Gergely Bunth: On quantum Rényi divergences

Topic - quantum extensions of Rényi divergences from several perspectives, applications in asymptotics of hypothesis testing, convertibility - asymptotic, catalytic, relative (sub)majorization, even in classical case....

Part I - Sec. 3: a variational formula that relates the classical Rényi divergence to the weighted (left) radius of two probability functions with respect to the relative entropy (Kullback-Leibler divergence). This formula can be easily extended to multivariate cases and coincides with the Hellinger transform if the weights are given by a probability distribution. In this part of the thesis, this formula is extended for families of quantum states (or positive semidefinite operators) in a natural way. Since there exist many versions of the quantum relative entropy, the multivariate formula may contain different quantum relative entropies for each state. It is proved that if the defining quantum relative entropies are monotone under pinchings, the corresponding divergence is a quantum extension of the classical Hellinger transform. In particular, in the two-variable case, this definition gives a quantum extension of the Rényi α-divergence if $\alpha \in (0, 1)$. Moreover, these quantities inherit the monotonicity and scaling properties of the involved quantum relative entropies. It is proved that none of the previously known quantu versions of D_α, that is the $\alpha - z$-divergences, measured or maximal Rényi divergences, are obtained in this way.

Part II -Sec. 4: This section is inspired by the fact that comparison of the elements in the spectrum characterizes asymptotic or catalytic majorization in preordered semirings. The spectrum consists of homomorphisms into certain fixed ordered semirings (reals with the usual structure and reals with the tropical structure (max instead of $+$). This is applied to ordered semirings consisting of pairs of continuous functions $\rho : X \to B(\mathcal{H})_>$ and $\sigma : Y \to B(\mathcal{H})_>$, $\mathcal{H}$ a finite dimensional Hilbert space (can be different in each pair) with X and Y compact Hausdorff spaces and with preorder given by relative submajorization with respect to completely positive trace nonincreasing maps.

In the classical case, (that is if of both σ and ρ range in $\sum_i a_i |i\rangle\langle i|$), all elements of the spectrum are obtained from Rényi divergences with $\alpha > 1$. It is shown that in the semiclassical case, that is, if all the elements in the range of σ commute, all the morphisms in the spectrum are obtained from sandwiched Rényi α divergences with $\alpha > 1$, so that the asymptotic or catalytic majorization can be characterized in this case. In general, some further elements of the spectrum are obtained by composing the sandwiched Rényi divergence with a quantum version of the geometric mean, which leads to a family of necessary consitions.

This is applied to several important cases, such as characterization the error exponents in the strong converse regime in the composite hypotesis testing, equivariant relative subjaoriztion, which leads to description of asymptotic transformations of by thermal processes, hypothesis testing with group symmetry or with reference frames, approximate joint transformations.

The ideas in this section also yield a new two-parameter family of quantum Rényi divergences - the γ-weighted sandwiched Rényi divergence.

Elements of the spectrum are closely related to quantum versions of Rényi α-divergences with $\alpha > 1$, in particular if all operators in the range of σ commute, then the spectrum is fully characterized by sandwiched Rényi divergences. It is proved that if the operators in the range of σ commute, the asymptotic majorization is characterized by sandwiched Rényi divergences for $\alpha > 1$, and a sufficient condition is obtained for catalytic majorization. In the general case, a set of necessary conditions is given.

The topic of the thesis is timely and important, given the importance of Rényi divergences in quantum information theory, resoure theories, convertibility, hypothesis testing. The obtained results are based on novel ideas and are rather strong, especially in the second part, builds on and extends known and recently obtained results on the importance of Rényi divergences in information theory and resource theories... blablabla...

Nevertheless, the thesis not written veryu well:

- p. 6, l. 6 from below: the sentence starting with "In a resource theory..." is rather strange. Also "sates" -> states

- p. 7: "...target operators are only bounds" ??

- $\mathcal{P}_f$ is used to denote two very different things: $\mathcal{P}_f(\mathcal{I})$ in p. 10 and the operator perspective function on p. 13. Though it should not cause confusion, a different notation would be better.

- p. 13, Theorem 2.2.1: "...so that the following..." -> "...such that the following..." would be better

- It seems that a definition of a quantum divergence is missing here. In Definition 2.2.6, the quantum Rényi divergence is defined merely by its values on jointly diagonalizable elements, where it should coincide with the classical Rényi divergence. However, unitary equivalence is repeatedly used in proofs. Also Lemma 3.2.1 is quite mysterious. In the paper [MBV22] a quantum divergence is defined first, as a unitarily invariant quantity. This seems to have been left out in the thesis.

- p.17, first displayed equation: $D_{\alpha,+\infty}$... in the second line, $(1-\alpha)$ is missing

- p. 17, paragraph above Ex. 2.2.11: I think that strict positivity of $D_{\alpha,z}$ is treated in [Mos23], Corollary III.28 only in the case $\alpha > 1$

- p. 20: "realation"

- p. 24, displayed equation under Eq. (2.40): I would say that here should be $\subseteq$ (instead of $\nsubseteq$)

- p. 25, Remark 2.2.25: Actually, it is the Hellinger transform of a set of probability distributions (P is a parameter)

- Properties of the quantum Rényi divergences are often refered to before their definition in Section 2.2.4. This is very confusing, especially when it concerns some commonly used expressions like "nonnegative", which is quite difficult to gues that this is actually some special property to be defined later. I strongly suggest to rearrange this part.

- p. 28, last paragraph: $\mathcal{D}_{\mathcal{H}}(D_P)$ etc, does not seem to be defined before.

- p. 29, displayed equations on the top of the page: V and V^* seem missing in line 3 and 4 of the equations

- p. 29: "isomeric"

- p. 29, definition of "regular" D_α: here $\kappa_{\rho,\sigma} > 0$ has no meaning

- p. 34, Theorem 2.3.36, part (ii)(c): $x \succcurlyeq_c y$ (instead of $ax \succcurlyeq_c ay$)
- p. 36., Remark 3.1.3: I think $P(x) \leq 0$, $x \in \mathcal{X}$ is not possible, since $\sum_x P(x) = 1$
- p. 36., Remark 3.1.7, equality for $R_{D^q,left}$: ω (instead of τ)
- p. 37, Lemma 3.1.9: Reference to Eq.(3.12) is used repeatedly, but I think it should be Eq. (3.3)
- I do not understand Remark 3.1.10
- Remark 3.1.13, also p. 40, first paragraph of Sec. 3.2): γ-weighted versions of D^{meas}, D^{Um} are mentioned here, but these were not defined
- p. 41, line 6 from below: S^0_+ should be S_+
- p. 42, last line: "Lemma 3.2.1..." perhaps should be "Lemma 3.2.2"
- p. 43, Remark 3.2.10: $Q_P^{G_P^{D^q}}$ very strange notation, it is not clear what it means. I think it comes from some notations in Sec. IV in [MBV22] that was not introduced here (as well as the references to γ-weighted versions...)
- p. 47, paragraph below the proof of Prop. 3.4.2: "...all barycentric Rényi divergences..." only those with D^q monotone under CPTP maps
- p. 56, Example 4.1.4: relative majorization should be defined (also proof of Prop. 4.2.3). Also its relation to relative submajorization for pairs of states should be clarified, or a reference to [BV21] (prop. 3.3) should be given. Also DPI (in proof of Prop. 4.2.3).
- p. 57, line above Def. 4.1.8: "to subsemiring" -> two subsemirings (?)
- top of p. 60, end of the proof of Lemma 4.2.4: better write out that $k = 1$ in the real case and $k = 0$ in the tropical case.
- p. 64, line 2: "realative". Also line 3 from below: in one case $\tilde{f}(c_1, d_1)$ should be $\tilde{f}(c_2, d_2)$.
- p. 65, Eq.(4.8) and the displayed equation above it: ρ should be $\rho(x)$ in some places
- p. 66, Prop 4.3.4: $\operatorname{Tr}\sigma(y')$ -> $\operatorname{Tr}\sigma(y') = 1$
- p. 67, Def. 4.3.5: "...a collection of maps $M : \bigtimes_{\mathcal{H}} C(Y, \mathcal{B}(\mathcal{H})_{>0} \to \bigtimes_{\mathcal{H}} C(Y', \mathcal{B}(\mathcal{H})_{>0}$... this makes no sense. I think what you mean is $M = \{M_{\mathcal{H}}\}$, where for each $\mathcal{H}$, $M_{\mathcal{H}} : C(Y, \mathcal{B}(\mathcal{H})_{>0} \to C(Y', \mathcal{B}(\mathcal{H})_{>0}$ (but this is not a map between the products...).
- p. 69, displayed equations on the top: the inequality sign should be opposite
- p. 84, Corollary 4.4.21: $\alpha \geq 0$ -> $\alpha \geq 1$ (?)
- p. 85, Prop. 4.4.23 (i): "trace-nonincreasing" - according to the proof, maybe "trace-preserving"?
- p. 88: Example 4.4.27: "Lemma 4.3.16" is actually a Proposition

- p. 88 Example 4.4.28 "...arbitrary an suppose..," -> "and"
- Ref. [JV18], [FF20] I think these were already published
- Incomplete references: [MBV22], [MH23a], [FFHT23], [HT23]
- some conclusion, possible future directions, further questions, ideas?

Overall evaluation

Some specific comments