On the properties $\alpha - z$ Rényi divergences on general von Neumann algebras

Fumio Hiai and Anna Jenčová

March 5, 2024

1 Introduction

2 Preliminaries

2.1 Basic definitions

Let $\mathcal{M}$ be a von Neumann algebra and let $\mathcal{M}^+$ be the cone of positive elements in $\mathcal{M}$. We denote the predual by $\mathcal{M}_*$, its positive part by $\mathcal{M}_*^+$ and the set of normal states by $\mathfrak{S}_*(\mathcal{M})$. For $\psi \in \mathcal{M}_*^+$, we will denote by $s(\psi)$ the support projection of ψ.

For $0 < p \leq \infty$, let $L_p(\mathcal{M})$ be the Haagerup L_p-space over $\mathcal{M}$ and let $L_p(\mathcal{M})$ its positive cone, [?]. We will use the identifications $\mathcal{M} \simeq L_\infty(\mathcal{M})$, $\mathcal{M}_* \ni \psi \leftrightarrow h_\psi \in L_1(\mathcal{M})$ and the notation $\mathrm{Tr}\, h_\psi = \psi(1)$ for the trace in $L_1(\mathcal{M})$. It this way, $\mathcal{M}_*^+$ is identified with the positive cone $L_1(\mathcal{M})^+$ and $\mathfrak{S}_*(\mathcal{M})$ with subset of elements in $L_1(\mathcal{M})^+$ with unit trace. Precise definitions and further details on the spaces $L_p(\mathcal{M})$ can be found in the notes [?].

2.2 The $\alpha - z$-Rényi divergences

In [? ?], the $\alpha - z$-Rényi divergence for $\psi, \varphi \in \mathcal{M}_*^+$ was defined as follows:

Definition 2.1. Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$ and let $\alpha, z > 0$, $\alpha \neq 1$. The $\alpha - z$-Rényi divergence is defined as

$$D_{\alpha,z}(\psi\|\varphi) := \frac{1}{\alpha - 1} \log \frac{Q_{\alpha,z}(\psi\|\varphi)}{\psi(1)},$$

where

$$Q_{\alpha,z} = \begin{cases} \mathrm{Tr}\left(h_\varphi^{\frac{1-\alpha}{2z}} h_\psi^{\frac{\alpha}{z}} h_\varphi^{\frac{1-\alpha}{2z}}\right)^z, & \text{if } 0 < \alpha < 1 \\ \|x\|_z^z, & \text{if } \alpha > 1 \text{ and} \\ & h_\psi^{\frac{\alpha}{z}} = h_\varphi^{\frac{\alpha-1}{2z}} x h_\varphi^{\frac{\alpha-1}{2z}}, \text{ with } x \in s(\varphi)L_z(\mathcal{M})s(\varphi) \\ \infty & \text{otherwise.} \end{cases}$$

In the case $\alpha > 1$, the following alternative form will be useful.

Lemma 2.2. *[? , Lemma 7] Let $\alpha > 1$ and $\psi, \varphi \in \mathcal{M}_*^+$. Then $Q_{\alpha,z}(\psi\|\varphi) < \infty$ if and only if there is some $y \in L_{2z}(\mathcal{M})s(\varphi)$ such that*

$$h_\psi^{\frac{\alpha}{2z}} = y h_\varphi^{\frac{\alpha-1}{2z}}.$$

Moreover, in this case, such y is unique and we have $Q_{\alpha,z}(\psi\|\varphi) = \|y\|_{2z}^{2z}$.

The standard Rényi divergence [? ? ?] is contained in this range as $D_\alpha(\psi\|\varphi) = D_{\alpha,1}(\psi\|\varphi)$. The sandwiched Rényi divergence is obtained as $\tilde{D}_\alpha(\psi\|\varphi) = D_{\alpha,\alpha}(\psi\|\varphi)$, see [? ? ? ?] for some alternative definitions and properties of $\tilde{D}_\alpha$. The definition in [?] and [?] is based on the Kosaki interpolation spaces $L_p(\mathcal{M}, \varphi)$ with respect to a state [?]. These spaces and complex interpolation method will be used frequently also in the present work.

Many of the properties of $D_{\alpha,z}(\psi\|\varphi)$ were extended from the finite dimensional case in [?]. In particular, a variational expression for $Q_{\alpha,z}$ in the case $0 < \alpha < 1$ was proved there, see part (i) in the theorem below. We will prove a similar variational expression also in the case when $\alpha > 1$.

Theorem 2.3 (Variational expressions). *Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$.*

(i) Let $0 < \alpha < 1$ and $\max\{\alpha, 1-\alpha\} \leq z$. Then

$$Q_{\alpha,z}(\psi\|\varphi) = \inf_{a \in \mathcal{M}^{++}} \left\{ \alpha \mathrm{Tr}\left((a^{\frac{1}{2}} h_\psi^{\frac{\alpha}{z}} a^{\frac{1}{2}})^{\frac{z}{\alpha}} \right) + (1-\alpha)\mathrm{Tr}\left((a^{\frac{1}{2}} h_\varphi^{\frac{1-\alpha}{z}} a^{\frac{1}{2}})^{\frac{z}{1-\alpha}} \right) \right\}.$$

(ii) Let $1 < \alpha$, $\max\{\frac{\alpha}{2}, \alpha - 1\} \leq z$. Then

$$Q_{\alpha,z}(\psi\|\varphi) = \sup_{a \in \mathcal{M}_+} \left\{ \alpha \mathrm{Tr}\left((a^{\frac{1}{2}} h_\psi^{\frac{\alpha}{z}} a^{\frac{1}{2}})^{\frac{z}{\alpha}} \right) - (\alpha-1)\mathrm{Tr}\left((a^{\frac{1}{2}} h_\varphi^{\frac{\alpha-1}{2z}} a^{\frac{1}{2}})^{\frac{z}{\alpha-1}} \right) \right\}.$$

Proof. For part (i) see [? , Theorem 1 (vi)]. The inequality $\geq$ in part (ii) holds for all α and z and was proved in [? , Theorem 2 (vi)]. We now prove the opposite inequality.

Assume first that $Q_{\alpha,z}(\psi\|\varphi) < \infty$, so that there is some $x \in s(\varphi)L_z(\mathcal{M})^+ s(\varphi)$ such that $h_\psi^{\frac{\alpha}{z}} = h_\varphi^{\frac{\alpha-1}{2z}} x h_\varphi^{\frac{\alpha-1}{2z}}$. Plugging this into the right hand side, we obtain

$$\begin{aligned}
&\sup_{a \in \mathcal{M}_+} \left\{ \alpha \mathrm{Tr}\left((a^{\frac{1}{2}} h_\psi^{\frac{\alpha}{z}} a^{\frac{1}{2}})^{\frac{z}{\alpha}} \right) - (\alpha-1)\mathrm{Tr}\left((a^{\frac{1}{2}} h_\varphi^{\frac{\alpha-1}{2z}} a^{\frac{1}{2}})^{\frac{z}{\alpha-1}} \right) \right\} \\
&= \sup_{a \in \mathcal{M}_+} \left\{ \alpha \mathrm{Tr}\left((a^{\frac{1}{2}} h_\varphi^{\frac{\alpha-1}{2z}} x h_\varphi^{\frac{\alpha-1}{2z}} a^{\frac{1}{2}})^{\frac{z}{\alpha}} \right) - (\alpha-1)\mathrm{Tr}\left((a^{\frac{1}{2}} h_\varphi^{\frac{\alpha-1}{2z}} a^{\frac{1}{2}})^{\frac{z}{\alpha-1}} \right) \right\} \\
&= \sup_{a \in \mathcal{M}_+} \left\{ \alpha \mathrm{Tr}\left((x^{\frac{1}{2}} h_\varphi^{\frac{\alpha-1}{2z}} a h_\varphi^{\frac{\alpha-1}{2z}} x^{\frac{1}{2}})^{\frac{z}{\alpha}} \right) - (\alpha-1)\mathrm{Tr}\left((h_\varphi^{\frac{\alpha-1}{2z}} a h_\varphi^{\frac{\alpha-1}{2z}})^{\frac{z}{\alpha-1}} \right) \right\} \\
&= \sup_{w \in L_{\frac{z}{\alpha-1}}(\mathcal{M})^+} \left\{ \alpha \mathrm{Tr}\left((x^{\frac{1}{2}} w x^{\frac{1}{2}})^{\frac{z}{\alpha}} \right) - (\alpha-1)\mathrm{Tr}\left(w^{\frac{z}{\alpha-1}} \right) \right\},
\end{aligned}$$

where we used the fact that $\mathrm{Tr}\,((h^*h)^p) = \mathrm{Tr}\,((hh^*)^p)$ for $p > 0$ and $h \in L_{\frac{p}{2}}(\mathcal{M})$, and Lemma A.1. Putting $w = x^{\alpha-1}$ we get

$$\sup_{w \in L_{\frac{z}{\alpha-1}}(\mathcal{M})^+} \left\{ \alpha \mathrm{Tr}\left((x^{\frac{1}{2}} w x^{\frac{1}{2}})^{\frac{z}{\alpha}} \right) - (\alpha-1)\mathrm{Tr}\left(w^{\frac{z}{\alpha-1}} \right) \right\} \geq \mathrm{Tr}\,(x^z) = \|x\|_z^z = Q_{\alpha,z}(\psi\|\varphi).$$

This finishes the proof in the case that $Q_{\alpha,z}(\psi\|\varphi)<\infty$. Note that this holds if $\psi\le\lambda\varphi$ for some $\lambda>0$. Indeed, since $\frac{\alpha}{2z}\in(0,1]$ by the assumption, we then have

$$h_\psi^{\frac{\alpha}{2z}}\le\lambda^{\frac{\alpha}{2z}}h_\varphi^{\frac{\alpha}{2z}},$$

hence by [? , Lemma A.58] there is some $b\in\mathcal{M}$ such that

$$h_\psi^{\frac{\alpha}{2z}}=bh_\varphi^{\frac{\alpha}{2z}}=yh_\varphi^{\frac{\alpha-1}{2z}},$$

where $y=bh_\varphi^{\frac{1}{2z}}\in L_{2z}(\mathcal{M})$. By Lemma 2.2 we get $Q_{\alpha,z}(\psi\|\varphi)=\|y\|_{2z}^{2z}<\infty$.

In the general case, the variational expression holds for $Q_{\alpha,z}(\psi\|\varphi+\epsilon\psi)$ for all $\epsilon>0$, so that we have

$$\begin{aligned}Q_{\alpha,z}(\psi\|\varphi+\epsilon\psi)&=\sup_{a\in\mathcal{M}_+}\left\{\alpha\mathrm{Tr}\left((a^{\frac12}h_\psi^{\frac{\alpha}{z}}a^{\frac12})^{\frac{z}{\alpha}}\right)-(\alpha-1)\mathrm{Tr}\left((a^{\frac12}h_{\varphi+\epsilon\psi}^{\frac{\alpha-1}{2z}}a^{\frac12})^{\frac{z}{\alpha-1}}\right)\right\}\\&\le\sup_{a\in\mathcal{M}_+}\left\{\alpha\mathrm{Tr}\left((a^{\frac12}h_\psi^{\frac{\alpha}{z}}a^{\frac12})^{\frac{z}{\alpha}}\right)-(\alpha-1)\mathrm{Tr}\left((a^{\frac12}h_{\varphi}^{\frac{\alpha-1}{2z}}a^{\frac12})^{\frac{z}{\alpha-1}}\right)\right\},\end{aligned}$$

where the inequality above follows by Lemma A.2. Therefore, since lower semicontinuity [? , Theorem 2 (iv)] gives

$$Q_{\alpha,z}(\psi\|\varphi)\le\liminf_{\epsilon\searrow0}Q_{\alpha,z}(\psi\|\varphi+\epsilon\psi)$$

the desired inequality follows.

□

Lemma 2.4. *Assume that $\lambda^{-1}\varphi\le\psi\le\lambda\varphi$ for some $\lambda>0$. Then the infimum in the variational expression in Theorem 2.3(i) is attained at a unique element $\bar a\in\mathcal{M}^{++}$. This element satisfies*

$$h_\psi^{\frac{\alpha}{2z}}ah_\psi^{\frac{\alpha}{2z}}=\big(h_\psi^{\frac{\alpha}{2z}}h_\varphi^{\frac{1-\alpha}{z}}h_\psi^{\frac{\alpha}{2z}}\big)^{\alpha}\tag{1}$$

$$h_\varphi^{\frac{1-\alpha}{2z}}a^{-1}h_\varphi^{\frac{1-\alpha}{2z}}=\big(h_\varphi^{\frac{1-\alpha}{2z}}h_\psi^{\frac{\alpha}{z}}h_\varphi^{\frac{1-\alpha}{2z}}\big)^{1-\alpha}.\tag{2}$$

Proof. We may assume that φ and hence also ψ is faithful. Following the proof of [? , Theorem 1 (vi)], we may use the assumptions and [? , Lemma A.58] to show that there are $b,c\in\mathcal{M}$ such that

$$h_\varphi^{\frac{1-\alpha}{2z}}=b\big(h_\varphi^{\frac{1-\alpha}{2z}}h_\psi^{\frac{\alpha}{z}}h_\varphi^{\frac{1-\alpha}{2z}}\big)^{\frac{1-\alpha}{2}},\qquad\big(h_\varphi^{\frac{1-\alpha}{2z}}h_\psi^{\frac{\alpha}{z}}h_\varphi^{\frac{1-\alpha}{2z}}\big)^{\frac{1-\alpha}{2}}=ch_\varphi^{\frac{1-\alpha}{2z}}.\tag{3}$$

Put $\bar a:=bb^*\in\mathcal{M}^{++}$, then we have $\bar a^{-1}=c^*c$ and $\bar a$ is indeed a minimizer of

$$Q_{\alpha,z}(\psi\|\varphi)=\inf_{a\in\mathcal{M}^{++}}\left\{\alpha\left\|h_\psi^{\frac{\alpha}{2z}}ah_\psi^{\frac{\alpha}{2z}}\right\|_{\frac{z}{\alpha}}^{\frac{z}{\alpha}}+(1-\alpha)\left\|h_\varphi^{\frac{1-\alpha}{2z}}a^{-1}h_\varphi^{\frac{1-\alpha}{2z}}\right\|_{\frac{z}{1-\alpha}}^{\frac{z}{1-\alpha}}\right\}.\tag{4}$$

We next observe that such minimizer is unique. Indeed, suppose that the infimum is attained at some $a_1,a_2\in\mathcal{M}^{++}$. Let $a_0:=(a_1+a_2)/2$. Since the map $L^p(\mathcal{M})\ni k\mapsto\|k\|_p^p$ is convex for any $p\ge1$ and $a_0^{-1}\le(a_1^{-1}+a_2^{-1})/2$, we have

$$\left\|h_\psi^{\frac{\alpha}{2z}}a_0h_\psi^{\frac{\alpha}{2z}}\right\|_{\frac{z}{\alpha}}^{\frac{z}{\alpha}}\le\frac12\left\{\left\|h_\psi^{\frac{\alpha}{2z}}a_1h_\psi^{\frac{\alpha}{2z}}\right\|_{\frac{z}{\alpha}}^{\frac{z}{\alpha}}+\left\|h_\psi^{\frac{\alpha}{2z}}a_2h_\psi^{\frac{\alpha}{2z}}\right\|_{\frac{z}{\alpha}}^{\frac{z}{\alpha}}\right\},$$

$$\begin{aligned}\left\|h_\varphi^{\frac{1-\alpha}{2z}}a_0^{-1}h_\varphi^{\frac{1-\alpha}{2z}}\right\|_{\frac{z}{1-\alpha}}^{\frac{z}{1-\alpha}}&\le\left\|h_\varphi^{\frac{1-\alpha}{2z}}\left(\frac{a_1^{-1}+a_2^{-1}}{2}\right)h_\varphi^{\frac{1-\alpha}{2z}}\right\|_{\frac{z}{1-\alpha}}^{\frac{z}{1-\alpha}}\\&\le\frac12\left\{\left\|h_\varphi^{\frac{1-\alpha}{2z}}a_1^{-1}h_\varphi^{\frac{1-\alpha}{2z}}\right\|_{\frac{z}{1-\alpha}}^{\frac{z}{1-\alpha}}+\left\|h_\varphi^{\frac{1-\alpha}{2z}}a_2^{-1}h_\varphi^{\frac{1-\alpha}{2z}}\right\|_{\frac{z}{1-\alpha}}^{\frac{z}{1-\alpha}}\right\}.\end{aligned}$$

Hence we have

$$\left\| h_\varphi^{\frac{1-\alpha}{2z}} a_0^{-1} h_\varphi^{\frac{1-\alpha}{2z}} \right\|_{\frac{z}{1-\alpha}} = \left\| h_\varphi^{\frac{1-\alpha}{2z}} \left(\frac{a_1^{-1} + a_2^{-1}}{2} \right) h_\varphi^{\frac{1-\alpha}{2z}} \right\|_{\frac{z}{1-\alpha}},$$

which implies that $a_0^{-1} = \frac{a_1^{-1}+a_2^{-1}}{2}$, as easily verified. From this we easily have $a_1 = a_2$.

The equality (2) is obvious from the second equality in (3) and $\bar{a}^{-1} = c^*c$. Since $Q_{\alpha,z}(\psi\|\varphi) = Q_{1-\alpha,z}(\varphi\|\psi)$, we see by uniqueness that the minimizer of the infimum expression for $Q_{1-\alpha,z}(\varphi\|\psi)$ (instead of (4)) is $\bar{a}^{-1}$ (instead of $\bar{a}$). This says that (1) is the equality corresponding to (2) when ψ, φ, α are replaced with $\varphi, \psi, 1-\alpha$, respectively.

□

3 Data processing inequality and reversibility of channels

Let $\gamma : \mathcal{N} \to \mathcal{M}$ be a normal positive unital map. Then the predual of γ defines a positive linear map $\gamma_* : L_1(\mathcal{M}) \to L_1(\mathcal{N})$ that preserves the trace, acting as

$$L_1(\mathcal{M}) \ni h_\rho \mapsto h_{\rho\circ\gamma} \in L_1(\mathcal{N}).$$

The support of γ will be denoted by $s(\gamma)$, recall that this is defined as the largest projection $p \in \mathcal{N}$ such that $\gamma(p) = 1$. For any $\rho \in \mathcal{M}_*^+$ we clearly have $s(\rho\circ\gamma) \leq s(\gamma)$, with equality if ρ is faithful. It follows that γ_* maps $L_1(\mathcal{M})$ to $s(\gamma)L_1(\mathcal{N})s(\gamma) \equiv L_1(s(\gamma)\mathcal{N}s(\gamma))$. For any $\rho \in \mathcal{M}_+^*$, $\rho \neq 0$, the map

$$s(\gamma)\mathcal{N}s(\gamma) \to s(\rho)\mathcal{M}s(\rho), \qquad a \mapsto s(\rho)\gamma(a)s(\rho)$$

is a faithful normal positive unital map, so using such restrictions we may always assume that both ρ and $\rho\circ\gamma$ are faithful.

The Petz dual of γ with respect to a faithful $\rho \in \mathcal{M}_*^+$ is a map $\gamma_\rho^* : \mathcal{M} \to \mathcal{N}$, introduced in [**?**]. It was proved that it is again normal, positive and unital, in addition, it is n-positive whenever γ is. As explained in [**?**] γ_ρ^* is determined by the equality

$$(\gamma_\rho^*)_*(h_{\rho\circ\gamma}^{\frac{1}{2}} b h_{\rho\circ\gamma}^{\frac{1}{2}}) = h_\rho^{\frac{1}{2}} \gamma(b) h_\rho^{\frac{1}{2}}, \tag{5}$$

for all $b \in \mathcal{N}^+$, here $(\gamma_\rho^*)_*$ is the predual map of γ_ρ^*. We also have

$$(\gamma_\rho^*)_*(h_{\rho\circ\gamma}) = (\gamma_\rho^*)_* \circ \gamma_*(h_\rho) = h_\rho$$

and $(\gamma_\rho^*)_{\rho\circ\gamma}^* = \gamma$.

3.1 Data processing inequality

In this paragraph we prove the data processing inequality (DPI) for $D_{\alpha,z}$ with respect to normal positive unital maps. In the case of the sandwiched divergences $\tilde{D}_\alpha$ with $1/2 \leq \alpha \neq 1$, DPI was proved in [**? ?**], see also [**?**] for an alternative proof in the case when the maps are also completely positive.

Lemma 3.1. *Let $\gamma : \mathcal{N} \to \mathcal{M}$ be a normal positive unital map and let $\rho \in \mathcal{M}_*^+$, $b \in \mathcal{N}^+$.*

(i) If $p \in [1/2, 1)$, then

$$\|h_{\rho\circ\gamma}^{\frac{1}{2p}} b h_{\rho\circ\gamma}^{\frac{1}{2p}}\|_p \leq \|h_\rho^{\frac{1}{2p}} \gamma(b) h_\rho^{\frac{1}{2p}}\|_p.$$

(ii) If $p \in [1,\infty]$, the inequality reverses.

Proof. Let us denote $\beta := \gamma^*_\rho$ and let $\omega \in \mathcal{M}_*^+$ be such that $h_\omega := h_{\rho\circ\gamma}^{\frac{1}{2}} b h_{\rho\circ\gamma}^{\frac{1}{2}} \in L_1(\mathcal{N})^+$. Then β is a normal positive unital map and we have

$$\beta_*(h_\omega) = h_\rho^{\frac{1}{2}}\gamma(b)h_\rho^{\frac{1}{2}}, \qquad \beta_*(h_{\rho\circ\gamma}) = h_\rho.$$

Let $p \in [1/2, 1)$, then

$$\begin{aligned}\|h_\rho^{\frac{1}{2p}}\gamma(b)h_\rho^{\frac{1}{2p}}\|_p^p &= \|h_\rho^{\frac{1-p}{2p}}\beta_*(h_\omega)h_\rho^{\frac{1-p}{2p}}\|_p^p = Q_{p,p}(\beta_*(h_\omega)\|h_\rho) = Q_{p,p}(\beta_*(h_\omega)\|\beta_*(h_{\rho\circ\gamma}))\\ &\geq Q_{p,p}(h_\omega\|h_{\rho\circ\gamma}) = \|h_{\rho\circ\gamma}^{\frac{1-p}{2p}}h_\omega h_{\rho\circ\gamma}^{\frac{1-p}{2p}}\|_p^p = \|h_{\rho\circ\gamma}^{\frac{1}{2p}}bh_{\rho\circ\gamma}^{\frac{1}{2p}}\|_p^p.\end{aligned}$$

Here we have used the DPI for the sandwiched Rényi divergence $D_{\alpha,\alpha}$ for $\alpha \in [1/2, 1)$, [? , Theorem 4.1]. This proves (i). The case (ii) was proved in [?] (see Eq. (22) therein), using the relation of the sandwiched Rényi divergence to the Kosaki L_p norms. In our setting, the proof can be written as

$$\begin{aligned}\|h_\rho^{\frac{1}{2p}}\gamma(b)h_\rho^{\frac{1}{2p}}\|_p^p &= Q_{p,p}(h_\rho^{\frac{1}{2}}\gamma(b)h_\rho^{\frac{1}{2}}\|h_\rho) = Q_{p,p}(\beta_*(h_\omega)\|\beta_*(h_{\rho\circ\gamma}))\\ &\leq Q_{p,p}(h_\omega\|h_{\rho\circ\gamma}) = \|h_{\rho\circ\gamma}^{\frac{1}{2p}}bh_{\rho\circ\gamma}^{\frac{1}{2p}}\|_p^p,\end{aligned}$$

here the inequality follows from the DPI for sandwiched Rényi divergence $D_{\alpha,\alpha}$ with $\alpha > 1$, [?]. □

Theorem 3.2 (DPI)**.** *Let $\psi, \varphi \in \mathcal{M}_*^+$, $\psi \neq 0$ and let $\gamma : \mathcal{N} \to \mathcal{M}$ be a normal positive unital map. Assume either of the following conditions:*

(i) $0 < \alpha < 1$, $\max\{\alpha, 1-\alpha\} \leq z$

(ii) $\alpha > 1$, $\max\{\alpha/2, \alpha - 1\} \leq z \leq \alpha$.

Then we have

$$D_{\alpha,z}(\psi\circ\gamma\|\varphi\circ\gamma) \leq D_{\alpha,z}(\psi\|\varphi).$$

Proof. Under the conditions (i), the DPI was proved in [? , Theorem 1 (viii)]. Since parts of this proof will be used below, we repeat it here.

Assume the conditions in (i) and put $p := \frac{z}{\alpha}$, $r := \frac{z}{1-\alpha}$, so that $p, r \geq 1$. For any $b \in \mathcal{N}^{++}$, we have by the Choi inequality [?] that $\gamma(b)^{-1} \leq \gamma(b^{-1})$, so that

$$\|h_\varphi^{\frac{1}{2r}}\gamma(b)^{-1}\varphi^{\frac{1}{2r}}\|_r \leq \|h_\varphi^{\frac{1}{2r}}\gamma(b^{-1})\varphi^{\frac{1}{2r}}\|_r.$$

Using the variational expression in Theorem 2.3 (i), we have

$$Q_{\alpha,z}(\psi\|\varphi) \leq \alpha\|h_\psi^{\frac{1}{2p}}\gamma(b)h_\psi^{\frac{1}{2p}}\|_p^p + (1-\alpha)\|h_\varphi^{\frac{1}{2r}}\gamma(b)^{-1}h_\varphi^{\frac{1}{2r}}\|_r^r \tag{6}$$

$$\leq \alpha\|h_\psi^{\frac{1}{2p}}\gamma(b)h_\psi^{\frac{1}{2p}}\|_p^p + (1-\alpha)\|h_\varphi^{\frac{1}{2r}}\gamma(b^{-1})h_\varphi^{\frac{1}{2r}}\|_r^r \tag{7}$$

$$\alpha\|h_{\psi\circ\gamma}^{\frac{1}{2p}}bh_{\psi\circ\gamma}^{\frac{1}{2p}}\|_p^p + (1-\alpha)\|h_{\varphi\circ\gamma}^{\frac{1}{2r}}b^{-1}h_{\varphi\circ\gamma}^{\frac{1}{2r}}\|_r^r, \tag{8}$$

here we used Lemma 3.1 (ii) for the last inequality. Since this holds for all $b \in \mathcal{N}^+ +$, it follows that $Q_{\alpha,z}(\psi\|\varphi) \leq Q_\alpha(\psi \circ \gamma\|\varphi \circ \gamma)$, which proves the DPI in this case.

Assume next the condition (ii), and put $p := \frac{z}{\alpha}$, $q := \frac{z}{\alpha-1}$, so that $p \in [1/2, 1)$ and $q \geq 1$. Using Theorem 2.3 (ii), we get for any $b \in \mathcal{N}^+$,

$$\begin{aligned} Q_{\alpha,z}(\psi\|\varphi) &\geq \alpha\|h_\psi^{\frac{1}{2p}}\gamma(b)h_\psi^{\frac{1}{2p}}\|_p^p - (\alpha-1)\|h_\varphi^{\frac{1}{2q}}\gamma(b)h_\varphi^{\frac{1}{2q}}\|_q^q \\ &\geq \alpha\|h_{\psi\circ\gamma}^{\frac{1}{2p}}bh_{\psi\circ\gamma}^{\frac{1}{2p}}\|_p^p - (\alpha-1)\|h_{\varphi\circ\gamma}^{\frac{1}{2r}}bh_{\varphi\circ\gamma}^{\frac{1}{2q}}\|_q^q, \end{aligned}$$

here we used both (i) and (ii) in Lemma 3.1. Again, since this holds for all $b \in \mathcal{N}^+$, we get the desired inequality.

□

3.2 Martingale convergence

3.3 Equality in DPI and reversibility of channels

In what follows, a channel is a normal 2-positive unital map $\gamma : \mathcal{N} \to \mathcal{M}$.

Definition 3.3. Let $\gamma : \mathcal{N} \to \mathcal{M}$ be a channel and let $\mathcal{S} \subset \mathcal{M}_*^+$. We say that γ is reversible (or sufficient) with respect to $\mathcal{S}$ if there exists a channel $\beta : \mathcal{M} \to \mathcal{N}$ such that

$$\rho \circ \gamma \circ \beta = \rho, \qquad \forall \rho \in \mathcal{S}.$$

The notion of sufficient channels was introduced by Petz [**?** **?**], who also obtained a number of conditions characterizing this situation. It particular, it was proved in [**?**] that sufficient channels can be characterized by equality in DPI for the relative entropy $D(\psi\|\varphi)$: if $D(\psi\|\varphi) < \infty$, then a channel γ is sufficient with respect to $\{\psi, \varphi\}$ if and only if

$$D(\psi \circ \gamma\|\varphi \circ \gamma) = D(\psi\|\varphi).$$

This result has been proved for a number of other divergence measures, including the standard Rényi divergences $D_{\alpha,1}$ with $0 < \alpha < 2$ ([]) and the sandwiched Rényi divergences $D_{\alpha,\alpha}$ for $\alpha > 1/2$ ([**?** **?**]). Our aim in this section is to prove that a similar statement holds for $D_{\alpha,z}$ for values of the parameters strictly contained in the DPI bounds of Theorem 3.2.

Throughout this section, we will assume that $\psi, \varphi \in \mathcal{M}_*^+$ are such that $s(\psi) \leq s(\varphi)$. As noted above, we may replace the channel γ by its restriction so that we may assume that both φ and $\varphi_0 := \varphi \circ \gamma$ are faithful.

Another important result of [**?**] shows that the Petz dual γ_φ^* is a universal recovery map, in the sense given in the proposition below.

Proposition 1. *Let $\varphi \in \mathcal{M}_*^+$ be faithful and let $\gamma : \mathcal{N} \to \mathcal{M}$ be a faithful channel. Then for any $\psi \in \mathcal{M}_*^+$, γ is reversible with respect to $\{\psi, \varphi\}$ if and only if $\psi \circ \gamma \circ \gamma_\varphi^* = \psi$.*

Consequently, there is a faithful normal conditional expectation $\mathcal{E}$ on $\mathcal{M}$ such that $\varphi \circ \mathcal{E} = \varphi$ and γ is sufficient with respect to $\{\psi, \varphi\}$ if and only if also $\psi \circ \mathcal{E} = \psi$.

Note that the range of the conditional expectation $\mathcal{E}$ in the above proposition is the set of fixed points of the channel $\gamma \circ \gamma_\varphi^*$.

3.3.1 The case $\alpha \in (0,1)$

Theorem 3.4. *Let $0 < \alpha < 1$ and $\alpha, 1-\alpha \le z$ where at least one of the inequalities is strict. Let $\psi, \varphi \in \mathcal{M}_*^+$ be such that $s(\psi) \le s(\varphi)$. Then γ is reversible with respect to $\{\psi, \varphi\}$ if and only if*

$$D_{\alpha,z}(\psi\|\varphi) = D_{\alpha,z}(\psi\circ\gamma\|\varphi\circ\gamma).$$

Proof. Let us denote $\psi_0 := \psi\circ\gamma$, $\varphi_0 := \varphi\circ\gamma$. Using restrictions as before, we may assume that both φ and φ_0 are faithful.

We first treat the case when $\lambda^{-1}\varphi \le \psi \le \lambda\varphi$ for some $\lambda > 0$, then ψ_0 and φ_0 also satisfy this condition an all the states $\psi, \varphi, \psi_0, \varphi_0$ are faithful. By Theorem 2.3 (i), there are some $\bar a \in \mathcal{M}^{++}$ and $\bar a_0 \in \mathcal{N}^{++}$ such that the infimum in the variational formula for $D_{\alpha,z}(\psi\|\varphi)$ resp. $D_{\alpha,z}(\psi_0\|\varphi_0)$ is attained. Using the inequalities in (6) - (8), we obtain

$$\begin{aligned} Q_{\alpha,z}(\psi\|\varphi) &\le \alpha\|h_\psi^{\frac{1}{2p}}\gamma(\bar a_0)h_\psi^{\frac{1}{2p}}\|_p^p + (1-\alpha)\|h_\varphi^{\frac{1}{2r}}\gamma(\bar a_0)^{-1}h_\varphi^{\frac{1}{2r}}\|_r^r \\ &\le \alpha\|h_{\psi_0}^{\frac{1}{2p}}\bar a_0 h_{\psi_0}^{\frac{1}{2p}}\|_p^p + (1-\alpha)\|h_{\varphi_0}^{\frac{1}{2r}}\bar a_0^{-1}h_{\varphi_0}^{\frac{1}{2r}}\|_r^r \\ &= Q_{\alpha,z}(\psi_0\|\varphi_0), \end{aligned}$$

where we again put $p = \frac{z}{\alpha}$, $r = \frac{z}{1-\alpha}$. Assume $D_{\alpha,z}(\psi\|\varphi) = D_{\alpha,z}(\psi_0\|\varphi_0)$, then all the above inequalities must be equalities.

This has several consequences. First, by uniqueness of $\bar a$ in Theorem 2.3 (i), we have $\gamma(\bar a_0) = \bar a$. Furthermore, by Lemma 3.1 (ii), we obtain that

$$\|h_\psi^{\frac{1}{2p}}\gamma(\bar a_0)h_\psi^{\frac{1}{2p}}\|_p^p = \|h_{\psi_0}^{\frac{1}{2p}}\bar a_0 h_{\psi_0}^{\frac{1}{2p}}\|_p^p, \qquad \|h_\varphi^{\frac{1}{2r}}\gamma(\bar a_0)^{-1}h_\varphi^{\frac{1}{2r}}\|_r^r = \|h_{\varphi_0}^{\frac{1}{2r}}\bar a_0^{-1}h_{\varphi_0}^{\frac{1}{2r}}\|_r^r.$$

By the assumptions, at least one of p and r must be strictly larger than 1. Assume that $r > 1$ (the case $p > 1$ is similar, even slightly easier). Since $h_\varphi^{\frac{1}{2r}}\gamma(\bar a_0)^{-1}h_\varphi^{\frac{1}{2r}} \le h_\varphi^{\frac{1}{2r}}\gamma(\bar a_0^{-1})h_\varphi^{\frac{1}{2r}}$, Lemma 3.1 and the equality above imply that

$$\|h_\varphi^{\frac{1}{2r}}\gamma(\bar a_0)^{-1}h_\varphi^{\frac{1}{2r}}\|_r^r = \|h_\varphi^{\frac{1}{2r}}\gamma(\bar a_0^{-1})h_\varphi^{\frac{1}{2r}}\|_r^r = \|h_{\varphi_0}^{\frac{1}{2r}}\bar a_0^{-1}h_{\varphi_0}^{\frac{1}{2r}}\|_r^r. \tag{9}$$

Using [? , Lemma 5.1], this shows that we must have

$$h_\varphi^{\frac{1}{2r}}\bar a^{-1}h_\varphi^{\frac{1}{2r}} = h_\varphi^{\frac{1}{2r}}\gamma(\bar a_0)^{-1}h_\varphi^{\frac{1}{2r}} = h_\varphi^{\frac{1}{2r}}\gamma(\bar a_0^{-1})h_\varphi^{\frac{1}{2r}}.$$

Put $h_\omega := h_\varphi^{\frac12}\bar a^{-1}h_\varphi^{\frac12}$, $h_{\omega_0} := h_{\varphi_0}^{\frac12}\bar a_0^{-1}h_{\varphi_0}^{\frac12}$. Then we have

$$(\gamma_\varphi^*)_*(h_{\omega_0}) = h_\varphi^{\frac12}\gamma(\bar a_0^{-1})h_\varphi^{\frac12} = h_\varphi^{\frac12}\gamma(\bar a_0)^{-1}h_\varphi^{\frac12} = h_\varphi^{\frac12}\bar a^{-1}h_\varphi^{\frac12} = h_\omega. \tag{10}$$

Using (9), we obtain

$$Q_{r,r}((\gamma_\varphi^*)_*(h_{\omega_0})\|(\gamma_\varphi^*)_*(h_{\varphi_0})) = \|h_\varphi^{\frac{1}{2r}}\gamma(\bar a_0^{-1})h_\varphi^{\frac{1}{2r}}\|_r^r = \|h_{\varphi_0}^{\frac{1}{2r}}\bar a_0^{-1}h_{\varphi_0}^{\frac{1}{2r}}\|_r^r = Q_{r,r}(h_{\omega_0}\|h_{\varphi_0}),$$

which by the properties of the sandwiched Rényi divergence [? , Thm.] implies that γ_φ^* is sufficient with respect to $\{\omega_0, \varphi_0\}$. By Proposition 1 and the fact that the Petz dual $(\gamma_\varphi^*)_{\varphi_0}^*$ is γ itself, this is equivalent to

$$\gamma_*\circ(\gamma_\varphi^*)_*(h_{\omega_0}) = h_{\omega_0},$$

so that by (10),

$$(\gamma_\varphi^*)_* \circ \gamma_*(h_\omega) = (\gamma_\varphi^*)_* \circ \gamma_* \circ (\gamma_\varphi^*)_*(h_{\omega_0}) = (\gamma_\varphi^*)_*(h_{\omega_0}) = h_\omega.$$

Hence γ is sufficient with respect to $\{\omega, \varphi\}$. Let $\mathcal{E}$ be the faithful normal conditional expectation as in Proposition 1. Then $\mathcal{E}$ preserves both h_ω and h_φ, which by [**?**] implies that

$$h_\omega = \mathcal{E}_*(h_\omega) = h_\varphi^{\frac{1}{2}} \mathcal{E}(\bar{a}^{-1}) h_\varphi^{\frac{1}{2}},$$

so that $\mathcal{E}(\bar{a}^{-1}) = \bar{a}^{-1}$. It follows that

$$\left(h_\varphi^{\frac{1}{2r}} h_\psi^{\frac{1}{p}} h_\varphi^{\frac{1}{2r}} \right)^{1-\alpha} = h_\varphi^{\frac{1}{2r}} \bar{a}^{-1} h_\varphi^{\frac{1}{2r}} \in L_r(\mathcal{E}(\mathcal{M}))$$

and consequently $|h_\psi^{\frac{1}{2p}} h_\varphi^{\frac{1}{2r}}| \in L_{2z}(\mathcal{E}(\mathcal{M}))$. Note that by the assumptions $2z > 1$, so that we may use the multiplicativity properties of the extension of $\mathcal{E}$ [**?**]. Let

$$h_\psi^{\frac{1}{2p}} h_\varphi^{\frac{1}{2r}} = u |h_\psi^{\frac{1}{2p}} h_\varphi^{\frac{1}{2r}}|$$

be the polar decomposition in $L_{2z}(\mathcal{M})$, then we have

$$u^* h_\psi^{\frac{1}{2p}} h_\varphi^{\frac{1}{2r}} = \mathcal{E}_{2z}(u^* h_\psi^{\frac{1}{2p}} h_\varphi^{\frac{1}{2r}}) = \mathcal{E}_{2p}(u^* h_\psi^{\frac{1}{2p}}) h_\varphi^{\frac{1}{2r}},$$

which implies that

$$\mathcal{E}_p(h_\psi^{\frac{1}{p}}) = \mathcal{E}_p(h_\psi^{\frac{1}{2p}} u u^* h_\psi^{\frac{1}{2p}}) = h_\psi^{\frac{1}{2p}} u u^* h_\psi^{\frac{1}{2p}} = h_\psi^{\frac{1}{p}}$$

Consequently, $\psi \circ \mathcal{E} = \psi$ and γ is sufficient with respect to $\{\psi, \varphi\}$.

□

3.3.2 The case $\alpha > 1$

A Haagerup L_p-spaces

The following lemmas are well known, proofs are given for completeness.

Lemma A.1. *For any $0 < p < \infty$ and $\varphi \in \mathcal{M}_*^+$, $h_\varphi^{\frac{1}{2p}} \mathcal{M}^+ h_\varphi^{\frac{1}{2p}}$ is dense in $L_p(\mathcal{M})^+$ with respect to the (quasi)-norm $\|\cdot\|_p$.*

Proof. We may assume that φ is faithful. By [**?** , Lemma 1.1], $\mathcal{M} h_\varphi^{\frac{1}{2p}}$ is dense in $L_{2p}(\mathcal{M})$ for any $0 < p < \infty$. Let $y \in L_p(\mathcal{M})^+$, then $y^{\frac{1}{2}} \in L_{2p}(\mathcal{M})$, hence there is a sequence $a_n \in \mathcal{M}$ such that $\|a_n h_\varphi^{\frac{1}{2p}} - y^{\frac{1}{2}}\|_{2p} \to 0$. Then also

$$\|h_\varphi^{\frac{1}{2p}} a_n^* - y^{\frac{1}{2}}\|_p = \|(a_n h_\varphi^{\frac{1}{2p}} - y^{\frac{1}{2}})^*\|_p = \|a_n h_\varphi^{\frac{1}{2p}} - y^{\frac{1}{2}}\|_p \to 0$$

and

$$\|h_\varphi^{\frac{1}{2p}} a_n^* a_n h_\varphi^{\frac{1}{2p}} - y\|_p = \|(h_\varphi^{\frac{1}{2p}} a_n^* - y^{\frac{1}{2}}) a_n h_\varphi^{\frac{1}{2p}} + y^{\frac{1}{2}} (a_n h_\varphi^{\frac{1}{2p}} - y^{\frac{1}{2}})\|_p$$

Since $\|\cdot\|_p$ is a (quasi)-norm, the above expression goes to 0 by the Hölder inequality.

□

Lemma A.2. *Let $\psi, \varphi \in \mathcal{M}_*^+$ with $\psi \leq \varphi$. Then for any $a \in \mathcal{M}$ and $p \in [1, \infty)$,*

$$\mathrm{Tr}\left((a^* h_\psi^{\frac{1}{p}} a)^p\right) \leq Tr\left((a^* h_\varphi^{\frac{1}{p}} a)^p\right)$$

Proof. Since $1/p \in (p, 1]$, it follows (see [**?** , Lemma B.7] and [**?** , Lemma 3.2]) that $h_\psi^{1/p} \leq h_\varphi^{1/p}$ as τ-measurable operators affiliated with $\mathcal{M} \rtimes_{\sigma^\phi} \mathbb{R}$ (in which $L_p(\mathcal{M})$ lives). Hence $a^* h_\psi^{1/p} a \leq a^* h_\varphi^{1/p} a$ in the same sense. Therefore, by [**?** , Lemma 2.5 (iii), Lemma 4.8], we have the statement. □