S. Chehade: Saturating the data processing inequality for $\alpha - z$ Rényi relative entropy

Referee report

The $\alpha - z$- Rényi relative entropies form an important class of information-theoretic quantities containing two most discussed quantum extensions of the classical α-Rényi relative entropy (the standard and sandwiched ones). After this class was introduced by Audenaert and Datta [2], a number of papers appeared discussing the properties of these quantities, in particular the data processing inequality (DPI) for different values of the parameters. The present paper studies the situation when the DPI is saturated. This question is important since it has been shown that in the cases of the two versions of the quantum Rényi relative entropy, equality in DPI implies reversibility of the channel, but is also interesting from the point of view of the structure and properties of the class of $\alpha - z$-RRE.

The author proves necessary and sufficient conditions for saturation of the DPI for the subset of parameters satisfying $1 < \alpha \leq 2$ and $\frac{\alpha}{2} \leq z \leq \alpha$. These conditions are of algebraic nature, of the form similar to that given in [21] for the sandwiched RRE. As they are, these results would be clearly worth publishing. Unfortunately, the paper is not fit for publication in the present form.

First of all, I must say, the paper is not well written and it is apparent that insufficient care was given to many details. There is a number of mistakes, typos, incorrect references, strange formulations and also some doubtful issues in the proofs, including statements that are clearly wrong. I did not check the more complicated proofs, since it is the responsibility of the author to prepare a manuscript readable also for the referee. I suggest to revise the paper following the points given below, and possibly resubmit.

An overall remark

The main proofs in the paper use the ideas and techniques from [21], where a variational formula of [13] is applied. A similar variational formula is developed also in the present work, using the results of at least three further papers, including another variational formula and an analytic function technique. So the argument is rather complicated. I suggest to explain the structure of the argument in the Introduction, describing the relations and differences to techniques used in the previous papers. It seems that the main ideas are the same, but there are some points that need different approaches. It would help the reader a lot to have this explained before diving into technicalities.

Main issues

1. p.2 : Corollary 1: there is a strange condition $\sigma \in \mathcal{Q}(\mathcal{H})$ hidden in the statement of the main result. The set $\mathcal{Q}(\mathcal{H})$ is not defined here and the condition can be easily overlooked by a less cautious reader. The definition appears later in the text, but it is not discussed what it means or how restrictive it might be. Such a discussion should appear in the Introduction.

2. p.8: the statement of Prop. 1 is somewhat confusing. Does it mean that the map given in Eq. (13) is concave on any convex subset of $\mathcal{I}(\mathcal{H})$? I also wonder about the structure of $\mathcal{I}(\mathcal{H})$: are there any convex subsets at all? Some thought, and explanation, should be devoted to this point.

3. It seems that the assumption $z > 1$ is used at several places (Lemma 5, Prop. 4, and subsequently in the main proofs). But, as far as I can see, this does not follow from the assumptions on α and z (if $1 < \alpha < 2$ and $\alpha/2 \leq z \leq \alpha$, then z can very well be less than 1). The interval $1 < z \leq \alpha \leq 2z$ appears in Concluding remarks, but this is not the same set of parameters as considered in the text.

4. p.15, the proof of Thm. 5 uses Prop. 4, where it is required that $\tilde{\sigma}$ belongs to a "convex subset of $\mathcal{I}(\mathcal{H})$". An argument should be given to show that this is true.

5. Prop. 5 and Prop. 6: Since any product state is also separable, there is clearly something wrong.

Minor points

1. Abstract: "necessary and algebraically sufficient conditions" ... (?) necessary and sufficient algebraic conditions

2. p. 1, the citation [30,31] here is not precise, these papers deals with sufficiency of quantum channels. Better references have to be found for the DPI (for quantum relative entropy and α-RRE).

3. p. 2: the main result is formulated as a Corollary, should it not be a Theorem?

4. p.2: main result: necessary and sufficient conditions are promised in the abstract, Corollary 1 gives only a necessary one.

5. p.2, Eq. (1) Λ is missing on the RHS

6. p.4, second displayed equations in Sec. 2.1.2: one $-z$ should be z

7. p.5, title of Sec. 2.1.3 : $\alpha - SSRD$

8. p.5, just above Sec. 2.1.4 [?] (reference is missing)

9. p.6, Remark 1: if ρ and σ are invertible then they must have the same supports

10. p.8: the remark that ...$(zX + H)^{\frac{p}{2}}$ is well defined should be written before (or just after, in any case somewhere around) the definition of $\mathcal{I}(\mathcal{H})$.

11. Some statements are rather obvious and do not need a proof (such as Lemmas 3 and 4, and also Prop. 3)

12. I suggest to use the same counter for Lemmas, Theorems, Propositions, etc. It would be easier to find them in the text.

13. Why are definitions given in Remarks? (Remarks 2 and 3).

14. p.11, Lemma 5: uniqueness of the maximizer is an important part of Lemma 5. It is also proved, but not explicitly present in its statement.

15. p. 18: Remark 4 does not make sense. I think the author wanted to say something else.