On the structure of states that saturate the Belavkin-Staszewski conditional mutual information

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Abstract

The purpose of these notes is to find a structural decomposition of the quantum states which are fixed points of the Belavkin-Staszewski relative entropy under data processing inequality.

1 Introduction

The main motivation for this manuscript arises from the fact that the set of points that saturate the DPI for the relative entropy is contained in that of the BS-entropy, but the converse is not true. There are counterexamples in both the general case (of $\rho, \sigma, \mathcal{T}$) [14, 15], as well as in the simplified tripartite case of the CMI and BS-CMI. In the latter case, we conclude that every quantum Markov chain is a BS recovery state, but there are BS recovery states which are not quantum Markov chains.

A natural question is then whether one can find a way to quantify the difference between the latter two sets in an exact or approximate way. In the exact way, we intend to study the measure of the set of quantum Markov chains in the set of BS recovery states. In the approximate way, we pose the natural question whether one can find a measure of the distance of a state to be a QMC that is lower bounded by the corresponding distance to a BSRS. Or more specifically, whether we can construct a lower bound for the CMI of a state in terms of one form of the BS-CMI, maybe up to some correction factor. In order to be able to tackle these problems, a necessary tool might be understanding the structural decomposition of states in both sets. This is the main finding of these notes.

2 Notation and preliminaries

2.1 Relative entropies

Let us consider a finite-dimensional Hilbert space $\mathcal{H}$ and $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ two quantum states on it. The *Umegaki relative entropy* [23] is defined as

$$D(\rho\|\sigma) := \begin{cases} \operatorname{tr}[\rho \log \rho - \rho \log \sigma] & \text{if } \ker \sigma \subseteq \ker \rho\,, \\ +\infty & \text{otherwise}\,, \end{cases}$$

and their *Belavkin-Staszewski (BS) entropy* [2] by

$$\widehat{D}(\rho\|\sigma) := \begin{cases} \operatorname{tr}\left[\rho \log \rho^{1/2}\sigma^{-1}\rho^{1/2}\right] & \text{if } \ker \sigma \subseteq \ker \rho\,, \\ +\infty & \text{otherwise}\,. \end{cases}$$

In the event of ρ and σ commuting, the two entropies coincide. Otherwise, the BS-entropy is strictly larger than the relative entropy [14].

Both notions above constitute quantum generalizations of the classical Kullback-Leibler divergence. The Umegaki relative entropy between two quantum states measures their distinguishability. Moreover, after the application of a quantum channel, i.e. a complete positive and trace-preserving linear map $\mathcal{T} : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{K})$, the distinguishability between those states can never increase. This phenomena is called *data-processing inequality* [20]:

$$D(\rho\|\sigma) \geq D(\mathcal{T}(\rho)\|\mathcal{T}(\sigma))\,. \tag{1}$$

However, there are situations in which, after the application of a quantum channel, the Umegaki relative entropy does not decrease. This *saturation* of the data-processing inequality was studied by Petz in [18, 19, 20], where he proved:

$$D(\rho\|\sigma) = D(\mathcal{T}(\rho)\|\mathcal{T}(\sigma)) \;\Leftrightarrow\; \rho = \sigma^{1/2}\mathcal{T}^*(\mathcal{T}(\sigma)^{-1/2}\mathcal{T}(\rho)\mathcal{T}(\sigma)^{-1/2})\sigma^{1/2}\,. \tag{2}$$

Note that the map in the right hand side is a quantum channel. It is called *Petz recovery map* and we denote it hereafter by $\mathcal{R}^\sigma_\mathcal{T}$. Eq. (2) then reads as an equivalence between saturation of the DPI for the relative entropy and ρ being a fixed point of $\mathcal{R}^\sigma_\mathcal{T} \circ \mathcal{T}$. This inequality has been multiple times strengthened by providing lower non-negative bounds to the difference between LHS and RHS of Eq. (1) in terms of various measures of the 'distance' from a state ρ to its Petz recovery map (or to a rotated version of it) [11, 22, 16], e.g. [10]

$$D(\rho\|\sigma) - D(\mathcal{T}(\rho)\|\mathcal{T}(\sigma)) \geq \left(\frac{\pi}{8}\right)^4 \left\|\rho^{-1}\right\|^{-2} \left\|\mathcal{T}(\rho)^{-1}\right\|^{-2} \|\mathcal{R}^\sigma_\mathcal{T} \circ \mathcal{T}(\rho) - \rho\|_1^4\,.$$

Let us move now to the setting of the Belavkin-Staszewski relative entropy, or BS-entropy for short. The data-processing inequality also holds for this quantity, namely for every $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ and every quantum channel $\mathcal{T} : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{K})$, we have

$$\widehat{D}(\rho\|\sigma) \geq \widehat{D}(\mathcal{T}(\rho)\|\mathcal{T}(\sigma))\,.$$

Additionally, saturation of the BS-entropy was proven in [5] to be equivalent to

$$\widehat{D}(\rho\|\sigma) = \widehat{D}(\mathcal{T}(\rho)\|\mathcal{T}(\sigma)) \;\Leftrightarrow\; \rho = \sigma\mathcal{T}^*(\mathcal{T}(\sigma)^{-1}\mathcal{T}(\rho)) =: \mathcal{B}^\sigma_\mathcal{T} \circ \mathcal{T}(\rho)\,.$$

The map $\mathcal{B}^\sigma_\mathcal{T}$ is trace preserving but, unfortunately, is not positive or even Hermitian-preserving in general. To deal with this issue, we can construct a new recovery condition for the BS-entropy by symmetrizing the latter one, given by:

$$\widehat{D}(\rho\|\sigma) = \widehat{D}(\mathcal{T}(\rho)\|\mathcal{T}(\sigma)) \;\Leftrightarrow\; \rho = (\sigma\mathcal{T}^*(\mathcal{T}(\sigma)^{-1}\mathcal{T}(\rho)\mathcal{T}(\sigma)^{-1})\sigma)^{1/2} =: \mathcal{B}^{\sigma,\mathrm{sym}}_\mathcal{T} \circ \mathcal{T}(\rho).$$

This equivalence will be shown in Theorem 6.1. The map $\mathcal{B}^{\sigma,\mathrm{sym}}_\mathcal{T}$ is positive, but it is not linear. Along the lines of the strengthened DPI for the relative entropy recalled above, some authors of the current draft proved in [5] the following inequality:

$$\widehat{D}(\rho\|\sigma) - \widehat{D}(\mathcal{T}(\rho)\|\mathcal{T}(\sigma)) \geq \left(\frac{\pi}{8}\right)^4 \left\|\rho^{-1/2}\sigma\rho^{-1/2}\right\|^{-4} \left\|\rho^{-1}\right\|^{-2} \left\|\mathcal{B}^\rho_\mathcal{T} \circ \mathcal{T}(\sigma) - \sigma\right\|_1^4\,. \tag{3}$$

2.2 Conditional mutual informations

Next, let us consider now a special case among the previous setting. Consider a tripartite Hilbert space $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and $\rho_{ABC} \in \mathcal{S}_+(\mathcal{H}_{ABC})$ a positive state. We define the *conditional mutual information* of ρ_{ABC} between A and C conditioned on B by

$$I_\rho(A:C|B) := S(\rho_{AB}) + S(\rho_{BC}) - S(\rho_{ABC}) - S(\rho_B)\,,$$

for $S(\rho_X) = -\operatorname{tr}[\rho_X \log \rho_X]$ the von Neumann entropy of ρ_X for $X \subset ABC$. The well-known property of strong subadditivity of the von Neumann entropy [17] is equivalent to the non-negativity of the conditional mutual information, which is furthermore known [20, 13] to vanish if, and only if,

$$\rho_{ABC} = \rho_{AB}^{1/2}\rho_B^{-1/2}\rho_{BC}\rho_B^{-1/2}\rho_{AB}^{1/2}\,, \tag{4}$$

i.e., whenever ρ_{ABC} is a *quantum Markov chain* (QMC).

In the same setting, we can define the *BS-conditional mutual information* (BS-CMI in short) of ρ_{ABC} between A and C conditioned on B in different forms, with the common ground that all vanish under the same conditions:

$$\begin{aligned}
\widehat{I}_\rho^{os}(A:C|B) &:= \widehat{D}(\rho_{ABC}\|\rho_{AB}\otimes\tau_C) - \widehat{D}(\rho_{BC}\|\rho_B\otimes\tau_C)\,,\\
\widehat{I}_\rho^{ts}(A:C|B) &:= \widehat{D}(\rho_{ABC}\|\rho_{AB}\otimes\rho_C) - \widehat{D}(\rho_{BC}\|\rho_B\otimes\rho_C)\,,\\
\widehat{I}_\rho^{rev}(A:C|B) &:= \widehat{D}(\rho_{AB}\otimes\tau_C\|\rho_{ABC}) - \widehat{D}(\rho_B\otimes\tau_C\|\rho_{BC})\,,
\end{aligned}$$

where $\tau_C = \mathbb{1}_C/d_C$. Translating the previous conditions of saturation of DPI into this setting, we have for $x \in \{os, ts, rev\}$

$$\begin{aligned}
\widehat{I}_\rho^{x}(A:C|B) = 0 \;\Leftrightarrow\; & \rho_{ABC} = \rho_{AB}\rho_B^{-1}\rho_{BC} =: \mathcal{B}(\rho_{BC})\\
\Leftrightarrow\; & \rho_{ABC} = (\rho_{AB}\rho_B^{-1}\rho_{BC}^2\rho_B^{-1}\rho_{AB})^{1/2} =: \mathcal{B}^{\mathrm{sym}}(\rho_{BC})\,.
\end{aligned} \tag{5}$$

This equivalence will be shown in Corollary 6.2. We call states that satisfy any of the conditions above *BS recovery states* (BSRS). The first condition has been used in the estimation of decay of correlations of Gibbs states of local 1D translation-invariant Hamiltonians at any positive temperature in the past [7]. In the recent paper [12], a reversed DPI based on the first equivalence has been used to show superexponential decay of the three BS-CMIs introduced above with the size of $|B|$ for Gibbs states of local 1D translation-invariant Hamiltonians at any positive temperature. Additionally, building on Eq. (3) for $\widehat{I}_\rho^{rev}(A:C|B)$ and an additional technical lemma, the following inequality was derived in the same paper

$$\widehat{I}_\rho^{rev}(A:C|B) \geq \left(\frac{\pi}{8}\right)^4 \|\rho_{BC}^{-1/2}\rho_{ABC}\rho_{BC}^{-1/2}\|_\infty^{-2}\|\Phi(\rho_{BC}) - \rho_{ABC}\|_1^4$$

for the map

$$\Phi(X) = \rho_B^{1/2}(\rho_B^{-1/2}\rho_{AB}\rho_B^{-1/2})^{1/2}\rho_B^{-1/2}X\rho_B^{-1/2}(\rho_B^{-1/2}\rho_{AB}\rho_B^{-1/2})^{1/2}\rho_B^{1/2}\,. \tag{6}$$

As an immediate consequence of this inequality, we have that $\widehat{I}_\rho^{rev}(A:C|B) = 0$ implies $\rho_{ABC} = \Phi(\rho_{BC})$, but the converse was left as an open question in [12]. We solve it in the positive in this draft, in Theorem 6.1, for general pairs of states ρ, σ and quantum channels $\mathcal{T}$.

3 Structure of states saturating BS-DPI

The starting point for the comparison between BS recovery states and quantum Markov chains is going to be their structural decomposition. For the quantum Markov chain case, this was studied in [13], obtaining that ρ_{ABC} is a quantum Markov chain between $A \leftrightarrow B \leftrightarrow C$ if, and only if

$$\rho_{ABC} = \bigoplus_n p_n \rho_{AB_n^L} \otimes \rho_{B_n^R C} \, .$$

Something similar, but more complicated, can be proven for BS recovery states.

Remark 3.1 *From the proof of Theorem 3.2 , we can extract that in (8), the matrix S_B can be expressed as $S_B = \rho_B^{\frac{1}{2}} U_B$ for some unitary matrix U_B. The associated expressions for M and N are given by*

$$\rho_{AB} = \rho_B^{\frac{1}{2}} U_B M U_B^* \rho_B^{\frac{1}{2}} \quad \rho_{BC} = \rho_B^{\frac{1}{2}} U_B N U_B^* \rho_B^{\frac{1}{2}},$$

by construction. As a consequence, on the one hand,

$$\rho_{AB}\rho_B^{-1}\rho_{BC} = \rho_B^{\frac{1}{2}} U_B M N U_B^* \rho_B^{\frac{1}{2}} = \rho_{ABC},$$

and on the other hand,

$$\rho_{ABC}^2 = \rho_{ABC}\rho_{ABC}^* = \rho_{AB}\rho_B^{-1}\rho_{BC}^2\rho_B^{-1}\rho_{AB},$$

which are the two equivalent conditions for the saturation of the BS-DPI stated in (5).

4 BS recovery states and quantum Markov chains

4.1 BSRS which are not QMC

In the previous sections, we have discussed the fact that the set of points that saturate the DPI for the relative entropy is contained in that of the BS-entropy, but the converse is not true [14, 15]. This can be translated to the simplified tripartite case of the conditional mutual information and the analogous BS quantities. In this case, we say that every quantum Markov chain is a BS recovery state, but there are BSRS which are not QMC. An example of this is presented below.

Example 4.1 *Consider a system with Hilbert spaces $\mathcal{H}_A = \mathcal{H}_B = \mathcal{H}_C = \mathbb{C}^2$ and let*

$$\rho_{ABC} = \frac{9}{47}\begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{4}{3} & 0 & -\frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{2}{3} & 0 & \frac{4}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{9} & 0 & \frac{1}{9} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{9} & 0 & \frac{4}{9} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} \end{pmatrix}$$

with marginals

$$\rho_{BC} = \frac{9}{47}\tau_A \otimes \begin{pmatrix} \frac{4}{9} & 0 & \frac{1}{9} & 0 \\ 0 & \frac{5}{3} & 0 & -\frac{2}{3} \\ \frac{1}{9} & 0 & \frac{10}{9} & 0 \\ 0 & -\frac{2}{3} & 0 & 2 \end{pmatrix}, \; \rho_{AB} = \frac{9}{47}\begin{pmatrix} \frac{5}{3} & -\frac{2}{3} & 0 & 0 \\ -\frac{2}{3} & 2 & 0 & 0 \\ 0 & 0 & \frac{4}{9} & \frac{1}{9} \\ 0 & 0 & \frac{1}{9} & \frac{10}{9} \end{pmatrix} \otimes \tau_C, \; \rho_B = \frac{9}{47}\tau_A \otimes \begin{pmatrix} \frac{19}{9} & -\frac{5}{9} \\ -\frac{5}{9} & \frac{28}{9} \end{pmatrix} \otimes \tau_C.$$

It can be checked that this state ρ_{ABC} satisfies the BS recovery condition

$$\rho_{ABC} = \rho_{AB}\rho_B^{-1}\rho_{BC},$$

but it is not a QMC since

$$\rho_{ABC} \neq \rho_{AB}^{\frac{1}{2}}\rho_B^{-\frac{1}{2}}\rho_{BC}\rho_B^{-\frac{1}{2}}\rho_{AB}^{\frac{1}{2}}.$$

A natural question is then how much larger the set of BSRS is with respect to the set of QMC. In order to explore this question, we will first characterize quantum Markov chains in terms of the matrix S_B in the decomposition (8).

Proposition 4.2 *A quantum state ρ_{ABC} is a QMC if, and only if, there exists a unitary matrix $V_B : \mathcal{H}_B \to \oplus_n \mathcal{H}_{B_n^L} \otimes \mathcal{H}_{B_n^R}$ such that*

$$S_B = V_B^* \bigoplus_{n=1}^{N} \left(\sqrt{p_n}\rho_{B_L^n}^{\frac{1}{2}} \otimes \rho_{B_R^n}^{\frac{1}{2}} \right).$$

Proof. Suppose that ρ_{ABC} is a QMC, that is, there exists a unitary matrix $V_B : \mathcal{H}_B \to \oplus_n \mathcal{H}_{B_n^L} \otimes \mathcal{H}_{B_n^R}$ such that

$$\rho_{ABC} = V_B^* \left(\bigoplus_{n=1}^{N} p_n \rho_{AB_n^L} \otimes \rho_{B_n^R C} \right) V_B,$$

where $(p_n)_{n\in\mathbb{N}}$ is a probability distribution. For every fixed $n \in \{1, \ldots, N\}$ define

$$M_{AB_n^L} = \rho_{B_n^L}^{-\frac{1}{2}} \rho_{AB_n^L} \rho_{B_n^R}^{-\frac{1}{2}}, \quad N_{AB_n^R} = \rho_{B_n^L}^{-\frac{1}{2}} \rho_{B_n^R C} \rho_{B_n^R}^{-\frac{1}{2}},$$

which satisfy

$$\operatorname{tr}_A M_{AB_n^L} = I_{B_n^L}, \quad \operatorname{tr}_C M_{B_n^R C} = I_{B_n^R}.$$

Then,

$$\begin{aligned}
\rho_{ABC} &= V_B^* \left(\bigoplus_{n=1}^{N} p_n \rho_{AB_n^L} \otimes \rho_{B_n^R C} \right) V_B \\
&= V_B^* \left(\bigoplus_{n=1}^{N} p_n (I_A \otimes \rho_{B_n^L}^{\frac{1}{2}} \otimes \rho_{B_n^R}^{\frac{1}{2}} \otimes I_C)(M_{AB_n^L} \otimes N_{B_n^R C})(I_A \otimes \rho_{B_n^L}^{\frac{1}{2}} \otimes \rho_{B_n^R}^{\frac{1}{2}} \otimes I_C)^* \right) V_B \\
&= (I_A \otimes S_B \otimes I_C) \left(\bigoplus_{n=1}^{N} M_{AB_n^L} \otimes N_{B_n^R C} \right) (I_A \otimes S_B^* \otimes I_C),
\end{aligned}$$

where $S_B = V_B^* \left(\bigoplus_{n=1}^{N} \sqrt{p_n} \rho_{B_n^L}^{\frac{1}{2}} \otimes \rho_{B_n^R}^{\frac{1}{2}} \right)$. Finally,

$$\|S_B\|_2^2 = \sum_{n=1}^{N} p_n \operatorname{tr}\left(\rho_{B_n^L} \otimes \rho_{B_n^R} \right) = 1.$$

The proof of the converse is analogous using the same constructions.

□

Remark 4.3 *The previous result shows us how to generate BSRS which are not QMC when both $\mathcal{H}_{B_n^L}$ and $\mathcal{H}_{B_n^R}$ are non-trivial for some $n \in \mathbb{N}$. For this purpose, it suffices to consider an invertible S_B such that $S_B S_B^*$ cannot be decomposed as direct sum of product matrices. This is always possible for non-trivial $\mathcal{H}_{B_n^L}$ and $\mathcal{H}_{B_n^R}$, since then the manifold*

$$\mathcal{G}_n = \left\{ V_n (X_{B_n^L} \otimes Y_{B_n^R}) V_n^* \ : \ X_{B_n^L}, Y_{B_n^R} > 0 \text{ and } V_n \text{ is unitary in } B(\mathcal{H}_{B_n^L} \otimes \mathcal{H}_{B_n^R}) \right\},$$

has strictly lower dimension (dimension here understood as manifold dimension) than the positive cone in $B(\mathcal{H}_{B_n^L} \otimes \mathcal{H}_{B_n^R})$.

4.2 QMC related to BSRS

The fact that every QMC is a BSRS, but the converse is not true, can be also understood in terms of vanishing conditional mutual informations. In particular, we have

$$I_\rho(A:C|B) = 0 \ \underset{\not\Leftarrow}{\Rightarrow} \ \widehat{I}_\rho^x(A:C|B) = 0\,,$$

for $x \in \{\text{os}, \text{ts}, \text{rev}\}$. Nevertheless, by virtue of the results from the previous sections, we can construct another state so that we can have equivalence between vanishing conditional mutual informations. This is the content of the next result, where we show that for every BSRS there is an associated QMC.

Theorem 4.4 *A state ρ_{ABC} is a BSRS if, and only if, $\eta_{ABC} = \frac{1}{d_B} \rho_B^{-\frac{1}{2}} \rho_{ABC} \rho_B^{-\frac{1}{2}}$ is a QMC. In particular, $\widehat{I}_\rho(A:C|B) = 0$ if, and only if, $I_\eta(A:C|B) = 0$.*

Proof. The structural decomposition of BSRS (8) can be rewritten for $S_B = \rho_B^{\frac{1}{2}} U_B^*$ as

$$\rho_B^{-\frac{1}{2}} \rho_{ABC} \rho_B^{-\frac{1}{2}} = U_B^* \left(\bigoplus_{n=1}^N M_n \otimes N_n \right) U_B.$$

for some unitary U_B. Renormalising the positive matrices M_n, N_n, for every $1 \le n \le N$, we convert them to the states

$$\eta_{AB_n^L} = \frac{1}{\operatorname{tr} M_n} M_n, \quad \eta_{B_n^R C} = \frac{1}{\operatorname{tr} N_n} N_n, \quad p_n = \operatorname{tr} M_n \operatorname{tr} N_n,$$

with $\sum_n p_n = d_B$. We can then define a new state

$$\eta_{ABC} := \frac{1}{d_B} \rho_B^{-\frac{1}{2}} \rho_{ABC} \rho_B^{-\frac{1}{2}} = U_B^* \left(\bigoplus_{n=1}^N \frac{p_n}{d_B} \eta_{AB_n^L} \otimes \eta_{B_n^R C} \right) U_B, \tag{8}$$

which is a QMC. Conversely, assume that η_{ABC} is a QMC, i.e. equation (11) holds. We need to check that the conditions (ii) and (iii) in Theorem 3.2 hold. If we take partial traces on the systems A and C in (11), we obtain the identity

$$\frac{1}{d_B} \bigoplus_{n=1}^N I_{B_n} = \bigoplus_{n=1}^N \frac{p_n}{d_B} \eta_{B_n^L} \otimes \eta_{B_n^R},$$

and in particular, for each sector n we have the condition

$$I_{B_n} = p_n \eta_{B_n^L} \otimes \eta_{B_n^R}.$$

If we trace now on B, we obtain that $p_n = d_{B_n} = d_{B_n^L} d_{B_n^R}$, which let us write

$$I_{B_n} = (d_{B_n}^L \eta_{AB_n^L}) \otimes (d_{B_n}^R \eta_{B_n^R C})$$

and as a consequence

$$d_{B_n}^L \eta_{AB_n^L} = \lambda I_{B_n^L}, \qquad d_{B_n}^R \eta_{AB_n^R} = \lambda^{-1} I_{B_n^R},$$

for $\lambda \in \mathbb{R} \setminus \{0\}$, but since $\eta_{AB_n^R}$ and $\eta_{AB_n^L}$ are quantum states $\lambda = 1$. □

Remark 4.5 *Theorem 4.4 shows that we can map any BSRS ρ_{ABC} to a QMC η_{ABC} satisfying $\eta_B = \tau_B$, but the converse is also true, i.e from any QMC with marginal τ_B we can also generate BSRS or QMC: Let η_{ABC} be a QMC with $\eta_B = \tau_B$, i.e. there exists a unitary U_B such that*

$$\eta_{ABC} = U_B^* \left(\bigoplus_{n=1}^{N} p_n \eta_{AB_L^n} \otimes \eta_{AB_R^n} \right) U_B.$$

In particular, tracing in AC,

$$d_B \eta_B = \bigoplus_{n=1}^{N} \left(\sqrt{p_n} d_{L^n} \eta_{B_L^n} \right) \otimes \left(\sqrt{p_n} d_{R^n} \eta_{B_R^n} \right) = \bigoplus_{n=1}^{N} I_{L^n} \otimes I_{R^n}.$$

Now, on the one hand by letting ρ_B be any state, if we define

$$\rho_{ABC} := d_B \rho_B^{\frac{1}{2}} \eta_{ABC} \rho_B^{\frac{1}{2}} = \underbrace{\rho_B^{\frac{1}{2}} U_B^*}_{S_B} \left[\bigoplus_{n=1}^{N} \underbrace{\left(\sqrt{p_n} d_{L^n} \eta_{AB_L^n} \right)}_{M_n} \otimes \underbrace{\left(\sqrt{p_n} d_{R^n} \eta_{B_R^n C} \right)}_{N_n} \right] \underbrace{U_B \rho_B^{\frac{1}{2}}}_{S_B^*},$$

η_{ABC} satisfies the conditions of Theorem 3.2, so ρ_{ABC} is a BSRS. On the other hand, if we take ρ_B of the form

$$\rho_B := U_B^* \left(\bigoplus_{n=1}^{N} \frac{1}{d_{B_n}} \rho_{B_L^n} \otimes \rho_{B_R^n} \right) U_B,$$

then $\rho_{ABC} := d_B \rho_B^{\frac{1}{2}} \eta_{ABC} \rho_B^{\frac{1}{2}}$ is a QMC.

Remark 4.6 *Theorem 4.4 provides a way to generate a QMC from any BSRS. However, the QMC obtained from two different BSRS could coincide.* ~~, and hence this is not a contradiction with ??.~~

From now on, we will denote by η_{ABC} the associated QMC for a BSRS ρ_{ABC} given by Theorem 4.4. We will see in the rest of this work how the relation between ρ_{ABC} and η_{ABC} allow us to get many interesting properties. The first one is the commutative of the marginals of η_{ABC}, $[\eta_{AB}, \eta_{BC}] = 0$:

$$d_B^2 \eta_{AB} \eta_{BC} = \rho_B^{-\frac{1}{2}} \rho_{AB} \rho_B^{-1} \rho_{BC} \rho_B^{-\frac{1}{2}} = \rho_B^{-\frac{1}{2}} \rho_{ABC} \rho_B^{-\frac{1}{2}} = \rho_B^{-\frac{1}{2}} \rho_{ABC}^* \rho_B^{-\frac{1}{2}} = \rho_B^{-\frac{1}{2}} \rho_{BC} \rho_B^{-1} \rho_{AB} \rho_B^{-\frac{1}{2}} = d_B^2 \eta_{BC} \eta_{AB},$$

where we have used (5). This fact allow us to generate many equivalent recovery conditions of different maps which satisfy different properties for the case $\sigma = \rho_{AB}$ and $\mathcal{T} = \mathrm{tr}_A$. The recovery condition obtained by using the map $\mathcal{B}^{\sigma}_{\mathrm{tr}_A}$ is trace preserving but not positive. $\mathcal{B}^{\rho_{AB}}_{\mathrm{tr}_A}$ acting on ρ_{BC} can also be rewritten as follows:

$$\begin{aligned}\mathcal{B}^{\rho_{AB}}_{\mathrm{tr}_A}(\rho_{BC}) &= \rho_{AB}\rho_B^{-1}\rho_{BC} \\ &= \rho_B^{\frac{1}{2}}(\rho_B^{-\frac{1}{2}}\rho_{AB}\rho_B^{-\frac{1}{2}})(\rho_B^{-\frac{1}{2}}\rho_{BC}\rho_B^{-\frac{1}{2}})\rho_B^{\frac{1}{2}} \\ &= \rho_B^{\frac{1}{2}}(\rho_B^{-\frac{1}{2}}\rho_{AB}\rho_B^{-\frac{1}{2}})^{\frac{1}{2}}\rho_B^{-\frac{1}{2}}\rho_{BC}\rho_B^{-\frac{1}{2}}(\rho_B^{-\frac{1}{2}}\rho_{AB}\rho_B^{-\frac{1}{2}})^{\frac{1}{2}}\rho_B^{\frac{1}{2}} \\ &=: \Phi(\rho_{BC}),\end{aligned}$$

since $[\rho_B^{-\frac{1}{2}}\rho_{AB}\rho_B^{-\frac{1}{2}}, \rho_B^{-\frac{1}{2}}\rho_{BC}\rho_B^{-\frac{1}{2}}] = 0$. The map Φ now is completely positive but not trace preserving. A very interesting recovery condition appears when we write the recovery condition for the map Φ in terms of the polar decomposition of $\rho_{AB}^{\frac{1}{2}}\rho_B^{-\frac{1}{2}}$, i.e.

$$\eta_{AB}^{\frac{1}{2}} = \frac{1}{\sqrt{d_B}}W_{AB}\rho_{AB}^{\frac{1}{2}}\rho_B^{-\frac{1}{2}} = \frac{1}{\sqrt{d_B}}\rho_B^{-\frac{1}{2}}\rho_{AB}^{\frac{1}{2}}W_{AB}^*. \tag{9}$$

then the condition is given by

$$\Phi(\rho_{BC}) = \rho_{AB}^{\frac{1}{2}}W_{AB}^*\rho_B^{-\frac{1}{2}}\rho_{BC}\rho_B^{-\frac{1}{2}}W_{AB}\rho_{AB}^{\frac{1}{2}},$$

which looks similarly to the Petz recovery map, but with the unitary matrix W_{AB} in between. Moreover, Φ satisfies the following trace inequality

$$\mathrm{tr}\,\Phi(X_{BC}) \leq d_A\|\rho_B^{-1}\|_\infty \,\mathrm{tr}\,X_{BC}.$$

Corollary 4.7 *Let ρ_{ABC} be a BSRS, the following are equivalent:*

1. ρ_{ABC} is a QMC.
2. $W_{AB}^*\eta_{BC}W_{AB} = \eta_{BC}$.
3. $\rho_{AB}^{-\frac{1}{2}}\rho_{ABC}\rho_{AB}^{-\frac{1}{2}} = \eta_{AB}^{-\frac{1}{2}}\eta_{ABC}\eta_{AB}^{-\frac{1}{2}}$.

Proof. The equivalence between 1. and 2. follows from the Petz recovery condition (4) and the fact that $\rho_{ABC} = \Phi(\rho_{BC})$. Now, condition 2. is equivalent to

$$\rho_{AB}^{-\frac{1}{2}}\rho_{ABC}\rho_{AB}^{-\frac{1}{2}} = W_{AB}\rho_{AB}^{-\frac{1}{2}}\rho_{ABC}\rho_{AB}^{-\frac{1}{2}}W_{AB}^*$$

and using the relations (12), we can write the RHS as

$$W_{AB}\rho_{AB}^{-\frac{1}{2}}\rho_{ABC}\rho_{AB}^{-\frac{1}{2}}W_{AB}^* = \frac{1}{d_B}\eta_{AB}^{-\frac{1}{2}}\rho_B^{-\frac{1}{2}}\rho_{ABC}\rho_B^{-\frac{1}{2}}\eta_{AB}^{-\frac{1}{2}} = \eta_{AB}^{-\frac{1}{2}}\eta_{ABC}\eta_{AB}^{-\frac{1}{2}}$$

□

Proposition 4.8 *Let ρ_{ABC} be a BSRS and define $\xi_{ABC} = \frac{1}{d_B}\rho_{AB}^{-\frac{1}{2}}\rho_{ABC}\rho_{AB}^{-\frac{1}{2}}$. Then there exists a unitary V_{AB} such that $V_{AB}\xi_{ABC}V_{AB}^*$ is a QMC. Moreover, a BSRS is a QMC if, and only if, the respecive ξ_{ABC} generated by ρ_{ABC} and η_{ABC} coincide.*

Proof. Let ρ_{ABC} be a BSRS. Then, there exists a decomposition

$$\rho_{ABC} = \rho_B^{-\frac{1}{2}} U_B \left(\bigoplus_{n=1}^{N} M_{AB_L^n} \otimes N_{B_R^n C} \right) U_B^* \rho_B^{-\frac{1}{2}}$$

satisfying the conditions of Theorem 3.2, and we can write the marginal $\rho_{AB}^{-\frac{1}{2}}$ using the polar decomposition,

$$\rho_{AB}^{-\frac{1}{2}} = \rho_B^{\frac{1}{2}} U_B \left(\bigoplus_{n=1}^{N} M_{AB_L^n}^{-\frac{1}{2}} \otimes N_{B_R^n C}^{-\frac{1}{2}} \right) U_B^* V_{AB}$$

for some unitary operator $V_{AB} \in B(\mathcal{H}_{AB})$. We can write then ξ_{ABC} using the block decomposition

$$\xi_{ABC} = \frac{1}{d_B} V_{AB}^* U_B \left(\bigoplus_{n=1}^{N} M_{AB_L^n}^{-\frac{1}{2}} M_{AB_L^n} M_{AB_L^n}^{-\frac{1}{2}} \otimes N_{B_R^n C}^{-\frac{1}{2}} N_{B_R^n C} N_{B_R^n C}^{-\frac{1}{2}} \right) U_B^* V_{AB},$$

i.e.

$$V_{AB} \xi_{ABC} V_{AB}^* = \frac{1}{d_B} U_B \left(\bigoplus_{n=1}^{N} M_{AB_L^n}^{-\frac{1}{2}} M_{AB_L^n} M_{AB_L^n}^{-\frac{1}{2}} \otimes N_{B_R^n C}^{-\frac{1}{2}} N_{B_R^n C} N_{B_R^n C}^{-\frac{1}{2}} \right) U_B^*,$$

and renormalizing similarly as we did in the proof of Theorem 4.4, the result holds. The last claim follows from previous Corollary. □

4.3 Approximate BSRS and QMC

~~The latter result gives~~ Theorem 4.4 provides an exact identification between quantum Markov chains and BS recovery states. A natural question is then whether an equivalence between approximate versions of these notions holds as well. We say that ρ_{ABC} is an ε-approximate quantum Markov chain if

$$I_\rho(A : C|B) \leq \varepsilon \,,$$

and analogously ρ_{ABC} is an ε-approximate BS recovery state if

$$\widehat{I}_\rho^{\text{rev}}(A : C|B) \leq \varepsilon \,.$$

To explore the connection between approximate quantum Markov chains and approximate BS recovery states, we first provide a general lower bound for the reversed BS-CMI of ρ_{ABC} in terms of the CMI of η_{ABC}, and conversely under some constraints.

Proposition 4.9 *Let ρ_{ABC} be a quantum state such that ρ_{BC} and ρ_B are invertible and with associated QMC η_{ABC}. Then,*

$$\widehat{I}_\rho^{rev}(A : C|B) \geq 2(\log \min\{d_A, d_C\} + 1)^{-8} \left(\frac{d_B \pi}{8 \| \rho_B^{-1} \|_\infty} \right)^4 \| \rho_{BC}^{-1/2} \rho_{ABC} \rho_{BC}^{-1/2} \|_\infty^{-2} I_\eta(A : C|B)^8 \,.$$

Conversely, if $[\eta_{AB}, \eta_{BC}] = 0$, then there exists a positive function f such that

$$\widehat{I}_\rho^{rev}(A : C|B) \leq f(\rho, d_A, d_B, d_C) I_\eta(A : C|B)^4$$

Proof. The trace norm between η_{ABC} and its recovery channel can be expressed in therms of ρ_{ABC} and Φ as follows:

$$\|\eta_{ABC} - \mathcal{P}_{B\to AB}(\eta_{BC})\|_1 = \frac{1}{d_B}\|\rho_B^{-1/2}\rho_{ABC}\rho_B^{-1/2} - \rho_B^{-1/2}\Phi(\rho_{BC})\rho_B^{-1/2}\|_1$$

Consider the completely positive and trace non increasing map

$$\mathcal{N}_\rho X = \frac{1}{\|\rho_B^{-1}\|_\infty}\rho_B^{-1/2}X\rho_B^{-1/2},$$

for $X \geq 0$. $\mathcal{N}_\rho$ satisfies then the DPI for the 1-distance and as a consequence,

$$\|\rho_{ABC} - \Phi(\rho_{BC})\|_1 \geq \frac{d_B}{\|\rho_B^{-1}\|_\infty}\|\eta_{ABC} - \mathcal{P}_{B\to AB}(\eta_{BC})\|_1.$$

To conclude this part, we make use now of

$$I_\eta(A:C|B) \leq 2(\log\min\{d_A, d_C\} + 1)\|\eta_{ABC} - \mathcal{P}_{B\to AB}(\eta_{BC})\|_1^{1/2},$$

which can be found in [6].

Conversely, since ρ_B is invertible, $\ker\mathcal{N}_\rho \neq 0$. Consider the operator matrix norm

$$\|\mathcal{N}_\rho\| = \sup_{\|Y\|_2\leq 1}\|\mathcal{N}_\rho Y\|_2$$

Since $\mathcal{N}_\rho$ is invertible,

$$\|\mathcal{N}_\rho Y\|_1 \geq \|\mathcal{N}_\rho Y\|_2 \geq \|\mathcal{N}_\rho^{-1}\|\|Y\|_2 \geq \frac{1}{rk(Y)}\|\mathcal{N}_\rho^{-1}\|\|Y\|_1 \geq \frac{1}{(d_Ad_Bd_C)^2}\|\mathcal{N}_\rho^{-1}\|\|Y\|_1$$

and by letting $Y = \rho_{ABC} - \Phi(\rho_{BC})$

$$\|\eta_{ABC} - \mathcal{P}_{B\to AB}(\eta_{BC})\|_1 \geq \frac{\|\mathcal{N}_\rho^{-1}\|}{d_B(d_Ad_Bd_C)^2}\|\rho_{ABC} - \Phi(\rho_{BC})\|_1$$

We obtain

$$\begin{aligned}
I_\eta(A:C|B) &\geq \left(\frac{\pi}{8}\right)^4\|\eta_{ABC}^{-1}\|_\infty^{-2}\|\eta_B^{-1}\|_\infty^{-2}\|\eta_{ABC} - \mathcal{P}_{B\to AB}(\eta_{BC})\|_1^4 \\
&\geq \left(\frac{\pi}{8}\right)^4\|\eta_{ABC}^{-1}\|_\infty^{-2}\|\eta_B^{-1}\|_\infty^{-2}\left(\frac{\|\mathcal{N}_\rho^{-1}\|}{d_B(d_Ad_Bd_C)^2}\right)^4\|\rho_{ABC} - \Phi(\rho_{BC})\|_1^4 \\
&= \left(\frac{\pi}{8}\right)^4\|\eta_{ABC}^{-1}\|_\infty^{-2}\|\eta_B^{-1}\|_\infty^{-2}\left(\frac{\|\mathcal{N}_\rho^{-1}\|}{d_B(d_Ad_Bd_C)^2}\right)^4\|\rho_{ABC} - \mathcal{B}_{\operatorname{tr}A}^{\rho_{AB}}(\rho_{BC})\|_1^4
\end{aligned}$$

since $[\eta_{AB}, \eta_{BC}] = 0$. Finally, we use Theorem 3.6 of [12] and obtain

$$\begin{aligned}
\widehat{I}_\rho^{rev}(A:C|B) &\leq \|\rho_{BC}^{-\frac{1}{2}}\rho_{ABC}\rho_{BC}^{-\frac{1}{2}}\|_\infty(\|\rho_B^{-1}\|_\infty\|\rho_B\|_\infty)^{\frac{1}{2}}\|\rho_{ABC}^{-1}\rho_{BC}\|_\infty\|\rho_{ABC}\rho_{BC}^{-1}\rho_B\rho_{AB}^{-1} - \mathbb{1}\|_\infty \\
&= \|\rho_{BC}^{-\frac{1}{2}}\rho_{ABC}\rho_{BC}^{-\frac{1}{2}}\|_\infty(\|\rho_B^{-1}\|_\infty\|\rho_B\|_\infty)^{\frac{1}{2}}\|\rho_{ABC}^{-1}\rho_{BC}\|_\infty\|\rho_{ABC}\mathcal{B}_{\operatorname{tr}A}^{\rho_{AB}}(\rho_{BC})^{-1} - \mathbb{1}\|_\infty \\
&\leq \|\rho_{BC}^{-\frac{1}{2}}\rho_{ABC}\rho_{BC}^{-\frac{1}{2}}\|_\infty(\|\rho_B^{-1}\|_\infty\|\rho_B\|_\infty)^{\frac{1}{2}}\|\rho_{ABC}^{-1}\rho_{BC}\|_\infty\|\mathcal{B}_{\operatorname{tr}A}^{\rho_{AB}}(\rho_{BC})^{-1}\|_\infty\|\rho_{ABC} - \mathcal{B}_{\operatorname{tr}A}^{\rho_{AB}}(\rho_{BC})\|_\infty \\
&\leq \|\rho_{BC}^{-\frac{1}{2}}\rho_{ABC}\rho_{BC}^{-\frac{1}{2}}\|_\infty(\|\rho_B^{-1}\|_\infty\|\rho_B\|_\infty)^{\frac{1}{2}}\|\rho_{ABC}^{-1}\rho_{BC}\|_\infty\|\mathcal{B}_{\operatorname{tr}A}^{\rho_{AB}}(\rho_{BC})^{-1}\|_\infty\|\rho_{ABC} - \mathcal{B}_{\operatorname{tr}A}^{\rho_{AB}}(\rho_{BC})\|_1
\end{aligned}$$

and the result holds.

$\square$

Open Question: How to bound $\widehat{I}_\rho^{rev}(A:C|B) \leq f(\|\rho_{ABC} - \Phi(\rho_{BC})\|_1)$ without the constraint?

Next, we can also prove an inequality in the converse direction without assuming any commutative condition on η_{ABC}. For that, we need to introduce the rotated version of Φ, namely

$$\Phi^{\text{rot}}(X) = \int_{-\infty}^{+\infty} dt\beta_0(t)\rho_B^{\frac{1-it}{2}}(\rho_B^{-1/2}\rho_{AB}\rho_B^{-1/2})^{\frac{1-it}{2}}\rho_B^{\frac{-1+it}{2}} X\rho_B^{\frac{-1-it}{2}}(\rho_B^{-1/2}\rho_{AB}\rho_B^{-1/2})^{\frac{1+it}{2}}\rho_B^{\frac{1+it}{2}},$$

with $\beta_0(t) = \frac{\pi}{2(\cosh(\pi t)+1)}$. The following result is an immediate consequence of the multivariate trace inequalities of Sutter et al. [22].

Proposition 4.10 *Let ρ_{ABC} be a positive quantum state and let $\eta_{ABC} = \frac{1}{d_B}\rho_B^{-1/2}\rho_{ABC}\rho_B^{-1/2}$. Then,*

$$\widehat{I}_\rho^{rev}(A:C|B) \leq \frac{1}{d_A}\left\|\rho_{BC}^{1/2}\right\|_\infty\left\|\rho_{ABC}^{-1}\rho_{BC}^{1/2}\right\|\left\|\Phi^{rot}(\rho_{BC}) - \rho_{ABC}\right\|_1.$$

Proof. We first rewrite $\widehat{I}_\rho^{rev}(A:C|B)$ as

$$\begin{aligned}
\widehat{I}_\rho^{rev}(A:C|B) &= \widehat{D}(\tau_A\otimes\rho_{BC}\|\rho_{ABC}) - \widehat{D}(\tau_A\otimes\rho_B\|\rho_{AB}) \\
&= \operatorname{tr}\Big[\tau_A\otimes\rho_{BC}\Big(\log\Big(\tau_A^{1/2}\otimes\rho_{BC}^{1/2}\,\rho_{ABC}^{-1}\,\tau_A^{1/2}\otimes\rho_{BC}^{1/2}\Big) - \log\Big(\tau_A^{1/2}\otimes\rho_B^{1/2}\,\rho_{AB}^{-1}\,\tau_A^{1/2}\otimes\rho_B^{1/2}\Big) \\
&\qquad - \log(\tau_A\otimes\rho_{BC}) + \log(\tau_A\otimes\rho_{BC}) - \log\rho_B + \log\rho_B)\Big] \\
&= -D(\tau_A\otimes\rho_{BC}\|\Omega),
\end{aligned}$$

where

$$\begin{aligned}
\Omega := \exp\Big\{&\log\Big(\tau_A^{1/2}\otimes\rho_B^{1/2}\,\rho_{AB}^{-1}\,\tau_A^{1/2}\otimes\rho_B^{1/2}\Big) - \log\Big(\tau_A^{1/2}\otimes\rho_{BC}^{1/2}\,\rho_{ABC}^{-1}\,\tau_A^{1/2}\otimes\rho_{BC}^{1/2}\Big) \\
&- \log(\tau_A\otimes\rho_{BC}) - \log\rho_B + \log\rho_B\Big\}.
\end{aligned}$$

Using that the relative entropy between two quantum states is always non-negative, and the extension of Golden-Thompson inequality from [22], and dropping the identities everywhere to ease notation, we have

$$\begin{aligned}
\widehat{I}_\rho^{rev}(A:C|B) &\leq \log\operatorname{tr}[\Omega] \\
&\leq \log\operatorname{tr}\Bigg[\int_{-\infty}^{+\infty} dt\beta_0(t)\Big(\rho_{BC}^{1/2}\,\rho_{ABC}^{-1}\,\rho_{BC}^{1/2}\Big)\rho_B^{\frac{1-it}{2}}\Big(\rho_B^{-1/2}\rho_{AB}\rho_B^{-1/2}\Big)^{\frac{1-it}{2}}\rho_B^{\frac{-1+it}{2}}\tau_A\otimes\rho_{BC} \\
&\qquad\cdot\rho_B^{\frac{-1-it}{2}}\Big(\rho_B^{-1/2}\rho_{AB}\rho_B^{-1/2}\Big)^{\frac{1+it}{2}}\rho_B^{\frac{1+it}{2}}\Bigg] \\
&= \log\operatorname{tr}\Big[\Big(\rho_{BC}^{1/2}\,\rho_{ABC}^{-1}\,\rho_{BC}^{1/2}\Big)\Phi^{\text{rot}}(\tau_A\otimes\rho_{BC}) - \tau_A\otimes\rho_{BC} + \tau_A\otimes\rho_{BC}\Big] \\
&\leq \operatorname{tr}\Big[\Big(\rho_{BC}^{1/2}\,\rho_{ABC}^{-1}\,\rho_{BC}^{1/2}\Big)\Phi^{\text{rot}}(\tau_A\otimes\rho_{BC}) - \tau_A\otimes\rho_{BC}\Big] \\
&\leq \frac{1}{d_A}\left\|\rho_{BC}^{1/2}\right\|_\infty\left\|\rho_{ABC}^{-1}\rho_{BC}^{1/2}\right\|\left\|\Phi^{\text{rot}}(\rho_{BC}) - \rho_{ABC}\right\|_1,
\end{aligned}$$

where we have used Hölder's inequality and $\log(x+1)\leq x$.

$\square$

Corollary 4.11 *Let ρ_{ABC} be a quantum state with invertible marginal ρ_B, and let $\eta_{ABC} = \frac{1}{d_B}\rho_B^{-1/2}\rho_{ABC}\rho_B^{-1/2}$. If ρ is an approximate BSRS, then η is an approximate QMC.*

5 Superexponential decay

An interesting application of the previous results is the study of the CMI decay associated to Gibbs states in a tripartite system ABC. In [7] it was proved that the CMI decays exponentially with the size of B while for the case of the BS-CMI, in [12] it was proved that the decay is superexponential. However, due to the connection established in Theorem 4.4 and together with the continuity bound of the previous results, we can prove superexponential decay for QMC η_{ABC} such that $\eta_B = \tau_B$.

Theorem 5.1 (Superexponential decay of CMI for η_{ABC}) *Let quantum spin system on $\mathbb{Z}$ with local, finite-range, translation-invariant interactions and ρ_{ABC} the associated Gibbs state. Then there exists a positive function $l \mapsto \varepsilon(l)$ with superexponential decay such that for every finite interval $I \subseteq \mathbb{Z}$ split into three subintervals $I = ABC$ and $\eta_{ABC} = \frac{1}{d_B}\rho_B^{-\frac{1}{2}}\rho_{ABC}\rho_B^{-\frac{1}{2}}$,*

$$I_\eta(A : C|B) \leq \mathcal{C}r(d_A, d_B, d_C)^{\frac{1}{2}} e^{\alpha(|A|+|B|)}\varepsilon(|B|),$$

where r is a rational function, $\mathcal{C}$ and α are constants only depending on inverse temperature β, strength J and range R of the pontential (see [12, Section 2.4]).

Proof. From Proposition 4.9, we can upper bound $I_\eta(A : C|B)$ as follows

$$I_\eta(A : C|B) \leq \sqrt{\frac{8(d_A + d_c + 1)^2}{d_B \pi \sqrt[4]{2}}} \left\|\rho_B^{-1}\right\|_\infty^{\frac{1}{2}} \left\|\rho_{BC}^{-\frac{1}{2}}\rho_{ABC}\rho_{BC}^{-\frac{1}{2}}\right\|_\infty^{\frac{1}{4}} \widehat{I}_\rho^{rev}(A : C|B)^{\frac{1}{8}}$$

On the one hand, by Lemma 3.5 of [12],

$$\left\|\rho_B^{-1}\right\|_\infty^{\frac{1}{2}} \leq \mathcal{C}_1 e^{\alpha_1 |B|}$$

and on the other hand, by Theorem 3.6 of [12]

$$\widehat{I}_\rho^{rev}(A : C|B)^{\frac{1}{8}} \leq \mathcal{C}_2 e^{\alpha_1(|A|+|B|)}\varepsilon(|B|),$$

with $\varepsilon(\,\cdot\,)$ a positive function with superexponential decay. Now to bound the last term, we use Theorem IX.2.1 of [4] and obtain first

$$\left\|\rho_{BC}^{-\frac{1}{2}}\rho_{ABC}\rho_{BC}^{-\frac{1}{2}}\right\|_\infty = \left\|\rho_{BC}^{-\frac{1}{2}}\rho_{ABC}^{\frac{1}{2}}\right\|_\infty^2 \leq \left\|\rho_{BC}^{-1}\rho_{ABC}\right\|_\infty$$

This should also be bounded similar as Lemma 3.5 of [12] □

Remark 5.2 *Since QMC are Gibbs states of local commuting Hamiltonians in 1D [8], by Remark 4.5 for any QMC η_{ABC} with $\eta_B = \tau_B$, it is possible to find the QMC (Gibbs state) ρ_{ABC} satisfying Theorem 5.1 under the due assumptions on the quantum spin system.*

6 BS recovery conditions

This section is devoted to showing several equivalent conditions for recoverability of states whenever there is saturation on the data-processing inequality for the Belavkin-Staszewski relative entropy.

Theorem 6.1 *Let ρ, σ be two quantum states, with σ invertible, and let $\mathcal{T}$ be a quantum channel. The following are equivalent:*

1. $\widehat{D}(\rho\|\sigma) = \widehat{D}(\mathcal{T}(\rho)\|\mathcal{T}(\sigma))$.

2. $\rho = \mathcal{B}^{\sigma}_{\mathcal{T}} \circ \mathcal{T}(\rho)$.

3. $\rho = \mathcal{B}^{\sigma,sym}_{\mathcal{T}} \circ \mathcal{T}(\rho)$.

Proof. 1. $\Leftrightarrow$ 2. This equivalence was proven in [5].
2. $\Rightarrow$ 3. Let us assume that 2. holds. Then, $\rho = \sigma\mathcal{T}^*(\mathcal{T}(\sigma)^{-1}\mathcal{T}(\rho))$, and hence

$$\rho^2 = \rho\rho^* = \sigma\mathcal{T}^*(\mathcal{T}(\sigma)^{-1}\mathcal{T}(\rho))\mathcal{T}^*(\mathcal{T}(\rho)\mathcal{T}(\sigma)^{-1})\sigma. \tag{10}$$

Now, as a consequence of Stinespring's dilation theorem, there is a Hilbert space $\mathcal{H}_E$ and an isometry $V : \mathcal{H}_1 \to \mathcal{H}_2 \otimes \mathcal{H}_E$ such that $\mathcal{T}(X) = \mathrm{Tr}_E(VXV^*)$. Note that the dual map is given by $\mathcal{T}^*(Y) = V^*(Y \otimes \mathbb{1}_E)V$. Replacing this in Eq. (13), we have

$$\rho^2 = \sigma V^*(\mathcal{T}(\sigma)^{-1}\mathcal{T}(\rho)) \otimes \mathbb{1}_E\, VV^*(\mathcal{T}(\rho)\mathcal{T}(\sigma)^{-1}) \otimes \mathbb{1}_E\, V\sigma\,,$$

and since V is an isometry and in particular $VV^* \leq \mathbb{1}$, the above implies

$$\begin{aligned}\rho^2 &\leq \sigma V^*(\mathcal{T}(\sigma)^{-1}\mathcal{T}(\rho)) \otimes \mathbb{1}_E(\mathcal{T}(\rho)\mathcal{T}(\sigma)^{-1}) \otimes \mathbb{1}_E\, V\sigma \\ &= \sigma\mathcal{T}^*\left(\mathcal{T}(\sigma)^{-1}\mathcal{T}(\rho)^2\mathcal{T}(\sigma)^{-1}\right)\sigma\,.\end{aligned} \tag{11}$$

We can conclude equality in the above inequality by multiplying the inequality by $\sigma^{-1/2}$ from both sides and combining it with the fact $\mathrm{tr}\big[\rho^2\sigma^{-1}\big] = \mathrm{tr}\big[\mathcal{T}(\rho)^2\mathcal{T}(\sigma)^{-1}\big]$ is equivalent to $\widehat{D}(\rho\|\sigma) = \widehat{D}(\mathcal{T}(\rho)\|\mathcal{T}(\sigma))$ from [14, Theorem 3.34]. Indeed,

$$\begin{aligned}0 &= \mathrm{tr}\big[\mathcal{T}(\rho)^2\mathcal{T}(\sigma)^{-1}\big] - \mathrm{tr}\big[\rho^2\sigma^{-1}\big] \\ &= \mathrm{tr}\Big[\mathcal{T}(\sigma)^{1/2}\mathcal{T}(\sigma)^{-1}\mathcal{T}(\rho)^2\mathcal{T}(\sigma)^{-1}\mathcal{T}(\sigma)^{1/2}\Big] - \mathrm{tr}\big[\rho^2\sigma^{-1}\big] \\ &= \mathrm{tr}\Big[\sigma^{1/2}\mathcal{T}^*\left(\mathcal{T}(\sigma)^{-1}\mathcal{T}(\rho)^2\mathcal{T}(\sigma)^{-1}\right)\sigma^{1/2}\Big] - \mathrm{tr}\Big[\sigma^{-1/2}\rho^2\sigma^{-1/2}\Big] \\ &= \Big\|\sigma^{1/2}\mathcal{T}^*\left(\mathcal{T}(\sigma)^{-1}\mathcal{T}(\rho)^2\mathcal{T}(\sigma)^{-1}\right)\sigma^{1/2} - \sigma^{-1/2}\rho^2\sigma^{-1/2}\Big\|_1\,,\end{aligned}$$

where in the last line we are using $\sigma^{1/2}\mathcal{T}^*\left(\mathcal{T}(\sigma)^{-1}\mathcal{T}(\rho)^2\mathcal{T}(\sigma)^{-1}\right)\sigma^{1/2} \geq \sigma^{-1/2}\rho^2\sigma^{-1/2}$ by Eq. (14).
3. $\Rightarrow$ 1. Because of the condition in 3., we have

$$\mathrm{tr}\big[\rho^2\sigma^{-1}\big] = \mathrm{tr}\big[\sigma\mathcal{T}^*\left(T(\sigma)^{-1}\mathcal{T}(\rho)^2\mathcal{T}(\sigma)^{-1}\right)\big] = \mathrm{tr}\big[\mathcal{T}(\rho)^2\mathcal{T}(\sigma)^{-1}\big]$$

and the proof is concluded by applying again [14, Theorem 3.34].

$\square$

At this point, we are ready to prove the last question left in the introduction: the reverse condition for the map Φ introduced in (6). Given now ρ_{ABC} a BSRS, we know from the previous discussion that η_{ABC} is a QMC, so it satisfies (4), which gives the recoverability condition

$$\begin{aligned}\rho_{ABC} &= \rho_B^{\frac{1}{2}}\left(\rho_B^{-\frac{1}{2}}\rho_{BC}\rho_B^{-\frac{1}{2}}\right)^{\frac{1}{2}}\rho_B^{-\frac{1}{2}}\rho_{AB}\rho_B^{-\frac{1}{2}}\left(\rho_B^{-\frac{1}{2}}\rho_{BC}\rho_B^{-\frac{1}{2}}\right)^{\frac{1}{2}}\rho_B^{\frac{1}{2}} \\ &= \Phi(\rho_{AB}).\end{aligned}$$

We can now combine the results obtained to characterise BSRS in terms of fixed points of three different maps.

We might suppress the following corollary

Corollary 6.2 *Let ρ_{ABC} be a quantum state such that ρ_{AB} is invertible, $\sigma = \rho_{AB} \otimes \tau_C$ and $\mathcal{T} = \mathrm{tr}_A$. The following are equivalent:*

1. *ρ_{ABC} is a BS recovery state.*
2. *$\rho_{ABC} = \mathcal{B}(\rho_{BC})$.*
3. *$\rho_{ABC} = \mathcal{B}^{sym}(\rho_{BC})$.*
4. *$\rho_{ABC} = \Phi(\rho_{AB})$.*
5. *$\rho_{ABC} = \mathcal{R}_W(\rho_{BC})$*

Proof. The equivalence between the first three points is a straightforward consequence of Theorem 6.1 particularized to a tripartite space $\mathcal{H}_{ABC}$, a state ρ_{ABC} on it, $\sigma_{ABC} = \rho_{AB} \otimes \tau_C$ and $\mathcal{T} = \mathrm{tr}_C$.

1. $\Rightarrow$ 4. This implication was already proven in [12], but we include it here for completeness. First, note that

$$\widehat{D}(\rho_{ABC}\|\rho_{AB} \otimes \tau_C) = \widehat{D}(\rho_{BC}\|\rho_B \otimes \tau_C) \;\Leftrightarrow\; \widehat{D}(\rho_{AB} \otimes \tau_C\|\rho_{ABC}) = \widehat{D}(\rho_B \otimes \tau_C\|\rho_{BC})\,.$$

By the strengthened data-processing inequality of the Belavkin-Staszewski relative entropy from [5], applied in the context of the reversed conditional mutual information, we have

$$\begin{aligned}\widehat{I}_\rho^{\mathrm{rev}}(A:C|B) &= \widehat{D}(\rho_{AB} \otimes \tau_C\|\rho_{ABC}) - \widehat{D}(\rho_B \otimes \tau_C\|\rho_{BC}) \\ &\geq \left(\frac{\pi}{4}\right)^4 \left\|\rho_{AB}^{-1/2}\rho_{ABC}\rho_{AB}^{-1/2}\right\|_\infty^{-2} \\ &\qquad \cdot \left\|\rho_{AB}^{1/2}\rho_B^{-1/2}\left(\rho_B^{-1/2}\rho_{BC}\rho_B^{-1/2}\right)^{1/2}\rho_B^{1/2} - \left(\rho_{AB}^{-1/2}\rho_{ABC}\rho_{AB}^{-1/2}\right)^{1/2}\rho_{AB}^{1/2}\right\|_2^4 .\end{aligned}$$

By [10, Lemma 2.2], the following inequality holds

$$2\|X - Y\|_2 \geq \|X^*X - Y^*Y\|_1$$

for any pair of operators X and Y such that $\mathrm{tr}[X^*X] = \mathrm{tr}[Y^*Y] = 1$. Denoting

$$X := \rho_{AB}^{1/2}\rho_B^{-1/2}\left(\rho_B^{-1/2}\rho_{BC}\rho_B^{-1/2}\right)^{1/2}\rho_B^{1/2}\,, \quad Y := \left(\rho_{AB}^{-1/2}\rho_{ABC}\rho_{AB}^{-1/2}\right)^{1/2}\rho_{AB}^{1/2}\,,$$

we note that $X^*X = \Phi(\rho_{AB}), Y^*Y = \rho_{ABC}$, and both have trace 1. Thus,

$$\widehat{I}_\rho^{\mathrm{rev}}(A:C|B) \geq \left(\frac{\pi}{8}\right)^4 \left\|\rho_{AB}^{-1/2}\rho_{ABC}\rho_{AB}^{-1/2}\right\|_\infty^{-2} \|\Phi(\rho_{AB}) - \rho_{ABC}\|_1^4\,,$$

and as a consequence, if $\widehat{I}_\rho^{\mathrm{rev}}(A:C|B) = 0$, then $\rho_{ABC} = \Phi(\rho_{AB})$.

4. $\Rightarrow$ 1. By $\rho_{ABC} = \Phi(\rho_{AB})$, we have

$$\underbrace{\rho_B^{-1/2}\rho_{ABC}\rho_B^{-1/2}}_{d_B\eta_{ABC}} = \Big(\underbrace{\rho_B^{-1/2}\rho_{BC}\rho_B^{-1/2}}_{d_B\eta_{BC}}\Big)^{1/2}\underbrace{\rho_B^{-1/2}\rho_{AB}\rho_B^{-1/2}}_{d_B\eta_{AB}}\Big(\underbrace{\rho_B^{-1/2}\rho_{BC}\rho_B^{-1/2}}_{d_B\eta_{BC}}\Big)^{1/2}\,,$$

for $\eta_{ABC} = \frac{1}{d_B}\rho_B^{-1/2}\rho_{ABC}\rho_B^{-1/2}$. This is equivalent to

$$\eta_{ABC} = \eta_{BC}^{1/2}\eta_B^{-1/2}\eta_{AB}\eta_B^{-1/2}\eta_{BC}^{1/2}\,,$$

and subsequently, to $I_\eta(A:C|B) = 0$. By Theorem 4.4, we this is equivalent to $\widehat{I}_\rho(A:C|B) = 0$, and thus to ρ_{ABC} being a BSRS.

1. $\Leftrightarrow$ 5. Follows from Corollary **??**. $\square$

7 Further questions we could to try answer

1. For quantum Markov chains, it holds that [21, Lemma 5.12]

$$I_\rho(A : C|B) \leq \inf_{\mu \in \mathrm{QMC}} D(\rho\|\mu) \,. \tag{13}$$

   Is a similar statement true for BSRS, for example,

$$\hat{I}_\rho(A : C|B) \leq \inf_{\mu \in \mathrm{BSRS}} \hat{D}(\rho\|\mu)\,?$$

   At the first look, the proof in [21] does not generalize because it uses that the CMI can be written as a sum of logarithms.

2. For quantum Markov chains, it is true that for every $\epsilon > 0$ there are states ρ such that $I_\rho(A : C|B) \leq \epsilon$, but

$$\min_{\mathrm{QMC}} \|\rho - \mu\|_1 \geq \text{const.}$$

   (see Proposition 5.9 of [21]). Is this still true for the BSRS, e.g., can we find for every ϵ a ρ such that $\widehat{I}_\rho(A : C|B) \leq \epsilon$ but

$$\min_{\mathrm{BSRS}} \|\rho - \mu\|_1 \geq \text{const.}\,?$$

   The proof in [21] uses the antisymmetric state and the chain rule for the CMI, so it seems not immediately clear how to generalize this.

3. What do the BSRS look like as a manifold? Are they connected and in which way? What is the dimension of the manifold? How many connected components does it have? Does it have a boundary on what is on it? What about geodesics? In principle, you could ask the same question for the QMC (or is this known)?

4. Can we find dimension independent lower bounds on the the BS-CMI in terms of how well the state can be recovered (does not have to be in trace norm)? For quantum Markov chains, there is for example the lower bound in terms of the fidelity by Fawzi and Renner [11] or Theorem 5.5 in [21], which states

$$I_\rho(A : C|B) \geq D_M(\rho\|\mathcal{R}^{\mathrm{rot}}_{B\to BC}(\rho_{AB}))$$

   where $\mathcal{R}^{\mathrm{rot}}_{B\to BC}$ is the rotated Petz recovery map.

5. Is there a direct way to prove that $\rho_{ABC} = \rho_{AB}\rho_B^{-1}\rho_{BC}$ implies the decomposition in 3.2? The other way round is not too difficult to see, see Remark 3.3.

6. Is there a version of Theorem 5.11 in [21], which gives a necessary condition for recovery, for the BSRS? The proof is rather long and needs (16), but on the other hand there are chain rules for the geometric Rényi divergencies [3].

7. We can prove superexponential decay of CMI for Gibbs states with $\rho_B = \tau_B$, as quantum approximate Markov chains are thermal. The idea is to use Proposition 4.9 jointly with [12, Theorem 3.6].

8. Can we give some kind of interpretation to the set of BSRS? For QMC, we can write them as Gibbs states of local Hamiltonians. This is probable a difficult question to answer since the BS relative entropy is also lacking an operational interpretation.

9. Given a state ρ, is there a way to compute its decomposition? Does the decomposition help to efficiently simulate these states? How to find the decomposition for QMC?

10. Can all BSRS be efficiently simulated in some way? This seems to be the case for QMC using the recovery map.

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