Various definitions

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April 11, 2024

1 Symmetric monoidal categories (SMC)

Monoidal category: A category C equipped with

- A functor $\otimes : C \times C \to C$;
- unit object $I \in C$;
- associator: natural iso $(a \otimes b) \otimes c \xrightarrow{\alpha_{a,b,c}} a \otimes (b \otimes c)$;
- left unitor: natural iso $I \otimes a \xrightarrow{\lambda_a} a$;
- right unitor: natural iso $a \otimes I \xrightarrow{\rho_a} a$
- **symmetric** if there is a symmetry: natural iso $a \otimes b \xrightarrow{\sigma_{a,b}} b \otimes a$ such that $\sigma_{b,a} = \sigma_{a,b}^{-1}$,

satisfying triangle, pentagon (+ symmetry) diagrams.

We will always assume that C is a SMC.

1.1 Closed SMC

A SMC C is **closed** if:

for every $b \in C$, the endofunctor $- \otimes b$ has a right adjoint $[b, -]$ (internal hom).

What does this mean?

(1) For all $a, c \in C$, $C(a \otimes b, c) \simeq C(a, [b, c])$, naturally in a, c.

(2) unit $\eta_a^b : a \to [b, a \otimes b]$, counit: $\epsilon_a^b : [b, a] \otimes b \to a$, natural transformations, triangle identities

‘ Relation of the two:

- Let i be the iso of (1):

$$\eta_a^b \in C(a, [b, a \otimes b]) \simeq C(a \otimes b, a \otimes b), \qquad \eta_a^b = i(id_{a\otimes b})$$
$$\epsilon_a^b \in C([b, a] \otimes b, a) \simeq C([b, a], [b, a]), \qquad \epsilon_a^b = i^{-1}(id_{[b,a]}).$$

- Conversely, from η^b, ϵ^b of (2), we define i as

$$g \in C(a \otimes b, c), \qquad i(g) = [b, g] \circ \eta^b_a$$

with inverse

$$h \in C(a, [b, c]), \quad i^{-1}(h) = \epsilon^b_c \circ (h \otimes b)$$

Equivalently: a SMC C is closed if and only if for all $b, c \in C$, there is an object $[b, c]$ and an **evaluation map** $eval_{b,c} : [b, c] \otimes b \to c$ that has the following **universal property**: for all $a \in C$ and $f : a \otimes b \to c$ there is a unique $h : a \to [b, c]$ such that

$$f = eval_{b,c} \circ (h \otimes b).$$

The evaluation map is the counit $eval_{b,c} = \epsilon^b_c$ above.

Internal hom is a functor $[-, -] : C^{op} \times C \to C$ and the isomorphism in (1) is natural in all 3 variables a, b, c. This follows by Yoneda (nLab).

1.2 Compact SMC

A SMC is **compact** if each object $a \in C$ has a dual $a^* \in C$ such that there are maps $\cup_a : I \to a^* \otimes a$ and $\cap_a : a \otimes a^* \to I$ satisfying the snake identities

$$(\cap_a \otimes id_a) \circ (id_a \otimes \cup_a) = id_a, \qquad (id_{a^*} \otimes \cap_a) \circ (\cup_a \otimes id_{a^*}) = id_{a^*}.$$

The following are easily seen by the snake identities (pictures):

(1) a^* is determined up to iso;

(1) $I^* \simeq I$, by the isomorphisms

$$\rho_{I^*} \circ \cup_I : I \to I^*, \qquad \cap_I \circ \lambda^{-1}_{I^*} : I^* \to I;$$

(2) $a^{**} \simeq a$, indeed, we may define $\cup_{a^*} : I \to a \otimes a^*$ and $\cap_{a^*} : a^* \otimes a$ as

$$\cup_{a^*} = \sigma_{a^*,a} \circ \cup_a, \qquad \cap_{a^*} = \cap_a \circ \sigma_{a^*,a},$$

so that a is dual to a^*, and use (1);

(3) if we fix a^* and $\cup_a$ ($\cap_a$), then $\cap_a$ ($\cup_a$) is uniquely determined;

(4) any assignment $a \mapsto a^*$ defines a functor $C \to C^{op}$ (if $f : a \to b$, we can use $\cup_a$ and $\cap_b$ to "bend the wires" to obtain a map $b^* \to a^*$, this is obviously functorial);

(5) $(a \otimes b)^* \simeq a^* \otimes b^*$, we can clearly put (using symmetry)

$$\cup_{a\otimes b} = \cup_a \otimes \cup_b, \qquad \cap_{a\otimes b} = \cap_a \otimes \cap_b$$

(5) C is closed, with $[b, c] = b^* \otimes c$: the iso $i : C(a \otimes b, c) \simeq C(a, b^* \otimes c)$ can be obtain (roughly) as

$$i(g) = g \circ \cup_b, \qquad i^{-1}(h) = \cap_b \circ h,$$

the maps are inverses by the snake identities, naturality is obvious, since i does nothing on a or c. The unit and counit of the adjunction are given as

$$\eta^b_a = a \otimes \cup_b : a \to b^* \otimes a \otimes b, \qquad \epsilon^b_a = \cap_b \otimes a : b^* \otimes a \otimes b \to a.$$

(6) Can we state a theorem like: C is compact if and only if for each $b \in C$ there is some $b^* \in C$ such that $b^* \otimes -$ is the right adjoint of $- \otimes b$ and ...? What should be the additional conditions?

2 Kleisli categories and monoidal monads

A **monad** on C is a triple (P, η, μ), where:

- $P : C \to C$ is an endofunctor;
- $\eta : Id_C \to P$, $\mu : P^2 \to P$ are natural transformations satisfying some triangles and squares.

2.1 Kleisli categories

The **Kleisli category** C_P has the same objects as C, with morphisms:

$$C_p(a, b) = C(a, P(b)),$$

the identity $id_a = \eta_a$ and for $f \in C_p(a, b)$, $g \in C_p(b, c)$, the composition is defined as

$$g \circ f := \mu_c \circ P(g) \circ f.$$

We have the following adjunction:

- the **left adjoint functor** $F_P : C \to C_P$ is defined as $a \mapsto a$ and for $f : a \to b$, we put $F_P(f) \in C_P(a, b) = C(a, P(b))$ as $\eta_b \circ f$;
- the **right adjoint functor** $G_P : C_P \to C$ is given as $a \mapsto P(a)$ and for $f \in C_P(a, b) = C(a, P(b))$ we put $G_P(f) \in C(P(a), P(b))$ as $G_P(f) = \mu_b \circ P(f)$.

This is indeed an adjunction, where the unit is given by η and the counit is determined as $\epsilon_a = id_{P(a)} \in C_P(P(a), a)$.

2.2 Monoidal monads

A monad is **monoidal** [**?**] if there are maps

$$\kappa_{a,b} : Pa \otimes Pb \to P(a \otimes b), \qquad a, b \in C,$$

natural in a, b and such that

- (P, η, κ) is a **monoidal functor**, that is, some diagrams involving P, α, λ, ρ, κ and η commute;
- additional diagrams containing μ commutes;
- **symmetric**: additionally a diagram with σ commutes.

A monoidal functor is **strict** if κ is iso.

Proposition 1. *[**?** , Prop. 1.2.2] There is a bijective correspondence between:*

(i) families of morphisms $\{\kappa_{a,b}\}$ such that (P, η, μ, κ) is a (symmetric) monoidal monad;

(ii) (symmetric) monoidal structures on C_P such that the left adjoint functor $F_P : C \to C_P$ is strict monoidal.

If (P, η, μ, κ) is a symmetric monoidal monad, we define the monoidal structure on C_P as follows. The functor

$$\otimes_P : C_P \times C_P \to C_P$$

is given as as $a \otimes_P b = a \otimes b$ on objects, and for $f \in C_P(a, c) = C(a, P(c))$ and $g \in C_P(b, d) = C(b, p(d))$, we define $f \otimes_P g \in C_P(a \otimes_P b, c \otimes_P d) = C(a \otimes b, P(c \otimes d))$ as

$$f \otimes_P g := \kappa_{b,d} \circ (f \otimes g).$$

The associator and unitors and symmetry in C_P can be defined from those in C by composition with η.

3 Compact Kleisli categories

Assume that C is a symmetric monoidal closed category. Assume that (P, η, μ, κ) is a monoidal monad, such that the category C_p with corresponding monoidal structure is compact. We will study some consequences of this.

3.1 First consequences

It follows that each object $a \in C_P$ has a dual object a^*, such that there is an isomorphism

$$i : C_P(a \otimes_P b^*, c) \simeq C_P(a, b \otimes_P c),$$

which is natural (in C_P) in a and c. This means that for any arrows $a \xrightarrow{f} a'$ and $a' \otimes b^* \xrightarrow{h'} c$ in C_P, we have

$$i(h' \circ_P (f \otimes_P id^P_{b^*})) = i(h') \circ_P f,$$

and similarly, gor $c \xrightarrow{g} c'$ and $a \otimes b^* \xrightarrow{h} c$ in C_P, we get

$$i(g \circ_P h) = (id^P_b \otimes_P g) \circ_P i(h).$$

By definition of the Kleisli category, i is an isomorphism (that is, a bijection of sets)

$$i : C(a \otimes b^*, P(c)) \simeq C(a, P(b \otimes c)).$$

We would like to show that i is natural in a and c also in the category C.

So let $f \in C(a, a')$ and let $\tilde{f} := \eta_{a'} \circ f \in C(a, P(a')) = C_P(a, a')$. Let $h' \in C(a' \otimes b^*, P(c))$, then

$$\begin{aligned} h' \circ_P (\tilde{f} \otimes_P b^*) &= \mu_c \circ P(h') \circ (\tilde{f} \otimes_P \eta_{b^*}) = \mu_c \circ P(h') \circ \kappa_{a',b^*} \circ (\eta_{a'} \otimes \eta_{b^*}) \circ (f \otimes b^*) \\ &= \mu_c \circ P(h') \circ \eta_{a' \otimes b^*} \circ (f \otimes b^*) = \mu_c \circ \eta_{P(c)} \circ h' \circ (f \otimes b^*) \\ &= h' \circ (f \otimes b^*), \end{aligned}$$

where we used that $\kappa_{a',b^*} \circ (\eta_{a'} \otimes \eta_{b^*}) = \eta_{a' \otimes b^*}$, naturality of η and the triangle identity. Similarly, we get for any $\bar{h}' \in C(a, P(b \otimes c))$,

$$\bar{h}' \circ_P f = \mu_{b \otimes c} \circ P(\bar{h}') \circ \eta_{a'} \circ f = \bar{h}' \circ f,$$

in particular, putting these together, this implies

$$i(h' \circ (f \otimes b^*)) = i(h' \circ_P (\tilde{f} \otimes_P id_{b^*})) = i(h') \circ_P \tilde{f} = i(h') \circ f.$$

Naturality in c is proved similarly.

It follows that there is an isomorphism

$$C(a, P(b \otimes c)) \simeq C(a \otimes b^*, P(c)) \simeq C(a, [b^*, P(c)]),$$

natural in a and c. By Yoneda, we get the isomorphism

$$P(b \otimes c) \simeq [b^*, P(c)],$$

natural in c. Putting $c = I$, we obtain

$$P(b) \simeq P(b \otimes I) \simeq [b^*, P(I)].$$

3.2 A construction of a monoidal monad

Fix an object $p \in C$ and assume that

- there is a bijective map $a \mapsto a^*$ on objects;
- for each $a \in C$, there is a morphism $\theta_a \in C(a, [a^*, p])$;
- for each morphism $f \in C(a, [b^*, p])$ there is some $\hat{f} \in C([a^*, p], [b^*, p])$

such that

(i) $\hat{\theta}_a = id_{[a^*,p]}$;

(ii) for $f \in C(a, [b^*, p])$, $\hat{f} \circ \theta_a = f$;

(iii) for $f \in C(a, [b^*, p])$ and $g \in C(b, [c^*, p])$,

$$(\hat{g} \circ f)^\wedge = \hat{g} \circ \hat{f}. \tag{1}$$

From this data, we may define a monad (P_p, θ, ν). Here the functor P_p acts as $a \mapsto [a^*, p]$ on objects and for $f \in C(a, b)$, we define $P_p(f) \in C([a^*, p], [b^*, p])$ by

$$P_p(f) := (\theta_b \circ f)^\wedge.$$

Moreover, ν is defined as $\nu_a = \hat{id}_{[a^*,p]}$. The fact that this is a monad follows easily from the properties (i)-(iii).

To make it monoidal, we add family of maps

$$\kappa_{a,b} : [a^*, p] \otimes [b^*, p] \to [(a \otimes b)^*, p],$$

such that

(iv) for all $a, b \in C$,
$$\theta_{a\otimes b} = \kappa \circ (\theta_a \otimes \theta_b);$$

(v) for $f \in C(a, [b^*, p])$ and $g \in C(c, [d^*, p])$,
$$\kappa_{b,d} \circ (\hat{f} \otimes \hat{g}) = (\kappa_{b,d} \circ (f \otimes g))^\wedge \circ \kappa_{a,c}.$$

Then one can check that $\kappa_{a,b}$ are natural in a, b and that
$$\nu_{a\otimes b} \circ P_p(\kappa_{a,b}) \circ \kappa_{[a^*,p],[b^*,p]} = \kappa_{a,b} \circ (\nu_a \otimes \nu_b).$$

We also need some properties with respect to α, λ, ρ and σ:

(vi) for all a, b, c,
$$\kappa_{a,b\otimes c} \circ (id_{[a^*,p]} \otimes \kappa_{b,c}) \circ \alpha_{[a^*,p],[b^*,p],[c^*,p]} = P_p(\alpha_{a,b,c}) \circ \kappa_{a\otimes b,c} \circ (\kappa_{a,b} \otimes id_{[c^*,p]})$$

(vii) for all a,
$$(\theta_a \circ \lambda_a)^\wedge \circ \kappa_{I,a} \circ (\theta_I \otimes id_{[a^*,p]}) = \lambda_{[a^*,p]}$$
$$(\theta_a \circ \rho_a)^\wedge \circ \kappa_{I,a} \circ (id_{[a^*,p]} \otimes \theta_I) = \rho_{[a^*,p]};$$

(viii) for all a, b,
$$(\theta_{b\otimes a} \circ \sigma_{a,b})^\wedge \circ \kappa_{a,b} = \kappa_{b,a} \circ \sigma_{[a^*,p],[b^*,p]}.$$

Then $(P_p, \theta, \nu, \kappa)$ is a monoidal monad, [**?**].

3.3 The Kleisli category C_p

The Kleisli category $C_p := C_{P_p}$ has the same objects as C, with morphisms $C_p(a, b) = C(a, [b^*, p])$, the identity is $id^p_a = \theta_a$ and for $f \in C_p(a, b)$, $g \in C_p(b, c)$, the composition is fiven as
$$g \circ_p f = \hat{g} \circ f.$$

Remark 1. Let j be the natural iso (in C):
$$j : C(a \otimes b, c) \simeq C(a, [b, c])$$

Note that $C_p(a, b)$ can be identified with $C(a \otimes b^*, p)$, with composition given by
$$j^{-1}(j(\psi)^\wedge \circ j(\varphi)), \qquad \varphi \in C(a \otimes b^*, p),\ \psi \in C(b \otimes c^*, p).$$

We equip C_p with the tensor product $\otimes_p$ defined by $a \otimes_p b = a \otimes b$ on objects and $f \otimes_p g = \kappa \circ (f \otimes g)$ on morphisms. Then $(C_p, \otimes_p, I)$ is a symmetric monoidal category, with the natural isomorphisms $\alpha, \lambda, \rho, \sigma$ extended by θ, that is, $\alpha^p := \theta \circ \alpha$, $\lambda^p := \theta \circ \lambda$, $\rho^p := \theta \circ \rho$, $\sigma^p := \theta \circ \sigma$.

3.4 When is C_p closed?

We need to define the internal hom $b \overset{p}{\multimap} c$, such that $b \overset{p}{\multimap} -$ is the right adjoint of $b \otimes_p -$ in C_p. In fact, it is enough to specify $b \overset{p}{\multimap} c$ on objects and to find an iso

$$C_p(a \otimes_p b, c) \simeq C_p(a, b \overset{p}{\multimap} c)$$

natural in a. As for the isomorphism, we must have

$$C(a \otimes b, [c^*, p]) \simeq C(a, [(b \overset{p}{\multimap} c)^*, p])$$

Since C is SMC, we have

$$C(a \otimes b, [c^*, p]) \simeq C((a \otimes b) \otimes c^*, p) \simeq C(a \otimes (b \otimes c^*), p) \simeq C(a, [b \otimes c^*, p])$$

and the isomorphisms are natural (in C) in all variables. This suggests to define $b \overset{p}{\multimap} c$ as the object such that $(b \overset{p}{\multimap} c)^* = b \otimes c^*$. Since $(-)^*$ is bijective, such an object exists and is unique.

As for naturality of the isomorphism, let us denote by i the resulting isomorphism

$$i : C(a \otimes b, [c^*, p]) \simeq C(a, [b \otimes c^*, p])$$

Let $f \in C_p(a', a) = C(a', [a^*, p])$. Then naturality means that we require

$$i(h \circ_p (f \otimes_p id_b^p)) = i(h) \circ_p f = i(h)^\wedge \circ f.$$

on the left hand side we obtain

$$h \circ_p (f \otimes_p id_b^p) = \hat{h} \circ \kappa_{a,b} \circ (f \otimes \theta_b) = \hat{h} \circ s_{a,b} \circ (f \otimes b),$$

where $\hat{h} \circ s_{a,b} \in C([a^*, p] \otimes b, [c^*, p])$. By naturality in C, we see that

$$i(\hat{h} \circ s_{a,b} \circ (f \otimes b)) = i(\hat{h} \circ s_{a,b}) \circ f,$$

where $i(\hat{h} \circ s_{a,b}) \in C([a^*, p], [b \otimes c^*, p])$. It follows that we need to have

$$i(\hat{h} \circ s_{a,b}) = i(h)^\wedge. \tag{2}$$