Report on AH11863

The authors have implemented several changes in their manuscript following my previous comments, which significantly improve the presentation. In particular, the aims of the paper are more clearly stated:
(A)- “we present a characterization of the structure of all implementations of the PR-box in the framework of GPTs”
(B)- “In any GPT, the pairs of measurements appearing in such implementations must be maximally incompatible and we show how the corresponding bipartite states are constructed from such pairs.”
(C)- In qubit channels: “we give a full description of all possible pairs of maximally incompatible two-outcome channel measurements and of all qubit PR-channels. In particular, we prove that all these channels are necessarily measure-and-prepare”

Also, the examples discussed throughout the sections are quite useful for the reader. Although I believe that the manuscript has improved, I still think that further changes need to be implemented, and some crucial questions need to be addressed – see my comments below.

Firstly, I don't see a definition or discussion on what a measure-and-prepare channel is – I suggest the authors to include a brief definition that explains the reader what types of measurements and of preparations are possible for such channels (e.g., local ones vs. joint ones).

Secondly, the paper is still quite dense, and the main results are not easily identifiable in the text. I suggest the authors to point in the introduction the particular places where the results are presented:
- (A) → Proposition 2,
- (B) → Proposition 2,
- (C) → Proposition 6.
These propositions might even be promoted to Theorems, since they capture the contribution of this paper.

Thirdly, I still struggle to fully capture what's new here – see my comments below about Sections IV and V: it looks like the authors there are framing known results in their notation. Is this the case? If so, sections IV and V don't really add to the paper, since they are independent of the tools required to prove the main result of Section VI, so I suggest the authors to remove them entirely, or to move them to an appendix on “interesting known examples”.

Finally, I'm concerned that the current presentation of the manuscript will prevent it from making an impact on the quantum information and quantum foundations community. More precisely, the authors seem keen to keep a presentation that is heavy in the mathematics and which does not discuss the conceptual aspects well enough. Take for instance the following examples: the authors have a very precise mathematical description of GPT notation in Section IIA, but they fail to present basic fundamental concepts in the Introduction (e.g., they seem to confuse the notion of a non-signalling theory, with that of a physical system, and with that of a box/statistical-correlations) -- see comments below. Therefore, I believe that more substantial editing needs to happen before the manuscript meets the quality requirements of Phys. Rev. A.

In the following provide a thorough list of the issues that need to be solved before considering publication:

1) Page 1, column 1: “Popescu and Rohrlich proved that the CHSH inequality may be violated even more by a no-signaling theory [5, 6]”
   → The first time that a PR box was placed within a no-signaling GPT was in the paper by

Barrett [Phys. Rev. A 75, 032304, 2007] which introduces the now called Boxworld GPT. Before then, a PR box was though of as a box whose correlations are compatible with the No Signalling principle, but who is not actually required to be feasible within a physical theory or GPT. I suggest the authors to rephrase the claim to make it more accurate.

2) Page 1, column 1: “The no-signaling theories that maximally violate CHSH inequality are called Popescu-Rohrlich boxes, or PR-boxes, and it was shown that the existence of a such systems in real world would have several rather interesting implications”
   → In this sentence the authors refer to PR boxes as both a “system” and a “non-signalling theory”. PR boxes are not a theory – they are a particular stochastic map from the set of input variables (x,y) to the set of outcome variables (a,b), characterised by the conditional probability distribution p(ab|xy) = \delta_{a+b = xy}/2. PR boxes, though, are correlations that can arise within a non-signalling GPT as shown in Ref. [Phys. Rev. A 75, 032304, 2007]. In addition, PR-boxes are not really “systems”; however one may think of the corresponding GPT state in Boxworld as the “system” associated to a PR box. I suggest the authors to revisit the claim (and their notation throughout the paper) to make it correct and clear.
3) Page 1, column 2: “From a realistic view-point it is important to remember that using a non-local channel may take some time and resources, but it is considered communication only if the channel is signaling.”
   → what does “it” refer to in the last part of the statement? What is the thing considered as communication? Do they mean that “a channel is considered to permit communication only if the channel is signalling”?
4) Page 1, column 2: “[...], we will call these channels the PR-channels.”
   → The full sentence is not grammatically correct. I suggest the authors to put a full stop before “we”.
5) Page 2, Column 1: “General probabilistic theories provide a framework that describes measurements of a physical systems in a general and mathematically clear way.”
   → The sentence is not technically incorrect, but GPT are much more than just that. I suggest the authors to rephrase the sentence as follows: “General probabilistic theories provide a framework that describes, amongst other things, measurements of a physical systems in a general and mathematically clear way.”
6) Beginning of section IIA: I understand that the authors want to focus on setting up the jargon/notation for their results. However, the average reader of PRA that comes from quantum information/foundations is not necessarily already in the GPT mind-frame or invested enough at this stage to tackle a 39-pages review on GPTs [16]. I suggest again the authors to include at the beginning of Section IIA, or right before it, a brief introduction about GPTs, to put all readers on the same page. What I'm thinking about is something at the level of the “Introduction to GPTs” section of [Phys. Rev. A 97, 042302 (2018)].
7) Page 2, column 2: “Although in general the measurements in the theory may be restricted, it this work we will assume that any effect [..]”
   → The assumption that the authors make in this work is called “No Restriction Hypothesis” in the literature. The authors should mention it by name, and cite [Phys. Rev. A 81, 062348 (2010)] for it.
8) Page 2, column 2 – Example 2: “[...] denote the set of states on [...]”
   → I believe the authors mean the set of normalized states.
9) Page 2, column 2 – Example 3: “This state space is also called the box world or a gbit [20] [...]”.
   → Boxworld is not the name of a state space, is the name of a GPT that has systems whose state spaces are squares. There might be instances in the literature where the name has been used wrongly, so I suggest the authors to not follow that trend. Indeed, there are GPTs that have square state spaces but which do not correspond to Boxworld – one simple example is to consider the GPT these spaces generate when the composition rule is no longer the max

tensor product.

10) Page 3, column 1, close to the top: “[...] , moreover, we can see that [...]”
→ Punctuation: I suggest to use full stop before “moreover” instead.

11) Page 3, end of column 1: “[...] we say that they are compatible if there is a four-outcome measurement [...]”
→ I suggest the authors to include a citation to the paper who introduced the notion of compatible measurements in GPTs.

12) Page 3, column 2: “This definition of compatibility of measurements generalizes the well-known notion of compatibility of POVMs in quantum theory”
→ I was confused the first time I read this sentence: what do they mean by generalisation? Is this a more general way to test for compatibility in quantum theory? I believe that the authors don't mean that, and that instead they mean “This definition of compatibility of measurements generalizes to GPTs the well-known notion of compatibility of POVMs in quantum theory”. If my understanding is correct, I suggest the authors to implement a similar rephrasing.

13) Page 4, column 2: “The joint state space, also called real tensor product in [26], denoted by $K \otimes\tilde{} K$, is the set of all bipartite states of a given theory.”
→ I believe that with this sentence (and following ones in the text) what the authors mean is the following: “In this manuscript we will denote the state space of a composite (bipartite) system as $K \otimes\tilde{} K$ (hereafter joint state space). The symbol $\otimes\tilde{}$ denotes the particular composition rule that the GPT comes equipped with, and will not be assigned any particular mathematical definition. The only necessary properties that $\otimes\tilde{}$ satisfies is that $K \otimes\tilde{} K$ is a state space and that $K \otimes\dot{} K \subseteq K \otimes\tilde{} K \subseteq K \otimes\hat{} K$.”.
If this is the case, I suggest the authors to implement a similar rephrasing, and remove the mention of real tensor products. The concept of “real tensor products” is not standard terminology in the community (it has only been used by these authors in a different paper); this terminology is confusing, and its introduction is not necessary for the paper (indeed, it's not used anywhere else in the manuscript). Also, Ref. [26] is only included in this manuscript as an example of this non-standard terminology, and is not referenced for any other purpose; hence I suggest the removal of the self-citation Ref. [26].

14) Title of section IID: “Review of CHSH inequality” → “Review of the CHSH inequality”

15) Page 4, column 2, bottom: “Any non-local box x is fully described by the outcome probabilities P_x(C=ε,D=η) [...]”
→ This formula is confusing: the authors use the same symbol (e.g., **A**) to denote both the choice of measurement and the corresponding outcome – namely, how can **A** be both an effect and a classical variable with value ε? Modern notation solves this issue and reads P_x(ε,η|C,D). I suggest the authors to change notation or to add a clarifying comment.

16) Page 5, column 1, after Eq. (9): “Any non-local box must satisfy these two conditions.”
→ There are nonlocal boxes that do not satisfy non-signalling; the pioneering work by Popescu and Rohrlich indeed asked what nonlocal boxes comply with the non-signalling condition, their intuition being it would single out quantum boxes (they disproved themselves). Also, the non-local boxes that one can realise experimentally in the lab are never non-signalling. The authors could correct the sentence by saying instead something along the lines of “In Bell experiments, it is of particular interest to focus on non-local boxes that satisfy these two conditions, i.e., those which comply with the No Signalling principle.” Also, I suggest the authors to include a citation to this No Signalling principle.

17) Page 5, column 1, next paragraph: “[..] implement a non-local box, which means precisely that the corresponding outcome probabilities satisfy the no-signaling conditions.”
→ In view of my previous comment, this claim is not correct. I'd suggest a rephrasing along the lines of: “[..] implement a non-local box whose corresponding outcome probabilities satisfy the no-signaling conditions.”.

18) Page 5, column 1, bottom: “[...] the maximal value reachable by a no-signaling theory is

X_CHSH = 4”
→What Popescu and Rohrlich showed is that the maximal value reachable by a non-signalling box is X_CHSH = 4. Popescu and Rohrlich never showed that such a conditional probability distribution could arise within a physical theory or GPT by performing space-like separeted measurements on a bipartite system. The authors should fix the sentence accordingly.

19) Page 5, column 2, second paragraph: “Indeed, by [20], the non- local boxes can be identified with elements of [..]”
→ This discussion applies to the particular GPT known as Boxworld, which may not be clear from the sentence. I suggest the authors to introduce the discussion with something like “Indeed, Ref. [20] defined the GPT called GNST (most commonly known as Box world) where the non-signalling boxes can be identified with elements of [...]”
20) Same paragraph: “The state corresponding to the PR-box is given by”
→ “In the GPT Boxworld, the state corresponding to the PR-box is given by”
21) Page 5, before Prop. 2: “as we have seen in Prop. 1, all maximally incompatible pairs are obtained from the square state space.”
→Do the authors mean “all maximally incompatible pairs are featured in GPTs whose state space is a square state space.”? If so, I suggest to make the statement clear.
22) Section IV: The novelty of the results introduced in this section is not clear. The authors give an example of a GPT whose state space is given by the classical channels, and that when Alice and Bob perform the measurements F_s,f (defined in that section) on a state (i.e., a channel) the input-output stochastic maps corresponds to a PR box.
However, according to the introduction:
- It was already known [11-14] that nonsignalling classical channels can produce PR boxes,
- These can be leveraged to construct a channel-based GPT that realises the PR box [13].
Therefore, is Section IV recasting known results in the language of the current manuscript? If this is indeed the case, the authors should make this clear in a new paragraph at the beginning of the section. If this is not the case, then could the authors please point me at the new result they introduce here?
23) Figure 1: the Figure shows how Alice and Bob can use a channel to generate correlations. However, what I had in mind in my previous review was a Figure that had a channel as a state, and where Alice and Bob could perform measurements on (like, the GPT state and effect). Maybe one could even add a diagram that tells how a channel can be seen as a state. I suggest that the authors add such a Figure to make things clearer. See the diagrams I include at the end of this report for reference.
24) Section V, the first part (up to where PPVMs are mentioned) – is this something new that the authors develop, or are they reviewing a framework that already exists? The authors should make this clear.
25) Section V: the maximally incompatible measurements that are constructed seem to also have been presented in Ref. [24], according to the first paragraph of Column 2 in Page 7.
26) Section V: similarly to the case of Classical Channels of Section IV (and my comment #25), The novelty of the results introduced in this section is not clear. Given the results of:
- Refs. [11-14]: nonsignalling channels can produce PR boxes,
- Ref. [13]: one can construct a channel-based GPT that realises the PR box,
- The form of the (maximally incompatible) measurements was developed in Ref. [24],
→ what is the result being found in this section?
27) Section VI: I suggest the authors to begin the section with a paragraph stating what they will do here; e.g., “In this section we will characterise the structure of the qubit quantum channels that one can generate PR-box correlations from.”
28) Proposition 6 seems to be one of the main results of the paper → I suggest to upgrade it to a theorem, or something that would highlight its relevance in the manuscript.
29) A main result of the paper is Prop. 6 in Section IV. What role do the sections IV and V play

in building up the setting for it? As it stands it looks like Sections IV and V are disconnected somehow from Section VI. The place where I see Section V being mentioned is in Example 6, however the reference is not necessary since the conclusions can be drawn from the results of Refs. [11-13] (like the authors mention in the text).

30) Conclusion, first sentence ends in “given that a certain state belongs to the joint state space”. This statement is too vague, what do the authors want to convey through this sentence?