On the structure of higher order quantum maps

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1 Preliminaries

1.1 The category FinVect

sec:fv

Let FinVect be the category of finite dimensional real vector spaces with linear maps. We will denote the usual tensor product by $\otimes$, then $(\mathrm{FinVect}, \otimes, I = \mathbb{R})$ is a symmetric monoidal category, with the associators, unitors and symmetries given by the obvious isomorphisms

$$\begin{aligned} \alpha_{U,V,W} &: (U \otimes V) \otimes W \simeq U \otimes (V \otimes W), \\ \lambda_V &: I \otimes V \simeq V, \qquad \rho_V : V \otimes I \simeq V, \\ \sigma_{U,V} &: U \otimes V \simeq V \otimes U. \end{aligned}$$

Let $(-)^* : V \mapsto V^*$ be the usual vector space dual, with duality denoted by $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}$. We will use the canonical identification $V^{**} = V$ and $(V_1 \otimes V_2)^* = V_1^* \otimes V_2^*$. With this duality, FinVect is compact closed. This means that for each object V, there are maps $\eta_V : I \to V^* \otimes V$ (the "cup") and $\epsilon_V : V \otimes V^* \to I$ (the "cap") such that the following snake identities hold:

$$(\epsilon_V \otimes V) \circ (V \otimes \eta_V) = V, \qquad (V^* \otimes \epsilon_V) \circ (\eta_V \otimes V^*) = V^*, \tag{1}$$ eq:snake

here we denote the identity map on the object V by V. Indeed, η_V can be identified with an element $\eta_V(1) \in V^* \otimes V$ and $\epsilon_V \in (V \otimes V^*)^* = V^* \otimes V$ is again an element of the same space. Choose a basis $\{e_i\}$ of V, let $\{e_i^*\}$ be the dual basis of V^*, that is, $\langle e_i^*, e_j \rangle = \delta_{i,j}$. Let us then define

$$\eta_V(1) = \epsilon_V := \sum_i e_i^* \otimes e_i.$$

It is easy to see that this definition does not depend on the choice of the basis, indeed, ϵ_V is the linear functional on $V \otimes V^*$ defined by

$$\langle \epsilon_V, x \otimes x^* \rangle = \langle x^*, x \rangle, \qquad x \in V, \ x^* \in V^*.$$

It is also easily checked that the snake identities (eq:snake 1) hold.

For two objects V and W in FinVect, we will denote the set of all morphisms (i.e. linear maps) $V \to W$ $L(V,W)$ by $\mathrm{FinVect}(V,W)$. Then $\mathrm{FinVect}(V,W)$ is itself a real linear space and we have the well-known identification $\mathrm{FinVect}(V,W) \simeq V^* \otimes W$. This can be given as follows: for each $f \in FV(V,W)$, we have $C_f := (V^* \otimes f)(\epsilon_V) = \sum_i e_i^* \otimes f(e_i) \in V^* \otimes W$. Conversely, since $\{e_i^*\}$ is a basis of V^*, any element $w \in V^* \otimes W$ can be uniquely written as $w = \sum_i e_i^* \otimes w_i$ for $w_i \in W$, and since $\{e_i\}$ is a basis of V, the assignment $f(e_i) := w_i$ determines a unique map $f : V \to W$. The relations between $f \in \mathrm{FinVect}(V, W)$ and $C_f \in V^* \otimes W$ can be also written as

$$\langle\, C_f, x \otimes y^* \,\rangle = \langle\, \epsilon_V, x \otimes f^*(y^*) \,\rangle = \langle\, f^*(y^*), x \,\rangle = \langle\, y^*, f(x) \,\rangle, \qquad x \in V,\ y^* \in W^*,$$

here $f^* : W^* \to V^*$ is the adjoint of f. Note that by compactness, the internal hom in FinVect satisfies $[V, W] \simeq V^* \otimes W$, so that in the case of FinVect, the object $[V, W]$ can be identified with the space of linear maps $\mathrm{FinVect}(V, W)$.

lassical *Example* 1. Let $V = \mathbb{R}^N$. In this case, we fix the canonical basis $\{|i\,\rangle,\ i = 1, \dots, N\}$. We will identify $(\mathbb{R}^N)^* = \mathbb{R}^N$, with duality $\langle\, x, y \,\rangle = \sum_i x_i y_i$, in particular, we identify $I = I^*$. We then have $\epsilon_V = \sum_i |i\,\rangle \otimes |i\,\rangle$ and if $f : \mathbb{R}^N \to \mathbb{R}^M$ is given by the matrix A in the two canonical bases, then $C_f = \sum_i |i\,\rangle \otimes A|i\,\rangle$ is the vectorization of A.

:quantum *Example* 2. Let $V = M_n^h$ be the space of $n \times n$ complex hermitian matrices. We again identify $(M_n^h)^* = M_n^h$, with duality $\langle\, A, B \,\rangle = \mathrm{Tr}\, A^T B$, where A^T is the usual transpose of the matrix A. Let us choose the basis in M_n^h, given as

$$\left\{ |j\,\rangle\langle\, k| + |k\,\rangle\langle\, j|,\ j \leq k,\ i\Big(|j\,\rangle\langle\, k| - |k\,\rangle\langle\, j|\Big),\ j < k \right\}.$$

Then one can check that

$$\left\{ \frac{1}{2}\Big(|j\,\rangle\langle\, k| + |k\,\rangle\langle\, j|\Big),\ j \leq k,\ \frac{i}{2}\Big(|k\,\rangle\langle\, j| - |j\,\rangle\langle\, k|\Big),\ j < k \right\}$$

is the dual basis and we have

$$\epsilon_V = \sum_{j,k} |j\,\rangle\langle\, k| \otimes |j\,\rangle\langle\, k|.$$

For any $f : M_n^h \to M_m^h$,

$$C_f = \sum_{j,k} |j\,\rangle\langle\, k| \otimes f(|j\,\rangle\langle\, k|)$$

is the Choi matrix of f.

1.2 Boolean functions

We will work with certain boolean functions, which are defined as functions from binary strings $\{0,1\}^*$ to $\{0,1\}$. We now list some basic notations used below.

For $s \in \{0,1\}$, we denote $\bar{s} := 1 - s$. For binary strings of fixed length n, that is, elements of $\{0,1\}^n$, we will denote by 0_n or just 0 the string $00\dots0$ and

$$e^i = \delta_{i,1} \dots \delta_{i,n}.$$

For $m, n \in \mathbb{N}$, we may identify $\{0,1\}^m \times \{0,1\}^n \simeq \{0,1\}^{m+n}$ by concatenation:

$$\{0,1\}^m \times \{0,1\}^n \ni (s^1, s^2) \mapsto s^1 s^2 = s_1^1 \dots s_m^1 s_1^2 \dots s_n^2 \in \{0,1\}^{m+n}.$$

With this identification, we will sometimes use the identification of $[m+n]$ as the disjoint union of indices $[m]\dot{\cup}[n]$, so the index set becomes $\{(1,1), \dots, (1,m), (2,1), \dots, (2,n)\}$, where $(1,i) = i$ and $(2,j) = m + j$, $i \in [m]$, $j \in [n]$, and for $s \in \{0,1\}^{m+n}$, we put $s_{(l,j)} = s_j^l$. We will use similar notations for any $n = n_1 + \dots + n_k$.

For any permutation $\sigma \in \mathscr{S}_n$, we will denote by the same symbol the action of σ on $\{0,1\}^n$ given as

$$\sigma(s_1 \dots s_n) = s_{\sigma^{-1}(1)} \dots s_{\sigma^{-1}(n)}.$$

Note that then for $\rho, \sigma \in \mathscr{S}_n$, we have $\rho(\sigma(s)) = (\rho \circ \sigma)(s)$.

Let us introduce the subset of boolean functions

$$\mathcal{F}_n := \{f : \{0,1\}^n \to \{0,1\},\ f(0_n) = 1\}.$$

With the poitwise ordering, $\mathcal{F}_n$ is a (finite) distributive lattice, with top element the constant 1 function and the bottom element $p_n := \chi_{0_n}$, the characteristic function of the zero string. Moreover, we have for $f, g \in \mathcal{F}_n$,

$$f \vee g = f + g - fg, \qquad f \wedge g = fg \tag{2}$$ eq:wedge

(with pointwise addition and multiplication). We also define complementation in $\mathcal{F}_n$ as

$$f^* := 1 - f + p_n.$$

It can be easily checked that with these structures $\mathcal{F}_n$ is a boolean algebra.

We now introduce some more operations in $\mathcal{F}_n$. For $f \in \mathcal{F}_n$ and any permutation $\sigma \in \mathscr{S}_n$, we see that $f \circ \sigma \in \mathcal{F}_n$. For $f \in \mathcal{F}_m$ and $g \in \mathcal{F}_n$, we define the function $f \otimes g \in \mathcal{F}_{m+n}$ as

$$(f \otimes g)(s^1 s^2) = f(s^1) g(s^2), \qquad s^1 \in \{0,1\}^m,\ s^2 \in \{0,1\}^n$$

As it is, this tensor product is not symmetric, but there is a permutation $\sigma \in \mathscr{S}_{m+n}$ such that $\sigma(s^1 s^2) = s^2 s^1$ and $(g \otimes f) = (f \otimes g) \circ \sigma$ for any $f \in \mathcal{F}_m$ and $g \in \mathcal{F}_n$.

fproduct **Lemma 1.** *For $f \in \mathcal{F}_m$, $g, h \in \mathcal{F}_n$, we have*

(i) $f \otimes g \leq (f^* \otimes g^*)^*$, with equality if and only if f and g are either both the top or both the bottom elements in $\mathcal{F}_m$ resp. $\mathcal{F}_n$

(ii) $f \otimes (g \vee h) = (f \otimes g) \vee (f \otimes h)$, $f \otimes (g \wedge h) = (f \otimes g) \wedge (f \otimes h)$.

Proof. The inequality in (i) is easily checked, since $(f \otimes g)(s^1 s^2)$ can be 1 only if $f(s^1) = g(s^2) = 1$. If both s^1 and s^2 are the zero strings, then $s^1 s^2 = 0_{m+n}$ and both sides are equal to 1. Otherwise, the condition $f(s^1) = g(s^2) = 1$ implies that $(f^* \otimes g^*)(s^1 s^2) = 0$, so that the right hand side must be 1. If f and g are both constant 1, then $(1 \otimes 1)^* = 1^* = p_{n+m} = 1^* \otimes 1^*$, in the case when both f and g are the minimal elements equality follows by duality. Finally, asume the equality holds and that $f \neq 1$, so that there is some s^1 such that $f(s^1) = 0$. But then $s^1 \neq 0_m$ and for any s^2,

$$0 = (f \otimes g)(s^1 s^2) = (f^* \otimes g^*)^*(s^1 s^2) = 1 - f^*(s^1) g^*(s^2) + p_{m+n}(s^1 s^2) = 1 - g^*(s^2),$$

which implies that $g(s^2) = 0$ for all $s^2 \neq 0_n$, that is, $g = p_n$. By the same argument, $f = p_m$ if $g \neq 1$, which implies that either $f = 1$ and $g = 1$, or $f = p_m$ and $g = p_n$.

To prove (ii), notice that for $s^1 \in \{0,1\}^n$ and $s^2 \in \{0,1\}^m$, we have by (eq:wedgevee_fun 2)

$$\begin{aligned} f(s^1)(g \vee h)(s^2) &= f(s^1)\big(g(s^2) + h(s^2) - g(s^2)h(s^2)\big) \\ &= f(s^1)g(s^2) + f(s^1)h(s^2) - f(s^1)^2 g(s^2)h(s^2) = f(s^1)g(s^2) \vee f(s^1)h(s^2), \end{aligned}$$

note that $f(s^1)^2 = f(s^1)$. The statement for $\wedge$ is also clear from (eq:wedgevee_fun 2).

□

We now describe an important example.

ex:ps *Example* 3. Let $S \subseteq [n]$ be any subset. We will denote

$$p_S(s) := \prod_{i \in S} \bar{s}_i.$$

It is clear that $p_S \in \mathcal{F}_n$ and $p_\emptyset = 1$, $p_{[n]} = p_n$. The following properties are also easy to see:

(i) if $S \subseteq T \subseteq [n]$, then $p_T \leq p_S$,

(ii) $p_S \wedge p_T = p_S p_T = p_{S \cup T}$, for $S, T \subseteq [n]$,

(iii) $p_S \vee p_T = p_S + p_T - p_{S \cup T}$, for $S, T \subseteq [n]$,

(iv) $p_S \circ \sigma = p_{\sigma^{-1}(S)}$, $S \subseteq [n]$, $\sigma \in \mathscr{S}_n$,

(iv) $p_S \otimes p_T = p_{S \dot{\cup} T}$, for $S \subseteq [m]$ and $T \subseteq [n]$.

We will use the above functions to introduce a convenient parametrization to $\mathcal{F}_n$, closely related to the Fourier transform.

hm:basis **Theorem 1.** *Any $f : \{0,1\}^n \to \{0,1\}$ can be written in the form*

$$f = \sum_{S \subseteq [n]} \hat{f}_S p_S$$

in a unique way, with $\hat{f}_S \in \mathbb{R}$ obtained as

$$\hat{f}_S = \sum_{\substack{s \in \{0,1\}^n \\ s_i = 1, \forall i \in S^c}} (-1)^{\sum_{i \in S} s_i} f(s).$$

Proof. Let $\mathcal{L}_n$ be the boolean algebra of all subsets of $[n]$. We may interpret f as the function $\mathcal{L}_n \to \mathbb{R}$, given by

$$\varphi : S \mapsto f(s), \qquad s \in \{0,1\}^n, \ s_i = 0 \iff i \in S.$$

By the Möbius inversion formula (see [Stanley, Sec. 3.7] for details), a function $g : \mathcal{L}_n \to \mathbb{R}$ satisfies

$$f(s) = \varphi(S) = \sum_{T \subseteq S} g(T), \qquad S \in \mathcal{L}_n$$

if and only if

$$g(S) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} \varphi(T) = \hat{f}_S.$$

Plugging this into the first equality, we get for any $s \in \{0,1\}^n$,

$$f(s) = \sum_{T : s_i = 0, \forall i \in T} \hat{f}_T = \sum_{T : p_T(s) = 1} \hat{f}_T = \sum_T \hat{f}_T p_T(s).$$

To show uniqueness, assume that $f = \sum_{T \subseteq [n]} c_T p_T$ for some $c_T \in \mathbb{R}$, $T \in \mathcal{L}_n$, and let s and S be connected as above. Then

$$\varphi(S) = f(s) = \sum_{T \subseteq [n]} c_T p_T(s) = \sum_{T \subseteq S} c_T,$$

so that we must have $c_T = \hat{f}_T$.

□

We can visualise $\mathcal{L}_n$ as a hypercube, and the coefficients $\hat{f}_S$ of f as labels for its vertices. The fact that the function f takes values in $\{0,1\}$ means that we must have

$$f(s) = \varphi(S) = \sum_{T \subseteq S} \hat{f}_T \in \{0,1\},$$

so that the sum of labels $\hat{f}_S$ over any face of the hypercube $\mathcal{L}_n$ containing the vertex $\emptyset$ must be 0 or 1. In particular, $\hat{f}_\emptyset = f(11\dots1) \in \{0,1\}$. Clearly, $f \in \mathcal{F}_n$ if and only if, in addition,

$$f(0) = \sum_{S \subseteq [n]} \hat{f}_S = 1.$$

p:mobius **Proposition 1.** *(i) For $f \in \mathcal{F}_n$ and $\sigma \in \mathscr{S}_n$,*

$$\widehat{(f \circ \sigma)}_S = \hat{f}_{\sigma(S)}, \qquad S \subseteq [n].$$

(ii) For $f \in \mathcal{F}_n$,

$$\widehat{f^*}_S = \begin{cases} 1 - \hat{f}_S & S = \emptyset \text{ or } S = [n], \\ -\hat{f}_S & \text{otherwise.} \end{cases}$$

(iii) For $f \in \mathcal{F}_m$, $g \in \mathcal{F}_n$, we have

$$\widehat{(f \otimes g)}_{S \dot\cup T} = \hat{f}_S \hat{g}_T, \qquad S \subseteq [m],\ T \subseteq [n].$$

(iv) For $f, g \in \mathcal{F}_n$, we have

$$\widehat{(f \wedge g)}_S = \sum_{T \subseteq S} \hat{f}_T \hat{g}_{S \setminus T}.$$

(v) For $f, g \in \mathcal{F}_n$, we have

$$\widehat{(f \vee g)}_S = \hat{f}_S + \hat{g}_S - \sum_{T \subseteq S} \hat{f}_T \hat{g}_{S \setminus T}.$$

Proof. All statements follow easily from the corresponding expressions and the uniqueness part in Theorem thm:basis 1.

□

2 The category of affine subspaces

2.1 The category Af

We now introduce the category Af, whose objects are of the form $X = (V_X, A_X)$, where V_X is an object in FinVect and $A_X \subseteq V_X$ a proper affine subspace, which means that $0 \notin A_X \neq \emptyset$. Morphisms $X \xrightarrow{f} Y$ in Af are linear maps $f : V_X \to V_Y$ such that $f(A_X) \subseteq A_Y$. For any object X, we put

$$L_X = \operatorname{Lin}(A_X) := \{a_1 - a_2,\ a_1, a_2 \in A_X\} = \{a - a_X,\ a \in A_X\}, \qquad S_X := \operatorname{Span}(A_X).$$

Here a_X is any element in A_X and L_X does not depend on this choice. Then L_X and S_X are linear subspaces such that $d_X := \dim(L_X) = \dim(S_X) - 1$. We will also denote $D_X = \dim(V_X)$. For any element $a_X \in A_X$, the affine subspace is determined as

$$A_X = a_X + L_X.$$

Let us now define the duality of affine subspaces as follows. Let V be an object in FinVect and let $C \subseteq V$ be any subset. Let

$$\tilde{C} := \{v^* \in V^*,\ \langle v^*, c\rangle = 1\}.$$

The following lemma collects some properties that are easily proven.

Lemma 2. *(i) $\tilde{C}$ is an affine subspace.*

(ii) $0 \in \tilde{C}$ if and only if $C = \emptyset$ and $\tilde{C} = \emptyset$ if and only if $0 \in \mathrm{Aff}(C)$.

(iii) Let $0 \notin \mathrm{Aff}(C)$, then $\mathrm{Aff}(C) = \tilde{\tilde{C}}$ and we have

$$\begin{aligned} \mathrm{Lin}(C) &= \mathrm{Lin}(\tilde{\tilde{C}}) = \tilde{C}^\perp = \mathrm{Span}(\tilde{C})^\perp, \qquad & \mathrm{Lin}(\tilde{C}) &= C^\perp = \mathrm{Span}(C)^\perp \\ \mathrm{Span}(C) &= C^{\perp\perp} = \mathrm{Lin}(\tilde{C})^\perp, & \mathrm{Span}(\tilde{C}) &= \mathrm{Lin}(C)^\perp. \end{aligned}$$

For any $\tilde{a}_X \in \tilde{A}_X$, the subspace A_X is determined as

$$A_X = S_X \cap \{\tilde{a}_X\}^\sim.$$

The relation between the subspaces L_X and S_X is given as

$$S_X = L_X \oplus \mathbb{R}a_X, \qquad L_X = S_X \cap \{\tilde{a}_X\}^\perp.$$

By Lemma 2 above, $\tilde{A}_X$ is a proper affine subspace in V_X^*, so that $X^* := (V_X^*, \tilde{A}_X)$ is an object in Af. We have $X^{**} = X$ and the corresponding subspaces are related as

$$L_{X^*} = S_X^\perp, \qquad S_{X^*} = L_X^\perp. \tag{3}$$

Note also that for $X \xrightarrow{f} Y$, the adjoint map satisfies $f^*(\tilde{A}_Y) \subseteq \tilde{A}_X$, so that $Y^* \xrightarrow{f^*} X^*$ and the duality $(-)^*$ is a full and faithful functor $\mathrm{Af}^{op} \to \mathrm{Af}$.

We will next introduce a monoidal structure $\otimes$ as follows. For two objects X and Y, we put $V_{X\otimes Y} = V_X \otimes V_Y$ and construct the affine subspace $A_{X\otimes Y}$ as the affine span of

$$A_X \otimes A_Y = \{a \otimes b,\ a \in A_X,\ b \in A_Y\}.$$

Fix any $\tilde{a}_X \in \tilde{A}_X$ and $\tilde{a}_Y \in \tilde{A}_Y$. Since $A_X \otimes A_Y \subset \{\tilde{a}_X \otimes \tilde{a}_Y\}^\sim$, the affine span of $A_X \otimes A_Y$ is a proper affine subspace and we have by Lemma 2

$$A_{X\otimes Y} := \mathrm{Aff}(A_X \otimes A_Y) = \{A_X \otimes A_Y\}^{\approx}.$$

Lemma 3. *For any $a_X \in A_X$, $a_Y \in A_Y$, we have*

$$\begin{aligned} L_{X\otimes Y} &= \mathrm{Lin}(A_X \otimes A_Y) = \mathrm{Span}(\{x \otimes y - a_X \otimes a_Y,\ x \in A_X,\ y \in A_Y\}) & (4)\\ &= (a_X \otimes L_Y) + (L_X \otimes a_Y) + (L_X \otimes L_Y) & (5) \end{aligned}$$

(here $+$ denotes the direct sum of subspaces). We also have

$$S_{X\otimes Y} = S_X \otimes S_Y.$$

Proof. The equality (4) follows from Lemma 2. For any $x \in A_X$, $y \in A_Y$ we have

$$x \otimes y - a_X \otimes a_Y = a_X \otimes (y - a_Y) + (x - a_X) \otimes a_Y + (x - a_X) \otimes (y - a_Y),$$

so that $L_{X \otimes Y} = \mathrm{Lin}(A_X \otimes A_Y)$ is contained in the subspace on the RHS of (5). Let d be the dimension of this subspace, then clearly

$$d_{X \otimes Y} \leq d \leq d_X + d_Y + d_X d_Y.$$

On the other hand, any element of S_X has the form tx for some $t \in \mathbb{R}$ and $x \in A_X$, so that it is easily seen that $S_X \otimes S_Y = S_{X \otimes Y}$. Hence

$$\begin{aligned} d_{X \otimes Y} &= \dim(L_{X \otimes Y}) = \dim(S_{X \otimes Y}) - 1 = \dim(S_X)\dim(S_Y) - 1 = (d_X + 1)(d_Y + 1) - 1 \\ &= d_X + d_Y + d_X d_Y. \end{aligned}$$

This completes the proof.

□

Lemma 4. *Let $I = (\mathbb{R}, \{1\})$. Then $(\mathrm{Af}, \otimes, I)$ is a symmetric monoidal category.*

Proof. Note that this structure is inherited from the symmetric monoidal structure in FinVect. To show that $\otimes$ is a functor, we have to check that for $X_1 \xrightarrow{f} Y_1$ and $X_2 \xrightarrow{g} Y_2$ in Af, we have $X_1 \otimes Y_1 \xrightarrow{f \otimes g} X_2 \otimes Y_2$ which amounts to showing that

$$(f \otimes g)(A_{X_1 \otimes Y_1}) \subseteq A_{X_2 \otimes Y_2}.$$

Let $x \in A_{X_1}$, $y \in A_{Y_1}$, then $f(x) \otimes g(y) \in A_{X_2} \otimes A_{Y_2} \subseteq A_{X_2 \otimes Y_2}$. Since $A_{X_1 \otimes Y_1}$ is the affine subspace generated by $A(X_1) \otimes A(Y_1)$, the above inclusion follows by linearity of $f \otimes g$.

It only remains to prove that the associators, unitors and symmetries from FinVect are morphisms in Af. We will prove this for the associators $\alpha_{X,Y,Z} : V_X \otimes (V_Y \otimes V_Z) \to (V_X \otimes V_Y) \otimes V_Z$, the other proofs are similar. We need to check that $\alpha_{X,Y,Z}(A_{X \otimes (Y \otimes Z)}) \subseteq A_{(X \otimes Y) \otimes Z}$. It is easily checked that $A_{X \otimes (Y \otimes Z)}$ is the affine span of elements of the form $x \otimes (y \otimes z)$, $x \in A_X$, $y \in A_Y$ and $z \in A_Z$, and we have

$$\alpha_{X,Y,Z}(x \otimes (y \otimes z)) = (x \otimes y) \otimes z \in A_{(X \otimes Y) \otimes Z}$$

for all such elements. The desired inclusion follows by linearity.

□

Theorem 2. *$(\mathrm{Af}, \otimes, I)$ is a *-autonomous category, with duality $(-)^*$, such that $I^* = I$.*

Proof. By Lemma 4, we have that $(\mathrm{Af}, \otimes I)$ is a symmetric monoidal category. We have also seen that the duality $(-)^*$ is a full and faithful contravariant functor. We only need to check the natural isomorphisms

$$\mathrm{Af}(X \otimes Y, Z^*) \simeq \mathrm{Af}(X, (Y \otimes Z)^*).$$

Since FinVect is compact, we have the natural isomorphisms

$$\mathrm{FinVect}(V_X \otimes V_Y, V_Z^*) \simeq \mathrm{FinVect}(V_X, V_Y^* \otimes V_Z^*),$$

determined by the equalities

$$\langle\, f(x \otimes y), z \,\rangle = \langle\, h(x), y \otimes z \,\rangle, \qquad x \in V_X,\ y \in V_Y,\ z \in V_Z,$$

for $f \in \mathrm{FinVect}(V_X \otimes V_Y, V_Z^*)$ and $h \in \mathrm{FinVect}(V_Z, V_Y^* \otimes V_Z^*)$. Since $A_{X\otimes Y}$ is an affine span of $A_X \otimes A_Y$, we see that f is in $\mathrm{Af}(X \otimes Y, Z^*)$ if and only if $f(x \otimes y) \in \tilde{A}_Z$ for all $x \in A_X$, $y \in A_Y$, that is,

$$1 = \langle\, f(x \otimes y), z\,\rangle = \langle\, h(x), y \otimes z\,\rangle \qquad \forall x \in A_X,\ \forall y \in A_Y,\ \forall z \in A_Z.$$

But this is equivalent to

$$h(x) \in (A_Y \otimes A_Z)^{\sim} = \tilde{A}_{Y\otimes Z}, \qquad \forall x \in A_X,$$

which means that $h \in \mathrm{Af}(X, (Y \otimes Z)^*)$.

□

A *-autonomous category is compact closed if it satisfies $(X \otimes Y)^* = X^* \otimes Y^*$. In general, $X \odot Y = (X^* \otimes Y^*)^*$ defines a dual symmetric monoidal structure that is different from $\otimes$. We next show that Af is not compact.

ncompact **Proposition 2.** *For objects in* Af*, we have* $(X \otimes Y)^* = X^* \otimes Y^*$ *exactly in one of the following situations:*

(i) $X \simeq I$ *or* $Y \simeq I$*,*

(ii) $d_X = d_Y = 0$*,*

(iii) $d_{X^*} = d_{Y^*} = 0$*.*

Proof. It is easily seen by definition that $A_{X^*} \otimes A_{Y^*} = \tilde{A}_X \otimes \tilde{A}_Y \subseteq \tilde{A}_{X\otimes Y} = A_{(X\otimes Y)^*}$. Hence the equality holds if and only if $d_{X^*\otimes Y^*} = d_{(X\otimes Y)^*}$. From Lemma 3, we see that

$$d_{X^*\otimes Y^*} = d_{X^*} + d_{Y^*} + d_{X^*}d_{Y^*}.$$

On the other hand, we have using (3) that $L_{(X\otimes Y)^*} = S_{X\otimes Y}^{\perp} = (S_X \otimes S_Y)^{\perp}$, so that

$$d_{(X\otimes Y)^*} = D_X D_Y - \dim(S_X)\dim(S_Y) = D_X D_Y - (d_X + 1)(d_Y + 1).$$

Taking into account that by (3) we have $d_{X^*} = D_X - d_X - 1$, similarly for d_{Y^*}, we obtain

$$d_{(X\otimes Y)^*} - d_{X^*\otimes Y^*} = d_X d_{Y^*} + d_{X^*} d_Y.$$

This is equal to 0 iff $d_X d_{Y^*} = d_Y d_{X^*} = 0$, which amounts to the conditions in the lemma.

□

In a *-autonomous category, the internal hom can be identified as $[X, Y] = (X \otimes Y^*)^*$. The underlying vector space is $V_{[X,Y]} = (V_X \otimes V_Y^*)^* = V_X^* \otimes V_Y$ and we have seen in Section 1.1 that we may identify this space with $\mathrm{FinVect}(V_X, V_Y)$, by $f \leftrightarrow C_f$. This property is extended to Af, in the following sense.

orphisms **Proposition 3.** *For any objects* X, Y *in* Af*, the map* $f \mapsto C_f$ *is a bijection of* $\mathrm{Af}(X, Y)$ *onto* $A_{[X,Y]}$*.*

Proof. Let $f \in \mathrm{FinVect}(V_X, V_Y)$. Since by definition $A_{[X,Y]} = \tilde{A}_{X\otimes Y^*}$ and $A_{X\otimes Y^*}$ is an affine span of $A_X \otimes A_Y^*$, we see that $C_f \in A_{[X,Y]}$ if and only if for all $x \in A_X$ and $y^* \in \tilde{A}_Y$, we have

$$1 = \langle\, C_f, x \otimes y^*\,\rangle = \langle\, y^*, f(x)\,\rangle.$$

This latter statement is clearly equivalent to $f(A_X) \subseteq A_Y$, so that $f \in \mathrm{Af}(X, Y)$. □

In the next result, we restrict the objects to spaces of hermitian matrices, as in Example 2 exm:quantum and morphisms to completely positive maps. We show that this restriction amounts to taking an intersection of $A_{[X,Y]}$ with the cone of positive semidefinite matrices. This, and subsequent examples, shows that for characterization of sets of quantum objects such as states, channels, combs and transformations between them, it is enough to work with the category Af.

An object X of Af will be called quantum if $V_X = M_n^h$ for some n and A_X is an affine subspace such that both A_X and $\tilde{A}_X$ have nonempty intersection with the interior of the positive cone $int(M_n^+)$ (recall that we identify $(M_n^h)^* = M_n^h$).

_quantum **Proposition 4.** *Let X, Y be quantum objects in* Af*. Then*

(i) X^* and $X \otimes Y$ are quantum objects as well

(ii) Let $V_X = M_n^h$, $V_Y = M_m^h$. Then for any $f \in \mathrm{FinVect}(M_n^h, M_m^h)$, we have $C_f \in A_{[X,Y]} \cap M_{mn}^+$ if and only if f is completely positive and

$$f(A_X \cap M_n^+) \subseteq A_Y \cap M_m^+.$$

Proof. The statement (i) is easily seen from $A_X \otimes A_Y \subseteq A_{X\otimes Y}$ and $\tilde{A}_X \otimes \tilde{A}_Y \subseteq \tilde{A}_{X\otimes Y}$, together with the fact that $int(M_n^+) \otimes int(M_m^+) \subseteq int(M_{mn}^+)$. To show (ii), let $C_f \in A_{[X,Y]} \cap M_{mn}^+$. By the prop:from_morphisms properties of the Choi isomorphism f is completely positive and by Proposition 3, $f(A_X) \subseteq A_Y$, this proves one implication. For the converse, note that we only need to prove that under the given assumptions, $f(A_X) \subseteq A_Y$, for which it is enough to show that $A_X \subseteq \mathrm{Aff}(A_X \cap M_n^+)$. To see this, pick some $a_X \in A_X \cap int(M_n^+)$. Any element in A_X can be written in the form $a_X + v$ for some $v \in L_X$. Since $a_X \in int(M_n^+)$, there is some $s > 0$ such that $a_\pm := a_X \pm sv \in M_n^+$, and since $\pm sv \in L_X$, we see that $a_\pm \in A_X \cap M_n^+$. It is now easily checked that

$$a_X + v = \frac{1+s}{2s} a_+ + \frac{s-1}{2s} a_- \in \mathrm{Aff}(A_X \cap M_n^+).$$

□

We can define classical obejcts in Af in a similar way, replacing M_n^h by $\mathbb{R}^N$ and the positive cone by $\mathbb{R}_+^N$. A similar statement holds in this case, with complete positivity replaced by positivity. We can similarly treat classical-to-quantum and quantum-to-classical maps as morphisms between these types of objects, satisfying appropriate positivity assumptions.

tum_maps *Example* 4. States, channels, combs, nonsignaling, etb, dual, process matrices

exm:qccq *Example* 5. POVMs, instruments, multimeters.

2.2 First order and higher order objects

We say that an object X in Af is first order if $d_X = D_X - 1$, equivalently, $S_X = V_X$. Another equivalent condition is $d_{X^*} = 0$, which means that A_X is determined by a single element $\tilde{a}_X \in V_X^*$ as

$$A_X = \{\tilde{a}_X\}^\sim, \qquad \tilde{A}_X = \{\tilde{a}_X\}.$$

In the case of first order quantum objects we additionally require that $\tilde{a}_X \in int(M_n^+)$, similarly for classical first order objects. Note that first order objects, resp. their duals, are exactly those satisfying condition (iii), resp. condition (ii), in Proposition prop:noncompact 2, in particular, $(X \otimes Y)^* = X^* \otimes Y^*$ for first order objects X and Y.

For a first order object $X = (V_X, \{\tilde{a}_X\}^{\sim})$, let us pick an element $a_X \in A_X$, then we have a direct sum decomposition

$$V_X = L_{X,0} \oplus L_{X,1},$$

where $L_{X,0} := \mathbb{R}\{a_X\}$, $L_{X,1} := \{\tilde{a}_X\}^{\perp} = L_X$. We also define the *conjugate object* $\tilde{X} = (V_X^*, \{a_X\}^{\sim})$, note that we always have $\tilde{a}_X \in A_{\tilde{X}}$ and with the choice $a_{\tilde{X}} = \tilde{a}_X$, we have $\tilde{\tilde{X}} = X$ and

$$L_{\tilde{X},u} = L_{X,\bar{u}}^{\perp}, \qquad u \in \{0,1\}. \tag{6}$$ eq:compl

These definitions depend on the choice of a_X, but we will assume below that this choice is fixed and that we choose $a_{\tilde{X}} = \tilde{a}_X$. For quantum objects we will assume that $a_X \in int(M_n^+)$.

Example 6. Example: quantum states, multiple of identity...

Higher order objects are those obtained from a finite set $\{X_1, \dots, X_n\}$ of first order objects by taking tensor products and duals, and applying any permuations of the spaces. The above is indeed a set, so that all the objects are different (though they may be isomorphic) and the ordering is not essential. We will also assume that the tensor unit is not contained in this set. Of course, any first order object is also higher order with $n = 1$. Note that we cannot say that such an object is automatically "of order n", as the following lemma shows.

ertensor **Lemma 5.** *Let X, Y be first order, then $X \otimes Y$ is first order as well.*

Proof. We have

$$S_{X \otimes Y} = S_X \otimes S_Y = V_X \otimes V_Y = V_{X \otimes Y}.$$

□

Example 7. (states quantum first order, channels, supermaps - quantum higher order)

Example 8. replacement $X^* \otimes Y$, quantum

2.3 Description of higher order objects

We start by noticing that there are certain objects in Af that can be constructed from a set of first order objects and functions in $\mathcal{F}_n$.

Let $X_1, \dots, X_n$ be first order objects in Af. Let $a_{X_i} \in A_{X_i}$ be fixed and let $\tilde{X}_i$ be the conjugate first order objects. Let us denote $V_i = V_{X_i}$ and

$$L_{i,u} := L_{X_i,u}, \qquad \tilde{L}_{i,u} := L_{\tilde{X}_i,u} \qquad u \in \{0,1\},\ i \in [n].$$

For $s \in \{0,1\}^n$, we define

$$L_s := L_{1,s_1} \otimes \cdots \otimes L_{n,s_n}, \qquad \tilde{L}_s := \tilde{L}_{1,s_1} \otimes \cdots \otimes \tilde{L}_{n,s_n},$$

then we have the direct sum decompositions

$$V := V_1 \otimes \cdots \otimes V_n = \bigoplus_{s \in \{0,1\}^n} L_s, \qquad V^* = V_1^* \otimes \cdots \otimes V_n^* = \bigoplus_{s \in \{0,1\}^n} \tilde{L}_s.$$

ma:Lperp **Lemma 6.** *For any $s \in \{0,1\}^n$, we have*

$$L_s^{\perp} = \bigoplus_{t \in \{0,1\}^n} \bar{\chi}_s(t) \tilde{L}_t, \qquad \tilde{L}_s^{\perp} = \bigoplus_{t \in \{0,1\}^n} \bar{\chi}_s(t) L_t.$$

Here $\chi_s : \{0,1\}^n \to \{0,1\}$ is the characteristic function of s, $\bar{\chi}_s = 1 - \chi_s$.

Proof. Using (eq:complement 6) and the direct sum decomposition of V_i^*, we get

$$\begin{aligned}
(L_{1,s_1} \otimes \cdots \otimes L_{n,s_n})^{\perp} &= \bigvee_j \left(V_1^* \otimes \cdots \otimes V_{j-1}^* \otimes \tilde{L}_{j,\bar{s}_j} \otimes V_{j+1}^* \otimes \cdots \otimes V_n^* \right) \\
&= \bigvee_j \left(\bigoplus_{\substack{t \in \{0,1\}^n \\ t_j \neq s_j}} \tilde{L}_{1,t_1} \otimes \cdots \otimes \tilde{L}_{n,t_n} \right) \\
&= \bigoplus_{\substack{t \in \{0,1\}^n \\ t \neq s}} \left(\tilde{L}_{1,t_1} \otimes \cdots \otimes \tilde{L}_{n,t_n} \right).
\end{aligned}$$

The proof of the other equality is the same.

□

lemma:Xf **Lemma 7.** *Put $a := a_1 \otimes \cdots \otimes a_n$, $\tilde{a} := \tilde{a}_1 \otimes \cdots \otimes \tilde{a}_n$. For $f \in \mathcal{F}_n$ define*

$$S_f := \bigoplus_{s \in \{0,1\}^n} f(s) L_s, \qquad A_f = A_f(X_1, \ldots, X_n) := S_f \cap \{\tilde{a}\}^{\sim}.$$

Then A_f is a proper affine subspace in V containing a. Moreover,

$$L_{A_f} = \bigoplus_{s \in \{0,1\}^n \setminus \{0\}} f(s) L_s, \qquad S_{A_f} = S_f$$

and the dual affine subspace satisfies

$$\tilde{A}_f(X_1, \ldots, X_n) = A_{f^*}(\tilde{X}_1, \ldots, \tilde{X}_n) = \bigoplus_{s \in \{0,1\}^n} f^*(s) \tilde{L}_s \cap \{a\}^{\sim}.$$

Proof. It is clear from definition that A_f is an affine subspace. Since $f(0) = 1$, the space S_f always contains the subspace $L_0 = L_{1,0} \otimes \cdots \otimes l_{n,0} = \mathbb{R}\{a\}$ and it is clear that $L_s \subseteq \{\tilde{a}\}^{\perp}$ for any $s \neq 0$. It follows that $a \in A_f$, so that $A_f \neq \emptyset$, and since $A_f \subseteq \{\tilde{a}\}^{\sim}$, we see that A_f is proper and $\tilde{a} \in \tilde{A}_f$. The expressions for L_{A_f} and S_{A_f} are immediate from the definition and $L_{A_f} = S_{A_f} \cap \{\tilde{a}\}^{\sim}$. To obtain the dual affine subspace, we compute using Lemma lemma:Lperp 6 and the fact that the subspaces form an independent decomposition,

$$\begin{aligned}
S_{\tilde{A}_f} = L_{A_f}^{\perp} &= \left(\bigoplus_{s \in \{0,1\}^n \setminus \{0\}} f(s) L_s \right)^{\perp} = \bigwedge_{\substack{s \in \{0,1\}^n \\ s \neq 0, f(s)=1}} L_s^{\perp} = \bigwedge_{\substack{s \in \{0,1\}^n \\ s \neq 0, f(s)=1}} \left(\bigoplus_{t \in \{0,1\}^n} \bar{\chi}_s(t) \tilde{L}_t \right) \\
&= \bigoplus_{t \in \{0,1\}^n} \left(\bigwedge_{\substack{s \in \{0,1\}^n \\ s \neq 0, f(s)=1}} \bar{\chi}_s(t) \tilde{L}_t \right) = \bigoplus_{t \in \{0,1\}^n} f^*(t) \tilde{L}_t.
\end{aligned}$$

To see the last equality, note that

$$\bigwedge_{\substack{s\in\{0,1\}^n\\ s\neq 0, f(s)=1}} \bar{\chi}_s(t) = \begin{cases} 1 & \text{if } t = 0 \\ 1 - f(t) & \text{if } t \neq 0 \end{cases} = f^*(t).$$

□

Since L_s, $s \in \{0,1\}$ is an independent decomposition, the map $f \mapsto S_f$, and hence also $f \mapsto A_f$, is injective. This map has the following further properties, which are easily checked:

(i) For the bottom and top elements in $\mathcal{F}_n$ we have

$$A_{p_n} = \{a\}, \qquad A_{1_n} = \{\tilde{a}\}^{\sim},$$

(ii) We have $f \leq g$ if and only if $A_f \subseteq A_g$,

(iii) $A_{f\wedge g} = A_f \cap A_g$,

(iv) $A_{f\vee g} = A_f \vee A_g := \mathrm{Aff}(A_f \cup A_g)$.

It follows that the set $\{A_f,\ f \in \mathcal{F}_n\}$ is a distributive lattice, with respect to the lattice operations $\cap$ and $\vee$.

Since all the affince subspaces $A_f \subseteq V$ are proper, there are objects $X_f := (V, A_f)$ in Af. The above relations can be rephrased as follows:

(i) $X_{1_n} = (V, \{\tilde{a}\}^{\sim})$ is a first order object, $X_{p_n} = (V^*, \{a\}^{\sim})^{\sim}$ is a dual first order object.

(ii) We have $f \leq g$ if and only if id_V is a morphism $X_f \to X_g$ in Af,

(iii) Let $f, g \leq h$. The following is a pullback diagram:

$$\begin{tikzcd} X_{f\wedge g} \arrow[r, "id_V"] \arrow[d, "id_V"'] & X_f \arrow[d, "id_V"] \\ X_g \arrow[r, "id_V"'] & X_h \end{tikzcd}$$

(iv) Let $h \leq f, g$. The following is a pushout diagram:

$$\begin{tikzcd} X_h \arrow[r, "id_V"] \arrow[d, "id_V"'] & X_f \arrow[d, "id_V"] \\ X_g \arrow[r, "id_V"'] & X_{f\vee g} \end{tikzcd}$$

In particular, it follows that $\{X_f,\ f \in \mathcal{F}_n\}$, is a distributive lattice, with pullbacks and pushouts as lattice operations. Furthermore, using the conjugate objects, we may construct

$$\tilde{X}_f := (V^*, A_f(\tilde{X}_1, \ldots, \tilde{X}_n))$$

and we see from Lemma 7 that

$$X_f^* = \tilde{X}_{f^*}, \qquad f \in \mathcal{F}_n. \tag{7}$$

We next observe that the higher order objects are of the form X_f, for some choice of the first order objects $X_1, \dots, X_n$ and a function f that belongs to a special subclass of $\mathcal{F}_n$. So assume that Y is a higher order object constructed from a set of distinct first order objects $Y_1, \dots, Y_n$, $Y_i = (V_{Y_i}, \{\tilde{a}_{Y_i}\}^\sim)$, we will write $Y \sim \{Y_1, \dots, Y_n\}$ in this case. Let us fix elements $a_{Y_i} \in A_{Y_i}$ and construct the objects $\tilde{Y}_i$.

By compactness of FinVect, we may assume (relabeling the objects if necessary) that the vector space of Y has the form

$$V_Y = V := V_1 \otimes \cdots \otimes V_n,$$

where V_i is either V_{Y_i} or $V_{Y_i}^*$, according to whether Y_i was subjected to taking duals an even or odd number of times. The indices such that the first case is true will be called the outputs and the subset of outputs in $[n]$ will be denoted by O, or O_Y, when we need to specify the object. The set $I = I_Y := [n] \setminus O_Y$ is the set of inputs. The reason for this terminology will become clear later.

Proposition 5. *For $i \in [n]$, let $X_i = Y_i$ if $i \in O_Y$ and $X_i = \tilde{Y}_i$ for $i \in I_Y$. There is a unique function $f \in \mathcal{F}_n$ such that*

$$Y = X_f = (V, A_f(X_1, \dots, X_n)).$$

Proof. Since the map $f \mapsto X_f$ is injective, uniqueness is clear. To show existence of this function, we will proceed by induction on n. For $n = 1$, the assertion is easily seen to be true, since in this case, we we have either $Y = Y_1$ or $Y = Y_1^*$. In the first case, $O = [1]$, $X_1 = Y_1$ and

$$S_Y = V_Y = V_1 = L_{1,0} \oplus L_{1,1} = 1(0)L_{1,0} \oplus 1(1)L_{1,1},$$

so in this case f is the constant 1. If $Y = Y_1^*$, we have $O = \emptyset$, $X_1 = \tilde{Y}_1$, and then

$$S_Y = \mathbb{R}\{\tilde{a}_{Y_1}\} = L_{1,0} = 1^*(0)L_{1,0} \oplus 1^*(1)L_{1,1},$$

so that $f = 1^*$.

Assume now that the assertion is true for all $m < n$. By construction, Y is either the tensor product $Y = Z_1 \otimes Z_2$, with

$$Z_1 \sim \{Y_1, \dots, Y_m\}, \qquad Z_2 \sim \{Y_{m+1}, \dots, Y_n\},$$

or Y is the dual of such a product. Let us assume the first case. It is clear that $O_{Z_1} \cup O_{Z_2} = O_Y$, and similarly for I, so that the corresponding objects $X_1, \dots, X_m$ and $X_{m+1}, \dots, X_n$ remain the same. By the induction assumption, there are functions $f_1 \in \mathcal{F}_m$ and $f_2 \in \mathcal{F}_{n-m}$ such that

$$S_Y = S_{Z_1} \otimes S_{Z_2} = \bigoplus_{\substack{s \in \{0,1\}^m \\ t \in \{0,1\}^{n-m}}} f_1(s) f_2(t) L_{1,s_1} \otimes \cdots \otimes L_{m,s_m} \otimes L_{m+1,t_1} \otimes \cdots \otimes L_{n,t_{n-m}}.$$

This implies the assertion, with $f = f_1 \otimes f_2$.

To finish the proof, it is now enough to observe that if the assertion holds for Y then it also holds for Y^*. So assume that $Y = X_f = (V, A_f(X_1, \dots, X_n))$ for some $f \in \mathcal{F}_n$, then $Y^* = X_f^* = \tilde{X}_{f^*} = (V^*, A_{f^*}(\tilde{X}_1, \dots, \tilde{X}_n)$. In is now enough to notice that $\tilde{X}_i = \tilde{\tilde{Y}}_i = Y_i$ if $i \in I_Y$ and $\tilde{X}_i = \tilde{Y}_i$ if $i \in O_Y$. Since by definition $O_{Y^*} = I_Y$, this proves the statement.

□

Let us stress that in general, the objects X_f depend on the choice of the elements a_{X_i}. From the above proof, it is clear that the description in Proposition 5 does not depend on the choice of the elements $a_{Y_i} \in A_{Y_i}$.

2.4 Type funtions and higher order objects

Let $\mathcal{T}_n \subseteq \mathcal{F}_n$ be defined as the subset generated from the constant function 1 on $\{0,1\}$ by taking duals and tensor products. For example, we have

$$\mathcal{T}_1 = \mathcal{F}_1 = \{1, 1^*\}, \quad \mathcal{T}_2 = \{1\otimes 1, (1\otimes 1)^*, 1\otimes 1^*, 1^*\otimes 1, (1^*\otimes 1)^*, (1\otimes 1^*)^*\},$$

etc. Elements of $\mathcal{T}_n$ will be called *type functions*. Similarly as for the higher order objects, the indexes in $[n]$ such that the corresponding component was subjected to taking the dual an even number of times will be called the outputs (of f) and denoted by $O = O_f$, indexes in $I = I_f := [n] \setminus O_f$ will be called inputs. From the proof of Proposition prop:boolean 5, it is easily seen that a higher order object is of the form $Y = X_f$ for a function $f \in \mathcal{T}_n$ with the same outputs (and of course also inputs) as Y. We next show that the converse is true.

type_hom **Proposition 6.** *Let $\{X_1, \dots, X_n\}$ be first order objects and let $f \in \mathcal{T}_n$. Then $Y = X_f$ is a higher order object with $O_Y = O_f$ and $Y \sim \{Y_1, \dots, Y_n\}$, where $Y_i = X_i$ for $i \in O_f$ and $Y_i = \tilde{X}_i$ for $i \in I_f$.*

Proof. As before, we will proceed by induction on n. For $n = 1$, we only have the possibilities $f = 1$ or $f = 1^*$. In the first case, $O = [1]$ and we get

$$S_f = 1L_{1,0} \oplus 1L_{1,1} = V_1,$$

so that $X_f = (V_1, \{\tilde{a}_1\}^\sim) = X_1$. In the second case, $O = \emptyset$ and

$$S_f = 1L_{1,0} = \mathbb{R}\{a_1\},$$

so that $X_f = (V_1, \{a_1\}) = \tilde{X}_1^*$. Assume next that the statement is true for all $m < n$ and assume that $f = f_1 \otimes f_2$ for some $f_1 \in \mathcal{F}_m$, $f_2 \in \mathcal{F}_{n-m}$, then it is easily seen that $Y = Z_1 \otimes Z_2$ for $Z_1 = X_{f_1}$ and $Z_2 = X_{f_2}$, constructed from $\{X_1, \dots, X_m\}$ resp. $\{X_{m+1}, \dots, X_n\}$. By the induction assumption, Z_1 and Z_2 are higher order objects, with $O_{Z_i} = O_{f_i}$, it follows that Y is a higher order object with $O_Y = O_{Z_1} \cup O_{Z_2} = O_{f_1} \cup O_{f_2} = O_f$.

Finally, assume that the statement is true for $f \in \mathcal{F}_n$, we will show that it holds for f^*. From eq:dualityXf (7), we see that $X_{f^*} = \tilde{X}_f$, which shows that $X_{f^*} \sim \{\tilde{Y}_1, \dots, \tilde{Y}_n\}$. Since taking duals will switch inputs and outputs, this finishes the proof.

□

Let $\{Y_1, \dots, Y_n\}$ be first order objects. The above results show that any of higher order object $Y \sim \{Y_1, \dots, Y_n\}$ with fixed set of outputs $O_Y = O$ satisfies $Y \simeq X_f$ for a unique type function $f \in \mathcal{T}_n$, $O_f = O$, and a fixed set of objects $\{X_1, \dots, X_n\}$, where the isomorphism is given by the action of some permutation in S_n on the space $V_1 \otimes \dots \otimes V_n$. Conversely, any object of this form has the above properties. A basic example of such a function is (see Section sec:boolean ??)

$$p_I(s) = \Pi_{i\in I}\bar{s}_i = \otimes_{i\in I} 1^*(s_i), \qquad s \in \{0,1\}^n,$$

where $I = [n] \setminus O$. Clearly, $p_I \in \mathcal{T}_n$ and the set of outputs of p_I is O. It is easy to see that $X_{p_I} \simeq Y_I^* \otimes Y_O$, where $Y_I = \otimes_{i\in I} Y_i$ and $Y_O = \otimes_{i\in O} Y_i$ are first order objects, so that the corresponding higher order object can be identified with the set of replacement channels $Y_I \to Y_O$. Similarly, the function $p_O^* \in \mathcal{T}_n$ has output set O and $X_{p_O^*} \simeq (Y_I \otimes Y_O^*)^* = [Y_I, Y_O]$, the set of all channels $Y_I \to Y_O$.

_setting **Lemma 8.** *Let $f \in \mathcal{T}_n$ and let $O_f = O$, $I = I_f$. Then*

$$p_I \leq f \leq p_O^*.$$

Proof. This is obviously true for $n = 1$. Indeed, in this case, $\mathcal{T}_1 = \mathcal{F}_1 = \{1 = p_\emptyset, 1^* = p_{[1]}\}$. If $f = 1$, then $O = [1]$, $I = \emptyset$ and

$$p_I = p_\emptyset = 1 = p_O^*,$$

the case $f = 1^*$ is obtained by taking complements. Assume that the assertion holds for $m < n$. Let $f \in \mathcal{T}_n$ and assume that $f = g \otimes h$ for some $g \in \mathcal{T}_m$, $h \in \mathcal{T}_{n-m}$. By the assumption,

$$p_{I_g} \otimes p_{I_h} \leq g \otimes h \leq p_{O_g}^* \otimes p_{O_h}^* \leq (p_{O_g} \otimes p_{O_f})^*,$$

the last inequality follows from Lemma lemma:fproduct 1. We have $O_f = O_g \cup (m + O_h)$, $I_f = I_g \cup (m + I_h)$, so that $p_{O_f} = p_{O_g} \otimes p_{O_h}$ and similarly for p_{I_f}. Now notice that any $f \in \mathcal{T}_n$ is either of the form $(f \otimes g) \circ \sigma$ or of the form $(f \otimes g)^* \circ \sigma$, for some permutation σ. Since the inequality is easily seen to be preserved by permutations, and reversed by duality which also swiches the input and output sets, the assertion is proved.

□

Combining this with the remarks below Lemma lemma:Xf 7, we get the following result (cf. cite). Recall that a bimorphism in a category is a morphism $X \xrightarrow{\varphi} Y$ which is both mono and epi, that is, such that for any pairs of arrows (with appropriate sources and targets) we have $\psi \circ f = \psi \circ g \iff f = g$ and $k \circ \psi = l \circ \psi \iff k = l$. It can be shown that the bimorphisms in Af are precisely those morphisms that are given by isomorphisms in FinVect.

:setting **Theorem 3.** *Let $Y \sim \{Y_1, \dots, Y_n\}$ be such that $O_Y = O$, $I_Y = I$. Then there exist bimorphisms*

$$Y_I^* \otimes Y_O \xrightarrow{\varphi} Y \xrightarrow{\psi} [Y_I, Y_O].$$

The bimorphisms are given by permutations.

We will see below that $\mathcal{T}_n$ is not a lattice for $n \geq 2$, so that for $f_1, f_2 \in \mathcal{T}_n$, neither of $f_1 \wedge f_2$ or $f_1 \vee f_2$ has to be a type function. Nevertheless, we have by the above results that if $O_{f_1} = O_{f_2}$,

$$Y_I^* \otimes Y_0 \xrightarrow{\varphi} X_{f_1 \wedge f_2} \xrightarrow{id_V} X_{f_1 \vee f_2} \xrightarrow{\psi} [Y_I, Y_O]$$

for some suitable bimorphisms φ, ψ, moreover, the objects $X_{f_1 \wedge f_2}$ and $X_{f_1 \vee f_2}$ are obtained as a pullback resp. pushout. It follows that although these objects may not be higher order objects themselves, they are included in some higher order object (e.g. $[Y_I, Y_O]$) with the same sets of inputs and outputs.

We finish this section by showing a simple way to obtain the output set of a type function.

_outputs **Proposition 7.** *For $f \in \mathcal{T}_n$, $i \in O_f$ if and only if $f(e^i) = 1$.*

Proof. Let $i \in O_f$, then by Lemma lemma:fh_setting 8, $p_{I_f}(e^i) = 1 \leq f(e^i)$, so that $f(e^i) = 1$. Conversely, if $f(e^i) = 1$, then by the other inequality in lemma lemma:fh_setting 8, $p_{O_f}(e^i) = 0$, whence $i \in O_f$.

□

3 Characterizations of type functions

We have the following description of the sets of type functions.

type_min **Proposition 8.** *The set $\mathcal{T}_n$ is the smallest subset in $\mathcal{F}_n$ such that:*

1. *$\mathcal{T}_n$ is invariant under permutations: if $f \in \mathcal{T}_n$, then $f \circ \sigma \in \mathcal{T}_n$ for any permutation $\sigma \in S_n$,*
2. *$\mathcal{T}_n$ is invariant under taking duals: if $f \in \mathcal{T}_n$ then $f^* \in \mathcal{T}_n$,*
3. *$\mathcal{T}_m \otimes \mathcal{T}_n \subseteq \mathcal{T}_{m+n}$,*
4. *$\mathcal{T}_1 = \{1, p_1\} = \mathcal{F}_1$.*

Proof. It is clear by construction that any system of subsets $\{\mathcal{S}_n\}_n$ with these properties must contain the type functions and that $\{\mathcal{T}_n\}_n$ itself has these properties.

□

Our goal is to find some characterization of the type functions. We start by looking at some examples and non-examples.

exm:T2 *Example* 9. The type functions for $n = 2$ are given as

$$s \quad \mapsto \quad 1, \quad \bar{s}_1\bar{s}_2, \quad \bar{s}_1, \quad 1 - \bar{s}_1 + \bar{s}_1\bar{s}_2,$$

and functions obtained from these by exchanging $s_1 \leftrightarrow s_2$, which gives 6 elements. It can be seen that $\mathcal{F}_n$ has 2^{2^n-1} elements, so that $\mathcal{F}_2$ has 8 elements in total. The two of them that are not type functions are

$$g(s) = 1 - \bar{s}_1 - \bar{s}_2 + 2\bar{s}_1 s_2, \qquad g^*(s) = \bar{s}_1 + \bar{s}_2 - \bar{s}_1\bar{s}_2.$$

This can be checked directly from Propositions 8 and 7. Indeed, if $g \in \mathcal{T}_2$, we would have $O_g = \emptyset$, so that $p_2 \leq g \leq p_\emptyset^* = p_2$, which is obviously not the case. Clearly, also the complement $g^* \notin \mathcal{T}_2$. Notice also that $g^* = p_{\{1\}} \vee p_{\{2\}}$, so that $\mathcal{T}_2$ is not a lattice. Since $\mathcal{F}_2$ can be identified as a sublattice in $\mathcal{F}_n$ for all $n \geq 2$ as $\mathcal{F}_2 \ni f \mapsto f \otimes 1_{n-2} \in \mathcal{F}_n$, we see that $\mathcal{T}_n$, $n \geq 2$ is a subposet in the distributive lattice $\mathcal{F}_n$ but itself not a lattice.

3.1 The poset $\mathcal{P}_f$

It will be convenient to use the representation

$$f = \sum_{S \subseteq [n]} \hat{f}_S p_S,$$

of a boolean function f obtained in Theorem 1. Let $\mathcal{P}_f$ be the subposet in $\mathcal{L}_n$ of elements such that $\hat{f}_S \neq 0$. We will show that if f is a type function, it is fully determined by $\mathcal{P}_f$.

We say that a poset $\mathcal{P}$ is graded of rank k if every maximal chain in $\mathcal{P}$ has the same length equal to k (recall that the length of a chain is defined as number of its elements -1). Equivalently, there is a unique rank function $\rho : \mathcal{P} \to \{0, 1, \ldots, k\}$ such that $\rho(S) = 0$ if S is a minimal element of $\mathcal{P}$ and $\rho(T) = \rho(S) + 1$ if T covers S, that is, $S \leq T$ and for any R such that $S \leq R \leq T$ we have $R = T$ or $R = S$.

Proposition 9. *Let $f \in \mathcal{T}_n$, then $\mathcal{P}_f$ is a graded poset with even rank $k \leq n$. If ρ is the rank function, then we have*

$$f = \sum_{S \in \mathcal{P}_f} (-1)^{\rho(S)} p_S.$$

See [Stanley] for details.

Then rank of $\mathcal{P}_f$ will be denoted by $r(f)$ and called the rank of f. Note that the assertion means that for $f \in \mathcal{T}_n$,

$$\hat{f}_S = \begin{cases} (-1)^{\rho(S)}, & \text{if } S \in \mathcal{P}_f \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We first note that the property in the statement is invariant under permutations and complements. Assume the statement holds for f and let us take any $\sigma \in \mathscr{S}_n$. From Proposition 1 that $\widehat{f \circ \sigma}_S = \hat{f}_{\sigma(S)}$ so that $S \mapsto \sigma(S)$ is an isomprphism of $\mathcal{P}_{f\circ\sigma}$ onto $\mathcal{P}_f$. Hence if $\mathcal{P}_f$ is graded with rank function ρ, then $\mathcal{P}_{f\circ\sigma}$ is graded with the same rank and has rank function $\rho \circ \sigma$. By the assumption we have

$$f \circ \sigma = \sum_{S \in \mathcal{P}_f} (-1)^{\rho(S)} p_S \circ \sigma = \sum_{S \in \mathcal{P}_f} (-1)^{\rho(S)} p_{\sigma^{-1}(S)} = \sum_{S \in \mathcal{P}_{f\circ\sigma}} (-1)^{\rho\circ\sigma(S)} p_S.$$

For the complement, we have from the assumption and Proposition 1(ii) that

$$f^* = (1 - \hat{f}_\emptyset)1 - \sum_{\substack{S \in \mathcal{P}_f \\ \emptyset \neq S, [n] \neq S}} (-1)^{\rho(S)} p_S + (1 - \hat{f}_{[n]}) p_n. \tag{8}$$

If $\emptyset \in \mathcal{P}_f$, then $\emptyset$ is the least element of $\mathcal{P}_f$, so that $\rho(\emptyset) = 0$ and therefore $\hat{f}_\emptyset = (-1)^0 = 1$. Similarly, if $[n] \in \mathcal{P}_f$, then $[n]$ is the largest element in $\mathcal{P}_f$, hence it is the last element in any maximal chain. It follows that $\rho([n]) = k$ and hence $\hat{f}_{[n]} = (-1)^k = 1$ (since k is even). Therefore the equality (8) implies that $\mathcal{P}_{f^*}$ differs from $\mathcal{P}_f$ only in the bottom and top elements: $\emptyset \in \mathcal{P}_f$ iff $\emptyset \notin \mathcal{P}_{f^*}$ and $p_n \in \mathcal{P}_f$ iff $p_n \notin \mathcal{P}_{f^*}$. It follows that $\mathcal{P}_{f^*}$ is graded as well, with rank equal to $k-2$, k or $k+2$, which in any case is even. Furthermore, let ρ^* be the rank function of f^*, then this also implies that for all $S \in \mathcal{P}_f$, $S \notin \{\emptyset, [n]\}$, we have $\rho^*(S) = \rho(S) \pm 1$, according to whether $\emptyset$ was added or removed. The statement now follows from (8).

We now proceed by induction on n. For $n = 1$, we have $\mathcal{L}_1 = \{\emptyset, [1]\}$ and $\mathcal{T}_1 = \{1, 1^*\}$. For $f = 1$, $\mathcal{P}_f = \{\emptyset\}$ is a singleton, which is clearly a graded poset, with rank $k = 0$ and trivial rank function ρ. We have

$$f = 1 = p_\emptyset = (-1)^{\rho(\emptyset)} p_\emptyset.$$

The proof for $f = 1^*$ is similar, replacing $\emptyset$ by $[1]$.

To finish the proof, assume that the statement is true for $m < n$ and let $f \in \mathcal{T}_n$. Then f is either a permutation of a product of some $f_1 \in \mathcal{F}_m$ and $f_2 \in \mathcal{T}_{n-m}$, or a dual of such an element. By the first part of the proof, we only need to prove that the statement holds for $f = f_1 \otimes f_2$. But in this case, by the induction assumption, $\mathcal{P}_{f_i}$ is graded with even rank k_i and rank function ρ_i. We also have

$$f = f_1 \otimes f_2 = \sum_{S \subseteq [m], T \subseteq [n-m]} (\widehat{f_1})_S (\widehat{f_2})_T p_S p_T = \sum_{S \subseteq [m], T \subseteq [n-m]} (-1)^{\rho_1(S) + \rho_2(T)} p_{S \dot\cup T}.$$

It follows that $\mathcal{P}_f = \mathcal{P}_{f_1} \times \mathcal{P}_{f_2}$ is the product of the two posets, which is a graded poset with rank $k = k_1 + k_2$ and rank function $\rho = \rho_1 + \rho_2$. This proves the statement.

□

We next show that the input and output sets of $f \in \mathcal{T}_n$ can be obtained from $\mathcal{P}_f$. For an index $i \in [n]$, let $M_{f,i}$ be the set of minimal elements of the subposet $\{S \in \mathcal{P}_f,\ i \in S\}$. Note that $M_{f,i}$ may be empty.

:pfinput **Proposition 10.** *Let $f \in \mathcal{T}_n$ and $i \in [n]$. Then*

1. *If $M_{f,i} \neq \emptyset$, then all elements in $M_{f,i}$ have the same rank, which will be denoted by $r_f(i)$. If $M_{f,i} = \emptyset$, we put $r_f(i) := r(f) + 1$.*

2. *$i \in O_f$ if and only if $r_f(i)$ is odd.*

Proof. Since $\mathcal{P}_f \simeq \mathcal{P}_{f\circ\sigma}$, it is quite clear that the two properties are preserved by permutations. We will show that they are preserved by complementation. Observe first that $M_{f,i} = \emptyset$ if and only if $M_{f^*,i} = \{[n]\}$, since $\mathcal{P}_{f^*}$ differs from $\mathcal{P}_f$ only up to adding/removing the least element $\emptyset$ and the greatest element $[n]$. If $M_{f,i}$ is empty, then $p_S(e^i) = 1$ for all $S \in \mathcal{P}_f$, so that $f(e^i) = f(0) = 1$ and $i \in O_f$, we also see that $r_f(i) = r(f) + 1$ is odd. If $\mathcal{P}_{f,i} = [n]$, then $r_f(i) = \rho_f([n]) = r(f)$ by definition of the rank, hence $r_f(i)$ is even. As we have seen, $i \in O_{f^*} = I_f$.

Let us assume that $M_{f,i}$ is not equal to $\emptyset$ or $\{[n]\}$. Then we must have $M_{f,i} = M_{f^*,i}$ and by the proof of Proposition 9 we have $\rho_{f^*}(S) = \rho_f(S) \pm 1$ for any S, depending only on the fact whether $\emptyset \in \mathcal{P}_f$. This implies that the properties are preserved by complementation.

We will now proceed by induction on n as before. Both assertions are quite trivial for $n = 1$, so assume the statements hold for $m < n$. It is enough to assume that $f = g \otimes h$ for some $g \in \mathcal{T}_m$ and $h \in \mathcal{T}_{n-m}$. Suppose without loss of generality that $i \in [m]$, then all elements of $M_{f,i}$ have the form $S\dot{\cup}T$, with $S \in M_{g,i}$ and T a minimal element in $\mathcal{P}_h$. Since $\rho_h(T) = 0$ for any minimal element $T \in \mathcal{P}_h$, we have by the induction assumption

$$\rho_f(S\dot{\cup}T) = \rho_g(S) + \rho_h(T) = \rho_g(S) = r_g(i).$$

The statement (ii) follows from the fact that $i \in O_f$ if and only if $i \in O_g$.

□

oro:free **Corollary 1.** *We have $\cap\mathcal{P}_f \in I_f$, $[n] \setminus \cup\mathcal{P}_f \in O_f$.*

Proof. If $i \in \cap\mathcal{P}_f$, then clearly $M_{f,i}$ is the set of minimal elements in $\mathcal{P}_f$, so that $r_f(i) = 0$ and $i \in I_f$ by Proposition 10. If $i \notin S$ for any $S \in \mathcal{P}_f$, then $M_{f,i} = \emptyset$ and $r_f(i) = r(f) + 1$ is odd. Hence $i \in O_f$.

□

Let us denote $F_{f,in} := \cap\mathcal{P}_f$, $F_{f,out} := [n] \setminus \cup\mathcal{P}_f$. Elements of these sets will be called free inputs resp. outputs. It is easily seen that we have

$$f = p_{F_{f,in}} \otimes g \otimes 1_{F_{f,out}}$$

for some type function g with no free inputs or outputs.

Examples/NOnexamples?

Obrazky?

3.2 Chains and combs

Let $\mathcal{P} = \{S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_N\}$ be a chain in $\mathcal{L}_n$. Then $\mathcal{P}$ is graded with rank $N-1$ and rank function $\rho(S_i) = i - 1$.

p:chains **Proposition 11.** *For a chain $\mathcal{P} = \{S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_N\}$, the function*

$$f = f_{\mathcal{P}} := \sum_i (-1)^{i-1} p_{S_i}$$

is a type function if and only if N odd.

Proof. By Proposition prop:graded 9, if $f \in \mathcal{T}_n$, then the rank of f must be even, so that N must be odd. We will show that the converse is true. We proceed by induction on N. For $N = 1$, we have $f = p_{S_1} \in \mathcal{T}_n$. Assume that the statement holds for all odd numbers $M < N$ and let $\mathcal{P}$ be a chain as above. Then we have

$$f = p_{S_1} \otimes g \otimes 1_{[n]\setminus S_N}$$

where g is the function for the chain $\emptyset = S'_1 \subsetneq S'_2 \subsetneq \cdots \subsetneq S'_N$, with $S'_i := S_i \setminus S_1$. Since f is a type function if g is, this shows that we may assume that the chain contains $\emptyset$ and $[n]$. But then

$$f = 1 + \sum_{j=2}^{N-1} (-1)^{j-1} p_{S_j} + p_n$$

and

$$f^* = 1 - f + p_n = \sum_{j=1}^{N-2} (-1)^{j-1} p_{T_j},$$

where $T_j := S_{j+1}$. By the induction assumption $f^* \in \mathcal{T}_n$, hence also $f = f^{**} \in \mathcal{T}_n$. $\square$

As we can see from Example exm:T2 9, all elements in $\mathcal{T}_2$ are chains. This is also true for $n = 3$. Indeed, up to a permutation that does not change the chain structure, any $f \in \mathcal{T}_3$ is either a product of two elements $g \in \mathcal{T}_2$ and $h \in \mathcal{T}_1$, or the dual of such a product. Since g must be a chain and $|\mathcal{P}_h| = 1$, their product must be a chain as well. Taking the dual of a chain only adds/removes the least/largest elements, so the dual of a chain must be a chain as well.

We now show that chains correspond to important higher order objects. Let k be odd and let $\mathcal{P} = \{\emptyset \subsetneq S_1 \subsetneq \cdots \subsetneq S_k \subsetneq [n]\}$. Let $f = f_{\mathcal{P}}$, then f is a type function. By Proposition prop:type_hom 6, for any first order objects $X_1, \ldots, X_n$, $Y = X_f$ is a higher order object such that $Y \sim \{Y_1, \ldots, Y_n\}$.

ns_combs **Proposition 12.** *Let $T_1 = S_1$, $T_i = S_i \setminus S_{i-1}$ for $i = 1, \ldots, k$ and $T_{k+1} = S_k^c$. For $S \subseteq [n]$, we denote $Y_S := \otimes_{i \in S} Y_i$. Then for $k = 1$, $Y \simeq [Y_{T_2}, Y_{T_1}]$ and for any odd $k > 1$,*

$$Y \simeq [Y_{T_{k+1}}, [[Y_{T_k}, [[....], Y_{T_2}]], Y_{T_1}]].$$

Remark (quantum) comb (examples)

Proof. It is easily checked by Proposition prop:fh_outputs 7 that $T_i \subseteq O_f$ if i is odd and $T_i \subseteq I_f$ otherwise. We therefore have

$$Y_{T_i} = \begin{cases} \otimes_{i \in T_i} X_i, & \text{if } i \text{ is odd,} \\ \otimes_{i \in T_i} \tilde{X}_i, & \text{if } i \text{ is even.} \end{cases}$$

Let $k = 1$, then $f = 1 - p_{T_1} + p_n$, and $f^* = p_{T_1} = p_{T_1} \otimes 1_{T_2}$. Then $X_f = \tilde{X}^*_{f^*}$, and we see that $\tilde{X}_{f^*} = X^*_{T_1} \otimes \tilde{X}_{T_2} = Y^*_{T_1} \otimes Y_{T_2}$. It follows that $X_f = (Y^*_{T_1} \otimes Y_{T_2})^* \simeq [Y_{T_2}, Y_{T_1}]$, where the isomorphism is given by swapping the spaces $V^*_{T_1}$ and V_{T_2}. Assume the assertion is true for $k-2$. As in the proof of Proposition 11, we see that

$$f^* = \sum_{i=1}^{k} (-1)^{i-1} p_{S_i} = p_{T_1} \otimes g \otimes 1_{T_{k+1}}$$

where $g = 1 - \sum_{i=2}^{k-1} (-1)^{i-1} p_{S'_i} + p_{S'_k}$ is the function for the chain $\mathcal{P}' = \{\emptyset \subsetneq S'_2 \subsetneq \cdots \subsetneq S'_k\}$ in $S'_k \simeq [n']$ for $n' = |S'_k|$, $S'_i = S_i \setminus S_1 = \cup_{j=2}^{i} T_j$. We have

$$X_f = \tilde{X}^*_{f^*} = (X^*_{Y_1} \otimes \tilde{X}_g \otimes \tilde{X}_{T_{k+1}})^* \simeq (\tilde{X}_{T_{k+1}} \otimes [\tilde{X}_g, X_{T_1}]^*)^* = [\tilde{X}_{T_{k+1}}, [\tilde{X}_g, X_{T_1}]].$$

Here we have used the fact that $\tilde{X}_{f^*}$ is constructed from $\tilde{X}_1, \dots, \tilde{X}_n$. By induction assumption, we get

$$[\tilde{X}_{T_{k+1}}, [\tilde{X}_g, X_{T_1}]] = [\tilde{X}_{T_{k+1}}, [[X_{T_k}, [[\dots], \tilde{X}_{T_2}]], X_{T_1}],$$

which is as required.

$\square$

Combs, picture, without $\emptyset$ or $[n]$? Free inputs and outputs!

3.3 Connecting chains: the causal product

We will introduce further operations of boolean functions. For $f_1 : \{0,1\}^m \to \{0,1\}$, $f_2 : \{0,1\}^n \to \{0,1\}$, we define

$$f_1 \lhd f_2 := f_1 \otimes 1_n + p_m \otimes (f_2 - 1), \qquad f_1 \rhd f_2 := 1_m \otimes f_2 + (f_1 - 1) \otimes p_n.$$

Lemma 9. *Let $f_1 \in \mathcal{F}_m$, $f_2 \in \mathcal{F}_n$. Then $f_1 \lhd f_2 \in \mathcal{F}_{n+m}$ and we have*

(i) $f_1 \lhd (f_2 \lhd f_3) = (f_1 \lhd f_2) \lhd f_3$,

(ii) $(f_1 \lhd f_2)^* = f_1^* \lhd f_2^*$,

(iii) $(f_1 \wedge f_2) \lhd f_3 = (f_1 \lhd f_3) \wedge (f_2 \lhd f_3)$ *and* $(f_1 \vee f_2) \lhd f_3 = (f_1 \lhd f_3) \vee (f_2 \lhd f_3)$,

(iv) $f_1 \lhd (f_2 \wedge f_3) = (f_1 \lhd f_2) \wedge (f_1 \lhd f_3)$ *and* $f_1 \lhd (f_2 \vee f_3) = (f_1 \lhd f_2) \vee (f_1 \lhd f_3)$.

Similar properties hold for $\rhd$. Moreover,

(v) $f_1 \rhd f_2 = (f_2 \lhd f_1) \circ \pi$, *where π is the permutation that acts on $\{0,1\}^m \times \{0,1\}^n$ as $\pi(s^1 s^2) = s^2 s^1$.*

(vi) $f_1 \otimes f_2 = (f_1 \lhd f_2) \wedge (f_1 \rhd f_2) = (f_1 \lhd f_2) \wedge (f_2 \lhd f_1) \circ \pi$.

Proof. The first assertion follows easily from

$$f_1 \lhd f_2(s^1 s^2) = \begin{cases} f_1(s^1), & \text{if } s^1 \neq 0_m, \\ f_2(s^2), & \text{if } s^1 = 0_m, \end{cases}$$

similarly for $\rhd$. The statements (i)–(v) follow by straightforward calculations and the equality above. To prove (vi), let $s^1 \in \{0,1\}^m$, $s^2 \in \{0,1\}^n$ and compute

$$\begin{aligned}(f_1 \lhd f_2) \wedge (f_1 \rhd f_2)(s^1 s^2) &= \big(f_1(s^1) + p_m(s^1)(f_2(s^2) - 1)\big)\big(f_2(s^2) + p_n(s^2)(f_1(s^1) - 1)\big) \\ &= f_1(s^1) f_2(s^2),\end{aligned}$$

the last equality follows from the fact that $f_i(s^i)(1 - f_i(s^i)) = 0$ (since $f_i(s^i) \in \{0,1\}$) and the fact that p_m is the least element in $\mathcal{F}_m$, so that $p_m(s^1)(f_1(s^1) - 1) = 0$.

$\square$

It is not clear that if f_1 and f_2 are type functions, then $f_1 \lhd f_2$ or $f_1 \rhd f_2$ are type functions as well. Nevertheless, we next show that this is true for chains.

d_chains **Proposition 13.** *Let $\mathcal{P}_1 = \{S_1 \subsetneq \cdots \subsetneq S_M\}$ be a chain in $[m]$ and $\mathcal{P}_2 = \{T_1 \subsetneq \cdots \subsetneq T_N\}$ a chain in $[n]$, with corresponding functions β_1 and β_2. Assume that both M and N are odd, so that β_1 and β_2 are type functions. Then $\beta = \beta_1 \lhd \beta_2$ is a type function corresponding to a chain of $M + N \pm 1$ elements in $[m+n]$, with $O_\beta = O_{\beta_1} \dot\cup O_{\beta_2}$, $I_\beta = I_{\beta_1} \dot\cup I_{\beta_2}$. A similar statement holds for $\rhd$.*

Proof. We have

$$\beta_1 = \sum_{j=1}^{M} (-1)^{j-1} p_{S_j}, \qquad \beta_2 = \sum_{k=1}^{N} (-1)^{k-1} p_{T_k},$$

so that

$$\beta = \sum_{j=1}^{M-1} (-1)^{j-1} p_{S_j} + (p_{S_M} - p_m + p_{[m]\dot\cup T_1}) + \sum_{k=2}^{N} (-1)^{k-1} p_{[m]\dot\cup T_k}.$$

The resulting funnction depends on whether $S_M = [m]$ and $T_1 = \emptyset$. If at least one of the equalities is true, then the expression in brackets is equal to $p_{[m]}$, p_{S_M} or $p_{[m]\dot\cup T_1}$ and β corresponds to a chain of $M + N - 1$ elements. If both $S_M \neq [m]$ and $T_1 \neq \emptyset$, then $p_{S_M} \neq p_m \neq p_{[m]\dot\cup T_1}$ and β corresponds to a chain of $M + N + 1$ elements.

For any $i \in [m]\dot\cup[n]$, we have $e^i_{m+n} = e^j_m 0_n$ or $e^i_{m+n} = 0_m e^k_n$ for some $j \in [m]$, $k \in [n]$. Then

$$\beta(e^i) = \beta_1(e^j_m) \ \text{ or } \beta(e^i) = \beta_2(e^k_n).$$

The statement on input/output indices follow from Lemma 8. lemma:fh_setting

$\square$

tructure **Theorem 4.** *For any $f \in \mathcal{T}_n$, there are chains $\beta_1 \in \mathcal{T}_{n_1}, \ldots, \beta_k \in \mathcal{T}_{n_k}$, $n = n_1 + \cdots + n_k$, such that*

$$f = \bigvee_{a \in A} \bigwedge_{b \in B} (\beta_{\pi^{-1}_{a,b}(1)} \lhd \ldots \lhd \beta_{\pi^{-1}_{a,b}(k)}) \circ \rho_{a,b}$$

for some finite index sets A and B, where $\pi_{a,b} \in \mathscr{S}_k$ and $\rho_{a,b} = \rho_{\pi_{a,b}} \circ \sigma$ for some $\sigma \in \mathscr{S}_n$ and $\rho_{\pi_{a,b}}(s^1 \ldots s^k) = s^{\pi^{-1}_{a,b}(1)} \ldots s^{\pi^{-1}_{a,b}(k)}$, $a \in A$, $b \in B$.

Proof. It is quite clear that the condition is invariant under permutations. Assume f can be written in the given form, then

$$f^* = \bigwedge_{a \in A} \bigvee_{b \in B} (\beta^*_{\pi^{-1}_{a,b}(1)} \lhd \ldots \lhd \beta^*_{\pi^{-1}_{a,b}(k)}) \circ \rho_{a,b}.$$

Since $\mathcal{F}_n$ is a distributive lattice, this can be rewritten as

$$f^* = \bigvee_{a^*\in B^{|A|}} \bigwedge_{b^*\in A} (\beta^*_{\pi^{-1}_{a^*,b^*}(1)} \lhd \ldots \lhd \beta^*_{\pi^{-1}_{a^*,b^*}(k)}) \circ \rho_{a^*,b^*},$$

where for $a^* = (b_a)_{a\in A}$ and $b^* = a$, we have $\pi_{a^*,b^*} = \pi_{a,b_a}$, and $\rho_{a^*,b^*} = \rho_{\pi_{a^*,b^*}} \circ \sigma$. Since β^*_j are chains in $[n_j]$, the assertion is true also for f.

Since for $n = 1$, $f \in \mathcal{T}_n$ is itself a (trivial) chain, the assertion is true in this case (note that it is also trivially true for $n = 2$ and $n = 3$, since all elements in $\mathcal{T}_2$ and $\mathcal{T}_3$ are chains). Proceeding by induction, it is now enough to show this form for $f = f_1 \otimes f_2$, where $f_1 \in \mathcal{T}_m$, $f_2 \in \mathcal{T}_{n-m}$ satisfy the conditions, so that

$$f_1 = \bigvee_{a\in A} \bigwedge_{b\in B} (\beta^1_{\pi^{-1}_{a,b}(1)} \lhd \ldots \lhd \beta^1_{\pi^{-1}_{a,b}(k_1)}) \circ \rho_{\pi_{a,b}} \circ \sigma_1, \qquad f_2 = \bigvee_{c\in C} \bigwedge_{d\in D} (\beta^2_{\pi^{-1}_{c,d}(1)} \lhd \ldots \lhd \beta^2_{\pi^{-1}_{c,d}(k_2)}) \circ \rho_{\pi_{c,d}} \circ \sigma_2.$$

for some chains $\beta^1_j \in \mathcal{T}_{m_j}$, $\sum_j m_j = m$, and $\beta^2_j \in \mathcal{T}_{l_j}$, $\sum_j l_j = n-m$ and permutations $\pi_{a,b} \in \mathscr{S}_{k_1}$, $\pi_{c,d} \in \mathscr{S}_{k_2}$, $\sigma_1 \in \mathscr{S}_m$, $\sigma_2 \in \mathscr{S}_{n-m}$. Using properties of the tensor product, we get

$$f_1 \otimes f_2 = \bigvee_{a\in A, c\in C} \bigwedge_{b\in B, d\in D} (\beta^1_{\pi^{-1}_{a,b}(1)} \lhd \ldots \lhd \beta^1_{\pi^{-1}_{a,b}(k_1)}) \otimes (\beta^2_{\pi^{-1}_{c,d}(1)} \lhd \ldots \lhd \beta^2_{\pi^{-1}_{c,d}(k_2)}) \circ (\rho_{\pi^1_{a,b}} \otimes \rho_{\pi^2_{c,d}}) \circ (\sigma_1 \otimes \sigma_2).$$

Let

$$\beta^{a,b}_1 := \beta^1_{\pi^{-1}_{a,b}(1)} \lhd \ldots \lhd \beta^1_{\pi^{-1}_{a,b}(k_1)}, \qquad \beta^{c,d}_2 := \beta^2_{\pi^{-1}_{c,d}(1)} \lhd \ldots \lhd \beta^2_{\pi^{-1}_{c,d}(k_2)}.$$

Then $\beta^{a,b}_1 \in \mathcal{T}_m$ and $\beta^{c,d}_2 \in \mathcal{T}_{n-m}$ are chains and we have

$$\beta^{a,b}_1 \otimes \beta^{c,d}_2 = (\beta^{a,b}_1 \lhd \beta^{c,d}_2) \wedge (\beta^{c,d}_{\xi^{-1}(1)} \lhd \beta^{a,b}_{\xi^{-1}(2)}) \circ \rho_\xi,$$

where ξ is the swap $1 \leftrightarrow 2$ and ρ_ξ acts on $[n] = [m]\dot\cup[n]$ as $\rho_\xi(s^1 s^2) = s^2 s^1$. Taking into account the decompositions $[m] = \dot\cup_j [m_j]$ and $[n-m] = \dot\cup_j [l_j]$, we see that $\rho_\xi = \rho_\varpi$ for a permutation $\varpi \in \mathscr{S}_{k_1+k_2}$ that swaps the two blocks $[k_1]$ and $[k_2]$.

Put $\beta_j := \beta^1_j$, $j = 1, \ldots, k_1$ and $\beta_{k_1+j} := \beta^2_j$, $j = 1, \ldots k_2$. Then β_j, $j = 1, \ldots, k := k_1 + k_2$ are chains, $\beta_j \in \mathcal{T}_{n_j}$, where $n_j := m_j$, $j = 1, \ldots, k_1$, $n_{k_1+j} := l_j$, $j = 1, \ldots, k_2$ and $\sum_{j=1}^k n_j = n$.

To get the permutation sets, let $A' = A \times C$, $B' = \{0, 1\} \times B \times D$. Let $\pi_{(a,c),(0,b,d)} = \pi_{a,b} \dot\cup \pi_{c,d}$ and $\pi_{(a,c),(1,b,d)} = \varpi \circ (\pi_{a,b} \dot\cup \pi_{c,d})$. Put also $\sigma = \sigma_1 \otimes \sigma_2$. Then

$$f = \bigvee_{a'\in A'} \bigwedge_{b'\in B'} (\beta_{\pi^{-1}_{a',b'}(1)} \lhd \ldots \lhd \beta_{\pi^{-1}_{a',b'}(k)}) \circ \rho_{\pi_{a',b'}} \circ (\sigma_1 \otimes \sigma_2).$$

The proof is complete.

□

Examples??

Let f be a function of the form as in Theorem thm:structure 4. Since all the chains $\beta_{a,b} := \beta_{\pi^{-1}_{a,b}(1)} \lhd \ldots \lhd \beta_{\pi^{-1}_{a,b}(k)} \circ \rho_{a,b}$ in the decomposition have the same input and output indices, they must satsfy the inequality $p_I \leq \beta_{a,b} \leq p^*_O$. But then the same is true for f. It follows that although we do not know whether f is a type function, the corresponding object X_f is always included in a set of channels.

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