E. Haapasalo: Compatibility of covariant quantum channels with emphasis on Weyl symmetry

Referee report

This paper studies conditions for compatibility of quantum channels which are covariant with respect to projective unitary representations of a locally compact amenable group, in particular of Weyl covariant channels. First, it is proved that for compatible covariant channels there is always a covariant joint channel. The special structure of covariant channels is then used to find the conditions. In another approach, some previous results coauthored by the author of the present paper are applied, using covariant Stinespring representations.

The question of compatibility of channels is of importance in quantum information theory, since it is one of the main features of quantum theory and is related to other quantum resources such as steering. Characterization of incompatibility for general channels is a difficult task, but is greatly simplified in the Weyl covariant case due to the special structure of such channels. Restriction to channels exhibiting this type of symmetry is reasonable and relevant. Thus the results of the paper are important and interesting and deserve publication, but, in my opinion, the paper is not very well written and needs more attention by the author before it can be published.

General remarks

1. The paper is not easy to read. The first 3 sections are quite fine, but from Sec. 4 onward, a reader not exactly familiar with this setting could easily get lost in technicalities. It is better to provide some lead, explain what is happening and why, before diving into computations. (see the more specific comments and suggestions provided below).

2. The context of Weyl covariance should be better introduced at the beginning of Sec. 4. Especially, the author should specify exactly what is new in the present setting and what is (an adaptation of) standard methods, with precise references. Later on, results of [4], [14] or [11] are used, which seem to be proved in a more specific settings, this should be explained.

3. The paper should be better organized, for example, a definition of a positive kernel before Lemma 4.3 would save a lot of space. There are also unnecessary repetitions and sources of confusion such as ambiguous notations that should be avoided.

4. A small issue: there are a few strange words and phrases here and there that need correction.

More specific comments and suggestions

1. page 3, Def. 2.1: better warn the reader about the input and output spaces of channels in $\mathbf{Ch}(\mathcal{H}, \mathcal{K})$, it could be confusing;

2. page 8, Prop. 3.6: this statement seems to be related to Prop. 4 in [11], there should be a comment on this;

3. It would be good to describe the cases $X = \mathbb{R}^N$ and $X = Z_d$ as examples at the beginning of Sec. 4 (before Prop. 4.1).

4. Lemma 4.3 and its proof: check the notations: T is used for a group homomorphism and a density operator, S is a symplectic form and a density operator (check also at other places). It is clear which is which from the context, but not so good for readability.

5. Many of the conditions (e.g. (4.3), (4.4), (4.5), (4.7)...) could be states as that something is a positive kernel $G \times G \to \mathbb{C}$, but positive kernels are defined much later. Better move this definition and the related definition of a function of positive type (and perhaps also the Fourier transform and Bochner theorem) before Lemma 4.3 and then state these conditions using this notion.

6. page 12: better state the form of channels in $\mathbf{Ch}_W^W$ as a theorem or proposition;

7. page 13, line 13: "This fact is of great importance, as we will see later on." specify where and why it is important;

8. page 14: Remark 4.8. begins by a repetition of a paragraph on p. 12, this is not necessary;

9. pages 15-18: the paper would gain a lot by some reorganization of the material here, so the reader can follow what is going on: e.g. start with the covariant Stinespring representation as in the proof of Thm 4.11, then find the characterization of $\mathbf{Ch}_W^W$ in Lemma 4.10, finih by Thm. 4.11.

10. a suggestion: Thm. 4.11 can be also formulated as follows: if we put $\tilde{\beta}(g,h) = \beta(g,h)\sqrt{\rho_\mu(g)\rho_\mu(h)}$, then $\tilde{\beta}$ is a positive kernel with $\tilde{\beta}^\Delta(g) = \rho_\mu(g)$ and

$$f_1(g) = \int e^{-iS(g,h)}\tilde{\beta}(h,h)dh, \qquad f_2(g) = \int \tilde{\beta}(h,gh)dh$$

this seems more appealing to me

11. page 20: characterization of covariant Gaussian channels is immediate from Theorem 4.6, skip the derivation above Thm. 5.1;

12. page 20: explain the implications of Remark 5.3 (after "However...");

13. page 21: why are the Gaussian channels defined again?

14. page 23: skip the (redundant) definition of a positive kernel before Corollary 5.3;

15. pages 23-24: the depolarizing channel is specified three times here.

Some typos

1. page 2, line 6: "...since describe changes..." maybe a pronoun is missing;

2. page 6, last line before Prop. 3.4: "...varied situations..." various;

3. page 7: "f_{α^V}" - $f_{\alpha_g^V}$ a subscript is missing (twice);
4. page 7, last line of the proof of Prop. 3.4: remove text in parentheses (not needed and confusing);