STRONG SUBADDITIVITY OF QUANTUM MECHANICAL ENTROPY FOR SEMIFINITE VON NEUMANN ALGEBRAS

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ABSTRACT. We show that for Segal entropy defined for states on an arbitrary von Neumann algebra with normal faithful semifinite trace strong subadditivity holds. We give proofs of some other related properties of this generalized entropy, in particular, the concavity of $S(\rho_{12}) - S(\rho_2)$, the subadditivity of entropy, and a generalization of Araki-Lieb inequality.

1. INTRODUCTION

The concept of entropy of a state on an arbitrary semifinite von Neumann algebra was first presented by Segal [14] in 1960. He defined it as $S(\rho) = -\tau(h_\rho \log h_\rho)$ for semifinite trace τ and bounded density matrix h_ρ. Umegaki [17] has extended this definition for all $h_\rho \in L^1(\mathcal{M}, \tau)$. Of course, von Neumann entropy defined on tracial operators from $\mathbb{B}(\mathcal{H})$ is its particular case. A number of useful properties of entropy are known for the classical and quantum systems.

The strong subadditivity (in short SSA) hypothesis for entropy, which is the main subject of this paper, has a long history. It was first conjectured by O.E. Lanford and D.W. Robinson in [4]. A proof of this hypothesis was given in 1973 by E.H. Lieb and M.B. Ruskai as a consequence of the celebrated Lieb inequallity which appeared in [5]. The theorem was stated (and proved) for the case of the full algebra $\mathbb{B}(\mathcal{H})$ of all bounded linear operators on a Hilbert space and the canonical trace. Later various generalisations of this theorem, as well as related results, turned up. In particular, M.B. Ruskai [13] extended the triangle inequality to the case of finite von Neumann algebras and bounded density matrices. So far, the strong subadditivity of entropy for semifinite algebras, even for bounded density matrices, has remained unproven, although it was stated as a conjecture in [13]. The present paper gives a proof of the strong subadditivity theorem for the Segal entropy, in addition, the proof employs neither the Golden-Thompson inequality nor Klein's inequality, as it does in the classical case [7]. Uhlmann in [16] has shown that strong subadditivity follows from the concavity of the function

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$C \to Tr\exp(K + \ln C)$ for $C, K \in \mathbb{B}(\mathcal{H})$. The concavity of the corresponding function in an arbitrary semifinite von Neumann algebra $\mathcal{M}$ is unclear. An alternate proof of strong subadditivity, still for the case of $\mathbb{B}(\mathcal{H})$, was later found by Epstein [2] too. Unfortunately, also his approach fails in the general case. The proof we present, employs the monotonicity of relative entropy property shown for general von Neumann algebras by Ohya and Petz and used by them in a proof of SSA in the finite dimensional case. In accordance with the fact that the SSA property is equivalent to the monotonicity of relative entropy in $\mathbb{B}(\mathcal{H})$, the appearance of this relation in the proof of SSA is rather unavoidable. Some other properties of entropy, are given here as consequences of the strong subadditivity theorem. In the case of $\mathbb{B}(\mathcal{H})$ their proofs employ the generalized Golden-Thompson inequality (still unknown for three operators affiliated with a semifinite algebra) and the structure of the predual of $\mathbb{B}(\mathcal{H})$. Also the existence of extremal points in the class of tracial operators, lacking in the predual of an arbitrary von Neumann algebra, seems to by crucial in the proofs of some of these properties in the $\mathbb{B}(\mathcal{H})$ case.

Since the strong subadditivity of entropy is fundamental in quantum information theory not only on account of its physical applications but also because of mathematical consequences, it seems important and interesting to have this property for the general case of Segal's entropy in semifinite von Neumann algebras.

2. Basic Notations

Let $\mathcal{M}$ be a von Neumann algebra acting on a complex Hilbert space $\mathcal{H}$, with the identity operator $\mathbb{1}$. $\mathcal{M}_*$ denotes the predual of $\mathcal{M}$, that is the space of all normal functionals on $\mathcal{M}$. Such functional ρ is said to be positive if $\rho(xx^*) \geq 0$ for every $x \in \mathcal{M}$, to be a state if $\rho(\mathbb{1}) = 1$.

Throughout this paper, it will be assumed that $\mathcal{M}$ has a normal faithful semifinite trace τ.

A densely defined closed operator a on $\mathcal{H}$, with the domain D_a, affiliated with $\mathcal{M}$ is called τ - measurable if for each $\varepsilon > 0$ there exists a projection $p \in \mathcal{M}$ such that $p\mathcal{H} \subset D_a$ and $\tau(\mathbb{1} - p) \leq \varepsilon$. This is equivalent to the property: there exists $\delta > 0$ such that $\tau(e_{[\delta,\infty)}(|a|)) < +\infty$ where $e_{[\delta,\infty)}(|a|)$ is the spectral projection of $|a|$ corresponding to the interval $[\delta, \infty)$.

The algebra of *measurable operators* $\widetilde{\mathcal{M}}$ is defined as a topological *-algebra of densely defined closed operators on $\mathcal{H}$ affiliated with $\mathcal{M}$ with strong addition $+$ and strong multiplication $\cdot$, i.e.

$$a + b = \overline{a+b}, \qquad a \cdot b = \overline{ab}, \qquad a, b \in \widetilde{\mathcal{M}},$$

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where $\overline{a+b}$ and $\overline{ab}$ are the closures of the corresponding operators defined by addition and composition respectively on the natural domains given by the intersections of the domains of the a and b and of the range of b and the domain of a respectively. The translation-invariant measure topology is defined by a fundamental system of neighbourhoods of 0, $\{N(\varepsilon,\delta) : \varepsilon,\delta > 0\}$, given by

$$N(\varepsilon,\delta) = \{a \in \widetilde{\mathcal{M}} : \text{there exists a projection } p \text{ in } \mathcal{M} \text{ such that} \\ ap \in \mathcal{M}, \quad \|ap\| \leq \varepsilon \quad \text{and} \quad \tau(\mathbb{1} - p) \leq \delta\}.$$

Accordingly, $a_n \to a$ in measure if for each $\varepsilon > 0$ there is a sequence $\{p_n\}$ of projections in $\mathcal{M}$ such that

$$\tau(\mathbb{1} - p_n) \to 0 \ \ and \ \ \|(a_n - a)p_n\| < \varepsilon.$$

The following "technical" form of convergence in measure is useful. Let

$$|a_n - a| = \int_0^\infty \lambda\, e_n(d\lambda)$$

be the spectral decomposition of $|a_n - a|$ with spectral measure e_n taking values in $\mathcal{M}$ since $a_n - a$, and thus $|a_n - a|$, are affiliated with $\mathcal{M}$. Then $a_n \to a$ *in measure* if and only if for each $\varepsilon > 0$

$$\tau(e_n([\varepsilon,\infty))) \to 0.$$

A stronger mode of convergence is convergence in Segal's sense. We say that $a_n \to a$ in Segal's sense if for each $\varepsilon > 0$ there is a projection p in $\mathcal{M}$ with $\tau(\mathbb{1} - p) < \varepsilon$ such that $(a_n - a)p \in \mathcal{M}$ and $\|(a_n - a)p\| \to 0$.

Denote by $L^1(\mathcal{M},\tau)$ the space of all measurable operators h with finite trace. The norm in $L^1(\mathcal{M},\tau)$ is defined as

$$\|h\|_1 = \tau(|h|).$$

If ρ is a normal state on $\mathcal{M}$, then there exists a so called density matrix h_ρ such that

$$\rho(x) = \tau(h_\rho x) \ \ for \ \ every \ \ x \in \mathcal{M}.$$

h_ρ is a self-adjoint, non negative operator belonging to $L^1(\mathcal{M},\tau)$, with spectral measure in $\mathcal{M}$. It has the $\|\cdot\|_1$-norm equal to one ($\tau(h_\rho) = 1$). By $s(x)$ we denote the support projection of a self-adjoint operator x.

Taking the operator function $f\colon x \to -x\log x$ on $L^1(\mathcal{M},\tau)$, we define the entropy S in Segal sense [14] of the state ρ as follows

$$S(\rho) := \tau(f(h_\rho)) = -\tau(h_\rho \log h_\rho)$$

Notice, that operator h_ρ can have the spectrum $\mathrm{sp}h_\rho = [0,+\infty)$ so the operator function $h_\rho \log h_\rho$ is expressed by

$$h_\rho \log h_\rho = h_\rho \cdot s(h_\rho) \log h_\rho,$$

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where $\cdot$ denotes the strong product of operators. Some properties of entropy were discussed in [10, 11, 12, 14, 17, 18].

3. CONCEPTS OF RELATIVE ENTROPY

In quantum information theory, the relative entropy is the most fundamental concept, having many important properties, and it is often taken as a starting point for the introduction of other kinds of entropy. Unfortunately, there are several definitions of relative entropy, which are not equivalent at first sight. The most general one, appropriate for the case of an arbitrary von Neumann algebra, expressed by means of the relative modular operator, was given by Araki [1] and employed by Ohya and Petz [11]. Ohya and Petz have also presented an alternative definition of relative entropy known as Kosaki's formula. A proof of equivalence of these two definitions can be found in [11, Theorem 5.11] as well as the statement that in the finite dimensional case they reduce to the formula $S(\omega, \varphi) = Tr(h_\omega(\log h_\omega - \log h_\varphi))$ [11, Chapter 5]. In [17] Umegaki defined quantum information between two states ω and φ such that

$$s(h_\omega) \leq s(h_\varphi),$$

on an arbitrary von Neumann algebra with semifinite trace τ in the similar way: $I(\omega, \varphi) = \tau(h_\omega(\log h_\omega - \log h_\varphi))$. It should be noted that this definition is a little formal, especially for a semifinite and not finite trace, since then the operators $\log h_\omega$ and $\log h_\varphi$ needn't be even measurable let alone the relation $h_\omega(\log h_\omega - \log h_\varphi) \in L^1(M, \tau)$. The proper formula for the information reads as

$$\begin{aligned} I(\omega, \varphi) =& \tau(h_\omega^{1/2}(\log h_\omega - log h_\varphi)h_\omega^{1/2}) \\ =& \tau(h_\omega^{1/2} \log h_\omega h_\omega^{1/2}) - \tau(h_\omega^{1/2} log h_\varphi)h_\omega^{1/2} \\ =& "\omega(\log h_\omega) - \omega(\log h_\varphi)" \end{aligned}$$

with an appriopriate definition of $\omega(\log h_\omega)$ and $\omega(\log h_\varphi)$, see [8] for a more detailed explanation. In [9] it is observed that for finite trace τ and states with bounded density matrices Araki's relative entropy is equivalent to Umegaki's information, however no proof of this important fact is given. A theorem that shows equivalence of Araki's and Umegaki's formulas for the relative entropy of arbitrary states on a semifinite von Neumann algebra is presented in [8]. It turns out that the equivalence, thought expected, requires not only a piece of involved analysis of unbounded operators but also a new result presenting a useful formula for the relative modular operator. As a consequence, the properties proved for Araki's relative entropy $S(\omega, \varphi)$ such as the ones given in [11, Corollary 5.12] may be applied to Umegaki's information $I(\omega, \varphi)$.

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It is worth noticing that as in Araki's relative entropy the information $I(\omega, \varphi)$ is uniquely determined as finite or $+\infty$ and is independent of the choice of the trace τ.

Taking into account equivalence between $I(\omega, \varphi)$ and $S(\varphi, \omega)$, where $S(\varphi, \omega)$ is Araki's relative entropy for states $\varphi, \omega \in \mathcal{M}_*$, we have the following

Theorem 1. *Let ω, φ be a normal state on a semifinite von Neumann algebra $\mathcal{M}$. Then*

a) $I(\omega, \varphi) = 0$ if and only if $\varphi = \omega$.

b) Let $\mathcal{N}$ be another von Neumann algebra and let $\alpha\colon \mathcal{N} \to \mathcal{M}$ be a unital normal Schwarz mapping. Then

$$I(\omega \circ \alpha, \varphi \circ \alpha) \leq I(\omega, \varphi)$$

(see[11, chapter 5]*).*

The proof refers to equvalence between Araki's relative entropy and Kosaki's formula. The monotonicity property is shown to hold for the last one.

4. Strong Subadditivity of Entropy

Consider the tensor product von Neumann algebra $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$. It is known [15] that for semifinite faithful traces τ_1 and τ_2 on $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively, $\tau_{12} := \tau_1 \otimes \tau_2$ is a semifinite faithful trace on $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$.

By π_1 and π_2 we denote the following natural counterparts of the partial traces in $\mathbb{B}(\mathcal{H})$:

$$\pi_1\colon (\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2)_* \to (\mathcal{M}_1)_*$$

$$\pi_2\colon (\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2)_* \to (\mathcal{M}_2)_*$$

given by the following formula: for any $\rho_{12} \in (\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2)_*$ and $x_1 \in \mathcal{M}_1$, $x_2 \in \mathcal{M}_2$

$$\pi_1(\rho_{12})(x_1) = \rho_{12}(x_1 \otimes \mathbb{1}) = \tau_{12}(h_{\rho_{12}}(x_1 \otimes \mathbb{1})),$$

$$\pi_2(\rho_{12})(x_2) = \rho_{12}(\mathbb{1} \otimes x_2) = \tau_{12}(h_{\rho_{12}}(\mathbb{1} \otimes x_2))$$

where $h_{\rho_{12}}$ is the density matrix of the state ρ_{12}.

Denote $\rho_1 := \pi_1(\rho_{12})$ and $\rho_2 := \pi_2(\rho_{12})$.

Then

$$\tau_{12}(h_{\rho_{12}}(x_1 \otimes \mathbb{1})) = \tau_1(h_{\rho_1} x_1)$$

and

$$\tau_{12}(h_{\rho_{12}}(\mathbb{1} \otimes x_2)) = \tau_2(h_{\rho_2} x_2).$$

Remark 2. We have

$$\rho_{12}(s(\rho_1) \otimes \mathbb{1}) = \rho_1(s(\rho_1)) = 1$$

thus

$$s(\rho_{12}) = s(h_{\rho_{12}}) \leq s(\rho_1) \otimes \mathbb{1},$$

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and similarly

$$s(\rho_{12}) = s(h_{\rho_{12}}) \leq \mathbb{1} \otimes s(\rho_2),$$

which gives

$$s(h_{\rho_{12}}) \leq s(\rho_1) \otimes s(\rho_2) = s(\rho_1 \otimes \rho_2) = s(h_{\rho_1 \otimes \rho_2}).$$

A similar line of reasoning we can perform for $\mathcal{M} = \mathcal{M}_1 \bar{\otimes} \mathcal{M}_2 \bar{\otimes} \mathcal{M}_3$ with semifinite trace $\tau_{123} = \tau_1 \otimes \tau_2 \otimes \tau_3$. Then for $i, j \in \{1, 2, 3\}$ $\pi_{ij} \colon \mathcal{M}_* \to (\mathcal{M}_i \bar{\otimes} \mathcal{M}_j)_*$ are partial traces and

$$\rho_{ij} := \pi_{ij}(\rho_{123}), \quad \rho_{123} \in (\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2 \bar{\otimes} \mathcal{M}_3)_*.$$

Of course

$$\pi_{12}(\rho_{123})(x_{12}) = \rho_{123}(x_{12} \otimes \mathbb{1}) = \tau_{123}(h_{\rho_{123}}(x_{12} \otimes \mathbb{1}))$$

for $x_{12} \in \mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$,

$$\pi_{23}(\rho_{123})(x_{23}) = \rho_{123}(\mathbb{1} \otimes x_{23}) = \tau_{123}(h_{\rho_{123}}(\mathbb{1} \otimes x_{23}))$$

for $x_{23} \in \mathcal{M}_2 \bar{\otimes} \mathcal{M}_3$,

$$\pi_2(\rho_{123})(x_2) = \rho_{123}(\mathbb{1} \otimes x_2 \otimes \mathbb{1}) = \tau_{123}(h_{\rho_{123}}(\mathbb{1} \otimes x_2 \otimes \mathbb{1}))$$

for $x_2 \in \mathcal{M}_2$.

Observe that we have e.g.

$$\pi_{23}^* \colon \mathcal{M}_2 \overline{\otimes} \mathcal{M}_3 \to \mathcal{M}_1 \overline{\otimes} \mathcal{M}_2 \overline{\otimes} \mathcal{M}_3$$

where

$$\pi_{23}^*(x_{23}) = \mathbb{1} \otimes x_{23}.$$

Lemma 3. *Let $\theta_{123} = \varphi_{12} \otimes \psi_3$. Then $\theta_{23} = \varphi_2 \otimes \psi_3$ and $\theta_2 = \varphi_2$.*

Proof. We have

$$\begin{aligned}
\theta_{23}(x_2 \otimes x_3) &= \theta_{123}(\mathbb{1} \otimes x_2 \otimes x_3) = \varphi_{12}(\mathbb{1} \otimes x_2)\psi_3(x_3) \\
&= \varphi_2(x_2)\psi_3(x_3) = \varphi_2 \otimes \psi_3(x_2 \otimes x_3), \\
\theta_2(x_2) &= \theta_{123}(\mathbb{1} \otimes x_2 \otimes \mathbb{1}) \\
&= \varphi_{12}(\mathbb{1} \otimes x_2)\psi_3(\mathbb{1}) = \varphi_{12}(\mathbb{1} \otimes x_2) = \varphi_2(x_2)
\end{aligned}$$

yielding the claim. □

Next two results concern arbitrary semifinite von Neumann algebras.

Lemma 4. *Let $\mathcal{M}$ be a von Neumann algebra with a normal faithful semifinite trace τ, let h and x be selfadjoint positive operators in $\widetilde{\mathcal{M}}$ such that $hx \in L^1(\mathcal{M}, \tau)$. Let for $\varepsilon > 0$, $x_\varepsilon = x(\mathbb{1} + \varepsilon x)^{-1}$. Then $hx_\varepsilon \in L^1(\mathcal{M}, \tau)$ and*

$$\lim_{\varepsilon \to 0} \tau(hx_\varepsilon) = \tau(hx).$$

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Proof. Observe first that

$$hx_\varepsilon = hx(\mathbb{1} + \varepsilon x)^{-1},$$

and since $hx \in L^1(\mathcal{M}, \tau)$ and $(\mathbb{1} + \varepsilon x)^{-1} \in \mathcal{M}$, we obtain $hx(\mathbb{1} + \varepsilon x)^{-1} \in L^1(\mathcal{M}, \tau)$. Further, we have

$$\tau(hx_\varepsilon) - \tau(hx) = \tau(h(x_\varepsilon - x)) = \tau(hx((\mathbb{1} + \varepsilon x)^{-1} - \mathbb{1})).$$

Let $\rho \in \mathcal{M}_*$ be the element uniquely corresponding to $hx \in L^1(\mathcal{M}, \tau)$ (the one whose 'density matrix' is hx). Then from the equality above and the fact that

$$\lim_{\varepsilon \to 0}((\mathbb{1} + \varepsilon x)^{-1} - \mathbb{1}) = 0 \quad \sigma\text{-weakly},$$

we get

$$\tau(hx_\varepsilon) - \tau(hx) = \tau(hx((\mathbb{1} + \varepsilon x)^{-1} - \mathbb{1})) = \rho((\mathbb{1} + \varepsilon x)^{-1} - \mathbb{1}) \to 0,$$

yielding the claim. □

The next result supplements Lemma 4.

Proposition 5. *Let $\mathcal{M}$ be a von Neumann algebra with a normal faithful semifinite trace τ, let h and x be selfadjoint positive operators in $\widetilde{\mathcal{M}}$ such that $h \in L^1(\mathcal{M}, \tau)$. Assume that there exists a finite limit $\lim_{\varepsilon \to 0} \tau(hx_\varepsilon)$. It follows that $h^{1/2}xh^{1/2} \in L^1(\mathcal{M}, \tau)$ and*

$$\lim_{\varepsilon \to 0} \tau(hx_\varepsilon) = \tau(h^{1/2}xh^{1/2}).$$

Proof. Since $x_\varepsilon \in \mathcal{M}$, and $h \in L^1(\mathcal{M}, \tau)$, we have

$$|h^{1/2}x_\varepsilon|^2 = x_\varepsilon h x_\varepsilon \in L^1(\mathcal{M}, \tau),$$

which means that $h^{1/2}x_\varepsilon \in L^2(\mathcal{M}, \tau)$. Since obviously $h^{1/2} \in L^2(\mathcal{M}, \tau)$, we obtain

$$\tau(hx_\varepsilon) = \tau(h^{1/2}(h^{1/2}x_\varepsilon)) = \tau(h^{1/2}x_\varepsilon h^{1/2}).$$

Let $\varepsilon_1 < \varepsilon_2$. Then for the spectral representation

$$x = \int_0^\infty \lambda\, e(d\lambda),$$

we get

$$\begin{aligned} x_{\varepsilon_1} - x_{\varepsilon_2} &= \int_0^\infty \Big(\frac{\lambda}{1 + \varepsilon_1\lambda} - \frac{\lambda}{1 + \varepsilon_2\lambda}\Big)\, e(d\lambda) \\ &= (\varepsilon_2 - \varepsilon_1) \int_0^\infty \frac{\lambda^2}{(1 + \varepsilon_1\lambda)(1 + \varepsilon_2\lambda)}\, e(d\lambda) \geq 0, \end{aligned}$$

consequently, since $h^{1/2}(x_{\varepsilon_1} - x_{\varepsilon_2})h^{1/2} \geq 0$, we get

$$\begin{aligned} &\|h^{1/2}(x_{\varepsilon_1} - x_{\varepsilon_2})h^{1/2}\|_1 = \tau(h^{1/2}(x_{\varepsilon_1} - x_{\varepsilon_2})h^{1/2}) \\ =&\tau(h^{1/2}x_{\varepsilon_1}h^{1/2}) - \tau(h^{1/2}x_{\varepsilon_2}h^{1/2}) = \tau(hx_{\varepsilon_1}) - \tau(hx_{\varepsilon_2}), \end{aligned}$$

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showing that $\{h^{1/2}x_\varepsilon h^{1/2} : \varepsilon > 0\}$ is Cauchy in $\|\cdot\|_1$-norm for $\varepsilon \to 0$. Denote

$$\|\cdot\|_1 - \lim_{\varepsilon\to 0} h^{1/2}x_\varepsilon h^{1/2} = z \in L^1(\mathcal{M},\tau). \tag{1}$$

We have

$$x - x_\varepsilon = \int_0^\infty \Big(\lambda - \frac{\lambda}{1+\varepsilon\lambda}\Big)\, e(d\lambda) = \varepsilon\int_0^\infty \frac{\lambda^2}{1+\varepsilon\lambda}\, e(d\lambda).$$

Take an arbitrary $\eta > 0$. There is $\delta > 0$ such that $\tau(e([0,\delta])^\perp) = \tau(e((\delta,+\infty))) < \eta$, and we get

$$\|(x - x_\varepsilon)e([0,\delta])\| = \Big\|\varepsilon\int_0^\delta \frac{\lambda^2}{1+\varepsilon\lambda}\, e(d\lambda)\Big\| \le \varepsilon\delta^2 \to 0 \quad \text{as} \quad \varepsilon\to 0,$$

which means that $x_\varepsilon \to x$ in *Segal's sense*. It follows that $x_\varepsilon \to x$ *in measure*. Since multiplication is continuous in the measure topology, we obtain

$$h^{1/2}x_\varepsilon h^{1/2} \to h^{1/2}xh^{1/2} \quad \textit{in measure.}$$

Since convergence in $\|\cdot\|_1$-norm implies convergence in measure, equality (1) yields

$$\|\cdot\|_1 - \lim_{\varepsilon\to 0} h^{1/2}x_\varepsilon h^{1/2} = h^{1/2}xh^{1/2}.$$

In particular, we get

$$\lim_{\varepsilon\to 0}\tau(hx_\varepsilon) = \lim_{\varepsilon\to 0}\tau(h^{1/2}x_\varepsilon h^{1/2}) = \tau(h^{1/2}xh^{1/2}). \tag{2}$$

□

Let $\mathcal{M}$ and $\mathcal{N}$ be arbitrary von Neumann algebras, and let x be a selfadjoint positive operator in $\widetilde{\mathcal{M}}$. Since

$$((\mathbb{1}_\mathcal{M}\otimes\varepsilon x)\otimes\mathbb{1}_\mathcal{N})^{-1} = (\mathbb{1}_\mathcal{M}\otimes\varepsilon x)^{-1}\otimes\mathbb{1}_\mathcal{N},$$

we have

$$\begin{aligned}(x\otimes\mathbb{1}_\mathcal{N})_\varepsilon &= (x\otimes\mathbb{1}_\mathcal{N})(\mathbb{1}_\mathcal{M}\otimes\mathbb{1}_\mathcal{N} + \varepsilon(x\otimes\mathbb{1}_\mathcal{N}))^{-1}\\ &= (x\otimes\mathbb{1}_\mathcal{N})(\mathbb{1}_\mathcal{M}\otimes\mathbb{1}_\mathcal{N} + \varepsilon x\otimes\mathbb{1}_\mathcal{N})^{-1}\\ &= (x\otimes\mathbb{1}_\mathcal{N})((\mathbb{1}_\mathcal{M}+\varepsilon x)\otimes\mathbb{1}_\mathcal{N})^{-1}\\ &= (x\otimes\mathbb{1}_\mathcal{N})((\mathbb{1}_\mathcal{M}+\varepsilon x)^{-1}\otimes\mathbb{1}_\mathcal{N})\\ &= x(\mathbb{1}_\mathcal{M}+\varepsilon x)^{-1}\otimes\mathbb{1}_\mathcal{N} = x_\varepsilon\otimes\mathbb{1}_\mathcal{N}.\end{aligned} \tag{3}$$

Proposition 6. *Let $\rho_{123} \in (\mathcal{M}_1\overline{\otimes}\mathcal{M}_2\overline{\otimes}\mathcal{M}_3)_*$ be a normal state with density matrix $h_{\rho_{123}} \in L^1(\mathcal{M}_1\overline{\otimes}\mathcal{M}_2\overline{\otimes}\mathcal{M}_3,\tau_{123})$ where $\tau_{123} = \tau_1\otimes\tau_2\otimes\tau_3$ is*

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a semifinite trace. Assume that $S(\rho_{12})$, $S(\rho_{23})$, $S(\rho_1)$ and $S(\rho_2)$ are finite. Then

$$S(\rho_{12}) = -\tau_{123}\big(h_{\rho_{123}}^{1/2}\log(h_{\rho_{12}}\otimes\mathbb{1})h_{\rho_{123}}^{1/2}\big), \tag{4}$$

$$S(\rho_{23}) = -\tau_{123}\big(h_{\rho_{123}}^{1/2}\log(\mathbb{1}\otimes h_{\rho_{23}})h_{\rho_{123}}^{1/2}\big), \tag{5}$$

$$\begin{aligned} S(\rho_1) &= -\tau_{123}\big(h_{\rho_{123}}^{1/2}\log(h_{\rho_1}\otimes\mathbb{1}\otimes\mathbb{1})h_{\rho_{123}}^{1/2}\big) \\ &= -\tau_{12}\big(h_{\rho_{12}}^{1/2}\log(h_{\rho_1}\otimes\mathbb{1})h_{\rho_{12}}^{1/2}\big), \end{aligned} \tag{6}$$

$$\begin{aligned} S(\rho_2) &= -\tau_{123}\big(h_{\rho_{123}}^{1/2}\log(\mathbb{1}\otimes h_{\rho_2}\otimes\mathbb{1})h_{\rho_{123}}^{1/2}\big) \\ &= -\tau_{12}\big(h_{\rho_{12}}^{1/2}\log(\mathbb{1}\otimes h_{\rho_2})h_{\rho_{12}}^{1/2}\big). \end{aligned} \tag{7}$$

Proof. We shall prove only that equality (4) holds, the remaining proofs being analogous. Denote by $\log^+$, and $\log^-$ the functions $\log^+ = \max\{\log, 0\}$, and $\log^- = \max\{-\log, 0\}$, respectively, so that $\log = \log^+ - \log^-$. On account of the finiteness of $S(\rho_{12})$, we have

$$-S(\rho_{12}) = \tau_{12}(h_{\rho_{12}}\log h_{\rho_{12}}) = \tau_{12}(h_{\rho_{12}}\log^+ h_{\rho_{12}}) - \tau_{12}(h_{\rho_{12}}\log^- h_{\rho_{12}}),$$

with both terms on the right hand side of the above equality being finite, i.e. $h_{\rho_{12}}\log^+ h_{\rho_{12}}, h_{\rho_{12}}\log^- h_{\rho_{12}} \in L^1(\mathcal{M}_1\overline{\otimes}\mathcal{M}_2, \tau_{12})$. On account of Lemma 3, Proposition 4 (i) and relation (3) we obtain

$$\begin{aligned} \tau_{12}(h_{\rho_{12}}\log^+ h_{\rho_{12}}) &= \lim_{\varepsilon\to 0}\tau_{12}(h_{\rho_{12}}(\log^+ h_{\rho_{12}})_\varepsilon) = \lim_{\varepsilon\to 0}\rho_{12}((\log^+ h_{\rho_{12}})_\varepsilon) \\ &= \lim_{\varepsilon\to 0}\rho_{123}((\log^+ h_{\rho_{12}})_\varepsilon\otimes\mathbb{1}) = \lim_{\varepsilon\to 0}\rho_{123}((\log^+ h_{\rho_{12}}\otimes\mathbb{1})_\varepsilon) \\ &= \lim_{\varepsilon\to 0}\tau_{123}(h_{\rho_{123}}(\log^+ h_{\rho_{12}}\otimes\mathbb{1})_\varepsilon) = \tau_{123}\big(h_{\rho_{123}}^{1/2}\log^+(h_{\rho_{12}}\otimes\mathbb{1})h_{\rho_{123}}^{1/2}\big). \end{aligned}$$

Analogously

$$\tau_{12}(h_{\rho_{12}}\log^- h_{\rho_{12}}) = \tau_{123}\big(h_{\rho_{123}}^{1/2}\log^-(h_{\rho_{12}}\otimes\mathbb{1})h_{\rho_{123}}^{1/2}\big),$$

which gives

$$\begin{aligned} -S(\rho_{12}) &= \tau_{123}\big(h_{\rho_{123}}^{1/2}\log^+(h_{\rho_{12}}\otimes\mathbb{1})h_{\rho_{123}}^{1/2}\big) + \\ &\quad - \tau_{123}\big(h_{\rho_{123}}^{1/2}\log^-(h_{\rho_{12}}\otimes\mathbb{1})h_{\rho_{123}}^{1/2}\big) \\ &= \tau_{123}\big(h_{\rho_{123}}^{1/2}(\log^+(h_{\rho_{12}}\otimes\mathbb{1}) - \log^-(h_{\rho_{12}}\otimes\mathbb{1}))h_{\rho_{123}}^{1/2}\big) \\ &= \tau_{123}\big(h_{\rho_{123}}^{1/2}\log(h_{\rho_{12}}\otimes\mathbb{1})h_{\rho_{123}}^{1/2}\big) \end{aligned}$$

yielding the claim. □

Now we are in a position to prove the main result of the paper.

Theorem (Strong subadditivity) 7. *Let $\rho_{123} \in (\mathcal{M}_1\overline{\otimes}\mathcal{M}_2\overline{\otimes}\mathcal{M}_3)_*$ be a normal state with density matrix $h_{\rho_{123}} \in L^1(\mathcal{M}_1\overline{\otimes}\mathcal{M}_2\overline{\otimes}\mathcal{M}_3, \tau_{123})$ where $\tau_{123} = \tau_1\otimes\tau_2\otimes\tau_3$ is a semifinite trace. Assume that $S(\rho_{123})$, $S(\rho_{12})$, $S(\rho_{23})$, $S(\rho_1)$ and $S(\rho_2)$ are finite. Then*

$$S(\rho_{123}) \le S(\rho_{12}) + S(\rho_{23}) - S(\rho_2).$$

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Proof. The line of the proof is similar to that in [11, Proposition 1.3] where the finite dimensional case is dealt with. By virtue of equalities (4), (5), (6) and (7), we have

$$
\begin{aligned}
&- S(\rho_{123}) + S(\rho_1) + S(\rho_{23}) = \tau_{123}(h_{\rho_{123}} \log h_{\rho_{123}}) + \\
&- \tau_{123}(h_{\rho_{123}}^{1/2}(\log h_{\rho_1} \otimes \mathbb{1} \otimes \mathbb{1}) h_{\rho_{123}}^{1/2}) + \\
&- \tau_{123}(h_{\rho_{123}}^{1/2}(\mathbb{1} \otimes \log h_{\rho_{23}}) h_{\rho_{123}}^{1/2}) \\
&= \tau_{123}(h_{\rho_{123}}^{1/2} \log h_{\rho_{123}} h_{\rho_{123}}^{1/2}) - \tau_{123}(h_{\rho_{123}}^{1/2} \log(h_{\rho_1} \otimes \mathbb{1} \otimes \mathbb{1}) h_{\rho_{123}}^{1/2}) + \\
&- \tau_{123}(h_{\rho_{123}}^{1/2} \log(\mathbb{1} \otimes h_{\rho_{23}}) h_{\rho_{123}}^{1/2}) = \tau_{123}(h_{\rho_{123}}^{1/2} \log h_{\rho_{123}} h_{\rho_{123}}^{1/2}) + \\
&- \tau_{123}(h_{\rho_{123}}^{1/2}(\log(h_{\rho_1} \otimes \mathbb{1} \otimes \mathbb{1}) + \log(\mathbb{1} \otimes h_{\rho_{23}})) h_{\rho_{123}}^{1/2}) \\
&= \tau_{123}(h_{\rho_{123}}^{1/2} \log h_{\rho_{123}} h_{\rho_{123}}^{1/2}) - \tau_{123}(h_{\rho_{123}} \log(h_{\rho_1} \otimes h_{\rho_{23}}) h_{\rho_{123}}^{1/2}) \\
&= \tau_{123}(h_{\rho_{123}}^{1/2}(\log h_{\rho_{123}} - \log(h_{\rho_1} \otimes h_{\rho_{23}})) h_{\rho_{123}}^{1/2}),
\end{aligned}
\tag{8}
$$

and analogously

$$
\begin{aligned}
&- S(\rho_{12}) + S(\rho_1) + S(\rho_2) = \tau_{12}(h_{\rho_{12}} \log h_{\rho_{12}}) + \\
&- \tau_{12}(h_{\rho_{12}}^{1/2}(\log h_{\rho_1} \otimes \mathbb{1}) h_{\rho_{12}}^{1/2}) - \tau_{12}(h_{\rho_{12}}^{1/2}(\mathbb{1} \otimes \log h_{\rho_2}) h_{\rho_{12}}^{1/2}) \\
&= \tau_{12}(h_{\rho_{12}}^{1/2}(\log h_{\rho_{12}} - \log(h_{\rho_1} \otimes h_{\rho_2}) h_{\rho_{12}}^{1/2}).
\end{aligned}
\tag{9}
$$

Now applying the definition of information, we obtain from relations (8) and (9)

$$
\begin{aligned}
&-S(\rho_{123}) + S(\rho_1) + S(\rho_{23}) \\
&= \tau_{123}(h_{\rho_{123}}^{1/2}(\log h_{\rho_{123}} - \log(h_{\rho_1} \otimes h_{\rho_{23}})) h_{\rho_{123}}^{1/2}) = I(\rho_{123}, \rho_1 \otimes \rho_{23}),
\end{aligned}
\tag{10}
$$

and

$$
\begin{aligned}
-S(\rho_{12}) + S(\rho_1) + S(\rho_2) &= \tau_{12}(h_{\rho_{12}}^{1/2}(\log h_{\rho_{12}} - \log(h_{\rho_1} \otimes h_{\rho_2}) h_{\rho_{12}}^{1/2}) \\
&= I(\rho_{12}, \rho_1 \otimes \rho_2).
\end{aligned}
\tag{11}
$$

Define a normal completely positive unital map

$$\alpha \colon \mathcal{M}_1 \overline{\otimes} \mathcal{M}_2 \to \mathcal{M}_1 \overline{\otimes} \mathcal{M}_2 \overline{\otimes} \mathcal{M}_3$$

by the formula

$$\alpha(x_{12}) = x_{12} \otimes \mathbb{1}, \quad x_{12} \in M_1 \overline{\otimes} M_2.$$

Then we have

$$\rho_{123}(\alpha(x_{12})) = \rho_{123}(x_{12} \otimes \mathbb{1}) = \rho_{12}(x_{12}),$$

and

$$
\begin{aligned}
\rho_1 \otimes \rho_{23}(\alpha(x_1 \otimes x_2)) &= \rho_1 \otimes \rho_{23}(x_1 \otimes x_2 \otimes \mathbb{1}) = \rho_1(x_1)\rho_{23}(x_2 \otimes \mathbb{1}) \\
&= \rho_1(x_1)\rho_2(x_2) = \rho_1 \otimes \rho_2(x_1 \otimes x_2),
\end{aligned}
$$

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thus

$$\rho_{123} \circ \alpha = \rho_{12} \qquad \text{and} \qquad (\rho_1 \otimes \rho_{23}) \circ \alpha = \rho_1 \otimes \rho_2.$$

On account of Theorem 1, we obtain

$$I(\rho_{12}, \rho_1 \otimes \rho_2) = I(\rho_{123} \circ \alpha, (\rho_1 \otimes \rho_{23}) \circ \alpha) \leq I(\rho_{123}, \rho_1 \otimes \rho_{23}),$$

i.e. by the equalities (10) and (11)

$$-S(\rho_{12}) + S(\rho_1) + S(\rho_2) \leq -S(\rho_{123}) + S(\rho_1) + S(\rho_{23}),$$

which yields the claim. □

Now we give a theorem known in the finite dimensional case of $\mathbb{B}(\mathcal{H})$ with the proof given by Lieb in [5]. The thesis of this theorem was also mentioned as a conjecture for semifinite trace in [13]. A property presented in this theorem can be viewed as a particular case of the strong subadditivity property. However, in the classical quantum mechanical case [7] its proof is straightforward. Based on [11, Corollary 5.6] and the equality

$$I(\rho, \phi) = S(\rho, \phi)$$

we have

Theorem 8 (Subadditivity of entropy). *For any $\rho_{12} \in (\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2)_*$*

$$S(\rho_{12}) \leq S(\rho_1) + S(\rho_2).$$

The equality holds if and only if $\rho_{12} = \rho_1 \otimes \rho_2$

Proof. Interchanging $\mathcal{M}_2$ and $\mathcal{M}_3$ in Theorem 7 (Strong subadditivity) and taking $\mathcal{M}_3 = \mathbb{C}\mathbb{1}$ one has

$$S(\rho_{12}) \leq S(\rho_1) + S(\rho_2). \tag{12}$$

Moreover

$$S(\rho_1) = -\tau_1(h_{\rho_1} \log h_{\rho_1}) = -\tau_{12}(h_{\rho_{12}} \log(h_{\rho_1} \otimes \mathbb{1}))$$

and

$$S(\rho_2) = -\tau_2(h_{\rho_2} \log h_{\rho_2}) = -\tau_{12}(h_{\rho_{12}} \log(\mathbb{1} \otimes h_{\rho_2})).$$

Then,

$$S(\rho_{12}) = S(\rho_1) + S(\rho_2)$$

is equivalent to

$$I(\rho_{12}, \rho_1 \otimes \rho_2) = 0.$$

On account of Theorem 1, the above equality is equivalent to

$$\rho_{12} = \rho_1 \otimes \rho_2.$$

□

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The following Araki-Lieb theorem was proved for the algebra $\mathbb{B}(\mathcal{H})$ of all bounded operators on Hilbert space $\mathcal{H}$ equipped with the canonical trace [6]. M.B. Ruskai in [13] has proved its weaker version in the case of a finite von Neumann algebra, and only for bounded density matrices. Here we give its version extended to an arbitrary semifinite von Neumann algebra, as a consequence of the strong subadditivity theorem. We begin with a proposition which is interesting in its own right.

Proposition 9. *Let h be a density matrix such that $h\log h \in L^1(\mathcal{M},\tau)$ and $h \in L^2(\mathcal{M},\tau)$. Then*

$$\tau(h\log h) \leq \log\tau(h^2).$$

Proof. Let

$$h = \int_0^\infty t\, e(dt)$$

be the spectral decomposition of h. Denote

$$h_n = \int_{1/n}^n t\, e(dt), \quad h_n' = \int_{1/n}^1 t\, e(dt), \quad h_n'' = \int_1^n t\, e(dt).$$

Then $h_n, h_n', h_n'' \in \mathcal{M}$ and $h_n = h_n' + h_n''$. We have

$$hh_n = \int_{1/n}^n t^2\, e(dt),$$

and since

$$\int_0^\infty t^2\, \tau(e(dt)) = \tau(h^2) < \infty,$$

we obtain from the Lebesgue Monotone Convergence Theorem

$$\begin{aligned} \|h^2 - hh_n\|_1 &= \left\| \int_0^{1/n} t^2\, e(dt) + \int_n^\infty t^2\, e(dt) \right\|_1 \\ &= \int_0^{1/n} t^2\tau(e(dt)) + \int_n^\infty t^2\, \tau(e(dt)) \to 0. \end{aligned}$$

For positive self-adjoint operator $a = \int_0^\infty t f(dt)$ affiliated with $\mathcal{M}$ we set $\tau(a) = \sup_n \tau(\int_0^n t f(dt)) = \int_0^\infty t\tau(f(dt))$, see [3]. Consequently,

$$\tau(hh_n) \to \tau(h^2). \tag{13}$$

Put

$$\log^+ t = \max\{\log t, 0\}, \quad \log^- t = \max\{-\log t, 0\},$$

thus $\log = \log^+ - \log^-$. We have

$$\log^+ h_n = \log h_n'', \qquad \log^- h_n = -\log h_n',$$

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and

$$h \log h_n' = \int_{1/n}^1 t \log t \, e(dt), \qquad h \log h_n'' = \int_1^n t \log t \, e(dt),$$
$$h \log^- h = -\int_0^1 t \log t \, e(dt), \quad h \log^+ h = \int_1^\infty t \log t \, e(dt).$$

Moreover, from the condition $h \log h \in L^1(\mathcal{M}, \tau)$, i.e.

$$\int_0^\infty t \log t \, \tau(e(dt)) = \int_0^1 t \log t \, \tau(e(dt)) + \int_1^\infty t \log t \, \tau(e(dt)) \text{—finite,}$$

we obtain, again on account of the Lebesgue Monotone Convergence Theorem,

$$\|h \log^- h - h \log^- h_n\|_1 = \|h \log^- h + h \log h_n'\|_1$$
$$= \left\| \int_0^{1/n} t \log t \, e(dt) \right\|_1 = -\int_0^{1/n} t \log t \, \tau(e(dt)) \to 0,$$

and

$$\|h \log^+ h - h \log^+ h_n\|_1 = \|h \log^+ h - h \log h_n''\|_1$$
$$= \left\| \int_n^\infty t \log t \, e(dt) \right\|_1 = \int_n^\infty t \log t \, \tau(e(dt)) \to 0.$$

Consequently,

(14)
$$\tau(h \log^- h_n) \to \tau(h \log^- h) \quad \text{and} \quad \tau(h \log^+ h_n) \to \tau(h \log^+ h).$$

Let ρ be the state with density matrix h, i.e.

$$\rho(x) = \tau(hx), \quad x \in \mathcal{M}.$$

Since log is an operator concave function, Jensen's inequality yields

(15)
$$\rho(\log h_n) \le \log \rho(h_n).$$

On account of the equality (13), we get

$$\rho(h_n) = \tau(hh_n) \to \tau(h^2),$$

and the relations (14) yield

$$\begin{aligned}\rho(\log h_n) &= \tau(h \log h_n) = \tau(h(\log^+ h_n - \log^- h_n)) \\ &= \tau(h \log^+ h_n) - \tau(h \log^- h_n) \to \tau(h \log^+ h) - \tau(h \log^- h) \\ &= \tau(h \log h).\end{aligned}$$

Now passing to the limit in the inequality (15) finishes the proof. $\square$

Theorem 10 (General Araki-Lieb Inequality). *Let $\rho_{123} \in (\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2 \bar{\otimes} \mathcal{M}_3)_*$ be a state with density matrix $h_{\rho_{123}} \in L^1(\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2 \bar{\otimes} \mathcal{M}_3, \tau_{123})$, where $\tau_{123} = \tau_1 \otimes \tau_2 \otimes \tau_3$ is a semifinite trace. Assume that $S(\rho_{123})$, $S(\rho_{12})$, $S(\rho_{23})$ are finite. Then*

$$S(\rho_{123}) \le S(\rho_{12}) + S(\rho_{23}) + \log \tau_2(h_{\rho_2}^2)$$

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Proof. If $\tau(h_\rho^2) = \infty$ then the thesis holds trivially. For $\tau(h_\rho^2) < \infty$ the theorem is a simple consequence of the strong subadditivity theorem and Proposition 9. $\square$

Remark 11. *A similar but a little weaker result was obtained by M.B. Ruskai in* [13]*. For states with bounded density matrices, and finite trace τ, instead of $\tau(h_\rho^2)$ she put* $\log||\rho||$*. Of course for $h_\rho \in \mathcal{M}$*

$$\log \tau(h_\rho^2) \leq \log(||\rho|| \cdot \tau(h_\rho)) = \log ||\rho||.$$

The following theorem was proved for trace-class operators $\hat{\rho}$ on a separable Hilbert space $\mathcal{H}$ and entropy of $\hat{\rho}$ defined as the von Neumann entropy

$$S(\hat{\rho}) = -Tr\hat{\rho}\ln\hat{\rho},$$

were Tr is the canonical trace on $\mathbb{B}(\mathcal{H})$. The result was given by E.Lieb and M.B. Ruskai in [6] in 1973 as a consequence of the concavity of the function $C \to Tr[\exp(k + \ln C)]$, for positive bounded C, and Klein's inequality (instead of Klein's inequality one could use the Peierls-Bogoliubov inequality). It was also mentioned in [6] that the strong subadditivity of entropy implies the result in question. Here we prove a version of this theorem generalized to an arbitrary semifinite von Neumann algebra. As in the case of $\mathbb{B}(\mathcal{H})$, we show that strong subadditivity implies the concavity of the function $\rho_{12} \to S(\rho_{12}) - S(\rho_2)$.

Theorem 12. *Let $\mathcal{M}_1, \mathcal{M}_2$ be von Neumann algebras with semifinite traces τ_1, τ_2 respectively. The function f given by*

$$f(\rho_{12}) = S(\rho_{12}) - S(\rho_2)$$

is concave on the set of normal states of $\mathcal{M}_1\overline{\otimes}\mathcal{M}_2$.

Proof. Let $\mathcal{M} = \mathcal{M}_1\bar{\otimes}\mathcal{M}_2\bar{\otimes}\mathbb{B}(\mathbb{C}^2)$. Denote $\rho_{12} := \alpha\rho'_{12} + (1-\alpha)\rho''_{12}$, and $\rho_{123} = \alpha\rho'_{12} \otimes \varphi_3 + (1-\alpha)\rho''_{12} \otimes \psi_3$, where $\alpha \in (0,1)$ and φ_3, ψ_3 are states on $\mathbb{B}(\mathbb{C}^2)$ with the density matrix h_{φ_3}, h_{ψ_3} being orthogonal, one-dimensional projections in $\mathbb{B}(\mathbb{C}^2)$.

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Since $S(\rho+\varphi) = S(\rho) + S(\varphi)$ for $h_\rho h_\varphi = 0$ [12]

$$\begin{aligned} S(\rho_{123}) &= S(\alpha\rho'_{12} \otimes \varphi_3) + S((1-\alpha)\rho''_{12} \otimes \psi_3) = \\ &\quad S(\alpha\rho'_{12}) + S((1-\alpha)\rho''_{12}) = \\ &\quad \alpha S(\rho'_{12}) + S(\alpha) + (1-\alpha)S(\rho''_{12}) + S(1-\alpha) \end{aligned}$$

$$S(\rho_{12}) = S(\alpha\rho'_{12} + (1-\alpha)\rho''_{12})$$

$$\begin{aligned} S(\rho_{23}) &= S(\alpha\rho'_{2} \otimes \varphi_3) + S((1-\alpha)\rho''_{2} \otimes \psi_3) = \\ &\quad S(\alpha\rho'_{2}) + S((1-\alpha)\rho''_{2}) = \\ &\quad \alpha S(\rho'_{2}) + S(\alpha) + (1-\alpha)S(\rho''_{2}) + S(1-\alpha) \end{aligned}$$

$$S(\rho_{2}) = S(\alpha\rho'_{2} + (1-\alpha)\rho''_{2})$$

were $S(\beta) = -\beta \log \beta$ for $\beta \in (0,1)$.

On account of the strong subadditivity theorem we have:

$$\begin{aligned} &S(\alpha\rho'_{12}) + S((1-\alpha)\rho''_{12}) + S(\alpha\rho'_{2} + (1-\alpha)\rho''_{2}) \le \\ &S(\alpha\rho'_{12} + (1-\alpha)\rho''_{12}) + S(\alpha\rho'_{2}) + S((1-\alpha)\rho''_{2}) \end{aligned}$$

Thus

$$\begin{aligned} &S(\alpha\rho'_{12} + (1-\alpha)\rho''_{12}) - S(\alpha\rho'_{2} + (1-\alpha)\rho''_{2}) \ge \\ &S(\alpha\rho'_{12}) + S((1-\alpha)\rho''_{12}) - [S(\alpha\rho'_{2}) + S((1-\alpha)\rho''_{2})] = \\ &\alpha S(\rho'_{12}) + S(\alpha) + (1-\alpha)S(\rho''_{12}) + S(1-\alpha) - \\ &[\alpha S(\rho'_{2}) + S(\alpha) + (1-\alpha)S(\rho''_{2}) + S(1-\alpha)] = \\ &\alpha S(\rho'_{12}) + (1-\alpha)S(\rho''_{12}) - [\alpha S(\rho'_{2}) + (1-\alpha)S(\rho''_{2})] \end{aligned}$$

The inequality means that

$$f(h_{\alpha\rho'_{12}+(1-\alpha)\rho''_{12}}) \ge \alpha f(h_{\rho'_{12}}) + (1-\alpha) f(h_{\rho''_{12}}),$$

which proves the concavity of f on the state space of $\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$. □

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