

Referee report

The paper studies incompatible pairs of quantum measurements using the framework of incompatibility witnesses based on quantum discrimination tasks as introduced in Ref. [12]. The authors concentrate on the special cases when both measurements and the witness are constructed from the same pair of MUBs but using different parameters for mixing with noise. This situation can be described by four parameters: r_x, r_y determine the witness and s_x, s_y determine the pair of POVMs. In this situation, a kind of incompatibility degree is discussed.

General comments

1. There seems to be a lot of confusion and misunderstanding. This appears already at the beginning of Sec. II.A where the very basic notions of POVMs and ensembles are introduced. Further, the description of the QSD scenario used for construction of the functionals that serve as witnesses does not make sense: the merged ensemble is labeled by elements of $X \cup Y$, whereas the joint measurement J has values in $X \times Y$. What is then the meaning of $P_{guess}(\mathcal{E}; J)$? This part is unreadable and the scenario that is basic for the paper can be understood only after reading Ref. [12].

The main object of study is the value of the functional (labelled by the parameters (r_x, r_y)) applied on the pair of POVMs labelled by (s_x, s_y) . By definition, a negative value is a proof of incompatibility of the POVMs and at the same time proves that the pair (r_x, r_y) indeed defines a witness. And here another confusion appears. The authors call this value "a nonlinear witness of (r_x, r_y, s_x, s_y) " and often write about "incompatible ensembles", without introducing such a notion. (It could be said that the ensembles are "incompatible" if they define a witness, but this is nowhere explained and also not standard). Overall there seems to be some confusion between ensembles and POVMs, which may be related to the fact that in this particular case these are given by the same operators (up to the noise parameter). It is also not clear how their notions of "inner incompatibility" and "general incompatibility" of the ensembles can be interpreted.

2. As for the contents: the authors compute the expressions needed for the evaluation of the witness on the given pair of POVMs in Eq. (8) (Appendix A 1) and also determine the boundary values of (s_x, s_y) corresponding to compatible POVMs (Eq. (10)/ Appendix A 2). Both these results can be found in Ref. [12] (in a much more readable form). In the rest of the paper, the incompatibility degree $D(r_x, r_y, s_x, s_y)$ is introduced as (minus the) value of the witness and it is shown that it can be expressed as a linear function of the difference vector $(s_x - s_{x_0}, s_y - s_{y_0})$ where (s_{x_0}, s_{y_0}) is the boundary point corresponding to the witness (r_x, r_y) . This observation leading to some geometric interpretation of the degree and its relation to the difference vector might be of some interest, but no further statements or results are presented.

Conclusion

The paper is not readable and brings little discernible new content. My suggestion is therefore to reject the paper.

A final remark: There are a lot of misspellings, such as "variables", "defination", "serval(?)", "brounary", etc. These are easily avoidable mistakes. As a sign of respect to future referees, I would suggest that the authors use a spell check before submitting papers.