α-z-Rényi divergences in von Neumann algebras: data-processing inequality, reversibility, and monotonicity properties in α, z

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Abstract

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1 Introduction

The $\alpha - z$-Rényi divergences were introduced in [**?**] as a two parameter family of quantum generalizations of the classical Rényi divergences. For a pair of density operators (or more generally positive operators) ρ and σ on a finite dimensional Hilbert space, the divergence $D_{\alpha,z}(\rho\|\sigma)$ is defined by

$$D_{\alpha,z}(\rho\|\sigma) = \frac{1}{\alpha-1}\log\frac{\mathrm{tr}(\rho^{\frac{\alpha}{2z}}\sigma^{\frac{1-\alpha}{z}}\rho^{\frac{\alpha}{2z}})}{\mathrm{tr}\rho},$$

where $0 < \alpha \neq 1$ and $z > 0$. This family includes the quantum relative entropy in the limit $\alpha \to 1$ and unifies the two most important quantum versions of Rényi divergences, which have already an established operational significance. Namely, the standard or Petz-type Rényi divergences, based on the Petz quasi-entropy [**?**] are obtained for $z = 1$, whereas $z = \alpha$ gives the sandwiched Rényi divergences introduced in []. These two quantities have important applications in...

A fundamental property of any information measure is the data processing inequality (DPI), that is, such a quantity must be nonincreasing under quantum channels, which are defined as completely positive trace preserving maps. For the Rényi divergences, this means that we require

$$D_{\alpha,z}(\Phi(\rho)\|\Phi(\sigma)) \leq D_{\alpha,z}(\rho\|\sigma)$$

for any quantum channel Φ and any pair of states ρ, σ. This requirement leads to a restriction on the values of the parameters. For the Petz-type divergences, the convexity theorem of Ando [**?**] shows that α is restricted to the interval $(0,1) \cup (1,2]$. The DPI in the sandwiched case holds for $\alpha \in [1/2,1) \cup (1,\infty)$, as was proved in [**?**] and independently in [**?**]. The proof in [**?**] was given only for the case $\alpha > 1$, but is interesting for us, because it utilizes complex interpolation methods and, as later observed in [**?**], it shows that DPI holds also for maps that are trace preserving and positive but not necessarily completely positive. Remarkably, by taking the limit $\alpha \searrow 1$, this proves this property also for the relative entropy.

The problem of finding the values of (α, z) for which the DPI holds was posed and studied already in [**?**]. The conjecture stated in [**?**] suggests the bounds

$$0 < \alpha < 1, \quad \max\{\alpha, 1-\alpha\} \leq z \qquad \text{or} \qquad \alpha > 1, \quad \max\{\alpha, \alpha-1\} \leq z \leq \alpha.$$

This conjecture was finally proved in [37].

Quantum divergences and their properties were mostly studied in the finite dimensional case, but there is a growing interest in extensions beyond this setting. Von Neumann algebras form a suitable framework for such extensions, since their are general enough to cover many finite or infinite dimensional situations and there are also strong mathematical tools and techniques available to deal with them. Some important quantum divergences were already extended to this setting, notably the Araki relative entropy [**?**]. The quasi-entropies were first introduced for states of von Neumann algebras [**?**] and studied in detail in [**?**]. The definition of sandwiched Rényi divergences in this setting, based on noncommutative L^p-spaces, was introduced independently in [4] using Araki-Masuda L^p-spaces and in [18, 19] using Kosaki L^p-spaces. A number of properties of the divergences including DPI was proved in these works and it was shown in [18] that the two definitions are indeed equivalent. See [14] for an overview of both the Petz-type and sandwiched Rényi divergences in the von Neumann algebra setting.

Sufficiency quantum channels with respect to a set of states was introduced by Petz [32, 33]. This definition was inspired by sufficient statistics for classical statistical models. The quantum counterpart in Petz's definition is that all the states in the given set can be recovered by another channel. It is an important result that this property is characterized by equality in the DPI for relative entropy, as well as for the transition probability (which is basically the Petz-type Rényi divergence for $\alpha = 1/2$). This result has been generalized and extended to a large class of divergences and other quantities satisfying the DPI []. In particular, this property for the standard Rényi divergence was proved in [] in the interval $(0,1)\cup(0,2)$ and in [18, 19] for sandwiched version with $\alpha \in (1/2,1)\cup(1,\infty)$. See also [14] for a summary. Various approximate versions were studied as well, in both the finitine dimensional [] and von Neumann algebra setting [].

The $\alpha - z$-Rényi divergences $D_{\alpha,z}$ in von Neumann algebras were introduced in [22] and a number of their important properties was subsequently proved in [21]. In particular, most of the important properties including the DPI (with respect to normal positive unital maps), monotonicity in the parameter z and a variational expression were proved in the case $0 < \alpha < 1$.

Inspired by the papers [21, 22], our aim in the present work is to complete their results on the properties of the $\alpha - z$-Rényi divergences. We first finish the proof of the variational expression for $\alpha > 1$ and prove the DPI, in the same bounds as in the finite dimensional case. Similarly to Zhang in [37], we apply the variational expressions to so this, but, not having the techniques of matrix analysis at our disposal, we use their relation to the sandwiched Rényi divergences for the proof. This allows us to show the DPI with respect to normal positive unital maps, similarly as in the case $0 < \alpha < 1$ in [21] and for the sandwiched Rényi divergences in [18, 19]. As as a consequence, we prove martingale convergence theorems for $D_{\alpha,z}$.

We further show that for the values of (α, z) inside the DPI bounds, equality in the DPI implies sufficiency of the channel, where by channel we here mean a normal 2-positive unital map. The proofs again use the variational expressions and their relation to the sandwiched Rényi divergences, together with the properties of conditional expectations.

especially the variational expression, DPI and monotonicity in z for $\alpha > 1$.

2 Preliminaries

Let $\mathcal{M}$ be a von Neumann algebra and let $\mathcal{M}^+$ be the cone of positive elements in $\mathcal{M}$. We denote the predual of $\mathcal{M}$ by $\mathcal{M}_*$, its positive part by $\mathcal{M}_*^+$ and the set of normal states by $\mathfrak{S}_*(\mathcal{M})$. For $\psi \in \mathcal{M}_*^+$, we will denote by $s(\psi)$ the support projection of ψ. We denote the unit of von Neumann algebras by $\mathbf{1}$.

For $0 < p \leq \infty$, let $L^p(\mathcal{M})$ be the *Haagerup L^p-space* [10, 35] over $\mathcal{M}$ and let $L^p(\mathcal{M})^+$ its positive cone. We note that $L^\infty(\mathcal{M}) = \mathcal{M}$, and use the identification $\mathcal{M}_* \ni \psi \leftrightarrow h_\psi \in L^1(\mathcal{M})$ (an order isomorphism), so that the tr-functional on $L^1(\mathcal{M})$ is defined by $\operatorname{tr} h_\psi := \psi(\mathbf{1})$ for $\psi \in \mathcal{M}_*$. It this way, $\mathcal{M}_*^+$ is identified with the positive cone $L^1(\mathcal{M})^+$, and $\mathfrak{S}_*(\mathcal{M})$ with the set of elements $h \in L^1(\mathcal{M})^+$ with $\operatorname{tr} h = 1$. Let $\|a\|_p$ denote the Haagerup (quasi-)norm of $a \in L^p(\mathcal{M})$, $0 < p \le \infty$. Precise definitions and further details on the spaces $L^p(\mathcal{M})$ can be found in [13, Chap. 9], or in the notes [35]. A short summary on the Haagerup L^p-spaces and some technical results that will be used below can be found in Appendix A.

In [21, 22], the α-z-Rényi divergence for $\psi, \varphi \in \mathcal{M}_*^+$ was defined as follows:

Definition 2.1. Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$, and let $\alpha, z > 0$, $\alpha \neq 1$. The *α-z-Rényi divergence* is defined as

$$D_{\alpha,z}(\psi\|\varphi) := \frac{1}{\alpha - 1} \log \frac{Q_{\alpha,z}(\psi\|\varphi)}{\psi(\mathbf{1})},$$

where

$$Q_{\alpha,z}(\psi\|\varphi) := \begin{cases} \operatorname{tr}\left(h_\varphi^{\frac{1-\alpha}{2z}} h_\psi^{\frac{\alpha}{z}} h_\varphi^{\frac{1-\alpha}{2z}}\right)^z, & \text{if } 0 < \alpha < 1, \\ \|x\|_z^z, & \text{if } \alpha > 1 \text{ and } h_\psi^{\frac{\alpha}{z}} = h_\varphi^{\frac{\alpha-1}{2z}} x h_\varphi^{\frac{\alpha-1}{2z}} \text{ with} \\ & x \in s(\varphi) L^z(\mathcal{M}) s(\varphi), \\ \infty, & \text{otherwise.} \end{cases}$$

Remark 2.2. As mentioned in Appendix A, the Haagerup $L^p(\mathcal{M})$ spaces are constructed inside the crossed product $\mathcal{R} := \mathcal{M} \rtimes_{\sigma^\omega} \mathbb{R}$ by the modular automorphism group σ^ω for a faithful normal semi-finite weight ω on $\mathcal{M}$. When two such weights ω_j, $j = 0, 1$, are given, we have the Haagerup L^p-spaces $L^p(\mathcal{M})^{(j)}$ constructed in $\mathcal{R}_j := \mathcal{M} \rtimes_{\sigma^{\omega_j}} \mathbb{R}$ with the canonical trace τ_j. Then there is an isomorphism $\kappa : \mathcal{R}_0 \to \mathcal{R}_1$ (extended to $\kappa : \widetilde{\mathcal{R}}_0 \to \widetilde{\mathcal{R}}_1$, the spaces of τ_j-measurable operators) such that $\tau_0 \circ \kappa^{-1} = \tau_1$, $\kappa\big(L^p(\mathcal{M})^{(0)}\big) = L^p(\mathcal{M})^{(1)}$ for $0 < p \le \infty$, and $\operatorname{tr}_0 \circ \kappa^{-1} = \operatorname{tr}_1$ on $L^1(\mathcal{M})^{(1)}$, where tr_j is the tr-functional on $L^1(\mathcal{M})^{(j)}$. (For more details on the isomorphism κ, see, e.g., [13, Remark 9.10].) Hence we note that the above definition of $D_{\alpha,z}(\psi\|\varphi)$ is indeed independent of the choice of the Haagerup L^p-spaces. From this fact, we also note that for $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$, with $s(\psi), s(\varphi) \le e$, where e is a projection in $\mathcal{M}$, the α-z-Rényi divergence $D_{\alpha,z}(\psi\|\varphi)$ is the same as that defined when ψ and φ regarded as functionals on the reduced von Neumann algebra $e\mathcal{M}e$.

In the case $\alpha > 1$, the following alternative form will be useful.

Lemma 2.3 ([21, Lemma 7]). *Let $\alpha > 1$ and $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$. Then $Q_{\alpha,z}(\psi\|\varphi) < \infty$ if and only if there is some $y \in L^{2z}(\mathcal{M}) s(\varphi)$ such that*

$$h_\psi^{\frac{\alpha}{2z}} = y h_\varphi^{\frac{\alpha-1}{2z}}.$$

Moreover, in this case, such y is unique and we have $Q_{\alpha,z}(\psi\|\varphi) = \|y\|_{2z}^{2z}$.

The *standard* or *Petz-type Rényi divergence* [12, 14, 31] is contained in this range as $D_\alpha(\psi\|\varphi) = D_{\alpha,1}(\psi\|\varphi)$. Also, the *sandwiched Rényi divergence* is obtained as $\tilde{D}_\alpha(\psi\|\varphi) = D_{\alpha,\alpha}(\psi\|\varphi)$; see [4, 14, 18, 19] for some alternative definitions and properties of $\tilde{D}_\alpha$. The definitions in [18, 19] are based on *Kosaki's interpolation L^p-spaces* $L^p(\mathcal{M}, \varphi)$ [23] with respect to φ. These spaces and the complex interpolation method are briefly summarized in Appendix C, and will be used frequently in the present work.

As have already been done by Kato in [21], many of the properties of $D_{\alpha,z}(\psi\|\varphi)$ are extended from the finite-dimensional case into the general von Neumann algebra case. In particular, the following variational expressions will be an important tool for our work.

Theorem 2.4 (Variational expressions)**.** *Let $\psi,\varphi\in\mathcal{M}_*^+$, $\psi\neq 0$.*

(i) Let $0<\alpha<1$, $\max\{\alpha,1-\alpha\}\le z$. Then

$$Q_{\alpha,z}(\psi\|\varphi)=\inf_{a\in\mathcal{M}^{++}}\left\{\alpha\mathrm{tr}\left((a^{\frac12}h_\psi^{\frac{\alpha}{z}}a^{\frac12})^{\frac{z}{\alpha}}\right)+(1-\alpha)\mathrm{tr}\left((a^{-\frac12}h_\varphi^{\frac{1-\alpha}{z}}a^{-\frac12})^{\frac{z}{1-\alpha}}\right)\right\}. \tag{2.1}$$

(ii) Let $\alpha>1$, $\max\{\alpha/2,\alpha-1\}\le z$. Then

$$Q_{\alpha,z}(\psi\|\varphi)=\sup_{a\in\mathcal{M}^{+}}\left\{\alpha\mathrm{tr}\left((a^{\frac12}h_\psi^{\frac{\alpha}{z}}a^{\frac12})^{\frac{z}{\alpha}}\right)-(\alpha-1)\mathrm{tr}\left((a^{\frac12}h_\varphi^{\frac{\alpha-1}{z}}a^{\frac12})^{\frac{z}{\alpha-1}}\right)\right\}. \tag{2.2}$$

Moreover, if $\psi\le\lambda\varphi$ for some $\lambda>0$, then (2.2) holds for all $z\ge\alpha-1>0$.

Proof. For part (i) see [21, Theorem 1(vi)]. The inequality $\ge$ in part (ii) holds for all α and z and was proved in [21, Theorem 2(vi)]. We now prove the opposite inequality.

Assume first that $Q_{\alpha,z}(\psi\|\varphi)<\infty$, so that there is some $x\in s(\varphi)L^z(\mathcal{M})^+s(\varphi)$ such that $h_\psi^{\frac{\alpha}{z}}=h_\varphi^{\frac{\alpha-1}{2z}}xh_\varphi^{\frac{\alpha-1}{2z}}$. Plugging this into the right-hand side of (2.2), we obtain

$$\begin{aligned}
&\sup_{a\in\mathcal{M}^+}\left\{\alpha\mathrm{tr}\left((a^{\frac12}h_\psi^{\frac{\alpha}{z}}a^{\frac12})^{\frac{z}{\alpha}}\right)-(\alpha-1)\mathrm{tr}\left((a^{\frac12}h_\varphi^{\frac{\alpha-1}{2z}}a^{\frac12})^{\frac{z}{\alpha-1}}\right)\right\}\\
&\quad=\sup_{a\in\mathcal{M}^+}\left\{\alpha\mathrm{tr}\left((a^{\frac12}h_\varphi^{\frac{\alpha-1}{2z}}xh_\varphi^{\frac{\alpha-1}{2z}}a^{\frac12})^{\frac{z}{\alpha}}\right)-(\alpha-1)\mathrm{tr}\left((a^{\frac12}h_\varphi^{\frac{\alpha-1}{2z}}a^{\frac12})^{\frac{z}{\alpha-1}}\right)\right\}\\
&\quad=\sup_{a\in\mathcal{M}^+}\left\{\alpha\mathrm{tr}\left((x^{\frac12}h_\varphi^{\frac{\alpha-1}{2z}}ah_\varphi^{\frac{\alpha-1}{2z}}x^{\frac12})^{\frac{z}{\alpha}}\right)-(\alpha-1)\mathrm{tr}\left((h_\varphi^{\frac{\alpha-1}{2z}}ah_\varphi^{\frac{\alpha-1}{2z}})^{\frac{z}{\alpha-1}}\right)\right\}\\
&\quad=\sup_{w\in L^{\frac{z}{\alpha-1}}(\mathcal{M})^+}\left\{\alpha\mathrm{tr}\left((x^{\frac12}wx^{\frac12})^{\frac{z}{\alpha}}\right)-(\alpha-1)\mathrm{tr}\left(w^{\frac{z}{\alpha-1}}\right)\right\},
\end{aligned} \tag{2.3}$$

where we have used the fact that $\mathrm{tr}\left((h^*h)^p\right)=\mathrm{tr}\left((hh^*)^p\right)$ for $p>0$, $h\in L^{\frac{p}{2}}(\mathcal{M})$, and Lemma A.1. Putting $w=x^{\alpha-1}$ we get

$$\sup_{w\in L^{\frac{z}{\alpha-1}}(\mathcal{M})^+}\left\{\alpha\mathrm{tr}\left((x^{\frac12}wx^{\frac12})^{\frac{z}{\alpha}}\right)-(\alpha-1)\mathrm{tr}\left(w^{\frac{z}{\alpha-1}}\right)\right\}\ge\mathrm{tr}(x^z)=\|x\|_z^z=Q_{\alpha,z}(\psi\|\varphi).$$

This finishes the proof of (2.2) in the case that $Q_{\alpha,z}(\psi\|\varphi)<\infty$. Note that this holds if $z\ge\alpha-1>0$ and $\psi\le\lambda\varphi$ for some $\lambda>0$. Indeed, since $\frac{\alpha-1}{z}\in(0,1]$ by the assumption, we then have $h_\psi^{\frac{\alpha-1}{z}}\le\lambda^{\frac{\alpha-1}{z}}h_\varphi^{\frac{\alpha-1}{z}}$. Hence by [14, Lemma A.58] there is some $b\in\mathcal{M}$ such that

$$h_\psi^{\frac{\alpha-1}{2z}}=bh_\varphi^{\frac{\alpha-1}{2z}},$$

so that $h_\psi^{\frac{\alpha}{2z}}=yh_\varphi^{\frac{\alpha-1}{2z}}$, where $y:=h_\varphi^{\frac{1}{2z}}b\in L^{2z}(\mathcal{M})$. By Lemma 2.3, $Q_{\alpha,z}(\psi\|\varphi)=\|y\|_{2z}^{2z}<\infty$. This shows the latter assertion too. (Note that this follows also from the order relations in [21, Theorem 2(iii)]).

In the general case, assume that $\max\{\alpha/2, \alpha-1\} \le z$. As shown above, the variational expression holds for $Q_{\alpha,z}(\psi\|\varphi+\varepsilon\psi)$ for all $\varepsilon > 0$, so that we have

$$\begin{aligned}Q_{\alpha,z}(\psi\|\varphi+\varepsilon\psi) &= \sup_{a\in\mathcal{M}^+}\left\{\alpha\mathrm{tr}\left((a^{\frac{1}{2}}h_\psi^{\frac{\alpha}{z}}a^{\frac{1}{2}})^{\frac{z}{\alpha}}\right)-(\alpha-1)\mathrm{tr}\left((a^{\frac{1}{2}}h_{\varphi+\varepsilon\psi}^{\frac{\alpha-1}{z}}a^{\frac{1}{2}})^{\frac{z}{\alpha-1}}\right)\right\}\\ &\le \sup_{a\in\mathcal{M}^+}\left\{\alpha\mathrm{tr}\left((a^{\frac{1}{2}}h_\psi^{\frac{\alpha}{z}}a^{\frac{1}{2}})^{\frac{z}{\alpha}}\right)-(\alpha-1)\mathrm{tr}\left((a^{\frac{1}{2}}h_{\varphi}^{\frac{\alpha-1}{z}}a^{\frac{1}{2}})^{\frac{z}{\alpha-1}}\right)\right\},\end{aligned}$$

where the inequality above follows by Lemma A.3. Therefore, since $z \ge \alpha/2$, from the lower semi-continuity of $Q_{\alpha,z}$ in [21, Theorem 2(iv)] we have

$$Q_{\alpha,z}(\psi\|\varphi) \le \liminf_{\varepsilon\searrow 0} Q_{\alpha,z}(\psi\|\varphi+\varepsilon\psi),$$

so that the desired inequality is obtained. □

We finish this section by investigation of the properties of the variational expression for $0 < \alpha < 1$, in the case when $\lambda^{-1}\varphi \le \psi \le \lambda\varphi$ for some $\lambda > 0$. This case will be denoted as $\psi \sim \varphi$.

Lemma 2.5. *Assume that $\psi \sim \varphi$. Then the infimum in (2.1) of Theorem 2.4(i) is attained at a unique element $\bar a \in \mathcal{M}^{++}$. This element satisfies*

$$h_\psi^{\frac{\alpha}{2z}}\bar a h_\psi^{\frac{\alpha}{2z}} = \left(h_\psi^{\frac{\alpha}{2z}}h_\varphi^{\frac{1-\alpha}{z}}h_\psi^{\frac{\alpha}{2z}}\right)^\alpha, \tag{2.4}$$

$$h_\varphi^{\frac{1-\alpha}{2z}}\bar a^{-1} h_\varphi^{\frac{1-\alpha}{2z}} = \left(h_\varphi^{\frac{1-\alpha}{2z}}h_\psi^{\frac{\alpha}{z}}h_\varphi^{\frac{1-\alpha}{2z}}\right)^{1-\alpha}. \tag{2.5}$$

Proof. We may assume that φ and hence also ψ are faithful. Following the proof of [21, Theorem 1(vi)], we may use the assumptions and [14, Lemma A.58] to show that there are $b, c \in \mathcal{M}$ such that

$$h_\varphi^{\frac{1-\alpha}{2z}} = b\Big(h_\varphi^{\frac{1-\alpha}{2z}}h_\psi^{\frac{\alpha}{z}}h_\varphi^{\frac{1-\alpha}{2z}}\Big)^{\frac{1-\alpha}{2}}, \qquad \Big(h_\varphi^{\frac{1-\alpha}{2z}}h_\psi^{\frac{\alpha}{z}}h_\varphi^{\frac{1-\alpha}{2z}}\Big)^{\frac{1-\alpha}{2}} = c h_\varphi^{\frac{1-\alpha}{2z}}. \tag{2.6}$$

With $\bar a := bb^* \in \mathcal{M}^{++}$ we have $\bar a^{-1} = c^*c$ and $\bar a$ is indeed a minimizer of (2.2), so that

$$Q_{\alpha,z}(\psi\|\varphi) = \Big\|h_\psi^{\frac{\alpha}{2z}}\bar a h_\psi^{\frac{\alpha}{2z}}\Big\|_{\frac{z}{\alpha}}^{\frac{z}{\alpha}} = \alpha\Big\|h_\psi^{\frac{\alpha}{2z}}\bar a h_\psi^{\frac{\alpha}{2z}}\Big\|_{\frac{z}{\alpha}}^{\frac{z}{\alpha}} + (1-\alpha)\Big\|h_\varphi^{\frac{1-\alpha}{2z}}\bar a^{-1}h_\varphi^{\frac{1-\alpha}{2z}}\Big\|_{\frac{z}{1-\alpha}}^{\frac{z}{1-\alpha}}. \tag{2.7}$$

(see the proof of [21, Theorem 1(vi)] for more detail). We next observe that such a minimizer is unique. Indeed, suppose that the infimum is attained at some $a_1, a_2 \in \mathcal{M}^{++}$. Let $a_0 := (a_1+a_2)/2$. Since the map $L^p(\mathcal{M}) \ni k \mapsto \|k\|_p^p$ is convex for any $p \ge 1$ and $a_0^{-1} \le (a_1^{-1}+a_2^{-1})/2$, we have

$$\Big\|h_\psi^{\frac{\alpha}{2z}}a_0 h_\psi^{\frac{\alpha}{2z}}\Big\|_{\frac{z}{\alpha}}^{\frac{z}{\alpha}} \le \frac{1}{2}\Big\{\Big\|h_\psi^{\frac{\alpha}{2z}}a_1 h_\psi^{\frac{\alpha}{2z}}\Big\|_{\frac{z}{\alpha}}^{\frac{z}{\alpha}} + \Big\|h_\psi^{\frac{\alpha}{2z}}a_2 h_\psi^{\frac{\alpha}{2z}}\Big\|_{\frac{z}{\alpha}}^{\frac{z}{\alpha}}\Big\}.$$

Moreover, using Lemma A.2 we have

$$\begin{aligned}\Big\|h_\varphi^{\frac{1-\alpha}{2z}}a_0^{-1}h_\varphi^{\frac{1-\alpha}{2z}}\Big\|_{\frac{z}{1-\alpha}}^{\frac{z}{1-\alpha}} &\le \Big\|h_\varphi^{\frac{1-\alpha}{2z}}\left(\frac{a_1^{-1}+a_2^{-1}}{2}\right)h_\varphi^{\frac{1-\alpha}{2z}}\Big\|_{\frac{z}{1-\alpha}}^{\frac{z}{1-\alpha}}\\ &\le \frac{1}{2}\Big\{\Big\|h_\varphi^{\frac{1-\alpha}{2z}}a_1^{-1}h_\varphi^{\frac{1-\alpha}{2z}}\Big\|_{\frac{z}{1-\alpha}}^{\frac{z}{1-\alpha}} + \Big\|h_\varphi^{\frac{1-\alpha}{2z}}a_2^{-1}h_\varphi^{\frac{1-\alpha}{2z}}\Big\|_{\frac{z}{1-\alpha}}^{\frac{z}{1-\alpha}}\Big\}.\end{aligned}$$

Hence the assumption of a_1, a_2 being a minimizer gives

$$\left\| h_\varphi^{\frac{1-\alpha}{2z}} a_0^{-1} h_\varphi^{\frac{1-\alpha}{2z}} \right\|_{\frac{z}{1-\alpha}} = \left\| h_\varphi^{\frac{1-\alpha}{2z}} \left(\frac{a_1^{-1} + a_2^{-1}}{2} \right) h_\varphi^{\frac{1-\alpha}{2z}} \right\|_{\frac{z}{1-\alpha}},$$

which implies that $a_0^{-1} = \frac{a_1^{-1}+a_2^{-1}}{2}$, as easily verified by Lemma A.2 again. From this we easily have $a_1 = a_2$.

The equality (2.5) is obvious from the second equality in (2.6) and $\bar{a}^{-1} = c^*c$. Since $Q_{\alpha,z}(\psi\|\varphi) = Q_{1-\alpha,z}(\varphi\|\psi)$, we see by uniqueness that the minimizer of the infimum expression for $Q_{1-\alpha,z}(\varphi\|\psi)$ (instead of (2.7)) is $\bar{a}^{-1}$ (instead of $\bar{a}$). This says that (2.4) is the equality corresponding to (2.5) when ψ, φ, α are replaced with $\varphi, \psi, 1-\alpha$, respectively. $\square$

To make the next lemma more readable, we will use the following notations:

$$p := \frac{z}{\alpha}, \qquad r := \frac{z}{1-\alpha}, \qquad \xi_p(a) := h_\psi^{\frac{1}{2p}} a h_\psi^{\frac{1}{2p}}, \qquad \eta_r(a) = h_\varphi^{\frac{1}{2r}} a^{-1} h_\varphi^{\frac{1}{2r}}.$$

We will also denote the function under the infimum in the variational expression in Theorem 2.4(i) by f, that is,

$$f(a) = \alpha\|\xi_p(a)\|_p^p + (1-\alpha)\|\eta_r(a)\|_r^r, \qquad a \in \mathcal{M}^{++}. \tag{2.8}$$

When $p \in (1,\infty)$, recall that $L^p(\mathcal{M})$ is uniformly convex (see [10], [23, Theorem 4.2]), so that the norm $\|\cdot\|_p$ is uniformly Fréchet differentiable (see, e.g., [2, Part 3, Chap. II]). Hence if $p, r > 1$, $a \mapsto \|\xi_p(a)\|_p^p$ and $a \mapsto \|\eta_r(a)\|_r^r$ are Fréchet differentiable on $\mathcal{M}^{++}$. Since differentiability of these functions is obvious when $p = 1$ and $r = 1$, we see that the function f is Fréchet differentiable on $\mathcal{M}^{++}$ for any $p, r \geq 1$, whose Fréchet derivative at a will be denoted by $\nabla f(a)$.

Lemma 2.6. *Assume that $\psi \sim \varphi$ and let $0 < \alpha < 1$, $\max\{\alpha, 1-\alpha\} \leq z$. Let $\bar{a} \in \mathcal{M}^{++}$ be as given in Lemma 2.5. If $p > 1$, then for every $C \geq Q_{\alpha,z}(\psi\|\varphi)$ and $\varepsilon > 0$ there is some $\delta > 0$ such that whenever $\|\xi_p(b)\|_p^p \leq C$ and $\|\xi_p(b) - \xi_p(\bar{a})\|_p \geq \varepsilon$, we have*

$$f(b) - Q_{\alpha,z}(\psi\|\varphi) \geq \delta.$$

A similar statement holds if $r > 1$.

Proof. By assumptions, $p, r \geq 1$. For $a, b \in \mathcal{M}^{++}$ and $s \in (0, 1/2)$, we have

$$\begin{aligned}
\|\xi_p(sb + (1-s)a)\|_p^p &= \|s\xi_p(b) + (1-s)\xi_p(a)\|_p^p \\
&= \left\|(1-2s)\xi_p(a) + 2s\frac{1}{2}(\xi_p(a) + \xi_p(b))\right\|_p^p \\
&\leq (1-2s)\|\xi_p(a)\|_p^p + 2s\left\|\frac{1}{2}(\xi_p(a) + \xi_p(b))\right\|_p^p.
\end{aligned}$$

Similarly,

$$\|\eta_r(sb + (1-s)a)\|_r^r \leq (1-2s)\|\eta_r(a)\|_r^r + 2s\left\|\frac{1}{2}(\eta_r(a) + \eta_r(b))\right\|_r^r,$$

where we have also used the fact that $(ta+(1-t)b)^{-1} \le ta^{-1}+(1-t)b^{-1}$ for $t \in (0,1)$ and Lemma A.2. It follows that

$$
\begin{aligned}
&\langle \nabla f(a), b-a\rangle \\
&\quad = \lim_{s\to 0^+} \frac{1}{s}[f(sb+(1-s)a)-f(a)] \\
&\quad \le 2\alpha\left[\left\|\frac{1}{2}(\xi_p(a)+\xi_p(b))\right\|_p^p - \|\xi_p(a)\|_p^p\right] + 2(1-\alpha)\left[\left\|\frac{1}{2}(\eta_r(a)+\eta_r(b))\right\|_r^r - \|\eta_r(a)\|_r^r\right] \\
&\quad = f(b)-f(a)-2\Bigg\{\alpha\left[\frac{1}{2}\|\xi_p(a)\|_p^p + \frac{1}{2}\|\xi_p(b)\|_p^p - \left\|\frac{1}{2}(\xi_p(a)+\xi_p(b))\right\|_p^p\right] \\
&\qquad\qquad + (1-\alpha)\left[\frac{1}{2}\|\eta_r(a)\|_r^r + \frac{1}{2}\|\eta_r(b)\|_r^r - \left\|\frac{1}{2}(\eta_r(a)+\eta_r(b))\right\|_r^r\right]\Bigg\}.
\end{aligned}
$$

Since $p, r \ge 1$, both terms in brackets in the last expression above are nonnegative. Let $\bar{a} \in \mathcal{M}^{++}$ be the minimizer as in Lemma 2.5, then $f(\bar{a}) = Q_{\alpha,z}(\psi\|\varphi)$ and $\nabla f(\bar{a}) = 0$, so that we get

$$
f(b) - Q_{\alpha,z}(\psi\|\varphi) \ge 2\alpha\left[\frac{1}{2}\|\xi_p(\bar{a})\|_p^p + \frac{1}{2}\|\xi_p(b)\|_p^p - \left\|\frac{1}{2}(\xi_p(\bar{a})+\xi_p(b))\right\|_p^p\right].
$$

Since $L^p(\mathcal{M})$ is uniformly convex, note (see, e.g., [36, Theorem 3.7.7]) that the function $h \mapsto \|h\|_p^p$ is uniformly convex on each bounded subset of $L^p(\mathcal{M})$. Hence for each $C > 0$ and $\varepsilon > 0$ there is some $\delta > 0$ such that for every h, k with $\|h\|_p^p, \|k\|_p^p \le C$ and $\|h-k\|_p \ge \varepsilon$, we have

$$
\frac{1}{2}\|h\|_p^p + \frac{1}{2}\|k\|_p^p - \left\|\frac{1}{2}(h+k)\right\|_p^p \ge \delta
$$

(see [36, p. 288, Exercise 3.3]). Since $\|\xi_p(\bar{a})\|_p = Q_{\alpha,z}(\psi\|\varphi)$ by (2.7), this proves the statement. The proof in the case $r > 1$ is similar. $\square$

3 Data processing inequality and martingale convergence

Let $\gamma : \mathcal{N} \to \mathcal{M}$ be a normal positive unital map. Then the predual of γ defines a positive linear map $\gamma_* : L^1(\mathcal{M}) \to L^1(\mathcal{N})$ that preserves the tr-functional, acting as

$$
L^1(\mathcal{M}) \ni h_\rho \mapsto h_{\rho\circ\gamma} \in L^1(\mathcal{N}).
$$

The support of γ will be denoted by $s(\gamma)$, recall that this is defined as the smallest projection $e \in \mathcal{N}$ such that $\gamma(e) = \mathbf{1}$ and in this case, $\gamma(a) = \gamma(eae)$ for any $a \in \mathcal{N}$. For any $\rho \in \mathcal{M}_*^+$ we clearly have $s(\rho\circ\gamma) \le s(\gamma)$, with equality if ρ is faithful. It follows that γ_* maps $L^1(\mathcal{M})$ to $s(\gamma)L^1(\mathcal{N})s(\gamma) \equiv L^1(s(\gamma)\mathcal{N}s(\gamma))$. For any $\rho \in \mathcal{M}_*^+$, $\rho \ne 0$, the map

$$
\gamma_0 : s(\rho\circ\gamma)\mathcal{N}s(\rho\circ\gamma) \to s(\rho)\mathcal{M}s(\rho), \qquad a \mapsto s(\rho)\gamma(a)s(\rho)
$$

is a faithful normal positive unital (i.e., $\gamma_0(\rho\circ\gamma) = s(\rho)$) map; see [13, Remark 6.7]. Moreover, for any $\varphi \in \mathcal{M}_*^+$ such that $s(\varphi) \le s(\rho)$, we have for any $a \in \mathcal{N}$,

$$
\varphi(\gamma_0(s(\gamma)as(\gamma))) = \varphi(s(\rho)\gamma(a)s(\rho)) = \varphi(\gamma(a)).
$$

Replacing γ by γ_0 and ρ by the restriction $\rho|_{s(\rho\circ\gamma)\mathcal{M}s(\rho\circ\gamma)}$, we may assume that both ρ and $\rho\circ\gamma$ are faithful, as far as we are concerned with $\varphi\in\mathcal{M}_*^+$ and $\varphi\circ\gamma\in\mathcal{N}_*^+$ with $s(\varphi)\le s(\rho)$.

The *Petz dual* of γ with respect to $\rho\in\mathcal{M}_*^+$ is a map $\gamma_\rho^*:\mathcal{M}\to\mathcal{N}$, introduced in [33] when ρ and $\rho\circ\gamma$ are faithful (hence so is γ as well). It was proved that γ_ρ^* is again normal, positive and unital, and in addition, it is n-positive whenever γ is. More generally, even though none of ρ, $\rho\circ\gamma$ and γ is faithful, letting $e:=s(\rho)$ and $e_0:=s(\rho\circ\gamma)$, we may use the restriction γ_0 as mentioned above to define the Petz dual $\gamma_\rho^*:e\mathcal{M}e\to e_0\mathcal{N}e_0$. As explained in [18], in this general setting, γ_ρ^* is determined by the equality

$$h_{\rho\circ\gamma}^{1/2}\gamma_\rho^*(a)h_{\rho\circ\gamma}^{1/2}=\gamma_*\big(h_\rho^{1/2}ah_\rho^{1/2}\big),\qquad a\in e\mathcal{M}e,$$

equivalently,

$$(\gamma_\rho^*)_*\big(h_{\rho\circ\gamma}^{1/2}bh_{\rho\circ\gamma}^{1/2}\big)=h_\rho^{1/2}\gamma(b)h_\rho^{1/2},\qquad b\in\mathcal{N}^+,\tag{3.1}$$

where γ_* and $(\gamma_\rho^*)_*$ are the predual maps of γ and γ_ρ^*, respectively. We also have

$$\rho\circ\gamma\circ\gamma_\rho^*=\rho,\qquad (\gamma_\rho^*)_{\rho\circ\gamma}^*=\gamma_0.\tag{3.2}$$

In the special case where γ is the inclusion map $\gamma:\mathcal{N}\hookrightarrow\mathcal{M}$ for a subalgebra $\mathcal{N}\subseteq\mathcal{M}$, the Petz dual is the *generalized conditional expectation* $\mathcal{E}_{\mathcal{N},\rho}:\mathcal{M}\to\mathcal{N}$, as introduced in [1]; see, e.g., [14, Proposition 6.5]. Hence $\mathcal{E}_{\mathcal{N},\rho}$ is a normal completely positive unital map with range in $\mathcal{N}$ and such that

$$\rho\circ\mathcal{E}_{\mathcal{N},\rho}=\rho.$$

3.1 Data processing inequality

In this subsection we prove the *data processing inequality* (*DPI*) for $D_{\alpha,z}$ with respect to normal positive unital maps. For standard Rényi divergence, that is, for $z=1$, the DPI is known to hold for $\alpha\in(0,1)\cup(1,2]$ under stronger positivity assumptions [12]. In the case of the sandwiched divergences $\tilde{D}_\alpha$ with $\alpha\in[1/2,1)\cup(1,\infty)$, DPI was proved in [18, 19]; see also [4] for an alternative proof in the case when the maps are assumed completely positive. In the finite-dimensional case, the DPI for $D_{\alpha,z}$ under completely positive maps was proved in [37], for α,z in the range specified as in Theorem 3.3 below.

The first part of the next lemma was essentially shown in [18, Proposition 3.12], while we give the proof for the convenience of the reader.

Lemma 3.1. *Let $\gamma:\mathcal{N}\to\mathcal{M}$ be a normal positive unital map. Let $\rho\in\mathcal{M}_*^+$, $\rho\ne0$, $e:=s(\rho)$ and $e_0:=s(\rho\circ\gamma)$. For any $p\ge1$, the map $\gamma_{\rho,p}^*:L^p(e_0\mathcal{N}e_0)\to L^p(e\mathcal{M}e)$, determined by*

$$\gamma_{\rho,p}^*\Big(h_{\rho\circ\gamma}^{\frac{1}{2p}}bh_{\rho\circ\gamma}^{\frac{1}{2p}}\Big)=h_\rho^{\frac{1}{2p}}\gamma(b)h_\rho^{\frac{1}{2p}},\qquad b\in\mathcal{N},\tag{3.3}$$

is a contraction such that

$$(\gamma_\rho^*)_*\Big(h_{\rho\circ\gamma}^{\frac{p-1}{2p}}kh_{\rho\circ\gamma}^{\frac{p-1}{2p}}\Big)=h_\rho^{\frac{p-1}{2p}}\gamma_{\rho,p}^*(k)h_\rho^{\frac{p-1}{2p}},\qquad k\in L^p(e_0\mathcal{N}e_0).\tag{3.4}$$

Moreover, if $\rho_n\in\mathcal{M}_*^+$ are such that $s(\rho)\le s(\rho_n)$ and $\|\rho_n-\rho\|_1\to0$, then for any $k\in L^p(e_0\mathcal{N}e_0)$ we have $\gamma_{\rho_n,p}^*(k)\to\gamma_{\rho,p}^*(k)$ in $L^p(\mathcal{M})$.

Proof. We use Kosaki's symmetric L^p-spaces $L^p(e_0\mathcal{N}e_0,\rho\circ\gamma)$ and $L^p(e\mathcal{M}e,\rho)$ (see (C.4) in Appendix C). The map $(\gamma_\rho^*)_*: L^1(e_0\mathcal{N}e_0)\to L^1(e\mathcal{M}e)$ is contractive with respect to $\|\cdot\|_1$. Its restriction to $h_{\rho\circ\gamma}^{1/2}\mathcal{N}h_{\rho\circ\gamma}^{1/2}$ ($\subseteq L^1(e_0\mathcal{N}e_0)$) is given by (3.1), which is also contractive with respect to $\|\cdot\|_{\infty,\rho\circ\gamma}$ and $\|\cdot\|_{\infty,\rho}$. Hence it follows from the Riesz–Thorin theorem that $(\gamma_\rho^*)_*$ is a contraction from $L^p(e_0\mathcal{N}e_0,\rho\circ\gamma)$ to $L^p(e\mathcal{M}e,\rho)$ for any $p\in(1,\infty)$. By (C.4) note that we have isometric isomorphisms

$$
\begin{aligned}
k\in L^p(e_0\mathcal{N}e_0)&\mapsto h_{\rho\circ\gamma}^{\frac{p-1}{2p}}kh_{\rho\circ\gamma}^{\frac{p-1}{2p}}\in L^p(e_0\mathcal{N}e_0,\rho\circ\gamma),\\
h\in L^p(e\mathcal{M}e)&\mapsto h_\rho^{\frac{p-1}{2p}}hh_\rho^{\frac{p-1}{2p}}\in L^p(e\mathcal{M}e,\rho).
\end{aligned}
$$

Hence we can define a contraction $\gamma_{\rho,p}^*: L^p(e_0\mathcal{N}e_0)\to L^p(e\mathcal{M}e)$ by (3.4). Then, for $k=h_{\rho\circ\gamma}^{\frac{1}{2p}}bh_{\rho\circ\gamma}^{\frac{1}{2p}}$ with $b\in\mathcal{N}$ we have

$$
h_\rho^{\frac{p-1}{2p}}\gamma_{\rho,p}^*\Big(h_{\rho\circ\gamma}^{\frac{1}{2p}}bh_{\rho\circ\gamma}^{\frac{1}{2p}}\Big)h_\rho^{\frac{p-1}{2p}}=(\gamma_\rho^*)_*\Big(h_{\rho\circ\gamma}^{\frac{1}{2}}bh_{\rho\circ\gamma}^{\frac{1}{2}}\Big)=h_\rho^{\frac{1}{2}}\gamma(b)h_\rho^{\frac{1}{2}},
$$

so that (3.3) holds. Since $h_{\rho\circ\gamma}^{\frac{1}{2p}}\mathcal{N}h_{\rho\circ\gamma}^{\frac{1}{2p}}$ is dense in $L^p(e_0\mathcal{N}e_0)$, this proves the first part of the statement.

Let ρ_n be a sequence as required and let $k\in L^p(e_0\mathcal{N}e_0)$. By the assumptions on the supports, $\gamma_{\rho_n,p}^*$ is well defined on k for all n. Further, we may assume that $k=h_{\rho\circ\gamma}^{\frac{1}{2p}}bh_{\rho\circ\gamma}^{\frac{1}{2p}}$ for some $b\in\mathcal{N}$, since the set of such elements is dense in $L^p(e_0\mathcal{N}e_0)$ and all the maps are contractions. Put $k_n:=h_{\rho_n\circ\gamma}^{\frac{1}{2p}}bh_{\rho_n\circ\gamma}^{\frac{1}{2p}}$, then we have

$$
\gamma_{\rho,p}^*(k)=h_\rho^{\frac{1}{2p}}\gamma(b)h_\rho^{\frac{1}{2p}},\qquad \gamma_{\rho_n,p}^*(k_n)=h_{\rho_n}^{\frac{1}{2p}}\gamma(b)h_{\rho_n}^{\frac{1}{2p}},
$$

and we have $k_n\to k$ in $L^p(\mathcal{N})$ and $\gamma_{\rho_n,p}^*(k_n)\to\gamma_{\rho,p}^*(k)$ in $L^p(\mathcal{M})$. Indeed, this follows by the Hölder inequality and the continuity of the map $L^1(\mathcal{M})^+\ni h\mapsto h^{\frac{1}{2p}}\in L^{2p}(\mathcal{M})^+$; see [24, Lemma 3.4]. Therefore

$$
\|\gamma_{\rho_n,p}^*(k)-\gamma_{\rho,p}^*(k)\|_p\le\|\gamma_{\rho_n,p}^*(k-k_n)\|_p+\|\gamma_{\rho_n,p}^*(k_n)-\gamma_{\rho,p}^*(k)\|_p\to 0,
$$

showing the latter statement. $\square$

Lemma 3.2. *Let $\gamma:\mathcal{N}\to\mathcal{M}$ be a normal positive unital map, and let $\rho\in\mathcal{M}_*^+$, $\rho\neq 0$, and $b\in\mathcal{N}^+$.*

(i) If $p\in[1/2,1)$, then

$$
\Big\|h_{\rho\circ\gamma}^{\frac{1}{2p}}bh_{\rho\circ\gamma}^{\frac{1}{2p}}\Big\|_p\le\Big\|h_\rho^{\frac{1}{2p}}\gamma(b)h_\rho^{\frac{1}{2p}}\Big\|_p.
$$

(ii) If $p\in[1,\infty]$, the inequality reverses.

Proof. Let us denote $\beta:=\gamma_\rho^*$ and let $\omega\in\mathcal{N}_*^+$ be such that $h_\omega:=h_{\rho\circ\gamma}^{\frac{1}{2}}bh_{\rho\circ\gamma}^{\frac{1}{2}}\in L^1(\mathcal{N})^+$. Then β is a normal positive unital map, and by (3.1) we have

$$
\beta_*(h_\omega)=h_\rho^{\frac{1}{2}}\gamma(b)h_\rho^{\frac{1}{2}},\qquad \beta_*(h_{\rho\circ\gamma})=h_\rho.
$$

Let $p \in [1/2, 1)$, then

$$\begin{aligned}\left\|h_\rho^{\frac{1}{2p}}\gamma(b)h_\rho^{\frac{1}{2p}}\right\|_p^p &= \left\|h_\rho^{\frac{1-p}{2p}}\beta_*(h_\omega)h_\rho^{\frac{1-p}{2p}}\right\|_p^p = Q_{p,p}(\beta_*(h_\omega)\|h_\rho) = Q_{p,p}(\beta_*(h_\omega)\|\beta_*(h_{\rho\circ\gamma})) \\ &\geq Q_{p,p}(h_\omega\|h_{\rho\circ\gamma}) = \left\|h_{\rho\circ\gamma}^{\frac{1-p}{2p}}h_\omega h_{\rho\circ\gamma}^{\frac{1-p}{2p}}\right\|_p^p = \left\|h_{\rho\circ\gamma}^{\frac{1}{2p}}bh_{\rho\circ\gamma}^{\frac{1}{2p}}\right\|_p^p.\end{aligned}$$

Here we have used the DPI for the sandwiched Rényi divergence $D_{\alpha,\alpha}$ for $\alpha \in [1/2, 1)$; see [19, Theorem 4.1]. This proves (i). The case (ii) is immediate from Lemma 3.1. This was proved also in [21] (see Eq. (22) therein), by using the same argument. □

Theorem 3.3 (DPI)**.** *Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$ and let $\gamma : \mathcal{N} \to \mathcal{M}$ be a normal positive unital map. Assume either of the following conditions:*

(i) $0 < \alpha < 1$, $\max\{\alpha, 1-\alpha\} \leq z$,

(ii) $\alpha > 1$, $\max\{\alpha/2, \alpha - 1\} \leq z \leq \alpha$.

Then we have

$$D_{\alpha,z}(\psi\circ\gamma\|\varphi\circ\gamma) \leq D_{\alpha,z}(\psi\|\varphi).$$

Proof. Under the conditions (i), the DPI was proved in [21, Theorem 1(viii)]. Since parts of this proof will be used below, we repeat it here.

Assume the conditions in (i) and put $p := \frac{z}{\alpha}$, $r := \frac{z}{1-\alpha}$, so that $p, r \geq 1$. For any $b \in \mathcal{N}^{++}$, we have by the Choi inequality [6] that $\gamma(b)^{-1} \leq \gamma(b^{-1})$, so that by Lemmas A.2 and 3.2(ii), we have

$$\left\|h_\varphi^{\frac{1}{2r}}\gamma(b)^{-1}\varphi^{\frac{1}{2r}}\right\|_r \leq \left\|h_\varphi^{\frac{1}{2r}}\gamma(b^{-1})\varphi^{\frac{1}{2r}}\right\|_r \leq \left\|h_{\varphi\circ\gamma}^{\frac{1}{2r}}b^{-1}h_{\varphi\circ\gamma}^{\frac{1}{2r}}\right\|_r^r. \tag{3.5}$$

Using the variational expression in Theorem 2.4(i), we have

$$\begin{aligned}Q_{\alpha,z}(\psi\|\varphi) &\leq \alpha\|h_\psi^{\frac{1}{2p}}\gamma(b)h_\psi^{\frac{1}{2p}}\|_p^p + (1-\alpha)\|h_\varphi^{\frac{1}{2r}}\gamma(b)^{-1}h_\varphi^{\frac{1}{2r}}\|_r^r \\ &\leq \alpha\|h_{\psi\circ\gamma}^{\frac{1}{2p}}bh_{\psi\circ\gamma}^{\frac{1}{2p}}\|_p^p + (1-\alpha)\|h_{\varphi\circ\gamma}^{\frac{1}{2r}}b^{-1}h_{\varphi\circ\gamma}^{\frac{1}{2r}}\|_r^r.\end{aligned}$$

Since this holds for all $b \in \mathcal{N}^{++}$, it follows that $Q_{\alpha,z}(\psi\|\varphi) \leq Q_\alpha(\psi\circ\gamma\|\varphi\circ\gamma)$, which proves the DPI in this case.

Assume next the condition (ii), and put $p := \frac{z}{\alpha}$, $q := \frac{z}{\alpha-1}$, so that $p \in [1/2, 1)$ and $q \geq 1$. Using Theorem 2.4(ii), we get for any $b \in \mathcal{N}^+$,

$$\begin{aligned}Q_{\alpha,z}(\psi\|\varphi) &\geq \alpha\|h_\psi^{\frac{1}{2p}}\gamma(b)h_\psi^{\frac{1}{2p}}\|_p^p - (\alpha-1)\|h_\varphi^{\frac{1}{2q}}\gamma(b)h_\varphi^{\frac{1}{2q}}\|_q^q \\ &\geq \alpha\|h_{\psi\circ\gamma}^{\frac{1}{2p}}bh_{\psi\circ\gamma}^{\frac{1}{2p}}\|_p^p - (\alpha-1)\|h_{\varphi\circ\gamma}^{\frac{1}{2r}}bh_{\varphi\circ\gamma}^{\frac{1}{2q}}\|_q^q,\end{aligned}$$

here we used both (i) and (ii) in Lemma 3.2. Again, since this holds for all $b \in \mathcal{N}^+$, we get the desired inequality. □

3.2 Martingale convergence

An important consequence of DPI is the martingale convergence property that will be proved in this subsection. Here assume that $\mathcal{M}$ is a σ-finite von Neumann algebra.

Theorem 3.4. *Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$, and let $\{\mathcal{M}_i\}$ be an increasing net of von Neumann subalgebras of $\mathcal{M}$ containing the unit of $\mathcal{M}$ such that $\mathcal{M} = \big(\bigcup_i \mathcal{M}_i\big)''$. Assume that α and z satisfy the DPI bounds (that is, condition (i) or (ii) in Theorem 3.3). Then we have*

$$D_{\alpha,z}(\psi\|\varphi) = \lim_i D_{\alpha,z}(\psi|_{\mathcal{M}_i}\|\varphi|_{\mathcal{M}_i}) \quad \textit{increasingly.} \tag{3.6}$$

Proof. Let $\varphi_i := \varphi|_{\mathcal{M}_i}$ and $\psi_i := \psi|_{\mathcal{M}_i}$. From Theorem 3.3 it follows that $D_{\alpha,z}(\psi\|\varphi) \geq D_{\alpha,z}(\psi_i\|\varphi_i)$ for all i and $i \mapsto D_{\alpha,z}(\psi_i\|\varphi_i)$ is increasing. Hence, to show (3.6), it suffices to prove that

$$D_{\alpha,z}(\psi\|\varphi) \leq \sup_i D_{\alpha,z}(\psi_i\|\varphi_i). \tag{3.7}$$

To do this, we may assume that φ is faithful. Indeed, assume that (3.7) has been shown when φ is faithful. For general $\varphi \in \mathcal{M}_*^+$, from the assumption of $\mathcal{M}$ being σ-finite, there exists a $\varphi_0 \in \mathcal{M}_*^+$ with $s(\varphi_0) = \mathbf{1} - s(\varphi)$. Let $\varphi^{(n)} := \varphi + n^{-1}\varphi_0$ and $\varphi_i^{(n)} := \varphi^{(n)}|_{\mathcal{M}_i}$ for each $n \in \mathbb{N}$. Thanks to the lower semi-continuity in [21, Theorems 1(iv) and 2(iv)] and the order relation [21, Theorems 1(iii) and 2(iii)] we have

$$\begin{aligned} D_{\alpha,z}(\psi\|\varphi) &\leq \liminf_{n\to\infty} D_{\alpha,z}(\psi\|\varphi^{(n)}) \\ &\leq \liminf_{n\to\infty} \sup_i D_{\alpha,z}(\psi_i\|\varphi_i^{(n)}) \\ &\leq \sup_i D_{\alpha,z}(\psi_i\|\varphi_i), \end{aligned}$$

proving (3.7) for general φ. Below we assume the faithfulness of φ and write $\mathcal{E}_{\mathcal{M}_i,\varphi}$ for the generalized conditional expectation from $\mathcal{M}$ to $\mathcal{M}_i$ with respect to φ. Then from the martingale convergence given in [17, Theorem 3], it follows that

$$\psi_i \circ \mathcal{E}_{\mathcal{M}_i,\varphi} = \psi \circ \mathcal{E}_{\mathcal{M}_i,\varphi} \to \psi \quad \text{in the norm,} \tag{3.8}$$

as well as

$$\varphi_i \circ \mathcal{E}_{\mathcal{M}_i,\varphi} = \varphi \circ \mathcal{E}_{\mathcal{M}_i,\varphi} = \varphi. \tag{3.9}$$

Using lower semi-continuity and DPI, we obtain

$$D_{\alpha,z}(\psi\|\varphi) \leq \liminf_i D_{\alpha,z}(\psi_i \circ \mathcal{E}_{\mathcal{M}_i,\varphi}\|\varphi) \leq \liminf_i D_{\alpha,z}(\psi_i\|\varphi_i) \leq \sup_i D_{\alpha,z}(\psi_i\|\varphi_i).$$

□

The next proposition is another martingale type convergence.

Proposition 3.5. *Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$, and let $\{e_i\}$ be an increasing net of projections in $\mathcal{M}$ such that $e_i \nearrow \mathbf{1}$. Either if $0 < \alpha < 1$ and $z > 0$, or if $\alpha > 1$ and $\max\{\alpha/2, \alpha - 1\} \leq z \leq \alpha$ (that is, α and z satisfy the DPI bounds when $\alpha > 1$), then*

$$D_{\alpha,z}(\psi\|\varphi) = \lim_i D_{\alpha,z}(e_i\psi e_i\|e_i\varphi e_i), \tag{3.10}$$

where $e_i\psi e_i := \psi|_{e_i\mathcal{M}e_i}$ and $e_i\varphi e_i := \varphi|_{e_i\mathcal{M}e_i}$.

Proof. Let $\psi_i, \varphi_i \in \mathcal{M}_*^+$ be such that $\psi_i := \psi(e_i \cdot e_i)$ and $\varphi_i := \varphi(e_i \cdot e_i)$. Since $s(\psi_i) \le e_i$ and $e_i\psi e_i = \psi_i|_{e_i\mathcal{M}e_i}$ and similarly for φ_i, we note (see Remark 2.2) that

$$D_{\alpha,z}(e_i\psi e_i\|e_i\varphi e_i) = D_{\alpha,z}(\psi_i|_{e_i\mathcal{M}e_i}\|\varphi_i|_{e_i\mathcal{M}e_i}) = D_{\alpha,z}(\psi_i\|\varphi_i)$$

for all $\alpha, z > 0$, $\alpha \ne 1$. Moreover, we have

$$\begin{aligned}\|\psi - \psi_i\| = \|h_\psi - e_n h_\psi e_n\|_1 &\le \|(\mathbf{1} - e_n)h_\psi\|_1 + \|e_n h_\psi(\mathbf{1} - e_n)\|_1 \\ &\le \|(\mathbf{1} - e_n)h_\psi^{1/2}\|_2\|h_\psi^{1/2}\|_2 + \|e_n h_\psi^{1/2}\|_2\|h_\psi^{1/2}(\mathbf{1} - e_n)\|_2 \\ &= \psi(\mathbf{1} - e_n)^{1/2}\psi(\mathbf{1})^{1/2} + \psi(e_n)^{1/2}\psi(\mathbf{1} - e_n)^{1/2} \to 0 \quad \text{as } n \to \infty,\end{aligned}$$

and similarly $\|\varphi - \varphi_i\|_1 \to 0$. Hence, when $0 < \alpha < 1$ and $z > 0$, the joint continuity of $Q_{\alpha,z}$ in [21, Theorem 1(iv)] gives (3.10).

Next, assume that $\alpha > 1$ and α, z satisfy the DPI bounds. Let $\mathcal{M}_i := e_i\mathcal{M}e_i \oplus \mathbb{C}(\mathbf{1} - e_i)$; then $\{\mathcal{M}_i\}$ is an increasing net of von Neumann subalgebras of $\mathcal{M}$ containing the unit of $\mathcal{M}$ with $\mathcal{M} = \left(\bigcup_i \mathcal{M}_i\right)''$. Since $\psi|_{\mathcal{M}_i} = e_i\psi e_i \oplus \psi(\mathbf{1} - e_i)$ and similarly for $\varphi|_{\mathcal{M}_i}$, it follows from Theorem 3.4 and [21, Theorems 1(ii) and 2(ii)] that

$$Q_{\alpha,z}(\psi\|\varphi) = \lim_i \left[Q_{\alpha,z}(e_i\psi e_i\|e_i\varphi e_i) + \psi(\mathbf{1} - e_i)^\alpha\varphi(\mathbf{1} - e_i)^{1-\alpha}\right].$$

Here, $\psi(\mathbf{1} - e_i)^\alpha\varphi(\mathbf{1} - e_i)^{1-\alpha}$ is defined with the usual conventions

$$0^{1-\alpha} := \begin{cases} 0 & (0 < \alpha < 1), \\ \infty & (\alpha > 1), \end{cases} \qquad \lambda \cdot \infty := \begin{cases} 0 & (\lambda = 0), \\ \infty & (\lambda > 0). \end{cases}$$

Then (3.10) holds if we show the following:

(1) If $Q_{\alpha,z}(\psi\|\varphi) = \infty$, then $\lim_i Q_{\alpha,z}(e_i\psi e_i\|e_i\varphi e_i) = \infty$.

(2) If $Q_{\alpha,z}(\psi\|\varphi) < \infty$, then $\lim_i \psi(\mathbf{1} - e_i)^\alpha\varphi(\mathbf{1} - e_i)^{1-\alpha} = 0$.

These two facts can be shown in the same way as in the proof of [12, Theorem 4.5], whose details are omitted here. ☐

4 Equality in DPI and reversibility of channels

In what follows, a channel is a normal 2-positive unital map $\gamma : \mathcal{N} \to \mathcal{M}$.

Definition 4.1. Let $\gamma : \mathcal{N} \to \mathcal{M}$ be a channel and let $\mathcal{S} \subset \mathcal{M}_*^+$. We say that γ is reversible (or sufficient) with respect to $\mathcal{S}$ if there exists a channel $\beta : \mathcal{M} \to \mathcal{N}$ such that

$$\rho \circ \gamma \circ \beta = \rho \quad \text{for all } \rho \in \mathcal{S}.$$

The notion of sufficient channels was introduced by Petz [32, 33], who also obtained a number of conditions characterizing this situation. It particular, it was proved in [33] that sufficient channels can be characterized by equality in the DPI for the relative entropy $D(\psi\|\varphi)$: if $D(\psi\|\varphi) < \infty$, then a channel γ is sufficient with respect to $\{\psi, \varphi\}$ if and only if

$$D(\psi \circ \gamma\|\varphi \circ \gamma) = D(\psi\|\varphi).$$

This characterization has been proved for a number of other divergence measures, including the standard Rényi divergences $D_{\alpha,1}$ with $0 < \alpha < 2$ and the sandwiched Rényi divergences $D_{\alpha,\alpha}$ for $\alpha > 1/2$ ([14, 18, 19]). Another important result of [33] shows that the Petz dual γ_φ^* is a universal recovery map, in the sense given in the proposition below.

Proposition 4.2. *Let $\gamma : \mathcal{N} \to \mathcal{M}$ be a channel and let $\varphi \in \mathcal{M}_*^+$ be such that both φ and $\varphi \circ \gamma$ are faithful. Then the following hold:*

(i) For any $\psi \in \mathcal{M}_*^+$, γ is sufficient with respect to $\{\psi, \varphi\}$ if and only if $\psi \circ \gamma \circ \gamma_\varphi^* = \psi$.

(ii) There is a faithful normal conditional expectation $\mathcal{E}$ from $\mathcal{M}$ onto a von Neumann subalgebra of $\mathcal{M}$ such that $\varphi\circ\mathcal{E} = \varphi$, and γ is sufficient with respect to $\{\psi, \varphi\}$ if and only if also $\psi\circ\mathcal{E} = \psi$.

Note (see [33, Theorem 2] and the proof of [33, Theorem 3]) that the range of the conditional expectation $\mathcal{E}$ in statement (ii) above is the set of fixed points of the channel $\gamma \circ \gamma_\varphi^*$.

Our aim in this section is to prove that equality in the DPI for $D_{\alpha,z}$ with values of the parameters (strictly) contained in the DPI bounds of Theorem 3.3 characterizes sufficiency of channels. Throughout this section, we use the notations $\psi_0 := \psi \circ \gamma$ and $\varphi_0 := \varphi \circ \gamma$. We also denote

$$p := \frac{z}{\alpha}, \qquad r := \frac{z}{1-\alpha}, \qquad q := -r = \frac{z}{\alpha - 1}.$$

4.1 The case $\alpha \in (0, 1)$

Here we study equality in the DPI for $D_{\alpha,z}$ with $\alpha \in (0, 1)$, for a pair of positive normal functionals $\psi, \varphi \in \mathcal{M}_*^+$ and a normal positive unital map $\gamma : \mathcal{N} \to \mathcal{M}$. We first prove some equality conditions in the case $\psi \sim \varphi$.

Proposition 4.3. *Let $0 < \alpha < 1$, $\max\{\alpha, 1 - \alpha\} \le z$. Assume that $\psi \sim \varphi$ and let $\gamma : \mathcal{N} \to \mathcal{M}$ be a normal positive unital map. Let $\bar{a} \in \mathcal{M}^{++}$ be the unique minimizer as in Lemma 2.5 for $Q_{\alpha,z}(\psi\|\varphi)$ and let $\bar{a}_0 \in \mathcal{N}^{++}$ be the minimizer for $Q_{\alpha,z}(\psi_0\|\varphi_0)$. The following conditions are equivalent:*

(i) $D_{\alpha,z}(\psi_0\|\varphi_0) = D_{\alpha,z}(\psi\|\varphi)$, i.e., $Q_{\alpha,z}(\psi_0\|\varphi_0) = Q_{\alpha,z}(\psi\|\varphi)$.

(ii) $\gamma(\bar{a}_0) = \bar{a}$ and $\left\|h_\psi^{\frac{1}{2p}}\gamma(\bar{a}_0)h_\psi^{\frac{1}{2p}}\right\|_p = \left\|h_{\psi_0}^{\frac{1}{2p}}\bar{a}_0h_{\psi_0}^{\frac{1}{2p}}\right\|_p$.

(iii) $\left\|h_\psi^{\frac{1}{2p}}\bar{a}h_\psi^{\frac{1}{2p}}\right\|_p = \left\|h_{\psi_0}^{\frac{1}{2p}}\bar{a}_0h_{\psi_0}^{\frac{1}{2p}}\right\|_p$.

(iv) $\gamma(\bar{a}_0^{-1}) = \bar{a}^{-1}$ and $\left\|h_\varphi^{\frac{1}{2r}}\gamma(\bar{a}_0^{-1})h_\varphi^{\frac{1}{2r}}\right\|_r = \left\|h_{\varphi_0}^{\frac{1}{2r}}\bar{a}_0^{-1}h_{\varphi_0}^{\frac{1}{2r}}\right\|_r$.

(v) $\left\|h_\varphi^{\frac{1}{2r}}\bar{a}^{-1}h_\varphi^{\frac{1}{2r}}\right\|_r = \left\|h_{\varphi_0}^{\frac{1}{2r}}\bar{a}_0^{-1}h_{\varphi_0}^{\frac{1}{2r}}\right\|_r$.

Proof. Since $\psi \sim \varphi$ by assumption and hence $\psi_0 \sim \varphi_0$, we have $s(\psi) = s(\varphi)$ and $s(\psi_0) = s(\varphi_0)$. Using restrictions explained in the beginning of Sec. 2, we may assume that all $\psi, \varphi, \psi_0, \varphi_0$ are faithful.

(i) $\Longrightarrow$ (ii) & (iv). By Lemma 3.2(ii)

$$\left\| h_\psi^{\frac{1}{2p}} \gamma(\bar{a}_0) h_\psi^{\frac{1}{2p}} \right\|_p \le \left\| h_{\psi_0}^{\frac{1}{2p}} \bar{a}_0 h_{\psi_0}^{\frac{1}{2p}} \right\|_p, \tag{4.1}$$

and by (3.5) we have

$$\left\| h_\varphi^{\frac{1}{2r}} \gamma(\bar{a}_0)^{-1} h_\varphi^{\frac{1}{2r}} \right\|_r \le \left\| h_\varphi^{\frac{1}{2r}} \gamma(\bar{a}_0^{-1}) h_\varphi^{\frac{1}{2r}} \right\|_r \le \left\| h_{\varphi_0}^{\frac{1}{2r}} \bar{a}_0^{-1} h_{\varphi_0}^{\frac{1}{2r}} \right\|_r. \tag{4.2}$$

From (4.1) and (4.2) it follows that

$$\begin{aligned} Q_{\alpha,z}(\psi\|\varphi) &\le \alpha \left\| h_\psi^{\frac{1}{2p}} \gamma(\bar{a}_0) h_\psi^{\frac{1}{2p}} \right\|_p^p + (1-\alpha) \left\| h_\varphi^{\frac{1}{2r}} \gamma(\bar{a}_0)^{-1} h_\varphi^{\frac{1}{2r}} \right\|_r^r \\ &\le \alpha \left\| h_{\psi_0}^{\frac{1}{2p}} \bar{a}_0 h_{\psi_0}^{\frac{1}{2p}} \right\|_p^p + (1-\alpha) \left\| h_{\varphi_0}^{\frac{1}{2r}} \bar{a}_0^{-1} h_{\varphi_0}^{\frac{1}{2r}} \right\|_r^r = Q_{\alpha,z}(\psi_0\|\varphi_0) = Q_{\alpha,z}(\psi\|\varphi). \end{aligned}$$

By uniqueness in Lemma 2.5 we find that $\gamma(\bar{a}_0) = \bar{a}$ and all the inequalities in (4.1) and (4.2) must become equalities. Since $\gamma(\bar{a}_0^{-1}) \ge \gamma(\bar{a}_0)^{-1}$, we see by Lemma A.2 that the equality in (4.2) yields $\gamma(\bar{a}_0^{-1}) = \gamma(\bar{a}_0)^{-1} = \bar{a}^{-1}$. Therefore, (ii) and (iv) hold.

The implications (ii) $\Longrightarrow$ (iii) and (iv) $\Longrightarrow$ (v) are obvious.

(iii) $\Longrightarrow$ (i). By (iii) with the equality (2.4) in Lemma 2.5 we have

$$\begin{aligned} Q_{\alpha,z}(\psi\|\varphi) &= \mathrm{tr}\Big(h_\psi^{\frac{1}{2p}} h_\varphi^{\frac{1}{r}} h_\psi^{\frac{1}{2p}} \Big)^z = \mathrm{tr}\Big(h_\psi^{\frac{1}{2p}} \bar{a} h_\psi^{\frac{1}{2p}} \Big)^p \\ &= \mathrm{tr}\Big(h_{\psi_0}^{\frac{1}{2p}} \bar{a}_0 h_{\psi_0}^{\frac{1}{2p}} \Big)^p = \mathrm{tr}\Big(h_{\psi_0}^{\frac{1}{2p}} h_{\varphi_0}^{\frac{1}{r}} h_{\psi_0}^{\frac{1}{2p}} \Big)^z = Q_{\alpha,z}(\psi_0\|\varphi_0). \end{aligned}$$

(v) $\Longrightarrow$ (i). Similarly, by (v) with the equality (2.5) in Lemma 2.5 we have

$$\begin{aligned} Q_{\alpha,z}(\psi\|\varphi) &= \mathrm{tr}\Big(h_\varphi^{\frac{1}{2r}} h_\psi^{\frac{1}{p}} h_\varphi^{\frac{1}{2r}} \Big)^z = \mathrm{tr}\Big(h_\varphi^{\frac{1}{2r}} \bar{a}^{-1} h_\varphi^{\frac{1}{2r}} \Big)^r \\ &= \mathrm{tr}\Big(h_{\varphi_0}^{\frac{1}{2r}} \bar{a}_0^{-1} h_{\varphi_0}^{\frac{1}{2r}} \Big)^r = \mathrm{tr}\Big(h_{\varphi_0}^{\frac{1}{2r}} h_{\psi_0}^{\frac{1}{p}} h_{\varphi_0}^{\frac{1}{2r}} \Big)^z = Q_{\alpha,z}(\psi_0\|\varphi_0). \end{aligned}$$

□

Remark 4.4. Note that the above conditions extend the results obtained in [27] and [38] in the finite-dimensional case. Indeed, as seen from (2.5), the first condition in (ii) with $\alpha = z$ is equivalent to the condition in [27, Theorem 1] (obtained under the more general assumption that $s(\psi) \le s(\varphi)$). Since $p = 1$ in this case and $\big\| h_\psi^{\frac{1}{2}} \gamma(a_0) h_\psi^{\frac{1}{2}} \big\|_1 = \mathrm{tr}\big(\gamma(\alpha_0) h_\psi\big) = \big\| h_{\psi_0}^{\frac{1}{2}} a_0 h_{\psi_0}^{\frac{1}{2}} \big\|_1$ for any $a_0 \in \mathcal{N}$, the second condition in (ii) is automatic. Moreover, (ii) extends the necessary condition in [38, Theorem 1.2(2)] to a necessary and sufficient one. While in both of these works γ was required to be completely positive, only positivity is enough for our result.

Theorem 4.5. *Let $0 < \alpha < 1$ and $\max\{\alpha, 1-\alpha\} \le z$. Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \ne 0$, and assume either that $\alpha < z$ and $s(\varphi) \le s(\psi)$, or that $1 - \alpha < z$ and $s(\psi) \le s(\varphi)$. Then a channel $\gamma : \mathcal{N} \to \mathcal{M}$ is reversible with respect to $\{\psi, \varphi\}$ if and only if*

$$D_{\alpha,z}(\psi\|\varphi) = D_{\alpha,z}(\psi \circ \gamma \| \varphi \circ \gamma).$$

Proof. This proof is a modification of the proof of [19, Theorem 5.1]. We will assume that $s(\varphi) \le s(\psi)$ and $\alpha < z$, that is, $p > 1$. In the other case we may exchange the roles of p, r and of ψ, φ by the equality $Q_{\alpha,z}(\psi\|\varphi) = Q_{1-\alpha,z}(\varphi\|\psi)$. As before, we may assume that both ψ and ψ_0 are faithful.

The strategy of the proof is to use known results in [18] for the sandwiched Rényi divergence $D_{p,p}$ with $p > 1$. For this, let $\mu, \omega \in \mathcal{M}_*^+$ be such that

$$h_\mu = \left| h_\varphi^{\frac{1}{2r}} h_\psi^{\frac{1}{2p}} \right|^{2z}, \qquad h_\omega = h_\psi^{\frac{p-1}{2p}} h_\mu^{\frac{1}{p}} h_\psi^{\frac{p-1}{2p}},$$

and notice that

$$Q_{z,\alpha}(\psi\|\varphi) = Q_{p,p}(\omega\|\psi).$$

Let $\mu_0, \omega_0 \in \mathcal{N}_*^+$ be similar functionals obtained from ψ_0, φ_0. Then we have the equality

$$Q_{p,p}(\omega_0\|\psi_0) = Q_{\alpha,z}(\psi_0\|\varphi_0) = Q_{\alpha,z}(\psi\|\varphi) = Q_{p,p}(\omega\|\psi). \tag{4.3}$$

Our first goal is to show that $\omega_0 = \omega \circ \gamma$, which implies by [18, Theorem 4.6] that γ is sufficient with respect to $\{\omega, \psi\}$. We then apply Proposition 4.2, and the properties of the extensions of the conditional expectation $\mathcal{E}$ to the Haagerup L^p-spaces proved in [20], see also Appendix A.

Let us remark here that if $\psi \sim \varphi$, it follows from (2.4) that $h_\omega = h_\psi^{\frac{1}{2}} \bar{a} h_\psi^{\frac{1}{2}}$ and $h_{\omega_0} = h_{\psi_0}^{\frac{1}{2}} \bar{a}_0 h_{\psi_0}^{\frac{1}{2}}$. Hence from (3.1) and condition (ii) in Proposition 4.3, we immediately have

$$(\gamma_\psi^*)_*(h_{\omega_0}) = h_\psi^{\frac{1}{2}} \gamma(\bar{a}_0) h_\psi^{\frac{1}{2}} = h_\omega, \quad \text{i.e.,} \quad \omega_0 \circ \gamma_\psi^* = \omega,$$

as well as $\psi_0 \circ \gamma_\psi^* = \psi$ by (3.2). These and (4.3) show that γ_ψ^* is sufficient with respect to $\{\omega_0, \psi_0\}$. By Proposition 4.2 and the fact that the Petz dual $(\gamma_\psi^*)_{\psi_0}^*$ is γ itself, this implies the desired equality

$$\omega \circ \gamma = \omega_0 \circ \gamma_\psi^* \circ \gamma = \omega_0.$$

In the case $\psi \not\sim \varphi$ some more work is required. Let $\psi_n := \psi + \frac{1}{n}\varphi$ and $\varphi_n := \varphi + \frac{1}{n}\psi$. Then all ψ_n, φ_n are faithful, $\psi_n \to \psi$, $\varphi_n \to \varphi$ in $\mathcal{M}_*^+$, and moreover, $\psi_n \sim \varphi_n$ for all n. Then $\psi_n \circ \gamma \sim \varphi_n \circ \gamma$, $\psi_n \circ \gamma \to \psi_0$, $\varphi_n \circ \gamma \to \varphi_0$ and by the joint continuity of $Q_{\alpha,z}$ in [21, Theorem 1(iv)], we have

$$\lim_n Q_{\alpha,z}(\psi_n \circ \gamma\|\varphi_n \circ \gamma) = Q_{\alpha,z}(\psi_0\|\varphi_0) = Q_{\alpha,z}(\psi\|\varphi) = \lim_n Q_{\alpha,z}(\psi_n\|\varphi_n).$$

Let $\bar{b}_n \in \mathcal{N}^{++}$ be the minimizer for the variational expression for $Q_{\alpha,z}(\psi_n \circ \gamma\|\varphi_n \circ \gamma)$ given in (2.1). Let also $\bar{a}_n$ be the minimizer for $Q_{\alpha,z}(\psi_n\|\varphi_n)$, and let $f_n : \mathcal{M}^{++} \to \mathbb{R}^+$ be the function minimized in the expression for $Q_{\alpha,z}(\psi_n\|\varphi_n)$ (see (2.8)). We then have

$$\begin{aligned} Q_{\alpha,z}(\psi_n \circ \gamma\|\varphi_n \circ \gamma) - f_n(\gamma(\bar{b}_n)) &= \alpha \left(\left\| h_{\psi_n\circ\gamma}^{\frac{1}{2p}} \bar{b}_n h_{\psi_n\circ\gamma}^{\frac{1}{2p}} \right\|_p^p - \left\| h_{\psi_n}^{\frac{1}{2p}} \gamma(\bar{b}_n) h_{\psi_n}^{\frac{1}{2p}} \right\|_p^p \right) \\ &\quad + (1-\alpha) \left(\left\| h_{\varphi_n\circ\gamma}^{\frac{1}{2r}} \bar{b}_n^{-1} h_{\varphi_n\circ\gamma}^{\frac{1}{2r}} \right\|_r^r - \left\| h_{\varphi_n}^{\frac{1}{2r}} \gamma(\bar{b}_n)^{-1} h_{\varphi_n}^{\frac{1}{2r}} \right\|_r^r \right) \ge 0, \end{aligned}$$

where the inequality follows from Lemma 3.2(ii) and (3.5). We obtain

$$Q_{\alpha,z}(\psi_n \circ \gamma\|\varphi_n \circ \gamma) - Q_{\alpha,z}(\psi_n\|\varphi_n) \ge f_n(\gamma(\bar{b}_n)) - Q_{\alpha,z}(\psi_n\|\varphi_n) \ge 0. \tag{4.4}$$

Now let $\mu_{n,0} \in \mathcal{N}_*^+$ and $\mu_n \in \mathcal{M}_*^+$ be such that (using (2.4) in Lemma 2.5)

$$h_{\mu_{n,0}}^{\frac{1}{p}} = \left| h_{\varphi_n\circ\gamma}^{\frac{1}{2r}} h_{\psi_n\circ\gamma}^{\frac{1}{2p}} \right|^{2\alpha} = h_{\psi_n\circ\gamma}^{\frac{1}{2p}} \bar{b}_n h_{\psi_n\circ\gamma}^{\frac{1}{2p}}, \qquad h_{\mu_n}^{\frac{1}{p}} = \left| h_{\varphi_n}^{\frac{1}{2r}} h_{\psi_n}^{\frac{1}{2p}} \right|^{2\alpha} = h_{\psi_n}^{\frac{1}{2p}} \bar{a}_n h_{\psi_n}^{\frac{1}{2p}}.$$

Then $h_{\mu_{n,0}}^{\frac{1}{p}} \to h_{\mu_0}^{\frac{1}{p}}$ in $L^p(\mathcal{N})$, this follows from the Hölder inequality and the fact [24] that the map $L^{2z}(\mathcal{N}) \ni h \mapsto |h|^{2\alpha} \in L^p(\mathcal{N})$ is continuous in the norm. Similarly, $h_{\mu_n}^{\frac{1}{p}} \to h_{\mu}^{\frac{1}{p}}$ in $L^p(\mathcal{M})$. Next, note that since $Q_{\alpha,z}(\psi_n\circ\gamma\|\varphi_n\circ\gamma)$ and $Q_{\alpha,z}(\psi_n\|\varphi_n)$ have the same limit, we see from (4.4) that $f_n(\gamma(\bar{b}_n)) - Q_{\alpha,z}(\psi_n\|\varphi_n) \to 0$. Moreover, by Lemma 3.2(ii) and (2.7) note that

$$\sup_n \left\| h_{\psi_n}^{\frac{1}{2p}} \gamma(\bar{b}_n) h_{\psi_n}^{\frac{1}{2p}} \right\|_p^p \le \sup_n \left\| h_{\psi_n\circ\gamma}^{\frac{1}{2p}} \bar{b}_n h_{\psi_n\circ\gamma}^{\frac{1}{2p}} \right\|_p^p = \sup_n Q_{\alpha,z}(\psi_n\circ\gamma\|\varphi_n\circ\gamma) < \infty.$$

Therefore, since $\big\| h_{\psi_n}^{\frac{1}{2p}} \gamma(\bar{b}_n) h_{\psi_n}^{\frac{1}{2p}} - h_{\mu_n}^{\frac{1}{p}} \big\|_p$ means $\|\xi_p(\gamma(\bar{b}_n)) - \xi_p(\bar{a}_n)\|_p$ defined for ψ_n (in place of ψ), it follows from Lemma 2.6 that $h_{\psi_n}^{\frac{1}{2p}} \gamma(\bar{b}_n) h_{\psi_n}^{\frac{1}{2p}} - h_{\mu_n}^{\frac{1}{p}} \to 0$ in $L^p(\mathcal{M})$. On the other hand, let $\gamma_{\psi_n,p}^*$, $\gamma_{\psi,p}^*$ be the contractions defined in Lemma 3.1. We then have

$$h_{\psi_n}^{\frac{1}{2p}} \gamma(\bar{b}_n) h_{\psi_n}^{\frac{1}{2p}} = \gamma_{\psi_n,p}^*\big(h_{\mu_{n,0}}^{\frac{1}{p}}\big)$$

and since $\gamma_{\psi_n,p}^*(k) \to \gamma_{\psi,p}^*(k)$ in $L^p(\mathcal{M})$ for any $k \in L^p(s(\psi\circ\gamma)\mathcal{N}s(\psi\circ\gamma))$ by Lemma 3.1, we have

$$\left\| \gamma_{\psi,p}^*\big(h_{\mu_0}^{\frac{1}{p}}\big) - \gamma_{\psi_n,p}^*\big(h_{\mu_{n,0}}^{\frac{1}{p}}\big) \right\|_p \le \left\| (\gamma_{\psi,p}^* - \gamma_{\psi_n,p}^*)\big(h_{\mu_0}^{\frac{1}{p}}\big) \right\|_p + \left\| \gamma_{\psi_n,p}^*\big(h_{\mu_0}^{\frac{1}{p}} - h_{\mu_{n,0}}^{\frac{1}{p}}\big) \right\|_p \to 0.$$

Putting all together, we obtain that

$$h_{\mu}^{\frac{1}{p}} = \lim_n h_{\mu_n}^{\frac{1}{p}} = \lim_n \gamma_{\psi_n,p}^*(h_{\mu_{n,0}}^{\frac{1}{p}}) = \gamma_{\psi,p}^*(h_{\mu_0}^{\frac{1}{p}}).$$

It follows from (3.4) that

$$(\gamma_\psi^*)_*(h_{\omega_0}) = h_\psi^{\frac{p-1}{2p}} \gamma_{\psi,p}^*(h_{\mu_0}^{\frac{1}{p}}) h_\psi^{\frac{p-1}{2p}} = h_\psi^{\frac{p-1}{2p}} h_\mu^{\frac{1}{p}} h_\psi^{\frac{p-1}{2p}} = h_\omega.$$

As we have seen in the case $\psi \sim \varphi$ above, this and (4.3) imply that

$$\omega\circ\gamma = \omega_0 \circ \gamma_\psi^* \circ \gamma = \omega_0.$$

Therefore, we have shown that γ is sufficient with respect to $\{\omega, \psi\}$.

Next, let $\mathcal{E}$ be the faithful normal conditional expectation onto the set of fixed points of $\gamma\circ\gamma_\psi^*$ (see a note after Proposition 4.2). Then by Proposition 4.2(ii), $\mathcal{E}$ preserves both ψ and ω. Since ψ is assumed faithful, we may construct the Haagerup L^p-spaces with respect to ψ, see Remark 2.2. We then have the extensions of $\mathcal{E}$ to any L^p-space as in Appendix A, such that by the bimodule property (A.2),

$$h_\psi^{\frac{p-1}{2p}} h_\mu^{\frac{1}{p}} h_\psi^{\frac{p-1}{2p}} = h_\omega = \mathcal{E}_*(h_\omega) = h_\psi^{\frac{p-1}{2p}} \mathcal{E}_p\big(h_\mu^{\frac{1}{p}}\big) h_\psi^{\frac{p-1}{2p}}.$$

It follows that $\big|h_\varphi^{\frac{1}{2r}} h_\psi^{\frac{1}{2p}}\big|^{2\alpha} = h_\mu^{\frac{1}{p}} \in L^p(\mathcal{E}(\mathcal{M}))$ and consequently $\big|h_\varphi^{\frac{1}{2r}} h_\psi^{\frac{1}{2p}}\big| = h_\mu^{\frac{1}{2z}} \in L^{2z}(\mathcal{E}(\mathcal{M}))^+$. Note that by the assumptions we must have $2z > 1$. Let $h_\varphi^{\frac{1}{2r}} h_\psi^{\frac{1}{2p}} = u\big|h_\varphi^{\frac{1}{2r}} h_\psi^{\frac{1}{2p}}\big|$ be the polar decomposition in $L^{2z}(\mathcal{M})$; then we have by using (A.2) again,

$$u^* h_\varphi^{\frac{1}{2r}} h_\psi^{\frac{1}{2p}} = \mathcal{E}_{2z}\big(u^* h_\varphi^{\frac{1}{2r}} h_\psi^{\frac{1}{2p}}\big) = \mathcal{E}_{2r}\big(u^* h_\varphi^{\frac{1}{2r}}\big) h_\psi^{\frac{1}{2p}},$$

which implies that $u^*h_\varphi^{\frac{1}{2r}} \in L^{2r}(\mathcal{E}(\mathcal{M}))$. Since ψ is faithful, we have

$$\ker\Big(\big(h_\varphi^{\frac{1}{2r}} h_\psi^{\frac{1}{2p}}\big)^*\Big) = \ker\big(h_\psi^{\frac{1}{2p}} h_\varphi^{\frac{1}{2r}}\big) = \ker h_\varphi^{\frac{1}{2r}} = \ker h_\varphi,$$

which implies that $uu^* = s(\varphi)$. Hence by uniqueness of the polar decomposition in $L^{2r}(\mathcal{M})$ and $L^{2r}(\mathcal{E}(\mathcal{M}))$, we obtain that $h_\varphi^{\frac{1}{2r}} \in L^{2r}(\mathcal{E}(\mathcal{M}))^+$ and $u \in \mathcal{E}(\mathcal{M})$. Therefore, we must have $h_\varphi \in L^1(\mathcal{E}(\mathcal{M}))$, so that $\varphi \circ \mathcal{E} = \varphi$ and γ is sufficient with respect to $\{\psi, \varphi\}$ by Proposition 4.2(ii) again. □

4.2 The case $\alpha > 1$

We now turn to the case $\alpha > 1$. Then within the DPI bounds, we have $p = \frac{z}{\alpha} \in [1/2, 1]$ and $q = \frac{z}{\alpha-1} \geq 1$, and we note that we always have $p < q$. Here we need to assume that $D_{\alpha,z}(\psi\|\varphi) < \infty$, so that by Lemma 2.3 there is some (unique) $y \in L^{2z}(\mathcal{M})$ such that

$$h_\psi^{\frac{1}{2p}} = y h_\varphi^{\frac{1}{2q}}.$$

By the proof of Theorem 2.4, we have the following variational expression

$$Q_{\alpha,z}(\psi\|\varphi) = \sup_{w \in L^q(\mathcal{M})^+} \big\{\alpha \mathrm{tr}\big((ywy^*)^p\big) - (\alpha-1)\mathrm{tr}\big(w^q\big)\big\}. \tag{4.5}$$

Indeed, we note that x in the proof of Theorem 2.4 is y^*y and $\mathrm{tr}\big((x^{\frac{1}{2}}wx^{\frac{1}{2}})^p\big)$ in expression (2.3) is rewritten as $\mathrm{tr}\big((|y|w|y|)^p\big) = \mathrm{tr}\big((ywy^*)^p\big)$. The supremum is attained at a unique point $\bar{w} = (y^*y)^{\alpha-1} \in L^q(\mathcal{M})^+$, uniqueness follows from strict concavity of the function $w \mapsto \alpha\mathrm{tr}\big((ywy^*)^p\big) - (\alpha-1)\mathrm{tr}\big(w^q\big)$.

By DPI, we have $D_{\alpha,z}(\psi_0\|\varphi_0) \leq D_{\alpha,z}(\psi\|\varphi) < \infty$, so that there is some (unique) $y_0 \in L^{2z}(\mathcal{N})$ such that

$$h_{\psi_0}^{\frac{1}{2p}} = y_0 h_{\varphi_0}^{\frac{1}{2q}}.$$

Since $D_{\alpha,z}(\psi\|\varphi) < \infty$ implies that $s(\psi) \leq s(\varphi)$, we may assume that both φ and φ_0 are faithful.

Lemma 4.6. *Let $\gamma : \mathcal{N} \to \mathcal{M}$ be a normal positive unital map. Let $\gamma^*_{\varphi,q} : L^q(\mathcal{N}) \to L^q(\mathcal{M})$ be the contraction as in lemma 3.1. Let $\bar{w} := (y^*y)^{\alpha-1} \in L^q(\mathcal{M})$ and $\bar{w}_0 := (y_0^*y_0)^{\alpha-1} \in L^q(\mathcal{N})$. Then equality in the DPI for $D_{\alpha,z}(\psi\|\varphi)$ holds if and only if*

$$\bar{w} = \gamma^*_{\varphi,q}(\bar{w}_0) \quad \textit{and} \quad \mathrm{tr}\big(\bar{w}_0^q\big) = \mathrm{tr}\big(\gamma^*_{\varphi,q}(\bar{w}_0)^q\big). \tag{4.6}$$

Proof. We first show that for any $w_0 \in L^q(\mathcal{N})^+$,

$$\mathrm{tr}\big((y\gamma^*_{\varphi,q}(w_0)y^*)^p\big) \geq \mathrm{tr}\big((y_0w_0y_0^*)^p\big). \tag{4.7}$$

Let us first assume that $w_0 = h_{\varphi_0}^{\frac{1}{2q}} b h_{\varphi_0}^{\frac{1}{2q}}$ for some $b \in \mathcal{N}^+$. Then $\gamma^*_{\varphi,q}(w_0) = h_\varphi^{\frac{1}{2q}}\gamma(b)h_\varphi^{\frac{1}{2q}}$. Therefore

$$\begin{aligned}\mathrm{tr}\big((y\gamma^*_{\varphi,q}(w_0)y^*)^p\big) &= \mathrm{tr}\Big(\big(yh_\varphi^{\frac{1}{2q}}\gamma(b)h_\varphi^{\frac{1}{2q}}y^*\big)^p\Big) = \mathrm{tr}\Big(\big(h_\psi^{\frac{1}{2p}}\gamma(b)h_\psi^{\frac{1}{2p}}\big)^p\Big) \geq \mathrm{tr}\Big(\big(h_{\psi_0}^{\frac{1}{2p}}bh_{\psi_0}^{\frac{1}{2p}}\big)^p\Big)\\ &= \mathrm{tr}\Big(\big(y_0h_{\varphi_0}^{\frac{1}{2q}}bh_{\varphi_0}^{\frac{1}{2q}}y_0^*\big)^p\Big) = \mathrm{tr}\big((y_0w_0y_0^*)^p\big),\end{aligned}$$

where the inequality is from Lemma 3.2(i). The proof of inequality (4.7) is finished by Lemma A.1. By using this and the fact that $\gamma^*_{\varphi,q}$ is a contraction, if follows from the variational expression in (4.5) that

$$\begin{aligned} Q_{\alpha,z}(\psi\|\varphi) &\geq \alpha\mathrm{tr}\big((y\gamma^*_{\varphi,q}(\bar{w}_0)y^*)^p\big) - (\alpha-1)\mathrm{tr}\big(\gamma^*_{\varphi,q}(\bar{w}_0)^q\big) \\ &\geq \alpha\mathrm{tr}\big((y_0\bar{w}_0y_0^*)^p\big) - (\alpha-1)\mathrm{tr}\big(\bar{w}_0^q\big) = Q_{\alpha,z}(\psi_0\|\varphi_0). \end{aligned}$$

Suppose that $Q_{\alpha,z}(\psi_0\|\varphi_0) = Q_{\alpha,z}(\psi\|\varphi)$, then both the inequalities must be equalities. Since $\bar{w} \in L^q(\mathcal{M})^+$ and $\bar{w}_0 \in L^q(\mathcal{N})^+$ are the unique elements such that the suprema in the respective variational expressions in (4.5) for $Q_{\alpha,z}(\psi\|\varphi)$ and $Q_{\alpha,z}(\psi_0\|\varphi_0)$ are attained, this proves (4.6). Conversely, if the equalities in (4.6) hold, then

$$Q_{\alpha,z}(\psi_0\|\varphi_0) = \mathrm{tr}\big((y_0^*y_0)^z\big) = \mathrm{tr}\big(\bar{w}_0^q\big) = \mathrm{tr}\big(\bar{w}^q\big) = \mathrm{tr}\big((y^*y)^z\big) = Q_{\alpha,z}(\psi\|\varphi).$$

□

Theorem 4.7. *Assume that $1 < \alpha < z+1$. Let $\gamma : \mathcal{N} \to \mathcal{M}$ be a channel and let $\psi, \varphi \in \mathcal{M}_*^+$ be such that $s(\psi) \leq s(\varphi)$ and $D_{\alpha,z}(\psi\|\varphi) < \infty$. Then $D_{\alpha,z}(\psi\circ\gamma\|\varphi\circ\gamma) = D_{\alpha,z}(\psi\|\varphi)$ if and only if γ is sufficient with respect to $\{\psi,\varphi\}$.*

Proof. Let $\bar{w}$ and $\bar{w}_0$ be as in Lemma 4.6. Let $\omega \in \mathcal{M}_*^+$ and $\omega_0 \in \mathcal{N}_*^+$ be such that

$$h_\omega = h_\varphi^{\frac{q-1}{2q}}\bar{w}h_\varphi^{\frac{q-1}{2q}}, \qquad h_{\omega_0} = h_{\varphi_0}^{\frac{q-1}{2q}}\bar{w}_0h_{\varphi_0}^{\frac{q-1}{2q}}.$$

Assume that the equality in DPI holds; then by Lemma 4.6 we have

$$Q_{q,q}(\omega_0\|\varphi_0) = \mathrm{tr}\big(\bar{w}_0^q\big) = \mathrm{tr}\big(\bar{w}^q\big) = Q_{q,q}(\omega\|\varphi).$$

and using also Lemma 3.1, we have

$$(\gamma^*_\varphi)_*(h_{\omega_0}) = h_\varphi^{\frac{1}{2\alpha}}\gamma^*_{\varphi,q}(\bar{w}_0)h_\varphi^{\frac{1}{2\alpha}} = h_\omega.$$

Similarly as in the proof of Theorem 4.5 (note that we have $q > 1$ by the assumption), this shows that γ is sufficient with respect to $\{\omega,\varphi\}$. Hence $\omega\circ\mathcal{E} = \omega$, where $\mathcal{E}$ is the conditional expectation onto the fixed points of $\gamma\circ\gamma^*_\varphi$. Using again the extensions of $\mathcal{E}$ and their properties in Appendix A,

$$h_\varphi^{\frac{q-1}{2q}}\bar{w}h_\varphi^{\frac{q-1}{2q}} = h_\omega = \mathcal{E}(h_\omega) = h_\varphi^{\frac{q-1}{2q}}\mathcal{E}(\bar{w})h_\varphi^{\frac{q-1}{2q}},$$

which implies that $\bar{w} = \mathcal{E}(\bar{w}) \in L^q(\mathcal{E}(\mathcal{M}))^+$. But then we also have

$$|y| = \bar{w}^{\frac{1}{2(\alpha-1)}} = \bar{w}^{\frac{q}{2z}} \in L^{2z}(\mathcal{E}(\mathcal{M}))^+$$

Let $y = u|y|$ be the polar decomposition of y; then we obtain from the definition of y that $uu^* = s(yy^*) = s(\psi)$. Furthermore, since

$$u^*h_\psi^{\frac{1}{2p}} = |y|h_\varphi^{\frac{1}{2q}} \in L^{2p}(\mathcal{E}(\mathcal{M})),$$

by uniqueness of the polar decomposition in $L^{2p}(\mathcal{M})$ and $L^{2p}(\mathcal{E}(\mathcal{M}))$, we obtain that $h_\psi^{\frac{1}{2p}} \in L^{2p}(\mathcal{E}(\mathcal{M}))^+$ and $u \in \mathcal{E}(\mathcal{M})$. Hence we must have $h_\psi \in L^1(\mathcal{E}(\mathcal{M}))$ so that $\psi\circ\mathcal{E} = \psi$ and γ is sufficient with respect to $\{\psi,\varphi\}$ by Proposition 4.2(ii). The converse is clear from DPI. □

5 Monotonicity in the parameter z

It is well known [4, 12, 18] that the standard Rényi divergence $D_{\alpha,1}(\psi\|\varphi)$ is monotone increasing in $\alpha \in (0,1)\cup(1,\infty)$ and the sandwiched Rényi divergence $D_{\alpha,\alpha}(\psi\|\varphi)$ is monotone increasing in $\alpha \in [1/2,1)\cup(1,\infty)$. It is also known [4, 12, 18] that

$$\lim_{\alpha\nearrow 1} D_{\alpha,1}(\psi\|\varphi) = \lim_{\alpha\nearrow 1} D_{\alpha,\alpha}(\psi\|\varphi) = D_1(\psi\|\varphi), \tag{5.1}$$

and if $D_{\alpha,1}(\psi\|\varphi) < \infty$ (resp., $D_{\alpha,\alpha}(\psi\|\varphi) < \infty$) for some $\alpha > 1$, then

$$\lim_{\alpha\searrow 1} D_{\alpha,1}(\psi\|\varphi) = D_1(\psi\|\varphi) \quad \Bigl(\text{resp., } \lim_{\alpha\searrow 1} D_{\alpha,\alpha}(\psi\|\varphi) = D_1(\psi\|\varphi)\Bigr), \tag{5.2}$$

where $D_1(\psi\|\varphi) := D(\psi\|\varphi)/\psi(\mathbf{1})$, the normalized relative entropy. In the rest of the paper we will discuss similar monotonicity properties of $D_{\alpha,z}$ in both parameters α, z and limits for $D_{\alpha,z}(\psi\|\varphi)$ as $\alpha\nearrow 1$, $\alpha\searrow 1$. We consider monotonicity in the parameter z in Sec. 4 and monotonicity in the parameter α in Sec. 5.

The next theorem in [21] shows the monotonicity of $D_{\alpha,z}$ in the parameter $z > 0$ when $0 < \alpha < 1$.

Theorem 5.1 ([21, Theorem 1(x)]). *For every $\psi,\varphi \in \mathcal{M}_*^+$, $\psi \neq 0$, and $0<\alpha<1$, the function $z \mapsto D_{\alpha,z}(\psi\|\varphi)$ is monotone increasing on $(0,\infty)$.*

5.1 The finite von Neumann algebra case

In this subsection we show monotonicity of $D_{\alpha,z}$ in the parameter z in the finite von Neumann algebra setting. Recall (see, e.g., [13, Example 9.11]) that if $(\mathcal{M},\tau)$ is a semi-finite von Neumann algebra $\mathcal{M}$ with a faithful normal semi-finite trace τ, then the Haagerup L^p-space $L^p(\mathcal{M})$ is identified with the L^p-space $L^p(\mathcal{M},\tau)$ with respect to τ. Hence one can define $Q_{\alpha,z}(\psi\|\varphi)$ for $\psi,\varphi \in \mathcal{M}_*^+$ by replacing, in Definition 2.1, $L^p(\mathcal{M})$ with $L^p(\mathcal{M},\tau)$ and $h_\psi \in L^1(\mathcal{M})^+$ with the Radon–Nikodym derivative $d\psi/d\tau \in L^1(\mathcal{M},\tau)^+$. Below we use the symbol h_ψ to denote $d\psi/d\tau$ as well. Note that τ on $\mathcal{M}^+$ is naturally extended to the positive part $\widetilde{\mathcal{M}}^+$ of the space $\widetilde{\mathcal{M}}$ of τ-measurable operators. By [8, Proposition 2.7] (also [13, Proposition 4.20]) we then have

$$\tau(a) = \int_0^\infty \mu_s(a)\,ds, \qquad a \in \widetilde{\mathcal{M}}^+, \tag{5.3}$$

where $\mu_s(a)$ is the generalized s-number of a (see [8]).

In the rest of this subsection, we assume that $(\mathcal{M},\tau)$ is a finite von Neumann algebra with a faithful normal finite trace τ. Note that in this setting, $\widetilde{\mathcal{M}}^+$ consists of all positive self-adjoint operators affiliated with $\mathcal{M}$.

Lemma 5.2. *Let $(\mathcal{M},\tau)$ be as mentioned above. For every $\psi,\varphi \in \mathcal{M}_*^+$ with $\psi \neq 0$ and for any $\alpha, z > 0$ with $\alpha \neq 1$,*

$$D_{\alpha,z}(\psi\|\varphi) = \lim_{\varepsilon\searrow 0} D_{\alpha,z}(\psi\|\varphi+\varepsilon\tau) \quad \textit{increasingly}, \tag{5.4}$$

and hence $D_{\alpha,z}(\psi\|\varphi) = \sup_{\varepsilon>0} D_{\alpha,z}(\psi\|\varphi+\varepsilon\tau)$.

Proof. *Case* $0<\alpha<1$. We need to prove that

$$Q_{\alpha,z}(\psi\|\varphi)=\lim_{\varepsilon\searrow 0}Q_{\alpha,z}(\psi\|\varphi+\varepsilon\tau)\quad\text{decreasingly.}\tag{5.5}$$

In the present setting we have by (5.3)

$$Q_{\alpha,z}(\psi\|\varphi)=\tau\Bigl(\bigl(h_\psi^{\alpha/2z}h_\varphi^{\frac{1-\alpha}{z}}h_\psi^{\alpha/2z}\bigr)^z\Bigr)=\int_0^\infty\mu_s\Bigl(h_\psi^{\alpha/2z}h_\varphi^{\frac{1-\alpha}{z}}h_\psi^{\alpha/2z}\Bigr)^z\,ds,\tag{5.6}$$

and similarly

$$Q_{\alpha,z}(\psi\|\varphi+\varepsilon\tau)=\int_0^\infty\mu_s\Bigl(h_\psi^{\alpha/2z}h_{\varphi+\varepsilon\tau}^{\frac{1-\alpha}{z}}h_\psi^{\alpha/2z}\Bigr)^z\,ds.$$

Since $h_{\varphi+\varepsilon\tau}^{\frac{1-\alpha}{z}}=(h_\varphi+\varepsilon\mathbf{1})^{\frac{1-\alpha}{z}}$ decreases to $h_\varphi^{\frac{1-\alpha}{z}}$ in the measure topology as $\varepsilon\searrow 0$, it follows that $h_\psi^{\alpha/2z}h_{\varphi+\varepsilon\tau}^{\frac{1-\alpha}{z}}h_\psi^{\alpha/2z}$ decreases to $h_\psi^{\alpha/2z}h_\varphi^{\frac{1-\alpha}{z}}h_\psi^{\alpha/2z}$ in the measure topology. Hence by [8, Lemma 3.4] we have $\mu_s\bigl(h_\psi^{\alpha/2z}h_{\varphi+\varepsilon\tau}^{\frac{1-\alpha}{z}}h_\psi^{\alpha/2z}\bigr)\searrow\mu_s\bigl(h_\psi^{\alpha/2z}h_\varphi^{\frac{1-\alpha}{z}}h_\psi^{\alpha/2z}\bigr)$ as $\varepsilon\searrow 0$ for almost every $s>0$. Since $s\mapsto\mu_s\bigl(h_\psi^{\alpha/2z}h_{\varphi+\tau}^{\frac{1-\alpha}{z}}h_\psi^{\alpha/2z}\bigr)$ is integrable on $(0,\infty)$, the Lebesgue convergence theorem yields (5.5).

Case $\alpha>1$. We need to prove that

$$Q_{\alpha,z}(\psi\|\varphi)=\lim_{\varepsilon\searrow 0}Q_{\alpha,z}(\psi\|\varphi+\varepsilon\tau)\quad\text{increasingly.}\tag{5.7}$$

For any $\varepsilon>0$, since $h_{\varphi+\varepsilon\tau}=h_\varphi+\varepsilon\mathbf{1}$ has the bounded inverse $h_{\varphi+\varepsilon\tau}^{-1}=(h_\varphi+\varepsilon\mathbf{1})^{-1}\in\mathcal{M}^+$, one can define $x_\varepsilon:=(h_\varphi+\varepsilon\mathbf{1})^{-\frac{\alpha-1}{2z}}h_\psi^{\alpha/z}(h_\varphi+\varepsilon\mathbf{1})^{-\frac{\alpha-1}{2z}}\in\widetilde{\mathcal{M}}^+$ so that

$$h_\psi^{\alpha/z}=(h_\varphi+\varepsilon\mathbf{1})^{\frac{\alpha-1}{2z}}x_\varepsilon(h_\varphi+\varepsilon\mathbf{1})^{\frac{\alpha-1}{2z}}.$$

In the present setting one can write by (5.3)

$$Q_{\alpha,z}(\psi\|\varphi+\varepsilon\tau)=\tau(x_\varepsilon^z)=\int_0^\infty\mu_s(x_\varepsilon)^z\,ds\quad(\in[0,\infty]).\tag{5.8}$$

Let $0<\varepsilon\le\varepsilon'$. Since $(h_\varphi+\varepsilon\mathbf{1})^{-\frac{\alpha-1}{z}}\ge(h_\varphi+\varepsilon'\mathbf{1})^{-\frac{\alpha-1}{z}}$, one has $\mu_s(x_\varepsilon)\ge\mu_s(x_{\varepsilon'})$ for all $s>0$, so that

$$Q_{\alpha,z}(\psi\|\varphi+\varepsilon\tau)\ge Q_{\alpha,z}(\psi\|\varphi+\varepsilon'\tau).$$

Hence $\varepsilon>0\mapsto D_{\alpha,z}(\psi\|\varphi+\varepsilon\tau)$ is decreasing.

First, assume that $s(\psi)\not\le s(\varphi)$. Then $\mu_{s_0}\Bigl(h_\psi^{\alpha/2z}s(\varphi)^\perp h_\psi^{\alpha/2z}\Bigr)>0$ for some $s_0>0$; indeed, otherwise, $h_\psi^{\alpha/2z}s(\varphi)^\perp h_\psi^{\alpha/2z}=0$ so that $s(\psi)\le s(\varphi)$. Hence we have

$$\mu_s(x_\varepsilon)=\mu_s\Bigl(h_\psi^{\alpha/2z}(h_\varphi+\varepsilon\mathbf{1})^{-\frac{\alpha-1}{z}}h_\psi^{\alpha/2z}\Bigr)\ge\varepsilon^{-\frac{\alpha-1}{z}}\mu_s\Bigl(h_\psi^{\alpha/2z}s(\varphi)^\perp h_\psi^{\alpha/2z}\Bigr)\nearrow\infty\quad\text{as }\varepsilon\searrow 0$$

for all $s\in(0,s_0]$. Therefore, it follows from (5.8) that $Q_{\alpha,z}(\psi\|\varphi+\varepsilon\tau)\nearrow\infty=Q_{\alpha,z}(\psi\|\varphi)$.

Next, assume that $s(\psi)\le s(\varphi)$. Take the spectral decomposition $h_\varphi=\int_0^\infty t\,de_t$ and define $y,x\in\widetilde{\mathcal{M}}^+$ by

$$y:=h_\varphi^{-\frac{\alpha-1}{z}}s(\varphi)=\int_{(0,\infty)}t^{-\frac{\alpha-1}{z}}\,de_t,\qquad x:=y^{1/2}h_\psi^{\alpha/z}y^{1/2}.$$

Since

$$h_\psi^{\alpha/z} = s(\varphi)h_\psi^{\alpha/z}s(\varphi) = h_\varphi^{\frac{\alpha-1}{2z}}y^{1/2}h_\psi^{\alpha/z}y^{1/2}h_\varphi^{\frac{\alpha-1}{2z}} = h_\varphi^{\frac{\alpha-1}{2z}}xh_\varphi^{\frac{\alpha-1}{2z}},$$

one has, similarly to (5.8),

$$Q_{\alpha,z}(\psi\|\varphi) = \tau(x^z) = \int_0^\infty \mu_s(x)^z\,ds. \tag{5.9}$$

We write $(h_\varphi+\varepsilon\mathbf{1})^{-\frac{\alpha-1}{z}}s(\varphi) = \int_{(0,\infty)}(t+\varepsilon)^{-\frac{\alpha-1}{z}}\,de_t$, and for any $\delta>0$ choose a $t_0>0$ such that $\tau(e_{(0,t_0)})<\delta$. Then, since $\int_{[t_0,\infty)}(t+\varepsilon)^{-\frac{\alpha-1}{z}}\,de_t \to \int_{[t_0,\infty)}t^{-\frac{\alpha-1}{z}}\,de_t$ in the operator norm as $\varepsilon\searrow 0$, we obtain $(h_\varphi+\varepsilon\mathbf{1})^{-\frac{\alpha-1}{z}}s(\varphi)\nearrow y$ in the measure topology (see [8, 1.5]), so that $h_\psi^{\alpha/2z}(h_\varphi+\varepsilon\mathbf{1})^{-\frac{\alpha-1}{z}}h_\psi^{\alpha/2z}\nearrow h_\psi^{\alpha/2z}yh_\psi^{\alpha/2z}$ in the measure topology as $\varepsilon\searrow 0$. Hence we have by [8, Lemma 3.4]

$$\mu_s(x_\varepsilon) = \mu_s\Big(h_\psi^{\alpha/2z}(h_\varphi+\varepsilon\mathbf{1})^{-\frac{\alpha-1}{z}}h_\psi^{\alpha/2z}\Big)\nearrow \mu_s\Big(h_\psi^{\alpha/2z}yh_\psi^{\alpha/2z}\Big) = \mu_s(x)$$

for all $s>0$. Therefore, by (5.8) and (5.9) the monotone convergence theorem yields (5.7). $\square$

Lemma 5.3. *Let $(\mathcal{M},\tau)$ and ψ,φ be as in Lemma 5.2, and let $0<z\le z'$. Then*

$$\begin{cases} D_{\alpha,z}(\psi\|\varphi)\le D_{\alpha,z'}(\psi\|\varphi), & 0<\alpha<1,\\ D_{\alpha,z}(\psi\|\varphi)\ge D_{\alpha,z'}(\psi\|\varphi), & \alpha>1.\end{cases}$$

Proof. The case $0<\alpha<1$ is in Theorem 5.1 for general von Neumann algebras. For the case $\alpha>1$, by Lemma 5.2 it suffices to show that, for every $\varepsilon>0$,

$$\tau\bigg(\Big(y_\varepsilon^{\frac{\alpha-1}{2z}}h_\psi^{\alpha/z}y_\varepsilon^{\frac{\alpha-1}{2z}}\Big)^z\bigg) \ge \tau\bigg(\Big(y_\varepsilon^{\frac{\alpha-1}{2z'}}h_\psi^{\alpha/z'}y_\varepsilon^{\frac{\alpha-1}{2z'}}\Big)^{z'}\bigg),$$

where $y_\varepsilon := (h_\varphi+\varepsilon\mathbf{1})^{-1}\in\mathcal{M}^+$. The above is equivalently written as

$$\tau\Big(\big|(h_\psi^{\alpha/2z'})^r(y_\varepsilon^{(\alpha-1)/2z'})^r\big|^{2z}\Big) \ge \tau\Big(\big|h_\psi^{\alpha/2z'}y_\varepsilon^{(\alpha-1)/2z'}\big|^{2zr}\Big),$$

where $r:=z'/z\ge 1$. Hence the desired inequality follows from Kosaki's ALT inequality [26, Corollary 3]. $\square$

When $(\mathcal{M},\tau)$ and ψ,φ are as in Lemma 5.2, one can define, thanks to Lemma 5.3, for any $\alpha\in(0,\infty)\setminus\{1\}$,

$$\begin{aligned} Q_{\alpha,\infty}(\psi\|\varphi) &:= \lim_{z\to\infty}Q_{\alpha,z}(\psi\|\varphi) = \inf_{z>0}Q_{\alpha,z}(\psi\|\varphi),\\ D_{\alpha,\infty}(\psi\|\varphi) &:= \frac{1}{\alpha-1}\log\frac{Q_{\alpha,\infty}(\psi\|\varphi)}{\psi(\mathbf{1})}\\ &= \lim_{z\to\infty}D_{\alpha,z}(\psi\|\varphi) = \begin{cases}\sup_{z>0}D_{\alpha,z}(\psi\|\varphi), & 0<\alpha<1,\\ \inf_{z>0}D_{\alpha,z}(\psi\|\varphi), & \alpha>1.\end{cases} \end{aligned} \tag{5.10}$$

If $h_\psi,h_\varphi\in\mathcal{M}^{++}$ (i.e., $\lambda^{-1}\tau\le\psi,\varphi\le\lambda\tau$ for some $\lambda>0$), then the Lie–Trotter formula gives

$$Q_{\alpha,\infty}(\psi\|\varphi) = \tau\big(\exp(\alpha\log h_\psi+(1-\alpha)\log h_\varphi)\big). \tag{5.11}$$

Lemma 5.4. *Let $(\mathcal{M},\tau)$ and ψ,φ be as in Lemma 5.2. Then for any $z>0$,*

$$\begin{cases} D_{\alpha,z}(\psi\|\varphi)\le D_1(\psi\|\varphi), & 0<\alpha<1,\\ D_{\alpha,z}(\psi\|\varphi)\ge D_1(\psi\|\varphi), & \alpha>1.\end{cases}$$

Proof. First assume that $h_\psi,h_\varphi\in\mathcal{M}^{++}$. Set self-adjoint $H:=\log h_\psi$ and $K:=\log h_\varphi$ in $\mathcal{M}$ and define $F(\alpha):=\log\tau\bigl(e^{\alpha H+(1-\alpha)K}\bigr)$ for $\alpha>0$. Then by (5.11), $F(\alpha)=\log Q_{\alpha,\infty}(\psi\|\varphi)$ for all $\alpha\in(0,\infty)\setminus\{1\}$, and we compute

$$\begin{aligned} F'(\alpha)&=\frac{\tau\bigl(e^{\alpha H+(1-\alpha)K}(H-K)\bigr)}{\tau\bigl(e^{\alpha H+(1-\alpha)K}\bigr)},\\ F''(\alpha)&=\frac{\tau\bigl(e^{\alpha H+(1-\alpha)K}(H-K)^2\bigr)\tau\bigl(e^{\alpha H+(1-\alpha)K}\bigr)-\bigl\{\tau\bigl(e^{\alpha H+(1-\alpha)K}(H-K)\bigr)\bigr\}^2}{\bigl\{\tau\bigl(e^{\alpha H+(1-\alpha)K}\bigr)\bigr\}^2}.\end{aligned}$$

Since $F''(\alpha)\ge 0$ on $(0,\infty)$ thanks to the Schwarz inequality, we see that $F(\alpha)$ is convex on $(0,\infty)$ and hence

$$D_{\alpha,\infty}(\psi\|\varphi)=\frac{F(\alpha)-F(1)}{\alpha-1}$$

is increasing in $\alpha\in(0,\infty)$, where for $\alpha=1$ the above right-hand side is understood as

$$F'(1)=\frac{\tau(e^H(H-K))}{\tau(e^H)}=\frac{\tau\bigl(h_\psi(\log h_\psi-\log h_\varphi)\bigr)}{\tau(h_\psi)}=D_1(\psi\|\varphi).$$

Hence by (5.10) the assertion holds when $h_\psi,h_\varphi\in\mathcal{M}^{++}$. Below we extend this to general $\psi,\varphi\in\mathcal{M}_*^+$.

Case $0<\alpha<1$. Let $\psi,\varphi\in\mathcal{M}_*^+$ and $z>0$. From [21, Theorem 1(iv)] and [14, Corollary 2.8(3)] we have

$$\begin{aligned} D_{\alpha,z}(\psi\|\varphi)&=\lim_{\varepsilon\searrow 0}D_{\alpha,z}(\psi+\varepsilon\tau\|\varphi+\varepsilon\tau),\\ D_1(\psi\|\varphi)&=\lim_{\varepsilon\searrow 0}D_1(\psi+\varepsilon\tau\|\varphi+\varepsilon\tau),\end{aligned}$$

so that we may assume that $\psi,\varphi\ge\varepsilon\tau$ for some $\varepsilon>0$. Take the spectral decompositions $h_\psi=\int_0^\infty t\,de_t^\psi$ and $h_\varphi=\int_0^\infty t\,de_t^\varphi$, and define $e_n:=e_n^\psi\wedge e_n^\varphi$ for each $n\in\mathbb{N}$. Then $\tau(e_n^\perp)\le\tau((e_n^\psi)^\perp)+\tau((e_n^\varphi)^\perp)\to 0$ as $n\to\infty$, so that $e_n\nearrow\mathbf{1}$. Hence by Proposition 3.5 one has $D_{\alpha,z}(e_n\psi e_n\|e_n\varphi e_n)\to D_{\alpha,z}(\psi\|\varphi)$. On the other hand, by [14, Proposition 2.10] one has $D_1(e_n\psi e_n\|e_n\varphi e_n)\to D_1(\psi\|\varphi)$ as well. Since $D_{\alpha,z}(e_n\psi e_n\|e_n\varphi e_n)\le D_1(e_n\psi e_n\|e_n\varphi e_n)$ holds by the above shown case, we obtain the desired inequality for general $\psi,\varphi\in\mathcal{M}_*^+$.

Case $\alpha>1$. We show the extension to general $\psi,\varphi\in\mathcal{M}_*^+$ by dividing four steps as follows, where $h_\psi=\int_0^\infty t\,e_t^\psi$ and $h_\varphi=\int_0^\infty t\,de_t^\varphi$ are the spectral decompositions.

(1) Assume that $h_\psi\in\mathcal{M}^+$ and $h_\varphi\in\mathcal{M}^{++}$. Set $\psi_n\in\mathcal{M}_*^+$ by $h_{\psi_n}=(1/n)e_{[0,1/n]}^\psi+\int_{(1/n,\infty)}t\,de_t^\psi$ ($\in\mathcal{M}^{++}$). Since $h_{\psi_n}^{\alpha/z}\searrow h_\psi^{\alpha/z}$ in the operator norm, we have by (5.6) and [8, Lemma 3.4]

$$\begin{aligned} Q_{\alpha,z}(\psi\|\varphi)&=\int_0^\infty\mu_s\Bigl((h_\varphi^{-1})^{\frac{\alpha-1}{2z}}h_\psi^{\alpha/z}(h_\varphi^{-1})^{\frac{\alpha-1}{2z}}\Bigr)^z\,ds\\ &=\lim_{n\to\infty}\int_0^\infty\mu_s\Bigl((h_\varphi^{-1})^{\frac{\alpha-1}{2z}}h_{\psi_n}^{\alpha/z}(h_\varphi^{-1})^{\frac{\alpha-1}{2z}}\Bigr)^z\,ds=\lim_{n\to\infty}Q_{\alpha,z}(\psi_n\|\varphi).\end{aligned}\tag{5.12}$$

From this and the lower semi-continuity of D_1 the extension holds in this case.

(2) Assume that $h_\psi \in \mathcal{M}^+$ and $h_\varphi \geq \delta\mathbf{1}$ for some $\delta > 0$. Set $\varphi_n \in \mathcal{M}_*^+$ by $h_{\varphi_n} = \int_{[\delta,n]} t\,de_t^\varphi + ne_{(n,\infty)}^\varphi$ $(\in \mathcal{M}^{++})$. Since $h_{\varphi_n}^{-\frac{\alpha-1}{z}} \searrow h_\varphi^{-\frac{\alpha-1}{z}}$ in the operator norm, we have by (5.6) and [8, Lemma 3.4] again

$$\begin{aligned} Q_{\alpha,z}(\psi\|\varphi) &= \int_0^\infty \mu_s\Big(h_\psi^{\alpha/2z} h_\varphi^{-\frac{\alpha-1}{z}} h_\psi^{\alpha/2z}\Big)^z\,ds \\ &= \lim_{n\to\infty} \int_0^\infty \mu_s\Big(h_\psi^{\alpha/2z} h_{\varphi_n}^{-\frac{\alpha-1}{z}} h_\psi^{\alpha/2z}\Big)^z\,ds = \lim_{n\to\infty} Q_{\alpha,z}(\psi\|\varphi_n). \end{aligned}$$

From this and (1) above the extension holds in this case too.

(3) Assume that ψ is general and $\varphi \geq \delta\tau$ for some $\delta > 0$. Set $\psi_n \in \mathcal{M}_*^+$ by $h_{\psi_n} = \int_{[0,n]} t\,de_t^\psi + ne_{(n,\infty)}^\varphi$ $(\in \mathcal{M}^+)$. Since $h_{\psi_n}^{\alpha/z} \nearrow h_\psi^{\alpha/z}$ in the measure topology, one can argue as in (5.12) with use of the monotone convergence theorem to see from (2) that the extension holds in this case too.

(4) Finally, from (3) with Lemma 5.2 and [14, Corollary 2.8(3)] it follows that the desired extension hods for general $\psi, \varphi \in \mathcal{M}_*^+$. $\square$

Note that the first inequality in (i) of the next proposition is a particular case of Theorem 5.1, while we include it here to make the statement complete.

Proposition 5.5. *Assume that $\mathcal{M}$ is a finite von Neumann algebra with a faithful normal finite trace τ. Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$. Then we have the following:*

(i) If $0 < \alpha < 1 < \alpha'$ and $0 < z \leq z' \leq \infty$, then

$$D_{\alpha,z}(\psi\|\varphi) \leq D_{\alpha,z'}(\psi\|\varphi) \leq D_1(\psi\|\varphi) \leq D_{\alpha',z'}(\psi\|\varphi) \leq D_{\alpha',z}(\psi\|\varphi).$$

(ii) For any $z \in [1,\infty]$,

$$\lim_{\alpha\nearrow 1} D_{\alpha,z}(\psi\|\varphi) = D_1(\psi\|\varphi). \tag{5.13}$$

(iii) If $D_{\alpha,\alpha}(\psi\|\varphi) < \infty$ for some $\alpha > 1$, then for any $z \in (1,\infty]$,

$$\lim_{\alpha\searrow 1} D_{\alpha,z}(\psi\|\varphi) = D_1(\psi\|\varphi). \tag{5.14}$$

Proof. (i) follows from Lemmas 5.3 and 5.4.

(ii) For every $z \in [1,\infty]$ and $\alpha \in (0,1)$, it follows from (i) above that

$$D_{\alpha,1}(\psi\|\varphi) \leq D_{\alpha,z}(\psi\|\varphi) \leq D_1(\psi\|\varphi).$$

Hence (5.13) follows since it holds for $D_{\alpha,1}$ by [12, Proposition 5.3(3)], as stated in (5.1).

(iii) Assume that $D_{\alpha,\alpha}(\psi\|\varphi) < \infty$ for some $\alpha > 1$. For every $z \in (1,\infty]$ and $\alpha \in (1,z]$, it follows from (i) that

$$D_1(\psi\|\varphi) \leq D_{\alpha,z}(\psi\|\varphi) \leq D_{\alpha,\alpha}(\psi\|\varphi).$$

Hence (5.14) follows since it holds for $D_{\alpha,\alpha}$ by [18, Proposition 3.8(ii)], as stated in (5.2). $\square$

In this subsection, in the specialized setting of finite von Neumann algebras, we have considered the monotonicity of $D_{\alpha,z}$ in the parameter z in an essentially similar way to the finite-dimensional case [28, 29]. By Theorem 5.1, for $0 < \alpha < 1$ this monotonicity holds in general von Neumann algebras. In the next subsection, we will extend it to this general setting also for $\alpha > 1$, under a restriction on z, by using the Haagerup reduction theorem and the complex interpolation method. Furthermore, we will extend the limits given in (5.13) and (5.14) in Sec. 5.3.

5.2 The general von Neumann algebra case

From now on let $\mathcal{M}$ be again a general von Neumann algebra. In the next proposition, we first extend Proposition 5.5(i) to general von Neumann algebras, based on Haagerup's reduction theorem [11], that is briefly explained in Appendix B for the convenience of the reader.

Proposition 5.6. *For every $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$, we have the following:*

(i) If $0 < \alpha < 1$ and $0 \le z \le z'$, then

$$D_{\alpha,z}(\psi\|\varphi) \le D_{\alpha,z'}(\psi\|\varphi) \le D_1(\psi\|\varphi).$$

(ii) If $\alpha > 1$ and $\max\{\alpha/2, \alpha-1\} \le z \le z' \le \alpha$, then

$$D_1(\psi\|\varphi) \le D_{\alpha,z'}(\psi\|\varphi) \le D_{\alpha,z}(\psi\|\varphi).$$

Proof. Since $D_{\alpha,z}(\psi\|\varphi)$ is the same as that for ψ, φ restricted to $s(\psi+\varphi)\mathcal{M}s(\psi+\varphi)$ (see Remark 2.2) and similarly for $D_1(\psi\|\varphi)$, we may assume that $\mathcal{M}$ is σ-finite. If $0 < \alpha < 1$ and $\max\{\alpha, 1-\alpha\} \le z$, then we have $D_{\alpha,z}(\psi\|\varphi) \le D_1(\psi\|\varphi)$ by Proposition 5.5(i) and Lemma B.2 in Appendix B. Combining this and Theorem 5.1 yields (i). Statement (ii) is again a consequence of Proposition 5.5(i) and Lemma B.2. ☐

For the proof of Proposition 5.6(ii) by use of Lemma B.2 (based on Theorem B.1) it is inevitable to restrict the parameter z to the DPI bounds when $\alpha > 1$. However, in the next theorem we show the $\alpha > 1$ counterpart of Theorem 5.1, showing the monotonicity of $D_{\alpha,z}$ in $z \ge \alpha/2$ when $\alpha > 1$. Nevertheless, we note that the inequalities between $D_{\alpha,z}$ and D_1 in (i) and (ii) of Proposition 5.6 are not included in Theorems 5.1 and 5.7. The proof below is based on Kosaki's interpolation L^p-spaces [23], briefly explained in Appendix C.

Theorem 5.7. *For every $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$, and $\alpha > 1$, the function $z \mapsto D_{\alpha,z}(\psi\|\varphi)$ is monotone decreasing on $[\alpha/2, \infty)$.*

Proof. Let $\alpha > 1$ and $z, z' \in [\alpha/2, \infty)$ be such that $z < z'$. We need to prove that $Q_{\alpha,z}(\psi\|\varphi) \ge Q_{\alpha,z'}(\psi\|\varphi)$. To do this, we may assume that $Q_{\alpha,z}(\psi\|\varphi) < \infty$. Hence by Lemma 2.3, there is some $y \in L^{2z}(\mathcal{M})$ such that

$$h_\psi^{\frac{\alpha}{2z}} = y h_\varphi^{\frac{\alpha-1}{2z}}, \qquad Q_{\alpha,z}(\psi\|\varphi) = \|y\|_{2z}^{2z}.$$

In particular, $e := s(\psi) \le s(\varphi)$, so that we may assume that φ is faithful. Let $\sigma \in \mathcal{M}_*^+$ be such that $s(\sigma) = \mathbf{1} - e$, and set $\psi_0 := \psi + \sigma$, so that ψ_0 is faithful too. Let us use for simplicity the notation L_L^p for Kosaki's left L^p-space $L^p(\mathcal{M}, \varphi)_L$ for $1 \le p \le \infty$; see (C.2) in Appendix C.

Consider the function

$$f(w) := h_{\psi_0}^{\frac{\alpha}{2z}w} e h_\varphi^{1-\frac{\alpha}{2z}w}, \qquad w \in S, \tag{5.15}$$

where $S := \{w \in \mathbb{C} : 0 \le \operatorname{Re} w \le 1\}$. Then, for $w = s + it$ with $0 \le s \le 1$ and $t \in \mathbb{R}$, since

$$f(s+it) = h_{\psi_0}^{\frac{\alpha}{2z}it} h_{\psi_0}^{\frac{\alpha}{2z}s} e h_\varphi^{1-\frac{\alpha}{2z}s} h_\varphi^{-\frac{\alpha}{2z}it},$$

we have

$$\begin{aligned} f(it) &= h_{\psi_0}^{\frac{\alpha}{2z}it} e h_\varphi^{-\frac{\alpha}{2z}it} h_\varphi \in L^\infty_L, \\ \|f(it)\|_{L^\infty_L} &= \left\| h_{\psi_0}^{\frac{\alpha}{2z}it} e h_\varphi^{-\frac{\alpha}{2z}it} \right\| = 1, \qquad t \in \mathbb{R}, \end{aligned}$$

and

$$\begin{aligned} f(1+it) &= h_{\psi_0}^{\frac{\alpha}{2z}it} h_\psi^{\frac{\alpha}{2z}} h_\varphi^{1-\frac{\alpha}{2z}} h_\varphi^{-\frac{\alpha}{2z}it} = \Big(h_{\psi_0}^{\frac{\alpha}{2z}it} y h_\varphi^{-\frac{\alpha}{2z}it} \Big) h_\varphi^{\frac{2z-1}{2z}} \in L^{2z}_L, \\ \|f(1+it)\|_{L^{2z}_L} &= \left\| h_{\psi_0}^{\frac{\alpha}{2z}it} y h_\varphi^{-\frac{\alpha}{2z}it} \right\|_{2z} = \|y\|_{2z}, \qquad t \in \mathbb{R}, \end{aligned}$$

where the last equality follows from [23, Lemma 10.1]. Furthermore, we observe that f is a bounded continuous function on S into $L^1(\mathcal{M})$ (see [23, Lemma 10.2]) and it is analytic in the interior (see, e.g., [13, Lemma 9.19 and Theorem 9.18(c)]). Therefore, f belongs to the set $\mathcal{F}'(L^\infty_L, L^{2z}_L)$ of $L^1(\mathcal{M})$-valued functions given in [23, Definition 1.4]. Since L^{2z}_L is reflexive thanks to $1 < \alpha \le 2z < \infty$, it follows from [23, Theorems 1.5 and Remark 3.4] that the set $\mathcal{F}'(L^\infty_L, L^{2z}_L)$ defines the interpolation space $C_\theta = C_\theta(L^\infty_L, L^{2z}_L)$ in [23, Definition 1.1]. Hence for any $\theta \in (0,1)$, we have $f(\theta) \in C_\theta$ and

$$\|f(\theta)\|_{C_\theta} \le \left(\sup_{t\in\mathbb{R}} \|f(it)\|_{L^\infty_L} \right)^{1-\theta} \left(\sup_{t\in\mathbb{R}} \|f(1+it)\|_{L^{2z}_L} \right)^{\theta} = \|y\|_{2z}^\theta,$$

where the inequality above is known from [3, Lemma 4.3.2(ii)]. By [23, Theorem 1.9] and the reiteration theorem (see [7] for the best result on that), $C_\theta = L^{2z/\theta}_L$ with equal norms, so that putting $\theta = z/z'$ we have

$$f(z/z') = h_\psi^{\frac{\alpha}{2z'}} h_\varphi^{1-\frac{\alpha}{2z'}} = y' h_\varphi^{\frac{2z'-1}{2z'}}$$

for some $y' \in L^{2z'}(\mathcal{M})$, and $\|y'\|_{2z'} \le \|y\|_{2z}^{z/z'}$. This implies that $h_\psi^{\frac{\alpha}{2z'}} = y' h_\varphi^{\frac{\alpha-1}{2z'}}$ so that $Q_{\alpha,z'}(\psi\|\varphi) = \|y'\|_{2z'}^{2z'} \le \|y\|_{2z}^{2z}$, and the assertion follows. $\square$

6 Monotonicity in the parameter α

In this section we show the monotonicity of $D_{\alpha,z}$ in the parameter α as well as limits of $D_{\alpha,z}$ as $\alpha \nearrow 1$ and $\alpha \searrow 1$.

6.1 The case $0 < \alpha < 1$ and all $z > 0$

The aim of this subsection is to prove the next theorem.

Theorem 6.1. *Let $\psi, \varphi \in \mathcal{M}_*^+$ and $z > 0$. Then we have*

(1) $\alpha \mapsto \log Q_{\alpha,z}(\psi\|\varphi)$ is convex on $(0,1)$,

(2) $\alpha \mapsto D_{\alpha,z}(\psi\|\varphi)$ is monotone increasing on $(0,1)$.

To prove this, we obtain a certain "log-majorization" result for positive τ-measurable operators, which might be meaningful in its own. Assume that $(\mathcal{R}, \tau)$ be a semi-finite von Neumann algebra $\mathcal{R}$ with a faithful normal semi-finite trace τ. Let $\widetilde{\mathcal{R}}$ denote the space of τ-measurable operators affiliated with $\mathcal{R}$. We consider operators $a \in \widetilde{\mathcal{R}}$ satisfying

$$a \in \mathcal{R} \ \text{ or } \ \mu_t(a) \le Ct^{-\gamma} \ (t > 0) \text{ for some } C, \gamma > 0, \tag{6.1}$$

where $\mu_t(a)$, $t > 0$, is the (tth) *generalized s-number* of a. For each $a \in \widetilde{\mathcal{R}}$ satisfying (6.1) we define (see [8])

$$\Lambda_t(a) := \exp \int_0^t \log \mu_s(a)\, ds, \qquad t > 0.$$

Note [8] that $\Lambda_t(a) \in [0, \infty)$, $t > 0$, are well defined whenever a satisfies (6.1). Also, note that if $a, b \in \widetilde{\mathcal{R}}$ satisfy (6.1), then $|a|^p$ ($p > 0$) and ab satisfy (6.1) too, as it is clear since $\mu_t(ab) \le \|a\|\mu_t(b)$ for $a \in \mathcal{R}$, $\mu_t(|a|^p) = \mu_t(a)^p$, and $\mu_t(ab) \le \mu_{t/2}(a)\mu_{t/2}(b)$; see [8, Lemma 2.5].

Proposition 6.2. *Let $a_j, b_j \in \widetilde{\mathcal{R}}^+$, $j = 1, 2$, satisfying (6.1) and assume that $a_1a_2 = a_2a_1$ and $b_1b_2 = b_2b_1$. Then for every $\theta \in (0,1)$ and any $t > 0$,*

$$\Lambda_t\Big((a_1^\theta a_2^{1-\theta})^{1/2}(b_1^\theta b_2^{1-\theta})(a_1^\theta a_2^{1-\theta})^{1/2}\Big) \le \Lambda_t\big(a_1^\theta b_1^\theta\big)\Lambda_t\big(a_2^{1-\theta} b_2^{1-\theta}\big). \tag{6.2}$$

In particular,

$$\Lambda_t\Big((a_1^{1/2}a_2^{1/2})^{1/2}(b_1^{1/2}b_2^{1/2})(a_1^{1/2}a_2^{1/2})^{1/2}\Big) \le \Lambda_t\big(a_1^{1/2}b_1a_1^{1/2}\big)^{1/2}\Lambda_t\big(a_2^{1/2}b_2a_2^{1/2}\big)^{1/2}. \tag{6.3}$$

Proof. Let $\theta \in (0,1)$ and $t > 0$ be arbitrary. For any $k \in \mathbb{N}$ we note that

$$\begin{aligned}
&\big((a_1^\theta a_2^{1-\theta})^{1/2}(b_1^\theta b_2^{1-\theta})(a_1^\theta a_2^{1-\theta})^{1/2}\big)^k \\
&\quad = (a_1^\theta a_2^{1-\theta})^{1/2}(b_1^\theta b_2^{1-\theta})(a_1^\theta a_2^{1-\theta})(b_1^\theta b_2^{1-\theta})\cdots(a_1^\theta a_2^{1-\theta})(b_1^\theta b_2^{1-\theta})(a_1^\theta a_2^{1-\theta})^{1/2} \\
&\quad = (a_1^\theta a_2^{1-\theta})^{1/2} b_2^{1-\theta}(b_1^\theta a_1^\theta)(a_2^{1-\theta} b_2^{1-\theta})\cdots(b_1^\theta a_1^\theta)(a_2^{1-\theta} b_2^{1-\theta})b_1^\theta(a_1^\theta a_2^{1-\theta})^{1/2}.
\end{aligned}$$

Since $(a_1^\theta a_2^{1-\theta})^{1/2} b_2^{1-\theta}$, $b_1^\theta a_1^\theta$, etc. are τ-measurable operators satisfying (6.1), we have by [8, Theorem 4.2(ii)]

$$\begin{aligned}
&\Lambda_t\big((a_1^\theta a_2^{1-\theta})^{1/2}(b_1^\theta b_2^{1-\theta})(a_1^\theta a_2^{1-\theta})^{1/2}\big)^k \\
&\quad \le \Lambda_t\big((a_1^\theta a_2^{1-\theta})^{1/2} b_2^{1-\theta}\big)\Lambda_t(b_1^\theta a_1^\theta)^{k-1}\Lambda_t\big(a_2^{1-\theta} b_2^{1-\theta}\big)^{k-1}\Lambda_t\big(b_1^\theta(a_1^\theta a_2^{1-\theta})^{1/2}\big),
\end{aligned}$$

so that

$$\Lambda_t\big((a_1^\theta a_2^{1-\theta})^{1/2}(b_1^\theta b_2^{1-\theta})(a_1^\theta a_2^{1-\theta})^{1/2}\big)$$
$$\le \Lambda_t\big((a_1^\theta a_2^{1-\theta})^{1/2} b_2^{1-\theta}\big)^{1/k}\Lambda_t(b_1^\theta a_1^\theta)^{1-\frac{1}{k}}\Lambda_t\big(a_2^{1-\theta}b_2^{1-\theta}\big)^{1-\frac{1}{k}}\Lambda_t\big(b_1^\theta(a_1^\theta a_2^{1-\theta})^{1/2}\big)^{1/k}.$$

Letting $k \to \infty$ gives (6.2). When $\theta = 1/2$, (6.2) is rewritten as (6.3). □

Remark 6.3. Since $\Lambda_t(a_j^r b_j^r) \le \Lambda_t(a_j b_j)^r$ for any $r \in (0,1)$ by [26], inequality (6.2) implies that

$$\Lambda_t\big((a_1^\theta a_2^{1-\theta})^{1/2}(b_1^\theta b_2^{1-\theta})(a_1^\theta a_2^{1-\theta})^{1/2}\big) \le \Lambda_t(a_1 b_1)^\theta \Lambda_t(a_2 b_2)^{1-\theta} = \Lambda_t\big(a_1 b_1^2 a_1\big)^{\frac{\theta}{2}}\Lambda_t\big(a_2 b_2^2 a_2\big)^{\frac{1-\theta}{2}}.$$

We are indeed interested in whether a stronger inequality

$$\Lambda_t\big((a_1^\theta a_2^{1-\theta})^{1/2}(b_1^\theta b_2^{1-\theta})(a_1^\theta a_2^{1-\theta})^{1/2}\big) \le \Lambda_t\big(a_1^{1/2} b_1 a_1^{1/2}\big)^\theta \Lambda_t\big(a_2^{1/2} b_2 a_2^{1/2}\big)^{1-\theta}$$

hold or not in the situation of Proposition 6.2. The last inequality is known to hold in the finite-dimensional setting, as shown in [15, Theorem 2.1].

Lemma 6.4. *Let $\psi, \varphi \in \mathcal{M}_*^+$ with $\psi \ne 0$, and assume that $s(\psi) \not\perp s(\varphi)$. Then for every $z > 0$, $Q_{\alpha,z}(\psi\|\varphi) > 0$ for all $\alpha \in (0,1)$, and $\alpha \mapsto Q_{\alpha,z}(\psi\|\varphi)$ is continuous on $(0,1)$.*

Proof. Assume that $Q_{\alpha,z}(\psi\|\varphi) = 0$ for some $z > 0$ and $\alpha \in (0,1)$. Then $h_\psi^{\alpha/2z} h_\varphi^{(1-\alpha)/2z} = 0$ as a τ-measurable operator affiliated with $\mathcal{R} := \mathcal{M} \rtimes_{\sigma^\omega} \mathbb{R}$ (see Appendix A). Since $s(\psi) = s(h_\psi^{\alpha/2z})$ and $s(\varphi) = s(h_\varphi^{(1-\alpha)/2z})$, it is easy to see that $s(\psi) \perp s(\varphi)$. Hence the first assertion follows.

Next, since $p > 0 \mapsto a^p \in \widetilde{\mathcal{R}}$ is differentiable in the measure topology for any $a \in \widetilde{\mathcal{R}}^+$ (see, e.g., [13, Lemma 9.19]), we see that $\alpha \mapsto h_\psi^{\alpha/2z} h_\varphi^{(1-\alpha)/z} h_\psi^{\alpha/2z}$ is differentiable (hence continuous) on $(0,1)$ in the measure topology. Hence by [13, Lemma 9.14], the function $\alpha \mapsto Q_{\alpha,z}(\psi\|\varphi) = \|h_\psi^{\alpha/2z} h_\varphi^{(1-\alpha)/z} h_\psi^{\alpha/2z}\|_z^z$ is continuous on $(0,1)$. Here, when $z < 1$, note [8, Theorem 4.9(iii)] that $|\,\|a\|_z^z - \|b\|_z^z| \le \|a-b\|_z^z$ for $a, b \in L^z(\mathcal{M})$. □

Proof of Theorem 6.1. We may assume that $s(\psi) \not\perp s(\varphi)$; otherwise, $Q_{\alpha,z}(\psi\|\varphi) = 0$ for all $\alpha \in (0,1)$. Then by Lemma 6.4, $Q_{\alpha,z}(\psi\|\varphi) \in (0,\infty)$ for all $\alpha \in (0,1)$, and $\alpha \mapsto Q_{\alpha,z}(\psi\|\varphi)$ is continuous on $(0,1)$. Hence $\alpha \mapsto D_{\alpha,z}(\psi\|\varphi)$ is continuous on $(0,1)$ too.

Let $\alpha_1, \alpha_2 \in (0,1)$ and $z > 0$. Let $(\mathcal{R}, \tau)$ be as in the proof of Lemma 6.4. Consider $a_j := h_\psi^{\alpha_j/z}$ and $b_j := h_\varphi^{(1-\alpha_j)/z}$ in $\widetilde{\mathcal{R}}^+$. Since $a_j \in L^{z/\alpha_j}(\mathcal{M})$, we note by [8, Lemma 4.8] that $\mu_t(a_j) = t^{-\alpha_j/z}\|a_j\|_{z/\alpha_j}$, $t > 0$, and hence a_j satisfies (6.1). Similarly, b_j does so. Therefore, we can apply (6.3) to a_j, b_j with $t = 1$ to obtain

$$\int_0^1 \log \mu_s\Big(h_\psi^{\frac{\alpha_1+\alpha_2}{4z}} h_\varphi^{\frac{2-\alpha_1-\alpha_2}{2z}} h_\psi^{\frac{\alpha_1+\alpha_2}{4z}}\Big)\, ds$$
$$\le \frac{1}{2}\Big[\int_0^1 \log \mu_s\Big(h_\psi^{\frac{\alpha_1}{2z}} h_\varphi^{\frac{1-\alpha_1}{z}} h_\psi^{\frac{\alpha_1}{2z}}\Big)\, ds + \int_0^1 \log \mu_s\Big(h_\psi^{\frac{\alpha_2}{2z}} h_\varphi^{\frac{1-\alpha_2}{z}} h_\psi^{\frac{\alpha_2}{2z}}\Big)\, ds\Big]. \tag{6.4}$$

Since $h_\psi^{\frac{\alpha_1+\alpha_2}{4z}} h_\varphi^{\frac{2-\alpha_1-\alpha_2}{2z}} h_\psi^{\frac{\alpha_1+\alpha_2}{4z}}$ is in $L^z(\mathcal{M})$, we have by [8, Lemma 4.8] again

$$\mu_s\Big(h_\psi^{\frac{\alpha_1+\alpha_2}{4z}} h_\varphi^{\frac{2-\alpha_1-\alpha_2}{2z}} h_\psi^{\frac{\alpha_1+\alpha_2}{4z}}\Big) = s^{-1/z}\Big\|h_\psi^{\frac{\alpha_1+\alpha_2}{4z}} h_\varphi^{\frac{2-\alpha_1-\alpha_2}{2z}} h_\psi^{\frac{\alpha_1+\alpha_2}{4z}}\Big\|_z$$

so that

$$\log \mu_s\Big(h_\psi^{\frac{\alpha_1+\alpha_2}{4z}} h_\varphi^{\frac{2-\alpha_1-\alpha_2}{2z}} h_\psi^{\frac{\alpha_1+\alpha_2}{4z}}\Big)^z = -\log s + \log Q_{\frac{\alpha_1+\alpha_2}{2},z}(\psi\|\varphi). \tag{6.5}$$

Similarly,

$$\log \mu_s\Big(h_\psi^{\frac{\alpha_j}{2z}} h_\varphi^{\frac{1-\alpha_j}{z}} h_\psi^{\frac{\alpha_j}{2z}}\Big)^z = -\log s + \log Q_{\alpha_j,z}(\psi\|\varphi), \qquad j=1,2. \tag{6.6}$$

Multiply z to both sides of (6.4) and insert (6.5) and (6.6) into it. Since $\int_0^1(-\log s)\,ds = 1$, we then arrive at

$$1+\log Q_{\frac{\alpha_1+\alpha_2}{2},z}(\psi\|\varphi) \le \frac{1}{2}\big[2+\log Q_{\alpha_1,z}(\psi\|\varphi)+\log Q_{\alpha_2,z}(\psi\|\varphi)\big],$$

which implies that $\alpha\mapsto \log Q_{\alpha,z}(\psi\|\varphi)$ is midpoint convex on $(0,1)$. Since midpoint convexity implies convexity for continuous functions, (1) holds. Moreover, by [21, Theorem 1(vii)] we find that $\lim_{\alpha\nearrow1} Q_{\alpha,z}(\psi\|\varphi)\le\psi(\mathbf{1})$. Therefore, (1) implies (2) from the defining formula of $D_{\alpha,z}$ in Definition 2.1. ☐

Remark 6.5. Theorem 6.1 can also be proved, though restricted to $z>1/2$, by use of the complex interpolation method based on Kosaki's interpolation L^p-spaces. It is worthwhile to sketch the proof here. It suffices to prove (1) of Theorem 6.1 as (2) is immediate from (1) as in the last part of the above proof of Theorem 6.1. Assume that $z>1/2$ and let $p:=2z$ and $q:=\frac{2z}{2z-1}$ with $1/p+1/q=1$. We may assume that $\mathcal{M}$ is σ-finite, by restricting ψ,φ to $s(\psi+\varphi)\mathcal{M}s(\psi+\varphi)$; see Remark 2.2. Choose $\rho,\sigma\in\mathcal{M}_*^+$ with $s(\rho)=\mathbf{1}-s(\psi)$ and $s(\sigma)=\mathbf{1}-s(\varphi)$, and put $\psi_0:=\psi+\rho$ and $\varphi_0:=\varphi+\sigma$, faithful functionals in $\mathcal{M}_*^+$. For $p\in(1,\infty)$ and $\eta\in(0,1)$ we write

$$L_L^p := L^p(\mathcal{M},\varphi_0)_L, \qquad L_R^p := L^p(\mathcal{M},\psi_0)_R, \qquad L_\eta^p := L^p(\mathcal{M},\psi_0,\varphi_0)_\eta,$$

(see (C.1)–(C.3) in Appendix C). Then by (C.6),

$$L_\eta^p = C_\eta(L_L^p, L_R^p) \tag{6.7}$$

with equal norms. We divide the proof into the cases $z\ge1$ and $1/2<z<1$.

For $z\ge1$, let $h_0:=h_\psi^{1/2}h_\varphi^{1/2}\in L^1(\mathcal{M})$. For each $\alpha\in(0,1)$ put $\eta:=\frac{z-\alpha}{2z-1}$; then we have $0<\eta<1$. Since

$$h_0 = h_\psi^{\frac{\eta}{q}}\Big(h_\psi^{\frac{\alpha}{2z}}h_\varphi^{\frac{1-\alpha}{2z}}\Big)h_\varphi^{\frac{1-\eta}{q}} = h_{\psi_0}^{\frac{\eta}{q}}\Big(h_\psi^{\frac{\alpha}{2z}}h_\varphi^{\frac{1-\alpha}{2z}}\Big)h_{\varphi_0}^{\frac{1-\eta}{q}},$$

we have $h_0\in L_\eta^p$ and $\|h_0\|_{p,\psi_0,\varphi_0,\eta}^p = Q_{\alpha,z}(\psi\|\varphi)$. Now let $\alpha_1,\alpha_2\in(0,1)$. For each $\theta\in(0,1)$ put $\alpha:=(1-\theta)\alpha_1+\theta\alpha_2$ and $\eta_j:=\frac{z-\alpha_j}{2z-1}$, $j=1,2$, so that $\eta:=\frac{z-\alpha}{2z-1}=(1-\theta)\eta_1+\theta\eta_2$. From (6.7) and the reiteration theorem, note that

$$L_\eta^p = C_\theta(L_{\eta_1}^p, L_{\eta_2}^p). \tag{6.8}$$

Since $h_0\in L_{\eta_1}^p\cap L_{\eta_2}^p$ as shown above, $\|h_0\|_{p,\psi_0,\varphi_0,\eta}\le\|h_0\|_{p,\psi_0,\varphi_0,\eta_1}^{1-\theta}\|h_0\|_{p,\psi_0,\varphi_0,\eta_2}^{\theta}$, which implies that

$$Q_{\alpha,z}(\psi\|\varphi)\le Q_{\alpha_1,z}(\psi\|\varphi)^{1-\theta}Q_{\alpha_2,z}(\psi\|\varphi)^{\theta}, \tag{6.9}$$

as desired.

Next, for $1/2<z<1$, denote $\Sigma=\Sigma(L_L^p,L_R^p):=L_L^p+L_R^p$ and let $\tilde{\mathcal{F}}(L_L^p,L_R^p)$ be the set of functions $f:S:=\{w\in\mathbb{C}:0\le\operatorname{Re}w\le1\}\to\Sigma$ satisfying

(i) f is bounded, continuous on S and analytic in the interior of S (with respect to the norm in Σ),

(ii) $f(it) \in L_L^p$ and $f(1+it) \in L_R^p$ for all $t \in \mathbb{R}$,

(iii) the maps $t \in \mathbb{R} \mapsto f(it) \in L_L^p$ and $t \in \mathbb{R} \mapsto f(1+it) \in L_R^p$ are continuous and

$$\max\left\{\sup_{t\in\mathbb{R}} \|f(it)\|_{L_L^p}, \sup_{t\in\mathbb{R}} \|f(1+it)\|_{L_R^p}\right\} < \infty.$$

Consider the function $f : S \to L^1(\mathcal{M})$ defined by

$$f(w) := h_\psi^{\frac{w}{q}+\frac{1-w}{p}} h_\varphi^{\frac{1-w}{q}+\frac{w}{p}}, \qquad w \in S. \tag{6.10}$$

Then we can manage to show that $f \in \tilde{\mathcal{F}}(L_L^p, L_R^p)$ and for each $\eta \in (0,1)$ and $t \in \mathbb{R}$, $f(\eta+it) \in L_\eta^p$ and $\|f(\eta+it)\|_{p,\varphi_0,\psi_0,\eta}^p = Q_{1-\eta,z}(\psi\|\varphi)$. Now let $\alpha_1, \alpha_2 \in (0,1)$. For each $\theta \in (0,1)$ let $\alpha := (1-\theta)\alpha_1 + \theta\alpha_2$ and $\eta_j := 1-\alpha_j$, $j = 1,2$, so that $\eta := 1-\alpha = (1-\theta)\eta_1 + \theta\eta_2$. With f given in (6.10) define $f_1(w) := f((1-w)\eta_1 + w\eta_2)$; then we have $f_1 \in \tilde{\mathcal{F}}(L_{\eta_1}^p, L_{\eta_2}^p)$. Since $L_\eta^p = C_\theta(L_{\eta_1}^p, L_{\eta_2}^p)$ by the reiteration theorem,

$$\|f(\eta)\|_{p,\varphi_0,\psi_0,\eta} = \|f_1(\theta)\|_{C_\theta(L_{\eta_1}^p, L_{\eta_2}^p)} \le \left(\sup_{t\mathbb{R}} \|f_1(it)\|_{L_{\eta_1}^p}\right)^{1-\theta} \left(\sup_{t\in\mathbb{R}} \|f_1(1+it)\|_{L_{\eta_2}^p}\right)^{\theta}.$$

which implies that $Q_{1-\eta,z}(\psi\|\varphi) \le Q_{1-\eta_1,z}(\psi\|\varphi)^{1-\theta} Q_{1-\eta_2,z}(\psi\|\varphi)^\theta$, as desired.

6.2 The case $1 < \alpha \le 2z$

In this subsection let us show monotonicity of $D_{\alpha,z}$ in the parameter $\alpha \in (1, 2z]$ when $z > 1/2$, based on the complex interpolation in a similar way as discussed in Remark 6.5.

Theorem 6.6. *Let $\psi, \varphi \in \mathcal{M}_*^+$ and $z > 1/2$. Then we have*

(1) $\alpha \mapsto \log Q_{\alpha,z}(\psi\|\varphi)$ is convex on $(1, 2z]$,

(2) $\alpha \mapsto D_{\alpha,z}(\psi\|\varphi)$ is monotone increasing on $(1, 2z]$.

Proof. Assume that $z > 1/2$ and let p, q, ψ_0 and φ_0 be defined in the same way as in the first paragraph of Remark 6.5. For each $\alpha \in (1, 2z]$ put $\eta := \frac{2z-\alpha}{2z-1} \in [0,1)$. Assume that $Q_{\alpha,z}(\psi\|\varphi) < \infty$ (hence $s(\psi) \le s(\varphi)$), so that there exists a unique $y \in s(\psi)L^p(\mathcal{M})s(\varphi)$ such that $h_\psi^{\frac{\alpha}{2z}} = y h_\varphi^{\frac{\alpha-1}{2z}}$. Since

$$h_\psi = h_\psi^{\frac{2z-\alpha}{2z}} y h_\varphi^{\frac{\alpha-1}{2z}} = h_{\psi_0}^{\frac{\eta}{q}} y h_{\varphi_0}^{\frac{1-\eta}{q}},$$

we have $h_\psi \in L_\eta^p$ and

$$\|h_\psi\|_{p,\psi_0,\varphi_0,\eta}^p = \|y\|_p^p = Q_{\alpha,z}(\psi\|\varphi),$$

where for $\eta = 0$ ($\alpha = 2z$) the left-hand side is $\|h_\psi\|_{L_L^p}^p$. Now let $\alpha_1, \alpha_2 \in (1, 2z]$ and $\alpha = (1-\theta)\alpha_1 + \theta\alpha_2$ for any $\theta \in (0,1)$. Put $\eta_j := \frac{2z-\alpha_j}{2z-1}$, $j = 1,2$, and $\eta := \frac{2z-\alpha}{2z-1} = (1-\theta)\eta + \theta\eta_1$. To show (1), it suffices to prove that (6.9) holds in the present situation. For this, we may assume that $Q_{\alpha_j,z}(\psi\|\varphi) < \infty$, $j = 1, 2$. Then we can use (6.8) similarly to the discussion in Remark 6.5 with h_ψ instead of h_0. Hence we have (6.9), and (1) follows.

As for (2), note that $h_\psi = h_{\psi_0}^{1/q} h_\psi^{1/p} \in L^p_R$ (see (C.3)) and $\|h_\psi\|^p_{L^p_R} = \|h_\psi^{1/p}\|_p^p = \psi(\mathbf{1})$. Assume that $1 < \alpha < \alpha_1 \le 2z$ and $Q_{\alpha_1,z}(\psi\|\varphi) < \infty$, so that $\alpha = (1-\theta)\alpha_1 + \theta$ for some $\theta \in (0,1)$. Let $\eta := \frac{2z-\alpha}{2z-1}$ and $\eta_1 := \frac{2z-\alpha_1}{2z-1}$. Since

$$L^p_\eta = C_\theta(L^p_{\eta_1}, L^p_R)$$

by the reiteration theorem, it follows that

$$Q_{\alpha,z}(\psi\|\varphi) \le Q_{\alpha_1,z}(\psi\|\varphi)^{1-\theta}\psi(\mathbf{1})^\theta.$$

Taking the logarithm and noting $\theta = \frac{\alpha_1-\alpha}{\alpha_1-1}$, we obtain $D_{\alpha,z}(\psi\|\varphi) \le D_{\alpha_1,z}(\psi\|\varphi)$, proving (2). □

6.3 Limits as $\alpha \nearrow 1$ and $\alpha \searrow 1$

The aim of this last subsection is to show the limits of $D_{\alpha,z}$ as $\alpha \nearrow 1$ and $\alpha \searrow 1$, extending the limits in (5.1) and (5.2).

Theorem 6.7. *Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \ne 0$. For every $z > 0$ we have*

$$\lim_{\alpha \nearrow 1} D_{\alpha,z}(\psi\|\varphi) = D_1(\psi\|\varphi).$$

Proof. Assume first that $z \in (0,1]$ and $0 \le 1-z < \alpha < 1$. Let $\beta := \frac{\alpha-1+z}{z}$; then $0 < \beta < 1$ and $\beta \nearrow 1$ as $\alpha \nearrow 1$. Hence the result follows from Lemma 6.8 below and (5.1) for $D_{\alpha,1}$. On the other hand, for the case $z \in [1,\infty)$ the result follows from Proposition 5.6(i) as (5.13) does from Proposition 5.5(i). □

Lemma 6.8. *Assume that $z \in (0,1]$ and $0 \le 1-z < \alpha < 1$. Let $\beta := \frac{\alpha-1+z}{z}$. Then for any $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \ne 0$,*

$$D_{\beta,1}(\psi\|\varphi) \le D_{\alpha,z}(\psi\|\varphi) \le D_{\alpha,1}(\psi\|\varphi).$$

Proof. Since the statement is trivial for $z = 1$, we may assume that $z \in (0,1)$. The second inequality follows from Proposition 5.6(i). For the first inequality, noting that $\beta \in (0,1)$ by assumption and by the Hölder inequality with $\frac{1}{2z} = \frac{1-z}{2z} + \frac{1}{2}$, we have

$$\begin{aligned} Q_{\alpha,z}(\psi\|\varphi) &= \left\| h_\psi^{\frac{\alpha}{2z}} h_\varphi^{\frac{1-\alpha}{2z}} \right\|_{2z}^{2z} = \left\| h_\psi^{\frac{1-z}{2z}} h_\psi^{\frac{\beta}{2}} h_\varphi^{\frac{1-\beta}{2}} \right\|_{2z}^{2z} \\ &\le \left\| h_\psi^{\frac{1-z}{2z}} \right\|_{\frac{2z}{1-z}}^{2z} \left\| h_\psi^{\frac{\beta}{2}} h_\varphi^{\frac{1-\beta}{2}} \right\|_2^{2z} = \psi(\mathbf{1})^{1-z} Q_{\beta,1}(\psi\|\varphi)^z, \end{aligned}$$

which proves the second inequality since $\alpha - 1 = z(\beta - 1)$. □

Theorem 6.9. *Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \ne 0$, and $z > 1/2$. Assume that $D_{\alpha,z}(\psi\|\varphi) < \infty$ for some $\alpha \in (1, 2z]$. Then we have*

$$\lim_{\alpha \searrow 1} D_{\alpha,z}(\psi\|\varphi) = D_1(\psi\|\varphi).$$

Proof. Assume that $z > 1/2$ and $D_{\alpha,z}(\psi\|\varphi) < \infty$ for some $\alpha \in (1, 2z]$. We may assume that φ is faithful. We utilize the function f on S given in (5.15), whose values are in L_L^{2z} as seen from the proof of Theorem 5.7. Since f is analytic in a neighborhood of $1/\alpha$, we have the expansion

$$f(w) = f\Big(\frac{1}{\alpha}\Big) + \Big(w - \frac{1}{\alpha}\Big)h + o\Big(w - \frac{1}{\alpha}\Big),$$

where $h \in L_L^{2z}$ is the derivative of f at $w = 1/\alpha$ and $\|o(\zeta)\|_{L_L^{2z}}/|\zeta| \to 0$ as $|\zeta| \to 0$. For each $\alpha' \in (1, \alpha)$ it follows that

$$f\Big(\frac{\alpha'}{\alpha}\Big) = f\Big(\frac{1}{\alpha}\Big) + \frac{\alpha'-1}{\alpha}\,h + o\Big(\frac{\alpha'-1}{\alpha}\Big) \quad \text{as } \alpha' \searrow 1.$$

Furthermore, as in the proof of Theorem 5.7, we have $f(\alpha'/\alpha) = h_\psi^{\frac{\alpha'}{2z}} h_\varphi^{1-\frac{\alpha'}{2z}} = y' h_\varphi^{\frac{2z-1}{2z}}$ for some $y' \in L^{2z}(\mathcal{M})$, so that $Q_{\alpha' z}(\psi\|\varphi) = \|y\|_{2z}^{2z} = \|f(\alpha'/\alpha)\|_{L_L^{2z}}$.

Now let us recall that L_L^{2z} is uniformly convex thanks to $2z > 1$ (see [10], [23, Theorem 4.2]), so that the norm $\|\cdot\|_{L_L^{2z}}$ is uniformly Fréchet differentiable (see, e.g., [2, Part 3, Chap. II]). We set $a_0 \in L_L^{\frac{2z}{2z-1}}$ with the unit norm by

$$a_0 := \Big(\frac{h_\psi}{\psi(\mathbf{1})}\Big)^{\frac{2z-1}{2z}} h_\varphi^{\frac{1}{2z}},$$

so that $\langle a_0, f(1/\alpha)\rangle = \|f(1/\alpha)\|_{L_L^{2z}}$, where the dual pairing of $L_L^{\frac{2z}{2z-1}}$ and L_L^{2z} is given in (C.5) in Appendix C with $p = \frac{2z}{2z-1}$. Then the uniform Fréchet differentiability of $\|\cdot\|_{L_L^{2z}}$ at $1/\alpha$ implies that

$$\langle a_0, h\rangle = \lim_{\alpha' \searrow 1} \frac{\|f(\alpha'/\alpha)\|_{L_L^{2z}} - \|f(1/\alpha)\|_{L_L^{2z}}}{\frac{\alpha'-1}{\alpha}} \tag{6.11}$$

and also

$$\begin{aligned}
\langle a_0, h\rangle &= \lim_{t\to 0} \frac{1}{it}\langle a_0, f((1/\alpha) + it) - f(1/\alpha)\rangle \\
&= \psi(\mathbf{1})^{-\frac{2z-1}{2z}} \lim_{t\to 0} \frac{1}{it}\Big\langle h_\psi^{\frac{2z-1}{2z}} h_\varphi^{\frac{1}{2z}}, h_\psi^{\frac{1}{2z}}\Big(h_{\psi_0}^{\frac{\alpha}{2z}it} h_\varphi^{-\frac{\alpha}{2z}it} - \mathbf{1}\Big) h_\varphi^{\frac{2z-1}{2z}}\Big\rangle \\
&= \psi(\mathbf{1})^{-\frac{2z-1}{2z}} \lim_{t\to 0} \frac{1}{it}\mathrm{tr}\Big[h_\psi^{\frac{2z-1}{2z}} h_\psi^{\frac{1}{2z}}\big(h_{\psi_0}^{\frac{\alpha}{2z}it} h_\varphi^{-\frac{\alpha}{2z}it} - \mathbf{1}\big)\Big] \\
&= \psi(\mathbf{1})^{-\frac{2z-1}{2z}} \frac{\alpha}{2z} \lim_{t\to 0} \mathrm{tr}\big[h_\psi\big(h_{\psi_0}^{it} h_\varphi^{-it} - \mathbf{1}\big)\big] \\
&= \psi(\mathbf{1})^{-\frac{2z-1}{2z}} \frac{\alpha}{2z} D(\psi\|\varphi),
\end{aligned} \tag{6.12}$$

where we have used (C.5) for the third equality and [30, Theorem 5.7] for the last equality. Since

$\psi(\mathbf{1}) = \|f(1/\alpha)\|_{L_L^{2z}}^{2z}$, it follows from (6.11) and (6.12) that

$$\begin{aligned}
D_{\alpha',z}(\psi\|\varphi) &= \frac{\log Q_{\alpha',z}(\psi\|\varphi) - \log\psi(\mathbf{1})}{\alpha'-1} = \frac{2z\log\|f(\alpha'/\alpha)\|_{L_L^{2z}} - 2z\log\|f(1/\alpha)\|_{L_L^{2z}}}{\alpha'-1} \\
&= \left(\frac{\log\|f(\alpha'/\alpha)\|_{L_L^{2z}} - \log\|f(1/\alpha)\|_{L_L^{2z}}}{\|f(\alpha'/\alpha)\|_{L_L^{2z}} - \|f(1/\alpha)\|_{L_L^{2z}}}\right)\frac{2z}{\alpha}\left(\frac{\|f(\alpha'/\alpha)\|_{L_L^{2z}} - \|f(1/\alpha)\|_{L_L^{2z}}}{\frac{\alpha'-1}{\alpha}}\right) \\
&\to \frac{1}{\psi(\mathbf{1})^{\frac{1}{2z}}}\,\frac{2z}{\alpha}\,\psi(\mathbf{1})^{-\frac{2z-1}{2z}}\,\frac{\alpha}{2z}\,D(\psi\|\varphi) = \frac{D(\psi\|\varphi)}{\psi(\mathbf{1})} = D_1(\psi\|\varphi)
\end{aligned}$$

as $\alpha' \searrow 1$, proving the statement. □

7 Concluding remarks

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Acknowledgments

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A Haagerup L^p-spaces

Let $\mathcal{R} := \mathcal{M} \rtimes_{\sigma^\omega} \mathbb{R}$ be the crossed product of $\mathcal{M}$ by the *modular automorphism group* σ_t^ω, $t \in \mathbb{R}$, for a faithful normal semi-finite weight ω on $\mathcal{M}$. Then $\mathcal{R}$ is a semi-finite von Neumann algebra with the canonical trace τ. Let θ_s, $s \in \mathbb{R}$, be the *dual action* on $\mathcal{R}$ having the τ-scaling property $\tau\circ\theta_s = e^{-s}\tau$, $s \in \mathbb{R}$; see [34, Chap. X] (also [13, Chap. 8]). Let $\widetilde{\mathcal{R}}$ denote the space of τ-measurable operators affiliated with $\mathcal{R}$; see [8] (also [13, Chap. 4]). For $0 < p \le \infty$, the *Haagerup L^p-space* $L^p(\mathcal{M})$ [10, 35] (also [13, Chap. 9]) is defined by

$$L^p(\mathcal{M}) := \{a \in \widetilde{\mathcal{R}} : \theta_s(a) = e^{-s/p}a,\ \ s \in \mathbb{R}\}.$$

In particular, $\mathcal{M} = L^\infty(\mathcal{M})$ and we have an order isomorphism $\mathcal{M}_* \cong L^1(\mathcal{M})$ given as $\psi \in \mathcal{M}_* \leftrightarrow h_\psi \in L^1(\mathcal{M})$, so that $\operatorname{tr} h_\psi = \psi(\mathbf{1})$, $\psi \in L^1(\mathcal{M})$, defines a positive linear functional tr on $L^1(\mathcal{M})$. For $0 < p < \infty$ the L^p-norm (quasi-norm for $0 < p < 1$) of $a \in L^p(\mathcal{M})$ is defined by $\|a\|_p := (\operatorname{tr}|a|^p)^{1/p}$, and the L^∞-norm $\|\cdot\|_\infty$ is the operator norm $\|\cdot\|$ on $\mathcal{M}$. For $1 \le p < \infty$, $L^p(\mathcal{M})$ is a Banach space whose dual Banach space is $L^q(\mathcal{M})$, where $1/p + 1/q = 1$, by the duality pairing

$$(a,b) \in L^p(\mathcal{M}) \times L^q(\mathcal{M}) \mapsto \operatorname{tr}(ab) = \operatorname{tr}(ba). \tag{A.1}$$

We will next recall the extensions of a conditional expectations to the Haagerup L^p-spaces, obtained in [20]. Let $\mathcal{N} \subseteq \mathcal{M}$ be a subalgebra such that $\sigma^\omega(\mathcal{N}) \subseteq \mathcal{N}$. Equivalently, there is a faithful normal conditional expectation $\mathcal{E}$ on $\mathcal{M}$ with range $\mathcal{N}$. We then have $\sigma^{\omega|_\mathcal{N}} = \sigma^\omega|_\mathcal{N}$ for the modular group of the restriction $\omega|_\mathcal{N}$ and we may identify $\mathcal{S} := \mathcal{N} \rtimes_{\sigma^{\omega_\mathcal{N}}} \mathbb{R}$ with a subalgebra in $\mathcal{R}$. It can be seen that

$$\tilde{\mathcal{E}} := \mathcal{E} \otimes id_{L_2(\mathbb{R})}$$

defines a conditional expectation of $\mathcal{R}$ onto $\mathcal{S}$. Further, let τ_0 be the canonical trace for $\mathcal{S}$, then we have $\tau_0 = \tau|_{\mathcal{S}}$ and $\tau = \tau \circ \tilde{\mathcal{E}} = \tau_0 \circ \tilde{\mathcal{E}}$. It follows that $\tilde{\mathcal{S}}$ can be identified with the subspace in $\tilde{\mathcal{R}}$ of elements affiliated with $\mathcal{S}$ and since the dual action for $\mathcal{S}$ is just a restriction of θ_s, $s \in \mathbb{R}$, we have a similar identification of $L^p(\mathcal{N})$ with a subspace in $L^p(\mathcal{M})$. By the results of [20, Sec. 2], $\mathcal{E}$ extends to a contractive projection $\mathcal{E}_p$ from $L^p(\mathcal{M})$ onto $L^p(\mathcal{N})$ for any $1 \leq p \leq \infty$, where

$$\mathcal{E}_\infty = \mathcal{E}, \qquad \mathcal{E}_1 = \mathcal{E}_*.$$

Moreover, for any $x \in L^p(\mathcal{M})$,

$$\mathcal{E}_p(x)^* = \mathcal{E}_p(x^*), \qquad x \geq 0 \implies \mathcal{E}_p(x) \geq 0$$

and for $1 \leq p, q, r \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{1}{s} \leq 1$, we have

$$\mathcal{E}_s(axb) = a\mathcal{E}_r(x)b, \qquad a \in L^p(\mathcal{N}),\ b \in L^q(\mathcal{N}),\ x \in L^r(\mathcal{M}), \tag{A.2}$$

see [20, Prop. 2.3].

We finish this section by some well known lemmas, proofs are given for completeness.

Lemma A.1. *For any $0 < p < \infty$ and $\varphi \in \mathcal{M}_*^+$, $h_\varphi^{\frac{1}{2p}} \mathcal{M}^+ h_\varphi^{\frac{1}{2p}}$ is dense in $L^p(\mathcal{M})^+$ with respect to the (quasi)-norm $\|\cdot\|_p$.*

Proof. We may assume that φ is faithful. By [20, Lemma 1.1], $\mathcal{M} h_\varphi^{\frac{1}{2p}}$ is dense in $L^{2p}(\mathcal{M})$ for any $0 < p < \infty$. Let $y \in L^p(\mathcal{M})^+$, then $y^{\frac{1}{2}} \in L^{2p}(\mathcal{M})$, hence there is a sequence $a_n \in \mathcal{M}$ such that $\|a_n h_\varphi^{\frac{1}{2p}} - y^{\frac{1}{2}}\|_{2p} \to 0$. Then also

$$\left\|h_\varphi^{\frac{1}{2p}} a_n^* - y^{\frac{1}{2}}\right\|_{2p} = \left\|(a_n h_\varphi^{\frac{1}{2p}} - y^{\frac{1}{2}})^*\right\|_{2p} = \left\|a_n h_\varphi^{\frac{1}{2p}} - y^{\frac{1}{2}}\right\|_{2p} \to 0$$

and

$$\left\|h_\varphi^{\frac{1}{2p}} a_n^* a_n h_\varphi^{\frac{1}{2p}} - y\right\|_p = \left\|(h_\varphi^{\frac{1}{2p}} a_n^* - y^{\frac{1}{2}}) a_n h_\varphi^{\frac{1}{2p}} + y^{\frac{1}{2}}(a_n h_\varphi^{\frac{1}{2p}} - y^{\frac{1}{2}})\right\|_p.$$

Since $\|\cdot\|_p$ is a (quasi)-norm, the above expression goes to 0 by the Hölder inequality. $\square$

Lemma A.2. *Let $0 < p \leq \infty$ and let $h, k \in L^p(\mathcal{M})^+$ be such that $h \leq k$. Then $\|h\|_p \leq \|k\|_p$. Moreover, if $1 \leq p < \infty$, then*

$$\|k - h\|_p^p \leq \|k\|_p^p - \|h\|_p^p.$$

Proof. The first statement follows from [8, Lemmas 2.5(iii) and 4.8]. The second statement is from [8, Lemma 5.1]. $\square$

Lemma A.3. *Let $\psi, \varphi \in \mathcal{M}_*^+$ with $\psi \leq \varphi$. Then for any $a \in \mathcal{M}$ and $p \in [1, \infty)$,*

$$\operatorname{tr}\Big(\big(a^* h_\psi^{1/p} a\big)^p\Big) \leq \operatorname{tr}\Big(\big(a^* h_\varphi^{1/p} a\big)^p\Big).$$

Proof. Since $1/p \in (0, 1]$, it follows (see [14, Lemma B.7] and [16, Lemma 3.2]) that $h_\psi^{1/p} \leq h_\varphi^{1/p}$. Hence $a^* h_\psi^{1/p} a \leq a^* h_\varphi^{1/p} a$. Therefore, by Lemma A.2, we have the statement. $\square$

B Haagerup's reduction theorem

In this appendix let us recall Haagerup's reduction theorem, which was presented in [11, Sec. 2] (a compact survey is also found in [9, Sec. 2.5]). Let $\mathcal{M}$ be a general σ-finite von Neumann algebra. Let ω be a faithful normal state of $\mathcal{M}$ and σ_t^ω ($t \in \mathbb{R}$) be the associated modular automorphism group. Consider the discrete additive group $G := \bigcup_{n\in\mathbb{N}} 2^{-n}\mathbb{Z}$ and define $\hat{\mathcal{M}} := \mathcal{M} \rtimes_{\sigma^\omega} G$, the crossed product of $\mathcal{M}$ by the action $\sigma^\omega|_G$. Then the dual weight $\hat\omega$ is a faithful normal state of $\hat{\mathcal{M}}$, and we have $\hat\omega = \omega \circ E_{\mathcal{M}}$, where $E_{\mathcal{M}} : \hat{\mathcal{M}} \to \mathcal{M}$ is the canonical conditional expectation (see, e.g., [13, Sec. 8.1], also [9, Sec. 2.5]).

Haagerup's reduction theorem is summarized as follows:

Theorem B.1 ([11])**.** *In the above setting, there exists an increasing sequence $\{\mathcal{M}_n\}_{n\ge1}$ of von Neumann subalgebras of $\hat{\mathcal{M}}$, containing the unit of $\hat{\mathcal{M}}$, such that the following hold:*

(i) Each $\mathcal{M}_n$ is finite with a faithful normal tracial state τ_n.

(ii) $\big(\bigcup_{n\ge1}\mathcal{M}_n\big)'' = \hat{\mathcal{M}}$.

(iii) For every n there exist a (unique) faithful normal conditional expectation $E_{\mathcal{M}_n} : \hat{\mathcal{M}} \to \mathcal{M}_n$ satisfying

$$\hat\omega \circ E_{\mathcal{M}_n} = \hat\omega, \qquad \sigma_t^{\hat\omega} \circ E_{\mathcal{M}_n} = E_{\mathcal{M}_n} \circ \sigma_t^{\hat\omega}, \quad t \in \mathbb{R}.$$

Moreover, for any $x \in \hat{\mathcal{M}}$, $E_{\mathcal{M}_n}(x) \to x$ in the σ-strong topology.

Furthermore, for any $\psi \in \mathcal{M}_*^+$ define $\hat\psi := \psi \circ E_{\mathcal{M}}$. Then by [17, Theorem 4] we have $\hat\psi \circ E_{\mathcal{M}_n} \to \hat\psi$ in the norm. Here we give the next lemma, which is used in Sec. 5.2.

Lemma B.2. *In the above situation, for any $\psi, \varphi \in \mathcal{M}_*^+$ with $\psi \ne 0$ let $\hat\psi := \psi \circ E_{\mathcal{M}}$ and $\hat\varphi := \varphi \circ E_{\mathcal{M}}$. If α and z satisfy the DPI bounds, i.e., condition (i) or (ii) in Theorem 3.3), then we have*

$$D_{\alpha,z}(\psi\|\varphi) = D_{\alpha,z}(\hat\psi\|\hat\varphi) = \lim_{n\to\infty} D_{\alpha,z}(\hat\psi|_{\mathcal{M}_n}\|\hat\varphi|_{\mathcal{M}_n}) \quad \textit{increasingly}, \tag{B.1}$$

$$D_1(\psi\|\varphi) = D_1(\hat\psi\|\hat\varphi) = \lim_{n\to\infty} D_1(\hat\psi|_{\mathcal{M}_n}\|\hat\varphi|_{\mathcal{M}_n}) \quad \textit{increasingly}. \tag{B.2}$$

Proof. Apply the DPI for $D_{\alpha,z}$ proved in Theorem 3.3 to the injection $\mathcal{M} \hookrightarrow \hat{\mathcal{M}}$ and to the conditional expectation $E_{\mathcal{M}} : \hat{\mathcal{M}} \to \mathcal{M}$. We then have the first equality in (B.1). By Theorem B.1 we can apply the martingale convergence in Theorem 3.4 to obtain the latter equality in (B.1) with increasing convergence. The assertion of D_1 in (B.2) is included in [9, Proposition 2.2], while this is an immediate consequence of the well-known martingale convergence and the DPI of the relative entropy [25]. □

C Kosaki's interpolation L^p-spaces

Assume that $\mathcal{M}$ is a σ-finite von Neumann algebra and let faithful $\psi_0, \varphi_0 \in \mathcal{M}_*^+$ be given. For each $\eta \in [0,1]$ consider an embedding $\mathcal{M} \hookrightarrow L^1(\mathcal{M})$ by $x \mapsto h_{\psi_0}^\eta x h_{\varphi_0}^{1-\eta}$. Defining $\|h_{\psi_0}^\eta x h_{\varphi_0}^{1-\eta}\|_\infty := \|x\|$ (the operator norm of x) on $h_{\psi_0}^\eta \mathcal{M} h_{\varphi_0}^{1-\eta}$ we have a pair $\big(h_{\psi_0}^\eta \mathcal{M} h_{\varphi_0}^{1-\eta}, L^1(\mathcal{M})\big)$ of compatible Banach spaces (see, e.g., [3]). For $1 < p < \infty$, *Kosaki's interpolation L^p-space* with respect to ψ_0, φ_0 and η [23] (also see [13, Sec. 9.3] for a compact survey) is defined as the complex interpolation Banach space:

$$L^p(\mathcal{M}, \psi_0, \varphi_0)_\eta := C_{1/p}\big(h_{\psi_0}^\eta \mathcal{M} h_{\varphi_0}^{1-\eta}, L^1(\mathcal{M})\big) \tag{C.1}$$

equipped with the interpolation norm $\|\cdot\|_{p,\psi_0,\varphi_0,\eta} := \|\cdot\|_{C_{1/p}}$. Then, Kosaki's theorem [23, Theorem 9.1] says that for every $\eta \in [0,1]$ and $p \in (1,\infty)$ with $1/p + 1/q = 1$,

$$\begin{aligned} &L^p(\mathcal{M}, \psi_0, \varphi_0)_\eta = h_{\psi_0}^{\eta/q} L^p(\mathcal{M}) h_{\varphi_0}^{(1-\eta)/q} \ (\subset L^1(\mathcal{M})), \\ &\|h_{\psi_0}^{\eta/q} a h_{\varphi_0}^{(1-\eta)/q}\|_{p,\psi_0,\varphi_0,\eta} = \|a\|_p, \qquad a \in L^p(\mathcal{M}), \end{aligned}$$

that is, $L^p(\mathcal{M}) \cong L^p(\mathcal{M}, \psi_0, \varphi_0)_\eta$ by the isometry $a \mapsto h_{\psi_0}^{\eta/q} a h_{\varphi_0}^{(1-\eta)/q}$. In the main body of this paper we use the special cases where $\eta = 0, 1$, that is,

$$L^p(\mathcal{M}, \varphi_0)_L := C_{1/p}\big(\mathcal{M} h_{\varphi_0}, L^1(\mathcal{M})\big) = L^p(\mathcal{M}) h_{\varphi_0}^{1/q}, \tag{C.2}$$

$$L^p(\mathcal{M}, \psi_0)_R := C_{1/p}\big(h_{\psi_0} \mathcal{M}, L^1(\mathcal{M})\big) = h_{\psi_0}^{1/q} L^p(\mathcal{M}), \tag{C.3}$$

which are called Kosaki's *left and right L^p-spaces*, respectively. Another special case we use is the *symmetric L^p-space* $L^p(\mathcal{M}, \varphi_0)$ where $\eta = 1/2$ and $\psi_0 = \varphi_0$, i.e.,

$$L^p(\mathcal{M}, \varphi_0) = C_{1/p}\big(h_{\varphi_0}^{1/2} \mathcal{M} h_{\varphi_0}^{1/2}, L^1(\mathcal{M})\big) = h_{\varphi_0}^{1/2q} L^p(\mathcal{M}) h_{\varphi_0}^{1/2q}, \tag{C.4}$$

whose interpolation norm is denoted by $\|\cdot\|_{p,\varphi_0}$. The L^p-L^q duality of Kosaki's L^p-spaces can be given by transforming the duality paring in (A.1); in particular, the duality pairing between $L^p(\mathcal{M}, \varphi_0)_L$ and $L^q(\mathcal{M}, \varphi_0)_L$ for $1 \le p < \infty$ and $1/p + 1/q = 1$ is written as

$$\langle a h_{\varphi_0}^{1;q}, b h_{\varphi_0}^{1/p} \rangle = \mathrm{tr}(ab), \qquad a \in L^p(\mathcal{M}),\ b \in L^q(\mathcal{M}). \tag{C.5}$$

Kosaki's non-commutative Stein–Weiss interpolation theorem [23, Theorem 11.1] says that for each $\eta \in (0,1)$ and $p \in (1,\infty)$, Kosaki's L^p-space $L^p(\mathcal{M}, \psi_0, \varphi_0)_\eta$ given in (C.1) is the complex interpolation space of the left and right L^p-spaces in (C.2) and (C.3) with equal norms, that is,

$$L^p(\mathcal{M}, \psi_0, \varphi_0)_\eta = C_{1/p}\big(h_{\psi_0}^\eta \mathcal{M} h_{\varphi_0}^{1-\eta}, L^1(\mathcal{M})\big) = C_\eta(L^p(\mathcal{M}, \varphi_0)_L, L^p(\mathcal{M}, \psi_0)_R). \tag{C.6}$$

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