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Compatibility of covariant quantum channels with emphasis on Weyl symmetry
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Compatibility of covariant quantum channels with emphasis on Weyl symmetry

Erkka Haapasalo

Abstract. Compatibility conditions of quantum channels featuring symmetry through covariance are studied. Compatibility here means the possibility of obtaining two or more channels through partial trace out of a broadcasting channel. We see that covariance conditions can be used to simplify compatibility conditions as the broadcasting channel can be assumed to be covariant in a particular way. A particular emphasis is on Weyl covariance and in determining compatibility conditions for Weyl-covariant channels. The concrete examples studied include the case of a non-compact continuous phase space and the case of a finite phase space.

1. Introduction

One of the features of quantum theory which is responsible for its radical deviation from classical physics is the fact that quantum measurement processes cannot typically be realized simultaneously with a joint device. This notion of *incompatibility* is widely studied for quantum observables (positive-operator-valued measures) [1, 2, 10, 13, 15, 16] and its negation in this context is often called as *joint measurability*. Incompatibility can be seen as a non-classical resource since there is an operational connection between the incompatibility of observables and detecting EPR-steering [17, 18]. Thus identifying incompatibility is important. In addition to observables, incompatibility has been studied for a wider range of quantum devices [8, 11, 12].

Since incompatibility is defined through negation, the impossibility of finding a joint measurement device, we must first be able to characterize the compatible device pairs or larger assemblages. This is, in general, a demanding task. However, in the presence of symmetry properties the problem

simplifies. Furthermore, symmetry is a pervasive feature in most physical measuring devices [19] and it is prudent to use this for our advantage.

This paper concentrates on covariant quantum channels and their compatibility properties. Quantum channels are an inalienable part of quantum measurements [3] since they describe changes of the quantum system in measurement processes or through time evolution. Compatibility of channels means that the channels have a broadcasting channel (a joint channel) from which the channels can be obtained as margins. Hence, compatibility of channels is closely connected to the concept of approximate quantum cloning. In this work, we utilize the symmetry properties of channels and show that in most physical situations, symmetry properties simplify determining compatible channels. This is due to the fact that the joint channels of covariant channels can be restricted to the smaller set of covariant joint channels. Similar methods and results have been successfully used before in the study of joint measurability [21]. As particular examples, we consider channels covariant under phase space shifts, the Weyl-covariant channels.

The paper is organized as follows: In Section 2, general notations are fixed and definitions for channels and their covariance are given. We go on to define compatibility in Section 3 and give a result stating that, in varied situations, joint channels of covariant channels can be restricted to covariant joint channels. Some compatibility conditions for covariant channels are presented in Subsection 3.1. Our main results deal with phase-space-covariant channels and their compatibility conditions which are discussed in the case of a generalized phase space in Section 4. The structure of covariant joint channels for Weyl covariance is discussed in Subsection 4.1 and in Subsection 4.2 compatibility conditions for Weyl-covariant channels arising from covariant Stinespring dilations are given. In Section 5, we discuss the consequences of the preceding section for two types of physical phase spaces, for a non-compact continuous phase space in Subsection 5.1 and for finite phase space in Subsection 5.2. In the former case, a particular emphasis is on Gaussian Weyl-covariant channels. Finally in Section 6 some of the earlier results are generalized to the multipartite case.

2. Definitions

In this short section we make the basic mathematical definitions used throughout this paper. Basics on Hilbert space operators and projective unitary representations are introduced. Most importantly, we give a formal definition for quantum channels and their covariance.

We let $\mathbb{N}$ be the set $\{1, 2, \ldots\}$ of positive natural numbers. The Hilbert spaces in this treatise are always assumed to be complex and the inner products are assumed to be linear in their second argument. For any Hilbert space $\mathcal{H}$, we denote by $\mathcal{L}(\mathcal{H})$, $\mathcal{T}(\mathcal{H})$, and respectively $\mathcal{U}(\mathcal{H})$ the sets of bounded linear operators, trace-class operators, and respectively unitary operators on $\mathcal{H}$. We denote by $\mathbb{1}_{\mathcal{H}}$ the identity operator on $\mathcal{H}$; we often omit the subscript

when there is no risk of confusion. Of special interest in quantum physics is the set $\mathcal{S}(\mathcal{H}) \subset \mathcal{T}(\mathcal{H})$ of positive trace-1 operators on $\mathcal{H}$, since these operators represent the *states* of the quantum system described by $\mathcal{H}$. Moreover, we denote the inner product, which is linear in the second argument throughout this treatise, of any Hilbert space by $\langle \cdot | \cdot \rangle$; there should be no confusion regarding which Hilbert space is in question in each situation.

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. We say that a linear map $\Phi : \mathcal{L}(\mathcal{K}) \to \mathcal{L}(\mathcal{H})$ is *completely positive* if for any $n \in \mathbb{N}$, $B_1, \ldots, B_n \in \mathcal{L}(\mathcal{K})$, and $\varphi_1, \ldots, \varphi_n \in \mathcal{H}$

$$\sum_{i,j=1}^n \langle \varphi_i | \Phi(B_i^* B_j) \varphi_j \rangle \geq 0.$$

If $\Phi(\mathbb{1}_\mathcal{K}) = \mathbb{1}_\mathcal{H}$, we say that Φ is *unital*. If Φ is (completely) positive and for any increasing net $(B_\alpha)_{\alpha \in A} \subset \mathcal{L}(\mathcal{K})$ of self-adjoint operators we have

$$\sup_{\alpha \in A} \Phi(B_\alpha) = \Phi\big(\sup_{\alpha \in A} B_\alpha \big)$$

(or, equivalently, Φ is (completely) positive and continuous with respect to the ultraweak topologies of $\mathcal{L}(\mathcal{K})$ and $\mathcal{L}(\mathcal{H})$), we say that Φ is *normal*.

Definition 2.1. A normal unital completely positive linear map $\Phi : \mathcal{L}(\mathcal{K}) \to \mathcal{L}(\mathcal{H})$ is called a *channel* and the set of such channels is denoted by $\mathbf{Ch}(\mathcal{H}, \mathcal{K})$.

Using the duality $\mathcal{T}(\mathcal{H})^* = \mathcal{L}(\mathcal{H})$, we may define the predual $\Phi_* : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K})$ for any channel $\Phi \in \mathbf{Ch}(\mathcal{H}, \mathcal{K})$ through

$$\mathrm{tr}\left[\Phi_*(T) B \right] = \mathrm{tr}\left[T \Phi(B) \right], \qquad T \in \mathcal{T}(\mathcal{H}), \quad B \in \mathcal{L}(\mathcal{K}).$$

Clearly Φ_* is fully determined by its restriction on $\mathcal{S}(\mathcal{H})$ and $\Phi_*(\rho) \in \mathcal{S}(\mathcal{K})$ for any $\rho \in \mathcal{S}(\mathcal{H})$. Indeed, the physical relevance of channels is in that they describe transformations of quantum states. This physical picture is also the reason for the ordering of the Hilbert spaces in the notation for the set of channels set up in Definition 2.1; $\mathcal{H}$ is here the true input space and $\mathcal{K}$ is the true output space. We continue working in the Heisenberg picture though as this is often mathematically the easier way. From the physical point of view, complete-positivity arises from the requirement that trivially extended maps should be positive as well. This means that, for any $n \in \mathbb{N}$, the map $\Phi \otimes \mathrm{id}_{\mathcal{M}_{n \times n}(\mathbb{C})} : \mathcal{L}(\mathcal{K}) \otimes \mathcal{M}_{n \times n}(\mathbb{C}) \to \mathcal{L}(\mathcal{H}) \otimes \mathcal{M}_{n \times n}(\mathbb{C})$ is required to be positive. Here we denote by $\mathcal{M}_{n \times n}(\mathbb{C})$ the algebra of $(n \times n)$-matrices with complex entries and by $\mathrm{id}_{\mathcal{M}_{n \times n}(\mathbb{C})}$ the identity map on this algebra. Moreover, the tensor product channel is the extension of the map $\mathcal{L}(\mathcal{K}) \times \mathcal{M}_{n \times n}(\mathbb{C}) \ni (B, M) \mapsto \big(\Phi(B), M \big) \in \mathcal{L}(\mathcal{H}) \times \mathcal{M}_{n \times n}(\mathbb{C})$.

Let $\mathcal{H}$ be a Hilbert space. It is well known that the automorphism group $\mathrm{Aut}\big(\mathcal{L}(\mathcal{H}) \big)$ of $\mathcal{L}(\mathcal{H})$ coincides with the projective group $\mathcal{U}(\mathcal{H})/(\mathbb{T}\mathbb{1}_\mathcal{H})$ where $\mathbb{T}$ is the set of modulus-1 complex numbers since, for any automorphism α on $\mathcal{L}(\mathcal{H})$, there is a unitary $U \in \mathcal{U}(\mathcal{H})$ defined up to phase such that $\alpha(A) = UAU^*$ for all $A \in \mathcal{L}(\mathcal{H})$. Let G be a topological group. An action (a homomorphism continuous with respect to the point-ultraweak topology of $\mathrm{Aut}\big(\mathcal{L}(\mathcal{H}) \big)$) $G \ni g \mapsto \alpha_g \in \mathrm{Aut}\big(\mathcal{L}(\mathcal{H}) \big)$ of G on $\mathcal{L}(\mathcal{H})$ is, according to

the above, described by a measurable map $u : G \to \mathcal{U}(\mathcal{H})/(\mathbb{T}\mathbb{1}_{\mathcal{H}})$ such that $\alpha_g(A) = U(g)AU(g)^*$, $g \in G$, $A \in \mathcal{L}(\mathcal{H})$, where $U(g)$ is a representative of $u(g)$ for all $g \in G$; for details, see Section VII.1 of [19]. These representatives can be chosen so that the resulting map $U : G \to \mathcal{U}(\mathcal{H})$ is a *projective unitary representation* [19, Theorem VII.7.5], i.e.,

- $U(e) = \mathbb{1}_{\mathcal{H}}$ (with e being the neutral element of G) and
- $U(gh) = m(g,h)U(g)U(h)$ for all g, $h \in G$ where $m : G \times G \to \mathbb{T}$ is a *G-multiplier*, i.e.,
  - $m(g,e) = 1 = m(e,g)$ for all $g \in G$ and
  - $m(g,h)m(gh,k) = m(g,hk)m(h,k)$ for all g, h, $k \in G$.

If the multiplier m above has the constant value 1, U is often called a(n ordinary) unitary representation. When an action $g \mapsto \alpha_g$ is thus associated with a projective unitary representation $U : G \to \mathcal{U}(\mathcal{H})$, we denote $\alpha_g =: \alpha_g^U$ for all $g \in G$.

Fix now Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, group G and projective representations $U : G \to \mathcal{U}(\mathcal{H})$ and $V : G \to \mathcal{U}(\mathcal{K})$. The following definition of covariant channels is the same as, e.g., in [9]:

Definition 2.2. We denote by $\mathbf{Ch}_U^V$ the subset of channels $\Phi \in \mathbf{Ch}(\mathcal{H}, \mathcal{K})$ such that

$$\Phi \circ \alpha_g^V = \alpha_g^U \circ \Phi, \qquad g \in G.$$

We call channels $\Phi \in \mathbf{Ch}_U^V$ (U, V)-*covariant*.

In typical situations the symmetry group G is locally compact and second countable (lcsc) and the representations are strongly continuous, i.e., the maps $g \mapsto U(g)\varphi$, $\varphi \in \mathcal{H}$, are continuous with respect to the locally compact topology of G and the natural topology of $\mathcal{H}$, and similarly for V.

Let G be a locally compact group and $L^\infty(G)$ be the space of (equivalence classes of) essentially bounded functions measurable with respect to any of the equivalent left (or right) Haar measure of G. Let us concentrate on the left-invariant case. A locally compact group G is said to be *amenable* if it allows an *invariant mean* $L^\infty(G) \ni f \mapsto \overline{f} \in \mathbb{C}$, i.e., $f \mapsto \overline{f}$ is a positive linear functional (i.e., linear map such that $\overline{f} \geq 0$ whenever $f \geq 0$,i.e., f is a pointwise-positive function) such that $\overline{1} = 1$ (where 1 on LHS stands for the function having the constant value 1), and, for any $f \in L^\infty(G)$ and $g \in G$, $\overline{f^g} = \overline{f}$ where $f^g \in L^\infty(G)$, $f^g(h) = f(g^{-1}h)$ for all $h \in G$. Compact groups are amenable. Indeed, when we fix the left Haar measure μ of a compact G with $\mu(G) = 1$, the map

$$L^\infty(G) \ni f \mapsto \overline{f} = \int_G f \, d\mu \in \mathbb{C}$$

is an invariant mean. According to the Markov-Kakutani fixed point theorem, also Abelian groups are amenable. In our most important examples, the symmetry groups will be amenable and invariant means will be used to make general channels covariant.

3. Compatibility of covariant channels

We now go on to give the central definition of this work: the notion of compatibility. The definition coincides with those made earlier in, e.g., [8, 11]. After this, we see how covariance in most physical situations can be used to restrict the set of joint channels of a compatible pair of covariant channels thus making the problem of finding whether a pair of channels is compatible or not more manageable. The central result of this section is Proposition 3.4 stating that a pair of covariant channels is compatible if and only if they are joined by a covariant channel.

Let $\mathcal{H}$, $\mathcal{K}_1$, and $\mathcal{K}_2$ be Hilbert spaces. The following definition is central for this work:

Definition 3.1. Let $\Psi \in \mathbf{Ch}(\mathcal{H}, \mathcal{K}_1 \otimes \mathcal{K}_2)$. Let us define the *margins* $\Psi_{(i)} \in \mathbf{Ch}(\mathcal{H}, \mathcal{K}_i)$, $i = 1, 2$, of Ψ through

$$
\begin{array}{rcll}
\Psi_{(1)}(A) & = & \Psi(A \otimes \mathbb{1}_{\mathcal{K}_2}), & A \in \mathcal{L}(\mathcal{K}_1), \\
\Psi_{(2)}(B) & = & \Psi(\mathbb{1}_{\mathcal{K}_1} \otimes B), & B \in \mathcal{L}(\mathcal{K}_2).
\end{array}
$$

Let $\Phi_i \in \mathbf{Ch}(\mathcal{H}, \mathcal{K}_i)$, $i = 1, 2$. The channels Φ_1 and Φ_2 are said to be *compatible* if there is a *joint channel* $\Psi \in \mathbf{Ch}(\mathcal{H}, \mathcal{K}_1 \otimes \mathcal{K}_2)$ for Φ_1 and Φ_2, i.e., $\Phi_i = \Psi_{(i)}$, $i = 1, 2$.

Suppose that $\Phi_i \in \mathbf{Ch}(\mathcal{H}, \mathcal{K}_i)$, $i = 1, 2$ are compatible and have a joint channel $\Psi \in \mathbf{Ch}(\mathcal{H}, \mathcal{K}_1 \otimes \mathcal{K}_2)$. Let us denote the partial trace over $\mathcal{K}_i$ on $\mathcal{T}(\mathcal{K}_1 \otimes \mathcal{K}_2)$ by tr_i, $i = 1, 2$. For any $\rho \in \mathcal{S}(\mathcal{H})$, we have

$$
\Phi_{1\,*}(\rho) = \Psi_{(1)\,*}(\rho) = \mathrm{tr}_2\big[\Psi_*(\rho)\big] \quad \Phi_{2\,*}(\rho) = \Psi_{(2)\,*}(\rho) = \mathrm{tr}_1\big[\Psi_*(\rho)\big].
$$

Thus, compatibility of two channels means that the channels can be seen as reduced dynamics of a broadcasting Schrödinger channel $\Psi_* : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{K}_1 \otimes \mathcal{K}_2)$.

Remark 3.2. Let us make a couple of observations on operations that preserve compatibility. Pick compatible channels $\Phi_i \in \mathbf{Ch}(\mathcal{H}, \mathcal{K}_i)$, $i = 1, 2$. Let us assume that $\Psi \in \mathbf{Ch}(\mathcal{H}, \mathcal{K}_1 \otimes \mathcal{K}_2)$ is a joint channel for Φ_1 and Φ_2. Channels of the form $\Phi_i' = \Gamma \circ \Phi_i \in \mathbf{Ch}(\mathcal{H}', \mathcal{K}_i)$, $i = 1, 2$, where $\mathcal{H}'$ is some Hilbert space and $\Gamma \in \mathbf{Ch}(\mathcal{H}, \mathcal{H}')$, are *pre-processings of Φ_1 and Φ_2 with a common preprocessor*. Such pre-processings are compatible as well. Indeed, the channel $\Gamma \circ \Psi$ is easily seen to be a joint channel for Φ_1' and Φ_2'.

Whenever $\mathcal{K}_i'$, $i = 1, 2$, are some Hilbert spaces and $\Delta_i \in \mathbf{Ch}(\mathcal{K}_i, \mathcal{K}_i')$ are channels, the channels $\tilde{\Phi}_i = \Phi_i \circ \Delta_i \in \mathbf{Ch}(\mathcal{H}, \mathcal{K}_i')$ are called as *post-processings of Φ_1 and Φ_2*. Such post-processings are also compatible. To see this, define the channel $\Delta_1 \otimes \Delta_2 \in \mathbf{Ch}(\mathcal{K}_1 \otimes \mathcal{K}_2, \mathcal{K}_1' \otimes \mathcal{K}_2')$ as the unique extension of the map $\mathcal{L}(\mathcal{K}_1') \times \mathcal{L}(\mathcal{K}_2') \ni (C, D) \mapsto \Delta_1(C) \otimes \Delta_2(D) \in \mathcal{L}(\mathcal{K}_1 \otimes \mathcal{K}_2)$ onto $\mathcal{L}(\mathcal{K}_1' \otimes \mathcal{K}_2')$. It is immediate that $\Psi \circ (\Delta_1 \otimes \Delta_2)$ is a joint channel for $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$. Thus, the set of compatible pairs of channels is closed under pre-processing with a common pre-processor and under post-processing.

The following lemma will be crucial in proving the main result, Proposition 3.4, of this section.

Lemma 3.3. *Let $\mathcal{H}$, $\mathcal{K}_1$, and $\mathcal{K}_2$ be Hilbert spaces. Suppose that $\Psi : \mathcal{L}(\mathcal{K}_1 \otimes \mathcal{K}_2) \to \mathcal{L}(\mathcal{H})$ is a positive linear map whose margins $\Psi_{(i)}$, $i = 1$, 2, are defined as in Definition 3.1. If $\Psi_{(i)}$, $i = 1$, 2 are normal, so is Ψ.*

Proof. Denote by $\mathbb{F}_i$ the set of finite-dimensional linear subspaces of $\mathcal{K}_i$, $i = 1$, 2. These are directed sets with respect to set inclusion. Consequently, the set $\mathbb{F}_1 \times \mathbb{F}_2$ is directed in a natural way. Denote by $P_{\mathcal{M}}$ the orthogonal projection onto the subspace $\mathcal{M} \in \mathbb{F}_i$, $i = 1$, 2. Using the normality of Φ_1 and Φ_2,

$$
\begin{aligned}
&\Psi(\mathbb{1}_{\mathcal{K}_1 \otimes \mathcal{K}_2} - P_{\mathcal{M}} \otimes P_{\mathcal{N}}) \\
={}& \Psi\big(\mathbb{1}_{\mathcal{K}_1 \otimes \mathcal{K}_2} - P_{\mathcal{M}} \otimes \mathbb{1}_{\mathcal{K}_2} + P_{\mathcal{M}} \otimes \mathbb{1}_{\mathcal{K}_2}(\mathbb{1}_{\mathcal{K}_1 \otimes \mathcal{K}_2} - \mathbb{1}_{\mathcal{K}_1} \otimes P_{\mathcal{N}})\big) \\
\leq{}& \Psi\big((\mathbb{1}_{\mathcal{K}_1 \otimes \mathcal{K}_2} - P_{\mathcal{M}} \otimes \mathbb{1}_{\mathcal{K}_2}) + (\mathbb{1}_{\mathcal{K}_1 \otimes \mathcal{K}_2} - \mathbb{1}_{\mathcal{K}_1} \otimes P_{\mathcal{N}})\big) \\
={}& \Psi_{(1)}(\mathbb{1}_{\mathcal{K}_1} - P_{\mathcal{M}}) + \Psi_{(2)}(\mathbb{1}_{\mathcal{K}_2} - P_{\mathcal{N}}) \underset{(\mathcal{M},\mathcal{N}) \in \mathbb{F}_1 \times \mathbb{F}_2}{\to} 0.
\end{aligned}
$$

Suppose that $(C_\alpha)_{\alpha \in A} \subset \mathcal{L}(\mathcal{K}_1 \otimes \mathcal{K}_2)$ is an increasing net of self-adjoint operators with the supremum C. Fix a positive $T \in \mathcal{T}(\mathcal{H})$, $\varepsilon > 0$, and $\alpha_0 \in A$. Choose $(\mathcal{M}, \mathcal{N}) \in \mathbb{F}_1 \times \mathbb{F}_2$, so that

$$
\operatorname{tr}\left[T\Psi(\mathbb{1}_{\mathcal{K}_1 \otimes \mathcal{K}_2} - P_{\mathcal{M}} \otimes P_{\mathcal{N}})\right] \leq \min\left\{ \frac{\varepsilon^2}{36\operatorname{tr}[T]\,\|C - C_{\alpha_0}\|^2},\, \frac{\varepsilon}{3\|C - C_{\alpha_0}\|} \right\},
$$

and $\alpha_1 \in A$, so that $\operatorname{tr}\left[T\Psi\big(P_{\mathcal{M}} \otimes P_{\mathcal{N}}(C - C_\alpha)P_{\mathcal{M}} \otimes P_{\mathcal{N}}\big)\right] \leq \varepsilon/3$ for all $\alpha \in A$, $\alpha \geq \alpha_1$; the first choice can be made based on the above estimation and the second one on the fact that Ψ restricted on a finite-dimensional sub-algebra is normal. We may now evaluate for any $\alpha \geq \alpha_0$, α_1

$$
\begin{aligned}
&\operatorname{tr}\left[T\Psi(C - C_\alpha)\right] \\
\leq{}& \operatorname{tr}\left[T\Psi(P_{\mathcal{M}} \otimes P_{\mathcal{N}}(C - C_\alpha)P_{\mathcal{M}} \otimes P_{\mathcal{N}}\right] \\
&+ \big|\operatorname{tr}\left[T\Psi\big((\mathbb{1}_{\mathcal{M}_1 \otimes \mathcal{M}_2} - P_{\mathcal{M}} \otimes P_{\mathcal{N}})(C - C_\alpha)P_{\mathcal{M}} \otimes P_{\mathcal{N}})\big)\right]\big| \\
&+ \big|\operatorname{tr}\left[T\Psi\big(P_{\mathcal{M}} \otimes P_{\mathcal{N}}(C - C_\alpha)(\mathbb{1}_{\mathcal{M}_1 \otimes \mathcal{M}_2} - P_{\mathcal{M}} \otimes P_{\mathcal{N}})\big)\right]\big| \\
&+ \operatorname{tr}\left[T\Psi\big((\mathbb{1}_{\mathcal{M}_1 \otimes \mathcal{M}_2} - P_{\mathcal{M}} \otimes P_{\mathcal{N}})(C - C_\alpha)(\mathbb{1}_{\mathcal{M}_1 \otimes \mathcal{M}_2} - P_{\mathcal{M}} \otimes P_{\mathcal{N}})\big)\right] \\
\leq{}& \operatorname{tr}\left[T\Psi(P_{\mathcal{M}} \otimes P_{\mathcal{N}}(C - C_\alpha)P_{\mathcal{M}} \otimes P_{\mathcal{N}}\right] \\
&+ 2\sqrt{\operatorname{tr}\left[T\Psi(\mathbb{1}_{\mathcal{K}_1 \otimes \mathcal{K}_2} - P_{\mathcal{M}} \otimes P_{\mathcal{N}})\right]} \times \\
&\times \sqrt{\operatorname{tr}\left[T\Psi\big(P_{\mathcal{M}} \otimes P_{\mathcal{N}}(C - C_\alpha)^2 P_{\mathcal{M}} \otimes P_{\mathcal{N}}\big)\right]} \\
&+ \|C - C_\alpha\|\operatorname{tr}\left[T\Psi(\mathbb{1}_{\mathcal{K}_1 \otimes \mathcal{K}_2} - P_{\mathcal{M}} \otimes P_{\mathcal{N}})\right] \\
\leq{}& \operatorname{tr}\left[T\Psi(P_{\mathcal{M}} \otimes P_{\mathcal{N}}(C - C_\alpha)P_{\mathcal{M}} \otimes P_{\mathcal{N}}\right] \\
&+ 2\|C - C_\alpha\|\sqrt{\operatorname{tr}\left[T\Psi(\mathbb{1}_{\mathcal{K}_1 \otimes \mathcal{K}_2} - P_{\mathcal{M}} \otimes P_{\mathcal{N}})\right]\operatorname{tr}[T]} \\
&+ \|C - C_\alpha\|\operatorname{tr}\left[T\Psi(\mathbb{1}_{\mathcal{K}_1 \otimes \mathcal{K}_2} - P_{\mathcal{M}} \otimes P_{\mathcal{N}})\right] \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.
\end{aligned}
$$

The second estimation above is based on the Cauchy-Schwarz inequality for the positive sesquilinear form $(C_1, C_2) \mapsto \operatorname{tr}\left[T\Psi(C_1^* C_2)\right]$ and on the inequality $D \leq \|D\|\mathbb{1}_{\mathcal{K}_1 \otimes \mathcal{K}_2}$ for any self-adjoint $D \in \mathcal{L}(\mathcal{K}_1 \otimes \mathcal{K}_2)$. Note also that, as $(C - C_\alpha)_{\alpha \in A}$ is a descending sequence of positive operators, also $(\|C - C_\alpha\|)_{\alpha \in A}$ is non-increasing, as one easily checks. Thus, Ψ is normal. $\qquad\square$

Fix Hilbert spaces $\mathcal{H}$, $\mathcal{K}_1$, and $\mathcal{K}_2$, a symmetry group G, and projective representations $U : G \to \mathcal{U}(\mathcal{H})$ and $V_i : G \to \mathcal{U}(\mathcal{K}_i)$, $i = 1, 2$. We are interested in the compatibility properties of pairs of channels within $\mathbf{Ch}_U^{V_1} \times \mathbf{Ch}_U^{V_2}$. Define the representation $V_{12} : G \to \mathcal{U}(\mathcal{K}_1 \otimes \mathcal{K}_2)$, $V_{12}(g) = V_1(g) \otimes V_2(g)$, $g \in G$. It follows easily that, whenever $\Psi \in \mathbf{Ch}_U^{V_{12}}$, then $\Psi_{(i)} \in \mathbf{Ch}_U^{V_i}$. However, in general, it does not hold that if $\Phi_i \in \mathbf{Ch}_U^{V_i}$, $i = 1, 2$, are compatible, they should have a joint channel $\Psi \in \mathbf{Ch}_U^{V_{12}}$. Next we will show however, that in various situations a covariant joint channel exists.

Proposition 3.4. *Suppose that $\mathcal{H}$ and $\mathcal{K}_i$, $i = 1, 2$, are separable Hilbert spaces, G is an amenable locally compact group, and $U : G \to \mathcal{U}(\mathcal{H})$ and $V_i : G \to \mathcal{U}(\mathcal{K}_i)$, $i = 1, 2$, are strongly continuous projective unitary representations. Any two compatible covariant channels $\Phi_i \in \mathbf{Ch}_U^{V_i}$, $i = 1, 2$, have a joint channel $\overline{\Psi} \in \mathbf{Ch}_U^{V_{12}}$.*

Proof. Let $\Phi_i \in \mathbf{Ch}_U^{V_i}$, $i = 1, 2$, be compatible and $\Psi \in \mathbf{Ch}(\mathcal{H}, \mathcal{K}_1 \otimes \mathcal{K}_2)$ be a joint channel for them which is not necessarily (U, V_{12})-covariant. Next, we shall 'covariantize' Ψ.

For any φ, $\psi \in \mathcal{H}$ and $C \in \mathcal{L}(\mathcal{K}_1 \otimes \mathcal{K}_2)$, define the function $f_{C,\varphi,\psi} : G \to \mathbb{C}$,

$$f_{C,\varphi,\psi}(g) = \langle U(g)^* \varphi | \Psi\big(V_{12}(g)^* C V_{12}(g)\big) U(g)^* \psi \rangle, \qquad g \in G.$$

It follows easily from the separability of the Hilbert spaces and the strong continuity of the representations that $f_{C,\varphi,\psi}$ are Haar measurable for all $C \in \mathcal{L}(\mathcal{K}_1 \otimes \mathcal{K}_2)$ and φ, $\psi \in \mathcal{H}$. Moreover,

$$|f_{C,\varphi,\psi}(g)| \le \|\varphi\| \|\psi\| \|C\|, \qquad C \in \mathcal{L}(\mathcal{K}_1 \otimes \mathcal{K}_2), \quad \varphi, \psi \in \mathcal{H}, \quad g \in G.$$

Thus $f_{C,\varphi,\psi} \in L^\infty(G)$ for all $C \in \mathcal{L}(\mathcal{K}_1 \otimes \mathcal{K}_2)$ and φ, $\psi \in \mathcal{H}$.

Pick an invariant mean $L^\infty(G) \ni f \mapsto \overline{f} \in \mathbb{C}$. It can be easily checked that $\overline{f_{C,\cdot,\cdot}} : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is a bounded sesquilinear form for every $C \in \mathcal{L}(\mathcal{K}_1 \otimes \mathcal{K}_2)$. Thus, there is a map $\overline{\Psi} : \mathcal{L}(\mathcal{K}_1 \otimes \mathcal{K}_2) \to \mathcal{L}(\mathcal{H})$ such that

$$\overline{f_{C,\varphi,\psi}} = \langle \varphi | \overline{\Psi}(C) \psi \rangle, \qquad C \in \mathcal{L}(\mathcal{K}_1 \otimes \mathcal{K}_2), \quad \varphi, \psi \in \mathcal{H}.$$

Reader may easily verify that $\overline{\Psi}$ is linear and unital. A simple calculation shows that $f_{C,\varphi,\psi}^g = f_{\alpha_g^{V_{12}}(C),U(g)\varphi,U(g)\psi}$ for all $g \in G$, $C \in \mathcal{L}(\mathcal{K}_1 \otimes \mathcal{K}_2)$, and φ, $\psi \in \mathcal{H}$. Using this and the defining feature of the invariant mean, we have

$$
\begin{aligned}
\langle \varphi | \overline{\Psi}\big(V_{12}(g) C V_{12}(g)^*\big) \psi \rangle &= \overline{f_{\alpha_g^{V_{12}}(C),\varphi,\psi}} = \overline{f_{C,U(g)^*\varphi,U(g)^*\psi}^g} \\
&= \overline{f_{C,U(g)^*\varphi,U(g)^*\psi}} = \langle \varphi | U(g) \overline{\Psi}(C) U(g)^* \psi \rangle
\end{aligned}
$$

for all $g \in G$, φ, $\psi \in \mathcal{H}$, and $C \in \mathcal{L}(\mathcal{K}_1 \otimes \mathcal{K}_2)$, implying that

$$\overline{\Psi} \circ \alpha_g^{V_{12}} = \alpha_g^U \circ \overline{\Psi}, \qquad g \in G.$$

Fix $n \in \mathbb{N}$ and $\varphi_1, \ldots, \varphi_n \in \mathcal{H}$ and $C_1, \ldots, C_n \in \mathcal{L}(\mathcal{K}_1 \otimes \mathcal{K}_2)$. Using the complete positivity of Ψ, we have for any $g \in G$

$$\sum_{i,j=1}^n f_{C_i^* C_j, \varphi_i, \varphi_j}(g) = \sum_{i,j=1}^n \langle U(g)^* \varphi_i | \Psi\big(V_{12}(g)^* C_i^* C_j V_{12}(g)\big) U(g)^* \varphi_j \rangle \ge 0.$$

Using the positivity of the invariant mean, it follows that

$$\sum_{i,j=1}^{n} \langle \varphi_i | \overline{\Psi}(C_i^* C_j) \varphi_j \rangle = \overline{\sum_{i,j=1}^{n} f_{C_i^* C_j, \varphi_i, \varphi_j}} \geq 0.$$

Define the functions $f_{(1),A,\varphi,\psi} \in L^\infty(G)$, $A \in \mathcal{L}(\mathcal{K}_1)$, φ, $\psi \in \mathcal{H}$,

$$f_{(1),A,\varphi,\psi}(g) = \langle U(g)^* \varphi | \Phi_1 \big(V_1(g)^* A V_1(g) \big) U(g)^* \psi \rangle, \qquad g \in G.$$

It follows from the covariance of Φ_1 that $f_{(1),A,\varphi,\psi}$ has the constant value $\langle \varphi | \Phi_1(A) \psi \rangle$. Thus, $\overline{f_{A \otimes \mathbb{1}_{\mathcal{K}_2}, \varphi, \psi}} = \overline{f_{(1),A,\varphi,\psi}} = \langle \varphi | \Phi_1(A) \psi \rangle$ for all $A \in \mathcal{L}(\mathcal{K}_1)$ and φ, $\psi \in \mathcal{H}$. Hence, for all φ, $\psi \in \mathcal{H}$ and $A \in \mathcal{L}(\mathcal{K}_1)$,

$$\langle \varphi | \overline{\Psi}(A \otimes \mathbb{1}_{\mathcal{K}_2}) \psi \rangle = \overline{f_{A \otimes \mathbb{1}_{\mathcal{K}_2}, \varphi, \psi}} = \langle \varphi | \Phi_1(A) \psi \rangle.$$

Similarly, $\overline{\Psi}(\mathbb{1}_{\mathcal{K}_1} \otimes B) = \Phi_2(B)$ for all $B \in \mathcal{L}(\mathcal{K}_2)$. Thus, $\overline{\Psi}$ is a covariant unital completely positive linear map whose margins coincide with Φ_1 and Φ_2. The normality of $\overline{\Psi}$ now follows immediately from Lemma 3.3. $\qquad\square$

Thus, under the conditions of Proposition 3.4, covariant channels are compatible if and only if they have a covariant joint channel. This greatly simplifies the compatibility conditions for covariant channels, as we will see in our examples.

3.1. Dilation-theoretic conditions for compatibility

In this subsection, we use dilation theory of channels to obtain compatibility conditions for covariant channels. Our aim is to characterize those covariant channels which are compatible with a fixed covariant channel. It is known that the channels Φ_2 which are compatible with a fixed channel Φ_1 are of the form $\Phi_1^c \circ \Phi_{1,2}$ with some channel $\Phi_{1,2}$ where Φ_1^c is (some of the equivalent) conjugate channel of Φ_1 [11]. Proposition 3.6 below is a covariant version of this result; a vital ingredient in the proof is the use of Proposition 3.4. Suppose that G is a group and $U : G \to \mathcal{U}(\mathcal{H})$ and $V : G \to \mathcal{U}(\mathcal{K})$ are projective unitary representations.

Definition 3.5. A triple $(\mathcal{L}, J, \overline{U})$ of a Hilbert space $\mathcal{L}$, an isometry $J : \mathcal{H} \to \mathcal{K} \otimes \mathcal{L}$, and a projective unitary representation $\overline{U} : G \to \mathcal{U}(\mathcal{L})$ is a *covariant Stinespring dilation* for a channel $\Phi \in \mathbf{Ch}_U^V$ if $\Phi(A) = J^*(A \otimes \mathbb{1}_{\mathcal{L}})J$, $A \in \mathcal{L}(\mathcal{K})$, and $JU(g) = \big(V(g) \otimes \overline{U}(g)\big)J$ for all $g \in G$. If additionally the vectors $(A \otimes \mathbb{1}_{\mathcal{L}})J\varphi$, $A \in \mathcal{L}(\mathcal{K})$, $\varphi \in \mathcal{H}$, span a dense subspace in $\mathcal{K}$, the covariant dilation $(\mathcal{L}, V, \overline{U})$ is called *minimal*.

Any channel $\Phi \in \mathbf{Ch}_U^V$ possesses a covariant Stinespring dilation and among them there is a minimal one which is unique up to unitary equivalence [9, Corollary 1]. Clearly, if U is associated with the multiplier m_U and V with m_V, the projective representation $\overline{U}$ in any of the covariant dilations is associated with the multiplier $(g, h) \mapsto m_U(g, h)\overline{m_V(g, h)}$.

Proposition 3.6. *Retain the assumptions of Proposition 3.4. Let $(\mathcal{L}, J, \overline{U})$ be the (essentially unique) covariant minimal Stinespring dilation for Φ_1. The channels Φ_1 and Φ_2 are compatible if and only if there is $\tilde{\Phi}_2 \in \mathbf{Ch}_{\overline{U}}^{V_2}$ such that*

$$\Phi_2(B) = J^*\big(\mathbb{1}_{\mathcal{K}_1} \otimes \tilde{\Phi}_2(B)\big)J, \qquad B \in \mathcal{L}(\mathcal{K}_2). \tag{3.1}$$

Proof. If Φ_1 and Φ_2 are compatible, according to Proposition 3.4, they are margins of a covariant joint channel $\Psi \in \mathbf{Ch}_U^{V_{1,2}}$. The existence of the channel $\tilde{\Phi}_2$ of the claim now follow from [9, Proposition 1, Remark 2].

Suppose that the channel $\tilde{\Phi}_2 \in \mathbf{Ch}_{\overline{U}}^{\overline{U}}$ of the claim exists. The map

$$\mathcal{L}(\mathcal{K}_1) \times \mathcal{L}(\mathcal{K}_2) \ni (A, B) \mapsto A \otimes \tilde{\Phi}_2(B) \in \mathcal{L}(\mathcal{K}_1 \otimes \mathcal{L})$$

uniquely extends into a channel defined on $\mathcal{L}(\mathcal{K}_1 \otimes \mathcal{K}_2)$. Thus, we may define the channel $\Psi \in \mathbf{Ch}(\mathcal{H}, \mathcal{K}_1 \otimes \mathcal{K}_2)$,

$$\Psi(A \otimes B) = J^*\big(A \otimes \tilde{\Phi}_2(B)\big)J, \qquad A \in \mathcal{L}(\mathcal{K}_1), \quad B \in \mathcal{L}(\mathcal{K}_2).$$

It is immediate that $\Psi_{(1)} = \Phi_1$, $\Psi_{(2)} = \Phi_2$. $\qquad\qquad\square$

Note that the above result is symmetric, i.e., we may switch the roles of Φ_1 and Φ_2. One may also verify that the channel Ψ in the latter part of the proof is actually $(U, V_{1,2})$-covariant, although it is not necessary for proving the claim.

4. General phase space and Weyl covariance

From now on we are primarily concerned with channels that preserve the symmetries of the phase spaces of the systems involved. In this section, we derive our main results on compatibility of phase-space-symmetric channels. After introducing the mathematical background, in Subsection 4.1, we characterize the covariant joint channels for a pair of phase-space-symmetric channels and, using this characterization, give the necessary and sufficient compatibility conditions for these symmetric channels. In Subsection 4.2, we take another approach and characterize the phase-space-symmetric channels which are compatible with a fixed phase-space-symmetric channel using Proposition 3.4.

The phase space of a quantum system can vary. This is why we consider a generalized phase space which includes the typical phase spaces (in the case of flat geometry). We assume that the position or configuration space of the system is modelled by a locally compact Abelian group $\mathbf{X}$ and the momentum space is associated with the dual group $\hat{\mathbf{X}}$ of unitary equivalence classes of irreducible representations of $\mathbf{X}$, i.e., the locally compact group characters on $\mathbf{X}$. In notation, we treat $\mathbf{X}$ as a multiplicative group (with neutral element 1) and $\hat{\mathbf{X}}$ as an additive group (with neutral element 0). The phase space itself is the group $G := \mathbf{X} \times \hat{\mathbf{X}}$ with the group law $(x, \xi)(y, \upsilon) = (xy, \xi + \upsilon)$. The neutral element of G is thus $e = (1, 0)$.

The most common examples of phase spaces include the continuous phase space of a particle with N spatial degrees of freedom where the configuration space is the self-dual $\mathbb{R}^N$. Another example is the case of a finite cyclic configuration space $\mathbb{Z}_d$ which is also self dual. These are listed below.

	position space	momentum space
general phase space	**X** locally compact Abelian group	**X̂** dual group
continuous phase space	$\mathbb{R}^N$ $N \in \mathbb{N}$	$\mathbb{R}^N$
finite phase space	$\mathbb{Z}_d = \mathbb{Z}/(d\mathbb{Z})$ $d \in \mathbb{N}$	$\mathbb{Z}_d$

We must place a particular regularity condition on our phase spaces: we require that there be a continuous map $(\cdot|\cdot) : \mathbf{X} \times \hat{\mathbf{X}} \to \mathbb{R}$ such that, for any $x \in \mathbf{X}$ and $\xi \in \hat{\mathbf{X}}$, $(x|\cdot) : \hat{\mathbf{X}} \to \mathbb{R}$ and $(\cdot|\xi) : \mathbf{X} \to \mathbb{R}$ are group homomorphisms and the dual pairing is given by

$$\langle x, \xi \rangle = e^{i(x|\xi)}, \qquad x \in \mathbf{X}, \quad \xi \in \hat{\mathbf{X}}.$$

This means that a continuous version of the logarithm of the dual pairing can be given in the form of a 'scalar product'. This allows us to define the skew-symmetric *symplectic form* $\omega : G \times G \to \mathbb{R}$; see the next table. Since, by the Pontryagin duality, G is self dual, it is easy to see that the form $(g, h) \mapsto e^{i\omega(g,h)}$ can be treated as the dual form of G with itself. Our continuous and finite examples satisfy these conditions as the table below illustrates. Note that the symplectic form for the continuous case is often described with the matrix

$$\Omega_N := \begin{pmatrix} 0 & \mathbb{1}_N \\ -\mathbb{1}_N & 0 \end{pmatrix}$$

given in the block form.

	dual form	symplectic form
general phase space	$\langle x, \xi \rangle = e^{i(x|\xi)}$, $x \in \mathbf{X}, \xi \in \hat{\mathbf{X}}$	$\omega(g, h) = (x|v) - (y|\xi)$, $g = (x, \xi) \in G, h = (y, v) \in G$
continuous phase space	$\langle \vec{q}, \vec{p} \rangle = e^{i\vec{q}^T \vec{p}}$, $\vec{q}, \vec{p} \in \mathbb{R}^N$	$\omega(\vec{z}, \vec{w}) = \vec{z}^T \Omega_N \vec{w}$, $\vec{z}, \vec{w} \in \mathbb{R}^N \times \mathbb{R}^N$
finite phase space	$\langle k, l \rangle = e^{i\frac{2\pi}{d}kl}$, $k, l \in \mathbb{Z}_d$	$\omega(\vec{m}, \vec{n}) = \frac{2\pi}{d}(k_1 l_2 - k_2 l_1)$, $\vec{m} = (k_1, l_1) \in \mathbb{Z}_d^2, \vec{n} = (k_2, l_2) \in \mathbb{Z}_d^2$

The (spin-0) particle associated with the general phase space G is described by the Hilbert space $\mathcal{H} := L^2(\mathbf{X})$, the space of (equivalence classes of) functions $\varphi : \mathbf{X} \to \mathbb{C}$ which are square-integrable with respect to a fixed (and hence any) Haar measure of $\mathbf{X}$. Translations of the configuration space correspond to a unitary representation U of $\mathbf{X}$ and momentum boosts to a representation V of $\hat{\mathbf{X}}$ on $\mathcal{H}$. These position shifts and momentum boosts are presented in the table below with the examples of a continuous phase space and a finite phase space. The Hilbert space in the continuous case is the space $L^2(\mathbb{R}^N)$ of Lebesgue-square-integrable functions on $\mathbb{R}^N$. The finite phase space case can be defined with respect to an orhonormal basis $\{|j\rangle\}_{j \in \mathbb{Z}_d} \subset \mathbb{C}^d$.

	position shift	momentum boost
general phase space	$\big(U(x)\varphi\big)(y) = \varphi(x^{-1}y),$ $\varphi \in L^2(\mathbf{X})$, a.a., $x, y \in \mathbf{X}$	$\big(V(\xi)\varphi\big)(y) = \langle y, \xi \rangle \varphi(y),$ $\varphi \in L^2(\mathbf{X})$, $\xi \in \hat{\mathbf{X}}$, a.a. $y \in \mathbf{X}$
continuous phase space	$\big(U(\vec{q})\psi\big)(\vec{x}) = \psi(\vec{x} - \vec{q}),$ $\psi \in L^2(\mathbb{R}^N)$, a.a. $\vec{q}, \vec{x} \in \mathbb{R}^n$	$\big(V(\vec{p})\psi\big)(\vec{x}) = e^{i\vec{x}^T \vec{p}}\psi(\vec{x}),$ $\psi \in L^2(\mathbb{R}^N)$, a.a. $\vec{p}, \vec{x} \in \mathbb{R}^N$
finite phase space	$U(k)|j\rangle = |j + k\rangle,$ $j, k \in \mathbb{Z}_d$	$V(l)|j\rangle = e^{i\frac{2\pi}{d}jl}|j\rangle,$ $j, l \in \mathbb{Z}_d$

The phase space translations manifest themselves through the map $W : G \to \mathcal{U}(\mathcal{H})$,
$$W(x, \xi) = e^{\frac{i}{2}(x|\xi)}U(x)V(\xi), \qquad (x, \xi) \in G.$$
It follows that
$$W(gh) = e^{\frac{i}{2}\omega(g,h)}W(g)W(h), \qquad g, h \in G,$$
i.e., W is a projective unitary representation, the so-called *Weyl representation*, with the multiplier $(g, h) \mapsto e^{\frac{i}{2}\omega(g,h)}$. Moreover, we have the canonical commutation relations
$$W(g)^* = W(g^{-1}), \quad W(g)W(h) = e^{-i\omega(g,h)}W(h)W(g), \qquad g, h \in G. \quad (4.1)$$
A variety of correspondences between classical phase spaces and quantum phase spaces has been identified in [20]. Although the results of [20] have been formulated for continuous phase spaces, a careful reader finds that most of the results actually hold in our general phase space situation.

Proposition 4.1. *The Weyl representation W is irreducible and operators $W(g)$, $g \in G$, span an ultraweakly dense operator system in $\mathcal{L}(\mathcal{H})$. Similarly, the projective representation $G \times G \ni (g, h) \mapsto W(g) \otimes W(h) \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$ is irreducible and the operators $W(g) \otimes W(h)$, $g, h \in G$ span an ultraweakly dense operator system in $\mathcal{L}(\mathcal{H} \otimes \mathcal{H})$.*

Proof. The claims for the latter representation follow immediately from the claims concerning W. Thus, we concentrate on the first half of the claim.

To see that W is irreducible, let $B \in \mathcal{L}(\mathcal{H})$ be such that $BW(g) = W(g)B$ for all $g \in G$. Thus, especially, $BV(\xi) = V(\xi)B$ for all $\xi \in \hat{\mathbf{X}}$, implying [6, Part II, Ch. 2, Sec. 5, Theorem 1] that there is an essentially bounded measurable function $b : \mathbf{X} \to \mathbb{C}$ such that $(B\varphi)(x) = b(x)\varphi(x)$ for all $\varphi \in \mathcal{H}$ and $x \in \mathbf{X}$. Similarly, $BU(x) = U(x)B$ for all $x \in \mathbf{X}$, implying $b(x^{-1}y) = b(y)$ for a.a. $(x,y) \in \mathbf{X}^2$. Using the Fubini theorem, we easily see that b is constant so that B is a scalar multiple of the identity. Thus, according to the Shur orthogonality condition, W is irreducible.

Denote by $\mathcal{D}'$ the commutant of $\mathcal{D} \subseteq \mathcal{L}(\mathcal{H})$, i.e., the set of all operators $E \in \mathcal{L}(\mathcal{H})$ such that $DE = ED$ for all $D \in \mathcal{D}$ and by $\mathcal{D}'' := (\mathcal{D}')'$ the double commutant of $\mathcal{D} \subseteq \mathcal{L}(\mathcal{H})$. According to the von Neumann density theorem [6, Part I, Ch. 3, Sec. 4, Theorem 2], the ultraweak closure of the linear hull of the range $\operatorname{ran} W$ of W coincides with the double commutant $(\operatorname{ran} W)'' = (\mathbb{C}\mathbb{1}_{\mathcal{H}})' = \mathcal{L}(\mathcal{H})$. □

Remark 4.2. In the sequel, particularly in the proof of Lemma 4.4, it is important to guarantee that the supports of the characteristic functions (or Fourier transforms) of states (which are functions on the phase space) cover the whole phase space. Let us prove this. Define the function $\hat{S} : G \to \mathbb{C}$, $\hat{S}(g) = \operatorname{tr}[SW(g)]$, $g \in G$, for every $S \in \mathcal{T}(\mathcal{H})$. For any compact sets $K_1 \subseteq \mathbf{X}$ and $K_2 \subseteq \hat{\mathbf{X}}$ there is $S \in \mathcal{T}(\mathcal{H})$ such that $\hat{S}(g) \neq 0$ for all $g \in K_1 \times K_2$. To see this, pick compact sets $K_1 \subseteq \mathbf{X}$ and $K_2 \subseteq \hat{\mathbf{X}}$. Using the continuity of the form $(\cdot|\cdot)$ and the compactness of K_1 and K_2, it can be shown that, for any fixed (small) $\varepsilon > 0$, there are compact neighbourhoods $\tilde{K}_1 \subseteq \mathbf{X}$ and $\tilde{K}_2 \in \hat{\mathbf{X}}$ of, respectively, 1 and 0 such that $(x|v - \xi) \in [-\varepsilon, \varepsilon]$ and $(y|\xi) \in [-\varepsilon, \varepsilon]$ for all $(x, \xi) \in \tilde{K}_1 \times \tilde{K}_2$ and $(y, v) \in K_1 \times K_2$. Fix a Haar measure dx of $\mathbf{X}$ and the associated Plancherel measure $d\xi$ for $\hat{\mathbf{X}}$. Define $S = |\check{\chi}_{\tilde{K}_2}\rangle\langle\check{\chi}_{\tilde{K}_1}|$, where χ_K is the characteristic function of a set K and $\check{\chi}_{\tilde{K}_2}$ is the inverse Fourier transform

$$\check{\chi}_{\tilde{K}_2}(x) = \int_{K_2} e^{-i(x|\xi)} \, d\xi, \qquad x \in \mathbf{X}.$$

It follows that

$$\hat{S}(y, v) = e^{\frac{i}{2}(y|v)} \int_{\tilde{K}_1} \int_{\tilde{K}_2} e^{i\left((x|v-\xi)+(y|\xi)\right)} \, d\xi \, dx \approx |\tilde{K}_1||\tilde{K}_2|e^{\frac{i}{2}(y|v)} \neq 0$$

for all $(y, v) \in K_1 \times K_2$, where $|K|$ is the size of the set K with respect to the appropriate measure.

4.1. Weyl-covariant channels and their joint channels

We are interested in channels $\Phi \in \mathbf{Ch}_W^W$, i.e., channels that behave covariantly under phase-space translations and compatibility conditions of pairs of such channels. The channels $\Phi \in \mathbf{Ch}_W^W$ have been completely characterized already in [4, 14] in the case of continuous and finite phase spaces and completely positive maps between CCR-algebras have been fully described in [5]. In Lemma 4.4 below we give a version of the result in [5] for normal channels adapted to the general phase space described above. From Lemma 4.4 we

obtain Corollary 4.7 giving a characterization of channels $\Phi \in \mathbf{Ch}_W^W$ which generalizes the characterizations of [4, 14] to the general phase space case, although this particular generalization is quite marginal. More importantly, from Lemma 4.4, we derive the general form of covariant joint channels for compatible pairs of channels within $\mathbf{Ch}_W^W$ in Theorem 4.6. Finally, using Proposition 3.4 and Theorem 4.6, we characterize covariant pairs of channels within $\mathbf{Ch}_W^W$.

In the following characterizations of covariant channels, the following notion often reoccurs.

Definition 4.3. Let Z be a non-empty set. We say that a function $F : Z \times Z \to \mathbb{C}$ is a *positive kernel*, if, for all $n \in \mathbb{N}$ and $z_1, \ldots, z_n \in Z$,

$$\big(F(z_i, z_j) \big)_{i,j=1}^n \geq 0.$$

Especially, if Z is an Abelian group, we say that a function $f : Z \to \mathbb{C}$ is a *function of positive type* if the function $Z \times Z \ni (z_1, z_2) \mapsto f(z_1^{-1} z_2) \in \mathbb{C}$ is a positive kernel.

Lemma 4.4. *Let $G_i = \mathbf{X}_i \times \hat{\mathbf{X}}_i$, $i = 1, 2$, be Abelian phase spaces (like the ones introduced above), $\omega_i : G_i \times G_i \to \mathbb{R}$ be the symplectic forms on G_i, $i = 1, 2$, $\mathcal{H}_i = L^2(\mathbf{X}_i)$, $i = 1, 2$, and $W_i : G_i \to \mathcal{U}(\mathcal{H}_i)$ be the Weyl representations associated with G_i, $i = 1, 2$. Suppose that $\delta : G_2 \to G_1$ is a continuous group homomorphism. A map $\Psi : \mathcal{L}(\mathcal{H}_2) \to \mathcal{L}(\mathcal{H}_1)$ defined through*

$$\Psi\big(W_2(g)\big) = f(g)[W_1 \circ \delta](g), \qquad g \in G_2, \tag{4.2}$$

with a function $f : G_2 \to \mathbb{C}$ is a channel if and only if f is continuous, $f(e_2) = 1$ (e_2 being the neutral element of G_2), and the function

$$G_2 \times G_2 \ni (g, h) \mapsto e^{\frac{i}{2}\left[\omega_2(g,h) - \omega_1\left(\delta(g), \delta(h)\right)\right]} f(g^{-1}h) \in \mathbb{C}$$

is a positive kernel.

Proof. Since $W(g)$, $g \in G_2$, are linearly independent and they span an ultra-weakly dense operator system in $\mathcal{L}(\mathcal{H}_2)$, Equation (4.2) defines a linear map. It remains to pinpoint the necessary and sufficient conditions for complete positivity, unitality, and normality the function f has to satisfy.

Pick $n, m \in \mathbb{N}$, $\gamma_{i,r} \in \mathbb{C}$, $g_{i,r} \in G_2$, $r = 1, \ldots, m$, and operators $C_i = \sum_{r=1}^m \gamma_{i,r} W_2(g_{i,r})$, $i = 1, \ldots, n$. Due to normality, complete positivity of Ψ is fully determined by the conditions

$$
\begin{aligned}
0 \quad \leq \quad & \sum_{i,j=1}^n \langle \varphi_i | \Psi(C_i^* C_j) \varphi_j \rangle = \sum_{i,j=1}^n \sum_{r,s=1}^m \overline{\gamma_{i,r}} \gamma_{j,s} \langle \varphi_i | \Psi\big(W_2(g_{i,r})^* W_2(g_{j,s})\big) \varphi_j \rangle \\
= \quad & \sum_{i,j=1}^n \sum_{r,s=1}^m \overline{\gamma_{i,r}} \gamma_{j,s} e^{\frac{i}{2}\omega_2(g_{i,r}, g_{j,s})} f(g_{i,r}^{-1} g_{j,s}) \langle \varphi_i | [W_1 \circ \delta](g_{i,r}^{-1} g_{j,s}) \varphi_j \rangle \\
= \quad & \sum_{i,j=1}^n \sum_{r,s=1}^m \overline{\gamma_{i,r}} \gamma_{j,s} e^{\frac{i}{2}\left[\omega_2(g_{i,r}, g_{j,s}) - \omega_1\left(\delta(g_{i,r}), \delta(g_{j,s})\right)\right]} f(g_{i,r}^{-1} g_{j,s}) \times \\
& \times \langle [W_1 \circ \delta](g_{i,r}) \varphi_i | [W_1 \circ \delta](g_{j,s}) \varphi_j \rangle.
\end{aligned}
$$

Careful reading reveals that these conditions equal the kernel condition of the claim.

For normality, let us define the Fourier transform [20] $\hat{T} : G_2 \to \mathbb{C}$ of a trace-class operator $T \in \mathcal{T}(\mathcal{H}_2)$, $\hat{T}(g) = \text{tr}\,[TW_2(g)]$, $g \in G$. A counterpart of the Bochner theorem holds [20, Proposition 3.4]: a function $F : G_2 \to \mathbb{C}$ is a Fourier transform of a positive trace-class operator $T \in \mathcal{T}(\mathcal{H}_2)$ if and only if F is continuous and of positive type.

Channel Ψ is normal if and only if $F_S : G_2 \ni g \mapsto \text{tr}\,[S\Psi(W_2(g))] \in \mathbb{C}$ is the Fourier transform of some positive $T \in \mathcal{T}(\mathcal{H}_2)$ for every positive $S \in \mathcal{T}(\mathcal{H}_1)$. The latter condition of the above Bochner theorem is easily seen to reduce to the kernel condition of the claim. It thus remains to require that F_S be continuous for every (positive) $S \in \mathcal{T}(\mathcal{H}_1)$. The result of Remark 4.2 can be used to find that, for any compact $K_1 \subseteq \mathbf{X}_1$ and $K_2 \subseteq \hat{\mathbf{X}}_1$ there is $S \in \mathcal{T}(\mathcal{H}_1)$ such that $\hat{S}$ is continuous and and non-zero on $K_1 \times K_2$. Since

$$F_S(g) = f(g)[\hat{S} \circ \delta](g), \qquad g \in G_2,$$

and $\hat{S} \circ \delta$ is continuous for every $S \in \mathcal{T}(\mathcal{H}_1)$, the above (together with the fact that G is locally compact) tells us that f has to be continuous. If f is continuous, the first condition of the modified Bochner theorem is automatically satisfied for any F_S, $S \in \mathcal{T}(\mathcal{H}_1)$. $\square$

Let us define an additional representation $W^{\otimes 2} : G \to \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$, $W^{\otimes 2}(g) = W(g) \otimes W(g)$, $g \in G$. Since the Hilbert space $\mathcal{H}$ is separable, the representation W is strongly continuous, and, as a locally compact Abelian group, G is amenable, Proposition 3.4 implies that two channels $\Phi_i \in \mathbf{Ch}_W^W$, $i = 1, 2$, are compatible if and only if they are margins of a channel $\Psi \in \mathbf{Ch}_W^{W^{\otimes 2}}$. Such covariant joint channels are also good candidates for optimal cloning channels due to the irreducibility of W. Before characterizing the covariant joint channels, let us define a class of functions that will shortly be seen to be associated with these channels.

Definition 4.5. We denote by $C_S^0(G \times G)$ the set of continuous functions $f : G \times G \to \mathbb{C}$ with $f(e, e) = 1$ such that the function

$$G^2 \times G^2 \ni (g_1, h_1; g_2, h_2) \mapsto e^{-\frac{i}{2}\left(\omega(g_1, h_2) + \omega(h_1, g_2)\right)} f(g_1^{-1}g_2, h_1^{-1}h_2) \in \mathbb{C}$$

is a positive kernel.

Theorem 4.6. *For any* $\Psi \in \mathbf{Ch}_W^{W^{\otimes 2}}$ *there is a unique* $f_\Psi \in C_S^0(G \times G)$ *such that*

$$\Psi(W(g) \otimes W(h)) = f_\Psi(g, h)W(gh), \qquad g, h \in G. \tag{4.3}$$

Moreover the map $\mathbf{Ch}_W^{W^{\otimes 2}} \ni \Psi \mapsto f_\Psi \in C_S^0(G \times G)$ *is bijective.*

Proof. Pick $\Psi \in \mathbf{Ch}_W^{W^{\otimes 2}}$. Fix $g, g', h \in G$. Using Equation (4.1) and the $(W, W^{\otimes 2})$-covariance of Ψ, we have

$$W(g')W(gh)^*\Psi\big(W(g) \otimes W(h)\big)$$
$$= e^{\frac{i}{2}\omega(g',gh)}W(gh)^*W(g')\Psi\big(W(g) \otimes W(h)\big)W(g')^*W(g')$$
$$= e^{\frac{i}{2}\omega(g',gh)}W(gh)^*\Psi\big(W(g')W(g)W(g')^* \otimes W(g')W(h)W(g')^*\big)W(g')$$
$$= W(gh)^*\Psi\big(W(g) \otimes W(h)\big)W(g').$$

Thus, for all $g, h \in G$, the operator $W(gh)^*\Psi\big(W(g) \otimes W(h)\big)$ commutes with the representation W and is thus a scalar multiple of the identity operator. Hence, there is a function $f_\Psi : G \times G \to \mathbb{C}$ such that Equation (4.3) holds.

The claim now follows from Lemma 4.4 upon noticing that $G \times G$ can be identified with the product phase space $\mathbf{X}^2 \times \hat{\mathbf{X}}^2$ and defining the homomorphism $\delta : G \times G \to G$, $\delta(g, h) = gh$, $g, h \in G$. $\qquad\square$

We may make a couple of observations on the functions $f \in C_S^0(G \times G)$ which highlight the limitations of compatibility of Weyl-covariant channels. Let us first show that any $f \in C_S^0(G \times G)$ is of positive type. Pick $f \in C_S^0(G \times G)$. We note that the positive kernel of Definition 4.5 remains positive even when we replace $g_i \to g_i^{-1}$ and $h_i \to h_i^{-1}$, $i = 1, 2$. After the change $g_1 \leftrightarrow g_2$ and $h_1 \leftrightarrow h_2$, we obtain that $G^2 \times G^2 \ni (g_1, h_1; g_2, h_2) \mapsto e^{\frac{i}{2}\big(\omega(g_1,h_2)+\omega(h_1,g_2)\big)}f(g_1^{-1}g_2, h_1^{-1}h_2) \in \mathbb{C}$ is a positive kernel. When $(a_{i,j})_{i,j=1}^n$ and $(b_{i,j})_{i,j=1}^n$ are positive definite, also the entrywise product $(a_{i,j}b_{i,j})_{i,j=1}^n$ is positive. Hence, the pointwise product of two positive kernels is also a positive kernel. Thus, the kernel condition of Definition 4.5 and its above modification imply that f is a function of positive type. We also immediately observe from the kernel condition of Definition 4.5 that, for any $f \in C_S^0(G \times G)$, $f(\cdot, e)$ and $f(e, \cdot)$ are of positive type.

Let us also note that the function $f : G \times G \to \mathbb{C}$, $f(g, h) = e^{-\frac{i}{2}\omega(g,h)}$, $g, h \in G$, is not in $C_S^0(G \times G)$. Indeed, if this was not the case, there would be a channel $\Psi \in \mathbf{Ch}_W^{W^{\otimes 2}}$,

$$\Psi\big(W(g) \otimes W(h)\big) = f(g, h)W(gh) = W(g)W(h), \qquad g, h \in G.$$

Since operators $W(g) \otimes W(h)$, $g, h \in G$, span an ultraweakly dense operator system, it follows that this channel would be the perfect cloner $A \otimes B \mapsto AB$, which is impossible. To directly see that $f \notin C_S^0(G \times G)$, pick $g, h \in G \setminus \{e\}$ such that $g \neq h$. If $f \in C_S^0(G \times G)$, the kernel condition in Definition 4.5 would immediately imply

$$\begin{pmatrix} 1 & e^{-\frac{i}{2}S(g,h)} \\ e^{-\frac{i}{2}S(g,h)} & 1 \end{pmatrix} \geq 0,$$

which is false. Thus, $f \notin C_S^0(G \times G)$. In fact, the above inequality also shows that f is not of positive type either.

We now turn our attention to Weyl-covariant channels $\Phi \in \mathbf{Ch}_W^W$ and their structure. Recall that the classical Bochner theorem states that a function $f : G \to \mathbb{C}$ is a continuous function of positive type such that $f(e) = 1$ if and only if there is a probability measure μ on the Borel σ-algebra of G such that f is the Fourier transform of μ, i.e.,

$$f(g) = \int_G e^{-i\omega(g,h)} \, d\mu(h), \qquad g \in G.$$

Recall that we may take $(g,h) \mapsto e^{i\omega(g,h)}$ as the dual form for G. As in the proof of Theorem 4.6, we see that, for any $\Phi \in \mathbf{Ch}_W^W$ there is a unique function $f_\Phi : G \to \mathbb{C}$ and $\Phi\big(W(g)\big) = f_\Phi(g)W(g)$ for all $g \in G$. According to Lemma 4.4, f_Φ is a continuous function of positive type with $f_\Phi(e) = 1$ and the map $\Phi \mapsto f_\Phi$ between $\mathbf{Ch}_W^W$ and the set of continuous functions $f : G \to \mathbb{C}$ of positive type with $f(e) = 1$ is bijective. Let $\Phi \in \mathbf{Ch}_W^W$ and μ_Φ be the probability measure on the Borel σ-algebra of G whose Fourier transform coincides with f_Φ. It follows that, for any $g \in G$,

$$
\begin{aligned}
\Phi\big(W(g)\big) &= f_\Phi(g)W(g) = \int_G e^{-i\omega(g,h)} \, d\mu_\Phi(h)W(g) \\
&= \int_G e^{-i\omega(g,h)} W(h)^* W(h) W(g) \, d\mu_\Phi(h) \\
&= \int_G W(h)^* W(g) W(h) \, d\mu_\Phi(h).
\end{aligned}
$$

Thus we arrive at the following characterization of $\mathbf{Ch}_W^W$ which is a slight generalization of the characterizations in [4, 14]:

Corollary 4.7. *For a channel $\Phi \in \mathbf{Ch}(\mathcal{H}, \mathcal{H})$, we have $\Phi \in \mathbf{Ch}_W^W$ if and only if there is a continuous function $f_\Phi : G \to \mathbb{C}$ of positive type with $f(e) = 1$ such that*

$$\Phi\big(W(g)\big) = f_\Phi(g)W(g), \qquad g \in G,$$

or, equivalently, there is a Borel probability measure μ_Φ on G such that

$$\Phi(B) = \int_G W(g)^* B W(g) \, d\mu_\Phi(g), \qquad B \in \mathcal{L}(\mathcal{H}).$$

We go on to give strict characterizations of compatibility of Weyl-covariant channels. Recalling Remark 3.2, we can make a simple observation.

Remark 4.8. Pick Φ_1, $\Phi_2 \in \mathbf{Ch}_W^W$ and let $\mu_1 := \mu_{\Phi_1}$ and $\mu_2 := \mu_{\Phi_2}$ be the Borel probability measures on G associated to Φ_1 and Φ_2 by Corollary 4.7. Equivalently, we may consider the continuous functions $f_i := f_{\Phi_i} : G \to \mathbb{C}$ of positive type with $f_i(e) = 1$, $i = 1, 2$, corresponding to these channels by Corollary 4.7. Let k_1, $k_2 \in G$. *The channels Φ_1 and Φ_2 are compatible if and only if the channels $\Phi_1^{k_1}$ and $\Phi_2^{k_2}$ associated, respectively, with the measures $\mu_1^{k_1}$ and $\mu_2^{k_2}$, where $\mu_i^{k_i}(X) = \mu_i(k_i^{-1}X)$ for all Borel sets $X \subseteq G$, $i = 1, 2$, are compatible.* The latter channels are equivalently associated with the functions $f_i^{k_i}$, $f_i^{k_i}(g) = e^{-i\omega(g,k_i)}f_i(g)$, $g \in G$, $i = 1, 2$. To see this,

define Δ_1, $\Delta_2 \in \mathbf{Ch}_W^W$, $\Delta_i(B) = W(k_i)^* B W(k_i)$, $B \in \mathcal{L}(\mathcal{H})$, $i = 1, 2$. It is easily checked that $\Phi_i^{k_i} = \Phi_i \circ \Delta_i$, $i = 1, 2$. Thus, $\Phi_1^{k_1}$ and $\Phi_2^{k_2}$ are post-processings of Φ_1 and Φ_2. Since Δ_1 and Δ_2 are unitary, Φ_1 and Φ_2 are also post-processings of $\Phi_1^{k_1}$ and $\Phi_2^{k_2}$. The claim now follows from Remark 3.2.

The first compatibility characterization given in this paper for Weyl-covariant channels follows directly from Proposition 3.4 and Theorem 4.6.

Corollary 4.9. *Let $\Phi_i \in \mathbf{Ch}_W^W$, $i = 1, 2$, and let $f_i : G \to \mathbb{C}$ be the corresponding continuous functions of positive type with $f_i(e) = 1$ such that $\Phi_i\big(W(g)\big) = f_i(g)W(g)$ for all $g \in G$, $i = 1, 2$. The channels Φ_1 and Φ_2 are compatible if and only if there is a function $f \in C_S^0(G \times G)$ such that*

$$f_1(g) = f(g, e), \quad f_2(g) = f(e, g), \qquad g \in G.$$

Proof. As already noted, Φ_1 and Φ_2 are compatible if and only if they are margins of a covariant channel $\Psi \in \mathbf{Ch}_W^{W^{\otimes 2}}$. Fix such a covariant joint channel $\Psi \in \mathbf{Ch}_W^{W^{\otimes 2}}$ and fix $f = f_\Psi$. It follows that, for all $g \in G$,

$$f_1(g)W(g) = \Phi_1\big(W(g)\big) = \Psi\big(W(g) \otimes W(e)\big) = f(g, e)W(g),$$

and similarly for f_2. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4.2. Compatibility conditions from covariant dilations

We now look for compatibility conditions for Weyl-covariant channels using Proposition 3.6. The goal of this section is to characterize all the Weyl-covariant channels which are compatible with a fixed $\Phi_1 \in \mathbf{Ch}_W^W$; this is done in Theorem 4.12. We have to impose a particular regularity assumption on Φ_1 though, namely, we assume that the probability measure μ_{Φ_1} associated to Φ_1 by Corollary 4.7 is absolutely continuous with respect to the Haar measure of G.

We start with some notational definitions. From now on, fix a Haar measure dg on G; the particular choice of measure is irrelevant. Whenever μ is a positive measure on the Borel σ-algebra of G which is absolutely continuous with respect to dg, denote the Radon-Nikodým derivative of μ with respect to dg by ρ_μ and $\operatorname{supp}\mu := \rho_\mu^{-1}\big((0, \infty]\big)$. From now on, all the 'almost all' (a.a.) phrases are to be understood with respect to dg.

Remark 4.10. Suppose that $\Phi_1 \in \mathbf{Ch}_W^W$ and denote by $\mu := \mu_{\Phi_1}$ the Borel probability measure associated to Φ_1 by Corollary 4.7. Assume that μ is absolutely continuous with respect to dg so that we may define the Radon-Nikodým derivative ρ_μ. Denote $\mathcal{L} := L^2(\mu)$ and define the isometry $J : \mathcal{H} \to \mathcal{H} \otimes \mathcal{L}$, $(J\varphi)(g) = \sqrt{\rho_\mu(g)}W(g)\varphi$ for all $\varphi \in \mathcal{H}$ and $g \in \operatorname{supp}\mu$; we identify here $\mathcal{H} \otimes \mathcal{L}$ with the space of (μ-equivalence classes of) functions $\psi : \operatorname{supp}\mu \to \mathcal{H}$ such that $\int_{\operatorname{supp}\mu} \|\psi(g)\|^2 \, d\mu(g) < \infty$. We also define the unitary representation $\overline{W} : G \to \mathcal{U}(\mathcal{L})$,

$$\big(\overline{W}(g)f\big)(h) = e^{i\omega(g,h)}f(h), \qquad g \in G, \quad h \in G.$$

One easily checks that $(\mathcal{L}, J, \overline{W})$ is a covariant Stinespring dilation for Φ_1. According to Proposition 3.6, in order to identify those channels $\Phi_2 \in \mathbf{Ch}_W^W$ which are compatible with Φ_1, we have to characterize the covariance structure $\mathbf{Ch}_{\overline{W}}^W$. This is done in the following lemma.

Lemma 4.11. *Suppose that μ is a probability measure on the Borel σ-algebra of G which is absolutely continuous with respect to dg and let $\mathcal{L} := L^2(\mu)$. Let the Hilbert space $\mathcal{L}$ and the representation $\overline{W}$ be as in Remark 4.10. For any $\tilde{\Phi} \in \mathbf{Ch}_{\overline{W}}^W$ there is $\zeta : G \times G \to \mathbb{C}$ such that $\zeta(\cdot, h)$ is continuous and $\zeta(e, h) = 1$ for a.a. $h \in G$, $\zeta(g, \cdot)$ is measurable and essentially bounded for all $g \in G$, the function*

$$G \times G \ni (g_1, g_2) \mapsto e^{\frac{i}{2}\omega(g_1, g_2)} \zeta(g_1^{-1} g_2, g_1 h) \in \mathbb{C} \tag{4.4}$$

is a positive kernel for a.a. $h \in G$, and

$$\big(\tilde{\Phi}\big(W(g)\big)f\big)(h) = \zeta(g, h)f(gh)\sqrt{\frac{\rho_\mu(gh)}{\rho_\mu(h)}}, \qquad g \in G, \quad h \in G, \quad f \in \mathcal{L}. \tag{4.5}$$

Moreover, for any $\zeta : G \times G \to \mathbb{C}$ continuous in the first argument, measurable and essentially bounded in the second argument, with $\zeta(e, \cdot) = 1$ a.e., and satisfying the condition that the function of Equation (4.4) be a positive kernel, the Equation (4.5) defines a $\tilde{\Phi} \in \mathbf{Ch}_{\overline{W}}^W$.

Proof. Pick $\tilde{\Phi} \in \mathbf{Ch}_{\overline{W}}^W$. It is easy to see that

$$\overline{W}(g')\tilde{\Phi}\big(W(g)\big) = e^{i\omega(g', g)}\tilde{\Phi}\big(W(g)\big)\overline{W}(g'), \qquad g, g' \in G.$$

Denote by $L^2(G)$ the Hilbert space of (equivalence classes of) functions which are square integrable with respect to dg. One can define a decomposable isometry $K : \mathcal{L} \to L^2(G)$, $(Kf)(h) = \rho_\mu(h)^{1/2}f(h)$, $f \in \mathcal{L}$, $h \in G$. Clearly, for all $F \in L^2(G)$, $(K^*F)(h) = \rho_\mu(h)^{-1/2}F(h)$ whenever $h \in \operatorname{supp}\mu$ and $(K^*F)(h) = 0$ otherwise. Define $\tilde{\Phi}_K := K\tilde{\Phi}(\cdot)K^*$ and the (canonical) diagonal representation $D : G \to \mathcal{U}\big(L^2(G)\big)$, $\big(D(g)F\big)(h) = e^{i\omega(g, h)}F(h)$, $g, h \in G$, $F \in L^2(G)$. It is easy to see that $K\overline{W}(g) = D(g)K$ and $D(g')\tilde{\Phi}_K\big(W(g)\big) = e^{i\omega(g', g)}\tilde{\Phi}_K\big(W(g)\big)\delta(g')$, $g, g' \in G$. According to [9, Lemma 1], this is equivalent with the existence of a function $\zeta : G \times G \to \mathbb{C}$ such that $\zeta(g, \cdot)$ is measurable and essentially bounded for all $g \in G$ and

$$\big(\tilde{\Phi}_K\big(W(g)\big)F\big)(h) = \zeta(g, h)F(gh), \qquad g \in G, \quad h \in G, \quad F \in L^2(G).$$

Since the Radon-Nikodým derivative of the measure $X \mapsto \mu(gX)$, $g \in G$, is $h \mapsto \rho_\mu(gh)$, the above is equivalent with Equation (4.5). The unitality of $\tilde{\Phi}$ implies $\zeta(e, h) = 1$ for a.a. $h \in G$.

We go on to the conditions the complete positivity of $\tilde{\Phi}$ sets on ζ. We again fix $n, m \in \mathbb{N}$, $f_1, \ldots, f_n \in \mathcal{L}$, $\gamma_{i,r} \in \mathbb{C}$, and $g_{i,r} \in G$, $i = 1, \ldots, n$, $r = 1, \ldots, m$, and define $C_i = \sum_{r=1}^m \gamma_{i,r}W(g_{i,r})$. The complete positivity is, again, fully characterized by inequalities of the following kind (where the

latter equality is obtained through a direct calculation using Equation (4.5) and the cocycle properties of the function $(g, h) \mapsto \rho_\mu(gh)$:

$$
\begin{aligned}
0 \;\leq\; & \sum_{i,j=1}^{n} \langle f_i | \tilde{\Phi}(C_i^* C_j) f_j \rangle \\
=\; & \sum_{i,j=1}^{n} \sum_{r,s=1}^{m} \overline{\gamma_{i,r}} \gamma_{j,s} e^{\frac{i}{2}\omega(g_{i,r}, g_{j,s})} \int_G \overline{f_i(g_{i,r}h)} f_j(g_{j,s}h) \zeta(g_{i,r}^{-1} g_{j,s}, g_{i,r}h) \times \\
& \times \sqrt{\rho_\mu(g_{i,r}h)\rho_\mu(g_{j,s}h)} \, dh \\
=\; & \sum_{i,j=1}^{n} \sum_{r,s=1}^{m} \overline{\gamma_{i,r}} \gamma_{j,s} e^{\frac{i}{2}\omega(g_{i,r}, g_{j,s})} \int_G \overline{(Kf_i)(g_{i,r}h)} (Kf_j)(g_{j,s}h) \times \\
& \times \zeta(g_{i,r}^{-1} g_{j,s}, g_{i,r}h) \, dh.
\end{aligned}
$$

After careful reading, one finds that this is equivalent with the kernel of Equation (4.4) being positive.

Let us concentrate on the normality of $\tilde{\Phi}$. We probe this question again using the Fourier transform $\widehat{\tilde{\Phi}_*(S)}$ similarly as in the proof of Lemma 4.4. For simplicity, suppose $S = |f\rangle\langle f|$, $f \in \mathcal{L}$. It follows, for every $g \in G$,

$$
\begin{aligned}
\widehat{\tilde{\Phi}_*(|f\rangle\langle f|)}(g) = \langle f | \tilde{\Phi}(W(g)) f \rangle \;&=\; \int_G \overline{f(h)} f(gh) \zeta(g,h) \sqrt{\rho_\mu(gh)\rho_\mu(h)} \, dh \\
&=\; \int_G \overline{(Kf)(h)} (Kf)(gh) \zeta(g,h) \, dh.
\end{aligned}
$$

As a function of g, the above expression has to be continuous and of positive type. The latter condition is easily seen to reduce to (4.4). Using polarization, we see that the functions $g \mapsto \int_G \overline{(Kf_1)(k)}(Kf_2)(gh)\zeta(g,h)\,dh = \langle Kf_1 | \tilde{\Phi}_K(W(g)) Kf_2 \rangle$, $f_1, f_2 \in \mathcal{L}$ also have to be continuous. Clearly we may replace Kf_i, $i = 1, 2$, with arbitrary functions $F_i \in L^2(G)$, $i = 1, 2$. Especially, for any compact $X \subset G$ and any $F \in C_0(G)$ (continuous and compactly supported functions on G), fixing $F_1 = \chi_X$ and $F_2 = F$, the map $g \mapsto \int_X \zeta(g,h) F(gh)\,dh$ is continuous. Since this holds for any compact X, we have that, for a.a. $h \in G$, the function $g \mapsto \zeta(g,h)F(gh)$ is continuous for any $F \in C_0(G)$. This implies that $\zeta(\cdot, h)$ is continuous for a.a. $h \in G$.

Thus, ζ satisfies the conditions of the claim. The reverse claim already follows from the proof thus far. $\qquad\square$

Any positive kernel $F : G \times G \to \mathbb{C}$ possesses a *Kolmogorov construction*, i.e., a pair $(\mathcal{M}, \eta)$ where $\mathcal{M}$ is a Hilbert space and $\eta : G \to \mathcal{M}$ is a function such that $\beta(g,h) = \langle \eta(g) | \eta(h) \rangle$ for all g, $h \in G$. Amongst these constructions there is an essentially unique minimal one where the vectors $\eta(g)$, $g \in G$, span a dense subspace of $\mathcal{M}$. Although the diagonal $\Delta = \{(g,g) \,|\, g \in G\} \subseteq G \times G$ is of zero measure, we may define the *diagonal values of F*: The diagonal value at $g \in G$ is given by $F^\Delta(g) := \|\eta(g)\|^2$, where $(\mathcal{M}, \eta)$ is any Kolmogorov

construction for F. We are finally ready to describe all the Weyl-covariant channels which are compatible with the channel Φ_1 of Remark 4.10.

Theorem 4.12. *Let Φ_1, $\Phi_2 \in \mathbf{Ch}_W^W$, and assume that the Borel probability measure $\mu = \mu_{\Phi_1}$ on G associated with Φ_1 by Corollary 4.7 is absolutely continuous with respect to dg. These channels are compatible if and only if there is a measurable positive kernel $\beta : G \times G \to \mathbb{C}$ such that $\beta(\cdot, g)$ and $\beta(g, \cdot)$ are continuous for a.a. $g \in G$, $\beta^\Delta(g) = 1$ for a.a. $g \in G$, and*

$$\Phi_2\big(W(g)\big) = \int_G \beta(h, gh) \sqrt{\rho_\mu(h)\rho_\mu(gh)}\, dh\, W(g), \qquad g \in G. \qquad (4.6)$$

Proof. Let $(\mathcal{L}, J, \overline{W})$ be the covariant Stinespring dilation for Φ_1 presented in Remark 4.10. According to Proposition 3.6, Φ_1 and Φ_2 are compatible if and only if there is $\tilde{\Phi}_2 \in \mathbf{Ch}_{\overline{W}}^W$ such that

$$\Phi_2(B) = J^*\big(\mathbb{1}_\mathcal{H} \otimes \tilde{\Phi}_2(B)\big)J, \qquad B \in \mathcal{L}(\mathcal{H}). \qquad (4.7)$$

Since channels $\tilde{\Phi} \in \mathbf{Ch}_{\overline{W}}^W$ are, according to Lemma 4.11, in one-to-one correspondence with the functions ζ as detailed in the claim of Lemma 4.11, there must exist such ζ for $\tilde{\Phi}_2$.

We have that β_k, $\beta_k(g, h) = e^{\frac{i}{2}\omega(g, h)}\zeta(g^{-1}h, gk)$, $g, h \in G$, is a positive kernel with $\beta_k^\Delta = 1$ a.e. for a.a. $k \in G$. Suppose that $k \in G$ is such that these conditions hold. Define $\beta = \beta_e$, whence it follows that $\beta(g, h) = e^{\frac{i}{2}\left(\omega(g, k) - \omega(h, k)\right)}\beta_k(gk^{-1}, hk^{-1})$ implying that β is a positive kernel with $\beta^\Delta(g) = 1$ for a.a. $g \in G$. It follows immediately that $\beta(g, \cdot)$ is continuous for a.a. $g \in G$. Hence also $g \mapsto \beta(g, h) = \overline{\beta(h, g)}$ is continuous for a.a. $h \in G$. Substituting the channel $\tilde{\Phi}_2 = \tilde{\Phi}$ of (4.5) with $\zeta(g, h) = e^{\frac{i}{2}\omega(g, h)}\beta(h, gh)$, $g \in G$, $h \in G$, into (4.7), we obtain Equation (4.6). On the other hand, $\zeta : G \times G \to \mathbb{C}$, $\zeta(g, h) = e^{\frac{i}{2}\omega(g, h)}\beta(h, gh)$, $g \in G$, $h \in G$, satisfies the conditions of the claim of Lemma 4.11. $\qquad \square$

Remark 4.13. We may rephrase Theorem 4.12 in the following form: *Suppose that $\Phi_1 \in \mathbf{Ch}_W^W$ is as in Theorem 4.12. The continuous function $f_2 : G \to \mathbb{C}$ of positive type with $f_2(e) = 1$ associated according to Corollary 4.7 with a channel $\Phi_2 \in \mathbf{Ch}_W^W$ which is compatible with Φ_1 can be written in the form*

$$f_2(g) = \int_G \beta(h, gh) \sqrt{\rho_\mu(h)\rho_\mu(gh)}\, dh, \qquad g \in G, \qquad (4.8)$$

where $\beta : G \times G \to \mathbb{C}$ is a positive kernel continuous separately in both arguments whose diagonal values are 1 a.e.

Suppose now that Φ_1, $\Phi_2 \in \mathbf{Ch}_W^W$ are associated with Borel probability measures measures $\mu_1 := \mu_{\Phi_1}$ and $\mu_2 := \mu_{\Phi_2}$ according to Corollary 4.7. Assume that both μ_1 and μ_2 are absolutely continuous with respect to dg and denote their Radon-Nikodým derivates with respect to dg by ρ_1 and, respectively, ρ_2. Also assume that $\hat{\rho}_2 = f_2 \in L^1(G)$ so that we may use the inverse Fourier transform to f_2 to obtain ρ_2. In this situation, Φ_1 and Φ_2 are

compatible if and only if there is a positive kernel $\beta : G \times G \to \mathbb{C}$ such that $\beta(\cdot, g)$ and $\beta(g, \cdot)$ are continuous and $\beta^\Delta(g) = 1$ for a.a. $g \in G$ and

$$\rho_2(k) = \int_G \int_G e^{i\omega(g^{-1}h, k)} \beta(g, h) \sqrt{\rho_1(g)\rho_1(h)} \, dg \, dh, \qquad k \in G. \qquad (4.9)$$

The above equation is obtained from (4.8) by taking the inverse Fourier transformation of f_2.

Especially if $\sqrt{\rho_1(\cdot)} \in L^1(G)$, any channel $\Phi_2 \in \mathbf{Ch}_W^W$ compatible with Φ_1 is associated with a measure μ_2 which is absolutely continuous with respect to dg and, consequently, compatibility is characterized by Equation (4.9). Indeed, if this is the case, we may evaluate for the continuous function f_2 of positive type associated with Φ_2

$$
\begin{aligned}
\int_G |f_2(g)| \, dg &= \int_G \left| \int_G \beta(h, hg) \sqrt{\rho_1(h)\rho_1(gh)} \, dh \right| dg \\
&\leq \int_G \int_G \underbrace{|\beta(h, gh)|}_{\leq 1 \text{ a.e.}} \sqrt{\rho_1(h)\rho_1(gh)} \, dh \, dg \\
&\leq \left(\int_G \sqrt{\rho_1(h)} \, dh \right)^2 < \infty.
\end{aligned}
$$

Thus, the inverse Fourier transform can be applied to f_2 yielding the density function ρ_2 such that $\Phi_2(B) = \int_G \rho_2(g) W(g)^* B W(g) \, dg$, $B \in \mathcal{L}(\mathcal{H})$.

It should be noted that, in Proposition 3.6, the correspondence between channels $\tilde{\Phi}_2 \in \mathbf{Ch}_{\overline{U}}^{V_2}$ and $\Phi_2 \in \mathbf{Ch}_U^{V_2}$ set up in (3.1) is many-to-one. Indeed, for any joint channel $\Psi \in \mathbf{Ch}_U^{V_1, 2}$ for compatible $\Phi_i \in \mathbf{Ch}_U^{V_i}$, $i = 1, 2$, there is a unique $\tilde{\Phi}_2 \in \mathbf{Ch}_{\overline{U}}^{V_2}$ such that $\Psi(A \otimes B) = V^*\big(A \otimes \tilde{\Phi}_2(B)\big)V$, $A \in \mathcal{L}(\mathcal{K}_1)$, $B \in \mathcal{L}(\mathcal{K}_2)$, when we fix a covariant minimal Stinespring dilation $(\mathcal{L}, V, \overline{U})$ for $\Phi_1 \in \mathbf{Ch}_U^{V_2}$. However, as any compatible pair of channels typically has infinitely many joint channels, there are typically many $\tilde{\Phi}_2$ such that Equation 3.1 is satisfied. From this it follows that, for any $\Phi_2 \in \mathbf{Ch}_W^W$ compatible with $\Phi_1 \in \mathbf{Ch}_W^W$ satisfying the conditions of Theorem 4.12, there are *a priori* several kernels β satisfying Equation (4.6) or, equivalently, (4.8).

5. Physical phase spaces

Next we shall adapt the results of Section 4 in a couple of physically motivated phase spaces. These are the continuous and finite phase spaces introduced in the beginning of Section 4.

5.1. Phase space $\mathbb{R}^N \times \mathbb{R}^N$

We now take a closer look at compatibility of Weyl-covariant channels of a spin-0 system in M- and N-dimensional Euclidean configuration spaces $\mathbf{X} = \mathbb{R}^m$, $m = N, M$. In Proposition 5.1 below, we characterize Weyl-covariant Gaussian channels. This helps to give concrete examples of functions in the

skew-positive set $C_S^0(G \times G)$ and thus to see that this set is, indeed, non-empty, at least in the typical situations. After this, as an example, we explore the compatibility conditions of two Weyl-covariant Gaussian channels.

The position shifts and momentum boosts reflected by the representations $U = U_m$ and $V = V_m$ are as in the table in the beginning of Section 4 and the Weyl representation is again given by $W_m(\vec{q}, \vec{p}) = U_m(\vec{q})V_m(\vec{p})$ for all $\vec{q}, \vec{p} \in \mathbb{R}^m$, $m = N, M$. Recall the matrices Ω_m appearing in the symplectic forms. We identify $(\mathbb{R}^m \times \mathbb{R}^m) \times (\mathbb{R}^m \times \mathbb{R}^m) = \mathbb{R}^{4m}$. The set $C_S^0(\mathbb{R}^{4m})$ consists of continuous functions $f : \mathbb{R}^{4m} \to \mathbb{C}$ such that $f(0,0) = 1$, and the function

$$\mathbb{R}^{2m} \times \mathbb{R}^{2m} \ni (\vec{z}_1, \vec{w}_1; \vec{z}_2, \vec{w}_2) \mapsto e^{-\frac{i}{2}\left(\vec{z}_1^T \Omega_m \vec{w}_2 + \vec{w}_1^T \Omega_m \vec{z}_2\right)} f(\vec{z}_1 - \vec{z}_2, \vec{w}_1 - \vec{w}_2) \in \mathbb{C} \tag{5.1}$$

is a positive kernel.

We will characterize the functions in $C_S^0(\mathbb{R}^{4N})$ corresponding to Gaussian channels in $\mathbf{Ch}_W^{W^{\otimes 2}}$. Gaussian channels have been studied in conjunction with compatibility questions earlier, e.g., in [10] albeit in the case of joint measurability of observables. Denote by $\mathcal{H}_m$ the Hilbert space $L^2(\mathbb{R}^m)$ which is the space where W_m operates, $m = N, M$. Recall that a channel $\Psi \in \mathbf{Ch}(\mathcal{H}_N, \mathcal{H}_M)$ is Gaussian, if and only if the Fourier transform $\widehat{\Psi_*(S)}$ is a Gaussian function for any positive $S \in \mathcal{T}(\mathcal{H}_N)$ such that $\hat{S}$ is Gaussian. Equivalently [7], there must be real-entry matrices $A \in \mathcal{M}_{2N \times 2M}(\mathbb{R})$ and $B \in \mathcal{M}_{2M \times 2M}(\mathbb{R})$ satisfying

$$B + i\Omega_M - iA^T \Omega_N A \geq 0 \tag{5.2}$$

and a vector $\vec{c} \in \mathbb{R}^{2M}$ such that $\Psi = \Psi_{A,B,\vec{c}}$,

$$\Psi_{A,B,\vec{c}}\left(W_M(\vec{z})\right) = e^{-\frac{1}{4}\vec{z}^T B \vec{z} - i\vec{c}^T \vec{z}} W(A\vec{z}), \qquad \vec{z} \in \mathbb{R}^{2M}. \tag{5.3}$$

The following result is obtained by directly checking the conditions a Gaussian channel with $M = 2N$ has to satisfy in order to be $(W_N, W_N^{\otimes 2})$-compatible. Note that we may associate $W_N^{\otimes 2}$ with W_{2N} and Ω_{2N} with $\Omega_N \oplus \Omega_N$.

Proposition 5.1. *For a Gaussian channel $\Psi \in \mathbf{Ch}(\mathcal{H}_N, \mathcal{H}_N \otimes \mathcal{H}_N)$, i.e., $\Psi = \Psi_{A,B,\vec{c}}$ where A, B, and $\vec{c}$ satisfy equations (5.2) and (5.3) with $M = 2N$, we have $\Psi \in \mathbf{Ch}_W^{W^{\otimes 2}}$ if and only if $A = \begin{pmatrix} \mathbb{1}_{2N} & \mathbb{1}_{2N} \end{pmatrix}$ and B is a symmetric matrix such that*

$$\begin{pmatrix} B_{1,1} & B_{1,2} - i\Omega \\ B_{1,2}^T + i\Omega^T & B_{2,2} \end{pmatrix} \geq 0 \tag{5.4}$$

when we write $B = (B_{i,j})_{i,j=1}^2$ in the block form. The function associated to Ψ by Equation (4.3) is given by

$$f_\Psi(\vec{z}) = e^{-\frac{1}{4}\vec{z}^T B \vec{z} - i\vec{c}^T \vec{z}}, \qquad \vec{z} \in \mathbb{R}^{4N}.$$

Moreover, a Gaussian channel $\Phi = \Phi_{A_0, B_0, \vec{c}_0} \in \mathbf{Ch}(\mathcal{H}_N, \mathcal{H}_N)$, where A_0, B_0, and $\vec{c}_0$ satisfy equations (5.2) and (5.3) with $A = A_0$, $B = B_0$, $\vec{c} = \vec{c}_0$, and $M = N$, is (W, W)-covariant if and only if $A_0 = \mathbb{1}_{2N}$ and $B_0 \geq 0$.

Especially, choosing $B = \alpha \mathbb{1}_{4N}$ for any $\alpha \geq 1$, inequality (5.4) is satisfied. Thus, the set $C_S^0(\mathbb{R}^{4N})$ contains a wide variety of Gaussian functions.

Example. Let us consider the compatibility of $\Phi_1, \Phi_2 \in \mathbf{Ch}_W^W$ where $\Phi_1 = \Phi_{\mathbb{1}_{2N}, B, \vec{c}}$ is Gaussian from the perspective of Theorem 4.12 and Remark 4.13. We assume that the measure associated with Φ_1 is absolutely continuous with respect to the Lebesgue measure. This is easily seen to be equivalent with B being of full rank; indeed the function $f_1 : \mathbb{R}^{2N} \to \mathbb{C}$, $f_1(\vec{v}) = e^{-\frac{1}{4}\vec{v}^T B \vec{v} - i\vec{c}^T \vec{v}}$, $\vec{v} \in \mathbb{R}^{2N}$, is in $L^1(\mathbb{R}^{2N})$ if and only if B does not have the eigenvalue 0. Taking the inverse Fourier transform of the preceding function, we obtain the density function $\rho_1 : \mathbb{R}^{2N} \to \mathbb{R}$,

$$\rho_1(\vec{w}) = \frac{1}{\pi^N \sqrt{\det B}} e^{-(\Omega\vec{w}+\vec{c})^T B^{-1}(\Omega\vec{w}+\vec{c})}, \qquad \vec{w} \in \mathbb{R}^{2N},$$

of the measure μ_1 associated with Φ_1 with respect to the Lebesgue measure. Note that $\sqrt{\rho_1(\cdot)} \in L^1(\mathbb{R}^{2N})$ implying that any Weyl-covariant channel compatible with Φ_1 is associated with a measure which is absolutely continuous with respect to the Lebesgue measure.

Suppose that Φ_2 is associated with the continuous function f_2 of positive type with $f_2(0) = 1$. According to Theorem 4.12, Φ_1 and Φ_2 are compatible if and only if there is a positive kernel $\beta : \mathbb{R}^{2N} \times \mathbb{R}^{2N} \to \mathbb{C}$ such that $\beta(\cdot, \vec{w})$ and $\beta(\vec{w}, \cdot)$ are continuous and $\beta^\Delta(\vec{w}) = 1$ for a.a. $\vec{w} \in \mathbb{R}^{2N}$ and, for all $\vec{v} \in \mathbb{R}^{2N}$,

$$
\begin{aligned}
f_2(\vec{v}) &= \frac{1}{\pi^N \sqrt{\det B}} \int_{\mathbb{R}^{2N}} \beta(\vec{w}, \vec{v} + \vec{w}) e^{-\frac{1}{2}(\Omega\vec{w}+\vec{c})^T B^{-1}(\Omega\vec{w}+\vec{c})} \times \\
&\quad \times e^{-\frac{1}{2}(\Omega\vec{v}+\Omega\vec{w}+\vec{c})^T B^{-1}(\Omega\vec{v}+\Omega\vec{w}+\vec{c})} \, d\vec{w} \\
&= \frac{e^{-\frac{1}{2}\vec{v}^T \Omega^T B^{-1}\Omega\vec{v}}}{\pi^N \sqrt{\det B}} \int_{\mathbb{R}^{2N}} \beta(\vec{w}-\vec{v}/2+\Omega\vec{c}, \vec{w}+\vec{v}/2+\Omega\vec{c}) \times \\
&\quad \times e^{-\vec{w}^T \Omega^T B^{-1}\Omega\vec{w}} \, d\vec{w}.
\end{aligned}
$$

Replacing the kernel β with the kernel $\beta' : (\vec{v}, \vec{w}) \mapsto \beta(\vec{v}+\Omega\vec{c}, \vec{w}+\Omega\vec{c})$, we see that we may omit the terms $\Omega\vec{c}$ in the equation above. If we further replace β' with $\tilde{\beta} : (\vec{v}, \vec{w}) \mapsto \beta'(\Omega^T B^{1/2}\vec{v}, \Omega^T B^{1/2}\vec{w})$, and make a change of variables, we find that the above Φ_1 and Φ_2 are compatible if and only if there is a positive kernel $\beta : \mathbb{R}^{2N} \times \mathbb{R}^{2N} \to \mathbb{C}$ such that $\beta(\cdot, \vec{v})$ and $\beta(\vec{v}, \cdot)$ are continuous and $\beta^\Delta(\vec{v}) = 1$ for a.a. $\vec{v} \in \mathbb{R}^{2N}$ and

$$f_2(\Omega^T B^{1/2}\vec{v}) e^{\frac{1}{2}\|\vec{v}\|^2} = \frac{1}{\pi^N} \int_{\mathbb{R}^{2N}} \beta(\vec{w}-\vec{v}/2, \vec{w}+\vec{v}/2) e^{-\|\vec{w}\|^2} \, d\vec{w}, \qquad \vec{v} \in \mathbb{R}^{2N}.$$

For two Gaussian channels $\Phi_1 = \Phi_{\mathbb{1}_{2N}, B, \vec{c}}$ and $\Phi_2 = \Phi_{\mathbb{1}_{2N}, C, \vec{d}}$, where B is of full rank, the above necessary and sufficient compatibility condition becomes

$$e^{\frac{1}{2}\|\vec{v}\|^2 - \frac{1}{4}\vec{v}^T B^{1/2}\Omega C \Omega^T B^{1/2}\vec{v} - i\vec{d}^T \Omega^T B^{1/2}\vec{v}} = \frac{1}{\pi^N} \int_{\mathbb{R}^{2N}} \beta(\vec{w}-\vec{v}/2, \vec{w}+\vec{v}/2) \frac{d\vec{w}}{e^{\|\vec{w}\|^2}}$$

for all $\vec{v} \in \mathbb{R}^{2N}$. When we replace β with β',

$$\beta'(\vec{v}, \vec{w}) = e^{i\vec{d}^T \Omega^T B^{1/2}(\vec{v}-\vec{w})} \beta(\vec{v}, \vec{w}), \qquad \vec{v}, \vec{w} \in \mathbb{R}^{2N},$$

we may omit the term $i\vec{d}^T\Omega^T B^{1/2}\vec{v}$ in the above equation. Thus, the above Gaussian Φ_1 and Φ_2 are compatible if and only if there is a positive kernel $\beta : \mathbb{R}^{2N} \times \mathbb{R}^{2N} \to \mathbb{C}$ such that $\beta(\cdot, \vec{v})$ and $\beta(\vec{v}, \cdot)$ are continuous and $\beta^\Delta(\vec{v}) = 1$ for a.a. $\vec{v} \in \mathbb{R}^{2N}$ and, for a.a. $\vec{v} \in \mathbb{R}^{2N}$,

$$e^{\frac{1}{2}\|\vec{v}\|^2 - \frac{1}{4}\vec{v}^T B^{1/2}\Omega C\Omega^T B^{1/2}\vec{v}} = \frac{1}{\pi^N} \int_{\mathbb{R}^{2N}} \beta(\vec{w} - \vec{v}/2, \vec{w} + \vec{v}/2)e^{-\|\vec{w}\|^2} \, d\vec{w}. \quad (5.5)$$

For the compatibility of the above Φ_1 and Φ_2 it is necessary that $\mathbb{1} - \frac{1}{2}B^{1/2}\Omega C\Omega^T B^{1/2} \leq 0$ or, equivalently,

$$\Omega C\Omega^T \geq 2B^{-1}. \qquad (5.6)$$

Indeed, if this was not the case, there would exist $\vec{v}_+ \in \mathbb{R}^{2N}$ such that, upon substituting $\vec{v} = \vec{v}_+$, the exponent on the LHS of Equation (5.5) is positive and, consequently, the LHS is greater than 1 whereas, since $\beta(\vec{v}, \vec{w}) \leq 1$ for a.a. $\vec{v}, \vec{w} \in \mathbb{R}^{2N}$, the RHS is bounded from the above by 1 for every $\vec{v} \in \mathbb{R}^{2N}$. Thus, in order to guarantee that two Gaussian channels are incompatible, it suffices to ensure that the inequality (5.6) is not satisfied.

There are also simple sufficient compatibility conditions for covariant Gaussian channels: Let $\Phi_i \in \mathbf{Ch}_W^W$, $i = 1, 2$, be Gaussian channels, i.e., there are positive $B_{i,i} \in \mathcal{M}_{2N \times 2N}(\mathbb{R})$ and vectors $\vec{c}_i \in \mathbb{R}^{2N}$ such that $\Phi_i = \Phi_{\mathbb{1}_{2N}, B_{i,i}, \vec{c}_i}$, $i = 1, 2$. We do not have to assume that $B_{1,1}$ or $B_{2,2}$ be of full rank. The channels Φ_1 and Φ_2 are compatible if there is $B_{1,2} \in \mathcal{M}_{2N \times 2N}(\mathbb{R})$ such that the inequality (5.4) holds. To see this, suppose that there is a real matrix $B_{1,2}$ such that the inequality (5.4) holds. Denote $B = (B_{i,j})_{i,j=1}^2$ and $\vec{c} = (\vec{c}_1, \vec{c}_2) \in \mathbb{R}^{4N}$. According to Proposition 5.1, we may define the Gaussian channel $\Psi := \Psi_{J^T, B, \vec{c}} \in \mathbf{Ch}_W^{W^{\otimes 2}}$. Considering images $\Psi(W_2(\vec{w}, 0)) = \Psi_{(1)}(W(\vec{w}))$ and $\Psi(W_2(0, \vec{w})) = \Psi_{(2)}(W(\vec{w}))$, $\vec{w} \in \mathbb{R}^{2N}$, one easily finds that $\Psi_{(i)} = \Phi_i$, $i = 1, 2$.

5.2. Phase space $\mathbb{Z}_d \times \mathbb{Z}_d$

In this subsection, we fix $d \in \mathbb{N}$ and study the Weyl-covariant channels in the case of the phase space $\mathbb{Z}_d \times \mathbb{Z}_d$ associated with a finite cyclic configuration space $\mathbf{X} = \mathbb{Z}_d$. Corollary 5.2 is a direct adaptation of Theorem 4.12 and this result is followed by some immediate observations resulting from it and connecting Corollary 5.2 to known facts about the compatibility of Weyl-covariant channels in the finite-phase space case. The latter part of this subsection is occupied by examples. In the first of them, we look at how much non-symmetric white noise can be added to a pair of unitary Weyl-covariant channels before the noisy pair becomes compatible; in the special case of symmetric noise, i.e., when the same amount of noise is added to both channels, the scenario is related to the noise robustness of incompatibility as described in [8]. In the second example, we give a simpler characterization of compatibility for Weyl-covariant channels in the qubit case. In this situation, Weyl-covariant channels coincide with the Pauli channels, i.e., the channels in the convex hull of the unitary Pauli channels $B \mapsto \sigma_j^* B\sigma_j$, $j = 0, 1, 2, 3$.

The details of the phase space $\mathbb{Z}_d \times \mathbb{Z}_d$ are presented in the table in the beginning of Section 4. Especially, the Weyl-representation W, which operates on the Hilbert space $\mathcal{H} := \mathbb{C}^d$, is again the product of the position space shift representation U and the momentum boost representation V. We may also formulate the counterpart of Corollary 4.7 for this finite phase space situation: A channel $\Phi \in \mathbf{Ch}(\mathcal{H}, \mathcal{H})$ is (W, W)-compatible if and only if there is a probability vector $p_\Phi : \mathbb{Z}_d^2 \to [0, 1]$ such that, for all $B \in \mathcal{L}(\mathcal{H})$,

$$\Phi(B) = \sum_{\vec{m} \in \mathbb{Z}_d^2} p_i(\vec{m}) W(\vec{m})^* B W(\vec{m}), \qquad (5.7)$$

or, equivalently, there is a function $f_\Phi : \mathbb{Z}_d^2 \to \mathbb{C}$ of positive type with $f_\Phi(0,0) = 1$ such that, for all $\vec{m} \in \mathbb{Z}_d^2$,

$$\Phi\big(W(\vec{m})\big) = f_\Phi(\vec{m}) W(\vec{m}). \qquad (5.8)$$

Let Φ_1, $\Phi_2 \in \mathbf{Ch}_W^W$ and assume that the functions $f_i := f_{\Phi_i}$, $i = 1$, 2, are as in Equation (5.8). According to Corollary 4.9, Φ_1 and Φ_2 are compatible if and only if there is a function $f : \mathbb{Z}_d^2 \times \mathbb{Z}_d^2 \to \mathbb{C}$ with $f(0,0;0,0) = 1$ such that the function

$$\mathbb{Z}_d^2 \times \mathbb{Z}_d^2 \ni (\vec{m}, \vec{r}; \vec{n}, \vec{s}) \mapsto e^{\frac{i}{2}\left(\omega(\vec{m},\vec{s}) + \omega(\vec{r},\vec{n})\right)} f(\vec{m} - \vec{n}, \vec{r} - \vec{s}) \in \mathbb{C}$$

is a positive kernel and $f_1(\vec{m}) = f(\vec{m}; 0, 0)$ and $f_2(\vec{m}) = f(0, 0; \vec{m})$ for all $\vec{m} \in \mathbb{Z}_d^2$. The following corollary, which immediately follows from Theorem 4.12 and Remark 4.13, is arguably a clearer condition though. Note that the regularity assumption made in Theorem 4.12 is automatically satisfied in this finite phase space case.

Corollary 5.2. *Let Weyl-covariant channels Φ_1, $\Phi_2 \in \mathbf{Ch}_W^W$ be associated with probability vectors $p_1 := p_{\Phi_1}$ and $p_2 := p_{\Phi_2}$ as in Equation (5.7). These channels are compatible if and only if there is a positive kernel $\beta : \mathbb{Z}_d^2 \times \mathbb{Z}_d^2 \to \mathbb{C}$ with $\beta(\vec{m}, \vec{m}) = 1$ for all $\vec{m} \in \mathbb{Z}_d^2$ such that*

$$p_2(\vec{r}) = \frac{1}{d^2} \sum_{\vec{m}, \vec{n} \in \mathbb{Z}_d^2} e^{i\omega(\vec{m} - \vec{n}, \vec{r})} \beta(\vec{m}, \vec{n}) \sqrt{p_1(\vec{m}) p_1(\vec{n})}, \qquad \vec{r} \in \mathbb{Z}_d^2. \qquad (5.9)$$

Corollary 5.2 gives us a recipe of finding all the channels $\Phi_2 \in \mathbf{Ch}_W^W$ compatible with a fixed $\Phi_1 \in \mathbf{Ch}_W^W$. Let us first take a look at two simple cases. Consider first the case where $\Phi_1(A) = W(\vec{m}_0)^* A W(\vec{m}_0)$ for all $A \in \mathcal{L}(\mathcal{H})$ with some fixed $\vec{m}_0 \in \mathbb{Z}_d^2$. Now $p_1(\vec{m}_0) = 1$ and $p_1(\vec{m}) = 0$ whenever $\vec{m} \neq \vec{m}_0$. When $\beta : \mathbb{Z}_d^2 \times \mathbb{Z}_d^2 \to \mathbb{C}$ is a positive kernel with $\beta(\vec{m}, \vec{m}) = 1$ for all $\vec{m} \in \mathbb{Z}_d^2$, it follows that the channel associated with β according to Corollary 5.2 is determined by the probability vector $p_2 : \mathbb{Z}_d^2 \to [0, 1]$,

$$p_2(\vec{r}) = \frac{1}{d^2} \sum_{\vec{m}, \vec{n} \in \mathbb{Z}_d^2} e^{i\omega(\vec{m} - \vec{n}, \vec{r})} \beta(\vec{m}, \vec{n}) \sqrt{p_1(\vec{m}) p_1(\vec{n})} = \frac{1}{d^2} \beta(\vec{m}_0, \vec{m}_0) = \frac{1}{d^2}$$

for all $\vec{r} \in \mathbb{Z}_d^2$. Hence, the only channel in $\mathbf{Ch}_W^W$ which is compatible with Φ_1 is the channel $\Phi^0 \in \mathbf{Ch}_W^W$ associated with the constant probability vector

$p_2(\vec{m}) = 1/d^2$ for all $\vec{m} \in \mathbb{Z}_d^2$. It follows that

$$\Phi^0(B) = \frac{1}{d}\mathrm{tr}\,[B]\,\mathbb{1}_{\mathcal{H}}, \qquad B \in \mathcal{L}(\mathcal{H}). \tag{5.10}$$

The channel Φ^0 is called as the *depolarizing channel*. The fact that Φ^0 is the only channel which the unitary channel Φ_1 is compatible with is a reflection of a well known fact (a version of the no-cloning principle): whenever Φ is a unitary channel, the only channels compatible with Φ are of the form $B \mapsto \mathrm{tr}\,[\sigma B]\,\mathbb{1}$ for some positive trace-1 operator σ and the only one of these within $\mathbf{Ch}_W^W$ is the depolarizing channel corresponding to $\sigma = d^{-1}\mathbb{1}_{\mathcal{H}}$.

On the other hand, let $\Phi_1 = \Phi^0$ be the depolarizing channel. Pick any probability vector $p_2 : \mathbb{Z}_d^2 \to [0, 1]$ and define the positive kernel $\beta : \mathbb{Z}_d^2 \times \mathbb{Z}_d^2 \to \mathbb{C}$, $\beta(\vec{m}, \vec{n}) = \sum_{\vec{s} \in \mathbb{Z}_d^2} p_2(\vec{s}) e^{i\omega(\vec{n}-\vec{m},\vec{s})}$, $\vec{m}, \vec{n} \in \mathbb{Z}_d^2$. It is easily checked that $\beta(\vec{m}, \vec{m}) = 1$ for all $\vec{m} \in \mathbb{Z}_d^2$ and

$$\frac{1}{d^2} \sum_{\vec{m},\vec{n} \in \mathbb{Z}_d^2} e^{i\omega(\vec{m}-\vec{n},\vec{r})} \beta(\vec{m}, \vec{n}) \sqrt{p_1(\vec{m})p_1(\vec{n})} = p_2(\vec{r}), \quad \vec{r} \in \mathbb{Z}_d^2.$$

Thus, Φ_1 is compatible with any channel $\Phi_2 \in \mathbf{Ch}_W^W$. Indeed, it is well known that Φ_1 is compatible with any channel, not only with covariant channels, as the channels of the form $B \mapsto \mathrm{tr}\,[\sigma B]\,\mathbb{1}$ (and the depolarizing channel in particular) are compatible with all other quantum devices, including channels.

Example. Let us consider an example where we mix white noise in the form of the completely depolarizing channel to two unitary Weyl-covariant channels and determine the conditions the mixing parameters have to satisfy so that the approximate unitary channels are compatible. Our target unitary channels are Φ_1 and Φ_2, $\Phi_i(B) = W(\vec{m}_i)^* B W(\vec{m}_i)$, $B \in \mathcal{L}(\mathcal{H})$, $i = 1, 2$, where $\vec{m}_1, \vec{m}_2 \in \mathbb{Z}_d^2$ are some fixed phase space points. These channels correspond to the point mass probability vectors $\delta_{\vec{m}_i}$, $i = 1, 2$, i.e., $\delta_{\vec{m}_i}(\vec{m}_i) = 1$ and $\delta_{\vec{m}_i}(\vec{m}) = 0$ otherwise, $i = 1, 2$. The white noise is represented by the depolarizing channel Φ^0 of Equation (5.10) corresponding to the uniform probability vector p^0, $p^0(\vec{m}) = d^{-2}$ for all $\vec{m} \in \mathbb{Z}_d^2$. Define the channels $\Phi_i^s \in \mathbf{Ch}_W^W$, $0 \le s \le 1$, $i = 1, 2$, $\Phi_i^s = (1-s)\Phi_i + s\Phi^0$, $i = 1, 2$. These correspond to the probability vectors $p_i^s = (1-s)\delta_{\vec{m}_i} + sp^0$, $0 \le s \le 1$, $i = 1, 2$.

Let us look at the compatibility conditions of channels Φ_1^s and Φ_2^t, $s, t \in [0, 1]$. According to Remark 4.8, we may simplify this task: we may simply assume that $\vec{m}_1 = 0 = \vec{m}_2$ and still obtain the compatibility conditions of the more general situation described above. Thus, we are looking at the conditions s and t have to satisfy so that the channels $\Phi^s = (1-s)\mathrm{id} + s\Phi^0$ and $\Phi^t = (1-t)\mathrm{id} + t\Phi^0$ are compatible. These channels correspond to the probability vectors $p^s = (1-s)\delta_0 + sp^0$ and $p^t = (1-t)\delta_0 + tp^0$.
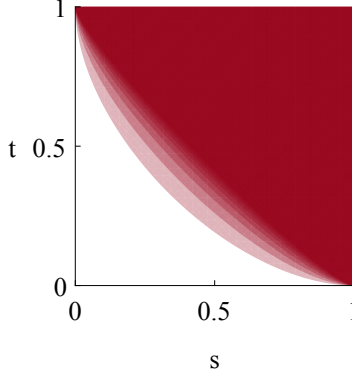
FIGURE 1. The area characterized by the inequality (5.12) of those noise parameters $(s, t) \in [0,1] \times [0,1]$ such that Φ_1^s and Φ_2^t are compatible is plotted here with varying dimension $d = 2, \ldots, 10$ (size of configuration space). Each darker shade indicates a rise in d by 1, i.e., the parameter values from the whole coloured area are the ones with which Φ_1^s and Φ_2^t are compatible in $d = 2$, those from the darker hue correspond to compatibility in $d = 3$, and so forth. We see that, as $d \to \infty$, the area of compatibility grows closer to just the upper right triangle. The white area is the area where Φ_1^s and Φ_2^t are incompatible in any dimension.

Direct substitution of p^s and p^t into Equation (5.9) yields for all $\vec{r} \in \mathbb{Z}_d^2$

$$(1-t)\delta_{\vec{r},0} + \frac{t}{d^2}$$

$$= \frac{1}{d^2}\left(1 - \frac{d^2-1}{d^2}s\right) + 2\frac{\sqrt{s}}{d^3}\sqrt{1 - \frac{d^2-1}{d^2}s} \sum_{\vec{m}\in\mathbb{Z}_d^2\setminus\{0\}} \mathrm{Re}\left(e^{i\omega(\vec{m},\vec{r})}\beta(\vec{m},0)\right)$$

$$+ \frac{s}{d^4} \sum_{\vec{m},\vec{n}\in\mathbb{Z}_d^2\setminus\{0\}} e^{i\omega(\vec{m}-\vec{n},\vec{r})}\beta(\vec{m},\vec{n}), \tag{5.11}$$

where $\delta_{\vec{m},\vec{r}}$ stands for the Kronecker symbol, i.e., $\delta_{\vec{m},\vec{n}} = 1$ if and only if $\vec{m} = \vec{n}$ and otherwise $\delta_{\vec{m},\vec{n}} = 0$. To find the boundary of the region of those $(s,t) \in [0,1]^2$ such that Φ^s and Φ^t are compatible, we look at the minimum value of t for each s such that Φ^s and Φ^t are compatible. We denote this value by $t_{\min}(s)$; this is found, e.g., by substituting $\vec{r} = 0$ in Equation (5.11), setting $1 - t_{\min}(s) + t_{\min}/d^2$ on the LHS and choosing β so that the RHS is maximized. This is due to the fact that the entry $p^t(0)$ is the largest of p^t and by maximizing this, the noise terms $p^t(\vec{r}) = t/d^2$, $\vec{r} \in \mathbb{Z}_d^2 \setminus \{0\}$, are minimized.

We have $|\beta(\vec{m},\vec{n})| \leq 1$ for all $\vec{m}$, $\vec{n} \in \mathbb{Z}_d^2$, and it is immediately seen that the above maximization task is solved by setting $\beta(\vec{m},\vec{n}) = 1$ for all $\vec{m}$, $\vec{n} \in$

$\mathbb{Z}_d^2$. Substituting this into Equation (5.11) where the LHS is $1 - t_{\min}(s) + t_{\min}/d^2$, through direct calculation we obtain $t_{\min}(s) = \left(\sqrt{1 - (1 - 1/d^2)s} - \sqrt{s}/d\right)^2$. This means that the above channels are compatible if and only if $\sqrt{t} \geq \sqrt{1 - d^{-2}(d^2 - 1)s} - \sqrt{s}/d$. Moving the last term on the RHS of this inequality to the LHS and squaring both sides of the resulting inequality we arrive at the following: the channels Φ^s and Φ^t (or equivalently the channels Φ_1^s and Φ_2^t of the beginning of this example) are compatible if and only if

$$s + \frac{2}{d}\sqrt{st} + t \geq 1. \tag{5.12}$$

We immediately notice that, as the dimension d increases, the inequality (5.12) grows ever closer to $s + t \geq 1$; this is also demonstrated in Figure 1. Pairs of unitary channels are the pairs whose incompatibility is the most resistant under added noise and, as dimension increases, we near the dimension-independent ultimate noise tolerance, since if the noise parameters satisfy $s + t \geq 1$, the channels Φ^s and Φ^t are guaranteed to be compatible [2, 8]. In the symmetric case where $s = t$, we see that the channels Φ_1^t and Φ_2^t are compatible if and only if $t \geq d/\left(2(d+1)\right)$. This result is in line with earlier findings [8].

Example. Let us consider the case of the simple phase space $\mathbb{Z}_2 \times \mathbb{Z}_2$. Naturally, the Hilbert space associated with the phase space is two dimensional. Let us fix a basis in $\mathbb{C}^2$ and define the Pauli matrices:

$$\sigma_0 = \mathbb{1}_2, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

When we define $W : \mathbb{Z}_2^2 \to \mathcal{L}(\mathbb{C}^2)$, $W(0,0) = \sigma_0$, $W(0,1) = \sigma_1$, $W(1,0) = \sigma_2$, and $W(1,1) = \sigma_3$, we notice that this map satisfies the CCR-relations (4.1) meaning that W is the Weyl representation associated with the phase space $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Any channel $\Phi \in \mathbf{Ch}_W^W$ is determined by a probability vector $p : \mathbb{Z}_2^2 \to [0,1]$ through $\Phi(B) = \sum_{k,l \in \mathbb{Z}_2} p(k,l)W(k,l)^* B W(k,l)$, $B \in \mathcal{L}(\mathbb{C}^2)$. According to Corollary 5.2, two channels $\Phi_1, \Phi_2 \in \mathbf{Ch}_W^W$ associated, respectively, with probability vectors $p_1, p_2 : \mathbb{Z}_2^2 \to [0,1]$ are compatible if and only if there is a positive matrix

$$B = \left(\beta(\vec{m},\vec{n})\right)_{\vec{m},\vec{n}\in\mathbb{Z}_2^2} = \begin{pmatrix} 1 & a & b & d \\ \bar{a} & 1 & c & e \\ \bar{b} & \bar{c} & 1 & f \\ \bar{d} & \bar{e} & \bar{f} & 1 \end{pmatrix}$$

such that Equation (5.9) is satisfied with $d = 2$.

We immediately notice that the phase terms $e^{i\omega(\vec{m}-\vec{n},\vec{r})}$ in (5.9) are all real, and taking the complex conjugate on both sides of this equation and, summing the resulting (true) equation with the original one, we find that we are free to assume that $B \in \mathcal{M}_{4\times 4}(\mathbb{R})$, i.e., $\beta(\vec{m},\vec{n}) \in \mathbb{R}$ for all $\vec{m}, \vec{n} \in \mathbb{Z}_2^2$. We are free to set $a = b = d = 0$ if $p_1(0,0) = 0$, $c = e = a = 0$ if $p_1(0,1) = 0$, $f = b = c = 0$ if $p_1(1,0) = 0$, and $d = e = f = 0$ if $p_1(1,1) = 0$; the resulting

matrix is positive whenever the original B is positive and satisfies Equation (5.9) similarly as B. Also note that $B \geq 0$ if and only if $\tilde{B} \geq 0$ where

$$
\tilde{B} \;=\; \begin{pmatrix} p_1(0,0) & \tilde{a} & \tilde{b} & \tilde{d} \\ \tilde{a} & p_1(0,1) & \tilde{c} & \tilde{e} \\ \tilde{b} & \tilde{c} & p_1(1,0) & \tilde{f} \\ \tilde{d} & \tilde{e} & \tilde{f} & p_1(1,1) \end{pmatrix}
$$

$$
:=\; \left(\sqrt{p_1(\vec{m})p_1(\vec{n})}\,\beta(\vec{m},\vec{n}) \right)_{\vec{m},\vec{n}\in\mathbb{Z}_2^2}
$$

is the entrywise product of B and $\left(\sqrt{p_1(\vec{m})p_1(\vec{n})} \right)_{\vec{m},\vec{n}\in\mathbb{Z}_2^2}$ (which is positive); also recall that $\beta(\vec{m},\vec{n}) \in \mathbb{R}$ for all $\vec{m},\,\vec{n} \in \mathbb{Z}_2^2$. We obtain the original B from $\tilde{B}$ by taking the entrywise product of $\tilde{B}$ and the positive $\left(c_{\vec{m},\vec{n}} \right)_{\vec{m},\vec{n}\in\mathbb{Z}_2^2}$ where $c_{\vec{m},\vec{n}} = \left(p_1(\vec{m})p_1(\vec{n}) \right)^{-1/2}$ whenever $p_1(\vec{m}) \neq 0 \neq p_1(\vec{n})$, and $c_{\vec{m},\vec{n}} = 0$ otherwise; note the above freedom in nullifying certain entries of B depending on the vanishing of p_1. Making these assumptions, we find, using (5.9),

$$
\begin{cases} p_2(0,0) + p_2(0,1) &=& \frac{1}{2} + \tilde{a} + \tilde{f}, \\ p_2(0,0) + p_2(1,0) &=& \frac{1}{2} + \tilde{b} + \tilde{e}, \\ p_2(0,0) + p_2(1,1) &=& \frac{1}{2} + \tilde{d} + \tilde{c}. \end{cases}
$$

From this we find that Φ_1 and Φ_2 are compatible if and only if there are $x,\,y,\,z \in \mathbb{R}$ such that

$$
\begin{pmatrix} p_1(0,0) & x & y & d(z) \\ x & p_1(0,1) & z & e(y) \\ y & z & p_1(1,0) & f(x) \\ d(z) & e(y) & f(x) & p_1(1,1) \end{pmatrix} \geq 0, \qquad (5.13)
$$

where

$$
\begin{cases} d(z) &=& \frac{1}{2}\big(p_2(0,0) - p_2(0,1) - p_2(1,0) + p_2(1,1)\big) - z, \\ e(y) &=& \frac{1}{2}\big(p_2(0,0) - p_2(0,1) + p_2(1,0) - p_2(1,1)\big) - y, \\ f(x) &=& \frac{1}{2}\big(p_2(0,0) + p_2(0,1) - p_2(1,0) - p_2(1,1)\big) - x. \end{cases}
$$

This is immediately seen as the existence of $x,\,y,\,z \in \mathbb{R}$ such that (5.13) is satisfied is equivalent with the existence of a positive $B = \big(\beta(\vec{m},\vec{n})\big)_{\vec{m},\vec{n}\in\mathbb{Z}_2^2}$ such that Equation (5.9) is satisfied.

6. Notes on the multipartite case

All the results of this paper except for Proposition 3.6, Theorem 4.12, Corollary 5.2, and their corollaries can be generalized in a straight-forward manner to the multipartite case.

Definition 6.1. Fix $m \in \mathbb{N}$ and Hilbert spaces $\mathcal{H}$ and $\mathcal{K}_i$, $i = 1,\ldots,m$. Suppose that $\Psi \in \mathbf{Ch}(\mathcal{H}, \mathcal{K}_1 \otimes \cdots \otimes \mathcal{K}_m)$ and define the i'th margin of Ψ, $i = 1,\ldots,m$,

$$
\Psi_{(i)}(B) = \Psi(\mathbb{1}_{\mathcal{K}_1} \otimes \cdots \otimes \mathbb{1}_{\mathcal{K}_{i-1}} \otimes B \otimes \mathbb{1}_{\mathcal{K}_{i+1}} \otimes \cdots \otimes \mathbb{1}_{\mathcal{K}_m}), \qquad B \in \mathcal{L}(\mathcal{K}_i).
$$

Channels $\Phi_i \in \mathbf{Ch}(\mathcal{H}, \mathcal{K}_i)$, $i = 1, \ldots, m$ are *compatible* if there is a *joint channel* $\Psi \in \mathbf{Ch}(\mathcal{H}, \mathcal{K}_1 \otimes \cdots \otimes \mathcal{K}_m)$ such that $\Phi_i = \Psi_{(i)}$, $i = 1, \ldots, m$.

Fix Hilbert spaces $\mathcal{H}$ and $\mathcal{K}_i$, $i = 1, \ldots, m$, a group G, and projective unitary representations $U : G \to \mathcal{U}(\mathcal{H})$ and $V_i : G \to \mathcal{U}(\mathcal{K}_i)$, $i = 1, \ldots, m$. Also define $V : G \to \mathcal{U}(\mathcal{K}_1 \otimes \cdots \mathcal{K}_m)$,

$$V(g) = V_1(g) \otimes \cdots \otimes V_m(g), \qquad g \in G.$$

With obvious modifications, the proofs of Lemma 3.3 and Proposition 3.4 can be adapted to the multipartite case. Hence, we have:

Proposition 6.2. *Suppose that $\mathcal{H}$ and $\mathcal{K}_i$, $i = 1, \ldots, m$, are separable Hilbert spaces, G is an amenable locally compact group, and $U : G \to \mathcal{U}(\mathcal{H})$ and $V_i : G \to \mathcal{U}(\mathcal{K}_i)$, $i = 1, \ldots, m$, are strongly continuous projective unitary representations. Any m-tuple $\Phi_i \in \mathbf{Ch}_U^{V_i}$, $i = 1, 2$, of compatible covariant channels has a joint channel $\overline{\Psi} \in \mathbf{Ch}_U^V$, where V is defined as above.*

Consider now the general phase space $G = \mathbf{X} \times \hat{\mathbf{X}}$ and the Weyl representation W of Section 4. Again, the compatibility properties hinge on the set $\mathbf{Ch}_W^{W^{\otimes m}}$, where $W^{\otimes m} : G \to \mathcal{U}(\mathcal{H}^{\otimes m})$,

$$W^{\otimes m}(g) = \underbrace{W(g) \otimes \cdots \otimes W(g)}_{m \text{ copies}}.$$

Definition 6.3. Denote by $C_S^0(G^N)$ the set of continuous functions $f : G^N \to \mathbb{C}$ with $f(e, \ldots, e) = 1$ such that the function

$$G^m \times G^m \ni (g_{1,1}, \ldots, g_{m,1}; g_{1,2}, \ldots, g_{m,2})$$

$$\mapsto f(g_{1,1}^{-1} g_{1,2}, \ldots, g_{m,1}^{-1} g_{m,2}) \prod_{\substack{k,l=1 \\ k \neq l}}^{m} e^{-\frac{i}{2}\omega(g_{k,1}, g_{l,2})} \in \mathbb{C}$$

is a positive kernel.

Again, the proofs of Proposition 4.1 and Theorem 4.6 are easily generalized and we obtain:

Theorem 6.4. *For any $\Psi \in \mathbf{Ch}_W^{W^{\otimes m}}$ there is a unique $f_\Psi \in C_S^0(G^N)$ such that*

$$\Psi\big(W(g_1) \otimes \cdots \otimes W(g_m)\big) = f_\Psi(g_1, \ldots, g_m) W(g_1 \cdots g_m), \qquad g_1, \ldots, g_m \in G.$$

Moreover, the map $\mathbf{Ch}_W^{W^{\otimes m}} \ni \Psi \mapsto f_\Psi \in C_S^0(G^N)$ is bijective.

Corollary 6.5. *Channels $\Phi_1, \ldots, \Phi_m \in \mathbf{Ch}_W^W$ are compatible if and only if there is $f \in C_S^0(G^N)$ such that*

$$f_1(g) = f(g, e, \ldots, e), \ldots, f_m(g) = f(e, \ldots, e, g), \qquad g \in G,$$

where $f_i : G \to \mathbb{C}$, $i = 1, \ldots, m$, are the continuous functions of positive type such that $f_i(e) = 1$ and $\Phi_i\big(W(g)\big) = f_i(g) W(g)$ for all $g \in G$, $i = 1, \ldots, m$.

Let us give a few notes on the continuous non-compact phase space ($\mathbf{X} = \mathbb{R}^N$) and retain the notation of Subsection 5.1. Define the representation $W_m : \mathbb{R}^{2mN} \to \mathcal{U}(\mathcal{H}^{\otimes m})$, $W_m(\vec{w}_1, \ldots, \vec{w}_m) = W(\vec{w}_1) \otimes \cdots \otimes W(\vec{w}_m)$, $\vec{w}_1, \ldots, \vec{w}_m \in \mathbb{R}^{2N}$.

Theorem 6.6. *A covariant channel* $\Psi \in \mathbf{Ch}_W^{W^{\otimes m}}$ *is Gaussian if and only if there is a symmetric matrix* $B \in \mathcal{M}_{2mN \times 2mN}(\mathbb{R})$, $B = (B_{k,l})_{k,l=1}^m$ *where* $B_{k,l} \in \mathcal{M}_{2N \times 2N}(\mathbb{R})$, $k, l = 1, \ldots, m$, *with*

$$
\begin{pmatrix}
B_{1,1} & B_{1,2} - i\Omega & \cdots & B_{1,m} - i\Omega \\
B_{1,2}^T + i\Omega^T & B_{2,2} & \cdots & B_{2,m} - i\Omega \\
\vdots & \vdots & \ddots & \vdots \\
B_{1,m}^T + i\Omega^T & B_{2,m}^T + i\Omega^T & \cdots & B_{m,m}
\end{pmatrix} \geq 0 \qquad (6.1)
$$

and a vector $\vec{c} \in \mathbb{R}^{2mN}$ *(that can be picked at random) such that*

$$
\Psi\big(W_m(\vec{z})\big) = e^{-\frac{1}{4}\vec{z}^T B \vec{z} - i\vec{c}^T \vec{z}} W(J_m^T \vec{z}), \qquad \vec{z} \in \mathbb{R}^{2mN},
$$

where $J_m : \mathbb{R}^{2N} \to \mathbb{R}^{2mN}$,

$$
\mathbb{R}^{2N} \ni \vec{w} \mapsto \underbrace{(\vec{w}, \ldots, \vec{w})}_{m \text{ copies}} \in \mathbb{R}^{2mN}.
$$

7. Conclusions

We have studied covariant channels and their compatibility conditions and have shown that covariant channels always have a covariant joint channel. These general results were utilized in an analysis of the compatibility conditions of Weyl-covariant channels, i.e., channels which behave symmetrically under phase space shifts. We have obtained necessary and sufficient conditions for compatibility of such channels involving the characteristic functions associated with the Weyl-covariant channels. Under some extra assumptions, we obtain a very descriptive compatibility condition which can be used as a recipe for generating all the covariant channels which are compatible with a fixed covariant channel. These results were investigated in the case of a non-compact continuous phase space and the case of a finite phase space with some illustrative examples. Finally, some notes on the multipartite case were presented.

We have seen that symmetries in the form of covariance properties can be used to greatly restrict the variety of possible joint channels of compatible channels, a results mirroring earlier results on quantum observables, particularly position and momentum [21]. The analysis of Weyl covariance was chosen here for the simplicity arising from the canonical commutation relations. The case of more general physical symmetries, such as Euclidean covariance, remains to be examined. The non-commutative groups involved in these studies will provide richer structures and hopefully interesting methods of establishing incompatibility of quantum channels.

Acknowledgements

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Erkka Haapasalo
Department of Physics and Center for Field Theory and Particle Physics,
Fudan University, Shanghai 200433, China
e-mail: erkka_haapasalo@fudan.edu.cn

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