Another note on equality in DPI for the BS relative entropy

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1 Equality conditions in QRE and BS-RE

Let $\mathcal{T}$ be a channel and let ρ, σ be states, σ invertible. According to [**? ?**], we have the following eqivalen conditions for equality in DPI.

QRE	BS-RE
$\sigma^{1/2}\mathcal{T}^*(\mathcal{T}(\sigma)^{-1/2}\mathcal{T}(\rho)\mathcal{T}(\sigma)^{1/2})\sigma^{1/2} = \rho$	$\sigma\mathcal{T}^*(\mathcal{T}(\sigma)^{-1}\mathcal{T}(\rho)) = \rho$
	$\sigma\mathcal{T}^*(\mathcal{T}(\sigma)^{-1}\mathcal{T}(\rho)^2\mathcal{T}(\sigma)^{-1})\sigma = \rho^2$
$\operatorname{Tr}\mathcal{T}(\rho)^{1/2}\mathcal{T}(\sigma)^{1/2} = \operatorname{Tr}\rho^{1/2}\sigma^{1/2}$	$\operatorname{Tr}\mathcal{T}(\rho)^2\mathcal{T}(\sigma)^{-1} = \operatorname{Tr}\rho^2\sigma^{-1}$
$\mathcal{T}^*(\mathcal{T}(\sigma)^{-1/2}\mathcal{T}(\rho)\mathcal{T}(\sigma)^{1/2}) = \sigma^{-1/2}\rho\sigma^{-1/2}$	$\mathcal{T}(\rho)\mathcal{T}(\sigma)^{-1}\mathcal{T}(\rho) = \rho\sigma^{-1}\rho$
$\sigma^{-1/2}\rho\sigma^{-1} \in \mathcal{F}_{(\mathcal{T}_\sigma\circ\mathcal{T})^*}$	$\sigma^{-1/2}\rho\sigma^{-1/2} \in \mathcal{M}_{\mathcal{T}_\sigma^*}$
$\sigma^{it-1/2}\rho\sigma^{-it-1/2} \in \mathcal{M}_{\mathcal{T}_\sigma^*}$, $\forall t \in \mathbb{R}$	

Proposition 1. *Assume that ρ_{ABC} is such that ρ_{AB} is invertible. Put $\eta_{AB} = \rho_B^{-1/2}\rho_{AB}\rho_B^{-1}$, $\eta_{BC} = \rho_B^{-1/2}\rho_{BC}\rho_B^{-1/2}$. The following are equivalent.*

(i) ρ_{ABC} *is a BS-QMC.*

(ii) $\rho_{ABC} = \rho_{AB}\rho_B^{-1}\rho_{BC}$.

(iii) η_{AB} *and* η_{BC} *commute, and* $\rho_{ABC} = \rho_B^{1/2}\eta_{AB}\eta_{BC}\rho_B^{1/2}$.

(iv) There is a decomposition and a unitary $U_B : \mathcal{H}_B \to \oplus_n \mathcal{H}_{B_L^n} \otimes \mathcal{H}_{B_R^N}$ *such that*

$$\rho_{ABC} = \rho_B^{1/2}U_B^*\left(\oplus_n \eta_{AB_L^n} \otimes \eta_{B_R^n C}\right)U_B\rho_B^{1/2}$$

for some $\eta_{AB_L^n} \in B(\mathcal{H}_{AB_L^n})^+$, $\eta_{B_R^n C} \in B(\mathcal{H}_{B_R^n C})^+$.

Moreover, a BS-QMC ρ_{ABC} is a QMC if and only if $\rho_B^{it}\eta_{AB}\rho_B^{-it}$ commutes with η_{BC} for all $t \in \mathbb{R}$.

Proof. The equivalence (i) $\iff$ (ii) was proved in []. If (ii) holds, then clearly $\rho_{ABC} = \rho_B^{1/2}\eta_{AB}\eta_{BC}\rho_B^{1/2} = \rho_{ABC}^*$. Since ρ_B is invertible, $[\eta_{AB}, \eta_{BC}] = 0$.

Assume (iii). Then η_{BC} commutes with all elements of the form

$$\eta_{AB}^{1/2} X_A \eta_{AB}^{1/2}, \qquad X_A \in B(\mathcal{H}_A).$$

Let $\Gamma(X_A) = \eta_{AB}^{1/2} X_A \eta_{AB}^{1/2}$, then η_{BC} must be in the commutant of $(\Gamma(B(\mathcal{H}_A)))$ in $B(\mathcal{H}_{ABC})$, which is equal to $\Gamma(B(\mathcal{H}_A))' \otimes B(\mathcal{H}_C)$. Since Γ defines a completely positive map $B(\mathcal{H}_A) \to B(\mathcal{H}_{AB})$, it follows by the Arveson commutant lifting theorem [**?** , 1.3.1] that any element $T_{AB} \in \Gamma(B(\mathcal{H}_A))'$ must commute with η_{AB} and be of the form $T_{AB} = I_A \otimes T_B$. Put

$$\mathcal{B} := \{T_B \in B(\mathcal{H}_B),\ T_B \text{ commutes with } \eta_{AB}\}, \tag{1}$$

then $\mathcal{B}$ is a *-subalgebra in $B(\mathcal{H}_B)$ and we must have $\eta_{BC} \in \mathcal{B} \otimes B(\mathcal{H}_C)$. It is also clear from the definition of $\mathcal{B}$ that $\eta_{AB} \in (I_A \otimes \mathcal{B})' = B(\mathcal{H}_A) \otimes \mathcal{B}'$.

For any subalgebra $\mathcal{B} \subseteq B(\mathcal{H}_B)$, there is a decomposition and a unitary U_B as in (iv) such that

$$\mathcal{B} = U_B \left(\oplus_n I_{B_L^n} \otimes B(\mathcal{H}_{B_R^n})\right) U_B^*, \qquad \mathcal{B}' = U_B \left(\oplus_n B(\mathcal{H}_{B_L^n}) \otimes I_{B_R^n}\right) U_B^*.$$

Since

$$\eta_{BC} \in (\mathcal{B} \otimes B(\mathcal{H}_C))^+ = U_B^* \left(\oplus_n I_{B_L^n} \otimes B(\mathcal{H}_{B_R^n C})^+\right) U_B,$$

we must have $\eta_{BC} = U_B^* \left(\oplus_n I_{B_L^n} \otimes \eta_{B_R^n C}\right) U_B$ for some $\eta_{B_R^n C} \in B(\mathcal{H}_{B_R^n C})^+$. Similarly, $\eta_{AB} = U_B^* \left(\oplus_n \eta_{AB_L^n} \otimes I_{B_R^n}\right) U_B$ for some $\eta_{AB_L^n} \in B(\mathcal{H}_{AB_L^n})^+$. The statement (iv) now follows from $\rho_{ABC} = \rho_B^{1/2}\eta_{AB}\eta_{BC}\rho_B^{1/2}$.

Suppose (iv) holds, then from

$$I_B = \mathrm{Tr}_{AC}\rho_B^{-1/2}\rho_{ABC}\rho_B^{-1/2} = U_B^* \left(\oplus_n \eta_{B_L^n} \otimes \eta_{B_R^n}\right) U_B$$

we infer that $\eta_{B_L^n} = I_{B_L^n}$ and $\eta_{B_R^n} = I_{B_R^n}$. It follows that $\rho_{AB} = \rho_B^{1/2} U_B^* \left(\oplus_n \eta_{AB_L^n} \otimes I_{B_R^n}\right) U_B \rho_B^{1/2}$ and similarly $\rho_{BC} = \rho_B^{1/2} U_B^* \left(\oplus_n I_{B_L^n} \otimes \eta_{B_R^n C}\right) U_B \rho_B^{1/2}$. The condition (ii) is immediate from this.

Assume now that ρ_{ABC} is a QMC. By [**?**], there is a decomposition and unitary $U_B : \mathcal{H}_B \to \oplus_n \mathcal{H}_{B_L^n} \otimes \mathcal{H}_{B_R^n}$, such that

$$\rho_{ABC} = U_B^* \left(\oplus_n p_n \rho_{AB_L^n} \otimes \rho_{B_R^n C}\right) U_B, \tag{2}$$

where $\rho_{AB_L^n} \in B(\mathcal{H}_{AB_L^n})$ and $\rho_{B_R^n C} \in B(\mathcal{H}_{B_R^n C})$ are states and $\{p_n\}_n$ is a probability distribution. It follows from this that

$$\rho_B = U_B^* \left(\oplus_n p_n \rho_{B_L^n} \otimes \rho_{B_R^n}\right) U_B, \tag{3}$$

and $\eta_{AB} = U_B^* \left(\oplus_n \rho_{B_L^n}^{-1/2} \rho_{AB_L^n} \rho_{B_L^n}^{-1/2} \otimes I_{B_R^n}\right) U_B$, $\eta_{BC} = U_B^* \left(\oplus_n I_{B_L^n} \otimes \rho_{B_R^n}^{-1/2} \rho_{B_R^n C} \rho_{B_R^n}^{-1/2}\right) U_B$. It is clear from this that ρ_{ABC} is a BS-QMC and that $\rho_B^{it}\eta_{AB}\rho_B^{-it}$ commutes with η_{BC} for all $t \in \mathbb{R}$.

For the converse, note that the condition implies that $\eta_{BC} \in \tilde{\mathcal{B}} \otimes B(\mathcal{H}_C)$, where

$$\tilde{\mathcal{B}} := \{T_B \in B(\mathcal{H}_B),\ T_B \text{ commutes with } \rho_B^{it}\eta_{AB}\rho_B^{-it},\ \forall t\}.$$

Then $\tilde{\mathcal{B}}$ is a subalgebra invariant under $\rho_B^{it} \cdot \rho_B^{-it}$. It also follows that $\eta_{AB} \in B(\mathcal{H}_A) \otimes \tilde{\mathcal{B}}'$, where the commutant $\tilde{\mathcal{B}}'$ is also invariant under $\rho_B^{it} \cdot \rho_B^{-it}$. Assume that $\tilde{\mathcal{B}}$ has a decomposition as in (**??**), then ρ_{ABC} has the form given in the statement (iv), but the invariance condition implies that ρ_B has the form (1). It folows that ρ_{ABC} has the form (**??**), so that it sis a QMC.

□