Optimization of Higher-order Quantum Structures -

Sequential Implementations

MINIMAL THESIS

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By

Soham Sau

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Principal Supervisor:

Doc. Mgr. Mário ZIMAN, PhD.

Institute of Physics,

Slovak Academy of Sciences

Co-supervisor:

Mgr. Michal SEDLÁK, PhD.

Institute of Physics,

Slovak Academy of Sciences

Abstract

In this thesis proposal, we have started with minimal exposure to the mathematical structures required for operational quantum theory. We have discussed the concepts of a state, observables, and channels. Then, we give a description of the measurement process and define quantum instruments. Quantum instruments describe a measurement where the post-measurement state is taken into account along with the measurement statistics. We have discussed the properties of quantum instruments and proved how they can implemented sequentially.

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I want to thank Srijita for constantly being present with me despite being so far away from me in space and time.

Finally, I'd like to give myself a pat on the back for always having a comeback from my mental and physical health issues.

Soham Sau

That which does not kill us makes us stronger.

– Friedrich Nietzsche

Contents

List of Figures

Keywords

POVMs, Measurements, Quantum Instruments,
Sequential implementation of Instruments.

Chapter 1

Introduction

The theory of quantum mechanics defined in Hilbert space has been the basis of successful and comprehensive research into practically all branches of physics for nearly many decades. Theoretical predictions and experimental results do not appear to be significantly in conflict with one another over the long run. Experiments have unequivocally demonstrated that quantum mechanics possesses a probabilistic structure. *Operational quantum theory* [1, 2] is a description of the theory that makes complete use of the intricate probabilistic structures of the theory. In this theory, the observables are described by positive operator-valued measures(POVMs)[3], whereas the traditional notion of observables, introduced by von-Neumann [4] as self-adjoint operators, is a special case of the POVMs, which are represented by the projection valued-measures(PVMs).

The theory allows us to advance to greater abstractness by incorporating a post-measurement condition into the scenario. The term *'quantum instrument'* [5, 6, 7, 8] refers to a framework that describes the process of measurement in which the system being measured is accessible after the measurement has taken place. This description is considered to be a higher-order

description of a measurement, which will be discussed in subsection 3.1.8. This concept is useful when the measured system will be utilized to obtain further information or other forms of interaction. Quantum instruments will be the subject of discussion in this minimal thesis proposal.

The primary goal of this minimal thesis proposal is to explore the *'sequentiality'* of quantum instruments, by which we mean - whether one can implement a quantum instrument in a sequential manner. This, in general, leads to the idea of implementing a measurement in a sequential manner, which, apart from being interesting theoretically, is important experimentally; as always, it is not possible to implement one complex measurement in one single shot.

The material is organized in the following way.
In the second chapter, we provide the mathematical basis of operational quantum theory. The third chapter gives the physical ideas about experiments. In that chapter, we formally define quantum instruments and other concepts surrounding them. In the fourth chapter, we introduced the notion of an adaptive sequence of instruments. Then, we address the question of the sequential implementation of quantum instruments in that chapter. Finally, we conclude the thesis and provide perspectives on further research.

Chapter 2

Mathematical Preliminaries of Quantum Theory

In this chapter, we will discuss the preliminary mathematical ideas in quantum theory, which are required to understand the thesis. The definitions are mostly referred from standard pieces of literature in the community. We refer the readers to those sources for comprehensive proof of these results. A few of the sources are [4, 8, 9, 10, 11].

2.1 The Hilbert space formalism

One of the fundamental purposes of a physical theory is to explain and predict events observed in real-life experiments. Quantum theory is a physical theory that is probabilistic in nature, unlike its classical counterpart. In any physical theory, one maps a physical quantity to its mathematical analog. Here, in quantum theory, the consistent mathematical formalism is obtained by the so-called *Hilbert space formalism*. Here, we will restrict

ourselves to Hilbert spaces of finite dimensions, d.

First, we start with the definition of Hilbert space.

Definition 2.1. *A Hilbert space, denoted by $\mathcal{H}$ is a complete inner product space.*

By *completeness of* $\mathcal{H}$, we mean every Cauchy sequence in $\mathcal{H}$ is convergent. An *inner product space* is a complex vector space with a defined inner product. More commonly, it is also known as the *scalar product*. The mathematical aspects of quantum theory have been formulated around different aspects of $\mathcal{H}$. In the mathematical literature, the elements of the Hilbert space are frequently referred to as vectors or specifically normalized vectors. A more physical term for normalized vectors are *pure states*, which will be explained later.

The elements of the Hilbert spaces are denoted by $|\ \rangle$, following the Bra ($\langle\ |$)-Ket($|\ \rangle$) notation, also known as the Dirac notation. So, for an element ψ of $\mathcal{H}$ is denoted as $|\psi\rangle$.

- *Orthonormal basis.* A set of elements $\{|\varphi_i\rangle\}_{i=1}^{d}$ is called an orthonormal basis of a d-dimensional Hilbert space $\mathcal{H}$ if, for the inner product $\langle\varphi_i|\varphi_j\rangle = \delta_{ij}$ and if $\forall\psi \in \mathcal{H}$ we have $\langle\psi|\varphi_i\rangle = 0, \forall i \implies \psi = 0$.

Let $\mathcal{L}(\mathcal{H})$ be the set of all linear operators on $\mathcal{H}$. By the term 'operators on $\mathcal{H}$' we mean a linear mapping from $\mathcal{H}$ to $\mathcal{H}$, which, for an operator A is denoted by $A : \mathcal{H} \mapsto \mathcal{H}$. If an operator B maps from a Hilbert space $\mathcal{H}$ to another Hilbert space $\mathcal{K}$, we denote, $B : \mathcal{H} \mapsto \mathcal{K}$, and $B \in \mathcal{L}(\mathcal{H}, \mathcal{K})$. In this thesis, we will use the notation $I_{\mathcal{H}}$ for the identity operator on $\mathcal{H}$, which means $I_{\mathcal{H}}\ |\psi\rangle = |\psi\rangle\ , \forall\ |\psi\rangle \in \mathcal{H}$.

All of the sets of operators which will be defined later are subsets of $\mathcal{L}(\mathcal{H})$. A subspace of $\mathcal{L}(\mathcal{H})$ is the *state space*, $\mathcal{S}(\mathcal{H})$, which can be thought of as

the space of all quantum states. Quantum states, denoted by ρ, which are also known as *density matrices*, belong to the space $\mathcal{S}(\mathcal{H})$.

Definition 2.2. *A quantum state is described by positive trace class operators with unit trace.*

Mathematically, we write the state space as

$$\mathcal{S}(\mathcal{H}) = \{\rho \ \in \mathcal{T}(\mathcal{H}) | \rho \geq O, Tr[\rho] = 1\} \tag{2.1}$$

where $\mathcal{T}(\mathcal{H})$ is the space of trace class operators in $\mathcal{H}$ and O is the zero operator on $\mathcal{H}$. By the term trace class operators, we mean such operators for which the trace can be defined. The state space forms a convex set, i.e., for any two states belonging to $\mathcal{S}(\mathcal{H})$, any convex combination of them will also belong to $\mathcal{S}(\mathcal{H})$. An extremal element of the convex set $\mathcal{S}(\mathcal{H})$ is called a *pure state*, which has the property of $Tr[\rho]^2 = 1$. Any other element of $\mathcal{S}(\mathcal{H})$ is called a *mixed state*. We can view pure states as operators $\rho \in \mathcal{S}(\mathcal{H})$ such that $\exists \psi \in \mathcal{H}$, such that $\rho = |\psi\rangle \langle\psi|$, i.e. ρ is a projector onto the normalized vector $|\psi\rangle$.

Any state $\rho \in \mathcal{S}(\mathcal{H})$ can be written as $\rho = \sum_x c_x P_x$, where $\{c_x\}$ is a sequence of positive numbers such that $\sum_x c_x = 1$ and $\{P_j\}$ is a sequence of associated orthogonal one-dimensional projections, i.e. $P_j P_k = \delta_{jk} P_j$. The property that the projections $\{P_j\}$ are orthogonal is referred to as *'canonical'*. A state may have several canonical convex decompositions. A unique canonical convex decomposition exists only when every number c_x is distinct.

- *Superpositions.* Pure states have a special structure, referred to as superposition, and it is frequently considered to be the fundamental

characteristic of quantum mechanics. In classical state space, there is no analogy that can be used to describe superposition. Suppose we consider two pure states and let them be linearly independent, $|\psi\rangle, |\phi\rangle, \in \mathcal{H}$, the state $|\alpha\rangle := \frac{1}{||a|\psi\rangle + b|\phi\rangle||}(a|\psi\rangle + b|\phi\rangle)$ is called as a superposition of $|\psi\rangle$ and $|\phi\rangle$, for two nonzero complex numbers a, b. It is important to note that while pure states are associated with rays, a particular superposition relies on the selected representatives of the equivalence class.

- *Spectral decomposition.* For a normal ($A^\dagger A = AA^\dagger$) trace class operator $A \in \mathcal{L}(\mathcal{H})$, there exists a set of complex numbers $\{\lambda_i\}$ and an orthonormal basis $\{\eta_i\}$ such that

$$A = \sum_i \lambda_i |\eta_i\rangle \langle\eta_i|$$

This is a well-known result in Hilbert's space operator theory, known as the Spectral decomposition.

- *Singular value decomposition.* Let A be an operator such that $A : \mathcal{H} \mapsto \mathcal{K}$. One can find orthonormal basis $\{|\eta_i\rangle\} \in \mathcal{K}$ and $\{|\xi_i\rangle\} \in \mathcal{H}$ such that

$$A = \sum_i s_i(A) |\eta_i\rangle \langle\xi_i|$$

The positive numbers $s_i(A)$s are called the singular values of A.

2.2 Kernel and Support of positive operators

Here, we will define the Kernel and Support of an operator. For an operator $F \in \mathcal{L}(\mathcal{H})$, $F \geq 0$, the kernel of F is the set $ker(F)$, such that

$$ker(F) = \{|\psi\rangle \in \mathcal{H} | \, F \, |\psi\rangle = 0\}$$

Physically, a kernel is that subspace of the state space where its action on the state results in a null vector. The support of ρ, denoted by $supp(\rho)$, is the orthocomplement space of the kernel, i.e., $supp(F) \equiv ker(F)^{\perp}$.

Here, we will discuss in detail the support of a sum of two positive operators. We will prove two lemmas, which will be used later to prove our results.

Let us denote the kernel of an operator C by, $ker(C)$.

Lemma 2.3. *Let $D \in \mathcal{L}(\mathcal{H})$, be a positive semidefinite operator. Then, $\langle\psi|D|\psi\rangle = 0$ for a vector $|\psi\rangle \in \mathcal{H}$ if and only if $|\psi\rangle \in ker(D)$.*

Proof. Clearly, $|\psi\rangle \in ker(D)$ implies $\langle\psi|D|\psi\rangle = 0$. For the converse implication, we consider spectral decomposition $D = \sum_{i=1}^{r_D} d_i \, |v_i\rangle \langle v_i|$ of operator D, in which $d_i > 0$ and vectors $\{|v_i\rangle\}_{i=1}^{r_D}$ are orthonormal. We can extend these vectors into an orthonormal basis $\{|v_i\rangle\}_{i=1}^{d} \in \mathcal{H}$. Thus, an arbitrary pure state $|\psi\rangle \in \mathcal{H}$ can be decomposed as $|\psi\rangle = \sum_{i=1}^{d} c_i \, |v_i\rangle = \sum_{i=1}^{r_D} c_i \, |v_i\rangle + |\psi'\rangle$, and $1 = \sum_{i=1}^{r_D} |c_i|^2 + \langle\psi'|\psi'\rangle$. Expectation value $\langle\psi|D|\psi\rangle = 0$ implies $\sum_{i=1}^{r_D} |c_i|^2 = 0$ and consequently $c_i = 0$ for $i = 1, \ldots, r_D$, $|\psi\rangle = |\psi'\rangle$. Due to $\langle a_i|\psi'\rangle = 0$ for $i = 1, \ldots, r_D$ we have $D \, |\psi\rangle = 0$. Thus, we proved $|\psi\rangle \in ker(D)$, which concludes the proof. □

Lemma 2.4. *For two positive-semidefinite operators $A, B \in \mathcal{L}(\mathcal{H})$,*

$$supp(A + B) = span(supp(A), supp(B)) \tag{2.2}$$

Proof. Let us expand A and B using their spectral decompositions. So we have, $A = \sum_i e_i |w_i\rangle \langle w_i|$ and $B = \sum_j f_j |u_j\rangle \langle u_j|$ where $e_i, f_j > 0$. Consider a state $|\psi\rangle \in ker(A+B)$. From Lemma (2.3), we infer that $\langle\psi|A+B|\psi\rangle = 0$. Due to $A, B \geq 0$ this implies $\langle\psi|A|\psi\rangle = \langle\psi|B|\psi\rangle = 0$. So $|\psi\rangle \in ker(A)$ as well as $|\psi\rangle \in ker(B)$. Thus $ker(A+B) = ker(A) \cap ker(B)$.

As we can write $ker(A) = (span(\{|w_i\rangle\}_{i=1}^{r_A}))^\perp$ and $ker(B) = (span(\{|u_i\rangle\}_{i=1}^{r_B}))^\perp$, we conclude that any $|\psi\rangle \in ker(A+B)$ must be orthogonal to all vectors $\{w_i\}_{i=1}^{r_A}, \{u_i\}_{i=1}^{r_B}$ and consequently also to their linear combinations. This proves $ker(A+B) \subseteq (span(supp(A), supp(B)))^\perp$. On the other hand, for every $|\psi\rangle \in (span(supp(A), supp(B)))^\perp$ we clearly have $(A+B)|\psi\rangle = 0$, so $|\psi\rangle \in ker(A+B)$ and we conclude Eq. (2.4) holds. □

2.2.1 State space of a two-dimensional system

In this subsection, we will discuss the concepts of the state space in the case of a more commonly used scenario, the two-dimensional system [11]. In quantum theory, the two-dimensional Hilbert space, or the system it represents, is referred to as a qubit. In the case of qubits, the Hilbert space $\mathcal{H}$ is isomorphic to $\mathbb{C}^2$. Utilizing the so-called *Bloch sphere representation* is a practical method of illustrating the state space for a two-dimensional Hilbert space.

Let us consider an orthonormal basis $\{|\vartheta\rangle, |\vartheta^\perp\rangle\}$ in $\mathbb{C}^2$. The standard operator basis consists of

$$\begin{aligned}\sigma_0 &= I \\ \sigma_x &= |\vartheta\rangle\langle\vartheta^\perp| + |\vartheta^\perp\rangle\langle\vartheta| \\ \sigma_y &= -i\,|\vartheta\rangle\langle\vartheta^\perp| + i\,|\vartheta^\perp\rangle\langle\vartheta| \\ \sigma_z &= |\vartheta\rangle\langle\vartheta| - |\vartheta^\perp\rangle\langle\vartheta^\perp|\end{aligned} \tag{2.3}$$

where $i = \sqrt{-1}$. They are called the *Pauli Operators* and they satisfy the orthogonality relations $Tr[\vartheta_0\vartheta_i] = 0$ and $Tr[\vartheta_i\vartheta_j] = 2\delta_{ij}$ for $i, j = x, y, z$. In this basis, a general qubit state can be written as

$$\rho = \frac{1}{2}(I + \vec{r}\cdot\vec{\sigma}) \tag{2.4}$$

where $\vec{r} \in \mathbb{R}^3$ and the eigenvalues of ρ are $\frac{1}{2}(1 \pm \|r\|)$. This condition is equivalent to the condition $\|\vec{r}\| \leq 1$. Hence, we can infer that the states form a sphere of unit radius in $\mathbb{R}^3$, known as the *Bloch sphere*. Figure 2.1 represents the so-called Bloch sphere.

Composite systems

Let us consider two quantum systems A and B, having Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. We represent the Hilbert space of the composite system of A and B as $\mathcal{H}_A \otimes \mathcal{H}_B$. Let, states ρ_A and ρ_B belongs to $\mathcal{S}(\mathcal{H}_A)$ and $\mathcal{S}(\mathcal{H}_B)$ respectively. We denote the state $\rho_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ as a state belonging to the composite system.

If one can write the state of the composite system as $\rho_{AB} = \sum_i p_i\,\rho_{Ai} \otimes \rho_{Bi}$,

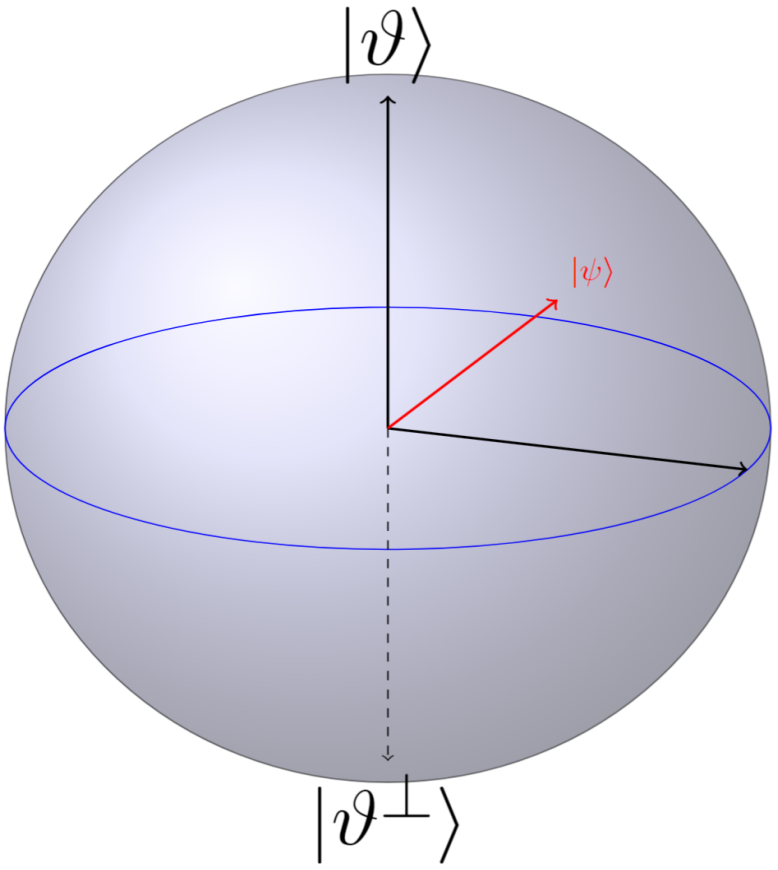

FIGURE 2.1: Diagrammatic representation of the Bloch-sphere. It represents the state space of a two-dimensional quantum system. $|\psi\rangle$ denotes a normalized (pure) state on the surface of the Bloch sphere

where p_i's are non zero real numbers, we call the state ρ_{AB} to be an *separable state*, else it is called a *entangled state*.

2.2.2 Effects

The idea of a state is fundamental to quantum theory. A state is typically understood to be a description of a collection of systems that are prepared identical to one another. On the other hand, an effect can be thought of as a particular type of measurement apparatus that results in either a "yes" or a "no" as the outcome.

Effects are considered to be affine mappings from $\mathcal{S}(\mathcal{H})$ to $[0, 1]$. Formally, for the bounded self-adjoint operator E, the probability of occurring of the event corresponding to E is $Tr[\rho E]$, $\forall \rho \in \mathcal{H}$, such that $0 \leq E \leq I_{\mathcal{H}}$. This is also known as the *Born rule*.

Let $\mathcal{L}_s$ denote the set of self-adjoint operators on $\mathcal{H}$. We denote $\mathcal{E}(\mathcal{H})$ as the set of *effects* in which every element E is such that

$$\mathcal{E}(\mathcal{H}) = \{E \in \mathcal{L}_s(\mathcal{H}) | O \leq E \leq I_{\mathcal{H}}\} \tag{2.5}$$

where $\mathcal{I}_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$.

The spaces $\mathcal{S}(\mathcal{H})$ and $\mathcal{E}(\mathcal{H})$ are the dual spaces of each other, by which we mean that one maps the other to a normalized probability distribution. The mathematical Hilbert space machinery is given an operational meaning as a result, and it can be compared with the results of actual experiments. If one were to fix the set of states first and then find the correspondence to effects, or vice versa, this would be possible from a mathematical standpoint.

Chapter 3

Observables, Measurements, and Experiments

We have devised the mathematical language required for the thesis in the previous chapter. Here, we will use those concepts to describe observables and channels. Then, we will show how it relates to experiments. First, we will start by describing an experiment [11].

3.1 Experiments from an operational perspective

Let's begin with the most basic scenario: Let us say we desire to find out more about a system we are looking into. We'll perform an *experiment* to gain further understanding of the system. *Operational* quantum theory [2], which is a statistical theory, predicts only the probabilities of measurement outcomes, not individual occurrences of particular outcomes. From now on, when we speak of quantum theory, we will be referring to Operational

quantum theory. The statistical basis of the measurement procedure is captured by the mathematical notion of an observable, which will be presented and examined in this chapter. Every experiment generates a series of results, each of which has a distinct probability of occurring based on the arrangement of the preparation and measurement equipment. Quantum theory contains features that both generalize and restrict conventional probability theory when used as a framework for calculating measurement outcome probabilities. In the sense that observables are described by positive operator-valued measures, it is a generalization, similar to how matrices are more general than numbers, and these are more general than probability measures. Nevertheless, the probability distributions that we can find in quantum measurements are naturally constrained by the laws of quantum theory. It can also be viewed in this way as a limitation of standard probability theory.

In general, experiments are divided into two parts: preparation and measurement. The division in an experiment can be arbitrary, but it does not affect the results. We assume a set of possible preparations and measurements, which can be combined to create an experiment with a probability distribution of measurement results. The preparations determine probability distributions for all possible measurements on the system. Alternatively, a fixed measurement procedure specifies an outcome probability distribution for each preparation. This is a brief explanation of what we call the *basic statistical framework*. Figure (3.1) gives a schematic representation of it.

3.1.1 Observables

When we perform an experiment, the final results obtained are interpreted as the probabilities of measurement outcomes. Suppose we consider a specific

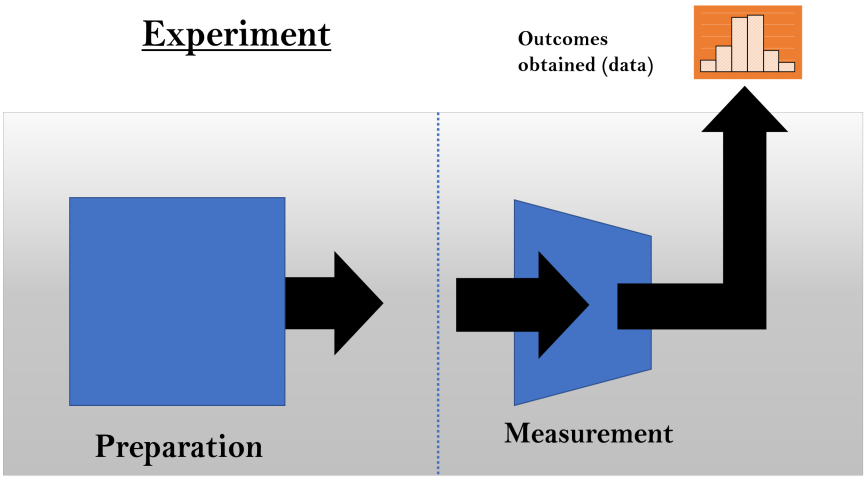

FIGURE 3.1: An experiment can be divided into a preparation and a measurement. We obtain the final output as the probability distribution of measurement outcomes.

physical event - we perform 'a measurement M which gives an outcome x'. This event is associated with an effect E, and the probability for the event to happen is given by the trace formula, $Tr[\rho E]$. Here, ρ is the density operator representing the input state of the measured system.

One way to imagine the entire measurement process is as a set of events, each of which has a distinct effect. We want to see which event is realized when we perform a measurement. What is known as an observable is the mathematical description of the possible events. As we shall see later, observables can be mathematically identified using what is known as Positive operator-valued measures (POVMs) [3, 11].

Definition 3.1. *An observable is a Positive Operator-Valued Measure (POVM), which is mapping* $A\colon x \mapsto A(x)$ *from a finite outcome set* Ω *to* $\mathcal{E}(\mathcal{H})$ *satisfying* $\sum_{x\in\Omega} A(x) = I_{\mathcal{H}}$. *If* $A(x)$ *satisfies an additional property* $A(x)^2 = A(x), \forall x \in \Omega$, *then* A *is known as a sharp observable, and it is mathematically identified as a Projection-Valued Measure (PVM).*

We denote the set of observables on $\mathcal{H}$ by $\mathcal{O}(\Omega, \mathcal{H})$. We can infer that PVMs are the special cases of POVMs, and POVMs are the general quantum measurements. All the effects of a PVM are projectors. PVMs are also called projective measurements. For any arbitrary PVM A, it can be easily shown that for all $i, j \in \Omega$ we have $A(i)A(j) = \delta_{i,j}$. PVMs are also known as *sharp* observables, and all other POVMs (which are not PVMs) are called *unsharp* observables.

The set of all observables defined on the same outcome set is a convex set. In an experiment, different observables can be statistically mixed by sampling them based on their respective weights and then be applied to measure the system.

Example: Qubit observable

Here, we will give an example of a commonly used observable. A qubit observable is an observable which is defined on the two-dimensional Hilbert space. Suppose we let the cardinal number of the outcome set of Ω_W is $|\Omega_W| = 3$, so one can represent W as

$$W_i = \frac{1}{2}(\alpha_i \mathbb{I}_2 + \vec{\alpha_i} \cdot \vec{\sigma_i}) \quad , i = 1, 2, 3$$

where α_i's are real numbers, $\vec{\alpha_i}$ are vectors in three dimensional real vectors, $\mathbb{I}_2$ is the identity operator on two dimensions and $||\vec{\alpha_i}|| \leq \alpha_i \leq 2$ - $||\vec{\alpha_i}||$. Here $\vec{\sigma} = (\mathbb{I}_2, \sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices in two dimensions, and they form a basis. The states here are the same as equations(2.3).

3.1.2 Post-processing of observables

It could be desirable to process the data and optimize the outcome obtained in each run of an experiment measuring the given POVM. For example, by analyzing the data and gaining the outcome statistics of another observable, one can see if it is possible to uncover some other property of the system. This is what is usually referred to as the postprocessing of observables.

Now, we give a mathematical definition of the postprocessing of an observable [12, 13].

Definition 3.2. POVM $B \in \mathcal{O}(\Omega_B, \mathcal{H})$ is called a postprocessing of POVM $A \in \mathcal{O}(\Omega_A, \mathcal{H})$, denoted by $A \to B$, if there exists a stochastic matrix ν_{kj} $k \in \Omega_A, j \in \Omega_B$ i.e $\nu_{kj} \geq 0$, $\forall k, \forall j$, satisfying $\sum_{j \in \Omega_B} \nu_{ab} = 1$ for all $k \in \Omega_A$, such that

$$B_j = \sum_{k \in \Omega_A} \nu_{kj} A_k \quad \forall b \in \Omega_B. \tag{3.1}$$

We say observables A and B are *postprocessing equivalent* if $A \to B$ and $B \to A$, denoted by $A \leftrightarrow B$.

The idea that the outcome statistics of B can be deterministically obtained from the statistics of A by some classical process represented by the stochastic matrix is captured by postprocessing. Postprocessing, as can be easily observed, induces a pre-order on the set of all observables on $\mathcal{H}$. This pre-order can then be extended to the equivalence classes of postprocessing equivalent observables, where it becomes a partial order.

Example

Here, we give an example of the postprocessing of observables, which is indeed quite a general example.

A trivial observable is an observable $\mathrm{T}^p \in \mathcal{O}(\Omega_{\mathrm{T}^p}, \mathcal{H})$ such that $\mathrm{T}^p(i) = p_i I$ for all $i \in \Omega_{\mathrm{T}^p}$ for some probability distribution $(p_i)_i$ on Ω_{T^p}.
A trivial observable can be postprocessed from any other observable $\mathrm{A} \in \mathcal{O}(\Omega_A, \mathcal{H})$ using the postprocessing matrix ν with $\nu_{yx} = p_x, \forall x \in \Omega_{\mathrm{T}^p}$ and $y \in \Omega_{\mathrm{A}}$. Thus, one can write $\mathrm{A} \to \mathrm{T}^p$ for any observable A and any probability distribution p.

3.1.3 Marginal POVM

We will now define a marginal POVM, a term that will be frequently used in this thesis.

Definition 3.3. *The marginal POVM $P' \in \mathcal{O}(\Omega, \mathcal{H})$ of a POVM $P \in \mathcal{O}(\Omega \times \lambda, \mathcal{H})$*

$$P'_x := \sum_{y \in \lambda} P_{x,y}$$

for all $x \in \Omega$. Here, we marginalize the POVM $P_{x,y}$ over the rightmost index, i.e., y.

For instance, the marginal POVM for $n = 3$ for $P'_{x=1}$ is given by $P_{11} + P_{12} + P_{13}$. Here, $\nu_{1,1} = \nu_{1,2} = \nu_{1,3} = 1$, and the matrix elements are 0 elsewhere. We can see that, in some sense, the marginalization ignores one index of the POVM.

3.1.4 Dilation of a POVM

It is well known that a PVM is a specific kind of POVM and that a POVM is considered a generalized measurement. Then, a natural question arises of whether it is possible to realize POVM as a PVM. The term 'realize' refers to whether something is physically possible or not. This question

again takes us back to the fact that we discussed in (3.1.1) that a POVM corresponds to an *unsharp* property, whether a PVM corresponds to a *sharp* property, i.e., projective measurement (PVM) can be detected with certainty and is physically implementable in a straightforward way.

It turns out that the answer to the above-mentioned question is positive.

Definition 3.4. *Consider a POVM* $\{\mathrm{A_i}\}_{i=1}^{n} \in \mathcal{O}(\Omega, \mathcal{H}_A)$. *There exists a PVM* $\{\mathrm{B_i}\}_{i=1}^{n} \in \mathcal{O}(\Omega, \mathcal{H}_B)$ *and an isometry* $V : \mathcal{H}_A \to \mathcal{H}_B$, *such that* $\mathrm{A} = V^\dagger B_i V$.

By isometry, we mean $V^\dagger V = I$.

This is known as the dilation of a POVM, or more specifically, the Naimark dilation [14]. The Naimark dilation confirms that each POVM is the projection of a PVM in an extended Hilbert space and establishes a distinct kind of connection between POVMs and PVMs. Physically speaking, POVMs can be considered as a projective measurement on an extended Hilbert space in which the original state resides on a proper subspace. Generally speaking, the extended Hilbert space must have enough additional dimensions depending on the POVM which we will try to realize via the Naimark dilation.

A common way of constructing PVM and isometry is to let $\mathcal{H}_B = \mathcal{H}_A \otimes \mathcal{H}_C$, $B_i = I_A \otimes |i\rangle_C \langle i|$ and $V = \sum_{i=1}^{n} \sqrt{A_i}_A \otimes |i\rangle_C$. We can see that if we extend the isometry to a unitary operation U such that $S = U(I_A \otimes |0\rangle_C)$. Given any arbitrary POVM A, there exist different Naimark extensions, i.e., the Naimark extension is not unique.

3.1.5 Quantum channels

In this subsection, we will talk about quantum operations and channels [10, 11].

When we refer to a quantum operation, we mean the most general transformation, which is probabilistic in nature and can be applied to a given quantum system. It is presumed that every operation can be applied to every state. An exemplary instance of an operation is the transmission of a photon through a fiber optic or polarization device, although with many practical constraints. When considering the probability observed for a specific state and observable, an operation can be interpreted as either a component of the preparations or a component of the measurement. This implies that an operation can be mathematically expressed in an equivalent way using either a mapping on states (Schrödinger picture) or a mapping on effects (Heisenberg picture). Our primary reference will be the Schrödinger picture.

It is possible for an operation to destroy one or more of the systems in the initial ensemble. To mathematically describe quantum operations, it is advantageous to introduce a set of subnormalized states. The fundamental concept in quantum theory is the probability rule, which dictates the probability distribution for every combination of a state and an observable. For subnormalized states, the predicted probabilities have a cumulative total that is less than or equal to 1. In this situation, we can expand the range of possible outcomes for the system's transformers (devices that impact the system) by including an additional outcome that represents the omission of any system in an experiment. Quantum channels describe the evolution of a quantum system.

As we have denoted before, $\mathcal{L}(\mathcal{H})$ and $\mathcal{L}(\mathcal{K})$ are the sets of bounded operators on $\mathcal{H}$ and $\mathcal{K}$ respectively. In the Schrödinger picture, a *quantum operation* is a map from states to states, or more formally, a function from the set of states, say, $\mathcal{S}(\mathcal{H})$ to set of subnormalized states, $\mathcal{S}'(\mathcal{H})$. No operation may change the statistical indistinguishably of distinct convex decompositions of a state ρ. Simply, it means that applying the operation to two states separately would be the same as applying the operation to the sum of the states. A quantum operation, therefore, preserves convex combinations. If we denote $\mathcal{M}$ as an operation,

$$\mathcal{M}(\sum_i \lambda_i \rho_i) = \lambda_i \mathcal{M}(\rho_i) \tag{3.2}$$

$\forall \rho \in \mathcal{S}(\mathcal{H})$, $0 \leq \lambda_{\leq} 1$ satisfying $\sum_i \lambda_i = 1$.
It is clear that to be a valid quantum operation, the following two mathematical properties need to be satisfied by $\mathcal{M}$,

1. $\mathcal{M}(\rho) \geq 0$
2. $Tr[\mathcal{M}(\rho)] \leq Tr[\rho]$

$\forall \rho \in \mathcal{S}(\mathcal{H})$. The first condition is called the *positivity* condition, and the second is called the *trace non-increasing* condition.
For this general operation to be a physical quantum operation, or more commonly, for a *quantum channel*, the second condition has to strictly satisfy the equality, i.e., $Tr[\mathcal{M}(\rho)] = Tr[\rho], \forall \rho \in \mathcal{S}(\mathcal{H})$. This is due to the trivial fact that the transformed state also $\in \mathcal{S}(\mathcal{H})$, and all states have their trace equal to 1.

Now, it may seem that these conditions are enough to define a quantum channel. However, there is an additional requirement, which becomes visible

only when we consider composite systems.

Let Z be any composite system where X and Y are subsystems of Z. We can extend any operation $\mathcal{M}_X$ acting on X as $(\mathcal{M}_X \otimes I_Y)$ acting on the composite system Z, where I_Y is the identity mapping defined on subsystem Y. If we act this operator on the state $\rho_{XY} = \rho_X \otimes \rho_Y$ then it should act as

$$(\mathcal{M}_X \otimes I_Y)\rho_{XY} = \mathcal{M}_X(\rho_X) \otimes \rho_Y$$

but this requirement is not trivial from the positivity of $\mathcal{M}_X$. Sheer positivity of $\mathcal{M}_X$ does not guarantee the positivity of the composite map $(\mathcal{M}_X \otimes I_Y)$. If the requirement of being positive holds for all extensions I_Y, then we classify the map $\mathcal{M}_X$ as *completely positive*. Otherwise, the map $\mathcal{M}_X$ leads to negative probabilities in some extension, which makes it physically impossible.

Finally, we give a definition of a quantum channel.

Definition 3.5. *A Quantum channel is represented as Completely Positive Trace-Preserving(CPTP) Map from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{K})$. We denote the set of quantum channels as $Ch(\mathcal{H}, \mathcal{K})$.*

3.1.6 Dilation of a quantum channel

Analogous to observables, a quantum channel also has a dilation theorem. In accordance with the Stinespring dilation theorem [15], it is possible to construct a channel by combining the basic operations. This can be done by performing a tensor product with a second Hilbert space of the system Hilbert space in a particular state, followed by a unitary transformation and a reduction to a subsystem. In regard to this, it is possible to consider

any quantum operation to be the result of a unitary evolution on a Hilbert space that is larger or dilated. The ancillary space of the system is typically referred to as the auxiliary system, to which one is required to couple the given system.

For a quantum channel $\mathcal{E} : \mathcal{H} \mapsto \mathcal{K}$, the Stinespring dilation of $\mathcal{N}$ is represented by the tuple $(\mathcal{H}_A, U)$ of an ancillary Hilbert space $\mathcal{H}_A$ and an isometry $U : \mathcal{H}_A \mapsto \mathcal{K}$, (satisfying $U^\dagger U = I_\mathcal{H}$) such that

$$\mathcal{E}(\rho) = Tr_{\mathcal{H}_A}[U\rho U^\dagger] \tag{3.3}$$

$\forall \rho \in \mathcal{S}(\mathcal{H})$. Here, U is said to be an isometry, as $U^\dagger U = I_\mathcal{H}$. This is a fundamental result of the Operator theory in Hilbert space.

3.1.7 Representations of Quantum Channels

In this subsection, we will discuss two common representations of a quantum channel.

Kraus representation

Another common way to express a quantum operation is using the Kraus representation [10], also called the operator-sum representation. This representation can be derived from the Stinespring dilation theorem. This representation is also known as the operator-sum representation.

Definition 3.6. *A linear map $\mathcal{N}$, is interpreted as a quantum operation $\mathcal{N} : \mathcal{L}(\mathcal{H}) \mapsto \mathcal{L}(\mathcal{K})$ is a quantum operation if and only if there exists bounded operators (also called Kraus operators) K_m, $m = 1, 2, \ldots, r_\mathcal{N}$ for*

some $r_{\mathcal{N}} \in \mathbb{N}$ *being a Kraus rank of* $\mathcal{N}$*, such that* $\mathcal{N}(\rho) = \sum_m K_m \rho {K_m}^\dagger$ *and* $\sum_m K_m^\dagger K_m \leq I$*. If the equality holds, the operation is a quantum channel.*

It is possible to choose $(dim(\mathcal{H})dim(\mathcal{K}))$ operators K_m. The operators $\{K_m\}$ are called the Kraus operators. It is important to note that the composition of a quantum operation is also a quantum operation.

■ *Example.* A common type of channel is a *unitary channel* $\mathcal{N}_{\mathcal{U}}$, defined as, $\mathcal{N}_{\mathcal{U}}(\rho) = U\rho U^\dagger$, where U is an unitary operator. Now, we can write $\mathcal{N}_{\mathcal{U}}(\rho) = \frac{1}{2}U\rho U^\dagger + \frac{1}{2}U\rho U^\dagger$. So, the corresponding Kraus operators are $K_1 = K_2 = \frac{1}{\sqrt{2}}U$, satisfying ${K_1}^\dagger K_1 + {K_2}^\dagger K_2 = 1$. All possible Kraus operators of $\mathcal{N}_{\mathcal{U}}$ are multiples of U.

Choi representation

Before discussing the representation, we will discuss an important result in linear algebra, which is widely used in quantum theory.

Choi's theorem

Let $\{|\psi_j\rangle\}$ be any orthonormal basis of the $d-$ dimensional Hilbert space $\mathcal{H}$ and $\mathcal{N}((|\psi_j\rangle \langle\psi_k|)$ be the $\{j,k\}^{th}$ element of the map $\mathcal{N}$.

For a positive linear map $\mathcal{N} : \mathcal{L}(\mathcal{H}) \mapsto \mathcal{L}(\mathcal{K})$, the following statements are equivalent:

1. The map $\mathcal{N}$ is completely positive
2. $\mathcal{N}$ is $d-$ positive, i.e $\mathcal{N} \otimes I_d$ is a positive map

3. the following matrix

$$\varphi(\mathcal{N}) = \begin{bmatrix} \mathcal{N}(|\psi_1\rangle\langle\psi_1|) & \cdots & \mathcal{N}(|\psi_1\rangle\langle\psi_d|) \\ \vdots & \ddots & \vdots \\ \mathcal{N}(|\psi_d\rangle\langle\psi_1|) & \cdots & \mathcal{N}(|\psi_d\rangle\langle\psi_d|) \end{bmatrix}$$

is positive. The $\varphi(\mathcal{N})$ is called the *Choi matrix* or *Choi operator* of $\mathcal{N}$.

Now, we will define the Choi representation [16].

Definition 3.7. *Any quantum operation* $\mathcal{N} : \mathcal{L}(\mathcal{H}) \mapsto \mathcal{L}(\mathcal{K})$ *is represented as*

$$\mathcal{J}(\mathcal{N}) := \sum_{j,k=0}^{d-1} |\psi_j\rangle\langle\psi_k| \otimes \mathcal{N}(|\psi_j\rangle\langle\psi_k|)$$

The operation $\mathcal{N}$ *is completely positive if and only if* $\mathcal{J}(\mathcal{N}) \geq 0$. *Also,* $\mathcal{N}$ *is trace-preserving (i.e.* $\mathcal{N}$ *is a channel) iff* $Tr_{\mathcal{K}}[\mathcal{J}(\mathcal{N})] = I_{\mathcal{H}}$.

This definition is also known as the *Choi–Jamiołkowski isomorphism*. Here, we have taken the dimension of $dim(\mathcal{H})$ to be d.

It is possible to relate numerous statements about states to statements about channels and vice versa using the Choi-Jamiolkowski isomorphism. More precisely, an isomorphism exists between the space of quantum channels and positive semidefinite operators on the tensor product space of the input and output Hilbert spaces. That illustrates the relationship between the two spaces.

3.1.8 Description of a measurement

There can be no physical theory without measurements since they are the only way of gaining information from the elements that the theory describes. It is possible that crucial aspects of the theory can be uncovered through a

comprehensive understanding and characterization of measurements. This is especially true in the field of quantum theory, where the discoveries of phenomena such as the unavoidable trade-off between information and disturbance or the intrinsic randomness of measurement outcomes have rendered our (classical) intuition imprecise and have rendered quantum theory puzzling to the majority of physicists even after a hundred years.

Here, we will describe a measurement at three different levels of abstraction [11].

- At the first level, a measurement is described only by the *observables*, and we focus only on the measurement outcome probabilities. So, the only possible information we can get is the probability distribution of the observable, which, as discussed earlier, is described by a POVM.
- At the second level, along with the measurement outcome probabilities, we also incorporate the post-measurement state. This leads to the notion of an *quantum instrument*. The concept of a quantum instrument will be elaborated in the next section.
- The *measurement model*, which is described at the third level of abstractness, involves the incorporation of more specific details, such as the probe observable of the measurement and the nature of the interaction between the measurement in question.

Here, the different levels of description of the measurement specify different levels of abstraction. One can think that a measurement model is a higher-order description of the measurement and that of the quantum instrument, as an instrument can be seen as a special case of the measurement model. Similarly, an observable can be viewed as a special case of a quantum

instrument.

Now, we will mathematically define the measurement model [11, 17]

Definition 3.8. *If $B \in (\Omega, \mathcal{H})$ is an observable associated with the Hilbert space $\mathcal{H}$, a measurement model is characterized by the quadruple $\mathcal{M} = \langle \mathcal{K}, \phi, \mathcal{N}, F \rangle$, where $\mathcal{K}$ is the Hilbert space associated with the probe system, ϕ is the initial state of the probe system, and $\mathcal{N}$ is describes the measurement interaction between the system and the probe, which is mathematically a channel from $\mathcal{L}(\mathcal{H} \otimes \mathcal{K}) \mapsto \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$ and F is an observable on the probe system having the same outcome space as B. The quadruple $\mathcal{M} = \langle \mathcal{K}, \phi, \mathcal{N}, F \rangle$ is a measurement model of B if for $i \in \Omega$ and $\rho \in \mathcal{S}(\mathcal{H})$*

$$Tr[\rho\, B_i] = Tr[\mathcal{N}(\rho \otimes \phi)(I \otimes F_i)]$$

This is known as the probability reproducibility condition.

3.1.9 Quantum Instruments

In this subsection, we will talk about quantum instruments, which were mentioned in the previous subsection.

In quantum physics, the concept of a *quantum instrument* [5, 11, 17] describes the measurement process, in which the process of measurement itself is viewed as a conditional state preparator. In addition, if we assume that a system is still in existence after a measurement has been taken, then it is possible to attempt to carry out additional measurements to acquire additional information from the system's initial state. In both cases, it is necessary for us to know not only the probabilities of the outcomes of the measurements but also the impact that the initial measurement had on the system and, as a result, on subsequent measurements.

The concept of a quantum instrument has proven helpful in describing a measurement procedure that takes the post-measurement state into account, along with capturing the measurement statistics. This is useful when the measured system will be utilized to obtain further information or other forms of interaction. Various properties of quantum instruments like , post-processing [13], dilation [18], incompatibility [19, 20, 21] have been investigated earlier. Quantum instruments offer a more experimentally grounded method than other existing approaches and can establish a unified operational basis for quantum theory [22]. Furthermore, they have undergone characterization and experimental analysis [23].
We will now mathematically define a quantum instrument.

Definition 3.9. *A quantum instrument is a mapping* $\mathcal{I} : x \mapsto \mathcal{I}_x$ *from a finite outcome set* Ω *to a set of quantum operations from* $\mathcal{L}(\mathcal{H})$ *to* $\mathcal{L}(\mathcal{K})$ *such that* $\sum_{x\in\Omega} \mathcal{I}_x \in Ch(\mathcal{H}, \mathcal{K})$ *is a quantum channel known as the* induced channel *of the quantum instrument. For an input state* $\rho \in \mathcal{S}(\mathcal{H})$*, the unnormalized conditional output state is given by* $\mathcal{I}_x(\rho)$*, where* x *is the corresponding measurement outcome, the probability of obtaining which is governed by* $Tr[A^{\mathcal{I}}(x)\rho] = Tr[\mathcal{I}_x(\rho)]$ *where* $A^{\mathcal{I}} \in \mathcal{O}(\Omega, \mathcal{H})$ *is the induced POVM of the instrument* $\mathcal{I}$*.*

The set of instruments with outcome set Ω from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{K})$ is denoted by $Ins(\Omega, \mathcal{H}, \mathcal{K})$. Figure (3.2) gives a schematic representation of a quantum instrument. As a quantum instrument is a collection of quantum operations, we can characterize it in Kraus representation by using each of the Kraus operators of the constituent operations. Similar to quantum operations, a composition of quantum instruments also forms a quantum instrument. It is evident that the induced POVM is unique to a given instrument; different instruments can have the same induced POVM. Quantum instruments offer

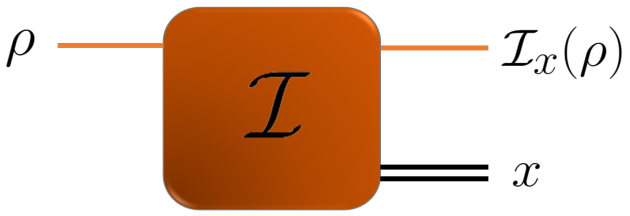

FIGURE 3.2: *Quantum instrument.* Here, the instrument takes ρ as an input state and produces $\mathcal{I}_x(\rho)$ as the output, which is conditioned on the classical outcome x.

a more experimentally grounded method than other existing approaches and can establish a unified operational basis for quantum theory [16]. Furthermore, they have undergone characterization and experimental analysis as well.

3.1.9.1 Some examples of Instruments

- *Trivial Instrument.* Let $A \in \mathcal{O}(\Omega, \mathcal{H})$ be a POVM, and let us fix a state $\phi \in \mathcal{S}(\mathcal{H})$. If $\rho \in \mathcal{S}(\mathcal{H})$ is a generic state, then

$$\mathcal{I}_x(\rho) = Tr[\rho A_x]\phi$$

  defines a trivial instrument. The term *trivial* means all measurements following a trivial instrument is trivialized, i.e., when we measure an observable B after measuring A, the *B-measurement* does not offer any information on the initial state ρ.

- *Trash-and-prepare instruments.* If $B = p_x I_{\mathcal{H}} \in \mathcal{O}(\Omega, \mathcal{H})$ be trivial POVM for some probability distribution $(p_x)_x$ over Ω, we define a trash-and-prepare instrument $\mathcal{P}^B \in Ins(\Omega, \mathcal{H}, \mathcal{K})$ such that for some

set of states $\{\phi_x\}_{x\in\Omega} \subset \mathcal{S}(\mathcal{K})$, we have,

$$\mathcal{P}_x^A(\rho) = Tr[\rho]p_x\phi_x$$

for all $\rho \in \mathcal{S}(\mathcal{H})$.

- *Lüders instrument.* A Lüders instrument $\mathcal{I}^A \in Ins(\Omega, \mathcal{H}, \mathcal{H})$ with a POVM $A \in \mathcal{O}(\Omega, \mathcal{H})$, is defined as

$$\mathcal{I}_x^A(\rho) = \sqrt{A_x}\rho\sqrt{A_x}$$

for all $x \in \Omega$ and $\rho \in \mathcal{S}(\mathcal{H})$. It is important to note that the input and the output space of a Lüders instrument are the same.

3.1.10 Ozawa's theorem

Now, we will define Ozawa's theorem, which is a famous result that was published in 1984.[18].

Ozawa's renowned theorem establishes the fundamental principles for the theory of measuring continuous observables. This theorem provides a comprehensive explanation of how to extend the description of measuring discrete quantum observables to continuous observables. It demonstrates how each measuring process of continuous observables results in a state change in the measured system caused by the measurement. It also establishes a direct relationship between completely positive instruments and the state changes determined by the measuring processes.

We will simply state the theorem and will not go into much detail. Let's recall a measurement model from section (3.1.8). It is known that a measurement

model $\mathcal{M}$ defines a unique instrument $\mathcal{I}^{\mathcal{M}}$, and we call it the instrument induced by measurement model $\mathcal{M}$.

Definition 3.10. *Ozawa's Theorem.* For any instrument $\mathcal{I} \in Ins(\Omega, \mathcal{H}, \mathcal{K})$ there exists a measurement model $\mathcal{M} = \langle \mathcal{K}, \phi, \mathcal{N}, F \rangle$, where $\phi \in \mathcal{S}(\mathcal{H})$ is a pure state, $\mathcal{N}$ is an unitary channel and F is a sharp observable such that $\mathcal{I} = \mathcal{I}^{\mathcal{M}}$.

This theorem tells us that we can extend an instrument to a measurement model that would include additional dimensions, and in addition, it is possible to choose the corresponding initial state of the measurement pointer state as a projector and the interaction channel as a unitary channel. This theorem can be thought of as the extension of the Stinespring dilation and the Neumark dilation.

Chapter 4

Sequential Implementation of Quantum Instruments

After building the required mathematical and theoretical background, we will start our discussion on the implementation of quantum instruments more precisely by presenting a study on the sequential implementation of quantum instruments. This chapter is based on a collaboration between my co-supervisor, Michal Sedlák, and me. By sequential implementation, we mean whether we can realize a single whole instrument into parts. This provides an intuition that a complex measurement can be realized as a sequence of other measurements.

We will start formulating the problem by describing an adaptive sequence of instruments.

4.1 Adaptive sequences of Quantum Instruments

We start by defining an adaptive sequence of quantum instruments. We will define an N step *adaptive sequence* instruments, $\mathcal{Q}$ by which we mean that the choice of each instrument in $\mathcal{Q}$ is dependent upon the classical outcome of the preceding instrument in the sequence. We will fix the following notation for our further definition.

Let $\{\mathcal{H}_0, \cdots, \mathcal{H}_N\}$ be an $(N+1)$ tuple of input Hilbert spaces and $\{\Omega_0, \cdots, \Omega_N\}$ be a $N-$tuple of outcome spaces. We define $\mathcal{Q}^k \equiv \{\mathcal{I}^{k,a_{k-1}}\} \in Ins(\Omega_k, \mathcal{H}_{k-1}, \mathcal{H}_k)$ $\forall a_{k-1} \in \Omega_{k-1}$ $\forall k, 1 \leq k \leq N$ (where we define $\Omega_0 \equiv \{1\}$). Here, the choice of the k^{th} instrument is dependent on a_{k-1}, which is the classical outcome of its preceding instrument. The whole adaptive sequence of instruments is represented by $\mathcal{Q}$, i.e $\mathcal{Q} \equiv (\mathcal{Q}^1, \cdots \mathcal{Q}^N)$. A finite sequence of instruments doesn't need to be adaptive but simply a composition of instruments. This situation corresponds to choosing as each element of the set $\mathcal{Q}^k$ equal, i.e., $\mathcal{I}^{k,a_{k-1}} = \mathcal{I}^k$.

Let $\mathcal{Q}$ be a N step adaptive sequence of instruments, the *Total Instrument* $\mathcal{Q}^N$, denoted by $\mathcal{T} \in Ins(\Omega_N, \mathcal{H}_0, \mathcal{H}_N)$ is

$$\mathcal{T}_{a_N} = \sum_{a_{N-1} \in \Omega_{N-1}} \cdots \sum_{a_1 \in \Omega_1} \left(\mathcal{I}_{a_N}^{N,a_{N-1}} \circ \cdots \circ \mathcal{I}_{a_1}^{1,1} \right) \tag{4.1}$$

The Total Instrument $\mathcal{T}$ of the adaptive sequence $\mathcal{Q}$ has the same outcome set as that of $\mathcal{Q}^N$. For $\mathcal{Q}$, we define the *Initial Instrument (II)* represented by $\mathcal{J} \in Ins(\Omega_{N-1}, \mathcal{H}_0, \mathcal{H}_{N-1})$ as $\mathcal{J} := \mathcal{I}^{1,1}$ and the set of *Residual Instruments (RIs)*, denoted by $\mathcal{W}^{a_{N-1}} \in Ins(\Omega_N, \mathcal{H}_{N-1}, \mathcal{H}_N)$, thus,

$$\mathcal{T}_{a_N} = \sum_{a_{N-1} \in \Omega_{N-1}} \mathcal{W}_{a_N}^{a_{N-1}} \circ \mathcal{J}_{a_{N-1}} \tag{4.2}$$

holds.

4.2 Sequential realization of Quantum Instruments

It is well known that a composition of instruments is an instrument. If we have a composition of instruments, they can be merged to form a single, unified instrument. In this section, we will investigate whether, given a single instrument, it is possible to realize it as an adaptive sequence of instruments. Suppose we have an instrument, which we refer to as our Total Instrument; we ask whether the Total Instrument can be decomposed into an adaptive sequence of constituent instruments. This question is of theoretical interest as it improves our understanding of the underlying framework of quantum instruments. By realizing an instrument sequentially, we tend to lower the implementational complexity of the measurement. This means the obtained results can help us devise new ways to implement complex quantum measurements with limited resources. Initially, we consider a scenario where an instrument could be written as a two-step adaptive sequence of instruments.

We begin by formulating a realization theorem for our claim.

Theorem 4.1. *Given a Total Instrument* $\mathcal{T} \in Ins(\Omega, \mathcal{H}, \mathcal{K})$ *and any POVM* $B \in \mathcal{O}(\Omega_B, \mathcal{H})$ *such that* $A^{\mathcal{T}} \to B$*, there exists an adaptive sequence of an Initial Instrument* $\mathcal{J} \in Ins(\Omega_B, \mathcal{H}, \mathcal{H})$ *and a set of Residual Instruments*

$\mathcal{W}^j \in Ins(\Omega, \mathcal{H}, \mathcal{K})\ \forall j \in \Omega_B$ *such that*

$$\mathcal{T}_k = \sum_{j \in \Omega_B} \mathcal{W}_k^j \circ \mathcal{J}_j \qquad \forall k \in \Omega \tag{4.3}$$

where the Initial Instrument $\mathcal{J}_j(\rho) = \sqrt{B_j}\rho\sqrt{B_j}, \forall \rho \in \mathcal{S}(\mathcal{H})$ *is the Lüders instrument of POVM* B*.*

Proof. Due to postprocessing relation $A^{\mathcal{T}} \to B$ (see Eq. (3.1)) there exist postprocessing matrix ν_{kj} $k \in \Omega_{A^{\mathcal{T}}}, j \in \Omega_B$. Let $\{{T'}_{k,m}\}$ be the set of the Kraus operators of the Total instrument, i.e.

$$\mathcal{T}_k = \sum_m {T'}_{k,m}\, \rho\, {T'}^{\dagger}_{k,m} \tag{4.4}$$

where $\sum_{k\in\Omega}\sum_m {T'}^{\dagger}_{k,m}{T'}_{k,m} = I_{\mathcal{H}}$.

We choose another set of Kraus operators $\{T_{k,jm}\}$ of $\mathcal{T}$, such that the two sets of Kraus operators are related by

$$T_{k,jm} = \sqrt{\nu_{kj}}\, T'_{k,m}, \tag{4.5}$$

where $k \in \Omega, j \in \Omega_B$, and $m = 1, \ldots, r_{\mathcal{T}_k}$.

It is straightforward to verify that, indeed, the action of both sets of Kraus operators is the same, i.e.

$$\mathcal{T}_k(\rho) = \sum_{j,m} T_{k,jm}\ \rho\ T^{\dagger}_{k,jm}(\rho) \quad \forall k \in \Omega, \rho \in \mathcal{S}(\mathcal{H}) \tag{4.6}$$

and the normalization condition is as well satisfied

$$\sum_{k,j,m} {T_{k,jm}}^{\dagger} T_{k,jm} = \sum_{k,m}\sum_j \nu_{kj}\ {T'}^{\dagger}_{k,m}{T'}_{k,m} = I_{\mathcal{H}}. \tag{4.7}$$

We observe that POVM elements B_j can be expressed via Kraus operators $T_{k,jm}$ as

$$B_j = \sum_{k,m} \nu_{kj} T'^{\dagger}_{k,m} T'_{k,m} = \sum_{k,m} T^{\dagger}_{k,jm} T_{k,jm}, \tag{4.8}$$

using equations (3.1) and (4.5). Let us define a projector Π_j onto the support of operator B_j. We denote the Moore-Penrose pseudo-inverse [**?**, **?**] of $Bj^{\frac{1}{2}}$ by $B_j^{-\frac{1}{2}}$. Thus,

$$\Pi_j B_j^{\pm\frac{1}{2}} = B_j^{\pm\frac{1}{2}}. \tag{4.9}$$

Further, we will denote the number of elements of the set Ω as $|\Omega|$. Next, let us define the Residual instruments by the choice of its Kraus operators $R^j_{k,m}$

$$R^j_{k,m} = T_{k,jm} B_j^{-\frac{1}{2}} \tag{4.10}$$

$$R^j_{k,0} = \frac{1}{|\Omega|}(I_{\mathcal{H}} - \Pi_j). \tag{4.11}$$

Let us note that we have defined an additional Kraus operator $R^j_{k,0}$, so in the following equations, index m starts from zero.

Valid quantum instruments must satisfy the normalization condition. This is expressed for Total instrument $\mathcal{T}$ by Eq. (4.7). Lüders instrument $\mathcal{J}$ is normalized by construction. Thus, it remains to prove that Kraus operators of residual instruments are properly normalized.

For every instrument $\mathcal{W}^j$ we get

$$\begin{aligned}\sum_{k,m} R^{j\,\dagger}_{k,m} R^j_{k,m} &= \sum_{k,m} {B_j}^{-\frac{1}{2}} {T_{k,jm}}^\dagger T_{k,jm} {B_j}^{-\frac{1}{2}} \\ &\quad + (I_\mathcal{H} - \Pi_j)(I_\mathcal{H} - \Pi_j) \\ &= \Pi_j + (I_\mathcal{H} - \Pi_j) = I_\mathcal{H}, \end{aligned} \tag{4.12}$$

where we used Eq. (4.8).

Next, using equations (4.10) and (4.11) and the definition of $\mathcal{J}$ from Theorem 4.1, we get

$$\begin{aligned}\sum_{j\in\Omega_B} (\mathcal{W}^j_k \circ \mathcal{J}_j)(\rho) &= \sum_{j,m} \Big(T_{k,jm} B_j^{-\frac{1}{2}} B_j^{\frac{1}{2}} \rho B_j^{\frac{1}{2}} B_j^{-\frac{1}{2}} T^\dagger_{k,jm} \\ &\quad + (I_\mathcal{H} - \Pi_j) B_j^{\frac{1}{2}} \rho B_j^{\frac{1}{2}} (I_\mathcal{H} - \Pi_k) \Big) \\ &= \sum_{j,m} T_{k,jm} \Pi_j \, \rho \, \Pi_j T^\dagger_{k,jm} \end{aligned} \tag{4.13}$$

$\forall \rho \in \mathcal{S}(\mathcal{H})$, where the term in the second line vanishes due to Eq. (4.9).

We can represent operators $T_{k,jm}$ in singular value decomposition

$$T_{k,jm} = \sum_v \lambda_v^{k,jm} \, |f_v^{k,jm}\rangle \langle e_v^{k,jm}| \tag{4.14}$$

where for every k, j, m $\{|e_v^{k,jm}\rangle\}_{v=1}^{d_\mathcal{K}}$ and $\{|f_v^{k,jm}\rangle\}_{v=1}^{d_\mathcal{H}}$ are orthonormal bases in $\mathcal{K}$ and $\mathcal{H}$ respectively, and $\lambda_v^{k,jm}$s are non-negative numbers. Using Eq. (4.8) we can write $B_j = \sum_{k,m} (\lambda_v^{k,jm})^2 \, |e_v^{k,jm}\rangle \langle e_v^{k,jm}|$, which can be written

as the sum of two positive semidefinite operators, i.e.

$$B_j = \left(\sum_{k',m'} (\lambda_v^{k',jm'})^2 \left| e_v^{k',jm'} \right\rangle \left\langle e_v^{k',jm'} \right| - \sum_{k,m} (\lambda_v^{k,jm})^2 \left| e_v^{k,jm} \right\rangle \left\langle e_v^{k,jm} \right| \right) + \sum_{k,m} (\lambda_v^{k,jm})^2 \left| e_v^{k,jm} \right\rangle \left\langle e_v^{k,jm} \right|$$

We can see that the support of $\sum_{k,m} (\lambda_v^{k,jm})^2 \left| e_v^{k,jm} \right\rangle \left\langle e_v^{k,jm} \right|$ is same as the input range of $T_{k,jm}$ which means the input range of $T_{k,jm}$ is included in the support of B_j. We refer the reader to Lemma (2.4) for more details.

Thus, $T_{k,jm}\Pi_j = T_{k,jm}$ and due to equation (4.6), equation (4.13) equals $\mathcal{T}(\rho)$. This means we have proved that $\mathcal{T}^k = \sum_{j \in \Omega_B} \mathcal{W}_k^j \circ \mathcal{J}_j$. Here, we have realized the Total Instrument $\mathcal{T}$ as an adaptive sequence of an Initial Instrument, $\mathcal{J}$ and a series of Residual Instruments, $\mathcal{W}$s. This completes the proof.

□

Remark 4.2. Let us note that outcomes j and k obtained in the sequential realization of the instrument $\mathcal{T}$ from Theorem 4.1 are not completely independent. By the proof of Theorem 4.1, especially its Eq. (4.5), we see that k can only take such values, for which the postprocessing matrix elements $\nu_{k,j} > 0$. In other words, the sequential realization is designed in such a way that some of the outcomes $k \in \Omega$ of the residual instrument $\mathcal{W}^j$ have zero probability of appearance for all states produced by quantum operation $\mathcal{J}_j$ of the Initial Instrument $\mathcal{J}$.

Here, we have proved that one can realize an instrument as a two-step

adaptive sequence consisting of an initial and a series of residual instruments, given the initial instrument is a Lüders instrument of the POVM B mentioned in the theorem and POVM B is a postprocessing of the induced POVM of the Total Instrument. We can call this feature the 'sequentiality' of quantum instruments.

4.2.1 Implementation of an Instrument as an $N-$step adaptive sequence of Instruments

Theorem (4.1) is an important result for us, and we are going to build the argument for decomposing an instrument into an N step adaptive sequence of instruments by using Theorem (4.1) recursively. We will explicitly use the notion of adaptive sequences to propose a corollary of Theorem (4.1) that provides a generalization of our result.

Corollary 4.3. *Given a Total instrument $\mathcal{T} \in Ins(\Omega_N, \mathcal{H}_0, \mathcal{H}_N)$ and the postprocessing relations $A^{\mathcal{T}} \to B^{N-1} \to \cdots \to B^1$ for the POVMs $B^k \in \mathcal{O}(\Omega_k, \mathcal{H}_0)$ for $k = 1, \cdots, N-1$, there exists an adaptive sequence of instruments $Q = (\mathcal{Q}^1, \cdots \mathcal{Q}^N)$ in which the instrument applied in the k^{th} step is denoted by $\mathcal{I}^{k,a_{k-1}} \in \mathcal{Q}^k$ and*

$$\mathcal{T}_{a_N} = \sum_{a_{N-1} \in \Omega_{N-1}} \cdots \sum_{a_1 \in \Omega_1} \mathcal{I}^{N,a_{N-1}}_{a_N} \circ \ldots \circ \mathcal{I}^{2,a_1}_{a_2} \circ \mathcal{I}^{1,1}_{a_1} \tag{4.15}$$

where the Initial instrument $\mathcal{I}^{1,1}$ is the Lüders instrument of POVM B^1.

Proof. We will apply the Theorem (4.1) recursively to prove our result. In order to clearly distinguish different uses of Theorem 4.1, we will add a fixed superscript to all involved objects. At first, we will use the postprocessing relation $A^{\mathcal{T}} \to B^{N-1}$ and decompose the Total Instrument $\mathcal{T}^N$ using

Theorem 4.1 into a two-step adaptive sequence, and we use the obtained Residual Instruments to define the last step of the N step adaptive sequence. Thus, $\mathcal{Q}^N = \{\mathcal{I}^{N,a_{N-1}} \equiv \mathcal{W}^{N,a_{N-1}}\}_{a_{N-1}\in\Omega_{N-1}}$. We denote the obtained initial instrument as $\mathcal{J}^N$. Next, we use the Initial Instrument $\mathcal{J}^N$ as the Total Instrument in Theorem 4.1, i.e. $\mathcal{T}^{N-1} = \mathcal{J}^N$. Thus, at this step $\mathcal{T}^{N-1} \in Ins(\Omega_{N-1}, \mathcal{H}_0, \mathcal{H}_0)$ and $A^{\mathcal{T}^{N-1}} = B^{N-1}$. Using the postprocessing relation $B^{N-1} \to B^{N-2}$ in Theorem 4.1, we obtain a new set of residual instruments, which we use to define the one-to-last step of the adaptive sequence.

Thus, $\mathcal{Q}^{N-1} = \{\mathcal{I}^{N-1,a_{N-2}} \equiv \mathcal{W}^{N-1,a_{N-2}}\}_{a_{N-2}\in\Omega_{N-2}}$. We remind that currently

$$\begin{aligned}
\sum_{a_{N-2}\in\Omega_{N-2}} \sum_{a_{N-1}\in\Omega_{N-1}} \mathcal{I}^{N,a_{N-1}}_{a_N} \circ \mathcal{I}^{N-1,a_{N-2}}_{a_{N-1}} \circ \mathcal{J}^{N-1}_{a_{N-2}} = \\
\sum_{a_{N-2}\in\Omega_{N-2}} \sum_{a_{N-1}\in\Omega_{N-1}} \mathcal{I}^{N,a_{N-1}}_{a_N} \circ \mathcal{W}^{N-1,a_{N-2}}_{a_{N-1}} \circ \mathcal{J}^{N-1}_{a_{N-2}} = \\
\sum_{a_{N-1}\in\Omega_{N-1}} \mathcal{I}^{N,a_{N-1}}_{a_N} \circ \mathcal{T}^{N-1}_{a_{N-1}} = \\
\sum_{a_{N-1}\in\Omega_{N-1}} \mathcal{I}^{N,a_{N-1}}_{a_N} \circ \mathcal{J}^{N}_{a_{N-1}} = \\
\sum_{a_{N-1}\in\Omega_{N-1}} \mathcal{W}^{N,a_{N-1}}_{a_N} \circ \mathcal{J}^{N}_{a_{N-1}} = \\
= \mathcal{T}^N_{a_N} \qquad (4.16)
\end{aligned}$$

By performing the described recursion process $(N-1)$ times in total, we determine the whole adaptive sequence of instruments if, in the last recursion step, we fix $\mathcal{Q}^1 = \{\mathcal{I}^{1,1} \equiv \mathcal{J}^2\}$.

□

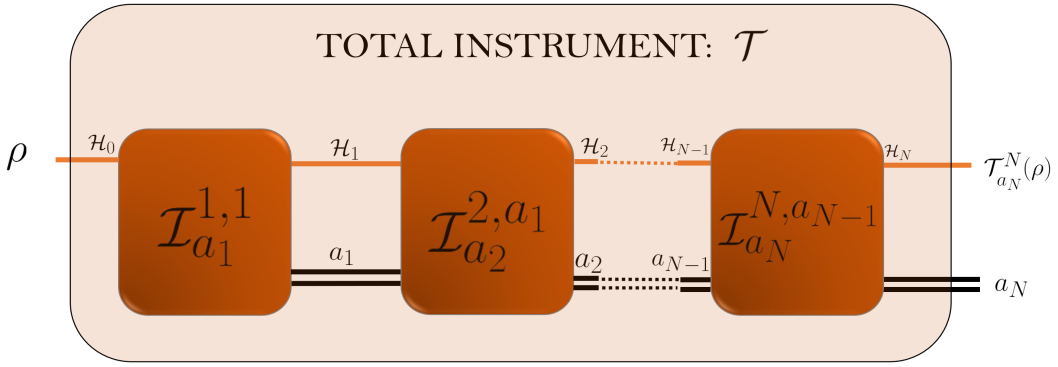

FIGURE 4.1: Schematic diagram of sequential realization of quantum instruments. It is shown that a Total Instrument $\mathcal{T}$ is decomposed into constituent instruments.

We can see figure (4.1) for a schematic representation. There, the corresponding Hilbert spaces are also mentioned. Here, we have proved that one can realize an instrument as a $N-$step adaptive sequence of constituent quantum instruments.

4.2.2 Instruments with $N-$tuple outcome space

Now, let us consider a situation in which the outcome space of the considered Total instrument $\mathcal{T}$ is an $N-$fold Cartesian product space $\Omega = \lambda_1 \times \ldots \times \lambda_N$. In this subsection, we will show that instrument $\mathcal{T}$ can be realized using an adaptive sequence of instruments $\mathcal{Q}$ such that its outcome $(a_1, \ldots, a_N) \in \Omega$ is produced by obtaining an element a_k at the k^{th} step of the adaptive sequence $\mathcal{Q}$.

We will do this by demonstrating that the considered scenario is a special case of Corollary 4.3. Thus, we define $\Omega_k = \lambda_1 \times \ldots \times \lambda_k$ and POVMs $B^k \in \mathcal{O}(\Omega_k, \mathcal{H})$ as

$$B^k_{a_1,\ldots,a_k} = \sum_{a_{k+1} \in \Omega_{k+1}} B^{k+1}_{a_1,\ldots,a_{k+1}}, \tag{4.17}$$

for all $k = 1, \ldots, N-1$ with $B^N \equiv A^{\mathcal{T}}$. Clearly, marginalization over the rightmost outcome, which relates POVMs B^{k+1} and B^k, can be understood as the postprocessing relation $B^{k+1} \to B^k$ by setting

$$\nu_{\vec{a}_{k+1}\,\vec{a}'_k} = \delta_{a_1 a'_1} \ldots \delta_{a_k a'_k} \quad \forall \vec{a}_{k+1} \in \Omega_{k+1} \;\; \forall \vec{a}'_k \in \Omega_k, \tag{4.18}$$

where we used the notation $\vec{a}_m \equiv (a_1, \ldots, a_m)$. Clearly, $\forall \vec{a}_{k+1} \sum_{\vec{a}'_k \in \Omega_k} \nu_{\vec{a}_{k+1}\,\vec{a}'_k} = 1$ and $\nu_{\vec{a}_{k+1}\,\vec{a}'_k}$ is nonzero only if $a_m = a'_m \; \forall m = 1, \ldots, k$. Due to Corollary 4.3, the Total Instrument $\mathcal{T}$ can be realized as an adaptive sequence of instruments in which outcomes at the k^{th} step belong to Ω_k and their probability of appearance depends on the POVM B^k. Thus, if outcome a'_1 is measured in the first step, then outcome $\vec{a}_2 = (a_1, a_2)$ will be obtained in the second step. Here, we must obtain an outcome such that $\nu_{\vec{a}_2\,\vec{a}'_1}$ is nonzero (see Remark 4.2). As we have mentioned above, this happens only if $a_1 = a'_1$. Thus, due to Eq.(4.18) we can simplify Eq. (4.15), which appears if we apply apply Corollary 4.3 to the considered scenario, as follows

$$\begin{aligned} \mathcal{T}_{\vec{a}_N} &= \sum_{\vec{a}_{N-1} \in \Omega_{N-1}} \cdots \sum_{\vec{a}_1 \in \Omega_1} \mathcal{I}^{N,\vec{a}_{N-1}}_{\vec{a}_N} \circ \ldots \circ \mathcal{I}^{2,\vec{a}_1}_{\vec{a}_2} \circ \mathcal{I}^{1,1}_{\vec{a}_1} \\ &= \sum_{a_{N-1} \in \lambda_{N-1}} \cdots \sum_{a_1 \in \lambda_1} \mathcal{I}^{N,\vec{a}_{N-1}}_{\vec{a}_N} \circ \ldots \circ \mathcal{I}^{2,\vec{a}_1}_{\vec{a}_2} \circ \mathcal{I}^{1,1}_{\vec{a}_1}, \end{aligned} \tag{4.19}$$

where in the first line, $\vec{a}_k$ stands for independent k tuple variables, while in the second line, the variables are $a_1, \ldots, a_N$ and $\vec{a}_k$ denotes k tuple $(a_1, \ldots, a_k)$ formed from them. In conclusion, the above findings can be summarized in the following corollary.

Corollary 4.4. *Given a Total instrument $\mathcal{T} \in Ins(\lambda_1 \times \ldots \times \lambda_N, \mathcal{H}_0, \mathcal{H}_N)$ there exists an adaptive sequence of instruments $\mathcal{Q} = (\mathcal{Q}^1, \cdots \mathcal{Q}^N)$ in which the instrument $\mathcal{I}^{k,\vec{a}_{k-1}} \in \mathcal{Q}^k$ applied in the k^{th} step determines the k^{th}*

element of the overall outcome $\vec{a}_N = (a_1, \ldots, a_N)$ *and*

$$\mathcal{T}_{\vec{a}_N} = \sum_{a_{N-1} \in \lambda_{N-1}} \cdots \sum_{a_1 \in \lambda_1} \mathcal{I}_{\vec{a}_N}^{N,\vec{a}_{N-1}} \circ \ldots \circ \mathcal{I}_{\vec{a}_2}^{2,\vec{a}_1} \circ \mathcal{I}_{\vec{a}_1}^{1,1}, \tag{4.20}$$

where the Initial instrument $\mathcal{I}^{1,1}$ *is the Lüders instrument.*

It is important to note here that we do not have to assume the postprocessing relations like in Corollary (4.3). This is due to the fact that the $N-$tuple outcome space allows for the marginalization of outcomes.

Finally, we have shown here that one can realize an instrument with a $N-$ fold Cartesian product output space as a $N-$ step adaptive sequence. This idea could potentially be useful to put the idea of sequentiality of instruments into practice in the real world, where input and output domains are constrained, and we would consider a scenario of an instrument with a discrete outcome space and a finite number of resources that are available for its realization.

Here, we need to implement the instruments of λ_k outcomes instead of a 'bigger' instrument with $\lambda_1\lambda_2 \ldots \lambda_N (k < N)$ outcomes. This, ideally, can save resources if the ancillary systems are reused.

Chapter 5

Discussions and Future Perspectives

In this minimal thesis proposal, at first, we describe the mathematical preliminaries for quantum theory and the operational perspective of measurement in quantum theory. Finally, we study the sequential decomposition of quantum instruments. We have introduced a notion of an adaptive sequence of instruments, and we have proved a realization theorem that shows that quantum instruments can be realized sequentially. The result extends the knowledge in the area of the foundations of quantum measurement theory in general from the operational point of view and motivates further research to develop the idea of *'sequentiality'* in quantum theory. It shows that it is possible to realize measurements in a sequential manner. On the theoretical level, one could potentially extend this of the idea of sequentiality to other higher-order quantum structures like superchannels and quantum combs.

In the future, we would like to study the conditions for optimality for the sequential implementation of instruments and POVMs and to provide a quantitative measure for the trade-off of spatial and temporal resources.

In the thesis, we have shown the sequentiality of any general instrument; it would be interesting to study particular classes of instruments like the Trash-and-Prepare and Measure-and-Prepare instruments.

As we know, POVMs are considered special cases of quantum instruments; hence the sequentiality of POVMs[24], form a special case of our described scenario. We plan to study specific POVMs, like the SIC-POVMs [25], for arbitrary dimensions and resolve different issues regarding them, like their existence itself. Also, that work would motivate one to study the Lüders instrument of SIC-POVMs and how their outcomes can processed for tomography.

Furthermore, from the practical perspective, this result is useful for implementing complex quantum measurements given a limited amount of resources. Specifically, we know that for NISQ (Noisy Intermediate-Scale Quantum)[26] devices, it is essential to reduce the spatial resources, and the ideas discussed above enable one to perform a complex quantum measurement through a series of sequential steps, making it a resource-efficient approach. For example, subsection (4.2.2) proves the claim for Instruments with N-tuple outcome space, and one can view the outcome space of a measurement associated with a NISQ device to be a cartesian product of finite (typically binary) outcome sets. Additionally, one can also investigate the ideas of how to perform those measurements via quantum circuits with basic quantum gates.

Also, an interesting sub-part of the problem for us would be to investigate what kind of sequential implementation can be done for projective measurements and whether they are advantageous or not. For example, a useful case to study will be a non-degenerate projective measurement or even their Lüders instrument.

Another perspective for future directions for us is designing programs or mathematical tools for implementing a given measurement in a sequence of other measurements where some of the sub-problems can be recast as semidefinite programming problems (allowing for guaranteed numerical precision of the solution) or more generic numerical optimizations.

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