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SPECTRUM AND ASYMPTOTIC BEHAVIOUR OF COMPLETELY POSITIVE MAPS ON $\mathscr{B}(H)$

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Abstract. Let $\mathscr{A}$ be the W^*-algebra $\mathscr{B}(H)$ of all bounded linear operators on a Hilbert space H, let T be an identity preserving, completely positive map on $\mathscr{A}$ and suppose the fixed space of T is one-dimensional and there exists a faithful normal T^*-invariant state on $\mathscr{A}$. Using the theory of semitopological semigroups the following is proved: (a) The peripheral point spectrum of T is the group Γ_h of all h-th roots of unity for some $h \geqslant 1$. (b) There exists a partially periodic map S on $\mathscr{A}$ (with the same properties as T) such that $\lim_n (T_*^n - S_*^n) = 0$ in the strong operator topology $\mathscr{L}_s(\mathscr{A}_*)$.

1. If $T = (t_{i,j})$ is a contraction on the finite-dimensional Banach space $E = \mathbf{C}^n$ then the set of unimodular eigenvalues $\sigma(T) \cap \Gamma$, Γ being the unit circle, determines the asymptotic behaviour of T, i. e. the convergence properties of $(T^n)_{n \in \mathbf{N}}$. Indeed, the point 1 is always a pole of the resolvent of order $\leqslant 1$ and the Cesàro means $\frac{1}{n} \Sigma_{k=0}^{n-1} T^k$ converge to a projection P onto the fixed space $F(T) := \{x \in E: Tx = x\}$. If additionally there exists a natural number $h \geqslant 1$ such that $\sigma(T^h) \cap \Gamma = \{1\}$ (e. g. $t_{i,j} \geqslant 0$ for all $1 \leqslant i, j \leqslant n$ ([9] I. Theorem 2.7)), the powers of T^h converge to a projection Q onto $F(T^h) = \{x \in E: Tx = \alpha x$ for some $\alpha = \alpha(x) \in \Gamma\}$. Let $S := T \circ Q$. Then it is easy to see that $\lim_n (T^n - S^n) = 0$. Since $(S\,|_{QE})^h = I\,|_{QE}$ and $S\,|_{ker\,Q} = 0$, the powers of T behave asymptotically like a "periodic" operator.

In order to extend these results to operators on infinite-dimensional Banach spaces E the first difficulty that arises is of topological nature, i. e. one has to distinguish between the uniform operator topology $\mathscr{L}(E)$, the strong operator topology $\mathscr{L}_s(E)$ and the weak operator topology $\mathscr{L}_w(E)$. The second difficulty is of spectral theoretical nature: In contrast to the finite-dimensional situation the spectral values of T need not to be eigenvalues nor are the singularities of the resolvent $R(.,T)$ of T necessarily poles.

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In this paper we use the theory of semitopological semigroups with a single generator to study the peripheral point spectrum and the asymptotic behaviour of completely positive maps on the W^*-algebra $\mathscr{B}(H)$ of all bounded linear operators on some Hilbert space H. More precisely, we prove the following:

Theorem 1. *Let $(\mathscr{B}(H), \varphi, T)$ be an irreducible W^*-dynamical system. Then the peripheral point spectrum of T is the group Γ_h of all h-th roots of unity for some $h \geqslant 1$.*

Theorem 2. *Let $(\mathscr{B}(H), \varphi, T)$ be an irreducible W^*-dynamical system. Then there exists an irreducible, partially periodic W^*-dynamical system $(\mathscr{B}(H), \varphi, S)$ such that $\lim_n (T_*^n - S_*^n) = 0$ in the strong operator topology $\mathscr{L}_s(\mathscr{B}(H)_*)$.*

Obviously, both results depend on the algebraic and topological nature of the W^*-algebra $\mathscr{B}(H)$. For a discussion of W^*-dynamical systems on general W^*-algebras we refer to [4], [6].

We call a triple $(\mathscr{A}, \varphi, T)$ a *W^*-dynamical system*, if $\mathscr{A}$ is a W^*-algebra with predual $\mathscr{A}_*$, φ is a faithful normal state on $\mathscr{A}$ and T is a completely positive, identity preserving map on $\mathscr{A}$ with $\varphi(Tx) = \varphi(x)$ for all x in $\mathscr{A}$. Since φ is faithful and invariant, T is weak $*$-continuous hence possesses a preadjoint $T_* \in \mathscr{L}(\mathscr{A}_*)$. We call a W^*-dynamical system *irreducible*, if the fixed space of T is one-dimensional and *partially periodic*, if T is a partially periodic map on $\mathscr{A}$. Recall that an operator S on a Banach space E is called partially periodic, if there exists $m_\circ \in \mathbf{N}$ such that $S(I - S^{m_\circ}) = 0$. Since $S^{m_\circ}(I - S^{m_\circ}) = 0$ it follows that $S^{m_\circ}$ is a projection. If $E_1 := S^{m_\circ}(E)$ and $E_0 := \ker S^{m_\circ}$, then $E = E_0 \oplus E_1$, $S|_{E_0} = 0$ and $(S|_{E_1})^{m_\circ} = I|_{E_1}$. In particular, every periodic operator is partially periodic with $E_0 = \{0\}$.

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2. In this section we give a proof of our spectral theoretical result. For this the following result is basic.

Proposition 2.1. *Let $(\mathscr{A}, \varphi, T)$ be a W^*-dynamical system. Then the cyclic semigroup $\mathscr{S}_* := \{T_*^n : n \in \mathbf{N}\}$ is relatively compact in the weak operator topology $\mathscr{L}_w(\mathscr{A}_*)$.*

Proof. We are using [10] Theorem III. 5. 4. (iii) to prove this assertion: Let $(p_n)_{n\in N}$ be a decreasing sequence of projections in $\mathscr{A}$ with $\inf_n p_n = 0$. From the Schwarz inequality for completely positive maps (see, e. g., [10] Corollary IV. 3. 8.) it follows

$$\varphi((T^k p_n)(T^k p_n)) \leqslant \varphi(T^k p_n) = \varphi(p_n)$$

for all k and n in $\mathbf{N}$. Since $\lim_n \varphi(p_n) = 0$ we obtain $\lim_n T^k(p_n) = 0$ uniformly in k in the σ-strong operator topology on $\mathscr{A}$ ([10] Proposition III. 5. 3.). Since this topology is finer than the weak*-topology, it follows $\lim_n \psi(T^k p_n) = 0$ uniformly in $k \in \mathbf{N}$ for all $\psi \in \mathscr{A}_*$. Thus the set $\mathscr{S}_*$ is relatively compact in $\mathscr{L}_w(\mathscr{A}_*)$. □

Therefore we can apply the theory of compact semitopological semigroups of operators with a single generator: the closure $\overline{\mathscr{S}}_*$ contains an unique minimal ideal $\mathscr{K}_*$ called the kernel, which is a compact group. The identity Q_* of $\mathscr{K}_*$ is a projection onto the closed linear span of all eigenvectors of T_* pertaining to peripheral eigenvalues. Moreover, the dual group $\widehat{\mathscr{K}}_*$ of $\mathscr{K}_*$ can be identified with the subgroup of the circle group generated by the peripheral point spectrum of T_*. For these facts we refer to [1] or, more adapted to our situation, [2] VII/4.

In the next proposition we make frequently use of the following fact: Let T be a completely positive contraction on a C^*-algebra $\mathscr{A}$ and let $x \in \mathscr{A}$ such that $T(xx^*) = T(x)T(x)^*$. Then $T(yx^*) = T(y)T(x)^*$ for all $y \in \mathscr{A}$. To see this, let for x and y in $\mathscr{A}$ be $B(x, y) := T(xy^*) - T(x)T(y)^*$. Then $B(\ ,\ )$ is a positive, sesquilinear map from $\mathscr{A} \times \mathscr{A}$ in $\mathscr{A}$ such that $B(x, x) = 0$ for some $x \in \mathscr{A}$ iff $B(x, y) = 0$ for all $y \in \mathscr{A}$ (since for all states ψ on $\mathscr{A}$ $\psi(B(x, x) = 0$ for some x in $\mathscr{A}$ iff $\psi(B(x, y) = 0$ for all $y \in \mathscr{A}$ by the Cauchy-Schwarz inequality). In particular if T^{-1} exists, is completely positive and contractive, then T is an *-automorphism on $\mathscr{A}$.

Proposition 2. 2. *Let $(\mathscr{A}, \varphi, T)$ be a W^*-dynamical system. Then the following assertions hold:*

(a) *The set of peripheral eigenvalues of T and T_* are equal.*

(b) *There exists a faithful normal conditional expectation Q of $\mathscr{A}$ onto the weak*-closed linear span $\mathscr{M}$ of all eigenvectors of T pertaining to the peripheral eigenvalues.*

Proof. (a) Since T is a contraction it follows $P\sigma(T_*) \cap \Gamma \subseteq P\sigma(T) \cap \Gamma$ ([3]

Proposition 3. 1.). Conversely, let $\alpha \in P\sigma(T) \cap \Gamma$ with normalized eigenvector $x_\alpha \in \mathscr{A}$. Using the Schwarz inequality for T and the faithfulness of φ it follows $T(x_\alpha x_\alpha^*) = x_\alpha x_\alpha^* = T(x_\alpha)\, T(x_\alpha)^*$. Thus $T(yx_\alpha^*) = \alpha^* T(y)\, x_\alpha^*$ for all $y \in \mathscr{A}$. Let φ_α be the normal linear form $(y \longmapsto (yx_\alpha^*))$ on $\mathscr{A}$. Then $\varphi_\alpha \neq 0$ and for all $y \in \mathscr{A}$ we obtain

$$(T_*\varphi_\alpha)(y) = \varphi_\alpha(Ty \cdot x_\alpha^*) = \alpha\varphi(T(yx_\alpha^*)) = \alpha \cdot \varphi_\alpha(y),$$

hence $\alpha \in P\sigma(T_*)$.

(b) Let Q_* be the unit of the topological group $\mathscr{K}_*$. Using [10] Corollary IV. 3. 4. and the complete positivity of T, it is easy to see that the operator $Q := (Q_*)^*$ is completely positive. Furthermore Q is faithful, i. e. $\ker Q \cap \mathscr{A}_+ = \{0\}$, since $Q_*\varphi = \varphi$. Thus Q is a conditional expectation and the range of Q is a W^*-subalgebra of $\mathscr{A}$ with $1 \in Q(\mathscr{A})$.

By the relative compactness of $\mathscr{S}_*$ there exists some ultrafilter $\mathscr{U}$ on N such that $\lim_{\mathscr{U}} T_*^n = Q_*$ in the weak operator topology. Letting $\alpha \in P\sigma(T) \cap \Gamma$ (considered as a subset of the dual group of $\mathscr{K}_*$), $0 \neq x_\alpha \in \mathscr{A}$ an eigenvector pertaining to α and $\varphi_\alpha \in \mathscr{A}_*$ as in (a), we obtain:

$$\varphi_\alpha = Q_*\varphi_\alpha = \lim_{\mathscr{U}} T_*^n \varphi_\alpha = (\lim_{\mathscr{U}} \alpha^n)\varphi_\alpha.$$

Thus $\lim_{\mathscr{U}} \alpha^n = 1$. Hence for all $\psi \in \mathscr{A}_*$

$$\psi(Qx_\alpha) = \psi(\lim_{\mathscr{U}} T^n x_\alpha) = (\lim_{\mathscr{U}} \alpha^n)\, \psi(x_\alpha) = \psi(x_\alpha).$$

Therefore $Qx_\alpha = x_\alpha$ or $x_\alpha \in Q(\mathscr{A})$.

Conversely let $\gamma \in \widehat{\mathscr{K}}_*$ and take $0 \neq x \in \mathscr{A}$. Then the element x_γ defined by

$$\psi(x_\gamma) := \int_{\mathscr{K}_*} (S_*\psi)(x) \langle S_*, \gamma\rangle \, dm(S_*) \qquad (\psi \in \mathscr{A}_*)$$

where m is the Haar measure on $\mathscr{K}_*$ belongs to $\mathscr{A}$ and $Tx_\gamma = \langle (Q \circ T)_*, \gamma\rangle x_\gamma$. Since $|\langle (Q \circ T)_*, \gamma\rangle| = 1$, x_γ belongs to $\mathscr{M}$. Thus the assertion is proved if we can show $Q(\mathscr{A}) \in \overline{\mathrm{lin}}\, \{x_\gamma : \gamma \in \widehat{\mathscr{K}}_*\}$. Suppose there exists $\psi_0 \in \mathscr{A}_*$ such that

$$\psi_0(x_\gamma) = \int_{\mathscr{K}_*} \psi_0(Sx) \overline{\langle S_*, \gamma\rangle}\, dm(S_*) = 0$$

for all $\gamma \in \widehat{\mathscr{K}}_*$. Since the mapping $(S_* \longmapsto \psi_0(Sx))$ is continuous and since the characters of $\widehat{\mathscr{K}}_*$ form a complete orthonormal basis of $L^2(\mathscr{K}_*, dm)$, we obtain $\psi_0(Sx) = 0$ for all $S_* \in \mathscr{K}_*$, in particular $\psi_0(Qx) = 0$. Therefore $\psi_0|_{Q(\mathscr{A})} = 0$, hence $Q(\mathscr{A}) \subseteq \overline{\mathrm{lin}}\, \{x_\gamma : \gamma \in \widehat{\mathscr{K}}_*\}$ by the Hahn-Banach theorem. □

Now we are prepared to give a proof of our spectral theoretical result.

Theorem 2. 3. *Let $(\mathscr{B}(H), \varphi, T)$ be an irreducible W^*-dynamical system. Then the peripheral point spectrum of T is the group Γ_h of all h-th roots of unity for some $h \geqslant 1$.*

Proof. Take $\alpha, \beta \in P\sigma(T) \cap \Gamma$ and let x_α and x_β be normalized eigenvectors of T pertaining to α (resp. β). Because of $T(x_\alpha x_\alpha^*) \geqslant x_\alpha x_\alpha^*$ and the faithfulness of φ it follows $x_\alpha x_\alpha^* \in F(T)$. Therefore x_α is unitary because $\| x_\alpha \| = 1$ and T is irreducible. Since $T(x_\alpha x_\beta^*) = T(x_\alpha)\, T(x_\beta)^* = \alpha\beta^* x_\alpha x_\beta^*$, the peripheral point spectrum of T is a subgroup of the circle group, the restriction of T to $\mathscr{M} = Q(\mathscr{B}(H))$ is an $*$-automorphism and the restriction of φ to $\mathscr{M}$ is a faithful normal trace. For the trace property note that $\varphi(x_\alpha x_\beta^*) = \varphi(T(x_\alpha x_\beta^*)) = \alpha\beta^* \varphi(x_\alpha x_\beta^*)$, thus $\varphi(x_\alpha x_\beta^*) = 0$ for $\alpha \neq \beta$ and $\varphi(x_\alpha x_\alpha^*) = \varphi(x_\alpha^* x_\alpha) = 1$. Therefore $\varphi(xy) = \varphi(yx)$ for all $x, y \in \operatorname{lin}\{u \in \mathscr{A} : u \text{ unitary}, Tu = \alpha u, |\alpha| = 1\}$. Thus φ is a trace on $\mathscr{M}$ (e.g. [8], 1. 8.).

Since $\mathscr{B}(H)$ is atomic and Q is a faithful normal conditional expactation, $\mathscr{M}$ is atomic ([10] Exercise V. 8. (a), p. 334), is of type I and is finite. Suppose the center $\mathscr{Z}$ of $\mathscr{M}$ is infinite-dimensional. Since $\mathscr{Z}$ is atomic it is isomorphic to ℓ^∞. T induces an irreducible $*$-automorphism S on $\mathscr{Z}$ and there exists a faithful normal linear form $\psi_0 \in \mathscr{Z}_*{}^+ (\cong \ell^1_+)$ such that $S_* \psi_0 = \psi_0$. But S is induced by some transformation τ of $\mathbf{N}$ onto $\mathbf{N}$. In fact, if $\delta_n(x) = \xi_n$ $(n \in \mathbf{N}, x = (\xi_n) \in \mathscr{Z})$, then $\delta_n \circ S$ is a normal, scalar valued $*$-homomorphism with $(\delta_n \circ S)(1) = 1$, hence of the form δ_m for some $m = \tau(n)$. Thus $S = S_\tau$. But since τ is bijective this conflicts with $S_* \psi = \psi$. Therefore the center of $\mathscr{M}$ is finite-dimensional.

Using [10] Theorem V. 1. 27. it follows that $\mathscr{M}$ is finite-dimensional. Thus the set of peripheral eigenvalues of T is a finite subgroup of Γ hence of the form Γ_h for some $h \geqslant 1$. □

Remarks. (1) For every natural number $h \leqslant \dim H$ there exists an irreducible W^*-dynamical system $(\mathscr{B}(H), \varphi, T)$ such that $P\sigma(T) \cap \Gamma = \Gamma_h$. Indeed, let $h \leqslant \dim H$ and take mutually orthogonal projections $p_1, \ldots, p_h$ in $\mathscr{B}(H)$ such that $\Sigma_{k=1}^h p_k = 1$. If π is a cyclic permutation of the set $\{1, \ldots, h\}$ of length h then the map S on $\operatorname{lin}\{p_k : 1 \leqslant k \leqslant h\}$ given by extension of the mapping $(p_k \longmapsto p_{\pi(k)})$ is completely positive, identity preserving and $F(S) = \mathbf{C}1$. Let Q be the faithful normal, conditional expactation on $\mathscr{B}(H)$ given by $(x \longmapsto \Sigma_{k=1}^h p_k x p_k)$ and let $T := S \circ Q$. If $\varphi := \tau \circ Q$ where τ is the normal state $(x \longmapsto \Sigma_{k=1}^h \gamma_k)$,

$x = \Sigma_{k=1}^{h} \gamma_k \cdot p_k \in \mathrm{lin}\{p_k : 1 \leqslant k \leqslant h\}$, then $(\mathscr{B}(H), \varphi, T)$ is an irreducible W^*-dynamical system such that $P\sigma(T) \cap \Gamma = \Gamma_h$.

(2) Let T be a completely positive and identity preserving map on a W^*-algebra $\mathscr{A}$ with preadjoint $T_* \in \mathscr{L}(\mathscr{A}_*)$ and suppose $P\sigma(T_*) \cap \Gamma \neq \emptyset$. Then there exists a positive linear form $\varphi \in \mathscr{A}_*$ such that $T_* \varphi = \varphi$. To see this let $\varphi_\alpha \in \mathscr{A}_*$ be a normalized eigenvector pertaining to the peripheral eigenvalue α of T_*. Then for all $x \in \mathscr{A}$: $|\varphi_\alpha(x)|^2 = |\varphi_\alpha(Tx)|^2 \leqslant |\varphi_\alpha|(TxTx^*) \leqslant (T_* |\varphi_\alpha|)(xx^*)$ and $\| \varphi_\alpha \| = \| |\varphi_\alpha| \| = |\varphi_\alpha|(T1) = \| T_* |\varphi_\alpha| \|$. Therefore $T_* |\varphi_\alpha| = |\varphi_\alpha|$ by [10] Proposition III. 4. 6. Assuming that T_* leaves no closed face $(\neq \{0\}, \mathscr{A}_*^+)$ of $\mathscr{A}_*^+$ invariant, every $0 \neq \varphi \in F(T_*)$ is faithful.

(3) If $(\mathscr{A}, \varphi, T)$ is a W^*-dynamical system it follows from Proposition 2. 1 and [9] Example III. 7. 3, that the Cesaro means $\frac{1}{n} \Sigma_{k=0}^{n-1} T_*^{\,k}$ of T_* converge in the strong operator topology on $\mathscr{L}(\mathscr{A}_*)$ to a projection of $\mathscr{A}_*$ onto the fixed space of T_*. Therefore a W^*-dynamical system $(\mathscr{A}, \varphi, T)$ is irreducible, iff $P\sigma(T_*) \cap \Gamma \neq \emptyset$ and T_* leaves no non trivial closed face of $\mathscr{A}_*^+$ invariant.

3. In this section we study the asymptotic behaviour of the powers T^n of an irreducible W^*-dynamical system $(\mathscr{B}(H), \varphi, T)$.

Proposition 3. 1. *Let $(\mathscr{B}(H), \varphi, T)$ be a W^*-dynamical system. Then the cyclic semi-group $\mathscr{S}_*$ is relatively compact in the strong operator topology $\mathscr{L}_s(\mathscr{B}(H)_*)$.*

Proof. Proposition 2. 1 shows that for all $0 \leqslant \psi \in \mathscr{A}_*$ the set $\{T_*^{\,n} \psi : n \in \mathrm{N}\}$ is weakly relatively compact. Using the [10] III. Corollary 5. 11. the assertion follows. □

From Proposition 3. 1 it follows that the compact semi-group $\bar{\mathscr{S}}_* \subseteq \mathscr{L}_s(\mathscr{B}(H)_*)$ has jointly continuous multiplication. If $\mathscr{K}_*$ is the kernel of $\bar{\mathscr{S}}_*$ then $\mathscr{K}_* = \bigcap_{k=1}^{\infty} \overline{\{T_*^{\,n} : n \geqslant k\}}$, every point $T_*^{\,n}$ is isolated in $\bar{\mathscr{S}}_*$ and $\bar{\mathscr{S}}_* = \mathscr{K}_* \cup \{T_*^{\,n} : n \in \mathrm{N}\}$ ([5], Theorem 3. 6). The next proposition shows how the topology on $\bar{\mathscr{S}}_*$ is related to the one on $\mathscr{K}_*$. (For a proof we refer to [5] Theorem 5. 3).

Proposition 3. 2. *Let $\bar{\mathscr{S}}$ be a compact topological semigroup with jointly continuous multiplication, generator t, kernel $\mathscr{K}$ and let q be the unit of $\mathscr{K}$. Let $\bar{\mathscr{S}}$ be topologized as follows:*

(a) *Every point* t^n, $n \in \mathbf{N}$, *is isolated.*

(b) *For every* $s \in \mathscr{K}$ *and an arbitrary neighbourhood* $U(s)$ *of* s *and* $n \in \mathbf{N}$ *we define the neighbourhood* $\widehat{U}_n(s)$ *as*

$$\widehat{U}_n(s) := U(s) \cup \{ t^k : k \geqslant n, \ (qt)^k \in U(s) \}.$$

Then the family of all sets $\widehat{U}_n(s)$ *for all neighbourhoods* $U(s)$ *of* $s \in \mathscr{K}$ *and all positive integers* n *defines a topology on* $\bar{\mathscr{S}}$ *which is equivalent to the given one.*

Using this topological result we are able to give a proof of our convergence theorem.

Theorem 3.3. *Let* $(\mathscr{B}(H), \varphi, T)$ *be an irreducible* W^**-dynamical system. Then there exists an irreducible, partially periodic* W^**-dynamical system* $(\mathscr{B}(H), \varphi, S)$ *such that* $\lim_n (T_*{}^n - S_*{}^n) = 0$ *in the strong operator topology* $\mathscr{L}_s(\mathscr{B}(H)_*)$.

Proof. It follows from Proposition 2.2 (a) and Theorem 2.3 that the kernel $\mathscr{K}_*$ of the compact topological semi-group $\bar{\mathscr{S}}_*$ generated by T_* in $\mathscr{L}_s(\mathscr{B}(H))$ is cyclic of order h for some positive integer h. Letting $S_* := Q_* \circ T_*$, Q_* the unit of $\mathscr{K}_*$, it follows that S_* is partially periodic, is irreducible and $\{ S_*{}^k : 1 \leqslant k \leqslant h \} = \mathscr{S}_*$. Since $\mathscr{K}_*$ carries the discrete topology it follows that the powers of $T_*{}^{kh}$ converge to $S_*{}^k$ in the strong operator topology by Proposition 3.2. Thus $\lim_n (T_*{}^n - S_*^n) = 0$ in $\mathscr{L}_s(\mathscr{B}(H))$. $\square$

The following follows immediately from Theorem 3.3.

Corollary 3.4. *Let* $(\mathscr{B}(H), \varphi, T)$ *be an irreducible* W^**-dynamical system with* $P\sigma(T) \cap \Gamma = \{1\}$. *Then* $\lim_n (T_*{}^n - 1 \otimes \varphi) = 0$ *in the strong operator topology* $\mathscr{L}_s(\mathscr{B}(H))$.

References

[1] K. DeLeeuw and I. Glicksberg: Applications of almost periodic compactifications, Acta Math., **105** (1961), 63–97.

[2] R. Derndinger, R. Nagel and G. Palm: 13 Lectures on ergodic theory, Preprint, Tübingen (1982).

[3] U. Groh: On the peripheral spectrum of uniformly ergodic positive operators on C^*-algebras, J. Operator Theory, **10** (1983), 31–37.

[4] U. Groh: Asymptotic behaviour of dynamical systems on W^*-algebras, Semesterbericht Funktionalanalysis Tübingen: Sommersemester (1982), 15–25.

[5] E. Hewitt: Compact, monothetic semigroups, Duke. Math. J., **23** (1956), 447–457.

[6] B. Kümmerer and R. Nagel: Mean ergodic semi-groups on W^*-algebras, Acta. Sci. Math., 41 (1979), 151–159.

[7] R. Nagel: Ergodic and mixing properties of linear operators, Proc. Royal Irish Acad. Sect. A, 74 (1974), 245–261.

[8] S. Sakai: C^*-algebras and W^*-algebras, Springer Verlag, Berlin-Heidelberg-New York, (1971).

[9] H. H. Schaefer: Banach lattices and positive operators, Springer Verlag, Berlin-Heidelberg-New York, (1974).

[10] M. Takesaki: Theory of operator algebras I, Springer Verlag, Berlin-Heidelberg-New York, (1979).