Monotonicity of $z \mapsto D_{\alpha,z}(\psi\|\varphi)$

1 Finite von Neumann algebra case

Assume that $(\mathcal{M},\tau)$ is a semi-finite von Neumann algebra $\mathcal{M}$ with a faithful normal semi-finite trace τ. Then the Haagerup L^p-space $L^p(\mathcal{M})$ is identified with the L^p-space $L^p(\mathcal{M},\tau)$ [6, Sec. 4.3] with respect to τ; see [15] and [6, Example 9.11] for details. Hence one can define $Q_{\alpha,z}(\psi\|\varphi)$ for $\psi,\varphi\in\mathcal{M}_*^+$ by replacing, in [11, Definition 1], $L^p(\mathcal{M})$ with $L^p(\mathcal{M},\tau)$ and $h_\psi\in L^1(\mathcal{M})_+$ for $\psi\in\mathcal{M}_*^+$ with the Radon–Nikodym derivative $d\psi/d\tau\in L^1(\mathcal{M},\tau)_+$. Below we use the symbol h_ψ in Haagerup's L^p-spaces to denote $d\psi/d\tau$ as well. Note that τ on $\mathcal{M}_+$ is naturally extended to the positive part $\widetilde{\mathcal{M}}_+$ of the space of τ-measurable operators. We then have (see [6, Proposition 4.20])

$$\tau(a)=\int_0^\infty \mu_s(a)\,ds\quad\text{for any } a\in\widetilde{\mathcal{M}}_+, \tag{1.1}$$

where $\mu_s(a)$ is the generalized s-number of a [1]. Also, note that if τ is finite, i.e., $\tau(\mathbf{1})<+\infty$, then $\widetilde{\mathcal{M}}_+$ consists of all positive self-adjoint operators affiliated with $\mathcal{M}$.

Throughout this section we assume that $\mathcal{M}$ is a finite von Neumann algebra with a faithful normal finite trace τ.

Lemma 1.1. *For every $\psi,\varphi\in\mathcal{M}_*^+$ with $\psi\neq 0$ and for any $\alpha,z>0$ with $\alpha\neq 1$,*

$$D_{\alpha,z}(\psi\|\varphi)=\lim_{\varepsilon\searrow 0} D_{\alpha,z}(\psi\|\varphi+\varepsilon\tau)\quad \textit{increasingly}, \tag{1.2}$$

and hence

$$D_{\alpha,z}(\psi\|\varphi)=\sup_{\varepsilon>0} D_{\alpha,z}(\psi\|\varphi+\varepsilon\tau). \tag{1.3}$$

Proof. Case $0<\alpha<1$. We need to prove that

$$Q_{\alpha,z}(\psi\|\varphi)=\lim_{\varepsilon\searrow 0} Q_{\alpha,z}(\psi\|\varphi+\varepsilon\tau)\quad\text{decreasingly}. \tag{1.4}$$

By the definition of $D_{\alpha,z}$ in [11, Definition 1] in the present setting we have, thanks to (1.1),

$$Q_{\alpha,z}(\psi\|\varphi)=\tau\Bigl(\Bigl(h_\psi^{\alpha/2z}h_\varphi^{\frac{1-\alpha}{z}}h_\psi^{\alpha/2z}\Bigr)^z\Bigr)=\int_0^\infty \mu_s\Bigl(h_\psi^{\alpha/2z}h_\varphi^{\frac{1-\alpha}{z}}h_\psi^{\alpha/2z}\Bigr)^z\,ds, \tag{1.5}$$

and similarly

$$Q_{\alpha,z}(\psi\|\varphi+\varepsilon\tau)=\int_0^\infty \mu_s\Bigl(h_\psi^{\alpha/2z}h_{\varphi+\varepsilon\tau}^{\frac{1-\alpha}{z}}h_\psi^{\alpha/2z}\Bigr)^z\,ds. \tag{1.6}$$

Since $h_{\varphi+\varepsilon\tau}^{\frac{1-\alpha}{z}}=(h_\varphi+\varepsilon\mathbf{1})^{\frac{1-\alpha}{z}}$ decreases to $h_\varphi^{\frac{1-\alpha}{z}}$ in the measure topology as $\varepsilon\searrow 0$, it follows that $h_\psi^{\alpha/2z}h_{\varphi+\varepsilon\tau}^{\frac{1-\alpha}{z}}h_\psi^{\alpha/2z}$ decreases to $h_\psi^{\alpha/2z}h_\varphi^{\frac{1-\alpha}{z}}h_\psi^{\alpha/2z}$ in the measure topology. Hence by [1, Lemma 3.4] we have

$$\mu_s\Bigl(h_\psi^{\alpha/2z}h_{\varphi+\varepsilon\tau}^{\frac{1-\alpha}{z}}h_\psi^{\alpha/2z}\Bigr)\searrow \mu_s\Bigl(h_\psi^{\alpha/2z}h_\varphi^{\frac{1-\alpha}{z}}h_\psi^{\alpha/2z}\Bigr)\quad\text{as }\varepsilon\searrow 0 \tag{1.7}$$

for a.e. $s > 0$. Since $s \mapsto \mu_s\Bigl(h_\psi^{\alpha/2z} h_{\varphi+\tau}^{\frac{1-\alpha}{z}} h_\psi^{\alpha/2z}\Bigr)$ is integrable on $(0,\infty)$, one can apply the Lebesgue convergence theorem with (1.5)–(1.7) to obtain (1.4).

Case $\alpha > 1$. We need to prove that

$$Q_{\alpha,z}(\psi\|\varphi) = \lim_{\varepsilon\searrow 0} Q_{\alpha,z}(\psi\|\varphi+\varepsilon\tau) \quad \text{increasingly.} \tag{1.8}$$

For any $\varepsilon > 0$, since $h_{\varphi+\varepsilon\tau} = h_\psi + \varepsilon\mathbf{1}$ has the bounded inverse $h_{\varphi+\varepsilon\tau}^{-1} = (h_\varphi+\varepsilon\mathbf{1})^{-1} \in \mathcal{M}_+$, one can define

$$x_\varepsilon := (h_\varphi+\varepsilon\mathbf{1})^{-\frac{\alpha-1}{2z}} h_\psi^{\alpha/z} (h_\varphi+\varepsilon\mathbf{1})^{-\frac{\alpha-1}{2z}} \in \widetilde{\mathcal{M}}_+,$$

so that

$$h_\psi^{\alpha/z} = (h_\varphi+\varepsilon\mathbf{1})^{\frac{\alpha-1}{2z}} x_\varepsilon (h_\varphi+\varepsilon\mathbf{1})^{\frac{\alpha-1}{2z}}.$$

In the present setting, the definition of $D_{\alpha,z}$ in [11, Definition 1] says that

$$Q_{\alpha,z}(\psi\|\varphi+\varepsilon\tau) = \begin{cases} \|x_\varepsilon\|_z^z & \text{it } x_\varepsilon \in L^z(\mathcal{M},\tau), \\ +\infty & \text{otherwise.} \end{cases}$$

With τ extended to $\widetilde{\mathcal{M}}_+$ and (1.1), the above can be written as

$$Q_{\alpha,z}(\psi\|\varphi+\varepsilon\tau) = \tau(x_\varepsilon^z) = \int_0^\infty \mu_s(x_\varepsilon)^z\,ds \ (\in [0,+\infty]). \tag{1.9}$$

Let $0 < \varepsilon \le \varepsilon'$. Since

$$(h_\varphi+\varepsilon\mathbf{1})^{-\frac{\alpha-1}{z}} \ge (h_\varphi+\varepsilon'\mathbf{1})^{-\frac{\alpha-1}{z}},$$

we have

$$\mu_s(x_\varepsilon) = \mu_s\bigl(h_\psi^{\alpha/2z}(h_\varphi+\varepsilon\mathbf{1})^{-\frac{\alpha-1}{z}} h_\psi^{\alpha/2z}\bigr) \ge \mu_s\bigl(h_\psi^{\alpha/2z}(h_\varphi+\varepsilon'\mathbf{1})^{-\frac{\alpha-1}{z}} h_\psi^{\alpha/2z}\bigr) = \mu_s(x_{\varepsilon'})$$

for all $s > 0$, so that

$$Q_{\alpha,z}(\psi\|\varphi+\varepsilon\tau) \ge Q_{\alpha,z}(\psi\|\varphi+\varepsilon'\tau).$$

Hence $\varepsilon > 0 \mapsto D_{\alpha,z}(\psi\|\varphi+\varepsilon\tau)$ is decreasing.

First, assume that $s(\psi) \not\le s(\varphi)$. Then $\mu_{s_0}(h_\psi^{\alpha/2z} s(\varphi)^\perp h_\psi^{\alpha/2z}) > 0$ for some $s_0 > 0$, since otherwise $h_\psi^{\alpha/2z} s(\varphi)^\perp h_\psi^{\alpha/2z} = 0$ so that $s(\psi) \le s(\varphi)$. Hence we have

$$\mu_s(x_\varepsilon) = \mu_s\bigl(h_\psi^{\alpha/2z}(h_\varphi+\varepsilon\mathbf{1})^{-\frac{\alpha-1}{z}} h_\psi^{\alpha/2z}\bigr) \ge \varepsilon^{-\frac{\alpha-1}{z}} \mu_s(h_\psi^{\alpha/2z} s(\varphi)^\perp h_\psi^{\alpha/2z}) \nearrow +\infty \quad \text{as } \varepsilon \searrow 0$$

for all $s \in (0, s_0]$. Therefore, it follows from (1.9) that $Q_{\alpha,z}(\psi\|\varphi+\varepsilon\tau) \nearrow +\infty = Q_{\alpha,z}(\psi\|\varphi)$.

Next, assume that $s(\psi) \le s(\varphi)$. Take the spectral decomposition $h_\varphi = \int_0^\infty t\,de_t$ and define $y, x \in \widetilde{\mathcal{M}}_+$ by

$$y := h_\varphi^{-\frac{\alpha-1}{z}} s(\varphi) = \int_{(0,\infty)} t^{-\frac{\alpha-1}{z}}\,de_t, \qquad x := y^{1/2} h_\psi^{\alpha/z} y^{1/2}.$$

Since

$$h_\psi^{\alpha/z} = s(\varphi) h_\psi^{\alpha/z} s(\varphi) = h_\varphi^{\frac{\alpha-1}{2z}} y^{1/2} h_\psi^{\alpha/z} y^{1/2} h_\varphi^{\frac{\alpha-1}{2z}} = h_\varphi^{\frac{\alpha-1}{2z}} x h_\varphi^{\frac{\alpha-1}{2z}},$$

the same reasoning in giving (1.9) shows that

$$Q_{\alpha,z}(\psi\|\varphi) = \tau(x^z) = \int_0^\infty \mu_s(x)^z\,ds. \tag{1.10}$$

We write

$$(h_\varphi+\varepsilon\mathbf{1})^{-\frac{\alpha-1}{z}}s(\varphi)=\int_{(0,\infty)}(t+\varepsilon)^{-\frac{\alpha-1}{z}}\,de_t,$$

and for any $\delta>0$ choose a $t_0>0$ such that $\tau(e_{(0,t_0)})<\delta$. Then, since

$$\int_{[t_0,\infty)}(t+\varepsilon)^{-\frac{\alpha-1}{z}}\,de_t\to\int_{[t_0,\infty)}t^{-\frac{\alpha-1}{z}}\,de_t\quad\text{in the operator norm as }\varepsilon\searrow 0,$$

we obtain $(h_\varphi+\varepsilon\mathbf{1})^{-\frac{\alpha-1}{z}}s(\varphi)\nearrow y$ in the measure topology, so that

$$h_\psi^{\alpha/2z}(h_\varphi+\varepsilon\mathbf{1})^{-\frac{\alpha-1}{z}}h_\psi^{\alpha/2z}\nearrow h_\psi^{\alpha/2z}yh_\psi^{\alpha/2z}\quad\text{in the measure topology as }\varepsilon\searrow 0.$$

Therefore, by [1, Lemma 3.4] we have

$$\mu_s(x_\varepsilon)=\mu_s\bigl(h_\psi^{\alpha/2z}(h_\varphi+\varepsilon\mathbf{1})^{-\frac{\alpha-1}{z}}h_\psi^{\alpha/2z}\bigr)\nearrow\mu_s(h_\psi^{\alpha/2z}yh_\psi^{\alpha/2z})=\mu_s(x)\tag{1.11}$$

for all $s>0$. It follows from (1.9)–(1.11) that (1.8) holds by the monotone convergence theorem. □

Lemma 1.2. *Let $(\mathcal{M},\tau)$ and ψ,φ be as above, and let $0<z\le z'$. Then*

$$D_{\alpha,z}(\psi\|\varphi)\le D_{\alpha,z'}(\psi\|\varphi)\quad\text{if }0<\alpha<1,\tag{1.12}$$

$$D_{\alpha,z}(\psi\|\varphi)\ge D_{\alpha,z'}(\psi\|\varphi)\quad\text{if }\alpha>1.\tag{1.13}$$

Proof. The case $0<\alpha<1$ in (1.12) has been shown in [11, Theorem 1(x)] for general von Neumann algebras.

By Lemma 1.1 and (1.9), for (1.13) it suffices to show that, for any $\varepsilon>0$,

$$\tau\Bigl(\Bigl[(h_\varphi+\varepsilon\mathbf{1})^{-\frac{\alpha-1}{2z}}h_\psi^{\alpha/z}(h_\varphi+\varepsilon\mathbf{1})^{-\frac{\alpha-1}{2z}}\Bigr]^z\Bigr)\ge\tau\Bigl(\Bigl[(h_\varphi+\varepsilon\mathbf{1})^{-\frac{\alpha-1}{2z'}}h_\psi^{\alpha/z'}(h_\varphi+\varepsilon\mathbf{1})^{-\frac{\alpha-1}{2z'}}\Bigr]^z\Bigr).$$

Letting $y:=(h_\varphi+\varepsilon\mathbf{1})^{-1}\in\mathcal{M}_+$ the above is equivalently written as

$$\tau\Bigl(\bigl|h_\psi^{\alpha/2z}y^{(\alpha-1)/2z}\bigr|^{2z}\Bigr)\ge\tau\Bigl(\bigl|h_\psi^{\alpha/2z'}y^{(\alpha-1)/2z'}\bigr|^{2z'}\Bigr),$$

that is,

$$\tau\Bigl(\bigl|(h_\psi^{\alpha/2z'})^r(y^{(\alpha-1)/2z'})^r\bigr|^{2z}\Bigr)\ge\tau\Bigl(\bigl|h_\psi^{\alpha/2z'}y^{(\alpha-1)/2z'}\bigr|^{2zr}\Bigr),$$

where $r:=z'/z\ge 1$. Hence the desired inequality follows from Kosaki's ALT inequality (see [13, Corollary 3]). □

When $(\mathcal{M},\tau)$ and ψ,φ are as in Lemma 1.1, one can define thanks to Lemma 1.2, for any $\alpha\in(0,\infty)\setminus\{1\}$,

$$Q_{\alpha,\infty}(\psi\|\varphi):=\lim_{z\to\infty}Q_{\alpha,\infty}(\psi\|\varphi)=\inf_{z>0}Q_{\alpha,z}(\psi\|\varphi),\tag{1.14}$$

$$\begin{aligned}D_{\alpha,\infty}(\psi\|\varphi)&:=\frac{1}{\alpha-1}\log\frac{Q_{\alpha,\infty}(\psi\|\varphi)}{\psi(\mathbf{1})}\\&=\lim_{z\to\infty}D_{\alpha,z}(\psi\|\varphi)=\begin{cases}\sup_{z>0}D_{\alpha,z}(\psi\|\varphi)&\text{if }0<\alpha<1,\\\inf_{z>0}D_{\alpha,z}(\psi\|\varphi)&\text{if }\alpha>1.\end{cases}\end{aligned}\tag{1.15}$$

If $h_\psi,h_\varphi\in\mathcal{M}_{++}$ (i.e., $\delta\tau\le\psi\le\delta^{-1}\tau$ and $\delta\tau\le\varphi\le\delta^{-1}\tau$ for some $\delta\in(0,1)$), then the Lie–Trotter formula gives

$$Q_{\alpha,\infty}(\psi\|\varphi)=\tau\bigl(\exp(\alpha\log h_\psi+(1-\alpha)\log h_\varphi)\bigr).\tag{1.16}$$

Lemma 1.3. *Let $(\mathcal{M},\tau)$ be as above. Then for every $\psi,\varphi\in\mathcal{M}_*^+$ with $\psi\ne 0$ and for any $z>0$,*

$$D_{\alpha,z}(\psi\|\varphi)\le D_1(\psi\|\varphi)\quad \text{if } 0<\alpha<1, \tag{1.17}$$
$$D_{\alpha,z}(\psi\|\varphi)\ge D_1(\psi\|\varphi)\quad \text{if } \alpha>1. \tag{1.18}$$

Proof. First, assume that $h_\psi,h_\varphi\in\mathcal{M}_{++}$. Set self-adjoint $H:=\log h_\psi$ and $K:=\log h_\varphi$ in $\mathcal{M}$ and define

$$F(\alpha):=\log\tau\bigl(e^{\alpha H+(1-\alpha)K}\bigr),\qquad \alpha>0.$$

Then by (1.16), $F(\alpha)=\log Q_{\alpha,\infty}(\psi\|\varphi)$ for all $\alpha\in(0,\infty)\setminus\{1\}$, and we compute

$$F'(\alpha)=\frac{\tau\bigl(e^{\alpha H+(1-\alpha)K}(H-K)\bigr)}{\tau\bigl(e^{\alpha H+(1-\alpha)K}\bigr)},$$
$$F''(\alpha)=\frac{\bigl[\tau\bigl(e^{\alpha H+(1-\alpha)K}(H-K)\bigr)\bigr]^2-\tau\bigl(e^{\alpha H+(1-\alpha)K}(H-K)^2\bigr)}{\bigl[\tau\bigl(e^{\alpha H+(1-\alpha)K}\bigr)\bigr]^2}.$$

Since $F''(\alpha)\ge 0$ on $(0,\infty)$ as verified by the Schwarz inequality, we see that $F(\alpha)$ is convex on $(0,\infty)$ and hence

$$D_{\alpha,\infty}(\psi\|\varphi)=\frac{F(\alpha)-F(1)}{\alpha-1}$$

is increasing in $\alpha\in(0,\infty)$, where for $\alpha=1$ the above RHS is understood as

$$F'(1)=\frac{\tau(e^H(H-K))}{\tau(e^H)}=\frac{\tau\bigl(h_\psi(\log h_\psi-\log h_\varphi)\bigr)}{\tau(h_\psi)}=D_1(\psi\|\varphi).$$

Hence the inequalities in (1.17) and (1.18) hold when $h_\psi,h_\varphi\in\mathcal{M}_{++}$. Below we extend these to general $\psi,\varphi\in\mathcal{M}_*^+$.

Case $0<\alpha<1$. Let $\psi,\varphi\in\mathcal{M}_*^+$ and $z>0$. From [11, Theorem 1(iv)] and [5, Corollary 2.8(3)] we have

$$D_{\alpha,z}(\psi\|\varphi)=\lim_{\varepsilon\searrow 0}D_{\alpha,z}(\psi+\varepsilon\tau\|\varphi+\varepsilon\tau),$$
$$D_1(\psi\|\varphi)=\lim_{\varepsilon\searrow 0}D_1(\psi+\varepsilon\tau\|\varphi+\varepsilon\tau),$$

so that we may assume that $\psi,\varphi\ge\varepsilon\tau$ for some $\varepsilon>0$. Take the spectral decompositions $\psi=\int_0^\infty t\,de_t^\psi$ and $h_\varphi=\int_0^\infty t\,de_t^\varphi$, and define $e_n:=e_n^\psi\wedge e_n^\varphi$ for each $n\in\mathbb{N}$. Then

$$\tau(e_n^\perp)\le\tau((e_n^\psi)^\perp)+\tau((e_n^\varphi)^\perp)\to 0\quad\text{as } n\to\infty,$$

so that $e_n\nearrow\mathbf{1}$. We set $\psi_n:=\psi(e_n\cdot e_n)$ and $\varphi_n:=\varphi(e_n\cdot e_n)$; then $h_{\psi_n}=e_nh_\psi e_n$ and $h_{\varphi_n}=e_nh_\varphi e_n$ are in $(e_n\mathcal{M}e_n)_{++}$. Note that

$$\begin{aligned}\|h_\psi-e_nh_\psi e_n\|_1&\le\|(\mathbf{1}-e_n)h_\psi\|_1+\|e_nh_\psi(\mathbf{1}-e_n)\|_1\\&\le\|(\mathbf{1}-e_n)h_\psi^{1/2}\|_2\|h_\psi^{1/2}\|_2+\|e_nh_\psi^{1/2}\|_2\|h_\psi^{1/2}(\mathbf{1}-e_n)\|_2\\&=\psi(\mathbf{1}-e_n)^{1/2}\psi(\mathbf{1})^{1/2}+\psi(e_n)^{1/2}\psi(\mathbf{1}-e_n)^{1/2}\to 0\quad\text{as } n\to\infty,\end{aligned}$$

and similarly $\|h_\varphi-e_nh_\varphi e_n\|_1\to 0$. Hence by [11, Theorem 1(iv)] one has $D_{\alpha,z}(e_n\psi e_n\|e_n\varphi e_n)\to D_{\alpha,z}(\psi\|\varphi)$. On the other hand, one has $D_1(e_n\psi e_n\|e_n\varphi e_n)\to D_1(\psi\|\varphi)$ by [5, Proposition 2.10]. Since $D_{\alpha,z}(e_n\psi e_n\|e_n\varphi e_n)\le D_1(e_n\psi e_n\|e_n\varphi e_n)$ holds by regarding $e_n\psi e_n,e_n\varphi e_n$

as functionals on the reduced von Neumann algebra $e_n\mathcal{M}e_n$, we obtain (1.17) for general $\psi,\varphi\in\mathcal{M}_*^+$.

Case $\alpha>1$. We extend (1.18) to general $\psi,\varphi\in\mathcal{M}_*^+$ by dividing four steps as follows, where $h_\psi=\int_0^\infty t\,e_t^\psi$ and $h_\varphi=\int_0^\infty t\,de_t^\varphi$ are the spectral decompositions.

(a) Assume that $h_\psi\in\mathcal{M}_+$ and $h_\varphi\in\mathcal{M}_{++}$. Set $\psi_n\in\mathcal{M}_*^+$ by $h_{\psi_n}=(1/n)e^\psi_{[0,1/n]}+\int_{(1/n,\infty)}t\,de_t^\psi$ $(\in\mathcal{M}_{++})$. Since $h_{\psi_n}^{\alpha/z}\searrow h_\psi^{\alpha/z}$ in the operator norem, we have by (1.5) and [1, Lemma 3.4]

$$\begin{aligned} Q_{\alpha,z}(\psi\|\varphi)&=\int_0^\infty \mu_s\bigl((h_\varphi^{-1})^{\frac{\alpha-1}{2z}}h_\psi^{\alpha/z}(h_\varphi^{-1})^{\frac{\alpha-1}{2z}}\bigr)^z\,ds\\ &=\lim_{n\to\infty}\int_0^\infty \mu_s\bigl((h_\varphi^{-1})^{\frac{\alpha-1}{2z}}h_{\psi_n}^{\alpha/z}(h_\varphi^{-1})^{\frac{\alpha-1}{2z}}\bigr)^z\,ds=\lim_{n\to\infty}Q_{\alpha,z}(\psi_n\|\varphi). \end{aligned}\tag{1.19}$$

By this and the lower semicontinuity of D, we see from the case $h_\psi,h_\varphi\in\mathcal{M}_{++}$ shown above that (1.18) holds also when $h_\psi\in\mathcal{M}_+$ and $h_\varphi\in\mathcal{M}_{++}$.

(b) Assume that $h_\psi\in\mathcal{M}_+$ and $h_\varphi\ge\delta\mathbf{1}$ for some $\delta>0$. Set $\varphi_n\in\mathcal{M}_*^+$ by $h_{\varphi_n}=\int_{[\delta,n]}t\,de_t^\varphi+ne^\varphi_{(n,\infty)}$ $(\in\mathcal{M}_{++})$. Since $h_{\varphi_n}^{-\frac{\alpha-1}{z}}\searrow h_\varphi^{-\frac{\alpha-1}{z}}$ in the operator norm, we have by (1.5) and [1, Lemma 3.4] again

$$\begin{aligned} Q_{\alpha,z}(\psi\|\varphi)&=\int_0^\infty \mu_s\bigl(h_\psi^{\alpha/2z}h_\varphi^{-\frac{\alpha-1}{z}}h_\psi^{\alpha/2z}\bigr)^z\,ds\\ &=\lim_{n\to\infty}\int_0^\infty \mu_s\bigl(h_\psi^{\alpha/2z}h_{\varphi_n}^{-\frac{\alpha-1}{z}}h_\psi^{\alpha/2z}\bigr)^z\,ds=\lim_{n\to\infty}Q_{\alpha,z}(\psi,\varphi_n). \end{aligned}$$

By this and the lower semicontinuity of D, it follows from (a) that (1.18) holds in this case too.

(c) Assume that ψ is general and $\varphi\ge\delta\tau$ for some $\delta>0$. Set $\psi_n\in\mathcal{M}_*^+$ by $h_{\psi_n}=\int_{[0,n]}t\,de_t^\psi+ne^\varphi_{(n,\infty)}$ $(\in\mathcal{M}_+)$. Since $h_{\psi_n}^{\alpha/z}\nearrow h_\psi^{\alpha/z}$ in the measure topology, one can argue as in (1.19), by use of the monotone convergence theorem, to see from (b) that (1.18) holds in this case too.

Finally, by Lemma 1.1 and [5, Corollary 2.8(3)], we see from (c) that (1.18) hods for general $\psi,\varphi\in\mathcal{M}_*^+$. □

In the next proposition, we summarize inequalities for $D_{\alpha,z}$ obtained so far in Lemmas 1.2 and 1.3.

Proposition 1.4. *Assume that $\mathcal{M}$ is a finite von Neumann algebra with a faithful normal finite trace τ. Then for every $\psi,\varphi\in\mathcal{M}_*^+$ with $\psi\ne0$, if $0<\alpha<1<\alpha'$ and $0<z\le z'\le\infty$, then*

$$D_{\alpha,z}(\psi\|\varphi)\le D_{\alpha,z'}(\psi\|\varphi)\le D_1(\psi\|\varphi)\le D_{\alpha',z'}(\psi\|\varphi)\le D_{\alpha',z}(\psi\|\varphi).$$

Corollary 1.5. *Let $(\mathcal{M},\tau)$ and ψ,φ be as in Proposition 1.4. Then for any $z\in[1,\infty]$,*

$$\lim_{\alpha\nearrow1}D_{\alpha,z}(\psi\|\varphi)=D_1(\psi\|\varphi).\tag{1.20}$$

Moreover, if $D_{\alpha,\alpha}(\psi\|\varphi)<+\infty$ for some $\alpha>1$ then for any $z\in(1,\infty]$,

$$\lim_{\alpha\searrow1}D_{\alpha,z}(\psi\|\varphi)=D_1(\psi\|\varphi).\tag{1.21}$$

Proof. Let $z \geq 1$. For every $\alpha \in (0,1)$, Proposition 1.4 gives

$$D_{\alpha,1}(\psi\|\varphi) \leq D_{\alpha,z}(\psi\|\varphi) \leq D_1(\psi\|\varphi).$$

Hence (1.20) follows since it holds for $D_{\alpha,1}$ (see [4, Proposition 5.3(3)]).

Next, assume that $D_{\alpha,\alpha}(\psi\|\varphi) < +\infty$ for some $\alpha > 1$. Let $z > 1$. For every $\alpha \in (1,z]$, Proposition 1.4 gives

$$D_1(\psi\|\varphi) \leq D_{\alpha,z}(\psi\|\varphi) \leq D_{\alpha,\alpha}(\psi\|\varphi).$$

Hence (1.21) follows since $\lim_{\alpha\searrow 1}(\psi\|\varphi) = D_1(\psi\|\varphi)$ holds (see [9, Proposition 3.8(ii)]). $\square$

Question 1.6. In the finite-dimensional setting, it is known (see [14, (II.16) and Proposition III.36]) that $\lim_{\alpha\to 1} D_{\alpha,z}(\rho\|\sigma) = D_1(\rho\|\sigma)$ for any positive semi-definite matrices ρ, σ and for every $z \in (0,\infty]$. So it is natural to conjecture that (1.20) and (1.21) hold for any $z \in (0,\infty]$. The differential method in the proof of [14, Proposition III.36] for matrices does not seem easy to deal with due to the unboundedness of h_ψ, h_φ, even though we have, for every τ-measurable operator $h \geq 0$,

$$\frac{d}{d\alpha}\, h^\alpha = h^\alpha \log h \quad \text{in the measure topology}, \qquad \alpha > 0,$$

and

$$\frac{d}{d\alpha}\, h^\alpha\Big|_{\alpha=0} = \log h \quad \text{in the measure topology}$$

if h is non-singular, as seen similarly to the proof of [6, Lemma 9.19].

Furthermore, it would be desirable to replace the assumption for the latter assertion of Corollary 1.5 with the assumption $D_{\alpha,z}(\psi\|\varphi) < +\infty$ for some $\alpha > 1$.

2 General von Neumann algebra case

In this section let $\mathcal{M}$ be a σ-finite general von Neumann algebra. Let us recall Haagerup's reduction theorem, which was presented in [3, Sec. 2] (a compact survey is also found in [2, Sec. 2.5]). Let ω be a faithful normal state of $\mathcal{M}$ and σ_t^ω ($t \in \mathbb{R}$) be the associated modular automorphism group. Consider the discrete additive group $G := \bigcup_{n\in\mathbb{N}} 2^{-n}\mathbb{Z}$ and define $\hat{\mathcal{M}} := \mathcal{M} \rtimes_{\sigma^\omega} G$, the crossed product of $\mathcal{M}$ by the action $\sigma^\omega|_G$. Then the dual weight $\hat\omega$ is a faithful normal state of $\hat{\mathcal{M}}$, and we have $\hat\omega = \omega \circ E_{\mathcal{M}}$, where $E_{\mathcal{M}} : \hat{\mathcal{M}} \to \mathcal{M}$ is the canonical conditional expectation (see, e.g., [6, Sec. 8.1], also [2, Sec. 2.5]).

Haagerup's reduction theorem is presented as follows:

Theorem 2.1 ([3]). *In the above setting, there exists an increasing sequence $\{\mathcal{M}_n\}_{n\geq 1}$ of von Neumann subalgebras of $\hat{\mathcal{M}}$, containing the unit of $\hat{\mathcal{M}}$, such that the following hold:*

(i) Each $\mathcal{M}_n$ is finite with a faithful normal tracial state τ_n.

(ii) $\big(\bigcup_{n\geq 1} \mathcal{M}_n\big)'' = \hat{\mathcal{M}}$.

(iii) For every n there exist a (unique) faithful normal conditional expectation $E_{\mathcal{M}_n} : \hat{\mathcal{M}} \to \mathcal{M}_n$ satisfying

$$\hat\omega \circ E_{\mathcal{M}_n} = \hat\omega, \qquad \sigma_t^{\hat\omega} \circ E_{\mathcal{M}_n} = E_{\mathcal{M}_n} \circ \sigma_t^{\hat\omega}, \quad t \in \mathbb{R}.$$

Moreover, for any $x \in \hat{\mathcal{M}}$, $E_{\mathcal{M}_n}(x) \to x$ strongly.

Furthermore, for any $\psi \in \mathcal{M}_*^+$, if we define $\hat{\psi} := \psi \circ E_{\mathcal{M}}$ then $\hat{\psi} \circ E_{\mathcal{M}_n} \to \hat{\psi}$ in the norm. (It is written in [2, Sec. 2.5] that this was proved in [3, Theorem 3.1]. However I can't see it there, though it follows from [8, Theorem 4] anyway.)

The next lemma will be useful in our discussions below.

Lemma 2.2. *In the above situation, for any $\psi, \varphi \in \mathcal{M}_*^+$ with $\psi \neq 0$ let $\hat{\psi} := \psi \circ E_{\mathcal{M}}$ and $\hat{\varphi} := \varphi \circ E_{\mathcal{M}}$ as above. Then for any $\alpha, z > 0$ with $\alpha \neq 1$ such that either*

$$\alpha \in (0,1), \qquad z \geq \max\{\alpha, 1-\alpha\},$$

or

$$\alpha > 1, \qquad \max\{\alpha/2, \alpha-1\} \leq z \leq \alpha,$$

we have

$$D_{\alpha,z}(\psi\|\varphi) = D_{\alpha,z}(\hat{\psi}\|\hat{\varphi}) = \lim_{n\to\infty} D_{\alpha,z}(\hat{\psi}|_{\mathcal{M}_n}\|\hat{\varphi}|_{\mathcal{M}_n}) \quad \textit{increasingly,} \tag{2.1}$$

as well as

$$D_1(\psi\|\varphi) = D_1(\hat{\psi}\|\hat{\varphi}) = \lim_{n\to\infty} D_1(\hat{\psi}|_{\mathcal{M}_n}\|\hat{\varphi}|_{\mathcal{M}_n}) \quad \textit{increasingly.} \tag{2.2}$$

Proof. Thanks to the DPI for $D_{\alpha,z}$ proved in [11, 10] applied to the injection $\mathcal{M} \hookrightarrow \hat{\mathcal{M}}$ and the conditional expectation $E_{\mathcal{M}} : \hat{\mathcal{M}} \to \mathcal{M}$ we have the first equality of (2.1). By [7, Theorem 0.1] we have the latter equality with increasing convergence in (2.1). The assertion of D_1 in (2.2) is included in [2, Proposition 2.2], while this is the well-known martingale convergence of the relative entropy (see [12]). $\square$

From Proposition 1.4 and Lemma 2.2 we have:

Proposition 2.3. *For every $\psi, \varphi \in \mathcal{M}_*^+$ the following hold:*

(1) If $0 < \alpha < 1$ and $\max\{\alpha, 1-\alpha\} \leq z \leq z'$,

$$D_{\alpha,z}(\psi\|\varphi) \leq D_{\alpha,z'}(\psi\|\varphi) \leq D_1(\psi\|\varphi).$$

(2) If $\alpha > 1$ and $\max\{\alpha/2, \alpha-1\} \leq z \leq z' \leq \alpha$,

$$D_1(\psi\|\varphi) \leq D_{\alpha,z'}(\psi\|\varphi) \leq D_{\alpha,z}(\psi\|\varphi).$$

Question 2.4. It should be conjectured that the monotonicity of $z \mapsto D_{\alpha,z}(\psi\|\varphi)$ holds true in general for all $z > 0$, although our proof using the martingale convergence in Lemma 2.2 is limited the parameter z to the DPI range for $D_{\alpha,z}$.

The next corollary follows from Proposition 2.3 as Corollary 1.5 has been shown from Proposition 1.4.

Corollary 2.5. *For any $z \geq 1$,*

$$\lim_{\alpha \nearrow 1} D_{\alpha,z}(\psi\|\varphi) = D_1(\psi\|\varphi). \tag{2.3}$$

Moreover, if $D_{\alpha,\alpha}(\psi\|\varphi) < +\infty$ for some $\alpha > 1$ then for any $z \in (1, \alpha]$,

$$\lim_{\alpha \searrow 1} D_{\alpha,z}(\psi\|\varphi) = D_1(\psi\|\varphi). \tag{2.4}$$

Again, we may conjecture that (2.3) and (2.4) hold true for general $z > 0$.

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