Michele Dall'Arno, Francesco Buscemi: Tight conic approximation of testing regions for quantum statistical models and measurements

Referee report

A testing region is the image of the set of states (or effects) under a measurement (or a set of quantum states, also called a statistical experiment) seen as linear mappings. In the present paper, the testing regions are approximated by ellipsoids, or other sets that the authors call d-cones. These approximations are optimal, at least in the information complete case, in the sense that the outer approximations are the minimal and the inner approximations the maximal possible.

The authors claim that their results are used to provide a semi-device independent test of the possibility to transform one measurement (or statistical experiment) to another by a quantum channel (simulability). But this is not exactly true, e.g. in Corollary 1, in general, only the existence of a positive map on the span of the POVM is proved, which may not even have a positive extension to all of $\mathcal{L}(\mathbb{C}^d)$, let alone being completely positive. The only case when existence of a channel can be shown is under precisely the same conditions as in Corollary 2 of Ref. [21], in higher dimensions or for measurements with more outcomes the condition does not imply simulability. The condition of the corollary also does not seem necessary, so it is not clear how it can serve as a simulability test. There is a similar problem with Corollary 2 dealing with the case of statistical experiments. The improvement that is provided in the present paper is that the conditions given in terms of approximating ellipsoids or d-cones are more tractable than the testing regions, but their applicability to simulability tests remains questionable.

Nevertheless, the approximations obtained in the paper could be of some interest, and I would suggest that the authors highlight some of the cases when the testing regions can be applied, which would go beyond the results of [21]. The paper is also not written very well and the arguments are unclear at some places. I think it would add value to the paper if the authors were more pedagogical, and provide more explanations on their arguments and underlying ideas, not just giving references to other results without further comments.

Some specific comments

- page 3, Definition 1: better explain here that Q^+ is the pseudoinverse, this is not yet clear at this point
- page 3, example of the SIC measurement: $\hat{\mathbf{u}}$ is not defined.
- page 4, Corollary 1: what is the "support of $\boldsymbol{\pi}_0$"?
- page 4, the definition of a d-cone is quite unclear. Also for the symmetric d-cone: it seems that the last ball should have radius 0, which should follow from $r(L) = r(0) = 0$, but then there is another ball at the origin. So actually there are $d + 1$ balls.
- page 6, Corollary 2: what is the "support of $\boldsymbol{\rho}_0$"?
- page 9, proof of Theorem 1: the paragraph starting with "The inner ellipsoid..." is very unclear and should be better explained.
- page 11, proof of Theorem 2: should be better explained.