J. Hwang, M. Jeong, S. Kim: Right Rényi mean and tensor product
Referee report

Matrix means appear are important in many areas, including matrix analysis, optimization, quantum information theory etc. The present paper studies the weighted right Rényi mean obtained from the quantum divergence $\Phi_{\alpha,z}$ which is a generalization of the Bures-Wasserstein metric and is related to the family of $\alpha - z$ -Rényi relative entropies. This is an important topic interesting for a wider audience.

The weighted right Rényi mean was obtained and characterized in Ref. [9] as a solution of certain matrix equation, the present paper discusses some of its properties: e.g. homogeneity, invariance under permutations and unitary conjugation, and behaviour with respect to tensor products. These properties are indeed expected to hold for any reasonable matrix mean, but the proofs are very easy or trivial, following directly from the characterizing equality and well known properties of the Kronecker (tensor) product of matrices. I am somewhat in doubt whether the content warrants publication. The authors should write some more about the importance of these properties and/or their applications.

The presentation is mostly clear, apart from some small language issues and some confusing statements, see the list below.

Specific remarks

1. The abstract (and elsewhere): "...we give the identity of two right Rényi means..." this is a strange sentence, it is better to write something like ,,... we give the expression for the tensor product of two right Rényi means"

2. p.2, second paragraph: "Recently..." remove one redundant "recently" from this sentence.

3. p.2, line 2 from below: "...positive probability vectors" does this means -strictly- positive probability vectors? Otherwise, just "probability vectors" is enough.

4. p.4, line 11: "...we obtain in [9]" better write "...it was obtained in [9]"

5. p.4, line 10 from below: "permutation on a n-letters" remove "a"

6. p.6, line 3: "...which is proved" better write "... which proves the statement." Similarly at the end of the proof of (6) on the same page: "...it is proved" could be replaced by "... the statement is proved" or "the proof is finished", etc.

7. the statement of Proposition 2.4 (1): this is very strange, since this would suggest that the equality holds for all X. This is better stated as that "the solution of $X = R_{\alpha,z}(\omega; A_1, \ldots, A_{n-1}, X)$ is .... ".

8. p.9, line 8: "It has been known ..." what is the meaning of this sentence? Does this mean that $\Phi_{\alpha,z}(X, A_i)$ is never strictly convex? Or only for some values of α and z? It is known that $Q_{\alpha,z}$ is -jointly- concave for e.g. $1/2 \leq \alpha \leq z \leq 1$, so it is also concave in the first variable. In any case, the authors should expand on their statement (and also support it by some examples or references).

Conclusion

The topic of the paper is interesting and important, although the results are rather easily obtained. The authors should emphasise the importance of the obtained properties and also address the issues in the list.