α-z-Rényi divergences in von Neumann algebras: data processing inequality, reversibility, and monotonicity properties in α, z

α-z-Rényi relative entropies on von Neumann algebras were recently introduced in Ref. [34] and some of their properties are established in Ref. [33]. The paper under review continues these studies by proving some of the remaining properties of these divergences. Here is a list of the main results of the paper:

- Theorem 2.4 gives a variational expression for the α-z-Rényi relative entropies for certain ranges of the parameters α, z. These variational expressions are generalizations of similar expression in the finite dimensional case.
- Based on Theorem 2.4, in Theorem 3.3 data processing inequality is established for all possible ranges of parameters α, z. This finishes the problem of data processing inequality for general von Neumann algebras, that prior to this work was known only in some special cases.
- In Section 4 the equality condition in the data processing inequality is established in Theorems 4.5 and 4.7.
- Monotonicity of the α-z-Rényi divergence in z (for a fixed α) is studied in Section 5. See, e.g., Theorem 5.7.
- The monotonicity problem in parameter α (for a fixed z) is investigated in Section 6, and the two cases of $\alpha < 1$ and $\alpha > 1$ are considered in Theorems 6.1 and 6.6 respectively.

The proofs in the paper are mostly based on Kosaki's interpolation theorem for L^p-spaces (see Section C) and some results/techniques developed in previous works, particularly for the case of sandwiched Rényi divergence for $z = \alpha$. In my opinion the proofs of Theorems 2.4 and 3.3 are relatively easy, yet the results in Sections 4-6 seem more interesting. In particular, as also mentioned in the paper, the results in Section 6 about monotonicity in parameter α are quite interesting and new even in the finite dimensional case. The proofs in this section are also compelling.

The paper is well-structured and well-written and the proofs are mostly clear.

I have mixed feelings about this paper. On the negative side, α-z-Rényi relative entropy have operational meanings only in the cases of $z = 1$ and $z = \alpha$ that have been studied in previous works. Also, as mentioned above, some of the proof techniques in the paper are very similar to what has been done previously. On the positive side, I found the results and techniques in Sections 4-6 interesting, which seem insightful even in the finite dimensional case as well as in the operationally meaningful special cases of $z = 1$ and $z = \alpha$.

Overall I think the paper is a nice contribution to the understanding of quantum Rényi relative entropies and would call for an accept, assuming that the authors apply the following comments in the revised version of their manuscript.

1. Page 2, last paragraph: I suggest to explain here what you mean by "sufficient statistics"
2. Page 7, last step of the proof of Lemma 2.6: please add here that you're using the operator convexity of $t \mapsto 1/t$
3. Page 8, 4th line of Section 3: "... denote by $s(\gamma)$, recall that ..."?
4. Page 14, statement of Proposition 4.2: following $\to$ followings

5. Page 16, first paragraph: I'm confused here about the assumption on γ. In the beginning of section it is assumed that γ is 2-positive, but here it says "positivity is enough."
6. Page 18, second line of Subsection 4.2: "[Better to write this assumption explicitly]"?
7. Page 20, Section 5: "We consider monotonicity in the parameter z in Sec. 4 and monotonicity in the parameter α in Sec. 5." Do you mean Section 5 and Section 6?
8. Page 21, Subsection 5.1: For readers' convenience please give the definition of the generalized s-number.
9. Page 21, third line of the proof of Lemma 5.2: By (6.3) do you mean (5.3)? It seems that in this section most references to equations are erroneous.
10. Page 33, Section 7, last line of the first paragraph: There is no Theorem 2.5!
11. Page 33, last paragraph: Please explain what you mean by "proper form"
12. Page 35, before the statement of Lemma A.1: "... well-known lemmas, proofs are given..."
13. Page 36, Theorem B.1(iii): exist $\rightarrow$ exists