SUFFICIENCY OF CHANNELS OVER VON NEUMANN ALGEBRAS

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LET M be a von Neumann algebra and N a von Neumann subalgebra of M. A mapping E: $M \to N$ is called conditional expectation into N if it possesses the following properties.

(i) $E(I) = I$ and $a \geq 0$ implies that $E(a) \geq 0$.
(ii) $E(ab) = aE(b)$ for all $a \in N$ and $b \in M$.

When ϕ is a faithful normal state on M then the conditional expectation preserving ϕ is unique whenever exists. The Takesaki Theorem ([**27**]) claims that it exists if and only if N is stable under the modular group of ϕ. Accardi and Cecchini ([**1**]) sharpened this result introducing a mapping E_ϕ: $M \to N$ in such a way that $E_\phi(ab) = aE_\phi(b)$ holds for every $b \in M$ and some $a \in N$ if and only if $\sigma_t^\phi(a) \in N$ for all $t \in \mathbb{R}$. In [**21**] we generalized their construction. Let N and M be von Neumann algebras and α: $N \to M$ a positive mapping. For a faithful normal state ϕ on M we defined a ϕ-dual α_ϕ^* of α mapping M into N. In the present paper we carry out the generalization of the Takesaki–Accardi–Cecchini theorem by characterizing the fixed point algebra of $\alpha_\phi^* \circ \alpha$ (Theorem 2).

The question naturally arises: What is the necessary and sufficient condition for the identity $\alpha_\phi^* = \alpha_\omega^*$ if ω and ϕ are different states? The answer forms the main result of the paper. Theorem 3 formulates several conditions.

The context of a mapping α: $N \to M$ and a family θ of states on M has information theoretic and statistical aspects. α may be regarded as a channel with a family θ of input states and for $\phi \in \theta$ the corresponding output state is $\phi \circ \alpha$. On the language of statistics (α, θ) is a statistical experiment ([**13**]). In mathematical physics α may be considered as a coarsegraining ([**24**] and references therein). So the same mathematical object arises in different fields. Here we deal with a statistical interpretation only because it fits very well into this framework. We call a channel (N, M, α) sufficient with respect to the family θ if there exists a positive mapping β: $M \to N$ such that $\phi \circ \beta \circ \alpha = \phi$ for all $\phi \in \theta$. We prove that α is sufficient for θ if and only if $\alpha_\phi^* = \alpha_\omega^*$ for every ϕ, $\omega \in \theta$. In particular, sufficiency is equivalent to pairwise sufficiency.

Preliminaries

Let M be a von Neumann algebra. We intend to use a convention in the whole paper: By a state we always mean a faithful normal positive (not necessarily normalized) functional. Dealing with several states on M we shall consider M in its standard form ([**10**], [**28**]). If M has the standard form (M, H, J, P) then M acts on the Hilbert space H, P is a cone in H such that every state ω has a unique vector representative Ω in P which is cyclic and separating for M. Given another normal state ϕ the densely defined quadratic form

$$a\Omega \to \phi(aa^*) \qquad (a \in M)$$

is closable and there exists an associated positive self-adjoint operator Δ. It is characterized by the following properties. $M\Omega$ is a core for $\Delta^{\frac{1}{2}}$ and $\|\Delta^{\frac{1}{2}}a\Omega\|^2 = \phi(aa^*)$ ([**18**], VI. § 2). Δ was called by Araki the relative modular operator of ϕ and ω and it is usually denoted by $\Delta(\phi, \omega)$ (see [**3**], [**4**]). Equivalently, $\Delta(\phi, \omega)$ is obtained from the polar decomposition of the closure $S_{\phi\omega}$ of the conjugate linear operator $a\Omega \to a^*\Phi$ (where Φ is the vector representative of ϕ from P). Namely, $S_{\phi\omega} = J\Delta(\phi, \omega)^{\frac{1}{2}}$. The operators J, $\Delta(\omega, \omega)$ and σ_t^ω are the standard ingredients of the Tomita–Takesaki theory with respect to $\Omega(\omega)$. (See, for example, [**5**], 2.5 or [**27**], Chapter 10). The modular group of ω is a one-parameter group of automorphisms of M and it looks like

$$\sigma_t^\omega(a) = \Delta(\omega, \omega)^{it} a \Delta(\omega, \omega)^{-it}.$$

Another Radon–Nikodym derivative like object for comparison of two states is the Radon–Nikodym cocyle discovered by Connes ([**7**], see also [**4**]). If ϕ is a faithful normal semi-finite weight then $[D\phi, D\omega]_t = u_t$ is a σ^ω-cocycle and

$$\sigma_t^\phi = u_t \sigma_t^\omega u_t^*.$$

The formula

$$[D\phi, D\omega]_t = \Delta(\phi, \omega)^{it} \Delta(\omega, \omega)^{-it}$$

forms a bridge between the two objects (see [**4**]).

Quasi-entropy was introduced in [**23**] (but implicitly used also in [**19**]). If f: $(0, \infty) \to \mathbb{R}$ is a continuous function and $\int \lambda \, dE_\lambda$ is the spectral resolution of $\Delta(\phi, \omega)$ then

$$S_f(\phi, \omega) = \int f(\lambda) \, d\langle E_\lambda \Omega, \Omega \rangle.$$

Monotonicity of quasi-entropies is the following assertion: If f is operator monotone on $\mathbb{R}^+$ and α: $N \to M$ is a 2-positive unital mapping then

$$S_f(\phi \circ \alpha, \omega \circ \alpha) \geq S_f(\phi, \omega).$$

A special quasi-entropy is the transition probability

$$P_A(\phi, \omega) = \langle \Delta(\phi, \omega)^{\frac{1}{2}}\Omega, \Omega \rangle = \langle \Phi, \Omega \rangle$$

introduced by Raggio in [23]. (The subscript A in P_A may stand for Araki, but on the other hand in the measure case sometimes $P_A(\phi, \omega)$ is called the affinity of the measures ϕ and ω.)

Channels

Let N and M be von Neumann algebras. The triple (N, M, α) will be called a channel if α: $N \to M$ is a linear mapping with the following properties.

(i) $\alpha(I) = I$.
(ii) α is w-continuous.
(iii) α is 2-positive.
(iv) if $a \in N_+$ and $\alpha(a) = 0$ then $a = 0$.

We note that (i) and (ii) imply that $\|\alpha\| = 1$ ([5], 3.2.6). When M acts on some Hilbert space H then (iii) is equivalent to the following.

(iii)′ $\sum_{i,j} \langle \alpha(a_i^* a_j)\xi_i, \xi_j \rangle \geq 0$ for every $a_i \in N$ and $\xi_i \in H$ $(i = 1, 2)$.

States on M will be called input states of the channel (N, M, α). If ϕ is an input state then $\phi \circ \alpha$ is the corresponding output state.

Now we fix a notation what we intend to use freely in the whole section. Whenever (N, M, α) is a channel with input states ϕ and ω, (M, H, J, P) $((N, H_0, J_0, P_0))$ stands for the standard form of M (N), and Φ, Ω (Φ_0, Ω_0) are the vector representatives of ϕ and ω $(\phi \circ \alpha, \omega \circ \alpha)$, respectively.

THEOREM 1. *Let (N, M, α) be a channel and ϕ an input state. There exists a channel (M, N, α_ϕ^*) characterized by the condition*

$$\langle \alpha(a)\Phi, Jb\Phi \rangle = \langle a\Phi_0, J_0\alpha_\phi^*(b)\Phi_0 \rangle$$

for all $a \in N$ and $b \in M$.

Proof. Let $b \in M_+$ be fixed for a while. The correspondence

$$\gamma_b \colon a \to \langle \alpha(a)\Phi, Jb\Phi \rangle$$

defines a positive normal functional on N. Clearly, $\gamma_b(a) \leq \|b\|\ \phi(\alpha(a))$ for every $a \in N_+$. According to the commutant Radon–Nikodym Theorem ([27], 5.19) there is an $x' \in (N')_+$ such that

$$\gamma_b(a) = \langle a\Phi_0, x'\Phi_0 \rangle = \langle a\Phi_0, J_0(J_0x'J_0)\Phi_0 \rangle.$$

By Tomita's theorem $J_0x'J_0 \in N$ and we define $\alpha_\phi^*(b)$ as $J_0x'J_0$. α_ϕ^* admits a linear extension to the whole M.

α_ϕ^* is unital and positive by construction. α_ϕ^* is continuous with respect to the weak operator topology and hence it is w-continuous. If $\alpha_\phi^*(b)=0$ then $\phi(b)=0$ and b must be 0 whenever b is positive.

α_ϕ^* is 2-positive if and only if

$$\sum_{i,j} \langle \alpha_\phi^*(b_i^*b_j)J_0a_i\Phi_0, J_0a_j\Phi_0\rangle \geqslant 0$$

for every $b_i \in M$ and $a_i \in N$ $(i=1,2)$. However,

$$\begin{aligned}\sum_{i,j} &\langle \alpha_\phi^*(b_i^*b_j)J_0a_i\Phi_0, J_0a_j\Phi_0\rangle \\ &= \sum_{i,j} \langle \alpha_\phi^*(b_i^*b_j)\Phi_0, J_0a_i^*a_j\Phi_0\rangle \\ &= \sum_{i,j} \langle \alpha(a_i^*a_j)\Phi_0, J_0b_i^*b_j\Phi_0\rangle \\ &= \sum_{i,j} \langle \alpha(a_i^*a_j)J_0b_i\Phi_0, J_0b_j\Phi_0\rangle\end{aligned}$$

and the last expression is positive due to the 2-positivity of α.

There are natural (that is, positive) embeddings i_ϕ: $M \to M_*$ and $i_{\phi\circ\alpha}$: $N \to N_*$. α_ϕ^* is exactly the mapping which makes the diagram

$$\begin{tikzcd} N \arrow[d, "i_{\phi\circ\alpha}"'] & M \arrow[l, "\alpha_\phi^*"'] \arrow[d, "i_\phi"] \\ N_* & M_* \arrow[l, "\alpha_*"'] \end{tikzcd}$$

commutative (α_* is the preadjoint of α). This point was emphasized in **[21]** where the dual was defined for slightly more general mappings. We ought to mention also the paper **[9]** where a dual (i.e., adjoint) is treated for not necessarily positive selfmappings of an algebra.

THEOREM 2. *Let (N, M, α) be a channel with an input state ϕ. Then for $a \in N$ the following conditions are equivalent.*

(i) $\alpha(a^*a)=\alpha(a)^*\alpha(a)$ *and* $\alpha(\sigma_t^{\phi\circ\alpha}(a))=\sigma_t^\phi(\alpha(a))$ *for every* $t \in \mathbb{R}$.

(ii) $\alpha_\phi^*\circ\alpha(a)=a$.

Furthermore, α restricted to $N_1=\{x \in N: \alpha_\phi^*\circ\alpha(x)=x\}$ is an isomorphism onto $M_1=\{y \in M: \alpha\circ\alpha_\phi^*(y)=y\}$.

Proof. First we treat the implication (i)$\to$(ii) and use the fixed notation. Recall that the operators

$$x\Phi_0 \to x^*\Phi_0 \qquad (x \in N)$$
$$y^\Phi \to y^*\Phi \qquad (y \in M)$$

are closable and their closures S_0 and S have polar decomposition $S_0 = J_0\Delta_0^{\frac{1}{2}}$ and $S = J\Delta^{\frac{1}{2}}$ ([5], 2.5.11). Being α a Schwarz mapping

$$x\Phi_0 \to \alpha(x)\Phi \qquad (x \in N)$$

admits an extension to a contraction V_α^ϕ. Since $\sigma_t^{\phi\circ\alpha}(x) = \Delta_0^{it}x\Delta_0^{-it}$ and $\sigma_t^\phi(x) = \Delta^{it}x\Delta^{-it}$ $(x \in N, y \in M)$ we have

$$V_\alpha^\phi S_0\Delta_0^{it}a\Phi_0 = \alpha(\sigma_t^{\phi\circ\alpha}(a)^*)\Phi = (\alpha(\sigma_t^{\phi\circ\alpha}(a))^*\Phi = S\Delta^{it}V_\alpha^\phi a\Phi_0.$$

Hence

$$V_\alpha^\phi J_0\Delta_0^{it+\frac{1}{2}}a\Phi_0 = J\Delta^{it+\frac{1}{2}}V_\alpha^\phi a\Phi_0$$

for all $t \in \mathbb{R}$. By analytical continuation at $t = i/2$ we obtain

$$V_\alpha^\phi J_0 a\Phi_0 = JV_\alpha^\phi a\Phi_0. \tag{1}$$

(ii) is equivalent to the condition

$$\langle \alpha(a)\Phi, J\alpha(x)\Phi \rangle = \langle a\Phi_0, J_0x\Phi_0 \rangle$$

for all $x \in N$. Since

$$\langle \alpha(a)\Phi, J\alpha(x)\Phi \rangle = \langle J_0(V_\alpha^\phi)^*JV_\alpha^\phi a\Phi_0, J_0x\Phi_0 \rangle$$

we have to check that

$$J_0a\Phi_0 = (V_\alpha^\phi)^*JV_\alpha^\phi a\Phi_0. \tag{2}$$

Straightforward computation gives

$$\begin{aligned}\|V_\alpha^\phi J_0a\Phi_0\|^2 &= \langle JV_\alpha^\phi a\Phi_0, JV_\alpha^\phi a\Phi_0 \rangle = \phi(\alpha(a)^*\alpha(a)) \\ &= \phi(\alpha(a^*a)) = \|J_0a\Phi_0\|^2.\end{aligned}$$

Therefore

$$(V_\alpha^\phi)^*V_\alpha^\phi J_0a\Phi_0 = J_0a\Phi_0 \tag{3}$$

must hold. From (1) and (3) the condition (2) follows.

To prove (ii)$\to$(i) first note that N_1 and M_1 are von Neumann subalgebras (cf. [20]). We have seen that that $a \in N_1$ is equivalent to (2). So if $a \in N_1$ then

$$\|a\Phi_0\| = \|(V_\alpha^\phi)^*JV_\alpha^\phi a\Phi_0\| \leq \|V_\alpha^\phi a\Phi_0\| \leq \|a\Phi_0\|$$

and $\|V_\alpha^\phi a\Phi_0\| = \|a\Phi_0\|$. In other words, $\phi(\alpha(a)^*\alpha(a)) = \phi(\alpha(a^*a))$. Since $\alpha(a)^*\alpha(a) \leq \alpha(a^*a)$ and ϕ is faithful, we infer $\alpha(a)^*\alpha(a) = \alpha(a^*a)$. According to [6] (see also [28], 9.2) we have

$$\alpha(xa) = \alpha(x)\alpha(a) \quad \text{and} \quad \alpha(a^*x) = \alpha(a)^*\alpha(x)$$

for every $x \in N$. In particular, $\alpha \mid N_1$ is an isomorphism, and evidently $\alpha(N_1) = M_1$.

With the notation $\phi' = \phi \mid M_1$ and $\psi' = \phi \circ \alpha \mid N_1$, an invocation to the KMS condition gives

$$\alpha(\sigma_t^{\psi'}(x)) = \sigma_t^{\phi'}(\alpha(x)) \qquad (t \in \mathbb{R}, x \in N_1).$$

By the mean ergodic theorem ([20]) there exists a conditional expectation of M onto M_1 (of N onto N_1) preserving ϕ ($\phi \circ \alpha$) and due to Takesaki Theorem ([29], see also [28], 10.1)

$$\sigma_t^{\psi'} = \sigma_t^{\phi \circ \alpha} \mid N_1 \quad \text{and} \quad \sigma_t^{\phi'} = \sigma_t^{\phi} \mid M_1.$$

The proof is complete.

THEOREM 3. *Let (N, M, α) be a channel. If ϕ and ω are states on M then the following conditions are equivalent.*

(i) $\alpha \circ \alpha_\omega^*([D\phi, D\omega]_t) = [D\phi, D\omega]_t$ *for every* $t \in \mathbb{R}$.
(ii) $P_A(\phi \circ \alpha, \omega \circ \alpha) = P_A(\phi, \omega)$.
(iii) $\alpha([D(\phi \circ \alpha), D(\omega \circ \alpha)]_t) = [D\phi, D\omega]_t$ *for every* $t \in \mathbb{R}$.
(iv) $\alpha_\omega^* = \alpha_\phi^*$
(v) $\phi \circ \alpha_\omega^* = \phi$.
(vi) $\omega \circ \alpha_\phi^* = \omega$.

Proof. Suppose (i) and let M_0 be the subalgebra generated by $\{[D\phi, D\omega]_t \colon t \in \mathbb{R}\}$. Then M_0 is stable under the modular group of ω and there is a conditional expectation $E\colon M \to M_0$ preserving ω. The converse of Connes' Theorem ([7] and [28], 5.1) guarantees a weight ϕ' on N_0 such that

$$[D\phi, D\omega]_t = [D\phi', D(\omega \mid M_0)]_t.$$

On the other hand

$$[D(\phi' \circ E), D\omega]_t = [D\phi', D(\omega \mid N_0)]_t$$

(see [28], 10.5) and $\phi' \circ E$ must be ϕ. Consequently, E preserves also ϕ. Applying the monotonicity theorem we have

$$\begin{aligned} P_A(\phi, \omega) &\le P_A(\phi \mid M_1, \omega \mid M_1) \le P_A(\phi \mid M_0, \omega \mid M_0) \le P_A(\phi, \omega) \\ &\le P_A(\phi \circ \alpha, \omega \circ \alpha) \le P_A(\phi \circ \alpha \mid N_1, \omega \circ \alpha \mid N_1) \end{aligned}$$

and since

$$P_A(\phi \circ \alpha \mid N_1, \omega \circ \alpha \mid N_1) = P_A(\phi \mid M_1, \omega \mid M_1)$$

(iii) can be concluded. (Recall that M_1 and N_1 were defined in Theorem 2.)

We shorten our notation to Δ (Δ_0) for the relative modular operator of ϕ and ω ($\phi \circ \alpha$ and $\omega \circ \alpha$). V_α^ϕ and V_α^ω are contractions of H_0 into H such that

$$V_\alpha^\phi(a\Phi_0) = \alpha(a)\Phi \quad \text{and} \quad V_\alpha^\omega(a\Omega_0) = \alpha(a)\Omega \qquad (a \in N).$$

Turning to (ii)→(iii) we use the formula

$$x^{\frac{1}{2}}=\frac{2}{\pi}\int_0^\infty x(1+t^2x)^{-1}\,\mathrm{d}t$$

and write

$$P_A(\phi,\omega)=\langle\Delta^{\frac{1}{2}}\Omega,\Omega\rangle=\frac{2}{\pi}\int_0^\infty\langle\Delta(1+t^2\Delta)^{-1}\Omega,\Omega\rangle\,\mathrm{d}t$$

$$=\pi^{-1}\int_0^\infty t^{-2}-t^{-4}\langle(t^{-2}+\Delta)^{-1}\Omega,\Omega\rangle\,\mathrm{d}t$$

Similarly

$$P_A(\phi\circ\alpha,\omega\circ\alpha)=\frac{2}{\pi}\int_0^\infty t^{-2}-t^{1}\langle(t^{-2}+\Delta_0)^{-1}\Omega_0,\Omega_0\rangle\,\mathrm{d}t.$$

We prove that

$$(t+\Delta_0)^{-1}\leqslant(V_\alpha^\omega)^*(t+\Delta)^{-1}V_\alpha^\omega \tag{4}$$

for $t>0$.

Δ is the associated positive selfadjoint operator to the densely defined closable quadratic form

$$q:\ a\Omega\to\phi(aa^*)\qquad(a\in M).$$

Let $\int_0^\infty\lambda\,\mathrm{d}E(\lambda)$ be the spectral decomposition of Δ and define $H_n=\int_0^n\lambda\,\mathrm{d}E(\lambda)$. Then $(t+H_n)^{-1}\to(t+\Delta)^{-1}$ strongly for all $t>0$.

The set $\{a\Omega_0\colon a\in N\}$ is a core for $\Delta_0^{\frac{1}{2}}$. Evidently,

$$\begin{aligned}\|\Delta_0^{\frac{1}{2}}a\Omega\|^2&=\phi\circ\alpha(aa^*)\geqslant\phi(\alpha(a)\alpha(a)^*)\\&=\|\Delta^{\frac{1}{2}}\alpha(a)\Omega\|^2\geqslant\|H_n^{\frac{1}{2}}V_\alpha^\omega(a\Omega_0)\|^2\end{aligned}$$

and we establish

$$(V_\alpha^\omega)^*H_nV_\alpha^\omega\leqslant\Delta_0.$$

Then

$$(t+\Delta_0)^{-1}\leqslant(t+(V_\alpha^\omega)^*H_nV_\alpha^\omega)^{-1}.$$

Since

$$(t+(V_\alpha^\omega)^*H_nV_\alpha^\omega)^{-1}\leqslant(V_\alpha^\omega)^*(t+H_n)^{-1}V_\alpha^\omega$$

(see [8], or [12]) we arrive at

$$(t+\Delta_0)^{-1} \leq (V_\alpha^\omega)^*(t+H_n)^{-1}V_\alpha^\omega$$

and letting $n \to \infty$ we complete the proof of (4).

Due to (4) we have

$$\langle (t+\Delta_0)^{-1}\Omega_0, \Omega_0\rangle \leq \langle (t+\Delta)^{-1}\Omega, \Omega\rangle$$

and $P_A(\phi\circ\alpha, \omega\circ\alpha) = P_A(\phi, \omega)$ must imply that

$$\langle (t+\Delta_0)^{-1}\Omega_0, \Omega_0\rangle = \langle (t+\Delta)^{-1}\Omega, \Omega\rangle \tag{5}$$

for almost all $t \in \mathbb{R}^+$ and by continuity for all $t \in \mathbb{R}^+$.

We can estimate

$$\begin{aligned}
&\|(V_\alpha^\omega)^*(t+\Delta)^{-1}\Omega - (t+\Delta_0)^{-1}\Omega_0\|^2 \\
&= \|((V_\alpha^\omega)^*(t+\Delta)^{-1}V_\alpha^\omega - (t+\Delta_0)^{-1})\Omega_0\|^2 \\
&\leq \|((V_\alpha^\omega)^*(t+\Delta)^{-1}V_\alpha^\omega - (t+\Delta_0)^{-1})^{\frac{1}{2}}\|^2 \\
&\quad \times \langle ((V_\alpha^\omega)^*(t+\Delta)^{-1}V_\alpha^\omega - (t+\Delta_0)^{-1})\Omega_0, \Omega_0\rangle
\end{aligned}$$

whenever $t > 0$. Referring to (5) we have

$$(V_\alpha^\omega)^*(t+\Delta)^{-1}\Omega = (t+\Delta_0)^{-1}\Omega_0$$

for all $t > 0$ and through analytic continuation for all $t \in \mathbb{C} - \mathbb{R}^-$. Derivating with respect to t we obtain

$$(V_\alpha^\omega)^*(t+\Delta)^{-2}\Omega = (t+\Delta_0)^{-2}\Omega_0.$$

Hence

$$\begin{aligned}
\|(V_\alpha^\omega)^*(t+\Delta)^{-1}\Omega\|^2 &= \langle (t+\Delta_0)^{-2}\Omega_0, \Omega_0\rangle \\
&= \langle (V_\alpha^\omega)^*(t+\Delta)^{-2}\Omega, \Omega_0\rangle = \|(t+\Delta)^{-1}\Omega\|^2.
\end{aligned}$$

Consequently,

$$V_\alpha^\omega(V_\alpha^\omega)^*(t+\Delta)^{-1}\Omega = (t+\Delta)^{-1}\Omega \qquad (t \in \mathbb{C} - \mathbb{R}^-).$$

We infer that

$$V_\alpha^\omega(t+\Delta_0)^{-1}\Omega_0 = V_\alpha^\omega(V_\alpha^\omega)^*(t+\Delta)^{-1}\Omega = (t+\Delta)^{-1}\Omega$$

and standard application of the Stone–Weierstrass theorem yields

$$V_\alpha^\omega \Delta_0^{it}\Omega_0 = \Delta^{it}\Omega. \tag{6}$$

Since

$$\alpha([D(\phi\circ\alpha), D(\omega\circ\alpha)]_t)\Omega = V_\alpha^\omega\Delta_0^{it}\Delta(\omega\circ\alpha, \omega\circ\alpha)^{-it}\Omega_0 = V_\alpha^\omega\Delta_0^{it}\Omega_0$$

and

$$\Delta^{it}\Omega = \Delta^{it}\Delta(\omega, \omega)^{-it}\Omega = [D\phi, D\omega]_t\Omega$$

cyclicity of Ω and (6) give

$$[D\phi, D\omega]_t = [D(\phi \circ \alpha), D(\omega \circ \alpha)]_t$$

and we conclude (ii).

We abbreviate $[D\phi, D\omega]_t([D(\phi \circ \alpha), D(\omega \circ \alpha)]_t)$ as $v_t(u_t)$. Assuming (iii) we have

$$\alpha(u_t^* u_t) = v_t^* v_t = \alpha(u_t)^* \alpha(u_t)$$

and

$$\sigma_t^\omega(\alpha(u_s)) = v_{t+s} v_t^* = \alpha(\sigma_t^{\omega \circ \alpha}(u_s))$$

is a consequence of the cocyle property. Because of Theorem 2 we establish (iii)$\to$(ii).

Continuing our proof of (iv) we benefit again from Theorem 2. Since $\alpha(au_t) = \alpha(a)v_t$ and $\alpha_\omega^*(bv_t) = \alpha_\omega^*(b)u_t$ for all $a \in N$, $b \in M$ and $t \in \mathbb{R}$ we have

$$\begin{aligned}\langle \alpha_\omega^*(b)\Delta_0^{it}\Omega_0, J_0 a \Delta_0^{it}\Omega_0\rangle &= \langle \alpha_\omega^*(b)u_t\Omega_0, J_0 a u_t \Omega_0\rangle \\ \langle \alpha_\omega^*(bv_t)\Omega_0, J_0 a u_t \Omega_0\rangle &= \langle bv_t\Omega, J\alpha(au_t)\Omega\rangle \\ &= \langle b\Delta^{it}\Omega, J\alpha(a)\Delta^{it}\Omega\rangle.\end{aligned}$$

Taking into account that $\Delta_0^{\frac{1}{2}}\Omega_0 = \Phi_0$ and $\Delta^{\frac{1}{2}}\Omega = \Phi$ we consider the analytical extension of the functions

$$\begin{aligned}&t \to \langle \alpha_\omega^*(b)\Delta_0^{it}\Omega_0, J_0 a \Delta_0^{it}\Omega_0\rangle \\ &t \to \langle b\Delta^{it}\Omega, J\alpha(a)\Delta^{it}\Omega\rangle\end{aligned}$$

to the strip $S = \{z \in \mathbb{C}\colon -\frac{1}{2} \le \operatorname{Im} z \le 0\}$ (which is analytic on $S^\circ = \{z \in \mathbb{C}\colon -\frac{1}{2} < \operatorname{Im} z < 0\}$, see [**27**], 9.15). Since they coincide on the real line we infer

$$\langle \alpha_\omega^*(b)\Phi_0, J_0 a \Phi_0\rangle = \langle b\Phi, J\alpha(a)\Phi\rangle$$

evaluating at $t = -i/2$. This means that $\alpha_\phi^* = \alpha_\omega^*$.

The implications (iv)$\to$(v) and (iv)$\to$(vi) are obvious and both (v) and (vi) imply (ii) by the monotonicity theorem.

If N is a subalgebra of M and α is the inclusion then α_ω^* reduces to the ω-conditional expectation of Accardi and Cecchini ([**1**]). In this special case we have the following.

Corollary. *Let N be a von Neumann subalgebra of M. Then the following conditions are equivalent.*

(i) $[D\phi, D\omega]_t \in N$ *for every* $t \in \mathbb{R}$.
(ii) $P_A(\phi \mid N, \omega \mid N) = P_A(\phi, \omega)$.
(iii) $[D\phi, D\omega]_t = [D(\phi \mid N), D(\omega \mid N)]_t$ *for every* $t \in \mathbb{R}$.
(iv) $E_\phi = E_\omega$.
(v) $\phi \circ E_\omega = \phi$.
(vi) $\omega \circ E_\phi = \omega$.

Sufficiency

Sufficiency is an important notion mathematical statistics ([13], [26]) and it has appeared in an operator algebra setup in the papers [30], [17], [14], [22]. A von Neumann algebra with a family of states corresponds to a statistical experiment (cf. [13]) and a channel as we defined is essentially the same as the randomization of an experiment ([13], [26]). Since many statistical notions do not have algebraic counterparts (yet), sufficiently of a channel can be defined by statistical isomorphism. Formally, a channel (N, M, α) is sufficient with respect to the family θ of input states if there exists a 2-positive mapping β: $M \to N$ such that $\phi \circ \alpha \circ \beta = \phi$ for all $\phi \in \theta$.

THEOREM 5. *Let (N, M, α) be a channel and ϕ, ω faithful normal states on M. Then the following conditions are equivalent.*

(i) *α is sufficient with respect to $\{\phi, \omega\}$.*
(ii) *$\alpha \circ \alpha^*_\omega([D\phi, D\omega]_t) = [D\phi, D\omega]_t$ for every $t \in \mathbb{R}$.*
(iii) *$P_A(\phi \circ \alpha, \omega \circ \alpha) = P_A(\phi, \omega)$.*
(iv) *$\alpha([D(\phi \circ \alpha), D(\omega \circ \alpha)]_t) = [D\phi, D\omega]_t$ for every $t \in \mathbb{R}$.*
(v) *$\alpha^*_\omega = \alpha^*_\phi$.*
(vi) *$\phi \circ \alpha^*_\omega = \phi$.*
(vii) *$\omega \circ \alpha^*_\phi = \omega$.*

Proof. (i)$\to$(iii) through the monotonicity theorem concerning P_A. The implication (v)$\to$(i) is trivial and the equivalence of (ii)–(vii) is stated in Theorem 3.

COROLLARY 6. *Let (N, M, α) be a channel with a family θ of faithful normal input states. Then the following conditions are equivalent.*

(i) *α is sufficient with respect to θ.*
(iii) *α is sufficient with respect to every $\{\phi, \omega\} \subset \theta$.*

Assume that $N \subset M$ and θ is a family of faithful normal states on M. We say that N is a sufficient subalgebra of M if the channel (N, M, id) is sufficient with respect to θ. (id: $N \to M$ is the inclusion mapping.) In other words, N is sufficient if there exists a 2-positive mapping β: $M \to N$ such that it leaves invariant all $\phi \in \theta$. (Note that this terminology differs from other ones, used for example in [14], [15], [24].)

The equivalence of the sufficiency and pairwise sufficiency is surprising at the first glance. It is not so if we recognise that a family of faithful normal states corresponds to a dominated experiment in the measure case (cf. [14], § 8). There exists a smallest sufficient subalgebra and in the light of Corollary 4 and Theorem 5 it is the von Neumann subalgebra generated by the set

$$\{[D\phi, D\omega]_t\colon\ t \in \mathbb{R} \text{ and } \phi, \omega \in \theta\}.$$

THEOREM 7. *Let (N, M, α) be a channel with a family θ of faithful normal input states. Then α is sufficient with respect to θ if and only if the subalgebra $\{a \in M\colon\ \alpha \circ \alpha_\omega^*(a) = a\} = M_1$ is sufficient with respect to θ.*

Proof. M_1 is sufficient if and only if $[D\phi, D\omega]_t \in M_1$ for every $t \in \mathbb{R}$ and $\phi, \omega \in \theta$. So Theorem 5 can be applied.

The inner perturbation of a state was studied by Araki ([**2**]). If $h = h^* \in M$, then for every faithful normal state ω there is a state ω^h determined by

$$[D\omega^h, D\omega]_t = e^{it(H+h)}e^{-itH}$$

where H stands for $\log \Delta(\omega, \omega)$. The main properties of the perturbed state are summarized in [**3**].

THEOREM 8. *Let (N, M, α) be a channel with a faithful normal input state ω. For $h = h^* \in M$ stand ω^h for the inner perturbation of ω by h. Then α is sufficient with respect to $\{\omega, \omega^h)$ if and only if $h \in M_1 = \{a \in M\colon\ \alpha \circ \alpha_\omega^*(a) = a\}$.*

Proof. Since $\alpha \circ \alpha_\omega^*$ preserves ω, by [**20**] the subalgebra M_1 admits a conditional expectation $E\colon\ M \to M_0$ preserving ω. Hence reference to Theorem 5 above and to Theorem 6 in [**24**] makes the proof complete.

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