Monotonicity of $\alpha \mapsto D_{\alpha,z}$ (2)

Here, we show the monotone increasing of $\alpha \mapsto D_{\alpha,z}(\psi\|\varphi)$ on $(0,\infty)$ for any $z>0$ when $\mathcal{M}$ is finite-dimensional. For the finite-dimensional case, we may assume that $\mathcal{M}=\mathbb{M}_n$, the $n\times n$ matrix algebra. For each $A\in\mathbb{M}_n$ we write $\lambda(A)=(\lambda_1(A),\dots,\lambda_n(A))$ for the eigenvalues of A in decreasing order (with multiplicities).

Lemma 0.1. *Let $A_j,B_j\in\mathbb{M}_n^+$, $j=1,2$, be such that $A_1A_2=A_2A_1$ and $B_1B_2=B_2B_1$. Then*

$$\lambda\big((A_1^{1/2}A_2^{1/2})^{1/2}(B_1^{1/2}B_2^{1/2})(A_1^{1/2}A_2^{1/2})^{1/2}\big)\prec_{\log}\lambda(A_1^{1/2}B_1A_1^{1/2})^{1/2}\lambda(A_2^{1/2}B_2A_2^{1/2})^{1/2}, \tag{0.1}$$

that is,

$$\prod_{i=1}^k\lambda_i\big((A_1^{1/2}A_2^{1/2})^{1/2}(B_1^{1/2}B_2^{1/2})(A_1^{1/2}A_2^{1/2})^{1/2}\big)\le\prod_{i=1}^k\lambda_i(A_1^{1/2}B_1A_1^{1/2})^{1/2}\lambda_i(A_2^{1/2}B_2A_2^{1/2})^{1/2} \tag{0.2}$$

for any $k=1,\dots,n$ with equality for $k=n$.

Proof. By continuity we may and do assume that A_j,B_j are all invertible. First prove the equality in (0.2) for $k=1$, equivalently,

$$\big\|(A_1^{1/2}A_2^{1/2})^{1/2}(B_1^{1/2}B_2^{1/2})(A_1^{1/2}A_2^{1/2})^{1/2}\big\|\le\big\|A_1^{1/2}B_1A_1^{1/2}\big\|^{1/2}\big\|A_2^{1/2}B_2A_2^{1/2}\big\|^{1/2}. \tag{0.3}$$

To do this, by replacing B_1,B_2 with b_1B_1,b_2B_2, respectively, for some $b_1,b_2>0$, it suffices to assume that $\big\|A_1^{1/2}B_1A_1^{1/2}\big\|=\big\|A_2^{1/2}B_2A_2^{1/2}\big\|=1$ so that $A_j^{1/2}B_jA_j^{1/2}\le I$, i.e., $B_j\le A_j^{-1}$ for $j=1,2$. We then have

$$B_1^{1/2}B_2^{1/2}=B_1\#B_2\le A_1^{-1}\#A_2^{-1}=(A_1^{1/2}A_2^{1/2})^{-1}$$

so that

$$(A_1^{1/2}A_2^{1/2})^{1/2}(B_1^{1/2}B_2^{1/2})(A_1^{1/2}A_2^{1/2})^{1/2}\le I.$$

This yields (0.3).

Next, for any $k=1,\dots,n$, consider the antisymmetric tensor powers $A_j^{\wedge k}$ and $B_j^{\wedge k}$ (see, e.g., [1, Sec. 4.6]). We then have (see [1, Lemma 4.6.3])

$$\begin{aligned}&\prod_{i=1}^k\lambda_i\big((A_1^{1/2}A_2^{1/2})^{1/2}(B_1^{1/2}B_2^{1/2})(A_1^{1/2}A_2^{1/2})^{1/2}\big)\\&\quad=\lambda_1\big(((A_1^{1/2}A_2^{1/2})^{1/2}(B_1^{1/2}B_2^{1/2})(A_1^{1/2}A_2^{1/2})^{1/2})^{\wedge k}\big)\\&\quad=\lambda_1\big(((A_1^{\wedge k})^{1/2}(A_2^{\wedge k})^{1/2}\big)^{1/2}\big((B_1^{\wedge k})^{1/2}(B_2^{\wedge k})^{1/2}\big)\big((A_1^{\wedge k})^{1/2}(A_2^{\wedge k})^{1/2}\big)^{1/2}\big),\end{aligned}$$

$$\begin{aligned}&\prod_{i=1}^k\lambda_i(A_1^{1/2}B_1A_1^{1/2})^{1/2}\lambda_i(A_2^{1/2}B_2A_2^{1/2})^{1/2}\\&\quad=\lambda_1\big((A_1^{1/2}B_1A_1^{1/2})^{\wedge k}\big)^{1/2}\ \lambda_1\big((A_2^{1/2}B_2A_2^{1/2})^{\wedge k}\big)^{1/2}\end{aligned}$$

$$= \lambda_1\big((A_1^{\wedge k})^{1/2} B_1^{\wedge k} (A_1^{\wedge k})^{1/2}\big)^{1/2} \lambda_1\big((A_2^{\wedge k})^{1/2} B_2^{\wedge k} (A_2^{\wedge k})^{1/2}\big)^{1/2}.$$

When applied to $A_j^{\wedge k}, B_j^{\wedge k}$, the above case implies the inequality for k in (0.2). Equality for $k = n$ is clear from

$$\begin{aligned}\det(A_1^{1/2}A_2^{1/2})^{1/2}(B_1^{1/2}B_2^{1/2})(A_1^{1/2}A_2^{1/2})^{1/2} &= (\det A_1)^{1/2}(\det B_1)^{1/2}(\det A_2)^{1/2}(\det B_2)^{1/2} \\ &= \big(\det A_1^{1/2}B_1A_1^{1/2}\big)^{1/2}\big(\det A_2^{1/2}B_2A_2^{1/2}\big)^{1/2}.\end{aligned}$$

□

Lemma 0.2. *Let $\rho, \sigma \in \mathbb{M}_n^+$ with $s(\rho) \not\perp s(\sigma)$. Then $\alpha \mapsto \log D_{\alpha,z}(\rho\|\sigma)$ is convex on $(0,1)$ for any $z > 0$. If furthermore $s(\rho) \le s(\sigma)$, then $\alpha \mapsto \log D_{\alpha,z}(\rho\|\sigma)$ is convex on $(1,\infty)$ for any $z > 0$.*

Proof. Since $s(\rho) \not\perp s(\sigma)$, note that $0 < Q_{\alpha,z}(\rho\|\sigma) < \infty$ for all $\alpha \in (0,1)$ and $z > 0$. Let $\alpha_1, \alpha_2 \in (0,1)$ and $z > 0$. Apply (0.1) to $A_j := \rho^{\alpha_j/z}$ and $B_j := \sigma^{(1-\alpha_j)/z}$. We then have

$$\lambda\big(\rho^{\frac{\alpha_1+\alpha_2}{4z}}\sigma^{\frac{2-\alpha_1-\alpha_2}{2z}}\rho^{\frac{\alpha_1+\alpha_2}{4z}}\big)^z \prec_{\log} \lambda\big(\rho^{\frac{\alpha_1}{2z}}\sigma^{\frac{1-\alpha_1}{z}}\rho^{\frac{\alpha_1}{2z}}\big)^{z/2}\lambda\big(\rho^{\frac{\alpha_2}{2z}}\sigma^{\frac{1-\alpha_2}{z}}\rho^{\frac{\alpha_2}{2z}}\big)^{z/2}.$$

Since log-majorization $\prec_{\log}$ implies weak majorization $\prec_w$ (see [1, Proposition 4.1.6]), we obtain

$$\begin{aligned}Q_{\frac{\alpha_1+\alpha_2}{2},z}(\rho\|\sigma) &= \mathrm{Tr}\,\big(\rho^{\frac{\alpha_1+\alpha_2}{4z}}\sigma^{\frac{2-\alpha_1-\alpha_2}{2z}}\rho^{\frac{\alpha_1+\alpha_2}{4z}}\big)^z \\ &= \sum_{i=1}^n \lambda_i\big(\rho^{\frac{\alpha_1+\alpha_2}{4z}}\sigma^{\frac{2-\alpha_1-\alpha_2}{2z}}\rho^{\frac{\alpha_1+\alpha_2}{4z}}\big)^z \\ &\le \sum_{i=1}^n \lambda_i\big(\rho^{\frac{\alpha_1}{2z}}\sigma^{\frac{1-\alpha_1}{z}}\rho^{\frac{\alpha_1}{2z}}\big)^{z/2}\lambda_i\big(\rho^{\frac{\alpha_2}{2z}}\sigma^{\frac{1-\alpha_2}{z}}\rho^{\frac{\alpha_2}{2z}}\big)^{z/2} \\ &\le \left[\sum_{i=1}^n \lambda_i\big(\rho^{\frac{\alpha_1}{2z}}\sigma^{\frac{1-\alpha_1}{z}}\rho^{\frac{\alpha_1}{2z}}\big)^z\right]^{1/2}\left[\sum_{i=1}^n \lambda_i\big(\rho^{\frac{\alpha_2}{2z}}\sigma^{\frac{1-\alpha_2}{z}}\rho^{\frac{\alpha_2}{2z}}\big)^z\right]^{1/2} \\ &= Q_{\alpha_1,z}(\rho\|\sigma)^{1/2}Q_{\alpha_2,z}(\rho\|\sigma)^{1/2}.\end{aligned}$$

This shows that $\alpha \in (0,1) \mapsto \log Q_{\alpha,z}(\rho\|\sigma)$ is midpoint convex. Since midpoint convexity implies convexity for continuous functions, the first assertion follows.

The proof of the latter assertion is similar by regarding $\sigma^{(1-\alpha_j)/z}$ as $(\sigma^{-1})^{(\alpha_j-1)/z}$ for $\alpha_j > 1$, where σ^{-1} is the generalized inverse of σ. □

Proposition 0.3. *Let $\rho, \sigma \in \mathbb{M}_n^+$ with $\rho \ne 0$. Then for every $z > 0$ the function $\alpha \mapsto D_{\alpha,z}(\rho\|\sigma)$ is monotone increasing on $(0,\infty)$, where $D_{1,z}(\rho\|\sigma) := D_1(\rho|\sigma) = \frac{D(\rho\|\sigma)}{\mathrm{Tr}\,\rho}$. In particular,*

$$D_{\alpha,z}(\rho\|\sigma) \le D_1(\rho\|\sigma) \le D_{\alpha',z}(\rho\|\sigma)$$

for all $\alpha \in (0,1)$ and $\alpha' \in (1,\infty)$.

Proof. It is known [3, Proposition III.36] that $\lim_{\alpha\to1} D_{\alpha,z}(\rho\|\sigma) = D_1(\rho\|\sigma)$ for any $z > 0$. So it suffices to show the monotone increasing of $\alpha \mapsto D_{\alpha,z}(\rho\|\sigma)$ on $(0,1)$ and on $(1,\infty)$ separately. If $s(\rho) \perp s(\sigma)$ then $D_{\alpha,z}(\rho\|\sigma) = \infty$ for all $\alpha > 0$. Hence assume that $s(\rho) \not\perp s(\sigma)$. Since

$$\lim_{\alpha\nearrow1} Q_{\alpha,z}(\rho\|\sigma) = \lim_{\alpha\nearrow1} \mathrm{Tr}\,\big(\rho^{\frac{\alpha}{2z}}\sigma^{\frac{1-\alpha}{z}}\rho^{\frac{\alpha}{2z}}\big)^z = \mathrm{Tr}\,(\rho^{1/2z}s(\sigma)\rho^{1/2z})^z \le \mathrm{Tr}\,\rho,$$

the first assertion of Lemma 0.2 shows that

$$D_{\alpha,z}(\rho\|\sigma)=\frac{\log Q_{\alpha,z}(\rho\|\sigma)-\log\operatorname{Tr}\rho}{\alpha-1}$$

is increasing on $(0,1)$.

Next, let us show the increasing of $\alpha\mapsto D_{\alpha,z}(\rho\|\sigma)$ on $(1,\infty)$. For this, since $D_{\alpha,z}(\rho\|\sigma)=\infty$ for all $\alpha\ge 1$ if $s(\rho)\not\le s(\sigma)$, we may assume that $s(\rho)\le s(\sigma)$. Since, with the generalized inverse σ^{-1},

$$\lim_{\alpha\searrow 1}Q_{\alpha,z}(\rho\|\sigma)=\lim_{\alpha\searrow 1}\operatorname{Tr}\left(\rho^{\frac{\alpha}{2z}}(\sigma^{-1})^{\frac{\alpha-1}{z}}\rho^{\frac{\alpha}{2z}}\right)^z=\operatorname{Tr}(\rho^{1/2z}s(\sigma)\rho^{1/2z})^z=\operatorname{Tr}\rho,$$

the result follows as above from the second assertion of Lemma 0.2. □

Remark 0.4. By a slight modification of the proof of Lemma 0.1 one can slightly extend the log-majorization in (0.1) as follows: for any $r\in(0,1)$,

$$\lambda\big((A_1^rA_2^{1-r})^{1/2}(B_1^rB_2^{1-r})(A_1^rA_2^{1-r})^{1/2}\big)\prec_{\log}\lambda(A_1^{1/2}B_1A_1^{1/2})^r\lambda(A_2^{1/2}B_2A_2^{1/2})^{1-r}. \tag{0.4}$$

In particular, when $A_2=B_2=I$, this log-majorization becomes Araki's log-majorization $\lambda(A_1^{r/2}B_1^rA_1^{r/2})\prec_{\log}\lambda((A_1^{1/2}B_1A_1^{1/2})^r)$ if $0<r<1$. Hence (0.4) is an extension of Araki's log-majorization, that seems new up to my knowledge and somewhat interesting from the matrix analysis point of view.

Remark 0.5. For our purpose, it is desirable to extend Lemma 0.1 to the von Neumann algebra setting. In view of Haagerup's reduction theory, our target is the case where $\mathcal{M}$ is a finite von Neumann algebra with a faithful normal finite trace τ. For this, Kosaki's proof [2] of the ALT inequality in the von Neumann algebra case might be helpful (?)

Remark 0.6. When $\mathcal{M}$ is an injective von Neumann algebra (in particular, $\mathcal{M}=B(\mathcal{H})$), it is immediate to see that Proposition 0.3 holds for every $\psi,\varphi\in\mathcal{M}_*^+$ with $\psi\ne 0$ and for any (α,z) in the DPI range of $D_{\alpha,z}$.

References

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