A : JBW-algebra, $P : A \to A$ normal positive, $P^2 = P$.

→ e - support of P : $\bigwedge \{ e \text{ projection}, \ P(e) = 1 \}$

Lemma: $e = 1$, then $P(A)$ is a JBW subalgebra and P has the property: $a, c \in P(A), \ b \in A \Rightarrow P(\{abc\}) = \{a P(b) c\}$

$$\boxed{a \in P(A), \ b \in A \Rightarrow P(a \circ b) = a \circ P(b)}$$

Proof: $P(a^2) = 0$; $P(a^2) = 0 \Leftrightarrow \{e a^2 e\} = 0$

$$U_e(a^2) = 0 \qquad a^2 \leq 1$$

[AS, Lemma 1.37] $\Updownarrow$

$$e \circ a^2 = 0 \Leftrightarrow (1-e) \circ a^2 = a^2$$

$$U_{1-e}(a^2) = 2(1-e) \circ ((1-e) \circ a^2) - (1-e) \circ a^2 = 2(1-e) \circ a^2 - a^2 = a^2$$

But if $e = 1 \Rightarrow a^2 = 0 \Rightarrow a = 0$.

By [ES, Coro. 1.5 + more preliminary], $P(A)$ is a JBW-subalgebra.

$a \in P(A), \ b \in A$,
$$P(a \circ b) = P(P(a) \circ b) \overset{[ES, \text{ Lemma } 1.1]}{=} P(P(a) \circ P(b))$$
$$= P(a) \circ P(b) = a \circ P(b)$$

Let $a, c \in P(A), \ b \in A$:

$$\{abc\} = (a \circ b) \circ c + (b \circ c) \circ a - (a \circ c) \circ b$$

$$P(\{abc\}) = P(a \circ b) \circ c + P(b \circ c) \circ a - (a \circ c) \circ P(b)$$
$$= (a \circ P(b)) \circ c + (P(b) \circ c) \circ a - (a \circ c) \circ P(b) = \{a P(b) c\}.$$

□

Lemma: $A_e = \{eAe\} = U_e(A)$, $P_e : A_e \to A_e$,

$$P_e := U_e \circ P|_{A_e}$$

$U_e P(\overset{= a}{U_e(a)})$ $\qquad a \in U_e(A)$, $P_e(a) = U_e \circ P(a)$

Then P_e is a positive unital normal projection on A_e, with support 1 (faithful).

Proof:

$$\leftarrow a = U_e(a)$$

$$P_e^2(a) = P_e(P_e(a)) = P_e(U_e \circ P(a)) =$$

$$= U_e \circ P(U_e \circ P(a)) = U_e \circ P^2(a) = U_e \circ P(a) = P_e(a)$$

idempotent, positive ✓

if s is the support of P_e $\Rightarrow$ $\boxed{P_e(s) = e}$ $\Rightarrow$

$s \in U_e(A)$, projection

$\cancel{P_e(e-s) = 0}$

$\cancel{U_e \circ P(e-s) = 0 \Rightarrow P(e-s) \le 1-e}$

$\cancel{1 - P(s) \le 1-e}$

$U_e(P(s)) = e$

$\Downarrow$

$e \le P(s)$

$P \ \big) \ e \le P(s)$

$1 \le P^2(s) = P(s) \le 1$

$\Rightarrow P(s) = 1 \Rightarrow s = e.$

$\square$

Lemma: There is a positive unital normal map $\Psi : A_e \to A_{1-e}$, such that

$$\Psi \circ P_e = \Psi \quad \text{and}$$

$$P = P_e \circ U_e + \Psi \circ U_e .$$

Conversely, any map of this form is a positive unital normal projection with support e.

Proposition: Let $P : A \to A$ be a positive unital normal projection with support e. Then there is a faithful unp P_e on $U_e(A)$ and a unital normal positive map $\Psi : U_e(A) \to U_{1-e}(A)$ such that

$$\Psi \circ P_e = \Psi \quad \text{and}$$

$$P = P_e \circ U_e + \Psi \circ U_e .$$

Any map of this form ...

$a \in A$:

$\boxed{\text{Proof}}$ $P(a) = U_e(P(a)) + U_{1-e}(P(a))$

$$= \underbrace{U_e \circ P}_{P_e}(U_e(a)) + \underbrace{U_{1-e} \circ P}_{\psi}(U_e(a))$$

$$\boxed{P(a) = P(U_e(a))}$$

$$P_e = U_e \circ P\big|_{U_e(A)} \;; \qquad \psi = U_{1-e} \circ P\big|_{U_e(A)}$$

$$\psi \circ P_e = \psi \circ U_e \circ P = U_{1-e} \circ P \circ \underline{U_e} \circ P = U_{1-e} \circ P = \psi \quad \square$$

Conversely, let Q ~~have~~ the form

$$Q = R \circ U_e + \psi \circ U_e,$$

where e is a projection in A, R is a faithful unital normal positive projection $U_e(A) \to U_e(A)$ and ψ is a unital normal positive map $U_e(A) \to U_{1-e}(A)$, and that $\psi \circ R = \psi$. Then Q is a unital normal positive projection on A with support e:

$\boxed{\text{Proof}}$: Q positive, normal $\to$ clear.

unital:

$$Q(1) = R \circ U_e(1) + \psi \circ U_e(1) = R(e) + \psi(e) = e + 1 - e = 1 \checkmark$$

idempotent:

$$Q^2(a) = (R \circ U_e + \psi \circ U_e)(R \circ U_e + \psi \circ U_e)(a)$$

$$= (R \circ U_e + \psi \circ U_e)\big(R(U_e(a)) + \psi(U_e(a))\big)$$

$$= (R \circ U_e + \psi \circ U_e)\big(R(U_e(a))\big) =$$

$$= R \circ U_e\big(R(U_e(a))\big) + \psi \circ U_e\big(R(U_e(a))\big)$$

$$= R^2(U_e(a)) + \Psi\circ R(U_e(a))$$

$$= R(U_e(a)) + \Psi(U_e(a)) = Q(a).$$

e is the support of Q:

$$Q(e) = R(e) + \Psi(e) = e + 1-e = 1$$

$$s \leq e$$

$$1 \overset{?}{=} Q(s) = R(s) + \Psi(s) \Rightarrow \begin{matrix} R(s) = e \\ \Psi(s) = 1-e \end{matrix}$$

$$\left(\text{since } \begin{matrix} R(s) \leq e \\ \Psi(s) \leq 1-e \end{matrix}\right)$$

$$\Rightarrow R(e-s) = 0, \ R \text{ is faithful} \Rightarrow e = s.$$

$\square$

Range of P and the Jordan product $*$:

$$P = R\circ V_e + \Psi\circ V_e:$$

$$a,b \in P(A) = \underbrace{R(A_e)}_{a\circ b} + \Psi(A_e) \qquad A_e = V_e(A)$$

$a\circ b$ if $a,b \in R(A_e)$
$a\circ b \in R(A_e)$

$a_1,b_1 \in R(A_e)$

$a_2,b_2 \in \Psi(A_e)$

$$\boxed{\begin{matrix} a = a_1 + a_2 \\ b = b_1 + b_2 \end{matrix}}$$

$$a\circ b = a_1\circ b_1 + \underbrace{a_2\circ b_2}$$

$$a * b = P(a\circ b) =$$

$$= P(a_1\circ b_1) = R(a_1\circ b_1) + \Psi(a_1\circ b_1)$$

$$= \underline{\underline{a_1\circ b_1 + \Psi(a_1\circ b_1)}}$$

D

$$\boxed{\text{WHAT IF } A \text{ is A JB-algebra.}}$$