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REVERSIBILITY CONDITIONS FOR QUANTUM OPERATIONS

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We give a list of equivalent conditions for reversibility of the adjoint of a unital Schwarz map, with respect to a set of quantum states. A large class of such conditions is given by preservation of distinguishability measures: F-divergences, L_1-distance, quantum Chernoff and Hoeffding distances. Here we summarize and extend the known results. Moreover, we prove a number of conditions in terms of the properties of a quantum Radon–Nikodym derivative and factorization of states in the given set. Finally, we show that reversibility is equivalent to preservation of a large class of quantum Fisher informations and χ^2-divergences.

Keywords: 2-positive maps; Schwarz maps; reversibility; f-divergences; Radon–Nikodym derivative; hypothesis testing; quantum Fisher information.

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1. Introduction

In the mathematical description of quantum mechanics, a quantum mechanical system is represented by a C^*-algebra $\mathcal{A} \subseteq B(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$. In the case that $\mathcal{H}$ is finite-dimensional, the physical states of the system are represented by density operators, that is, positive operators with unit trace. The evolution of the system is described, in the Schrödinger picture, by a transformation T on the states. Here T is usually required to be a completely positive trace preserving map on the algebra.

Let $\mathcal{S}$ be a set of states, then $\mathcal{S}$ can be seen as carrying some information. If $\mathcal{S}$ undergoes a quantum operation T, then some information can be lost. If $\mathcal{S}$ represents a code which is sent through a noisy channel $T : \mathcal{A} \to \mathcal{B}$, then the resulting code $T(\mathcal{S})$ might contain less information than $\mathcal{S}$. In the framework of quantum statistics, $\mathcal{S}$ represents a prior knowledge on the state of the system and the task of the statistician is to make some inference on the true state. But if, say, $\mathcal{S}$ is a family of states on the bipartite system $\mathcal{A} \otimes \mathcal{B}$ and only the system $\mathcal{A}$ is accessible, then the statistician has to work with the restricted states which might be distinguished with less precision. However, it might happen in some situations that the original information can be recovered, in the sense that there is a quantum operation S such that $S \circ T(\sigma) = \sigma$ for all $\sigma \in \mathcal{S}$. In this case we say that T is reversible for $\mathcal{S}$. Such maps are also called sufficient for $\mathcal{S}$, which comes from the well-known notion of sufficiency in classical statistics.

The information loss under quantum operations is expressed in the monotonicity property of distinguishability measures: quantum f-divergences [28] like relative entropy, the L_1-distance, quantum Chernoff and Hoeffding distances [2], etc., which means that these measures are non-increasing under quantum operations. It is quite clear that if T is reversible for $\mathcal{S}$, then T must preserve all of these measures on $\mathcal{S}$. It was an important observation in [27] that preservation of the relative entropy, along with other equivalent conditions, is equivalent to reversibility. These results were then extended in the papers [29, 15, 16]; see also [24]. The very recent paper [13] extends the monotonicity results to the case that T is the adjoint of a subunital Schwarz map and proves that reversibility is equivalent to preservation of a large class of quantum f-divergences, as well as distinguishability measures related to quantum hypothesis testing: the quantum Chernoff and Hoeffding distances. In the present paper, we find conditions for reversibility in terms of the L_1-distance and complete the results for the Chernoff and Hoeffding distances and L_1-distance for n copies of the states, giving an answer to some of the questions left open in [13]. Moreover, we find a class of quantum Fisher informations, such that preservation of elements in this class is equivalent to reversibility. We also prove reversibility conditions in terms of a quantum Radon–Nikodym derivative, and a quantum version of the factorization theorem of classical statistics.

The various equivalent reversibility conditions are interesting also from the opposite point of view, when we are interested in the equality conditions for the divergences in the first place. This was used, for example, for a characterization of the quantum Markov property [10, 15, 19, 20], conditions for nullity of the quantum discord [8, Lemma 8.12], [6], conditions for strict decrease of Holevo quantity [31] and the equality conditions in certain Minkowski type quantum inequalities and related quantities, [18].

In a preliminary section, we deal with the properties of positive maps, 2-positive maps and Schwarz maps, and their duals with respect to a state. In particular, we find a new characterization of 2-positivity in terms of generalized Schwarz inequality and we show that a unital positive map has the property that its duals with respect to all states are Schwarz maps, if and only if it is 2-positive. Then we proceed to the various reversibility conditions: we list the already known conditions related to f-divergences and give an example of a (non-quadratic and strictly convex) operator convex function f, such that preservation of the corresponding f-divergence does not imply reversibility. Further, we prove reversibility conditions in terms of a quantum Radon–Nikodym derivative and certain factorization conditions on the states. In Secs. 3.4 and 3.5, we deal with the L_1-distance, quantum Chernoff and Hoeffding distances. In the last section, we give the reversibility conditions in terms of the quantum Fisher information.

2. Preliminaries

Let $\mathcal{H}$ be a finite-dimensional Hilbert space and let $\mathcal{A} \subseteq B(\mathcal{H})$ be a C^*-algebra. We denote by $\mathcal{A}^+$ the positive cone in $\mathcal{A}$ and by $\mathcal{S}(\mathcal{A})$ the set of states on $\mathcal{A}$. For $a \in \mathcal{A}^+$, we denote by $\operatorname{supp} a$ the projection onto the support of a, that is, $\operatorname{supp} a$ is the smallest projection p satisfying $ap = a$.

A positive linear functional τ on $\mathcal{A}$ such that $\tau(ab) = \tau(ba)$ for all $a, b \in \mathcal{A}$ (equivalently, $\tau(a^*a) = \tau(aa^*)$ for all $a \in \mathcal{A}$) is called a trace. We will also require that τ is faithful, then any linear functional φ on $\mathcal{A}$ has the form

$$\varphi(a) = \tau(a\rho_\varphi), \quad a \in \mathcal{A}$$

for a unique operator $\rho_\varphi \in \mathcal{A}$, and φ is a state if and only if $\rho_\varphi \geq 0$ and $\tau(\rho_\varphi) = 1$. In this case, ρ_φ is called the density operator of φ with respect to τ. Conversely, any operator $\rho \in \mathcal{A}^+$ with $\tau(\rho) = 1$ defines a state φ_ρ on $\mathcal{A}$ with density ρ. Moreover, if τ is faithful, then

$$\langle a, b\rangle_\tau = \tau(a^*b), \quad a, b \in \mathcal{A}$$

defines an inner product in $\mathcal{A}$.

Clearly, $\mathcal{A}$ inherits the trace $\mathrm{Tr} = \mathrm{Tr}_{\mathcal{H}}$ from $B(\mathcal{H})$, but in general, there exists different faithful traces on $\mathcal{A}$ even if we require $\tau(I) = \mathrm{Tr}(I)$. We will consider general traces only in Sec. 2.4, in the rest of the paper we always assume that $\tau = \mathrm{Tr} = \mathrm{Tr}_{\mathcal{H}}$ for a fixed representation $\mathcal{A} \subseteq B(\mathcal{H})$. Accordingly, the density operators with respect to Tr will be referred to simply as density operators and we will identify $\mathcal{S}(\mathcal{A})$ with the set $\{\rho \in \mathcal{A}^+, \mathrm{Tr}\,\rho = 1\}$. We will also denote $\langle a, b\rangle := \langle a, b\rangle_{\mathrm{Tr}}$ the restriction of the Hilbert–Schmidt inner product in $B(\mathcal{H})$.

2.1. *Positive maps*

Let $\mathcal{B} \subseteq B(\mathcal{K})$ be a finite-dimensional C^* algebra and let $T : \mathcal{A} \to \mathcal{B}$ be a positive map. Let T^* be the adjoint of T, with respect to the Hilbert–Schmidt inner product. We will say that T is faithful if $T(a) = 0$ for $a \geq 0$ implies $a = 0$.

Lemma 1. *Suppose that $T : \mathcal{A} \to \mathcal{B}$ is a positive map. The following are equivalent.*

(i) *$T(\rho)$ is invertible for any positive invertible ρ.*
(ii) *$T(\rho)$ is invertible for some positive invertible ρ.*
(iii) *T^* is faithful.*

Proof. The implication (i) $\Rightarrow$ (ii) is trivial. Suppose (ii) and let $a \geq 0$ be such that $T^*(a) = 0$. Then $0 = \mathrm{Tr}\, T^*(a)\rho = \mathrm{Tr}\, aT(\rho)$, hence $a = 0$.

Suppose (iii) and let ρ be any positive invertible element. Let $q := \operatorname{supp} T(\rho)$. Then $0 = \operatorname{Tr} T(\rho)(I - q) = \operatorname{Tr} \rho T^*(I - q)$, this implies $I - q = 0$, hence (i) holds. $\square$

Lemma 2. *Let $T : \mathcal{A} \to \mathcal{B}$ be a positive map, such that $T^*(I) \leq I$. Let ρ and σ be positive operators and let $p = \operatorname{supp} \rho$, $p_0 = \operatorname{supp} T(\rho)$, $q = \operatorname{supp} \sigma$ and $q_0 = \operatorname{supp} T(\sigma)$. Then*

(i) *$T^*(I - p_0) \leq I - p$.*
(ii) *if $q \leq p$ then $q_0 \leq p_0$.*
(iii) *$T(p\mathcal{A}p) \subseteq p_0\mathcal{B}p_0$.*
(iv) *if T^* is unital, then $T^*(p_0) \geq p$.*

Proof. Note that for $0 \leq a \leq I$ and any positive ω, $a \leq I - \operatorname{supp} \omega$ if and only if $\operatorname{Tr} a\omega = 0$. We have

$$\operatorname{Tr} \rho T^*(I - p_0) = \operatorname{Tr} T(\rho)(I - p_0) = 0$$

which implies (i). Moreover, suppose $q \leq p$, then by (i),

$$0 \leq \operatorname{Tr} T(\sigma)(I - p_0) = \operatorname{Tr} \sigma T^*(I - p_0) \leq \operatorname{Tr} \sigma(I - p) = 0$$

this proves (ii). Let a be a positive element in $p\mathcal{A}p$, then $\operatorname{supp} a \leq p$, hence by (ii), $\operatorname{supp} T(a) \leq p_0$, so that $T(a) \in p_0\mathcal{B}p_0$. Since $p\mathcal{A}p$ is generated by its positive cone, this implies (iii).

Finally, (iv) follows directly from (i) if T^* is unital. $\square$

We say that T is n-positive if the map

$$T_{(n)} := id_n \otimes T : M_n(\mathbb{C}) \otimes \mathcal{A} \to M_n(\mathbb{C}) \otimes \mathcal{B}$$

is positive, and T is completely positive if it is n-positive for all n. The adjoint T^* is n-positive if and only if T is n-positive.

2.2. 2-*positive maps and Schwarz maps*

We say that T is a Schwarz map if it satisfies the Schwarz inequality

$$T(a^*a) \geq T(a)^*T(a), \quad a \in \mathcal{A}. \tag{1}$$

This implies that T is positive and subunital, that is, $T(I) \leq I$. It is well-known that a unital 2-positive map is a Schwarz map [25, Proposition 3.3].

Let $c \in \mathcal{A}^+$ and $a \in \mathcal{A}$. We define $a^*c^{-1}a := \lim_{\varepsilon \to 0} a^*(c + \varepsilon I)^{-1}a$, if the limit exists. Note that this is the case if and only if the range of a is contained in the range of c and then $a^*c^{-1}a = ac^-a$, where c^- denotes the generalized inverse of c.

Lemma 3. *Let $a, b, c \in \mathcal{A}$. Then the block matrix $M = \left(\begin{smallmatrix} a & b \\ b^* & c \end{smallmatrix}\right)$ is positive if and only if $c \geq 0$, $bc^{-1}b^*$ is defined and satisfies $a \geq bc^{-1}b^*$.*

Proof. The proof for the case that c is invertible can be found in [4]. For the general case, note that $M \geq 0$ if and only if $\begin{pmatrix} a & b \\ b^* & c+\varepsilon I \end{pmatrix}$ is positive for all $\varepsilon > 0$. By the first part of the proof, this is equivalent to $c \geq 0$ and $a \geq b(c+\varepsilon I)^{-1}b^*$ for all $\varepsilon > 0$. Since $b(c+\varepsilon I)^{-1}b^*$ is an increasing net of positive operators, the limit exists if and only if it is bounded from above, this proves the lemma. □

Let $c \in \mathcal{A}$ be a positive invertible element. Then we say that T satisfies the generalized Schwarz inequality for c if for all $a \in \mathcal{A}$, $T(a)^*T(c)^{-1}T(a)$ is defined and satisfies [21]

$$T(a^*c^{-1}a) \geq T(a)^*T(c)^{-1}T(a), \quad a \in \mathcal{A}. \tag{2}$$

Note that the condition that $T(a)^*T(c)^{-1}T(a)$ is defined is satisfied if T^* is subunital, by Lemma 2(iii).

The next proposition gives a characterization of 2-positivity of maps in terms of the generalized Schwarz inequality, which might be interesting in its own right:

Proposition 1. *Let $T : \mathcal{A} \to \mathcal{B}$ be a positive map. Then T is 2-positive if and only if T satisfies the generalized Schwarz inequality for every positive invertible $c \in \mathcal{A}$.*

Proof. Let $M = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ be a positive element in $M_2(\mathcal{A})$. Let $\varepsilon > 0$ and denote $M_\varepsilon := \begin{pmatrix} a & b \\ b^* & c+\varepsilon I \end{pmatrix}$. Then $M_\varepsilon \geq 0$ and it is clear that $T_{(2)}(M) \geq 0$ if and only if $T_{(2)}(M_\varepsilon) \geq 0$ for all $\varepsilon > 0$. Hence we may suppose that c is invertible. In this case, $M \geq 0$ if and only if $c \geq 0$ and $a - bc^{-1}b^* \geq 0$, by Lemma 3. Then

$$M = \begin{pmatrix} a - bc^{-1}b^* & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} bc^{-1}b^* & b \\ b^* & c \end{pmatrix}$$

where both summands are positive. Since T is positive, this implies that T is 2-positive if and only if for all $b \in \mathcal{A}$ and invertible $c \in \mathcal{A}^+$,

$$T_{(2)} \begin{pmatrix} bc^{-1}b^* & b \\ b^* & c \end{pmatrix} = \begin{pmatrix} T(bc^{-1}b^*) & T(b) \\ T(b^*) & T(c) \end{pmatrix} \geq 0.$$

Again by Lemma 3, this is equivalent to the generalized Schwarz inequality for c. □

2.3. *The map T_ρ*

Let $\rho \in \mathcal{S}(\mathcal{A})$. We define a sesquilinear form in $\mathcal{A}$ by

$$\langle a, b\rangle_\rho = \operatorname{Tr} a^*\rho^{1/2}b\rho^{1/2}, \quad a, b \in \mathcal{A}.$$

Then $\langle\cdot,\cdot\rangle_\rho$ defines an inner product in $p\mathcal{A}p$, where $p = \operatorname{supp}\rho$.

Let $T : \mathcal{A} \to \mathcal{B}$ be a positive and trace preserving map, so that $T(\rho)$ is a density operator in $\mathcal{B}$. Let $p_0 = \operatorname{supp} T(\rho)$, then by Lemma 2(iii), $T(p\mathcal{A}p) \subseteq p_0\mathcal{B}p_0$.

The map $T_\rho : p\mathcal{A}p \to p_0\mathcal{B}p_0$ is defined by

$$T_\rho(b) = T(\rho)^{-1/2}T(\rho^{1/2}b\rho^{1/2})T(\rho)^{-1/2}, \quad b \in p\mathcal{A}p.$$

Note that $T_\rho(a)$ is the unique element in $p_0\mathcal{B}p_0$ satisfying

$$\langle T^*(b), a\rangle_\rho = \langle b, T_\rho(a)\rangle_{T(\rho)}, \quad b \in \mathcal{B}, \tag{3}$$

so that T_ρ is the dual of the unital map T^*, defined in [26]. Note also that T_ρ is positive and unital and its adjoint $T_\rho^* : p_0\mathcal{B}p_0 \to p\mathcal{A}p$,

$$T_\rho^*(b) = \rho^{1/2}T^*(T(\rho)^{-1/2}bT(\rho)^{-1/2})\rho^{1/2}$$

satisfies

$$T_\rho^* \circ T(\rho) = \rho \tag{4}$$

by Lemma 2(iv).

It can be shown that T is n-positive if and only if T_ρ is n-positive. We will now investigate the case when T_ρ is a Schwarz map.

Lemma 4. *Let $T : \mathcal{A} \to \mathcal{B}$ be a positive trace preserving map and suppose that ρ is an invertible density operator. Then T_ρ is a Schwarz map if and only if T satisfies the generalized Schwarz inequality for $c = \rho$.*

Proof. T_ρ satisfies the Schwarz inequality (1) if and only if

$$T(\rho^{1/2}b^*b\rho^{1/2}) \geq T(\rho^{1/2}b^*\rho^{1/2})T(\rho)^{-1}T(\rho^{1/2}b\rho^{1/2}), \quad b \in \mathcal{A}.$$

Putting $a = \rho^{1/2}b\rho^{1/2}$, we see that this is equivalent to

$$T(a^*\rho^{-1}a) \geq T(a)^*T(\rho)^{-1}T(a), \quad a \in \mathcal{A}. \qquad \square$$

The above lemma, together with Proposition 1, implies the following result. Its importance will become clear at the beginning of Sec. 3.

Proposition 2. *Let $T : \mathcal{A} \to \mathcal{B}$ be a positive trace preserving map. Then T_ρ is a Schwarz map for any invertible density operator ρ if and only if T is 2-positive.*

2.4. *Multiplicative domain and fixed points*

This section contains some known results on the multiplicative domains and sets of fixed points of unital Schwarz maps and related decompositions of the density operators. We include the proofs partly for the convenience of the reader, and partly because we need a particular form of some of the results (mainly Theorem 2(v) and 2(vi)) which might be difficult to find explicitly in the literature.

Let $\mathcal{B} \subset \mathcal{A} \subseteq B(\mathcal{H})$ be a C^*-subalgebra. We will denote by $\mathcal{A}'$ the commutant of $\mathcal{A}$, that is the set of all elements in $B(\mathcal{H})$, commuting with $\mathcal{A}$. Then $\mathcal{A}'$ is a C^*-subalgebra in $B(\mathcal{H})$. The relative commutant of $\mathcal{B}$ in $\mathcal{A}$ is the subalgebra $\mathcal{B}' \cap \mathcal{A}$. A conditional expectation $E : \mathcal{A} \to \mathcal{B}$ is a positive linear map, such that $E(bac) = bE(a)c$ for all $a \in \mathcal{A}$, $b, c \in \mathcal{B}$. Such a map is always completely positive. There exists a unique trace preserving conditional expectation $E : \mathcal{A} \to \mathcal{B}$, determined by

$\mathrm{Tr}(ab) = \mathrm{Tr}(E(a)b)$ for $a \in \mathcal{A}$, $b \in \mathcal{B}$ (that is, E is the adjoint of the embedding $\mathcal{B} \hookrightarrow \mathcal{A}$ with respect to $\langle\cdot,\cdot\rangle$).

Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a unital Schwarz map. Let us denote

$$\mathcal{M}_\Phi := \{a \in \mathcal{A},\ \Phi(a^*a) = \Phi(a)^*\Phi(a),\ \Phi(aa^*) = \Phi(a)\Phi(a)^*\}.$$

It is known that [13, Lemma 3.9]

$$\mathcal{M}_\Phi = \{a \in \mathcal{A},\ \Phi(ab) = \Phi(a)\Phi(b),\ \Phi(ba) = \Phi(b)\Phi(a), \forall\, b \in \mathcal{A}\}.$$

This implies that $\mathcal{M}_\Phi$ is a subalgebra in $\mathcal{A}$, called the multiplicative domain of Φ. The restriction of Φ to $\mathcal{M}_\Phi$ is a *-homomorphism.

Let now $\Phi : \mathcal{A} \to \mathcal{A}$ be a unital Schwarz map and suppose that there is an invertible density operator $\rho \in \mathcal{S}(\mathcal{A})$, such that $\Phi^*(\rho) = \rho$. Let us denote by $\mathcal{F}_\Phi$ the set of fixed points of Φ, that is,

$$\mathcal{F}_\Phi := \{a \in \mathcal{A},\ \Phi(a) = a\}$$

and let φ_ρ denote the state $\varphi_\rho(a) = \mathrm{Tr}\,\rho a$ for $a \in \mathcal{A}$.

Theorem 1. (i) *$\mathcal{F}_\Phi$ is a subalgebra in $\mathcal{M}_\Phi$.*

(ii) *There exists a conditional expectation $E_\Phi : \mathcal{A} \to \mathcal{F}_\Phi$, such that $E_\Phi^*(\rho) = \rho$.*

(iii) *$\rho^{it}\mathcal{F}_\Phi\rho^{-it} \subseteq \mathcal{F}_\Phi$ for all $t \in \mathbb{R}$.*

(iv) *Let us fix a faithful trace τ in $\mathcal{F}_\Phi$. Then we have a decomposition*

$$\rho = \rho^A\rho^B,$$

where $\rho^A \in \mathcal{F}_\Phi$ is the density operator with respect to τ of the restriction of φ_ρ to $\mathcal{F}_\Phi$ and $\rho^B \in \mathcal{F}_\Phi' \cap \mathcal{A}$ is a positive invertible element such that $\Phi^*(\rho^B) = \rho^B$.

Proof. (i) Let $a \in \mathcal{F}_\Phi$, then since Φ is a Schwarz map, $\Phi(a^*a) \geq \Phi(a)^*\Phi(a) = a^*a$. But we have $\mathrm{Tr}\,\rho(\Phi(a^*a) - a^*a) = 0$, so that $\Phi(a^*a) = a^*a$, similarly $\Phi(aa^*) = aa^*$, hence $a \in \mathcal{M}_\Phi$. Let now $a, b \in \mathcal{F}_\Phi$, then $\Phi(ab) = \Phi(a)\Phi(b) = ab$ and obviously $\Phi(a + b) = a + b$, $\Phi(I) = I$, so that $\mathcal{F}_\Phi$ is a subalgebra.

(ii) Let $E_\Phi := \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \Phi^k$, then by the ergodic theorem, E_Φ is a conditional expectation onto the fixed point subalgebra $\mathcal{F}_\Phi$. It is obvious that $E_\Phi^*(\rho) = \rho$.

(iii) Is equivalent to (ii) by Takesaki's theorem [34].

(iv) It was shown in [15] that for any subalgebra satisfying (iii), there is a decomposition $\rho = \rho^A\rho^B$, where ρ^A is the density of the restriction of φ_ρ to $\mathcal{F}_\Phi$ with respect to τ and ρ^B is a positive invertible element in the relative commutant $\mathcal{F}_\Phi' \cap \mathcal{A}$. For any $a \in \mathcal{A}$,

$$\mathrm{Tr}\,\Phi(a)\rho = \mathrm{Tr}\,\Phi(a)\rho^A\rho^B = \mathrm{Tr}\,\Phi(a\rho^A)\rho^B = \mathrm{Tr}\,a\rho^A\Phi^*(\rho^B)$$

so that $\rho^A\rho^B = \rho = \Phi^*(\rho) = \rho^A\Phi^*(\rho^B)$, this implies $\rho^B = \Phi^*(\rho^B)$. □

Theorem 2. *Let $\rho \in \mathcal{A}$ be an invertible density operator and let $T : \mathcal{A} \to \mathcal{B}$ be a trace preserving map, such that both T^* and T_ρ are Schwarz maps and $\rho_0 := T(\rho)$ is invertible. Denote $\Phi := T^* \circ T_\rho$ and $\tilde{\Phi} := T_\rho \circ T^*$. Then*

(i) *$\mathcal{F}_{\tilde{\Phi}}$ is a subalgebra in $\mathcal{M}_{T^*}$ and $\mathcal{F}_\Phi$ is a subalgebra in $\mathcal{M}_{T_\rho}$.*
(ii) *The restriction of T^* is a *-isomorphism from $\mathcal{F}_{\tilde{\Phi}}$ onto $\mathcal{F}_\Phi$, and its inverse is the restriction of T_ρ.*
(iii) *$\mathcal{F}_\Phi$ is a subalgebra in $T^*(\mathcal{M}_{T^*})$.*
(iv) *$\rho^{it}\mathcal{F}_\Phi\rho^{-it} \subseteq \mathcal{F}_\Phi$ and $\rho_0^{it}\mathcal{F}_{\tilde{\Phi}}\rho_0^{-it} \subseteq \mathcal{F}_{\tilde{\Phi}}$, for all $t \in \mathbb{R}$.*
(v) *$T(\mathcal{F}'_\Phi \cap \mathcal{A}) \subseteq \mathcal{F}'_{\tilde{\Phi}} \cap \mathcal{B}$*
(vi) *There are decompositions*

$$\rho = T^*(\rho_0^A)\rho^B, \quad \rho_0 = \rho_0^A T(\rho^B)$$

where $\rho_0^A \in \mathcal{F}_{\tilde{\Phi}}^+$ and $\rho^B \in \mathcal{F}'_\Phi \cap \mathcal{A}^+$ is such that $\Phi^*(\rho^B) = \rho^B$.

Proof. Note that we have $\Phi^*(\rho) = \rho$ and $\tilde{\Phi}^*(\rho_0) = T \circ \Phi^*(\rho) = \rho_0$. Moreover, since $T_\rho^*(\rho_0) = \rho$, T_ρ is faithful by Lemma 1.

By Theorem 1(i), $\mathcal{F}_{\tilde{\Phi}}$ is a subalgebra in $\mathcal{M}_{\tilde{\Phi}}$. It is easy to see that, since T_ρ is faithful, $\mathcal{M}_{\tilde{\Phi}} \subseteq \mathcal{M}_{T^*}$. The second inclusion in (i) is proved similarly.

By (i), the restriction of T^* is a *-homomorphism on $\mathcal{F}_{\tilde{\Phi}}$. Since $\Phi \circ T^* = T^* \circ \tilde{\Phi}$ and $\tilde{\Phi} \circ T_\rho = T_\rho \circ \Phi$, we have $T^*(\mathcal{F}_{\tilde{\Phi}}) \subseteq \mathcal{F}_\Phi$, $T_\rho(\mathcal{F}_\Phi) \subseteq \mathcal{F}_{\tilde{\Phi}}$ and $T_\rho \circ T^*(a) = a$ for $a \in \mathcal{F}_{\tilde{\Phi}}$, this proves (ii).

(iii) Follows from (i) and (ii).

(iv) Follows from Theorem 1(iii).

To prove (v), let $b \in \mathcal{F}_{\tilde{\Phi}}$, $a \in \mathcal{F}'_\Phi \cap \mathcal{A}$ and $c \in \mathcal{B}$. Then

$$\begin{aligned}\operatorname{Tr} cbT(a) &= \operatorname{Tr} T^*(cb)a = \operatorname{Tr} T^*(c)T^*(b)a = \operatorname{Tr} T^*(c)aT^*(b) = \operatorname{Tr} T^*(b)T^*(c)a \\ &= \operatorname{Tr} T^*(bc)a = \operatorname{Tr} bcT(a) = \operatorname{Tr} cT(a)b\end{aligned}$$

so that $T(a) \in \mathcal{F}'_{\tilde{\Phi}} \cap \mathcal{B}$, where we used the fact that $b \in \mathcal{M}_{T^*}$, $T^*(b) \in \mathcal{F}_\Phi$ and cyclicity of the trace.

To prove (vi), let τ be the restriction of Tr to $\mathcal{F}_\Phi$. By (ii), $\tilde{\tau} := \tau \circ T^*$ defines a faithful trace on $\mathcal{F}_{\tilde{\Phi}}$. By Theorem 1(iv), we have the decompositions

$$\rho = \rho^A \rho^B, \quad \rho_0 = \rho_0^A \rho_0^B$$

where $\rho^A(\rho_0^A)$ is the density of the restriction of $\varphi_\rho(\varphi_{\rho_0})$ to $\mathcal{F}_\Phi(\mathcal{F}_{\tilde{\Phi}})$ with respect to τ ($\tilde{\tau}$). Let now $a \in \mathcal{F}_\Phi$, then

$$\begin{aligned}\tau(aT^*(\rho_0^A)) &= \tau(\Phi(a)T^*(\rho_0^A)) = \tau(T^*(T_\rho(a))T^*(\rho_0^A)) = \tau(T^*(T_\rho(a)\rho_0^A)) \\ &= \tilde{\tau}(T_\rho(a)\rho_0^A) = \operatorname{Tr} T_\rho(a)\rho_0 = \operatorname{Tr} a\rho = \tau(a\rho^A).\end{aligned}$$

It follows that $\rho^A = T^*(\rho_0^A)$. If $b \in \mathcal{B}$, then

$$\operatorname{Tr} T^*(b)\rho = \operatorname{Tr} T^*(b)T^*(\rho_0^A)\rho^B = \operatorname{Tr} T^*(b\rho_0^A)\rho^B = \operatorname{Tr} b\rho_0^A T(\rho^B)$$

so that $\rho_0 = \rho_0^A T(\rho^B)$. □

3. Conditions for Reversibility

Let $\mathcal{A} \subseteq B(\mathcal{H})$ and $\mathcal{B} \subseteq B(\mathcal{K})$ be finite-dimensional C^*-algebras. Let $\mathcal{S} \subset \mathcal{S}(\mathcal{A})$ be a set of density operators and let $T\colon \mathcal{A} \to \mathcal{B}$ be such that T^* is a unital Schwarz map. We say that T is reversible (or sufficient) for $\mathcal{S}$ if there is a map $S\colon \mathcal{B} \to \mathcal{A}$, such that S^* is a unital Schwarz map and

$$S \circ T(\sigma) = \sigma, \quad \sigma \in \mathcal{S}. \tag{5}$$

In this section, we study various conditions for reversibility. If not stated otherwise, we assume that the following two conditions hold:

(1) $\mathcal{S}$ contains an invertible element ρ and $T(\rho)$ is invertible as well.
(2) $T\colon \mathcal{A} \to \mathcal{B}$ is such that both T^* and T_ρ are unital Schwarz maps.

In the original approach of [29], the map T and the recovery map S were both required to be 2-positive. The possibility of weakening this assumption was discussed in [13, Remark 5.8], where the question was raised whether it is enough to assume that T^* is a unital Schwarz map for the map T_ρ to be a Schwarz map as well. Proposition 2 above shows that this is not the case, in fact, it follows that if Condition 2 holds for any density ρ, then T must be 2-positive. Moreover, as we will see in Theorem 4, regarding reversibility of T, Condition 2 is not more general than assuming that T is a completely positive map.

On the other hand, note that the Condition 1 is not restrictive. Indeed, for $\mathcal{S} \subset \mathcal{S}(\mathcal{A})$ there always exists a (finite) convex combination ρ of elements in $\mathcal{S}$, such that $\operatorname{supp}\sigma \leq \operatorname{supp}\rho =: p$ for all $\sigma \in \mathcal{S}$. Moreover, T is reversible for $\mathcal{S}$ if and only if it is reversible for the closed convex hull $\bar{co}(\mathcal{S})$, therefore, we may always suppose that $\rho \in \mathcal{S}$. By Lemma 2, we also have $p_0 := \operatorname{supp} T(\rho) \geq \operatorname{supp} T(\sigma)$ for all $\sigma \in \mathcal{S}$. Hence $\mathcal{S} \subset \mathcal{S}(p\mathcal{A}p)$ and $T(\mathcal{S}) \subset \mathcal{S}(p_0\mathcal{B}p_0)$.

Let $\tilde{T}$ be the restriction of T to $p\mathcal{A}p$, then $\tilde{T}$ maps $p\mathcal{A}p$ into $p_0\mathcal{B}p_0$, by Lemma 2. We have $\tilde{T}(\sigma) = T(\sigma)$ for $\sigma \in \mathcal{S}$. Again by Lemma 2,

$$\tilde{T}^*(p_0) = pT^*(p_0)p = p,$$

so that $\tilde{T}^*$ is a unital Schwarz map. Note also that $\tilde{T}_\rho = T_\rho$. It follows that if T satisfies Condition 2, then $\tilde{T}$ satisfies both 1 and 2. Moreover, T is reversible for $\mathcal{S} \subset \mathcal{S}(\mathcal{A})$ if and only if $\tilde{T}$ is reversible for $\mathcal{S} \subset \mathcal{S}(p\mathcal{A}p)$. Indeed, let $\tilde{S}$ be the restriction of S to $p_0\mathcal{B}p_0$, where $S\colon \mathcal{B} \to \mathcal{A}$ is the adjoint of a unital Schwarz map satisfying (5). Then $\tilde{S}$ maps $p_0\mathcal{B}p_0$ into $p\mathcal{A}p$, $\tilde{S}^*$ is a unital Schwarz map and $\tilde{S} \circ \tilde{T}(\sigma) = S \circ T(\sigma) = \sigma$ for all $\sigma \in \mathcal{S}$. Conversely, let $\tilde{S}\colon p_0\mathcal{B}p_0 \to p\mathcal{A}p$ be the adjoint of a unital Schwarz map, such that $\tilde{S} \circ \tilde{T}(\sigma) = \sigma$ for $\sigma \in \mathcal{S}$, then we extend $\tilde{S}$ to a map $S\colon \mathcal{B} \to \mathcal{A}$ by

$$S(b) = \tilde{S}(p_0 b p_0) + [\operatorname{Tr} b(1 - p_0)]\rho \quad b \in \mathcal{B}.$$

Then S^* is a unital Schwarz map and $S \circ T(\sigma) = \tilde{S} \circ \tilde{T}(\sigma) = \sigma$ for every $\sigma \in \mathcal{S}$. Moreover, S is n-positive whenever $\tilde{S}$ is n-positive.

The above constructions can be easily illustrated in the trivial case when $\mathcal{S} = \{\rho\}$. Then both T and $\tilde{T}$ are always reversible, the recovery map being T^*_ρ for $\tilde{T}$, and an extension of T^*_ρ for T.

3.1. *Quantum f-divergences*

Let $f : [0, \infty) \to \mathbb{R}$ be a function. Recall that f is operator convex if $f(\lambda A + (1 - \lambda)B) \leq \lambda f(A) + (1-\lambda) f(B)$ for any $\lambda \in [0, 1]$ and any positive matrices A, B of any dimension. It was proved in [13] that any operator convex function has an integral representation of the form

$$f(x) = f(0) + ax + bx^2 + \int_{(0,\infty)} \left(\frac{x}{1+t} - \frac{x}{x+t} \right) d\mu_f(t), \quad x \in [0, \infty)$$

where $a \in \mathbb{R}$, $b \geq 0$ and μ_f is a non-negative measure on $(0, \infty)$ satisfying $\int (1 + t)^{-2} d\mu_f(t) < \infty$.

Let now σ and ρ be two density operators and suppose that $\operatorname{supp} \sigma \leq \operatorname{supp} \rho$. Let $\Delta_{\sigma,\rho} = L_\sigma R_\rho^{-1}$ be the relative modular operator, note that $\Delta_{\sigma,\rho}(a) = \sigma a \rho^{-1}$ for any $a \in \mathcal{A}$. Let $f : [0, \infty) \to \mathbb{R}$ be an operator convex function. The f-divergence of σ with respect to ρ is defined by

$$S_f(\sigma, \rho) = \langle \rho^{1/2}, f(\Delta_{\sigma,\rho}) \rho^{1/2} \rangle$$

see [13] also for the case of arbitrary pairs of density operators. A well-known example is the relative entropy $S(\sigma, \rho) = \operatorname{Tr} \sigma(\log \sigma - \log \rho)$, which corresponds to the operator convex function $f(x) = x \log x$. Another example is given by $S_s(\sigma, \rho) = 1 - \operatorname{Tr} \sigma^s \rho^{1-s}$, this corresponds to the function $f_s(x) = 1 - x^s$, which is operator convex for $s \in [0, 1]$.

Let $T^* : \mathcal{B} \to \mathcal{A}$ be a unital Schwarz map. Then any f-divergence is monotone under T [13], in the sense that

$$S_f(T(\sigma), T(\rho)) \leq S_f(\sigma, \rho).$$

Theorem 3 ([13]). *Under the Conditions* 1 *and* 2, *the following are equivalent.*

(i) *T is reversible for $\mathcal{S}$.*
(ii) *$S(T(\sigma), T(\rho)) = S(\sigma, \rho)$ for all $\sigma \in \mathcal{S}$.*
(iii) *$T^*(T(\sigma)^{it} T(\rho)^{-it}) = \sigma^{it} \rho^{-it}$ for $\sigma \in \mathcal{S}$, $t \in \mathbb{R}$.*
(iv) *$\operatorname{Tr} T(\sigma)^s T(\rho)^{1-s} = \operatorname{Tr} \sigma^s \rho^{1-s}$ for all $\sigma \in \mathcal{S}$ and some $s \in (0, 1)$.*
(v) *$S_f(T(\sigma), T(\rho)) = S_f(\sigma, \rho)$ for all $\sigma \in \mathcal{S}$ and some operator convex function f with $|\operatorname{supp} \mu_f| \geq \dim(\mathcal{H})^2 + \dim(\mathcal{K})^2$, where $|X|$ denotes the number of elements in the set X.*
(vi) *Equality holds in* (v) *for all operator convex functions.*
(vii) *Equality holds in* (iv) *for all $s \in [0, 1]$.*
(viii) *$T^*_\rho \circ T(\sigma) = \sigma$ for all $\sigma \in \mathcal{S}$.*

Remark 1. The equivalence of (i)–(iii) and (viii) was first proved in [27], for the case when all states are faithful and T is the restriction to a subalgebra, and subsequently for any unital 2-positive map in [29], in the more general setting of von Neumann algebras, see also [15, 16], where Conditions (iv) and (vii) were proved.

The following example shows that, unlike the classical case, preservation of an f-divergence with strictly operator convex f is in general not sufficient for reversibility. This solves another open problem of [13], showing that the support condition in Theorem 3(v) cannot be completely removed.

Example 1. The function $f(x) = (1 + x)^{-1}$, $x \geq 0$ is operator convex and the corresponding measure μ_f is concentrated in the point $t = 1$, $\mu_f(\{1\}) = 1$. We have

$$S_f(\sigma, \rho) = \operatorname{Tr} \rho(L_\sigma + R_\rho)^{-1}(\rho).$$

We will show that the equality $S_f(T(\sigma), T(\rho)) = S_f(\sigma, \rho)$ does not imply reversibility of T.

Let $\mathcal{A}$ be a matrix algebra and let $\sigma \in \mathcal{A}$ be an invertible density matrix. Let $p \in \mathcal{A}$ be a projection such that $\sigma p \neq p\sigma$ and $\operatorname{Tr} p\sigma = \lambda \neq 1/2$. Let $\mathcal{B} \subset \mathcal{A}$ be the abelian subalgebra generated by p and let $T : \mathcal{A} \to \mathcal{B}$ be the trace preserving conditional expectation, then $T(\sigma)$ is the density of the restriction of σ to $\mathcal{B}$. Put $x := (1 - \lambda)p + \lambda(I - p) \in \mathcal{B}$ and $\rho := (I - x)^{-1}\sigma x$. Then

$$\rho = c^{-1} x\sigma x \geq 0$$

where $c = \lambda(1 - \lambda)$, and

$$\operatorname{Tr} \rho = c^{-1} \operatorname{Tr} \sigma x^2 = 1$$

so that ρ is an invertible density matrix as well. Moreover, we also have $T(\rho) = (I - x)^{-1}T(\sigma)x$, so that

$$(L_\sigma + R_\rho)^{-1}(\rho) = x = (L_{T(\sigma)} + R_{T(\rho)})^{-1}(T(\rho))$$

and the equality $S_f(T(\sigma), T(\rho)) = S_f(\sigma, \rho)$ holds. On the other hand, we have from Theorem 5(iv) below that T is reversible if and only if $\sigma = \sigma^A \rho^B$ and $\rho = \rho^A \rho^B$ for some $\sigma^A, \rho^A \in \mathcal{B}^+$ and $\rho^B \in \mathcal{A}^+$. It follows that both σ^A and ρ^A commute with ρ^B and, since $\mathcal{B}$ is abelian, this implies that σ^A and ρ^A commute with σ. But this is possible only if ρ^A and σ^A are constants. It follows that we must have $\sigma = \rho$ and it is easy to see that this implies that σ commutes with x, which is not possible by the construction of x.

3.2. *The commutant Radon–Nikodym derivative*

Let ρ, σ be density operators in $\mathcal{A}$ and suppose that $\operatorname{supp} \sigma \leq \operatorname{supp} \rho =: p$. The commutant Radon–Nikodym derivative of σ with respect to ρ is defined by

$$d(\sigma, \rho) = \rho^{-1/2} \sigma \rho^{-1/2}.$$

Then $d = d(\sigma, \rho)$ is the unique element in $p\mathcal{A}p$, satisfying

$$\operatorname{Tr} \sigma a = \langle I, a \rangle_\sigma = \langle d, a \rangle_\rho. \tag{6}$$

Moreover, $d \geq 0$ and $\|d\|$ is the smallest number λ satisfying $\sigma \leq \lambda\rho$, note that $\|d\| \geq 1$ and $\|d\| = 1$ if and only if $\rho = \sigma$.

Lemma 5. *Let* $\operatorname{supp} \sigma \leq \operatorname{supp} \rho$. *Let* $T : \mathcal{A} \to \mathcal{B}$ *be a trace preserving positive map. Then*

$$d(T(\sigma), T(\rho)) = T_\rho(d(\sigma, \rho)).$$

Proof. Directly by definition of T_ρ and $d(\sigma, \rho)$. □

The following simple lemma provides a useful tool for the analysis of reversibility. Note also that it gives a reversibility condition also for the case when both T and the reverse map S are only required to be positive and trace preserving.

Lemma 6. *Let ρ be invertible and let $T : \mathcal{A} \to \mathcal{B}$ be a trace preserving positive map. Then $T^*_\rho \circ T(\sigma) = \sigma$ if and only if $T^*(d(T(\sigma), T(\rho))) = d(\sigma, \rho)$.*

Proof. For $a \in \mathcal{A}$, we have by (3) and (6) that

$$\begin{aligned} \langle T^*(d(T(\sigma), T(\rho))), a \rangle_\rho &= \langle d(T(\sigma), T(\rho)), T_\rho(a) \rangle_{T(\rho)} = \operatorname{Tr} T_\rho(a) T(\sigma) \\ &= \operatorname{Tr} a T^*_\rho \circ T(\sigma). \end{aligned}$$

It follows that $T^*(d(T(\sigma), T(\rho))) = d(\sigma, \rho)$ if and only if $\operatorname{Tr} a T^*_\rho \circ T(\sigma) = \operatorname{Tr} a\sigma$ for all $a \in \mathcal{A}$. □

Now we are able to characterize reversibility in terms of the Radon–Nikodym derivative. While (ii) or (iii) give easy conditions for reversibility, Condition (iv) will be necessary for the proof of Theorem 6 below. The last two conditions are not really new, but will be useful in proving Theorem 7.

Theorem 4. *Suppose the Conditions* 1 *and* 2 *hold. Let us denote* $\Phi = T^* \circ T_\rho$. *Then the following are equivalent.*

(i) *T is reversible for $\mathcal{S}$.*
(ii) *$T^*(d(T(\sigma), T(\rho))) = d(\sigma, \rho)$, for all $\sigma \in \mathcal{S}$.*
(iii) *$d(\sigma, \rho) \in \mathcal{F}_\Phi$, for all $\sigma \in \mathcal{S}$.*
(iv) *$\rho^{it} d(\sigma, \rho) \rho^{-it} \in T^*(\mathcal{M}_{T^*})$, for all $\sigma \in \mathcal{S}$ and $t \in \mathbb{R}$.*
(v) *There is a trace preserving completely positive map $\hat{S} : \mathcal{B} \to \mathcal{A}$, such that $\hat{S} \circ T(\sigma) = \sigma$, $\sigma \in \mathcal{S}$.*
(vi) *There are trace preserving completely positive maps $\hat{T} : \mathcal{A} \to \mathcal{B}$ and $\hat{S} : \mathcal{B} \to \mathcal{A}$, such that $\hat{T}(\sigma) = T(\sigma)$, $\hat{S} \circ T(\sigma) = \sigma$, $\sigma \in \mathcal{S}$.*

Proof. By Lemma 6, (ii) is equivalent to $T^*_\rho \circ T(\sigma) = \sigma$ for $\sigma \in \mathcal{S}$, which is equivalent to (i) by Theorem 3(viii). (iii) is the same as (ii), by Lemma 5. Since by Theorem 2(iii), $\mathcal{F}_\Phi$ is a subalgebra in $T^*(\mathcal{M}_{T^*})$ and $\rho^{it}\mathcal{F}_\Phi\rho^{-it} \subseteq \mathcal{F}_\Phi$ for all $t \in \mathbb{R}$, (iii) implies (iv).

Suppose (iv) and let $\mathcal{A}_1$ be the subalgebra generated by $\{\rho^{it}d(\sigma,\rho)\rho^{-it},\ t \in \mathbb{R},\ \sigma \in \mathcal{S}\}$. Then $\mathcal{A}_1 \subseteq T^*(\mathcal{M}_{T^*})$. Let $E : \mathcal{A} \to \mathcal{A}_1$ be the trace preserving conditional expectation. Then its adjoint is the embedding $E^* : \mathcal{A}_1 \hookrightarrow \mathcal{A}$ and since $\rho^{it}\mathcal{A}_1\rho^{-it} \subseteq \mathcal{A}_1$ for all $t \in \mathbb{R}$, the map E_ρ is the ρ-preserving conditional expectation, [1]. Hence

$$E^*(d(E(\sigma), E(\rho))) = d(E(\sigma), E(\rho)) = E_\rho(d(\sigma,\rho)) = d(\sigma,\rho).$$

By the equivalence of (ii) and (i) and Theorem 3(viii) (for the map E), $E^*_\rho \circ E(\sigma) = \sigma$ for all $\sigma \in \mathcal{S}$.

Let F^* denote the embedding $\mathcal{M}_{T^*} \hookrightarrow \mathcal{B}$, then, as above, its adjoint $F = F^{**} : \mathcal{B} \to \mathcal{M}_{T^*}$ is the trace preserving conditional expectation. Let us define the map $\bar{T} : \mathcal{M}_{T^*} \to T^*(\mathcal{M}_{T^*})$ by $\bar{T} := T^* \circ F^*$. Then since T^* is faithful by Lemma 1, $\bar{T}$ is injective, so that $\bar{T}$ is a *-isomorphism and there is an inverse map $R = (\bar{T})^{-1} : T^*(\mathcal{M}_{T^*}) \to \mathcal{M}_{T^*}$. Define the map $\hat{S} : \mathcal{B} \to \mathcal{A}$ by $\hat{S} := E^*_\rho \circ R^* \circ F$. Then $\hat{S}$ is completely positive and trace preserving. Moreover, $T^* \circ \hat{S}^* = T^* \circ F^* \circ R \circ E_\rho = \bar{T} \circ R \circ E_\rho = E^* \circ E_\rho$, so that $\hat{S} \circ T(\sigma) = (E^* \circ E_\rho)^*(\sigma) = \sigma$ and (v) holds.

Suppose (v). Let $\mathcal{S}_0 := T(\mathcal{S})$ and let $\sigma_0 = T(\sigma)$ for $\sigma \in \mathcal{S}$. Then since $\hat{S}(\sigma_0) = \sigma$ and $T \circ \hat{S}(\sigma_0) = T(\sigma) = \sigma_0$, the map $\hat{S}$ is reversible for $\mathcal{S}_0$. Hence by Theorem 3(viii), the map $\hat{T} := \hat{S}^*_{\rho_0}$ is completely positive and satisfies $\hat{T}(\sigma) = \sigma_0$, this proves (vi). The implication (vi) $\to$ (i) is clear. □

Remark 2. Note that by the proof of (v), the completely positive maps $\hat{T}$ and $\hat{S}$ can always be given as adjoints of a composition of a conditional expectation and a *-isomorphism.

Corollary 1. *Under the Conditions* 1 *and* 2, T *is reversible for* $\mathcal{S}$ *if and only if* T *is reversible for* $\tilde{\mathcal{S}} := \bigcup\{\rho^{is}\mathcal{S}\rho^{-is},\ s \in \mathbb{R}\}$.

Proof. Suppose T is reversible for $\mathcal{S}$. Let $\sigma \in \mathcal{S}$ and let $d = d(\sigma,\rho)$. Then $d \in \mathcal{F}_\Phi$ and therefore also $d(\rho^{is}\sigma\rho^{-is},\rho) = \rho^{is}d\rho^{-is} \in \mathcal{F}_\Phi$, for all $s \in \mathbb{R}$. □

3.3. *Factorization*

In this section, we give a characterization of reversibility in terms of the structure of states in $\mathcal{S}$. More precisely, we show that the elements in $\mathcal{S}$ must have the form of a product of two positive operators, such that T^* is multiplicative on one of them and the other does not depend on σ. This can be viewed as a quantum version of the classical factorization theorem for sufficient statistics, see, e.g., [33]. The first such factorization result was proved in [22], see also [13, Theorem 6.1]. Similar conditions for the infinite dimensional case are proved in [15, Theorem 6].

Theorem 5. *Assume Conditions* 1 *and* 2. *Let* $\Phi = T^* \circ T_\rho$ *and* $\tilde{\Phi} = T_\rho \circ T^*$. *Then the following are equivalent.*

(i) T *is reversible for* $\mathcal{S}$.

(ii) *There is a positive invertible element* $\rho^B \in \mathcal{F}'_\Phi \cap \mathcal{A}$, *such that for each* $\sigma \in \mathcal{S}$,

$$\sigma = T^*(\sigma_0^A)\rho^B, \quad T(\sigma) = \sigma_0^A T(\rho^B),$$

with some $\sigma_0^A \in \mathcal{F}^+_{\tilde{\Phi}}$.

(iii) *There is an element* $\rho^B \in \mathcal{A}^+$, *such that for each* $\sigma \in \mathcal{S}$,

$$\sigma = T^*(\sigma_0^A)\rho^B, \quad T(\sigma) = \sigma_0^A T(\rho^B),$$

with some $\sigma_0^A \in \mathcal{B}^+$.

(iv) *There is an element* $\rho^B \in \mathcal{A}^+$, *such that each* $\sigma \in \mathcal{S}$ *has the form*

$$\sigma = \sigma^A \rho^B,$$

where σ^A *is a positive element in* $T^*(\mathcal{M}_{T^*})$.

Proof. Let us denote $\sigma_0 := T(\sigma)$ for $\sigma \in \mathcal{S}$. Suppose (i) and let

$$\rho = T^*(\rho_0^A)\rho^B, \quad \rho_0 = \rho_0^A T(\rho^B)$$

be the decomposition from Theorem 2(vi). Then, by Theorems 2 and 4, we have for $\sigma \in \mathcal{S}$,

$$\begin{aligned}\sigma &= \rho^{1/2} d(\sigma,\rho)\rho^{1/2} = \rho^{1/2} T^*(d(\sigma_0,\rho_0))\rho^{1/2} \\ &= T^*(\rho_0^A)^{1/2} T^*(d(\sigma_0,\rho_0)) T^*(\rho_0^A)^{1/2}\rho^B \\ &= T^*((\rho_0^A)^{1/2} d(\sigma_0,\rho_0)(\rho_0^A)^{1/2})\rho^B = T^*(\sigma_0^A)\rho^B\end{aligned}$$

where we put $\sigma_0^A := (\rho_0^A)^{1/2} d(\sigma_0,\rho_0)(\rho_0^A)^{1/2}$. Since $d(\sigma_0,\rho_0) = T_\rho(d(\sigma,\rho)) \in \mathcal{F}^+_{\tilde{\Phi}}$, σ_0^A is a positive element in $\mathcal{F}_{\tilde{\Phi}}$. Moreover, $\sigma_0^A = T(\rho^B)^{-1/2}\sigma_0 T(\rho^B)^{-1/2}$, hence

$$\sigma_0 = \sigma_0^A T(\rho^B),$$

where we used Theorem 2(v). This proves (ii). It is clear that (ii) implies (iii).

Suppose (iii). Then for $a \in \mathcal{B}$,

$$\operatorname{Tr} a\sigma_0 = \operatorname{Tr} a\sigma_0^A T(\rho^B) = \operatorname{Tr} T^*(a\sigma_0^A)\rho^B.$$

On the other hand,

$$\operatorname{Tr} a\sigma_0 = \operatorname{Tr} T^*(a)\sigma = \operatorname{Tr} T^*(a)T^*(\sigma_0^A)\rho^B.$$

Putting $a = \sigma_0^A$, we obtain

$$\operatorname{Tr} T^*((\sigma_0^A)^2)\rho^B = \operatorname{Tr} T^*(\sigma_0^A)^2\rho^B.$$

Since ρ is invertible, the decomposition implies that ρ^B must be invertible as well, hence by Schwarz inequality, $T^*((\sigma_0^A)^2) = T^*(\sigma_0^A)^2$. This implies that $\sigma_0^A \in \mathcal{M}_{T^*}$, which proves (iv) with $\sigma^A := T^*(\sigma_0^A)$.

Finally, suppose (iv). Let $\sigma \in \mathcal{S}$. Since both σ^A and ρ^B are positive and so is their product σ, they must commute. It follows that

$$w_t := \sigma^{it}\rho^{-it} = (\sigma^A)^{it}(\rho^A)^{-it} \in T^*(\mathcal{M}_{T^*})$$

for all $t \in \mathbb{R}$, where $\rho = \rho^A\rho^B$ is the decomposition for ρ. We have $\rho^{is}w_t\rho^{-is} = w_s^* w_{t+s} \in T^*(\mathcal{M}_{T^*})$ for all $t, s \in \mathbb{R}$. By analytic continuation for $t = -i/2$, we get $\rho^{is}\sigma^{1/2}\rho^{-1/2}\rho^{-is} \in T^*(\mathcal{M}_{T^*})$, hence also $\rho^{is}d(\sigma,\rho)\rho^{-is} \in T^*(\mathcal{M}_{T^*})$ for all s. By Theorem 4(v), this implies (i). □

The next corollary shows that the recovery map T_ρ does not depend on the choice of ρ. For faithful states, this was proved already in [29].

Corollary 2. *Suppose the Conditions* 1 *and* 2 *hold. Then T is reversible for $\mathcal{S}$ if and only if $T_\sigma = T_\rho|_{\operatorname{supp}\sigma\mathcal{A}\operatorname{supp}\sigma}$ for all $\sigma \in \mathcal{S}$.*

Proof. Let $\sigma \in \mathcal{S}$, $q := \operatorname{supp}\sigma$, $q_0 := \operatorname{supp}T(\sigma)$ and suppose that T is reversible for $\mathcal{S}$. Let us denote $w = \sigma^{1/2}\rho^{-1/2}$, $w_0 = T(\sigma)^{1/2}T(\rho)^{-1/2}$. By Theorem 5(ii) and Theorem 2, we have

$$w_0 = (\sigma_0^A)^{1/2}(\rho_0^A)^{-1/2} \in \mathcal{F}_{\tilde{\Phi}}$$

and

$$w = T^*(w_0) \in \mathcal{F}_\Phi, \quad w_0 = T_\rho(w).$$

Then for $a \in q\mathcal{A}q$,

$$\begin{aligned} T_\sigma(a) &= T(\sigma)^{-1/2}T(\sigma^{1/2}a\sigma^{1/2})T(\sigma)^{-1/2} \\ &= (w_0^{-1})^* T_\rho(w^*aw)w_0^{-1} = (w_0^{-1})^* T_\rho(w)^* T_\rho(a)T_\rho(w)w_0^{-1} \\ &= q_0 T_\rho(a) q_0. \end{aligned}$$

Since ρ^B is invertible, we must have $q_0 = \operatorname{supp}\sigma_0^A \in \mathcal{F}_{\tilde{\Phi}}$ and $q = \operatorname{supp}T^*(\sigma_0^A) = T^*(q_0)$. Hence also $T_\rho(q) = q_0$ and $q_0T_\rho(a)q_0 = T_\rho(qaq) = T_\rho(a)$.

Conversely, since T_ρ is unital, the equality $T_\sigma = T_\rho|_{q\mathcal{A}q}$ implies that $T_\sigma^* = T_\rho^*|_{q_0\mathcal{B}q_0}$ by Lemma 2, so that $T_\rho^* \circ T(\sigma) = T_\sigma^* \circ T(\sigma) = \sigma$ and T is reversible for $\mathcal{S}$. □

3.4. *Quantum hypothesis testing*

Let σ and ρ be density operators in $\mathcal{A}$. Let us consider the problem of testing the hypothesis $H_0 = \rho$ against the alternative $H_1 = \sigma$. Any test is represented by an operator $0 \le M \le I$, which corresponds to rejecting the hypothesis. Then we have the error probabilities

$$\alpha(M) = \operatorname{Tr}\rho M, \quad \beta(M) = \operatorname{Tr}\sigma(1 - M).$$

For $s \in [0,1)$, we define the Bayes optimal test to be a minimizer of the expression

$$s\alpha(M) + (1-s)\beta(M) = (1-s)(1 - \mathrm{Tr}(\sigma - t\rho)M), \quad t = \frac{s}{1-s}. \tag{7}$$

Then the minimal Bayes error probability is

$$\Pi_s := \min_{0 \le M \le I} \{s\alpha(M) + (1-s)\beta(M)\} = s\alpha(M_{\frac{s}{1-s}}) + (1-s)\beta(M_{\frac{s}{1-s}})$$

where M_t maximizes the expression $\mathrm{Tr}(\sigma - t\rho)M$ over all $0 \le M \le I$. Below we formulate the quantum version of the Neyman–Pearson lemma. The obtained Bayes optimal tests are called the (quantum) NP tests for (ρ, σ).

If $a \in \mathcal{A}$ is a self adjoint operator, we denote by a_+ the positive part of a, that is, $a_+ = \sum_{i,\lambda_i>0} p_i$, where $a = \sum_i \lambda_i p_i$ is the spectral decomposition of a.

Lemma 7 ([14, 11]). *For $t \ge 0$, let $P_{t,+} := \mathrm{supp}(\sigma - t\rho)_+$ and let $P_{t,0}$ be the projection onto the kernel of $\sigma - t\rho$. Then $0 \le M_t \le I$ is a Bayes optimal test if and only if*

$$M_t = P_{t,+} + X_t$$

with $0 \le X_t \le P_{t,0}$. The minimal Bayes error probability is

$$\Pi_s = \frac{1}{2}(1 - \|(1-s)\sigma - s\rho\|_1).$$

Let now $T : \mathcal{A} \to \mathcal{B}$ be a trace preserving positive map. Let $s \in (0,1)$, $t = s(1-s)^{-1}$ and let Π_s^0 be the minimal Bayes error probability for testing the hypothesis $H_0 = T(\rho)$ against $H_1 = T(\sigma)$. For $N \in \mathcal{B}$, $0 \le N \le I$, we have

$$\mathrm{Tr}(T(\sigma) - tT(\rho))N = \mathrm{Tr}(\sigma - t\rho)T^*(N) \le \max_{0 \le M \le I} \mathrm{Tr}(\sigma - t\rho)M$$

so that $\Pi_s^0 \ge \Pi_s$, this is equivalent to the fact that

$$\|T(\sigma - t\rho)\|_1 \le \|\sigma - t\rho\|_1. \tag{8}$$

In [17], equality in (8) was investigated for a pair of invertible density operators, in the case when T is the restriction to a subalgebra. If equality holds for all $t \ge 0$, then the subalgebra must contain some Bayes optimal test for all $s \in [0,1]$, such subalgebras are called 2-sufficient. It was shown that in some cases, 2-sufficiency is equivalent to sufficiency, that is, reversibility of T for $\{\sigma, \rho\}$. From another point of view, this condition was studied also in [5] and it was shown that for a completely positive trace preserving map, the equality implies reversibility for certain sets $\mathcal{S}$.

Since the L_1-norm is one of the basic distance measures on states, equivalence between equality in (8) and reversibility is an important open question. We will show below (Theorem 6) that this equivalence holds if equality in (8) is required for all σ in the extended family $\tilde{\mathcal{S}} = \bigcup\{\rho^{is}\mathcal{S}\rho^{-is},\ s \in \mathbb{R}\}$. Moreover, Theorem 7 shows this equivalence if equality in (8) holds for n copies of the states, for all n.

We will suppose below that ρ is invertible.

Lemma 8 ([17, Lemma 4]). *$P_{t,0} \neq 0$ if and only if t is an eigenvalue of $d(\sigma, \rho)$. Moreover, the rank of $P_{t,0}$ is equal to the multiplicity of t.*

Lemma 9. *The function $t \mapsto P_{t,+}$ is right-continuous. Moreover,*

$$\lim_{s \to t^-} P_{s,+} = P_{t,+} + P_{t,0}, \quad t \geq 0.$$

Proof. Let $\rho(t) := \sigma - t\rho$ for $t \in \mathbb{R}$. Let $\lambda_1^\downarrow(t), \dots, \lambda_N^\downarrow(t)$ denote the decreasingly ordered eigenvalues of $\rho(t)$ (with multiplicities). For $t_1, t_2 \in \mathbb{R}$, we have $\rho(t_1) = \rho(t_2) + (t_2 - t_1)\rho$. By Weyl's perturbation theorem [3, Corollary III.2.6], this implies that

$$\max_j |\lambda_j^\downarrow(t_1) - \lambda_j^\downarrow(t_2)| \leq |t_1 - t_2| \|\rho\|.$$

Moreover, since ρ is invertible, we obtain by [3, Corollary III.2.2] that

$$\lambda_j^\downarrow(t_2) < \lambda_j^\downarrow(t_2) + (t_2 - t_1)\lambda_N^\downarrow(\rho) \leq \lambda_j^\downarrow(t_1)$$

when $t_1 < t_2$, where $\lambda_N^\downarrow(\rho)$ denotes the smallest eigenvalue of ρ. Hence the functions $t \mapsto \lambda_j^\downarrow(t)$ are continuous and strictly decreasing.

It is clear that for $t < 0$ all $\lambda_j^\downarrow(t)$ are strictly positive, and that $\lambda_j^\downarrow(t) = 0$ for some index j if and only if $P_{t,0} \neq 0$. Let $0 \leq t_1 < \cdots < t_n$ be the eigenvalues of $d(\sigma, \rho)$ and put $t_0 := 0$, $t_{n+1} := \infty$. Then there are indices $i_k \in \{1, \dots, N\}$, $k = 1, \dots, n$, such that $N = i_1 > i_2 > \cdots > i_n > i_{n+1} := 0$ and for every $t \in [t_{k-1}, t_k)$ the strictly positive eigenvalues of $\rho(t)$ are given by $\lambda_1^\downarrow(t), \dots, \lambda_{i_k}^\downarrow(t)$.

Let $t \in [t_{k-1}, t_k)$ and let $\gamma(t)$ be a circle, contained entirely in the open half-plane of complex numbers having strictly positive real parts and enclosing all $\lambda_1^\downarrow(t), \dots, \lambda_{i_k}^\downarrow(t)$. By continuity of $\lambda_j^\downarrow$, there is some $\delta > 0$ such that $\gamma(t)$ encloses $\lambda_1^\downarrow(s), \dots, \lambda_{i_k}^\downarrow(s)$ for all $s \in (t - \delta, t + \delta)$ and $[t, t + \delta) \subset [t_{k-1}, t_k)$. Then

$$P_{s,+} = \frac{1}{2i\pi} \oint_{\gamma(t)} (zI - \rho(s))^{-1} dz, \quad s \in [t, t + \delta).$$

This implies that $t \mapsto P_{t,+}$ is right-continuous. Let now $t \in (t_{k-1}, t_k)$, then we can find $\delta > 0$ as above, but such that, moreover, $(t - \delta, t + \delta) \subset (t_{k-1}, t_k)$. In this case,

$$P_{s,+} = \frac{1}{2i\pi} \oint_{\gamma(t)} (zI - \rho(s))^{-1} dz, \quad s \in (t - \delta, t + \delta)$$

so that $t \mapsto P_{t,+}$ is continuous at t. Suppose $t = t_{k-1}$, then by definition of i_k and t_{k-1}, we must have

$$\lambda_j^\downarrow(t_{k-1}) \begin{cases} > 0 & j \leq i_k, \\ = 0 & j = i_k + 1, \dots, i_{k-1}, \\ < 0 & j > i_{k-1}. \end{cases}$$

Let γ_k' be a circle in the complex plane, enclosing $\lambda_1^\downarrow(t_{k-1}),\dots,\lambda_{i_k}^\downarrow(t_{k-1})$ and 0, but such that the closed disc encircled by γ_k' does not contain any other eigenvalue of $\rho(t_{k-1})$. Then there is some $\delta>0$ such that $(t_{k-1}-\delta,t_{k-1})\subset[t_{k-2},t_{k-1})$ and

$$P_{s,+}=\frac{1}{2i\pi}\oint_{\gamma_k'}(zI-\rho(s))^{-1}dz,\quad s\in(t_{k-1}-\delta,t_{k-1}).$$

It follows that $\lim_{s\to t_{k-1}^-}P_{s,+}=P_{t_{k-1},+}+P_{t_{k-1},0}$. Since $P_{t,0}=0$ for $t\notin\{t_1,\dots,t_n\}$, this proves the assertion. □

Let us denote $Q_{t,+}:=\mathrm{supp}(T(\sigma)-tT(\rho))_+$ and $Q_{t,0}$ the projection onto the kernel of $T(\sigma)-tT(\rho)$.

Lemma 10. *Let $T:\mathcal{A}\to\mathcal{B}$ be a trace preserving positive map and suppose that both ρ and $T(\rho)$ are invertible. The following are equivalent.*

(i) $\|T(\sigma)-tT(\rho)\|_1=\|\sigma-t\rho\|_1$, *for all* $t\in\mathbb{R}$.
(ii) $P_{t,+}=T^*(Q_{t,+})$, $P_{t,0}=T^*(Q_{t,0})$ *for* $t\in\mathbb{R}$.

Proof. Since $Q_{t,+}$ is an NP test for $(T(\rho),T(\sigma))$, (i) implies that

$$\mathrm{Tr}(T(\sigma)-tT(\rho))Q_{t,+}=\mathrm{Tr}(\sigma-t\rho)T^*(Q_{t,+})=\max_{0\le M\le I}\mathrm{Tr}(\sigma-t\rho)M$$

so that $T^*(Q_{t,+})$ is an NP test for (ρ,σ). By Lemma 7, there is some $0\le X_t\le P_{t,0}$, such that $T^*(Q_{t,+})=P_{t,+}+X_t$. It follows that $P_{t,+}=T^*(Q_{t,+})$ holds for all t such that $P_{t,0}=0$, that is, for $t\in\mathbb{R}\backslash\{t_1,\dots,t_n\}$. Since $t\mapsto P_{t,+}$ and $t\mapsto T^*(Q_{t,+})$ are right continuous, it follows that $T^*(Q_{t,+})=P_{t,+}$ for all t. On the other hand, by Lemma 9 we have for all t

$$P_{t,+}+P_{t,0}=\lim_{s\to t^-}P_{s,+}=\lim_{s\to t^-}T^*(Q_{s,+})=T^*(Q_{t,+})+T^*(Q_{t,0})$$

hence $P_{t,0}=T^*(Q_{t,0})$ for all t. The converse is obvious. □

Theorem 6. *Assume the Conditions* 1 *and* 2. *Then*

(i) *T is reversible for $\mathcal{S}$ if and only if*

$$\|\sigma-t\rho\|_1=\|T(\sigma)-tT(\rho)\|_1,\quad\sigma\in\tilde{\mathcal{S}},\ \ t\ge0.\tag{9}$$

(ii) *Suppose that $\rho^{is}\mathcal{S}\rho^{-is}\subseteq\mathcal{S}$ for all $s\in\mathbb{R}$. Then T is reversible for $\mathcal{S}$ if and only if*

$$\|\sigma-t\rho\|_1=\|T(\sigma)-tT(\rho)\|_1,\quad\sigma\in\mathcal{S},\ \ t\ge0.\tag{10}$$

(iii) *Suppose that $\mathcal{B}$ is abelian. Then T is reversible for $\mathcal{S}$ if and only if* (10) *holds. Moreover, in this case all elements in $\mathcal{S}$ commute.*
(iv) *Suppose that all elements in $\mathcal{S}$ commute with ρ. Then T is reversible for $\mathcal{S}$ if and only if* (10) *holds.*

Proof. (i) By Corollary 1, T is reversible for $\mathcal{S}$ if and only if it is reversible for $\tilde{\mathcal{S}}$. By monotonicity (8), we get (9).

For the converse, let $\sigma \in \tilde{\mathcal{S}}$. Then by Lemma 10, (9) implies that $P_{t,0} = T^*(Q_{t,0})$ for the corresponding projections for σ and ρ. This implies that $Q_{t,0} \in \mathcal{M}_{T^*}$ and $P_{t,0} \in T^*(\mathcal{M}_{T^*})$.

Let $t_1, \ldots, t_n$ be the eigenvalues of $d = d(\sigma, \rho)$ and let $F_1, \ldots, F_n$ be the corresponding eigenprojections. Denote $P_i := P_{t_i,0}$. Then we have $(d - t_i)\rho^{1/2}P_i = \rho^{-1/2}(\sigma - t_i\rho)P_i = 0$ and this implies

$$d\rho^{1/2} \sum_i P_i = \rho^{1/2} \sum_i t_i P_i.$$

Moreover, any vector in the range of $\rho^{1/2}P_i\rho^{1/2}$ is an eigenvector of d, so that $\operatorname{supp}(\rho^{1/2}P_i\rho^{1/2}) \leq F_i$ and by Lemma 8, $\operatorname{rank}(F_i) = \operatorname{rank}(P_i) = \operatorname{rank}(\rho^{1/2}P_i\rho^{1/2})$. It follows that $\sum_i P_i$ is invertible, so that $d(\sigma, \rho) = \rho^{1/2}c\rho^{-1/2}$, with

$$c := \sum_i t_i P_i \left(\sum_j P_j\right)^{-1} \in T^*(\mathcal{M}_{T^*}).$$

It follows that for $s \in \mathbb{R}$ and $\sigma \in \mathcal{S}$, $\rho^{is-1/2}d(\sigma, \rho)\rho^{1/2-is} \in T^*(\mathcal{M}_{T^*})$. By analytic continuation, we get $\rho^{it}d(\sigma, \rho)\rho^{-it} \in T^*(\mathcal{M}_{T^*})$ for all $t \in \mathbb{R}$, which implies that T is reversible for $\mathcal{S}$, by Theorem 4.

(ii) Clearly follows from (i).

(iii) Let $\sigma \in \mathcal{S}$ and let $P_{t,0}$ and $Q_{t,0}$ be the corresponding projections. Note that since $\mathcal{B}$ is commutative, $Q_{t,0}$ must commute for all t. Suppose that (10) holds, then $P_{t,0} = T^*(Q_{t,0})$ and, since then $Q_{t,0} \in \mathcal{M}_{T^*}$, this implies that all $P_{t,0}$ commute as well. As in the proof of (i), $d(\sigma, \rho) = \rho^{1/2}c\rho^{-1/2}$, where we now have $c \geq 0$. This implies that $d(\sigma, \rho)\rho = \rho^{1/2}c\rho^{1/2} \geq 0$, hence $d(\sigma, \rho)\rho = \rho d(\sigma, \rho)$ and therefore also $\sigma\rho = \rho\sigma$. This implies that $\rho^{is}\sigma\rho^{-is} = \sigma$ and the statement follows by (ii). The converse implication is clear.

(iv) Follows from (ii). □

3.5. *Quantum Chernoff and Hoeffding distances*

Let $n \in \mathbb{N}$ and suppose we are given n identical copies of the states $\rho^{\otimes n}, \sigma^{\otimes n} \in \mathcal{S}(\mathcal{A}^{\otimes n})$. Consider the problem of testing the hypothesis $H_0 = \rho^{\otimes n}$ against $H_1 = \sigma^{\otimes n}$. Then the minimum Bayes error probability is

$$\Pi_{s,n} = \frac{1}{2}(1 - \|(1 - s)\sigma^{\otimes n} - s\rho^{\otimes n}\|_1).$$

It is an important result of [2] that as $n \to \infty$, the probabilities $\Pi_{s,n}$ decay exponentially fast and the rate of convergence is given by

$$\lim_n -\frac{1}{n} \log \Pi_{s,n} = -\log\left(\inf_{0 \leq u \leq 1} \operatorname{Tr} \sigma^u \rho^{1-u}\right) =: C(\sigma, \rho) \tag{11}$$

for any $s \in [0,1]$, where we put $x^0 = \operatorname{supp} x$ for any positive $x \in \mathcal{A}$. The quantity $C(\sigma,\rho)$ is called the quantum Chernoff distance. Note that C is related to the convex quantum f-divergence $S_u(\sigma,\rho)$, but one can show that C itself is not an f-divergence [13]. Nevertheless, if T is the adjoint of a unital Schwarz map, then C satisfies monotonicity:

$$C(\sigma,\rho) \geq C(T(\sigma), T(\rho))$$

and, moreover, $C(\sigma,\rho) = 0$ if and only if $\sigma = \rho$.

Let us consider again the problem of testing the hypothesis $H_0 = \rho$ against the alternative $H_1 = \sigma$. Let $0 \leq M \leq I$ be a test. Differently from the Bayesian approach, in the asymmetric approach the error probability $\alpha(M)$ is bounded, $\alpha(M) \leq \epsilon$ for some fixed $\epsilon > 0$. The error probability $\beta(M)$ is then minimalized over all tests, under this constraint,

$$\beta_\epsilon := \inf\{\beta(M),\, 0 \leq M \leq I,\, \alpha(M) \leq \epsilon\}.$$

Suppose we have n independent copies of the states $\sigma^{\otimes n}$ and $\rho^{\otimes n}$ and let $M_n \in \mathcal{A}^{\otimes n}$. Here we require that the probabilities $\alpha(M_n)$ decay exponentially as $n \to \infty$. Let $r > 0$ and put

$$\beta_{r,n} := \inf\{\beta(M_n),\ 0 \leq M_n \leq I,\ \alpha(M_n) \leq e^{-nr}\}.$$

The following equality was proved in [9, 23]: For $r > 0$,

$$\lim_n -\frac{1}{n}\log \beta_{r,n} = \sup_{0\leq u<1} \frac{-ur - \log \operatorname{Tr} \rho^u \sigma^{1-u}}{1-u} =: H_r(\rho,\sigma).$$

The limit expression is called the quantum Hoeffding distance. Similarly as the Chernoff distance, H_r is not an f-divergence [13], but it is related to S_u. This implies the monotonicity

$$H_r(T(\sigma), T(\rho)) \leq H_r(\sigma,\rho)$$

for T the adjoint of a unital Schwarz map. Moreover, by [12], see also [13],

$$H_0(\sigma,\rho) := \lim_{r\to 0} H_r(\sigma,\rho) = S(\sigma,\rho) = \operatorname{Tr}\sigma(\log\sigma - \log\rho)$$

holds if $\operatorname{supp}\sigma \leq \operatorname{supp}\rho$.

Suppose that $q := \operatorname{supp}\sigma \leq \operatorname{supp}\rho$, then the function $[0,\infty) \ni r \mapsto H_r(\sigma,\rho)$ has the following properties [12], see also [2]:

The function is convex and lower semicontinuous, for $r \in [0, S_\sigma(\rho,\sigma)]$ it is strictly convex and decreasing, and for $r \geq S_\sigma(\rho,\sigma)$ it has a constant value $H_r(\sigma,\rho) = -\log \operatorname{Tr} q\rho$, here

$$S_\sigma(\rho,\sigma) = -\log \operatorname{Tr} q\rho + \frac{1}{\operatorname{Tr} q\rho}\operatorname{Tr}\rho(\log\rho - \log\sigma)q.$$

Note that if $\operatorname{supp}\sigma = \operatorname{supp}\rho$, then $S_\sigma(\rho,\sigma) = S(\rho,\sigma)$.

Proposition 3 ([13]). *Let σ and ρ be two density operators in $\mathcal{A}$ such that* $\operatorname{supp}\sigma = \operatorname{supp}\rho$. *Let $T : \mathcal{A} \to \mathcal{B}$ be the adjoint of a unital Schwarz map and suppose that one of the following conditions holds*:

(i) $C(\sigma, \rho) = C(T(\sigma), T(\rho))$.
(ii) $H_r(\sigma, \rho) = H_r(T(\sigma), T(\rho))$ *for some* $r \in [0, S(T(\rho), T(\sigma))]$.

Then $T^*_\rho \circ T(\sigma) = \sigma$.

Theorem 7. *Assume the Conditions* 1 *and* 2. *Then the following are equivalent.*

(i) T *is reversible for* $\mathcal{S}$.
(ii) $C(\sigma, \rho) = C(T(\sigma), T(\rho))$ *for all* $\sigma \in co(\mathcal{S})$.
(iii) $\|\sigma^{\otimes n} - t\rho^{\otimes n}\|_1 = \|T(\sigma)^{\otimes n} - tT(\rho)^{\otimes n}\|_1$ *for all* $\sigma \in \mathcal{S}$, $t \geq 0$ *and* $n \in \mathbb{N}$.
(iv) $H_r(\sigma, \rho) = H_r(T(\sigma), T(\rho))$ *for all* $\sigma \in \mathcal{S}$ *and* $r \geq 0$.

Suppose moreover that there is some $\mathcal{S}_0 \subset \mathcal{S}$, *such that* $\mathcal{S} \subseteq \overline{co}(\mathcal{S}_0 \cup \{\rho\})$ *and* $T(\rho) \notin \overline{T(\mathcal{S}_0)}$. *Then there exists some* $r_0 > 0$ *such that* (i)–(iv) *are equivalent to*

(v) $H_r(\sigma, \rho) = H_r(T(\sigma), T(\rho))$ *for all* $\sigma \in co(\mathcal{S})$ *and some* $r \in [0, r_0]$.

Proof. Since T is reversible for $\mathcal{S}$ if and only if it is reversible for $co(\mathcal{S})$, (i) implies (ii) by monotonicity of C. Conversely, suppose (ii) and let $\sigma \in \mathcal{S}$, then $\sigma_1 := \frac{1}{2}(\sigma + \rho)$ is an invertible element in $co(\mathcal{S})$. Proposition 3 now implies that $T^*_\rho \circ T(\sigma_1) = \sigma_1$ and by (4), we have also $T^*_\rho \circ T(\sigma) = \sigma$.

Further, suppose (i), then by Theorem 4(vi), there are trace preserving completely positive maps $\hat{T} : \mathcal{A} \to \mathcal{B}$ and $\hat{S} : \mathcal{B} \to \mathcal{A}$, such that $\hat{T}(\sigma) = T(\sigma)$, $\hat{S} \circ T(\sigma) = \sigma$, $\sigma \in \mathcal{S}$. It follows that $T(\sigma)^{\otimes n} = \hat{T}(\sigma)^{\otimes n} = \hat{T}^{\otimes n}(\sigma^{\otimes n})$ and $\sigma^{\otimes n} = \hat{S}^{\otimes n}(T(\sigma)^{\otimes n})$, for all $\sigma \in \mathcal{S}$, where $\hat{T}^{\otimes n}$ and $\hat{S}^{\otimes n}$ are completely positive and trace preserving. By monotonicity of the L_1-norm, this implies (iii). The implications (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) were already proved in [13].

Suppose now that the additional condition holds. Let us choose some $\varepsilon \in (0, 1)$ and put

$$r_0 := \inf_{\sigma \in \mathcal{S}_0} S(T(\rho), T(\varepsilon\rho + (1 - \varepsilon)\sigma)).$$

Then if $r_0 = 0$, there exists a sequence $\sigma_n \in \mathcal{S}_0$, such that $S(T(\rho), T(\varepsilon\rho + (1 - \varepsilon)\sigma_n)) \to 0$. This implies that $T(\sigma_n) \to T(\rho)$, so that $T(\rho) \in \overline{T(\mathcal{S}_0)}$, which is not possible. Hence $r_0 > 0$.

Suppose (v) holds and let $\sigma \in \mathcal{S}_0$. Denote $\sigma_\varepsilon = \varepsilon\rho + (1 - \varepsilon)\sigma$. Then $0 \leq r \leq S(T(\rho), T(\sigma_\varepsilon))$. Since σ_ε is invertible, we can apply Proposition 3, which implies that $T^*_\rho \circ T(\sigma_\varepsilon) = \sigma_\varepsilon$ and therefore also $T^*_\rho \circ T(\sigma) = \sigma$ for all $\sigma \in \mathcal{S}_0$. Since $\mathcal{S} \subseteq \overline{co}(\mathcal{S}_0 \cup \{\rho\})$, this implies (i). The implication (i) $\Rightarrow$ (v) follows by monotonicity. $\square$

Remark 3. Note that if all elements in $\mathcal{S}$ are invertible, then we may replace $co(\mathcal{S})$ by $\mathcal{S}$ in (ii) and by $\mathcal{S}_0$ in (v), where we put $r_0 := \inf_{\sigma \in \mathcal{S}_0} S(T(\rho), T(\sigma))$.

3.6. Quantum Fisher information and χ^2-divergence

Let us denote by $\mathcal{D}$ the set of invertible density operators in $\mathcal{A}$. Then $\mathcal{D}$ is a differentiable manifold, where the tangent space at each point $\rho \in \mathcal{D}$ is the vector space $\mathcal{T}_\rho$ of traceless self-adjoint elements in $\mathcal{A}$.

A monotone metric on $\mathcal{D}$ is a Riemannian metric λ_ρ, satisfying

$$\lambda_\rho(x,x) \geq \lambda_{T(\rho)}(T(x),T(x)), \quad x \in \mathcal{T}_\rho, \quad \rho \in \mathcal{D} \tag{12}$$

for any completely positive trace preserving map $T : \mathcal{A} \to \mathcal{B}$.

It was proved by Petz in [30] that any monotone metric has the form

$$\lambda_\rho(x,y) = \mathrm{Tr}(J_\rho^f)^{-1}(x)y$$

with $J_\rho^f = f(\Delta_\rho)R_\rho$, where $\Delta_\rho := \Delta_{\rho,\rho} = L_\rho R_\rho^{-1}$, and $f : (0,\infty) \to (0,\infty)$ an operator monotone function satisfying the symmetry $f(t) = tf(t^{-1})$. Under the normalization condition $f(1) = 1$, the restriction of λ_ρ to the submanifold of diagonal elements in $\mathcal{D}$ coincides with the classical Fisher information for probability measures on a finite set, moreover, the monotonicity condition (12) characterizes the classical Fisher information up to multiplication by a constant. Accordingly, any monotone metric with the above normalization is called a quantum Fisher information.

The operator J_ρ^f satisfies [28, 24]

$$J_{T(\rho)}^f \geq TJ_\rho^f T^*$$

for any operator monotone (not necessarily symmetric or normalized) function f and $T : \mathcal{A} \to \mathcal{B}$ the adjoint of a unital Schwarz map. This is equivalent to [30]

$$(J_\rho^f)^{-1} \geq T^*(J_{T(\rho)}^f)^{-1}T, \tag{13}$$

which implies that the monotonicity (12) holds for all such f and T.

A related quantity is the quantum version of the χ^2-divergence, which was introduced in [32] as

$$\chi^2_{1/f}(\sigma,\rho) = \lambda_\rho^f(\sigma-\rho,\sigma-\rho)$$

where λ_ρ^f is a monotone metric.

Let now $f : (0,\infty) \to (0,\infty)$ be operator monotone. Then $t \mapsto f(t)^{-1}$ is a non-negative operator monotone decreasing function on $(0,\infty)$. By [7], for each such function there is a positive Borel measure ν_f with support in $[0,\infty)$ and $\int_0^\infty (1+s^2)^{-1}d\nu_f(s) < \infty$, $\int_0^\infty s(1+s^2)^{-1}d\nu_f(s) < \infty$, such that

$$f(t)^{-1} = \int_0^\infty \frac{1}{s+t}d\nu_f(s) = \int_0^\infty f_s(t)^{-1}d\nu_f(s)$$

where $f_s(t) = s + t$, $t \in \mathbb{R}^+$. Then it follows that

$$(J_\rho^f)^{-1} = f(L_\rho R_\rho^{-1})^{-1} R_\rho^{-1} = \int_0^\infty (sR_\rho + L_\rho)^{-1} d\nu_f(s) = \int_0^\infty (J_\rho^s)^{-1} d\nu_f(s) \quad (14)$$

where $J_\rho^s := J_\rho^{f_s} = sR_\rho + L_\rho$.

Lemma 11. *Let $T : \mathcal{A} \to \mathcal{B}$ be the adjoint of a unital Schwarz map. Let $x \in \mathcal{A}$. Then for $s \geq 0$,*

$$\operatorname{Tr} x^*(sR_\rho + L_\rho)^{-1}(x) \geq \operatorname{Tr} T(x)^*(sR_{T(\rho)} + L_{T(\rho)})^{-1}(T(x)) \quad (15)$$

and equality holds if and only if

$$(sR_\rho + L_\rho)^{-1}(x) = T^*[(sR_{T(\rho)} + L_{T(\rho)})^{-1}(T(x))]. \quad (16)$$

Proof. Since the function f_s is operator monotone, the inequality (15) follows from (13) for $f = f_s$. If equality holds for some $x \in \mathcal{A}$, then

$$\langle x, ((J_\rho^s)^{-1} - T^*(J_{T(\rho)}^s)^{-1}T)(x)\rangle = 0$$

which again by (13) is equivalent to $((J_\rho^s)^{-1} - T^*(J_{T(\rho)}^s)^{-1}T)(x) = 0$. □

It follows from the above lemma and the integral representation (14) that $\lambda_\rho^f(x,x) = \lambda_{T(\rho)}^f(T(x),T(x))$ if and only if (16) holds for all $s \in \operatorname{supp}\nu_f$, that is,

$$(s + \Delta_\rho)^{-1}(x\rho^{-1}) = T^*[(s + \Delta_{T(\rho)})^{-1}(T(x)T(\rho)^{-1})], \quad s \in \operatorname{supp}\nu_f. \quad (17)$$

Let now $x \in \mathcal{T}_\rho$. Then since ρ is invertible, there exists some interval $I \ni 0$ such that $\sigma_u := \rho + ux \in \mathcal{S}(\mathcal{A})$ for $u \in I$. Let us denote by $I_{\rho,x}$ the largest such interval and let $\mathcal{S}_{\rho,x} := \{\sigma_u, u \in I_{\rho,x}\}$.

Proposition 4. *Let $\rho \in \mathcal{D}$, $x \in \mathcal{T}_\rho$ and $T : \mathcal{A} \to \mathcal{B}$ be such that T and $\mathcal{S}_{\rho,x}$ satisfy the Conditions* 1 *and* 2*. Then the following are equivalent.*

(i) $\lambda_\rho^f(x,x) = \lambda_{T(\rho)}^f(T(x),T(x))$ *for a monotone metric such that* $|\operatorname{supp}\nu_f| \geq |\operatorname{spec}(\Delta_\rho) \cup \operatorname{spec}(\Delta_{T(\rho)})|$.
(ii) $\rho^{it}x\rho^{-it-1} = T^*(T(\rho)^{it}T(x)T(\rho)^{-it-1})$, $t \in \mathbb{R}$.
(iii) $\rho^{-1/2}x\rho^{-1/2} = T^*(T(\rho)^{-1/2}T(x)T(\rho)^{-1/2})$.
(iv) T *is reversible for* $\mathcal{S}_{\rho,x}$.
(v) $\lambda_\rho^f(x,x) = \lambda_{T(\rho)}^f(T(x),T(x))$ *for any monotone metric* λ_ρ^f.

Proof. Note that by the assumptions, $T(\rho)$ must be invertible. Suppose (i), then (17) holds and by [13, Lemma 5.2], this implies that

$$h(\Delta_\rho)x\rho^{-1} = h(\Delta_{T(\rho)})T(x)T(\rho)^{-1}$$

for any complex-valued function h on $\operatorname{spec}(\Delta_\rho) \cup \operatorname{spec}(\Delta_{T(\rho)})$. In particular, for $h(\lambda) = \lambda^{it}$, we get (ii). We have (ii) $\Rightarrow$ (iii) by analytic continuation for $t = i/2$. Suppose (iii) and let $\sigma_u \in \mathcal{S}_{\rho,x}$. Then

$$\begin{aligned} T^*(d(T(\sigma_u), T(\rho))) &= T^*(I + uT(\rho)^{-1/2}T(x)T(\rho)^{-1/2}) \\ &= I + u\rho^{-1/2}x\rho^{-1/2} = d(\sigma_u, \rho) \end{aligned}$$

and by Theorem 4, this implies (iv). (iv) implies (v) by monotonicity of Fisher information. The implication (v) $\Rightarrow$ (i) is trivial. □

Let $\mathcal{S} \subset \mathcal{S}(\mathcal{A})$ and let $\operatorname{Lin}(\mathcal{S}) = \operatorname{span}\{\sigma_1 - \sigma_2 : \sigma_1, \sigma_2 \in \mathcal{S}\}$. Then $\operatorname{Lin}(\mathcal{S})$ is a vector subspace in the real vector space of self-adjoint traceless operators.

Theorem 8. *Suppose that the Conditions* 1 *and* 2 *hold. Then the following are equivalent.*

(i) *T is reversible for $\mathcal{S}$.*
(ii) *$\lambda^f_\rho(x,x) = \lambda^f_{T(\rho)}(T(x), T(x))$ for all $x \in \operatorname{Lin}(\mathcal{S})$ and all monotone metrics.*
(iii) *$\chi^2_{1/f}(\sigma, \rho) = \chi^2_{1/f}(T(\sigma), T(\rho))$ for all $\sigma \in \mathcal{S}$ and all χ^2-divergences.*
(iv) *The equality in* (ii) *holds for some symmetric positive operator monotone function f such that $|\operatorname{supp}\mu_f| \geq \dim(\mathcal{H})^2 + \dim(\mathcal{K})^2$.*
(v) *The equality in* (iii) *holds for some f as in* (iv).

Proof. (i) implies (ii) by monotonicity of Fisher information and the implication (ii) $\Rightarrow$ (iii) is clear. We also have (ii) $\Rightarrow$ (iv) and both (iv) and (iii) imply (v). It is therefore enough to prove (v) $\Rightarrow$ (i). So suppose (v) and let $\sigma \in \mathcal{S}$. Put $x = \sigma - \rho$ in Proposition 4(iii), then it follows that $T^*(d(T(\sigma), T(\rho))) = d(\sigma, \rho)$ for $\sigma \in \mathcal{S}$ which implies (i) by Theorem 4. □

Remark 4. An important example of a quantum Fisher information, respectively χ^2-divergence, is given by $f(t) = \frac{1}{2}f_1(t) = \frac{1}{2}(1+t)$. In this case, ν_f is concentrated in $t = 1$ and $\lambda^f_\rho(x,y) = 2\operatorname{Tr} y(L_\rho + R_\rho)^{-1}(x)$ is called the Bures metric. It is the smallest element in the family of quantum Fisher informations. The simple example below shows that preservation of the Bures metric does not imply reversibility, so that, once again, the support condition in (iv) respectively (v) of the above theorem cannot be dropped.

So let $y = y^* \in \mathcal{A}$ be such that $\rho y \neq y\rho$ and $\operatorname{Tr}\rho y = 0$, and let $\mathcal{C} \subset \mathcal{A}$ be the commutative subalgebra generated by y. Then $z := \rho y + y\rho \in \mathcal{T}_\rho$ and, by replacing y by ty for some $t > 0$ if necessary, we may suppose that $\sigma := \rho + z \in \mathcal{D}$. Let $T : \mathcal{A} \to \mathcal{C}$ be the trace preserving conditional expectation, then $T(\sigma) = T(\rho) + T(z) = T(\rho) + T(\rho)y + yT(\rho)$. This implies that

$$(L_\rho + R_\rho)^{-1}(\sigma - \rho) = y = (L_{T(\rho)} + R_{T(\rho)})^{-1}(T(\sigma) - T(\rho))$$

which implies that $\chi^2_{1/f}(\sigma, \rho) = \chi^2_{1/f}(T(\sigma), T(\rho))$.

On the other hand, if T is reversible, then by Theorem 5(iv), ρ and σ must commute. But we have $[\sigma, \rho] = [\rho^2, y] \neq 0$.

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