INTEGRAL FORMULA FOR QUANTUM RELATIVE ENTROPY IMPLIES DATA PROCESSING INEQUALITY

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Abstract. Integral representations of quantum relative entropy, and of the directional second and higher order derivatives of von Neumann entropy, are established, and used to give simple proofs of fundamental, known data processing inequalities: the Holevo bound on the quantity of information transmitted by a quantum communication channel, and, much more generally, the monotonicity of quantum relative entropy under trace-preserving positive linear maps — complete positivity of the map need not be assumed. The latter result was first proved by Müller-Hermes and Reeb, based on work of Beigi.

In the last section, we consider any 'divergence' that is non-increasing under quantum measurements, such as the concavity of von Neumann entropy, or various known quantum divergences. An elegant argument due to Hiai, Ohya, and Tsukada is used to show that the infimum of such a 'divergence' on pairs of quantum states with prescribed trace distance is the same as the corresponding infimum on pairs of binary classical states.

1. Introduction

Half a century ago, Alexander Holevo proved his famous inequality: the quantity of information transmitted by a quantum communication channel using a given ensemble of quantum states is bounded from above by the extent to which von Neumann entropy is concave on the ensemble. One of the main ingredients in Holevo's proof is an explicit, closed formula for the directional second derivative S'' of the von Neumann entropy.

Since that time, the Holevo bound has become an important building block of the vast theory of quantum information. Generalizations and alternative proofs, often using advanced methods of that theory, have been given. Almost simultaneously with Holevo's work, Elliott Lieb and Mary Beth Ruskai [8] established the strong subadditivity of von Neumann entropy, which quickly led to Göran Lindblad's proof [9] of monotonicity of quantum relative entropy under completely positive trace-preserving linear maps — a generalization of Holevo's inequality. Much later, a further generalization was proved by A. Müller-Hermes and D. Reeb [10], based on work of S. Beigi [2]: quantum relative entropy cannot increase under a trace-preserving positive linear map — complete positivity of the map need not be assumed.

Key words and phrases. Quantum relative entropy, Data processing inequality, Holevo bound, von Neumann entropy, concavity.

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In this paper, we return to more classical methods of analysis and linear algebra. In Section 4, we prove the alternative formula

$$-\frac{1}{2}S(\rho+t\sigma)''(0)=\int_{-\infty}^{\infty}\frac{dt}{|t|^3}\operatorname{tr}^-(\rho+t\sigma)$$

for the directional second derivative of von Neumann entropy, which then leads to similar formulas for directional derivatives of higher order. Note that tr^- stands for the sum of absolute values of negative eigenvalues.

Before that, in Section 3, we establish a similar formula for the quantum relative entropy. The simplest form of this formula is

$$D(\rho\|\sigma)=\int_{-\infty}^{\infty}\frac{dt}{|t|(t-1)^2}\operatorname{tr}^-((1-t)\rho+t\sigma),$$

which holds for any two quantum states.

We then show in Section 5 that our formulae lead to the above mentioned monotonicity of quantum relative entropy and to the Holevo inequality in a very simple way.

Finally, in Section 6, we consider any 'divergence-like' quantity that is non-increasing under quantum measurements, such as the concavity of von Neumann entropy, sandwiched Rényi divergences, or the more general optimized quantum f-divergences. We use a simple construction due to Hiai, Ohya, and Tsukada [3] from the year 1981 to show that the infimum of such a 'divergence' on pairs with prescribed trace distance is the same for (arbitrary dimensional) quantum states as for binary classical states. For example, the Holevo inequality is used to obtain a new, tight lower bound on the concavity of von Neumann entropy, improving on a lower bound given by Isaac Kim in 2014.

Notations and terminology. We write log for the natural logarithm. Partial derivatives will be denoted by putting the corresponding variable in the subscript. A $'$ means differentiation with respect to t.

The set of n-square matrices with complex entries is written $M_n(\mathbb{C})$. The identity matrix is $\mathbf{1}$. A complex matrix A is *psdh* if it is positive semi-definite Hermitian, written $A\geq 0$. For a Hermitian matrix A, we write $A=A^+-A^-$, where $A^\pm\geq 0$ and $A^+A^-=A^-A^+=0$. We define $|A|=A^++A^-$. We write $\operatorname{tr}^\pm A=\operatorname{tr}A^\pm$ and $\|A\|_1=\operatorname{tr}|A|$ for the sum of absolute values of positive/negative eigenvalues and all eigenvalues of A, respectively.

A *density matrix*, or *quantum state*, is a psdh matrix with trace 1. The *von Neumann entropy* of a psdh matrix ρ (of arbitrary trace) is $S(\rho)=-\operatorname{tr}\rho\log\rho$. The *quantum relative entropy* (Umegaki [11]) of two psdh matrices ρ and σ (of arbitrary trace) is

$$D(\rho\|\sigma)=\begin{cases}\operatorname{tr}\rho(\log\rho-\log\sigma) \text{ if } \operatorname{im}\rho\subseteq\operatorname{im}\sigma\\ +\infty \text{ otherwise.}\end{cases}$$

2. Preliminaries

2.1. **Derivative of matrix logarithm.** Let $X(t)$ be a differentiable curve whose values are positive definite matrices. By [1, formula (7)], we have

$$(\log X)' = \int_0^\infty (X + r\mathbf{1})^{-1} X' (X + r\mathbf{1})^{-1}\, dr.$$

From this, we infer

Lemma 1. *If, for a given t, $X(t)$ commutes with a matrix Y, then*

$$\operatorname{tr} Y (\log X)'(t) = \operatorname{tr} Y X'(t) X(t)^{-1}.$$

2.2. **Eigenvalues of matrix pencils.** It will be useful to study the negative real eigenvalues of the linear matrix pencil $A(t) = (1-t)\rho + t\sigma$, where $A(0) = \rho \geq 0$ and $A(1) = \sigma$ is Hermitian. Firstly, we show that negative real eigenvalues of $A(t)$ can only occur for real t.

Lemma 2. *If $A(t)e = -re$ for a unit vector e and a positive real number r, then $t((\rho - \sigma)e, e) > 0$, and therefore t is real.*

Proof. We have $-r = (A(t)e, e) = (1-t)(\rho e, e) + t(\sigma e, e)$, whence $t((\rho - \sigma)e, e) = (\rho e, e) + r > 0$. □

Define the bivariate polynomial

$$f(t, r) = \det(A(t) + r\mathbf{1}). \tag{1}$$

We have $f(t, r) = 0$ if and only if $-r$ is an eigenvalue of $A(t)$. In this case, the ratio of partial derivatives is given by

Lemma 3. *If $A(t)e = -re$ for a unit vector e, then*

$$f_t(t, r) = ((\sigma - \rho)e, e) f_r(t, r).$$

Proof. The equality is easily seen after including e in an orthonormal eigenbasis of $A(t)$. □

These two lemmas imply that any negative simple eigenvalue of $A(t)$ gets more negative as t moves farther away from zero. More precisely, we have

Corollary 4. *If $f(t, r) = 0$ and $r > 0$, then t is real, and*

$$t f_r(t, r) f_t(t, r) \leq 0.$$

Equality holds if and only if $f_r(t, r) = 0$, i.e., if $-r$ is a multiple eigenvalue of $A(t)$.

We have $f(t, r) = f_r(t, r) = 0$ if and only if $-r$ is a multiple eigenvalue of $A(t)$. As a polynomial in r, f has a discriminant whose value is a polynomial in t. The discriminant is zero for a given value of t if and only if the matrix $A(t)$ has a multiple eigenvalue. This happens either for finitely many t or for all t.

3. Quantum relative entropy

Let ρ and σ be psdh matrices. We wish to prove an integral formula for the quantum relative entropy $D(\rho\|\sigma)$. In the first part of the proof, we join the pair (ρ,σ) to infinity in the direction of the identity matrix. This is done in the following two lemmas.

Lemma 5. $\lim_{r\to\infty} D(\rho+r\mathbf{1}\|\sigma+r\mathbf{1}) = \operatorname{tr}(\rho-\sigma)$.

Proof. $D(\rho+r\mathbf{1}\|\sigma+r\mathbf{1}) = rD(\mathbf{1}+\rho/r\|\mathbf{1}+\sigma/r) \sim r\operatorname{tr}(\rho-\sigma)/r$. □

We set $A(t)=(1-t)\rho+t\sigma$ for this section. The notation (1) introduced in Subsection 2.2 will be used.

Lemma 6. (a) *For all $r>0$, we have*

$$\frac{d}{dr}D(\rho+r\mathbf{1}\|\sigma+r\mathbf{1}) =$$
$$= \operatorname{tr}\log(\rho+r\mathbf{1}) - \operatorname{tr}\log(\sigma+r\mathbf{1}) + \operatorname{tr}(\sigma-\rho)(\sigma+r\mathbf{1})^{-1} =$$
$$= \log f(0,r) - \log f(1,r) + (\log f)'(1,r)$$

(b) *If $\operatorname{im}\rho\subseteq\operatorname{im}\sigma$, then the expression above is $o(1/r)$ as $r\to 0$ or $r\to\infty$.*

Proof. (a) On the left hand side, we differentiate the product in the argument of tr to get

$$\operatorname{tr}\left(\log(\rho+r\mathbf{1}) - \log(\sigma+r\mathbf{1}) + (\rho+r\mathbf{1})\left((\rho+r\mathbf{1})^{-1}-(\sigma+r\mathbf{1})^{-1}\right)\right).$$

Apply the identity

$$\mathbf{1}-(\rho+r\mathbf{1})(\sigma+r\mathbf{1})^{-1} = (\sigma-\rho)(\sigma+r\mathbf{1})^{-1}$$

to arrive at the middle expression in the Lemma. Then use $\sigma-\rho=A'(1)$, $\sigma=A(1)$,

$$\operatorname{tr}A'(1)(A(1)+r\mathbf{1})^{-1} = \operatorname{tr}(\log(A+r\mathbf{1}))'(1)$$

and finally $\operatorname{tr}\log=\log\det$ to get to the bottom line.

(b) When $r\to 0$, the first two terms are $O(\log(1/r))$ and the third term is $O(1)$. When $r\to\infty$, the third term is $\sim\operatorname{tr}(\sigma-\rho)/r$, and the sum of the first two terms is $\operatorname{tr}\log(\mathbf{1}+\rho/r)-\operatorname{tr}\log(\mathbf{1}+\sigma/r)\sim\operatorname{tr}(\rho-\sigma)/r$. □

From these two lemmas, we have

$$D(\rho\|\sigma)+\operatorname{tr}(\sigma-\rho) =$$
$$= -\int_0^\infty \left(\log f(0,r)-\log f(1,r)+(\log f)'(1,r)\right)\,dr =$$
$$= \int_0^\infty r\left((\log f)_r(0,r)-(\log f)_r(1,r)+(\log f)'_r(1,r)\right)\,dr$$

if $\operatorname{im}\rho\subseteq\operatorname{im}\sigma$. Let

$$g(t)=\frac{1}{t}-\frac{1}{t-1}+\frac{1}{(t-1)^2}=\frac{1}{t(t-1)^2},$$

then

$$D(\rho\|\sigma)+\operatorname{tr}(\sigma-\rho) = \int_0^\infty r\cdot(\operatorname{Res}_{t=0}+\operatorname{Res}_{t=1})(g\cdot(\log f)_r)\,dr. \tag{2}$$

For a fixed $r > 0$, observe that $g \cdot (\log f)_r = gf_r/f$, as a rational function of t in the complex plane, is holomorphic except where $t = 0$, $t = 1$, or $f = 0$. The latter case occurs if and only if $-r$ is an eigenvalue of $A(t)$. If $-r$ is a simple eigenvalue of $A(t)$, then $f_t(t, r) \neq 0$ by Corollary 4, and the residue of gf_r/f at t is gf_r/f_t. By the Residue Theorem, the sum of all residues is zero because the function is $O\left(|t|^{-3}\right)$ as $t \to \infty$, so the contour integrals on circles $|t| = T$ tend to zero as $T \to \infty$. The right hand side of (2) therefore becomes

$$-\int_0^\infty r \sum_{f(t,r)=0} \frac{gf_r}{f_t}\, dr = \int_{-\infty}^\infty \frac{dt}{|t|(t-1)^2} \sum_{f(t,r)=0} r^+,$$

at least if not all $A(t)$ have multiple eigenvalues. Indeed, in this case only finitely many $A(t)$ have multiple eigenvalues, and for a simple negative eigenvalue $-r$, Corollary 4 implies that $t \in \mathbb{R}$, $f_r(t, r) \neq 0$, and $|dr/dt| = |f_t/f_r| = -(\operatorname{sgn} t) f_t/f_r$ as we move along the algebraic plane curve $f = 0$.

The last sum that has appeared is $\operatorname{tr}^- A(t)$. We conclude:

Theorem 7. *Let $\rho, \sigma \in M_n(\mathbb{C})$ be psdh matrices. Then*

$$D(\rho\|\sigma) = \operatorname{tr}(\rho - \sigma) + \int_{-\infty}^\infty \frac{dt}{|t|(t-1)^2} \operatorname{tr}^- A(t) =$$

$$= \int_{-\infty}^0 \frac{dt}{|t|(|t|+1)^2} (\operatorname{tr}^+ A(t) - \operatorname{tr}\rho) + \int_1^\infty \frac{dt}{t(t-1)^2} \operatorname{tr}^- A(t),$$

where $A(t) = (1-t)\rho + t\sigma$, and $\operatorname{tr}^\pm$ stands for the sum of absolute values of positive and negative eigenvalues, respectively.

Proof. First equality: Both sides are $+\infty$ unless $\operatorname{im}\rho \subseteq \operatorname{im}\sigma$, which we henceforth assume. Restricting our attention to the image of σ, we may assume that σ is positive definite to begin with. Since both sides are then continuous, we may change σ a little bit so that it has no multiple eigenvalues. The preceding discussion then applies and the first equality is proved.

Second equality: For $0 \leq t \leq 1$, we have $A(t) \geq 0$ and therefore $\operatorname{tr}^- A(t) = 0$. For $t < 0$, observe that

$$\operatorname{tr}^+ A(t) - \operatorname{tr}\rho = \operatorname{tr}^- A(t) + |t| \operatorname{tr}(\rho - \sigma).$$

Then use $\int_{-\infty}^0 dt/(t-1)^2 = 1$ to conclude. □

From Theorem 7, we immediately recover the well-known fact that $D(\rho\|\sigma)$ is a convex function of the pair (ρ, σ), and it is nonnegative whenever $\operatorname{tr}\rho \geq \operatorname{tr}\sigma$.

4. Higher order derivatives of von Neumann entropy

In this section, ρ is a psdh matrix and σ is a Hermitian matrix with $\operatorname{im}\sigma \subseteq \operatorname{im}\rho$. We wish to find an integral formula for $S(\rho + t\sigma)^{(m)}(0)$ when $m \geq 2$. When $m = 2$, $\operatorname{tr}\rho = 1$, and $\operatorname{tr}\sigma = 0$, an explicit formula for this quantity, in terms of the spectral decomposition of ρ, has been given by A. S. Holevo in his seminal paper [4, Lemma 4]. The fact that our integral formula yields the same value seems non-obvious. The proof given below does not rely on Holevo's explicit formula.

Theorem 8. *Let $\rho, \sigma \in M_n(\mathbb{C})$ with $\rho \geq 0$, $\sigma^* = \sigma$, and $\operatorname{im}\sigma \subseteq \operatorname{im}\rho$.*

(a) *For all $m \geq 2$, we have*

$$-\frac{1}{m!} S(\rho + t\sigma)^{(m)}(0) = \int_{-\infty}^{\infty} \frac{dt}{|t|t^m} \operatorname{tr}^-(\rho + t\sigma), \tag{3}$$

where tr^- stands for the sum of absolute values of negative eigenvalues.

(b) *When $m \geq 2$ is even, the quantity (3) is nonnegative and convex as a function of the pair (ρ, σ).*

Proof. (a) Case $m = 2$: We have

$$-S(\rho + t\sigma)''(0) = \lim_{t \to 0} \frac{1}{t^2} \left(2S(\rho) - S(\rho + t\sigma) - S(\rho - t\sigma)\right).$$

In the parentheses here, we have

$$D(\rho + t\sigma \| \rho) + D(\rho - t\sigma \| \rho).$$

By Theorem 7, we have

$$D(\rho \pm t\sigma \| \rho) \mp t \operatorname{tr}\sigma = \int_{-\infty}^{\infty} \frac{ds}{|s|(s-1)^2} \operatorname{tr}^-(\rho \pm (1-s)t\sigma).$$

Putting $u = (1-s)t$, i.e. substituting $s = 1 - u/t$, this becomes

$$t^2 \int_{-\infty}^{\infty} \frac{du}{|t-u|u^2} \operatorname{tr}^-(\rho \pm u\sigma) \sim t^2 \int_{-\infty}^{\infty} \frac{du}{|u|^3} \operatorname{tr}^-(\rho + u\sigma)$$

as $t \to 0$ by Lebesgue's Dominated Convergence Theorem. Case $m = 2$ follows.

If the statement holds for m, then

$$\begin{aligned}-\frac{1}{m!} S(\rho + t\sigma)^{(m)}(u) &= \int_{-\infty}^{\infty} \frac{dt}{|t|t^m} \operatorname{tr}^-(\rho + (t+u)\sigma) = \\ &= \int_{-\infty}^{\infty} \frac{dt}{|t-u|(t-u)^m} \operatorname{tr}^-(\rho + t\sigma)\end{aligned}$$

for small $|u|$. Use Lebesgue's Theorem to differentiate w.r.t. u at 0 under the integral sign, and get Theorem 8(a) for $m + 1$.

(b) Observe that $\operatorname{tr}^-(\rho + t\sigma)$ is nonnegative and convex as a function of the pair (ρ, σ). For m even, $|t|t^m \geq 0$ for all t. □

5. Data processing inequalities

Let $\mathcal{E} : M_n(\mathbb{C}) \to M_{n'}(\mathbb{C})$ be a trace-nonincreasing positive linear map. *Positivity* means that psdh matrices are taken to psdh matrices (and therefore Hermitian matrices are taken to Hermitian matrices). *Trace-nonincreasing* means that $\operatorname{tr}\mathcal{E}A \leq \operatorname{tr}A$ for all $A \geq 0$. An important example is given by a *positive operator valued measure*, or *partition of unity*: psdh matrices $E_1, \ldots, E_k$ summing to $\mathbf{1}$, which give rise to a completely positive, trace-preserving linear map, the *quantum measurement*

$$\mathcal{E} : A \mapsto \operatorname{diag}(\operatorname{tr}E_1A, \ldots, \operatorname{tr}E_kA).$$

Lemma 9. (a) *For any trace-nonincreasing positive linear map $\mathcal{E}$ and any Hermitian matrix A, we have $\operatorname{tr}^\pm \mathcal{E}A \leq \operatorname{tr}^\pm A$.*

(b) *Equality holds in the statement (a) if and only if* $\operatorname{tr} \mathcal{E}A^{\pm} = \operatorname{tr} A^{\pm}$ *and* $\mathcal{E}A^{+}\mathcal{E}A^{-} = 0$.

(c) *For a quantum measurement, the condition of equality in (a) is that for all i, we have $E_iA^{+} = 0$ or $E_iA^{-} = 0$.*

Proof. It suffices to treat the + case because passing from A to $-A$ interchanges tr^{+} and tr^{-} as well as A^{+} and A^{-}.

(a), (b) We have

$$\operatorname{tr}^{+} \mathcal{E}A = \operatorname{tr}^{+} \mathcal{E}(A^{+} - A^{-}) = \operatorname{tr}^{+}(\mathcal{E}A^{+} - \mathcal{E}A^{-}) \leq$$
$$\leq \operatorname{tr}^{+} \mathcal{E}A^{+} = \operatorname{tr} \mathcal{E}A^{+} \leq \operatorname{tr} A^{+} = \operatorname{tr}^{+} A.$$

(c) For a quantum measurement, the condition of equality is that there be no i with $\operatorname{tr} E_iA^{\pm} > 0$ for both signs. □

5.1. **Quantum relative entropy.** From Theorem 7 and Lemma 9, we recover (for the finite-dimensional case) the **data processing inequality**

$$D(\mathcal{E}\rho\|\mathcal{E}\sigma) \leq D(\rho\|\sigma) \tag{4}$$

for any trace-nonincreasing positive linear map $\mathcal{E}$ and any psdh matrices ρ and σ such that $\operatorname{tr} \mathcal{E}\rho = \operatorname{tr} \rho$. Note that *complete* positivity of $\mathcal{E}$ is not assumed. Equality holds in (4) if and only if every affine combination

$$A = A(t) = (1 - t)\rho + t\sigma$$

of ρ and σ has $\mathcal{E}A^{+}\mathcal{E}A^{-} = 0$, and $\operatorname{tr} \mathcal{E}A(t)^{\pm} = \operatorname{tr} A(t)^{\pm}$ whenever $\pm t < 0$. For a quantum measurement, the condition of equality is that for all i and all affine combinations A, we should have $E_iA^{+} = 0$ or $E_iA^{-} = 0$.

The inequality (4), in this generality, was first proved by A. Müller-Hermes and D. Reeb [10]. They also covered the infinite-dimensional case. Their approach was based on the work of S. Beigi [2] establishing the data processing inequality for sandwiched Rényi divergences, with respect to quantum channels (completely positive trace-preserving linear maps).

5.2. **The Holevo bound.** In [4], A. S. Holevo used his explicit formula for S'' to prove his celebrated upper bound on the quantity of information transmitted by a quantum communication channel. We shall now show how Theorem 8 quickly leads to a generalization of the same bound, which, however, also follows from (4).

Let $\mathcal{E}$ be a trace-nonincreasing positive linear map. From Theorem 8 and Lemma 9, we see that

$$-S^{(m)}(\mathcal{E}\rho + t\mathcal{E}\sigma)(0) \leq -S^{(m)}(\rho + t\sigma)(0)$$

for any psdh matrix ρ, any Hermitian matrix σ satisfying $\operatorname{im} \sigma \subseteq \operatorname{im} \rho$, and any even $m \geq 2$. We have equality if and only if every combination

$$A = A(t) = \rho + t\sigma$$

has $\mathcal{E}A^{+}\mathcal{E}A^{-} = 0$ and $\operatorname{tr} \mathcal{E}A^{-} = \operatorname{tr} A^{-}$. For a quantum measurement $\mathcal{E}$, the condition of equality is that for all i and t we should have $E_iA(t)^{+} = 0$ or $E_iA(t)^{-} = 0$.

In particular, $S - S \circ \mathcal{E}$ is a concave function on psdh matrices.

Define the *Holevo quantity*

$$\chi(\rho_1, \ldots, \rho_l; q_1, \ldots, q_l) := S\left(\sum_{j=1}^{l} q_j\rho_j\right) - \sum_{j=1}^{l} q_j S(\rho_j).$$

From Theorem 8(b), we recover the well-known fact that the Holevo quantity is nonnegative and convex as a function of $(\rho_1, \ldots, \rho_l)$.

By Jensen's inequality, for any psdh matrices $\rho_1, \ldots, \rho_l$, and any weights $q_1, \ldots, q_l > 0$ summing to 1, we have

$$\chi(\mathcal{E}\rho_1, \ldots, \mathcal{E}\rho_l; q_1, \ldots, q_l) \le \chi(\rho_1, \ldots, \rho_l; q_1, \ldots, q_l), \tag{5}$$

with equality if and only if $\mathcal{E}A^+\mathcal{E}A^- = 0$ and $\operatorname{tr}\mathcal{E}A^- = \operatorname{tr}A^-$ for every affine combination A of $\rho_1, \ldots, \rho_l$.

In words: the Holevo quantity is non-increasing under trace-preserving positive linear maps. Note that complete positivity of the map need not be assumed.

When $\mathcal{E}$ is a quantum measurement, and each ρ_j has trace 1, (5) is Holevo's inequality. The left hand side is the *mutual information* between the random input j (whose distribution is given by the probabilities q_j) and the measurement output i (whose conditional distribution is given by the conditional probabilities $\operatorname{tr}E_i\rho_j$ once j has occurred). We have equality in (5) if and only if for all i and all affine combinations A of the ρ_j, we have $E_iA^+ = 0$ or $E_iA^- = 0$.

6. Lower bounds on generalized divergences

It was shown by F. Hiai, M. Ohya, and M. Tsukada [3] that the minimum of the quantum relative entropy for two quantum states with prescribed trace distance is attained on binary classical states. In this section, we shall use their method prove the analogous result for any quantity that depends on two quantum states and is non-increasing under quantum measurements. As examples of such quantities, we have already discussed in Section 5 the quantum relative entropy and the concavity of von Neumann entropy, but the sandwiched Rényi divergence with parameter $\alpha > 1$ is also non-increasing, not just under quantum measurements, but under quantum channels (completely positive trace-preserving maps), as was shown by S. Beigi [2]. More generally, M. M. Wilde [12] proved the same for optimized quantum f-divergences. An alternative proof (with respect to trace-preserving positive linear maps satisfying a certain Schwarz-type inequality) was given by H. Li [7].

Let ρ_0 and ρ_1 be density matrices of dimension ≥ 2. Set

$$\rho_1 - \rho_0 = \|\rho_1 - \rho_0\|_1\sigma,$$

where σ is traceless Hermitian with 1-norm 1. Let $\mathbb{C}^n = V_+ \oplus V_-$ be an orthogonal decomposition with $\sigma V_\pm \subseteq V_\pm$ and $\pm\sigma \ge 0$ on $V_\pm$. Let $E_\pm$ be the orthogonal projection onto $V_\pm$, and let $\mathcal{E}$ be the quantum measurement given by these two projections. Then, for any density matrix ρ, we have $\mathcal{E}\rho = \operatorname{diag}(\operatorname{tr}E_+\rho, \operatorname{tr}E_-\rho)$, whence

$$\|\mathcal{E}\rho_1 - \mathcal{E}\rho_0\|_1 = 2\operatorname{tr}E_+(\rho_1 - \rho_0) = 2\|\rho_1 - \rho_0\|_1 \operatorname{tr}E_+\sigma = \|\rho_1 - \rho_0\|_1.$$

We have $D(\mathcal{E}\rho_0\|\mathcal{E}\rho_1) \le D(\rho_0\|\rho_1)$ and

$$\chi(\mathcal{E}\rho_0, \mathcal{E}\rho_1; q_0, q_1) \le \chi(\rho_0, \rho_1; q_0, q_1) \tag{6}$$

for any positive q_0 and q_1 summing to 1. In both of these inequalities, equality holds if and only if the density matrices ρ_0, ρ_1 and $|\sigma|$ are linearly dependent, i.e., lie on a line. Unique such ρ_0 and ρ_1 exist for any prescribed σ and any prescribed values of $0 \le \operatorname{tr} E_+\rho_0 \le \operatorname{tr} E_+\rho_1 \le 1$.

More generally, let Δ be any function depending on two density matrices and satisfying the data processing inequality $\Delta(\mathcal{E}\rho_0\|\mathcal{E}\rho_1) \le \Delta(\rho_0\|\rho_1)$ for any quantum measurement $\mathcal{E}$ with two possible outcomes $+$ and $-$. Then we arrive at

Theorem 10. *For any quantum states (density matrices) $\rho_0, \rho_1 \in M_n(\mathbb{C})$, there exist binary classical states (diagonal 2-square density matrices) ρ_0' and ρ_1' such that $\|\rho_1' - \rho_0'\|_1 = \|\rho_1 - \rho_0\|_1$ and $\Delta(\rho_0'\|\rho_1') \le \Delta(\rho_0\|\rho_1)$.*

For the case of the Holevo quantity, with the notations above, we have $S(\mathcal{E}\rho) = h(\operatorname{tr} E_+\rho)$, where $h(x) = -x\log x - (1-x)\log(1-x)$ is the binary entropy function. Therefore, the mutual information is

$$\chi(\mathcal{E}\rho_0, \mathcal{E}\rho_1; q_0, q_1) = I(t_0, t_1; q_0, q_1) := h(q_0 t_0 + q_1 t_1) - q_0 h(t_0) - q_1 h(t_1),$$

where $t_j = \operatorname{tr} E_+\rho_j$. Observe that

$$t_1 - t_0 = \operatorname{tr} E_+(\rho_1 - \rho_0) = \|\rho_1 - \rho_0\|_1 \operatorname{tr} E_+\sigma = \|\rho_1 - \rho_0\|_1/2.$$

We arrive at

Theorem 11. *For any nonnegative q_0 and q_1 summing to 1, we have*

$$\chi(\rho_0, \rho_1; q_0, q_1) \ge$$
$$\ge \min\{I(t_0, t_1; q_0, q_1) : 0 \le t_0 \le t_1 \le 1, t_1 - t_0 = \|\rho_1 - \rho_0\|_1/2.\}$$

This theorem and the possibility of equality in (6) tell us that for the Holevo quantity, or 'quantum entropy concavity' $\chi(\rho_0, \rho_1; q_0, q_1)$, the largest lower bound that depends only on $\rho_1 - \rho_0$ and q_1 is the 'minimal classical binary entropy concavity', i.e., the minimum in Theorem 11. It does not seem possible to compute this minimum exactly. There are various ways to get weaker but more explicit lower bounds. A simple way is to use the convexity and symmetry of $-h''(x) = 1/x + 1/(1-x)$ to prove that the minimum is

$$\ge 4q_0q_1\left(h\left(\frac{1}{2}\right) - h\left(\frac{2 + \|\rho_1 - \rho_0\|_1}{4}\right)\right),$$

with equality if and only if $q_1 = 1/2$ or $\rho_0 = \rho_1$. Note that $h(1/2) = \log 2$. For $\rho_0 \ne \rho_1$ and $q_0q_1 > 0$, this weaker lower bound on

$$\chi(\rho_0, \rho_1; q_0, q_1)$$

is still strictly greater than the previously known lower bound

$$q_0q_1\|\rho_1 - \rho_0\|_1^2/2$$

due to I. H. Kim [5]. This is because $-h''$ is minimal, with value 4, only at 1/2.

For lower bounds depending on other parameters of ρ_0 and ρ_1, and also for upper bounds, see [6].

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