# On the properties $\alpha - z$ Rényi divergences on general von Neumann algebras

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## 1 Introduction

## 2 Preliminaries

### 2.1 Basic definitions

Let $\mathcal{M}$ be a von Neumann algebra and let $\mathcal{M}^+$ be the cone of positive elements in $\mathcal{M}$. We denote the predual by $\mathcal{M}_*$, its positive part by $\mathcal{M}_*^+$ and the set of normal states by $\mathfrak{S}_*(\mathcal{M})$. For $\psi \in \mathcal{M}_*^+$, we will denote by $s(\psi)$ the support projection of $\psi$.

For $0 < p \leq \infty$, let $L_p(\mathcal{M})$ be the Haagerup $L_p$-space over $\mathcal{M}$ and let $L_p(\mathcal{M})$ its positive cone, [? ]. We will use the identifications $\mathcal{M} \simeq L_\infty(\mathcal{M})$, $\mathcal{M}_* \ni \psi \leftrightarrow h_\psi \in L_1(\mathcal{M})$ and the notation $\mathrm{Tr}\, h_\psi = \psi(1)$ for the trace in $L_1(\mathcal{M})$. It this way, $\mathcal{M}_*^+$ is identified with the positive cone $L_1(\mathcal{M})^+$ and $\mathfrak{S}_*(\mathcal{M})$ with subset of elements in $L_1(\mathcal{M})^+$ with unit trace. Precise definitions and further details on the spaces $L_p(\mathcal{M})$ can be found in the notes [? ].

### 2.2 The $\alpha - z$-Rényi divergences

In [2? ], the $\alpha - z$-Rényi divergence for $\psi, \varphi \in \mathcal{M}_*^+$ was defined as follows:

**Definition 1.** Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$ and let $\alpha, z > 0$, $\alpha \neq 1$. The $\alpha - z$-Rényi divergence is defined as

$$D_{\alpha,z}(\psi\|\varphi) := \frac{1}{\alpha - 1} \log \frac{Q_{\alpha,z}(\psi\|\varphi)}{\psi(1)},$$

where

$$Q_{\alpha,z} = \begin{cases} \mathrm{Tr}\, \left(h_\varphi^{(1-\alpha)/2z} h_\psi^{\alpha/z} h_\varphi^{(1-\alpha)/2z}\right)^z, & \text{if } 0 < \alpha < 1 \\ \|x\|_z^z, & \text{if } \alpha > 1 \text{ and} \\ & h_\psi^{\alpha/z} = h_\varphi^{(\alpha-1)/2z} x h_\varphi^{(\alpha-1)/2z}, \text{ with } x \in s(\varphi) L_z(\mathcal{M}) s(\varphi) \\ \infty & \text{otherwise.} \end{cases}$$

In the case $\alpha > 1$, the following alternative form will be useful.

**Lemma 1.** *[2] Let $\alpha > 1$ and $\psi, \varphi \in \mathcal{M}_*^+$. Then $Q_{\alpha,z}(\psi\|\varphi) < \infty$ if and only if there is some $y \in L_{2z}(\mathcal{M})s(\varphi)$ such that*

$$h_\psi^{\alpha/2z} = y h_\varphi^{(\alpha-1)/2z}.$$

*Moreover, in this case, such $y$ is unique and we have $Q_{\alpha,z}(\psi\|\varphi) = \|y\|_{2z}^{2z}$.*

The standard Rényi divergence [? ? ? ] is contained in this range as $D_\alpha(\psi\|\varphi) = D_{\alpha,1}(\psi\|\varphi)$. The sandwiched Rényi divergence is obtained as $\tilde{D}_\alpha(\psi\|\varphi) = D_{\alpha,\alpha}(\psi\|\varphi)$, see [1? ? ? ] for some alternative definitions and properties of $\tilde{D}_\alpha$. The definition in [? ] and [1] is based on the Kosaki interpolation spaces $L_p(\mathcal{M}, \varphi)$ with respect to a state [? ]. These spaces and complex interpolation method will be used frequently also in the present work.

Many of the properties of $D_{\alpha,z}(\psi\|\varphi)$ were extended from the finite dimensional case in [2]. In particular, the following variational expressions will be an important tool for our work.

**Theorem 1.** *Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$.*

*(i) Let $0 < \alpha < 1$ and $\max\{\alpha, 1-\alpha\} \leq z$. Then*

$$Q_{\alpha,z}(\psi\|\varphi) = \inf_{a\in\mathcal{M}^{++}} \left\{\alpha \mathrm{Tr}\left((a^{1/2}h_\psi^{\alpha/z}a^{1/2})^{z/\alpha}\right) + (1-\alpha)\mathrm{Tr}\left((a^{-1/2}h_\varphi^{(1-\alpha)/2z}a^{-1/2})^{z/(1-\alpha)}\right)\right\}.$$

*Moreover, if $\lambda^{-1}\varphi \leq \psi \leq \lambda\varphi$ for some $\lambda > 0$, then the infimum is attained at some $a_0 \in \mathcal{M}^{++}$ satisfying*

$$h_\varphi^{(1-\alpha)/2z} a_0^{-1} h_\varphi^{(1-\alpha)/2z} = \left(h_\varphi^{(1-\alpha)/2z} h_\psi^{\alpha/z} h_\varphi^{(1-\alpha)/2z}\right)^{1-\alpha}$$

*and*

$$\mathrm{Tr}\left((h_\psi^{\alpha/2z} a_0 h_\psi^{\alpha/2z})^{z/\alpha}\right) = \mathrm{Tr}\left((h_\varphi^{(1-\alpha)/2z} h_\psi^{\alpha/z} h_\varphi^{(1-\alpha)/2z})^z\right).$$

*(ii) Let $1 < \alpha \leq 2z$, then*

$$Q_{\alpha,z}(\psi\|\varphi) = \sup_{a\in\mathcal{M}_+} \left\{\alpha \mathrm{Tr}\left((a^{1/2}h_\psi^{\alpha/z}a^{1/2})^{z/\alpha}\right) - (\alpha-1)\mathrm{Tr}\left((a^{1/2}h_\varphi^{(\alpha-1)/2z}a^{1/2})^{z/(\alpha-1)}\right)\right\}.$$

*Proof.* For part (i) see [2, Theorem 1 (vi)] and its proof. The inequality $\geq$ in part (ii) holds for all $\alpha$ and $z$ and was proved in [2, Theorem 2 (vi)]. We now prove the opposite inequality.

Assume first that $Q_{\alpha,z}(\psi\|\varphi) < \infty$, so that there is some $x \in s(\varphi)L_z(\mathcal{M})^+s(\varphi)$ such that $h_\psi^{\alpha/z} = h_\varphi^{(\alpha-1)/2z} x h_\varphi^{(\alpha-1)/2z}$. Plugging this into the right hand side, we obtain

$$\begin{aligned}
&\sup_{a\in\mathcal{M}_+} \left\{\alpha \mathrm{Tr}\left((a^{1/2}h_\psi^{\alpha/z}a^{1/2})^{z/\alpha}\right) - (\alpha-1)\mathrm{Tr}\left((a^{1/2}h_\varphi^{(\alpha-1)/2z}a^{1/2})^{z/(\alpha-1)}\right)\right\}\\
&= \sup_{a\in\mathcal{M}_+} \left\{\alpha \mathrm{Tr}\left((a^{1/2}h_\varphi^{(\alpha-1)/2z} x h_\varphi^{(\alpha-1)/2z} a^{1/2})^{z/\alpha}\right) - (\alpha-1)\mathrm{Tr}\left((a^{1/2}h_\varphi^{(\alpha-1)/2z}a^{1/2})^{z/(\alpha-1)}\right)\right\}\\
&= \sup_{a\in\mathcal{M}_+} \left\{\alpha \mathrm{Tr}\left((x^{1/2}h_\varphi^{(\alpha-1)/2z} a h_\varphi^{(\alpha-1)/2z} x^{1/2})^{z/\alpha}\right) - (\alpha-1)\mathrm{Tr}\left((h_\varphi^{(\alpha-1)/2z} a h_\varphi^{(\alpha-1)/2z})^{z/(\alpha-1)}\right)\right\}\\
&= \sup_{w\in L_{z/(\alpha-1)}(\mathcal{M})^+} \left\{\alpha \mathrm{Tr}\left((x^{1/2}wx^{1/2})^{z/\alpha}\right) - (\alpha-1)\mathrm{Tr}\left(w^{z/(\alpha-1)}\right)\right\},
\end{aligned}$$

where we used the fact that $\mathrm{Tr}\,((a^*a)^p) = \mathrm{Tr}\,((aa^*)^p)$ for $p > 0$ and $a \in L_{p/2}(\mathcal{M})$ and the fact that the set of elements of the form $h_\varphi^{(\alpha-1)/2z} a h_\varphi^{(\alpha-1)/2z}$ with $a \in \mathcal{M}^+$ is dense in the positive cone $L_{z/(\alpha-1)}(\mathcal{M})^+$. Putting $w = x^{\alpha-1}$ we get

$$\sup_{w\in L_{z/(\alpha-1)}(\mathcal{M})^+} \left\{\alpha \mathrm{Tr}\left((x^{1/2}wx^{1/2})^{z/\alpha}\right) - (\alpha-1)\mathrm{Tr}\left(w^{z/(\alpha-1)}\right)\right\} \geq \mathrm{Tr}\,(x^z) = \|x\|_z^z = Q_{\alpha,z}(\psi\|\varphi).$$

This finishes the proof in the case that $Q_{\alpha,z}(\psi\|\varphi)<\infty$. Note that this holds if $\psi\le\lambda\varphi$ for some $\lambda>0$. Indeed, since $\alpha/2z\in(0,1]$ by the assumption, we then have

$$h_\psi^{\alpha/2z}\le\lambda^{\alpha/2z}h_\varphi^{\alpha/2z},$$

hence by [? , Lemma A.58] there is some $b\in\mathcal{M}$ such that

$$h_\psi^{\alpha/2z}=bh_\varphi^{\alpha/2z}=yh_\varphi^{(\alpha-1)/2z},$$

where $y=bh_\varphi^{1/2z}\in L_{2z}(\mathcal{M})$. By Lemma 1 we get $Q_{\alpha,z}(\psi\|\varphi)=\|y\|_{2z}^{2z}<\infty$.

In the general case, note that lower semicontinuity [2], we have

$$Q_{\alpha,z}(\psi\|\varphi)\le\liminf_{\epsilon\searrow 0}Q_{\alpha,z}(\psi\|\varphi+\epsilon\psi)$$

and by the previous paragraph, the variational expression holds for $Q_{\alpha,z}(\psi\|\varphi)$ for all $\epsilon>0$. The proof is finished by using norm continuity of the map $L_1(\mathcal{M})^+\ni h\mapsto h^{1/p}\in L_p(\mathcal{M})^+$ for $p>1$.

I am not entirely sure about this proof, since the convergence of the expressions

$$\mathrm{Tr}\,\left((a^{1/2}(h_\varphi+\epsilon h_\psi)^{(\alpha-1)/2z}a^{1/2})^{z/(\alpha-1)}\right)\to\mathrm{Tr}\,\left((a^{1/2}h_\varphi^{(\alpha-1)/2z}a^{1/2})^{z/(\alpha-1)}\right)$$

also depends on $\|a\|$, which is not bounded. But probably I misinterpreted something, or I am just being stupid.

□

# 3 Data processing inequality and reversibility of quantum channels

Let $\gamma:\mathcal{N}\to\mathcal{M}$ be a normal positive unital map. Then the predual of $\gamma$ defines a positive linear map $\gamma_*:L_1(\mathcal{M})\to L_1(\mathcal{N})$ that preserves the trace. The support of $\gamma$ will be denoted by $s(\gamma)$, recall that this is the largest projection $p\in\mathcal{N}$ such that $\gamma(p)=1$. For any $\rho\in\mathcal{M}_*^+$ we clearly have $s(\rho\circ\gamma)\le s(\gamma)$, with equality if $\rho$ is faithful. It follows that $\gamma_*$ maps $L_1(\mathcal{M})$ to $s(\gamma)L_1(\mathcal{N})s(\gamma)\equiv L_1(s(\gamma)\mathcal{N}s(\gamma))$. For any $\rho\in\mathcal{M}_+^*$, $\rho\neq 0$, the map

$$s(\gamma)\mathcal{N}s(\gamma)\to s(\rho)\mathcal{M}s(\rho),\qquad a\mapsto s(\rho)\gamma(a)s(\rho)$$

is a faithful normal positive unital map, so using such restrictions we may always assume that both $\rho$ and $\rho\circ\gamma$ are faithful.

The Petz dual of $\gamma$ with respect to a faithful $\rho\in\mathcal{M}_*^+$ is a map $\gamma_\rho^*:\mathcal{M}\to\mathcal{N}$, introduced in [? ? ]. It was proved that it is again normal, positive and unital, in addition, it is $n$-positive whenever $\gamma$ is. As explained in [? ] $\gamma_\rho^*$ is determined by the equality

$$(\gamma_\rho^*)_*(h_{\rho\circ\gamma}^{1/2}bh_{\rho\circ\gamma}^{1/2})=h_\rho^{1/2}\gamma(b)h_\rho^{1/2},\tag{1}$$

for all $b\in\mathcal{N}^+$, here $(\gamma_\rho^*)_*$ is the predual map of $\gamma_\rho^*$. We also have

$$(\gamma_\rho^*)_*(h_{\rho\circ\gamma})=(\gamma_\rho^*)_*\circ\gamma_*(h_\rho)=h_\rho.$$

### 3.1 Data processing inequality

In this paragraph we prove the data processing inequality (DPI) for $D_{\alpha,z}$ with respect to normal positive unital maps. In the case of the sandwiched divergences $\tilde{D}_\alpha$ with $1/2 \leq \alpha \neq 1$, DPI was proved in [1? ], see also [? ] for an alternative proof in the case when the maps are also completely positive.

**Lemma 2.** *Let $\gamma : \mathcal{N} \to \mathcal{M}$ be a normal positive unital map and let $\rho \in \mathcal{M}_*^+$, $b \in \mathcal{N}^+$.*

*(i) If $p \in [1/2, 1)$, then*

$$\|h_{\rho\circ\gamma}^{\frac{1}{2p}} b h_{\rho\circ\gamma}^{\frac{1}{2p}}\|_p \leq \|h_\rho^{\frac{1}{2p}} \gamma(b) h_\rho^{\frac{1}{2p}}\|_p.$$

*(ii) If $p \in [1, \infty]$, the inequality reverses.*

*Proof.* Let us denote $\beta := \gamma_\rho^*$ and let $\omega \in \mathcal{M}_*^+$ be such that $h_\omega := h_{\rho\circ\gamma}^{1/2} b h_{\rho\circ\gamma}^{1/2} \in L_1(\mathcal{N})^+$. Then $\beta$ is a normal positive unital map and we have

$$\beta_*(h_\omega) = h_\rho^{1/2}\gamma(b)h_\rho^{1/2}, \qquad \beta_*(h_{\rho\circ\gamma}) = h_\rho.$$

Let $p \in [1/2, 1)$, then

$$\begin{aligned}\|h_\rho^{\frac{1}{2p}} \gamma(b) h_\rho^{\frac{1}{2p}}\|_p^p &= \|h_\rho^{\frac{1-p}{2p}} \beta_*(h_\omega) h_\rho^{\frac{1-p}{2p}}\|_p^p = Q_{p,p}(\beta_*(h_\omega)\|h_\rho) = Q_{p,p}(\beta_*(h_\omega)\|\beta_*(h_{\rho\circ\gamma}))\\ &\geq Q_{p,p}(h_\omega\|h_{\rho\circ\gamma}) = \|h_{\rho\circ\gamma}^{\frac{1-p}{2p}} h_\omega h_{\rho\circ\gamma}^{\frac{1-p}{2p}}\|_p^p = \|h_{\rho\circ\gamma}^{\frac{1}{2p}} b h_{\rho\circ\gamma}^{\frac{1}{2p}}\|_p^p.\end{aligned}$$

Here we have used the DPI for the sandwiched Rényi divergence $D_{\alpha,\alpha}$ for $\alpha \in [1/2, 1)$, [1, Theorem 4.1]. This proves (i). The case (ii) was proved in [2] (see Eq. (22) therein), using the relation of the sandwiched Rényi divergence to the Kosaki $L_p$ norms as

$$\begin{aligned}\|h_\rho^{1/2p}\gamma(b)h_\rho^{1/2p}\|_p^p &= Q_{p,p}(h_\rho^{1/2}\gamma(b)h_\rho^{1/2}\|h_\rho) = Q_{p,p}(\beta_*(h_\omega)\|\beta_*(h_{\rho\circ\gamma}))\\ &\leq Q_{p,p}(h_\omega\|h_{\rho\circ\gamma}) = \|h_{\rho\circ\gamma}^{\frac{1}{2p}} b h_{\rho\circ\gamma}^{\frac{1}{2p}}\|_p^p,\end{aligned}$$

here the inequality follows from the DPI for sandwiched Rényi divergence $D_{p,p}$ with $p > 1$, [? ]. □

**Theorem 2** (DPI)**.** *Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$ and let $\gamma : \mathcal{N} \to \mathcal{M}$ be a normal positive unital map. Assume either of the following conditions:*

*(i) $0 < \alpha < 1$, $\max\{\alpha, 1-\alpha\} \leq z$*

*(ii) $\alpha > 1$, $\max\{\alpha/2, \alpha - 1\} \leq z \leq \alpha$.*

*Then we have*

$$D_{\alpha,z}(\psi\circ\gamma\|\varphi\circ\gamma) \leq D_{\alpha,z}(\psi\|\varphi).$$

*Proof.* Under the conditions (i), the DPI was proved in [2, Theorem 1 (viii)]. There an additional assumption was used, namely that $\mathcal{N}$ is $\sigma$-finite, to construct a faithful map out of $\gamma$. This can be done using the restriction described above, so the additional condition is not needed.

We next prove that DPI holds under the condition (ii).

□

Let $\psi$ be a faithful normal state on a von Neumann algebra $\mathcal{M}$. We will prove the following inequality:

$$\|h_{\psi\circ\gamma}^{\frac{1}{2p}} b h_{\psi\circ\gamma}^{\frac{1}{2p}}\|_p \leq \|h_\psi^{\frac{1}{2p}} \gamma(b) h_\psi^{\frac{1}{2p}}\|_p \tag{2}$$

for all $p \in [1/2, 1]$, all $b \in \mathcal{N}^+$ and any unital positive map $\gamma : \mathcal{N} \to \mathcal{M}$ (Eq. (19) in [2]). This then implies DPI for the $\alpha - z$-Rényi divergence for $\alpha/2, \alpha - 1 \leq z \leq \alpha$.

Let $\gamma_\psi^*$ be the Petz dual of $\gamma$ with respect to $\psi$, then its predual satisfies

$$(\gamma_\psi^*)_*(h_{\psi\circ\gamma}^{1/2} b h_{\psi\circ\gamma}^{1/2}) = h_\psi^{1/2} \gamma(b) h_\psi^{1/2}$$

(this is eq. (21) in [2]). Put $h_\omega := h_{\psi\circ\gamma}^{1/2} b h_{\psi\circ\gamma}^{1/2} \in L_1(\mathcal{N})^+$. We then have, using Thm. 4.1 in [1]

$$\begin{aligned}\|h_\psi^{\frac{1}{2p}} \gamma(b) h_\psi^{\frac{1}{2p}}\|_p^p &= \|h_\psi^{\frac{1-p}{2p}} (\gamma_\psi^*)_*(h_\omega) h_\psi^{\frac{1-p}{2p}}\|^p = \tilde{Q}_p((\gamma_\psi^*)_*(h_\omega)\|(\gamma_\psi^*)_*(h_{\psi\circ\gamma})) \\ &\geq \tilde{Q}_p(h_\omega\|h_{\psi\circ\gamma}) = \|h_{\psi\circ\gamma}^{\frac{1-p}{2p}} h_\omega h_{\psi\circ\gamma}^{\frac{1-p}{2p}}\|^p = \|h_{\psi\circ\gamma}^{\frac{1}{2p}} b h_{\psi\circ\gamma}^{\frac{1}{2p}}\|_p^p.\end{aligned}$$

# References


[1] A. Jenčová, Rényi relative entropies and noncommutative $L_p$-spaces II, Ann. Henri Poincaré 22, 3235–3254 (2021)

[2] Shinya Kato, On $\alpha - z$-Rényi divergence in the von Neumann algebra setting, arXiv:2311.01748.