Li Gao, Haojian Li, Iman Marvian, and Cambyse Rouzé: Sufficient statistic and recoverability via quantum Fisher information

Referee report

The aim of the paper is a characterization of recoverability of quantum channels through the quantum Fisher information. More precisely, a quantum Fisher information is defined as a Riemannian metric on the manifold of density matrices of full rank, which is nonincreasing under quantum channels. For a large class of such metrics, called regular in the paper, it is shown that preservation of the metric is equivalent to sufficiency of the channels for the given parametrized family of states (that is, the channel is reversible on this family). The most commonly known quantum Fisher information, the SLD QFI, does not have this property. The approximate case is also treated and recoverability bounds are given in terms of the QFI. Similar questions are studied for the closely related χ^2-divergences. Some applications to quantum thermodynamics and the resource theory of asymmetry are discussed.

Overall evaluation

There is a number of works dealing with recoverability of channels and its characterization by decrease of the relative entropy or other divergences and related quantities. The question whether a similar characterization can be achieved by the quantum Fisher information is a very natural one. The paper is well written and easy to read, the proofs are clearly explained and correct. I would have no doubts to recommend its publication to a high quality journal. But I am not entirely convinced that the present results are significant and strong enough to meet the standards of Communications in Mathematical Physics. I give a brief description of the results below.

A more detailed description

The equality case, that is the exact recoverability or sufficiency of a channel, has been treated before in the cited paper [Jen12], at least for the χ^2-divergences or linearly parametrized families of states. The present paper generalizes this result to arbitrary (smoothly) parametrized families of full rank states (Theorem 5.2). The fact that the SLD QFI does not characterize sufficiency of channels was also already suggested in [Jen12].

The approximate case is more tricky, and seems to be treated for the first time here. In Sec. 4, the difference $\gamma_\rho(A) - \gamma_{\Phi(\rho)}(\Phi(A))$ for a strongly regular monotone metric γ and a channel Φ gives an upper bound on $\|A - \mathcal{R}^t_{\rho,\Phi}(\Phi(A))\|_1$, here $\mathcal{R}^t_{\rho,\Phi}$ is the rotated Petz map that serves as a universal recovery map in the exact case. This bound is obtained from the integral representation of the operator (anti)monotone function and follows an idea somewhat similar to the one used for quasi-entropies in the paper arXiv:1710.08080 by E. Carlen and A. Vershynina (this paper was not cited in the manuscript, but I think just by mistake). Usefulness of the general bound obtained in Lemma 4.4 is not immediately clear, but from there some more practically looking bounds are obtained in the special cases of the BKM (Cor. 4.6) and the x^α-metrics (Cor. 4.7). These bounds are far from universal, in fact, the only universal bound is found in Sec. 4.2 for the $x^{\frac{1}{2}}$-metric, which is a nice result. It is obtained in a quite different way, and it is not compared to the one coming from Corollary 4.7 (which is probably quite far from that of Theorem 4.9). Optimality questions for the obtained bounds are not discussed.

In Sec. 5, a recoverability result for a smooth 1-dimensional family of states is obtained by bounding the decrease in the relative entropy by that of the BKM metric, essentially using the fact that the BKM metric can be defined by differentiation of the relative entropy. In Secs. 1.6 and 6, the equality results are applied to show existence of a time-translation invariant or a covariant recovery map by equality in the Fisher information matrix.

Some minor specific comments

1. p. 8, lines 55-60: $L(\theta)$ or L_θ?
2. p. 9, first sentence ("Note that..."): check the conditions on the supports, for example, if ρ_θ is invertible, then clearly $\mathrm{supp}(|\dot\rho_\theta|) \subseteq \mathrm{supp}(\rho_\theta)$ but I_{RLD} is finite.
3. p. 9, line 22-23: $\mathcal{L}_{\theta_o}$ should be $\mathcal{L}_o$ (twice)
4. p. 10, line 41-42: "One problem remains open..." from Remark 3.7 it seems to follow that the answer is no(?).
5. p. 13, Lemma 4.1: $\Phi(\rho)$ might be singular, so A should be restricted
6. p. 16, line 44: "Right operator mean metric" perhaps RLD?
7. p. 23, Remark 5.4: reference to Eq.(5.2) maybe there should be (5.1)?
8. References [CV20a] and [CV20b] are the same. Maybe [CV20b] should be to another paper by Carlen and Vershynina (mentioned above)?