On the category of affine subspaces

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1 Once more from the top

We present some important categories.

1.1 The category FinVect

Let FinVect be the category of finite dimensional real vector spaces with linear maps. Then $(\mathrm{FinVect}, \otimes, \mathbb{R})$ is a symmetric monoidal category, with the usual tensor product of vector spaces. With the usual duality $(-)^* : V \mapsto V^*$ of vector spaces, FinVect is compact closed. Put

$$e_U : U \otimes U^* \to \mathbb{R}, \qquad e_u(u \otimes u^*) = \langle\, u^*, u \,\rangle,$$

then e_U is the cap for the duality of U and U^*. The corresponding element $\eta_U \in (U \otimes U^*) = U^* \otimes U$ is the cup, given by

$$\eta_U = \sum_i e_i^* \otimes e_i$$

where $\{e_i\}$ is a basis of U and $\{e_i^*\}$ the dual basis of U^*, determined by $\langle\, e_i^*, e_j \,\rangle = \delta_{ij}$. It is easily verified that η_U does not depend on the choice of the basis $\{e_i\}$.

By compactness the internal hom is $[U, V] = U^* \otimes V$ and the evaluation map $U \otimes [U, V] \to V$ is given by

$$eval_{U,V} = e_U \otimes V : U \otimes U^* \otimes V \to V.$$

For any $w \in U^* \otimes V$, we obtain a linear map $\hat{w} : U \to V$ by

$$\hat{w}(u) = (e_U \otimes V)(u \otimes w),$$

(we write V for the identity map id_V). Conversely, for any $f : U \to V$ we define $\tilde{f} \in U^* \otimes V$ as

$$\tilde{f} = (f^* \otimes V)(\eta_V).$$

Note that this gives the usual identification

$$\langle\, \hat{w}(u), v^* \,\rangle = \langle\, w, u \otimes v^* \,\rangle, \qquad u \in U,\ v^* \in V^*$$

between maps $U \to V$ and elements of $U^* \otimes V$. Put $\circ_{U,V,W} := U^* \otimes e_V \otimes W$, then $\circ_{U,V,W}$ is a linear map

$$[U, V] \otimes [V, W] \to [U, W]$$

which corresponds to composition of maps: for $f : U \to V$ and $g : V \to W$, we get

$$\circ_{U,V,W} : \tilde{f} \otimes \tilde{g} \mapsto (g \circ f)^{\sim}.$$

Similarly, e_V (tensored with identity maps and composed with symmetries as necessary) defines a partial composition map

$$[U, V \otimes X] \otimes [V \otimes Y, W] \to [U \otimes Y, X \otimes W].$$

This can be depicted graphically in a nice way.

1.2 Affine subspaces

A subset $A \subseteq V$ of a finite dimensional vector space V is an affine subspace if $\sum_i \alpha_i a_i \in A$ whenever all $a_i \in A$ and $\sum_i \alpha_i = 1$. We say that A is proper if $0 \neq A$ and $A \neq \emptyset$. We will always mean that an affine subspace is proper (if not explicitly stated otherwise).

An affine subspace can be determined in two ways:

(i) Let $L \subseteq V$ be a linear subspace and $a_0 \neq L$. Then

$$A = a_0 + L$$

is an affine subspace. Note that $a_0 \in A$ and $A \cap L = \emptyset$. Conversely, any affine subspace A can be given in this way, with a_0 an arbitrary element in A and

$$L = Lin(A) := \{a_1 - a_2,\ a_1, a_2 \in A\} = \{a - a_0,\ a \in A\}.$$

(ii) Let $S \subseteq V$ be a linear subspace and $a_0^* \in V^* \setminus S^\perp$. Then

$$A = \{a \in S, \langle\, a_0^*, a\,\rangle = 1\}$$

is an affine subspace. Conversely, any affine subspace A is given in this way, with $S = span(A)$ and a_0^* an arbitrary element in

$$\tilde{A} = \{a^* \in V^*,\ \langle\, a^*, a\,\rangle = 1,\ \forall a \in A\}.$$

For an affine subspace A, $\tilde{A}$ is an affine subspace as well and we have $\tilde{\tilde{A}} = A$. More generally, if $\emptyset \neq C \subseteq A$ is any subset of an affine subspace A, then $\tilde{C}$ is an affine subspace and $\tilde{\tilde{C}}$ is the smallest affine subspace containing C, that is,

$$\tilde{\tilde{C}} = \{\sum_i \alpha_i c_i,\ c_i \in C,\ \sum_i \alpha_i = 1\}.$$

In this case, we may write $\tilde{\tilde{C}}$ as

$$\tilde{\tilde{C}} = c_0 + Lin(C) = c_0 + span(\{c_1 - c_2,\ c_1, c_2 \in C\})$$

with an arbitrary element $c_0 \in C$, or as

$$\tilde{\tilde{C}} = \{c \in span(C),\ \langle\, a_0^*, c\,\rangle = 1\}$$

for an arbitrary element $a_0^* \in \tilde{A}$. We clearly have

$$Lin(\tilde{C}) = C^\perp = span(C)^\perp, \qquad Lin(C) = Lin(\tilde{\tilde{C}}) = \tilde{C}^\perp = span(\tilde{C})^\perp$$

and by duality also

$$span(C) = C^{\perp\perp} = Lin(\tilde{C})^\perp, \qquad span(\tilde{C}) = Lin(C)^\perp.$$

1.3 The category Af

The objects of Af are of the form $X = (V_X, A_X, a_X, \tilde{a}_X)$, where V_X is in FinVect, $A_X \subseteq V_X$ an affine subspace, $a_X \in A_X$ and $\tilde{a}_X \in \tilde{A}_X$ are some elements. Morphisms $X \to Y$ are linear maps $f : V_X \to V_Y$ such that $f(A_X) \subseteq A_Y$. Note that by definition A_X is proper for any object X. We may also add two special objects: the initial object $\emptyset := (\{0\}, \emptyset, -, 0)$ and the terminal object $0 := (\{0\}, \{0\}, 0, -)$, here the affine subspaces are obviously not proper. The products and coproducts with these element do not work, however!

For any object X, we also put

$$L_X := Lin(A_X) \qquad S_X := span(A_X), \qquad d_X := \dim(L_X), \qquad D_X := \dim(V_X).$$

Note that X is uniquely determined also when A_X is replaced by L_X or S_X.

1.3.1 Limits and colimits

Limits and colimits should be obtained from those in FinVect, we have to spectify the other structures and check whether the corresponding arrows are in Af.

Let X, Y be two objects in Af. We define their product as

$$X \times Y := (V_X \times V_Y, A_X \times A_Y, (a_X, a_Y), \frac{1}{2}(\tilde{a}_X, \tilde{a}_Y)),$$

where

$$A_X \times A_Y := \{(x, y) \in V_X \times V_Y,\ x \in A_X, y \in A_Y\}$$

is the direct product of A_X and A_Y. It is easily verified that this is indeed an affine subspace and the usual projections $\pi_X : V_X \times V_Y \to V_X$ and $\pi_Y : V_X \times V_Y \to V_Y$ are in Af. Moreover, for $f : Z \to X$ and $g : Z \to Y$, the map $f \times g(z) = (f(z), g(z))$ is also clearly a morphism $Z \to X \times Y$ in Af. We have

$$L_{X\times Y} = L_X \times L_Y, \qquad S_{X\times Y} = (S_X \times S_Y) \wedge \{(\tilde{a}_X, -\tilde{a}_Y)\}^{\perp}.$$

The coproduct is defined as

$$X \oplus Y = (V_X \times V_Y, A_X \oplus A_Y, \frac{1}{2}(a_X, a_Y), (\tilde{a}_X, \tilde{a}_Y)),$$

where

$$A_X \oplus A_Y := \{(tx, (1-t)y),\ x \in A_X, y \in A_Y,\ t \in \mathbb{R}\}$$

is the direct sum. To check that this is an affine subspace, let $x_i \in A_X$, $y_i \in A_Y$, $s_i \in \mathbb{R}$ and let $\sum_i \alpha_i = 1$, then

$$\sum_i \alpha_i(s_i x_i, (1-s_i)y_i) = (\sum_i s_i\alpha_i x_i, \sum_i (1-s_i)\alpha_i y_i) = (sx, (1-s)y) \in A_X \oplus A_Y,$$

where $s = \sum_i s_i\alpha_i$, $x = s^{-1}\sum_i s_i\alpha_i x_i$ if $s \neq 0$ and is arbitrary in A_X otherwise, similarly $y = (1-s)^{-1}\sum_i (1-s_i)\alpha_i y_i$ if $s \neq 1$ and is arbitrary otherwise. The usual embeddings $p_X : V_X \to V_X \times V_Y$ and $p_Y : V_Y \to V_X \times V_Y$ are easily seen to be morphsims in Af.

Let $f:X\to Z$, $g:Y\to Z$ be any morphisms in Af and consider the map $V_X\times V_Y\to Z$ given as $f\oplus g(u,v)=f(u)+g(v)$. We need to show that it preserves the affine subspaces. So let $x\in A_X$, $y\in A_Y$, then since $f(x),g(y)\in A_Z$, we have for any $s\in\mathbb{R}$,

$$f\oplus g(sx,(1-s)y)=sf(x)+(1-s)g(y)\in A_Z.$$

We also have

$$L_{X\oplus Y}=(L_X\times L_Y)\vee\mathbb{R}\{(a_X,-a_Y)\},\qquad S_{X\oplus Y}=S_X\times S_Y.$$

Let us turn to equalizers. So let $f,g:X\to Y$ and let

$$V_E=\{v\in V_X,\ f(v)=g(v)\}.$$

Let $h:Z\to X$ equalize f,g, then $h(V_Z)\subseteq V_E$ and $h(A_Z)\subseteq A_X\cap V_E$, so that $A_X\cap V_E$ must be nonempty. In this case,

$$E=(V_E,A_E:=V_E\cap A_X,a_E,\tilde a_E:=\tilde a_X)$$

with the inclusion map $V_E\hookrightarrow V_X$ is an equalizer of f,g for any choice of $a_E\in A_E$ (note that choosing another a_E gives us an isomorphic object in Af). If the intersection $V_E\cap A_X$ is empty, then the only equalizing arrow for f and g is $\emptyset\to X$, which is then the equalizer.

For the coequalizer, let V_Q be the quotient space $V_Q:=V_Y|_{Im(f-g)}$ and let $q:V_Y\to V_Q$ be the quotient map. If some $h:Y\to Z$ coequalizes f and g, then h maps $Im(f-g)$ to 0, so that $Im(f-g)\cap A_Y=\emptyset$, unless Z is the terminal object. It is easily checked that if $Im(f-g)\cap A_Y=\emptyset$, then

$$Q=(V_Q,A_Q:=q(A_Y),a_Q:=q(a_Y),\tilde a_Q)$$

together with the quotient map q is the coequalizer of f and g for any choice of $\tilde a_Q\in\tilde A_Q$. If the intersection is nonempty, then the unique coeqalizing arrow is $Y\to 0$, which is then the coequalizer.

Let us mention pullbacks and pushouts. Since pullbacks can be obtained from products and equalizers, we see that we have a similar situation: if a pullback is "well defined", then it coincides with the pullback in FinVect, otherwise it is trivial. More precisely, if $f:X\to Z$ and $g:Y\to Z$, then we put

$$V_P:=\{(x,y)\in V_X\times V_Y,f(x)=g(y)\}.$$

If $V_P\cap A_X\times A_Y\neq\emptyset$, that is, there are some $x\in A_X$ and $y\in A_Y$ such that $f(x)=g(y)$, then

$$(V_P,A_P:=(A_X\times A_Y)\cap V_P,a_P,\frac{1}{2}(\tilde a_X,\tilde a_Y))$$

with the two projections is a pullback of f and g for any choice of $a_P\in A_P$, otherwise the pullback is just the initial object $\emptyset$.

Similarly, let $f:Z\to X$, $g:Z\to Y$, then let V_Q be the quotient of $V_X\times V_Y$ by the subspace

$$\{(f(z),-g(z)),\ x\in V_Z\}.$$

If this subspace does not contain any element of $A_X\oplus A_Y$, that is, there is no $z\in V_Z$ such that for some $t\in\mathbb{R}$,

$$f(tz)\in A_X,\qquad g((t-1)z)\in A_Y,$$

then

$$Q=(V_Q,A_Q:=q(A_X\oplus A_Y),\frac{1}{2}q(a_X,a_Y),\tilde a_Q)$$

with maps $x\mapsto q(x,0)$ and $y\mapsto q(0,y)$ is the pushout of f and g. Otherwise the pushout is just 0.

1.3.2 Tensor products

Let X, Y be objects in Af. Let us define

$$A_{X\otimes Y} := \{x\otimes y, x\in A_X, y\in A_Y\}^{\approx}.$$

In other words, $A_{X\otimes Y}$ is the affine subspace in $V_X\otimes V_Y$ containing $A_X\otimes A_Y$. We have

$$\begin{aligned} L_{X\otimes Y} &= Lin(A_X\otimes A_Y) = span(\{x\otimes y - a_X\otimes a_Y,\ x\in A_X,\ y\in A_Y\}) \\ &= (a_X\otimes L_Y) + (L_X\otimes a_Y) + (L_X\otimes L_Y) \end{aligned} \tag{1}$$

(here + denotes the direct sum of subspaces). We also have

$$S_{X\otimes Y} = S_X\otimes S_Y.$$

Proof. Let $x\in A_X$, $y\in A_Y$, then

$$x\otimes y - a_X\otimes a_Y = a_X\otimes(y-a_Y) + (x-a_X)\otimes a_Y + (x-a_X)\otimes(y-a_Y),$$

so that $L_{X\otimes Y} = Lin(A_X\otimes A_Y)$ is contained in the subspace on the RHS of (1). Let d be the dimension of this subspace, then clearly

$$d_{X\otimes Y}\le d\le d_X + d_Y + d_Xd_Y.$$

On the other hand, any element of S_X has the form tx for some $t\in\mathbb{R}$ and $x\in A_X$, so that it is easily seen that $S_X\otimes S_Y = S_{X\otimes Y}$. Hence

$$\begin{aligned} d_{X\otimes Y} &= \dim(L_{X\otimes Y}) = \dim(S_{X\otimes X}) - 1 = \dim(S_X)\dim(S_Y) - 1 = (d_X+1)(d_Y+1) - 1 \\ &= d_X + d_Y + d_Xd_Y. \end{aligned}$$

This completes the proof. □

For X, Y in Af, put

$$X\otimes Y := (V_X\otimes V_Y, A_{X\otimes Y}, a_X\otimes a_Y, \tilde a_X\otimes\tilde a_Y).$$

Also let $I := (\mathbb{R}, \{1\}, \{1\}, \{1\})$. Then $(\mathrm{Af}, \otimes, I)$ is a symmetric monoidal category. We only have to check that the associators, unitors and symmetries from FinVect are morphisms in Af. We leave this for some other day.

1.3.3 Duality

We define $X^* := (V_X^*, \tilde A_X, \tilde a_X, a_X)$. Note that we have

$$L_{X^*} = S_X^{\perp},\qquad S_{X^*} = L_X^{\perp}.$$

It follows that

$$d_{X^*} = D_X - d_X - 1.$$

It is easily seen that $(-)^*$ defines a full and faithful functor $\mathrm{Af}^{op}\to\mathrm{Af}$, moreover, $X^{**} = X$ (if we us the canonical identification of any V in FinVect with its second dual).

Theorem 1. *$(\mathrm{Af}, \otimes, I)$ is a *-autonomous category, with duality $(-)^*$.*

Proof. ...

$\square$

Let us define the dual tensor product by $\odot$, that is

$$X \odot Y = (X^* \otimes Y^*)^*.$$

We then have

$$\begin{aligned} L_{X\odot Y} &= S^{\perp}_{X^*\otimes Y^*} = (S_{X^*} \otimes S_{Y^*})^{\perp} = (L^{\perp}_X \otimes L^{\perp}_Y)^{\perp} \\ S_{X\odot Y} &= L^{\perp}_{X^*\otimes Y^*} = (\tilde{a}_X \otimes S^{\perp}_Y)^{\perp} \wedge (S^{\perp}_X \otimes \tilde{a}_Y)^{\perp} \wedge (S^{\perp}_X \otimes S^{\perp}_Y)^{\perp} \end{aligned}$$

In particular,

$$d_{X\odot Y} = D_X d_Y + d_X D_Y - d_X d_Y.$$

Lemma 1. *Let X, Y be nontrivial. Then $X \otimes Y = X \odot Y$ if and only if $D_X = d_X + 1$ and $D_Y = d_Y + 1$.*

Proof. It is easy to see that (when identifying $X = X^{**}$), we have $A_X \otimes A_Y \subseteq \tilde{A}_{X^*\otimes Y^*}$, hence $A_{X\otimes Y} \subseteq A_{X\odot Y}$. We see from the above computatons that

$$d_{X\odot Y} - d_{X\otimes Y} = d_X(D_Y - 1) + (D_X - 1)d_Y - 2d_X d_Y \geq 0,$$

with equality if and only if the conditions of the lemma hold.

$\square$

The internal hom has the form

$$[X, Y] = (X \otimes Y^*)^* = X^* \odot Y.$$

We then have

$$L_{[X,Y]} = (S_X \otimes L^{\perp}_Y)^{\perp}, \qquad S_{[X,Y]} = (\tilde{a}_X \otimes S^{\perp}_Y)^{\perp} \wedge (L_X \otimes \tilde{a}_Y)^{\perp} \wedge (L_X \otimes S^{\perp}_Y)^{\perp}$$

and

$$d_{[X,Y]} = D_X D_Y - (d_X + 1)(D_Y - d_Y).$$

We say $X \hookrightarrow Y$ (X is embedded in Y) if $V_X = V_Y$ and $A_X \subseteq A_Y$, $a_X = a_Y$, $\tilde{a}_X = \tilde{a}_Y$.

1.3.4 The category AfH

It is easily seen that the following are equivalent:

1. $D_X = d_X + 1$;
2. $S_X = V_X$;
3. $L_X = \{\tilde{a}_X\}^{\perp}$;
4. $S_{X^*} = \mathbb{R}\tilde{a}_X$;

5. $L_{X^*} = \{0\}$.

We say that an object X is first order if any of these conditions is fulfilled. A channel is an object $[X, Y]$ where X and Y are first order.

Lemma 2. *An object Z is embedded in a channel $[X, Y]$ if and only if $V_Z = V_X^* \otimes V_Y$, $a_Z = \tilde{a}_X \otimes a_Y$, $\tilde{a}_Z = a_X \otimes \tilde{a}_Y$ and*

...

We define AfH as the full subcategory of Af containing all first order objects and closed under (finite products,) duals and tensor products .