Variational expression of $Q_{\alpha,z}$ for $0<\alpha<1$

Let $0<\alpha<1$ and $z>0$. Let $\psi, \varphi \in M_*^+$. Kato proved

$$Q_{\alpha,z}(\psi\|\varphi) \le \inf_{a\in M^{++}} \Big[\alpha \,\mathrm{tr}\Big(\big(a^{1/2} h_\psi^{\frac{\alpha}{z}} a^{1/2}\big)^{\frac{z}{\alpha}}\Big) + (1-\alpha)\,\mathrm{tr}\Big(\big(a^{-1/2} h_\varphi^{\frac{1-\alpha}{z}} a^{-1/2}\big)^{\frac{z}{1-\alpha}}\Big)\Big].$$

With $r := \max\left\{\frac{\alpha}{z}, \frac{1-\alpha}{z}\right\}$, for any $\varepsilon>0$ set $\psi_\varepsilon, \varphi_\varepsilon \in M_*^+$ by

$$h_{\psi_\varepsilon} := (h_\psi^r + \varepsilon h_\varphi^r)^{1/r}, \qquad h_{\varphi_\varepsilon} := (h_\varphi^r + \varepsilon h_\psi^r)^{1/r}.$$

Then we have

$$h_{\psi_\varepsilon}^r \ge h_\psi^r, \quad h_{\psi_\varepsilon}^r \ge \varepsilon h_\varphi^r, \quad h_{\varphi_\varepsilon}^r \ge h_\varphi^r, \quad h_{\varphi_\varepsilon}^r \ge \varepsilon h_\psi^r,$$

so that

$$h_{\psi_\varepsilon}^r = h_\psi^r + \varepsilon h_\varphi^r \le \varepsilon^{-1} h_{\varphi_\varepsilon}^r + \varepsilon h_{\varphi_\varepsilon}^r = (\varepsilon^{-1}+\varepsilon) h_{\varphi_\varepsilon}^r,$$

$$h_{\varphi_\varepsilon}^r = h_\varphi^r + \varepsilon h_\psi^r \le \varepsilon^{-1} h_{\psi_\varepsilon}^r + \varepsilon h_{\psi_\varepsilon}^r = (\varepsilon^{-1}+\varepsilon) h_{\psi_\varepsilon}^r.$$

Hence, since $\frac{\alpha}{rz} \le 1$,

$$(\varepsilon^{-1}+\varepsilon)^{-\frac{\alpha}{rz}} h_{\varphi_\varepsilon}^{\frac{\alpha}{z}} \le h_{\psi_\varepsilon}^{\frac{\alpha}{z}} \le (\varepsilon^{-1}+\varepsilon)^{\frac{\alpha}{rz}} h_{\varphi_\varepsilon}^{\frac{\alpha}{z}}.$$

By the second paragraph of the proof of (iv) of [Kato, Theorem 1] we obtain

$$Q_{\alpha,z}(\psi_\varepsilon\|\varphi_\varepsilon) = \inf_{a\in M^{++}} \Big[\alpha\, \mathrm{tr}\Big(\big(a^{1/2} h_{\psi_\varepsilon}^{\frac{\alpha}{z}} a^{1/2}\big)^{\frac{z}{\alpha}} \Big) + (1-\alpha)\, \mathrm{tr}\Big(\big(a^{-1/2} h_{\varphi_\varepsilon}^{\frac{1-\alpha}{z}} a^{-1/2}\big)^{\frac{z}{1-\alpha}} \Big) \Big].$$

Since $h_{\psi_\varepsilon}^{\frac{\alpha}{z}} \ge h_{\psi}^{\frac{\alpha}{z}}$ so that $a^{1/2} h_{\psi_\varepsilon}^{\frac{\alpha}{z}} a^{1/2} \ge a^{1/2} h_{\psi}^{\frac{\alpha}{z}} a^{1/2}$, we have, by Lemma A.2 of our paper,

$$\| a^{1/2} h_{\psi_\varepsilon}^{\frac{\alpha}{z}} a^{1/2} \|_{z/\alpha} \ge \| a^{1/2} h_{\psi}^{\frac{\alpha}{z}} a^{1/2} \|_{z/\alpha},$$

i.e.,

$$\mathrm{tr}\Big(\big(a^{1/2} h_{\psi_\varepsilon}^{\frac{\alpha}{z}} a^{1/2}\big)^{\frac{z}{\alpha}} \Big) \ge \mathrm{tr}\Big(\big(a^{1/2} h_{\psi}^{\frac{\alpha}{z}} a^{1/2}\big)^{\frac{z}{\alpha}} \Big)$$

and similarly,

$$\mathrm{tr}\Big(\big(a^{-1/2} h_{\varphi_\varepsilon}^{\frac{1-\alpha}{z}} a^{-1/2}\big)^{\frac{z}{1-\alpha}} \Big) \ge \mathrm{tr}\Big(\big(a^{-1/2} h_{\varphi}^{\frac{1-\alpha}{z}} a^{-1/2}\big)^{\frac{z}{1-\alpha}} \Big)$$

for any $a \in M^{++}$. Therefore,

$$Q_{\alpha,z}(\psi_\varepsilon\|\varphi_\varepsilon) \ge \inf_{a\in M^{++}} \Big[\alpha\, \mathrm{tr}\Big(\big(a^{1/2} h_{\psi}^{\frac{\alpha}{z}} a^{1/2}\big)^{\frac{z}{\alpha}} \Big) + (1-\alpha) \Big(\big(a^{-1/2} h_{\varphi}^{\frac{1-\alpha}{z}} a^{-1/2}\big)^{\frac{z}{1-\alpha}} \Big) \Big].$$

Letting $\varepsilon \searrow 0$ gives

$$Q_{\alpha,z}(\psi\|\varphi) \ge \inf_{a\in M^{++}} \Big[\qquad \Big].$$