Quantum Uncertainty Principles for Measurements with Interventions

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Heisenberg's uncertainty principle implies fundamental constraints on what properties of a quantum system can we simultaneously learn. However, it typically assumes that we probe these properties via measurements at a single point in time. In contrast, inferring causal dependencies in complex processes often requires interactive experimentation - multiple rounds of interventions where we adaptively probe the process with different inputs to observe how they affect observed outputs. Here we demonstrate universal uncertainty principles for general interactive measurements involving arbitrary rounds of interventions. As a case study, we show that they imply an uncertainty trade-off between measurements compatible with different causal dependencies.

Introduction – The most powerful means of learning is through *interactive measurements*. When toddlers attempt to learn of their environment, they do not merely observe. Instead, they probe their environment through active interventions – performing various actions, observing resulting reactions and adapting future actions based on such observations. Such interactive measurements are essential to fully infer causation, allowing a toddler to learn whether one event caused the other or if they emerged from some common-causes [1]. Indeed, interactive measurements permeate diverse sciences. Whether using reinforcement learning to explore optimal strategies in competitive games or sending data packets to probe the characteristics of a network – adaptive intervention is critical [2–4].

The world, however, is fundamentally quantum. Ultimately, all systems obey Heisenberg's uncertainty principle [5]. Certain observable properties that cannot be simultaneously determined with certainty – the iconic case being the position and momentum of a free particle. Yet, current theories typically concern only measurements where there is no active intervention. Could such a fundamental uncertainty principle also exist when multiple preceding interventions (see Fig. 1)? How would this uncertainty principle on interactive measurements interplay with interventions aimed to discern causal structure?

We explore these questions by deriving the uncertainty principle for interactive measurements. These principles then pinpoint when two interactive measurements are non-compatible – and quantify the necessary trade-offs in the certainty of their measurement outcomes. Our results make no assumptions on the number of interventions or the causal structure of processes we probe, and encompasses previous uncertainty relations for states and channels [6–8]. We apply them to interactive measurements compatible with direct-cause vs common-cause, showing that they satisfy an uncertainty trade-off analogous to position and momentum.

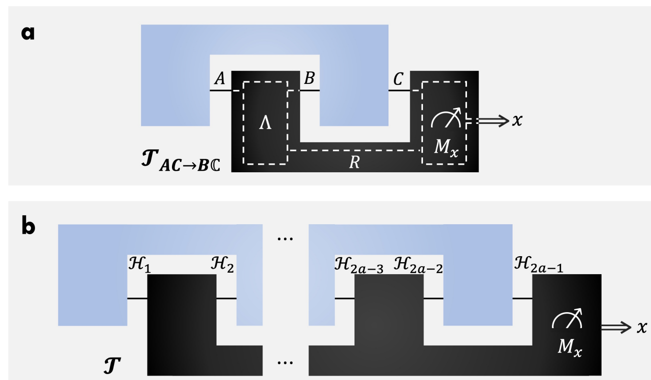

FIG. 1. **Interactive Measurements:** The circuit fragment $\mathcal{T}_{AC\to B\mathbb{C}}$, color black in (a), describes the an interactive measurement with one round of intervention on a dynamica process (blue box). Such a measurement consists of a quantum channel $\Lambda_{A\to BR}$ interacting A with memory R, followed by a POVM $M_{CR} := \{M_x\}_x$ with x standing for the outcome. More generally, an interactive measurement with $a-1$ interventions is illustrated in (b). Such circuit fragment appear in diverse contexts, including quantum illumination, quantum agents and non-Markovian open systems [9–12]

Framework – Interactive measurement describes an agent probing some dynamical process Φ. Consider an illustrative example where Φ represents a single qubit that evolves while in possible contact with other systems (e.g., a non-Markovian environment). Now suppose an agent, Alice, can access this qubit at different points in time, say t_X and t_Y, and she wishes to distinguish between two possible scenarios:

(i) The system at t_X is a *direct cause* of the system at t_Y. That is, the system's qubit is undergoing Markovian evolution, such that its state at t_Y depends solely on the state at t_X (Fig. 2b-i).

(ii) The system at t_X and t_Y share a *common cause*. That is, the system's qubit is initially correlated with some ancilla, which subsequently discards its own qubit after t_X and replaces it with the ancilla (Fig. 2b-ii).

Simply measuring the qubit at t_X and t_Y is generally insufficient. Except in a few special cases [13, 14], the agent cannot eliminate either scenario regardless of how much measurement statistics she collects. The idiom, 'correlations do not imply causation' applies. Indeed, things only get more complex when we allow superposition of causal orders (see Fig. 2b-iii).

To infer causal order reliably, interventions are required. Consider first the case with a single intervention, represented by applying some two-body interaction Λ between the system and some reference memory R, followed by a joint measurement M_x (see Fig. 1a). For convenience, systems before and after intervention, at time t_A and t_B, are given distinct labels A and B, whereas system followed by a measurement at t_C is labelled by C. Both the dynamics and the agent are treated mathematically via supermaps [15–18]. We provide a quick review in [19] Sec. IA.

Interventions bestow agents significantly more freedom. For instance, Λ could represent first measuring A in some basis, and then preparing it in some state that depends on the measurement outcome. Each interactive measurement yields a classical outcome, from which the agent can potentially infer relevant information about Φ. We then introduce *eigencircuits* of $\mathcal{T}$ as the circuit that always yields a definite outcome when tested by $\mathcal{T}$, analogous to eigenstates of projective measurements [20]

We can further extend this framework to the general scenario involving $a-1$ interventions prior to the final measurement (see Fig. 1b). Here, we label the system before and after the k^{th} intervention ($1 \leqslant k \leqslant a-1$) as $\mathcal{H}_{2k-1}$ and $\mathcal{H}_{2k}$. Potential causal orders can be far more varied. For example, all the systems could form a Markov chain, such that $\mathcal{H}_{2k+1}$ depends only on the state of $\mathcal{H}_{2k}$. Alternatively, a subset of them could share a common-cause. The most general interactive measurement then consists of preparing a memory state on R, followed by $a-1$ rounds of memory-system interactions, and a joint POVM on system and agent memory. Such models appear in diverse contexts, including models of thermalization [11], non-Markovianity [12], quantum interactive agents [3, 9], contextuality and retrodiction [21, 22].

Uncertainty Principles – In conventional quantum theory, certain observables are mutually incompatible. Given an observable $\mathcal{O}$ whose outcomes o_k occurs with probability p_k, we can quantify the uncertainty by employing the Shannon Entropy $H(\mathcal{O}) := -\sum_k p_k \log p_k$. The entropic uncertainty principle then states that there exists mutually non-compatible observables $\mathcal{O}_1$ and $\mathcal{O}_2$,

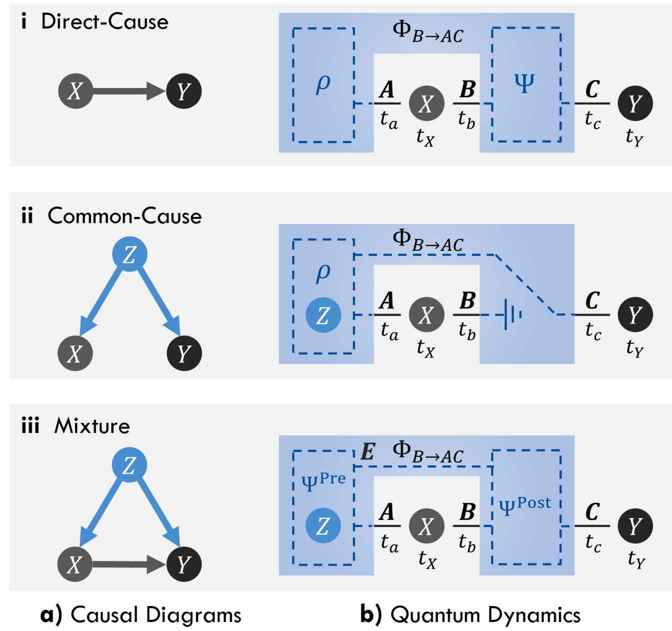

FIG. 2. **Quantum Description of Causal Structures:** There are three possible causal structures for two events X and Y, all of which can be expressed by a quantum dynamic process $\Phi_{B\to AC}$. In (i) direct-cause, $\Phi_{B\to AC}$ involves preparing a state A to be observed at X, whose output is sent directly to Y via quantum channel from B to C. In (ii) common-cause, correlations between X and Y can be attributed to measurements on some pre-prepared correlated state ρ_{AC} (event Z). Most generally (iii), $\Phi_{B\to AC}$ consists of a state-preparation process $\Psi^{\rm Pre}_{\mathbb{C}\to AE}$ and a post-processing quantum channel $\Psi^{\rm Post}_{BE\to C}$ (b-iii; E is an ancillary system). This then corresponds to a (possibly coherent) mixture of direct and common cause.

such that the joint uncertainty $H(\mathcal{O}_1)+H(\mathcal{O}_2)$ is always lower-bounded by some state-independent $C>0$ [23–37]. Our main result is to derive entropic bounds for general interactive measurements:

Theorem 1. *Given two interactive measurements $\mathcal{T}_1$ and $\mathcal{T}_2$ acting on some dynamical process Φ. The entropy of their measurement outcomes [38], when summed, satisfies*

$$H(\mathcal{T}_1)_\Phi + H(\mathcal{T}_2)_\Phi \geqslant C(\mathcal{T}_1, \mathcal{T}_2), \tag{1}$$

where $C(\mathcal{T}_1,\mathcal{T}_2)$ – measuring incompatibility between $\mathcal{T}_1$ and $\mathcal{T}_2$ – is non-negative and independent of Φ. $C(\mathcal{T}_1,\mathcal{T}_2)$ can be explicitly computed and is strictly non-zero whenever $\mathcal{T}_1$ and $\mathcal{T}_2$ has no common eigencircuit.

In [19] Sec. II, we illustrate a choice of $C(\mathcal{T}_1,\mathcal{T}_2)$ that reduces to $\log(1/c)$ when $\mathcal{T}_1$ and $\mathcal{T}_2$ are standard quantum measurements. Meanwhile, just as there exist many alternative bounds beyond $\log(1/c)$ [39–43], there are many other valid bounds for $H(\mathcal{T}_1)_\Phi + H(\mathcal{T}_2)_\Phi$ (See [19] Sec. II). Here we focus on a choice of $C(\mathcal{T}_1,\mathcal{T}_2)$ that can give tighter bounds in causal inference settings.

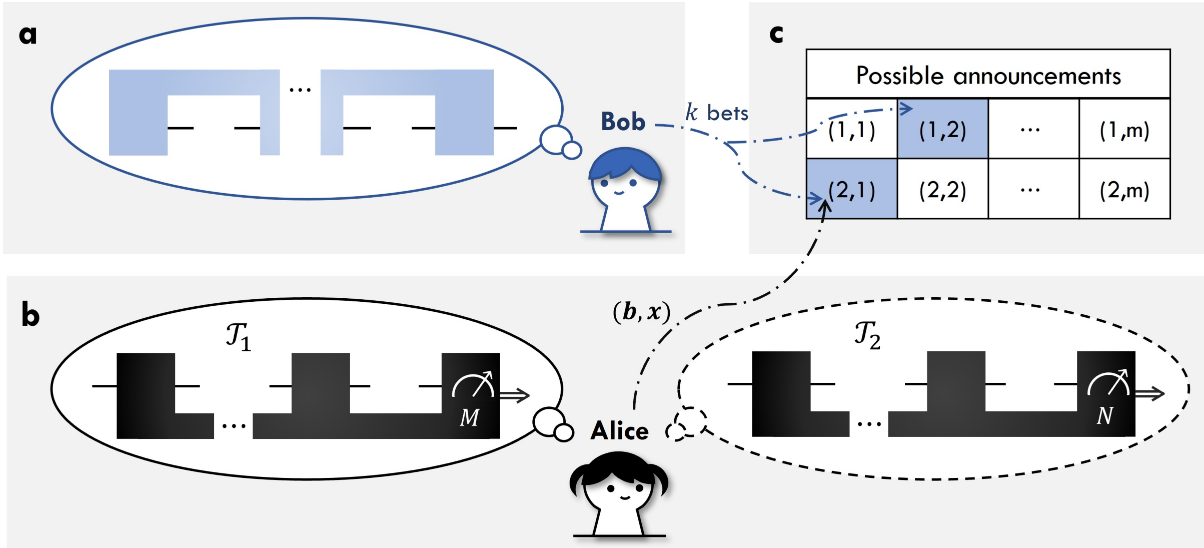

FIG. 3. **The Quantum Roulette:** The quantum roulette is a game that aids in interpreting lower bounds for the combined uncertainty of two general interactive measurements $\{\mathcal{T}_b\}_{b=1,2}$. $\mathcal{T}_1$ and $\mathcal{T}_2$ – picture in (b). Now introduce a quantum 'roulette table' with $2 \times m$ grid of cells (c), labelled (b, x) with $x = 1, \ldots, m$. In the k^{th} order game, Bob begins with k chips, of which he can allocate to k of these cells. Bob then supplies Alice with a dynamical process Φ (a). Alice selects a b at random, and measures Φ with $\mathcal{T}_b$ to obtain outcome x. Bob wins if he has a chip on the cell (b, x). Theorem 1 then relates Bob's winning probabilities with the incompatibility between $\mathcal{T}_1$ and $\mathcal{T}_2$.

Our formulation of $C(\mathcal{T}_1, \mathcal{T}_2)$ carries direct operational meaning in a guessing game which we refer to as the *quantum roulette*. The two-party game consists of (1) Alice, an agent that probes any supplied dynamical process using one of two possible interactive measurements, $\mathcal{T}_1$ and $\mathcal{T}_2$. (2) Bob, who can engineer various dynamical processes for Alice to probe (see Fig. 3). In each round, Alice and Bob begin with a 'roulette table', whose layout consists of all tuples (b, x), where $b \in \{1, 2\}$ and x are all possible measurement outcomes of $\mathcal{T}_1$ and $\mathcal{T}_2$. Bob begins with k chips, which he can use to place bets on k of the possible tuples and supplies Alice with any Φ of his choosing. Alice will then select some $b \in \{1, 2\}$ at random and probe Φ with $\mathcal{T}_b$. She finally announces both b and the resulting measurement outcome x. Bob wins if one of his chips is on (b, x).

Let p_k denote Bob's maximum winning probability. Naturally $p_0 = 0$ and p_k increases monotonically with k, tending to 1. We define a probability vector $\mathbf{w}$ with elements $w_k = p_k - p_{k-1}$, $k = 1, 2, \ldots$, representing the increase in Bob's probability of winning with k rather than $k-1$ chips. In [19] Sec. IIA, we show that

$$C(\mathcal{T}_1, \mathcal{T}_2) := 2H(\mathbf{w}) - 2 \tag{2}$$

is a lower bound for $H(\mathcal{T}_1)_\Phi + H(\mathcal{T}_2)_\Phi$.

This game gives an operational criterion of non-compatibility for interactive measurements. When two observables are compatible, $H(\mathbf{w}) = 1$. This aligns with the scenario that $\mathbf{w} = (0.5, 0.5, 0, \ldots, 0)$, which occurs when Bob's success rate is limited only by his uncertainty of which measurement Alice makes. That is, placing one counter ensures Bob can correctly predict the outcome of $\mathcal{T}_1$ and two counters gives him perfect prediction regardless of b. We see this is only possible if $\mathcal{T}_1$ and $\mathcal{T}_2$ share at least one common eigencircuit. Thus, $H(\mathcal{T}_1)_\Phi + H(\mathcal{T}_2)_\Phi$ is strictly greater than 0 whenever $\mathcal{T}_1$ and $\mathcal{T}_2$ share no common eigencircuit.

Causal Uncertainty Relations – The central relevance of interventions in causal inference make it an appropriate case study. Consider an agent probing a d-level open quantum system via a intervention at time t_X (where we label the system X), prior to measurement at time t_Y (where we label the system Y). We then introduce two families of interactive measurements the agent can adopt: $\mathcal{M}_{\text{CC}}$ and $\mathcal{M}_{\text{DC}}$, as depicted in Fig. 4. Each $\mathcal{T}_1 \in \mathcal{M}_{\text{CC}}$ is a *maximal common-cause indicator*, such that its eigencircuits imply that X and Y are actually two arms of some maximally entangled state (Fig. 2b-ii). Meanwhile, each $\mathcal{T}_2 \in \mathcal{M}_{\text{DC}}$ is a *maximal direct-case indicator*, whose eigencircuit involve a lossless channel from X to Y (i.e., Fig. 2b-i where Ψ is unitary). In Supplemental Material Sec. III, we establish the following *causal uncertainty relation*:

$$H(\mathcal{T}_1) + H(\mathcal{T}_2) \geqslant 2 \log d, \tag{3}$$

for any $\mathcal{T}_1 \in \mathcal{M}_{\text{CC}}$ and $\mathcal{T}_2 \in \mathcal{M}_{\text{CC}}$. Furthermore, this bound can be saturated.

Consider the application of this uncertainty to a specific parametrized quantum circuits $\Phi_{\alpha,\beta}$ (Fig. 5a) de-

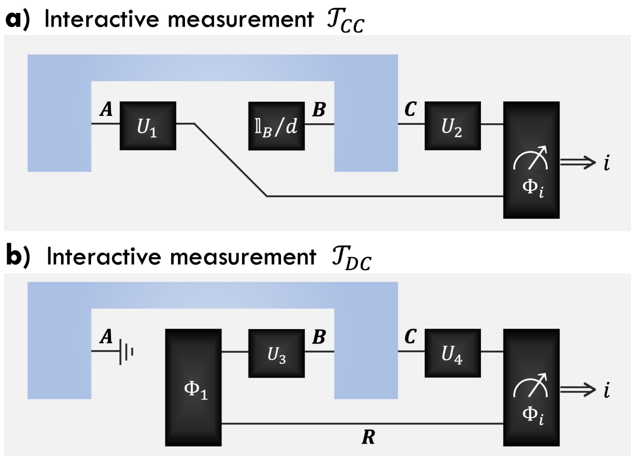

FIG. 4. **Maximal Common-Cause and Direct-Cause Indicators:** We introduce (a) $\mathcal{M}_{\rm CC} = \{\mathcal{T}_{\rm CC}(U_1, U_2)\}$ and (b) $\mathcal{M}_{\rm DC} = \{\mathcal{T}_{\rm DC}(U_3, U_4)\}$ as two respective families of interactive measurements with a single intervention. Here, system A, B and C are d-level quantum systems (qudits), and each U_k, $k = 1, 2, 3, 4$ is some single-qudit unitary, and $|\Phi_1\rangle := \sum_{k=0}^{d-1} |kk\rangle / \sqrt{d}$. Measurements are done with respect to a maximally entangling basis $\{\Phi_i\}_i$ with d^2 possible outcomes. The two measurement families are incompatible, and satisfy the causal uncertainty relation in Eq. 3.

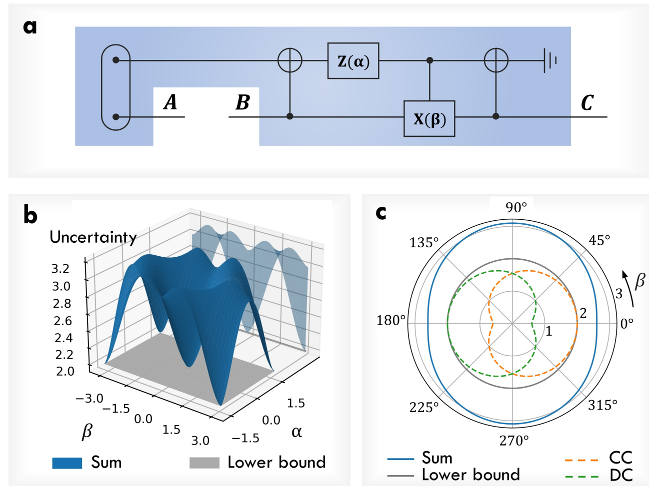

FIG. 5. **Causal Uncertainty Relations on Non-Markovian Dynamics:** Consider a single qubit – bottom rail of the circuit in (a) – undergoing non-Markovian evolution, Here $Z(\theta)$ and $X(\theta)$ represent single-qubit rotation gates in X and Z axis. (b) illustrates the combined uncertainty $H(\mathcal{T}_1) + H(\mathcal{T}_2)$, where $\mathcal{T}_1 \in M_{\rm CC}$ and $\mathcal{T}_2 \in M_{\rm DC}$ are respectively common-cause and direct-cause indicators in Fig 4 with all U_k set to the identity. Observe this never goes below the fundamental lower bound of 2 (gray plane). (c) illustrates $H(\mathcal{T}_1)$ (green dashed), $H(\mathcal{T}_2)$ (red dashed) and their sum (blue solid) for $\alpha = -\pi/4$ and various values of β, corresponding to various coherent superpositions of common-cause and direct-cause circuits.

scribing a single qubit undergoing non-Markovian evolution. Fig. 5b then demonstrates the combined uncertainty $H(\mathcal{T}_1) + H(\mathcal{T}_2)$ for various values of α and β, including cases where they saturates the lower bound of 2. We also note that unlike classical processes, which must be either purely common-cause, or purely direct-cause, or a probabilistic mixture of both – quantum processes can feature richer causal dependencies [44]. Fig. 5c depicts this for the cross-section of $\alpha = \pi/4$. Such circuits include the coherent superposition of direct and common cause as a special case. Our causal uncertainty relation also applies to these uniquely quantum causal structures.

Discussion – The most powerful means of learning the dynamics of environmental processes involves interactive measurement – a procedure in which we can intervene by injecting (possible entangled) quantum states into the process over multiple time-steps before observing the final output. Here, we derive entropic uncertainty relations that governs all interactive measurements, bounding their joint uncertainty whenever such measurement outcomes are non-compatible. In context of causal inference, they predict a uniquely quantum entropic trade-off between measurements that probe for direct and common cause. More generally, our relations encompass all possible means for an agent to interact and learn about a target quantum system and thus include previously studied uncertainty relations on states and channels as special cases.

One potential application of such relations is the metrology of unknown quantum processes. Here, full tomography of a general quantum process is impractical. Even a single non-Markovian qubit measured at two different times requires 54 different interactive measurements [45]. Our result may help us ascertain specific properties of a process while avoiding this costly procedure. In [19] Sec. VB, we illustrate how our causal uncertainty relations imply that a single interactive measurement can rule out specific causal structures. Indeed, quantum illumination and adaptive sensing can both cast as measuring desired properties of a candidate quantum process, and thus could benefit from such an approach [46, 47].

Interactive measurements through repeated interventions also merge is other settings. In quantum open systems, sequential intervention provides a crucial toolkit for characterizing non-Markovian noise [12, 48, 49]. Meanwhile, in reinforcement learning, quantum agents that continuously probe an environment show enhancements in enacting or learning complex adaptive behaviour [3, 9, 50]. Investigating uncertainty relations specific to such contexts has exciting potential, perhaps revealing new means of probing non-Markovian dynamics, or fundamental constraints on how well an agent can simultaneously optimize two different rewards..

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[20] Consider an interactive measurement $\mathcal{T}_{AC\to B\mathbb{C}} = \{M_x \circ \Lambda_{A\to BR}\}_x$ with a single interaction as shown in Fig. 1a. Its eigencircuit is a dynamical process $\Phi_{B\to AC}$ satisfying $\mathrm{Tr}[M_x \cdot \Lambda_{A\to BR}(\Phi_{B\to AC})] = 1$ for some measurement outcome x. Further details are in Supplemental Material Sec. IA.

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