DYNAMICS ON NONCOMMUTATIVE ORLICZ SPACES

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Abstract. Quantum dynamical maps are defined and studied for quantum statistical physics based on Orlicz spaces. This complements earlier work [21] where we made a strong case for the assertion that statistical physics of regular systems should properly be based on the pair of Orlicz spaces $\langle L^{\cosh -1}, L\log(L+1)\rangle$, since this framework gives a better description of regular observables, and also allows for a well-defined entropy function. In the present paper we "complete" the picture by addressing the issue of the dynamics of such a system, as described by a Markov semigroup corresponding to some Dirichlet form. Specifically we show that even in the most general non-commutative contexts, completely positive Markov maps satisfying a natural Detailed Balance condition, canonically admit an action on a large class of quantum Orlicz spaces. This is achieved by the development of a new interpolation strategy for extending the action of such maps to the appropriate intermediate spaces of the pair $\langle L^\infty, L^1\rangle$. As a consequence, we obtain that quantum dynamics in the form of Markov semigroups described by some Dirichlet form, naturally extends to the context proposed in [21].

Contents

1. Introduction

In our previous paper [21] we have argued that statistical physics of regular systems, both classical and quantum, should be based on the pair of Orlicz spaces $\langle L^{\cosh -1}, L\log(L+1)\rangle$.

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We remind the reader that a regular observable is characterized by the property of finiteness of all its moments. Although at first sight this property can be regarded as restrictive, the important point to note here is that the standard formulation of classical (quantum) theory based on the pair of Banach spaces $\langle L^\infty, L^1\rangle$ ($\langle B(\mathcal{H}), \mathfrak{F}(\mathcal{H})\rangle$, respectively) with pure deterministic time evolution, as well as such important classes of Lévy processes as Wiener and Poisson processes do satisfy the above regularity requirements. Moreover, the proposed extension of the allowed family of observables includes regular *unbounded* observables.

However, to get a fully-fledged theory, a description of dynamics should be provided. In particular, one wants to describe dynamical semigroups within the proposed scheme based on the pair of Orlicz spaces $\langle L^{\cosh -1}, L\log(L+1)\rangle$. The main difficulty in carrying out such a description is that the standard interpolation theory must be adapted to a framework based on non-commutative Orlicz spaces. We will do this in sections devoted to the study of quantum maps on the distinguished Orlicz spaces.

The paper is organized as follows: We start by revising relevant material from [21] for the sake of context, in the process also making our exposition more self-contained. We then we give a brief exposition of the aspects of crossed products relevant to the exposition. Section 4 is devoted to the study of criteria which allow for the extension of Markov maps on the underlying von Neumann algebra, to a large class of Orlicz spaces. Along the way we will also compare the various strategies of achieving this extension. To illustrate the utility of the theory, a class of quantum dynamical maps which fit naturally with the developed techniques will be considered in Section 5. The last section contains some conclusions and remarks.

2. Notation, terminology, and previous results

We follow notation used in [21] and [22]. Let (X,μ) be a measure space. We denote $L^1(X,\mu) = \{f : \int_X |f|d\mu < \infty\}$, while $L^\infty(X,\mu)$ stands for the essentially bounded, measurable functions on X. Their non-commutative analogues are: $\mathfrak{F}_T(\mathcal{H})$ - the trace class operators on a Hilbert space $\mathcal{H}$, and $B(\mathcal{H})$ – all linear bounded operators on $\mathcal{H}$. We remind the reader that $L^p(X,\Sigma,m)$ spaces $(1 \le p < \infty)$, (X,Σ,m) a measure space, may be regarded as spaces of measurable functions conditioned by the functions $t \mapsto |t|^p$ $(1 \le p < \infty)$. The more general category of Orlicz spaces is defined as spaces of measurable functions conditioned by a more general class of convex functions; the so-called Young's functions. A function $\Phi : [0,\infty) \to [0,\infty]$ is a Young's function if Φ is convex, $\lim_{u\to 0+}\Phi(u) = \Phi(0) = 0$, $\lim_{u\to\infty}\Phi(u) = \infty$, and Φ is non-constant on $(0,\infty)$. It is worth pointing out that such functions have a nice integral representation, for details see [21] and the references given there.

Let L^0 be the space of measurable functions on some σ-finite measure space (X,Σ,μ). We will always assume that the considered measures are σ-finite.

Definition 2.1. The Orlicz space L^Ψ (being a Banach space) associated with Ψ is defined to be the set

$$L^\Psi \equiv L^\Psi(X,\Sigma,\mu) = \{f \in L^0 : \Psi(\lambda|f|) \in L^1 \quad \text{for some} \quad \lambda = \lambda(f) > 0\}. \tag{2.1}$$

Orlicz spaces may be normed in one of two (ultimately equivalent) ways, namely either by the so-called Luxemburg-Nakano norm

$$\|f\|_\psi = \inf\{\lambda > 0 : \|\psi(|f|/\lambda)\|_1 \le 1\}.$$

or the equivalent Orlicz norm, given by the formula

$$\|f\|^O_\psi = \sup\{|\int_X fg\,dm| : g \in L^{\psi^*}, \|g\|_\psi \leq 1\}.$$

Following convention, we will write L^Ψ when the Luxemburg-Nakano norm is used, and L_Ψ if the Orlicz norm is used.

The basic Orlicz spaces used in this paper are $L\log(L+1)$, and $L^{\cosh -1}$ defined by Young's functions: $x \mapsto x\log(x+1)$, and $x \mapsto \cosh(x) - 1$ respectively.

A natural question that arises, is *what can be said about uniqueness of the correspondence: Young's function $\Psi \mapsto L^\Psi$-Orlicz space.* To answer this question one needs the concept of equivalent Young's functions. To define this concept we firstly introduce an order relation on Young's functions by saying that $F_1 \succ F_2$ if and only if $F_1(bx) \geq F_2(x)$ for $x \geq 0$ and some $b > 0$, and then say that the functions F_1 and F_2 are equivalent, $F_1 \approx F_2$, if $F_1 \prec F_2$ and $F_1 \succ F_2$. One has (see [28])

Theorem 2.2. *Let Φ_i, $i = 1, 2$ be a pair of equivalent Young's function. Then as Banach spaces, $L^{\Phi_1} = L^{\Phi_2}$.*

Consequently, on condition that equivalence is preserved, *one can "replace" complicated Young's function's with simpler versions!*

Our main results concerning classical statistical physics, stated and proved in [21] (see also [22]) are:

Theorem 2.3. *The dual pair $\langle L^{\cosh -1}, L\log(L+1)\rangle$ provides the basic mathematical ingredient for a description of a general, classical regular system.*

Turning to the quantum case, as a first step, one should define the quantum counterpart of measurable functions L^0. In the quantum world there is no known space that is a direct analogue of L^0. But one is able to define a quantum analogue of the space of all measurable functions which are bounded, except on a set of finite measure. This space turns out to be more than adequate for our purposes. So to this end let $\mathfrak{M} \subset B(\mathcal{H})$ be a semifinite von Neumann algebras equipped with an fns (faithful normal semifinite) trace τ. The space of all τ-measurable operators is defined as follows. Let a be a densely defined closed operator on $\mathcal{H}$ with domain $\mathcal{D}(a)$ and let $a = u|a|$ be its polar decomposition. One says that a is affiliated with $\mathfrak{M}$ (denoted $a\eta\mathfrak{M}$) if u and all the spectral projections of $|a|$ belong to $\mathfrak{M}$. Then a is τ-measurable if $a\eta\mathfrak{M}$, and for each $\delta > 0$, there exists a projection $e \in \mathfrak{M}$ such that $e\mathcal{H} \subset \mathcal{D}(a)$ and $\tau(1-e) \leq \delta$. We denote by $\widetilde{\mathfrak{M}}$ the set of all τ-measurable operators. The algebra $\widetilde{\mathfrak{M}}$ (equipped with the topology of convergence in measure) is a substitute for L^0 in the quantum world (for details see [25], [33], and [30]).

Following the Dodds, Dodds, de Pagter approach [3] we need the concept of generalized singular values. Namely, given an element $f \in \widetilde{\mathfrak{M}}$ and $t \in [0, \infty)$, the generalized singular value $\mu_t(f)$ is defined by $\mu_t(f) = \inf\{s \geq 0 : \tau(\mathbb{I} - e_s(|f|)) \leq t\}$ where $e_s(|f|)$ $s \in \mathbb{R}$ is the spectral resolution of $|f|$. The function $t \to \mu_t(f)$ will generally be denoted by $\mu(f)$. For details on the generalized singular values see [5]. Here, we note only that this directly extends classical notions where for any $f \in L^0$, the function $(0, \infty) \to [0, \infty] : t \to \mu_t(f)$ is known as the decreasing rearrangement of f.

The key ingredient of the Dodds, Dodds, de Pagter approach is the concept of a Banach Function Space. To define this concept, let $L^0(0, \infty)$ stand for measurable functions on $(0, \infty)$ and L^0_+ denote $\{f \in L^0(0, \infty); f \geq 0\}$. A function norm

ρ on $L^0(0, \infty)$ is defined to be a mapping $\rho : L^0_+ \to [0, \infty]$ satisfying

- $\rho(f) = 0$ iff $f = 0$ a.e.
- $\rho(\lambda f) = \lambda\rho(f)$ for all $f \in L^0_+, \lambda > 0$.
- $\rho(f + g) \leq \rho(f) + \rho(g)$ for all $f, g \in L^0_+$.
- $f \leq g$ implies $\rho(f) \leq \rho(g)$ for all $f, g \in L^0_+$.

Such a ρ may be extended to all of L^0 by setting $\rho(f) = \rho(|f|)$, in which case we may then define $L^\rho(0,\infty) = \{f \in L^0(0,\infty) : \rho(f) < \infty\}$. If now $L^\rho(0,\infty)$ turns out to be a Banach space when equipped with the norm $\rho(\cdot)$, we refer to it as a Banach Function space. If $\rho(f) \leq \liminf_n \rho(f_n)$ whenever $(f_n) \subset L^0$ converges almost everywhere to $f \in L^0$, we say that ρ has the Fatou Property. If less generally this implication only holds for $(f_n) \cup \{f\} \subset L^\rho$, we say that ρ is lower semi-continuous. If further the situation $f \in L^\rho$, $g \in L^0$ and $\mu_t(f) = \mu_t(g)$ for all $t > 0$, forces $g \in L^\rho$ and $\rho(g) = \rho(f)$, we call L^ρ rearrangement invariant (or symmetric).

By employing generalized singular values and Banach Function Spaces, Dodds, Dodds and de Pagter [3] formally defined the noncommutative space $L^\rho(\widetilde{\mathfrak{M}}) \equiv L^\rho(\mathfrak{M}, \tau) \equiv L^\rho(\mathfrak{M})$ to be

$$L^\rho(\mathfrak{M}) = \{f \in \widetilde{\mathfrak{M}} : \mu(f) \in L^\rho(0,\infty)\}$$

and showed that if ρ is lower semicontinuous and $L^\rho(0,\infty)$ rearrangement-invariant, $L^\rho(\mathfrak{M})$ is a Banach space when equipped with the norm $\|f\|_\rho = \rho(\mu(f))$.

In the context of semifinite algebras, we then obtained the following quantized version of Theorem 2.3 (see [21] and [22] for details).

Theorem 2.4. *The dual pair of quantum Orlicz spaces* $(L^{\cosh -1}, L\log(L+1))$ *provides the basic mathematical ingredient for a description of a general, quantum regular system.*

Subsequent to this, the utility of the space $L\log(L+1)$ as a home for states with good entropy, was explored for general (so also type III) von Neumann algebras [23].

However no formalism for quantum mechanics is complete without a good theory of dynamics. The objective of the present paper is therefore to show that even for a the most general von Neumann algebras, the above framework admits a good theory of Markov dynamics.

3. Crossed products

In contexts where one has to deal with a von Neumann algebras which do not have a trace, one of course does not have access to the elegant theory of Dodds, Dodds, and de Pagter. In such cases one needs to follow the philosophy of Haagerup and Terp, which makes essential use of the notion of crossed products of von Neumann algebras to posit quantum L^p-spaces. For the sake of the reader we briefly review some of the essential facts regarding continuous crossed products, before going on to analyse the behaviour of quantum dynamical maps with respect to these crossed products.

Let $\mathfrak{M}$ be a von Neumann algebra acting on the Hilbert space $\mathcal{H}$, and let ν be a fixed faithful normal seimifinite weight on $\mathfrak{M}$. The crossed product algebra $\mathcal{M} = \mathfrak{M} \rtimes_\sigma \mathbb{R}$ is then defined to be the von Neumann algebra, acting on $L^2(\mathbb{R}, \mathcal{H})$, which is generated by operators $\pi(x)$ and $\lambda(t)$, defined by (cf [2], [14])

$$(\pi(x)\xi)(s) = \sigma_{-s}(x)\xi(s), \tag{3.1}$$

$$(\lambda(t)\xi)(s) = \xi(s-t), \tag{3.2}$$

where $x \in \mathfrak{M}$, $t \in \mathbb{R}$, and $\xi \in L^2(\mathbb{R}, \mathcal{H})$ - the space $\mathcal{H}$-valued square-integrable functions on $\mathbb{R}$ (specifically $\xi \in L^2(\mathbb{R}, \mathcal{H})$ if $\int_{\mathbb{R}} \|\xi(t)\|_2^2 \, dt < \infty$). One may then define a dual action of $\mathbb{R}$ on $\mathcal{M}$ in the form of a one-parameter group of automorphisms (θ_s), by means of the prescription

$$\theta_s(a) = a, \quad \theta_s(\lambda(t)) = e^{-ist}\lambda(t) \text{ for all } a \in \mathfrak{M} \text{ and } s, t \in \mathbb{R}. \tag{3.3}$$

It turns out that $\mathfrak{M} \rtimes_\sigma \mathbb{R}$ *is a semifinite von Neumann algebra* which admits a faithful normal semifinite trace satisfying the condition that $\tau \circ \theta_s = e^{-s}\tau$ for all $s \in \mathbb{R}$. (See [11, Lemma 5.2], [32, Lemma 8.2] for details.)

On noting that the $\lambda(t)$'s are unitaries (whence $\lambda^*(t) = \lambda(-t)$, it follows from the definition of the crossed product that

1. $\lambda(t)\pi(x)\lambda^*(t) = \pi(\sigma_t(x))$ for $x \in \mathfrak{M}$ and $t \in \mathbb{R}$.
2. $\pi(x)\lambda(t)\pi(y)\lambda(s) = \pi(x\sigma_t(y))\lambda(ts)$,
3. $(\pi(x)\lambda(t))^* = \pi(\sigma_{-t}(x^*))\lambda(-t)$,
4. $\mathfrak{M} \rtimes_\sigma \mathbb{R}$ is the closure of the *-algebra of linear combinations of products $\lambda(s)\pi(x)$ with $x \in \mathfrak{M}$ and $s \in \mathbb{R}$.

4. Completely positive maps on quantum Orlicz spaces

The basic aim of this section is to provide conditions under which Markov (positive normal maps) on von Neumann algebras, allow for an extension to a large class of Orlicz spaces; in particular to the Orlicz space $L^{\cosh -1}$ which is a natural home for the regular observables (see [17], [21], and [22]). Ultimately we wish to show that an important class of quantum maps originally defined on the von Neumann algebra generated by bounded observables, also gives well defined time evolution of regular observables on this Orlicz space.

In achieving this extension, an appropriate extension of some form of interpolation scheme to the setting of type III von Neumann algebras will be needed, but the question that arises is exactly which mode of interpolation will be suited to the task? To clarify the picture (and justify some of the subsequent investigations), we will for the reader's convenience, make some preliminary observations:

Observation 4.1.
1. *Classically the exact interpolation spaces for the couple $\langle L^1, L^\infty\rangle$ coincide with the rearrangement invariant spaces on $\mathbb{R}$ and $\langle L^1, L^\infty\rangle$ is a Calderón couple (for all details see Chapter 26, Interpolation of Banach spaces by N. Kalton, S. Montgomery-Smith in* [13]*).*
2. *Dodds, Dodds, de Pagter have shown (see Theorem 3.2 in* [4]*) that the classical interpolation scheme for fully symmetric Banach function spaces on $\mathbb{R}^+$ can be extended to the context of semifinite von Neumann algebras equipped with a faithful normal semifinite trace.*
3. *Whilst complex interpolation has a proven track record in constructing and studying L^p-spaces corresponding to type III algebras, there is as yet no complex interpolative way to construct Orlicz spaces.*
4. *We therefore need a scheme built on the philosophy of the first point above, but which also takes note of the specific structure of Orlicz spaces corresponding to type III algebras. Some sort of hybrid of the two approaches therefore seems to be appropriate. This objective will be finally achieved in Theorem 4.11. (For a description of the structure of the pair $\langle L^1 \cap L^\infty, L^1 + L^\infty\rangle$ in the general setting, see* [16]*.)*

Before proceeding with the study of interpolation in the non-commutative setting, let us pause to clarify the description of the relevant quantum spaces.

When starting with a general von Neumann algebra $\mathfrak{M}$, we gain access to noncommutative measurability, by passing to the (much larger) semifinite von Neumann algebra $\mathcal{M} = \mathfrak{M} \rtimes_\sigma \mathbb{R}$ (see Sections 2 and 3). We emphasize that no information is lost in this transition, since the original algebra $\mathfrak{M}$, can be canonically embedded in the larger algebra $\mathcal{M}$, see Remark 6.1 in [21].

To describe the manner in which Orlicz spaces are constructed for type III algebras, we need some preliminaries. Let $\mathfrak{M}$ be a von Neumann algebra with fns weight ν. Further, let $h = \frac{d\tilde{\nu}}{d\tau}$ where $\tilde{\nu}$ is the dual weight of ν on the crossed product $\mathcal{M}$ and $\tau \equiv \tau_{\mathcal{M}}$ is its canonical trace. We will write $\mathfrak{n}_\nu$ for $\{a \in \mathfrak{M}; \nu(a^*a) < \infty\}$. For any Orlicz space $L^\Psi(\mathbb{R})$, the associated fundamental function on $[0, \infty)$, is defined by $t \to \|\chi_E\|_\Psi$ where E is a measurable subset of $(\mathbb{R}, \lambda)$ for which $\lambda(E) = t$. To distinguish the two case, we will write φ_Ψ for the fundamental function induced by it's Luxemburg norm, and $\tilde{\varphi}_\Psi$ for the fundamental function corresponding to the Orlicz norm. Finally, a complementary Orlicz function Ψ^* may be defined by setting $\Psi^*(u) = \sup_{v>0}\{uv - \Psi(v)\}$. A crucial fact underlying the definition given below, is that classically the space $L_{\Psi^*}(\mathbb{R})$ is the Köthe dual of $L^\Psi(\mathbb{R})$ with $\tilde{\varphi}_{\Psi^*}(t)\varphi_\Psi(t) = t$ for all $t \geq 0$ [1].

Definition 4.2 ([16])**.** We define the Orlicz space $L^\Psi(\mathfrak{M})$ to be

$$L^\Psi(\mathfrak{M}) = \{a \in \tilde{\mathcal{M}} : [e\tilde{\varphi}_{\Psi^*}(h)^{\frac{1}{2}}]a[\tilde{\varphi}_{\Psi^*}(h)^{\frac{1}{2}}f] \in L^1(\mathfrak{M}) \text{ for all projections e, f} \in \mathfrak{n}_\nu\}.$$

In the case where $\mathfrak{M}$ is semifinite and the canonical weight a trace, it is common to rather denote the associated spaces by either of $L^\Psi(\mathfrak{M}, \tau)$, or $L^\Psi(\widetilde{\mathfrak{M}})$. If Haagerup's strategy is used, we write $L^\Psi(\mathfrak{M})$ as above.

As noted earlier, if we are primarily interested in demonstrating the existence of a quantum map on the $L^p(\mathfrak{M})$-spaces ($1 \leq p \leq \infty$), then theorems like Theorem 5.1 of [12] (which is built on the technology of complex interpolation) will suffice. We briefly review the essentials of that approach. Let T be a positive normal map on $\mathfrak{M}$ which in this case need not be completely positive. For the sake of simplicity of exposition, we will here assume that $\mathfrak{M}$ is σ-finite, with ν a faithful normal state. Here we will follow the exposition of [9],[10], [15], and [33]. There is an operator $h \in L^1(\mathfrak{M})$ (where $\widetilde{\nu}$ is the dual weight of ω and $h \equiv h_\nu \equiv \frac{d\widetilde{\nu}}{d\tau}$) associated with the state ν. One may then use this operator to define embeddings of $\mathfrak{M}$ into $L^p(\mathfrak{M})$ as follows:

$$\iota_p : \mathfrak{M} \ni a \mapsto h_\nu^{\frac{1}{2p}} a h_\nu^{\frac{1}{2p}}. \tag{4.1}$$

Further, let T also be unital in its action on $\mathfrak{M}$. Define $T^{(p)}$ by

$$T^{(p)}(\iota_p(a)) = \iota_p(Ta), \tag{4.2}$$

for $a \in \mathfrak{M}$. Note that $\iota_p(\mathfrak{M})$ is dense in $L^p(\mathfrak{M})$ (see Lemma 1.6 in [9]). Therefore, $T^{(p)}$ is densely defined. (Similar conclusions hold in the general non-σ-finite case where we have a weight instead of a state. But in that case the embedding ι_p should be defined on the subalgebra span$\{y^*x : x, y \in \mathfrak{M}; \nu(x^*x) < \infty, \nu(y^*y) < \infty\}$ rather than the full algebra.)

Moreover by Theorem 5.1 in [12] (see also [9, Proposition 2.2]), whenever $\nu \circ T \leq \gamma\nu$ for some $\gamma > 0$, the map $T^{(p)}$ will extend to a positive bounded map on $L^p(\mathfrak{M})$.

However we are interested in demonstrating the existence of quantum maps on spaces quite different from L^p-spaces (here and subsequently in this section L^p stands for Haagerup's L^p space). In proving that we do have such a map on the space of regular observables, the primary difficulty we need to overcome, is that the current versions of the complex method only really work for L^p spaces. So some ingenuity is needed if we are to be successful. The assumption that seems to help to bridge this gap, is the requirement that T also be completely positive. We pause to point out that all of the theory developed in this section holds true for general von Neumann algebras, not just σ-finite ones. So unless otherwise stated, we will for the remainder of this section assume that $\mathfrak{M}$ is a possibly non-σ-finite algebra equipped with a faithful normal semifinite weight ν. We will write σ_t^ν for the modular group associated with the weight ν. We recall the following theorem from [12]. A refinement of this result proves to the crucial ingredient in our theory.

Theorem 4.3 ([12, Theorem 4.1])**.** *Let $\mathfrak{M}$ be as before, and assume that $T : \mathfrak{M} \to \mathfrak{M}$ is a completely bounded normal map such that*

$$T \circ \sigma_t^\nu = \sigma_t^\nu \circ T, \quad t \in \mathbb{R}.$$

Then T admits a unique bounded normal extension $\widetilde{T}$ on $\mathfrak{M} \rtimes_{\sigma^\nu} \mathbb{R}$ such that $\|T\| = \|\widetilde{T}\|$ and

$$\widetilde{T}(\lambda(s)\pi_\alpha(x)) = \lambda(s)\pi_\alpha(T(x)), \quad x \in \mathfrak{M}, s \in \mathbb{R}.$$

Moreover, $\widetilde{T}$ satisfies the following properties:

1. *Let B be the von Neumann subalgebra on $L^2(\mathbb{R}, H)$ generated by all $\lambda(s)$, $s \in \mathbb{R}$. Then*
$$\widetilde{T}(a\pi_\alpha(x)b) = a\pi_\alpha(T(x))b \quad \text{for all } a, b \in B.$$
2. *$\widetilde{T} \circ \sigma_t^{\widehat{\nu}} = \sigma_t^{\widehat{\nu}} \circ \widetilde{T}$ for all $t \in \mathbb{R}$ where $\widehat{\nu}$ is the dual weight of ν.*
3. *If T is positive, then so is $\widetilde{T}$.*
4. *Assume in addition that $\nu \circ T \leq \nu$. Then $\widehat{\nu} \circ \widetilde{T} \leq \widehat{\nu}$.*

Remark 4.4. We investigate the well-definiteness of the map $\widetilde{T}$ in the case where $\mathfrak{M}$ is a σ-finite von Neumann algebra in standard form, and ν a state with cyclic and separating vector Ω. To have a well defined linear map $\widetilde{T}$ on $\mathfrak{M} \rtimes_\sigma \mathbb{R}$ one wishes to have

$$\widetilde{T}(\widetilde{x}) = \widetilde{T}(\sum_i \lambda(s_i)\pi(x_i)) = \sum_i \lambda(s_i)\pi(T(x_i)), \tag{4.3}$$

where $\widetilde{x} = \sum_i \lambda(s_i)\pi(x_i) \in \mathfrak{M} \rtimes_\sigma \mathbb{R}$. But to guarantee the well definiteness of the linear map $\widetilde{T}$, so to have $\widetilde{T}(0) = 0$, one should be able to show that

$$\sum_i \lambda(s_i)\pi(x_i) = 0 \tag{4.4}$$

implies

$$\sum_i \lambda(s_i)\pi(T(x_i)) = 0 \tag{4.5}$$

for $s_i \in \mathbb{R}$ and $x_i \in \mathfrak{M}$. To this end let us consider (4.4) in detail. Namely, note that (4.4) implies $\| \sum_i \lambda(s_i)\pi(x_i)\xi\| = 0$ for any $\xi \in L^2(\mathbb{R}, \mathcal{H})$. Taking $\xi(t)$ to be of the form $\xi(t) = f(t)\Omega$ with $f \in L^2(\mathbb{R})$, one has

$$\begin{aligned}(4.6)\qquad 0 &= \int_{\mathbb{R}} \sum_{i,j} \Big(\lambda(s_i)\pi(x_i)f(t)\Omega, \lambda(s_j)\pi(x_j)f(t)\Omega\Big)dt \\ &= \int_{\mathbb{R}} \sum_{i,j} \big(\sigma_{s_i}(x_i)\Omega, \sigma_{s_j}(x_j)\Omega\big)\overline{f(t-s_i)}f(t-s_j)dt \\ &= \int_{\mathbb{R}} \sum_{i,j} \nu(\sigma_{s_i}(x_i^*)\sigma_{s_j}(x_j))\overline{f(t-s_i)}f(t-s_j)dt.\end{aligned}$$

Further note that on setting $a_{i,j} \equiv \int_{\mathbb{R}} \overline{f(t-s_i)}f(t-s_j)dt$, one obtains the positive definite matrix $a_{i,j}$. As $f \in L^2(\mathbb{R})$ is an arbitrary function one can then expect that $a_{i,j}$ is an arbitrary positive definite matrix. As the matrix a is positive $a \equiv \{a_{i,j}\} \geq 0$, it should then be of the form $a = b^*b$, see Lemma 3.1 in Chapter IV [31]. Therefore $a_{i,j} = \sum_k \overline{b_{k,i}}\ b_{k,j}$. Consequently, the fact that $\| \sum_i \lambda(s_i)\pi(x_i)\xi\| = 0$, combined with our choice of the vector ξ leads to

$$\begin{aligned}(4.7)\qquad 0 &= \sum_{i,j} \nu(\sigma_{s_i}(x_i^*)\sigma_{s_j}(x_j))a_{i,j} = \sum_{i,j,k} \nu(\sigma_{s_i}(x_i^*)\sigma_{s_j}(x_j))\overline{b_{k,i}}\ b_{k,j} \\ &= \sum_k \nu(\Big(\sum_i b_{k,i}\sigma_{s_i}(x_i)\Big)^* \Big(\sum_j b_{k,j}\sigma_{s_j}(x_j)\Big)).\end{aligned}$$

But, Ω is cyclic and separating, so ν is a faithful state. Thus, one gets

$$(4.8)\qquad \sum_i b_{k,i}\sigma_{s_i}(x_i) = 0.$$

To see how big the family of matrices $\{b_{i,j}\}$ is, let us take a basis in $L^2(\mathbb{R})$, for example consisting of $H_n(t)$-Hermite polynomials, and note that

$$(4.9)\qquad a_{i,j} = \sum_n \int_{\mathbb{R}} \overline{f(t-s_i)}H_n(t)dt \int_{\mathbb{R}} \overline{H_n(t)}f(t-s_j)dt \equiv \sum_n \overline{b_{n,i}}b_{n,j}.$$

Then (4.8) can be rewritten as

$$(4.10)\qquad \sum_i \int_{\mathbb{R}} \overline{H_n(t)}f(t-s_i)dt\ \sigma_{s_i}(x_i) = 0$$

for any $f \in L^2(\mathbb{R})$ and any n.

However, we note that $\{b_{n,i} \equiv \int_{\mathbb{R}} \overline{H_n(t)}f(t-s_i)dt\}$, for fixed i, is an arbitrary element in l_2-space. Thus, we see at once that $\{\lambda(s_i)\pi(x_i)\}$ are linearly independent and the map $\widetilde{T}$ is well defined.

In addition to the properties noted by Haagerup, Junge and Xu, the extension $\widetilde{T}$ also satisfies the following requirement. The establishment of this additional property, proves to be crucial.

Proposition 4.5. *Let T and $\widetilde{T}$ be as before. If each of (1)-(4) holds, then $\tau \circ \widetilde{T} \leq \tau$ where τ is the canonical trace on $\mathcal{M} = \mathfrak{M} \rtimes_{\sigma^\nu} \mathbb{R}$.*

Proof. Let $h = \frac{d\widehat{\nu}}{d\tau}$. By equation (1.1) of [12] and page 2130 of [12], the action of σ_t^ν is induced by $a \to h^{it}ah^{-it}$. In the language of [27], $\widehat{\nu}$ is then of the form $\tau(h\cdot) = \widehat{\nu}(\cdot)$ (see [27, Theorem 5.12]). So by [27, Proposition 4.3], we have that $\tau = \widehat{\nu}(h^{-1}\cdot)$. From the proof of Theorem 7.4 of [27] considered alongside the discussion following Proposition 4.1 of [27], it is clear that this means that for any $a \in (\mathfrak{M} \rtimes_{\sigma^\nu} \mathbb{R})_+$, have that $\tau(a) = \lim_{\epsilon \searrow 0} \widehat{\nu}([h^{-1/2}(\mathbb{1} + \epsilon h^{-1})^{-1/2}]a[h^{-1/2}(\mathbb{1} + \epsilon h^{-1})^{-1/2}])$. By the Borel functional calculus for affiliated operators, we have that $h^{-1}(\mathbb{1} + \epsilon h^{-1})^{-1} = (h + \epsilon\mathbb{1})^{-1}$ for each $\epsilon > 0$. So this formula becomes $\tau(a) = \lim_{\epsilon \searrow 0} \widehat{\nu}((h + \epsilon\mathbb{1})^{-1/2} a (h + \epsilon\mathbb{1})^{-1/2})$. We also have that the von Neumann algebra B generated by the $\lambda(t)$'s, agrees with the commutative von Neumann algebra generated by h. Now for any $\epsilon > 0$, $(\epsilon\mathbb{1} + h)^{-1/2} \equiv (\epsilon + h)^{-1/2}$ is bounded, and so belongs to B. So it is a simple matter to use parts (1) and (4) of Theorem 4.3 to see that

$$\begin{aligned} \widehat{\nu}((\epsilon + h)^{-1/2}\widetilde{T}(a)(\epsilon + h)^{-1/2}) &= \widehat{\nu}(\widetilde{T}((\epsilon + h)^{-1/2}a(\epsilon + h)^{-1/2})) \\ &\leq \widehat{\nu}((\epsilon + h)^{-1/2}a(\epsilon + h)^{-1/2}). \end{aligned}$$

Letting ϵ decrease to zero, now yields $\tau(\widetilde{T}(a)) \leq \tau$, as required. □

With the above corollary at our disposal we may at this point appeal to Yeadon's ergodic theorem for positive maps [35] (recently extended and significantly sharpened by Haagerup, Junge and Xu [12, Theorem 5.1]) to see that $\widetilde{T}$ extends canonically to a bounded map on $L^1(\mathcal{M}, \tau)$. Since $\mathcal{M}$ is semifinite, the full power of the DDdP approach is therefore at our disposal, and we may use real interpolation (see [4]) to extend the action of $\widetilde{T}$ to a large class of rearrangement invariant Banach function spaces associated with $\mathcal{M}$. Specifically $\widetilde{T}$ canonically induces an action on each Orlicz space $L^\Phi(\mathcal{M}, \tau)$.

Whilst this fact is worthy of noting, we are here interested in the action of $\widetilde{T}$ on spaces associated with $\mathfrak{M}$, not $\mathcal{M}$. For this purpose we need the following observation:

Proposition 4.6. *Let T be a completely positive map on $\mathfrak{M}$ satisfying $\nu \circ T \leq \nu$ and let $\widetilde{T}$ be its completely positive extension to $\mathcal{M} = \mathfrak{M} \rtimes_{\sigma^\nu} \mathbb{R}$. Then $\widetilde{T}$ canonically induces a map on the space $(L^\infty + L^1)(\mathcal{M}, \tau)$.*

Proof. If we apply [12, Theorem 5.1] to Proposition 4.5, it is clear that $\widetilde{T}$ canonically induces a map on $L^1(\mathcal{M}, \tau)$. So this claim follows from the theory of Dodds, Dodds and de Pagter [4]. □

Proposition 4.7. *Let T be a completely positive map on $\mathfrak{M}$ satisfying $\nu \circ T \leq \nu$ and let $\widetilde{T}$ be its completely positive extension to $\mathcal{M} = \mathfrak{M} \rtimes_{\sigma^\nu} \mathbb{R}$. For the sake of simplicity we will also write $\widetilde{T}$ for the extension to $(L^\infty + L^1)(\mathcal{M}, \tau)$. In its action on $(L^\infty + L^1)(\mathcal{M}, \tau)$, it satisfies the condition that $\widetilde{T}(ab) = a\widetilde{T}(b)$ for all $a \in B$ and all $b \in (L^\infty + L^1)(\mathcal{M}, \tau)$. (Here B is von Neumann subalgebra generated by all $\lambda(s)$, $s \in \mathbb{R}$.)*

Proof. The stated property is known to hold for $\mathcal{M}$. Since $\mathcal{M} \cap L^1(\mathcal{M}, \tau_{\mathcal{M}})$ is norm-dense in $L^1(\mathcal{M}, \tau_{\mathcal{M}})$, the continuity of $\widetilde{T}$ on $L^1(\mathcal{M}, \tau_{\mathcal{M}})$, then ensures that it also holds for $L^1(\mathcal{M}, \tau_{\mathcal{M}})$. □

The reason for proving the above, is that all the type III Orlicz spaces with upper fundamental index less that 1, live inside $(L^\infty + L^1)(\mathcal{M}, \tau)$. We first proceed to define the fundamental indices of an Orlicz space. These were introduced by Zippin [36].

Definition 4.8. Let $L^\Psi(\mathfrak{M})$ be an Orlicz space, and let φ_ψ be the fundamental function of the space $L^\Psi(0,\infty)$. (Here we consider the Luxemburg, norm but the case of the Orlicz norm is completely analogous.) Let $M_\psi(t) = \sup_{s>0} \frac{\varphi_\Psi(st)}{\varphi_\Psi(s)}$. (If we use the fundamental function of the space $L_\Psi(0,\infty)$ equipped with the Orlicz norm, we will write $\widetilde{M}_\psi(t)$ for this function.) Then the lower and upper fundamental indices of $L^\Psi(\mathfrak{M})$ are defined to be

$$\underline{\beta}_{L^\Psi} = \sup_{0<t\leq 1} \frac{\log M_\psi(s)}{\log s} \quad \text{and} \quad \overline{\beta}_{L^\Psi} = \inf_{1<t} \frac{\log M_\psi(s)}{\log s}$$

respectively.

The following result is a variant of [16, Theorem 3.13], where the Boyd indices were used to prove a similar theorem.

Proposition 4.9. *Let $L^\Psi(\mathfrak{M})$ be an Orlicz space with upper fundamental index strictly less than 1. Then $L^\psi(\mathfrak{M}) \subset (L^\infty + L^1)(\mathcal{M}, \tau_\mathcal{M})$ where $\mathcal{M} = \mathfrak{M} \rtimes_{\sigma^\nu} \mathbb{R}$. Moreover the canonical topology on $L^\psi(\mathfrak{M})$ then agrees with the subspace topology inherited from $(L^\infty + L^1)(\mathcal{M}, \tau_\mathcal{M})$.*

Proof. It is clear from exercise 14 of [1, Chapter 3] (see also Corollary 8.15, page 275 in [1]) that $0 < \underline{\beta}_{\Psi^*} = 1 - \overline{\beta}_\Psi = \sup_{0<t\leq 1} \frac{\log \widetilde{M}_{\psi^*(t)}}{\log t} = \lim_{t\to 0^+} \frac{\log \widetilde{M}_{\psi^*(t)}}{\log t}$. So given any $n \in \mathbb{N}$, there must exist $1 \geq t_0 > 0$ such that

$$\frac{\log(\widetilde{M}_{\psi^*(t)})}{\log t} \geq \frac{n-1}{n} \underline{\beta}_{\Psi^*}$$

for all $0 < t \leq t_0$. This can be shown to imply the fact that

$$\widetilde{M}_{\psi^*(t)} \leq t^{\frac{n-1}{n}\underline{\beta}_{\Psi^*}}.$$

It is an exercise to see that

$$\frac{\widetilde{\varphi}_{\Psi^*(s)}}{\widetilde{\varphi}_{\Psi^*(s/t)}} \leq \sup_{r>0} \frac{\widetilde{\varphi}_{\Psi^*(rt)}}{\widetilde{\varphi}_{\Psi^*(r)}} = \widetilde{M}_{\psi^*(t)}.$$

On selecting $s_t < 0$ so that $e^{s_t} = t$, this in turn ensures that

$$d_{s_t} = \widetilde{\varphi}_{\Psi^*}(e^{-s_t}h)^{-1}\widetilde{\varphi}_{\Psi^*}(h) \leq t^{\frac{n-1}{n}\underline{\beta}_{\Psi^*}}.$$

Let $x \in L^\Psi(\mathfrak{M})$ be given. We remind the reader that then $\mu_1(x) = \sup_{0<t\leq 1} t\mu_t(x)$ and $t\mu_t(x) = \mu_1(d_{s_t}^{1/2} x d_{s_t}^{1/2})$ for any $0 < t \leq 1$ (see [16, Theorem 3.10] and its proof). For $0 < t \leq t_0$ we then clearly have that $t\mu_t(x) \leq \|d_{s_t}\|\mu_1(a) \leq t^{\frac{n-1}{n}\overline{\beta}_\Psi}$. Finally use the fact that $t \to \mu_t(x)$ is decreasing, to see that for $n > 1$

$$\begin{aligned} \mu_1(x) &\leq \int_0^1 \mu_r(x)\,dr = \int_0^1 \frac{r\mu_r(x)}{r}\,dr \\ &\leq \left[\int_0^{t_0} \left(\frac{1}{r}\right)^{1-\frac{n-1}{n}\underline{\beta}_{\Psi^*}} dt + \int_{t_0}^1 \frac{1}{r}\,dr\right] \sup_{0<t\leq 1} t\mu_t(x). \end{aligned}$$

Hence $\mu_1(x) = \sup_{0<t\leq 1} t\mu_t(x)$ is equivalent to the canonical norm on $(L^\infty + L^1)(\mathcal{M}, \tau_\mathcal{M})$, namely $\int_0^1 \mu_r(x)\,dr$. The claim follows □

What we still need is an alternative criterion for identifying the elements of $(L^\infty + L^1)(\mathcal{M}, \tau)$ that belong to some $L^\Psi(\mathfrak{M})$. The following lemma succeeds in this regard, by greatly simplifying the recipe for constructing Orlicz spaces for type III algebras.

Lemma 4.10. *Let $\mathcal{M} = \mathfrak{M} \rtimes_{\sigma^\varphi} \mathbb{R}$, let θ_s be the dual action of $\mathbb{R}$ on $\mathcal{M}$, and let $h = \frac{d\widehat{\varphi}}{d\tau}$. Given some Young's function Ψ, the maps $v_s = \varphi_\Psi(e^{-s}h)\varphi_\Psi(h)^{-1}$ are bounded for each $s \in \mathbb{R}$. Moreover $a \in \widetilde{\mathcal{M}}$ belongs to $L^\Psi(\mathfrak{M})$ if and only if for any $s \in \mathbb{R}$ we have that $\theta_s(a) = v_s^{1/2} a v_s^{1/2}$.*

Proof. We show that the maps $d_s = \widetilde{\varphi}_{\Psi^*}(e^{-s}h)^{-1}\widetilde{\varphi}_{\Psi^*}(h)$ are bounded for each $s \in \mathbb{R}$, with $a \in \widetilde{\mathcal{M}}$ belonging to $L^\Psi(\mathfrak{M})$ if and only if $\theta_s(a) = e^{-s}d_s^{1/2}ad_s^{1/2}$ for any $s \in \mathbb{R}$. The lemma will then follow from the observation that for any $s \in \mathbb{R}$, we have that $v_s = e^{-s}d_s$. To see this, observe that it follows from [1, Theorem II.5.2 & Corollary IV.8.15] that $e^{-s}\frac{\widetilde{\varphi}_{\Psi^*}(t)}{\widetilde{\varphi}_{\Psi^*}(e^{-s}t)} = \frac{\varphi_\Psi(e^{-s}t)}{\varphi_\Psi(t)}$ for any $t > 0$

To see the claim regarding boundedness, observe that if $s < 0$, we may conclude from [1, Corollary II.5.3] that $\widetilde{\varphi}_{\Psi^*}(t) \leq \widetilde{\varphi}_{\Psi^*}(e^{-s}t)$ for all $t > 0$, or equivalently that $\frac{\widetilde{\varphi}_{\Psi^*}(t)}{\widetilde{\varphi}_{\Psi^*}(e^{-s}t)} \leq 1$. The map d_s will then clearly be contractive. If on the other hand $s \geq 0$, then again by [1, Corollary II.5.3], we have that $\frac{\widetilde{\varphi}_{\Psi^*}(t)}{t} \leq \frac{\widetilde{\varphi}_{\Psi^*}(e^{-s}t)}{e^{-s}t}$ for all $t > 0$, or equivalently that $\frac{\widetilde{\varphi}_{\Psi^*}(t)}{\widetilde{\varphi}_{\Psi^*}(e^{-s}t)} \leq e^s$. So in this case $\|d_s\| \leq e^s$.

Next let $a \in L^\Psi(\mathfrak{M})$ be given. It was shown in the proof of [16, 3.10], that we then have that $\theta_s(a) = e^{-s}d_s^{1/2}ad_s^{1/2}$ for any $s \geq 0$. So for $s \geq 0$, we must then have that $a = \theta_{-s}(\theta_s(a)) = e^{-s}\theta_{-s}(d_s^{1/2})\theta_{-s}(a)\theta_{-s}(d_s^{1/2})$ or equivalently that $\theta_{-s}(a) = e^s\theta_{-s}(d_s^{1/2})^{-1}a\theta_{-s}(d_s^{1/2})^{-1}$. Now observe that

$$\theta_{-s}(d_s) = \widetilde{\varphi}_{\Psi^*}(e^{-s}\theta_{-s}(h))^{-1}\widetilde{\varphi}_{\Psi^*}(\theta_{-s}(h)) = \widetilde{\varphi}_{\Psi^*}(h)^{-1}\widetilde{\varphi}_{\Psi^*}(e^s h) = d_{-s}^{-1}.$$

In the case $s \geq 0$, we therefore also have that $\theta_{-s}(a) = e^s(d_{-s}^{1/2})^{-1}ad_{-s}^{1/2}$.This proves the "only if" part of the equivalence.

For the converse assume that the stated condition regarding the action of θ_s holds for some $a \in \widetilde{\mathcal{M}}$. Note that the action of the θ's extends to operators affiliated to $\mathcal{M}$. So for any projection $e \in \mathfrak{M}$ of finite weight, we know that $e.\widetilde{\varphi}_{\Psi^*}(h)$ is closable with $\tau_{\mathcal{M}}$-dense domain. It is easy to conclude that

$$\begin{aligned}\theta_s(e[\widetilde{\varphi}_{\Psi^*}(h)\chi_{[0,n]}(h)]) &= \theta_s(e)\theta_s(\widetilde{\varphi}_{\Psi^*}(h)\chi_{[0,n]}(h) \\ &= e\widetilde{\varphi}_{\Psi^*}(e^{-s}h)\chi_{[0,n]}(e^{-s}h) \\ &= e\widetilde{\varphi}_{\Psi^*}(e^{-s}h)\chi_{[0,e^s n]}(h).\end{aligned}$$

But then the operators $\theta_s([e\widetilde{\varphi}_{\Psi^*}(h)])$ and $[e.\widetilde{\varphi}_{\Psi^*}(e^{-s}h)]$ must agree on the dense subspace $\cup_{\lambda<\infty}\chi_{[0,\lambda](h)}(\mathfrak{H})$. Hence $[e.\theta_s(\widetilde{\varphi}_{\Psi^*}(h)] = [e.\widetilde{\varphi}_{\Psi^*}(e^{-s}h)]$ [8, Lemma 2.1]. By duality $\theta_s(\widetilde{\varphi}_{\Psi^*}(h)e) = \widetilde{\varphi}_{\Psi^*}(e^{-s}h)e$. Hence $d_s^{1/2}\theta_s(\widetilde{\varphi}_{\Psi^*}(h)e) = \widetilde{\varphi}_{\Psi^*}(h)e$. By duality we then also have that $\theta_s([e\widetilde{\varphi}_{\Psi^*}(h)])d_s^{1/2} = [e\widetilde{\varphi}_{\Psi^*}(h)]$. Consequently for any two projections $e, f \in \mathfrak{M}$ of finite weight, we have that

$$\begin{aligned}\theta_s([f\widetilde{\varphi}_{\Psi^*}(h)]a(\widetilde{\varphi}_{\Psi^*}(h)e)) &= \theta_s([f\widetilde{\varphi}_{\Psi^*}(h)])\theta_s(a)\theta_s(\widetilde{\varphi}_{\Psi^*}(h)e) \\ &= e^{-s}\theta_s([f\widetilde{\varphi}_{\Psi^*}(h)])d_s^{1/2}ad_s^{1/2}\theta_s(\widetilde{\varphi}_{\Psi^*}(h)e) \\ &= e^{-s}[f\widetilde{\varphi}_{\Psi^*}(h)]a(\widetilde{\varphi}_{\Psi^*}(h)e).\end{aligned}$$

for all $s \in \mathbb{R}$. Hence by definition $a \in L^\Psi(\mathfrak{M})$ (see [16]). $\square$

We are finally ready to prove that CP Markov dynamics on $\mathfrak{M}$, canonically extend to a large class of quantum Orlicz spaces. We remind the reader that a map T on $\mathfrak{M}$, is called sub-Markov, if the situation $0 \leq a \leq \mathbb{1}$ $(a \in \mathfrak{M})$, ensures that $0 \leq T(a) \leq \mathbb{1}$. In the case where $\mathfrak{M}$ is σ-finite and ν a state, the density $h = \frac{d\widetilde{\nu}}{d\tau_{\mathcal{M}}}$ is actually an element of $L^1(\mathfrak{M})$, and hence $\varphi_\Psi(h) \in L^\Psi(\mathfrak{M})$. In this setting we say that a map S on $L^\Psi(\mathfrak{M})$ is sub-L^Ψ-Markov, if the situation $0 \leq a \leq \varphi_\Psi(h)$ $(a \in L^\Psi(\mathfrak{M}))$, ensures that $0 \leq S(a) \leq \varphi_\Psi(h)$.

Theorem 4.11. *Let $\mathcal{M} = \mathfrak{M} \rtimes_{\sigma^\nu} \mathbb{R}$, and let Ψ be a Young's function. Let $T : \mathfrak{M} \to \mathfrak{M}$ be a completely positive normal map such that $\nu \circ T \leq \nu$, and*

$$T \circ \sigma_t^\nu = \sigma_t^\nu \circ T, \quad t \in \mathbb{R},$$

with $\widetilde{T}$ denoting the map induced by T on $(L^\infty + L^1)(\mathcal{M}, \tau)$, and suppose that $\overline{\beta}_{L^\Psi} < 1$.

- *Then $\widetilde{T}$ restricts to a bounded map T_Ψ on $L^\Psi(\mathfrak{M})$.*
- *In the case where $\mathfrak{M}$ is σ-finite and ν a state, the map T_Ψ will in it's action on $L^\Psi(\mathfrak{M})$, map elements of the form $\varphi_\Psi(h)^{1/2} a \varphi_\Psi(h)^{1/2}$ $(a \in \mathfrak{M})$ onto elements of the form $\varphi_\Psi(h)^{1/2} T(a) \varphi_\Psi(h)^{1/2}$. In particular T_Ψ will be sub-L^Ψ-Markov whenever T is sub-Markov. Moreover for any sub-L^Ψ-Markov map S on $L^\Psi(\mathfrak{M})$, we may then find a sub-Markov map S_∞ on $\mathfrak{M}$ such that $S(\varphi_\Psi(h)^{1/2} a \varphi_\Psi(h)^{1/2}) = \varphi_\Psi(h)^{1/2} S_\infty(a) \varphi_\Psi(h)^{1/2}$ for all $a \in \mathfrak{M}$.*

Proof. If $\overline{\beta}_{L^\Psi} < 1$, then $L^\Psi(\mathfrak{M})$ lives inside $(L^\infty + L^1)(\mathcal{M}, \tau)$, and also gets its topology from $(L^\infty + L^1)(\mathcal{M}, \tau)$. So all one needs to do to prove the first claim, is to note

- that if $a \in (L^\infty + L^1)(\mathcal{M}, \tau)$, then also $v_s^{1/2} a v_s^{1/2} \in (L^\infty + L^1)(\mathcal{M}, \tau)$,
- then that $\widetilde{T}(v_s^{1/2} a v_s^{1/2}) = v_s^{1/2} \widetilde{T}(a) v_s^{1/2}$ by Proposition 4.7,
- and then simply apply the Lemma.

For the second claim, note that by the preceding proposition, $L^\Psi(\mathfrak{M})$ lives inside $(L^1 + L^\infty)(\mathcal{M})$. Given $a \in \mathfrak{M}$, we will then have that $\varphi_\Psi(h)^{1/2} a \varphi_\Psi(h)^{1/2} \in L^\Psi(\mathfrak{M}) \subset (L^1 + L^\infty)(\mathcal{M})$. On applying Proposition 4.7 to $\widetilde{T}$, it follows that $\chi_{[0,n]}(h) \widetilde{T}(\varphi_\Psi(h)^{1/2} a \varphi_\Psi(h)^{1/2}) \chi_{[0,n]}(h) = \widetilde{T}([\varphi_\Psi(h)^{1/2} \chi_{[0,n]}(h)] a [\varphi_\Psi(h)^{1/2} \chi_{[0,n]}(h)])$ for each $n \in \mathbb{N}$. But since $[\varphi_\Psi(h)^{1/2} \chi_{[0,n]}(h)] \in B$, it follows from the definition of $\widetilde{T}$ that

$$\begin{aligned} \widetilde{T}([\varphi_\Psi(h)^{1/2} \chi_{[0,n]}(h)] a [\varphi_\Psi(h)^{1/2} \chi_{[0,n]}(h)]) &= [\varphi_\Psi(h)^{1/2} \chi_{[0,n]}(h)] T(a) [\varphi_\Psi(h)^{1/2} \chi_{[0,n]}(h)] \\ &= \chi_{[0,n]}(h) [[\varphi_\Psi(h)^{1/2} T(a) \varphi_\Psi(h)^{1/2}] \chi_{[0,n]}(h). \end{aligned}$$

Hence for each n, we have

$$\begin{aligned} \chi_{[0,n]}(h) T_\Psi(\varphi_\Psi(h)^{1/2} a \varphi_\Psi(h)^{1/2}) \chi_{[0,n]}(h) &= \chi_{[0,n]}(h) \widetilde{T}(\varphi_\Psi(h)^{1/2} a \varphi_\Psi(h)^{1/2}) \chi_{[0,n]}(h) \\ &= \chi_{[0,n]}(h) [\varphi_\Psi(h)^{1/2} T(a) \varphi_\Psi(h)^{1/2}] \chi_{[0,n]}(h). \end{aligned}$$

This is enough to prove the second claim.

We proceed with the proof of the third claim. To this end let $a \in L^\Psi(\mathfrak{M})$ be given with $0 \leq a \leq \varphi_\Psi(h)$. It then follows from [29, Lemma 2.2 d] that there exists a contractive element $x \in \mathcal{M}^+$ such that $a = \varphi_\Psi(h)^{1/2} x \varphi_\Psi(h)^{1/2}$. For any $s \in \mathbb{R}$ we may then apply θ_s to both sides. Recall that in Lemma 4.10 we had that

$v_s = \varphi_\Psi(e^{-s}h)\varphi_\Psi(h)^{-1}$. So by that Lemma, we will then have that

$$\begin{aligned}\varphi_\Psi(e^{-s}h)^{1/2}x\varphi_\Psi(e^{-s}h)^{1/2} &= v_s^{1/2}(\varphi_\Psi(h)^{1/2}x\varphi_\Psi(h)^{1/2})v_s^{1/2}\\ &= \theta_s(\varphi_\Psi(h)^{1/2}x\varphi_\Psi(h)^{1/2})\\ &= \theta_s(\varphi_\Psi(h)^{1/2})\theta_s(x)\theta_s(\varphi_\Psi(h)^{1/2})\\ &= \varphi_\Psi(e^{-s}h)^{1/2}\theta_s(x)\varphi_\Psi(e^{-s}h)^{1/2}.\end{aligned}$$

Equivalently $x = \theta_s(x)$ for each $s \in \mathbb{R}$, which ensures that $x \in \mathfrak{M}$. Since x is positive and contractive, we have that $0 \leq x \leq \mathbb{1}$ and hence that $0 \leq T(x) \leq \mathbb{1}$. But then by the second part of the proof, $T_\Psi(a) = \varphi_\Psi(h)^{1/2}T(x)\varphi_\Psi(h)^{1/2} \leq \varphi_\Psi(h)$, as required.

It remains to prove the final claim. The proof uses the fact verified above that if we are given $a \in L^\Psi(\mathfrak{M})$ with $0 \leq a \leq \varphi_\Psi(h)$, then $a = \varphi_\Psi(h)^{1/2}x\varphi_\Psi(h)^{1/2}$ for some contractive element of $x \in \mathfrak{M}^+$, and is to all intents and purposes a minor modification of the second part of the proof of [9, Proposition 2.5]. The one fact we need to verify for that proof to adapt to the present context, is that if for some $b \in \mathfrak{M}$ we have that $0 = \varphi_\Psi(h)^{1/2}b\varphi_\Psi(h)^{1/2}$, then $b = 0$. This can be seen to follow by noting that if $0 = \varphi_\Psi(h)^{1/2}b\varphi_\Psi(h)^{1/2}$, then $0 = \widetilde{\varphi}_{\Psi^*}(h)^{1/2}(\varphi_\Psi(h)^{1/2}b\varphi_\Psi(h)^{1/2})\widetilde{\varphi}_{\Psi^*}(h)^{1/2} = h^{1/2}bh^{1/2}$, and applying [9, Lemma 1.3]. □

Example 4.12. The upper fundamental index of the space $L^{\cosh -1}(0,\infty)$ is $\frac{1}{2}$. Isomorphic Orlicz spaces share the same indices. So it is sufficient to prove this for a space isomorphic to $L^{\cosh -1}(0,\infty)$. We show how to construct such space before proving the claim. It is easy to see that the graphs of e^t and $\frac{e^2}{4}t^2$ are tangent at $t = 2$. This fact ensures that

$$\Psi_e(t) = \begin{cases} \frac{e^2}{4}t^2 & \text{if} \quad 0 \leq t \leq 2 \\ e^t & \text{if} \quad 2 < t \end{cases}$$

is a Young's function. Using Maclaurin series it is easy to see that $\lim_{t\to 0+} \frac{\cosh(t)-1}{\Psi_e(t)} = \frac{2}{e^2}$. Since we also have that $\lim_{t\to\infty} \frac{\cosh(t)-1}{\Psi_e(t)} = \lim_{t\to\infty} \frac{e^t+e^{-t}-2}{2e^t} = \frac{1}{2}$, it is clear that $\Psi_e \approx \cosh -1$. (To see this note that the limit formulae ensure that we may find $0 < \alpha < \beta < \infty$ so that $\frac{1}{e^2} < \frac{\cosh(t)-1}{\Psi_e(t)} < \frac{3}{e^2}$ on $[0,\alpha]$, and $\frac{1}{4} < \frac{\cosh(t)-1}{\Psi_e(t)} < \frac{3}{4}$ on $[\beta,\infty)$. Since the function $\frac{\cosh(t)-1}{\Psi_e(t)}$ has a both a minimum and maximum on the interval $[\alpha,\beta]$, a combination of these facts ensures that we can find positive constants $0 < m < M < \infty$ so that $m\Psi_e(t) < \cosh(t) - 1 < M\Psi_e(t)$ for all $t \in [0,\infty)$.) This is clearly enough to ensure that $L^{\Psi_e}(0,\infty) \equiv L^{\cosh -1}(0,\infty)$.

It remains to compute the fundamental indices of $L^{\Psi_e}(0,\infty)$. We will use the formulas in Remark 2.3 of [36] to compute these indices. We will assume that $L^{\Psi_e}(0,\infty)$ is equipped with the Luxemburg norm. Since

$$\Psi_e^{-1}(t) = \begin{cases} \frac{2}{e}t^{1/2} & \text{if} \quad 0 \leq t \leq e^2 \\ \log(t) & \text{if} \quad e^2 < t \end{cases},$$

it now follows from [1, 4.8.17] that the fundamental function of $L^{\Psi_e}(0,\infty)$ is given by

$$\varphi_e(t) = \begin{cases} \frac{e}{2}t^{1/2} & \text{if} \quad t \geq e^{-2} \\ \frac{1}{-\log(t)} & \text{if} \quad t < e^{-2} \end{cases}$$

We proceed to compute the function $M_{\Psi_e}(s) = \sup_{t>0} \frac{\varphi_e(st)}{\varphi_e(t)}$. In computing this function, we first consider the case where $0 < s \leq 1$. Since φ_e is increasing, we then have that

$$\frac{\varphi_e(st)}{\varphi_e(t)} \leq \frac{\varphi_e(t)}{\varphi_e(t)} = 1$$

for any $t > 0$. Since we also have that

$$\lim_{t\to 0} \frac{\varphi_e(st)}{\varphi_e(t)} = \lim_{t\to 0} \frac{\log(t)}{\log(st)} = \lim_{t\to 0} \frac{\log(t)}{\log(s) + \log(t)} = 1,$$

it is clear that $M_{\Psi_e}(s) = 1$ in this case.

Now let s be given with $s > 1$. We then have that

$$\frac{\varphi_e(st)}{\varphi_e(t)} = \begin{cases} s^{1/2} & \text{if } \ t > \frac{1}{e^2} \\ -\frac{e}{2}s^{1/2}t^{1/2}\log(t) & \text{if } \ \frac{1}{e^2} > t > \frac{1}{se^2} \\ \frac{\log(t)}{\log(st)} & \text{if } \ t < \frac{1}{se^2} \end{cases}$$

It is not too difficult to see that the function $t \to -\frac{e}{2}s^{1/2}t^{1/2}\log(t)$ has a maximum of $s^{1/2}$ at $t = e^{-2}$ on the interval $(0, 1)$. So for $t \in (\frac{1}{se^2}, \frac{1}{e^2})$, the supremum of the above quotient is $s^{1/2}$. Finally consider the function

$$t \to \tfrac{\log(t)}{\log(st)} = \tfrac{\log(t)}{\log(s)+\log(t)} = 1 - \tfrac{\log(s)}{\log(s)+\log(t)} = 1 - \tfrac{\log(s)}{\log(st)}.$$

It is easy to see that

$$\frac{d}{dt}(1 - \tfrac{\log(s)}{\log(s)+\log(t)}) = \frac{\log(s)}{t(\log(st))^2} > 0$$

on $t \in (0, \frac{1}{se^2})$. Hence on $(0, \frac{1}{se^2}]$, $t \to \frac{\log(t)}{\log(st)}$ attains a maximum of $1 + \frac{1}{2}\log(s)$ at $t = \frac{1}{se^2}$. Using the fact that $1 + \log(t) \leq t$, it is now easy to see that $1 + \frac{1}{2}\log(s) = 1 + \log(s^{1/2}) \leq s^{1/2}$. Putting all these facts together leads to the conclusion that $M_{\Psi_e}(s) = \sup_{t>0} \frac{\varphi_e(st)}{\varphi_e(t)} = s^{1/2}$ in this case. We therefore have that

$$\overline{\beta}_{\Psi_e} = \lim_{s\to\infty} \frac{\log M_{\Psi_e}(s)}{\log s} = \lim_{s\to\infty} \frac{\log s^{1/2}}{\log s} = \frac{1}{2}$$

as claimed. Similarly

$$\underline{\beta}_{\Psi_e} = \lim_{s\to 0+} \frac{\log M_{\Psi_e}(s)}{\log s} = \lim_{s\to 0+} \frac{\log 1}{\log s} = 0.$$

Corollary 4.13. *If T is a completely positive map on $\mathfrak{M}$ satisfying $\nu \circ T \leq \nu$, then $\widetilde{T}$ canonically induces an action on $L^{\cosh -1}(\mathfrak{M})$.*

Of course the question now arises as to how the maps $T^{(p)}$ on $L^p(\mathfrak{M})$ (defined earlier), compare to the extension of $\widetilde{T}$ to $L^p(\mathfrak{M})$ by means of the above process, and ultimately also how the work of Goldstein and Lindsay, and Haagerup, Junge and Xu, compare to ours. This relationship is clarified by the following corollary to Theorem 4.11.

Corollary 4.14. *In the case $\Psi(t) = t^p$ $(p > 1)$, the maps induced by $\widetilde{T}$ on $L^p(\mathfrak{M})$, are exactly the maps $T^{(p)}$ constructed in* [12, Theorem 5.1].

Proof. For the sake of clarity of exposition, we restrict attention to the σ-finite case, assuming that ν is in fact a state. The claim may then be proven by simply replacing $\varphi_\Psi(h)^{1/2}$ with $h^{1/(2p)}$ in the proof of the second claim of Theorem 4.11. $\square$

5. Utility of the theory for mathematical physics

In mathematical physics the set of observables of a given quantum system will for the most part lead to a σ-finite von Neumann algebra $\mathfrak{M}$ on a separable Hilbert space, and hence in this second part of the paper, we will restrict to this context. Our primary goal in this section, is to demonstrate the utility of the preceding theory for mathematical physics, by showing that a large class of natural quantum maps fulfil the criteria of the preceding section, and hence allow for an extension to the space $L^{\cosh -1}(\mathfrak{M})$. But what exactly is a "natural quantum map"?

In standard quantum statistical mechanics the starting point is a triple $(\mathcal{A}, T_t, \omega)$ consisting of a C^*-algebra, a family of dynamical maps $\{T_t\}$, and a time invariant faithful state ω. The dynamics of the family can also be expressed at the Hilbert space level. Specifically on passing to the GNS-construction $(\pi_\omega, \mathcal{H}_\omega, \Omega)$, it is easy to see that the action of each T_t on $\mathcal{A}$, in a very natural way induces an action $\hat{T}_t$ on the dense set $\pi_\omega(A)\Omega \subset \mathcal{H}_\omega$ which is defined by

$$\hat{T}_t\pi_\omega(a)\Omega = \pi_\omega(T_t(a))\Omega.$$

The question arises as to whether $\hat{T}_t$ is bounded or even closable. The additional assumption of complete positivity leads to an easy proof of the contractivity of $\hat{T}_t$. There are however alternative conditions similarly ensuring the contractivity of $\hat{T}_t$. A condition of this type which arises from physical considerations, is the so-called detailed balance condition (DBC for short). At a physical level DBC may also be thought of as a condition corresponding to the microscopic reversibility of the system. This condition too ensures the contractivity of $\hat{T}_t$. In fact it does more; the assumption of DBC also ensures that each $\hat{T}_t$ commutes with the modular operator and that at least one of the conditions in Theorem 4.3 is then automatically satisfied. Thus the assumption that the quantum maps $\{T_t\}$ satisfy DBC, ensures that at the Hilbert space level, the maps $\{\hat{T}_t\}$ induce very regular dynamics. From the perspective of Physics, this elegant duality reflects the duality between the Heisenberg and Schrödinger pictures in the description of quantum systems. In this regard we point out that since the publication of Tolman's book [34] and the paper of Glauber [7], DBC is frequently used in statistical physics. (See for example [6] and [24], and the references therein.)

5.1. **Detailed Balance Condition.**

Recall that for us a positive, normal, unital map $T : \mathfrak{M} \to \mathfrak{M}$ is deemed to be a *Markov map*. The class of Markov maps seems to be too general to describe the most interesting genuine dynamics. Hence, to select more regular maps we define:

Definition 5.1. A Markov map satisfies the Detailed Balance Condition (for brevity DBC) with respect to a state ω on $\mathfrak{M}$ if the following conditions are satisfied (see [19], [20])

$$\omega(x^*T(y)) = \omega(\Theta(y^*)T\Theta(x)) \tag{5.1}$$

for any $x, y \in \mathfrak{M}$, where Θ is a reversing operation, i.e. an antilinear Jordan morphism on $\mathfrak{M}$ such that $\Theta^2 =$ identity map, and $\omega(\Theta(xy)) = \omega(\Theta(x)\Theta(y))$.

DBC implies that (see [20]):

$$\omega(T(x)) = \omega(x), \quad x \in \mathfrak{M}. \tag{5.2}$$

and that

$$\widehat{T}x\Omega = T(x)\Omega \tag{5.3}$$

defines a bounded operator on $\mathcal{H}$ which commutes with the modular operator Δ. Moreover, it is an easy observation to make that

$$T \circ \sigma_t = \sigma_t \circ T, \quad \text{for any t} \in \mathbb{R}. \tag{5.4}$$

To see this note that for any $x \in \mathfrak{M}$ and any $y' \in \mathfrak{M}'$ ($\mathfrak{M}'$ stands for the commutator of $\mathfrak{M}$) one has:

$$T(\sigma_t(x))y'\Omega = y\prime T(\sigma_t(x))\Omega = y'\widehat{T}\sigma_t(x)\Omega = y'\widehat{T}\Delta^{it}x\Omega = y'\sigma_t(T(x))\Omega = \sigma_t(T(x))y'\Omega$$

which proves the claim.

Before proceeding further let us pause to make some important remarks on the DBC.

Remark 5.2.
1. There are various versions of DBC. For example, one can use the more general form of DBC which was given in [24]. However, the form given here has a more "transparent" physical interpretation. In particular, to the best of our knowledge, only the form of DBC given in Definition 5.1 leads to a one-to-one correspondence between dynamical semigroups on the set of observables and semigroups on the Hilbert space of (state) vectors respectively (see [20]).
2. Frequently, DBC is related to KMS symmetry. However it is important to note that only DBC forces the map T to commute with the autormorphism group, and this property is essential in our analysis.
3. **But**, in general, tensor product structure is not respected by DBC. Namely, if a (positive) map $T : \mathfrak{M} \to \mathfrak{M}$ satisfies DBC then $T \otimes id : \mathfrak{M} \otimes \mathfrak{N} \to \mathfrak{M} \otimes \mathfrak{N}$, where $\mathfrak{N}$ is a *-algebra, does not need to be a positive map. Therefore, one can not expect that an extension of a positive map T on the tensor product structure will satisfy DBC. Consequently, to get well defined dynamical maps on the crossed products, a further selection of positive maps should be done. To this end, *complete positivity will be assumed additionally.*
4. For a recent account on DBC we refer the reader to [6].

5.2. **Extension of dynamical maps to crossed products and their corresponding algebra of τ-measurable operators.** In our study of quantum maps, we need to canonically extend a dynamical map T defined on a von Neumann algebra $\mathfrak{M}$ to a corresponding map which is defined on a certain noncommutative Orlicz space. As a first step we have to extend T to the corresponding crossed product. Here, we will follow the definition given in [12].

Definition 5.3. Let $T : \mathfrak{M} \to \mathfrak{M}$ be a positive, unital map satisfying DBC with respect to a faithful, normal state $\omega(\cdot) = (\Omega, \cdot\, \Omega)$. Define

$$\widetilde{T}(\lambda(t)\pi(x)) = \lambda(t)\pi(T(x)), \tag{5.5}$$

for $t \in \mathbb{R}$, and $x \in \mathfrak{M}$.

To formulate and then to prove results concerning $\widetilde{T}$, we need some preliminaries. Firstly note that if $x \in \mathfrak{M}$ then we can define an operator $\widetilde{x}$ on $L^2(\mathbb{R}, \mathcal{H})$ by $(\widetilde{x}\xi)(s) = x\xi(s)$ for $\xi \in C_c(\mathbb{R}, \mathcal{H})$ (see [2]). The important point to make here is that the form of $\xi = \xi_0 \otimes f$ with $\xi_0 \in \mathcal{H}$ and $f \in C_c(\mathbb{R})$ leads to

$$x\xi(s) = xf(s)\xi_0 = f(s)x\xi_0 = (x\xi_0 \otimes f)(s). \tag{5.6}$$

Consequently, $\widetilde{x} = x \otimes \mathbb{I}$ and $x \otimes \mathbb{I}$ maps $C_c(\mathbb{R}, \mathcal{H})$ into $C_c(\mathbb{R}, \mathcal{H})$. Furthermore, one can write

$$((x \otimes \mathbb{I})\xi)(s) = x\xi(s). \tag{5.7}$$

Turning to measurable operators, we recall that $\mathcal{M} \equiv \mathfrak{M} \rtimes_\sigma \mathbb{R}$ is a semifinite von Neumann algebra equipped with a canonical normal faithful semifinite trace τ (cf Section 3). It is worth pointing out that invariance of τ with respect to the extension $\widetilde{T}$ of a natural physical map, indicates that the action of $\widetilde{T}$ can be lifted to the τ-measurable operators $\widetilde{\mathcal{M}}$. We are now in a position to formulate and to prove the main result of this section.

Theorem 5.4. *Let $T : \mathfrak{M} \to \mathfrak{M}$ be a completely positive, unital map satisfying DBC with respect to a faithful, normal state $\omega(\cdot) = (\Omega, \cdot\, \Omega)$. Then*

1. *$\widetilde{T}$ is a bounded linear map on $\mathfrak{M} \rtimes_\sigma \mathbb{R}$.*
2. *$\tau \circ \widetilde{T} = \tau$.*

Proof. We remind the reader that any completely positive map T is also completely bounded and $\|T(\mathbb{I})\| = \|T\| = \|T\|_{cb}$, where $\|\cdot\|_{cb}$ stands for the cb-norm, see Proposition 3.6 in [26]. Moreover, all finite linear combinations of $\lambda(s)\pi(x)$, $s \in \mathbb{R}$ $x \in \mathfrak{M}$ form a *-dense involutive subalgebra of $\mathfrak{M} \rtimes_\sigma \mathbb{R}$. Thus, the statement that $\widetilde{T}$ is the well defined map on $\mathfrak{M} \rtimes_\sigma \mathbb{R}$ follows from Theorem 4.1 in [12] and this proves the first claim.

To prove the second claim, we need some facts from the book of van Daele [2].

(1) $L^2(\mathbb{R}, \mathcal{H})$ can be canonically identified with $\mathcal{H} \otimes L^2(\mathbb{R})$ by

$$(U(\xi_0 \otimes f))(s) = f(s)\xi_0, \tag{5.8}$$

for any $\xi_0 \in \mathcal{H}$ and $f \in C_c(\mathbb{R})$ ($C_c(\mathbb{R})$ - the space of continuous complex valued functions on $\mathbb{R}$ with compact supports), see Proposition 2.2 in [2].

(2) Let λ_t denote the left translation by $-t$ in $L^2(\mathbb{R})$. Then , see Proposition 2.8 in [2],

$$\lambda(t) = \mathbb{I} \otimes \lambda_t. \tag{5.9}$$

(3) $\mathfrak{M} \rtimes_\sigma \mathbb{R}$ is spatially isomorphic to the von Neumann algebra on $\mathcal{H} \otimes L^2(\mathbb{R})$ generated by the operators

$$\{x \otimes \mathbb{I}, \Delta^{is} \otimes \lambda_s; x \in \mathfrak{M}, s \in \mathbb{R}\},$$

see Proposition 2.12 in [2].

The canonical faithful semi-finite trace on $(\mathfrak{M} \rtimes_\sigma \mathbb{R})^+$, can in this case then be defined by means of the formula

$$\tau = \sup_K \tau_K, \tag{5.10}$$

where $\tau_K(a) = (\xi_K, a\xi_K)$, $a \in \mathfrak{M} \rtimes_\sigma \mathbb{R}$. Here $\xi_K = \Omega \otimes \mathcal{F}^* f_K$, where $f_K(s) = \chi_K(s) \exp \frac{s}{2}$, and $\mathcal{F}$ stands for the Fourier transform on $L^2(\mathbb{R})$. K denotes a compact subset in $\mathbb{R}$.

Furthermore, see Lemma 3.3 in [2], if $f \in C_c(\mathbb{R})$ and it has support in the compact set K, then

$$\tau_K(\pi(x)\lambda(s)\lambda(f)) = 2\pi\widehat{f}(i+s)\omega(x), \tag{5.11}$$

where $\lambda(f) = \mathbb{I} \otimes \lambda_f$, $\lambda_f = \mathcal{F}^* m_f \mathcal{F}$, and m_f is the multiplication operator by f in $L^2(\mathbb{R})$.

So for $f = \chi_K$ (χ_K stands for the indicator function of a compact subset $K \subset \mathbb{R}$) we have by (5.11) that

$$\begin{aligned} \tau_K(\lambda(s)\pi(x)\lambda(\chi_K)) &= \tau_K(\pi(\sigma_s(x))\lambda(s)\lambda(\chi_K)) \\ &= \omega(\sigma_s(x)) \int e^{-its} e^t \chi_K(t) dt \\ &= \omega(x) \int_K e^{-its} e^t \chi_K(t) dt \\ &= \tau_K(\lambda(s)\pi(x)), \end{aligned} \tag{5.12}$$

where we have used the invariance of ω with respect to the modular automorphism and the formula (5.11). The last equality follows from

$$\lambda(\chi_K)\xi_K = \mathbb{I} \otimes \mathcal{F}^* m_{\chi_K} \mathcal{F} \cdot \ \Omega \otimes \mathcal{F}^* f_K = \xi_K.$$

Secondly

$$\tau_K \circ \widetilde{T}(\lambda(s)\pi(x)) = \tau_K(\lambda(s)\pi(T(x))) = \tau_K(\lambda(s)\pi(x)), \tag{5.13}$$

where we have used Definition 5.3, (5.12), and the invariance of ω with respect to T. Thirdly, we note that

$$(\lambda(s)\pi(x))^* \lambda(s)\pi(x) = \pi(x^* x), \tag{5.14}$$

and hence as $\lambda(t)\pi(x)\lambda(t)^* = \pi(\sigma_t(x))$:

$$\begin{aligned} \Big(\sum_i \lambda(s_i)\pi(x_i)\Big)^* \Big(\sum_j \lambda(s_j)\pi(x_j)\Big) &= \sum_{i,j} \pi(x_i)^* \lambda(s_j - s_i)\pi(x_j) \\ &= \sum_{ij} \lambda(s_j - s_i)\pi(\sigma_{s_i - s_j}(x_i)^* x_j), \end{aligned} \tag{5.15}$$

for any $s, t \in \mathbb{R}$ and $x, y \in \mathfrak{M}$. Define $\widetilde{T}$ as in Remark 4.4, i.e

$$\widetilde{T}(\widetilde{x}) = \widetilde{T}(\sum_i \lambda(s_i)\pi(x_i)) = \sum_i \lambda(s_i)\pi(T(x_i)), \tag{5.16}$$

where $\widetilde{x} = \sum_i \lambda(s_i)\pi(x_i) \in \mathfrak{M} \rtimes_\sigma \mathbb{R}$. As

$$\begin{aligned} \widetilde{x}^* \widetilde{x} &= (\sum_i \lambda(s_i)\pi(x_i))^* (\sum_j \lambda(s_j)\pi(x_j)) \\ &= \sum_{ij} \lambda(s_j - s_i)\pi(\sigma_{s_i - s_j}(x_i)^* x_j), \end{aligned} \tag{5.17}$$

one has

$$\begin{aligned} \tau_K \circ \widetilde{T}(\widetilde{x}^* \widetilde{x}) &= \tau_K(\sum_{i,j} \lambda(s_i - s_j)\pi \circ T(\sigma_{s_i - s_j}(x_i)^* x_j)) \\ &= \sum_{i,j} \tau_K(\lambda(s_i - s_j)\pi(\sigma_{s_i - s_j}(x_i)^* x_j) = \tau_K(\widetilde{x}^* \widetilde{x}), \end{aligned} \tag{5.18}$$

where we have used (5.13). But since τ_K increases to the trace τ over $\mathfrak{M} \rtimes_\sigma \mathbb{R}$ as K increases, the trace τ is also invariant with respect to $\widetilde{T}$. □

6. Conclusions and final remarks

By focusing attention on observables, algebras and states, we proposed a new formalism for Statistical Mechanics, both classical and quantum, in [21] and [22]. It is based on two distinguished Orlicz spaces $L^{\cosh -1}$ and $L\log(L+1)$, and proves to be a canonical extension of the traditional formalism for elementary quantum mechanics; for details see [22].

However in general, physical systems are dynamical, i.e. they evolve in time. So a state (respectively an observable) can exhibit changes brought about by the passage of time. *With this aim in mind, we have in this paper defined and examined quantum maps which are able to describe dynamical processes within this same formalism.*

(1) The following axiomatic framework for the study of (quantum) statistical analysis now emerges from the foregoing analysis,
   - A von Neumann algebra $\mathfrak{M}$ and an associated faithful normal semifinite weight ν corresponding to the given quantum system.
   - The pair of spaces $\langle L\log(L+1)(\mathfrak{M}), L^{\cosh -1}(\mathfrak{M})\rangle$ as respective homes for good states and observables. [21]

   In particular, it was shown that a large class of physical maps satisfy the basic requirements necessary for the extension to and examination of maps on quantum Orlicz spaces.

(2) In closing we remind the reader that Dirichlet forms are a natural source of Markov semigroups. The extension of this theory to von Neumann algebras therefore provides an alternative source of quantum maps on noncommutative Orlicz spaces. However this topic and the role of such quantum maps in Statistical Physics exceed the scope of this paper. For a recent account of this topic, we refer the reader to [18].

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