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Markov triplets on CCR-algebras

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Abstract. The paper contains a detailed computation about the algebra of canonical commutation relation, the representation of the Weyl unitaries, the quasi-free states and their von Neumann entropy. The Markov triplet is defined by constant entropy increase. The Markov property of a quasi-free state is described by the representing block matrix. The proof is based on results on the statistical sufficiency in the non-commutative case. The relation to classical Gaussian Markov triplets is also described.

Introduction

The notion of quasi-free state was developed in the framework of the C*-algebraic approach to the canonical commutation relation (CCR) [14], [10], [7], [16]. The CCR-algebra is generated by the Weyl unitaries (satisfying a commutation relation, therefore Weyl algebra is an alternative terminology). The quasi-free states on CCR can be regarded as analogues of Gaussian distributions in classical probability: The n-point functions can be computed from the 2-point functions and in a kind of central limit theorem the limiting state is quasi-free and it maximizes the von Neumann entropy when the 2-point function is fixed [17]. The quasi-free states are quite tractable, for example the von Neumann entropy has an explicit expression [8], [7].

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The Markov property was invented by Accardi in the non-commutative (or quantum probabilistic) setting [1], [2], [3], [4]. This Markov property is based on a completely positive, identity preserving map, so-called quasi-conditional expectation and it was formulated in the tensor product of matrix algebras. A state of a tensor product system is Markovian if and only if the von Neumann entropy increase is constant. This property and a possible definition of the Markov condition was suggested in [18]. A remarkable property of the von Neumann entropy is the strong subadditivity [13], [9], [15], [19] which plays an important role in the investigations of quantum system's correlations. The above mentioned constant increase of the von Neumann entropy is the same as the equality for the strong subadditivity of von Neumann entropy.

A CCR (or Weyl) algebra is parametrized by a Hilbert space, so we use the notation $\mathrm{CCR}(\mathcal{H})$ when $\mathcal{H}$ is the Hilbert space. Assume that φ_{123} is a state on the composite system $\mathrm{CCR}(\mathcal{H}_1) \otimes \mathrm{CCR}(\mathcal{H}_2) \otimes \mathrm{CCR}(\mathcal{H}_3)$. Denote by $\varphi_{12}, \varphi_{23}$ the restrictions to the first two and to the second and third factors, similarly φ_2 is the restriction to the second factor. The Markov property is defined as

$$S(\varphi_{123}) - S(\varphi_{12}) = S(\varphi_{23}) - S(\varphi_2),$$

where S denotes the von Neumann entropy. When φ_{123} is quasi-free, it is given by a positive operator (corresponding to the 2-point function) and the main goal of the present paper is to describe the Markov property in terms of this operator. The paper [20] studies a similar question for the CAR algebra and [5] is about multivariate Gaussian distributions. Although the multivariate Gaussian case (in classical probability) is rather different from the present non-commutative setting, we use the same block matrix formalism (and the paper [5] was actually a preparation of this problem). The proof of the main result uses the description of sufficient statistics in the non-commutative case. A quasi-free state is described by a block matrix and the Markov property is formulated by the entries. A Markovian quasi-free state induces multivariate Gaussian restrictions, but they are very special in that framework.

The paper is organized as follows. The preliminary section contains some crucial properties of the Weyl unitaries, the Fock space, the CCR algebra and quasi-free states. This is written for the sake of completeness, the results are known but not well-accessible in the literature, cf. [6], [7], [10]. The main point is the von Neumann entropy formula which is well-known for the CCR quasi-free state. In the next section we investigate the quasi-free Markov triplets. We obtain a necessary and sufficient condition described in the block matrix approach: The block matrix should be block diagonal. There are nontrivial Markovian quasi-free states which are not a product in the time localization. The existence of such state is interesting, because it is in contrast to the CAR case [20]. However, the first and the third subalgebras are always independent. Finally we prove that commuting field operators form a classical Gaussian Markov triplet.

1. CCR algebras and quasi-free states

1.1. Introduction to Weyl unitaries. In this part the basis of Hermite functions of the Hilbert space $L^2(\mathbb{R})$ is described in details, the creation, annihilation operators and the Weyl unitaries are constructed.

The *Hermite polynomials*

$$H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \qquad (n = 0, 1, \dots) \tag{1}$$

are orthogonal in the Hilbert space $L^2(\mathbb{R}, e^{-x^2}\,dx)$, they satisfy the recursion

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0 \tag{2}$$

and the differential equation

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0. \tag{3}$$

The normalized Hermite polynomials

$$\tilde{H}_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x) \tag{4}$$

form an orthonormal basis. From this basis of $L^2(\mathbb{R}, e^{-x^2}\,dx)$, we can get easily a basis in $L^2(\mathbb{R})$:

$$\varphi_n(x) := e^{-x^2/2} \tilde{H}_n(x)\,. \tag{5}$$

These are called the *Hermite functions.* In terms of the Hermite functions equation (2) becomes

$$x\varphi_n(x) = \frac{\sqrt{n}\varphi_{n-1}(x) + \sqrt{n+1}\varphi_{n+1}(x)}{\sqrt{2}}. \tag{6}$$

If the operators a and a^+ are defined as

$$a\varphi_n = \sqrt{n}\varphi_{n-1}, \qquad a^+\varphi_n = \sqrt{n+1}\varphi_{n+1} \tag{7}$$

with $a\varphi_0 = 0$ and the multiplication by the variable x is denoted by Q, then (6) is

$$Q = \frac{1}{\sqrt{2}}(a + a^+). \tag{8}$$

From the equation

$$\frac{\partial}{\partial x}\Big(H_n(x)e^{-x^2/2}\Big) = H_n'(x)e^{-x^2/2} - xH_n(x)e^{-x^2/2},$$

one can obtain

$$P\varphi_n := \frac{1}{\mathrm{i}}\varphi_n' = \frac{\sqrt{n}\varphi_{n-1} - \sqrt{n+1}\varphi_{n+1}}{\mathrm{i}\sqrt{2}}, \tag{9}$$

that is

$$P = \frac{\mathrm{i}}{\sqrt{2}}(a^+ - a). \tag{10}$$

Therefore,

$$a = \frac{1}{\sqrt{2}}(Q + \mathrm{i}P), \qquad a^+ = \frac{1}{\sqrt{2}}(Q - \mathrm{i}P).$$

Lemma 1.1. *For $z \in \mathbb{C}$ the identity*

$$e(z) := \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\varphi_n(x) = \pi^{-1/4} \exp\Big(-\frac{z^2 + x^2}{2}\Big) \exp(zx\sqrt{2})$$

holds. Moreover,

$$e(z) = e^{za^+}\varphi_0, \qquad \|e(z)\| = e^{|z|^2/2}.$$

Proof. The identity can be deduced from the generator function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}H_n(x) = \exp(2xt - t^2) \tag{11}$$

of the Hermite polynomials. ∎

The above $e(z)$ is called the *exponential vector.*

For $z \in \mathbb{C}$, the operator $\mathrm{i}(za - \bar{z}a^+)$ is defined originally on the linear combinations of the basis vectors φ_n and it is a symmetric operator. It can be proven that its closure is self-adjoint, therefore $\exp(za - \bar{z}a^+)$ becomes a unitary.

$$W(z) := e^{za - \bar{z}a^+} \tag{12}$$

is called the *Weyl unitary.* Note that

$$W(z) = \exp \mathrm{i}\sqrt{2}(\alpha P + \beta Q)$$

if $z = \alpha + \mathrm{i}\beta$. Multiple use of the identity

$$e^{\mathrm{i}(tQ+uP)} = \exp(\mathrm{i}tu/2)e^{\mathrm{i}tQ}e^{\mathrm{i}uP} = \exp(-\mathrm{i}tu/2)e^{\mathrm{i}uP}e^{\mathrm{i}tQ} \tag{13}$$

gives the following result.

Theorem 1.1.
$$W(z)W(z') = W(z+z')\exp(\mathrm{i}\ \mathrm{Im}(\bar{z}z'))$$
for $z, z' \in \mathbb{C}$.

With straightforward computation one gets the following.

Lemma 1.2.
$$e^{za-\bar{z}a^+}\varphi_0 = e^{-|z|^2/2}e^{za^+}\varphi_0 = \frac{e(z)}{\|e(z)\|}.$$

The functions

$$L_n^\alpha(x) = \sum_{k=0}^{n} \frac{(-1)^k (n+\alpha)!}{k!(n-k)!(\alpha+k)!} x^k \qquad (\alpha > -1) \tag{14}$$

are called the *associated Laguerre polynomials.* We write simply $L_n(x)$ for $\alpha = 0$.

Theorem 1.2. *For* $n \geq m$
$$\langle \varphi_m, W(z)\varphi_n \rangle = e^{-|z|^2/2}\sqrt{\frac{m!}{n!}}\, z^{n-m} L_m^{n-m}(|z|^2)$$
holds.

Proof. First note that definition (7) implies

$$a^k (a^+)^n \varphi_0 = \begin{cases} \frac{n!}{(n-k)!}(a^+)^{n-k}\varphi_0, & \text{if } k \leq n, \\ 0, & \text{if } k > n. \end{cases} \tag{15}$$

If $[A,B]$ commutes with A and B, then the formula $e^A e^B = e^{[A,B]/2}e^{A+B}$ holds. Since $[-\bar{z}a^+, za] = |z|^2 I$, we can write

$$\begin{aligned} W(z)\varphi_n &= e^{za-\bar{z}a^+}\varphi_n = e^{-|z|^2/2}e^{-\bar{z}a^+}e^{za}\varphi_n \\ &= \frac{e^{-|z|^2/2}e^{-\bar{z}a^+}}{\sqrt{n!}} \sum_{k=0}^{\infty} \frac{z^k}{k!} a^k (a^+)^n \varphi_0 \\ &= \frac{e^{-|z|^2/2}e^{-\bar{z}a^+}}{\sqrt{n!}} \sum_{k=0}^{n} \frac{z^k n!}{k!(n-k)!} (a^+)^{n-k}\varphi_0. \end{aligned}$$

Now we can compute the matrix elements:

$$\langle\varphi_m, W(z)\varphi_n\rangle = \frac{e^{-|z|^2/2}}{\sqrt{m!\,n!}}\sum_{k=0}^{n}\sum_{\ell=0}^{\infty}\frac{(-\bar z)^\ell z^k n!}{\ell!\,k!\,(n-k)!}\langle (a^+)^m\varphi_0, (a^+)^{n-k+\ell}\varphi_0\rangle$$

$$= \frac{e^{-|z|^2/2}}{\sqrt{m!\,n!}}\sum_{k=0}^{n}\sum_{\ell=0}^{m}\frac{(-\bar z)^\ell z^k n!\,m!}{\ell!\,k!\,(n-k)!(m-\ell)!}\langle (a^+)^{m-\ell}\varphi_0, (a^+)^{n-k}\varphi_0\rangle$$

$$= e^{-|z|^2/2}\sum_{k=0}^{n}\sum_{\ell=0}^{m}\frac{(-\bar z)^\ell z^k}{\ell!\,k!}\sqrt{\frac{m!\,n!}{(n-k)!\,(m-\ell)!}}\langle\varphi_{m-\ell},\varphi_{n-k}\rangle$$

$$= e^{-|z|^2/2}\sum_{k=0}^{n}\sum_{\ell=0}^{m}\frac{(-\bar z)^\ell z^k}{\ell!\,k!}\sqrt{\frac{m!\,n!}{(n-k)!\,(m-\ell)!}}\delta_{m-l,n-k},$$

where $\delta_{k,\ell}$ denotes the Kronecker symbol. For $n \ge m$, we get non-vanishing elements if and only if $k = n-m+\ell$, where $n-m \le k \le n$ and by the formula (14) we obtain

$$\begin{aligned}\langle\varphi_m, W(z)\varphi_n\rangle &= e^{-|z|^2/2}\sqrt{\frac{m!}{n!}}\sum_{l=0}^{m}\frac{(-1)^\ell|z|^{2\ell}z^{n-m}n!}{\ell!(m-l)!(n-m+l)!}\\ &= e^{-|z|^2/2}\sqrt{\frac{m!}{n!}}z^{n-m}L_m^{n-m}(|z|^2),\end{aligned}$$

as we stated. ■

Note that the case $m \ge n$ can be read out from the theorem, since

$$\langle\varphi_m, W(z)\varphi_n\rangle = \overline{\langle\varphi_n, W(-z)\varphi_m\rangle}.$$

The case $m = n$ involves the Laguerre polynomials. The analogue of (11) is the formula

$$\sum_{n=0}^{\infty} t^n L_n(x) = \frac{1}{1-t}\exp\Big(-\frac{xt}{1-t}\Big) \tag{16}$$

which holds for $|t| < 1$ and $x \in \mathbb{R}^+$. This formula is used to obtain

$$\sum_{n=0}^{\infty}\mu^n(1-\mu)\langle\varphi_n, W(z)\varphi_n\rangle = \exp\Big(-\frac{|z|^2}{2}\frac{1+\mu}{1-\mu}\Big) \tag{17}$$

for $0 < \mu < 1$. Note that

$$D = \sum_{n=0}^{\infty} \mu^n (1-\mu) |\varphi_n\rangle\langle\varphi_n| \tag{18}$$

is a statistical operator (in spectral decomposition). In the corresponding state the self-adjoint operator

$$\frac{za - \bar{z}a^+}{\mathrm{i}} = (-\mathrm{i}z)a + \overline{(-\mathrm{i}z)}a^+$$

has Gaussian distribution.

1.2. The Fock space. Let $\mathcal{H}$ be a Hilbert space. If π is a permutation of the numbers $\{1, 2, \ldots, n\}$, then on the n-fold tensor product $\mathcal{H}^{\otimes n} := \mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}$ we have a unitary U_π such that

$$U_\pi(f_1 \otimes f_2 \otimes \cdots \otimes f_n) = f_{\pi(1)} \otimes f_{\pi(2)} \otimes \cdots \otimes f_{\pi(n)}.$$

The operator

$$P_n(f_1 \otimes f_2 \otimes \cdots \otimes f_n) := \frac{1}{n!} \sum_{\pi} f_{\pi(1)} \otimes f_{\pi(2)} \otimes \cdots \otimes f_{\pi(n)}$$

is a projection onto the symmetric subspace

$$\mathcal{H}^{\vee n} := \{g \in \mathcal{H}^{\otimes n} : U_\pi g = g \text{ for every } \pi\}$$

which is the linear span of the vectors

$$|f_1, f_2, \ldots, f_n\rangle \equiv f_1 \vee f_2 \vee \cdots \vee f_n := \frac{1}{\sqrt{n!}} \sum_{\pi} f_{\pi(1)} \otimes f_{\pi(2)} \otimes \cdots \otimes f_{\pi(n)},$$

where $f_1, f_2, \ldots, f_n \in \mathcal{H}$. Obviously,

$$f_1 \vee f_2 \vee \cdots \vee f_n = f_{\pi(1)} \vee f_{\pi(2)} \vee \cdots \vee f_{\pi(n)}$$

for any permutation π.

Assume that $e_1, e_2, \ldots, e_m$ is a basis in $\mathcal{H}$. When we consider a vector

$$e_{i(1)} \vee e_{i(2)} \vee \cdots \vee e_{i(n)}$$

in the symmetric tensor power $\mathcal{H}^{\vee n}$, we may assume that $1 \le i(1) \le i(2) \le \cdots \le i(n) \le m$. A vector e_t may appear several times, assume that its multiplicity is r_t, that is, $r_t := \{\ell : i(\ell) = t\}$. The norm of the vector is $\sqrt{r_1!\, r_2! \cdots r_m!}$ and

$$\left\{ \frac{1}{\sqrt{r_1!\, r_2! \cdots r_m!}}\, e_{i(1)} \vee e_{i(2)} \vee \cdots \vee e_{i(n)} \;:\; 1 \le i(1) \le i(2) \le \cdots \le i(n) \le m \right\} \tag{19}$$

is an orthonormal basis in $\mathcal{H}^{\vee n}$. Another notation is

$$|e_1^{r_1}, e_2^{r_2}, \dots, e_m^{r_m}\rangle \equiv e_{i(1)} \vee e_{i(2)} \vee \cdots \vee e_{i(n)}.$$

The *symmetric Fock space* is the direct sum

$$\mathcal{F}(\mathcal{H}) := \mathbb{C}\Phi \oplus \mathcal{H}^{\vee 1} \oplus \mathcal{H}^{\vee 2} \oplus \cdots$$

where Φ is called the vacuum vector and in this spirit the summand $\mathcal{H}^{\vee n}$ is called the n-particle subspace. Since $\mathcal{H}^{\vee 1}$ is identical with $\mathcal{H}$, the Hilbert space $\mathcal{F}(\mathcal{H})$ is an extension of $\mathcal{H}$. The union of the vectors (19) (for every n) is a basis of the Fock space.

Lemma 1.3. *If $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, then $\mathcal{F}(\mathcal{H}) = \mathcal{F}(\mathcal{H}_1) \otimes \mathcal{F}(\mathcal{H}_2)$.*

Proof. It is enough to see that

$$(\mathcal{H}_1 \oplus \mathcal{H}_2)^{n\vee} = \mathcal{H}_1^{\vee n} \oplus (\mathcal{H}_1^{\vee(n-1)} \otimes \mathcal{H}_2) \oplus \cdots \oplus (\mathcal{H}_1 \otimes \mathcal{H}_2^{\vee(n-1)}) \oplus \mathcal{H}_2^{\vee n}.$$

If $e_1, e_2, \dots, e_m$ is a basis in $\mathcal{H}_1$ and $f_1, f_2, \dots, f_k$ is a basis in $\mathcal{H}_2$, then the (non-normalized) basis vector

$$e_{i(1)} \vee e_{i(2)} \vee \cdots \vee e_{i(t)} \vee f_{j(1)} \vee f_{j(2)} \vee \cdots \vee f_{j(n-t)}$$

can be identified with

$$e_{i(1)} \vee e_{i(2)} \vee \cdots \vee e_{i(t)} \otimes f_{j(1)} \vee f_{j(2)} \vee \cdots \vee f_{j(n-t)}$$

which is a basis vector in $\mathcal{H}_1^{\vee t} \otimes \mathcal{H}_2^{\vee(n-t)}$.

■

For $f \in \mathcal{H}$ the *creation operator* $a^+(f)$ is defined as

$$a^+(f)|f_1, f_2, \dots, f_n\rangle = |f, f_1, f_2, \dots, f_n\rangle. \tag{20}$$

$a^+(f)$ is linear in the variable f and it maps the n-particle subspace into the $(n+1)$-particle subspace. Its adjoint is the *annihilation operator* which acts as

$$a(f)|f_1, f_2, \dots, f_n\rangle = \sum_{i=1}^{n} \langle f, f_i\rangle |f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_n\rangle. \tag{21}$$

Given an operator $A \in B(\mathcal{H})$ acting on the one-particle space, we can extend it to the Fock space as follows:

$$\mathcal{F}(A)|f_1, f_2, \dots, f_n\rangle = \sum_{i=1}^{n} |f_1, \dots, f_{i-1}, Af_i, f_{i+1}, \dots, f_n\rangle. \tag{22}$$

The next lemma can be shown by simple computation.

Lemma 1.4. *For $f, g \in \mathcal{H}$, we have $\mathcal{F}(|f\rangle\langle g|) = a^+(f)a(g)$.*

Another possibility for extension, or second quantization, of an operator $U \in B(\mathcal{H})$ is given by

$$\Gamma(U)|f_1, f_2, \dots, f_n\rangle = |Uf_1, Uf_2, \dots, Uf_n\rangle. \tag{23}$$

It is easy to see that

Lemma 1.5. $\Gamma(U_1U_2) = \Gamma(U_1)\Gamma(U_2)$ *and* $\Gamma(U^*) = \Gamma(U)^*$.

If U is a unitary, then $\Gamma(U)$ is unitary as well. Moreover, if $U(t)$ is a continuous one-parameter group with generator A, then the generator of the continuous one-parameter group $\Gamma(U(t))$ on $\mathcal{F}(\mathcal{H})$ is the closure of $\mathcal{F}(A)$. To show an example, we note that the statistical operator (18) is $(1-\mu)\Gamma(\mu)$ in the case of a one-dimensional $\mathcal{H}$.

Lemma 1.6. *Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $U = U_1 \oplus U_2$. Then $\Gamma(U_1 \oplus U_2) = \Gamma(U_1) \otimes \Gamma(U_2)$.*

1.3. The algebra of the canonical commutation relation. Let $\mathcal{H}$ be a finite-dimensional Hilbert space. Assume that for every $f \in \mathcal{H}$ a unitary operator $W(f)$ is given such that the relations

$$W(f_1)W(f_2) = W(f_1 + f_2)\exp(\mathrm{i}\,\sigma(f_1, f_2)), \tag{24}$$

$$W(-f) = W(f)^* \tag{25}$$

hold for $f_1, f_2, f \in \mathcal{H}$ with $\sigma(f_1, f_2) := \mathrm{Im}\langle f_1, f_2\rangle$.

The abstract C*-algebra generated by these unitaries is unique and denoted by $\mathrm{CCR}(\mathcal{H})$. The relation (24) shows that $W(f_1)$ and $W(f_2)$ commute if f_1 and f_2 are orthogonal. Therefore for an n-dimensional $\mathcal{H}$, the algebra $\mathrm{CCR}(\mathcal{H})$ is an n-fold tensor product

$$\mathrm{CCR}(\mathbb{C}) \otimes \cdots \otimes \mathrm{CCR}(\mathbb{C}).$$

Since $W(tf)W(sf) = W((t+s)f)$ for $t, s \in \mathbb{R}$, the mapping $t \mapsto W(tf)$ is a one-parameter unitary group which is not norm continuous since $\|W(f_1) - W(f_2)\| \geq \sqrt{2}$ when $f_1 \neq f_2$ [16].

The C*-algebra $\mathrm{CCR}(\mathcal{H})$ has a very natural state

$$\omega(W(f)) := \exp(-\|f\|^2/2) \tag{26}$$

which is called the *Fock state.* The GNS-representation of $\mathrm{CCR}(\mathcal{H})$ is called the *Fock representation* and it leads to the Fock space $\mathcal{F}(\mathcal{H})$ with cyclic vector Φ. If f_1 and f_2 are orthogonal vectors, then

$$\begin{aligned}\omega(W(f_1)W(f_2)) &= \omega(W(f_1+f_2)) = \exp(-\|f_1+f_2\|^2/2)\\ &= \exp(-\|f_1|^2/2)\exp(-\|f_2\|^2/2) = \omega(W(f_1))\omega(W(f_2)).\end{aligned}$$

Therefore ω is a product state and it follows that the GNS Hilbert space is a tensor product. (This is another argument to justify Lemma 1.3.) We shall identify the abstract unitary $W(f)$ with the representing unitary acting on the tensor product GNS-space $\mathcal{F}(\mathcal{H})$. The map

$$t \mapsto \pi_\Phi(W(tf))$$

is an so-continuous 1-parameter group of unitaries, and according to the Stone theorem

$$\pi_\Phi(W(tf)) = \exp(\mathrm{i}tB(f))$$

for a self-adjoint operator $B(f)$, called the *field operator.* Let

$$B^{\pm}(f) = \frac{1}{2}(B(f) \mp \mathrm{i}B(\mathrm{i}f)).$$

Then

$$[B^-(f), B^+(g)] = \langle g, f\rangle$$

is the canonical commutation relation for the *creation operator* $B^+(g)$ and the *annihilation operator* $B^-(f)$. When $\mathcal{H} = \mathbb{C}$, then

$$W(z) = \exp\mathrm{i}(a(z) + a^+(z)),$$

where $a^+(z) = \mathrm{i}\bar{z}a^+$.

1.4. Quasi-free states. The Fock state (26) can be generalized by choosing a positive operator $A \in B(\mathcal{H})$:

$$\omega_A(W(f)) := \exp(-\|f\|^2/2 - \langle f, Af\rangle). \tag{27}$$

This is called the *quasi-free state* [14].

Example 1.1. Assume that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and write the positive mapping $A \in B(\mathcal{H})$ in the form of a block matrix:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

If $f \in \mathcal{H}_1$, then

$$\omega_A(W(f \oplus 0)) = \exp(-\|f\|^2/2 - \langle f, A_{11}f\rangle).$$

Therefore the restriction of the quasi-free state ω_A to $\mathrm{CCR}(\mathcal{H}_1)$ is the quasi-free state $\omega_{A_{11}}$.

If

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix},$$

then $\omega_A = \omega_{A_{11}} \otimes \omega_{A_{22}}$.

■

Example 1.2. Assume that $\mathcal{H}$ is one-dimensional and let $A = \lambda > 0$. We can read out from formulas (17) and (18) that the statistical operator of ω_λ in the Fock representation is

$$D_\lambda = \sum_{n=0}^{\infty} \frac{1}{1+\lambda}\left(\frac{\lambda}{1+\lambda}\right)^n |\varphi_n\rangle\langle\varphi_n|. \tag{28}$$

(Note the $\mu = \lambda/(1+\lambda)$ in (18).) Moreover,

$$\omega_\lambda(a^+a) = \lambda. \tag{29}$$

One can easily compute the von Neumann entropy of the state ω_λ from the eigenvalues of the statistical operator D_λ:

$$S(\omega_\lambda) = \eta(\lambda) - \eta(\lambda+1), \tag{30}$$

where $\eta(\lambda) = -\lambda \log \lambda$.

The case of finite-dimensional $\mathcal{H}$ can be reduced to the one-dimensional by the spectral decomposition of the operator A.

■

Assume that ω is a state of $\mathrm{CCR}(\mathcal{H})$. If $C_\omega(f,g) := \omega(a^+(f)a(g))$ can be defined, then it will be called the 2-*point function* of ω.

Theorem 1.3. *Assume that the spectral decomposition of* $0 \le A \in B(\mathcal{H})$ *is*

$$A = \sum_{i=1}^{m} \lambda_i |e_i\rangle\langle e_i|. \tag{31}$$

Then the statistical operator of the quasi-free state ω_A *in the Fock representation is*

(32)

$$D_A = \Big(\prod_{i=1}^{m} \frac{1}{1+\lambda_i}\Big) \sum_{r_j} \Big(\prod_{i=1}^{m} \Big(\frac{\lambda_i}{1+\lambda_i}\Big)^{r_i} \frac{1}{r_i!}\Big) |e_1^{r_1}, e_2^{r_2}, \dots, e_m^{r_m}\rangle\langle e_1^{r_1}, e_2^{r_2}, \dots, e_m^{r_m}|,$$

where summation is over $n = 0, 1, 2, \dots$ *and the decompositions* $n = r_1 + r_2 + \dots + r_m$. *Moreover,*

$$\omega_A(a^+(f)a(g)) = \langle g, Af\rangle \qquad (f, g \in \mathcal{H})$$

and

$$S(\omega_A) = \operatorname{Tr} \eta(A) - \operatorname{Tr} \eta(A + I).$$

Proof. The basic idea is the decomposition

$$\omega_A = \omega_{\lambda_1} \otimes \omega_{\lambda_2} \otimes \dots \otimes \omega_{\lambda_m} \tag{33}$$

when the space $\mathcal{H}$ is decomposed into the direct sum of the one-dimensional subspaces $\mathbb{C}|e_i\rangle$ and $\mathcal{F}(\mathcal{H})$ and $\mathrm{CCR}(\mathcal{H})$ become tensor product. The statistical operator of ω_{λ_i} is

$$D_{\lambda_i} = \sum_{r_i=0}^{\infty} \frac{1}{1+\lambda_i} \Big(\frac{\lambda_i}{1+\lambda_i}\Big)^{r_i} \frac{1}{r_i!} |e_i^{r_i}\rangle\langle e_i^{r_i}|,$$

the tensor product is exactly the stated matrix.

When we want to check the 2-point function, it is enough to consider the case $f = g = e_i$. This is OK due to (29).

The von Neumann entropy is deduced from (30) and (33). ∎

If (31) holds, then

$$\Gamma(A(I+A)^{-1})|e_1^{r_1}, e_2^{r_2}, \dots, e_m^{r_m}\rangle = \prod_{i=1}^{m} \Big(\frac{\lambda_i}{1+\lambda_i}\Big)^{r_i} |e_1^{r_1}, e_2^{r_2}, \dots, e_m^{r_m}\rangle$$

and we have

$$D_A = \frac{1}{c_A} \Gamma(A(I+A)^{-1}), \quad \text{where} \quad c_A = \operatorname{Tr} \Gamma(A(I+A)^{-1}). \tag{34}$$

This leads to the following result.

Theorem 1.4. *Let ω_A and ω_B be quasi-free states of* $\mathrm{CCR}(\mathcal{H})$ *which correspond to the operators* $0 \le A, B \in B(\mathcal{H})$. *Their Connes cocycle is*

$$[D\omega_A, D\omega_B]_t = u_t \Gamma\left((A(I+A)^{-1})^{\mathrm{i}t}(B(I+B)^{-1})^{-\mathrm{i}t}\right) \tag{35}$$

where $u_t = (\mathrm{Tr}\,\Gamma(A(I+A)^{-1}))^{-\mathrm{i}t}(\mathrm{Tr}\,\Gamma(B(I+B)^{-1}))^{\mathrm{i}t}$.

Theorem 1.5. *Let ω be a state of* $\mathrm{CCR}(\mathcal{H})$ *such that its 2-point function is* $\omega(a^+(f)a(g)) = \langle g, Af\rangle$ $(f, g \in \mathcal{H})$ *for a positive operator* $A \in B(\mathcal{H})$. *Then* $S(\omega) \le S(\omega_A)$ *and equality implies* $\omega = \omega_A$.

Proof. Consider the one-dimensional case when $A = \lambda$. We compute the relative entropy $S(\omega||\omega_\lambda)$:

$$\begin{aligned} S(\omega||\omega_\lambda) &= -S(\omega) - \omega(\log D_\lambda) \\ &= -S(\omega) - \log(1+\lambda)\,\omega\Big(\sum_{n=0}^{\infty} |\varphi_n\rangle\langle\varphi_n|\Big) - \log\frac{\lambda}{1+\lambda}\,\omega\Big(\sum_{n=0}^{\infty} n|\varphi_n\rangle\langle\varphi_n|\Big) \\ &= -S(\omega) - \log(1+\lambda) - \lambda\log\frac{\lambda}{1+\lambda} = -S(\omega) + S(\omega_\lambda). \end{aligned}$$

Since the relative entropy $S(\omega||\omega_\lambda) > 0$ if ω and ω_λ are different, the statement is obtained.

The general case can be proved by similar computation. The result was also obtained in connection with the central limit theorem [17]. ■

2. Markov triplets

Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ be a finite-dimensional Hilbert space and consider the Fock representation of $\mathrm{CCR}(\mathcal{H}) \equiv \mathrm{CCR}(\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3)$ on $\mathcal{F}(\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3)$. Instead of the C*-algebra, we work with the weak closure in the Fock representation: $\mathcal{A}_{123} := B(\mathcal{F}(\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3)) \equiv B(\mathcal{F}(\mathcal{H}_1)) \otimes B(\mathcal{F}(\mathcal{H}_2)) \otimes B(\mathcal{F}(\mathcal{H}_3)))$. This algebra has subalgebras

$$\begin{aligned} \mathcal{A}_{12} &:= B(\mathcal{F}(\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus 0)) \equiv B(\mathcal{F}(\mathcal{H}_1)) \otimes B(\mathcal{F}(\mathcal{H}_2)) \otimes \mathbb{C}I, \\ \mathcal{A}_{23} &:= B(\mathcal{F}(0 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3)) \equiv \mathbb{C}I \otimes B(\mathcal{F}(\mathcal{H}_2)) \otimes B(\mathcal{F}(\mathcal{H}_3)), \\ \mathcal{A}_2 &:= B(\mathcal{F}(0 \oplus \mathcal{H}_2 \oplus 0)) \equiv \mathbb{C}I \otimes B(\mathcal{F}(\mathcal{H}_2)) \otimes \mathbb{C}I. \end{aligned}$$

Assume that D_{123} is a statistical operator in $\mathcal{A}_{123}$ and we denote by D_{12}, D_2, D_{23} its reductions in the subalgebras $\mathcal{A}_{12}, \mathcal{A}_2, \mathcal{A}_{23}$, respectively. These subalgebras form a *Markov triplet* with respect to the state D_{123} if

$$S(D_{123}) - S(D_{23}) = S(D_{12}) - S(D_2), \tag{36}$$

where S denotes the von Neumann entropy and we assume that both sides are finite in the equation. The state ω corresponding to the statistical operator D_{123} is called the *Markov state.*

Condition (36) is equivalent to several other conditions, see, for example, Chapter 9 of [19] about the details and proofs. In most studies about the strong subadditivity of the von Neumann entropy and the equality case (36), the Hilbert space is assumed to be finite-dimensional. In our setting the Fock space is always infinite-dimensional, so [12] might be the optimal reference. Here we prefer an equivalent formulation in terms of Connes cocyles:

$$[D\omega_{123}, D(\varphi_1 \otimes \omega_{12})]_t = [D(\omega_{12} \otimes \varphi_3), D(\varphi_1 \otimes \omega_2 \otimes \varphi_3)]_t \tag{37}$$

for every real t, where φ_1 and φ_3 are arbitrary states. (In terms of the statistical operators in (36) condition (37) has the equivalent form $D_{123}^{\mathrm{i}t} D_{23}^{-\mathrm{i}t} = D_{12}^{\mathrm{i}t} D_2^{-\mathrm{i}t}$.)

Let

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{38}$$

be a positive operator on $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ and assume that D_{123} is the statistical operator of the quasi-free state ω_A. ω_A restricted to $\mathcal{A}_{23}$ is a quasi-free state induced by the operator

$$\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}.$$

Set

$$D = \begin{bmatrix} I & 0 & 0 \\ 0 & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{bmatrix}, \qquad B = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & I \end{bmatrix}, \qquad C = \begin{bmatrix} I & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Then equality (37) may become

$$[D\omega_A, D\omega_D]_t = [D\omega_B, D\omega_C]_t. \tag{39}$$

According to Theorem 1.4 this is the condition

$$\Gamma\left((A(I+A)^{-1})^{\mathrm{i}t}(D(I+D)^{-1})^{-\mathrm{i}t}\right) = \lambda_t \Gamma\left((B(I+B)^{-1})^{\mathrm{i}t}(C(I+C)^{-1})^{-\mathrm{i}t}\right)$$

with a set of numbers λ_t.

One can see from formula (23) that $\Gamma(U) = \lambda\Gamma(V)$ for a $\lambda \in \mathbb{C}$ implies $\lambda = 1$ and $U = V$. Therefore condition (37) becomes the following.

Theorem 2.1. *For a quasi-free state ω_A the Markov property* (36) *is equivalent to the condition*

$$A^{\mathrm{it}}(I+A)^{-\mathrm{it}}D^{-\mathrm{it}}(I+D)^{\mathrm{it}}=B^{\mathrm{it}}(I+B)^{-\mathrm{it}}C^{-\mathrm{it}}(I+C)^{\mathrm{it}} \tag{40}$$

for every real t.

The problem is the solution of this equation. Note that if condition (40) holds for every real t, then analytic continuation gives all complex t.

Corollary 2.1. *If A gives a Markov triplet, then U^*AU gives a Markov triplet as well when $U=\mathrm{Diag}(U_1,U_2,U_3)$ with unitaries $U_i\in B(\mathcal{H}_i)$, $1\le i\le 3$.*

Example 2.1. The following matrix satisfies condition (40):

$$A=\begin{bmatrix} A_{11} & \begin{bmatrix} a & 0\end{bmatrix} & 0 \\ \begin{bmatrix} a^* \\ 0\end{bmatrix} & \begin{bmatrix} c & 0 \\ 0 & d\end{bmatrix} & \begin{bmatrix} 0 \\ b\end{bmatrix} \\ 0 & \begin{bmatrix} 0 & b^*\end{bmatrix} & A_{33}\end{bmatrix}=\begin{bmatrix} \begin{bmatrix} A_{11} & a \\ a^* & c\end{bmatrix} & 0 \\ 0 & \begin{bmatrix} d & b \\ b^* & A_{33}\end{bmatrix}\end{bmatrix}, \tag{41}$$

where the parameters a,b,c,d (and 0) are matrices. This is a block diagonal matrix, $A=\mathrm{Diag}(A_1,A_2)$, so we have $f(A)=\mathrm{Diag}(f(A_1),f(A_2))$ for a function f. The matrices B, C and D are block diagonal as well:

$$\begin{aligned} B &=\mathrm{Diag}(A_1,\mathrm{Diag}(d,I)),\\ C &=\mathrm{Diag}(\mathrm{Diag}(I,c),\mathrm{Diag}(d,I)),\\ D &=\mathrm{Diag}(\mathrm{Diag}(I,c),A_2).\end{aligned}$$

Therefore,

$$f(A)g(D)=\mathrm{Diag}(f(A_1)\,\mathrm{Diag}(g(I),g(c)),f(A_2)g(A_2))$$

and

$$f(B)g(C)=\mathrm{Diag}(f(A_1)\,\mathrm{Diag}(g(I),g(c)),\mathrm{Diag}(f(d)g(d),f(I)g(I))).$$

If $fg=1$, then $f(A)g(D)=f(B)g(C)$.

Note that

$$A=\begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & A_{23} \\ 0 & A_{32} & A_{33}\end{bmatrix} \text{ and } A=\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & A_{33}\end{bmatrix}$$

are particular cases.

On the basis of the previous corollary we can use block diagonal unitaries to have further examples.

■

Theorem 2.2. *The condition*

$$A^{-1}(I+A)D(I+D)^{-1} = B^{-1}(I+B)C(I+C)^{-1} \tag{42}$$

implies that

$$A_{13} = A_{12}A_{22}^{-1}A_{23} \quad \textit{and} \quad A_{13} = A_{12}(A_{22}+I)^{-1}A_{23}. \tag{43}$$

Proof. In the computation of the inverse of a block matrix, the following formula is very useful. If P and S are square matrices and S is invertible, then

$$M^{-1} = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}^{-1} = \begin{bmatrix} (M/S)^{-1} & -(M/S)^{-1}QS^{-1} \\ -S^{-1}R(M/S)^{-1} & S^{-1}+S^{-1}R(M/S)^{-1}QS^{-1} \end{bmatrix}, \tag{44}$$

where $(M/S) \equiv P - QS^{-1}R$ is the Schur complement of S in M (see [11, Section 7.7], actually, the checking is a simple multiplication). If P is also invertible, the equation

$$S^{-1}+S^{-1}R(M/S)^{-1}QS^{-1} = (M/P)^{-1} \tag{45}$$

also holds, where $(M/P) \equiv S - RP^{-1}Q$ is the Schur complement of P in M. For solving (42), we partition the block matrix A in the following way:

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} P & Q \\ Q^* & S \end{bmatrix},$$

where we used the fact that A is positive self-adjoint and used the notations

$$S = \begin{bmatrix} A_{22} & A_{23} \\ A_{23}^* & A_{33} \end{bmatrix},$$

$P = A_{11}$ and $Q = [A_{12} \quad A_{13}]$. With the help of (44) we get

$$A^{-1}(I+A) = \begin{bmatrix} I+(A/S)^{-1} & -(A/S)^{-1}QS^{-1} \\ -S^{-1}Q^*(A/S)^{-1} & I+(A/P)^{-1} \end{bmatrix}. \tag{46}$$

Similarly we write D in 2×2 matrix form:

$$D = \begin{bmatrix} I & 0 & 0 \\ 0 & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & S \end{bmatrix},$$

where 0 denotes the block matrix $[0 \quad 0]$, or its transpose. Now we can compute the left-hand side of (42):
(47)
$$A^{-1}(I+A)D(I+D)^{-1} = \begin{bmatrix} \frac{1}{2}\left[I+(A/S)^{-1}\right] & -(A/S)^{-1}Q(I+S)^{-1} \\ -\frac{1}{2}S^{-1}Q^*(A/S)^{-1} & \left[I+(A/P)^{-1}\right]S(I+S)^{-1} \end{bmatrix}.$$

With a similar procedure we have

$$B = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} P & \tilde{Q} \\ \tilde{Q}^* & \tilde{S} \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} I & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \tilde{S} \end{bmatrix},$$

where

$$\tilde{S} = \begin{bmatrix} A_{22} & 0 \\ 0 & I \end{bmatrix}, \qquad \tilde{Q} = [A_{12} \quad 0],$$

and the same remark concern for the 0 block matrices as above. We get for the right-hand side of (42):
(48)
$$B^{-1}(I+B)C(I+C)^{-1} = \begin{bmatrix} \frac{1}{2}[I+(B/\tilde{S})^{-1}] & -(B/\tilde{S})^{-1}\tilde{Q}(I+\tilde{S})^{-1} \\ -\frac{1}{2}\tilde{S}^{-1}\tilde{Q}^*(B/\tilde{S})^{-1} & \left[I+(B/P)^{-1}\right]\tilde{S}(I+\tilde{S})^{-1} \end{bmatrix}.$$

From the equality between (47) and (48) we have equations for the block matrices. The equality of $(1,1)$ elements implies $(A/S) = (B/\tilde{S})$. This and the equality of $(1,2)$ elements gives the equation $Q(I+S)^{-1} = \tilde{Q}(I+\tilde{S})^{-1}$, which lead us to $A_{13} = A_{12}(A_{22}+I)^{-1}A_{23}$. From the $(2,1)$ elements we have $S^{-1}Q^* = \tilde{S}^{-1}\tilde{Q}^*$, this implies the other necessary condition. The $(2,2)$ elements will be equal automatically when these conditions hold.

■

According to [5], (43) means that the $(1,3)$ element of A^{-1} and $(A+I)^{-1}$ are 0. It is interesting that if we take the determinant of equation (40), then we have

$$\begin{aligned}(\mathrm{Det}\,A)(\mathrm{Det}\,C)(\mathrm{Det}\,D)^{-1}(\mathrm{Det}\,B)^{-1} \\ = (\mathrm{Det}(I+A))(\mathrm{Det}(I+C))(\mathrm{Det}(I+D))^{-1}(\mathrm{Det}(I+B))^{-1}.\end{aligned}$$

According to Theorem 5 in [5], both sides are smaller than or equal to 1 and (43) is equivalent to the condition that both sides are exactly 1.

Let X be the inverse of the block matrix (38) and suppose that (42) holds. A tedious computation yields that

$$\begin{aligned}
X_{11} &= \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1},\\
X_{12} &= -\left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1} A_{12}A_{22}^{-1},\\
X_{13} &= 0,\\
X_{22} &= \left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1} + A_{22}^{-1}A_{23}\left(A_{33} - A_{32}A_{22}^{-1}A_{23}\right)^{-1} A_{32}A_{22}^{-1},\\
X_{23} &= -A_{22}^{-1}A_{23}\left(A_{33} - A_{32}A_{22}^{-1}A_{23}\right)^{-1},\\
X_{33} &= \left(A_{33} - A_{32}A_{22}^{-1}A_{23}\right)^{-1}.
\end{aligned}$$

The next example shows that conditions (43) are not sufficient, in contrast to the classical Gaussian Markov triplets [5].

Example 2.2. The matrix

$$A = \begin{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ -\frac{2}{7} & -\frac{2}{7} \end{bmatrix} & \begin{bmatrix} \frac{1}{14} & \frac{1}{14} \\ -\frac{1}{49} & -\frac{1}{49} \end{bmatrix} \\ \begin{bmatrix} 1 & -\frac{2}{7} \\ 1 & -\frac{2}{7} \end{bmatrix} & \begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ -\frac{2}{7} & -\frac{2}{7} \end{bmatrix} \\ \begin{bmatrix} \frac{1}{14} & -\frac{1}{49} \\ \frac{1}{14} & -\frac{1}{49} \end{bmatrix} & \begin{bmatrix} 1 & -\frac{2}{7} \\ 1 & -\frac{2}{7} \end{bmatrix} & \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix}$$

is positive and fulfills conditions (43), but (40) does not hold. Indeed, numerical computation shows that

$$\log A(I+A)^{-1} + \log C(I+C)^{-1} \neq \log B(I+B)^{-1} + \log D(I+D)^{-1},$$

or an alternative argument is that the matrix is different from (41), cf. Theorem 2.3.

This example shows that condition

$$D_{123}D_{23}^{-1} = D_{12}D_{2}^{-1} \tag{49}$$

is weaker than

$$D_{123}^{\mathrm{it}}D_{23}^{-\mathrm{it}} = D_{12}^{\mathrm{it}}D_{2}^{-\mathrm{it}} \qquad (t \in \mathbb{R}). \tag{50}$$

Note that in the finite-dimensional case (50) is equivalent to

$$D_{123}^{1/2}D_{23}^{-1/2} = D_{12}^{1/2}D_{2}^{-1/2},$$

see Chapter 9 of [19].

■

In the notation

$$K := A^{-1}(I+A),\ L := D^{-1}(I+D),\ M := B^{-1}(I+B),\ N := C^{-1}(I+C)$$

condition (40) becomes

$$K^{-\mathrm{it}}L^{\mathrm{it}} = M^{-\mathrm{it}}N^{\mathrm{it}}. \tag{51}$$

Theorem 2.3. *The Markov property* (36) *is satisfied if and only if there is a projection* $P \in B(\mathcal{H})$ *such that* $P|\mathcal{H}_1 \equiv I$, $P|\mathcal{H}_3 \equiv 0$ *and* $PA = AP$. *In other words,* A *is block diagonal in the form* (41).

Proof. We write the matrices of the relation (51) in block form:

$$K = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}, \quad L = \begin{bmatrix} 2 & 0 & 0 \\ 0 & L_{22} & L_{23} \\ 0 & L_{32} & L_{33} \end{bmatrix},$$
$$M = \begin{bmatrix} M_{11} & M_{12} & 0 \\ M_{21} & M_{22} & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad N = \begin{bmatrix} 2 & 0 & 0 \\ 0 & N_{22} & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Suppose that the Markov property is satisfied, and we use it in the form (51). Since $K = I + A^{-1}$, the block diagonal structure (41) of A is equivalent to that property of K. We shall work on K.

Let $\mathcal{C}$ be the subalgebra generated by the set $\{K^{\mathrm{i}t}L^{-\mathrm{i}t} : t \in \mathbb{R}\}$. By the factorization result in [12], there are positive matrices $\tilde{X}, \tilde{Y} \in \mathcal{C}$ and $0 \le \tilde{Z} \in B(\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3)$, such that

$$K = \tilde{X}\tilde{Z}, \qquad L = \tilde{Y}\tilde{Z}, \qquad \tilde{Z}\tilde{X} = \tilde{X}\tilde{Z}, \qquad \tilde{Z}\tilde{Y} = \tilde{Y}\tilde{Z}. \tag{52}$$

Since (51) implies that $\mathcal{C} \subseteq B(\mathcal{H}_1 \oplus \mathcal{H}_2) \oplus \mathbb{C}I$, we have $\tilde{X}$ and $\tilde{Y}$ in the above form

$$\tilde{X} := \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \quad \text{and} \quad \tilde{Y} := \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix}.$$

We write $\tilde{Z}$ in a similar block form:

$$\tilde{Z} = \begin{bmatrix} Z & z \\ z^* & \tilde{Z}_{33} \end{bmatrix},$$

where $Z \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)$ and $z^* = [\tilde{Z}_{31}, \tilde{Z}_{32}]$. Then

$$K = \tilde{X}\tilde{Z} = \begin{bmatrix} XZ & Xz \\ z^* & \tilde{Z}_{33} \end{bmatrix}, \quad \text{and} \quad L = \tilde{Y}\tilde{Z} = \begin{bmatrix} YZ & Yz \\ z^* & \tilde{Z}_{33} \end{bmatrix}. \tag{53}$$

This implies that

$$Xz = Yz = z = \begin{bmatrix} 0 \\ L_{23} \end{bmatrix}, \tag{54}$$

Z commutes with X and Y and

$$[K_{31}, K_{32}, K_{33}] = [z^*, \tilde{Z}_{33}] = [0, L_{32}, L_{33}].$$

In particular, $K_{31} = K_{13} = 0$ and $K_{23} = L_{23}$.

By (53) and (54), we have

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} 0 \\ L_{23} \end{bmatrix} = XZz = ZYz = \begin{bmatrix} 2 & 0 \\ 0 & L_{22} \end{bmatrix} z = \begin{bmatrix} 0 \\ L_{22}L_{23} \end{bmatrix}$$

and we get $K_{12}K_{23} = K_{12}L_{23} = 0$.

If the range of L_{23} is $\mathcal{H}_2$, then $K_{12} = 0$, and if $L_{23} = 0$, then $K_{23} = 0$, so in both cases K is block diagonal.

Suppose now that the range of L_{23} is not $\mathcal{H}_2$ and $L_{23} \neq 0$. Then there is a decomposition $\mathcal{H}_2 = \mathcal{K}_a \oplus \mathcal{K}_b$, where $\mathcal{K}_b$ is the range of L_{23}. Next we work in the frame of the decomposition $(\mathcal{H}_1 \oplus \mathcal{K}_a) \oplus \mathcal{K}_b$.

For each vector $\xi \in \mathcal{K}_b$, we have $X\xi = Y\xi = \xi$. It follows that there are matrices $X_1, Y_1 \in B(\mathcal{H}_1 \oplus \mathcal{K}_a)$, such that

$$X = \begin{bmatrix} X_1 & 0 \\ 0 & I \end{bmatrix}, \quad Y = \begin{bmatrix} Y_1 & 0 \\ 0 & I \end{bmatrix}.$$

If we write

$$Z = \begin{bmatrix} Z_1 & z_1 \\ z_1^* & Z_{33} \end{bmatrix}, \qquad Z_1 \in B(\mathcal{H}_1 \oplus \mathcal{K}_a) \text{ and } Z_{33} \in B(\mathcal{K}_b),$$

then

$$XZ = \begin{bmatrix} X_1Z_1 & X_1z_1 \\ z_1^* & Z_{33} \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12}^1 & K_{13}^1 \\ K_{21}^1 & K_{22}^1 & K_{23}^1 \\ K_{31}^1 & K_{32}^1 & K_{33}^1 \end{bmatrix}$$

and

$$YZ = \begin{bmatrix} Y_1Z_1 & Y_1z_1 \\ z_1^* & Z_{33} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & L_{22}^1 & L_{23}^1 \\ 0 & L_{32}^1 & L_{33}^1 \end{bmatrix}$$

with the block decompositions of the matrices in $B(\mathcal{H}_1 \oplus \mathcal{K}_a \oplus \mathcal{K}_b)$ on the right-hand sides. This implies that Z_1 commutes with both X_1 and Y_1, $K_{33}^1 = Z_{33} = L_{33}^1$ and

$$X_1z_1 = Y_1z_1 = z_1 = \begin{bmatrix} 0 \\ L_{23}^1 \end{bmatrix} \tag{55}$$

and we are in a similar situation as before. (Compare with the relations (52), (53) and (54).) We also get $K_{12}^1 L_{23}^1 = 0$ exactly as before. Note that now we can write

$$K = \begin{bmatrix} K_{11} & K_{12}^1 & 0 & 0 \\ K_{21}^1 & K_{22}^1 & L_{23}^1 & 0 \\ 0 & L_{32}^1 & L_{33}^1 & L_{23} \\ 0 & 0 & L_{32} & L_{33} \end{bmatrix}.$$

Again, if $L_{23}^1 = 0$ or if the range of L_{23}^1 is $\mathcal{K}_a$, then the matrix K is block diagonal. If this condition does not hold, then the above procedure can be repeated: we decompose the subspace $\mathcal{K}_a = \mathcal{K}_a^2 \oplus \mathcal{K}_b^2$, where $\mathcal{K}_b^2$ is the range of L_{23}^1, and write X_1, Y_1 in the block-diagonal form, using (55), etc.

After repeating this procedure n times, we get the matrix K in the form

$$K = \begin{bmatrix} K_{11} & K_{12}^n & 0 & 0 & 0 \\ K_{21}^n & K_{22}^n & L_{23}^n & 0 & 0 \\ 0 & L_{32}^n & L_{33}^n & L_{34}^{n'} & 0 \\ 0 & 0 & L_{43}^{n'} & L_{44}^{n'} & L_{23} \\ 0 & 0 & 0 & L_{32} & L_{33} \end{bmatrix}$$

in $B(\mathcal{H}_1 \oplus \mathcal{K}_a^n \oplus \mathcal{K}_b^n \oplus \mathcal{K}_b^{n'} \oplus \mathcal{H}_3)$, where $\mathcal{K}_b^{n'} = \oplus_{k=1}^{n-1} \mathcal{K}_b^k$, $\mathcal{K}_b^1 \equiv \mathcal{K}_b$ and $K_{12}^n L_{23}^n = 0$. Since $\mathcal{H}_2$ is finite-dimensional, there must be some n, such that the matrix L_{23}^n is either 0 or has range $\mathcal{K}_a^n$. In both cases, the matrix K has a block diagonal form, and so does the matrix A.

■

The CCR Markov triplets have some similarity to Markov states on a product algebra $M_n(\mathbb{C}) \otimes (M_u(\mathbb{C}) \otimes M_t(\mathbb{C})) \otimes M_n(\mathbb{C})$ $(n = ut)$. If ω_1 is a state on $M_n(\mathbb{C}) \otimes M_u(\mathbb{C})$ and ω_2 is a state on $M_t(\mathbb{C}) \otimes M_n(\mathbb{C})$, then $\omega_1 \otimes \omega_2$ is Markovian, but there are other Markov states, however they are constructed essentially by this idea [9].

3. Connection to classical Gaussians

Markov triplets in the classical Gaussian case were studied in [5]. The present non-commutative situation has some relation to the classical Gaussian.

Lemma 3.1. *Let $e_1, e_2, \dots, e_k$ be linearly independent unit vectors in $\mathcal{H}$ such that $\langle e_i, e_j \rangle$ is real, $1 \le i \le j \le k$. Then the Weyl unitaries $W(te_j) = \exp(tiB(e_j))$ commute. With respect to a quasi-free state* (27)*, the joint distribution of the field operators $B(e_1), B(e_2), \dots, B(e_k)$ is Gaussian.*

Assume that $f_1, f_2, \dots, f_k$ are orthonormal vectors and $Sf_i = e_i$ for a linear mapping S. The covariance is the matrix of the linear operator $S^*(I + 2A)S$ in the basis $f_1, f_2, \dots, f_k$.

Proof. The characteristic function of the joint distribution is

$$\begin{aligned}(t_1, t_2, \dots, t_j) &\mapsto \omega_A(\exp(\mathrm{i}t_1 B_1)\exp(\mathrm{i}t_2 B_2)\cdots\exp(\mathrm{i}t_k B_k)) \\ &= \omega_A(W(t_1e_1 + t_2e_2 + \cdots + t_ke_k)) \\ &= \exp\Big(-\frac{1}{2}\Big(\sum_{i,j} t_it_j\langle e_i, (I+2A)e_j\rangle\Big)\Big) \\ &= \exp\Big(-\frac{1}{2}\Big(\sum_{i,j} t_it_j\langle Sf_i, (I+2A)Sf_j\rangle\Big)\Big).\end{aligned}$$

This gives the result. ■

Next we assume that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ and assume that $\dim \mathcal{H}_i = k$ $(1 \le i \le 3)$. Choose pairwise orthogonal unit vectors f_j, $1 \le j \le 3k$, such that

$$f_{(i-1)k+r} \in \mathcal{H}_i, \qquad 1 \le i \le 3, \quad 0 \le r \le k-1,$$

and unit vectors e_j, $1 \le j \le 3k$, such that

$$e_{(i-1)k+r} \in \mathcal{H}_i, \qquad 1 \le i \le 3, \quad 0 \le r \le k-1, \tag{56}$$

and

$$\langle e_t, e_u\rangle \text{ is real for any } 1 \le t, u \le 3k. \tag{57}$$

There is an invertible block diagonal matrix $S = \mathrm{Diag}(S_1, S_2, S_3)$ such that $Sf_j = e_j$, $1 \le j \le 3k$.

The Weyl unitaries $W(te_j) = \exp(ti B(e_j))$ commute. The joint distribution of the field operators $B(e_j)$ is Gaussian with covariance block matrix $S^*(I+2A)S$. It follows from [5] that the classical (multi-valued) Gaussian triplet

$$(B(e_1), \dots, B(e_k)), \quad (B(e_{k+1}), \dots, B(e_{2k})), \quad (B(e_{2k+1}), \dots, B(e_{3k}))$$

is Markovian if and only if

$$[S^*(I+2A)S]_{13} = [S^*(I+2A)S]_{12}[S^*(I+2A)S]_{22}^{-1}[S^*(I+2A)S]_{23}.$$

Since

$$[S^*(I+2A)S]_{13} = S_1^*(I+2A)_{13}S_3$$

and

$$\begin{aligned}&[S^*(I+2A)S]_{12}[S^*(I+2A)S]_{22}^{-1}[S^*(I+2A)S]_{23} \\ &\qquad = S_1^*(I+2A)_{12}(I+2A)_{22}^{-1}(I+2A)_{23}S_3,\end{aligned}$$

the matrix S can be removed from the condition and we have the equivalent form $(I+2A)_{13} = (I+2A)_{12}(I+2A)_{22}^{-1}(I+2A)_{23}$ which means that the $(1,3)$ element of $(I+2A)^{-1}$ is 0. If the quasi-free state induced by A gives a Markov triplet, then A is the form of (41) due to Theorem 2.3. In particular, $(A^{-1})_{13} = 0$ and reference to [5] gives the following result.

Theorem 3.1. *Let ω be a quasi-free state on* $\mathrm{CCR}(\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3)$ *which is Markovian with respect to the decomposition* $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$. *Assume that* e_j, $1 \le j \le 3k$, *are unit vectors such that* (56) *and* (57) *hold. Then the classical* (*multi-valued*) *Gaussian triplet*

$$(B(e_1), \dots, B(e_k)), \quad (B(e_{k+1}, \dots, B(e_{2k})), \quad (B(e_{2k+1}, \dots, B(e_{3k}))$$

is Markovian, moreover, $(B(e_1), \dots, B(e_k))$*and* $(B(e_{2k+1}, \dots, B(e_{3k}))$ *are independent.*

In a final remark we compare the Markov condition for the classical multivariate Gaussian triplet with the CCR case. The classical condition is $A_{12}A_{22}^{-1}A_{23} = A_{13}$. The CCR condition can be formulated as $A_{12}f(A_{22})A_{23} = A_{13}$ with any continuous function f. (This implies immediately that $A_{13} = 0$.) Therefore, the CCR condition is much more restrictive.

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