



# Incompatibility in General Probabilistic Theories, Generalized Spectrahedra, and Tensor Norms

Andreas Bluhm<sup>1</sup>, Anna Jenčová<sup>2</sup>, Ion Nechita<sup>3</sup>

<sup>1</sup> QMATH, Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, 2100 Copenhagen, Denmark. E-mail: [bluhm@math.ku.dk](mailto:bluhm@math.ku.dk)

<sup>2</sup> Mathematical Institute, Slovak Academy of Sciences, Bratislava, Slovakia. E-mail: [jenca@mat.savba.sk](mailto:jenca@mat.savba.sk)

<sup>3</sup> Laboratoire de Physique Théorique, Université de Toulouse, CNRS, UPS, Toulouse, France.  
 E-mail: [nechita@irsamc.ups-tlse.fr](mailto:nechita@irsamc.ups-tlse.fr)

Received: 22 March 2021 / Accepted: 8 March 2022

Published online: 21 June 2022 – © The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2022

**Abstract:** In this work, we investigate measurement incompatibility in general probabilistic theories (GPTs). We show several equivalent characterizations of compatible measurements. The first is in terms of the positivity of associated maps. The second relates compatibility to the inclusion of certain generalized spectrahedra. For this, we extend the theory of free spectrahedra to ordered vector spaces. The third characterization connects the compatibility of dichotomic measurements to the ratio of tensor crossnorms of Banach spaces. We use these characterizations to study the amount of incompatibility present in different GPTs, i.e. their compatibility regions. For centrally symmetric GPTs, we show that the compatibility degree is given as the ratio of the injective and the projective norm of the tensor product of associated Banach spaces. This allows us to completely characterize the compatibility regions of several GPTs, and to obtain optimal universal bounds on the compatibility degree in terms of the 1-summing constants of the associated Banach spaces. Moreover, we find new bounds on the maximal incompatibility present in more than three qubit measurements.

## Contents

1. Introduction . . . . .	1126
2. Main Results . . . . .	1128
3. Notation . . . . .	1133
4. General Probabilistic Theories . . . . .	1134
4.1 Definitions . . . . .	1134
4.2 Base norms and order unit norms . . . . .	1135
4.3 Centrally symmetric GPTs . . . . .	1136
4.4 Measurements and compatibility . . . . .	1137
5. Compatible Measurements and Positive Maps . . . . .	1139
6. Inclusion of Generalized Spectrahedra and Compatibility of Measurements in GPTs . . . . .	1144

6.1	The GPT jewel	1144
6.2	Spectrahedral inclusion and compatibility	1148
6.3	Compatibility region and inclusion constants	1150
6.4	Inclusion constants from symmetrization	1151
7.	Tensor Crossnorms and Compatibility of Effects	1153
7.1	$E_g$ as a centrally symmetric GPT	1153
7.2	Effects and tensor crossnorms	1154
7.3	Incompatibility witnesses	1158
8.	Compatibility of Effects in Centrally Symmetric GPTs	1161
9.	Examples of Compatibility Regions	1166
9.1	Hypercubes	1166
9.2	Euclidean balls	1168
9.3	Cross polytopes	1169
9.4	Relation to 1-summing norms	1170
9.5	Upper bounds for centrally symmetric GPTs	1171
10.	Discussion of Results	1172
10.1	Centrally symmetric GPTs	1172
10.1.1	Hypercubes	1172
10.1.2	Euclidean balls	1172
10.1.3	Cross polytopes	1174
10.2	General centrally symmetric GPTs	1175
10.3	Other GPTs	1176
10.3.1	Quantum mechanics	1176
10.3.2	General bounds	1177
	Acknowledgments	1177
	Appendix A: Necessary Background	1177
	A.1. Convex cones	1177
	A.2. Tensor products of cones	1178
	A.3. Positive maps	1178
	A.4. Tensor products of Banach spaces	1181
	Appendix B: An Extension Theorem for Ordered Vector Spaces	1183
	Appendix C: Generalized Spectrahedra	1185
	Appendix D: Basic Results on Compatibility Regions and Degree	1191
	Appendix E: Conic Programming for Compatibility in GPTs	1193
	E.1. Background	1193
	E.2. Map extension	1194
	E.3. Computing $\  \cdot \ _c$	1196
	References	1196

## 1. Introduction

Many of the phenomena that distinguish quantum mechanics from a classical theory can be traced back to the incompatibility of measurements [Hei27, Boh28]. Two measurements are incompatible if there does not exist a third one which implements them simultaneously. The position and momentum observables are typical examples of this behavior. This non-classicality present in a collection of measurements is necessary for many tasks in quantum information theory, because compatible measurements cannot

exhibit non-locality in terms of violation of a Bell inequality [Fin82, BCP+14] or steering [UBGP15]. In this sense, incompatibility is a resource for quantum processing tasks similar to entanglement [HKR15].

It is thus natural to ask how much of this resource is available in a given situation. As noise can destroy incompatibility in the same way as it can destroy entanglement, *noise robustness* is a natural way to quantify incompatibility [DFK19]. While many works have investigated noise robustness for concrete measurements (see [HMZ16] for a review), one can ask this question more generally, leading to the *compatibility regions* studied in [BHSS13, HSTZ14, BN18, BN20]:

*How much incompatibility can be found for a quantum system of a certain dimension, a certain number of measurements and a certain number of measurement outcomes?*

In fact, the incompatibility of measurements is not restricted to quantum mechanics, but is present in all non-classical theories [Plá16]. These theories can conveniently be described in the framework of *general probabilistic theories* (GPTs). We refer the reader to [Lam18] for a good introduction. An example of a GPT is a theory which has a state space isomorphic to a square (see e.g. [BHSS13, JP17, Jen18] or Example 4.7 for  $n = 2$ ). Such a GPT can be used to model a theory containing PR boxes [PR94] which maximally violate the CHSH inequality. One of the motivation behind the study of GPTs is to characterize quantum mechanics among the probabilistic physical theories (see [BBLW06, BBLW07, WPGF09, SB14] for some examples). Thus, it is interesting to study measurement incompatibility and compatibility regions not only in quantum mechanics but in GPTs [BHSS13, Gud13, JP17, Jen18].

The main theme of this work is to provide several equivalent characterizations of compatibility in GPTs, from different perspectives, and using different mathematical theories. We combine ideas from [Jen18] on incompatibility in GPTs with ideas from [BN18, BN20] on (free) spectrahedra and we relate them to the theory of tensor norms in Banach spaces. We characterize compatibility of GPT measurements in five different ways (see Table 1), unearthing powerful connections between the underlying mathematical theories. Our first main contribution is thus *conceptual*, placing the measurement compatibility problem at the intersection of functional analysis, free convexity theory, and Banach space theory. We expect that this realization will help identifying other points of contact between these fields of mathematics and quantum information theory, allowing for progress on both fronts. Our second main contribution is *applicative*: in the second part of the paper, we exploit these new connections and use results for tensor norms from Banach space theory to new bounds on the robustness of compatibility in GPTs, by relating ratios of different tensor norms to amount of noise needed to break all forms of incompatibility of dichotomic GPT measurements. From this perspective, our work pursues a similar approach to [ALP+20], where similar ideas were used in the setting of XOR non-local games.

In Sect. 2, we start with an overview of the main results of this work. Subsequently, we introduce the basic notation in Sect. 3. In Sect. 4, we present a thorough discussion of GPTs. In the next sections, we find several equivalent characterization of measurement compatibility. First, we make the connection between compatibility of measurements and positivity of associated maps in Sect. 5. The findings in this section complement the connection to entanglement breaking maps in [Jen18]. Sect. 6 connects compatibility of measurements to the inclusion of certain generalized spectrahedra, thereby extending the results in [BN18, BN20]. After this section, we focus primarily on dichotomic measurements. Sect. 7 connects compatibility of measurements to norms on their associated tensors. In particular, Sect. 7.3 introduces a form of incompatibility witnesses

and connects them to related notions in [Jen18,BN20]. In Sect. 8, we focus on GPTs with symmetric state space, the so-called centrally symmetric GPTs. For this class, the connection to tensor norms has a particularly nice form. In particular, the noise robustness of incompatible measurements is connected to ratios of injective and projective tensor norms of certain Banach spaces. This connection to tensor norms allows us to find in Sect. 9 concrete bounds on the compatibility regions of certain GPTs of interest, in several cases characterizing them completely; on the way, we expose a relation to 1-summing norms. We collect the results concerning compatibility regions in Sect. 10 and compare them to previously known bounds in the literature.

This is complemented by the appendices, in which we review the necessary background and prove some technical results. Appendix A contains a review of results on convex cones (Appendices A.1 and A.2), positive maps (Appendix A.3) and Banach spaces (Appendix A.4). Moreover, Appendix B presents technical results on the extension of positive maps in certain spaces, which is important for the rest of this work. Appendix C extends the theory of free spectrahedra, which was developed in [HKM13,DDOSS17,HKMS19] among others, to ordered vector spaces. These first few appendices therefore lay the technical groundwork for the rest of the paper. Appendix D complements Sect. 4 by giving additional basic results relating different compatibility regions. Finally, we present in Appendix E some conic programs to compute quantities from the main text.

## 2. Main Results

The aim of this work is to study measurement compatibility in general probabilistic theories (GPTs). The GPTs provide a framework for the study of physical theories permitting probabilistic mixtures. Important examples of GPTs are classical probability theory, quantum mechanics and the GPT of quantum channels.

Any GPT is built on basic operational notions of *states* (or preparation procedures) and *effects* (or dichotomic measurements) of the theory, which are identified with certain positive elements in a pair of dual ordered vector spaces,  $(V, V^+)$  and  $(A, A^+)$ , respectively. The space of effects,  $(A, A^+)$ , contains a distinguished order unit  $\mathbb{1}$ , corresponding to the trace in quantum mechanics. Elements of the set  $K := \{v \in V : \mathbb{1}(v) = 1\}$  represent states of the theory and elements  $f \in A$  for which  $0 \leq f \leq \mathbb{1}$  correspond to effects. Composite systems are described by tensor products of the corresponding ordered vector spaces. Note that the definition of such a tensor product is not unique and depends on the theory in question. Nevertheless, there is a minimal and maximal tensor product cone, and the cone describing composite systems in any theory must lie between these two. In particular, states in the minimal cone are called *separable*, whereas all the other states are *entangled*.

We will present the framework of GPTs in more details in Sect. 4. Throughout this work, we assume that the space  $V$  (and of course also  $A$ ) is finite dimensional and we will study finite tuples of measurements with a finite set of outcomes, labelled by the set  $[k]$ . Such measurements are represented by  $k$ -tuples of effects  $f_1, \dots, f_k$  such that  $\sum_i f_i = \mathbb{1}$ .

This work is concerned with the question of *compatibility* of a given set of measurements, that is, the existence of a joint measurement  $h$  for a given  $g$ -tuple of measurements  $f = (f^{(1)}, \dots, f^{(g)})$ , such that all  $f^{(i)}$  are marginals of  $h$  (Definition 4.9): if the outcome set of each measurement is specified by the  $g$ -tuple  $\mathbf{k} = (k_1, \dots, k_g) \in \mathbb{N}^g$ , then

**Table 1.** An overview of the five-fold characterization of compatibility of GPT measurements

Equivalent formulations of compatibility in GPTs		
<i>Input:</i> a $g$ -tuple of measurements $(f^{(1)}, \dots, f^{(g)}) \in A^{\mathbf{k}} \iff$ a linear map $\Phi^{(f)} \iff$ a tensor $\phi^{(f)}$		
Formulation	Result	Keywords and Methods
(a) $\Phi^{(f)}$ has a pos. ext. $(\mathbb{R}^{\mathbf{k}}, \mathbb{R}_+^{\mathbf{k}}) \rightarrow (A, A^+)$	Theorem 5.3	Functional analysis, map extension, entanglement breaking, polysimplex
(b) $\Phi^{(f)}$ is entanglement breaking		
(c) $\varphi^{(f)} \in \text{Ran}(J_{\mathbf{k}} \otimes \text{id})$		
(d) $\mathcal{D}_{\text{GPT}}(\mathbf{k}; V, V^+) \subseteq \mathcal{D}_f(\mathbf{k}; V, V^+)$	Theorem 6.11	Free convexity, generalized free spectrahedron
(e) $\ \bar{\varphi}^{(f)}\ _{\rho} \leq 1$ (dichotomic measurements)	Theorem 7.2	Banach space, tensor norm

$h$  has outcomes in  $[k_1] \times \dots [k_g]$  and

$$\forall i \in [g], \forall j \in [k_i], \quad f_j^{(i)} = \sum_{m_1, \dots, m_{i-1}, m_{i+1}, \dots, m_g} h_{m_1, \dots, m_{i-1}, j, m_{i+1}, \dots, m_g}.$$

The main insight of this paper is a five-fold characterization of compatibility, from three different perspectives. We summarize the different characterizations of compatibility in Table 1, and we detail the points (a)–(e) below.

The basis for all these characterization is a construction of a GPT with state space isomorphic to the polysimplex  $\Pi_{i=1}^g \Delta_{k_i}$  (Sect. 5). It is an easy observation that the polysimplex is isomorphic to a subset in  $\mathbb{R}^{\mathbf{k}} := \mathbb{R}^{k_1 \cdots k_g}$ , obtained as  $J_{\mathbf{k}}(\Delta_{\mathbf{k}})$ , where  $\Delta_{\mathbf{k}}$  is the standard simplex in  $\mathbb{R}^{\mathbf{k}}$  and  $J_{\mathbf{k}}$  is an orthogonal projection whose action on  $\Delta_{\mathbf{k}}$  can be interpreted as “mapping all elements of the simplex to its marginals”. The space of effects for this GPT is then identified with the subspace  $E_{\mathbf{k}} \subseteq \mathbb{R}^{\mathbf{k}}$  which is the range of  $J_{\mathbf{k}}$ , endowed with the positive cone  $E_{\mathbf{k}}^+ = E_{\mathbf{k}} \cap \mathbb{R}_+^{\mathbf{k}}$  inherited from the simplicial cone  $\mathbb{R}_+^{\mathbf{k}}$  in  $\mathbb{R}^{\mathbf{k}}$ . We identify the dual space as  $E_{\mathbf{k}}^* = E_{\mathbf{k}}$ , the dual cone then becomes the projection of the simplicial cone  $(E_{\mathbf{k}}^+)^* = J_{\mathbf{k}}(\mathbb{R}_+^{\mathbf{k}})$ .

To a given  $g$ -tuple of measurements  $f = (f^{(1)}, \dots, f^{(g)})$  with outcomes determined by  $\mathbf{k}$ , we then associate a linear map  $\Phi^{(f)} : E_{\mathbf{k}} \rightarrow A$ . By the standard identification of linear maps with elements in the tensor products, there is a related vector  $\varphi^{(f)} \in E_{\mathbf{k}}^* \otimes A$ . It is then shown that such a map is positive if and only if it corresponds to a  $g$ -tuple of measurements, equivalently,  $\varphi^{(f)} \in (E_{\mathbf{k}}^+)^* \otimes_{\max} A^+$ .

As our first main result, we show in Theorem 5.3 that compatibility of  $f$  is characterized by either of the following equivalent conditions:

- (a)  $\Phi^{(f)}$  has a positive extension to a map  $(\mathbb{R}^{\mathbf{k}}, \mathbb{R}_+^{\mathbf{k}}) \rightarrow (A, A^+)$ ;
- (b)  $\Phi^{(f)}$  is *entanglement breaking*, in very much the same sense as the entanglement breaking channels in quantum theory;
- (c)  $\varphi^{(f)} \in (J_{\mathbf{k}} \otimes \text{id})(\mathbb{R}_+^{\mathbf{k}} \otimes A^+)$ .

These conditions are closely related to the results of [Jen18]. Note that the ordered vector space  $(E_{\mathbf{k}}, E_{\mathbf{k}}^+)$  plays a universal role, since it only depends on the number of measurements and the numbers of their outcomes, but not on the GPT under study.

The next part of the paper is motivated by the effort to extend the results of [BN18] and [BN20] to GPTs. For this we need some generalization of the free spectrahedra considered e.g. in [HKM13, DDOSS17, HKMS19]. A *generalized spectrahedron* as defined in Appendix C is determined by a tuple  $(a_1, \dots, a_g) \in M^g$ , where  $(M, M^+)$  is an

ordered vector space. Given another ordered vector space  $(L, L^+)$  and a tensor cone  $C$  in the tensor product  $M \otimes L$ , the generalized spectrahedron is

$$\mathcal{D}_a(L, C) := \left\{ (v_1, \dots, v_g) \in L^g : \sum_{i=1}^g a_i \otimes v_i \in C \right\}.$$

For a usual free spectrahedron,  $M^+$ ,  $L^+$ , and  $C$  are the cones of positive semidefinite matrices of different dimensions. Note that the generalized spectrahedron can be seen as a representation of the cone  $C$  in the space  $L^g \cong \mathbb{R}^g \otimes L$  and if  $a$  is a basis of  $M$ , it is isomorphic to  $C$ . In Sect. 6.1 we introduce the generalized spectrahedron  $\mathcal{D}_{\text{GPT}\diamond}(\mathbf{k}; V, V^+)$ , called the *GPT jewel*, given by  $(M, M^+) = (E_{\mathbf{k}}, E_{\mathbf{k}}^+)$ ,  $a = w$  is a certain fixed basis of  $E_{\mathbf{k}}$  (see the definition on p. 14) and the maximal tensor cone. It is a universal object corresponding to the matrix jewel defined in [BN20] for quantum mechanics. We also define the generalized spectrahedron  $\mathcal{D}_f(\mathbf{k}; V, V^+)$  determined by the measurements under study (Definition 6.8).

The main result of this part is the following equivalent condition for compatibility of  $f$ , see Theorem 6.11:

$$(d) \quad \mathcal{D}_{\text{GPT}\diamond}(\mathbf{k}; V, V^+) \subseteq \mathcal{D}_f(\mathbf{k}; V, V^+).$$

This shows that the conditions (a)–(c) can be expressed by inclusion of generalized spectrahedra and gives a connection between the results of [BN18, BN20] and [Jen18].

Another characterization is obtained if we restrict to dichotomic measurements ( $k_1 = \dots = k_g = 2$ ). We use the symmetry of the cone  $(E_g^+)^*$ , where  $E_g := E_{(2,2,\dots,2)}$ , and the decomposition  $E_g = \mathbb{R}1_{2^g} \oplus \ell_\infty^g$  to relate compatibility of effects to *reasonable crossnorms* on the tensor product  $\ell_\infty^g \otimes A$  (where we always endow  $A$  with its order unit norm). Namely, for any  $f = (f_1, \dots, f_g) \in A^g$  we put

$$\bar{\varphi}^{(f)} := \varphi^{(f)} - 2^{-g} 1_{2^g} \otimes \mathbb{1} \cong \sum_i e_i \otimes (2f_i - \mathbb{1}) \in \ell_\infty^g \otimes A.$$

Then  $g$ -tuples of effects are determined by the condition  $\|\bar{\varphi}^{(f)}\|_\varepsilon \leq 1$ , where  $\|\cdot\|_\varepsilon$  is the injective crossnorm in  $\ell_\infty^g \otimes A$ . As another main result, we find a reasonable crossnorm  $\|\cdot\|_c$  in  $\ell_\infty^g \otimes A$ , such that compatibility of a  $g$ -tuple  $f$  of dichotomic measurements is equivalent to

$$(e) \quad \|\bar{\varphi}^{(f)}\|_c \leq 1,$$

see Theorem 7.2. These different viewpoints will be helpful in determining the amount of incompatibility available in a GPT. We also show in the appendices that the conditions (a) and (e) can be checked by conic programs.

Any tuple of measurements  $(f^{(1)}, \dots, f^{(g)})$  can be made compatible if one adds enough noise. The amount of noise needed for this is the basis for the commonly used definitions of compatibility regions/degrees. In this work, we will consider *white noise*, i.e. the effects of the noisy measurements are of the form

$$\tilde{f}_j^{(i)} = s_i f_j^{(i)} + (1 - s_i) \frac{\mathbb{1}}{k_i}. \quad (1)$$

Here,  $k_i$  is the number of outcomes of the  $i$ -th measurement and  $s_i \in [0, 1]$  is a parameter quantifying the amount of noise added to the  $i$ -th measurement. In other words, the device corresponding to the noisy measurement  $\tilde{f}^{(i)}$  carries out the measurement  $f^{(i)}$

with probability  $s_i$ . With probability  $(1 - s_i)$ , the device outputs a random number in  $\{1, \dots, k_i\}$  where each outcome has equal probability. The minimal amount of noise such that the measurements become compatible is a measure of their incompatibility. Therefore, we are interested in the set  $\Gamma(f) \subseteq [0, 1]^g$  of  $s_1, \dots, s_g$  such that the noisy measurements  $\tilde{f}^{(i)}$  in Eq. (1) are compatible. The set of  $s \in [0, 1]^g$  such that this is true for *any* collection of  $g$  measurements with outcome sets given by  $\mathbf{k}$  will be called the *compatibility region* for the GPT and denoted by  $\Gamma(\mathbf{k}; V, V^+) := \bigcap_f \Gamma(f)$ . The compatibility region is a measure of the amount of incompatibility available in the GPT.

The largest  $s \in [0, 1]$  such that  $(s, \dots, s) \in \Gamma(f)$  is the compatibility degree of  $f$ , denoted by  $\gamma(f)$ . This corresponds to the situation when the same amount of noise is added to each measurement in  $f$ . The value such that this is true for all collections of measurements with outcomes given by  $\mathbf{k}$  is the *compatibility degree* of the GPT:

$$\gamma(\mathbf{k}; V, V^+) := \min_f \gamma(f) = \max\{s \in [0, 1], (s, \dots, s) \in \Gamma(\mathbf{k}, V, V^+)\}.$$

Consider the set of vectors  $s \in [0, 1]^g$  that can be used to scale the GPT jewel  $\mathcal{D}_{\text{GPT}\diamond}$  such that it is contained in  $\mathcal{D}_f(\mathbf{k}; V, V^+)$  for any collection of  $g$  measurements  $f^{(i)}$  with  $k_i$  outcomes. This set is the set of *inclusion constants* of  $\mathcal{D}_{\text{GPT}\diamond}$ , denoted by  $\Delta(\mathbf{k}; V, V^+)$ . Following along the lines of [BN18, BN20], we prove that the compatibility region of a GPT is equal to the set of inclusion constants of the corresponding jewel, see Theorem 6.18:

$$\Gamma(\mathbf{k}; V, V^+) = \Delta(\mathbf{k}; V, V^+).$$

We can get more results for dichotomic measurements. In this case we write  $\Gamma(g; V, V^+)$  for the compatibility region, similarly  $\gamma(g; V, V^+)$  for the compatibility degree. The GPT jewel in this case is called the *GPT diamond* and denoted by  $\mathcal{D}_{\text{GPT}\diamond}(g; V, V^+)$ . Using the compatibility characterization (e) we immediately obtain a direct relation of the compatibility measures to the norm  $\|\cdot\|_c$ , in particular,

$$\gamma(f) = 1/\|\bar{\varphi}^{(f)}\|_c.$$

From this relation, we find in Theorem 7.6 that we have

$$\Gamma(g; V, V^+) = \{s \in [0, 1]^g, \|s \cdot \varphi\|_c \leq 1, \forall \varphi \in \ell_\infty^g \otimes A, \|\varphi\|_\varepsilon \leq 1\}, \quad (2)$$

where for  $\varphi = \sum_i e_i \otimes \varphi_i$  we put  $s \cdot \varphi = \sum_i e_i \otimes s_i \varphi_i$ . In particular, the compatibility degree satisfies

$$\gamma(g; V, V^+) = 1/\max_{\|\varphi\|_\varepsilon \leq 1} \|\varphi\|_c \geq 1/\rho(\ell_\infty^g, A) \geq 1/\min\{g, \dim(V)\}, \quad (3)$$

where the quantity  $\rho(X, Y)$  for a pair of Banach spaces  $X, Y$  was introduced in [ALP+20, Eq. (15)]:

$$\rho(X, Y) = \max_{z \in X \otimes Y} \frac{\|z\|_{X \otimes_\pi Y}}{\|z\|_{X \otimes_\varepsilon Y}}.$$

A characterization which is dual to the one above can be obtained using *incompatibility witnesses*. In this work, an incompatibility witness is a tuple  $z = (z_1, \dots, z_g)$  such that there is a state  $z_0 \in K$  such that  $(z_0, z_1, \dots, z_g) \in \mathcal{D}_{\text{GPT}\diamond}(g; V, V^+)$  (Definition 7.11 and Proposition 7.13). The set of incompatibility witnesses is denoted  $\mathcal{P}_{\text{GPT}\diamond}(g; V, V^+)$



and we show that it can be identified with the unit ball of the dual norm to  $\|\cdot\|_c$ . For the relation of this definition of incompatibility witnesses to the definitions in [Jen18] and [BN20], see Propositions 7.13 and 7.14. An incompatibility witness  $z$  certifies incompatibility for some collection of effects if and only if we have  $\sum_i \|z_i\|_V$  strictly larger than one. Theorem 7.18 gives another characterization of the compatibility region as the set of vectors  $s \in [0, 1]^g$  which shrink all incompatibility witnesses such that they cannot detect incompatibility anymore:

$$\Gamma(g; V, V^+) = \left\{ (s_1, \dots, s_g) \in [0, 1]^g : \sum_{i=1}^g s_i \|z_i\|_V \leq 1 \ \forall (z_1, \dots, z_g) \in \mathcal{P}_{\text{GPT}\diamond}(g; V, V^+) \right\}.$$

Let us remark that there is another notion of an incompatibility witness for quantum measurements, which has an operational interpretation in terms of success probabilities in certain state discrimination tasks, [CHT19]. Such witnesses, among other concepts, were extended to GPTs in [Kur20] (even in infinite dimensions). The connection to the witnesses considered in this paper is left for future work.

Finally, we consider a special class of GPTs for which our results have a simpler form: the *centrally symmetric* GPTs. The state spaces of these GPTs are the unit balls of some norm on a vector space  $\bar{V}$  and we have  $V \cong \mathbb{R} \oplus \bar{V}$ , see Sect. 4.3 for more details. Important examples are the Bloch ball describing 2-level systems in quantum mechanics, or the hypercubic GPT. In Theorem 8.5, we show that for this class of GPTs, we can replace in Eq. (2) the space  $A$  by  $\bar{A}$  and the norm  $\|\cdot\|_c$  by the projective norm in  $\ell_\infty^g \otimes_\pi \bar{A}$ , where  $\bar{A}$  is the dual Banach space to  $\bar{V}$ :

$$\begin{aligned} \Gamma(g; V, V^+) &= \{s \in [0, 1]^g : \|s \cdot \bar{z}\|_{\ell_1^g \otimes_\pi \bar{V}} \leq 1, \ \forall \|\bar{z}\|_{\ell_1^g \otimes_\varepsilon \bar{V}} \leq 1\} \\ &= \{s \in [0, 1]^g : \|s \cdot \bar{\varphi}\|_{\ell_\infty^g \otimes_\pi \bar{A}} \leq 1, \ \forall \|\bar{\varphi}\|_{\ell_\infty^g \otimes_\varepsilon \bar{A}} \leq 1\}. \end{aligned}$$

In particular, the compatibility degree is

$$\gamma(g; V, V^+) = 1/\rho(\ell_\infty^g, \bar{A}) \geq 1/\min\{g, \dim(\bar{A})\}.$$

This allows us to exactly compute or at least to find some new bounds for the compatibility degree for some cases of centrally symmetric GPTs (Sects. 9 and 10). In particular, the lower bound is attained by the hypercubic GPT (Corollary 9.2). Note that this bound is larger than the lower bound in Eq. (3). The tightness of the general lower bound  $\max\{1/g, 1/\dim(V)\}$  remains an open question.

To get a better understanding of our results, let us consider compatibility for a pair of qubit effects. Qubits are described by a centrally symmetric GPT, where the corresponding normed space is  $\ell_2^3$ . Each qubit effect has the form

$$f = \frac{1}{2}(\alpha I + \mathbf{a} \cdot \sigma),$$

where  $\sigma = (\sigma_1, \sigma_2, \sigma_3)$  are Pauli matrices and  $\mathbf{a} \in \mathbb{R}^3$  is a vector such that  $\|\mathbf{a}\|_2 \leq \alpha \leq 2 - \|\mathbf{a}\|_2$ . The effects are called *unbiased* if  $\alpha = 1$ . For a pair of unbiased effects  $f_1 = \frac{1}{2}(I + \mathbf{a} \cdot \sigma)$  and  $f_2 = \frac{1}{2}(I + \mathbf{b} \cdot \sigma)$ , we have

$$\|\bar{\varphi}^{(f)}\|_c = \|e_1 \otimes \mathbf{a} + e_2 \otimes \mathbf{b}\|_{\ell_\infty^2 \otimes_\pi \ell_2^3} = \frac{1}{2}(\|\mathbf{a} + \mathbf{b}\|_2 + \|\mathbf{a} - \mathbf{b}\|_2),$$



(see Sect. 9.2) so that the condition (e) becomes the well known compatibility condition for two unbiased qubit effects obtained in [Bus86, BH08]. This condition was extended for pairs of unbiased effects in some centrally symmetric GPTs (with regular even sided polygons as state spaces) in [Kob17]. Our results generalize the condition to  $g$ -tuples of arbitrary effect in any GPT.

We also put forward a connection to 1-summing norms:

$$\lim_{g \rightarrow \infty} \gamma(g; V, V^+) = \frac{1}{\pi_1(\bar{V})},$$

where  $\pi_1$  is the 1-summing norm of the Banach space  $(\bar{V}, \|\cdot\|_{\bar{V}})$  (Sect. 9.4). This relation allows us to prove new lower bounds for the compatibility degree of qubits, which are described by the Bloch ball, corresponding to  $\bar{V} = \ell_2^3$  (see Eq. (41)).

### 3. Notation

In this section, we will only fix the notation for the different objects we will use in this paper and refer the reader for background and technical results on some of the tools we will use to the appendices.

In this paper, we will assume all vector spaces to be finite dimensional and over the real field. For brevity, we will often write  $[n] := \{1, \dots, n\}$  for  $n \in \mathbb{N}$ . Let  $\mathcal{M}_n(\mathbb{C})$  denote the  $n \times n$  matrices with complex entries. The real vector space of Hermitian matrices will be written as  $\mathcal{M}_n^{\text{sa}}(\mathbb{C})$ . For a vector space  $L$  and an element  $a \in L^g \cong L \times \dots \times L$ ,  $g \in \mathbb{N}$ , we will write  $a_i, i \in [g]$ , for its components without specifying this in advance, if no confusion can arise. For a convex set  $K \subseteq \mathbb{R}^g$ , let us write

$$K^\circ := \{x \in \mathbb{R}^g : \langle x, k \rangle \leq 1 \ \forall k \in K\}$$

for the polar set of  $K$ . Moreover, let us write for the direct sum of two convex sets  $K_i \in \mathbb{R}^{g_i}, g_i \in \mathbb{N}, i \in [2]$ ,

$$K_1 \oplus K_2 := \text{conv}\{(x, 0); (0, y) : x \in K_1, y \in K_2\} \subseteq \mathbb{R}^{g_1+g_2}. \quad (4)$$

If  $K_1, K_2$  are polytopes containing 0, it holds that  $K_1 \oplus K_2 = (K_1^\circ \times K_2^\circ)^\circ$  [Bre97, Lemma 2.4], where

$$K_1 \times K_2 := \{(x, y) : x \in K_1, y \in K_2\} \subseteq \mathbb{R}^{g_1+g_2} \quad (5)$$

is the cartesian product of the two convex sets. We will denote the probability simplex in  $\mathbb{R}^k$  (i.e. the probability distributions on  $k$  symbols) by  $\Delta_k$ . For a convex cone on  $L$ , we will write  $L^+$ . For simplicity, we will drop “convex” and talk simply about cones. Likewise, we will write  $(L^+)^*$  for the dual cone on the dual vector space  $L^*$ . For the (pre-)ordered vector space defined by the cone  $L^+$ , we will write  $(L, L^+)$ . If  $L^+$  is a proper cone, we will call speak of a *proper ordered vector space*. We refer the reader to Appendix A.1 for the terminology we will use regarding convex cones and for some basic results. Concerning tensor products of cones  $L_1^+, L_2^+$ , we will write  $L_1^+ \otimes_{\min} L_2^+$  for the minimal and  $L_1^+ \otimes_{\max} L_2^+$  for the maximal tensor product (see Appendix A.2). Any cone between these two extreme cases we will call a *tensor cone* for  $L_1^+$  and  $L_2^+$ . It is well-known that linear maps  $\Phi : L_1 \mapsto L_2^*$  are in one-to-one correspondence with tensors  $\varphi^\Phi \in L_1^* \otimes L_2^*$  and linear forms  $s_\Phi : L_1 \otimes L_2 \mapsto \mathbb{R}$ . See Appendix A.3 for a recap and technical results concerning positive maps between (pre-)ordered vector spaces. We

**Table 2.** Notation table

Notation	Description	Introduced
<b>Convex geometry</b>		
$K$	Convex body	Sect. 3
$K_1 \oplus K_2$	Direct sum of convex sets	Eq. (4)
$K_1 \times K_2$	Cartesian product of convex sets	Eq. (5)
$L$	$\mathbb{R}$ -vector space	Sect. 3
$L^+$	Convex cone	Appendix A.1
$(L^+)^*$	dual cone	Eq. (42)
$\mathbb{1}$	Order unit	Appendix A.1
$L_1^+ \otimes_{\min} L_2^+$	Minimal tensor product of cones	Eq. (43)
$L_1^+ \otimes_{\max} L_2^+$	Maximal tensor product of cones	Eq. (44)
$\chi_L$	Evaluation tensor	Eqs. (45), (46)
<b>Banach spaces</b>		
$(X, \ \cdot\ _X)$	Banach space	Sect. 10.3.2
$\ \cdot\ _{X \otimes_{\pi} Y}$	Projective tensor norm	Definition A.11
$\ \cdot\ _{X \otimes_{\varepsilon} Y}$	Injective tensor norm	Definition A.12
$\ \cdot\ _c$	Compatibility norm	Eq. (25)
<b>GPTs</b>		
$(V, V^+, \mathbb{1})$	a GPT triple	Sect. 4.1
$(A, A^+, \mathbb{1})$	Cone of effects	Sect. 4.1
$\ \cdot\ _V, \ \cdot\ _A$	Base norm, resp. order unit norm	Definition 4.4
$\Gamma(\mathbf{k}; V, V^+)$	Compatibility region	Definition 4.10
$\gamma(\mathbf{k}; V, V^+)$	Compatibility degree	Definition 4.12
<b>Generalized spectrahedra</b>		
$P_{\mathbf{k}}$	The polysimplex	Eq. (6)
$\mathcal{D}_a(L, C)$	Generalized spectrahedron	Definition C.1
$\mathcal{D}_{\text{GPT}}(\mathbf{k}; L, L^+)$	the GPT jewel	Definition 6.1
$\mathcal{D}_{\text{GPT}\diamond}(g; L, L^+)$	the GPT diamond	Definition 6.1
$\mathcal{D}_f(\mathbf{k}; L, L^+)$	Spectrahedron associated to a measurement tuple	Definition 6.8
$\Delta_a(L, C_M, C_N)$	Inclusion constant set	Definition C.11

write  $\chi_L$  for the tensor associated to the identity map  $\text{id}_L : L \rightarrow L$ . Finally, we will use tensor products of Banach spaces. For Banach spaces  $(X, \|\cdot\|_X)$ ,  $(Y, \|\cdot\|_Y)$  we will write  $\|\cdot\|_{X \otimes_{\pi} Y}$  for their projective tensor norm and  $\|\cdot\|_{X \otimes_{\varepsilon} Y}$  for their injective tensor norm, sometimes abbreviated to  $\|\cdot\|_{\pi}$  and  $\|\cdot\|_{\varepsilon}$  (see Appendix A.4 for a short review). We write  $\rho(X, Y)$  for the maximal ratio between projective and injective norm (Eq. 53).

We gather in Table 2, for the convenience of the reader, the important notation used in this work, together with a reference to the exact place in the text where it is defined or where it first appears.

## 4. General Probabilistic Theories

In this section, we introduce the class of physical theories we are interested in, the so-called *general probabilistic theories* (GPTs). They form the framework that is used to describe states and measurement outcomes of arbitrary (physical) theories. It is within this generalized setting that we would like to study questions about the compatibility of measurements. In this brief introduction, we mostly follow the exposition in [Lam18].

**4.1. Definitions.** Any GPT corresponds to a triple  $(V, V^+, \mathbb{1})$ , where  $V$  is a vector space with a proper cone  $V^+$  and  $\mathbb{1}$  is an order unit in the dual cone  $A^+ = (V^+)^* \subset V^* = A$ . We assume here that  $V$  is finite dimensional. For a short review of the terminology used to describe convex cones and some basic results, see Appendix A.1. The set of states of

the system is identified as the subset

$$K := \{v \in V^+, \langle \mathbb{1}, v \rangle = 1\}.$$

Note that  $K$  is compact and convex and is a base of the cone  $V^+$ .

*Example 4.1.* Any classical system is described by the triple  $\text{CM}_d := (\mathbb{R}^d, \mathbb{R}_+^d, 1_d)$ ,  $d \in \mathbb{N}$ , where  $\mathbb{R}_+^d$  denotes the set of elements with non-negative coordinates and  $1_d = (1, 1, \dots, 1) = \sum_i e_i \in \mathbb{R}^d$ ; here  $e_1, \dots, e_d$  denotes the standard basis on  $\mathbb{R}^d$ . Then  $(\mathbb{R}^d)^* = \mathbb{R}^d$  with duality given by the standard inner product and the simplicial cone  $\mathbb{R}_+^d$  is self-dual. The classical state space is the probability simplex

$$\Delta_d = \left\{ x = (x_1, \dots, x_d) \in \mathbb{R}^d, x_i \geq 0, \sum_{i=1}^d x_i = 1 \right\} = \{x \in \mathbb{R}_+^d, \langle x, 1_d \rangle = 1\}.$$

*Example 4.2.* Quantum mechanics corresponds the triple  $\text{QM}_d := (\mathcal{M}_d^{\text{sa}}(\mathbb{C}), \text{PSD}_d, \text{Tr})$ ,  $d \in \mathbb{N}$ , where  $\text{PSD}_d$  is the cone of  $d \times d$  positive semidefinite complex, self-adjoint matrices, and  $\text{Tr}$  is the usual, un-normalized, trace. As in the case of classical systems described above, the  $\text{PSD}_d$  cone is self-dual. The quantum state space is the set of density matrices

$$\mathcal{S}_d := \{\rho \in \mathcal{M}_d^{\text{sa}}(\mathbb{C}) : \rho \geq 0, \text{Tr } \rho = 1\}.$$

*Example 4.3.* The hypercube GPT has the unit ball of  $\ell_\infty^n$  as a state space,  $n \in \mathbb{N}$ . It is described by the triple  $\text{HC}_n := (\mathbb{R}^{n+1}, C_n, \mathbb{1})$ , where

$$C_n = \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} : x_0 \geq \max_{i \in [n]} |x_i|\} \quad \text{and} \quad \mathbb{1}(x_0, x_1, \dots, x_n) = x_0.$$

Note also that any compact convex subset in a (finite-dimensional) vector space can be represented as a base of some proper cone. Indeed, let  $K$  be such a set and let  $A = A(K)$  be the set of affine functions  $K \rightarrow \mathbb{R}$ . Then  $A$  is a finite dimensional vector space and the subset  $A^+ = A(K)^+$  is a proper cone in  $A$ . Let  $\mathbb{1} = 1_K$  be the constant function, then  $\mathbb{1}$  is an order unit. Put  $V = A^*$ ,  $V^+ = (A^+)^*$ , then  $V^+$  is a proper cone in  $V$  and  $K$  is affinely isomorphic to the base of  $V^+$ , determined by  $\mathbb{1}$ .

**4.2. Base norms and order unit norms.** In GPTs, we have natural norms induced by the cones in the state space and the space of effects. They make the space  $V$  into a base norm space and the space  $A$  into an order unit space. For more details about the following, see the excellent [Lam18, Chapter 1.6].

**Definition 4.4.** Given a GPT  $(V, V^+, \mathbb{1})$ , define the following norm on  $V$ , called a *base norm*

$$\|x\|_V = \inf\{\mathbb{1}(y) + \mathbb{1}(z) : y, z \in V^+ \text{ s.t. } x = y - z\}$$

as well as a norm on  $A := V^*$ , called an *order unit norm*

$$\|\alpha\|_A = \inf\{t \geq 0 : \alpha \in t[-\mathbb{1}, \mathbb{1}]\}.$$

Using the above definition, one can characterize positivity in  $V$  using metric properties:

$$x \in V^+ \iff \|x\|_V = \mathbb{1}(x).$$

The base norm and the order unit norm are dual to each other, i.e.

$$\|x\|_V = \sup_{\|\alpha\|_A \leq 1} |\langle \alpha, x \rangle|.$$

*Example 4.5.* For the classical GPT  $\mathbf{CM}_d$ , the base norm and the order unit norm correspond to the  $\ell_1$  and  $\ell_\infty$  norms, respectively:

$$\|x\|_V = \|x\|_1 = \sum_{i=1}^d |x_i| \quad \text{and} \quad \|\alpha\|_A = \|\alpha\|_\infty = \max_{i \in [d]} |\alpha_i|.$$

For quantum mechanics, we obtain the Schatten 1 and  $\infty$  norms, respectively:

$$\begin{aligned} \|x\|_V &= \|x\|_1 = \text{Tr} \sqrt{x^2} \\ \|\alpha\|_A &= \|\alpha\|_\infty, \end{aligned}$$

where the  $\|\cdot\|_\infty$  norm is the usual operator norm.

**4.3. Centrally symmetric GPTs.** For some GPTs such as the hypercube  $\mathbf{HC}_n$ , we have more structure we can use. We review in this section what it means for a GPT  $(V, V^+, \mathbb{1})$  to be *centrally symmetric* in the sense of [LPW18, Definition 25]. In this case, the vector space  $V$  admits a decomposition  $V = \mathbb{R}v_0 \oplus \bar{V}$ , and we shall write  $x = (x_0, \bar{x})$  for a vector  $x = x_0v_0 \oplus \bar{x}$ . We then have

$$x = (x_0, \bar{x}) \in V^+ \iff \|\bar{x}\|_{\bar{V}} \leq x_0,$$

where  $\|\cdot\|_{\bar{V}}$  denotes the norm on  $\bar{V}$ . The decomposition  $V = \mathbb{R}v_0 \oplus \bar{V}$  is such that the form  $\mathbb{1}$  is given by  $\mathbb{1}(x) = x_0$ . The base norm is given as  $\|x\|_V = \max(|x_0|, \|\bar{x}\|_{\bar{V}})$ . The dual ordered vector space  $(A, A^+) = (V^*, (V^+)^*)$  admits a similar description in terms of the dual norm  $\|\cdot\|_{\bar{A}}$  in the dual space  $\bar{A} = \bar{V}^*$ :  $A = \mathbb{R}\mathbb{1} \oplus \bar{A}$  and  $a = (a_0, \bar{a}) \in A^+ \iff \|\bar{a}\|_{\bar{A}} \leq a_0$ . Note that for centrally symmetric spaces, both  $A$  and  $V$  possess distinguished order units.

**Lemma 4.6.** *Let  $(V, V^+, \mathbb{1})$  be a centrally symmetric GPT. The order unit norm on  $A$  satisfies*

$$\|\varphi\|_A = |\varphi_0| + \|(\varphi_1, \dots, \varphi_g)\|_{\bar{A}}$$

for all  $\varphi \in A$  with coordinates  $\varphi = \varphi_0\mathbb{1} + \sum_{i=1}^g \varphi_i\alpha_i$ , where  $\varphi_0, \varphi_i \in \mathbb{R}$  for all  $i \in [g]$  and  $\{\mathbb{1}, \alpha_i\}_{i \in [g]}$  is a basis of  $A$ .

*Proof.* By definition of the order unit norm of a vector  $\varphi \in A$ ,  $\|\varphi\|_A$  is the minimal  $t \geq 0$  such that  $\varphi \in t[-\mathbb{1}, \mathbb{1}]$ .

$$\begin{aligned}
\varphi \in t[-\mathbb{1}, \mathbb{1}] &\iff t\mathbb{1} - \varphi \in A^+ \wedge t\mathbb{1} + \varphi \in A^+ \\
&\iff (t - \varphi_0)\mathbb{1} - \sum_{i=1}^g \varphi_i \alpha_i \in A^+ \wedge (t + \varphi_0)\mathbb{1} + \sum_{i=1}^g \varphi_i \alpha_i \in A^+ \\
&\iff (t - \varphi_0) \geq \|(\varphi_1, \dots, \varphi_g)\|_{\bar{A}} \wedge (t + \varphi_0) \geq \|(\varphi_1, \dots, \varphi_g)\|_{\bar{A}} \\
&\iff t \geq |\varphi_0| + \|(\varphi_1, \dots, \varphi_g)\|_{\bar{A}}.
\end{aligned}$$

This proves the assertion.  $\square$

*Example 4.7.* The hypercube GPT  $\text{HC}_n$  is centrally symmetric, with the base norm and the order unit norm given by

$$\begin{aligned}
\|x\|_V &= \max(|x_0|, \|(x_1, \dots, x_n)\|_\infty) = \|x\|_\infty = \max_{0 \leq i \leq n} |x_i| \\
\|\alpha\|_A &= |\alpha_0| + \|(\alpha_1, \dots, \alpha_n)\|_1 = \|\alpha\|_1 = \sum_{i=0}^n |\alpha_i|.
\end{aligned}$$

Such GPTs can be used to model theories containing PR boxes [PR94].

*Example 4.8.* Qubits (i.e. quantum mechanics for two-level systems) form a centrally symmetric GPT, with the unit ball of  $\mathbb{R}^3$  as a state space: the *Bloch ball*. One can decompose any qubit state  $\rho$  as

$$\rho = \frac{1}{2} (I_2 + r \cdot \sigma),$$

where  $I_2$  is the identity matrix and  $\sigma = (\sigma_X, \sigma_Y, \sigma_Z)$  is the vector of Pauli matrices and  $r \in \mathbb{R}^3$  is a vector of norm at most one.

**4.4. Measurements and compatibility.** We have already introduced states and norms in our GPT. In this section, we show how the elements of  $A$  can represent measurements in this theory and we study their compatibility. We work under the no-restriction hypothesis [JL13], which states that all affine functions  $K \rightarrow [0, 1]$  correspond to physical effects.

Let  $(V, V^+, \mathbb{1})$  be a GPT. Any measurement of the system is determined by assigning the corresponding outcome probabilities to any state. We will only consider measurements with a finite number of outcomes, usually labeled by the set  $[k]$ ,  $k \in \mathbb{N}$ .

There are several equivalent ways to describe a measurement. First of all, to each outcome  $i \in [k]$  and  $\rho \in K$ , let  $f_i(\rho)$  denote the probability of obtaining the outcome  $i$  if the system is in the state  $\rho$ . Then,  $f_i$  is an affine map  $K \rightarrow [0, 1]$ ,  $i \in [k]$ , and corresponds to some  $f_i \in A^+$ ,  $f_i \leq \mathbb{1}$ . Such elements are called *effects*. Thus, the measurement is described by a tuple of effects  $f = (f_1, \dots, f_k)$  which must satisfy  $\sum_{i=1}^k f_i = \mathbb{1}$ . The measurement can be also described as an affine map into the probability simplex  $f : K \rightarrow \Delta_k$ , given by

$$\rho \mapsto \sum_{i=1}^k f_i(\rho) e_i.$$

Here,  $e_i$  are the vertices (or extremal points) of the simplex. We will use the same notation also for the unique extension of this map to  $(V, V^+) \rightarrow (\mathbb{R}^k, \mathbb{R}_+^k)$  (where the vertices correspond to elements of the standard basis). There is also an associated positive unital map  $\Phi_f : (\mathbb{R}^k, \mathbb{R}_+^k) \rightarrow (A, A^+)$ , determined by

$$\Phi_f(e_i) = f_i,$$

which is clearly the adjoint map of  $f$ . A measurement  $f = (f_1, \dots, f_k)$  in which the effects are multiples of the order unit  $f_i = p_i \mathbb{1}$  is called *trivial*; the vector  $(p_1, \dots, p_k) \in \mathbb{R}^k$  is then a probability vector.

We turn now to the analysis of the notion of compatibility of several measurements. We shall consider sets of  $g$  measurements, having possibly different numbers of outcomes  $k_1, \dots, k_g$ . We shall write  $\mathbf{k} := (k_1, \dots, k_g) \in \mathbb{N}^g$ .

**Definition 4.9.** Let  $f^{(1)}, \dots, f^{(g)}$  be measurements,  $f^{(i)} = (f_1^{(i)}, \dots, f_{k_i}^{(i)})$ ,  $i = 1, \dots, g$ . We will say that the collection  $(f^{(1)}, \dots, f^{(g)})$  is *compatible* if all  $f^{(i)}$  can be obtained as marginals of a single *joint measurement*. More precisely, there is a measurement  $h$  with outcomes labeled by  $[k_1] \times \dots \times [k_g]$  such that

$$f_j^{(i)} = \sum_{m_1, \dots, m_{i-1}, m_{i+1}, \dots, m_g} h_{m_1, \dots, m_{i-1}, j, m_{i+1}, \dots, m_g}, \quad j \in [k_i], i \in [g].$$

The *noise robustness of incompatibility* with respect to *white noise* can be described as the amount of white noise that has to be mixed with the measurements to make the collection  $f = (f^{(1)}, \dots, f^{(g)})$  compatible, see [HKR15]. This leads to the following definition of the compatibility region for  $f$ :

$$\Gamma(f) := \{s \in [0, 1]^g : s_i f^{(i)} + (1 - s_i) \mathbb{1}/k_i \text{ are compatible measurements}\}.$$

By considering these regions for all collections of measurements with  $\mathbf{k}$  outcomes, we obtain a characterization of the amount of incompatibility available in the given GPT.

**Definition 4.10.** Given a GPT  $(V, V^+, \mathbb{1})$ ,  $g \in \mathbb{N}$  and  $\mathbf{k} \in \mathbb{N}^g$ , we define the *compatibility region for  $\mathbf{k}$  outcomes* as

$$\begin{aligned} \Gamma(\mathbf{k}; V, V^+) &:= \{s \in [0, 1]^g : s_i f^{(i)} + (1 - s_i) \mathbb{1}/k_i \text{ are compatible measurements for all} \\ &\quad \text{collections } f^{(i)} \in A^{k_i}, i \in [g], \text{ of } g \text{ measurements with } \mathbf{k} \text{ outcomes}\} \\ &= \bigcap_{f \text{ with } \mathbf{k} \text{ outcomes}} \Gamma(f). \end{aligned}$$

If  $\mathbf{k} = 2^{\times g}$ , we will just write  $\Gamma(g; V, V^+)$ .

It is easy to see that  $\Gamma(\mathbf{k}; V, V^+)$  is convex and that  $\Gamma(\mathbf{k}; V, V^+) = [0, 1]^g$  if and only if all  $\mathbf{k}$  outcome measurements are compatible. If  $k_i \geq 2$  for all  $i \in [g]$ , then [Plá16] shows that this is the case if and only if the state space is isomorphic to a simplex.

The compatibility region is always non-empty as the next proposition shows. The intuitive argument is that we can add as many trivial measurements as we want without affecting the compatibility of a set of measurements and that every measurement is compatible with itself.

**Proposition 4.11.** *Let  $(V, V^+, \mathbb{1})$  be a GPT. Then, for any  $g \in \mathbb{N}$  and  $\mathbf{k} \in \mathbb{N}^g$ , it holds that*

$$\left\{ s \in [0, 1]^g : \sum_{i=1}^g s_i \leq 1 \right\} \subseteq \Gamma(\mathbf{k}; V, V^+).$$

*Proof.* We need to prove that  $e_i \in \Gamma(\mathbf{k}; V, V^+)$  for all  $i \in [g]$ , where  $\{e_i\}_{i=1}^g$  is the standard basis in  $\mathbb{R}^g$ . The statement then follows by convexity. Without loss of generality, let us consider  $e_1$ . Let  $f \in A^{k_1}$  be a measurement. Then,  $h$  with effects

$$h_{i_1, \dots, i_g} = \frac{1}{k_2 \cdots k_g} f_{i_1} \quad \forall i_j \in [k_j], j \in [g]$$

is a joint measurement for  $f$  and the trivial measurements with effects  $\mathbb{1}/k_i$  for all  $i \in [g] \setminus \{1\}$ . This proves the assertion.  $\square$

The elements in the compatibility region with all coefficients equal corresponds to the notion of a compatibility degree for  $f$ :

$$\gamma(f) := \max\{s \in [0, 1], (s, \dots, s) \in \Gamma(f)\}.$$

**Definition 4.12.** Let  $(V, V^+, \mathbb{1})$  be a GPT,  $g \in \mathbb{N}$ ,  $\mathbf{k} \in \mathbb{N}^g$ . We define the *compatibility degree* for  $\mathbf{k}$  outcomes as

$$\begin{aligned} \gamma(\mathbf{k}; V, V^+) &:= \max\{s \in [0, 1] : sf^{(i)} + (1-s)\mathbb{1}/k_i \text{ are compatible measurements} \\ &\quad \text{for all collections } f^{(i)} \in A^{k_i}, i \in [g], \text{ of } g \text{ measurements} \\ &\quad \text{with } \mathbf{k} \text{ outcomes}\} \\ &= \min_{f \text{ with } \mathbf{k} \text{ outcomes}} \gamma(f). \end{aligned}$$

If  $\mathbf{k} = 2^{\times g}$ , we will just write  $\gamma(g; V, V^+)$ .

The compatibility degree decreases with the number of measurements, i.e.,  $\gamma(\mathbf{k}; V, V^+) \geq \gamma(\mathbf{k}'; V, V^+)$  for  $g \leq g'$ . The intuitive argument is that subsets of compatible measurements remain compatible. Moreover, the more outcomes we have, the more difficult it becomes for all measurements to be compatible, i.e.,  $\Gamma(\mathbf{k}'; V, V^+) \subseteq \Gamma(\mathbf{k}; V, V^+)$  and in particular  $\gamma(\mathbf{k}'; V, V^+) \leq \gamma(\mathbf{k}; V, V^+)$  for  $\mathbf{k}' \geq \mathbf{k}$  entrywise. For the proofs of these and related results, see Appendix D.

## 5. Compatible Measurements and Positive Maps

In this section, we prove that the compatibility of a collection of measurements can be characterized by the properties of a certain associated positive map.

Let  $f = (f^{(1)}, \dots, f^{(g)})$  be a collection of measurements for a GPT  $(V, V^+, \mathbb{1})$ . Assume that for  $i \in [g]$ ,  $g \in \mathbb{N}$ , the measurement  $f^{(i)}$  has  $k_i$  outcomes and let us denote  $\mathbf{k} := (k_1, \dots, k_g) \in \mathbb{N}^g$ . The collection of  $g$  affine maps  $f^{(i)} : K \rightarrow \Delta_{k_i}$  can be naturally identified with an affine map of the state space into the *polysimplex*

$$P_{\mathbf{k}} := \Delta_{k_1} \times \cdots \times \Delta_{k_g}, \quad (6)$$



defined as

$$\rho \mapsto (f^{(1)}(\rho), \dots, f^{(g)}(\rho)) \in P_{\mathbf{k}}.$$

Conversely, we can obtain a collection of  $g$  measurements from any such map by composition with the projections  $m_i : P_{\mathbf{k}} \rightarrow \Delta_{k_i}$  to the  $i$ -th component of  $P_{\mathbf{k}}$ , for  $i \in [g]$ .

We next find a suitable representation of  $P_{\mathbf{k}}$  as a base of the positive cone in an ordered vector space. As we have seen in Sect. 4, we can find such a representation by considering affine functions  $P_{\mathbf{k}} \rightarrow \mathbb{R}$ . Since  $P_{\mathbf{k}}$  is a convex polytope, any affine function on  $P_{\mathbf{k}}$  is determined by its value on each of the  $k_1 \cdots k_g$  vertices of  $P_{\mathbf{k}}$ . Therefore,  $A(P_{\mathbf{k}})$  is isomorphic to a subspace

$$E_{\mathbf{k}} \subseteq \mathbb{R}^{\mathbf{k}} := \mathbb{R}^{k_1 \cdots k_g} \cong \bigotimes_{i \in [g]} \mathbb{R}^{k_i} \quad (7)$$

and it is clear that the positive cone  $A(P_{\mathbf{k}})^+ \cong E_{\mathbf{k}}^+ := E_{\mathbf{k}} \cap \mathbb{R}_{+}^{\mathbf{k}}$ . The unit  $1_{P_{\mathbf{k}}}$  corresponds to the vector  $1_{\mathbf{k}} = 1_{k_1} \otimes \cdots \otimes 1_{k_g}$  of all ones, which is clearly contained in  $E_{\mathbf{k}} \cap \text{int}(\mathbb{R}_{+}^{\mathbf{k}})$ . Some technical results on such ordered vector spaces can be found in Appendix B. The main result is that positive functionals on tensor products with these spaces can be extended to positive functionals on  $(\mathbb{R}^{\mathbf{k}}, \mathbb{R}_{+}^{\mathbf{k}})$  (Proposition B.3), which follows from the M. Riesz extension theorem.

Let  $J_{\mathbf{k}}$  denote the orthogonal projection  $\mathbb{R}^{\mathbf{k}} \rightarrow E_{\mathbf{k}}$ . By Proposition B.4, we may put  $E_{\mathbf{k}}^* \cong E_{\mathbf{k}}$  and  $(E_{\mathbf{k}}^+)^* \cong J_{\mathbf{k}}(\mathbb{R}_{+}^{\mathbf{k}})$ . Then  $P_{\mathbf{k}}$  is represented as the base of  $(E_{\mathbf{k}}^+)^*$  determined by the unit functional  $1_{\mathbf{k}}$ , that is

$$P_{\mathbf{k}} \cong \{x \in J_{\mathbf{k}}(\mathbb{R}_{+}^{\mathbf{k}}), \langle 1_{\mathbf{k}}, x \rangle = 1\} = \{J_{\mathbf{k}}y, y \in \mathbb{R}_{+}^{\mathbf{k}}, \langle 1_{\mathbf{k}}, y \rangle = 1\} = J_{\mathbf{k}}(\Delta_{\mathbf{k}}),$$

where

$$\Delta_{\mathbf{k}} := \Delta_{k_1 \cdots k_g} \cong \otimes_i \Delta_{k_i}.$$

The vertex  $(e_{i_1}^{(k_1)}, \dots, e_{i_g}^{(k_g)})$  of  $P_{\mathbf{k}}$  corresponds precisely to  $J_{\mathbf{k}}(e_{i_1}^{(k_1)} \otimes \cdots \otimes e_{i_g}^{(k_g)})$ , where  $e_j^{(k)}$  denote elements of the standard basis in  $\mathbb{R}^k$ .

We have therefore constructed a linear and order isomorphism between the order unit spaces  $(A(P_{\mathbf{k}}), A(P_{\mathbf{k}})^+, 1_{P_{\mathbf{k}}})$  and  $(E_{\mathbf{k}}, E_{\mathbf{k}}^+, 1_{\mathbf{k}})$ , respectively the base norm spaces  $(V(P_{\mathbf{k}}), V(P_{\mathbf{k}})^+)$  and  $(E_{\mathbf{k}}^*, (E_{\mathbf{k}}^+)^* = J_{\mathbf{k}}(\mathbb{R}_{+}^{\mathbf{k}}))$ . We will use this identification throughout this section. The following result is easily checked.

**Lemma 5.1.** *The restriction*

$$J_{\mathbf{k}}|_{\Delta_{\mathbf{k}}} : \Delta_{\mathbf{k}} \rightarrow J_{\mathbf{k}}(\Delta_{\mathbf{k}}) \cong P_{\mathbf{k}}$$

*defines a collection  $(j^{(1)}, \dots, j^{(g)})$  of measurements on  $\Delta_{\mathbf{k}}$ , where  $j^{(i)}(p)$  is the  $i$ -th marginal of  $p$ , for any  $p \in \Delta_{\mathbf{k}} \cong \otimes_i \Delta_{k_i}$ . In other words, for a probability distribution  $p \in \Delta_{\mathbf{k}}$ , we have  $J_{\mathbf{k}}(p) = (p^{(1)}, \dots, p^{(g)}) \in P_{\mathbf{k}}$ , where  $p^{(i)} = j^{(i)}(p)$  is the  $i$ -th marginal of  $p$ .*

To give a more explicit description of  $E_{\mathbf{k}}$ , observe first that each projection  $m^i : P_{\mathbf{k}} \rightarrow \Delta_{k_i}$  onto the  $i$ -th component is a measurement on  $P_{\mathbf{k}}$  and the corresponding effects are given by the elements of  $E_{\mathbf{k}}^+$  of the form

$$\eta_j^{(i)} := 1_{k_1} \otimes \cdots \otimes 1_{k_{i-1}} \otimes e_j^{(k_i)} \otimes 1_{k_{i+1}} \otimes \cdots \otimes 1_{k_g}, \quad j \in [k_i], \quad i \in [g].$$

By the results of [Jen18], these elements generate the extremal rays of  $E_{\mathbf{k}}^+$  and consequently  $E_{\mathbf{k}}$  is spanned by the set  $\{\eta_j^{(i)}\}$ . We choose a basis of  $E_{\mathbf{k}}$  given as

$$w := \{1_{\mathbf{k}}, w_j^{(i)}, \quad j \in [k_i - 1], \quad i \in [g]\},$$

where

$$w_j^{(i)} := \underbrace{1_{k_1} \otimes \cdots \otimes 1_{k_{i-1}}}_{i-1 \text{ factors}} \otimes v_j^{(k_i)} \otimes \underbrace{1_{k_{i+1}} \otimes \cdots \otimes 1_{k_g}}_{g-i \text{ times}} \in \mathbb{R}^{\mathbf{k}}$$

and  $v_1^{(k)}, \dots, v_{k-1}^{(k)}$  are vectors in  $\mathbb{R}^k$  defined as

$$v_j^{(k)}(l) := -\frac{2}{k} + 2\delta_{l,j}, \quad \forall j \in [k-1], \quad \forall l \in [k].$$

The set  $\{1_k, v_j^{(k)}, j \in [k-1]\}$  is a basis of  $\mathbb{R}^k$  and the dual basis of  $\mathbb{R}^k \cong (\mathbb{R}^k)^*$  is  $\{\frac{1}{k}1_k, v_j^{(*k)}, j \in [k-1]\}$ , where

$$v_j^{(*k)} := \frac{1}{2} \left( e_j^{(k)} - e_k^{(k)} \right), \quad j \in [k-1].$$

It is then clear that

$$w^* := \left\{ \frac{1}{k_1 \dots k_g} 1_{\mathbf{k}}, w_j^{(*i)}, \quad j \in [k_i - 1], \quad i \in [g] \right\},$$

where

$$w_j^{(*i)} := \frac{k_i}{k_1 \dots k_g} 1_{k_1} \otimes \cdots \otimes 1_{k_{i-1}} \otimes v_j^{(*k_i)} \otimes 1_{k_{i+1}} \otimes \cdots \otimes 1_{k_g},$$

is the basis of  $E_{\mathbf{k}}^* \cong E_{\mathbf{k}}$  which is dual to  $w$ . This choice of basis may seem strange at first, but it will be convenient in Sect. 6.

Let  $\Phi : E_{\mathbf{k}} \rightarrow A$  be a linear map, then  $\Phi$  is determined by elements  $p_0, p_j^{(i)} \in A$ , such that

$$\Phi(1_{\mathbf{k}}) = p_0, \quad \Phi(w_j^{(i)}) = p_j^{(i)}, \quad j \in [k_i - 1], \quad i \in [g].$$

We can identify such maps with tensors and positive maps with some elements of a tensor cone (see ‘‘Sects. A.2 and A.3’’). Then the corresponding element  $\varphi^{\Phi} \in E_{\mathbf{k}}^* \otimes A$  is

$$\varphi^{\Phi} = \frac{1}{k_1 \dots k_g} 1_{\mathbf{k}} \otimes p_0 + \sum_{i=1}^g \sum_{j=1}^{k_i-1} w_j^{(i*)} \otimes p_j^{(i)}.$$

**Proposition 5.2.** *Let  $\Phi$  and  $p_0, p_j^{(i)} \in A$  be as above. Then  $\Phi : (E_{\mathbf{k}}, E_{\mathbf{k}}^+) \rightarrow (A, A^+)$  is positive and unital if and only if  $p_0 = \mathbb{1}$  and there is a collection  $(f^{(1)}, \dots, f^{(g)})$  of measurements,  $f^{(i)} : K \rightarrow \Delta_{k_i}$ , such that*

$$p_j^{(i)} = 2f_j^{(i)} - \frac{2}{k_i}\mathbb{1}, \quad j \in [k_i - 1], \quad i \in [g].$$

*The adjoint map  $\Phi^* : (V, V^+) \rightarrow (E_{\mathbf{k}}^*, (E_{\mathbf{k}}^+)^*)$  satisfies*

$$\Phi^*(\rho) = (f^{(1)}(\rho), \dots, f^{(g)}(\rho)), \quad \rho \in K. \quad (8)$$

*Proof.* It is clear that  $\Phi$  is unital if and only if  $p_0 = \mathbb{1}$ . Moreover,  $\Phi$  is positive if and only if  $\varphi^\Phi \in (E_{\mathbf{k}}^+)^* \otimes_{\max} A^+$  (see Appendix A.3). Since the elements  $\eta_j^{(i)}$  generate extremal rays of  $E_{\mathbf{k}}^+$ , it is enough to verify that

$$\langle \varphi^\Phi, \eta_j^{(i)} \otimes v \rangle \geq 0, \quad \forall v \in V^+, \quad j \in [k_i], \quad i \in [g].$$

From this condition and

$$\langle w_{j'}^{(*i')}, \eta_j^{(i)} \rangle = \frac{1}{2} \delta_{i,i'} (\delta_{j,j'} - \delta_{j,k_i}),$$

we get that for  $i \in [g]$ ,

$$\frac{1}{k_i}\mathbb{1} + \frac{1}{2}p_j^{(i)} \in A^+, \quad j \in [k_i - 1], \quad \frac{1}{k_i}\mathbb{1} - \frac{1}{2} \sum_{j=1}^{k_i-1} p_j^{(i)} \in A^+.$$

We now put

$$f_j^{(i)} := \frac{1}{k_i}\mathbb{1} + \frac{1}{2}p_j^{(i)}, \quad j \in [k_i - 1], \quad f_{k_i}^{(i)} := \mathbb{1} - \sum_{j=1}^{k_i-1} f_j^{(i)},$$

then  $(f^{(1)}, \dots, f^{(g)})$  is a collection of measurements of the required form. Since the  $\eta_j^{(i)}$  correspond to the effects of the projection map onto the  $i$ -th component of  $P_{\mathbf{k}}$ , the last statement follows from

$$\langle \Phi^*(v), \eta_j^{(i)} \rangle = \langle \varphi^\Phi, \eta_j^{(i)} \otimes v \rangle = \langle f_j^{(i)}, v \rangle, \quad j \in [k_i], \quad i \in [g].$$

□

Let  $f = (f^{(1)}, \dots, f^{(g)})$  be a collection of measurements with outcome spaces specified by the vector  $\mathbf{k} = (k_1, \dots, k_g)$ . By the above result, we clearly have that there is an associated positive unital map  $\Phi^{(f)} : E_{\mathbf{k}} \rightarrow A$ , defined as

$$\begin{aligned} \Phi^{(f)} : E_{\mathbf{k}} &\rightarrow A \\ \mathbb{1}_{\mathbf{k}} &\mapsto \mathbb{1} \\ w_j^{(i)} &\mapsto 2f_j^{(i)} - \frac{2}{k_i}\mathbb{1} \quad \forall j \in [k_i - 1], \quad i \in [g]. \end{aligned} \quad (9)$$

The corresponding element in  $(E_{\mathbf{k}}^+)^* \otimes_{\max} A^+$  is given by

$$\varphi^{(f)} = \frac{1}{k_1 \dots k_g} 1_{\mathbf{k}} \otimes 1 + \sum_{i \in [g]} \sum_{j \in [k_i - 1]} w_j^{(*i)} \otimes (2f_j^{(i)} - \frac{2}{k_i} 1). \quad (10)$$

Note that the action of the map  $\Phi^{(f)}$  can alternatively be defined as

$$\begin{aligned} \Phi^{(f)} : E_{\mathbf{k}} &\rightarrow A \\ \eta_j^{(i)} &\mapsto f_j^{(i)} \quad \forall j \in [k_i], i \in [g] \end{aligned}$$

at the cost that this is only well-defined for measurements since the  $\eta_j^{(i)}$  are not all linearly independent.

We now prove the main result of this section; see [Jen18, Theorem 1 and Proposition A1] for a result similar to the equivalence of (2)–(5).

**Theorem 5.3.** *Let  $f = (f^{(1)}, \dots, f^{(g)})$ ,  $\Phi^{(f)}$  and  $\varphi^{(f)}$  be as above. The following are equivalent.*

- (1) *There is a positive map  $\tilde{\Phi} : (\mathbb{R}^{\mathbf{k}}, \mathbb{R}_+^{\mathbf{k}}) \rightarrow (A, A^+)$  extending  $\Phi^{(f)}$ .*
- (2)  *$\Phi^{(f)}$  is entanglement breaking.*
- (3)  *$\varphi^{(f)} \in (J_{\mathbf{k}} \otimes \text{id})(\mathbb{R}_+^{\mathbf{k}} \otimes A^+)$ .*
- (4) *There is some measurement  $h : K \rightarrow \Delta_{\mathbf{k}}$  such that  $(\Phi^{(f)})^*|_K = J_{\mathbf{k}} \circ h$ .*
- (5) *The collection  $f$  is compatible.*

*Proof.* The statements (1)–(3) are equivalent by Proposition B.8. Moreover, we have (4)  $\iff$  (5) by Lemma 5.1 and Eq. (8).

Assume (1), then since  $\Phi^{(f)}$  is unital and  $1_{\mathbf{k}} \in E_{\mathbf{k}}$ ,  $\tilde{\Phi}$  is unital as well. Hence there is some measurement  $h : K \rightarrow \Delta_{\mathbf{k}}$  such that  $\tilde{\Phi}^*|_K = h$ . For  $\rho \in K$  and  $e \in E_{\mathbf{k}}$ , we have

$$\langle (\Phi^{(f)})^*(\rho), e \rangle = \langle \rho, \Phi^{(f)}(e) \rangle = \langle \rho, \tilde{\Phi}(e) \rangle = \langle h(\rho), e \rangle = \langle J_{\mathbf{k}}(h(\rho)), e \rangle,$$

which proves (4). Finally, assume (4), then we have

$$\begin{aligned} \langle \Phi^{(f)}(e), v \rangle &= \langle e, (\Phi^{(f)})^*(v) \rangle = \langle e, J_{\mathbf{k}} \circ h(v) \rangle = \langle e, h(v) \rangle = \langle \Phi_{(h)}(e), v \rangle, \\ e &\in E_{\mathbf{k}}, v \in V, \end{aligned}$$

so that  $\Phi_{(h)}$  is a positive extension of  $\Phi^{(f)}$ . Here, we have identified the affine map  $h$  on  $K$  with its unique extension to a linear map on  $V$ . This finishes the proof.  $\square$

*Remark 5.4.* We note in Appendix B that map extension can be verified using conic programming (see Sect. E.2 in the Appendix). The same is true for the existence of a joint measurement (see [WPGF09, Plá16]).

## 6. Inclusion of Generalized Spectrahedra and Compatibility of Measurements in GPTs

In this section, we will show that the compatibility of measurements in a GPT can be phrased as an inclusion problem of generalized spectrahedra, which we introduce in this work. Recall that a spectrahedron [RG95, Vin14] is a convex set that can be represented by a linear matrix inequality, that is by positive semidefinite constraints. We define a generalized spectrahedron as a convex subset of some vector space which can be represented by positivity conditions with respect to some abstract cone. Namely, for  $L$ ,  $M$  two finite-dimensional vector spaces, a cone  $C \subseteq M \otimes L$  and a  $g$ -tuple of elements  $a = (a_1, \dots, a_g) \in M^g$ , we define the *generalized spectrahedron*

$$\mathcal{D}_a(L, C) := \{(v_1, \dots, v_g) \in L^g : \sum_{i=1}^g a_i \otimes v_i \in C\}. \quad (11)$$

We develop the theory of generalized free spectrahedra in Appendix C. Most importantly, Proposition C.9 connects the inclusion of generalized free spectrahedra to the positivity of a certain map between the elements defining the two generalized free spectrahedra, which generalizes results from free spectrahedra in [HKM13, HKMS19, BN18].

**6.1. The GPT jewel.** In this section, we define the universal generalized spectrahedron we will consider to relate the compatibility of measurements in GPTs to inclusion problems of generalized spectrahedra. It plays the same role as the matrix diamond in [BN18] and the matrix jewel in [BN20] and is in fact a generalization of both.

Recall from the previous section that, for a positive integer  $g$  and a  $g$ -tuple of positive integers  $\mathbf{k} \in \mathbb{N}^g$ , we define the vector subspace  $E_{\mathbf{k}} \subseteq \mathbb{R}^{k_1 \cdots k_g}$  as the linear span of the basis  $w = \{1_{\mathbf{k}}, w_j^{(i)}, i \in [g], j \in [k_i - 1]\}$ , where  $1_{\mathbf{k}}$  is the all-ones vector and

$$w_j^{(i)} = 1_{k_1} \otimes \cdots \otimes 1_{k_{i-1}} \otimes \underbrace{\left(2e_j^{(k_i)} - \frac{2}{k_i} 1_{k_i}\right)}_{v_j^{(k_i)}} \otimes 1_{k_{i+1}} \otimes \cdots \otimes 1_{k_g}$$

for  $i \in [g]$  and  $j \in [k_i - 1]$ . In particular,  $\dim E_{\mathbf{k}} = 1 - g + \sum_{i=1}^g k_i$ .

**Definition 6.1.** Let  $g \in \mathbb{N}$ ,  $\mathbf{k} \in \mathbb{N}^g$  and let  $(L, L^+)$  be a proper ordered vector space. The  $(\mathbf{k}; L, L^+)$ -*jewel* is defined as

$$\mathcal{D}_{\text{GPT}\heartsuit}(\mathbf{k}; L, L^+) := \mathcal{D}_w(L, E_{\mathbf{k}}^+ \otimes_{\max} L^+).$$

For  $k_1 = \dots = k_g = 2$ , the above has an especially easy form and we write

$$\mathcal{D}_{\text{GPT}\heartsuit}(g; L, L^+) := \mathcal{D}_{\text{GPT}\heartsuit}(2^{\times g}; L, L^+)$$

for short. In this case,  $w_1^{(i)} = c_i$  for all  $i \in [g]$ , with

$$c_i = (1, 1) \otimes \cdots \otimes (1, 1) \otimes (+1, -1) \otimes (1, 1) \otimes \cdots \otimes (1, 1) \in \mathbb{R}^{2^g}.$$

We call this object the  $(g; L, L^+)$ -*diamond*.

Since  $w$  is a basis of  $E_{\mathbf{k}}$ , we see that the  $(\mathbf{k}, L, L^+)$ -jewel fully describes the tensor product  $E_{\mathbf{k}}^+ \otimes_{\max} L^+$ . The next lemma gives a more explicit description. First, fix  $\{e_{\kappa}\}_{\kappa \in [k_1] \times \dots \times [k_g]}$ , the standard basis of  $\mathbb{R}^{\mathbf{k}}$ . One has the following decompositions

$$\forall i \in [g], j \in [k_i - 1], \quad w_j^{(i)} = \sum_{\kappa \in [k_1] \times \dots \times [k_g]} w_j^{(i)}(\kappa) e_{\kappa},$$

where by the definition of  $w_j^{(i)}$ ,

$$w_j^{(i)}(\kappa) = v_j^{(k_i)}(\kappa_i) = -\frac{2}{k_i} + 2\delta_{\kappa_i, j}. \quad (12)$$

**Lemma 6.2.** *Let  $(L, L^+)$  be a proper ordered vector space. Moreover, let  $\mathbf{k} \in \mathbb{N}^g$ ,  $z_0$ ,  $z_j^{(i)} \in L$ , where  $i \in [g]$ ,  $j \in [k_i - 1]$ . Then,*

$$z_0 + \sum_{i=1}^g \sum_{j=1}^{k_i-1} w_j^{(i)}(\kappa) z_j^{(i)} \in L^+ \quad \forall \kappa \in [k_1] \times \dots \times [k_g] \quad (13)$$

if and only if

$$1_{\mathbf{k}} \otimes z_0 + \sum_{i=1}^g \sum_{j=1}^{k_i-1} w_j^{(i)} \otimes z_j^{(i)} \in E_{\mathbf{k}}^+ \otimes_{\max} L^+.$$

In particular, for  $\mathbf{k} = 2^{\times g}$ , Eq. (13) has the form

$$z_0 + \sum_{i=1}^g \varepsilon_i z_i \in L^+ \quad \forall \varepsilon \in \{\pm 1\}^g.$$

*Proof.* Let  $z_0, z_j^{(i)} \in L$ ,  $i \in [g]$ ,  $j \in [k_i - 1]$  and put

$$y := 1_{\mathbf{k}} \otimes z_0 + \sum_{i=1}^g \sum_{j=1}^{k_i-1} w_j^{(i)} \otimes z_j^{(i)} = \sum_{\kappa \in [k_1] \times \dots \times [k_g]} e_{\kappa} \otimes (z_0 + \sum_{i=1}^g \sum_{j=1}^{k_i-1} w_j^{(i)}(\kappa) z_j^{(i)}).$$

The standard basis  $\{e_{\kappa}\}_{\kappa \in [k_1] \times \dots \times [k_g]}$  of  $\mathbb{R}^{\mathbf{k}} \cong (\mathbb{R}^{\mathbf{k}})^*$  is self-dual and the elements  $e_{\kappa}$  define the extremal rays of the cone  $\mathbb{R}_+^{\mathbf{k}} \cong (\mathbb{R}_+^{\mathbf{k}})^*$ . Therefore,  $y \in E_{\mathbf{k}}^+ \otimes_{\max} L^+$  if and only if for all  $\beta \in (L^+)^*$  and  $\kappa \in [k_1] \times \dots \times [k_g]$ ,

$$\langle e_{\kappa} \otimes \beta, y \rangle = \langle \beta, z_0 + \sum_{i=1}^g \sum_{j=1}^{k_i-1} w_j^{(i)}(\kappa) z_j^{(i)} \rangle \geq 0$$

by Proposition B.5. Equivalently,  $z_0 + \sum_{i=1}^g \sum_{j=1}^{k_i-1} w_j^{(i)}(\kappa) z_j^{(i)} \in (L^+)^{**}$  for all  $\kappa \in [k_1] \times \dots \times [k_g]$ . Since  $L$  is finite dimensional and  $L^+$  is closed, we have that  $(L^+)^{**} = L^+$  by the bipolar theorem.  $\square$

**Remark 6.3.** The above lemma yields a more appealing form of the  $(g; L, L^+)$ -diamond, namely

$$\mathcal{D}_{\text{GPT}\diamond}(g; L, L^+) = \left\{ (z_0, \dots, z_g) \in L^{g+1} : \forall \varepsilon \in \{\pm 1\}^g, z_0 + \sum_{i=1}^g \varepsilon_i z_i \in L^+ \right\}. \quad (14)$$

**Remark 6.4.** Lemma 6.2 shows that  $\mathcal{D}_{\text{GPT}\diamond}(\mathbf{k}; L, L^+)$  is a maximal generalized spectrahedron, in the sense of Definition C.4. If  $\mathbf{k} = 2^{\times g}$ , the corresponding closed convex cone is generated by the unit ball of  $\ell_1^g$ , as can be seen from Remark 6.3.

**Remark 6.5.** It can be seen by Lemma 6.2 that for any  $z = (z_0, z_j^{(i)})_{i,j} \in \mathcal{D}_{\text{GPT}\diamond}(\mathbf{k}; L, L^+)$  we must have  $z_0 \in L^+$ . Indeed, let

$$z_\kappa := z_0 + \sum_{i=1}^g \sum_{j=1}^{k_i-1} w_j^{(i)}(\kappa) z_j^{(i)} \quad \kappa \in [k_1] \times \dots \times [k_g]. \quad (15)$$

Then  $z_\kappa \in L^+$  and one can see using Eq. (12) that  $z_0$  is the barycenter of  $\{z_\kappa\}_\kappa$ :

$$z_0 = \frac{1}{k_1 \dots k_g} \sum_{\kappa \in [k_1] \times \dots \times [k_g]} z_\kappa. \quad (16)$$

Note also that we obtain  $z_j^{(i)}$  from  $z_\kappa$  by

$$z_j^{(i)} = \frac{1}{2} (z_{k_1, \dots, k_{i-1}, j, k_{i+1}, \dots, k_g} - z_{k_1, \dots, k_g}), \quad i \in [g], j \in [k_i - 1]. \quad (17)$$

In the case of the GPT  $(V, V^+, \mathbb{1})$  (or if we have fixed an order unit in  $(L^+)^*$ ), the  $(\mathbf{k}; V, V^+)$  jewel can be normalized by assuming that  $z_0 \in K$  (that is,  $\mathbb{1}(z_0) = 1$ ). Indeed, this follows from Proposition 6.6 below. The resulting set is then convex, closed and bounded. Indeed, boundedness can be seen easily from

$$\frac{1}{k_1 \dots k_g} \sum_{\kappa} \|z_\kappa\|_V = \frac{1}{k_1 \dots k_g} \sum_{\kappa} \mathbb{1}(z_\kappa) \leq \mathbb{1}(z_0) = 1,$$

so that the norm  $\|z_\kappa\|_V$  is bounded. The normalized jewel can be used in all the results below.

**Proposition 6.6.** *Let  $(V, V^+, \mathbb{1})$  be a GPT and  $g \in \mathbb{N}$ ,  $\mathbf{k} \in \mathbb{N}^g$ . Moreover, let  $z = (0, z_j^{(i)})_{i,j} \in \mathcal{D}_{\text{GPT}\diamond}(\mathbf{k}; V, V^+)$ . Then,  $z_j^{(i)} = 0$  for all  $j \in [k_i - 1]$ ,  $i \in [g]$ .*

*Proof.* Let  $z$  be as in the statement and let  $z_\kappa$  be as in Eq. (15). Then since  $z_0 = 0$  and all  $z_\kappa \in V^+$ , Eq. (16) implies that  $z_\kappa = 0$  for all  $\kappa \in [k_1] \times \dots \times [k_g]$ . The statement now follows from Eq. (17).  $\square$

**Example 6.7.** Let us discuss now the particular forms of the GPT diamond in the case of classical, quantum, and hypercubic GPTs.

In the case of the classical GPT  $\text{CM}_d$ , a  $(g+1)$ -tuple  $(z_0, z_1, \dots, z_g)$  of vectors from  $\mathbb{R}^d$  is an element of  $\mathcal{D}_{\text{GPT}\diamond}(g; \mathbb{R}^d, \mathbb{R}_+^d)$  if, for all  $j \in [d]$ , and for all sign choices  $\varepsilon \in \{\pm 1\}^g$ ,

$$z_0(j) + \sum_{i=1}^g \varepsilon_i z_i(j) \geq 0 \iff z_0(j) \geq \sum_{i=1}^g |z_i(j)| \iff z_0(j) \geq \|z(j)\|_1.$$



For quantum mechanics  $\text{QM}_d$ , the condition reads, for self-adjoint matrices  $z_0, \dots, z_g \in \mathcal{M}_d^{\text{sa}}(\mathbb{C})$ ,

$$z_0 + \sum_{i=1}^g \varepsilon_i z_i \in \text{PSD}_d,$$

for all sign vectors  $\varepsilon \in \{\pm 1\}^g$ . From Remark 6.5, we know that  $z_0$  is positive semidefinite. We claim that

$$\mathcal{D}_{\text{GPT}\diamond}(g; \mathcal{M}_d^{\text{sa}}(\mathbb{C}), \text{PSD}_d) = \{(z_0, z_0^{1/2} \tilde{z} z_0^{1/2}) : z_0 \in \text{PSD}_d \text{ and } \tilde{z} \in \mathcal{D}_{\diamond, g}(d)\}, \quad (18)$$

where  $\mathcal{D}_{\diamond, g}(d)$  is the “standard”, quantum mechanical matrix diamond, used in [BN18] (see also [DDOSS17]):

$$\mathcal{D}_{\diamond, g}(d) = \left\{ X \in (\mathcal{M}_d^{\text{sa}}(\mathbb{C}))^g : \forall \varepsilon \in \{\pm 1\}^g, \sum_{i=1}^g \varepsilon_i X_i \leq I_d \right\}.$$

To show that Eq. (18) holds, we prove the two inclusions. Let us start by showing “ $\supseteq$ ”. For  $\tilde{z} \in \mathcal{D}_{\diamond, g}(d)$ , we have

$$I_d + \sum_{i=1}^g \varepsilon_i \tilde{z}_i \geq 0 \implies z_0^{1/2} (I_d + \sum_{i=1}^g \varepsilon_i \tilde{z}_i) z_0^{1/2} \geq 0 \iff z_0 + \sum_{i=1}^g \varepsilon_i z_0^{1/2} \tilde{z}_i z_0^{1/2} \geq 0.$$

For the reverse inclusion, note that, given  $(z_0, \dots, z_g) \in \mathcal{D}_{\text{GPT}\diamond}(g; \mathcal{M}_d^{\text{sa}}(\mathbb{C}), \text{PSD}_d)$ , we have, for all  $\varepsilon$ ,

$$z_0 \geq \pm \sum_{i=1}^g \varepsilon_i z_i,$$

hence the support of the self-adjoint matrix  $\sum_{i=1}^g \varepsilon_i z_i$  is included in the support of  $z_0$ . By restricting all the matrices to the support of  $z_0$ , we have, for all  $\varepsilon$ ,

$$0 \leq z_0 + \sum_{i=1}^g \varepsilon_i z_i = z_0^{1/2} \left( I + \sum_{i=1}^g \varepsilon_i \tilde{z}_i \right) z_0^{1/2} \implies I + \sum_{i=1}^g \varepsilon_i \tilde{z}_i \geq 0,$$

where  $\tilde{z}_i := z_0^{-1/2} z_i z_0^{-1/2}$ . Hence, the usual matrix diamond and the GPT diamond for  $\text{QM}_d$  differ only in the choice of the free term  $z_0$ : in the former case, one fixes  $z_0 = I_d$ , while in the latter case  $z_0$  is free.

In the case of the hypercubic GPT  $\text{HC}_n$ , the situation is similar to the classical GPT:

$$\begin{aligned} (z_0, \dots, z_g) \in \mathcal{D}_{\text{GPT}\diamond}(g; \mathbb{R}^{n+1}, C_n) &\iff \forall j \in [n], \quad z_0(0) \pm z_0(j) \\ &\geq \sum_{i=1}^g |z_i(0) \pm z_i(j)|. \end{aligned}$$

**6.2. Spectrahedral inclusion and compatibility.** Having introduced the universal GPT jewel and diamond, we use them to characterize compatibility of measurements in GPTs via generalized spectrahedral inclusion. For the proofs, we will make heavy use of Proposition B.5, relating  $E_{\mathbf{k}}^+ \otimes_{\max} L^+$  and  $\mathbb{R}_{+}^{\mathbf{k}} \otimes L^+$ , and Proposition B.8, characterizing the extendibility of certain positive maps.

**Definition 6.8.** Let  $(V, V^+, \mathbb{1})$  be a GPT. We define the following generalized spectrahedron: for the preordered vector space  $(L, L^+)$  and  $\mathbf{k} \in \mathbb{N}^g$ ,  $f_j^{(i)} \in A$ ,  $i \in [g]$ ,  $j \in [k_i - 1]$ ,

$$\mathcal{D}_f(\mathbf{k}; L, L^+) := \left\{ \left( z_0, z_j^{(i)} \right)_{ij} \in L^{1-g+\sum_i k_i} : \mathbb{1} \otimes z_0 + \sum_{i=1}^g \sum_{j=1}^{k_i-1} \left( 2f_j^{(i)} - \frac{2}{k_i} \mathbb{1} \right) \otimes z_j^{(i)} \in A^+ \otimes_{\min} L^+ \right\}. \quad (19)$$

In the case where  $\mathbf{k} = 2^{\times g}$ , we will write  $\mathcal{D}_f(g; L, L^+)$  for simplicity.

The following key result connects the inclusion of the GPT jewel  $\mathcal{D}_{\text{GPT}\diamond}(\mathbf{k}; L, L^+)$  inside a given  $\mathcal{D}_f$  defined above to the positivity of a linear map between two tensor cones. This establishes a bridge between generalized spectrahedral inclusion and measurement compatibility in GPTs.

**Proposition 6.9.** Let  $\mathbf{k} \in \mathbb{N}^g$  and let  $(L, L^+)$  be a proper ordered vector space. Then, the inclusion  $\mathcal{D}_{\text{GPT}\diamond}(\mathbf{k}; L, L^+) \subseteq \mathcal{D}_f(\mathbf{k}; L, L^+)$  holds if and only if  $\Phi^{(f)} \otimes \text{id}_L : (E_{\mathbf{k}} \otimes L, E_{\mathbf{k}}^+ \otimes_{\max} L^+) \rightarrow (A \otimes L, A^+ \otimes_{\min} L^+)$  is positive, where  $\Phi^{(f)}$  is defined as in Eq. (9).

*Proof.* This follows directly from Propositions B.5 and C.9.  $\square$

We have two important special cases.

**Proposition 6.10.** Let  $\mathbf{k} \in \mathbb{N}^g$  and let  $(V, V^+, \mathbb{1})$  be a GPT. Then,  $\mathcal{D}_{\text{GPT}\diamond}(\mathbf{k}; \mathbb{R}, \mathbb{R}_+) \subseteq \mathcal{D}_f(\mathbf{k}; \mathbb{R}, \mathbb{R}_+)$  if and only if  $\{f_1^{(i)}, \dots, f_{k_i}^{(i)}\}$  form measurements for all  $i \in [g]$ , where  $f_{k_i}^{(i)} := \mathbb{1} - \sum_{j=1}^{k_i-1} f_j^{(i)}$ .

*Proof.* This follows immediately from Propositions 6.9 and 5.2.

An alternative proof is as follows. Let  $E_i := \text{span}\{1_{k_i}, v_j^{(k_i)}, j \in [k_i - 1]\}$ . Corollaries B.6 and C.18 yield that the inclusion is true if and only if

$$\mathcal{D}_{(1_{k_i}, v^{(k_i)})}(\mathbb{R}, E_i^+) \subseteq \mathcal{D}_{(\mathbb{1}, f^{(i)})}(\mathbb{R}, A^+) \quad \forall i \in [g].$$

From [BN20, Lemma 4.3], it follows that the extreme rays of  $\mathcal{D}_{(1_{k_i}, v^{(k_i)})}(\mathbb{R}, E_i^+)$  are

$$\mathbb{R}_+(1, \frac{k_i}{2} e_j) \quad \forall j \in [k_i - 1], \quad \mathbb{R}_+(1, -\frac{k_i}{2} (1, \dots, 1)).$$

Here,  $\{e_j\}_{j \in [k_i - 1]}$  is the standard basis in  $\mathbb{R}^{k_i - 1}$ . Thus, the inclusion is equivalent to

$$k_i f_j^{(i)} \in A^+ \quad \forall j \in [k_i - 1] \quad \wedge \quad k_i \mathbb{1} - \sum_{j=1}^{k_i-1} k_i f_j^{(i)} \in A^+$$

for all  $i \in [g]$ . Dividing by  $k_i$  proves the assertion.  $\square$

**Theorem 6.11.** *Let  $\mathbf{k} \in \mathbb{N}^g$  and let  $(V, V^+, \mathbb{1})$  be a GPT. Then,  $\mathcal{D}_{\text{GPT}\diamond}(\mathbf{k}; V, V^+) \subseteq \mathcal{D}_f(\mathbf{k}; V, V^+)$  if and only if  $\{f_1^{(i)}, \dots, f_{k_i}^{(i)}\}_{i \in [g]}$  are compatible measurements, where  $f_{k_i}^{(i)} := \mathbb{1} - \sum_{j=1}^{k_i-1} f_j^{(i)}$  for all  $i \in [g]$ .*

*Proof.* Follows by Propositions 6.9, B.8 and Theorem 5.3.  $\square$

The characterization of extendable maps in Proposition B.8 gives us another condition for compatibility.

**Corollary 6.12.** *Let  $(V, V^+, \mathbb{1})$  be a GPT. Elements  $\{f_1^{(i)}, \dots, f_{k_i}^{(i)}\} \in A^{k_i}$  form compatible measurements for all  $i \in [g]$  if and only if*

$$\forall z \in \mathcal{D}_{\text{GPT}\diamond}(\mathbf{k}; V, V^+), \quad \mathbb{1}(z_0) \geq \sum_{i=1}^g \sum_{j=1}^{k_i-1} \langle \frac{2}{k_i} \mathbb{1} - 2f_j^{(i)}, z_j^{(i)} \rangle. \quad (20)$$

*Proof.* The equation above is equivalent to the positivity of the linear form  $s_\Phi$  from Proposition B.8.  $\square$

*Remark 6.13.* Note that the last condition appears much weaker than the requirement from Theorem 6.11. Indeed, the condition in the above theorem reads

$$y_{f,z} := \mathbb{1} \otimes z_0 + \sum_{i=1}^g \sum_{j=1}^{k_i-1} \left[ 2f_j^{(i)} - \frac{2}{k_i} \mathbb{1} \right] \otimes z_j^{(i)} \in A^+ \otimes_{\min} V^+,$$

$$\forall z = (z_0, z_j^{(i)})_{ij} \in \mathcal{D}_{\text{GPT}\diamond}(\mathbf{k}; V, V^+),$$

whereas the condition in Eq. (20) requires only evaluation of  $y_{f,z}$  against the tensor  $\chi_V$  from Eq. (46) (instead of all elements of  $V^+ \otimes_{\max} A^+$ ). Note that this means that if  $C \subset A \otimes V$  is a tensor cone such that  $\chi_V \in C^*$ , then we may replace the cone  $A^+ \otimes_{\min} V^+$  in Definition 6.8 (with  $L = V$ ) by  $C$ . If we have a reasonable family of tensor cones  $C_L \subset A \otimes L$  for a family of proper ordered vector spaces  $(L, L^+)$  containing  $(V, V^+)$  and such that  $C_V = C$ , then we may replace the cones in Definition 6.8 for all such  $L$  (cf. Proposition A.7). This would lead to somewhat different definitions, but equivalent in the two extreme cases ( $L = \mathbb{R}$  and  $L = V$ ). Compare this to [BN18, Theorem 5.3].

The “intermediate” cases of Proposition 6.9 can be also related to compatibility, as follows. Let  $(L, L^+)$  be an ordered vector space with  $L^+$  closed and let  $\Psi : (A, A^+) \rightarrow (L^*, (L^+)^*)$  be a positive map such that  $\Psi(\mathbb{1}) \in \text{int}((L^+)^*)$ , then  $\Psi(\mathbb{1})$  is an order unit in  $(L^*, (L^+)^*)$  and we can think of the triple  $(L, L^+, \Psi(\mathbb{1}))$  as describing a GPT. Clearly, if  $f = \{f_1, \dots, f_k\}$  is a measurement for  $(V, V^+, \mathbb{1})$ , then  $\Psi(f) = \{\Psi(f_1), \dots, \Psi(f_k)\}$  is a measurement for  $(L, L^+, \Psi(\mathbb{1}))$ .

**Proposition 6.14.** *Let  $(L, L^+)$  be a proper ordered vector space. Then  $\mathcal{D}_{\text{GPT}\diamond}(\mathbf{k}; L, L^+) \subseteq \mathcal{D}_f(\mathbf{k}; L, L^+)$  if and only if for any positive map  $\Psi : (A, A^+) \rightarrow (L^*, (L^+)^*)$  such that  $\Psi(\mathbb{1}) \in \text{int}(L^+)^*$ , the elements  $\Psi(f_j^{(i)})$ ,  $j \in [k_i]$ ,  $i \in [g]$  form a collection of compatible measurements on  $(L, L^+, \Psi(\mathbb{1}))$ .*

*Proof.* We start by observing that for a proper cone  $C$ ,  $\varphi \in \text{int } C^*$  if and only if  $\varphi(c) > 0$  for all  $c \in C \setminus \{0\}$  [AT07, Theorem 3.5]. Let  $\psi \in \text{int}(V^+ \otimes_{\max} (L^+)^*)$ . Then,  $\langle \psi, \alpha \otimes$

$v\rangle > 0$  for all  $\alpha \in A^+$ ,  $v \in L^+$  such that  $\alpha \otimes v \neq 0$ . Going to the associated positive map  $\Psi : (A, A^+) \rightarrow (L^*, (L^+)^*)$ , we have

$$\langle \psi, \alpha \otimes v \rangle = \langle \Psi(\alpha), v \rangle.$$

Thus,  $\Psi(\mathbb{1}) \in \text{int}(L^+)^*$  for all  $\psi \in \text{int}(V^+ \otimes_{\max} (L^+)^*)$ .

For  $z = (z_0, z_j^{(i)})_{ij} \in \mathcal{D}_{\text{GPT}\diamond}(\mathbf{k}; L, L^+)$  let  $y_z$  be the corresponding element in  $E_{\mathbf{k}}^+ \otimes_{\max} L^+$ . Then the inclusion is equivalent to

$$\langle \psi, (\Phi^{(f)} \otimes \text{id})(y_z) \rangle \geq 0, \quad \forall z \in \mathcal{D}_{\text{GPT}\diamond}(\mathbf{k}; L, L^+), \quad \forall \psi \in V^+ \otimes_{\max} (L^+)^*. \quad (21)$$

Let  $\Psi : (A, A^+) \rightarrow (L^*, (L^+)^*)$  be the positive map corresponding to  $\psi$ . Then

$$\langle \psi, (\Phi^{(f)} \otimes \text{id})(y_z) \rangle = \langle \Psi(\mathbb{1}), z_0 \rangle + \sum_{i=1}^g \sum_{j=1}^{k_i-1} \langle 2\Psi(f_j^{(i)}) - \frac{2}{k_i} \Psi(\mathbb{1}), z_j^{(i)} \rangle.$$

By density, we can restrict to  $\psi$  such that  $\Psi(\mathbb{1}) \in \text{int}(L^+)^*$  in Eq. (21) since the set of such elements contains  $\text{int}(V^+ \otimes_{\max} (L^+)^*)$  as argued above. The statement now follows by Corollary 6.12.  $\square$

*Remark 6.15.* The set of positive maps  $\Psi$ , or vectors  $\psi \in V^+ \otimes_{\max} (L^+)^*$ , can be restricted to extremal elements. In general, however, extremal positive maps are difficult to characterize. If the generalized spectrahedra  $\mathcal{D}_f(\mathbf{k}, L, L^+)$  are defined as in Remark 6.13 by a family of cones  $\mathcal{C}_L \subset A \otimes L$ , then it is enough to consider maps such that  $\Psi \otimes \text{id}$  is positive with respect to this family and using extremal maps with this property might be more convenient.

As an example, consider the case of quantum systems  $\text{QM}_d$  from Example 4.2. Here, the canonical tensor  $\chi_V$  is the maximally entangled state which is clearly an element in  $\text{PSD}_{d^2}$ . So if we restrict the intermediate cases to  $(L, L^+) = (\mathcal{M}_k^{\text{sa}}(\mathbb{C}), \text{PSD}_k)$ , we can replace the minimal tensor product  $\text{PSD}_d \otimes_{\min} \text{PSD}_k$  by the larger cone  $\text{PSD}_{dk}$ . Accordingly, we can reduce to extremal completely positive maps  $\mathcal{M}_d(\mathbb{C}) \rightarrow \mathcal{M}_k(\mathbb{C})$ , which are of the form  $X \mapsto A^* X A$  for some  $A : \mathbb{C}^k \rightarrow \mathbb{C}^d$ . The requirement  $\Psi(\mathbb{1}) \in \text{int} \text{PSD}_k$  is met if  $A^* A > 0$ . If  $k \leq d$ , it is easy to see that  $A^* f^{(i)} A$  are compatible for all  $A$  if and only if they are compatible for all isometries  $A : \mathbb{C}^k \rightarrow \mathbb{C}^d$ , see [BN18, Theorem 5.3].

**6.3. Compatibility region and inclusion constants.** We can now relate the compatibility region  $\Gamma(\mathbf{k}, V, V^+)$  (Definition 4.10) to the inclusion constants defined in Appendix C (Definition C.11). Recall that the inclusion constants are the tuples  $s \in [0, 1]^g$  making the implication

$$\mathcal{D}_a(\mathbb{R}, M^+) \subseteq \mathcal{D}_b(\mathbb{R}, N^+) \implies (1, s) \cdot \mathcal{D}_a(L, C_M) \subseteq \mathcal{D}_b(L, C_N)$$

true for  $M^+, N^+$  containing order units. The following is a restriction of the set of inclusion constants, where we require the coefficients by which we scale to be the same on some elements.

**Definition 6.16.** Given a GPT  $(V, V^+, \mathbb{1})$ ,  $g \in \mathbb{N}$  and  $\mathbf{k} \in \mathbb{N}^g$ , we define the set of inclusion constants for the  $(\mathbf{k}; V, V^+)$ -jewel as

$$\begin{aligned} \Delta(\mathbf{k}; V, V^+) &:= \{s \in [0, 1]^g : \forall f_j^{(i)} \in A, j \in [k_i - 1], i \in [g], \\ &\quad \mathcal{D}_{\text{GPT}\diamond}(\mathbf{k}; \mathbb{R}, \mathbb{R}^+) \subseteq \mathcal{D}_f(\mathbf{k}; \mathbb{R}, \mathbb{R}^+) \\ &\implies (1, s_1^{\times(k_1-1)}, \dots, s_g^{\times(k_g-1)}) \cdot \mathcal{D}_{\text{GPT}\diamond}(\mathbf{k}; V, V^+) \subseteq \mathcal{D}_f(\mathbf{k}; V, V^+)\}. \end{aligned}$$

If  $\mathbf{k} = 2^{\times g}$ , we will just write  $\Delta(g; V, V^+)$ .

*Remark 6.17.* The notation  $\Delta(\mathbf{k}; V, V^+)$  introduced above should not be confused with the notation  $\Delta_k$  for the  $(k - 1)$ -dimensional probability simplex, used extensively in Sect. 5.

We can now prove that the set of inclusion constants for the GPT jewel is precisely the compatibility region of the GPT introduced in Definition 4.10. The result below connects the operationally defined compatibility region with the geometrical set of inclusion constants.

**Theorem 6.18.** *Given a GPT  $(V, V^+, \mathbb{1})$ , it holds that  $\Gamma(\mathbf{k}; V, V^+) = \Delta(\mathbf{k}; V, V^+)$ .*

*Proof.* From Proposition 6.10 we infer that the inclusion  $\mathcal{D}_{\text{GPT}\diamond}(\mathbf{k}; \mathbb{R}, \mathbb{R}^+) \subseteq \mathcal{D}_f(\mathbf{k}; \mathbb{R}, \mathbb{R}^+)$  holds if and only if the  $\{f_1^{(i)}, \dots, f_{k_i}^{(i)}\}$  are measurements for all  $i \in [g]$ . Here,  $f_{k_i}^{(i)} = \mathbb{1} - \sum_{j=1}^{k_i-1} f_j^{(i)}$  for all  $i \in [g]$ . The statement then is an easy consequence of Theorem 6.11 and the following equivalence:

$$\begin{aligned} (1, s_1^{\times(k_1-1)}, \dots, s_g^{\times(k_g-1)}) \cdot \mathcal{D}_{\text{GPT}\diamond}(\mathbf{k}; V, V^+) &\subseteq \mathcal{D}_f(\mathbf{k}; V, V^+) \\ \iff \mathbb{1} \otimes z_0 + \sum_{i=1}^g \sum_{j=1}^{k_i-1} s_i \left( 2f_j^{(i)} - \frac{2}{k_i} \mathbb{1} \right) \otimes z_j^{(i)} &\in A^+ \otimes_{\min} V^+ \quad \forall z \in \mathcal{D}_{\text{GPT}\diamond}(\mathbf{k}; V, V^+) \\ \iff \mathcal{D}_{\text{GPT}\diamond}(\mathbf{k}; V, V^+) &\subseteq \mathcal{D}_{f'}(\mathbf{k}; V, V^+), \end{aligned}$$

where  $(f')_j^{(i)} = s_i f_j^{(i)} + (1 - s_i) \frac{1}{k_i} \mathbb{1}$  for  $j \in [k_i - 1], i \in [g]$ . □

**6.4. Inclusion constants from symmetrization.** The aim of this section is to show that we can obtain bounds on the inclusion set of the GPT jewel from consideration of the inclusion set of the GPT diamond. As the latter has more symmetries, it is much easier to work with. Using Theorem 6.18, we obtain bounds on measurements with  $\mathbf{k}$  outcomes derived from the compatibility region for dichotomic measurements. This section is inspired by [BN20, Section 7].

The following is the GPT analogue of Theorem 7.2 of [BN20]. Its proof is very similar.

**Theorem 6.19.** *Let  $g \in \mathbb{N}$ ,  $k_j \in \mathbb{N}$  for all  $j \in [g]$ . Let  $(V, V^+, \mathbb{1})$  be a GPT. Then,*

$$\left( (k_1 - 1)^{-2}, \dots, (k_g - 1)^{-2} \right) \cdot \Delta \left( \sum_{i=1}^g (k_i - 1); V, V^+ \right) \subseteq \Delta(\mathbf{k}; V, V^+).$$

*Proof.* Let  $\bar{k} = \sum_{i=1}^g (k_i - 1)$ . First, we shall find conditions for  $\lambda_i \in [0, 1]$  and  $\mu_i \in [0, 1]$ ,  $i \in [g]$ , such that

$$\lambda \cdot \mathcal{D}_{\text{GPT}\diamond}(\bar{k}; \mathbb{R}, \mathbb{R}^+) \subseteq \mathcal{D}_{\text{GPT}\diamond}(\mathbf{k}; \mathbb{R}, \mathbb{R}^+)$$

and

$$\mu \cdot \mathcal{D}_{\text{GPT}\diamond}(\mathbf{k}; \mathbb{R}, \mathbb{R}^+) \subseteq \mathcal{D}_{\text{GPT}\diamond}(\bar{k}; \mathbb{R}, \mathbb{R}^+),$$

where  $\lambda = (1, \lambda_1^{\times(k_1-1)}, \dots, \lambda_g^{\times(k_g-1)})$  and  $\mu = (1, \mu_1^{\times(k_1-1)}, \dots, \mu_g^{\times(k_g-1)})$ .

By Corollary C.18, it is enough to consider

$$(1, \lambda_i^{\times(k_i-1)}) \cdot \mathcal{D}_{\text{GPT}\diamond}((k_i - 1); \mathbb{R}, \mathbb{R}^+) \subseteq \mathcal{D}_{\text{GPT}\diamond}(k_i; \mathbb{R}, \mathbb{R}^+)$$

and

$$(1, \mu_i^{\times(k_i-1)}) \cdot \mathcal{D}_{\text{GPT}\diamond}(k_i; \mathbb{R}, \mathbb{R}^+) \subseteq \mathcal{D}_{\text{GPT}\diamond}(k_i - 1; \mathbb{R}, \mathbb{R}^+).$$

Note that  $\mathcal{D}_{1_{\mathbf{k}} \otimes a}(L, E_{\mathbf{k}} \otimes_{\max} N^+) = \mathcal{D}_a(L, N^+)$  by Lemma 6.2, where  $(L, L^+)$  and  $(M, M^+)$  are proper ordered vector,  $N^+$  a tensor cone for  $M^+$  and  $L^+$  and  $1_{\mathbf{k}} \otimes a = (1_{\mathbf{k}} \otimes a_1, \dots, 1_{\mathbf{k}} \otimes a_n) \in (E_{\mathbf{k}} \otimes M)^n$ ,  $n \in \mathbb{N}$ . From Lemma 4.3 of [BN20], we know that the extreme rays of  $\mathcal{D}_{\text{GPT}\diamond}(k_i - 1; \mathbb{R}, \mathbb{R}^+)$  are

$$\mathbb{R}_+(1, \pm e_j), \quad j \in [k_i - 1]$$

while the extreme rays of  $\mathcal{D}_{\text{GPT}\diamond}(k_i; \mathbb{R}, \mathbb{R}^+)$  are

$$\mathbb{R}_+\left(1, \frac{k_i}{2}e_j\right), \quad j \in [k_i - 1] \quad \text{and} \quad \mathbb{R}_+\left(1, -\frac{k_i}{2}\underbrace{(1, \dots, 1)}_{k_i-1}\right).$$

We find thus that  $(1, \pm \lambda_i e_j) \in \mathcal{D}_{\text{GPT}\diamond}(k_i; \mathbb{R}, \mathbb{R}^+)$  if

$$\pm \lambda_i e_j \in \left[-\frac{k_i}{2(k_i - 1)}e_j, \frac{k_i}{2}e_j\right]$$

since

$$-\frac{k_i}{2(k_i - 1)}e_j = \frac{1}{k_i - 1} \left( \sum_{l \in [k_i-1] \setminus j} \frac{k_i}{2}e_l - \frac{k_i}{2}(1, \dots, 1) \right).$$

Thus,  $\lambda_i \leq k_i/(2(k_i - 1))$ . Considering the  $\ell_1$  norm of the extreme points of  $\mathcal{D}_{\text{GPT}\diamond}(k_i; \mathbb{R}, \mathbb{R}^+) \cap \{(x_0, \dots, x_{k_i-1}) : x_0 = 1\}$ , we infer moreover that  $\mu_i \leq 2/(k_i(k_i - 1))$ . Proposition C.6 and Remark 6.4 together imply furthermore that from

$$\mu \cdot \mathcal{D}_{\text{GPT}\diamond}(\mathbf{k}; \mathbb{R}, \mathbb{R}^+) \subseteq \mathcal{D}_{\text{GPT}\diamond}(\bar{k}; \mathbb{R}, \mathbb{R}^+),$$

follows

$$\mu \cdot \mathcal{D}_{\text{GPT}\diamond}(\mathbf{k}; V, V^+) \subseteq \mathcal{D}_{\text{GPT}\diamond}(\bar{k}; V, V^+).$$

Let  $s \in \Delta(\bar{k}; V, V^+)$  and let  $\circ$  denote the entrywise product of vectors. Let  $f_j^{(i)} \in A$  for all  $i \in [g]$  and  $j \in [k_i - 1]$ . Then, by the above, we have the implication

$$\lambda \cdot \mathcal{D}_{\text{GPT}\diamond}(\bar{k}; \mathbb{R}, \mathbb{R}^+) \subseteq \mathcal{D}_{\text{GPT}\diamond}(\mathbf{k}; \mathbb{R}, \mathbb{R}^+) \subseteq \mathcal{D}_f(\mathbf{k}; \mathbb{R}, \mathbb{R}^+)$$

$$\implies \mu \circ \lambda \circ s \cdot \mathcal{D}_{\text{GPT}\diamond}(\mathbf{k}; V, V^+) \subseteq \lambda \circ s \cdot \mathcal{D}_{\text{GPT}\diamond}(\bar{k}; V, V^+) \subseteq \mathcal{D}_f(\mathbf{k}; V, V^+),$$

from which  $\mu \circ \lambda \circ s \in \Delta(\mathbf{k}, V, V^+)$  and the assertion follows since  $\lambda_i \cdot \mu_i \leq (k_i - 1)^{-2}$ .  $\square$

## 7. Tensor Crossnorms and Compatibility of Effects

In this section, we will only consider dichotomic measurements, that is, with outcomes in  $\{1, 2\}$ ; in other words, we set  $k_1 = \dots = k_g = 2$ . Any such measurement is determined by an effect in  $A^+$  related to the probability of obtaining the outcome 1. We will say that a collection of effects is compatible if the corresponding collection of binary measurements is compatible. For simplicity, we will write  $E_g$  for  $E_{\mathbf{k}}$  with  $\mathbf{k} = 2^{\times g}$ .

**7.1.  $E_g$  as a centrally symmetric GPT.** Let us begin by studying more in-depth the space  $E_g$  and see that it gives itself rise to a GPT. Recalling the notations of Sects. 5 and 6.1, the distinguished basis of  $E_g$  is  $w = \{1_g, c_i\}$ , where

$$\begin{aligned} 1_g &= (1, 1, \dots, 1) \\ c_i &= (1, 1)^{\otimes(i-1)} \otimes (1, -1) \otimes (1, 1)^{\otimes(g-i)} \end{aligned}$$

and the dual basis  $w^*$  in the space  $E_g^* \cong E_g$  is  $w^* = \{\check{1}_g, \check{c}_i\}$ , where

$$\check{1}_g := 2^{-g} 1_g, \quad \check{c}_i := 2^{-g} c_i.$$

Let us point out that  $(E_g, E_g^+, \check{1}_g)$  is a centrally symmetric GPT in the sense of [LPW18, Definition 25] corresponding to the  $\ell_1$  norm in  $\mathbb{R}^g$  (see also Sect. 4.3). Indeed, we clearly have  $E_g = \mathbb{R}1_g \oplus \bar{E}_g$ , where

$$\bar{E}_g = \text{span}\{c_i, i \in [g]\} = \text{span}\{\check{c}_i, i \in [g]\}$$

and an element  $a1_g + \sum_i x_i c_i \in E_g$  is in  $E_g^+$  if and only if

$$a + \sum_{i=1}^g \varepsilon_i x_i \geq 0, \quad \forall \varepsilon \in \{\pm 1\}^g,$$

as can be seen e.g. from Remark 6.3 (with  $L = \mathbb{R}$ ). This is equivalent to

$$a \geq \sum_{i=1}^g |x_i| = \left\| \sum_i x_i e_i^{(g)} \right\|_1,$$

here  $\{e_i^{(g)}\}$  is the canonical basis in  $\mathbb{R}^g$  and  $\|\cdot\|_1$  is the  $\ell_1$  norm. It follows that the subspace  $\bar{E}_g$  endowed with the restriction of the base norm  $\max\{|a|, \sum_i |x_i|\}$  can be identified with  $\ell_1^g$ , via the isometry

$$c_i \mapsto e_i^{(g)}, \quad i \in [g].$$

By duality, positivity in  $(E_g^*, (E_g^+)^*)$  is characterized by the  $\ell_\infty$  norm and the map

$$\check{c}_i \mapsto e_i^{(g)}, \quad i \in [g]$$

is an isometry of  $(\bar{E}_g, \|\cdot\|_{\check{1}_g})$  onto  $\ell_\infty^g$ . These identifications will be used throughout.



Since the  $\ell_1$  and  $\ell_\infty$  norms are invariant under sign changes of the coordinates, the map

$$\sigma_\varepsilon : a1_g + \sum_i x_i c_i \mapsto a1_g + \sum_i \varepsilon_i x_i c_i \quad (22)$$

is an order isomorphism of both  $(E_g, E_g^+)$  and  $(E_g^*, (E_g^+)^*)$ , for any sign vector  $\varepsilon \in \{\pm 1\}^g$ . This implies that for any proper ordered vector space  $(L, L^+)$  and  $z \in E_g \otimes L$  we have

$$\begin{aligned} z \in E_g^+ \otimes_{\min} L^+ &\iff (\sigma_\varepsilon \otimes \text{id})(z) \in E_g^+ \otimes_{\min} L^+ \\ z \in E_g^+ \otimes_{\max} L^+ &\iff (\sigma_\varepsilon \otimes \text{id})(z) \in E_g^+ \otimes_{\max} L^+ \end{aligned}$$

for any  $\varepsilon \in \{\pm 1\}^g$ , and similarly for  $(E_g^+)^*$ .

**7.2. Effects and tensor crossnorms.** Now we can make the connection between the compatibility of effects and norms on their corresponding tensors. Let  $(V, V^+, \mathbb{1})$  be a GPT. In addition, let  $f = (f_1, \dots, f_g)$  be  $g$ -tuple of elements in  $A$  and let

$$\varphi^{(f)} = \check{1}_g \otimes \mathbb{1} + \underbrace{\sum_{i=1}^g \check{c}_i \otimes (2f_i - \mathbb{1})}_{\bar{\varphi}^{(f)}}. \quad (23)$$

By the results of Sect. 5,  $f$  is a collection of effects if and only if  $\varphi^{(f)} \in (E_g^+)^* \otimes_{\max} A^+$  and  $f$  is compatible if and only if  $\varphi^{(f)} \in (E_g^+)^* \otimes_{\min} A^+$ . Let  $\sigma_\varepsilon$  for  $\varepsilon \in \{\pm 1\}^g$  be as in Eq. (22). Then

$$(\sigma_\varepsilon \otimes \text{id})(\varphi^{(f)}) = \check{1}_g \otimes \mathbb{1} + \sum_{i=1}^g \check{c}_i \otimes \varepsilon_i (2f_i - \mathbb{1}) = \varphi^{(f_\varepsilon)}, \quad (24)$$

where  $(f_\varepsilon)_i = f_i$  if  $\varepsilon_i = 1$  and  $(f_\varepsilon)_i = 1 - f_i$  if  $\varepsilon_i = -1$ . The invariance of the maximal and minimal tensor products under  $\sigma_\varepsilon \otimes \text{id}$  is a manifestation of the fact that a relabelling of measurement outcomes defines again a measurement and that such a relabelling does not change compatibility of the effects under study.

Our aim in this section is to make a connection between compatibility and reasonable crossnorms of Banach spaces (see Appendix A.4). The results of [LPW18] imply that any base norm on a bipartite GPT is a reasonable crossnorm.

**Proposition 7.1.** *Consider the GPT  $(V_A \otimes V_B, V_A^+ \otimes V_B^+, \mathbb{1}_A \otimes \mathbb{1}_B)$ , where  $(V_\#, V_\#^+, \mathbb{1}_\#)$  are GPTs for  $\# \in \{A, B\}$  and  $V_A^+ \otimes V_B^+$  is some proper tensor cone for  $V_A^+, V_B^+$ . Then, the base norm  $\|\cdot\|_{V_A^+ \otimes V_B^+}$  is a reasonable crossnorm.*

*Proof.* Let  $x \in V_A \otimes V_B$ . From [LPW18, Equation (26)], it holds that  $\|x\|_{V_A^+ \otimes V_B^+} \leq \|x\|_{V_A^+ \otimes_{\min} V_B^+}$ . Together with [LPW18, Proposition 22], this implies that  $\|x\|_{V_A^+ \otimes V_B^+} \leq \|x\|_\pi$ . Moreover, [LPW18, Proposition 22] implies  $\|x\|_\varepsilon \leq \|x\|_{\text{LO}}$ , where we refer to [LPW18] for the definition of the latter norm. From the discussion after [LPW18, Definition 6], it follows that  $\|x\|_{\text{LO}} \leq \|x\|_{V_A^+ \otimes V_B^+}$ . Thus,

$$\|x\|_\varepsilon \leq \|x\|_{V_A^+ \otimes V_B^+} \leq \|x\|_\pi.$$

Therefore, the assertion follows from point (a) of Proposition A.14.  $\square$

We now return to compatible tuples of effects. In this paragraph,  $A$  and  $\bar{E}_g$  will always be endowed with the order unit norms  $\|\cdot\|_{\mathbb{1}}$  and  $\|\cdot\|_{\check{\mathbb{1}}_g}$ .

**Theorem 7.2.** *Let  $f = (f_1, \dots, f_g)$  be a  $g$ -tuple of elements in  $A$  and let*

$$\bar{\varphi}^{(f)} = \sum_{i=1}^g \check{c}_i \otimes (2f_i - \mathbb{1}).$$

*Then we have*

- (1) *Let  $\|\cdot\|_{\varepsilon}$  be the injective crossnorm in  $\ell_{\infty}^g \otimes A$ . Then  $f$  is a collection of effects if and only if*

$$\|\bar{\varphi}^{(f)}\|_{\varepsilon} \leq 1.$$

- (2) *There is a reasonable crossnorm  $\|\cdot\|_c$  in  $\ell_{\infty}^g \otimes A$ , called the compatibility norm, such that  $f$  is a compatible collection of effects if and only if*

$$\|\bar{\varphi}^{(f)}\|_c \leq 1.$$

*Proof.* (1) can be easily observed directly from

$$\|\bar{\varphi}^{(f)}\|_{\varepsilon} = \left\| \sum_{i=1}^g \check{c}_i \otimes (2f_i - \mathbb{1}) \right\|_{\varepsilon} = \max_{i \in [g]} \|2f_i - \mathbb{1}\|_{\mathbb{1}},$$

since  $\ell_{\infty} \otimes_{\varepsilon} X \cong \ell_{\infty}(X)$  as Banach spaces [Rya02, Example 3.3]. Recall that  $\|x\|_{\mathbb{1}} = \inf\{t \geq 0 : x \in t[-\mathbb{1}, \mathbb{1}]\}$ . For (2), note that we have by Theorem 5.3 and Eq. (24) that  $f$  is a compatible collection of effects if and only if

$$\check{\mathbb{I}}_g \otimes \mathbb{1} \pm \bar{\varphi}^{(f)} \in (E_g^*)^+ \otimes_{\min} A^+,$$

which is equivalent to

$$\|\bar{\varphi}^{(f)}\|_{\check{\mathbb{I}}_g \otimes \mathbb{1}} \leq 1.$$

Here  $\|\cdot\|_{\check{\mathbb{I}}_g \otimes \mathbb{1}}$  is the order unit norm for the tensor product GPT  $(E_g \otimes V, E_g^+ \otimes_{\max} V^+, \check{\mathbb{I}}_g \otimes \mathbb{1})$ . As the dual to the base norm which is a reasonable crossnorm by Proposition 7.1,  $\|\cdot\|_{\check{\mathbb{I}}_g \otimes \mathbb{1}}$  is a reasonable crossnorm in  $E_g^* \otimes A$ . Hence we may define the norm  $\|\cdot\|_c$  as the restriction of the order unit norm to  $\bar{E}_g \otimes A$ .

Note that part (1) can be proved similarly using the order unit norm for the GPT  $(E_g \otimes V, E_g^+ \otimes_{\min} V^+, \check{\mathbb{I}}_g \otimes \mathbb{1})$ . The result follows from the fact that by [LPW18, Proposition 22] the base norm in this case is equal to the projective norm of the two base norm spaces.  $\square$

**Question 7.3.** *The symmetry of  $E_g$  is crucial in the above proof. For general  $E_{\mathbf{k}}$ , however, such symmetry is not present, so the proof does not immediately extend. It is an interesting question whether there is a characterization similar to Theorem 7.2 also for measurements with more than two outcomes.*

We derive now a more explicit form of the crossnorm  $\|\cdot\|_c$ .

**Proposition 7.4.** *Let  $\bar{\varphi} \in \ell_\infty^g \otimes A$ , then*

$$\|\bar{\varphi}\|_c = \inf \left\{ \left\| \sum_j h_j \right\|_{\mathbb{1}}, \bar{\varphi} = \sum_j z_j \otimes h_j, \|z_j\|_\infty = 1, h_j \in A^+ \right\}. \quad (25)$$

*Proof.* Note that  $\bar{\varphi} = \bar{\varphi}^{(f)}$  for some  $g$ -tuple of elements in  $A$ . By definition,

$$\begin{aligned} \|\bar{\varphi}^{(f)}\|_c &= \inf \{ \lambda > 0, \lambda \check{1}_g \otimes \mathbb{1} \pm \bar{\varphi}^{(f)} \in (E_g^+)^* \otimes_{\min} A^+ \} \\ &= \inf \{ \lambda > 0, \lambda \check{1}_g \otimes \mathbb{1} - \bar{\varphi}^{(f)} \in (E_g^+)^* \otimes_{\min} A^+ \} \\ &= \inf \{ \lambda > 0, \lambda \check{1}_g \otimes \mathbb{1} - \bar{\varphi}^{(f)} = \sum_j x_j \otimes h_j, x_j \in (E_g^+)^*, h_j \in A^+ \}, \end{aligned}$$

where we have used Eq. (24) in the second equality. Assume that  $\lambda \check{1}_g \otimes \mathbb{1} - \bar{\varphi}^{(f)} = \sum_j x_j \otimes h_j$  for some  $\lambda > 0$  and  $x_j \in (E_g^+)^*, h_j \in A^+$ . We may identify any  $x_j \in (E_g^+)^*$  with a pair  $(a_j, z_j)$  with  $\|z_j\|_\infty \leq a_j$ , and then

$$\sum_j x_j \otimes h_j = \check{1}_g \otimes \sum_j a_j h_j + \sum_j z_j \otimes h_j.$$

By some easy reshuffling, we may always assume that  $a_j = 1$ . Hence

$$\bar{\varphi}^{(f)} = \check{1}_g \otimes \left( \lambda \mathbb{1} - \sum_j h_j \right) + \sum_j z_j \otimes h_j$$

and since  $\bar{\varphi}^{(f)} \in \bar{E}_g \otimes A$ , we must have  $\lambda \mathbb{1} = \sum_j h_j$ . Since  $\|z_j\|_\infty \leq 1$ , we may normalize  $z_j$  and obtain

$$\bar{\varphi}^{(f)} = \sum_j \|z_j\|_\infty^{-1} z_j \otimes \|z_j\|_\infty h_j, \quad \|z_j\|_\infty h_j \in A^+, \quad \sum_j \|z_j\|_\infty h_j \leq \sum_j h_j = \lambda \mathbb{1}.$$

This finishes the proof.  $\square$

**Remark 7.5.** Note that the unit ball of  $\|\cdot\|_c$  can be written as the set

$$\left\{ \sum_i z_i \otimes h_i, \|z_i\|_\infty \leq 1, h_i \in A^+, \sum_i h_i = \mathbb{1} \right\}.$$

Alternatively, we can identify the unit ball of  $\|\cdot\|_c$  with

$$\{(v_1, \dots, v_g) \in A^g : (\mathbb{1}, v_1, \dots, v_g) \in \mathcal{D}_{\min}(\mathcal{C}; A, A^+)\},$$

where

$$\mathcal{C} := \{(x, \bar{x}) : x \in \mathbb{R}, \bar{x} \in \mathbb{R}^g, x \geq \|\bar{x}\|_\infty\}$$

and  $\mathcal{D}_{\min}$  is as in Definition C.3.

It is almost immediate from the definition that  $\|\cdot\|_c$  can be computed using a conic program (see Appendix E.3 for a nicer dual formulation).

We next describe the compatibility region as the set of inclusion constants for the crossnorms  $\|\cdot\|_\varepsilon$  and  $\|\cdot\|_c$ . For any element  $\bar{\varphi} = \sum_i \check{c}_i \otimes p_i$  and any  $s \in \mathbb{R}^g$ , we define

$$s.\varphi := \sum_i \check{c}_i \otimes s_i p_i.$$

Given a  $g$ -tuple of elements  $f = (f_1, \dots, f_g) \in A^g$  and  $\bar{\varphi}^{(f)} = \sum_i \check{c}_i \otimes (2f_i - \mathbb{1})$ , we have

$$s.\bar{\varphi}^{(f)} = \sum_i \check{c}_i \otimes s_i (2f_i - \mathbb{1}) = \sum_i \check{c}_i \otimes (2(s_i f_i + (1 - s_i)\mathbb{1}/2) - \mathbb{1}) = \bar{\varphi}^{(f_s)},$$

where  $f_s = (s_1 f_1 + (1 - s_1)\mathbb{1}/2, \dots, s_g f_g + (1 - s_g)\mathbb{1}/2)$ . If  $f$  is a  $g$ -tuple of effects, then  $f_s$  is a  $g$ -tuple of effects as well, obtained by mixing each  $f_i$  with the trivial effect  $\mathbb{1}/2$ . From the definition of the compatibility region and Theorem 7.2, we obtain

$$\Gamma(f) = \{s \in [0, 1]^g, \|s.\bar{\varphi}^{(f)}\|_c \leq 1\}, \quad (26)$$

in particular

$$\gamma(f) = 1/\|\bar{\varphi}^{(f)}\|_c. \quad (27)$$

The following result is now immediate.

**Theorem 7.6.** *We have*

$$\Gamma(g; V, V^+) = \{s \in [0, 1]^g, \|s.\varphi\|_c \leq 1, \forall \varphi \in \ell_\infty^g \otimes A, \|\varphi\|_\varepsilon \leq 1\}.$$

*In particular, the compatibility degree satisfies*

$$\gamma(g; V, V^+) = 1/\max_{\|\varphi\|_\varepsilon \leq 1} \|\varphi\|_c.$$

**Example 7.7.** For quantum mechanics  $\text{QM}_d$  ( $A^+ = \text{PSD}_d$ ,  $\mathbb{1} = \text{Tr}$ ), the expression for the unit ball of  $\|\cdot\|_c$  from Remark 7.5 becomes the minimal matrix convex set (of a fixed dimension)  $\mathcal{W}_d^{\min}(B_{\ell_\infty^g})$  (over the unit ball  $B_{\ell_\infty^g}$ , cf. [FNT17, Section 4] or [PSS18, Eq. (1.4)]). Since the unit ball of the injective norm  $\|\cdot\|_\varepsilon$  in  $\ell_\infty^g \otimes (\mathcal{M}_d^{\text{sa}}(\mathbb{C}), \|\cdot\|_\infty)$  is the maximal matrix convex set  $\mathcal{W}_d^{\max}(B_{\ell_\infty^g})$ , the expressions for  $\Gamma$  and  $\gamma$  in Theorem 7.6 correspond precisely to the inclusion constants for minimal and maximal matrix convex sets.

This gives the following lower bound on the compatibility degree, which uses that  $\|\cdot\|_c$  is a reasonable crossnorm, thus being upper bounded by the projective tensor norm.

**Corollary 7.8.** *We have*

$$\gamma(g; V, V^+) \geq 1/\rho(\ell_\infty^g, A),$$

where  $\rho(\ell_\infty^g, A)$  as in Eq. (53).

**Question 7.9.** *It is not clear whether this bound is tight in general. In the case when  $g \leq \dim(V)$ , it is known that the bound  $1/g$  is attained by the hypercube GPT, see [Jen18]. We will find below a tight bound in the special case of centrally symmetric GPTs (see Sect. 8), which is larger than the above bound.*

**7.3. Incompatibility witnesses.** In this section, we will consider different notions of witnesses. We will start with objects certifying that the elements of  $A$  under study are effects before considering objects certifying compatibility. This viewpoint is dual to the one we have hitherto adopted in this section.

We introduce the set of *effect witnesses*

$$\mathcal{Q}_{\text{GPT}\diamond}(g; V, V^+) := \left\{ z \in V^g : \sum_{i=1}^g \|z_i\|_V \leq 1 \right\}. \quad (28)$$

Note that the set above is the unit ball of  $\ell_1^g \otimes_{\pi} (V, \|\cdot\|_V)$  (see Eq. (51)).

**Proposition 7.10.** *Elements  $f_1, \dots, f_g \in A$  are effects (i.e.  $f_i, 1 - f_i \in A^+$  for all  $i \in [g]$ ) if and only if*

$$\sum_{i=1}^g \langle 2f_i - 1, z_i \rangle \leq 1 \quad \forall (z_1, \dots, z_g) \in \mathcal{Q}_{\text{GPT}\diamond}(g; V, V^+).$$

*Proof.* This is straightforward from Theorem 7.2, the duality of the injective and projective norms and the fact that

$$\langle z, \bar{\varphi}^{(f)} \rangle = \sum_{i=1}^g \langle 2f_i - 1, z_i \rangle.$$

□

Similarly, using the results of the previous section, we introduce the following definition:

**Definition 7.11** (Strict incompatibility witnesses). Let  $\|\cdot\|_{c^*}$  be the dual norm to  $\|\cdot\|_c$  in  $\mathbb{R}^g \otimes V$  and let  $\mathcal{P}_{\text{GPT}\diamond}(g; V, V^+)$  denote the unit ball of  $\|\cdot\|_{c^*}$ . An element  $z = (z_1, \dots, z_g) \in \mathcal{P}_{\text{GPT}\diamond}(g; V, V^+)$  is called a *strict incompatibility witness* if

$$\|z\|_{\pi} = \sum_i \|z_i\|_V > 1.$$

The idea behind this definition is clear: a strict incompatibility witness is a functional such that the value  $\langle z, \bar{\varphi}^{(f)} \rangle \leq 1$  for all compatible  $g$ -tuples of effects but there exists some  $f$  with  $\langle z, \bar{\varphi}^{(f)} \rangle > 1$ , so that  $z$  witnesses incompatibility of  $f$ .

**Proposition 7.12.** *Effects  $f_1, \dots, f_g \in A^+$  are compatible if and only if  $\langle z, \bar{\varphi}^{(f)} \rangle \leq 1$  for all strict incompatibility witnesses  $z \in \mathcal{P}_{\text{GPT}\diamond}(g; V, V^+)$ .*

*Proof.* It is clear from Theorem 7.2 that  $\langle z, \bar{\varphi}^{(f)} \rangle \leq 1$  for all  $z \in \mathcal{P}_{\text{GPT}\diamond}(g; V, V^+)$  if and only if the  $f_i$  are compatible effects. The assertion follows since for effects  $f_i$ ,  $\langle z, \bar{\varphi}^{(f)} \rangle \leq 1$  for all  $z \in \mathcal{Q}_{\text{GPT}\diamond}(g; V, V^+)$ . □

A notion of incompatibility witness was also introduced in [BN20]. Translated to the GPT setting, this would be an element  $z = (z_0, z_1, \dots, z_g)$  from  $\mathcal{D}_{\text{GPT}\diamond}(g; V, V^+)$  and  $z$  detects incompatibility for some  $g$ -tuple of effects  $f$  if

$$z \notin \mathcal{D}_f(g; V, V^+).$$

We next show that this notion is closely related to the one introduced here.

**Proposition 7.13.** *Let  $z = (z_1, \dots, z_g) \in V^g$ . Then  $z \in \mathcal{P}_{\text{GPT}\diamond}(g; V, V^+)$  if and only if there is some  $z_0 \in K$  such that*

$$(z_0, z_1, \dots, z_g) \in \mathcal{D}_{\text{GPT}\diamond}(g; V, V^+).$$

Moreover, in this case,

$$(z_0, z_1, \dots, z_g) \notin \mathcal{D}_f(g; V, V^+)$$

for a tuple of effects  $f$  if and only if there exists a positive map  $Y : (V, V^+) \rightarrow (V, V^+)$  such that  $\langle (\text{id} \otimes Y)(z), \bar{\varphi}^{(f)} \rangle > 1$ .

*Proof.* Note that the set

$$\{z \in V^g, \exists z_0 \in K, (z_0, z) \in \mathcal{D}_{\text{GPT}\diamond}(g; V, V^+)\}$$

is convex, closed and contains 0, therefore, by Corollary 6.12 and Theorem 7.2, it is the polar of the unit ball  $B_{\|\cdot\|_c}$ . This implies the first statement.

Furthermore, we have that

$$0 > s_{\Phi(f)} \left( 1_g \otimes z_0 + \sum_{i=1}^g c_i \otimes z_i \right) = \langle \chi, 1 \otimes z_0 + \sum_{i=1}^g (2f_i - 1) \otimes z_i \rangle = 1 - \langle z, \bar{\varphi}^{(f)} \rangle.$$

implies that  $(z_0, z) \notin \mathcal{D}_f(g; V, V^+)$ . Conversely,  $(z_0, z) \notin \mathcal{D}_f(g; V, V^+)$  implies that there is a  $y \in V^+ \otimes_{\max} A^+$  such that

$$0 > \langle y, 1 \otimes z_0 + \sum_{i=1}^g (2f_i - 1) \otimes z_i \rangle.$$

We can find a positive map  $Y' : (V, V^+) \rightarrow (V, V^+)$  such that  $y = (Y' \otimes \text{id})(\chi_V) = (\text{id} \otimes Y'^*)(\chi_V)$ . Observe that we must have  $1(Y'(z_0)) > 0$ . Indeed, assume that  $1(Y'(z_0)) = 0$ , then since  $Y'$  is positive, we obtain from (16) and (17) that  $Y'(z_i) = 0$  for all  $i$ , which would imply  $\langle y, 1 \otimes z_0 + \sum_{i=1}^g (2f_i - 1) \otimes z_i \rangle = 0$ . By normalizing  $Y = (1/1(Y'(z_0)))Y'$  we obtain that  $Y(z_0) \in K$ . It follows thus that

$$\begin{aligned} 0 &> \langle (\text{id} \otimes Y^*)(\chi_V), 1 \otimes z_0 + \sum_{i=1}^g (2f_i - 1) \otimes z_i \rangle \\ &= s_{\Phi(f)} \left( 1_g \otimes Y(z_0) + \sum_{i=1}^g c_i \otimes Y(z_i) \right) = 1 - \langle (\text{id} \otimes Y)z, \bar{\varphi}^{(f)} \rangle. \end{aligned}$$

□

A third notion of incompatibility witness has been introduced in [Jen18]. There, an incompatibility witness is a positive map  $W : (V(P_{\mathbf{k}}), V(P_{\mathbf{k}})^+) \rightarrow (V, V^+)$  such that  $\text{Tr}((\Phi^{(f)})^* W) < 0$ ; recall that  $P_{\mathbf{k}} = \Delta_{k_1} \times \dots \times \Delta_{k_g}$  is the polysimplex from Sect. 5. Using the order isomorphism between  $(V(P_{\mathbf{k}}), V(P_{\mathbf{k}})^+)$  and  $(E_{\mathbf{k}}^*, (E_{\mathbf{k}}^*)^+)$ , these correspond to positive maps  $W : (E_{\mathbf{k}}^*, (E_{\mathbf{k}}^*)^+) \rightarrow (V, V^+)$  such that  $\text{Tr}[(\Phi^{(f)})^* W] < 0$ . We make the connection to elements of  $\mathcal{P}_{\text{GPT}\diamond}(g; V, V^+)$  below.

**Proposition 7.14.** *There is a one-to-one correspondence between  $z \in \mathcal{D}_{\text{GPT}\diamond}(g; V, V^+)$  and positive maps  $W : (E_g^*, (E_g^*)^+) \rightarrow (V, V^+)$ . Moreover,  $z_0 \in K$  if and only if  $\mathbb{1}(W(\check{\mathbb{I}}_g)) = 1$ . Restricting to such  $W$ ,*

$$\text{Tr}[(\Phi^{(f)})^* W] < 0 \quad \Longleftrightarrow \quad \langle z, \bar{\varphi}^f \rangle > 1$$

for a collection of effects  $f$ .

*Proof.* The map  $W : (E_g^*, (E_g^*)^+) \rightarrow (V, V^+)$  is positive if and only if  $\varphi^W \in E_g^+ \otimes_{\max} V^+$ . We can decompose

$$\varphi^W = 1_g \otimes z_0 + \sum_{i=1}^g c_i \otimes z_i$$

and  $\varphi^W \in E_g^+ \otimes_{\max} V^+$  if and only if  $z \in \mathcal{D}_{\text{GPT}\diamond}(g; V, V^+)$ . This proves the first assertion. We have  $\mathbb{1}(W(\check{\mathbb{I}}_g)) = \langle \varphi^W, \check{\mathbb{I}}_g \otimes \mathbb{1} \rangle$  from Eq. (49), which proves the second assertion. Moreover,  $\text{Tr}[(\Phi^{(f)})^* W] = \text{Tr}[W(\Phi^{(f)})^*] < 0$  if and only if  $\langle \varphi^W, \varphi^{\Phi^{(f)}} \rangle < 0$  by Lemma A.10. Furthermore,

$$\langle \varphi^W, \varphi^{\Phi^{(f)}} \rangle = \mathbb{1}(z_0) + \sum_{i \in [g]} \langle 2f_i - \mathbb{1}, z_i \rangle = 1 - \langle z, \bar{\varphi}^f \rangle.$$

This proves the last assertion.  $\square$

Lastly, we can also connect incompatibility witnesses of [Jen18] and [BN20] directly with each other. To clarify this point was one of the motivations for this work.

**Corollary 7.15.** *There is a one-to-one correspondence between  $z \in \mathcal{D}_{\text{GPT}\diamond}(\mathbf{k}; V, V^+)$  and positive maps  $W : (E_{\mathbf{k}}^*, (E_{\mathbf{k}}^*)^+) \rightarrow (V, V^+)$ . Moreover,*

$$\begin{aligned} \exists \text{ positive map } Y : (V, V^+) \rightarrow (V, V^+) \text{ s.t. } \text{Tr}[(\Phi^{(f)})^* YW] < 0 \\ \Longleftrightarrow \quad z \neq \mathcal{D}_f(\mathbf{k}; V, V^+) \end{aligned}$$

for a collection of measurements  $f$ .

*Proof.* This follows from Propositions 7.13 and 7.14, realizing that similar statements can be obtained for  $\mathcal{D}_{\text{GPT}\diamond}(\mathbf{k}; V, V^+)$  instead of  $\mathcal{D}_{\text{GPT}\diamond}(g; V, V^+)$ .  $\square$

The following characterization of elements of the dual unit ball follows easily from Remark 6.3:

**Corollary 7.16.** *Let  $z \in V^g$ , then  $z \in \mathcal{P}_{\text{GPT}\diamond}(g; V, V^+)$  if and only if there is some  $z_0 \in K$  such that*

$$z_0 + \sum_i \varepsilon_i z_i \in V^+, \quad \forall \varepsilon \in \{\pm 1\}^g.$$

**Definition 7.17.** The set

$$\Pi(g; V, V^+) := \left\{ (s_1, \dots, s_g) \in [0, 1]^g : \sum_{i=1}^g s_i \|z_i\|_V \leq 1 \quad \forall (z_1, \dots, z_g) \in \mathcal{P}_{\text{GPT}\diamond, g} \right\} \quad (29)$$

is called the  $(g; V, V^+)$ -blind region.



**Theorem 7.18.** *It holds that*

$$\Pi(g; V, V^+) = \Gamma(g; V, V^+).$$

*Proof.* The statement follows from Theorem 7.6 and duality of the norms.  $\square$

**Example 7.19.** In quantum mechanics  $\text{QM}_d$ , we see by the relation (described in Example 6.7) of the corresponding GPT diamond  $\mathcal{D}_{\text{GPT}\diamond, g}$  to the matrix diamond  $\mathcal{D}_{\diamond, g}(d)$  that we have

$$\mathcal{P}_{\text{GPT}\diamond, g} = \{(\rho^{1/2} X_1 \rho^{1/2}, \dots, \rho^{1/2} X_g \rho^{1/2}), X \in \mathcal{D}_{\diamond, g}(d), \rho \in \mathcal{S}_d\}.$$

It follows that any element  $X$  in the matrix diamond defines a family of incompatibility witnesses. Moreover, some of these witnesses are strict if and only if we have

$$\|X\|_\sigma := \sup_{\rho \in \mathcal{S}_d} \sum_i \|\rho^{1/2} X_i \rho^{1/2}\|_1 > 1.$$

It follows that we can write the Eq. (29) as

$$\Pi(g, \mathcal{M}_d^{\text{sa}}, \text{PSD}_d) = \{(s_1, \dots, s_g) \in [0, 1]^g, \|s \cdot X\|_\sigma \leq 1 \forall X \in \mathcal{D}_{\diamond, g}(d)\}.$$

Note that the matrix diamond is precisely the unit ball of the injective norm  $\|\cdot\|_\varepsilon$  in  $\ell_1^g \otimes (\mathcal{M}_d^{\text{sa}}(\mathbb{C}), \|\cdot\|_\infty)$ , so that similarly as in Example 7.7 it is the maximal matrix convex set  $\mathcal{W}_d^{\text{max}}(B_{\ell_1^g})$ . From the results of Example 7.7 and duality of  $\ell_\infty^g$  and  $\ell_1^g$ , one would expect that the unit ball of the norm  $\|\cdot\|_\sigma$  is precisely the minimal matrix convex set  $\mathcal{W}_d^{\text{min}}(B_{\ell_1^g})$ . One can prove directly that this is indeed the case.

## 8. Compatibility of Effects in Centrally Symmetric GPTs

While Theorem 7.2 nicely characterizes compatible effects in terms of their associated tensors, it is in general hard to compute the crossnorm  $\|\cdot\|_c$ . In this section, we will show that in centrally symmetric GPTs, one can substitute, in certain situations, the norm  $\|\cdot\|_c$  by the *projective tensor norm*. This can be done, for example, in the case of the most incompatible effects, see Remark 8.7.

In this section we will assume that  $(V, V^+, \mathbb{1})$  is a *centrally symmetric* GPT. In this setting, the vector space of un-normalized states decomposes as  $V = \mathbb{R}v_0 \oplus \bar{V}$ , and we shall write  $V \ni z = (z^\circ, \bar{z})$  to denote the vector  $z = z^\circ v_0 + \bar{z}$ , with  $z^\circ \in \mathbb{R}$  and  $\bar{z} \in \bar{V}$ . The decomposition is such that  $\mathbb{1}(z) = z^\circ$ . Similarly, any element of  $A$  has the form  $(\alpha, \bar{f})$  for  $\alpha = \langle f, v_0 \rangle \in \mathbb{R}$  and  $\bar{f} \in \bar{A}$ . See also Sect. 4.3.

Let  $f_1, \dots, f_g \in A$ ,  $f_i = (\alpha_i, \bar{f}_i)$ . Then each  $f_i$  is an effect if and only if  $\|\bar{f}_i\|_{\bar{A}} \leq \min\{\alpha_i, (1 - \alpha_i)\}$ , which implies that we must have  $\|\bar{f}_i\|_{\bar{A}} \leq 1/2$ . Hence if  $f_i$  is an effect, then also  $(1/2, \bar{f}_i)$  is an effect. Effects of the form  $(1/2, \bar{f})$  are called *unbiased*. See also Remark 8.3 below.

For  $f = (f_1, \dots, f_g)$ ,  $f_i = (\alpha_i, \bar{f}_i)$ , we have

$$\bar{\varphi}^{(f)} = y_f \otimes \mathbb{1} + \bar{\xi}^{(f)}, \quad y_f := \sum_i (2\alpha_i - 1) \check{c}_i, \quad \bar{\xi}^{(f)} := 2 \sum_i \check{c}_i \otimes \bar{f}_i.$$

Let us denote by  $\|\cdot\|_{\bar{\varepsilon}}$  and  $\|\cdot\|_{\bar{\pi}}$  the injective and projective norm on the space  $\ell_\infty^g \otimes (\bar{A}, \|\cdot\|_{\bar{A}})$ ; importantly, note that  $\dim \bar{A} = \dim A - 1$ .

**Proposition 8.1.** *Let  $\bar{\varphi} = y \otimes \mathbb{1} + \bar{\xi}$ , for some  $y \in \ell_\infty^g$  and  $\bar{\xi} \in \ell_\infty^g \otimes \bar{A}$ . Then*

$$\begin{aligned} \max\{\|y\|_\infty, \|\bar{\xi}\|_{\bar{\varepsilon}}\} &\leq \|\bar{\varphi}\|_{\varepsilon} \leq \|y\|_\infty + \|\bar{\xi}\|_{\bar{\varepsilon}} \\ \max\{\|y\|_\infty, \|\bar{\xi}\|_{\bar{\pi}}\} &\leq \|\bar{\varphi}\|_c \leq \|y\|_\infty + \|\bar{\xi}\|_{\bar{\pi}}. \end{aligned}$$

*Proof.* Let  $y = \sum_i y_i \check{c}_i$  and  $\bar{\xi} = \sum_i \check{c}_i \otimes \bar{g}_i$ . Then

$$\|\bar{\varphi}\|_{\varepsilon} = \max_i \|(y_i, \bar{g}_i)\|_{\mathbb{1}} = \max_i (|y_i| + \|\bar{g}_i\|_{\bar{A}}).$$

since  $\ell_\infty^g \otimes_{\varepsilon} X \cong \ell_\infty^g(X)$  [Rya02, Example 3.3] and the inequalities for the injective norm are immediate.

Next, assume that  $\|\bar{\varphi}\|_c \leq 1$ . By Proposition 7.4 that there are some  $z_j \in \ell_\infty^g$ ,  $\|z_j\|_\infty = 1$  and  $h_j = a_j \mathbb{1} + \bar{h}_j$  such that  $\|\bar{h}_j\|_{\bar{A}} \leq a_j$ ,  $\|\sum_j \bar{h}_j\|_{\bar{A}} \leq 1 - \sum_j a_j$  and

$$\bar{\varphi} = \sum_j z_j \otimes h_j = \sum_j a_j z_j \otimes \mathbb{1} + \sum_j z_j \otimes \bar{h}_j.$$

It follows that

$$y = \sum_j a_j z_j, \quad \bar{\xi} = \sum_j z_j \otimes \bar{h}_j.$$

Hence  $\|y\|_\infty \leq \sum_j a_j \leq 1$ , moreover,  $\sum_j \|\bar{h}_j\|_{\bar{A}} \leq \sum_j a_j \leq 1$ , so that  $\|\bar{\xi}\|_{\bar{\pi}} \leq 1$ . This proves the first inequality for  $\|\cdot\|_c$ . For the second inequality, note that since  $\|\cdot\|_c$  is a crossnorm, we have

$$\|\bar{\varphi}\|_c \leq \|y \otimes \mathbb{1}\|_c + \|\bar{\xi}\|_c \leq \|y\|_\infty + \|\bar{\xi}\|_{\bar{\pi}}.$$

In the last inequality we also used the fact that

$$\|\bar{\xi}\|_{\pi} \leq \|\bar{\xi}\|_{\bar{\pi}}$$

since the inclusion  $\bar{A} \rightarrow A$  is an isometry. □

**Corollary 8.2.** *We have, for a tuple  $f \in A^g$ :*

$$f \text{ effects} \implies \|\bar{\xi}^{(f)}\|_{\bar{\varepsilon}} = 2 \max_{i=1}^g \|\bar{f}_i\|_{\bar{A}} \leq 1 \quad (30)$$

and

$$f \text{ compatible effects} \implies \|\bar{\xi}^{(f)}\|_{\bar{\pi}} \leq 1.$$

If, moreover, the effects  $f_i$  are unbiased, the reverse implications hold.

*Proof.* The implications follows easily by Theorem 7.2 and Proposition 8.1. If the  $f_i$  are unbiased,  $y_f = 0$ . We moreover have for  $\bar{\xi} \in \ell_\infty^g \otimes \bar{A}$  that

$$\|\bar{\xi}\|_{\varepsilon} = \|\bar{\xi}\|_{\bar{\varepsilon}} \quad \text{and} \quad \|\bar{\xi}\|_c = \|\bar{\xi}\|_{\pi} = \|\bar{\xi}\|_{\bar{\pi}}.$$

Thus, Proposition 8.1 implies the last statement. See also [Lam18, Proposition 2.25] or [LPW18, Proposition 27]. □

**Remark 8.3.** Note that the reverse implication in Eq. (30) is true if and only if the effect  $f$  is unbiased (we are considering here  $g = 1$ ). Indeed, an element  $f = f_0 \mathbb{1} + \bar{f} \in A$  is an effect if and only if

$$\begin{aligned} f \in A^+ &\iff \|\bar{f}\|_{\bar{A}} \leq f_0 \quad \text{and} \\ \mathbb{1} - f \in A^+ &\iff \|\bar{f}\|_{\bar{A}} \leq 1 - f_0, \end{aligned}$$

which give together  $\|\bar{f}\|_{\bar{A}} \leq \min(f_0, 1 - f_0)$ . However, Eq. (30) gives  $\|\bar{f}\|_{\bar{A}} \leq 1/2$  which is identical to the previous condition if and only if  $f_0 = \langle f, v_0 \rangle = 1/2$ , i.e. if the effect is unbiased.

We now turn to incompatibility witnesses. Let us denote

$$\begin{aligned} \mathcal{P}'_{\text{GPT}\diamond}(g; V, V^+) &:= \{(z_1, \dots, z_g) \in \mathcal{P}_{\text{GPT}\diamond}(g; V, V^+), z_i = (0, \bar{z}_i), i \in [g]\} \\ &= (0 \oplus \bar{V})^g \cap \mathcal{P}_{\text{GPT}\diamond}(g; V, V^+). \end{aligned}$$

**Lemma 8.4.** Let  $\bar{z} = (\bar{z}_1, \dots, \bar{z}_g) \in \bar{V}^g$  and let  $z = ((0, \bar{z}_1), \dots, (0, \bar{z}_g))$ . Then

$$\|z\|_{c^*} = \|z\|_{\varepsilon} = \|\bar{z}\|_{\bar{\varepsilon}}, \quad \|z\|_{\pi} = \|\bar{z}\|_{\bar{\pi}}$$

here  $\varepsilon$ , resp.  $\bar{\varepsilon}$ , is the injective crossnorm in  $\ell_1^g \otimes (V, \|\cdot\|_V)$ , resp.  $\ell_1^g \otimes (\bar{V}, \|\cdot\|_{\bar{V}})$ , similarly for  $\pi$  and  $\bar{\pi}$ . In particular,

$$\begin{aligned} z \in \mathcal{P}'_{\text{GPT}\diamond}(g; V, V^+) &\iff (v_0, (0, \bar{z}_1), \dots, (0, \bar{z}_g)) \in \mathcal{D}_{\text{GPT}\diamond}(g; V, V^+) \\ &\iff \|\bar{z}\|_{\bar{\varepsilon}} \leq 1. \end{aligned}$$

*Proof.* We have

$$\|z\|_{\pi} = \sum_{i=1}^g \|z_i\|_V = \sum_{i=1}^g \|\bar{z}_i\|_{\bar{V}} = \|\bar{z}\|_{\bar{\pi}}.$$

Further,

$$\begin{aligned} \|z\|_{\varepsilon} \leq 1 &\iff \left\| \sum_{i=1}^g \varepsilon_i(0, \bar{z}_i) \right\|_V = \left\| \sum_{i=1}^g \varepsilon_i \bar{z}_i \right\|_{\bar{V}} \leq 1, \quad \forall \varepsilon \in \{\pm 1\}^g \\ &\iff \|\bar{z}\|_{\bar{\varepsilon}} \leq 1. \end{aligned}$$

We have used Eq. (52). On the other hand, using Proposition 7.13, for any  $\varepsilon \in \{\pm 1\}^g$ ,

$$\begin{aligned} \left\| \sum_{i=1}^g \varepsilon_i \bar{z}_i \right\|_{\bar{V}} \leq 1, \quad \forall \varepsilon \in \{\pm 1\}^g &\iff v_0 + \sum_i \varepsilon_i(0, \bar{z}_i) \in V^+, \quad \forall \varepsilon \in \{\pm 1\}^g \\ &\iff (v_0, (0, \bar{z}_1), \dots, (0, \bar{z}_g)) \in \mathcal{D}_{\text{GPT}\diamond}(g; V, V^+) \\ &\implies z \in \mathcal{P}'_{\text{GPT}\diamond}(g; V, V^+) \iff \|z\|_{c^*} \leq 1. \end{aligned}$$

Since  $\|\cdot\|_{c^*}$  is a crossnorm, we have

$$z \in \mathcal{P}'_{\text{GPT}\diamond}(g; V, V^+) \iff \|z\|_{c^*} \leq 1 \implies \|z\|_{\varepsilon} \leq 1,$$

this finishes the proof.  $\square$

Remarkably, we have the following result where we write  $\bar{A}$  and  $\bar{V}$  for the Banach spaces  $(\bar{A}, \|\cdot\|_{\bar{A}})$  and  $(\bar{V}, \|\cdot\|_{\bar{V}})$ .

**Theorem 8.5.** *For a centrally symmetric GPT  $(V, V^+, \mathbb{1})$ , we have*

$$\begin{aligned} \Gamma(g; V, V^+) &= \Pi'(g; V, V^+) := \{s \in [0, 1]^g : \|s \cdot \bar{z}\|_{\ell_1^g \otimes_\pi \bar{V}} \leq 1, \forall \|\bar{z}\|_{\ell_1^g \otimes_\varepsilon \bar{V}} \leq 1\} \\ &= \{s \in [0, 1]^g : \|s \cdot \bar{\varphi}\|_{\ell_\infty^g \otimes_\pi \bar{A}} \leq 1, \forall \|\bar{\varphi}\|_{\ell_\infty^g \otimes_\varepsilon \bar{A}} \leq 1\}. \end{aligned}$$

In particular, the compatibility degree is

$$\gamma(g; V, V^+) = 1/\rho(\ell_\infty^g, \bar{A}) = 1/\rho(\ell_1^g, \bar{V}),$$

where the quantity  $\rho$  is as in Eq. (53):

$$\rho(\ell_\infty^g, \bar{A}) = \max_{\bar{\varphi}} \frac{\|\bar{\varphi}\|_{\ell_\infty^g \otimes_\pi \bar{A}}}{\|\bar{\varphi}\|_{\ell_\infty^g \otimes_\varepsilon \bar{A}}}.$$

*Proof.* By Theorem 7.18, to prove the first equality it is enough to show that for a centrally symmetric GPT we have

$$\Pi(g; V, V^+) = \Pi'(g; V, V^+).$$

The inclusion  $\Pi \subseteq \Pi'$  is clear from Lemma 8.4. To prove the reverse inclusion, let us define, for a tuple  $s \in [0, 1]^g$ ,

$$\begin{aligned} \beta_s &:= \max_{z \in \mathcal{P}_{\text{GPT}\diamond}(g; V, V^+)} \|s \cdot z\|_{\ell_1^g \otimes_\pi V} = \max_{z \in \mathcal{P}_{\text{GPT}\diamond}(g; V, V^+)} \sum_{i=1}^g s_i \|z_i\|_V \\ \beta'_s &:= \max_{z \in \mathcal{P}'_{\text{GPT}\diamond}(g; V, V^+)} \|s \cdot z\|_{\ell_1^g \otimes_\pi V} = \max_{\|\bar{z}\|_{\ell_1^g \otimes_\varepsilon \bar{V}} \leq 1} \sum_{i=1}^g s_i \|\bar{z}_i\|_{\bar{V}}. \end{aligned}$$

Obviously,  $s \in \Pi(g; V, V^+)$  if and only if  $\beta_s \leq 1$ , the same being true for the primed variants. We shall prove that  $\beta_s \leq \beta'_s$ , for all  $s \in [0, 1]^g$ . To this end, fix some  $s$  and let  $z \in \mathcal{P}_{\text{GPT}\diamond}(g; V, V^+)$ ,  $z_i = (z_i^\circ, \bar{z}_i)$ , achieve the maximum in  $\beta_s$ . We consider the set partition  $[g] = I \sqcup J$ , where  $i \in I \iff z_i^\circ > \|\bar{z}_i\|_{\bar{V}}$ . We have thus

$$\beta_s = \sum_{i \in I} s_i |z_i^\circ| + \sum_{j \in J} s_j \|\bar{z}_j\|_{\bar{V}}. \quad (31)$$

Let us denote by  $s_I$ , resp.  $s_J$ , the restriction of the  $g$ -tuple  $s$  to the index set  $I$ , resp.  $J$ . Putting

$$\lambda := \sum_{i \in I} |z_i^\circ|,$$

we have  $\sum_{i \in I} s_i |z_i^\circ| \leq \lambda \|s_I\|_\infty \leq \lambda \beta'_{s_I}$ , where the last inequality follows from the fact that for any  $\bar{v} \in \bar{V}$  with  $\|\bar{v}\|_{\bar{V}} \leq 1$ , we have  $\|(0, \dots, 0, \bar{v}, 0, \dots, 0)\|_{\bar{\varepsilon}} \leq 1$ .

Note that  $\lambda \leq 1$ , since  $z \in \mathcal{P}_{\text{GPT}\diamond}(g; V, V^+)$  implies using Proposition 7.13 that  $z_0 + \sum_{i=1}^g \varepsilon_i z_i \in V^+$  for some  $z_0 \in K$  and all  $\varepsilon \in \{\pm 1\}^g$ . Therefore, an application of  $\mathbb{1}$  yields  $\sum_{i=1}^g |z_i^\circ| \leq 1$ .

Let us now focus on the second term in the RHS of Eq. (31). First, note that if the set  $J$  is empty we are done:  $\beta_s \leq \lambda\beta'_s \leq \beta'_s$ . Assume now  $\lambda \in [0, 1)$ . Using Lemma 8.6 below, we have that

$$\frac{1}{1-\lambda} ((0, \bar{z}_j))_{j \in J} \in \mathcal{P}'_{\text{GPT}\diamond}(|J|; V, V^+),$$

and thus

$$\sum_{j \in J} s_j \|\bar{z}_j\|_{\bar{V}} \leq (1-\lambda)\beta'_{s_J}$$

by the definition of  $\beta'_{s_J}$ . We have proven, up to this point, that  $\beta_s \leq \lambda\beta'_{s_I} + (1-\lambda)\beta'_{s_J}$ . To conclude, we need to show that the function  $\beta'$  is “concave”, which we do next. Let  $\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{|I|} \in \bar{V}$  be such that  $\|\bar{x}_0\|_{\bar{V}} \leq 1$  and

$$\begin{aligned} ((1, \bar{x}_0), (0, \bar{x}_1), \dots, (0, \bar{x}_{|I|})) &\in \mathcal{D}_{\text{GPT}\diamond}(|I|; V, V^+) \quad \text{and} \\ \beta'_{s_I} &= \sum_{i=1}^{|I|} s_i \|\bar{x}_i\|_{\bar{V}}. \end{aligned}$$

Consider similar elements  $\bar{y}_0, \bar{y}_1, \dots, \bar{y}_{|J|} \in \bar{V}$ . The claim follows from Proposition 7.13 and the following fact:

$$\begin{aligned} ((1, \lambda\bar{x}_0 + (1-\lambda)\bar{y}_0), \lambda(0, \bar{x}_1), \dots, \lambda(0, \bar{x}_{|I|}), (1-\lambda)(0, \bar{y}_1), \dots, (1-\lambda)(0, \bar{y}_{|J|})) \\ \in \mathcal{D}_{\text{GPT}\diamond}(g; V, V^+). \end{aligned}$$

□

**Lemma 8.6.** *Let  $z \in \mathcal{P}_{\text{GPT}\diamond}(g; V, V^+)$ , and consider a subset  $J \subseteq [g]$  such that  $\sum_{i \notin J} |z_i^\circ| < 1$ . Then,*

$$\frac{1}{1 - \sum_{i \notin J} |z_i^\circ|} ((0, \bar{z}_j))_{j \in J} \in \mathcal{P}'_{\text{GPT}\diamond}(|J|; V, V^+).$$

*Proof.* Let us put  $I := [g] \setminus J$ . From the hypothesis  $z \in \mathcal{P}_{\text{GPT}\diamond}(g; V, V^+)$ , using Proposition 7.13 we have that, for any signs  $\varepsilon_i, \eta_j$ ,

$$1 + \sum_{i \in I} \varepsilon_i z_i^\circ + \sum_{j \in J} \eta_j z_j^\circ \geq \left\| \bar{z}_0 + \sum_{i \in I} \varepsilon_i \bar{z}_i + \sum_{j \in J} \eta_j \bar{z}_j \right\|_{\bar{V}},$$

where  $\bar{z}_0 \in \bar{V}$  is such that  $(z_0, z_1, \dots, z_g) \in \mathcal{D}_{\text{GPT}\diamond}(g; V, V^+)$ . Writing

$$\sum_{j \in J} \eta_j \bar{z}_j = \frac{1}{2} \left[ \left( \bar{z}_0 + \sum_{i \in I} \varepsilon_i \bar{z}_i + \sum_{j \in J} \eta_j \bar{z}_j \right) - \left( \bar{z}_0 + \sum_{i \in I} \varepsilon_i \bar{z}_i + \sum_{j \in J} (-\eta_j) \bar{z}_j \right) \right],$$

we get

$$\left\| \sum_{j \in J} \eta_j \bar{z}_j \right\|_{\bar{V}} \leq \frac{1}{2} \left[ \left( 1 + \sum_{i \in I} \varepsilon_i z_i^\circ + \sum_{j \in J} \eta_j z_j^\circ \right) + \left( 1 + \sum_{i \in I} \varepsilon_i z_i^\circ + \sum_{j \in J} (-\eta_j) z_j^\circ \right) \right]$$

$$= 1 + \sum_{i \in I} \varepsilon_i z_i^\circ.$$

The conclusion follows from the expression for the injective norm in Eq. (52), Lemma 8.4 and by setting, for all  $i \in I$ ,  $\varepsilon_i = -\text{sign}(z_i^\circ)$ .  $\square$

*Remark 8.7.* Note that Theorem 8.5 and Proposition 8.1 imply that minimal compatibility degree is attained on tuples of unbiased effects.

## 9. Examples of Compatibility Regions

In this section, we use the connection to tensor crossnorms exhibited in Sects. 7 and 8 to give bounds on the compatibility region of various GPTs of interest. We have seen that, in order to find the most incompatible  $g$ -tuples of effects, one needs to compare the injective with the  $c$ -tensor norms for an arbitrary GPT, and the injective with the projective tensor norms for centrally symmetric GPTs. We are thus interested in the ratio

$$\frac{1}{\rho(\ell_1^g, X)} = \min_{z \in \ell_1^g \otimes X} \frac{\|z\|_{\ell_1^g \otimes_\varepsilon X}}{\|z\|_{\ell_1^g \otimes_\pi X}} = \min_{z \in \ell_\infty^g \otimes X^*} \frac{\|z\|_{\ell_\infty^g \otimes_\varepsilon X^*}}{\|z\|_{\ell_\infty^g \otimes_\pi X^*}}.$$

Proposition 12 of [ALP+20] implies the generic bound

$$\rho(\ell_1^g, X) \leq \min\{g, \dim X\}.$$

Thus,

$$\gamma(g; V, V^+) \geq \frac{1}{\min\{g, \dim V\}} \quad (32)$$

for a GPT  $(V, V^+, \mathbb{1})$  by Corollary 7.8. If  $(V, V^+, \mathbb{1})$  is a centrally symmetric GPT of dimension  $n + 1$ , we find by Theorem 8.5 that

$$\gamma(g; V, V^+) \geq \frac{1}{\min\{g, n\}}. \quad (33)$$

For concrete GPTs, however, we can often find better bounds using Theorem 8.5. This is the topic of the remainder of this section. If  $(V, V^+, \mathbb{1})$  is the centrally symmetric GPT giving rise to the Banach space  $X$  as in Sect. 4.3, we will write  $\Gamma(g; X)$  and  $\gamma(g; X)$  for brevity.

**9.1. Hypercubes.** Let us consider the hypercube GPT  $\text{HC}_n$  from Example 4.7, where the state space  $K$  is the  $n$ -dimensional hypercube, the unit ball of the Banach space  $\ell_\infty^n$ . We have  $\dim V = n + 1$ .

Let us compute the injective norm  $\ell_1^g \otimes_\varepsilon \ell_\infty^n$  of a vector  $z$  using duality:

$$\|z\|_{\ell_1^g \otimes_\varepsilon \ell_\infty^n} = \sup_{y: \|y\|_{\ell_\infty^g \otimes_\pi \ell_1^n} \leq 1} |\langle y, z \rangle|.$$

But the extremal points of the unit ball of the Banach space  $\ell_\infty^g \otimes_\pi \ell_1^n$  are the vectors  $\varepsilon \otimes e_j$  [Rya02, Proposition 2.2], where  $\varepsilon \in \{\pm 1\}^g$  is an extremal point of the unit ball of  $\ell_\infty^g$  and the  $e_j$  are the standard basis vectors,  $j \in [n]$ . We have thus

$$\|z\|_{\ell_1^g \otimes_\varepsilon \ell_\infty^n} = \sup_{j, \varepsilon} \left| \sum_{i=1}^g \varepsilon_i z_{ij} \right|,$$

where  $z_i = \sum_{j=1}^n z_{ij} e_j$  (see also Eq. (52)). Choosing the appropriate  $\varepsilon$  solves easily the maximization problem at fixed  $j$ . We have thus, in the case of the hypercube,

$$\begin{aligned} \|z\|_{\ell_1^g \otimes_\varepsilon \ell_\infty^n} &= \sup_{j \in [n]} \sum_{i=1}^g |z_{ij}| \\ \|z\|_{\ell_1^g \otimes_\pi \ell_\infty^n} &= \sum_{i=1}^g \sup_{j \in [n]} |z_{ij}|, \end{aligned}$$

where the second expression follows from Eq. (51).

**Proposition 9.1.** *For the centrally symmetric GPT with hypercube state space, the compatibility region is the enlarged simplex*

$$\Gamma(g; \ell_\infty^n) = \{s \in [0, 1]^g : \forall I \subseteq [g] \text{ s.t. } |I| \leq n, \sum_{i \in I} s_i \leq 1\}.$$

Before proving the result, let us point out that when  $g \leq n$ , the enlarged simplex above is just the usual probability simplex  $\Delta_g$ ; when  $g > n$ , the set is strictly larger than  $\Delta_g$ , see Fig. 1.

*Proof.* Let us first show the “ $\supseteq$ ” inclusion. Take  $s \in [0, 1]^g$  with the property in the statement. For  $i \in [g]$ , choose

$$J(i) \in \operatorname{argmax}_{j \in [n]} |z_{ij}|.$$

Let us write  $J([g]) = \{J(i) : i \in [g]\}$  for the range of  $J$ . We also choose, for every  $j \in J([g])$ ,

$$i_*(j) \in \operatorname{argmax}_{i \in J^{-1}(j)} s_i;$$

note that  $i_*$  is an injective function, since  $J^{-1}(j) \cap J^{-1}(j') = \emptyset$  for  $j \neq j'$ . We have then, for  $z$  such that  $\|z\|_{\ell_1^g \otimes_\varepsilon \ell_\infty^n} \leq 1$ ,

$$\begin{aligned} \|s \cdot z\|_{\ell_1^g \otimes_\pi \ell_\infty^n} &= \sum_{i=1}^g s_i |z_{iJ(i)}| = \sum_{j \in J([g])} \sum_{i \in J^{-1}(j)} s_i |z_{ij}| \leq \sum_{j \in J([g])} s_{i_*(j)} \sum_{i \in J^{-1}(j)} |z_{ij}| \\ &\leq \sum_{j \in J([g])} s_{i_*(j)} \leq 1, \end{aligned}$$

where the last inequality follows from the hypothesis and the fact that  $|J([g])| \leq n$ .

For the reverse implication, fix a subset  $I \in [g]$  of cardinality at most  $n$ . For  $z_{ij} = 1$  if  $i = j \in I$  and 0 elsewhere, we have  $\|z\|_{\ell_1^g \otimes_\varepsilon \ell_\infty^n} \leq 1$  and

$$\|s \cdot z\|_{\ell_1^g \otimes_\pi \ell_\infty^n} = \sum_{i \in I} s_i.$$

The assertion follows again from Theorem 8.5.  $\square$

**Corollary 9.2.** *The largest diagonal element in  $\Gamma(g; \ell_\infty^n)$  is  $s = 1/\min(g, n)$ .*

**9.2. Euclidean balls.** We consider now the case of a centrally symmetric GPT with the unit ball of  $\ell_2^n$  as the state space. This GPT is of particular interest because for  $n = 3$ , one recovers the state space of qubits: the Bloch ball (see Example 4.8). In this setting, we have, using Eq. (52),

$$\|z\|_{\ell_1^g \otimes_\varepsilon \ell_2^n} = \sup_{y: \|y\|_{\ell_\infty^g \otimes_\pi \ell_2^n} \leq 1} |\langle y, z \rangle| = \sup_{\|y\|_2 \leq 1} \sum_{i=1}^g |\langle y, z_i \rangle| = \sup_{\varepsilon \in \{\pm 1\}^g} \left\| \sum_{i=1}^g \varepsilon_i z_i \right\|_2.$$

**Proposition 9.3.** *For all  $g, n$ ,*

$$\text{QC}_g := \left\{ s \in [0, 1]^g : \sum_{i=1}^g s_i^2 \leq 1 \right\} \subseteq \Gamma(g; \ell_2^n).$$

*If  $g \leq n$ , we have equality above.*

*Proof.* Consider  $s \in [0, 1]^g$  with the property that  $\sum_{i=1}^g s_i^2 \leq 1$ . Let  $z \in \ell_1^g \otimes \ell_2^n$  be such that  $\|z\|_{\ell_1^g \otimes_\varepsilon \ell_2^n} \leq 1$ . We have

$$\|s \cdot z\|_{\ell_1^g \otimes_\pi \ell_2^n} = \sum_{i=1}^g s_i \|z_i\|_2 \leq \left( \sum_{i=1}^g s_i^2 \right)^{1/2} \left( \sum_{i=1}^g \|z_i\|_2^2 \right)^{1/2}$$

by Cauchy-Schwarz and then use the expression for the injective norm to get

$$\sum_{i=1}^g \|z_i\|_2^2 = \frac{1}{2^g} \sum_{\varepsilon \in \{\pm 1\}^g} \left\| \sum_{i=1}^g \varepsilon_i z_i \right\|_2^2 \leq 1.$$

For the upper bound in the case  $g \leq n$ , use  $z_i = s_i e_i$ , where  $e_i$  is the standard basis of  $\mathbb{R}^g$ . The assertion follows from the previous calculations using Theorem 8.5.  $\square$

Let us consider now pairs of effects ( $g = 2$ ). The projective crossnorm in  $\ell_\infty^2 \otimes X$  has an explicit expression (see e.g. the proof of [ALP+20, Proposition 15]): for  $\varphi = e_1 \otimes x_1 + e_2 \otimes x_2$ , we have

$$\|\varphi\|_\pi = \frac{1}{2} (\|x_1 + x_2\|_X + \|x_1 - x_2\|_X).$$

Using this expression, we obtain from (27) and Corollary 8.2:

1. For any GPT  $(V, V^+, \mathbb{1})$  and any pair  $f = (f_1, f_2)$  of effects the compatibility degree is

$$\gamma(f) \leq \|f_1 - f_2\|_A + \|f_1 - (\mathbb{1} - f_2)\|_A.$$

2. For a centrally symmetric GPT and a pair  $f = ((1/2, \bar{f}_1), (1/2, \bar{f}_2))$ , we have

$$\gamma(f) = \|\bar{f}_1 + \bar{f}_2\|_{\bar{A}} + \|\bar{f}_1 - \bar{f}_2\|_{\bar{A}}.$$

This is a generalization of a known result for qubits ( $n = 3$ ), see e.g. [BH08, Proposition 4].



**9.3. Cross polytopes.** Let us now consider the GPT with the  $n$ -dimensional  $\ell_1$  unit ball as state space. Again,  $\dim V = n + 1$ .

Let

$$x = \sum_{i=1}^g e_i \otimes x^{(i)} \in \ell_1^g \otimes \ell_1^n,$$

where  $\{e_i\}_{i \in [g]}$  is the canonical basis of  $\mathbb{R}^g$ . In this case, we have by Eq. (51)

$$\|x\|_{\ell_1^g \otimes_{\pi} \ell_1^n} = \sum_{i=1}^g \|x^{(i)}\|_1 = \sum_{i=1}^g \sum_{j=1}^n |x_j^{(i)}|.$$

Moreover, by Eq. (52)

$$\|x\|_{\ell_1^g \otimes_{\varepsilon} \ell_1^n} = \sup_{\varepsilon \in \{\pm 1\}^g} \left\| \sum_{i=1}^g \varepsilon_i x^{(i)} \right\|_1 = \sup_{\varepsilon \in \{\pm 1\}^g} \sum_{j=1}^n \left| \sum_{i=1}^g \varepsilon_i x_j^{(i)} \right|.$$

**Proposition 9.4.** *For all  $n, g \in \mathbb{N}$ , it holds that*

$$\gamma(g; \ell_1^n) \geq \frac{1}{g 2^{g-1}} \left( \left\lfloor \frac{g}{2} \right\rfloor + 1 \right) \binom{g}{\lfloor \frac{g}{2} \rfloor + 1}.$$

*This bound is achieved for  $n \geq 2^{g-1}$ .*

*Proof.* We will again use Theorem 8.5 to relate the ratio of norms to the compatibility degree. Let us now consider  $y^{(i)} \in \mathbb{R}^g$ ,  $i \in [n]$ , such that  $y_j^{(i)} = x_i^{(j)}$ . Then,

$$\|x\|_{\ell_1^g \otimes_{\pi} \ell_1^n} = \sum_{i=1}^g \sum_{j=1}^n |y_i^{(j)}| \quad \|x\|_{\ell_1^g \otimes_{\varepsilon} \ell_1^n} = \sup_{\varepsilon \in \{\pm 1\}^g} \sum_{j=1}^n |\langle \varepsilon, y^{(j)} \rangle|.$$

Our aim is to estimate  $\inf\{\|x\|_{\ell_1^g \otimes_{\varepsilon} \ell_1^n} : \|x\|_{\ell_1^g \otimes_{\pi} \ell_1^n} = 1\}$ . We find

$$\|x\|_{\ell_1^g \otimes_{\varepsilon} \ell_1^n} \geq \frac{1}{2^g} \sum_{\varepsilon \in \{\pm 1\}^g} \sum_{j=1}^n |\langle \varepsilon, y^{(j)} \rangle| =: f(y^{(1)}, \dots, y^{(n)}).$$

Let us therefore minimize  $f(y^{(1)}, \dots, y^{(n)})$  under the constraint imposed by the projective norm. Without loss of generality, we can assume  $y^{(j)} \geq 0$  for all  $j \in [n]$ . Since  $f$  is convex and invariant under permutation of the  $y^{(j)}$  we have

$$\begin{aligned} f(y^{(1)}, \dots, y^{(n)}) &= \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} f(y^{(\sigma(1))}, \dots, y^{(\sigma(n))}) \\ &\geq f\left(\frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} (y^{(\sigma(1))}, \dots, y^{(\sigma(n))})\right) = f(\bar{y}, \dots, \bar{y}), \end{aligned}$$

where  $\mathfrak{S}_n$  is the symmetric group on  $n$  symbols and  $\bar{y} = 1/n \sum_{i=1}^n y^{(i)}$ . We thus need only minimize over  $\bar{y} \in \mathbb{R}_+^g$  such that  $\|\bar{y}\|_1 = 1/n$ . Since

$$\bar{y} \mapsto \sum_{\varepsilon \in \{\pm 1\}^g} |\langle \varepsilon, \bar{y} \rangle|$$

is Schur convex, the minimum is achieved at  $\bar{y} = 1/(gn)(1, \dots, 1)$ . Thus,

$$\|x\|_{\ell_1^g \otimes_{\pi} \ell_1^n} \geq \frac{\sum_{\varepsilon \in \{\pm 1\}^g} \left| \sum_{i=1}^g \varepsilon_i \right|}{2^g g}.$$

For  $n \geq 2^{g-1}$ , it can be checked that  $\{y^{(j)}\}_{j \in [n]} = \{\{y^{(\varepsilon)}\}_{\varepsilon \in \{\pm 1\}^{g-1}}, 0, \dots, 0\}$  achieves this bound, where  $y^{(\varepsilon)} = 1/(g2^{g-1})(1, \varepsilon_1, \dots, \varepsilon_{g-1})$ . It is finally easy to compute that

$$\begin{aligned} \sum_{\varepsilon \in \{\pm 1\}^g} \left| \sum_{i=1}^g \varepsilon_i \right| &= \sum_{k=0}^g |g - 2k| \binom{g}{k} \\ &= \left( \left\lfloor \frac{g}{2} \right\rfloor + 1 \right) \binom{g}{\lfloor \frac{g}{2} \rfloor + 1}. \end{aligned}$$

□

**9.4. Relation to 1-summing norms.** A different way to obtain bounds on the  $\rho(\ell_1^g, X)$  quantity, which were mentioned at the beginning of this section, is to use results on 1-summing norms of Banach spaces; this method was already used in [ALP+20]. The 1-summing norm of (the identity operator of) a Banach space  $X$  is the smallest constant  $c$  with the property that for all  $g \geq 1$  and  $z_1, \dots, z_g \in X$  we have

$$\sum_{i=1}^g \|z_i\|_X \leq c \sup_{\|y\|_{X^*} \leq 1} \sum_{i=1}^g |\langle y, z_i \rangle|.$$

We denote it by  $\pi_1(X)$ .

**Proposition 9.5.** *The largest  $s$  such that*

$$s(1, \dots, 1) \in \bigcap_{g \geq 1} \Gamma(g; V, V^+)$$

*is  $s = 1/\pi_1(\bar{V})$ , in the case where  $(V, V^+, \mathbb{1})$  is a centrally symmetric GPT. For a general GPT  $(V, V^+, \mathbb{1})$ ,*

$$\gamma(g; V, V^+) \geq \frac{1}{\pi_1(V)},$$

*where  $V$  is the corresponding base norm space.*

*Proof.* It is clear from the expressions for the injective (Eq. (52)) and projective (Eq. (51)) norm that  $\sup_{g \in \mathbb{N}} \rho(\ell_1^g, X) = \pi_1(X)$ . The assertion follows from Theorem 8.5 and Corollary 7.8. □

The following constants were computed in [Gor69, Theorem 2]:

- $\pi_1(\ell_1^n) = \frac{2^{n-1}n}{(1+\lfloor n/2 \rfloor)(\lfloor \frac{n}{1+\lfloor n/2 \rfloor} \rfloor)} \sim \sqrt{\frac{\pi n}{2}}$
- $\pi_1(\ell_2^n) = \frac{\sqrt{\pi} \Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} \sim \sqrt{\frac{\pi n}{2}}$
- $\pi_1(\ell_\infty^n) = n$ ,

where  $\sim$  expresses the asymptotic behavior for large  $n \in \mathbb{N}$ .

*Remark 9.6.* As seen above, the 1-summing norms give bounds on the compatibility degree of arbitrarily many dichotomic measurements. The asymptotic growth in the  $\ell_1^n$  and the  $\ell_2^n$  case show that the bounds improve over the naive bound  $1/n$ .

**Proposition 9.7.** *Let  $d \in \mathbb{N}$  and let  $S_1^d$  be the Banach space of self-adjoint matrices equipped with the Schatten 1 norm. Then,*

$$\pi_1(S_1^d) \leq cd,$$

where  $c$  is a constant. In particular, we can take  $c = 7.79$ .

*Proof.* By [GG71], it follows that  $\pi_1(X)\lambda(X) = n$  for any  $n$ -dimensional Banach space  $X$  with enough symmetries in the sense of [TJ89, §16]. Here,  $\lambda(X)$  is the projection constant of  $X$  [TJ89, §32]. From [TJ89, Proposition 32.7], it follows that

$$\lambda(X) \geq c'[C_2(X)(1 + \log C_2(X))^{1/2}]^{-1}\sqrt{n},$$

where  $c'$  is a universal constant. Here,  $C_2(X)$  is the Rademacher cotype 2 constant [TJ89, §4]. The space  $X = S_1^d$  has enough symmetries, has dimension  $n = d^2$  and  $C_2(S_1^d) \leq \sqrt{e}$  [TJ74]. This proves the assertion. A more careful analysis of the proof of Theorem 10.14 and Proposition 32.7 of [TJ89] shows that we can take [TJ89, Equation (10.17)]

$$c = \inf_{q \in (2, \infty)} (\sqrt{q} C_2(S_1^d))^{\frac{q}{q-2}}. \quad (34)$$

With  $C_2(S_1^d) \leq \sqrt{e}$ , we obtain the desired bound on  $c$  through numerical optimization.  $\square$

**9.5. Upper bounds for centrally symmetric GPTs.** For centrally symmetric GPTs, the results of [ALP+20] yield also upper bounds on the compatibility degree. In particular, we can use [ALP+20, Prop. 15], [ALP+20, Lemma 19] and [ALP+20, Theorem 6]:

- (1) For any centrally symmetric GPT with  $\dim(\bar{V}) \geq 2$ , we have

$$\gamma(2; V, V^+) \leq 1/\sqrt{2}. \quad (35)$$

- (2) For any centrally symmetric GPT with  $\dim(\bar{V}) \geq g$ , we have

$$\gamma(g; V, V^+) \leq \sqrt{2/g} \quad (36)$$

- (3) For a centrally symmetric GPT related to  $\ell_2^n$ , we have for  $g \geq n$

$$\gamma(g; V, V^+) \leq 1/\sqrt{n}. \quad (37)$$

- (4) There exists a constant  $c > 0$  such that for any centrally symmetric GPT with  $\dim(\bar{V}) = n$

$$\gamma(g; V, V^+) \leq c \frac{\log(\min\{g, n\})}{(\min\{g, n\})^{1/8}}. \quad (38)$$

*Remark 9.8.* If one drops the condition that the GPT  $(V, V^+, \mathbb{1})$  be centrally symmetric, no general non-trivial upper bounds for the compatibility degree can be derived. Indeed, for the classical mechanics GPT  $\text{CM}_d$ , since all crossnorms agree [ALP19, ALPP19], we have, for all  $g, d \in \mathbb{N}$ ,

$$\Gamma(g; \text{CM}_d) = [0, 1]^g \quad \text{and} \quad \gamma(g; \text{CM}_d) = 1.$$

The central symmetry condition rules out simplicial cones, forcing all incompatibility regions for  $g \geq 2$  to be non-trivial.

## 10. Discussion of Results

In this section, we collect the results on the different compatibility regions for GPTs of interest obtained in this work and compare them to previously known bounds. We focus on results for dichotomic measurements. Note that one can always lift the results for dichotomic measurements to bounds on measurements with more outcomes using the symmetrization techniques from Theorem 6.19.

### 10.1. Centrally symmetric GPTs.

*10.1.1. Hypercubes* If the state space is the unit ball of the Banach space  $\ell_\infty^n$ , Proposition 9.1 characterizes the compatibility region completely. We find that the compatibility region is

$$\Gamma(g; \ell_\infty^n) = \{s \in [0, 1]^g : \forall I \subseteq [g] \text{ s.t. } |I| \leq n, \sum_{i \in I} s_i \leq 1\}.$$

We have plotted this enlarged simplex in Fig. 1. In particular, this implies that

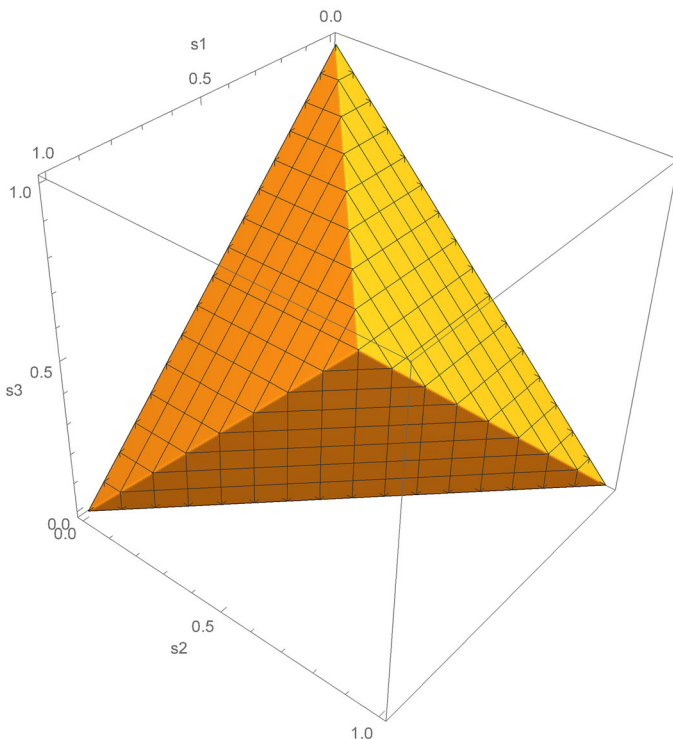
$$\gamma(g; \ell_\infty^n) = \frac{1}{\min\{g, n\}}. \quad (39)$$

Before this work, this result was shown for  $g = 2$  and  $n = 2$  in [BHSS13, Proposition 2] (see also [JP17]) and generalized to  $g \leq n, n \in \mathbb{N}$  in [Jen18, Corollary 5].

*10.1.2. Euclidean balls* For Euclidean balls, i.e. if the state space is the unit ball of  $\ell_2^n$ , Proposition 9.3 shows that

$$\Gamma(g; \ell_2^n) = \left\{ s \in [0, 1]^g : \sum_{i=1}^g s_i^2 \leq 1 \right\} =: \text{QC}_g \quad \forall g, n \in \mathbb{N} \text{ s.t. } g \leq n.$$

In particular,  $\gamma(g; \ell_2^n) = 1/\sqrt{g}$  if  $g \leq n$ . Thus, we have a complete characterization for few measurements.



**Fig. 1.** The compatibility region  $\Gamma(g; \ell_\infty^n)$  for  $n = 2$ ,  $g = 3$ . The polyhedron is the convex hull of the five points  $(0, 0, 0)$ ,  $(1, 0, 0)$ ,  $(0, 1, 0)$ ,  $(0, 0, 1)$ , and  $(1/2, 1/2, 1/2)$ ; it strictly contains the simplex  $\Delta_3$

For  $g > n$ , Proposition 9.3 shows that

$$\Gamma(g; \ell_2^n) \supseteq \text{QC}_g. \quad (40)$$

Moreover, combining this with Eq. (37) and the 1-summing constants in Sect. 9.4, we obtain

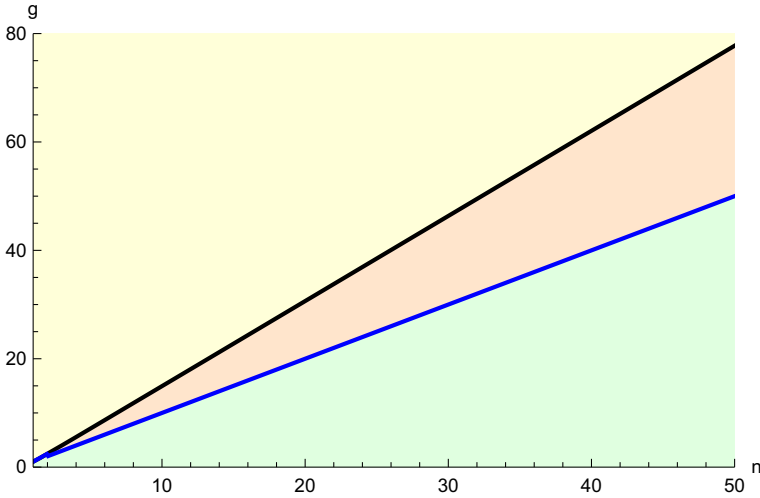
$$\frac{1}{\sqrt{n}} \geq \gamma(g; \ell_2^n) \geq \max \left\{ 1/\sqrt{g}, \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi} \Gamma(\frac{n+1}{2})} \right\} \quad \forall g, n \in \mathbb{N} \text{ s.t. } g \geq n.$$

Hence, for  $g > (\sqrt{\pi} \Gamma(\frac{n+1}{2}) / \Gamma(\frac{n}{2}))^2$  the inclusion in Eq. (40) is strict. See Fig. 2 for a phase diagram of our findings.

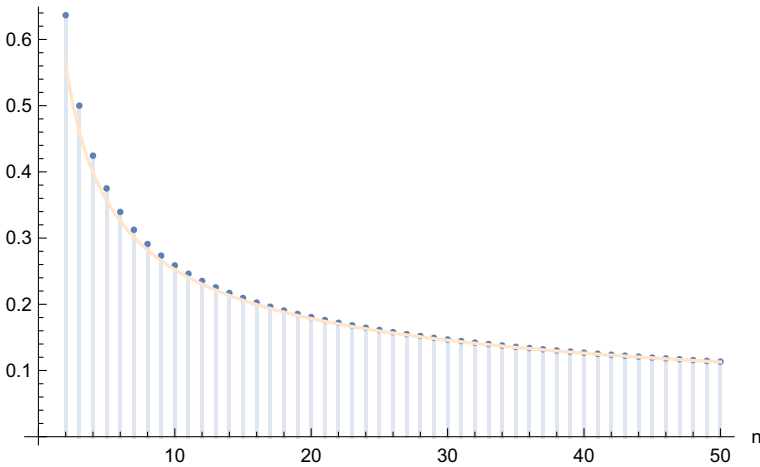
It is easy to compute that

$$h(n) := \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi} \Gamma(\frac{n+1}{2})} = \begin{cases} \frac{4^m}{\pi^m} \binom{2m}{m}^{-1} & n = 2m \\ 4^{-m} \binom{2m}{m} & n = 2m + 1 \end{cases}$$

and asymptotically, the bound behaves like  $\sqrt{2/(\pi n)}$ . See Fig. 3 for a plot of the function. Prior to this work, only results for  $n = 3$  were known, see Sect. 10.3.1 below.



**Fig. 2.** The blue function is  $n \mapsto n$ . The black function is  $n \mapsto h(n)^{-2}$ . In the green region, we know  $\Gamma(g; \ell_2^n) = \text{QC}_g$ . In the yellow region, we know  $\Gamma(g; \ell_2^n) \supsetneq \text{QC}_g$ . In the orange region, we only know  $\Gamma(g; \ell_2^n) \supseteq \text{QC}_g$ , but not whether the inclusion is strict

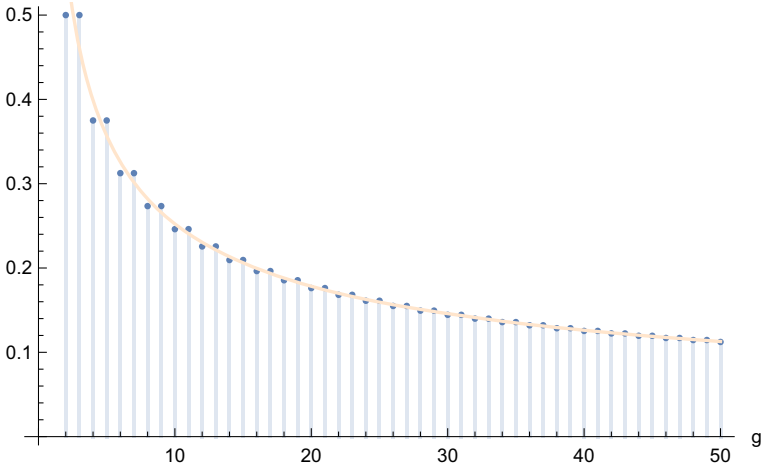


**Fig. 3.** The functions  $h(n)$  (in blue) and  $\sqrt{2/(\pi n)}$  (in orange), for  $n \in \{2, \dots, 50\}$

**10.1.3. Cross polytopes** If the state space is the unit ball of  $\ell_1^n$ , we have only bounds on the compatibility degree. For  $n \geq 2^{g-1}$ , Proposition 9.4 proves that the compatibility degree is given as

$$\gamma(g; \ell_1^n) = \frac{1}{g2^{g-1}} \left( \left\lfloor \frac{g}{2} \right\rfloor + 1 \right) \binom{g}{\left\lfloor \frac{g}{2} \right\rfloor + 1} =: f(g).$$

Asymptotically,  $f(g)$  behaves like  $\sqrt{2/(\pi g)}$ . See Fig. 4 for a plot of this function. For  $n < 2^{g-1}$ , Proposition 9.4 and the 1-summing constant in Sect. 9.4 merely provide lower



**Fig. 4.** The functions  $f(g)$  (in blue) and  $\sqrt{2/(\pi g)}$  (in orange), for  $g \in \{2, \dots, 50\}$

bounds on the compatibility degree. Combined with Eq. (35), we obtain for  $n \geq 2$ :

$$\frac{1}{\sqrt{2}} \geq \gamma(g; \ell_1^n) \geq \max\{f(n), f(g)\}.$$

For  $n \geq g$ , the upper bound can be improved to  $\sqrt{2/g} \geq \gamma(g; \ell_1^n)$  using Eq. (36). Using [ALP+20, Equation (61)], Proposition 30 of [LPW18] shows that in general  $\sqrt{2/\min\{n, g\}} \geq \gamma(g; \ell_1^n)$ . It remains an open question to find any bounds on  $\Gamma(g; \ell_1^n)$  for non-diagonal elements. Our lower bounds improve over the lower bounds derived in [LPW18].

**10.2. General centrally symmetric GPTs.** For general centrally symmetric GPTs, we have only bounds on the compatibility degree. Let  $(V, V^+, \mathbb{1})$  be a centrally symmetric GPT where  $V$  has dimension  $n + 1$ . Equations (33) and (35) imply for  $n \geq 2$ :

$$\frac{1}{\sqrt{2}} \geq \gamma(g; V, V^+) \geq \frac{1}{\min\{n, g\}}.$$

The upper bound shows that all centrally symmetric GPTs with  $n \geq 2$  have incompatible dichotomic measurements (otherwise we would have  $\gamma(g; V, V^+) = 1$ ), which is consistent with the findings of [Plá16]. For  $n \geq g$ , the upper bound can be improved to  $\sqrt{2/g} \geq \gamma(g; V, V^+)$  using Eq. (36). Moreover, Eq. (38) implies that there exists a constant  $c > 0$  such that

$$\gamma(g; V, V^+) \leq c \frac{\log(\min\{g, n\})}{(\min\{g, n\})^{1/8}}.$$

This bound is useful for us in the regime  $n < g$  and large  $n$ .

### 10.3. Other GPTs.

**10.3.1. Quantum mechanics** For quantum systems of dimension 2, the state space corresponds to the unit ball of  $\ell_2^3$  via the Bloch ball and is therefore centrally symmetric. We can hence use the results outlined in Sect. 10.1.2 for  $n = 3$ . For  $g = 2$  and  $g = 3$ , we have that

$$\Gamma(g; \text{QM}_2) = \text{QC}_g.$$

This recovers the results of [Bus86, BA07, BH08, PG11, BHSS13, BLPY16, BN18] (see Section 9.1 of [BN18] for the contributions of the respective works). In particular, the compatibility degree is  $\gamma(g; \text{QM}_2) = 1/\sqrt{g}$  in this case.

For  $g \geq 4$ , we still have

$$\Gamma(g; \text{QM}_2) \supseteq \text{QC}_g,$$

which recovers the result of [BN18]. However, we find that the compatibility degree is bounded by

$$\forall g \geq 4, \quad 0.58 \approx \frac{1}{\sqrt{3}} \geq \gamma(g; \text{QM}_2) \geq \frac{1}{2}. \quad (41)$$

The upper bound follows from Eq. (37) and the lower bound from the 1-summing norm of  $\ell_2^3$ . This improves over the lower bound  $\max\{1/\sqrt{g}, 1/4\}$  obtained in [BN18] and recovers the best known upper bound imposed by the compatibility degree for  $g = 3$  (see Proposition D.1). Numerics using the SDP in [WPGF09] indicate that the SIC-POVM constructed by choosing vertices of a tetrahedron on the Bloch sphere has  $\gamma(f) = 1/\sqrt{3}$ , whereas we can find random examples with  $\gamma(f) \leq 0.56 < 1/\sqrt{3}$  for  $g = 4$ . Thus,  $\gamma(4; \text{QM}_2) < 1/\sqrt{3}$ . However, the exact value of  $\gamma(4; \text{QM}_2)$  remains presently open.

**Question 10.1.** *We know, from Proposition 9.5, that*

$$\lim_{g \rightarrow \infty} \gamma(g; \text{QM}_2) = \inf_{g \geq 1} \gamma(g; \text{QM}_2) = \frac{1}{\pi_1(\ell_2^3)} = \frac{1}{2}.$$

*Is the lower bound in Eq. (41) optimal, that is,*

$$\forall g \geq 4 \quad \gamma(g; \text{QM}_2) = \frac{1}{2} \iff \gamma(4; \text{QM}_2) = \frac{1}{2} \quad ?$$

Beyond qubits, Proposition 9.7 gives

$$\gamma(g; \text{QM}_d) \geq \frac{1}{7.79d} \quad \forall g \in \mathbb{N}.$$

This bound has the same dimension dependence as the bound  $\gamma(g; \text{QM}_d) \geq \frac{1}{2d}$  derived in [BN18], but the constant is worse. One could therefore hope to improve the lower bound on the compatibility degree using better bounds on  $C_2(S_1^d)$ . However, even putting the cotype 2 constant to 1 in Eq. (34), which is the minimal value possible, gives a constant of approximately 3.81, which is still larger than 2. Thus, we need a better way to compute  $\pi_1(S_1^d)$  in order to improve over the bound on  $\gamma(g; \text{QM}_d)$  in [BN18].



**10.3.2. General bounds** For general GPTs, we have the generic lower bound in Eq. (32):

$$\gamma(g; V, V^+) \geq \frac{1}{\min\{g, \dim V\}}.$$

While the bound in terms of  $g$  was known (see e.g. [HMZ16]), the bound in terms of the dimension seems to be new.

**Acknowledgments.** A.B. and I.N. would like to thank Guillaume Aubrun for helpful discussions and for explaining to us how to bound the 1-summing norm of  $S_1$ . Moreover, the authors would like to thank Guillaume Aubrun also for organizing the workshop GPT & QIT in Lyon. Furthermore, A.B. and A.J. would like to thank Milán Mosonyi for organizing the QIMP 2018 conference, during which the idea for this project was born. A.B. acknowledges support from the VILLUM FONDEN via the QMATH Centre of Excellence (Grant no. 10059) and from the QuantERA ERA-NET Cofund in Quantum Technologies implemented within the European Union's Horizon 2020 Programme (QuantAlgo project) via the Innovation Fund Denmark. A.J. was supported by the grants APVV-16-0073 and VEGA 2/0142/20. I.N. was supported by the ANR project ESQuisses, Grant Number ANR-20-CE47-0014-01.

**Publisher's Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

#### Declarations

**Data Availability Statement** Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

## Appendix A: Necessary Background

**A.1. Convex cones.** Let  $L$  be a finite-dimensional real vector space. A subset  $L^+ \subseteq L$  is a *convex cone* if  $\lambda x + \mu y \in L^+$  for all  $x, y \in L^+$  and all  $\lambda, \mu \in \mathbb{R}_+$ . Often, we drop “convex” and talk simply about “cones”. To avoid pathologies, we assume all cones to be non-empty. A cone will be called *generating* if  $L = L^+ - L^+$ . Moreover, it is *pointed* if  $L^+ \cap (-L^+) = \{0\}$  (sometimes this is called *salient* instead). A *proper* cone is a convex cone which is closed, pointed and generating. A *base* of a cone  $C$  is a convex set  $K \subset C$  such that for every  $x \in C$ , there is a unique  $\lambda \geq 0$  such that  $x \in \lambda K$ . The *dual cone* of  $L^+$  is the closed convex cone

$$(L^+)^* := \{f \in L^* : f(x) \geq 0 \forall x \in L^+\}, \quad (42)$$

where  $L^*$  is the dual vector space of  $L$ . Two cones  $L_1^+$  and  $L_2^+$  living in vector spaces  $L_1$  and  $L_2$ , respectively, are *isomorphic* if there is a linear bijection  $\Theta : L_1 \rightarrow L_2$  such that  $\Theta(L_1^+) = L_2^+$ . A cone  $L^+$  is called *simplicial* if it is isomorphic to  $\mathbb{R}_+^d$ , where  $d = \dim(V^+)$ .

A *preordered vector space* is a tuple  $(L, L^+)$ , where  $L$  is a vector space and  $L^+$  is a convex cone. If  $L^+$  is additionally pointed, it is an *ordered vector space*. If  $V^+$  is a proper cone, we call  $(V, V^+)$  a *proper ordered vector space*. We write  $y \geq x$  for  $x, y \in V$  to mean that  $y - x \in V^+$ . If  $(V, V^+)$  is a preordered vector space, then  $(V^*, (V^+)^*)$  is its dual preordered vector space. If  $(V, V^+)$  is a proper ordered vector space, so is  $(V^*, (V^+)^*)$ .

An *order unit*  $\mathbb{1} \in V^+$  is an element such that for every  $v \in V$ , there is a  $\lambda > 0$  such that  $v \in \lambda[-\mathbb{1}, \mathbb{1}]$ . By [AT07, Lemma 1.7],  $\mathbb{1} \in V^+$  is an order unit if and only if  $\mathbb{1} \in \text{int } V^+$ .

**Lemma A.1.** *Let  $(L, L^+)$  be a preordered vector space and let  $\mathbb{1}$  be an order unit. Then  $L^+ = L$  if and only if  $-\mathbb{1} \in L^+$ .*

*Proof.* One implication is clear. Thus, let  $-\mathbb{1} \in L^+$ . Let  $v \in L$ . Since  $\mathbb{1}$  is an order unit, there is a  $\lambda > 0$  such that  $v + \lambda\mathbb{1} \in L^+$ . Since  $-\lambda\mathbb{1} \in L^+$ , it follows that  $v \in L^+$ . This proves the assertion since  $v \in L$  was arbitrary.  $\square$

The following facts about cones and their duals will be useful:

**Theorem A.2** (Bipolar theorem, [Roc70, Theorem 14.1]). *Every non-empty closed convex cone  $C \subseteq L$  satisfies  $C^{**} \cong C$ , where we have identified  $L \cong L^{**}$ .*

**Lemma A.3** ([Mul97]). *Let  $C \subseteq L$  be a convex cone. The cone  $C$  is generating if and only if  $\text{int } C \neq \emptyset$ . If  $\text{cl } C$  is pointed, then  $C^*$  is generating. If  $C$  is generating, then  $C^*$  is pointed. In particular, if  $C$  is proper, then  $C^*$  is proper.*

**A.2. Tensor products of cones.** There are in general infinitely many natural ways to define the tensor product of two cones  $L_1^+ \subseteq L_1$  and  $L_2^+ \subseteq L_2$ . Among these, there is a minimal and a maximal choice: The *minimal tensor product* of  $L_1^+$  and  $L_2^+$  is the cone

$$L_1^+ \otimes_{\min} L_2^+ := \text{conv}\{x \otimes y : x \in L_1^+, y \in L_2^+\}, \quad (43)$$

whereas the *maximal tensor product* of  $L_1^+$  and  $L_2^+$  is defined as

$$L_1^+ \otimes_{\max} L_2^+ := ((L_1^+)^* \otimes_{\min} (L_2^+)^*)^*. \quad (44)$$

It can be seen that if  $L_1^+$  and  $L_2^+$  are proper,  $L_1^+ \otimes_{\min} L_2^+$  and  $L_1^+ \otimes_{\max} L_2^+$  are proper as well [ALP19, Fact S23]. Moreover,  $L_1 \otimes_{\min} L_2$  is closed if  $L_1$  and  $L_2$  are [AS17, Exercise 4.14]. We call  $C$  a *tensor cone* for  $L_1^+$  and  $L_2^+$  if

$$L_1^+ \otimes_{\min} L_2^+ \subseteq C \subseteq L_1^+ \otimes_{\max} L_2^+.$$

From the recent work [ALPP19], we know that the tensor product of two cones is unique if and only if one of the cones is simplicial. This solves a longstanding open problem from [NP69, Bar81]:

**Theorem A.4** ([ALPP19, Theorem A]). *Let  $L_1^+$  and  $L_2^+$  be proper cones. Then,  $L_1^+ \otimes_{\min} L_2^+ = L_1^+ \otimes_{\max} L_2^+$  if and only if  $L_1^+$  or  $L_2^+$  is simplicial.*

**A.3. Positive maps.** Let  $(L, L^+)$  be a preordered vector space and let  $(L^*, (L^+)^*)$  be its dual. Consider the identity map  $\text{id}_L : L \rightarrow L$  and the associated canonical evaluation tensor  $\chi_L \in L \otimes L^*$ , defined by the following remarkable property:

$$\forall v \in L, \forall \alpha \in L^*, \quad \langle \chi_L, \alpha \otimes v \rangle = \alpha(v). \quad (45)$$

Using coordinates, we have

$$\chi_L = \sum_{i=1}^{\dim L} v_i \otimes \alpha_i \in L \otimes L^* \quad (46)$$

for  $\{v_i\}_{i=1}^{\dim L}$  a basis of  $L$  and  $\{\alpha_i\}_{i=1}^{\dim L}$  the corresponding dual basis in  $L^*$  (we have  $\alpha_i(v_j) = \delta_{ij}$ ). Let  $(M, M^+)$  be another preordered vector space and let  $\Phi : M \rightarrow L^*$  be a linear map. We define the linear functional  $s_\Phi : M \otimes L \rightarrow \mathbb{R}$  as

$$s_\Phi(z) = \langle \chi_L, (\Phi \otimes \text{id})(z) \rangle, \quad z \in M \otimes L. \quad (47)$$

We then have

$$\Phi(w) = \sum_{i=1}^{\dim L} s_\Phi(w \otimes v_i) \alpha_i, \quad \forall w \in M. \quad (48)$$

Since  $(M \otimes L)^* \cong M^* \otimes L^*$ ,  $s_\Phi$  corresponds to an element  $\varphi^\Phi \in M^* \otimes L^*$ , determined as

$$\langle \varphi^\Phi, w \otimes v \rangle = s_\Phi(w \otimes v) = \langle \Phi(w), v \rangle, \quad w \in M, v \in L. \quad (49)$$

One can check that

$$\varphi^\Phi = (\text{id} \otimes \Phi)(\chi_{M^*}) = (\Phi^* \otimes \text{id})(\chi_L). \quad (50)$$

Here,  $\Phi^* : L \rightarrow M^*$  is the dual linear map. Note that we have  $\chi_L = \varphi^{\text{id}_{L^*}}$ . In this paper, we shall often switch between the three different equivalent points of view:

$$\begin{array}{ccccc} \text{linear map} & \longleftrightarrow & \text{linear form} & \longleftrightarrow & \text{tensor} \\ \Phi : M \rightarrow L^* & & s_\Phi : M \otimes L \rightarrow \mathbb{R} & & \varphi^\Phi \in M^* \otimes L^* \end{array}$$

We say that  $\Phi$  is a *positive map*  $(M, M^+) \rightarrow (L^*, (L^+)^*)$  if  $\Phi(M^+) \subseteq (L^+)^*$ . It is quite clear that this happens if and only if  $s_\Phi : (M \otimes L, M^+ \otimes_{\min} L^+) \rightarrow \mathbb{R}$  is positive, equivalently,  $\varphi^\Phi \in (M^+)^* \otimes_{\max} (L^+)^*$ ; this gives the well-known correspondence between positive maps and the maximal tensor product. In particular, we have

**Lemma A.5.** *The identity map  $\text{id}_L : (L, L^+) \rightarrow (L, L^+)$  is positive. Moreover,  $\chi_L \in L^+ \otimes_{\max} (L^+)^*$ .*

*Proof.* The first assertion is obvious. The second one follows from Eq. (45).  $\square$

For a positive linear map  $\Psi : (M, M^+) \rightarrow (L, L^+)$ , the dual map  $\Psi^* : (L^*, (L^+)^*) \rightarrow (M^*, (M^+)^*)$  is positive as well. Moreover, it holds that for preordered vector spaces  $(L_i, L_i^+)$ ,  $(M_i, M_i^+)$  and positive maps  $\Phi_i : (L_i, L_i^+) \rightarrow (M_i, M_i^+)$ ,  $i \in [2]$ , both

$$\Phi_1 \otimes \Phi_2 : (L_1 \otimes L_2, L_1^+ \otimes_{\min} L_2^+) \rightarrow (M_1 \otimes M_2, M_1^+ \otimes_{\min} M_2^+)$$

and

$$\Phi_1 \otimes \Phi_2 : (L_1 \otimes L_2, L_1^+ \otimes_{\max} L_2^+) \rightarrow (M_1 \otimes M_2, M_1^+ \otimes_{\max} M_2^+)$$

are positive [Jen18, Section II.C]. Both statements also follow from Lemma A.6 below.

The minimal tensor product is associated with a special kind of maps. We first prove the following lemma, clarifying what happens under reordering of tensor products of cones.

**Lemma A.6.** *Let  $A^+, B^+, C^+, D^+$  be cones. Then*

$$(A^+ \otimes_{\min} B^+) \otimes_{\min} (C^+ \otimes_{\max} D^+) \subseteq (A^+ \otimes_{\min} C^+) \otimes_{\max} (B^+ \otimes_{\min} D^+).$$

*Proof.* Consider arbitrary  $a \in A^+$ ,  $b \in B^+$  and  $e \in C^+ \otimes_{\max} D^+$ . We have to check that for any  $\varphi \in (A^*)^+ \otimes_{\max} (C^*)^+$  and  $\psi \in (B^*)^+ \otimes_{\max} (D^*)^+$ , we have

$$\langle a \otimes b \otimes e, \varphi \otimes \psi \rangle \geq 0.$$

We shall use (twice) the following fact: given cones  $X^+, Y^+$  in vector spaces  $X, Y$ ,  $z \in X^+ \otimes_{\max} Y^+$  and  $\sigma \in (X^+)^*$ , the vector  $y = \langle \sigma, z \rangle \in Y$  defined by

$$\langle \tau, y \rangle = \langle \sigma \otimes \tau, z \rangle, \quad \forall \tau \in Y^*$$

is positive (i.e.  $y \in \overline{Y^+} \cong (Y^+)^{**}$ ). This corresponds to the fact that the evaluation of a positive map at a positive element is positive, and we leave its proof as an exercise for the reader.

Using the fact above, and writing  $\gamma := \langle a, \varphi \rangle \in (C^+)^*$  and  $\delta := \langle b, \psi \rangle \in (D^+)^*$ , we have

$$\langle a \otimes b \otimes e, \varphi \otimes \psi \rangle = \langle e, \gamma \otimes \delta \rangle \geq 0,$$

proving the statement of the lemma.  $\square$

**Proposition A.7.** *Let  $\Phi : (M, M^+) \rightarrow (L^*, (L^+)^*)$  be a positive map between preordered vector spaces. The following conditions are equivalent:*

- (1) *The map  $\Phi \otimes \text{id}_L : (M \otimes L, M^+ \otimes_{\max} \overline{L^+}) \rightarrow (L^* \otimes L, (L^+)^* \otimes_{\min} \overline{L^+})$  is positive.*
- (2) *The map  $s_\Phi : (M \otimes L, M^+ \otimes_{\max} L^+) \rightarrow \mathbb{R}$  is positive.*
- (3)  $\varphi^\Phi \in (M^+)^* \otimes_{\min} (L^+)^*$
- (4) *The map  $\Phi \otimes \text{id}_N : (M \otimes N, M^+ \otimes_{\max} N^+) \rightarrow (L^* \otimes N, (L^+)^* \otimes_{\min} N^+)$  is positive, for any preordered vector space  $(N, N^+)$  with  $N^+$  closed.*

*Proof.* Since both (2)  $\iff$  (3) and (4)  $\implies$  (1) are trivial, we only prove (1)  $\implies$  (2) and (3)  $\implies$  (4). The first implication follows immediately from Lemma A.5. For the second implication, use Lemmas A.5 and A.6 to prove that

$$\begin{aligned} \varphi^\Phi \otimes \chi_N &\in ((M^+)^* \otimes_{\min} (L^+)^*) \otimes_{\min} ((N^+)^* \otimes_{\max} N^+) \subseteq (M^+ \otimes_{\max} N^+)^* \\ &\quad \otimes_{\max} ((L^+)^* \otimes_{\min} N^+), \end{aligned}$$

which is precisely the desired conclusion, since  $\varphi^\Phi \otimes \chi_N$  is the tensor corresponding to the map  $\Phi \otimes \text{id}_N$ .  $\square$

**Definition A.8.** Let  $\Phi : (M, M^+) \rightarrow (L^*, (L^+)^*)$  be a positive linear map between preordered vector spaces. We say that  $\Phi$  is *entanglement breaking* (EB) if any of the equivalent conditions in Proposition A.7 holds.

*Remark A.9.* Point (1) of Proposition A.7 was used as a definition in [Jen18]. Note that this definition of entanglement breaking maps agrees with the usual one used in quantum mechanics [HSR03] for  $L^*$ ,  $M$  being the vector space of Hermitian matrices and their respective cones being the cones of positive semidefinite elements. This can be seen from the point (3) of Proposition A.7, which states that the corresponding Choi matrix is separable [Wat18, Exercise 6.1].

We conclude this section with a small lemma connecting the trace of the composition of two maps and the inner product of the corresponding tensors. Recall that the *trace* of a linear operator  $\Phi : L \rightarrow L$  is defined as

$$\text{Tr}\Phi = \sum_{j=1}^{\dim L} \alpha_j(\Phi(v_j)) = \langle \chi_L, \varphi^\Phi \rangle,$$

where  $\{v_j\}_{j=1}^{\dim L}$ ,  $\{\alpha_j\}_{j=1}^{\dim L}$ , are dual bases of  $L$ ,  $L^*$ .

**Lemma A.10.** *Let  $\Psi : M \rightarrow L$ ,  $\Phi : M^* \rightarrow L^*$  be two linear maps between vector spaces. Then,*

$$\text{Tr}[\Psi\Phi^*] = \langle \varphi^\Phi, \varphi^\Psi \rangle.$$

*Proof.* Using the definition of the trace and Eq. (50), we have

$$\begin{aligned} \text{Tr}[\Psi\Phi^*] &= \langle \chi_L, \varphi^{\Psi\Phi^*} \rangle \\ &= \langle \chi_L, (\Phi\Psi^* \otimes \text{id}_L)(\chi_{L^*}) \rangle \\ &= \langle (\Phi^* \otimes \text{id}_{L^*})(\chi_L), (\Psi^* \otimes \text{id}_L)(\chi_{L^*}) \rangle \\ &= \langle \varphi^\Phi, \varphi^\Psi \rangle. \end{aligned}$$

□

**A.4. Tensor products of Banach spaces.** Besides the tensor product of convex cones, we will also need the tensor product of Banach spaces. See [Rya02] for a good introduction. Let  $X$ ,  $Y$  be two Banach spaces with norms  $\|\cdot\|_X$  and  $\|\cdot\|_Y$ , respectively. Then, there are usually infinitely many natural norms that can be used to turn the vector space  $X \otimes Y$  into a Banach space. There are two choices of norms which are minimal and maximal in a sense as we shall see next.

**Definition A.11** (*Projective tensor norm*). The *projective norm* of an element  $z \in X \otimes Y$  is defined as

$$\|z\|_{X \otimes_\pi Y} := \inf \left\{ \sum_i \|x_i\|_X \|y_i\|_Y : z = \sum_i x_i \otimes y_i \right\}.$$

Let  $X^*$  and  $Y^*$  be the dual spaces of  $X$  and  $Y$ , respectively, and let their norms be  $\|\cdot\|_{X^*}$  and  $\|\cdot\|_{Y^*}$ .

**Definition A.12** (*Injective tensor norm*). Let  $z = \sum_i x_i \otimes y_i \in X \otimes Y$ . Then, its *injective norm* is

$$\|z\|_{X \otimes_\epsilon Y} := \sup \left\{ \left| \sum_i \varphi(x_i) \psi(y_i) \right| : \|\varphi\|_{X^*} \leq 1, \|\psi\|_{Y^*} \leq 1 \right\}.$$

If  $X$  and  $Y$  are clear from the context, we will sometimes only write  $\|\cdot\|_\pi$  and  $\|\cdot\|_\varepsilon$ , respectively. Importantly,  $\|\cdot\|_\varepsilon$  and  $\|\cdot\|_\pi$  are dual norms, i.e.

$$\|z\|_{X \otimes_\varepsilon Y} = \sup_{\|\varphi\|_{X^* \otimes_\pi Y^*} \leq 1} |\varphi(z)|$$

and vice versa.

In some cases, the projective and injective norms have simpler expressions. The projective norm  $\ell_1^g \otimes_\pi X$  of a vector

$$\mathbb{R}^g \otimes X \ni z = \sum_{i=1}^g e_i \otimes z_i$$

with  $\{e_i\}_{i \in [g]}$  the standard basis is given by (see e.g. [Rya02, Example 2.6])

$$\|z\|_{\ell_1^g \otimes_\pi X} = \sum_{i=1}^g \|z_i\|_X. \quad (51)$$

The injective norm  $\ell_1^g \otimes_\varepsilon X$  is (see e.g. [Rya02, Example 3.4])

$$\|z\|_{\ell_1^g \otimes_\varepsilon X} = \sup_{\|y\|_{X^*} \leq 1} \sum_{i=1}^g |\langle y, z_i \rangle| = \sup_{\varepsilon \in \{\pm 1\}^g} \left\| \sum_{i=1}^g \varepsilon_i z_i \right\|_X. \quad (52)$$

We will later be interested in the maximal ratio between the projective and the injective norms introduced in [ALP+20, Equation (15)]:

$$\rho(X, Y) = \max_{\|z\|_\varepsilon \leq 1} \|z\|_\pi \quad (53)$$

where the maximum runs over all  $z \in X \otimes Y$ .

The injective and the projective norms both belong to a class of tensor norms which are the reasonable crossnorms.

**Definition A.13** ([Rya02]). Let  $X$  and  $Y$  be two Banach spaces. We say that a norm  $\|\cdot\|_\alpha$  on  $X \otimes Y$  is a *reasonable crossnorm* if it has the following properties:

- (1)  $\|x \otimes y\|_\alpha \leq \|x\|_X \|y\|_Y$  for all  $x \in X, y \in Y$ ,
- (2) For all  $\varphi \in X^*$ , for all  $\psi \in Y^*$ ,  $\varphi \otimes \psi$  is bounded on  $X \otimes Y$  and  $\|\varphi \otimes \psi\|_{\alpha^*} \leq \|\varphi\|_{X^*} \|\psi\|_{Y^*}$ ,

where  $\|\cdot\|_{\alpha^*}$  is the dual norm to  $\|\cdot\|_\alpha$ .

The injective and projective norms are the smallest and largest reasonable crossnorms we can put on  $X \otimes Y$ , respectively:

**Proposition A.14** ([Rya02, Proposition 6.1]). *Let  $X$  and  $Y$  be Banach spaces.*

(a) *A norm  $\|\cdot\|_\alpha$  on  $X \otimes Y$  is a reasonable crossnorm if and only if*

$$\|z\|_{X \otimes_\varepsilon Y} \leq \|z\|_\alpha \leq \|z\|_{X \otimes_\pi Y}$$

*for all  $z \in X \otimes Y$ .*

(b) *If  $\|\cdot\|_\alpha$  is a reasonable crossnorm on  $X \otimes Y$ , then  $\|x \otimes y\|_\alpha = \|x\|_X \|y\|_Y$  for every  $x \in X$  and every  $y \in Y$ . Furthermore, for all  $\varphi \in X^*$  and all  $\psi \in Y^*$ , the norm  $\|\cdot\|_{\alpha^*}$  satisfies  $\|\varphi \otimes \psi\|_{\alpha^*} = \|\varphi\|_{X^*} \|\psi\|_{Y^*}$ .*

## Appendix B: An Extension Theorem for Ordered Vector Spaces

In this section, we prove an extension theorem which we use in Sect. 5 to relate the compatibility of measurements in a GPT to properties of an associated map (in particular Proposition B.8). Along the way, we will use the extension theorem to study tensor products of certain cones.

Let  $E \subseteq \mathbb{R}^d$  be a subspace containing a point with positive coordinates:

$$E \cap \text{ri}(\mathbb{R}_+^d) \neq \emptyset. \quad (54)$$

We set  $E^+ := E \cap \mathbb{R}_+^d$ . Note that  $E^+$  is proper in  $E$  (see [Roc70, Corollary 6.5.1] to conclude that  $E^+$  is generating).

We shall use the following key extension theorem (see e.g. [Cas05, Theorem 1]):

**Theorem B.1** (M. Riesz extension theorem). *Let  $(X, X^+)$  be a preordered vector space,  $Y \subseteq X$  a linear subspace, and  $\varphi : Y \rightarrow \mathbb{R}$  a positive linear form on  $(Y, Y^+)$ , where  $Y^+ := Y \cap X^+$ . Assume that for every  $x \in X$ , there exists  $y \in Y$  such that  $x \leq y$ . Then, there exists a positive linear form  $\tilde{\varphi} : X \rightarrow \mathbb{R}$  such that  $\tilde{\varphi}|_Y = \varphi$ .*

*Remark B.2.* Note that [Cas05, Theorem 1] states the theorem only for ordered vector spaces. However, the same proof works if  $X^+$  is an arbitrary convex cone.

We prove now the main result of this section; see also Remark B.7 for an equivalent formulation. Note that the tensor product of  $\mathbb{R}_+^d$  with a proper cone  $L^+$  is unique since  $\mathbb{R}_+^d$  is simplicial.

**Proposition B.3.** *Let  $(L, L^+)$  be a proper ordered vector space, and  $E \subseteq \mathbb{R}^d$  as in Eq. (54). Any positive linear form*

$$\varphi : (E \otimes L, (E \otimes L) \cap (\mathbb{R}_+^d \otimes L^+)) \rightarrow \mathbb{R} \quad (55)$$

*can be extended to a positive linear form  $\tilde{\varphi} : \mathbb{R}^d \otimes L \rightarrow \mathbb{R}$ .*

*Proof.* We shall use Theorem B.1 with  $X = \mathbb{R}^d \otimes L$ ,  $X^+ = \mathbb{R}_+^d \otimes L^+$  and  $Y = E \otimes L$ . We have to show that for any  $x \in \mathbb{R}^d \otimes L$ , there is a  $y \in E \otimes L$  such that  $y - x \in \mathbb{R}_+^d \otimes L^+$ . It is enough to consider simple tensors of the form  $x = r \otimes v$ , where  $r \in \mathbb{R}^d$  and  $v \in L$ ; the general case will follow by linearity. Since  $L^+$  is generating, there are  $v_+, v_- \in L^+$  such that  $v = v_+ - v_-$ . Furthermore, from the assumption (54),  $E$  contains a vector with strictly positive coordinates  $e$ , hence there exist  $\lambda_{\pm} > 0$  such that  $\lambda_+ e - r \geq 0$  and  $\lambda_- e + r \geq 0$ . Then,

$$\lambda_+ e \otimes v_+ + \lambda_- e \otimes v_- - r \otimes v = \lambda_+ e \otimes v_+ + \lambda_- e \otimes v_- - r \otimes v_+ + r \otimes v_- \in \mathbb{R}_+^d \otimes L^+$$

and  $\lambda_{\pm} e \otimes v_{\pm} \in E \otimes L$ . Thus, we can choose  $y = \lambda_+ e \otimes v_+ + \lambda_- e \otimes v_-$ .  $\square$

We can now identify the dual proper ordered vector space  $(E^*, (E^+)^*)$ .

**Proposition B.4.** *Let us identify the dual vector space  $E^* \cong E$ , with duality given by the standard inner product in  $\mathbb{R}^d$ . We then have  $(E^+)^* = J(\mathbb{R}_+^d)$ , where  $J : \mathbb{R}^d \rightarrow E$  is the orthogonal projection onto  $E$ . For a proper ordered vector space  $(L, L^+)$ , we have*

$$(E^+)^* \otimes_{\min} L^+ = (J \otimes \text{id}_L)(\mathbb{R}_+^d \otimes L^+).$$

*Proof.* Let  $r \in \mathbb{R}_+^d$ , then for  $e \in E^+ \subseteq \mathbb{R}_+^d$

$$\langle J(r), e \rangle = \langle r, e \rangle \geq 0,$$

since  $\mathbb{R}_+^d$  is self-dual, so that  $J(\mathbb{R}_+^d) \subseteq (E^+)^*$ . By Proposition B.3, any  $\varphi \in (E^+)^*$  extends to a positive form  $\tilde{\varphi} : \mathbb{R}^d \rightarrow \mathbb{R}$ , so that  $\varphi = J(\tilde{\varphi})$ , which implies the reverse inclusion. If  $(L, L^+)$  is a proper ordered vector space, then  $(E^+)^* \otimes_{\min} L^+$  is a cone of elements of the form

$$\sum_j \varphi_j \otimes v_j = \sum_j J(\tilde{\varphi}_j) \otimes v_j = (J \otimes \text{id}_L) \left( \sum_j \tilde{\varphi}_j \otimes v_j \right)$$

for  $v_j \in L^+$  and  $\tilde{\varphi}_j \in \mathbb{R}_+^d$ , proving the last statement.  $\square$

We now provide a useful characterization of the maximal tensor product of  $E^+$  with  $L^+$ , identifying at the same time the cone appearing in Eq. (55).

**Proposition B.5.** *For  $E$  as above and  $(L, L^+)$  a proper ordered vector space, it holds that*

$$E^+ \otimes_{\max} L^+ = (E \otimes L) \cap (\mathbb{R}_+^d \otimes L^+).$$

*Proof.* The inclusion “ $\subseteq$ ” follows from the monotonicity of the max tensor product with respect to each factor. To show the reverse inclusion “ $\supseteq$ ”, we have to prove that for any  $z \in (E \otimes L) \cap (\mathbb{R}_+^d \otimes L^+)$ , and for any  $\beta \in (E^+)^*$ ,  $\alpha \in (L^+)^*$ , we have that  $\langle \beta \otimes \alpha, z \rangle \geq 0$ . By Proposition B.4,  $\beta \in (E^+)^*$  implies that  $\beta = J(\tilde{\beta})$  for a positive form  $\tilde{\beta} : \mathbb{R}^d \rightarrow \mathbb{R}$ . Since  $z \in \mathbb{R}_+^d \otimes L^+$ , we have a decomposition  $z = \sum_{i=1}^d r_i \otimes v_i$ , where  $r_i \in \mathbb{R}_+^d$  and  $v_i \in L^+$ . This yields

$$\langle \beta \otimes \alpha, z \rangle = \langle \tilde{\beta} \otimes \alpha, z \rangle = \sum_{i=1}^d \tilde{\beta}(r_i) \alpha(v_i) \geq 0,$$

finishing the proof.  $\square$

**Corollary B.6.** *For  $E_1, E_2$  satisfying Eq. (54) (but not necessarily of the same dimension), it holds that*

$$E_1^+ \otimes_{\max} E_2^+ = (E_1 \otimes E_2) \cap (\mathbb{R}_+^{d_1} \otimes \mathbb{R}_+^{d_2}).$$

*Proof.* The proof is almost the same as for Proposition B.5, only that the functionals on both subspaces need to be extended.  $\square$

**Remark B.7.** Using Proposition B.5, one can restate Proposition B.3 as follows: Any positive linear form  $\varphi : (E \otimes L, E^+ \otimes_{\max} L^+) \rightarrow \mathbb{R}$  can be extended to a positive linear form  $\tilde{\varphi} : \mathbb{R}^d \otimes L \rightarrow \mathbb{R}$  for proper  $L^+$ .

We now study extendability of general positive maps on  $(E, E^+)$ . The proof technique is inspired by the finite dimensional version of Arveson’s extension theorem [Pau03, Theorem 6.2].

**Proposition B.8.** *Let  $E \subseteq \mathbb{R}^d$  be a subspace such that  $E \cap \text{ri}(\mathbb{R}_+^d) \neq \emptyset$  and  $E^+ = E \cap \mathbb{R}_+^d$ . Let  $J$  be the orthogonal projection onto  $E$  and let  $(L, L^+)$  be an proper ordered vector space. Finally, let  $\Phi : E \rightarrow L^*$  be a linear map. The following are equivalent:*



- (1) There exists a positive extension  $\tilde{\Phi} : (\mathbb{R}^d, \mathbb{R}_+^d) \rightarrow (L^*, (L^+)^*)$  of  $\Phi$ .
- (2) The linear map  $\Phi \otimes \text{id}_L : (E \otimes L, E^+ \otimes_{\max} L^+) \rightarrow (L^* \otimes L, (L^+)^* \otimes_{\min} L^+)$  is positive.
- (3) The form  $s_\Phi : (E \otimes L, E^+ \otimes_{\max} L^+) \rightarrow \mathbb{R}$  is positive.
- (4)  $\varphi^\Phi \in (J \otimes \text{id}_{L^*})(\mathbb{R}_+^d \otimes (L^+)^*)$ .
- (5)  $\Phi$  is entanglement breaking.

*Proof.* By Propositions A.7 and B.4, the statements (2) - (5) are equivalent. It is therefore enough to show that (1)  $\implies$  (2) and (3)  $\implies$  (1). We start by showing that the existence of the positive extension (1) implies the positivity of the map  $\Phi \otimes \text{id}_L$  in (2). Both  $\tilde{\Phi}$  and  $\text{id}_L$  are positive maps. Therefore,  $\tilde{\Phi} \otimes \text{id}_L : (\mathbb{R}^d \otimes L, \mathbb{R}_+^d \otimes_{\min} L^+ = \mathbb{R}_+^d \otimes_{\max} L^+) \rightarrow (L^* \otimes L, (L^+)^* \otimes_{\min} L^+)$  is positive. The claim follows by Proposition B.5, since  $\Phi \otimes \text{id}_L$  is a restriction of this map to  $E \otimes L$ .

It remains to show that (3)  $\implies$  (1). Using Proposition B.3 and Remark B.7, we extend the form  $s_\Phi$  to  $\tilde{s}_\Phi : \mathbb{R}^d \otimes L \rightarrow \mathbb{R}$ . Let  $\tilde{\Phi} : (\mathbb{R}^d, \mathbb{R}_+^d) \rightarrow (L^*, (L^+)^*)$  be the positive map related to  $\tilde{s}_\Phi$  as in Eq. (48). It remains to check that  $\tilde{\Phi}$  is indeed an extension of  $\Phi$ . Let  $v_j$  and  $\alpha_j$  be elements of a basis of  $L$  and its dual basis for all  $j \in [\dim L]$ . For any  $e \in E$ , we compute

$$\tilde{\Phi}(e) = \sum_{j=1}^{\dim L} \tilde{s}_\Phi(e \otimes v_j) \alpha_j = \sum_{j=1}^{\dim L} s_\Phi(e \otimes v_j) \alpha_j = \Phi(e),$$

finishing the proof.  $\square$

*Remark B.9.* Note that the existence of a positive extension of  $\Phi : E \rightarrow L^*$  can be checked using conic programming, see Sect. E.2 in the Appendix.

## Appendix C: Generalized Spectrahedra

In this section, we will generalize some of the theory of (free) spectrahedra to the setting of ordered vector spaces by allowing for more general cones than the cone of positive semidefinite matrices. We reformulate the compatibility of measurements as an inclusion problem for generalized spectrahedra in Sect. 6. We define a generalized spectrahedron as a convex subset of some vector space which can be represented by positivity conditions with respect to some abstract cone.

**Definition C.1.** Let  $L, M$  be two finite-dimensional vector spaces and consider a cone  $C \subseteq M \otimes L$ . For a  $g$ -tuple of elements  $a = (a_1, \dots, a_g) \in M^g$ , we define the *generalized spectrahedron*

$$\mathcal{D}_a(L, C) := \{(v_1, \dots, v_g) \in L^g : \sum_{i=1}^g a_i \otimes v_i \in C\}.$$

*Remark C.2.* It is easy to see that any generalized spectrahedron is a convex cone and that the generalized spectrahedron is closed if  $C$  is. In fact, note that the  $g$ -tuple  $a \in M^g$  defines a linear map  $a : \mathbb{R}^g \rightarrow M$ , by  $x \mapsto \sum_i x_i a_i$ . The generalized spectrahedron  $\mathcal{D}_a(L, C)$  is the largest cone in  $\mathbb{R}^g \otimes L \cong L^g$  that makes the map

$$a \otimes \text{id}_L : (\mathbb{R}^g \otimes L, \mathcal{D}_a(L, C)) \rightarrow (M \otimes L, C)$$

positive. If  $a$  is a basis of  $M$ , then the corresponding map  $a \otimes \text{id}_L$  is an isomorphism through which the cones  $C$  and  $\mathcal{D}_a(L, C)$  are affinely isomorphic.

Note that usual spectrahedra correspond to the choice  $L = \mathbb{R}$ ,  $M = \mathcal{M}_n^{\text{sa}}(\mathbb{C})$ , and  $C$  being the positive semidefinite cone. Free spectrahedra are the union over  $d \geq 1$  of generalized spectrahedra for  $L = \mathcal{M}_d^{\text{sa}}(\mathbb{C})$  and  $C$  being the PSD cone of  $dn \times dn$  matrices; note that there exists no natural notion of generalized *free* spectrahedra, since there are no canonical sequences of cones  $(C_d)_{d \geq 1}$ , with  $C_1 = C$ .

We will now consider generalized spectrahedra which are in some sense minimal and maximal. The definitions are inspired by the corresponding notions for matrix convex sets, see [DDOSS17, PSS18].

**Definition C.3.** Let  $\mathcal{C} \subset \mathbb{R}^g$  be a closed convex cone. Let  $(L, L^+)$  be a finite-dimensional preordered vector space. Then

$$\mathcal{D}_{\min}(\mathcal{C}; L, L^+) := \left\{ \sum_i x^{(i)} \otimes h_i \in \mathbb{R}^g \otimes L \cong L^g : x^{(i)} \in \mathcal{C}, h_i \in L^+ \forall i \right\} \cong \mathcal{C} \otimes_{\min} L^+$$

is the *minimal* generalized spectrahedron corresponding to  $\mathcal{C}$ .

**Definition C.4.** Let  $\mathcal{C} \subset \mathbb{R}^g$  be a closed convex cone. Let  $(L, L^+)$  be a finite-dimensional preordered vector space with  $L^+$  closed. Then,

$$\mathcal{D}_{\max}(\mathcal{C}; L, L^+) := \left\{ (v_1, \dots, v_g) \in L^g : \sum_{i=1}^g h_i v_i \in L^+ \forall h \in \mathbb{R}^g \text{ s.t. } \sum_{i=1}^g h_i c_i \geq 0 \forall c \in \mathcal{C} \right\} \cong \mathcal{C} \otimes_{\max} L^+$$

is the *maximal* generalized spectrahedron corresponding to  $\mathcal{C}$ .

In order to show that these sets are indeed generalized spectrahedra and to justify the identifications with minimal and maximal cones, we prove a lemma.

**Lemma C.5.** Let  $(L, L^+)$  be a preordered vector space. Let  $e = (e_1, \dots, e_g)$  be the canonical basis of  $\mathbb{R}^g$ . Then,

$$\mathcal{D}_{\min}(\mathcal{C}; L, L^+) \cong \mathcal{D}_e(L, \mathcal{C} \otimes_{\min} L^+).$$

If  $L^+$  is closed, then

$$\mathcal{D}_{\max}(\mathcal{C}; L, L^+) \cong \mathcal{D}_e(L, \mathcal{C} \otimes_{\max} L^+).$$

*Proof.* By definition,  $(v_1, \dots, v_g) \in \mathcal{D}_{\min}(\mathcal{C}; L, L^+)$  if and only if  $v_j = \sum_i x_j^{(i)} h_i$  for some  $x^{(i)} \in \mathcal{C}$  and  $h_i \in L^+$ . Moreover,  $\sum_{j=1}^g e_j \otimes v_j \in \mathcal{C} \otimes_{\min} L^+$  if and only if

$$\sum_{j=1}^g e_j \otimes v_j = \sum_i x^{(i)} \otimes h_i \quad x^{(i)} \in \mathcal{C}, h_i \in L^+.$$

A comparison of coordinates proves the first assertion. Let now  $\sum_{i=1}^g e_i \otimes v_i \in \mathcal{C} \otimes_{\max} L^+$ . By definition, this is true if and only if

$$\sum_{i=1}^g \varphi(e_i) v_i \in L^+ \quad \forall \varphi \in \mathcal{C}^*$$

since  $L^+$  is closed. Realizing that for  $c = \sum_{i=1}^g c_i e_i$ , we can write  $\varphi(c) = \sum_{i=1}^g c_i h_i \geq 0$  with  $h_i = \varphi(e_i)$  for all  $i \in [g]$ , proves one inclusion. The second assertion follows by identifying  $(h_1, \dots, h_g)$  as in the assertion with some  $\varphi \in \mathcal{C}$ .  $\square$

The following proposition justifies that we speak of minimal and maximal cones.

**Proposition C.6.** *Let  $\mathcal{C} \subset \mathbb{R}^g$  be a closed convex cone. Let  $(L, L^+)$ ,  $(M, M^+)$  be two finite-dimensional preordered vector spaces with a tensor cone  $C \subset M \otimes L$ . Let  $M^+$ ,  $L^+$  be closed convex cones. If  $a \in M^g$  such that  $\mathcal{D}_a(\mathbb{R}, M^+) = \mathcal{C}$ , then*

$$\mathcal{D}_{\min}(\mathcal{C}; L, L^+) \subseteq \mathcal{D}_a(L, C) \subseteq \mathcal{D}_{\max}(\mathcal{C}; L, L^+).$$

Moreover, for a closed convex  $\mathcal{C}'$  such that  $\mathcal{C}' \subseteq \mathcal{C}$ , it holds that

$$\mathcal{D}_{\min}(\mathcal{C}'; L, L^+) \subseteq \mathcal{D}_{\min}(\mathcal{C}; L, L^+) \quad \text{and} \quad \mathcal{D}_{\max}(\mathcal{C}'; L, L^+) \subseteq \mathcal{D}_{\max}(\mathcal{C}; L, L^+).$$

*Proof.* Let  $v \in \mathcal{D}_{\min}(\mathcal{C}; L, L^+)$ . Then,  $v = (v_1, \dots, v_g)$  where

$$v_j = \sum_i x_j^{(i)} h_i \quad \forall j \in [g] \quad (56)$$

for some  $x^{(i)} \in \mathcal{C}$  and  $h_i \in L^+$ . Thus, we can rewrite

$$\sum_{j=1}^g a_j \otimes v_j = \sum_i \left( \sum_{j=1}^g x_j^{(i)} a_j \right) \otimes h_i.$$

Since  $\mathcal{D}_a(\mathbb{R}, M^+) = \mathcal{C}$ , it holds that

$$\sum_{j=1}^g x_j^{(i)} a_j \in M^+ \quad \forall i.$$

Thus, since  $C$  is a tensor cone,

$$\sum_{j=1}^g a_j \otimes v_j \in M^+ \otimes_{\min} L^+ \subseteq C.$$

Therefore, we can conclude that  $v \in \mathcal{D}_a(L, C)$ , which proves the first assertion.

For the second assertion, let  $v = (v_1, \dots, v_g) \in \mathcal{D}_a(L, C)$ . Thus

$$\sum_{j=1}^g a_j \otimes v_j \in C \subseteq M^+ \otimes_{\max} L^+,$$

since  $C$  is a tensor cone. Thus,

$$\sum_{j=1}^g \varphi(a_j) \psi(v_j) \geq 0 \quad \forall \varphi \in (M^+)^*, \psi \in (L^+)^*.$$

Using that  $M^+ \cong (M^+)^{**}$  and that  $\mathcal{D}_a(\mathbb{R}, M^+) = \mathcal{C}$ , it follows that  $(\psi(v_1), \dots, \psi(v_g)) \in \mathcal{C}$  for all  $\psi \in (L^+)^*$ . Hence, for all  $\psi \in (L^+)^*$ , it holds that

$$\sum_{i=1}^g h_i \psi(v_i) \geq 0 \quad \forall h \in \mathbb{R}^g \text{ s.t. } \sum_{i=1}^g h_i c_i \geq 0 \quad \forall c \in \mathcal{C}.$$

Using that  $L^+ \cong (L^+)^{**}$ , it follows that  $v \in \mathcal{D}_{\max}(\mathcal{C}; L, L^+)$ . The inclusions in the last assertion follow directly from the Definitions C.3 and C.4.  $\square$

*Remark C.7.* In view of Remark C.2, the first inclusion of the above proposition follows directly from the fact that the map  $a : (\mathbb{R}^g, \mathcal{C}) \rightarrow (M, M^+)$  is positive and for any positive map, its tensor product with the identity map is positive with respect to the minimal tensor cone. Since we have  $\mathcal{C}^* = a^*((M^+)^*)$ , the second inclusion is a consequence of the same property of  $a^*$ .

*Remark C.8.* Assume that  $M^+$  and  $L^+$  are proper cones. By the results of [ALPP19], Proposition C.6 implies that the generalized spectrahedron  $\mathcal{D}_a(L, \mathcal{C})$  does not depend on the choice of the tensor cone  $\mathcal{C}$  if and only if at least one of the cones  $\mathcal{C} = \mathcal{D}_a(\mathbb{R}, M^+)$  or  $L^+$  is simplicial.

As for free spectrahedra, we can connect the inclusion of generalized spectrahedra to the positivity of an associated map [HKM13, DDOSS17, HKMS19].

**Proposition C.9.** *Let  $L$ ,  $M$  and  $N$  be finite-dimensional vector spaces and let  $C_M \subset M \otimes L$ ,  $C_N \subset N \otimes L$  be two cones. Moreover, let  $a \in M^g$ ,  $b \in N^g$  be two tuples, where  $a$  consists of linearly independent elements which span the subspace  $M' \subseteq M$ . We define a map  $\Phi : M' \rightarrow N$ ,  $\Phi(a_i) = b_i$  for all  $i \in [g]$ . Then,*

$$\mathcal{D}_a(L, C_M) \subseteq \mathcal{D}_b(L, C_N)$$

*if and only if  $\Phi \otimes \text{id}_L : (M' \otimes L, M' \otimes L \cap C_M) \rightarrow (N \otimes L, C_N)$  is positive.*

*Proof.* Let us assume the inclusion. Consider  $z \in M' \otimes L$ . Thus, we can write

$$z = \sum_{i=1}^g a_i \otimes z_i,$$

where  $z_i \in L$  for all  $i \in [g]$ . If  $z \in C_M$ , then  $(z_1, \dots, z_g) \in \mathcal{D}_a(L, C_M)$  and hence

$$(\Phi \otimes \text{id}_L)(z) = \sum_{i=1}^g b_i \otimes z_i.$$

The right hand side is in  $C_N$  since  $\mathcal{D}_a(L, C_M) \subseteq \mathcal{D}_b(L, C_N)$ . Conversely, let  $\Phi \otimes \text{id}_L$  be positive. Let  $(v_1, \dots, v_g) \in \mathcal{D}_a(L, C_M)$ . Then,

$$\sum_{i=1}^g a_i \otimes v_i \in (M' \otimes L) \cap C_M$$

and the assertion follows from an application of  $\Phi \otimes \text{id}_L$  to this element.  $\square$

As for free spectrahedra, it is possible to look at what inclusion with respect to some preordered vector space  $(L, L^+)$  implies for the inclusion with respect to  $(\mathbb{R}, \mathbb{R}_+)$ .

**Proposition C.10.** *Let  $(M, M^+)$ ,  $(N, N^+)$  and  $(L, L^+)$  be preordered vector spaces, where  $N^+$  is closed and  $L^+$  contains at least one element which is not in  $-L^+$ . Moreover, let  $C_M \subset M \otimes L$  and  $C_N \subset N \otimes L$  be tensor cones and  $a \in M^g$ ,  $b \in N^g$  for some  $g \in \mathbb{N}$ . Then,*

$$\mathcal{D}_a(L, C_M) \subseteq \mathcal{D}_b(L, C_N) \implies \mathcal{D}_a(\mathbb{R}, M^+) \subseteq \mathcal{D}_b(\mathbb{R}, N^+)$$

*Proof.* Let  $v \in L^+$ ,  $v \notin -\overline{L^+}$  and  $x \in \mathcal{D}_a(\mathbb{R}, M^+)$ . Then,  $(x_1 v, \dots, x_g v) \in \mathcal{D}_a(L, C_M)$ , since  $C_M$  contains in particular  $M^+ \otimes_{\min} L^+$ . Thus,

$$\left( \sum_{i=1}^g x_i b_i \right) \otimes v \in C_N.$$

This implies that  $\sum_{i=1}^g x_i b_i \in (N^+)^{**} \cong N^+$ , since  $C_N \subseteq N^+ \otimes_{\max} L^+$  and we can find  $\varphi \in (L^+)^*$  such that  $\varphi(v) > 0$  since  $\psi(v) = 0$  for all  $\psi \in L^*$  implies  $-v \in \overline{L^+}$ . Hence  $x \in \mathcal{D}_b(\mathbb{R}, N^+)$ .  $\square$

In general,  $\mathcal{D}_a(\mathbb{R}, M^+) \subseteq \mathcal{D}_b(\mathbb{R}, N^+)$  does not imply  $\mathcal{D}_a(L, C_M) \subseteq \mathcal{D}_b(L, C_N)$ . However, if  $M$  and  $N$  contain order units, the implication can be made true by shrinking the left hand side.

**Definition C.11.** Let  $(M, M^+)$ ,  $(N, N^+)$  and  $(L, L^+)$  be preordered vector spaces where  $M^+$  and  $N^+$  contain order units  $\mathbb{1}_M$  and  $\mathbb{1}_N$ , respectively. Moreover, let  $C_M \subset M \otimes L$  and  $C_N \subset N \otimes L$  be tensor cones and  $a \in M^g$ ,  $b \in N^g$  for some  $g \in \mathbb{N}$ . The set of inclusion constants for  $\mathcal{D}_a(L, C_M)$  and  $C_N$  is defined as

$$\begin{aligned} \Delta_a(L, C_M, C_N) &:= \{s \in [0, 1]^g : \forall b \in N^g, \mathcal{D}_{(\mathbb{1}_M, a)}(\mathbb{R}, M^+) \subseteq \mathcal{D}_{(\mathbb{1}_N, b)}(\mathbb{R}, N^+) \\ &\implies (1, s) \cdot \mathcal{D}_{(\mathbb{1}_M, a)}(L, C_M) \subseteq \mathcal{D}_{(\mathbb{1}_N, b)}(L, C_N)\}. \end{aligned}$$

Here,  $(1, s) \cdot \mathcal{D}_{(\mathbb{1}_M, a)}(L, C_M) := \{(v_0, s_1 v_1, \dots, s_g v_g) : v \in \mathcal{D}_{(\mathbb{1}_M, a)}(L, C_M)\}$ .

**Question C.12.** Are there natural conditions which would entail  $\{s : \sum_i s_i \leq 1\} \subseteq \Delta_a(L, C_M, C_N)$ ? This can be done for free spectrahedra, see [HKMS19, Theorem 1.4] and [DDOSS17, Section 8].

The following definition is motivated by the matrix range introduced in [Arv72] and generalized in [DDOSS17].

**Definition C.13.** Let  $(L, L^+)$  be a preordered vector space with order unit  $\mathbb{1}_L$  and consider a tuple  $a \in L^g$ . Then, the functional range of  $a$  is the set

$$\mathcal{W}(a) = \{(\varphi(a_1), \dots, \varphi(a_g)) : \varphi \in (L^+)^*, \varphi(\mathbb{1}_L) = 1\} \subseteq \mathbb{R}^g.$$

**Proposition C.14.** Let  $(L, L^+)$  be a preordered vector space with order unit  $\mathbb{1}_L$  and let  $a \in L^g$ . If  $V^+$  is proper, then  $\mathcal{W}(a)$  is compact and convex. The set  $\mathcal{W}(a)$  is non-empty if and only if  $L^+ \neq L$ .

*Proof.* For any  $a_i, i \in [g]$ , there is a  $t_i \geq 0$  such that  $a_i \in t_i[-\mathbb{1}_L, \mathbb{1}_L]$ . Thus,  $|\varphi(a_i)| \leq t_i$  for all  $\varphi \in (L^+)^*$ ,  $\varphi(\mathbb{1}_L) = 1$  and boundedness of  $\mathcal{W}(a)$  follows. Let  $x^{(n)}$  be a sequence in  $\mathcal{W}(a)$  converging to  $x$ . With any  $x^{(n)}$ , we can associate a  $\varphi_n \in (L^+)^*$ ,  $\varphi_n(\mathbb{1}_L) = 1$ . Since  $(L^+)^*$  is closed by definition, there is a map  $\varphi \in (L^+)^*$ ,  $\varphi(\mathbb{1}_L) = 1$  such that  $x = (\varphi(a_1), \dots, \varphi(a_n))$  and  $x \in \mathcal{W}(a)$ . This follows from the Bolzano-Weierstrass theorem and the fact that the unital positive functionals form a compact set (consider the order unit norm on  $L$ ). Convexity follows from the fact that the set of  $\varphi$  as in the statement is convex.

The set  $\mathcal{W}(a)$  is empty if and only if there are no unital functionals in  $L^*$ . Let  $L = L^+$ . Then, it is easy to see that  $(L^+)^* = \{0\}$ . Conversely, if  $L^+ \neq L$ , then  $-\mathbb{1}_L \notin L^+$  by Lemma A.1. Thus, the functional  $\varphi$  on  $\mathbb{R}\mathbb{1}_L$  such that  $\varphi(\mathbb{1}_L) = 1$  can be extended to an element in  $(L^+)^*$  by Theorem B.1.  $\square$

**Proposition C.15.** *Let  $(L, L^+)$  be a proper ordered vector space with order unit  $\mathbb{1}_L$ . Furthermore, let  $a \in L^{\mathbb{R}}$ . Let  $\mathcal{C}_a := \{x \in \mathbb{R}^{\mathbb{R}} : (1, -x) \in \mathcal{D}(\mathbb{1}_L, a)(\mathbb{R}, L^+)\}$ . Then,  $\mathcal{W}(a)^\circ = \mathcal{C}_a$ . If  $0 \in \mathcal{W}(a)$ , then  $\mathcal{C}_a^\circ = \mathcal{W}(a)$ .*

*Proof.* Since  $L^+$  is closed,  $L^+ \cong (L^+)^{**}$  and

$$1 + \sum_i x_i a_i \in L^+ \iff 1 + \sum_i x_i \varphi(a_i) \geq 0 \quad \forall \varphi \in (L^+)^*, \varphi(\mathbb{1}_L) = 1.$$

Note that the only map in  $(L^+)^*$  with  $\varphi(\mathbb{1}_L) = 0$  is the constant map, since  $\mathbb{1}_L$  is an order unit. Thus, it is enough to check unital maps for the  $\Leftarrow$  implication. This proves the first assertion. The second assertion follows from the bipolar theorem for convex sets ([AS17, Equation (1.10)]) since  $\mathcal{W}(a)$  is closed.  $\square$

**Proposition C.16.** *Let  $(L, L^+)$  be a proper ordered vector space with order unit  $\mathbb{1}_L$ . Furthermore, let  $a \in L^{\mathbb{R}}$ . Then,  $\mathcal{C}_a$  is bounded if and only if  $0 \in \text{int } \mathcal{W}(a)$ .*

*Proof.* This follows from the fact that for convex sets  $K \subseteq \mathbb{R}^n$ ,  $K^\circ$  is bounded if and only if  $0 \in \text{int } K$  [AS17, Exercise 1.14], combined with Proposition C.15.  $\square$

**Proposition C.17.** *Let  $(M, M^+)$  and  $(N, N^+)$  be two proper ordered vector spaces containing order units  $\mathbb{1}_M$  and  $\mathbb{1}_N$ . Let  $a \in M^k$  and  $b \in N^l$  be such that  $\mathcal{C}_a, \mathcal{C}_b$  are polytopes for  $k, l \in \mathbb{N}$ . Then, for any closed tensor cone  $C_{MN}$ ,*

$$\mathcal{D}(\mathbb{1}_M \otimes \mathbb{1}_N, a \otimes \mathbb{1}_N, \mathbb{1}_M \otimes b)(\mathbb{R}, C_{MN}) = \mathbb{R}_+ \{(1, -z) : z \in \mathcal{C}_a \oplus \mathcal{C}_b\}.$$

*Proof.* Let  $-x \in \mathcal{C}_a$ ,  $-y \in \mathcal{C}_b$ . Then,  $(1, x, 0) \in \mathcal{D}(\mathbb{1}_M \otimes \mathbb{1}_N, a \otimes \mathbb{1}_N, \mathbb{1}_M \otimes b)(\mathbb{R}, C_{MN})$ , because  $(\mathbb{1}_M + \sum_i x_i a_i) \otimes \mathbb{1}_N \in M^+ \otimes_{\min} N^+$ , and likewise  $(1, 0, y) \in \mathcal{D}(\mathbb{1}_M \otimes \mathbb{1}_N, a \otimes \mathbb{1}_N, \mathbb{1}_M \otimes b)(\mathbb{R}, C_{MN})$ , such that “ $\supset$ ” holds in the assertion. Conversely, let  $(c, x, y) \in \mathcal{D}(\mathbb{1}_M \otimes \mathbb{1}_N, a \otimes \mathbb{1}_N, \mathbb{1}_M \otimes b)(\mathbb{R}, C_{MN})$ , where  $x \in \mathbb{R}^k$  and  $y \in \mathbb{R}^l$ . Then,

$$c\mathbb{1}_M \otimes \mathbb{1}_N + \left(\sum_{i=1}^k x_i a_i\right) \otimes \mathbb{1}_N + \mathbb{1}_M \otimes \left(\sum_{j=1}^l y_j b_j\right) \in C_{MN}. \quad (57)$$

Boundedness of  $\mathcal{C}_a$  and  $\mathcal{C}_b$  implies by Proposition C.16 that  $0 \in \text{int } \mathcal{W}(a)$ ,  $0 \in \text{int } \mathcal{W}(b)$ . Let  $\varphi, \psi$  be positive unital functionals which send  $a$  and  $b$  to zero, respectively. Then,  $\varphi \otimes \psi \in (C_{MN})^*$ . An application of this map to Eq. (57) implies  $c \geq 0$ .

Let  $c = 0$ . Then, by applying  $\varphi \otimes \beta$ ,  $\alpha \otimes \psi$  for  $\alpha \in (M^+)^*$  and  $\beta \in (N^+)^*$ , it holds that  $\sum_i x_i a_i \in M^+$ ,  $\sum_j y_j b_j \in N^+$ . Since  $\mathcal{C}_a$  is bounded,  $\{a_i\}_i$  is a set of linearly independent elements. The same is true for  $\{b_j\}_j$ . Without loss of generality, let  $x \neq 0$ . Then  $\sum_i x_i a_i \neq 0$  and  $(1, -\lambda x) \in \mathcal{C}_a$  for all  $\lambda \geq 0$ , which is a contradiction to  $\mathcal{C}_a$  being bounded. Thus  $c = 0$  implies  $x = 0 = y$ .

Thus, we can set  $c = 1$  without loss of generality. Let now for  $\varphi' \in (M^+)^*$ ,  $\psi' \in (N^+)^*$ ,  $\varphi'(\mathbb{1}_M) = 1 = \psi'(\mathbb{1}_N)$ . An application of  $\varphi' \otimes \psi'$  to Eq. (57) implies  $\mathcal{C}^\circ \supset \mathcal{W}(a) \times \mathcal{W}(b)$ , where  $\mathcal{C} := \{z : (1, -z) \in \mathcal{D}(\mathbb{1}_M \otimes \mathbb{1}_N, a \otimes \mathbb{1}_N, \mathbb{1}_M \otimes b)(\mathbb{R}, C_{MN})\}$ . Since  $\mathcal{C}$  is closed and contains 0, we obtain  $\mathcal{C} \subset (\mathcal{W}(a) \times \mathcal{W}(b))^\circ = (\mathcal{C}_a^\circ \times \mathcal{C}_b^\circ)^\circ = \mathcal{C}_a \oplus \mathcal{C}_b$  with Proposition C.15.  $\square$

**Corollary C.18.** *Let  $(M, M^+)$  and  $(N, N^+)$  be two proper ordered vector spaces with closed cones containing order units  $\mathbb{1}_M$  and  $\mathbb{1}_N$ . Let  $a \in M^k$  and  $b \in N^l$  be such that  $\mathcal{C}_a, \mathcal{C}_b$  are polytopes for  $k, l \in \mathbb{N}$ . Let moreover  $(L, L^+)$  be another preordered vector space with order unit  $\mathbb{1}_L$  and  $h_1 \in L^k$ ,  $h_2 \in L^l$ . Then, for any closed tensor cone  $C_{MN}$ ,*

$$\mathcal{D}(\mathbb{1}_M \otimes \mathbb{1}_N, a \otimes \mathbb{1}_N, \mathbb{1}_M \otimes b)(\mathbb{R}, C_{MN}) \subseteq \mathcal{D}(\mathbb{1}_L, h_1, h_2)(\mathbb{R}, L^+)$$

if and only if

$$\mathcal{D}_{(\mathbb{1}_M, a)}(\mathbb{R}, M^+) \subseteq \mathcal{D}_{(\mathbb{1}_L, h_1)}(\mathbb{R}, L^+) \quad \wedge \quad \mathcal{D}_{(\mathbb{1}_N, b)}(\mathbb{R}, N^+) \subseteq \mathcal{D}_{(\mathbb{1}_L, h_2)}(\mathbb{R}, L^+).$$

## Appendix D: Basic Results on Compatibility Regions and Degree

In this section, we will prove some basic results on the relation between different compatibility regions, which complements Sect. 4. We will start by showing that the compatibility degree decreases with the number of measurements.

**Proposition D.1.** *Let  $(V, V^+, \mathbb{1})$  be a GPT and let  $g, g' \in \mathbb{N}$  be such that  $g \leq g'$ . Let  $\mathbf{k}' \in \mathbb{N}^{g'}$  and  $\mathbf{k} = (k'_1, \dots, k'_g)$ . Then,*

$$\gamma(\mathbf{k}; V, V^+) \geq \gamma(\mathbf{k}'; V, V^+).$$

*Proof.* Let  $f^{(i)} \in A^{k'_i}, i \in [g']$ , be a collection of measurements. Let  $s = \gamma(\mathbf{k}'; V, V^+)$  and let  $\tilde{f}^{(i)}$  be the corresponding noisy measurements defined by

$$\tilde{f}_j^{(i)} = s f_j^{(i)} + (1-s) \frac{\mathbb{1}}{k'_i} \quad \forall j \in [k'_i], i \in [g'].$$

Then, the  $\tilde{f}^{(i)}, i \in [g']$ , are compatible with joint measurement  $h'$ . Setting

$$h_{i_1, \dots, i_g} = \sum_{j \in [k'_j], j \in [g'] \setminus [g]} h'_{i_1, \dots, i_{g'}}$$

yields a joint measurement for  $\tilde{f}^{(i)}, i \in [g]$ . Since the measurements were arbitrary, it follows that  $s \leq \gamma(\mathbf{k}; V, V^+)$ .  $\square$

Intuitively, the more outcomes we have, the more difficult it becomes for all measurements to be compatible. This is the content of the next proposition, which is the GPT version of [BN20, Proposition 3.35]. The proof is very similar.

**Proposition D.2.** *Let  $g \in \mathbb{N}, \mathbf{k}', \mathbf{k} \in \mathbb{N}^g$  and let  $(V, V^+, \mathbb{1})$  be a GPT. Let  $\mathbf{k}' \geq \mathbf{k}$ , where the inequality is meant to hold entrywise. Then*

$$\Gamma(\mathbf{k}'; V, V^+) \subseteq \Gamma(\mathbf{k}; V, V^+).$$

In particular,  $\gamma(\mathbf{k}'; V, V^+) \leq \gamma(\mathbf{k}; V, V^+)$ .

*Proof.* Let  $(f_1^{(i)}, \dots, f_{k'_i}^{(i)}) \in A^{k'_i}$  be measurements for all  $i \in [g]$ . Let  $s \in \Gamma(\mathbf{k}'; V, V^+)$ . Then, the measurements

$$F^{(i)} := \left( s_i f_1^{(i)} + (1-s_i) \frac{\mathbb{1}}{k'_i}, \dots, s_i f_{k'_i}^{(i)} + (1-s_i) \frac{\mathbb{1}}{k'_i}, \underbrace{(1-s_i) \frac{\mathbb{1}}{k'_i}, \dots, (1-s_i) \frac{\mathbb{1}}{k'_i}}_{k'_i - k_i} \right) \in A^{k'_i}$$

are compatible. We will show that this is still true if we replace, for some fixed  $l \in [g]$ , the  $l$ -th measurement  $F^{(l)}$  by

$$G^{(l)} := \left( s_l f_1^{(l)} + (1-s_l) \frac{\mathbb{1}}{k_l}, \dots, s_l f_{k_l}^{(l)} + (1-s_l) \frac{\mathbb{1}}{k_l} \right) \in A^{k_l}.$$

An iterative application of this procedure then shows that  $s \in \Gamma(\mathbf{k}; V, V^+)$  and the assertion follows.

Let  $H_{i_1, \dots, i_g}$  be the effects of the joint measurement for the  $F^{(i)}$ , where  $i_j \in [k'_j]$ ,  $j \in [g]$ . Let us define

$$R_{i_1, \dots, i_{l-1}, k_l+1, i_{l+1}, \dots, i_g} := \sum_{i_l=k_l+1}^{k'_l} H_{i_1, \dots, i_g}.$$

Then, let us define

$$h_{i_1, \dots, i_g} = H_{i_1, \dots, i_g} + \frac{1}{k_l} R_{i_1, \dots, i_{l-1}, k_l+1, i_{l+1}, \dots, i_g} \quad i_l \in [k_l], \quad i_j \in [k'_j] \quad \forall j \in [g] \setminus \{l\}.$$

It is easy to verify that  $h$  is a measurement. Moreover, for  $m \in [g]$ ,  $m \neq l$ ,  $q \in [k'_m]$ ,

$$\sum_{\substack{i_j \in [k'_j], j \in [g] \setminus \{m, l\}, \\ i_l \in [k_l], i_m = q}} h_{i_1, \dots, i_g} = F_q^{(m)}$$

and for  $p \in [k_l]$ ,

$$\begin{aligned} \sum_{i_j \in [k'_j], j \in [g] \setminus \{l\}, i_l = p} h_{i_1, \dots, i_g} &= \sum_{i_j \in [k'_j], j \in [g] \setminus \{l\}, i_l = p} H_{i_1, \dots, i_g} \\ &\quad + \frac{1}{k_l} \sum_{i_j \in [k'_j], j \in [g] \setminus \{l\}} R_{i_1, \dots, i_{l-1}, k_l+1, i_{l+1}, \dots, i_g} \\ &= F_p^{(l)} + \frac{1}{k_l} \sum_{j=k_l+1}^{k'_l} F_j^{(l)} \\ &= s_l f_p^{(l)} + \frac{(1-s_l)}{k'_l} \mathbb{1} + \frac{(1-s_l)(k'_l - k_l)}{k_l k'_l} \mathbb{1} \\ &= s_l f_p^{(l)} + \frac{(1-s_l)}{k_l} \mathbb{1} = G_p^{(l)}. \end{aligned}$$

Thus,  $h$  is the desired joint measurement for the  $F^{(i)}$ ,  $i \in [g] \setminus \{l\}$  and  $G^{(l)}$ . □

We have seen in the last proposition how the compatibility regions are related for measurements with a different number of outcomes within the same theory. We will now show that sometimes the compatibility regions of different GPTs can be related.

**Proposition D.3.** *Let  $g \in \mathbb{N}$ ,  $\mathbf{k} \in \mathbb{N}^g$ . Let  $(V_1, V_1^+, \mathbb{1}_{A_1})$  and  $(V_2, V_2^+, \mathbb{1}_{A_2})$  be two GPTs. Let  $\Psi : (V_1, V_1^+) \rightarrow (V_2, V_2^+)$  be a positive map such that  $\Psi^*(\mathbb{1}_{A_2}) = \mathbb{1}_{A_1}$ . If  $\Psi$  is a retraction, i.e. if there is a positive map  $\Theta : (V_2, V_2^+) \rightarrow (V_1, V_1^+)$  such that  $\Psi \circ \Theta = \text{id}_{V_2}$ , then the GPT  $(V_1, V_1^+, \mathbb{1}_{A_1})$  is “less compatible” than  $(V_2, V_2^+, \mathbb{1}_{A_2})$  in the measurement setting  $\mathbf{k}$ :  $\Gamma(\mathbf{k}; V_1, V_1^+) \subseteq \Gamma(\mathbf{k}; V_2, V_2^+)$ .*



*Proof.* Let  $s \in \Gamma(\mathbf{k}; V_1, V_1^+)$ . Let  $f^{(i)} \in A_2^{k_i}, i \in [g]$ , be a collection of measurements. Then, the  $\Psi^*(f_j^{(i)}) \in A_1^+, j \in [k_i]$ , form a collection of measurements as well. Moreover, the measurements given by

$$h_j^{(i)} = s_i \Psi^*(f_j^{(i)}) + (1 - s_i) \frac{\mathbb{1}_{A_1}}{k_i} \quad \forall i \in [g], j \in [k_i]$$

are compatible by the assumption on  $s$ . Since,  $\Psi$  is a retraction, it holds that  $\Theta^* \circ \Psi^* = \text{id}_{A_2}$ . Since  $\Psi^*(\mathbb{1}_{A_2}) = \mathbb{1}_{A_1}$ , also  $\Theta^*(\mathbb{1}_{A_1}) = \mathbb{1}_{A_2}$ . Thus, the image of the joint measurement for the  $h^{(i)}$  under  $\Theta^*$  is again measurement in  $(V_2, V_2^+, \mathbb{1}_{A_2})$ . Furthermore, it is a joint measurement for the noisy measurements

$$s_i f_j^{(i)} + (1 - s_i) \frac{\mathbb{1}_{A_2}}{k_i} \quad \forall i \in [g], j \in [k_i],$$

since  $\Theta^*(\Psi^*(f_j^{(i)})) = f_j^{(i)}$  for all  $i \in [g], j \in [k_i]$ .  $\square$

*Remark D.4.* An example for this situation is  $(V_1, V_1^+, \mathbb{1}_{A_1}) = \text{QM}_d$  for some  $d \in \mathbb{N}$  and  $(V_2, V_2^+, \mathbb{1}_{A_2}) = \text{CM}_d$ , where  $\Psi^*$  embeds the probability distributions in  $\text{CM}_d$  as diagonal matrices. Then,  $\Theta^*$  projects onto the diagonal entries of the matrix.

## Appendix E: Conic Programming for Compatibility in GPTs

*E.1. Background.* To begin, let us briefly recapitulate the theory of conic programming. We follow [GM12, Section 4]. All vector spaces we consider will be finite dimensional. We will equip them with an inner product choosing a pair of dual bases for the vector space and its dual.

**Definition E.1** (*Conic program* [GM12, Definition 4.6.1]). Let  $L^+ \subseteq L, M^+ \subseteq M$  be closed convex cones, let  $b \in M, c \in L^*$  and let  $A : L \rightarrow M$  be a linear operator. A *conic program* is an optimization problem of the form

$$\begin{aligned} & \text{maximize} && \langle c, x \rangle \\ & \text{subject to} && b - A(x) \in M^+ \\ & && x \in L^+ \end{aligned}$$

The dual problem is then given by [GM12, Section 4.7]

$$\begin{aligned} & \text{minimize} && \langle b, y \rangle \\ & \text{subject to} && A^*(y) - c \in (L^+)^* \\ & && y \in (M^+)^* \end{aligned}$$

Weak duality always hold, i.e. the value of the primal problem is upper bounded by the value of the dual program if the dual program is feasible. A sufficient condition for strong duality to hold is the following version of Slater's condition:

**Theorem E.2** ([GM12, Theorem 4.7.1]). *If the conic program in Definition E.1 is feasible, has finite value  $\gamma$  and has an interior point  $\tilde{x}$ , then the dual program is also feasible and has the same value  $\gamma$ .*

If  $M^+ \neq \{0\}$ ,  $\tilde{x}$  is an interior point if  $\tilde{x} \in \text{int}(L^+)$  and  $b - A(\tilde{x}) \in \text{int}(M^+)$  [GM12, Definition 4.6.4].

**E.2. Map extension.** In this section, we will show that the existence of a positive extension of a positive map can be checked using a conic program. We can give the following generalization of the results in [HJRW12].

**Theorem E.3.** *Let  $E \subseteq \mathbb{R}^d$  be a  $g$ -dimensional subspace such that  $E \cap \text{ri}(\mathbb{R}_+^d) \neq \emptyset$  and  $E^+ = E \cap \mathbb{R}_+^d$ . Moreover, let  $(V, V^+, \mathbb{1})$  be a GPT. Finally, let  $\Phi : E \rightarrow A$  be given by a basis  $\{e_i\}_{i \in [g]} \subset E$  of  $E$ ,  $g \in \mathbb{N}$ , and  $\{f_i\}_{i \in [g]} \subset A$  such that  $\Phi(e_i) = f_i$  for all  $i \in [g]$ . Then, there exists a positive extension  $\tilde{\Phi} : (\mathbb{R}^d, \mathbb{R}_+^d) \rightarrow (A, A^+)$  of  $\Phi$  if and only if the conic program*

$$\begin{aligned} & \text{maximize} && -\langle s, (h_1^+ - h_1^-, \dots, h_g^+ - h_g^-) \rangle \\ & \text{subject to} && \sum_{i \in [g]} e_i \otimes (h_i^+ - h_i^-) \in E^+ \otimes_{\max} V^+ \\ & && 1 - \langle \mathbb{1}, h_i^\pm \rangle \geq 0 \\ & && h_i^\pm \in V^+ \quad \forall i \in [g] \end{aligned}$$

has value 0. Here,  $s : V^g \rightarrow \mathbb{R}$  is given as

$$\langle s, h_1, \dots, h_g \rangle = \langle \chi_V, \sum_{i \in [g]} f_i \otimes (h_i^+ - h_i^-) \rangle = \sum_{i \in [g]} f_i (h_i^+ - h_i^-).$$

*Proof.* Let  $z \in E \otimes V$ . Using that the  $e_i$  form a basis, we can write

$$z = \sum_{i \in [g]} e_i \otimes z_i,$$

where  $z_i \in V$  for all  $i \in [g]$ . Since  $V^+$  is proper, we can decompose each  $z_i = z_i^+ - z_i^-$ , where  $z_i^\pm \in V^+$ . Then,  $\langle s, (z_1, \dots, z_g) \rangle = s_\Phi(z)$ , where  $s_\Phi$  is defined as in Proposition B.8. By linearity, it suffices to restrict to  $z$  such that  $\langle \mathbb{1}, z_i^\pm \rangle \leq 1$  for all  $i \in [g]$  in order to check positivity of  $s_\Phi$ . Thus, the conic program has value 0 if and only if  $s_\Phi$  is positive. The assertion then follows from Proposition B.8.  $\square$

**Remark E.4.** Of course, we could write down a trivial conic program for map extension just checking whether the corresponding tensor  $\varphi^\Phi$  is in  $(E^+)^* \otimes_{\min} A^+$  using Proposition B.8. However, checking membership in this cone might be hard in practice. If  $E^+$  is a polyhedral cone,

$$\sum_{i \in [g]} e_i \otimes (h_i^+ - h_i^-) \in E^+ \otimes_{\max} V^+$$

can be checked as

$$\sum_{i \in [g]} \langle \alpha_j, e_i \rangle (h_i^+ - h_i^-) \in V^+ \quad \forall j,$$

where the  $\alpha_j$  are the extremal rays of  $(E^+)^*$  (compare to Lemma 6.2). This is arguably easier since it does not involve checking the membership in a tensor cone directly. Moreover, it recovers the result that in quantum mechanics, where  $V^+ = \text{PSD}_d$ : compatibility via map extension can be checked using a semidefinite program [HJRW12].

**Theorem E.5.** *The conic program in Theorem E.3 is feasible and satisfies strong duality.*

*Proof.* In the following, we identify  $z = \sum_{i \in [2g]} z_i \otimes \mu_i \in L \otimes \mathbb{R}^{2g}$  with the vector  $(z_1, \dots, z_{2g})$ , where  $\{\mu_i\}_{i \in [2g]}$  is an orthonormal basis of  $\mathbb{R}^{2g}$ . Comparing the conic program to Definition E.1, we identify

$$\begin{aligned} M &= (E \otimes V) \times \mathbb{R}^{2g} \\ M^+ &= (E^+ \otimes_{\max} V^+) \times \mathbb{R}_+^{2g} \\ L &= V \otimes \mathbb{R}^{2g} \\ L^+ &= (V^+) \otimes \mathbb{R}_+^{2g} \\ c &= -s \\ b &= (0, \underbrace{1, \dots, 1}_{2g}) \in (E \otimes V) \times \mathbb{R}^{2g} \\ A(h_1^\pm, \dots, h_g^\pm) &= \left( - \sum_{i \in [g]} e_i \otimes (h_i^+ - h_i^-), \langle \mathbb{1}, h_1^\pm \rangle, \dots, \langle \mathbb{1}, h_g^\pm \rangle \right). \end{aligned}$$

It can be verified that dual conic program is thus given by

$$\begin{aligned} &\text{minimize} && \sum_{i \in [2g]} y_i \\ &\text{subject to} && s + \sum_{i \in [2g]} y_i \mathbb{1} \otimes \mu_i^* + B(z) \in A^+ \otimes \mathbb{R}_+^{2g} \\ &&& z \in (E^+)^* \otimes_{\min} A^+ \\ &&& y_i \in \mathbb{R}_+ \quad \forall i \in [2g]. \end{aligned}$$

Here,  $B(z) \in A \otimes \mathbb{R}^{2g}$  is given as  $B(z)(h_1, \dots, h_g) = \langle z, \sum_{i \in [g]} e_i \otimes h_i \rangle$  and  $\mu_i^*$  is the dual basis of  $\varepsilon_i$ . Letting  $y_1 = \dots = y_{2g}$  and realizing that  $\mathbb{1} \otimes (1, \dots, 1)$  is an order unit in  $A^+ \otimes \mathbb{R}_+^{2g}$ , for any  $z \in (E^+)^* \otimes_{\min} A^+$  we can find a  $y_1 > 0$  such that

$$s + y_1 \mathbb{1} \otimes (1, \dots, 1) + B(z) \in \text{int} \left( A^+ \otimes \mathbb{R}_+^{2g} \right).$$

This is true, since the order unit is an interior point of  $A^+ \otimes \mathbb{R}_+^{2g}$ , hence there is a  $y_1$  such that

$$\frac{1}{y_1} (s + B(z)) + \mathbb{1} \otimes (1, \dots, 1) \in \text{int} \left( A^+ \otimes \mathbb{R}_+^g \right).$$

Since the interior points of  $A^+ \otimes \mathbb{R}_+^g$  are those points  $w$  such that  $\langle w, x \rangle > 0$  for all  $x \in V^+ \otimes \mathbb{R}_+^{2g} \setminus \{0\}$ , multiplication by  $y_1$  preserves the fact that the point is in  $\text{int} \left( A^+ \otimes \mathbb{R}_+^{2g} \right)$ . Therefore, the dual problem has an interior point. This also implies that the value of the dual program is finite, since it is lower bounded by 0. The remarks at the beginning of [GM12, Section 4.7] imply that Theorem E.2 still applies if we interchange the primal and the dual problem. Thus, the assertion follows.  $\square$

*E.3. Computing  $\|\cdot\|_c$ .* Finally, we note that the norm  $\|\cdot\|_c$ , introduced in Theorem 7.2 and Proposition 7.4, can also be computed by a conic program, namely: for any  $\bar{\varphi} \in E_g^* \otimes A$ ,  $-\|\bar{\varphi}\|_c$  is the value of the following conic program

$$\begin{aligned} & \text{maximize} && -\lambda \\ & \text{subject to} && \lambda \check{1}_g \otimes \mathbb{1} - \bar{\varphi} \in (E_g^+)^* \otimes_{\min} A^+ \\ & && \lambda \in \mathbb{R}_+ \end{aligned}$$

Since  $\|\bar{\varphi}\|_c$  is finite and since  $\lambda \check{1}_g \otimes \mathbb{1} - \bar{\varphi} \in \text{int}((E_g^+)^* \otimes_{\min} A^+)$  for  $\lambda$  large enough since  $\check{1}_g \otimes \mathbb{1} \in (E_g^+)^* \otimes_{\min} A^+$  is an order unit, strong duality holds by Theorem E.2. The dual conic program is

$$\begin{aligned} & \text{minimize} && -\langle \bar{\varphi}, y \rangle \\ & \text{subject to} && 1 - \langle \check{1}_g \otimes \mathbb{1}, y \rangle \geq 0 \\ & && y \in E_g^+ \otimes_{\max} V^+ \end{aligned}$$

The membership in  $E_g^+ \otimes_{\max} V^+$  can be decided only evaluating the equations in Eq. (13) by Lemma 6.2. So we can give the alternative formulation of the dual conic program which only checks membership in  $V^+$ :

$$\begin{aligned} & \text{minimize} && -\langle \bar{\varphi}, 1_g \otimes y_0 + \sum_{i=1}^g c_i \otimes (y_i^+ - y_i^-) \rangle \\ & \text{subject to} && 1 - \langle \mathbb{1}, y_0 \rangle \geq 0 \\ & && y_0 + \sum_{i=1}^g \varepsilon_i (y_i^+ - y_i^-) \in V^+ \quad \forall \varepsilon \in \{\pm 1\}^g \\ & && y_0, y_i^\pm \in V^+ \quad \forall i \in [g]. \end{aligned}$$

Due to Theorem 7.2, this recovers the result that in quantum mechanics, where  $V^+ = \text{PSD}_d$ , compatibility of effects can be checked using a semidefinite program (see e.g. [WPGF09]). It also recovers the result that compatibility in GPTs can be checked with a conic program [Plá16], although the one we give here is different.

## References

- [ALP19] Aubrun, G., Lami, L., Palazuelos, C.: Universal entangleability of non-classical theories. arXiv preprint [arXiv:1910.04745](https://arxiv.org/abs/1910.04745), (2019)
- [ALP+20] Aubrun, G., Lami, L., Palazuelos, C., Szarek, S.J., Winter, A.: Universal gaps for XOR games from estimates on tensor norm ratios. *Commun. Math. Phys.* **375**, 679–724 (2020)
- [ALPP19] Aubrun, G., Lami, L., Palazuelos, C., Pláivala, M.: Entangleability of cones. *Geom. Funct. Anal.* **31**, 181–205 (2021)
- [Arv72] Arveson, W.: Subalgebras of  $C^*$ -algebras II. *Acta Math.* **128**, 271–308 (1972)
- [AS17] Aubrun, G., Szarek, S.J.: Alice and Bob meet Banach. Number 223 in *Mathematical Surveys and Monographs*. American Mathematical Society (2017)
- [AT07] Aliprantis, C.D., Tourky, R.: *Cones and Duality*, volume 84 of *Graduate Studies in Mathematics*. American Mathematical Society (2007)
- [BA07] Brougham, T., Andersson, E.: Estimating the expectation values of spin-1/2 observables with finite resources. *Phys. Rev. A* **76**, 052313 (2007)
- [Bar81] Barker, G.P.: Theory of cones. *Linear Algebra Appl.* **39**, 263–291 (1981)

- [BBLW06] Barnum, H., Barrett, J., Leifer, M., Wilce, A.: Cloning and broadcasting in generic probabilistic theories. arXiv preprint [arXiv:quant-ph/0611295](https://arxiv.org/abs/quant-ph/0611295) (2006)
- [BBLW07] Barnum, H., Barrett, J., Leifer, M., Wilce, A.: Generalized no-broadcasting theorem. *Phys. Rev. Lett.* **99**, 240501 (2007)
- [BCP+14] Brunner, N., Cavalcanti, D., Pironio, S., Scarani, V., Wehner, S.: Bell nonlocality. *Rev. Mod. Phys.* **86**, 419–478 (2014)
- [BH08] Busch, P., Heinosaari, T.: Approximate joint measurements of qubit observables. *Quant. Inf. Comput.* **8**(8), 797–818 (2008)
- [BHSS13] Busch, P., Heinosaari, T., Schultz, J., Stevens, N.: Comparing the degrees of incompatibility inherent in probabilistic physical theories. *EPL (Europhysics Letters)* **103**(1), 10002 (2013)
- [BLPY16] Busch, P., Lahti, P., Pellonpää, J.-P., Ylinen, K.: *Quantum Measurement*. Springer, New York (2016)
- [BN18] Bluhm, A., Nechita, I.: Joint measurability of quantum effects and the matrix diamond. *J. Math. Phys.* **59**(11), 112202 (2018)
- [BN20] Bluhm, A., Nechita, I.: Compatibility of quantum measurements and inclusion constants for the matrix jewel. *SIAM J. Appl. Algebra Geom.* **4**(2), 255–296 (2020)
- [Boh28] Bohr, N.: The quantum postulate and the recent development of atomic theory. *Nature* **121**(3050), 580–590 (1928)
- [Bre97] Bremner, D.D.: On the complexity of vertex and facet enumeration for complex polytopes. Ph.D. thesis, School of Computer Science, McGill University, Montréal, Canada (1997)
- [Bus86] Busch, P.: Unsharp reality and joint measurements for spin observables. *Phys. Rev. D* **33**(8), 2253 (1986)
- [Cas05] Castillo, R.E.: A note on Krein’s theorem. *Lect. Mat.* **26**, 5–9 (2005)
- [CHT19] Carmeli, C., Heinosaari, T., Toigo, A.: Quantum incompatibility witnesses. *Phys. Rev. Lett.* **122**(13), 130402 (2019)
- [DDOSS17] Kenneth, R., Davidson, A.D.-O., Shalit, O.M., Solel, B.: Dilations, inclusions of matrix convex sets, and completely positive maps. *Int. Math. Res. Not.* **13**, 4069–4130 (2017)
- [DFK19] Designolle, S., Farkas, M., Kaniewski, J.: Incompatibility robustness of quantum measurements: a unified framework. *New J. Phys.* **21**(11), 113053 (2019)
- [Fin82] Fine, A.: Hidden variables, joint probability, and the Bell inequalities. *Phys. Rev. Lett.* **48**(5), 291–295 (1982)
- [FNT17] Fritz, T., Netzer, T., Thom, A.: Spectrahedral containment and operator systems with finite-dimensional realization. *SIAM J. Appl. Algebra Geom.* **1**(1), 556–574 (2017)
- [GG71] Garling, D.J.H., Gordon, Y.: Relations between some constants associated with finite dimensional Banach spaces. *Israel J. Math.* **9**, 346–361 (1971)
- [GM12] Gärtner, B., Matoušek, J.: *Approximation Algorithms and Semidefinite Programming*. Springer, New York (2012)
- [Gor69] Gordon, Y.: On  $p$ -absolutely summing constants of Banach spaces. *Israel J. Math.* **7**(2), 151–163 (1969)
- [Gud13] Gudder, S.: Compatibility for probabilistic theories. *Mathematica Slovaca* **66**(2), 449–458 (2016)
- [Hei27] Heisenberg, W.: Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik. *Z. Phys.* **43**(3), 172–198 (1927)
- [HJRW12] Heinosaari, T., Jivulescu, M.A., Reeb, D., Wolf, M.M.: Extending quantum operations. *J. Math. Phys.* **53**, 102208 (2012)
- [HKM13] Helton, J.W., Klep, I., McCullough, S.: The matricial relaxation of a linear matrix inequality. *Math. Program.* **138**(1–2), 401–445 (2013)
- [HKMS19] Helton, J.W., Klep, I., McCullough, S., Schweighofer, M.: Dilations, linear matrix inequalities, the matrix cube problem and beta distributions. *Memoirs Am. Math. Soc.* 257(1232), (2019)
- [HKR15] Heinosaari, T., Kiukas, J., Reitzner, D.: Noise robustness of the incompatibility of quantum measurements. *Phys. Rev. A* **92**, 022115 (2015)
- [HMZ16] Heinosaari, T., Miyadera, T., Ziman, M.: An invitation to quantum incompatibility. *J. Phys. A: Math. Theor.* **49**(12), 123001 (2016)
- [HSR03] Horodecki, M., Shor, P.W., Ruskai, M.B.: Entanglement breaking channels. *Rev. Math. Phys.* **15**(06), 629–641 (2003)
- [HSTZ14] Heinosaari, T., Schultz, J., Toigo, A., Ziman, M.: Maximally incompatible quantum observables. *Phys. Rev. A* **378**(24–25), 1695–1699 (2014)
- [Jen18] Jenčová, A.: Incompatible measurements in a class of general probabilistic theories. *Phys. Rev. A* **98**(1), 012133 (2018)
- [JL13] Janotta, P., Lal, R.: Generalized probabilistic theories without the no-restriction hypothesis. *Phys. Rev. A* **87**, 052131 (2013)
- [JP17] Jenčová, A., Plávala, M.: Conditions on the existence of maximally incompatible two-outcome measurements in general probabilistic theory. *Phys. Rev. A* **96**, 022113 (2017)

- [Kob17] Kobayshi, M.: Coexistence on reflecting hyperplane in generalized probability theories. *J. Math. Phys.* **58**(8), 082203 (2017)
- [Kur20] Kuramochi, Y.: Compact convex structure of measurements and its applications to simulability, incompatibility, and convex resource theory of continuous-outcome measurements. arXiv preprint [arXiv:2002.03504](https://arxiv.org/abs/2002.03504), (2020)
- [Lam18] Lami, L.: Non-classical correlations in quantum mechanics and beyond. PhD thesis. arXiv preprint [arXiv:1803.02902](https://arxiv.org/abs/1803.02902), (2018)
- [LPW18] Lami, L., Palazuelos, C., Winter, A.: Ultimate data hiding in quantum mechanics and beyond. *Commun. Math. Phys.* **361**(2), 661–708 (2018)
- [Mul97] Mulansky, B.: Multivariate Approximation and Splines, volume 125 of International Series of Numerical Mathematics, chapter Tensor products of convex cones, pp. 167–176. Birkhäuser Verlag (1997)
- [NP69] Namioka, I., Phelps, R.R.: Tensor products of compact convex sets. *Pacific J. Math.* **31**(2) (1969)
- [Pau03] Paulsen, V.: Completely Bounded Maps and Operator Algebras, volume 78 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge (2003)
- [PG11] Pal, R., Ghosh, S.: Approximate joint measurement of qubit observables through an Arthur-Kelly model. *J. Phys. A: Math. Theor.* **44**(48), 485303 (2011)
- [Plá16] Plávala, M.: All measurements in a probabilistic theory are compatible if and only if the state space is a simplex. *Phys. Rev. A* **94**, 042108 (2016)
- [PR94] Popescu, S., Rohrlich, D.: Quantum nonlocality as an axiom. *Found. Phys.* **24**, 379–385 (1994)
- [PSS18] Passer, B., Shalit, O.M., Solel, B.: Minimal and maximal matrix convex sets. *J. Funct. Anal.* **274**, 3197–3253 (2018)
- [RG95] Ramana, M., Goldman, A.J.: Some geometric results in semidefinite programming. *J. Global Optim.* **7**(1), 33–50 (1995)
- [Roc70] Rockafellar, R.T.: *Convex Analysis*. Number 28 in Princeton Mathematical Series. Princeton University Press, Princeton (1970)
- [Rya02] Ryan, A.R.: *Introduction to tensor products of Banach spaces*. Springer, New York (2002)
- [SB14] Stevens, N., Busch, P.: Steering, incompatibility, and Bell-inequality violations in a class of probabilistic theories. *Phys. Rev. A* **89**, 022123 (2014)
- [TJ74] Tomczak-Jaegermann, N.: The moduli of smoothness and convexity and the Rademacher averages of the trace classes  $S_p$  ( $1 \leq p \leq \infty$ ). *Studia Mathematica* **50**(2), 163–182 (1974)
- [TJ89] Tomczak-Jaegermann, N.: *Banach-Mazur Distances and Finite-dimensional Operator Ideals*. Longman Scientific & Technical, Harlow (1989)
- [UBGP15] Uola, R., Budroni, C., Gühne, O., Pellonpää, J.-P.: One-to-one mapping between steering and joint measurability problems. *Phys. Rev. Lett.* **115**(23), 230402 (2015)
- [Vin14] Vinzant, C.: What is... a spectrahedron? *Not. Am. Math. Soc.* **61**(5), 492–494 (2014)
- [Wat18] Watrous, J.: *The Theory of Quantum Information*. Cambridge University Press, Cambridge (2018)
- [WPGF09] Wolf, M.M., Pérez-García, D., Fernández, C.: Measurements incompatible in quantum theory cannot be measured jointly in any other no-signaling theory. *Phys. Rev. Lett.* **103**, 230402 (2009)

Communicated by A. Childs