Optimal covariant quantum measurements

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Abstract. We discuss symmetric quantum measurements and the associated covariant observables modelled, respectively, as instruments and positive-operator-valued measures. The emphasis of this work are the optimality properties of the measurements, namely, extremality, informational completeness, and the rank-1 property which contrast the complementary class of (rank-1) projection-valued measures. The first half of this work concentrates solely on finite-outcome measurements symmetric w.r.t. finite groups where we derive exhaustive characterizations for the pointwise Kraus-operators of covariant instruments and necessary and sufficient extremality conditions using these Kraus-operators. We motivate the use of covariance methods by showing that observables covariant with respect to symmetric groups contain a family of representatives from both of the complementary optimality classes of observables and show that even a slight deviation from a rank-1 projection-valued measure can yield an extreme informationally complete rank-1 observable. The latter half of this work derives similar results for continuous measurements in (possibly) infinite dimensions. As an example we study covariant phase space instruments, their structure, and extremality properties.

Keywords: quantum measurements, positive-operator-valued measures, quantum instruments, covariance, optimal measurements, extremality

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1. Introduction

Unlike in the classical theory of measurements, in quantum theory, it is essential to describe, not only the outcome statistics, but also how the measurement (or, more precisely, the registering of an outcome) changes the system being measured. The outcome statistics are described by a quantum observable which is modelled by a normalized positive-operator-valued measure where the outcome probabilities are given by the Born rule. The more complete description of a measurement, also taking into account the conditional state transformations, is given by an instrument which can be viewed as a state-transformation-valued measure [4, 11, 12]. To classify observables and measurements and to assign physical meaning to them, it is common to require that the observables and instruments reflect the symmetries of the classical outcome spaces and those of the quantum systems involved. These are commonly described through symmetry properties; see [11, Chapter 4], [22, Chapters 3 and 4], and [30]. For example, phase space measurements are covariant under translations of the phase space and the displacement operators (the Weyl-representation) mediating these shifts in the Hilbert space. Given the physical relevance of covariant observables and instruments, it is important to understand their structure. This is one of the key goals of this work.

When carrying out a measurement, one has to consider the cost of the measurement setting. For this, it is important to characterize the optimal observables [2, 3, 16]. However, there are several optimality criteria which sometimes cannot be simultaneously satisfied. The measurement may be informative enough to determine the pre-measurement state or to determine how the system evolves after the measurement. The measurement may be free from noise caused by classically manipulating the outcome statistics of a genuinely more informative observable (post-processing cleanness) or from quantum noise caused by manipulating the pre-measurement state before measuring a cleaner observable (pre-processing cleanness). These optimality criteria (post-processing cleanness in particular) often interact in joint measurement settings which manifests as measurement uncertainty relations [1, 5] and information-disturbance trade-off relations [17, 18] which place restrictions on how optimal different parts of the measurement can be. The conflicting optimality properties give rise to mutually exclusive optimality classes [16] and in this work we exhibit how these classes are represented in covariance structures. In addition to the above optimality modes, we can impose the condition on extremality, i.e., require that the measurement be an extreme point of a relevant set of measurements. A measurement device can be a member of a number of different convex sets, meaning that there are different extremality properties with different physical interpretations. In this work, we identify extreme points of entire sets of devices as well as the extreme points of the restricted sets of covariant devices. As extreme devices minimize concave optimality measures and maximize convex measures (e.g., mutual information), these devices are often optimal for specific tasks, e.g., state discrimination tasks [23, Section I.2.4, Theorem 2.22]. Extreme devices are also free from classical randomness arising from mixing different measurement schemes.

In what follows, we formalize the notions discussed above: covariance, optimality, and extremality. We first make some initial observations on optimal measurements and the structure of covariant observables and instruments. We also detail the importance of observables covariant w.r.t. a symmetric group and present a relevant family of optimal covariant qutrit observables in Example 1. After this, in Section 3, we see that covariant instruments can be described by pointwise Kraus-operators given by a set of single-point Kraus operators of very particular form. Earlier studies [6, 8] on covariance structures have shown that covariant observables and instruments can be dilated into a canonical systems of imprimitivity and these results have earlier been used to derive structure results for covariant measurements [15, 20, 21], but our results are more specific in that they give clear conditions for the single-point Kraus operators (called in our work as 'intertwiners') and also allow very nice necessary and sufficient characterizations for extremality in the form of linear-independence conditions. After this, we consider the consequences of these results for covariant POVMs and channels. Motivated by the importance of the symmetric group, we give generalizations of the results of Example 1 for general symmetric groups and corresponding covariant POVMs in Example 2. We will see that, in general we can determine a family of observables covariant w.r.t. the symmetric group in any finite-dimensional system where all the optimality classes are represented and that representatives from these disjoint classes can be chosen arbitrarily close one another. After this, we generalize many of these results for measurements with continuous value spaces and possibly infinite-dimensional input and output systems.

2. Basic definitions and observations

Let us concentrate on a quantum system described by the Hilbert space $\mathcal{H}$. In quantum mechanical description, observables are represented as normalized positive operator valued measures and states are density operators, i.e. trace-1 positive operators. In the complete description of a measurement, we need to specify how the detection of an outcome x affects the input state ρ and this is done by an instrument $\mathcal{I}$, originally introduced by Davies and Lewis [11, 12], which is a collection of completely positive linear maps. In the first half of this work, we concentrate on the case of finite value spaces and finite-dimensional Hilbert spaces.

Definition 1. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and $\mathbb{X}$ be a finite set.

(i) A collection $\mathsf{M} = (\mathsf{M}_x)_{x\in\mathbb{X}}$ of linear operators on $\mathcal{H}$ is a *positive-operator-valued measure (POVM)* if $\mathsf{M}_x \geq 0$ for all $x \in \mathbb{X}$. If M_x is an orthogonal projection for all $x \in \mathbb{X}$, we say that M is a *projection-valued measure (PVM).*

(ii) If a POVM $\mathsf{M} = (\mathsf{M}_x)$ is normalized, i.e., $\sum_{x\in\mathbb{X}} \mathsf{M}_x = \mathbb{1}_{\mathcal{H}}$, we say that M is an *observable.* A normalized PVM is also called as a sharp observable.

(iii) A collection $\mathcal{I} = (\mathcal{I}_x)_{x\in\mathbb{X}}$ of 'superoperators' is a *quantum-operation-valued measure (QOVM)* if, for each $x \in \mathbb{X}$, $\mathcal{I}_x$ is a linear map on (trace-class) operators of $\mathcal{H}$ with values in the set of (trace-class) operators of $\mathcal{K}$ which is completely positive, i.e., $\mathcal{I}_x \otimes \mathrm{id}_n$ is positive for all $n \in \mathbb{N}$ where id_n is the identity map on the algebra of $(n \times n)$-matrices with complex entries.

(iv) If a QOVM $\mathcal{I} = (\mathcal{I}_x)_{x\in\mathbb{X}}$ is normalized, i.e, $\sum_{x\in\mathbb{X}} \mathcal{I}_x$ is trace preserving, then $\mathcal{I}$ is called as an *instrument.*

(v) If $\mathrm{tr}\,[\rho\mathsf{M}_x] = \mathrm{tr}\,[\mathcal{I}_x(\rho)]$ for all states ρ on $\mathcal{H}$ and all $x \in \mathbb{X}$, for an instrument [QOVM] $\mathcal{I}$ and an observable [POVM] M, we say that $\mathcal{I}$ *measures* M or $\mathcal{I}$ is an M-*instrument [*M-*QOVM].*

For an observable $\mathsf{M} = (\mathsf{M}_x)_{x\in\mathbb{X}}$, the number $\mathrm{tr}\,[\rho\mathsf{M}_x]$ is interpreted as a probability to get the value x in the measurement of M when the system is prepared in the state ρ. For an instrument $\mathcal{I} = (\mathcal{I}_x)_{x\in\mathbb{X}}$, $\mathcal{I}_x(\rho)$ is a (non-normalized) output state conditioned by x, and $\sum_{x\in\mathbb{X}} \mathcal{I}_x(\rho)$ is the unconditioned total state. Note that the output states $\mathcal{I}_x(\rho)$ may reside in a different Hilbert space $\mathcal{K}$. For more details on quantum measurement theory, we refer to [4].

As stated in Introduction, observables are charaterized by symmetries. For example, position observables transform covariantly under the position shifts (translations) generated by the momentum operator. In addition to the sharp position (i.e. the spectral measure of the position operator), there are infinitely many unsharp position POVMs which all are smearings of the sharp one. To define a symmetric or covariant POVM, one must start by fixing a symmetry of the outcome space. For this, we need an appropriate (finite) symmetry group G which acts on $\mathbb{X}$, i.e. any $g \in G$ 'transforms' or 'shifts' an outcome x into $gx \in \mathbb{X}$. The neutral element $e \in G$ does nothing: $ex = x$ (and $ge = g = eg$). Moreover, we let $\mathbb{X}$ be a G-space, i.e., $g(hx) = (gh)x$ for all g, $h \in G$ and $x \in \mathbb{X}$. We can make further definitions on our G-space $\mathbb{X}$:

- **Orbits**: Let $\mathcal{O}$ be the set of the orbits $Gx = \{gx \mid g \in G\} \subseteq \mathbb{X}$. Thus, the outcome space is the disjoint union of the orbits, $\mathbb{X} = \biguplus \mathcal{O} = \biguplus_{\Omega\in\mathcal{O}} \Omega$.
- **Representative of an orbit**: For any $\Omega \in \mathcal{O}$, we fix $x_\Omega \in \Omega$ (i.e. $\Omega = Gx_\Omega$). From this it follows that, for all $x \in \Omega \in \mathcal{O}$, we can further fix $g_x \in G$ such that $x = g_x x_\Omega$. Note that these choices are not in general unique, but we keep these elements fixed throughout this work.
- **Stability subgroups**: We define the stability subgroup $G_x := \{g \in G \mid gx = x\}$ of any $x \in \mathbb{X}$. One easily finds that, whenever x and x' are in the same orbit, G_x and $G_{x'}$ are isomorphic. Thus, any orbit has an essentially unique stability subgroup H_Ω which we can choose to coincide with G_{x_Ω}.

Also, G is assumed to act on the operator space of the system (the quantum input): any operator A in the Heisenberg picture transforms into $\alpha_g(A) := U(g)AU(g)^*$ where $U(g)$ is a unitary operator and $g \mapsto \alpha_g$ is a group homomorphism of G into the automorphism group of the operator algebra. (Note that, in the Schrödinger picture, any density operator ρ transforms to $U(g)^*\rho U(g)$ under the action of $g \in G$.) This means that we may choose $g \mapsto U(g)$ to be a projective unitary representation, i.e., there is a multiplier or 2-cocycle $m : G \times G \to \mathbb{T}$ such that $U(gh) = m(g,h)U(g)U(h)$ for all g, $h \in G$. The 2-cocycle conditions read

$$m(e,g) = m(g,e) = 1, \qquad m(g,h)m(gh,k) = m(g,hk)m(h,k)$$

for all g, h, $k \in G$. In this setting, we make the following definition:

Definition 2. We say that an observable (or a POVM) $\mathsf{M} = (\mathsf{M}_x)_{x\in\mathbb{X}}$ is $(\mathbb{X}, U)$-*covariant* if

$$\mathsf{M}_{gx} = U(g)\mathsf{M}_x U(g)^*, \qquad g \in G,\ x \in \mathbb{X}, \tag{1}$$

$(\mathbb{X}, U)$-covariance means that, for any unit vector $\psi \in \mathcal{H}$, the shifted probability distribution $x \mapsto \langle\psi|\mathsf{M}_{gx}\psi\rangle$ is the same as $x \mapsto \langle\psi_g|\mathsf{M}_x\psi_g\rangle$ where $\psi_g = U(g)^*\psi$ is the symmetrically transformed input state. Thus, changing the initial state should only move the probability distribution without deforming its shape. One can see the condition (1) as a generalization of canonical quantization of the classical variable x [20], or as the definition of the generalized imprimitivity system [8, 11, 26, 30].

Entire measurement settings can be symmetric in the sense that applying symmetry transformations on input states is the same as registering transformed values and obtaining conditional output states which are symmetrically transformed. We keep the above finite G-space $\mathbb{X}$ and the input representation U fixed and introduce output system symmetries via a projective unitary representation $g \mapsto V(g)$ operating on the output system Hilbert space $\mathcal{K}$. We may define symmetry in measurements in the following way:

Definition 3. We say that an instrument (or a QOVM) $\mathcal{I} = (\mathcal{I}_x)_{x\in\mathbb{X}}$ is $(\mathbb{X}, U, V)$–*covariant* if

$$\mathcal{I}_{gx}\big(U(g)\rho U(g)^*\big) = V(g)\mathcal{I}_x(\rho)V(g)^* \tag{2}$$

for all $x \in \mathbb{X}$, $g \in G$, and all input states ρ.

It easily follows that the observable [POVM] measured by an $(\mathbb{X}, U, V)$–covariant instrument [QOVM] is $(\mathbb{X}, U)$–covariant. Moreover, if M is an $(\mathbb{X}, U)$–covariant POVM and V is a projective representation of the same group G in any Hilbert space $\mathcal{K}$ there exists an $(\mathbb{X}, U, V)$–covariant M–QOVM. Namely, for any $\Omega \in \mathcal{O}$, choose a state σ'_Ω of $\mathcal{K}$ and define the H_Ω–invariant state $\sigma_\Omega := (\#H_\Omega)^{-1}\sum_{h\in H_\Omega} V(h)\sigma'_\Omega V(h)^*$ and a QOVM

$$\mathcal{I}^{\rm nuc}_x(\rho) := \mathrm{tr}\,[\rho\mathsf{M}_x]\, V(g_x)\sigma_\Omega V(g_x)^*$$

for all $x \in \Omega \in \mathcal{O}$; if M is normalized, $\mathcal{I}^{\rm nuc}$ is an instrument. Operationally, in the measurement of M with $\mathcal{I}^{\rm nuc}$, if x is obtained (with the probability $\mathrm{tr}\,[\rho\mathsf{M}_x]$) then the output state is $\sigma_x = V(g_x)\sigma_\Omega V(g_x)^*$ which does not depend on the input state ρ. Such an instrument is called measure-and-prepare or nuclear [10].

In the following theorem, we give an initial simple structure result for covariant POVMs and observables. Note that this result is well known and the version in transitive value spaces is given, e.g., in [22, Theorem 4.2.3].

Theorem 1. *A POVM* M *is covariant if and only if* $\mathsf{M}_x = U(g_x)K_\Omega U(g_x)^*$ *for all* $x \in \Omega \in \mathcal{O}$ *where* K_Ω *is a positive operator such that* $K_\Omega U(h) = U(h)K_\Omega$, $h \in H_\Omega$. *Now* M *is normalized exactly when* $K := \sum_{\Omega\in\mathcal{O}}\sum_{x\in\Omega} U(g_x)K_\Omega U(g_x)^* = \mathbb{1}$.

Proof. For all $\Omega \in \mathcal{O}$, $x \in \Omega$, and $h \in H_\Omega$, one gets $(g_x h)x_\Omega = g_x x_\Omega = x \in \Omega$. Using this, we obtain from (1), for any $(\mathbb{X}, U)$-covariant POVM $\mathsf{M} = (\mathsf{M}_x)_{x\in\mathbb{X}}$,

$$\mathsf{M}_x = \mathsf{M}_{g_x x_\Omega} = U(g_x)\mathsf{M}_{x_\Omega}U(g_x)^* = \mathsf{M}_{(g_x h)x_\Omega} = U(g_x)U(h)\mathsf{M}_{x_\Omega}U(h)^*U(g_x)^*.$$

so that $U(h)\mathsf{M}_{x_\Omega} = \mathsf{M}_{x_\Omega}U(h)$. By denoting $K_\Omega := \mathsf{M}_{x_\Omega}$ we are done. □

Note that if the covariant POVM M of Theorem 1 is not normalized (i.e., $K \neq \mathbb{1}$) but K is invertible, one can define a normalized covariant POVM (i.e., an observable) as the collection of effects $K^{-1/2}\mathsf{M}_x K^{-1/2}$, $x \in \mathbb{X}$. Indeed, $U(g)KU(g)^* = K$ so that K and thus $K^{-1/2}$ commutes with any $U(g)$. Note that the eigenvalues of K (and $K^{-1/2}$) are positive. Moreover, we note that, in some situations, there are only trivial solutions M for (1). For example, if there is only one orbit, $\mathcal{O} = \{\mathbb{X}\}$, and the subrepresentation $h \mapsto U(h)$ of $H_{\mathbb{X}}$ is irreducible, then $K_{\mathbb{X}} = k\mathbb{1}$, $k \geq 0$ (by Schur's lemma). Thus, $\mathsf{M}_x = k\mathbb{1}$ for all $x \in \mathbb{X}$.

We also obtain a similar preliminary characterization for covariant QOVMs and instruments which we will further refine later in this work.

Theorem 2. *A QOVM or an instrument $\mathcal{I}$ is $(\mathbb{X}, U, V)$–covariant if and only if*

$$\mathcal{I}_x(\rho) = V(g_x)\Lambda_\Omega\big(U(g_x)^*\rho U(g_x)\big)V(g_x)^*$$

for all $x \in \Omega \in \mathcal{O}$ where Λ_Ω is a completely positive linear map such that $\Lambda_\Omega\big(U(h)\rho U(h)^*\big) = V(h)\Lambda_\Omega(\rho)V(h)^*$ for all $h \in H_\Omega$ and all input states ρ. Clearly, the normalization condition $\sum_{x\in\mathbb{X}} \mathrm{tr}\,[\mathcal{I}_x(\rho)] \equiv 1$ holds if and only if $\sum_{\Omega\in\mathcal{O}}\sum_{x\in\Omega} \mathrm{tr}\,\big[\Lambda_\Omega\big(U(g_x)^*\rho U(g_x)\big)\big] \equiv 1$.

One easily sees that we may choose $\Lambda_\Omega = \mathcal{I}_{x_\Omega}$ for any $\Omega \in \mathcal{O}$, and the theorem immediately follows using Equation (2). Furthermore, the normalization condition above simplifies to $\sum_{\Omega\in\mathcal{O}}(\#H_\Omega)^{-1}\sum_{g\in G} \mathrm{tr}\,\big[\Lambda_\Omega\big(U(g)^*\rho U(g)\big)\big] \equiv 1$ where $\#S$ is the number of elements in a set S.

Typically there are infinitely many covariant observables so we can ask which are the optimal ones which satisfy the condition (1). The following six optimality criteria for an observable $\mathsf{M} = (\mathsf{M}_x)_{x\in\mathbb{X}}$ have been previously studied [2, 3, 16]:

(1) **Determination of past**: M determines the past of the system or is *informationally complete (IC)* if its outcome statistics fully determine the pre-measurement state, i.e., for any two input states ρ and σ, $\mathrm{tr}\,[\rho\mathsf{M}_x] = \mathrm{tr}\,[\sigma\mathsf{M}_x]$ for all $x \in \mathbb{X}$ implies $\rho = \sigma$.

(2) **Determination of future**: M determines the future of the system if any M–instrument is nuclear.

(3) **Determination of values**: M determines its values if, for any $x \in \mathbb{X}$ and $\varepsilon > 0$, there is an input state ρ such that $\mathrm{tr}\,[\rho\mathsf{M}_x] > 1 - \varepsilon$.

(4) **Pre-processing cleanness**: M is pre-processing clean if it cannot be obtained from a strictly less noisy observable by first pre-processing the input state, i.e., whenever $\mathsf{N} = (\mathsf{N}_x)_{x\in\mathbb{X}}$ is an observable in a possibly different Hilbert space $\mathcal{K}$ and Φ is a quantum channel (i.e., a completely positive trace-preserving map) with input $\mathcal{K}$ and output $\mathcal{H}$ such that $\mathsf{M}_x = \Phi^*(\mathsf{N}_x)$ for all $x \in \mathbb{X}$, then there is a channel Ψ with input $\mathcal{H}$ and output $\mathcal{K}$ such that $\mathsf{N}_x = \Psi^*(\mathsf{M}_x)$ for all $x \in \mathbb{X}$.

(5) **Post-processing cleanness**: M is post-processing clean if it cannot be obtained by first measuring a strictly more informative observable and then classically

manipulating the outcome data. This means that, whenever there is an observable $\mathsf{N} = (\mathsf{N}_y)_{\mathbb{Y}}$ with a possibly different value set $\mathbb{Y}$ but in the same Hilbert space $\mathcal{H}$ and a probability (Markov) matrix $(p_{x|y})_{x\in\mathbb{X},\,y\in\mathbb{Y}}$ (i.e., $p_{x|y} \geq 0$ for all $x \in \mathbb{X}$ and $y \in \mathbb{Y}$ and $\sum_{x\in\mathbb{X}} p_{x|y} = 1$ for all $y \in \mathbb{Y}$) such that $\mathsf{M}_x = \sum_{y\in\mathbb{Y}} p_{x|y}\mathsf{N}_y$ for all $x \in \mathbb{X}$, there also exists a probability matrix $(q_{y|x})$ such that $\mathsf{N}_y = \sum_{x\in\mathbb{X}} q_{y|x}\mathsf{M}_x$.

In [16], we showed that properties (2) and (5) are both equivalent with the observable M being of rank 1 (i.e. $\mathsf{M}_x = |d_x\rangle\langle d_x|$ or $\mathsf{M}_x = 0$ for any x). There is also another source of classical noise, namely, the mixing of POVMs. This corresponds to the sixth optimality property of being extreme. In the definition below, we also describe the extreme instruments.

Definition 4. We say that an observable $\mathsf{M} = (\mathsf{M}_x)_{x\in X}$ is

(i) *extreme* if it is an extreme point of the convex set of all observables in $\mathcal{H}$ with the value space $\mathbb{X}$, i.e., if $\mathsf{M}_x = t\mathsf{M}_x^+ + (1-t)\mathsf{M}_x^-$ for all $x \in \mathbb{X}$, where $\mathsf{M}^\pm = (\mathsf{M}_x^\pm)_{x\in\mathbb{X}}$ are observables in $\mathcal{H}$ and $t \in (0,1)$, then $\mathsf{M}^+ = \mathsf{M}^-$ and

(ii) an *extreme observable of the* $(\mathbb{X}, U)$*-covariance structure* if it is an extreme point of the convex set of all $(\mathbb{X}, U)$-covariant observables.

We say that an instrument $\mathcal{I} = (\mathcal{I}_x)_{x\in\mathbb{X}}$ (with the input Hilbert space $\mathcal{H}$ and output Hilbert space $\mathcal{K}$) is

(I) *extreme* if it is an extreme point of the convex set of all instruments with the input Hilbert space $\mathcal{H}$, output Hilbert space $\mathcal{K}$, and value set $\mathbb{X}$, i.e., if $\mathcal{I}_x(\rho) = t\mathcal{I}_x^+(\rho) + (1-t)\mathcal{I}_x^-(\rho)$ for all $x \in \mathbb{X}$ and all input states ρ, where $\mathcal{I}^\pm = (\mathcal{I}_x^\pm)_{x\in\mathbb{X}}$ are instruments (with the input Hilbert space $\mathcal{H}$ and output Hilbert space $\mathcal{K}$) and $t \in (0,1)$, then $\mathcal{I}^+ = \mathcal{I}^-$ and

(II) an *extreme instrument of the* $(\mathbb{X}, U, V)$*-covariance structure* if it is an extreme point of the convex set of all $(\mathbb{X}, U, V)$-covariant instruments.

Extreme observables cannot be presented as convex mixtures of observables ('coin tossing between measurements') and, thus, they are free from this type of noise. Extreme elements of the covariance structure do not exhibit noise of this type caused by mixing other covariant observables. Naturally, extreme observables are extreme also within the covariance structure but a covariance structure might not support a single extreme observable. Sharp observables are automatically extreme and they are also free from quantum noise of pre-processing. The third important property of sharp observables is that they determine their values with probabilistic certainty.

Thus, one essentially ends up to two mutually exclusive classes of optimal POVMs:

(a) projection valued rank-1 POVMs and

(b) informationally complete extreme (rank-1) POVMs.

We emphasise that a covariance system characterised by (1) might not allow rank-1, extreme, PVM, or IC solutions. In the worst case, none such optimal solutions exist (e.g. a system with only a trivial solution, an example of which was given just after Theorem 1).

In the D-dimensional Hilbert space $\mathcal{H}$, any IC extreme observable (is rank-1 and) has exactly D^2 non-zero effects $\mathsf{M}_x = |d_x\rangle\langle d_x|$ which form a linearly independent set. Similarly, any rank-1 sharp observable $\mathsf{M}_x = |d_x\rangle\langle d_x|$ has D (linearly independent) non-zero projections which form the usual 'basis measurement.' Indeed, now $\langle d_x|d_y\rangle = \delta_{xy}$ for non-zero vectors d_x and d_y. Since our optimality classes (a) and (b) are clearly disjoint (i.e. the determination of the values and the past are complementary properties) we cannot force any observable to be optimal in all six ways above. What one can do is to assume that some optimality criteria hold only approximately and there are 'continuous' transformation from one class to the other class of properties. We will exhibit examples of this kind of transformations which also preserve covariance.

The common criterion in both optimality classes (a) and (b) is the rank-1 property which we assume from now on. Clearly, a covariant observable M is of rank 1 if and only if, for any orbit $\Omega \in \mathcal{O}$, its 'seed' is of the form $K_\Omega = |d_\Omega\rangle\langle d_\Omega|$ where d_Ω is a common eigenvector for all unitary operators $U(h)$, $h \in H_\Omega$, or $d_\Omega = 0$. Indeed, $U(h)|d_\Omega\rangle\langle d_\Omega| = |d_\Omega\rangle\langle d_\Omega|U(h)$ exactly when $U(h)d_\Omega = cd_\Omega$, $c \in \mathbb{T} := \{c \in \mathbb{C}\,|\,|c| = 1\}$. If $H_\Omega \ni h \mapsto U(h)$ is irreducible then $d_\Omega = 0$ as otherwise $\mathbb{C}d_\Omega$ would be a non-trivial invariant subspace. Hence, we may choose $d_x = U(g_x)d_\Omega$, $x \in \Omega$. If $d_\Omega = 0$ then all operators $\mathsf{M}_x = |d_x\rangle\langle d_x|$ vanish in the orbit Ω so that the outcomes of that orbit are never registered in any measurement of M. In this case, one can redefine $\mathbb{X}$ to be the union of all orbits where M is not zero.

If M belongs to class (a) (i.e. is a sharp observable) then it has exactly D non-zero (mutually orthogonal) unit vectors d_x. For example, if there is only one orbit $\Omega = \mathbb{X}$ and $H_\mathbb{X} = \{e\}$ then both G and $\mathbb{X}$ has exactly D elements (i.e. any $x = g_x x_\Omega$ where g_x is unique) we may take any orthonormal basis $\{d_x\}_{x\in\mathbb{X}}$ of a D-dimensional Hilbert space and define a unitary representation $U(g) := \sum_{x\in\mathbb{X}} |d_{gx}\rangle\langle d_x|$ to get a covariant rank-1 PVM $\mathsf{M}_x := |d_x\rangle\langle d_x|$. In this case, we see that (1) cannot have a extreme IC solution (since we would need D^2 non-zero effects). However, one can extend the covariance structure in such a way that it may also admit an extreme IC solution: We extend the group action $G \times \mathbb{X} \ni (g, x) \mapsto gx \in \mathbb{X}$ to the Cartesian product $\mathbb{X}^2 := \mathbb{X} \times \mathbb{X}$ into $G \times \mathbb{X}^2 \ni \big(g, (x, y)\big) \mapsto g(x, y) := (gx, gy) \in \mathbb{X}^2$ and interpret any covariant observable $\mathsf{M} = (\mathsf{M}_x)_{x\in\mathbb{X}}$ as a covariant observable $\mathsf{G} = (\mathsf{G}_{(x,y)})_{(x,y)\in\mathbb{X}^2}$ with $\mathsf{G}_{(x,y)} = \delta_{x,y}\mathsf{M}_x$. Note that U remains the same. Clearly, G is supported on the diagonal $\{(x, x)\,|\,x \in \mathbb{X}\} \cong \mathbb{X}$ and it can be seen as a (trivial) joint measurement of M with itself; recall that a POVM $(\mathsf{G}_{(x,y)})$ is a joint observable for POVMs (A_x) and (B_y) if $\sum_y \mathsf{G}_{(x,y)} = \mathsf{A}_x$ and $\sum_x \mathsf{G}_{(x,y)} = \mathsf{B}_y$. A question is whether there is a covariant extreme IC solution for this enlarged system. We readdress this problem in Example 1 in the following subsection

2.1. The relevance of covariance structures involving symmetric groups

Next we will see that any covariant rank-1 POVM is a projection (and postprocessing) of a rank-1 PVM and that this PVM (extension) can be assumed to be covariant w.r.t. the symmetric group $\mathrm{Sym}(G)$ in a particular way to be determined shortly. Indeed, let $\mathsf{M}_x = U(g_x)|d_\Omega\rangle\langle d_\Omega|U(g_x)^*$, $x \in \Omega \in \mathcal{O}$, be a covariant rank-1 POVM which need not be normalized since we can normalize it later (see Remark 1). In order to see that M can be obtained from a rank-1 PVM through post-processing and projecting we take the following steps:

(i) Define a new (finite) outcome space $\mathbb{X}' := \mathcal{O} \times G$ and a POVM

$$\mathsf{M}'_{\Omega,g} := \frac{1}{\#H_\Omega} U(g)|d_\Omega\rangle\langle d_\Omega|U(g)^*, \qquad \Omega \in \mathcal{O}, \quad g \in G,$$

Clearly, $\mathsf{M}'_{\Omega,g_x} = \mathsf{M}'_{\Omega,g_x h} = \mathsf{M}_x/\#H_\Omega$, $x \in \Omega$, $h \in H_\Omega$, so that if M is normalized then M' is also normalized to $\mathbb{1}$ and M is a post-processing of M',

$$\mathsf{M}_x = \sum_{h \in H_\Omega} \mathsf{M}'_{\Omega,g_x h}, \qquad x \in \Omega,$$

that is, any measurement of M' can be viewed as a measurement of M. Note that M' is also covariant when $\mathbb{X}'$ is equipped with the G-action $g(\Omega, g') := (\Omega, gg')$ and the orbits are $\{\Omega\} \times G$, $\Omega \in \mathcal{O}$.

(ii) Consider then a covariant Naĭmark dilation of M' (which is minimal if and only if $d_\Omega \neq 0$ for all $\Omega \in \mathcal{O}$ (i.e. $\mathsf{M}_x \neq 0$ for all $x \in \mathbb{X}$): The $(\#\mathcal{O}\#G)$–dimensional dilation space is spanned by orthonormal vectors $|\Omega, g\rangle$, $\Omega \in \mathcal{O}$, $g \in G$. Now

$$J := \sum_{\Omega \in \mathcal{O}} \frac{1}{\sqrt{\#H_\Omega}} \sum_{g \in G} |\Omega, g\rangle\langle d_\Omega|U(g)^*$$

and the canonical (rank-1) PVM $\mathsf{Q}_{\Omega,g} := |\Omega, g\rangle\langle \Omega, g|$ are such that

$$\mathsf{M}'_{\Omega,g} \equiv J^* \mathsf{Q}_{\Omega,g} J.$$

Clearly, M' is normalized if and only if J is an isometry (i.e. $J^*J = \mathbb{1}$). Thus, any measurement of the normalized POVM M' can be seen as a measurement of Q when the states are restricted to the range (subspace) of the Naĭmark projection JJ^*. Note that also Q is covariant. Indeed, if m is the Schur multiplier (2-cocycle) of the projective unitary representation $g \mapsto U(g)$ one can define a multiplier (left regular) representation

$$V(g) := \sum_{\Omega \in \mathcal{O}} \sum_{g' \in G} \overline{m(g, g')} |\Omega, gg'\rangle\langle \Omega, g'|$$

such that $V(gg') = m(g, g')V(g)V(g')$, $V(g)J = JU(g)$ and $\mathsf{Q}_{g(\Omega,g')} = V(g)\mathsf{Q}_{\Omega,g'}V(g)^*$.

(iii) We can extend the group G and assume that the multiplier $m(g,g')\equiv 1$. Indeed, as shown in Appendix A, one can suppose that there exists a (minimal) positive integer $p\le \#G$ such that $m(g,g')^p=1$ for all $g,\, g'\in G$ and $m(e,e)=1$. Define then the (multiplicative) cyclic group $\langle t\rangle=\{1,t,t^2,\ldots,t^{p-1}\}$ where $t:=\exp(2\pi i/p)$ so that $m(g,g')\in\langle t\rangle$, i.e. $m(g,g')=t^{q(g,g')}$ where $q(g,g')\in\{0,1,\ldots,p-1\}$. Now a central extension group (induced by m) is a finite set $G_m:=G\times\langle t\rangle$ equipped with the multiplication $(g,t^k)(g',t^\ell):=\big(gg',\overline{m(g,g')}t^{k+\ell}\big)$. Since $m(g,e)=m(e,g)=m(e,e)=1$ one sees that $(e,1)$ is the identity element of G_m and $(g,t^k)^{-1}=\Big(g^{-1},\overline{m(g,g^{-1})}^{-1}t^{-k}\Big)$. Defining unitary operators $\tilde U(g,t^k):=t^kU(g)$ one gets the unitary representation of G_m, i.e. $\tilde U\big((g,t^k)(g',t^\ell)\big)=\tilde U(g',t^\ell)\tilde U(g,t^k)$ with the constant cocycle. Futhermore, the action gx extends trivially: $(g,t^k)x:=gx$ and we get

$$\mathsf{M}_{(g,t^k)x}=\mathsf{M}_{gx}=U(g)\mathsf{M}_xU(g)^*=\tilde U(g,t^k)\mathsf{M}_x\tilde U(g,t^k)^*.$$

Hence, M can be seen as a covariant POVM with respect to the larger group G_m. Note that if already $m(g,g')\equiv 1$ one has $p=1$, $\langle t\rangle=\{1\}$ and $G_m\cong G$ via $(g,1)\mapsto g$. To conclude, one can replace G with G_m (and elements g with pairs (g,t^k)) everywhere in items (1) and (2) and put $m(g,g')\equiv 1$.

(iv) If $m(g,g')\equiv 1$ then $V(g)=\sum_{\Omega\in\mathcal{O}}\sum_{g'\in G}|\Omega,gg'\rangle\langle\Omega,g'|$ is just a permutation $\pi(g')=gg'$ acting on the basis vectors $|\Omega,g'\rangle$ for a fixed Ω. Thus, one can view G as a subgroup of the symmetric group $\mathrm{Sym}(G)$ of bijective maps $\pi:G\to G$. Especially, V extends to the unitary representation $\overline{V}(\pi):=\sum_{\Omega\in\mathcal{O}}\sum_{g'\in G}|\Omega,\pi(g')\rangle\langle\Omega,g'|$, $\pi\in\mathrm{Sym}(G)$, which is a direct sum of the representations $\pi\mapsto\sum_{g'\in G}|\Omega,\pi(g')\rangle\langle\Omega,g'|$. Note that the PVM $\mathsf{Q}_{\Omega,g}=|\Omega,g\rangle\langle\Omega,g|$ of item (2) is also covariant with respect to the larger group $\mathrm{Sym}(G)$: $\mathsf{Q}_{\Omega,\pi(g)}=\overline{V}(\pi)\mathsf{Q}_{\Omega,g}\overline{V}(\pi)^*$. Finally, we can simply number the elements of G, $G=\{g_1,g_2,\ldots,g_{\#G}\}$, and identify G (respectively, $\mathrm{Sym}(G)$) with $\{1,2,\ldots,\#G\}$ (resp. the permutations of the integers in question).

Above, we have a method for constructing optimal observables. Namely, one can start from item (iv) and go backwards, i.e., start with the rank-1 PVM (sharp observable) $\mathsf{Q}=(\mathsf{Q}_n)_{n=1}^D$, where $\mathsf{Q}_n^D:=|n\rangle\langle n|$, $n\in\mathbb{X}_D:=\{1,\ldots,D\}$, which is covariant with respect to the symmetric group $S_D=\mathrm{Sym}(\mathbb{X}_D)$ which act in an D–dimensional Hilbert space with an orthonormal basis $\{|1\rangle,\,|2\rangle,\ldots,|D\rangle\}$ via the representation $U(\pi)=\sum_{n=1}^D|\pi(n)\rangle\langle n|$. Note that, in item (iv), $D=\#G$ and $|n\rangle=|\Omega,g_n\rangle$. Next, we can project this PVM onto a subspace and relabel and post-process the resulting POVM and thus obtain any POVM covariant w.r.t. to any group G such that $\#G\le D$.

The steps taken above show that POVMs covariant w.r.t. the symmetric groups are crucial for understanding covariant finite observables in finite dimensional Hilbert spaces. However, in this setting, the optimality class (b) is excluded. These optimal observables are naturally obtained after properly projecting and post-processing the rank-1 PVMs of item (iv). The question still arises, can we find optimal observables of class (b) even in the setting of item (iv) above by extending the value space. To get an IC extreme POVM we first enlarge the outcome space $\mathbb{X}_D$ to the Cartesian product $\mathbb{X}_D^2 = \{(n,m)\,|\,1 \le n,\, m \le D\}$ where S_D acts via $\pi(n,m) := \big(\pi(n),\pi(m)\big)$. Identify $\mathbb{X}_D$ with the diagonal of $\mathbb{X}_D^2$. Note that we obtain all finite covariant POVMs through projections and relabelings from this larger $(\mathbb{X}_D^2, U)$–covariance structure as well where $U : \pi \mapsto \sum_{n=1}^D |\pi(n)\rangle\langle n|$ as the sharp observables supported by the diagonal are already sufficient for this. In Example 2, we define a continuous family of covariant rank-1 IC extreme observables (with outcome space $\mathbb{X}_D^2$) with the end point Q^D. We want to stress that the connective POVMs are also extreme and thus they are not (classical) convex mixtures. In dimension three ($D = 3$) this is an easy exercise which we demonstrate next.

Example 1. Consider the permutation group S_3 of a three element set $\mathbb{X}_3 = \{1,2,3\}$. Its generators are permutations (12) and (13). The other permutations are $e = (1) = (12)(12)$, $(123) = (13)(12)$, $(132) = (12)(13)$, and $(23) = (12)(13)(12)$. By definition, S_3 operates on $\{1,2,3\}$ by permuting its elements (e.g. $(23)1 = 1$, $(23)2 = 3$ and $(23)3 = 2$). As before, S_3 operates also on the nine element set $\mathbb{X}_3^2 = \{1,2,3\} \times \{1,2,3\}$ [e.g. $(23)(1,3) := \big((23)1,(23)3\big) = (1,2)$]. Let the Hilbert space be three dimensional, fix its orthonormal basis $\{|1\rangle, |2\rangle, |3\rangle\}$ and define a unitary representation by $U(\pi) = \sum_{n=1}^3 |\pi(n)\rangle\langle n|$, $\pi \in S_3$, that is,

$$
\begin{aligned}
&U(12) = |2\rangle\langle 1| + |1\rangle\langle 2| + |3\rangle\langle 3|, \;\; U(13) = |3\rangle\langle 1| + |2\rangle\langle 2| + |1\rangle\langle 3|,\\
&U(1) = |1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3|, \quad\; U(123) = |2\rangle\langle 1| + |3\rangle\langle 2| + |1\rangle\langle 3|,\\
&U(132) = |3\rangle\langle 1| + |1\rangle\langle 2| + |2\rangle\langle 3|, U(23) = |1\rangle\langle 1| + |3\rangle\langle 2| + |2\rangle\langle 3|.
\end{aligned}
$$

(i) We have $\mathbb{X}_3^2 = \Omega \uplus \Omega'$ where the orbits are $\Omega = \{(1,1),(2,2),(3,3)\} \cong \mathbb{X}_3$ and $\Omega' = \{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}$ from where we pick points $x_\Omega = (1,1)$ and $x_{\Omega'} = (1,2)$.

(ii) Stability subgroups are $H_\Omega = \{(1),(23)\}$ and $H_{\Omega'} = \{(1)\}$.

(iii) Since $H_{\Omega'}$ is trivial, its seed $K_{\Omega'}$ can be an arbitrary positive operator. On the other hand, the seed $K_\Omega \ge 0$ must commute with

$$U(23) = |1\rangle\langle 1| + |3\rangle\langle 2| + |2\rangle\langle 3| = 1 \cdot (|1\rangle\langle 1| + |\varphi_+^{23}\rangle\langle\varphi_+^{23}|) - 1 \cdot |\varphi_-^{23}\rangle\langle\varphi_-^{23}|,$$

where the eigenvectors are of the form $\varphi_\pm^{ij} := 2^{-1/2}(|i\rangle \pm |j\rangle)$, $i,j \in \{1,2,3\}$, so that

$$K_\Omega = a|1\rangle\langle 1| + b|1\rangle\langle\varphi_+^{23}| + \bar{b}|\varphi_+^{23}\rangle\langle 1| + c|\varphi_+^{23}\rangle\langle\varphi_+^{23}| + d|\varphi_-^{23}\rangle\langle\varphi_-^{23}|,$$

where the complex numbers satisfy the following conditions: $a,\, c,\, d \ge 0$ and $ac \ge |b|^2$.

(iv) Choose $g_{(1,1)} = (1)$, $g_{(2,2)} = (12)$ and $g_{(3,3)} = (13)$ for Ω and $g_{(1,2)} = (1)$, $g_{(2,1)} = (12)$, $g_{(1,3)} = (23)$, $g_{(3,1)} = (132)$, $g_{(2,3)} = (123)$, and $g_{(3,2)} = (13)$ for Ω'.

(v) Finally, we normalize the following covariant POVM (where $a,\, c,\, d \ge 0$ and

$ac \geq |b|^2)$

$$\begin{aligned}
\mathsf{M}_{(1,1)} =& K_\Omega = a|1\rangle\langle 1| + b|1\rangle\langle\varphi_+^{23}| + \bar{b}|\varphi_+^{23}\rangle\langle 1| + c|\varphi_+^{23}\rangle\langle\varphi_+^{23}| + d|\varphi_-^{23}\rangle\langle\varphi_-^{23}|,\\
\mathsf{M}_{(2,2)} =& U(12)K_\Omega U(12)^*\\
=& a|2\rangle\langle 2| + b|2\rangle\langle\varphi_+^{13}| + \bar{b}|\varphi_+^{13}\rangle\langle 2| + c|\varphi_+^{13}\rangle\langle\varphi_+^{13}| + d|\varphi_-^{13}\rangle\langle\varphi_-^{13}|,\\
\mathsf{M}_{(3,3)} =& U(13)K_\Omega U(13)^*\\
=& a|3\rangle\langle 3| + b|3\rangle\langle\varphi_+^{21}| + \bar{b}|\varphi_+^{21}\rangle\langle 3| + c|\varphi_+^{21}\rangle\langle\varphi_+^{21}| + d|\varphi_-^{21}\rangle\langle\varphi_-^{21}|,
\end{aligned}$$

$$\begin{aligned}
\mathsf{M}_{(1,2)} &= K_{\Omega'} \geq 0, & \mathsf{M}_{(3,1)} &= U(132)K_{\Omega'}U(132)^*,\\
\mathsf{M}_{(2,1)} &= U(12)K_{\Omega'}U(12)^*, & \mathsf{M}_{(2,3)} &= U(123)K_{\Omega'}U(123)^*,\\
\mathsf{M}_{(1,3)} &= U(23)K_{\Omega'}U(23)^*, & \mathsf{M}_{(3,2)} &= U(13)K_{\Omega'}U(13)^*.
\end{aligned}$$

If the operators $\mathsf{M}_{(n,m)}$ are linearly independent (resp. rank-1) then the normalized operators $K^{-1/2}\mathsf{M}_{(n,m)}K^{-1/2}$, $K = \sum_{n,m=1}^3 \mathsf{M}_{(n,m)}$, are also linearly independent (resp. rank-1).

Note that the matrices of the first three operators are

$$\mathsf{M}_{(1,1)} = \begin{pmatrix} a & b' & b' \\ \overline{b'} & c' & c' \\ \overline{b'} & c' & c' \end{pmatrix} + d' \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix},$$

$$\mathsf{M}_{(2,2)} = \begin{pmatrix} c' & \overline{b'} & c' \\ b' & a & b' \\ c' & \overline{b'} & c' \end{pmatrix} + d' \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix},$$

$$\mathsf{M}_{(3,3)} = \begin{pmatrix} c' & c' & \overline{b'} \\ c' & c' & \overline{b'} \\ b' & b' & a \end{pmatrix} + d' \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $b' = 2^{-1/2}b$, $c' = c/2$, and $d' = d/2$ (now $ac' \geq |b'|^2$).

- M is rank-1 iff $K_\Omega = |d_\Omega\rangle\langle d_\Omega|$ and $K_{\Omega'} = |d_{\Omega'}\rangle\langle d_{\Omega'}|$. Now $K_\Omega = |d_\Omega\rangle\langle d_\Omega| \neq 0$ iff either $ac = |b|^2 \neq 0$ (i.e. $ac' = |b'|^2$) and $d = 0$, or $a = b = c = 0$ and $d > 0$.
- M is a rank-1 PVM if $a = 1$ and $b = c = d = 0$ and $K_{\Omega'} = 0$ (i.e. $\mathsf{M}_{(n,m)} = \delta_{nm}|n\rangle\langle n|$). Note that we can always choose the basis such that a rank-1 PVM is the corresponding diagonal 'basis measurement.'
- A rank-1 M is IC extreme (after normalization) iff the nine effects $\mathsf{M}_{(n,m)}$ are linearly independent. By direct calculation, this happens if we choose $K_\Omega = |1\rangle\langle 1|$ and $K_{\Omega'} = |d_{\Omega'}\rangle\langle d_{\Omega'}|$ where $d_{\Omega'} = \alpha\big(e^{-i\pi/8}|1\rangle + e^{i\pi/8}|2\rangle\big)$, $\alpha > 0$. For the properly normalized POVM, see Example 2.

To conclude, we have a continuous (α–indexed) family of covariant rank-1 IC extreme POVMs whose ($\alpha = 0$) end point is a covariant rank-1 PVM. The POVMs with $\alpha > 0$ and $\alpha = 0$ represent the two complementary optimality classes. It is interesting to see that in the case $\alpha \approx 0$ we get an IC POVM which is 'almost' a PVM. $\triangle$

Using similar methods as above, we may extend an $(\mathbb{X}, U, V)$–covariant QOVM into an instrument whose values are described by $\mathcal{O}$ and G and whose symmetries are simply described by permutations of the elements of G. Let m_U (resp. m_V) be the multiplier associated with U (resp. with V). In particular, through a similar group extension method, picking a (minimal) positive integer $p \leq \#G$ such that $m_U(g,h)^p = 1 = m_V(g,h)^p$ for all $g,\, h \in G$, we may essentially assume that U and V are ordinary unitary representations, i.e., $m_U(g,h) = 1 = m_V(g,h)$ for all $g,\, h \in G$. In the following section, we will further concentrate on covariant QOVMs and instruments enabling a more detailed analysis of covariant observables as well.

3. Instruments covariant with respect to a finite group

In this section, we take a closer look at covariant instruments covariant w.r.t. a finite group, give a thorough description of their structure and associate particular single-point Kraus operators of these instruments with minimal Stinespring dilations. The results obtained are then used in the subsequent Subsection 3.1 to characterize the extreme covariant instruments.

We fix Hilbert spaces $\mathcal{H}$ (input system) and $\mathcal{K}$ (output system) and a finite set $\mathbb{X}$ (measurement outcomes). We denote by $\mathcal{L}(\mathcal{H})$ the set of (bounded) linear operators on $\mathcal{H}$ and by $\mathcal{U}(\mathcal{H})$ the group of unitary operators on $\mathcal{H}$. We use the same notations for the output system Hilbert space $\mathcal{K}$ and, moreover, denote by $\mathcal{L}(\mathcal{H},\mathcal{K})$ the set of (bounded) linear operators defined on $\mathcal{H}$ and taking values in $\mathcal{K}$. As in this and a couple of the following sections we concentrate on finite dimensional systems, we can disregard the notion of boundedness for now. We assume $\mathbb{X}$ to be a G-space for a finite group G, and retain the related notations fixed earlier. Let us fix an orbit $\Omega \in \mathcal{O}$. We denote by $\hat{H}_\Omega$ the representation dual of H_Ω, i.e., the set of unitary equivalence classes of irreducible unitary representations of H_Ω. We pick a representative for every element of $\hat{H}_\Omega$ and we denote these representatives typically by $\eta : H_\Omega \to \mathcal{U}(\mathcal{K}_\eta)$ and the corresponding equivalence class we denote by $[\eta]$. This convention should cause no confusion. We denote, for any $[\eta] \in \hat{H}_\Omega$, the dimension of $\mathcal{K}_\eta$ by $D_\eta \in \mathbb{N} := \{1, 2, 3, \ldots\}$ and fix an orthonormal basis $\{e_{\eta,i}\}_{i=1}^{D_\eta}$ for $\mathcal{K}_\eta$. We denote, for any $[\eta] \in \hat{H}_\Omega$,

$$\eta_{i,j}(h) := \langle e_{\eta,i} | \eta(h) e_{\eta,j} \rangle, \qquad i,\, j = 1, \ldots, D_\eta, \quad h \in H_\Omega.$$

As we identify Ω with G/H_Ω, we pick a section $s_\Omega : \Omega \to G$ (i.e., $s_\Omega(x)H_\Omega$ corresponds to x for any $x \in \Omega$) such that $s_\Omega(x_\Omega) = e$.‡ Using these, we define, for all $[\eta] \in \hat{H}_\Omega$, the cocycles $\zeta^\eta : G \times \Omega \to \mathcal{U}(\mathcal{K}_\eta)$ through

$$\zeta^\eta(g, x) = \eta\big(s_\Omega(x)^{-1} g^{-1} s_\Omega(gx)\big), \qquad g \in G, \quad x \in \Omega,$$

‡ Note that we have used the notation g_x for $s_\Omega(x)$ for all $x \in \Omega$ in Section 2, but this notation would be slightly cumbersome in the following discussion. Also recall that we have fixed a reference point $x_\Omega \cong H_\Omega = G_{x_\Omega}$ for any orbit $\Omega \cong G/H_\Omega$.

and define the cocycle $\zeta^\pi : G \times \Omega \to \mathcal{U}(\mathcal{H}_\pi)$ in exactly the same way whenever $\pi : H_\Omega \to \mathcal{U}(\mathcal{H}_\pi)$ is a unitary representation in some Hilbert space $\mathcal{H}_\pi$. Note that the cocycle conditions

$$\zeta^\pi(gh, x) = \zeta^\pi(h, x)\zeta^\pi(g, hx), \qquad \zeta^\pi(e, x) = \mathbb{1}_{\mathcal{H}_\pi} \tag{3}$$

hold for any g, $h \in G$ and $x \in \Omega$. In addition, for any $h \in H_\Omega$, $\zeta^\pi(h^{-1}, x_\Omega) = \pi(h)$. Finally, we denote by $\zeta^\eta_{i,j} : G \times \Omega \to \mathbb{C}$ the matrix element functions of ζ^η in the basis $\{e_{\eta,i}\}_{i=1}^{D_\eta}$ for any $[\eta] \in \hat{H}_\Omega$.

We say that a quadruple $(\mathcal{M}, \mathsf{P}, \overline{U}, J)$ consisting of a Hilbert space $\mathcal{M}$, a sharp observable $\mathsf{P} = (\mathsf{P}_x)_{x\in\mathbb{X}}$ in $\mathcal{M}$, a unitary representation $\overline{U} : G \to \mathcal{U}(\mathcal{M})$, and an linear map $J : \mathcal{H} \to \mathcal{K} \otimes \mathcal{M}$ is an $(\mathbb{X}, U, V)$*–covariant minimal Stinespring dilation* for an $(\mathbb{X}, U, V)$–covariant QOVM (or, more specifically, instrument) $\mathcal{I} = (\mathcal{I}_x)_{x\in\mathbb{X}}$ if

(i) $\mathcal{I}^*_x(B) = J^*(B \otimes \mathsf{P}_x)J$ for all $x \in \mathbb{X}$ and $B \in \mathcal{L}(\mathcal{K})$, where $\mathcal{I}^*_x$ is the Heisenberg dual operation for $\mathcal{I}_x$ (i.e., $\mathrm{tr}\,[\mathcal{I}^*_x(B)\rho] := \mathrm{tr}\,[B\mathcal{I}_x(\rho)]$ for all $B \in \mathcal{L}(\mathcal{K})$ and all input states ρ),

(ii) $JU(g) = \big(V(g) \otimes \overline{U}(g)\big)J$ for all $g \in G$,

(iii) $\overline{U}(g)\mathsf{P}_x\overline{U}(g)^* = \mathsf{P}_{gx}$ for all $g \in G$ and $x \in \mathbb{X}$, and

(iv) vectors $(B \otimes \mathsf{P}_x)J\varphi$, $B \in \mathcal{L}(\mathcal{K})$, $x \in \mathbb{X}$, $\varphi \in \mathcal{H}$, span $\mathcal{K} \otimes \mathcal{M}$.

Recall that any QOVM $\mathcal{I}$ has a [minimal] Stinespring dilation $(\mathcal{M}, \mathsf{P}, J)$ satisfying item (i) [and item (iv)] above. We construct the representation $\overline{U}$ satisfying items (ii) and (iii) for any covariant instrument in Appendix B for completeness. There we also show (using Mackey's theory of imprimitivity) that a $(\mathbb{X}, U, V)$-covariant minimal dilation $(\mathcal{M}, \mathsf{P}, \overline{U}, J)$ for an $(\mathbb{X}, U, V)$-covariant QOVM $\mathcal{I} = (\mathcal{I}_x)_{x\in\mathbb{X}}$ can be given the following form: There is, for each $\Omega \in \mathcal{O}$, a (finite-dimensional) Hilbert space $\mathcal{M}^\Omega$ and $\mathcal{H}^\Omega$ such that $\mathcal{M}^\Omega = \mathbb{C}^{\#\Omega} \otimes \mathcal{H}^\Omega$, and $\mathcal{M} = \bigoplus_{\Omega\in\mathcal{O}} \mathcal{M}^\Omega$. Moreover, for each $\Omega \in \mathcal{O}$, there is a unitary representation $\pi^\Omega : H_\Omega \to \mathcal{U}(\mathcal{H}^\Omega)$ such that

$$\big(\overline{U}^\Omega(g)f\big)(x) = \zeta^\Omega(g^{-1}, x)f(g^{-1}x), \qquad g \in G, \quad f \in \mathcal{M}^\Omega, \quad x \in \Omega, \tag{4}$$

where $\zeta^\Omega := \zeta^{\pi^\Omega}$ is the cocycle associated with π^Ω. Note that we identify $\mathcal{M}^\Omega$ with the Hilbert space of functions $f : \Omega \to \mathcal{H}^\Omega$. Furthermore, for each $\Omega \in \mathcal{O}$, $\mathsf{P}^\Omega := (\mathsf{P}_x)_{x\in\Omega}$ is a sharp observable in $\mathcal{M}^\Omega$ and

$$\mathsf{P}^\Omega_x f = f(x), \qquad x \in \Omega, \quad f \in \mathcal{M}^\Omega. \tag{5}$$

In total, $(\overline{U}, \mathsf{P})$ is a direct sum of the *canonical systems of imprimitivity* $(\overline{U}^\Omega, \mathsf{P}^\Omega)$ over $\Omega \in \mathcal{O}$.

To elaborate Theorem 2, we present a useful definition. From now on, the paradoxical notation $m = 1, \ldots, 0$ means that the set of indices m is empty, and sums of the form $\sum_{m=1}^{0}(\cdots)$ vanish.

Definition 5. Given, for any $\Omega \in \mathcal{O}$ and $[\eta] \in \hat{H}_\Omega$, a number $M_\eta \in \{0\} \cup \mathbb{N}$, we say that operators $L^\Omega_{\eta,i,m} \in \mathcal{L}(\mathcal{H},\mathcal{K})$ constitute a *set of* $(\mathbb{X}, U, V)$*–intertwiners* if, for all orbits Ω, $[\eta] \in \hat{H}_\Omega$, $i = 1, \ldots, D_\eta$, $m = 1, \ldots, M_\eta$, and $h \in H_\Omega$,

$$L^\Omega_{\eta,i,m} U(h) = \sum_{j=1}^{D_\eta} \eta_{i,j}(h) V(h) L^\Omega_{\eta,j,m}. \tag{6}$$

This set of $(\mathbb{X}, U, V)$–intertwiners is *normalized* if

$$\sum_{\Omega\in\mathcal{O}} \sum_{g\in G} \sum_{[\eta]\in\hat{H}_\Omega} \sum_{i=1}^{D_\eta} \sum_{m=1}^{M_\eta} \frac{1}{\#H_\Omega} U(g) L^{\Omega\,*}_{\eta,i,m} L^\Omega_{\eta,i,m} U(g)^* = \mathbb{1}_\mathcal{H}. \tag{7}$$

The (normalized) set $\{L^\Omega_{\eta,i,m} \,|\, m = 1, \ldots, M_\eta,\ i = 1, \ldots, D_\eta,\ [\eta] \in \hat{H}_\Omega,\ \Omega \in \mathcal{O}\}$ of $(\mathbb{X}, U, V)$–intertwiners is *minimal* if, for any orbit $\Omega \in \mathcal{O}$, the set

$$\{L^\Omega_{\eta,i,m} \,|\, m = 1, \ldots, M_\eta,\ i = 1, \ldots, D_\eta,\ [\eta] \in \hat{H}_\Omega\}$$

is linearly independent.

Note that, whenever $M_{\eta'} = 0$ for some $[\eta'] \in \hat{H}_\Omega$, the set of intertwiners $L^\Omega_{\eta,i,m}$ does not contain operators where η' appears as an index. This is to avoid zero operator as an intertwiner: If $L^\Omega_{\eta',i,m}$ would be required to be zero for some $[\eta']$ and all $i = 1, \ldots, D_\eta$, by choosing $M_{\eta'} = 0$ we can exclude these intertwiners from the set. Avoiding zero operators is important for the linear independence of a minimal set of intertwiners which is an important feature as we will see in Lemma 1 shortly. The following theorem exhaustively determines the $(\mathbb{X}, U, V)$–covariant instruments. It also gives a recipe for constructing covariant instruments and indicates that covariant instruments have the structure conjectured in Section III [21]; in this reference, the form of covariant instruments (QOVMs) of the theorem below is conjectured for general type-I groups. While the conjecture remains to be proven in this generality, in the subsequent Theorem 5 we prove this structure result for a varied class of covariant (continuous) instruments.

Theorem 3. *For any* $(\mathbb{X}, U, V)$*–covariant QOVM [instrument]* $\mathcal{I} = (\mathcal{I}_x)_{x\in\mathbb{X}}$*, there is a [normalized] minimal set*

$$\{L^\Omega_{\eta,i,m} \,|\, m = 1, \ldots, M_\eta,\ i = 1, \ldots, D_\eta,\ [\eta] \in \hat{H}_\Omega,\ \Omega \in \mathcal{O}\}$$

of $(\mathbb{X}, U, V)$*–intertwiners, where* $M_\eta \in \mathbb{N} \cup \{0\}$ *for all* $[\eta] \in \hat{H}_\Omega$ *and* $\Omega \in \mathcal{O}$*, such that, for all* $\Omega \in \mathcal{O}$*,* $g \in G$*, and input states* ρ *on* $\mathcal{H}$*,*

$$\mathcal{I}_{gH_\Omega}(\rho) = \sum_{[\eta]\in\hat{H}_\Omega} \sum_{i=1}^{D_\eta} \sum_{m=1}^{M_\eta} V(g) L^\Omega_{\eta,i,m} U(g)^* \rho U(g) L^{\Omega\,*}_{\eta,i,m} V(g)^*. \tag{8}$$

On the other hand, whenever $\{L^\Omega_{\eta,i,m} \,|\, m = 1, \ldots, M_\eta,\ i = 1, \ldots, D_\eta,\ [\eta] \in \hat{H}_\Omega,\ \Omega \in \mathcal{O}\}$*, where* $M_\eta \in \mathbb{N} \cup \{0\}$ *for any* $[\eta] \in \hat{H}_\Omega$ *and* $\Omega \in \mathcal{O}$*, is a [normalized] set of* $(\mathbb{X}, U, V)$*–intertwiners, Equation (8) determines an* $(\mathbb{X}, U, V)$*–covariant QOVM [instrument]* $\mathcal{I} = (\mathcal{I}_x)_{x\in\mathbb{X}}$*.*

Note that, for the QOVM $\mathcal{I}$ of Equation (8), and for any orbit $\Omega \in \mathcal{O}$, the map Λ_Ω of Theorem 2 is given by $\Lambda_\Omega(\rho) = \sum_{[\eta]\in\hat{H}_\Omega} \sum_{i=1}^{D_\eta} \sum_{m=1}^{M_\eta} L^\Omega_{\eta,i,m} \rho L^{\Omega *}_{\eta,i,m}$ for any input state ρ.

Proof. Let us first fix an $(\mathbb{X}, U, V)$–covariant QOVM $\mathcal{I} = (\mathcal{I}_x)_{x\in\mathbb{X}}$ and equip it with a minimal $(\mathbb{X}, U, V)$–covariant Stinespring's dilation $(\mathcal{M}, \mathsf{P}, \overline{U}, J)$ so that $(\overline{U}, \mathsf{P})$ is a system of imprimitivity. As detailed before the statement of this theorem, we represent this system of imprimitivity as a direct sum of the canonical systems $(\overline{U}^\Omega, \mathsf{P}^\Omega)$ of imprimitivity defined in Equations (4) and (5).

Let us fix an orbit $\Omega \in \mathcal{O}$. According to the Peter-Weyl theorem, for each $[\eta] \in \hat{H}_\Omega$, there is a Hilbert space $\mathcal{M}_\eta$ such that $\mathcal{H}^\Omega = \bigoplus_{[\eta]\in\hat{H}_\Omega} \mathcal{K}_\eta \otimes \mathcal{M}_\eta$ and $\pi^\Omega(g) = \bigoplus_{[\eta]\in\hat{H}_\Omega} \eta(g) \otimes \mathbb{1}_{\mathcal{M}_\eta}$ for all $g \in G$. Denote the dimension of $\mathcal{M}_\eta$ by M_η and pick an orthonormal basis $\{f_{\eta,m}\}_{m=1}^{M_\eta} \subset \mathcal{M}_\eta$. Let $\{\delta_x\}_{x\in\Omega}$ be the natural basis of $\mathbb{C}^{\#\Omega}$. Thus, $\{\delta_x \otimes e_{\eta,i} \otimes f_{\eta,m} \,|\, x \in \Omega,\ [\eta] \in \hat{H}_\Omega,\ i = 1, \ldots, D_\eta,\ m = 1, \ldots, M_\eta\}$ is an orthonormal basis of $\mathcal{H}^\Omega$ and the union of these bases over Ω is an orthonormal basis for $\mathcal{M}$. Define, for $x \in \Omega$, $[\eta] \in \hat{H}_\Omega$, $i = 1, \ldots, D_\eta$, and $m = 1, \ldots, M_\eta$, the isometry $V_{x,\eta,i,m} : \mathcal{K} \to \mathcal{K} \otimes \mathcal{M}^\Omega \subseteq \mathcal{K} \otimes \mathcal{M}$ through $V_{x,\eta,i,m}\psi = \psi \otimes \delta_x \otimes e_{\eta,i} \otimes f_{\eta,m}$ for all $\psi \in \mathcal{K}$. Clearly, $V_{x,\eta,i,m} B V^*_{x,\eta,i,m} = B \otimes |\delta_x \otimes e_{\eta,i} \otimes f_{\eta,m}\rangle\langle \delta_x \otimes e_{\eta,i} \otimes f_{\eta,m}|$ for all $B \in \mathcal{L}(\mathcal{K})$. Denoting $K_{x,\eta,i,m} := V^*_{x,\eta,i,m} J$, we find, for all $x \in \Omega$ and $B \in \mathcal{L}(\mathcal{K})$,

$$
\begin{aligned}
\mathcal{I}^*_x(B) = J^*(B \otimes \mathsf{P}_x)J &= \sum_{[\eta]\in\hat{H}_\Omega} \sum_{i=1}^{D_\eta} \sum_{m=1}^{M_\eta} J^*(B \otimes |\delta_x \otimes e_{\eta,i} \otimes f_{\eta,m}\rangle\langle \delta_x \otimes e_{\eta,i} \otimes f_{\eta,m}|)J \\
&= \sum_{[\eta]\in\hat{H}_\Omega} \sum_{i=1}^{D_\eta} \sum_{m=1}^{M_\eta} J^* V_{x,\eta,i,m} B V^*_{x,\eta,i,m} J = \sum_{[\eta]\in\hat{H}_\Omega} \sum_{i=1}^{D_\eta} \sum_{m=1}^{M_\eta} K^*_{x,\eta,i,m} B K_{x,\eta,i,m}. \qquad (9)
\end{aligned}
$$

Clearly, $\overline{U}^\Omega(g)(\delta_x \otimes e_{\eta,i} \otimes f_{\eta,m}) = \delta_{gx} \otimes \zeta^\eta(g^{-1}, gx) e_{\eta,i} \otimes f_{\eta,m}$ for all $g \in G$, $x \in \Omega$, $[\eta] \in \hat{H}_\Omega$, $i = 1, \ldots, D_\eta$, and $m = 1, \ldots, M_\eta$. Using this and the intertwining properties of J, we find that, for all $\varphi \in \mathcal{H}$, $\psi \in \mathcal{K}$, $g \in G$, $x \in \Omega$, $[\eta] \in \hat{H}_\Omega$, $i = 1, \ldots, D_\eta$, and $m = 1, \ldots, M_\eta$,

$$
\begin{aligned}
&\langle \psi | K_{x,\eta,i,m} U(g) \varphi \rangle = \langle V_{x,\eta,i,m}\psi | J U(g) \varphi \rangle = \langle V_{x,\eta,i,m}\psi | \big(V(g) \otimes \overline{U}(g)\big) J \varphi \rangle \\
=& \langle V(g)^*\psi \otimes \overline{U}(g)^*(\delta_x \otimes e_{\eta,i} \otimes f_{\eta,m}) | J\varphi \rangle \\
=& \langle V(g)^*\psi \otimes \delta_{g^{-1}x} \otimes \zeta^\eta(g, g^{-1}x) e_{\eta,i} \otimes f_{\eta,m} | J\varphi \rangle \\
=& \sum_{j=1}^{D_\eta} \langle V(g)^*\psi \otimes \delta_{g^{-1}x} \otimes \zeta^\eta(g, g^{-1}x) e_{\eta,i} \otimes f_{\eta,m} | (\mathbb{1}_\mathcal{K} \otimes \mathbb{1}_{\mathbb{C}^{\#\Omega}} \otimes |e_{\eta,j}\rangle\langle e_{\eta,j}| \otimes \mathbb{1}_{\mathcal{M}_\eta}) J\varphi \rangle \\
=& \sum_{j=1}^{D_\eta} \overline{\zeta^\eta_{j,i}(g, g^{-1}x)} \langle V(g)^*\psi \otimes \delta_{g^{-1}x} \otimes e_{\eta,i} \otimes f_{\eta,m} | J\varphi \rangle \\
=& \sum_{j=1}^{D_\eta} \zeta^\eta_{i,j}(g^{-1}, x) \langle \psi | V(g) K_{g^{-1}x,\eta,j,m} \varphi \rangle,
\end{aligned}
$$

where we have used the fact that $\zeta^\eta(g, g^{-1}x)^* = \zeta^\eta(g^{-1}, x)$ which follows from the cocycle conditions. This means that

$$K_{x,\eta,i,m}U(g) = \sum_{j=1}^{D_\eta} \zeta^\eta_{i,j}(g^{-1}, x)V(g)K_{g^{-1}x,\eta,j,m}. \tag{10}$$

As earlier, let x_Ω be a representative for Ω such that $H_\Omega = G_{x_\Omega}$, i.e., $x_\Omega = H_\Omega$ in the identification $\Omega = G/H_\Omega$. For all $[\eta] \in \hat{H}_\Omega$, $i = 1, \ldots, D_\eta$, and $m = 1, \ldots, M_\eta$, define $L^\Omega_{\eta,i,m} := K_{x_\Omega,\eta,i,m}$. Recall that, for all $h \in H_\Omega$ and $[\eta] \in \hat{H}_\Omega$, $\zeta^\eta(h^{-1}, x_\Omega) = \eta(h)$. Using Equation (10), we now have for all $[\eta] \in \hat{H}_\Omega$, $i = 1, \ldots, D_\eta$, $m = 1, \ldots, M_\eta$, and $h \in H_\Omega$,

$$L^\Omega_{\eta,i,m}U(h) = \sum_{j=1}^{D_\eta} \zeta^\eta_{i,j}(h^{-1}, x_\Omega)V(h)K_{h^{-1}x_\Omega,\eta,j,m} = \sum_{j=1}^{D_\eta} \eta_{i,j}(h)V(h)L^\Omega_{\eta,j,m}.$$

Thus, we obtain Equation (6).

Let us check that the operators $L^\Omega_{\eta,i,m}$ are linearly independent. To show this, let us first note that vectors $(B \otimes \mathsf{P}_{x_\Omega})J\varphi$, $B \in \mathcal{L}(\mathcal{K})$, $\varphi \in \mathcal{H}$, span $\mathcal{K} \otimes \mathsf{P}_{x_\Omega}\mathcal{M} = \mathcal{K} \otimes \left(\bigoplus_{[\eta]\in\hat{H}_\Omega} \mathcal{K}_\eta \otimes \mathcal{M}_\eta\right)$; this follows immediately from the minimality of $(\mathcal{M}, \mathsf{P}, J)$. Let $\beta_{\eta,i,m} \in \mathbb{C}$, $[\eta] \in \hat{H}_\Omega$, $i = 1, \ldots, D_\eta$, $m = 1, \ldots, M_\eta$, and define $v := \sum_{[\eta]\in\hat{H}_\Omega} \sum_{i=1}^{D_\eta} \sum_{m=1}^{M_\eta} \beta_{\eta,i,m} e_{\eta,i} \otimes f_{\eta,m} \in \bigoplus_{[\eta]\in\hat{H}_\Omega} \mathcal{K}_\eta \otimes \mathcal{M}_\eta$. Let us assume that $\sum_{[\eta]\in\hat{H}_\Omega} \sum_{i=1}^{D_\eta} \sum_{m=1}^{M_\eta} \beta_{\eta,i,m} L^\Omega_{\eta,i,m} = 0$. Fix a non-zero $\psi_0 \in \mathcal{K}$ so that, for all $\varphi \in \mathcal{H}$ and $B \in \mathcal{L}(\mathcal{K})$,

$$\begin{aligned}
0 &= \sum_{[\eta]\in\hat{H}_\Omega} \sum_{i=1}^{D_\eta} \sum_{m=1}^{M_\eta} \beta_{\eta,i,m} \langle B^*\psi_0 | L^\Omega_{\eta,i,m}\varphi\rangle \\
&= \sum_{[\eta]\in\hat{H}_\Omega} \sum_{i=1}^{D_\eta} \sum_{m=1}^{M_\eta} \beta_{\eta,i,m} \langle B^*\psi_0 \otimes \delta_{x_\Omega} \otimes e_{\eta,i} \otimes f_{\eta,m} | J\varphi\rangle \\
&= \langle B^*\psi_0 \otimes \delta_{x_\Omega} \otimes v | J\varphi\rangle = \langle \psi_0 \otimes v | (B \otimes \mathsf{P}_{x_\Omega})J\varphi\rangle.
\end{aligned}$$

According to the observation we made before picking the coefficients $\beta_{\eta,i,m}$, this means that $\psi_0 \otimes v = 0$ and, since $\psi_0 \neq 0$, we have $v = 0$. This is equivalent with the vanishing of the coefficients $\beta_{\eta,i,m}$, proving the linear independence of $\{L^\Omega_{\eta,i,m} \,|\, [\eta] \in \hat{H}_\Omega,\ i = 1, \ldots, D_\eta,\ m = 1, \ldots, M_\eta\}$.

Again identifying $\Omega = G/H_\Omega$ and $x_\Omega = H_\Omega$, from (10) we obtain

$$K_{gH_\Omega,\eta,i,m} = \sum_{j=1}^{D_\eta} \zeta^\eta_{i,j}(g^{-1}, gH_\Omega)V(g)K_{H_\Omega,\eta,j,m}U(g)^* = \sum_{j=1}^{D_\eta} \zeta^\eta_{i,j}(g^{-1}, gH_\Omega)V(g)L^\Omega_{\eta,j,m}U(g)^*. \tag{11}$$

Indeed, it is easy to see directly that the RHS of Equation (11) is invariant in substitutions $g \mapsto gh$ where $h \in H_\Omega$. Using the Schrödinger version of Equation

(9), Equation (11), and the easily proven fact that, for any $[\eta] \in \hat{H}_\Omega$, $g \in G$, and $j,\, k = 1, \ldots, D_\eta$, $\sum_{i=1}^{D_\eta} \zeta^\eta_{i,j}(g^{-1}, gH_\Omega)\overline{\zeta^\eta_{i,k}(g^{-1}, gH_\Omega)} = \delta_{j,k}$, where $\delta_{j,k}$ is the Kronecker symbol (i.e., $\delta_{j,k} = 1$ if $j = k$ and, otherwise, $\delta_{j,k} = 0$), we find, for all input states ρ and $g \in G$,

$$
\begin{aligned}
\mathcal{I}_{gH_\Omega}(\rho) &= \sum_{[\eta]\in\hat{H}_\Omega} \sum_{i=1}^{D_\eta} \sum_{m=1}^{M_\eta} K_{gH_\Omega,\eta,i,m}\rho K^*_{gH_\Omega,\eta,i,m} \\
&= \sum_{[\eta]\in\hat{H}_\Omega} \sum_{i,j,k=1}^{D_\eta} \sum_{m=1}^{M_\eta} \zeta^\eta_{i,j}(g^{-1}, gH_\Omega)\overline{\zeta^\eta_{i,k}(g^{-1}, gH_\Omega)} V(g) L^\Omega_{\eta,j,m} U(g)^*\rho U(g) L^{\Omega\,*}_{\eta,k,m} V(g) \\
&= \sum_{[\eta]\in\hat{H}_\Omega} \sum_{i=1}^{D_\eta} \sum_{m=1}^{M_\eta} V(g) L^\Omega_{\eta,i,m} U(g)^*\rho U(g) L^{\Omega\,*}_{\eta,i,m} V(g)^*,
\end{aligned}
$$

implying Equation (8).

Let us now assume that $\mathcal{I}$ is an instrument and move on to proving Equation (7). Let us first note that, for any orbit Ω, $[\eta] \in \hat{H}_\Omega$, $m = 1, \ldots, M_\eta$, and $h \in H_\Omega$, we find, using the already established Equation (6),

$$
\begin{aligned}
\sum_{i=1}^{D_\eta} U(h) L^{\Omega\,*}_{\eta,i,m} L^\Omega_{\eta,i,m} U(h)^* &= \sum_{i,j,k=1}^{D_\eta} \overline{\eta_{i,j}(h^{-1})}\eta_{i,k}(h^{-1}) L^{\Omega\,*}_{\eta,j,m} V(h)V(h)^* L^\Omega_{\eta,k,m} \\
&= \sum_{j,k=1}^{D_\eta} \langle \eta(h)^* e_{\eta,j} | \eta(h)^* e_{\eta,k}\rangle L^{\Omega\,*}_{\eta,j,m} L^\Omega_{\eta,k,m} = \sum_{i=1}^{D_\eta} L^{\Omega\,*}_{\eta,i,m} L^\Omega_{\eta,i,m}.
\end{aligned}
$$

Using the above observation and the dual (Heisenberg) version of the already established Equation (8), we find

$$
\begin{aligned}
\mathbb{1}_\mathcal{H} &= \sum_{x\in\mathbb{X}} \mathcal{I}^*_x(\mathbb{1}_\mathcal{K}) = \sum_{\Omega\in\mathcal{O}} \sum_{x\in\Omega} \mathcal{I}^*_{s^\Omega(x)H_\Omega}(\mathbb{1}_\mathcal{K}) \\
&= \sum_{\Omega\in\mathcal{O}} \sum_{x\in\Omega} \sum_{[\eta]\in\hat{H}_\Omega} \sum_{i=1}^{D_\eta} \sum_{m=1}^{M_\eta} U\big(s^\Omega(x)\big) L^{\Omega\,*}_{\eta,i,m} L^\Omega_{\eta,i,m} U\big(s^\Omega(x)\big)^* \\
&= \sum_{\Omega\in\mathcal{O}} \sum_{x\in\Omega} \sum_{h\in H_\Omega} \sum_{[\eta]\in\hat{H}_\Omega} \sum_{i=1}^{D_\eta} \sum_{m=1}^{M_\eta} \frac{1}{\#H_\Omega} U\big(s^\Omega(x)h\big) L^{\Omega\,*}_{\eta,i,m} L^\Omega_{\eta,i,m} U\big(s^\Omega(x)h\big)^* \\
&= \sum_{\Omega\in\mathcal{O}} \sum_{g\in G} \sum_{[\eta]\in\hat{H}_\Omega} \sum_{i=1}^{D_\eta} \sum_{m=1}^{M_\eta} \frac{1}{\#H_\Omega} U(g) L^{\Omega\,*}_{\eta,i,m} L^\Omega_{\eta,i,m} U(g)^*,
\end{aligned}
$$

implying Equation (7). The final converse claim follows from Theorem 2 upon noting that the operation Λ_Ω defined just after the statement of this theorem with a (minimal) set of $(\mathbb{X}, U, V)$–intertwiners $L_{\eta,i,m}$ satisfies the conditions of Theorem 2 by using Equation (6) [and (7)]. $\square$

Remark 1. Suppose that, for any orbit $\Omega \in \mathcal{O}$ and $[\eta] \in \hat{H}_\Omega$, $M_\eta \in \{0\} \cup \mathbb{N}$ and $L^\Omega_{\eta,i,m} \in \mathcal{L}(\mathcal{H},\mathcal{K})$, $i = 1,\dots,D_\eta$, $m = 1,\dots,M_\eta$, constitute a (minimal) non-normalized set of $(\mathbb{X}, U, V)$-intertwiners. This means that

$$K := \sum_{\Omega\in\mathcal{O}} \sum_{g\in G} \sum_{[\eta]\in\hat{H}_\Omega} \sum_{i=1}^{D_\eta} \sum_{m=1}^{M_\eta} \frac{1}{\#H_\Omega} U(g) L^{\Omega\,*}_{\eta,i,m} L^\Omega_{\eta,i,m} U(g)^*$$

does not necessarily coincide with $\mathbb{1}_\mathcal{H}$. Since, due to its definition, K commutes with U, i.e., $U(g)K = KU(g)$ for all $g \in G$, we may define, for any orbit Ω, $[\eta] \in \hat{H}_\Omega$, $i = 1,\dots,D_\eta$, and $m = 1,\dots,M_\eta$, the new operator $\tilde{L}^\Omega_{\eta,i,m} := L^\Omega_{\eta,i,m} K^{-1/2}$ (where $K^{-1/2}$ is the square root of the generalized inverse of K) which still satisfy Equation (6) (with $L^\Omega_{\eta,i,m}$ replaced with $\tilde{L}^\Omega_{\eta,i,m}$) and which now, additionally, satisfy

$$\sum_{\Omega\in\mathcal{O}} \sum_{g\in G} \sum_{[\eta]\in\hat{H}_\mathcal{O}} \sum_{i=1}^{D_\eta} \sum_{m=1}^{M_\eta} \frac{1}{\#H_\Omega} U(g) \tilde{L}^{\Omega\,*}_{\eta,i,m} \tilde{L}^\Omega_{\eta,i,m} U(g)^* = \operatorname{supp} K$$

where $\operatorname{supp} K$ is the support projection of K. Thus we obtain an $(\mathbb{X}, \tilde{U}, V)$–covariant instrument through Equation (8) (with $L^\Omega_{\eta,i,m}$ replaced with $\tilde{L}^\Omega_{\eta,i,m}$) for a possibly smaller input Hilbert space $(\operatorname{supp} K)(\mathcal{H}) =: \tilde{\mathcal{H}}$ which is an invariant subspace for U where the restriction of U we denote by $\tilde{U}$. Naturally, if U is irreducible, we have $K \in \mathbb{C}\mathbb{1}_\mathcal{H}$ so that $\tilde{\mathcal{H}} = \mathcal{H}$ or $\tilde{\mathcal{H}} = \{0\}$; the latter case is possible only in the highly reduced case where $L^\Omega_{\eta,i,m}$ all vanish (which is hardly interesting). △

In the proof of Theorem 3, we saw that, from a minimal covariant Stinespring dilation of a covariant QOVM [instrument] $\mathcal{I}$, we obtain a minimal [normalized] set of $(\mathbb{X}, U, V)$–intertwiners defining $\mathcal{I}$ through Equation (8). The following lemma gives the converse result: a *minimal* set of intertwiners can be used to define a minimal covariant Stinespring dilation for a covariant QOVM. This result will be very useful when giving extremality conditions for covariant instruments.

Lemma 1. *Let $\mathcal{I}$ be an $(\mathbb{X}, U, V)$–covariant QOVM [instrument] defined through Equation (8) by a minimal [normalized] set of $(\mathbb{X}, U, V)$–intertwiners consisting of $L^\Omega_{\eta,i,m} \in \mathcal{L}(\mathcal{H},\mathcal{K})$ for all $\Omega \in \mathcal{O}$, $[\eta] \in \hat{H}_\Omega$, $i = 1,\dots,D_\eta$, and $m = 1,\dots,M_\eta$ where $M_\eta \in \{0\} \cup \mathbb{N}$. Defining*

$$K_{gH_\Omega,\eta,i,m} := \sum_{j=1}^{D_\eta} \zeta^\eta_{i,j}(g^{-1}, gH_\Omega) V(g) L^\Omega_{\eta,j,m} U(g)^* \tag{12}$$

for all $\Omega \in \mathcal{O}$, $g \in G$, $[\eta] \in \hat{H}_\Omega$, $i = 1,\dots,D_\eta$, and $m = 1,\dots,M_\eta$ and setting

$$\mathcal{M} := \bigoplus_{\Omega\in\mathcal{O}} \mathbb{C}^{\#\Omega} \otimes \Big(\bigoplus_{[\eta]\in\hat{H}_\Omega} \mathcal{K}_\eta \otimes \mathbb{C}^{M_\eta} \Big),$$

the linear map $J : \mathcal{H} \to \mathcal{K} \otimes \mathcal{M}$

$$J\varphi = \sum_{\Omega\in\mathcal{O}} \sum_{x\in\Omega} \sum_{[\eta]\in\hat{H}_\Omega} \sum_{i=1}^{D_\eta} \sum_{m=1}^{M_\eta} K_{x,\eta,i,m}\varphi \otimes \delta_x \otimes e_{\eta,i} \otimes f_{\eta,m}, \qquad \varphi \in \mathcal{H},$$

where $\{\delta_x\}_{x\in\mathbb{X}}$ is the natural basis for $\mathbb{C}^{\#\mathbb{X}} \supseteq \mathbb{C}^{\#\Omega}$ and $\{f_{\eta,m}\}_{m=1}^{M_\eta}$ is some orthonormal basis of $\mathbb{C}^{M_\eta}$, the sharp observable $\mathsf{P} = (\mathsf{P}_x)_{x\in\mathbb{X}}$,

$$\mathsf{P}_x = |\delta_x\rangle\langle\delta_x| \otimes \left(\bigoplus_{[\eta]\in\hat{H}_\Omega} \mathbb{1}_{\mathcal{K}_\eta} \otimes \mathbb{1}_{\mathbb{C}^{M_\eta}} \right), \qquad x \in \Omega \in \mathcal{O},$$

and the unitary representation $\overline{U} : G \to \mathcal{U}(\mathcal{M})$ through

$$\overline{U}(g)(\delta_x \otimes e_{\eta,i} \otimes f_{\eta,m}) = \delta_{gx} \otimes \zeta^\eta(g^{-1}, gx)e_{\eta,i} \otimes f_{\eta,m}$$

for all $g \in G$, $x \in \Omega \in \mathcal{O}$, $[\eta] \in \hat{H}_\Omega$, $i = 1,\ldots,D_\eta$, and $m = 1,\ldots,M_\eta$, the quadruple $(\mathcal{M}, \mathsf{P}, \overline{U}, J)$ is a minimal $(\mathbb{X}, U, V)$–covariant Stinespring dilation for $\mathcal{I}$.

Proof. Let us start by proving that $(\mathcal{M}, \mathsf{P}, J)$ is a minimal Stinespring dilation for $\mathcal{I}$. The fact that $\mathcal{I}^*_x(B) = J^*(B \otimes \mathsf{P}_x)J$ for all $x \in \mathbb{X}$ and $B \in \mathcal{L}(\mathcal{K})$ is proven through a simple direct calculation. Let us concentrate on the minimality claim. Let us first show that, for any $x \in \Omega \in \mathcal{O}$, the set $\{K_{x,\eta,i,m} \,|\, [\eta] \in \hat{H}_\Omega,\ i = 1,\ldots,D_\eta, m = 1,\ldots,M_\eta\} =: \mathbf{K}_x$ is linearly independent. Let us fix an orbit $\Omega \in \mathcal{O}$, and $g \in G$ and let $\beta_{\eta,i,m} \in \mathbb{C}$, $[\eta] \in \hat{H}_\Omega$, $i = 1,\ldots,D_\eta$, $m = 1,\ldots,M_\eta$, be such that $\sum_{[\eta]\in\hat{H}_\Omega}\sum_{i=1}^{D_\eta}\sum_{m=1}^{M_\eta} \beta_{\eta,i,m}K_{gH_\Omega,\eta,i,m} = 0$. Using Equation (12), we obtain

$$\begin{aligned} 0 &= \sum_{[\eta]\in\hat{H}_\Omega}\sum_{i=1}^{D_\eta}\sum_{m=1}^{M_\eta} \beta_{\eta,i,m}K_{gH_\Omega,\eta,i,m} = \sum_{[\eta]\in\hat{H}_\Omega}\sum_{i,j=1}^{D_\eta}\sum_{m=1}^{M_\eta} \zeta^\eta_{i,j}(g^{-1}, gH_\Omega)\beta_{\eta,i,m}V(g)L^\Omega_{\eta,j,m}U(g)^* \\ &= V(g)\left[\sum_{[\eta]\in\hat{H}_\Omega}\sum_{j=1}^{D_\eta}\sum_{m=1}^{M_\eta}\left(\sum_{i=1}^{D_\eta}\zeta^\eta_{i,j}(g^{-1}, gH_\Omega)\beta_{\eta,i,m}\right)L^\Omega_{\eta,j,m}\right]U(g)^* = 0. \end{aligned}$$

Since $\{L^\Omega_{\eta,i,m} \,|\, [\eta] \in \hat{H}_\Omega,\ i = 1,\ldots,D_\eta,\ m = 1,\ldots,M_\eta\}$ is linearly independent, it immediately follows that, for all $[\eta] \in \hat{H}_\Omega$, $j = 1,\ldots,D_\eta$, and $m = 1,\ldots,M_\eta$, $\sum_{i=1}^{D_\eta}\zeta^\eta_{i,j}(g^{-1}, gH_\Omega)\beta_{\eta,i,m} = 0$. Thus, we obtain

$$0 = \sum_{i,j=1}^{D_\eta}\overline{\zeta^\eta_{k,j}(g^{-1}, gH_\Omega)}\zeta^\eta_{i,j}(g^{-1}, gH_\Omega)\beta_{\eta,i,m} = \sum_{i=1}^{D_\eta}\delta_{i,k}\beta_{\eta,i,m} = \beta_{\eta,k,m}$$

for any $[\eta] \in \hat{H}_\Omega$, $k = 1,\ldots,D_\eta$, and $m = 1,\ldots,M_\eta$, proving that $\mathbf{K}_x$ is linearly independent.

Let us assume that $\Psi \in \mathcal{K}\otimes\mathcal{M}$ is such that $\langle\Psi|(B\otimes\mathsf{P}_x)J\varphi\rangle = 0$ for all $B \in \mathcal{L}(\mathcal{K})$, $x \in \mathbb{X}$, and $\varphi \in \mathcal{H}$. For any $x \in \Omega \in \mathcal{O}$, $[\eta] \in \hat{H}_\Omega$, $i = 1,\ldots,D_\eta$, and $m = 1,\ldots,M_\eta$, there is $\psi_{x,\eta,i,m} \in \mathcal{K}$ such that

$$\Psi = \sum_{\Omega\in\mathcal{O}}\sum_{x\in\Omega}\sum_{[\eta]\in\hat{H}_\Omega}\sum_{i=1}^{D_\eta}\sum_{m=1}^{M_\eta}\psi_{x,\eta,i,m}\otimes\delta_x\otimes e_{\eta,i}\otimes f_{\eta,m}.$$

Thus, we have, for all $B \in \mathcal{L}(\mathcal{K})$, $x \in \Omega \in \mathcal{O}$, and $\varphi \in \mathcal{H}$,

$$0 = \langle \Psi | (B \otimes \mathsf{P}_x) J\varphi \rangle = \sum_{[\eta] \in \hat{H}_\Omega} \sum_{i=1}^{D_\eta} \sum_{m=1}^{M_\eta} \langle \psi_{x,\eta,i,m} | B K_{x,\eta,i,m} \varphi \rangle,$$

implying, upon substituting $B = |\psi\rangle\langle\psi'|$, that, for all ψ, $\psi' \in \mathcal{K}$, $x \in \Omega \in \mathcal{O}$, and $\varphi \in \mathcal{H}$, $\sum_{[\eta]\in\hat{H}_\Omega} \sum_{i=1}^{D_\eta} \sum_{m=1}^{M_\eta} \langle \psi_{x,\eta,i,m} | \psi \rangle \langle \psi' | K_{x,\eta,i,m} \varphi \rangle = 0$. Since $\mathbf{K}_x$ is linearly independent for any $x \in \Omega \in \mathcal{O}$, this means that, for all $x \in \Omega \in \mathcal{O}$, $[\eta] \in \hat{H}_\Omega$, $i = 1, \ldots, D_\eta$, $m = 1, \ldots, M_\eta$, and $\psi \in \mathcal{K}$, $\langle \psi_{x,\eta,i,m} | \psi \rangle = 0$. This, of course, means that, for all $x \in \Omega \in \mathcal{O}$, $[\eta] \in \hat{H}_\Omega$, $i = 1, \ldots, D_\eta$, and $m = 1, \ldots, M_\eta$, $\psi_{x,\eta,i,m} = 0$, i.e., $\Psi = 0$, proving the minimality.

As in the proof of Theorem 3, we can show that Equation (10) holds so that we have, for all $g \in G$ and $\varphi \in \mathcal{H}$,

$$\begin{aligned}
JU(g)\varphi &= \sum_{\Omega\in\mathcal{O}} \sum_{x\in\Omega} \sum_{[\eta]\in\hat{H}_\Omega} \sum_{i=1}^{D_\eta} \sum_{m=1}^{M_\eta} K_{x,\eta,i,m} U(g)\varphi \otimes \delta_x \otimes e_{\eta,i} \otimes f_{\eta,m} \\
&= \sum_{\Omega\in\mathcal{O}} \sum_{x\in\Omega} \sum_{[\eta]\in\hat{H}_\Omega} \sum_{i,j=1}^{D_\eta} \sum_{m=1}^{M_\eta} \zeta^\eta_{i,j}(g^{-1}, x) V(g) K_{g^{-1}x,\eta,j,m}\varphi \otimes \delta_x \otimes e_{\eta,i} \otimes f_{\eta,m} \\
&= \sum_{\Omega\in\mathcal{O}} \sum_{x\in\Omega} \sum_{[\eta]\in\hat{H}_\Omega} \sum_{i,j=1}^{D_\eta} \sum_{m=1}^{M_\eta} \zeta^\eta_{i,j}(g^{-1}, gx) V(g) K_{x,\eta,j,m}\varphi \otimes \delta_{gx} \otimes e_{\eta,i} \otimes f_{\eta,m} \\
&= \sum_{\Omega\in\mathcal{O}} \sum_{x\in\Omega} \sum_{[\eta]\in\hat{H}_\Omega} \sum_{j=1}^{D_\eta} \sum_{m=1}^{M_\eta} V(g) K_{x,\eta,j,m}\varphi \otimes \delta_{gx} \otimes \zeta^\eta(g^{-1}, gx) e_{\eta,j} \otimes f_{\eta,m} \\
&= \big(V(g) \otimes \overline{U}(g)\big) \sum_{\Omega\in\mathcal{O}} \sum_{x\in\Omega} \sum_{[\eta]\in\hat{H}_\Omega} \sum_{j=1}^{D_\eta} \sum_{m=1}^{M_\eta} K_{x,\eta,j,m}\varphi \otimes \delta_x \otimes e_{\eta,i} \otimes f_{\eta,m} \\
&= \big(V(g) \otimes \overline{U}(g)\big) J\varphi.
\end{aligned}$$

Proving that $\overline{U}(g)\mathsf{P}_x\overline{U}(g)^* = \mathsf{P}_{gx}$ for all $g \in G$ and $x \in \mathbb{X}$ is straightforward. □

3.1. Extreme instruments covariant with respect to a finite group

We retain the definitions and assumptions we have made in the beginning of this section regarding the finite group G, the G-space $\mathbb{X}$, the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, and the unitary representations U and V. Using Theorem 3 and Lemma 1, we next determine extremality conditions for $(\mathbb{X}, U, V)$–covariant instruments using the necessary and sufficient extremality conditions established in [15]. In this earlier work, the extremality conditions were described as requirements on the minimal dilations, but next we will describe extremality conditions using minimal intertwiners. Since the description of intertwiners is shallower than that of minimal dilations (since no ancillary system is involved), conditions in the context of intertwiners are arguably more accessible than the earlier ones. Indeed, we shall see that extreme covariant instruments are characterized by a rather simple linear-independence condition. We remind the reader of the modes of extremality described in Definition 4. The extremality mode relevant in the theorem below is the one described in item (II) of said Definition.

Theorem 4. *Let $\mathcal{I}$ be an $(\mathbb{X}, U, V)$–covariant instrument defined through Equation (8) by a minimal set of $(\mathbb{X}, U, V)$–intertwiners consisting of $L^{\Omega}_{\eta,i,m} \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ for all $\Omega \in \mathcal{O}$, $[\eta] \in \hat{H}_\Omega$, $i = 1, \ldots, D_\eta$, and $m = 1, \ldots, M_\eta$ where $M_\eta \in \{0\} \cup \mathbb{N}$. The instrument $\mathcal{I}$ is an extreme instrument of the $(\mathbb{X}, U, V)$–covariance structure if and only if the set*

$$\left\{ \sum_{g\in G} \sum_{i=1}^{D_\eta} U(g) L^{\Omega *}_{\eta,i,m} L^{\Omega}_{\eta,i,n} U(g)^* \,\middle|\, m, n = 1, \ldots, M_\eta,\ [\eta] \in \hat{H}_\Omega,\ \Omega \in \mathcal{O} \right\}$$

is linearly independent.

Proof. Let $(\mathcal{M}, \mathsf{P}, \overline{U}, J)$ be the minimal $(\mathbb{X}, U, V)$–covariant Stinespring dilation for $\mathcal{I}$ as defined in Lemma 1. Denote, for brevity, for any orbit $\Omega \in \mathcal{O}$,

$$\mathcal{H}^\Omega := \bigoplus_{[\eta]\in\hat{H}_\Omega} \mathcal{K}_\eta \otimes \mathbb{C}^{M_\eta}.$$

According to the results of [15], $\mathcal{I}$ is an extreme instrument of the $(\mathbb{X}, U, V)$–covariance structure if and only if, for $E \in \mathcal{L}(\mathcal{M})$ the conditions $E\mathsf{P}_x = \mathsf{P}_x E$ for all $x \in \mathbb{X}$, $E\overline{U}(g) = \overline{U}(g)E$ for all $g \in G$, and $J^*(\mathbb{1}_\mathcal{K} \otimes E)J = 0$ imply $E = 0$; note that for this extremality characterization it is vital that the dilation is minimal. Let $E \in \mathcal{L}(\mathcal{M})$ be such that $E\mathsf{P}_x = \mathsf{P}_x E$ for all $x \in \mathbb{X}$ and $E\overline{U}(g) = \overline{U}(g)E$ for all $g \in G$. The first condition is equivalent with the existence of $E_x \in \mathcal{L}(\mathcal{H}^\Omega)$, $x \in \Omega \in \mathcal{O}$, such that $E(\delta_x \otimes v) = \delta_x \otimes E_x v$ for all $v \in \mathcal{H}^\Omega$. Denoting, for all $g \in G$ and $x \in \Omega \in \mathcal{O}$, $\zeta^\Omega(g, x) := \bigoplus_{[\eta]\in\hat{H}_\Omega} \zeta^\eta(g, x) \otimes \mathbb{1}_{M_\eta}$, the second condition is easily seen to be equivalent with

$$\zeta^\Omega(g^{-1}, gx)E_x = E_{gx}\zeta^\Omega(g^{-1}, gx), \qquad x \in \Omega \in \mathcal{O}, \quad g \in G. \tag{13}$$

Identifying $\Omega = G/H_\Omega$, we obtain $E_{gH_\Omega} = \zeta^\Omega(g^{-1}, gH_\Omega)E_{H_\Omega}\zeta^\Omega(g^{-1}, gH_\Omega)^*$ for any orbit Ω. Note that, defining, for all orbits Ω and $h \in H_\Omega$, $\zeta^\Omega(h^{-1}, H_\Omega) =: \pi^\Omega(h)$, we determine a unitary representation $\pi^\Omega : H_\Omega \to \mathcal{U}(\mathcal{H}^\Omega)$ such that

$$\pi^\Omega(h) = \bigoplus_{[\eta]\in\hat{H}_\Omega} \eta(h) \otimes \mathbb{1}_{M_\eta}. \tag{14}$$

Using Equation (13), we have $\pi^\Omega(h)E_{H_\Omega} = \zeta^\Omega(h^{-1}, H_\Omega)E_{H_\Omega} = E_{hH_\Omega}\zeta^\Omega(h^{-1}, H_\Omega) = E_{H_\Omega}\pi^\Omega(h)$ for all $\Omega \in \mathcal{O}$ and $h \in H_\Omega$. The decomposition in Equation (14) implies now that $E_{H_\Omega} = \bigoplus_{[\eta]\in\hat{H}_\Omega} \mathbb{1}_{\mathcal{K}_\eta} \otimes E_\eta$ for all $\Omega \in \mathcal{O}$ where $E_\eta \in \mathcal{L}(\mathbb{C}^{M_\eta})$ for all $[\eta] \in \hat{H}_\Omega$. We now have $E_{gH_\Omega} = \zeta^\Omega(g^{-1}, gH_\Omega)E_{H_\Omega}\zeta^\Omega(g^{-1}, gH_\Omega)^* = \bigoplus_{[\eta]\in\hat{H}_\Omega} \zeta^\eta(g^{-1}, gH_\Omega)\zeta^\eta(g^{-1}, gH_\Omega)^* \otimes$

$E_\eta = \bigoplus_{[\eta]\in\hat{H}_\Omega} \mathbb{1}_{\mathcal{K}_\eta} \otimes E_\eta = E_{H_\Omega}$ for any orbit Ω and $g \in G$. Thus,

$$E = \sum_{\Omega\in\mathcal{O}} \sum_{x\in\Omega} |\delta_x\rangle\langle\delta_x| \otimes \left(\bigoplus_{[\eta]\in\hat{H}_\Omega} \mathbb{1}_{\mathcal{K}_\eta} \otimes E_\eta \right) \tag{15}$$

In the same way as in the proof of Theorem 3, we see that, for any orbit $\Omega \in \mathcal{O}$, $h \in H_\Omega$, $[\eta] \in \hat{H}_\Omega$, and $m, n = 1, \ldots, M_\eta$, $\sum_{i=1}^{D_\eta} U(h) L^{\Omega\,*}_{\eta,i,m} L^{\Omega}_{\eta,i,n} U(h)^* = \sum_{i=1}^{D_\eta} L^{\Omega\,*}_{\eta,i,m} L^{\Omega}_{\eta,i,n}$. Recall the section $s_\Omega : \Omega \to G$ such that $s_\Omega(x_\Omega) = e$. Using the above observation and Equation (15), we have, for any $\varphi \in \mathcal{H}$,

$$\langle J\varphi|(\mathbb{1}_{\mathcal{K}} \otimes E)J\varphi\rangle = \sum_{\Omega\in\mathcal{O}} \sum_{x\in\Omega} \sum_{[\eta]\in\hat{H}_\Omega} \langle J\varphi|(\mathbb{1}_{\mathcal{K}} \otimes |\delta_x\rangle\langle\delta_x| \otimes \mathbb{1}_{\mathcal{K}_\eta} \otimes E_\eta)J\varphi\rangle$$

$$= \sum_{\Omega\in\mathcal{O}} \sum_{x\in\Omega} \sum_{[\eta]\in\hat{H}_\Omega} \sum_{i=1}^{D_\eta} \sum_{m,n=1}^{M_\eta} \langle K_{x,\eta,i,m}\varphi \otimes f_{\eta,m}|K_{x,\eta,i,n}\varphi \otimes E_\eta f_{\eta,n}\rangle$$

$$= \sum_{\Omega\in\mathcal{O}} \sum_{x\in\Omega} \sum_{[\eta]\in\hat{H}_\Omega} \sum_{i=1}^{D_\eta} \sum_{m,n=1}^{M_\eta} \langle f_{\eta,m}|E_\eta f_{\eta,n}\rangle \langle K_{s_\Omega(x)H_\Omega,\eta,i,m}\varphi|K_{s_\Omega(x)H_\Omega,\eta,i,n}\varphi\rangle$$

$$= \sum_{\Omega\in\mathcal{O}} \sum_{x\in\Omega} \sum_{[\eta]\in\hat{H}_\Omega} \sum_{i,j,k=1}^{D_\eta} \sum_{m,n=1}^{M_\eta} \langle f_{\eta,m}|E_\eta f_{\eta,n}\rangle \overline{\zeta^\eta_{i,j}\big(s_\Omega(x)^{-1}, x\big)} \zeta^\eta_{i,k}\big(s_\Omega(x)^{-1}, x\big) \times$$
$$\times \langle L^\Omega_{\eta,j,m} U\big(s_\Omega(x)\big)^*\varphi|L^\Omega_{\eta,k,n} U\big(s_\Omega(x)\big)^*\varphi\rangle$$

$$= \sum_{\Omega\in\mathcal{O}} \sum_{x\in\Omega} \sum_{[\eta]\in\hat{H}_\Omega} \sum_{i=1}^{D_\eta} \sum_{m,n=1}^{M_\eta} \langle f_{\eta,m}|E_\eta f_{\eta,n}\rangle \langle L^\Omega_{\eta,i,m} U\big(s_\Omega(x)\big)^*\varphi|L^\Omega_{\eta,i,n} U\big(s_\Omega(x)\big)^*\varphi\rangle$$

$$= \sum_{\Omega\in\mathcal{O}} \sum_{x\in\Omega} \sum_{h\in H_\Omega} \sum_{[\eta]\in\hat{H}_\Omega} \sum_{i=1}^{D_\eta} \sum_{m,n=1}^{M_\eta} \frac{1}{\#H_\Omega} \langle f_{\eta,m}|E_\eta f_{\eta,n}\rangle \langle L^\Omega_{\eta,i,m} U\big(s_\Omega(x)h\big)^*\varphi|L^\Omega_{\eta,i,n} U\big(s_\Omega(x)h\big)^*\varphi\rangle$$

$$= \sum_{\Omega\in\mathcal{O}} \sum_{g\in G} \sum_{[\eta]\in\hat{H}_\Omega} \sum_{i=1}^{D_\eta} \sum_{m,n=1}^{M_\eta} \frac{1}{\#H_\Omega} \langle f_{\eta,m}|E_\eta f_{\eta,n}\rangle \langle L^\Omega_{\eta,i,m} U(g)^*\varphi|L^\Omega_{\eta,i,n} U(g)^*\varphi\rangle$$

$$= \sum_{\Omega\in\mathcal{O}} \sum_{[\eta]\in\hat{H}_\Omega} \sum_{m,n=1}^{M_\eta} \beta^\Omega_{\eta,m,n} \sum_{g\in G} \sum_{i=1}^{D_\eta} \langle L^\Omega_{\eta,i,m} U(g)^*\varphi|L^\Omega_{\eta,i,n} U(g)^*\varphi\rangle,$$

where we have denoted $\beta^\Omega_{\eta,m,n} := (\#H_\Omega)^{-1}\langle f_{\eta,m}|E_\eta f_{\eta,n}\rangle$, for all orbits $\Omega \in \mathcal{O}$, $[\eta] \in \hat{H}_\Omega$, and $m, n = 1, \ldots, M_\eta$. From this observation the claim immediately follows. □

Suppose now that U is irreducible. Now for any minimal set of $(\mathbb{X}, U, V)$–intertwiners $L^\Omega_{\eta,i,m}$, $\Omega \in \mathcal{O}$, $[\eta] \in \hat{H}_\Omega$, $i = 1, \ldots, D_\eta$, $m = 1, \ldots, M_\eta$, where $M_\eta \in \{0\} \cup \mathbb{N}$ for all $[\eta] \in \hat{H}_\Omega$ and any orbit $\Omega \in \mathcal{O}$, we have

$$\sum_{g\in G} \sum_{i=1}^{D_\eta} U(g) L^{\Omega\,*}_{\eta,i,m} L^\Omega_{\eta,i,n} U(g)^* = \beta^\Omega_{\eta,m,n} \mathbb{1}_{\mathcal{H}}$$

with some $\beta^{\Omega}_{\eta,m,n} \in \mathbb{C}$ for any orbit $\Omega \in \mathcal{O}$, $[\eta] \in \hat{H}_{\mathcal{O}}$, and $m, n = 1, \ldots, M_\eta$. Thus, the corresponding $(\mathbb{X}, U, V)$–covariant instrument $\mathcal{I}$ is an extreme instrument in the $(\mathbb{X}, U, V)$–covariance structure if and only if there is only one orbit Ω_0 and only one $[\eta_0] \in \hat{H}_{\Omega_0}$ such that $L^{\Omega_0}_{\eta,i,m} \neq 0$ for some $i \in \{1, \ldots, D_{\eta_0}\}$ in which case $m_{\eta_0} = 1$, i.e., the only possibly non-zero minimal $(\mathbb{X}, U, V)$–intertwiners are $L^{\Omega_0}_{\eta_0,i,1}$, $i = 1, \ldots, D_{\eta_0}$ with a unique orbit Ω_0 and a unique $[\eta_0] \in \hat{H}_{\Omega_0}$. This means that the instrument $\mathcal{I}$ is supported totally on Ω_0. If we now equip $\mathcal{I}$ with the minimal $(\mathbb{X}, U, V)$–covariant Stinespring dilation $(\mathcal{M}, \mathsf{P}, \overline{U}, J)$ of Lemma 1, the representation $\overline{U}$ only consists of the transitive part $\overline{U}^{\Omega_0}$ (see Equation (4) in Appendix B). Moreover the multiplicity m_{η_0} of $[\eta_0]$ is 1 meaning that $\overline{U}$ is irreducible. This means that, when U is irreducible and we give an $(\mathbb{X}, U, V)$–covariant instrument $\mathcal{I}$ an $(\mathbb{X}, U, V)$–covariant minimal dilation $(\mathcal{M}, \mathsf{P}, \overline{U}, J)$, where $\mathcal{M}$, P, and $\overline{U}$ have the decomposition of Equations (4) and (5) into transitive constituents over $\mathcal{O}$ where $\overline{U}^{\Omega}$ is induced from $\pi^{\Omega} : H_\Omega \to \mathcal{U}(\mathcal{H}^{\Omega})$ for each $\Omega \in \mathcal{O}$, the instrument $\mathcal{I}$ is an extreme instrument of the $(\mathbb{X}, U, V)$–covariance structure if and only if only one of these constituents, corresponding to a fixed $\Omega_0 \in \mathcal{O}$, is non-zero and the corresponding π^{Ω} is irreducible. See Proposition 1 for a generalization of this fact in the single-orbit (transitive) case.

Remark 2. Let us next take a quick look at the extremality mode (I) of Definition 4. This extremality property also depends on the minimal Stinespring dilation of the instrument and, if the instrument $\mathcal{I}$ is $(\mathbb{X}, U, V)$–covariant, we can use the minimal dilation presented in Lemma 1. It follows that the condition can be formulated as a property of the Kraus operators $K_{x,\eta,i,m}$ of the instrument obtained through Equation (12) from the minimal $(\mathbb{X}, U, V)$–intertwiners $L^{\Omega}_{\eta,i,m}$, associated with the instrument $\mathcal{I}$: it follows that the instrument $\mathcal{I}$ is extreme if and only if the set of operators $K^*_{x,\eta,i,m} K_{x,\vartheta,j,n}$, $x \in \mathbb{X}$, $[\eta]$, $[\vartheta] \in \hat{H}_{Gx}$, $i = 1, \ldots, D_\eta$, $j = 1, \ldots, D_\vartheta$, $m = 1, \ldots, M_\eta$, $n = 1, \ldots, M_\vartheta$, is linearly independent. Naturally, an extreme instrument is also an extreme instrument of the $(\mathbb{X}, U, V)$–covariance structure; in Appendix C we see how this can be seen directly using the respective extremality characterizations. $\triangle$

4. Observables and channels covariant with respect to a finite group

In this section, we concentrate on covariant observables and channels (or POVMs and QOVMs in general) and derive characterizations for them and their extremality using Theorems 3 and 4. We will also generalize Example 1 to derive a continuous family of extreme rank-1 observables with representatives from the two mutually exclusive optimality classes. Let us retain the finite group G and the G-space structure of the value space $\mathbb{X}$ and the representation $U : G \to \mathcal{U}(\mathcal{H})$ of the preceding section. We may view an $(\mathbb{X}, U)$–covariant POVM as a particular $(\mathbb{X}, U, V)$–covariant QOVM with the trivial output space $\mathbb{C}$ where V is the trivial representation of G. Using this observation and Theorems 3 and 4, we obtain the following result characterizing the $(\mathbb{X}, U)$–covariant POVMs and observables (and thus elaborating on Theorem 1) and the extreme observables of the $(\mathbb{X}, U)$–covariance structure. As the result is a direct corollary, we do not give a separate proof for it. Note that extreme points of sets of covariant observables have also been studied in [9, 14, 15, 24]. Also the non-covariant results presented in [28] can be seen as corollaries of the following extremality characterization (in the case where every orbit is a singleton).

Corollary 1. *Let* $\mathsf{M} = (\mathsf{M}_x)_{x\in\mathbb{X}}$ *be an* $(\mathbb{X}, U)$*–covariant POVM. For any orbit* $\Omega \in \mathcal{O}$*, there is an operator* $K_\Omega \in \mathcal{L}(\mathcal{H})$ *such that, for any* $g \in G$*,*

$$\mathsf{M}_{gH_\Omega} = U(g)K_\Omega U(g)^*. \tag{16}$$

For any $\Omega \in \mathcal{O}$*, the above operator* K_Ω *has the following structure: For all* $[\eta] \in \hat{H}_\Omega$ *there is a number* $M_\eta \in \{0\} \cup \mathbb{N}$ *and a linearly independent set*

$$\{d^\Omega_{\eta,i,m} \in \mathcal{H} \,|\, [\eta] \in \hat{H}_\Omega,\ i = 1, \ldots, D_\eta,\ m = 1, \ldots, M_\eta\}$$

such that, for any $[\eta] \in \hat{H}_\Omega$*,* $i = 1, \ldots, D_\eta$*,* $m = 1, \ldots, M_\eta$*, and* $h \in H_\Omega$*,*

$$U(h)d^\Omega_{\eta,i,m} = \sum_{j=1}^{D_\eta} \eta_{j,i}(h)d^\Omega_{\eta,j,m} \tag{17}$$

and

$$K_\Omega = \sum_{[\eta]\in\hat{H}_\Omega} \sum_{i=1}^{D_\eta} \sum_{m=1}^{M_\eta} |d^\Omega_{\eta,i,m}\rangle\langle d^\Omega_{\eta,i,m}|. \tag{18}$$

Furthermore, if M *is an observable,*

$$\sum_{\Omega\in\mathcal{O}} \sum_{g\in G} \sum_{[\eta]\in\hat{H}_\Omega} \sum_{i=1}^{D_\eta} \sum_{m=1}^{M_\eta} \frac{1}{\#H_\Omega} |U(g)d^\Omega_{\eta,i,m}\rangle\langle U(g)d^\Omega_{\eta,i,m}| = \mathbb{1}_\mathcal{H}. \tag{19}$$

This observable is an extreme observable of the $(\mathbb{X}, U)$*–covariance structure if and only if the set*

$$\left\{ \sum_{g\in G} \sum_{i=1}^{D_\eta} |U(g)d^\Omega_{\eta,i,m}\rangle\langle U(g)d^\Omega_{\eta,i,n}| \,\middle|\, m, n = 1, \ldots, M_\eta,\ [\eta] \in \hat{H}_\Omega,\ \Omega \in \mathcal{O} \right\}$$

is linearly independent. Moreover, when $d^\Omega_{\eta,i,m}$*,* $\Omega \in \mathcal{O}$*,* $[\eta] \in \hat{H}_\Omega$*,* $i = 1, \ldots, D_\eta$*,* $m = 1, \ldots, M_\eta$*, where* $M_\eta \in \{0\} \cup \mathbb{N}$*, are vectors satisfying Equations (17) [and (19)],*§ *Equations (16) and (18) define an* $(\mathbb{X}, U)$*–covariant POVM [observable].*

Example 2. We continue to study the situation introduced in Example 1 and generalize it to a general finite dimension $D \geq 2$. The Hilbert space of our system is $\mathcal{H}_D \simeq \mathbb{C}^D$, the symmetry group is the permutation group $S_D = \mathrm{Sym}\big(\{1, 2, \ldots, D\}\big)$ which operates in the value space $\mathbb{X}_D^2 = \{1, \ldots, D\}^2$ of our measurements through $S_D \times \mathbb{X}_D^2 \ni \big(\pi, (m, n)\big) \mapsto \pi(m, n) = \big(\pi(m), \pi(n)\big) \in \mathbb{X}_D^2$ and in $\mathcal{H}_D$ through the unitary

§ Sometimes the vectors $U(g)d^\Omega_{\eta,i,m}$ are called *generalized coherent states.*

representation $U : S_D \to \mathcal{U}(\mathcal{H}_D)$ defined w.r.t. a fixed orthonormal basis $\{|n\rangle\}_{n=1}^D$ of $\mathcal{H}_D$ via $U(\pi)|n\rangle = |\pi(n)\rangle$ for all $\pi \in S_D$ and $n = 1, \ldots, D$. Note that U is not irreducible as $\psi_0 := D^{-1/2}\big(|1\rangle + \cdots + |D\rangle\big)$ is invariant under U and thus U can be restricted to the orthogonal complement $\{\psi_0\}^\perp$. This restriction is irreducible and is called as the standard representation of S_D.

The set $\mathbb{X}_D^2$ splits into two orbits, the diagonal $\Omega = \{(1,1), (2,2), \ldots, (D,D)\}$ and the off-diagonal $\Omega' = \mathbb{X}_D^2 \setminus \Omega$. Picking the reference points $x_\Omega = (1,1)$ and $x_{\Omega'} = (1,2)$, the stability subgroup H_Ω is easily seen to be the subgroup of those $\pi \in S_D$ such that $\pi(1) = 1$ and the stability subgroup $H_{\Omega'}$ is easily seen to consist of those $\pi' \in S_D$ such that $\pi'(1) = 1$ and $\pi'(2) = 2$. Hence, $H_\Omega \simeq \mathrm{Sym}\big(\{2, 3, \ldots, D\}\big) \simeq S_{D-1}$ and $H_{\Omega'} \simeq \mathrm{Sym}\big(\{3, 4, \ldots, D\}\big) \simeq S_{D-2}$; if $D = 2$ then $H_{\Omega'}$ is the single-element group. It follows that, for any $(\mathbb{X}_D^2, U)$–covariant observable $\mathsf{M} = \big(\mathsf{M}_{(m,n)}\big)_{(m,n)\in\mathbb{X}_D^2}$, there are positive kernels K_Ω and $K_{\Omega'}$ such that $U(\pi)K_\Omega = K_\Omega U(\pi)$ for all $\pi \in H_\Omega$ and $U(\pi')K_{\Omega'} = K_{\Omega'}U(\pi')$ for all $\pi' \in H_{\Omega'}$ and $\mathsf{M}_{(\pi(1),\pi(1))} = U(\pi)K_\Omega U(\pi)^*$ for all $\pi \in S_D$ (defining the diagonal values) and $\mathsf{M}_{(\pi(1),\pi(2))} = U(\pi)K_{\Omega'}U(\pi)^*$ for all $\pi \in S_D$ (defining the off-diagonal values). Furthermore, there are non-negative integers M_η and $M_{\eta'}$, $[\eta] \in \hat{H}_\Omega$, $[\eta'] \in \hat{H}_{\Omega'}$, and two linearly independent sets $\{d_{\eta,i,m} \,|\, [\eta] \in \hat{H}_\Omega,\ i = 1, \ldots, D_\eta,\ m = 1, \ldots, M_\eta\}$ and $\{d'_{\eta',k,r} \,|\, [\eta'] \in \hat{H}_{\Omega'},\ k = 1, \ldots, D_{\eta'},\ r = 1, \ldots, M_{\eta'}\}$ of vectors from $\mathcal{H}_D$ such that $U(\pi)d_{\eta,i,m} = \sum_{j=1}^{D_\eta} \eta_{j,i}(\pi)d_{\eta,j,m}$ for all $\pi \in H_\Omega$, $[\eta] \in \hat{H}_\Omega$, $i = 1, \ldots, D_\eta$, and $m = 1, \ldots, M_\eta$ and $U(\pi')d'_{\eta',k,r} = \sum_{\ell=1}^{D_{\eta'}} \eta'_{\ell,k}(\pi')d'_{\eta',\ell,r}$ for all $\pi' \in H_{\Omega'}$, $[\eta'] \in \hat{H}_{\Omega'}$, $k = 1, \ldots, D_{\eta'}$, and $r = 1, \ldots, M_{\eta'}$.

In exactly the same way as in Example 1, we obtain (rank-1) PVMs (or sharp observables) when concentrating on the diagonal orbit. Let us construct a family of rank-1 informationally complete extreme $(\mathbb{X}_D^2, U)$–covariant observables. As we are interested in the rank-1 case, we only concentrate on the characters (1-dimensional irreducible representations) of the stability subgroups and unit multiplicities in the above framework. Let us make things simple by just assuming that the characters involved are just the trivial characters $\zeta_0 \in \hat{H}_\Omega$ and $\zeta'_0 \in \hat{H}_{\Omega'}$, i.e., $\langle \pi, \zeta_0\rangle = 1 = \langle \pi', \zeta'_0\rangle$ for all $\pi \in H_\Omega$ and $\pi' \in H_{\Omega'}$. It follows that we only have single vectors $d_\Omega := d_{\zeta_0,1,1}$ and $d_{\Omega'} := d'_{\zeta'_0,1,1}$ which satisfy $U(\pi)d_\Omega = \langle \pi, \zeta_0\rangle d_\Omega = d_\Omega$ for all $\pi \in H_\Omega$ and $U(\pi')d_{\Omega'} = \langle \pi', \zeta'_0\rangle d_{\Omega'} = d_{\Omega'}$ for all $\pi' \in H_{\Omega'}$.

Note that we do not have to overly worry about the normalization of the vectors d_Ω and $d_{\Omega'}$ for the moment as we can carry out the normalization afterwards according to Remark 1, i.e., we may freely concentrate on covariant POVMs. Let us make the ansatz $d_\Omega = |1\rangle$ and $d_{\Omega'} = d_{\Omega'}(\alpha) := \alpha(e^{-i\pi/8}|1\rangle + e^{i\pi/8}|2\rangle)$ where $\alpha \geq 0$; indeed these are valid choices as they comply with the above necessary conditions. The case $\alpha = 0$ corresponds to a rank-1 sharp observable supported by the diagonal Ω. For now, let us assume that $\alpha > 0$. For any $m \leq D$, we obtain $|m\rangle\langle m|$ as an operator $U(\pi)|1\rangle\langle 1|U(\pi)^*$ for some $\pi \in S_D$. Let $(m,n) \in \Omega'$ and choose $\pi \in S_D$ such that $\pi(1) = m$ and $\pi(2) = n$. It now follows that

$$U(\pi)|d_{\Omega'}(\alpha)\rangle\langle d_{\Omega'}(\alpha)|U(\pi)^* = \alpha^2\big(|m\rangle\langle m| + e^{-i\pi/4}|m\rangle\langle n| + e^{i\pi/4}|n\rangle\langle m| + |n\rangle\langle n|\big).$$

Through linear combinations with operators from the diagonal, we now obtain the operators

$$A := e^{-i\pi/4}|m\rangle\langle n| + e^{i\pi/4}|n\rangle\langle m|, \qquad B := e^{i\pi/4}|m\rangle\langle n| + e^{-i\pi/4}|n\rangle\langle m|,$$

where B is obtained by reversing the roles of m and n, and ultimately $2^{-3/2}\big[(A+B)+i(A-B)\big] = |m\rangle\langle n|$. All in all, the operators $U(\pi)|1\rangle\langle 1|U(\pi)^*$ and $U(\pi)|d_{\Omega'}(\alpha)\rangle\langle d_{\Omega'}(\alpha)|U(\pi)^*$, where $\pi\in S_D$, span the whole of $\mathcal{L}(\mathcal{H}_D)$. Following Remark 1, we may define

$$\begin{aligned} K(\alpha) &= \frac{1}{\#H_\Omega}\sum_{\pi\in S_D} U(\pi)|1\rangle\langle 1|U(\pi)^* + \frac{1}{\#H_{\Omega'}}\sum_{\pi\in S_D} U(\pi)|d_{\Omega'}(\alpha)\rangle\langle d_{\Omega'}(\alpha)|U(\pi)^* \\ &= \Big[\big(2D-2-\sqrt{2}\big)\alpha^2+1\Big]\mathbb{1} + \sqrt{2}\alpha^2 D|\psi_0\rangle\langle\psi_0| \\ &= \Big[\big(2D-2-\sqrt{2}\big)\alpha^2+1\Big]\big(\mathbb{1}-|\psi_0\rangle\langle\psi_0|\big) + \Big[\big(2+\sqrt{2}\big)\big(D-1\big)\alpha^2+1\Big]|\psi_0\rangle\langle\psi_0| \end{aligned}$$

where the second equality is obtained through direct calculation and the final formula is the spectral resolution of $K(\alpha)$; recall the isotropic vector ψ_0 defined in the beginning of this example. Note that any operator commuting with U has a spectral resolution like this recalling the decomposition of U into the trivial character operating in the 1-dimensional subspace spanned by ψ_0 and to the standard representation operating in $\{\psi_0\}^\perp$. Hence, we have the normalizer

$$K(\alpha)^{-1/2} = \Big[\big(2D-2-\sqrt{2}\big)\alpha^2+1\Big]^{-1/2}\big(\mathbb{1}-|\psi_0\rangle\langle\psi_0|\big) + \Big[\big(2+\sqrt{2}\big)\big(D-1\big)\alpha^2+1\Big]^{-1/2}|\psi_0\rangle\langle\psi_0|$$

and we may define the $(\mathbb{X}_D^2, U)$–covariant rank-1 observable $\mathsf{M}^\alpha = (\mathsf{M}^\alpha_{(m,n)})_{(m,n)\in\mathbb{X}_D^2}$ for all $\alpha\geq 0$ through

$$\begin{aligned} \mathsf{M}^\alpha_{(m,m)} =& K(\alpha)^{-1/2}|m\rangle\langle m|K(\alpha)^{-1/2}, \\ \mathsf{M}^\alpha_{(m,n)} =& U(\pi)K(\alpha)^{-1/2}|d_{\Omega'}(\alpha)\rangle\langle d_{\Omega'}(\alpha)|K(\alpha)^{-1/2}U(\pi)^* \\ =& \alpha^2 K(\alpha)^{-1/2}\big(|m\rangle\langle m| + e^{-i\pi/4}|m\rangle\langle n| + e^{i\pi/4}|n\rangle\langle m| + |n\rangle\langle n|\big)K(\alpha)^{-1/2} \end{aligned}$$

for all $m\neq n$ where $\pi\in S_D$ is such that $\pi(1)=m$ and $\pi(2)=n$. Whenever $\alpha>0$, using our observations just before introducing $K(\alpha)$ and the fact that $K(\alpha)^{-1/2}$ commutes with U, the range of M^α spans $\mathcal{L}(\mathcal{H}_D)$ showing that M^α is informationally complete. Since M^α has D^2 non-zero outcomes when $\alpha>0$, this also implies that the set $\{\mathsf{M}^\alpha_{(m,n)}\,|\,(m,n)\in\mathbb{X}_D^2\}$ is linearly independent. Hence, as a rank-1 observable, M^α is also extreme within the convex set of all observables with a finite outcome space and operating in $\mathcal{H}_D$ [13]. In the case $\alpha=0$, one gets the rank-1 sharp observable $\mathsf{M}^0_{(m,n)} = \delta_{m,n}|m\rangle\langle m|$. To conclude, both of the mutually exclusive classes of optimal observables are represented within the $(\mathbb{X}_D^2, U)$–covariance structure and they are arbitrarily close one another when $\alpha\approx 0$. It is easy to see that in the limit $\alpha\to\infty$, the diagonal effects of M^α vanish so that the limit rank-1 POVM is not informationally complete. The limit observable is a sharp observable only if $D=2$. We observe that the marginal observables $(\mathsf{A}^\alpha_m)_{m=1}^D$ and $(\mathsf{B}^\alpha_n)_{n=1}^D$ (defined by $\mathsf{A}^\alpha_m := \sum_{n=1}^D \mathsf{M}^\alpha_{(m,n)}$ and $\mathsf{B}^\alpha_n := \sum_{m=1}^D \mathsf{M}^\alpha_{(m,n)}$) are $(\mathbb{X}_D, U)$–covariant (e.g. $U(\pi)\mathsf{A}^\alpha_m U(\pi)^* = \mathsf{A}_{\pi(m)}$) but they are not of rank 1 except in the case $\alpha = 0$ when they coincide with the basis measurement $\big(|m\rangle\langle m|\big)_{m=1}^D$ and M^0 is their only possible joint measurement.

We notice that, when $\alpha = 0$ or $\alpha = (2+\sqrt{2})^{-1/2} =: \alpha_0$, $K(\alpha)^{-1/2}$ is particularly simple. According to the above discussion, M^{α_0} is an example of a rank-1 extreme informationally complete observable in the $(\mathbb{X}_D^2, U)$–covariance structure. In a straightforward manner, we find that, for all m, $n = 1, \ldots, D$, $\mathsf{M}^{\alpha_0}_{(m,n)} = |d_{m,n}\rangle\langle d_{m,n}|$ where

$$d_{m,n} = \frac{1}{\sqrt{2D}}\big(e^{-i\pi/8}|m\rangle + e^{i\pi/8}|n\rangle\big) - \frac{1}{D}\left(\sqrt{1+1/\sqrt{2}} - 1\right)\psi_0.$$

$\triangle$

In addition to a POVM, a quantum measurement associated with an instrument $\mathcal{I} = (\mathcal{I}_x)_{x\in\mathbb{X}}$ with input Hilbert space $\mathcal{H}$ and output space $\mathcal{K}$ also defines the total unconditioned state transformation $\sum_{x\in\mathbb{X}} \mathcal{I}_x$ from the set $\mathcal{S}(\mathcal{H})$ of input states to the set of output states $\mathcal{S}(\mathcal{K})$. This transformation is also known as a *channel*, a trace-preserving completely positive (affine) map. A channel is a special case of a *quantum operation*, i.e., a completely positive map without the condition of being trace preserving. We immediately see that any quantum operation [channel] can be viewed as a QOVM [an instrument] with a single outcome. Let us again assume that G is a finite group and that $U : G \to \mathcal{U}(\mathcal{H})$ and $V : G \to \mathcal{U}(\mathcal{K})$ are unitary representations mediating the input and output symmetries. We say that a quantum operation or channel $\Phi : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{K})$ is *(U, V)–covariant* if, for all $g \in G$ and $\rho \in \mathcal{S}(\mathcal{H})$,

$$\Phi\big(U(g)\rho U(g)^*\big) = V(g)\Phi(\rho)V(g)^*.$$

Furthermore, we say that a (U, V)–covariant channel Φ is an *extreme channel of the (U, V)–covariance structure* if Φ is an extreme point of the convex set of all (U, V)–covariant channels. Clearly, a (U, V)–covariant channel is an example of an $(\mathbb{X}, U, V)$–covariant instrument where $\mathbb{X} = \{x_0\}$ is a singleton where G acts trivially. The following is again a direct corollary of Theorems 3 and 4 and the above observation.

Corollary 2. *Let Φ be a (U, V)–covariant quantum operation [channel]. There is, for any $[\vartheta] \in \hat{G}$, a number $M_\vartheta \in \{0\} \cup \mathbb{N}$, and a linearly independent set*

$$\{L_{\vartheta,i,m} \in \mathcal{L}(\mathcal{H}, \mathcal{K}) \,|\, [\vartheta] \in \hat{G},\ i = 1, \ldots, D_\vartheta,\ m = 1, \ldots, M_\vartheta\}$$

of operators such that, for any $[\vartheta] \in \hat{G}$, $i = 1, \ldots, D_\vartheta$, $m = 1, \ldots, M_\vartheta$, and $g \in G$,

$$L_{\vartheta,i,m}U(g) = \sum_{j=1}^{D_\vartheta} \vartheta_{i,j}(g)V(g)L_{\vartheta,j,m} \tag{20}$$

and, for any $\rho \in \mathcal{S}(\mathcal{H})$,

$$\Phi(\rho) = \sum_{[\vartheta]\in\hat{G}} \sum_{i=1}^{D_\vartheta} \sum_{m=1}^{M_\vartheta} L_{\vartheta,i,m}\rho L^*_{\vartheta,i,m}. \tag{21}$$

If Φ is a channel, then

$$\sum_{[\vartheta]\in\hat{G}}\sum_{i=1}^{D_\vartheta}\sum_{m=1}^{M_\vartheta} L_{\vartheta,i,m}^* L_{\vartheta,i,m} = \mathbb{1}_{\mathcal{H}}, \tag{22}$$

This channel is an extreme channel of the (U,V)–covariance structure if and only if the set

$$\left\{ \sum_{i=1}^{D_\vartheta} L_{\vartheta,i,m}^* L_{\vartheta,i,n} \,\middle|\, m,\, n = 1,\ldots,M_\vartheta,\ [\vartheta]\in\hat{G} \right\}$$

is linearly independent. Moreover, given a set of linear operators $L_{\vartheta,i,m}$, $[\vartheta]\in\hat{G}$, $i=1,\ldots,D_\vartheta$, $m=1,\ldots,M_\vartheta$, where $M_\vartheta\in\{0\}\cup\mathbb{N}$, satisfying Equation (20) [and (22)], Equation (21) defines a (U,V)–covariant quantum operation [channel].

Suppose that $L_{\vartheta,i,m}:\mathcal{H}\to\mathcal{K}$, $[\vartheta]\in\hat{G}$, $i=1,\ldots,D_\vartheta$, $m=1,\ldots,M_\vartheta$, where $M_\vartheta\in\{0\}\cup\mathbb{N}$, satisfy the condition of Equation (20). It easily follows that, for any $[\vartheta]\in\hat{G}$ and $m,\,n=1,\ldots,M_\vartheta$, the operator $\sum_{i=1}^{D_\vartheta} L_{\vartheta,i,m}^* L_{\vartheta,i,n}$ commutes with U. This is why we may omit the G-summations in the normalization condition of Equation (22), the channel characterization of Equation (21), and the operators essential for the extremality characterization of Corollary 2.

5. Covariant continuous instruments associated with a compact stability subgroup

We now concentrate on continuous quantum measurements possibly in infinite-dimensional systems and their symmetry properties. We will derive results closely paralleling Theorems 3 and 4 and Lemma 1 in this setting. However, we no longer can apply similar easy 'hands-on' methods of the preceding sections enabled by the finiteness of value spaces and Hilbert spaces. However, on a deeper level, our proofs are still, in a way, analogical to the earlier proofs but, in order to facilitate our proofs, we import some pre-established results from [15]. We now explicitly define $\mathcal{L}(\mathcal{H})$ as the algebra of bounded operators on the Hilbert space $\mathcal{H}$ and $\mathcal{T}(\mathcal{H})$ as the trace class on $\mathcal{H}$. We start by giving a generalization for Definition 1. In the sequel, we say that a map $\Phi:\mathcal{T}(\mathcal{H})\to\mathcal{T}(\mathcal{K})$ is a *quantum operation* if it is linear and its Heisenberg dual $\Phi^*:\mathcal{L}(\mathcal{K})\to\mathcal{L}(\mathcal{H})$, defined through $\mathrm{tr}\,[\rho\Phi^*(B)]=\mathrm{tr}\,[\Phi(\rho)B]$ for all $\rho\in\mathcal{T}(\mathcal{H})$ and $B\in\mathcal{L}(\mathcal{K})$, is completely positive.

Definition 6. Let $(\mathbb{X},\Sigma)$ be a measurable space (i.e., $\mathbb{X}\neq\emptyset$ and Σ is a σ-algebra of subsets of $\mathbb{X}$) and $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces.

(i) A map $\mathsf{M}:\Sigma\to\mathcal{L}(\mathcal{H})$ is a *positive-operator-valued measure (POVM)* if, for all $X\in\Sigma$, $\mathsf{M}(X)\geq 0$, $\mathsf{M}(\emptyset)=0$, and, for any disjoint sequence $X_1,X_2,\ldots\in\Sigma$, $\mathsf{M}\big(\cup_{i=1}^\infty X_i\big)=\sum_{i=1}^\infty \mathsf{M}(X_i)$ where the series converges weakly. This POVM M is a *projection-valued measure (PVM)* if $\mathsf{M}(X)$ is an orthogonal projection for all $X\in\Sigma$.

(ii) A POVM $\mathsf{M} : \Sigma \to \mathcal{L}(\mathcal{H})$ is an *observable* if is is normalized, i.e., $\mathsf{M}(\mathbb{X}) = \mathbb{1}_{\mathcal{H}}$. A normalized PVM is called as a *sharp observable*.

(iii) We say that a map $\mathcal{I} : \Sigma \times \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K})$ is a *quantum operation-valued measure (QOVM)* (with the value space $(\mathbb{X}, \Sigma)$, input space $\mathcal{H}$, and output space $\mathcal{K}$) if, for any $X \in \Sigma$, $\mathcal{I}(X, \cdot)$ is a quantum operation, $\mathcal{I}(\emptyset, \rho) \equiv 0$ and, for any disjoint sequence $X_1, X_2, \ldots \in \Sigma$ and any $\rho \in \mathcal{T}(\mathcal{H})$, $\mathcal{I}\big(\cup_{i=1}^{\infty} X_i, \rho\big) = \sum_{i=1}^{\infty} \mathcal{I}(X_i, \rho)$ where the sum converges w.r.t. the trace norm topology.

(iv) A QOVM $\mathcal{I} : \Sigma \times \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K})$ is an *instrument* if it is normalized, i.e., $\mathcal{I}(\mathbb{X}, \cdot)$ is trace preserving.

(v) An instrument $\mathcal{I} : \Sigma \times \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K})$ *measures the observable* $\mathsf{M} : \Sigma \to \mathcal{L}(\mathcal{H})$ or is an M*–instrument* if $\mathrm{tr}\,[\rho \mathsf{M}(X)] = \mathrm{tr}\,[\mathcal{I}(X, \rho)]$ for all $\rho \in \mathcal{T}(\mathcal{H})$ and $X \in \Sigma$.

For any instrument [QOVM] $\mathcal{I} : \Sigma \times \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K})$, we also define the Heisenberg instrument [QOVM] $\mathcal{I}^* : \Sigma \times \mathcal{L}(\mathcal{K}) \to \mathcal{L}(\mathcal{H})$ through

$$\mathrm{tr}\,[\rho \mathcal{I}^*(X, B)] = \mathrm{tr}\,[\mathcal{I}(X, \rho) B], \qquad \rho \in \mathcal{T}(\mathcal{H}), \quad B \in \mathcal{L}(\mathcal{K}), \quad X \in \Sigma,$$

i.e., for all $X \in \Sigma$, $\mathcal{I}^*(X, \cdot)$ is the Heisenberg dual operation of $\mathcal{I}(X, \cdot)$. If $\mathbb{X}$ is a topological space, we denote the corresponding Borel σ-algebra by $\mathcal{B}(\mathbb{X})$; there is never any ambiguity about which is the topology concerned, so the topology is not specifically indicated in this notation.

Let G be a group. We say that a set $\mathbb{X}$ is a [transitive] G-space if there is a map $G \times \mathbb{X} \ni (g, x) \mapsto gx \in \mathbb{X}$ such that $ex = x$ for all $x \in \mathbb{X}$ and $(gh)x = g(hx)$ for all g, $h \in G$ and $x \in \mathbb{X}$ [and, for any x, $y \in \mathbb{X}$, there is $g \in G$ such that $gx = y$]. Suppose that $(\mathbb{X}, \Sigma)$ is a measurable space where $\mathbb{X}$ is a G-space and that, for any $g \in G$, the map $x \mapsto gx$ is measurable. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and $U : G \to \mathcal{U}(\mathcal{H})$ and $V : G \to \mathcal{U}(\mathcal{K})$ be unitary representations. We define the covariance of QOVMs, POVMs, instruments, and observables analogously to Definitions 2 and 3; in particular, an instrument [QOVM] $\mathcal{I} : \Sigma \times \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K})$ is (Σ, U, V)*–covariant* if, for any $X \in \Sigma$, $\rho \in \mathcal{T}(\mathcal{H})$, and $g \in G$,

$$\mathcal{I}\big(gX, U(g)\rho U(g)^*\big) = V(g)\mathcal{I}(X, \rho)V(g)^*.$$

In the special case $\mathcal{K} = \mathbb{C}$, the set of (Σ, U, V)–covariant instruments [QOVMs] simplifies to the set of (Σ, U)–covariant observables [POVMs] $\mathsf{M} : \Sigma \to \mathcal{L}(\mathcal{H})$ for which

$$U(g)\mathsf{M}(X)U(g)^* = \mathsf{M}(gX), \qquad g \in G, \quad X \in \Sigma.$$

For any (Σ, U, V)–covariant instrument [QOVM] $\mathcal{I}$, there is a quadruple $(\mathcal{M}, \mathsf{P}, \overline{U}, J)$ consisting of a Hilbert space $\mathcal{M}$, a sharp observable $\mathsf{P} : \Sigma \to \mathcal{L}(\mathcal{M})$, a unitary representation $\overline{U} : G \to \mathcal{U}(\mathcal{M})$, and a linear map $J : \mathcal{H} \to \mathcal{K} \otimes \mathcal{M}$ so that

(i) $\mathcal{I}^*(X, B) = J^*\big(B \otimes \mathsf{P}(X)\big)J$ for all $X \in \Sigma$ and $B \in \mathcal{L}(\mathcal{K})$,

(ii) $JU(g) = \big(V(g) \otimes \overline{U}(g)\big)J$ for all $g \in G$,

(iii) $\overline{U}(g)\mathsf{P}(X)\overline{U}(g)^* = \mathsf{P}(gX)$ for all $g \in G$ and $X \in \Sigma$, and

(iv) the vectors $\big(B \otimes \mathsf{P}(X)\big)J\varphi$, $B \in \mathcal{L}(\mathcal{K})$, $X \in \Sigma$, $\varphi \in \mathcal{H}$, span a dense subspace of $\mathcal{K} \otimes \mathcal{M}$.

The existence of a triple $(\mathcal{M}, \mathsf{P}, J)$ satisfying items (i) and (iv) above is well known [27], and the existence of the unitary representation $\overline{U}$ satisfying items (ii) and (iii) is proven essentially in the same way as in the finite-outcome and finite-dimensional case which is studied in Appendix B. The quadruple $(\mathcal{M}, \mathsf{P}, \overline{U}, J)$ of items (i), (ii), and (iii) is called as a (Σ, U, V)-*covariant dilation for* $\mathcal{I}$ and if item (iv) also holds, then this dilation is *minimal*. As a special case, we obtain a (Σ, U)-covariant dilation $(\mathcal{M}, \mathsf{P}, \overline{U}, J)$ for a (Σ, U)-covariant observable M when we set $\mathcal{K} = \mathbb{C}$ and let V be trivial.

Let G be a locally compact second-countable group which is Hausdorff. If Ω is locally compact, second countable, and Hausdorff and Ω is a transitive G-space such that the map $G \times \Omega \ni (g, \omega) \mapsto g\omega \in \Omega$ is continuous, there is a closed subgroup $H \leq G$ such that Ω is homeomorphic with G/H (space of left cosets) and, in this identification, the G-action is of the form

$$g(g'H) = (gg')H, \qquad g,\, g' \in G.$$

From now on, we assume that $\mathcal{H}$ and $\mathcal{K}$ are separable Hilbert spaces, G is a locally compact and second-countable group which is Hausdorff, $H \leq G$ is a closed subgroup, and $U : G \to \mathcal{U}(\mathcal{H})$ and $V : G \to \mathcal{U}(\mathcal{K})$ are strongly continuous‖ unitary representations. We will concentrate on $\big(\mathcal{B}(G/H), U, V\big)$–covariant instruments and $\big(\mathcal{B}(G/H), U, V\big)$–covariant dilations which we will call, for short, $(G/H, U, V)$–*covariant*. In the same context, we call $\big(\mathcal{B}(G/H), U\big)$–covariant observables as $(G/H, U)$–*covariant*. Note that, we are now restricting to the transitive, i.e., single-orbit case. We also fix a quasi-G–invariant measure $\mu : \mathcal{B}(G/H) \to [0, \infty]$ and a measurable section $s : G/H \to G$ for the factor projection $g \mapsto gH$ such that $s(H) = e$. It is well known [30] that, fixing a left Haar measure μ_G for G, there is a $(\mu_G \times \mu)$-measurable function $\rho : G \times G/H \to (0, \infty)$ coinciding $(\mu_G \times \mu)$-a.e. with the function $(g, x) \mapsto (d\mu_g/d\mu)(x)$ where $\mu_g(X) = \mu(gX)$ for all $X \in \mathcal{B}(G/H)$ and $\rho(gh, x) = \rho(g, hx)\rho(h, x)$ for $(\mu_G \times \mu_G \times \mu)$-a.a. $(g, h, x) \in G \times G \times G/H$. As in Section 3, we define, for any unitary representation $\pi : H \to \mathcal{U}(\mathcal{H}_\pi)$, the cocycle $\zeta^\pi : G \times G/H \to \mathcal{U}(\mathcal{H}_\pi)$ through $\zeta^\pi(g, x) = \pi\big(s(x)^{-1}g^{-1}s(gx)\big)$ for all $g \in G$ and $x \in G/H$. The cocycle conditions (3) still hold. In this setting, a $(G/H, U, V)$–covariant instrument [or a QOVM in general] $\mathcal{I}$ can be given a minimal $(G/H, U, V)$–covariant dilation $(\mathcal{M}, \mathsf{P}, \overline{U}, J)$ where P and $\overline{U}$ constitute a canonical system of imprimitivity [6, 15]. This means the following: There is a strongly continuous unitary representation $\pi : H \to \mathcal{U}(\mathcal{H}_\pi)$ in some separable Hilbert space $\mathcal{H}_\pi$ such that $\mathcal{M} = L^2_\mu \otimes \mathcal{H}_\pi$ (which we identify with the Hilbert space of μ-equivalence classes of μ-square-integrable functions $F : G/H \to \mathcal{H}_\pi$), $\mathsf{P} = \mathsf{P}^G_\pi$ defined through

$$\big(\mathsf{P}^G_\pi(X)F\big)(x) = \chi_X(x)F(x), \qquad X \in \mathcal{B}(G/H), \quad F \in L^2_\mu \otimes \mathcal{H}_\pi, \quad x \in G/H, \tag{23}$$

‖ That is, e.g., $g \mapsto U(g)\varphi$ is continuous for any $\varphi \in \mathcal{H}$.

and $\overline{U} = U_\pi^G$ defined through

$$\big(U_\pi^G(g)F\big)(x) = \sqrt{\rho(g^{-1},x)}\zeta^\pi(g^{-1},x)F(g^{-1}x), \qquad g\in G, \quad F\in L_\mu^2\otimes\mathcal{H}_\pi, \quad x\in G/H. \tag{24}$$

The representation U_π^G is called as the representation *induced from* π and $(\mathsf{P}_\pi^G, U_\pi^G)$ is the *canonical system of imprimitivity associated to* π; note that P_π^G is a $(G/H, U_\pi^G)$–covariant PVM.

In many cases, the $(G/H,U)$-covariant observables are well known; these include the situations where G is Abelian, H is compact (and the measure diagonalizing U is absolutely continuous w.r.t. the Plancherel measure), or U is square integrable. In what follows, we utilize the properties these covariant observables have to characterize the covariant instruments. This is why we make the following assumptions, which hold in all the above mentioned cases, on the $(G/H,U)$-covariant observables as this allows us to derive very specific forms for covariant instruments without having to prove the same results separately for all the special cases:

(a) There is a dense subspace $\mathcal{D}$ of $\mathcal{H}$ which is U–invariant, i.e., $U(g)\mathcal{D}\subseteq\mathcal{D}$ for all $g\in G$.

(b) There is a norm $\|\cdot\|_1:\mathcal{D}\to[0,\infty)$ so that $(\mathcal{D},\|\cdot\|_1)$ is a separable normed space. Moreover, for all $g\in G$ and $\varphi\in\mathcal{D}$, $\|U(g)\varphi\|_1=\|\varphi\|_1$.

(c) For any $(G/H,U)$–covariant POVM M, there is a strongly continuous unitary representation $\pi_0: H\to\mathcal{H}_{\pi_0}$ in a separable Hilbert space $\mathcal{H}_{\pi_0}$ and a linear operator $\Theta:\mathcal{D}\to\mathcal{H}_{\pi_0}$ such that $\|\Theta\varphi\|\le\|\varphi\|_1$ for all $\varphi\in\mathcal{D}$, $\Theta U(h)=\pi_0(h)\Theta$ for all $h\in H$ and, defining the linear map $J:\mathcal{H}\to L_\mu^2\otimes\mathcal{H}_{\pi_0}$ through $(J\varphi)(gH)=\pi_0\big(s(gH)^{-1}g\big)\Theta U(g)^*\varphi$ for all $\varphi\in\mathcal{D}$ and $g\in G$, $(L_\mu^2\otimes\mathcal{H}_{\pi_0}, \mathsf{P}_{\pi_0}^G, U_{\pi_0}^G, J)$ is a minimal $(G/H,U)$–covariant Naĭmark dilation for M.

We next summarize how these conditions are seen to hold in the special case listed before making the above assumptions. First, suppose that G Abelian and define $\mathcal{D}\subseteq\mathcal{H}$ and the norm $\|\cdot\|_1$ on $\mathcal{D}$ in the same way as in [14]: In this setting, there are $(G/H,U)$-covariant POVMs (and, consequently, $(G/H,U,V)$-covariant QOVMs) if and only if there is a standard measure $\nu:\mathcal{B}(\hat{G}/H^\perp)\to[0,\infty]$ (where $\hat{G}$ is the dual group and $H^\perp\le\hat{G}$ is the annihilator of H, i.e., the subgroup of those $\eta\in\hat{G}$ such that $\langle h,\eta\rangle=1$ for all $h\in H$) such that the decomposing measure $\mu_U:\mathcal{B}(\hat{G})\to[0,\infty]$ for U is given by $\int_{\hat{G}} f\,d\mu_U=\int_{\hat{G}/H^\perp}\int_{H^\perp} f(\gamma+\eta)\,d\mu_{H^\perp}(\eta)\,d\nu(\gamma+H^\perp)$ for all $f\in L^1_{\mu_U}$ where $\mu_{H^\perp}$ is a fixed Haar measure for the group $H^\perp$ [7, Theorem 4.2]. The decomposing measure μ_U is defined as the measure such that there is a measurable field $\gamma\mapsto\mathcal{H}(\gamma)$ of Hilbert spaces such that $\mathcal{H}$ can be identified with the direct-integral Hilbert space $\int_{\hat{G}}^\oplus\mathcal{H}(\gamma)\,d\mu_U(\gamma)$ such that $\big(U(g)\varphi\big)(\gamma)=\langle g,\gamma\rangle\varphi(\gamma)$ for all $g\in G$, $\varphi\in\mathcal{H}$, and $\gamma\in\hat{G}$. The domain $\mathcal{D}\subset\mathcal{H}$ is now the subspace of those $\varphi\in\mathcal{H}$ such that

$$\|\varphi\|_1:=\sqrt{\int_{\hat{G}/H^\perp}\left(\int_{H^\perp}\|\varphi(\gamma+\eta)\|\,d\mu_{H^\perp}\right)^2 d\nu(\gamma+H^\perp)}<\infty.$$

It is easily seen that conditions (a) and (b) hold. We next fix an infinite-dimensional separable Hilbert space $\mathcal{M}$. For any measurable field $W : \gamma \mapsto W(\gamma)$ of isometries $W(\gamma) : \mathcal{H}(\gamma) \to \mathcal{M}$, we define $\Theta_W : \mathcal{D} \to L^2_\nu \otimes \mathcal{M}$ through

$$(\Theta_W \varphi)(\gamma + H^\perp) = \int_{H^\perp} W(\gamma + \eta)\varphi(\gamma + \eta)\, d\mu_{H^\perp}(\eta), \qquad \varphi \in \mathcal{D}, \quad \gamma \in \hat{G},$$

where we identify $L^2_\nu \otimes \mathcal{M}$ with the space of (equivalence classes of) ν-square-integrable functions $f : \hat{G}/H^\perp \to \mathcal{M}$. Condition (c) is now the content of Theorem 3.1 of [14] and the proof thereof, namely, for any $(G/H, U)$-covariant observable there is a measurable field W of isometries such that condition (c) is satisfied with $\Theta = \Theta_W$. The same holds, in a straightforward manner for covariant POVMs.

Secondly, let G be unimodular and H be compact and assume that the decomposing measure μ_U for U is absolutely continuous with respect to the Plancherel measure $\mu_{\hat{G}}$. Denoting, for all $[\gamma] \in \hat{G}$, the Hilbert space of the selected representative γ by $\mathcal{K}(\gamma)$ (i.e., $\gamma : G \to \mathcal{U}\big(\mathcal{K}(\gamma)\big)$), there is a measurable field $\gamma \mapsto \mathcal{M}(\gamma)$ of Hilbert spaces such that $\mathcal{H}$ can be identified with the direct integral $\int_{\hat{G}}^\oplus \mathcal{K}(\gamma) \otimes \mathcal{M}(\gamma)\, d\mu_U([\gamma])$. Moreover, define the subspace $\mathcal{D} \subseteq \mathcal{H}$ and the norm $\|\cdot\|_1$ in the same way as in [15]: For measurable fields $\hat{G} \ni \gamma \mapsto \zeta(\gamma) \in \mathcal{K}(\gamma)$ and $\hat{G} \ni \gamma \mapsto \xi(\gamma) \in \mathcal{M}(\gamma)$, we denote by $\zeta \star \xi$ the field $\gamma \mapsto \zeta(\gamma) \otimes \xi(\gamma)$. We define $\mathcal{D}$ as the linear hull product form fields $\zeta \star \xi$ such that

$$\|\zeta \star \xi\|_1 := \int_{\hat{G}} \|\zeta(\gamma)\| \|\xi(\gamma)\|\, d\mu_{\hat{G}}(\gamma) < \infty$$

where $\mu_{\hat{G}} : \mathcal{B}(\hat{G}) \to [0, \infty]$ is the Placherel measure associated to a fixed Haar measure over G. The norm $\mathcal{D} \ni \varphi \mapsto \|\varphi\|_1 = \int_{\hat{G}} \|\varphi(\gamma)\|\, d\mu_{\hat{G}}(\gamma) \in [0, \infty)$ is now well defined. These choices are easily seen to satisfy conditions (a) and (b) and in Theorem 3 of [15] and the proof thereof we see that the operator Θ of condition (c) can be found for any $(G/H, U)$-covariant POVM. However, since the construction of this operator is somewhat complicated, we do not go into this in more detail here.

Thirdly the conditions (a), (b), and (c) hold is the case like that above, except that U is square integrable, as the results of [25] and Section 6.1 of [15] show. We shall not go into details here, but we should emphasize that, in this case, $\mathcal{D} = \mathcal{H}$ and $\|\cdot\|_1$ coincides with the usual Hilbert norm; in fact, the operators Θ of item (c) above are, in this case Hilbert-Schmidt operators; see Theorem 5 of [15].

Using conditions (a), (b), and (c), one can prove a counterpart of Theorem 4 of [15] using same methods as we will employ shortly. However, in order to obtain more interesting results, we have to assume that

(d) $H \leq G$ is compact.

It hence follows that the dual $\hat{H}$ is countable. As earlier, we pick, for any $[\eta] \in \hat{H}$ a representative $\eta : H \to \mathcal{U}(\mathcal{K}_\eta)$ and denote by D_η the dimension of $\mathcal{K}_\eta$ (which is finite). For any $[\eta] \in \hat{H}$, we also fix an orthonormal basis $\{e_{\eta,i}\}_{i=1}^{D_\eta} \subset \mathcal{K}_\eta$ and denote

$$\eta_{i,j}(h) := \langle e_{\eta,i} | \eta(h) e_{\eta,j} \rangle, \qquad i,\, j = 1, \ldots, D_\eta, \quad h \in H.$$

Moreover, for any $[\eta] \in \hat{H}$, we define the functions $\zeta^\eta_{i,j} : G \times G/H \to \mathbb{C}$ through the matrix elements of ζ^η in the basis $\{e_{\eta,i}\}_{i=1}^{D_\eta}$. Since H is compact, G/H allows an essentially unique regular G–invariant measure $\mu : \mathcal{B}(G/H) \to [0,\infty]$, i.e., $\mu(gX) = \mu(X)$ for all $X \in \mathcal{B}(G/H)$. We keep this measure fixed in the sequel implying that we may assume $\rho \equiv 1$ in the definition (24) of the induced representation.

Let us make a useful definition. Below, we say that, given a set A, a set $\{L_a\}_{a\in A}$ of linear operators $L_a : \mathcal{D} \to \mathcal{K}$ is $(\mathcal{K},\mathcal{D})$-*weakly independent* if, for $(\beta_a)_{a\in A} \in \ell^2_A$, the condition $\sum_{a\in A} \beta_a \langle\psi|L_a\varphi\rangle = 0$ for any $\psi \in \mathcal{K}$ and any $\varphi \in \mathcal{D}$ implies $\beta_a = 0$ for all $a \in A$. Moreover, the notation $m = 1,\ldots,M$ is to be taken as usual when $M \in \mathbb{N}$; if $M = 0$, this means that the set of indices m discussed is empty; and if $M = \infty$, the set of indices m is the entire $\mathbb{N}$.

Definition 7. We say that, given $M_\eta \in \mathbb{N} \cup \{0,\infty\}$ for any $[\eta] \in \hat{H}$, a set

$$\{L_{\eta,i,m} \,|\, [\eta] \in \hat{H},\ i = 1,\ldots,D_\eta,\ m = 1,\ldots,M_\eta\}$$

of linear operators $L_{\eta,i,m} : \mathcal{D} \to \mathcal{K}$ is a *set of* $(G/H,U,V)$*–intertwiners* if

$$L_{\eta,i,m}U(h) = \sum_{j=1}^{D_\eta} \eta_{i,j}(h)V(h)L_{\eta,j,m} \tag{25}$$

for all $[\eta] \in \hat{H}$, $i = 1,\ldots,D_\eta$, $m = 1,\ldots,M_\eta$, and $h \in H$,

$$\sum_{[\eta]\in\hat{H}} \sum_{i=1}^{D_\eta} \sum_{m=1}^{M_\eta} \|L_{\eta,i,m}\varphi\|^2 \le \|\varphi\|_1^2, \qquad \varphi \in \mathcal{D}, \tag{26}$$

and there is a number $M \ge 0$ such that

$$\int_{G/H} \sum_{[\eta]\in\hat{H}} \sum_{i=1}^{D_\eta} \sum_{m=1}^{M_\eta} \|L_{\eta,i,m}U(g)^*\varphi\|^2 \, d\mu(gH) \le M\|\varphi\|^2, \qquad \varphi \in \mathcal{D}. \tag{27}$$

This set of $(G/H,U,V)$-covariant intertwiners is *minimal* if it is $(\mathcal{K},\mathcal{D})$-weakly independent. A set of $(G/H,U,V)$-intertwiners $L_{\eta,i,m} : \mathcal{D} \to \mathcal{K}$ ($[\eta] \in \hat{H}$, $i = 1,\ldots,D_\eta$, $m = 1,\ldots,M_\eta$ where $M_\eta \in \mathbb{N} \cup \{0,\infty\}$ for all $[\eta] \in \hat{H}$) is *normalized* if

$$\int_{G/H} \sum_{[\eta]\in\hat{H}} \sum_{i=1}^{D_\eta} \sum_{m=1}^{M_\eta} \|L_{\eta,i,m}U(g)^*\varphi\|^2 \, d\mu(gH) = \|\varphi\|^2, \qquad \varphi \in \mathcal{D}. \tag{28}$$

Note that, using Equation (25), the integrand in Equations (27) and (28) is found to be invariant in the replacement $g \to gh$ whenever $h \in H$ in exactly the same way as earlier in Section 3; this allows us to interpret the integrand as a function on G/H. Note also that Equation (28) (whenever it holds) already implies the weaker Equation (27). The following theorem is a generalized version of Theorem 3 (albeit in the single-orbit case).

Theorem 5. *Let $\mathcal{I}:\mathcal{B}(G/H)\times\mathcal{T}(\mathcal{H})\to\mathcal{T}(\mathcal{K})$ be a $(G/H,U,V)$–covariant QOVM [instrument]. There are, for any $[\eta]\in\hat{H}$, $M_\eta\in\mathbb{N}\cup\{0,\infty\}$ and a minimal [normalized] set $\{L_{\eta,i,m}:\mathcal{D}\to\mathcal{K}\,|\,m=1,\ldots,M_\eta,\ i=1,\ldots,D_\eta,\ [\eta]\in\hat{H}\}$ of $(G/H,U,V)$–intertwiners such that, for all $X\in\mathcal{B}(G/H)$, $B\in\mathcal{L}(\mathcal{K})$, and $\varphi\in\mathcal{D}$,*

$$\langle\varphi|\mathcal{I}^*(X,B)\varphi\rangle=\int_X\sum_{[\eta]\in\hat{H}}\sum_{i=1}^{D_\eta}\sum_{m=1}^{M_\eta}\langle V(g)L_{\eta,i,m}U(g)^*\varphi|BV(g)L_{\eta,i,m}U(g)^*\varphi\rangle\,d\mu(gH).\tag{29}$$

On the other hand, given $M_\eta\in\mathbb{N}\cup\{0,\infty\}$ for any $[\eta]\in\hat{H}$ and a [normalized] set of $(G/H,U,V)$–intertwiners consisting of $L_{\eta,i,m}:\mathcal{D}\to\mathcal{K}$, $[\eta]\in\hat{H}$, $i=1,\ldots,D_\eta$, $m=1,\ldots,M_\eta$, Equation (29) defines a $(G/H,U,V)$–covariant QOVM [instrument].

Proof. Let $\mathcal{I}:\mathcal{B}(G/H)\times\mathcal{T}(\mathcal{H})\to\mathcal{T}(\mathcal{K})$ be an $(\mathbb{X},U,V)$–covariant QOVM [instrument] and equip it with a $(G/H,U,V)$–covariant minimal Stinespring dilation $(L^2_\mu\otimes\mathcal{H}_\pi,\mathsf{P}^G_\pi,U^G_\pi,J)$ where $\pi:H\to\mathcal{U}(\mathcal{H}_\pi)$ is strongly continuous unitary representation in a separable Hilbert space $\mathcal{H}_\pi$. Let $\mathsf{M}:\mathcal{B}(G/H)\to\mathcal{L}(\mathcal{H})$ be the (possibly continuous) POVM [observable] measured by $\mathcal{I}$, i.e., $\mathrm{tr}\,[\rho\mathsf{M}(X)]=\mathrm{tr}\,[\mathcal{I}(X,\rho)]$ for all $\rho\in\mathcal{T}(\mathcal{H})$ and $X\in\mathcal{B}(G/H)$. According to item (c) (see also [8] and [15]), there is a strongly continuous unitary representation $\pi_0:H\to\mathcal{U}(\mathcal{H}_{\pi_0})$ in a separable Hilbert space $\mathcal{H}_{\pi_0}$, and a linear map $\Theta:\mathcal{D}\to\mathcal{H}_{\pi_0}$ (with $\|\Theta\varphi\|\le\|\varphi\|_1$ for all $\varphi\in\mathcal{D}$ and $\Theta U(h)=\pi_0(h)\Theta$ for all $h\in H$) such that we may define the minimal $(G/H,U)$-covariant minimal Naĭmark dilation $(L^2_{\mu_0}\otimes\mathcal{H}_{\pi_0},\mathsf{P}^G_{\pi_0},U^G_{\pi_0},J_0)$ where $(J_0\varphi)(gH)=\pi_0\big(s(gH)^{-1}g\big)\Theta U(g)^*\varphi$ for all $\varphi\in\mathcal{D}$ and $gH\in G/H$.

According to Proposition 6 of [15], there is an isometry $\Lambda:\mathcal{H}_{\pi_0}\to\mathcal{K}\otimes\mathcal{H}_\pi$ with the property $\Lambda\pi_0(h)=\big(V(h)\otimes\pi(h)\big)\Lambda$ for all $h\in H$ such that, defining the decomposable isometry $W:L^2_\mu\otimes\mathcal{H}_{\pi_0}\to\mathcal{K}\otimes L^2_\mu\otimes\mathcal{H}_\pi$ through $(Wf)(x)=W(x)f(x)$ for all $f\in L^2_\mu\otimes\mathcal{H}_{\pi_0}$ and $x\in G/H$, where

$$W(gH)=\big(V(g)\otimes\zeta^\pi(g^{-1},gH)\big)\Lambda\zeta^{\pi_0}(g^{-1},gH)^*,\qquad g\in G,\tag{30}$$

we have $J=WJ_0$. Through simple calculation, we find that this means $(J\varphi)(gH)=J(gH)\varphi$ for all $\varphi\in\mathcal{D}$ and $gH\in G/H$ where

$$J(gH)=W(gH)\zeta^{\pi_0}(g^{-1},gH)\Theta U(g)^*=\big(V(g)\otimes\zeta^\pi(g^{-1},gH)\Lambda\Theta U(g)^*,\qquad g\in G,$$

where we have used Equation (30).

According to the Peter-Weyl theorem, for each $[\eta]\in\hat{H}$, there is a separable Hilbert space $\mathcal{M}_\eta$ so that $\mathcal{H}_\pi=\bigoplus_{[\eta]\in\hat{H}}\mathcal{K}_\eta\otimes\mathcal{M}_\eta$ and $\pi(h)=\bigoplus_{[\eta]\in\hat{H}}\eta(h)\otimes\mathbb{1}_{\mathcal{M}_\eta}$ for all $h\in H$. Denote, for each $[\eta]\in\hat{H}$, $M_\eta:=\dim\mathcal{M}_\eta\in\{0,\infty\}\cup\mathbb{N}$, and let $\{f_{\eta,m}\}_{m=1}^{M_\eta}$ be an orthonormal basis for $\mathcal{M}_\eta$ for all $[\eta]\in\hat{H}$. Defining, for all $[\eta]\in\hat{H}$, $i=1,\ldots,D_\eta$, and $m=1,\ldots,M_\eta$, the isometry $V_{\eta,i,m}:\mathcal{K}\to\mathcal{K}\otimes\mathcal{H}_\pi$ through $V_{\eta,i,m}\psi=\psi\otimes e_{\eta,i}\otimes f_{\eta,m}$ for all $\psi\in\mathcal{K}$, we denote $L_{\eta,i,m}:=V^*_{\eta,i,m}\Lambda\Theta$.

Proving that the set consisting of the operators $L_{\eta,i,m}$ is $(\mathcal{K},\mathcal{D})$-weakly independent is carried out in exactly the same way as the corresponding proof in Section 3. Pick $[\eta]\in\hat{H}$, $i=1,\ldots,D_\eta$, $m=1,\ldots,M_\eta$, and $h\in H$. We have

$$\begin{aligned} L_{\eta,i,m}U(h) &= V^*_{\eta,i,m}\Lambda\Theta U(h) = V^*_{\eta,i,m}\Lambda\pi_0(h)\Theta = V^*_{\eta,i,m}\big(V(h)\otimes\pi(h)\big)\Lambda\Theta \\ &= V(h)V^*_{\eta,i,m}\big(\mathbb{1}_\mathcal{K}\otimes\pi(h)\big)\Lambda\Theta = \sum_{j=1}^{D_\eta}\eta_{i,j}(h)V(h)L_{\eta,j,m} \end{aligned}$$

where we have used $\big(\mathbb{1}_\mathcal{K}\otimes\pi(h)\big)V_{\eta,i,m} = \sum_{j=1}^{D_\eta}\eta_{j,i}(h)V_{\eta,j,m}$ (which is easily proven) in the final equality, thus proving Equation (25). Using the Pythagorean theorem and the fact that Λ is an isometry, we find $\sum_{[\eta]\in\hat{H}}\sum_{i=1}^{D_\eta}\sum_{m=1}^{M_\eta}\|L_{\eta,i,m}\varphi\|^2 = \sum_{[\eta]\in\hat{H}}\sum_{i=1}^{D_\eta}\sum_{m=1}^{M_\eta}\|V^*_{\eta,i,m}\Lambda\Theta\varphi\|^2 = \|\Lambda\Theta\varphi\|^2 = \|\Theta\varphi\|^2 \le \|\varphi\|_1^2$ for all $\varphi\in\mathcal{D}$, implying Inequality (26). Using the fact that, for all $\varphi\in\mathcal{D}$ and $g\in G$, $(J\varphi)(gH) = J(gH)\varphi$, we find, for all $\varphi\in\mathcal{D}$, $X\in\mathcal{B}(G/H)$, and $B\in\mathcal{L}(\mathcal{K})$,

$$\begin{aligned} \langle\varphi|\mathcal{I}^*(X,B)\varphi\rangle &= \langle J\varphi|\big(B\otimes\mathsf{P}(X)\big)J\varphi\rangle = \int_X\langle J(gH)\varphi|(B\otimes\mathbb{1}_{\mathcal{H}_\pi})J(gH)\varphi\rangle\,d\mu(gH) \\ &= \int_X\langle\big(V(g)\otimes\zeta^\pi(g^{-1},gH)\big)\Lambda\Theta U(g)^*\varphi|\big(BV(g)\otimes\zeta^\pi(g^{-1},gH)\big)\Lambda\Theta U(g)^*\varphi\rangle\,d\mu(gH) \\ &= \int_X\langle\Lambda\Theta U(g)^*\varphi|\big(V(g)^*BV(g)\otimes\mathbb{1}_{\mathcal{H}_\pi}\big)\Lambda\Theta U(g)^*\varphi\rangle\,d\mu(gH) \\ &= \int_X\sum_{[\eta]\in\hat{H}}\sum_{i=1}^{D_\eta}\sum_{m=1}^{M_\eta}\langle\Lambda\Theta U(g)^*\varphi|V_{\eta,i,m}V(g)^*BV(g)V^*_{\eta,i,m}\Lambda\Theta U(g)^*\varphi\rangle\,d\mu(gH) \\ &= \int_X\sum_{[\eta]\in\hat{H}}\sum_{i=1}^{D_\eta}\sum_{m=1}^{M_\eta}\langle V(g)L_{\eta,i,m}U(g)^*\varphi|BV(g)L_{\eta,i,m}U(g)^*\varphi\rangle\,d\mu(gH), \end{aligned}$$

proving Equation (29). The proof of the converse claim is straight-forward and is left for the reader; note that Equation (28) corresponds to the normalization condition $\mathcal{I}^*(G/H,\mathbb{1}_\mathcal{K}) = \mathbb{1}_\mathcal{H}$ for an instrument and Equation (27) corresponds to the boundedness of $\mathcal{I}^*(G/H,\mathbb{1}_\mathcal{K})$ for a QOVM. □

We again have the following elaboration for the final claim of Theorem 5 stating that we may construct minimal covariant dilations from minimal sets of intertwiners.

Lemma 2. *Let* $\{L_{\eta,i,m}:\mathcal{D}\to\mathcal{K}\,|\,m=1,\ldots,M_\eta,\ i=1,\ldots,D_\eta,\ [\eta]\in\hat{H}\}$ *be a minimal [normalized] set of* $(G/H,U,V)$*–intertwiners where* $M_\eta\in\{0,\infty\}\cup\mathbb{N}$ *for each* $[\eta]\in\hat{H}$ *and define, for all* $[\eta]\in\hat{H}$, $i=1,\ldots,D_\eta$, $m=1,\ldots,M_\eta$, *and* $g\in G$,

$$K_{\eta,i,m}(gH) := \sum_{j=1}^{D_\eta}\zeta^\eta_{i,j}(g^{-1},gH)V(g)L_{\eta,j,m}U(g)^*. \tag{31}$$

For each $[\eta]\in\hat{H}$, *let* $\mathcal{M}_\eta$ *be an* M_η*-dimensional Hilbert space with the orthonormal basis* $\{f_{\eta,m}\}_{m=1}^{M_\eta}$ *and define the strongly continuous unitary representation* $\pi:H\to\mathcal{U}(\mathcal{H}_\pi)$

where $\mathcal{H}_\pi = \bigoplus_{[\eta]\in\hat{H}} \mathcal{K}_\eta \otimes \mathcal{M}_\eta$ *and* $\pi(h) = \bigoplus_{[\eta]\in\hat{H}} \eta(h) \otimes \mathbb{1}_{\mathcal{M}_\eta}$ *for all* $h \in H$ *and the linear map* $J : \mathcal{H} \to \mathcal{K} \otimes L^2_\mu \otimes \mathcal{H}_\pi$ *such that, for all* $\varphi \in \mathcal{D}$ *and* $g \in G$,

$$(J\varphi)(gH) = \sum_{[\eta]\in\hat{H}} \sum_{i=1}^{D_\eta} \sum_{m=1}^{M_\eta} K_{\eta,i,m}(gH)\varphi \otimes e_{\eta,i} \otimes f_{\eta,m},$$

the quadruple $(L^2_\mu \otimes \mathcal{H}_\pi, \mathsf{P}^G_\pi, U^G_\pi, J)$ *is a minimal* $(G/H, U, V)$*–covariant Stinespring dilation for the* $(G/H, U, V)$*–covariant QOVM [instrument]* $\mathcal{I}$ *defined through Equation (29).*

Proof. Let $M_\eta \in \{0, \infty\} \cup \mathbb{N}$ for each $[\eta] \in \hat{H}$ and suppose that operators $L_{\eta,i,m} : \mathcal{D} \to \mathcal{K}$, $[\eta] \in \hat{H}$, $i = 1, \ldots, D_\eta$, $m = 1, \ldots, M_\eta$, constitute a [normalized] minimal set of $(G/H, U, V)$–intertwiners and define the representation π and the linear map J as in the claim. Direct calculation utilizing Equation (28) shows that $\|J\varphi\| \le \sqrt{M}\|\varphi\|$ for all $\varphi \in \mathcal{D}$ where $M \ge 0$ is the number in the condition for a set of intertwiners given around Equation (27). Since $\mathcal{D}$ is a dense subspace of $\mathcal{H}$, this means that J indeed can be extended into a linear map $J : \mathcal{H} \to \mathcal{K} \otimes L^2_\mu \otimes \mathcal{H}_\pi$. Thus, equation $\mathcal{I}^*(X, B) = J^*\big(B \otimes \mathsf{P}(X)\big)J$ for all $X \in \mathcal{B}(G/H)$ and $B \in \mathcal{L}(\mathcal{K})$ defines a QOVM [instrument] $\mathcal{I} : \mathcal{B}(G/H) \times \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K})$. Checking $JU(g) = \big(V(g) \otimes U^G_\pi(g)\big)J$ for all $g \in G$ and $X \in \mathcal{B}(G/H)$ is straight-forward and is left for the reader. Let us concentrate on showing that the vectors $\big(B \otimes \mathsf{P}^G_\pi(X)\big)J\varphi$, $B \in \mathcal{L}(\mathcal{K})$, $X \in \mathcal{B}(G/H)$, $\varphi \in \mathcal{H}$, span a dense subspace of $\mathcal{K} \otimes L^2_\mu \otimes \mathcal{H}_\pi$.

Proving that the set $\{K_{\eta,i,m}(x) \,|\, m = 1, \ldots, M_\eta,\ i = 1, \ldots, D_\eta,\ [\eta] \in \hat{H}\}$ is $(\mathcal{K}, \mathcal{D})$-weakly independent for any $x \in G/H$ is carried out in essentially the same way as in the proof of Lemma 1. Let $\Psi \in \mathcal{K} \otimes L^2_\mu \otimes \mathcal{H}_\pi$ be such that $\langle \Psi|\big(B \otimes \mathsf{P}^G_\pi(X)\big)J\varphi\rangle = 0$ for all $B \in \mathcal{L}(\mathcal{K})$, $X \in \mathcal{B}(G/H)$, and $\varphi \in \mathcal{H}$. We may assume that, for any $[\eta] \in \hat{H}$, $i = 1, \ldots, D_\eta$, and $m = 1, \ldots, M_\eta$, there is a field $G/H \ni x \mapsto \psi_{\eta,i,m}(x) \in \mathcal{K}$ such that $\Psi(x) = \sum_{[\eta]\in\hat{H}} \sum_{i=1}^{D_\eta} \sum_{m=1}^{M_\eta} \psi_{\eta,i,m}(x) \otimes e_{\eta,i} \otimes f_{\eta,m}$ for all $x \in G/H$, so that we may assume that $\sum_{[\eta]\in\hat{H}} \sum_{i=1}^{D_\eta} \sum_{m=1}^{M_\eta} \|\psi_{\eta,i,m}(x)\|^2 = \|\Psi(x)\|^2 < \infty$ for all $x \in G/H$. Essentially in the same way as in the proof of Lemma 1, we find that, for any $B \in \mathcal{L}(\mathcal{K})$, $X \in \mathcal{B}(G/H)$, and $\varphi \in \mathcal{D}$,

$$0 = \langle \Psi|\big(B \otimes \mathsf{P}^G_\pi(X)\big)J\varphi\rangle = \int_X \sum_{[\eta]\in\hat{H}} \sum_{i=1}^{D_\eta} \sum_{m=1}^{M_\eta} \langle \psi_{\eta,i,m}(x)|BK_{\eta,i,m}(x)\varphi\rangle \, d\mu(x).$$

Substituting above $B = |\psi\rangle\langle\psi'|$ where ψ, $\psi' \in \mathcal{K}$ and varying $X \in \mathcal{B}(G/H)$, we find that, for any $\varphi \in \mathcal{D}$ and ψ, $\psi' \in \mathcal{K}$, there is a μ-measurable set $N_{\varphi,\psi,\psi'} \subset G/H$ such that $\mu(N_{\varphi,\psi\psi'}) = 0$ and

$$\sum_{[\eta]\in\hat{H}} \sum_{i=1}^{D_\eta} \sum_{m=1}^{M_\eta} \langle \psi_{\eta,i,m}(x)|\psi\rangle\langle\psi'|K_{\eta,i,m}(x)\varphi\rangle = 0 \tag{32}$$

for all $x \in (G/H) \setminus N_{\varphi,\psi,\psi'}$. Let $C_\mathcal{D} \subset \mathcal{D}$ be a countable set which is dense in $\mathcal{D}$ w.r.t. the 1-norm (recall the assumption (b) we made in the beginning of this section) and $C_\mathcal{K} \subset \mathcal{K}$ be a countable set dense in $\mathcal{K}$ w.r.t. the usual Hilbert space topology. Define $N := \bigcup\{N_{\varphi,\psi,\psi'}\,|\,\varphi \in C_\mathcal{D},\ \psi,\ \psi' \in C_\mathcal{K}\}$. Clearly, $\mu(N) = 0$. Pick $\varphi \in \mathcal{D}$, $\psi,\ \psi' \in \mathcal{K}$ and let $(\varphi_r)_{r=1}^\infty \subset C_\mathcal{D}$, $(\psi_r)_{r=1}^\infty \subset C_\mathcal{K}$, and $(\psi'_r)_{r=1}^\infty \subset C_\mathcal{K}$ be sequences such that $\lim_{r\to\infty}\|\varphi-\varphi_r\|_1 = \lim_{r\to\infty}\|\psi-\psi_r\| = \lim_{r\to\infty}\|\psi'-\psi'_r\| = 0$. Using the Pythagorean theorem, the Cauchy-Schwarz inequality and Equation (26), we may easily evaluate, for any $x \in G/H$,

$$
\begin{aligned}
&\left|\sum_{[\eta]\in\hat{H}}\sum_{i=1}^{D_\eta}\sum_{m=1}^{M_\eta}\big(\langle\psi_{\eta,i,m}(x)|\psi\rangle\langle\psi'|K_{\eta,i,m}(x)\varphi\rangle - \langle\psi_{\eta,i,m}(x)|\psi_r\rangle\langle\psi'_r|K_{\eta,i,m}(x)\varphi_r\rangle\big)\right|\\
\le&\|\Psi(x)\|\|\psi-\psi_r\|\|\psi'\|\|\varphi\|_1 + \|\Psi(x)\|\|\psi_r\|\|\psi'-\psi'_r\|\|\varphi\|_1 + \|\Psi(x)\|\|\psi_r\|\|\psi'_r\|\|\varphi-\varphi_r\|_1\\
\le&\|\Psi(x)\|\|\psi-\psi_r\|\|\psi'\|\|\varphi\|_1 + \|\Psi(x)\|\big(\|\psi-\psi_r\|+\|\psi\|\big)\|\psi'-\psi'_r\|\|\varphi\|_1\\
&+\|\Psi(x)\|\big(\|\psi-\psi_r\|+\|\psi\|\big)\big(\|\psi'-\psi'_r\|+\|\psi'\|\big)\|\varphi-\varphi_r\|_1 \overset{r\to\infty}{\to} 0,
\end{aligned}
$$

From this, it immediately follows that, Equation (32) holds for all $\varphi \in \mathcal{D}$, ψ, $\psi' \in \mathcal{K}$, and $x \in (G/H)\setminus N$. Since, for any $x \in G/H$ and $\psi \in \mathcal{K}$, the sequence $\big(\langle\psi|\psi_{\eta,i,m}(x)\rangle\,\big|\,i = 1,\ldots,D_\eta,\ m = 1,\ldots,M_\eta,\ [\eta] \in \hat{H}\big)$ is square summable, and the set of operators $K_{\eta,i,m}(x)$, $[\eta] \in \hat{H}$, $i = 1,\ldots,D_\eta$, $m = 1,\ldots,M_\eta$, is $(\mathcal{K},\mathcal{D})$-weakly independent, it follows that $\langle\psi|\psi_{\eta,i,m}(x)\rangle = 0$ for all $[\eta] \in \hat{H}$, $i = 1,\ldots,D_\eta$, $m = 1,\ldots,M_\eta$, $\psi \in \mathcal{K}$, and $x \in (G/H)\setminus N$. This means that $\psi_{\eta,i,m}(x) = 0$ for all $[\eta] \in \hat{H}$, $i = 1,\ldots,D_\eta$, $m = 1,\ldots,M_\eta$, and $x \in (G/H)\setminus N$. Hence, $\Psi(x) = 0$ for μ-a.a. $x \in G/H$, i.e., $\Psi = 0$ finalizing the proof. □

5.1. Extreme continuous covariant instruments associated with a compact stability subgroup

We next derive necessary and sufficient conditions for a covariant instrument to be extreme within the covariance structure. We also briefly discuss extremality of such instruments in the set of all instruments. The results are analogous to Theorem 4 where we have to replace linear independence with a suitable generalization. Throughout this section, we keep the notations introduced in the beginning of this section fixed and continue to assume that G is locally compact, second countable, and Hausdroff and that $H \le G$ is compact. We also assume that the conditions (a), (b), (c), and (d) stated in the beginning of this section still hold. Extreme instruments of the $(G/H,U,V)$–covariance structure are still defined as the extreme points of the convex set of all $(G/H,U,V)$–covariant instruments and an instrument $\mathcal{I}: \mathcal{B}(G/H)\times\mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K})$ is extreme (with no specifier) if it is an extreme point of the convex set of all instruments with value space $\big(G/H,\mathcal{B}(G/H)\big)$, input Hilbert space $\mathcal{H}$ and output Hilbert space $\mathcal{K}$. Initially (in the following theorem) we are mainly interested in the extreme points of the covariance structure. Note that we define the convex combination $\mathcal{I} = t\mathcal{I}^+ + (1-t)\mathcal{I}^-$ of instruments $\mathcal{I}^\pm: \mathcal{B}(G/H)\times\mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K})$ with $0 \le t \le 1$ through $\mathcal{I}(X,\rho) = t\mathcal{I}^+(X,\rho) + (1-t)\mathcal{I}^-(X,\rho)$ for all $X \in \mathcal{B}(G/H)$ and $\rho \in \mathcal{T}(\mathcal{H})$.

Using Theorem 5 and Lemma 2 and earlier extremality characterizations from [15], we may describe all the extreme instruments of the $(G/H,U,V)$–covariance structure. For this, we make a couple of technical definitions. Pick, for all $[\eta]\in\hat H$, $M_\eta\in\{0,\infty\}\cup\mathbb{N}$, and let $L_{\eta,i,m}:\mathcal{D}\to\mathcal{K}$, $[\eta]\in\hat H$, $i=1,\ldots,D_\eta$, $m=1,\ldots,M_\eta$, constitute a minimal normalized set of $(G/H,U,V)$–intertwiners. Using the Cauchy-Schwarz inequality (in its different forms) and Equation (28), we have, for any $\varphi\in\mathcal{D}$, $[\eta]\in\hat H$, m, $n=1,\ldots,M_\eta$,

$$
\begin{aligned}
&\left|\int_{G/H}\sum_{i=1}^{D_\eta}\langle L_{\eta,i,m}U(g)^*\varphi|L_{\eta,i,n}U(g)^*\varphi\rangle\,d\mu(gH)\right|\\
&\le\int_{G/H}\sum_{i=1}^{D_\eta}\|L_{\eta,i,m}U(g)^*\varphi\|\|L_{\eta,i,n}U(g)^*\varphi\|\,d\mu(gH)\\
&\le\int_{G/H}\sqrt{\sum_{i=1}^{D_\eta}\|L_{\eta,i,m}U(g)^*\varphi\|^2\sum_{j=1}^{D_\eta}\|L_{\eta,j,n}U(g)^*\varphi\|^2}\,d\mu(gH)\\
&\le\sqrt{\int_{G/H}\sum_{i=1}^{D_\eta}\|L_{\eta,i,m}U(g)^*\varphi\|^2\,d\mu(gH)\int_{G/H}\sum_{j=1}^{D_\eta}\|L_{\eta,j,n}U(g)^*\varphi\|^2\,d\mu(gH)}\\
&\le\int_{G/H}\sum_{[\eta]\in\hat H}\sum_{i=1}^{D_\eta}\sum_{r=1}^{M_\eta}\|L_{\eta,i,r}U(g)^*\varphi\|^2\,d\mu(gH)=\|\varphi\|^2,
\end{aligned}
$$

meaning that $\mathcal{D}^2\ni(\varphi,\psi)\mapsto\int_{G/H}\sum_{i=1}^{D_\eta}\langle L_{\eta,i,m}U(g)^*\varphi|L_{\eta,i,n}U(g)^*\psi\rangle\,d\mu(gH)\in\mathbb{C}$ is a bounded sesquilinear form for all $[\eta]\in\hat H$ and m, $n=1,\ldots,M_\eta$ (and, thus, extends to $\mathcal{H}^2$); we denote the corresponding bounded linear operator as

$$\int_{G/H}\sum_{i=1}^{D_\eta}U(g)L_{\eta,i,m}^*L_{\eta,i,n}U(g)^*\,d\mu(gH)\in\mathcal{L}(\mathcal{H}).$$

Moreover, given sets $A\neq\emptyset$ and $B_a\neq\emptyset$ for any $a\in A$, we say that a set consisting of $B_{a,b,c}\in\mathcal{L}(\mathcal{H})$, b, $c\in B_a$, $a\in A$, is strongly independent if, for any decomposable bounded operator $\bigoplus_{a\in A}(\beta_{a,b,c})_{b,c\in B_a}\in\bigoplus_{a\in A}\mathcal{L}(\ell^2_{B_a})\subset\mathcal{L}\left(\bigoplus_{a\in A}\ell^2_{B_a}\right)$, the condition $\sum_{a\in A}\sum_{b,c\in B_a}\beta_{a,b,c}B_{a,b,c}=0$ (where the series is required to converge strongly) implies $\beta_{a,b,c}=0$ for all $a\in A$ and b, $c\in B_a$.

Theorem 6. *Let $\mathcal{I}$ be a $(G/H,U,V)$–covariant instrument defined through Equation (29) by a minimal normalized set of $(G/H,U,V)$–intertwiners consisiting of $L_{\eta,i,m}:\mathcal{D}\to\mathcal{K}$, $[\eta]\in\hat H$, $i=1,\ldots,D_\eta$, $m=1,\ldots,M_\eta$, where $M_\eta\in\{0,\infty\}\cup\mathbb{N}$. This instrument is an extreme instrument of the $(G/H,U,V)$–covariance structure if and only if the set*

$$\left\{\int_{G/H}\sum_{i=1}^{D_\eta}U(g)L_{\eta,i,m}^*L_{\eta,i,n}U(g)^*\,d\mu(gH)\,\middle|\,m,\,n=1,\ldots,M_\eta,\ [\eta]\in\hat H\right\}$$

is strongly independent.

Proof. Let $(L^2_\mu \otimes \mathcal{H}_\pi, \mathsf{P}^G_\pi, U^G_\pi, J)$ be the minimal $(G/H, U, V)$–covariant Stinespring dilation for $\mathcal{I}$ defined by $L_{\eta,i,m}$, $[\eta] \in \hat{H}$, $i = 1, \ldots, D_\eta$, $m = 1, \ldots, M_\eta$ as in Lemma 2. According to [15], $\mathcal{I}$ is an extreme instrument of the $(G/H, U, V)$–covariance structure if and only if, for $E \in \mathcal{L}(L^2_\mu \otimes \mathcal{H}_\pi)$, the conditions $E\mathsf{P}^G_\pi(X) = \mathsf{P}^G_\pi(X)E$ for all $X \in \mathcal{B}(G/H)$, $EU^G_\pi(g) = U^G_\pi(g)E$ for all $g \in G$, and $J^*(\mathbb{1}_\mathcal{K} \otimes E)J = 0$ imply $E = 0$. This is why we next focus on characterizing the intersection of the commutant of the range of P^G_π and that of the range of U^G_π.

Suppose that $E \in \mathcal{L}(L^2_\mu \otimes \mathcal{H}_\pi)$ commutes with P^G_π and U^G_π. The former condition implies that there is a (weakly) μ-measurable field $G/H \ni x \mapsto E(x) \in \mathcal{L}(\mathcal{H}_\pi)$ such that $(EF)(x) = E(x)F(x)$ for all $F \in L^2_\mu \otimes \mathcal{H}_\pi$ and $x \in G/H$. Fix a left Haar measure μ_G for G. Requiring that $EU^G_\pi(g) = U^G_\pi(g)E$ for all $g \in G$ easily yields that, for all $g \in G$, there is $N_g \in \mathcal{B}(G/H)$ such that $\mu(N_g) = 0$ and

$$E(x)\zeta^\pi(g, x) = \zeta^\pi(g, x)E(gx) \tag{33}$$

for all $x \in (G/H) \setminus N_g$.

Denote by N the set of those $(g, x) \in G \times G/H$ such that Equation (33) does not hold. Since $\mathcal{H}_\pi$ is separable, this is easily seen to be a Borel set. Using the Fubini theorem, we get

$$(\mu_G \times \mu)(N) = \int_N d(\mu_G \times \mu) = \int_G \underbrace{\int_{G/H} \chi_N(g, x)\, d\mu(x)}_{=0}\, d\mu_G(g) = 0,$$

implying that Equation (33) holds for $(\mu_G \times \mu)$-a.a. $(g, x) \in G \times G/H$. Using the Fubini theorem for a second time, we find $0 = (\mu_G \times \mu)(N) = \int_N d(\mu_G \times \mu) = \int_{G/H} \int_G \chi_N(g, x)\, d\mu_G(g)\, d\mu(x)$ and, since $\int_G \chi_N(g, x)\, d\mu_G(g) \geq 0$ for all $x \in G/H$, this means that $\int_G \chi_N(g, x)\, d\mu_G(g) = 0$ for μ-a.a. $x \in G/H$. This means that we may pick $x_0 \in G/H$ with the property $\chi_N(g, x_0) = 0$ for μ_G-a.a. $g \in G$. This means that, for μ_G-a.a. $g \in G$,

$$E(x_0)\zeta^\pi(g, x_0) = \zeta^\pi(g, x_0)E(gx_0). \tag{34}$$

Since G is locally compact and second countable, we may assume that the set $Y \in \mathcal{B}(G)$ of those $g \in G$ such that Equation (34) holds (and whose complement is μ_G-null) is a countable union of compact sets, implying that $X := \{gH \mid g \in Y\}$ is a Borel-measurable subset of G/H. The pre-image of $(G/H) \setminus X$ under the factor projection $g \mapsto gH$ is contained within the μ_G-null $G \setminus Y$. Since, according to Corollary V.5.16 of [30], a set $Z \in \mathcal{B}(G/H)$ is μ-null if and only if its pre-image under the factor projection is μ_G-null, we have that $\mu\big((G/H) \setminus X\big) = 0$. It now follows from the above and Equation (34), for all $g \in G$ such that $gs(x_0)^{-1} \in Y$, i.e., for μ_G-a.a. $g \in G$,

$$\begin{aligned}
E(gH) &= E(gs(x_0)^{-1}x_0) = \zeta^\pi\big(gs(x_0)^{-1}, x_0\big)^* E(x_0)\zeta^\pi\big(gs(x_0)^{-1}, x_0\big) \\
&= \Big(\zeta^\pi\big(s(x_0)^{-1}, x_0\big)\zeta^\pi(g, H)\Big)^* E(x_0)\zeta^\pi\big(s(x_0)^{-1}, x_0\big)\zeta^\pi(g, H) \\
&= \zeta^\pi(g, H)^* E_0 \pi(g, H) = \pi\big(g^{-1}s(gH)\big)^* E_0 \pi\big(g^{-1}s(gH)\big)
\end{aligned} \tag{35}$$

where we have denoted $E_0 := \zeta^\pi\big(s(x_0)^{-1}, x_0\big)^* E(x_0)\zeta^\pi\big(s(x_0)^{-1}, x_0\big)$.

Denote by N_1 the μ_G-measurable subset of those $g \in G$ such that Equation (34) does not hold. Since we have, for every $f \in L^1(G)$, $\int_G f\, d\mu_G = \int_{G/H}\int_H f(gh)\, d\mu_H(h)\, d\mu(gH)$, where μ_H is the essentially unique left Haar measure on H, we have

$$0 = \mu_G(N_1) = \int_G \chi_{N_1}\, d\mu_G = \int_{G/H} \underbrace{\int_H \chi_{N_1}(gh)\, d\mu_H(h)}_{\geq 0}\, d\mu(gH),$$

implying that, for μ_G-a.a. $g \in G$ (i.e., for μ-a.a. $gH \in G/H$) $\int_H \chi_{N_1}(gh)\, d\mu_H(h) = 0$. It follows that there is $g_0 \in G$ such that $\chi_{N_1}(g_0 h) = 0$ for μ_H-a.a. $h \in H$. Since $\mu_G(N_1) = 0$, we may assume that $g_0 \in G \setminus N_1$. Thus, we find that, for μ_H-a.a. $h \in H$, $\pi(h)E_0\pi(h)^* = \pi\big(g_0^{-1}s(g_0H)\big)\pi\big(h^{-1}g_0^{-1}s(g_0H)\big)^* E_0 \pi\big(h^{-1}g_0^{-1}s(g_0H)\big)\pi\big(g_0^{-1}s(g_0H)\big)^* = \pi\big(g_0^{-1}s(g_0H)\big)E(g_0H)\pi\big(g_0^{-1}s(g_0H)\big)^* = E_0$ where we have used the fact that $g_0 \in G\setminus N_1$ in the final equality. Using the strong continuity of π, this means that $E_0\pi(h) = \pi(h)E_0$ for all $h \in H$. Using Equation (35), this means that $E(x) = E_0$ for μ-a.a. $x \in G/H$. Thus the intersection of the commutant of the range of P_π^G and that of the range of U_π^G is included within the set of those operators $E \in \mathcal{L}(L^2_\mu \otimes \mathcal{H}_\pi)$ defined by some $E_0 \in \mathcal{L}(\mathcal{H}_\pi)$ commuting with the range of π through $(EF)(x) = E_0F(x)$ for all $F \in L^2_\mu \otimes \mathcal{H}_\pi$ and $x \in G/H$. The converse inclusion is immediate. Thus the intersection we are studying corresponds to the commutant of the range of π.

Let $E \in \mathcal{L}(L^2_\mu \otimes \mathcal{H}_\pi)$ commute with P_π^G and U_π^G and let E_0 be the corresponding operator in the commutant of π. Using the definition $\pi(h) = \bigoplus_{[\eta]\in\hat{H}} \eta(h) \otimes \mathbb{1}_{\mathcal{M}_\eta}$ for all $h \in H$, we find that there is a bounded sequence $\hat{H} \ni [\eta] \mapsto E_\eta \in \mathcal{L}(\mathcal{M}_\eta)$ such that $E(x) = E_0 = \bigoplus_{[\eta]\in\hat{H}} \mathbb{1}_{\mathcal{K}_\eta} \otimes E_\eta$ for μ-a.a. $x \in G/H$. Define, for all $[\eta] \in \hat{H}$, $i = 1, \ldots, D_\eta$, and $m = 1, \ldots, M_\eta$, the isometry $V_{\eta,i,m} : \mathcal{K} \to \mathcal{K} \otimes \mathcal{H}_\pi$ as earlier. Denoting $\beta_{\eta,m,n} := \langle f_{\eta,m}|E_\eta f_{\eta,n}\rangle$ for all $[\eta] \in \hat{H}$ and $m, n = 1, \ldots, M_\eta$, we find that, for any $\varphi \in \mathcal{D}$,

$$\begin{aligned}
&\langle J\varphi|(\mathbb{1}_\mathcal{K} \otimes E)J\varphi\rangle = \int_{G/H} \langle (J\varphi)(x)|(\mathbb{1}_\mathcal{K} \otimes E_0)(J\varphi)(x)\rangle\, d\mu(x)\\
&= \int_{G/H} \sum_{[\eta],[\vartheta]\in\hat{H}} \sum_{i=1}^{D_\eta}\sum_{j=1}^{D_\vartheta}\sum_{m=1}^{M_\eta}\sum_{n=1}^{M_\vartheta} \langle V^*_{\eta,i,m}(J\varphi)(x)|\underbrace{V^*_{\eta,i,m}(\mathbb{1}_\mathcal{K} \otimes E_0)V_{\vartheta,j,n}}_{=\langle f_{\eta,m}|E_\eta f_{\eta,n}\rangle \delta_{[\eta],[\vartheta]}\delta_{i,j}} V^*_{\vartheta,j,n}(J\varphi)(x)\rangle\, d\mu(x)\\
&= \sum_{[\eta]\in\hat{H}} \sum_{m,n=1}^{M_\eta} \langle f_{\eta,m}|E_\eta f_{\eta,n}\rangle \int_{G/H} \sum_{i=1}^{D_\eta} \langle K_{\eta,i,m}(x)\varphi|K_{\eta,i,n}(x)\varphi\rangle\, d\mu(x)\\
&= \sum_{[\eta]\in\hat{H}} \sum_{m,n=1}^{M_\eta} \beta_{\eta,m,n} \int_{G/H} \sum_{i=1}^{D_\eta} \langle L_{\eta,i,m}U(g)^*\varphi|L_{\eta,i,n}U(g)^*\varphi\rangle\, d\mu(gH)
\end{aligned}$$

where the final equality follows in a straight-forward way from the definition of the operators $K_{\eta,i,m}(x)$ as they appear in Lemma 2 and $\sum_{i=1}^{D_\eta} \overline{\zeta^\eta_{i,j}(g^{-1}, gH)}\zeta^\eta_{i,k}(g^{-1}, gH) = \delta_{j,k}$ for all $[\eta] \in \hat{H}$, $j, k = 1, \ldots, D_\eta$, and $g \in G$. Noting that the set of decomposable bounded operators in $\bigoplus_{[\eta]\in\hat{H}}\ell^2_{B_\eta}$, where B_η is the set of indices $m=1,\ldots,M_\eta$ for any $[\eta]\in\hat{H}$, coincides with the set of $\bigoplus_{[\eta]\in\hat{H}}\big(\langle f_{\eta,m}|E_\eta f_{\eta,n}\rangle\big)_{m,n=1}^{M_\eta}$, where $\hat{H}\ni[\eta]\mapsto E_\eta\in\mathcal{L}(\mathcal{M}_\eta)$ is a bounded sequence, the claim now follows from the extremality characterization stated at the beginning of this proof. □

Remark 3. Recall the stricter conditions for an instrument to being extreme (i.e., an extreme point within the set of all instruments with the same value space and input and output Hilbert spaces) made in the beginning of this subsection. Let $\mathcal{I}$ be a $(G/H,U,V)$–covariant instrument defined through Equation (29) by a minimal set of $(G/H,U,V)$–intertwiners $L_{\eta,i,m}$, $[\eta]\in\hat{H}$, $i=1,\ldots,D_\eta$, $m=1,\ldots,M_\eta$. For brevity, let us denote the set of indices (η,i,m), where $[\eta]\in\hat{H}$, $i=1,\ldots,D_\eta$, and $m=1,\ldots,M_\eta$, by B. Using the minimal Stinespring dilation of Lemma 2 and an earlier extremality characterization given in [29] and recalling the section $s:G/H\to G$, we find that the above $\mathcal{I}$ is an extreme instrument if and only if, for a family $\{f^\beta_\gamma\}_{\beta,\gamma\in B}\subset L^\infty_\mu$ such that $G/H\ni x\mapsto\big(f^\beta_\gamma(x)\big)_{\beta,\gamma\in B}\in\mathcal{L}(\ell^2_B)$ is μ-essentially bounded, the condition

$$\int_{G/H}\sum_{\beta,\gamma\in B}f^\beta_\gamma(x)\langle L_\beta(U\circ s)(x)^*\varphi|L_\gamma(U\circ s)(x)^*\varphi\rangle\,d\mu(x)=0$$

for all $\varphi\in\mathcal{D}$ implies $f^\beta_\gamma(x)=0$ for all $\beta,\gamma\in B$ and μ-a.a. $x\in G/H$. This fact is proven in Appendix D. △

Let us give an extremality condition which is particularly convenient when the input representation U is irreducible. We formulate this result, not using minimal intertwiners, but using a minimal covariant dilation of a $(G/H,U,V)$–covariant instrument into a canonical system of imprimitivity. Note that we do not have assume that H is compact.

Proposition 1. *Let $\mathcal{I}$ be a $(G/H,U,V)$–covariant instrument and let $\pi:H\to\mathcal{U}(\mathcal{H}_\pi)$ be a strongly continuous unitary representation, where $\mathcal{H}_\pi$ is separable, and $J:\mathcal{H}\to\mathcal{K}\otimes L^2_\mu\otimes\mathcal{H}_\pi$ be an isometry such that $(L^2_\mu\otimes\mathcal{H}_\pi,\mathsf{P}_\pi,U_\pi,J)$ is a minimal $(G/H,U,V)$–covariant Stinespring dilation for $\mathcal{I}$. If π is irreducible, then $\mathcal{I}$ is an extreme instrument of the $(G/H,U,V)$–covariance structure. If U is irreducible, also the converse claim holds.*

Proof. For the duration of this proof, define the map $\mathcal{L}(\mathcal{H}_\pi)\ni E\mapsto E^\bullet\in\mathcal{L}(L^2_\mu\otimes\mathcal{H}_\pi)$ through $(E^\bullet f)(x)=Ef(x)$ for all $E\in\mathcal{L}(\mathcal{H}_\pi)$, $f\in L^2_\mu\otimes\mathcal{H}_\pi$, and $x\in G/H$. Suppose first that π is irreducible. This means that the commutant $(\mathrm{ran}\,\pi)'$ of the range of π is $\mathbb{C}\mathbb{1}_{\mathcal{H}_\pi}$. The commutant $(\mathrm{ran}\,U_\pi)'$ of the range of U_π is, according to the proof of Theorem 6, the image of $(\mathrm{ran}\,\pi)'$ under the map $E\mapsto E^\bullet$. Clearly, this means that $(\mathrm{ran}\,U_\pi)'=\mathbb{C}\mathbb{1}_{L^2_\mu\otimes\mathcal{H}_\pi}$. (This just means that, when π is irreducible, then also U_π is irreducible which is well known.) Obviously, the map $(\mathrm{ran}\,U_\pi)'\ni D\mapsto J^*(\mathbb{1}_\mathcal{K}\otimes D)J\in\mathcal{L}(\mathcal{H})$ is now injective, meaning that $\mathcal{I}$ is an extreme instrument of the $(G/H,U,V)$–covariance structure.

Suppose then that U is irreducible and $\mathcal{I}$ is an extreme instrument of the $(G/H,U,V)$–covariance structure. Using the intertwining property $JU(g)=\big(V(g)\otimes$

$U_\pi(g)\big)J$ for all $g\in G$ and the fact that $(\mathrm{ran}\,U_\pi)'$ is the image of $(\mathrm{ran}\,\pi)'$ under $E\mapsto E^\bullet$, it easily follows that $U(g)J^*(\mathbb{1}_\mathcal{K}\otimes E^\bullet)J=J^*(\mathbb{1}_\mathcal{K}\otimes E^\bullet)JU(g)$ for all $g\in G$ and $E\in(\mathrm{ran}\,\pi)'$, implying that, for all $E\in(\mathrm{ran}\,\pi)'$, there is $z(E)\in\mathbb{C}$ such that $J^*(\mathbb{1}_\mathcal{K}\otimes E^\bullet)J=z(E)\mathbb{1}_\mathcal{H}$, i.e., $0=J^*\mathbb{1}_\mathcal{K}\otimes\big(E^\bullet-z(E)\mathbb{1}_{L^2_\mu\otimes\mathcal{H}_\pi}\big)J=J^*\mathbb{1}_\mathcal{K}\otimes\big(E-z(E)\mathbb{1}_{\mathcal{H}_\pi}\big)^\bullet J$. Since $\mathcal{I}$ is an extreme instrument of the $(G/H,U,V)$–covariance structure, the extremality condition given in [15] (which has also appeared in the proof of Theorem 6) implies that $E=z(E)\mathbb{1}_{\mathcal{H}_\pi}$ for all $E\in(\mathrm{ran}\,\pi)'$, i.e., $(\mathrm{ran}\,\pi)'=\mathbb{C}\mathbb{1}_{\mathcal{H}_\pi}$ meaning that π is irreducible. □

Remark 4. Let, for each $[\eta]\in\hat H$, $M_\eta\in\{0,\infty\}\cup\mathbb{N}$, and let $L_{\eta,i,m}:\mathcal{D}\to\mathcal{K}$, $[\eta]\in\hat H$, $i=1,\ldots,D_\eta$, $m=1,\ldots,M_\eta$, constitute a minimal set of $(G/H,U,V)$–intertwiners. Define, for all $[\eta]\in\hat H$ and $i=1,\ldots,D_\eta$, the isometry $V_{\eta,i}:\mathcal{K}\to\mathcal{K}_\eta\otimes\eta$ through $V_{\eta,i}\psi=e_{\eta,i}\otimes\psi$ for all $\psi\in\mathcal{K}$. This allows us to define the operators $B_{\eta,m}:\mathcal{D}\to\mathcal{K}_\eta\otimes\mathcal{K}$ for all $[\eta]\in\hat H$ and $m=1,\ldots,M_\eta$ through

$$B_{\eta,m}=\sum_{i=1}^{D_\eta}V_{\eta,i}L_{\eta,i,m}.$$

Thus, $L_{\eta,i,m}=V_{\eta,i}^*B_{\eta,m}$ for all $[\eta]\in\hat H$, $i=1,\ldots,D_\eta$, and $m=1,\ldots,M_\eta$ and one easily finds that

$$B_{\eta,m}U(h)=\big(\eta(h)\otimes V(h)\big)B_{\eta,m},\qquad [\eta]\in\hat H,\quad m=1,\ldots,M_\eta,\quad h\in H.\tag{36}$$

This intertwining property can often be easier to verify that the property of Equation (25) using Clebsch-Gordan methods.

The instrument defined by the intertwiners $L_{\eta,i,m}$, $[\eta]\in\hat H$, $i=1,\ldots,D_\eta$, $m=1,\ldots,M_\eta$, (whenever this set is normalized) is an extreme instrument of the $(G/H,U,V)$–covariance structure if and only if the set

$$\left\{\int_G U(g)B_{\eta,m}^*B_{\eta,n}U(g)^*\,d\mu_G(g)\,\middle|\,m,\,n=1,\ldots,M_\eta,\ [\eta]\in\hat H\right\}$$

is strongly independent. The operators

$$\int_G U(g)B_{\eta,m}^*B_{\eta,n}U(g)^*\,d\mu_G(g)=\int_{G/H}U(g)B_{\eta,m}^*B_{\eta,n}U(g)^*\,d\mu(gH)$$

are defined in the same way as the integrated operators in the claim of Theorem 6. The above equality follows from Equation (36) upon choosing μ so that the associated left Haar measure μ_H of H (i.e., the left Haar measure of H such that $\int_G f\,d\mu_G=\int_{G/H}\int_H f(gh)\,d\mu_H(h)\,d\mu(gH)$ for all $f\in L^1(G)$) is normalized, i.e., $\mu_H(H)=1$. Similarly, we have

$$\int_{G/H}\sum_{i=1}^{D_\eta}U(g)L_{\eta,i,m}^*L_{\eta,i,n}U(g)^*\,d\mu(gH)=\int_G\sum_{i=1}^{D_\eta}U(g)L_{\eta,i,m}^*L_{\eta,i,n}U(g)^*\,d\mu_G(g)$$

for all $[\eta]\in\hat H$ and $m,\,n=1,\ldots,M_\eta$ which can be substituted in the claim of Theorem 6. In particular, these operators commute with the representation U. △

Example 3. We finally study the case of covariant phase space measurements and the corresponding instruments. The pre-measurement system is a quantum system with N degrees of freedom and associated with the Hilbert space $L^2(\mathbb{R}^N)$ and the post-measurement system has N' degrees of freedom and is associated with the Hilbert space $L^2(\mathbb{R}^{N'})$ in the position representation. The position shifts act on the states by shifting the argument of a state vector associated with a pure state, i.e., through the unitary representation $U_N : \mathbb{R}^N \to \mathcal{U}\big(L^2(\mathbb{R}^N)\big)$, $\big(U_N(\vec{q})\varphi\big)(\vec{x}) = \varphi(\vec{x}-\vec{q})$ for all $\vec{q} \in \mathbb{R}^N$, $\varphi \in L^2(\mathbb{R}^N)$, and a.a. $\vec{x} \in \mathbb{R}^N$. The momentum boosts are hence associated with the unitary representation $V_N : \mathbb{R}^N \to \mathcal{U}\big(L^2(\mathbb{R}^N)\big)$, $V_N(\vec{p}) = \mathcal{F}^* U_N(\vec{p})\mathcal{F}$ for all $\vec{p} \in \mathbb{R}^N$, where $\mathcal{F}$ is the unitary Fourier transform operator, i.e., for all $\vec{p} \in \mathbb{R}^N$, $\varphi \in L^2(\mathbb{R}^N)$, and a.a. $\vec{x} \in \mathbb{R}^N$, $\big(V_N(\vec{p})\varphi\big)(\vec{x}) = e^{i\vec{x}^T\vec{p}}\varphi(\vec{x})$. By defining

$$W_N(\vec{q},\vec{p}) := e^{\frac{i}{2}\vec{q}^T\vec{p}} U_N(\vec{q})V_N(\vec{p}), \qquad \vec{q},\,\vec{p} \in \mathbb{R}^N,$$

we are able to encapsulate position shifts and momentum boosts into phase space translations giving rise to a projective unitary representation $W_N : \mathbb{R}^N \times \mathbb{R}^N \to \mathcal{U}\big(L^2(\mathbb{R}^N)\big)$. Indeed, one easily checks that, upon defining the $(2N \times 2N)$-matrix

$$S_N := \begin{pmatrix} 0 & \mathbb{1}_N \\ -\mathbb{1}_N & 0 \end{pmatrix}$$

in the block form and denoting the phase space points by $\vec{z} = (\vec{q},\vec{p}) \in \mathbb{R}^{2N}$, we have

$$W_N(\vec{z}+\vec{w}) = e^{\frac{i}{2}\vec{z}^T S_N \vec{w}} W_N(\vec{z})W_N(\vec{w}), \qquad \vec{z},\,\vec{w} \in \mathbb{R}^N. \tag{37}$$

This projective representation is called as the *Weyl representation.* In quantum optics literature, the operators $W_N(\vec{z})$, $\vec{z} \in \mathbb{R}^N$, are associated to the *displacement operators.*

Let us next introduce the *Weyl-Heisenberg group* H_N which coincides, as a set, with $\mathbb{R}^{2N} \times \mathbb{T}$ and whose group law is given by

$$(\vec{z},s)(\vec{w},t) = (\vec{z}+\vec{w}, st e^{-i\vec{z}^T S_N \vec{w}}), \qquad \vec{z},\,\vec{w} \in \mathbb{R}^{2N}, \quad s,\,t \in \mathbb{T}.$$

Let us also define the map $D_N : H_N \to \mathcal{U}\big(L^2(\mathbb{R}^N)\big)$ through

$$D_N(\vec{z},s) = \overline{s} W_N(\vec{z}), \qquad \vec{z} \in \mathbb{R}^{2N}, \quad s \in \mathbb{T}.$$

Using Equation (37), one easily sees that D_N is an ordinary strongly continuous unitary representation. In fact, H_N can be seen as a central extension of the additive group $\mathbb{R}^{2N}$ by the multiplier $(\vec{z},\vec{w}) \mapsto e^{-i\vec{z}^T S_N \vec{w}}$ and D_N as the lifting of the Weyl representation W_N to H_N.

Let Y be a real $(2N \times 2N')$–matrix such that $Y^T S_{N'} Y = S_N$. We let $U = D_N$ and define $V : H_N \to \mathcal{U}\big(L^2(\mathbb{R}^{N'})\big)$ through $V(\vec{w},s) := D_{N'}(Y\vec{w},s)$ for all $\vec{w} \in \mathbb{R}^{2N}$ and $s \in \mathbb{T}$. One may easily check that V is an ordinary unitary representation as well. The value space of the measurements we are interested in is $\mathbb{R}^{2N}$, so that the stability subgroup is $H := \{0\} \times \mathbb{T}$. Since the restrictions $U|_H$ and $V|_H$ coincide and have values in the respective centres of $\mathcal{L}\big(L^2(\mathbb{R}^N)\big)$ and $\mathcal{L}\big(L^2(\mathbb{R}^{N'})\big)$, the intertwining property of Equation (6) becomes irrelevant. Moreover, there is only one $\eta \in \hat{H}$ (the trivial one) appearing in this scenario. This means that the relevant (minimal) sets of $(\mathbb{R}^{2N}, U, V)$–intertwiners are (weakly independent) sets $\{L_m\}_{m=1}^M \subset \mathcal{L}\big(L^2(\mathbb{R}^N), L^2(\mathbb{R}^{N'})\big)$, with $M \in \mathbb{N} \cup \{\infty\}$, of Hilbert-Schmidt operators such that $\sum_{m=1}^M \mathrm{tr}\,[L_m^* L_m] < \infty$; this set of intertwiners is normalized if and only if $\sum_{m=1}^M \mathrm{tr}\,[L_m^* L_m] = \pi^{-N}$. Indeed, perusing [25] and Section 6.1 of [15], we see that, in item (a) of the beginning of this section, we may choose $\mathcal{D} = L^2(\mathbb{R}^N)$ and, in item (b), $\|\cdot\|_1$ can be chosen as the ordinary Hilbert norm so that the intertwiners are simply bounded operators. The Hilbert-Schmidt property follows from the square-integrability of U, i.e., for all unit vectors φ, $\psi \in L^2(\mathbb{R}^N)$,

$$\int_{\mathbb{T}} \int_{\mathbb{R}^{2N}} |\langle \varphi | U(\vec{z}, s)\psi\rangle|^2 \, d\vec{z}\, ds = \int_{\mathbb{R}^{2N}} |\langle \varphi | W_N(\vec{z})\psi\rangle|^2 \, d\vec{z} = \pi^N$$

which, in turn, implies, according to Lemma 2 of [25] that, for positive $A \in \mathcal{L}\big(L^2(\mathbb{R}^N)\big)$ and $T \in \mathcal{T}\big(L^2(\mathbb{R}^N)\big)$, the function $\mathbb{R}^{2N} \ni \vec{z} \mapsto \mathrm{tr}\,[W_N(\vec{z}) T W_N(\vec{z})^* A]$ is Lebesgue-integrable if and only if $A \in \mathcal{T}\big(L^2(\mathbb{R}^N)\big)$ in which case $\int_{\mathbb{R}^{2N}} \mathrm{tr}\,[W_N(\vec{z}) T W_N(\vec{z})^* A]\, d\vec{z} = \pi^N \mathrm{tr}\,[T]\,\mathrm{tr}\,[A]$.

We say that an instrument [or a QOVM, in general] $\mathcal{I} : \mathcal{B}(\mathbb{R}^{2N}) \times \mathcal{T}\big(L^2(\mathbb{R}^N)\big) \to \mathcal{T}\big(L^2(\mathbb{R}^{N'})\big)$ is a *covariant phase space instrument [QOVM]* if it is $(\mathbb{R}^{2N}, U, V)$–covariant, i.e., for all $\vec{z} \in \mathbb{R}^{2N}$, $X \in \mathcal{B}(\mathbb{R}^{2N})$, and $\rho \in \mathcal{S}\big(L^2(\mathbb{R}^N)\big)$,

$$\mathcal{I}\big(X + \vec{z}, W_N(\vec{z})\rho W_N(\vec{z})^*\big) = W_{N'}(Y\vec{z}) \mathcal{I}(X, \rho) W_{N'}(Y\vec{z})^*.$$

For any covariant phase space instrument [QOVM] $\mathcal{I}$ there is $M \in \mathbb{N} \cup \{\infty\}$ and a minimal [normalized] set $\{L_m\}_{m=1}^M$ of $(\mathbb{R}^{2N}, U, V)$–intertwiners like those above such that

$$\mathcal{I}(X, \rho) = \int_X \sum_{m=1}^M W_{N'}(Y\vec{z}) L_m W_N(\vec{z})^* \rho W_N(\vec{z}) L_m^* W_{N'}(Y\vec{z})^* \, d\vec{z}$$

for all $X \in \mathcal{B}(\mathbb{R}^{2N})$ and $\rho \in \mathcal{S}\big(L^2(\mathbb{R}^N)\big)$. The observable [POVM] measured by $\mathcal{I}$ is easily seen to coincide with M_S,

$$\mathsf{M}_S(X) = \frac{1}{\pi^N} \int_X W_N(\vec{z}) S W_N(\vec{z})^* \, d\vec{z}, \qquad X \subseteq \mathbb{R}^{2N} \text{ (measurable)},$$

defined by $S = \pi^N \sum_{m=1}^M L_m^* L_m \in \mathcal{T}\big(L^2(\mathbb{R}^N)\big)$. Moreover, in the normalized case, this covariant phase space instrument $\mathcal{I}$ is an extreme point of the $(\mathbb{R}^{2N}, U, V)$–covariance structure if and only if $M = 1$. Indeed, if $M = 1$, extremality follows immediately from Theorem 6. If, on the other hand, $M > 1$, then, using Lemma 2 of [25], we have that $\int_{\mathbb{R}^{2N}} W_N(\vec{z}) L_m^* L_n W_N(\vec{z})\, d\vec{z}$ is a multiple of the identity for $1 \leq m, n \leq M$. According to Theorem 6, $\mathcal{I}$ cannot be an extreme instrument of the $(\mathbb{R}^{2N}, U, V)$–covariance structure.

According to Remark 3, a covariant phase space instrument $\mathcal{I}$ associated with the normalized set intertwiners L_m, $m = 1, \ldots, M \in \mathbb{N} \cup \{\infty\}$ is an extreme instrument if and only if, for $\{f_{m,n}\}_{m,n=1}^M \subset L^\infty(\mathbb{R}^{2N})$ such that $\mathbb{R}^{2N} \ni \vec{z} \mapsto \big(f_{m,n}(\vec{z})\big)_{m,n=1}^M \in \mathcal{L}(\ell^2_{\mathbb{N}_M})$

(where $\mathbb{N}_M$ is the set of indices $m = 1, \ldots, M$) is an essentially bounded field, the condition

$$\int_{\mathbb{R}^{2N}} \sum_{m,n=1}^{M} f_{m,n}(\vec{z}) W_N(\vec{z}) L_m^* L_n W_N(\vec{z})^* \, d\vec{z} = 0$$

implies $f_{m,n} = 0$ for all $m, n = 1, \ldots, M$. However, this extremality characterization is greatly simplified recalling that an extreme instrument is also an extreme instrument of the convex subset of covariant phase space instruments and thus only has one intertwiner, i.e., $M = 1$. This can also be proven directly: Assume that the covariant phase space instrument associated with the minimal set $\{L_m\}_{m=1}^M$ of intertwiners is an extreme instrument. We make the counter assumption that $M \geq 2$, so that L_1 and L_2 are non-zero, implying that $\|L_1\|_{HS} \neq 0 \neq \|L_2\|_{HS}$ where $\|K\|_{HS} = \sqrt{\mathrm{tr}\,[K^* K]}$ is the Hilbert-Schmidt norm of the Hilbert-Schmidt operator K. Let us define the constant functions $f_{1,1} \equiv \|L_1\|_{HS}^{-2}$, $f_{2,2} \equiv -\|L_2\|_{HS}^{-2}$, and $f_{m,n} \equiv 0$ otherwise for $m, n = 1, \ldots, M$. Using Lemma 2 of [25], it easily follows that

$$\int_{\mathbb{R}^{2N}} \sum_{m,n=1}^{M} f_{m,n}(\vec{z}) W_N(\vec{z}) L_m^* L_n W_N(\vec{z})^* \, d\vec{z} = 0 \quad \Longrightarrow \quad (f_{m,n})_{m,n=1}^{M} \equiv 0,$$

where the final implication following from the extremality characterization clearly does not hold. Thus, $M = 1$. It finally follows that a covariant phase space instrument $\mathcal{I}$ is an extreme instrument if and only if (any) minimal set of intertwiners associated with $\mathcal{I}$ is a singleton $\{L\}$ and, for any $f \in L^\infty(\mathbb{R}^{2N})$,

$$\int_{\mathbb{R}^{2N}} f(\vec{z}) W_N(\vec{z}) L^* L W_N(\vec{z})^* \, d\vec{z} = 0 \quad \Longrightarrow \quad f \equiv 0.$$

We note that a covariant phase space instrument is an extreme instrument if and only if its pointwise Kraus rank [29] is 1 and the covariant phase space observable it measures is an extreme POVM [16]. △

6. Conclusions

In this work we have presented a comprehensive study of covariant quantum measurements studied in the form of POVMs and instruments. We have given exhaustive characterizations for these covariant measurement devices and for their extremality properties. In particular, in Examples 1 and 2, we have introduced a parametrized family $\{\mathsf{M}^\alpha\}_{\alpha \geq 0}$ of POVMs covariant w.r.t. the symmetric group S_D in dimension D where M^0 is a rank-1 PVM and, whenever $\alpha > 0$, M^α is extreme (within the set of all POVMs) rank-1 informationally complete POVM. Since being a rank-1 PVM and a rank-1 extreme informationally complete POVM are complementary properties for optimal quantum observables according to [16], we observe the remarkable fact that these complementary classes are just a ‘small deviation’ away from each other in the sense that even a small positive value of α produces a POVM in the second optimality class whereas M^0 is firmly in the first class.

There are several questions that remain to be studied in the field of symmetric quantum measurements. Recall the definitions of modes of optimality made in Section 2. The post-processing-clean observables have been identified in [16] as the rank-1 observables. Since it might happen that there is no rank-1 covariant POVM, it is reasonable to study the maximality w.r.t. the post-processing pre-order restricted to the class of $(\mathbb{X},U)$–covariant observables where the G-space $\mathbb{X}$ may vary. Without restricting generality, we may assume that the probability matrices involved are G-equivariant. Indeed, suppose that $\mathbb{X}$ and $\mathbb{Y}$ are G-spaces and M [resp. N] is a $(\mathbb{X},U)$–covariant [resp. $(\mathbb{Y},U)$–covariant] observable such that $\mathsf{M}_x = \sum_{y\in\mathbb{Y}} p'_{x|y}\mathsf{N}_y$ for some probability matrix $(p'_{x|y})$. Define the probability matrix $p_{x|y} := (\#G)^{-1}\sum_{g\in G} p'_{gx|gy}$ which is equivariant: $p_{x|gy} = p_{g^{-1}x|y}$. Since $\mathsf{M}_x = U(g)^*\mathsf{M}_{gx}U(g) = \sum_{y\in\mathbb{Y}} p'_{gx|y}U(g)^*\mathsf{N}_yU(g) = \sum_{y'\in\mathbb{Y}} p'_{gx|gy'}U(g)^*\mathsf{N}_{gy'}U(g) = \sum_{y'\in\mathbb{Y}} p'_{gx|gy'}\mathsf{N}_{y'}$ one gets $\sum_{y\in\mathbb{Y}} p_{x|y}\mathsf{N}_y = (\#G)^{-1}\sum_{g\in G}\sum_{y'\in\mathbb{Y}} p'_{gx|gy'}\mathsf{N}_{y'} = \mathsf{M}_x$. Another important problem arises in the case where there are no rank-1 covariant POVMs: Might it happen that the only covariant instruments measuring a covariant observable M are nuclear (i.e. determine the future) although M is not of rank 1? Without the requirement of covariance, an observable determines the future if and only if it is of rank 1, implying that post-processing maximality and determination of the future are identical properties. Whether this result also holds for the respective optimality properties restricted to covariance structures is still an open problem.

Determination of the past, i.e. informational completeness, is often closely tied to covariance. Indeed, most of the relevant informationally complete observables, e.g. the covariant phase space observable generated by the vacuum, arise from covariance structures. However, it remains to be determined under which conditions does a covariance structure contain informationally complete observables. Similarly, whether a covariance structure allows a sharp observable is an interesting question which, however, has been solved in the case of an Abelian symmetry group [14, 19].

Recall that an observable M determines its values if $\|\mathsf{M}_x\| = 1$ for all outcomes x; this is called as the norm-1 property and follows easily from the definition of value determination. Value determination within covariance structures is a further valid avenue of research. In [16], it was shown that value determination is related to (although not exactly the same as) pre-processing purity: an observable $\mathsf{M} = (\mathsf{M}_x)_x$ is pre-processing pure if and only if, $\mathsf{M}_x = \mathsf{P}_x \oplus \mathsf{E}_x$ where $(\mathsf{P}_x)_{x\in\mathbb{X}}$ is a sharp observable in a subspace $\mathcal{H}_0 \subseteq \mathcal{H}$ and $(\mathsf{E}_x)_{x\in\mathbb{X}}$ is some other observable in the orthogonal complement of $\mathcal{H}_0$ [16]. Within a covariance structure, we can restrict the quantum noise arising from pre-processing into covariant channels: If $\mathsf{M}_x = \Phi^*(\mathsf{N}_x)$ where M [resp. N] is $(\mathbb{X},U)$–covariant [resp. $(\mathbb{X},V)$–covariant] then $\mathsf{M}_x = \tilde\Phi^*(\mathsf{N}_x)$ where the covariant channel $\tilde\Phi$ is defined by $\tilde\Phi(\rho) = (\#G)^{-1}\sum_{g\in G} V(g)^*\Phi\big(U(g)\rho U(g)^*\big)V(g)$. How to characterize the pre-processing-clean observables in this restricted form of covariant pre-processing is left as a future research problem.

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Appendix A. 2-cocycles for finite groups

Fix a finite group G and let $m : G \times G \to \mathbb{T}$ be a 2-cocycle, i.e. it satisfies the cocycle condition $m(g, hk)m(h, k) \equiv m(gh, k)m(g, h)$. Define a function

$$t(g) := \prod_{h \in G} m(g, h) \in \mathbb{T}$$

so that, for all $g,\, h \in G$,

$$\frac{t(g)t(h)}{t(gh)} = \prod_{k \in G} \frac{m(g, hk)m(h, k)}{m(gh, k)} = m(g, h)^{\#G}.$$

Hence, we have the least positive integer $p \leq \#G$ such that $m(g, h)^p \equiv t'(g)t'(h)/t'(gh)$ for some function $t' : G \to \mathbb{T}$. Write $t'(g) = e^{ip\varphi(g)}$ where φ is real valued and define a new 2-cocycle m' via $m'(g, h) := e^{i\varphi(gh)}e^{-i\varphi(g)}e^{-i\varphi(h)}m(g, h)$. Hence, $m'(g, h)^p \equiv 1$. By defining a 2-cocycle $m''(g, h) := m'(g, h)/m'(e, e)$ we also have $m''(g, h)^p \equiv 1$ and, in addition, $m''(e, e) = 1$.

One can replace the projective unitary representation $g \mapsto U(g)$ with the new projective unitary representation $U'(g) := m'(e, e)e^{i\varphi(g)}U(g)$. Indeed, $U(gh) = m(g, h)U(g)U(h)$ implies $U'(gh) = m''(g, h)U'(g)U'(h)$. Furthermore, the covariance condition $\mathsf{M}_{gx} = U(g)\mathsf{M}_x U(g)^*$ equals with $\mathsf{M}_{gx} = U'(g)\mathsf{M}_x U'(g)^*$ so that, without restricting generality, we may assume that the multiplier m of U satisfies $m(e, e) = 1$ and $m(g, h)^p \equiv 1$ for some (minimal) integer $p > 0$.

Appendix B. Covariant Stinespring dilations

Let us make the same assumptions as in Section 3 and fix an $(\mathbb{X}, U, V)$–covariant instrument [or a QOVM, in general] $\mathcal{I} = (\mathcal{I}_x)_{x \in \mathbb{X}}$ and a minimal Stinespring dilation $(\mathcal{M}, \mathsf{P}, J)$ for $\mathcal{I}$. We first show that there is a unitary representation $\overline{U} : G \to \mathcal{U}(\mathcal{M})$ such that $JU(g) = \big(V(g) \otimes \overline{U}(g)\big)J$ for all $g \in G$. In the sequel, we denote, for all $Y \subseteq \mathbb{X}$, $\mathcal{I}_Y := \sum_{x \in Y} \mathcal{I}_x$. Let us pick $n \in \mathbb{N}$, $B_1, \ldots, B_n \in \mathcal{L}(\mathcal{K})$, $x_1, \ldots, x_n \in \mathbb{X}$, and $\varphi_1, \ldots, \varphi_n \in \mathcal{H}$ and define $\xi := \sum_{i=1}^n (B_i \otimes \mathsf{P}_{x_i})J\varphi_i$ and $\xi_g := \sum_{i=1}^n (B_i V(g)^* \otimes \mathsf{P}_{gx_i})JU(g)\varphi_i$ for all

$g \in G$. Using the $(\mathbb{X}, U, V)$–covariance, we have

$$
\begin{aligned}
\|\xi_g\|^2 &= \sum_{i,j=1}^{n} \langle JU(g)\varphi_i | \big(V(g)B_i^* B_j V(g)^* \otimes \mathsf{P}_{gx_i}\mathsf{P}_{gx_j}\big) JU(g)\varphi_j \rangle \\
&= \sum_{i,j=1}^{n} \langle U(g)\varphi_i | \mathcal{I}^*_{\{gx_i\}\cap\{gx_j\}}\big(V(g)B_i^* B_j V(g)^*\big) U(g)\varphi_j \rangle \\
&= \sum_{i,j=1}^{n} \langle \varphi_i | \mathcal{I}^*_{\{x_i\}\cap\{x_j\}}(B_i^* B_j)\varphi_j \rangle = \|\xi\|^2
\end{aligned}
$$

for all $g \in G$. The minimality of $(\mathcal{M}, \mathsf{P}, J)$ implies that we may define, for each $g \in G$, a unique isometry $\tilde{U}(g) \in \mathcal{L}(\mathcal{K} \otimes \mathcal{M})$ such that $\tilde{U}(g)(B \otimes \mathsf{P}_x)J = \big(BV(g)^* \otimes \mathsf{P}_{gx}\big)JU(g)$ for all $B \in \mathcal{L}(\mathcal{K})$ and $x \in \mathbb{X}$. It is easily checked (using again the minimality) that $\tilde{U}(gh) = \tilde{U}(g)\tilde{U}(h)$ for all $g,\, h \in G$ from whence it easily follows that $\tilde{U} : G \to \mathcal{U}(\mathcal{K}\otimes\mathcal{M})$ is a unitary representation.

Let $\xi \in \mathcal{K} \otimes \mathcal{M}$ be as above and pick $g \in G$ and $B \in \mathcal{L}(\mathcal{K})$. Using covariance, we get

$$
\begin{aligned}
\langle \xi | \tilde{U}(g)(B \otimes \mathbb{1}_{\mathcal{M}})\xi \rangle &= \sum_{i,j=1}^{n} \langle (B_i \otimes \mathsf{P}_{x_i})J\varphi_i | (BB_j V(g)^* \otimes \mathsf{P}_{gx_j})JU(g)\varphi_j \rangle \\
&= \sum_{i,j=1}^{n} \langle \varphi_i | \mathcal{I}^*_{\{x_i\}\cap\{gx_j\}}\big(B_i^* B B_j V(g)^*\big) U(g)\varphi_j \rangle \\
&= \sum_{i,j=1}^{n} \langle \varphi_i | \mathcal{I}^*_{\{gg^{-1}x_i\}\cap\{gx_j\}}\big(V(g)V(g)^* B_i^* B B_j V(g)^*\big) U(g)\varphi_j \rangle \\
&= \sum_{i,j=1}^{n} \langle U(g)^*\varphi_i | \mathcal{I}^*_{\{g^{-1}x_i\}\cap\{x_j\}}\big(V(g)^* B_i^* B B_j\big)\varphi_j \rangle \\
&= \sum_{i,j=1}^{n} \langle \big(B^* B_i V(g) \otimes \mathsf{P}_{g^{-1}x_i}\big) JU(g)^*\varphi_i | (B_j \otimes \mathsf{P}_{x_j})J\varphi_j \rangle \\
&= \sum_{i,j=1}^{n} \langle \tilde{U}(g)^*(B^* B_i \otimes \mathsf{P}_{x_i})J\varphi_i | (B_j \otimes \mathsf{P}_{x_j})J\varphi_j \rangle \\
&= \langle \xi | (B \otimes \mathbb{1}_{\mathcal{M}})\tilde{U}(g)\xi \rangle
\end{aligned}
$$

which, together with the minimality, implies that $\tilde{U}(g)(B \otimes \mathbb{1}_{\mathcal{M}}) = (B \otimes \mathbb{1}_{\mathcal{M}})\tilde{U}(g)$ for all $g \in G$ and $B \in \mathcal{L}(\mathcal{K})$. This means that there is a unique unitary representation $\overline{U} : G \to \mathcal{U}(\mathcal{M})$ such that $\tilde{U}(g) = \mathbb{1}_{\mathcal{K}} \otimes \overline{U}(g)$ for all $g \in G$. Furthermore, for any $g \in G$,

$x \in \mathbb{X}$, and ξ as above,

$$\begin{aligned}\big(\mathbb{1}_{\mathcal{K}} \otimes \overline{U}(g)\mathsf{P}_x\overline{U}(g)^*\big)\xi &= \sum_{i=1}^n \tilde{U}(g)(\mathbb{1}_{\mathcal{K}} \otimes \mathsf{P}_x)\tilde{U}(g)^*(B_i \otimes \mathsf{P}_{x_i})J\varphi_i \\ &= \sum_{i=1}^n \tilde{U}(g)\big(B_iV(g) \otimes \mathsf{P}_x\mathsf{P}_{g^{-1}x_i}\big)JU(g)^*\varphi_i \\ &= \sum_{i=1}^n \tilde{U}(g)\big(B_iV(g) \otimes \mathsf{P}_{\{x\}\cap\{g^{-1}x_i\}}\big)JU(g)^*\varphi_i \\ &= \sum_{i=1}^n (B_i \otimes \mathsf{P}_{\{gx\}\cap\{x_i\}})J\varphi_i = (\mathbb{1}_{\mathcal{K}} \otimes \mathsf{P}_{gx})\xi.\end{aligned}$$

Minimality again implies that $\overline{U}(g)\mathsf{P}_x\overline{U}(g)^* = \mathsf{P}_{gx}$ for all $g \in G$ and $x \in \mathbb{X}$.

It follows that the pair $(\overline{U}, \mathsf{P})$ is an example of an imprimitivity system. Let us define, for each orbit $\Omega \in \mathcal{O}$, the Hilbert space $\mathcal{M}^\Omega := \big(\sum_{x\in\Omega}\mathsf{P}_x\big)\mathcal{M}$ the map $\overline{U}^\Omega : G \to \mathcal{U}(\mathcal{M}^\Omega)$, $\overline{U}^\Omega(g) = \sum_{x\in\Omega}\mathsf{P}_x\overline{U}(g)|_{\mathcal{M}^\Omega}$ for all $g \in G$, and the PVM $\mathsf{P}^\Omega = (\mathsf{P}^\Omega_x)_{x\in\Omega} := (\mathsf{P}_x)_{x\in\Omega}$ in $\mathcal{M}^\Omega$. It easily follows that $\overline{U}^\Omega$ is still a unitary representation and $\overline{U}^\Omega(g)\mathsf{P}^\Omega_x\overline{U}^\Omega(g)^* = \mathsf{P}^\Omega_{gx}$ for all $g \in G$ and $x \in \Omega$. This means that, for any orbit Ω, $(\overline{U}^\Omega, \mathsf{P}^\Omega)$ is a transitive system of imprimitivity as G acts transitively in any orbit. Mackey's imprimitivity theorem tells us that, for any orbit Ω, we may assume (possibly by tweaking the isometry J) that there is a (finite-dimensional) Hilbert space $\mathcal{H}^\Omega$ and a unitary representation $\pi^\Omega : H_\Omega \to \mathcal{U}(\mathcal{H}^\Omega)$ such that $\mathcal{M}^\Omega = \mathbb{C}^{\#\Omega} \otimes \mathcal{H}^\Omega$ and Equations (4) and (5) hold.

Appendix C. Extremality within the set of all instruments

Let us now directly see how the extremality characterization within the set of all instruments presented in Remark 2 implies the extremality within the set of $(\mathbb{X}, U, V)$–covariant instruments. We continue to use the notations fixed in Section 3. Let us assume that an $(\mathbb{X}, U, V)$–covariant instrument $\mathcal{I} = (\mathcal{I}_x)_{x\in\mathbb{X}}$ is an extreme instrument. Let

$$\{L^\Omega_{\eta,i,m}\,|\,m = 1,\ldots,M_\eta,\ i = 1,\ldots,D_\eta,\ [\eta] \in \hat{H}_\Omega,\ \Omega \in \mathcal{O}\}$$

be a minimal set of $(\mathbb{X}, U, V)$–intertwiners, where $M_\eta \in \{0\} \cup \mathbb{N}$ for all $\Omega \in \mathcal{O}$ and $[\eta] \in \hat{H}_\Omega$. Let $\beta^\Omega_{\eta,m,n} \in \mathbb{C}$, $\Omega \in \mathcal{O}$, $[\eta] \in \hat{H}_\Omega$, m, $n = 1, \ldots, M_\eta$, be such that

$$\sum_{\Omega\in\mathcal{O}}\sum_{g\in G}\sum_{[\eta]\in\hat{H}_\Omega}\sum_{i=1}^{D_\eta}\sum_{m,n=1}^{M_\eta}\beta^\Omega_{\eta,m,n}L^{\Omega\,*}_{\eta,i,m}L^\Omega_{\eta,i,n} = 0.$$

Denote $\gamma_{x,\eta,\vartheta,i,j,m,n} = (\#H_{Gx})\beta^{Gx}_{\eta,m,n}$ for all $x \in \mathbb{X}$ whenever $[\eta] = [\vartheta] \in \hat{H}_{Gx}$, $i = j \in \{1, \ldots, D_\eta\}$, and m, $n = 1, \ldots, M_\eta$. Otherwise, $\gamma_{x,\eta,\vartheta,i,j,m,n} = 0$. Using similar tricks as earlier (and denoting by $\delta_{j,k}$ the Kronecker symbol, i.e., $\delta_{j,k} = 1$ if $j = k$ and, otherwise, $\delta_{j,k} = 0$), we find

$$\sum_{\Omega\in\mathcal{O}}\sum_{x\in\Omega}\sum_{[\eta],[\vartheta]\in\hat{H}_\Omega}\sum_{i=1}^{D_\eta}\sum_{j=1}^{D_\vartheta}\sum_{m=1}^{M_\eta}\sum_{n=1}^{M_\vartheta}\gamma_{x,\eta,\vartheta,i,j,m,n}K^*_{x,\eta,i,m}K_{x,\vartheta,j,n}$$

$$=\sum_{\Omega\in\mathcal{O}}\sum_{x\in\Omega}\sum_{[\eta]\in\hat{H}_\Omega}\sum_{i=1}^{D_\eta}\sum_{m,n=1}^{M_\eta}(\#H_\Omega)\beta^\Omega_{\eta,m,n}K^*_{x,\eta,i,m}K_{x,\eta,i,n}$$

$$=\sum_{\Omega\in\mathcal{O}}\sum_{x\in\Omega}\sum_{[\eta]\in\hat{H}_\Omega}\sum_{j,k=1}^{D_\eta}\sum_{m,n=1}^{M_\eta}(\#H_\Omega)\underbrace{\sum_{i=1}^{D_\eta}\overline{\zeta^\eta_{i,j}\big(s_\Omega(x)^{-1},x\big)}\zeta^\eta_{i,k}\big(s_\Omega(x)^{-1},x\big)}_{=\delta_{j,k}}\times$$

$$\times\,\beta^\Omega_{\eta,m,n}U\big(s_\Omega(x)\big)L^{\Omega\,*}_{\eta,j,m}L^\Omega_{\eta,k,n}U\big(s_\Omega(x)\big)^*$$

$$=\sum_{\Omega\in\mathcal{O}}\sum_{x\in\Omega}\sum_{[\eta]\in\hat{H}_\Omega}\sum_{i=1}^{D_\eta}\sum_{m,n=1}^{M_\eta}(\#H_\Omega)\beta^\Omega_{\eta,m,n}U\big(s_\Omega(x)\big)L^{\Omega\,*}_{\eta,i,m}L^\Omega_{\eta,i,n}U\big(s_\Omega(x)\big)^*$$

$$=\sum_{\Omega\in\mathcal{O}}\sum_{x\in\Omega}\sum_{h\in H_\Omega}\sum_{[\eta]\in\hat{H}_\Omega}\sum_{i=1}^{D_\eta}\sum_{m,n=1}^{M_\eta}\beta^\Omega_{\eta,m,n}U\big(s_\Omega(x)h\big)L^{\Omega\,*}_{\eta,i,m}L^\Omega_{\eta,i,n}U\big(s_\Omega(x)h\big)^*$$

$$=\sum_{\Omega\in\mathcal{O}}\sum_{g\in G}\sum_{[\eta]\in\hat{H}_\Omega}\sum_{i=1}^{D_\eta}\sum_{m,n=1}^{M_\eta}\beta^\Omega_{\eta,m,n}U(g)L^{\Omega\,*}_{\eta,i,m}L^\Omega_{\eta,i,n}U(g)^*=0.$$

Using the extremality of $\mathcal{I}$, we now find that $\gamma_{x,\eta,\vartheta,i,j,m,n} = 0$ for all orbits $\Omega\in\mathcal{O}$, $x\in\Omega$, $[\eta]$, $[\vartheta]\in\hat{H}_\Omega$, $i=1,\ldots,D_\eta$, $j=1,\ldots,D_\vartheta$, $m=1,\ldots,M_\eta$, and $n=1,\ldots,M_\vartheta$, implying that $\beta^\Omega_{\eta,m,n}=0$ for all $\Omega\in\mathcal{O}$, $[\eta]\in\hat{H}_\Omega$, and m, $n=1,\ldots,M_\eta$. Thus, $\mathcal{I}$ is also an extreme instrument of the $(\mathbb{X},U,V)$–covariance structure.

Appendix D. Extremality within the set of all instruments: the continuous case

We now prove the extremality characterization of Remark 3. We fix the $(G/H,U,V)$–covariant instrument $\mathcal{I}$ of said Remark and retain the notation and definitions therein. Let $(L^2_\mu\otimes\mathcal{H}_\pi,\mathsf{P}^G_\pi,U^G_\pi,J)$ be the minimal $(G/H,U,V)$–covariant Stinespring dilation for $\mathcal{I}$ constructed in Lemma 2. According to [29], $\mathcal{I}$ is extreme if and only if, for $E\in\mathcal{L}(L^2_\mu\otimes\mathcal{H}_\pi)$ such that $\mathsf{P}^G_\pi(X)E = E\mathsf{P}^G_\pi(X)$ for all $X\in\mathcal{B}(G/H)$, the condition $J^*(\mathbb{1}_\mathcal{K}\otimes E)J=0$ implies $E=0$. Let us fix $E\in\mathcal{L}(L^2_\mu\otimes\mathcal{H}_\pi)$ such that $\mathsf{P}^G_\pi(X)E=E\mathsf{P}^G_\pi(X)$ for all $X\in\mathcal{B}(G/H)$. It follows that there is a μ-measurable field $G/H\ni x\mapsto E(x)\in\mathcal{L}(\mathcal{H}_\pi)$ such that $(DF)(x)=D(x)F(x)$ for all $F\in L^2_\mu\otimes\mathcal{H}_\pi$ and $x\in G/H$. We define $f^\beta_\gamma\in L^\infty_\mu$ through $f^{\eta,i,m}_{\vartheta,j,n}(x)=\langle e_{\eta,i}\otimes f_{\eta,m}|E(x)(e_{\vartheta,j}\otimes f_{\vartheta,n})\rangle$ for all $x\in G/H$ and (η,i,m), $(\vartheta,j,n)\in B$. Using Equation (25), we have, for all $(\eta,i,m)\in B$ and $g\in G$,

$$\sum_{k=1}^{D_\eta}\zeta^\eta_{i,k}(g^{-1},gH)V(g)L_{\eta,k,m}U(g)^*=(V\circ s)(gH)L_{\eta,i,m}(U\circ s)(gH)^*.$$

Using this and the definitions of Lemma 2, we get, for all $\varphi \in \mathcal{D}$,

$$
\begin{aligned}
&\langle J\varphi|(\mathbb{1}_{\mathcal{K}}\otimes E)J\varphi\rangle = \int_{G/H}\big\langle (J\varphi)(x)\big|\big(\mathbb{1}_{\mathcal{K}}\otimes E(x)\big)(J\varphi)(x)\big\rangle\,d\mu(x)\\
&=\int_{G/H}\sum_{[\eta],[\vartheta]\in\hat{H}}\sum_{i,k=1}^{D_\eta}\sum_{j,l=1}^{D_\vartheta}\sum_{m=1}^{M_\eta}\sum_{n=1}^{M_\vartheta}\overline{\zeta^{\eta}_{i,k}(g^{-1},gH)}\zeta^{\vartheta}_{j,l}(g^{-1},gH)\times\\
&\times\langle V(g)L_{\eta,k,m}U(g)^*\varphi|V(g)L_{\vartheta,l,n}U(g)^*\varphi\rangle\langle e_{\eta,i}\otimes f_{\eta,m}|E(gH)(e_{\vartheta,j}\otimes f_{\vartheta,n})\rangle\,d\mu(gH)\\
&=\int_{G/H}\sum_{\beta,\gamma\in B}f^{\beta}_{\gamma}(x)\langle L_\beta(U\circ s)(x)^*\varphi|L_\gamma(U\circ s)(x)^*\varphi\rangle\,d\mu(x).
\end{aligned}
$$

Noticing that $G/H\ni x\mapsto \big(f^{\beta}_{\gamma}(x)\big)_{\beta,\gamma\in B}\in\mathcal{L}(\ell^2_B)$ is μ-essentially bounded and that any family $\{f^{\beta}_{\gamma}\}_{\beta,\gamma\in B}\subset L^\infty_\mu$ with this property can be reached with a μ-essentially bounded μ-measurable field $G/H\ni x\mapsto E(x)\in\mathcal{L}(\mathcal{H}_\pi)$ through $f^{\eta,i,m}_{\vartheta,j,n}(x)=\langle e_{\eta,i}\otimes f_{\eta,m}|E(x)(e_{\vartheta,j}\otimes f_{\vartheta,n})\rangle$ for all $x\in G/H$ and (η,i,m), $(\vartheta,j,n)\in B$ and using the fact that such bounded fields of operators exactly correspond to bounded operators commuting with P^G_π, we obtain the desired extremality characterization. Also note that, using familiar countability arguments, $E(x)=0$ for μ-a.a. $x\in G/H$ for a μ-essentially bounded μ-measurable field $G/H\ni x\mapsto E(x)\in\mathcal{L}(\mathcal{H}_\pi)$ is equivalent with $f^{\beta}_{\gamma}(x)=0$ for μ-a.a. $x\in G/H$ and all β, $\gamma\in B$ where $\{f^{\beta}_{\gamma}\}_{\beta,\gamma\in B}\subset L^\infty_\mu$ is defined as above and the μ-null set of those $x\in G/H$ for which $f^{\beta}_{\gamma}(x)\neq 0$ does not have to depend on β, $\gamma\in B$.

[1] A. Barchielli, M. Gregoratti, and A. Toigo, Measurement uncertainty relations for discrete observables: Relative entropy formulation, *Commun. Math. Phys.* **357** 1253-1304 (2018)

[2] F. Buscemi, M. Keyl, G.M. D'Ariano, P. Perinotti, and R.F. Werner, Clean positive operator valued measures, *J. Math. Phys.* **46**, 082109 (2005)

[3] P. Busch and P. Lahti, The determination of the past and the future of a physical system in quantum mechanics, *Found. Phys.* **19**, 633-678 (1989)

[4] P. Busch, P. Lahti, J.-P. Pellonpää, and K. Ylinen, *Quantum Measurement, Text and Monographs is Physics* (Springer, Berlin, 2016)

[5] P. Busch, P. Lahti, and R.F. Werner, Heisenberg uncertainty for qubit measurements, *Phys. Rev. A* **89** 012129 (2014)

[6] C. Carmeli, T. Heinosaari, A. Toigo, Covariant quantum instruments, *J. Funct. Anal.* **257**, 3353-3374 (2009)

[7] G. Cassinelli, E. De Vito, and A. Toigo, Positive operator valued measures covariant with respect to an Abelian group, *J. Math. Phys.* **45**, 418 (2004)

[8] U. Cattaneo, On Mackey's imprimitivity theorem, *Comment. Math. Helvetici* **54**, 629-641 (1979)

[9] G. Chiribella and G.M. D'Ariano, Extremal covariant measurements, *J. Math. Phys.* **47**, 092107 (2006)

[10] H. Cycon and K.-E. Hellwig, Conditional expectations in generalized probability theory, *J. Math. Phys.* **18**, 1154-1161 (1977)

[11] E.B. Davies, *Quantum Theory of Open Systems* (Academic Press, London, 1976)

[12] E.B. Davies and J.T. Lewis: An operational approach to quantum probability, *Commun. Math. Phys.* **17**, 239-260 (1970)

[13] E. Haapasalo, T. Heinosaari, and J.-P. Pellonpää, Quantum measurements on finite dimensional systems: relabeling and mixing, *Quantum Inf. Process.* **11**, 1751-1763 (2012)

[14] E. Haapasalo and J.-P. Pellonpää, Extreme covariant quantum observables in the case of an Abelian symmetry group and a transitive value space, *J. Math. Phys.* **52**, 122102 (2011)

[15] E. Haapasalo and J.-P. Pellonpää, Covariant KSGNS construction and quantum instruments, *Rev. Math. Phys.* **29**, 1-47 (2017)

[16] E. Haapasalo and J.-P. Pellonpää, Optimal quantum observables, *J. Math. Phys.* **58**, 122104 (2017)

[17] A.K. Hashagen and M.M. Wolf, Universality and optimality in the information–disturbance tradeoff, *Ann. Henri Poincaré* **20** 219-258 (2019)

[18] T. Heinosaari and T. Miyadera, Qualitative noise-disturbance relation for quantum measurements, *Phys. Rev. A* **88**, 042117 (2013)

[19] A.S. Holevo, Generalized imprimitivity systems for abelian groups, *Soviet. Math.* (Iz. VUZ) **27**(2), 53-80 (1983)

[20] A.S. Holevo, On a generalization of canonical quantization, *Math. USSR Izvestiya* **28**, 175-188 (1987)

[21] A.S. Holevo, Radon-Nikodym derivatives of quantum instruments, *J. Math. Phys.* **39**, 1373-1387 (1998)

[22] A.S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (Publications of the Scuola Normale Superiore, Pisa, 2011)

[23] A.S. Holevo, *Quantum Systems, Channels, Information: A Mathematical Introduction* (de Gruiter, Berlin, 2012)

[24] A.S. Holevo and J.-P. Pellonpää, Extreme covariant observables for type I symmetry groups, *Found. Phys.* 39, 625-641 (2009)

[25] J. Kiukas, P. Lahti, and K. Ylinen, Normal covariant quantization maps, *Math. Anal. Appl.* **319**, 783-801 (2006)

[26] G.W. Mackey, *Unitary Group Representations in Physics, Probability, and Number Theory* (Addison-Wesley, Reading, Massachusetts, 1978, 1989)

[27] M. Ozawa, Quantum measuring processes of continuous observables, *J. Math. Phys.* **25**, 79-87 (1984)

[28] K.R. Parthasarathy, Extremal decision rules in quantum hypothesis testing, *Infinite Dimens. Anal.* **2**, 557-568 (1999)

[29] J.-P. Pellonpää, Quantum instruments: I. Extreme instruments, *J. Phys. A: Math. Theor.* **46**, 025302 (2013)

[30] V.S. Varadarajan, *Geometry of Quantum Theory* (Springer, New York, 1985)