Response to reviewers on
"Optimal covariant quantum measurements"

We are grateful for the reviewers' diligent work and corrections and suggestions on our manuscript. We have tried to implement their suggestions and done extensive re-editing on our paper. Below, we describe in detail the changes and our responses to each issue raised by the reviewers. After writing this response, we noticed that we had accidentally switched the order of the reviewers: 'Reviewer 1' below refers to 'referee 2' as indicated in the decision letter from J. Phys. A: Math. Theor. and vice versa. We hope that this does not cause too much confusion.

Reviewer 1

- Reviewer 1: "The paper lacks a proper introduction, clarifying the questions asked and their motivation, describing (in an understandable way, without much technical details) the results obtained and putting them into context of previous works. The authors should state precisely what is new here compared to other works also cited here, including the previous works of the authors. A description of the structure of the paper is also advisable."
  - Response: Our initial idea was to give a technically simple introduction which goes 'straight into business' but in a way that, e.g., a graduate student could follow with minimal background knowledge. We must admit that we have probably failed in this. This is why we have introduced a more traditional introduction discussing the broad topics treated in our work and their relevance in quantum measurement theory and giving basic references to pre-existing results. Finally, a detailed description of the work done is given in the end of the new introduction highlighting the novel aspects of our work. The old introduction is now 'Basic definitions and observations'. This section has also been modified to make it more reader friendly; see especially our response to the second reviewer on this aspects.
- Reviewer 1: "The paper is rather technical and the form of presentation chosen by the authors makes it difficult to read. The authors should provide some guidance for the reader, on what is going on and why. Moreover, I find some parts of the paper rather confusing and it seems that some reorganization is needed. Overall, changes are necessary to increase readability of the paper. Some suggestions are in the specific comments below, but these do not cover all of the problematic places."
  - Response: We have added more guiding paragraphs especially in the beginnings of all the sections and subsections. We have also separated the parts dealing with the extreme points of the covariance structures and extreme instruments within covariance structures into two new subsections, 'Extreme instruments covariant with respect to a finite group' and 'Extreme continuous covariant instruments associated

with a compact stability subgroup'. See more on changes intended to increase readability in our responses to the specific comments of reviewer 1. These are also discussed in our responses to reviewer 2.

- Reviewer 1: "There is a lot of symbols introduced in the paper, either plain or calligraphic, adorned with upper and lower indices, which are difficult to keep track of. In this situation a typo can cause severe problems for the reader. The authors should thoroughly check the paper for typos. A few are suggested below."
  - Response: We agree that our notations are, in places, a bit complex. We have double checked our notations and corrected typos, especially those found by the reviewers.
- Reviewer 1: "A strange statement: "...there is necessarily no nontrivial solution M for (1).""
  - Response: We have removed the double negative and replaced it with the following: "Moreover, we note that, in some situations, there are only trivial solutions M for (1)."
- Reviewer 1: "It is not clear what exactly is constructed in the steps (i) - (iv). Namely, it is not clear what "the above type covariant rank -1 PVM" or "rank-1 PVM as above" in the last paragraph on p. 5 refers to."
  - Response: We have rewritten the paragraphs preceding and following the steps (i) – (iv) to make our point clearer. It is emphasized before the steps that the methods presented in the steps can be used to extend a POVM in a finite-dimensional Hilbert space and covariant w.r.t. a finite group into a PVM (or sharp observable) which is of a very particular form and covariant w.r.t. a symmetric group. The idea is to show that the POVMs (or PVMs) covariant w.r.t. symmetric groups can be used to generate all POVMs (in finite dimensions) covariant w.r.t. any finite group. After the steps (i) – (iv), we move on to study POVMs covariant w.r.t. symmetric groups motivated by their relevance established in the steps. In a separate paragraph (the second one after the steps) we emphasize that we want to study the informationally complete POVMs (and other optimal POVMs) in this setting and, for that, we extend the value space into the Cartesian product of the symmetric group with itself. This sets the scene for Example 1.
- Reviewer 1: "the steps (i)-(iv) are referred to as (1)-(4)."
  - Response: The change in numbering was caused by a clash with the journal style file. This has been fixed.
- Reviewer 1: "As I understand, Example 1 gives a way to construct a one-parameter family of covariant rank-1 extreme POVMs, containing both a PVM and IC POVMs. The relation between this example and the construction (i) - (iv) is not very clearly explained. Especially, I find the paragraph before Example 1 rather confusing."
  - Response: There is no direct connection to steps (i) – (iv), but the steps rather motivate the study of POVMs covariant w.r.t. symmetric

groups. Example 1 re-introduces IC POVMs in the setting of POVMs covariant w.r.t. symmetric groups as these are lost in the extension steps (i) – (iv). We have tried to make this clear in the paragraphs following the steps to make a clearer separation between the steps and Example 1.

- Reviewer 1: "p. 11 - It is suggested that a conjecture in [16] is resolved, if so, this should be better explained."
  - Response: We have added more information on where the conjecture is made in the reference. Moreover, we highlight that the conjecture addresses a more general problem and that a more general result is found later in our paper (Theorem 5).
- Reviewer 1: "proof of Thm. 3 uses the representation of the imprimitivity system given in Appendix B. I strongly suggest to give its form and the introduced notations explicitly before the statement of Thm. 3."
  - Response: We have done this in the new version. The derivation of the canonical system of imprimitivity is kept in the appendix but the form (the PVMs P^Ω and the representations $\overline{U}^\Omega$) of this system is given after the introduction of covariant dilations in the main text.
- Reviewer 1: "There seems to be a typo: the computations from the second paragraph of the proof of Thm 3 onward make sense only if one replaces $\mathcal{M}^\Omega$ by $\mathcal{H}^\Omega$."
  - Response: This is true. We have made this replacement in the proof. The only exception is in the definition of the isometries $V_{x,\eta,i,m}$ in the second paragraph of this proof where having $\mathcal{M}^\Omega$ instead of $\mathcal{H}^\Omega$ is better although the difference here is slight.
- Reviewer 1: "p. 12, 1 line before last - e_{η_i} should be $e_{\eta,i}$."
  - Response: Corrected
- Reviewer 1: "ζ should be ζ^η"
  - Response: Corrected
- Reviewer 1: "Better also explain somewhere that one may put $M_\eta = 0$ if $L^\Omega_{\eta,i,m}$ would have to be 0"
  - Response: We have added some text to this effect just before Theorem 3.
- Reviewer 1: "p. 18 - there is some discrepancy in the definition of $\mathcal{M}^\Omega$ in the proof of Thm. 4 and in Appendix B (similar to the above)."
  - Response: Corrected
- Reviewer 1: "p. 25 - $\mathcal{I}(\Omega, \cdot)$ should probably be $\mathcal{I}(\mathbb{X}, \cdot)$"
  - Corrected
- Reviewer 1: "p. 27 - I find this page quite confusing. What is the very particular form of the minimal covariant dilation mentioned in the first paragraph? What is the role of the assumptions (a)-(c)? What was proved in [4,12] and how it is related to the results of Thm. 5? I would also suggest to give a more comprehensible description of some of the examples at the bottom of p. 27 (and perhaps only mention the others)"

  - Response: The wording in the first paragraph is changed to emphasize that the results in [4,12] imply that a covariant instrument can be dilated into a canonical system of imprimitivity and then the details of this dilation are introduced. The assumptions (a) – (c) are motivated in the new version by first stating the three exemplary cases where these conditions hold and stating that, assuming these conditions, we can treat a wide class, including in particular the three special cases, of covariance structures in one go. After introducing (a) – (c), the first two cases (Abelian group, compact stability subgroup) are treated explicitly but shortly.
- Reviewer 1: "p. 27, assumption (c) - it seems that $\mathcal{H}^g_{\pi_0}$ should be $L^2_\mu \otimes \mathcal{H}_{\pi_0}$? Also clarify that $x = gH$"
  - Response: Both errors are now corrected.
- Reviewer 1: "p. 29, proof of Thm. 5 - the existence of the minimal covariant Naimark dilation of M at the beginning of the proof should follow by assumption (c). Why the reference to [5] and [12]?"
  - Response: In the new version we directly apply assumption (c). The reference to [5,12] was a relic of an earlier version where we had not yet adopted the generalized view of assumptions (a) – (d).
- Reviewer 1: "p. 29, proof of Thm. 5 - here P_0 should perhaps be $\mathsf{P}^G_{\pi_0}$"
  - Response: Corrected

Reviewer 2

- Reviewer 2 points out that the referee guidelines of J. Math. Phys. A: Math. Theor require mathematical papers to have proper physical motivation and to give background information for the new results and to clearly delineate the new results. The reviewer states that they cannot find anything like this in the introduction.
  - Response: We acknowledge that the first version of the introduction did not find its mark. We have written a new introduction where we present known results and detail how our new results relate to them. We also describe the physical relevance of the different optimality properties of measurements more closely and have added references to relevant earlier results dealing with this (e.g. measurement uncertainty relations and information-disturbance relations)
- Reviewer 2: "I have the impression that the construction of pp. 5-7 is important for the aims of the paper, but I cannot be sure of this: there is no explicit indication of this. A better division in sections and subsections of the whole paper would be of help. Example 1 has at least a starting point and an end, denoted by a triangle, but many other results are dispersed in the discussion, without any evidence."
  - Response: As detailed in our response to reviewer 1, we have tried to make the relevance of the steps (i) – (iv) clearer in the paragraphs preceding and following these steps; they show that, in finite-dimensions

when the symmetry group is finite, all covariance systems can be seen as subsystems of a covariance system w.r.t. symmetric groups in a particular way. After the steps (i) – (iv), we have explained that we want to study optimal observables in general in the case of covariance w.r.t. symmetric groups and this motivates example 1.

- Reviewer 2: "Some examples are elaborated in the article, but it is at all not clear if they are completely abstract examples or if they have some physical meaning. This has to do with the need of giving some idea of possible applications and with connections with the existing literature."
  - Response: As detailed in our first response to reviewer 2, we have added physical motivations and references to our new introduction.
- Reviewer 2: "Also the results in the paper are often stated without any connection to the earlier literature. Just to give a single example, Theorem 1 is a very well known fact when the G-space $\mathbb{X}$ is transitive (see e.g. [H11], Theorem 4.2.3)."
  - Response: Amongst the literature added, we have attributed the transitive version of this result to the reference cited by the reviewer. The non-transitive case is only a minor generalization.
- Reviewer 2: "In the whole paper I find two "official" definitions (p. 11 and p. 28), but the paper is full of definitions which can be lost very easily, because dispersed in the general presentation."
  - Response: We have added "official" definitions for observables, POVMs, instruments, and quantum-operations-valued measures (QOVMs) (see the meaning of this in a subsequent response), their covariance conditions, and their (two kinds of) extremality in the 'Basic definitions and observations' section which is the old and edited introduction. Not all these definitions are repeated officially in the continuous case as they are so similar. Also definitions of the orbits, their representatives, and stability subgroups are added in itemized environments so that the reader can easily find them. Moreover, definitions of the 5 optimality properties, other than extremality, are similarly enumerated and explained more in detail for the reader's convenience. Covariance and extreme channels are still not "officially" defined as the discussion on channels is located in the short final part of Section 4 and these properties are not discussed after this point. These can also given full definitions if needed.
- Reviewer 2: "Often, too many different concepts and results are mixed up together. For instance, at p. 3, the paragraph before Theorem 1 contains: a) the definition of orbit; b) the introduction of the notations x_Ω, g_x, H_Ω, K_Ω c) the proof of Theorem 1. This is too much: when one is giving a look to the paper it is impossible to find where definitions are given and notations introduced."
  - Response: points a) and b) are now dealt with by itemizing them so that reader can find the definition of these notions and notations

more easily. The proof of Theorem 1 is now given "officially" after the statement of the theorem.

- Reviewer 2: "Pay attention to typos and English language. I find "ja" at p.7, 8 (2 times), 9. Check the use of "a" and "an", the use of "as"."
  - Response: We have reread the paper and corrected the typos according to our ability. Specifically, the Finnish conjunction "ja" (which appeared also in a couple of other places) has now been changed to "and"; these were remnants of an earlier working draft from where we introduced formulas and, it seems, also conjunctions between formulas.
- Reviewer 2: "The sentence after Theorem 1 starts with "Note that if M is not normalized...". But M is normalized by definition! The authors mean that, by taking a certain expression given in the theorem, then one can construct a POVM by normalization. This needs to be written in a clearer way and perhaps inserted in the formulation of the theorem."
  - Response: We have solved this problem by making a distinction in out formal definition of observables, instruments etc. (The new Definition 1) between POVMs and observables: POVMs are positive-operator-valued measures which are not necessarily normalized whereas observables as normalized POVMs. This is, admittedly, a little aberrant from the language commonly used but we defend this distinction by the fact that the acronym 'POVM' has no 'N' in it for 'normalized'. We have made a similar distinction between quantum-operation-valued measure (QOVM) and an instrument in the same definition. These distinctions are carried out over the whole new version of the manuscript. This also affects the definition of (normalized) sets of intertwiners.
- Reviewer 2: "In the same paragraph I find "Moreover, we note that there is necessarily no non trivial solution M for (1)." This sentence seems to me to say that only trivial POVM can be covariant, while the authors probably wanted to say that sometimes M turns out to be trivial, when too many conditions are required."
  - Response: We have removed the double negative in this sentence to make our message clearer. What we mean is, in fact, what the reviewer first read from it: the covariance condition might be so restrictive that it only allows trivial solutions; an example of such a situation is given in the same paragraph. However, the second possible meaning given by the reviewer is undoubtedly also true although we did not mean it here. We hope that the new wording is clearer.