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On spectra of some completely positive maps *

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Abstract Let $\sum_{i=1}^{\infty} A_iA_i^*$ and $\sum_{i=1}^{\infty} A_i^*A_i$ converge in the strong operator topology. We study the map $\Phi_{\mathcal{A}}$ defined on the Banach space of all bounded linear operators $\mathcal{B}(\mathcal{H})$ by $\Phi_{\mathcal{A}}(X)=\sum_{i=1}^{\infty} A_iXA_i^*$ and its restriction $\Phi_{\mathcal{A}}|_{\mathcal{K}(\mathcal{H})}$ to the Banach space of all compact operators $\mathcal{K}(\mathcal{H})$. We first consider the relationship between the boundary eigenvalues of $\Phi_{\mathcal{A}}|_{\mathcal{K}(\mathcal{H})}$ and its fixed points. Also, we show that the spectra of $\Phi_{\mathcal{A}}$ and $\Phi_{\mathcal{A}}|_{\mathcal{K}(\mathcal{H})}$ are the same sets. In particular, the spectra of two completely positive maps involving the unilateral shift are described.

Keywords: Completely positive maps, the fixed points, spectra

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1 Introduction and preliminaries

Let $\mathcal{B}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ be the Banach space of all bounded linear operators and all compact operators on a separable complex Hilbert space $\mathcal{H}$, respectively. For $A\in\mathcal{B}(\mathcal{H})$, we write $A\geq 0$ if A is a positive operator, meaning $\langle Ax,x\rangle\geq 0$ for all $x\in\mathcal{H}$. Suppose that $\mathcal{T}(\mathcal{H})$ represents the set of all trace class operators on $\mathcal{H}$. Then the predual of $\mathcal{B}(\mathcal{H})$ is identified with $\mathcal{T}(\mathcal{H})$. As usual, $|A|:=(A^*A)^{\frac{1}{2}}$ is the absolute value of operator $A\in\mathcal{B}(\mathcal{H})$ and the one-rank operator $x\otimes y$ is defined by $x\otimes y(z)=\langle z,y\rangle x$ for vectors $x,y,z\in\mathcal{H}$. Let $S^+:=\frac{|S|+S}{2}$ and $S^-:=\frac{|S|-S}{2}$ be the positive and negative parts of a self-adjoint operator $S\in\mathcal{B}(\mathcal{H})$.

Let $\mathcal{X}$ be a Banach space and $B\in\mathcal{B}(\mathcal{X})$. We shall denote by $N(B)$, $R(B)$, $\sigma_{\mathrm{p}}(B)$ and $\sigma(B)$ the null space, the range, the point spectrum and the spectrum of B, respectively. It is known that $\partial\sigma(B)\subseteq\sigma_{\mathrm{ap}}(B)$, where $\partial\sigma(B)$ is the boundary of $\sigma(B)$ and $\sigma_{\mathrm{ap}}(B)$ is the approximate point spectrum of B. That is,

$$\sigma_{\mathrm{ap}}(B)=\{\lambda:\ \text{there exists } x_n\in\mathcal{X} \text{ with } \|x_n\|=1 \text{ such that } \lim_{n\to\infty}\|(B-\lambda I)x_n\|=0\}.$$

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Moreover, the right spectrum of the operator B is the set

$$\sigma_{\mathrm{r}}(B) = \{\lambda : B - \lambda I \text{ is not surjective on } \mathcal{X}\}.$$

For an integer $n \geq 1$, $\mathcal{M}_n(\mathcal{B}(\mathcal{H}))$ is the von Neumann algebra of $n \times n$ matrices whose entries are in $\mathcal{B}(\mathcal{H})$. Let $\Phi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ be a linear map. Then Φ is a positive map if $\Phi(A) \geq 0$ for all $A \geq 0$. Also, for each positive integer n, Φ induces a linear map $\Phi_n : \mathcal{M}_n(\mathcal{B}(\mathcal{H})) \longrightarrow \mathcal{M}_n(\mathcal{B}(\mathcal{H}))$ by the formula

$$\Phi_n([A_{i,j}]) = [\Phi(A_{i,j})], \qquad \text{for } [A_{i,j}] \in \mathcal{M}_n(\mathcal{B}(\mathcal{H})).$$

The map Φ is said to be n positive if the map Φ_n is positive. If Φ_n is positive for all $n \geq 1$, then Φ is called completely positive. Moreover, Φ is normal if Φ is continuous with respect to the ultraweak (σ-weak) topology. The normal completely positive maps on $\mathcal{B}(\mathcal{H})$ were characterized by Kraus in [6, Theorem 3.3] as follows.

Kraus theorem. Let Φ on $\mathcal{B}(\mathcal{H})$ be contractive ($\|\Phi\| \leq 1$). Then Φ is a normal completely positive map if and only if there exists a sequence $\{A_i\}_{i=1}^{\infty}$ of $\mathcal{B}(\mathcal{H})$ with $\sum_{i=1}^{\infty} A_i A_i^* \leq I$ such that

$$\Phi(X) = \sum_{i=1}^{\infty} A_i X A_i^* \qquad \text{for all} \quad X \in \mathcal{B}(\mathcal{H}).$$

The sequence $\{A_i\}_{i=1}^{\infty}$ is not necessary unique and $\sum_{i=1}^{\infty} A_i A_i^* \leq I$ in the strong operator topology. The family $\{A_i\}_{i=1}^{\infty}$ is called Kraus operators of Φ.

Let

$$\mathcal{A} = \{A_i : \ A_i \in \mathcal{B}(\mathcal{H}) \text{ for } i = 1, 2, 3, \cdots\} \ \text{ and } \ \mathcal{A}' = \{S : \ SB = BS \ \text{ for } \ B \in \mathcal{A}\}.$$

If $\sum_{i=1}^{\infty} A_i A_i^*$ and $\sum_{i=1}^{\infty} A_i^* A_i$ converge in the strong operator topology, then $\Phi_{\mathcal{A}}$ and its dual $\Phi_{\mathcal{A}}^{\dagger}$ defined on the Banach space $\mathcal{B}(\mathcal{H})$ are represented by

$$\Phi_{\mathcal{A}}(X) := \sum_{i=1}^{\infty} A_i X A_i^* \quad \text{and} \quad \Phi_{\mathcal{A}}^{\dagger} := \sum_{i=1}^{\infty} A_i^* X A_i, \qquad \text{for all } X \in \mathcal{B}(\mathcal{H}), \tag{1.1}$$

respectively. Then we conclude from Kraus theorem that $\Phi_{\mathcal{A}}$ and its dual $\Phi_{\mathcal{A}}^{\dagger}$ are normal completely positive maps.

The study of positive maps and completely positive maps is essential and useful in both mathematics and quantum theory. Many interesting results of (completely) positive maps in operator algebras were obtained in [2,3-5,10]. As fixed states of quantum operations are useful in the theory of quantum error correction, fixed points of completely positive maps were considered from different aspects (see [1,7-9,13,19-20]). In [14], Popescu studied the inequality $\Phi_{\mathcal{A}}(B) \leq B$ and the equation $\Phi_{\mathcal{A}}(B) = B$ by using the minimal isometric dilation and Poisson transforms. Arias et al. in [1] have considered some sufficient condition under which the fixed points of a unital quantum operation is the commutants of it's Kraus operators. Also, the Poisson boundary (the set of all fixed points of compact operators) of a unital quantum operation was characterized in [7]. Moreover, Prunaru has presented the canonical decompositions and lifting theorems to provide a description of the fixed points of completely positive maps in [15,16] and Magajna has given some properties of completely positive maps and its restriction to the Hilbert-Schmidt class in [12].

In this paper, we first consider the relationship between the boundary eigenvalues of $\Phi_{\mathcal{A}}|_{\mathcal{K}(\mathcal{H})}$ and its fixed points. We show that $\Phi_{\mathcal{A}}(A) = \omega A$ if and only if

$$\Phi_{\mathcal{A}}(|A|) = |\omega||A| \quad \text{and} \quad A_i A = e^{\sqrt{-1}\theta} A A_i \quad \text{for } i = 1, 2, \cdots,$$

where $A \in \mathcal{K}(\mathcal{H})$, $\omega \in \mathbb{C}$ with

$$|\omega| = \max\{\|\sum_{i=1}^{\infty} A_i A_i^*\|, \quad \|\sum_{i=1}^{\infty} A_i^* A_i\|\} \quad \text{and} \quad \theta := arg(\omega)$$

is the argument of the complex number ω. Also, we get that the spectra of $\Phi_{\mathcal{A}}$ and $\Phi_{\mathcal{A}}|_{\mathcal{K}(\mathcal{H})}$ are the same sets. That is,

$$\sigma(\Phi_{\mathcal{A}}|_{\mathcal{K}(\mathcal{H})}) = \sigma(\Phi_{\mathcal{A}}^{\dagger}|_{\mathcal{T}(\mathcal{H})}) = \sigma(\Phi_{\mathcal{A}}).$$

In particular, the spectra of the completely positive maps

$$\Theta(X) = V^* X V \quad \text{and} \quad \Upsilon(X) = \Theta(X) + (I - VV^*)X(I - VV^*)$$

are described, where V is the unilateral shift.

2 Main results

To show our main results, we need the following lemmas.

Lemma 1. Let $\Phi_{\mathcal{A}}(I) \le I$ and $\Phi_{\mathcal{A}}^{\dagger}(I) \le I$. If $B \in \mathcal{B}(\mathcal{H})$ is a compact self-adjoint operator such that $\Phi_{\mathcal{A}}(B) \ge B$, then $B \in \mathcal{A}'$ and $\Phi_{\mathcal{A}}(B) = B$.

Proof. Since B is a compact self-adjoint operator, it follows that $B = B^+ - B^-$, where B^+ and B^- are compact positive operators. Then the assumption $\Phi_{\mathcal{A}}(B) \ge B$ implies that

$$\Phi_{\mathcal{A}}(B^+) - \Phi_{\mathcal{A}}(B^-) \ge B^+ - B^-.$$

Let P be an orthogonal projection onto the subspace $\overline{R(B^+)}$ and $P^{\perp} = I - P$. Clearly, $P^{\perp} B^+ = 0$ and $PB^- = 0$. Thus

$$P\Phi_{\mathcal{A}}(B^+)P - P\Phi_{\mathcal{A}}(B^-)P \ge PB^+P - PB^-P = B^+,$$

so

$$\sum_{i=1}^{\infty} (PA_iP)B^+(PA_i^*P) = P\Phi_{\mathcal{A}}(B^+)P \ge B^+. \tag{2.1}$$

It is easy to verify that

$$\sum_{i=1}^{\infty} (PA_iP)(PA_i^*P) \le P\Phi_{\mathcal{A}}(I)P \le I \quad \text{and} \quad \sum_{i=1}^{\infty} (PA_i^*P)(PA_iP) \le P\Phi_{\mathcal{A}}^{\dagger}(I)P \le I.$$

Then inequality (2.1) and [8, Lemma 1] imply that

$$P\Phi_{\mathcal{A}}(B^+)P = B^+, \tag{2.2}$$

so $P\Phi_{\mathcal{A}}(B^-)P = 0$. That is,

$$P\Phi_{\mathcal{A}}(B^-) = \Phi_{\mathcal{A}}(B^-)P = 0. \tag{2.3}$$

By a similar way as above, we can get that

$$P^{\perp}\Phi_{\mathcal{A}}(B^{+}) = \Phi_{\mathcal{A}}(B^{+})P^{\perp} = 0. \tag{2.4}$$

Combining equations (2.2)-(2.4), we conclude that

$$\Phi_{\mathcal{A}}(B^{+}) = (P + P^{\perp})\Phi_{\mathcal{A}}(B^{+})(P + P^{\perp}) = B^{+}$$

and

$$\Phi_{\mathcal{A}}(B^{-}) = (P + P^{\perp})\Phi_{\mathcal{A}}(B^{-})(P + P^{\perp}) = B^{-}.$$

Thus $\Phi_{\mathcal{A}}(B) = B$, so $B \in \mathcal{A}'$ follows from [8, Theorem 3].□

The following lemma is an extension of [11, Proposition 1], which has shown the same result under the assumption of $\Phi_{\mathcal{A}}(I) = I$.

Lemma 2. Let $\Phi_{\mathcal{A}}(I) \leq I$ and $A \in \mathcal{B}(\mathcal{H})$. If $\Phi_{\mathcal{A}}(A^*A) = \Phi_{\mathcal{A}}(A^*)\Phi_{\mathcal{A}}(A)$, then $AA_i^* = A_i^*\Phi_{\mathcal{A}}(A)$ for $i = 1, 2, \cdots$.

Proof. Let

$$V = \begin{pmatrix} A_1, A_2, \cdots, A_n, \cdots \end{pmatrix} : \oplus_{i=1}^{\infty}\mathcal{H} \to \mathcal{H}.$$

Then

$$V^* = \begin{pmatrix} A_1^* \\ A_2^* \\ \vdots \\ A_n^* \\ \vdots \end{pmatrix} : \mathcal{H} \to \oplus_{i=1}^{\infty}\mathcal{H} \quad \text{and} \quad VV^* = \sum_{i=1}^{\infty} A_iA_i^* \leq I.$$

Moreover, $\Phi_{\mathcal{A}}(A) = V\widetilde{A}V^*$, where $\widetilde{A} \in \mathcal{B}(\oplus_{i=1}^{\infty}\mathcal{H})$ has the operator matrix form

$$\widetilde{A} = \mathrm{diag}(A, A, \cdots, A, \cdots).$$

If $\Phi_{\mathcal{A}}(A^*A) = \Phi_{\mathcal{A}}(A^*)\Phi_{\mathcal{A}}(A)$, then

$$V\widetilde{A^*}\widetilde{A}V^* = V\widetilde{A^*}V^*V\widetilde{A}V^*,$$

so

$$V\widetilde{A^*}(\widetilde{I} - V^*V)\widetilde{A}V^* = 0.$$

Obviously,

$$\widetilde{I} - V^*V \geq 0$$

follows from the fact of $I - VV^* \geq 0$. Thus $(\widetilde{I} - V^*V)\widetilde{A}V^* = 0$, which implies

$$\widetilde{A}V^* = V^*V\widetilde{A}V^* = V^*\Phi_{\mathcal{A}}(A).$$

Therefore,

$$\begin{pmatrix} AA_1^* \\ AA_2^* \\ \vdots \\ AA_n^* \\ \vdots \end{pmatrix} = \begin{pmatrix} A_1^*\Phi_{\mathcal{A}}(A) \\ A_2^*\Phi_{\mathcal{A}}(A) \\ \vdots \\ A_n^*\Phi_{\mathcal{A}}(A) \\ \vdots \end{pmatrix},$$

that is $AA_i^* = A_i^*\Phi_{\mathcal{A}}(A)$ for $i = 1, 2, \cdots$.□

Lemma 3. Let $B_i \in \mathcal{B}(\mathcal{H})$ be positive with $B_{i+1} \geq B_i \geq 0$ for $i = 1, 2, \cdots$. If B_n converges to B in the strong operator topology, then $\lim_{n\to\infty} \|B_n\| = \|B\|$.

Proof. Clearly, $B \geq B_i \geq 0$, which implies that $\|B_i\| \leq \|B\|$ for $i = 1, 2, \cdots$. For any $\varepsilon > 0$, there exists a unit vector $x \in \mathcal{H}$ such that $\langle Bx, x\rangle > \|B\| - \frac{\varepsilon}{2}$. Since B_n converges to B in the strong operator topology, it follows $\lim_{n\to\infty} \langle B_n x, x\rangle = \langle Bx, x\rangle$, so there exists N such that

$$\langle Bx, x\rangle - \frac{\varepsilon}{2} < \langle B_n x, x\rangle \leq \langle Bx, x\rangle \qquad \text{for } n > N.$$

Thus

$$\|B\| - \varepsilon < \langle B_n x, x\rangle \leq \|B_n\| \leq \|B\| \qquad \text{for } n > N.$$

That is $\lim_{n\to\infty} \|B_n\| = \|B\|$.□

The following lemma is well known. For the convenience of the reader, we provide a simple proof.

Lemma 4. Let $\Phi_{\mathcal{A}}(I) \leq I$ and $C \in \mathcal{B}(\mathcal{H})$. Then $\Phi_{\mathcal{A}}(C^*C) \geq \Phi_{\mathcal{A}}(C^*)\Phi_{\mathcal{A}}(C)$.

Proof. It is easy to see that

$$\begin{pmatrix} I & C \\ C^* & C^*C \end{pmatrix} \geq 0.$$

Then $\Phi_{\mathcal{A}}(I) \leq I$ and the fact that $\Phi_{\mathcal{A}}$ is completely positive imply

$$\begin{pmatrix} I & \Phi_{\mathcal{A}}(C) \\ \Phi_{\mathcal{A}}(C^*) & \Phi_{\mathcal{A}}(C^*C) \end{pmatrix} \geq \begin{pmatrix} \Phi_{\mathcal{A}}(I) & \Phi_{\mathcal{A}}(C) \\ \Phi_{\mathcal{A}}(C^*) & \Phi_{\mathcal{A}}(C^*C) \end{pmatrix} \geq 0,$$

so $\Phi_{\mathcal{A}}(C^*C) \geq \Phi_{\mathcal{A}}(C^*)\Phi_{\mathcal{A}}(C)$. □

The following is one of main results, which can be seen as an extension of [7, Theorem 3.1] and [8, Theorem 3].

Theorem 5. Let $\sum_{i=1}^{\infty} A_i A_i^*$ and $\sum_{i=1}^{\infty} A_i^* A_i$ converge in the strong operator topology. If $A \in \mathcal{K}(\mathcal{H})$ and $\omega \in \mathbb{C}$ with $|\omega| = \max\{\|\sum_{i=1}^{\infty} A_i A_i^*\|, \|\sum_{i=1}^{\infty} A_i^* A_i\|\}$, then the following statements are equivalent

(a) $\Phi_{\mathcal{A}}(A) = \omega A$;

(b) $\Phi_{\mathcal{A}}(|A|) = |\omega||A|$ and $A_i A = e^{\sqrt{-1}\theta} A A_i$ for $i = 1, 2, \cdots$, where $\theta = arg(\omega)$ is the argument of the complex number ω;

(c) $\Phi_{\mathcal{A}}(|A|) = |\omega||A|$ and $A_i A = e^{\sqrt{-1}\theta} U A_i |A|$ for $i = 1, 2, \cdots$, where $A = U|A|$ is the polar decomposition of A;

(d) $\Phi_{\mathcal{A}}(|A^*|) = |\omega||A^*|$ and $AA_i^* = e^{\sqrt{-1}\theta} A_i^* A$ for $i = 1, 2, \cdots$.

Proof. Setting

$$B_i = \frac{A_i}{|\omega|^{\frac{1}{2}}} \qquad \text{for } i = 1, 2, \cdots$$

and

$$\Phi(X) = \sum_{i=1}^{\infty} B_i X B_i^* \qquad \text{for all } \ X \in \mathcal{B}(\mathcal{H}),$$

we conclude that

$$\Phi(I) = \sum_{i=1}^{\infty} B_i B_i^* \leq I \quad \text{and} \quad \Phi^{\dagger}(I) = \sum_{i=1}^{\infty} B_i^* B_i \leq I.$$

(a)$\Longrightarrow$(b). Let $A \in \mathcal{K}(\mathcal{H})$ and $\Phi_{\mathcal{A}}(A) = \omega A$. Then $\Phi(A) = e^{\sqrt{-1}\theta}A$, which implies $\Phi(A^*) = e^{-\sqrt{-1}\theta}A^*$, so

$$\Phi(A^*)\Phi(A) = A^*A.$$

Thus $\Phi(A^*A) \geq A^*A$ follows from Lemma 4. Moreover, we conclude from Lemma 1 that

$$\Phi(A^*A) = A^*A \quad \text{and} \quad A^*AB_i = B_iA^*A \ \text{ for } i = 1, 2, \cdots.$$

Hence,

$$|A| \in \{B_i : \ i = 1, 2, \cdots\}'$$

and

$$(A^*A)\sum_{i=1}^{\infty} B_iB_i^* = \sum_{i=1}^{\infty} B_i(A^*A)B_i^* = A^*A.$$

Then

$$A^*A(I - \sum_{i=1}^{\infty} B_iB_i^*) = 0,$$

which implies

$$|A|(I - \sum_{i=1}^{\infty} B_iB_i^*) = 0,$$

so

$$|\omega||A| = |\omega||A|\sum_{i=1}^{\infty} B_iB_i^* = |\omega|\sum_{i=1}^{\infty} B_i|A|B_i^* = |\omega|\Phi(|A|) = \Phi_{\mathcal{A}}(|A|).$$

In a similar way as above, it follows from $\Phi_{\mathcal{A}}(A^*) = \overline{\omega}A^*$ that

$$\Phi(A^*) = e^{-\sqrt{-1}\theta}A^* \quad \text{and} \quad \Phi(AA^*) = AA^* = \Phi(A)\Phi(A^*),$$

so Lemma 2 yields that for $i = 1, 2, \cdots$,

$$A^*B_i^* = B_i^*\Phi(A^*) = e^{-\sqrt{-1}\theta}B_i^*A^*.$$

Therefore,

$$A_iA = e^{\sqrt{-1}\theta}AA_i \quad \text{ for } i = 1, 2, \cdots.$$

(b)$\Longrightarrow$(c). If $\Phi_{\mathcal{A}}(|A|) = |\omega||A|$, then $\Phi(|A|) = |A|$, so

$$|A| \in \{B_i : \ i = 1, 2, \cdots\}'$$

follows from Lemma 1. Thus

$$|A| \in \{A_i : \ i = 1, 2, \cdots\}',$$

which implies

$$A_iA = e^{\sqrt{-1}\theta}AA_i = e^{\sqrt{-1}\theta}U|A|A_i = e^{\sqrt{-1}\theta}UA_i|A| \quad \text{ for } i = 1, 2, \cdots.$$

(c)$\Longrightarrow$(a) and (d)$\Longrightarrow$(a) are clear.

(a)$\Longrightarrow$(d). Obviously, (a) implies $\Phi_{\mathcal{A}}(A^*) = \overline{\omega}A^*$. Then in a similar way to (a)$\Longrightarrow$(b), we conclude that

$$\Phi_{\mathcal{A}}(|A^*|) = |\omega||A^*| \quad \text{and} \quad A_iA^* = e^{-\sqrt{-1}\theta}A^*A_i \ \text{ for } \ i = 1, 2, \cdots,$$

so $AA_i^* = e^{\sqrt{-1}\theta} A_i^* A$ for $i = 1, 2, \cdots$. □

The following corollary is clear from Theorem 5.

Corollary 6. Let $\Phi_{\mathcal{A}}(I) \leq I$ and $\Phi^\dagger_{\mathcal{A}}(I) \leq I$. If $A \in \mathcal{K}(\mathcal{H})$ and $\omega \in \mathbb{C}$ with $|\omega| = 1$, then the following statements are equivalent

(a) $\Phi_{\mathcal{A}}(A) = \omega A$.

(b) $\Phi_{\mathcal{A}}(|A^*|) = |A^*|$ and $AA_i^* = \omega A_i^* A$ for $i = 1, 2, \cdots$.

(c) $\Phi_{\mathcal{A}}(|A|) = |A|$ and $A_i A = \omega A A_i$ for $i = 1, 2, \cdots$.

The following corollary is an extension of [11, Corollary 1], which showed the result for $A \in \mathcal{T}(\mathcal{H})$.

Corollary 7. Let $\Phi_{\mathcal{A}}(I) = I$ and $\Phi^\dagger_{\mathcal{A}}(I) = I$. If $A \in \mathcal{K}(\mathcal{H})$ and $\omega \in \mathbb{C}$ with $|\omega| = 1$, then the following statements are equivalent

(a) $\Phi_{\mathcal{A}}(A) = \omega A$

(b) $A_i A = \omega A A_i$ for $i = 1, 2, \cdots$.

(c) $AA_i^* = \omega A_i^* A$ for $i = 1, 2, \cdots$.

(d) $\Phi^\dagger_{\mathcal{A}}(A) = \overline{\omega} A$.

Proof. (a)$\Longrightarrow$(b) follows from Corollary 6.

(b)$\Longrightarrow$ (a). Since $\Phi_{\mathcal{A}}(I) = I$, we have

$$\Phi_{\mathcal{A}}(A) = \sum_{i=1}^{\infty} A_i A A_i^* = \omega A \sum_{i=1}^{\infty} A_i A_i^* = \omega A.$$

In a similar way, we get that (a)$\Longleftrightarrow$ (c). Obviously, (d)$\Longleftrightarrow$ $\Phi^\dagger_{\mathcal{A}}(A^*) = \omega A^*$. We conclude from (a)$\Longleftrightarrow$(b) that

$$\Phi^\dagger_{\mathcal{A}}(A^*) = \omega A^* \Longleftrightarrow A_i^* A^* = \omega A^* A_i^* \Longleftrightarrow A_i A = \omega A A_i,$$

for $i = 1, 2, \cdots$. Thus (d)$\Longleftrightarrow$(b). □

Corollary 8. Let $\Phi_{\mathcal{A}}(I) = I$ and $\Phi^\dagger_{\mathcal{A}}(I) \leq I$. If $A \in \mathcal{K}(\mathcal{H})$ and $\omega \in \mathbb{C}$ with $|\omega| = 1$ satisfy $\Phi_{\mathcal{A}}(A) = \omega A$, then $\Phi_{\mathcal{A}}(A^n) = \omega^n A^n$ for $n = 1, 2, \cdots$.

Proof. We conclude from Theorem 5 that $\Phi_{\mathcal{A}}(A) = \omega A$ implies

$$A_i A = \omega A A_i \quad \text{for } i = 1, 2, \cdots .$$

Thus

$$A_i A^n = \omega^n A^n A_i \quad \text{for } n = 1, 2, \cdots ,$$

so

$$\Phi_{\mathcal{A}}(A^n) = \sum_{i=1}^{\infty} A_i A^n A_i^* = \omega^n A^n \sum_{i=1}^{\infty} A_i A_i^* = \omega^n A^n \quad \text{for} \;\; n = 1, 2, \cdots$$

□

The following is another main result, which considers the spectra and the norms of $\Phi_{\mathcal{A}}$ and $\Phi_{\mathcal{A}}|_{\mathcal{K}(\mathcal{H})}$.

Theorem 9. Let $\sum_{i=1}^{\infty} A_i A_i^*$ and $\sum_{i=1}^{\infty} A_i^* A_i$ converge in the strong operator topology. Then

(a) $\Phi_{\mathcal{A}}(\mathcal{K}(\mathcal{H})) \subseteq \mathcal{K}(\mathcal{H})$ and $\Phi^\dagger_{\mathcal{A}}(\mathcal{K}(\mathcal{H})) \subseteq \mathcal{K}(\mathcal{H})$.

(b) $\|\Phi_{\mathcal{A}}|_{\mathcal{K}(\mathcal{H})}\| = \|\sum_{i=1}^{\infty} A_i A_i^*\|$ and $\|\Phi^\dagger_{\mathcal{A}}|_{\mathcal{K}(\mathcal{H})}\| = \|\sum_{i=1}^{\infty} A_i^* A_i\|$.

(c) $r(\Phi_{\mathcal{A}}|_{\mathcal{K}(\mathcal{H})}) = r(\Phi_{\mathcal{A}})$ and $r(\Phi^\dagger_{\mathcal{A}}|_{\mathcal{K}(\mathcal{H})}) = r(\Phi^\dagger_{\mathcal{A}})$, where $r(T)$ is the spectral radius of the operator $T \in \mathcal{B}(\mathcal{X})$.

(d) $\|\Phi_{\mathcal{A}}^{\dagger}\Phi_{\mathcal{A}}|_{\mathcal{K}(\mathcal{H})}\| = \|\sum_{j=1}^{\infty} A_j^*(\sum_{i=1}^{\infty} A_iA_i^*)A_j\|$.

(e) $\Phi_{\mathcal{A}}(\mathcal{T}(\mathcal{H})) \subseteq \mathcal{T}(\mathcal{H})$ and $\Phi_{\mathcal{A}}^{\dagger}(\mathcal{T}(\mathcal{H})) \subseteq \mathcal{T}(\mathcal{H})$.

(f) $\sigma(\Phi_{\mathcal{A}}|_{\mathcal{K}(\mathcal{H})}) = \sigma(\Phi_{\mathcal{A}}^{\dagger}|_{\mathcal{T}(\mathcal{H})}) = \sigma(\Phi_{\mathcal{A}})$.

Proof. (a) Since every compact operator can be approximated by sums of rank one operators in the operator norm topology and $\Phi_{\mathcal{A}}$ is bounded, we only need to show that $\sum_{i=1}^{\infty} A_ix\otimes A_iy \in \mathcal{K}(\mathcal{H})$ holds for all $x, y \in \mathcal{H}$. It is clear that

$$\begin{aligned}
(\textstyle\sum_{i=1}^{\infty} \|A_ix\otimes A_iy\|)^2 &\leq (\textstyle\sum_{i=1}^{\infty} \|A_ix\|^2)\sum_{i=1}^{\infty}\|A_iy\|^2 \\
&= (\textstyle\sum_{i=1}^{\infty}\langle A_i^*A_ix, x\rangle)\sum_{i=1}^{\infty}\langle A_i^*A_iy, y\rangle \\
&= \langle\textstyle\sum_{i=1}^{\infty} A_i^*A_ix, x\rangle\langle\sum_{i=1}^{\infty} A_i^*A_iy, y\rangle \\
&< +\infty,
\end{aligned}$$

as $\sum_{i=1}^{\infty} A_i^*A_i$ converges in the strong operator topology. Thus $\sum_{i=1}^{\infty} A_ix\otimes A_iy$ converges in the operator norm topology, so $\sum_{i=1}^{\infty} A_ix\otimes A_iy \in \mathcal{K}(\mathcal{H})$ for all $x, y \in \mathcal{H}$.

(b) It follows from (a) that $\mathcal{K}(\mathcal{H})$ is an invariant subspace of the operator $\Phi_{\mathcal{A}}$, which is seen as an operator on Banach space $\mathcal{B}(\mathcal{H})$. Thus $\Phi_{\mathcal{A}}|_{\mathcal{K}(\mathcal{H})}$ is an operator on Banach space $\mathcal{K}(\mathcal{H})$, so

$$\|\Phi_{\mathcal{A}}|_{K(\mathcal{H})}\| = \sup\{\frac{\|\Phi_{\mathcal{A}}(C)\|}{\|C\|}:\ 0\neq C\in\mathcal{K}(\mathcal{H})\} \leq \sup\{\frac{\|\Phi_{\mathcal{A}}(X)\|}{\|X\|}:\ 0\neq X\in\mathcal{B}(\mathcal{H})\}.$$

Using Russo-Dye theorem, we get that

$$\|\Phi_{\mathcal{A}}|_{\mathcal{K}(\mathcal{H})}\| \leq \sup\{\frac{\|\Phi_{\mathcal{A}}(X)\|}{\|X\|}:\ 0\neq X\in\mathcal{B}(\mathcal{H})\} = \|\Phi_{\mathcal{A}}(I)\| = \|\sum_{i=1}^{\infty} A_iA_i^*\|.$$

Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$ and P_n be the orthogonal projection onto the subspace $\overline{\vee\{e_i : i = 1, 2, \cdots, n\}}$ for $n = 1, 2, 3, \cdots$. Then P_n converges to the unit operator I in the ultraweak topology, so $\Phi_{\mathcal{A}}(P_n)$ converges to $\Phi_{\mathcal{A}}(I)$ in the ultraweak topology. Thus Lemma 3 yields that

$$\|\Phi_{\mathcal{A}}|_{\mathcal{K}(\mathcal{H})}\| \geq \lim_{n\to\infty}\|\Phi_{\mathcal{A}}(P_n)\| = \|\Phi_{\mathcal{A}}(I)\|,$$

so $\|\Phi_{\mathcal{A}}|_{\mathcal{K}(\mathcal{H})}\| = \|\sum_{i=1}^{\infty} A_iA_i^*\|$ as desired. Similarly, we have

$$\|\Phi_{\mathcal{A}}^{\dagger}|_{\mathcal{K}(\mathcal{H})}\| = \|\Phi_{\mathcal{A}}^{\dagger}(I)\| = \|\sum_{i=1}^{\infty} A_i^*A_i\|.$$

(c) We conclude from (a) and (b) that

$$\|(\Phi_{\mathcal{A}}|_{\mathcal{K}(\mathcal{H})})^n\| = \|(\Phi_{\mathcal{A}})^n|_{\mathcal{K}(\mathcal{H})}\| = \|(\Phi_{\mathcal{A}})^n(I)\| = \|(\Phi_{\mathcal{A}})^n\|$$

for $n = 1, 2, 3, \cdots$. Using the spectra radius formula, we get that

$$r(\Phi_{\mathcal{A}}|_{\mathcal{K}(\mathcal{H})}) = \lim_{n\to\infty}\|(\Phi_{\mathcal{A}}|_{\mathcal{K}(\mathcal{H})})^n\|^{\frac{1}{n}} = \lim_{n\to\infty}\|(\Phi_{\mathcal{A}})^n\|^{\frac{1}{n}} = r(\Phi_{\mathcal{A}}).$$

The equation $r(\Phi_{\mathcal{A}}^{\dagger}|_{\mathcal{K}(\mathcal{H})}) = r(\Phi_{\mathcal{A}}^{\dagger})$ is shown in a similar way. Moreover, (d) is clear from (b), and (e) is easy to verify.

(f) It is well-known that $\mathcal{T}(\mathcal{H})$ is a Banach space which is isometric isomorphic to the dual space of $\mathcal{K}(\mathcal{H})$ and its dual space is isometric isomorphic to $\mathcal{B}(\mathcal{H})$. Then we conclude from a direct calculation that

$$(\Phi_{\mathcal{A}}|_{\mathcal{K}(\mathcal{H})})^* = \Phi_{\mathcal{A}}^{\dagger}|_{\mathcal{T}(\mathcal{H})} \quad\text{and}\quad (\Phi_{\mathcal{A}}^{\dagger}|_{\mathcal{T}(\mathcal{H})})^* = \Phi_{\mathcal{A}},$$

where $(\Phi_{\mathcal{A}}|_{\mathcal{K}(\mathcal{H})})^*$ is the adjoint operator of $\Phi_{\mathcal{A}}|_{\mathcal{K}(\mathcal{H})}$. Thus $(\Phi_{\mathcal{A}}|_{\mathcal{K}(\mathcal{H})})^{**} = \Phi_{\mathcal{A}}$, which implies that $\sigma(\Phi_{\mathcal{A}}|_{\mathcal{K}(\mathcal{H})}) = \sigma(\Phi_{\mathcal{A}})$ as $\sigma(T) = \sigma(T^*)$ for the bounded operator T on a nontrivial Banach space. □

Remark 1. Let $\sum_{i=1}^{\infty} A_i A_i^*$ converge in the strong operator topology. Then the inclusion relation of $\Phi_{\mathcal{A}}(\mathcal{K}(\mathcal{H})) \subseteq \mathcal{K}(\mathcal{H})$ is not sufficient for the convergence of $\sum_{i=1}^{\infty} A_i^* A_i$ in the strong operator topology. Here, we provide a counterexample. Define the sequence of operators $V_n \in \mathcal{K}(\mathcal{H})$ by

$$V_n = \frac{1}{\sqrt{n}} e_n \otimes e_1 \quad \text{for } n = 1, 2, \cdots,$$

where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$. Then

$$\sum_{i=1}^{\infty} V_i V_i^* = \sum_{n=1}^{\infty} \frac{1}{n} e_n \otimes e_n \in \mathcal{K}(\mathcal{H}) \quad \text{and} \quad \sum_{i=1}^{\infty} V_i^* V_i \notin \mathcal{B}(\mathcal{H}).$$

It is easy to verify that

$$\Lambda(X) = \sum_{i=1}^{\infty} V_i X V_i^* \in \mathcal{K}(\mathcal{H}) \qquad \text{for all } X \in \mathcal{B}(\mathcal{H})$$

(or see [18, Corollary 2.4]). Thus $\Lambda(\mathcal{K}(\mathcal{H})) \subseteq \mathcal{K}(\mathcal{H})$ and $\sum_{i=1}^{\infty} V_i^* V_i$ is not convergent in the strong operator topology. Furthermore, if $\Gamma : \mathcal{K}(\mathcal{H}) \to \mathcal{K}(\mathcal{H})$ is completely positive, then there exists a sequence of Kraus operators $\{B_i\}_{i=1}^{\infty}$ for Γ such that $\sum_{i=1}^{\infty} B_i B_i^*$ is convergent in the strong operator topology and

$$\Gamma(X) = \sum_{i=1}^{\infty} B_i X B_i^* \qquad \text{for } X \in \mathcal{K}(\mathcal{H}).$$

Thus Γ can be extended to a (unique) completely positive map on $\mathcal{B}(\mathcal{H})$ ([18, Proposition 2.1]).

Using the proof of Theorem 9 (a), we conclude that $\Phi_{\mathcal{A}}(\mathcal{K}(\mathcal{H})) \subseteq \mathcal{K}(\mathcal{H})$ if and only if

$$\Phi_{\mathcal{A}}(x \otimes x) \subseteq \mathcal{K}(\mathcal{H}) \quad \text{for all} \quad x \in \mathcal{H},$$

which is equivalent to that $\forall x \in \mathcal{H}$ and $\forall \varepsilon > 0$, there exists a positive integer N such that $\forall n > N$ and all integers p,

$$\| \sum_{i=1}^{p} A_{n+i} x \otimes A_{n+i} x \| < \varepsilon$$

holds. □

The following corollary is related to [17, Corollary 2.2], which shows this result for a finite dimensional Hilbert space.

Corollary 10. Let $\Phi_{\mathcal{A}}(I) = I$ and $\Phi_{\mathcal{A}}^{\dagger}(I) \leq I$. Then

$$\sigma_{\mathrm{p}}(\Phi_{\mathcal{A}}^{\dagger} \Phi_{\mathcal{A}}|_{\mathcal{K}(\mathcal{H})}) \cap \{z \in \mathbb{C} : |z| = 1\} \subseteq \{1\} \subseteq \sigma_{\mathrm{ap}}(\Phi_{\mathcal{A}}|_{\mathcal{K}(\mathcal{H})}).$$

Proof. Suppose that $0 \neq B \in \mathcal{K}(\mathcal{H})$ and $\omega \in \mathbb{C}$ with $|\omega| = 1$ satisfy $(\Phi_{\mathcal{A}}^{\dagger} \Phi_{\mathcal{A}})(B) = \omega B$. Then Corollary 6 implies that

$$A_j^* A_i B = \omega B A_j^* A_i \quad \text{for } i, j = 1, 2, \cdots,$$

so

$$\sum_{j=1}^{\infty} A_j A_j^* A_i B = \omega \sum_{j=1}^{\infty} A_j B A_j^* A_i \qquad \text{for} \ \ i = 1, 2, \cdots .$$

Thus

$$A_i B = \omega \Phi_{\mathcal{A}}(B) A_i \quad \text{for } i = 1, 2, \cdots ,$$

which yields

$$\Phi_{\mathcal{A}}(B) = \sum_{i=1}^{\infty} A_i B A_i^* = \omega \sum_{i=1}^{\infty} \Phi_{\mathcal{A}}(B) A_i A_i^* = \omega \Phi_{\mathcal{A}}(B).$$

Hence, $\omega = 1$ follows from the fact of $\Phi_{\mathcal{A}}(B) \neq 0$. Moreover, Theorem 9 implies that

$$1 \in \sigma(\Phi_{\mathcal{A}}) = \sigma(\Phi_{\mathcal{A}}|_{\mathcal{K}(\mathcal{H})}) \quad \text{and} \quad r(\Phi_{\mathcal{A}}|_{\mathcal{K}(\mathcal{H})}) = r(\Phi_{\mathcal{A}}) \leq 1,$$

so

$$1 \in \partial\sigma(\Phi_{\mathcal{A}}|_{\mathcal{K}(\mathcal{H})}) \subseteq \sigma_{\text{ap}}(\Phi_{\mathcal{A}}|_{\mathcal{K}(\mathcal{H})}).$$

□

In [16, Corollary 2.2], Prunaru have shown that if $\Phi_{\mathcal{A}}(I) \leq I$, then $1 \in \sigma_{\text{p}}(\Phi_{\mathcal{A}})$ if and only if $\lim\limits_{n\to\infty} \Phi_{\mathcal{A}}^{(n)}(I)$ converges to the nonzero operator in the strong operator topology. In the following, we consider the similar problem for $1 \in \sigma_{\text{p}}(\Phi_{\mathcal{A}}|_{\mathcal{K}(\mathcal{H})})$.

Proposition 11. Let $\Phi_{\mathcal{A}}(I) \leq I$ and $\Phi_{\mathcal{A}}^{\dagger}(I) \leq I$. Then the following statements are equivalent:

(a) $1 \in \sigma_{\text{p}}(\Phi_{\mathcal{A}}|_{\mathcal{K}(\mathcal{H})})$;

(b) There is a finite rank orthogonal projection $P \neq 0$ such that $\Phi_{\mathcal{A}}(P) = P$;

(c) There is a finite rank orthogonal projection P such that $\Phi_{\mathcal{A}}(P) \leq P$ and $\lim\limits_{n\to\infty} \Phi_{\mathcal{A}}^{(n)}(P)$ converges to the nonzero operator in the strong operator topology;

(d) There exists a compact self-adjoint operator $B \neq 0$ such that $\Phi_{\mathcal{A}}(B) \geq B$;

(e) $1 \in \sigma_{\text{p}}(\Phi_{\mathcal{A}}^{\dagger}|_{\mathcal{K}(\mathcal{H})})$.

Proof. (a)$\Longrightarrow$(b) and (a)$\Longleftrightarrow$(e) follow from the proof of [8, Theorem 3].

(b)$\Longrightarrow$(c) and (a)$\Longrightarrow$(d) are obvious. By Lemma 1, we get that (d)$\Longrightarrow$(a) holds.

(c)$\Longrightarrow$(a). Let

$$0 \neq S = \lim_{n\to\infty} \Phi_{\mathcal{A}}^{(n)}(P).$$

Then it is clear that

$$S \leq P \quad \text{and} \quad \Phi_{\mathcal{A}}(S) = S.$$

Moreover, S is a compact operator, since $R(P)$ is finite dimensional. □

In the following examples, we mainly give some specific calculation of the spectra of two completely positive maps. The first example has been used in [8,11] for studying the fixed points (states) of quantum operations.

Example 1. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$. Define the unilateral shift $V \in \mathcal{B}(\mathcal{H})$ by $V(e_i) = e_{i+1}$ $i = 1, 2, \ldots$. Then $V^*(e_1) = 0$ and $V^*(e_j) = e_{j-1}$ for $j \geq 2$. It is easy to see that $V^*V = I$ and VV^* is an orthogonal projection from $\mathcal{H}$ onto subspace $\bigvee\{e_i : i \geq 2\}$. Moreover, the map $\Theta : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ defined by

$$\Theta(B) = V^* B V, \qquad \text{for } B \in \mathcal{B}(\mathcal{H})$$

is a unital completely positive map.

Let $\psi_n = \frac{1}{\sqrt{n}}(e_1 + e_2 \cdots e_n)$ for $n = 1, 2, 3 \cdots$. Then $\psi_n \otimes \psi_n \in \mathcal{K}(\mathcal{H})$ and

$$\Theta(\psi_n \otimes \psi_n) = V^*(\psi_n \otimes \psi_n)V = (\psi_n - \frac{1}{\sqrt{n}}e_n) \otimes (\psi_n - \frac{1}{\sqrt{n}}e_n),$$

so

$$\lim_{n\to\infty} \|\Theta(\psi_n \otimes \psi_n) - \psi_n \otimes \psi_n\| = \lim_{n\to\infty} \| - \psi_n \otimes \frac{1}{\sqrt{n}}e_n - \frac{1}{\sqrt{n}}e_n \otimes \psi_n + \frac{1}{n}e_n \otimes e_n\| = 0.$$

That is, $1 \in \sigma_{\mathrm{ap}}(\Theta|_{\mathcal{K}(\mathcal{H})})$.

Claim 1.

$$\sigma(\Theta) = \sigma_{\mathrm{p}}(\Theta) = \mathbb{D} \quad \text{and} \quad \sigma_{\mathrm{p}}(\Theta|_{\mathcal{K}(\mathcal{H})}) = \mathbb{T},$$

where $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ is closed unit disk and $\mathbb{T} = \{z \in \mathbb{C} : |z| < 1\}$ is open unit disk.

Let $|\mu| \leq 1$ and

$$A_\mu = e_1 \otimes e_1 + \sum_{k=1}^{\infty} \mu^k e_{k+1} \otimes e_{k+1}.$$

Then it is easy to verify that $\Theta(A_\mu) = \mu A_\mu$ and $A_\mu \in \mathcal{K}(\mathcal{H})$ for $|\mu| < 1$. It follows from Theorem 9 that $r(\Theta) \leq 1$, so

$$\sigma(\Theta) = \sigma_{\mathrm{p}}(\Theta) = \mathbb{D}.$$

Suppose that $|\omega| = 1$ and $0 \neq A \in \mathcal{K}(\mathcal{H})$ such that $\Theta(A) = \omega A$. We conclude from Theorem 5 that $\Theta(|A|) = |A|$, so $V^*|A|V = |A|$. Hence,

$$\langle |A|e_j, e_j\rangle = \langle V^*|A|Ve_j, e_j\rangle = \langle |A|e_{j+1}, e_{j+1}\rangle \quad \text{for } j = 1, 2, 3 \cdots .$$

Then the assumption of $A \in \mathcal{K}(\mathcal{H})$ implies that $\langle |A|e_j, e_j\rangle = 0$ for $j = 1, 2, 3 \cdots$, which yields $|A| = 0$. This contradiction comes from the equation $\Theta(A) = \omega A$. Therefore, $\omega \notin \sigma_{\mathrm{p}}(\Theta|_{\mathcal{K}(\mathcal{H})})$, so

$$\sigma_{\mathrm{p}}(\Theta|_{\mathcal{K}(\mathcal{H})}) = \mathbb{T} \quad \text{and} \quad \sigma(\Theta|_{\mathcal{K}(\mathcal{H})}) = \mathbb{D}.$$

Claim 2.

$$\{1\} \subseteq \sigma_{\mathrm{r}}(\Theta) \subseteq \{z \in \mathbb{C} : |z| = 1\} \quad \text{and} \quad \sigma_{\mathrm{r}}(\Theta|_{\mathcal{K}(\mathcal{H})}) = \{z \in \mathbb{C} : |z| = 1\}.$$

Let $|\mu| < 1$. We need to show that $\Theta^\dagger - \mu\Phi_I$ is surjective on the Banach space $\mathcal{B}(\mathcal{H})$ and $\Theta^\dagger|_{\mathcal{K}(\mathcal{H})} - \mu\Phi_I|_{\mathcal{K}(\mathcal{H})}$ is surjective on the Banach space $\mathcal{K}(\mathcal{H})$, where Φ_I is the identity on $\mathcal{B}(\mathcal{H})$.

For any $Y \in \mathcal{B}(\mathcal{H})$, it is easy to see that

$$\sum_{n=1}^{\infty} \|\mu^{n-1}V^nY(V^n)^*\| \leq \|Y\| \sum_{n=1}^{\infty} |\mu|^{n-1} < +\infty.$$

Thus $\sum\limits_{n=1}^{\infty} \mu^{n-1}V^nY(V^n)^*$ converges in the operator norm topology. Obviously,

$$V^*[\sum_{n=1}^{\infty} \mu^{n-1}V^nY(V^n)^*]V - \mu \sum_{n=1}^{\infty} \mu^{n-1}V^nY(V^n)^* = Y,$$

so

$$\sigma_{\mathrm{r}}(\Theta) \subseteq \{z \in \mathbb{C} : |z| = 1\}.$$

Since $Z \in \mathcal{K}(\mathcal{H})$ implies $\sum\limits_{n=1}^{\infty} \mu^{n-1} V^n Z (V^n)^* \in \mathcal{K}(\mathcal{H})$, it follows

$$\sigma_{\mathrm{r}}(\Theta|_{\mathcal{K}(\mathcal{H})}) \subseteq \{z \in \mathbb{C} : |z| = 1\}.$$

Moreover,

$$\sigma_{\mathrm{r}}(\Theta|_{\mathcal{K}(\mathcal{H})}) \supseteq \sigma(\Theta|_{\mathcal{K}(\mathcal{H})}) \backslash \sigma_{\mathrm{p}}(\Theta|_{\mathcal{K}(\mathcal{H})}) = \{z \in \mathbb{C} : |z| = 1\}.$$

Thus

$$\sigma_{\mathrm{r}}(\Theta|_{\mathcal{K}(\mathcal{H})}) = \{z \in \mathbb{C} : |z| = 1\}.$$

If $W \in \mathcal{B}(\mathcal{H})$ satisfies

$$\Theta(W) - W = V^* W V - W = I,$$

then

$$\langle W e_{n+1}, e_{n+1} \rangle = \langle W e_n, e_n \rangle + 1 \quad \text{for } \ n = 1, 2, 3 \cdots .$$

so

$$\langle W e_{n+1}, e_{n+1} \rangle = \langle W e_1, e_1 \rangle + n \quad \text{for } \ n = 1, 2, 3 \cdots .$$

This is a contradiction. Hence $1 \in \sigma_{\mathrm{r}}(\Theta)$.

Claim 3.

$$\sigma_{\mathrm{p}}(\Theta^\dagger) = \emptyset \quad \text{and} \quad \sigma_{\mathrm{r}}(\Theta^\dagger) \supseteq \mathbb{T}.$$

Indeed, if $0 \in \sigma_{\mathrm{p}}(\Theta^\dagger)$, then there exits $0 \neq B \in \mathcal{B}(\mathcal{H})$ such that

$$\Theta^\dagger(B) = V B V^* = 0,$$

which implies that

$$B = V^* V B V^* V = V^* \Theta^\dagger(B) V = 0.$$

This is a contradiction, so $0 \notin \sigma_{\mathrm{p}}(\Theta^\dagger)$.

Furthermore, if $0 \neq \lambda \in \sigma_{\mathrm{p}}(\Theta^\dagger)$, then

$$\Theta^\dagger(C) = V C V^* = \lambda C$$

for some $0 \neq C \in \mathcal{B}(\mathcal{H})$. Thus,

$$V C C^* V^* = V C V^* V C^* V^* = \Theta^\dagger(C) \Theta^\dagger(C^*) = |\lambda|^2 C C^*,$$

which yields that

$$|\lambda|^2 \langle C C^* e_1, e_j \rangle = \langle V C C^* V^* e_1, e_j \rangle = 0 \quad \text{for } \ j = 1, 2, 3 \cdots ,$$

so $CC^* e_1 = 0$. It is easy to see that

$$|\lambda|^2 \langle C C^* e_{n+1}, e_j \rangle = \langle V C C^* V^* e_{n+1}, e_j \rangle = \langle V C C^* e_n, e_j \rangle \quad \text{for } \ n = 1, 2, 3 \cdots .$$

Then we know from mathematical induction that

$$C C^* e_n = 0 \quad \text{for } n = 1, 2, 3 \cdots .$$

Hence $CC^* = 0$. This contradiction yields $\lambda \notin \sigma_{\mathrm{p}}(\Theta^\dagger)$, so $\sigma_{\mathrm{p}}(\Theta^\dagger) = \emptyset$.

To get $\sigma_{\rm r}(\Theta^\dagger) \supseteq \mathbb{T}$, we suppose that $0 \neq |\mu| < 1$ and an operator $X \in \mathcal{B}(\mathcal{H})$ satisfy the equation

$$\Theta^\dagger(X) - \mu X = VXV^* - \mu X = e_1 \otimes e_1.$$

Then

$$\langle e_1, e_j\rangle = \langle (e_1 \otimes e_1)e_1, e_j\rangle = \langle (VXV^* - \mu X)e_1, e_j\rangle = -\mu\langle Xe_1, e_j\rangle \quad \text{for } j = 1, 2, 3 \cdots,$$

so

$$\langle Xe_1, e_1\rangle = -\frac{1}{\mu} \quad \text{and} \quad \langle Xe_1, e_j\rangle = 0 \quad \text{for } j = 2, 3, 4, \cdots.$$

It is easy to see that for $n, j = 2, 3, 4, \cdots$,

$$0 = \langle (e_1 \otimes e_1)e_n, e_j\rangle = \langle (VXV^* - \mu X)e_n, e_j\rangle = \langle Xe_{n-1}, e_{j-1}\rangle - \mu\langle Xe_n, e_j\rangle.$$

Using mathematical induction again, we get that

$$\langle Xe_n, e_n\rangle = -\frac{1}{\mu^n} \quad \text{and} \quad \langle Xe_n, e_j\rangle = 0 \quad \text{for } n \neq j. \tag{2.5}$$

Since $0 \neq |\mu| < 1$, it follows

$$\lim_{n\to\infty} |\langle Xe_n, e_n\rangle| = +\infty.$$

This is a contradiction. Obviously, there is not an operator $X \in \mathcal{B}(\mathcal{H})$ such that

$$\Theta^\dagger(X) = VXV^* = e_1 \otimes e_1.$$

Thus $\mu \in \sigma_{\rm r}(\Theta^\dagger|_{\mathcal{K}(\mathcal{H})})$ and $\mu \in \sigma_{\rm r}(\Theta^\dagger)$ for all $|\mu| < 1$.

Claim 4.

$$\sigma_{\rm r}(\Theta^\dagger) = \sigma_{\rm r}(\Theta^\dagger|_{\mathcal{K}(\mathcal{H})}) = \mathbb{D}.$$

It follows from claim 3 that

$$\sigma_{\rm p}(\Theta^\dagger|_{\mathcal{K}(\mathcal{H})}) = \sigma_{\rm p}(\Theta^\dagger) = \emptyset \quad \text{and} \quad \sigma_{\rm r}(\Theta^\dagger|_{\mathcal{K}(\mathcal{H})}) \supseteq \mathbb{T}.$$

Clearly, Theorem 9 implies

$$r(\Theta^\dagger|_{\mathcal{K}(\mathcal{H})}) = r(\Theta^\dagger) \leq 1,$$

so

$$\sigma(\Theta^\dagger) = \sigma(\Theta^\dagger|_{\mathcal{K}(\mathcal{H})}) = \mathbb{D}.$$

Thus

$$\sigma_{\rm r}(\Theta^\dagger) = \sigma(\Theta^\dagger)\backslash\sigma_{\rm p}(\Theta^\dagger) = \mathbb{D}$$

and

$$\sigma_{\rm r}(\Theta^\dagger|_{\mathcal{K}(\mathcal{H})}) = \sigma(\Theta^\dagger|_{\mathcal{K}(\mathcal{H})})\backslash\sigma_{\rm p}(\Theta^\dagger|_{\mathcal{K}(\mathcal{H})}) = \mathbb{D}.$$

Combining claims 1-4, we get that

$$\sigma_{\rm p}(\Theta^\dagger|_{\mathcal{K}(\mathcal{H})}) = \sigma_{\rm p}(\Theta^\dagger) = \emptyset, \qquad \sigma_{\rm p}(\Theta|_{\mathcal{K}(\mathcal{H})}) \cap \sigma_{\rm r}(\Theta|_{\mathcal{K}(\mathcal{H})}) = \emptyset$$

and

$$\sigma(\Theta) = \sigma_{\rm p}(\Theta) = \sigma_{\rm r}(\Theta^\dagger|_{\mathcal{K}(\mathcal{H})}) = \sigma(\Theta|_{\mathcal{K}(\mathcal{H})}) = \sigma(\Theta^\dagger|_{\mathcal{K}(\mathcal{H})}) = \sigma(\Theta^\dagger) = \mathbb{D}.$$

Furthermore, we conclude from the proof of Claim 1 that

$$\sigma(\Theta|_{\mathcal{T}(\mathcal{H})}) = \mathbb{D} \quad \text{and} \quad \sigma_{\mathrm{p}}(\Theta|_{\mathcal{T}(\mathcal{H})}) = \mathbb{T}.$$

Also, the proof of Claim 3 implies that

$$\sigma(\Theta^\dagger|_{\mathcal{T}(\mathcal{H})}) = \sigma_{\mathrm{r}}(\Theta^\dagger|_{\mathcal{T}(\mathcal{H})}) = \mathbb{D}.$$

Moreover, it is easy to verify that

$$\sigma(\Theta^\dagger\Theta) = \sigma_r(\Theta^\dagger\Theta) = \sigma_p(\Theta^\dagger\Theta|_{\mathcal{K}(\mathcal{H})}) = \sigma(\Theta^\dagger\Theta|_{\mathcal{K}(\mathcal{H})}) = \sigma_r(\Theta^\dagger\Theta|_{\mathcal{K}(\mathcal{H})}) = \{0,1\}.$$

Indeed, $\Theta^\dagger\Theta$ is an idempotent operator on the Banach space $\mathcal{B}(\mathcal{H})$ such that the kernel space

$$N(\Theta^\dagger\Theta) = \{\lambda e_1 \otimes x + \mu y \otimes e_1 : \quad x, y \in \mathcal{H} \text{ and } \lambda, \mu \in \mathbb{C}\} = N(\Theta^\dagger\Theta|_{\mathcal{K}(\mathcal{H})})$$

and the range

$$R(\Theta^\dagger\Theta) = \{VV^*XVV^* : \quad X \in \mathcal{B}(\mathcal{H})\} \supsetneqq R(\Theta^\dagger\Theta|_{\mathcal{K}(\mathcal{H})}).$$

□

Example 2. Let $\Upsilon : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be

$$\Upsilon(B) = \Theta(B) + P_1BP_1 = V^*BV + P_1BP_1 \quad \text{for } B \in \mathcal{B}(\mathcal{H}),$$

where $P_1 = e_1 \otimes e_1$, V and Θ are the same as above Example 1. It is easy to see that $\Upsilon - \Theta$ is an idempotent operator on the Banach space $\mathcal{B}(\mathcal{H})$ and

$$R(\Upsilon - \Theta) = \{P_1XP_1 : \quad X \in \mathcal{B}(\mathcal{H})\}.$$

Thus

$$\mathcal{B}(\mathcal{H}) = R(\Upsilon - \Theta) \dotplus N(\Upsilon - \Theta)$$

and with respect to the above space decomposition,

$$\Upsilon - \Theta = \begin{pmatrix} \widetilde{I} & 0 \\ 0 & 0 \end{pmatrix} : \quad R(\Upsilon - \Theta) \dotplus N(\Upsilon - \Theta),$$

where $\widetilde{I}$ is identity on the Banach space $R(\Upsilon - \Theta)$. Since

$$\Theta(P_1XP_1) = 0 \quad \text{for all} \quad X \in \mathcal{B}(\mathcal{H}),$$

it follows

$$\Theta = \begin{pmatrix} 0 & \Theta_1 \\ 0 & \Theta_2 \end{pmatrix} : \quad R(\Upsilon - \Theta) \dotplus N(\Upsilon - \Theta), \tag{2.6}$$

so

$$\Upsilon = \begin{pmatrix} \widetilde{I} & \Theta_1 \\ 0 & \Theta_2 \end{pmatrix} : \quad R(\Upsilon - \Theta) \dotplus N(\Upsilon - \Theta). \tag{2.7}$$

Moreover, equation (2.6) and Example 1 imply that

$$\mathbb{D} = \sigma(\Theta) = \{0\} \cup \sigma(\Theta_2),$$

which yields

$$\sigma(\Theta_2)\backslash\{0\} = \mathbb{D}\backslash\{0\}.$$

Clearly,

$$\sigma(\Upsilon)\backslash\{0\} = [\{1\} \cup \sigma(\Theta_2)]\backslash\{0\} = \sigma(\Theta_2)\backslash\{0\} = \mathbb{D}\backslash\{0\}$$

follows from equation (2.7). Also,

$$\Upsilon(P_1 - e_2 \otimes e_2) = V^*(P_1 - e_2 \otimes e_2)V + P_1 = 0,$$

which induces $0 \in \sigma(\Upsilon)$. Thus

$$\sigma(\Upsilon|_{\mathcal{K}(\mathcal{H})}) = \sigma(\Upsilon) = \mathbb{D}.$$

It is easy to verify that

$$\Upsilon^\dagger(B) = \Theta^\dagger(B) + P_1BP_1 = VBV^* + P_1BP_1, \qquad \text{for } B \in \mathcal{B}(\mathcal{H}).$$

Furthermore, in a way similar to Claims 3-4 of Example 1, we get that

$$\sigma_{\mathrm{p}}(\Upsilon^\dagger) = \emptyset \quad \text{and} \quad \sigma(\Upsilon^\dagger) = \sigma_{\mathrm{r}}(\Upsilon^\dagger) = \mathbb{D}.$$

□

Remark 2. Let $\Delta : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be

$$\Delta(X) = UXU^* \quad \text{for } X \in \mathcal{B}(\mathcal{H}),$$

where $U \in \mathcal{B}(\mathcal{H})$ is a unitary operator. Then

$$\sigma(\Delta) = \sigma(\Delta|_{\mathcal{K}(\mathcal{H})}) \subseteq \{z \in \mathbb{C} : |z| = 1\}.$$

Indeed, for all $X \in \mathcal{B}(\mathcal{H})$ and complex numbers λ with $|\lambda| < 1$, we conclude that

$$\|\Delta(X) - \lambda X\| \geq \|\Delta(X)\| - \|\lambda X\| = (1 - |\lambda|)\|X\|,$$

so $\lambda \notin \sigma_{\mathrm{ap}}(\Delta)$. Since $\sigma_{\mathrm{ap}}(\Delta|_{\mathcal{K}(\mathcal{H})}) \subseteq \sigma_{\mathrm{ap}}(\Delta)$, it follows $\lambda \notin \sigma_{\mathrm{ap}}(\Delta|_{\mathcal{K}(\mathcal{H})})$. To get $\lambda \notin \sigma_{\mathrm{r}}(\Delta|_{\mathcal{K}(\mathcal{H})})$, we only need to show that $(\Delta|_{\mathcal{K}(\mathcal{H})})^* - \lambda\widetilde{I}$ is injective on the Banach space $\mathcal{T}(\mathcal{H})$, where $\widetilde{I}$ is the identity map on $\mathcal{T}(\mathcal{H})$. Suppose that $A \in \mathcal{T}(\mathcal{H})$ satisfies

$$[(\Delta|_{\mathcal{K}(\mathcal{H})})^* - \lambda\widetilde{I}](A) = 0.$$

Then for all $Y \in \mathcal{K}(\mathcal{H})$, we have

$$tr[Y(U^*AU - \lambda A)] = tr[(\Delta(Y) - \lambda Y)A] = 0,$$

which yields $U^*AU - \lambda A = 0$. Thus the assumption of $|\lambda| < 1$ implies that $A = 0$, so $\lambda \notin \sigma_{\mathrm{r}}(\Delta|_{\mathcal{K}(\mathcal{H})})$. Therefore,

$$\sigma(\Delta) = \sigma(\Delta|_{\mathcal{K}(\mathcal{H})}) \subseteq \{z \in \mathbb{C} : |z| = 1\}$$

follows from Theorem 9.

Let $\{r_i\}_{i=-\infty}^{+\infty}$ be a dense subset of the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ and $\{e_i\}_{i=-\infty}^{+\infty}$ be an orthonormal basis of $\mathcal{H}$. Setting

$$U_0e_i = r_ie_i, \quad \text{for} \quad i = 0, \pm1, \pm2, \cdots,$$

we conclude that U_0 is a unitary operator and

$$\sigma(\Psi) = \sigma(\Psi|_{\mathcal{K}(\mathcal{H})}) = \{z \in \mathbb{C} : |z| = 1\},$$

where $\Psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ has the form

$$\Psi(X) = U_0 X U_0^* \quad \text{for } X \in \mathcal{B}(\mathcal{H}).$$

Also, if $\{\lambda_i\}_{i=1}^m \subseteq \{z \in \mathbb{C} : |z| = 1\}$ and Q_i (for $i = 1, 2 \cdots m$) are orthogonal projections with $\sum_{i=1}^m Q_i = I$, then it is easy to see that

$$\sigma_{\mathrm{p}}(\Omega) = \sigma(\Omega|_{\mathcal{K}(\mathcal{H})}) = \sigma(\Omega) = \sigma_{\mathrm{p}}(\Omega|_{\mathcal{K}(\mathcal{H})}) = \{\lambda_i \overline{\lambda_j} : \quad i, j = 1, 2 \cdots m\},$$

where $\Omega : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ has the form

$$\Omega(X) = \sum_{i=1}^m \sum_{j=1}^m \lambda_i \overline{\lambda_j} Q_i X Q_j \quad \text{for } X \in \mathcal{B}(\mathcal{H}).$$

Furthermore, suppose that $W \in \mathcal{B}(\mathcal{H})$ is the bilateral shift. That is,

$$W e_i = e_{i+1}, \quad \text{for} \quad i = 0, \pm 1, \pm 2, \cdots.$$

Define the map $\Pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by

$$\Pi(X) = W X W^* \quad \text{for } X \in \mathcal{B}(\mathcal{H}).$$

Then

$$\sigma_{\mathrm{p}}(\Pi) = \sigma(\Pi|_{\mathcal{K}(\mathcal{H})}) = \sigma_{\mathrm{p}}(\Pi^\dagger) = \sigma(\Pi^\dagger|_{\mathcal{K}(\mathcal{H})}) = \{z \in \mathbb{C} : |z| = 1\} \tag{2.8}$$

and

$$\sigma_{\mathrm{p}}(\Pi|_{\mathcal{K}(\mathcal{H})}) = \sigma_{\mathrm{p}}(\Pi^\dagger|_{\mathcal{K}(\mathcal{H})}) = \emptyset. \tag{2.9}$$

Indeed, it is clear that $\sum_{i=-\infty}^{+\infty} \mu^i e_i \otimes e_i \in \mathcal{B}(\mathcal{H})$ and

$$\Pi(\sum_{i=-\infty}^{+\infty} \mu^i e_i \otimes e_i) = \frac{1}{\mu} \sum_{i=-\infty}^{+\infty} \mu^i e_i \otimes e_i$$

for $\mu \in \{z \in \mathbb{C} : |z| = 1\}$. Thus equation (2.8) holds. Also, equation (2.9) follows from Theorem 5 and the fact that $A \in \mathcal{K}(\mathcal{H})$ with $WAW^* = A$ implies $A = 0$. □

Remark 3. It is well-known that c_0 (the Banach space of null sequences) is an uncomplemented closed subspace of l_∞ (the Banach space of bounded sequences). Based on this result, we obtain that $\mathcal{K}(\mathcal{H})$ is not complemented in the Banach space $\mathcal{B}(\mathcal{H})$. Conversely, we suppose that $\mathcal{K}(\mathcal{H})$ is complemented in the Banach space $\mathcal{B}(\mathcal{H})$. Let $\widetilde{M} \in \mathcal{B}(\mathcal{B}(\mathcal{H}))$ be the idempotent operator such that $R(\widetilde{M}) = \mathcal{K}(\mathcal{H})$. Define the operator $\widetilde{N} \in \mathcal{B}(\mathcal{B}(\mathcal{H}))$ by

$$\widetilde{N}(X) = \sum_{i=1}^{+\infty} \langle X e_i, e_i \rangle e_i \otimes e_i \qquad \text{for all } X \in \mathcal{B}(\mathcal{H}),$$

where $\{e_i\}_{i=1}^\infty$ is an orthonormal basis of $\mathcal{H}$. Clearly, $\widetilde{N} \in \mathcal{B}(\mathcal{B}(\mathcal{H}))$ is also an idempotent operator. It is easy to see that

$$\widetilde{N}\widetilde{M}(X) = \sum_{i=1}^{+\infty} \langle \widetilde{M}(X) e_i, e_i \rangle e_i \otimes e_i = \widetilde{N}\widetilde{M}\widetilde{N}\widetilde{M}(X) \qquad \text{for all } X \in \mathcal{B}(\mathcal{H}),$$

which implies that $\widetilde{N}\widetilde{M}$ is an idempotent operator. Thus

$$\mathcal{B}(\mathcal{H}) = R(\widetilde{N}\widetilde{M}) \dotplus N(\widetilde{N}\widetilde{M}),$$

which yields

$$R(\widetilde{N}) = R(\widetilde{N}\widetilde{M}) \dotplus N(\widetilde{N}\widetilde{M}) \cap R(\widetilde{N}).$$

That is, $R(\widetilde{N}\widetilde{M})$ is a complemented subspace $R(\widetilde{N})$. Moreover, it is easy to verify that $R(\widetilde{N}\widetilde{M}) \simeq c_0$ and $R(\widetilde{N}) \simeq l_\infty$, where $\simeq$ is the isometric isomorphism between two Banach spaces. Thus we get that c_0 is a complemented subspace l_∞. This is a contradiction.$\Box$

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