Advances in Operator Theory

On the geometry of an order unit space

--Manuscript Draft--

Manuscript Number:	AIOT-D-23-00203R1
Full Title:	On the geometry of an order unit space
Article Type:	Original Paper
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Response to Reviewers:	

On the geometry of an order unit space

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Abstract. We introduce the notion of *skeleton* with a head in a non-zero real vector space. We prove that skeletons with a head describe order unit spaces geometrically. Next, we prove that periphery consists of boundary elements of the positive cone of norm one. We discuss some elementary properties of the skeleton. We also find a condition under which V contains a copy of ℓ_∞^n for some $n \in \mathbb{N}$ as an order unit subspace.

Mathematics Subject Classification (2010). Primary: 46B40; Secondary: 46B20, 46L05.

Keywords. Order unit space, skeleton, ∞-orthogonality, lead point.

1. Introduction

Let X be a normed linear space and let $x, y \in X$. We say that x is ∞-*orthogonal* to y, (we write, $x \perp_\infty y$), if $\|x + ky\| = \max\{\|x\|, \|ky\|\}$ for all $k \in \mathbb{R}$. It was proved in [11] that if (V, e) is an order unit space and if $u, v \in V^+ \setminus \{0\}$, then $u \perp_\infty v$ if and only if $\| \|u\|^{-1} u + \|v\|^{-1} v\| = 1$. For $u, v \in V^+$, we say that u is *absolutely* ∞-*orthogonal* to v (we write $u \perp_\infty^a v$) if $u_1 \perp_\infty v_1$ whenever $0 \leq u_1 \leq u$ and $0 \leq v_1 \leq v$.

In a unital C*-algebra A, an element $p \in A$ is called a projection if $p^2 = p = p^*$ or equivalently, $p, 1 - p \in A^+$ with $p(1 - p) = 0$. In [12, Theorem 2.1], the author proved that p is a projection if and only if $p, 1 - p \in A^+$ and $p \perp_\infty^a (1 - p)$. Any von Neumann algebra has plenty of projections which play a key role in describing the algebras. However, on the contrast, a C*-algebra may not have any non-trivial projection. In this paper, we weaken the notion of projections and consider instead that of *skeletal elements*. An element $a \in A_{sa}$ is said to be *skeletal*, if $a, 1 - a \in A^+$ and $a \perp_\infty (1 - a)$. Let M be a self-adjoint subspace of A containing 1. In this paper, we have proved that the set of all skeletal elements of M_{sa} together with 1 describe M_{sa} as an order unit space. More precisely, the *skeleton* (the set of all skeletal elements) describe an order unit space.

Order unit spaces dominate the interface of commutative and non-commutative C*-algebras. In early 1940's, Stone, Kakutani, Krein and Yosida proved independently that if an order unit space (V, e) is a vector lattice in its order structure, then it is unitally lattice isomorphic to a dense lattice subspace of $C_{\mathbb{R}}(X)$ for some suitable compact Hausdorff space X [1, Theorem II.1.10]. (See the notes after Section1, Chapter II of [1] for the details.) However, in 1951, Sherman proved that the self-adjoint part of a C*-algebra A is a vector lattice in its order structure if and only if A is commutative [16]. The same year, Kadison proved that the infimum of a pair of self-adjoint operators on a complex Hilbert space exists if and only if they are comparable [9]. This result is known as Kadison's anti-lattice theorem.

After these two papers, it became imperative to study the order structure from a different point of view. To continue the endeavour, in the same year, Kadison proved in another paper that a unital self-adjoint subspace of a unital C*-algebra is an order unit space [10]. (Much later in 1977, Choi and Effros proved that a unital self-adjoint subspace of a unital C*-algebra is precisely a matrix order unit space [4].) Soon after Kadison underscored the importance of order unit spaces as a possible model for non-commutative ordered spaces, there was a flux of research in the study of order unit spaces and their duals. Some early prominent references are Bonsall, Edwards, Ellis, Asimov and Ng, besides many others. (See [2, 3, 5, 6, 15]. We refer to [1, 8] for more references and details.)

The dual of an order unit space is a base normed space which is defined through the geometric notion of a *base* in an ordered vector space. On the other hand, the notion of an order unit is order theoretic in nature. The main aim of this paper is to study a set of geometric properties that determines order unit spaces. We begin with the following:

Definition 1.1. Let X be a non-zero real vector space and let $e \in X$ with $e \neq 0$. Then a subset $S \subset X \setminus \{0, e\}$ is said to be a *skeleton* with e as its head, if the following conditions are satisfied.

1. If $u \in S$, then $e - u \in S$;
2. If $u, v \in S$ and $\lambda, \mu \in \mathbb{R}$ with $\lambda > 0$ and $\mu > 0$, there exist $w \in S$ and $\alpha, \beta \in \mathbb{R}^+$ such that $\lambda u + \mu v = \alpha e + \beta w$; and
3. If $e = \sum_{i=1}^{n} \alpha_i u_i$ for some $u_1, \dots, u_n \in S$ and $\alpha_1, \dots, \alpha_n > 0$ with $n \geq 2$, then $\sum_{i \neq j} \alpha_i \geq 1$ for all $j = 1, \dots, n$.

In this paper, we prove the following characterization of order unit spaces.

Theorem 1.2. *Let X be a non-zero real vector space, $e \in X$ with $e \neq 0$ and let S be a skeleton in X with e as its head. If V is the linear span of $S \cup \{e\}$ and if V^+ is the cone generated by $S \cup \{e\}$, then (V, V^+, e) is an order unit space such that*

$$S = \{v \in V^+ : \|v\| = \|e - v\| = 1\}.$$

(Here $\| \cdot \|$ is the order unit norm on V.)

We also prove the converse of this result.

Theorem 1.3. *Let (V, e) be an order unit space. Put*

$$(S_V) := \{u \in V : \|u\| = \|e - u\| = 1\}.$$

Then S_V is a skeleton in V with e as its head such that (V, e) is the order unit space generated by S_V.

Let (V, e) be an order unit space and consider its generating skeleton S_V with its head e. Put $C_V := \bigcup_{u \in S}[e, u]$. Then $S_V = C_V \bigcap (e - C_V)$. We prove that $S_V = C_V \bigcap Bd(V^+)$. (Here $Bd(V^+)$ denotes the boundary of V^+.) We note that in a unital C*-algebra A, S_A includes all the projections of A. We discuss some elementary properties of the skeleton. Using these properties, we prove that every order unit space (V, e) of dimension more than 1 contains a copy of ℓ_∞^2 as an order unit subspace. We also prove that V is a union of these copies in such a way any two such subspaces meet precisely at the *axis* $\mathbb{R}e$. Further, we find a condition under which V contains a copy of ℓ_∞^n for some $n \in \mathbb{N}$ as an order unit subspace.

The scheme of the paper is as follows. In Section 2, we discuss some of the properties of skeleton in a non-zero real vector space and prove Theorem 1.2. In Section 3, we prove Theorem 1.3. In Section 4, we study some elementary properties of the skeleton corresponding to an order unit space. In Section 5, we find a condition under which an order unit space would contain a copy of ℓ_∞^n for some $n \in \mathbb{N}$ as an order unit subspace besides some other results.

At the end of this section we mention the following conventions. In this paper, we have considered two types of intervals, namely convex intervals and order intervals.

If X is a (real) vector space, then for $x, y \in X$ we write

$$[x, y] := \{(1 - \alpha)x + \alpha y : \alpha \in [0, 1]\}.$$

If (V, V^+) is a real ordered vector space, then for $u, v \in V$ with $u \leq v$, we write

$$[u, v]_o := \{w \in V : u \leq w \leq v\}.$$

2. The skeleton

In this section we shall prove Theorem 1.2. We begin with some preliminary results. Throughout in this section, we shall assume that X is a non-zero real vector space and S is a skeleton in X with $e \neq 0$ as its head (see Definition 1.1). First of all, we prove some of the easy consequences of Definition 1.1.

Lemma 2.1. *1. $S \bigcap [0, 1]e = \emptyset$.*

2. If $u, v \in S$ and $\alpha, \beta \in \mathbb{R}^+$, then there exist $w \in S$ and $\lambda, \mu \in \mathbb{R}^+$ with $\lambda \leq \min\{\alpha, \beta\}$ and $\lambda + \mu \leq \alpha + \beta$ such that $\alpha u + \beta v = \lambda e + \mu w$.

3. Let $u, v \in S$ be such that $e = \alpha u + \beta v$ for some $\alpha, \beta \in \mathbb{R}^+$. Then $\alpha = 1 = \beta$.

4. Let $u, v \in S$ such that $\alpha u = \beta v$ for some $\alpha, \beta \in \mathbb{R}$.

(a) Then $\alpha\beta \geq 0$;

(b) $\alpha = 0$ if and only if $\beta = 0$;

(c) *If $\alpha\beta > 0$. Then $u = v$.*

5. *Let $u_1, \dots, u_n$ be distinct elements of S and $\alpha_0, \alpha_1, \dots, \alpha_n \in \mathbb{R}^+$ such that $\alpha_0 e + \sum_{i=1}^n \alpha_i u_i = 0$. Then $\alpha_i = 0$ for each $i = 0, 1, \dots, n$.*

Proof. (1): Let $\alpha e \in S$ for some $\alpha \in (0,1)$. Then by 1.1(1), $(1-\alpha)e \in S$. Since $e = \alpha e + (1-\alpha)e$, by 1.1(3), we get $\alpha \geq 1$ and $1 - \alpha \geq 1$ which is absurd.

(2): Let $\alpha_1 \in (0,1)$. Then by 1.1(2), $\alpha_1 u + (1-\alpha_1)v = \lambda e + \mu w$ for some $w \in$ and $\lambda, \mu \in \mathbb{R}^+$. Thus $(\lambda + \mu)e = \alpha_1 u + (1-\alpha_1)v + \mu(e - w)$. Now by 1.1(3), we get $\lambda + \mu \leq 1$, $\lambda + \mu \leq \alpha_1 + \mu$ and $\lambda + \mu \leq 1 - \alpha_1 + \mu$. Thus $\lambda \leq \min\{\alpha_1, 1-\alpha_1\}$ and $\lambda + \mu \leq 1$. For arbitrary $\alpha, \beta \in \mathbb{R}^+$, we consider $\alpha_1 := \frac{\alpha}{\alpha+\beta}$.

(3): By 1.1(3), we have $\alpha \geq 1$ and $\beta \geq 1$. Now, $\alpha(e-u) + \beta(e-v) = (\alpha+\beta-1)e$ and by 1.1(1), $e-u, e-v \in S$. Thus invoking 1.1(3) once again, we get $\alpha \geq \alpha+\beta-1$ and $\beta \geq \alpha+\beta-1$. Thus $\alpha \leq 1$ and $\beta \leq 1$ so that $\alpha = 1 = \beta$.

(4)(a): Let $\alpha\beta < 0$. For definiteness, we assume that $\alpha > 0$ and $\beta < 0$. Then $\alpha(e-u) - \beta(e-v) = (\alpha-\beta)e$ with $\alpha - \beta > 0$. Thus by condition 1.1(3), we get $\alpha, -\beta \geq \alpha - \beta$. But then $\alpha = 0 = \beta$ which contradicts the assumption. Thus $\alpha\beta \geq 0$.

(4)(b) follows immediately as $0 \notin S$.

(4)(c): Now assume that $\alpha\beta > 0$. For definiteness, we assume that $\alpha > 0$ and $\beta > 0$. Further, without any loss of generality, we may assume that $\alpha \leq \beta$. Put $\frac{\alpha}{\beta} = \lambda$. Then $0 < \lambda \leq 1$ and $\lambda u = v$. Thus $e = \lambda u + (e - v)$. Now by 1.1(3), we get $\lambda \geq 1$ so that $\lambda = 1$. Thus $u = v$.

(5): We have $\sum_{i=1}^n \alpha_i(e - u_i) = (\sum_{i=0}^n \alpha_i)e$. Assume, if possible that $\sum_{i=0}^n \alpha_i > 0$. Then by 1.1(3), we get $\sum_{i=0}^n \alpha_i \leq (\sum_{i=1}^n \alpha_i) - \alpha_j$ for all $j = 1, \dots, n$. In other words, $\alpha_0 + \alpha_j = 0$ for all $j = 1, \dots, n$. Therefore, $\alpha_j = 0$ for every $j = 0, 1, \dots, n$. □

Lemma 2.2. *Let $u, v \in S$ be such that $\alpha e + \beta u = \gamma e + \delta v$ for some $\alpha, \beta, \delta, \gamma \in \mathbb{R}$ with $\beta, \delta \geq 0$. Then $\alpha = \gamma$ and we have either $\beta = 0 = \delta$ or $u = v$.*

Proof. First, we show that $\alpha = \gamma$. If $\alpha < \gamma$, then $(\gamma - \alpha + \delta)e = \beta u + \delta(e - v)$. By Lemma 2.1(5), we must have $\gamma - \alpha + \delta > 0$ as $\delta \geq 0$. Thus by 1.1(3), we have $\gamma - \alpha + \delta \leq \delta$. But then we arrive at a contradiction, $\gamma \leq \alpha$. Thus $\alpha \geq \gamma$. Now, by symmetry, we have $\gamma \geq \alpha$ so that $\alpha = \gamma$. Thus $\beta u = \delta v$. The rest of the proof follows from Lemma 2.1(4). □

Proposition 2.3. *For $u \in S$ we consider*

$$K(u) := \mathrm{co}\{0, e, u\} = \bigcup_{\alpha \in [0,1]} \alpha[e, u].$$

(a) *For $u, v \in S$ with $u \neq v$, we have $K(u) \cap K(v) = [0,1]e$.*

(b) *For $u \in S$, we have $K(e-u) = e - K(u)$.*

Proof. (a): Let $w \in K(u) \cap K(v)$. Then there exist $\alpha, \beta, \gamma, \delta \in [0,1]$ with $\alpha + \beta \leq 1$ and $\gamma + \delta \leq 1$ such that $w = \alpha e + \beta u = \gamma e + \delta v$. If possible, assume that $\alpha \neq \gamma$. For definiteness, we let $\alpha > \gamma$. Then $(\alpha-\gamma+\beta)e = \beta(e-v)+\delta v$. Thus by 1.1(3), we get $\beta \geq \alpha - \gamma + \beta$ so that $\alpha \leq \gamma$. This contradicts the assumption. Hence $\alpha = \gamma$ so that $\beta u = \delta v$. Now by Lemma 2.1(4), we have $\beta = 0 = \delta$ as $u \neq v$. Thus $w \in [0,1]e$, that is, $K(u) \cap K(v) \subset [0,1]e$. As $[0,1]e \subset K(x)$ for any $x \in S$, the proof is complete.

(b): Let $w \in K(e-u)$. Then $w = \alpha e + \beta(e-u)$ for some $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$. Thus $e - w = (1-\alpha-\beta)e + \beta u$. Since $1-\alpha-\beta, \beta \in [0,1]$ and $1-\alpha-\beta+\beta = 1-\alpha \leq 1$, we get that $e - w \in K(u)$. Thus $K(e-u) \subset e - K(u)$ for all $u \in S$. So for any $u \in S$, we also have $K(u) = K(e-(e-u)) \subset e - K(e-u)$. Therefore, $K(e-u) = e - K(u)$ for all $u \in S$. ☐

Corollary 2.4. *For $u, v \in S$ with $u \neq v$, we have $[e,u] \bigcap [e,v] = \{e\}$.*

Proof. Note that if $\alpha e \in [e,u]$ for some $u \in S$ and $\alpha \in [0,1]$, say, $\alpha e = (1-\lambda)e + \lambda u$, then $\lambda(e-u) = (1-\alpha)e$. As $e - u \in S$ by 1.1(1) and as $S \bigcap \mathbb{R}e = \emptyset$, we must have $\lambda = 0 = 1 - \alpha$. Thus $[e,u] \bigcap [e,v] = \{e\}$. Now as $[e,u] \subset K(u)$, the result follows from Proposition 2.3. ☐

Let E be a convex subset of a real vector space X with $0 \in E$. An element $x \in E$ is called a *lead point* of E, if for any $y \in E$ and $\lambda \in [0,1]$ with $x = \lambda y$, we have $\lambda = 1$ and $y = x$. The set of all lead points of E is denoted by $Lead(E)$.

A non-empty set E of a real vector space V is said to be *linearly compact*, if for any $x, y \in E$ with $x \neq y$, we have, the intersection of E with the line through x and y, $\{\lambda \in \mathbb{R} : (1-\lambda)x + \lambda y \in E\}$, is compact (in $\mathbb{R}$). Note that if E is convex, the above intersection is an interval. Following [7, Proposition 3.2], we may conclude that if E is a linearly compact convex set with $0 \in E$, then $Lead(E)$ is non-empty and for each $x \in E, x \neq 0$, there exist a unique $u \in Lead(E)$ and a unique $0 < \alpha \leq 1$ such that $x = \alpha u$.

Theorem 2.5. *Let X be a non-zero real vector space and let S be a skeleton in X with $e \neq 0$ as its head. Consider $K = \bigcup_{u \in S} K(u)$ and $C = \bigcup_{u \in S} [e,u]$. Then K is convex set containing 0 and e such that $Lead(K) = C$. Moreover, e is an extreme point of K.*

Proof. Let $x, y \in K$. Then there are $u, v \in S$ and $\alpha, \beta, \gamma, \delta \in [0,1]$ with $\alpha + \beta \leq 1$ and $\gamma + \delta \leq 1$ such that $x = \alpha e + \beta u$ and $y = \gamma e + \delta v$. Then for $\lambda \in [0,1]$, we have

$$\begin{aligned} \lambda x + (1-\lambda)y &= (\lambda\alpha + (1-\lambda)\gamma)e + \lambda\beta u + (1-\lambda)\delta v \\ &= (\lambda\alpha + (1-\lambda)\gamma)e + k(\lambda_1 u + (1-\lambda_1)v) \end{aligned}$$

where $k := \lambda\beta + (1-\lambda)\delta$ and $\lambda_1 := \frac{\lambda\beta}{\lambda\beta + (1-\lambda)\delta} \in [0,1]$. By Lemma 2.1(2), we can find $w \in S$ and $\eta, \kappa \in [0,1]$ with $\eta \leq \min\{\lambda_1, 1-\lambda_1\}$ and $\eta + \kappa \leq 1$ such that $\lambda_1 u + (1-\lambda_1)v = \eta e + \kappa w$. Thus

$$\lambda x + (1-\lambda)y = (\lambda\alpha + (1-\lambda)\gamma)e + k(\eta e + \kappa w) = \alpha_1 e + \beta_1 w$$

where

$$\alpha_1 = \lambda\alpha + (1-\lambda)\gamma + k\eta = \lambda\alpha + (1-\lambda)\gamma + (\lambda\beta + (1-\lambda)\delta)\eta$$

and

$$\beta_1 = k\kappa = (\lambda\beta + (1-\lambda)\delta)\kappa.$$

Now $\alpha_1 \geq 0$, $\beta_1 \geq 0$ and

$$\begin{aligned} \alpha_1 + \beta_1 &= \lambda\alpha + (1-\lambda)\gamma + (\lambda\beta + (1-\lambda)\delta)(\eta + \kappa) \\ &\leq \lambda\alpha + (1-\lambda)\gamma + \lambda\beta + (1-\lambda)\delta \\ &= \lambda(\alpha+\beta) + (1-\lambda)(\gamma+\delta) \\ &\leq 1 \end{aligned}$$

for $\alpha + \beta \leq 1$, $\gamma + \delta \leq 1$ and $\eta + \kappa \leq 1$. Thus $\lambda x + (1-\lambda)y \in K(w) \subset K$. Hence K is a convex set containing 0 and e.

Next, we show that $Lead(K) = C$. Let $v \in K$ $v \neq 0$. If $v = \alpha e$ then $0 < \alpha \leq 1$ and $e \in C$. Now assume that $v \notin [0,1]e$. Then there exists a unique $u \in S$ such that $v \in K(u)$. In other words, $v = \alpha e + \beta u$ for some $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$. Since $v \neq 0$, we have $\beta > 0$. Set $w = (\alpha+\beta)^{-1}v$. Then $w \in C$ and $v = (\alpha+\beta)w$. Thus every element of K can be written as a positive scalar multiple of an element of C. We show that $C = Lead(K)$.

Let $u \in C$ and assume that $u = \alpha w$ for some $w \in K$ and $\alpha \in [0,1]$. Let $u \in [e, v]$ for some $v \in S$, say $u = \lambda v + (1-\lambda)e = e - \lambda(e - v)$. As $w \in K$, we have $w = \gamma e + \delta x$ for some $x \in S$ and $\gamma, \delta \in [0,1]$ with $\gamma + \delta \leq 1$. Then $e - \lambda(e-v) = \alpha(\gamma e + \delta x)$ or equivalently, $(1 - \alpha\gamma)e = \lambda(e - v) + \alpha\delta x$. Since $\alpha, \gamma \in [0,1]$, we have $1 - \alpha\gamma \geq 0$. If $\alpha\gamma = 1$, then $\alpha = 1$ and we have $u = w$. So we assume that $1 - \alpha\gamma > 0$. Thus by 1.1(3), we get $1 - \alpha\gamma \leq \alpha\delta$. Therefore, $1 \leq \alpha(\gamma + \delta) \leq \alpha \leq 1$ for $\gamma + \delta \leq 1$. So we have $\alpha = 1$ and $u = w$ once again. Hence $C \subset Lead(K)$.

Conversely, let $u \in Lead(K)$. If $u = \alpha e$ for some $\alpha \in [0,1]$, then by the definition of Lead, we have $\alpha = 1$ and $u = e \in C$. So we assume that $u \notin [0,1]e$. Then as above, there exists $x \in C$ and $\lambda \in [0,1]$ such that $u = \lambda x$. As $x \in Lead(K)$, we must have $\lambda = 1$ and $u = x \in C$. Hence $Lead(K) \subset C$ and consequently, $Lead(K) = C$.

Finally, we show that e is an extreme point of K. Let $e = \alpha u + (1-\alpha)v$ for some $u, v \in K$ and $0 < \alpha < 1$. Recall that $K = \bigcup_{x \in S} K(x)$. Also $K(x) = \bigcup_{\alpha \in [0,1]} \alpha[e, x]$ and $e - x \in S$ for all $x \in S$. Thus there exist $u_1, v_1 \in S$ and $\gamma, \delta, \lambda, \mu \in [0,1]$ such that $u = \gamma(e - \lambda u_1)$ and $v = \delta(e - \mu v_1)$. Simplifying, we have

$$(1 - \alpha\lambda - (1-\alpha)\delta)\, e + \alpha\gamma\lambda u_1 + (1-\alpha)\delta\mu v_1 = 0.$$

If $u_1 = v_1$, by Lemma 2.1(5),we get

$$(i) \qquad 1 - \alpha\lambda - (1-\alpha)\delta = 0$$

and

$$(ii) \qquad \alpha\gamma\lambda + (1-\alpha)\delta\mu = 0.$$

Since $\alpha, \lambda, \delta \in [0,1]$, we get

$$1 = \alpha\gamma + (1-\alpha)\delta \leq 1$$

so that $\gamma = 1 = \delta$. Thus by (ii) we get $\alpha\lambda + (1-\alpha)\mu = 0$. Since $0 < \alpha < 1$, we may conclude that $\lambda = 0 = \mu$. Thus $u = e = v$.

If $u_1 \neq v_1$, invoking Lemma 2.1(5) once again, we get (i) and $\alpha\gamma\lambda = 0 = (1-\alpha)\delta\mu$. Now as above, we may again conclude that $u = e = v$. Hence e is an extreme point of K. $\square$

Remark 2.6. It is easy to note that $K = \mathrm{co}(S)$. Thus $K \cap \mathbb{R}e = [0,1]e$. Also, by Proposition 2.3, we have $e - K = K$. Thus 0 is also an extreme point of K.

Proposition 2.7. *Let X be a non-zero real vector space and let S be a skeleton in X with $e \neq 0$ as its head and let $u \in S$. Then $\alpha e + \beta u \in K$ if and only if $\alpha, \alpha+\beta \in \mathbb{R}^+$ and* $\max\{\alpha, \alpha+\beta\} \leq 1$.

Proof. Let $\alpha e + \beta u \in K$. If $\alpha e + \beta u = \lambda e$ for some $\lambda \in [0,1]$, then $(\lambda - \alpha)e = \beta u$. As $u \in S$, we must have $\lambda = \alpha$ and $\beta = 0$. Thus $0 \leq \alpha + \beta = \alpha \leq 1$. So we assume that $\alpha e + \beta u \notin \mathbb{R}e$. Then by Theorem 2.5, there exists a unique $x \in C$ and $0 < \lambda \leq 1$ such that $\alpha e + \beta u = \lambda x$. Consequently, we can also find $w \in S$ and $1 \leq \theta < 1$ such that $x = \theta e + (1-\theta)w$. Thus $\alpha e + \beta u = \lambda\theta e + \lambda(1-\theta)w$. Now we show that $\alpha \geq 0$.

Assume, if possible, that $\alpha < 0$. Then $\lambda - \alpha > 0$ and we have

$$(\lambda - \alpha)e = (\lambda\theta + \lambda(1-\theta) - \alpha)e = \beta u + \lambda(1-\theta)(e-w).$$

If $\beta \geq 0$, then by Lemma 2.1(3), we have $\lambda - \alpha = \beta = \lambda(1-\theta)$. But then $\alpha = \lambda\theta \geq 0$ which is a contradiction. Thus $\beta < 0$ and we have

$$(\lambda - \alpha - \beta)e = -\beta(e-u) + \lambda(1-\theta)(e-w).$$

Again invoking Lemma 2.1(3), we conclude that $\lambda - \alpha - \beta = -\beta = \lambda(1-\theta)$. This leads to another contradiction $\alpha = \lambda \geq 0$. Hence $\alpha \geq 0$.

Next, we aim to prove that $\alpha + \beta \geq 0$ and assume to the contrary that $\alpha + \beta < 0$, that is, $0 \leq \alpha < -\beta$. Now $(\alpha - \lambda\theta)e = -\beta u + (1-\lambda)\theta w$ so by Lemma 2.1(5), we must have $\alpha - \lambda\theta > 0$ as $\beta < 0$. By Lemma 2.1(3), we have $\alpha - \lambda\theta = -\beta = (1-\lambda)\theta$. Thus $\alpha + \beta = \lambda\theta \geq 0$ contradicting the assumption $\alpha + \beta < 0$. Thus $\alpha + \beta \geq 0$.

Since $\alpha e + \beta u \in K \setminus \mathbb{R}e$, there exists a unique $w \in S$ such that $\alpha e + \beta u = \gamma e + \delta w$ where $\gamma, \delta \in \mathbb{R}^+$ with $\gamma + \delta \leq 1$.

Let $\beta \leq 0$. Then $(\alpha - \gamma)e = -\beta u + \delta w$ so by Lemma 2.1(5), we must have $\alpha - \gamma \geq 0$. If $\alpha = \gamma$, then we further get $-\beta = 0 = \delta$ so that

$$0 \leq \alpha + \beta = \alpha = \gamma = \gamma + \delta \leq 1.$$

If $\alpha > \gamma$, then by Lemma 2.1(3), we get $\alpha - \gamma = -\beta = \delta$. Thus $\alpha = \gamma + \delta \leq 1$ and $\alpha + \beta = \gamma \leq \gamma + \delta \leq 1$.

Next, let $\beta > 0$. Then $(\alpha + \beta - \gamma)e = \beta(e-u) + \delta w$. Thus by Lemma 2.1(5), we get $\alpha + \beta - \gamma \geq 0$. If $\alpha + \beta = \gamma$, we further get $\beta = 0 = \delta$, contradicting $\beta > 0$. Thus $\alpha + \beta > \gamma$. Invoking Lemma 2.1(3), we have $\alpha + \beta - \gamma = \beta = \delta$. Hence $0 \leq \alpha \leq \alpha + \beta = \gamma + \delta \leq 1$.

Conversely, we assume that $0 \leq \alpha, \alpha + \beta \leq 1$. When $\beta \geq 1$, we have $\alpha e + \beta u \in K(u) \subset K$ for $\alpha, \beta \geq 0$ and $\alpha + \beta \leq 1$. When $\beta < 0$, we can write $\alpha e + \beta u = (\alpha + \beta)e - \beta(e-u) \in K(e-u) \subset K$ for $\alpha + \beta, -\beta \geq 0$ and $\alpha = (\alpha + \beta) - \beta = \alpha \leq 1$. $\square$

Now we prove the characterization of order unit spaces.

Proof of Theorem 1.2. Put $C = \bigcup_{u \in S}[e, u]$ and $K = \text{co}(C)$. Then by Theorem 2.5, K is a convex set containing 0 with $C = Lead(K)$. Also, by Proposition 2.3, $K = \bigcup_{\alpha \in [0,1]} \alpha C$. Thus $V^+ = \bigcup_{\lambda \in \mathbb{R}^+} \lambda C = \bigcup_{n \in \mathbb{N}} nK$ and $V = V^+ - V^+$. We prove that

$$K = \{v \in V^+ : 0 \leq v \leq e\}. \qquad (*)$$

Let $v \in K$. Then $v \in V^+$ so that $0 \leq v$. Let $v = \alpha u$ for some $u \in C$ and $\alpha \in [0, 1]$. If $u = e$, then $v = \alpha e \leq e$. So let $u \neq e$. Then there exists $w \in S$ and $\lambda \in [0, 1]$ such that $u = (1 - \lambda)e + \lambda w$. Thus

$$e - v = (1 - \alpha)e + \alpha\lambda(e - w) \in K(e - w) \subset V^+$$

for $e - w \in S$ and $1 - \alpha, \alpha\lambda \geq 0$ with $1 - \alpha + \alpha\lambda \leq 1$. Therefore, $K \subset \{v \in V^+ : 0 \leq v \leq e\}$.

Conversely, assume that $0 \leq u \leq e$. Then $u, e - u \in V^+$. Thus there exist $v, w \in C$ and $\alpha, \beta \geq 0$ such that $u = \alpha v$ and $e - u = \beta w$. Then $e = \alpha v + \beta w$. As $C = \bigcup_{x \in S}[e, x]$ and as $e - x \in S$ whenever $x \in S$, we can find $v_1, w_1 \in S$ and $\lambda, \mu \in [0, 1]$ such that $v = e - \lambda v_1$ and $w = e - \mu w_1$. Thus $e = (\alpha + \beta)e - \alpha\lambda v_1 - \beta\mu w_1$. We show that $\alpha, \beta \in [0, 1]$.

Case 1. $\alpha + \beta \leq 1$.

In this case, we have $(1 - \alpha - \beta)e + \alpha\lambda v_1 + \beta\mu w_1 = 0$ with e, v_1 and w_1 having coefficients from $\mathbb{R}^+$. Thus by Lemma 2.1(5), we get $1 - \alpha - \beta = 0 = \alpha\lambda = \beta\mu$. As $\alpha, \beta \geq 0$, we must have $\alpha, \beta \in [0, 1]$.

Case 2. $\alpha + \beta > 1$.

In this case, we have $(\alpha + \beta - 1)e = \alpha\lambda v_1 + \beta\mu w_1$ with e, v_1 and w_1 having coefficients from $\mathbb{R}^+$. Thus by Lemma 2.1(3), we get $1 - \alpha - \beta = \alpha\lambda = \beta\mu$. Since $\lambda, \mu \in [0, 1]$ and $\alpha, \beta \in \mathbb{R}^+$, we can conclude that $\alpha, \beta \in [0, 1]$.

Thus in any case, $\alpha \in [0, 1]$ so that $u = \alpha v \in K$ as $v \in C$. This proves $(*)$.

Since $V = V^+ - V^+$, it follows from $(*)$ that e is an order unit for V. We prove that V^+ is proper. Let $\pm u \in V^+$. Then there exist $v, w \in C$ and $\alpha, \beta \geq 0$ such that $u = \alpha v$ and $-u = \beta w$. Thus $\alpha v + \beta w = 0$. We show that $\alpha = 0 = \beta$. Assume, if possible, that $\alpha > 0$. Then $\beta > 0$ too, for $v \neq 0$. Put $k = \frac{\alpha}{\alpha + \beta}$. Then $0 < k < 1$ and we have $ku + (1 - k)v = 0$ so that $e = k(e - u) + (1 - k)(e - v)$. As $u, v \in C$, we have $e - u, e - v \in K$. As e is an extreme point of K by Theorem 2.5, we deduce that $e - u = e = e - v$, that is, $u = 0 = v$ which is absurd. Thus $\alpha = 0$. Therefore, V^+ is proper.

Next, we show that V^+ is Archimedean. Let $v \in V$ be such that $ke + v \in V^+$ for all $k > 0$. Then $v_n := \left(\frac{1}{1+\|v\|}\right)\left(\frac{1}{n+1}e + \frac{n}{n+1}v\right) \in V^+$ for all $n \in \mathbb{N}$. Since

$$\|v_n\| \leq \left(\frac{1}{1 + \|v\|}\right)\left(\frac{1 + n\|v\|}{1 + n}\right) < 1,$$

we have $v_n \in K$ for every n. Fix n and find $w_n \in C$ and $\lambda_n \in [0, 1]$ such that $v_n = \lambda_n w_n$. As $w_n \in C$, we can find $u_n \in S$ and $\mu_n \in [0, 1]$ such that $w_n =$

$(1-\mu_n)e+\mu_n u_n$. Thus $v_n = \lambda_n(1-\mu_n)e+\lambda_n\mu_n u_n$. Put $\lambda_n(1-\mu_n) = \alpha_n$ and $\lambda_n\mu_n = \beta_n$. Then $\alpha_n, \beta_n \in \mathbb{R}^+$ with $\alpha_n+\beta_n \le 1$ such that $v_n = \alpha_n e+\beta_n u_n$.

Set $u := u_1, \alpha = 2\alpha_1(1+\|v\|)$ and $\beta = 2\beta_1(1+\|v\|)$. Then $v_1 = \frac{e+v}{2(1+\|v\|)}$. Thus $e+v = \alpha e+\beta u$, or equivalently, $v = (\alpha-1)e+\beta u$ where $\alpha, \beta \in \mathbb{R}^+$ with $\alpha+\beta \le 2(1+\|v\|)$. It follows that

$$\begin{aligned}\alpha_n e+\beta_n u_n = v_n &= \frac{1}{(1+\|v\|)}\left(\frac{e+nv}{n+1}\right)\\ &= \frac{(n\alpha-n+1)e+n\beta u}{(n+1)(1+\|v\|)},\end{aligned}$$

that is, $\alpha_n e+\beta_n u_n = \gamma_n e+\delta_n u$ for all n where $\gamma_n = \frac{(n\alpha-n+1)}{(n+1)(1+\|v\|)}$ and $\delta_n = \frac{n\beta}{(n+1)(1+\|v\|)}$. Since $\beta_n, \delta_n \in \mathbb{R}^+$, by Lemma 2.2 we have $\frac{(n\alpha-n+1)}{(n+1)(1+\|v\|)} = \gamma_n = \alpha_n \in [0,1]$ for every $n \in \mathbb{N}$. Taking limit as $n \to \infty$, we may conclude that $0 \le \alpha-1 \le 1+\|v\| (\le 1+2\|v\|)$. Also $\alpha-1+\beta \le 1+2\|v\|$. Now by Proposition 2.7, $v = (\alpha-1)e+\beta u \in (1+2\|v\|)K \subset V^+$. Hence V^+ is Archimedean.

Now it follows that (V, V^+, e) is an order unit space. We also note that $K = \{v \in V^+ : \|v\| \le 1\}$. Thus

$$C = Lead(K) = \{v \in V^+ : \|v\| = 1\}.$$

We show that $S = C\cap(e-C)$. Since $S \subset C$ and since $e-S = S$ by condition 1.1(1), we have $S \subset C \bigcap (e-C)$. Now let $w \in C\bigcap(e-C)$. Then $w, e-w \in C$. Find $u, v \in S$ and $\alpha, \beta \in [0,1]$ such that

$$w = \alpha u+(1-\alpha)e = e-\alpha(e-u)$$

and

$$e-w = \beta v+(1-\beta)e = e-\beta(e-v)$$

so that $e = \alpha(e-u)+\beta(e-v)$. Now by Lemma 2.1(3), we get $\alpha = 1 = \beta$ whence $w = u \in S$. Thus $S = \{v \in V^+ : \|v\| = 1 = \|e-v\|\}$. ☐

3. The concrete skeleton

In this section, we shall prove Theorem 1.3. First we discuss some properties of the positive elements with norm one in an order unit space. For an order unit space (V, e), the set of states of V is denoted by $S(V)$.

Proposition 3.1. *Let (V, e) be an order unit space of dimension ≥ 2. Consider $C_V = \{u \in V^+ : \|u\| = 1\}$.*

1. *Fix $u \in C_V$ with $u \ne e$ and consider the one dimensional affine subspace*

$$L(u) = \{u_\lambda := e-\lambda(e-u) : \lambda \in \mathbb{R}\}$$

of V. Then
   - *(a) $\{u_\lambda : \lambda \in \mathbb{R}\}$ is decreasing;*
   - *(b) $u_\lambda \in V^+$ if and only if $\lambda\|e-u\| \le 1$;*
   - *(c) $\|u_\lambda\| = \max\{1, |\lambda\|e-u\|-1|\}$ for every $\lambda \in \mathbb{R}$;*

   (d) *There exists a unique* $\bar{u} \in L(u)$ *such that* $\bar{u}, e - \bar{u} \in C_V$.

2. *For* $u, v \in C_V$, *we have either* $L(u) \bigcap L(v) = \{e\}$ *or* $L(u) = L(v)$.

Proof. $(1)(a)$ follows from the construction of u_λ.

$(1)(b)$: Since $\|u\| = 1$, we have $e - u \in V^+$. Let $\lambda \geq 0$. Then $u_\lambda \in V^+$ if and only if $\pm\lambda(e - u) \leq e$ or equivalently, $\lambda\|e - u\| \leq 1$. If $\lambda < 0$, then $u_\lambda \in V^+$ trivially.

$(1)(c)$: First we note that since $e - u \in V^+ \setminus \{0\}$, there exists $f_u \in S(V)$ such that $\|e - u\| = f_u(e - u) = 1 - f_u(u)$. Also then, $f_u(u) \leq f(u)$ for all $f \in S(V)$ with $f_u(u) < 1$. Fix $\lambda \in \mathbb{R}$.

Case 1. $\lambda \geq 0$. Then for $k \in \mathbb{R}$, we have $u_\lambda \leq ke$, that is, $\lambda u \leq (k-1+\lambda)e$ if and only if $k \geq 1$. Next, for $l \in \mathbb{R}$, we have $le + u_\lambda \in V^+$, that is, $(l + 1 - \lambda)e + \lambda u \in V^+$ if and only if $l + 1 - \lambda + \lambda f_u(u) \geq 0$ as $f_u(u) \leq f(u)$ for all $f \in S(V)$. In other words, $le + u_\lambda \in V^+$ if and only if $l \geq \lambda\|e - u\| - 1$. Thus for $\lambda \geq 0$, we have

$$\|u_\lambda\| = \inf\{\alpha > 0 : \alpha e \pm u_\lambda \in V^+\} = \max\{1, \lambda\|e - u\| - 1\}.$$

Case 2. $\lambda < 0$. Then $le + u_\lambda \in V^+$ for all $l \geq 0$. Next, for $k \in \mathbb{R}$, we have $u_\lambda \leq ke$, that is, $(k - 1 + \lambda)e - \lambda u \geq 0$ if and only $k - 1 + \lambda - \lambda f_u(u) \geq 0$ for $f_u(u) \leq f(u)$ for all $f \in S(V)$ and $-\lambda > 0$. Thus $u_\lambda \leq ke$ if and only if $k \geq -\lambda\|e - u\| + 1$. Therefore, for $\lambda < 0$, we have

$$\|u_\lambda\| = \inf\{\alpha > 0 : \alpha e \pm u_\lambda \in V^+\} = 1 - \lambda\|e - u\|.$$

Summing up, for any $\lambda \in \mathbb{R}$, we have

$$\|u_\lambda\| = \max\{1, |\lambda\|e - u\| - 1|\}.$$

$(1)(d)$: Put $\bar{u} = e - \|e - u\|^{-1}(e - u)$. Then by (c), $\|\bar{u}\| = 1$. Also by construction, $\|e - \bar{u}\| = 1$. Next, assume that $u_\lambda \in L(u)$ is such that $u_\lambda, e - u_\lambda \in C_V$. Then, as $\|e - u_\lambda\| = 1$, we get $|\lambda|\|e - u\| = 1$. If $\lambda = -\|e - u\|^{-1}$, then $\|u_\lambda\| = 2$ so we must have $\lambda = \|e - u\|^{-1}$. Thus $u_\lambda = \bar{u}$.

(2): Let $w \in L(u) \bigcap L(v)$ with $w \neq e$. Then there are $\lambda, \mu \in \mathbb{R} \setminus \{0\}$ such that $w = e - \lambda(e - u) = e - \mu(e - v)$. Thus $\lambda(e - u) = \mu(e - v)$. Let $\alpha \in \mathbb{R}$. Then

$$e - \alpha(e - u) = e - \alpha\lambda\mu^{-1}(e - v)$$

so that $L(u) \subset L(v)$. Now by symmetry, we have $L(u) = L(v)$. $\square$

The following statements can be verified easily.

Corollary 3.2. *Under the assumptions of Proposition 3.1, we have*

1. $\|u_\lambda\| = 1$ *if and only if* $0 \leq \lambda\|e - v\| \leq 2$;
2. $\{\|u_\lambda\| : \lambda \in (-\infty, 0]\}$ *is strictly decreasing;*
3. $\{\|u_\lambda\| : \lambda \in \left[\frac{2}{\|e-u\|}, \infty\right)\}$ *is strictly increasing:*
4. $C_V \bigcap (e - C_V) = \{\bar{u} : u \in C_V\}$;
5. $C(u) := L(u) \bigcap C_V = [e, \bar{u}]$; *and*
6. $L(u) \bigcap (e - C_V) = \{\bar{u}\}$.

Let (V, e) be an order unit space and assume that $u, v \in V^+ \setminus \{0\}$. Let us recall that $u \perp_\infty v$ if and only if $\| \|u\|^{-1} u + \|v\|^{-1} v\| = 1$ [11, Theorem 3.3].

Lemma 3.3. *Let (V, e) be an order unit space and let $u \in C_V$. Then the following statements are equivalent:*

1. *$u \in S_V := C_V \cap (e - C_V)$;*
2. *$u \perp_\infty (e - u)$;*
3. *there exists $v \in C_V$ such that $u \perp_\infty v$;*
4. *there exists $v \in C_V$ such that $u + v \in C_V$;*
5. *there exists a state f of V such that $f(u) = 0$.*

Proof. If $u \in S_V$, then $u, e - u \in V^+$ with $\|u\| = 1 = \|e - u\|$. Also then $\|u + e - u\| = \|e\| = 1$ so that $u \perp_\infty (e - u)$. Thus (1) implies (2). Also, (2) implies (3) trivially.

Next, let $u \perp_\infty v$ for some $v \in C_V$. Then $\|u+v\| = 1$ so that $u+v \in C_V$. That is, (3) implies (4).

Now, if $u + v \in C_V$, then $u + v \leq e$. Thus $v \leq e - u \leq e$ and we have

$$1 = \|v\| \leq \|e - u\| \leq \|e\| = 1.$$

Therefore, $u \in S_V$ that is, (4) implies (1).

Again, if $u \in S_V$, then $e - u \in C_V$. Thus there exists a state f of V such that $1 = f(e - u) = 1 - f(u)$, or equivalently, $f(u) = 0$. Therefore, (1) implies (5).

Conversely, if $f(u) = 0$ for some state f on V, then $f(e-u) = 1$ so that $\|e - u\| \geq 1$. Also, as $0 \leq u \leq e$, we have $0 \leq e - u \leq e$ so that $\|e - u\| \leq 1$. Thus $e - u \in C_V$ so $u \in S_V$. Hence (5) implies (1). □

Proof of Theorem 1.3. First we prove that S_V is a skeleton in V with e as its head. By construction, we have $e - u \in S_V$ whenever $u \in S_V$. Also we note that $S_V \subset V^+$. In fact, if $u \in S_V$, then $u, e - u \leq e$.

Let $u, v \in S_V$ and $\lambda \in (0, 1)$. Then $x := \lambda u + (1 - \lambda)v \neq 0$. Put $x_1 = \|x\|^{-1} x$. Then $x_1 \in C_V$. Thus by Corollary 3.2(5), there exists a $w \in S_V$ and $\alpha_1 \in [0, 1]$ such that $x_1 = \alpha_1 e + (1 - \alpha_1) w$. Now it follows that $\lambda u + (1 - \lambda) v = \alpha e + \beta w$ where $\alpha = \alpha_1 \|\lambda u + (1 - \lambda) v\|$ and $\beta = (1 - \alpha_1)\|\lambda u + (1 - \lambda) v\|$. Here $\alpha, \beta \in \mathbb{R}^+$ with $\alpha + \beta = \|\lambda u + (1 - \lambda) v\| \leq 1$.

Next, let $e = \sum_{i=1}^n \alpha_i$ for some $u_1, \dots, u_n \in S_V$ and $\alpha_1, \dots, \alpha_n > 0$ with $n \geq 2$. By Lemma 3.3, There exist $f_1, \dots, f_n \in S(V)$ such that $f_i(u_i) = 0$ for $i = 1, \dots, n$. Thus

$$1 = f_j(e) = f_j\left(\sum_{i=1}^n \alpha_i u_i\right) = \sum_{i \neq j} \alpha_i f_j(u_i) \leq \sum_{i \neq j} \alpha_i.$$

Hence S_V is a skeleton in V with e as its head. Also then $C_V = \bigcup_{u \in S_V} [e, u]$ and $K_V = \mathrm{co}(C_V) = \bigcup_{\alpha \in [0,1]} \alpha C_V = [0, e]_o$. Thus $V^+ = \bigcup_{\lambda \in \mathbb{R}^+} \lambda C_V$ so that S_V generates (V, V^+, e). □

4. Some properties of the concrete skeleton

Recall that if (V, e) is an order unit space, then the corresponding (concrete) skeleton S_V with e as its head is given by

$$S_V := C_V \cap (e - C_V) = \{v \in V^+ : \|v\| = 1 = \|e - v\|\}.$$

Here $C_V := \{v \in V^+ : \|v\| = 1\}$.

In this section, we discuss some of the properties and examples of the (concrete) skeleton corresponding to an order unit space.

Theorem 4.1. *Let (V, e) be an order unit space and let $Bd(V^+)$ denote the $\|\cdot\|$-boundary of V^+. Then $S_V = C_V \bigcap Bd(V^+)$.*

Proof. Let $u \in S_V$. Then by Lemma 3.3, $u \in C_V$ and there exists $f \in S(V)$ such that $f(u) = 0$. Thus for any $\epsilon > 0$, we have $f(u - \epsilon e) = -\epsilon < 0$ so that $u - \epsilon e \notin V^+$. Hence $u \in \overline{(V \setminus V^+)}$ and consequently, $u \in Bd(V^+)$.

Conversely, let $v \in Bd(V^+)$. Then $v \in V^+$ so that $f(v) \geq 0$ for all $f \in S(V)$. We show that $f(v) = 0$ for some $f \in S(V)$.

Assume to the contrary that $f(v) > 0$ for all $f \in S(V)$. Then for each $f \in S(V)$, there exists $\epsilon_f > 0$ such that $f(v) > \epsilon_f$. Put

$$U_f := \{g \in V^* : g(v) > \epsilon_f\}.$$

Then $\{U_f : f \in S(V)\}$ is a weak*-open cover for $S(V)$. Since $S(V)$ is weak*-compact, there exist $f_1, \dots, f_n \in S(V)$, for some $n \in \mathbb{N}$, such that $S(V) \subset \bigcup_{i=1}^{n} U_{f_i}$. Put $\epsilon = \min\{\epsilon_{f_1}, \dots, \epsilon_{f_n}\}$. If $f \in S(V)$, then $f \in U_{f_i}$ for some i, $1 \leq i \leq n$. Thus $f(v) > \epsilon_{f_i} \geq \epsilon$ for all $f \in S(V)$. Let $w \in V$ with $\|v - w\| < \epsilon$. Then $v - w \leq \epsilon e$ so, for any $f \in S(V)$, we have $f(v) - f(w) \leq \epsilon f(e) = \epsilon$. In other words, $f(w) \geq f(v) - \epsilon > 0$ for all $f \in S(V)$. Hence $w \in V^+$. But this contradicts the assumption that $v \in Bd(V^+)$. So, there must exist an $f \in S(V)$ such that $f(v) = 0$. Now, by Lemma 3.3, we have $C_V \bigcap Bd(V^+) \subset S_V$ which completes the proof. $\square$

Remark 4.2. We have $Bd(V^+) = \bigcup_{u \in S_V} [0, \infty)u$. Also, for $u, v \in S_V$ with $u \neq v$, we have $[0, \infty)u \bigcap [0, \infty)v = \{0\}$. Thus the $\|\cdot\|$-boundary of V^+ is the disjoint union of the rays passing through S_V.

Proposition 4.3. *Let (V, e) be an order unit space and let $u, v \in S_V$ with $u \neq v$. Then the following statements are equivalent:*

1. *$(u, v) \bigcap S_V \neq \emptyset$;*
2. *there are states f and g of V such that $f(u) = 1 = f(v)$ and $g(u) = 0 = g(v)$;*
3. *$[u, v] \subset S_V$.*

Proof. Let $w \in (u, v) \bigcap S_V$. Then $w = (1-\alpha)u + \alpha v \in S_V$ for some $\alpha \in (0, 1)$. Find $f, g \in S(V)$ such that $f((1-\alpha)u + \alpha v) = 1$ and $g((1-\alpha)u + \alpha v) = 0$. Thus

$$1 = f((1-\alpha)u + \alpha v) = (1-\alpha)f(u) + \alpha f(v) \leq (1-\alpha) + \alpha = 1$$

so that $f(u) = 1 = f(v)$. Again

$$0 = g((1-\alpha)u + \alpha v) = (1-\alpha)g(u) + \alpha g(v)$$

so that $(1-\alpha)g(u) = 0 = \alpha g(v)$. Since $0 < \alpha < 1$, we get $g(u) = 0 = g(v)$.

Let $\lambda \in [0,1]$ and consider $x = (1-\lambda)u + \lambda v$. Then $x \in V^+$ with $\|x\| \le 1$. Also as

$$f(x) = (1-\lambda)f(u) + \lambda f(v) = 1 - \lambda + \lambda = 1,$$

we note that $x \in C_V$. Further,

$$g(x) = (1-\lambda)g(u) + \lambda g(v) = 0$$

so that $x \in S_V$. $\square$

Corollary 4.4. *Let $u, v \in S_V$ with $u \le v$. Then $[u, v] \subset S_V$.*

Proof. Put $w = \frac{1}{2}(u+v)$. Then $u \le w \le v$ and $e - v \le e - w \le e - u$. Since $u, v, e-u, e-v \in C_V$, we get that $w, e-w \in C_V$. Thus $w \in S_V$. Now by Proposition 4.3, we may conclude that $[u, v] \subset S_V$. $\square$

Now the next result follows immediately.

Corollary 4.5. *Let (V, e) be an order unit space and assume that $u_1, u_2 \in S_V$ with $u_1 \perp_\infty u_2$. Then*

1. $[u_1, e-u_2] \bigcup [u_2, e-u_1] \subset S_V$;
2. $((u_1, u_2) \bigcup (e-u_1, e-u_2)) \bigcap S_V = \emptyset$.

4.1. Direct sum of order unit spaces.

Next, we turn to describe $C_{\ell_\infty^n}$ and $S_{\ell_\infty^n}$.

Lemma 4.6. *Let (V_1, e_1) and (V_2, e_2) be any two order unit spaces. Consider $V = V_1 \times V_2$, $V^+ = V_!^+ \times V_2^+$ and $e = (e_1, e_2)$. Then (V, e) is also an order unit space and we have*

1. $C_V = (C_{V_1} \times [0, e_2]_o) \bigcup ([0, e_1]_o \times C_{V_2})$ *and*
2. $S_V = (S_{V_1} \times [0, e_2]_o) \bigcup ([0, e_1]_o \times S_{V_2}) \bigcup (C_{V_1} \times (e_2 - C_{V_2})) \bigcup ((e_1 - C_{V_1}) \times C_{V_2})$.

Proof. For $(v_1, v_2) \in V$, we have $\|(v_1, v_2)\| = \max\{\|v_1\|, \|v_2\|\}$. Thus $(u_1, u_2) \in C_V$ if and only if $u_1 \in V_1^+$, $u_2 \in V_2^+$ and $\max\{\|u_1\|, \|u_2\|\} = 1$. Therefore, $C_V = (C_{V_1} \times [0, e_2]_o) \bigcup ([0, e_1]_o \times C_{V_2})$. Now, as $(u_1, u_2) \in S_V$ if and only $(u_1, u_2), (e_1 - u_1, e_2 - u_2) \in C_V$, we may deduce that $S_V = (S_{V_1} \times [0, e_2]_o) \bigcup ([0, e_1]_o \times S_{V_2}) \bigcup (C_{V_1} \times (e_2 - C_{V_2})) \bigcup ((e_1 - C_{V_1}) \times C_{V_2})$. $\square$

Replace V_2 by $\mathbb{R}$. As $C_{\mathbb{R}} = \{1\}$ and $S_{\mathbb{R}} = \emptyset$, we may conclude the following:

Corollary 4.7. *Let (V, e) be an order unit space. Consider $\hat{V} = V \times \mathbb{R}$, $\hat{V}^+ = V^+ \times \mathbb{R}^+$ and $\hat{e} = (e, 1)$. Then $(\hat{V}, \hat{e})$ is an order unit space and we have*

1. $C_{\hat{V}} = (C_V \times [0,1]) \bigcup ([0, e_2]_o \times \{1\})$ *and*
2. $S_{\hat{V}} = (S_V \times [0,1]) \bigcup (C_V \times \{0\}) \bigcup ((e - C_V) \times \{1\})$.

Again using $C_1 = \{1\}$, $R_1 = \emptyset$ and following the induction on n, we can easily obtain the skeleton of ℓ_∞^n with the help of Corollary 4.7.

Corollary 4.8. *Fix $n \in \mathbb{N}$, $n \geq 2$. Put $C_n := C_{\ell_\infty^n}$ and $S_n := S_{\ell_\infty^n}$. Then*

1. $C_n = \{(\alpha_1, \dots, \alpha_n) : \min\{\alpha_i\} \geq 0$ *and* $\max\{\alpha_i\} = 1\}$ *and*
2. $S_n = \{(\alpha_1, \dots, \alpha_n) : \min\{\alpha_i\} = 0$ *and* $\max\{\alpha_i\} = 1\}$.

4.2. Adjoining a normed linear space to an order unit space

Let (V, e) be an order unit space and let X be a real normed linear space. Consider $V_X := V \oplus_1 X$ and put $V_X^+ = \{(v, x) : \|x\|e \leq v\}$ and $e_X = (e, 0)$. It was shown in [14] that (V_X, V_X^+, e_X) is an order unit space in such a way that the order unit norm coincides with the ℓ_1-norm on V_X. Here we describe the canopy and its periphery corresponding to V_X. For this purpose, we introduce the following notion.

Definition 4.9. Let (V, e) be an order unit space. Then $u \in V^+$ is said to be a *semi-skeletal* element if $u = \alpha(e - w) + (1 - \alpha)w$ for some $w \in S_V$ and $\alpha \in [0, 1]$. When $\alpha = \frac{1}{2}$, then $u = \frac{1}{2}e$ which is called the *central semi-skeletal* element. The set of all semi-skeletal elements of V is denoted by S_V^s.

Theorem 4.10. *Let (V, V^+, e) be an order unit space and let X a real normed linear space. Consider the corresponding order unit space $(V \oplus_1 X, V_X^+, e_X)$. Then the canopy and the periphery of $V \oplus_1 X$ are given by*

$$C_{V \oplus_1 X} = \{(u, x) \in V_X^+ : \|x\|e \leq u \text{ and } \|u\| + \|x\| = 1\}$$

and

$$S_{V \oplus_1 X} = \{(u, x) : u \in S_V^s \text{ and } \|x\| + \|u\| = 1\}.$$

Proof. Let $w \in S_V$, $\alpha \in [0, 1]$ and put $u = \alpha(e - w) + (1 - \alpha)w$. Then $u \in V^+$ and $\|u\| = \max\{\alpha, 1 - \alpha\}$. Let $x \in X$ with $\|x\| = 1 - \|u\|$. For definiteness, we assume that $\alpha \geq \frac{1}{2}$. Then $\|u\| = \alpha$ and $\|x\| = 1 - \alpha$. Now, as

$$\begin{aligned} u - \|x\|e &= \alpha(e - w) + (1 - \alpha)w - (1 - \alpha)e \\ &= (2\alpha - 1)(e - w) \in V^+ \end{aligned}$$

and $\|(u, x)\|_1 = 1$, we have $(u, x) \in C_{V \oplus_1 X}$. Further, as $e - u = (1 - \alpha)(e - w) + \alpha w$, we have $\|e - u\| = \alpha$ so that $\|e_X - (u, x)\|_1 = 1$. Thus $(u, x) \in S_{V \oplus_1 X}$.

Conversely, we assume that $(u, x) \in S_{V \oplus_1 X}$. Then $(u, x), (e - u, -x) \in V_X^+$ and we have $\|(u, x)\|_1 = 1 = \|(e - u, -x)\|_1$. Thus $\|x\|e \leq u$, $\|-x\|e \leq e - u$ and

$$\|u\| + \|x\| = 1 = \|e - u\| + \| - x\|.$$

Now, it follows that $\|u\| = \|e - u\| = 1 - \|x\|$ whence $(1 - \|u\|)e \leq u$. Thus

$$\begin{aligned} \|u - (1 - \|u\|)e\| &= \sup\{\phi\,(u - (1 - \|u\|)e) : \phi \in S(V)\} \\ &= \sup\{\phi(u) : \phi \in S(V)\} - (1 - \|u\|) \\ &= 2\|u\| - 1. \end{aligned}$$

If $2\|u\| - 1 = 0$, that is $\|u\| = \frac{1}{2}$, then $u = \frac{1}{2}e$ and is a (centrally) semi-skeletal element, for $(1 - \|u\|)e \leq u \leq \|u\|e$. So we may assume that $2\|u\| - 1 > 0$. Put $w = \frac{u - (1 - \|u\|)e}{2\|u\| - 1}$. Then $w \in V^+$ and consequently, $w \in C_V$.

Find $\phi \in S(V)$ such that $f\phi(e-u) = \|e-u\| = \|u\|$. Then $\phi(w) = 0$ so that $w \in S_V$. Further, we have $u = \|x\|(e-w)+(1-\|x\|)w$ for $\|x\| = 1-\|u\|$. Thus u is again a semi-skeletal element. $\square$

Remark 4.11. We can deduce from the proof of Theorem 4.10 that

$$S_V^s = \{u \in [0,e] : \|u\| = \|e-u\|\} = \{u \in V : \|u\| = \|e-u\| \leq 1\}.$$

5. Some applications

Lemma 5.1. *Let (V,e) be an order unit space of dimension ≥ 2 (so that $S_V \neq \emptyset$). Assume that $u \in S_V$ and consider*

$$P_u := \{\alpha e + \beta u : \alpha, \beta \in \mathbb{R}\}.$$

Then P_u is a unitally order isomorphic to ℓ_∞^2.

Proof. Consider the mapping $\chi : P_u \to \ell_\infty^2$ given by $\chi(\alpha e + \beta u) = (\alpha, \alpha+\beta)$ for all $\alpha, \beta \in \mathbb{R}$. Then χ is a unital bijection. We show that χ is an order isomorphism. We first assume that $\alpha e + \beta u \in V^+$. Since $u \in S_V$, we can find $f, g \in S(V)$ such that $f(u) = 1$ and $g(u) = 0$. Thus $0 \leq f(\alpha e + \beta u) = \alpha + \beta$ and $0 \leq g(\alpha e + \beta u) = \alpha$. Thus $(\alpha, \alpha+\beta) \in \ell_\infty^{2+}$. Conversely, we assume that $(\alpha, \alpha+\beta) \in \ell_\infty^{2+}$. Then $\alpha e + \beta u = \alpha(e-u) + (\alpha+\beta)u \in V^+$. $\square$

Let X be a (real) normed linear space and let S be a non-empty subset of X. We say that $\perp_\infty$ is *additive* in S, if for $x, y, z \in S$ with $x \perp_\infty y$ and $x \perp_\infty z$, we have $x \perp_\infty (y+z)$.

Theorem 5.2. *Let (V,e) be an order unit space with $\dim(V) \geq 2$. Then V contains a copy of ℓ_∞^2 as an order unit subspace. Moreover, V contains a copy of ℓ_∞^n $(n \geq 2)$ as an order unit subspace if and only if there exist $u_1, \dots, u_n \in S_V$ such that $u_i \perp_\infty u_j$ for all $i, j \in \{1, \dots, n\}$ with $i \neq j$, $\sum_{i=1}^n u_i = e$ and $\perp_\infty$ is additive in the linear span of $u_1, \dots, u_n$.*

Proof. It follows from Lemma 5.1 that V contains a copy of ℓ_∞^2 as an order unit subspace.

Next, we assume that W is an order unit subspace of V and $\Gamma : \ell_\infty^n \to W$ is a surjective unital order isomorphism. Put $\gamma(e_i) = u_i$ for $i = 1, \dots, n$ where $\{e_1, \dots, e_n\}$ is the standard unit basis of ℓ_∞^n. Then $u_1, \dots, u_n \in C_V$ with $\sum_{i=1}^n u_i = e$. Consider the bi-orthonormal system $\{f_1, \dots, f_n\}$ in ℓ_1^n so that $f_i(e_j) = \delta_{ij}$. Then $\{f_1 \circ \Gamma^{-1}, \dots, f_n \circ \Gamma^{-1}\}$ is the set of pure states of W. We can extend $f_i \circ \Gamma^{-1}$ to a pure state g_i of V for each $i = 1, \dots, n$. Then $g_i(u_j) = \delta_{ij}$ so that $u_1, \dots, u_n \in S_V$ and we have $u_i \perp_\infty u_j$ for all $i, j \in \{1, \dots, n\}$ with $i \neq j$. Also, if $\alpha_1, \dots, \alpha_n \in \mathbb{R}$, then

$$\begin{aligned} \left\| \sum_{i=1}^n \alpha_i u_i \right\| &= \left\| \sum_{i=1}^n \alpha_i e_i \right\|_\infty \\ &= \max\{|\alpha_i| : 1 \leq i \leq n\} \\ &= \max\{\|\alpha_i u_i\| : 1 \leq i \leq n\}. \end{aligned}$$

Thus $\perp_\infty$ is additive in the linear span of $u_1, \dots, u_n$.

Conversely, we assume that there exist $u_1, \dots, u_n \in S_V$ such that $u_i \perp_\infty u_j$ for all $i, j \in \{1, \dots, n\}$ with $i \neq j$, $\sum_{i=1}^n u_i = e$ and $\perp_\infty$ is additive in the linear span U of $u_1, \dots, u_n$. Define $\Phi : U \to \ell_\infty^n$ by $\Phi(\sum_{i=1}^n \alpha_i u_i) = (\alpha_i)$. Then Φ is a unital linear bijection. Also

$$\begin{aligned} \left\| \Phi \left(\sum_{i=1}^n \alpha_i u_i \right) \right\| &= \|(\alpha_i)\|_\infty \\ &= \max\{|\alpha_i| : 1 \leq i \leq n\} \\ &= \max\{\|\alpha_i u_i\| : 1 \leq i \leq n\} \\ &= \left\| \sum_{i=1}^n \alpha_i u_i \right\| \end{aligned}$$

as $\perp_\infty$ is additive on U. Thus that Φ is an isometry. Now, being a unital linear surjective isometry, Φ is a unital order isomorphism. $\square$

Remark 5.3. An order unit space of dimension 2 is unitally isometric to ℓ_∞^2. However, we shall show in a forthcoming paper that for the dimension greater than 2, there exist non-isometric order unit spaces of the same dimension.

Proposition 5.4. *Let (V, e) be an order unit space and let $u, v \in S_V$. Then, either $P_u = P_v$ or $P_u \bigcap P_v = \mathbb{R}e$.*

Proof. Let $w \in P_u \bigcap P_v$ be such that $w \notin \mathbb{R}e$. Without any loss of generality, we assume that $\|w\| = 1$. Consider $w_1 := \frac{1}{2}(e+w)$. Then $w_1 \in V^+$. Also then $w_1 \in P_u \bigcap P_v$. So, without any loss of generality again, we further assume that $w \in V^+$, that is, $w \in C_V$.

Since $w \in P_u$, we have $w = \alpha e + \beta u$ for some $\alpha, \beta \in \mathbb{R}$. As $w \in C_V$, we have $\alpha \geq 0$, $\alpha + \beta \geq 0$ and $\max\{\alpha, \alpha + \beta\} = 1$. Since $w \notin \mathbb{R}e$, we have $\beta \neq 0$. If $\beta > 0$, then we have $1 = \alpha + \beta > \alpha \geq 0$. Thus $w = \alpha e + (1 - \alpha)u \in [e, u]$. Next if $\beta < 0$, then $1 = \alpha > \alpha + \beta \geq 0$ so that $-1 \leq \beta < 0$. Thus $w = e + \beta u = e - (-\beta)(e - (e - u)) \in [e, e - u]$. Summing up, we have $w \in [e, u] \bigcup [e, e - u]$. Similarly, as $w \in P_v$, we also have $w \in [e, v] \bigcup [e, e - v]$. Thus $w \in ([e, u] \bigcup [e, e - u]) \bigcap ([e, v] \bigcup [e, e - v])$. Since $w \neq e$, using Corollary 2.4, we conclude that one of the equalities $[e, u] = [e, v]$ or $[e, u] = [e, e - v]$, or $[e, e - u] = [e, v]$, or $[e, e - u] = [e, e - v]$ hold. In other words, either $u = v$ or $u = e - v$. In both the situations, we have $P_u = P_v$. $\square$

Theorem 5.5. *Let (V, e) be an order unit space with $\dim V \geq 2$. Then $V = \bigcup\{P_u : u \in S_V\}$ in such a way that $\bigcap\{P_u : u \in S_V\} = \mathbb{R}e$ and if $v \in V$ with $v \notin \mathbb{R}e$, then there exists a unique $w \in S_V$ such that $v \in P_w = P_{(e-w)}$.*

Proof. Let $v \in V$ with $v \notin \mathbb{R}e$. For simplicity, we assume that $\|v\| = 1$. Put $v_1 = \frac{1}{2}(e + v)$ and $v_2 = \frac{1}{2}(e - v)$. Then $v_1, v_2 \in V^+$ and $v_1, v_2 \notin \mathbb{R}e$. Also then $v = v_1 - v_2$ and $1 = \|v\| = \max\{\|v_1\|, \|v_2\|\}$. Replacing v by $-v$, if required, we further assume that $0 < \|v_2\| \leq \|v_1\| = 1$, that is, $\|e + v\| = 2$. Thus we can find $f \in S(V)$ such that $1 + f(v) = f(e + v) = 2$. Therefore, $f(v) = 1$. Put $w = \|e - v\|^{-1}(e - v)$. Then $w \in C_V$. Further, $f(e - w) =$

$1 - \|e-v\|^{-1}(1-f(v)) = 1$ so that $e - w \in C_V$. Thus $w \in S_V$. Now $v = e - \|e-v\| w$ so that $v \in P_w = P_{(e-w)}$.

Uniqueness of w follows from Proposition 5.4. $\square$

Remark 5.6. Let $v \in V$ with $v \notin \mathbb{R}e$. Find $f \in S(V)$ such that $\|v\| = |f(v)|$. If $\|v\| = f(v)$, then $\|e + \|v\|^{-1} v\| = 2$. Thus v has a unique representation $v = \lambda e + \mu w$ in P_w where $w = \|\|v\|e - v\|^{-1}(\|v\|e - v) \in S_V$ and $\lambda = \|v\|$ and $\mu = -\|\|v\|e - v\|$. When $\|v\| = -f(v)$, we replace v by $-v$.

Acknowledgment

The author is grateful to the referee for valuable suggestions. The author was partially supported by Science and Engineering Research Board, Department of Science and Technology, Government of India sponsored Mathematical Research Impact Centric Support project (reference no. MTR/2020/000017).

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RESPONSE TO THE REFEREE'S REPORT

First of all, I thank the referee for a detailed report and a fair opinion. Please find below my response to the referee's comments and observations. I have thoroughly revised the paper as per suggestions made by the referee and including my own observations.

I begin with commenting on the introduction of several terms. As this paper is formulated to look at the order unit spaces from a new point of view, some new terms are inevitable. However, as observed by the referee, I have abandoned the terms periphery, peripheral elements, canopy, summit and kept terms skeleton with a head, semi-skeletal point (replacing semi-peripheral point), lead point (this term was introduced in another paper and used so far in two paper besides this one). I have corrected all the typos indicated by the referee and to be best of my own observations. I have corrected the proofs wherever referee found them wrong or inadequate.

Specific comments

1. The notion "absolutely ∞-orthogonal" has been used in the second paragraph of the paper to describe projections in a C*-algebra.
2. Observation incorporated.
3. It was a typo; corrected.
4. The sentence has been rephrased.
5. An alternative proof has been given.
6. Proof included.
7. Details included.
8. Proof included.
9. Proof abridged.
10. Suggestion incorporated.
11. Statement amended.
12. The referred result has been recalled before stating the result.
13. Explained.
14. The two types of intervals have been distinguished and explained in the introduction of the paper.
15. Suggestion incorporated.
16. A new proof has be given.
17. Observation agreed.
18. Observation agreed.
19. Explanation given before the statement of the result.
20. The result sponged.
21. The remark sponged.

All the typos have been corrected. Notations for the two types of intervals have been mentioned and explained at the end of introduction. The notation for the state space has been discussed in a suitable place of the paper.