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MONOTONE σ-COMPLETE RC-GROUPS

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Abstract

An RC-group is a unital group G with a distinguished compression base with respect to which G satisfies the Rickart projection and general comparison properties. We prove that a monotone σ-complete RC-group is a union of subgroups each of which is a lattice-ordered Dedekind σ-complete RC-group.

1. *Introduction*

If L is an orthomodular lattice (OML), then L is covered by its own blocks (maximal compatible subsets of L) and each block in L is a Boolean sublattice of L [**24**]. In this paper, we obtain an analogous result for an archimedean RC-group G (Definition 4.3 below), the role of the OML blocks being played by certain lattice-ordered unital subgroups of G called C-blocks (Definition 5.1 below). If G is monotone σ-complete, then so are the C-blocks, and each C-block has a representation as a group of functions on a basically disconnected compact Hausdorff space.

The partially ordered additive group $\mathbb{G}(\mathfrak{H})$ of bounded self-adjoint operators on a Hilbert space $\mathfrak{H}$ is an important example of a monotone σ-complete RC-group, and it provides much of the motivation for the developments in this paper. As a partially ordered set, $\mathbb{G}(\mathfrak{H})$ is an *anti-lattice*, that is, operators $A, B \in \mathbb{G}(\mathfrak{H})$ have an infimum in $\mathbb{G}(\mathfrak{H})$ only if $A \leqslant B$ or $B \leqslant A$ [**21**]. However, $\mathbb{G}(\mathfrak{H})$ (and more generally, the self-adjoint part of any AW*-algebra) is covered by subgroups that are lattice ordered. Recall that there are profound connections between commutativity of operators and lattice-ordered subgroups of operator algebras [**29**].

A second source of motivation for our work derives from algebraic logic and involves the Lindenbaum–Tarski algebras, called *MV-algebras*, associated with Łukasiewicz multi-valued logics [**4**] and employed by Mundici in the classification of AF C*-algebras (see [**23**, **25**]). In [**23**], Mundici proved that MV-algebras are the same thing as unit intervals in unital ℓ-groups. If the MV-algebra E is the unit interval in a unital ℓ-group G, then by [**17**, Proposition 16.9], E is monotone σ-complete if and only if G is monotone σ-complete, and by [**16**], E is a Heyting MV-algebra if and only if G is an RC-group.

In what follows, we use additive notation for abelian groups, we use the terminology of [**17**] for partially ordered abelian groups, and we adopt the nomenclature in [**2**] for effect algebras. For background material involving orthomodular posets

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and orthomodular lattices, see [**24**]. For the reader's convenience, we give a brief synopsis of pertinent terminology and notation.

Let G be a partially ordered abelian group with positive cone $G^+ = \{g \in G : 0 \leqslant g\}$. Recall that G is *directed* if and only if $G = G^+ - G^+$. We say that G is *unperforated* if and only if, for all $g \in G$ and every $n \in \mathbb{N} := \{1, 2, 3, \dots\}$, the condition $0 \leqslant ng$ implies $0 \leqslant g$. We say that G is *archimedean* if and only if, whenever $g, h \in G$ and $ng \leqslant h$ for all $n \in \mathbb{N}$, then $g \leqslant 0$. An element $u \in G^+$ is called an *order unit* in G if and only if, for every $g \in G$, there exists $n \in \mathbb{N}$ such that $g \leqslant nu$. If G has an order unit, then G is directed.

Let X be a partially ordered set (poset). If $a, b, c \in X$, and we write $a \wedge_X b = c$, we mean that the infimum (greatest lower bound) $a \wedge_X b$ of $\{a, b\}$, *calculated in* X, exists and equals c. A similar convention applies to the supremum (least upper bound) $a \vee_X b$ and to infinite infima and suprema. The poset X is called *monotone σ-complete* if and only if every ascending (descending) sequence in X that is bounded above (below) in X has a supremum (an infimum) in X. We say that X is *Dedekind σ-complete* if and only if every sequence in X that is bounded above (below) in X has a supremum (an infimum) in X. If $Y \subseteq X$, we understand that Y carries the *induced* partial order, that is, that Y is partially ordered by the restriction to Y of the partial order on X.

Definition 1.1. If X is a poset and $Y \subseteq X$, we say that Y is *sup/inf-closed in* X if and only if Y is closed under the calculation of existing suprema and infima in X, that is, whenever $\emptyset \neq Z \subseteq Y$ and $s := \bigvee_X \{z : z \in Z\}$ (respectively, $t := \bigwedge_X \{z : z \in Z\}$) exists in X, then $s \in Y$ (respectively, $t \in Y$).

If Y is sup/inf-closed in X and X is monotone (respectively, Dedekind) σ-complete, then Y is monotone (respectively, Dedekind) σ-complete.

Let G be a partially ordered abelian group. If, as a partially ordered set, G forms a lattice (that is, $g \wedge_G h$ and $g \vee_G h$ exist for all $g, h \in G$), then G is said to be *lattice ordered*, or an *ℓ-group*. We shall only be considering abelian partially ordered groups, so by an ℓ-group, we always mean an abelian lattice-ordered group. If G is an ℓ-group, then G is unperforated [**17**, Proposition 1.22] and, therefore, it is torsion free as an abelian group. As the mapping $g \mapsto -g$ on G is order reversing and of period two, there is a *duality* on G whereby properties of suprema are converted to properties of infima and *vice versa*. In particular, G is Dedekind (respectively, monotone) σ-complete if and only if every sequence (respectively, every ascending sequence) in G that is bounded above in G has a supremum in G. Also, if G is directed and monotone σ-complete, then G is Dedekind σ-complete if and only if G is an ℓ-group.

A *unital group* is a directed abelian group G with a distinguished element $u \in G^+$, called the *unit*, such that the set $E(G) := \{e \in G : 0 \leqslant e \leqslant u\}$, called the *unit interval*, generates G^+ in the sense that every element in G^+ is a finite sum of (not necessarily distinct) elements of $E(G)$. If G is understood, we often write E rather than $E(G)$ for the unit interval in G. The unit u in a unital group G is automatically an order unit in G. If G is a unital group, then the unit interval E generates G^+ and $G = G^+ - G^+$, whence E generates G as a group. Obviously, the unit interval E in a unital group G is sup/inf-closed in G, hence, if G is monotone or Dedekind σ-complete, then so is E.

The unit interval E in a unital group G with unit u forms an *effect algebra* with unit u under the restriction of $+$ to E [**2**], hence we refer to elements of E as *effects*. An effect $e \in E$ is said to be *sharp* if and only if the only effect $f \in E$ with $f \leqslant e$ and $f \leqslant u - e$ is $f = 0$ [**18**].

Let E be the unit interval in a unital group G with unit u. A subset $S \subseteq E$ is a *sub-effect algebra* of E if and only if (1) $0, u \in S$, (2) $s \in S \Rightarrow u - s \in S$ and (3) for all $s, t \in S$, $s + t \in E \Rightarrow s + t \in S$. If S is a sub-effect algebra of E and $s, t \in S$, then $s \leqslant t \Leftrightarrow t - s \in S$.

Definition 1.2 [**14**, Definition 1]. A sub-effect algebra S of the unit interval E in a unital group G is said to be *normal* if and only if, for all $e, f, d \in E$ with $e + f + d \in E$, we have $e + d, f + d \in S \Rightarrow d \in S$.

If S is a sub-effect algebra of E, we say that elements $s, t \in S$ are *Mackey compatible in* S, in symbols sC_St, if and only if there are elements $s_1, t_1, d \in S$ such that $s_1 + t_1 + d \in S$, $s = s_1 + d$, and $t = t_1 + d$ [**22**, p. 70]. If S is a normal sub-effect algebra of E and $s, t \in S$, then $sC_St \Leftrightarrow sC_Et$.

2. *Examples*

The additive group $\mathbb{R}$ of real numbers, totally ordered as usual, is an archimedean, Dedekind σ-complete, unital ℓ-group with unit 1, with the standard unit interval $[0, 1]$ as its unit interval, and with 0 and 1 as its only sharp elements. As totally ordered subgroups of $\mathbb{R}$, the rational numbers $\mathbb{Q}$ with unit 1 and the integers $\mathbb{Z}$ with unit 1 are archimedean, unital ℓ-groups, and $\mathbb{Z}$ (but not $\mathbb{Q}$) is Dedekind σ-complete.

The unital groups in Examples 2.1–2.3 below will help to fix ideas and illustrate the ensuing developments.

Example 2.1. Let $\mathfrak{H}$ be a Hilbert space and define $\mathbb{G}(\mathfrak{H})$ to be the additive abelian group of all bounded self-adjoint operators on $\mathfrak{H}$. Under the usual partial order, $\mathbb{G}(\mathfrak{H})$ is an archimedean, unperforated, unital group with the identity operator $\mathbf{1}$ as the unit, and by Vigier's theorem [**28**, p. 263], it is monotone σ-complete. The unit interval $\mathbb{E}(\mathfrak{H})$ in $\mathbb{G}(\mathfrak{H})$ is called the standard effect algebra on $\mathfrak{H}$, and the set $\mathbb{P}(\mathfrak{H})$ consisting of all projection operators $P = P^2 = P^*$ on $\mathfrak{H}$ is a normal sub-effect algebra of $\mathbb{E}(\mathfrak{H})$. If $P \in \mathbb{E}(\mathfrak{H})$, then $P \in \mathbb{P}(\mathfrak{H}) \Leftrightarrow P$ is sharp. As an effect algebra in its own right, $\mathbb{P}(\mathfrak{H})$ forms a complete OML. If $P \in \mathbb{P}(\mathfrak{H})$, the mapping $J_P : \mathbb{G}(\mathfrak{H}) \to \mathbb{G}(\mathfrak{H})$ defined by $J_P(A) := PAP$ for all $A \in \mathbb{G}(\mathfrak{H})$ is an order-preserving group endomorphism called the Naimark compression determined by P.

If $\mathcal{F}$ is a field of subsets of a nonempty set X, then, partially ordered by set inclusion, $\mathcal{F}$ is a Boolean algebra under the usual set operations. In the following example, we exhibit an archimedean ℓ-group with unit interval isomorphic to $\mathcal{F}$.

Example 2.2. Let $\mathcal{F}$ be a field of subsets of a nonempty set X, and define $\mathbb{G}(X, \mathcal{F}, \mathbb{Z})$ to be the partially ordered additive group, with pointwise addition and partial order, of all bounded functions $f : X \to \mathbb{Z}$ such that $f^{-1}(n) \in \mathcal{F}$ for all $n \in \mathbb{Z}$. Then, with the constant function $1(x) \equiv 1$ as unit, $\mathbb{G}(X, \mathcal{F}, \mathbb{Z})$ is an archimedean unital ℓ-group, and the unit interval $\mathbb{E}(X, \mathcal{F}, \mathbb{Z})$ in $\mathbb{G}(X, \mathcal{F}, \mathbb{Z})$ is isomorphic to the Boolean algebra of sets $\mathcal{F}$ under the mapping $p \mapsto p^{-1}(1)$ for all $p \in \mathbb{E}(X, \mathcal{F}, \mathbb{Z})$. If $p \in \mathbb{E}(X, \mathcal{F}, \mathbb{Z})$, the mapping $J_p : \mathbb{G}(X, \mathcal{F}, \mathbb{Z}) \to \mathbb{G}(X, \mathcal{F}, \mathbb{Z})$ defined by the pointwise product $J_p(f) := pf$ for all $f \in \mathbb{G}(X, \mathcal{F}, \mathbb{Z})$ is an order-preserving group endomorphism called the Boolean compression determined by p.

The unital ℓ-group $\mathbb{G}(X, \mathcal{F}, \mathbb{Z})$ in Example 2.2 is monotone σ-complete if and only if $\mathcal{F}$, as a Boolean algebra, is σ-complete. If, for every $x \in X$, the singleton set $\{x\}$ belongs to the field $\mathcal{F}$, then $\mathcal{F}$ is a σ-complete Boolean algebra if and only if $\mathcal{F}$ is a σ-field. If B is a Boolean algebra and $X(B)$ is the Stone space of B, then B is isomorphic to the field $\mathcal{F}$ of compact open subsets of $X(B)$; hence B is isomorphic to the unit interval $\mathbb{E}(X(B), \mathcal{F}, \mathbb{Z})$ in the archimedean ℓ-group $\mathbb{G}(X(B), \mathcal{F}, \mathbb{Z})$.

If X is a topological space, then $C(X, \mathbb{R})$ denotes the set of all continuous functions $f : X \to \mathbb{R}$ organized into a partially ordered real vector space with pointwise operations and pointwise partial order. As an additive abelian group, $C(X, \mathbb{R})$ is an archimedean ℓ-group. The constant function $1(x) \equiv 1$ on X, denoted simply by 1, is an order unit in $C(X, \mathbb{R})$ if and only if all continuous real-valued functions on X are bounded.

Example 2.3. If X is a compact Hausdorff space, then $C(X, \mathbb{R})$ is an archimedean unital ℓ-group with unit 1. Denote the unit interval in $C(X, \mathbb{R})$ by $E(X)$ and denote the set of sharp elements in $E(X)$ by $P(X)$. If $p \in C(X, \mathbb{R})$, then $p \in P$ if and only if there is a compact open subset K of X such that $p = \chi_K$ (the characteristic set function of K). The set $P(X)$ is a normal sub-effect algebra of $E(X)$ and, under the mapping $p \mapsto p^{-1}(1)$, $P(X)$ is isomorphic to the Boolean algebra of all compact open subsets of X. If $p \in P(X)$, the mapping $J_p : C(X, \mathbb{R}) \to C(X, \mathbb{R})$ defined by the pointwise product $J_p(f) := pf$ for all $f \in C(X, \mathbb{R})$ is an order-preserving group endomorphism called the p-compression on $C(X, \mathbb{R})$.

Let X be a compact Hausdorff space and let $\mathcal{F}$ be the field of compact open subsets of X. With pointwise operations and the supremum norm, $C(X, \mathbb{R})$ is actually a commutative lattice-ordered Banach algebra. Alternatively, it is a lattice-ordered order-unit Banach space with 1 as the order unit [**1**, p. 69], and the supremum and order-unit norms coincide. Recall that X is *basically disconnected* if and only if the closure of every open F_σ set in X remains open, and X is *extremally disconnected* if and only if the closure of every open set in X remains open. We note that extremally disconnected $\Rightarrow$ basically disconnected $\Rightarrow$ totally disconnected. If X is extremally disconnected (respectively, basically disconnected), then $\mathcal{F}$ is a complete (respectively, a σ-complete) Boolean algebra. Conversely, if B is a Boolean algebra and $X(B)$ is the Stone space of B, then B is σ-complete (respectively, complete) if and only if $X(B)$ is basically disconnected (respectively, totally disconnected).

The compact Hausdorff space X is basically disconnected if and only if $C(X, \mathbb{R})$ is monotone σ-complete [**17**, Corollary 9.3]. (Also, see Theorem 4.8 below.) The compact Hausdorff space X is extremally disconnected if and only if $C(X, \mathbb{R})$ is a commutative AW*-algebra.

3. *CB-groups*

Definition 3.1. Let G be a unital group with unit u and unit interval E. A mapping $J : G \to G$ is called a *retraction* with *focus* p on G if and only if J is an order-preserving group endomorphism, $p = J(u) \in E$, and for all $e \in E$, $e \leqslant p \Rightarrow J(e) = e$. A retraction $J : G \to G$ is said to be *direct* if and only if $g \in G^+ \Rightarrow J(g) \leqslant g$. A retraction J on G is called a *compression* if and only if $J^{-1}(0) \cap E = \{e \in E : e + J(u) \in E\}$ [**11**]. Two retractions J and J' on G are called *quasicomplements* of each other if and only if, for all $g \in G^+$, $J(g) = g \Leftrightarrow J'(g) = 0$ and $J'(g) = g \Leftrightarrow J(g) = 0$.

If J is a retraction on G, then J is an idempotent, that is, $J \circ J = J$ and its focus is a sharp element of E [**11**, Lemmas 2.2 and 2.8]. If J and J' are quasicomplements, they are necessarily compressions [**11**, Lemma 3.2(iii)].

Lemma 3.2. *Let J be a direct retraction on the unital group G and define $J' : G \to G$ by $J'(g) := g - J(g)$ for all $g \in G$. Then:*
(i) *J' is a direct retraction on G;*
(ii) *J and J' are quasicomplementary compressions on G;*
(iii) *if $\widetilde{J}$ is a retraction on G with the same focus as J, then $\widetilde{J} = J$.*

Proof. Parts (i) and (ii) follow from [**11**, Theorem 2.8]. To prove (iii), let u be the unit in G, let E be the unit interval in G, let $p := J(u) = \widetilde{J}(u)$ be the common focus of J and $\widetilde{J}$, and let $e \in E$. As E generates G as a group, it will be sufficient to prove that $J(e) = \widetilde{J}(e)$. As $0 \leqslant J(e) \leqslant J(u) = p$, we have $\widetilde{J}(J(e)) = J(e)$. Also, $0 \leqslant \widetilde{J}(J'(e)) \leqslant \widetilde{J}(J'(u)) = \widetilde{J}(u - J(u)) = \widetilde{J}(u-p) = 0$, hence $\widetilde{J}(J'(e)) = 0$. Consequently, $\widetilde{J}(e) = \widetilde{J}(J(e) + J'(e)) = J(e)$. □

Definition 3.3. By a *compression base* for the unital group G with unit interval E [**14**, **15**], we mean a family $(J_p)_{p \in P}$ of compressions on G, indexed by a normal sub-effect algebra P of E, such that (i) each $p \in P$ is the focus of J_p and (ii) if $p, q, r \in P$ and $p + q + r \in E$, then $J_{p+r} \circ J_{q+r} = J_r$. A compression base $(J_p)_{p \in P}$ for G is *proper* if and only if every direct compression on G belongs to the family $(J_p)_{p \in P}$; it is *direct* if and only if it is the family of all direct compressions on G; and it is *total* if and only if every retraction on G is a compression and belongs to the family $(J_p)_{p \in P}$.

If G is a unital group with unit u and $(J_p)_{p \in P}$ is a compression base for G, then for each $p \in P$, we have $u - p \in P$, and J_{u-p} is the unique compression in the compression base that is a quasicomplement of J_p.

In Example 2.1, if $P \in \mathbb{P}(\mathfrak{H})$, then the Naimark compression J_P is indeed a compression on $\mathbb{G}(\mathfrak{H})$, the family $(J_P)_{P \in \mathbb{P}(\mathfrak{h})}$ is a total compression base for $\mathbb{G}(\mathfrak{H})$, and the only direct compressions on $\mathbb{G}(\mathfrak{H})$ are $J_{\mathbf{0}}$ and $J_{\mathbf{1}}$. In Example 2.2, if $p \in \mathbb{E}(X, \mathcal{F}, \mathbb{Z})$ then the Boolean compression J_p is a direct compression on $\mathbb{G}(X, \mathcal{F}, \mathbb{Z})$ and the family $(J_p)_{p \in \mathbb{E}(X,\mathcal{F},\mathbb{Z})}$ is a direct and total compression base for $\mathbb{G}(X, \mathcal{F}, \mathbb{Z})$. In Example 2.3, if $p \in P(X)$, then the p-compression J_p is a direct compression on $C(X, \mathbb{R})$ and the family $(J_p)_{p \in P(X)}$ is a direct and total compression base for $C(X, \mathbb{R})$.

Definition 3.4. A *compression-based group*, for short a *CB-group*, is a unital group G together with a distinguished compression base $(J_p)_{p\in P}$ for G. If G is a CB-group with compression base $(J_p)_{p\in P}$, then the normal sub-effect algebra P of E is called the set of *projections* in G. A CB-group with a proper, a direct, or a total compression base is called a *proper*, a *direct*, or a *total* CB-group, respectively.

Every unital group G can be organized into a direct CB-group simply by taking the compression base to consist of all direct compressions on G, indexed by their own foci. (Note that the zero mapping $g \mapsto 0$ and the identity mapping $g \mapsto g$ are direct compressions on G.) In a direct CB-group G, the set P of projections is a Boolean algebra.

Definition 3.5 [**10**, Definition 3.3]. A *compressible group* is a unital group for which every retraction is determined by its focus and every retraction has a quasicomplementary retraction.

By [**11**, Corollary 4.6], the unital group $\mathbb{G}(\mathfrak{H})$ in Example 2.1 is a compressible group. If G is a compressible group, then every retraction on G is a compression, and G is organized into a total CB-group by taking the compression base to be the family of all retractions (hence compressions) on G [**14**, Theorem 2.3]. Conversely, if G is a total CB-group, then G is necessarily a compressible group.

Theorem 3.6. *Let G be an ℓ-group and let u be an order unit in G. Then:*
(i) *with u as the unit, G is a unital group;*
(ii) *every retraction on the unital group G is a direct compression, hence G is a compressible group;*
(iii) *if J is a (necessarily direct) compression on the unital group G, $p := J(u)$ is the focus of J, $g \in G^+$, $n \in \mathbb{N}$, and $g \leqslant nu$, then $J(g) = g \wedge_G np$;*
(iv) *if the unital ℓ-group G is organized into a direct CB-group, then it is a total CB-group, the set P of projections in E coincides with the set of sharp elements in E, and, as a sub-effect algebra of E, P is a Boolean algebra;*
(v) *G is monotone σ-complete if and only if its unit interval E is monotone σ-complete.*

Proof. For (i), see [**25**]. For (ii) and (iv), see [**10**, Theorem 3.5]. Part (iii) follows from [**17**, Proposition 8.3]. For (v), see [**17**, Proposition 16.9]. ☐

Corollary 3.7. *If G is a unital ℓ-group and also a CB-group, then G is proper $\Leftrightarrow$ G is direct $\Leftrightarrow$ G is total.*

In what follows, we shall refer to a unital ℓ-group that is a proper (hence direct and total) CB-group as a *proper ℓ-group*. For instance, the unital ℓ-groups $\mathbb{G}(X, \mathcal{F}, \mathbb{Z})$ in Example 2.2 and $C(X, \mathbb{R})$ in Example 2.3 are proper (hence direct and total) ℓ-groups with compression bases $(J_p)_{p\in\mathbb{E}(X,\mathcal{F},\mathbb{Z})}$, and $(J_p)_{p\in P(X)}$, respectively.

Standing Assumption 3.8. Henceforth we assume that G is a CB-group with unit u, unit interval E, and projection base $(J_p)_{p\in P}$.

Organized into an effect algebra in its own right, P *is an orthomodular poset* (OMP) [**14**, Theorem 2]. The notion of compatibility in the following definition is closely related to Mackey compatibility (see Theorem 3.11).

Definition 3.9 [**10**, Definition 2.6].
(i) If $p \in P$, let $C(p) := \{g \in G : g = J_p(g) + J_{u-p}(g)\}$. Elements $g \in C(p)$ are said to be *compatible* with the projection p.
(ii) If $\emptyset \neq M \subseteq P$, define $C(M) := \bigcap_{p\in M} C(p)$.

In Example 2.1, a self-adjoint operator is compatible with a projection if and only if it commutes with that projection. In Examples 2.2 and 2.3, every element is compatible with every projection.

Lemma 3.10.
(i) *If $p \in P$, then J_p is a direct compression on G if and only if $G = C(p)$.*
(ii) *If G is an ℓ-group, then $G = C(P)$.*
(iii) *G is a direct CB-group if and only if G is a proper CB-group and $G = C(P)$.*

Proof. (i) If J_p is a direct compression on G and $g \in G$, then $g \in C(p)$ by Lemma 3.2. Conversely, if $G = C(p)$ and $g \in G^+$, then $0 \leqslant J_{u-p}(g)$, hence $J_p(g) \leqslant J_p(g) + J_{u-p}(g) = g$, so J_p is direct. Part (ii) follows from (i) and Theorem 3.6, and (iii) also follows from (i). □

Let $p \in P$ and $\emptyset \neq M \subseteq P$. Then $C(p)$ is a subgroup of G, $C(p) = C(u-p)$, and $0, u, p, u-p \in C(p)$. In fact, $C(p)$ *is a unital group with unit u, and the family of all restrictions to $C(p)$ of compressions J_q with $q \in P \cap C(p)$ is a compression base for $C(p)$* [**15**, Theorem 3.3]. Also, $C(M)$ is a subgroup of G, $u \in C(M)$, and since u is an order unit in G, it is also an order unit in $C(M)$, hence $C(M)$ is directed. However, there is no *a priori* reason to assume that $C(M)$ is a unital group unless it happens that the unit interval $E \cap C(M)$ in $C(M)$ generates $C(M)^+ = G^+ \cap C(M)$.

Theorem 3.11. *If $p, q \in P$, then the following conditions are mutually equivalent:* (i) $J_p \circ J_q = J_q \circ J_p$; (ii) $J_p(q) = J_q(p)$; (iii) $J_p(q) \leqslant q$; (iv) pC_Eq; (v) pC_Pq; (vi) $\exists\, r \in P$ *such that* $J_p \circ J_q = J_r$; (vii) $J_p(q) \in P$; (viii) $q \in C(p)$; (ix) $p \in C(q)$.

Proof. See [**15**, Theorem 2.7]. □

As a consequence of Theorem 3.11, if $p, q \in P$, then $p \in C(q)$ if and only if p and q are Mackey compatible in P, or equivalently, in E. Therefore, *for $p, q \in P$ we usually write the condition $p \in C(q)$ in the form pCq.* In what follows, we shall be focusing on the OMP P and its properties, so *we shall write existing suprema and infima in P without using subscripts.* (We continue to use subscripts to signify existing suprema and infima in other subsets of G.)

Corollary 3.12. *Let $p, q \in P$. Then:*
(i) *$pCq \Leftrightarrow qCp$;*
(ii) *if pCq, then $p \wedge q$ exists in P, and it is also the infimum $p \wedge_E q$ of p and q in E;*
(iii) *if pCq, then $J_p \circ J_q = J_q \circ J_p = J_{p\wedge q}$ and $J_p(q) = J_q(p) = p \wedge q$;*
(iv) *$p \leqslant q \Leftrightarrow J_p \circ J_q = J_q \circ J_p = J_p$.*

THEOREM 3.13. *Let $p_1, p_2, \ldots, p_n \in P$ with $p := \sum_{i=1}^n p_i$. Then:*
(i) *if $p \in E$ and $g \in \bigcap_{i=1}^n C(p_i)$, then $p \in P$, $g \in C(p)$, $J_p(g) = \sum_{i=1}^n J_{p_i}(g)$, and $p = p_1 \vee p_2 \vee \ldots \vee p_n$;*
(ii) *if $p_1, p_2, \ldots, p_n$ are pairwise compatible, then $p_1 \wedge p_2 \wedge \ldots \wedge p_n$ and $p_1 \vee p_2 \vee \ldots \vee p_n$ exist in P and*

$$\bigcap_{i=1}^{n} C(p_i) \subseteq C(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \cap C(p_1 \vee p_2 \vee \ldots \vee p_n).$$

Proof. See [**15**, Theorems 2.9 and 2.10]. □

THEOREM 3.14. *If $p, q, r \in P$, pCq, pCr, and qCr, then $pC(q \vee r)$, $(p \wedge q)C(p \wedge r)$, and $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$.*

Proof. By Theorem 3.13, we have $pC(q \vee r)$ and $pC(q \wedge r)$, hence $J_p(q \vee r) = p \wedge (q \vee r)$ and $J_p(q \wedge r) = p \wedge q \wedge r$ by Corollary 3.12. As qCr, there are projections $q_1, r_1, d \in P$ such that $q = q_1 + d$, $r = r_1 + d$, $q \vee r = q_1 + r_1 + d \in P$, and $d = q \wedge r \in P$. Therefore, $J_p(d) = J_p(q \wedge r) = p \wedge q \wedge r$. By Corollary 3.12 again, $p \wedge q = J_p(q) = J_p(q_1) + J_p(d) = J_p(q_1) + (p \wedge q \wedge r)$ and $p \wedge r = J_p(r) = J_p(r_1) + J_p(d) = J_p(r_1) + (p \wedge q \wedge r)$, and it follows that $J_p(q_1) = (p \wedge q) - (p \wedge q \wedge r) \in P$ and $J_p(r_1) = (p \wedge r) - (p \wedge q \wedge r) \in P$. Therefore, $J_p(q_1) + J_p(r_1) + J_p(d) = J_p(q_1 + r_1 + d) = J_p(q \vee r) = p \wedge (q \vee r) \leqslant u$ with $p \wedge q = J_p(q_1) + J_p(d)$ and $p \wedge r = J_p(r_1) + J_p(d)$, so $(p \wedge q)C(p \wedge r)$ and $(p \wedge q) \vee (p \wedge r) = p \wedge (q \vee r)$. □

By Theorems 3.13 and 3.14, the OMP P is *regular* [**20**] in the sense that sets of pairwise compatible elements of P belong to Boolean sub-effect algebras of P.

LEMMA 3.15. *If $p \in P$, then $J_p(G) = \{g \in G : J_p(g) = g\}$ is sup/inf-closed in G.*

Proof. As J_p is idempotent, we have $J_p(G) = \{g \in G : J_p(g) = g\}$. Let $\emptyset \neq H \subseteq J_p(G)$ and suppose that $s := \bigvee_G \{h : h \in H\}$ exists in G. Since $h \leqslant s$ for all $h \in H$, we have $h = J_p(h) \leqslant J_p(s)$ for all $h \in H$, whence $s \leqslant J_p(s)$. Choose and fix $h_0 \in H$. Then $0 \leqslant s - h_0 \leqslant J_p(s) - h_0 = J_p(s - h_0)$, whence $0 \leqslant J_{u-p}(s - h_0) \leqslant J_{u-p}(J_p(s - h_0)) = 0$, that is, $J_{u-p}(s - h_0) = 0$. Therefore, $s - h_0 = J_p(s - h_0) = J_p(s) - h_0$, and it follows that $s = J_p(s) \in J_p(G)$. The fact that $J_p(G)$ is closed under the calculation of infima in G follows by duality. □

THEOREM 3.16. *Let G be monotone σ-complete, let $\emptyset \neq M \subseteq P$, and suppose that $(g_i)_{i \in \mathbb{N}}$ is an ascending sequence in $C(M)$ that is bounded above in G. Then $\bigvee_G \{g_i : i \in \mathbb{N}\} \in C(M)$. Therefore, $C(M)$ is monotone σ-complete.*

Proof. It will suffice to prove the theorem for the case $M = \{p\}$ with $p \in P$. Thus, assume that $(g_i)_{i \in \mathbb{N}}$ is an ascending sequence in $C(p)$ and $g_i \leqslant b \in G$ for all $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, define $h_i := J_p(g_i)$ and $k_i := J_{u-p}(g_i)$. Since $g_i \in C(p)$, we have $g_i = h_i + k_i$ for all $i \in \mathbb{N}$. Also, $(h_i)_{i \in \mathbb{N}}$ is an ascending sequence in $J_p(G)$, bounded above in G by $J_p(b)$, and $(k_i)_{i \in \mathbb{N}}$ is an ascending sequence in $J_{u-p}(G)$, bounded above in G by $J_{u-p}(b)$. As G is monotone σ-complete, the suprema $h := \bigvee_G \{h_i : i \in \mathbb{N}\}$ and $k := \bigvee_G \{k_i : i \in \mathbb{N}\}$ exist in G. By Lemma 3.15, $h = J_p(h)$

and $k = J_{u-p}(k)$, whence $J_{u-p}(h) = J_{u-p}(J_p(h)) = 0$ and $J_p(k) = J_p(J_{u-p}(k)) = 0$, and it follows that $J_p(h+k) + J_{u-p}(h+k) = h+k$, that is, $h+k \in C(p)$.

Clearly, $h+k$ is an upper bound in G for $\{h_i + k_i : i \in \mathbb{N}\}$. Suppose $c \in G$ with $h_i + k_i \leqslant c$ for all $i \in \mathbb{N}$. Fix $i \in \mathbb{N}$. If $j \in \mathbb{N}$ and $m = \max\{i, j\}$, then $h_i + k_j \leqslant h_m + k_m \leqslant c$, whence $k_j \leqslant c - h_i$, and it follows that $k \leqslant c - h_i$, so $h_i \leqslant c - k$. But i is an arbitrary positive integer, and therefore $h \leqslant c - k$ (that is, $h + k \leqslant c$), and we conclude that $h + k = \bigvee_G\{h_i + k_i : i \in \mathbb{N}\} = \bigvee_G\{g_i : i \in \mathbb{N}\}$, so $\bigvee_G\{g_i : i \in \mathbb{N}\} \in C(p)$. □

4. *The Rickart projection and general comparability properties*

We maintain our Standing Assumption 3.8, so that G is a CB-group with unit u, unit interval E, and compression base $(J_p)_{p\in P}$.

Definition 4.1. The CB-group G has the *Rickart projection property* if and only if there is a mapping $' \colon G \to P$, called the *Rickart mapping*, such that, for all $g \in G$ and all $p \in P$, $p \leqslant g' \Leftrightarrow g \in C(p)$ with $J_p(g) = 0$.

In Example 2.1, the compressible group $\mathbb{G}(\mathfrak{H})$ has the Rickart projection property, the Rickart mapping being given by $A \mapsto A'$, where A' is the projection onto the null space of $A \in \mathbb{G}(\mathfrak{H})$. In Example 2.2, $\mathbb{G}(X, \mathcal{F}, \mathbb{Z})$ has the Rickart projection property, the Rickart mapping being given by $f \mapsto f' = \chi_K$, where χ_K is the characteristic set function of $K := f^{-1}(0)$ for all $f \in \mathbb{G}(X, \mathcal{F}, \mathbb{Z})$.

Theorem 4.2. *Suppose that G has the Rickart projection property. Then we have the following.*

(i) *P is an orthomodular lattice (OML) and, for $p, q \in P$, $p \wedge q = J_p((J_p(q'))')$.*
(ii) *If $e \in E$, then $e'' := (e')'$ is the smallest element in $\{p \in P : e \leqslant p\}$.*
(iii) *P is sup/inf-closed in E.*
(iv) *If the unit interval E is monotone σ-complete, then P is a σ-complete OML.*
(v) *Let $\emptyset \neq M \subseteq P$. Then if the supremum $s := \bigvee\{p : p \in M\}$ exists in P (respectively, if the infimum $t := \bigwedge\{p : p \in M\}$ exists in P), then $C(M) \subseteq C(s)$ (respectively, $C(M) \subseteq C(t)$).*

Proof. (i) See [**15**, Theorem 5.4]. (ii) See [**15**, Theorem 5.3 (viii)]. (iii) Let $\emptyset \neq M \subseteq P$ and suppose that $t = \bigwedge_E\{p : p \in M\}$. As $t \leqslant p$ for all $p \in M$, (ii) implies that $t'' \leqslant p$ for all $p \in M$, whence $t'' \leqslant t$. But, by (ii) again, $t \leqslant t''$, so $t = t'' \in P$. That P is closed under the computation of suprema in E follows by duality. Property (iv) follows from (iii) and the fact that P is a lattice. (v) Suppose $s := \bigvee\{p : p \in M\}$ exists in P, and $g \in C(M)$. We have to prove that $g \in C(s)$. Let $h := g + J_{u-s}(g) - J_s(g)$ and let $p \in M$. As $p \leqslant s$, we have pCs, $pC(u-s)$, $p \wedge s = p$, and $p \wedge (u-s) = 0$, whence $J_p(h) = 0$. Now $pC(u-s)$ implies that $(u-p)C(u-s)$, hence, since $g \in C(p)$, we have

$$\begin{aligned} J_p(J_{u-s}(g)) + J_{u-p}(J_{u-s}(g)) &= J_{u-s}(J_p(g)) + J_{u-s}(J_{u-p}(g)) \\ &= J_{u-s}(J_p(g) + J_{u-p}(g)) = J_{u-s}(g), \end{aligned}$$

that is, $J_{u-s}(g) \in C(p)$. Similarly, since pCs and $g \in C(p)$, it follows that $J_s(g) \in C(p)$. As $g, J_{u-s}(g), J_s(g) \in C(p)$, we have $h \in C(p)$. Therefore, as $J_p(h) = 0$, the Rickart projection property implies that $p \leqslant h' \in P$. Since p is an arbitrary element of M, it follows that $s = \bigvee\{p : p \in M\} \leqslant h'$, whence that $h \in C(s)$ and $J_s(h) = 0$. But then $h = J_s(h) + J_{u-s}(h) = J_{u-s}(h) = 2J_{u-s}(g)$ (that is, $g + J_{u-s}(g) - J_s(g) = 2J_{u-s}(g)$), whereupon $g = J_s(g) + J_{u-s}(g)$ (that is, $g \in C(s)$). By duality and the fact that $C(r) = C(u-r)$ for all $r \in P$, we also have $t = \bigwedge\{p : p \in M\} \Rightarrow C(M) \subseteq C(t)$. □

Definition 4.3 [**15**, Definition 4.1]. Let $g \in G$.

(i) $CPC(g) := C(\{p \in P : g \in C(p)\})$.

(ii) $P^{\pm}(g) := \{p \in P \cap CPC(g) : g \in C(p) \text{ and } J_{u-p}(g) \leqslant 0 \leqslant J_p(g)\}$.

(iii) G has the *general comparability property* or, for short, is a *comparability group* if and only if $P^{\pm}(g) \neq \emptyset$ for all $g \in G$. If G is a comparability group and also an ℓ-group, then we call G a *comparability ℓ-group*.

(iv) G is an *RC-group* if and only if it has both the Rickart property and the general comparability property. If G is an RC-group and also an ℓ-group, then we call G an *RCℓ-group*.

In Example 2.1, if $A \in \mathbb{G}(\mathfrak{H})$, then $B \in CPC(A)$ if and only if B 'double commutes' with A, that is, if and only if B commutes with every bounded operator on $\mathfrak{H}$ that commutes with A. Of course, if $G = C(P)$ (for example, in Example 2.2), we have $G = CPC(g)$ for all $g \in G$.

If $p \in P^{\pm}(g)$, then p 'splits' $g = J_p(g) + J_{u-p}(g)$ into a 'positive part' $J_p(g) \geqslant 0$ and a 'negative part' $J_{u-p}(g) \leqslant 0$. If G is a comparability group and $p, q \in P^{\pm}(g)$, then $J_p(g) = J_q(g)$ and $J_{u-p}(g) = J_{u-q}(g)$ [**15**, Theorem 4.2], hence we can and do define $g^+ := J_p(g)$, $g^- := -J_{u-p}(g)$, and $|g| := g^+ + g^-$ for any choice of $p \in P^{\pm}(g)$. Thus, *if G is a comparability group and $g \in G$, we have $g^+, g^-, |g| \in G^+$ with $g = g^+ - g^-$.*

Consider the total CB-group $\mathbb{G}(\mathfrak{H})$ in Example 2.1. Let $A \in \mathbb{G}(\mathfrak{H})$ and let $A = |A|S = S|A|$ be the polar decomposition of A, where $|A| = \sqrt{A^2}$, S is self-adjoint, and $S^2 \in \mathbb{P}(\mathfrak{H})$ is the projection onto the orthogonal complement of the null space of A. Then $P := (S^2 + S)/2 \in \mathbb{P}(\mathfrak{H})$, P double commutes with A, and $J_{\mathbf{1}-P}(A) = (\mathbf{1} - P)A \leqslant \mathbf{0} \leqslant PA = J_P(A)$, so $\mathbb{G}(\mathfrak{H})$ has the general comparability property, and it follows that $\mathbb{G}(\mathfrak{H})$ is an RC-group. We note that $A^+ = (|A| + A)/2$ and $A^- = (|A| - A)/2$.

It is easy to see that the proper ℓ-group $\mathbb{G}(X, \mathcal{F}, \mathbb{Z})$ in Example 2.2 is an RCℓ-group.

Theorem 4.4. *Let G be a comparability group. Then:*

(i) *if $p \in E$, then $p \in P \Leftrightarrow p$ is sharp;*

(ii) *G is unperforated and torsion free;*

(iii) *G is archimedean if and only if for all $a, b \in G^+$, $na \leqslant b$ for all positive integers n only if $a = 0$;*

(iv) *if G is monotone σ-complete, then G is archimedean;*

(v) *G is a proper CB-group.*

Proof. For (i), see [**15**, Lemma 6.9], and for (ii) and (iii), see [**15**, Lemma 4.6]. To prove (iv), we use (iii). Thus, assume that $a, b \in G^+$ and $na \leqslant b$ for all $n \in \mathbb{N}$. Then $(na)_{n\in\mathbb{N}}$ is an ascending sequence bounded from above in G, therefore the supremum $s := \bigvee_G \{na : n \in \mathbb{N}\}$ exists in G. But $na = a + (n-1)a$ implies $s = a + s$, hence $a = 0$. To prove (v), suppose that J is a direct compression with focus p on G. Then p is a sharp element of E, so $p \in P$ by (i), and it follows from Lemma 3.2(iii) that $J = J_p$. □

Corollary 4.5. *Every comparability ℓ-group is a proper ℓ-group, hence it is a direct and a total CB-group.*

Proof. Combine Theorem 4.4(v) and Theorem 3.6. □

Theorem 4.6. *Let G be a proper ℓ-group. Then:*
(i) *P is sup/inf-closed in G;*
(ii) *if G is monotone σ-complete, then G is an RCℓ-group.*

Proof. (i) By Theorem 3.6, every sharp effect is the focus of a direct compression on G and by Corollary 3.7, G is a total ℓ-group. Let $M \subseteq P$ and suppose that $s = \bigvee_G \{p : p \in M\}$ exists in G. Evidently, $s \in E$, hence it will be sufficient to prove that s is sharp. Thus, suppose $e \in E$ with $e \leqslant s$ and $e \leqslant u - s$. Every ℓ-group satisfies the generalized distributive law, so $e = e \wedge_G s = \bigvee_G \{e \wedge_G p : p \in M\}$. For each $p \in M$, J_p is a direct compression on G, hence $e \wedge_G p = J_p(e)$ by Theorem 3.6(iii). But, for each $p \in M$, $0 \leqslant e \leqslant u - s \leqslant u - p$, so $e \wedge_G p = J_p(e) = 0$, and it follows that $e = 0$.

(ii) Let G be monotone σ-complete. By [**17**, Theorem 9.9] and Theorem 3.6(iv), G is a comparability group. By [**17**, Lemma 9.8], for each $h \in G^+$, there exists a projection $h^\# \in P$ such that $J_{h^\#}(h) = h$ and, for every projection $q \in P$, $J_q(h) = h \Rightarrow h^\# \leqslant q$. If $q \in P$ and $h^\# \leqslant q$, then by Corollary 3.12, $J_q(h) = J_q(J_{h^\#}(h)) = J_{h^\#}(h) = h$, hence $h^\# \leqslant q \Leftrightarrow J_q(h) = h$. Define $h' := u - h^\#$. If $p \in P$, then, replacing q by $u - p$, we find that $p \leqslant h' \Leftrightarrow J_{u-p}(h) = h \Leftrightarrow J_p(h) = 0$. Now let $g \in G$ and put $g' := |g|'$. By [**10**, Lemma 6.2(viii)], we have $p \leqslant g' \Leftrightarrow J_p(|g|) = 0 \Leftrightarrow g \in C(p)$ with $J_p(g) = 0$, hence G has the Rickart projection property. □

Corollary 4.7. *If G is a monotone σ-complete proper ℓ-group, then P is a σ-complete Boolean algebra.*

Proof. Assume the hypotheses. By Corollary 3.7, G is a direct CB-group, hence by Theorem 3.6(iv), P is a Boolean algebra. By Theorem 4.6(ii), G has the Rickart projection property. As G is monotone σ-complete, so is E, hence P is σ-complete by Theorem 4.2(iv). □

Theorem 4.8. *If X is a compact Hausdorff space, then the following conditions are mutually equivalent:*
(i) *X is basically disconnected;*
(ii) *$C(X, \mathbb{R})$ (Example 2.3) is monotone σ-complete;*
(iii) *$C(X, \mathbb{R})$ is an RCℓ-group.*

Proof. By [**17**, Corollary 9.3], we have (i) $\Leftrightarrow$ (ii). By Theorem 4.6, (i) $\Rightarrow$ (iii). To prove that (iii) $\Rightarrow$ (i), assume that $C(X, \mathbb{R})$ is an RC-group and let F be an open F_σ subset of X. We have to show that the closure $\overline{F}$ of F is open.

By [**19**, Theorem C, p. 217], there exists $f \in E(X)$ such that $F = \{x \in X : 0 < f(x)\}$. There is a compact open set $D \subseteq X$ such that $f'' = \chi_D$, the characteristic set function of D. As $f \in E(X)$, we have $f \leqslant f'' = \chi_D$ [**15**, Theorem 5.3(vii)], hence $F \subseteq D$, so $\overline{F} \subseteq D$. As D is open, it will be sufficient to prove that $D \subseteq \overline{F}$.

Suppose there exists $d \in D$ such that $d \notin \overline{F}$. Then there exists an open $U \subseteq X$ such that $d \in U$ and $U \cap F = \emptyset$, that is, $f = 0$ on U. By Urysohn's lemma, there exists $g \in E(X)$ such that $g(d) = 1$ and $g = 0$ on $X \setminus U$. As $C(X, \mathbb{R})$ satisfies general comparability, there is a compact open subset K of X such that $g \leqslant f$ on K and $f \leqslant g$ on $X \setminus K$. Now $x \in F \Rightarrow x \notin U \Rightarrow g(x) = 0$, whence $x \in F \Rightarrow g(x) < f(x) \Rightarrow x \notin X \setminus K \Rightarrow x \in K$. Therefore, $f \leqslant \chi_K$, and it follows that $\chi_D = f'' \leqslant \chi_K$ [**15**, Theorem 5.3(viii)]; hence $D \subseteq K$. Thus, we arrive at the contradiction $1 = g(d) \leqslant f(d) = 0$. $\square$

Without a requirement that ensures the existence of a good supply of projections, for example, general comparability, the Rickart projection property *per se* may not be a very stringent condition. For instance, if $[0, 1] \subseteq \mathbb{R}$ is the standard unit interval, then $C([0, 1], \mathbb{R})$ is a direct ℓ-group with $\{J_0, J_1\}$ as the (total) compression base, and $C([0, 1], \mathbb{R})$ has the Rickart projection property, but $[0, 1]$ is not basically disconnected.

5. *Blocks and C-blocks*

We maintain our Standing Assumption 3.8, so that G is a CB-group with unit u, unit interval E, and compression base $(J_p)_{p\in P}$.

Definition 5.1. A subset B of P is called a *block* in P if and only if B is a maximal set of pairwise compatible elements of P. A subgroup of G having the form $C(B)$, where B is a block in P, is called a *C-block*.

Lemma 5.2. *Let $B \subseteq P$ be a block. Then:*

(i) *B is a normal sub-effect algebra of E;*
(ii) *B is a Boolean algebra and, for $p, q \in B$, $u - p$ is the Boolean complement of p in B, $p \wedge_B q = p \wedge q$, and $p \vee_B q = p \vee q$;*
(iii) *if G has the Rickart projection property, then B is sup/inf-closed in P, E, and G;*
(iv) *if G is monotone σ-complete and has the Rickart projection property, then P is a σ-complete OML and B is a σ-complete Boolean algebra.*

Proof. (i) Obviously, $0, u \in B$. If $p \in B$, then pCq for all $q \in B$, hence $(u-p)Cq$ for all $q \in B$, hence the maximality of B implies that $u - p \in B$. Suppose $p, q \in B$ and $p+q \leqslant u$. Then, as $p, q \in P$ and P is an OMP, we have $p+q = p \vee q$; hence by Theorem 3.13 and the maximality of B, it follows that $p+q \in B$. Therefore, B is a sub-effect algebra of E. Suppose that $e, f, d \in E$ with $e+f+d \leqslant u$, $p := e+d \in B$, and $q := f + d \in B$. As $p, q \in P$ and P is a normal sub-effect algebra of E, it follows that $d \in P$, hence that $e = p - d, f = q - d \in P$. As P is an OMP, we have $d = p \wedge q$, hence $d \in B$ by Theorem 3.13 and the maximality of B, hence B is a normal sub-effect algebra of E.

Part (ii) follows directly from Theorems 3.13 and 3.14.

(iii) Suppose $M \subseteq B$ and $p = \bigvee\{b : b \in M\}$ exists in P. Then by Theorem 4.2(v), $B \subseteq C(M) \subseteq C(p)$, so $p \in B$ by maximality, and it follows that B is sup/inf-closed in P. By Theorem 4.2(iii), P is sup/inf-closed in E, whence B is sup/inf-closed in E. As E is always sup/inf-closed in G, it follows that B is sup/inf-closed in G.

(iv) Assume the hypotheses of (iv). By (iii), B is sup/inf-closed in G, hence B is monotone σ-complete. But a monotone σ-complete Boolean algebra is a σ-complete Boolean algebra. □

In Example 2.1, the C-blocks in $\mathbb{G}(\mathfrak{H})$ are the intersections with $\mathbb{G}(\mathfrak{H})$ of the maximal commutative self-adjoint subalgebras of the algebra $\mathbb{B}(\mathfrak{H})$ of all bounded linear operators on $\mathfrak{H}$.

Lemma 5.3. *Let $B \subseteq P$ be a block. Then:*

(i) $0, u \in B \subseteq C(B)$ *and* $C(B)$ *is a directed subgroup of* G*;*
(ii) $P \cap C(B) = B$*;*
(iii) $p \in B \Rightarrow J_p(C(B)) \subseteq C(B)$*;*
(iv) $g \in C(B) \Rightarrow P \cap CPC(g) \subseteq B$.

Proof. (i) The fact that $0, u \in B \subseteq C(B)$ is clear, and since u is an order unit in G, it is an order unit in $C(B)$, hence $C(B)$ is directed. (ii) Obviously, $B \subseteq P \cap C(B)$, and $P \cap C(B) \subseteq B$ follows from the maximality of B. (iii) Let $p, q \in P \cap C(B) = B$ and let $g \in C(B)$. Then, by Corollary 3.12, $J_q(J_p(g)) + J_{u-q}(J_p(g)) = J_p(J_q(g) + J_{u-q}(g)) = J_p(g)$, hence $J_p(g) \in C(B)$. (iv) If $g \in C(B)$ and $p \in P \cap CPC(g)$, then for all $q \in B$, we have $g \in C(q)$, hence pCq, and $p \in B$ follows from the maximality of B. □

If G is a comparability group and $g, h \in G$, then the *pseudo meet* $g \sqcap h$ and *pseudo join* $g \sqcup h$ are defined by $g \sqcap h := g - (g-h)^+$ and $g \sqcup h := g + (h-g)^+$ [**13**, Definition 5.2].

Theorem 5.4. *Let G be a comparability group, let $B \subseteq P$ be a block, and let $g, h \in C(B)$. Then:*

(i) $g^+, g^-, |g|, g \sqcap h, g \sqcup h \in C(B)$*;*
(ii) $C(B)$ *is a unital ℓ-group with unit u, hence a direct ℓ-group;*
(iii) $g \wedge_{C(B)} h = g \sqcap h$ *and* $g \vee_{C(B)} h = g \sqcup h$*;*
(iv) *the family $(\bar{J}_p)_{p \in B}$ of restrictions $\bar{J}_p$ to $C(B)$ of compressions J_p with $p \in B$ is a compression base for $C(B)$;*
(v) *$C(B)$ is a comparability group with respect to the compression base $(\bar{J}_p)_{p \in B}$;*
(vi) *the compression base $(\bar{J}_p)_{p \in B}$ coincides with the direct compression base for the proper ℓ-group $C(B)$;*
(vii) *if G is a RC-group, then $C(B)$ is a RCℓ-group;*
(viii) *if G is monotone σ-complete, then $C(B)$ is a Dedekind σ-complete proper ℓ-group.*

Proof. (i) Let $p \in P^{\pm}(g)$. By Lemma 5.3(iv), $p \in B$, hence $g^+ = J_p(g) \in C(B)$ by Lemma 5.3(iii). Thus, $g^- = (-g)^+ \in C(B)$, $|g| = g^+ + g^- \in C(B)$, $g \sqcap h = g - (g-h)^+ \in C(B)$ and $g \sqcup h = g + (h-g)^+ \in C(B)$.

(ii) Let $g, h \in C(B)$ and choose $p \in P^{\pm}(g-h)$. By Lemma 5.3(iv), $p \in B$, hence $g, h \in C(p)$. By Lemma 5.3(iii), $J_p(h), J_{u-p}(g) \in C(B)$. Since $J_{u-p}(g-h) \leqslant 0 \leqslant J_p(g-h)$, we have $J_{u-p}(g) \leqslant J_{u-p}(h)$ and $J_p(h) \leqslant J_p(g)$. Let $a := J_p(h)+J_{u-p}(g)$. Then $a \in C(B)$, $a \leqslant J_p(g) + J_{u-p}(g) = g$ and $a \leqslant J_p(h) + J_{u-p}(h) = h$, so a is a lower bound in $C(B)$ for g and h. Suppose that $b \in C(B)$ and $b \leqslant g, h$. Then $J_p(b) \leqslant J_p(h)$ and $J_{u-p}(b) \leqslant J_{u-p}(g)$, so $b = J_p(b) + J_{u-p}(b) \leqslant a$. Thus a is the infimum of g and h in $C(B)$, and it follows that $C(B)$ is an ℓ-group. As u is an order unit in $C(B)$, it follows that $C(B)$ is a unital ℓ-group with unit u, hence it is a proper ℓ-group.

(iii) By [**15**, Theorem 6.6(iii)], $g \sqcap h$ is a maximal lower bound in G for g and h, hence, since $g \sqcap h \in C(B)$ by (i), we have $g \wedge_{C(B)} h = g \sqcap h$. Likewise, $g \vee_{C(B)} h = g \sqcup h$.

(iv) The unit interval in the compressible CB-group $C(B)$ is $E \cap C(B)$. By Lemma 5.2(i), B is a normal sub-effect algebra of E, hence B is a normal sub-effect algebra of $E \cap C(B)$. If $p \in B$, then by Lemma 5.3(iii), $\bar{J}_p \colon C(B) \to C(B)$, and it is obvious that $\bar{J}_p$ is a retraction on $C(B)$, hence, since $C(B)$ is an ℓ-group, $\bar{J}_p$ is a direct compression on $C(B)$. Clearly, $(\bar{J}_p)_{p \in B}$ is a compression base for $C(B)$.

(v) If $q \in B$ and $g \in C(B)$, then $g \in C(q)$, hence $g = J_q(g) + J_{u-q}(g) = \bar{J}_q(g) + \bar{J}_{u-q}(g)$ and, therefore, g is compatible with q in the CB-group $C(B)$ with compression base $(\bar{J}_p)_{p \in B}$. For $g \in C(B)$ choose $p \in P^{\pm}(g)$. By Lemma 5.3(iv), $p \in B$, and we have $\bar{J}_{u-p}(g) \leqslant 0 \leqslant \bar{J}_p$. Therefore, with respect to the compression base $(\bar{J}_p)_{p \in B}$, $C(B)$ is a comparability group.

(vi) As $C(B)$ is an ℓ-group and a comparability group, Corollary 4.5 implies that $(\bar{J}_p)_{p \in B}$ coincides with the direct compression base for $C(B)$.

(vii) By [**15**, Theorem 6.10(i)], $g' \in CPC(g)$ for all $g \in G$, whence $g \in C(B) \Rightarrow g' \in B$, and it is clear that the restriction to $C(B)$ of the Rickart mapping $g \mapsto g'$ on G is a Rickart mapping for $C(B)$.

(viii) Follows from (ii) and Theorem 3.16. □

COROLLARY 5.5. *Let G be a monotone σ-complete comparability group and let $(p_i)_{i \in \mathbb{N}}$ be a sequence of projections in P such that, for each $n \in \mathbb{N}$, $p_1 \vee p_2 \vee \ldots \vee p_n$ exists in P. Then $s := \bigvee_G \{p_i : i \in \mathbb{N}\}$ exists in G and $s = \bigvee \{p_i : i \in \mathbb{N}\} \in P$.*

Proof. For each $n \in \mathbb{N}$, let $q_n := p_1 \vee p_2 \vee \ldots \vee p_n$. Then $(q_n)_{n \in \mathbb{N}}$ is an ascending sequence in P, so $\bigvee_G \{q_n : n \in \mathbb{N}\}$ exists in G, and it is clear that $\bigvee_G \{p_i : i \in \mathbb{N}\} = \bigvee_G \{q_n : n \in \mathbb{N}\}$, so let $s := \bigvee_G \{q_n : n \in \mathbb{N}\}$. As $(q_n)_{n \in \mathbb{N}}$ is an ascending sequence in P, we have $q_n C q_m$ for all $n, m \in \mathbb{N}$, so by Zorn's lemma, there is a maximal set B of pairwise compatible projections, that is, a block in P, with $q_n \in B$ for all $n \in \mathbb{N}$. By Theorem 3.16, $s = \bigvee_G \{q_n : n \in \mathbb{N}\} \in C(B)$, and therefore $s = \bigvee_{C(B)} \{q_n : n \in \mathbb{N}\}$. By Theorem 5.4, $C(B)$ is an ℓ-group, hence by Theorem 4.6 applied to the ℓ-group $C(B)$, we have $s \in B \subseteq P$. □

DEFINITION 5.6. Let G be a RC-group and let $g \in G$. If $\lambda \in \mathbb{Q}$, write $\lambda = m/n$ with $m, n \in \mathbb{Z}$, $n > 0$, and define $p_{g,\lambda} := ((ng - mu)^+)'$. By [**12**, Lemma 4.1], $p_{g,\lambda} \in P$ is well-defined, and we refer to the family $(p_{g,\lambda})_{\lambda \in \mathbb{Q}}$ as the *rational spectral resolution* of g in G.

In Example 2.1, if $A \in \mathbb{G}(\mathfrak{H})$, and if $(P_\lambda)_{\lambda \in \mathbb{R}}$ is the spectral resolution of A, then $(P_\lambda)_{\lambda \in \mathbb{Q}}$ is the rational spectral resolution of A in $\mathbb{G}(\mathfrak{H})$.

Theorem 5.7. *Let G be an archimedean RC-group, let $g \in G$, and let $(p_{g,\lambda})_{\lambda\in\mathbb{Q}}$ be the rational spectral resolution of g in G. Then:*

(i) $\lambda \in \mathbb{Q} \Rightarrow p_{g,\lambda} \in CPC(g)$ *and* $g \in C(p_{g,\lambda})$*;*
(ii) *if $p \in P$, then $pCp_{g,\lambda}$ for all $\lambda \in \mathbb{Q}$ if and only if $g \in C(p)$;*
(iii) *there is a block B in P such that $p_{g,\lambda} \in B$ for all $\lambda \in \mathbb{Q}$, and for any such block B, we have $g \in C(B)$.*

Proof. (i) See [**12**, Theorem 4.5(i)]. (ii) See [**12**, Theorem 4.9].

(iii) By (i), the projections in the family $(p_{g,\lambda})_{\lambda\in\mathbb{Q}}$ are pairwise compatible, hence by Zorn's lemma, there is a block B containing the family $(p_{g,\lambda})_{\lambda\in\mathbb{Q}}$. Furthermore, if B is any such block and $p \in B$, then $g \in C(p)$, and it follows that $g \in C(B)$. □

Let G be an archimedean RC-group, let $g \in G$, and let $q \in P$. By [**12**, Theorems 4.12 and 5.9(iii)] the rational spectral resolution of q is given by $p_{q,\lambda} = 0$ if $\lambda < 0$, $p_{q,\lambda} = u - q$ if $0 \leqslant \lambda < 1$, and $p_{q,\lambda} = u$ if $1 \leqslant \lambda$, for all $\lambda \in \mathbb{Q}$. Therefore, $g \in C(q)$ if and only if $p_{g,\lambda}Cp_{q,\mu}$ for all $\lambda, \mu \in \mathbb{Q}$. Hence we can and do extend the notion of compatibility as follows.

Definition 5.8. Let G be an archimedean RC-group and let $g, h \in G$. We say that g and h are *compatible* and write gCh if and only if $p_{g,\lambda}Cp_{h,\mu}$ for all $\lambda, \mu \in \mathbb{Q}$.

Theorem 5.9. *Let G be an archimedean RC-group. Then a subset of G is a C-block if and only if it is a maximal set of pairwise compatible elements of G.*

Proof. Let B be a block in P. If $g \in C(B)$, then $p \in B \Rightarrow g \in C(p) \Rightarrow pCp_{g,\lambda}$ for all $\lambda \in \mathbb{Q}$, whence $p_{g,\lambda} \in B$ for all $\lambda \in \mathbb{Q}$ by the maximality of B. Consequently, any two elements of $C(B)$ are compatible. If $h \in G$ and hCg for all $g \in C(B)$, then hCp for all $p \in B$, so $h \in C(B)$. Therefore, $C(B)$ is a maximal set of pairwise compatible elements of G.

Conversely, suppose H is a maximal set of pairwise compatible elements of G, and let $B := H \cap P$. If $g \in H$ and $\lambda \in \mathbb{Q}$, then $p_{g,\lambda}Ch$ for all $h \in H$, so $p_{g,\lambda} \in H$ by maximality. Conversely, if $g \in G$ and $p_{g,\lambda} \in H$ for all $\lambda \in \mathbb{Q}$, then gCh for all $h \in H$, whence $g \in H$ by maximality. Therefore, $g \in H \Leftrightarrow p_{g,\lambda} \in H$ for all $\lambda \in \mathbb{Q}$, and it follows that $H = C(B)$ where $B := H \cap P$. □

The following theorem summarizes our main results.

Theorem 5.10. *If G is an archimedean RC-group, then P is an OML, P is covered by its blocks, the blocks in P are the maximal Mackey compatible subsets of P, and they are Boolean algebras. Likewise, G is covered by its C-blocks, which are in bijective correspondence with the blocks in P; the C-blocks in G are the maximal compatible subsets of G and are archimedean lattice-ordered RC-groups. If G is a monotone σ-complete RC-group, then G is archimedean and the C-blocks in G are Dedekind σ-complete archimedean total RCℓ-groups. Furthermore, every monotone σ-complete proper ℓ-group is a Dedekind σ-complete archimedean total RCℓ-group.*

6. *Dedekind σ-complete proper ℓ-groups*

In [**17**, Chapter 9], a faithful continuous-function representation is given for Dedekind σ-complete unital ℓ-groups, hence for the C-blocks in a monotone σ-complete RC-group. Our purpose in this final section is to indicate how this develops and to sketch an alternative but related representation in terms of g-tribes. We maintain our Standing Assumption 3.8, so that G is a CB-group with unit u, unit interval E, and compression base $(J_p)_{p\in P}$. To avoid triviality, *we also assume that* $G \neq \{0\}$ (*that is,* $u \neq 0$).

If X is a nonempty set, we understand that the function space $\mathbb{R}^X$ consisting of all functions $f\colon X \to \mathbb{R}$ is organized into a locally convex, archimedean, partially ordered, directed, and lattice-ordered real topological vector space with pointwise operations, pointwise partial order, and the topology of pointwise convergence.

Definition 6.1. A *state* on G is an order-preserving additive group homomorphism $\omega\colon G \to \mathbb{R}$ such that $\omega(u) = 1$ [**17**, p. 60]. The *state space* of G is the convex subset $\Omega \subseteq \mathbb{R}^G$ consisting of all states on G, with the relative topology inherited from $\mathbb{R}^G$. A state $\omega \in \Omega$ is said to be *discrete* if and only if $\omega(G)$ is an additive cyclic subgroup of $\mathbb{R}$ [**17**, p. 70]. A state $\omega \in \Omega$ is said to be *σ-additive* if and only if, whenever $(g_n)_{n\in\mathbb{N}}$ is an ascending sequence in G and $g = \bigvee_G\{g_n : n \in \mathbb{N}\}$, it follows that $\omega(g) = \sup\{\omega(g_n) : n \in \mathbb{N}\}$. The set of all extreme points of the convex set Ω, called the *extreme boundary* of Ω, is denoted by $\partial_{\mathrm{e}}\Omega$, and it is understood that $\partial_{\mathrm{e}}\Omega$ carries the relative topology inherited from Ω (that is, the topology of pointwise convergence).

By [**17**, Corollary 4.4 and Proposition 6.7], Ω is a nonempty convex compact Hausdorff space. By [**17**, Theorem 4.14], G is archimedean if and only if $G^+ = \{g \in G : 0 \leqslant \omega(g)$ for all $\omega \in \Omega\}$. As a consequence of the Krein–Mil'man theorem, Ω is the closed convex hull of $\partial_{\mathrm{e}}\Omega$.

Suppose that B is a block in P. In the literature, there are a number of different (topologically equivalent) definitions of the Stone space of the Boolean algebra B. For our purposes, we define the *Stone space* $X(B)$ of B to be the set of all functions $\gamma \in \mathbb{R}^B$ such that $\gamma(B) \subseteq \{0,1\}$, $\gamma(u) = 1$, and $\gamma(p+q) = \gamma(p) + \gamma(q)$ whenever $p, q, p+q \in B$. The functions $\gamma \in X(B)$ are in bijective correspondence with the maximal proper ideals in B under $\gamma \leftrightarrow \gamma^{-1}(0)$. With the topology of pointwise convergence inherited from $\mathbb{R}^B$, the Stone space $X(B)$ is a compact Hausdorff totally-disconnected topological space, and B is isomorphic to the field of compact open subsets of $X(B)$ under the correspondence $p \leftrightarrow \{\gamma \in X : \gamma(p) = 1\}$ for all $p \in B$.

Theorem 6.2. *Let G be a proper ℓ-group with state space Ω, let P be the Boolean algebra of projections in E, and let $X(P)$ be the Stone space of P. If $\omega \in \partial_{\mathrm{e}}\Omega$, let $\omega|_P$ be the restriction of ω to P. Then:*

(i) $\omega \in \partial_{\mathrm{e}}\Omega \Rightarrow \omega|_P \in X(P)$;

(ii) *if G is a RCℓ-group, then the mapping $\omega \mapsto \omega|_P$ is a homeomorphism of $\partial_{\mathrm{e}}\Omega$ onto $X(P)$, hence $\partial_{\mathrm{e}}\Omega$ is a compact subset of Ω;*

(iii) *if G is monotone σ-complete, then G is a RCℓ-group and both $\partial_{\mathrm{e}}\Omega$ and $X(P)$ are basically disconnected.*

Proof. (i) See [**17**, Lemma 8.10(d)]. (ii) See [**17**, Theorem 8.14]. Part (iii) follows from (ii) and Theorems 4.2 and 4.6. □

DEFINITION 6.3. Denote by $\operatorname{Aff}(\Omega)$ the vector subspace of $C(\Omega, \mathbb{R})$ consisting of the affine functions on Ω, that is, $\operatorname{Aff}(\Omega)$ is the vector space over $\mathbb{R}$ of all continuous functions $f\colon \Omega \to \mathbb{R}$ such that, for all $t \in [0,1] \subseteq \mathbb{R}$ and all $\alpha, \beta \in \Omega$, $f(t\alpha + (1-t)\beta) = tf(\alpha) + (1-t)f(\beta)$. If $g \in G$, define $\widehat{g} \in \operatorname{Aff}(\Omega)$ by evaluation (that is, $\widehat{g}(\omega) := \omega(g)$ for all $\omega \in \Omega$), and define $\widetilde{g}$ to be the restriction of $\widehat{g}$ to $\partial_{\mathrm{e}}\Omega$. Also define $\widehat{G} := \{\widehat{g} : g \in G\}$ and $\widetilde{G} := \{\widetilde{g} : g \in G\}$.

As is easily verified, $\operatorname{Aff}(\Omega)$ is a (supremum) norm-closed vector subspace of the real Banach space $C(\Omega, \mathbb{R})$, hence $\operatorname{Aff}(\Omega)$ is a real Banach space. Under pointwise partial order, $\operatorname{Aff}(\Omega)$ is an archimedean partially ordered real vector space, and the constant function 1 is an order unit in $\operatorname{Aff}(\Omega)$, hence $\operatorname{Aff}(\Omega)$ is directed. If $0 \leqslant f \in \operatorname{Aff}(\Omega)$, there exists $n \in \mathbb{N}$ with $f \leqslant n \cdot 1$, so with $e := (1/n)f$, we have $0 \leqslant e \leqslant 1$ and $f = e + e + \ldots + e$ (n summands), hence $\operatorname{Aff}(\Omega)$ is an archimedean unital group under addition. As such, $\operatorname{Aff}(\Omega)$ has its own state space; however, by [**17**, Theorem 7.1], nothing new is achieved by passing to the state space of $\operatorname{Aff}(\Omega)$ as it is affinely homeomorphic to Ω under a natural evaluation mapping. With 1 as the order unit, $\operatorname{Aff}(\Omega)$ is an order-unit Banach space [**1**, p. 69], and the order-unit norm coincides with the supremum norm.

THEOREM 6.4. *Both the mappings $g \mapsto \widehat{g}$ from G to $\operatorname{Aff}(\Omega)$ and $g \mapsto \widetilde{g}$ from G to $C(\partial_{\mathrm{e}}(\Omega), \mathbb{R})$ are order-preserving additive group homomorphisms. Moreover, the following conditions are mutually equivalent:*
(i) *G is archimedean;*
(ii) *$g \mapsto \widehat{g}$ is an isomorphism of G onto $\widehat{G}$ (as ordered groups);*
(iii) *$g \mapsto \widetilde{g}$ is an isomorphism of G onto $\widetilde{G}$ (as ordered groups).*

Proof. See [**17**, Theorem 7.7]. □

If G is archimedean, the isomorphism $g \mapsto \widehat{g}$ of G onto $\widehat{G} \subseteq \operatorname{Aff}(\Omega)$ and the isomorphism $g \mapsto \widetilde{g}$ of G onto $\widetilde{G} \subseteq C(\partial_{\mathrm{e}}\Omega, \mathbb{R})$ provide faithful function-representations for G, and the compression base for G can be reproduced faithfully both in $\widehat{G}$ and in $\widetilde{G}$. This raises the question of how to give a perspicuous intrinsic characterization of $\widehat{G}$ in $\operatorname{Aff}(\Omega)$ and of $\widetilde{G}$ in $C(\partial_{\mathrm{e}}(\Omega), \mathbb{R})$. Unfortunately, answers to these questions are known only 'under fairly restrictive hypotheses on G' [**17**, p. 118]. However, for the situation of interest to us here, the answers are known for the case in which G is a Dedekind σ-complete proper ℓ-group.

THEOREM 6.5. *Let G be a Dedekind σ-complete proper ℓ-group. Then:*
(i) $\widehat{G} = \{f \in \operatorname{Aff}(\Omega) : f(\omega) \in \omega(G)$ *for all discrete* $\omega \in \Omega\}$*;*
(ii) $\widetilde{G} = \{f \in C(\partial_{\mathrm{e}}\Omega, \mathbb{R}) : f(\omega) \in \omega(G)$ *for all discrete* $\omega \in \partial_{\mathrm{e}}\Omega\}$*.*

Proof. See [**17**, Corollaries 9.14 and 9.15]. □

Suppose G is a Dedekind σ-complete proper ℓ-group. Even with Theorem 6.5 at our disposal, there remains the issue of how to calculate existing suprema and infima of countably infinite subsets of $\widehat{G}$ or $\widetilde{G}$, since these are not necessarily the pointwise suprema and infima. We shall now sketch an alternative representation for G in terms of a *g-tribe* of functions (first introduced in [**6**]) in which existing suprema and infima are calculated pointwise. (The 'g' in g-tribe stands for 'group'.)

DEFINITION 6.6. Let X be a nonempty set. A *g-tribe* on X is a subgroup T of $\mathbb{R}^X$ such that (i) every function $f \in T$ is bounded; (ii) $1 \in T$; and (iii) whenever $(f_n)_{n\in\mathbb{N}}$ is a sequence in T and there exists $f \in T$ with $f_n \leqslant f$ for all $n \in \mathbb{N}$, then $\bigvee_{F(X)}\{f_n : n \in \mathbb{N}\} \in T$.

Note that Definition 6.6 does not require that T is a sup/inf-closed subgroup of $\mathbb{R}^X$ since the condition that $(f_n)_{n\in\mathbb{N}}$ is bounded by $f \in T$ is stronger than the condition that it is bounded by $f \in \mathbb{R}^X$. With the partial order induced from $\mathbb{R}^X$ (that is, the pointwise partial order), a g-tribe T on X is a Dedekind σ-complete unital ℓ-group, hence we can and do organize it into a proper ℓ-group in such a way that the Boolean algebra P_T of projections in T is the set of all sharp elements of T. As T is Dedekind σ-complete, P_T is a σ-complete Boolean algebra. A projection in P_T necessarily has the form χ_A, where $A \subseteq X$, and the set $\mathcal{B}(T) := \{A \subseteq X : \chi_A \in P_T\}$ is a σ-field of subsets of X isomorphic to P_T under $A \leftrightarrow \chi_A$.

The following theorem [**6**; **8**, Theorem 7.1.24] can be regarded as a generalization of the Loomis–Sikorski representation theorem for σ-complete Boolean algebras.

THEOREM 6.7. *Let (G, u) be a Dedekind σ-complete unital ℓ-group with unit u and let T be the g-tribe on $\partial_{\mathrm{e}}\Omega$ generated by $\widetilde{G}$. Then there is a uniquely determined surjective order-preserving group homomorphism $\eta\colon T \to G$ such that $\eta(1) = u$, η preserves all existing countable suprema and infima, and, if $f \in T$, $g \in G$, then $\eta(f) = g$ if and only if $\{\omega \in \partial_{\mathrm{e}}\Omega : f(\omega) \neq \widetilde{g}(\omega)\}$ is a meager subset of $\partial_{\mathrm{e}}\Omega$.*

Let G be a Dedekind σ-complete proper ℓ-group. The triple $(\partial_{\mathrm{e}}\Omega, T, \eta)$ in Theorem 6.7 is called the *canonical representation* of G. In the next two propositions, we describe some properties of canonical representations. First we need a lemma.

LEMMA 6.8. *Let T be a g-tribe. Then for every $f \in T$ and every characteristic function $\chi_A \in T$ we have $f \cdot \chi_A \in T$ (here '$\cdot$' means the pointwise multiplication of functions).*

Proof. Assume first that $0 \leqslant f \leqslant 1$. Then $f \cdot \chi_A = f \wedge \chi_A \in T$. Now assume that $f \geqslant 0$. Since T is a unital ℓ-group, we have $f = f_1 + f_2 + \ldots + f_n$, $0 \leqslant f_i \leqslant 1$, $i = 1, 2, \ldots, n$. Then $f \cdot \chi_A = \sum_{i=1}^n f_i \cdot \chi_A \in T$. Since T is directed, for any $f \in T$, $f = f_1 - f_2$, $f_1, f_2 \in T^+$, hence $f \cdot \chi_A = f_1 \cdot \chi_A - f_2 \cdot \chi_A \in T$. ☐

PROPOSITION 6.9. *Let (G, u) be a Dedekind σ-complete unital ℓ-group with unit u, and let $(\partial_{\mathrm{e}}\Omega, T, \eta)$ be its canonical representation (obtained by the Loomis–Sikorski Theorem 6.7). Then $\eta(T^+) = G^+$.*

Proof. Since η is order preserving, we have $\eta(T^+) \subseteq G^+$. To prove the inverse inclusion, we need to prove that every $g \in G^+$ has a representative $f \in T^+$ such that $\eta(f) = g$. Let $g \in G^+$. Since η is surjective, there is $f_1 \in T$ such that $\eta(f_1) = g$. Put $M := \{\omega \in \partial_{\mathrm{e}}\Omega : f_1(\omega) < 0\}$. Then $f_1 = f_1 \cdot \chi_M + f_1 \cdot \chi_{M^c}$, $f_1 \cdot \chi_M < 0$, $f_1 \cdot \chi_{M^c} \geqslant 0$, and $f_1 \cdot \chi_M, f_1 \cdot \chi_{M^c} \in T$ by Lemma 6.8. Let $p := \eta(\chi_M)$. Since η preserves lattice operations, p is a characteristic element of G and, by [**27**], $J_p(g) = \eta(f_1 \cdot \chi_M) \leqslant 0$, while $J_{u-p}(g) = \eta(f_1 \cdot \chi_{M^c}) \geqslant 0$. So we have $g^- = -J_p(g)$, $g^+ = J_{u-p}(g)$, and since $g \in G^+$, we have $g^- = 0$. It follows that $\eta(f_1 \cdot \chi_M) = 0$. Therefore, we may replace f_1 by $f := f_1 \cdot \chi_{M^c} \in T^+$, and $\eta(f) = g$. □

Proposition 6.10. *Let (G, u) be a Dedekind σ-complete unital ℓ-group with unit u and let $(\partial_e\Omega, T, \eta)$ be its canonical representation. Then $H := \ker(\eta)$ is an ℓ-ideal of T and G is isomorphic with the quotient T/H.*

Proof. By [**17**, Definition, p. 8], an ideal of a partially ordered group G is any directed convex subgroup of G. Let $H := \ker(\eta) = \{f \in T : \eta(f) = 0\}$. Since η is an order-preserving group homomorphism, H is a convex subgroup of T. Moreover, if $(g_i)_{i\in\mathbb{N}}$ is a nondecreasing sequence of elements in H, bounded above by an $f \in T$, then $\bigvee g_i =: g$ exists in T and belongs to H.

To prove that H is directed, take $h \in H$. Since T is directed, we have $h = f_1 - f_2$ for some $f_1, f_2 \in T^+$. We have to prove that there are $h_1, h_2 \in T^+ \cap H$ such that $h = h_1 - h_2$. By the Loomis–Sikorski Theorem 6.7, $h \in H$ if and only if $\chi_{N(h)} \in H$ (equivalently, if and only if $N(h)$ is a meager set), where $N(h) = \{\omega : f(\omega) \neq 0\}$ is the carrier of h. We may write $h = h \cdot \chi_{N(h)}$, since $h = 0$ on $N(h)^c$. By Lemma 6.8, $h \cdot \chi_{N(h)} \in T$, and $h = h \cdot \chi_{N(h)} = f_1 \cdot \chi_{N(h)} - f_2 \cdot \chi_{N(h)}$, $f_1 \cdot \chi_{N(h)}, f_2 \cdot \chi_{N(h)} \in T$.

Observe that for any $g, \chi_A \in T$, there is $n > 0$ such that $-n\chi_A \leqslant g \cdot \chi_A \leqslant n\chi_A$, so that $\chi_A \in H$ implies $g \cdot \chi_A \in H$.

Applying this observation and the fact that $\chi_{N(h)} \in H$, we obtain that $f_1 \cdot \chi_{N(h)}, f_2 \cdot \chi_{N(h)} \in H \cap T^+$, which entails that H is directed, hence an ideal of T. According to [**17**, Corollary 1.14], G/H is a lattice-ordered group. We have, for $x, y \in T$, that $x \sim_H y$ if and only if $x - y \in H$, which is equivalent with $N(x-y) \in H$. This yields $x \sim_H y$ if and only if $\eta(x) = \eta(y)$, and hence $T/H \equiv G$. □

For the canonical representation $(\partial_e\Omega, T, \eta)$, we have $\eta(P_T) = P$ (see [**7**]). By a generalization of the Butnariu–Klement theorem [**5**], every $f \in T$ is $\mathcal{B}(T)$-measurable. Suppose that m is a σ-additive state on T, and define the σ-additive probability measure μ on the σ-field $\mathcal{B}(T)$ by $\mu(A) := m(\chi_A)$ for all $A \in \mathcal{B}(T)$. Then, by [**9**],

$$f \in T \Rightarrow m(f) = \int_{\partial_e\Omega} f(\omega)\mu(d\omega).$$

If $\mathcal{F}$ is the σ-field of real Borel sets, then a (*sharp real*) *observable* for G is a mapping $\Lambda\colon \mathcal{F} \to P$ such that (i) $\Lambda(\mathbb{R}) = u$ and (ii) if $(M_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint sets in $\mathcal{F}$, then $\Lambda(\bigcup_{n\in\mathbb{N}} M_n) = \bigvee_P\{\Lambda(M_n) : n \in \mathbb{N}\}$. Each such observable Λ is uniquely determined by its *spectral resolution*; that is, the mapping $\mathbb{R} \to P$ defined by $\lambda \mapsto \Lambda((-\infty, \lambda])$. For the general theory of observables (sharp and unsharp), see [**3**, **24**, **30**].

Theorem 6.11. *Let G be a Dedekind σ-complete proper ℓ-group and let $(\partial_e\Omega, T, \eta)$ be the canonical representation of G. Then we have the following.*

(i) *G is a total RC-group, P is the set of sharp elements in E, and P is a σ-complete Boolean algebra.*

(ii) *The Rickart mapping on G is given by*

$$g' = \eta(\widetilde{g}^{-1}(0)) \quad \textit{for every } g \in G.$$

(iii) *For every $g \in G$, there exists a sharp real observable Λ_g on G such that, for every σ-additive state ω on G,*

$$\omega(g) = \int_{\mathbb{R}} \lambda\omega(\Lambda_g(d\lambda)),$$

and this observable is determined by its spectral resolution

$$\Lambda_g((-\infty,\lambda]) = \eta(\widetilde{g}^{-1}(-\infty,\lambda]) \quad \textit{for every } \lambda \in \mathbb{R}.$$

(iv) *If $\lambda \in \mathbb{Q}$, then $p_{g,\lambda} = \Lambda_g((-\infty,\lambda])$. Moreover, if $\lambda = m/n$ with $m \in \mathbb{Z}$, $n \in \mathbb{N}$, then λ is a rational eigenvalue of Λ_g; that is, $\Lambda_g(\{\lambda\}) \neq 0$, if and only if $(ng - mu)' \neq 0$.*

(v) *For all $g, h \in G$, $\Lambda_{g+h} = \Lambda_g + \Lambda_h$, $\Lambda_{g\wedge h} = \Lambda_g \wedge \Lambda_h$, $\Lambda_{g\vee h} = \Lambda_g \vee \Lambda_h$, where the operations on the right-hand sides are defined by the functional calculus for observables.*

Proof. (i) By Corollary 3.7, G is total; by Theorem 3.6(iv) P is the set of sharp effects in G; and by Corollary 4.7, P is a σ-complete Boolean algebra.

(ii) Follows from [**27**, Theorem 4.2].

(iii) The mapping $\Lambda_g(X) = \eta(\widetilde{g}^{-1}(X))$, where X is a real Borel set, defines a sharp real observable with the desired properties ([**27**, Theorem 1.2]).

(iv) Follows from [**27**, Theorem 4.4].

(v) See [**26**, **27**].

□

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