Letters in Mathematical Physics

Approximate recoverability and relative entropy II: 2-positive channels of general von Neumann algebras

--Manuscript Draft--

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<tr><td>Response to Reviewers:</td><td>We thank the referee for his careful reading of the manuscript and his useful comments. We have addressed them as follows:

\medskip
\noindent
1. We have taken care of this by adapting the notations and wording in the introduction more closely to those used in the main proof. \\
2. This has been deleted.\\
3. Done.\\
4. The referee might have been misled here by typos in eq. (26) and (27). We have set these typos straight and added some further explanations regarding the origin of the support projector in eq. (27) after (26). \\
5. We have inserted these references.\\
6. This has been corrected.\\
7. We have substituted ``rotated Petz map'' \\
8. We have exchanged $\Delta_{\eta_\cA,\psi_\cA}$ for $\Delta_{\eta,\psi;\cA}$ throughout, and have done the same for similar other expressions of this type.\\
9. corrected\\
10. We have moved this to appendix C. \\
11. corrected\\
12. We agree with the referee here. However, ref. 7 and the proposed argument implicitly uses duality between L_p and $L_q, 1/p+1/q=1$, which has been proven by Araki and Masada when the reference state $|\psi\rangle$ is faithful. The authors of ref. 7 say that they have convinced themselves that this goes through for general $|\psi\rangle$. We have no doubts about this but have preferred to give an elementary proof not using duality. \\
13. Has been added in the statement of the lemmas\\
14. corrected\\
15. corrected\\
16. corrected\\

\medskip

We have also taken the opportunity to include ref. [22] and refer to this reference for a</td></tr>
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	proof of lemma 8. In eq. (7), we have corrected a sign and in eq. (6) we have included a support projection which is necessary if ω_ψ is not faithful (this has likewise been adjusted in a few other places, without essential changes to the proof).

Approximate recoverability and relative entropy II: 2-positive channels of general von Neumann algebras

Thomas Faulkner[1]*, Stefan Hollands[2]†
[1] *University of Illinois at Urbana-Champaign, IL*
[2] *ITP, Universität Leipzig, MPI-MiS Leipzig and KITP, Santa Barbara*

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Abstract

We generalize our results in paper I in this series to quantum channels between general von Neumann algebras, proving the approximate recoverability of states which undergo a small change in relative entropy through the channel. To this end, we derive a strengthened form of the quantum data processing inequality for the change in relative entropy of two states under a channel between two von Neumann algebras. Compared to the usual inequality, there is an explicit lower bound involving the fidelity between the original state and a recovery channel.

1 Introduction

The relative entropy between two density operators ρ, σ, defined as

$$S(\rho|\sigma) = \mathrm{Tr}[\rho(\ln\rho - \ln\sigma)], \tag{1}$$

is an asymptotic measure of their distinguishability. Classically, $e^{-NS(\{p_i\}|\{q_i\})}$ approaches for large N the probability for a sample of size N of letters, distributed according to the true distribution $\{p_i\}$, when calculated according to an incorrect guess $\{q_i\}$. In the non-commutative setting, the relative entropy was later generalized to pairs ω_ψ, ω_η of normal, positive functionals on a general von Neumann algebra by Araki [2, 3] using relative modular hamiltonians.

By far the most fundamental property of the relative entropy – from which in fact essentially all others follow – is its monotonicity under a channel. A channel, as defined in this paper, is a normal, unital, 2-positive Schwarz map between von Neumann algebras. By well-known theorems of the Stinespring-type, this includes in the case of

*tomf@illinois.edu
†stefan.hollands@uni-leipzig.de

finite-dimensional type-I von Neumann algebras the familiar operations of a (i) a unitary time evolution of the density matrix, (ii) a von Neumann measurement followed by post-selection, (iii) forgetting part of the system (partial trace). The fundamental property is that if $T : \mathcal{B} \to \mathcal{A}$ is such a channel and its application to a state functional ω on $\mathcal{A}$ is given by pull-back $\omega \circ T$, then always

$$S(\omega_\psi|\omega_\eta) \geqslant S(\omega_\psi \circ T|\omega_\eta \circ T), \tag{2}$$

where ω_ψ, ω_η are any two positive, normal, unital functionals[1] on the von Neumann algebra $\mathcal{A}$. In quantum information theory, T is related to data processing, so (2) is sometimes called the data-processing inequality (DPI). It implies for instance the strong subadditivity property of the von Neumann entropy [31]. After important special cases were proven by [32, 2], the data-processing inequality was demonstrated in the setting of general von Neumann algebras by Uhlmann [48]. It can be interpreted as saying that the distinguishability between ω_η, ω_ψ cannot increase under the channel T.

Suppose the distinguishability decreases only by a small amount under T. Then one would like to say that, given ω_η, T, the state ω_ψ can be recovered "with correspondingly high fidelity" from $\omega_\psi \circ T$. For this, one should establish an improved lower bound on the quantity $S(\omega_\psi|\omega_\eta) - S(\omega_\psi \circ T|\omega_\eta \circ T)$ in terms of the fidelity between the original state ω_ψ, and a recovered version of $\omega_\psi \circ T$. Starting with the pioneering work by [15], this issue has been considered by several authors such as [27, 6, 50, 10, 24, 44, 45], partly based on earlier characterizations of the case of equality in (2) [39, 40, 41, 23].

A particularly attractive lower bound in the DPI has recently been given by Junge et al. [27] for von Neumann algebras of type I. They consider a certain recovery channel α that is closely related to the Petz-map (see e.g. [38], prop. 8.3 and sec. 4 below) which is constructed in a canonical way out of T and ω_η alone, and which hence does not depend on ω_ψ. They then show that there is a lower bound in the DPI involving the log-fidelity,

$$S(\omega_\psi|\omega_\eta) - S(\omega_\psi \circ T|\omega_\eta \circ T) \geqslant -\ln F(\omega_\psi|\omega_\psi \circ T \circ \alpha)^2, \tag{3}$$

where F is the "square root" fidelity between two states (20).

It is interesting to note that, while [27] prove it only for type I von Neumann algebras[2], the ingredients of this inequality formally make sense for channels between arbitrary von Neumann algebras. Thus, it is a natural question whether the inequality generalizes to that setting. We are motivated in particular by recent applications of the improved form of the DPI in high energy physics [13, 12], where such questions are natural in the algebraic approach to quantum field theory [18]. In this context, the von Neumann algebras under consideration are of type III_1 [9] and in many cases a distinguished reference state ω_η exists, namely the vacuum. On the other hand, the proof by [27] strongly uses the special properties of type I von Neumann algebras. In this paper, we give a proof of (3) for general σ-finite von Neumann algebras, see thm. 2. While we were finishing this manuscript [28] appeared which reported results that have some overlap with ours, albeit with different methods. Our result generalizes paper I [16], where the case of an

[1] In the case of finite-dimensional type I von Neumann algebras, these would correspond to density matrices as in $\omega_\psi(a) = \mathrm{Tr}(a\omega_\psi)$.

[2] I.e., direct sums of matrix algebras or the algebra of all bounded operators on a separable Hilbert space.

embedding T of general σ-finite von Neumann algebras was considered – corresponding to a "partial trace" of density operators in the type I context; see also [22] for a variant of the inequality (3) for general von Neumann algebras.

This paper is organized as follows. In section 2 we recall how the ingredients in our inequality (3) are defined in the setting of general von Neumann algebras. Then in section 3 we state the main results indicating also some generalizations. In section 4 we consider some basic examples, and in section 5 we give the proof of our main theorems.

Notations and conventions: Calligraphic letters $\mathcal{A}, \mathcal{M}, \ldots$ denote von Neumann algebras. Calligraphic letters $\mathscr{H}, \mathscr{K}, \ldots$ denote linear spaces. A frequently used notation is $\mathbb{S}_a = \{z \in \mathbb{C} \mid 0 < \mathrm{Re}(z) < a\}$. For vectors in a Hilbert space, we prefer to use the physicist's "ket"-notation $|\psi\rangle$ and we write the scalar product as

$$(|\psi\rangle, |\psi'\rangle)_{\mathscr{H}} =: \langle\psi|\psi'\rangle. \tag{4}$$

It is by assumption anti-linear in the first entry. For norms of vectors in a Hilbert space, we do not use the ket notation but write $\||\psi\rangle\| =: \|\psi\|$. The norm of a bounded linear operator T on $\mathscr{H}$ is defined as $\|T\| = \sup_{|\psi\rangle:\|\psi\|=1} \|T\psi\|$ in this notation. Our von Neumann algebras are assumed to be σ-finite, i.e. the existence of a faithful normal state is required.

2 Relative entropy, data processing inequality and Petz map

2.1 Relative entropy

Let $(\mathcal{M}, J, \mathscr{P}^\natural_{\mathcal{M}}, \mathscr{H})$ be a von Neumann algebra in standard form acting on a Hilbert space $\mathscr{H}$, with natural cone $\mathscr{P}^\natural_{\mathcal{M}}$ and modular conjugation J. See paper I [16] for our notations and [8, 46] as general references for explanations of these terms . As in paper I [16], we use relative modular operators $\Delta_{\eta,\psi}$ in our constructions. The relative modular operator depends on a vector $|\psi\rangle \in \mathscr{P}^\natural_{\mathcal{M}}$ and a state (i.e., positive, linear, normalized, ultraweakly continuous map) $\omega_\eta : \mathcal{M} \to \mathbb{C}$. A vector $|\eta\rangle \in \mathscr{H}$ may be picked representing this state as in $\omega_\eta(m) = \langle\eta|m\eta\rangle$, but $\Delta_{\eta,\psi}$ does not depend on the highly non-unique choice of vector representative $|\eta\rangle$.

According to [2, 3], if $\pi^{\mathcal{M}}(\eta) \geqslant \pi^{\mathcal{M}}(\psi)$, the relative entropy between two state functionals ω_ψ, ω_η on $\mathcal{M}$ may be defined by

$$S(\omega_\psi|\omega_\eta) = -\lim_{\alpha\to 0^+} \frac{\langle\psi|\Delta^\alpha_{\eta,\psi}\psi\rangle - 1}{\alpha}, \tag{5}$$

(where the vector representative $|\psi\rangle$ is chosen in the natural cone $\mathscr{P}^\natural_{\mathcal{M}}$); otherwise, it is by definition infinite. For a thorough discussion of quantum divergences such as S in the context of general von Neumann algebras see e.g. [20, 38] as general references.

For $t \in \mathbb{R}$, the Connes-cocycle is a partial isometry from $\pi^{\mathcal{M}}(\eta)\mathcal{M}\pi^{\mathcal{M}}(\psi)$ satisfying [see e.g. [4], thm. C.1 (β2)]

$$(D\eta : D\psi)_t \pi^{\mathcal{M}'}(\psi) = \Delta^{it}_{\eta,\psi}\Delta^{-it}_{\psi,\psi}. \tag{6}$$

Like the relative entropy, the Connes-cocycle does not depend on the choice of natural cone. In fact, in terms of the Connes-cocycle, the relative entropy may also be defined as

$$S(\omega_\psi|\omega_\eta) = i\frac{\mathrm{d}}{\mathrm{d}t}\omega_\psi[(D\eta : D\psi)_t]|_{t=0}. \tag{7}$$

The derivative exists whenever $S(\omega_\psi|\omega_\eta) < \infty$ [38], thm. 5.7. In the case of the matrix algebra $M_n(\mathbb{C})$, where ω_η and ω_ψ are identified with density matrices, the relative entropy as defined here is equivalent to the usual expression (1).

2.2 Data processing inequality

The basic situation studied in this paper is the following. $\mathcal{B}, \mathcal{A}$ are two von Neumann algebras, assumed to be in standard form. They act on Hilbert spaces $\mathscr{K}, \mathscr{H}$, with corresponding natural cones $\mathscr{P}^\natural_\mathcal{A}, \mathscr{P}^\natural_\mathcal{B}$ and associated anti-linear unitary maps $J_\mathcal{A}, J_\mathcal{B}$ (so that $J_\mathcal{A}|\xi\rangle = |\xi\rangle$ for $|\xi\rangle \in \mathscr{P}^\natural_\mathcal{A}$, and similarly for $\mathcal{B}$) . A *channel* is a normal, i.e. ultra weakly continuous, linear mapping

$$T : \mathcal{B} \to \mathcal{A} \tag{8}$$

such that $T(1) = 1$ and such that any non-negative self-adjoint element of $\mathcal{B}$ gets mapped to such an element of $\mathcal{A}$, i.e. $T(b^*b) \geqslant 0$. T is not required to be a homomorphism (but could be). We will use the following standard terminology.

Definition 1. *1. A channel T is called a Schwarz map if for any $a \in \mathcal{B}$ we have Kadison's property*

$$T(a^*)T(a) \leqslant T(a^*a). \tag{9}$$

2. A channel T is called 2-positive if any non-negative element from the 2×2 matrix algebra $\mathcal{B} \otimes M_2(\mathbb{C})$ with entries in $\mathcal{B}$ gets mapped under $T \otimes 1_{M_2}$ to a non-negative element of $\mathcal{A} \otimes M_2(\mathbb{C})$.

A 2-positive map is also a Schwarz map, see e.g. [42], thm. E. In particular, both properties follow if $T = \iota$ is an embedding of von Neumann algebras, i.e. if $\mathcal{B}$ is a von Neumann subalgebra of $\mathcal{A}$ under a $*$-homomorphism ι. That case was treated in paper I [16].

If T is a linear, normal mapping $T : \mathcal{B} \to \mathcal{A}$, then we define its adjoint $\mathcal{A}_* \to \mathcal{B}_*$, operating between the corresponding spaces of normal linear functionals by duality. Let us assume that we have two normal state functionals ω_η, ω_ψ on $\mathcal{A}$ induced by vectors from $\mathscr{H}$. Then the pull backs $\omega_\eta \circ T, \omega_\psi \circ T$ give normal state functionals on $\mathcal{B}$. We shall write $|\eta_\mathcal{A}\rangle \in \mathscr{H}, |\eta_\mathcal{B}\rangle \in \mathscr{K}$ for the unique vector representatives in the natural cones $\mathscr{P}^\natural_\mathcal{A}, \mathscr{P}^\natural_\mathcal{B}$, respectively, so that

$$\omega_\eta(a) = \langle \eta_\mathcal{A}|a\eta_\mathcal{A}\rangle, \quad \omega_\eta \circ T(b) = \langle \eta_\mathcal{B}|b\eta_\mathcal{B}\rangle \quad (a \in \mathcal{A}, b \in \mathcal{B}). \tag{10}$$

A similar notation is adopted when $|\eta\rangle$ is replaced by $|\psi\rangle$.

The quantum data processing inequality (DPI) [48] states that

$$S(\omega_\psi|\omega_\eta) \geqslant S(\omega_\psi \circ T|\omega_\eta \circ T). \tag{11}$$

One of our main results, thm. 1, will be an improved, explicit, lower bound on this inequality in the spirit of [27], generalizing our earlier paper I [16]where the special case of inclusions T was treated.

2.3 Petz recovery map

We now recall the definition of the Petz map in the case of general von Neumann algebras, discussed in more detail in [38], sec. 8. Let $T : \mathcal{B} \to \mathcal{A}$ be a unital, normal, and 2-positive (or Schwarz) map between two von Neumann algebras. Let ω_η be a normal state functional on $\mathcal{A}$ whose vector represenative in the natural cone is $|\eta_\mathcal{A}\rangle$, with pull-back $\omega_\eta \circ T$ to $\mathcal{B}$, whose vector representative in the natural cone is $|\eta_\mathcal{B}\rangle$. In the rest of this work, we will restrict our statements and proofs for simplicity to the case that both ω_η and $\omega_\eta \circ T$ are faithful, meaning that $|\eta_\mathcal{A}\rangle, |\eta_\mathcal{B}\rangle$ are cyclic and separating for $\mathcal{A}, \mathcal{B}$, respectively.

However, we note from the beginning that all of our results can be generalized to non-faithful ω_η and $\omega_\eta \circ T$ by the following argument from paper I [16]: If ω_η is not faithful, we consider instead the von Neumann subalgebra $\mathcal{A}_\pi = \pi^\mathcal{A}(\eta)\mathcal{A}\pi^\mathcal{A}(\eta)\pi^{\mathcal{A}'}(\eta)$, noting that this does not change the relative entropy $S(\omega_\psi|\omega_\eta)$ since by definition $\pi^\mathcal{A}(\psi) \leqslant \pi^\mathcal{A}(\eta)$ unless the entropy is infinite (in which case all of our inequalities are trivial). Likewise, letting $\pi^\mathcal{B}(\eta)$ be the support projection of $\omega_\eta \circ T$ in $\mathcal{B}$, if $\omega_\eta \circ T$ is not faithful, we pass to $\mathcal{B}_\pi = \pi^\mathcal{B}(\eta)\mathcal{B}\pi^\mathcal{B}(\eta)\pi^{\mathcal{B}'}(\eta)$. Finally, we pass from $T : \mathcal{B} \to \mathcal{A}$ to $T_\pi(\,.\,) = \pi^\mathcal{A}(\eta)T(\,.\,)\pi^\mathcal{A}(\eta)\pi^{\mathcal{A}'}(\eta)$ which is again a normal, unital 2-positive (or Schwarz) map $T_\pi : \mathcal{B}_\pi \to \mathcal{A}_\pi$. By such kinds of constructions, we can and will restrict ourselves from now on to states such that both $\omega_\eta \circ T, \omega_\eta$ are faithful.

Now we recall the definition of the Petz-map. The "KMS"-scalar product on $\mathcal{A}$ regarded as a vector space is defined as

$$\langle a_1, a_2 \rangle_\eta = \langle \eta_\mathcal{A} | a_1^* \Delta_{\eta,\mathcal{A}}^{1/2} a_2 \eta_\mathcal{A} \rangle, \quad a_1, a_2 \in \mathcal{A}. \tag{12}$$

We can likewise use $|\eta_\mathcal{B}\rangle$ to define a KMS scalar product for $\mathcal{B}$. Then the KMS scalar products on $\mathcal{A}, \mathcal{B}$ allow us to define the adjoint $T^+ : \mathcal{A} \to \mathcal{B}$ of a normal, unital and 2-positive channel $T : \mathcal{B} \to \mathcal{A}$ by results of Petz, see [38] prop. 8.3, who shows that T^+ is well-defined, normal, unital, and 2-positive. The rotated Petz map, which we call $\alpha_{\eta,T}^t : \mathcal{A} \to \mathcal{B}$, is defined by conjugating this with the respective modular flows, i.e.

$$\alpha_{\eta,T}^t = \varsigma_{\eta,\mathcal{B}}^t \circ T^+ \circ \varsigma_{\eta,\mathcal{A}}^{-t} \tag{13}$$

where $\varsigma_{\eta,\mathcal{A}}^t = \mathrm{Ad}\Delta_{\eta,\mathcal{A}}^{it}$ is the modular flow for $\mathcal{A}, |\eta_\mathcal{A}\rangle$ etc. An equivalent definition is:

Definition 2. *For $b \in \mathcal{B}, a \in \mathcal{A}$, and T is unital, normal, and 2-positive, the rotated Petz map $\alpha_{\eta,T}^t : \mathcal{A} \to \mathcal{B}$ is defined implicitly by the identity:*

$$\langle b\eta_\mathcal{B} | J_\mathcal{B} \Delta_{\eta,\mathcal{B}}^{it} \alpha_{\eta,T}^t(a) \eta_\mathcal{B} \rangle = \langle T(b)\eta_\mathcal{A} | J_\mathcal{A} \Delta_{\eta,\mathcal{A}}^{it} a \eta_\mathcal{A} \rangle. \tag{14}$$

The following is then a trivial consequence of [38], prop. 8.3:

Lemma 1. *The map $\alpha_{\eta,T}^t : \mathcal{A} \to \mathcal{B}$ is well-defined, normal, unital, and 2-positive for all $t \in \mathbb{R}$.*

For the case of finite dimensional type I factors, the definition of the rotated Petz map is easily seen to coincide with that given by [27].

3 Main theorems

In this section we state our main theorems. Let $T : \mathcal{B} \to \mathcal{A}$ be a normal, unital Schwarz map between two von Neumann algebras in a standard form and let ω_ψ, ω_η be normal state functionals on $\mathcal{A}$. Let $|\psi_\mathcal{A}\rangle$ be the vector representative in the natural cone of ω_ψ and $|\psi_\mathcal{B}\rangle$ that of $\omega_\psi \circ T$. We introduce following [42, 40] a linear map $V_\psi : \mathscr{K} \to \mathscr{H}$ by

$$V_\psi b|\psi_\mathcal{B}\rangle := T(b)|\psi_\mathcal{A}\rangle \quad (b \in \mathcal{B}). \tag{15}$$

As it stands, the definition is actually consistent only when $\omega_\psi \circ T$ is faithful (i.e., that $|\psi_\mathcal{B}\rangle$ is cyclic and separating). In the general case, one can define [42] instead

$$V_\psi(b|\psi_\mathcal{B}\rangle + |\zeta\rangle) := T(b)|\psi_\mathcal{A}\rangle \quad (b \in \mathcal{B}, \pi^{\mathcal{B}'}(\psi)|\zeta\rangle = 0). \tag{16}$$

In either case, it easily follows from Kadison's property (9) that V_ψ is a contraction $\|V_\psi\| \leqslant 1$, see e.g. [42], proof of thm. 4.

The first main result is a strengthened version of the DPI. To this end, we introduce a vector valued function

$$t \mapsto |\Gamma_\psi(1/2 + it)\rangle := \Delta^{1/2+it}_{\eta,\psi;\mathcal{A}} V_\psi \Delta^{-1/2-it}_{\eta,\psi;\mathcal{B}} |\psi_\mathcal{B}\rangle \quad (t \in \mathbb{R}), \tag{17}$$

the existence and properties of which are established in lem. 3 and lem. 4 below. Here $\Delta_{\eta,\psi;\mathcal{A}}$ is the relative operator associated with $|\psi_\mathcal{A}\rangle, |\eta_\mathcal{A}\rangle$, and similarly for $\mathcal{A}$ replaced by $\mathcal{B}$. In particular, the representation (24) shows in conjunction with Stone's theorem (see e.g. [14], sec. 5.3) applied to the exponential factors $\Delta^{it}_{\eta,\psi;\mathcal{A}}, \Delta^{-it}_{\eta,\psi;\mathcal{B}}$ that this function is strongly continuous.

Theorem 1. *Let ω_η, ω_ψ be normal states on the von Neumann algebra $\mathcal{A}$, ω_η faithful, and let $T : \mathcal{B} \to \mathcal{A}$ be a normal, unital Schwarz map such that $\omega_\eta \circ T$ is a faithful state on the von Neumann algebra $\mathcal{B}$. Then for $q \in [1, 2]$ we have*

$$S(\omega_\psi|\omega_\eta) - S(\omega_\psi \circ T|\omega_\eta \circ T) \geqslant - \int_{-\infty}^{\infty} p(t) \ln \|\Gamma_\psi(1/2 + it)\|^2_{q,\psi} \, \mathrm{d}t \ . \tag{18}$$

Here $\|\zeta\|_{q,\psi}$ denotes the Araki-Masuda-Hölder $L_q(\mathcal{A}', \psi)$-norm relative to $|\psi\rangle, \mathcal{A}'$, and $p(t) := \pi \left[1 + \cosh(2\pi t)\right]^{-1}$, which is a probability density.

The proof is given in sec. 5 and is similar to that of thm. 1, paper I [16]. We will therefore only provide complete details where the proof deviates significantly or where we would like to make shortcuts.

Remarks: 1) The non-commutative Hölder spaces on a von Neumann algebra $\mathcal{M}$ in standard form, weighted by a vector $|\psi\rangle$ in the natural cone, called $L_p(\mathcal{M}, \psi)$, were defined and analyzed in [4] (for $p \in [1, \infty]$). Their definition is recalled for convenience in app. B.

2) In paper I [16], this result was established in the case when $T = \iota : \mathcal{B} \subset \mathcal{A}$ is an inclusion of von Neumann algebras.

3) The integrand of (18) is pointwise non-positive, as can be seen from lem. 3, and $\|\Gamma_\psi(1/2+it)\|_{q,\psi}$ is continuous in t, because the vector $|\Gamma_\psi(1/2+it)\rangle$ is strongly continuous

and the L_q-norm is continuous with respect to the norm topology in $\mathscr{H}$, see e.g. app. B. This implies that the integral in (18) is definite in the Lebesgue sense, but might be $-\infty$.

To state our second main theorem, we define

$$\alpha = \int_{-\infty}^{\infty} p(t)\alpha^t_{\eta,T}\,\mathrm{d}t, \tag{19}$$

where $\alpha^t_{\eta,T}$ is the rotated Petz map associated with T and $|\eta\rangle$ as discussed in sec. 2.3. This map is a 2-positive channel – called "recovery channel" – provided that T has these properties. We also define the "square root" fidelity $F = F_{\mathcal{A}}$ [48, 1] between two normal states on $\mathcal{A}$ as usual by

$$\|\zeta\|_{1,\psi,\mathcal{A}'} = \sup\{|\langle\zeta|a'\psi\rangle| : a' \in \mathcal{A}', \|a'\| = 1\} =: F_{\mathcal{A}}(\omega_\zeta|\omega_\psi), \tag{20}$$

where the characterization of F as an L_1-norm is well-known and demonstrated in the required generality e.g. in paper I [16], lem. 3 (1).

Theorem 2. *Let $T : \mathcal{B} \to \mathcal{A}$ be a unital, normal, and 2-positive map between two von Neumann algebras. Let ω_η, ω_ψ be normal states functionals on $\mathcal{A}$ such that both $\omega_\eta, \omega_\eta \circ T$ are faithful. Then*

$$S(\omega_\psi|\omega_\eta) - S(\omega_\psi \circ T|\omega_\eta \circ T) \geqslant -\ln F(\omega_\psi|\omega_\psi \circ T \circ \alpha)^2. \tag{21}$$

Remarks: 1) In the case of von Neumann algebras of type I, our result reduces to that by Junge et al. [27].
2) The case of an inclusion $T = \iota : \mathcal{B} \subset \mathcal{A}$ was treated separately already in paper I.
3) By one of the Fuchs-van-der-Graff inequalities, see e.g. paper I [16], lem. 3 (3) for the case of general von Neumann algebras, and the elementary bound $-\ln x \geqslant 1-x, x \in (0, 1]$, we obtain an upper bound on the norm distance between the original and recovered state:

$$S(\omega_\psi|\omega_\eta) - S(\omega_\psi \circ T|\omega_\eta \circ T) \geqslant \tfrac{1}{4}\|\omega_\psi \circ T \circ \alpha - \omega_\psi\|^2. \tag{22}$$

Proof. Consider the linear operator $V_\eta : \mathscr{K} \to \mathscr{H}$ defined by

$$V_\eta(\pi^{\mathcal{B}}(\psi)b|\eta_{\mathcal{B}}\rangle + |\zeta\rangle) = \pi^{\mathcal{A}}(\psi)T(b)|\eta_{\mathcal{A}}\rangle. \tag{23}$$

Here, $b \in \mathcal{B}, |\zeta\rangle \in [\pi^{\mathcal{B}}(\psi)\mathcal{B}\eta_{\mathcal{B}}]^\perp$. It has been shown in [42], proof of thm. 4 that $\|V_\eta\| \leqslant 1$, which uses the 2-positivity of T, and that $V_\eta \bar{S}_{\eta,\psi;\mathcal{B}} \subset \bar{S}_{\eta,\psi;\mathcal{A}} V_\psi$, where we mean the closures of the Tomita operators. We find:

$$\begin{aligned}
|\Gamma_\psi(1/2 + it)\rangle =& \Delta^{1/2+it}_{\eta,\psi;\mathcal{A}} V_\psi \Delta^{-1/2-it}_{\eta,\psi;\mathcal{B}} |\psi_{\mathcal{B}}\rangle \\
=& \Delta^{it}_{\eta,\psi;\mathcal{A}} J_{\mathcal{A}} \bar{S}_{\eta,\psi;\mathcal{A}} V_\psi \Delta^{-1/2-it}_{\eta,\psi;\mathcal{B}} |\psi_{\mathcal{B}}\rangle \\
=& \Delta^{it}_{\eta,\psi;\mathcal{A}} J_{\mathcal{A}} V_\eta \bar{S}_{\eta,\psi;\mathcal{B}} \Delta^{-1/2-it}_{\eta,\psi;\mathcal{B}} |\psi_{\mathcal{B}}\rangle \\
=& \Delta^{it}_{\eta,\psi;\mathcal{A}} J_{\mathcal{A}} V_\eta J_{\mathcal{B}} \Delta^{-it}_{\eta,\psi;\mathcal{B}} |\psi_{\mathcal{B}}\rangle.
\end{aligned} \tag{24}$$

Let γ_t be the non-normalized linear functional on $\mathcal{A}$ corresponding to $|\Gamma_\psi(1/2 + it)\rangle$. We claim, compare paper I [16], thm. 4 (2):

Lemma 2. *We have $\gamma_t(a_+) \leqslant \omega_\psi \circ T \circ \alpha^t_{\eta,T}(a_+)$ for $t \in \mathbb{R}, a_+ \in \mathcal{A}_+$.*

Proof. In (14), set $b = B^*B, a \equiv a_+ = A^*A$. We get:

$$\langle \Delta^{-it}_{\eta,\mathcal{B}} J_\mathcal{B} B \eta_\mathcal{B} | \alpha^t_{\eta,T}(A^*A) \Delta^{-it}_{\eta,\mathcal{B}} J_\mathcal{B} B \eta_\mathcal{B} \rangle = \langle \Delta^{-it}_{\eta,\mathcal{A}} J_\mathcal{A} A \eta_\mathcal{A} | T(B^*B) \Delta^{-it}_{\eta,\mathcal{A}} J_\mathcal{A} A \eta_\mathcal{A} \rangle. \tag{25}$$

Now, let $b' \in \mathcal{B}'$, and let $|\psi_\mathcal{B}\rangle$ be the vector representative in the natural cone of the state $\omega_\psi \circ T$ on $\mathcal{B}$. Then define

$$B := J_\mathcal{B} \Delta^{it}_{\eta,\psi;\mathcal{B}} b' \Delta^{-it}_{\eta,\mathcal{B}} J_\mathcal{B}. \tag{26}$$

We have $B \in \mathcal{B}$ because $\Delta^{it}_{\eta,\psi;\mathcal{B}} \Delta^{-it}_{\eta,\mathcal{B}}$ is a partial isometry in $\mathcal{B}'$ and $\Delta^{it}_{\eta,\mathcal{B}} b' \Delta^{-it}_{\eta,\mathcal{B}} \in \mathcal{B}'$, $J_\mathcal{B} \mathcal{B}' J_\mathcal{B} = \mathcal{B}$, by Tomita-Takesaki theory. Then (25) yields, using that $v' = \Delta^{-it}_{\eta,\mathcal{B}} \Delta^{it}_{\eta,\psi;\mathcal{B}}$ is a partial isometry in $\mathcal{B}'$ such that $(v')^* v' = \pi^{\mathcal{B}'}(\psi)$, and using Kadison's property (9):

$$\begin{aligned}
&\langle \pi^{\mathcal{B}'}(\psi) b' \eta_\mathcal{B} | \alpha^t_{\eta,T}(A^*A) \pi^{\mathcal{B}'}(\psi) b' \eta_\mathcal{B} \rangle \\
&= \langle \Delta^{-it}_{\eta,\mathcal{B}} J_\mathcal{B} B \eta_\mathcal{B} | \alpha^t_{\eta,T}(A^*A) \Delta^{-it}_{\eta,\mathcal{B}} J_\mathcal{B} B \eta_\mathcal{B} \rangle \\
&\geqslant \langle \Delta^{-it}_{\eta,\mathcal{A}} J_\mathcal{A} A \eta_\mathcal{A} | T(B^*) T(B) \Delta^{-it}_{\eta,\mathcal{A}} J_\mathcal{A} A \eta_\mathcal{A} \rangle \\
&\geqslant \langle \Delta^{-it}_{\eta,\mathcal{A}} J_\mathcal{A} A \eta_\mathcal{A} | T(B^*) \pi^\mathcal{A}(\psi) T(B) \Delta^{-it}_{\eta,\mathcal{A}} J_\mathcal{A} A \eta_\mathcal{A} \rangle \\
&= \langle \pi^\mathcal{A}(\psi) T(B) \eta_\mathcal{A} | \Delta^{-it}_{\eta,\mathcal{A}} J_\mathcal{A} A^* A J_\mathcal{A} \Delta^{it}_{\eta,\mathcal{A}} \pi^\mathcal{A}(\psi) T(B) \eta_\mathcal{A} \rangle \\
&= \langle V_\eta B \eta_\mathcal{B} | J_\mathcal{A} \Delta^{-it}_{\eta,\psi;\mathcal{A}} A^* A \Delta^{it}_{\eta,\psi;\mathcal{A}} J_\mathcal{A} V_\eta B \eta_\mathcal{B} \rangle \\
&= \langle \Delta^{it}_{\eta,\psi;\mathcal{A}} J_\mathcal{A} V_\eta J_\mathcal{B} \Delta^{it}_{\eta,\psi;\mathcal{B}} b' \eta_\mathcal{B} | (A^*A) \Delta^{it}_{\eta,\psi;\mathcal{A}} J_\mathcal{A} V_\eta J_\mathcal{B} \Delta^{it}_{\eta,\psi;\mathcal{B}} b' \eta_\mathcal{B} \rangle.
\end{aligned} \tag{27}$$

The set of vectors $\{b'|\eta_\mathcal{B}\rangle : b' \in \mathcal{B}'\}$ is dense because $|\eta_\mathcal{B}\rangle$ is the the vector representative of $\omega_\eta \circ T$ in the natural cone of $\mathcal{B}$ which is cyclic and separating as $\omega_\eta \circ T$ is assumed faithful. Hence the set of vectors $b'|\eta_\mathcal{B}\rangle$ can be used to approximate $|\psi_\mathcal{B}\rangle$ in norm. Then, since all operators in this inequality are bounded (in particular V_η, which is a contraction), (27) gives under this approximation

$$\gamma_t(A^*A) \leqslant \langle \psi_\mathcal{B} | \alpha^t_{\eta,T}(A^*A) \psi_\mathcal{B} \rangle, \tag{28}$$

which is equivalent to the desired bound. □

By this lemma, together with paper I [16], lem. 3 (1), and the monotonicity of the fidelity [48, 1] we have $\|\Gamma_\psi(1/2 + it)\|_{1,\psi} = F(\gamma_t|\omega_\psi) \leqslant F(\omega_\psi \circ T \circ \alpha^t_{\eta,T}|\omega_\psi)$, with L_1-norm relative to $\mathcal{A}'$ and $|\psi\rangle$. Thm. 1 for the case $q = 1$ therefore gives us

$$S(\omega_\psi|\omega_\eta) - S(\omega_\psi \circ T|\omega_\eta \circ T) \geqslant -2 \int_{-\infty}^{\infty} p(t) \ln F(\omega_\psi \circ T \circ \alpha^t_{\eta,T}|\omega_\psi)\, \mathrm{d}t. \tag{29}$$

Since ln and F are concave, we can further use the Jensen inequality to pull the integral inside the fidelity, see paper I [16], lem. 10 and proof of thm. 2 for the identical details of these arguments. This completes the proof. □

Remark: The attentive reader will have noticed that thm. 1 can evidently be combined with lem. 2 also for $q \in (1, 2)$. Firstly, it has been shown by [20], thm. 3.16 (7) that

$$\omega_{\zeta_1} \geqslant \omega_{\zeta_2} \implies \|\zeta_1\|_{q,\psi,\mathcal{A}'} \geqslant \|\zeta_2\|_{q,\psi,\mathcal{A}'}. \tag{30}$$

Indeed, the first condition gives $|\zeta_2\rangle = a'|\zeta_1\rangle$ with $a' \in \mathcal{A}'$ such that $\|a'\| \leqslant 1$. The second relation then follows from invariance of the $L_q(\mathcal{A}', \psi)$-norm [4, 7], $\|\zeta_2\|_{q,\psi,\mathcal{A}'} = \|a'\zeta_1\|_{q,\psi,\mathcal{A}'} \leqslant \|a'\| \, \|\zeta_1\|_{q,\psi,\mathcal{A}'}$. Therefore, lem. 2 and thm. 1 with $q = 2s$ give

$$S(\omega_\psi|\omega_\eta) - S(\omega_\psi \circ T|\omega_\eta \circ T) \geqslant \frac{1-s}{s} \int_{-\infty}^{\infty} p(t) D_s(\omega_\psi \circ T \circ \alpha^t_{\eta,T}|\omega_\psi) \, dt. \tag{31}$$

Here, $D_s, s \in (1/2, 1)$ are the "sandwiched Renyi divergences" [37, 51], defined by

$$D_s(\omega_\zeta|\omega_\psi) = (s-1)^{-1} \ln \|\zeta\|^{2s}_{2s,\psi,\mathcal{A}'} \tag{32}$$

(norm relative to $\mathcal{A}'$). D_s is known to be jointly convex [20], thm. 3.6 (5)

$$\sum_i \lambda_i D_s(\omega_{\zeta_i}|\omega_{\psi_i}) \geqslant D_s(\sum_i \lambda_i \omega_{\zeta_i}| \sum_j \lambda_j \omega_{\psi_j}), \tag{33}$$

where the sum is finite and $\lambda_i \geqslant 0, \sum \lambda_i = 1$. This suggests that we can use again the Jensen inequality to pull the integral inside D_s in (31). We would get, for $s \in (1/2, 1)$

$$S(\omega_\psi|\omega_\eta) - S(\omega_\psi \circ T|\omega_\eta \circ T) \geqslant \frac{1-s}{s} D_s(\omega_\psi \circ T \circ \alpha|\omega_\psi). \tag{34}$$

To assess the potential use of this inequality it would be interesting to know whether the right side has an operational meaning in terms of hypothesis testing, as has been established for the sandwiched Renyi-divergence of index $s > 1$, see [36].

4 Examples

4.1 Relation with conditional expectation

Let $\mathcal{B} \subset \mathcal{A}$ be an inclusion of von Neumann factors. Consider the reference state σ fixed by a conditional expectation $E : \mathcal{A} \to \mathcal{B}$. We work out the estimate in Thm. 2 in this case. The rotated Petz map reads (see [16] and also [5], [38])

$$\alpha^t_\sigma = \iota \circ \varsigma^t_{\sigma;\mathcal{B}} \circ j_\mathcal{B} \mathrm{Ad}_{V^\star} j_\mathcal{A} \circ \varsigma^{-t}_{\sigma;\mathcal{A}} \tag{35}$$

Here $j_\mathcal{A} = \mathrm{Ad} J_\mathcal{A}$ and $\varsigma^t_{\sigma,\mathcal{A}}$ is the modular flow of σ on $\mathcal{A}$ etc. In the above expression $\mathcal{B}$ is represented on a different Hilbert space since we do not always have a common cyclic and separating vector for $\mathcal{A}, \mathcal{B}$. In particular in the case where σ is fixed by a conditional expectation, we do not have such a vector. V is the isometry that relates the two Hilbert spaces and is defined via $V(b\,|\eta_\mathcal{B}\rangle) = \iota(b)\,|\eta_\mathcal{A}\rangle$ (following [16] we distinguish these representations by writing explicitly the inclusion $\iota(\mathcal{B}) \subset \mathcal{A}$.) If σ is fixed by E then we can dramatically simplify the recovery map using Takesaki's theorem [47] to cancel all the modular operators. In particular one finds:

$$\alpha^t_\sigma = \iota \circ \mathrm{Ad}_{V^\star} = E \tag{36}$$

See, for example, Eq 3.3-3.5 of [17] for more details on this case (to compare to this paper send $\iota \to \beta$ and $\mathcal{A}, \mathcal{B} \to \mathcal{M}, \mathcal{N}$). The integral in the recovery channel (19) is now trivial.

It is well known that the Petz map for an inclusion reduces to a conditional expectation if the defining state is fixed by a conditional expectation [39]. This remains true for the rotated Petz map, def. 2. So the estimate becomes:

$$S(\rho\,|\,\sigma\circ E) - S(\rho|_{\mathcal{B}}\,|\,\sigma|_{\mathcal{B}}) \geqslant -\ln F(\rho\circ E, \rho)^2 \tag{37}$$

Notice however that this estimate can be derived in a more elementary manner. We have the well known equality (see [38]):

$$S(\rho\,|\,\sigma\circ E) - S(\rho|_{\mathcal{B}}\,|\,\sigma|_{\mathcal{B}}) = S(\rho\,|\rho\circ E) \tag{38}$$

So (37) simply follows from the monotonicity of the "sandwiched Renyi divergence" as a function of the Renyi parameter s [7].

4.2 Markov semi-groups

Let $\{T_t\}_{t\geqslant 0} : \mathcal{A} \to \mathcal{A}$ be a weakly *-continuous semi-group of 2-positive unital maps such that $T_0 = id$, leaving invariant a reference state σ. On physical grounds, further conditions are often imposed on such a "Markov semi-group" expressing a kind of detailed balance condition, see [30, 34, 33]. One such condition that is natural from a physical viewpoint in many cases is that

$$\sigma(a^* T_t(b)) = \sigma(\Theta(b^*) T_t \circ \Theta(a)) \quad (t \geqslant 0), \tag{39}$$

where Θ is some anti-linear anti-automorphism[3] of $\mathcal{A}$ leaving σ invariant, such that $\Theta^2 = id$, often having the interpretation of a parity or PCT map. This can be seen to imply that T_t commutes with the modular group ς^s_η of $(\mathcal{A}, \sigma)$, see [35], lem. 4.8. This further gives (by analytic continuation and the KMS property)

$$\sigma(a^* \varsigma_\sigma^{-s+i/2} \circ T_t \circ \varsigma^s_\sigma(b)) = \sigma(a^* T_t \circ \varsigma_\sigma^{+i/2}(b)) = \sigma(\Theta \circ T_t \circ \Theta(a^*)\varsigma_\sigma^{+i/2}(b)), \tag{40}$$

showing in view of (14) that the rotated Petz map of T_t is given by $\Theta \circ T_t \circ \Theta$.

Thus, in the present case, the recovery channel (19) for T_t is (for given $t \geqslant 0$) $\alpha = \Theta \circ T_t \circ \Theta$. Then thm. 2 gives a formula for the entropy production by the Markov process up to time t expressed by the following proposition:

Proposition 1. *Let $\{T_t\}_{t\geqslant 0} : \mathcal{A} \to \mathcal{A}$ be a Markov semi-group leaving invariant a normal state σ of the von Neumann algebra $\mathcal{A}$ fulfilling the above detailed balance condition. Then for all $t \geqslant 0$,*

$$S(\rho|\sigma) - S(\rho\circ T_t|\sigma) \geqslant -2\ln F(\rho\circ T_t \circ \Theta \circ T_t \circ \Theta, \rho). \tag{41}$$

5 Proof of theorem 1

The proof is along similar lines as that of paper I [16], thm. 1 but we take the opportunity to present alternative proofs in some cases to obtain simplifications. The argument

[3] This means $\Theta(\lambda a) = \bar{\lambda}\Theta(a), \Theta(a)\Theta(b) = \Theta(ba)$ for $a, b \in \mathcal{A}$. It actually suffices that Θ is a Jordan map, see [35] for details.

is divided into several steps. First we consider a special case involving an additional assumption. This is then removed in a second step.

We shall first assume that there exists $c > 0$ such that

$$\omega_\psi \leqslant c\omega_\eta. \tag{42}$$

By definition, since T is positivity-preserving, we then also have $\omega_\psi \circ T \leqslant c\omega_\eta \circ T$. Following the conventions introduced above around (10), we use notations such as $|\eta_\mathcal{B}\rangle, |\eta_\mathcal{A}\rangle$, in this proof, and similarly for $|\psi\rangle$. In the same spirit, we will write $\Delta_{\eta,\psi;\mathcal{B}} = \Delta_{\eta,\psi;\mathcal{B}}, \Delta_{\eta,\psi;\mathcal{A}} = \Delta_{\eta_\mathcal{A},\psi_\mathcal{A}}$ etc.

Throughout this proof, we will repeatedly make use of the following well-known notations and facts related to non-negative, self-adjoint operators.

1. If A is a non-negative, self-adjoint operator with domain $\mathscr{D}(A)$ on a Hilbert space $\mathscr{H}$, then complex powers A^z are defined via the spectral theorem on $s(A)\mathscr{H}$ and by 0 on $(1 - s(A))\mathscr{H}$. Here $s(A)$ is the support projection of A.

2. If A, B are two non-negative self-adjoint operators on $\mathscr{H}$, we write $A \leqslant B$ if $\mathscr{D}(B^{1/2}) \subset \mathscr{D}(A^{1/2})$ and $\langle\zeta|A\zeta\rangle \leqslant \langle\zeta|B\zeta\rangle$ for $|\zeta\rangle \in \mathscr{D}(B^{1/2})$.

3. In this situation, we have $\mathscr{R}(A^{1/2}) \subset \mathscr{D}(B^{-1/2})$ and $\|B^{-1/2}A^{1/2}\zeta\| \leqslant \|\zeta\|$ for all $|\zeta\rangle \in \mathscr{D}(A^{1/2})$.

4. Let $\mathbb{S}_{1/2} = \{z \in \mathbb{C} \mid 0 < \mathrm{Re}(z) < 1/2\}$. Suppose $|F(z)\rangle$ is an $\mathscr{H}$-valued function on $\bar{\mathbb{S}}_{1/2}$ (overline denoting the closure) which is holomorphic on $\mathbb{S}_{1/2}$ and bounded and weakly continuous on $\bar{\mathbb{S}}_{1/2}$. Let A be a non-negative, self-adjoint operator on $\mathscr{H}$ such that $\sup_{t\in\mathbb{R}} \|A^{1/2+it}F(1/2 + it)\| = C_1 < \infty, \sup_{t\in\mathbb{R}} \|A^{it}F(it)\| = C_0 < \infty$. Then $A^z|F(z)\rangle$ is an $\mathscr{H}$-valued function on the strip $\bar{\mathbb{S}}_{1/2}$ which is holomorphic on $\mathbb{S}_{1/2}$, bounded in norm (by $\max\{C_0, C_1\}$) and weakly continuous on $\bar{\mathbb{S}}_{1/2}$.

Item 3) follows straightforwardly from the definitions and item 4) is shown in [11], lem. 2.1.

Step 1): First, we analyze some properties of the family of operators appearing in (17). The following lemma does not use the majorization condition (42) yet.

Lemma 3. *The operator* $\Delta^z_{\eta,\psi;\mathcal{A}} V_\psi \Delta^{-z}_{\eta,\psi;\mathcal{B}}$ *satisfies* $\|\Delta^z_{\eta,\psi;\mathcal{A}} V_\psi \Delta^{-z}_{\eta,\psi;\mathcal{B}}\| \leqslant 1$ *for* $z \in \bar{\mathbb{S}}_{1/2}$.

Proof. It follows from the Schwarz inequality (9) that $V^*_\psi \Delta_{\eta,\psi;\mathcal{A}} V_\psi \leqslant \Delta_{\eta,\psi;\mathcal{B}}$, see [42], proof of thm. 4. Let $\alpha \in [0, 1]$. By the operator monotonicity of the function $\mathbb{R}_+ \ni x \mapsto x^\alpha \in \mathbb{R}_+$, it follows $(V^*_\psi \Delta_{\eta,\psi;\mathcal{A}} V_\psi)^\alpha \leqslant \Delta^\alpha_{\eta,\psi;\mathcal{B}}$, see e.g. [43], Ex. 24, p. 315. Furthermore, since V_ψ is a contraction it follows from Jensen's operator inequality (see e.g. [38], lem. 1.2, or [42], thm. C) $V^*_\psi \Delta^\alpha_{\eta,\psi;\mathcal{A}} V_\psi \leqslant (V^*_\psi \Delta_{\eta,\psi;\mathcal{A}} V_\psi)^\alpha$, so $V^*_\psi \Delta^\alpha_{\eta,\psi;\mathcal{A}} V_\psi \leqslant \Delta^\alpha_{\eta,\psi;\mathcal{B}}$. This is the same as saying that $\|\Delta^{\alpha/2}_{\eta,\psi;\mathcal{A}} V_\psi \Delta^{-\alpha/2}_{\eta,\psi;\mathcal{B}}\zeta\| \leqslant \|\zeta\|$ for $|\zeta\rangle \in \mathscr{R}(\Delta^{\alpha/2}_{\eta,\psi;\mathcal{B}}) \oplus (1 - \pi^{\mathcal{B}'}(\psi_\mathcal{B})\pi^\mathcal{B}(\eta_\mathcal{B}))\mathscr{H}$ which is dense. The statement follows because $\Delta^{it}_{\eta,\psi;\mathcal{A}}, \Delta^{it}_{\eta,\psi;\mathcal{B}}$ are isometric when t is real. $\square$

Similarly as in paper I, we introduce a vector-valued function by

$$|\Gamma_\psi(z)\rangle \equiv \Delta^z_{\eta,\psi;\mathcal{A}} V_\psi \Delta^{-z}_{\eta,\psi;\mathcal{B}} |\psi_\mathcal{B}\rangle \in \mathscr{H}. \tag{43}$$

This vector-valued function is well-defined by the preceding lemma at first for any $z \in \bar{\mathbb{S}}_{1/2}$. In fact, it is analytic in $\mathbb{S}_{1/2}$ and weakly continuous and bounded on $\bar{\mathbb{S}}_{1/2}$, by the same arguments as in paper I [16], thm. 4 (1), or the next lemma. With the goal of extending the domain of analyticity of this vector, we let Π_Λ be the projector onto the spectral subspace of the modular operator $\ln \Delta_{\psi_\mathcal{A}}$, associated with the interval $(-\Lambda, \Lambda)$, for some large "cutoff" Λ which we will let got to infinity. Then we define the new vector valued function $\Pi_\Lambda|\Gamma_\psi(z)\rangle$, at first on the one-sided strip $\bar{\mathbb{S}}_{1/2}$. The following two lemmas, which also rely on the majorization condition (42), give an analytic continuation to the two-sided strip $\{z : -1/2 < \mathrm{Re} z < 1/2\}$. They express the same as paper I, lem. 6 (1,2) but we give a somewhat different proof.

Lemma 4. *The function $\Pi_\Lambda|\Gamma_\psi(z)\rangle$ has a unique extension to $\{z : -1/2 \leqslant \mathrm{Re} z \leqslant 1/2\}$ which is weakly continuous and bounded and which is holomorphic in the interior $\{z : -1/2 < \mathrm{Re} z < 1/2\}$.*

Proof. We consider separately the following cases:

(1) $1/2 \geqslant \mathrm{Re} z \geqslant 0$: Then by lem. 5, $(D\eta_\mathcal{B} : D\psi_\mathcal{B})_{iz} \in \mathcal{B}$, and we may write, at first formally,

$$\Pi_\Lambda|\Gamma_\psi(z)\rangle = \Pi_\Lambda \Delta^z_{\eta,\psi;\mathcal{A}}(T[(D\eta_\mathcal{B} : D\psi_\mathcal{B})_{iz}]|\psi\rangle) \tag{44}$$

using the definition of V_ψ and of the (formally analytically extended) Connes cocycle (6), $(D\eta_\mathcal{B} : D\psi_\mathcal{B})_{iz}\pi^{\mathcal{B}'}(\psi) = \Delta^{-z}_{\eta,\psi;\mathcal{B}}\Delta^z_{\psi,\mathcal{B}}$. We analyze the three factors on the right side as follows. Π_Λ is simply a bounded, z-independent operator. By lem. 5, $|F(z)\rangle = T[(D\eta_\mathcal{B} : D\psi_\mathcal{B})_{iz}]|\psi\rangle$ is bounded and weakly continuous for $z \in \bar{\mathbb{S}}_{1/2}$ and holomorphic for $z \in \mathbb{S}_{1/2}$. By lem. 3 (or alternatively Tomita-Takesaki theory, since $T[(D\eta_\mathcal{B} : D\psi_\mathcal{B})_{iz}] \in \mathcal{A}$), $\|\Delta^z_{\eta,\psi;\mathcal{A}}F(z)\|$ is bounded for $z \in \bar{\mathbb{S}}_{1/2}$. Thus, item 4) (which is [11], lem. 2.1) shows that $\Pi_\Lambda|\Gamma_\psi(z)\rangle$ is holomorphic for $z \in \mathbb{S}_{1/2}$ as well as weakly continuous and bounded for $z \in \bar{\mathbb{S}}_{1/2}$.

(2) $-1/2 \leqslant \mathrm{Re} z \leqslant 0$: We may write, at first formally,

$$\Pi_\Lambda|\Gamma_\psi(z)\rangle = (\Pi_\Lambda\Delta^z_{\psi_\mathcal{A}})(D\eta_\mathcal{A} : D\psi_\mathcal{A})^*_{-i\bar{z}}(V_\psi\Delta^{-z}_{\eta_\mathcal{B},\psi_\mathcal{B}}|\psi_\mathcal{B}\rangle). \tag{45}$$

We analyze the three factors on the right side as follows. By construction, $\Pi_\Lambda\Delta^z_{\psi_\mathcal{A}}$ is a holomorphic family of operators for $z \in -\mathbb{S}_{1/2}$, as well strongly continuous and norm bounded, with operator norm not exceeding $e^{2\Lambda|\mathrm{Re} z|}$. $(D\eta_\mathcal{A} : D\psi_\mathcal{A})^*_{-i\bar{z}}$ is holomorphic for $z \in -\mathbb{S}_{1/2}$, as well as weakly continuous and norm-bounded for $z \in -\bar{\mathbb{S}}_{1/2}$ by lem. 5. $\Delta^{-z}_{\eta,\psi;\mathcal{B}}|\psi_\mathcal{B}\rangle$ is holomorphic for $z \in -\mathbb{S}_{1/2}$, as well as strongly continuous and norm-bounded for $z \in -\bar{\mathbb{S}}_{1/2}$, by Tomita-Takesaki theory. This shows that $\Pi_\Lambda|\Gamma_\psi(z)\rangle$ is holomorphic for $z \in -\mathbb{S}_{1/2}$ as well as weakly continuous and bounded for $z \in -\bar{\mathbb{S}}_{1/2}$.

Cases (1) and (2) together give the statement by an application of the operator-valued edge-of-the-wedge theorem because the weakly continuous functions defined in cases (1) and (2) clearly match for $z = it, t \in \mathbb{R}$. □

Lemma 5. *Suppose $\mathcal{A}$ is a von Neumann algebra in standard form and $|\eta\rangle, |\psi\rangle$ are vectors in the natural cone such that $\omega_\psi \leqslant c\omega_\eta$ for some $c < \infty$. Then $z \mapsto \Delta^{-z}_{\eta,\psi}\Delta^z_\psi$ is holomorphic for $z \in \mathbb{S}_{1/2}$ as well as weakly continuous and norm-bounded in $\bar{\mathbb{S}}_{1/2}$, which defines an analytic extension $(D\eta : D\psi)_{iz}$ of the Connes-cocycle (6). For $z \in \bar{\mathbb{S}}_{1/2}$ one has $(D\eta : D\psi)_{iz} \in \pi^\mathcal{A}(\eta)\mathcal{A}\pi^\mathcal{A}(\psi)$.*

Proof. This fact is well known and we include the proof in appendix C only for completeness taking into account that the support projection of $|\psi\rangle$ may be non-trivial. □

Step 2): Let $1/p_\theta = (1-2\theta)/2 + 2\theta/q$ for $\theta \in [0, 1/2]$, which interpolates between $p_0 = 2, p_{1/2} = q \in [1, 2]$. Very similarly as in paper I, the relationship between the function (43) and the quantum data-processing inequality is that

$$\lim_{\theta \to 0^+} \left(-\frac{1}{\theta} \ln \|\Pi_\Lambda \Gamma_\psi(\theta)\|^2_{p_\theta, \psi, \mathcal{A}'} \right) = S(\psi_\mathcal{A}|\eta_\mathcal{A}) - S(\psi_\mathcal{B}|\eta_\mathcal{B}). \tag{46}$$

Here and in the following, $S(\psi_\mathcal{A}|\eta_\mathcal{A}) \equiv S_\mathcal{A}(\omega_\psi|\omega_\eta)$ and $S(\psi_\mathcal{B}|\eta_\mathcal{B}) \equiv S_\mathcal{B}(\omega_\psi \circ T|\omega_\eta \circ T)$ are used as a shorthand. Furthermore, on the left side of (46), we mean the Araki-Masuda $L_{p_\theta}(\mathcal{A}', \psi)$-norm, i.e. that relative to the commutant algebra $\mathcal{A}'$ and the vector $|\psi\rangle$ in the natural cone. In the following, we will often drop the subscript $\mathcal{A}'$ to simplify the notation.

For finite dimensional quantum systems, (46) is a straightforward consequence of the definitions. In the general von Neumann context, we could prove (46) in basically the same way as in paper I, but here we can take certain shortcuts since we are not interested in reproving paper I, thm. 3: Setting first $|\zeta_\theta\rangle = \Pi_\Lambda|\Gamma_\psi(\theta)\rangle / \|\Pi_\Lambda \Gamma_\psi(\theta)\|$, the homogeneity of the L_p-norm gives

$$\begin{aligned} \lim_{\theta \to 0^+} \left(-\frac{1}{\theta} \ln \|\Pi_\Lambda \Gamma_\psi(\theta)\|^2_{p_\theta, \psi} \right) &= \lim_{\theta \to 0^+} \left(-\frac{1}{\theta} \ln \|\Pi_\Lambda \Gamma_\psi(\theta)\|^2 \right) + \lim_{\theta \to 0^+} \left(-\frac{1}{\theta} \ln \|\zeta_\theta\|^2_{p_\theta, \psi} \right) \\ &=: A + B. \end{aligned} \tag{47}$$

Note that $\lim_{\theta \to 0} \|\zeta_\theta - \psi\|^2/\theta = 0$ by construction as $\Pi_\Lambda|\Gamma_\psi(z)\rangle$ is strongly holomorphic at $z = 0$ (by lem. 4) and $\Pi_\Lambda|\Gamma_\psi(0)\rangle = |\psi\rangle$ by definition. The term B is treated with the next two lemmas playing the same role as paper I, lem. 4 and thm. 3 (2).

Lemma 6. *Let $\mathcal{M}$ be a von Neumann algebra in standard form acting on $\mathscr{H}$. For two vectors $|\zeta\rangle, |\psi\rangle \in \mathscr{H}$ such that $|\psi\rangle$ is in the natural cone, $\|\psi\| = 1$, and $p \in [1, 2]$, we have*

$$|\langle \zeta|\psi\rangle| \leqslant \|\zeta\|_{p,\psi} \leqslant \|\zeta\|, \tag{48}$$

where the L_p-norm is relative to $\mathcal{M}$.

Proof. (1) First inequality. The statement holds trivially for $p = 2$, because $\|\zeta\|_{2,\psi} = \|\pi^{\mathcal{M}'}(\psi)\zeta\|$, so

$$|\langle \zeta|\psi\rangle| = |\langle \zeta|\pi^{\mathcal{M}'}(\psi)\psi\rangle| \leqslant \|\pi^{\mathcal{M}'}(\psi)\zeta\| = \|\zeta\|_{2,\psi}. \tag{49}$$

For $p = 1$, the inequality follows by the "sup" variational definition of the fidelity (20) (relative to $\mathcal{M}'$), which is equal to the L_1-norm, as proven in the generality needed here in paper I, lem. 3 (1). So we can assume $p \in (1, 2)$ in the rest of the proof.

Let $1/p' = 1 - 1/p$ be the dual Hölder index, so $p' \in (2, \infty)$, $1/p - 1/2 =: \alpha \in (0, 1/2)$, $1/2 - 1/p' =: \alpha' \in (0, 1/2)$. For any fixed $\varepsilon > 0$, we can chose a unit vector $|\phi\rangle$ so that $\pi^\mathcal{M}(\phi) \geqslant \pi^\mathcal{M}(\zeta)$ and $\|\Delta^{-\alpha}_{\phi,\psi}\zeta\| \leqslant \|\zeta\|_{p,\psi} + \varepsilon$ by the variational definition of the L_p norm.

Then we have

$$
\begin{aligned}
|\langle\zeta|\psi\rangle| =&|\langle\Delta_{\phi,\psi}^{-\alpha}\zeta|\Delta_{\phi,\psi}^{\alpha'}\psi\rangle| \\
\leqslant&(\|\zeta\|_{p,\psi}+\varepsilon)\|\Delta_{\phi,\psi}^{\alpha'}\psi\| \\
\leqslant&(\|\zeta\|_{p,\psi}+\varepsilon)\sup\{\|\Delta_{\phi,\psi}^{it}\psi\|,\|\Delta_{\phi,\psi}^{1/2+it}\psi\|\}_{t\in\mathbb{R}} \\
=&\|\zeta\|_{p,\psi}+\varepsilon,
\end{aligned}
\tag{50}
$$

using [11], lem. 2.1 to go to the third line (basically the Hadamard three lines theorem). This proves (2) in the remaining cases since $\varepsilon>0$ is arbitrary.

(2) Second inequality. See e.g. [7], lem. 8, or the following explicit and simple argument. For $p=1$ or $p=2$, the inequality is trivial, by the similar arguments as in step (1). We can assume that $|\zeta\rangle$ is in the natural cone because the L_p-norm and the Hilbert space norm are unchanged if we replace $|\zeta\rangle$ by its unique representer in the natural cone, see e.g. app. B, and by rescaling and homogneity of the L_p norm we can assume $\|\zeta\|=1$. Then $J\Delta_{\zeta,\psi}^{-\alpha}J=\Delta_{\psi,\zeta}^{\alpha}$ see [4], thm. C.1, and we get from the "inf" variational definition of the L_p-norm that

$$
\|\zeta\|_{p,\psi}\leqslant\|\Delta_{\zeta,\psi}^{-\alpha}\zeta\|=\|J\Delta_{\psi,\zeta'}^{\alpha}J\zeta'\|=\|\Delta_{\psi,\zeta}^{\alpha}\zeta\|. \tag{51}
$$

We have $\|\Delta_{\psi,\zeta}^{it}\zeta\|\leqslant\|\zeta\|=1$ and $\|\Delta_{\psi,\zeta'}^{1/2+it}\zeta\|=\|J\Delta_{\psi,\zeta}^{1/2}\zeta\|=\|\psi\|=1$ for $t\in\mathbb{R}$. Therefore, by [11], lem. 2.1, $\|\Delta_{\psi,\zeta}^{\alpha}\zeta\|\leqslant 1$ and the proof is complete. □

In view of $\mathrm{Re}\langle\zeta|\psi\rangle\leqslant|\langle\zeta|\psi\rangle|$, the inequalities of lem. 6 give for $p\in[1,2]$, $\|\zeta\|=\|\psi\|=1$

$$
\frac{1}{2}\|\psi-\zeta\|^2=1-\mathrm{Re}\langle\zeta|\psi\rangle\geqslant 1-\|\zeta\|_{p,\psi}\geqslant 0. \tag{52}
$$

This immediately implies the following lemma

Lemma 7. *Let $\mathcal{M}$ be a von Neumann algebra in standard form acting on $\mathscr{H}$, let $|\psi\rangle$ be a unit vector in the natural cone and let $|\zeta_\theta\rangle\in\mathscr{H}$ be a 1-parameter family of unit vectors such that*

$$
\lim_{\theta\to 0}\frac{\|\zeta_\theta-\psi\|^2}{\theta}=0. \tag{53}
$$

Then we have

$$
\lim_{\theta\to 0}\frac{1}{\theta}\ln\|\zeta_\theta\|_{p_\theta,\psi}=0 \tag{54}
$$

for any 1-parameter family $p_\theta\in[1,2]$, where the L_{p_θ}-norm is relative to $\mathcal{M}$.

Proof. This follows from (52) because if we substitute $|\zeta\rangle\to|\zeta_\theta\rangle, p\to p_\theta$, divide by θ, then we see that $\lim_{\theta\to 0}\theta^{-1}|\ \|\zeta_\theta\|_{p_\theta,\psi}-1\ |=0$. □

Lem. 7 applied to $|\zeta_\theta\rangle$ and $\mathcal{M}=\mathcal{A}'$ implies that $B=0$ in (47). Now we study the term A in that equation. By the chain rule – noting that the derivative exists in view of lem. 4 –

$$
\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}\theta}\|\Pi_\Lambda\Gamma_\psi(\theta)\|^2\Big|_{\theta=0} &= -2\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{Re}\langle\psi_{\mathcal{A}}|\Pi_\Lambda\Gamma_\psi(\theta)\rangle\Big|_{\theta=0} \\
&= -2i\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi_{\mathcal{B}}|(D\eta_{\mathcal{A}}:D\psi_{\mathcal{A}})_t^{-1}V_\psi(D\eta_{\mathcal{B}}:D\psi_{\mathcal{B}})_t\psi_{\mathcal{B}}\rangle\Big|_{t=0}.
\end{aligned}
\tag{55}
$$

Here we use the Connes-cocycle (6) for $t \in \mathbb{R}$, and we used that the derivative can be also calculated by analytic continuation $\theta \to it$ by analyticity in view of lem. 4. (Here and below, the inverse Connes-cocycle is defined in view of [4], thm. C.1 ($\gamma 1, \gamma 3$), as $(D\eta_{\mathcal{A}} : D\psi_{\mathcal{A}})_t^{-1} = (D\eta_{\mathcal{A}} : D\psi_{\mathcal{A}})_t^* \in \pi^{\mathcal{A}}(\psi)\mathcal{A}$, considering $(D\eta_{\mathcal{A}} : D\psi_{\mathcal{A}})_t$ as an operator with domain $\pi^{\mathcal{A}}(\psi)\mathscr{H}$.)

The Connes-cocycles $(D\eta_{\mathcal{A}} : D\psi_{\mathcal{A}})_t^{-1}, (D\eta_{\mathcal{B}} : D\psi_{\mathcal{B}})_t$ ($t \in \mathbb{R}$) are elements of $\mathcal{A}, \mathcal{B}$. We are in particular allowed to use the definition of V_ψ, eq. (15), and $\Delta^{\theta}_{\psi,\mathcal{A}}|\psi_{\mathcal{A}}\rangle = |\psi_{\mathcal{A}}\rangle$. We obtain:

$$-\frac{\mathrm{d}}{\mathrm{d}\theta}\|\Pi_\Lambda \Gamma_\psi(\theta)\|^2\bigg|_{\theta=0} = -2i\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi_{\mathcal{A}}|(D\eta_{\mathcal{A}} : D\psi_{\mathcal{A}})_t^{-1} T[(D\eta_{\mathcal{B}} : D\psi_{\mathcal{B}})_t]\psi_{\mathcal{A}}\rangle\bigg|_{t=0}. \tag{56}$$

Finally, we distribute the derivatives on the right side and apply the definition of the relative entropy in terms of the Connes-cocycle (7), showing in view of (47) and $B = 0$ that (46) holds.

Step 3): In view of step 2), we should next look for an upper bound on $\|\Pi_\Lambda \Gamma_\psi(\theta)\|_{p_\theta,\psi}$ (with L_p-norm relative to $\mathcal{A}'$ and the vector $|\psi\rangle$ in the natural cone of $\mathcal{A}$). The basic tool is the following lemma.

Lemma 8. *Let $\mathcal{M}$ be a von Neumann algebra in standard form acting on $\mathscr{H}$, $|\psi\rangle$ a not necessarily cyclic and separating vector in the natural cone. Let $|G(z)\rangle$ be a $\mathscr{H}$-valued holomorphic function on the strip $\mathbb{S}_{1/2} = \{0 < \mathrm{Re} z < 1/2\}$ that is uniformly bounded and weakly continuous in the closure $\bar{\mathbb{S}}_{1/2}$. For $0 < \theta < 1/2$ with $p_0, p_{1/2} \in [1, 2]$, put*

$$\frac{1}{p_\theta} = \frac{1-2\theta}{p_0} + \frac{2\theta}{p_1} \tag{57}$$

Then we have

$$\begin{aligned} &\ln \|G(\theta)\|_{p_\theta,\psi} \\ &\leqslant \int_{-\infty}^{\infty} \mathrm{d}t \left((1-2\theta)\alpha_\theta(t) \ln \|G(it)\|_{p_0,\psi} + (2\theta)\beta_\theta(t) \ln \|G(1/2+it)\|_{p_{1/2},\psi} \right), \end{aligned} \tag{58}$$

where

$$\alpha_\theta(t) = \frac{\sin(2\pi\theta)}{(1-2\theta)(\cosh(2\pi t) - \cos(2\pi\theta))}, \qquad \beta_\theta(t) = \frac{\sin(2\pi\theta)}{2\theta(\cosh(2\pi t) + \cos(2\pi\theta))}. \tag{59}$$

A proof of this lemma has been given in paper I, lem. 9 in the case when $|\psi\rangle$ is cyclic and separating. This condition has been removed in [22].
In view of lem. 4, we can apply lem. 8 to $|G(z)\rangle = \Pi_\Lambda|\Gamma_\psi(z)\rangle$, $p_0 = 2, p_{1/2} = q$ and $\mathcal{M} = \mathcal{A}'$. We can furthermore use that

$$\|\Pi_\Lambda \Gamma_\psi(it)\|_{2,\psi,\mathcal{A}'} = \|\pi^{\mathcal{A}}(\psi)\Pi_\Lambda \Gamma_\psi(it)\| \leqslant \|\Delta^{it}_{\eta,\psi;\mathcal{A}} V_\psi \Delta^{-it}_{\eta,\psi;\mathcal{B}} \psi_{\mathcal{B}}\| \leqslant \|V_\psi\| \leqslant 1, \tag{60}$$

since V_ψ is a contraction. Then we find using (46) that

$$\begin{aligned} &S(\psi_{\mathcal{A}}|\eta_{\mathcal{A}}) - S(\psi_{\mathcal{B}}|\eta_{\mathcal{B}}) \\ &\geqslant -2\pi \int_{-\infty}^{\infty} [1+\cosh(2\pi t)]^{-1} \ln \|\Pi_\Lambda \Delta^{1/2+it}_{\eta,\psi;\mathcal{A}} V_\psi \Delta^{-1/2-it}_{\eta,\psi;\mathcal{B}} \psi_{\mathcal{B}}\|_{q,\psi,\mathcal{A}'} \mathrm{d}t. \end{aligned} \tag{61}$$

We can let $\Lambda \to \infty$, using the continuity of the L_q-norm in the norm-topology on the Hilbert space $\mathscr{H}$, for $1 \leqslant q \leqslant 2$, [4], lem. 6.1 (2), thus allowing us to replace Π_Λ with 1. This completes the proof of the theorem under the majorization hypothesis (42).

Step 4): We shall now remove the majorization hypothesis (42) by the same method as in paper I. This hypothesis was crucially used only in lem. 5, which in turn was used in order to show that the regularized vector $\Pi_\Lambda|\Gamma_\psi(z)\rangle$ is holomorphic near $z = 0$ as proven in lem. 4. Since we now no longer want to assume the majorization hypothesis, we shall introduce a "Gaussian" regularization. For $P > 0$, we consider the scaled Gaussians

$$\hat{g}_P(x) = e^{-x^2/(2P)} \quad \Longrightarrow \quad g_P(k) = \sqrt{P/\pi}e^{-Pk^2/2}. \tag{62}$$

Then we define a (not necessarily normalized) vector $|\psi_{P\mathcal{A}}\rangle$ in the natural cone of $\mathcal{A}$ by its associated state functional ($a \in \mathcal{A}$)

$$\omega_{\psi_{P\mathcal{A}}}(a) := \langle \hat{g}_P(\ln \Delta_{\eta,\psi;\mathcal{A}})\psi_\mathcal{A}|a\hat{g}_P(\ln \Delta_{\eta,\psi;\mathcal{A}})\psi_\mathcal{A}\rangle. \tag{63}$$

Likewise, we define a vector $|\psi_{P\mathcal{B}}\rangle$ in the natural cone of $\mathcal{B}$ by its associated state functional ($b \in \mathcal{B}$) $\omega_{\psi_{P\mathcal{B}}}(b) = \omega_{\psi_{P\mathcal{A}}}(T(b))$. It follows from the spectral theorem that $\hat{g}_P(\ln \Delta_{\eta,\psi;\mathcal{A}})|\psi_\mathcal{A}\rangle \to |\psi_\mathcal{A}\rangle$, so $\omega_{\psi_{P\mathcal{A}}} \to \omega_{\psi_\mathcal{A}}$ as $P \to \infty$ in norm. The corresponding vectors in the natural cone then satisfy $\|\psi_\mathcal{A} - \psi_{P\mathcal{A}}\| \to 0$, in view of the well-known inequality $\|\zeta - \eta\|\|\zeta + \eta\| \geqslant \|\omega_\zeta - \omega_\eta\| \geqslant \|\zeta - \eta\|^2$ for vectors $|\zeta\rangle, |\eta\rangle$ in the natural cone. By the same argument then also $\omega_{\psi_{P\mathcal{B}}} \to \omega_{\psi_\mathcal{B}}$ in norm, hence also $\|\psi_\mathcal{B} - \psi_{P\mathcal{B}}\| \to 0$.

By lem. 9 in app. A, the analysis of the previous steps 1)–3) is applicable and the theorem (i.e. (61)) holds for ω_{ψ_P} in place of ω_ψ. Furthermore, lem. 9 5) shows that $\limsup_{P\to\infty} S(\psi_{P\mathcal{A}}|\eta_\mathcal{A}) = S(\psi_\mathcal{A}|\eta_\mathcal{A})$. On the other hand, the usual lower semi-continuity of the relative entropy [3], thm. 3.7 (1) gives us $S(\psi_\mathcal{B}|\eta_\mathcal{B}) \leqslant \liminf_{P\to\infty} S(\psi_{P\mathcal{B}}|\eta_\mathcal{B})$, and we see that

$$\begin{aligned} & S(\psi_\mathcal{A}|\eta_\mathcal{A}) - S(\psi_\mathcal{B}|\eta_\mathcal{B}) \\ \geqslant & -2\pi \limsup_{P\to\infty} \int_{-\infty}^{\infty} [1 + \cosh(2\pi t)]^{-1} \ln \|\Gamma_{\psi_P}(1/2 + it)\|_{q,\psi_P,\mathcal{A}'} \mathrm{d}t \\ \geqslant & -2\pi \int_{-\infty}^{\infty} [1 + \cosh(2\pi t)]^{-1} \limsup_{P\to\infty} \left(\ln \|\Gamma_{\psi_P}(1/2 + it)\|_{q,\psi_P,\mathcal{A}'} \right) \mathrm{d}t. \end{aligned} \tag{64}$$

In the second line we used the Fatou lemma and the fact $\ln \|\Gamma_{\psi_P}(1/2 + it)\|_{q,\psi_P,\mathcal{A}'} \leqslant \ln \|\Gamma_{\psi_P}(1/2 + it)\| \leqslant 0$ which follows from (17), $\|V_{\eta,P}\| \leqslant 1$ [cf. (23) with the replacement $\psi \to \psi_P$], $\|\psi_P\| \leqslant \|\psi\| = 1$ [cf. lem. 9 1) in app. A] and the fact that the Hilbert space norm dominates the L_q-norm [cf. lem. 6].

Step 5): We now analyze the lim-sup on the right side of (64). What must be shown is that the lower bound in (64) is not smaller than the lower bound (18) in the statement of the theorem. There are two sub-steps: (i) showing that $\lim_P |\Gamma_{\psi_P}(1/2+it)\rangle = |\Gamma_\psi(1/2+it)\rangle$ in the strong sense for any $t \in \mathbb{R}$, and (ii) showing that $\limsup_P \|\zeta\|_{q,\psi_P,\mathcal{A}'} \leqslant \|\zeta\|_{q,\psi,\mathcal{A}'}$ for any $|\zeta\rangle \in \mathscr{H}$. Using these two sub-steps and the shorthands $|\Gamma_{\psi_P}(1/2 + it)\rangle \equiv |\zeta_P\rangle, |\Gamma_\psi(1/2 + it)\rangle \equiv |\zeta\rangle$, we conclude

$$\limsup_{P\to\infty} \|\zeta_P\|_{q,\psi_P,\mathcal{A}'} \leqslant \limsup_{P\to\infty} (\|\zeta\|_{q,\psi_P,\mathcal{A}'} + \|\zeta - \zeta_P\|_{q,\psi_P,\mathcal{A}'}) \leqslant \|\zeta\|_{q,\psi,\mathcal{A}'}. \tag{65}$$

Here, we used the triangle inequality for the L_p-norm in the range $p \in [1,2]$ [cf. app. B] and $\|\zeta - \zeta_P\|_{q,\psi_P,\mathcal{A}'} \to 0$ because the L_p-norm is dominated by the Hilbert space norm [cf. lem. 6] – together with (i) and (ii). (65) and (64) together give the statement (18) of the theorem. Thus, to conclude the proof, we must demonstrate sub-steps (i) and (ii).

Sub-step (i): [3], lem. 4.1 shows that $\Delta^{1/2}_{\psi_P,\eta;\mathcal{A}} \to \Delta^{1/2}_{\psi,\eta;\mathcal{A}}$ in the strong resolvent sense, using also that $\pi^{\mathcal{B}'}(\eta) = \pi^{\mathcal{A}'}(\eta) = 1$ since $|\eta_{\mathcal{A}}\rangle, |\eta_{\mathcal{B}}\rangle$ are cyclic and separating by assumption. Therefore, for any bounded continuous function $f : \mathbb{R}_+ \to \mathbb{C}$, $f(\Delta_{\psi_P,\eta;\mathcal{A}}) \to f(\Delta_{\psi,\eta;\mathcal{A}})$ strongly by [14], prop. 10.1.9, and therefore, for example, $(i + \ln \Delta_{\psi_P,\eta;\mathcal{A}})^{-1} \to (i + \ln \Delta_{\psi,\eta;\mathcal{A}})^{-1}$ in the strong sense. By [14], prop. 10.1.8, $\Delta^{it}_{\psi_P,\eta;\mathcal{A}} \to \Delta^{it}_{\psi,\eta;\mathcal{A}}$ strongly, and likewise for the expressions with $\mathcal{B}$ in place of $\mathcal{A}$. The representation (24) can be stated using results of [4], thm. C.1 and the support properties of the modular operators as

$$|\Gamma_{\psi_P}(1/2 + it)\rangle = J_{\mathcal{A}} \Delta^{-it}_{\psi_P,\eta;\mathcal{A}} W_\eta \Delta^{it}_{\psi_P,\eta;\mathcal{B}} |\psi_{P\mathcal{B}}\rangle \tag{66}$$

with $W_\eta(b|\eta_{\mathcal{B}}\rangle) := T(b)|\eta_{\mathcal{A}}\rangle$, which is a well defined contraction that does not depend on P. We therefore have $|\Gamma_{\psi_P}(1/2 + it)\rangle \to |\Gamma_\psi(1/2 + it)\rangle$ strongly because each factor in this expression converges strongly.

Sub-step (ii): This is lem. 9 4) in app. A.

□

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A Properties of $|\psi_P\rangle$

The following lemma summarizes key properties of $|\psi_P\rangle$ which was defined by $v'_P \hat{g}_P(\ln \Delta_{\eta,\psi})|\psi\rangle = |\psi_P\rangle$, where $v'_P \in \mathcal{A}'$ is the partial isometry which puts $|\psi_P\rangle$ into the natural cone.

We have already noted that $\omega_{\psi_P} \to \omega_\psi$ as $P \to \infty$ in norm. Further properties are (here $\Delta_{\eta,\psi}$ refers to the algebra $\mathcal{A}$):

Lemma 9. *1. There is $a_P \in \mathcal{A}$ such that $|\psi_P\rangle = v'_P a_P |\psi\rangle$ and $\|a_P\| \leqslant 1$ for all P.*

2. For $0 \leqslant \alpha \leqslant 1$ and any $|\eta\rangle$ in the natural cone of $\mathcal{A}$, we have

$$\Delta^\alpha_{\psi_P,\eta} \geqslant a_P \Delta^\alpha_{\psi,\eta} a^*_P \tag{67}$$

3. For some $c_P < \infty$, we have $\omega_{\psi_P} \leqslant c_P \omega_\eta$.

4. $\lim_{P\to\infty} \|\zeta\|_{p,\psi_P,\mathcal{A}'} \leqslant \|\zeta\|_{p,\psi,\mathcal{A}'}$ for any $|\zeta\rangle \in \mathscr{H}$ and $p \in [1,2]$.

5. *For $|\eta\rangle$ in the natural cone, we have $\lim_{P\to\infty} S_{\mathcal{A}}(\psi_P|\eta) = S_{\mathcal{A}}(\psi|\eta)$.*

Proof. 1) This follows from paper I [16], lem. 5 (1).

2) An easy calculation gives that $\Delta_{\psi_P,\eta} = a_P \Delta_{\psi,\eta} a_P^*$. Then, since $\|a_P\| \leqslant 1$ by 1), it is legitimate to use Jensen's inequality (see e.g. [38], lem. 1.2, or [42], thm. C) applied to the operator monotone function x^α. This gives the desired inequality.

3) This follows from paper I [16], lem. 5 (4,5).

4) This follows from the continuity property of the L_p-norms in the range $p \in [1,2]$, see app. B and the fact that $\|\psi - \psi_P\| \to 0$.

5) This follows from paper I [16], thm. 5. Alternatively, we can argue using 2) starting with the following representation of the relative entropy:

$$S_{\mathcal{A}}(\psi_P|\eta) = \lim_{\alpha\to 1^-} \frac{1}{\alpha-1}\{\ln\langle\eta|\Delta^\alpha_{\psi_P,\eta}\eta\rangle - \ln\langle\eta|\Delta_{\psi_P,\eta}\eta\rangle\}. \tag{68}$$

Now we use 2), and $\bar{S}_{\psi,\eta} = J\Delta^{1/2}_{\psi,\eta}$:

$$\begin{aligned} S(\psi_P|\eta) &\leqslant \lim_{\alpha\to 1^-} \frac{1}{\alpha-1} \ln \frac{\langle a_P^*\eta|\Delta^\alpha_{\psi,\eta} a_P^*\eta\rangle}{\|a_P\psi\|^2} \\ &= \lim_{\alpha\to 1^-} \frac{1}{\alpha-1} \ln \frac{\langle a_P\psi|J\Delta^{\alpha-1}_{\psi,\eta} J a_P\psi\rangle}{\|a_P\psi\|^2} \\ &= \lim_{\alpha\to 1^-} \frac{1}{\alpha-1} \ln \frac{\langle a_P\psi|\Delta^{1-\alpha}_{\eta,\psi} a_P\psi\rangle}{\|a_P\psi\|^2}. \end{aligned} \tag{69}$$

We write out again the definition of a_P in this expression, paper I [16], proof of lem. 5 (1), obtaining

$$\begin{aligned} S(\psi_P|\eta) &\leqslant -\lim_{\theta\to 0^+} \frac{1}{\theta} \ln \frac{\langle \hat{g}_P(\Delta_{\eta,\psi})\psi|\Delta^\theta_{\eta,\psi}\hat{g}_P(\Delta_{\eta,\psi})\psi\rangle}{\|\hat{g}_P(\Delta_{\eta,\psi})\psi\|^2} \\ &= -\langle \hat{g}_P(\Delta_{\eta,\psi})\psi|(\ln\Delta_{\eta,\psi})\hat{g}_P(\Delta_{\eta,\psi})\psi\rangle. \end{aligned} \tag{70}$$

The term in the second line can be written in terms of the spectral measure $E_{\eta,\psi}(k)\mathrm{d}k$ of $\ln\Delta_{\eta,\psi}$ and then we can argue just as in paper I [16], thm. 5 (2) to show that $\limsup_{P\to\infty} S(\psi_P|\eta) \leqslant S(\psi|\eta)$. The opposite relation $\liminf_{P\to\infty} S(\psi_P|\eta) \geqslant S(\psi|\eta)$ follows from the usual lower-semi continuity of the relative entropy [3]. □

B Weighted L_p-norms [4], [7]

The weighted L_p-norms and -spaces were defined and investigated by [4] relative to a fixed cyclic and separating vector $|\psi\rangle \in \mathscr{H}$ in the a natural cone of a standard representation of a von Neumann algebra $\mathcal{M}$. The generalization to arbitrary $|\psi\rangle \in \mathscr{H}$ in the a natural cone is given in [7], who also prove several further results.

Here we need the L_p spaces in the range $1 \leqslant p < 2$. Then $L_p(\mathcal{M},\psi)$ is defined as the completion of $\mathscr{H}$ with respect to the following L_p-norm:

$$\|\zeta\|_{p,\psi} = \inf\{\|\Delta^{(1/2)-(1/p)}_{\phi,\psi}\zeta\| : \|\phi\| = 1, \pi^{\mathcal{M}}(\phi) \geqslant \pi^{\mathcal{M}}(\zeta)\}. \tag{71}$$

As in the main text, we use the general notation ω_ψ (resp. ω'_ψ) for the normal state functional on $\mathcal{M}$ (resp. $\mathcal{M}'$) induced by a normalized state $|\psi\rangle \in \mathscr{H}$, but the L_p norms $\|\zeta\|_{p,\psi}$ are always defined using the unique [4], thm. C.1, vector representative in the natural cone.

The L_p-norms satisfy $\|m\zeta\|_{p,\psi} \leqslant \|m\|\|\zeta\|_{p,\psi}$ for all $m \in \mathcal{M}$. There exists $u \in \mathcal{M}$ and $|\zeta'\rangle$ in the natural cone such that $u^*u = \pi^{\mathcal{M}}(\zeta'), uu^* = \pi^{\mathcal{M}}(\zeta)$ by [4], thm. C.1, and this implies that $\|\zeta\|_{p,\psi} = \|\zeta'\|_{p,\psi}$ and that the $L_p(\mathcal{M},\psi)$-norms can be viewed as functionals of $\omega'_\zeta \in \mathscr{S}(\mathcal{M}')$, the normal state functional on $\mathcal{M}'$ induced by the vector $|\zeta\rangle \in \mathscr{H}$. By composition with the map $\mathscr{S}(\mathcal{M}') \ni \omega'_\psi \mapsto |\psi\rangle \in \mathscr{P}^\natural_{\mathcal{M}}$ sending a functional on $\mathcal{M}'$ to its unique representer in the natural cone, the L_p norms may thus be viewed as functionals $\mathscr{S}(\mathcal{M}') \times \mathscr{S}(\mathcal{M}') \ni (\omega'_\zeta, \omega'_\psi) \mapsto \|\zeta\|_{p,\psi}$.

It is known ([20], thm. 3.16 (3) and [25], prop. 3.10) that for $p \in [1,2]$ the map $(\omega'_\psi, \omega'_\zeta) \mapsto \|\zeta\|^p_{p,\psi}$ is continuous in norm. These references work with a different approach to L_p-norms developed in [25, 26] and in [20]. But these definitions can be shown to be equivalent to those used by [4, 7], see [25], thm. 3.3 and [26], prop 2.4 and thm. 3.1. Thus, we may use results on the L_p-norms in their setting in the setting of [4, 7], too.

The mentioned continuity property allows one to obtain to a certain extent properties of the L_p-norms for general vectors $|\psi\rangle$ in the natural cone from the case when $|\psi\rangle$ is cyclic and separating, treated mostly in [4]: Since we assume σ-finite von Neumann algebras, there exists a cyclic and separating vector $|\eta\rangle$ in the natural cone, and we can use this to define $\omega_{\psi_\varepsilon} = \omega_\psi + \varepsilon\omega_\eta$, where $\varepsilon \geqslant 0$, and where $|\psi_\varepsilon\rangle$ is the unique vector representative in the natural cone, which is hence cyclic and separating for $\varepsilon > 0$. By well-known properties of the natural cone, we have $\lim_{\varepsilon\to 0} \|\psi - \psi_\varepsilon\| = 0$, so the continuity property may be applied.

For example, the triangle inequality for the L_p-norms, proven in [4], thm. 1 (1)

$$\|\zeta + \eta\|_{p,\psi} \leqslant \|\zeta\|_{p,\psi} + \|\eta\|_{p,\psi}, \tag{72}$$

extends to the case when $|\psi\rangle$ is not cyclic and separating.

C Proof of lem. 5

The condition $\omega_\psi \leqslant c\omega_\eta$ implies that

$$\langle a\psi|\Delta_{\eta,\psi} a\psi\rangle = \langle \eta|a\pi^{\mathcal{A}}(\psi)a^*\eta\rangle \geqslant c^{-1}\langle\psi|a\pi^{\mathcal{A}}(\psi)a^*\psi\rangle = c^{-1}\langle a\psi|\Delta_\psi a\psi\rangle \tag{73}$$

where $a \in \mathcal{A}$. Since the support of Δ_ψ is $\pi^{\mathcal{A}}(\psi)\pi^{\mathcal{A}'}(\psi)$ and that of $\Delta_{\eta,\psi}$ is $\pi^{\mathcal{A}}(\eta)\pi^{\mathcal{A}'}(\psi)$, we get $\langle\zeta|\Delta_\psi\zeta\rangle \leqslant c\langle\zeta|\Delta_{\eta,\psi}\zeta\rangle$ also for any $|\zeta\rangle$ from $\mathcal{A}|\psi\rangle \oplus (1-\pi^{\mathcal{A}'}(\psi))\mathscr{H}$. But $\mathcal{A}|\psi\rangle \oplus (1-\pi^{\mathcal{A}'}(\psi))\mathscr{H}$ is a core for the self-adjoint operator $\Delta^{1/2}_{\eta,\psi}$, so we have $c\Delta_{\eta,\psi} \geqslant \Delta_\psi$.

Next, by operator monotonicity of the function $x^\alpha, \alpha \in [0,1]$ (see e.g. [43], Ex. 24, p. 315), we get $\Delta^\alpha_\psi \leqslant c^\alpha \Delta^\alpha_{\eta,\psi}$, and thereby $\|\Delta^{-z}_{\eta,\psi}\Delta^z_\psi\| \leqslant c^{2\mathrm{Re}(z)}$ when $z \in \bar{\mathbb{S}}_{1/2}$ by item 3). If $a \in \mathcal{A}$, standard Tomita-Takesaki theory tells us that $\Delta^z_\psi a|\psi\rangle$ is holomorphic inside $\mathbb{S}_{1/2}$ and strongly continuous and bounded on $\bar{\mathbb{S}}_{1/2}$. Item 4) in sec. 5 [which is [11], lem. 2.1] tells us that $\Delta^{-z}_{\eta,\psi}\Delta^z_\psi a|\psi\rangle$ is holomorphic in $\mathbb{S}_{1/2}$ and weakly continuous and bounded in norm on $\bar{\mathbb{S}}_{1/2}$.

Since the supports satisfy $s(\Delta_\psi) = \pi^{\mathcal{A}}(\psi)\pi^{\mathcal{A}'}(\psi) \leqslant \pi^{\mathcal{A}}(\eta)\pi^{\mathcal{A}'}(\psi) = s(\Delta_{\eta,\psi})$, we have the same statements for $\Delta_{\eta,\psi}^{-z}\Delta_\psi^z|\zeta\rangle$ when $|\zeta\rangle \in \mathcal{A}|\psi\rangle \oplus (1-\pi^{\mathcal{A}'}(\psi))\mathscr{H}$ which is dense in $\mathscr{H}$. By approximating a general vector in $\mathscr{H}$ with such $|\zeta\rangle$ and using that $\|\Delta_{\phi,\psi}^{-z}\Delta_\psi^z\| \leqslant c^{2\mathrm{Re}(z)}$ when $z \in \bar{\mathbb{S}}_{1/2}$, one can see $\Delta_{\eta,\psi}^{-z}\Delta_\psi^z|\zeta\rangle$ is weakly continuous and norm bounded in $\bar{\mathbb{S}}_{1/2}$ and analytic in $\mathbb{S}_{1/2}$. For example, to prove analyticity, let $|\zeta\rangle \in \mathscr{H}$ and $|\chi\rangle \in \mathcal{A}|\psi\rangle \oplus (1 - \pi^{\mathcal{A}'}(\psi))\mathscr{H}$ such that $\|\zeta - \chi\| < \varepsilon$. For $z_0 \in \mathbb{S}_{1/2}$ we first have for sufficiently small $r > 0$,

$$\left\| \Delta_{\eta,\psi}^{-z_0}\Delta_\psi^{z_0}|\zeta\rangle - \frac{1}{2\pi i}\int_{\partial B_r(z_0)} dz\,(z-z_0)^{-1}\Delta_{\eta,\psi}^{-z}\Delta_\psi^z|\zeta\rangle \right\| \leqslant 2c\|\chi - \eta\|, \tag{74}$$

where we have written $|\zeta\rangle = |\chi\rangle + (|\zeta\rangle - |\chi\rangle)$ and used that $\Delta_{\eta,\psi}^{-z}\Delta_\psi^z|\chi\rangle$ is already known to satisfy the Cauchy formula. Then, the violation of the Cauchy formula is $< 2c\varepsilon$, and since $\varepsilon > 0$ can be arbitrarily small, it is zero, hence $\Delta_{\eta,\psi}^{-z}\Delta_\psi^z|\zeta\rangle$ is analytic for $z \in \mathbb{S}_{1/2}$. By an application of the Banach-Steinhaus principle combined with the Cauchy formula, $\Delta_{\eta,\psi}^{-z}\Delta_\psi^z$ is analytic for $z \in \mathbb{S}_{1/2}$ as an operator valued function.

We use this operator valued function to define an analytic extension $(D\eta : D\psi)_{iz}$ of the Connes cocycle (6) e.g. by analytic continuation of [4], Eq. (C.30). Then, by [4], thm. C.1 ($\beta 2$), we know that $(D\eta : D\psi)_{iz} \in \pi^{\mathcal{A}}(\eta)\mathcal{A}\pi^{\mathcal{A}}(\psi)$, for $z \in i\mathbb{R}$, so for any $a' \in \mathcal{A}'$, $z \mapsto [a', (D\eta : D\psi)_{iz}]$ is a holomorphic function in $\mathbb{S}_{1/2}$ taking values in the bounded operators with a bounded, vanishing limit as $\mathrm{Re}z \to 0^+$. Then, by the by the operator-valued edge of the wedge theorem, the function vanishes identically and we get $(D\eta : D\psi)_{iz} \in \mathcal{A}'' = \mathcal{A}$ when z is in the closure of $\mathbb{S}_{1/2}$. By a similar argument, we find in fact that $(D\eta : D\psi)_{iz} \in \pi^{\mathcal{A}}(\eta)\mathcal{A}\pi^{\mathcal{A}}(\psi)$ for $z \in \bar{\mathbb{S}}_{1/2}$.

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