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Extension of Jones' Theory on Index to Arbitrary Factors

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V. Jones' theory on index of (II_1-) subfactors is generalized to arbitrary factors. The main technical tools are Connes' spatial theory and Haagerup's theory on operator-valued weights. © 1986 Academic Press, Inc

0. INTRODUCTION

In [3], V. Jones developed the striking theory on index of II_1- (sub) factors. The purpose of the present article is to show that based on Connes' spatial theory [1] and Haagerup's theory on operator-valued weights [2], one can carry out similar analysis for arbitrary factors.

More precisely, let M be an arbitrary factor with a subfactor N. We assume the existence of a normal conditional expectation $E: M \to N$. (Otherwise, N is "very small" in M so that the "index $[M:N]$" should be $+\infty$.) For this E, Index E will be defined. When M and N are II_1- factors, the index of the canonical conditional expectation determined by the unique normalized trace on M is exactly Jones' index $[M:N]$ [3] based on the coupling constant.

For our Index E, the analysis in [3] is valid so that among other things we have

$$\text{Index } E \in \left\{4\cos^2\frac{\pi}{n};\, n = 3, 4,...\right\} \cup [4, \infty].$$

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F. Goodman, and V. Jones for fruitful discussions. Also he is indebted to Professor A. Kishimoto for pointing out an error in the previous version of the article. As the reader might note, the present work is an abstract theory. Further results as well as examples will be published elsewhere.

1. Preliminaries

Here we collect basic facts on Connes' spatial theory [1] and Haagerup's theory on operator-valued weights [2].

1.1. *Spatial theory of von Neumann Algebras*

Let M be a von Neumann algebra on a Hilbert space $\mathfrak{h}$ and ψ be a normal faithful semifinite (n.f.s.) weight on the commutant. We use the following standard notations:

$$\mathfrak{n}_\psi = \{x \in M';\ \psi(x^*x) < +\infty\},$$

$\mathfrak{h}_\psi$ = the Hilbert space completion of $\mathfrak{n}_\psi$ with respect to $x \to \psi(x^*x)^{1/2}$,

Λ_ψ = the canonical injection of $\mathfrak{n}_\psi$ into $\mathfrak{h}_\psi$,

π_ψ = the regular representation of M' on $\mathfrak{h}_\psi$ $(\pi_\psi(x)\Lambda_\psi(y) = \Lambda_\psi(x, y),\ x \in M',\ y \in \mathfrak{n}_\psi)$.

For a vector ξ in $\mathfrak{h}$, we define the operator from $\mathfrak{h}_\psi$ to $\mathfrak{h}$ by

$$\mathfrak{D}(R^\psi(\xi)) = \Lambda_\psi(\mathfrak{n}_\psi),$$

$$R^\psi(\xi)\Lambda_\psi(x) = x\xi, \qquad x \in \mathfrak{n}_\psi.$$

The vector ξ is said to be ψ-bounded if $R^\psi(\xi)$ is bounded. In this case, the extended bounded linear operator from $\mathfrak{h}_\psi$ to $\mathfrak{h}$ is still denoted by $R^\psi(\xi)$, and we set

$$\theta^\psi(\xi, \xi) = R^\psi(\xi)R^\psi(\xi)^*$$

(which is a bounded operator on $\mathfrak{h}$). The set $D(\mathfrak{h}; \psi)$ of all ψ-bounded vectors in $\mathfrak{h}$ forms a dense subspace. It is easily checked that

$$xR^\psi(\xi) \subseteq R^\psi(\xi)\pi_\psi(x), \qquad \xi \in \mathfrak{h},\ x \in M'.$$

It follows that $\theta^\psi(\xi, \xi)$, $\xi \in D(\mathfrak{h}; \psi)$, belongs to M_+. Even if $\xi \in \mathfrak{h}$ is not in $D(\mathfrak{h}, \psi)$, $\theta^\psi(\xi, \xi)$ still makes sense as an element in $\hat{M}_+$, the extended positive part of M [2]. (Details can be found on p. 66 of [6].) We also remark that

$$\theta^{\psi}(x\xi, x\xi) = x\theta^{\psi}(\xi, \xi)x^*, \qquad x \in M.$$

(When ξ is not in $D(\mathfrak{h}; \psi)$, the right side should be understood as a form product.) There always exists a family $\{\xi_\alpha\}_{\alpha \in A}$ in $D(\mathfrak{h}; \psi)$ such that

$$1 = \sum_{\alpha \in A} \theta^{\psi}(\xi_\alpha, \xi_\alpha).$$

(The convergence is in the strong operator topology.)

Let ϕ be a n.f.s. weight on M (extended to $\hat{M}_+$). We define $q_\phi : \mathfrak{h} \to [0, \infty]$ by

$$q_\phi(\xi) = \phi(\theta^{\psi}(\xi, \xi)).$$

It is a lower semicontinuous (hence, closable) quadratic form (on $D(q_\phi)$). Thus, by Friedrichs' theorem, there exists a unique positive self adjoint operator $d\phi/d\psi$ on $\mathfrak{h}$ such that

$$\begin{aligned} \overline{q_\phi}(\xi) &= \|(d\phi/d\psi)^{1/2}\xi\|^2 \in [0, \infty] \\ &= (d\phi/d\psi\ \xi \mid \xi) \qquad \text{as a form.} \end{aligned}$$

This $d\phi/d\psi$ is referred to as the spatial derivative of ϕ relative to ψ. Among other interesting properties, this satisfies

$$\begin{gathered} (d\phi/d\psi)^{-1} = d\psi/d\phi, \\ (d\phi/d\psi)^{it}x(d\phi/d\psi)^{-it} = \sigma_t^{\phi}(x), \qquad x \in M, \end{gathered}$$

where σ_t^ϕ denotes the modular automorphism group on M induced by ϕ.

1.2. *Operator-Valued Weights*

Let N be a von Neumann subalgebra in M. An operator-valued weight $E: M \to N$ is a map $M_+ \to \hat{N}_+$ (which can be uniquely extended to the map $\hat{M}_+ \to \hat{N}_+$) such that

(a) E is additive,

(b) $E(axa^*) = aE(x)a^*$, $x \in M_+$, $a \in N$.

In the obvious way, normality, faithfulness, and semifiniteness of an operator-valued weight can be defined. In this article, operator-valued weights are always assumed to be n.f.s. The set of all (n.f.s.) operator-valued weights from M to N is denoted by $P(M, N)$. Haagerup showed that

$$P(M, N) \neq \varnothing \Leftrightarrow P(N', M') \neq \varnothing$$

and constructed an order-reversing bijection between $P(M, N)$ and $P(N', M')$. In his approach, it is not clear if the above bijection can be constructed in a canonical way. However, based on the spatial theory (1.1), it is possible to construct the canonical order-reversing bijection from $P(M, N)$ onto $P(N', M')$ (denoted by $E \to E^{-1}$). For a given $E \in P(M, N)$, the canonical $E^{-1} \in P(N', M')$ is characterized by

$$d(\phi \circ E)/d\psi = d\phi/d(\psi \circ E^{-1}).$$

Here, ϕ and ψ are n.f.s. weights on N and M', respectively. (And E^{-1} depends only on E.) In particular, we have

$$(E^{-1})^{-1} = E,$$
$$(F \circ E)^{-1} = E^{-1} \circ F^{-1},$$

where F is an operator-valued weight from N to its subalgebra.

Let ω be a n.f.s. weight on M. In the obvious way, it can be considered as an operator-valued weight from M to $\mathbb{C}1$. Since $\{\mathbb{C}1\}' = B(\mathfrak{h})$, we get $\omega^{-1} \in P(B(\mathfrak{h}), M')$. For a rank-one operator $\xi \otimes \xi$, $\xi \in \mathfrak{h}((\xi \otimes \xi)\zeta = (\zeta \mid \xi)\xi, \zeta \in \mathfrak{h})$, we have

$$\omega^{-1}(\xi \otimes \xi) = \theta^{\omega}(\xi, \xi) \in \hat{M}'_{+}.$$

Let J be a unitary involution on $\mathfrak{h}$ and $E \in P(M, N)$. Then $\tilde{E} \in P(JMJ, JNJ)$ is defined by

$$\begin{aligned} \tilde{E}(x) &= JE(Jx^*J)^*J \\ &= JE(JxJ)J, \qquad x \in JM_{+}J = (JMJ)_{+}. \end{aligned}$$

Since $(JMJ)' = JM'J \subseteq (JNJ)' = JN'J$, $\tilde{E}$ determines $(\tilde{E})^{-1} \in P(JN'J, JM'J)$. On the other hand, $E^{-1} \in P(N', M')$ induces $(E^{-1})^{\sim} \in P(JN'J, JM'J)$ by

$$\begin{aligned} (E^{-1})^{\sim}(x) &= JE^{-1}(Jx^*J)^*J \\ &= JE^{-1}(JxJ)J, \qquad x \in JN'_{+}J. \end{aligned}$$

LEMMA 1.3. *We have* $(\tilde{E})^{-1} = (E^{-1})^{\sim}$.

Proof. Let ψ, χ be n.f.s. weights on M and M', respectively. Define n.f.s. weights $\tilde{\psi}, \tilde{\chi}$ on JMJ and $JM'J$, respectively, by

$$\tilde{\psi}(x) = \psi(JxJ), \qquad x \in JM_{+}J,$$
$$\tilde{\chi}(x) = \chi(JxJ), \qquad x \in JM'_{+}J.$$

Since

$$\tilde{\chi}((JxJ)^*(JxJ)) = \tilde{\chi}(Jx^*xJ) = \chi(x^*x),$$

the map $x \in \mathfrak{n}_\chi \to JxJ \in \mathfrak{n}_{\tilde{\chi}}$ gives rise to the (antilinear) isometry I from $\mathfrak{h}_\chi$ onto $\mathfrak{h}_{\tilde{\chi}}$. For each $\xi \in \mathfrak{h}_\chi$ and $x \in M'$, we have

$$R^{\tilde{\chi}}(J\xi)\Lambda_{\tilde{\chi}}(JxJ) = Jx\xi,$$

which means that

$$\begin{aligned} R^{\tilde{\chi}}(J\xi)I &= JR^\chi(\xi),\\ \theta^{\tilde{\chi}}(J\xi, J\xi) &= R^{\tilde{\chi}}(J\xi)II^*R^{\tilde{\chi}}(J\xi)^*\\ &= JR^\chi(\xi)R^\chi(\xi)^*J = J\theta^\chi(\xi,\xi)J. \end{aligned}$$

We thus get

$$\begin{aligned} \tilde{\psi}(\theta^{\tilde{\chi}}(J\xi, J\xi)) &= \psi(JJ\theta^\chi(\xi,\xi)JJ)\\ &= \psi(\theta^\chi(\xi,\xi)). \end{aligned}$$

By the definition of a spatial derivative, we get

$$d\tilde{\psi}/d\tilde{\chi} = J\, d\psi/d\chi\, J. \tag{1}$$

Let ϕ be a n.f.s. weight on N so that $\phi \circ E$ is a n.f.s. weight on M. As before we define n.f.s. weights $\tilde{\phi}$, $(\phi \circ E)^\sim$ on JNJ and JMJ, respectively. Also we get the n.f.s. weight $(\chi \circ E^{-1})^\sim$ on $JN'J$. We obviously have

$$\begin{aligned} (\phi \circ E)^\sim &= \tilde{\phi} \circ \tilde{E},\\ (\chi \circ E^{-1})^\sim &= \tilde{\chi} \circ (E^{-1})^\sim. \end{aligned}$$

Apply J from the left and the right to

$$d(\phi \circ E)/d\chi = d\phi/d(\chi \circ E^{-1}).$$

Then (1) (applied for M, M' and N, N') implies

$$d(\phi \circ E)^\sim/d\tilde{\chi} = d\tilde{\phi}/d(\chi \circ E^{-1})^\sim.$$

The left side is equal to

$$d(\tilde{\phi} \circ \tilde{E})/d\tilde{\chi} = d\tilde{\phi}/d(\tilde{\chi} \circ (\tilde{E})^{-1})$$

while the right side is equal to

$$d\tilde{\phi}/d(\tilde{\chi} \circ (E^{-1})^\sim).$$

We thus get

$$d\tilde{\phi}/d(\tilde{\chi}\circ(\tilde{E})^{-1}) = d\tilde{\phi}/d(\tilde{\chi}\circ(E^{-1})^{\sim}),$$

that is, $(\tilde{E})^{-1} = (E^{-1})^{\sim}$. Q.E.D.

2. Definition of Index E

From now on, let M be a (σ-finite) factor on a Hilbert space $\mathfrak{h}$ with a subfactor N. We also fix a normal conditional expectation E from M onto N. In this section, we will define Index E (which is a priori a scalar in $[1, \infty]$).

Since E is an operator-valued waight, as in 1.2 we get $E^{-1} \in P(N', M')$. For any unitary u in M', we have

$$uE^{-1}(1)u^* = E^{-1}(u1u^*) = E^{-1}(1).$$

Since M is a factor, this means that $E^{-1}(1)$ is a scalar (possibly $+\infty$).

Definition 2.1. Index E is the scalar $E^{-1}(1)$.

Here, the sizes of M', N', and E^{-1} depend on the action of M on a given Hilbert space. However, thanks to the next result, the above definition is legitimate.

Theorem 2.2. *The number $E^{-1}(1)$ does not depend on $\mathfrak{h}$.*

Proof. An algebraic isomorphism is a composition of an induction, an amplification, and a spatial isomorphism.

Let p be a nonzero projection in M'. (Since M is a factor, the central support of p is 1) and $x \in M \to xp \in Mp$ is isomorphic. Also, $x \in N \to xp \in Np$ is isomorphic. Here, $Np \subsetneqq Mp$ act on $p\mathfrak{h}$. We fix a normal faithful state ϕ on N. The state $\tilde{\phi}$ on Np, the conditional expectation $\tilde{E}$: $Mp \to Np$ are defined in the obvious way by using the above-mentioned isomorphisms. Also the state $\phi\circ E$ on M induces $(\phi\circ E)^{\sim}$ and $(\phi\circ E)^{\sim} = \tilde{\phi}\circ\tilde{E}$.

For each $\xi \in \mathfrak{h}$ and $x \in N$ we get

$$R^{\tilde{\phi}}(p\xi)\Lambda_{\tilde{\phi}}(xp) = xp\xi = px\xi.$$

The map $\Lambda_{\tilde{\phi}}(xp) \to \Lambda_{\phi}(x)$ induces the isometry from $\mathfrak{h}_{\tilde{\phi}}$ onto $\mathfrak{h}_{\phi}$. Identifying $\mathfrak{h}_{\tilde{\phi}}$ with $\mathfrak{h}_{\phi}$, we have

$$\begin{aligned} &pD(\mathfrak{h}, \phi) \subseteqq D(p\mathfrak{h}, \tilde{\phi}), \\ &R^{\tilde{\phi}}(p\xi) = pR^{\phi}(\xi), \qquad \xi \in D(\mathfrak{h}, \phi). \end{aligned} \tag{2}$$

The same result is also valid for $\phi\circ E$ and $(\phi\circ E)^{\sim}$. We now fix a family $\{\xi_\alpha\}_{\alpha\in A}$ in $D(\mathfrak{h}, \phi)$ such that

$$1=\sum_{\alpha\in A}\theta^{\phi}(\xi_\alpha, \xi_\alpha).$$

In terms of this family, Index E is computed by

$$\begin{aligned}E^{-1}(1)&=\sum_{\alpha\in A}E^{-1}(\theta^{\phi}(\xi_\alpha, \xi_\alpha))\\&=\sum_{\alpha\in A}E^{-1}\circ\phi^{-1}(\xi_\alpha\otimes\xi_\alpha)\\&=\sum_{\alpha\in A}(\phi\circ E)^{-1}(\xi_\alpha\otimes\xi_\alpha)\\&=\sum_{\alpha\in A}\theta^{\phi\circ E}(\xi_\alpha, \xi_\alpha)\in[0, \infty].\end{aligned}$$

To compute Index $\tilde{E}$, we compare $\tilde{E}^{-1}(p)$ with p. Since

$$p=\sum_{\alpha\in A}\theta^{\phi}(\xi_\alpha, \xi_\alpha)p=\sum_{\alpha\in A}\theta^{\tilde{\phi}}(p\xi_\alpha, p\xi_\alpha)$$

by (2), we get

$$\begin{aligned}\tilde{E}^{-1}(p)&=\sum_{\alpha\in A}\tilde{E}^{-1}(\theta^{\tilde{\phi}}(p\xi_\alpha, p\xi_\alpha))\\&=\sum_{\alpha\in A}\theta^{\tilde{\phi}\circ\tilde{E}}(p\xi_\alpha, p\xi_\alpha) && \text{(as the previous computation)}\\&=\sum_{\alpha\in A}\theta^{(\phi\circ E)^{\sim}}(p\xi_\alpha, p\xi_\alpha)\\&=\sum_{\alpha\in A}\theta^{\phi\circ E}(\xi_\alpha, \xi_\alpha)p && \text{(by (2) for } \phi\circ E \text{ and } (\phi\circ E)^{\sim}).\end{aligned}$$

Therefore, $\tilde{E}^{-1}(p)$ and $E^{-1}(1)$ are the same scalar.

To consider an amplification, let us assume that $\mathfrak{k}$ be a Hilbert space. Let $\tilde{E}$ be the expectation $E\otimes\mathrm{Id}_{\mathbb{C}1_{\mathfrak{k}}}$ from $M\otimes\mathbb{C}1_{\mathfrak{k}}\cong M$ onto $N\otimes\mathbb{C}1_{\mathfrak{k}}\cong N$. It is easy to check

$$\tilde{E}^{-1}=E^{-1}\otimes\mathrm{Id}_{B(\mathfrak{k})}.$$

Thus, $E^{-1}(1)$ and $\tilde{E}^{-1}(1_{\mathfrak{k}\otimes\mathfrak{k}})$ are the same scalar.

Finally, the number $E^{-1}(1)$ is not changed by a spatial isomorphism.

Q.E.D.

This result will also be used to verify the fact that the index is kept when one goes up to "an extenxion of M by E."

3. Properties

When Index $E < +\infty$, the operator-valued weight $E^{-1} \in P(N', M')$ is a scalar multiple of a conditional expectation. In fact, by setting $\tau = (E^{-1}(1))^{-1}$, we have

$$(\tau E^{-1})(x) = (\tau E^{-1})(x1) = x(\tau E^{-1})(1) = x, \qquad x \in M',$$

that is, τE^{-1} is a conditional expectation from N' onto M'.

Let $\phi \in N_*^+$ be a fixed normal faithful state. By representing M on $\mathfrak{h} = \mathfrak{h}_{\phi \circ E}$, we assume $\phi \circ E = \omega_{\xi_0}$ with a cyclic and separating vector ξ_0. Define e_N by

$$e_N(x\xi_0) = E(x)\xi_0, \qquad x \in M.$$

It extends to a (bounded) operator on $\mathfrak{h}$ (still denoted by e_N), which is a projection in N'. (It is possible to show that the projection e_N does not depend on a choice of $\phi \in N_*^+$.)

Lemma 3.1. $E^{-1}(e_N) = 1$. *In particular, Index* $E \geqq 1$, *and the equality holds if and only if* $M = N$.

Proof. Let $\mathfrak{k} = \overline{N\xi_0}$. Then N leaves $\mathfrak{k}$ invariant so that N acts on $\mathfrak{k}$ in the obvious way. This action is standard and $\phi = \omega_{\xi_0}$ (see [5] for details). Also e_N is exactly the orthogonal projection from $\mathfrak{h}$ onto $\mathfrak{k}$.

For each $\xi \in \mathfrak{h}$, we get

$$R^{\phi \circ E}(\xi): \quad x\xi_0 (= \Lambda_{\phi \circ E}(x)) \in M\xi_0 \to x\xi \in \mathfrak{h}.$$

In particular, we have $\theta^{\phi \circ E}(\xi_0, \xi_0) = 1 \ (= 1_{\mathfrak{h}})$. Also, for each $\xi \in \mathfrak{k}$, we get

$$R^{\phi}(\xi): \quad x\xi_0 (= \Lambda_{\phi}(x)) \in N\xi_0 \to x\xi \in \mathfrak{h}.$$

Thus, $R^{\phi}(\xi_0): \mathfrak{k} \to \mathfrak{h}$ is the inclusion map and $R^{\phi}(\xi_0)^* = e_N$, $\theta^{\phi}(\xi_0, \xi_0) = e_N$. Therefore, $(\phi \circ E)^{-1} = E^{-1} \circ \phi^{-1}$ implies

$$\begin{aligned} E^{-1}(e_N) &= E^{-1}(\theta^{\phi}(\xi_0, \xi_0)) \\ &= E^{-1} \circ \phi^{-1}(\xi_0 \otimes \xi_0) \\ &= (\phi \circ E)^{-1}(\xi_0 \otimes \xi_0) \\ &= \theta^{\phi \circ E}(\xi_0, \xi_0) = 1. \end{aligned}$$

We also get

$$\begin{aligned} \text{Index } E = 1 &\Leftrightarrow e_N = 1 \qquad (e_N \leqq 1 \text{ and } E^{-1} \text{ is faithful}) \\ &\Leftrightarrow \mathfrak{h} = \mathfrak{k} \\ &\Leftrightarrow M = N. \end{aligned}$$

Q.E.D.

In the rest of the section we will assume that M acts on $\mathfrak{h} = \mathfrak{h}_{\phi \circ E}$ (standardly). The identical arguments as [3, p. 8] show:

LEMMA 3.2. (i) $e_N x e_N = E(x) e_N$, $x \in M$.

(ii) *For* $x \in M$, $x \in N$ if and only if $e_N x = x e_N$.

(iii) $N' = \langle M', e_N \rangle$, the algebra generated by M' and e_N.

(iv) $J e_N J = e_N$, where J is determined by $\phi \circ E$ (see [5]).

(v) $\langle M, e_N \rangle = J N' J$.

(vi) Operators of the form $a_0 + \sum_{i=1}^n a_i e_N b_i$; $a_i\ b_i \in M'$ form a dense *-subalgebra of N'. (Actually, a_0 is not necessary; see [4].)

LEMMA 3.3. [4]. *When* Index $E < +\infty$, *for each* $x \in N'$, *there exists a unique* $\tilde{x} \in M'$ *such that* $x e_N = \tilde{x} e_N$.

Proof. The conditional expectation τE^{-1} at the beginning of the section is denoted by $E_{M'}$. Lemma 3.1 means that $E_{M'}(e_N) = \tau$.

The uniqueness follows from

$$\begin{aligned} \tau^{-1} E_{M'}(x e_N) &= \tau^{-1} E_{M'}(\tilde{x} e_N) \\ &= \tau^{-1} \tilde{x} E_{M'}(e_N) = \tilde{x}. \end{aligned}$$

To show the existence of $\tilde{x}$, at first we assume $x = a_0 + \sum_{i+1}^n a_i e_N b_i$; $a_i, b_i \in M'$. Then Lemma 3.2(i), (iv) imply

$$x e_N = a_0 e_N + \sum_{i=1}^n a_i J E(J b_i J) J e_N.$$

Therefore, $\tilde{x} = a_0 + \sum_{i=1}^n a_i J E(J b_i J) J$ does a job. For a generic x, choose a net $\{x_\iota\}$ of elements of the form described in Lemma 3.2(vi) such that $x_\iota \to x$ strongly. For each ι, we can find $\tilde{x}_\iota \in M'$ such that $x_\iota e_N = \tilde{x}_\iota e_N$. By the first part of the proof we actually have the explicit expression of $\tilde{x}_\iota$,

$$\tilde{x}_\iota = \tau^{-1} E_{M'}(x_\iota e_N).$$

By setting $\tilde{x} = \tau^{-1} E_{M'}(x e_N) \in M'$, we have $x e_N = \tilde{x} e_N$ by continuity. Q.E.D.

When Index $E = +\infty$, Lemma 3.3 fails.

COROLLARY 3.4. *When Index $E < +\infty$, for any projection p in N' there exists a sequence $\{a_i\}$ in M' such that*

$$p = \sum_{i=1}^{\infty} a_i e_N a_i^*.$$

Proof. Let $\{p_i\}_{i=1,2,\dots}$ be projections (or 0) in N' satisfying $p = \sum_{i=1}^{\infty} p_i$, $P_1 \lesssim e_N$, and $p_i \sim e_N$ or 0, $i = 2, 3, \dots$. Therefore,

$$p = \sum_{i=1}^{\infty} u_i e_N u_i^*,$$

where u_i, $i = 1, 2, \dots$, are partial isometries (or 0) in N'. The corollary follows by applying the previous lemma to $u_i \in N'$. Q.E.D.

If M is of type III and Index $E < +\infty$, with little more effort, one can prove the following: There exists u in M such that for any $x \in M$ there exists a unique $y \in N$ with $x = uy$. Furthermore, this u satisfies $E(u^*u) = 1$ and $1 = ue_N u^*$.

When M, N are II_1-factors, M possesses the unique normalized trace tr_M (whose restriction tr_N to N is the unique normalized trace on N). There exists a canonical normal conditional expectation $E: M \to N$ satisfying $\mathrm{tr}_N \circ E = \mathrm{tr}_M$. Also, representing M on $\mathfrak{h}_{\mathrm{tr}_M}$, we get a unit trace vector, and this gives rise to the unique normalized trace $\mathrm{tr}_{M'}$. It is straightforward to check

$$d\mathrm{tr}_M / d\mathrm{tr}_{M'} = 1.$$

We thus get

$$1 = d(\mathrm{tr}_N \circ E)/d\mathrm{tr}_{M'} = d\mathrm{tr}_N / d(\mathrm{tr}_{M'} \circ E^{-1}).$$

The last equation in 1.1 implies that $\mathrm{tr}_{M'} \circ E^{-1}$ (on N') is tracial. By Lemma 3.1, we have

$$\mathrm{tr}_{M'} \circ E^{-1}(e_N) = 1.$$

On the other hand, the normalized trace $\mathrm{tr}_{N'}$ on N' is related to Jones' index $[M:N]$ by

$$[M:N] = \dim_N(L^2(M; \mathrm{tr}_M)) = \mathrm{tr}_{N'}(e_N)^{-1}.$$

Since N' is a factor, we must have

$$\mathrm{tr}_{N'} = [M:N]^{-1}\, \mathrm{tr}_{M'} \circ E^{-1}.$$

Hence,

$$\begin{aligned} 1 = \operatorname{tr}_{N'}(1) &= [M:N]^{-1} \operatorname{tr}_{M'}(E^{-1}(1)) \\ &= [M:N]^{-1} \operatorname{tr}_{M'}((\operatorname{Index} E)1) \\ &= [M:N]^{-1} \operatorname{Index} E, \end{aligned}$$

that is, Index $E = [M:N]$.

4. Local Index

A theory of local index in the II_1-set-up is relatively simple (although it is important). In our set-up, it is more technical, the reason being that a projection in the relative commutant may or may not be invariant under a certain modular automorphism group.

Let p be a nonzero projection in the relative commutant $M \cap N'$. It is easy to see that $E(p)$ and $E^{-1}(p)$ are scalars. The reduced von Neumann algebra pMp and its subalgebra Np act on $p\mathfrak{h}$. For each $x \in pMp$, we set

$$E_p(x) = E(p)^{-1} E(x)p \in Np.$$

Lemma 4.1. *E_p is a normal conditional expectation from pMp onto Np.*

Proof. Normality, projection property, and bimodule property can be easily checked. The faithfulness follows from the fact that $x \in N \to xp \in Np$ is isomorphic. Q.E.D.

Let $\phi \in N_*^+$ be a fixed faithful state and $\psi = \phi \circ E \in M_*^+$. The algebraic isomorphism between N and Np induces the state $\tilde{\phi}$ on Np. Let ψ' be the restriction of ψ to pMp. For each $x \in pMp$, we compute

$$\begin{aligned} \tilde{\phi} \circ E_p(x) &= \tilde{\phi}(E(p)^{-1} E(x)p) \\ &= \phi(E(p)^{-1} E(x)) = E(p)^{-1} \psi(x). \end{aligned}$$

We thus get

$$\begin{aligned} \tilde{\phi} \circ E_p &= E(p)^{-1}\psi', \\ (\tilde{\phi} \circ E_p)^{-1} &= E(p)\psi'^{-1}. \end{aligned} \tag{3}$$

Here, ψ'^{-1} is in $P(pB(\mathfrak{h})p, M'p)$. As usual, we assume $\mathfrak{h} = \mathfrak{h}_\psi$ and pick up a family $\{\xi_\alpha\}_{\alpha \in A}$ in $D(\mathfrak{h}, \phi)$ such that

$$1 = \sum_{\alpha \in A} \theta^\phi(\xi_\alpha, \xi_\alpha).$$

For $x \in N$, we have

$$R^{\tilde{\phi}}(p\xi_\alpha)\Lambda_{\tilde{\phi}}(xp) = xp\xi_\alpha = px\xi_\alpha$$
$$= pR^{\phi}(\xi_\alpha)\Lambda_\phi(x).$$

Identifying $\mathfrak{h}_\phi$ and $\mathfrak{h}_{\tilde{\phi}}$ ($\Lambda_\phi(x) = \Lambda_{\tilde{\phi}}(xp)$), we get

$$p\xi_\alpha \in D(p\mathfrak{h}, \tilde{\phi}), \qquad R^{\tilde{\phi}}(p\xi_\alpha) = pR^{\phi}(\xi_\alpha),$$

and

$$p = \sum_{\alpha \in A} p\theta^{\phi}(\xi_\alpha, \xi_\alpha)p = \sum_{\alpha \in A} \theta^{\tilde{\phi}}(p\xi_\alpha, p\xi_\alpha). \tag{4}$$

The modular automorphism σ_t^E ($= \sigma_t^{\phi \circ E}|_{M \cap N'}$ which does not depend on ϕ) on $M \cap N'$ may or may not leave p invariant. We begin with the case $\sigma_t^E(p) = p$, $t \in \mathbb{R}$. Note that

$$\sigma_t^{\psi}(pMp) = p\sigma_t^{\psi}(M)p = pMp,$$
$$\sigma_t^{\psi}(Np) = \sigma_t^{\psi}(N)p = \sigma_t^{\phi}(N)p = Np.$$

The first fact implies that $\sigma_t^{\psi'}$ is just the restriction of σ_t^{ψ} to pMp while the second means $\sigma_t^{\psi'}(Np) = Np$. Therefore, by Takesaki's result [5], there exists a (unique) normal conditional expectation F: $pMp \to Np$ characterized by

$$\psi'(xF(y)z) = \psi'(xyz), \qquad y \in pMp,\ x, z \in Np.$$

This F is exactly E_p constructed above. In fact one can directly check the above equation for E_p.

PROPOSITION 4.2. *When $\sigma_t^E(p) = p$, $t \in \mathbb{R}$, we have* Index $E_p = E(p)E^{-1}(p)$.

Proof. We consider $\mathfrak{k} = \mathfrak{h}_{\psi'}$ as a (closed) subspace of $\mathfrak{h} = \mathfrak{h}_\psi$ in the obvious way (by identifying $\Lambda_{\psi'}(x)$, $x \in pMp$, with $\Lambda_\psi(x)$). We will investigate the (not necessarily bounded) operator $R^{\psi}(p\xi_\alpha)$.

For a vector $\Lambda_\psi(x)$, $x \in M$, in $\mathfrak{D}(R^{\psi}(p\xi_\alpha)) \subseteqq \mathfrak{h}$, we consider the following decomposition:

$$\Lambda_\psi(x) = \Lambda_\psi(pxp) + (\Lambda_\psi(x) - \Lambda_\psi(pxp)).$$

This is an orthogonal decomposition of $\Lambda_\psi(x)$ into $\mathfrak{k} \oplus (\mathfrak{h} \ominus \mathfrak{k})$. In fact, for each $\Lambda_{\psi'}(pyp) = \Lambda_\psi(pyp)$, $y \in M$, we get

$$\begin{aligned}(\Lambda_\psi(x) - \Lambda_\psi(pxp) \mid \Lambda_\psi(pyp)) &= \psi(py^*px) - \psi(py^*pxp) \\ &= 0\end{aligned}$$

due to $\sigma_t^\psi(p) = \sigma_t^E(p) = p$. In other words, the projection $P\colon \mathfrak{h} \to \mathfrak{k}$ (different from p) and $1 - P$ leave $\mathfrak{D}(R^\psi(p\xi_\alpha))$ invariant, and

$$\begin{aligned}P\Lambda_\psi(x) &= \Lambda_\psi(pxp) \qquad (= \Lambda_{\psi'}(pxp)),\\ (1-P)\Lambda_\psi(x) &= \Lambda_\psi(x - pxp).\end{aligned}$$

Therefore, the following matrix notation is valid:

$$R^\psi(p\xi_\alpha) = [A, B]\left(\begin{bmatrix} \mathfrak{k} \\ \mathfrak{h} \ominus \mathfrak{k}\end{bmatrix} \to \mathfrak{h}\right).$$

Note that

$$\begin{aligned}A\Lambda_\psi(pxp) &= R^\psi(p\xi_\alpha)\Lambda_\psi(pxp) = pxp\xi_\alpha \in p\mathfrak{h} \qquad (= R^{\psi'}(p\xi_\alpha)\Lambda_{\psi'}(pxp)),\\ B\Lambda_\psi(x - pxp) &= R^\psi(p\xi_\alpha)\Lambda_\psi(x - pxp) = (x - pxp)p\xi_\alpha \\ &= (1-p)xp\xi_\alpha \in (1-p)\mathfrak{h}.\end{aligned}$$

Therefore, $R^\psi(p\xi_\alpha)$ can further be written as

$$R^\psi(p\xi_\alpha) = \begin{bmatrix} R^{\psi'}(p\xi_\alpha) & 0 \\ 0 & B\end{bmatrix}\left(\begin{bmatrix} \mathfrak{k} \\ \mathfrak{k} \ominus \mathfrak{h}\end{bmatrix} \to \begin{bmatrix} p\mathfrak{h} \\ (1-p)\mathfrak{h}\end{bmatrix}\right).$$

We thus get

$$\theta^\psi(p\xi_\alpha, p\xi_\alpha) = \begin{bmatrix} \theta^{\psi'}(p\xi_\alpha, p\xi_\alpha) & 0 \\ 0 & BB^*\end{bmatrix}\left(\begin{bmatrix} p\mathfrak{h} \\ (1-p)\mathfrak{h}\end{bmatrix} \to \begin{bmatrix} p\mathfrak{h} \\ (1-p)\mathfrak{h}\end{bmatrix}\right)$$

(as an element in $\hat{M}'_+$). It commutes with

$$p = \begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix}$$

so that we get

$$\theta^\psi(p\xi_\alpha, p\xi_\alpha)p = \theta^{\psi'}(p\xi_\alpha, p\xi_\alpha). \tag{5}$$

Now the local index ($=$Index E_p) can be computed by

$$\begin{aligned}
E_p^{-1}(p) &= E_p^{-1}\left(\sum_{\alpha\in A} \theta^{\tilde{\phi}}(p\xi_\alpha, p\xi_\alpha)\right) && \text{by (4)}\\
&= \sum_{\alpha\in A} E_p^{-1}\circ(\tilde{\phi})^{-1}(p\xi_\alpha\otimes p\xi_\alpha)\\
&= \sum_{\alpha\in A} (\tilde{\phi}\circ E_p)^{-1}(p\xi_\alpha\otimes p\xi_\alpha)\\
&= E(p)\sum_{\alpha\in A} \theta^{\psi'}(p\xi_\alpha, p\xi_\alpha) && \text{by (3)}\\
&= E(p)\sum_{\alpha\in A} \theta^{\psi}(p\xi_\alpha, p\xi_\alpha)p && \text{by (5)}\\
&= E(p)\sum_{\alpha\in A} \{E^{-1}\circ\phi^{-1}(p\xi_\alpha\otimes p\xi_\alpha)\}p && (\text{since } \psi=\phi\circ E)\\
&= E(p)\sum_{\alpha\in A} \{E^{-1}(p\phi^{-1}(\xi_\alpha\otimes\xi_\alpha)p)\}p && (\text{since } p\in N',\\
& && \quad \phi^{-1}\in P(B(\mathfrak{h}), N'))\\
&= E(p)E^{-1}\left(p\sum_{\alpha\in A} \theta^{\phi}(\xi_\alpha, \xi_\alpha)p\right)p\\
&= E(p)E^{-1}(p)p.
\end{aligned}$$

Therefore, E_p^{-1} $(p=\mathrm{Id}_{p\mathfrak{h}})$ is p multiplied by the scalar $E(p)E^{-1}(p)$.

Q.E.D.

We now drop the assumption $\sigma_t^E(p)=p$. The equality in Proposition 4.2 fails. Instead we get

PROPOSITION 4.3. *If* Index $E<\infty$, *then* Index $E_p \leqq E(p)E^{-1}(p)$.

Proof. The notations ϕ, ψ, ψ' in the previous proof will be kept, and M acts on $\mathfrak{h}=\mathfrak{h}_\psi$. Let $\psi=\omega_{\xi_0}$ with a cyclic and separating vector ξ_0. As in the proof of Lemma 3.1, we have

$$\xi_0\in D(\mathfrak{h}, \phi), \qquad \theta^{\phi}(\xi_0, \xi_0)=e_N.$$

More generally, for $z\in M'$ (actually for $z\in N'$) one can easily prove

$$z\xi_0\in D(\mathfrak{h}, \phi), \quad \theta^{\phi}(z\xi_0, z\xi_0)=ze_Nz^*. \tag{6}$$

Note that $D(\mathfrak{h}, \psi)=M'\xi_0$ and $R^{\psi}(z\xi_0)=z$, $z\in M'$. As before we regard $\mathfrak{k}=\mathfrak{h}_{\psi'}$ as a closed subspace of $\mathfrak{h}$, and write

$$R^{\psi}(z\xi_0)\ (=z)=\begin{bmatrix} A & B\\ C & D\end{bmatrix}\left(\begin{bmatrix}\mathfrak{k}\\ \mathfrak{h}\ominus\mathfrak{k}\end{bmatrix}\to\begin{bmatrix}p\mathfrak{h}\\ (1-p)\mathfrak{h}\end{bmatrix}\right).$$

Here all of A, B, C, D are bounded operators between the relevant spaces. The set of all $\Lambda_\psi(pxp)=\Lambda_{\psi'}(pxp)$, $x\in M$, forms a dense subspace of $\mathfrak{k}$. (This is exactly $\mathscr{D}(R^{\psi'}(pz\xi_0))$.) We compute

$$\begin{aligned}pR^{\psi}(z\xi_0)\Lambda_\psi(pxp)&=pxpz\xi_0\\&=R^{\psi'}(pz\xi_0)\Lambda_{\psi'}(pxp).\end{aligned}$$

This means $A\supseteqq R^{\psi'}(pz\xi_0)$, and hence

$$pz\xi_0\in D(p\mathfrak{h},\psi'),\qquad R^{\psi'}(pz\xi_0)=A$$

due to the boundedness of A. Since $(1-p)\,R^{\psi}(z\xi_0)\,\Lambda_\psi(pxp)=(1-p)pxpz\xi_0=0$, we get $C=0$. We have thus shown

$$R^{\psi}(z\xi_0)=\begin{bmatrix}R^{\psi'}(pz\xi_0) & B\\ 0 & D\end{bmatrix}\left(\begin{bmatrix}\mathfrak{k}\\ \mathfrak{h}\ominus\mathfrak{k}\end{bmatrix}\to\begin{bmatrix}p\mathfrak{h}\\ (1-p)\mathfrak{h}\end{bmatrix}\right),$$

$$\theta^{\psi}(z\xi_0,z\xi_0)=zz^*=\begin{bmatrix}\theta^{\psi'}(pz\xi_0,pz\xi_0)+BB^* & BD^*\\ DB^* & DD^*\end{bmatrix}$$

$$\left(\begin{bmatrix}p\mathfrak{h}\\ (1-p)\mathfrak{h}\end{bmatrix}\to\begin{bmatrix}p\mathfrak{h}\\ (1-p)\mathfrak{h}\end{bmatrix}\right).$$

Since $zz^*\in M'$ commutes with p, we must have $BD^*=0$ and

$$\begin{aligned}zz^*p&=\theta^{\psi'}(pz\xi_0,pz\xi_0)+BB^*\\&\geqq\theta^{\psi'}(pz\xi_0,pz\xi_0)\end{aligned}\tag{7}$$

for any $z\in M'$. We now estimate

$$\begin{aligned}E_p^{-1}(pze_Nz^*p)&=E_p^{-1}(p\theta^{\phi}(z\xi_0,z\xi_0)p) && \text{by (6)}\\&=E_p^{-1}(\theta^{\tilde\phi}(pz\xi_0,pz\xi_0)) && \text{(similar argument as (4))}\\&=E(p)\theta^{\psi'}(pz\xi_0,pz\xi_0) && \text{by (3)}\\&\leqq E(p)zz^*p && \text{by (7)}\\&=E(p)E^{-1}(ze_Nz^*)p && \text{(Lemma 3.1)}.\end{aligned}$$

Therefore, applying Corollary 3.4 to $p\in M\cap N'\subseteqq N'$, we get $E_p^{-1}(p)\leqq E(p)E^{-1}(p)p$ by the above estimate and the normality of E_p^{-1} and E^{-1}.

Q.E.D.

THEOREM 4.4 (i) *If $\{p_i\}$ is a partition of $(M \cap N')_E$, the fixed point subalgebra of $M \cap N'$ under σ_t^E, $t \in \mathbb{R}$, then we have*

$$\text{Index } E = \sum_i E(p_i)^{-1} \text{Index } E_{p_i}.$$

(ii) *If* Index $E < 4$, *then* $M \cap N' = \mathbb{C}1$.

Proof. (i) Summing up $E^{-1}(p_i) = E(p_i)^{-1}$ Index E_{p_i} (Proposition 4.2) over i, we get (i).

(ii) We will prove

$$M \cap N' \neq \mathbb{C}1 \Rightarrow \text{Index } E \geqq 4.$$

Assume Index $E < \infty$ and $M \cap N' \neq \mathbb{C}1$. Choose a projection p ($\neq 0, 1$) in $M \cap N'$. Then Proposition 4.3 and Lemma 3.1 imply

$$\begin{aligned} \text{Index } E &= E^{-1}(p) + E^{-1}(1-p) \\ &\geqq E(p)^{-1} \text{ Index } E_p + E(1-p)^{-1} \text{ Index } E_{1-p} \\ &\geqq E(p)^{-1} + E(1-p)^{-1}. \end{aligned}$$

But, since $E(p)$ and $E(1-p)$ are scalars summing up to 1, we get Index $E \geqq 4$. Q.E.D.

5. RESTRICTION ON INDEX VALUES

As usual, let $M \supseteqq N$ be factors and $E = E_N: M \to N$ be a given normal conditional expectation. We assume Index $E \in (1, \infty)$ and set

$$\tau = E^{-1}(1)^{-1} \in (0, 1).$$

As before, M acts on $\mathfrak{h}_{\phi \circ E}$ ($\phi \in N_*^+$ faithful) and we have the projection $e_N \in N'$, $Je_N J = e_N$, and $\langle M, e_N \rangle = JN'J$ (Lemma 3.2). By transferring the conditional expectation $\tau E^{-1}: N' \to M'$ (at the beginning of Sect. 3) by $J \cdot J$ we get the conditional expectation $E_M: \langle M, e_N \rangle \to M$,

$$E_M(x) = \tau J E^{-1}(JxJ)J, \qquad x \in \langle M, e_N \rangle.$$

We remark that

$$E_M(e_N) = \tau. \tag{8}$$

Also (whichever space $\langle M, e_N\rangle$ acts on: see Theorem 2.2) we have

$$\begin{aligned}\text{Index } E_M &= \text{Index } (\tau E^{-1}) \qquad \text{(Lemma 1.3)}\\ &= \tau^{-1}(E^{-1})^{-1}(1)\\ &= \tau^{-1}E(1) = \tau^{-1}\\ &= \text{Index } E\end{aligned}$$

thanks to $(E^{-1})^{-1} = E$.

LEMMA 5.1. *e_N is in the centralizer of $\phi \circ E_N \circ E_M$.*

Proof. Due to Lemma 3.2(vi), it suffices to check

(a) $\phi \circ E_N \circ E_M(ae_N) = \phi \circ E_N \circ E_M(e_N a)$,

(b) $\phi \circ E_N \circ E_M(ae_N be_N) = \phi \circ E_N \circ E_M(e_N ae_N b)$,

for $a, b \in M$.

First, both sides of (a) are $\tau\phi \circ E_N(a)$ by (8). Second, the left side of (b) is equal to

$$\begin{aligned}&\phi \circ E_N \circ E_M(aE_N(b)e_N) \qquad \text{(Lemma 3.2(i))}\\ &= \tau\phi \circ E_N(aE_N(b))\\ &= \tau\phi(E_N(a)E_N(b)).\end{aligned}$$

Similar computations also show that the right side of (b) is the same.

Q.E.D.

So far all the algebras have been acting on $\mathfrak{h}_{\phi \circ E_N}$. We now let $\langle M, e_N\rangle$ (and its subalgebras) act on $\mathfrak{h}_{\phi \circ E_N \circ E_M}$. As before we get the projection $e_M \in M'$ and

$$e_M = Je_M J, \qquad \langle M, e_N, e_M\rangle = JM'J,$$

where this J is the modular involution on $\mathfrak{h}_{\phi \circ E_N \circ E_M}$. As remarked before, Index E_M = Index $E_N = \tau^{-1}$ and thus we can construct the conditional expectation $E_{\langle M, e_N\rangle}$: $\langle M, e_N, e_M\rangle \to \langle M, e_N\rangle$ by sending $\tau E_M^{-1}: M' \to \langle M, e_N\rangle'$ to $\langle M, e_N, e_M\rangle$ by the new J.

As in [3] we get:

LEMMA 5.2. (i) $e_N e_M e_N = \tau e_N$, $e_M e_N e_M = \tau e_M$.

(ii) $e_N \wedge e_M = e_N \wedge e_M^{\perp} = e_M^{\perp} \wedge e_M = 0$.

Letting $\langle M, e_N, e_M\rangle$ (and its subalgebras) act on $\mathfrak{h}_{\phi \circ E_N \circ E_M \circ E_{\langle M, e_N\rangle}}$, we get the next projection in $\langle M, e_N\rangle'$. In particular, this projection commutes with e_N. Repeating this process, we obtain projections

$$e_N = e_1, \qquad e_M = e_2, e_3, e_4, \ldots,$$

and a tower of factors

$$M_0 = M, M_1 = \langle M_0, e_1 \rangle, \qquad M_2 = \langle M_1, e_2 \rangle, \ldots.$$

We also get conditional expectations

$$E_i : M_{i+1} \to M_i, \qquad i = 0, 1, 2, \ldots,$$

satisfying Index $E_i =$ Index $E = \tau^{-1}$ and $E_i(e_{i+1}) = \tau$.

First we remark (Lemma 5.2) that

$$e_i e_{i \pm 1} e_i = \tau e_i,$$
$$e_i e_j = e_j e_i \qquad \text{if } |i-j| \geqq 2.$$

Also, Lemma 5.3 implies that e_{i+1} is in the centralizer of the state $\phi \circ E_N \circ E_0 \circ E_1 \circ \cdots \circ E_j$ on $M_j, j \geqq i$. Let A_j be the subalgebra generated by $1, e_1, e_2, \ldots, e_j$ (in M_j) and Tr_j be the restriction of $\phi \circ E_N \circ E_0 \circ E_1 \circ \cdots \circ E_{j-1}$ to A_j (which is indeed a trace on A_j). So we get the tower (A_j, Tr_j), $A_j \subseteqq A_{j+1}$, and Tr_j is exactly the restriction of Tr_{j+1} to A_j. Also $E_i(e_{i+1}) = \tau$ implies that

$$\mathrm{Tr}_j(e_j w) = \tau \mathrm{Tr}_{j-1}(w)$$

for a word w on $1, e_1, \ldots, e_{j-1}$.

Thus, we can repeat the beautiful analysis in [3, Sect. 4], for (A_j, Tr_j) and get

THEOREM 5.4. Index $E \in \{4 \cos^2 \pi/n; n = 3, 4, \ldots\} \cup [4, \infty]$.

We were able to construct traces on A_j's. But without having them, one can still conclude the above theorem thanks to [7].

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