Report on

Rényi relative entropies and noncommutative L_p-spaces II

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A variant of the α-Rényi divergence, called the α-sandwiched Rényi divergence, has recently played a vital role in quantum information theory. In two papers in *Ann. Henri Pincaré* 19 (2018), Berta–Scholz–Tomamichel and the author of the present paper under review extended the notion of the sandwiched Rényi divergence to the general von Neumann algebra setting, independently in different approaches. The present paper is a continuation of those two papers.

In the previous papers, Berta–Scholz–Tomamichel used Araki–Masuda's L_p-norms to define the relevant quantity $\tilde{Q}_\alpha(\psi\|\varphi)$ and the sandwiched Rényi divergence

$$\tilde{D}_\alpha(\psi\|\varphi) = \frac{1}{\alpha-1}\log\tilde{Q}_\alpha(\psi\|\varphi)$$

for normal states φ, ψ of a von Neumann algebra and for $\alpha \in [1/2, \infty) \setminus \{1\}$. On the other hand, the present author used Kosaki's *symmetric* L_p-spaces to define $\tilde{Q}_\alpha(\psi\|\varphi)$ and $\tilde{D}_\alpha(\psi\|\varphi)$ for $\alpha \in (1, \infty)$, and proved that both definitions are equivalent for $\alpha \in (1, \infty)$. In the first part of the present paper, she proposes yet another approach by using Kosaki's *right* L_p-spaces to define $\tilde{D}_\alpha$ for all $\alpha \in [1/2, \infty) \setminus \{1\}$, and proves that the new definition of $\tilde{D}_\alpha$ is equivalent to that of Berta–Scholz–Tomamichel for all $\alpha \in [1/2, 1\infty) \setminus \{1\}$ (hence also equivalent to her previous definition for $\alpha \in (1, \infty)$.

The DPI (Data processing inequality) is, among others, the most important property of quantum divergences, which says that the quantum divergence is monotone under quantum channels (i.e., completely positive and trace preserving maps) or sometimes under general positive and trace preserving maps. In the previous papers, Berta–Scholz–Tomamichel showed the DPI for $\tilde{D}_\alpha$ under quantum channels for all $\alpha \in [1/2, \infty) \setminus \{1\}$, and the present author showed the DPI under general positive trace preserving maps for $\alpha \in (1, \infty)$. In this paper, she proves that $\tilde{D}_\alpha$ satisfies the DPI for general positive trace preserving maps also for $\alpha \in [1/2, 1)$, by cleverly using the variational expression of $\tilde{Q}_\alpha(\psi\|\varphi)$ for $\alpha \in [1/2, 1)$ recently given by Hiai. This DPI for $\tilde{D}_\alpha$ for $\alpha \in [1/2, 1)$ is new even in the finite-dimensional case. Moreover, she extends Hiai's variational expression of $\tilde{Q}_\alpha(\psi\|\varphi)$ to $\alpha \in (1, \infty)$, thus completing the extension of Frank and Lieb's variational expression of $\tilde{Q}_\alpha$ in the finite-dimensional case to the von Neumann algebra case.

In connection with the DPI, an interesting problem is whether the equality case $\tilde{D}(\Phi(\psi)\|\Phi(\varphi)) = \tilde{D}_\alpha(\psi\|\varphi)$ for $\tilde{D}_\alpha$ under a quantum channel or a more general 2-positive trace preserving map Φ implies the sufficiency (or reversibility) of Φ for $\{\psi, \varphi\}$. In the previous paper, the author showed the sufficiency result for a 2-positive Φ when $\alpha \in (1, \infty)$. In this paper, she extends this sufficiency result to the case $\alpha \in (1/2, 1)$, thus completing the sufficiency problem with respect to $\tilde{D}_\alpha$.

In the paper under review, the author improves and completes the results of her previous paper quite successfully. The results established are very strong and the proofs are correct.

Major comments.

(1) In the previous paper the author used Kosaki's *symmetric* L_p-spaces to deal with $\tilde{D}_\alpha$ for $\alpha \in (1, \infty)$, but in this paper she uses Kosaki's *right* L_p-spaces to define $\tilde{D}_\alpha$ for all $\alpha \in [1/2, \infty) \setminus \{1\}$ and to compare them with Berta, Scholz and Tomamichel's definition. The difference between the definitions in the previous and the present papers with Kosaki's *symmetric* and *right* L_p-spaces should be stressed.

(2) Kosaki's *right* L_p-spaces are used in Section 2. But the author's previous definition with Kosaki's *symmetric* L_p-spaces is sometimes used in discussions in later sections; for example, in the proof of Proposition 3.4. This situation seems rather complicated for the reader who doesn't know the previous paper well. Although the author gives an excuse by writing "all notations, definitions and results will be used without separate introduction", I think that the author should briefly recall the previous definition with Kosaki's *symmetric* $L_p(\mathcal{M}, \varphi)$. Also, it should be better to explain $L_p(\mathcal{M})$, $L_p^R(\mathcal{M}, \varphi_0)$ and $L_p(\mathcal{M}, \varphi)$ more explicitly. cf. Item 21 below.

(3) The structure of the proof of Theorem 5.1 is rather complicated. I think that the author had better write more explicitly why the proof has to start with the case that Φ is a quantum channel.

Minor corrections/suggestions. All of the comments below are not essential, but they might be useful to improve the presentation.

1. Page 1, line -5: We showed that $\to$ We showed in [9] that

2. Page 2, line -6: $C_p \to C_{1/p}$

3. Page 3, line -10: The phrase "closed in $\|\cdot\|_{p,\varphi}$" is rather strange, because the norm $\|\cdot\|_{p,\varphi}$ is defined only on the relevant subspace in $eL^2(\mathcal{M})$. Maybe, this should be "complete with respect to $\|\cdot\|_{p,\varphi}$".

4. Page 3, line -2: This is the characterization theorem of $L_p^R(\mathcal{M}, \varphi_0)$ in [11, Theorem 9.1]. Of course, the expression looks a type of polar decomposition, but the phrase "By the polar decomposition" is not suitable.

5. Page 4, line 8: $h_{\varphi_0}^{1/2}\xi \to h_\varphi^{1/2}\xi$ [seems better]

6. Page 4, line 9: The same comment as Item 4 about "the polar decomposition".

7. Page 4, Proposition 2.2: "If" $\to$ "if" (two places) (or) "such that" $\to$ "such that:"

8. Page 4, Proposition 2.3: "Then" $\to$ "Then:" ; $\to$. (at the end of (i)) "if" $\to$ "If" (at the beginning of (ii) and (iii))

9. Page 4, line -2: Note that a $\tilde{\xi} \in L_2(\mathcal{M})$ with $\|\tilde{\xi}\|_{q,\varphi}^{(2)} = 1$ such that the supremum in (ii) is attained is unique up to a multiple constant with absolute 1. So it is better to write here as: there is a unique element $\tilde{\xi} \in L_2(\mathcal{M})$ with $\|\tilde{\xi}\|_{q,\varphi}^{(2)} = 1$ such that $\|\xi\|_{p,\varphi}^{(2)} = (\xi, \tilde{\xi})$.

10. Page 5, line 4: $\|h_\varphi^{1/2-1/p}\eta\|_q \to \|h_\varphi^{1/q-1/2}\eta\|_q$ [seems better]

11. Page 5, line 10: It might be better to note here that $\|h_\varphi^{1/2}\eta\|_{q,\varphi_0}^R \le \|h_{\varphi_0}^{1/2}\eta\|_{q,\varphi_0}^R$ for any $\eta \in L_2(\mathcal{M})$.

12. Page 5, line -6: For the spatial derivative, the original paper [A. Connes, On the spatial theory of von Neumann algebras, J. Funct. Anal. 35 (1980), 153–164] should be referred to.

13. Page 6, line 13: Put a comma after "$2 \le p \le \infty$"

14. Page 6, line -5: the Hölder inequality (or) Hölder's inequality

15. Page 7, line 1: Better to give a proof of $u^*u \le s(\omega)$.

16. Page 7, line 14: It seems better to write as:
Since $h_\psi^{1/2-1/p}h_\omega^{1/2} = h_\psi^{1/2-1/p}k^*h_\varphi^{1/2-1/p}$, we have by [9, (A.3)]
$$\|\Delta_{\psi,\varphi}^{1/2-1/p}h_\omega^{1/2}\|_2 = \|h_\psi^{1/2-1/p}k^*\|_2 = \|kh_\psi^{1/2-1/p}\|_2 \le \|k\|_p,$$

17. Page 7, line -13: $s(\psi) \ge s(\omega_\eta) \to s(\psi) \ge s(\omega_{\eta^*})$ (?)

18. Page 7, line -12: It seems that the argument around here can be written more simply as: For any $\eta \in L_2(\mathcal{M})$ with $\|\eta\|_{q,\varphi}^{(2)} = \|\eta\|_{q,\varphi}^{BST} \le 1$ we have by (2)
$$|(h_\omega^{1/2}, \eta)| \le \|h_\omega^{1/2}\|_{p,\varphi}^{BST}\|\eta\|_{q,\varphi}^{BST} \le \|h_\omega^{1/2}\|_{p,\varphi}^{BST}.$$

19. Page 8, line -6: $u_t \in \mathcal{M}$ is a partial isometry $\to$ $u_t \in \mathcal{M}$ is a contraction. It is clear that u_t is a partial isometry when either $s(\psi) \le s(\varphi)$ or $s(\varphi) \le s(\psi)$. But I am not sure if this is the case in general. If then, $\|u_t\| = 1 \to \|u_t\| \le 1$ in the next line.

20. Page 8, line -5: $h_\varphi u_t \to h_{\varphi_0} u_t$ [seems better]

21. Page 9, line -8: The symmetric Kosaki's L_p-space appears here at the first time. So it is better to explain the meaning of $L_\alpha(\mathcal{M}, \varphi)$ with a non-faithful φ.

22. Page 9, Proof of Proposition 3.4: Better to add the proof in the case where $h_\psi \notin L_\alpha(\mathcal{M}, \varphi)$; then the right hand side of the expression of (i) is ∞.

23. Page 10, (4): Probably, the right-hand side of (4) should be $|h_\psi^{1/2} h_\varphi^{1/2\alpha-1/2}|^{2\alpha}$.

24. Page 11, (2): We will give → We will first give [better]

25. Page 11, line −8: $\mu(1)^{-1/q}\langle \eta, h_\psi^{1/2}\rangle \to \mu(1)^{-1/q}\langle h_\psi^{1/2}, \eta\rangle$

26. Page 11, line −4: Here it is better to add: Note that $h_\omega^{1/2} = |\eta^*| = \eta v$ with the polar decomposition $\eta^* = v|\eta^*|$ and hence $\|\eta\|_{q,\varphi}^{(2)} = \|h_\omega^{1/2}\|_{q,\varphi}^{(2)}$.

27. Page 12, line 7: Better to add: $i_{p,\varphi}(k) = h_\varphi^{1/2q} k h_\varphi^{1/2q}$ $(1/p + 1/q = 1)$

28. Page 12, line 12: $l \in L_q(\mathcal{M}, \varphi) \to l \in L_q(\mathcal{M})$

29. Page 12, line 14: Better to mention that Φ_φ is the predual map of Φ_φ^* and $\Phi_\varphi(\Phi(\varphi)) = \varphi$. Also, better to cite [9, (17)] for the claim "Since also Φ_φ is ...... between $L_p(\mathcal{M}, \varphi)$ and $L_q(\mathcal{M}, \varphi)$,"

30. Page 12, line −11: Better to mention that $\Phi^* : \mathcal{N} \to \mathcal{M}$ is the dual map of Φ. [The different maps Φ_φ^*, Φ_φ, and Φ^* arise from Φ, and it is difficult for the reader to recognize them.]

31. Page 12, line −3: I think that Lemma 5.1 of [T. Fack and H. Kosaki, Generalized s-numbers of τ-measurable operators, Pacific J. Math. 123 (1986), 269–300] is more suitable than [12, Lemma 3.3], because the latter is for Kosaki's $L_p(\mathcal{M}, \varphi_0)$, but (11) here is for Haagerup's $L_p(\mathcal{M})$.

32. Page 14, line 1: Better to cite [9, Theorem 4.2] as well as [9, Lemma 4.3].

33. Page 14, line 7: I think that $u^*u = s(h_\psi^{1/2}) = s(\psi)$ rather than $u^*u \geq s(\psi)$.

34. Page 14, line 9: It seems that [10, Proposition 2.3 (iii)] is used here. If so, better to cite this.

35. Page 14, the last sentence: Here the author writes that the proof is similar to that for quantum channels above. Moreover, in the last of the proof of Theorem 5.1, it is written that the proof is finished exactly as in the case $\psi \sim \varphi$. Since this part of the proof is essential, for the convenience of the reader it seems better to give more explanation here, although the discussion is similar to the proof for quantum channels.

36. Page 15, line 7: $> 0 \to \geq 0$ (at the end of this displayed line)

37. Page 15, line 8: independent from → independent of (?)

38. Page 15, line 8: φ and $\psi \to \varphi, \psi$ and y whenever $\|\varphi\|_1, \|\psi\|_1, \|\xi(y)\|_\gamma \leq c$ for some $c > 0$ [The second part of Lemma 5.2 is a well-known fact for a uniformly convex Banach space, say $L_\gamma(\mathcal{M})$, but two elements of $L_\gamma(\mathcal{M})$ are assumed to be in a bounded subset. Note that if $\|\varphi\|_1, \|\psi\|_1 \leq c$, then $\|h_\mu^{1/\gamma}\| \leq c^{1/\gamma}$.

39. Page 15, line -13: Put a period.

40. Page 15, the last line: Put a period.

41. Page 16, line 4: Better to change to: In the above we have used Lemma 3.5 for the equality and (13) for the inequality.

42. Page 16, (19): Put a period.

43. Page 16, line -10: $\alpha \geq 2/3 \to 2/3 \leq \alpha < 1$ [better]

44. Page 16, line -9: Theorem 5.2 in Fack and Kosaki's paper referred to in Item 31 is better than [12, Theorem 6.6], although Kosaki's $L_\gamma(\mathcal{M}, \varphi_0)$ is isometric to $L_\gamma(\mathcal{M})$.

45. Page 17, Lemma 5.5: Better to write this lemma as Proposition 5.5. [This is not used as a lemma.]

46. Page 17, line 5: by Lemma 3.5 $\to$ by Lemma 3.5 and the equality $\xi_{\alpha,\varphi}(\Phi^*(\bar{y})) = \Phi^*_{\alpha^*,\varphi}(h_\nu^{1/\gamma})$

47. Page 17, References: Update [7].