EXPLANATIONS CONCERNING THE REFEREES' REMARKS

General

I have changed a little the title and layout of the paper. I have also added some comments, explanations and literature items.

Referee #1

For normal conditional expectations, that a commuting square implies the generalized SSA as shown follows from (Petz 1991, https://doi.org/10.1007/BF02571350, Theorem 12), which goes even beyond the setting of general von Neumann algebras.

The Petz result, formulated in the setting of C*-algebras, concerns relative entropies and can be applied to our situation in the case of a finite von Neumann algebra with a normal faithful unital trace τ. This is due to the relation between the relative entropy $S(\omega, \tau)$ and the Segal entropy $H(\omega)$:

$$S(\omega, \tau) = H(\omega), \quad \text{(in Petz's notation } S(\varphi, \omega) = H(\omega) \text{ with } \varphi = \tau\text{)},$$

and the assumption made there (Theorem 12) that *φ is a separating state* which amounts to saying that τ is faithful. I have added an appropriate comment at the end.

The definition of conditional expectation as a map $\mathbb{E}\colon \mathcal{M} \to \mathcal{N}$ from the algebra $\mathcal{M}$ onto its subalgebra $\mathcal{N}$ adopted in the paper is in a sense 'canonical' in operator algebras. I am not sure what is meant by saying that *the restriction map is taken as the conditional expectation* since for the conditional expectation in the above sense we always have $\mathbb{E}|\mathcal{N} = \mathrm{id}\,|\mathcal{N}$. Maybe the issue is as follows: *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be von Neumann algebras with normal semifinite faithful traces τ_1 and τ_2, and let $\beta\colon \mathcal{M}_1 \to \mathcal{M}_2$ be a positive unital ultraweakly continuous map such that $\tau_1 = \tau_2 \circ \beta$. Then we can define a transpose map $\alpha\colon \mathcal{M}_2 \to \mathcal{M}_1$ by the formula*

$$\tau_1(b_1 \alpha(a_2)) = \tau_2(\beta(b_1) a_2), \quad (b_1 \in \mathcal{M}_1^+,\ a_2 \in \mathcal{M}_2^+).$$

The above is taken almost word for word (including notation) from M. Ohya & D. Petz, *Quantum Entropy and Its Use*, Springer, Berlin, 2004, pp. 116–117 just before Proposition 7.3 and in the first two lines of its proof. (Incidentally, neither the possibility of defining such a map α is proved nor its properties are mentioned.) Now if we take as $\mathcal{M}_1$ a von Neumann subalgebra of a von Neumann algebra $\mathcal{M}_2$, $\beta = \iota\colon \mathcal{M}_1 \hookrightarrow \mathcal{M}_2$ — the embedding of $\mathcal{M}_1$ into $\mathcal{M}_2$, τ_2 — a normal faithful semifinite trace on

$\mathcal{M}_2$, and $\tau_1 = \tau_2|\mathcal{M}_1$ under the assumption that $\tau_2|\mathcal{M}_1$ is semifinite, then we shall obtain as the map transpose to ι the conditional expectation $\mathbb{E}$ from $\mathcal{M}_2$ onto $\mathcal{M}_1$ such that $\tau_2 \circ \mathbb{E} = \tau_2$, which follows from the equality

$$\begin{aligned}\tau_1(b_1\mathbb{E}a_2) &= \tau_1(\mathbb{E}(b_1a_2)) = (\tau_2|\mathcal{M}_1)(\mathbb{E}(b_1a_2)) = ((\tau_2|\mathcal{M}_1) \circ \mathbb{E})(b_1a_2)) \\ &= (\tau_2 \circ \mathbb{E})(b_1a_2)) = \tau_2(b_1a_2) = \tau_2(\iota(b_1)a_2).\end{aligned}$$

In the same way, the map transpose to the conditional expectation $\mathbb{E}\colon \mathcal{M}_2 \to \mathcal{M}_1$ such that $\tau_2 \circ \mathbb{E} = \tau_2$ will be the embedding ι. The problem is that this definition of conditional expectation as a transpose (or dual) map as above is possible only in the semifinite case while the definition of conditional expectation as a projection of norm one onto a subalgebra works in full generality. (Of course, from this general definition follow a number of nontrivial consequences as e.g. complete positivity or the module property.)

To clarify the proof of Lemma 1, the module property of conditional expectation has been stated.

I have added a simple argument that if for two unitary groups (v_t) and (w_t), (v_tw_t) is also a unitary group, then (v_t) and (w_t) commute, and described precisely how this commutation property transfers to generators.

Referee #2

I have changed a little the title of the paper, the point is that the new one refers to an inequality which can be considered as an *SSA-like inequality* but, strictly speaking, it is not such because the SSA inequality is the one about entropies in tensor products.

The structure of states which satisfy the SSA of quantum entropy with equality given in P. Hayden et al., *Structure of states which satisfy strong subadditivity of quantum entropy with equality*, Comm. Math. Phys. **246** (2004), 359–374, depends heavily on formulating the problem in the tensor product setup (as is the case with the SSA), even putting aside the finiteness of dimension which is crucial for the description od this structure. Whether it might be possible to state the problem of equality in the setting of commuting squares, and possibly even for semifinite von Neumann algebras (and solve it!) is a more complicated question, at least, I can't see a clear way to the solution.