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Assemblages and steering in general probabilistic theories

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Abstract

We study steering in the framework of general probabilistic theories. We show that for dichotomic assemblages, steering can be characterized in terms of a certain tensor cross norm, which is also related to a steering degree given by steering robustness. Another contribution is the observation that steering in general probabilistic theories (GPTs) can be conveniently treated using Choquet theory for probability measures on the state space. In particular, we find a variational expression for universal steering degree for dichotomic assemblages and obtain conditions characterizing unsteerable states analogous to some conditions recently found for the quantum case. The setting also enables us to rather easily extend the results to infinite dimensions and arbitrary numbers of measurements with arbitrary outcomes.

Keywords: quantum steering, general probabilistic theories, tensor cross norms, Choquet order

1. Introduction

EPR steering was first described by Schrödinger in 1936 [33] as a bipartite scenario where one party can steer the state of a distant party by performing local measurements, in a way that cannot be explained by classical correlations. A precise definition and a systematic treatment was given in [19, 36], where quantum steering was interpreted in operational terms, as the possibility of certifying entanglement when one party is untrusted. In this way, steering became an important resource in quantum information theory and has attracted a lot of attention due to applications as well as for its relations to other nonclassical phenomena, such as entanglement, Bell nonlocality or incompatibility of measurements. In recent years, methods for characterizing and quantifying steering were developed in the literature that can be efficiently evaluated by SDPs, see [11] for an overview. For a recent review of quantum steering see [35].

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Quantum steering can be described as follows. Assume Alice and Bob share a bipartite state ρ_{AB}. After Alice performs a measurement M_x on her side and obtains the result a, Bob is left with the conditional (nonnormalized) state $\rho_{a|x} = (M_{a|x} \otimes id)(\rho_{AB})$. The assemblage $\{\rho_{a|x}\}$ of conditional states admits a local hidden state (LHS) model if there is an ensemble of states $\{\lambda(\omega), \rho_\omega\}$ from which $\rho_{a|x}$ are obtained by a set of conditional probabilities $\{p(a|x,\omega)\}$ as

$$\rho_{a|x} = \sum_\omega \lambda(\omega) p(a|x,\omega)\rho_\omega, \quad \forall\ a, x.$$

In this case Bob is not convinced that the state ρ_{AB} was entangled. If no LHS model exists, Bob can be sure of both entanglement and incompatibility of Alice's measurements. However, there are entangled states that are unsteerable, which means that the assemblages obtained by any measurement always have a LHS model, this was already observed in [19, 36]. Some steerability criteria were recently obtained in [23–25] through a geometric approach.

Nonclassical phenomena are often studied in a broader framework of general probabilistic theories (GPTs) [6], which has been used for better understanding the operational features of Bell nonlocality, incompatibility of measurements, steering and their relations [4, 5, 9, 17, 28, 29]. GPTs include classical and quantum theory, as well as PR boxes exhibiting maximal violations of Bell inequalities [31], quantum channels [18] or post quantum steering [12]. Another motivation comes from quantum foundations, where the aim is to characterize quantum theory among physical theories, e.g. [13, 16, 22].

In this work, we use further assumptions on the GPTs, namely the no-restriction hypothesis and local tomography assumption. In this setting, the full strength of duality of ordered vector spaces and their tensor products can be put into work and the relations of nonclassical phenomena to some mathematical concepts can be revealed. For example, nonclassical correlations can be expressed in terms of tensor cross norms in Banach spaces [3]. In [7], incompatibility of measurements in GPTs is characterized and quantified using different mathematical points of view: extendability of maps, theory of generalized spectrahedra and tensor cross norms.

The aim of the present work is to study steering in GPTs. We first restrict to dichotomic steering, where Alice uses only dichotomic measurements. In this case, we show that steering is characterized by tensor cross norms in very much the same way as obtained in [7]. In fact, the connection is immediate from the formal relation of the two notions of steering and incompatibility: in the framework of GPTs, an assemblage can be associated with a family of measurements in a certain dual GPT, see section 3.6. Nevertheless, we give different proofs here, more clearly related to the structure of tensor products with hypercubic state spaces.

In the second part, we show that steering can be represented using Choquet order on the set of Radon probability measures over the state space. More precisely, the assemblages are represented by sets of (simple) probability measures and a LHS model is given by another probability measure that is an upper bound for the assemblage in Choquet order. We then obtain a convenient description of LHS models and steering using classical results in Choquet theory, which naturally includes infinite assemblages. In particular, we show that we can always use an LHS model concentrated on the extreme boundary of the state space, equivalently, the LHS model is given by a boundary measure. Further, we may assume that this measure is invariant under the group of transformations leaving the assemblage invariant. Using this representation, we find a variational expression for the steering degree for dichotomic assemblages that is independent of the size of the assemblage. In the quantum case, this expression coincides with the quantum steering constants obtained in [8] and relates to the one-summing constant for centrally symmetric state spaces. We also find characterizations of (one way) unsteerable states in a GPT, which are similar to those obtained in [24, 25] in the quantum case.

2. Notations and preliminaries

In this paper we follow the general assumptions and formalism of GPTs used in [21]. We further assume the no-restriction hypothesis and tomographic locality described below. See also e.g. [30] for another recent exposition with a convenient diagrammatic presentation.

A system of a general probabilistic theory (GPT) is a triple $(V, V^+, \mathbb{1})$, where V is a real vector space of finite dimension, V^+ is a closed convex cone which is pointed ($V^+ \cap (-V^+) = \{0\}$) and separating ($V = V^+ - V^+$) and $\mathbb{1}$ is a distinguished element of the dual vector space V^*, called the unit effect. In V^*, the positive cone $(V^*)^+$ is defined as the dual cone to V^+:

$$(V^*)^+ = (V^+)^* := \{f \in V^*,\ \langle f, v\rangle \geqslant 0,\ \forall\ v \in V^+\},$$

here $\langle \cdot, \cdot \rangle$ denotes the duality $V^* \times V \to \mathbb{R}$. We will use the notation $\leqslant$ for the ordering induced by V^+ or $(V^*)^+$ in their respective spaces. In this ordering, the unit effect $\mathbb{1}$ is assumed to be an order unit, which means that for all $f \in V^*$ there is some $\alpha > 0$ such that $\alpha\mathbb{1} \pm f \in (V^*)^+$, or, equivalently, that $\mathbb{1}$ is an interior element in $(V^*)^+$. The order unit norm in V^* is defined as

$$\|f\|_{\mathbb{1}} = \inf\{\lambda < 0,\ \lambda\mathbb{1} \pm f \in (V^*)^+\}.$$

The subset

$$K := \{\rho \in V^+,\ \langle \mathbb{1}, \rho\rangle = 1\}$$

is interpreted as the set of states and called the state space. Clearly, K is a compact convex subset of V and a base of the cone V^+. Extremal points of K are called pure states, the set of all pure states, that is, the extreme boundary of K, will be denoted by $\partial_e K$. The dual of $\|\cdot\|_{\mathbb{1}}$ in V is the base norm

$$\|v\|_V = \inf\{\langle \mathbb{1}, v_+ + v_-\rangle,\ v = v_+ - v_-,\ v_+, v_- \in V^+\} = \sup_{\pm f \leqslant \mathbb{1}} \langle f, v\rangle.$$

Elements of the unit interval $E = \{f \in V^*,\ 0 \leqslant f \leqslant \mathbb{1}\}$ are called effects. The effects can be identified with affine maps $K \to [0, 1]$ and are interpreted as dichotomic (or yes–no) measurements of the system: the value $\langle f, \rho\rangle \in [0, 1]$ gives the probability that the measurement represented by the effect f gives outcome 'yes' if the system is in the state $\rho \in K$. Similarly, measurements with n outcomes are represented by collections $f_1, \ldots, f_n \in E$, $\sum_i f_i = \mathbb{1}$, where $\langle f_i, \rho\rangle$ is interpreted as the probability of ith outcome in the state ρ. We assume here the no-restriction hypothesis, which means that also conversely, any such collection of effects corresponds to a measurement.

We now describe composite systems. Let $(V_A, V_A^+, \mathbb{1}_A)$ and $(V_B, V_B^+, \mathbb{1}_B)$ be two systems. Under the assumption of tomographic locality, that is, that all bipartite states can be distinguished by locally prepared effects, the composite system is given by a triple $(V_{AB}, V_{AB}^+, \mathbb{1}_{AB})$, where $V_{AB} = V_A \otimes V_B$, $\mathbb{1}_{AB} = \mathbb{1}_A \otimes \mathbb{1}_B$ and V_{AB}^+ is a positive cone in V_{AB} satisfying

$$V_A^+ \otimes_{\min} V_B^+ \subseteq V_{AB}^+ \subseteq V_A^+ \otimes_{\max} V_B^+.$$

Here $V_A^+ \otimes_{\min} V_B^+$ is the minimal cone

$$V_A^+ \otimes_{\min} V_B^+ = \left\{ \sum_i v_{i,A} \otimes v_{i,B},\ v_{i,A} \in V_A^+, v_{i,B} \in V_B^+ \right\}$$

containing all separable states and $V_A^+ \otimes_{\max} V_B^+$ is the maximal cone

$$V_A^+ \otimes_{\max} V_B^+ = \{v_{AB} \in V_{AB},\ \langle f_A \otimes f_B, v_{AB}\rangle \geqslant 0,\ \forall\ f_A \in (V_A^*)^+,\ f_B \in (V_B^*)^+\}$$

for which all separately prepared measurements are valid. We have corresponding definitions of the minimal and maximal positive cones in the tensor product of the dual spaces and the duality relations hold:

$$(V_A^*)^+ \otimes_{\min} (V_B^*)^+ = (V_A^+ \otimes_{\max} V_B^+)^*, \qquad (V_A^*)^+ \otimes_{\max} (V_B^*)^+ = (V_A^+ \otimes_{\min} V_B^+)^*.$$

The state space K_{AB} satisfies

$$K_A \otimes_{\min} K_B \subseteq K_{AB} \subseteq K_A \otimes_{\max} K_B.$$

As it was recently proved in [2], the inclusion $K_A \otimes_{\min} K_B \subseteq K_A \otimes_{\max} K_B$ is strict, unless K_A or K_B is a simplex. The states in $K_A \otimes_{\min} K_B$ are called separable, all other states in $K_A \otimes_{\max} K_B$ are entangled.

Some of the basic examples are described below. In all the examples, we identify $V^* = V$. For $V = \mathbb{R}^n$, the duality $\langle \cdot, \cdot \rangle$ will always be given by the standard inner product in $\mathbb{R}^n$.

Example 2.1. **Classical theory.** In the classical GPT, the systems have the form $(\mathbb{R}^n, \mathbb{R}^n_+, (1, \ldots, 1))$, where $\mathbb{R}^n_+$ is the simplicial cone generated by the positive half-axes. The state space is the simplex $\Delta_n = \{(x_1, \ldots, x_n), x_i \geqslant 0, \sum_i x_i = 1\}$. For any state space K we have $\Delta_n \otimes_{\max} K = \Delta_n \otimes_{\min} K = K^{\oplus n}$, the convex direct sum of n-copies of K.

Example 2.2. **Quantum theory.** Here $V = M_n^{sa}$ is the space of n by n complex Hermitian matrices, $V^+ = M_n^+$ is the cone of positive semidefinite matrices and the duality is given by $\langle X, Y\rangle = \mathrm{Tr}\, XY$. The unit effect is $\mathbb{1} = I$, the identity matrix. The state space is the set of density matrices $D_n = \{\rho \in M_n^+, \mathrm{Tr}\ \rho = 1\}$. The tensor product of the cones M_m^+ and M_n^+ is the cone M_{mn}^+ of positive definite matrices in $M_{mn}^{sa} = M_m^{sa} \otimes M_n^{sa}$.

Example 2.3. Note that for any compact convex subset S in the Euclidean space $\mathbb{R}^g$ we can construct a triple $(V_S, V_S^+, \mathbb{1}_S)$ with the state space K_S affinely isomorphic[1] to S: By an affine transformation, we may always assume that $0 \in S$ and that S spans $\mathbb{R}^g$. Put $V_S = \mathbb{R}^{g+1}$ and $K_S = \{(1, x), x \in S\}$. We define V_S^+ as the cone generated by K_S and put $\mathbb{1}_S = (1, 0)$.

Example 2.4. **Centrally symmetric state spaces.** In the previous example, assume that S is the unit ball of a norm $\|\cdot\|$ in $\mathbb{R}^g$. We will denote the corresponding triple as $(V_{\|\cdot\|}, V_{\|\cdot\|}^+, (1, 0))$ and the state space by $K_{\|\cdot\|}$. Then we obtain

$$K_{\|\cdot\|} = \{(1, x),\ \|x\| \leqslant 1\}, \qquad V_{\|\cdot\|}^+ = \{(t, x),\ \|x\| \leqslant t\}.$$

We will identify the dual space as $V_{\|\cdot\|}^* = V_{\|\cdot\|} = \mathbb{R}^{g+1}$. With the dual norm in $\mathbb{R}^g$

$$\|\varphi\|^* := \sup_{\|x\| \leqslant 1} |\langle \varphi, x\rangle|, \quad \varphi \in \mathbb{R}^g,$$

the dual cone becomes

$$\left(V_{\|\cdot\|}^*\right)^+ = V_{\|\cdot\|^*}^+ = \{(t, \varphi),\ \|\varphi\|^* \leqslant t\}.$$

[1] For convex sets C and D, an affine isomorphism is a bijective map $C \to D$ preserving the convex structure.

Moreover, the central element $(1, 0)$ is an order unit in both spaces. In this case we have for the base norm (in $V_{\|\cdot\|}$) $\|(s, x)\|_{V_{\|\cdot\|}} = \max\{|s|, \|x\|\}$ and the order unit norm (in $V^*_{\|\cdot\|}$) becomes $\|(t, \varphi)\|_{(1,0)} = |t| + \|\varphi\|^*$.

We now look at the tensor product of a centrally symmetric system $(V_{\|\cdot\|}, V^+_{\|\cdot\|}, (1, 0))$ with any other triple $(V, V^+, \mathbb{1})$. We will use the obvious identifications $\mathbb{R}^{g+1} \otimes V \cong V^{g+1} \cong V \oplus V^g$, so the first copy is distinguished. Let $(y_0, y) \in V^{g+1}$, $y = (y_1, \ldots, y_g)$, then it is easily checked that $(y_0, y) \in V^+_{\|\cdot\|} \otimes_{\min} V^+$ if and only if

$$y = \sum_j x_j \otimes z_j, \quad \|x_j\| = 1, \ z_j \in V^+, \quad \sum_j z_j \leqslant y_0 \tag{1}$$

and $(y_0, y) \in V^+_{\|\cdot\|} \otimes_{\max} V^+$ if and only if

$$\sum_i \varphi_i y_i \leqslant y_0, \quad \forall \ \varphi \in \mathbb{R}^g, \ \|\varphi\|^* = 1. \tag{2}$$

Note that using this condition for $\pm\varphi$, we obtain that $y_0 \in V^+$ and $y_0 = 0$ only if $y = 0$.

An important centrally symmetric example is the qubit system (M_2^{sa}, M_2^+, I_2), where the state space is isomorphic to the unit ball in $\ell_2^3 = (\mathbb{R}^3, \| \cdot \|_2)$. This is the unique quantum system which is centrally symmetric. We will also frequently use the hypercubic systems obtained from $\ell_\infty^g = (\mathbb{R}^g, \| \cdot \|_\infty)$, we will denote the corresponding triple as $(V_g, V_g^+, \mathbb{1}_g)$. The state space

$$K_g := \{(1, z_1, \ldots, z_g), \ |z_i| \leqslant 1, \ \forall \ i\}$$

is isomorphic to the hypercube $[-1, 1]^g$.

Example 2.5. **Boxworld.** This type of GPTs was introduced in [6]. Boxworld is a toy theory that describes black boxes with finitely many inputs and outputs. Here the basic systems have state spaces isomorphic to a Cartesian product of simplexes $S_{\mathbf{k}} := \Delta_{k_1} \times \cdots \times \Delta_{k_g}$, here $\mathbf{k} = (k_1, \ldots, k_g) \in \mathbb{N}^g$. The corresponding triple $(V_{\mathbf{k}}, V^+_{\mathbf{k}}, \mathbb{1}_{\mathbf{k}})$ can be constructed as in example 2.3. We will use the notation $K_{\mathbf{k}}$ for the corresponding state spaces. Apart from trivial cases, this state space is not a simplex. See [17] for more information on the structure of $V_{\mathbf{k}}$ and $V^+_{\mathbf{k}}$ and their duals. Maximal tensor products of such systems describe the most general nonlocal correlations satisfying the no-signaling conditions, whereas the minimal tensor products correspond to local correlations.

The simplest (nontrivial) example is the gbit, where $g = k_1 = k_2 = 2$. In this case, the state space is isomorphic to a square $[-1, 1]^2$. The maximal tensor product of two gbits exhibits maximal violation of the CHSH inequality, [30]. More generally, for $k_1 = \cdots = k_g = 2$ the state space is isomorphic to a hypercube. These are the only systems in this GPT that have centrally symmetric state spaces (example 2.4).

We now turn to composite systems $(V_{AB}, V^+_{AB}, \mathbb{1}_{AB})$. We will consider the norms obtained in V_{AB} from the tensor product of the Banach spaces V_A and V_B equipped with their respective base norms. Let $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ be (finite dimensional) Banach spaces and let $(X^*, \| \cdot \|^*_X)$, $(Y^*, \| \cdot \|^*_Y)$ be the Banach space duals. Let $\| \cdot \|$ be a norm on the tensor product $X \otimes Y$ and let $\| \cdot \|^*$ be the dual norm in the dual space $(X \otimes Y)^* = X^* \otimes Y^*$. Then $\| \cdot \|$ is called a (reasonable) cross norm if both $\| \cdot \|$ and $\| \cdot \|^*$ are multiplicative on simple tensors:

$$\|x \otimes y\| = \|x\|_X \|y\|_Y, \quad \|\varphi \otimes \psi\|^* = \|\varphi\|^*_X \|\psi\|^*_Y,$$
$$\forall \ x \in X, y \in Y, \varphi \in X^*, \psi \in Y^*.$$

Equivalently, we have the inequality

$$\|z\|_{\epsilon(X,Y)} \leqslant \|z\| \leqslant \|z\|_{\pi(X,Y)}, \quad \forall\, z \in X \otimes Y.$$

Here $\epsilon(X, Y)$ denotes the injective cross norm given by

$$\|z\|_{\epsilon(X,Y)} = \sup\,\{\langle \varphi \otimes \psi, z\rangle,\ \varphi \in X^*, \psi \in Y^*, \|\varphi\|_X^* \leqslant 1,\ \|\psi\|_Y^* \leqslant 1\} \tag{3}$$

and $\pi(X, Y)$ is the projective cross norm

$$\|z\|_{\pi(X,Y)} = \inf\left\{\sum_i \|x_i\|_X \|y_i\|_Y,\ z = \sum_i x_i \otimes y_i,\ x_i \in X, y_i \in Y\right\}. \tag{4}$$

See [32] for more details.

It was proved in [3] that the base norm $\|\cdot\|_{V_{AB}}$ for the composite system $(V_{AB}, V_{AB}^+, \mathbb{1}_{AB})$ is a reasonable cross norm for the Banach spaces $(V_A, \|\cdot\|_{V_A})$ and $(V_B, \|\cdot\|_{V_B})$, and that it coincides with the projective cross norm $\pi(A, B) := \pi(V_A, V_B)$ if V_{AB}^+ is the separable cone $V_{AB}^+ = V_A^+ \otimes_{\min} V_B^+$. For completeness, we give a proof for the following equivalent formulation of the latter statement.

Theorem 2.6. *A bipartite state $\rho_{AB} \in K_A \otimes_{\max} K_B$ is separable if and only if $\|\rho_{AB}\|_{\pi(A,B)} \leqslant 1$.*

Proof. Assume ρ_{AB} is separable, then $\rho_{AB} = \sum_i \lambda_i \rho_{i,A} \otimes \rho_{i,B}$ for $\rho_{i,A} \in K_A$, $\rho_{i,B} \in K_B$ and probabilities λ_i. By definition of the projective cross norm,

$$\|\rho_{AB}\|_{\pi(A,B)} \leqslant \sum_i \lambda_i \|\rho_{i,A}\|_{V_A} \|\rho_{i,B}\|_{V_B} = 1.$$

Assume the converse and let $y_{i,A} \in V_A$ and $y_{i,B} \in V_B$ be such that $\rho_{AB} = \sum_i y_{i,A} \otimes y_{i,B}$ and $\sum_i \|y_{i,A}\|_{V_A} \|y_{i,B}\|_{V_B} \leqslant 1$. Then $y_{i,A} = y_{i,A}^+ - y_{i,A}^-$ with $y_{i,A}^\pm \in V_A^+$ and $\|y_{i,A}\|_{V_A} = \|y_{i,A}^+\|_{V_A} + \|y_{i,A}^-\|_{V_A}$, similarly for $y_{i,B}$. It follows that $\rho_{AB} = \rho_{AB}^+ - \rho_{AB}^-$, with

$$\rho_{AB}^+ = \sum_i (y_{i,A}^+ \otimes y_{i,B}^+ + y_{i,A}^- \otimes y_{i,B}^-) \in V_A^+ \otimes_{\min} V_B^+$$

and

$$\rho_{AB}^- = \sum_i (y_{i,A}^+ \otimes y_{i,B}^- + y_{i,A}^- \otimes y_{i,A}^+) \in V_A^+ \otimes_{\min} V_B^+.$$

We obtain

$$\begin{aligned}
1 &= \langle \mathbb{1}_{AB}, \rho_{AB}\rangle = \langle \mathbb{1}_{AB}, \rho_{AB}^+ - \rho_{AB}^-\rangle \leqslant \langle \mathbb{1}_{AB}, \rho_{AB}^+ + \rho_{AB}^-\rangle \\
&= \langle \mathbb{1}_A \otimes \mathbb{1}_B, \rho_{AB}^+ + \rho_{AB}^-\rangle \\
&= \sum_i \left(\|y_{i,A}^+\|_{V_A} \|y_{i,B}^+\|_{V_B} + \|y_{i,A}^-\|_{V_A} \|y_{i,B}^-\|_{V_B} + \|y_{i,A}^+\|_{V_A} \|y_{i,B}^-\|_{V_B} + \|y_{i,A}^-\|_{V_A} \|y_{i,B}^+\|_{V_B}\right) \\
&= \sum_i \|y_{i,A}\|_{V_A} \|y_{i,B}\|_{V_B} \leqslant 1.
\end{aligned}$$

It follows that $\langle \mathbb{1}_{AB}, \rho^-_{AB}\rangle = 0$, whence $\rho^-_{AB} = 0$ and $\rho_{AB} = \rho^+_{AB}$ is separable. $\square$

An operational characterization of the injective norm in terms of XOR game strategies can be found in [3].

3. Steering and tensor norms

3.1. Conditional states and assemblages

Let $\sigma_{AB} \in K_{AB}$ be a state of the composite system $(V_{AB}, V^+_{AB}, \mathbb{1}_{AB})$ and let $\{f_x\}_{x=1}^g$ be a (finite) collection of measurements on the system $(V_A, V^+_A, \mathbb{1}_A)$, with effects $f_{a|x}$ and outcomes a in some finite set of outcomes Ω_x. Viewing any effect $0 \leqslant f \leqslant \mathbb{1}_A$ as an affine function over the state space K_A, we may define the conditional states

$$\rho_{a|x} := (f_{a|x} \otimes \mathrm{id}_B)(\sigma_{AB}) \in V^+_B.$$

For all $x = 1, \ldots, g$, we have $\sum_{a\in\Omega_x} \rho_{a|x} = \sigma_B = (\mathbb{1}_A \otimes \mathrm{id}_B)(\sigma_{AB})$. More generally, any collection

$$\left\{\rho_{a|x} \in V^+_B,\ \sum_{a\in\Omega_x} \rho_{a|x} = \sigma_B \in K_B,\ \forall\ x = 1, \ldots, g\right\}$$

is called an assemblage in the system $(V_B, V^+_B, \mathbb{1}_B)$ with barycenter σ_B. The tuple $\mathbf{k} = (k_1, \ldots, k_g)$ with $k_x = |\Omega_x|$, $x = 1, \ldots, g$ determines the shape of the assemblage. We will say that the assemblage is dichotomic if $k_x = 2$ for all x, in this case we will use $\Omega_x = \{+, -\}$ as the set of labels.

The next result shows that if we do not restrict the choice of the system V_A and allow maximal tensor products, all assemblages in the system V_B can be obtained as conditional states for a bipartite state and some collection of measurements on V_A of a corresponding shape. More precisely, all assemblages in V_B can be obtained from a bipartite state in the maximal tensor product with a Boxworld system determined only by the shape of the assemblage, by applying a canonical set of measurements (example 2.5).

Theorem 3.1. *Let $\mathbf{k} = (k_1, \ldots, k_g)$ and let $K_{\mathbf{k}}$ be the corresponding Boxworld state space. For any system $(V, V^+, \mathbb{1})$ with state space K, the set of all assemblages of shape $\mathbf{k}$ is isomorphic to the tensor product $K_{\mathbf{k}} \otimes_{\max} K$. In particular, there is a canonical set of measurements $\{p_x\}$ (obtained from the projections $\pi_x : \Pi_{y=1}^g \Delta_{k_y} \to \Delta_{k_x}$ onto the xth component), such that for any assemblage $\{\rho_{a|x}\}$ in V^+ of shape $\mathbf{k}$, there is a (unique) state $\xi \in K_{\mathbf{k}} \otimes_{\max} K$ such that*

$$\rho_{a|x} = (p_{a|x} \otimes \mathrm{id}_K)(\xi), \quad \forall\ a, x.$$

Proof. We will give a proof for the case of dichotomic assemblages, mostly to introduce some notations needed later. For the proof in general see [17].

So let $\mathbf{k} = (2, \ldots, 2)$. By the isomorphism $(\lambda, 1-\lambda) \mapsto 2\lambda - 1$, we will identify $S_{\mathbf{k}} = \Delta_2^g \simeq [-1, 1]^g$. Let $(V_g, V^+_g, \mathbb{1}_g)$ be the system with centrally symmetric state space $K_g \simeq [-1, 1]^g$, see example 2.4. It is easily seen from (2) (where in this case $\|\cdot\| = \|\cdot\|_\infty$ so that $\|\cdot\|^* = \|\cdot\|_1$) that the maximal tensor product $K_g \otimes_{\max} K$ is the subset in V^{g+1} of elements of the form $(\sigma, y_1, \ldots, y_g)$ with $\sigma \in K$ and $\sigma \pm y_x \in V^+$. Clearly, $\{\frac{1}{2}(\sigma \pm y_x)\}$ is a dichotomic assemblage with barycenter σ. Conversely, for any assemblage $\{\rho_{\pm|x}\}$ with barycenter σ, $(\sigma, y_1, \ldots, y_g)$ with $y_x := \rho_{+,x} - \rho_{-|x}$ is an element of $K_g \otimes_{\max} K$ and it is clear that this established a one-to-one correspondence. The effects $p_{\pm|x}$ have the form $\frac{1}{2}(1, \pm e_x)$, where $\{e_x\}_{x=1}^g$ is the standard basis in $\mathbb{R}^g$. $\square$

The isomorphism between assemblages and $K_{\mathbf{k}} \otimes_{\max} K$ defined in the previous theorem will be denoted by $\Sigma_{\mathbf{k}}$. We stress that the state space $K_{\mathbf{k}}$ and the set of measurements $\{p_x\}$ in the above theorem only depend on the shape $\mathbf{k}$ of the assemblages, so they are universal for all GPTs. For dichotomic assemblages with g elements, we will use the notation $\Sigma_g := \Sigma_{\mathbf{k}}$ and by the above proof, we have for an assemblage $\{\rho_{\pm|x}\}$ with barycenter σ and $\xi \in K_g \otimes_{\max} K$

$$\Sigma_g(\{\rho_{\pm|x}\}) = (\sigma, y_1, \ldots, y_g), \quad y_x = \rho_{+|x} - \rho_{-|x}, \tag{5}$$

$$\Sigma_g^{-1}(\xi) = \{(p_{\pm|x} \otimes \mathrm{id}_K)(\xi)\}_{x=1}^{g}, \quad p_{\pm|x} = \frac{1}{2}(1, \pm e_x). \tag{6}$$

It is well known that any assemblage in quantum theory can be obtained with $V_A \simeq V_B$. In the general case, this depends on the properties of the system V_B as well as the choice of the tensor product. For more information see [5].

3.2. Local hidden state models and steering

Following the operational definition of quantum steering [36], we say that an assemblage of conditional states $\{\rho_{a|x} = (f_{a|x} \otimes \mathrm{id}_B)(\sigma_{AB})\}$ admits a LHS model if there is some (finite) set Λ, a probability measure $q \in \mathcal{P}(\Lambda)$, conditional probabilities $q(a|x, \lambda)$ and elements $\rho_\lambda \in K$ such that

$$\rho_{a|x} = \sum_{\lambda \in \Lambda} q(\lambda) q(a|x, \lambda) \rho_\lambda, \qquad a \in \Omega_x, \; x = 1, \ldots, g. \tag{7}$$

We say that a bipartite state σ_{AB} is $(A \to B)$ steerable if there is a set $\{f_x\}$ of measurements on the system A such that the assemblage $\{\rho_{a|x}\}$ does not admit a LHS model. If no such collection of measurements exists, the state is unsteerable. We may restrict the set of measurements, so we say that, for example, a state is $(A \to B)$ unsteerable by dichotomic measurements if all corresponding dichotomic assemblages admit a LHS.

For a general assemblage $\{\rho_{a|x}\}$ satisfying (7) we will also say that the assemblage is classical. We will show later that we may equivalently formulate the condition in (7) with the set Λ replaced by the (possibly infinite) set $\partial_e K$ of pure states, common for all LHS models.

Theorem 3.2. *Let $\{\rho_{a|x}\}$ be an assemblage of shape $\mathbf{k}$. Then the assemblage is classical if and only if the state $\Sigma_{\mathbf{k}}(\{\rho_{a|x}\})$ is separable.*

Proof. We again prove the statement for dichotomic assemblages that we focus on in this section. For a general proof, see [17].

Let $\{\rho_{\pm,x}\}$ be a dichotomic assemblage with barycenter σ, then by (5), $\Sigma_g(\{\rho_{\pm|x}\}) = (\sigma, y_1, \ldots, y_g)$, with $y_x = \rho_{+|x} - \rho_{-|x}$. We will first prove that the assemblage admits a LHS model (7) if and only if

$$y_x = \sum_\lambda h_\lambda(x) \phi_\lambda, \qquad h_\lambda(x) \in [-1, 1], \; x = 1, \ldots, g, \quad \sum_\lambda \phi_\lambda = \sigma. \tag{8}$$

Indeed, assume the LHS model, then we may put $h_\lambda(x) := q(+|x, \lambda) - q(-|a, \lambda)$ and $\phi_\lambda := q(\lambda)\rho_\lambda$, note that $\sum_\lambda \phi_\lambda = \sigma$. Conversely, assume (8) holds, then we may put $q(\pm|x, \lambda) := \frac{1}{2}(1 \pm h_\lambda(x))$ and by normalizing ϕ_λ, we obtain

$$\rho_{\pm|x} = \frac{1}{2}(\sigma \pm y_x) = \sum_\lambda q(\lambda) q(\pm|x, \lambda) \rho_\lambda,$$

which defines a LHS model for the assemblage.

We now note that (8) is equivalent to $(\sigma, y) \in K_g \otimes_{\min} K$: put $h_\lambda = (h_\lambda(1), \ldots, h_\lambda(g)) \in \mathbb{R}^g$, then $\|h_\lambda\|_\infty \leqslant 1$ and we have $y = \sum_\lambda h_\lambda \otimes \phi_\lambda$. By an easy reshuffling, this can be seen to be equivalent to the characterization of $K_g \otimes_{\min} K$ in (1), see example 2.4. □

Our first characterization of steering by tensor cross norms follows immediately from theorems 3.1, 3.2 and 2.6.

Corollary 3.3. *Let $\{\rho_{a|x}\}$ be an assemblage in $(V, V^+, \mathbb{1})$ of shape $\mathbf{k}$ and let $K_A = K_{\mathbf{k}}$, $K_B = K$. Let $\xi_{AB} = \Sigma_{\mathbf{k}}(\{\rho_{a|x}\}) \in K_A \otimes_{\max} K_B$. The assemblage is classical if and only if $\|\xi_{AB}\|_{\pi(A,B)} \leqslant 1$.*

Remark 3.4. Note that in the setting of corollary 3.3, we have the following equivalent statements for any bipartite state $\sigma_{AB} \in K_A \otimes_{\max} K_B$:

(a) σ_{AB} is unsteerable;
(b) The assemblage $\Sigma_g^{-1}(\sigma_{AB})$ admits a LHS model;
(c) σ_{AB} is separable.

It is well known and immediately seen that a separable state is always unsteerable, see also [17, section V.A] for an easy argument based on the theorems 3.1 and 3.2. On the other hand, there are unsteerable entangled quantum states.

3.3. Dichotomic assemblages and tensor cross norms

We now concentrate on dichotomic assemblages with a given barycenter σ. The set of all such assemblages with g elements will be denoted by $\mathcal{A}^g_{2,\sigma}$. We will show that under the assumption that $\sigma \in \mathrm{int}(K)$, classical elements in $\mathcal{A}^g_{2,\sigma}$ can be characterized by means of tensor cross norms in $\ell^g_\infty \otimes V$, if we choose an appropriate norm in V, depending on σ.

Since $\sigma \in \mathrm{int}(V^+)$, it is an order unit in the ordering of V. Let us denote the corresponding order unit norm by $\|\cdot\|_\sigma$. Its dual in V^* is the base norm with respect to the base of $(V^*)^+$, given as

$$K^\sigma = \{h \in A^+,\ \langle h, \sigma\rangle = 1\}. \tag{9}$$

This base norm will be denoted by $\|\cdot\|^\sigma$.

By (5), any dichotomic assemblage with barycenter σ can be identified with $(\sigma, y_1, \ldots, y_g) \in V^{1+g}$ such that $\pm y_x \leqslant \sigma$. Since σ is fixed, any element in $\mathcal{A}^g_{2,\sigma}$ is characterized by $y = (y_1, \ldots, y_g) \in V^g \simeq \mathbb{R}^g \otimes V$ such that $\|y_x\|_\sigma \leqslant 1$. The following is an easy observation from the definition of the injective norm.

Proposition 3.5. *Let $y \in V^g$. Then $\{\frac{1}{2}(\sigma \pm y_x)\}$ is an assemblage if and only if $\|y\|_{\epsilon,\sigma} \leqslant 1$, where $\|\cdot\|_{\epsilon,\sigma}$ is the injective cross norm in the tensor product $\ell^g_\infty \otimes (V, \|\cdot\|_\sigma)$.*

Proof. This follows from (3): for $y = (y_1, \ldots, y_g) \in V^g \equiv \mathbb{R}^g \otimes V$, we have

$$\|y\|_{\epsilon,\sigma} = \max\{\langle \psi \otimes f, y\rangle,\ \|\psi\|_1 \leqslant 1, \|f\|^\sigma \leqslant 1\}$$
$$= \max\left\{\|\sum_x \psi_x y_x\|_\sigma,\ \|\psi\|_1 \leqslant 1\right\} = \max_x \|y_x\|_\sigma,$$

here $\psi = (\psi_1, \ldots, \psi_x) \in \mathbb{R}^g$. □

We now define a new norm in $\ell_\infty^g \otimes V$: for $y = (y_1, \ldots, y_g) \in V^g \simeq \mathbb{R}^g \otimes V$, put

$$\|y\|_{\text{steer},\sigma} := \inf\left\{ \|\sum_j \phi_j\|_\sigma,\ y = \sum_j z_j \otimes \phi_j,\ \|z_j\|_\infty = 1,\ \phi_j \in V^+ \right\}.$$

Proposition 3.6. *$\|\cdot\|_{\text{steer},\sigma}$ is a reasonable cross norm on $\ell_\infty^g \otimes (V, \|\cdot\|_\sigma)$.*

Proof. It is easily seen that $\|\cdot\|_{\text{steer},\sigma}$ is a norm, so it suffices to show that for $y \in V^g$, $\|y\|_{\epsilon,\sigma} \leqslant \|y\|_{\text{steer},\sigma} \leqslant \|y\|_{\pi,\sigma}$, where $\|\cdot\|_{\pi,\sigma}$ is the projective norm. Assume that $y = \sum_j z_j \otimes \phi_j$ with $z_j = (z_{j,1}, \ldots, z_{j,g}) \in \mathbb{R}^g$, $\|z_j\|_\infty = 1$ and $\phi_j \in V^+$. Then $y_x = \sum_j z_{j,x}\phi_j$ so that

$$-\|\sum_j \phi_j\|_\sigma \sigma \leqslant -\sum_j \phi_j \leqslant -\sum_j |z_{j,x}|\phi_j \leqslant y_x \leqslant \sum_j |z_{j,x}|\phi_j \leqslant \sum_j \phi_j \leqslant \|\sum_j \phi_j\|_\sigma \sigma$$

hence $\|y_x\|_\sigma \leqslant \|\sum_j \phi_j\|_\sigma$. Since this is true for all x and all such expressions of y, this implies that $\max_x \|y_x\|_\sigma \leqslant \|y\|_{\text{steer},\sigma}$ and the first inequality is proved. For the second inequality, let $y = \sum_{j=1}^n \tilde{z}_j \otimes \psi_j$ with $\tilde{z}_j \in \ell_\infty^g$ and $\psi_j \in V$. Put $z_j = \|\tilde{z}_j\|_\infty^{-1}\tilde{z}_j$ and let $\psi_j^\pm = \frac{1}{2}(\|\psi_j\|_\sigma \sigma \pm \psi_j) \in V^+$. Then $\psi_j = \psi_j^+ - \psi_j^-$ and we have

$$y = \sum_j z_j \otimes \|\tilde{z}_j\|_\infty \psi_j^+ + \sum_j (-z_j) \otimes \|\tilde{z}_j\|_\infty \psi_j^- = \sum_{k=1}^{2n} w_k \otimes \zeta_k,$$

where $w_j = z_j$, $w_{n+j} = -z_j$, and $\zeta_j = \|\tilde{z}_j\|_\infty \psi_j^+$, $\zeta_{n+j} = \|\tilde{z}_j\|_\infty \psi_j^-$, for $j = 1, \ldots, n$. By definition of $\|\cdot\|_{\text{steer},\sigma}$, we obtain

$$\|y\|_{\text{steer},\sigma} \leqslant \|\sum_{k=1}^{2n} \zeta_k\|_\sigma = \|\sum_j \|\tilde{z}_j\|_\infty (\psi_j^+ + \psi_j^-)\|_\sigma \leqslant \sum_j \|\tilde{z}_j\|_\infty \|\psi_j\|_\sigma.$$

This implies the second inequality. □

Theorem 3.7. *Let $y \in V^g$ be such that $\{\frac{1}{2}(\sigma \pm y_x)\}$ is an assemblage. Then the assemblage is classical if and only if $\|y\|_{\text{steer},\sigma} \leqslant 1$.*

Proof. By theorem 3.2, the assemblage is classical if and only if $(\sigma, y) \in K_g \otimes_{\min} K$. It is immediate from (1) that this is equivalent to $\|y\|_{\text{steer},\sigma} \leqslant 1$. □

3.4. Steering witnesses

Let $\xi \in K_{\mathbf{k}} \otimes_{\max} K$ be any element. It is clear from the above results and the definition of the minimal and maximal tensor products that the corresponding assemblage of shape $\mathbf{k}$ is classical if and only if $\langle w, \xi \rangle \geqslant 0$ for any $w \in (V_{\mathbf{k}}^*)^+ \otimes_{\max} (V^*)^+$. Therefore any element $w \in (V_{\mathbf{k}}^*)^+ \otimes_{\max} (V^*)^+$ defines a steering witness. We say that a steering witness is strict if there is some $\xi \in K_{\mathbf{k}} \otimes_{\max} K$ such that $\langle w, \xi \rangle < 0$, which means that $w \notin (V_{\mathbf{k}}^*)^+ \otimes_{\min} (V^*)^+$. Hence the set of all strict steering witnesses is $(V_{\mathbf{k}}^*)^+ \otimes_{\max} (V^*)^+ \setminus (V_{\mathbf{k}}^*)^+ \otimes_{\min} (V^*)^+$. From now on we restrict to dichotomic assemblages.

Proposition 3.8. *An element $(w_0, w_1, \ldots, w_g) \in (V^*)^{g+1} \simeq \mathbb{R}^{g+1} \otimes V^*$ is a steering witness if and only if*

$$\sum_{x=1}^g \epsilon_x w_x \leqslant w_0, \quad \forall\ \epsilon \in \{\pm 1\}^g. \tag{10}$$

In this case, we always have $w_0 \in (V^*)^+$ and $w_0 = 0$ implies $w_x = 0$ for all x. A steering witness is strict if and only if there exist some $\sigma \in K$ and elements $y_x \in V$, $\|y_x\|_\sigma \leqslant 1$ such that

$$\langle w_0, \sigma\rangle + \sum_x \langle w_x, y_x\rangle < 0.$$

Proof. The first part follows from example 2.4 and the fact that the extremal points $\epsilon \in [-1, 1]^g$ are precisely the elements of $\{\pm 1\}^g$. The last statement follows from the description of elements in $K_g \otimes_{\max} K$ in the proof of theorem 3.1. □

Let now $\sigma \in \mathrm{int}(V^+)$ and let us restrict to assemblages in $\mathcal{A}^g_{2,\sigma}$. Let $w = (w_0, w_1, \ldots, w_g)$ be a steering witness. Since σ is an interior point, we have by proposition 3.8 that $\langle w_0, \sigma\rangle > 0$ unless $w_0 = 0$ and in this case $w = 0$. Therefore we may restrict to witnesses with $\langle w_0, \sigma\rangle = 1$ and then the value of the witnesses on elements in $\mathcal{A}^g_{2,\sigma}$ is determined by the g-tuple $(w_1, \ldots, w_g) \in (V^*)^g$.

Proposition 3.9. *Let $\sigma \in \mathrm{int}(K)$ and let $w \in (V^*)^g$. The following are equivalent:*

(*a*) *There is some $w_0 \in (V^*)^+$, such that $\langle w_0, \sigma\rangle = 1$ and (w_0, w) is a steering witness.*
(*b*) *$\sum_{x=1}^g |\langle w_x, y_x\rangle| \leqslant 1$ for all $y \in V^g$ such that $\|y\|_{\mathrm{steer},\sigma} \leqslant 1$.*

Proof. Assume (a) and let $\|y\|_{\mathrm{steer},\sigma} \leqslant 1$. Note that by definition of the norm, we have for any $\epsilon \in \{\pm 1\}^g$,

$$\|(\epsilon_1 y_1, \ldots, \epsilon_g y_g)\|_{\mathrm{steer},\sigma} = \|(y_1, \ldots, y_g)\|_{\mathrm{steer},\sigma},$$

so that $(\sigma, \epsilon_1 y_1, \ldots, \epsilon_g y_g)$ corresponds to a classical assemblage. Therefore we have

$$0 \leqslant \langle (w_0, w_1, \ldots, w_g), (\sigma, \epsilon_1 y_1, \ldots, \epsilon_g y_g)\rangle = 1 + \sum_x \epsilon_x \langle w_x, y_x\rangle, \quad \forall\, \epsilon \in \{\pm 1\}^g,$$

this proves (b).

For the converse, choose any element $w_0' \in V^*$ such that $\langle w_0', \sigma\rangle = 1$. Let $(\sigma, y) \in K_g \otimes_{\min} K$, then $\|y\|_{\mathrm{steer},\sigma} \leqslant 1$ and by (b)

$$\langle (w_0', w_1, \ldots, w_g), (\sigma, y_1, \ldots, y_g)\rangle = 1 + \sum_x \langle w_x, y_x\rangle \geqslant 1 - \sum_x |\langle w_x, y_x\rangle| \geqslant 0.$$

It follows that $w' = (w_0', w_1, \ldots, w_g)$ defines a positive functional on the subspace

$$\mathcal{L} = \{(y_0, y_1, \ldots, y_g) \in V^{g+1},\ y_0 \in \mathbb{R}\sigma\}$$

with the cone $\mathcal{L} \cap (V_g^+ \otimes_{\min} V^+)$. Since this subspace contains the interior element $(\sigma, 0) \in \mathrm{int}(V_g^+ \otimes_{\min} V^+)$, w' extends to an element $w \in (V_g^*)^+ \otimes_{\max} (V^*)^+$. It is easily checked that $w = (w_0, w_1, \ldots, w_g)$ with $w_0 \in (V^*)^+$ and

$$\langle w_0, \sigma\rangle = \langle w, (\sigma, 0)\rangle = \langle w', (\sigma, 0)\rangle = \langle w_0', \sigma\rangle = 1.$$

This finishes the proof. □

We will denote the set of $w \in (V^*)^g$ satisfying the above conditions by $\mathcal{W}^g_{2,\sigma}$, note that this is the unit ball in $\ell_1^g \otimes V^*$ with respect to the cross norm dual to $\|\cdot\|_{\mathrm{steer},\sigma}$. By the above result, up to multiplication by a positive constant, $\mathcal{W}^g_{2,\sigma}$ is the set of steering witnesses for assemblages in $\mathcal{A}^g_{2,\sigma}$. We will further say that such a witness is strict if it has a negative value on some assemblage in $\mathcal{A}^g_{2,\sigma}$. For the following characterization of strict witnesses, recall that the base norm $\|\cdot\|^\sigma$ in V^* is the dual to $\|\cdot\|_\sigma$ and has the form

$$\|h\|^\sigma = \sup_{\pm y \leqslant \sigma} \langle h, y\rangle.$$

Proposition 3.10. *The steering witness $w \in \mathcal{W}_{2,\sigma}$ is strict if and only if*

$$\|w\|_{\pi,\sigma} = \sum_{x=1}^{g} \|w_x\|^\sigma > 1,$$

here $\|\cdot\|_{\pi,\sigma}$ is the projective cross norm in the tensor product $\ell_1^g \otimes (V^*, \|\cdot\|^\sigma)$.

Proof. It is easy to check directly that $\|w\|_{\pi,\sigma} = \sum_{x=1}^g \|w_x\|^\sigma$. By proposition 3.8, w is a strict steering witness if and only if there are some $y_x \in V$, $\|y_x\|_\sigma \leqslant 1$ such that

$$0 > 1 + \sum_x \langle w_x, y_x\rangle = 1 - \sum_x \langle w_x, -y_x\rangle \geqslant 1 - \sum_x \|w_x\|^\sigma.$$

□

3.5. Steering degree

The steering degree of assemblages of shape $\mathbf{k}$ can be quantified by the amount of noise that needs to be mixed with the assemblage $\{\rho_{a|x}\}$ in order to obtain a classical assemblage, this is also called steering robustness [27]. The noise is represented by assemblages of the same shape $\mathbf{k}$. Steering robustness is a monotone in various resource theories of steering, [14] or [37]. See [10, 11] for some variants, as well as some other possibilities of quantifying steering in (quantum) assemblages.

The set of assemblages of a given shape inherits its convex structure from the isomorphism with $K_{\mathbf{k}} \otimes_{\max} K$. As noise, we will use a single trivial assemblage, of the form $\{\omega_{a|x}\sigma\}$ with $\omega_{a|x} = k_x^{-1}$ for all $a \in \Omega_x$. The steering degree of $\{\rho_{a|x}\}$ is defined as

$$s(\{\rho_{a|x}\}) = \sup\{s \in [0,1],\ \{s\rho_{a|x} + (1-s)k_x^{-1}\sigma\} \text{ is classical}\}.$$

We also define $s_{\mathbf{k},\sigma}(K)$ as the infimum of the steering degrees of all assemblages with shape $\mathbf{k}$ and barycenter σ. Since we assume K fixed, we will skip it from the notation. Restricting to dichotomic assemblages, we now show that the steering degree can be expressed using the norm $\|\cdot\|_{\text{steer},\sigma}$. In this case we denote $s_{\mathbf{k},\sigma} \equiv s_{g,\sigma}$.

Theorem 3.11. *Let $\{\rho_{\pm|x}\} \in \mathcal{A}^g_{2,\sigma}$ and let $y_x = \rho_{+|x} - \rho_{-|x}$, $x = 1, \ldots, g$. Then*

$$s(\{\rho_{\pm|x}\}) = \|(y_1, \ldots, y_g)\|^{-1}_{\text{steer},\sigma}. \tag{11}$$

For the steering degree $s_{g,\sigma}$, we have

$$s_{g,\sigma} = \sup_{y \in \mathbb{R}^g \otimes V} \frac{\|y\|_{\epsilon,\sigma}}{\|y\|_{\text{steer},\sigma}} \geqslant \sup_{y \in \mathbb{R}^g \otimes V} \frac{\|y\|_{\epsilon,\sigma}}{\|y\|_{\pi,\sigma}}. \tag{12}$$

Dually, in terms of the steering witnesses, we obtain

$$s_{g,\sigma} = \sup\left\{ s \in [0,1],\ \sum_{x=1}^{g} s\|w_x\|^\sigma \leqslant 1, \forall (w_1, \ldots, w_g) \in \mathcal{W}^g_{2,\sigma} \right\}. \tag{13}$$

Proof. Let $s \in [0,1]$, then the tensor element corresponding to the mixed assemblage $\{s\rho_{\pm|x} + (1-s)\frac{1}{2}\sigma\}$ is $(\sigma, sy_1, \dots, sy_g)$. By theorem 3.7 we have

$$s(\{\rho_{\pm|x}\}) = \sup\{s \in [0,1],\ s\|(y_1,\dots,y_g)\|_{\mathrm{steer},\sigma} \leqslant 1\} = \|(y_1,\dots,y_g)\|^{-1}_{\mathrm{steer},\sigma}.$$

The equality and inequality in (12) now follow from proposition 3.5 and proposition 3.6, (13) follows from proposition 3.9 and the definition of the norm $\|\cdot\|^\sigma$. □

In section 4.2 below we will characterize the universal (that is, independent of g) steering degree for dichotomic assemblages with barycenter σ:

$$s_\sigma := \max\{s \in [0,1],\ (\sigma, sy) \in K_g \otimes_{\min} K,\ \forall (\sigma, y) \in K_g \otimes_{\max} K,\ \forall g\} \tag{14}$$

$$= \max\left\{ s \in [0,1],\ s\sum_i \|w_i\|^\sigma \leqslant 1,\ \forall w \in \mathcal{W}^g_{2,\sigma}, \forall g \right\} \tag{15}$$

$$= \inf_{g \in \mathbb{N}} s_{g,\sigma}. \tag{16}$$

3.6. Relation to compatibility of measurements

Let $\{f_x\}_{x=1}^g$ be a collection of measurements with outcomes in Ω_x, with effects $f_{a|x}$. We say that the collection is compatible if all f_x are marginals of a joint measurement h with outcomes in $Y = \Pi_x \Omega_x$:

$$f_{a|x} = \sum_{(a_1,\dots,a_g) \in Y, a_x = a} h_{a_1,\dots,a_g}, \quad \forall\ a, x.$$

In [7] a compatibility degree was studied, defined as a robustness degree with respect to the set of trivial measurements with $f_{a|x} = k_x^{-1}\mathbb{1}$ for all x and a. The aim of the present section is to remark that such collections of measurements and existence of a joint measurement for them are mathematically equivalent to assemblages and existence of LHS models. This simple observation shows a link to the results obtained in [7] on a connection between compatibility, and related notions of witnesses and degree, and minimal/maximal tensor products of cones and tensor norms.

So let σ be an arbitrary interior point in the state space K. Since $\sigma \in \mathrm{int}(V^+)$, it defines a strictly positive functional over the dual space V^* with cone $(V^*)^+$. We may therefore (formally) define a triple $(V^*, (V^*)^+, \sigma)$, with the state space K^σ (see (9)) being a base of $(V^*)^+$. Note that we always have $\mathbb{1} \in K^\sigma$, moreover, the corresponding base norm (in V^*) and order unit norm (in V) are $\|\cdot\|^\sigma$ and $\|\cdot\|_\sigma$ considered above.

The following is rather straightforward.

Lemma 3.12. *Finite sets of measurements on $(V, V^+, \mathbb{1})$ correspond precisely to assemblages for $(V^*, (V^*)^+, \sigma)$ with barycenter $\mathbb{1}$. Also conversely, assemblages for $(V, V^+, \mathbb{1})$ with barycenter $\sigma \in \mathrm{int}(K)$ correspond precisely to finite sets of measurements on $(V^*, (V^*)^+, \sigma)$. Moreover, the measurements are compatible if and only if the corresponding assemblage is classical.*

Let us now look at the norm characterizing steering for assemblages for the system $(V^*, (V^*)^+, \sigma)$ with barycenter $\mathbb{1}$: the norm $\|\cdot\|_{\mathbb{1}}$ on V^* is the usual order unit norm and the dual norm $\|\cdot\|^{\mathbb{1}}$ in V is the base norm $\|\cdot\|_V$. The results of the previous sections correspond to the results in [7] for compatibility of dichotomic measurements, in particular the norm $\|\cdot\|_{\mathrm{steer},\mathbb{1}}$ on $\ell_\infty^g \otimes V^*$ becomes precisely the compatibility norm $\|\cdot\|_c$ as in [7].

The above relations have some immediate consequences, obtained from the results in [7] and duality relations:

(a) The steering degree for any $(V, V^+, \mathbb{1})$ with $\dim(V) = d$ is lower bounded by

$$s_{g,\sigma} \geqslant 1/\min\{g, d\}.$$

(b) In the centrally symmetric case, assume the system $(V_{\|\cdot\|}, V^+_{\|\cdot\|}, (1, 0))$ given by a norm $\|\cdot\|$ in $\mathbb{R}^n$. If the barycenter σ is the central point $(1, 0)$, then the steering degrees $s_{\mathbf{k},(1,0)}$ are the same as the compatibility degrees for the system given by the dual norm $(V_{\|\cdot\|^*}, V^+_{\|\cdot\|^*}, (1, 0))$. In particular, for dichotomic assemblages, we have

$$s_{g,(1,0)} = \sup_{y\in\mathbb{R}^{gn}} \frac{\|y\|_{\epsilon(\ell_\infty^g,(\mathbb{R}^n,\|\cdot\|))}}{\|y\|_{\pi(\ell_\infty^g,(\mathbb{R}^n,\|\cdot\|))}},$$

the ratio of the injective and projective tensor cross norms for the spaces ℓ_∞^g and $(\mathbb{R}^n, \|\cdot\|)$ (cf [7, theorem 10.5]). We also have the tight lower bound

$$s_{g,(1,0)} \geqslant 1/\min\{g, d-1\},$$

attained for the state spaces isomorphic to cross-polytopes (unit balls of the ℓ_1-norm).

(c) The cone V^+ is called (weakly) self-dual if it is affinely isomorphic to its dual $(V^*)^+$. The isomorphism extends to a linear isomorphism $\Psi : V \to V^*$ and by some easy manipulations we may assume that $\Psi = \Psi^*$ and $\Psi(\sigma) = \mathbb{1}$ for some $\sigma \in \mathrm{int}(K)$. Then Ψ defines an isomorphism $(V, V^+, \mathbb{1}) \to (V^*, (V^*)^+, \sigma)$, in the sense that it is a linear bijection $V \to V^*$ that maps K onto K^σ. It is clear that Ψ also maps measurements on $(V, V^+, \mathbb{1})$ onto measurements on $(V^*, (V^*)^+, \sigma)$ and Ψ and Ψ^{-1} both preserve compatibility of measurements. Lemma 3.12 now implies that for the system $(V, V^+, \mathbb{1})$ the steering degrees for the barycenter σ are equal to compatibility degrees.

(d) In the quantum case, there exists an isomorphism as above for any invertible density matrix σ, given by the map $M_n^{sa} \ni X \mapsto \sigma^{-1/2} X \sigma^{-1/2}$, mapping classical assemblages onto compatible measurements and relating the steering degrees to the compatibility degrees, [20, 34]. It follows that the steering degrees for all invertible barycenters are the same and equal to the compatibility degrees, see also [8]. Note that if the barycenter σ is not invertible then we can restrict to a quantum system of lower dimension.

4. Steering and Choquet order

In this section, we show how the properties of probability measures on compact convex sets can be used to characterize steering in GPTs. Although for simplicity the dimension of the systems is assumed to be finite, we remark that most of the results can be extended as stated or with slight technical modifications also to some infinite dimensional cases.

Let $(V, V^+, \mathbb{1})$ be a system with state space K. We now briefly introduce spaces of functions and measures over K. Let $C(K)$ denote the (infinite dimensional) Banach space of continuous functions $f : K \to \mathbb{R}$ with maximum norm $\|f\|_{\max} = \max_{\rho\in K}|f(\rho)|$. This space has a natural positive cone $C(K)^+$ of functions having nonnegative values. With respect to the corresponding ordering, $C(K)$ is also a lattice, with $f \vee g$ and $f \wedge g$ the pointwise maximum and minimum for $f, g \in C(K)$.

Let us denote by $P(K) \subseteq C(K)$ the set of convex functions and by $A(K) \subseteq C(K)$ the set of affine functions in $C(K)$. That is,

$$P(K) = \{f \in C(K),\ f(t\rho_1 + (1-t)\rho_2) \leqslant tf(\rho_1) + (1-t)f(\rho_2),\ t \in [0,1]\}$$
$$A(K) = \{f \in C(K),\ f(t\rho_1 + (1-t)\rho_2) = tf(\rho_1) + (1-t)f(\rho_2),\ t \in [0,1]\}.$$

Then $A(K)$ is a linear subspace in $C(K)$. Any $h \in V^*$ defines an element in $A(K)$ by restriction and it can be seen that any function $A(K)$ uniquely extends to a linear functional in V^*. This identification $V^* \equiv A(K)$ will be used from now on. We then have $(V^*)^+ = A(K)^+ = C(K)^+ \cap A(K)$ and the unit $\mathbb{1}$ is identified with the constant unit function over K. Note that the order unit norm $\|\cdot\|_{\mathbb{1}}$ in V^* coincides with the maximum norm inherited from $C(K)$.

The set $P(K)$ is a convex cone and we have $A(K) \subseteq P(K) \subseteq C(K)$. The cone $P(K)$ is closed in norm and also closed under suprema: $f \vee g \in P(K)$ if $f, g \in P(K)$. The set of finite suprema

$$h_1 \vee \ldots \vee h_n(\rho) = \max_i \langle h_i, \rho\rangle, \quad \rho \in K$$

with $h_1, \ldots, h_n \in V^*$, $n \in \mathbb{N}$, is norm-dense in $P(K)$. The cone $P(K)$ is also generating in $C(K)$: the subspace $P(K) - P(K)$ is dense in $C(K)$.

The dual Banach space $C^*(K)$ can be identified with the space $\mathcal{M}(K)$ of signed Borel measures over K with finite total variation, as $\mu(f) = \int_K f\, \mathrm{d}\mu$. Let us denote by $\mathcal{M}(K)^+$ the cone of finite positive measures and by $\mathcal{P}(K)$ the set of probability measures over K. Then $\mathcal{M}(K)^+$ generates $\mathcal{M}(K)$ and $\mathcal{P}(K)$ is a base of $\mathcal{M}(K)^+$ and is compact in the weak*-topology inherited from the Banach space duality with $C(K)$. For any $\sigma \in K$, the probability measure concentrated in σ belongs to $\mathcal{P}(K)$ and is denoted by δ_σ. A measure of the form $\sum_{i=1}^n c_i\delta_{\rho_i}$, $\rho_i \in K$, $c_i \in \mathbb{R}$ is called simple. For $\sigma \in K$, we denote by $\mathcal{P}_\sigma(K)$ the subset of probability measures $\mu \in \mathcal{P}(K)$ with barycenter $\bar{\mu} = \int_K \rho\, \mathrm{d}\mu(\rho) = \sigma$. Note that the barycenter $\bar{\mu}$ is an element of K characterized by

$$\langle h, \bar{\mu}\rangle = \int \langle h, \rho\rangle \mathrm{d}\mu, \quad \forall\ h \in V^*.$$

The Choquet order on $\mathcal{M}(K)$ is defined as the dual of the ordering in $C(K)$ obtained from the cone $P(K)$: we have $\nu \prec \mu$ if

$$\int f\, \mathrm{d}\nu \leqslant \int f\, \mathrm{d}\mu, \quad \forall\ f \in P(K).$$

Applying this to $f \in A(K)$, we see that $\nu \prec \mu$ implies $\bar{\mu} = \bar{\nu}$ for $\mu, \nu \in \mathcal{P}(K)$. For a simple probability measure $\nu = \sum_i \lambda_i\, \delta_{\sigma_i}$ and $\mu \in \mathcal{P}(K)$, we have the following characterization of the ordering [1, corollary I.3.4]:

$$\nu \prec \mu \iff \exists \mu_i \in \mathcal{P}_{\sigma_i}(K),\ \mu = \sum_i \lambda_i \mu_i. \tag{17}$$

That is, μ is obtained by refinements of the point masses of ν, so we may imagine that μ is supported 'closer' to the extreme boundary. Indeed, a positive measure is maximal with respect to $\prec$ if and only if it is a boundary measure, that is, it is concentrated on the set of pure states $\partial_e K$. We will denote the set of all boundary measures in $\mathcal{P}_\sigma(K)$ by $\mathcal{P}^b_\sigma(K)$. Further, any positive measure is upper bounded in $\prec$ by a positive boundary measure, in particular, any element $\sigma \in K$ is the barycenter of some measure $\mu \in \mathcal{P}^b_\sigma(K)$. For further details see e.g. [1, 26].

4.1. Boundary measures and LHS models

Let $\{\rho_{a|x}\}$ be an assemblage with barycenter σ. Observe that a LHS model of the form (7) for $\{\rho_{a|x}\}$ can be expressed as

$$\rho_{a|x} = \int_K q(a|x,\rho)\rho\, \mathrm{d}\mu(\rho),$$

where $\mu = \sum_{\lambda\in\Lambda} q(\lambda)\delta_{\rho_\lambda} \in \mathcal{P}_\sigma(K)$.

We will now map the assemblage $\{\rho_{a|x}\}$ onto a set of simple probability measures $\{\nu_x\}_{x\in X}$ in $P_\sigma(K)$: put $\lambda_{a|x} := \langle \mathbb{1}, \rho_{a|x}\rangle$ and $\sigma_{a|x} := \lambda_{a|x}^{-1}\rho_{a|x} \in K$ (if $\lambda_{a|x} = 0$ we may pick any state $\sigma_{a|x} \in K$). For any x, put

$$\mu_x := \sum_{a\in\Omega_x} \lambda_{a|x}\delta_{\sigma_{a|x}},$$

then μ_x is a simple probability measure and we have $\bar{\mu}_x = \sum_a \lambda_{a|x}\sigma_{a|x} = \sigma$.

Conversely, from any (finite) set of simple probability measures with the same barycenter we easily obtain an assemblage. Note that this map, in contrast with the isomorphism $\Sigma_{\mathbf{k}}$, is not-one-to-one, and is also not affine, in the sense that it does not match the convex structure on assemblages inherited from $K_{\mathbf{k}} \otimes_{\max} K$ to the convex structure in $\mathcal{P}(K)$. Nevertheless, we will show that existence of a LHS model can be expressed in terms of the Choquet order in $\mathcal{P}_\sigma(K)$.

Proposition 4.1. *Let $\nu \in \mathcal{P}_\sigma(K)$ be a simple measure, $\nu = \sum_a \lambda_a\delta_{\sigma_a}$. Then $\nu \prec \mu$ for some $\mu \in \mathcal{P}_\sigma(K)$ if and only if there are measurable functions $q(a|\cdot) : K \to [0, 1]$ such that $\sum_a q(a|\rho) = 1, \forall\, \rho \in K$ and*

$$\lambda_a\sigma_a = \int_K q(a|\rho)\rho\, \mathrm{d}\mu(\rho).$$

Proof. Since ν is a simple measure, we can use (17) to characterize the Choquet order.

So assume that $\nu \prec \mu$, so that $\mu = \sum_a \lambda_a\mu_a$ for some $\mu_a \in \mathcal{P}_{\sigma_a}(K)$. Since each μ_a is absolutely continuous with respect to μ, we may put $q(a|\cdot) = \lambda_a \frac{\mathrm{d}\mu_a}{\mathrm{d}\mu}$, where $\frac{\mathrm{d}\mu_a}{\mathrm{d}\mu}$ is the Radon–Nikodym derivative. Then $q(a|\cdot)$ are nonnegative Borel functions such that $\sum_a q(a|\cdot) = \sum_a \lambda_a \frac{\mathrm{d}\mu_a}{\mathrm{d}\mu} = 1$ μ-almost surely. We may therefore assume that $\sum_a q(a|\rho) = 1$ by suitably replacing the values of the functions on a subset with μ-measure zero. Then

$$\lambda_a\sigma_a = \lambda_a\bar{\mu}_a = \lambda_a \int_K \rho\, \mathrm{d}\mu_a(\rho) = \int_K \rho q(a|\rho)\mathrm{d}\mu(\rho).$$

Assume the converse, then $\lambda_a = \int_K q(a|\rho)\mathrm{d}\mu$ so that there is a probability measure μ_a such that $\frac{\mathrm{d}\mu_a}{\mathrm{d}\mu} = \lambda_a^{-1} q(a|\cdot)$. We then have $\sigma_a = \int \rho\, \mathrm{d}\mu_a(\rho) = \bar{\mu}_a$ and $\sum_a \lambda_a\mu_a = \mu$, so that $\nu \prec \mu$. □

We now extend the definition of an assemblage in the sense that we no longer assume the parameter set X to be finite. We say that an assemblage $\{\{\rho_{a|x}\}_{a\in\Omega_x}\}_{x\in X}$ admits a LHS model (or is classical) if there is some $\mu \in \mathcal{P}_\sigma(K)$ and measurable functions $q(a|x,\cdot) : K \to [0, 1]$, $\sum_a q(a|x,\cdot) = 1$ for all $x \in X$, such that

$$\rho_{a|x} = \int_K \rho q(a|x,\rho)\mathrm{d}\mu(\rho), \quad a \in \Omega_x,\ x \in X. \tag{18}$$

Let $T : K \to K$ be an affine map. We say that an assemblage is invariant under T if the corresponding set of simple probability measures $\{\nu_x\}$ is invariant under T, that is, for every $x \in X$ there is some $x' \in X$ such that $\nu_x^T (= \nu_x \circ T^{-1}) = \nu_{x'}$. Note that in this case T must preserve the barycenter of the assemblage.

The following theorem collects some observations for this definition of a LHS model and its relation to Choquet order. Note that the statement (c) shows that for finite assemblages this definition of a LHS model coincides with the previous one from section 3.2.

Theorem 4.2. *Let $\{\rho_{a|x}\}_{x\in X}$ be an assemblage with barycenter σ and let $\{\nu_x\}_{x\in X}$ be the corresponding simple measures in $\mathcal{P}_\sigma(K)$. Then*

(*a*) *The assemblage is classical if and only if all ν_x have a common upper bound in Choquet order.*

(*b*) *Any measure μ such that $\{\nu_x\}_{x\in X} \prec \mu$ defines some LHS model for the assemblage. We may always assume that μ is concentrated on the set of pure states.*

(*c*) *If X is a finite set and $\{\nu_x\}_{x\in X} \prec \mu$, then we may assume that μ is simple.*

(*d*) *The assemblage is classical if and only if any finite sub-assemblage $\{\rho_{a|x}\}_{x\in F}$, $F \subseteq X$, $|F| < \infty$ is classical.*

(*e*) *Assume that $\{\nu_x\}_{x\in X} \prec \mu$. If the assemblage is invariant under an affine bijection $T : K \to K$ then we may assume that μ is invariant under T ($\mu^T = \mu$).*

Proof. The statement (a) follows immediately from proposition 4.1, (b) follows from proposition 4.1 and the fact that any measure in $\mathcal{P}(K)$ is upper bounded (in the Choquet order) by a boundary measure. For (c), let $\rho_{a|x} = \lambda_{a|x}\sigma_{a|x}$ for $\sigma_{a|x} \in K$. Assume that $\{\nu_x\}_{x\in X} \prec \mu$ and $|X| < \infty$. As in the proof of proposition 4.1, we then have convex decompositions $\sum_a \lambda_{a|x}\mu_{a|x} = \mu$, $\bar{\mu}_{a|x} = \sigma_{a|x}$, for each $x \in X$. Since $\mathcal{P}(K)$ is a Choquet simplex (e.g. [1, corollary II.4.2]), all the decompositions have a common refinement [1, propostion II.3.3]: there are probability measures μ_ω indexed by $\omega \in \Omega = \Pi_{x\in X}\Omega_x$ and some probability measure q over Ω such that $\mu = \sum_{\omega\in\Omega} q(\omega)\mu_\omega$ and

$$\lambda_{a|x}\mu_{a|x} = \sum_{\omega,\omega_x=a} q(\omega)\mu_\omega = \sum_\omega d(a|x,\omega)q(\omega)\mu_\omega,$$

where $d(a|x,\omega) = 1$ if $\omega_x = a$ and is 0 otherwise. Put $\rho_\omega := \bar{\mu}_\omega$, then we obtain

$$\rho_{a|x} = \lambda_{a|x}\sigma_{a|x} = \lambda_{a|x}\bar{\mu}_{a|x} = \sum_\omega d(a|x,\omega)q(\omega)\rho_\omega,$$

which is a LHS model with a simple measure $\mu := \sum_\omega q(\omega)\delta_{\rho_\omega}$.

To prove (d), assume that any finite sub-assemblage $\{\rho_{a|x}\}_{x\in F}$ is classical. By (a), this is equivalent to the fact that for any finite $F \subseteq X$, the subset

$$M_F := \{\mu \in \mathcal{P}_\sigma(K),\ \nu_x \prec \mu,\ \forall\ x \in F\}$$

is nonempty and it is easily seen from the definition of Choquet order that M_F is also closed, in the topology of $\mathcal{P}(K)$. Moreover, since for any finite collection $F_i \subseteq X$, $|F_i| < \infty$, $i = 1, \ldots, n$, we have $\bigcap_i M_{F_i} = M_{\cup_i F_i}$, we see that

$$\{M_F,\ F \subseteq X \text{ is finite}\}$$

is a collection of closed subsets in $\mathcal{P}(K)$ with the finite intersection property. The statement now follows by compactness of $\mathcal{P}(K)$.

To prove (e), let $\{\nu_x\} \prec \mu$ and let $T : K \to K$ be an affine bijection preserving $\{\nu_x\}_{x\in X}$, then for any $f \in P(K)$ and $x \in X$,

$$\int_K f\,\mathrm{d}\nu_x = \int_K f \circ T^{-1}\,\mathrm{d}\nu_x^T \leqslant \int_K f \circ T^{-1}\,\mathrm{d}\mu = \int_K f\,\mathrm{d}\mu^{T^{-1}}$$

so that $\nu_x \prec \mu^{T^{-1}}$. The set of all affine bijections $K \to K$ form a compact group (in the topology of pointwise convergence) of which the elements preserving the assemblage form a compact subgroup G. It is easily checked that the map $G \to \mathcal{P}(K)$ given by $S \mapsto \mu^S$ is continuous. Let m be the Haar measure for G and let $\mu_m = \int_G \mu^S\,\mathrm{d}m(S)$, then μ_m is invariant under T and we have for any $f \in P(K)$ and $x \in X$:

$$\int_K f\,\mathrm{d}\mu_m = \int_G\int_K f(\rho)\mathrm{d}\mu^S(\rho)\mathrm{d}m(S) \geqslant \int f\,\mathrm{d}\nu_x.$$

□

We now give some further characterization of the Choquet order in the case of simple measures.

Proposition 4.3. *Let $\mu, \nu \in \mathcal{P}(K)$ and assume that $\nu = \sum_{a=1}^k \lambda_a\delta_{\sigma_a}$ is simple. Then $\nu \prec \mu$ if and only if for all $h_1, \dots, h_k \in V^*$ we have*

$$\sum_a \lambda_a\langle h_a, \sigma_a\rangle \leqslant \int_K (h_1 \vee \dots \vee h_k)(\rho)\mathrm{d}\mu.$$

If both measures have the same barycenter, it is enough to assume that $\sum_a h_a = 0$.

Proof. Assume the inequality holds for all $h_1, \dots, h_k \in V^*$. Let $f \in P(K)$, then there are affine functions $h_a \in A(K) = V^*$, $a = 1, \dots, k$, such that $h_a \leqslant f$ and $f(\sigma_a) = \langle h_a, \sigma_a\rangle$. Then we have $h_1 \vee \cdots \vee h_k \leqslant f$ and therefore

$$\int f\,\mathrm{d}\nu = \sum_a \lambda_a f(\sigma_a) = \sum_a \lambda_a\langle h_a, \sigma_a\rangle \leqslant \int_K h_1 \vee \dots \vee h_k(\rho)\mathrm{d}\mu \leqslant \int_K f\,\mathrm{d}\mu.$$

For the converse, assume that $\nu \prec \mu$, then

$$\sum_a \lambda_a\langle h_a, \sigma_a\rangle \leqslant \sum_a \lambda_a\left(\bigvee_{a'} h_{a'}\right)(\sigma_a) = \int\left(\bigvee_{a'} h_{a'}\right)\mathrm{d}\nu \leqslant \int\left(\bigvee_{a'} h_{a'}\right)\mathrm{d}\mu,$$

the last inequality follows from the fact that $\bigvee_{a'} h_{a'} \in P(K)$. To prove the last statement, note that if $\bar{\nu} = \bar{\mu}$, then we may replace each h_a by $\tilde{h}_a = h_a + h$ for some $h \in V^*$, since then $\bigvee \tilde{h}_{a'} = \bigvee h_{a'} + h$ and $\sum_a \lambda\langle h, \sigma_a\rangle = \langle h, \bar{\nu}\rangle = \langle h, \bar{\mu}\rangle = \int h d\mu$, so that the inequality is preserved. □

4.2. Dichotomic assemblages and steering degree

We now restrict our attention to dichotomic assemblages $\{\rho_{\pm,x}\}$, in which case the corresponding simple measures are supported in up to two points, we will denote the set of all such probability measures with barycenter σ by $\mathcal{P}_{2,\sigma}(K)$.

Lemma 4.4. *For $\nu \in \mathcal{P}_{2,\sigma}(K)$ and $\mu \in \mathcal{P}_\sigma(K)$, we have $\nu \prec \mu$ if and only if*

$$\int_K |\langle h, \rho\rangle| \mathrm{d}\nu(\rho) \leqslant \int_K |\langle h, \rho\rangle| \mathrm{d}\mu(\rho), \quad \forall\ h \in V^*.$$

Proof. Let $\nu = \lambda\delta_{\sigma_+} + (1-\lambda)\delta_{\sigma_-}$. By putting $k = 2$ in the last part of proposition 4.3, we see that $\nu \prec \mu$ if and only if

$$\lambda\langle h, \sigma_+\rangle - (1-\lambda)\langle h, \sigma_-\rangle \leqslant \int (h \vee -h)\mathrm{d}\mu = \int |\langle h, \rho\rangle| \mathrm{d}\mu, \quad \forall\ h \in V^*.$$

Assume the inequality in the statement holds, then $\nu \prec \mu$ follows from

$$\lambda\langle h, \sigma_+\rangle - (1-\lambda)\langle h, \sigma_-\rangle \leqslant \lambda|\langle h, \sigma_+\rangle| + (1-\lambda)|\langle h, \sigma_-\rangle| = \int |\langle h, \rho\rangle| \mathrm{d}\nu.$$

The converse follows from the fact that $\rho \mapsto |\langle h, \rho\rangle|$ defines a function in $P(K)$. □

Lemma 4.5. *For any $h \in V^*$ and $\mu \in \mathcal{P}_\sigma(K)$, we have*

$$\int_K |\langle h, \rho\rangle| \mathrm{d}\mu \leqslant \|h\|^\sigma = \max_{\nu \in \mathcal{P}_{2,\sigma}(K)} \int_K |\langle h, \rho\rangle| \mathrm{d}\nu.$$

Proof. Let $K_\pm = \{\rho \in K, \pm\langle h, \rho\rangle \geqslant 0\}$, then

$$\int_K |\langle h, \rho\rangle| \mathrm{d}\mu(\rho) = \int_{K_+} \langle h, \rho\rangle \mathrm{d}\mu - \int_{K_-} \langle h, \rho\rangle \mathrm{d}\mu = \langle h, \mu_+ - \mu_-\rangle,$$

where $\mu_\pm = \int_{K_\pm} \rho\, \mathrm{d}\mu(\rho)$. Since $\mu_+ + \mu_- = \sigma$, we have $\pm(\mu_+ - \mu_-) \leqslant \sigma$, so that $\langle h, \mu_+ - \mu_-\rangle \leqslant \|h\|^\sigma$. Since $\|\cdot\|^\sigma$ is a base norm with respect to the order unit σ, there are some $y_\pm \in V^+$ such that $y_+ + y_- = \sigma$ and

$$\begin{aligned}\|h\|^\sigma &= \langle h, y_+ - y_-\rangle = |\langle h, y_+\rangle| + |\langle h, y_-\rangle| \\ &= s_+|\langle h, \sigma_+\rangle| + s_-|\langle h, \sigma_-\rangle| = \int_K |\langle h, \rho\rangle| \mathrm{d}\nu(\rho),\end{aligned}$$

where $s_\pm = \langle \mathbb{1}, y_\pm\rangle$, $\sigma_\pm = s_\pm^{-1} y_\pm$ and $\nu = s_+\delta_{\sigma_+} + s_-\delta_{\sigma_-} \in \mathcal{P}_{2,\sigma}(K)$. □

We also have an alternative characterization of the witness set $\mathcal{W}^g_{2,\sigma}$.

Lemma 4.6. *Let $w \in (V^*)^g$. Then $w \in \mathcal{W}^g_{2,\sigma}$ if and only if for all $\mu \in \mathcal{P}_\sigma(K)$,*

$$\sum_x \int_K |\langle w_x, \rho\rangle| \mathrm{d}\mu(\rho) \leqslant 1.$$

Proof. Assume that $(w_1, \ldots, w_g) \in \mathcal{W}^g_{2,\sigma}$, then by proposition 3.8 there is some $w_0 \in (V^*)^+$, $\langle w_0, \sigma\rangle = 1$ such that $\sum_x \epsilon_x w_x \leqslant w_0$ for all $\epsilon \in \{\pm 1\}^g$. For any $\rho \in K$, there is some $\epsilon \in \{\pm 1\}^g$ such that

$$\sum_x |\langle w_x, \rho\rangle| = \sum_x \epsilon_x \langle w_x, \rho\rangle \leqslant \langle w_0, \rho\rangle.$$

For any $\mu \in \mathcal{P}_\sigma(K)$ we obtain

$$\sum_x \int_K |\langle w_x, \rho\rangle| \mathrm{d}\mu \leqslant \int_K \langle w_0, \rho\rangle \mathrm{d}\mu = \langle w_0, \sigma\rangle = 1.$$

For the converse, let $\{\rho_{\pm|x}\}_{x=1}^{g}$ be a classical dichotomic assemblage. Note that $\Sigma_g(\{\rho_{\pm|x}\}) = (\sigma, y)$ with $y_x = \rho_{+|x} - \rho_{-|x}$. Let ν_x, $x = 1, \ldots, g$ be the corresponding simple probability measures, then there is some $\mu \in \mathcal{P}_\sigma(K)$ such that $\nu_x \prec \mu$ for $x = 1, \ldots, g$. By the definition of Choquet order

$$\sum_x |\langle w_x, y_x\rangle| \leqslant \sum_x |\langle w_x, \rho_{+|x}\rangle| + |\langle w_x, \rho_{-|x}\rangle|$$
$$= \sum_x \int_K |\langle w_x, \rho\rangle| \mathrm{d}\nu_x \leqslant \sum_x \int_K |\langle w_x, \rho\rangle| \mathrm{d}\mu \leqslant 1.$$

The assertion now follows from proposition 3.9. □

We now obtain an expression for the universal steering degree s_σ, independent of the size of the assemblage.

Theorem 4.7. *For $\mu \in \mathcal{P}_\sigma(K)$, let*

$$c_\mu := \inf_{h\in A, \|h\|^\sigma = 1} \int_K |\langle h, \rho\rangle| \mathrm{d}\mu(\rho).$$

Then $c_\mu \leqslant s_\sigma$. There exists a boundary measure $\mu \in \mathcal{P}_\sigma(K)$ satisfying $c_\mu = s_\sigma$, that is invariant under any affine bijection $T : K \to K$ such that $T(\sigma) = \sigma$.

Proof. Let $\mu \in \mathcal{P}_\sigma(K)$. Note that c_μ is the largest $c \in [0, 1]$ such that $c\|h\|^\sigma \leqslant \int_K |\langle h, \rho\rangle| \mathrm{d}\mu(\rho)$, for all $h \in V^*$. Let $(w_1, \ldots, w_g) \in \mathcal{W}_{2,\sigma}^{g}$, then by lemma 4.6 we have

$$c_\mu \sum_x \|w_x\|^\sigma \leqslant \sum_x \int_K |\langle w_x, \rho\rangle| \mathrm{d}\mu(\rho) \leqslant 1,$$

so that $c_\mu \leqslant s_\sigma$ by (15).

We now prove existence of the measure such that equality is attained. Let $s = s_\sigma$. By definition of s_σ and theorem 4.2(d), we see that

$$\left\{\rho_\pm^s := s\rho_{\pm|\nu} + (1-s)\frac{1}{2}\sigma,\ \rho_\pm \in V^+,\ \rho_+ + \rho_- = \sigma\right\}$$

is a classical dichotomic assemblage. Moreover, this assemblage is clearly invariant under the group G_σ of affine bijections that preserve σ, so that using again theorem 4.2 and proposition 4.3, we see that there is some measure $\mu \in \mathcal{P}_\sigma^b(K)$, invariant under G_σ and such that

$$\int |\langle h, \rho\rangle| \mathrm{d}\mu \geqslant \langle h, \rho_+^s - \rho_-^s\rangle = s\langle h, \rho_+ - \rho_-\rangle,$$
$$\forall\ h \in V^*,\ \forall\ \rho_\pm \in V^+,\ \rho_+ + \rho_- = \sigma.$$

Let $h \in V^*$, then since $\|\cdot\|^\sigma$ is a base norm, there are some $\rho_\pm \in V^+$, $\rho_+ + \rho_- = \sigma$ such that $\|h\|^\sigma = \langle h, \rho_+ - \rho_-\rangle$, this implies

$$\int_K |\langle h, \rho\rangle| \mathrm{d}\mu \geqslant s\|h\|^\sigma.$$

It follows that $c_\mu \geqslant s$. By the first part of the proof, we now have $c_\mu = s_\sigma$. □

Example 4.8. As observed in (4) on p 12, for a quantum system (M_n^{sa}, M_n^+, I), s_σ is the same for all invertible σ. So we may choose $\sigma = n^{-1}I$ and then $\|h\|^\sigma = n^{-1}\|h\|_{\mathrm{Tr}}$, where $\|h\|_{\mathrm{Tr}}$

is the trace norm. The universal dichotomic steering degree (and compatibility degree) is then given by

$$s_\sigma = \inf_{\|h\|_{\mathrm{Tr}}=n} \int_{\mathcal{P}_n} |\langle \psi, h\psi\rangle| \mathrm{d}\mu(\psi),$$

where μ is the unique unitarily invariant measure on the set $\mathcal{P}_n$ of pure states. This corresponds to the results of [8], where the infimum was also evaluated.

Example 4.9. Let $(V, V^+, \mathbb{1})$ be the centrally symmetric system given by the norm $\|\cdot\|$ in $\mathbb{R}^n$ and let σ be the central element $\sigma = (1, 0)$. Note that in this case the dichotomic assemblages are the same as sets of dichotomic measurements for the centrally symmetric system given by the norm $\|\cdot\|^*$, so by the results of [7] we have that $s_{(0,1)} = \pi_1^{-1}$ where π_1 is the one-summing constant for the norm $\|\cdot\|^*$.

We now check this in our setting. The base K^σ of $(V^*)^+$ is the dual state space isomorphic to the unit ball of the dual norm

$$K^{(1,0)} = K^* = \{(1, \psi),\ \|\psi\|^* \leqslant 1\}$$

and the base norm is then $\|(t, \varphi)\|^{(1,0)} = \max\{|t|, \|\varphi\|^*\}$. The boundary measures $\nu \in \mathcal{P}^b_{(1,0)}(K)$ which are invariant under affine bijections preserving $(1, 0)$ correspond to probability measures on B concentrated on the extreme boundary $\partial_e B$ that are invariant under isometries of $\|\cdot\|$. Let $(t, \varphi) \in V^*$, $\|(t, \varphi)\|^{(1,0)} = 1$, then for any such ν,

$$\begin{aligned} 1 &\geqslant \int_K |\langle (t,\varphi), (1, x)\rangle| \mathrm{d}\nu = \int_B |t + \langle \varphi, x\rangle| \mathrm{d}\nu \\ &= \frac{1}{2}\int_B |t + \langle \varphi, x\rangle| + |t - \langle \varphi, x\rangle| \mathrm{d}\nu \\ &= \int_B \max\{|t|, |\langle \varphi, x\rangle|\} \mathrm{d}\nu \geqslant \int_B |\langle \varphi, x\rangle| \mathrm{d}\nu = \int_K |\langle (0,\varphi), (1, x)\rangle| \mathrm{d}\nu \end{aligned}$$

(the second equality holds since ν is invariant under the map $x \mapsto -x$). If $\|\varphi\|^* \leqslant |t|$, then we have $|\langle \varphi, x\rangle| \leqslant |t|$ for all $x \in B$ and the integral is equal to $|t| = 1$. If $\|\varphi\|^* \geqslant |t|$, then $\|\varphi\|^* = 1$. It follows that infimum in the definition of c_ν is attained at an element with $t = 0$ and we have

$$c_\nu = \inf_{\|\varphi\|^*=1} \int_B |\langle \varphi, x\rangle| \mathrm{d}\mu = \inf_{\|\varphi\|^*=1} \int_{\overline{\partial_e B}} |\langle \varphi, x\rangle| \mathrm{d}\mu \leqslant \pi_1^{-1}.$$

Here the inequality follows from [15, theorem 1], moreover, equality is attained for some invariant probability measure ν_0 on $\overline{\partial_e B}$. This measure corresponds to some measure in $\mathcal{P}(K)$, which we also denote by ν_0, and invariance implies that its barycenter must be $(1, 0)$. It follows from our results that there is some invariant boundary measure μ such that

$$\pi_1^{-1} = c_{\nu_0} \leqslant s_{(1,0)} = c_\mu \leqslant \pi_1^{-1},$$

so that, indeed, $\pi_1^{-1} = s_{(1,0)}$.

5. Unsteerable states in GPTs

Let $(V_A, V_A^+, \mathbb{1}_A)$ and $(V_B, V_B^+, \mathbb{1}_B)$ be two systems and let $\sigma_{AB} \in K_A \otimes_{\max} K_B$. For a measurement $f = \{f_a\}$ on the system V_A, let $\{\rho_{a|f} := (f_a \otimes \mathrm{id}_B)(\sigma_{AB})\}$ be the ensemble of conditional states and let $\nu_f \in \mathcal{P}_{\sigma_B}(K_B)$ denote the corresponding simple measure. By theorem 4.2, σ_{AB} is $(A \to B)$ unsteerable (by a specified set $\mathcal{F}$ of measurements) if and only if $\{\rho_{a|f}\}_{f\in\mathcal{F}}$ is classical: there is some measure $\mu \in \mathcal{P}^b_{\sigma_B}(K_B)$ such that for all $f \in \mathcal{F}$,

$$\{\nu_f\} \prec \mu.$$

Assume that $U_A : K_A \to K_A$ and $U_B : K_B \to K_B$ are affine bijections such that $U_A \otimes U_B$ preserves σ_{AB}. If also U_A preserves the specified family of measurements, then since

$$U_B((f_a \otimes \mathrm{id}_B)(\sigma_{AB})) = (f_a \circ U_A^{-1} \circ U_A \otimes U_B)(\sigma_{AB}) = (f_a \circ U_A^{-1} \otimes \mathrm{id}_B)(\sigma_{AB}),$$

the assemblage is invariant under U_B, so we may assume that $\mu = \mu^{U_B}$ is an invariant boundary probability measure.

Note that the state σ_{AB} determines a linear map $S_{A\to B} : V_A^* \to V_B$ by

$$\langle h_B, S_{A\to B}(h_A)\rangle \;=\; \langle h_A \otimes h_B, \sigma_{AB}\rangle, \qquad h_A \in V_A^*,\; h_B \in V_B^*.$$

As before, let $K_A^{\sigma_A} = \{h_A \in (V_A^*)^+,\; \langle h_A, \sigma_A\rangle = 1\}$, then notice that $S_{A\to B}$ maps $K_A^{\sigma_A}$ into K_B. Similarly, there is a linear map $S_{B\to A} : V_B^* \to V_A$ that maps $K_B^{\sigma_B}$ into K_A and is the adjoint of $S_{A\to B}$. The map $S_{A\to B}$ maps any set of measurements $\{f_{a|x}\}$ to an assemblage with barycenter σ_B: $\{\rho_{a|x} = (f_{a|x} \otimes \mathrm{id}_B)(\sigma_{AB}) = S_{A\to B}(f_{a|x})\}$. If $\sigma_B \in \mathrm{int}(K_B)$, then such an assemblage defines a set of measurements for the system $(V_B^*, (V_B^*)^+, \sigma_B)$ by lemma 3.12. Note that if σ_B is not an interior point, then we may restrict to the system $(F_B, F_B^+, \mathbb{1}|_{F_B})$, where F_B^+ is the face of V_B^+ generated by σ_B and $F_B = F_B^+ - F_B^+$ is the generated subspace.

The following connection of unsteerable states an incompatibility breaking maps is now immediate.

Proposition 5.1. *The state σ_{AB} is unsteerable by a family of measurements $\mathcal{F}$ if and only if $S_{A\to B}$ is $\mathcal{F}$-incompatibility breaking.*

Combining with proposition 4.3, we obtain the following condition (this should be compared with the condition in [24, theorem 1] in the quantum case).

Theorem 5.2. *Let $\mathcal{F}$ be a set of measurements with k outcomes. A state σ_{AB} is $\mathcal{F}$-unsteerable if and only if for any boundary measure $\mu \in \mathcal{P}^b_{\sigma_B}(K)$, any measurement $f = \{f_a\}_{a=1}^k \in \mathcal{F}$ and any collection $\{g_a\}_{a=1}^k$ in V_B^*, we have*

$$\sum_a \langle f_a \otimes g_a, \sigma_{AB}\rangle \left(= \sum_a \langle g_a, S_{A\to B}(f_a)\rangle \right) \leqslant \int (\vee_a g_a)\mathrm{d}\mu.$$

If moreover $U_A : K_A \to K_A$ and $U_B : K_B \to K_B$ are affine bijections such that $(U_A \otimes U_B)(\sigma_{AB}) = \sigma_{AB}$ and U_B preserves $\mathcal{F}$, then we may assume that $\mu^{U_B} = \mu$.

If $\mathcal{F}$ is the set of dichotomic measurements, we obtain the following characterization by properties of the map $S_{B\to A}$. Note that the inequality is similar to the principal radius [35, equation (32)] for qubit systems.

Theorem 5.3. *A state $\sigma_{AB} \in K_A \otimes_{\max} K_B$ is $(A \to B)$ unsteerable by dichotomic measurements if and only if there is a boundary measure $\mu \in \mathcal{P}^b_{\sigma_B}(K_B)$ such that for all $h \in V_B^*$,*

$$\|S_{B\to A}(h)\|_{V_A} \leqslant \int_K |\langle h, \rho\rangle| \mathrm{d}\mu(\rho).$$

If there are affine bijections $U_A : K_A \to K_A$ and $U_B : K_B \to K_B$ such that $(U_A \otimes U_B)(\sigma_{AB}) = \sigma_{AB}$, then we may assume that $\mu^{U_B} = \mu$. In particular, this is true if $\|S_{B\to A}(h)\|_{V_A} \leqslant s_{\sigma_B}\|h\|^{\sigma_B}$ for all h.

Proof. Let $\mathcal{F}_2$ be the set of all dichotomic measurements. For $\{f_\pm\} \in \mathcal{F}_2$, let $\rho_{\pm,f} = (f_\pm \otimes \mathrm{id})(\sigma_{AB})$. Then σ_{AB} is unsteerable if and only if the assemblage $\{\rho_{\pm,f}\}_{f\in\mathcal{F}_2}$ is classical, that is, there is some $\mu \in \mathcal{P}^b_{\sigma_B}(K_B)$, such that for all $h \in V_B^*$ and $f \in \mathcal{F}_2$,

$$\langle h, \rho_{+|f} - \rho_{-|f}\rangle \leqslant \int |\langle h, \rho\rangle \mathrm{d}\mu.$$

Moreover, if $(U_A \otimes U_B)(\sigma_{AB}) = \sigma_{AB}$ then the assemblage is invariant under U_B so the measure μ can be also taken invariant. Now note that the expression on the left is equal to

$$\langle ((f_+ - f_-) \otimes h)(\sigma_{AB})\rangle \;=\; \langle f_+ - f_-, S_{B\to A}(h)\rangle$$

and the supremum over all dichotomic measurements is equal to $\|S_{B\to A}(h)\|_{V_A}$. The last statement follows from theorem 4.7. □

6. Conclusions

We have studied steering in the setting of GPTs. For dichotomic measurements, we proved that steering can be characterized and quantified in terms of certain Banach space tensor cross norms, analogously to compatibility of dichotomic measurements. In the general case, we have shown that steering can be conveniently treated using the classical Choquet theory for probability measures on the compact and convex state space.

We used this setting for some alternative characterization of LHS models. For dichotomic assemblages with a fixed barycenter, we found a variational expression for the universal steering degree that generalizes the expressions known from quantum systems and centrally symmetric systems. We also considered characterizations of bipartite states that are unsteerable and obtained conditions similar to those recently proved for quantum systems.

A characterization of classical assemblages with a fixed barycenter by a tensor cross norm as in theorem 3.7 is possible only for dichotomic assemblages. Such assemblages have certain symmetries not present in the general case, where we only have a tensor norm characterization from corollary 3.3. A characterization analogous to theorem 3.7 for general assemblages is a natural question for further investigation.

Our results can be immediately applied to the study of compatible measurements, in particular a similar formula can be found for the g-independent incompatibility degree for dichotomic measurements that was only lower bounded by the one-summing constant in [7]. A more precise study of the consequences of the results in section 4 to compatibility of measurements is left for future work.

Observe also that similar results hold also for compact convex subsets in arbitrary (infinite dimensional) locally convex spaces and can be easily extended beyond finite outcome measurements.

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Data availability statement

No new data were created or analysed in this study.

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