On the category of affine subspaces

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We present some important categories.

0.1 The category FinVect

Let FinVect be the category of finite dimensional real vector spaces with linear maps. Then $(\mathrm{FinVect}, \otimes, \mathbb{R})$ is a symmetric monoidal category, with the usual tensor product of vector spaces. With the usual duality $(-)^* : V \mapsto V^*$ of vector spaces, FinVect is compact closed. Put

$$e_U : U \otimes U^* \to \mathbb{R}, \qquad e_U(u \otimes u^*) = \langle u^*, u \rangle,$$

then e_U is the cap (counit) for the duality of U and U^*. The unit (cup) of the duality is

$$\eta_U : \mathbb{R} \to U^* \otimes U, \qquad \eta_U(1) = \sum_i e_i^* \otimes e_i,$$

where $\{e_i\}$ is a basis of U and $\{e_i^*\}$ the dual basis of U^*, determined by $\langle e_i^*, e_j \rangle = \delta_{ij}$. It is easily verified that η_U does not depend on the choice of the basis $\{e_i\}$. We will identify $\eta_U := \eta_U(1)$. Note that η_U is just e_U seen as an element in $(U \otimes U^*)^* = U^* \otimes U$.

By compactness, the internal hom is $[U, V] = U^* \otimes V$. Note that we have the usual identification

$$\langle f(u), v^* \rangle = \langle w, u \otimes v^* \rangle, \qquad u \in U, \ v^* \in V^* \tag{1}$$

between maps $f : U \to V$ and elements $w \in U^* \otimes V = [U, V]$, so the internal hom is identified with the set of all morphisms $U \to V$. The above relation of f and w can be also written as

$$f(u) = (e_U \otimes V)(u \otimes w), \qquad w = (f^* \otimes V)(\eta_V),$$

(we write V for the identity map id_V). Here f^* is the usual adjoint map $f^* : V^* \to U^*$.

The composition $\circ$ defines a map

$$[U, V] \otimes [V, W] \to [U, W],$$

using the identification $[U, V] = U^* \otimes V$, $\circ$ is identified with

$$U^* \otimes V \otimes V^* \otimes W \xrightarrow{U^* \otimes e_V \otimes W} U^* \otimes W.$$

Using symmetry and e_V, we can also define partial compositions through V in an obvious way. This can be depicted graphically in a nice way.

0.2 Affine subspaces

A subset $A \subseteq V$ of a finite dimensional vector space V is an affine subspace if $\sum_i \alpha_i a_i \in A$ whenever all $a_i \in A$ and $\sum_i \alpha_i = 1$. We say that A is proper if $0 \neq A$ and $A \neq \emptyset$. We will always mean that an affine subspace is proper (if not explicitly stated otherwise).

An affine subspace can be determined in two ways:

(i) Let $L \subseteq V$ be a linear subspace and $a_0 \notin L$. Then

$$A = a_0 + L$$

is a proper affine subspace. Note that $a_0 \in A$ and $A \cap L = \emptyset$. Conversely, any proper affine subspace A can be given in this way, with a_0 an arbitrary element in A and

$$L = Lin(A) := \{a_1 - a_2, \ a_1, a_2 \in A\} = \{a - a_0, \ a \in A\}.$$

(ii) Let $S \subseteq V$ be a linear subspace and $a_0^* \in V^* \setminus S^\perp$. Then

$$A = \{a \in S, \langle a_0^*, a \rangle = 1\}$$

is a proper affine subspace. Conversely, any proper affine subspace A is given in this way, with $S = span(A)$ and a_0^* an arbitrary element in

$$\tilde{A} = \{a^* \in V^*, \ \langle a^*, a \rangle = 1, \ \forall a \in A\}.$$

For an affine subspace A, $\tilde{A}$ is an affine subspace as well. If A is proper, then $\tilde{A}$ is proper and we have $\tilde{\tilde{A}} = A$. More generally, if $\emptyset \neq C \subseteq A$ is any subset of a proper affine subspace A, then $\tilde{C}$ is a proper affine subspace and $\tilde{\tilde{C}}$ is the smallest affine subspace containing C, that is,

$$\tilde{\tilde{C}} = \{\sum_i \alpha_i c_i, \ c_i \in C, \ \sum_i \alpha_i = 1\}.$$

In this case, we may write $\tilde{\tilde{C}}$ as

$$\tilde{\tilde{C}} = c_0 + Lin(C) = c_0 + span(\{c_1 - c_2, \ c_1, c_2 \in C\})$$

with an arbitrary element $c_0 \in C$, or as

$$\tilde{\tilde{C}} = \{c \in span(C), \ \langle a_0^*, c \rangle = 1\}$$

for an arbitrary element $a_0^* \in \tilde{A}$. We clearly have

$$Lin(\tilde{C}) = C^\perp = span(C)^\perp, \qquad Lin(C) = Lin(\tilde{\tilde{C}}) = \tilde{C}^\perp = span(\tilde{C})^\perp$$

and by duality also

$$span(C) = C^{\perp\perp} = Lin(\tilde{C})^\perp, \qquad span(\tilde{C}) = Lin(C)^\perp.$$

0.3 The category Af

The objects of Af are of the form $X = (V_X, A_X, a_X, \tilde{a}_X)$, where V_X is in FinVect, $A_X \subseteq V_X$ an affine subspace, $a_X \in A_X$ and $\tilde{a}_X \in \tilde{A}_X$ are some elements. Morphisms $X \to Y$ are linear maps $f : V_X \to V_Y$ such that $f(A_X) \subseteq A_Y$. Note that by definition A_X is proper for any object X. We may also add two special objects: the initial object $\emptyset := (\{0\}, \emptyset, -, 0)$ and the terminal object $0 := (\{0\}, \{0\}, 0, -)$, here the affine subspaces are obviously not proper.

For any object X, we also put

$$L_X := Lin(A_X) \qquad S_X := span(A_X), \qquad d_X := \dim(L_X), \qquad D_X := \dim(V_X).$$

Note that X is uniquely determined also when A_X is replaced by L_X or S_X.

0.3.1 Limits and colimits

Limits and colimits should be obtained from those in FinVect, we have to spectify the other structures and check whether the corresponding arrows are in Af.

First, note that $\{0\}$ is both initial and terminal in FinVect. In Af, it is easily seen that $\emptyset$ is **initial** and 0 is **terminal** in Af.

Let X, Y be two objects in Af. Assume first that both are proper. We define their **product** as

$$X \times Y := (V_X \times V_Y, A_X \times A_Y, (a_X, a_Y), \frac{1}{2}(\tilde{a}_X, \tilde{a}_Y)),$$

where

$$A_X \times A_Y := \{(x, y) \in V_X \times V_Y, \ x \in A_X, y \in A_Y\}$$

is the direct product of A_X and A_Y. It is easily verified that this is indeed an affine subspace and the usual projections $\pi_X : V_X \times V_Y \to V_X$ and $\pi_Y : V_X \times V_Y \to V_Y$ are in Af. Moreover, for $f : Z \to X$ and $g : Z \to Y$, the map $f \times g(z) = (f(z), g(z))$ is also clearly a morphism $Z \to X \times Y$ in Af. We have

$$L_{X \times Y} = L_X \times L_Y, \qquad S_{X \times Y} = (S_X \times S_Y) \wedge \{(\tilde{a}_X, -\tilde{a}_Y)\}^\perp.$$

We put $X \times \emptyset = \emptyset$, with the unique morphisms $\pi_X : \emptyset \to X$ and $\pi_\emptyset : \emptyset \to \emptyset$. If $Y \xrightarrow{f} X$ and $Y \xrightarrow{g} \emptyset$, then it is clear that $Y = \emptyset$, this shows that this is indeed the product. Further, put $X \times 0 = X$, with $\pi_X = id_X$ and $\pi_0 : X \xrightarrow{!} 0$. It is also readily verified that this is the product.

The **coproduct** for proper objects X, Y is defined as

$$X \oplus Y = (V_X \times V_Y, A_X \oplus A_Y, \frac{1}{2}(a_X, a_Y), (\tilde{a}_X, \tilde{a}_Y)),$$

where

$$A_X \oplus A_Y := \{(tx, (1-t)y), \ x \in A_X, y \in A_Y, \ t \in \mathbb{R}\}$$

is the direct sum. To check that this is an affine subspace, let $x_i \in A_X$, $y_i \in A_Y$, $s_i \in \mathbb{R}$ and let $\sum_i \alpha_i = 1$, then

$$\sum_i \alpha_i(s_i x_i, (1-s_i)y_i) = (\sum_i s_i \alpha_i x_i, \sum_i (1-s_i)\alpha_i y_i) = (sx, (1-s)y) \in A_X \oplus A_Y,$$

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where $s = \sum_i s_i \alpha_i$, $x = s^{-1} \sum_i s_i \alpha_i x_i$ if $s \neq 0$ and is arbitrary in A_X otherwise, similarly $y = (1 - s)^{-1} \sum_i (1 - s_i) \alpha_i y_i$ if $s \neq 1$ and is arbitrary otherwise. The usual embeddings $p_X : V_X \to V_X \times V_Y$ and $p_Y : V_Y \to V_X \times V_Y$ are easily seen to be morphsims in Af.

Let $f : X \to Z$, $g : Y \to Z$ be any morphisms in Af and consider the map $V_X \times V_Y \to Z$ given as $f \oplus g(u, v) = f(u) + g(v)$. We need to show that it preserves the affine subspaces. So let $x \in A_X$, $y \in A_Y$, then since $f(x), g(y) \in A_Z$, we have for any $s \in \mathbb{R}$,

$$f \oplus g(sx, (1 - s)y) = sf(x) + (1 - s)g(y) \in A_Z.$$

We also have

$$L_{X \oplus Y} = (L_X \times L_Y) \vee \mathbb{R}\{(a_X, -a_Y)\}, \qquad S_{X \oplus Y} = S_X \times S_Y.$$

Similarly as in the case of products, it is verified that $X \oplus \emptyset = X$ and $X \oplus 0 = 0$. (All the statements for coproducts can be obtained from duality defined below).

One can also discuss equalizers and coequalizers, here we only note that these may be trivial even for proper objects. We next consider pullbacks and pushouts, in some special cases that will be needed below.

Let X, Y be proper objects. We say that $X \xrightarrow{f} Y$ is an inclusion if the corresponding arrow in FinVect is an isomorphism and $f(a_X) = a_Y$, $f^*(\tilde{a}_Y) = \tilde{a}_X$. Assume that X, Y, Z_0, Z_1 are proper objects such that there are inclusions

$$Z_0 \xrightarrow{f_0} X \xrightarrow{f_1} Z_1, \qquad Z_0 \xrightarrow{g_0} Y \xrightarrow{g_1} Z_1.$$

Note that in particular $\psi_0 := f_0 \circ g_0^{-1}$ is an isomorphism of V_Y onto V_X and $\psi_0(a_Y) = a_X$, $\psi_0^*(\tilde{a}_X) = \tilde{a}_Y$. The **pushout** of f_0, g_0 is the object $X \sqcup_{f_0, g_0} Y = (V_X, A_{X \sqcup_{f_0, g_0} Y}, a_X, \tilde{a}_X)$, with

$$A_{X \sqcup_{f_0, g_0} Y} = \{sa + (1 - s)\psi_0(b), \ a \in A_X, \ b \in A_Y, \ s \in \mathbb{R}\},$$

together with the inclusions given by the linear maps $id : V_X \to V_X$ and $\psi_0 : V_Y \to V_X$. Indeed, these are clearly morphisms in Af, and we have

$$id \circ f_0 = f_0 = \psi_0 \circ g_0.$$

Also, if $X \xrightarrow{i} Z$ and $Y \xrightarrow{j} Z$ are such that $i \circ f_0 = j \circ g_0$, then $i = i \circ id$, $j = i \circ \psi_0$, so i is a morphism $X \sqcup_{f_0, g_0} Y \to Z$, obviously unique, with the required properties. We have

$$L_{X \sqcup_{f_0, g_0} Y} = L_X \vee \psi_0(L_Y), \qquad S_{X \sqcup_{f_0, g_0} Y} = S_X \vee \psi_0(S_Y).$$

Similarly, it can be shown that the **pullback** of f_1, g_1 is $X \sqcap_{f_1, g_1} Y = (V_X, A_X \cap \phi_1(A_Y), a_X, \tilde{a}_X)$ where $\phi_1 = f_1^{-1} g_1$, and the inclusions given by id_X and ϕ_1^{-1}.

0.3.2 Tensor products

Let X, Y be objects in Af. Let us define

$$A_{X \otimes Y} := \{x \otimes y, x \in A_X, y \in A_Y\}^{\approx}.$$

In other words, $A_{X \otimes Y}$ is the smallest affine subspace in $V_X \otimes V_Y$ containing $A_X \otimes A_Y$. For proper objects, we have

$$L_{X \otimes Y} = Lin(A_X \otimes A_Y) = span(\{x \otimes y - a_X \otimes a_Y, \ x \in A_X, \ y \in A_Y\})$$
$$= (a_X \otimes L_Y) + (L_X \otimes a_Y) + (L_X \otimes L_Y) \tag{2}$$

(here $+$ denotes the direct sum of subspaces). We also have

$$S_{X \otimes Y} = S_X \otimes S_Y.$$

Proof. Let $x \in A_X$, $y \in A_Y$, then

$$x \otimes y - a_X \otimes a_Y = a_X \otimes (y - a_Y) + (x - a_X) \otimes a_Y + (x - a_X) \otimes (y - a_Y),$$

so that $L_{X \otimes Y} = Lin(A_X \otimes A_Y)$ is contained in the subspace on the RHS of (2). Let d be the dimension of this subspace, then clearly

$$d_{X \otimes Y} \leq d \leq d_X + d_Y + d_X d_Y.$$

On the other hand, any element of S_X has the form tx for some $t \in \mathbb{R}$ and $x \in A_X$, so that it is easily seen that $S_X \otimes S_Y = S_{X \otimes Y}$. Hence

$$d_{X \otimes Y} = \dim(L_{X \otimes Y}) = \dim(S_{X \otimes X}) - 1 = \dim(S_X) \dim(S_Y) - 1 = (d_X + 1)(d_Y + 1) - 1$$
$$= d_X + d_Y + d_X d_Y.$$

This completes the proof.

$\square$

For X, Y in Af, put

$$X \otimes Y := (V_X \otimes V_Y, A_{X \otimes Y}, a_X \otimes a_Y, \tilde{a}_X \otimes \tilde{a}_Y).$$

Note that by this definition, $X \otimes \emptyset = \emptyset$ and $X \otimes 0 = 0$ unless $X = \emptyset$, in which case $\emptyset \otimes 0 = \emptyset$.

Also let $I := (\mathbb{R}, \{1\}, \{1\}, \{1\})$.

Lemma 1. $(\text{Af}, \otimes, I)$ *is a symmetric monoidal category.*

Proof. We have to check that for $X_1 \xrightarrow{f} Y_1$ and $X_2 \xrightarrow{g} Y_2$ in Af, we have $X_1 \otimes Y_1 \xrightarrow{f \otimes g} X_2 \otimes Y_2$, which amounts to showing that

$$(f \otimes g)(A_{X_1 \otimes Y_1}) \subseteq A_{X_2 \otimes Y_2}.$$

Let $x \in A_{X_1}$, $y \in A_{Y_1}$, then $f(x) \otimes g(y) \in A_{X_2} \otimes A_{Y_2} \subseteq A_{X_2 \otimes Y_2}$. Since $A_{X_1 \otimes Y_1}$ is the affine subspace generated by $A(X_1) \otimes A(Y_1)$, the above inclusion follows by linearity of $f \otimes g$.

It only remains to prove that the associators, unitors and symmetries from FinVect are morphisms in Af. Indeed, let $\alpha_{X,Y,Z} : V_X \otimes (V_Y \otimes V_Z) \to (V_X \otimes V_Y) \otimes V_Z$ be the associator in FinVect. We need to check that $\alpha_{X,Y,Z}(A_{X \otimes (Y \otimes Z)}) \subseteq A_{(X \otimes Y) \otimes Z}$. It is easily checked that $A_{X \otimes (Y \otimes Z)}$ is the affine span of elements of the form $x \otimes (y \otimes z)$, $z \in A_X$, $y \in A_Y$ and $z \in A_Z$, and we have

$$\alpha_{X,Y,Z}(x \otimes (y \otimes z)) = (x \otimes y) \otimes z \in A_{(X \otimes Y) \otimes Z}$$

for all such elements. The desired inclusion follows by linearity. The proof of the other inclusions is similar.

$\square$

0.3.3 Duality

We define $X^* := (V_X^*, \tilde{A}_X, \tilde{a}_X, a_X)$. Note that we have

$$L_{X^*} = S_X^\perp, \qquad S_{X^*} = L_X^\perp, {}' \qquad d_{X^*} = D_X - d_X - 1.$$

It is easily seen that $(-)^*$ defines a full and faithful functor $\text{Af}^{op} \to \text{Af}$, moreover, $X^{**} = X$ (if we us the canonical identification of any V in FinVect with its second dual).

Theorem 1. $(\mathrm{Af}, \otimes, I)$ *is a *-autonomous category, with duality* $(-)^*$, *such that* $I^* = I$.

Proof. We only need to check the natural isomorphisms

$$\mathrm{Af}(X \otimes Y, Z^*) \simeq \mathrm{Af}(X, (Y \otimes Z)^*).$$

Since FinVect is compact, we have the natural isomorphisms

$$\mathrm{FinVect}(V_X \otimes V_Y, V_Z^*) \simeq \mathrm{FinVect}(V_X, V_Y^* \otimes V_Z^*),$$

determined by the equalities

$$\langle\, f(x \otimes y), z \,\rangle = \langle\, h(x), y \otimes z \,\rangle, \qquad x \in V_X,\ y \in V_Y,\ z \in V_Z,$$

for $f \in \mathrm{FinVect}(V_X \otimes V_Y, V_Z^*)$ and $h \in \mathrm{FinVect}(V_Z, V_Y^* \otimes V_Z^*)$. Since $A_{X \otimes Y}$ is an affine span of A_X and A_Y, we see that f is in Af if and only if $f(x \otimes y) \in \tilde{A}_Z$, that is,

$$1 = \langle\, f(x \otimes y), z \,\rangle = \langle\, h(x), y \otimes z \,\rangle \qquad \forall x \in A_X,\ y \in A_Y,\ z \in A_Z.$$

But this is equivalent to

$$h(x) \in (A_Y \otimes A_Z)^{\check{}} = \tilde{A}_{Y \otimes Z}, \qquad \forall x \in A_X,$$

which means that $h \in \mathrm{Af}$.

$\square$

0.3.4 The dual tensor product

Let us define the dual tensor product by $\odot$, that is

$$X \odot Y = (X^* \otimes Y^*)^*.$$

We then have

$$L_{X \odot Y} = S_{X^* \otimes Y^*}^{\perp} = (S_{X^*} \otimes S_{Y^*})^{\perp} = (L_X^{\perp} \otimes L_Y^{\perp})^{\perp}$$
$$S_{X \odot Y} = L_{X^* \otimes Y^*}^{\perp} = (\tilde{a}_X \otimes S_Y^{\perp})^{\perp} \wedge (S_X^{\perp} \otimes \tilde{a}_Y)^{\perp} \wedge (S_X^{\perp} \otimes S_Y^{\perp})^{\perp}$$

In particular,

$$d_{X \odot Y} = D_X d_Y + d_X D_Y - d_X d_Y.$$

Lemma 2. *Let* X, Y *be nontrivial. Then* $X \otimes Y = X \odot Y$ *exactly in one of the following situations:*

1. $X \simeq I$ *or* $Y \simeq I$,

2. $d_X = d_Y = 0$,

3. $D_X = d_X + 1$ *and* $D_Y = d_Y + 1$ *(Objects with this property will be called first order).*

Proof. It is easy to see that (when identifying $X = X^{**}$), we have $A_X \otimes A_Y \subseteq \tilde{A}_{X^* \otimes Y^*}$, hence $A_{X \otimes Y} \subseteq A_{X \odot Y}$. We see from the above computatons that

$$d_{X \odot Y} - d_{X \otimes Y} = d_X(D_Y - 1) + (D_X - 1)d_Y - 2d_X d_Y \geq 0,$$

with equality if and only if the conditions of the lemma hold.

$\square$

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To each object $X = (V_X, A_X, a_X, \tilde{a}_X)$ we may define two object

$$X_{\min} := (V_X, \{a_X\}, a_X, \tilde{a}_X), \qquad X_{\max} := (V_X, \{\tilde{a}_X\}^\sim, a_X, \tilde{a}_X).$$

It is easily seen that $X_{\min} = (X_{\max}^*)^*$ and $X_{\max} = (X_{\min}^*)^*$, moreover, $X_{\max}$ and $(X_{\min})^*$ are first order objects. We have the inclusions

$$X_{\min} \xrightarrow{id} X \xrightarrow{id} X_{\max}.$$

We also have the inclusions

$$X \otimes Y \to X \odot Y \to X_{\max} \odot Y_{\max} = X_{\max} \otimes Y_{\max}$$

and

$$X \otimes Y \to X_{\max} \otimes Y \to X_{\max} \otimes Y_{\max}, \quad X \otimes Y \to X \otimes Y_{\max} \to X_{\max} \otimes Y_{\max}.$$

We can therefore define pullbacks and pushouts, which then becomes

$$(X_{\max} \otimes Y) \sqcap (X \otimes Y_{\max}) = X \otimes Y, \qquad (X_{\max} \otimes Y) \sqcup (X \otimes Y_{\max}) = X_{\max} \otimes Y_{\max}.$$

Hence we may decompose $X \odot Y$ into two parts

$$X \prec Y := (X \odot Y) \sqcap (X_{\max} \otimes Y), \qquad X \succ Y := (X \odot Y) \sqcap (X \otimes Y_{\max}).$$

Lemma 3. *We have*

$$L_{X \prec Y} = V_X \otimes L_Y + L_X \otimes a_Y, \qquad S_{X \prec Y} = V_X \otimes L_Y + S_X \otimes a_Y.$$

Similarly,

$$L_{X \succ Y} = L_X \otimes V_Y + a_X \otimes L_Y, \qquad S_{X \succ Y} = L_X \otimes V_Y + a_X \otimes S_Y.$$

Proof. We have

$$L_{X \prec Y} = L_{X \odot Y} \cap L_{X_{\max} \otimes Y} = (L_X^\perp \otimes L_Y^\perp)^\perp \cap (V_X \otimes L_Y + \{\tilde{a}_X\}^\perp \otimes a_Y).$$

Since $L_X^\perp \otimes L_Y^\perp \subseteq (V_X \otimes L_Y)^\perp$, we see that $V_X \otimes L_Y \subseteq L_{X \prec Y}$. On the other hand, let $x \in V_X$, then it is easy to see that $x \otimes a_Y \in L_{X \prec Y}$ if and only if $x \in L_X^{\perp\perp} = L_X$, indeed, since $\tilde{a}_Y \in S_{Y^*} = L_Y^\perp$, we must have

$$\langle x^*, x \rangle = \langle x^* \otimes \tilde{a}_Y, x \otimes a_Y \rangle = 0, \qquad \forall x^* \in L_X^\perp.$$

The proof for $\succ$ is similar.

$\square$

Note that these forms do not depend on the choice of the elements a_X, a_Y...!

Lemma 4. *We have*

$$X \otimes Y = (X \prec Y) \sqcap (X \succ Y), \qquad X \odot Y = (X \prec Y) \sqcup (X \succ Y).$$

Proof. We have

$$(X \prec Y) \sqcap (X \succ Y) = ((X \odot Y) \sqcap (X_{\max} \otimes Y)) \sqcap ((X \odot Y) \sqcap (X \otimes Y_{\max}))$$
$$= (X \odot Y) \sqcap ((X_{\max} \otimes Y) \sqcap (X \otimes Y_{\max})) = (X \odot Y) \sqcap (X \otimes Y) = X \otimes Y.$$

The second equality follows easily from Lemma 3

$\square$

We next show that $- \prec -$ and $- \succ -$ define a functor $\mathrm{Af} \times \mathrm{Af} \to \mathrm{Af}$. Let $X_1 \xrightarrow{f} Y_1$ and $X_2 \xrightarrow{g} Y_2$, we will show that $X_1 \prec Y_1 \xrightarrow{f \otimes g} X_2 \prec Y_2$. For this, we need to prove that $(f \otimes g)(A_{X_1 \prec X_2}) \subseteq A_{X_2 \prec Y_2}$ and $(f \otimes g)(A_{X_1 \succ X_2}) \subseteq A_{X_2 \succ Y_2}$. This is clear from Lemma 3.

Lemma 5. $(\mathrm{Af}, \prec, I)$ *is a monoidal category.*

Proof. It is easily checked from Lemma 3 that $\alpha_{X,Y,X}(L_{(X \prec Y) \prec Z}) = L_{X \prec (Y \prec Z)}$ and clearly also $\alpha_{X,Y,Z}(a_X \otimes a_Y \otimes a_Z) = a_X \otimes a_Y \otimes a_Z$, so that α is the associator. Since $I \otimes X = I \odot X$ and $X \odot I = X \otimes I$, we have $I \prec X = I \otimes X$ and $X \prec I = X \otimes I$, so λ and ρ are the unitors. But note that $\sigma_{X,Y}(A_{X \prec Y}) = A_{Y \succ X}$, so this structure is not symmetric. $\square$

We have $(X \prec Y)^* = X^* \prec Y^*$. Indeed, by duality,

$$(X \prec Y)^* = ((X \odot Y) \sqcap (X_{\max} \odot Y))^* = (X \odot Y)^* \sqcup (X_{\max} \otimes Y)^*$$
$$= (X^* \otimes Y^*) \sqcup (X^*_{\max} \odot Y^*)$$

0.3.5 Internal hom

The internal hom has the form

$$[X, Y] = (X \otimes Y^*)^* = X^* \odot Y. \tag{3}$$

We then have

$$L_{[X,Y]} = (S_X \otimes L_Y^\perp)^\perp = (V_X^* \otimes L_Y) \vee (L_{X^*} \otimes V_Y),$$
$$S_{[X,Y]} = (\tilde{a}_X \otimes S_Y^\perp)^\perp \wedge (L_X \otimes \tilde{a}_Y)^\perp \wedge (L_X \otimes S_Y^\perp)^\perp = (V_X^* \otimes L_Y) \vee (L_{X^*} \otimes V_Y) \vee (\tilde{a}_X \otimes a_Y).$$

and

$$d_{[X,Y]} = D_X D_Y - (d_X + 1)(D_Y - d_Y).$$

As we have seen in FinVect, the space $V_{[X,Y]} = V_X^* \otimes V_Y$ is identified with the space of all linear maps $V_X \to V_Y$, by (1). We will show that $A_{[X,Y]}$ corresponds to the affine subspace of maps mapping A_X into A_Y, that is, morphisms in Af. Indeed, we see from (1) that f is in Af if and only if

$$\langle f(x), y^* \rangle = \langle w, x \otimes y^* \rangle = 1, \qquad x \in A_X, \ y^* \in \tilde{A}_Y,$$

which is equivalent to $w \in (A_X \otimes \tilde{A}_Y)^\sim = \tilde{A}_{X \otimes Y^*}$.

0.3.6 The no signaling product

For two objects X, Y we define

$$X \prec Y := (V_X \otimes V_Y, A_{X \prec Y}, a_X \otimes a_Y, \tilde{a}_X \otimes \tilde{a}_Y),$$

where $A_{X \prec Y}$ is determined by

$$S_{X \prec Y} = V_X \otimes S_Y \cap S_{X \odot Y} = (V_X \otimes S_Y) \cap (L_{X \odot Y} \vee \{a_X \otimes a_Y\}) = (V_X \otimes S_Y) \cap ((V_X \otimes L_Y) \vee (L_X \otimes V_Y) \vee \{a_X \cdot a_Y\})$$

Lemma 6.

We may similarly define $X \succ Y$. Setting $f \prec g = f \otimes g$ for $X_1 \xrightarrow{f} X_2$, $Y_1 \xrightarrow{g} Y_2$, we see that $(- \prec -)$ is functorial. Indeed, to show that $X_1 \prec Y_1 \xrightarrow{f \otimes g} X_2 \prec Y_2$, we need to show that $f \otimes g(A_{X_1 \prec Y_1}) \subseteq A_{X_2 \prec Y_2}$. Assume $w \in A_{X_1 \prec Y_1}$, that is, $w \in V_{X_1} \otimes S_{Y_1}$ and $\langle w, \tilde{a}_{X_1} \otimes \tilde{a}_{Y_1} \rangle = 1$.

8

0.3.7 Dualizable (nuclear) objects

An object in Af is nuclear if the natural map $X^* \otimes X \to [X, X]$ is an isomorphism (santocanale). That is, the inclusion $X^* \otimes X \subseteq X^* \odot X$ that comes from the embedding

$$\tilde{A}_X \otimes A_X \subseteq (A_X \otimes \tilde{A}_X)^{\sim}$$

becomes an equality. As we have seen in Lemma 2, for proper objects we have $X^* \otimes X = X^* \odot X$ if and only if

$$d_X + 1 = D_X = D_{X^*} = d_{X^*} + 1 = D_X - d_X.$$

It follows that $d_X = 0$ and $D_X = 1$, so that $X \simeq I$. Hence the tensor unit is the unique dualizable (or nuclear) object in Af.

0.3.8 No signaling

We say that $X \xrightarrow{f} Y$ is no signaling if

$$y^* \circ f = \tilde{a}_Y \circ f, \qquad \forall y \in Y^* = [Y, I],$$

in other words

$$y \circ f = 0, \qquad \forall y^* \in L_{Y^*} = S_Y^{\perp}.$$

Taking $w \in A_{[X,Y]}$ be the corresponding elements, this means that

$$\langle\, w, x \otimes y^* \,\rangle = 0, \qquad \forall x \in V_X, \ y^* \in S_Y^{\perp},$$

in other words

$$w \in (V_X \otimes S_Y^{\perp})^{\perp} = V_X^* \otimes S_Y,$$

so that

$$w \in A_{[X,Y]} \cap (V_X^* \otimes S_Y).$$

Since $a_{[X,Y]} = \tilde{a}_X \otimes a_Y \in V_X^* \otimes S_Y$, we have that

$$a_{[X,Y]} - w \in L_{[X,Y]} \cap V_X^* \otimes S_Y = (S_X \otimes L_Y^{\perp})^{\perp} \cap V_X^* \otimes S_Y = (V_X^* \otimes L_Y) + (L_{X^*} \otimes a_Y).$$

We can also define no signaling in the oposite way, that is,

$$f(x) = f(a_X), \qquad \forall x \in A_X.$$

This is of course the same as

$$f(x) = 0, \qquad \forall x \in L_X,$$

or

$$\langle\, w, x \otimes y^* \,\rangle = 0, \qquad \forall x \in L_X, \ y^* \in V_Y^*,$$

that is,

$$w \in (L_X \otimes V_Y^*)^{\perp} = L_X^{\perp} \otimes V_Y = S_{X^*} \otimes V_Y.$$

It follows that

$$\tilde{a}_X \otimes a_Y - w \in L_{[X,Y]} \cap S_{X^*} \otimes V_Y = L_{X^*} \otimes V_Y + \tilde{a}_X \otimes L_Y.$$

0.4 Once more on the monoidal structures

0.4.1 Tensor product

We have

$$L_{X \otimes Y} = (a_X \otimes L_Y) + (L_X \otimes a_Y) + (L_X \otimes L_Y) = (S_X \otimes L_Y) + (L_X \otimes a_Y) = (L_X \otimes S_Y) + (a_X \otimes L_Y)$$

A closed symmetric monoidal structure. We have

$$L_{X_{\max} \otimes Y} = V_X \otimes L_Y + \{\tilde{a}_X\}^\perp \otimes a_Y, \qquad L_{X_{\min} \otimes Y} = a_X \otimes L_Y.$$

Lemma 7. *We have*
$$X \otimes Y = (X_{\max} \otimes Y) \sqcap (X \otimes Y_{\max}).$$

Proof. This is easy, since

$$S_{X_{\max} \otimes Y} \cap S_{X \otimes Y_{\max}} = (V_X \otimes S_Y) \cap (S_X \otimes V_Y) = S_X \otimes S_Y.$$

$\square$

0.4.2 Dual product

By definition, $X \odot Y = (X^* \otimes Y^*)^*$. We have

$$L_{X \odot Y} = (L_X^\perp \otimes L_Y^\perp)^\perp = (V_X \otimes L_Y) \vee (L_X \otimes V_Y).$$

We have
$$L_{X_{\max} \odot Y} = (V_X \otimes L_Y) \vee (\{\tilde{a}_X\}^\perp \otimes V_Y), \qquad L_{X_{\min} \odot Y} = V_X \otimes L_Y.$$

Lemma 8. *We have*

$$X_{\max} \otimes Y_{\max} = (X_{\max} \odot Y) \sqcup (X \odot Y_{\max})$$
$$X \odot Y = (X_{\min} \odot Y) \sqcup (X \odot Y_{\min})$$
$$X \otimes Y = (X_{\min} \otimes Y) \sqcup (X \otimes Y_{\min}) \sqcup (X_{\min} \odot Y \sqcap X \odot Y_{\min})$$

Proof. The first is easy, the seconf follows from Lemma 7 by duality, the third is also easy.

$\square$

0.5 The no signalling product

Let us define $X \prec Y := (X \odot Y) \sqcap (X_{\max} \otimes Y)$. We have

$$L_{X \prec Y} = V_X \otimes L_Y + L_X \otimes a_Y = L_{X_{\min} \odot Y} + L_{Y \otimes Y_{\min}}.$$

So that
$$X \prec Y := (X \odot Y) \sqcap (X_{\max} \otimes Y) = (X_{\min} \odot Y) + (X \otimes Y_{\min})$$

Lemma 9. *We have*

$$(X \otimes Y) \sqcup (X_{\min} \odot Y) = (X \odot Y) \sqcap (X_{\max} \otimes Y)$$
$$= (X_{\min} \odot Y) + (X \otimes Y_{\min}) = (X_{\max} \otimes Y) \sqcap (X \odot Y_{\max}).$$

Let us denote the above object by $X \prec Y$. Then $A_{X \prec Y}$ is the set of elements in $V_X \otimes V_Y$ such that $\langle w, \cdot \otimes y^* \rangle$ is a fixed element in A_X, independently of $y^* \in \tilde{A}_Y$.

Blbe uvedenie, definicia!

Proof. We see that $A_{X \prec Y} \subseteq A_{X \odot Y}$, moreover,

$$A_{X \prec Y} = \{w \in A_{X \odot Y}, \ \langle w, id_X \otimes y^* \rangle = 0, \ \forall y^* \in L_{Y^*}\}.$$

In other words, since clearly $a_X \otimes a_Y \in A_{X \prec Y}$,

$$
\begin{aligned}
L_{X \prec Y} &= \{w - a_X \otimes a_Y, \ w \in A_{X \prec Y}\} = L_{X \odot Y} \cap (V_X^* \otimes L_Y^*)^\perp = (L_X^\perp \otimes S_{Y^*})^\perp \cap (V_X^* \otimes L_{Y^*})^\perp \\
&= ((L_X^\perp \otimes S_{Y^*}) \vee (V_X^* \otimes L_{Y^*}))^\perp = ((V_X^* \otimes L_{Y^*}) + (S_X^* \otimes \tilde{a}_Y))^\perp = S_{(X \prec Y)^*}^\perp
\end{aligned}
$$

But also

$$
\begin{aligned}
L_{X \odot Y} \cap (V_X^* \otimes L_Y^*)^\perp &= ((V_X \otimes L_Y) \vee L_X \otimes V_Y) \cap (V_X \otimes S_Y) \\
&= ((V_X \otimes L_Y) + (L_X \otimes a_Y) + (L_X \otimes S_Y')) \cap ((V_X \otimes L_Y) + (L_X \otimes a_Y) + (L_X' \otimes a_Y)) \\
&= ((V_X \otimes L_Y) + (L_X \otimes a_Y).
\end{aligned}
$$

First note that we have

$$L_{X_{\min} \odot Y} = V_X \otimes L_Y, \qquad L_{X \otimes Y_{\min}} = L_X \otimes a_Y$$

and therefore

$$L_{X \otimes Y_{\min}} \cap L_{X_{\min} \odot Y} = \{0\}.$$

Further,

$$L_{X \otimes Y} = (S_X \otimes L_Y) + (L_X \otimes a_Y) = (S_X \otimes L_Y) + L_{X \otimes Y_{\min}}$$

and

$$L_{X \odot Y} = (V_X \otimes L_Y) \vee (L_X \otimes V_Y), \qquad L_{X_{\max} \otimes Y} = (V_X \otimes L_Y) + (\{\tilde{a}_X\}^\perp \otimes a_Y)$$

We have

$$S_{(X \otimes Y) \sqcup (X_{\min} \odot Y)} = S_{X \otimes Y} \vee S_{X_{\min} \odot Y} = S_{X \otimes Y} \vee L_{X_{\min} \odot Y}$$

$\square$

0.6 The category AfH

The category AfH will be constructed as a subcategory in Af.

0.6.1 First order objects

It is easily seen that the following are equivalent:

1. $D_X = d_X + 1$;

2. $S_X = V_X$;

3. $L_X = \{\tilde{a}_X\}^\perp$;

4. $S_{X^*} = \mathbb{R}\tilde{a}_X$;

5. $L_{X^*} = \{0\}$.

We say that an object X is first order if any of these conditions is fulfilled. We have seen that for proper objects, $X \otimes Y = X \odot Y$ if and only if both X and Y are first order. We also have

Lemma 10. *X is first order if and only if $X \otimes X = X \odot X$.*

'

Lemma 11. *Let X, Y be first order, then $X \otimes Y$ is first order.*

Proof. We have
$$S_{X \otimes Y} = S_X \otimes S_Y = V_X \otimes V_Y = V_{X \otimes Y}.$$
□

0.6.2 Channels

A channel is an object $[X, Y]$ where X and Y are first order. As we have seen,
$$X^* \otimes Y \subseteq X^* \odot Y = [X, Y].$$

If X is first order, $\tilde{A}_X = \{\tilde{a}_X\}$ and the elements of $A_{X^* \otimes Y} = \tilde{a}_X \otimes A_Y$ are identified with channels of the form
$$f(x) = \langle \tilde{a}_X, x \rangle y, \qquad x \in V_X,$$
for some $y \in A_Y$. Such maps will be called replacement channels.

Lemma 12. *Let X, Y be first order and let $w \in V_X^* \otimes V_Y$. Then $w \in A_{[X,Y]}$ if and only if*
$$\circ_Y : w_{X^* Y} \otimes \tilde{a}_Y \mapsto \tilde{a}_X.$$

Proof. Let $f : V_X \to V_Y$ be the map corresponding to w, then
$$\circ_Y(w \otimes \tilde{a}_Y) = (V_X^* \otimes e_{V_Y})(w \otimes \tilde{a}_Y) = \tilde{a}_Y \circ f,$$
where $\tilde{a}_Y \in V_Y^*$ is seen as a map $V_y \to \mathbb{R}$. So $\tilde{a}_Y \circ f : V_X \to \mathbb{R}$ is an element in V_X^*. We know that $w \in A_{[X,Y]}$ iff $f(A_X) \subseteq A_Y$, which is equivalent to $\tilde{a}_Y \circ f(x) = 1$ for all $x \in A_X$, so that $\tilde{a}_Y \circ f \in \tilde{A}_X = \{\tilde{a}_X\}$, since X is first order. □

Lemma 13. *Let Y be first order and $w \in V_X^* \otimes V_Y$. Then $w \in A_{[X,Y]}$ if and only if*
$$\circ_Y(w_{X^* Y} \otimes \tilde{a}_Y) \in \tilde{A}_X.$$

Moreover,
$$\tilde{A}_{[X,Y]} = A_X \otimes \{\tilde{a}_Y\}.$$

Proof. Since Y is first order, we have $A_{Y^*} = \tilde{A}_Y = \{\tilde{a}_Y\}$ and by (3)
$$\tilde{A}_{[X,Y]} = A_{X \otimes Y^*} = A_X \otimes \{\tilde{a}_Y\}.$$

As in the above proof, let $f : V_X \to V_Y$ be the map corresponding to w. Then $\tilde{a}_Y \circ f \in V_X^*$ and $w \in A_{[X,Y]}$ iff $f(A_X) \subseteq A_Y$. This means that
$$\tilde{a}_Y \circ f(x) = 1, \qquad \forall x \in A_X,$$
which means that $\tilde{a}_Y \circ f \in \tilde{A}_X$.

□

0.6.3 AfH

The category AfH is the full subcategory in Af created from first order objects by taking tensor products and duals. We will add more later. We will use the notation V_{XY^*} for $V_X \otimes V_Y^*$, etc.

Any object X in AfH is created from first order objects $X_1, \ldots, X_k$, so that $V_X = \tilde{V}_{X_1} \otimes \cdots \otimes \tilde{V}_{X_k}$, where $\tilde{V}_{X_i}$ is either V_{X_i} or $V_{X_i}^*$, $i = 1, \ldots, k$. We will next show that any object is a set of channels that contains all replacement channels.

Proposition 1. *Let X be an object in* AfH*. Then there are first order objects Y_I and Y_O and inclusions f, g such that*

$$Y_I^* \otimes Y_O \xrightarrow{f} X \xrightarrow{g} [Y_I, Y_O]. \tag{4}$$

Proof. Let X be first order, then since I is first order,

$$I^* \otimes X = I \otimes X \xrightarrow{\lambda_X} X \xrightarrow{\lambda_X^{-1}} I \otimes X = I \odot X = [I, X].$$

Clearly, $f = \lambda_X$ and $g = \lambda_X^{-1}$ are inclusions. Now assume that Z satisfies (4) and let $X = Z^*$. Taking duals and composing with symmetries, we get

$$Y_O^* \otimes Y_I \xrightarrow{\sigma_{V_{Y,O}^*, V_{Y_I}}} Y_I \otimes Y_0^* = [Y_I \otimes Y_O]^* \xrightarrow{g^*} X \xrightarrow{f^*} (Y_I^* \otimes Y_O)^* \xrightarrow{\sigma_{V_{Y_I}, V_{X_O}^*}} (Y_O \otimes Y_I)^* = [Y_O, Y_I].$$

Since the compositions of f^* and g^* with symmetries are inclusions, we see that X satisfies (4).

Next, let X_1 and X_2 satisfy (4) with some first order objects Y_I^i, Y_O^i and inclusions f^i, g^i, $i = 1, 2$, and let $X = X_1 \otimes X_2$. We then have, using the appropriate symmetries

$$Y_I^1 Y_I^2 \otimes (Y_O^1 Y_O^2)^* \xrightarrow{\sigma_{Y_I^2, Y_O^1}} Y_I^1 \otimes (Y_O^1)^* \otimes Y_I^2 \otimes (Y_O^1)^* \xrightarrow{f^1 \otimes f^2} X \xrightarrow{g^1 \otimes g^2} [Y_I^1, Y_O^1] \otimes [Y_I^2, Y_O^2] \xrightarrow{\sigma_{Y_O^1, Y_O^2}} [Y_I^1 Y_I^2, I_O^1 Y_O^2].$$

Perhaps the last arrow needs some checking, so lt us do it properly. We need to show that for $w \in A_{[Y_I^1, Y_O^1] \otimes [Y_I^2, Y_O^2]}$, we have $\sigma_{Y_O^1, Y_O^2}(w) \in A_{[Y_I^1 Y_I^2, I_O^1 Y_O^2]}$, but this is clear using Lemma 12. $\qquad \square$

The pair (Y_I, Y_O) for an object X will be called the setting of X. For objects of the same setting we may take pullbacks and pushouts of the corresponding inclusions.

Pullbacks are intersections, pushouts the affine mixture.

Channels into (from) products and coproducts

We define AfH as the full subcategory of Af containing all first order objects and closed under (finite products,) duals and tensor products .