

LL18554 is being considered for publication in Physical Review Letters.

Limitations of classically-simulable measurements for quantum state discrimination  
by Chengkai Zhu, Zhiping Liu, Chenghong Zhu, et al.

Dear Dr. Jencova,

You helped us evaluate this submission to Physical Review Letters before. The authors now have resubmitted their manuscript and we would appreciate your advice again, as you are ideally suited to assess the changes and responses. The abstract is below.

Please let us know as soon as possible whether you can review it again.

To download the manuscript, obtain more information, or send a report, please log into our referee server at:  
<https://referees.aps.org/reviews/LL18554-6eab1ca-943792>

To accept to review, visit:  
<https://referees.aps.org/reviews/LL18554-6eab1ca-943792/promise>

To decline to review, visit:  
<https://referees.aps.org/reviews/LL18554-6eab1ca-943792/decline>

**We append copies of reports from you and the other referees on the previous version of the manuscript and of the authors' response.**

**One or more pieces of previous correspondence that we need to send to you is in PDF format. Please find the individual files attached to this email.**

**Supplemental Material associated with this manuscript is available via our referee server.**

Thank you for your help.

Yours sincerely,

Stojan Rebic, Ph.D.  
Senior Associate Editor  
Physical Review Letters  
Email: [prl@aps.org](mailto:prl@aps.org)

**ADDITIONAL MATERIAL AVAILABLE (SEE FULL REFERRAL LETTER):**

- [Supplemental Material intended for publication \(via referee server\)](#)
- Second Report of Referee A – LL18554/Zhu
- Second Report of Referee B – LL18554/Zhu
- Report of Referee C – LL18554/Zhu
- Resubmission letter from author – LL18554/Zhu
- Marked manuscript from author – LL18554/Zhu
- Memo: Guidelines to referees for Physical Review Letters

# Limitations of Classically-Simulable Measurements for Quantum State Discrimination

Chengkai Zhu,<sup>1,\*</sup> Zhiping Liu,<sup>1,2,\*</sup> Chenghong Zhu,<sup>1</sup> and Xin Wang<sup>1,†</sup>

<sup>1</sup>*Thrust of Artificial Intelligence, Information Hub,*

*The Hong Kong University of Science and Technology (Guangzhou), Guangzhou 511453, China*

<sup>2</sup>*National Laboratory of Solid State Microstructures, School of Physics and Collaborative  
Innovation Center of Advanced Microstructures, Nanjing University, Nanjing 210093, China*

(Dated: May 17, 2024)

In the realm of fault-tolerant quantum computing, stabilizer operations play a pivotal role, characterized by their remarkable efficiency in classical simulation. This efficiency sets them apart from non-stabilizer operations within the quantum computational theory. In this paper, we investigate the limitations of classically-simulable measurements in distinguishing quantum states. We demonstrate that any pure magic state and its orthogonal complement of odd prime dimensions cannot be unambiguously distinguished by stabilizer operations, regardless of how many copies of the states are supplied. We also reveal intrinsic similarities and distinctions between the quantum resource theories of magic states and entanglement in quantum state discrimination. The results emphasize the inherent limitations of classically-simulable measurements and contribute to a deeper understanding of the quantum-classical boundary.

*Introduction.*—The computational power of quantum computers, including a substantial speed-up over their classical counterparts in solving certain number-theoretic problems [1–3] and simulating quantum systems [4, 5], can only be unlocked with a scalable quantum computing solution. Fault-tolerant quantum computation (FTQC) provides a scheme to overcome obstacles of physical implementation such as decoherence and inaccuracies [6–8].

A cornerstone of the FTQC resides in stabilizer circuits, comprised exclusively of the Clifford gates. It is well-known that the stabilizer circuits can be efficiently classically simulated [9], and therefore do not confer any quantum computational advantage. However, *magic states* are quantum states that cannot be prepared using the stabilizer formalism [10], and can promote the stabilizer circuits to universal quantum computation via state injection [11–13]. In this context, the magic states and non-stabilizer operations characterize the computational power of universal quantum computation.

While extensive research has explored the stabilizerness of quantum states and gates within circuits [14, 15], a crucial yet underexplored facet is the stabilizerness of quantum measurements [16] – a critical process for reliably decoding classical information encoded in quantum states. In general, it is not applicable for one to access the physical properties of a locally interacting quantum many-body system by classical simulation. However, when information is encoded in a stabilizer state, the decoding process via stabilizer measurements remains efficiently classically simulable [17]. This prompts a fundamental question: can stabilizer measurements perfectly decode all tasks, or are there inherent limitations? Investigating the distinction in decoding capabilities between stabilizer measurements, which are classically efficiently simulable, and other measurements becomes paramount for understanding the intricate relationship between classical information encoded in quantum states and the measurement process.

The ability to retrieve classical information from quantum systems varies significantly with different measurements. One celebrated example is the quantum nonlocality without

entanglement [18]. In essence, global measurements can always perfectly distinguish mutually orthogonal quantum states, while there is a set of product states that cannot be distinguished via local quantum operations and classical communications (LOCC). This distinction between global and local measurements has garnered substantial attention, proving to be intricately linked with quantum entanglement theory and the concept of nonlocality [19–26]. This primitive gap between distinct classes of measurements makes quantum state discrimination (QSD) a crucial aspect of fundamental physics [27, 28], where it can be used to test the principles and nature of quantum mechanics. Moreover, QSD has led to fruitful applications in quantum cryptography [29–31], quantum dimension witness [32, 33] and quantum data hiding [34–36].

Inspired by the intrinsic behavior of different measurements in entanglement theory, we raise a natural and important question for understanding the limit and power of the classically-simulable measurements. In particular, is there a sharp gap between the classically-simulable measurements and those that could potentially promote universal quantum computation? If such a gap exists, it will imply considerable advantages that the resource of magic states can provide to the measurement in quantum information processing.

In this Letter, we give an affirmative answer to this question. We show that any pure magic state and its orthogonal complement cannot be unambiguously distinguished via Positive Operator-Valued Measures (POVMs) having positive discrete Wigner functions, which are classically-simulable and strictly include stabilizer measurements [17, 37], no matter how many copies of the states are supplied. We also demonstrate an exponential decay on the asymptotic minimal error probability for distinguishing the Strange state and its orthogonal complement via POVMs having positive discrete Wigner functions, where the Strange state is a representative qutrit magic state defined as  $|\mathbb{S}\rangle := (|1\rangle - |2\rangle)/\sqrt{2}$  [10].

In addition, we show that every set of orthogonal pure stabilizer states can be unambiguously distinguished via POVMs

having positive discrete Wigner functions, indicating there is no similar phenomenon as the unextendible product basis (UPB) in entanglement theory. Moreover, we demonstrate that even with the assistance of one or two copies of any qutrit magic state, the Strange state and its orthogonal complement remain indistinguishable via POVMs having positive discrete Wigner functions. It is different from entanglement theory where a single copy of the Bell state is always sufficient to perfectly distinguish a pure entangled state and its orthogonal complement using PPT POVMs [38].

*Preliminaries.*— To characterize the stabilizerness of quantum states and operations, we first recall the definition of the discrete Wigner function [39–41]. Throughout the paper, we study the Hilbert space  $\mathcal{H}_d$  with an odd dimension  $d$ , and if the dimension is not prime, it should be understood as a tensor product of Hilbert spaces each having an odd prime dimension. Let  $\mathcal{L}(\mathcal{H}_d)$  be the space of linear operators mapping  $\mathcal{H}_d$  to itself and  $\mathcal{D}(\mathcal{H}_d)$  be the set of density operators acting on  $\mathcal{H}_d$ . It is worth noting that qudit-based quantum computing is gaining increasing significance, as numerous problems in the field are awaiting further exploration [42].

Given a standard computational basis  $\{|j\rangle\}_{j=0,\dots,d-1}$ , the unitary boost and shift operators  $X, Z \in \mathcal{L}(\mathcal{H}_d)$  are defined by  $X|j\rangle = |j \oplus 1\rangle, Z|j\rangle = w^j|j\rangle$ , where  $w = e^{2\pi i/d}$  and  $\oplus$  is the addition in  $\mathbb{Z}_d$ . The discrete phase space of a single  $d$ -level system is  $\mathbb{Z}_d \times \mathbb{Z}_d$ . At each point  $\mathbf{u} = (a_1, a_2) \in \mathbb{Z}_d \times \mathbb{Z}_d$ , the discrete Wigner function of a state  $\rho$  is defined as  $W_\rho(\mathbf{u}) := \frac{1}{d} \text{Tr}[A_{\mathbf{u}}\rho]$  where  $A_{\mathbf{u}}$  is the phase-space point operator given by  $A_{\mathbf{0}} := \frac{1}{d} \sum_{\mathbf{u}} T_{\mathbf{u}}, A_{\mathbf{u}} := T_{\mathbf{u}} A_{\mathbf{0}} T_{\mathbf{u}}^\dagger$  and  $T_{\mathbf{u}} = \tau^{-a_1 a_2} Z^{a_1} X^{a_2}, \tau = e^{(d+1)\pi i/d}$ . We say a state  $\rho$  has positive discrete Wigner functions (PWFs) if  $W_\rho(\mathbf{u}) \geq 0, \forall \mathbf{u} \in \mathbb{Z}_d \times \mathbb{Z}_d$  and briefly call it PWF state. Let  $\mathbf{E} = \{E_j\}_{j=0}^{n-1}$  be an  $n$ -valued POVM acting on  $\mathcal{H}_d$  with  $\sum_{j=0}^{n-1} E_j = \mathbb{1}$ . The discrete Wigner function of each effect  $E_j$  is given by  $W(E_j|\mathbf{u}) = \text{Tr}[E_j A_{\mathbf{u}}]$ .  $\mathbf{E}$  is said to be a PWF POVM if each  $E_j$  has PWFs. More details can be found in appendix.

In odd prime dimensions, quantum circuits with initial states and all subsequent quantum operations having PWFs, which strictly include stabilizer (STAB) operations, admit efficient classical simulations [17, 37], extending the Gottesman-Knill theorem. On the contrary, negativity in Wigner functions is usually regarded as an indication of ‘non-classicality’ [43, 44] and identified as a computational resource. Thus, PWF POVMs are recognized as classically-simulable measurements [45]. The exclusive applicability of these results to odd prime dimensions may stem from the unique property that only quantum systems of such dimensions exhibit covariance of the Wigner function w.r.t. Clifford operations [46]. It’s worth noting that there exist mixed magic states with PWFs, rendering them useless for magic state distillation [13]. These states are termed *bound universal states* [47], analogous to states with a positive partial transpose (PPT) in entanglement distillation [48]. Therefore, PWF POVMs strictly include all STAB POVMs as

$$\text{STAB POVMs} \subsetneq \text{PWF POVMs} \subsetneq \text{All POVMs}.$$

*Asymptotic limits of PWF POVMs for a pure state and its orthogonal complement.*— Our primary aim is to elucidate the constraints inherent in measurements that can be efficiently classically simulated. QSD describes a general process of extracting classical information from quantum systems via measurements. To distinguish two states, one usually performs a two-outcome POVM on the received state and then determines which state it is according to the measurement outcome.

It is well-known that the asymptotic regime of QSD can unravel the underlying mechanism of entanglement [22, 25, 49]. The limit of local measurements exhibits a fundamental distinction between pure and mixed states [22]. Moreover, the asymptotic error probability in QSD is interlinked with the quantum relative entropy, Petz’s Rényi divergence [50], and the sandwiched Rényi divergence [51, 52]. Notably, in the regime of many copies, greater flexibility and options exist for the potential POVMs. However, we shall show a wide range of quantum states that cannot be unambiguously distinguished via PWF POVMs, including STAB POVMs, no matter how many copies are supplied.

**Theorem 1** *Let  $\rho_0 \in \mathcal{D}(\mathcal{H}_d)$  be a pure magic state and  $\rho_1 = (\mathbb{1} - \rho_0)/(d-1)$  be its orthogonal complement, where  $\mathbb{1}$  is the identity matrix. Then for any integer  $n \in \mathbb{Z}^+$ ,  $\rho_0^{\otimes n}$  and  $\rho_1^{\otimes n}$  cannot be unambiguously distinguished by PWF POVMs.*

Theorem 1 reveals a significant disparity in the ability of PWF POVMs and other measurements in QSD. It indicates that the classical information you are allowed to extract from the encoded states is limited when the measurements allowed are restricted to those classically-simulable ones. The limitation of the classically-simulable measurements cannot be overcome even by increasing the number of copies of the states.

From the angle of quantum resource theories (QRTs) [53], this theorem unravels the challenge of distinguishing a pure resourceful state and its orthogonal complement via free operations in the QRT of magic states. This parallels a phenomenon in entanglement theory where any pure entangled state and its orthogonal complement cannot be unambiguously distinguished via PPT POVMs with an arbitrary number of copies provided [38, 49, 54]. However, perfect distinguishability is achievable through global measurements. Notably, Takagi and Regula introduced a quantifier of resourcefulness for measurements [55], demonstrating that resourceful measurements can outperform free measurements in certain QSD tasks [56]. Here, our result further specifies the constraints of free measurements within the QRT of magic states, revealing that free operations cannot distinguish a pure resourceful state and its orthogonal complement, even in the many-copy regime.

The proof of Theorem 1 relies on Lemma 2 which identifies the feature of *PWF unextendible* subspaces, and a fact that the orthogonal complement of any pure magic state is PWF since  $-1/d \leq W_\rho(\mathbf{u}) \leq 1/d, \forall \rho \in \mathcal{D}(\mathcal{H}_d), \forall \mathbf{u}$  [46]. We call a subspace  $\mathcal{S} \subseteq \mathcal{H}_d$  *PWF unextendible* if there is no

PUF state  $\rho$  whose support is a subspace of  $\mathcal{S}^\perp$ , and PUF extendible otherwise. A subspace  $\mathcal{S} \subseteq \mathcal{H}_d$  is called *strongly PUF unextendible* if for any positive integer  $n$ ,  $\mathcal{S}^{\otimes n}$  is PUF unextendible. As a simple example, if we let  $\mathcal{S}^\perp$  be a one-dimensional subspace spanned by the strange state  $|\mathbb{S}\rangle$ , then  $\mathcal{S}$  is (strongly) PUF unextendible. In fact, the unextendibility of subspaces indicates the distinguishability of quantum states. It is well-known that a UPB for a multipartite quantum system indicates indistinguishability under LOCC operations [26].

**Lemma 2** *For a PUF unextendible subspace  $\mathcal{S} \subseteq \mathcal{H}_d$ , if there is a PUF state  $\rho \in \mathcal{D}(\mathcal{S})$  such that  $\text{supp}(\rho) = \mathcal{S}$ , then  $\mathcal{S}$  is strongly PUF unextendible.*

We note that Lemma 2 implies that for a set of orthogonal quantum states  $\{\rho_1, \dots, \rho_n\}$ , if there is a  $\rho_i$  whose support is strongly PUF unextendible, then  $\{\rho_1, \dots, \rho_n\}$  cannot be unambiguously distinguished by PUF POVMs no matter how many copies are used. This leads to and generalizes the result of Theorem 1. We sketch the proof of Lemma 2 as follows.

First, we demonstrate that  $\mathcal{S}^{\otimes 2}$  is PUF unextendible through a proof by contradiction. Suppose  $\rho_s \in \mathcal{D}(\mathcal{S})$  is a PUF state such that  $\text{supp}(\rho_s) = \mathcal{S}$ . If there is a PUF state  $\sigma$  supporting on  $(\mathcal{S}^{\otimes 2})^\perp$ , then we have  $\text{Tr}[\sigma(\rho_s \otimes \rho_s)] = 0$  which leads to  $\text{Tr}[\rho_s \text{Tr}_2[\sigma(\mathbb{1} \otimes \rho_s)]] = 0$ . It is easy to check that  $\sigma' = \text{Tr}_2[\sigma(\mathbb{1} \otimes \rho_s)]$  is a positive semi-definite operator with PWFs if it is non-zero. If it is zero, we can check that  $\text{Tr}_1 \sigma$  is a positive semi-definite operator with PWFs. In either case, we will get a PUF state supported on  $\mathcal{S}^\perp$ , a contradiction to the PUF unextendibility of  $\mathcal{S}$ . Hence, we conclude that  $\mathcal{S}^{\otimes 2}$  is PUF unextendible. Using a similar technique, we can conclusively demonstrate that  $\mathcal{S}^{\otimes n}$  is PUF unextendible for any positive integer  $n$ . The details can be found in appendix.

*Asymptotic limits of PUF POVMs for mixed states.*— Followed by Lemma 2, we note that Theorem 1 displays a special case of a strongly PUF unextendible subspace. The orthogonal complement of a pure magic state turns out to be a PUF state which lies in a  $d-1$  dimensional PUF unextendible subspace. This prompts an intriguing inquiry into the minimal dimension of such subspace. Notably, we will show there is a much smaller strongly PUF unextendible subspace, indicating the presence of mixed magic states that cannot be unambiguously distinguished from their orthogonal complements via PUF POVMs in the many-copy scenario.

**Proposition 3** *There exists a strongly PUF unextendible subspace  $\mathcal{S} \subseteq \mathcal{H}_d$  of dimension  $(d+1)/2$ .*

This proposition implies there is a  $(d-1)/2$ -dimensional subspace in which all states are magic states. The detailed proof is in appendix and we give a simple example as follows.

**Example 1** Consider a qudit system with  $d = 5$ . We have the

following basis that spans  $\mathcal{H}_5$ .

$$\begin{aligned} |v_0\rangle &= |0\rangle, \\ |v_1\rangle &= (|1\rangle + |2\rangle + |3\rangle + |4\rangle)/2, \\ |v_2\rangle &= (-|1\rangle + |2\rangle + |3\rangle - |4\rangle)/2, \\ |v_3\rangle &= (|1\rangle - |2\rangle + |3\rangle - |4\rangle)/2, \\ |v_4\rangle &= (|1\rangle + |2\rangle - |3\rangle - |4\rangle)/2. \end{aligned} \quad (1)$$

Let  $\rho_0 = (|v_0\rangle\langle v_0| + |v_1\rangle\langle v_1| + |v_2\rangle\langle v_2|)/3$ ,  $\rho_1 = (|v_3\rangle\langle v_3| + |v_4\rangle\langle v_4|)/2$ , and  $\mathcal{S}_0 = \text{supp}(\rho_0)$ ,  $\mathcal{S}_1 = \text{supp}(\rho_1)$ . Followed by the idea in the proof of Proposition 3, one can check that there is no PUF state in  $\mathcal{S}_1$ , and  $\rho_0$  is a PUF state. Thus,  $\mathcal{S}_0$  is a strongly PUF unextendible subspace.  $\rho_0$  and  $\rho_1$  cannot be unambiguously distinguished by PUF POVMs, no matter how many copies of them are supplied.

More generally, we establish an easy-to-compute criterion for identifying the circumstances under which two quantum states cannot be unambiguously distinguished by PUF POVMs in the many-copy scenario.

**Theorem 4** *Given  $\rho_0, \rho_1 \in \mathcal{D}(\mathcal{H}_d)$ , if any of them has strictly positive discrete Wigner functions, i.e.,  $W_{\rho_i}(\mathbf{u}) > 0, \forall \mathbf{u}$ , then for any integer  $n \in \mathbb{Z}^+$ ,  $\rho_0^{\otimes n}$  and  $\rho_1^{\otimes n}$  cannot be unambiguously distinguished by PUF POVMs.*

Theorem 4 is of broad applicability for both pure and mixed states. The indistinguishability can be checked through a simple computation of the discrete Wigner functions, streamlining the conventional method by analyzing exponentially large Hilbert space.

*Minimum error discrimination by PUF POVMs.*— After characterizing the limits of PUF POVMs, we further study the minimum error QSD to unveil the capabilities inherent in PUF POVMs. For states  $\rho_0$  and  $\rho_1$  with prior probability  $p$  and  $1-p$ , respectively, we denote  $P_e^{\text{PUF}}(\rho_0, \rho_1, p)$  as the optimal error probability of distinguishing them by PUF POVMs. Mathematically, this optimal error probability can be expressed via semidefinite programming (SDP) [57] as follows.

$$P_e^{\text{PUF}} = \min_{E_0, E_1} (1-p) \text{Tr}(E_0 \rho_1) + p \text{Tr}(E_1 \rho_0), \quad (2a)$$

$$\text{s.t. } E_0 \geq 0, E_1 \geq 0, E_0 + E_1 = \mathbb{1}, \quad (2b)$$

$$W(E_0|\mathbf{u}) \geq 0, W(E_1|\mathbf{u}) \geq 0, \forall \mathbf{u}, \quad (2c)$$

where Eq. (2c) ensures  $\{E_0, E_1\}$  is a PUF POVM. We provide the dual SDP in appendix. For  $\rho_0$  to be the Strange state and  $\rho_1$  to be its orthogonal complement, we demonstrate the following asymptotic error behavior.

**Proposition 5** *Let  $\rho_0$  be the Strange state  $|\mathbb{S}\rangle\langle\mathbb{S}|$  and  $\rho_1 = (\mathbb{1} - |\mathbb{S}\rangle\langle\mathbb{S}|)/2$  be its orthogonal complement. For  $n \in \mathbb{Z}^+$ , we have*

$$P_e^{\text{PUF}}(\rho_0^{\otimes n}, \rho_1^{\otimes n}, \frac{1}{2}) = \frac{1}{2^{n+1}}. \quad (3)$$

*The optimal PUF POVM is  $\{E, \mathbb{1}-E\}$ , where  $E = (|\mathbb{K}\rangle\langle\mathbb{K}| + |\mathbb{S}\rangle\langle\mathbb{S}|)^{\otimes n}$  and  $|\mathbb{K}\rangle = (|1\rangle + |2\rangle)/\sqrt{2}$ .*



We remark what we obtain here is the optimal error probability using PWF POVMs to distinguish  $n$  copies of the Strange state and its orthogonal complement. We first find the protocol above for the desired error probability and then utilize the dual SDP of (2) to establish the optimality of this protocol. The detailed proof is provided in appendix. It can be seen that the optimal error probability will exponentially decay with respect to the number of copies supplied. Nevertheless, it is important to note that the error persists for all finite values of  $n$ , aligning with the indistinguishability established in Theorem 1.

We further discuss the relationship between Proposition 5 and the Chernoff exponent in hypothesis testing. The celebrated quantum Chernoff theorem [50, 58, 59] establishes that  $\xi_C(\rho_0, \rho_1) := \lim_{n \rightarrow \infty} -\frac{1}{n} \log P_e(\rho_0^{\otimes n}, \rho_1^{\otimes n}, p) = -\min_{0 \leq s \leq 1} \log \text{Tr}[\rho_0^{1-s} \rho_1^s]$ , where  $P_e(\rho_0^{\otimes n}, \rho_1^{\otimes n}, p)$  is the average error of distinguishing  $\rho_0$  and  $\rho_1$  via global measurements,  $\xi_C(\rho_0, \rho_1)$  is the so-called Chernoff exponent. The Chernoff exponent concerning a specific class of measurements, e.g., {LOCC, PPT, SEP}, is defined in [49]. The authors proved that the Chernoff bounds in these cases are indeed faithful by showing an exponential decay of  $P_e^X(\rho_0, \rho_1, p)$  where  $X \in \{\text{LOCC, PPT, SEP}\}$ . Similarly, Proposition 5 may give an insight that the Chernoff bound concerning PWF measurements is also faithful.

Proposition 5 also implies applications in quantum data hiding [34, 60, 61]. Despite the original data-hiding setting where pairs of states of a bipartite system are perfectly distinguishable via general entangled measurements yet almost indistinguishable under LOCC, it is conceivable to extend data-hiding techniques to broader contexts dictated by specific physical circumstances [55]. As discussed in [55], one may consider the scenario that information is encoded in a way that Pauli measurements have less capability of decoding it than arbitrary measurements. Then only the party with the ability to generate magic can reliably retrieve the message. Here, we define  $\|\cdot\|_{\text{PWF}}$  and  $R(\text{PWF})$  as the *distinguishability norm* and the *data-hiding ratio* [61] associated with PWF POVMs, respectively. Proposition 5 directly gives a lower bound on the *data-hiding ratio* against PWF POVMs as follows.

$$R(\text{PWF}) = \max \frac{\|p\rho - (1-p)\sigma\|_{\text{All}}}{\|p\rho - (1-p)\sigma\|_{\text{PWF}}} \geq \frac{1}{1-2^{-n}}. \quad (4)$$

We also observe that a potential correlation between  $R(\text{PWF})$  and the generalized robustness of measurement [55] merits further investigation, with preliminary evidence provided in appendix.

*Distinctions between QRT of magic states and entanglement in QSD tasks.*— The asymptotic limits of PWF POVMs share similarities with LOCC operations, both of which are considered free within their respective resource theories. Whereas, there are fundamental distinctions between the QRT of magic states and entanglement, considering the QSD tasks. In Table I, we display a comparison between the QRT of magic states and entanglement in QSD, including their similarities and the following distinctions.

	QRT of magic states	QRT of entanglement
Asymptotic limits of free POVMs	✓	✓
Existence of UPB phenomenon	✗	✓
Perfect discrimination with the aid of one copy of maximal resource	✗	✓

TABLE I. **Comparison between the QRT of magic states and entanglement.** The second row represents if any resourceful pure state and its orthogonal complement are indistinguishable by free measurements in the many-copy scenario. The third row represents whether there is a UPB phenomenon. The last row represents whether the assistance of one copy of the maximally resourceful state is sufficient for perfect discrimination.

Recall that in entanglement theory, the UPB is an incomplete orthogonal product basis whose complementary subspace contains no product state [26]. It shows examples of orthogonal product states that cannot be perfectly distinguished by LOCC operations. Correspondingly, we may imagine whether there is a similar ‘UPB’ phenomenon in the QRT of magic states. That is if there is an incomplete orthogonal stabilizer basis whose complementary subspace contains no stabilizer state. We show that this is not the case as follows.

**Theorem 6** *For a subspace  $\mathcal{S} \in \mathcal{H}_d$ , if  $\mathcal{S}$  has a set of basis  $\{|\psi_i\rangle\}_{i=0}^{n-1}$  where every  $|\psi_i\rangle$  is a stabilizer state, then  $\mathcal{S}$  is PWF extendible.*

A direct consequence of this theorem is that any set of orthogonal pure stabilizer states  $\{|\psi_i\rangle\}_{i=0}^{n-1}$  can be unambiguously distinguished via PWF POVMs as we can choose  $E_i = |\psi_i\rangle\langle\psi_i|$  for  $i = 0, 2, \dots, n-1$  and  $E_n = \mathbb{1} - \sum_{i=0}^{n-1} |\psi_i\rangle\langle\psi_i|$ . Therefore, we confirm the absence of an analogous UPB phenomenon in the QRT of magic states.

Besides, it was shown that one copy of the Bell state is always sufficient for perfectly distinguishing any pure state  $\rho_0$  and its orthogonal complement  $\rho_1$  via PPT POVMs [38], i.e., distinguishing  $\rho_0 \otimes \Phi_2^+$  and  $\rho_1 \otimes \Phi_2^+$ . However, things are different in the QRT of magic states where we find the Strange state and its orthogonal complement cannot be perfectly distinguished by PWF POVMs with the assistance of one or two copies of any qutrit magic state.

**Proposition 7** *Let  $\rho_0$  be the Strange state  $|\mathbb{S}\rangle\langle\mathbb{S}|$  and  $\rho_1 = (\mathbb{1} - |\mathbb{S}\rangle\langle\mathbb{S}|)/2$  be its orthogonal complement.  $\rho_0 \otimes \tau^{\otimes k}$  and  $\rho_1 \otimes \tau^{\otimes k}$  cannot be perfectly distinguished by PWF POVMs for any qutrit magic state  $\tau$  and  $k = 1$  or  $2$ .*

The main idea is to analyze the minimal *mana* [62]  $\tau^{\otimes k}$  must have to perfectly distinguish  $\rho_0 \otimes \tau^{\otimes k}$  and  $\rho_1 \otimes \tau^{\otimes k}$  by PWF POVMs. A similar result can be obtained for the Norell state  $|\mathbb{N}\rangle := (-|0\rangle + 2|1\rangle - |2\rangle)/\sqrt{6}$  [10]. Hence, we have witnessed the distinctions of the QRT of magic states and entanglement in regard to the resource cost for perfect discrimination.

*Concluding remarks.*— We have explored the limitations of PWF POVMs which can be efficiently classically simulated and strictly include all stabilizer measurements. Our results show that the QRT of magic states and entanglement exhibit significant similarities and distinctions in quantum state discrimination.

These results have implications in various fields, including connections between the QRT of magic states and quantum data hiding [36, 55, 60, 61]. It remains interesting to further study the limits of stabilizer measurements or classically-simulable ones in quantum channel discrimination [63–65] and other operational tasks [66–68]. Note that as it is still open whether all operations with negative discrete Wigner functions are useful for magic state distillation [10], a comprehensive characterization of the quantum-classical boundary of measurements is still needed. Additionally, it is interesting to study the limitations of stabilizer measurements in a multi-qubit system [69–72], and recent advances in generalized phase-space simulation methods for qubits [73, 74] offer potential avenues to explore this, which we will leave to future work.

*Acknowledgments.*— We would like to thank the anonymous referees for their helpful comments. This work was supported by the Start-up Fund (No. G0101000151) from The Hong Kong University of Science and Technology (Guangzhou), the Guangdong Provincial Quantum Science Strategic Initiative (No. GDZX2303007), the National Key R&D Program of China (No. 2024YFE0102500), and the Education Bureau of Guangzhou Municipality.

---

\* Chengkai Zhu and Zhiping Liu contributed equally to this work.

† felixxinwang@hkust-gz.edu.cn

- [1] P. W. Shor, *SIAM Journal on Computing* **26**, 1484 (1997).
- [2] L. K. Grover, “A fast quantum mechanical algorithm for database search,” (1996), [arXiv:quant-ph/9605043](https://arxiv.org/abs/quant-ph/9605043) [quant-ph].
- [3] A. M. Childs and W. van Dam, *Rev. Mod. Phys.* **82**, 1 (2010).
- [4] S. Lloyd, *Science* **273**, 1073 (1996).
- [5] A. M. Childs, D. Maslov, Y. Nam, N. J. Ross, and Y. Su, *Proceedings of the National Academy of Sciences* **115**, 9456 (2018).
- [6] P. W. Shor, “Fault-tolerant quantum computation,” (1997), [arXiv:quant-ph/9605011](https://arxiv.org/abs/quant-ph/9605011) [quant-ph].
- [7] E. T. Campbell, B. M. Terhal, and C. Vuillot, *Nature* **549**, 172 (2017).
- [8] E. Knill, *Nature* **434**, 39 (2005).
- [9] D. Gottesman, *Stabilizer codes and quantum error correction* (California Institute of Technology, 1997).
- [10] V. Veitch, S. A. Hamed Mousavian, D. Gottesman, and J. Emerson, *New Journal of Physics* **16**, 1 (2014).
- [11] X. Zhou, D. W. Leung, and I. L. Chuang, *Physical Review A* **62** (2000), [10.1103/physreva.62.052316](https://doi.org/10.1103/physreva.62.052316).
- [12] D. Gottesman and I. L. Chuang, *Nature* **402**, 390 (1999).
- [13] S. Bravyi and A. Kitaev, *Physical Review A* **71** (2005), [10.1103/physreva.71.022316](https://doi.org/10.1103/physreva.71.022316).
- [14] E. T. Campbell, H. Anwar, and D. E. Browne, *Physical Review X* **2** (2012).
- [15] M. Yoganathan, R. Jozsa, and S. Strelchuk, *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* **475**, 20180427 (2019).
- [16] M. A. Nielsen and I. L. Chuang, *Quantum computation and quantum information* (Cambridge university press, 2010).
- [17] A. Mari and J. Eisert, *Physical Review Letters* **109**, 1 (2012).
- [18] C. H. Bennett, D. P. DiVincenzo, C. A. Fuchs, T. Mor, E. Rains, P. W. Shor, J. A. Smolin, and W. K. Wootters, *Physical Review A* **59**, 1070 (1999).
- [19] D. Leung, A. Winter, and N. Yu, *Reviews in Mathematical Physics* **33**, 2150013 (2021).
- [20] S. Bandyopadhyay, A. Cosentino, N. Johnston, V. Russo, J. Watrous, and N. Yu, *IEEE Transactions on Information Theory* **61**, 3593 (2015).
- [21] A. M. Childs, D. Leung, L. Mančinska, and M. Ozols, *Communications in Mathematical Physics* **323**, 1121 (2013).
- [22] S. Bandyopadhyay, *Physical Review Letters* **106**, 1 (2011).
- [23] J. Calsamiglia, J. I. De Vicente, R. Muñoz-Tapia, and E. Bagan, *Physical Review Letters* **105**, 1 (2010).
- [24] S. Halder, M. Banik, S. Agrawal, and S. Bandyopadhyay, *Physical Review Letters* **122**, 40403 (2019).
- [25] J. Walgate, A. J. Short, L. Hardy, and V. Vedral, *Physical Review Letters* **85**, 4972 (2000).
- [26] C. H. Bennett, D. P. DiVincenzo, T. Mor, P. W. Shor, J. A. Smolin, and B. M. Terhal, *Physical Review Letters* **82**, 5385 (1999).
- [27] J. Bae and L.-C. Kwek, *Journal of Physics A: Mathematical and Theoretical* **48**, 083001 (2015).
- [28] W. H. G. Correa, L. Lami, and C. Palazuelos, *IEEE Transactions on Information Theory* **68**, 7306 (2022).
- [29] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, *Reviews of Modern Physics* **74**, 145 (2002).
- [30] R. Cleve, D. Gottesman, and H.-K. Lo, *Physical Review Letters* **83**, 648 (1999).
- [31] A. Leverrier and P. Grangier, *Physical Review Letters* **102** (2009).
- [32] N. Brunner, M. Navascués, and T. Vértesi, *Physical Review Letters* **110** (2013).
- [33] M. Hendrych, R. Gallego, M. Mičuda, N. Brunner, A. Acín, and J. P. Torres, *Nature Physics* **8**, 588 (2012).
- [34] B. M. Terhal, D. P. DiVincenzo, and D. W. Leung, *Phys. Rev. Lett.* **86**, 5807 (2001).
- [35] T. Eggeling and R. F. Werner, *Phys. Rev. Lett.* **89**, 097905 (2002).
- [36] W. Matthews, S. Wehner, and A. Winter, *Communications in Mathematical Physics* **291**, 813 (2009).
- [37] V. Veitch, C. Ferrie, D. Gross, and J. Emerson, *New Journal of Physics* **14**, 1 (2012).
- [38] N. Yu, R. Duan, and M. Ying, *IEEE Transactions on Information Theory* **60**, 2069 (2014).
- [39] W. K. Wootters, *Annals of Physics* **176**, 1 (1987).
- [40] D. Gross, *Journal of Mathematical Physics* **47** (2006).
- [41] D. Gross, *Applied Physics B* **86**, 367 (2006).
- [42] Y. Wang, Z. Hu, B. C. Sanders, and S. Kais, *Frontiers in Physics* **8** (2020).
- [43] E. F. Galvão, *Physical Review A* **71** (2005).
- [44] C. Cormick, E. F. Galvão, D. Gottesman, J. P. Paz, and A. O. Pittenger, *Physical Review A* **73** (2006).
- [45] M. Howard, J. Wallman, V. Veitch, and J. Emerson, *Nature* **510**, 351 (2014).
- [46] H. Zhu, *Phys. Rev. Lett.* **116**, 040501 (2016).
- [47] E. T. Campbell and D. E. Browne, *Physical Review Letters* **104** (2010).
- [48] M. Horodecki, P. Horodecki, and R. Horodecki, *Phys. Rev.*

- Lett. **80**, 5239 (1998).
- [49] H.-C. Cheng, A. Winter, and N. Yu, *Communications in Mathematical Physics*, **1** (2023).
  - [50] K. M. R. Audenaert, M. Nussbaum, A. Szkoła, and F. Verstraete, *Communications in Mathematical Physics* **279**, 251 (2008).
  - [51] T. Ogawa and H. Nagaoka, in *Asymptotic Theory of Quantum Statistical Inference* (WORLD SCIENTIFIC, 2005) pp. 28–42.
  - [52] M. Mosonyi and T. Ogawa, *Communications in Mathematical Physics* **334**, 1617 (2014).
  - [53] E. Chitambar and G. Gour, *Reviews of Modern Physics* **91** (2019).
  - [54] Y. Li, X. Wang, and R. Duan, *Physical Review A* **95**, 052346 (2017).
  - [55] R. Takagi and B. Regula, *Physical Review X* **9** (2019), 10.1103/physrevx.9.031053.
  - [56] M. Oszmaniec and T. Biswas, *Quantum* **3**, 133 (2019).
  - [57] S. P. Boyd and L. Vandenberghe, *Convex optimization* (Cambridge university press, 2004).
  - [58] K. M. R. Audenaert, J. Calsamiglia, R. Muñoz-Tapia, E. Bagan, L. Masanes, A. Acín, and F. Verstraete, *Physical Review Letters* **98** (2007).
  - [59] F. Hiai and D. Petz, *Communications in mathematical physics* **143**, 99 (1991).
  - [60] D. DiVincenzo, D. Leung, and B. Terhal, *IEEE Transactions on Information Theory* **48**, 580 (2002).
  - [61] L. Lami, C. Palazuelos, and A. Winter, *Communications in Mathematical Physics* **361**, 661 (2018).
  - [62] X. Wang, M. M. Wilde, and Y. Su, *Physical Review Letters* **124**, 090505 (2018).
  - [63] S. Pirandola, R. Laurenza, C. Lupo, and J. L. Pereira, *npj Quantum Information* **5** (2019).
  - [64] X. Wang and M. M. Wilde, *Physical Review Research* **1** (2019).
  - [65] R. Takagi, B. Regula, K. Bu, Z.-W. Liu, and G. Adesso, *Physical Review Letters* **122** (2019).
  - [66] R. Uola, T. Bullock, T. Kraft, J.-P. Pellonpää, and N. Brunner, *Physical Review Letters* **125** (2020).
  - [67] A. F. Ducuara and P. Skrzypczyk, *Physical Review Letters* **125** (2020).
  - [68] P. Lipka-Bartosik, A. F. Ducuara, T. Purves, and P. Skrzypczyk, *PRX Quantum* **2** (2021).
  - [69] M. Howard and E. Campbell, *Physical Review Letters* **118** (2017).
  - [70] S. Bravyi, G. Smith, and J. A. Smolin, *Physical Review X* **6**, 1 (2016).
  - [71] J. R. Seddon and E. T. Campbell, *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* **475**, 20190251 (2019).
  - [72] O. Hahn, A. Ferraro, L. Hultquist, G. Ferrini, and L. García-Álvarez, *Physical Review Letters* **128** (2022).
  - [73] M. Zurel, C. Okay, and R. Raussendorf, *Physical Review Letters* **125** (2020), 10.1103/physrevlett.125.260404.
  - [74] R. Raussendorf, J. Bermejo-Vega, E. Tyhurst, C. Okay, and M. Zurel, *Physical Review A* **101** (2020), 10.1103/physreva.101.012350.