On the structure of higher order quantum maps

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1 Affine subspaces and higher order maps

1.1 The category FinVect

sec:fv

The category FinVect is a basic example of a compact closed category, whose structure underlies all the results in this paper. For basics on category theory and symmetric monoidal categories, see [MacLane]. For compact closed categories and the use of categories in quantum theory, see [book]. [Kelly compact closed]

Let FinVect be the category of finite dimensional real vector spaces with linear maps. We will denote the usual tensor product by $\otimes$, then $(\mathrm{FinVect}, \otimes, I = \mathbb{R})$ is a symmetric monoidal category, with the associators, unitors and symmetries given by the obvious isomorphisms

$$\begin{aligned} \alpha_{U,V,W} &: (U \otimes V) \otimes W \simeq U \otimes (V \otimes W), \\ \lambda_V &: I \otimes V \simeq V, \qquad \rho_V : V \otimes I \simeq V, \\ \sigma_{U,V} &: U \otimes V \simeq V \otimes U. \end{aligned}$$

Let $(-)^* : V \mapsto V^*$ be the usual vector space dual, with duality denoted by $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}$. We will use the canonical identification $V^{**} = V$ and $(V_1 \otimes V_2)^* = V_1^* \otimes V_2^*$. With this duality, FinVect is compact closed. This means that for each object V, there are morphisms $\eta_V : I \to V^* \otimes V$ (the "cup") and $\epsilon_V : V \otimes V^* \to I$ (the "cap") such that the following snake identities hold:

$$(\epsilon_V \otimes V) \circ (V \otimes \eta_V) = V, \qquad (V^* \otimes \epsilon_V) \circ (\eta_V \otimes V^*) = V^*, \tag{1}$$ eq:snake

here we denote the identity map on the object V by V.

Let us identify these morphisms. First, η_V is a linear map $\mathbb{R} \to V^* \otimes V$, which can be identified with the element $\eta_V(1) \in V^* \otimes V$ and $\epsilon_V \in (V \otimes V^*)^* = V^* \otimes V$ is again an element of the same space. Choose a basis $\{e_i\}$ of V, let $\{e_i^*\}$ be the dual basis of V^*, that is, $\langle e_i^*, e_j \rangle = \delta_{i,j}$. Let us define

$$\eta_V(1) = \epsilon_V := \sum_i e_i^* \otimes e_i.$$

It is easy to see that this definition does not depend on the choice of the basis, indeed, ϵ_V is the linear functional on $V \otimes V^*$ defined by

$$\langle \epsilon_V, x \otimes x^* \rangle = \langle x^*, x \rangle, \qquad x \in V,\ x^* \in V^*.$$

It is also easily checked that the snake identities (eq:snake 1) hold.

For two objects V and W in FinVect, we will denote the set of all morphisms (i.e. linear maps) $V \to W$ by $\mathrm{FinVect}(V,W)$. Then $\mathrm{FinVect}(V,W)$ is itself a real linear space and we have the well-known identification $\mathrm{FinVect}(V,W) \simeq V^* \otimes W$. This can be given as follows: for each $f \in \mathrm{FinVect}(V,W)$, we have $C_f := (V^* \otimes f)(\epsilon_V) = \sum_i e_i^* \otimes f(e_i) \in V^* \otimes W$. Conversely, since $\{e_i^*\}$ is a basis of V^*, any element $w \in V^* \otimes W$ can be uniquely written as $w = \sum_i e_i^* \otimes w_i$ for $w_i \in W$, and since $\{e_i\}$ is a basis of V, the assignment $f(e_i) := w_i$ determines a unique map $f : V \to W$. The relations between $f \in \mathrm{FinVect}(V,W)$ and $C_f \in V^* \otimes W$ can be also written as

$$\langle C_f, x \otimes y^* \rangle = \langle \epsilon_V, x \otimes f^*(y^*) \rangle = \langle f^*(y^*), x \rangle = \langle y^*, f(x) \rangle, \qquad x \in V,\ y^* \in W^*,$$

here $f^* : W^* \to V^*$ is the adjoint of f. Note that by compactness, the *internal hom* in FinVect satisfies $V \multimap W \simeq V^* \otimes W$, so that in the case of FinVect, the object $V \multimap W$ can be identified with the space of linear maps $\mathrm{FinVect}(V,W)$.

The following two examples are most important for us.

:classical *Example* 1. Let $V = \mathbb{R}^N$. In this case, we fix the canonical basis $\{|i\rangle,\ i = 1, \dots, N\}$. We will identify $(\mathbb{R}^N)^* = \mathbb{R}^N$, with duality $\langle x, y \rangle = \sum_i x_i y_i$, in particular, we identify $I = I^*$. We then have $\epsilon_V = \sum_i |i\rangle \otimes |i\rangle$ and if $f : \mathbb{R}^N \to \mathbb{R}^M$ is given by the matrix A in the two canonical bases, then $C_f = \sum_i |i\rangle \otimes A|i\rangle$ is the vectorization of A.

xm:quantum *Example* 2. Let $V = M_n^h$ be the space of $n \times n$ complex hermitian matrices. We again identify $(M_n^h)^* = M_n^h$, with duality $\langle A, B \rangle = \mathrm{Tr}\, A^T B$, where A^T is the usual transpose of the matrix A. Let us choose the basis in M_n^h, given as

$$\Big\{ |j\rangle\langle k| + |k\rangle\langle j|,\ j \le k,\ i\Big(|j\rangle\langle k| - |k\rangle\langle j|\Big),\ j < k \Big\}.$$

Then one can check that

$$\Big\{ \frac{1}{2}\Big(|j\rangle\langle k| + |k\rangle\langle j|\Big),\ j \le k,\ \frac{i}{2}\Big(|k\rangle\langle j| - |j\rangle\langle k|\Big),\ j < k \Big\}$$

is the dual basis and we have

$$\epsilon_V = \sum_{j,k} |j\rangle\langle k| \otimes |j\rangle\langle k|.$$

For any $f : M_n^h \to M_m^h$,

$$C_f = \sum_{j,k} |j\rangle\langle k| \otimes f(|j\rangle\langle k|)$$

is the Choi matrix of f.

1.2 The category Af

We now introduce the category Af, whose objects are of the form $X = (V_X, A_X)$, where V_X is an object in FinVect and $A_X \subseteq V_X$ is a proper affine subspace, see Appendix sec:app_affine B for definitions and basic properties. Morphisms $X \xrightarrow{f} Y$ in Af are linear maps $f : V_X \to V_Y$ such that $f(A_X) \subseteq A_Y$. For two objects X and Y and a linear map $f : V_X \to V_Y$, we write $X \xrightarrow{f} Y$ with the meaning that f is a morphism in Af. In particular, if $V_X = V_Y = V$, then $X \xrightarrow{id_V} Y$ means that $A_X \subseteq A_Y$. The set of all morphisms $X \xrightarrow{f} Y$ in Af will be denoted by $\mathrm{Af}(X, Y)$.

For any object X, we put

$$L_X := \mathrm{Lin}(A_X), \quad S_X := \mathrm{Span}(A_X), \quad D_X = \dim(V_X), \quad d_X = \dim(L_X).$$

We have

$$A_X = a + L_X = S_X \cap \{\tilde{a}\}^{\sim}, \tag{2}$$ eq:ALS

for any choice of elements $a \in A_X$ and $\tilde{a} \in A_X^*$.

We will next introduce a monoidal structure $\otimes$ as follows. For two objects X and Y, we put $V_{X\otimes Y} = V_X \otimes V_Y$ and construct the affine subspace $A_{X\otimes Y}$ as the affine span of

$$A_X \otimes A_Y = \{a \otimes b,\ a \in A_X,\ b \in A_Y\}.$$

Fix any $\tilde{a}_X \in A_X^*$ and $\tilde{a}_Y \in A_Y^*$. Since $A_X \otimes A_Y \subseteq \{\tilde{a}_X \otimes \tilde{a}_Y\}^*$, the affine span of $A_X \otimes A_Y$ is a proper affine subspace and we have by Lemma lemma:dual 15

$$A_{X\otimes Y} := \mathrm{Aff}(A_X \otimes A_Y) = \{A_X \otimes A_Y\}^{**}.$$

sor_spaces **Lemma 1.** *For any $a_X \in A_X$, $a_Y \in A_Y$, we have*

$$\begin{aligned} L_{X\otimes Y} &= \mathrm{Lin}(A_X \otimes A_Y) = \mathrm{Span}(\{x \otimes y - a_X \otimes a_Y,\ x \in A_X,\ y \in A_Y\}) && (3) \quad \texttt{eq:lxy1} \\ &= (a_X \otimes L_Y) + (L_X \otimes a_Y) + (L_X \otimes L_Y) && (4) \quad \texttt{eq:lxy} \end{aligned}$$

(here + denotes the direct sum of subspaces). We also have

$$S_{X\otimes Y} = S_X \otimes S_Y.$$

Proof. The equality (eq:lxy1 3) follows from Lemma lemma:dual 15. For any $x \in A_X$, $y \in A_Y$ we have

$$x \otimes y - a_X \otimes a_Y = a_X \otimes (y - a_Y) + (x - a_X) \otimes a_Y + (x - a_X) \otimes (y - a_Y),$$

so that $L_{X\otimes Y} = \mathrm{Lin}(A_X \otimes A_Y)$ is contained in the subspace on the RHS of (eq:lxy 4). Let d be the dimension of this subspace, then clearly

$$d_{X\otimes Y} \le d \le d_X + d_Y + d_X d_Y.$$

On the other hand, any element of S_X has the form tx for some $t \in \mathbb{R}$ and $x \in A_X$, so that it is easily seen that $S_X \otimes S_Y = S_{X\otimes Y}$. Hence

$$\begin{aligned} d_{X\otimes Y} &= \dim(L_{X\otimes Y}) = \dim(S_{X\otimes Y}) - 1 = \dim(S_X)\dim(S_Y) - 1 = (d_X + 1)(d_Y + 1) - 1 \\ &= d_X + d_Y + d_X d_Y. \end{aligned}$$

This completes the proof. $\square$

a:monoidal **Lemma 2.** *Let $I = (\mathbb{R}, \{1\})$. Then $(\mathrm{Af}, \otimes, I)$ is a symmetric monoidal category.*

Proof. Note that this structure is inherited from the symmetric monoidal structure in FinVect. To show that $\otimes$ is a functor, we have to check that for $X_1 \xrightarrow{f} Y_1$ and $X_2 \xrightarrow{g} Y_2$ in Af, we have $X_1 \otimes Y_1 \xrightarrow{f \otimes g} X_2 \otimes Y_2$ which amounts to showing that

$$(f \otimes g)(A_{X_1 \otimes Y_1}) \subseteq A_{X_2 \otimes Y_2}.$$

Let $x \in A_{X_1}$, $y \in A_{Y_1}$, then $f(x) \otimes g(y) \in A_{X_2} \otimes A_{Y_2} \subseteq A_{X_2 \otimes Y_2}$. Since $A_{X_1 \otimes Y_1}$ is the affine subspace generated by $A_{X_1} \otimes A_{Y_1}$, the above inclusion follows by linearity of $f \otimes g$.

It only remains to prove that the associators, unitors and symmetries from FinVect are morphisms in Af. We will prove this for the associators $\alpha_{X,Y,Z} : V_X \otimes (V_Y \otimes V_Z) \to (V_X \otimes V_Y) \otimes V_Z$, the other proofs are similar. We need to check that $\alpha_{X,Y,Z}(A_{X \otimes (Y \otimes Z)}) \subseteq A_{(X \otimes Y) \otimes Z}$. It is easily checked that $A_{X \otimes (Y \otimes Z)}$ is the affine span of elements of the form $x \otimes (y \otimes z)$, $x \in A_X$, $y \in A_Y$ and $z \in A_Z$, and we have

$$\alpha_{X,Y,Z}(x \otimes (y \otimes z)) = (x \otimes y) \otimes z \in A_{(X \otimes Y) \otimes Z}$$

for all such elements. The desired inclusion follows by linearity. □

coro:dual

By Corollary 4, the dual affine subspace A_X^* is a proper affine subspace in V_X^*, so that $X^* := (V_X^*, A_X^*)$ is an object in Af. We have $X^{**} = X$ and the corresponding subspaces are related as

$$L_{X^*} = S_X^{\perp}, \qquad S_{X^*} = L_X^{\perp}. \tag{5}$$ eq:duality

It is easily seen that for any $X \xrightarrow{f} Y$, the adjoint map satisfies $f^*(A_Y^*) \subseteq A_X^*$, so that $Y^* \xrightarrow{f^*} X^*$ and the duality $(-)^*$ is a full and faithful functor $\mathrm{Af}^{op} \to \mathrm{Af}$. However, as we will see below, Af with this monoidal structure and duality is not compact closed. Nevertheless, we next show that it is *-autonomous, which will be crucial for the structure of higher order objects studied further. For details on *-autonomous categories, see [Barr kniha,...].

Theorem 1. *$(\mathrm{Af}, \otimes, I)$ is a *-autonomous category, with duality $(-)^*$, such that $I^* = I$.*

lemma:monoidal

Proof. By Lemma 2, we have that $(\mathrm{Af}, \otimes, I)$ is a symmetric monoidal category. We have also seen that the duality $(-)^*$ is a full and faithful contravariant functor. We only need to check the natural isomorphisms

$$\mathrm{Af}(X \otimes Y, Z^*) \simeq \mathrm{Af}(X, (Y \otimes Z)^*).$$

Since FinVect is compact, we have the natural isomorphisms

$$\mathrm{FinVect}(V_X \otimes V_Y, V_Z^*) \simeq \mathrm{FinVect}(V_X, V_Y^* \otimes V_Z^*),$$

determined by the equalities

$$\langle\, f(x \otimes y), z \,\rangle = \langle\, \hat{f}(x), y \otimes z \,\rangle, \qquad x \in V_X,\ y \in V_Y,\ z \in V_Z,$$

for $f \in \mathrm{FinVect}(V_X \otimes V_Y, V_Z^*)$ and the corresponding morphism $\hat{f} \in \mathrm{FinVect}(V_Z, V_Y^* \otimes V_Z^*)$. Since $A_{X\otimes Y}$ is an affine span of $A_X \otimes A_Y$, we see that f is in $\mathrm{Af}(X \otimes Y, Z^*)$ if and only if $f(x \otimes y) \in A_Z^*$ for all $x \in A_X$, $y \in A_Y$, that is,

$$1 = \langle\, f(x \otimes y), z \,\rangle = \langle\, \hat{f}(x), y \otimes z \,\rangle \qquad \forall x \in A_X,\ \forall y \in A_Y,\ \forall z \in A_Z.$$

But this is equivalent to

$$\hat{f}(x) \in (A_Y \otimes A_Z)^* = A^*_{Y\otimes Z}, \qquad \forall x \in A_X,$$

which means that $\hat{f} \in \mathrm{Af}(X, (Y \otimes Z)^*)$.

□

A *-autonomous category is compact closed if it satisfies $(X \otimes Y)^* = X^* \otimes Y^*$. In general, $X \odot Y = (X^* \otimes Y^*)^*$ defines a dual symmetric monoidal structure that is different from $\otimes$. We next show that Af is not compact closed.

Proposition 1. *For objects in* Af*, we have $(X \otimes Y)^* = X^* \otimes Y^*$ exactly in one of the following situations:*

(i) $X \simeq I$ *or* $Y \simeq I$*,*

(ii) $d_X = d_Y = 0$*,*

(iii) $d_{X^*} = d_{Y^*} = 0$*.*

Proof. Since FinVect is compact, we have $V_{(X\otimes Y)^*} = (V_X \otimes V_Y)^* = V_X^* \otimes V_Y^* = V_{X^*\otimes Y^*}$. It is also easily seen by definition that $A_{X^*} \otimes A_{Y^*} = A_X^* \otimes A_Y^* \subseteq A^*_{X\otimes Y} = A_{(X\otimes Y)^*}$, so that we always have $A_{X^*\otimes Y^*} \subseteq A_{(X\otimes Y)^*}$. Hence the equality holds if and only if $d_{X^*\otimes Y^*} = d_{(X\otimes Y)^*}$. From Lemma 1, we see that

$$d_{X^*\otimes Y^*} = d_{X^*} + d_{Y^*} + d_{X^*} d_{Y^*}.$$

On the other hand, we have using (5) that $L_{(X\otimes Y)^*} = S^\perp_{X\otimes Y} = (S_X \otimes S_Y)^\perp$, so that

$$d_{(X\otimes Y)^*} = D_X D_Y - \dim(S_X)\dim(S_Y) = D_X D_Y - (d_X + 1)(d_Y + 1).$$

Taking into account that by (5) we have $d_{X^*} = D_X - d_X - 1$, similarly for d_{Y^*}, we obtain

$$d_{(X\otimes Y)^*} - d_{X^*\otimes Y^*} = d_X d_{Y^*} + d_{X^*} d_Y.$$

This is equal to 0 iff $d_X d_{Y^*} = d_Y d_{X^*} = 0$, which amounts to the conditions in the lemma.

□

In a *-autonomous category, the internal hom can be identified as $X \multimap Y = (X \otimes Y^*)^*$. The underlying vector space is $V_{X\multimap Y} = (V_X \otimes V_Y^*)^* = V_X^* \otimes V_Y$ and we have seen in Section 1.1 that we may identify this space with $\mathrm{FinVect}(V_X, V_Y)$, by $f \leftrightarrow C_f$. This property is extended to Af, in the following sense.

Proposition 2. *For any objects X, Y in* Af*, the map $f \mapsto C_f$ is a bijection of* $\mathrm{Af}(X,Y)$ *onto $A_{X\multimap Y}$. In particular, A_X^* can be identified with* $\mathrm{Af}(X,I)$.

Proof. Let $f \in \mathrm{FinVect}(V_X, V_Y)$. Since by definition $A_{X\multimap Y} = A^*_{X\otimes Y^*} = (A_X \otimes A_Y^*)^*$, we see that $C_f \in A_{X\multimap Y}$ if and only if for all $x \in A_X$ and $\tilde{y} \in A_Y^*$, we have

$$1 = \langle C_f, x \otimes \tilde{y} \rangle = \langle \tilde{y}, f(x) \rangle.$$

This latter statement is clearly equivalent to $f(A_X) \subseteq A_Y$, so that $f \in \mathrm{Af}(X,Y)$.

□

We now show a property that will be crucial for the use of Af for description of higher order maps in quantum theory. For this, we restrict to objects such that the underlying vector spaces are spaces of hermitian matrices, as in Example 2. We also restrict morphisms between such spaces to completely positive maps. We show that this restriction amounts to taking an intersection of $A_{X\multimap Y}$ with the cone of positive semidefinite matrices. This, and subsequent examples, shows that for characterization of sets of quantum objects such as states, channels, combs and transformations between them, it is enough to work with the category Af.

An object X of Af will be called *quantum* if $V_X = M_n^h$ for some n and A_X is an affine subspace such that both A_X and A_X^* contain a positive multiple of the identity matrix E_n[1]. (recall that we identify $(M_n^h)^* = M_n^h$).

Proposition 3. *Let X, Y be quantum objects in* Af*. Then*

(i) X^* and $X \otimes Y$ are quantum objects as well.

(ii) Let $V_X = M_n^h$, $V_Y = M_m^h$. Then for any $f \in$ $\mathrm{FinVect}(M_n^h, M_m^h)$*, we have $C_f \in A_{X\multimap Y} \cap M_{mn}^+$ if and only if f is completely positive and*

$$f(A_X \cap M_n^+) \subseteq A_Y \cap M_m^+.$$

Proof. The statement (i) is easily seen from $A_X \otimes A_Y \subseteq A_{X\otimes Y}$ and $A_X^* \otimes A_Y^* \subseteq A^*_{X\otimes Y}$. To show (ii), let $C_f \in A_{X\multimap Y} \cap M_{mn}^+$. By the properties of the Choi isomorphism f is completely positive and by Proposition 2, $f(A_X) \subseteq A_Y$, this proves one implication. For the converse, note that we only need to prove that under the given assumptions, $f(A_X) \subseteq A_Y$, for which it is enough to show that $A_X \subseteq \mathrm{Aff}(A_X \cap M_n^+)$. To see this, let $c_X E_n \in A_X$ for $c_X > 0$. Any element in A_X can be written in the form $c_X E_n + v$ for some $v \in L_X$. Since $c_X E_n \in int(M_n^+)$, there is some $s > 0$ such that $a_\pm := c_X E_n \pm sv \in M_n^+$, and since $\pm sv \in L_X$, we see that $a_\pm \in A_X \cap M_n^+$. It is now easily checked that

$$c_X E_n + v = \frac{1+s}{2s} a_+ + \frac{s-1}{2s} a_- \in \mathrm{Aff}(A_X \cap M_n^+).$$

□

[1] We use the notation E_n, and not I_n, to avoid the slight chance that it might be confused with the monoidal unit.

We can define classical objects in Af in a similar way, replacing M_n^h by $\mathbb{R}^N$ and the positive cone by $\mathbb{R}^N_+$, and we require that both A_X and A_X^* contains a positive multiple of the unit vector $e_N = (1, \ldots, 1) \in \mathbb{R}^N$. A similar statement holds in this case, with complete positivity replaced by positivity. We can similarly treat classical-to-quantum and quantum-to-classical maps as morphisms between these types of objects, satisfying appropriate positivity assumptions.

uantum_maps *Example* 3 (States, channels and combs). The basic example of a quantum object corresponds to the set of quantum states. Let

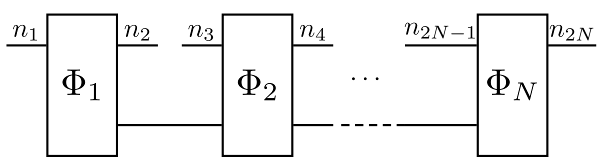
$$\mathcal{A}_n := \{T \in M_n^h,\ \mathrm{Tr}\,[T] = 1\}.$$

Then $\mathcal{S}_n := (M_n^h, \mathcal{A}_n)$ is an object in Af, and it is a quantum object, since we have $E_n \in \mathcal{A}_n^* = \{E_n\}$ and $\frac{1}{n}E_n \in \mathcal{A}_n$. The set $\mathcal{A}_n \cap M_n^+$ is the set of *quantum states*. By Proposition prop:ihom_morphisms 2, $\mathcal{C}_{m,n} := \mathcal{S}_m \multimap \mathcal{S}_n$ is a quantum object as well, with underlying vector space M_{mn}^h, and Proposition prop:ihom_quantum 3 shows that $A_{\mathcal{C}_{m,n}} \cap M_{mn}^+$ is the set of Choi matrices of completely positive maps $f : M_m \to M_n$ mapping states to states, i. e. *quantum channels*. Note that the dual object $\mathcal{C}_{m,n}^* = \mathcal{S}_m \otimes \mathcal{S}_n^*$ represents the set of Choi matrices of replacement channels $M_n \to M_m$, that is, channels that map any state in M_n to a fixed state in M_m. We also have $\mathcal{S}_n^* = \mathcal{S}_n \multimap I = \mathcal{C}_{n,1}$ and $\mathcal{S}_n = I \multimap \mathcal{S}_n = \mathcal{C}_{n,1}$.

We can proceed inductively as follows. Put

$$\mathcal{C}^2_{m,n,k,l} = \mathcal{C}_{m,n} \multimap \mathcal{C}_{k,l} = (\mathcal{S}_m \multimap \mathcal{S}_n) \multimap (\mathcal{S}_k \multimap \mathcal{S}_l),$$

then $A_{\mathcal{C}^2_{m,n,k,l}} \cap M_{mnkl}^+$ is the set of Choi matrices of completely positive maps $M_{mn} \to M_{kl}$, mapping Choi matrices of channels to Choi matrices of channels, or *quantum 2-combs*, see [paviani]. The 1-combs coincide with quantum channels. The *quantum $(N+1)$-combs* for any N is the set of Choi matrices of compeletely positive maps mapping N-combs to 1-combs. The corresponding objects in Af are then given as $\mathcal{C}^N_{n_1,\ldots,n_{2N}} = \mathcal{C}^{N-1}_{n_2,\ldots,n_{2N-2}} \multimap \mathcal{C}_{n_1,n_{2N}}$. By Proposition prop:ihom_quantum 3, these are all quantum objects. It was proved in [paviani] that the completely positive maps corresponding to N-combs have the form



for some channels $\Phi_1, \ldots, \Phi_N$. Paviani, networks, etc

_classical *Example* 4 (Partially classical maps). We may similarly define the basic classical object as

$$\mathcal{P}_k := (\mathbb{R}^k, \{x \in \mathbb{R}^k,\ \sum_i x_i = 1\}).$$

In this case, $\mathcal{A}_{\mathcal{P}_k} \cap \mathbb{R}^k_+$ is the probability simplex. We then obtain further useful objects by combining with the quantum objects. For example, it can be easily seen that $\mathcal{S}_n \multimap \mathcal{P}_k$ intersected with the cone $M_n^+ \otimes \mathbb{R}^n_+$ corresponds to k-outcome *measurements*. Similarly, we obtain k-outcome *quantum instruments* from $\mathcal{S}_m \multimap (\mathcal{S}_n \otimes \mathcal{P}_k)$, *quantum multimeters* from $(\mathcal{S}_m \otimes \mathcal{P}_k) \multimap \mathcal{P}_l$, *quantum testers* from $\mathcal{C}^N_{n_1,\ldots,n_{2N}} \multimap \mathcal{P}_k$, etc.

1.3 First order and higher order objects

We say that an object X in Af is *first order* if $d_X = D_X - 1$, equivalently, $S_X = V_X$. Another equivalent condition is $d_{X^*} = 0$, which means that A_X is determined by a single element $\tilde{a}_X \in V_X^*$ as

$$A_X = \{\tilde{a}_X\}^*, \qquad A_X^* = \{\tilde{a}_X\}.$$

Note that first order objects, resp. their duals, are exactly those satisfying condition (iii), resp. condition (ii), in Proposition 1, in particular, $(X \otimes Y)^* = X^* \otimes Y^*$ for first order objects X and Y.

Higher order objects in Af are objects obtained from a finite set $\{X_1, \ldots, X_n\}$ of first order objects by taking tensor products and duals. The above is indeed a set, so that all the objects are different (though they may be isomorphic) and the ordering is not essential. We will also assume that the monoidal unit I is not contained in this set. By definition of $X \multimap Y$, and since we may identify $X \multimap I$ with X^*, we see that higher order objects are also generated by applying the internal hom inductively on $\{X_1, \ldots, X_n\}$ if we allow $X_i = I$ for some i. It follows that the objects introduced in Examples 3 and 4 are indeed higher order objects in Af according to the above definition.

Of course, any first order object is also higher order with $n = 1$. Note that we cannot say that a higher order object generated from $\{X_1, \ldots, X_n\}$ is automatically "of order n", as the following lemma shows.

Lemma 3. *Let X, Y be first order, then $X \otimes Y$ is first order as well.*

Proof. We have $S_{X\otimes Y} = S_X \otimes S_Y = V_X \otimes V_Y = V_{X\otimes Y}$.

□

As we have seen, higher order objects are obtained by applying the internal hom iteratively. The following properties of such iterations are easily seen from the definition and properties of $\multimap$.

Lemma 4. *Let X, Y, Z be any objects in* Af*. Then we have*

(i) $Z \multimap (X \multimap Y) \simeq (Z \otimes X) \multimap Y \simeq X \multimap (Z \multimap Y)$.

(ii) If $X = (V_X, \{\tilde{a}_X\}^*)$ and $Y = (V_Y, \{\tilde{a}_Y\}^*)$ are first order, then $Z \multimap (X \multimap Y)$ is determined as

$$A_{Z\multimap(X\multimap Y)} = \{w \in V_Z^* \otimes V_X^* \otimes V_Y, (id \otimes \tilde{a}_Y)(w) \in A_Z^* \otimes \tilde{a}_X\}.$$

Note also that since we identify $X^{**} = X$ for any object X, the isomorphisms in (i) above are given by the symmetries in FinVect, that is, by permutations of the components in the tensor products of the underlying vector spaces. To save some parentheses, we also assume that the internal homs associate to the right, so we write $X \multimap Y \multimap Z$ instead of $X \multimap (Y \multimap Z)$.

Example 5 (Channels and Combs). Let X and Y be first order objects in Af. As we have seen, $C_1(X, Y) := X \multimap Y$ is then a higher order object, called a *channel* or *1-comb* (We slightly abuse the terminology here). We will inductively construct higher order objects in Af, similarly as in Example 3. An *N-comb* over first order objects $X_1, \dots, X_{2N}$ is an object

$$\begin{aligned} C_N(X_1, \dots, X_{2N}) &:= C_{N-1}(X_2, \dots, X_{2N-2}) \multimap (X_1 \multimap X_{2N}) \\ &\simeq X_1 \multimap C_{N-1}(X_2, \dots, X_{2N-2} \multimap X_{2N} \\ &\simeq X_1 \multimap (X_2 \multimap \dots \multimap (X_N \multimap X_{N+1}) \multimap \dots \multimap X_{2N-1}) \multimap X_{2N} \end{aligned}$$

where the isomorphisms follow by Lemma 4. The subspace A_{C_N} for an N-comb C_N can be found inductively, using Lemma 4 (ii). If $X_1, \dots, X_{2N}$ are quantum objects, then $\tilde{a}_{X_i}$ is always a multiple of the identity, so we obtain the characterization of quantum combs in [paviani].

2 Combinatorial description of higher order objects

In this section, we discuss a combinatiorial description of higher order objects similar to that of [Pavia]. We will use the definitions and results given in Appendix A.4.

For a first order object $X = (V_X, \{\tilde{a}_X\}^\sim)$, let us pick an element $a_X \in A_X$. We have a direct sum decomposition

$$V_X = L_{X,0} \oplus L_{X,1},$$

where $L_{X,0} := \mathbb{R}\{a_X\}$, $L_{X,1} := \{\tilde{a}_X\}^\perp = L_X$. We define the *conjugate object* as $\tilde{X} = (V_X^*, \{a_X\}^\sim)$. Note that we always have $\tilde{a}_X \in A_{\tilde{X}}$ and with the choice $a_{\tilde{X}} = \tilde{a}_X$, we obtain $\tilde{\tilde{X}} = X$ and

$$L_{\tilde{X},u} = L_{X,1-u}^\perp, \qquad u \in \{0,1\}. \tag{6}$$

These definitions depend on the choice of a_X, but we will assume below that this choice is fixed and that we choose $a_{\tilde{X}} = \tilde{a}_X$. Since we will always work with a finite set of objects at a time, this will not create any problems.

A first order quantum or classical object is the set of states $\mathcal{S}_n$ or the set of probability distributions $\mathcal{P}_N$, see Examples 3, 4. In these cases, a_X will be chosen as the appropriate multiple of the identity. Note that then

$$L_{X,0} = L_{\tilde{X},0} = \mathbb{R}\{E_n\}, \qquad L_{X,1} = L_{\tilde{X},1} = \mathcal{T}_n := \{T \in M_n^h, \mathrm{Tr}\,[T] = 0\}$$

(similarly for $\mathcal{P}_N$).

Let $X_1, \dots, X_n$ be first order objects in Af. Let $a_{X_i} \in A_{X_i}$ be fixed and let $\tilde{X}_i$ be the conjugate first order objects. Let us denote $V_i = V_{X_i}$ and

$$L_{i,u} := L_{X_i,u}, \qquad \tilde{L}_{i,u} := L_{\tilde{X}_i,u} \qquad u \in \{0,1\},\ i \in [n].$$

For a string $s \in \{0,1\}^n$, we define

$$L_s := L_{1,s_1} \otimes \dots \otimes L_{n,s_n}, \qquad \tilde{L}_s := \tilde{L}_{1,s_1} \otimes \dots \otimes \tilde{L}_{n,s_n},$$

then we have the direct sum decompositions

$$V := V_1 \otimes \dots \otimes V_n = \sum_{s \in \{0,1\}^n} L_s, \qquad V^* = V_1^* \otimes \dots \otimes V_n^* = \sum_{s \in \{0,1\}^n} \tilde{L}_s$$

(here $\sum$ denotes the direct sum).

lemma:Lperp **Lemma 5.** *For any $s \in \{0,1\}^n$, we have*

$$L_s^\perp = \sum_{t \in \{0,1\}^n} (1 - \chi_s(t)) \tilde{L}_t, \qquad \tilde{L}_s^\perp = \sum_{t \in \{0,1\}^n} (1 - \chi_s(t)) L_t.$$

Here $\chi_s : \{0,1\}^n \to \{0,1\}$ is the characteristic function of s.

Proof. Using (6) eq:complement and the direct sum decomposition of V_i^*, we get

$$\begin{aligned}
(L_{1,s_1} \otimes \cdots \otimes L_{n,s_n})^\perp &= \bigvee_j \left(V_1^* \otimes \cdots \otimes V_{j-1}^* \otimes \tilde{L}_{j,1-s_j} \otimes V_{j+1}^* \otimes \cdots \otimes V_n^* \right) \\
&= \bigvee_j \left(\sum_{\substack{t \in \{0,1\}^n \\ t_j \neq s_j}} \tilde{L}_{1,t_1} \otimes \cdots \otimes \tilde{L}_{n,t_n} \right) \\
&= \sum_{\substack{t \in \{0,1\}^n \\ t \neq s}} \left(\tilde{L}_{1,t_1} \otimes \cdots \otimes \tilde{L}_{n,t_n} \right).
\end{aligned}$$

The proof of the other equality is the same.

$\square$

lemma:Xf **Lemma 6.** *Put $a := a_1 \otimes \cdots \otimes a_n$, $\tilde{a} := \tilde{a}_1 \otimes \cdots \otimes \tilde{a}_n$. For $f \in \mathcal{F}_n$ define*

$$S_f = S_f(X_1, \ldots, X_n) := \sum_{s \in \{0,1\}^n} f(s) L_s, \qquad A_f = A_f(X_1, \ldots, X_n) := S_f \cap \{\tilde{a}\}^\sim.$$

Then A_f is a proper affine subspace in V containing a. Moreover,

$$\mathrm{Lin}(A_f) = \sum_{s \in \{0,1\}^n \setminus \{\theta_n\}} f(s) L_s, \qquad \mathrm{Span}(A_f) = S_f.$$

The map $f \mapsto A_f$ is injective and has the following further properties.

(i) The dual affine subspace satisfies

$$\tilde{A}_f(X_1, \ldots, X_n) = A_{f^*}(\tilde{X}_1, \ldots, \tilde{X}_n).$$

(ii) Let $\sigma \in \mathscr{S}_n$ and let the corresponding symmetry $\otimes_i V_i \to \otimes_i V_{\sigma^{-1}(i)}$ be also denoted by σ. Then we have

$$A_f(X_{\sigma(1)}, \ldots, X_{\sigma(n)}) = \sigma^{-1}(A_{f \circ \sigma}(X_1, \ldots, X_n)).$$

(iii) Let $f_1 \in \mathcal{F}_{n_1}$, $f_2 \in \mathcal{F}_{n_2}$, $n_1 + n_2 = n$. Then

$$S_{f_1 \otimes f_2}(X_1, \ldots, X_n) = S_{f_1}(X_1, \ldots, X_{n_1}) \otimes S_{f_2}(X_{n_1+1}, \ldots, X_n)$$

Proof. It is clear from definition that A_f is an affine subspace. Since $f(\theta_n) = 1$, the space S_f always contains the subspace $L_0 = L_{1,0} \otimes \cdots \otimes L_{n,0} = \mathbb{R}\{a\}$ and it is clear that $L_s \subseteq \{\tilde{a}\}^\perp$ for any $s \neq \theta_n$. It follows that $a \in A_f$, so that $A_f \neq \emptyset$, and since $A_f \subseteq \{\tilde{a}\}^\sim$, we see that A_f is proper and $\tilde{a} \in \tilde{A}_f$. The expressions for $\mathrm{Lin}(A_f)$ and $\mathrm{Span}(A_f)$ are immediate from the definition and (16).

Injectivity of the map $f \mapsto A_f$ is clear from the fact that L_s, $s \in \{0,1\}$ is an independent decomposition. To prove (i), we compute using Lemma 5 and the fact that the subspaces form an independent decomposition,

$$\mathrm{Span}(\tilde{A}_f) = \mathrm{Lin}(A_f)^\perp = \Bigg(\sum_{s\in\{0,1\}^n\setminus\{0\}} f(s)L_s\Bigg)^\perp = \bigwedge_{\substack{s\in\{0,1\}^n\\ s\neq 0, f(s)=1}} L_s^\perp = \bigwedge_{\substack{s\in\{0,1\}^n\\ s\neq 0, f(s)=1}} \Bigg(\sum_{t\in\{0,1\}^n} (1-\chi_s(t))\tilde{L}_t\Bigg)$$
$$= \sum_{t\in\{0,1\}^n} \Bigg(\bigwedge_{\substack{s\in\{0,1\}^n\\ s\neq 0, f(s)=1}} (1-\chi_s(t))\tilde{L}_t\Bigg) = \sum_{t\in\{0,1\}^n} f^*(t)\tilde{L}_t.$$

To see the last equality, note that

$$\bigwedge_{\substack{s\in\{0,1\}^n\\ s\neq 0, f(s)=1}} (1-\chi_s(t)) = \begin{cases} 1 & \text{if } t = \theta_n \\ 1 - f(t) & \text{if } t \neq \theta_n \end{cases} = f^*(t).$$

To show (ii), compute

$$\sigma(S_f(X_{\sigma(1)},\dots,X_{\sigma(n)})) = \sigma(\sum_s f(s)L_{\sigma(1),s_1}\otimes\cdots\otimes L_{\sigma(n),s_n})$$
$$= \sum_s f(s)L_{1,s_{\sigma^{-1}(1)}}\otimes\cdots\otimes L_{n,s_{\sigma^{-1}(n)}} = S_{f\circ\sigma}(X_1,\dots,X_n).$$

It folows that

$$A_f(X_{\sigma(1)},\dots,X_{\sigma(n)}) = S_f(X_{\sigma(1)},\dots,X_{\sigma(n)}) \cap \{\sigma^{-1}(\tilde{a})\}^\sim = \sigma^{-1}(A_{f\circ\sigma}(X_1,\dots,X_n)).$$

The statement (iii) is easily seen from the definitions. □

Since all the affince subspaces $A_f \subseteq V$ are proper, we may form the objects

$$X_f = X_f(X_1,\dots,X_n) := (V, A_f(X_1,\dots,X_n))$$

in Af. The following properties follow easily from the above Lemma.

p:Xf_const **Proposition 4.** *Let $X_1,\dots,X_n$ be first order objects. The map $\mathcal{F}_n \ni f \mapsto X_f(X_1,\dots,X_n)$ is injective and we have*

(i) For the least and the largest element in $\mathcal{F}_n$,

$$X_{p_{[n]}} = \tilde{X}_1^* \otimes\cdots\otimes \tilde{X}_n^*, \qquad X_{1_n} = X_1\otimes\cdots\otimes X_n,$$

(ii) $X^*_f(X_1, \dots, X_n) = X_{f^*}(\tilde{X}_1, \dots, \tilde{X}_n)$,

(iii) $X_{f_1 \otimes f_2}(X_1, \dots, X_n) = X_{f_1}(X_1, \dots, X_{n_1}) \otimes X_{f_2}(X_{n_1+1}, \dots, X_n)$,

(iv) the symmetry $\sigma \in \mathscr{S}_n$ *is an isomorphism* $X_f(X_{\sigma(1)}, \dots, X_{\sigma(n)}) \xrightarrow{\sigma} X_{f \circ \sigma}(X_1, \dots, X_n)$.

It follows from the independence of L_s, $s \in \{0,1\}^n$, that the subspaces S_f form a distributive sublatice in the lattice of subspaces of V and we clearly have $f \leq g$ if and only if $S_f \subseteq S_g$ and $S_{f \wedge g} = S_f \cap S_g$, $S_{f \vee g} = S_f \vee S_g$. The following proposition shows the corresponding properties of X_f, in categorical terms. We skip the easy proof.

Proposition 5. *Let* $f, g, h \in \mathcal{F}_n$.

(i) $f \leq g$ *if and only if* $X_f \xrightarrow{id_V} X_g$ *in* Af.

(ii) Let $k \leq f, g \leq h$ *then the following diagram is a pullback resp. pushout:*

$$\begin{tikzcd} X_{f\wedge g} \arrow[r, "id_V"] \arrow[d, "id_V"'] & X_f \arrow[d, "id_V"] \\ X_g \arrow[r, "id_V"'] & X_h \end{tikzcd} \qquad \begin{tikzcd} X_h \arrow[r, "id_V"] \arrow[d, "id_V"'] & X_f \arrow[d, "id_V"] \\ X_g \arrow[r, "id_V"'] & X_{f\vee g} \end{tikzcd}$$

Our goal is to show that the higher order objects are precisely those of the form $Y = X_f(X_1, \dots, X_n)$ for some choice of the first order objects $X_1, \dots, X_n$ and a function f that belongs to a special subclass $\mathcal{T}_n \subseteq \mathcal{F}_n$. The elements of this subclass will be called the *type functions*, or *types*, and are defined as those functions in $\mathcal{F}_n$ that can be obtained by taking the constant function 1_1 in each coordinate and then repeatedly applying duals and tensor products of such functions in any order. The set of indices for which the corresponding coordinate was subjected to taking the dual an odd number of times will be called the *inputs* (of f) and denoted by $I = I_f$, indices in $O = O_f := [n] \setminus I_f$ will be called *outputs*. The reason for this terminology will become clear later. It is easy to observe that if $f \in \mathcal{T}_n$, then $O_{f^*} = I_f$ and $I_{f^*} = O_f$. Further, for $f_1 \in \mathcal{T}_{n_1}$, $f_2 \in \mathcal{T}_{n_2}$, we have $O_{f_1 \otimes f_2} = O_{f_1} \oplus O_{f_2}$ and $I_{f_1 \otimes f_2} = I_{f_1} \oplus I_{f_2}$, see (eq:disu 13) for the definition.

We have the following description of the sets of type functions.

p:type_min **Proposition 6.** *The system* $\{\mathcal{T}_n\}_{n \in \mathbb{N}}$ *is the smallest system of sets such that*

1. $\mathcal{T}_1 = \mathcal{F}_1$, $\mathcal{T}_n \subseteq \mathcal{F}_n$ *for all* n,
2. *For* $[n] = [n_1] \oplus [n_2]$, $\mathcal{T}_{n_1} \otimes \mathcal{T}_{n_2} \subseteq \mathcal{T}_n$,
3. $\mathcal{T}_n$ *is invariant under permutations: if* $f \in \mathcal{T}_n$, *then* $f \circ \sigma \in \mathcal{T}_n$ *for any* $\sigma \in \mathscr{S}_n$,
4. $\mathcal{T}_n$ *is invariant under complementation: if* $f \in \mathcal{T}_n$ *then* $f^* \in \mathcal{T}_n$.

Proof. It is clear by construction that any system of subsets $\{\mathcal{S}_n\}_n$ with these properties must contain the type functions and that $\{\mathcal{T}_n\}_n$ itself has these properties. $\square$

Assume that Y is a higher order object constructed from a set of distinct first order objects $Y_1, \dots, Y_n$, $Y_i = (V_{Y_i}, \{\tilde{a}_{Y_i}\}^{\sim})$. Let us fix elements $a_{Y_i} \in A_{Y_i}$ and construct the conjugate objects $\tilde{Y}_i$. By compactness of FinVect, we may assume (relabeling the objects if necessary) that the vector space of Y has the form

$$V_Y = V := V_1 \otimes \cdots \otimes V_n,$$

where V_i is either V_{Y_i} or $V_{Y_i}^*$, according to whether Y_i was subjected to taking duals an even or odd number of times. Similarly as for the type functions, the indices such that the first case is true will be called the outputs and the subset of outputs in $[n]$ will be denoted by O, or O_Y, when we need to specify the object. The set $I = I_Y := [n] \setminus O_Y$ is the set of inputs.

Theorem 2. *Let Y be a higher order object, constructed from first order objects $Y_1, \dots, Y_n$. For $i \in [n]$, let $X_i = Y_i$ if $i \in O_Y$ and $X_i = \tilde{Y}_i$ for $i \in I_Y$. There is a unique function $f \in \mathcal{T}_n$, with $O_f = O_Y$, such that*

$$Y = X_f = (V, A_f(X_1, \dots, X_n)).$$

Conversely, let $X_1, \dots, X_n$ be first order objects and let $f \in \mathcal{T}_n$. Then $Y = X_f$ is a higher order object with $O_Y = O_f$, with underlying first order objects $Y_1, \dots, Y_n$, where $Y_i = X_i$ for $i \in O_f$ and $Y_i = \tilde{X}_i$ for $i \in I_f$.

Proof. Since the map $f \mapsto X_f$ is injective, uniqueness is clear. To show existence of this function, we will proceed by induction on n. For $n = 1$, the assertion is easily seen to be true, since in this case, we we have either $Y = Y_1$ or $Y = Y_1^*$. In the first case, $O = [1]$, $X_1 = Y_1$ and

$$S_Y = V_Y = V_1 = L_{1,0} \oplus L_{1,1} = 1(0)L_{1,0} \oplus 1(1)L_{1,1},$$

so in this case $f \in \mathcal{T}_1$ is the constant 1. If $Y = Y_1^*$, we have $O = \emptyset$, $X_1 = \tilde{Y}_1$, and then

$$S_Y = \mathbb{R}\{\tilde{a}_{Y_1}\} = L_{1,0} = 1^*(0)L_{1,0} \oplus 1^*(1)L_{1,1},$$

so that $f = 1^* = p_{[1]} \in \mathcal{T}_1$. It is clear that $_f = O_Y$ in both cases.

Assume now that the assertion is true for all $m < n$. By construction, Y is either the tensor product $Y = Z_1 \otimes Z_2$, with Z_1 constructed from $Y_1, \dots, Y_m$ and Z_2 from $Y_{m+1}, \dots, Y_n$, or Y is the dual of such a product. Let us assume the first case. It is clear that $O_{Z_1} \oplus O_{Z_2} = O_Y$, and similarly for I, so that the corresponding objects $X_1, \dots, X_m$ and $X_{m+1}, \dots, X_n$ remain the same. By the induction assumption, there are functions $f_1 \in \mathcal{T}_m$ and $f_2 \in \mathcal{T}_{n-m}$ such that $O_{f_1} = O_{Z_1}$, $O_{f_2} = O_{Z_2}$ and, by Proposition 4(iii),

$$Y = Z_1 \otimes Z_2 = X_{f_1}(X_1, \dots, X_m) \otimes X_{f_2}(X_{m+1}, \dots, X_n) = X_{f_1 \otimes f_2}(X_1, \dots, X_n)$$

This implies the assertion, with $f = f_1 \otimes f_2 \in \mathcal{T}_n$ and $O_f = O_{f_1} \oplus O_{f_2} = O_Y$. To finish the proof, it is now enough to observe that if the assertion holds for Y then it also holds for Y^*. So assume that $Y = X_f(X_1, \dots, X_n)$ for some $f \in \mathcal{T}_n$, then by Proposition 4(ii), $Y^* = X_f^* = \tilde{X}_{f^*}(\tilde{X}_1, \dots, \tilde{X}_n)$. By the construction of conjugate objects, we have $\tilde{X}_i = \tilde{\tilde{Y}}_i = Y_i$ if $i \in I_Y$ and $\tilde{X}_i = \tilde{Y}_i$ if $i \in O_Y$. Since by definition and the assumption, $O_{Y^*} = I_Y = I_f = O_{f^*}$, this proves the statement.

The converse is proved by a similar induction argument, using Proposition 4. $\square$

Let us stress that in general, the objects X_f depend on the choice of the elements a_{X_i}. From the above proof, it is clear that the description in Theorem 2 does not depend on the choice of the elements $a_{Y_i} \in A_{Y_i}$. Furthermore, if all the first order objects are quantum, we have $S_f(X_1,\dots,X_n) = S_f(\tilde{X}_1,\dots,\tilde{X}_n)$ and both a and $\tilde{a}$ are some, possibly different, multiple of the identity. The spaces $A_f(X_1,\dots,X_n)$ and $A_f(\tilde{X}_1,\dots,\tilde{X}_n)$ differ only by this multiple.

3 The type functions

The aim of this section is to gain some understanding into the structure and properties of the set of type functions. We start by an important example.

Example 6. Let $T \subseteq [n]$. It is easily seen that the function p_T (see Example 9 in Appendix A.4) is a type function, since we have

$$p_T(s) = \Pi_{j\in T}(1-s_j) = \Pi_{j\in T}1^*(s_j) = (\otimes_{j\in T}1^*)(s).$$

By definition, T is the set of inputs for p_T. Let $\mathcal{S} = \{X_1,\dots,X_n\}$ be a set of first order objects. Let $k = |T|$ and let $\sigma \in \mathscr{S}_n$ be such that $\sigma^{-1}(T) = [k]$. Then $p_T \circ \sigma = p_k \otimes 1_{n-k}$. By Proposition 4, it follows that we have the isomorphism

$$X_{p_T}(X_1,\dots,X_n) \overset{\sigma}{\simeq} X_{p_k\otimes 1_{n-k}}(X_{\sigma^{-1}(1)},\dots,X_{\sigma^{-1}(n)}) = \tilde{X}_T^* \otimes X_{[n]\setminus T},$$

here $\tilde{X}_T = \otimes_{j\in T}\tilde{X}_j$ and $X_{[n]\setminus T} = \otimes_{j\in[n]\setminus T}X_j$ are first order object by Lemma 3. It follows that p_T describes replacement channels with set of input indices T. By duality, we obtain the isomorphisms

$$X_{p_T^*}(X_1,\dots,X_n) = X^*_{p_T}(\tilde{X}_1,\dots,\tilde{X}_n) \overset{\sigma}{\simeq} (X_T^* \otimes \tilde{X}_{[n]\setminus T})^* \overset{\rho}{\simeq} [\tilde{X}_{[n]\setminus T}, X_T],$$

where ρ is the transposition in $\mathscr{S}_2$. It follows that $p_T^* = 1 - p_T + p_n$ corresponds to all channels with output indices T.

Lemma 7. *Let $f \in \mathcal{T}_n$ and let $O_f = O$, $I = I_f$. Then*

$$p_I \le f \le p_O^*.$$

Proof. This is obviously true for $n = 1$. Indeed, in this case, $\mathcal{T}_1 = \mathcal{F}_1 = \{1 = p_\emptyset, 1^* = p_1\}$. If $f = 1$, then $O = [1]$, $I = \emptyset$ and

$$p_I = p_\emptyset = 1 = p_O^*,$$

the case $f = 1^*$ is obtained by taking complements. Assume that the assertion holds for $m < n$. Let $f \in \mathcal{T}_n$ and assume that $f = g \otimes h$ for some $g \in \mathcal{T}_m$, $h \in \mathcal{T}_{n-m}$. By the assumption,

$$p_{I_g} \otimes p_{I_h} \le g \otimes h \le p^*_{O_g} \otimes p^*_{O_h} \le (p_{O_g} \otimes p_{O_h})^*,$$

the last inequality follows from Lemma 13. With the decomposition $[n] = [m][m+1,n]$, we have $O_f = O_g \oplus O_h$, $I_f = I_g \oplus I_h$, so that by Lemma 14, $p_{O_f} = p_{O_g} \otimes p_{O_h}$ and similarly for

p_{I_f}. Now notice that any $f \in \mathcal{T}_n$ is either of the form $(f \otimes g) \circ \sigma$ or of the form $(f \otimes g)^* \circ \sigma$, for some permutation σ. Since the inequality is easily seen to be preserved by permutations, and reversed by duality which also swiches the input and output sets, the assertion is proved. $\square$

Combining this with Proposition 4, we get the following result (cf. cite).

ro:setting **Corollary 1.** *Let Y be a higher order objects constructed from first order objects $Y_1, \dots, Y_n$, $O_Y = O$, $I_Y = I$. Then there are $\sigma_1, \sigma_2 \in \mathscr{S}_n$ such that we have the morphisms*

$$Y_I^* \otimes Y_O \xrightarrow{\sigma_1} Y \xrightarrow{\sigma_2} [Y_I, Y_O].$$

We also obtain a simple way to identify the output indices of a type function.

fh_outputs **Proposition 7.** *For $f \in \mathcal{T}_n$, $j \in O_f$ if and only if $f(e^j) = 1$, here $e^j = \delta_{1,j} \dots \delta_{n,j}$.*

Proof. Let $i \in O_f$, then by Lemma 7, $p_{I_f}(e^i) = 1 \leq f(e^i)$, so that $f(e^i) = 1$. Conversely, if $f(e^i) = 1$, then by the other inequality in lemma 7, $p_{O_f}(e^i) = 0$, whence $i \in O_f$. $\square$

exm:T2 *Example* 7. The type functions for $n = 2$ are given as (writing $\bar{u} = 1 - u$ for $u \in \{0, 1\}$, and $s = s_1 s_2$):

$$1_2(s) = 1, \quad p_2(s) = \bar{s}_1 \bar{s}_2, \quad p_{[1]}(s) = \bar{s}_1, \quad p^*_{[1]}(s) = 1 - \bar{s}_1 + \bar{s}_1 \bar{s}_2,$$

and functions obtained from these by permutation, which gives 6 different elements. We have seen that $\mathcal{F}_n$ has $2^{2^n - 1}$ elements, so that $\mathcal{F}_2$ has 8 elements in total. The two remaining functions are

$$g(s) = 1 - \bar{s}_1 - \bar{s}_2 + 2\bar{s}_1 s_2, \qquad g^*(s) = \bar{s}_1 + \bar{s}_2 - \bar{s}_1 \bar{s}_2.$$

It can be checked directly from Lemma 7 and Proposition 7 that g is not a type function. Indeed, if $g \in \mathcal{T}_2$, we would have $O_g = \emptyset$, so that $p_2 \leq g \leq p^*_\emptyset = p_2$, which is obviously not the case. Clearly, also the complement $g^* \notin \mathcal{T}_2$. Notice also that $g^* = p_{[1]} \vee p_{[2]}$, so that $\mathcal{T}_2$ is not a lattice.

Since $\mathcal{F}_2$ can be identified as a sublattice in $\mathcal{F}_n$ for all $n \geq 2$ as $\mathcal{F}_2 \ni f \mapsto f \otimes 1_{n-2} \in \mathcal{F}_n$, the above example shows that $\mathcal{T}_n$, $n \geq 2$ is a subposet in the distributive lattice $\mathcal{F}_n$ but itself not a lattice, so that for $f_1, f_2 \in \mathcal{T}_n$, none of $f_1 \wedge f_2$ or $f_1 \vee f_2$ has to be a type function. Nevertheless, we have by the above results that all type functions with the same output indices are contained in the interval $p_I \leq f \leq p^*_O$, which is a distributive lattice. Elements of such an interval will be called *subtypes*. It is easily seen that for $n = 2$ all subtypes are type functions, but it is not difficult to find a subtype for $n = 3$ which is not in $\mathcal{T}_3$. The objects corresponding to subtypes are not necessarily higher order objects, but are embedded in $[Y_I, Y_O]$ and contain the replacement channels. If f_1 and f_2 have the same output set, then $f_1 \vee f_2$ and $f_1 \wedge f_2$ are subtypes. By Proposition ??, the corresponding objects can be obtained by pushouts resp. pullbacks of the higher order objects corresponding to f_1 and f_2.

3.1 The poset $\mathcal{P}_f$

By Theorem thm:mobius 6, any boolean function has a unique expression of the form

$$f = \sum_{T \subseteq [n]} \hat{f}_T p_T,$$

where $\hat{f}$ is the Möbius transform of f. Using this, we introduce a poset related to f, which will be useful for description of the structure of f. We will need the definitions and basic results in Appendix app:poset A.2.

Let $\mathcal{P}_f$ be the subposet in the distributive lattice 2^n, of elements such that $\hat{f}_T \neq 0$. The main result of this paragraph is that any type function $f \in \mathcal{T}_n$ is fully determined by $\mathcal{P}_f$. We first need to show how some of the operations on type functions are reflected on $\mathcal{P}_f$.

lemma:Pf **Lemma 8.** *Let $f \in \mathcal{T}_n$.*

(i) If $\sigma \in \mathscr{S}_n$, then $S \mapsto \sigma^{-1}(S)$ is an isomorphism of $\mathcal{P}_f$ onto $\mathcal{P}_{f\circ\sigma}$.

(ii) For $g \in \mathcal{T}_m$ and the decomposition $[n+m] = [n] \oplus [m]$, we have $\mathcal{P}_{f\otimes g} \simeq \mathcal{P}_f \times \mathcal{P}_g$, with the isomorphism given by $(S,T) \mapsto S \oplus T$ and

$$(\widehat{f \otimes g})_{(S,T)} = \hat{f}_S \hat{g}_T.$$

Proof. The statement is proved using Lemma lemma:PSPT 14. We have

$$f \circ \sigma = \sum_{S \subseteq [n]} \hat{f}_S p_S \circ \sigma = \sum_{S \subseteq [n]} \hat{f}_S p_{\sigma^{-1}(S)} = \sum_{S \subseteq [n]} \hat{f}_{\sigma(S)} p_S.$$

The statement (i) now follows by uniqueness of the Möbius transform. Similarly, for $s = s^1 s^2$,

$$\begin{aligned} f \otimes g(s) = f(s^1)g(s^2) &= \sum_{S \subseteq [n]} \sum_{T \subseteq [m]} \hat{f}_S \hat{g}_T p_S(s^1) p_T(s^2) = \sum_{S \subseteq [n]} \sum_{T \subseteq [m]} \hat{f}_S \hat{g}_T (p_S \otimes p_T)(s) \\ &= \sum_{S \subseteq [n]} \sum_{T \subseteq [m]} \hat{f}_S \hat{g}_T (p_{S \oplus T})(s). \end{aligned}$$

This proves (ii).

$\square$

thm:graded **Theorem 3.** *Let $f \in \mathcal{T}_n$, then $\mathcal{P}_f$ is a graded poset with even rank $r(f) := r(\mathcal{P}_f) \leq n$. Moreover, we have*

$$f = \sum_{S \in \mathcal{P}_f} (-1)^{\rho_f(S)} p_S,$$

where ρ_f is the rank function of f.

Proof. We will proceed by induction on n. Assume that $n = 1$. Then $2^n = \{\emptyset, [1]\}$ and $\mathcal{T}_1 = \{1, 1^*\}$. For both type functions, $\mathcal{P}_f$ is a singleton, which is clearly a graded poset, with rank $k = 0$ and trivial rank function $\rho_f \equiv 0$. We have

$$f = 1 = p_\emptyset = (-1)^{\rho(\emptyset)} p_\emptyset.$$

The statement for $f = 1^*$ follows by duality.

Assume next that the statement holds for all $m < n$ and let $f \in \mathcal{T}_n$. By construction, it is enough to show that the property is invariant under permutations and complement and that it holds for any f of the form $f = f_1 \otimes f_2$ for type functions $f_1 \in \mathcal{T}_{n_1}$, $f_2 \in \mathcal{T}_{n_2}$. So assume f has the desired property and let $\sigma \in \mathscr{S}_n$. It is clear by the isomorphism in Lemma 8 (i) that $\mathcal{P}_{f\circ\sigma}$ is a graded poset as well, with the same even rank as f and rank function $\rho_{f\circ\sigma} = \rho_f \circ \sigma$. Then

$$f \circ \sigma = \sum_{S \subseteq [n]} (-1)^{\rho_f(S)} p_S \circ \sigma = \sum_{S \subseteq [n]} (-1)^{\rho_f \circ \sigma(S)} p_S.$$

Further, we have

$$\begin{aligned} f^* = 1 - f + p_n &= (1 - \hat{f}_\emptyset) p_\emptyset - \sum_{\emptyset, [n] \neq S \subseteq [n]} \hat{f}_S p_S + (1 - \hat{f}_{[n]}) p_n \\ &= (1 - \hat{f}_\emptyset) 1 - \sum_{\substack{S \in \mathcal{P}_f \\ \emptyset \neq S, [n] \neq S}} (-1)^{\rho_f(S)} p_S + (1 - \hat{f}_{[n]}) p_n. \end{aligned} \tag{7}$$

eq:dual_rank

If $\emptyset \in \mathcal{P}_f$, then $\emptyset$ is the least element of $\mathcal{P}_f$, so that $\rho_f(\emptyset) = 0$ and therefore $\hat{f}_\emptyset = (-1)^0 = 1$. Similarly, if $[n] \in \mathcal{P}_f$, then $[n]$ is the largest element in $\mathcal{P}_f$, hence it is the last element in any maximal chain. It follows that $\rho_f([n]) = k$ and hence $\hat{f}_{[n]} = (-1)^k = 1$ (since k is even). Therefore the equality (7) implies that $\mathcal{P}_{f^*}$ differs from $\mathcal{P}_f$ only in the bottom and top elements: $\emptyset \in \mathcal{P}_f$ iff $\emptyset \notin \mathcal{P}_{f^*}$ and $[n] \in \mathcal{P}_f$ iff $[n] \notin \mathcal{P}_{f^*}$. It follows that $\mathcal{P}_{f^*}$ is graded as well, with rank equal to $k-2$, k or $k+2$, which in any case is even. Furthermore, this also implies that for all $S \in \mathcal{P}_f$, $S \notin \{\emptyset, [n]\}$, we have $\rho_{f^*}(S) = \rho_f(S) \pm 1$, according to whether $\emptyset$ was added or removed. The statement now follows from (7).

Now assume that $f = f_1 \otimes f_2$. By the induction assumption, both $\mathcal{P}_{f_1}$ and $\mathcal{P}_{f_2}$ are graded posets. By Lemma 8, $\mathcal{P}_f \simeq \mathcal{P}_{f_1} \times \mathcal{P}_{f_2}$, so that $\mathcal{P}_f$ is a graded poset as well, with rank function $\rho_f(S,T) = \rho_{f_1}(S) + \rho_{f_2}(T)$ and rank $r(f) = r(f_1) + r(f_2)$. By Lemma 8 (ii), we get

$$f = \sum_{S \subseteq [n_1], T \subseteq [n_2]} (\widehat{f_1})_S (\widehat{f_2})_T p_S \otimes p_T = \sum_{S \subseteq [n_1], T \subseteq [n_2]} (-1)^{\rho_{f_1}(S) + \rho_{f_2}(T)} p_{S \oplus T}.$$

This finishes the proof.

□

remark:n *Remark* 1. Notice that we need to assume n to be known. Indeed, for any m and f, $\mathcal{P}_f$ and $\mathcal{P}_{f \otimes 1_m}$ are the same, but the two type functions are different. In particular, the corresponding constructions of higher order objects are different.

In the course of the above proof, we have also shown the following.

pro:pf_dual **Corollary 2.** *Let $f \in \mathcal{T}_n$. Then $\mathcal{P}_{f^*} \setminus \{\emptyset, [n]\} = \mathcal{P}_f \setminus \{\emptyset, [n]\}$ and $\emptyset \in \mathcal{P}_f$ if and only if $\emptyset \notin \mathcal{P}_{f^*}$. The same holds for $[n]$.*

We introduce *labels* for the elements of $\mathcal{P}_f$ in the following way. For $S \in \mathcal{P}_f$, put

$$L_S := \{i \in [n] \ : \ i \in S, \ \forall S' \subsetneq S, i \notin S'\}.$$

In other words, i is a label for S if S is a minimal element in the subposet of elements containing i in $\mathcal{P}_f$. We will use the notation $L_{S,f}$ if the function f has to be specified. It is easily seen that for $S \in \mathrm{Min}(\mathcal{P}_f)$, $L_S = S$ and for any $S \in \mathcal{P}_f$, $S = \cup_{S' \subseteq S} L_{S'}$. It follows that $f \in \mathcal{T}_n$ (with known n) is fully determined by the order relation on $\mathcal{P}_f$ and the label sets. All the information about f can be therefore obtained from the *labelled Hasse diagram* of $\mathcal{P}_f$.

Examples (n=2,3,4). Hasse diagrams

Example 8 ($\mathcal{T}_3$). As we can see from Example exm:T2 7, all elements in $\mathcal{T}_2$ are chains. This is also true for $n = 3$. Indeed, up to a permutation that does not change the chain structure, any $f \in \mathcal{T}_3$ is either a product of two elements $g \in \mathcal{T}_2$ and $h \in \mathcal{T}_1$, or the dual of such a product. Since g must be a chain and $|\mathcal{P}_h| = 1$, their product must be a chain as well. Taking the dual of a chain only adds/removes the least/largest elements, so the dual of a chain must be a chain as well.

Let us denote

$$I_f^F := \cap_{S \in \mathrm{Min}(\mathcal{P}_f)} L_S, \qquad O_f^F := [n] \setminus \cup_{S \in \mathcal{P}_f} L_S.$$

It is easily checked by Proposition prop:fh_outputs 7 that any $i \in O_f^F$ is an output index, since in this case we have $f(e^i) = f(\theta_n) = 1$. Such elements will be called the *free outputs* of f. If f has some free outputs, then necessarily $[n] \notin \mathcal{P}_f$. Similarly, any $j \in I_f^F$ is an input of f, since j must be contained in any $T \in \mathcal{P}_f$, so that $p_T(e^j) = 0$ for all $T \in \mathcal{P}_f$ and consequently $f(e^j) = 0$. Such elements will be called *free inputs* of f. The elements of $I_f^F \cup O_f^F$ will be called *free indices* of f. It is clear that up to a permutation, $f = p_k \otimes g \otimes 1_l$, where $k = |I_f^F|$, $l = |O_f^F|$ and $g \in \mathcal{T}_{n-k-l}$ has no free indices. As posets, $\mathcal{P}_f \simeq \mathcal{P}_g$, with labels

$$L_{S,f} = \begin{cases} \sigma(L_{S,g}), & \text{if } S \notin \mathrm{Min}(\mathcal{P}_f) \\ \sigma(L_{S,g}) \cup I_f^F, & \text{otherwise,} \end{cases}$$

for some $\sigma \in \mathscr{S}_n$. Clearly, $n - k - l$ has to be specified for g.

The two distinguished elements $\emptyset$ and $[n]$, if present in $\mathcal{P}_f$, can be easily recognized from its structure as a labelled poset. Indeed, $\emptyset \in \mathcal{P}_f$ if and only if it is the smallest element in $\mathcal{P}_f$ and has an empty label. Similarly, $[n] \in \mathcal{P}_f$ if and only if it is the largest element and $\cup_{S \in \mathcal{P}_f} L_S = [n]$. The basic operations on type functions are obtained as follows.

coro:Pf **Corollary 3.** *Let $f \in \mathcal{T}_n$, $g \in \mathcal{T}_m$. Then*

(i) For $\sigma \in \mathscr{S}_n$, $\mathcal{P}_{f \circ \sigma} \simeq \mathcal{P}_f$, with the labels changed as $L_S \mapsto \sigma^{-1}(L_S)$.

(ii) $\mathcal{P}_{f^*}$ is obtained from from $\mathcal{P}_f$ by adding/removing $\emptyset$ and $[n]$. If $[n]$ is added, then $L_{[n],f^*} = O_f^F$. All other elements and labels remain the same.

(iii) Assume the decomposition $[n+m] = [n] \oplus [m]$. Then $\mathcal{P}_{f \otimes g} \simeq \mathcal{P}_f \times \mathcal{P}_g$, with label sets

$$L_{(S,T)} = \begin{cases} L_S \cup (n + L_T), & \textit{if } S \in \mathrm{Min}(\mathcal{P}_f), \ T \in \mathrm{Min}(\mathcal{P}_g) \\ L_S, & \textit{if } S \notin \mathrm{Min}(\mathcal{P}_f), \ T \in \mathrm{Min}(\mathcal{P}_g) \\ n + L_T, & \textit{if } S \in \mathrm{Min}(\mathcal{P}_f), \ T \notin \mathrm{Min}(\mathcal{P}_g) \\ \emptyset, & \textit{otherwise.} \end{cases}$$

Proof. We only need to prove the statements on the label sets. This is quite clear in (i). In (ii), if $[n] \in \mathcal{P}_{f^*}$, then the only new indices not apearing below $[n]$ can be the free outputs of f. In (iii), assume that $i \in L_{(S,T)}$, then $S \oplus T$ must be a minimal element in $\mathcal{P}_{f\otimes g}$ containing i. Hence, either $i \in S$ or $i \in n+T$. In the first case, $i \in (S' \oplus T') \leq (S \oplus T)$ whenever $i \in S' \leq S$ and $T' \leq T$, so we must have $i \in L_S$ and $T \in \mathrm{Min}(\mathcal{P}_g)$. Similarly, for $i \in n+T$, we get $i \in n + L_T$ and $S \in \mathrm{Min}(\mathcal{P}_f)$.

□

We next show that the input and output sets of $f \in \mathcal{T}_n$ can be easily recognized from the labels in $\mathcal{P}_f$.

rop:pfinput **Proposition 8.** *Let $f \in \mathcal{T}_n$ and $i \in [n]$. Then*

(i) All $S \in \mathcal{P}_f$ such that $i \in L_S$ have the same rank, which will be denoted by $r_f(i)$. If $i \in O_f^F$, we put $r_f(i) := r(f) + 1$.

(ii) $i \in O_f$ if and only if $r_f(i)$ is odd.

Proof. As before, we proceed by induction on n. Both assertions are quite trivial for $n = 1$, so assume the statements hold for $m < n$. It is easily seen that the properties are invariant under permutations. Assume (i) and (ii) hold for $f \in \mathcal{T}_n$ and consider f^*. If $i \in L_{[n],f^*}$, then i cannot be contained in the label set of any other element, so (i) is true. Also, by Corollary coro:Pf 3, $L_{[n],f^*} = O_f^F$, so that i is an input of f. Since $[n]$ is the largest element of $\mathcal{P}_f$, $\rho_f([n]) = r(f)$ is even, so that (ii) holds as well. By duality, both statements hold if $i \in O_f^F$. In all other cases, $i \in L_{S,f}$ if and only if $i \in L_{S,f^*}$, so (i) is true for f^*. By the proof of Proposition thm:graded 3 we have $\rho_{f^*}(S) = \rho_f(S) \pm 1$ for any S, depending only on the fact whether $\emptyset \in \mathcal{P}_f$. This implies that (i) and (ii) are preserved by complementation.

It is now enough to assume that $f = g \otimes h$ for some $g \in \mathcal{T}_m$ and $h \in \mathcal{T}_{n-m}$. Suppose without loss of generality that $i \in [m]$, then $i \in L_{S\oplus T,f}$ if and only if $i \in L_{S,g}$ and $T \in \mathrm{Min}(h)$. Since then $\rho_h(T) = 0$, we have by the induction assumption

$$\rho_f(S \oplus T) = \rho_g(S) + \rho_h(T) = \rho_g(S) = r_g(i).$$

The statement (ii) follows from the fact that $i \in O_f$ if and only if $i \in O_g$.

□

3.2 Chains and combs

We have seen that for some type functions the poset $\mathcal{P}_f$ is a chain, which is also a basic example of a graded poset. A chain in 2^n has the form $\mathcal{C} = \{S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_N\}$, $S_i \subseteq [n]$. Note that the length of the chain $\mathcal{C}$ is $N-1$. It is clear that $\mathcal{C}$ is graded with rank $N-1$ and rank function $\rho(S_i) = i - 1$.

rop:chains **Proposition 9.** *For a chain $\mathcal{C} = \{S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_N\}$, the function*

$$f = f_{\mathcal{C}} := \sum_{i=1}^{N} (-1)^{i-1} p_{S_i}$$

is a type function if and only if N odd. In this case, we say that f is a chain type.

Proof. By Proposition thm:graded 3, if $f \in \mathcal{T}_n$, then the rank of f must be even, so that N must be odd. We will show that the converse is true. We proceed by induction on N. For $N = 1$, we have $f = p_{S_1} \in \mathcal{T}_n$. Assume that the statement holds for all odd numbers $M < N$ and let $\mathcal{C}$ be a chain as above. It is easily checked that

$$f \approx p_{n_1} \otimes g \otimes 1_{n-n_N},$$

where $n_i := |S_i|$ and $g \in \mathcal{F}_{n_N - n_1}$ is the function for a chain $\mathcal{C}'$ in $2^{n_N - n_1}$ of the form $\mathcal{C}' := \{\emptyset \subsetneq S_2' \subsetneq \cdots \subsetneq [n_N - n_1]\}$. Since f is a type function if g is, this shows that we may assume that the chain $\mathcal{C}$ contains $\emptyset$ and $[n]$. But then

$$f = 1 + \sum_{j=2}^{N-1} (-1)^{j-1} p_{S_j} + p_n$$

and

$$f^* = 1 - f + p_n = \sum_{j=1}^{N-2} (-1)^{j-1} p_{S_{j+1}},$$

By the induction assumption $f^* \in \mathcal{T}_n$, hence also $f = f^{**} \in \mathcal{T}_n$. $\square$

Let $f \in \mathcal{T}_n$ be a chain type and let $\mathcal{P}_f = \{S_1 \subsetneq \cdots \subsetneq S_N\}$ be the corresponding chain. There is a decomposition of $[n]$ given as

$$T_0 := S_1, \quad T_j := S_{j+1} \setminus S_j, \; j = 1, \dots, N-1, \quad T_N := [n] \setminus S_N.$$

It is clear that the label sets are given as $L_{S_j} = T_{j-1}$, $j = 1, \dots, N$ and it can be easily seen from Proposition prop:pfinput 8 that

$$I_f = \bigcup_{j=0}^{(N-1)/2} T_{2j}, \qquad O_f = \bigcup_{j=0}^{(N-1)/2} T_{2j+1} \cup O_f^F, \qquad I_f^F = T_0, \qquad O_f^F = T_N \tag{8}$$ eq:chain_i

(note that N must be odd). As we have seen, $f \approx p_{n_1} \otimes g \otimes 1_{n-n_N}$ and g is a chain type with no free indices. By Proposition prop:Xf_const 4, we have for any collection of first order objects

$$X_f(X_1, \dots, X_n) \overset{\sigma}{\simeq} \tilde{X}^*_{I_f^F} \otimes X_g(X_{\sigma^{-1}(1)}, \dots, X_{\sigma^{-1}(n_N - n_1)}) \otimes X_{O_f^F},$$

for some $\sigma \in \mathscr{S}_n$. We will show below that chain types correspond to an important kind of higher order objects.

ains_combs **Proposition 10.** *Let $f \in \mathcal{T}_n$ be a chain type with $\mathcal{P}_f = \{\emptyset = S_1 \subsetneq \cdots \subsetneq S_N = [n]\}$, with label sets $T_i = L_{S_{i-1}}$, $i = 1, \dots, n$. Let $Y = X_f(X_1, \dots, X_n)$ for some first order objects $X_1, \dots, X_n$. Then for $N \geq 3$, Y is an $(N-1)/2$-comb. More precisely, let $Y_1, \dots, Y_n$ be such that $Y_i = X_i$ for $i \in O_f$ and $Y_i = \tilde{X}_i$ for $i \in I_f$. Then*

$$Y \overset{\sigma}{\simeq} [Y_{T_{N-1}}, [[Y_{T_{N-2}}, [[\dots, [[Y_{T_{\frac{N+1}{2}}}, Y_{T_{\frac{N-1}{2}}}], Y_{T_{\frac{N-3}{2}}}]], \dots, Y_{T_2}]], Y_{T_1}]]$$

where we put $Y_T = \otimes_{j \in T} Y_j$ and $\sigma \in \mathscr{S}_n$.

Proof. Let $Y_1, \dots, Y_n$ be as assumed, then by (8),

$$Y_{T_i} = \begin{cases} \otimes_{j\in T_i} X_j, & \text{if } i \text{ is odd,} \\ \otimes_{j\in T_i} \tilde{X}_i, & \text{if } i \text{ is even.} \end{cases}$$

We will proceed by induction on N. Let $N = 3$, then $f = 1 - p_{S_2} + p_n$, and we see from Example 6 that $Y \simeq [Y_{T_2}, Y_{T_1}]$. Assume the assertion is true for $N-2$. As in the proof of Proposition 9, we see that

$$f^* = \sum_{i=1}^{N-2} (-1)^{i-1} p_{S_{i+1}} \approx p_{n_2} \otimes g \otimes 1_{n-n_{N-1}}$$

where $g \in \mathcal{T}_{n_{N-1}-n_2}$ is the chain type for a chain $\{\emptyset \subsetneq \sigma(S_3 \setminus S_2) \subsetneq \cdots \subsetneq \sigma(S_{N-1} \setminus S_2) = [n_{N-1} - n_2]\}$, for some $\sigma \in \mathscr{S}_n$ such that $\sigma(S_2) = [n_2]$ and $\sigma(T_{N-1}) = \sigma([n] \setminus S_{N-1}) = [n - n_{N-1}]$. By Proposition 4, we see that

$$X_f(X_1, \dots, X_n) = X^*_{f^*}(\tilde{X}_1, \dots, \tilde{X}_n) \overset{\sigma}{\simeq} (Y_{T_{N-1}} \otimes \tilde{X}_g \otimes Y^*_{T_1})^* \overset{\pi}{\simeq} [Y_{T_{N-1}}, [\tilde{X}_g, Y_{T_1}]]$$

where $\tilde{X}_g = X_g(\tilde{X}_{\sigma^{-1}(1)}, \dots, \tilde{X}_{\sigma^{-1}(n_{N-1})})$ and $\rho \in \mathscr{S}_3$. Since g satisfies the induction assumption, and $\tilde{\tilde{X}}_i = X_i$, we obtain

$$\tilde{X}_g \overset{\sigma'}{\simeq} [Y_{T_{N-2}}, [[\dots, [[Y_{T_{\frac{N+1}{2}}}, Y_{T_{\frac{N-1}{2}}}], Y_{T_{\frac{N-3}{2}}}]], \dots, Y_{T_2}]],$$

for some permutaton σ'. This proves the result.

□

Combs, picture
Other examples

3.3 Connecting chains: the causal product

It is easy to see that two chains can be appended to create a single chain using the ordinal sum, and any chain of more than one elements can be decomposed as an ordinal sum of chains. Such operations are trickier for chain types, since the chains have to be of even length. The next operation on boolean functions will be suitable for such considerations.

For a fixed decomposition $[n] = [n_1] \oplus [n_2]$ and functions $f_1 : \{0,1\}^{n_1} \to \{0,1\}$, $f_2 : \{0,1\}^{n_2} \to \{0,1\}$, we define their *causal product* as

$$f_1 \lhd f_2 := f_1 \otimes 1_{n_2} + p_{n_1} \otimes (f_2 - 1_{n_2}).$$

For $s^1 \in \{0,1\}^{n_1}$ and $s^2 \in \{0,1\}^{n_2}$, this function acts as

$$(f_1 \lhd f_2)(s^1 s^2) = f_1(s^1) + p_{n_1}(s^1)(f_2(s^2) - 1) = \begin{cases} f_1(s^1), & \text{if } s^1 \neq \theta_{n_1}, \\ f_2(s^2), & \text{if } s^1 = \theta_{n_1}. \end{cases} \tag{9}$$

The following properties are immediate from (9).

Lemma 9. *Let $f_1, g_1 \in \mathcal{F}_{n_1}$, $f_2, g_2 \in \mathcal{F}_{n_2}$. Then $f_1 \lhd f_2 \in \mathcal{F}_{n_1+n_2}$ and we have*

(i) $(f_1 \lhd f_2)^* = f_1^* \lhd f_2^*$,

(ii) $(f_1 \vee g_1) \lhd (f_2 \vee g_2) = (f_1 \lhd f_2) \vee (g_1 \lhd g_2) = (f_1 \lhd g_2) \vee (g_1 \lhd f_2)$,

(iii) $(f_1 \wedge g_1) \lhd (f_2 \wedge g_2) = (f_1 \lhd f_2) \wedge (g_1 \lhd g_2) = (f_1 \lhd g_2) \wedge (g_1 \lhd f_2)$.

Moreover, for any $f_3 \in \mathcal{F}_{n_3}$, and for the decomposition $[n] = [n_1] \oplus [n_2] \oplus [n_3]$, we have

$$(f_1 \lhd f_2) \lhd f_3 = f_1 \lhd (f_2 \lhd f_3).$$

We can also combine f_1 and f_2 in the opposite order:

$$f_2 \lhd f_1 := 1_{n_1} \otimes f_2 + (f_1 - 1_n) \otimes p_{n_2},$$

so that

$$(f_2 \lhd f_1)(s^1 s^2) = f_2(s^2) + p_{n_2}(s^2)(f_1(s^1) - 1_{n_1}) = \begin{cases} f_2(s^2), & \text{if } s^2 \neq \theta_{n_2}, \\ f_1(s^1), & \text{if } s^2 = \theta_{n_2}. \end{cases} \tag{10}$$

Of course, this product has similar properties as listed in the above lemma. To avoid any confusion, we have to bear in mind the fixed decomposition $[n] = [n_1] \oplus [n_2]$ and that for the concatenation $s = s^1 s^2$, f_i acts on s^i.

Lemma 10. *In the situation as above, we have*

$$f_1 \otimes f_2 = (f_1 \rhd f_2) \wedge (f_2 \rhd f_1).$$

Proof. This is again by straightforward computation from (9) and (10): let $s^1 \in \{0,1\}^{n_1}$, $s^2 \in \{0,1\}^{n_2}$ and compute

$$\begin{aligned} (f_1 \lhd f_2) \wedge (f_2 \lhd f_1)(s^1 s^2) &= \big(f_1(s^1) + p_{n_1}(s^1)(f_2(s^2) - 1)\big) \big(f_2(s^2) + p_{n_2}(s^2)(f_1(s^1) - 1)\big) \\ &= f_1(s^1) f_2(s^2), \end{aligned}$$

the last equality follows from the fact that $f_i(s^i)(1 - f_i(s^i)) = 0$ (since $f_i(s^i) \in \{0,1\}$) and the fact that p_{n_1} is the least element in $\mathcal{F}_{n_1}$, so that $p_{n_1}(s^1)(f_1(s^1) - 1) = p_{n_1}(s^1) - p_{n_1}(s^1) = 0$. □

For the smallest and the largest element in $\mathcal{F}_n$, the causal product behaves as follows.

Lemma 11. *Let $f \in \mathcal{F}_{n_1}$ and let $n_2 \in \mathbb{N}$. Then for the decomposition $[n_1 + n_2] = [n_1] \oplus [n_2]$,*

$$f \lhd 1_{n_2} = f \otimes 1_{n_2} \leq 1_{n_2} \lhd f$$

and

$$p_{n_2} \lhd f = f \otimes p_{n_2} \leq f \lhd p_{n_2}.$$

In particular,

$$(p_{n_1} \otimes 1_{n_2})^* = 1_{n_1} \lhd p_{n_2} = 1 - p_{[n_1]} + p_{n_1+n_2}$$

is the chain type for $\{\emptyset \subsetneq [n_1] \subsetneq [n_1 + n_2]\}$. Similar properties hold for the decomposition $[n_1 + n_2] = [n_2] \oplus [n_1]$.

Proof. Immediate from the definition of the causal product and Lemma lemma:causal_tensor 10.

□

Using the last part of Lemma lemma:causal_product 9, for a decomposition $[n] = \oplus_i [n_i]$ and $f_i \in \mathcal{F}_{n_i}$, we may define the function $f_1 \lhd \ldots \lhd f_k \in \mathcal{F}_n$. Note that we have for $s = s^1 \ldots s^k$,

$$
\begin{aligned}
(f_1 \lhd \ldots \lhd f_k)(s) &= f_1(s^1) + p_{n_1}(s^1)(f_2(s^2) - 1) + \cdots + p_{n_1}(s^1) \ldots p_{n_{k-1}}(s^{k-1})(f_k(s^k) - 1) \\
&= \begin{cases} f_1(s_1) & \text{if } s^1 \neq \theta_{n_1} \\ f_2(s^2) & \text{if } s^1 = \theta_{n_1}, s^2 \neq \theta_{n_2} \\ \ldots & \\ f_k(s^k) & \text{if } s^1 = \theta_{n_1}, \ldots, s^{k-1} = \theta_{n_{k-1}}. \end{cases}
\end{aligned}
$$

For any permutation $\pi \in \mathscr{S}_k$, we define $f_{\pi^{-1}(1)} \lhd \ldots \lhd f_{\pi^{-1}(k)} \in \mathcal{F}_n$ in an obvious way.

We will show that the causal product is related to the ordinal sum $\star$ of the corresponding posets.

rtl_ordinal **Proposition 11.** *Let $f \in \mathcal{T}_n$, $g \in \mathcal{T}_m$ and consider the decomposition $[n+m] = [n] \oplus [m]$. Replace the labels of $\mathcal{P}_g$ by their translations $L_S \mapsto n + L_S = \{n + i,\ i \in L_S\}$. Then*

(a) If $[n] \in \mathcal{P}_f$ and $\emptyset \in \mathcal{P}_g$, then $\mathcal{P}_{f \lhd g} = \mathcal{P}_f \star (\mathcal{P}_g \setminus \{\emptyset\})$, with all labels remaining the same.

(b) If $[n] \in \mathcal{P}_f$ and $\emptyset \notin \mathcal{P}_g$, then $\mathcal{P}_{f \lhd g} = (\mathcal{P} \setminus \{[n]\}) \star \mathcal{P}_g$, where the labels of $[n]$ are added to the labels of elements in $\mathrm{Min}(\mathcal{P}_g)$.

(c) If $[n] \notin \mathcal{P}_f$ and $\emptyset \in \mathcal{P}_g$, then $\mathcal{P}_{f \lhd g} = \mathcal{P}_f \star (\mathcal{P}_g \setminus \{\emptyset\})$, where the free outputs of f are added to the label sets of elements in $\mathrm{Min}(\mathcal{P}_g \setminus \emptyset)$.

(d) If $[n] \notin \mathcal{P}_f$ and $\emptyset \notin \mathcal{P}_g$, then $\mathcal{P}_{f \lhd g} = \mathcal{P}_f \star \{\bullet\} \star \mathcal{P}_g$, where $\{\bullet\}$ is a one-element poset with label $L_\bullet = O_f^F$.

Proof. By definition of the causal product, we have

$$
f \lhd g = \sum_{S \in \mathcal{P}_f \setminus \{[n]\}} \hat{f}_S p_S + (\hat{f}_{[n]} - 1 + \hat{g}_\emptyset) p_{[n]} + \sum_{T \in \mathcal{P}_g \setminus \{\emptyset\}} \hat{g}_T p_{[n] \oplus T}.
$$

The term in brackets can be equal to 1, -1, or 0, depending on whether $[n] \in \mathcal{P}_f$ and $\emptyset \in \mathcal{P}_g$. The statement is now immediate.

□

It is not clear that if f and g are type functions, then $f \lhd g$ is a type function as well. Nevertheless, it can be seen from the above result that if both f and g are chain types with chains of N and M elements, respectively, then $f \lhd g$ is a chain type for a chain with $M + N \pm 1$ elements. Note also that that this construction can be interpreted as appending the two chains in the respective order. Our next result shows that if f or g is a chain type, we always obtain a type function.

end_chain_f **Proposition 12.** *Let $f \in \mathcal{T}_{n_1}$ and let $\beta \in \mathcal{T}_{n_2}$ be a chain type. Then both $f \lhd \beta$ and $\beta \lhd f$ are types, with outputs $O = O_f \oplus O_\beta$ and inputs $I = I_f \oplus I_\beta$.*

Proof. Let $\beta = \sum_{k=1}^{N}(-1)^{k-1}p_{S_k}$ for some odd N and $S_1 \subsetneq \cdots \subsetneq S_N \subseteq [n_2]$. We will proceed by induction on N. Suppose $N = 1$. If $S_1 = \emptyset$, then $\beta = 1_{n_2}$ and we have by Lemma 11

$$f \lhd 1_{n_2} = f \otimes 1_{n_2} \in \mathcal{T}_{n_1+n_2}$$

and

$$1_{n_2} \lhd f = (p_{n_2} \lhd f^*)^* = (f \otimes p_{n_2})^* \in \mathcal{T}_{n_1+n_2}.$$

Assume that $S_1 = [n_2]$, then $\beta = p_{n_2}$ and the assertion follows by duality. If $\emptyset \neq S_1 \subsetneq [n_2]$, then we have $\beta \approx p_{m_1} \otimes 1_{m_2} = p_{m_1} \lhd 1_{m_2}$ for $m_1 = |S_1|$, $m_1 + m_2 = n_2$. Then

$$\beta \lhd f \approx p_{m_1} \lhd (1_{m_2} \lhd f) \in \mathcal{T}_{n_1+n_2}, \quad f \lhd \beta \approx (f \lhd p_{m_1}) \lhd 1_{m_2} \in \mathcal{T}_{n_1+n_2},$$

by the first part of the proof and Lemma 9.

Assume next that the assertion holds for all odd numbers $M < N$. Using Proposition 11 (d), we see that $\beta \approx \beta_1 \lhd \beta_2$, where β_1 is an $N-2$-element chain type and β_2 is a one-element chain type. Then

$$\beta \lhd f \approx \beta_1 \lhd (\beta_2 \lhd f), \qquad f \lhd \beta \approx (f \lhd \beta_1) \lhd \beta_2$$

are type functions, by the induction assumption.

To prove the statement on the output and input indices, note that for any $i \in [n_1] \oplus [n_2]$, we have $e^i_{n_1+n_2} = e^j_{n_1}\theta_{n_2}$ or $e^i_{n_1+n_2} = \theta_{n_1}e^k_{n_2}$ for some $j \in [n_1]$, $k \in [n_2]$. Then

$$f \lhd \beta(e^i) = f(e^j_{n_1}) \text{ or } f \lhd \beta(e^i) = \beta(e^k_{n_2}).$$

The statement on input/output indices follow from Lemma 7. The proof for $\beta \lhd f$ is similar. □

3.4 The structure of type functions

Our main result here is the following structure theorem for the type functions.

Theorem 4. *Let $f \in \mathcal{T}_n$. Then there is a decomposition $[n] = \oplus_{i=1}^k [n_i]$, chain types $\beta_1 \in \mathcal{T}_{n_1}, \ldots, \beta_k \in \mathcal{T}_{n_k}$ such that $O_f = \oplus_j O_{\beta_j}$, $I_f = \oplus_j I_{\beta_j}$, finite index sets A, B and permutations $\pi_{a,b} \in \mathscr{S}_k$, $a \in A$, $b \in B$ such that*

$$f \approx \bigvee_{a\in A}\bigwedge_{b\in B}(\beta_{\pi^{-1}_{a,b}(1)} \lhd \ldots \lhd \beta_{\pi^{-1}_{a,b}(k)}) = \bigwedge_{b\in B}\bigvee_{a\in A}(\beta_{\pi^{-1}_{a,b}(1)} \lhd \ldots \lhd \beta_{\pi^{-1}_{a,b}(k)}).$$

Proof. We will once again proceed by induction on n. Since any element in $\mathcal{T}_n$ for $n \leq 3$ is a chain type, the statement clearly holds in this case. Assume the condition holds for all $m < n$. The condition is obviously invariant under permutations. Assume f can be written in the given form, then

$$f^* \approx \bigwedge_{a\in A}\bigvee_{b\in B}(\beta^*_{\pi^{-1}_{a,b}(1)} \lhd \ldots \lhd \beta^*_{\pi^{-1}_{a,b}(k)}) = \bigvee_{b\in B}\bigwedge_{a\in A}(\beta^*_{\pi^{-1}_{a,b}(1)} \lhd \ldots \lhd \beta^*_{\pi^{-1}_{a,b}(k)}).$$

Since β^*_j is a chain type for each j, this proves the statement for f^*.

It is now enough to show this form for $f = f_1 \otimes f_2$, where $f_1 \in \mathcal{T}_m$, $f_2 \in \mathcal{T}_{n-m}$ with $[n] = [m] \oplus [n-m]$. By the induction assumption, f_1 amd f_2 satisfy the conditions, so that

$$f_1 \approx \bigvee_{a\in A} \bigwedge_{b\in B} (\beta^1_{\pi^{-1}_{a,b}(1)} \lhd \ldots \lhd \beta^1_{\pi^{-1}_{a,b}(k_1)}) = \bigwedge_{b\in B} \bigvee_{a\in A} (\beta^1_{\pi^{-1}_{a,b}(1)} \lhd \ldots \lhd \beta^1_{\pi^{-1}_{a,b}(k_1)}),$$
$$f_2 \approx \bigvee_{c\in C} \bigwedge_{d\in D} (\beta^2_{\tau^{-1}_{c,d}(1)} \lhd \ldots \lhd \beta^2_{\tau^{-1}_{c,d}(k_2)}) = \bigwedge_{d\in D} \bigvee_{c\in C} (\beta^2_{\tau^{-1}_{c,d}(1)} \lhd \ldots \lhd \beta^2_{\tau^{-1}_{c,d}(k_2)})$$

for some chain types $\beta^1_j \in \mathcal{T}_{m_j}$, $[m] = \oplus_{j=1}^{k_1}[m_j]$, and $\beta^2_j \in \mathcal{T}_{l_j}$, $[n-m] = \oplus_{j=1}^{k_2}[l_j]$ and permutations $\pi_{a,b} \in \mathscr{S}_{k_1}$, $\tau_{c,d} \in \mathscr{S}_{k_2}$. Let

$$\beta_1^{a,b} := \beta^1_{\pi^{-1}_{a,b}(1)} \lhd \ldots \lhd \beta^1_{\pi^{-1}_{a,b}(k_1)}, \qquad \beta_2^{c,d} := \beta^2_{\tau^{-1}_{c,d}(1)} \lhd \ldots \lhd \beta^2_{\tau^{-1}_{c,d}(k_2)}.$$

Using the properties of the tensor product (Lemma 13)ii), we get from Lemma 10

$$f \approx \big(\bigvee_{a\in A} \bigwedge_{b\in B} \beta_1^{a,b}\big) \otimes \big(\bigvee_{c\in C} \bigwedge_{d\in D} \beta_2^{c,d}\big) = \bigvee_{a,c} \bigwedge_{b,d} (\beta_1^{a,b} \otimes \beta_2^{c,d}) = \bigvee_{a,c} \bigwedge_{b,d} (\beta_1^{a,b} \lhd \beta_2^{c,d}) \wedge (\beta_2^{c,d} \lhd \beta_1^{a,b})$$

On the other hand, using Lemma 10 and Lemma 9, we get

$$\begin{aligned} f &\approx \big(\bigwedge_{b\in B} \bigvee_{a\in A} \beta_1^{a,b}\big) \otimes \big(\bigwedge_{d\in D} \bigvee_{c\in C} \beta_2^{c,d}\big) \\ &= \left[\big(\bigwedge_{b\in B} \bigvee_{a\in A} \beta_1^{a,b}\big) \lhd \big(\bigwedge_{d\in D} \bigvee_{c\in C} \beta_2^{c,d}\big)\right] \wedge \left[\big(\bigwedge_{d\in D} \bigvee_{c\in C} \beta_2^{c,d}\big) \lhd \big(\bigwedge_{b\in B} \bigvee_{a\in A} \beta_1^{a,b}\big)\right] \\ &= \big(\bigwedge_{b,d} \bigvee_{a,c} \beta_1^{a,b} \lhd \beta_2^{c,d}\big) \wedge \big(\bigwedge_{b,d} \bigvee_{a,c} \beta_2^{c,d} \lhd \beta_1^{a,b}\big). \end{aligned}$$

We have the decomposition $[n] = \oplus_{j=1}^k [n_j]$, with $k = k_1 + k_2$ and $n_j = m_j$, $j = 1, \ldots, k_1$, $n_j = l_{j-k_1}$, $j = k_1+1, \ldots, k$, and chain types $\beta_j \in \mathcal{T}_{n_j}$, $\beta_j = \beta^1_j$ for $j = 1, \ldots, k_1$ and $\beta_j = \beta^2_{j-k_1}$ for $j = k_1+1, \ldots, k$. To get the permutation sets, let $A' = A \times C$, $B' = B \times D \times \mathscr{S}_2$ and define $\pi_{a',b'}$ in $\mathscr{S}_k$ as the block permutation with respect to the decomposition $[k] = [k_1] \oplus [k_2]$ (see Appendix A.1)

$$\pi_{(a,c),(b,d,\lambda)} = \rho_\lambda \circ (\pi_{a,b} \oplus \tau_{c,d}).$$

This finishes the proof.

□

Remark 2. In general, it is not clear for which sets of permutations such a combination of chain types will be a chain type. Nevertheless, since all the connected chains have the same input and output indices, a function of the form as in Theorem 4 will always be a subtype. That is, the objects corresponding to such a function will describe a set of channels, obtained by taking pullback and pushouts of combs.

3.5 The labelled poset $\mathcal{P}_f^0$

Let $\mathcal{P}_f^0$ be the subposet in $\mathcal{P}_f$, consisting of the elements with nonempty labels and possibly $\emptyset$. We will show that any $f \in \mathcal{T}_n$ is fully determined by $\mathcal{P}_f^0$ (and n). This is convenient, because $\mathcal{P}_f^0$ is much smaller and easier to visualise that $\mathcal{P}_f$. More importantly, from $\mathcal{P}_f^0$, one can find a decomposition of f as products and complements of other functions. In particular, it is possible to obtain from $\mathcal{P}_f^0$ some choice of the chain types $\beta_1, \dots, \beta_k$ in the decomposition in Theorem 4.

We will start by some basic properties of $\mathcal{P}_f^0$. Some further properties, and more technical parts of the proofs, can be found in Appendix C.

Recall that for all $T \in \mathcal{P}_f$, we have $T = \cup_{T' \in \mathcal{P}_f^0, T' \subseteq T} L_{T'}$.

a:p0_basic **Lemma 12.** *Let $f \in \mathcal{T}_n$.*

(i) $\mathrm{Min}(\mathcal{P}_f^0) = \mathrm{Min}(\mathcal{P}_f)$.

(ii) $\mathcal{P}_f^0$ *is a chain* $\iff$ $\mathcal{P}_f$ *is a chain* $\iff$ $\mathcal{P}_f^0 = \mathcal{P}_f$.

(iii) $\mathcal{P}_f^0$ *has a largest element if and only if f or f^* has a free output. In this case the corresponding set is the largest element in $\mathcal{P}_f$.*

Proof. (i) Obvious. For (ii), assume that $\mathcal{P}_f^0$ is a chain and let $S, T \in \mathcal{P}_f$. Let $i \in S \setminus T$ and $j \in T$, and let $S' \subseteq S$ and $T' \subseteq T$ be such that $S', T' \in \mathcal{P}_f^0$ and $i \in L_{S'}$, $j \in L_{T'}$. Since $\mathcal{P}_f^0$ is a chain, S' and T' are comparable. If $S' \leq T'$, then $S' \subseteq T' \subseteq T$, so that $i \in T$, which is not possible. Hence $T' \subseteq S' \subseteq S$, for all $T' \in \mathcal{P}_f^0$, $T' \subseteq T$. Hence $T \subseteq S$, and $\mathcal{P}_f$ is a chain. It is clear that then $\mathcal{P}_f^0 = \mathcal{P}_f$.

If $\mathcal{P}_f$ is not a chain, then there are some type functions f_1, f_2 such that $f = f_1 \otimes f_2$ or $f = (f_1 \otimes f_2)^*$. Moreover, the ranks of f_1 and f_2 are at least 2. It follows that both $\mathcal{P}_{f_1 \otimes f_2}$ and $\mathcal{P}_{(f_1 \otimes f_2)^*}$ contain an element $S \oplus T$, where $S \in \mathcal{P}_{f_1}$, $T \in \mathcal{P}_{f_2}$ but none of the two elements is minimal. Then there is some $S' \in \mathcal{P}_{f_1}$ and $T' \in \mathcal{P}_{f_2}$ such that $S' \oplus T$, $S \oplus T' \subseteq S \oplus T$, so that no element of $S \oplus T$ is a label. Hence $S \oplus T \notin \mathcal{P}_f^0$, so that $\mathcal{P}_f \neq \mathcal{P}_f^0$.

To prove (iii) let T be the largest element in P_f^0. Then

$$\cup \mathcal{P}_f = \cup L_S \subseteq T \subseteq \cup \mathcal{P}_f.$$

If follows that $T = \cup \mathcal{P}_f$ is the largest element in $\mathcal{P}_f$. If $T \neq [n]$, then clearly, f has some free outputs. If $T = [n]$, then since $[n] \in \mathcal{P}_f^0$, we have $\emptyset \neq L_{[n],f} = O^F_{f^*}$, so that f^* has free outputs. $\square$

pf0_largest **Proposition 13.** *Assume that $\mathcal{P}_f^0$ has a largest element. Then either f is a chain type, or there is some $h \in \mathcal{T}_m$ such that $\mathcal{P}_h^0$ has no largest element, and a chain type $\beta \in \mathcal{T}_{n-m}$ such that $f \approx h \lhd \beta$.*

Proof. Let T be the largest element in $\mathcal{P}_f^0$. If $T = [n]$, then by Lemma 12(iii) and its proof, f^* has some free outputs. Hence $f^* \approx h_1 \otimes 1_{k_1}$ for some $h_1 \in \mathcal{T}_{n-k_1}$ with no free outputs. It follows that $f \approx (h_1 \otimes 1_{k_1})^* = (h_1 \lhd 1_{k_1})^* = h_1^* \lhd p_{k_1}$.

If $T \neq [n]$, then, since T is also the largest element in $\mathcal{P}_f$, we see that f has free outputs, so that $f \approx h_1 \otimes 1_{k_1} = h_1 \lhd 1_1$, where $h_1 \in \mathcal{T}_{n-k_1}$ is a type function with no free outputs. Since clearly $\mathcal{P}^0_{h_1} = \mathcal{P}^0_f$, T is the largest element in $\mathcal{P}^0_{h_1}$, but this time $T = [n-k_1]$, so that we may use the first part of the proof. We obtain that there is some k_2 and a type function $h_2 \in \mathcal{T}_{n-k_1-k_2}$ with no free outputs such that

$$f \approx h_1 \lhd 1_{k_1} = h_2^* \lhd p_{k_2} \lhd 1_{k_1}$$

So far, we have written f in the form $f = h \lhd \beta$, where β is a chain type and $h \in \mathcal{T}_{n-k}$ for $k > 0$ is such that h^* has no free outputs. It follows that if $\mathcal{P}^0_h$ has a largest element, then it cannot be equal to $[n-k]$. Hence h must have some free outputs, and we may proceed as above, replacing f by h. Since n is decreasing at each step, we either get to $n-k \leq 3$, in which case h must be a chain and therefore also $f = h \lhd \beta$ is a chain, or $\mathcal{P}^0_h$ has no largest element.

□

In the situation of the above Proposition, if f is not a chain, $\mathcal{P}^0_h$ and β can be seen from $\mathcal{P}^0_f$ as follows. Since h has no free outputs and does not contain $[m]$, we obtain from Proposition 11 that $\mathcal{P}^0_f \approx \mathcal{P}^0_h \star (\mathcal{P}_\beta \setminus \{\emptyset\})$, with the same sets of labels. It follows that there exists a largest element in $\mathcal{P}^0_f$ with the property that it covers more than one element. Let this element be S and let $T_1, \dots, T_k$ be the elements covered by S. There is a chain $S = S_1 \lneq \cdots \lneq S_K$, where S_K is the largest element in $\mathcal{P}^0_f$. We then have $\mathcal{P}^0_h \simeq \cup_j T_j^{\downarrow}$. If K is even, add an element S_0 with empty label at the bottom of the chain to obtain a chain of even length. This then corresponds to the chain type.

Proposition 14. *Let $f \in \mathcal{T}_n$ be such that either f or f^* has a free input. Then either f is a chain type, or there is some chain type $\beta \in \mathcal{T}_k$ and some $h \in \mathcal{T}_{n-k}$ such that h and h^* have no free inputs and $f \approx \beta \lhd h$.*

Proof. Assume f has a free input, $f \approx p_k \otimes h = p_k \lhd h$, where $h \in \mathcal{P}_{n-k}$ has no free inputs. On the other hand, if f^* has a free input, then, similarly, $f^* \approx p_k \lhd g$ for some $g \in \mathcal{T}_{n-k}$ with no free inputs, hence $f \approx (p_k \lhd g)^* = 1_k \lhd g^*$. Repeating the process, we get the result after finitely many steps.

□

Under the assumptions of the proposition above, assume that f is not a chain type. Suppose that $\mathcal{P}^0_f$ has a least element S_1 and let S be the smallest elemet in $\mathcal{P}^0_f$ with the property that it is covered by more than one element. Let these elements be $T_1, \dots, T_l$ and let $S_1 \lneq \cdots \lneq S_K$ be a chain such that $S_K = S$. Put $L := \cap_j L_{T_j}$. Using Proposition 11 as before, we have the following situations. If $L = \emptyset$, then put $k = |S_K|$. If K is odd, then the chain corresponds to a chain type $\beta \in \mathcal{T}_k$ and $\mathcal{P}^0_g = \cup_j T_j^{\uparrow} \cup \{\emptyset\}$. If K is even, then $\mathcal{P}^0_g = \cup_j T_j^{\uparrow}$ and β is the chain type for the chain $S_1 \subsetneq \cdots \subsetneq S_{K-1}$ (with free outputs in L_{S_K}). If $L \neq \emptyset$, then add an element S_{K+1} at the end of the chain, with label $L_{S_{K+1}} = L$ and put $k = |S_{K+1}|$. If K is odd, then $\beta \in \mathcal{T}_k$ is the chain type for the chain $S_1 \lneq \cdots \lneq S_K$ (with free outputs in L) and $\mathcal{P}^0_g = \cup_j T_j^{\uparrow} \cup \{\emptyset\}$, with labels of T_j replaced by $L_{T_j} \setminus L$. If K is even, then $\mathcal{P}^0_g = \cup_j T_j^{\uparrow}$ and β is the chain type for $S_1 \subsetneq \cdots \subsetneq S_{K+1}$.

If $\mathcal{P}^0_f$ has no least element, then by the assumptions f must have free inputs (since $\mathcal{P}^0_{f^*}$ has least element $\emptyset$). Hence $f \approx p_k \lhd g$, for $g \in \mathcal{T}_{n-k}$ with no free inputs. Note also that we have $\emptyset \notin \mathcal{P}_g$, so that g^* has no free inputs as well.

We will now deal with the case when f and f^* have no free indices. Let $\mathcal{P}$ be a poset with labels in $[n]$ and let $\mathcal{P}_1, \mathcal{P}_2 \subseteq \mathcal{P}$ be nonempty. We will say that $\mathcal{P}_1$ and $\mathcal{P}_2$ are *independent components* of $\mathcal{P}$ if $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$ (direct sum of posets) and $L_S \cap L_T = \emptyset$ for any $S \in \mathcal{P}_1$ and $T \in \mathcal{P}_2$. In this case, we will write

$$\mathcal{P} = \mathcal{P}_1 +^L \mathcal{P}_2.$$

Proposition 15. *Let $f \in \mathcal{T}_n$ be such that f and f^* have no free indices. Assume that $\emptyset \notin \mathcal{P}_f$.*

(i) If $f^* \approx f_1 \otimes f_2$ for some type functions f_1 and f_2, then

$$\mathcal{P}^0_f \simeq (\mathcal{P}^0_{f_1} \setminus \emptyset) +^L (\mathcal{P}^0_{f_2} \setminus \emptyset).$$

(ii) If $\mathcal{P}^0_f = \mathcal{P}_1 +^L \mathcal{P}_2$ for some labelled subposets $\mathcal{P}_1$ and $\mathcal{P}_2$, then there are some type functions f_1 and f_2 such that $\mathcal{P}_1 = (\mathcal{P}^0_{f_1} \setminus \emptyset)$, $\mathcal{P}_2 = (\mathcal{P}^0_{f_2} \setminus \emptyset)$ and $f^* \approx f_1 \otimes f_2$.

(iii) If $f \approx f_1 \otimes f_2$ for type functions f_1 and f_2, then no decomposition of $\mathcal{P}^0_f$ into independent components exists.

Proof. Under the assumptions, $\mathcal{P}_{f^*}$ contains $\emptyset$ and has no largest element. Therefore $\mathcal{P}^0_f = \mathcal{P}^0_{f^*} \setminus \{\emptyset\}$ and $\mathcal{P}^0_f$ has no largest or least element. Assume that $f_1 \in \mathcal{T}_{n_1}$ and $f_2 \in \mathcal{T}_{n_2}$ are such that $f^* = f_1 \otimes f_2$ for the decomposition $[n] = [n_1] \oplus [n_2]$. Since $\emptyset \in \mathcal{P}_{f^*} = \mathcal{P}_{f_1 \otimes f_2}$, we see by Corollary 3 (iii) that both $\mathcal{P}_{f_1}$ and $\mathcal{P}_{f_2}$ must contain $\emptyset$ and $\mathcal{P}^0_{f^*}$ consists of $\mathcal{P}^0_{f_1}$ and $\mathcal{P}^0_{f_2}$ glued at $\emptyset$, with labels of $\mathcal{P}^0_{f_2}$ translated by n_1. Since $\mathcal{P}^0_f = \mathcal{P}^0_{f^*} \setminus \{\emptyset\}$, the assertion (i) follows.

For (iii), assume that $f = f_1 \otimes f_2$ and $\mathcal{P}^0_f = \mathcal{P}_1 +^L \mathcal{P}_2$. Note that none of the functions can be a 1-element chain, since then f would have free inputs or outputs. Let $\mathrm{Min}(\mathcal{P}_{f_1}) = \{U_1, \dots, U_k\}$, $\mathrm{Min}(\mathcal{P}_{f_2}) = \{V_1, \dots, V_l\}$. By Corollary 3(iii), we have $\mathrm{Min}(\mathcal{P}_f) = \mathrm{Min}(\mathcal{P}_{f_1}) \times \mathrm{Min}(\mathcal{P}_{f_2})$. For some i and j, let $(U_i, V_j) \in \mathcal{P}_1$. Since (U_i, V_j) and (T, V_j) are comparable for any $T \in \mathcal{P}^0_{f_1}$, $U_i \leq T$, we must have $(T, V_j) \in \mathcal{P}_1$ for all such T. By Lemma 16, there is some T that covers U_i. But then $L_{(T,V_j)} = L_{T,V_{j'}} = L_T$ for all j', so that $(T, V_{j'}) \in \mathcal{P}_1$ for all j'. Since $(U_i, V_{j'}) \leq (T, V_{j'})$ for all j', this implies that $(U_i, V_{j'}) \in \mathcal{P}_1$ for all j'. By the same reasoning with V_j, we get that all $(U_i, V_j) \in \mathcal{P}_1$, which is not posssible.

For (ii), assume that $\mathcal{P}^0_f = \mathcal{P}_1 +^L \mathcal{P}_2$. Since either f or f^* is, up to a permutation, a tensor product of type functions and we cannot have $f \approx f_1 \otimes f_2$ by (iii), it must hold that $f^* \approx f_1 \otimes f_2$. But then by (i) $\mathcal{P}^0_f = \mathcal{P}'_1 +^L \mathcal{P}'_2$, with $\mathcal{P}'_i = \mathcal{P}^0_{f_i} \setminus \{\emptyset\}$. Let

$$\mathcal{P}^0_f = \mathcal{Q}_1 +^L \dots +^L \mathcal{Q}_M$$

be the finest decomposition into independent components. Then there are some $C, D \subset [M]$ such that

$$\mathcal{P}_1 = +^L_{i \in C} \mathcal{Q}_i, \quad \mathcal{P}_2 = +^L_{i \in [M] \setminus C} \mathcal{Q}_i, \quad \mathcal{P}'_1 = +^L_{i \in D} \mathcal{Q}_i, \quad \mathcal{P}'_2 = +^L_{i \in [M] \setminus D} \mathcal{Q}_i.$$

Assume that $M \geq 3$, otherwise each $\mathcal{P}_i$ is one of $\mathcal{P}'_j$ and we are done. Then D or $[M] \setminus D$ has at least two elements. So assume $|D| \geq 2$. Then $\mathcal{P}'_1$ has no largest element, which implies that $\mathcal{P}'_1 = \mathcal{P}^0_{f_1} \setminus \{\emptyset\} = \mathcal{P}^0_{f_1^*}$ and f_1^* satisfies the assumptions in (iii). Hence $f_1 \approx g_1 \otimes g_2$ for some type functions g_1 and g_2 and we obtain that

$$\mathcal{P}^0_{g_1} \setminus \{\emptyset\} = +^L_{i \in D'} \mathcal{Q}_i, \quad \mathcal{P}^0_{g_2} \setminus \{\emptyset\} = +^L_{i \in D \setminus D'} \mathcal{Q}_i$$

for some $D' \subset D$. Continuing in this way, we obtain that for any $i \in [M]$ there is some type function g_i such that $\emptyset \in \mathcal{P}^0_{g_i}$ and $\mathcal{Q}_i = \mathcal{P}^0_{g_i} \setminus \{\emptyset\}$. It follows that

$$\mathcal{P}_1 = +^L_{i \in C}(\mathcal{P}^0_{g_i} \setminus \{\emptyset\}) = \mathcal{P}^0_{\otimes_{i \in D} g_i} \setminus \{\emptyset\}$$

and $f_1 \approx \otimes_{i \in D} g_i$, similarly, $\mathcal{P}_2 = \mathcal{P}^0_{\otimes_{i \in [M] \setminus D} g_i} \setminus \{\emptyset\}$ and $f_2 \approx \otimes_{i \in [M] \setminus D} g_i$.

□

It follows that in the situation of the above Proposition, if $f^* \approx g_1 \otimes \cdots \otimes g_k$, we can identify $\mathcal{P}^0_{g_l}$ by looking at the independent components of $\mathcal{P}^0_f$. The case when $\mathcal{P}^0_f$ has no independent components is somewhat more complicated. By the lemma above, this is the case when $f \approx f_1 \otimes \cdots \otimes f_k$. The proof of the following result is in Appendix C.

ree_product **Proposition 16.** *Let $f \in \mathcal{T}_n$ be such that f and f^* have no free indices and $\emptyset \notin \mathcal{P}^0_f$. Assume $f \approx f_1 \otimes \cdots \otimes f_k$ is a finest decomposition of f as a product. Then the labelled posets $\mathcal{P}^0_{f_l}$, $l = 1, \ldots, k$ can be obtained from the structure of $\mathcal{P}^0_f$, up to a permutation on the labels.*

thm:pf0 **Theorem 5.** *Every type function $f \in \mathcal{T}_n$ is fully determined by the labelled poset $\mathcal{P}^0_f$.*

Proof. We will proceed by induction on n. If f is a chain type, the assertion follows from Lemma 12 and Proposition 3, so that the assertion holds for $n \leq 3$. Assume it is true for all $m < n$ and let $f \in \mathcal{T}_n$. By Propositions 13 and 14 and remarks below them, if f or f^* has some free indices, then we have $f = \beta_1 \lhd h \lhd \beta_2$, where β_1, β_2 are chain types and $h \in \mathcal{T}_m$ is such that h and h^* do not have any free indices. Moreover, the chain types and $\mathcal{P}^0_h$ can be obtained from $\mathcal{P}^0_f$. Since $m < n$, h is determined by $\mathcal{P}^0_h$ by the induction assumtions, so we are done.

If both f and f^* have no free indices, we may assume that $\emptyset \notin \mathcal{P}_f$, otherwise we replace f by f^*. Then if $\mathcal{P}^0_f$ has independent components, we have $f^* = f_1 \otimes f_2$ for some type functions $f_i \in \mathcal{T}_{n_i}$, $i = 1, 2$ and $n = n_1 + n_2$, such that the components have the form $\mathcal{P}^0_{f_i} \setminus \{\emptyset\}$. Since $n_1, n_2 < n$, we are done. If $\mathcal{P}^0_f$ has no independent components, then $f = f_1 \otimes \cdots \otimes f_k$ for some $f_i \in \mathcal{T}_{m_i}$, $i = 1, \ldots, k$, $m_1 + \cdots + m_k = n$, and we obtin $\mathcal{P}^0_{f_l}$ for all $l = 1, \ldots, k$ from $\mathcal{P}^0_f$ by Proposition 16. Again, the assertion follows by applying induction assumption on each f_l.

□

It also folows that we can obtain a choice of set of chain types appearing in the decomposition of Theorem 4 from $\mathcal{P}^0_f$. If $f = \beta_1 \lhd h \lhd \beta_2$ for chain types β_1 and β_2 and a type function h, then we can construct such a set by adding β_1, β_2 to a collection of chains for h. If h and h^* have no free indices, then if $\mathcal{P}^0_h$ has independent components, then $h^* = h_1 \otimes \cdots \otimes h_k$, so the set of chain types of h is obtained by putting together the complements of all chain types for the functions h_i. Similarly, if $h = g_1 \otimes \cdots \otimes g_k$, we collect all chain types for all the components g_j (without taking complements this time). Each h_i or g_j can be decomposed as above, until we necessarily get to the situation when all the components are chain types themselves.

4 Conclusions

A Some basic definitions

For $m \leq n \in \mathbb{N}$, we will denote the corresponding interval $\{m, m+1, \ldots, n\}$ by $[m, n]$. For $m = 1$, we will simplify to $[n] := [1, n]$. Let $\mathscr{S}_n$ denote the set of all permutations of $[n]$.

A.1 Block permutations

sec:permut

For $n_1, n_2 \in \mathbb{N}$, $n_1 + n_2 = n$, we will denote by $[n] = [n_1] \oplus [n_2]$ the decomposition of $[n]$ as a concatenation of two intervals

$$[n] = [n_1][n_1 + 1, n_1 + n_2].$$

Similarly, for $n = \sum_{j=1}^{k} n_j$, we have the decomposition

$$[n] = \oplus_{j=1}^{k} [n_j] = [m_1, m_1 + n_1][m_2, m_2 + n_2] \ldots [m_k, m_k + n_k],$$

where $m_j := \sum_{l=1}^{j-1} n_j$ (so $m_1 = 0$). Note that the order of $n_1, \ldots, n_k$ in this decomposition is fixed.

We have two kinds of special permutations related to the above decomposition. For $\sigma_j \in \mathscr{S}_{n_j}$, we denote by $\oplus_j \sigma_j \in \mathscr{S}_n$ the permutation that acts as

$$m_j + l \mapsto m_j + \sigma_j(l), \qquad l = 1, \ldots, n_j,\ j = 1, \ldots, k.$$

On the other hand, we have for any $\lambda \in \mathscr{S}_k$ a unique permutation $\rho_\lambda \in \mathscr{S}_n$ such that ρ_λ^{-1} acts as

$$[m_1, m_1 + n_1][m_2, m_2 + n_2]...[m_k, m_k + n_k] \mapsto [m_{\lambda(1)} + n_{\lambda(1)}][m_{\lambda(2)} + n_{\lambda(2)}] \ldots [m_{\lambda(k)} + n_{\lambda(k)}]$$

Note that we have

$$\rho_\lambda \circ (\oplus_j \sigma_j) = (\oplus_j \sigma_{\lambda(j)}) \circ \rho_\lambda.$$

(These permutations come from the operadic structure on the set of all permutations $\mathscr{S}_*$. See [Leinster, Higher operads] for the definition of and operad.)

A.2 Partially ordered sets

app:poset

An overall reference for this section is [stanley].

A partially ordered set, or a *poset*, is a set $\mathcal{P}$ endowed with a reflexive, antisymmetric and transitive relation $\leq$, called the partial order. We will only encounter the situation when $\mathcal{P}$

is finite. A basic example of a poset is the set $\mathcal{P}(X)$ of all subsets of a finite set X, ordered by inclusion. If $X = [n]$, we will denote $\mathcal{P}(X)$ by 2^n.

A subposet in a poset $\mathcal{P}$ is a $\mathcal{Q} \subseteq \mathcal{P}$ endowed with the partial order relation inherited from $\mathcal{P}$. For any subset $\mathcal{R} \subseteq \mathcal{P}$, we define two special subposets in $\mathcal{P}$ as

$$\mathcal{R}^{\downarrow} = \{p \in \mathcal{P},\ p \leq r \text{ for some } r \in \mathcal{R}\},\ \mathcal{R}^{\uparrow} = \{p \in \mathcal{P},\ p \geq r \text{ for some } r \in \mathcal{R}\}.$$

The set of minimal elements in $\mathcal{P}$ will be denoted by $\mathrm{Min}(\mathcal{P})$. For elements $p, q \in \mathcal{P}$, we say that q covers p, in notation $p \ll q$, if $p \leq q$ and for any r such that $p \leq r \leq q$ we have $r = p$ or $r = q$. If p covers a minimal element, we will say that p is a minimal covering element.

A totally ordered subposet $\mathcal{C} \subseteq \mathcal{P}$ is called a chain in $\mathcal{P}$. Such a chain is maximal if it is not contained in any other chain in $\mathcal{P}$. The length of a chain $\mathcal{C}$ is defined as $|\mathcal{C}| - 1$.

We say that a poset $\mathcal{P}$ is graded of rank k if every maximal chain in $\mathcal{P}$ has the same length equal to k. Equivalently, there is a unique rank function $\rho : \mathcal{P} \to \{0, 1, \ldots, k\}$ such that $\rho(p) = 0$ if p is a minimal element of $\mathcal{P}$ and $\rho(q) = \rho(p) + 1$ if $p \ll q$. Basic examples of graded posets are chains, antichains and 2^n.

If $\mathcal{P}$ and $\mathcal{Q}$ are posets with disjoint sets, their direct sum $\mathcal{P} + \mathcal{Q}$ is a poset defined as the disjoint union $\mathcal{P} \cup \mathcal{Q}$, such that the order is preserved in each component and elements in different components are incomparable. Another way to compose $\mathcal{P}$ and $\mathcal{Q}$ is the ordinal sum $\mathcal{P} \star \mathcal{Q}$, where the underlying set is again the union $\mathcal{P} \cup \mathcal{Q}$ and the order in each component is preserved, but for $p \in \mathcal{P}$ and $q \in \mathcal{Q}$ we have $p \leq q$. A third way to compose posets that we will use is the direct product $\mathcal{P} \times \mathcal{Q}$, where the underlying set is the cartesian product $\mathcal{P} \times \mathcal{Q}$, with $(p_1, q_1) \leq (p_2, q_2)$ if $p_1 \leq p_2$ in $\mathcal{P}$ and $q_1 \leq q_2$ in $\mathcal{Q}$. If $\mathcal{P}_1$ and $\mathcal{P}_2$ are graded posets with rank functions ρ_1 and ρ_2, then $\mathcal{P}_1 \times \mathcal{P}_2$ is graded as well, with rank function ρ given as

$$\rho(p_1, p_2) = \rho_1(p_1) + \rho_2(p_2).$$

The Hasse diagram of a finite poset $\mathcal{P}$ is a graph whose vertices are elements of $\mathcal{P}$ and there is an edge between p and q if $p \ll q$, and if $p \lneq r$, then r is drawn above p. Two posets are isomorphic if and only if they have the same Hasse diagrams.

A.3 Binary strings

A binary string of length n is a sequence $s = s_1 \ldots s_n$, where $s_i \in \{0, 1\}$. Such a string can be interpreted as an element $\{0, 1\}^n$, but also as a map $[n] \to \{0, 1\}$, or a subset in $[n] := \{1, \ldots, n\}$. It will be convenient to use all these interpretations, but we will distinguish between them. The strings in $\{0, 1\}^n$ will be denoted by small letters, whereas the corresponding subsets of $[n]$ will be denoted by the corresponding capital letters. More specifically, for $s \in \{0, 1\}^n$ and $T \subseteq [n]$, we denote

$$S := \{i \in [n],\ s_i = 0\}, \qquad t := t_1 \ldots t_n,\ t_j = 0 \iff j \in T. \tag{11}$$ eq:string_s

As usual, the set of all subsets of $[n]$ will be denoted by 2^n. With the inclusion ordering and complementation $S^c := [n] \setminus S$, 2^n is a boolean algebra, with the smallest element $\emptyset$ and largest element $[n]$.

The group $\mathscr{S}_n$ has an obvious action on $\{0,1\}^n$. Indeed, for a string s interpreted as a map $[n] \to 2$, we may define the action of $\sigma \in \mathscr{S}_n$ by precomposition as

$$\sigma(s) := s \circ \sigma^{-1} = s_{\sigma^{-1}(1)} \dots s_{\sigma^{-1}(n)}.$$

Note that in this way we have $\rho(\sigma(s)) = (\rho \circ \sigma)(s)$. For a decomposition $[n] = \oplus_{j=1}^k [n_j]$, we have a corresponding decomposition of any string $s \in \{0,1\}^n$ as a concatenation of strings

$$s = s^1 \dots s^k, \qquad s^j \in \{0,1\}^{n_j}.$$

For permutations $\sigma_j \in \mathscr{S}_{n_j}$ and $\lambda \in \mathscr{S}_k$, we have

$$\rho_\lambda \circ (\oplus_j \sigma_j)(s^1 \dots s^k) = \rho_\lambda(\sigma_1(s^1) \dots \sigma_k(s^k)) = \sigma_{\lambda(1)}(s^{\lambda(1)})\sigma_{\lambda(2)}(s^{\lambda(2)}) \dots \sigma_{\lambda(k)}(s^{\lambda(k)}).$$

ec:boolean

A.4 Boolean functions and the Möbius transform

A function $f : \{0,1\}^n \to \{0,1\}$ is called a boolean function. The set of boolean functions, with pointwise ordering and complementation given by the negation $\bar{f} = 1 - f$, is a boolean algebra that can be identified with 2^{2^n}. We will denote the maximal element (the constant 1 function) by 1_n. Similarly, we denote the constant zero function by 0_n. For boolean functions f, g, the pointwise minima and maxima will be denoted by $f \wedge g$ and $f \vee g$. It is easily seen that we have

$$f \vee g = f + g - fg, \qquad f \wedge g = fg, \tag{12}$$ eq:wedgeve

all the operations are pointwise. We now introduce and important example.

ex:pS *Example* 9. For $S \subseteq [n]$, we define

$$p_S(t) = \Pi_{j \in S}(1 - t_j), \qquad t \in \{0,1\}^n.$$

That is, $p_S(t) = 1$ if and only if $S \subseteq T$. In particular, $p_\emptyset = 1_n$ and $p_{[n]}$ is the characteristic function of the zero string. Clearly, for $S, T \subseteq [n]$ we have $p_{S \cup T} = p_S p_T = p_S \wedge p_T$, in particular, $p_S = \Pi_j p_{\{j\}}$.

By the Möbius transform, all boolean functions can be expressed as combinations of the functions p_S, $S \subseteq [n]$ from the previous example.

thm:mobius **Theorem 6.** *Any $f : \{0,1\}^n \to 2$ can be expressed in the form*

$$f = \sum_{S \subseteq [n]} \hat{f}_S p_S$$

in a unique way. The coefficients $\hat{f}_S \in \mathbb{R}$ obtained as

$$\hat{f}_S = \sum_{\substack{t \in \{0,1\}^n \\ t_j = 1, \forall j \in S^c}} (-1)^{\sum_{j \in S} t_j} f(t).$$

Proof. By the Möbius inversion formula (see [Stanley, Sec. 3.7] for details), functions $f, g : 2^n \to \mathbb{R}$ satisfy

$$f(S) = \sum_{T \subseteq S} g(T), \qquad S \in 2^n$$

if and only if

$$g(S) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} f(T).$$

We now express this in terms of the corresponding strings s and t. It is easily seen that $T \subseteq S$ if and only if $s_j = 0$ for all $j \in T$, equivalently, $t_j = 1$ for all $j \in S^c$. Moreover, in this case we have $|S \setminus T| = \sum_{j \in S} t_j$. This shows that $g(S) = \hat{f}_S$, as defined in the statement. The first equality now gives

$$f(s) = f(S) = \sum_{T \subseteq S} g(T) = \sum_{T : s_i = 0, \forall i \in T} \hat{f}_T = \sum_{T : p_T(s) = 1} \hat{f}_T = \sum_{T \subseteq [n]} \hat{f}_T p_T.$$

For uniqueness, assume that $f = \sum_{T \subseteq [n]} c_T p_T$ for some coefficients $c_T \in \mathbb{R}$. Then

$$f(s) = \sum_{T : p_T(s) = 1} c_T = \sum_{T \subseteq S} c_T.$$

Uniqueness now follows by uniqueness in the Möbius inversion formula.

□

A.5 The boolean algebra $\mathcal{F}_n$

Let us introduce the subset of boolean functions

$$\mathcal{F}_n := \{f : \{0,1\}^n \to 2,\ f(\theta_n) = 1\},$$

where we use θ_n to denote the zero string $00\ldots0$. In other words, $\mathcal{F}_n$ is the interval of all elements greater than $p_{[n]}$ in the boolean algebra 2^{2^n} of all boolean functions. With the pointwise ordering, $\mathcal{F}_n$ is a distributive lattice, with top element 1_n and bottom element $p_n := p_{[n]}$. We also define complementation in $\mathcal{F}_n$ as

$$f^* := 1_n - f + p_n.$$

It can be easily checked that with these structures $\mathcal{F}_n$ is a boolean algebra, though it is not a subalgebra of 2^{2^n}.

Note that $p_S \in \mathcal{F}_n$ for any $S \subseteq [n]$, so in particular for $[k]$, with $k \leq n$. If $k < n$, we denote these functions as before by $p_{[k]}$, using the notation p_n only for the distinguished top element.

We now introduce some more operations in $\mathcal{F}_n$. For $f \in \mathcal{F}_n$ and any permutation $\sigma \in \mathscr{S}_n$, we clearly have $f \circ \sigma \in \mathcal{F}_n$. Further, let $f \in \mathcal{F}_{n_1}$ and $g \in \mathcal{F}_{n_2}$. With the decomposition $[n_1 + n_2] = [n_1] \oplus [n_2]$ and the corresponding concatenation of strings $s = s^1 s^2$, we define the function $f \otimes g \in \mathcal{F}_{n_1+n_2}$ as

$$(f \otimes g)(s^1 s^2) = f(s^1) g(s^2), \qquad s^1 \in \{0,1\}^{n_1},\ s^2 \in \{0,1\}^{n_2}.$$

Let $\lambda \in \mathscr{S}_2$ be the transposition, then we have for any $f \in \mathcal{F}_{n_1}$ and $g \in \mathcal{F}_{n_2}$

$$(g \otimes f) = (f \otimes g) \circ \rho_\lambda,$$

where ρ_λ is the block permutation defined in Section A.1.

If $f, g \in \mathcal{F}_n$ are such that $g = f \circ \sigma$ for some $\sigma \in \mathscr{S}_n$, we write $f \approx g$. It is easily observed that if $f_1 \approx g_1$ and $f_2 \approx g_2$, then $f_1 \otimes f_2 \approx g_1 \otimes g_2$ and if $f \approx g$ then also $f^* \approx g^*$.

We now show some further important properties of these operations.

lemma:fproduct **Lemma 13.** *For $f \in \mathcal{F}_{n_1}$ and $g, h \in \mathcal{F}_{n_2}$, we have*

(i) $f \otimes g \leq (f^* \otimes g^*)^*$, with equality if and only if either $f = 1_{n_1}$ and $g = 1_{n_2}$, or $f = p_{n_1}$ and $g = p_{n_2}$.

(ii) $f \otimes (g \vee h) = (f \otimes g) \vee (f \otimes h)$, $f \otimes (g \wedge h) = (f \otimes g) \wedge (f \otimes h)$.

Proof. The inequality in (i) is easily checked, since $(f \otimes g)(s^1 s^2)$ can be 1 only if $f(s^1) = g(s^2) = 1$. If both s^1 and s^2 are the zero strings, then $s^1 s^2 = \theta_{n_1+n_2}$ and both sides are equal to 1. Otherwise, the condition $f(s^1) = g(s^2) = 1$ implies that $(f^* \otimes g^*)(s^1 s^2) = 0$, so that the right hand side must be 1. If f and g are both constant 1, then

$$(1_{n_1} \otimes 1_{n_2})^* = 1^*_{n_1+n_2} = p_{n_1+n_2} = p_{n_1} \otimes p_{n_2} = 1^*_{n_1} \otimes 1^*_{n_2},$$

in the case when both f and g are the minimal elements equality follows by duality. Finally, asume the equality holds and that $f \neq 1_{n_1}$, so that there is some s^1 such that $f(s^1) = 0$. But then $s^1 \neq \theta_{n_1}$, so that $f^*(s_1) = 1$ and for any s^2,

$$0 = (f \otimes g)(s^1 s^2) = (f^* \otimes g^*)^*(s^1 s^2) = 1 - f^*(s^1) g^*(s^2) + p_{n_1+n_2}(s^1 s^2) = 1 - g^*(s^2),$$

which implies that $g(s^2) = 0$ for all $s^2 \neq \theta_{n_2}$, that is, $g = p_{n_2}$. By the same argument, $f = p_{n_1}$ if $g \neq 1_{n_2}$, which implies that either $f = 1_{n_1}$ and $g = 1_{n_2}$, or $f = p_{n_1}$ and $g = p_{n_2}$.

The statement (ii) is easily proved from (12).

□

Consider the decomposition $[n] = [n_1] \oplus [n_2]$ and let $S \subseteq [n_1]$, $T \subseteq [n_2]$. We then denote by $S \oplus T$ the disjoint union

$$S \oplus T := S \cup (n_1 + T) = S \cup \{n_1 + j,\ j \in T\}. \tag{13}$$ eq:disu

We summarize some easy properties of the basic functions p_S, $S \subseteq [n]$.

lemma:PSPT **Lemma 14.** *(i) For $S, T \subseteq [n]$, we have $S \subseteq T \iff p_T \leq p_S \iff p_S p_T = p_S$.*

(ii) For $S \subseteq [n]$, $\sigma \in \mathscr{S}_n$, $p_S \circ \sigma = p_{\sigma^{-1}(S)}$.

(iii) For $S \subseteq [n_1]$ and $T \subseteq [n_2]$, $p_S \otimes p_T = p_{S \oplus T}$.

Let $f \in \mathcal{F}_n$ and let $\hat{f}$ be the Möbius transform. Note that since f has values in $\{0, 1\}$, we have by the proof of Theorem 6

$$\forall S \in 2^n, \quad \sum_{T \subseteq S} \hat{f}_T = f(s) \in \{0, 1\}; \qquad \sum_{T \in 2^n} \hat{f}_T = f(\theta_n) = 1.$$

Proposition 17. *(i) For $f \in \mathcal{F}_n$ and $\sigma \in \mathscr{S}_n$, $\widehat{(f \circ \sigma)}_S = \hat{f}_{\sigma(S)}$, $\qquad S \subseteq [n]$.*

(ii) For $f \in \mathcal{F}_n$, $\widehat{f^*}_S = \begin{cases} 1 - \hat{f}_S & S = \emptyset \text{ or } S = [n], \\ -\hat{f}_S & \text{otherwise.} \end{cases}$

(iii) For $f \in \mathcal{F}_{n_1}$, $g \in \mathcal{F}_{n_2}$, we have $\widehat{(f \otimes g)}_{S \oplus T} = \hat{f}_S \hat{g}_T$, $S \subseteq [n_1]$, $T \subseteq [n_2]$.

Proof. All statements follow easily from Lemma 14 and the uniqueness part in Theorem 6. □

B Affine subspaces

Let V be a finite dimensional real vector space. A subset $A \subseteq V$ is an *affine subspace* in V if for any choice of $a_1, \dots, a_k \in A$ and $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ such that $\sum_i \alpha_i = 1$, we have $\sum_i \alpha_i a_i \in A$. It is clear that $A = \emptyset$ is trivially an affine subspace. Moreover, any linear subspace in V is an affine subspace, and an affine subspace A is linear if and only if $0 \in A$. If $A \neq \emptyset$ and also $0 \notin A$, we say that A is *proper*.

A proper affine subspace $A \subseteq V$ can be determined in two ways. Let

$$\mathrm{Lin}(A) := \{a_1 - a_2,\ a_1, a_2 \in A\}.$$

It is easily verified that $\mathrm{Lin}(A)$ is a linear subspace, moreover, for any $a \in A$, we have

$$\mathrm{Lin}(A) = \{a_1 - a,\ a_1 \in A\}, \qquad A = a + \mathrm{Lin}(A). \tag{14}$$

We put $\dim(A) := \dim(\mathrm{Lin}(A))$, the dimension of A.

Let V^* be the vector space dual of V and let $\langle \cdot, \cdot \rangle$ be the duality. For a subset $C \subseteq V$, put

$$C^* := \{v^* \in V^*,\ \langle v^*, a \rangle = 1,\ \forall a \in A\}.$$

Let $\tilde{a} \in A^*$ be any element and let $\mathrm{Span}(A)$ be the linear span of A in V. We then have

$$A = \mathrm{Span}(A) \cap \{\tilde{a}\}^*, \tag{15}$$

independently of $\tilde{a}$. The relation between the two expressions for A, given by (14) and (15) is obtained as

$$\mathrm{Span}(A) = \mathrm{Lin}(A) + \mathbb{R}\{a\}, \qquad \mathrm{Lin}(A) = \mathrm{Span}(A) \cap \{\tilde{a}\}^{\perp}, \tag{16}$$

independently of $a \in A$ or $\tilde{a} \in A^*$. Here $+$ denotes the direct sum of the vector spaces and $C^{\perp}$ denotes the annihilator of a set C. The following lemma is easily proven.

Lemma 15. *Let $C \subseteq V$ be any subset. Then C^* is an affine subspace in V^* and we have*

$$0 \in C^* \iff C = \emptyset, \qquad C^* = \emptyset \iff 0 \in \mathrm{Aff}(C).$$

Assume $C \neq \emptyset$ and $0 \notin \mathrm{Aff}(C)$. Then *(i)* C^* *is proper and we have* $\mathrm{Lin}(C^*) = C^\perp = \mathrm{Span}(C)^\perp$*,*

(ii) $\mathrm{Aff}(C) = C^{**}$ *and for any* $c_0 \in C$*, we have*

$$\mathrm{Lin}(C) := \mathrm{Span}\{c_1 - c_2,\ c_1, c_2 \in C\} = \mathrm{Span}\{c - c_0,\ c \in C\} = \mathrm{Lin}(C^{**}).$$

coro:dual **Corollary 4.** *Let* $A \subseteq V$ *be a proper affine subspace. Then*

(i) A^* *is a proper affine subspace in* V^* *and* $A^{**} = A$*.*

(ii) $\mathrm{Lin}(A^*) = \mathrm{Span}(A)^\perp$, $\mathrm{Span}(A^*) = \mathrm{Lin}(A)^\perp$*.*

(iii) $\dim(A^*) = \dim(V) - \dim(A) - 1$*.*

The proper affine subspace A^* in the above Corollary will be called the *affine dual* of A. Note that the dual depends on the choice of the ambient vector space V.

C Labelled posets and type functions

app:pf0

We start by showing some basic properties of $\mathcal{P}_f^0$, $f \in \mathcal{T}_n$. By definition, $\mathcal{P}_f^0$ is a poset whose elements are labelled by subsets in $[n]$. The elements in $\mathcal{P}_f^0$ will be denoted by capital letters $S, T, R, \dots$, but they will not be viewed as subsets of $[n]$. The label set of T will be denoted by L_T. The order relation in $\mathcal{P}_f^0$ will be denoted as $\leq$.

The labelling in $\mathcal{P}_f^0$ has some immediate properties: if $S, T \in \mathcal{P}_f^0$, then $S \leq T$ implies that $L_S \cap L_T = \emptyset$ and if $S, T \in \mathrm{Min}(\mathcal{P}_f^0)$, then $L_S \subseteq L_T$ implies $S = T$. Also, the smallest element, if present, is the only element in $\mathcal{P}_f^0$ that may have an empty label set.

0_mincover **Lemma 16.** *If* $\mathcal{P}_f^0$ *has more than one element, then any* $S \in \mathrm{Min}(\mathcal{P}_f^0)$ *is covered by at least one element.*

Proof. We will proceed by induction on n. The assertion is clearly true for chains, so for $n \leq 3$. Assume it holds for all $m < n$ and let $f \in \mathcal{T}_n$. Assume that $S \in \mathrm{Min}(\mathcal{P}_f)$ is not covered by any element. If $S = \emptyset$, then $\mathcal{P}_f^0 = \{\emptyset\}$. Otherwise, $\mathcal{P}_f^0$ does not contain $\emptyset$ and has no largest element, so that $\emptyset \in \mathcal{P}_{f^*}^0$ and $\mathcal{P}_f^0 = \mathcal{P}_{f^*}^0 \setminus \{\emptyset\}$.

If $f \approx f_1 \otimes f_2$, then $S = (S_1, S_2)$ for $S_i \in \mathrm{Min}(\mathcal{P}_{f_i}^0)$ and it is clear that both S_1 and S_2 cannot be covered by any element. By the induction assumption, $\mathcal{P}_{f_1}^0 = \{S_1\}$ and $\mathcal{P}_{f_2}^0 = \{S_2\}$, so that $\mathcal{P}_f^0 = \{(S_1, S_2)\}$. Assume that $f^* \approx f_1 \otimes f_2$, then $\emptyset \in \mathcal{P}_{f_i}^0$ for $i = 1, 2$. Since S covers $\emptyset$ in $\mathcal{P}_{f^*}^0$ and is not covered by any $T \in \mathcal{P}_{f^*}^0$, it follows from Corollary coro:Pf 3 that we have the same situation in one of $\mathcal{P}_{f_i}^0$, say in $\mathcal{P}_{f_1}^0$, there is some S_1 such that $\emptyset \ll S_1$ but no $T_1 \in \mathcal{P}_{f_1}^0$ covers S_1. Then $\mathcal{P}_{f_1}^0$ has no largest element, so that $\mathcal{P}_{f_1^*}^0 = \mathcal{P}_{f_1}^0 \setminus \{\emptyset\}$ has no largest element as well. By the induction assumption, $\mathcal{P}_{f_1^*}^0 = \{S_1\}$, which is not possible.

□

0_covermin **Lemma 17.** *Any element* $T \in \mathcal{P}_f^0$ *can cover at most one minimal element.*

Proof. We will proceed by induction on n. Since the assertion is trivial for chains, it holds for $n \leq 3$. Assume it is true for $m < n$ and let $f \in \mathcal{T}_n$. Let T be an element that covers $T_1, \ldots, T_k \in \mathrm{Min}(\mathcal{P}^0_f)$, $k > 1$, so that we must have $\emptyset \notin \mathcal{P}_f$. Then T is not the largest element, otherwise by Lemma 12, T would be the largest element in $\mathcal{P}_f$ in which case the rank of f would be 1, which is not possible. It follows that T and $T_1, \ldots, T_k$ are all contained in $\mathcal{P}^0_{f^*}$.

Assume that $f \approx f_1 \otimes f_2$, then any T_i is of the form $T_i = (T_1^i, T_2^i)$, with $T^i_j \in \mathrm{Min}(\mathcal{P}^0_{f_j})$. If $i \neq i'$, we may assume that, say, $T_1^i \neq T_1^{i'}$. Since T covers both T_i and $T_{i'}$, we must have $T_2^i = T_2^{i'}$ and $T = (S, T_2^i)$ for some $S \in \mathcal{P}^0_{f_1}$ such that $T_1^i, T_1^{i'} \ll S$. By the induction assumption, this is not possible. If $f^* \approx f_1 \otimes f_2$, then using Corollary 3, we may assume that there are $S, S_1, \ldots, S_k \in \mathcal{P}^0_{f_1}$ such that $\emptyset \ll S_i \ll S$. If S is the largest element in $\mathcal{P}^0_{f_1}$, then by Proposition 13 and the remarks below it, $f_1 \approx h \lhd \beta$ for a chain type β and h with no largest element, and $\mathcal{P}^0_h = \{S_1, \ldots, S_k\}^{\downarrow}$. By the properties of S_i, it follows that $\mathrm{Min}(\mathcal{P}^0_{h^*}) = \mathrm{Max}(\mathcal{P}^0_{h^*}) = \{S_1, \ldots, S_k\}$, which is impossible for $k > 1$ by Lemma 16. Since S cannot be the largest element, $S_1, \ldots, S_k$ are minimal elements in $\mathcal{P}^0_{f_1^*}$ covered by $S \in \mathcal{P}^0_{f_1^*}$, which is not possible by the induction assumption.

□

We now proceed towards the proof of Proposition 16. Let $f \in \mathcal{T}_n$ be such that f and f^* have no free indices and $\emptyset \notin \mathcal{P}_f$. Assume that $\mathcal{P}^0_f$ has no independent component, so that $f \approx f_1 \otimes \cdots \otimes f_k$ for some type functions $f_1, \ldots, f_k$ and $[n] = [n_1] \oplus \cdots \oplus [n_k]$. Assume that this is a finest decomposition of this form, so that no f_l is a product. These assumptions will be kept throughout this section. Note that for any l, f_l cannot have free indices, since these would be also free indices of f.

By Corollary 3 (i) and (iii), any element in Φ^0_f can be written as $T = (T_1, \ldots, T_k)$, with all $T_l \in \mathrm{Min}(\mathcal{P}^0_{f_l})$ except possibly one index l_0. Application of a permutation is manifested only on the labels sets.

Lemma 18. *There is a decomposition $C_1, \ldots, C_k$ of $[n]$ and a bijection $\varphi_l : [n_l] \to C_l$ such that for any $T = (T_1, \ldots, T_k) \in \mathcal{P}^0_f$*

$$L_{(T_1,\ldots,T_k)} = \begin{cases} \varphi_l(L_{T_l}) & \textit{if } \exists\, l, T_l \notin \mathrm{Min}(\mathcal{P}_{f_l}) \\ \cup_l \varphi_l(L_{T_l}), & \textit{otherwise.} \end{cases}$$

Proof. Let $\sigma \in \mathscr{S}_n$ be such that $f \circ \sigma = f_1 \otimes \cdots \otimes f_k$. By Corollary 3,

$$L_{(T_1,\ldots,T_k),f} = \sigma^{-1}(L_{(T_1,\ldots,T_k),\otimes_l f_l}) = \begin{cases} \sigma^{-1}(m_l + L_{T_l}) & \text{if } \exists\, l, T_l \notin \mathrm{Min}(\mathcal{P}_{f_l}) \\ \cup_l \sigma^{-1}(m_l + L_{T_l}), & \text{otherwise,} \end{cases}$$

where $m_l = \sum_{i=1}^{l-1} n_i$. Put $C_l = \sigma^{-1}(m_l + [n_l])$ and $\varphi_l(i) = \sigma^{-1}(m_l + i)$.

□

To ease the subsequent notations, we will replace the labels of $T_l \in \mathcal{P}^0_{f_l}$ with $\varphi_l(L_{T_l}) \subseteq C_l$. Any $i \in C_l$ is thus connected to f_l. We will refer to the inclusion of an index i in C_l as coloring i by a color $l \in \{1, \ldots, k\}$.

:color_min **Lemma 19.** *Assume that the coloring is known for each index in the label sets of minimal or minimal covering elements. Then we can reconstruct all $\mathcal{P}^0_{f_l}$ from $\mathcal{P}^0_f$.*

Proof. Let $U \in \mathrm{Min}(\mathcal{P}^0_f)$, $U = (Z_1, \ldots, Z_k)$ with $Z_l \in \mathrm{Min}(\mathcal{P}_{f_l})$. By the assumption, we know the coloring of any $i \in L_U = \cup_l L_{Z_l}$. Since L_{Z_l} is a label set of f_l if and only if $L_{Z_l} \subseteq C_l$, we obtain all label sets of minimal elements in $\mathcal{P}^0_{f_l}$. Similarly, all labels of minimal covering elements in f_l are contained in C_l.

For any $U = (Z_1, \ldots, Z_k) \in \mathrm{Min}(\mathcal{P}_f)$, let $\tilde{L}^l_U := L_U \cap (\cup_{l' \neq l} C_{l'}) = \{i \in L_U, i \notin C_l\}$. Then $\tilde{L}^l_U = \cup_{l' \neq l} L_{Z_{l'}}$. It follows by the properties of the minimal label sets that $\tilde{L}^l_U \subseteq \tilde{L}^l_{U'}$ implies $\tilde{L}^l_U = \tilde{L}^l_{U'}$. Consequently, $\tilde{L}^l_U \subseteq L_{U'}$ if and only if $U' = (Z_1, \ldots, Z'_l, \ldots, Z_k)$ for some $Z'_l \in \mathrm{Min}(\mathcal{P}^0_{f_l})$.

Fix a minimal element U and consider the subposet in $\mathcal{P}^0_f$, given as

$$\mathcal{P}_l := \{U' \in \mathrm{Min}(\mathcal{P}^0_f),\ \tilde{L}^l_U \subseteq L_{U'}\}^{\uparrow}.$$

From Corollary coro:Pf 3 (iii), we see that after removing the minimal elements of $\mathcal{P}_l$, the poset decomposes into independent components, one of which corresponds to $\mathcal{P}^0_{f_l}$ with removed minimal elements. This component can be recognized by the labels of minimal covering elements (which are now minimal elements in the component), colored by l. To this component, we add back the minimal elements in $\mathcal{P}_l$, with the order relations as in $\mathcal{P}^0_f$ but with labels $L_{U'} \setminus \tilde{L}^l_U$.

□

We now show how to obtain the coloring of labels of minimal and minimal covering elements. For any type function g and any label set L for g, we introduce the following sets:

$$\begin{aligned}
\mathcal{U}^g &:= \{L_U,\ U \in \mathrm{Min}(\mathcal{P}^0_g)\} \\
\mathcal{V}^g_L &:= \cap\{L_U, U \ll_g L,\ U \in \mathrm{Min}(\mathcal{P}^0_g)\} \\
\mathcal{W}^g_L &:= \mathcal{U}^g \setminus (\cup\{L_U, U \ll_g L,\ U \in \mathrm{Min}(\mathcal{P}^0_g)\}).
\end{aligned}$$

where we write $U \ll_g L$ if $U \ll T$ in $\mathcal{P}^0_g$ with $L_T = L$.

Now let $L_1, \ldots, L_M$ be all the different label sets for minimal covering elements in $\mathcal{P}^0_f$. Put $\mathcal{V}_i := \mathcal{W}^f_{L_i}$ and $\mathcal{W}_i := \mathcal{W}^f_{L_i}$. Then there is some l such that $\mathcal{V}_i, \mathcal{W}_i, L_i \subseteq C_l$. Indeed, let T be a minimal covering element such that $L_T = L_i$. Then $T = (T_1, \ldots, T_k)$, where all $T_{l'}$ are minimal in $\mathcal{P}_{f_{l'}}$ except a single index l, for which T_l is minimal covering in $\mathcal{P}^0_{f_l}$. We then have $L_i = L_T = L_{T_l} \subseteq C_l$. If $U = (V_1, \ldots, V_k) \ll T$, then necessarily $V_{l'} = T_{l'}$ for $l' \neq l$ and $V_l \ll T_l$, so that $V_l \ll_{f_l} L_i$. We obtain

$$\mathcal{V}_i = \cap\{\cup_{l'} L_{S_{l'}},\ S_{l'} \in \mathrm{Min}(\mathcal{P}_{f_{l'}}),\ S_l \ll_{f_l} L_i\} = \cup_{l \neq l'} I^F_{f_{l'}} \cup \mathcal{V}^{f_l}_{L_i} = \mathcal{V}^{f_l}_{L_i}.$$

Similarly, we obtain that $\mathcal{W}_i = \mathcal{W}^{f_l}_{L_i}$. It is clear from this that $\mathcal{V}_i, \mathcal{W}_i, L_i$ must all be colored by the same color.

For all i, let us denote $C'_i := \mathcal{V}_i \cup \mathcal{W}_i \cup L_i$ and define $i \sim j$ if there are some T, S that have a common upper bound in $\mathcal{P}^0_f$ and $L_T = L_i$, $L_S = L_j$, or if $C'_i \cap C'_j \neq \emptyset$. Take the transitive closure of this relation (also denoted by $\sim$). We next prove several claims:

Claim 1. For any $p \in \mathcal{U} := \mathcal{U}^f$, there is some $i \in [M]$ such that $p \in \mathcal{V}_i$, so that all labels of minimal elements are colored. For this, let $U = (V_1, \ldots, V_l)$ be a minimal element such that $p \in L_U$, then $p \in L_{V_l}$ for exactly one l. By Lemma lemma:pf0_mincover 16, V_l is covered by at least one $S \in \mathcal{P}^0_{f_l}$, so that $U \ll L_T$ for $T = (T_1, \ldots, T_k)$, with $T_{l'} = V_{l'}$ for $l' \neq l$ and $T_l = S$. Hence $L_S = L_i$ for some $i \in [M]$ and we have seen that in this case, $\mathcal{V}_i = \mathcal{V}^{f_l}_{L_i}$. Hence it is enough to show that $p \in \mathcal{V}^{f_l}_L$ for some label set L of a minimal covering element in $\mathcal{P}^0_{f_l}$.

By the assumptions f_l cannot have free inputs, and since $\emptyset \neq V_l \in \mathrm{Min}(\mathcal{P}^0_{f_l})$, we see that $\mathcal{P}^0_{f_l}$ must have more than one minimal element. Further, $\emptyset \in \mathcal{P}_{f^*_l}$, so that f^*_l has no free inputs as well. By Proposition prop:pf0_largest 13 and the remarks below it, $f_l \approx h \lhd \beta$ where h is a type function such that h and h^* have no free indices and β is the chain type for a chain on top of $\mathcal{P}^0_{f_l}$. Using Lemma lemma:pf0_covermin 17, we see that this chain cannot contain any minimal covering element. It follows that V_l is a minimal element in $\mathcal{P}^0_h$ with the same minimal covering label sets as in $\mathcal{P}^0_{f_l}$. It follows that $\mathcal{V}^{f_l}_L = \mathcal{V}^h_L$ for any minimal covering label set L.

If h is again a product, we continue this process, until we get to a situation such that $\mathcal{P}^0_h$ has independent components. In this case, all minimal elements U with $p \in L_U$ and all label sets that cover them are connected to one component. If this component has a least element U, then $p \in L_U = \mathcal{V}^h_L$ for any L such that $U \ll L$, and we are done. If not, let g be a type function such that this component is equal to $\mathcal{P}^0_g \setminus \{\emptyset\}$. Arguing as above about the top elements, we obtain that all minimal and minimal covering elements in the component are contained in $\mathcal{P}^0_{g^*}$. Therefore, we have $\mathcal{V}^h_L = \mathcal{V}^{g^*}_L$. If p is a free input of g^*, then $p \in \mathcal{V}^{g^*}_L$ for any minimal covering label L of g^*. Otherwise, let q be a type function with no free indices such that $g^* = p_r \lhd q$, then $\mathcal{P}^0_{g^*}$ and $\mathcal{P}^0_q$ are the same as posets, with the same label sets except that the free indices of g^* are added to the labels of minimal elements in $\mathcal{P}_q$. We may therefore continue the same process with $\mathcal{P}^0_q$. Since the number of minimal elements is decreasing, we get to a situation when all components have a least element. Hence $p \in \mathcal{V}_i$ for some i.

Claim 2. If $\emptyset \notin \mathcal{P}_{f_l}$ and $L_i, L_j \subseteq C_l$, then $i \sim j$. Since $\mathcal{V}_i$ and $\mathcal{W}_i$ have the same color as L_i, it then follows using Claim 1 that all indices in label sets of minimal and minimal covering elements in C_l will have the same color. If $\mathcal{P}^0_{f_l}$ has a largest element, then its label is an upper bound of both L_i and L_j, so that $i \sim j$. Assmue that $\mathcal{P}_{f_l}$ has no largest element, then f_l has no free indices. Since the decomposition $f = f_1 \otimes \cdots \otimes f_k$ is the finest decomposition of f as a product, f_l cannot be a product. By Proposition prop:nofree_components 15, we obtain that $\mathcal{P}^0_{f_l}$ must have independent components $\mathcal{P}^0_{f_l} = \mathcal{P}_1 +^L \mathcal{P}_2$. Assume that $L_i \in \mathcal{P}_1$, then any minimal element it covers must be in $\mathcal{P}_1$ too. Hence $\{V \ll_{f_l} L_i,\ V \in \mathrm{Min}(\mathcal{P}_{f_l})\} \subseteq \mathcal{P}_1$, so that $\mathcal{W}_i$ contains all indices of minimal elements in $\mathcal{P}_2$. In particular, $\mathcal{V}_{i'} \subseteq \mathcal{W}_i$ for all $L_{i'}$ in $\mathcal{P}_2$, so that $i \sim i'$ for all such i'. It is easily concluded that this proves the claim.

Claim 3. If $i \sim j$, then $L_i, L_j \in C_l$ for some l. Similarly as before, this implies that if some indices in $\mathcal{U} \cup (\cup_i L_i)$ have the same color, then they belong to the same function f_l. We have seen that if $L_i \in C_l$ and $L_j \in C_{l'}$ with $l \neq l'$, then the sets C'_i and C'_j are contained in separated sets of labels, so we cannot have $C'_i \cap C'_j \neq \emptyset$. It is also clear that if L_i and $L_{i'}$ have a common upper bound, then we must be connected to the same function f_l, so that $L_{i'} \in C_l$. Hence $i \not\sim j$.

Claim 4. We have $L_i \subseteq C_l$ with $\emptyset \in \mathcal{P}^0_{f_l}$ if and only if $\mathcal{V}_i = \mathcal{W}_i = \emptyset$. If U is any minimal element in $\mathcal{P}^0_f$, then two such label sets L_i and L_j are connected with different functions if and only if they appear in different independent components of the labelled poset $U^{\uparrow} \setminus \{U\}$. Indeed, the first statement is clear from the definition of $\mathcal{V}_i$ and $\mathcal{W}_i$, and Claim 1. For the second statement, note that $U^{\uparrow} \setminus \{U\}$ contains $\mathcal{P}^0_{f_l} \setminus \{\emptyset\}$ for any $\mathcal{P}_{f_l}$ containing $\emptyset$ as one of its independent components. If $\mathcal{P}^0_{f_l}$ has a largest element, then $\mathcal{P}^0_{f_l} \setminus \{\emptyset\}$ cannot have independent components. Otherwise we have $\mathcal{P}^0_{f_l} \setminus \{\emptyset\} = \mathcal{P}^0_{f_l^*}$ and since f_l is not a product, $\mathcal{P}^0_{f_l^*}$ cannot have any independent components as well, by Proposition prop:nofree_components 15.

Proof of Proposition prop:nofree_product *16.* Let C'_i, $i = 1, \dots, M$ be as described above. Assume that the equivalence relation $\sim$ has k' equivalence classes, then pick some of the colors $1, \dots, k'$ for each equivalence class $[i]_\sim$ and use it for all indices in $\cup_{j \in [i]_\sim} C'_j$. Take all L_i such that $\mathcal{V}_i = \mathcal{W}_i = \emptyset$ and use the procedure described in Claim 4 to merge some of the equivalence classes if necessary. Claims 1-4 show that for all label indices for minimal and minimal covering elements, we obtain a coloring that matches the decomposition of f as a tensor product $f \approx f_1 \otimes \cdots \otimes f_k$. Using Lemma lemma:color_min 19, we get all the labelled posets $\mathcal{P}^0_{f_l}$, with labels transformed by the bijections φ_l. Applying any bijection $C_l \to [n_j]$ on the label sets, we obtain $\mathcal{P}^0_{f_l}$ up to apermutation of the labels.

□