Fuzzy Sets and Systems

Fuzzy observables and the universal family of fuzzy events

--Manuscript Draft--

Manuscript Number:	FSS-D-21-00091R1
Article Type:	Full Length Article (FLA)
Keywords:	Fuzzy Observables, Fuzzy sets, Positive Operator Valued Measures, Markov Kernels, Foundations of Quantum mechanics, Quantum Structures

1. Abstract

We prove the existence of a universal family of fuzzy events $\mathcal{F} = \{(\mathbb{R}, \mu^U_\Delta)\}_{\Delta\in\mathcal{B}(X)}$ such that every fuzzy observable F is the fuzzy version of a sharp observable E^F with the fuzzification realized by $\mathcal{F}$. That is a consequence of the existence of a universal Markov kernel μ^U such that every commutative POVM F is the fuzzy version of a spectral measure E^F with the fuzzification realized by μ^U. We provide a general proof for POVMs defined on the σ-algebra of a Hausdorff, locally compact, second countable topological space which is based on a modified version of the transferring principle introduced by Lebesgue, Riesz, de la Vallée-Poussin and Jessen; both E^F and μ^U are obtained by means of a bijective map between the infinite dimensional cube and the unit interval.

Then, we show that every weak Markov kernel is functionally subordinated to μ^U; in some sense the randomization encoded in μ^U includes all the possible randomizations. Finally, we show that the universal family of fuzzy events corresponding to the universal Markov kernel includes all the possible fuzzy events of which a POVM can represent the probabilities. Moreover, the probabilities associated to a fuzzy observable F coincide with the probabilities of the universal family of fuzzy events with respect to E^F.

Cover Letter

Dear Editor, we are sending you the revised version of the paper ‘’Fuzzy observables and the universal family of fuzzy events”. We thanks the referee’s for their observations that we have carefully followed trying to improve the paper. Here is a detailed list of the changes we have done.

First referee report

1) *It is not clear to me if the current results are as relevant to physical examples as some previous works*

   In order to answer this question, we first note that in the phase space formulation of quantum mechanics (E. Prugovecki, S.T. Ali, F.E. Schroeck) the POVMs are defined on the phase space which is a simplectic space which is different from $\mathbb{R}$ (with the exception of the case of the Heisenberg group symmetries in dimension one). Therefore the answer is in the positive, we believe the results we provided in the paper can (in principle) be of interest in some physical applications as well.

2) *I spotted some instances of 'application' where it would be better to use 'funtion' or 'map*

   “Application” has been replaced by “map” all over the paper.

3) *As proved in the paper, every weak Markov kernel is functionally subordinate to a universal Markov kernel. If a weak Markov kernel is given, is it possible to practical to find the function that gives the relation, or is the result only about existence of such function?*

We think this is a relevant point deserving a future work. The paper provides an existence proof but a possible algorithmic construction can come from the generalization of some previous results. We outline a possible direction: By theorem 3.8 in Ref. [1], there is a function g such that $\int g(t)\, dF(t) = A^F$ where A^F is a generator for $\mathcal{A}^W(F)$. Then, by equation (15) in the old version of the paper, $A^F = \int g(t)\, dF(t) = \int f(t) \int \gamma_{dt}(\lambda)\, dE^B_\lambda = \int\int g(t)\gamma_{dt}(\lambda)\, dE^B_\lambda = \int dE^B(\lambda)\big[\int g(t)\gamma_{dt}(\lambda)\big] = \int f(\lambda) dE^B_\lambda$, where $f(\lambda) := \int g(t)\gamma_{dt}(\lambda)$. On the other hand, g can be defined by an algorithmic procedure (see [2]).

Clearly, the previous reasoning needs to be made rigorous. But its actual weakness is due to the fact that the results in [1, 2] holds in the case of real POVMs F such that $F(\Delta)$ is discrete for every Δ. Those results must be generalized to the case of a locally compact, second countable, Hausdorff, topological space without the restriction due to the discreteness of the operators in the range of F.

Second referee report

1) *It should be clearly explained what is essentially new here compared to the previous works. What are the main technical obstacles the authors had to tackle to obtain the present results?*

1.1) The construction in section 2.1 of the paper differs from the one in [3] not only for some details related to the choice of the partitions but mainly because the one in [3] does not use the transferring principle in its full generality, i.e., in order to prove the existence of a one-to-one map $f : [0,1]^\omega \to [0,1]$ from the infinite cube to the interval $[0,1]$ that can be used in order to define the sharp version E^F of F. We had to add a constraint (see lines 194-197 of the old version of the paper) in order to ensure that the function f can be defined. This constraint has been replaced by the choice of lexicographical order in the new version of the paper.

The existence of f is relevant from the mathematical viewpoint since it gives full generality to the construction of the sharp version E^F. Indeed, the sharp version E^F of F can be defined through f as $E^F(i_l^{(n)}) = \prod_j E_j(f_j^{-1}(i_l^n))$, $i_l^{(n)} \in \mathcal{I}$ (see equation (6) in the old version of the paper). In the paper by Cattaneo and Nisticó, the function f is not introduced and the connection between E^F and F is not encrypted in a one-to-one function between $[0,1]$ and $[0,1]^\omega$.

1.2) Note also that in [3] the problem of the existence of a σ-additive set function $\omega_{(\cdot)}(\lambda)$, $\lambda \in \sigma(A^F)$, is not taken into consideration and only the measurability of $\omega_\Delta(\cdot)$, for every Δ is proved. In Ref. [4], only the existence of an additive set function $\omega_{(\cdot)}(\lambda)$, $\lambda \in \sigma(A^F)$ is proved; the **algorithmic** definition of a σ-additive set function $\omega_{(\cdot)}(\lambda)$ is left as an open problem (see section III, part C in [4]).

The algorithmic construction we provide in the present paper puts an end to this question. Actually we prove a quite stronger result since we show that $\omega^U_{(\cdot)}(\lambda)$ is σ-additive for every λ in a set of measure one with respect to E^F for every POVM F.

That is crucial in order to show the existence of a universal Markov kernel. All of that required a **new self-contained algorithmic more general proof for the existence of the** ω^U_Δ. See item 1.3) below and the comment added in the introduction to section 2.4 of the revised version of the paper.

1.3) The proof of the existence of the universal Markov kernel in the real case, $X = \mathbb{R}$, has been proved in [6] and is based both on the additivity of $\omega_\Delta(\lambda)$ proved in [4] and on corollary 1 in [5] which uses the ordered structure of $\mathbb{R}$. While an extension of the proof of the additivity to the general case seems to be straightforward, it seems to us, that the extension of corollary1 is not so easy. **We circumvented the problem** by proving (see page 18 of the old version of the paper) that there is a subset $I \subset [0,1]$, $E^F(I) = \mathbf{1}$, such that, for every $\lambda \in I$, $\omega^U_{(\cdot)}(\lambda)$ is σ-additive on a ring $\mathcal{R}(\mathcal{S})$ which generates $\mathcal{B}(X)$, the Borel σ-algebra of X. Then, transfinite induction is used in order to prove that $\omega^U_{(\cdot)}(\lambda)$ can be extended to $\mathcal{B}(X)$. See the comment added in the introduction to section 2.4 of the revised version of the paper.

1.4) We also remark that the paper adds some new interpretative result as well. In particular, section 3 is devoted to explain/describe the meaning of the universal Markov kernel in the framework of fuzzy set theory. In other words it translates the concept of “universality” in the language of fuzzy sets.

2) *The constructions leading to the transferring principle should be*

written in a clearer and more explicit way, so that the computations can be more easily understood.

We have written a new version of the construction which should be (we think) more clear and more explicit as suggested by the referee. We also illustrated the procedure in the case $n = 1, 2, 3$ in order to explicitly illustrate the construction.

3) *The proofs should be better written. For example at many places, the authors use subindices like r_l that are in fact treated as pairs of indices (as on p. 12, where some expressions with index r_l^k are summed over r and k, etc), this is quite confusing. There is also a number of typos.*

   The notation has been completely changed in order to avoid the uses of indexes of the kind r_l^k with summation over r and k. The proofs have been modified as well. First, they have been written in the new notation but they have also been simplified. For example the proof of the additivity of E^F on $\mathcal{I}$ has been greatly reduced and simplified. Moreover, it has been illustrated by means of a drawing in the case of dimension 2. The proof of the σ-additivity (see pages 16-17 of the revised version) has been changed and clarified as well. Several typos have been corrected.

References

[1] Beneduci, R.: *Infinite sequences of linear functionals, positive operator-valued measures and Naimark extension theorem*, Bull.

Lond. Math. Soc. **42** (2010), 441—451.

[2] Beneduci, R.: *Stochastic matrices and a property of the infinite sequences of linear functionals*, Linear Algebra and its Applications, **43**, (2010) 1224-1239.

[3] G. Cattaneo, G. Nisticò, From unsharp to sharp quantum observable: The general Hilbert space case, J. Math. Phys. 41 (2000) 4365.

[4] R. Beneduci, G. Nisticó, Sharp reconstruction of unsharp quantum observables, J. Math. Phys. 44 (2003) 5461.

[5] R. Beneduci, A geometrical characterizations of commutative positive operator valued measures, J. Math. Phys. 47 (2006) 062104.

[6] R. Beneduci, Universal Markov Kernels for Quantum Observables, Geometric Methods in Physics. XXXVI Workshop 2017, Trends in Mathematics, 21-29 (2018).

[7] R. Beneduci, Universal Randomization of Quantum Observables, Int. J. Theor. Phys., doi.org/10.1007/s10773-019-04090-y (2019).

Sincerely yours,

Roberto Beneduci

Dipartimento di Fisica

Universitá della Calabria

Via P. Bucci cubo 31-C

87036 Arcavacata di Rende (Cosenza) Italia

Office: 0039-0984496415

Mobile: 0039-3492671248

e-mail: roberto.beneduci@unical.it

Fuzzy observables and the universal family of fuzzy events

Roberto Beneduci[a,*], Tommaso Gentile[b]

[a] *Dipartimento di Fisica, Universitá della Calabria*
and
Istituto Nazionale di Fisica nucleare, gruppo c. Cosenza
via P. Bucci, cubo 31C, Rende (CS), Italia
[b] *Dipartimento di Matematica, Universitá della Calabria*
via P. Bucci, cubo 30B, Rende (CS), Italia

Abstract

We prove the existence of a universal family of fuzzy events $\mathcal{F} = \{(\mathbb{R}, \mu^U_\Delta)\}_{\Delta\in\mathcal{B}(X)}$ such that every fuzzy observable F is the fuzzy version of a sharp observable E^F with the fuzzification realized by $\mathcal{F}$. That is a consequence of the existence of a universal Markov kernel μ^U such that every commutative POVM F is the fuzzy version of a spectral measure E^F with the fuzzification realized by μ^U. We provide a general proof for POVMs defined on the σ-algebra of a Hausdorff, locally compact, second countable topological space which is based on a modified version of the transferring principle introduced by Lebesgue, Riesz, de la Vallée-Poussin and Jessen; both E^F and μ^U are obtained by means of a bijective map between the infinite dimensional cube and the unit interval.

Then, we show that every weak Markov kernel is functionally subordinated to μ^U; in some sense the randomization encoded in μ^U includes all the possible randomizations. Finally, we show that the universal family of fuzzy events corresponding to the universal Markov kernel includes all the possible fuzzy events of which a POVM can represent the probabilities. Moreover, the probabilities associated to a fuzzy observable F coincide with the probabilities of the universal family of fuzzy events with respect to E^F.

*Corresponding author
Email address: `roberto.beneduci@unical.it` (Roberto Beneduci)

Key words: Fuzzy Observables, Fuzzy sets, Positive Operator Valued Measures, Markov Kernels, Foundations of Quantum mechanics, Quantum Structures

2010 MSC: 81P15, 28Bxx, 47A60, 03E72

1. Introduction

In the present section we briefly recall the concept of fuzzy observable and its connections with fuzzy sets. As a consequence of the statistical nature of the measurement process [19], quantum observables are represented by positive operator valued measures [12, 15, 19, 27, 29, 32]. Spectral measures are particular examples of POVMs and, by the spectral theorem, they are in a one-to-one correspondence with self-adjoint operators. Therefore, the analysis of the measurement process reveals that the set of self-adjoint operators is too small in order to represent quantum observables. A very relevant feature of positive operator valued measures (POVMs) is that there are couples of non-commuting POVMs which are the marginals of a joint POVM (they are jointly measurable). We know that in the particular case of self-adjoint operators joint measurability and commutativity coincide [12, 25].

As a relevant example one can consider the spectral measures of the position and momentum operators, E^Q and E^P for which a joint spectral measure does not exist since they do not commute. Nevertheless, it is possible to randomize E^Q and E^F by means of Markov kernels μ^Q and μ^P. That provides two POVMs F^Q and F^P for which a joint POVM does exist [6, 9, 15, 29]. In particular, for every Δ in the Borel σ-algebra of the reals, $\mathcal{B}(\mathbb{R})$,

$$F^Q(\Delta) = \int_{\mathbb{R}} \mu^Q_\Delta(\lambda)\, dE^Q_\lambda, \qquad (1)$$
$$F^P(\Delta) = \int_{\mathbb{R}} \mu^P_\Delta(\lambda)\, dE^P_\lambda.$$

The POVMs F^Q and F^P are called the unsharp or fuzzy version of E^Q and E^P respectively [2, 17, 29] and are commutative since they are contained in the commutative von Neumann algebras generated by E^Q and E^P respectively. It is relevant that there is a third POVM, F, of which F^Q and F^P are the marginals [6, 15, 29, 32],

$$F(\Delta_q \times \mathbb{R}) = F^Q(\Delta_q)$$
$$F(\mathbb{R} \times \Delta_p) = F^P(\Delta_p).$$

That allows a representation of quantum mechanics on a phase space which should be interpreted as a stochastic phase space whose points are fuzzy points [29].

There are other examples of the power of POVMs as the mathematical representative of quantum observables [1, 2, 12, 15, 19, 27, 29, 32].

The previous example provides some insight about the relevance of commutative POVMs to quantum physics. We add that they model certain standard forms of noise in quantum measurements and provide optimal approximators as marginals in joint measurements of incompatible observables (e.g., Position and Momentum) [13].

In (1) we obtained the fuzzy position and momentum POVMs (that are commutative POVMs) as the randomization of the sharp position and momentum operators with the randomization realized through Markov kernels. That is a general proprety of commutative POVMs, i.e., every commutative POVM $F : \mathcal{B}(X) \to \mathcal{L}_s^+(\mathcal{H})$ from the σ-algebra of a topological space X to the space of linear positive self-adjoint operators on a Hilbert space $\mathcal{H}$ is the random version of a spectral measure E^F (the sharp version of F); the randomization being represented by a Markov kernel μ [8, 10, 16, 20, 21],

$$\langle \psi, F(\Delta)\psi \rangle := \int \mu_\Delta(\lambda)\, d\langle \psi, E_\lambda \psi \rangle, \quad \Delta \in \mathcal{B}(X), \quad \psi \in \mathcal{H}. \tag{2}$$

Consider for example the unsharp position observable in equation (1). It can be represented as follows [15],

$$\langle\psi, F^Q(\Delta)\psi\rangle := \int_{\mathbb{R}} \mu_\Delta(\lambda)\, d\langle\psi, E^Q_\lambda \psi\rangle, \quad \Delta\in\mathcal{B}(\mathbb{R}), \quad \psi\in L^2(\mathbb{R}), \tag{3}$$
$$\mu_\Delta(\lambda) := \int_{\mathbb{R}} \chi_\Delta(\lambda - y)\, f(y)\, dy, \quad \lambda\in\mathbb{R}$$

where, f is a positive, bounded, Borel function such that $\int_{\mathbb{R}} f(y)dy = 1$, while E^Q is the spectral measure corresponding to the position operator

$$Q : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$
$$(Q\psi)(x) := x\psi(x), \quad a.a. \quad x\in\mathbb{R}.$$

The quantity $\langle\psi, E^Q(\Delta)\psi\rangle$ can be interpreted as the probability that a perfectly accurate measurement (sharp measurement) of the position gives a result in Δ. A possible interpretation of equation (3) is that, due to measurement imprecision[1], the outcomes of the measurement of E^Q are randomized: if the sharp value of the outcome of the measurement of E^Q is λ then the apparatus produces with probability $\mu_\Delta(\lambda)$ a reading in Δ. As a result, the probability of an outcome in Δ is given by $\langle\psi, F^Q(\Delta)\psi\rangle$ so that F^Q represents an unsharp measurement of E^Q.

An interpretation of equation (2) in the framework of fuzzy sets theory [34, 35] has been suggested in Ref. [17]. A fuzzy set is a pair $A = (\mathbb{R}, \mu_A)$ where $\mu_A : \mathbb{R} \to [0, 1]$ is a membership function. The value $\mu_A(x)$ is interpreted as the membership degree of x in A. A fuzzy event is a fuzzy set such that the membership function μ_A is Borel measurable. If ν is a probability measure on $\mathbb{R}$, the probability of a fuzzy event with respect to ν is defined as

$$P(A) = \int \mu_A(x)\, d\nu(x).$$

Going back to equation (3), we have the following interpretation in terms of fuzzy sets: the Markov kernel μ provides a family of fuzzy events $\{(\mathbb{R}, \mu_\Delta)\}_{\Delta\in\mathcal{B}(\mathbb{R})}$.

[1]Which can be thought to be intrinsic to the quantum measurement process and then to be unavoidable

For every point $x \in \mathbb{R}$, the family of membership functions $\{\mu_\Delta\}_{\Delta\in\mathcal{B}(\mathbb{R})}$ defines a probability measure. That is a very peculiar situation with respect to the general definition of fuzzy events. For every $\psi \in \mathcal{H}$, the expression

$$\langle\psi, F^Q(\Delta)\psi\rangle = \int_{\mathbb{R}} \mu_\Delta(x)\, d\langle\psi, E^Q_x\psi\rangle$$

can then be interpreted as the probability of the fuzzy event $(\mathbb{R}, \mu_\Delta)$ with respect to the probability measure $\langle\psi, E^Q(\cdot)\psi\rangle$. Such a probability coincides with the probability of the event Δ (which is not a fuzzy event) with respect to the probability $\langle\psi, F^Q(\cdot)\psi\rangle$. In other words, the unsharp observable F^Q gives the probabilities of the fuzzy events $(\mathbb{R}, \mu_\Delta)$ with respect to the probability measures corresponding to E^Q.

The present paper focuses on the analysis of the fuzzification process that connects E and F. In particular, we prove that there is a universal Markov kernel μ^U that connects every commutative POVM F (and then every fuzzy observable) to its sharp version E^F (we just require the POVM to be defined on a Hausdorff, locally compact topological space whose topology is countably generated) and that every Markov kernel is functionally subordinated to the universal Markov kernel. That generalizes some previous results [4, 6] where the existence of a universal Markov kernel is proved in the case $X = \mathbb{R}$. Moreover, we provide a general procedure for the construction of both the sharp version and the universal Markov kernel μ^U which is based on a modified version of Jessen's transfering principle [22]. The transferring principle was introduced by Lebesgue [26], Riesz [30] and de la Vallée Poussin [33] in the n-dimensional case. Later Jessen generalized the transferring principle in order to define integration of functions with a countable number of variables [22].

Concerning the possible interpretations of the universal Markov kernel we have that, if we interpret a Markov kernel as a measure of the randomization due to the measurement imprecision, than the existence of the universal Markov kernel means that there is a randomization which includes all the others (see sections 2.4). On the other hand, if we interpret a Markov kernel as a family of fuzzy events, then the existence of the universal Markov kernel μ^U means that every POVM F gives the probabilities of the fuzzy events, $\{(\mathbb{R}, \mu^U_\Delta)\}_{\Delta\in\mathcal{B}(X)}$, with respect to E^F and that $\{(\mathbb{R}, \mu^U_\Delta)\}_{\Delta\in\mathcal{B}(X)}$ can be interpreted as a universal family of fuzzy events which includes (up to functional subordination) all the possible fuzzy events of which a POVM can represent the probabilities (see section 3).

Now, we briefly recall the main definitions and properties of POVMs. In what follows, $\mathcal{B}(X)$ denotes the Borel σ-algebra of a topological space X and $\mathcal{L}_s^+(\mathcal{H})$ the space of all bounded positive self-adjoint linear operators acting in a Hilbert space $\mathcal{H}$ with scalar product $\langle\cdot,\cdot\rangle$.

Definition 1. A Positive Operator Valued measure (for short, POVM) is a map $F : \mathcal{B}(X) \to \mathcal{L}_s^+(\mathcal{H})$ such that:

$$F\big(\bigcup_{n=1}^{\infty}\Delta_n\big) = \sum_{n=1}^{\infty} F(\Delta_n).$$

where, $\{\Delta_n\}$ is a countable family of disjoint sets in $\mathcal{B}(X)$ and the series converges in the weak operator topology. It is said to be normalized if

$$F(X) = \mathbf{1}$$

where $\mathbf{1}$ is the identity operator.

Definition 2. A POVM is said to be commutative if

$$\big[F(\Delta_1), F(\Delta_2)\big] = \mathbf{0}, \quad \forall\,\Delta_1\,,\Delta_2 \in \mathcal{B}(X). \tag{4}$$

Definition 3. A POVM is said to be orthogonal if $\Delta_1 \cap \Delta_2 = \emptyset$ implies

$$F(\Delta_1)F(\Delta_2) = \mathbf{0} \tag{5}$$

where $\mathbf{0}$ is the null operator. An orthogonal POVM is called projection valued measure (for short, PVM).

Note that if F is an orthogonal POVM, the operators $F(\Delta)$ are projection operators.

Definition 4. A Spectral measure is a real, normalized PVM.

In quantum mechanics, non-orthogonal normalized POVMs represent **generalised** or **unsharp** or **fuzzy** observables while PVMs represent **standard** or **sharp** observables.

We recall that $\langle \psi, F(\Delta)\psi \rangle$ is interpreted as the probability that a measurement of the observable represented by F gives a result in Δ.

The following theorem gives a characterization of commutative POVMs as the randomization of spectral measures with the randomization realized by means of Markov kernels.

Definition 5. Let Λ be a topological space. A Markov kernel is a map $\mu : \Lambda \times \mathcal{B}(X) \to [0,1]$ such that,

1. $\mu_\Delta(\cdot)$ is a measurable function for each $\Delta \in \mathcal{B}(X)$,
2. $\mu_{(\cdot)}(\lambda)$ is a probability measure for each $\lambda \in \Lambda$.

Definition 6. Let ν be a measure on Λ. A map $\mu : (\Lambda, \nu) \times \mathcal{B}(X) \to [0,1]$ is a weak Markov kernel with respect to the measure ν if:

1. $\mu_\Delta(\cdot)$ is a measurable function for each $\Delta \in \mathcal{B}(X)$,
2. for every $\Delta \in \mathcal{B}(X)$, $0 \leq \mu_\Delta(\lambda) \leq 1, \quad \nu - a.e.$,
3. $\mu_\emptyset(\lambda) = 0, \mu_\Lambda(\lambda) = I \quad \nu - a.e.$,
4. for any sequence $\{\Delta_i\}_{i\in\mathbb{N}}$, $\Delta_i \cap \Delta_j = \emptyset$,

$$\sum_i \mu_{(\Delta_i)}(\lambda) = \mu_{(\cup_i \Delta_i)}(\lambda), \quad \nu - a.e.$$

In the following the symbol $\mathcal{A}(F)$ denotes the von Neumann algebra generated by the POVM F, i.e., the von Neumann algebra generated by the set $\{F(\Delta)\}_{\mathcal{B}(X)}$. Analogously $\mathcal{A}(B)$ denotes the von Neumann algebra generated by the self-adjoint operator B. Hereafter, we assume that X is a Hausdorff, locally compact, second countable topological space. The connection between commutative POVMs and randomization of spectral measures has been pointed out by several authors [7, 8, 10, 14, 16, 20, 21]. One of the possible formulation of their equivalence is provided in the following theorem.

Theorem 1.1 ([8, 10])**.** *A POVM $F : \mathcal{B}(X) \to \mathcal{L}_s^+(\mathcal{H})$ is commutative if and only if there exists a bounded self-adjoint operator $A = \int \lambda \, dE_\lambda$ with spectrum $\sigma(A) \subset [0,1]$, and a Markov Kernel $\mu : \sigma(A) \times \mathcal{B}(X) \to [0,1]$ such that*

1) $F(\Delta) = \int_{\sigma(A)} \mu_\Delta(\lambda) \, dE_\lambda, \quad \Delta \in \mathcal{B}(X)$.

2) $\mathcal{A}(F) = \mathcal{A}(A)$.

3) *there are a ring $\mathcal{R}(\mathcal{S}) \subset \mathcal{B}(X)$ and a set $\Gamma \subset \sigma(A)$, $E(\Gamma) = \mathbf{1}$, such that $\mu_\Delta(\lambda)$ is continuous for every $\Delta \in \mathcal{R}(\mathcal{S})$ and $\lambda \in \Gamma$. In particular, $\mu : \Gamma \times \mathcal{B}(X) \to [0,1]$ is a Feller Markov kernel.*

The operator A (or equivalently, the spectral measure E) introduced in theorem 1.1 is called the **sharp version** of F and is unique up to almost everywhere bijections [7, 8]. The POVM F is said to be a smearing of the spectral measure E or equivalently a smearing of A. It can be interpreted as a fuzzy version of E [10].

Other equivalent versions of these result exist [3, 5, 18, 21]. In [21] and [5] their equivalence has been proved.

2. Universal Markov kernel

In the present section we prove the existence of a universal Markov kernel for commutative POVMs (see subsection 2.4). That generalizes the result in [4, 6] where the existence of a universal Markov kernel has been proved for real commutative POVMs.

The proof is based on a modified version of the algorithm developed by Lebesgue, Riesz, Jessen and Sz-Nagy [22, 26, 28, 31] that they used in order to prove the transferring principle. In particular it was used by Jessen in order to introduce integration in infinite dimensional spaces. Later, the algorithm was used by Sz-Nagy in order to prove a theorem previously proved by von

Neumann, i.e., that to an arbitrary family $\{A_i\}_{i\in I}$ of bounded commuting self-adjoint operators there corresponds a self-adjoint operator A and a family of measurable functions $\{f_i\}_{i\in I}$ such that $A_i = f_i(A)$, for all $i \in I$ (see section 130 in Ref. [30]). A similar algorithm has been used in Ref. [14] in order to show that every real commutative POVM is the smearing of a spectral measure (see also [7]).

In particular, Jessen [22] proposed a quite general procedure to define a bijective function from a subset of the infinite dimensional torus to a subset of the unit interval (transferring principle). Here we modify Jessen's algorithm in order to obtain a bijective function $f : [0,1]^\omega \to [0,1]$ from the infinite dimentional cube to the unit interval and make use of such a function in order to define the sharp version of a commutative POVM $F : \mathcal{B}(X) \to \mathcal{L}_s(\mathcal{H})$ where X is a second countable topological space X. That generalizes some previous results, uses explicitly and in its full generality the transferring principle and provides a more powerful and compact formulation based on the bijective function $f : [0,1]^\omega \to [0,1]$. Then, we proceed to prove the existence of the universal Markov kernel on a Hausdorff, locally compact, second countable space.

2.1. Transferring Principle

We diverge from Jessen's construction in order to obtain a one-to-one function between $[0,\,1]^\omega$ and $[0,\,1]$. In particular, we use left-closed subintervals and provide two related nets of dissections of $[0,\,1]^\omega$ and $[0,\,1]$ in the following recursive way.

Set $D_1 := \{[0,\,1]^\omega\}$ and $d_1 := \{[0,\,1]\}$.

Suppose now that the dissections D_n and d_n are given. We write for them $D_n = \{I_1^{(n)}, \dots, I_{k_n}^{(n)}\}$ and $d_n = \{i_1^{(n)}, \dots, i_{h_n}^{(n)}\}$, where $I_l^{(n)} = b_{l,\,1}^{(n)} \times b_{l,\,2}^{(n)} \times \cdots \times b_{l,\,\lambda(l,\,n)}^{(n)} \times [0,\,1]^\omega$. We assume that the following conditions hold:

1. each set $b_{l,\,j}^{(n)}$ is a left-closed interval with the exception of those sets with right endpoint equal to 1;

2. $b_{l,\,\lambda(l,\,n)-1}^{(n)} \neq [0,\,1]$;

3. $b^{(n)}_{l,\,\lambda(l,\,n)} = [0,\,1]$.

At step $n+1$ the dissections are defined as follows. The dissection D_{n+1} is obtained by subdividing each $b^{j}_{l,\,j}$ into 2 subintervals of the same size and the dissection d_{n+1} is obtained by subdividing each interval $i^{(n)}_l$ into 2^n subintervals of the same size.

In order to clarify our construction we will give explicitly the first terms of the two nets. We write $D_1 = \{[0,\,1] \times [0,\,1]^\omega\}$, so

$$\begin{aligned} D_2 &= \left\{ \left[0,\,\frac{1}{2}\right) \times [0,\,1]^\omega\,,\ \left[\frac{1}{2},\,1\right] \times [0,\,1]^\omega \right\} = \\ &= \left\{ \left[0,\,\frac{1}{2}\right) \times [0,\,1] \times [0,\,1]^\omega\,,\ \left[\frac{1}{2},\,1\right] \times [0,\,1] \times [0,\,1]^\omega \right\} \end{aligned}$$

and

$$d_2 = \left\{ \left[0,\,\frac{1}{2}\right)\,,\ \left[\frac{1}{2},\,1\right] \right\}$$

Thus we have $k_2 = h_2 = 2$, $\lambda(l,\,2) = 2$ for each l, $b^{(2)}_{1,\,1} = \left[0,\,\frac{1}{2}\right)$, $b^{(2)}_{1,\,2} = [0,\,1]$, $b^{(2)}_{2,\,1} = \left[\frac{1}{2},\,1\right]$ and $b^{(2)}_{2,\,2} = [0,\,1]$.

The dissection D_3 is the following.

$$\begin{aligned} D_3 = \Bigg\{ &\left[0,\,\frac{1}{4}\right) \times \left[0,\,\frac{1}{2}\right) \times [0,\,1]^\omega,\ \left[0,\,\frac{1}{4}\right) \times \left[\frac{1}{2},\,1\right] \times [0,\,1]^\omega, \\ &\left[\frac{1}{4},\,\frac{1}{2}\right) \times \left[0,\,\frac{1}{2}\right) \times [0,\,1]^\omega,\ \left[\frac{1}{4},\,\frac{1}{2}\right) \times \left[\frac{1}{2},\,1\right] \times [0,\,1]^\omega, \\ &\left[\frac{1}{2},\,\frac{3}{4}\right) \times \left[0,\,\frac{1}{2}\right) \times [0,\,1]^\omega,\ \left[\frac{1}{2},\,\frac{3}{4}\right) \times \left[\frac{1}{2},\,1\right] \times [0,\,1]^\omega, \\ &\left[\frac{3}{4},\,1\right] \times \left[0,\,\frac{1}{2}\right) \times [0,\,1]^\omega,\ \left[\frac{3}{4},\,1\right] \times \left[\frac{1}{2},\,1\right] \times [0,\,1]^\omega \Bigg\} \end{aligned}$$

The corresponding dissection d_3 is the following

$$d_3 = \left\{ \left[0,\,\frac{1}{8}\right),\ \left[\frac{1}{8},\,\frac{1}{4}\right),\ \left[\frac{1}{4},\,\frac{3}{8}\right),\ \left[\frac{3}{8},\,\frac{1}{2}\right),\ \left[\frac{1}{2},\,\frac{5}{8}\right),\ \left[\frac{5}{8},\,\frac{3}{4}\right),\ \left[\frac{3}{4},\,\frac{7}{8}\right),\ \left[\frac{7}{8},\,1\right] \right\},$$

so $k_3 = h_3 = 8$, $\lambda(l,\,3) = 3$ for each l. Moreover notice that each $I^{(3)}_1$ has the same Lebesgue measure, equal to $\frac{1}{8}$, and the same holds for each $i^{(3)}_1$.

It is easy to prove that, for each $n \in \mathbb{N}$ and for each $l = 1,\,2,\,\ldots,\,k_n$, $k_n = h_n = 2^{\frac{n(n-1)}{2}}$, $\lambda(l,\,n) = n$ and the intervals $I^{(n)}_l$ and $i^{(n)}_l$ have the same Lebesgue measure.

In D_n we can order the intervals $I_l^{(n)}$ according to the lexicographical order in $[0, 1]^\omega$ and, at the same time, the intervals $i_l^{(n)}$ are taken according to the total order in $[0, 1]$. With such a choice we have the following useful property, that can be easily proved by using an induction. Let $x_l^{(k)}$ be the point in $[0, 1]^\omega$ whose first k coordinates are the left endpoints of the intervals $b_{l,j}^{(k)}$ and the others are equal to 0, and let m be a non negative integer. Then it holds that

$$x_l^{(k)} = x^{(k+m)}_{2^{mk+\binom{m}{2}}l+1}, \forall m \in \mathbb{N}$$

More in general the two sequences of dissections satisfy the following property. Let $\{I_n\}$ be a sequence of subintervals of $[0, 1]^\omega$ such that $I_n \in D_n$ and $I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$. Then the corresponding intervals i_n are such that $i_1 \supset i_2 \supset \cdots \supset i_n \supset \cdots$ and the cardinality of their intersections are

$$\left|\bigcap_n i_n\right| = \left|\bigcap_n I_n\right| \leq 1.$$

Nevertheless, for every point $x \in [0, 1]^\omega$, there is exactly one sequence of nested intervals $\{I_n\}$, $I_n \in D_n$, whose intersection is $\{x\}$. Analogously, every point in $[0, 1]$ is the intersection of a unique sequence of nested intervals, $\{i_n\}$, $i_n \in d_n$. Notice that to the point $x_l^{(k)}$ correspond the sequence $\{I_n\}$ such that, for each $m \in \mathbb{N}$, $I_{k+m} = I^{(k+m)}_{2^{mk+\binom{m}{2}}l+1}$.

Now, let $\{I_n\}$, $I_n \in D_n$, be the descending chain of intervals corresponding to the point $x \in [0, 1]^\omega$. Let $\{i_n\}$, $i_n \in d_n$ be the corresponding chain of descending intervals contained in $[0, 1]$. The previous property ensures the existence of a point $y_x \in [0, 1]$ such that $y_x \in i_n$, for all $n \in \mathbb{N}$. We then have the following theorem which slightly modify the transferring principle introduced by Jessen (see [22] for more details).

Theorem 2.1 (Transferring Principle)**.** *There exists a one-to-one function*

$$f : [0, 1]^\omega \to [0, 1],$$

with the property that corresponding sets have always the same Lebesgue measure. Moreover, trivially, $f^{-1}(i_l^{(n)}) = I_l^{(n)}$.

At variance with Jessen's construction we have used half-open intervals and added more constraints in the definition of the dissections. That ensured the bijectivity of the function $f : [0,\, 1]^\omega \to [0,\, 1]$ that in Jessen's version is bijectvie on a subset of $[0,\, 1]^\omega$.

2.2. The Sharp Version

Now, we can use the transferring principle in order to define the spectral measure (sharp version) E^F associated to a commutative POVM F. Here we diverge from Ref. [14] since we do not limit ourselves to introduce, for every n, a map between D_n and d_n but we use the transferring principle in its full generality and meaning, i.e., we show that there is a function $f : [0,1]^\omega \to [0,1]$ which epitomizes the link between E^F and F. Moreover, we extend the results in [14] to the case of a Hausdorff, locally compact, second countable space. Let X be a second countable space and $\mathcal{S}$ a countable basis for the topology of X (we suppose $\emptyset, X \in \mathcal{S}$). Let $\mathcal{R}(\mathcal{S})$ be the ring generated by $\mathcal{S}$ and $\mathcal{B}(X)$ the Borel σ-algebra generated by $\mathcal{R}(\mathcal{S})$.

Let $\{\Delta_n\}_{n\in\mathbb{N}}$ be an enumeration of $\mathcal{R}(\mathcal{S})$. Let F be a commutative POVM on $\mathcal{B}(X)$ and E_n the spectral resolution of $F(\Delta_n)$. Let $f_j^{-1}(x)$ be the j-th coordinate of $f^{-1}(x)$, $x \in [0,1]$. In particular $f_j^{-1}(i_l^{(n)}) = b_{l,\,j}^{(n)}$ where $b_{l,\,j}^{(n)}$ is the j-th edge of the subinterval $I_l^{(n)} = b_{l,\,1}^{(n)} \times b_{l\,2}^{(n)} \times \cdots \times b_{l,\,n}^{(n)}$.

We can define a projection valued map on the set of subintervals of $[0,\, 1]$ as follows. Note that $\mathcal{I} := \cup_{n=1}^{\infty} d_n$ is a semiring. Then, thanks to the commutativity of the POVM F,

$$E_0^F : \mathcal{I} \to \mathcal{L}_s^+(\mathcal{H})$$
$$E_0^F(i_l^{(n)}) := \prod_{j=1}^{n} E_j[f_j^{-1}(i_l^{(n)})] = \prod_{j=1}^{n} E_j(b_{l,\,j}^{(n)}) \tag{6}$$

defines a projection valued map. It is straightforward to show that $E_0^F(i_1^{(n)})E_0^F(i_2^{(n)}) = \mathbf{0}$ if $l \neq j$ and that $E_0^F([0,\, 1]) = \mathbf{1}$ (concerning this last property, note that $f_j^{-1}([0,\, 1]) = [0,\, 1]$).

Moreover, E_0^F is additive on $\mathcal{I}$. Indeed, assume $i_l^{(n)} = \bigsqcup_{j=1}^m i_{r_j}^{(k_j)}$, with $k_j > n$, for all j. This case can be reduced to one in which the intervals are all in the same dissection as follows. Let $k = max\{k_j\}$. Then, each set $i_{r_j}^{(k_j)}$ can be decomposed as the disjoint union of sets $i_p^{(k)}$ from D_k and the additivity of E^F on $i_p^{(k)}$ implies the additivity on the family of sets $\{i_{r_j}^{(k_j)}\}_j$. Finally, by using induction on $k-n$ one can prove that the additivity in the case $i_l^{(n)} = \bigsqcup_{j=1} i_j^{(k)}$ is equivalent to the additivity in the case $k-n=1$. Let us prove the additivity in this last case. Let $I_l^{(n)} = b_{l,1}^{(n)} \times b_{l,2}^{(n)} \times \cdots \times b_{l,n}^{(n)} \times [0,1]^\omega$. Then

$$i_l^{(n)} = \bigsqcup_{s=2^n(l-1)+1}^{l\,2^n} i_s^{(n+1)}$$

For each s, we write, as usual,

$$I_s^{(n+1)} = b_{s,1}^{(n+1)} \times b_{s,2}^{(n+1)} \times \cdots \times b_{s,n+1}^{(n+1)} \times [0,1]^\omega$$

Moreover we notice that

$$b_{l,j}^{(n)} = b_{2^n(l-1)+1,\,j}^{(n+1)} \sqcup b_{2^n\,l,\,j}^{(n+1)} = b_{2^n(l-1)+1,\,j}^{(n+1)} \sqcup b_{2^n(l-1)+2^{n-j+1},\,j}^{(n+1)}, \ \forall j.$$

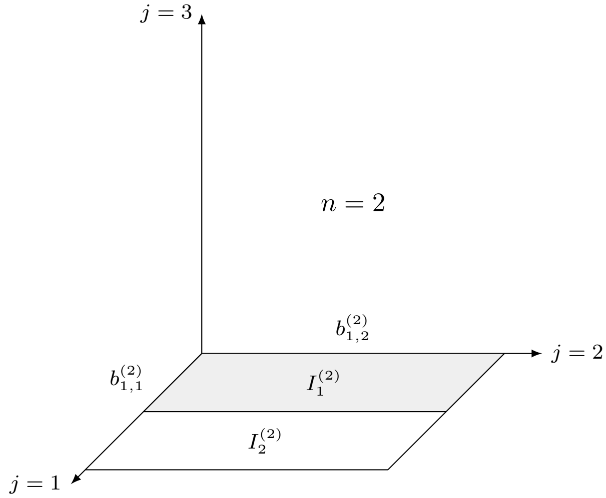
Thus, by definition of E_0^F,

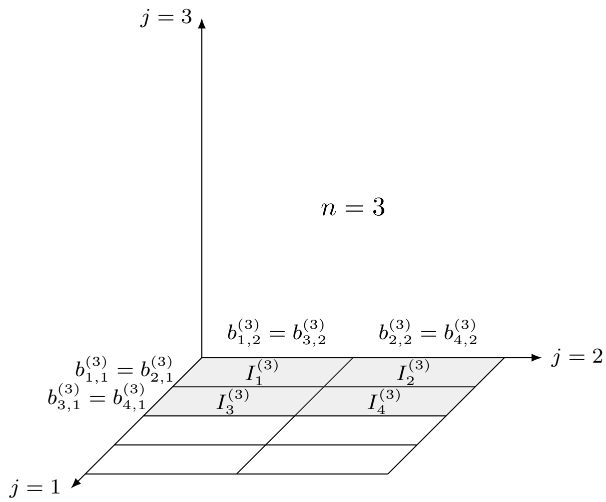
$$E_0^F(i_l^{(n)}) = \prod_{j=1}^n E_j(b_{l,j}^{(n)})$$

and

$$E_0^F(i_s^{(n+1)}) = \prod_{k=1}^{n+1} E_k(b_{s,k}^{(n+1)})$$

For example, in the case $n=2$ we have,





$$I_1^{(2)} = I_1^{(3)} + I_2^{(3)} + I_3^{(3)} + I_4^{(3)}$$

and

$$b^{(2)}_{1,1} = b^{(3)}_{1,1} + b^{(3)}_{4,1}, \;\; b^{(2)}_{1,2} = b^{(3)}_{1,2} + b^{(3)}_{2,2} = b^{(3)}_{3,2} + b^{(3)}_{4,2}$$

$$b^{(3)}_{1,1} = b^{(3)}_{2,1}, \;\; b^{(3)}_{3,1} = b^{(3)}_{4,1}, \;\; b^{(3)}_{3,2} = b^{(3)}_{1,2}$$

so that,

$$
\begin{aligned}
&E_1(b^{(3)}_{1,1})E_2(b^{(3)}_{1,2}) + E_1(b^{(3)}_{2,1})E_2(b^{(3)}_{2,2}) + E_1(b^{(3)}_{3,1})E_2(b^{(3)}_{3,2}) + E_1(b^{(3)}_{4,1})E_2(b^{(3)}_{4,2}) \\
&= E_1(b^{(3)}_{1,1})[E_2(b^{(3)}_{1,2}) + E_2(b^{(3)}_{2,2})] + E_1(b^{(3)}_{4,1})[E_2(b^{(3)}_{3,2}) + E_2(b^{(3)}_{4,2})] \\
&= E_1(b^{(3)}_{1,1})E_2(b^{(2)}_{1,2}) + E_1(b^{(3)}_{4,1})E_2(b^{(2)}_{1,2}) \\
&= [E_1(b^{(3)}_{1,1}) + E_1(b^{(3)}_{4,1})]E_2(b^{(2)}_{1,2}) = E_1(b^{(2)}_{1,1})E_2(b^{(2)}_{1,2}).
\end{aligned}
$$

In the general case,

$$
\begin{aligned}
\sum_{s=2^n(l-1)+1}^{2^n\, l} E_0^F(i_s^{(n+1)}) &= \sum_{s=2^n(l-1)+1}^{2^n\, l} \prod_{k=1}^{n+1} E_k(b^{(n+1)}_{s,k}) \\
&= \sum_{s=2^n(l-1)+1}^{2^n\, l} \prod_{k=1}^{n} E_k(b^{(n+1)}_{s,k}) \\
&= \left[E_1(b^{(n+1)}_{2^n(l-1)+1,1}) + E_1(b^{(n+1)}_{2^n\, l,1})\right] \sum_{s=2^n(l-1)+1}^{2^n(l-1)+2^{n-1}} \prod_{k=2}^{n} E_k(b^{(n+1)}_{s,k}) \\
&= E_1(b^{(n)}_{l,1}) \sum_{s=2^n(l-1)+1}^{2^n(l-1)+2^{n-1}} \prod_{k=2}^{n} E_k(b^{(n+1)}_{s,k}) \\
&= E_1(b^{(n)}_{l,1}) \left[E_2(b_{2^n(l-1)+1,2}) + E_2(b_{2^n(l-1)+2^{n-1},2})\right] \sum_{s=2^n(l-1)+1}^{2^n(l-1)+2^{n-2}} \prod_{k=3}^{n} E_k(b^{(n+1)}_{s,k}) \\
&= E_1(b^{(n)}_{l,1})E_2(b^{(n)}_{l,2}) \sum_{s=2^n(l-1)+1}^{2^n(l-1)+2^{n-2}} \prod_{k=3}^{n} E_k(b^{(n+1)}_{s,k}) \\
&= \cdots \\
&= E_1(b^{(n)}_{l,1})E_2(b^{(n)}_{l,2}) \cdots E_{n-1}(b^{(n)}_{l,n-1}) \sum_{s=2^n(l-1)+1}^{2^n(s-1)+1} \prod_{k=n}^{n} E_k(b^{(n+1)}_{s,k}) \\
&= E_1(b^{(n)}_{l,1})E_2(b^{(n)}_{l,2}) \cdots E_n(b^{(n)}_{l,n}) = E_0^F(i_l^{(n)})
\end{aligned}
\tag{7}
$$

Now, we can define a spectral measure as follows. Let $\lambda \in [0,1]$. For every

$n \in \mathbb{N}$, let $C_\lambda^n := \{l \,|\, i_l^{(n)} \subset [0,\lambda)\}$ and define

$$\widetilde{E}_0^F(\lambda) := \lim_{n\to\infty} \sum_{l \in C_\lambda^n} E_0^F(i_l^{(n)}). \tag{8}$$

Note that, for every $\lambda \in [0,1]$, $\widetilde{E}_0^F(\lambda)$ is a projection operator. Moreover, $\widetilde{E}_0^F(0) = \mathbf{0}$, $\widetilde{E}_0^F(1) = \mathbf{1}$ and $\widetilde{E}_0^F(\lambda_1) \le \widetilde{E}_0^F(\lambda_2)$ whenever $\lambda_1 < \lambda_2$. We also define $\widetilde{E}_0^F(\lambda) = \mathbf{0}$, $\lambda < 0$ and $\widetilde{E}_0^F(\lambda) = \mathbf{1}$, $\lambda > 1$. Note also that, by the additivity of E_0^F, $\widetilde{E}_0^F(\lambda) = \sum_{l\in C_\lambda^n} E_0^F(i_l^{(n)})$ if λ is the right extreme of an interval in d_n.

Now, let λ be the right extreme of an interval in d_j and, for every $n > j$, let β_n be the right extreme of an interval in d_n such that there is an index l_n for which $[0,\beta_n) \cup i_{l_n}^{(n)} = [0,\lambda)$. Then, $\widetilde{E}_0^F(\beta_n) + E_0^F(i_{l_n}^{(n)}) = \widetilde{E}_0^F(\lambda)$. Moreover, $\lim_{n\to\infty} E_0^F(i_{l_n}^{(n)}) = \lim_{n\to\infty} \prod_{k=1}^n E_k[f_k^{-1}(i_{l_n}^{(n)})] = \mathbf{0}$ since $i_{l_n}^{(n)} \downarrow \emptyset$ and, by construction, $f_j^{-1}(i_{l_n}^{(n)}) \downarrow \emptyset$ for every $j \in \mathbb{N}$. Therefore,

$$\widetilde{E}_0^F(\lambda) = \lim_{n\to\infty} \widetilde{E}_0^F(\beta_n). \tag{9}$$

Note also that if $i_l^{(r)} = [\lambda, \lambda')$ then,

$$\widetilde{E}_0^F(\lambda') - \widetilde{E}_0^F(\lambda) = E_0^F(i_l^{(r)}). \tag{10}$$

Now, the family of projections $\{E_\lambda^F\}$,

$$E_\lambda^F := \lim_{\beta\to\lambda^-} \widetilde{E}_0^F(\beta), \quad \lambda \in \mathbb{R}, \tag{11}$$

defines a spectral family such that

$$E^F(i_l^{(r)}) = E_{\lambda'}^F - E_\lambda^F = E_0^F(i_l^{(r)}), \qquad i_l^{(r)} = [\lambda, \lambda'). \tag{12}$$

In order to prove (12), for every $n > r$, let β_n be the right extreme of an interval in d_n such that there is an index l_n for which $[0,\beta_n) \cup i_{l_n}^{(n)} = [0,\lambda)$ and γ_n the right extreme of an interval in d_n such that there is an index j_n for which $[0,\gamma_n) \cup i_{j_n}^{(n)} = [0,\lambda')$. Then, by (9) and (10),

$$E^F_{\lambda'} - E^F_{\lambda} = \lim_{n\to\infty} \widetilde{E}^F_0(\gamma_n) - \lim_{n\to\infty} \widetilde{E}^F_0(\beta_n) = \widetilde{E}^F_0(\lambda') - \widetilde{E}^F_0(\lambda) = E^F_0(i^{(n)}_l).$$

To the spectral family E^F_λ, there corresponds a spectral measure $E^F : \mathcal{B}([0,1]) \to \mathcal{L}^+_s(\mathcal{H})$. Indeed, $E^F([\lambda,\lambda') := E^F_{\lambda'} - E^F_\lambda$ is σ-additive on $\mathcal{I}$. Then it can be extended to the ring $\mathcal{R}(\mathcal{I})$ and then to $\mathcal{B}([0,1])$ (see theorem 7 in Ref. [11]).

By (6), (8) and (11), for every $B \in \mathcal{R}(\mathcal{I})$, $E^F(B) \in \mathcal{A}^W(F)$ where $\mathcal{A}^W(F)$ denotes the von Neumann algebra generated by $\{F(\Delta)\}_{\Delta\in\mathcal{B}(X)}$. Actually we can prove the following lemma.

Lemma 2.1. $\mathcal{A}^W(E^F) \subset \mathcal{A}^W(F)$.

Proof. We have, $\mathcal{A}^W\big(\{E^F(\Delta)\}_{\Delta\in\mathcal{R}(\mathcal{I})}\big) \subset \mathcal{A}^W(F)$. Moreover, $\mathcal{I}$ is a basis for the topology of $[0,1]$ and by proposition 3.1 in the appendix, $\mathcal{A}^W\big(\{E^F(\Delta)\}_{\mathcal{R}(\mathcal{I})}\big) = \mathcal{A}^W(E^F)$. □

2.3. The kernel

In order to define the universal Markov kernel we first define a family of functions $\omega^U_{\Delta_j}$, $\Delta_j \in \mathcal{R}(\mathcal{S})$, whose integral with respect to E^F gives $F(\Delta_j)$. For each $j \in \{1, 2, \dots, n\}$, let $\nu_j(i^{(n)}_l)) = \sup f_j^{-1}(i^{(n)}_l)$, and $\chi_{i^{(n)}_l}(\lambda)$ the characteristic function of the interval $i^{(n)}_l$. The sequence of non-increasing functions

$$\omega^{(n)}_{\Delta_j}(\lambda) := \sum_{l=1}^{2^{\binom{n}{2}}} \nu_j(i^{(n)}_l)\, \chi_{i^{(n)}_l}(\lambda) \geq 0$$

converges uniformly to a function $\omega^U_{\Delta_j}(\lambda) := \lim_{n\to\infty} \omega^{(n)}_{\Delta_j}(\lambda)$. Let $B^n_{k,j} := \{l \,|\, \nu_j(i^{(n)}_l) = \frac{k}{2^{n-j}}\}$. We have,

$$\int_0^1 \omega^U_{\Delta_j}(\lambda)\, dE^F_\lambda = \lim_{n\to\infty}\int_0^1 \omega^{(n)}_{\Delta_j}(\lambda)\, dE^F_\lambda = \lim_{n\to\infty}\sum_{j=1}^{2^{\binom{n}{2}}} \nu_j((i^{(n)}_j)E^F(i^{(n)}_l) =$$

$$= \lim_{n\to\infty}\sum_{k=1}^{2^{n-j}} \frac{k}{2^{n-j}} \sum_{l\in B^n_{k,j}} \prod_r E_r(f_r^{-1}(i^{(n)}_l)) =$$

$$= \lim_{n\to\infty}\sum_{k=1}^{2^{n-j}} \frac{k}{2^{n-j}}\, E_j\left(\left[\frac{k-1}{2^{n-j}}, \frac{k}{2^{n-j}}\right)\right) \sum_{l\in B^n_{k,j}} \prod_{r\neq j} E_r(f_r^{-1}(i^{(n)}_l))$$

$$= \lim_{n\to\infty}\sum_{k=1}^{2^{n-j}} \frac{k}{2^{n-j}}\, E_j\left(\left[\frac{k-1}{2^{n-j}}, \frac{k}{2^{n-j}}\right)\right)$$

$$= \int_0^1 \lambda\, E_j(d\lambda) = F(\Delta_j) \tag{13}$$

where we have used equation (12) and the identity $\sum_{j\in B^n_{k,j}} \prod_{r\neq j} E_r(f_r^{-1}(i^{(n)}_l)) = \mathbf{1}$ which can be derived by noting that the set $B^n_{k,j}$ includes all the indexes l such that the j-th edge of $f^{-1}(i^{(n)}_l)$ is fixed to be $[\frac{k-1}{2^{n-j}}, \frac{k}{2^{n-j}})$. Then, the other edges of $\{f^{-1}(i^{(n)}_l)\}_{l\in B^n_{k,j}}$ are arbitrary. Hence,

$$\left\{ \bigtimes_{i\neq j} f_i^{-1}(i^{(n)}_l)\right\}_{l\in B^n_{k,j}} = \left\{ \bigtimes_{i\neq j} \left[\frac{k_i-1}{2^{n-i}}, \frac{k_i}{2^{n-i}},\right)\right\}_{k_i=1,\dots,2^{n-i}}.$$

and (supposing $j\neq n-1$)

$$\begin{aligned}\left\{ \bigtimes_{i\neq j} \left[\frac{k_i-1}{2^{n-i}}, \frac{k_i}{2^{n-i}}\right)\right\}_{k_i=1,\dots,2^{n-i}} &= \left\{ \bigtimes_{i\neq j,n-1} \left[\frac{k_i-1}{2^{n-i}}, \frac{k_i}{2^{n-i}}\right)\right\}_{k_i=1,\dots,2^{n-i}} \bigtimes \left[0, \frac{1}{2}\right) \\ &\bigcup \left\{ \bigtimes_{i\neq j,n-1} \left[\frac{k_i-1}{2^{n-i}}, \frac{k_i}{2^{n-i}}\right)\right\}_{k_i=1,\dots,2^{n-i}} \bigtimes \left[\frac{1}{2}, 1\right] \\ &= \left\{ \bigtimes_{i\neq j,n-1} \left[\frac{k_i-1}{2^{n-i}}, \frac{k_i}{2^{n-i}}\right)\right\}_{k_i=1,\dots,2^{n-i}} \bigtimes [0,\, 1]\end{aligned}$$

so that

$$\sum_{l\in B^n_{k,j}} \prod_{r\neq j} E_r(f_r^{-1}(i^{(n)}_l)) = \sum_{l\in B^n_{k,j}} \prod_{r\neq j,n-1} E_r(f_r^{-1}(i^{(n)}_l)).$$

In the case $j = n-1$,

$$
\begin{aligned}
\left\{ \bigtimes_{i\neq j} \left[\frac{k_i-1}{2^{n-i}}, \frac{k_i}{2^{n-i}}\right)\right\}_{k_i=1,\dots,2^{n-i}} &= \left\{ \bigtimes_{i\neq j, n-2} \left[\frac{k_i-1}{2^{n-i}}, \frac{k_i}{2^{n-i}}\right)\right\}_{k_i=1,\dots,2^{n-i}} \times \left[0, \frac{1}{4}\right) \\
&\bigcup \left\{ \bigtimes_{i\neq j, n-2} \left[\frac{k_i-1}{2^{n-i}}, \frac{k_i}{2^{n-i}}\right)\right\}_{k_i=1,\dots,2^{n-i}} \times \left[\frac{1}{4}, \frac{1}{2}\right) \\
&\bigcup \left\{ \bigtimes_{i\neq j, n-2} \left[\frac{k_i-1}{2^{n-i}}, \frac{k_i}{2^{n-i}}\right)\right\}_{k_i=1,\dots,2^{n-i}} \times \left[\frac{1}{2}, \frac{3}{4}\right) \\
&\bigcup \left\{ \bigtimes_{i\neq j, n-2} \left[\frac{k_i-1}{2^{n-i}}, \frac{k_i}{2^{n-i}}\right)\right\}_{k_i=1,\dots,2^{n-i}} \times \left[\frac{3}{4}, 1\right] \\
&= \left\{ \bigtimes_{i\neq j, n-2} \left[\frac{k_i-1}{2^{n-i}}, \frac{k_i}{2^{n-i}}\right)\right\}_{k_i=1,\dots,2^{n-i}} \times [0,1]
\end{aligned}
$$

so that

$$
\sum_{l\in B^n_{k,j}} \prod_{r\neq j} E_r(f_r^{-1}(i_l^{(n)})) = \sum_{l\in B^n_{k,j}} \prod_{r\neq j, n-2} E_r(f_r^{-1}(i_l^{(n)})).
$$

By iterating the procedure, one arrives at $\sum_{l\in B^n_{k,j}} \prod_{r\neq j} E_r(f_r^{-1}(i_l^{(n)})) = \mathbf{1}$.

It is worth remarking that the functions $\omega^U_{\Delta_j}$ do not depend on the POVM F. That is analogous to what has been observed for the construction in Ref. [14] and is at the root of the proof of the existence of a universal Markov kernel (see below).

In the following, we use the symbol $\mathcal{D}(X)$ to denote the set of POVMs from the Borel σ-algebra $\mathcal{B}(X)$ to $\mathcal{L}^+_s(\mathcal{H})$. We then have a map $\omega^U : [0,1]\times\mathcal{R}(\mathcal{S}) \to [0,1]$ such that, for every $F \in \mathcal{D}(X)$, $F(\Delta) = \int \omega^U_\Delta(\lambda)\, dE^F_\lambda$, $\Delta \in \mathcal{R}(\mathcal{S})$.

Lemma 2.2. $\mathcal{A}^W(E^F) = \mathcal{A}^W(F)$.

Proof. By lemma 2.1, $\mathcal{A}^W(E^F) \subset \mathcal{A}^W(F)$. By equation (13) $\mathcal{A}^W(\{F(\Delta)\}_{\Delta\in\mathcal{R}(\mathcal{S})}) \subset \mathcal{A}^W(E^F)$. By proposition 3.1 in the appendix $\mathcal{A}^W(\{F(\Delta)\}_{\Delta\in\mathcal{R}(\mathcal{S})}) = \mathcal{A}^W(F)$. □

We have proved the following proposition.

Proposition 2.2. *There is a map $\omega^U : [0,1]\times\mathcal{R}(\mathcal{S}) \to [0,1]$ with the following property. For every $F \in \mathcal{D}(X)$, there is a spectral measure E^F with spectrum in $[0,1]$ which generates $\mathcal{A}^W(F)$ and is such that*

$$F(\Delta) = \int \omega^U_\Delta(\lambda)\, dE^F_\lambda, \quad \Delta \in \mathcal{R}(\mathcal{S}).$$

2.4. The Universal Markov Kernel

We are now ready to prove the existence of the universal Markov kernel. That includes the proof of the σ-additivity of the set function $\omega^U_{(\cdot)}(\lambda)$ in 2.3. Note that in [14] the problem of the σ-additivity of $\omega_{(\cdot)}(\lambda) : \mathcal{B}(\mathbb{R}) \to [0,1]$ (which correspond to the function $\omega^U_{(\cdot)}(\lambda) : \mathcal{B}(X) \to [0,1]$ of the present paper in the particular case $X = \mathbb{R}$) is not taken into consideration and only the measurability of $\omega_\Delta(\cdot)$, for every $\Delta \in \mathcal{B}(\mathbb{R})$, is proved. Furtheremore, in Ref. [7] only the finite additivity of $\omega_{(\cdot)}(\lambda)$ is proved; the algorithmic definition of σ-additive set functions $\omega_{(\cdot)}(\lambda)$ is left as an open problem (see section III, part C in [7]). The algorithmic construction we provide in the present paper puts an end to this question. Actually we prove a quite stronger result since we show that $\omega^U_{(\cdot)}(\lambda)$ is σ-additive for every λ in a set of measure one with respect to E^F for every POVM F. That is crucial in order to show the existence of a universal Markov kernel. All of that requires a new self-contained and more general algorithmic proof of the existence of the functions ω^U_Δ. Concerning this last point, note that the proof of the existence of the universal Markov kernel in the real case, $X = \mathbb{R}$, has been proved in [6] and is based both on the additivity of $\omega_\Delta(\lambda)$ (proved in [7]) and on corollary 1 in [8] which uses the ordered structure of $\mathbb{R}$. While an extension of the proof of the additivity to the general case seems to be straightforward, it seems to us that the extension of corollary 1 is not so easy. We circumvented the problem by proving that there is a subset $I \subset [0,1]$, $E^F(I) = \mathbf{1}$, such that, for every $\lambda \in I$, $\omega^U_{(\cdot)}(\lambda)$ is σ-additive on a ring $\mathcal{R}(\mathcal{S})$ which generates $\mathcal{B}(X)$, the Borel σ-algebra of X. Then, transfinite induction is used in order to prove that $\omega^U_{(\cdot)}(\lambda)$ can be extended to $\mathcal{B}(X)$.

Theorem 2.3. *There are a subset $I \subset [0,1]$ and a Markov kernel $\mu^U : \mathcal{B}(X) \times I \to [0,1]$ such that, for every $F \in \mathcal{D}(X)$, $E^F(I) = \mathbf{1}$ and*

$$F(\Delta) = \int \mu^U_\Delta(\lambda)\, dE^F_\lambda$$

where E^F is the spectral measure whose existence has been proved in subsection 2.2

Proof. Let $F \in \mathcal{D}(X)$. By theorem 1.1 there are a self-adjoint operator B with spectrum in $[0,1]$ and a Markov kernel μ such that B generates $\mathcal{A}^W(F)$ and $\int \mu_\Delta(\lambda)\, dE^B_\lambda = F(\Delta)$ for every $\Delta \in \mathcal{B}(X)$. Let $\nu(\cdot) = \langle \psi_0, E^B(\cdot)\psi_0\rangle$ where E^B is the spectral measure corresponding to B and ψ_0 is a separating vector for $\mathcal{A}^W(F)$.

By lemma 2.2 there is a spectral measure E^F which generates $\mathcal{A}^W(F)$. Let A^F be the corresponding self-adjoint operator and $\sigma(A^F) \subset [0,1]$ its spectrum. We have

$$\int_{\sigma(A^F)} \omega^U_{\Delta_i}(\lambda)\, dE^F_\lambda = F(\Delta_i), \quad i \in \mathbb{N}.$$

Since B and A^F generate $\mathcal{A}^W(F)$, there are two Borel functions $f_F : \sigma(B) \to \sigma(A^F)$ and $h_F : \sigma(A^F) \to \sigma(B)$ such that $E^F(G) = E^B(f_F^{-1}(G))$, $G \in \mathcal{B}([0,1])$ and $E^B(D) = E^F(h_F^{-1}(D))$, $D \in \mathcal{B}([0,1])$. Then, there is a set N such that $E^B(N) = \mathbf{1}$ and $(h_F \circ f_F)(\lambda) = \lambda$ for every $\lambda \in N$. In other words $f_F : N \to f_F(N)$ is injective. By Souslin's theorem ([24], page 440-442) $f_F(N)$ is a Borel set and $E^F(f_F(N)) = E^B[f_F^{-1}(f_F(N))] = \mathbf{1}$.

Then, by the change of measure principle,

$$\int_{\sigma(B)} \omega^U_{\Delta_i}(f_F(\lambda))\, dE^B_\lambda = \int_{\sigma(A^F)} \omega^U_{\Delta_i}(\lambda)\, dE^F_\lambda = F(\Delta_i) = \int_{\sigma(B)} \mu_{\Delta_i}(\lambda)\, dE^B_\lambda$$

Therefore, $\omega^U_{\Delta_i}(f_F(\lambda)) = \mu_{\Delta_i}(\lambda)$, $E^B - a.e.$. Since, E^B and ν are mutually absolutely continuous,

$$\omega^U_{\Delta_i}(f_F(\lambda)) = \mu_{\Delta_i}(\lambda), \quad \nu - a.e. \tag{14}$$

Let $M^F \subset \sigma(B)$, $\nu(M^F) = 1$ be such that

$$\omega^U_{\Delta_i}(f_F(\lambda)) = \mu_{\Delta_i}(\lambda), \quad \lambda \in M^F, i \in \mathbb{N}. \tag{15}$$

Thus, thanks to the σ-additivity of μ, $(\omega^U_{(\cdot)} \circ f_F)(\lambda))$ is σ-additive on $\mathcal{R}(\mathcal{S})$ for every $\lambda \in N^F := M^F \cap N$. As a consequence, $\omega^U_{(\cdot)}(\lambda)$ is σ-additive on $\mathcal{R}(\mathcal{S})$ for every $\lambda \in f_F(N^F)$. Note that $E^F[f_F(N^F)] = E^B[f_F^{-1}(f_F(N^F))] = \mathbf{1}$.

By repeating the reasoning for every $F \in \mathcal{D}(X)$, one proves that the set $I := \cup_{F\in\mathcal{D}(X)} f_F(N^F)$ is such that $\omega^U_{(\cdot)}(\lambda)$ is σ-additive on $\mathcal{R}(\mathcal{S})$ for every $\lambda \in I$.

In what follows we need the set I to be measurable. Thus, if I is not a Borel set we enlarge the Borel σ-algebra in order to include I. In particular, we consider the σ-algebra $\mathfrak{S}$ generated by I and $\mathcal{B}([0,1])$.

Since, $\forall F \in \mathcal{D}(X)$, $[0,1] \setminus I \subset [0,1] \setminus f_F(N^F)$ and $E^F\big([0,1] \setminus f_F(N^F)\big) = \mathbf{0}$, the set $[0,1] \setminus I$ is a subset of a E^F-null set for any $F \in \mathcal{D}(X)$. Then, each PVM E^F can be extended to $\mathfrak{S}$. The extension $\widetilde{E}^F : \mathfrak{S} \to \mathcal{L}_s(\mathcal{H})$ satisfies the following relations:

$$\begin{aligned} \widetilde{E}^F(\sigma(A^F)) = \widetilde{E}^F(I) = \mathbf{1} \\ \widetilde{E}^F(I \cap \Delta) = E^F(\Delta), \quad \forall \Delta \in \mathcal{B}[0,1] \\ A^F = \int_{[0,1]} \lambda \, d\widetilde{E}^F_\lambda. \end{aligned}$$

The space $([0,1], \mathfrak{S})$ is a measurable space and I is a measurable subset of $\mathfrak{S}$. Moreover, for each $\Delta \in \mathcal{R}(\mathcal{S})$, the function $\omega^U_\Delta : ([0,1], \mathfrak{S}) \to ([0,1], \mathcal{B}[0,1])$ is $\mathfrak{S}$-measurable and

$$\int_{[0,1]} \omega^U_\Delta(\lambda)\, d\widetilde{E}^F_\lambda = \int_{[0,1]} \omega^U_\Delta(\lambda)\, dE^F_\lambda = F(\Delta), \quad \forall F \in \mathcal{D}(X). \tag{16}$$

Now, for every $\lambda \in I$, the measure $\omega^U_{(\cdot)}(\lambda) : \mathcal{R}(\mathcal{S}) \to [0,1]$ can be extended to the Borel σ-algebra $\mathcal{B}(X)$. Let $\mu_{(\cdot)}(\lambda) : \mathcal{B}(X) \to [0,1]$ denotes such an extension. We want to show that, for each $\Delta \in \mathcal{B}(X)$, μ_Δ is $\mathfrak{S}$-measurable and $\int \mu^U_\Delta(\lambda)\, dE_\lambda = F(\Delta)$. That can be proved by using transfinite induction. Let Δ be an open set. Then, there is an increasing sequence of open sets $\Delta_{k_i} \in \mathcal{S}$ such that $\Delta_{k_i} \uparrow \Delta$. Then, for every λ, $\omega^U_{\Delta_{k_i}}(\lambda) = \mu^U_{\Delta_{k_i}}(\lambda) \uparrow \mu^U_\Delta(\lambda)$ so that μ^U_Δ is $\mathfrak{S}$-measurable. Let Δ be a G_δ set. Then there is a decreasing sequence of open sets G_i such that $G_i \downarrow \Delta$. Moreover, for every λ, $\mu^U_{G_i}(\lambda) \downarrow \mu^U_\Delta(\lambda)$ so that μ^U_Δ is $\mathfrak{S}$-measurable.

Let G_0 be the family of open subsets of X, ω_1 the first uncountable ordinal and G_α, $\alpha < \omega_1$ the Borel hierarchy [24]. In particular, $G_1 = G_\delta$, $G_2 = G_{\delta\sigma}$, $G_3 = G_{\delta\sigma\delta}, \dots$ and $G_\alpha = (\cup_{\beta<\alpha} G_\beta)_\sigma$ for each limit ordinal α. By means of the same reasoning that we used in the case of open and G_δ sets, one can prove the $\mathfrak{S}$-measurability of μ_Δ for every Δ of the kind $G_{\delta,\sigma}, G_{\delta\sigma\delta} \dots$. Analogously, if μ^U_Δ is $\mathfrak{S}$-measurable for each $\Delta \in G_\alpha$ then, μ^U_Δ is $\mathfrak{S}$-measurable for each Δ in $G_{\alpha+1}$. Indeed, each set in $G_{\alpha+1}$ is either the countable union or the countable intersection of sets in G_α and the previous reasoning can be used. If α is a limit ordinal and μ^U_Δ is $\mathfrak{S}$-measurable for each $\Delta \in G_\beta$, $\beta < \alpha$, then, μ^U_Δ is $\mathfrak{S}$-measurable for each $\Delta \in G_\alpha = (\cup_{\beta<\alpha} G_\beta)_\sigma$. Indeed, each set in G_α is the countable union of sets in $\cup_{\beta<\alpha} G_\beta$ and the previous reasoning can be used. Therefore, by transfinite induction, $\mu^U_\Delta(\cdot) : I \to [0,1]$ is $\mathfrak{S}$-measurable for each $\Delta \in \cup_{\alpha<\omega_1} G_\alpha = \mathcal{B}(X)$.

Moreover, since $\mu^U_\Delta = \omega^U_\Delta$, $\Delta \in \mathcal{R}(\mathcal{S})$, the POVM $F'(\Delta) = \mu^U_\Delta(A^F)$, $\Delta \in \mathcal{B}(X)$, coincides with $F(\Delta)$ for every $\Delta \in \mathcal{R}(\mathcal{S})$. Since $F : \mathcal{R}(\mathcal{S}) \to \mathcal{E}(\mathcal{H})$ has a unique extension [11] to the Borel σ-algebra $\mathcal{B}(X)$, it must be $F(\Delta) = F'(\Delta) = \mu^U_\Delta(A^F)$.

□

Definition 7. The Markov kernel $\mu^U : \mathcal{B}(X) \times I \to [0,1]$ whose existence has been proved in theorem 2.3 is called the universal Markov kernel.

2.5. Functional subordination

The following definition establishes a weak functional relationship between weak Markov kernels. In the following Λ and Γ denote compact subsets of $[0,1]$ while ν denotes a probability measure.

Definition 8. A weak Markov kernel $\gamma : (\Lambda, \nu) \times \mathcal{B}(X) \to [0,1]$ is functionally subordinated to a weak Markov kernel $\mu : \big(\Gamma, \nu\big) \times \mathcal{B}((X) \to [0,1]$ if there is a measurable function $f : \Lambda \to \Gamma$ such that, $\gamma_\Delta(\lambda) = \mu_\Delta(f(\lambda))$, ν-a.e.

The following theorem has been established in references [4, 6] in the case of real POVMs. Once theorem 2.3 has been proved, the proof can be straightforwardly extended to the case of an arbitrary POVM. For completness we provide the proof below.

Theorem 2.4. *Every weak Markov kernel* $\gamma : \big(\Lambda, \nu\big) \times \mathcal{B}(X) \to [0,1]$ *is functionally subordinated to the universal Markov kernel.*

Proof. We proceed as in the proof of theorem 6 in Ref. [4]. Without loss of generality, we can assume Λ to be the support of ν. Let $L_\infty(\Lambda, \nu)$ be the space of essentially bounded measurable functions (with two functions identified if they coincide up to ν-null sets) and $\mathcal{A}_\nu$ the von Neumann algebra of multiplication operators on $\mathcal{H} = L_2(\Lambda, \nu)$ which corresponds to $L_\infty(\Lambda, \nu)$. In particular, for every function $f \in L_\infty(\Lambda, \nu)$ there is a multiplication operator

$$M_f : L_2(\Lambda, \nu) \to L_2(\Lambda, \nu)$$

$$[M_f(h)](x) = f(x)h(x).$$

The generator of $\mathcal{A}_\nu$ is $B := M_x$, $[Bh](x) = [M_x(h)](x) = xh(x)$, $x \in \Lambda$. The spectrum of B, $\sigma(B)$, coincides with the support, Λ, of ν and the spectral measure corresponding to B is $E^B(\Delta) = M_{\chi_\Delta}$. Moreover, ν is a scalar-valued spectral measure for B, i.e., ν and E^B are mutually absolutely continuous.

Now, we can define a commutative POVM,

$$F(\Delta) := \gamma_\Delta(B) = \int_{\sigma(B)} \gamma_\Delta(\lambda)\, dE^B_\lambda = M_{\gamma_\Delta}, \quad \Delta \in \mathcal{B}(X). \tag{17}$$

By lemma 2.2, there is a generator A^F of $\mathcal{A}^W(F)$ with spectral resolution E^F and a Markov kernel μ^U such that

$$\int_{\sigma(A^F)} \mu^U_\Delta(\lambda)\, dE^F_\lambda = F(\Delta), \quad \Delta \in \mathcal{B}(X).$$

Since B generates $\mathcal{A}_\nu \supset \mathcal{A}^W(F)$, there is a Borel function $f : \sigma(B) \to \sigma(A^F)$ such that $E^F(G) = E^B(f^{-1}(G))$, $G \in \mathcal{B}([0,1])$. Then,

$$\int_{\sigma(B)} \mu^U_\Delta(f(\lambda))\, dE^B_\lambda = \int_{\sigma(A^F)} \mu^U_\Delta(\lambda)\, dE^F_\lambda = F(\Delta) = \int_{\sigma(B)} \gamma_\Delta(\lambda)\, dE^B_\lambda$$

Therefore, $\mu^U_\Delta(f(\lambda)) = \gamma_\Delta(\lambda)$, $E^B - a.e.$. Since, E^B and ν are mutually absolutely continuous,

$$\mu^U_\Delta(f(\lambda)) = \gamma_\Delta(\lambda), \quad \nu - a.e.$$

□

3. The universal family of fuzzy events

We know that a commutative POVM is the randomization of a spectral measure (theorem 1.1). We have shown that the universal Markov kernel μ^U is the source of the randomness for every commutative POVM F: if we take two different commutative POVMs F_1 and F_2, they will be the random version of two spectral measures E_1 and E_2 with the randomization realized by μ^U in both the cases.

The commutative POVM F_1 can also be seen as the randomization of a spectral measure E'_1 with the randomization realized by a Markov kernel μ'. We have shown that there is a function f such that $\mu' = \mu^U \circ f$. Thus, the randomization which characterizes every other Markov kernel μ' can be replicated by μ^U by a change in the variable $\mu^U \circ f$.

In the introduction we noted that a Markov kernel $\mu : \mathbb{R} \times \mathcal{B}(X) \to [0,1]$ provides a family of fuzzy events $\{(\mathbb{R}, \mu_\Delta)\}_{\Delta \in \mathcal{B}(X)}$. For every point $x \in \mathbb{R}$, the family $\{\mu_\Delta(x)\}_{\Delta \in \mathcal{B}(X)}$ defines a probability measure. Moreover, we pointed out that the expression

$$\langle \psi, F(\Delta)\psi \rangle = \int_{\mathbb{R}} \mu_\Delta(\lambda)\, d\langle \psi, E_\lambda \psi \rangle \tag{18}$$

can be interpreted as the probability of the fuzzy event $(\mathbb{R}, \mu_\Delta)$ with respect to the probability measure $\langle \psi, E(\cdot)\psi \rangle$. Such a probability coincides with the probability of the event Δ (which is not a fuzzy event) with respect to the probability $\langle \psi, F(\cdot)\psi \rangle$. On the other hand we proved that for every commutative POVM F,

$$\langle\psi, F(\Delta)\psi\rangle = \int_{\mathbb{R}} \mu^U_\Delta(\lambda)\, d\langle\psi, E^F_\lambda\psi\rangle, \quad \psi \in \mathcal{H}.$$

where E^F is the sharp version of F and μ^U is the universal Markov kernel. Therefore, the existence of the universal Markov kernel μ^U means that every POVM F gives the probabilities of the same fuzzy events $(\mathbb{R}, \mu^U_\Delta)$, $\Delta \in \mathcal{B}(X)$, with respect to the probability measures corresponding to E^F.

In other words, $\langle\psi, F_1(\Delta)\psi\rangle$ and $\langle\psi, F_2(\Delta)\psi\rangle$ are the probabilities of the same fuzzy event, $(\mathbb{R}, \mu^U_\Delta)$, with respect to the probability measures $\langle\psi, E^{F_1}(\cdot)\psi\rangle$ and $\langle\psi, E^{F_2}\psi\rangle$ respectively.

If we now recall that every Markov kernel μ is functionally subordinated to the universal Markov kernel, $\mu_\Delta = \mu^U_\Delta \circ f$, we see that, in a certain sense, $\{(\mathbb{R}, \mu^U_\Delta)\}_{\Delta\in\mathcal{B}(X)}$ is a universal family of fuzzy events since it includes (up to functional subordination) all the possible fuzzy events of which a POVM can represent the probabilities. Indeed, from equation (18) we see that F gives the probabilities of the fuzzy events $\{(\mathbb{R}, \mu_\Delta)\}_{\Delta\in\mathcal{B}(X)}$ with respect to E which, by functional subordination, coincides with the probabilities of the fuzzy events $\{(\mathbb{R}, \mu^U_\Delta \circ f)\}_{\Delta\in\mathcal{B}(X)}$ with respect to E,

$$\langle\psi, F(\Delta)\psi\rangle = \int_{\mathbb{R}} \mu^U_\Delta(f(\lambda))\, d\langle\psi, E_\lambda\psi\rangle$$

Appendix

Proposition 3.1. *Let X be second countable. Let $\mathcal{S}$ be a basis for the topology of X. Let $\mathcal{R}(\mathcal{S})$ be the ring generated by $\mathcal{S}$. Let $F : \mathcal{B}(X) \to \mathcal{L}^+_s(\mathcal{H})$ be a POVM. Then, the von Neumann algebra $\mathcal{A}^W(\mathcal{R}(\mathcal{S}))$ generated by $\{F(\Delta)\}_{\Delta\in\mathcal{R}(\mathcal{S})}$ coincides with the von Neumann algebra $\mathcal{A}^W(F)$.*

Proof. Let $G \subset X$ be an open set. Then, there is an increasing sequence of sets $\Delta_k \in \mathcal{S}$ such that $\Delta_k \uparrow G$. By the continuity of F, $F(\Delta_k) \uparrow F(G)$. Since $F(\Delta_k) \in \mathcal{A}^W(\mathcal{R}(\mathcal{S}))$ and $\mathcal{A}^W(\mathcal{R}(\mathcal{S}))$ is weakly closed, $F(G) \in \mathcal{A}^W(\mathcal{R}(\mathcal{S}))$. Let $\Delta \subset X$ be a G_δ set. Then there is a decreasing sequence of open sets G_k such that $G_k \downarrow \Delta$ and, by the continuity of F and the weak closure of $\mathcal{A}^W(\mathcal{R}(\mathcal{S}))$, $F(G_k) \downarrow F(\Delta) \in \mathcal{A}^W(\mathcal{R}(\mathcal{S}))$. Let G_0 be the family of open subsets of X, ω_1 the first uncountable ordinal and G_α, $\alpha < \omega_1$ the Borel hierarchy [24]. In particular, $G_1 = G_\delta$, $G_2 = G_{\delta\sigma}$, $G_3 = G_{\delta\sigma\delta}, \ldots$ and $G_\alpha = (\cup_{\beta<\alpha} G_\beta)_\sigma$ for each limit ordinal α. Suppose $F(\Delta) \in \mathcal{A}^W(\mathcal{R}(\mathcal{S}))$ for each $\Delta \in G_\alpha$. Then, $F(\Delta) \in \mathcal{A}^W(\mathcal{R}(\mathcal{S}))$ for each $\Delta \in G_{\alpha+1}$. Indeed, each set in $G_{\alpha+1}$ is either the countable union or the countable intersection of sets in G_α and the previous reasoning can be used. If α is a limit ordinal and $F(\Delta) \in \mathcal{A}^W(\mathcal{R}(\mathcal{S}))$ for each $\Delta \in G_\beta$, $\beta < \alpha$, then $F(\Delta) \in \mathcal{A}^W(\mathcal{R}(\mathcal{S}))$ for each $\Delta \in G_\alpha = (\cup_{\beta<\alpha} G_\beta)_\sigma$. Indeed, each set in G_α is the countable union of sets in $\cup_{\beta<\alpha} G_\beta$ and the previous reasoning can be used. Therefore, by transfinite induction, $F(\Delta) \in \mathcal{A}^W(\mathcal{R}(\mathcal{S}))$ for each $\Delta \in \cup_{\alpha<\omega_1} G_\alpha = \mathcal{B}(X)$ so that $\mathcal{A}^W(\mathcal{R}(\mathcal{S})) = \mathcal{A}^W(F)$.

□

Acknowledgment

The present work has been realized in the framework of the activities of the INDAM (Istituto Nazionale di Alta Matematica).
The authors have been supported by *PO Calabria FSE 2007/2013 - Obiettivo Operativo M2 - Piano per le Risorse Umane - Piano d'Azione 2011-2013*, for the I.N.F.N. Project "*SIC-POVM and Quantum Information*".

References

References

[1] S.T. Ali, La Rivista Nuovo Cimento 8 (1985) 1-127.

[2] S.T. Ali, G.G. Emch, Fuzzy observables in quantum mechanics, J. Math. Phys. 15 (1974) 176.

[3] S.T. Ali, A geometrical property of POV-measures and systems of covariance, Lecture Notes in Mathematics 905 (1982) 207-228.

[4] R. Beneduci, Universal Randomization of Quantum Observables, Int. J. Theor. Phys., doi.org/10.1007/s10773-019-04090-y (2019).

[5] R. Beneduci, Commutative POVM-measures: from the Choquet representation to the Markov kernel and back, Russian Journal of Mathematical Physics, 25 (2018) 158-182.

[6] R. Beneduci, Universal Markov Kernels for Quantum Observables, Geometric Methods in Physics. XXXVI Workshop 2017, Trends in Mathematics, 21-29 (2018).

[7] R. Beneduci, G. Nisticó, Sharp reconstruction of unsharp quantum observables, J. Math. Phys. 44 (2003) 5461.

[8] R. Beneduci, A geometrical characterizations of commutative positive operator valued measures, J. Math. Phys. 47 (2006) 062104.

[9] R. Beneduci, Joint measurability through Naimark's dilation theorem, Rep. Math. Phys. 79 (2017) 197-213.

[10] R. Beneduci, Positive Operator Valued Measures and Feller Markov Kernels, J. Math. Anal. Appl. 442 (2016) 50-71.

[11] S.K. Berberian, Notes on Spectral theory, Van Nostrand Mathematical Studies, New York, 1966.

[12] P. Busch, M. Grabowski, P. Lahti, Operational quantum physics, Lecture Notes in Physics 31 (1995).

[13] P. Busch, P. Lahti, R. Werner, Proof of Heisenberg's Error-Disturbance Relation, Phys. Rev. Lett. 111 (2013) 160405.

[14] G. Cattaneo, G. Nisticò, From unsharp to sharp quantum observable: The general Hilbert space case, J. Math. Phys. 41 (2000) 4365.

[15] E.B. Davies, Quantum mechanics of Open Systems, Academic Press, London, 1976.

[16] A. Dvurečenskij, Smearing of Observables and Spectral Measures on Quantum Structures, Found. Phys. 43 (2013) 210-224.

[17] T. Heinonen, P. Lahti, K.Ylinen, Covariant Fuzzy Observables and Coarse-Graining, Rep. Math. Phys. 53 (2004) 425-441.

[18] A.S. Holevo, An analog of the theory of statistical decisions in non-commutative probability theory, Trans, Moscow Math. Soc. 26 (1972) 133.

[19] A.S. Holevo, Probabilistics and statistical aspects of quantum theory, North Holland, Amsterdam, 1982.

[20] A. Jenčová, S. Pulmannová, How sharp are PV measures, Rep. Math. Phys. 59 (2007) 257-266.

[21] A. Jenčová, S. Pulmannová, Characterizations of Commutative POV Measures, Found. Phys. 39 (2009) 613-624.

[22] B. Jessen, The theory of integration in a space of an infinite number of dimensions, Acta Mathematica 63 (1935) 249-323.

[23] A. N. Kolmogorov, S. V. Fomin, Introductory Real Analysis, Dover, New York, 1970.

[24] K. Kuratowski, A. Mostowski, Set Theory with an introduction to descriptive set theory, North-Holland, New York, 1976.

[25] P. Lahti, Coexistence and joint measurability in quantum mechanics, Int. J. Theor. Phys., 42 (2003) 893-906.

[26] E. Lebesgue, Sur l'intégration des fonctions discontinues, Annales scientifiques de l'É.N.S. 3^e serie, tome 27 (1910) 361-450.

[27] G. Ludwig, Foundations of quantum mechanics I, Springer-Verlag, New York, 1983.

[28] B. Sz-Nagy, Spektraldarstellung linearer transformationens des Hilbertschen Raumes, Ergebnisse, d. Math, V/5, Berlin, 1942.

[29] E. Prugovečki, Stochastic Quantum Mechanics and Quantum Spacetime, D. Reidel Publishing Company, Dordrecht, Holland, 1984.

[30] F. Riesz, B. Sz. Nagy, Functional Analysis, Dover, 1990.

[31] F. Riesz, Untersuchungen *ü*ber systeme integrierbarer funktionen, Math. Annalen 69 (1910) 449-497.

[32] F.E. Schroeck Jr., Quantum Mechanics on Phase Space, Kluwer Academic Publishers, Dordrecht, 1996.

[33] de la Vallée-Poussin, Sur la transformation d'une intégrale multiple en une intégrale simple, Ann. Soc. Sci. Bruxelles 35 (1911) 189-190.

[34] L.A. Zadeh, Fuzzy Sets, Information and Control 8 (1965) 338-353.

[35] L.A. Zadeh, Probability Measures on Fuzzy Events, J. Math. Anal. Appl. 23 (1968) 421-427.