STRONG CONVERSE RATE FOR STATE DISCRIMINATION IN GENERAL VON NEUMANN ALGEBRAS

MARIUS JUNGE, NICHOLAS LARACUENTE

By definition Quantum information theory is the merger of Information Theory and Quantum Mechanics. One of the first tasks in Quantum Information Theory were to confirm Shannon's striking connection between entropic terms and transmission rates of channels. This applies in particular to converse rates for quantum capacity with or without entanglement assistance, private capacity see [1]. Since then families of relative entropy such as Petz's family of relative entropies and sandwiched relative entropies provide fundamental tools in quantum information theory. The aim of this contribution is to enforce the connection between *state discrimination* and the family of *Sandwiched relative entropies.* This quest for an operational interpretation of relative entropies continues the deep work of Mosonyi and his collaborators [2, 3].

For the sake of concreteness less us consider the state discrimination problem: Let h and k be two unit vectors in a Hilbert space. We want to determine wether $h = k$. As so often in quantum mechanics we can only use measurements and compare

$$\rho(T) \;:=\; (h, Th) \stackrel{?}{=} (k, Tk) \;=:\; \eta(T)$$

for measurements $0 \leq T \leq 1$. Any experiment is eventually restricted in their access to measurements T. In finite dimensional quantum information theory h could be the purification of a density on Alice's system A and we can only use $T \in L(A)$ the linear maps on the Alice's Hilbert space this implementing $\rho(T) = tr(dT)$ for a density. In high energy physics with modelling a black hole only observables associated to a region on the boundary may be accessible to draw conclusion on this part of the system. Let us recall that a von Neumann algebra is a subalgebra of $\mathbb{B}(H)$, the bounded operator on a Hilbert space which is closed under *, and also closed in the weak operator topology, or equivalently a dual Banach space. By restricting $T \in N$ we can find the information given by the following matrix

	ρ	η
T	$\rho(T)$	$\eta(T)$
1-T	$\rho(1-T)$	$\eta(1-T)$

.

Here the restriction is to $T \in N$. Equivalently our discrimination tool is restricted to a quantum to classical measurement channel

$$\phi_T(\rho) \;=\; (\rho(T), \rho(1-T))$$

and it will be paramount to find good T's. However, equality is fragile. The classical method in quantum information theory is to consider tensor copies $h^{\otimes_n}$, $k^{\otimes_n}$ of h and k, respectively and use tests $T_n \in N^{\otimes_n}$. Strong converse results are concerned with asymptotic exponential bounds and consider the operational set

$$B(\rho,\eta) \;=\; \left\{(R,r)\middle|\exists_{n_0}\forall_{n \,\geq\, n_0}\exists_{0\leq T_n\leq 1, T_n\in N^{\otimes n}} \;:\; \eta^{\otimes_n}(T_n) \;\leq\; e^{-rn} \text{ and } \rho^{\otimes_n}(T_n) \;\geq\; e^{-Rn}\right\} .$$

This limit procedure, as unpractical as it may appear, is standard when comparing entropic quantities with rates. The entropic terms relevant for this discrimination problem are the sandwiched relative enropies

$$D_\alpha(\rho\|\sigma) \;=\; \frac{\alpha}{\alpha-1}\ln\|d_\eta^{\frac{1-\alpha}{2\alpha}}d_\rho d_\eta^{\frac{1-\alpha}{2\alpha}}\|_\alpha$$

where d_η, d_ρ are the density of η and ρ, respectively, and $\|x\|_\alpha \;=\; [tr|d_x|^\alpha]^{1/\alpha}$ is the α norm. Admittedly, not every von Neumann algebra has a trace. However, thanks to the work of Araki, Petz, Haagerup, Kosaki and others [4, 5, 6, 7, 8, 9], all these expressions make perfect sense without a trace, see the technical text for details. A familiar concept in convexity theory is the definition of the Legendre transform

$$f^*(y) \;=\; \sup\{xy - f(y)|y>0\} .$$

For convex functions $f^{**} = f$, a standard trick showing duality results for function spaces. This can be modified for functions on $(0,1)$. As observed in [3] the correct transformation is $\alpha \;=\; \frac{1}{1-u}$. Instead of one sandwiched Rényi entropy, Mosonyi and his coauthors consider the modified Legendre transform of all relative entropies

$$H_r^*(\rho\|\eta) \;=\; \sup_{\alpha>1}\frac{\alpha}{\alpha-1}r - \ln\|d_\eta^{\frac{1-\alpha}{2\alpha}}d_\rho d_\eta^{\frac{1-\alpha}{2\alpha}}\|_\alpha \;=\; \sup_{\alpha>1}\frac{\alpha}{\alpha-1}(r-D_\alpha) .$$

Since it is possible to recover the function $\alpha \mapsto D_\alpha$ from the Legendre transform, it is fair to say that the family of sandwiched relative entropies has an operational meaning if the dual function has which is collaborated by the following result.

Theorem 0.1. *Let η be a normal faithful state on a von Neumann algebra such that $D_\infty(\rho|\eta)$ is finite. Then*

$$H_r^*(\rho\|\eta) \;=\; B_r(\rho\|\eta) \;:=\; \sup\{R|\forall_{r'<r} \;:\; (R,r')\in B(\rho\|\eta)\} .$$

As so often in quantum information theory this theorem has been proven in different degrees of generality. For matrix algebras, i.e. finite dimensional type I factors $N = \mathbb{B}(H)$,

H finite, this is due to [2]. This was extended to the type I algebra $N = \mathbb{B}(H)$ by (see) and for so-called hyperfinite von Neumann algebras by Mosonyi and Hia. In this contribution we remove the additional assumption hyperfinite. All previous proofs eventually reduced the problem to the finite dimensional case, and an incredible versatile pinching trick. Hyperfinite von Neumann algebras are those which admit a family finite dimensional C^*-algebras which is dense in the weak operator algebra topology. Let us provide two justifications for exploring this less traditional extension of quantum information theory. First, in random matrix theory distributional properties of unitary or gaussian examples are always considered in the large limit. The hope is of course that for sufficiently large matrices the limit situation approximatively occurs. This justifies performing calculation in the limit algebra itself. In the cases of independently chosen unitaries $(u_j)_{j=1}^m$ in $U(n)$ one can consider the algebra obtained by considering noncommutative polynomials in

$$p(u_1(\omega), u_1^*(\omega), ..., u_m(\omega), u_m^*(\omega))$$

together with the expected normalized trace. The resulting algebra is the very well-studied reduced C^* algebra of the free group, see [10, 11] (Voiculescu, Haagerup & Thjorbornsen). This algebra admits a nice trace and hence relative entropies can easily be defined. Our other justification stems from the impact of quantum information theory on high energy physics and potentially the theory of black holes. In this context the AdS/CFT correspondence requires to put a quantum field theory on the boundary and an algebra of observables in the bulk. Quantum field theories require to work with so called type III_1 algebras. Some of them may or may not be hyperfinite. However, the construction of appropriate test in sniffing out sandwiched relative entropy should depend on having access to these matrical approximation inside a given complicated von Neumann algebra. Our approach to finding these test, hopefully of more general interest, is to look outside the algebra instead of inside using Haagerup's reduction method [12]. The philosophy is easy: If the given von Neumann algebra does not admit a trace, it can still embedded one a larger von Neumann algebra admitting a normal conditional expectation which is a good limit of tracial ones

$$M \subset M \rtimes G \; = \; \overline{\bigcup_k M_k} \, .$$

Conditional expectation are projection from a large algebra to a smaller one. In order to find good tests, we first extend the states to the crossed product with the discrete group, then restrict them to M_k where a trace is present and we find a density. Using discretization, we open the door to the standard pinching trick and ultimately reduce the problem to states on commutative Neumann algebras. Adding more conditional expectations to existing proof methods is the new technical ingredient for avoiding the injectivity (= hyperfinite) assumption.

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MARIUS JUNGE AND NICHOLAS LARACUENTE

Abstract. We extend results showing the operational interpretation of sandwiched relative Rényi entropy in the strong converse of hypothesis testing from the hyperfinite setting to general von Neumann algebras. Our results extend earlier works of Mosonyi and Ogawa (2015) and of Mosonyi and Hiai (2023). We use a reduction method of Haagerup, Junge, and Xu to approximate results in an arbitrary von Neumann algebra by those in finite von Neumann algebras. Within these finite von Neumann algebras, we may apply the quantum method of types. Generalizing beyond the hyperfinite setting shows that the operational meaning of sandwiched Rényi entropy is not restricted to the matrix setting. Furthermore, applicability in general von Neumann algebras opens potential new connections to random matrix theory and the quantum information theory of black holes.

1. Introduction

Recall the hypothesis testing problem: given some number of copies of an unknown state ω with the promise that $\omega \in \{\rho, \sigma\}$, one may construct tests, two-outcome measurements to distinguish the identity of ω. Hypothesis testing is a long-studied problem in quantum information theory [1, 2, 3, 4, 5]. The set of tests is denoted by $\mathbb{T} := \{T | 0 \leq T \leq 1\}$. As a two-outcome POVM, the outcome probabilities are given on normal states by $\rho \mapsto (\rho(T), 1 - \rho(T))$. The error probability of the first kind or type I error probability is given by

$$\alpha_n(T_n) := \mathrm{tr}(\rho^{\otimes n}(\mathbf{1}^{\otimes n} - T_n)) ,$$

representing the probability that the state is mistakenly identified as ρ when it is actually η. The error probability of the second kind or type II error probability is given by

$$\beta_n(T_n) := \mathrm{tr}(\eta^{\otimes n} T_n) ,$$

representing the probability that the state is mistakenly identified as η when it is actually ρ. The nth minimum type I error probability of Hoeffding type is defined [6] as

$$\alpha^*_{e^{-nr}}(\rho^{\otimes n} \| \eta^{\otimes n}) := \min_{0 \leq T_n \leq 1} \left\{ \rho^{\otimes n}(\mathbf{1} - T_n) : \eta^{\otimes n}(T_n) \leq e^{-nr} \right\} ,$$

where the minimum is taken in all tests in $\mathcal{M}^{\bar{\otimes} n}$ with $\eta^{\otimes n} T_n) \leq e^{-nr}$. Correspondingly,

$$1 - \alpha^*_{e^{-nr}}(\rho^{\otimes n} \| \eta^{\otimes n}) = \max_{0 \leq T_n \leq 1} \left\{ \rho^{\otimes n}(T_n) : \eta^{\otimes n}(T_n) \leq e^{-nr} \right\} \tag{1}$$

[†]Corresponding author: Nicholas LaRacuente, nlaracu@iu.edu

is the nth maximum type I success probability. To interpret the Hoeffding error/success probabilities, note that one can always avoid the possibility of one kind of error, trivally by always guessing one state or the other. With many copies, one may soften this restriction to require merely that the probability of a type II error decraeses exponentially with the number of copies. In the many-copy limit or Shannon regime, the probability of the type I error may converge to zero while the probability of a type II error approaches a fixed value. The parameter r in 1.1 bounds the rate at which errors of the second kind go to zero. As in [7], we denote

$$B(\rho,\eta) \ := \ \big\{(R,r)\big|\exists_{n_0}\forall_{n\ \geq\ n_0}\exists_{0\leq T_n\leq 1, T_n\in N^{\otimes n}} \ : \ \eta^{\otimes_n}(T_n) \ \leq \ e^{-rn} \text{ and } \rho^{\otimes_n}(T_n) \ \geq \ e^{-Rn}\big\} \ ,$$

and

$$B_r(\rho\|\eta) \ := \ \sup\{R|\forall_{r'<r} \ : \ (R,r')\in B(\rho\|\eta)\} \ .$$

One may then observe that

$$B_r(\rho\|\eta) = -\lim_{n\to\infty}\frac{1}{n}\log\Big\{1-\alpha^*_{e^{-nr}}(\rho^{\otimes n}\|\eta^{\otimes n})\Big\} \ .$$

The sandwiched Rényi relative entropy [8, 9] is given by

$$D^*_\alpha(\rho\|\eta) = \|\rho^*\rho\|_{L_p^{1/2}(\eta)} \ ,$$

where the norm is in the Kosaki $L_p^{1/2}$ space weighted by η [10, 11]. The Hoeffding anti-divergence with respect to the sandiwched Rényi relative entropy is given for a pair of states ρ,η by

$$H^*_r(\rho\|\eta) = \sup_{\alpha>1}\frac{\alpha-1}{\alpha}\Big\{r - D^*_\alpha(\rho\|\eta)\Big\} \ .$$

The main result of this paper is then:

Theorem 1.1. *For any $r\in\mathbb{R}^+$ and states $\rho,\eta\in\mathcal{M}_*$ on any von Neumann algebra $\mathcal{M}$ for which $a\rho\leq\eta\leq a\rho^{-1}$ for some $a>0$,*

$$B_r(\rho\|\eta) = H^*_r(\rho\|\eta) \ .$$

The primary implication of Theorem 1.1 is an operational interpretation of the sandiwched relative Rényi entropies beyond the setting of matrix algebras or their limits. The hyperfinite case of 1.1 was shown by Mosonyi and Hiai as [6, Theorem 3.9]. Similarly, this Theorem generalizes Mosonyi and Ogawa's [7, Theorem 4.10]. These follow classical results of Han and Kobayashi [12] and Csiszár [13]. Since the trace is necessarily infinite on all elements of a type III von Neumann algebra, the usual notion of Shannon entropy is never finite and therefore of little use. The relative entropy, however, admits a variety of trace-free definitions and naturally generalizes the notion of entropy to general von Neumann algebras. Indeed, One may define quantum relative entropy and its Rényi generalizations in terms of the relative modular operator, which is at the heart of von Neumann algebra theory. The sandwiched Rényi relative entropy

2. Background

Here we briefly recall the notation of von Neumann algebras and Kosaki spaces, as well as the relative entropy and generalized divergences. For more detail, we refer the reader to Mosonyi and Hiai's recent work in hyperfinite algebras [6].

For a faithful state η, by $L_p^{1/2}(\eta)$ we denote the Kosaki L_p, the completion of $\mathcal{N}$ with norm

$$\|\eta^{1/2} X \eta^{1/2}\|_{L_p^{1/2}(\eta)} = \|\eta^{1/2-1/2p} X \eta^{1/2-1/2p}\|_{L_p} ,$$

where L_p is the associated Haagerup L_p space [10, 14]. In von Neumann algebras with finite trace, we may take the usual trace and Schatten norms. In von Neumann algebras that lack even a semifinite trace, the Kosaki L_p and Haagerup norms are still valid and allow us to extend the definitions of relative entropies and related quantities [11, 15]. Following [16], we recall a generalized, parameterized, pre-logarithm f-divergence

$$Q_r^f(\rho\|\eta) := \|\eta^{1/2} f(\Delta_{\rho|\eta}^{1/r})^{1/2*} f(\Delta_{\rho|\eta}^{1/r})^{1/2} \eta^{1/2}\|_{L_r^{1/2}(\eta)} . \tag{2}$$

The corresponding entropy expression is

$$D_r^f(\rho\|\eta) := -2r \ln Q_r^f(\rho\|\eta) .$$

In finite dimension, the density matrix is naturally and trivially identified with a corresponding quantum state. In a semifinite von Neumann algebra $\mathcal{M}$, one may also associate via an intervible mapping a state $\rho \in \mathcal{M}_*$ with a density operator d_ρ for which $\mathrm{tr}(d_\rho X) = \rho(X)$ for all X. In semifinite von Neumann algebras, recall the function

$$g_{\rho,\eta}(z) := d_\rho^z d_\eta^{-z} ,$$

where d_ρ and d_η are respectively the densities corresponding to states ρ and η. This function commutes with the modular automorphism group with respect to η, so it belongs to $\pi(\mathcal{M}) \cong \mathcal{M}$ in the crossed product.

$$Q_r^f(\rho\|\eta) = \|f(g_{\rho|\eta}^{1/r})^{1/2*} f(g_{\rho|\eta}^{1/r})^{1/2}\|_{L_r^{1/2}(\eta)} . \tag{3}$$

Within f-divergences, our focus is on divergences of the form $Q_{x\mapsto x^q, r}$. The main cases of interest are $q = r = p$, yielding the sandwiched relative Rényi p-entropies, and $r = 1, q = 1/p$, yielding the Petz-Rényi relative p-entropies. In either of these cases, $g_{\rho\ \eta}^{q/2r*}$ is a holomorphic function of $p \in [1, \infty)$ assuming Equation (6). Hence $g_{\rho|\eta}^{q/2r*} g_{\rho|\eta}^{q/2r}$ is holomorphic in p.

Remark 2.1. *Although the modular operator is not assured to be holomorphic in type III_1, the sandwiched Rényi entropy D_α^* arises from setting*

$$Q_p^* = \|\eta^{1/2} \Delta_{\rho|\eta}^{1/2*} \Delta_{\rho|\eta}^{1/2} \eta^{1/2}\|_{L_p^{1/2}(\eta)} . \tag{4}$$

The sandwiched relative Rényi entropy is known to obey a data processing inequality [17]. *Furthermore, both $Q_\alpha^*(\rho\|\eta)$ and $D_\alpha^*(\rho\|\eta)$ are monotonically non-decreasing in α for every pair of states ρ and η. When $\rho \leq C\eta$, there exists a ν for which $\rho^{1/2} = \nu\eta^{1/2}$. One may thereby also write* *the sandwiched relative entropy as* $\|\eta^{1/2}\nu^*\nu\eta^{1/2}\|_{L_p^{1/2}(\eta)}$. *In these expressions, the p-dependence is relegated to the norm weighting, which is analytic for* $p \in (0, \infty)$.

The generalized Hoeffding anti-divergence is defined for $r \in \mathbb{R}$ by

$$H_r^f(\rho\|\eta) := \sup_{\alpha>1} \frac{\alpha-1}{\alpha}\Big\{r - D_\alpha^f(\rho\|\eta)\Big\} . \tag{5}$$

Remark 2.2. *The Hoeffding anti-divergence with respect to the Sandwiched relative entropy, which we denote* H_r^*, *is given equivalently by*

$$H_r^*(\rho\|\eta) = \sup_{\alpha>1} \frac{\alpha-1}{\alpha} r - \ln Q_\alpha^*(\rho\|\eta) .$$

We recall a Theorem of Mosonyi and Hiai [6, Theorem 3.9] showing the desired result in hyperfinite von Neumann algebras:

Theorem 2.3 (Mosonyi-Hiai)**.** *Assume that* $\mathcal{M}$ *is an injective (hyperfinite) von Neumann algebra. Let* $\rho, \eta \in \mathcal{M}_*^+$ *be states such that* $D_{\alpha_0}^*(\rho\|\eta) < +\infty$ *for some* $\alpha_0 > 1$. *Then*

$$\lim_{n\to\infty} -\frac{1}{n}\log\Big\{1 - \alpha_{e^{-nr}}^*(\rho^{\otimes n}\|\eta^{\otimes n})\Big\} = H_r^*(\rho\|\eta) .$$

This connection to the Holevo type I error probability yields an operational interpretation of the sandwiched Rényi entropies as strong converse rates in hypothesis testing. The main goal of the current manuscript is to remove the requirement of injectivity or hyperfiniteness.

As a technical tool, recall Stein's interpolation theorem on compatible Banach spaces:

Theorem 2.4 (Stein's Interpolation, [18])**.** *Let* (A_0, A_1) *and* (B_0, B_1) *be two couples of Banach spaces that are each compatible. Let* $\{T_z | z \in S\} \subset \mathbb{B}(A_0 + A_1, B_0 + B_1)$ *be a bounded analytic family of maps such that*

$$\{T_{it}|\, t \in \mathbb{R}\} \subset \mathbb{B}(A_0, B_0) \ , \ \{T_{1+it}|\, t \in \mathbb{R}\} \subset \mathbb{B}(A_1, B_1) .$$

Suppose $\Lambda_0 = \sup_t \| T_{it} \|_{\mathcal{B}(A_0,B_0)}$ *and* $\Lambda_1 = \sup_t \| T_{1+it} \|_{\mathcal{B}(A_1,B_1)}$ *are both finite, then for* $0 < \theta < 1$, T_θ *is a bounded linear map from* $(A_0, A_1)_\theta$ *to* $(B_0, B_1)_\theta$ *and*

$$\| T_\theta \|_{\mathbb{B}((A_0,A_1)_\theta,(B_0,B_1)_\theta)} \le \Lambda_0^{1-\theta}\Lambda_1^\theta .$$

Kosaki L_p space innately support interpolation, and were in fact originally defined as interpolation spaces [10]. Another important tool is the Haagerup reduction method [14].

3. Proof of Main Results

The proof of Theorem 1.1 follows successive, arbitrarily precise approximations. First, using the Haagerup reduction, we approximate relative entropy in arbitrary von Neumann algebras by that in finite von Neumann algebras. Within finite von Neumann algebras, there is an easy identification between densities and states. We may then approximate each density by one with finite spectrum. Finite spectrum enables the method of types as described in [1], which exploits the fact that for many copies of the same states, though the dimension grows exponentially in copy-number, the number of distinct eigenvalues grows only polynomially. Ultimately, the method of types yields a projection to a commuting von Neumann algebra that approximately preserves the distinguishability of states. Because commutative von Neumann algebars are automatically hyperfinite, we may then apply results of [6] to obtain a sequence of tests achieving the desired values. Since these tests apply to states in the original von Neumann algebra, the same value is achieved there. The converse follows almost immediatel from known results for relative entropy. Below we explain this derivation in detail.

We often assume and make use of a semidefinite order comparison assumption between two states ρ and η:

$$a\rho \leq \eta \leq a\rho^{-1} \tag{6}$$

for some $a > 0$.

3.1. **Approximation via Finite Algebras.**

Lemma 3.1. *Let η be a state and $\mathcal{E}_n : \mathcal{N} \to \mathcal{N}_n$ a sequence of η-preserving conditional expectations such that for every state $\omega \in \mathcal{N}_*$, $\lim_n \mathcal{E}_n(\omega)$ converges to ω in the $\mathcal{N}_*$ norm. Then $\lim_{n\to\infty} Q_p^*(\mathcal{E}_n(\rho)\|\mathcal{E}_n(\eta))$ converges uniformly in p on any compact interval $p \in [1, p_{\max}]$ and for any state ρ satisfying Equation* (6).

Proof. Let $\gamma_n = \nu_{\mathcal{E}_n(\rho)|\eta}\eta^{1/2}$ such that $\mathcal{E}_n(\rho)^{1/2} = \nu_{\mathcal{E}_n(\rho)|\eta}\eta^{1/2}$ when $\mathcal{E}_n(\rho)^{1/2} \leq C\eta^{1/2}$ for some $C > 0$. Recall that if $\rho^{1/2} \leq C\eta^{1/2}$, then $\mathcal{E}_n(\rho)^{1/2} \leq C\mathcal{E}_n(\eta)^{1/2} = \eta^{1/2}$. Since $\nu_{\rho|\eta}\eta^{1/2} = \rho^{1/2}$, convergence of the sequence $(\mathcal{E}_n(\rho))_{n=1}^{\infty}$ implies convergence of the sequence $(\nu_{\mathcal{E}_n(\rho)|\mathcal{E}_n(\eta)}\mathcal{E}_n(\eta)^{1/2})_{n=1}^{\infty}$. The Kosaki norm $\|\gamma_n^*\gamma_n\|_{L_p^{1/2}(\eta)}$ thereby converges. What remains is to show that this convergence can be made uniform for $p \in [1, p_{\max}]$.

Via Stein's interpolation as in Theorem 2.4 and using Remark 2.1,

$$Q^*_{p_{\max}/(1+(p_{\max}-1)\theta)}(\rho\|\eta) \leq \|\gamma^*\gamma\|^{1-\theta}_{L^{1/2}_{p_{\max}}(\eta)} \|\gamma^*\gamma\|^{\theta}_{L_1^{1/2}(\eta)}$$

for $\theta \in [0, 1]$. We therefore have that $Q_p^*(\rho\|\eta)$ converges at a rate upper bounded by that of $\max_\theta \|\gamma^*\gamma\|^{1-\theta}_{L^{1/2}_{p_{\max}}(\eta)} \|\gamma^*\gamma\|^{\theta}_{L_1^{1/2}(\eta)}$ for all $p \in [1, p_{\max}]$, where we may take a maximum rather than supremum over θ due to compactness of the interval. ■

Lemma 3.2. *Let $\rho \leq C\eta$ for some $C > 0$. Then*

$$\lim_{p\to\infty} Q_p^*(\rho\|\eta) \;=\; Q_\infty^*(\rho\|\eta)\ .$$

Proof. We first show that for $2 < p < \infty$

$$\mathrm{Prob}_\eta(v > \lambda)\lambda^p \;\leq\; \|v\eta^{1/2p}\|_p^p\ .$$

Indeed,

$$L_p \;=\; [L_\infty, L_2]_{2/p} \subset L_{2/p,\infty}$$

(see [10]). Then we can find $v = x_1 + x_2$ such that

$$\|x_1\|_\infty + t\|x_2\|_2 \;\leq\; t^{2/p}\|x\|_p\ .$$

For $p = 2$ the chebychev inequality is trivial and hence there exists e with $\|x_2 e\|_\infty \le \mu$ and

$$\varphi(1-e)\mu^2 \le \|x_2\|_2^2 \, .$$

We choose $\mu = t\|x_2\|_2$ and get

$$\|xe\|_\infty \;\le\; t^{2/p}\|x\|_p$$

and

$$\varphi(1-e) \le t^{-2}$$

For a given λ we choose $\lambda = t^{2/p}\|x\|_p$, i.e $t^{2/p} \;=\; \frac{\lambda}{\|x\|_p}$. This gives $t^{-2} = \frac{\|x\|_p^p}{\lambda^p}$.

With the help of the Chebychev inequality it is not hard to conclude. Assume that $\lim_p \|x\|_p \le \gamma < \|x\|_\infty$. Let $(1+\varepsilon)\gamma < \|x\|_\infty$. Then we can find e_p such that

$$\varphi(1-e_p)((1+\varepsilon)\gamma)^p \;\le\; \gamma^p \, .$$

and $xe_p \;\le\; (1+\varepsilon)\gamma$. Since $(1+\varepsilon)^{-p}$ converges to 0, we deduce that e_p converges to 1 in the strong operator topology and hence

$$\|x\|_\infty \;\le\; \lim_p \|xe_p\|_\infty \;\le\; (1+\varepsilon)\gamma \, .$$

This contradiction concludes the proof. ∎

Remark 3.3. *The scale of L_p norms is continuous. Indeed, since $\|\cdot\|_{L_p^{1/2}}$ is monotone in p, we just have to argue that $\|x\|_p \;\le\; \limsup_{q\to p}\|x\|_q = \gamma$. Recall that*

$$\|x\|_p \;\le\; \|x\|_\infty^{1-q/p}\|x\|_q^{q/p} \;\le\; \|x\|_\infty^{1-q/p}\gamma^{q/p}$$

holds for $\frac{1}{p} = \frac{1-\theta}{\infty} + \frac{\theta}{q}$. Sending $q \to p$ yields $\|x\|_p \le \gamma$.

Lemma 3.4. *Consider a von Neumann algebra $\mathcal{N}$ and any states $\rho, \eta \in \mathcal{N}_*$ obeying Equation (6). Then*

- *For any $r \le D_1(\rho\|\eta)$, $H_r^*(\rho\|\eta) = 0$, and the supremum is achieved at $\alpha = 1$. Hence for every $\alpha_0 \in (1,\infty)$,*

$$H_r^*(\rho\|\eta) \le \sup_{\alpha\in(1,\alpha_0]} \frac{\alpha-1}{\alpha}\Big(r - D_\alpha^*(\rho\|\eta)\Big) \, .$$

- *For any $r \in (D^*(\rho\|\eta), D_\infty^*(\rho\|\eta))$, there exists an $\alpha_0 \in (1,\infty)$ for which*

$$H_r^*(\rho\|\eta) \le \sup_{\alpha\in(1,\alpha_0]} \frac{\alpha-1}{\alpha}\Big(r - D_\alpha^*(\rho\|\eta)\Big) \, .$$

- *For any $r \ge D_\infty^*(\rho\|\eta)$ and $\delta > 0$, there exists an $\alpha_0 \in (1,\infty)$ for which*

$$H_r^*(\rho\|\eta) \le (1-\delta) \sup_{\alpha\in(1,\alpha_0]} \frac{\alpha-1}{\alpha}\big(r - D_\alpha^*(\rho\|\eta)\big) \, ,$$

  and

$$H_r^*(\rho\|\eta) \le \sup_{\alpha\in(1,\alpha_0]} \Big(\frac{\alpha-1}{\alpha} r - \ln Q_\alpha^*(\rho\|\eta)\Big) + \delta r \, .$$

Proof. Recall Equation (5). Moreover, recall that $D_\alpha(\rho\|\eta)$ is non-decreasing in $\alpha \in [1,\infty]$. If $r \leq D_1(\rho\|\eta)$, then it is clear that $r - D^*_\alpha(\rho\|\eta) < 0$ for all $\alpha \geq 1$. Also, within this range,

$$\frac{\alpha-1}{\alpha}\big(r - D^*_\alpha(\rho\|\eta)\big)$$

is monotonically decreasing in α, because increasing α makes the $D^*_\alpha(\rho\|\eta)$ larger by the monotonicy of relative entropy in α, and increasing α makes $(\alpha-1)/\alpha$ larger, multiplying a negative number. Therefore, the supremum is achieved at $\alpha = 1$.

If $r \in (D^*(\rho\|\eta), D^*_\infty(\rho\|\eta))$, there exists an α_0 such that for all $1 < \alpha \leq \alpha_0$, $r - D^*_\alpha(\rho\|\eta) > 0$ and for all $\alpha > \alpha_0$, $r - D_\alpha(\rho\|\eta) < 0$. Hence the supremum is achieved with $\alpha \in (1, \alpha_0]$.

If $r \geq D_\infty(\rho\|\eta)$, then the supremum in Equation (5) may require arbitrarily large α, including $\alpha = \infty$. For every $\alpha \in (1,\infty]$ and $\delta > 0$, there exists an $\alpha' \in (1,\infty)$ such that for every $\alpha'' \geq \alpha'$,

$$\frac{\alpha''-1}{\alpha''} \geq (1-\delta) \geq (1-\delta)\frac{\alpha-1}{\alpha} \tag{7}$$

Therefore,

$$\begin{aligned}(1-\delta)H^*_r(\rho\|\eta) &= (1-\delta)\frac{\alpha-1}{\alpha}\big(r - D^*_\alpha(\rho\|\eta)\big)\\ &\leq \frac{\alpha'-1}{\alpha'}\big(r - D^*_\alpha(\rho\|\eta)\big)\end{aligned}$$

for some α achieving the supremum. If $\alpha < \alpha'$, then

$$(1-\delta)H^*_r(\rho\|\eta) \leq (1-\delta)\sup_{\alpha\in(1,\alpha']}\frac{\alpha-1}{\alpha}\big(r - D^*_\alpha(\rho\|\eta)\big) \leq \sup_{\alpha\in(1,\alpha']}\frac{\alpha-1}{\alpha}\big(r - D^*_\alpha(\rho\|\eta)\big)\ .$$

Of $\alpha > \alpha'$, then by monotonicity, $D^*_\alpha(\rho\|\eta) \leq D^*_{\alpha'}(\rho\|\eta)$. Recall also Lemma 3.2. Therefore,

$$\frac{\alpha'-1}{\alpha'}\big(r - D^*_\alpha(\rho\|\eta)\big) \geq \frac{\alpha'-1}{\alpha'}\big(r - D^*_{\alpha'}(\rho\|\eta)\big)\ ,$$

completing the relative bound. For the absolute bound, Equation (7) implies that

$$\frac{\alpha'-1}{\alpha'}r - D^*_{\alpha'}(\rho\|\eta) \geq \sup_{\alpha\geq\alpha'}\Big(\frac{\alpha-1}{\alpha}(1-\delta)r - \ln Q^*_\alpha\Big)\ .$$

By the monotonicity of $Q^*_\alpha(\rho\|\eta)$ in α, we are free to replace $Q^*_\alpha(\rho\|\eta)$ by $Q^*_{\alpha'}(\rho\|\eta)$. Therefore,

$$\begin{aligned}&\sup_{\alpha\in(1,\alpha')}\Big(\frac{\alpha-1}{\alpha}r - \ln Q^*_\alpha(\rho\|\eta)\Big) - \delta r\\ \geq&\max\Big\{\sup_{\alpha\in(1,\alpha')}\Big(\frac{\alpha-1}{\alpha}r - \ln Q^*_\alpha(\rho\|\eta)\Big), \frac{\alpha'-1}{\alpha'}r - \ln Q^*_{\alpha'}(\rho\|\eta)\Big\}\\ \geq&\sup_{\alpha>1}\Big(\frac{\alpha-1}{\alpha}r - \ln Q^*_\alpha(\rho\|\eta)\Big) - \delta r\\ \geq&H^*_r(\rho\|\eta) - \delta r\ .\end{aligned}$$

∎

Lemma 3.5. *(Haagerup Reduction) Let $(\mathcal{M}, \eta)$ repsectively be a von Neumann algebra and normal, faithful state. Let $G = \bigcup_n 2^{-n}\mathbb{Z} \subset \mathbb{R}$, and $\mathcal{M} \rtimes G$ denote the crossed product with respect to η. Let $\hat{\eta}$ denote the canonical embedding of η in $(\mathcal{M} \rtimes G)_*$. Let*

$$H_r^{*\leq \alpha_0}(\rho\|\eta) := \sup_{\alpha \in (1,\alpha_0]} \frac{\alpha - 1}{\alpha}\big(r - D_\alpha^*(\rho\|\eta)\big) \ .$$

There for every $\alpha_0 \in (1, \infty)$ and $\rho \in \mathcal{M}_*$ obeying Equation (6), there exists a sequence of finite von Neumann algebras $(\mathcal{M}_n)_{n=1}^\infty$ $\mathcal{M}_n \subseteq \mathcal{M} \rtimes G$ such that

$$H_r^{*\leq \alpha_0}(\rho\|\eta) = \lim_{n\to\infty} H_r^{*\leq \alpha_0}(\mathcal{E}_n(\hat{\rho})\|\hat{\eta}) \ ,$$

where $\hat{\rho}$ denotes the canonical embedding of ρ into $(\mathcal{M} \rtimes G)_*$.

Proof. The Haagerup reduction [14, Theorem 2.1] yields a sequnce of von Neumann algebras and conditional expectations $(\mathcal{E}_k)_{n=1}^\infty : \mathcal{M}_k \to \mathcal{M} \rtimes G$ for which

i) $\mathcal{E}$ and $\tilde{\eta} = \eta \circ \mathcal{E}$ are faithful.
ii) There exists an increasing family of subalgebras $\mathcal{M}_k$ and normal conditional expectations $\mathcal{E}_k : \mathcal{M} \rtimes G \to \mathcal{M}_k$ such that $\tilde{\eta} F_k = \tilde{\eta}$;
iii) $\lim_k \|F_k(\psi) - \psi\|_{(\mathcal{M}\rtimes G)_*} = 0$ for every normal state $\psi \in (\mathcal{M} \rtimes G)_*$;
iv) For every k there exists a normal faithful trace trace $\tau_k(x) = \tilde{\eta}(d_k(x))$ such that $d_k \in (\mathcal{M}_k)_*^+$, and $a_k \leq d_k \leq a_k^{-1}$ for some scalars $a_k \in \mathbb{R}^+$. Hence $\tilde{M_k}$ is of type II_1.

Let $\rho_n := \mathcal{E}_n(\hat{\rho})$, and recall that

$$H_r^{*\leq \alpha_0}(\rho\|\eta) = \sup_{\alpha \in (1,\alpha_0]} \Big(\frac{\alpha - 1}{\alpha} r - \ln Q_\alpha^*(\rho\|\eta)\Big) \ .$$

By Lemma 3.1, $\lim_{n\to\infty} Q_\alpha^*(\rho_n\|\eta_n) = Q_\alpha^*(\hat{\rho}\|\eta)$ uniformly on α in the compact interval $[1, \alpha_0]$. Equation (6) implies that $Q_\alpha^* \in (0, \infty)$, so the logarithm is continuous. Therefore,

$$\lim_{n\to\infty} H_r^{*\leq \alpha_0}(\rho_n\|\hat{\eta}) = H_r^{*\leq \alpha_0}(\hat{\rho}\|\hat{\eta}) \ .$$

Furthermore, because the embedding from $\mathcal{M}_*$ into $(\mathcal{M} \rtimes G)_*$ is faithful,

$$H_r^{*\leq \alpha_0}(\hat{\rho}\|\hat{\eta}) = H_r^{*\leq \alpha_0}(\rho\|\eta) \ .$$

■

3.2. **Approximation via Finite Spectrum.**

Remark 3.6. *In a finite von Neumann algebra, there is a canonical, invertible mapping between a state ρ and its density d_ρ with respect to the finite trace. In this section and thereafter, we interpret expressions that mix states with densities as applying the canonical mapping where needed. For example, we may interpret $D_\alpha^*(\rho\|d_\eta)$ as $D_\alpha^*(d_\rho\|d_\eta)$. It also holds by data processing and the invertibility of this map that $D_\alpha^*(\rho\|\eta) = D_\alpha^*(d_\rho\|d_\eta)$ for all ρ, η.*

Lemma 3.7. *Let $\delta < d_\varphi < \delta^{-1}$ be a bounded element in a finite von Neumann algebra. Then there exists a sequence of conditional expectations $(F_k)_{k=1}^\infty$ such that $F_k(d_\varphi)$ has finite spectrum,*

$$|D_p^*(\rho\|\eta) - D_p^*(\rho\|F_k(\eta))| \;\le\; \frac{1}{k}$$

for all $p \in (1,\infty]$, and

$$|H_r^*(\rho\|\varphi) - H_r^*(\rho\|F_k(\varphi))| \;\le\; \frac{1}{k}\,.$$

For fixed k, $|spec(F_k(d_\varphi))| \;\le\; 2k\delta^{-2}$, where $spec(X)$ denotes the spectrum of an operator X.

Proof. Recall the spectral decomposition

$$d_\varphi(\mu) = \int_0^\infty x d\mu(x)\,.$$

Let $a > 1$, and define

$$d_a^- = \sum_j a^j 1_{[a^j,a^{j+1})}(d_\varphi),\; d_a^+ = \sum_j a^{j+1} 1_{[a^j,a^{j+1})}(d_\varphi)\,.$$

Then

$$d_a^- \le d_\varphi \le d_a^+ \le a d_a^-\,.$$

Thanks to the boundedness assumption, only finitely many of the spectral projections $e_j = 1_{[a^j,a^{j+1})}$ are non-trivial. Let $M_a = \mathrm{span}(e_j)$ be the commutative, finite subalgebra generated by these projections and F_a be the conditional expectation to that subalgebra. Then $d_a^- \le F_a(d_\varphi) \le d_a^+$, so $a^{-1}F_a(d_\varphi) \le d_\varphi \le aF_a(d_\varphi)$. For any $g \ge 0$ and $d > 0$,

$$\|d^{-1/2p'} g d^{-1/2p'}\|_p = \|g^{1/2} d^{-1/2p'}\|_{2p}^2 = \|g^{1/2} d^{-1/p'} g^{1/2}\|_p\,.$$

Now we note that $d_1 \le K d_2$ implies

$$\|g^{1/2} d_2^{-1/p'} g^{1/2}\|_p \;\le\; K^{1/p'} \|g^{1/2} d_1^{-1/p'} g^{1/2}\|_p\,.$$

In our situation, we deduce

$$\|d_\varphi^{-1/2p'} g d_\varphi^{-1/2p'}\|_p^p \;\le\; a^{p/p'} \|F_a(d_\varphi)^{-1/2p'} g F_a(d_\varphi)^{-1/2p'}\|_p^p \;\le\; a^{2p/p'} \|d_\varphi^{-1/2p'} g d_\varphi^{-1/2p'}\|_p^p\,.$$

Taking the logarithm, we get

$$\big|\ln \|d^{-1/2p'} g d^{-1/2p'}\|_p - \ln \|F_a(d_\varphi)^{-1/2p'} g F_a(d_\varphi)^{-1/2p'}\|_p\big| \;\le\; \frac{1}{p'} \ln a\,.$$

Recall that $D_p^*(\rho\|\eta) = p' \ln Q_p^*(\rho\|\eta)$. Therefore,

$$\big|D_p^*(\rho\|\eta) - D_p^*(\rho\|\eta)\|_p\big| \;\le\; \ln a$$

Moreoever, $p = \alpha \in (1,\infty]$, we see that $\frac{1}{p'} = \frac{\alpha-1}{\alpha} \le 1$. Now, we may choose $a = 1 + \frac{1}{k}$, so $\ln a \le \frac{1}{k}$. Setting $g = \rho = \nu\eta^{1/2}$ and $d = d_\varphi$ as in Remark 2.1,

$$\big|\ln Q_\alpha^*(\rho\|\eta) - \ln Q_\alpha^*(\rho\|F_{1/k}(\eta))\big| \;\le\; \frac{1}{k}\,.$$

An upper bound for $H_r^*(\rho\|F_k(\eta))$ over $H_r^*(\rho\|\eta)$ follows from assuming that $\ln Q_\alpha^*(\rho\mathcal{F}_k(\eta))$ is $1/k$ smaller than $\ln Q_\alpha^*(\rho\|\eta)$ for the value of α already achieving the supremum as in Remark 2.2. By similar argument, the most negative decrease is $1/k$. Hence $1/k$ bounds the H_r^* difference.

According to our boundedness assumption, $|\sigma(F_a(d_\varphi))| \leq j_2 - j_1$ for some $j_1, j_2 \in \mathbb{N}$ such that $a^{-j_1} \leq \delta$ and $a^{j_2} \leq \delta^{-1}$. This means

$$(j_2 - j_1)\ln a \; = \; \delta^{-2} \, .$$

For our choice, $j_2 - j_1 \leq 2k\delta^{-2}$. ■

The next lemma is a version of the well-known method of types in (quantum) information theory. For many copies of a density with finite spectrum, though the dimension scales exponentially in the number of copies, the number of eigenvalues scales only polynomially [1].

Lemma 3.8. *Let N be a finite von Neumann algebra and $d = \sum_{j=1}^K d_j e_j$ a finite density with finite spectrum, where $(e_j)_{j=1}k$ is a family of projections for which $\sum_{j=1}^K e_j = 1$. Then...*

i) *Let F be the conditional expectation onto d', the algebra of operators that commute with d. Then F has cp-order index $\leq K$.*
ii) *The conditional expectation onto $d^{\otimes n}$ has complete index $\leq (n+1)^K$.*
iii)

$$\begin{aligned} H_r(\rho|d) \; &= \; \frac{1}{n}H_{rn}(\rho^{\otimes n}|d^{\otimes n}) \leq \frac{1}{n}H_{rn}(F_n(\rho^{\otimes n})|d^{\otimes n}) \\ &\leq \frac{1}{n}H_{rn}(\rho^{\otimes n}|d^{\otimes n}) + K\frac{\log(n+1)}{n} \\ &= \; H_r(\rho|d) + K\frac{\log(n+1)}{n} \, . \end{aligned}$$

Proof. The conditional expectation is given by

$$F(x) \; = \; \sum_j e_j x e_j \; ,$$

Let $x \geq 0$, $N \subset \mathbb{B}(H)$, and h be any vector in H. We note the subspace decomposition $h = \sum_{j=1}^K e_j h$. Therefore,

$$\begin{aligned} (h, xh) \; &= \; \sum_{j,j'} (x^{1/2}e_j h, x^{1/2}e_{j'}h) \\ &\leq \sum_{j,j'} \|x^{1/2}e_j h\| \|x^{1/2}e_{j'}h\| \; = \; \Big(\sum_{j=1}^K (h, e_j x e_j h)^{1/2}\Big)^2 \\ &\leq \; K\sum_j (h, F(x)h) \, . \end{aligned}$$

As the inequality holds for arbitrary $h \in H$, it implies that $KF(x) \geq_{cp} x$, proving the first assertion. For the second assertion, we observe that, thanks to commutativity

$$d^{\otimes n} \;=\; \sum_{j_1+\cdots+j_K=n} d_1^{j_1}\cdots d_K^{j_K} \sum_{\#l|i_l=1=j_1,\ldots\#l|i_l=K=j_k} e_{i_1}\otimes\cdots\otimes e_{i_n}\;.$$

In $\mathbb{R}^K$ we consider the simplex $S_{n+1}=\{(x_1,...,x_K)|0\leq x_j,\sum_j x_j\leq n+1\}$. This contains our discrete points and an $p+\overset{\circ}{S}_1$, where $\overset{\circ}{S}_1$ denotes the interior, around it. Thus

$$\#\{(j_1,...,j_k)|\sum_l j_l=n\} \;\leq\; \frac{vol(S_{n+1})}{vol(S_1)} \;=\; (n+1)^K\;.$$

For the last assertion, we fix K, d and $n\in\mathbb{N}$ and the conditional expectation F onto $\{d^{\otimes n}\}'$ and use the notation $\rho_n=\rho^{\otimes n}$, $d_n=d^{\otimes n}$. We recall that

$$Q_\alpha^*(F(\rho^{\otimes n})\|F(d^{\otimes n})) \;\leq\; Q_\alpha^*(\rho^{\otimes n}\|d^{\otimes n})\;.$$

On the other hand $\rho_n\leq \mathrm{Ind}_n F(\rho_n)$ implies

$$\|d_n^{-1/2p'}\rho_n d_n^{-1/2p'}\|_p \;\leq\; \mathrm{Ind}\|d_n^{-1/2p'}F(\rho_n)d_n^{-1/2p'}\|_p\;,$$

where $\mathrm{Ind}=(n+1)^K$. This implies

$$\ln Q_\alpha^*(F(\rho_n)\|d_n) \;\leq\; \ln Q_\alpha^*(\rho_n\|d_n) \;\leq\; \ln Q_\alpha^*(F(\rho_n)\|d_n)+K\ln(n+1)\;.$$

Since Q_α^* is uniformly bounded in α, H_r^* is correspondingly bounded in the reverse direction. ■

3.3. **Finding Tests.**

Lemma 3.9. *Let $n_0\in\mathbb{N}$ and assume that*

$$B_{rn_0}(\rho^{\otimes n_0}\|\eta^{\otimes n_0}) \;\leq\; H_{rn_0}^*(\rho^{\otimes n_0}\|\eta^{\otimes n_0})\;.$$

Then

$$B_r(\rho\|\eta) \;\leq\; H_r^*(\rho\|\eta)\;.$$

Proof. Recall that

$$D_\alpha^*(\rho^{\otimes n_0}\|\eta^{\otimes n_0}) \;=\; n_0\;.$$

This implies

$$H_{n_0r}^*(\rho^{\otimes n_0}\|\eta^{\otimes n_0}) \;=\; n_0H_r^*(\rho\|\eta)\;.$$

Let $H_r^*(\rho\|\eta)<R+\delta$. Then we can find an k_1 and tests $T_{kn_0}\in N^{\otimes kn_0}$ such that

$$\liminf_k \frac{-1}{k}\ln\eta^{kn_0}(T_{kn_0}) \;\geq\; n_0r$$

and

$$\limsup_k -\frac{1}{k}\rho^{kn_0}(T_{kn_0}) \;\leq\; n_0R\;. \tag{8}$$

For a given $n\in\mathbb{N}$ we choose $kn_0<n\;\leq\;(k+1)n_0$. We may consider the test

$$T_n \;=\; T_{kn_0}\otimes 1^{n-kn_0}$$

and note that
$$\frac{-1}{n}\ln \eta^{\otimes n}(T_n) \;=\; \frac{k(n)n_0}{n}\frac{-1}{k(n)n_0}\ln \eta^{\otimes_{k(n)n_0}}(T_{k(n)n_0}) \;.$$
Note that $n-k(n) \;\le\; n_0$ implies $\lim_{n\to\infty}\frac{n-k(n)n_0}{n}=1$. Thus for the limes inferior, we deduce indeed
$$\liminf_n \frac{-1}{n}\ln \eta^{\otimes n}(T_n) \;\ge\; \frac{n_0 r}{n_0} \;=\; r \;.$$
The same calculation shows that
$$\limsup \frac{-1}{n}\ln \rho^{\otimes n}(T_n) \;\le\; R+\delta$$
using again $\lim_{n\to\infty}\frac{k(n)n_0}{n}=1$ and divding (8) by n_0. ■

Remark 3.10. The notation B_r is motivated by the earlier work of Mosonyi [7]. In the joint paper with Hiai [6], the authors prefer to work with $1-\alpha^*_{e^{-rn}}(\rho^{\otimes n}\|\eta^{\otimes n}) \;=\; \sup\{\rho^{\otimes n}(T_n)|\eta^{\otimes n}(T_n) \;\le\; e^{-rn}\}$ as defined in Equation (1). We can easily adapt the Lemma for this slightly stronger condition. Indeed, assume that $\eta^{kn_0}(T_{kn_0}) \;\le\; e^{-rkn_0}$. Then
$$\eta^n(T_n) \;=\; \eta^{\otimes k(n)n_0}(T_{k(n)n_0}) \;\le\; e^{-rk(n)n_0} \;=\; e^{-rn}e^{r(n-k(n)n_0)} \;.$$
Define $\theta_n = e^{-r(n-k(n)n_0)}$. The $\eta^{\otimes n}(\theta_n T_n) \;\le\; e^{-rn}$. Thus taking the logarithm we get
$$\begin{aligned}\liminf \frac{-1}{n}\ln \rho^{\otimes n}(\theta_n T_n) \;&=\; \lim \frac{-1}{n}\ln\theta_n + \liminf_n \frac{k(n)n_0}{n}\frac{1}{k(n)n_0}\rho^{k(n)n_0}(T_{k(n)n_0}) \\ &=\; \frac{1}{n_0}\liminf_k \frac{-1}{k}\ln \rho^{kn_0}(T_{kn_0}) \;.\end{aligned}$$
Indeed, the first limit is 0 and the multiplicative term in the second part converges to 1. Thus we have proved that
$$\liminf_k \frac{-1}{k}\ln(1-\alpha^*_{e^{-rn_0k}}(\rho^{\otimes_{kn_0}}\|\eta^{\otimes_{kn_0}})) < Rn_0$$
implies
$$\liminf_n \frac{-1}{n}\ln(1-\alpha^*_{e^{-rn}}(\rho^{\otimes n}\|\eta^{\otimes n})) < R \;.$$

Lemma 3.11. *Let ρ,η be states on $\mathcal{M}_*$ satisfying Equation* (6). *Let $G=\bigcup_n 2^{-n}\mathbb{Z}\subset\mathbb{R}$, $\mathcal{M}\rtimes G$ denote the crossed product with respect to η, and $\iota:\mathcal{M}_*\to(\mathcal{M}\rtimes G)_*$ denote the canonical embedding on states. Then*
$$\limsup_{n\to\infty}\log(\rho^{\otimes n}(T_n)) \le H^*_r(\rho\|\eta) \;.$$

Proof. By Lemma 3.4, for any $r\in\mathbb{R}$, and states $\rho,\eta\in\mathcal{M}_*$, and any $\epsilon_1>0$, there exists an α_0 such that
$$|H^{*\le\alpha_0}_r(\rho\|\eta) - H^*_r(\rho\|\eta)| \le \epsilon_1 \;. \tag{9}$$
By Lemma 3.5, there exists a sequence of von Neumann algebras $(\mathcal{M}_k \subseteq \mathcal{M}\rtimes G)_{k=1}^\infty$ with respective restriction maps $(\mathcal{E}_k)$ such that for any ϵ_1, there exists some k_0 that for all $k\ge k_0$,
$$|H^{*\le\alpha_0}_r(\rho\|\eta) - H^{*\le\alpha_0}_r(\mathcal{E}_k(\iota(\rho))\|\iota(\eta))| \le \epsilon_2 \;. \tag{10}$$

Combining with Equation (9),

$$|H_r^*(\rho\|\eta) - H_r^*(\mathcal{E}_k(\iota(\rho))\|\iota(\eta))| \leq \epsilon_2 + 2\epsilon_1 \tag{11}$$

for sufficiently large k and ϵ_0. Clearly $\mathcal{E}_k(\iota(\rho)) \in (\mathcal{M}_k)_*$, and since $\iota(\eta)$ is invariant under $\mathcal{E}_k$, $\iota(\eta) = \mathcal{E}_k(\iota(\eta)) \in (\mathcal{M}_k)_*$. Let d_η be the same density of η in every $\mathcal{M}_k$. By Lemma 3.7, there exists a sequence of conditional expectations $(F_{k,l})_{l=1}^\infty$ such that for each l,

$$|H_r^*(\mathcal{E}_k(\iota(\rho))\|\iota(\eta)) - H_r^*(\mathcal{E}_k(\iota(\rho))\|F_{k,l}(\iota(\eta)))| \;\leq\; \frac{1}{l}\;, \tag{12}$$

and each $|\mathrm{spec}(F_{k,l}(d_\varphi))| \;\leq\; 2l\delta^{-2}$. Let $d_{\eta,k,l} := F_{k,l}(d_\eta)$. Let $\tilde{F}_{k,l,n}$ be the projection onto the commutant algebra of $F_{k,l}(\iota(\eta))^{\otimes n}$ By Lemma 3.8, for every $n \in \mathbb{N}$,

$$H_r^*(\mathcal{E}_k(\iota(\rho))\|d_{\eta,k,l}) \leq \frac{1}{n} H_{rn}^*(F_{k,l,n}(\mathcal{E}_k(\iota(\rho))^{\otimes n})\|d_{\eta,k,l}^{\otimes n}) \leq H_r^*(\mathcal{E}_k(\iota(\rho))\|d_{\eta,k,l}) + K\frac{\log(n+1)}{n}\;. \tag{13}$$

Because $F_{k,l,n}(\mathcal{E}_k(\iota(\rho))^{\otimes n})$ and $d_{\eta,k,l}^{\otimes n}$ commute, they are contained within the predual of a commuting algebra. That algebra is automatically hyperfinite. Therefore, Hiai and Mosonyi's Theorem 2.3 applies, yielding for every k, l, and n that

$$\lim_{m\to\infty} -\frac{1}{nm}\log\left\{1 - \alpha_{e^{-mnr}}^*(F_{k,l,m}(\mathcal{E}_k(\iota(\rho))^{\otimes n})\|d_{\eta,k,l}^{\otimes n})\right\} = H_{rn}^*(F_{k,l,n}(\mathcal{E}_k(\iota(\rho))^{\otimes n})\|d_{\eta,k,l}^{\otimes n})\;.$$

Using Equation (13),

$$H_{rn}^*(F_{k,l,n}(\mathcal{E}_k(\iota(\rho))^{\otimes n})\|d_{\eta,k,l}^{\otimes n}) \leq H_{rn}^*(\mathcal{E}_k(\iota(\rho))^{\otimes n}\|d_{\eta,k,l}^{\otimes n}) + K\frac{\log(n+1)}{n}\;.$$

The definition of $1 - \alpha_{e^{-rm}}(\cdot,\cdot)$ as in Equation (1) then implies that there exists a sequence of tests $(T_{k,l,n,m})_{m=1}^\infty$ such that for each m, $\eta^{\otimes nm}(T_{k,l,n,m}) \leq e^{-mnr}$, and

$$-\lim_{m\to\infty} \rho^{\otimes nm}(T_{k,l,n,m}) = H_{rn}^*(F_{k,l,n}(\mathcal{E}_k(\iota(\rho))^{\otimes n})\|d_{\eta,k,l}^{\otimes n})\;. \tag{14}$$

If a sequence of tests $(\tilde{T}_{nm})$ achieves in the limit as $n \to \infty$ a particular type I and II error rates on $F_{k,l,n}(\mathcal{E}_k(\iota(\rho))^{\otimes n})$ and $d_{\eta,k,l}^{\otimes n} = F_{k,l,n}(d_{\eta,k,l}^{\otimes n})$, then there exists a sequence of tests including those projections in the tests that achieve the same error rates on $\mathcal{E}_k(\iota(\rho))^{\otimes n}$ and $d_{\eta,k,l}^{\otimes n}$. Therefore,

$$\begin{aligned}&\lim_{m\to\infty} -\frac{1}{nm}\log\left\{1 - \alpha_{e^{-mnr}}^*(\mathcal{E}_k(\iota(\rho))^{\otimes n}\|d_{\eta,k,l}^{\otimes n})\right\}\\ &\leq \lim_{m\to\infty} -\frac{1}{nm}\log\left\{1 - \alpha_{e^{-mnr}}^*(F_{k,l,m}(\mathcal{E}_k(\iota(\rho))^{\otimes n})\|d_{\eta,k,l}^{\otimes n})\right\}\;.\end{aligned}$$

Furthermore, for any $\epsilon_3 > 0$, we may choose l sufficiently large that both

$$\begin{aligned}&\Big|\lim_{m\to\infty} -\frac{1}{nm}\log\left\{1 - \alpha_{e^{-mnr}}^*(\mathcal{E}_k(\iota(\rho))^{\otimes n}\|d_{\eta,k,l}^{\otimes n})\right\}\\ &- \lim_{m\to\infty} -\frac{1}{nm}\log\left\{1 - \alpha_{e^{-mnr}}^*(\mathcal{E}_k(\iota(\rho))^{\otimes n}\|d_{\eta}^{\otimes n})\right\}\Big| \leq \epsilon_3\;,\end{aligned} \tag{15}$$

and

$$\left|H_{rn}^*(\mathcal{E}_k(\iota(\rho))^{\otimes n}\|d_{\eta,k,l}^{\otimes n}) - H_{rn}^*(\mathcal{E}_k(\iota(\rho))^{\otimes n}\|d_{\eta}^{\otimes n})\right| \leq \epsilon_3\;.$$

Therefore,

$$\lim_{m\to\infty} -\frac{1}{nm}\log\left\{1-\alpha^*_{e^{-mnr}}(\mathcal{E}_k(\iota(\rho))^{\otimes n}\|d_\eta^{\otimes n})\right\} \leq H^*_{rn}(\mathcal{E}_k(\iota(\rho))^{\otimes n}\|d_\eta^{\otimes n}) + K\frac{\log(n+1)}{n} + 2\epsilon_3 \ .$$

Via Lemma 3.9 and Remark 3.10, it then follows that for any $\epsilon_4 > 0$, we may choose an n_0 and l_0 for which

$$\lim_{m\to\infty} -\frac{1}{m}\log\left\{1-\alpha^*_{e^{-mr}}(\mathcal{E}_k(\iota(\rho))\|d_\eta)\right\} \leq H^*_r(\mathcal{E}_k(\iota(\rho))\|d_\eta) + \epsilon_4 \ .$$

Then using Equation (11),

$$\lim_{m\to\infty} -\frac{1}{m}\log\left\{1-\alpha^*_{e^{-mr}}(\mathcal{E}_k(\iota(\rho))\|d_\eta)\right\} \leq H^*_r(\iota(\rho)\|d_\eta) + \epsilon_4 + 2\epsilon_1 + \epsilon_2 \ .$$

Each of the approximation constants can be taken arbitrarily small. Although they exist in the larger algebra $\mathcal{M} \rtimes G$, the tests (T_n) are valid tests on states in $\mathcal{M}_*$ as embedded via ι into $(\mathcal{M} \rtimes G)_*$. ∎

Lemma 3.12. *For any states $\rho, \eta \in \mathcal{M}_*$ on von Neumann algebra $\mathcal{M}$ and any $n \in \mathbb{N}$,*

$$-\frac{1}{n}\log\left\{1-\alpha^*_{e^{-nr}}(\rho^{\otimes n}\|\eta^{\otimes n})\right\} \geq H^*_r(\rho\|\eta) \ .$$

Proof. Within finite dimension, this result essentially follows the proof of Mosonyi and Ogawa's [7, Lemma 4.7]. To adapt that proof, we first note that expressions of the form $\mathrm{tr}(\rho^{\otimes n}T_n)$ can be trivially replaced by $\rho^{\otimes n}(T_n)$. Second, and more substantially, their proof uses [7, Lemma 3.3], which shows monotonicty of sandwiched Rényi relative entropy under measurements. It is now well-known that Rényi relative entropy obeys the data processing inequality [17], subsuming monotonicity under measurement. Hence the result holds unmodified in the general von Neumann algebra setting. ∎

3.4. **Proof the the Main Theorem 1.1.**

Proof. The Theorem follows from combining Lemma 3.11 with Lemma 3.12.

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Email address: nlaracu@iu.edu