A note on equality in DPI for the BS relative entropy

Anna Jenčová

May 14, 2024

Let $\rho, \sigma \in B(\mathcal{H})^+$. The Belavkin-Staszewski relative entropy is defined as

$$\hat{D}(\rho\|\sigma) := \operatorname{Tr} \rho \log(\rho^{1/2}\sigma^{-1}\rho^{1/2}) = \operatorname{Tr} \sigma f(\sigma^{-1/2}\rho\sigma^{-1/2}),$$

with $f(t) = t \log t$. By [**?** , Cor. 3.31], $\hat{D}$ is nonincreasing under positive trace preserving maps $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$, and by [**?** , Thm. 3.34 (h)], the equality

$$\hat{D}(\Phi(\rho)\|\Phi(\sigma)) = \hat{D}(\rho\|\sigma) \tag{1}$$

holds if and only if $d := \sigma^{-1/2}\rho\sigma^{-1/2}$ satisfies $\Phi_\sigma(d^2) = \Phi_\sigma(d)^2$, where

$$\Phi_\sigma(X) = \Phi(\sigma)^{-1/2}\Phi(\sigma^{1/2}X\sigma^{1/2})\Phi(\sigma)^{-1/2}, \qquad X \in B(\mathcal{H})$$

is the Petz dual of Φ with respect to σ. Note that Φ_σ is positive and unital. If Φ is completely positive, we may use the following fact.

Lemma 1. *Let $\Psi : B(\mathcal{H}) \to B(\mathcal{K})$ be a completely positive unital map with Kraus representation $\Psi(\cdot) = \sum_i K_i^*(\cdot)K_i$. Then the multiplicative domain of Ψ has the form*

$$\mathcal{M}_\Psi = \{K_iK_j^*,\ i,j\}',$$

(here C' denotes the commutant of a subset $C \subseteq B(\mathcal{H})$).

This implies the following result. Assume that $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ has the form $\Phi(\cdot) = \sum_{i=1}^n L_i^*(\cdot)L_i$, for some $L_i : \mathcal{K} \to \mathcal{H}$ such that $\sum_i L_iL_i^* = I_\mathcal{H}$. Then the equality (1) holds if and only if d commutes with all elements of the form

$$\sigma^{1/2}L_i\Phi(\sigma)^{-1}L_j^*\sigma^{1/2}, \qquad i,j = 1,\dots,n.$$

Let us apply this in the special case when $\rho = \rho_{ABC} \in B(\mathcal{H}_{ABC})^+$, $\sigma = \rho_{AB} \otimes \tau_C$ and $\Phi = \operatorname{Tr}_A$, here $\tau_C = \dim(\mathcal{H}_C)^{-1}I_C$ is the maximally mixed state. The condition then becomes that d must commute with all elements of the form

$$\rho_{AB}^{1/2}(|i\rangle\langle j|_A \otimes \rho_B^{-1})\rho_{AB}^{1/2} \otimes I_C, \qquad i,j = 1,\dots \dim(\mathcal{H}_A).$$

We may clearly replace d by $\tilde{d} = (\rho_{AB}^{-1/2} \otimes I_C)\rho_{ABC}(\rho_{AB}^{-1/2} \otimes I_C)$. The condition is then

$$\tilde{d} \in \mathcal{R} \otimes B(\mathcal{H}_C),$$

where $\mathcal{R} = \Gamma(B(\mathcal{H}_A))'$, with $\Gamma : B(\mathcal{H}_A) \to B(\mathcal{H}_{AB})$ is a cpmpletely positive map given as

$$\Gamma(X_A) = V^*(X_A \otimes I_B)V, \qquad X_A \in B(\mathcal{H}_A),$$

here $V = (I_A \otimes \rho_B^{-1/2})\rho_{AB}^{1/2}$. Assume that ρ_{AB} is invertible. By Arveson's commutant lifting theorem [**?** , Thm. 1.3.1], for every $T \in \mathcal{R}$ there is a unique $T_1 \in B(\mathcal{H}_B)$ such that $(I_A \otimes T_1)V = VT$ and the map $T \mapsto T_1$ is a *-isomorphism of $\mathcal{R}$ onto $(I_A \otimes B(\mathcal{H}_B)) \cap \{VV^*\}'$.

Note that $VV^* = (I_A \otimes \rho_B^{-1/2})\rho_{AB}(I_A \otimes \rho_B^{-1/2})$, so that the map $T \mapsto T_1$ is given as

$$\mathrm{Tr}_A[VTV^*] = \mathrm{Tr}_A[(I_A \otimes T_1)VV^*] = T_1 \mathrm{Tr}_A[VV^*] = T_1.$$

The inverse map $T_1 \mapsto T$ is obtained from the polar decomposition of V: $V = HW$, where $H = (VV^*)^{1/2}$ and W is a unitary. Then T_1 commutes with H and we have

$$VW^*(I_A \otimes T_1)W = H(I_A \otimes T_1)W = (I_A \otimes T_1)HW = (I_A \otimes T_1)V,$$

so that $T = W^*(I_A \otimes T_1)W$.

This leads to the following construction of states ρ_{ABC} such that

$$\hat{D}(\rho_{ABC} \| \rho_{AB} \otimes \tau_C) = \hat{D}(\rho_{BC} \| \rho_B \otimes \tau_C).$$

Let $M \in B(\mathcal{H}_{AB})^{++}$ be any element such that $\mathrm{Tr}_A[M] = I_B$. Let $\mathcal{T} \subseteq B(\mathcal{H}_B)$ be such that $I_A \otimes \mathcal{T} = \{M\}' \cap I_A \otimes B(\mathcal{H}_B)$. Take any state $\rho_B \in B(\mathcal{H}_B)^{++}$ and put

$$\rho_{AB} = (I_A \otimes \rho_B^{1/2})M(I_A \otimes \rho_B^{1/2}),$$

then clearly ρ_{AB} is a state and $\mathrm{Tr}_A \rho_{AB} = \rho_B$. Let W be the unitary such that

$$(I_A \otimes \rho_B)^{1/2} M^{1/2} = \rho_{AB}^{1/2} W^*.$$

Now choose any positive element $\tilde{d} \in W^*(I_A \otimes \mathcal{T})W \otimes B(\mathcal{H}_C)$ such that $\mathrm{Tr}_C \tilde{d} = I_{AB}$ and put

$$\rho_{ABC} = (\rho_{AB}^{1/2} \otimes I_C)\tilde{d}(\rho_{AB}^{1/2} \otimes I_C).$$

Then we have $\mathrm{Tr}_C \rho_{ABC} = \rho_{AB}$, $\mathrm{Tr}_{AC} \rho_{ABC} = \rho_{AB}$.

Since $\mathcal{T}$ is a subalgebra in $B(\mathcal{H}_B)$, there is a unitary U_B and a decomposition

$$\mathcal{T} = U_B \left(\bigoplus_n I_{B_n^L} \otimes B(\mathcal{H}_{B_n^R}) \right) U_B^*,$$

so that

$$(W \otimes I_C)\tilde{d}(W^* \otimes I_C) = (I_A \otimes U_B \otimes I_C) \left(\oplus_n I_{AB_n^L} \otimes d_{B_n^R C}\right) (I_A \otimes U_B^* \otimes I_C)$$

for some positive elements $d_{B_n^R C} \in B(\mathcal{H}_{B_n^R} \otimes \mathcal{H}_C)^+$. Moreover, from $M \subseteq (I \otimes \mathcal{T})'$, we get

$$M = (I_A \otimes U_B) \left(\oplus_n M_{AB_n^L} \otimes I_{B_n^R}\right) (I_A \otimes U_B^*)$$

for $M_{AB_n^L} \in B(\mathcal{H}_A \otimes \mathcal{H}_{B_n^L})^+$. Putting all together, we get

$$\begin{aligned}
\rho_{ABC} &= (\rho_{AB}^{1/2} W^* W \otimes I_C)\tilde{d}(W^* W \rho_{AB}^{1/2} \otimes I_C) \\
&= ((I_A \otimes \rho_B^{1/2})M^{1/2} \otimes I_C)(I_A \otimes U_B \otimes I_C) \left(\oplus_n I_{AB_n^L} \otimes d_{B_n^R C}\right) (I_A \otimes U_B^* \otimes I_C)(M^{1/2}(I_A \otimes \rho_B^{1/2}) \otimes I_C) \\
&= (I_A \otimes \rho_B^{1/2} U_B \otimes I_C) \left(\oplus_n M_{AB_n^L} \otimes d_{B_n^R C}\right) (I_A \otimes U_B^* \rho_B^{1/2} \otimes I_C).
\end{aligned}$$

1 The structure of ρ_{ABC}

Proposition 1. *Let ρ_{ABC} be a state (such that ρ_{AB} is invertible). The equality*

$$\hat{D}(\rho_{ABC}\|\rho_{AB}\otimes\tau_C)=\hat{D}(\rho_{BC}\|\rho_B\otimes\tau_C) \tag{2}$$

holds if and only if there are:

(i) Hilbert spaces $\mathcal{H}_{B_n^L}$, $\mathcal{H}_{B_n^R}$ such that $\mathcal{H}_B\simeq\oplus_n(\mathcal{H}_{B_n^L}\otimes\mathcal{H}_{B_n^R})$,

(ii) positive (invertible) elements $M_n\in B(\mathcal{H}_A\otimes\mathcal{H}_{B_n^L})$ such that $\mathrm{Tr}\,_A M_n=I_{B_n^L}$,

(iii) positive elements $d_n\in B(\mathcal{H}_{B_n^R}\otimes\mathcal{H}_C)$ such that $\mathrm{Tr}\,_C d_n=I_{B_n^R}$,

(iv) an (invertible) operator $S_B:\oplus_n(\mathcal{H}_{B_n^L}\otimes\mathcal{H}_{B_n^R})\to\mathcal{H}_B$ such that $\mathrm{Tr}\,[S_BS_B^*]=1$

such that

$$\rho_{ABC}=(I_A\otimes S_B\otimes I_C)\left(\oplus_n M_n\otimes d_n\right)(I_A\otimes S_B^*\otimes I_C)$$

Proof. Assume that ρ_{ABC} has this form. Then

$$\rho_{AB}=\mathrm{Tr}\,_C\rho_{ABC}=(I_A\otimes S_B)\left(\oplus_n M_n\otimes I_{B_n^R}\right)(I_A\otimes S_B^*),\qquad \rho_B=S_BS_B^*.$$

Let us denote $M:=\oplus_n M_n\otimes I_{B_n^R}$, $d:=\oplus_n I_{B_n^L}\otimes d_n$. Using polar decompositions, there is some unitary $W\in B(\mathcal{H}_{AB})$ such that

$$(I_A\otimes S_B)M^{1/2}W^*=\rho_{AB}^{1/2}=WM^{1/2}(I_A\otimes S_B^*).$$

It follows that

$$(\rho_{AB}^{-1/2}\otimes I_C)\rho_{ABC}(\rho_{AB}^{-1/2}\otimes I_C)=(W\otimes I_C)(I_A\otimes d)(W^*\otimes I_C)$$

and

$$\rho_{AB}=WM^{1/2}(I_A\otimes S_B^*S_B)M^{1/2}W^*.$$

We may clearly replace τ_C by I_C in the equality (2), since this only adds a constant to both sides. We get

$$\begin{aligned}\hat{D}(\rho_{ABC}\|\rho_{AB}\otimes I_C)&=\mathrm{Tr}\,(\rho_{AB}\otimes I_C)f((W\otimes I_C)(I_A\otimes d)(W^*\otimes I_C))\\&=\mathrm{Tr}\,[(M^{1/2}(I_A\otimes S_B^*S_B)M^{1/2}\otimes I_C)f(I_A\otimes d)]\\&=\mathrm{Tr}\,[(M(I_A\otimes S_B^*S_B)\otimes I_C)f(I_A\otimes d)]=\mathrm{Tr}\,[(S_B^*S_B\otimes I_C)f(d)],\end{aligned}$$

here $f(t)=t\log t$ and we have used the fact that $M\otimes I_C$ commutes with $I_A\otimes d$.

We also have

$$\rho_{BC}=(S_B\otimes I_C)d(S_B^*\otimes I_C)$$

and with the polar decomposition $S_B=\rho_B^{1/2}U_B$, we get

$$(\rho_B^{-1/2}\otimes I_C)\rho_{BC}(\rho_B^{-1/2}\otimes I_C)=(U_B\otimes I_C)d(U_B^*\otimes I_C).$$

It follows that

$$\hat{D}(\rho_{BC}\|\rho_B\otimes I_C)=\mathrm{Tr}\,[(\rho_B\otimes I_C)f((U_B\otimes I_C)d(U_B^*\otimes I_C))]=\mathrm{Tr}\,[(S_B^*S_B\otimes I_C)f(d)]=\hat{D}(\rho_{ABC}\|\rho_{AB}\otimes I_C).$$

For the converse, assume that (2) holds. Put $R := (\rho_{AB}^{-1/2}\otimes I_C)\rho_{ABC}(\rho_{AB}^{-1/2}\otimes I_C)$, so that $R \geq 0$ and $\mathrm{Tr}_C[R] = I_{AB}$. Moreover, R must be in the multiplicative domain of the map

$$\Phi_\sigma(X_{ABC}) = (\rho_B^{-1/2}\otimes I_C)\mathrm{Tr}_A[(\rho_{AB}^{1/2}\otimes I_C)X(\rho_{AB}^{1/2}\otimes I_C)](\rho_B^{-1/2}\otimes I_C) = \sum_i L_i^* X L_i,$$

where the Kraus operators have the form

$$L_i = (\rho_{AB}^{1/2}(|i\rangle_A \otimes I_B)\rho_B^{-1/2}) \otimes I_C.$$

By Lemma 1, the operator R must commute with all elements of the form

$$\rho_{AB}^{1/2}(|i\rangle\langle j|_A \otimes \rho_B^{-1})\rho_{AB}^{1/2}\otimes I_C, \qquad i,j = 1,\dots \dim(\mathcal{H}_A).$$

This means that

$$R \in \mathcal{R}\otimes B(\mathcal{H}_C),$$

where $\mathcal{R} = \Gamma(B(\mathcal{H}_A))'$, with $\Gamma : B(\mathcal{H}_A) \to B(\mathcal{H}_{AB})$ is a completely positive map given as

$$\Gamma(X_A) = V^*(X_A\otimes I_B)V, \qquad X_A \in B(\mathcal{H}_A),$$

with $V := (I_A \otimes \rho_B^{-1/2})\rho_{AB}^{1/2}$. Since ρ_{AB} is invertible by the assumption, Arveson's commutant lifting theorem [**?** , Thm. 1.3.1] says that for every $T \in \mathcal{R}$ there is a unique $T_1 \in B(\mathcal{H}_B)$ such that $(I_A\otimes T_1)V = VT$ and the map $T \mapsto T_1$ is a *-isomorphism of $\mathcal{R}$ onto the subalgebra $\mathcal{R}_1 \subseteq B(\mathcal{H}_B)$ given by

$$(I_A \otimes B(\mathcal{H}_B)) \cap \{VV^*\}' = I_A \otimes \mathcal{R}_1.$$

Note that $M := VV^* = (I_A\otimes\rho_B^{-1/2})\rho_{AB}(I_A\otimes\rho_B^{-1/2})$ satisfies $\mathrm{Tr}_A[M] = I_B$, so that this *-isomorphism is defined by

$$\mathrm{Tr}_A[VTV^*] = \mathrm{Tr}_A[(I_A\otimes T_1)VV^*] = T_1\mathrm{Tr}_A[VV^*] = T_1.$$

The inverse map $\mathcal{R}_1 \to \mathcal{R}$ is obtained from the polar decomposition $V = M^{1/2}W$, where W is a unitary. For any $T_1 \in \mathcal{R}_1$, $I_A\otimes T_1$ commutes with $M^{1/2}$ and we have

$$VW^*(I_A\otimes T_1)W = M^{1/2}(I_A\otimes T_1)W = (I_A\otimes T_1)M^{1/2}W = (I_A\otimes T_1)V,$$

so that $T = W^*(I_A\otimes T_1)W$. It follows that $\mathcal{R} = W^*(I_A\otimes\mathcal{R}_1)W$ and hence

$$R \in \mathcal{R}\otimes B(\mathcal{H}_C) = (W^*\otimes I_C)(I_A\otimes\mathcal{R}_1\otimes B(\mathcal{H}_C))(W\otimes I_C).$$

It follows that there is some positive element $N \in \mathcal{R}_1\otimes B(\mathcal{H}_C)$ such that

$$R = (W^*\otimes I_C)(I_A\otimes N)(W\otimes I_C). \tag{3}$$

Moreover, since $\mathrm{Tr}_C[R] = I_{AB}$, we must have $\mathrm{Tr}_C[N] = I_B$. Note also that

$$M\otimes I_C \in (I_A\otimes\mathcal{R}_1)'\otimes I_C = (I_A\otimes\mathcal{R}_1\otimes B(\mathcal{H}_C))',$$

so that $M\otimes I_C$ commutes with $I_A\otimes N$. To finish the proof, we write

$$\rho_{ABC} = (\rho_{AB}\otimes I_C)R(\rho_{AB}\otimes I_C)$$

and

$$\rho_{AB} = (I_A \otimes \rho_B^{1/2})V = (I_A \otimes \rho_B^{1/2})M^{1/2}W.$$

Combining this with (3), we obtain

$$\rho_{ABC} = (I_A \otimes \rho_B^{1/2})(M \otimes I_C)(I_A \otimes N)(I_A \otimes \rho_B^{1/2}).$$

Since $\mathcal{R}_1 \subseteq B(\mathcal{H}_B)$ is a subalgebra, there are Hilbert spaces as in (i) and a unitary $U_B : \mathcal{H}_B \to \oplus_n \mathcal{H}_{B_n^L} \otimes \mathcal{H}_{B_n^R}$ such that

$$\mathcal{R}_1 = U_B^* \left(\bigoplus_n I_{B_n^L} \otimes B(\mathcal{H}_{B_n^R}) \right) U_B,$$

Using this decomposition, we see that there are elements M_n as in (ii) such that $M = (I_A \otimes U_B^*)(\oplus_n M_n \otimes I_{B_n^R})(I_A \otimes U_B)$ and similarly, there are elements N_n as in (iii) such that $N = (U_B^* \otimes I_C)(\oplus_n I_{B_n^L} \otimes N_n)(U_B \otimes I_C)$. Now we see that ρ_{ABC} has the required form, with $S_B = \rho_B^{1/2} U_B^*$.

□