L. Gao, M.M. Wilde: Recoverability for optimized quantum f-divergences

Referee report

In this paper, a refinement of the data processing inequality is obtained, for quantum f-divergences and optimized f-divergences, in terms of the Petz recovery map and its rotated or universal versions. The authors apply a similar method as in the works [CV18,CV20a] by Carlen and Vershynina, using the integral representation of the operator anti-monotone function f and properties of the relative modular operator. The results on f-divergences partially improve the known results, while he inequalities for optimized f-divergences are completely new. In particular, recoverability results are obtained for the sandwiched Rényi relative entropy. As a consequence, reversibility (exact recoverability) of quantum channels is characterized by equalities in the optimized divergences. Moreover, the definition of optimized f-divergences is extended to von Neumann algebras, as well as the recoverability and reversibility results.

Refinements and extensions of the data processing inequality for various types of divergences provide an important line of research in quantum information theory and this work is a valuable contribution. Also the extension to von Neumann algebras fits into current research in this field. The paper is mostly well written and readable, but there are some parts that could be improved, as detailed below. There is also some number of typos, some are listed below, but the manuscript should be carefully checked. Apart from these minor issues, I recommend the paper for publication.

1. p. 8, last paragraph of Sec. 3.1: it should be specified that $d\lambda < C_{a,b} d\mu(\lambda)$ on (a, b)

2. p. 11, line 1: It would be better to write $\Phi : L_1(\mathcal{M}_1) \to L_1(\mathcal{M}_2)$, regarding the notations of the paper. Also line 6: $R_{\Phi,\rho}(\Phi(\sigma)) = \sigma$ and $R_{\Phi,\rho} : L_1(\mathcal{M}_2) \to L_1(\mathcal{M}_1)$.

3. p.12, Eq. (12): $R^{1/2}$ should be $R_\rho^{1/2}$

4. The proof of Lemma 4.17: here the spatial derivative is used without any warning or reference to the Appendix. Besides, I think that the readers not familiar with modular theory would have problems understanding this proof. The use of the spatial derivative here does not correspond to the definition in the Appendix (the role of the state and the vector are reversed), which might be rather confusing. In finite dimensions, the statement can be proved directly, using the known form of the relative modular operator and the Petz map, and I suggest to use the direct proof here, since it is more accessible to readers interested mostly in the finite dimensional case.

5. Definition 5.1: I suggest (here and elsewhere) to stress that one can use a *-representation of $\mathcal{M}$ where the states have vector representations. Otherwise it seems that for a concrete von Neumann subalgebra of $B(H)$ the optimized f-divergence is defined only for vector states, which would be rather restrictive e.g. if $\mathcal{M} = B(H)$, as in Example 5.5. This may be confusing.

6. p. 46, displayed equation just above Prop. 5.3: $\tilde{f}$ should be f.

7. p. 47, paragraph just above Sec. 5.2: perhaps some more explanation or a reference is needed for the statement that "...$\langle h_\rho^{1/2} | f(\Delta(\sigma, \omega)) | h_\rho^{1/2} \rangle$ is convex over σ and concave over ω"

8. p. 48, Example 5.5, line -6: perhaps there should be $\tilde{D}_\alpha$ instead of D_α (in the line just above $\lim_{\alpha\to 1^+} \tilde{D}_\alpha(\rho\|\sigma) = D(\rho\|\sigma)$). Also, probably just a technicality, but there is a condition that ρ is majorized by σ in Thm. 3 of [BST18], which is not the same as that $\tilde{D}_\alpha(\rho\|\sigma) < \infty$. Moreover, what if $\tilde{D}_\alpha = \infty$ for all $\alpha > 1$?

9. p.52: a typo "Let $\mathcal{L} \subset B(H)$ denote the von Neumann subalgebra in $B(K)$..."

10. p. 53, beginning of Sec. 5.4: "...generalized conditional expectation... whose adjoint is the Petz map". As far as I can see, in the definition below, the Petz map for the inclusion $\mathcal{N} \hookrightarrow \mathcal{M}$ is the generalized conditional expectation (not its adjoint).

11. p. 54, case ii): it should be assumed that $s(\rho) \leq e$.

12. p. 56, proof of Lemma 5.12: I am convinced that the proof is correct, but it is not well written. For example, one cannot assume that $h_{\omega_{\mathcal{N}}}^{1/2} = bh_{\rho_{\mathcal{N}}}^{1/2}$ for some $b \in \mathcal{N}$, but the the fact that $\mathcal{N}h_{\rho_{\mathcal{N}}}^{1/2}$ is dense in $L_2(\mathcal{N})$ can be used to prove that $V_\rho a = aV_\rho$ for $a \in \mathcal{N}$, which is all that is needed here. Also I find the argument using the spatial derivative rather confusing. One can just say that the operator $S_{\sigma,\boldsymbol{\omega}}$ only depends on the vector state induced by $\boldsymbol{\omega}$ on the commutant $\mathcal{M}'$, which has the same vector representative in $L_2(\mathcal{M})^+$ as the state given by the vector $J\boldsymbol{\omega}$ on $\mathcal{M}$.

13. p. 56, Lemma 5.13: definition of $|c_t\rangle$, R_ρ should be E_ρ?

14. p. 57, the proof of Lemma 5.13: in the second set of displayed equation, one equation seems to appear twice. In the last set of displayed equations, $\sigma_{\mathcal{N}}$, $\rho_{\mathcal{N}}$ should be σ, ρ.

15. p. 58: $\sigma_{\mathcal{N}}$ and σ are replaced at two places: on the first line an again in a similar situation in the last paragraph.

16. p. 58: polar decomposition in $L_2(\mathcal{M})$ is not explained

17. end of p. 59: some argument or reference should be given for the statement that Eq. (25) ”remains valid in Haagerup L_p-spaces”.