(W^{\otimes n})$. In the 2-variable case this is known to coincide with the sandwiched Rényi divergence (at least for $\alpha\ge 1/2$ [32, 75]), and therefore we conjecture that

$$D_{P,P(0)}(W)=\overline{D}_P^{\mathrm{meas}}(W)$$

holds also in the multi-variate case when $P(0)>1$. A dual quantity to the measured one is given by the *maximal Rényi divergence*, defined as $D_P^{\max}(W):=\inf_w D_P(w)$, where the infimum goes over all commuting families $(w_x)_{x\in\mathcal{X}}$ such that $W_x=\Phi(w_x)$, $x\in\mathcal{X}$, for some quantum channel Φ. These were originally introduced in the 2-variable case in [65], where it was shown that for $\alpha\in(0,1)\cup(1,2]$ they can be given explicitly as $Q_\alpha^{\max}(\rho\|\sigma)=\operatorname{Tr}\sigma\#_\alpha\rho$, where $\sigma\#_\alpha\rho$ is the Kubo-Ando α-weighted geometric mean of ρ and σ. Here it may be expected that a relation of the form $Q_P^{\max}(W)=\operatorname{Tr}G_P^q(W)$ holds, where G_P^q is some multi-variate extension of the Kubo-Ando weighted geometric means; one natural candidate would be the Karcher mean. A closely related, but not completely identical question is whether

$$-\log Q_P^{\max}(W)=\sum_{x\in\mathcal{X}}P(x)D^{\max}\left(\frac{G_P^q(W)}{\operatorname{Tr}G_P^q(W)}\middle\|W_x\right)$$

with some multi-variate non-commutative weighted geometric mean G_P^q, as is the case for 2 variables [72]. It is easy to see that $D_P^{\rm meas}$, $\overline{D}_P^{\rm meas}$, and $D_P^{\rm max}$ satisfy the data processing inequality for any P for which the classical D_P does; the non-trivial question is the additivity of $\overline{D}_P^{\rm meas}$ and $D_P^{\rm max}$. (We remark that additivity of the maximal Rényi divergence seems to be an open question even in the 2-variable case when $\alpha > 2$.)

It is worth noting that, as emphasized above, some of these approaches are very closely related to the notions of barycenters and multi-variate matrix means, which have been the subject of intensive research in recent years in matrix analysis (see, e.g., [1, 2, 12, 13, 14, 35, 50, 55, 56, 64, 67, 89, 90, 91]), and we plan to draw on these results. However, there are also important differences: for instance, in the usual approach, weighted matrix geometric means are required to satisfy a number of properties, like concavity or monotonicity in the arguments on the level of operators, which seems far too restrictive for multi-variate Rényi divergences, where it is sufficient to have such properties for a scalar quantity (i.e., the trace of a more generally defined matrix mean).

Most of the above approaches can be fully or partially followed also when the states are given by density operators on infinite-dimensional Hilbert spaces, or more generally, as normalized positive linear functionals on a von Neumann algebra. Moreoever, at least the second, the third and the sixth approaches can also be considered in GPTs; 2-variable relative entropies and Rényi divergences may be obtained for instance from the integral formulas given in [28, 41], or using measured or maximal versions, or their regularizations (the latter may depend on the choice of the tensor product). We will also consider the above problems in these more general settings.

1.2.3 Methodology for Objective 3

Our starting point will be an upper bound on the minimum of the sum of the two error probabilities, given by the PI and Audenaert in [5] as $|\mathcal{I}|\cdot|\mathcal{J}|\cdot e^{-\frac{n}{2}C(\{\rho_i\}_{i\in\mathcal{I}}\|\{\sigma_j\}_{j\in\mathcal{J}})}$, where $C(\{\rho_i\}_{i\in\mathcal{I}}\|\{\sigma_j\}_{j\in\mathcal{J}}) := \min_{i,j}\max_{\alpha\in[0,1]}(1-\alpha)D_\alpha(\rho_i\|\sigma_j)$ is the worst-case pairwise *Chernoff divergence*. This gives the asymptotic bounds

$$\frac{1}{2}C(\{\rho_i\}_{i\in\mathcal{I}}\|\{\sigma_j\}_{j\in\mathcal{J}}) \le \mathrm{c}(\{\rho_i\}_{i\in\mathcal{I}}\|\{\sigma_j\}_{j\in\mathcal{J}}) \le C(\{\rho_i\}_{i\in\mathcal{I}}\|\{\sigma_j\}_{j\in\mathcal{J}}) \tag{1.6}$$

on the optimal exponent $\mathrm{c}(\{\rho_i\}_{i\in\mathcal{I}}\|\{\sigma_j\}_{j\in\mathcal{J}})$ of the sum of the two error probabilities (*Chernoff exponent*) whenever $\mathcal{I}$ and $\mathcal{J}$ are finite sets. Clearly, the above upper bound on the sum of the single-copy error probabilities becomes meaningless when at least one of the sets has infinite cardinality; however, the lower bound in (1.6) may still hold, and we expect this to be the case. It is not known whether the factor $1/2$ in the lower bound above is optimal; however, an example due to the PI and coauthors suggests that this is probably the case [38]. It is also not known whether the lower bound in (1.6) still holds when the sets have infinite cardinality. Our first goal in this project point will be answering these questions. Second, we we will look for analogous bounds for the other exponents. The right approach here seems to be to scale the plane of achievable error exponent pairs by straight lines of constant slope, analogously to [92], and obtain bounds on the resulting variants of the direct exponents; our preliminary results suggest that this should be possible, leading to similar bounds as above with the prefactor in the lower bound depending on the slope of the line. These should then be ideally refined to finite-copy bounds, at least for finite $\mathcal{I}$ and $\mathcal{J}$. We will also look for regularized entropic expressions of other error exponents, analogous to the one given for the Stein exponent in [10]. Apart from their conceptual significance, these may serve as intermediate steps towards proving single-letter bounds, e.g., by using convexity estimates of the relevant divergences, and the exact equalities might also be helpful in proving optimality of such bounds.

We will also aim at determining the exact strong converse exponent of composite binary i.i.d. state discrimination, and prove the strong converse property of Stein's lemma. We expect that the latter will follow from a strengthened version of the quantitative continuity bounds for the sandwiched Rényi α-divergences around $\alpha = 1$ [95, 96], which we explain here in some detail, as such a strengthening would be important in itself, with several potential applications in non-i.i.d. problems.

It is known [73] that for a fixed pair of states ρ,σ, the family of Petz-type Rényi divergences is monotone increasing and continuous; in particular, the limit $\alpha \to 1$ gives the relative entropy $D(\rho\|\sigma)$. A quantitative continuity bound around $\alpha = 1$ was given in [95, 96] as

$$D_\alpha(\rho\|\sigma) \le D(\rho\|\sigma) + (\alpha-1)(4\cosh c)(\log\eta(\rho,\sigma))^2, \tag{1.7}$$

valid for any positive constant c and $1 < \alpha \le 1 \le 1+\delta_c(\rho,\sigma)$, where $\delta_c(\rho,\sigma) := \min\left\{\frac{1}{2}, \frac{c}{2\log\eta(\rho,\sigma)}\right\}$, and $\eta(\rho,\sigma) := 1 + \operatorname{Tr}\rho^{1/2}\sigma^{1/2} + \operatorname{Tr}\rho^{3/2}\sigma^{-1/2}$ (we assume that the support of ρ is dominated by the support of σ). As a consequence of the Araki-Lieb-Thirring inequality [4, 63], $D_\alpha(\rho\|\sigma)$ may be replaced in (1.7) with $D_\alpha^*(\rho\|\sigma)$. This implies that for i.i.d. states $\rho_n = \rho^{\otimes n}, \sigma_n = \sigma^{\otimes n}$,

$$D_\alpha^*(\rho_n\|\sigma_n) - D(\rho_n\|\sigma_n) = n\left(D_\alpha^*(\rho\|\sigma) - D(\rho\|\sigma)\right) \le n(\alpha-1)(4\cosh c)(\log\eta(\rho,\sigma))^2,$$

for any $1 \le \alpha \le 1 + \delta_c(\rho,\sigma)$, where the upper bound scales linearly in n, and the range of valid α's does not change with n. On the contrary, a direct application of (1.7) to $\rho = \rho_n, \sigma = \sigma_n$, yields

$$D_\alpha^*(\rho_n\|\sigma_n) - D(\rho_n\|\sigma_n) \le (\alpha-1)(4\cosh c)(\log\eta(\rho_n,\sigma_n))^2,$$

for $1 \le \alpha \le 1 + \delta_c(\rho_n,\sigma_n)$, where the upper bound scales quadratically in n, and the range of valid α's decreases linearly with n. This kind of asymptotics is highly undesirable when we want to move to non-iid states, and therefore it would be important to find continuity bounds that scale better with increasing system size, at least for specific types of correlated states. In particular, sufficiently strong continuity bounds for Rényi divergences of the form $(1/n)D_\alpha^*(\rho^{\otimes n}\|\int\sigma^{\otimes n}\,d\nu(\sigma))$ with an arbitrary probability measure ν would settle the strong converse problem of the composite i.i.d. Stein's lemma described in Section 1.1.5. Our preliminary results show that for the local restrictions of translation-invariant and gauge-invariant fermionic quasi-free lattice states, continuity bounds hold where the upper bound scales linearly with the system size, and the range of valid α's is independent of it, and are even universal in the states, as long as their symbols are uniformly bounded away from 0 and 1 (see [74] for details on the latter). Our aim in this project point will be to prove such well-behaved continuity results for other types of correlated state pairs, in particular, for the case of an i.i.d. first and an averaged i.i.d. second argument, described above.

1.2.4 General considerations

Use and management of research data and other research outputs: The objectives of the project are analytical results in mathematics, which will be published in peer-reviewed journals and presented at conferences. No research data is expected to be produced.

Open science principles: As is common in the field, and has been the practice of the PI all through his research career, all publications stemming from he project will be uploaded on the arXiv before submission to a journal and after acceptance, and hence will be freely available to view for anyone.

Multi- and interdisciplinary approach: The proposed project is interdisciplinary, at the interface of matrix analysis, functional analysis, information theory, and mathematical physics, addressing problems and using tools from, and promising results relevant for, each of these fields.

Gender equality in research: We are committed to creating a safe and encouraging environment for anyone participating in the project in any form.

1.3 RELEVANCE, QUALITY AND NOVELTY OF THE PROJECT

Determining exact trade-off relations and identifying the corresponding (Rényi) information measures have been in the center of research in information theory, both classical and quantum. In the quantum case, non-commutativity of density operators results in an infinity of possible quantum extensions of the classical information measures, and it is a fundamental problem to identify among them the ones that are operationally relevant. In the case of the 2-variable Rényi divergences this lead to distinguishing the Petz-type and the sandwiched Rényi divergences as the operationally relevant ones (to a large extent due to the work of the PI and coauthors [37, 71, 75, 77, 78]); however, understanding the exact parameter ranges where these divergences are relevant and the role they play in the quantification of trade-off relations is still the subject of active research (see, e.g., the series of very recent papers [58, 59, 60, 61, 62]), with many important questions still open. Relatedly, the need to understand the mathematical properties of these information measures have been a driving force in matrix analysis (and in more general settings, in the applications of operator algebra techniques) for several decades, with very intensive activity to this day.

The study of multi-variate quantum Rényi divergences, on the other hand, started only very recently; apart from the recent preprint [72] by the PI and coauthors, there seems to be only one more paper [29] to consider possible definitions and mathematical properties of such quantities. On the operational side, two papers from the past year appeared in the quantum information theory community addressing such problems [25, 66], but only giving definitive solutions in the classical case, which also highlights that many fundamental questions are still open even in the classical case. All these demonstrate a rising interest in the subject, but also that research in this direction is in a very preliminary status, therefore this seems to be the right time to push it further. The proposed research on multi-variate quantum Rényi divergences thus clearly goes beyond the current state of the art, with the additional edge that the PI is among the first and so far very few to have worked on the subject.

While hypothesis testing results for simple i.i.d. hypotheses are conceptually very important, realistic scenarios call for more complicated models, where either correlations are present, or the hypotheses are not specified by simple states/channels, but only by some sets to which they belong. Obtaining exact trade-off relations for such models is known to be notoriously difficult in the quantum case, and even when they are available, they might be given in terms of complicated regularized formulas that are not feasible to evaluate in practice, even numerically. Hence, the proposed research on composite state discrimination addresses some fundamental and long-standing open problems in quantum information theory. The novelty of the proposed approach lies in the application of techniques developed recently by the PI and coauthors to tackle such problems [80], which gives a very realistic chance to go beyond the current state of the art.

Finally, the possibility of non-trivial super-exponential asymptotics in correlated state discrimination problems seems to be a completely new phenomenon, discovered by the PI and coauthors in the recent paper [17]. Understanding it better promises new ways of looking at correlations and criticality in many-body quantum systems, as well as the possible asymptotic behaviours of entropic quantities in such systems, which is of fundamental interest not only for information theory but also for statistical physics.

1.3.1 Relation to the European Research Area

The proposed research is in the field of quantum Shannon theory, one of the theoretical/mathematical foundations of the emergent quantum technologies, which form a high priority research area within the EU and worldwide. The successful execution of the project is likely to create new intra-European research links and long-lasting collaborations between researchers in Bratislava and Budapest via the PI's professional contacts in the latter, thus contributing to European competitiveness. The relocation of the PI to Bratislava for the purposes of the project fits perfectly into the ERA's "ambition to create a single, borderless market for research, innovation and technology across the EU", encouraging "the free circulation of researchers and knowledge", as stated on the website of ERA (link).

1.4 EXCELLENCE OF THE RESEARCHER

The PI of the present proposal, Dr. Mosonyi, has obtained MSc degrees in both mathematics (2000) and physics (2004), and a PhD in theoretical physics (2005) under the supervision of Prof. Mark Fannes and Prof. Dénes Petz. He has been working in research since completing his PhD, meaning a total of eighteen years of postdoctoral research experience, of which he has spent ten years working in leading research centers in Japan, Singapore, the United Kingdom, Spain, and Germany, collaborating with prominent researchers of his field. Since his return to Hungary in 2016, he has successfully built a quantum information theory research group at the Institute of Mathematics of the Budapest University of Technology and Economics.

Since the beginning of his research career, Dr. Mosonyi has been working on mathematical problems in quantum information theory; in particular, quantum state discrimination, classical-quantum channel coding, and the mathematical study of quantum divergences and their applications. He has published his research in 33 papers, many of them in top-ranking journals in mathematical physics and information theory, including 4 papers in Communications in Mathematical Physics and 7 papers in IEEE Transactions on Information Theory, the leading journals of mathematical physics and information theory, respectively. The number of yearly citations to his work has been steadily increasing over his career, with the total amount of independent citations having had doubled in the past five years (from 376 to 773 on MTMT,[2] and from 722 to 1557 on google scholar[3]), clearly showing the long-term high impact of his work.

Dr. Mosonyi is an internationally recognized member of the quantum information theory research community, and an internationally leading expert in the mathematical study of quantum divergences and their applications in quantum Shannon theory. His professional recognition is clearly shown, among others, by various invitations to speak at conferences and to serve on the editorial boards of journals. In the past five years, he has been invited to speak at 9 conferences, as well as to give a tutorial on entropies at the 2019 edition of the Quantum Information Processing (QIP) conference series, the most selective and prestigious annual conference on (mainly theoretical) quantum information science. From 2018 to 2021, he served as an associate editor for quantum information theory at the IEEE Transactions on Information Theory, the most prominent forum of dissemination of results in quantum Shannon theory. From 2016 to 2021 he served as an editor for the newly established online journal Quantum (according to scimago, it is currently in the top 5% of journals in the category Physics and Astronomy),

[2] The official academic bibliography database of Hungary (link).

[3] link to google scholar

and has been serving on the steering board of the same journal since 2021. He has also been on the editorial advisory board of the Journal of Mathematical Physics since 2016.

Since the beginning of his research career, Dr. Mosonyi has been very actively and successfully applying in competitive funding schemes for the funding of his research activities. He twice won the Junior Research Fellowship of the Erwin Schrödinger Institute for Mathematical Physics in Vienna, a two-year research fellowship of the Japan Society for the Promotion of Science (JSPS) with a corresponding research grant, hosted at Tohoku University, Japan, a very prestigious two-year Marie Curie International Incoming fellowship of the European Commission, hosted at the University of Bristol, and a three-year individual Bolyai Fellowship of the Hungarian Academy of Sciences (shortened by one year after receiving a Lendület grant). He has won two research grants of the Hungarian National Research, Development and Innovation Office as the PI of a research group (K 124152, 2017–2022 and KH 129601, 2018–2021), and since three years ago, he has also been a project leader in the Quantum Information National Laboratory, a large consortial research grant for the advancement of quantum technologies in Hungary.

Most importantly, he received a five-year Lendület grant of the Hungarian Academy of Sciences in 2018 to establish the MTA-BME "Lendület" (Momentum) Quantum Information Theory research group at the Institute of Mathematics at BME; this grant just ended before the submission of this proposal. The MTA-BME QIT research group was very successful; the results of the project have appeared in 55 publications: 46 in peer-reviewed journals, 2 in conference proceedings, and 7 in online preprints that are currently under review at journals. All the publications appeared in highly ranked journals, with many in the top journals in information theory (IEEE Transactions on Information Theory, 9 papers and 1 more under review), and in mathematical and theoretical physics (Communications in Mathematical Physics, 3 papers, Annales Henri Poincare, 1 paper, Quantum, 5 papers). The results of the group were presented at 41 conference and invited seminar talks, of which 4 were given at QIP, with 2 based on papers coauthored by the PI. The most important annual conference series in quantum Shannon theory is Beyond IID in Information Theory, where the group members presented 4 talks, all of them based on papers (co)authored by the PI.[4] All these demonstrate the PI's ability to identify relevant problems at the forefront of research in quantum information theory, secure funding for his research ideas, and put together, conduct, and manage a research group producing research output of the highest quality.

Curriculum Vitae

Personal information

First and last name: Milán Mosonyi
Identifier: ORCID 0000-0002-5973-5533
Date of birth: 12 January 1976
Nationality: Hungarian
Website: https://qi.math.bme.hu/
https://math.bme.hu/ mosonyi/

Education

04/2005 PhD in Theoretical Physics
Faculty of Science, Catholic University of Leuven, Belgium

06/2004 MSc in Physics
Faculty of Natural Sciences (TTK), Budapest University of Technology and Economics (BME), Hungary

06/2004 MSc in Mathematics
Faculty of Natural Sciences (TTK),
Eötvös Loránd University of Sciences (ELTE), Hungary

Current positions

09/2023 – Research fellow
Alfréd Rényi Institute of Mathematics, Budapest, Hungary

02/2012 – Associate professor
Budapest University of Technology and Economics, Hungary

[4] Details of the group's achievements can be found on the group webpage (link).

Previous positions

06/2005 – 01/2012	Assistant professor Budapest University of Technology and Economics, Hungary

On leave from the Budapest University of Technology and Economics:

09/2015 – 08/2016	Postdoctoral Research Fellow, Technische Universität München
09/2013 – 08/2015	Postdoctoral Research Fellow, Universitat Autònoma de Barcelona
09/2011 – 08/2013	Marie Curie Research Fellow at the School of Mathematics, University of Bristol
05/2009 – 08/2011	Research Fellow at the Centre for Quantum Technologies, National University of Singapore
12/2006 – 11/2008	JSPS Postdoctoral Research Fellow, Tohoku University, Sendai, Japan

Scholarships and awards

09/2016 – 08/2018	Bolyai János Research Fellowship of the Hungarian Academy of Sciences
09/2011 – 08/2013	European Commission Marie Curie International Incoming Fellowship: "Quantum Information Theory and Statistics (QUANTSTAT)"
01/2009 – 03/2009	Junior Research Fellowship of the Erwin Schrödinger Institute, Vienna
12/2006 – 11/2008	Postdoctoral Research Fellowship of the Japan Society for the Promotion of Science (JSPS) and Grant-in-Aid for JSPS Fellows $18 \cdot 06916$
01/2009 – 03/2009	Junior Research Fellowship of the Erwin Schrödinger Institute, Vienna
09/2000 – 08/2003	State scholarship for PhD students, Hungary

Student and post-docs supervision

2017–2023	2 PhD supervisions (ongoing) at BME TTK 3 MSc students (1 at ELTE TTK, 2 at BME TTK) 3 BSc students (2 at ELTE TTK, 1 at BME TTK)

Teaching activities

2000–2023, BME TTK

- Lecture course on Quantum Information Theory (BSc, MSc, PhD, 2015, 2016, 2017, 2018, 2019, 2020, 2022, 2023 autumn)
- Lecture course on Advanced Quantum Information Theory (BSc, MSc, PhD, 2018, 2019, 2020, 2021 spring).
- Lecture course on Matrix Analysis and Quantum Information Theory (BSc, MSc, PhD, 2017 spring, 2021, autumn)
- Lecture course on Matrix Analysis (BSc, MSc, PhD, 2022, 2023 autumn)
- Lecture course on Advanced Measure Theory (BSc, MSc, 2017, autumn)
- Lecture course on Measure Theory and Complex Functions (BSc, 2017, spring)
- Lecture course on Advanced Functional Analysis (MSc and PhD; twice between 2000 and 2006).
- Exercise classes in Analysis and Functional Analysis for Mathematics and Engineering students (BSc, several times between 2000 and 2006)
- Calculus and Linear Algebra lectures and exercise classes for foreign students in Engineering (BSc, several times between 2000 and 2006)

2015–2016, Technische Universtät München

- Exercise class in Functional Analysis for Mathematics students (BSc and MSc).

2011–2013, University of Bristol

- Lecture course on Information Theory for final-year Mathematics and Engineering undergraduates.
- Calculus tutorials for students in Mathematics (BSc, first and second semester course).

1998–2000, ELTE TTK

- Exercise classes in Analysis for teacher training students in Mathematics (BSc, four semesters course; once).
- Exercise class in Complex Function Theory for Physics students (BSc, 2000).

Organisation of scientific meetings

2023 Main organizer of the Quantum Information Theory and Mathematical Physics workshop, Budapest, 28–30 June, 2023. cca. 20 participants (website)

2022 Rényi 100 conference, organizer of the invited session on Quantum Information Theory, 20–23 June 2022; cca. 150 participants (website)

2022 Main organizer of the Quantum Information Theory and Mathematical Physics workshop, Budapest, 14–17 June, 2022. cca. 20 participants (website)

2019 Main organizer of the Quantum Information Theory and Mathematical Physics workshop, Budapest, 2–5 September, 2019. cca. 20 participants (website)

2019 Program committee member, 19th Asian Quantum Information Science Conference (AQIS 2019), Seoul, Korea, August 19–23, 2019; (website)

2018 Main organizer of the Quantum Information Theory and Mathematical Physics workshop, Budapest, 20–23 September, 2018. cca. 20 participants (website)

2017 Main organizer of the Quantum Information Theory and Mathematical Physics workshop, Budapest, 30 August–2 September, 2017. cca. 20 participants (website)

2016 Main organizer of the Quantum Information Theory and Mathematical Physics workshop, Budapest, 16–19. September, 2016. cca. 20 participants (website)

2014 Program committee member, 11th Central European Quantum Information Processing Workshop, 5–8 June 2014 (website)

Reviewing activities

Editorial work

- IEEE Transacions on Information Theory, associate editor for Quantum Information Theory, October 2018 - September 2021; (website)
- Quantum, editor, October 2016 - July 2021, then member of the Steering Board (website)
- Journal of Mathematical Physics, member of the Editorial Advisory Board, May 2016 - (website)

Referee work

- For journals: Communications in Mathematical Physics; Reviews in Mathematical Physics; Letters in Mathematical physics; Journal of Mathematical Physics; IEEE Transactions on Information Theory; Journal of Physics A; Linear Algebra and its Applications; International Journal of Quantum Information Theory; Infinite Dimensional Analysis, Quantum Probability and Related Topics; Periodica Mathematica Hungarica
- Member of the board of referees for the Hungarian National Agency for Research, Development and Innovation (NKFIH), section of Mathematics, 2018–2021

PhD evaluation

- Referee of the doctoral thesis of Oskari Kerppo, University of Turku, 2022.
- Referee of the doctoral thesis of Martin Plávala, Comenius University Bratislava, 2019.
- Referee of the doctoral thesis of Kun Fang, University of Technology Sydney, 2018.
- Referee of the doctoral thesis of Tamás Kói, BME, 2017.
- Referee for the departmental defense of the doctoral thesis of Gábor Balló, University of Pannonia, 2013.
- Examiner at the general doctoral exam of Fanni Sélley, BME, 16 October 2018.
- Examiner at the general doctoral exam of Mihály Csirik, ELTE, 5 April 2017.

Major collaborations

Fumio Hiai, Tohoku University. Topics: state discrimination, quantum Rényi divergences and f-divergences, reversibility problems. Ongoing since 2007.

Tomohiro Ogawa, University of Electro-communications. Topics: state discrimination, classical-quantum channel coding. Ongoing since 2007.

Overview of the researcher's most important projects in the last 5 years

Project name/identification	Source of funding	Budget (EUR) [1]	Project period	The role of the researcher in the project
MTA-BME "Lendület" (Momentum) Quantum Information Theory research group (website)	Hungarian Academy of Sciences	430.000	2018–2023	PI
NRDI K124152 Trade-off relations and divergences in quantum information theory	NRDI[2]	20.000	2017–2022	PI
NRDI KH129601 Mathematical analysis of quantum communication problems	NRDI	50.000	2018–2021	PI
Quantum Information National Laboratory (website)	NRDI	14.000.000	2020–2025	Project leader[3]

[1] All values are approximate, based on the HUF/EUR exchange rate at the time of submission.

[2]NRDI: Hungarian National Research, Development and Innovation Office

[3]This will be terminated if the present application is successful and the project starts at MISAS.

Overview of the researcher's most important outputs

Output name/identification	Type of output	Short description	The role of the researcher
Milán Mosonyi, Tomohiro Ogawa. Quantum hypothesis testing and the operational interpretation of the quantum Rényi relative entropies. *Communications in Mathematical Physics*, 334(3):1617–1648, 2015.	Research paper	We give an explicit single-copy expression for the strong converse exponent of quantum binary state discrimination in terms of the sandwiched Rényi divergences, thereby establishing the operational significance of these quantities.	Author
Milán Mosonyi. The strong converse exponent of discriminating infinite-dimensional quantum states. *Communications in Mathematical Physics*, 400:83–132, 2023.	Research paper	We extend the above result to pairs of density operators on infinite-dimensional Hilbert spaces by showing a finite-dimensional approximability property of the sandwiched Rényi divergences.	Author
Milán Mosonyi, Tomohiro Ogawa. Strong converse exponent for classical-quantum channel coding. *Communications in Mathematical Physics*, 355(1):373–426, 2017	Research paper	We give a single-letter expression for the strong converse exponent of classical-quantum channel coding in terms of the sandwiched Rényi divergence radius of the channel, thereby also giving an operational interpretation of this information measure.	Author
Milán Mosonyi. Coding theorems for compound problems via quantum Rényi divergences. *IEEE Transactions on Information Theory*, 61(6):2997–3012, 2015.	Research paper	We develop a simple and robust technique based on a subadditivity property of the sandwiched Rényi divergences to extend single-object coding theorems to multiple-object ones, i.e., where the primary object (state to compress, channel to use) is not specified exactly, but only by its membership in a larger set.	Author
Milán Mosonyi, Dénes Petz. Structure of sufficient quantum coarse grainings; *Letters in Mathematical Physics*, 67: 19–30, 2004.	Research paper	We show that a quantum channel preserves the relative entropy of two states if and only if they satisfy a very strong factorization property.	Author

The main research directions of the PI are the mathematical study of quantum (Rényi) divergences and their applications in quantum information theory, in particular, in state discrimination and channel coding. His most important research achievements (with coauthors) include the following:

- The determination of the direct exponent of binary state discrimination in terms of regularized Petz-type Rényi divergences of various physically relevant non-i.i.d. states [39, 40], including translation- and gauge-invariant quasi-free states of fermionic [74] and bosonic lattices [69].

- The determination of the strong converse exponent of binary state discrimination in terms of sandwiched Rényi divergences for translation-invariant product states on finite-dimensional Hilbert spaces [75], infinite-dimensional Hilbert spaces [71] and injective von Neumann algebras [37], as well as for various correlated states on spin chains with finite-dimensional local algebra [76], and for a discrimination problem in between state- and channel discrimination [21].

- The determination of the strong converse exponent of classical-quantum channel coding in terms of the sandwiched Rényi divergences radius [77], and in terms of the weighted sandwiched Rényi divergences radius for constant composition coding [78].

- Algebraic characterizations of reversibility of a quantum operation on a set of states, and its detection by quantum divergences [79, 33, 36].

1.5 EXCELLENCE OF THE APPLICANT/HOST ORGANISATION

The Mathematical Institute of the Slovak Academy of Sciences is a scientific institute focused mainly on basic research in mathematics and theoretical informatics. The Institute has a long tradition in several important branches of pure and applied mathematics and participated in a number of successful projects in both basic and applied research, including projects of Frame Projects of EU, Structural projects of EU, and projects of domestic agencies APVV and VEGA. The researchers of the Institute belong to the top in their respective fields, in a world-wide context, and are engaged in multiple collaborations with experts from internationally renowned institutions. In collaboration with the Comenius University, the Institute organises a PhD study program and many young scientists and students use the Slovak fellowship program SAIA for short term study stays at the institute.

The researchers of the institute are well connected with experts in their respective fields internationally. In the field of quantum information theory, the Institute has a close connection to neighbouring institutions such as the Slovak Technical University in Bratislava, Palacký University in Olomouc or the Research Center for Quantum Information at the Institute of Physics of SAS, through extensive collaborations and numerous joint projects. The institute organizes several successful and long-term established seminars, such as the seminar on Quantum structures, with invited talks by distinguished scientists. The institute is well equipped with all the standard hardware and software needed for mathematics research, including quality equipment for online presentation and communication. The library of Mathematical Institute SAS belongs to the best mathematical libraries in Slovakia, with access to the most important scientific databases. Furthermore, the Slovak Academy of Sciences provides access to a supercomputer.

The main professional contact and planned collaborator of the PI at MISAS will be Anna Jenčová, who is an internationally renowned expert in quantum information theory, quantum foundations, and generalized probabilistic theories, among others. Most importantly for the proposal, she is one of the top experts worldwide in the study of quantum divergences, in particular, their monotonicity and reversibility properties, and especially in the most general von Neumann algebra setting [43, 44, 45, 46, 47, 48]. We envision close and intensive collaboration between the PI and Anna Jenčová, which will be the main foundation of the two-way knowledge transfer between the host institute and the PI; on the one hand, the PI and the project will largely benefit from the expertise of Anna Jenčová in the above detailed subjects, while on the other hand the PI brings his expertise in quantum Shannon theory, which complements and enriches the current pool of expertise of the host institute.

2 Impact

2.1 THE WIDER IMPACT OF THE PROJECT

Scientific impact: The proposed activity in the project is fundamental research in basic science, at the interface of mathematics, information theory, and mathematical physics, and hence its most significant impact is expected to be of scientific nature. Identifying the operationally relevant information measures and establishing their mathematical properties have been central problems in information theory since its conception, and therefore it seems reasonable to expect that the results of the project on multi-variate Rényi divergences will have significant impact on quantum information theory and on matrix analysis well beyond the scope of the project and on every time scale from short to medium to long run, analogously to the study of 2-variable quantum divergences. Progress in the problems related to Objectives 1 and 3 will require the development of robust tools and proof methods that are applicable in correlated and composite settings, which are likely to find applications to many other "non-i.i.d." problems both in quantum information theory and in quantum statistical physics, again forecasting an impact of the results of the project well beyond its scope and duration.

Economic and societal impact: Quantum information science has been a priority research area on the international (e.g., European flagship project, or QuantERA) and on the national level in most developed countries in the past years, due to high expectations regarding its potential technological applications (including cryptography, sensing, simulation of physical systems, or computing), which might have a transformative effect on technology, economy, and ultimately also on society. While our project is of foundational nature and not directly related to applications, information measures derived from quantum divergences and information theoretic techniques play important roles in the theoretical analysis of the above applications, through which our results might ultimately have an impact on them.

Impact on the researcher: Collaboration with Anna Jenčová at the host institute is guaranteed to have a substantial positive impact on the skills of the PI, partly via deepening and broadening his knowledge of operator algebraic techniques, and partly by learning generalized probabilistic theories, from a top expert in these subjects. These should help broadening and diversifying the research profile of the PI, resulting in a long-term positive impact on his research. Careerwise, the project would allow the PI to focus solely on research for two years, working on his own research agenda. Successful completion of the project should significantly contribute towards his ability to secure further funding for his research in competitive funding schemes, including ERC, and maintain a research group working under his guidance.

Impact on the host organization: The PI will bring expertise and research topics to the host institute that are complementary to the existing ones, therefore broadening the scope of research at the institute. The planned invitation of visitors and organization of workshops within the project (see work package 4 below) will provide convenient opportunities to the researchers in the host institute to interact with leading experts in quantum Shannon theory, and these activities will also greatly enhance the visibility of the institute in the quantum information theory community. Successful collaboration on the project with Anna Jenčová (and possibly other researchers at the institute) is likely to create long-term professional relations that would be maintained well beyond the duration of the project and even if the PI continues his career elsewhere, and lead to further fruitful collaborative work.

Potential negative impact: The project is of theoretical/foundational nature, and we cannot foresee any negative impact worth mentioning.

Direct benefits measurable within the monitored data: The main type of monitored data will be quality publications, of which we expect 6-9 papers in highly ranked journals.

Potential obstacles to the planned impact of the project: One of the most important impact that research in mathematics can have is developing notions and tools that can be utilized by other researchers in their work. This impact may be considerably reduced if others come up with more efficient tools that replace the previous ones. It is, however, neither possible nor desirable to prevent this from happening.

2.2 MEASURES TO MAXIMISE IMPACT - DISSEMINATION AND COMMUNICATION, EXPLOITATION OF RESULTS

The major expected impact of the project is on the work of other researchers, which is best maximized by publishing the results of the project in highly ranked journals, and presenting them at prestigious conferences as well as at smaller events and seminars to experts in the particular research direction. Accordingly, we will aim at publishing our results in leading journals in mathematical physics and information theory, in particular, in Communications in Mathematical Physics, Annales Henri Poincaré, and IEEE Transactions on Information Theory. Taking into account the publication record of the PI, this seems completely realistic. The two largest conferences most relevant to the field of the proposal are QIP and Beyond IID in Information Theory, and we are going to submit our work to these conferences; here again, past record shows a very realistic chance of success. On top of these, we will seek opportunities to present our results at smaller events and seminars to experts working in related directions.

In accordance with the above, results of the project may be best utilized after the completion of the project by continuing research building on the findings of the project and possibly extending the scope of the activities by involving a larger group coordinated by the PI. In order to maximize this type of impact, the PI will submit one or more major grant applications for the funding of a research group for after the completion of the present project.

3 Implementation

3.1 PROJECT PLAN AND DELIVERABLES

We plan to start the project on 01 September 2024, and complete it on 30 June 2026, amounting to a total of 22 months. 3 work packages are assigned to the three main research topics (Super-exponential state discrimination, Multi-variate Rényi divergences, and Trade-off relations for composite hypothesis testing). These are logically interrelated and use closely related mathematical tools, therefore they can be pursued in parallel along the whole duration of the project. 1 additional work package is assigned to organize two workshops and additional seminar talks at the host institute, and 1 work package to the preparation of at least one grant application in the second half of the project, to secure funding for the continued research activities of the PI and to pursue new research goals that are likely to emerge as a result of the present project.

We plan to use the available 2.000 EUR/month contribution to research team to hire a postdoc (R2 researcher). The monthly salary of the postdoc will depend on their level of experience and the form of employment (full time/part time). The duration of the employment will be determined by the available budget and the time needed to start the employment (advertisement of the job, evaluation of applications, etc.); we tentatively calculate with 16 person months for the postdoc.

The 1.300 EUR/month research cost will be used partly to cover research trips of the PI and the postdoc (conference participation, collaborations), the expenses of hosting guest researchers from other institutes, and the organizations of two workshops at the host institute.

3.1.1 Work packages

Note: All indirect and research costs in the following tables are given excluding VAT.

Work package number	1
Title of the work package	Super-exponential
Start of implementation of the work package (Mx Month)	M1
End of implementation of the work package (Mx month)	M22
Involvement (expressed in Person Months)	PI: 7 person months, postdoc: 6 person months
Personnel costs (in EUR)	Total: 50 261 EUR, of which PI: 33 761 EUR (4 823 EUR/person month x 7) postdoc: 16 500 (2 750 EUR/person month x 6)
Other eligible costs, excluding personnel costs (in EUR excluding VAT)	Total: 9 884.2 EUR, of which Indirect costs: 5967.5 EUR (852.5 EUR/PI person month x 7) Research costs: 3916.7 EUR (calculating with 1 - 2 000 EUR/ conference or research trip)
Objectives	
Find examples of correlated states for which the error probabilities in asymptotic state discrimination decrease super-exponentially, find necessary and sufficient conditions for such behaviour, give bounds on the error asymptotics, find sharp trade-off relations, and connect the error asymptotics to measures of correlation.	
Description of the work package	
Task 1: Construct examples with gauge-invariant and non-gauge invariant fermionic and bosonic lattice systems with super-exponential decrease of the discrimination errors. Task 2: Find the correct scale of regularization for Rényi divergences, express trade-off relations in terms of the regularized Rényi divergences. Task 3: Investigate the possibility of super-exponential state discrimination of (pure) finitely correlated states. Task 4: Find quantitative relations between the error asymptotics and correlations in the states. Task 5: Research visits at other research institutes to enhance dissemination, collaboration, and knowledge transfer. Task 6: Publication of our findings in peer-reviewed journals. Task 7: Presentation of our findings at international conferences and seminar talks.	
Deliverables	
2-3 papers and 2-3 conference talks.	

Work package number	2
Title of the work package	Multi-Rényi
Start of implementation of the work package (Mx Month)	M1
End of implementation of the work package (Mx month)	M22
Involvement (expressed in Person Months)	7
Personnel costs (in EUR)	Total: 50 261 EUR, of which PI: 33 761 EUR (4 823 EUR/person month x 7) postdoc: 16 500 (2 750 EUR/person month x 6)
Other eligible costs, excluding personnel costs (in EUR excluding VAT)	Total: 9 884.2 EUR, of which Indirect costs: 5967.5 EUR (852.5 EUR/PI person month x 7) Research costs: 3916.7 EUR (calculating with 1 - 2 000 EUR/ conference or research trip)
Objectives	
Define multi-variate quantum Rényi divergences with good mathematical properties, find non-trivial applications in quantum information theoretic tasks.	
Description of the work package	
Task 1: Analyse different constructions for multi-variate quantum Rényi divergences, establish their mathematical properties (additivity, monotonicity, continuity properties), identify reversibility witnesses. Task 2: Find multi-variate extensions of the Petz-type and the sandwiched Rényi divergences. Task 3: Give trade-off inequalities and exact trade-off relations in terms of multi-variate Rényi divergences in state/channel discrimination and exclusion, and in other quantum information theoretic problems. Task 4: Apply multi-variate Rényi divergences in state conversion problems. Task 5: Research visits at other research institutes to enhance dissemination, collaboration, and knowledge transfer. Task 6: Publication of our findings in peer-reviewed journals. Task 7: Presentation of our findings at international conferences and seminar talks.	
Deliverables	
2-3 papers and 2-3 conference talks.	

Work package number	3
Title of the work package	Composite
Start of implementation of the work package (Mx Month)	M1
End of implementation of the work package (Mx month)	M22
Involvement (expressed in Person Months)	6
Personnel costs (in EUR)	Total: 39 938 EUR, of which PI: 28 938 EUR (4 823 EUR/person month x 6) postdoc: 11 000 (2 750 EUR/person month x 4)
Other eligible costs, excluding personnel costs (in EUR excluding VAT)	Total: 8 615 EUR, of which Indirect costs: 5 115 EUR (852.5 EUR/PI person month x 6) Research costs: 3 500 EUR (calculating with 1 - 2 000 EUR/ conference or research trip)
Objectives	
Give sharp inequalities between various exponents of composite state discrimination and pairwise error exponents, establish the strong converse property of Stein's lemma for composite state discrimination.	
Description of the work package	
Task 1: Find a sharp lower bound on the composite Chernoff exponent in terms of the worst-case pairwise Chernoff divergences for infinite sets of states. Task 2: Find sharp lower bounds on the direct exponents in terms of the worst-case pairwise Hoeffding divergences, for finite as well as for infinite sets of states. Task 3: Investigate the possibility of an improved continuity bound of the sandwiched Rényi α-divergences in the parameter α around $\alpha = 1$. Task 4: Prove the strong converse of Stein's lemma for composite state discrimination. Task 5: Extend the above results to channel discrimination where possible. Task 6: Research visits at other research institutes to enhance dissemination, collaboration, and knowledge transfer. Task 7: Publication of our findings in peer-reviewed journals. Task 8: Presentation of our findings at international conferences and seminar talks.	
Deliverables	
2-3 papers and 2-3 conference talks.	

Work package number	4
Title of the work package	Knowledge transfer
Start of implementation of the work package (Mx Month)	M1
End of implementation of the work package (Mx month)	M22
Involvement (expressed in Person Months)	1
Personnel costs (in EUR)	4 823 EUR (1 person month)
Other eligible costs, excluding personnel costs (in EUR excluding VAT)	Total: 13 352.5 EUR, of which Indirect costs: 852.5 EUR (1 PI person month) Research costs: 12 500 EUR, of which 2 x 5 000 EUR for the two workshops (accommodation of invited speakers, catering) 2 500 EUR for the local expenses of visitors
Objectives	
Increase the international visibility of the host institute in the field of quantum information theory. Knowledge transfer.	
Description of the work package	
Task 1: Invitation of researchers from other institutes to give seminar talks and to exchange ides. Task 2: Organization of an international workshop at the host institute at around half time of the project. Task 3: Organization of an international workshop at the host institute at the end of the project.	
Deliverables	
Seminar talks by invited speakers, two international workshops.	

Work package number	5
Title of the work package	Grant application
Start of implementation of the work package (Mx Month)	M12
End of implementation of the work package (Mx month)	M22
Involvement (expressed in Person Months)	1
Personnel costs (in EUR)	4 823 EUR (1 person month)
Other eligible costs, excluding personnel costs (in EUR excluding VAT)	Indirect costs: 852.5 (1 PI person month)
Objectives	
Submission of grant application(s) to secure funding for the continuation of the research activities of the PI.	
Description of the work package	
Task: Put together and submit one or more grant applications for the support of a research group.	
Deliverables	
Submitted application(s).	

3.1.2 List of work packages

Work package number	**Title of the work package**	**Start of activities**	**End of activities**
1	Super-exponential	M1	M22
2	Multi-Rényi	M1	M22
3	Composite	M1	M22
4	Knowledge transfer	M1	M22
5	Grant application	M12	M22

3.1.3 List of deliverables

The project is of theoretical nature in quantum Shannon theory, and therefore the main deliverables will be publications in peer-reviewed journals and conference presentations. We remark that in this field very few conferences have published proceedings, and papers in such proceedings are generally considered to be of lower value than papers in well-established journals of the field, and therefore we will primarily aim at publishing our results in the latter form.

Deliverable number	Deliverable	Work package number	Type	Access and dissemination	Method of verification	Delivery (project implementation month)
D1	Research paper(s) on super-exponential state discrimination	WP1	Publication	P	Peer review	M11
D2	Conference presentation(s) on super-exponential state discrimination	WP1	Presentation	P	Conference website(s), recording(s)	M11
D3	Research paper(s) on multi-variate Rényi divergences	WP2	Publication	P	Peer review	M11
D4	Conference presentation(s) on multi-variate Rényi divergences	WP2	Presentation	P	Conference website(s), recording(s)	M11
D5	Research paper(s) on composite state discrimination	WP3	Publication	P	Peer review	M11
D6	Conference presentation(s) on composite state discrimination	WP3	Presentation	P	Conference website(s), recording(s)	M11
D7	Seminar talks by invited researchers	WP4	Presentation	N	Institute website, recording(s)	M11
D8	Workshop at the host institute	WP4	Presentation	N	Institute website, recordings.	M11
D9	Interim report	WP1–WP4	Report	N	Progress evaluation	M11
D10	Research paper(s) on super-exponential state discrimination	WP1	Publication	P	Peer review	M22
D11	Conference presentation(s) on super-exponential state discrimination	WP1	Presentation	P	Conference website(s), recording(s)	M22
D12	Research paper(s) on multi-variate Rényi divergences	WP2	Publication	P	Peer review	M22
D13	Conference presentation(s) on multi-variate Rényi divergences	WP2	Presentation	P	Conference website(s), recording(s)	M22
D14	Research paper(s) on composite state discrimination	WP3	Publication	P	Peer review	M22
D15	Conference presentation(s) on composite state discrimination	WP3	Presentation	P	Conference website(s), recording(s)	M22
D16	Seminar talks by invited researchers	WP4	Presentation	N	Institute website, recording(s)	M22
D17	Workshop at the host institute	WP4	Presentation	N	Institute website, recordings.	M22
D18	Grant application	WP5	Grant application	N	Submission confirmation	M22
D19	Final report	WP1–WP4	Report	N	Final evaluation	M22

3.1.4 List of milestones

Milestone number	Milestone	Work package number	Method of verification	Expected time to reach the milestone (project month)
M1	Construction of new examples for super-exponential state discrimination.	WP1	Publication	11
M2	Identifying the multi-variate extensions of the Petz-type and the sandwiched Rényi divergences.	WP2	Publication	11
M3	Establishing optimal lower bounds on the Chernoff and Hoeffding exponents in composite state discrimination.	WP3	Publication	11
M4	Exact trade-off relations for super-exponential state discrimination of quasi-free sates.	WP1	Publication	20
M5	Exact trade-off relations for quantum state exclusion.	WP2	Publication	20
M6	Proof of the strong converse property for the composite Stein's lemma.	WP3	Publication	20
M7	Submission of a grant proposal for a research group.	WP5	Confirmation of submission	22

3.2 IMPLEMENTATION RISKS AND PROPOSED MEASURES

The two main risks of realistic chance is that we get stuck with one or more of the objectives, or that someone else happens to work on the same problems, and arrives at some of our expected results before us. Regarding the first risk, it is first of all important to note that the different objectives can be pursued in parallel, and therefore it is unlikely that the planned research gets stuck completely at any given time. If progress with some of the objectives is temporarily hindered and we don't see how to proceed to fully achieve the objectives then first of all we will consider relaxations of the problems, e.g., consider multi-variate of composite problems assuming that some of the states commute, and second, we will seek collaboration on those particular problems with other researchers who may contribute to the research with complementary expertise to ours. Given that the PI is very well connected with other researchers in the quantum Shannon theory research community, finding such collaborations should be easy. Regarding the second main risk, it is important to note that tackling the problems we set out to study requires a very special set of expertise, and therefore the number of researchers who may be working on exactly the same problems is expected to be very low. Our strategy to decrease this risk will be to maintain regular communication with other researchers with related interest (e.g., via mutual visits, as outlined in the work packages), and seek collaboration when our research seems to be heading the same direction.

3.2.1 Risks of implementation

Description of the risk of implementation	Probability of Risk	Severity of Risk	Work package	Proposed measures for risk mitigation or elimination
Getting stuck with one of the objectives.	Medium.	Medium.	WP 1-3	Consider relaxations, bring in complementary expertise via collaborations.
Competition by other research groups.	Medium.	Medium.	WP 1-3	Maintain communication with potential competitors, seek collaborations.
Travel restrictions due to a new pandemic.	Low.	Low.	WP 4	Use online tools, move visits and workshops to restriction-free periods.

3.3 OPERATIONAL CAPACITY OF THE APPLICANT/HOST ORGANISATION

Research in mathematics requires very little material resources, and the host institute can provide everything that might be needed in physical infrastructure and in professional assistance with the grant management.

3.3.1 Description of the research/innovation infrastructure of the applicant/host organisation that is necessary for the implementation of the project

Name of infrastructure or equipment	Short description
Office equipment	A desk and a desktop computer, printer, quality hardware for video conferencing
DEVANA Supercomputer	The Slovak Academy of Sciences provides access to supercomputer DEVANA with available total performance about 800 Tflop/s (from 1.1. 2024)
Library services	Access to the mathematical library of MISAS and library services
Access to databases	The Slovak Academy of Sciences provides access to scientific databases such as the Web of Science, Scopus (from 1.1. 2024), Springerlink, ScienceDirect, Mathematical Reviews etc.

3.3.2 List of the five most important projects of the applicant/host organisation and their relevance to the proposed project (in the last 5 years)

Project name/identification	Programme/scheme/grant provider	Short description
Mathematical support of quantum technologies/ NFP313011T683	EU Operational Programme Research and Innovation/ ITMS-2014+	The main goal of the project is the stabilisation of a quality research team and to realize independent research in the area of quantum technologies. The goals are reached by research of mathematical structures and functions.
Probabilistic, Algebraic and Quantum Mechanical Methods of Uncertainty Determination/ APVV-20-0069	Slovak Research and Development Agency (APVV)	Investigation of uncertainty in quantum structures and elsewhere, by combination of methods of algebra, probability theory, functional analysis, category theory and fuzzy mathematics. Joint with STU.
Mathematical models of non-classical events and uncertainty/ VEGA 2/0142/20	Scientific Grant Agency of the Ministry of Education of the Slovak Republic and SAS (VEGA)	Mathematical models for quantum structures, quantum information theory and uncertainty. Joint with STU.
Designing quantum higher order structures/ APVV-22-0570	Slovak Research and Development Agency (APVV)	Theoretical study and design of higher order maps - quantum networks. Joint with Institute of Physics, SAS.
Probabilistic, Algebraic and Quantum-Mechanical Aspects of Uncertainty/ APVV-16-0073	Slovak Research and Development Agency (APVV)	The aim of the project is to obtain original research results concerning description of uncertainty related to quantum structures. We focus on the study of total and partial algebraic structures derived from mathematical foundations of quantum mechanics, as well as on the description of quantum states, channels and more general processes and their estimation and discrimination procedures.

3.3.3 List of maximum five most important outputs of the applicant/host organisation relevant to the submitted project

Output name/identification	Type of output (e.g., publication, dataset, software, patent, service, product, etc.)	Short description
A. Jenčová, Rényi relative entropies and noncommutative Lp-spaces II, Ann. Henri Poincaré 22, 3235-3254 (2021), https://doi.org/10.1007/s00023-021-01074-9	Publication (article)	In the setting of von Neumann algebras, sandwiched Rényi relative entropies for $1/2 < \alpha < 1$ are defined and their properties are studied using noncommutative Lp-spaces.
A. Jenčová, Rényi Relative Entropies and Noncommutative Lp-Spaces, Annales Henri Poincaré 19, 2513-2542 (2018), https://doi.org/10.1007/s00023-018-0683-5	Publication (article)	In the setting of von Neumann algebras, sandwiched Rényi relative entropies for $\alpha > 1$ are defined and their properties are studied using noncommutative Lp-spaces.
A. Jenčová, Reversibility conditions for quantum operations, Reviews in Mathematical Physics, 24(07), 1250016, 2012. https://doi.org/10.1142/S0129055X1250016X	Publication (article)	Sufficient (reversible) channels are characterized by preservation of various quantities, such as quasi-entropies, quantum Fisher information and the L1-distance.
A. Jenčová, Dénes Petz; Sufficiency in quantum statistical inference; Commun. Math. Phys. 263(2006), 259-276, https://doi.org/10.1007/s00220-005-1510-7	Publication (article)	A paper studying sufficient (reversible) channels and their characterizations in the setting of von Neumann algebras.
Dvurecenskij, Pulmannová - New Trends in Quantum Structures, Dordrecht : Kluwer Academic Publishers; Bratislava : Ister Science, 2000. 541+xvi pp. https://doi.org/10.1007/978-94-017-2422-7. ISBN 0-7923-6471-6	Publication (book)	A fundamental and widely cited monograph describing structures arising in the mathematical description of quantum theory and related areas.

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