Decoherence free algebra and periodicity for a quantum channel

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1 Introduction

Quantum channels are basic tools in quantum theory. They are seen as the counterpart of Markov operators in the non commutative models and they are generally used to represent the evolution of an open quantum system in discrete time models. For instance, in the quantum information world, a channel is typically used to/for...... while in statistical problems....

In this paper a quantum channel will be a unital normal completely positive operator on the algebra $B(\mathcal{H})$ of the bounded operators on a Hilbert space $\mathcal{H}$. A main object of interest will be the decoherence free algebra (DFA) of the channel, that we shall denote by $\mathcal{N}$, together with all the properties that can be connected to it. The DFA for dynamical semigroups of channels was already a very popular object in the 70's and 80's, extensively used in order to study ergodic/asymptotic properties of the semigroup (see FrigerioVerri, Robinson, Evans....) More recently the DFA appeared again in the literature and was studied because of the interest in reversible subsystems and in relation with environmental decoherence.... Most of these previous studies are generally concentrated in the case of a continuous time Markov see
- Fagnola& Rebolledo, where we find a characterization in terms of the Lindblad form of the generator
- CSU or Hellmich, where the link with environmental decoherence and other forms of decompositions is treated
- Batkai et al : Abstract "We consider semigroups of operators on a W^*-algebra and prove, under appropriate assumptions, the existence of a Jacobs-DeLeeuw-Glicksberg type decomposition. This decomposition splits the algebra into a "stable" and "reversible" part with respect to the semigroup and yields, among others, a structural approach to the Perron-Frobenius spectral theory for completely positive operators on W^*-algebras."

From these starting points, we developed our results, mainly dealing with the properties of the DFA for a quantum channel acting on $B(\mathcal{H})$ with an invariant faithful state .
- a main initial result about the atomicity of the DFA
- we exploit this property and so the minimal central projections of the DFA to deduce ad-hoc decompositions of the invariant states and of the Kraus operators
- moreover we can study the periodic behavior of the channel (we start from the irreducible case and then go to the general one); this allows to arrive to a simplified description of the channel.

To do this, we also recover some similar results obtained for the fixed points algebra.

Cite DFSU and say few words on the similarities and differences between the discrete and continuous time cases.

Finally, we describe some examples, using the family of OQRWs. In the last one, we try to throw a glance to a case without invariant state.

2 Multiplicative domain and fixed points

Let $\mathcal{H}$ be a separable Hilbert space and let $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$ be a unital normal cp map. Then Φ has a representation of the form

$$\Phi(A) = \sum_{k=1}^{\infty} V_k^* A V_k, \qquad A \in B(\mathcal{H}), \tag{1}$$

where the Kraus operators $V_k \in B(\mathcal{H})$ are such that $\sum_k V_k^* V_k = I$.

We will consider the following sets of operators:

$$\begin{aligned}
\mathcal{M}(\Phi) &:= \{A \in B(\mathcal{H}), \Phi(A^*A) = \Phi(A)^*\Phi(A), \Phi(AA^*) = \Phi(A)\Phi(A)^*\} \\
&= \{A \in B(\mathcal{H}), \Phi(AB) = \Phi(A)\Phi(B), \Phi(BA) = \Phi(B)\Phi(A),\ \forall B \in B(\mathcal{H})\} \\
\mathcal{N}(\Phi) &:= \cap_n \mathcal{M}(\Phi^n) \\
\mathcal{F}(\Phi) &:= \{A \in B(\mathcal{H}), \Phi(A) = A\}
\end{aligned}$$

Then $\mathcal{M}(\Phi)$ is called the multiplicative domain, $\mathcal{N}(\Phi)$ is the decoherence free subalgebra and $\mathcal{F}(\Phi)$ is the fixed point domain of Φ. Since the map Φ will be fixed throughout, we use the notations $\mathcal{M} = \mathcal{M}(\Phi)$, $\mathcal{N} = \mathcal{N}(\Phi)$ and $\mathcal{F} = \mathcal{F}(\Phi)$.

We now collect some basic facts about these sets. The proofs are included for the convenience of the reader.

Proposition 1. *$\mathcal{M} = \{V_j V_k^*,\ j, k = 1, 2, \ldots\}'$, where $\{\ \}'$ denotes the commutant.*

Proof. It will be convenient to use the Stinespring representation of Φ: Let $\mathcal{K}$ be a separable Hilbert space and let $\{e_n\}$ be an orthonormal basis. Put

$$V = \sum_j V_j \otimes |e_j\rangle,$$

then $V \in B(\mathcal{H}, \mathcal{H} \otimes \mathcal{K})$, $V^*V = \sum_j V_j^* V_j = I$ and

$$\Phi(A) = V^*(A \otimes I)V, \qquad A \in B(\mathcal{H}).$$

Let $P = VV^*$, then $P \in B(\mathcal{H} \otimes \mathcal{K})$ is a projection and we have $A \in \mathcal{M}$ if and only if $A \otimes I$ commutes with P. Indeed, suppose $A \in \mathcal{M}$, then

$$V^*(A^*A \otimes I)V = V^*(A^* \otimes I)P(A \otimes I)V.$$

It follows that $P(A^* \otimes I)(1 - P)(A \otimes I)P = 0$, hence $(1 - P)(A \otimes I)P = 0$, so that

$$(A \otimes I)P = P(A \otimes I)P.$$

Similarly, we get the same for A^* and this implies that

$$P(A \otimes I) = P(A \otimes I)P = (A \otimes I)P.$$

The converse is easy. Now notice that $P = \sum_{j,k} V_j V_k^* \otimes |e_j\rangle\langle e_k|$, this implies the statement. $\square$

By this characterization, we can see that $\mathcal{M}$ is a von Neumann subalgebra in $B(\mathcal{H})$ (see also [?]) and it is clear that the restriction of Φ is a *-homomorphism $\mathcal{M} \to B(\mathcal{H})$. Consequently, $\mathcal{N}$ is a von Neumann subalgebra as well and the restriction of Φ defines a *-endomorphism of $\mathcal{N}$.

Remark 1. *Notice that $\Phi|_{\mathcal{N}}$ is not always a *-automorphism. Indeed, $\mathcal{N}$ can have, for instance, a non-trivial intersection with the kernel of Φ. Since this intersection is a subalgebra, it then contains a nonzero projection $0 \neq P \in Ker(\Phi) \cap \mathcal{N}$. On the other hand, any projection in $Ker(\Phi)$ is necessarily in $\mathcal{N}$, so that this happens if and only if Φ is not faithful. But even if Φ is faithful, $\Phi|_{\mathcal{N}}$ needs not be a *-automorphism, example?*

Proposition 2. *We have the following characterizations of $\mathcal{N}$:*

(i) $\mathcal{N} = \{V_{i_1} \dots V_{i_n} V_{j_1}^* \dots V_{j_n}^*,\ i_k, j_k = 1, 2, \dots;\ n \in \mathbb{N}\}'$.

(ii) $\mathcal{N}$ is the von Neumann algebra generated by the preserved projections, i.e. by the set

$$\{Q \in B(\mathcal{H}) : \Phi^n(Q) \text{ is a projection } \forall\, n \geq 0\}.$$

Proof. (i) is immediate from Proposition 1. (ii) holds since $P \in \mathcal{M}$ if and only if $\Phi(P)$ is a projection.

□

In contrast, the set of fixed points is in general not a subalgebra. Some example?

Proposition 3. *$\mathcal{F}$ is a von Neumann algebra if and only if it is contained in $\mathcal{N}$. In this case, we have*

$$\mathcal{F} = \{V_j, V_j^*,\ j = 1, 2 \dots\}'$$

Proof. The first statement is quite obvious. Assume now that $\mathcal{F}$ is a von Neumann algebra and let $A \in \mathcal{F}$. Then

$$0 = \Phi(A^*A) - A^*A = (V_jA - AV_j)^*(V_jA - AV_j),$$

this implies $AV_j = V_jA$. Similarly, we obtain $AV_j^* = V_j^*A$. It follows that $\mathcal{F} \subseteq \{V_j, V_j^*,\ j = 1, 2 \dots\}'$ The converse inclusion is clear.

□

3 Maps with a faithful invariant state

In this section, we assume that there is a faithful normal state $\rho \in \mathfrak{S}(\mathcal{H})$ for Φ. The following results are well known.

Proposition 4. *Assume that there is a faithful normal invariant state for Φ. Then*

(i) $\mathcal{F}$ is a von Neumann subalgebra.

*(ii) The restriction $\Phi|_{\mathcal{N}}$ is a *-automorphism.*

Proof. If ρ is a faithful invariant state, then for any $A \in \mathcal{F}$, $\rho(\Phi(A^*A) - A^*A) = 0$. Since $\Phi(A^*A) - A^*A \geq 0$ by the Schwarz inequality for cp maps, this implies that $\Phi(A^*A) = A^*A = \Phi(A)^*\Phi(A)$, hence $\mathcal{F} \subseteq \mathcal{N}$. The statement (i) now follows by Proposition 3. For (ii) see [?]? or maybe Robinson?

□

In the presence of a faithful invariant state, there is another special subalgebra investigated in the literature, e.g. [3, 15]. More precisely, let $\mathbf{S}$ be the closure of the set of channels $\{\Phi^n,\ n \in \mathbb{N}\}$ in the point-ultraweak topology. If there exists a faithful normal invariant state ρ for Φ, then for any $\varphi \in B(\mathcal{H})_*$, the set [15, Proposition 2.1]

$$\{\Phi_*^n \varphi,\ n \in \mathbb{N}\}$$

is weakly relatively compact, equivalently, the set $\mathbf{S}$ consists of normal operators and is a compact semitopological semigroup. It is proved in [3, 15, 16] that the set

$$\mathcal{M}_r := \overline{\operatorname{span}\{x \in \mathcal{M},\ \Phi(x) = \lambda x,\ |\lambda| = 1\}}^{w*} = \{x \in \mathcal{M}, T(x^*x) = T(x)^*T(x),\ \forall T \in \mathbf{S}\}$$

is a von Neumann subalgebra, called the reversible subspace (see [16, Theorem 2.1] and [3, Theorem 1.2]). Moreover, there is a conditional expectation $F \in \mathbf{S}$ with range $\mathcal{M}_r$, such that $F_*(\rho) = \rho$ (see the proof of [15, Proposition 2.2]) .

It is quite clear that we have $\mathcal{M}_r \subseteq \mathcal{N}$ and that equality holds in finite dimensions. We next show that equality holds for channels on $B(\mathcal{H})$ (or more generally on atomic von Neumann algebras).

Theorem 1. *Assume that a uncp map $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$ admits a faithful normal invariant state ρ. Then $\mathcal{N} = \mathcal{M}_r$.*

Proof. Let $\mathcal{B}_1$, $\mathcal{N}_1$ and $\mathcal{R}_1$ be the unit balls of $B(\mathcal{H})$, $\mathcal{N}$ and $\mathcal{M}_r$, respectively. Then

$$\mathcal{R}_1 \subseteq \mathcal{N}_1 \subseteq \bigcap_n \Phi^n(\mathcal{B}_1).$$

Indeed, the first inclusion follows from $\mathcal{M}_r \subseteq \mathcal{N}$ and the second from the fact that the restriction $\Phi|_{\mathcal{N}}$ is an automorphism. We will show that $\mathcal{R}_1 = \bigcap_n \Phi^n(\mathcal{B}_1)$, which implies the statement. (This proof is inspired by [1].)

We will use a Hahn-Banach separation argument. So let $x \in \bigcap_n \Phi^n(\mathcal{B}_1) \setminus \mathcal{R}_1$. Since $\mathcal{R}_1 \subset B(\mathcal{H})$ is convex and compact in the weak*-topology, there exists some $\psi \in B(\mathcal{H})_*$ such that

$$\psi(x) > \sup_{y \in \mathcal{R}_1} \psi(y) = \|\psi|_{\mathcal{M}_r}\|_1 = \|F_*\psi\|_1.$$

For each $n \in \mathbb{N}$, there is some $y_n \in \mathcal{B}_1$ such that $x = \Phi^n(y_n)$ and we have

$$\|\Phi_*^n \psi\|_1 \geq (\Phi_*^n \psi)(y_n) = \psi(x).$$

Note that since Φ_* is a contraction, $\{\|\Phi_*^n \psi\|_1\}_n$ is a bounded nonincreasing sequence and we have

$$\lim_n \|\Phi_*^n \psi\|_1 \geq \psi(x) > \|F_*\psi\|_1.$$

On the other hand, for any $\varphi \in B(\mathcal{H})_*$, the orbit

$$\mathbf{S}_*\varphi := \{S_*\varphi,\ S \in \mathbf{S}\} = \{\Phi_*^n \varphi,\ n \in \mathbb{N}\}^{-w}$$

is weakly compact. Since $F \in \mathbf{S}$, $\mathbf{S}_*\varphi$ contains $F_*\varphi$ and since $B(\mathcal{H})_*$ is a separable Banach space, the weak topology on the orbit is a metric topology ([10, Theorem V.6.3]). Hence there is a subsequence of $\Phi^n_*\varphi$ converging weakly to $F_*\varphi$.

Let Φ^{n_k} be such that $\Phi^{n_k}_*\psi \to F_*\psi$ and let $\varphi_1, \dots, \varphi_4 \in B(\mathcal{H})^+_*$ be such that $\psi = \sum_i c_i\varphi_i$. Then we may assume that $\Phi^{n_k}_*\varphi_i$ are all weakly convergent, restricting to subsequences if necessary ([10, Theorem V.6.1]). By [18, Corollary III.5.11], $\Phi^{n_k}_*\varphi_i$ are all norm convergent. It follows that $\Phi^{n_k}_*\psi \to F_*\psi$ in norm, so that

$$\lim \|\Phi^n_*\psi\|_1 = \lim \|\Phi^{n_k}_*\psi\|_1 = \|F_*\psi\|_1,$$

a contradiction.

□

Corollary 1. *Let $\mathcal{M} = B(\mathcal{H})$ and let ρ be a faithful normal invariant state. Then $\mathcal{N}$ is the range of a conditional expectation $E_{\mathcal{N}}$ preserving ρ.*

Remark 2. *Note that by Corollary 1, the subalgebra $\mathcal{N}$ is atomic. Note also that Theorem 1 and Corollary 1 hold for uncp maps on any atomic von Neumann algebra $\mathcal{M}$. The same proof can be used also in continuous time case.*

Remark 3. *In the situation of Theorem 1, we always have environmental decoherence according to [5] with decomposition $\mathcal{N}$ and $\mathcal{M}_s$ maybe we should recall here some definition (?)... This is an almost direct consequence stated for instance in [9, Proposition 31].*

??? This second part should be moved in the next subsection????

Moreover, in this case, we have some standard tools to study the velocity of decoherence. One is a kind of spectral gap inequality, in order to characterize the situation when the dissipative elements converge to 0 exponentially fast and with uniform rate.

Indeed, calling ρ the invariant faithful state, we can search for a non negative constant ϵ such that $\|\Phi^n(x) - \Phi^n(Fx)\|_{2,\rho} \leq e^{-\epsilon n}\|x - Fx\|_{2,\rho}$. Finding a strictly positive ϵ, uniform in n and x, would give the uniform exponential convergence to the decoherence-free algebra. A common technique, also for finite classical Markov chains, consists in using continuous time generators. The same ideas can be applied to our case of interest:
- first, we consider the operator Φ^d, where d is the period of the channel, so that the algebra $\mathcal{N}$ is the fixed space of the new operator
- second, we consider the infinitesimal Lindblad generator $\mathcal{L} := (\Phi^d - 1)$, inheriting the invariant faithful states of Φ, and we compute the spectral gap of $\mathcal{L}$ with the usual standard techniques (see...).

Also entropy inequalities and log-Sobolev could be mentioned here (?????)

3.1 Irreducible maps

We say that the map Φ is irreducible if there are no nontrivial subharmonic projections, that is, if $P \in B(\mathcal{H})$ is a projection such that $\Phi(P) \geq P$ then $P = 0$ or $P = I$. If there is faithful normal invariant state, this clearly happens if and only if $\mathcal{F} = \mathbb{C}I$.

Definition 1. *Period of Φ. Let Φ be an irreducible uncp map. Then the period d is the maximal integer d such that there exists a resolution of the identity $Q_0, \dots Q_{d-1}$ verifying $\Phi(Q_j) = Q_{j-1}$ for all j (subtraction on indices are modulo d).*

Proposition 5 (Groh [15] and Batkai et al [3, Propositions 6.1 and 6.2])**.** *Let Φ be an irreducible uncp map on $B(\mathcal{H})$ with an invariant faithful state. Then the peripheral point spectrum of Φ is the group of all the d-th roots of unity for some $d \geq 1$ and all the eigenvalues in the peripheral point spectrum are simple. Moreover there exists a unitary operator U such that $U^d = 1$ and $\Phi(U^n) = \exp(i2\pi n/d)U^n$.*

(recall also Evans & Hoegh-Krohn 78, where this result was proven for the finite dimensional case)

Corollary 2. *Let Φ be an irreducible uncp map on $B(\mathcal{H})$ with an invariant faithful state. Then Φ has finite period, the cyclic resolution of Φ is unique and $\mathcal{N}$ is an abelian algebra spanned by the cyclic projections of Φ.*

Proof. Calling ω the primary d-th root of unity and U a pertaining eigenvector, we have that U is a unitary operator satisfying $U^d = 1$ and $\Phi(U^n) = \omega^n U^n$ by previous proposition. It follows that U^n is the unique (up to multiplicative constants) eigenvector associated with the eigenvalue ω^n. By Theorem 1,

$$\mathcal{N} = \mathcal{M}_r = \text{span}\,\{I, U, \dots U^{d-1}\} = \{U, U^*\}''.$$

In particular, it follows that the abelian subalgebra generated by U is finite dimensional and therefore U admits a spectral representation

$$U = \sum_{j=0}^{d-1} \omega^j Q_j$$

for some orthogonal projections Q_j summing up to 1. We immediately deduce that, since $\Phi(U) = \omega U$, then $\Phi(Q_j) = Q_{j-1}$ for all j, so that $Q_0, ..Q_{d-1}$ is a cyclic decomposition of Φ and we have

$$\mathcal{N} = \{U, U^*\}'' = \text{span}\,\{Q_0, \dots, Q_{d-1}\}.$$

To prove uniqueness, assume that $P_0, ...P_{d-1}$ is another cyclic resolution of Φ. Then we can construct the unitary operator

$$V = \sum_{j=0}^{d-1} \omega^j P_j,$$

which is an eigenvalue for ω. Since the eigenvalues are simple, we must have $V = zU$ for some $z \in \mathbb{C}$, $|z| = 1$ and it is easy to see that for each n we must have $P_n = Q_j$ for some $j = 0, 1, \dots, d-1$.

□

Proposition 6. *Suppose Φ is an irreducible uncp map with an invariant faithful state and let $Q_0, \dots, Q_{d-1}$ be the cyclic resolution for Φ. Then*

1. *$\mathcal{F}(\Phi^m)$ is a subalgebra of $\mathcal{N}$ for any m;*
2. *$\mathcal{F}(\Phi^d) = \mathcal{N}$ and d is the smallest integer with this property;*
3. *$\mathcal{F}(\Phi^m) \neq \mathbb{C}1$ if and only if $GCD(m, d) > 1$.*

Moreover, denote by $\Phi^d_{|k}$ the restriction of Φ^d to the subalgebra $Q_kB(\mathcal{H})Q_k$, then $\Phi^d_{|k}$ is irreducible, positive recurrent and aperiodic, and consequently ergodic.

Proof. Let $\rho \in \mathfrak{S}(\mathcal{H})$ be the unique faithful invariant state of Φ, then ρ is also invariant for Φ^m, so that by Propositions 3 and 4, $\mathcal{F}(\Phi^m)$ is a subalgebra in $\mathcal{N}(\Phi^m)$. Note that for any $n \in \mathbb{N}$ and $X \in \mathcal{M}(\Phi^n)$, we have from Schwartz inequality for ucp maps that

$$\begin{aligned}\Phi^n(X^*X) &= \Phi(\Phi^{n-1}(X^*X)) \geq \Phi(\Phi^{n-1}(X)^*\Phi^{n-1}(X)) \\ &\geq \Phi^n(X)^*\Phi^n(X) = \Phi^n(X^*X).\end{aligned}$$

Using the fact that $\Phi^{n-1}(X^*X) - \Phi^{n-1}(X)^*\Phi^{n-1}(X) \geq 0$ and that ρ is a faithful invariant state, we obtain that $\Phi^{n-1}(X^*X) = \Phi^{n-1}(X)^*\Phi^{n-1}(X)$. This implies that $\mathcal{M}(\Phi^n) \subseteq \mathcal{M}(\Phi^{n-1})$ for all n and hence

$$\mathcal{N}(\Phi^m) = \cap_n \mathcal{M}(\Phi^{mn}) = \cap_n \mathcal{M}(\Phi^n) = \mathcal{N}.$$

This proves 1.

By definition of cyclic decomposition, we have $Q_j \in \mathcal{F}(\Phi^d)$ for all j, this implies $\mathcal{N} \subseteq \mathcal{F}(\Phi^d)$. The converse inclusion holds by part 1. If $n < d$, then $\Phi^n(Q_{d-1}) = Q_{d-n-1} \neq Q_{d-1}$, so that $Q_{d-1} \notin \mathcal{F}(\Phi^n)$ and hence $\mathcal{F}(\Phi^n) \neq \mathcal{N}$, this proves 2.

Assume now that $\mathcal{F}(\Phi^m) \neq \mathbb{C}1$, then there is some nontrivial minimal projection $P \in \mathcal{F}(\Phi^m)$, which by part 1. must be of the form $P = Q_{j_1} + \cdots + Q_{j_k}$ for some (distinct) indices $0 \leq j_i \leq d-1$ and $k < d$. Let $P_i = \Phi^i(P)$, $i = 0, \ldots, m-1$, then all P_i are minimal projections in $\mathcal{F}(\Phi^m)$, so that for $i \neq j$, either $P_iP_j = 0$ or $P_i = P_j$. By rearranging the indices if necessary, we may assume that $P_0, \ldots, P_{l-1}$ are mutually orthogonal and all other P_i are contained in $\{P_0, \ldots, P_{l-1}\}$. Then $\sum_{i=0}^{m-1} P_i = \sum_{j=0}^{l-1} n_jP_j$ for some integers n_j. On the other hand, we have $\sum_{i=0}^{m-1} P_i \in \mathcal{F} = \mathbb{C}1$ since Φ is irreducible. It follows that $n_1 = \cdots = n_{l-1} =: n$ and

$$\sum_{i=0}^{m-1} P_i = nI = n\sum_{j=0}^{l-1} P_j.$$

This implies $m = nl$. Further, $\sum_{j=0}^{l-1} P_j = I$ implies that $d = kl$ by the definition of P_j. Note also that $l > 1$ since otherwise we would have $\Phi(P) = P$, which is not possible. Conversely, assume that $GCD(m, d) = l > 1$ and let $d = kl$. Put $P = Q_0 + Q_l + \cdots + Q_{(k-1)l}$, then clearly P is a projection, $P \neq 0, 1$ and $\Phi^l(P) = P$ and also $\Phi^m(P) = P$, since m is a multiple of l, so that $P \in \mathcal{F}(\Phi^m)$ and $\mathcal{F}(\Phi^m) \neq \mathbb{C}1$.

To prove the last statement, observe that $\Phi^d_{|k}$ is positive recurrent because the restriction of the Φ-invariant state will give a faithful $\Phi^d_{|k}$-invariant state. By contradiction, if $\Phi^d_{|k}$ is reducible, then we have a non trivial $\Phi^d_{|k}$-harmonic projection Q, $0 < Q < Q_k$, i.e. such that $\Phi^d(Q) = Q$. But then this Q is in $\mathcal{N}$ and, by positivity $\Phi^j(Q)$ is a projection bounded above by $\Phi^j(Q_k) = Q_{k-j}$. We deduce that $\sum_{j=0}^{d-1} \Phi^j(Q) < \sum_{j=0}^{d-1} Q_{k-j} = 1$ is a non trivial projection and a fixed point for Φ and this contradicts the irreducibility of Φ.

Similarly, for the period, we know that $\Phi^d_{|k}$ has finite period by Groh; we call its period d_k, with cyclic decomposition $R_0, \ldots R_{d_k-1}$, $R_0 + \cdots + R_{d_k-1} = Q_k$. R_0 is a fixed point for Φ^{dd_k}, so it belongs to $\mathcal{N}$ and $\Phi^j(R_0)$, $j = 0, \ldots d \cdot d_k - 1$, will give a cyclic decomposition for Φ. So $dd_k = d$ which implies $d_k = 1$ and $R_0 = Q_k$.

□

4 Reducible maps

Let Φ be a uncp map on $B(\mathcal{H})$, fixed throughout. We assume that Φ admits a faithful normal invariant state ρ. By Corollary 1, $\mathcal{N}$ is the range of a faithful normal conditional expectation $E_{\mathcal{N}}$. Therefore, $\mathcal{N}$ must be type I with discrete center, [19]. On the other hand, it is known [12] that the limit

$$\lim_n \frac{1}{n}\sum_{k=0}^{n-1}\Phi^k \tag{2}$$

exists in the point-ultraweak topology and gives a faithful normal conditional expectation $E_{\mathcal{F}}$ onto $\mathcal{F}$, satisfying

$$E_{\mathcal{F}}\circ\Phi = \Phi\circ E_{\mathcal{F}} = E_{\mathcal{F}}. \tag{3}$$

Hence $\mathcal{F}$ is an atomic von Neumann subalgebra of $\mathcal{N}$. In this section, we study the structure of the two algebras.

We first describe a general form of a faithful normal conditional expectation on $B(\mathcal{H})$.

Lemma 1. *Let $E : B(\mathcal{H}) \to B(\mathcal{H})$ be a faithful normal conditional expectation and let $\mathcal{R} = E(B(\mathcal{H}))$ be its range. Then*

(i) $\mathcal{R}$ is atomic, so that there is a direct sum decomposition $\mathcal{H} = \bigoplus_j \mathcal{H}_j$, Hilbert spaces $\mathcal{H}_j^L$, $\mathcal{H}_j^R$ and unitaries $U_j : \mathcal{H}_j \to \mathcal{H}_j^L \otimes \mathcal{H}_j^R$ such that

$$\mathcal{R} = \bigoplus_j U_j^*(B(\mathcal{H}_j^L)\otimes I_{\mathcal{H}_j^R})U_j;$$

(ii) the orthogonal projections P_j onto $\mathcal{H}_j$ are minimal central projections in $\mathcal{R}$ and

$$E(A) = \sum_j E(P_jAP_j);$$

(iii) identifying $P_jB(\mathcal{H})P_j$ with $B(\mathcal{H}_j)$, the restriction of E to $P_jB(\mathcal{H})P_j$ is determined by

$$E(U_j^*(A_j\otimes B_j)U_j) = U_j^*(A_j\otimes \mathrm{Tr}\,[\rho_jB_j]I_{\mathcal{H}_j^R})U_j,$$

where each $\rho_j \in \mathfrak{S}(\mathcal{H}_j^R)$ is a (fixed) faithful normal state;

(iv) a normal state $\omega \in \mathfrak{S}(\mathcal{H})$ is invariant under E if and only if

$$\omega = \oplus_j\lambda_jU_j^*(\omega_j^L\otimes\rho_j)U_j,$$

where ρ_j are as in (iii), $\{\lambda_j\}$ are probabilities and $\omega_j^L \in \mathfrak{S}(\mathcal{H}_j^L)$.

Proof. The range $\mathcal{R}$ is atomic by [19]. Let $\{P_j\}$ be the minimal central projections in $\mathcal{R}$ and let $\mathcal{H}_j = P_j\mathcal{H}$. Since $\mathcal{R}P_j$ is a type I factor acting on $\mathcal{H}_j$, there are Hilbert spaces $\mathcal{H}_j^L$, $\mathcal{H}_j^R$ and a unitary $U_j : \mathcal{H}_j \to \mathcal{H}_j^L\otimes\mathcal{H}_j^R$ such that

$$\mathcal{R}P_j = U_j^*(B(\mathcal{H}_j^L)\otimes I_{\mathcal{H}_j^R})U_j,$$

this proves (i). By the properties of conditional expectations,

$$E(P_jAP_k) = P_jE(A)P_k = P_jP_kE(A)$$

for any $A \in B(\mathcal{H})$, this proves (ii). It also follows that under the identification in (iii), $E(B(\mathcal{H}_j)) \subseteq B(\mathcal{H}_j)$ for all j and the restriction E_j of E is a faithful normal conditional expectation on $B(\mathcal{H}_j)$, with range $U_j^*(B(\mathcal{H}_j^L) \otimes I_{\mathcal{H}_j^R})U_j$. Let $A_j \in B(\mathcal{H}_j^L)$, $B_j \in B(\mathcal{H}_j^R)$, then we have

$$E(U_j^*(A_j \otimes B_j)U_j) = U_j^*(A_j \otimes I)U_j E(U_j^*(I \otimes B_j)U_j) = E(U_j^*(I \otimes B_j)U_j)U_j^*(A_j \otimes I)U_j, \quad (4)$$

it follows that $E(U_j^*(I \otimes B_j)U_j)$ commutes with all elements in $U_j^*(B(\mathcal{H}_j^L) \otimes I))U_j$, so that there is some $\rho_j(B_j) \in \mathbb{C}$ such that $E(U_j^*(I \otimes B_j)U_j) = \rho_j(B_j)P_j$. It is clear that $B_j \mapsto \rho_j(B_j)$ defines a normal state on $B(\mathcal{H}_j^R)$, which must be faithful since E is. This proves (iii).

Finally, let $\omega \in \mathfrak{S}(\mathcal{H})$. It is clear that if $\omega \circ E = \omega$, then we must have $\omega = \lambda_j \omega_j$ for some $\omega_j \in \mathfrak{S}(\mathcal{H}_j)$ and $\lambda_j = \operatorname{Tr} P_j \omega$. Let $\omega_j^L \in \mathfrak{S}(\mathcal{H}_j^L)$ be determined by $\omega_j^L(A_j) = \omega_j(U_j^*(A_j \otimes I)U_j)$. Then ω_j, and consequently also ω, is invariant under E if and only if for all $A_j \in B(\mathcal{H}_j^L)$ and $B_j \in B(\mathcal{H}_j^R)$,

$$\omega_j(U_j^*(A_j \otimes B_j)U_j) = \omega_j \circ E(U_j^*(A_j \otimes B_j)U_j) = \omega_j^L(A_j)\rho_j(B_j) = (\omega_j^L \otimes \rho_j)(A_j \otimes B_j).$$

□

Let us now turn to the algebras $\mathcal{F}$ and $\mathcal{N}$. We begin with the central projections. Let $\mathcal{Z}(\mathcal{F})$ and $\mathcal{Z}(\mathcal{N})$ denote the center of $\mathcal{F}$ and $\mathcal{N}$, and let $\mathcal{Z} = \mathcal{Z}(\mathcal{F}) \cap \mathcal{Z}(\mathcal{N})$. Clearly, $\mathcal{Z}$ is a discrete abelian von Neumann algebra. Let $\{Z_1, Z_2, \dots\}$ be minimal projections in $\mathcal{Z}$ and put $\mathcal{N}_i := Z_i \mathcal{N}$. Note that identifying $Z_i B(\mathcal{H}) Z_i$ with $B(Z_i \mathcal{H})$, we have $\Phi(B(Z_i\mathcal{H})) \subseteq B(Z_i\mathcal{H})$, so that $\Phi_i := \Phi|_{B(Z_i\mathcal{H})}$ is a uncp map on $B(Z_i\mathcal{H})$, with $\mathcal{N}(\Phi_i) = \mathcal{N}_i$.

Proposition 7. *For each i, there is some $d_i \in \mathbb{N}$ and minimal projections $Q_0^i, \dots, Q_{d_i-1}^i \in \mathcal{Z}(\mathcal{N})$ forming a cyclic resolution of identity for Φ_i. That is, $Z_i = \sum_{m=0}^{d_i-1} Q_m^i$ and*

$$\Phi(Q_m^i) = Q_{m \oplus_{d_i} 1}^i.$$

The number d_i will be called the period of Φ_i.

Remark 4. *Note that if Φ is irreducible, $\mathcal{Z}$ is trivial, so that there is a unique component, and the notion of a period is the same as in Definition 1. For reducible Φ, we will see later in Proposition 9 that d_i is the period of all irreducible restrictions of the component Φ_i.*

Proof. Let $Q_0^i, Q_1^i, \dots$ be minimal central projections in $\mathcal{N}_i$, then clearly all Q_m^i are minimal central projections in $\mathcal{N}$ and we have $\sum_m Q_m^i = Z_i$. Since the restriction of Φ_i to $\mathcal{N}_i$ is a *-automorphism, $\Phi(Q_m^i) = \Phi_i(Q_m^i)$ is a minimal central projection as well. Put

$$d_i := \inf\{m, \Phi^m(Q_0^i) = Q_0^i\},$$

then since Φ preserves the faithful state ρ, $d_i < \infty$. Assume that the projections are numbered so that

$$Q_m^i = \Phi^m(Q_0^i), \qquad m = 0, \dots, d_i - 1.$$

Put $Q^i := \sum_{m=0}^{d_i-1} Q_m^i$, then obviously $Q^i \in \mathcal{Z}(\mathcal{N})$ and $\Phi(Q^i) = Q^i$, so that $Q^i \in \mathcal{Z}$. Since also $Q^i \leq Z_i$ and Z_i is minimal in $\mathcal{Z}$, we must have $Q^i = Z_i$.

□

We now describe the action of Φ_i on one component $\mathcal{N}_i$. For simplicity, we drop the index i, this correspond to assuming that there is only one such component, so that $\mathcal{Z}$ is trivial. Let the period of Φ be d. In this case, the center of $\mathcal{N}$ has dimension d and is generated by the minimal cyclic projections $Q_0, \ldots, Q_{d-1}$.

Since $\mathcal{N}$ is the range of $E_{\mathcal{N}}$, we may use Lemma 1 to describe its structure. Let us denote $\mathcal{K}_m := Q_m\mathcal{H}$, then there are Hilbert spaces $\mathcal{K}_m^L, \mathcal{K}_m^R$, $m = 0, \ldots, d-1$ and unitaries $S_m : \mathcal{K}_m \to \mathcal{K}_m^L \otimes \mathcal{K}_m^R$ such that

$$\mathcal{N} = \bigoplus_{m=0}^{d-1} S_m^*(B(\mathcal{K}_m^L) \otimes I_m^R)S_m. \tag{5}$$

Here we put $I_m^R = I_{\mathcal{K}_m^R}$ to simplify notations, we will use a similar notation for $I_{\mathcal{K}_m^L}$. Let also $\rho_m \in \mathfrak{S}(\mathcal{K}_m^R)$ denote the states determining $E_{\mathcal{N}}$, as in Lemma 1 (iii).

Proposition 8. *Assume that $\mathcal{Z}$ is trivial and let the period of Φ be d. Let $\oplus = \oplus_d$ denote addition modulo d. Then there are*

(a) unitaries $T_m : \mathcal{K}_m^L \to \mathcal{K}_{m\oplus 1}^L$, $m = 0, \ldots, d-1$;

(b) uncp maps $\Xi_m : B(\mathcal{K}_m^R) \to B(\mathcal{K}_{m\oplus 1}^R)$, $m = 0, \ldots, d-1$;

such that for all m,

(i) $\rho_{m\oplus 1} \circ \Xi_m = \rho_m$;

(ii) $\Xi_{m\oplus(d-1)} \circ \cdots \circ \Xi_{m\oplus 1} \circ \Xi_m$ is irreducible and aperiodic;

(iii) the restriction $\Phi|_{B(\mathcal{K}_m)}$ is a uncp map $B(\mathcal{K}_m) \to B(\mathcal{K}_{m\oplus 1})$, determined as

$$\Phi(S_m^*(A_m \otimes B_m)S_m) = S_{m\oplus 1}^*(T_m A_m T_m^* \otimes \Xi_m(B_m))S_{m\oplus 1};$$

(iv) Φ has a Kraus representation $\Phi(A) = \sum_k V_k^* A V_k$, such that

$$V_k = \sum_m S_m^*(T_m^* \otimes L_{m,k})S_{m\oplus 1},$$

where $\Xi_m = \sum_k L_{m,k}^* \cdot L_{m,k}$ is a Kraus representation of Ξ_m.

Proof. Let $A_m \in B(\mathcal{K}_m^L)$. Since $\Phi(Q_m\mathcal{N}) = Q_{m\oplus 1}\mathcal{N}$, we have

$$\Phi(S_m^*(A_m \otimes I_m^R)S_m) = S_{m\oplus 1}^*(A'_m \otimes I_{m\oplus 1}^R)S_{m\oplus 1}$$

for some $A'_m \in B(\mathcal{K}_{m\oplus 1}^L)$ and the map $A_m \mapsto A'_m$ defines a *-isomorphism of $B(\mathcal{K}_m^L)$ onto $B(\mathcal{K}_{m\oplus 1}^L)$. Hence there is a unitary operator $T_m : \mathcal{K}_m^L \to \mathcal{K}_{m\oplus 1}^L$, such that $A'_m = T_m A_m T_m^*$. Moreover, by the multiplicativity properties of Φ on $\mathcal{N}$, we have $\Phi(Q_m A) = Q_{m\oplus 1}\Phi(A)$ for all $A \in B(\mathcal{H})$, and for all $B_m \in B(\mathcal{K}_m^R)$,

$$\begin{aligned}\Phi(S_m^*(A_m \otimes B_m)S_m) &= \Phi(S_m^*(A_m \otimes I_m^R)S_m)\Phi(S_m^*(I_m^L \otimes B_m)S_m)\\ &= \Phi(S_m^*(I_m^L \otimes B_m)S_m)\Phi(S_m^*(A_m \otimes I_m^R)S_m).\end{aligned}$$

It follows that $\Phi(S_m^*(I_m^L \otimes B_m)S_m)$ is an element in $B(\mathcal{K}_{m\oplus 1})$, commuting with all elements in $S_{m\oplus 1}^*(B(\mathcal{K}_{m\oplus 1}^L) \otimes I_{m\oplus 1}^R)S_{m\oplus 1}$, so that

$$\Phi(S_m^*(I_m^L \otimes B_m)S_m) = S_{m\oplus 1}^*(I_{m\oplus 1}^L \otimes B'_m)S_{m\oplus 1}$$

for some $B'_m \in B(\mathcal{K}^R_{m\oplus 1})$. It is clear that $B_m \mapsto B'_m$ defines a uncp map $\Xi_m : B(\mathcal{K}^R_m) \to B(\mathcal{K}^R_{m\oplus 1})$. Putting al together proves (iii).

To see (ii), let $\tilde{\Xi}_m$ be the given composition and let $R_m \in B(\mathcal{K}^R_m)$ be a projection that is fixed or periodic for $\tilde{\Xi}_m$. Then $S^*_m(I^L_m \otimes R_m)S_m$ is in $\mathcal{N}$, so that R_m must be trivial. Further, note that since $E_{\mathcal{N}} \in \mathbf{S}$, it must commute with Φ. For $B_m \in B(\mathcal{K}^R_m)$, we have by Lemma 1

$$\Phi \circ E_{\mathcal{N}}(S^*_m(I^L_m \otimes B_m)S_m) = \rho_m(B_m)\Phi(Q_m) = \rho_m(B_m)Q_{m\oplus 1}$$

and

$$E_{\mathcal{N}} \circ \Phi(S^*_m(I^L_m \otimes B_m)S_m) = E_{\mathcal{N}}(S^*_{m\oplus 1}(I^L_{m\oplus 1} \otimes \Xi_m(B_m))S_{m\oplus 1}) = \rho_{m\oplus 1}(\Xi_m(B_m))Q_{m\oplus 1},$$

so that (i) holds.

Finally, let $\Phi = \sum_k V^*_k \cdot V_k$ be any Kraus representation of Φ. Then we have

$$\Phi(A) = \sum_{m,n=0}^{d-1} \Phi(Q_m A Q_n) = \sum_{m,n=0}^{d-1} Q_{m\oplus 1}\Phi(Q_m A Q_n)Q_{n\oplus 1},$$

so that we may assume that each V_k has the form $V_k = \sum_m V_{k,m}$, with $V_{k,m} = Q_m V_k Q_{m\oplus 1}$. Moreover, for each m, $\sum_k V^*_{k,m} \cdot V_{k,m}$ is a Kraus representation of the restriction $\Phi|_{B(\mathcal{K}_m)}$.

Let $\Xi_m = \sum_l K^*_{m,l} \cdot K_{m,l}$ be a minimal Kraus representation. It follows from (iii) that

$$\Phi|_{B(\mathcal{K}_m)} = \sum_l S^*_{m\oplus 1}(T_m \otimes K^*_{m,l})S_m \cdot S^*_m(T^*_m \otimes K_{m,l})S_{m\oplus 1}$$

is another Kraus representation of $\Phi|_{B(\mathcal{K}_m)}$, hence there are some $\{\eta^j_{k,l}\}$ such that $\sum_i \eta^j_{i,k}\bar{\eta}^j_{i,l} = \delta_{k,l}$ and

$$V_{k,m} = \sum_l \eta^j_{k,l} S^*_m(T^*_m \otimes K_{m,l})S_{m\oplus 1} = S^*_m(T^*_m \otimes L_{m,k})S_{m\oplus 1},$$

where $L_{m,k} := \sum_l \eta^m_{k,l} K_{m,l}$, this proves (iv).

□

Note that by identifying

$$\mathcal{H} = \bigoplus_m \mathcal{K}_m \simeq \sum_m \mathcal{K}_m \otimes |m\rangle$$

and

$$\mathcal{K} := \bigoplus_m \mathcal{K}^L_m \otimes \mathcal{K}^R_m \simeq \sum_m \mathcal{K}^L_m \otimes \mathcal{K}^R_m \otimes |m\rangle,$$

(5) can be written as

$$\mathcal{N} = S^*(\sum_{m=0}^{d-1} B(\mathcal{K}^L_m) \otimes I^R_m \otimes |m\rangle\langle m|)S,$$

where $S : \mathcal{H} \to \mathcal{K}$ is a unitary given as $S = \sum_m S_m \otimes |m\rangle\langle m|$. We will also use the notation

$$\mathcal{K}^R := \bigoplus_m \mathcal{K}^R_m \simeq \sum_m \mathcal{K}^R_m \otimes |m\rangle$$

and put $I^R := I_{\mathcal{K}^R}$. We are now ready to describe the subalgebra $\mathcal{F}$. In the following proposition, we keep the notations of Prop. 8.

Proposition 9. *Let us denote*

$$\tilde{T}_m : \mathcal{K}_0^L \to \mathcal{K}_{m\oplus 1}^L, \quad \tilde{T}_m := T_m \dots T_0,\ m = 0, \dots, d-1; \quad \tilde{T}_{-1} := I_0^L$$

and let $T : \mathcal{K}_0^L \otimes \mathcal{K}^R \to \mathcal{K}$ *be the unitary defined as*

$$T = \sum_{m=0}^{d-1} \tilde{T}_{m-1} \otimes I_m^R \otimes |m\rangle\langle m|.$$

(i) The operator $\tilde{T}_{d-1} \in \mathcal{U}(\mathcal{K}_0^L)$ *has a discrete spectrum. Let* R_j *be its minimal spectral projections and let* $\mathcal{L}_j := R_j \mathcal{K}_0^L$*, then*

$$\mathcal{F} = S^* T (\bigoplus_j B(\mathcal{L}_j) \otimes I^R) T^* S;$$

(ii) Let $\sigma_j \in \mathfrak{S}(\mathcal{K}^R)$ *be the faithful normal states corresponding to* $E_{\mathcal{F}}$ *as in Lemma 1 (iii) and (iv). Then*

$$\sigma_j \equiv \sigma := \frac{1}{d} \sum_{m=0}^{d-1} \rho_m \otimes |m\rangle\langle m|, \qquad \forall j;$$

(iii) Invariant states $\xi \in \mathfrak{S}(\mathcal{H})$ *for* Φ *are precisely those of the form*

$$\xi = S^* T \,(\omega \otimes \sigma)\, T^* S,$$

where $\omega = \sum_j \lambda_j \omega_j \otimes |j\rangle\langle j|$ *for some probabilities* $\{\lambda_j\}$ *and states* $\omega_j \in \mathfrak{S}(\mathcal{L}_j)$*.*

(iv) Let $P_j := S^* T (R_j \otimes I^R) T^* S$ *be the minimal central projections in* $\mathcal{F}$*. The restrictions* $\Phi|_{B(P_j\mathcal{H})}$ *have the form*

$$\Phi|_{B(P_j\mathcal{H})}(S^*T(A_j \otimes B)T^*S) = S^*T(A_j \otimes \Psi_j(B))T^*S, \quad A_j \in B(\mathcal{L}_j), B \in B(\mathcal{K}^R),$$

where Ψ_j *are irreducible uncp maps on* $B(\mathcal{K}^R)$*. Moreover, all* Ψ_j *coincide on block-diagonal elements of the form* $\sum_m B_{mm} \otimes |m\rangle\langle m|$ *and we have*

$$\Psi_j(\sum_m B_{mm} \otimes |m\rangle\langle m|) = \sum_m \Xi_m(B_{mm}) \otimes |m \oplus 1\rangle\langle m \oplus 1|.$$

In particular, for all j*,* Ψ_j *has period* d*,* $\mathcal{N}(\Psi_j) = span\{I_m^R \otimes |m\rangle\langle m|, m = 0, \dots, d-1\}$ *and* σ *of (ii) is the unique invariant state.*

Proof. Since $\mathcal{F} \subseteq \mathcal{N}$, we may apply Proposition 8. It can be easily checked that an element of $\mathcal{N}$ is in $\mathcal{F}$ if and only if it is of the form

$$S^*T(A \otimes I^R)T^*S$$

with $A \in \mathcal{A} := \{\tilde{T}_{d-1}\}' \cap B(\mathcal{H}_0^L)$. Note that the commutant $\mathcal{A}' := \{\tilde{T}_{d-1}\}'' \cap B(\mathcal{H}_0^L) = \mathcal{Z}(\mathcal{A})$ is abelian. Further, we have $\mathcal{F} \simeq \mathcal{A}$ and since $\mathcal{F}$ is atomic, $\mathcal{A}$ must be such as well, so that $\{\tilde{T}_{d-1}\}'' \cap B(\mathcal{H}_0^L)$ must be discrete. This proves (i).

By Lemma 1, there are some states $\sigma_j \in \mathfrak{S}(\mathcal{K}^R)$ such that

$$E_{\mathcal{F}}(S^*T(R_j \otimes B)T^*S) = \sigma_j(B)P_j, \tag{6}$$

where $B \in B(\mathcal{K}^R)$ and $P_j := S^*T(R_j \otimes I^R)T^*S$ are the minimal central projections in $\mathcal{F}$. Moreover, since $E_{\mathcal{F}}$ is given by (2) and satisfies (3), we see that a state ξ is invariant for Φ if and only if it is invariant for $E_{\mathcal{F}}$. Consequently, by Lemma 1 (iv), any state of the form $\psi = T^*S(\omega_j \otimes \sigma_j)S^*T$ with $\omega_j \in \mathfrak{S}(\mathcal{L}_j)$ is an invariant state for Φ. It follows that for any $m = 0, \dots d-1$,

$$\begin{aligned}\sigma_j(I_m \otimes |m\rangle\langle m|) &= \psi(S^*T(R_j \otimes I_m \otimes |m\rangle\langle m|)T^*S) = \psi \circ \Phi(S^*T(R_j \otimes I_m \otimes |m\rangle\langle m|)T^*S)\\ &= \psi(S^*T(R_j \otimes I_{m\oplus 1} \otimes |m \oplus 1\rangle\langle m \oplus 1|)T^*S) = \sigma_j(I_{m\oplus 1} \otimes |m \oplus 1\rangle\langle m \oplus 1|)\end{aligned}$$

so that $\sigma_j(I_m \otimes |m\rangle\langle m|) = 1/d$. Let now $B = \sum_{m,n} B_{mn} \otimes |m\rangle\langle n| \in B(\mathcal{K}^R)$. Since $E_{\mathcal{N}} \in \mathbf{S}$, we obtain from (3) that also $E_{\mathcal{F}} \circ E_{\mathcal{N}} = E_{\mathcal{N}} \circ E_{\mathcal{F}} = E_{\mathcal{F}}$. Using Lemma 1 (ii) for $E_{\mathcal{N}}$, we get

$$\begin{aligned}E_{\mathcal{F}}(S^*T(R_j \otimes B)T^*S) &= \sum_m E_{\mathcal{F}} \circ E_{\mathcal{N}}(Q_m S^*T(R_j \otimes B)T^*SQ_m)\\ &= \sum_m E_{\mathcal{F}} \circ E_{\mathcal{N}}(S_m^*(\tilde{T}_{m-1}R_j\tilde{T}_{m-1}^* \otimes B_{mm})S_m)\\ &= \sum_m \rho_m(B_{mm})E_{\mathcal{F}}(S_m^*(\tilde{T}_{m-1}R_j\tilde{T}_{m-1}^* \otimes I_m^R)S_m)\\ &= \sum_m \rho_m(B_{mm})E_{\mathcal{F}}(S^*T(R_j \otimes I_m^R \otimes |m\rangle\langle m|)T^*S) = \frac{1}{d}\sum_m \rho_m(B_{mm})P_j.\end{aligned}$$

This and (6) proves (ii) and (iii).

Finally, we prove (iv). We see by the multiplicativity properties of Φ on $\mathcal{N}$ that $\Phi(B(P_j\mathcal{H})) \subseteq B(P_j\mathcal{H})$ and that the restrictions have the given form with some uncp map Ψ_j on $B(\mathcal{K}^R)$. Since any fixed point of Ψ_j is related to a fixed point of Φ, we can see that it must be trivial, so that Ψ_j are irreducible. For any $B_m \in B(\mathcal{K}_m^R)$, we have by Proposition 8,

$$\begin{aligned}\Phi(S^*T(R_j \otimes B_m \otimes |m\rangle\langle m|)T^*S &= \Phi(S_m^*(\tilde{T}_{m-1}R_j\tilde{T}_{m-1}^* \otimes B_m)S_m)\\ &= S_{m\oplus 1}^*(\tilde{T}_mR_j\tilde{T}_m^* \otimes \Xi_m(B_m))S_{m\oplus 1})\\ &= S^*T(R_j \otimes \Xi_m(B_m) \otimes |m \oplus 1\rangle\langle m \oplus 1|)T^*S.\end{aligned}$$

It follows that $I_m^R \otimes |m\rangle\langle m| \in \mathcal{N}(\Psi_j)$ for all m and j. Hence any minimal projection in $\mathcal{N}(\Psi_j)$ must be of the form $Q \otimes |m\rangle\langle m|$ for some $m = 0, \dots, d-1$ and some projection $Q \in B(\mathcal{K}_m^R)$. But then it easily follows that $I_m \otimes Q$ is is in $\mathcal{N}$, so that we must have $Q = I_m^R$. Further, observe that from $\Phi(Q_mAQ_n) = Q_{m\oplus 1}\Phi(A)Q_{n\oplus 1}$ we get

$$\Psi_j(\sum_{m,n} B_{mn} \otimes |m\rangle\langle n|) = \sum_{m,n} \Psi_{j,mn}(B_{mn}) \otimes |m \oplus 1\rangle\langle n \oplus 1|,$$

where $\Psi_{j,mm} = \Xi_m$ for all j and m. Hence by Proposition 8 (i) and (ii)

$$\sigma(\Psi_j(\sum_{m,n} B_{mn} \otimes |m\rangle\langle n|)) = \frac{1}{d}\sum_m \rho_{m\oplus 1}(\Xi_m(B_{mm})) = \frac{1}{d}\sum_m \rho_m(B_{mm}) = \sigma(\sum_{m,n} B_{mn} \otimes |m\rangle\langle n|).$$

□

5 Open quantum random walks

In this section we discuss an important example of uncp maps.

Let $\mathcal{H} = \oplus_{i\in V}\mathfrak{h}_i$, where V is a countable set of vertices and $\mathfrak{h}_i$ are separable Hilbert spaces. Note that we may express $\mathcal{H}$ as $\mathcal{H} = \sum_{i\in V}\mathfrak{h}_i \otimes |i\rangle$. An open quantum random walk (OQRW) [2]... is a completely positive trace preserving map $\mathfrak{M}$ on the space $\mathcal{I}_1(\mathcal{H})$ of trace-class operators, of the form

$$\mathfrak{M} : \rho \mapsto \sum_{i,j} V_{i,j}\rho V_{i,j}^*,$$

where $V_{i,j} = L_{i,j} \otimes |i\rangle\langle j|$ and $L_{i,j}$ are bounded operators $\mathfrak{h}_j \to \mathfrak{h}_i$ satisfying

$$\sum_{i\in V} L_{i,j}^* L_{i,j} = I_j, \qquad \forall j \in V. \tag{7}$$

Put $\Phi = \mathfrak{M}^*$, then Φ is a uncp map. Note that any operator $A \in B(\mathcal{H})$ can be written as

$$A = \sum_{i,j\in V} A_{i,j} \otimes |i\rangle\langle j|,$$

where $A_{i,j}$ is an operator $\mathfrak{h}_j \to \mathfrak{h}_i$. We next investigate the multiplicative domain $\mathcal{M}$ and the decoherence-free subalgebra $\mathcal{N}$ of Φ.

Proposition 10. *Let Φ be an OQRW. Then $A \in \mathcal{M}$ if and only if*

$$A_{i,i} L_{i,j} L_{k,j}^* = L_{i,j} L_{k,j}^* A_{k,k}, \quad \forall i, j, k \tag{8}$$

and

$$A_{l,i} L_{i,j} = 0 = L_{l,j}^* A_{l,i}, \quad \forall i, j, l \in V, \ i \neq l \tag{9}$$

Proof. It is easy to see from Proposition 1 that $A \in \mathcal{M}$ if and only if A commutes with all operators of the form $L_{i,j} L_{k,j}^* \otimes |i\rangle\langle k|$, $i, j, k \in V$. This is equivalent to (8), together with

$$A_{l,i} L_{i,j} L_{k,j}^* = 0 = L_{k,j} L_{l,j}^* A_{l,i}, \quad \forall i, j, k, l \in V, \ l \neq i \tag{10}$$

It is clear that (9) implies (10). For the converse, multiply the first equality of (10) by $L_{k,j}$ from the right and sum over $k \in V$, then (7) implies the first equality of (9). The second equality is proved similarly.

□

To obtain $\mathcal{N}$, we invoke the notation of [7] of the sets $\mathcal{P}_n(i, j)$ of paths from i to j of length n and operators L_π for $\pi \in \mathcal{P}_n(i, j)$. Namely, if $\pi = (i_0 = i, i_1, \ldots, i_n = j)$, then

$$L_\pi = L_{i_n,i_{n-1}} L_{i_{n-1},i_{n-2}} \ldots L_{i_1,i_0}$$

Since the Kraus operators of Φ^n are operators of the form $L_\pi \otimes |i\rangle\langle j|$ for $\pi \in \mathcal{P}_n(j, i)$, the next result can be proved exactly as the previous one.

Proposition 11. *$A \in \mathcal{N}$ if and only if for all $i, j, k, l \in V$, $l \neq i$ and $n \in \mathbb{N}$,*

$$A_{i,i} L_\pi L_{\pi'}^* = L_\pi L_{\pi'}^* A_{k,k}, \quad \forall \pi \in \mathcal{P}_n(j, i), \ \pi' \in \mathcal{P}_n(j, k) \tag{11}$$

and

$$A_{l,i} L_{i,j} = 0 = L_{l,j}^* A_{l,i}. \tag{12}$$

Due to the characterization in the previous proposition, we can deduce a decomposition of the decoherence-free algebra in block diagonal and block off-diagonal operators.

Corollary 3. *$\mathcal{N} = \mathcal{N}_D \oplus \mathcal{N}_{OD}$ where:*

$$\begin{aligned}
\mathcal{N}_D &= \{A = \sum_{i\in V} A_{ii} \otimes |i\rangle\langle i|, A \in \mathcal{N}\} \\
&= \{A = \sum_{i\in V} A_{ii} \otimes |i\rangle\langle i| \,:\, A_{i,i} L_\pi L_{\pi'}^* = L_\pi L_{\pi'}^* A_{k,k}, \forall i,k \in V, \forall (\pi,\pi') \in \cup_{j,n}(\mathcal{P}_n(j,i) \times \mathcal{P}_n(j,k))\} \\
\mathcal{N}_{OD} &= \{A = \sum_{i\neq j\in V} A_{ij} \otimes |i\rangle\langle j|, A \in \mathcal{N}\} \\
&= \{A = \sum_{i\neq j\in V} A_{ij} \otimes |i\rangle\langle j| \,:\, A_{i,j} L_{j,l} = 0 = L_{i,l}^* A_{i,j},\ \forall i \neq j, l \in V\}.
\end{aligned}$$

When $\mathcal{N}_{OD}$ is non-trivial, it means that $\mathcal{N} \cap \ker \Phi$ is non trivial and, since it is a von Neumann algebra, then it will contain some diagonal projections. Indeed, if x is in $\mathcal{N} \cap \ker \Phi \setminus \{0\}$, then x^*x is a positive element in $\mathcal{N} \cap \ker \Phi \setminus \{0\}$ and its block-diagonal part $(x^*x)_D := \sum_{i\in V} (x^*x)_{ii} \otimes |i\rangle\langle i|$ is also a positive operator in

$$\mathcal{N} \cap \ker \Phi \cap \{\text{block diagonal operators}\} \setminus \{0\} = \mathcal{N}_D \cap \ker \Phi \setminus \{0\}$$

which is also a von Neumann algebra and so it has to contain a non trivial projection. Summing up, we deduce

$$\mathcal{N}_{OD} \neq \{0\} \quad \Rightarrow \quad \mathcal{N} \cap \ker \Phi \neq \{0\} \quad \Leftrightarrow \quad \mathcal{N}_D \cap \ker \Phi \neq \{0\}.$$

A projection P is in $\mathcal{N}_D \cap \ker \Phi$ if and only if $P = \sum_{i\in V} P_i \otimes |i\rangle\langle i|$ with $P_i L_{ij} = 0$ for all i and j. Such a P exists and is non trivial iff there exists an index i such that $W_i := \cap_j \text{Range}(L_{ij})^\perp \neq \{0\}$. Then we take P_i the projection on W_i. Of course, this cannot happen if Φ admits a faithful normal invariant state, since then Φ is faithful and there can be no projections in $\ker \Phi$.

Example 1. *We consider an OQRW with $V = \{0,1\}$ and $\mathfrak{h}_0 = \mathfrak{h}_1 = \mathbb{C}^2$. Let $L_{00} = L_{11} = 0$, $L_{01} = U$, $L_{10} = V$ with U, V unitaries on $\mathbb{C}^2$. For $\rho = \sum_{k,l=0,1} \rho_{k,l} \otimes |k\rangle\langle l|$, we can explicitly write the action of Φ as*

$$\Phi(\rho) = (U^*\rho_{11}U) \otimes |0\rangle\langle 0| + (V^*\rho_{00}V) \otimes |1\rangle\langle 1|.$$

It is easy to see that the trace state is invariant for Φ, so that we have a positive recurrent OQRW. Moreover $\mathcal{N}$ is block diagonal due to Corollary 3. Using previous results, we can state that $\mathcal{N}$ consists of all block diagonal operators, i.e.

$$\mathcal{N} = \{A \otimes |0\rangle\langle 0| + B \otimes |1\rangle\langle 1| \,,\ A, B \in M_2(\mathbb{C})\}$$

with minimal central projections $I \otimes |0\rangle\langle 0|, I \otimes |1\rangle\langle 1|$. While the fixed points' algebra $\mathcal{F}$ is given by

$$\mathcal{F} = \{A_0 \otimes |0\rangle\langle 0| + U^*A_0U \otimes |1\rangle\langle 1| \,,\ A_0 \in M_2(\mathbb{C})\ s.t.\ [A_0, UV]\}$$

Now, if UV is a multiple of the identity operator, $\mathcal{F} = \mathcal{N}$ and this is not particularly interesting for us here; while, in the other cases, $\mathcal{F}$ is a commutative algebra strictly included in $\mathcal{N}$, we can denote by w_0, w_1 the norm 1 eigenvectors of UV (unique up to a multiplication) and the minimal central projections for $\mathcal{F}$ will be

$$|w_j\rangle\langle w_j| \otimes |0\rangle\langle 0| + |U^*w_j\rangle\langle U^*w_j| \otimes |1\rangle\langle 1|, \quad j = 0, 1.$$

This gives a very simple example of a case where the minimal central projections of $\mathcal{F}$ and $\mathcal{N}$ are different.

5.1 An example with generalized Pauli operators

Let $V = \{0, 1\}$ and $\mathfrak{h}_0 = \mathfrak{h}_1 = \mathfrak{h}$. Let $L_{00} = L_{11} = \sqrt{\alpha}U_0$, $L_{01} = L_{10} = \sqrt{1-\alpha}U_1$, with $\alpha \in (0, 1)$ and U_0, U_1 unitaries on $\mathfrak{h}$. Explicitly, Φ acts as

$$\Phi(A) = [\alpha(U_0^* A_{00} U_0) + (1-\alpha)U_1^* A_{11} U_1] \otimes |0\rangle\langle 0| + [(1-\alpha)U_1^* A_{00} U_1 + \alpha U_0^* A_{11} U_0] \otimes |1\rangle\langle 1|$$

for $A = \sum_{k,l} A_{k,l} \otimes |k\rangle\langle l| \in \mathfrak{h} \otimes \mathbb{C}^2$. Assume $d := \dim(\mathfrak{h}) < \infty$, then $(2d)^{-1} I_{\mathfrak{h}} \otimes I_2$ is a faithful invariant state for Φ. We next investigate the fixed points and decoherence free subalgebra in the case when U_0 and U_1 are generalized Pauli operators.

Let $\{|j\rangle,\ j = 0, \ldots, d-1\}$ denote an ONB in $\mathfrak{h}$ and let $\oplus$ be addition modulo d. Put $\omega = e^{i2\pi/d}$ and define the operators Z and X as

$$Z|j\rangle = \omega^j |j\rangle$$
$$X|j\rangle = |j \oplus 1\rangle$$

Then Z and X are unitaries satisfying the commutation relation

$$ZX = \omega XZ.$$

Let us also denote

$$W(p) = Z^p X^{-p}, \qquad p \in \mathbb{Z},$$

then $W(p)$ satisfy the relations

$$W(p)W(q) = W(q)W(p) = \omega^{pq} W(p+q). \tag{13}$$

Let Φ be an OQRW as above, with $U_0 = Z$, $U_1 = X$. We first find the fixed point subalgebra of Φ, this can be done using Proposition 3. We see that

$$\mathcal{F} = \{Z \otimes |0\rangle\langle 0|,\ Z \otimes |1\rangle\langle 1|,\ X \otimes |0\rangle\langle 1|, X \otimes |1\rangle\langle 0|\}'$$

and from this, we get

$$\mathcal{F} = \{\begin{pmatrix} A & 0 \\ 0 & XAX^*, \end{pmatrix},\ A \in \{Z, X^2\}'\}. \tag{14}$$

The condition $A \in \{Z, X^2\}'$ implies that A is diagonal in the basis $\{|j\rangle\}$ and

$$A = X^2 A (X^*)^2 \implies \sum_j a_j |j\rangle\langle j| = \sum_j a_j |j \oplus 2\rangle\langle j \oplus 2|,$$

so that $a_j = a_{j\oplus 2}$ for $j = 0, \ldots, d-1$.

Assume now that d is odd. Then it follows that $a_j = a_0$ for all j, so that $\mathcal{F}$ is trivial. Hence, in this case, Φ is irreducible. Put $W = W(1) = ZX^*$, then

$$Z^* W Z = X^* W X = \omega W$$

It follows that $\Phi(W \otimes I_2) = \omega(W \otimes I_2)$, so $\tilde{W} := W \otimes I_2$ is an eigenvector related to the peripheral eigenvalue ω. The eigenvalues of W are ω^k, $k = 0, \ldots, d-1$, each with an eigenvector $|x_k\rangle$. Hence the period of Φ is d and we have the cyclic decomposition

$$\{Q_m = |x_m\rangle\langle x_m| \otimes I_2,\ m = 0, \ldots, d-1\}.$$

By the results of Section 3.1, $\mathcal{N}$ is spanned by $\{Q_0, \dots, Q_{d-1}\}$.

We next turn to the more interesting case when d is even. Put $q = d/2$. Then we see that (14) holds, with $A = a_+P_+ + a_-P_-$, where $a_+, a_- \in \mathbb{C}$ and

$$P_+ = \sum_{k=0}^{q-1} |2k\rangle\langle 2k|, \qquad P_- = \sum_{k=0}^{q-1} |2k+1\rangle\langle 2k+1|.$$

So $\mathcal{F}$ is isomorphic to the abelian algebra spanned by these two projections. Note that we have $XP_+X^* = P_-$, $XP_-X^* = P_+$, so that we may write

$$\mathcal{F} = \mathrm{span}\{\tilde{P}_+ := \begin{pmatrix} P_+ & 0 \\ 0 & P_- \end{pmatrix}, \tilde{P}_- := \begin{pmatrix} P_- & 0 \\ 0 & P_+ \end{pmatrix}\}.$$

Let us compute $\mathcal{N}$ using Proposition 11. Note first that for $\pi \in \mathcal{P}_n(i,j)$,

$$L_\pi = xZ^{n-l}X^l,$$

where $x \neq 0$ is some constant and $l \in \mathbb{N}$ is even if and only if $i = j$. It follows that if $\pi \in \mathcal{P}_n(j,i)$, $\pi' \in \mathcal{P}_n(j,k)$, we have

$$L_\pi L_{\pi'}^* = yZ^pX^{-p} = yW(p),$$

where $0 \neq y \in \mathbb{C}$ and $|p|$ is even iff $k = i$. Since all $L_{i,j}$ are (nonzero) multiples of unitary operators, we must have $\mathcal{N}_{OD} = \{0\}$. From the conditions on the diagonal blocks, we obtain that $A_{i,i}$ must commute with $W(p)$ for all even $|p|$ and $A_{i,i} = W(p)^*A_{j,j}W(p)$ for all $|p|$ odd if $i \neq j$. Using (13), we obtain that

$$\mathcal{N} = \{\begin{pmatrix} A & 0 \\ 0 & WAW^* \end{pmatrix}, \ A \in \{W(2)\}'\}.$$

It follows that $\mathcal{N}$ is isomorphic to the algebra $\{W(2)\}'$. One can see by (13) and $d = 2q$ that $W(2)^q = I$, so that the eigenvalues of $W(2)$ are the q-th roots of unity, that is, $\mu_m = \omega^{2m}$, $m = 0, \dots, q-1$. Let us denote

$$|m,+\rangle := \frac{1}{\sqrt{q}} \sum_{l=0}^{q-1} \omega^{2l(m-l+1)}|2l\rangle, \qquad |m,-\rangle := \frac{1}{\sqrt{q}} \sum_{l=0}^{q-1} \omega^{2l(m-l)}|2l \oplus 1\rangle$$

then one can check that

$$Q_m := |m,+\rangle\langle m,+| + |m,-\rangle\langle m,-|$$

is the eigenprojection corresponding to the eigenvalue μ_m. Since W commutes with $W(2)$ by (13), we have $WQ_mW^* = Q_m$, so that the center of $\mathcal{N}$ is spanned by the projections

$$\tilde{Q}_m := Q_m \otimes I_2, \qquad m = 0, \dots, q-1.$$

Further, it is easily checked that for $m = 0, \dots, q-1$, we have

$$Z|m,+\rangle = |m \oplus_q 1, +\rangle, \qquad Z|m,-\rangle = \omega|m \oplus_q 1, -\rangle$$

and

$$X|m,+\rangle = |m \oplus_q 1, -\rangle, \qquad X|m,-\rangle = \bar{\omega}^{2(m+1)}|m \oplus_q 1, +\rangle.$$

Since the action of Φ on elements of $\mathcal{N}$ has the form

$$\Phi\begin{pmatrix} A & 0 \\ 0 & WAW^* \end{pmatrix} = \begin{pmatrix} Z^*AZ & 0 \\ 0 & X^*AX \end{pmatrix},$$

we obtain $\Phi(\tilde{Q}_m) = \tilde{Q}_{m\ominus_q 1}$. It follows that there is a unique cycle of length q and consequently only one component $\mathcal{N}_{[1]} = \mathcal{N}$, with period q. Note that here the order of the projections in the cyclic decomposition is reversed.

We will identify the objects described in Section 4 for this special case. We have $\mathcal{K}_m^L = Q_m\mathfrak{h}$, $\mathcal{K}_m^R = \mathbb{C}^2$ and $\mathcal{K}^R = \sum_m \mathbb{C}^2 \otimes |m\rangle\langle m| \simeq \mathfrak{h}$. Put $S_m = I_m^L \otimes |0\rangle\langle 0| + W^*|_{\mathcal{K}_m^L} \otimes |1\rangle\langle 1|$, $m = 0, \dots, q-1$, then we have

$$\mathcal{N} = \oplus_m S_m^*(B(\mathcal{K}_m^L) \otimes I_2)S_m.$$

Let us compute the states ρ_m and maps Ξ_m defined in Proposition 8. Let $\Delta, \bar{\Delta} : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ be given by

$$\Delta\begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} = \begin{pmatrix} b_{00} & 0 \\ 0 & b_{11} \end{pmatrix}, \quad \bar{\Delta}\begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} = \begin{pmatrix} b_{11} & 0 \\ 0 & b_{00} \end{pmatrix}.$$

It is easily checked that for each m, the map $\Xi_m : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ is defined as

$$\Xi_m = \Xi := \alpha\Delta + (1-\alpha)\bar{\Delta}.$$

It follows by Proposition 8 (i) that the states ρ_m must all be equal to the unique invariant state $\rho = \frac{1}{2}I_2$ of Ξ.

Let us now turn to Proposition 9. The unitaries $T_m : \mathcal{K}_m^L \to \mathcal{K}_{m\ominus_q 1}^L$ are given by the restrictions $T_m = Z^*|_{\mathcal{K}_m^L}$ and hence $\tilde{T}_{m-1} = Z^{-m}|_{\mathcal{K}_0^L}$, $m = 0, \dots, q-1$. In particular, the unitary $\tilde{T}_{q-1}$ has two eigenvalues ± 1, with eigenvectors $|0, \pm\rangle$, so that

$$\tilde{T}_{q-1} = |0,+\rangle\langle 0,+| - |0,-\rangle\langle 0,-|.$$

The subalgebra $\{\tilde{T}_{q-1}\}' \cap B(\mathcal{K}_0^L)$ of Proposition 9 is the abelian subalgebra spanned by the projections $|0,\pm\rangle\langle 0,\pm|$. Note that we have

$$\sum_{m=0}^{q-1} \tilde{T}_{m-1}|0,\pm\rangle\langle 0,\pm|\tilde{T}_{m-1}^* = \sum_{m=0}^{q-1} Z^{-m}|0,\pm\rangle\langle 0,\pm|Z^m = \sum_{m=0}^{q-1} |m,\pm\rangle\langle m,\pm| = P_\pm,$$

so that

$$S^*T(|0,\pm\rangle\langle 0,\pm| \otimes I^R)T^*S = \tilde{P}_\pm,$$

which corresponds to Prop. 9 (i). For $s \in [0,1]$, put $\omega_s := s|0,+\rangle\langle 0,+| + (1-s)|0,-\rangle\langle 0,-|$, then we can see from Prop. 9 (ii) and (iii) that the invariant states of Φ are precisely those of the form

$$\xi_s := S^*T(\omega_s \otimes \frac{1}{d}I_d)T^*S = \frac{1}{d}(s\tilde{P}_+ + (1-s)\tilde{P}_-).$$

Finally, let $\Psi_\pm$ be the irreducible channels on $B(\mathcal{K}^R)$ corresponding to the restrictions of Φ by the projections $\tilde{P}_\pm$ as in Prop. 9 (iv). Let X_q, Z_q be the generalized Pauli operators on the q-dimensional Hilbert space with standard basis $\{|m\rangle\}$. One can check that we have

$$\Psi_+ = (\alpha\Delta + (1-\alpha)\bar{\Delta}) \otimes (X_q \cdot X_q^*) = \Xi \otimes (X_q \cdot X_q^*)$$

and

$$\Psi_- = \alpha\Delta \otimes (X_q \cdot X_q^*) + (1-\alpha)\bar{\Delta} \otimes (X_q Z_q^* \cdot Z_q X_q^*).$$

5.2 Homogeneous OQRWs

We could also consider the special case of homogeneous nearest neighbor OQWs on $\mathbb{Z}$, $\mathbb{Z}^d$, $\mathbb{Z}_d$ with a finite dimensional local space $\mathfrak{h} = \mathfrak{h}_i$ the homogeneous case, V is a group (I shall concentrate on the cases mentioned above), $\mathfrak{h} = \mathfrak{h}_i$ for all i and the transition operators are translation invariant, i.e. $L_{ij} = L_{i+n,j+n} =: L_n$ for any i, j, n.

We can define the local operator $\mathcal{L}$, acting on $L^1(\mathfrak{h})$

$$\mathcal{L}(\rho) = \sum_n L_n \rho L_n^*$$

This $\mathcal{L}$ has at least one invariant state ρ^{inv}.

For homogeneous OQWs, we have an invariant faithful state only in the case the group is finite (i.e. only $\mathbb{Z}_d$ among the ones mentioned above). In other cases, we shall have an invariant weight ω, that we can define on positive operators in the following way

$$\omega(\sum x_{ij} \otimes |i\rangle\langle j|) = \sum_j \mathrm{Tr}\,(\rho^{\text{inv}} x_{jj}).$$

ω is faithful iff ρ^{inv} is faithful.

When this happens, by Proposition 3, $\mathcal{F}(\mathfrak{M}^*) = \{L_s, s \in S\}'$ will be an algebra included in $\mathcal{N}$.

We consider a set of generators S for the group V: $S = +1, -1$ for $\mathbb{Z}$ or $\mathbb{Z}_d$, $S = \{\pm v_0, ..., \pm v_{d-1}\}$ where $(v_0, ..., v_{d-1})$ is a canonical basis for the case $V = \mathbb{Z}^d$. In the nearest neighbor case we shall have $L_{i-1,i} = L_-$, $L_{i+1,i} = L_+$ and all the other $L_{ij} = 0$.

An immediate application of Proposition .... will give us the following.

Corollary 4. *Characterization of $\mathcal{M}$ for homogeneous OQWs on $\mathbb{Z}$ (or $\mathbb{Z}_d$).*

$A \in \mathcal{M}$ if and only if

$$A_{i,i} L_+ L_-^* = L_+ L_-^* A_{i-2,i-2}, \quad A_{i-2,i-2} L_- L_+^* = L_- L_+^* A_{i,i}, \quad A_{i,i} \in \{|L_+^*|, |L_-^*|\}' \quad \forall i,$$

and

$$A_{ik} L_- = A_{ik} L_+ = L_-^* A_{ik} = L_+^* A_{ik} = 0, \quad \forall i, k \in V,\ i \neq k \tag{15}$$

In particular, when at least one transition operator is invertible, $\mathcal{M}$ contains only block-diagonal operators.

We shall concentrate on the case $V = \mathbb{Z}$ in particular; at least the case $V = \mathbb{Z}_d$ should easily follow being careful to distinguish the cases when d is even or odd.

Let $\mathfrak{M}$ be a OQRW and $\rho \in \mathfrak{S}(\mathcal{H})$. Then ρ is invariant for $\mathfrak{M}$ if and only if $\rho = \sum_i \rho_i \otimes |i\rangle\langle i|$ and

$$\rho = \sum_i \rho_i \otimes |i\rangle\langle i|, \qquad \rho_k = \sum_i L_{k,i} \rho_i L_{k,i}^*,\ \forall k \in V \tag{16}$$

Suppose that there is some faithful normal invariant state ρ. Let $\mathcal{F} = \mathcal{F}_\Phi$. Then it is clear that any $A \in \mathcal{F}$ must be block-diagonal, that is $A = \sum_i A_i \otimes |i\rangle\langle i|$. By Proposition 3, we obtain that $A \in \mathcal{F}$ if and only if

$$A_i L_{i,j} = L_{i,j} A_j, \qquad A_j L_{i,j}^* = L_{i,j}^* A_i, \quad \forall i, j \in V. \tag{17}$$

Moreover, by Proposition 9, we can see that $\mathcal{M} = \mathcal{N} = \mathcal{F}$ if and only if

$$L_{i,j} \in \{\tilde{L}_\pi,\ \pi \in \mathcal{P}_{odd}(j,i)\}'', \qquad \forall i,j \in V, \tag{18}$$

where $\mathcal{P}_{odd}(j,i)$ is the set of paths $j \to i$ with odd length and for $\pi = (j, i_1, \dots, i_{n-1}, i) \in \mathcal{P}_{odd}(j,i)$, we define

$$\tilde{L}_\pi := L_{i,i_{n-1}} L^*_{i_{n-2},i_{n-1}} \dots L_{i_2,i_1} L^*_{j,i_1}.$$

Proposition 12. *Consider a homogeneous, irreducible OQRW with local space $\mathbb{C}^2$ and with transition operators which are invertible matrices. Then*

$$\mathcal{N} = span\{P_{odd}, P_{even}\}$$

unless there exists an orthonormal basis $\{f_0, f_1\}$ such that L_- and L_+ are one diagonal and one off-diagonal in this basis.

In the last case, $\mathcal{N}$ is generated by the cyclic projections

$$P_{\epsilon,\delta} = \sum_j (|f_\epsilon\rangle\langle f_\epsilon| \otimes |4j+\delta\rangle\langle 4j+\delta| + |f_{1-\epsilon}\rangle\langle f_{1-\epsilon}| \otimes |4j+2+\delta\rangle\langle 4j+2+\delta|),$$

with $\epsilon, \delta = 0, 1$ and the period is 4. Otherwise the period is 2 with cyclic projections P_{odd}, P_{even}.

The period was already computed in [8].

Proof. By Corollary 3, we know that the decoherence free algebra $\mathcal{N}$ consists only of block-diagonal operators. Then a projection P in $\mathcal{N}$ will have the form

$$P = \sum_j P_j \otimes |j\rangle\langle j|,$$

where, by Corollary 4, satisfy at least the conditions

$$P_j \in \{|L_+^*|, |L_-^*|\}', \qquad P_{j-1} L_- L_+^* = L_- L_+^* P_{j+1} \qquad \forall\, j. \tag{19}$$

We can write the action of Φ explicitly, in particular

$$\begin{aligned} \Phi(P) &= \sum_j (L_+^* P_{j+1} L_+ + L_-^* P_{j-1} L_-) \otimes |j\rangle\langle j|, \\ \Phi^2(P) &= \sum_j (L_+^{*2} P_{j+2} L_+^2 + L_-^{*2} P_{j-2} L_-^2 + L_-^* L_+^* P_j L_+ L_- + L_+^* L_-^* P_j L_- L_+) \otimes |j\rangle\langle j|. \end{aligned} \tag{20}$$

By these relations, it is easily deduced that $\Phi^n(P_{odd})$ is equal to P_{odd} for even n and to P_{even} for odd n (and similarly for $\Phi^n(P_{even})$). In particular, $\Phi^n(P_{odd}), \Phi^n(P_{even})$ are always projections and this allows us to conclude that P_{odd} and P_{even} belong to $\mathcal{N}$.... reference? Moreover, they are trivially central, i.e., for any other projection P in $\mathcal{N}$, $PP_{odd} = P_{odd}P$ and $PP_{even} = P_{even}P$.

When there exists an orthonormal basis $\{f_0, f_1\}$ such that L_- and L_+ are one diagonal and one off-diagonal in this basis, it is easy to see that the projections $P_{\epsilon,\delta}$ in the statement are cyclic. It is a little more complicated to see that these cyclic projections can exist only in that case and anyway no other minimal projection can then appear.

So now we want to consider, for a homogeneous irreducible OQRW, whether there exists a projection P in $\mathcal{N} \setminus \mathrm{span}\{P_{odd}, P_{even}\}$. We shall see that this is not possible, unless we are in the special case described in the statement.

If such a P exists, then $P = PP_{odd} + PP_{even}$ and the two addends are both in $\mathcal{N}$, so, by homogeneity, it will be sufficient to search for a projection P in $\mathcal{N}$ such that $P = PP_{even}$ and $0 < P < P_{even}$. Then we consider $P = \sum_j P_{2j} \otimes |2j\rangle\langle 2j|$.

Relations (19) imply that all the P_{2j}'s have the same rank (since the transition operators are invertible). Then, if P is different from 0 and from P_{even}, the only possibility is that P_{2j} is a rank one projection for any j. Calling u a norm one vector such that $P_0 = |u\rangle\langle u|$, and denoting $V := L_- L_+^*$, we deduce

$$P = \sum_j |V^{-j}u\rangle\langle V^{*j}u| \otimes |2j\rangle\langle 2j|,$$

where $V^{-j}u \parallel V^{*j}u$ because any P_{2j} is a projection and, due to the first condition in (19), $V^{*j}u$ is a common eigenvector of $|L_+^*|$ and $|L_-^*|$ for any j.

Similar considerations will hold for $\Phi^n(P)$, but considering only odd vertices instead of even vertices when n is odd. Indeed, starting with $n = 1$ (for $\Phi^n(P)$ we simply proceed inductively),
- $\Phi(P)$ is a projection in $\mathcal{N}$, $\Phi(P) \leq P_{odd}$ due to the fact that $0 \leq P \leq P_{even}$ and Φ is positive,
- moreover , when $P \neq P_{even}$ then $\Phi(P) \neq P_{odd}$ by irreducibility; indeed, if we had for instance $P \neq P_{even}$ and $\Phi(P) = P_{odd}$, then $P_{even} - P$ would be a non-zero projection in the kernel of Φ and this contradicts irreducibility.

Then, using (20), we need that

$$\Phi^2(P)(1 \otimes |0\rangle\langle 0|) = (L_+^{*2} P_2 L_+^2 + L_-^{*2} P_{-2} L_-^2 + L_-^* L_+^* P_0 L_+ L_- + L_+^* L_-^* P_0 L_- L_+)$$

is a one dimensional projection. This implies in particular that $L_-^* L_+^* u \parallel L_+^* L_-^* u$, so that u is an eigenvector for $(L_+^* L_-^*)^{-1} L_-^* L_+^*$.

Also, calling $u^\perp$ a norm one vector orthogonal to u, $P' := P_{even} - P = \sum_j |V^{-j}u^\perp\rangle\langle V^{*j}u^\perp| \otimes |2j\rangle\langle 2j|$, will be a projection in $\mathcal{N}$ and so $u^\perp$ will satisfy the same conditions as u.

Summing up, we have that u and $u^\perp$ should be two distinct eigenvectors for the operators

$$|L_+^*|, \qquad |L_-^*|, \qquad W := (L_+^* L_-^*)^{-1} L_-^* L_+^*. \tag{21}$$

Now, we claim that, due to irreducibility, the previous operators cannot be all proportional to the identity and we postpone of some lines the proof of this claim.

This fact implies that, either such vectors u and $u^\perp$ do not exist, and so $\mathcal{N} =$ ...., or they can be chosen in a unique way, up to multiplicative constants, as the orthonormal basis which diagonalize all the three operators above. In the latter case, we now look at the form of $\Phi(P)$ given in (20) and we see that

$$\Phi(P)(1 \otimes |j\rangle\langle j|) = L_+^* P_{j+1} L_+ + L_-^* P_{j-1} L_-$$

should be a one dimensional projection on a vector v which should be an eigenvector of the same three operators. This implies that

$$L_\epsilon^* u, L_\epsilon^* u^\perp \in \operatorname{span}\{u\} \cup \operatorname{span}\{u^\perp\}, \qquad \epsilon = +, -.$$

This implies that the operators L_+ and L_- should be either diagonal or off-diagonal in the basis $\{u, u^\perp\}$; but they cannot be both diagonal nor both off-diagonal, because this would contradict irreducibility. So the conclusion follows choosing $\{f_0, f_1\} = \{u, u^\perp\}$.

Finally, we go back to prove the claim. By contradiction, we suppose that all the operators in (21) are proportional to the identity, so that

$$L_+ = c_+ U_+, \qquad L_- = c_- U_- \qquad W = c1$$

for some complex numbers c_+, c_-, c and unitary operators U_+, U_-. Then we can rewrite

$$W = c1 = U_- U_+ U_-^* U_+^* \qquad \Rightarrow \qquad U_- = c U_+ U_- U_+^*$$

But now write the diagonal form of the unitary U_+, $U_+^* = \sum_{k=0,1} \lambda_k |v_k\rangle\langle v_k|$, with λ_0, λ_1 in the unit circle and $\{v_0, v_1\}$ orthonormal basis, and consider

$$\langle v_k, U_- v_j\rangle = c\langle v_k, U_+ U_- U_+^* v_j\rangle = c\overline{\lambda}_k \lambda_j \langle v_k, U_- v_j\rangle \qquad \text{for } j,k = 0,1.$$

This implies $c = 1$ and $\lambda_0 = \lambda_1$ which requires that U_+ and so L_+ are proportional to the identity. But this contradicts irreducibility. □

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