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Quantum Uncertainty Principles for Measurements with Interventions

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Heisenberg's uncertainty principle implies fundamental constraints on what properties of a quantum system can we simultaneously learn. However, it typically assumes that we probe these properties via measurements at a single point in time. In contrast, inferring causal dependencies in complex processes often requires interactive experimentation - multiple rounds of interventions where we adaptively probe the process with different inputs to observe how they affect outputs. Here we demonstrate universal uncertainty principles for general interactive measurements involving arbitrary rounds of interventions. As a case study, we show that they imply an uncertainty trade-off between measurements compatible with different causal dependencies.

Introduction – We learn about physical systems through measurement, and the uncertainty principle fundamentally limits what we can simultaneously learn [1]. Quantum mechanics states the existence of incompatible measurements (e.g., position and momentum of a free particle), such that predicting both outcomes to absolute precision is impossible [2–4]. Subsequent use of information theory then led to various entropic uncertainty relations that quantified uncertainty using entropic measures [5], culminating with universal uncertainty relations that provide general constraints of the joint probabilities of incompatible measurements [6–9].

Yet these relations pertain to only passive measurements, where a system is left to evolve freely before observation (see Fig. 1a). In contrast, the most powerful means of learning involve intervention. When toddlers learn of their environment, they do not merely observe. Instead, they actively intervene – performing various actions, observing resulting reactions and adapting future actions based on observations. Such *interactive measurements* are essential to fully infer causation, so we may know if one event caused another or if both emerged from some common-causes [10]. Indeed, interactive measurements permeate diverse sciences. Whether using reinforcement learning to identify optimal strategic behaviour or sending data packets to probe the characteristics of a network [11–13]. Such interactive measurement process also describe many quantum protocols, including quantum illumination, quantum-enhanced agents and non-Markovian open systems [14–17].

Could uncertainty principles also fundamentally constrain such interactive measurements (see Fig. 1b-d)? How would such principle interplay with interventions aimed to discern causal structure? Here, we explore these questions by deriving a universal uncertainty principle t-

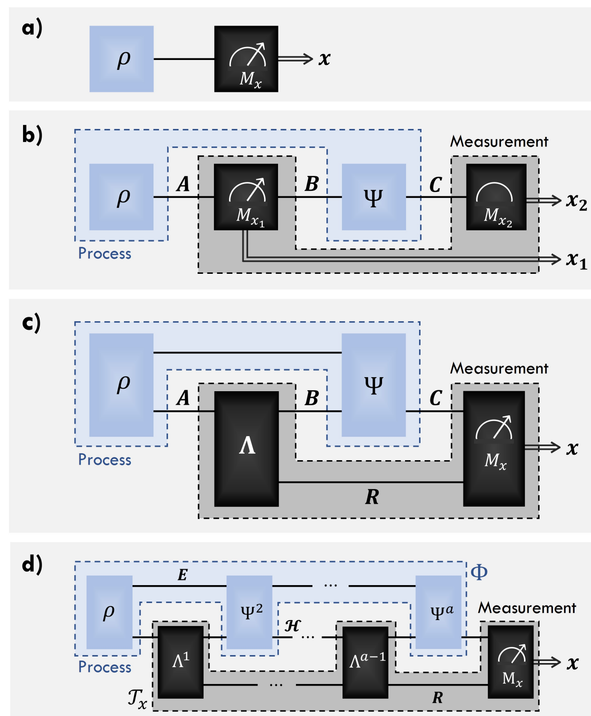

FIG. 1. **Interactive Measurements:** Our uncertainty relations apply to all interactive measurements. These include standard passive measurements on some quantum state (a) and thus encompass standard uncertainty relations. Our framework also embraces scenario (b), a quantum system is measured at two points in time, arriving at the framework of pseudo-density matrix [18]. We consider cases on non-Markovian processes where the initial system and its evolution might be connected by a quantum memory (c), such interactive measurements can involve coherent interacting the system with a quantum register R and doing some joint measurement at a subsequent time-step. Most generally, our results pertain any interactive measurement $\mathcal{T}_x$ with interventions at a different time-steps as illustrated in (d).

hat constrains the joint measurement probabilities of interactive measurements. This principle then pinpoints when two interactive measurements are non-compatible – and quantifies the necessary trade-offs in the certainty of their measurement outcomes. Our results make no assumptions on the number of interventions or the causal structure of processes we probe, and encompass previous uncertainty relations for states and channels as special cases [19–21]. We apply them to interactive measurements compatible with direct-cause vs common-cause, showing that they satisfy an uncertainty trade-off analogous to position and momentum.

Framework – The premise of an interactive measurement consists of an agent that wishes to probe the dynamics of some unknown quantum process Φ. Here Φ can be modelled as an open quantum system, consisting of a system accessible to the agent with $\mathcal{H}$, that is coupled with some generally non-Markovian environment E (see blue shaded region in Fig. 1d). Initially, the $\mathcal{H}$-E system is in some joint state ρ. At each time-step k, the system and environment jointly evolve under Ψ^k. Φ is then completely defined by the set $\{\Psi^k\}_{k=2}^a$ and the initial state ρ, where a represent to the number of time-steps. In literature, Φ offers the most general representations of non-Markovian quantum stochastic processes [22] and is also closely related to concepts of higher-order quantum maps, adaptive agents, and causal networks [23–28].

Interactive measurements then represent the most general means for an agent to determine properties of Φ (see black shaded region in Fig. 1d): the agent initializes some internal memory register R; between time-steps (i.e., before Ψ^k with $2 \leqslant k \leqslant a$), the agent performs an *intervention* – some general quantum operation Λ^k that interacts her memory R with the accessible system $\mathcal{H}$; after $a-1$ such interventions, the agent finally makes a joint measurement with respect to some positive operator valued measure (POVM) $M := \{M_x\}$ on the joint $\mathcal{H}$-R system to obtain some outcome x. Thus, each interactive measurement $\mathcal{T}$ is completely defined by set of interventions $\{\Lambda^k\}_{k=1}^{a-1}$ and POVM M. Just as a conventional POVM measurement on a quantum state induces some probability distribution over measurement outcomes, so does an interactive measurement on a quantum process. Analogous to eigenstates, we say Φ is an *eigencircuit* of $\mathcal{T}$ if Φ always yields a definite outcome when measured by $\mathcal{T}$.

We make two remarks. (1) The interactive measurements encompass *everything* an agent can possibly do causally. Notably, R can also store classical information. For example, making a projective measurement and conditioning future action on the system based on the result of these measurements. (2) Both Φ and $\mathcal{T}$ have succinct representations using Choi-Jamiołkowski operators, often referred to as quantum combs [24, 25] or process tensors [17]. We provide a rigorous mathematical treatment in supplemental material Sec. IB and IC [29].

Uncertainty Principles – In conventional quantum theory, certain observables are mutually incompatible. Given an observable $\mathcal{O}$ whose outcomes o_k occurs with probability p_k, we can quantify the uncertainty by the Shannon Entropy $H(\mathcal{O}) := -\sum_k p_k \log p_k$. The entropic uncertainty principle then states that there exists mutually non-compatible observables $\mathcal{O}_1$ and $\mathcal{O}_2$, such that the joint uncertainty $H(\mathcal{O}_1) + H(\mathcal{O}_2)$ is always lower-bounded by some state-independent $C > 0$ [30–32]

Can we identify similar uncertainty relations for general interactive measurements? We answer this question by employing majorization [33]. Consider two probability vectors $\mathbf{x}$ and $\mathbf{y}$, whose elements x_k and y_k are arranged in non-increasing order. We say $\mathbf{x}$ is majorized by $\mathbf{y}$, written as $\mathbf{x} \prec \mathbf{y}$, if $\sum_{k=1}^i x_k \leqslant \sum_{k=1}^i y_k$ holds for all index i. The rationale is that majorization maintains significant connections with entropy since $\mathbf{x} \prec \mathbf{y}$ implies that $H(\mathbf{x}) \geqslant H(\mathbf{y})$. In fact, $\mathbf{x} \prec \mathbf{y}$ implies $f(\mathbf{x}) \geqslant f(\mathbf{y})$ for a large class of functions known as *Schur-concave functions*. Such functions align with those that remain non-decreasing under random relabeling of measurement outcomes, and have been proposed as the most general class of uncertainty quantifiers [6]. Thus, majorization constraints on outcome probabilities for conventional quantum measurements are referred to as universal uncertainty relations [6–9]. Here, we establish such a *universal uncertainty relation for general interactive measurements* (See supplementary materials Sec. IID, for Proof [29]):

Lemma 1. *Consider two distinct interactive measurements $\mathcal{T}_1$ and $\mathcal{T}_2$ on some dynamical process Φ, with outcomes described by probability distributions $\mathbf{p}$ and $\mathbf{q}$. There then exists a probability vector $\mathbf{v}(\mathcal{T}_1, \mathcal{T}_2)$ such that*

$$\frac{1}{2}\mathbf{p} \oplus \frac{1}{2}\mathbf{q} \prec \mathbf{v}(\mathcal{T}_1, \mathcal{T}_2). \tag{1}$$

Here the vector-type bound $\mathbf{v}(\mathcal{T}_1, \mathcal{T}_2)$ is independent Φ, and hence captures the essential incompatibility between $\mathcal{T}_1$ and $\mathcal{T}_2$. Meanwhile, $\oplus$ represents the concatenation of vectors. For example, $(1, 0) \oplus (1/2, 1/2) = (1, 0, 1/2, 1/2)$.

Our result for interactive measurement is also universal in this sense. In particular, they imply an infinite family of uncertainty relations, namely $f(\mathbf{p}/2 \oplus \mathbf{q}/2) \geqslant f(\mathbf{v}(\mathcal{T}_1, \mathcal{T}_2))$ for any Schur-concave function f (including Rényi entropies). Choosing f as the Shannon entropy, Lem. 1 then results in entropic bounds for general interactive measurements (see [29] Sec. IID for details):

Theorem 1. *Given two interactive measurements $\mathcal{T}_1$ and $\mathcal{T}_2$ acting on some dynamical process Φ. The entropy of their measurement outcomes [34], when summed, satisfies*

$$H(\mathcal{T}_1)_\Phi + H(\mathcal{T}_2)_\Phi \geqslant C(\mathcal{T}_1, \mathcal{T}_2), \tag{2}$$

where $C(\mathcal{T}_1, \mathcal{T}_2)$ – measuring incompatibility between $\mathcal{T}_1$ and $\mathcal{T}_2$ – is non-negative and independent of Φ. $C(\mathcal{T}_1, \mathcal{T}_2)$ can be explicitly computed and is strictly non-zero whenever $\mathcal{T}_1$ and $\mathcal{T}_2$ have no common eigencircuit.

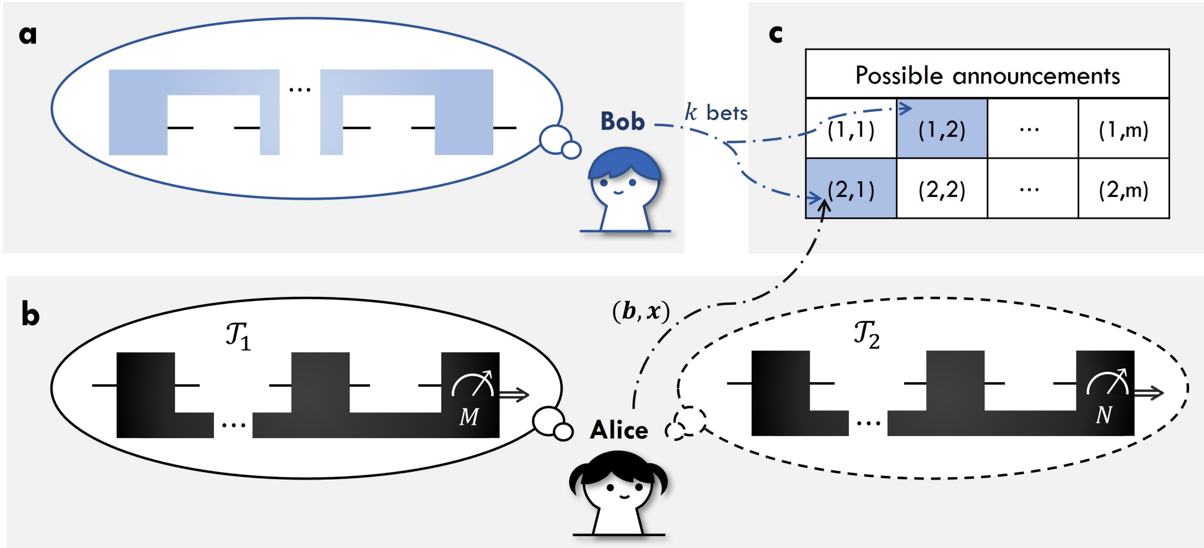

FIG. 2. **The Quantum Roulette:** The quantum roulette is a game that aids in interpreting lower bounds for the combined uncertainty of two general interactive measurements $\{\mathcal{T}_b\}_{b=1,2}$. $\mathcal{T}_1$ and $\mathcal{T}_2$ – picture in (b). Now introduce a quantum 'roulette table' with $2 \times m$ grid of cells (c), labelled (b, x) with $x = 1, \ldots, m$. In the k^{th} order game, Bob begins with k chips, of which he can allocate to k of these cells. Bob then supplies Alice with a dynamical process Φ (a). Alice selects a b at random, and measures Φ with $\mathcal{T}_b$ to obtain outcome x. Bob wins if he has a chip on the cell (b, x). Lemma 1 and theorem 1 then relate Bob's winning probabilities with the incompatibility between $\mathcal{T}_1$ and $\mathcal{T}_2$.

In [29] Sec. IID, we illustrate a choice of $C(\mathcal{T}_1, \mathcal{T}_2)$ that reduces to $\log(1/c)$ when $\mathcal{T}_1$ and $\mathcal{T}_2$ are standard quantum measurements. Here c stands for the maximal overlap between measurements [5]. Meanwhile, just as there exist many alternative bounds beyond $\log(1/c)$ [35–46], there are many other valid bounds for $H(\mathcal{T}_1)_\Phi + H(\mathcal{T}_2)_\Phi$ (See [29] Sec. IID). Here we focus on a choice of $C(\mathcal{T}_1, \mathcal{T}_2)$ that can give tighter bounds in causal inference settings. More results are presented in [29] Sec. IIC.

Our formulations of $\mathbf{v}(\mathcal{T}_1, \mathcal{T}_2)$ and $C(\mathcal{T}_1, \mathcal{T}_2)$ carry direct operational meaning in a guessing game which we refer to as the *quantum roulette*. The two-party game consists of (1) Alice, an agent that probes any supplied dynamical process using one of two possible interactive measurements, $\mathcal{T}_1$ and $\mathcal{T}_2$. (2) Bob, who can engineer various dynamical processes for Alice to probe (see Fig. 2). In each round, Alice and Bob begin with a 'roulette table', whose layout consists of all tuples (b, x), where $b \in \{1, 2\}$ and x are all possible measurement outcomes of $\mathcal{T}_1$ and $\mathcal{T}_2$. Bob begins with k chips, which he can use to place bets on k of the possible tuples and supplies Alice with any Φ of his choosing. Alice will then select some $b \in \{1, 2\}$ at random and probe Φ with $\mathcal{T}_b$. She finally announces both b and the resulting measurement outcome x. Bob wins if one of his chips is on (b, x).

Let p_k denote Bob's maximum winning probability. Naturally $p_0 = 0$ and p_k increases monotonically with k, tending to 1. We define a probability vector $\mathbf{w}$ with elements $w_k = p_k - p_{k-1}$, $k = 1, 2, \ldots$, representing the increase in Bob's probability of winning with k rather than $k-1$ chips. In [29] Sec. IID, we show that $\mathbf{v}(\mathcal{T}_1, \mathcal{T}_2) := \mathbf{w}$ and $C(\mathcal{T}_1, \mathcal{T}_2) := 2H(\mathbf{w}) - 2$ are bounds for $\mathbf{p}/2 \oplus \mathbf{q}/2$ and $H(\mathcal{T}_1)_\Phi + H(\mathcal{T}_2)_\Phi$ respectively.

This game gives an operational criterion of non-compatibility for interactive measurements. When two observables are compatible, $H(\mathbf{w}) = 1$. This aligns with the scenario that $\mathbf{w} = (0.5, 0.5, 0, \ldots, 0)$, which occurs when Bob's success rate is limited only by his uncertainty of which measurement Alice makes. That is, placing one counter ensures Bob can correctly predict the outcome of $\mathcal{T}_1$ and two counters gives him perfect prediction regardless of b. We see this is only possible if $\mathcal{T}_1$ and $\mathcal{T}_2$ share at least one common eigencircuit. Thus, $H(\mathcal{T}_1)_\Phi + H(\mathcal{T}_2)_\Phi$ is strictly greater than 0 whenever $\mathcal{T}_1$ and $\mathcal{T}_2$ share no common eigencircuit.

Causal Uncertainty Relations – The central relevance of interventions in causal inference makes it an appropriate illustrative case example [47]. Consider the case where Φ represents a d-level system (the accessible qudit) that evolves while in possible contact with other systems (e.g. a non-Markovian environment E). Now suppose an agent, Alice, can access this qudit at two different points in time, say t_X and t_Y. In general the quantum process Φ can fall under three scenarios [48]:

(i) The system at t_X is a *direct cause* of the system at t_Y: the qudit at t_Y is the output of some quantum map acting on the qubit at t_X (Fig. 3b-i).

(ii) The system at t_X and t_Y share a *common cause*: the qudit at t_X is correlated with an environmental qudit E. E is measured at time t_Y and (Fig. 3b-ii).

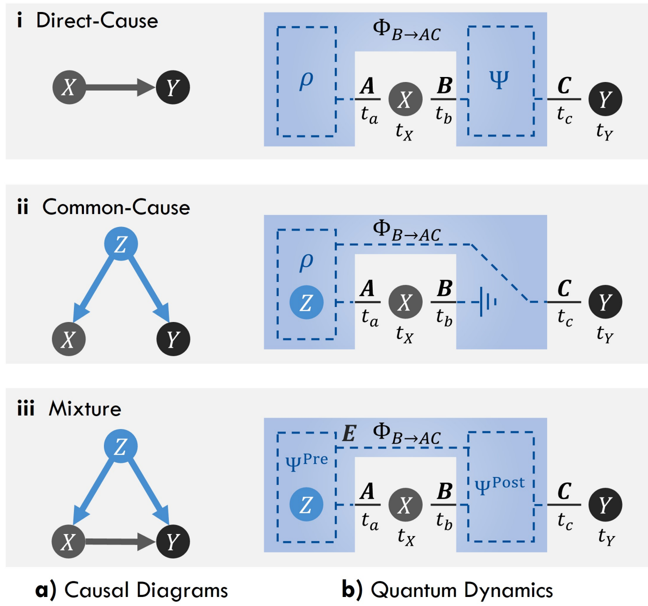

FIG. 3. **Quantum Description of Causal Structures:** There are three possible causal structures for two events X and Y, all of which can be expressed by a quantum dynamic process $\Phi_{B\to AC}$. In (i) direct-cause, $\Phi_{B\to AC}$ involves preparing a state A to be observed at X, whose output is sent directly to Y via quantum channel from B to C. In (ii) common-cause, correlations between X and Y can be attributed to measurements on some pre-prepared correlated state ρ_{AC} (event Z). Most generally (iii), $\Phi_{B\to AC}$ consists of a state-preparation process $\Psi^{\rm Pre}_{\mathbb{C}\to AE}$ and a post-processing quantum channel $\Psi^{\rm Post}_{BE\to C}$ (b-iii; E is an ancillary system). This then corresponds to a (possibly coherent) mixture of direct and common cause.

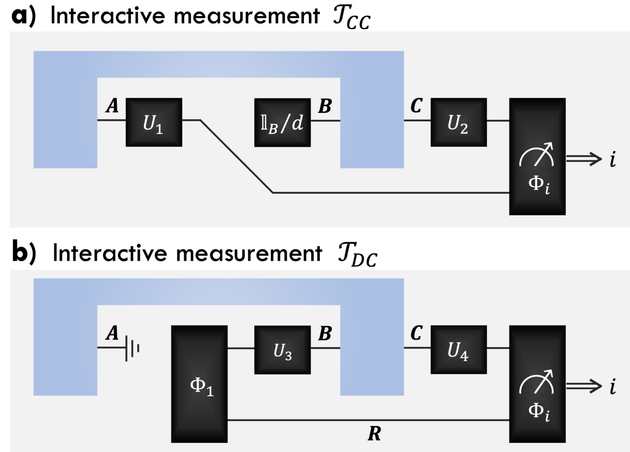

FIG. 4. **Maximal Common-Cause and Direct-Cause Indicators:** We introduce (a) $\mathcal{M}_{\rm CC} = \{\mathcal{T}_{\rm CC}(U_1, U_2)\}$ and (b) $\mathcal{M}_{\rm DC} = \{\mathcal{T}_{\rm DC}(U_3, U_4)\}$ as two respective families of interactive measurements with a single intervention. Here, system A, B and C are d-level quantum systems (qudits), and each U_k, $k = 1, 2, 3, 4$ is some single-qudit unitary, and $|\Phi_1\rangle := \sum_{k=0}^{d-1} |kk\rangle/\sqrt{d}$. Measurements are done with respect to a maximally entangling basis $\{\Phi_i\}_i$ with d^2 possible outcomes. The two measurement families are incompatible, and satisfy the causal uncertainty relation in Eq. 3.

(iii) Mixture of both, corresponding to a general non-Markovian quantum process. (Fig. 3b-iii).

We now introduce two families of interactive measurements: $\mathcal{M}_{\rm CC}$ and $\mathcal{M}_{\rm DC}$, as depicted in Fig. 4. Each $\mathcal{T}_1 \in \mathcal{M}_{\rm CC}$ is a *maximal common-cause indicator*, such that its eigencircuits imply that X and Y are actually two arms of some maximally entangled state (Fig. 3b-ii). Meanwhile, each $\mathcal{T}_2 \in \mathcal{M}_{\rm DC}$ is a *maximal direct-cause indicator*, whose eigencircuit involve a lossless channel from X to Y (i.e., Fig. 3b-i where Ψ is unitary). In [29] Sec. IIIA, we establish the following *causal uncertainty relation*:

$$H(\mathcal{T}_1) + H(\mathcal{T}_2) \geqslant 2\log d, \tag{3}$$

for any $\mathcal{T}_1 \in \mathcal{M}_{\rm CC}$ and $\mathcal{T}_2 \in \mathcal{M}_{\rm CC}$. Here $H(\mathcal{T}_i)$ $(i = 1, 2)$ is the Shannon entropy of the probability distribution associated with outcomes when $\mathcal{T}_i$ is measured. Furthermore, this bound can be saturated.

Consider the application of this uncertainty to a specific parameterized quantum circuit $\Phi_{\alpha,\beta}$ (Fig. 5a) describing a single qubit undergoing non-Markovian evolution. Fig. 5b then demonstrates the combined uncertainty $H(\mathcal{T}_1) + H(\mathcal{T}_2)$ for various values of α and β, including cases where they saturate the lower bound of 2. We also note that unlike classical processes, which must be either purely common-cause, or purely direct-cause, or a probabilistic mixture of both – quantum processes can feature richer causal dependencies [49]. Fig. 5c depicts this for the cross-section of $\alpha = \pi/4$. Such circuits include the coherent superposition of direct and common cause as a special case. Our causal uncertainty relation also applies to these uniquely quantum causal structures.

Discussion – The most powerful means of learning involves interactive measurement – a procedure in which we can intervene by injecting (possible entangled) quantum states into the process over multiple time-steps before observing the final output [50–52]. Here, we derive entropic uncertainty relations that govern all interactive measurements, bounding their joint uncertainty whenever such measurement outcomes are non-compatible. In the context of causal inference, they predict a uniquely quantum entropic trade-off between measurements that probe for direct and common cause. More generally, our relations encompass all possible means for an agent to interact and learn about a target quantum system and thus include previously studied uncertainty relations on states and channels as special cases.

One potential application of such relations is the metrology of unknown quantum processes [53–55]. Here,

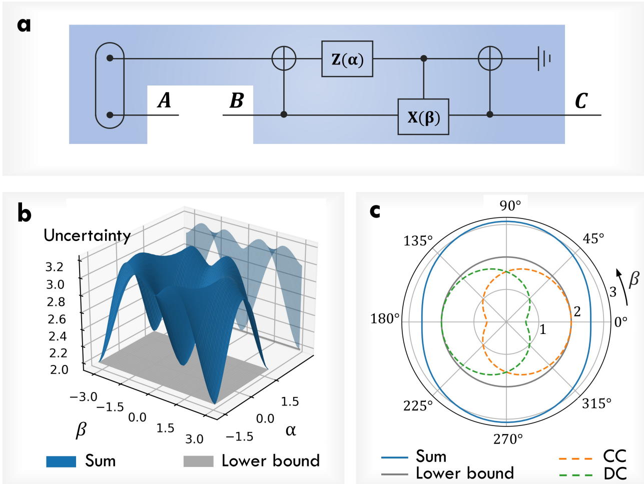

FIG. 5. **Causal Uncertainty Relations on Non-Markovian Dynamics:** Consider a single qubit – bottom rail of the circuit in (a) – undergoing non-Markovian evolution. Here the systems are initialized in maximally entangled state, $Z(\theta)$ and $X(\theta)$ represent single-qubit rotation gates in X and Z axis. (b) illustrates the combined uncertainty $H(\mathcal{T}_1) + H(\mathcal{T}_2)$, where $\mathcal{T}_1 \in M_{\mathrm{CC}}$ and $\mathcal{T}_2 \in M_{\mathrm{DC}}$ are respectively common-cause and direct-cause indicators in Fig. 4 with all U_k set to the identity. Observe this never goes below the fundamental lower bound of 2 (gray plane). (c) illustrates $H(\mathcal{T}_1)$ (green dashed), $H(\mathcal{T}_2)$ (red dashed) and their sum (blue solid) for $\alpha = -\pi/4$ and various values of β, corresponding to various coherent superpositions of common-cause and direct-cause circuits.

full tomography of a general quantum process is impractical. Even a single non-Markovian qubit measured at two different times requires 54 different interactive measurements [56]. Our result may help us ascertain specific properties of a process while avoiding this costly procedure. In [29] Sec. IVB, we illustrate how our causal uncertainty relations imply that a single interactive measurement can rule out specific causal structures. Indeed, quantum illumination and adaptive sensing can both cast as measuring desired properties of a candidate quantum process, and thus could benefit from such an approach.

Interactive measurements through repeated interventions also merge in other settings. In quantum open systems, sequential intervention provides a crucial toolkit for characterizing non-Markovian noise [57]. Meanwhile, in reinforcement learning, quantum agents that continuously probe an environment show enhancements in enacting or learning complex adaptive behaviour [58]. Investigating uncertainty relations specific to such contexts has exciting potential, perhaps revealing new means of probing non-Markovian dynamics, or fundamental constraints on how well an agent can simultaneously optimize two different rewards.

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